Teacher's Copy

Algebra 2 Unit 1

Polynomial, Rational, and Radical Relationships

Eureka Math

Table of Contents[1]

Polynomial, Rational, and Radical Relationships

[1] Each lesson is ONE day, and ONE day is considered a 45-minute period.

Algebra II • Module 1

Polynomial, Rational, and Radical Relationships

OVERVIEW

In this module, students draw on their foundation of the analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property (**A-SSE.B.2, A-APR.A.1**). Students connect multiplication of polynomials with multiplication of multi-digit integers, and division of polynomials with long division of integers (**A-APR.A.1**, **A-APR.D.6**). Students identify zeros of polynomials, including complex zeros of quadratic polynomials, and make connections between zeros of polynomials and solutions of polynomial equations (**A-APR.B.3**). Students explore the role of factoring, as both an aid to the algebra and to the graphing of polynomials (**A-SSE.2**, **A-APR.B.2**, **A-APR.B.3**, **F-IF.C.7c**). Students continue to build upon the reasoning process of solving equations as they solve polynomial, rational, and radical equations, as well as linear and non-linear systems of equations (**A-REI.A.1**, **A-REI.A.2**, **A-REI.C.6**, **A-REI.C.7**). The module culminates with the fundamental theorem of algebra as the ultimate result in factoring. Students pursue connections to applications in prime numbers in encryption theory, Pythagorean triples, and modeling problems.

An additional theme of this module is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers. Students use appropriate tools to analyze the key features of a graph or table of a polynomial function and relate those features back to the two quantities that the function is modeling in the problem (**F-IF.C.7c**).

Focus Standards

Reason quantitatively and use units to solve problems.

N-Q.A.2[2] Define appropriate quantities for the purpose of descriptive modeling.★

Perform arithmetic operations with complex numbers.

N-CN.A.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

[2] This standard will be assessed in Algebra II by ensuring that some modeling tasks (involving Algebra II content or securely held content from previous grades and courses) require the student to create a quantity of interest in the situation being described (i.e., this is not provided in the task). For example, in a situation involving periodic phenomena, the student might autonomously decide that amplitude is a key variable in a situation and then choose to work with peak amplitude.

N-CN.A.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations.

N-CN.C.7 Solve quadratic equations with real coefficients that have complex solutions.

Interpret the structure of expressions.

A-SSE.A.2[3] Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

Understand the relationship between zeros and factors of polynomials.

A-APR.B.2[4] Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

A-APR.B.3[5] Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

A-APR.C.4 Prove[6] polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.*

Rewrite rational expressions.

A-APR.D.6[7] Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

Understand solving equations as a process of reasoning and explain the reasoning.

A-REI.A.1[8] Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

---

[3] In Algebra II, tasks are limited to polynomial, rational, or exponential expressions. Examples: see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$. In the equation $x^2 + 2x + 1 + y^2 = 9$, see an opportunity to rewrite the first three terms as $(x+1)^2$, thus recognizing the equation of a circle with radius 3 and center $(-1, 0)$. See $(x^2 + 4)/(x^2 + 3)$ as $((x^2+3) + 1)/(x^2+3)$, thus recognizing an opportunity to write it as $1 + 1/(x^2 + 3)$.

[4] Include problems that involve interpreting the Remainder Theorem from graphs and in problems that require long division.

[5] In Algebra II, tasks include quadratic, cubic, and quadratic polynomials and polynomials for which factors are not provided. For example, find the zeros of $(x^2 - 1)(x^2 + 1)$.

[6] Prove *and apply* (in preparation for Regents Exams).

[7] Include rewriting rational expressions that are in the form of a complex fraction.

[8] In Algebra II, tasks are limited to simple rational or radical equations.

A-REI.A.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Solve equations and inequalities in one variable.

A-REI.B.4[9] Solve quadratic equations in one variable.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Solve systems of equations.

A-REI.C.6[10] Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

A-REI.C.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.*

Analyze functions using different representations.

F-IF.C.7 Graph functions expressed symbolically and show key features of the graph (by hand in simple cases and using technology for more complicated cases).★

c. Graph polynomial functions, identifying zeros when suitable factorizations are available and showing end behavior.

Translate between the geometric description and the equation for a conic section.

G-GPE.A.2 Derive the equation of a parabola given a focus and directrix.

Extension Standards

The (+) standards below are provided as an extension to Module 1 of the Algebra II course to provide coherence to the curriculum. They are used to introduce themes and concepts that will be fully covered in the Precalculus course. *Note: None of the (+) standards below will be assessed on the Regents Exam or PARCC Assessments until Precalculus.*

Use complex numbers in polynomial identities and equations.

N-CN.C.8 (+) Extend polynomial identities to the complex numbers. *For example, rewrite $x^2 + 4$ as $(x + 2i)(x - 2i)$.*

---

[9] In Algebra II, in the case of equations having roots with nonzero imaginary parts, students write the solutions as a ± bi, where *a* and *b* are real numbers.

[10] In Algebra II, tasks are limited to 3 x 3 systems.

N-CN.C.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Rewrite rational expressions.

A-APR.C.7 (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Foundational Standards

Use properties of rational and irrational numbers.

N-RN.B.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Reason quantitatively and use units to solve problems.

N-Q.A.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.★

Interpret the structure of expressions.

A-SSE.A.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1 + r)^n$ *as the product of P and a factor not depending on P.*

Write expressions in equivalent forms to solve problems.

A-SSE.B.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

a. Factor a quadratic expression to reveal the zeros of the function it defines.

Perform arithmetic operations on polynomials.

A-APR.A.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Create equations that describe numbers or relationships.

A-CED.A.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions*.★

A-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

A-CED.A.3 Represent constraints by equations or inequalities and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods*.★

A-CED.A.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning used in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R*.★

Solve equations and inequalities in one variable.

A-REI.B.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

A-REI.B.4 Solve quadratic equations in one variable.

a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

Solve systems of equations.

A-REI.C.5 Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

Represent and solve equations and inequalities graphically.

A-REI.D.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

A-REI.D.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★

Translate between the geometric description and the equation for a conic section.

G-GPE.A.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Focus Standards for Mathematical Practice

MP.1 **Make sense of problems and persevere in solving them.** Students discover the value of equating factored terms of a polynomial to zero as a means of solving equations involving polynomials. Students solve rational equations and simple radical equations, while considering the possibility of extraneous solutions and verifying each solution before drawing conclusions about the problem. Students solve systems of linear equations and linear and quadratic pairs in two variables. Further, students come to understand that the complex number system provides solutions to the equation $x^2 + 1 = 0$ and higher-degree equations.

MP.2 **Reason abstractly and quantitatively.** Students apply polynomial identities to detect prime numbers and discover Pythagorean triples. Students also learn to make sense of remainders in polynomial long division problems.

MP.4 **Model with mathematics.** Students use primes to model encryption. Students transition between verbal, numerical, algebraic, and graphical thinking in analyzing applied polynomial problems. Students model a cross-section of a riverbed with a polynomial, estimate fluid flow with their algebraic model, and fit polynomials to data. Students model the locus of points at equal distance between a point (focus) and a line (directrix) discovering the parabola.

MP.7 **Look for and make use of structure.** Students connect long division of polynomials with the long-division algorithm of arithmetic and perform polynomial division in an abstract setting to derive the standard polynomial identities. Students recognize structure in the graphs of polynomials in factored form and develop refined techniques for graphing. Students discern the structure of rational expressions by comparing to analogous arithmetic problems. Students perform geometric operations on parabolas to discover congruence and similarity.

MP.8 **Look for and express regularity in repeated reasoning.** Students understand that polynomials form a system analogous to the integers. Students apply polynomial identities to detect prime numbers and discover Pythagorean triples. Students recognize factors of expressions and develop factoring techniques. Further, students understand that all quadratics can be written as a product of linear factors in the complex realm.

Terminology

New or Recently Introduced Terms

- **A Square Root of a Number** (*A square root of a number* x is a number whose square is x. In symbols, a square root of x is a number a such that $a^2 = x$. Negative numbers do not have any real square roots, zero has exactly one real square root, and positive numbers have two real square roots.)
- **The Square Root of a Number** (Every positive real number x has a unique positive square root called *the square root* or *principal square root* of x; it is denoted $\sqrt{x}$. The square root of zero is zero.)
- **Pythagorean Triple** (A *Pythagorean triple* is a triplet of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. The triplet $(3, 4, 5)$ is a Pythagorean triple but $(1, 1, \sqrt{2})$ is not, even though the numbers are side lengths of an isosceles right triangle.)

- **End Behavior** (Let f be a function whose domain and range are subsets of the real numbers. The end behavior of a function f is a description of what happens to the values of the function
  - as x approaches positive infinity, and
  - as x approaches negative infinity.)
- **Even Function** (Let f be a function whose domain and range is a subset of the real numbers. The function f is called *even* if the equation, $f(x) = f(-x)$, is true for every number x in the domain. Even-degree polynomial functions are sometimes even functions, such as $f(x) = x^{10}$, and sometimes not, such as $g(x) = x^2 - x$.)
- **Odd Function** (Let f be a function whose domain and range is a subset of the real numbers. The function f is called *odd* if the equation, $f(-x) = -f(x)$, is true for every number x in the domain. Odd-degree polynomial functions are sometimes odd functions, such as $f(x) = x^{11}$, and sometimes not, such as $h(x) = x^3 - x^2$.)
- **Rational Expression** (A *rational expression* is either a numerical expression or a variable symbol, or the result of placing two previously generated rational expressions into the blanks of the addition operator (__+__), the subtraction operator (__−__), the multiplication operator (__×__), or the division operator (__÷__).)
- **Parabola** (A *parabola* with *directrix line* L and *focus point* F is the set of all points in the plane that are equidistant from the point F and line L.)
- **Axis of Symmetry** (The *axis of symmetry of a parabola* given by a focus point and a directrix is the perpendicular line to the directrix that passes through the focus.)
- **Vertex of a Parabola** (The *vertex of a parabola* is the point where the axis of symmetry intersects the parabola.)
- **Dilation at the Origin** (A dilation at the origin D_k is a horizontal scaling by $k > 0$ followed by a vertical scaling by the same factor k. In other words, this dilation of the graph of $y = f(x)$ is the graph of the equation $y = kf\left(\frac{1}{k}x\right)$. A dilation at the origin is a special type of a dilation.)

Familiar Terms and Symbols[11]

- Sequence
- Arithmetic Sequence
- Numerical Symbol
- Variable Symbol
- Algebraic Expression
- Numerical Expression
- Polynomial Expression
- Monomial
- Degree of a Monomial
- Binomial

[11] These are terms and symbols students have seen previously.

