Math 253: Quiz 1 **26 January, 2023**

Name: Solutions

/ 25

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. [8 points] Suppose we have three vectors, $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{j} + 3\mathbf{k}$, $\mathbf{c} = -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$. Compute the following quantities which are either scalars or vectors. You can write the vectors using either component notation or standard unit vector notation.

a) $\|\mathbf{a}\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

b) $(\mathbf{a} \cdot \mathbf{b})\,\mathbf{c} = (0 - 1 + 3)\,\vec{c} = 2\vec{c}$

$= -2\hat{\imath} + 4\hat{\jmath} - 8\hat{k} = \langle -2, 4, -8 \rangle$

c) a unit vector in the direction of $\mathbf{b}$:

$\mathbf{u} = \dfrac{\vec{b}}{\|\vec{b}\|} = \dfrac{\langle 0, 1, 3 \rangle}{\sqrt{0^2+1^2+3^2}} = \dfrac{1}{\sqrt{10}}\langle 0, 1, 3 \rangle$

$= \dfrac{1}{\sqrt{10}}\hat{\jmath} + \dfrac{3}{\sqrt{10}}\hat{k}$

d) the vector projection of $\mathbf{b}$ onto $\mathbf{a}$:

$\mathbf{w} = \text{proj}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\vec{b}\cdot\vec{a}}{\|\vec{a}\|}\,\dfrac{\vec{a}}{\|\vec{a}\|} = \dfrac{(\vec{b}\cdot\vec{a})\,\vec{a}}{\|\vec{a}\|^2}$

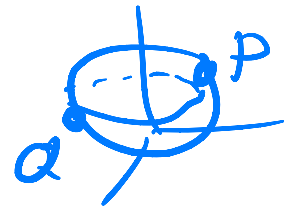
$= \dfrac{2\langle 1, -1, 1 \rangle}{3} = \left\langle \dfrac{2}{3}, -\dfrac{2}{3}, \dfrac{2}{3} \right\rangle$

$= \dfrac{2}{3}\hat{\imath} - \dfrac{2}{3}\hat{\jmath} + \dfrac{2}{3}\hat{k}$

2. [6 points] Find the equation of the sphere which has diameter PQ, where $P = (2, -1, -3)$ and $Q = (-2, 5, -1)$.

