SOLUTIONS

Worksheet: The four subspaces

For each matrix A below I show $R =$ `rref`(A), i.e. from Matlab. Answer the following questions:

dimensions in green

- what are the dimensions of the four subspaces $C(A^\top), C(A), N(A), N(A^\top)$?
- find a basis for each of the first three subspaces $C(A^\top), C(A), N(A)$

1.

$$A = \begin{bmatrix} 8 & 1 & 15 \\ 3 & 5 & 1 \\ 4 & 9 & -1 \end{bmatrix} \qquad \to \qquad R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$[m=3, n=3, r=2]$

2 $C(A^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 8 \\ 1 \\ 15 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \right\}$

2 $C(A) = \text{span}\left\{ \begin{bmatrix} 8 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} \right\}$

1 $N(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

1 $N(A^T)$

2.

$$A = \begin{bmatrix} 12 & -10 & 5 \\ -9 & -1 & -5 \\ 1 & 3 & 12 \end{bmatrix} \qquad \to \qquad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$[m=3, n=3, r=3]$

3 $C(A^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 12 \\ -10 \\ 5 \end{bmatrix}, \begin{bmatrix} -9 \\ -1 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 12 \end{bmatrix} \right\} = \mathbb{R}^3$

3 $C(A) = \text{span}\left\{ \begin{bmatrix} 12 \\ -9 \\ 1 \end{bmatrix}, \begin{bmatrix} -10 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -5 \\ 12 \end{bmatrix} \right\} = \mathbb{R}^3$

0 $N(A) = Z$

0 $N(A^T) = Z$

3.

$$A = \begin{bmatrix} 2 & -1 & 5 & 2 \\ 2 & 1 & -1 & 6 \end{bmatrix} \qquad \rightarrow \qquad R = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -3 & 2 \end{bmatrix}$$

[$m=2, n=4, r=2$]

2 $C(A^T) = \text{span}\left\{\begin{bmatrix}2\\-1\\5\\2\end{bmatrix}, \begin{bmatrix}2\\1\\-1\\6\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}0\\1\\-3\\2\end{bmatrix}\right\}$

2 $C(A) = \text{span}\left\{\begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$, 2 $N(A) = \text{span}\left\{\begin{bmatrix}-1\\3\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\-2\\0\\1\end{bmatrix}\right\}$

0 $N(A^T) = Z$

4.

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \\ 5 & -1 \\ 2 & 6 \end{bmatrix} \qquad \rightarrow \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

[$m=4, n=2, r=2$]

2 $C(A^T) = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$

2 $C(A) = \text{span}\left\{\begin{bmatrix}2\\-1\\5\\2\end{bmatrix}, \begin{bmatrix}2\\1\\-1\\6\end{bmatrix}\right\}$

0 $N(A) = Z$ 2 $N(A^T)$

5. For A in problem **4**, $\texttt{rref}(A^{\top}) = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -3 & 2 \end{bmatrix}$. From this, find a basis for $N(A^{\top})$.

$N(A^T) = \text{span}\left\{\begin{bmatrix}-1\\3\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\-2\\0\\1\end{bmatrix}\right\}$