

SOLUTIONS

Name: _____

Math 314 Linear Algebra (Bueler)

Friday, 29 April 2022

Final Exam

No book, electronics, calculator, or internet access. Allowed notes: one sheet of letter paper ($= 8.5'' \times 11''$ paper), with anything written on both sides. 150 points possible. 125 minutes maximum.

1. Consider this 3×5 matrix and its row-reduced echelon form ($R = \text{rref}(A)$):

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 & -1 \\ 0 & 2 & 4 & 0 & -2 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \quad \rightarrow \quad R = \begin{bmatrix} 0 & 1 & 0 & -2 & -5 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- (a) (3 pts) What is the rank of A ?

2

- (b) (3 pts) What is the dimension of the null space of A ?

3

- (c) (7 pts) Find a basis for each of these 3 subspaces associated to A :

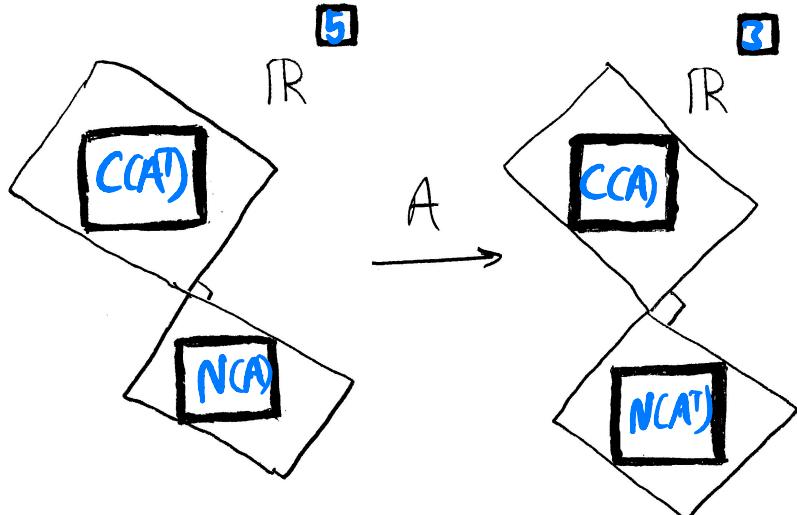
row space $C(A^\top)$, column space $C(A)$, null space $N(A)$

$$C(A^\top) = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}, \quad C(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \right\},$$

$$N(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- (d) (5 pts)

Fill-in the **6 bold boxes** in the “big picture” at right. Give either the name of the subspace or the dimension which applies to the specific matrix A :



2. Suppose A is any m by n matrix.

(a) (7 pts) Show that the null space $N(A)$ is a subspace.

If \vec{x} and \vec{y} are in $N(A)$, and c, d are real numbers, then $A(c\vec{x} + d\vec{y}) = cA\vec{x} + dA\vec{y} = c\vec{0} + d\vec{0} = \vec{0}$.

Thus $c\vec{x} + d\vec{y}$ is in $N(A)$, so $N(A)$ is a subspace.

(b) (7 pts) Show that the null space $N(A)$ and the row space $C(A^T)$ are orthogonal.

If \vec{x} is in $N(A)$ and $\vec{y} = A^T \vec{w}$ is in $C(A^T)$, then $\vec{x}^T \vec{y} = \vec{x}^T A^T \vec{w} = (A\vec{x})^T \vec{w} = \vec{0}^T \vec{w} = 0$. Thus \vec{x}, \vec{y} are orthogonal, so $N(A), C(A^T)$ are orthogonal.

3. (7 pts) Find the general solution of this linear system of two equations, and show your work:

$$3x_1 - x_2 = 4$$

$$-6x_1 + 2x_2 = -8$$

$$R_2 \leftarrow R_2 + 2R_1$$

$$\left[\begin{array}{cc|c} 3 & -1 & 4 \\ 0 & 0 & 0 \end{array} \right]$$

$$\vec{x}_p = \begin{bmatrix} 4/3 \\ 0 \end{bmatrix} \quad \left. \begin{array}{l} \\ \end{array} \right\}$$

$$\vec{s}_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$$\vec{x} = \vec{x}_p + t\vec{s}_1 = \begin{bmatrix} 4/3 + 1/3t \\ t \end{bmatrix}$$

set free variable x_2 to 0 for \vec{x}_p , 1 for \vec{s}_1

4. (a) (10 pts) Use the Gauss-Jordan elimination to compute the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -2 & -6 \\ 1 & 4 & 4 \end{bmatrix}. \text{ Show your work.}$$

$$\begin{array}{c} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -2 & -2 & -6 & 0 & 1 & 0 \\ 1 & 4 & 4 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\quad} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array} \right] \\ \xrightarrow{\quad} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \end{array} \right] \xrightarrow{\quad} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \end{array} \right] \\ \xrightarrow{\quad} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & -1 & 1 \end{array} \right] \xrightarrow{\quad} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 2 & -3 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & -1 & 1 \end{array} \right] \\ \text{I} \qquad \qquad \qquad \text{II} \qquad \qquad \qquad A^{-1} \end{array}$$

$$A^{-1} = \boxed{\begin{bmatrix} 8 & 2 & -3 \\ 1 & \frac{1}{2} & 0 \\ -3 & -1 & 1 \end{bmatrix}}$$

