Worksheet: If IEEE 754 had a 12-bit standard ...

A model floating point system $\mathbb{F}$ is in Lecture 13 of the textbook (L. Trefethen and D. Bau, *Numerical Linear Algebra*, SIAM Press 1997). Practical systems are implemented in bits and in hardware. The actual IEEE 754 standards for 32-bit single precision and 64-bit double precision representations are cumbersome, so for convenience we pretend here that the standard has a 12-bit version. It might look like this:

s	e_1	e_2	e_3	e_4	b_1	b_2	b_3	b_4	b_5	b_6	b_7

These 12 bits are organized as follows to represent a *nonzero* number:

$$x = (-1)^s \, (1.b_1b_2b_3b_4b_5b_6b_7)_2 \; 2^{(e_1e_2e_3e_4)_2 - (0111)_2}$$

Note that $(1.b_1b_2b_3b_4b_5b_6b_7)_2$ is called the *mantissa*. The power on the 2 is the *exponent*. The special offset $(0111)_2$, equal to 7 in base ten, is called the *exponent bias*. We also define some exceptional cases:

- exponent bits $(0000)_2$ are used for the number zero or subnormal numbers
- exponent bits $(1111)_2$ are used for the other exceptions: $\pm\infty$ and NaN

(No further details of the $(1111)_2$ exceptions will be considered here.) *Normal* numbers have exponents in this range: $1_{10} = (0001)_2 \le (e_1e_2e_3e_4)_2 \le (1110)_2 = 6_{10}$.

(a) What is the largest real number that this system can represent? Show the bits.

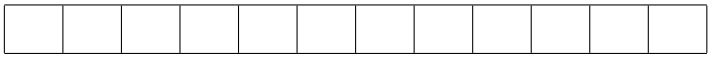
(b) What is the smallest positive number that this system can represent? (*I.e. what is the first normal number to the right of zero?*) Show the bits.

(c) If we define $\epsilon_{\text{machine}}$ as the gap between 1 and the next representable number greater than 1, what is the value of $\epsilon_{\text{machine}}$ in this system?

(d) What is the representation of zero? Show the bits.

(e) What is the representation of 4? Show the bits.

(f) What is the largest representable number which is smaller than 8? Show the bits.



(g) In the interval $[4, 8)$, how many numbers can be represented?

(h) Including zero, exactly how many distinct normal numbers can be represented in this system? (*Exclude subnormal numbers, and exclude exceptions using exponent* $(1111)_2$*, e.g.* $\pm\infty$ *and NaN.*)

(i) Show the bits of one subnormal number.