

SOLUTIONS

Name: _____

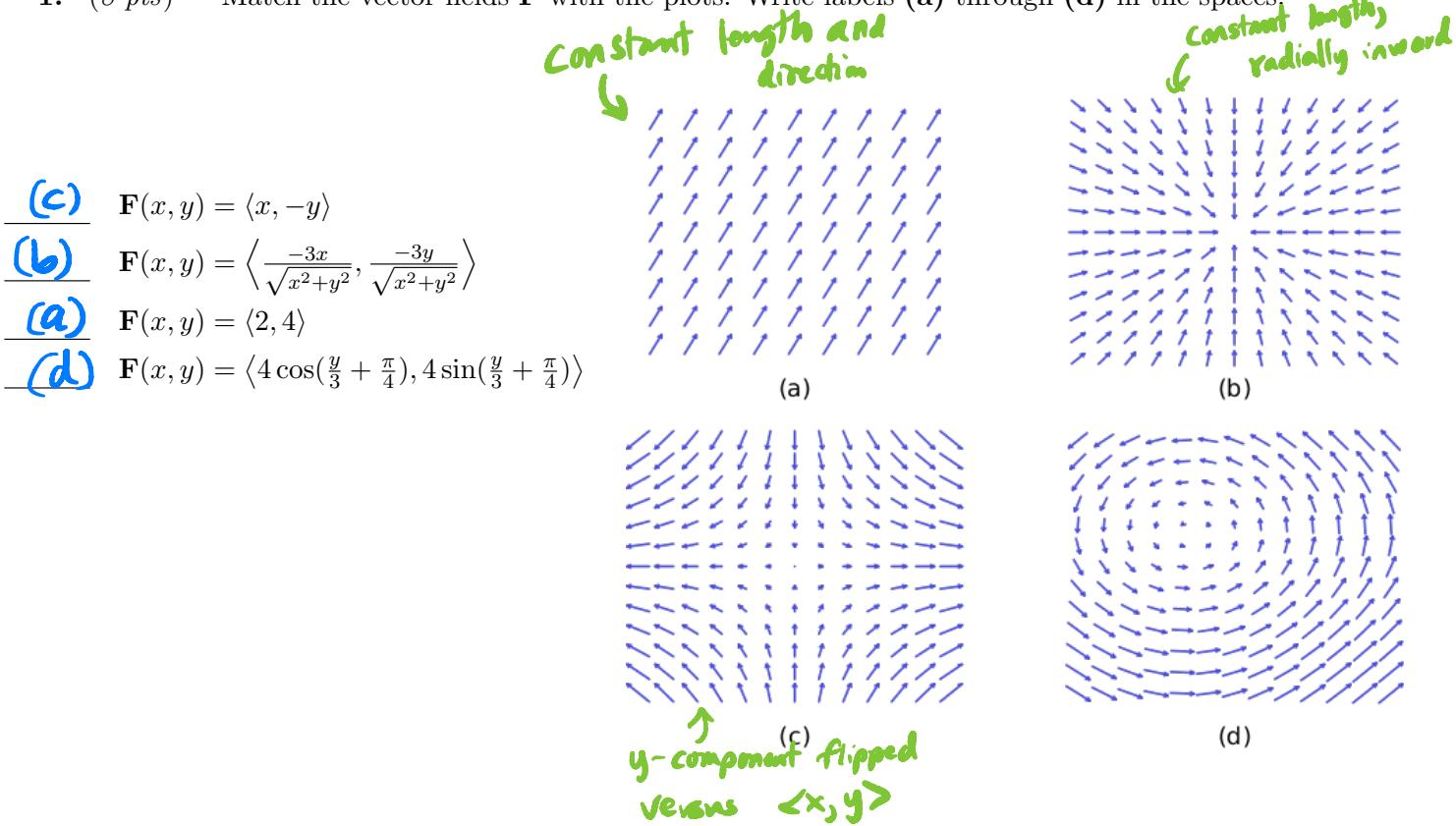
Math 253 Calculus III (Bueler)

Wednesday, 3 May 2023

Final Exam

No book, electronics, calculator, or internet access. $\frac{1}{2}$ sheet of notes allowed; double-sided okay! 125 points possible. 120 minutes maximum.

