

# Conservation and DG

4 April 2024

- Poisson strong form again:

$$-\nabla^2 u = f \quad \text{on } \Omega$$

- recall  $\nabla^2 u = \nabla \cdot \nabla u$

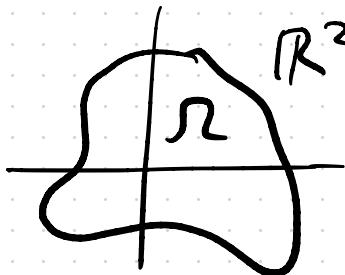
"discontinuous Galerkin"

integrate  $\otimes$  over any  $S \subset \Omega$ :

$$\int_S -\nabla \cdot \nabla u = \int_S f$$

div.  
num.  $\int$

$$-\int_{\partial S} \nabla u \cdot \hat{n} = \int_S f \quad \textcircled{1}$$



multiply ① by any  $v$  and integrate over  $\Omega$ :

$$\int_{\Omega} -(\nabla^2 u) v = \int_{\Omega} f v$$

div.  
thm

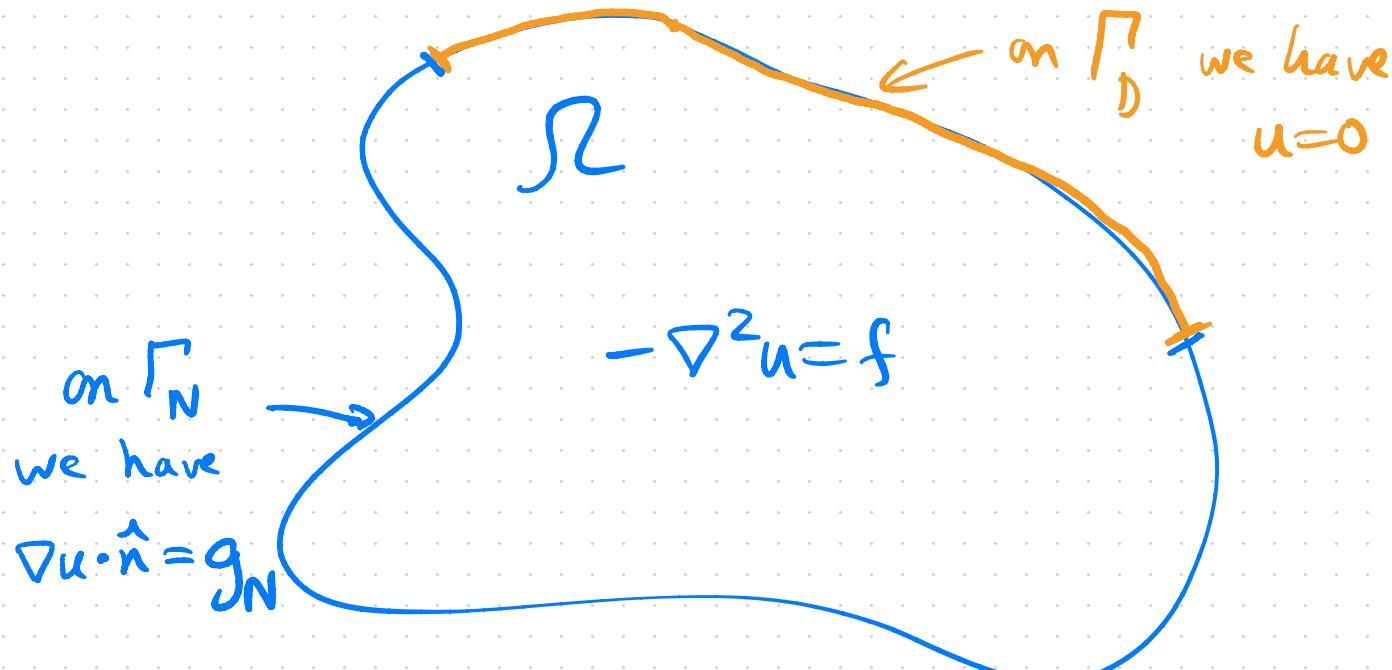
$$\int_{\Omega} \nabla u \cdot \nabla v - \int_{\partial\Omega} v \nabla u \cdot \hat{n} = \int_{\Omega} f v \quad ②$$