- Trinomial
- Coefficient of a Monomial
- Terms of a Polynomial
- Like Terms of a Polynomial
- Standard Form of a Polynomial in One Variable
- Degree of a Polynomial in One Variable
- Equivalent Polynomial Expressions
- Polynomial Identity
- Function
- Polynomial Function
- Degree of a Polynomial Function
- Constant Function
- Linear Function
- Quadratic Function
- Discriminant of a Quadratic Function
- Cubic Function
- Zeros or Roots of a Function
- Increasing/Decreasing
- Relative Maximum
- Relative Minimum
- Graph of f
- Graph of $y = f(x)$

Suggested Tools and Representations

- Graphing Calculator
- Wolfram Alpha Software
- Geometer's Sketchpad Software

Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic B	Constructed response with rubric	N-Q.A.2, A-SSE.A.2, A-APR.B.2, A-APR.B.3, A-APR.C.4, A-REI.A.1, A-REI.B.4b, F-IF.C.7c
End-of-Module Assessment Task	After Topic D	Constructed response with rubric	N-Q.A.2, A.SSE.A.2, A.APR.B.2, A-APR.B.3, A-APR.C.4, A-APR.D.6, A-REI.A.1, A-REI.A.2, A-REI.B.4b, A-REI.C.6, A-REI.C.7, F-IF.C.7c, G-GPE.A.2

Topic C:

Solving and Applying Equations—Polynomial, Rational, and Radical

A-APR.D.6, A-REI.A.1, A-REI.A.2, A-REI.B.4b, A-REI.C.6, A-REI.C.7, G-GPE.A.2

Focus Standard:	A-APR.D.6	Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
	A-REI.A.1	Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.
	A-REI.A.2	Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.
	A-REI.B.4	Solve quadratic equations in one variable. b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.
	A-REI.C.6	Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.
	A-REI.C.7	Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.
	G-GPE.A.2	Derive the equation of a parabola given a focus and directrix.

Instructional Days:	14
Lesson 22:	Equivalent Rational Expressions (S)[1]
Lesson 23:	Comparing Rational Expressions (S)
Lesson 24:	Multiplying and Dividing Rational Expressions (P)
Lesson 25:	Adding and Subtracting Rational Expressions (P)
Lesson 26:	Solving Rational Equations (P)
Lesson 27:	Word Problems Leading to Rational Equations (P)
Lesson 28:	A Focus on Square Roots (P)
Lesson 29:	Solving Radical Equations (P)
Lesson 30:	Linear Systems in Three Variables (P)
Lesson 31:	Systems of Equations (E)
Lesson 32:	Graphing Systems of Equations (S)
Lesson 33:	The Definition of a Parabola (S)
Lesson 34:	Are all Parabolas Congruent? (P)
Lesson 35:	Are all Parabolas Similar? (S)

In Topic C, students continue to build upon the reasoning used to solve equations and their fluency in factoring polynomial expressions. In Lesson 22, students expand their understanding of the division of polynomial expressions to rewriting simple rational expressions (**A-APR.D.6**) in equivalent forms. In Lesson 23, students learn techniques for comparing rational expressions numerically, graphically, and algebraically. In Lessons 24–25, students learn to rewrite simple rational expressions by multiplying, dividing, adding, or subtracting two or more expressions. They begin to connect operations with rational numbers to operations on rational expressions. The practice of rewriting rational expressions in equivalent forms in Lessons 22–25 is carried over to solving rational equations in Lessons 26 and 27. Lesson 27 also includes working with word problems that require the use of rational equations. In Lessons 28–29, we turn to radical equations. Students learn to look for extraneous solutions to these equations as they did for rational equations.

In Lessons 30–32, students solve and graph systems of equations including systems of one linear equation and one quadratic equation and systems of two quadratic equations. Next, in Lessons 33–35, students study the definition of a parabola as they first learn to derive the equation of a parabola given a focus and a directrix and later to create the equation of the parabola in vertex form from the coordinates of the vertex and the location of either the focus or directrix. Students build upon their understanding of rotations and translations from Geometry as they learn that any given parabola is congruent to the one given by the equation $y = ax^2$ for some value of a and that all parabolas are similar.

[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

Lesson 22: Equivalent Rational Expressions

Student Outcomes

- Students define rational expressions and write them in equivalent forms.

Lesson Notes

In this module, we have been working with polynomials and polynomial functions. In elementary school, students mastered arithmetic operations with integers before advancing to performing arithmetic operations with rational numbers. Just as a rational number is built from integers, a rational expression is built from polynomial expressions. A precise definition of a rational expression is included at the end of the lesson.

Informally, a rational expression is any expression that is made by a finite sequence of addition, subtraction, multiplication, and division operations on polynomials. After algebraic manipulation, a rational expression can always be written as $\frac{P}{Q}$, where P is any polynomial and Q is any polynomial except the zero polynomial. Remember that constants, such as 2, and variables, such as x, count as polynomials, so the rational numbers are also considered to be rational expressions. Standard **A-APR.C.6** focuses on rewriting rational expressions in equivalent forms, and in the next three lessons, we apply that standard to write complicated rational expressions in the simplified form $\frac{P}{Q}$. However, the prompt "simplify the rational expression" does not only mean putting expressions in the form $\frac{P}{Q}$ but also any form that is conducive to solving the problem at hand. The skills developed in Lessons 22–25 are necessary prerequisites for addressing standard **A-REI.A.2**, solving rational equations, which is the focus of Lessons 26 and 27.

Classwork

Opening Exercise (8 minutes)

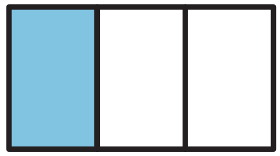
The Opening Exercise serves two purposes: (1) to reactivate prior knowledge of equivalent fractions, and (2) as a review for students who struggle with fractions. We want students to see that the process they use to reduce a fraction to lowest terms is the same they will use to reduce a rational expression to lowest terms. To begin, pass out 2–3 notecard-sized slips of paper to each student or pair of students.

- We are going to start with a review of how to visualize equivalent fractions.

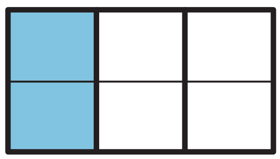
> **Opening Exercise**
>
> **On your own or with a partner, write two fractions that are equivalent to $\frac{1}{3}$, and use the slips of paper to create visual models to justify your response.**

Use the following to either walk through the exercise for scaffolding or as an example of likely student responses.

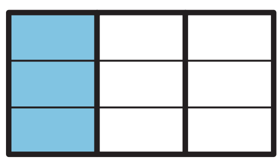
- We can use the following area model to represent the fraction $\frac{1}{3}$. Because the three boxes have the same area, shading one of the three boxes shows that $\frac{1}{3}$ of the area in the figure is shaded.



- Now, if we draw a horizontal line dividing the columns in half, we have six congruent rectangles, two of which are shaded so that $\frac{2}{6}$ of the area in the figure is shaded.



- In the figure below, we have now divided the original rectangle into nine congruent sub-rectangles, three of which are shaded so that $\frac{3}{9}$ of the area in the figure is shaded.



Scaffolding:

Students who are already comfortable with fractions can instead reduce the following rational expressions to lowest terms.

$$\frac{5}{15}, \frac{27}{36}, \frac{\sqrt{75}}{5}, \frac{\pi^4}{\pi^2}$$

In any case, do not spend too much time on these exercises; instead, use them as a bridge to reducing rational expressions that contain variables.

- Let's suppose that the area of the original rectangle is 1. In walking the class through the example, point out that the shaded area in the first figure is $\frac{1}{3}$, the shaded area in the second figure is $\frac{2}{6}$, and the shaded area in the third figure is $\frac{3}{9}$. Since the area of the shaded regions are the same in all three figures, we see that $\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$. Thus, $\frac{1}{3}$, $\frac{2}{6}$, and $\frac{3}{9}$ are equivalent fractions.

If students have come up with different equivalent fractions, then incorporate those into the discussion of equivalent areas, noting that the shaded regions are the same for every student.

MP.8

- Now, what if we were to choose any positive integer n and draw lines across our figure so that the columns are divided into n pieces of the same size? What is the area of the shaded region?

Give students time to think and write, and ask them to share their answers with a partner. Anticipate that students will express the generalization in words or suggest either $\frac{1}{3}$ or $\frac{n}{3n}$. Both are correct and, ideally, both will be suggested.

- Thus, we have the rule:

> If a, b, and n are integers with $n \neq 0$ and $b \neq 0$, then
> $$\frac{na}{nb} = \frac{a}{b}.$$

> *Scaffolding:*
> Students may also express the generalization in words.

The result summarized in the box above is also true for real numbers a, b, and n as well as for polynomial and rational expressions.

- Then $\frac{2}{6} = \frac{2(1)}{2(3)} = \frac{1}{3}$ and $\frac{3}{9} = \frac{3(1)}{3(3)} = \frac{1}{3}$.
- We say that a rational number is simplified, or reduced to lowest terms, when the numerator and denominator do not have a factor in common. Thus, while $\frac{1}{3}$, $\frac{2}{6}$, and $\frac{3}{9}$ are equivalent, only $\frac{1}{3}$ is in lowest terms.

Discussion (10 minutes)

- Which of the following are rational numbers, and which are not?

$$\frac{3}{4}, 3.14, \pi, \frac{5}{0}, -\sqrt{17}, 23, \frac{1+\sqrt{5}}{2}, -1, 6.022 \times 10^{23}, 0$$

  - *Rational:* $\frac{3}{4}, 3.14, 23, -1, 6.022 \times 10^{23}, 0$
  - *Not rational:* $\pi, \frac{5}{0}, -\sqrt{17}$, $\frac{1+\sqrt{5}}{2}$

Of the numbers that were not rational, were they all irrational numbers?

  - *No. Since division by zero is undefined, $\frac{5}{0}$ is neither rational nor irrational.*

- Today we learn about rational expressions, which are related to the polynomials we've been studying. Just as the integers are the foundational building blocks of rational numbers, polynomial expressions are the foundational building blocks for rational expressions. Based on what we know about rational numbers, give an example of what you think a rational expression is.

> *Scaffolding:*
> Relate the new ideas of rational expressions back to the more familiar ideas of rational numbers throughout this lesson.

Ask students to write down an example and share it with their partner or small group. Allow groups to debate and present one of the group's examples to the class.