1. (8 pts) Match the vector fields \mathbf{F} with the plots. Write labels (a) through (d) in the spaces.



2. (10 pts) Determine whether \mathbf{F} is a conservative vector field. If it is, find f so that $\mathbf{F} = \nabla f$.

$$\mathbf{F}(x, y) = (ye^x + \sin y)\mathbf{i} + (e^x + x \cos y)\mathbf{j}$$

$$P, Q$$

$$P_y = e^x + \cos y \quad Q_x = e^x + \cos y$$

yes, conservative

$$f_x = P = ye^x + \sin y$$

$$f(x, y) = ye^x + x \sin y + g(y)$$

$$e^x + x \cos y = Q = f_y = e^x + x \cos y + g'(y) \Leftrightarrow g'(y) = 0$$

$$\Leftrightarrow g(y) = C$$

$$f(x, y) = ye^x + x \sin y$$

"+C" {optional}

$$P = x^2 + y, \quad Q = 3x - y^2$$

3. (10 pts) Suppose $\mathbf{F}(x, y) = \langle x^2 + y, 3x - y^2 \rangle$. Calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for any positively-oriented, closed, simple curve C enclosing a region D which has area 6. (Hint. Green's Theorem)

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P dx + Q dy \stackrel{GT}{=} \iint_D Q_x - P_y dA$$

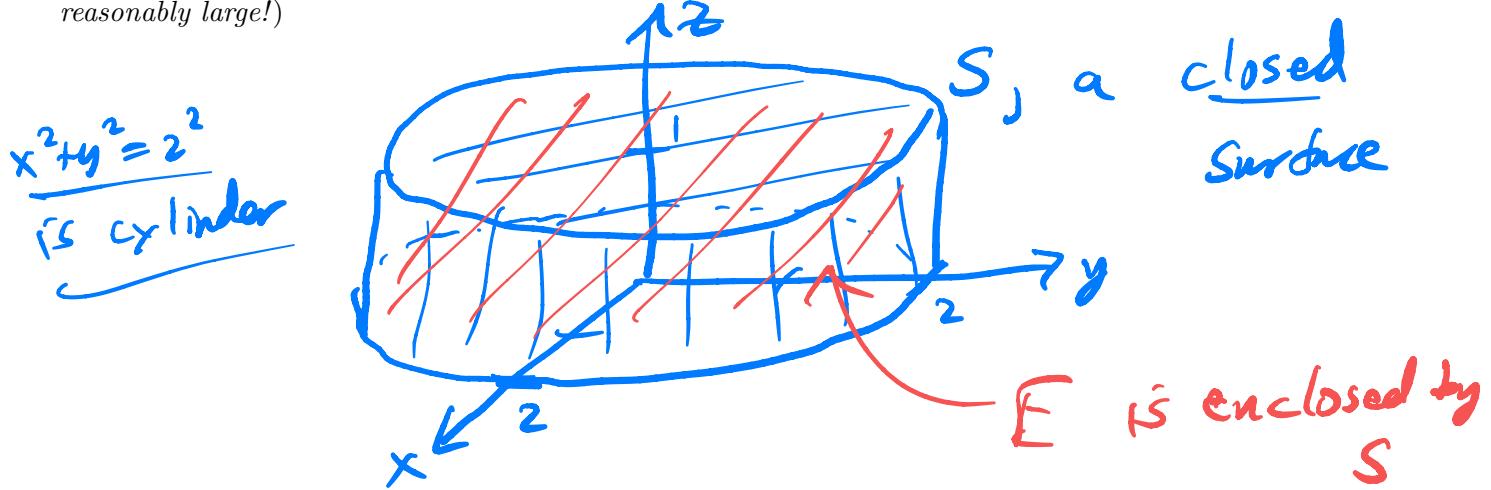
$$= \iint_D 3 - 1 dA = 2 A_D = \boxed{12}$$