(b) (5 pts) At the first stages of elimination in part (a) you applied row operations which can be written as elimination matrices. Specifically, give the matrices E_{21} and E_{31} which you used.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \leftarrow \text{used as: } [E_{21}, A | E_{21}, I]$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \leftarrow \text{then: } [E_{31}, E_{21}, A | E_{31}, E_{21}, I] \\ \text{etc.}$$

5. Is each statement about square matrices **True** or **False**? If False, provide a counterexample.

(a) (3 pts) If Q is an orthogonal matrix then Q is invertible.

true

$$\left(\begin{array}{l} Q^T Q = I \\ \text{so} \\ Q^{-1} = Q^T \end{array} \right)$$

(b) (3 pts) If P is a projection matrix then P is invertible.

false

$$P = 0 \text{ satisfies } P^2 = P$$

(c) (3 pts) If P is a permutation matrix then P is invertible.

true

$$\left(P \text{ has } \det(P) = \pm 1, \text{ for example} \right)$$

(d) (3 pts) If A is a diagonalizable matrix then A is invertible.

false

$$A = 0 \text{ is diagonalizable: } 0 = I O I^{-1}$$

(e) (3 pts) If all eigenvalues of a matrix A are zero then $A = 0$.

false

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(f) (3 pts) If any eigenvalue of A is zero then A is not invertible.

true

$$\left(\begin{array}{l} \text{if } A\vec{v} = 0\vec{v} = \vec{0} \text{ for } \vec{v} \neq 0 \\ \text{then } N(A) \neq \{0\} \end{array} \right)$$

(g) (3 pts) If $A^2 = 0$ then $A = 0$.

false

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

6. (a) (10 pts) Form and solve the normal equations: Show your work.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 0 \\ 2 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$A^T A \vec{x} = A^T \vec{b} \quad \left\{ \begin{array}{l} \text{normal} \\ \text{eqns} \end{array} \right.$$

$$A^T A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 10 & -3 \\ -3 & 2 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 10 & -3 & 3 \\ -3 & 2 & -1 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 10 & -3 & 3 \\ 0 & 11/10 & -1/10 \end{array} \right]$$

$$R_2 \leftarrow R_2 + \frac{3}{10} R_1$$

$$x_2 = (-1/10) / (11/10) = -1/11$$

$$10x_1 - 3(-1/11) = 3 \Leftrightarrow 10x_1 = 3 - 3/11 \Leftrightarrow x_1 = \frac{(30)}{11}/10 = 3/11$$

$$\vec{x} = \begin{bmatrix} -1/11 \\ 3/11 \end{bmatrix}$$

(b) (4 pts) In part (a), is the vector \mathbf{b} in the subspace $C(A)$? Explain your reasoning.

no. if \vec{b} were in $C(A)$ then $A\vec{x} = \vec{b}$

exactly because normal equations are $A\vec{x} = P\vec{b}$

and $P\vec{b} = \vec{b}$ if \vec{b} in $C(A)$. but $A\vec{x} = \begin{bmatrix} 4/11 \\ 3/11 \\ 6/11 \\ 7/11 \end{bmatrix} \neq \vec{b}$

7. (7 pts) Recall that the projection matrix P which projects onto the column space $C(A)$ of a matrix A with linearly-independent columns (full column rank) is $P = A(A^\top A)^{-1}A^\top$. Show that P is a projection matrix.

$$P^2 = A(A^\top A)^{-1}\underbrace{A^\top A}_{=I}(A^\top A)^{-1}A^\top = A(A^\top A)^{-1}A^\top = P$$

so P is a projection

8. (a) (7 pts) Suppose that a square matrix B is diagonalizable, that is, suppose $B = X\Lambda X^{-1}$ where X is invertible and Λ is diagonal. Give a formula which shows why it is easy to compute B^{100} .

$$\begin{aligned} B^{100} &= X \underbrace{\Lambda}_{\pm} \underbrace{X^{-1}}_{\pm} X \underbrace{\Lambda}_{\pm} \underbrace{X^{-1}}_{\pm} \cdots \underbrace{X \Lambda X^{-1}}_{\pm} \\ &= X \underbrace{\Lambda^{100}}_{\text{easy:}} X^{-1} \quad \Lambda^{100} = \begin{bmatrix} \lambda_1^{100} & & \\ & \ddots & \\ & & \lambda_n^{100} \end{bmatrix} \end{aligned}$$

- (b) (7 pts) Describe how to compute $\det(B)$ if $B = X\Lambda X^{-1}$ is diagonalized.

$$\begin{aligned} \det(B) &= \det(X) \det(\Lambda) \det(X^{-1}) \\ &= \det(X) (\lambda_1 \cdots \lambda_n) \frac{1}{\det(X)} \\ &= \lambda_1 \cdots \lambda_n = \prod_{i=1}^n \lambda_i \end{aligned}$$