- Recall that a rational number is a number that we can write as $\frac{p}{q}$, where p and q are integers and q is nonzero. We can consider a new type of expression, called a *rational expression*, which is made from polynomials by adding, subtracting, multiplying, and dividing them. Any rational expression can be expressed as $\frac{P}{Q}$, where P and Q are polynomial expressions and Q is not zero, even though it may not be presented in this form originally.

Remind students that numbers are also polynomial expressions, which means that rational numbers are included in the set of rational expressions.

- The following are examples of rational expressions. Notice that we need to exclude values of the variables that make the denominators zero so that we do not divide by zero.
- $\frac{31}{47}$
    - *The denominator is never zero, so we do not need to exclude any values.*
- $\frac{ab^2}{3a-2b}$
    - *We need $3a \neq 2b$.*
- $\frac{5x+1}{3x^2+4}$
    - *The denominator is never zero, so we do not need to exclude any values.*
- $\frac{3}{b^2-7}$
    - *We need $b \neq \sqrt{7}$ and $b \neq -\sqrt{7}$.*

Have students create a Frayer model in their notebooks, such as the one provided. Circulate around the classroom to informally assess student understanding. Since a formal definition of rational expressions has not yet been given, there is some leeway on the description and characteristics sections, but make sure that nothing they have written is incorrect. Ask students to share their characteristics, examples, and non-examples to populate a class model on the board.

Description *An expression that can be written as $\frac{P}{Q}$, where P and Q are polynomials and Q is not zero.*	Characteristics *Follows similar rules as rational numbers do.*
Rational Expression	
$\frac{3}{5}$, $\frac{x^2-4x}{x+1}$ *with $x \neq -1$* $\frac{x^2y}{2}$, $\frac{a^2+b^2}{(a+2)(a-1)}$, *with $a \neq -2, 1$* Examples	$\frac{x+1}{0}$ *(cannot divide by 0)* $\frac{2^x}{3x}$ *(2^x is not a polynomial)* Non-examples

It is important to note that the excluded values of the variables remain even after simplification. This is because the two expressions would not be equal if the variables were allowed to take on these values. Discuss with a partner when the following are not equivalent and why:

- $\frac{2x}{3x}$ and $\frac{2}{3}$
    - *These are equivalent everywhere except at $x = 0$. At $x = 0$, $\frac{2x}{3x}$ is undefined, whereas $\frac{2}{3}$ is equal to $\frac{2}{3}$.*

> *Scaffolding:*
> Encourage struggling students to plug in various values of the variables to see that the expressions *are* equivalent for almost all values of the variables. But for values in which the denominator of one expression is equal to zero, they are not equivalent.

- $\frac{3x(x-5)}{4(x-5)}$ and $\frac{3x}{4}$
  - *At $x = 5$, $\frac{3x(x-5)}{4(x-5)}$ is undefined, whereas $\frac{3x}{4} = \frac{3(5)}{4} = \frac{15}{4}$.*
- $\frac{x-3}{x^2-x-6}$ and $\frac{1}{x+2}$
  - *At $x = 3$, $\frac{x-3}{x^2-x-6}$ is undefined, whereas $\frac{1}{x+2} = \frac{1}{3+2} = \frac{1}{5}$.*
- Summarize with your partner or in writing any conclusions you can draw about equivalent rational expressions. Circulate around the classroom to assess understanding.

Example 1 (6 minutes)

Example 1

Consider the following rational expression: $\frac{2(a-1)-2}{6(a-1)-3a}$. Turn to your neighbor and discuss the following: For what values of a is the expression undefined?

Sample the students' answers. When they suggest that the denominator cannot be zero, give the class a minute to work out that the denominator is zero when $a = 2$.

$$\frac{2(a-1)-2}{6(a-1)-3a}$$

$$\begin{aligned} 6(a-1)-3a &= 0 \\ 6a-6-3a &= 0 \\ 3a-6 &= 0 \\ a &= 2 \end{aligned}$$

- Let's reduce the rational expression $\frac{2(a-1)-2}{6(a-1)-3a}$ with $a \neq 2$ to lowest terms. Since no common factor is visible in the given form of the expression, we first simplify the numerator and denominator by distributing and combining like terms.

$$\begin{aligned} \frac{2(a-1)-2}{6(a-1)-3a} &= \frac{2a-2-2}{6a-6-3a} \\ &= \frac{2a-4}{3a-6} \end{aligned}$$

- Next, we factor the numerator and denominator, and divide both by any common factors. This step shows clearly why we had to specify that $a \neq 2$.

$$\begin{aligned} \frac{2(a-1)-2}{6(a-1)-3a} &= \frac{2a-4}{3a-6} \\ &= \frac{2(a-2)}{3(a-2)} \\ &= \frac{2}{3} \end{aligned}$$

Scaffolding:

Students may need to be reminded that although $(a-1)$ appears in the numerator and denominator, it is not a common *factor* to the numerator and denominator, and thus, we cannot simplify the expression by dividing by $(a-1)$.

- As long as $a \neq 2$, we see that $\frac{2(a-1)-2}{6(a-1)-3a}$ and $\frac{2}{3}$ are equivalent rational expressions.

If we allow a to take on the value of 2, then $\frac{2(a-1)-2}{6(a-1)-3a}$ is undefined. However, the expression $\frac{2}{3}$ is always defined so these expressions are not equivalent.

Exercise 1 (10 minutes)

Allow students to work on the following exercises in pairs.

Exercise 1

Reduce the following rational expressions to lowest terms, and identify the values of the variable(s) that must be excluded to prevent division by zero.

a. $\frac{2(x+1)+2}{(2x+3)(x+1)-1}$

$\frac{2(x+1)+2}{(2x+3)(x+1)-1} = \frac{2x+4}{2x^2+5x+2} = \frac{2(x+2)}{(2x+1)(x+2)} = \frac{2}{2x+1}$, *for* $x \neq -2$ *and* $x \neq -\frac{1}{2}$.

b. $\frac{x^2-x-6}{5x^2+10x}$

$\frac{x^2-x-6}{5x^2+10x} = \frac{(x+2)(x-3)}{5x(x+2)} = \frac{x-3}{5x}$, *for* $x \neq 0$ *and* $x \neq -2$.

c. $\frac{3-x}{x^2-9}$

$\frac{3-x}{x^2-9} = \frac{-(x-3)}{(x-3)(x+3)} = -\frac{1}{x+3}$, *for* $x \neq 3$ *and* $x \neq -3$.

d. $\frac{3x-3y}{y^2-2xy+x^2}$

$\frac{3x-3y}{y^2-2xy+x^2} = \frac{-3(y-x)}{(y-x)(y-x)} = -\left(\frac{3}{y-x}\right)$, *for* $y \neq x$.

Closing (5 minutes)

The precise definition of a rational expression is presented here for teacher reference and may be shared with students at your discretion. Discussion questions for closing this lesson follow the definition. Notice the similarity between the definition of a rational expression given here and the definition of a polynomial expression given in the closing of Lesson 5 earlier in this module.

> **Rational Expression:** A *rational expression* is either a numerical expression or a variable symbol or the result of placing two previously generated rational expressions into the blanks of the addition operator (__+__), the subtraction operator (__−__), the multiplication operator (__×__), or the division operator (__÷__).

Have students discuss the following questions with a partner and write down their conclusions. Circulate around the room to assess their understanding.

- How do you reduce a rational expression of the form $\frac{P}{Q}$ to lowest terms?
  - *Factor the polynomial expressions in the numerator and denominator, and divide any common factors from both the numerator and denominator.*
- How do you know which values of the variable(s) to exclude for a rational expression?
  - *Any value of the variable(s) that makes the denominator zero at any point of the process must be excluded.*

Lesson Summary

- **If $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{n}$ are integers with $\boldsymbol{n \neq 0}$ and $\boldsymbol{b \neq 0}$, then**
$$\frac{na}{nb} = \frac{a}{b}.$$
- **The rule for rational expressions is the same as the rule for integers but requires the domain of the rational expression to be restricted (i.e., no value of the variable that can make the denominator of the original rational expression zero is allowed).**

Exit Ticket (6 minutes)

Name ____ Date ____________________

Lesson 22: Equivalent Rational Expressions

Exit Ticket

1. Find an equivalent rational expression in lowest terms, and identify the value(s) of the variables that must be excluded to prevent division by zero.

$$\frac{x^2 - 7x + 12}{6 - 5x + x^2}$$

2. Determine whether or not the rational expressions $\frac{x+4}{(x+2)(x-3)}$ and $\frac{x^2+5x+4}{(x+1)(x+2)(x-3)}$ are equivalent for $x \neq -1$, $x \neq -2$, and $x \neq 3$. Explain how you know.

Exit Ticket Sample Solutions

1. **Find an equivalent rational expression in lowest terms, and identify the value(s) of the variables that must be excluded to prevent division by zero.**

   If $x \neq 3$ and $x \neq 2$, then we have

$$\frac{x^2 - 7x + 12}{6 - 5x + x^2} = \frac{(x-4)(x-3)}{(x-3)(x-2)} = \frac{x-4}{x-2}.$$

2. **Determine whether or not the rational expressions $\frac{x+4}{(x+2)(x-3)}$ and $\frac{x^2+5x+4}{(x+1)(x+2)(x-3)}$ are equivalent for $x \neq -1$, $x \neq -2$, and $x \neq 3$. Explain how you know.**

   Since $\frac{x^2+5x+4}{(x+1)(x+2)(x-3)} = \frac{(x+1)(x+4)}{(x+1)(x+2)(x-3)} = \frac{x+4}{(x+2)(x-3)}$ as long as $x \neq -1$, $x \neq -2$, and $x \neq 3$, the rational expressions $\frac{x+4}{(x+2)(x-3)}$ and $\frac{x^2+5x+4}{(x+1)(x+2)(x-3)}$ are equivalent.