4. (10 pts) Evaluate the line integral if C is the line segment from $(0, 0)$ to $(5, 4)$:

$$\begin{aligned} \int_C xe^y ds &= \int_0^1 x(t) e^{y(t)} \frac{ds}{dt} dt & \vec{r}(t) = t \langle 5, 4 \rangle \\ &= \int_0^1 5t e^{4t} \sqrt{25+16} dt & 0 \leq t \leq 1 \\ &= 5\sqrt{41} \int_0^1 t e^{4t} dt & \|\vec{r}'(t)\| = \sqrt{25+16} \\ &= 5\sqrt{41} \left[\frac{1}{4} t e^{4t} \right]_0^1 - \int_0^1 \frac{1}{4} e^{4t} dt & \text{integrate by parts: } u = t, v = \frac{1}{4} e^{4t}, du = dt, dv = e^{4t} dt \\ &= 5\sqrt{41} \left(\frac{1}{4} e^4 - \frac{1}{16} [e^{4t}]_0^1 \right) = 5\sqrt{41} \left(\frac{1}{4} e^4 - \frac{1}{16} e^4 + \frac{1}{16} \right) \\ &= \boxed{5\sqrt{41} \left(\frac{3}{16} e^4 + \frac{1}{16} \right)} \end{aligned}$$

5. Consider the closed surface S which is formed by the equations $x^2 + y^2 = 4$, $z = 0$, and $z = 1$.

(a) (4 pts) Sketch the surface S . (Remember to label the axes and indicate scale on each axis. Sketch reasonably large!)



- (b) (6 pts) Now suppose $\mathbf{F} = \langle x, y, z^2 - 1 \rangle$ is a vector field. Use the divergence theorem to compute

$$\iint_S \mathbf{F} \cdot \mathbf{N} dS = \iiint_E \nabla \cdot \tilde{\mathbf{F}} dV = \iiint_E 1+z dV$$

$$= 2 \int_0^{2\pi} \int_0^2 \int_0^1 1+z r dz dr d\theta = 4\pi \left(\int_0^2 r dr \right) \left(\int_0^1 1+z dz \right)$$

$$= 4\pi \cdot \frac{2^2}{2} \cdot \left[z + \frac{z^2}{2} \right]_0^1 = 8\pi \left(1 + \frac{1}{2} \right) = 8\pi \cdot \frac{3}{2}$$

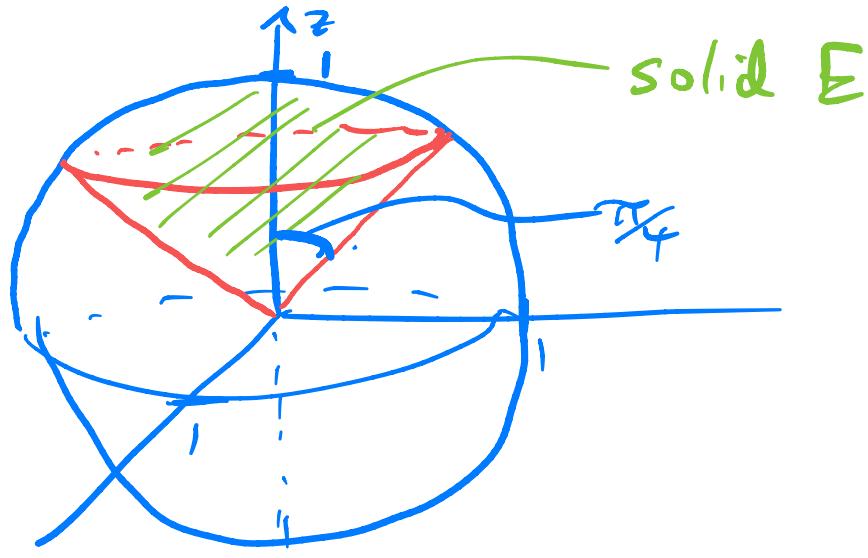
$$\neq 12\pi$$

6. (8 pts) Find a unit vector that is orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$.

$$\vec{v} = \langle 1, 1, 0 \rangle \times \langle 1, 0, 1 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \langle 1, -1, -1 \rangle$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1, -1, -1 \rangle}{\sqrt{3}} = \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$$

7. (a) (4 pts) Sketch the solid which is above the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 1$. (Label the axes and indicate scale on each axis. Sketch reasonably large!)



