

## Homework #11

Due Wednesday 13 April, 2022 at 11:59pm.

Submit as a single PDF via Gradescope; see the Canvas page

[canvas.alaska.edu/courses/7017](https://canvas.alaska.edu/courses/7017)

Textbook Problems from Strang, *Intro Linear Algebra*, 5th ed. will be graded for completion. Answers/solutions to these Problems are linked at

[bueler.github.io/math314/resources.html](https://bueler.github.io/math314/resources.html)

The **P** Problems will be graded for correctness. When grading these Problems, I will expect you to write explanations using complete sentences!

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*Put these Textbook Problems first on your PDF, in this order.*

**from Problem Set 6.1, pages 297–302:** # 1, 5, 8, 14, 16, 21, 24, 33

**from Problem Set 6.2, pages 313–317:** # 1, 2, 4, 7, 15, 20

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*Put these **P** Problems next on your PDF, in this order.*

**P52. (a)** Compute the eigenvalues and eigenvectors of  $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ .

**(b)** Find the eigenvalues and eigenvectors of the matrices the matrices  $A + I$  and  $A^{-1}$ . Confirm that the eigenvectors are the same as those for  $A$ .

**(c)** Apply the functions  $f(x) = x + 1$  and  $g(x) = 1/x$  to the eigenvalues of  $A$ , and confirm this gives the same eigenvalues computed in **(b)**.

*This illustrates a more general rule, that functions of a matrix, like  $f(A) = A + I$  and  $g(A) = A^{-1}$ , have eigenvalues which are computed by the same function, but on complex numbers.*

**P53. (a)** Compute the eigenvalues of the matrices

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

**(b)** The matrix  $B$  is rank one. To consider this case, suppose  $u, v$  are nonzero (column) vectors in  $\mathbb{R}^n$ . Let  $C = uv^\top$ . (Recall this is the general formula for a rank one matrix!) Show that  $Cu = \lambda u$  and find  $\lambda$ . Also show that if  $w$  is any vector orthogonal to  $v$  then  $Cw = 0$ . What can you conclude about the eigenvalues of rank one matrices?

**P54.** 2 by 2 rotation matrices have the form

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Using this form, find a non-identity matrix  $A$  with the property that  $A^3 = I$ . Now compute the eigenvalues of  $A$ , and confirm that the eigenvalues are certain complex numbers  $\lambda$ , on the unit circle, which satisfy  $\lambda^3 = 1$ .

**P55.** Answer true or false; if true give an explanation and if false give a counterexample. Assume  $A$  is 3 by 3.

- (a) If the eigenvalues of  $A$  are 2, 2, 5 then  $A$  is invertible.
- (b) If the eigenvalues of  $A$  are 2, 2, 5 then  $A$  is diagonalizable.

**P56.** The following four diagonalizable  $2 \times 2$  matrices look rather similar to each other, for instance in terms of the sizes of the entries:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 7 \\ -3 & -4 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 6.9 \\ -3 & -4 \end{bmatrix}$$

However, their *powers* act in completely different ways. Looking at the eigenvalues will explain it.

(a) Compute the eigenvalues of the 4 matrices and put all 8 values on a single graph in the complex plane. (*The eigenvalues of real matrices are generally complex!*) Show the unit circle on your graph, and then show the eigenvalues as 8 clearly-labeled dots in the complex plane; consider using a color for each matrix. Draw the picture well enough so that you can tell where all the eigenvalues are relative to the unit circle.

(b) Recall that if  $M$  is a square and diagonalizable matrix with eigenvalues  $\lambda_i$  and eigenvectors  $x_i$ , so that  $M = X\Lambda X^{-1}$  is the diagonalization, then

$$M^k = X\Lambda^k X^{-1}.$$

(Remember the calculation:  $M^2 = X\Lambda X^{-1}X\Lambda X^{-1} = X\Lambda^2 X^{-1}$ .) That is, powers of  $A$  can be computed by putting the power on the diagonal matrix  $\Lambda$ .

Based on where the eigenvalues in part (a) are, relative to the unit circle, match  $A, B, C, D$  to these descriptions:

1.  $M^{100} = I$
2.  $M^{100} = -M$
3.  $M^{100}$  has tiny entries
4.  $M^{100}$  has enormous entries

(Hint. A power of a complex number  $x + iy = re^{i\theta}$  which is on the unit circle just rotates around the circle. In general,  $(re^{i\theta})^k = r^k e^{ik\theta}$ .)

(c) Use Matlab to compute the 100th power of the 4 matrices to confirm your results.