Problem Set Sample Solutions

1. **Find an equivalent rational expression in lowest terms, and identify the value(s) of the variable that must be excluded to prevent division by zero.**

a.	$\frac{16n}{20n}$	$\frac{4}{5}$;	$n \neq 0$
b.	$\frac{x^3y}{y^4x}$	$\frac{x^2}{y^3}$;	$x \neq 0$ *and* $y \neq 0$
c.	$\frac{2n-8n^2}{4n}$	$\frac{1-4n}{2}$;	$n \neq 0$
d.	$\frac{db+dc}{db}$	$\frac{b+c}{b}$;	$b \neq 0$ *and* $d \neq 0$.
e.	$\frac{x^2-9b^2}{x^2-2xb-3b^2}$	$\frac{x+3b}{x+b}$;	$x \neq 3b$ *and* $x \neq -b$
f.	$\frac{3n^2-5n-2}{2n-4}$	$\frac{3n+1}{2}$;	$n \neq 2$
g.	$\frac{3x-2y}{9x^2-4y^2}$	$\frac{1}{3x+2y}$;	$y \neq \frac{3}{2}x$ *and* $y \neq -\frac{3}{2}x$
h.	$\frac{4a^2-12ab}{a^2-6ab+9b^2}$	$\frac{4a}{a-3b}$;	$a \neq 3b$
i.	$\frac{y-x}{x-y}$	-1;	$x \neq y$
j.	$\frac{a^2-b^2}{b+a}$	$a - b$;	$a \neq -b$
k.	$\frac{4x-2y}{3y-6x}$	$-\frac{2}{3}$;	$y \neq 2x$
l.	$\frac{9-x^2}{(x-3)^3}$	$-\frac{3+x}{(x-3)^2}$;	$x \neq 3$

m. $\frac{x^2-5x+6}{8-2x-x^2}$ $-\frac{x-3}{4+x}$; $x \neq 2$ *and* $x \neq -4$

n. $\frac{a-b}{xa-xb-a+b}$ $\frac{1}{x-1}$; $x \neq 1$ *and* $a \neq b$

o. $\frac{(x+y)^2-9a^2}{2x+2y-6a}$ $\frac{x+y+3a}{2}$; $a \neq \frac{x+y}{3}$

p. $\frac{8x^3-y^3}{4x^2-y^2}$ $\frac{4x^2+2xy+y^2}{2x+y}$; $y \neq 2x$ *and* $y \neq -2x$

2. Write a rational expression with denominator $6b$ that is equivalent to

a. $\frac{a}{b}$.

$\frac{6a}{6b}$

b. one-half of $\frac{a}{b}$.

$\frac{3a}{6b}$

c. $\frac{1}{3}$.

$\frac{2b}{6b}$

3. Remember that algebra is just another way to perform arithmetic but with variables replacing numbers.

a. Simplify the following rational expression: $\frac{(x^2y)^2(xy)^3z^2}{(xy^2)^2yz}$.

$$\frac{(x^2y)^2(xy)^3z^2}{(xy^2)^2yz} = \frac{x^4y^2 \cdot x^3y^3 \cdot z^2}{x^2y^4 \cdot yz} = \frac{x^7y^5z^2}{x^2y^5z} = x^5z$$

b. Simplify the following rational expression without using a calculator: $\frac{12^2 \cdot 6^3 \cdot 5^2}{18^2 \cdot 15}$.

$$\frac{12^2 \cdot 6^3 \cdot 5^2}{18^2 \cdot 15} = \frac{4^2 \cdot 3^2 \cdot 6^3 \cdot 5^2}{2^2 \cdot 9^2 \cdot 3 \cdot 5} = \frac{2^4 \cdot 3^2 \cdot 2^3 \cdot 3^3 \cdot 5^2}{2^2 \cdot 3^4 \cdot 3 \cdot 5} = \frac{2^7 \cdot 3^5 \cdot 5^2}{2^2 \cdot 3^5 \cdot 5} = 2^5 \cdot 5 = 32 \cdot 5 = 160$$

MP.7

c. How are the calculations in parts (a) and (b) similar? How are they different? Which expression was easier to simplify?

Both simplifications relied on using the rules of exponents. It was easier to simplify the algebraic expression in part (a) because we did not have to factor any numbers, such as 18, 15, and 12. However, if we substitute $x = 2$, $y = 3$, and $z = 5$, these two expressions have the exact same structure. Algebra allows us to do this calculation more quickly.

Lesson 23: Comparing Rational Expressions

Student Outcomes

- Students compare rational expressions by writing them in different but equivalent forms.

Lesson Notes

The skills developed in Lessons 22–25 are prerequisites for addressing standard **A-REI.A.2**, solving rational equations, which is the focus of Lessons 26 and 27. In this lesson, students extend comparisons of rational numbers to comparing rational expressions and using numerical, graphical, and algebraic analysis. Although students use graphing calculators to compare certain rational expressions, learning to graph rational functions is not the focus of this lesson.

Classwork

Opening Exercise (10 minutes)

The Opening Exercise serves two purposes: (1) to reactivate prior knowledge of comparing fractions and (2) as a review for students who struggle with fractions. We want students to see that the same process is used to compare fractions and to compare rational expressions.

As done in the previous lesson, give students slips of notecard-sized paper on which to make visual arguments for which fraction is larger. Each student (or pair of students) should get at least two slips of paper. This exercise leads us to the graphical analysis employed in the last example of the lesson.

> **Opening Exercise**
>
> **Use the slips of paper you have been given to create visual arguments for which fraction is larger.**

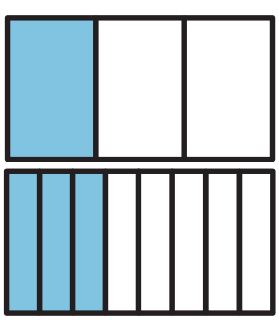
Scaffolding:

Students already comfortable with fractions may wish to only briefly review the visual representations; however, it is important for each student to be aware of the three methods of comparison, in general, graphical (or visual), numerical, and algebraic (by finding a common denominator).

In any case, do not spend too much time on these exercises, but instead use them as a bridge to comparing rational expressions that contain variables.

Ask students to make visual arguments as to whether $\frac{1}{3}$ or $\frac{3}{8}$ is larger. Use the following as either scaffolding for struggling students or as an example of student work.

- We can use the following area models to represent the fraction $\frac{1}{3}$ and $\frac{3}{8}$ as we did in Lesson 22.



- We see that these visual representations of the fractions give us strong evidence that $\frac{1}{3} < \frac{3}{8}$.
- Discuss with your neighbor another way to make a comparison between the two fractions and why we might not want to always rely on visual representations.
  - *Students should suggest finding decimal approximations of fractions and converting the fractions to equivalent fractions with common denominators. Reasons for not using visual representations may include the difficulty with fractions with large denominators or with irrational parts.*

Once students have had a chance to discuss alternative methods, ask them to choose one of the two methods to verify that the visual representations above are accurate.

  - *Decimal approximations: We have* $\frac{1}{3} \approx 0.333$ *and* $\frac{3}{8} = 0.375$*; thus,* $\frac{1}{3} < \frac{3}{8}$.
  - *Common denominators: We have* $\frac{1}{3} = \frac{8}{24}$ *and* $\frac{3}{8} = \frac{9}{24}$*. Since* $\frac{8}{24} < \frac{9}{24}$*, we know that* $\frac{1}{3} < \frac{3}{8}$.
- Discuss with your partner the pros and cons of both methods before discussing as a class.
  - *Decimal approximations are quick with a calculator but may take a while if long division is needed. Many students prefer decimals to fractions, but they use approximations of the numbers instead of the exact values of the numbers. Common denominators use the actual numbers but are still working with fractions.*
- Just as we can determine whether two rational expressions are equivalent in a similar way as we can with rational numbers, we can extend our ideas of comparing rational numbers to comparing rational expressions.

Exercises 1–5 (11 minutes)

As the students work on Exercises 1–5, circulate through the class to assess their understanding.

Exercises 1–5

We will start by working with positive integers. Let m and n be positive integers. Work through the following exercises with a partner.

1. Fill out the following table.

n	$\frac{1}{n}$
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$
4	$\frac{1}{4}$
5	$\frac{1}{5}$
6	$\frac{1}{6}$

2. **Do you expect $\frac{1}{n}$ to be bigger or smaller than $\frac{1}{n+1}$? Do you expect $\frac{1}{n}$ to be bigger or smaller than $\frac{1}{n+2}$? Explain why.**

From the table, as n increases, $\frac{1}{n}$ decreases. This means that if you add 1 to n, then you will get a smaller number. That is, $\frac{1}{n} > \frac{1}{n+1} > \frac{1}{n+2}$.

3. **Compare the rational expressions $\frac{1}{n}$, $\frac{1}{n+1}$, and $\frac{1}{n+2}$ for $n = 1$, 2, and 3. Do your results support your conjecture from Exercise 2? Revise your conjecture if necessary.**

For $n = 1$, $\frac{1}{n} = 1$, $\frac{1}{n+1} = \frac{1}{2}$, and $\frac{1}{n+2} = \frac{1}{3}$, and we know $1 > \frac{1}{2} > \frac{1}{3}$.

For $n = 2$, we have $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, and we know $\frac{1}{2} > \frac{1}{3} > \frac{1}{4}$.

For $n = 3$, we have $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$, and we know $\frac{1}{3} > \frac{1}{4} > \frac{1}{5}$.

This supports the conjecture that $\frac{1}{n} > \frac{1}{n+1} > \frac{1}{n+2}$.

4. **From your work in Exercises 1 and 2, generalize how $\frac{1}{n}$ compares to $\frac{1}{n+m}$, where m and n are positive integers.**

Since m is a positive integer being added to n, the denominator will increase, which will decrease the value of the rational expression overall. That is, $\frac{1}{n} > \frac{1}{n+m}$ for positive integers m and n.

5. **Will your conjecture change or stay the same if the numerator is 2 instead of 1? Make a conjecture about what happens when the numerator is held constant, but the denominator increases for positive numbers.**

It will stay the same because this would be the same as multiplying the inequality by 2, and multiplication by a positive number does not change the direction of the inequality. If the numerator is held constant and the denominator increases, you are dividing by a larger number, so you get a smaller number overall.

Example 1 (11 minutes)

- Suppose we want to compare the values of the rational expressions $\frac{x+1}{x}$ and $\frac{x+2}{x+1}$ for positive values of x. What are some ways to do this?

Ask students to suggest some methods of comparison. If needed, guide them to the ideas of using a numerical comparison through a table of values and a graphical comparison of the related rational functions $y = \frac{x+1}{x}$ and $y = \frac{x+2}{x+1}$ for $x > 0$.

- Let's start our comparison of $\frac{x+1}{x}$ and $\frac{x+2}{x+1}$ by looking at a table of values.

Have students complete the table below using their calculators and rounding to four decimal places.

