Review Topics
for in-class **Midterm Quiz 1** on **Wednesday 11 October 2023**

The Midterm Quiz will cover Lectures 1, 2, 3, 4, 5, 6, 7, 8 in Trefethen & Bau. The problems will be of these types: state definitions, state theorems, state algorithms (as pseudocode or Matlab), describe or illustrate geometrical ideas, basic applications of theorems, quick calculations, prove simple theorems/corollaries.

Definitions. Know how to define:

- matrix-vector product; matrix-matrix product
- adjoint; hermitian; transpose
- inner product; outer product
- unitary
- $\|\cdot\|_p$ of a vector in $\mathbb{C}^m$ for any $1 \le p \le \infty$
- induced matrix norm $\|\cdot\|$
- Frobenius matrix norm $\|\cdot\|_F$
- projector; orthogonal projector
- eigenvalue; eigenvector
- singular value

Matrix Factorizations and Constructions. Know the properties of the factors in each factorization below. (*For example, $\hat U$ has orthonormal columns and is of same size as A in the $m \ge n$ case of the reduced SVD factorization $A = \hat U \hat\Sigma V^*$.*) Assume $A \in \mathbb{C}^{m\times n}$ unless otherwise stated. Be able to use the factorization in simple calculations. Be able to compute the factorization by hand in sufficiently small and simple cases.

- full SVD: $A = U\Sigma V^*$
- reduced SVD, $m \ge n$: $A = \hat U \hat\Sigma V^*$
- full QR, $m \ge n$: $A = QR$
- reduced QR, $m \ge n$: $A = \hat Q \hat R$
- eigenvalue, $m = n$: $A = X\Lambda X^{-1}$ $\leftarrow$ *not always possible!*
- orthogonal projector onto range(A): $P = \hat Q \hat Q^* = A\,(A^*A)^{-1}\,A^*$

Algorithms. Be able to state these algorithms, including the amount of work to leading order.

- matrix-vector and matrix-matrix products
- Alg. 7.1: classical Gram-Schmidt for QR
- Alg. 8.1: modified Gram-Schmidt for QR
- high-level algorithm to solve $Ax = b$ when invertible $A = U\Sigma V^*$ has known SVD
- high-level algorithm to solve $Ax = b$ when invertible $A = QR$ has known QR decomposition

Facts and Formulas. Know as facts. Be able to prove unless otherwise stated.

- Cauchy-Schwarz: $|x^* y| \le \|x\| \, \|y\|$ [*proof not required*]
- invariance of $\|\cdot\|_2$ and $\|\cdot\|_F$ matrix norms under unitaries
- $\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$
- $\|A\|_2 = \sigma_1$
- for $A \in \mathbb{C}^{m\times n}$ with $m \ge n$, A has full rank if and only if A^*A is nonsingular
- rank(A) is number of nonzero singular values (in exact arithmetic)
- if $m = n$ then $|\det(A)| = \prod_{i=1}^m \sigma_i$
- the singular values of A are the square roots of the eigenvalues of A^*A
- if P is an (orthogonal) projector then $I - P$ is an (orthogonal) projector
- if P is an orthogonal projector then $I - 2P$ is unitary

Ideas. Please be comfortable with the following ideas! Some ideas correspond to theorems, but otherwise it is just a perspective or paradigm.

- L1 and L2: how to think about Ax, $A^{-1}b$, Qx, Q^*b
- L4: the image of the unit sphere under any $m \times n$ matrix is a hyperellipsoid
- L5: sums like this are optimal (in what sense?) approximations of A:

$$A_\nu = \sum_{j=1}^{\nu} \sigma_j u_j v_j^*$$

- L6: given A, the orthogonal projector onto range(A) is constructable . . . know the formulas . . . full rank versus not full rank?
- L7: construction of orthogonal functions (e.g. orthogonal polynomials) is an application of the Gram-Schmidt process, and/or of $A = QR$, when the columns of A are infinitely long
- L8: the leading-order number of operations in some algorithm can be counted by geometric arguments