Homework #6

Due Monday 21 February, 2022 at 11:59pm.

Submit as a single PDF via Gradescope, linked from the Canvas page
`canvas.alaska.edu/courses/7017`
Textbook Problems from Strang, *Intro Linear Algebra*, 5th ed. will be graded for completion. Answers/solutions are linked at
`bueler.github.io/math314/resources.html`
P Problems will be graded for correctness. When grading these, I expect you to write explanations using complete sentences!

Put these Textbook Problems first on your PDF, in this order.

from Problem Set 3.1, pages 130–133: # 4, 5, 12, 14, 17, 20, 23, 27

from Problem Set 3.2, pages 141–147: # 1, 2, 7, 10, 15, 17, 22

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*Put these **P** Problems next on your PDF, in this order.*

P21. If no zero pivots appear along the way, elimination can factor a symmetric matrix S into $S = LDL^\top$ where L is lower triangular, with ones on the diagonal as usual, and D is a diagonal matrix. The calculation proceeds essentially the same as an LU factorization, but once we see U we can, because of symmetry, "pull out" the diagonal entries from U, and the remaining upper triangular matrix will be the transpose of the L factor. Specifically, the diagonal matrix D is formed from the pivots.

Do this factorization on the following matrices:

$$S = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}, \quad S = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Check the factorization $S = LDL^\top$ by multiplying back!

P22. **(a)** How many entries of a symmetric 5×5 matrix S can be chosen independently?

(b) Large, symmetric matrices, of size $m \times m$, require about half the storage (memory) on a computer as general matrices of the same size if the entries are saved in a careful scheme. Explain.

(c) A *skew-symmetric* matrix A is one for which $A^\top = -A$. How many entries of a skew-symmetric 5×5 matrix A can be chosen independently?

P23. **(a)** Assume A is $m \times n$. Explain why the product $A^\top A$ is always defined; what size is it? Then explain why

$$(A^\top A)_{ij} = \sum_{k=1}^{m} a_{ki}\, a_{kj}$$

(*Hint. Start from the general formula for* $(AB)_{ij}$, *but then specialize to the current case.*)

(b) Show that if A is not a zero matrix then $A^\top A$ is also not a zero matrix.

(c) Find a nonzero matrix A so that $A^2 = 0$. Then calculate $A^\top A$ and confirm it is *not* zero.

P24. Which of the following subsets of $\boldsymbol{R}^3$ are actually subspaces? Explain.

(a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_3$.

(b) The plane of vectors with $b_1 = 1$.

(c) The vectors with $b_1 b_2 b_3 = 0$.

(d) All linear combinations of $\boldsymbol{v} = (3, 1, 0)$ and $\boldsymbol{w} = (2, 2, 2)$.

(e) All vectors which satisfy $b_1 + b_2 + b_3 = 0$.

(f) All vectors with $b_1 \ge b_2 \ge b_3$.

P25. In each part, describe the smallest subspace of the 2×2 matrix space M which contains the given matrices. (*Hint. Answer by giving a parameterized general form for a matrix in the subspace.*)

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

P26. Is it possible to construct a matrix A whose column space contains $(1, 1, 0)$ and $(0, 1, 1)$, and whose nullspace contains $(1, 0, 1)$ and $(0, 0, 1)$. Explain your answer. (*Hint. What size is A? How many pivots and how many free variables?*)

P27. If A is 4×4 and invertible, describe the nullspace of the 4×8 matrix $B = [A \;\; A]$.