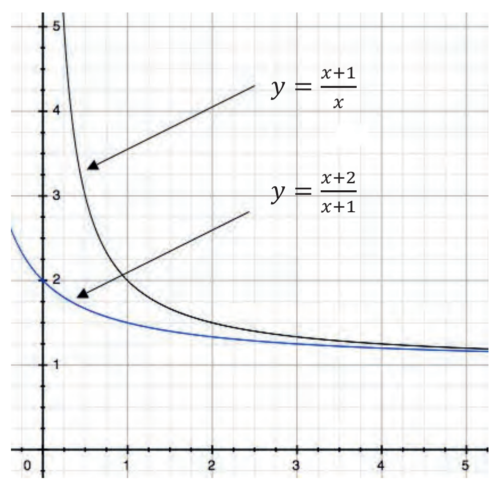
Example 1

x	$\frac{x+1}{x}$	$\frac{x+2}{x+1}$
0.5	3	1.6667
1	2	1.5000
1.5	1.6667	1.4000
2	1.5	1.3333
5	1.2	1.1667
10	1.1	1.0909
100	1.01	1.0099

Discussion

- From the table of values, it appears that $\frac{x+1}{x} > \frac{x+2}{x+1}$ for positive values of x. However, we have only checked 7 values of x, so we cannot yet say that this is the case for every positive value of x. How else can we compare the values of these two expressions?
  - *Students should suggest graphing the functions $y = \frac{x+1}{x}$ and $y = \frac{x+2}{x+1}$.*

Have students graph the two functions $y = \frac{x+1}{x}$ and $y = \frac{x+2}{x+1}$ on their calculators, and ask them to share their observations. Does the graph verify the conclusions we drew from the table above?



- It seems from both the table of data and from the graph that $\frac{x+1}{x} > \frac{x+2}{x+1}$ for positive values of x, but we have not shown it conclusively. How can we use algebra to determine if this inequality is always true?
  - *We want to compare $\frac{x+1}{x}$ and $\frac{x+2}{x+1}$.*

Ask students what they need to do before comparing fractions such as $\frac{3}{8}$ and $\frac{2}{7}$. Wait for someone to suggest that they need to find a common denominator.

- Let's take a step back and see how we would compare the two fractions $\frac{3}{8}$ and $\frac{2}{7}$. First, we find the common denominator.

Wait for a student to volunteer that the common denominator is 56.

- Next, we rewrite each fraction as an equivalent fraction with denominator 56:

$$\frac{3}{8} = \frac{21}{56} \text{ and } \frac{2}{7} = \frac{16}{56}.$$

- Since $21 > 16$, and 56 is a positive number, we know that $\frac{21}{56} > \frac{16}{56}$; thus, we know that $\frac{3}{8} > \frac{2}{7}$.
- The process for comparing rational expressions is the same as the process for comparing fractions. As is always the case with inequalities, we need to be careful about changing the inequality if we multiply or divide by a negative number.
- What is the common denominator of the two expressions $\frac{x+1}{x}$ and $\frac{x+2}{x+1}$?
  - $x(x+1)$
- First, multiply the numerator and denominator of the first expression by $(x+1)$:

$$\boldsymbol{\frac{x+1}{x} = \frac{(x+1)(x+1)}{x(x+1)}}$$
$$\boldsymbol{\frac{x+1}{x} = \frac{x^2+2x+1}{x(x+1)}}.$$

- Next, multiply the numerator and denominator of the second expression by x:

$$\frac{x+2}{x+1} = \frac{x(x+2)}{x(x+1)}$$
$$\frac{x+2}{x+1} = \frac{x^2+2x}{x(x+1)}.$$

- Clearly, we have

$$x^2 + 2x + 1 > x^2 + 2x,$$

  and since x is always positive, we know that the denominator $x(x+1)$ is always positive. Thus, we see that

$$\frac{x^2+2x+1}{x(x+1)} > \frac{x^2+2x}{x(x+1)},$$

  so we have established that $\frac{x+1}{x} > \frac{x+2}{x+1}$ for all positive values of x.
- For rational expressions, numerical and visual comparisons can provide evidence that one expression is larger than another for specified values of the variable. However, finding common denominators and doing the algebra to show that one is larger than the other is the conclusive way to show that the values of one rational expression are consistently larger than the values of another.

Closing (6 minutes)

Ask students to do a side-by-side comparison of the different methods for comparing rational numbers to the extended method used for comparing rational expressions.

Rational Numbers	Rational Expressions
▪ Visually: Use area models or number lines to represent fractions and compare their relative sizes. Difficult with large numbers. ▪ Numerically: Perform the division to find a decimal approximation to compare the sizes. ▪ Algebraically: Find equivalent fractions with common denominators and compare their numerators.	▪ Visually: Use a graphing utility to graph functions representing each expression and compare their relative heights. Easy with technology but inconclusive. ▪ Numerically: Compare several values of the functions to see their relative sizes. Straightforward but tells us even less than graphing. ▪ Algebraically: Find equivalent fractions with common denominators and compare their numerators. Best way but special care needs to be taken with values that may be negative.

- How do you compare two rational expressions of the form $\frac{P}{Q}$?
  - *Before comparing the expressions, find equivalent rational expressions with the same denominator. Then we can compare the numerators for values of the variable that do not cause the positive/negative signs to switch. Numerical and graphical analysis may be used to help understand the relative sizes of the expressions.*

> **Lesson Summary**
>
> **To compare two rational expressions, find equivalent rational expression with the same denominator. Then we can compare the numerators for values of the variable that do not cause the rational expression to change from positive to negative or vice versa.**
>
> **We may also use numerical and graphical analysis to help understand the relative sizes of expressions.**

Exit Ticket (7 minutes)

Name ______________________________ Date ____________________

Lesson 23: Comparing Rational Expressions

Exit Ticket

Use the specified methods to compare the following rational expressions: $\frac{x+1}{x^2}$ and $\frac{1}{x}$ for $x > 0$.

1. Fill out the table of values.

x	$\frac{x+1}{x^2}$	$\frac{1}{x}$
1		
10		
25		
50		
100		
500		

2. Graph $y = \frac{x+1}{x^2}$ and $y = \frac{1}{x}$ for positive values of x.

3. Find the common denominator and compare numerators for positive values of x.

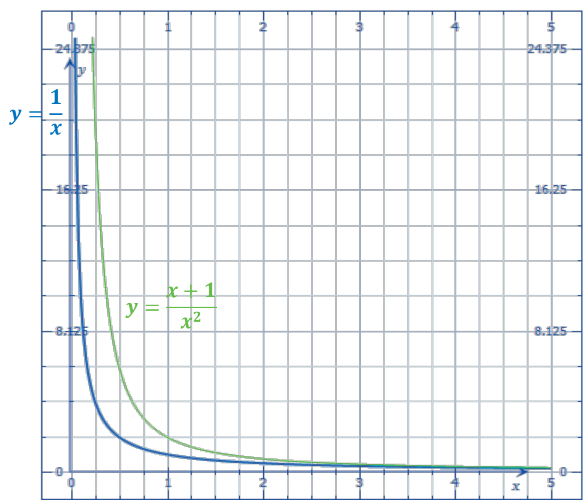
Exit Ticket Sample Solutions

Use the following methods to compare the following rational expressions: $\frac{x+1}{x^2}$ and $\frac{1}{x}$.

1. Fill out the table of values.

x	$\frac{x+1}{x^2}$	$\frac{1}{x}$
1	$\frac{2}{1} = 2$	$\frac{1}{1} = 1$
10	$\frac{11}{100} = 0.11$	$\frac{1}{10} = 0.1$
25	$\frac{26}{625} = 0.0416$	$\frac{1}{25} = 0.04$
50	$\frac{51}{2500} = 0.0204$	$\frac{1}{50} = 0.02$
100	$\frac{101}{10000} = 0.0101$	$\frac{1}{100} = 0.01$
500	$\frac{501}{250000} = 0.002004$	$\frac{1}{500} = 0.002$

2. Graph $y = \frac{x+1}{x^2}$ and $y = \frac{1}{x}$ for positive values of x.



3. **Find the common denominator and compare numerators for positive values of x.**

The common denominator of x and x^2 is x^2.

$$\frac{1}{x} = \frac{1}{x} \cdot \frac{x}{x} = \frac{x}{x^2}$$
$$\frac{x+1}{x^2} = \frac{x+1}{x^2}$$

For any value of x, x^2 is positive. Since

$$x + 1 > x,$$

we then have,

$$\frac{x+1}{x^2} > \frac{x}{x^2}$$
$$\frac{x+1}{x^2} > \frac{1}{x}.$$

Problem Set Sample Solutions

1. **For parts (a)–(d), rewrite each rational expression as an equivalent rational expression so that each expression has a common denominator.**

   a. $\frac{3}{5}, \frac{9}{10}, \frac{7}{15}, \frac{7}{21}$

   $\frac{18}{30}, \frac{27}{30}, \frac{14}{30}, \frac{10}{30}$

   b. $\frac{m}{sd}, \frac{s}{dm}, \frac{d}{ms}$

   $\frac{m^2}{msd}, \frac{s^2}{sdm}, \frac{d^2}{dms}$

   c. $\frac{1}{(2-x)^2}, \frac{3}{(2x-5)(2-x)}$

   $\frac{(2x-5)}{(2-x)^2(2x-5)}, \frac{-3(2-x)}{(2x-5)(2-x)^2}$

   d. $\frac{3}{x^2-x}, \frac{5}{x}, \frac{2x+2}{2x^2-2}$

   $\frac{3(x+1)}{x(x-1)(x+1)}, \frac{5(x-1)(x+1)}{x(x-1)(x+1)}, \frac{x(x+1)}{x(x-1)(x+1)}$

2. **If x is a positive number, for which values of x is $x < \frac{1}{x}$?**

   Before we can compare two rational expressions, we need to express them as equivalent expressions with a common denominator. Since $x \neq 0$, we have $x = \frac{x^2}{x}$. Then $x < \frac{1}{x}$ exactly when $\frac{x^2}{x} < \frac{1}{x}$, which happens when $x^2 < 1$. The only positive real number values of x that satisfy $x^2 < 1$ are $0 < x < 1$.

3. Can we determine if $\frac{y}{y-1} > \frac{y+1}{y}$ for all values $y > 1$? Provide evidence to support your answer.

Before we can compare two rational expressions, we need to express them as equivalent expressions with a common denominator. Since $y > 1$, neither denominator is ever zero. Then $\frac{y}{y-1} = \frac{y^2}{y(y-1)}$ and $\frac{y+1}{y} = \frac{(y+1)(y-1)}{y(y-1)} = \frac{y^2-1}{y(y-1)}$. Since $y^2 > y^2 - 1$ for all values of y, we know that $\frac{y^2}{y(y-1)} > \frac{y^2-1}{y(y-1)}$. Then we can conclude that $\frac{y}{y-1} > \frac{y+1}{y}$ for all values $y > 1$.

