

Midterm Exam 1

No book, notes, electronics, calculator, or internet access. 100 points possible. 65 minutes maximum.

1. Consider the following linear system $Ax = b$:

$$\begin{aligned} 2x_1 + x_2 &= 6 \\ -2x_1 + 3x_2 + 2x_3 &= 0 \\ 2x_1 + 9x_2 + 9x_3 &= 13 \end{aligned}$$

$$\leftarrow A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 3 & 2 \\ 2 & 9 & 9 \end{bmatrix}$$

(a) (10 pts) Solve the linear system by *elimination* and then *back-substitution*. Use the standard algorithm. Show your work, and in particular show, as an intermediate stage, the triangular system which you get after elimination.

elimination

$$\begin{bmatrix} 2 & 1 & 0 & 6 \\ -2 & 3 & 2 & 0 \\ 2 & 9 & 9 & 13 \end{bmatrix}$$

$$R_2 \leftarrow R_2 + R_1$$

$$\begin{bmatrix} 2 & 1 & 0 & 6 \\ 0 & 4 & 2 & 6 \\ 2 & 9 & 9 & 13 \end{bmatrix}$$

$$l_{21} = -1$$

$$R_3 \leftarrow R_3 - R_1$$

$$\begin{bmatrix} 2 & 1 & 0 & 6 \\ 0 & 4 & 2 & 6 \\ 0 & 8 & 9 & 7 \end{bmatrix}$$

$$l_{31} = 1$$

$$R_3 \leftarrow R_3 - 2R_2$$

$$\begin{bmatrix} 2 & 1 & 0 & 6 \\ 0 & 4 & 2 & 6 \\ 0 & 0 & 5 & -5 \end{bmatrix}$$

$$l_{32} = 2$$

triangular system
 $U\vec{x} = \vec{z}$

back-subst.

$$5x_3 = -5 \quad \therefore x_3 = -1$$

$$4x_2 + 2x_3 = 6 \quad \therefore x_2 = \frac{6 - 2(-1)}{4} = 2$$

$$2x_1 + x_2 = 6 \quad \therefore x_1 = \frac{6 - (2)}{2} = 2$$

$$\therefore \vec{x} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

(b) (4 pts) From part (a), what elimination matrix E_{32} does the row operation which generated a zero in the (3,2) location?

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & z & 1 \end{bmatrix}$$

(c) (4 pts) From part (a), what three numbers were the pivots?

2, 4, 5

(d) (6 pts) The computation in part (a) can be regarded as factoring $A = LU$. What lower triangular matrix L and upper triangular matrix U were computed?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

(e) (2 pts) Multiply LU and confirm you get the original matrix A .

$$LU = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 3 & 2 \\ 2 & 9 & 9 \end{bmatrix} = A \quad \checkmark$$

2. (10 pts) Suppose A is an invertible $n \times n$ matrix which has a known LU factorization into a lower triangular matrix L and an upper triangular matrix U . That is, suppose $A = LU$. Explain the steps, and name the algorithms, which you would use to solve a linear system $A\mathbf{x} = \mathbf{b}$. (Hint. The elimination stage has already been done. Don't propose to redo it!)

[already done: ① $A = LU$ so $LU\vec{x} = \vec{b}$]

- ② solve $L\vec{c} = \vec{b}$ by forward substitution
 ③ solve $U\vec{x} = \vec{c}$ by back substitution

3. (8 pts) I have claimed that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Show (confirm) that this formula is correct. (Hint. A matrix multiplication suffices.)

$$A^{-1}A = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \checkmark$$

4. True or false? Circle one. Give a short justification if true, and a counterexample if false.

- (a) (3 pts) A matrix with two equal columns is not invertible.

TRUE
FALSE

An invertible matrix must have linearly-independent columns.

- (b) (3 pts) Every triangular matrix with 1's down the main diagonal is invertible.

TRUE
FALSE

Since all pivots are nonzero, we can invert the matrix.

- (c) (3 pts) If A is symmetric, so $a_{ij} = a_{ji}$, then A is invertible.

TRUE
FALSE

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible, but it is symmetric

[the zero matrix works here too]

- (d) (3 pts) If AB and BA are defined then A and B are square.

TRUE
FALSE

$A = \begin{bmatrix} 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are not square,

but $AB = \begin{bmatrix} 5 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ are defined

- (e) (3 pts) If A and B are square matrices of the same size then $(A + B)^2 = A^2 + 2AB + B^2$.

TRUE
FALSE

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ show $AB \neq BA$,

$$\text{so } (A+B)^2 - (A^2 + 2AB + B^2) = A^2 + AB + BA + B^2 - A^2 - 2AB - B^2$$

$$= BA - AB \neq 0.$$

5. (a) (10 pts) Invert this matrix by the Gauss-Jordan method:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Please show your work!

