SOLUTIONS

Worksheet: Using the row-reduced echelon form

For each linear system $A\mathbf{x} = \mathbf{b}$ below I applied Matlab's `rref()` command to the augmented matrix $[A\ \mathbf{b}]$ to get the row-reduced echelon form $[R\ \mathbf{d}]$. Interpret it to answer the following questions:

- what is the **rank** of A?
- **find special solutions** which span the nullspace $N(A)$
- **identify vectors** which span the column space $C(A)$
- **write down** the general solution to the system $A\mathbf{x} = \mathbf{b}$

1.

$$\begin{aligned} 8x_1 + x_2 + 15x_3 &= -22 \\ 3x_1 + 5x_2 + x_3 &= 1 \\ 4x_1 + 9x_2 - x_3 &= 6 \end{aligned} \qquad \Longrightarrow \qquad [R\ \mathbf{d}] = \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- rank $= 2$ (2 pivots) $\iff$ $x_1 + 2x_3 = -3$, $x_2 - x_3 = 2$
- $\vec{s}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ so $N(A) = \text{span}\{\vec{s}_1\}$
- pivot columns show $C(A) = \text{span}\left\{\begin{bmatrix} 8 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}\right\}$
- $\vec{x}_p = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$ so $\underline{\vec{x} = \vec{x}_p + t_1 \vec{s}_1}$ for any t_1 (∞ solns)

2.

$$\begin{aligned} 12x_1 - 10x_2 + 5x_3 &= -6 \\ -9x_1 - x_2 - 5x_3 &= -32 \\ x_1 + 3x_2 + 12x_3 &= 38 \end{aligned} \qquad \Longrightarrow \qquad [R\ \mathbf{d}] = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

- rank $= 3$ $\iff$ $x_1 = 2$, $x_2 = 4$, $x_3 = 2$
- $N(A) = Z = \{\vec{0}\}$
- $C(A) = \text{span}\left\{\begin{bmatrix} 12 \\ -9 \\ 1 \end{bmatrix}, \begin{bmatrix} -10 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -5 \\ 12 \end{bmatrix}\right\}$
- $\underline{\vec{x} = \vec{d} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}}$ (only one solution)

3.

$$\begin{aligned} 2x_1 - x_2 + 5x_3 + 2x_4 &= 5 \\ 2x_1 + x_2 - x_3 + 6x_4 &= 7 \end{aligned} \qquad \Longrightarrow \qquad [R\ \mathbf{d}] = \begin{bmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & -3 & 2 & 1 \end{bmatrix}$$

$$\Longleftrightarrow \quad \begin{aligned} x_1 \quad + x_3 + 2x_4 &= 3 \\ x_2 - 3x_3 + 2x_4 &= 1 \end{aligned}$$

- rank $= 2$
- $\vec{s}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \end{bmatrix}$, $\vec{s}_2 = \begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$, and $N(A) = \text{Span}\{\vec{s}_1, \vec{s}_2\}$
- $C(A) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2$
- $\vec{x}_p = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ so $\underline{\vec{x} = \vec{x}_p + t_1 \vec{s}_1 + t_2 \vec{s}_2}$ for any t_1, t_2 (∞ solns)

4.

$$\begin{aligned} 2x_1 + 2x_2 &= 4 \\ -x_1 + x_2 &= 4 \\ 5x_1 - x_2 &= -8 \\ 2x_1 + 6x_2 &= 16 \end{aligned} \qquad \Longrightarrow \qquad [R\ \mathbf{d}] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Longleftrightarrow \quad \begin{aligned} x_1 \quad &= -1 \\ x_2 &= 3 \end{aligned}$$

- rank $= 2$
- $N(A) = Z = \{\vec{0}\}$
- $C(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 6 \end{bmatrix} \right\}$
- $\underline{\vec{x} = \vec{x}_p = \begin{bmatrix} -1 \\ 3 \end{bmatrix}}$ (one solution)

5. In problem **4** there are four equations in two unknowns. In typical cases there would be no solutions at all. Show a representative $[R\ \mathbf{d}]$ when there are no solutions.

$$[R\ \vec{d}] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & -7 \end{bmatrix} \Big\}$$ inconsistent equations in general