4. For positive x, determine when the following rational expressions have negative denominators:

   a. $\frac{3}{5}$

   5 is never less than 0.

   b. $\frac{x}{5-2x}$

   $5 - 2x < 0$ when $5 < 2x$, which is equivalent to $\frac{5}{2} < x$.

   c. $\frac{x+3}{x^2+4x+8}$

   For any real number x, $x^2 + 4x + 8$ is never negative. One way to see this is that $x^2 + 4x + 8 = (x+2)^2 + 4$, which is the sum of two positive numbers.

   d. $\frac{3x^2}{(x-5)(x+3)(2x+3)}$

   For positive x, $x + 3$ and $2x + 3$ are always positive. The number $x - 5$ is negative when $x < 5$, so the denominator is negative when $x < 5$.

5. Consider the rational expressions $\frac{x}{x-2}$ and $\frac{x}{x-4}$.

   a. Evaluate each expression for $x = 6$.

   If $x = 6$, then $\frac{x}{x-2} = 1.5$ and $\frac{x}{x-4} = 3$.

   b. Evaluate each expression for $x = 3$.

   If $x = 3$, then $\frac{x}{x-2} = 3$ and $\frac{x}{x-4} = -3$.

   c. Can you conclude that $\frac{x}{x-2} < \frac{x}{x-4}$ for all positive values of x? Explain how you know.

   No, because $\frac{x}{x-2} > \frac{x}{x-4}$ when $x = 3$, it is not true that $\frac{x}{x-2} < \frac{x}{x-4}$ for every positive value of x.

MP.3

d. EXTENSION: Raphael claims that the calculation below shows that $\frac{x}{x-2} < \frac{x}{x-4}$ for all values of x, where $x \neq 2$ and $x \neq 4$. Where is the error in the calculation?

Starting with the rational expressions $\frac{x}{x-2}$ and $\frac{x}{x-4}$, we need to first find equivalent rational expressions with a common denominator. The common denominator we will use is $(x-4)(x-2)$. We then have

$$\frac{x}{x-2} = \frac{x(x-4)}{(x-4)(x-2)}$$
$$\frac{x}{x-4} = \frac{x(x-2)}{(x-4)(x-2)}.$$

Since $x^2 - 4x < x^2 - 2x$ for $x > 0$, we can divide each expression by $(x-4)(x-2)$. We then have $\frac{x(x-4)}{(x-4)(x-2)} < \frac{x(x-2)}{(x-4)(x-2)}$, and we can conclude that $\frac{x}{x-2} < \frac{x}{x-4}$ for all positive values of x.

The error in logic in this calculation is that the denominator $(x-4)(x-2)$ is not always a positive number for all positive values of x. In fact, if $2 < x < 4$ then $(x-4)(x-2) < 0$. Thus, even though $x^2 - 4x < x^2 - 2x$ when $x > 0$, the inequality $\frac{x^2-4x}{(x-4)(x-2)} < \frac{x^2-2x}{(x-4)(x-2)}$ is not valid for every positive value of x.

MP.2

6. Consider the populations of two cities within the same state where the large city's population is P and the small city's population is Q. For each of the following pairs, state which of the expressions has a larger value. Explain your reasoning in the context of the populations.

a. $P + Q$ and P

The value of $P + Q$ is larger than P. The expression $P + Q$ represents the total population of the two cities and P represents the population of the larger city. Since these quantities are populations of cities, we can assume they are greater than zero.

b. $\frac{P}{P+Q}$ and $\frac{Q}{P+Q}$

The value of $\frac{P}{P+Q}$ is larger. As stated in part (a), $P + Q$ represents the total population of the two cities. Hence, $\frac{P}{P+Q}$ and $\frac{Q}{P+Q}$ represent each city's respective fraction of the total population. Since $P > Q$, $\frac{P}{P+Q} > \frac{Q}{P+Q}$.

c. $2Q$ and $P + Q$

The value of $P + Q$ is larger than the value of $2Q$. The population of the smaller of the two cities is represented by Q, so $2Q$ represents a population twice the size of the smaller city, but $2Q = Q + Q < P + Q$.

d. $\frac{P}{Q}$ and $\frac{Q}{P}$

The value of $\frac{P}{Q}$ is larger. These expressions represent the ratio between the populations of the cities. For instance, the larger city is $\frac{P}{Q}$ times larger than the smaller city. Since $P > Q$, $\frac{P}{Q} > 1 > \frac{Q}{P}$, we can say that there are $\frac{P}{Q}$ people in the larger city for every one person in the smaller city.

MP.2

e. $\frac{P}{P+Q}$ and $\frac{1}{2}$

The value of $\frac{P}{P+Q}$ is larger. Since P is the population of the larger city, the first city represents more than half of the total.

f. $\frac{P+Q}{P}$ and $P-Q$

The value of $P-Q$ is larger. The expression $P-Q$ represents the difference in population between the two cities. The expression $\frac{P+Q}{P}$ can represent the ratio of how much larger the total is compared to the population of the larger city, but we know that P represents more than half of the total; therefore, $\frac{P+Q}{P}$ cannot be larger than 2. Without the context, we could not say that $P-Q$ is larger than 2, but in the context of the problem, since P is the population of a large city and Q is the population of a small city, $P-Q>2$. Thus, $P-Q>\frac{P+Q}{P}$.

g. $\frac{P+Q}{2}$ and $\frac{P+Q}{Q}$

The value of $\frac{P+Q}{2}$ is larger. The sum divided by the number of cities represents the average population of the two cities and will be significantly higher than the ratio represented by $\frac{P+Q}{Q}$. Alternatively, Q is much larger than 2, so $\frac{P+Q}{2}<\frac{P+Q}{Q}$.

h. $\frac{1}{P}$ and $\frac{1}{Q}$

The value of $\frac{1}{Q}$ is larger. The expression $\frac{1}{Q}$ represents the proportion of the population of the second city a single citizen represents. Similarly for $\frac{1}{P}$, since the second city has a smaller population, each individual represents a larger proportion of the whole than in the first city.

Lesson 24: Multiplying and Dividing Rational Expressions

Student Outcomes

- Students multiply and divide rational expressions and simplify using equivalent expressions.

Lesson Notes

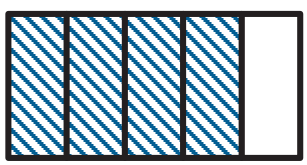
This lesson quickly reviews the process of multiplying and dividing rational numbers using techniques students already know and translates that process to multiplying and dividing rational expressions (MP.7). This enables students to develop techniques to solve rational equations in Lesson 26 (**A-APR.D.6**). This lesson also begins developing facility with simplifying complex rational expressions, which is important for later work in trigonometry. Teachers may consider treating the multiplication and division portions of this lesson as two separate lessons.

Classwork

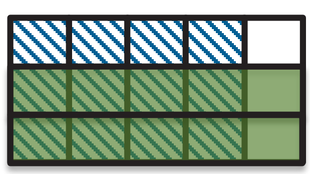
Opening Exercise (5 minutes)

Distribute notecard-sized slips of paper to students, and ask them to shade the paper to represent the result of $\frac{2}{3} \cdot \frac{4}{5}$. Circulate around the classroom to assess student proficiency. If many students are still struggling to remember the area model after the scaffolding, present the problem to them as shown. Otherwise, allow them time to do the multiplication on their own or with their neighbor and then progress to the question of the general rule.

- First, we represent $\frac{4}{5}$ by dividing our region into five vertical strips of equal area and shading 4 of the 5 parts.



- Now we need to find $\frac{2}{3}$ of the shaded area. So we divide the area horizontally into three parts of equal area, and then shade two of those parts.



- Thus, $\frac{2}{3} \cdot \frac{4}{5}$ is represented by the region that is shaded twice. Since 8 out of 15 subrectangles are shaded twice, we have $\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$. With this in mind, can we create a general rule about multiplying rational numbers?

Scaffolding:

If students do not remember the area model for multiplication of fractions, have them discuss it with their neighbor. If necessary, use an example like $\frac{1}{2} \cdot \frac{1}{2}$ to see if they can scale this to the problem presented.

If students are comfortable with multiplying rational numbers, omit the area model and ask them to determine the following products.

- $\frac{2}{3} \cdot \frac{3}{8} = \frac{1}{4}$
- $\frac{1}{4} \cdot \frac{5}{6} = \frac{5}{24}$
- $\frac{4}{7} \cdot \frac{8}{9} = \frac{32}{63}$

Allow students to come up with this "rule" based on the example and prior experience. Have them discuss their thoughts with their neighbor and write the rule.

> If a, b, c, and d are integers with $c \neq 0$ and $d \neq 0$, then
> $$\frac{a}{c} \cdot \frac{b}{d} = \frac{ab}{cd}.$$

The rule summarized above is also valid for real numbers.

Discussion (2 minutes)

- To multiply rational expressions, we follow the same procedure we use when multiplying rational numbers: we multiply together the numerators and multiply together the denominators. We finish by reducing the product to lowest terms.

> **If a, b, c, and d are rational expressions with $b \neq 0$, $d \neq 0$, then**
> $$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Lead students through Examples 1 and 2, and ask for their input at each step.

Example 1 (4 minutes)

Give students time to work on this problem and discuss their answers with a neighbor before proceeding to a whole class discussion.

> **Example 1**
>
> **Make a conjecture about the product $\frac{x^3}{4y} \cdot \frac{y^2}{x}$. What will it be? Explain your conjecture and give evidence that it is correct.**

- We begin by multiplying the numerators and denominators.

$$\frac{x^3}{4y} \cdot \frac{y^2}{x} = \frac{x^3y^2}{4yx}$$

- Identify the greatest common factor (GCF) of the numerator and denominator. The GCF of x^3y^2 and $4xy$ is xy.

$$\frac{x^3}{4y} \cdot \frac{y^2}{x} = \frac{(xy)x^2y}{4(xy)}$$

Scaffolding:

- To assist students in making the connection between rational numbers and rational expressions, show a side-by-side comparison of a numerical example from a previous lesson like the one shown.

$\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$	$\frac{x^3}{4y} \cdot \frac{y^2}{x} = ?$

- If students are struggling with this example, include some others, such as
  - $\frac{x^2}{3} \cdot \frac{6}{x}$
  - $\frac{y}{x^2} \cdot \frac{y^4}{x}$

- Finally, we divide the common factor xy from the numerator and denominator to find the reduced form of the product:

$$\frac{x^3}{4y} \cdot \frac{y^2}{x} = \frac{x^2 y}{4}.$$

Note that we are intentionally avoiding using the phrases "cancel xy" or "cancel the common factor" in this lesson. We want to highlight that it is division that allows us to simplify these expressions. Ambiguous words like "cancel" can lead students to simplify $\frac{\sin x}{x}$ to $\sin$—they "canceled" the x!