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

The Pivots

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

∴

$$A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(check: $A A^{-1} = I \checkmark$)

- (b) (2 pts) What is the determinant of the matrix in part (a)?

$$\det(A) = 1 \cdot 1 \cdot 1 = 1$$

(product of pivots)

6. (6 pts) Find the angle between these vectors. (Hint. You can write the answer in terms of an inverse trigonometric function, but otherwise everything should be simplified.)

$$\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\cos \theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|} = \frac{2 - 1 - 2}{\sqrt{9} \sqrt{3}} = \frac{-1}{3\sqrt{3}}$$

$$\theta = \arccos\left(\frac{-1}{3\sqrt{3}}\right)$$

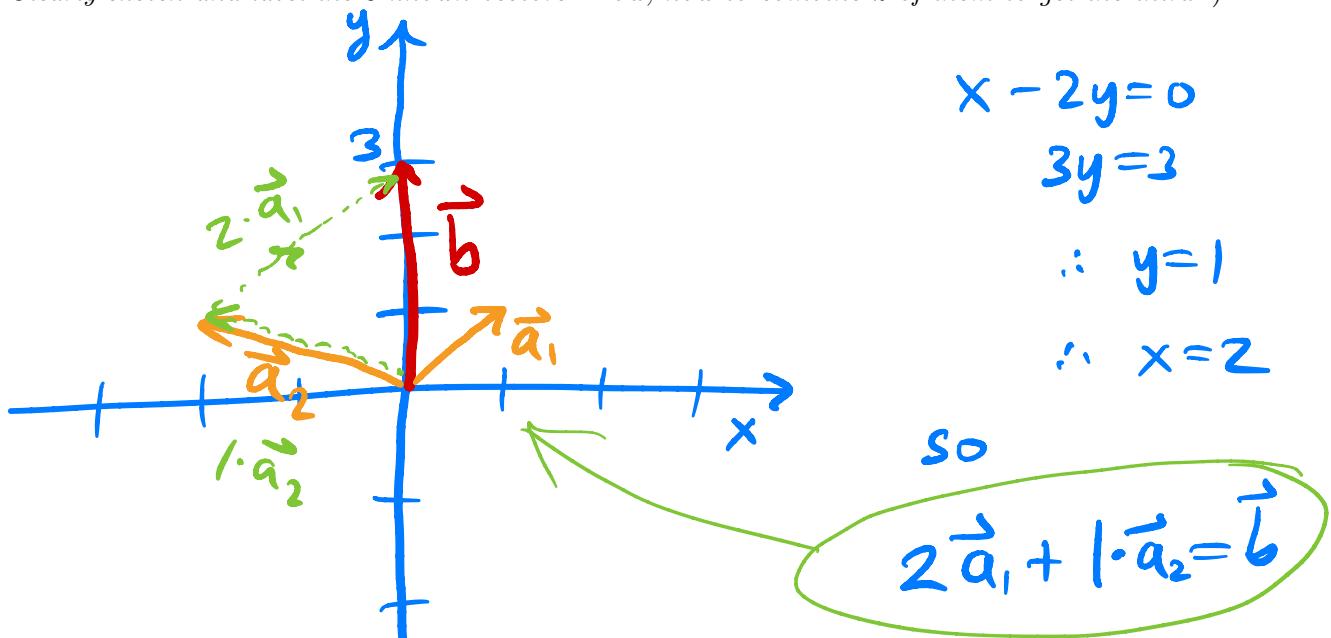
note $\frac{\pi}{2} < \theta < \pi$

since $\cos \theta < 0$

7. (6 pts) Sketch the column picture of this linear system. Noting that the solution values (x, y) play a particular role in this picture, find them and then show the solution in the sketch:

$$\begin{aligned} x - 2y &= 0 \\ x + y &= 3 \end{aligned} \iff A\vec{x} = \vec{b}$$

(Hint. Clearly sketch and label the 3 known vectors. Now, how to combine 2 of them to get the third?)



8. Consider the linear system

$$ax - 2y = 1$$

$$x + 4y = 3$$

- (a) (4 pts) For which number a does elimination break down permanently, so that there are no solutions?

if $\boxed{a = -\frac{1}{2}}$

then permanent
breakdown

$$\begin{array}{r} -\frac{1}{2}x - 2y = 1 \\ x + 4y = 3 \\ \hline -\frac{1}{2}x - 2y = 1 \\ R_2 \leftarrow R_2 + 2R_1 \quad 0x + 0y = 5 \end{array} \leftarrow \text{no soln}$$

- (b) (4 pts) For which number a does elimination break down temporarily, so that a row swap allows a solution?

if $\boxed{a=0}$ then system

$$\begin{array}{l} -2y = 1 \\ x + 4y = 3 \end{array}$$

requires row swap for a solution (by back-subst.)

- (c) (3 pts) For the value of a found in part (b), solve the system.

after swap: $x + 4y = 3$
 $-2y = 1$

back-subst.

$$\boxed{y = -\frac{1}{2}}$$

$$\boxed{x = \frac{3 - 4(-\frac{1}{2})}{1} = 5}$$

Extra Credit. (3 pts) Find the quadratic polynomial $p(x) = a + bx + cx^2$ which passes through the points $(-1, 1)$, $(1, 5)$, $(3, 17)$.

Set-up linear system for coefficients and solve:

$$a - b + c = 1$$

$$a + b + c = 5 \rightarrow$$

$$a + 3b + 9c = 17$$

$$\begin{bmatrix} 1 & -1 & 1 & | & 1 \\ 0 & 2 & 0 & | & 4 \\ 0 & 4 & 8 & | & 16 \\ \hline 1 & -1 & 1 & | & 1 \\ 0 & 2 & 0 & | & 4 \\ 0 & 0 & 8 & | & 8 \end{bmatrix}$$

$$\boxed{\begin{array}{l} a=2 \\ b=2 \\ c=1 \end{array}}$$

9. (6 pts) Suppose $L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$. What is L^{-1} ?

do Gauss-Jordan, but only the elimination part:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ a & 1 & 0 & 0 & 1 & 0 \\ b & c & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & c & 1 & -b & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & 0 & 1 & -b+ac & -c & 1 \end{bmatrix}$$

so

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{bmatrix}$$

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