- (b) (6 pts) Use spherical coordinates to find the volume of the solid in part (a).

$$\begin{aligned}
 V &= \iiint_E 1 \, dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \quad \text{dV in spherical} \\
 &= 2\pi \left(\int_0^1 \rho^2 \, d\rho \right) \left(\int_0^{\pi/4} \sin \phi \, d\phi \right) \\
 &= 2\pi \cdot \frac{1}{3} \cdot [-\cos \phi]_0^{\pi/4} = \frac{2\pi}{3} \left(-\frac{1}{\sqrt{2}} + 1 \right) \\
 &= \boxed{\frac{2\pi}{3} \left(1 - \frac{1}{\sqrt{2}} \right)}
 \end{aligned}$$

8. (10 pts) Find an equation of the plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$. Simplify to the form $ax + by + cz + d = 0$.

$$\vec{n} = \langle 5, -1, -1 \rangle$$

plane: $\vec{n} \cdot (\vec{r} - \vec{r}_0) = \langle 5, -1, -1 \rangle \cdot \langle x-1, y+1, z+1 \rangle = 0$

$$\Leftrightarrow 5(x-1) - (y+1) - (z+1) = 0$$

$$\Leftrightarrow 5x - y - z - 7 = 0$$

9. Suppose $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} + (z^2 + xy)\mathbf{k}$.

- (a) (5 pts) Compute and simplify the divergence $\nabla \cdot \mathbf{F}$.

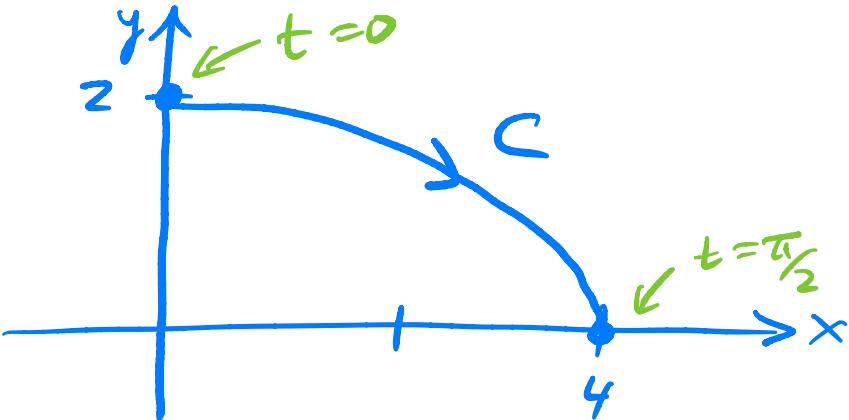
$$\nabla \cdot \vec{F} = 1 + 2y + 2z$$

- (b) (5 pts) Compute and simplify the curl $\nabla \times \mathbf{F}$.

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y^2 & z^2 + xy \end{vmatrix} = (x-0)\hat{i} - (y-0)\hat{j} + (0-0)\hat{k} \\ &= \langle x, -y, 0 \rangle \end{aligned}$$

10. (a) (5 pts) Sketch the plane curve C with the given vector equation, indicating its orientation. (Label the axes and indicate scale on each axis. Sketch reasonably large!)

$$\mathbf{r}(t) = 4 \sin t \mathbf{i} + 2 \cos t \mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2}$$



- (b) (5 pts) Compute the unit tangent vector field $\mathbf{T}(t)$ for the curve C in part (a).

$$\begin{aligned}\vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}'(t)}{\|\vec{\mathbf{r}}'(t)\|} = \frac{\langle 4\cos t, -2\sin t \rangle}{\sqrt{4^2\cos^2 t + 2^2\sin^2 t}} \quad \left. \begin{array}{l} \text{either} \\ \text{is} \\ \text{fine} \end{array} \right\} \\ &= \frac{1}{\sqrt{3\cos^2 t + 1}} \langle 2\cos t, -\sin t \rangle\end{aligned}$$

11. (9 pts) Compute and simplify the linearization $L(x, y)$ of the function at the point:

$$f(x, y) = \sqrt{xy}, \quad (1, 4)$$

$$\begin{aligned}f = (xy)^{1/2} \Rightarrow f_x &= \frac{1}{2}(xy)^{-1/2} \cdot y = \frac{\sqrt{y}}{2\sqrt{x}} \\ f_y &= \frac{1}{2}(xy)^{-1/2} \cdot x = \frac{\sqrt{x}}{2\sqrt{y}}\end{aligned}$$

$$\begin{aligned}L(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0) \\ &= 2 + \frac{\sqrt{2}}{2 \cdot 1}(x-1) + \frac{1}{2 \cdot 2}(y-2) \\ &= \boxed{(2 + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{4}(y-2))}\end{aligned}$$

12. Suppose

$$w = xy + yz + zx, \quad x = r \cos \theta, \quad y = r \sin \theta, \quad z = r\theta$$

(a) (5 pts) State the chain rule for $\frac{\partial w}{\partial \theta}$ which applies in this case.