It is important to understand why we are allowed to divide the numerator *and* denominator by x. The rule $\frac{na}{nb} = \frac{a}{b}$ works for rational expressions as well. Performing a simplification such as $\frac{x}{x^3 y} = \frac{1}{x^2 y}$ requires doing the following steps: $\frac{x}{x^3 y} = \frac{x \cdot 1}{x \cdot x^2 y} = \frac{x}{x} \cdot \frac{1}{x^2 y} = 1 \cdot \frac{1}{x^2 y} = \frac{1}{x^2 y}$.

Example 2 (3 minutes)

Before walking students through the steps of this example, ask them to try to find the product using the ideas of the previous example.

> **Example 2**
>
> **Find the following product:** $\left(\frac{3x-6}{2x+6}\right) \cdot \left(\frac{5x+15}{4x+8}\right)$.

First, we can factor the numerator and denominator of each rational expression.

MP.7

- Identify any common factors in the numerator and denominator.

$$\left(\frac{3x-6}{2x+6}\right) \cdot \left(\frac{5x+15}{4x+8}\right) = \left(\frac{3(x-2)}{2(x+3)}\right) \cdot \left(\frac{5(x+3)}{4(x+2)}\right)$$
$$= \frac{15(x-2)(x+3)}{8(x+3)(x+2)}$$

The GCF of the numerator and denominator is $x+3$.

Then, we can divide the common factor $(x+3)$ from the numerator and denominator and obtain the reduced form of the product.

$$\left(\frac{3x-6}{2x+6}\right) \cdot \left(\frac{5x+15}{4x+8}\right) = \frac{15(x-2)}{8(x+2)}.$$

Exercises 1–3 (5 minutes)

Students can work in pairs on the following three exercises. Circulate around the class to informally assess their understanding. For Exercise 1, listen for key points such as "factoring the numerator and denominator can help" and "multiplying rational expressions is similar to multiplying rational numbers."

Exercises 1–3

1. **Summarize what you have learned so far with your neighbor.**

   Answers will vary.

2. **Find the following product and reduce to lowest terms: $\left(\frac{2x+6}{x^2+x-6}\right)\cdot\left(\frac{x^2-4}{2x}\right)$.**

$$\left(\frac{2x+6}{x^2+x-6}\right)\cdot\left(\frac{x^2-4}{2x}\right)=\left(\frac{2(x+3)}{(x+3)(x-2)}\right)\cdot\left(\frac{(x-2)(x+2)}{2x}\right)$$
$$=\frac{2(x+3)(x-2)(x+2)}{2x(x+3)(x-2)}$$

   The factors 2, $x+3$, and $x-2$ can be divided from the numerator and the denominator in order to reduce the rational expression to lowest terms.

$$\left(\frac{2x+6}{x^2+x-6}\right)\cdot\left(\frac{x^2-4}{2x}\right)=\frac{x+2}{x}$$

3. **Find the following product and reduce to lowest terms: $\left(\frac{4n-12}{3m+6}\right)^{-2}\cdot\left(\frac{n^2-2n-3}{m^2+4m+4}\right)$.**

$$\left(\frac{4n-12}{3m+6}\right)^{-2}\cdot\left(\frac{n^2-2n-3}{m^2+4m+4}\right)=\left(\frac{3m+6}{4n-12}\right)^{2}\cdot\left(\frac{n^2-2n-3}{m^2+4m+4}\right)$$
$$=\frac{3^2(m+2)^2(n-3)(n+1)}{4^2(n-3)^2(m+2)^2}$$
$$=\frac{9(n+1)}{16(n-3)}$$

Scaffolding:

Students may need to be reminded how to interpret a negative exponent. If so, ask them to calculate these values.

- $3^{-2}=\frac{1}{3^2}=\frac{1}{9}$
- $\left(\frac{2}{5}\right)^{-3}=\left(\frac{5}{2}\right)^{3}=\frac{5^3}{2^3}=\frac{125}{8}$
- $\left(\frac{x}{y^2}\right)^{-5}=\left(\frac{y^2}{x}\right)^{5}=\frac{(y^2)^5}{x^5}=\frac{y^{10}}{x^5}$

Discussion (5 minutes)

Recall that division of numbers is equivalent to multiplication of the numerator by the reciprocal of the denominator. That is, for any two numbers a and b, where $b\neq 0$, we have

$$\frac{a}{b}=a\cdot\frac{1}{b},$$

where the number $\frac{1}{b}$ is the multiplicative inverse of b. But, what if b is itself a fraction? How do we evaluate a quotient such as $\frac{3}{5}\div\frac{4}{7}$?

- How do we evaluate $\frac{3}{5}\div\frac{4}{7}$?

Have students work in pairs to answer this and then discuss.

By our rule above, $\frac{3}{5}\div\frac{4}{7}=\frac{3}{5}\cdot\frac{1}{4/7}$. But, what is the value of $\frac{1}{4/7}$? Let x represent $\frac{1}{4/7}$, which is the multiplicative inverse of $\frac{4}{7}$. Then we have

$$x\cdot\frac{4}{7}=1$$
$$4x=7$$
$$x=\frac{7}{4}.$$

Scaffolding:

Students may be better able to generalize the procedure for dividing rational numbers by repeatedly dividing several examples, such as $\frac{1}{2}\div\frac{3}{4}$, $\frac{2}{3}\div\frac{7}{10}$, and $\frac{1}{5}\div\frac{2}{9}$. After dividing several of these examples, ask students to generalize the process (MP.8).

Since we have shown that $\frac{1}{4/7} = \frac{7}{4}$, we can continue our calculation of $\frac{3}{5} \div \frac{4}{7}$ as follows:

$$\begin{aligned}\frac{3}{5} \div \frac{4}{7} &= \frac{3}{5} \cdot \frac{1}{4/7} \\ &= \frac{3}{5} \cdot \frac{7}{4} \\ &= \frac{21}{20}.\end{aligned}$$

This same process applies to dividing rational expressions, although we might need to perform the additional step of reducing the resulting rational expression to lowest terms. Ask students to generate the rule for division of rational numbers.

> If a, b, c, and d are integers with $b \neq 0$, $c \neq 0$, and $d \neq 0$, then
> $$\frac{a}{c} \div \frac{b}{d} = \frac{a}{c} \cdot \frac{d}{b}.$$

The result summarized in the box above is also valid for real numbers.

Now that we know how to divide rational numbers, how do we extend this to divide rational expressions?

- Dividing rational expressions follows the same procedure as dividing rational numbers: we multiply the first term by the reciprocal of the second. We finish by reducing the product to lowest terms.

> **If a, b, c, and d are rational expressions with $b \neq 0$, $c \neq 0$, and $d \neq 0$, then**
> $$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

Example 3 (3 minutes)

As in Example 2, ask students to apply their knowledge of rational number division to rational expressions by working on their own or with a partner. Circulate to assist and assess understanding. Once students have made attempts to divide, use the scaffolded questions to develop the concept as necessary.

> **Example 3**
>
> **Find the quotient and reduce to lowest terms:** $\frac{x^2-4}{3x} \div \frac{x-2}{2x}$.

Scaffolding:

A side-by-side comparison may help as before.

$\frac{3}{5} \div \frac{4}{7} = \frac{21}{20}$	$\frac{x^3}{4y} \div \frac{y^2}{x} = ?$

For struggling students, give

- $\frac{x^2y}{4} \div \frac{xy^2}{8} = \frac{2x}{y}$
- $\frac{3y^2}{z-1} \div \frac{12y^5}{(z-1)^2} = \frac{(z-1)}{4y^3}$.

For advanced students, give

- $\frac{x-3}{x^2+x-2} \div \frac{x^2-x-6}{x-1} = \frac{1}{(x+2)^2}$
- $\frac{x^2-2x-24}{x^2-4} \div \frac{x^2+3x-4}{x^2+x-2} = \frac{x-6}{x-2}$.

- First, we change the division of $\frac{x^2-4}{3x}$ by $\frac{x-2}{2x}$ into multiplication of $\frac{x^2-4}{3x}$ by the multiplicative inverse of $\frac{x-2}{2x}$.

$$\frac{x^2-4}{3x} \div \frac{x-2}{2x} = \frac{x^2-4}{3x} \cdot \frac{2x}{x-2}$$

- Then, we perform multiplication as in the previous examples and exercises. That is, we factor the numerator and denominator and divide any common factors present in both the numerator and denominator.

$$\begin{aligned}\frac{x^2-4}{3x} \div \frac{x-2}{2x} &= \frac{(x-2)(x+2)}{3x} \cdot \frac{2x}{x-2} \\ &= \frac{2(x+2)}{3}\end{aligned}$$

Exercise 4 (3 minutes)

Allow students to work in pairs or small groups to evaluate the following quotient.

Exercises 4–5

4. **Find the quotient and reduce to lowest terms: $\frac{x^2-5x+6}{x+4} \div \frac{x^2-9}{x^2+5x+4}$.**

$$\begin{aligned}\frac{x^2-5x+6}{x+4} \div \frac{x^2-9}{x^2+5x+4} &= \frac{x^2-5x+6}{x+4} \cdot \frac{x^2+5x+4}{x^2-9} \\ &= \frac{(x-3)(x-2)}{x+4} \cdot \frac{(x+4)(x+1)}{(x-3)(x+3)} \\ &= \frac{(x-2)(x+1)}{(x+3)}\end{aligned}$$

Discussion (4 minutes)

What do we do when the numerator and denominator of a fraction are themselves fractions? We call a fraction that contains fractions a *complex fraction*. Remind students that the fraction bar represents division, so a complex fraction represents division between rational expressions.

Allow students the opportunity to simplify the following complex fraction.

$$\frac{12/49}{27/28}$$

Allow students to struggle with the problem before discussing solution methods.

$$\begin{aligned}\frac{12/49}{27/28} &= \frac{12}{49} \div \frac{27}{28} \\ &= \frac{12}{49} \cdot \frac{28}{27} \\ &= \frac{3 \cdot 4}{7 \cdot 7} \cdot \frac{4 \cdot 7}{3^3} \\ &= \frac{4^2}{7 \cdot 3^2} \\ &= \frac{16}{63}\end{aligned}$$

Notice that in simplifying the complex fraction above, we are merely performing division of rational numbers, and we already know how to do that. Since we already know how to divide rational expressions, we can also simplify rational expressions whose numerators and denominators are rational expressions.

Exercise 5 (4 minutes)

Allow students to work in pairs or small groups to simplify the following rational expression.

