SOLUTIONS

Worksheet: Double and triple integrals!

1. Suppose $A = \{(x,y) \,|\, 1 \le x^2 + y^2 \le 4\}$. Write the double integral as an iterated integral, and evaluate it:

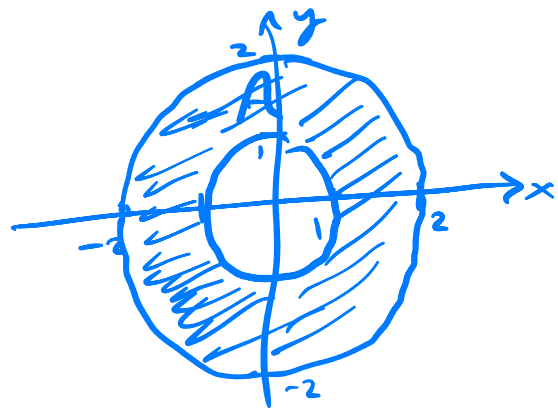
$$\iint_A \sqrt{x^2+y^2}\, dA =$$

$x^2+y^2=r^2$

$\sqrt{x^2+y^2}=r$

$dA = r\,dr\,d\theta$

(*Hint.* Sketch A. You can do the integral in polar coordinates!)



$$= \int_{\theta=0}^{2\pi}\int_{r=1}^{2} r \cdot r\, dr\, d\theta$$

$$= 2\pi \int_1^2 r^2\, dr = 2\pi \left[\frac{r^3}{3}\right]_1^2 = 2\pi\left(\frac{8}{3}-\frac{1}{3}\right)$$

$$= \frac{14\pi}{3}$$

2. The set $E = [0,1]\times[1,2]\times[2,3]$ is a cube. Write the triple integral as an iterated integral, and evaluate it:

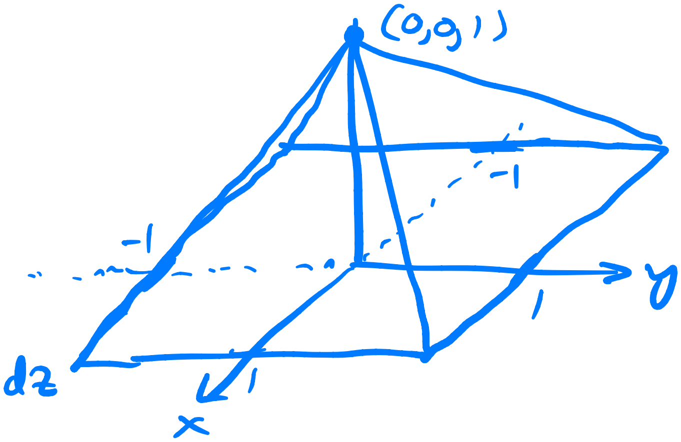
$$\iiint_E x + y\, dV = \int_{x=0}^{1}\int_{y=1}^{2}\int_{z=2}^{3} x+y\, dz\, dy\, dx$$

$$= \int_0^1\int_1^2 (x+y)(3-2)\, dy\, dx = \int_0^1 \left(\int_1^2 x+y\, dy\right) dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2}\right]_1^2 dx = \int_0^1 \left[2x+2-\left(x+\frac{1}{2}\right)\right] dx$$

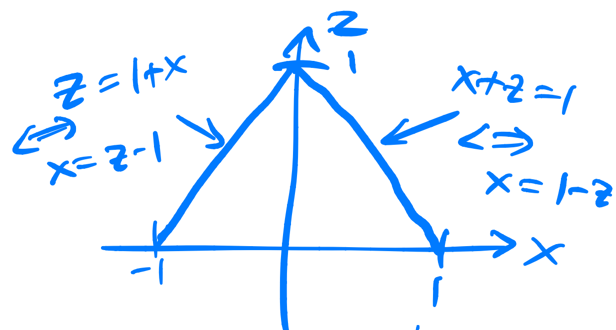
$$= \int_0^1 x + \frac{3}{2}\, dx = \left[\frac{x^2}{2}+\frac{3}{2}x\right]_0^1 = \frac{1}{2}+\frac{3}{2} = 2$$

3. A right pyramid R has a base in the x, y plane which is the square $[-1,1] \times [-1,1]$, and its tip is at the point $(0,0,1)$. Its density increases as one approaches the tip, namely $\rho(x,y,z) = 1+z$, in mass per volume units. Find the total mass.



$$M = \iiint_R 1+z \, dV$$

$$= \int_{z=0}^{z=1} \int_{x=z-1}^{x=1-z} \int_{y=z-1}^{y=1-z} 1+z \, dy\,dx\,dz$$



$$= \int_0^1 (1+z)\left(\text{area of square } [z-1,1-z]\times[z-1,1-z]\right) dz$$

$$= \int_0^1 (1+z)\left((1-z-z+1)^2\right) dz = 4\int_0^1 (1+z)(1-z)\,dz = 4\int_0^1 1-z^2\,dz$$

$$= 4\left[z - \frac{z^3}{3}\right]_0^1 = 4\cdot\frac{2}{3} = \frac{8}{3}$$

4. Find the volume of the sphere of radius one by setting-up either a double or a triple integral, and evaluating it. Of course, the answer you should get is $V = 4\pi/3$.

$$V = 2\int_{\theta=0}^{2\pi}\int_{r=0}^{1} \sqrt{1-r^2}\, r\,dr\,d\theta = 4\pi \int_0^1 (1-r^2)^{1/2}\, r\,dr$$

$u = 1-r^2$

$$= 4\pi \int_1^0 u^{1/2} \frac{du}{-2} = 2\pi\int_0^1 u^{1/2}\,du = 2\pi\left[\frac{2}{3}u^{3/2}\right]_0^1 = \frac{4\pi}{3}$$

$$V = \int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\int_{z=-\sqrt{1-x^2-y^2}}^{z=+\sqrt{1-x^2-y^2}} 1\,dz\,dy\,dx$$

wise to switch to polar here

$$= \int_{-1}^{1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} 2\sqrt{1-x^2-y^2}\,dy\,dx = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} 2\sqrt{1-r^2}\,r\,dr\,d\theta$$

$$= \dots = 4\pi/3$$