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial \theta}$$

(b) (5 pts) Compute $\frac{\partial w}{\partial \theta}$ when $r = 2$ and $\theta = \frac{\pi}{2}$. Simplify the answer to a number.

$$\frac{\partial w}{\partial x} = y+z, \quad \frac{\partial w}{\partial y} = x+z, \quad \frac{\partial w}{\partial z} = y+x \quad | \quad \begin{array}{l} x=0 \\ y=2 \\ z=\pi \end{array}$$

$$\begin{aligned} \therefore \left(\frac{\partial w}{\partial \theta} \right) &= (y+z) \cdot (-r \sin \theta) + (x+z) \cdot (r \cos \theta) + (y+x) \cdot r \\ &= (2+\pi) \cdot (-2 \cdot 1) + (0+\pi) \cancel{(2 \cdot 0)} + (0+2) \cdot 2 \\ &= -4 - 2\pi + 4 \quad \text{---} \quad -2\pi \end{aligned}$$

Extra Credit I. (2 pts) Stokes' theorem says that $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N} dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$ for any surface S in 3D with oriented boundary C . Explain, in sentences and equations, using correct notation, the situation in which this theorem becomes Green's theorem. (A sketch may help, too.)

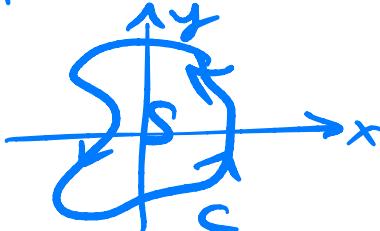
Suppose S is a surface in the xy plane.

Then $\vec{N} = \hat{k}$ is a normal direction,

and $\nabla \times \vec{F} = \langle Q_z - R_y, R_x - P_z, Q_x - P_y \rangle$,

so $(\nabla \times \vec{F}) \cdot \hat{k} = Q_x - P_y$. So

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{N} dS = \iint_S Q_x - P_y dA = \oint_C \vec{F} \cdot d\vec{r} \quad \text{Green's theorem}$$



13. (a) (5 pts) Find all critical points of the function:

$$f(x, y) = 2 - x^4 + 2x^2 - y^2$$

$$\nabla f = \langle -4x^3 + 4x, -2y \rangle$$

$$\begin{aligned} -4x^3 + 4x &= 0 \\ -2y &= 0 \\ \Leftrightarrow x(-x^2 + 1) &= 0 \\ y &= 0 \\ \Leftrightarrow x &= 0 \text{ or } x = \pm 1 \\ \text{and } y &= 0 \end{aligned}$$

(0, 0)

(+1, 0)

(-1, 0)

- (b) (5 pts) Find all local maxima, local minima, and saddle points of the function in part (a).

$$\begin{aligned} D &= f_{xx} f_{yy} - f_{xy}^2 = (-12x^2 + 4)(-2) - 0 \\ &= 8(3x^2 - 1) \end{aligned}$$

<u>crit.pt.</u>	<u>D</u>	<u>f_{xx}</u>	<u>type</u>
(0, 0)	-	-	Saddle
(+1, 0)	+	-	local max
(-1, 0)	+	-	local max

Extra Credit II. (2 pts) Assume $\mathbf{F}(x, y)$ is a conservative vector field defined on a open, connected region D . Fix any point (a, b) in D . Explain why the formula $f(x, y) = \int_{(a,b)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r}$ defines a function on D .

Since \vec{F} is conservative, $\vec{F} = \nabla f$ for some $f(x, y)$. Therefore

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} \stackrel{\text{FTLI}}{\downarrow} f(\text{end } c) - f(\text{start } c)$$

is path-independent. So \otimes makes sense: it defines one value $f(x, y)$. [And $\nabla f = \vec{F}$ in fact.]