will assume  $u=0$  or  $f_0$   
and  $\nabla u \cdot \hat{n} = g_N$  on  $\Gamma_N$

idea: exact solution satisfies both ① (for all  $S$ )  
and ② (for all  $v$ )

Q. Which do we want to use as the basis  
of a numerical scheme?

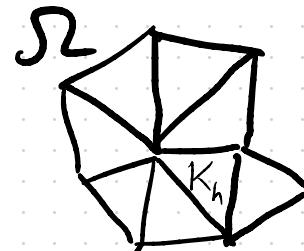
# a convenient Poisson problem :



$$\partial\Omega = \Gamma_D \cup \Gamma_N$$

## Finite volumes

$$-\int_{\partial S} \nabla u \cdot \hat{n} = \int_S f \quad \forall S \subset \Omega$$



→  
approx

$$-\int_{\partial K_h} \nabla u_h \cdot \hat{n} = \int_{K_h} f \quad \forall \text{cells } K_h$$

## Finite elements

$$\int_{\Omega} \nabla u \cdot \nabla v - \int_{\Gamma_N} v g_N = \int_{\Omega} f v \quad \forall v \in H_0^1(\Omega)$$

→  
approx

$$\int_{\Omega} \nabla u_h \cdot \nabla v_h - \int_{\Gamma_N} v_h g_N = \int_{\Omega} f v_h \quad \forall v \in V_h$$

Summary :

strong form

$$-\nabla^2 u = f$$

integrate over  $K$

multiplies by  $v$   
and integrate over  $\Omega$

Conservation over cells

$$-\int_K \nabla u \cdot n^1 = \int_K f$$

weak form over  $\Omega$

$$\int_{\Omega} \nabla u \cdot \nabla v - \int_{\Gamma_N} v g_n = \int_{\Omega} f v$$

## Element-wise

### weak form:

#### strong form

$$-\nabla^2 u = f$$

↓ mult. by  $v$  and integrate over one element  $K$

#### element-wise weak form

$$\int_K \nabla u \cdot \nabla v - \int_{\partial K} v \nabla u \cdot \hat{n} = \int_K f v$$

take  $v=1$  on  $K$

↓ Sum over elements  
and cancel along interior edges ("facets")

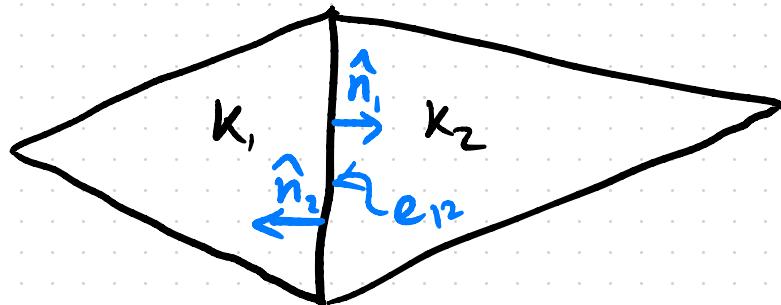
#### conservation over cells

$$-\int_{\partial K} \nabla u \cdot \hat{n} = \int_K f$$

#### weak form over $\Omega$

$$\int_{\Omega} \nabla u \cdot \nabla v - \int_{\Gamma_N} v g_n = \int_{\Omega} f v$$

re Cancelling in adjacent cells: let  $S = K_1 \cup K_2$



$$\left. \begin{aligned} \int_{K_1} \nabla u \cdot \nabla v - \int_{\partial K_1} v \nabla u \cdot \hat{n} &= S_{fv} \\ \end{aligned} \right\}$$

