Math 253: Quiz 10 SOLUTIONS **Thursday 20 April, 2023**

Name: ______________________________ / 25

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. [4 points] Is $\mathbf{F} = 2x\cos y\,\mathbf{i} - x^2\sin y\,\mathbf{j}$ conservative? If it is, find a potential function.

$P = 2x\cos y \to P_y = -2x\sin y$

$Q = -x^2\sin y \to Q_x = -2x\sin y$ ✓ yes

$f_x = P = 2x\cos y$

$f(x,y) = x^2\cos y + g(y)$

$-x^2\sin y = Q = f_y = -x^2\sin y + g'(y)$

$0 = g'(y)$

$g(y) = c$

potential: $f(x,y) = x^2\cos y$ $\underline{+C}$ optional

2. [4 points] Is $\mathbf{F} = \langle \sin y, -x\cos y, x\rangle$ conservative? If it is, find a potential function.

method 1:

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \sin y & -x\cos y & x \end{vmatrix} = (0-0)\hat{i} - (1-0)\hat{j} + (-\cos y - \cos y)\hat{k}$$

not zero

also not zero

$\nabla\times\vec{F} \neq \vec{0}$

$\therefore$ no

method 2:

$P = \sin y$ $\quad P_y = \cos y$ $\quad P_z = 0$

$Q = -x\cos y$ $\quad Q_x = -\cos y$ $\quad Q_z = 0$

$R = x$ $\quad R_x = 1$ $\quad R_y = 0$

$\therefore$ no

3. **[4 points]** Is the following statement **true** or **false**? Explain in one or two sentences, justifying with any theorem that might apply.

> If a vector field $\mathbf{F}(x,y,z)$ is conservative on the open and connected region D, then line integrals of $\mathbf{F}$ are path independent on D, regardless of the shape of D.

If C_1, C_2 are paths with the same endpoints, and if $\vec{F} = \nabla f$, then

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} \nabla f\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

same endpoints

FTC LI

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_2} \nabla f\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

by fund. thm. line integrals, so $\vec{F}$ is path independent.

True.

4. **[4 points]** Suppose $f(x,y,z) = xyz^2 - yz$ and C is a straight line from $(0,1,2)$ to $(1,1,1)$. Evaluate the integral using the Fundamental Theorem for Line Integrals:

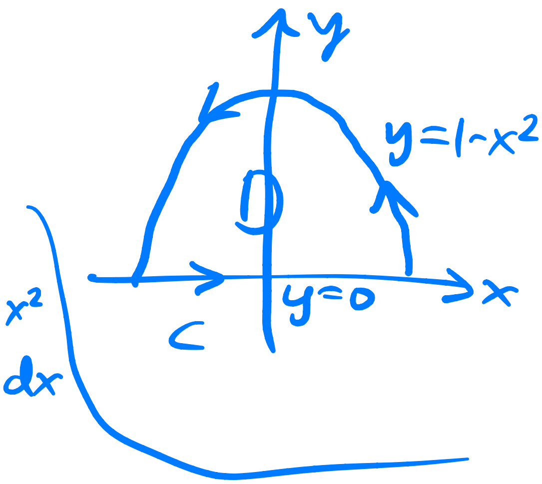
$$\int_C \nabla f \cdot d\mathbf{r} = f(1,1,1) - f(0,1,2)$$

$$= (1-1) - (0-2)$$

$$= 2$$

5. [4 points] Suppose C is the boundary of the region lying between the graphs of $y=0$ and $y=1-x^2$, and assume that C is oriented in the counterclockwise direction. Compute using Green's theorem:

$$\oint_C \overbrace{x^2}^{P}\,dx+\overbrace{3yx}^{Q}\,dy=$$



$$=\iint_D Q_x-P_y\,dA$$

$$=\int_{x=-1}^{1}\int_{y=0}^{1-x^2}3y-0\,dy\,dx=\int_{-1}^{1}\left[\frac{3}{2}y^2\right]_{y=0}^{y=1-x^2}dx$$

$$=\frac{3}{2}\int_{-1}^{1}(1-x^2)^2\,dx=\frac{3}{2}\int_{-1}^{1}1-2x^2+x^4\,dx$$

$$=\frac{3}{2}\left[x-\frac{2}{3}x^3+\frac{x^5}{5}\right]_{-1}^{1}=\frac{3}{2}\left[2-\frac{2}{3}\cdot 2+\frac{2}{5}\right]=\frac{8}{5}$$

6. [5 points] Suppose D is **any** simply-connected region in the plane, and let C be its boundary, oriented in the counterclockwise direction. Compute using Green's theorem and simplify as far as possible:

$$\oint_C \underbrace{-y}_{P}\,dx+\overbrace{x}^{Q}\,dy=\iint_D Q_x-P_y\,dA$$

$$=\iint_D 1-(-1)\,dA=\iint_D 2\,dA$$

$$=2\,(\text{area of } D)$$

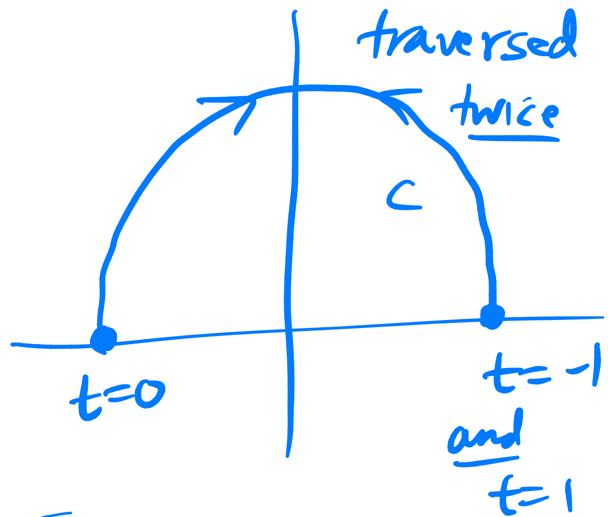
Extra Credit. [1 point] Suppose C is the parameterized curve $\mathbf{r}(t) = \langle \cos(1-t^2), \sin(1-t^2) \rangle$ for $-1 \le t \le 1$, whose graph is the upper half of the unit circle. Suppose $\mathbf{F} = \langle xe^y, \sin(x+y) \rangle$. Compute the line integral, explaining your steps:

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

(**Hint.** Do not apply brute force. Think about the curve.)

C_1: same $\vec{r}(t)$, $-1 \le t \le 0$

C_2: same $\vec{r}(t)$, $0 \le t \le 1$



$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_{C_1} \vec{F} \cdot d\vec{r} \underbrace{- \int_{C_1} \vec{F} \cdot d\vec{r}}_{C_2 \text{ is } C_1 \text{ with reversed direction}} = 0$$

EXTRA SPACE FOR ANSWERS