9. (7 pts) Let $Q = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$. Show that Q is an orthogonal matrix.

$$Q^T Q = \frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1+3 & -\sqrt{3}+\sqrt{3} \\ -\sqrt{3}+\sqrt{3} & 1+3 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

10. (a) (5 pts) Recall that 2 by 2 rotation matrices have the form

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Using this form, find a non-identity matrix A with the property that $A^4 = I$, and verify this property for your matrix.

$$\theta = \pi/2 : A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\therefore A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A^4 = (A^2)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

- (b) (5 pts) Compute the eigenvalues of the matrix A which you found in part (a).

$$p(\lambda) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$$\lambda_1 = +i, \quad \lambda_2 = -i$$

11. (a) (10 pts) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}. \quad p(\lambda) = \det \begin{bmatrix} 2-\lambda & 2 & 2 \\ 2 & -\lambda & 0 \\ 2 & 0 & -\lambda \end{bmatrix}$$

$$= (2-\lambda)(\lambda^2 - 0) - 2(-2\lambda - 0) + 2(0 + 2\lambda)$$

$$= (2-\lambda)\lambda^2 + 4\lambda + 4\lambda = \lambda(\lambda^2 - 2\lambda - 8)$$

$$\Rightarrow (\lambda^2 - 2\lambda - 8) = \lambda(\lambda - 4)(\lambda + 2) \therefore \lambda = -2, 0, 4$$

$\lambda_1 = -2$: $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \vec{x} = \vec{0} \rightarrow \begin{bmatrix} 4 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \vec{x} = \vec{0} \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$

$$\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 0$: $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \vec{x} = \vec{0} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$

$$\vec{x}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda_3 = 4$: $\begin{bmatrix} -2 & 2 & 2 \\ 2 & -4 & 0 \\ 2 & 0 & -4 \end{bmatrix} \vec{x} = \vec{0} \rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$

$$\vec{x}_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

(b) (3 pts) Is the matrix in part (a) diagonalizable? Give a brief justification. (Hint. No further calculations are necessary.)

yes. A is symmetric

12. Consider the transformations from $\mathbf{V} = \mathbb{R}^2$ to $\mathbf{W} = \mathbb{R}^2$. For each one, is it linear? (Show it is, or give a counterexample.) Then give a simplified formula for $T(T(\mathbf{v}))$.

(a) (5 pts) $T(\mathbf{v}) = -\mathbf{v} + (1, 1)$

not linear: $T(\vec{0}) = (1, 1) \neq \vec{0}$

$$\begin{aligned} T(T(\vec{v})) &= T(-\vec{v} + (1, 1)) = -(-\vec{v} + (1, 1)) + (1, 1) \\ &= \vec{v} - (1, 1) + (1, 1) = \vec{v} \end{aligned}$$

(b) (5 pts) $T(\mathbf{v}) = \frac{1}{2}(v_1 + v_2, v_1 + v_2)$

$$\begin{aligned} \text{Linear: } T(c\vec{v} + d\vec{w}) &= T((av_1 + dw_1, av_2 + dw_2)) \\ &= \frac{1}{2}(av_1 + dw_1 + av_2 + dw_2, av_1 + dw_1 + av_2 + dw_2) \\ &= \frac{1}{2}(av_1 + av_2, av_1 + av_2) + \frac{1}{2}(dw_1 + dw_2, dw_1 + dw_2) \\ &= a \cdot \frac{1}{2}(v_1 + v_2, v_1 + v_2) + d \cdot \frac{1}{2}(w_1 + w_2, w_1 + w_2) \\ &= aT(\vec{v}) + dT(\vec{w}) \end{aligned}$$

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$$T(T(\vec{v})) = T\left(\left(\frac{v_1+v_2}{2}, \frac{v_1+v_2}{2}\right)\right)$$

T is a projection

$$= \frac{1}{2}\left(\frac{v_1+v_2}{2} + \frac{v_1+v_2}{2}, \frac{v_1+v_2}{2} + \frac{v_1+v_2}{2}\right)$$

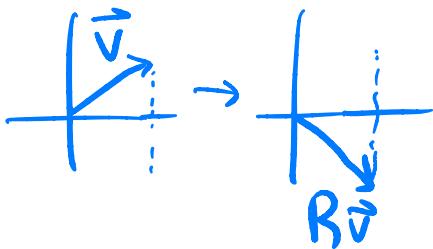
$$= \frac{1}{2}(v_1 + v_2, v_1 + v_2) = T(\vec{v})$$

~~q~~ (sorry)

Extra Credit. (3 pts) The matrix $Q = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$ in problem ~~2~~ is neither a rotation matrix nor a reflection matrix. But it can be factored into the product of a rotation matrix and a reflection matrix. Do so.

$$Q = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{R = \text{reflection across } x\text{-axis:}} \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}}_{S = \text{rotation by } \frac{\pi}{3} \text{ radians}}$$

$R = \text{reflection across } x\text{-axis:}$



$S = \text{rotation by } \frac{\pi}{3} \text{ radians}$

$$= \begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix}$$

↑ see problem 10

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