$$2r = \|\overrightarrow{PQ}\| = \|\langle -4, 6, 2\rangle\|$$

$$= \sqrt{4^2+6^2+2^2} = \sqrt{54} \Rightarrow r = \frac{\sqrt{54}}{2}$$

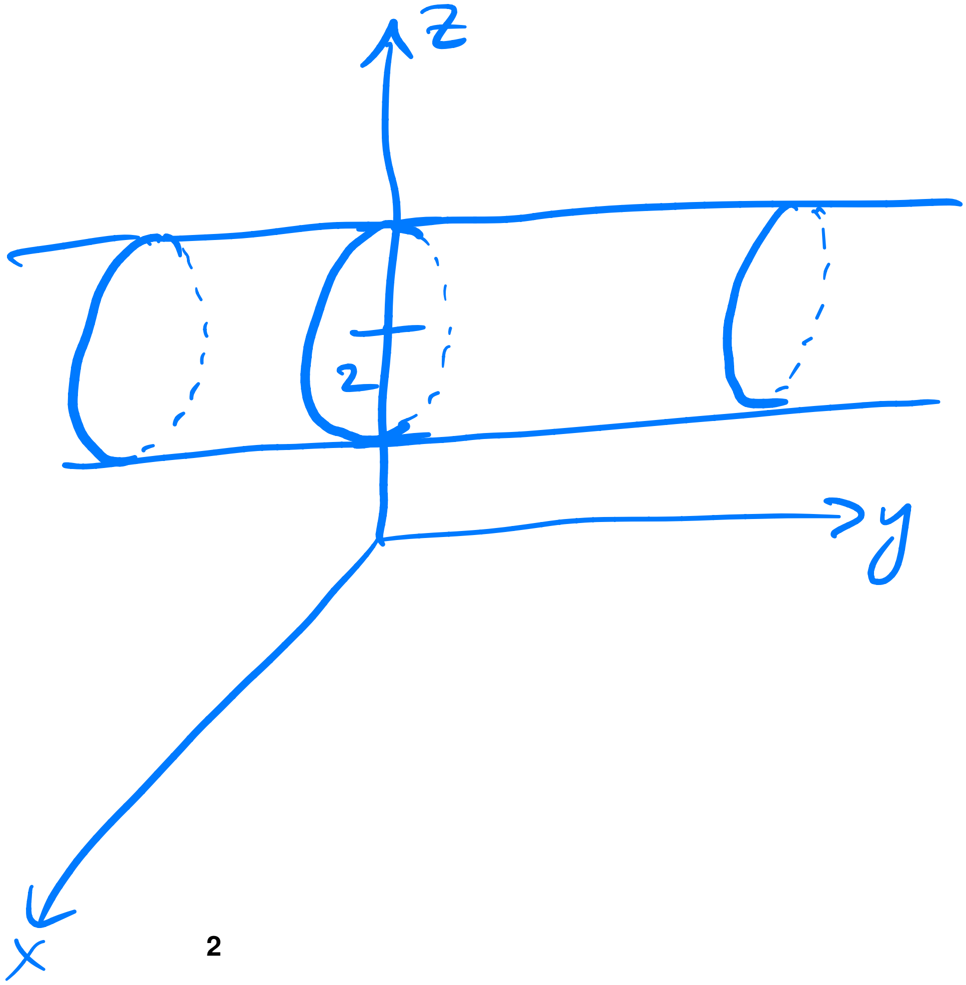


$$C = \tfrac{1}{2}(P+Q) = \left(\frac{2+-2}{2}, \frac{-1+5}{2}, \frac{-3+-1}{2}\right) = (0, 2, -2)$$

$$(x-0)^2 + (y-2)^2 + (z+2)^2 = \frac{27}{2}$$

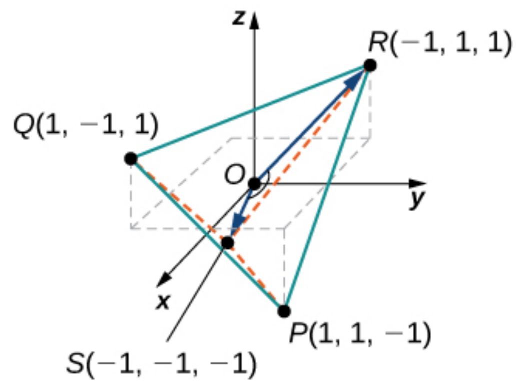
3. [6 points] Describe the set of points in three dimensional space that satisfies $x^2 + (z-2)^2 = 1$, and sketch a graph of the surface. (*Please make your graph at least two inches tall, label the axes, and put at least one scale value, a labeled tick mark, along each axis.*)

$x^2+(z-2)^2=1$ is circle in x, z plane, so it is a cylinder along a line parallel to the y-axis



4. [5 points] A methane molecule (figure) has a carbon atom situated at the origin and four hydrogen atoms located at points $P(1,1,-1)$, $Q(1,-1,1)$, $R(-1,1,1)$, and $S(-1,-1,-1)$. Find the angle θ between vectors $\overrightarrow{OS}$ and $\overrightarrow{OR}$.

Hint. It is just fine if your answer has an arccos() *in it, but otherwise it should be simplified. I know you do not have a calculator!*



$$\overrightarrow{OS} = \langle -1, -1, -1 \rangle$$

$$\overrightarrow{OR} = \langle -1, 1, 1 \rangle$$

$$\cos\theta = \frac{\overrightarrow{OS} \cdot \overrightarrow{OR}}{\|\overrightarrow{OS}\|\,\|\overrightarrow{OR}\|} = \frac{1-1-1}{\sqrt{3}\,\sqrt{3}}$$

$$= \frac{-1}{3}$$

$$\theta = \arccos\left(-1/3\right)$$

Extra Credit. [1 point] Show that $\|\mathbf{v}-\mathbf{u}\|^2 = \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{u}\|^2$.

$$\|\vec{v}-\vec{u}\|^2 = (\vec{v}-\vec{u})\cdot(\vec{v}-\vec{u})$$

$$= \vec{v}\cdot\vec{v} - \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{u}$$

$$= \|\vec{v}\|^2 - \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{v} + \|\vec{u}\|^2$$

$$= \|\vec{v}\|^2 - 2\,\vec{u}\cdot\vec{v} + \|\vec{u}\|^2$$

EXTRA SPACE FOR ANSWERS