$$\left. \begin{aligned} \int_{K_2} \nabla u \cdot \nabla v - \int_{\partial K_2} v \nabla u \cdot \hat{n} &= S_{fv} \\ \end{aligned} \right\}$$

add along  $e_{12}$ , and use continuity of  $\nabla u$ :  
 $\hat{n}_1 = -\hat{n}_2$ :

$$\begin{aligned} \int_S \nabla u \cdot \nabla v - \int_S v \nabla u \cdot \hat{n} \\ = S_{fv} \end{aligned}$$

def: a numerical scheme for Poisson equation  
is conservative if there is an approximation

$$\sigma_h \approx -\nabla u_h$$

for numerical solution  $u_h$ , with  $\sigma_h$  single-valued over all edges in  $\partial\Gamma_h$ , so that

$$\sum_K \sigma_h \cdot \hat{n} = \sum_K f$$

lemma: (a) a convergent FE scheme based on  
the element-wise weak form

$$\int_K \nabla u_h \cdot \nabla v_h - \int_{\partial K} v_h \nabla u_h \cdot \hat{n} = \int_K f v_h$$

is conservative if  $v_h = \underbrace{\prod_K}_{\text{characteristic function on } K}$  is an allowed  
test function

(b) if an FE scheme is conservative then

c)  $\int_{\Gamma_D} \sigma_h \cdot \hat{n} = \int_N g_N + \int_{\Omega} f \quad \left. \right\} \begin{matrix} \text{numerically} \\ \text{checkable} \\ \text{global} \\ \text{conservation} \end{matrix}$

demo I will demo 3 Poisson solvers

poisson-pCG.py

our usual "primal" CG1 (piecewise linear) FE scheme, with check on  $\mathcal{C}$  at end

poisson-pDG.py

"primal" DGO FE scheme  
... and I don't know how to  
check  $\mathcal{C}$

poisson-mDG.py

"mixed" DGO FE scheme,  
with check on  $\mathcal{C}$  at end

- primal means we solve for  $u_h$
- mixed means we solve for  $u_h$   
and a variable  $\sigma_h$  which  
approximates  $\sigma_h = -\nabla u_h$

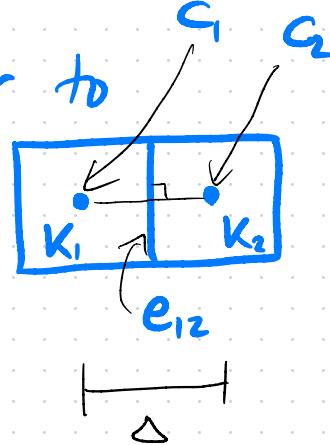
## primal DGO:

- if faces of cells are perpendicular to lines between cell centers

then

$$\begin{aligned} & \int_{e_{12} \subset K_1} v_h \nabla u_h \cdot \hat{n} + \int_{e_{12} \subset K_2} v_h \nabla u_h \cdot \hat{n} \\ & \cong \int_{e_{12}} \frac{u_h(c_1) - u_h(c_2)}{\Delta} (v_h(c_1) - v_h(c_2)) \end{aligned}$$

“two-point flux approximation”



- "DG0" means  $u_h$  and  $v_h$  are constant on each element
- for DG0,  $\nabla u_h = 0$  and  $\nabla v_h = 0$  in element, so element-wise weak form is just
 
$$-\int_{\partial K} v_h \nabla u_h \cdot \hat{n} = \int_K f v_h$$
- summing over all elements and using result on previous slide gives this expression:

$$F = (\text{jump}(u) / \Delta t) * \text{jump}(v) * \underbrace{\int_S}_{\text{means integral over interior edges}} - f * v * dx$$

- this is essentially a finite volume method

irritating fact 1:

Dirichlet conditions

in this method must be imposed weakly  
(see code)

irritating fact 2:

I cannot figure out a good way to  
do a global conservation check:

c

$$\oint_{\Gamma_D} \sigma_h \cdot \hat{n} = \int_N g_N + \int_S f$$

not clear  
how to do  
this integral

## Mixed DG:

- quite different, and not as irritating
- again  $U_h$  and  $V_h$  are constant on each element  
(at least in  $k=1$  case below)
- start from strong form  $-\nabla^2 u = f$   
but write it as system:  
$$\begin{aligned}\sigma &= -\nabla u && \left. \begin{aligned} u &\text{ scalar} \\ \nabla \cdot \sigma &= f \end{aligned} \right\} \sigma = (\sigma_1, \sigma_2) \text{ vector}\end{aligned}$$
- multiply first equation by vector test function  $\omega$  and second equation by scalar test function  $v_i$  and integrate over  $\Omega$ :

$$\int_{\Omega} \sigma \cdot \omega + \int_{\Omega} \nabla u \cdot \omega = 0$$

$$\int_{\Omega} (\nabla \cdot \sigma) v = \int_{\Omega} fv$$

- Integrate by parts to move  $\nabla$  off of  $u$ :

$$\int_{\Omega} \sigma \cdot \omega - \int_{\Omega} u (\nabla \cdot \omega) + \int_{\partial \Omega} u \omega \cdot \hat{n} = 0$$

$$\int_{\Omega} (\nabla \cdot \sigma) v = \int_{\Omega} fv$$

- since  $u=0$  on  $\Gamma_0$ , that part of } boundary integral disappears } Dirichlet becomes natural!

- to enforce  $\nabla u \cdot n = -\sigma \cdot n = g_N$  on  $\Gamma_N$  requires  
a Dirichlet condition on  $\sigma$ :  

$$\begin{cases} \sigma \cdot n = g_N \\ \omega \cdot n = 0 \end{cases} \text{ on } \Gamma_N$$

Neumann  
becomes  
essential!

- final mixed weak form:

$$\begin{aligned} \int_{\Omega} \sigma \cdot \omega - \int_{\Omega} u (\nabla \cdot \omega) &= 0 \\ \int_{\Omega} (\nabla \cdot \sigma) v &= \int_{\Omega} f v \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} M$$

$$\begin{aligned} F &= \text{dot}(\sigma, \omega) * dx - u * \text{div}(\omega) * dx \\ &\quad + \text{div}(\sigma) * v * dx - f * v * dx \end{aligned}$$

- Solving  $\textcircled{M}$  accurately/stably requires careful choice of elements:

$$\left. \begin{array}{c} RT_k \times DG_{k+1} \\ \text{to } \sigma, w \text{ here} \\ \hline BDM_k \times DG_{k+1} \\ \text{to } u, v \text{ here} \end{array} \right\} \text{for triangular elements}$$

- see literature and "Periodic Table of the Finite Elements"

## results:

- poisson-pCG.py is as expected, but when we measure global conservation  $\textcircled{C}$ , it is off by much more than rounding error
- poisson-pDG.py seems to work, but no apparent way to check  $\textcircled{C}$
- poisson-mDG.py works well and global conservation checks out:
  - $\textcircled{C}$  holds to within rounding error

try  
varying  
 $m, k$

## References

- [firedrakeproject.org/demos/poisson-mixed.py](http://firedrakeproject.org/demos/poisson-mixed.py)  
explains how poisson-mDG.py  
works
- [github.com/tlroy/thermal porous](https://github.com/tlroy/thermal porous)  
explains how poisson-pDG.py  
works

“Mixed formulation of Poisson equation”

see intro/Intro.pdf

## Further reading:

- Hughes et al (2000). The continuous Galerkin method is locally conservative. *J. Comput. Phys.* 163(2), 467–488
- Periodic Table of the Finite Elements,  
at  
[z.umn.edu/femtable](http://z.umn.edu/femtable)