> *Scaffolding:*
>
> For struggling students, give a simpler example, such as
>
> $$\frac{\left(\frac{2x}{3y}\right)}{\left(\frac{6x}{4y^2}\right)} = \frac{2x}{3y} \div \frac{6x}{4y^2}$$
> $$= \frac{2x}{3y} \cdot \frac{4y^2}{6x}$$
> $$= \frac{4}{9}y.$$

5. Simplify the rational expression.

$$\frac{\left(\frac{x+2}{x^2-2x-3}\right)}{\left(\frac{x^2-x-6}{x^2+6x+5}\right)}$$

$$\frac{\frac{x+2}{x^2-2x-3}}{\frac{x^2-x-6}{x^2+6x+5}} = \frac{x+2}{x^2-2x-3} \div \frac{x^2-x-6}{x^2+6x+5}$$

$$= \frac{x+2}{x^2-2x-3} \cdot \frac{x^2+6x+5}{x^2-x-6}$$

$$= \frac{x+2}{(x-3)(x+1)} \cdot \frac{(x+5)(x+1)}{(x-3)(x+2)}$$

$$= \frac{x+5}{(x-3)^2}$$

Closing (3 minutes)

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. In particular, ask students to articulate the processes for multiplying and dividing rational expressions and simplifying complex rational expressions either verbally or symbolically.

> Lesson Summary
>
> **In this lesson we extended multiplication and division of rational numbers to multiplication and division of rational expressions.**
>
> - **To multiply two rational expressions, multiply the numerators together and multiply the denominators together, and then reduce to lowest terms.**
> - **To divide one rational expression by another, multiply the first by the multiplicative inverse of the second, and reduce to lowest terms.**
> - **To simplify a complex fraction, apply the process for dividing one rational expression by another.**

Exit Ticket (4 minutes)

Name ______________________________ Date ______________

Lesson 24: Multiplying and Dividing Rational Expressions

Exit Ticket

Perform the indicated operations and reduce to lowest terms.

1. $\dfrac{x-2}{x^2+x-2} \cdot \dfrac{x^2-3x+2}{x+2}$

2. $\dfrac{\left(\dfrac{x-2}{x^2+x-2}\right)}{\left(\dfrac{x^2-3x+2}{x+2}\right)}$

Exit Ticket Sample Solutions

Perform the indicated operations and reduce to lowest terms.

1. $\frac{x-2}{x^2+x-2} \cdot \frac{x^2-3x+2}{x+2}$

$$\frac{x-2}{x^2+x-2} \cdot \frac{x^2-3x+2}{x+2} = \frac{x-2}{(x-1)(x+2)} \cdot \frac{(x-1)(x-2)}{x+2} = \frac{(x-2)^2}{(x+2)^2}$$

2. $\frac{\left(\frac{x-2}{x^2+x-2}\right)}{\left(\frac{x^2-3x+2}{x+2}\right)}$

$$\frac{\left(\frac{x-2}{x^2+x-2}\right)}{\left(\frac{x^2-3x+2}{x+2}\right)} = \frac{x-2}{x^2+x-2} \div \frac{x^2-3x+2}{x+2}$$
$$= \frac{x-2}{x^2+x-2} \cdot \frac{x+2}{x^2-3x+2}$$
$$= \frac{x-2}{(x-1)(x+2)} \cdot \frac{x+2}{(x-2)(x-1)}$$
$$= \frac{1}{(x-1)^2}$$

Problem Set Sample Solutions

1. **Complete the following operations:**

a. Multiply $\frac{1}{3}(x-2)$ by 9.

$3x-6$

b. Divide $\frac{1}{4}(x-8)$ by $\frac{1}{12}$.

$3x-24$

c. Multiply $\frac{1}{4}\left(\frac{1}{3}x+2\right)$ by 12.

$x+6$

d. Divide $\frac{1}{3}\left(\frac{2}{5}x-\frac{1}{5}\right)$ by $\frac{1}{15}$.

$2x-1$

e. Multiply $\frac{2}{3}\left(2x+\frac{2}{3}\right)$ by $\frac{9}{4}$.

$3x+1$

f. Multiply $0.03(4-x)$ by 100.

$12-3x$

2. **Simplify each of the following expressions.**

a. $\left(\frac{a^3b^2}{c^2d^2} \cdot \frac{c}{ab}\right) \div \frac{a}{c^2d^3}$

$abcd$

b. $\frac{a^2+6a+9}{a^2-9} \cdot \frac{3a-9}{a+3}$

3

c. $\frac{6x}{4x-16} \div \frac{4x}{x^2-16}$

$\frac{3(x+4)}{8}$

d. $\frac{3x^2-6x}{3x+1} \cdot \frac{x+3x^2}{x^2-4x+4}$

$\frac{3x^2}{x-2}$

e. $\frac{2x^2-10x+12}{x^2-4} \cdot \frac{2+x}{3-x}$

-2

f. $\frac{a-2b}{a+2b} \div (4b^2-a^2)$

$-\frac{1}{(a+2b)^2}$

g. $\frac{d+c}{c^2+d^2} \div \frac{c^2-d^2}{d^2-dc}$

$-\frac{d}{c^2+d^2}$

h. $\frac{12a^2-7ab+b^2}{9a^2-b^2} \div \frac{16a^2-b^2}{3ab+b^2}$

$\frac{b}{4a+b}$

i. $\left(\frac{x-3}{x^2-4}\right)^{-1} \cdot \left(\frac{x^2-x-6}{x-2}\right)$

$(x+2)^2$

j. $\left(\frac{x-2}{x^2+1}\right)^{-3} \div \left(\frac{x^2-4x+4}{x^2-2x-3}\right)$

$\frac{(x-3)(x+1)(x^2+1)^3}{(x-2)^5}$

k. $\frac{6x^2-11x-10}{6x^2-5x-6} \cdot \frac{6-4x}{25-20x+4x^2}$

$-\frac{2}{2x-5}$, *or* $\frac{2}{5-2x}$

l. $\frac{3x^3-3a^2x}{x^2-2ax+a^2} \cdot \frac{a-x}{a^3x+a^2x^2}$

$-\frac{3}{a^2}$

3. Simplify the following complex rational expressions.

a. $\frac{\left(\frac{4a}{6b^2}\right)}{\left(\frac{20a^3}{12b}\right)}$ $\qquad \frac{2}{5a^2b}$

b. $\frac{\left(\frac{x-2}{x^2-1}\right)}{\left(\frac{x^2-4}{x-6}\right)}$ $\qquad \frac{x-6}{(x+2)(x^2-1)}$

c. $\frac{\left(\frac{x^2+2x-3}{x^2+3x-4}\right)}{\left(\frac{x^2+x-6}{x+4}\right)}$ $\qquad \frac{1}{x-2}$

4. Suppose that $x = \frac{t^2+3t-4}{3t^2-3}$ and $y = \frac{t^2+2t-8}{2t^2-2t-4}$, for $t \neq 1$, $t \neq -1$, $t \neq 2$, and $t \neq -4$. Show that the value of x^2y^{-2} does not depend on the value of t.

$$\begin{aligned}
x^2y^{-2} &= \left(\frac{t^2+3t-4}{3t^2-3}\right)^2 \left(\frac{t^2+2t-8}{2t^2-2t-4}\right)^{-2} \\
&= \left(\frac{t^2+3t-4}{3t^2-3}\right)^2 \div \left(\frac{t^2+2t-8}{2t^2-2t-4}\right)^2 \\
&= \left(\frac{t^2+3t-4}{3t^2-3}\right)^2 \left(\frac{2t^2-2t-4}{t^2+2t-8}\right)^2 \\
&= \left(\frac{(t-1)(t+4)}{3(t-1)(t+1)}\right)^2 \left(\frac{2(t-2)(t+1)}{(t-2)(t+4)}\right)^2 \\
&= \frac{4(t-1)^2(t+4)^2(t-2)^2(t+1)^2}{9(t-1)^2(t+1)^2(t-2)^2(t+4)^2} \\
&= \frac{4}{9}
\end{aligned}$$

Since $x^2y^{-2} = \frac{4}{9}$, the value of x^2y^{-2} does not depend on t.

5. Determine which of the following numbers is larger without using a calculator, $\frac{15^{16}}{16^{15}}$ or $\frac{20^{24}}{24^{20}}$. (Hint: We can compare two positive quantities a and b by computing the quotient $\frac{a}{b}$. If $\frac{a}{b} > 1$, then $a > b$. Likewise, if $0 < \frac{a}{b} < 1$, then $a < b$.)

$$\begin{aligned}\frac{15^{16}}{16^{15}} \div \frac{20^{24}}{24^{20}} &= \frac{15^{16}}{16^{15}} \cdot \frac{24^{20}}{20^{24}}\\ &= \frac{3^{16} \cdot 5^{16}}{(2^4)^{15}} \cdot \frac{(2^3)^{20} \cdot 3^{20}}{(2^2)^{24} \cdot 5^{24}}\\ &= \frac{2^{60} \cdot 3^{36} \cdot 5^{16}}{2^{108} \cdot 5^{24}}\\ &= \frac{3^{36}}{2^{48} \cdot 5^{8}}\\ &= \frac{9^{18}}{2^{40} \cdot 10^{8}}\\ &= \left(\frac{9}{10}\right)^{8} \cdot \left(\frac{9^{10}}{2^{40}}\right)\\ &= \left(\frac{9}{10}\right)^{8} \cdot \left(\frac{9}{16}\right)^{10}\end{aligned}$$

Since $\frac{9}{16} < 1$, and $\frac{9}{10} < 1$, we know that $\left(\frac{9}{10}\right)^{8} \cdot \left(\frac{9}{16}\right)^{10} < 1$. Thus, $\frac{15^{16}}{16^{15}} \div \frac{20^{24}}{24^{20}} < 1$, and we know that $\frac{15^{16}}{16^{15}} < \frac{20^{24}}{24^{20}}$.

6. (Optional) One of two numbers can be represented by the rational expression $\frac{x-2}{x}$, where $x \neq 0$ and $x \neq 2$.

a. Find a representation of the second number if the product of the two numbers is 1.

Let the second number be y. Then $\left(\frac{x-2}{x}\right) \cdot y = 1$, so we have

$$\begin{aligned}y &= 1 \div \left(\frac{x-2}{x}\right)\\ &= 1 \cdot \left(\frac{x}{x-2}\right)\\ &= \frac{x}{x-2}.\end{aligned}$$

b. Find a representation of the second number if the product of the two numbers is 0.

Let the second number be z. Then $\left(\frac{x-2}{x}\right) \cdot z = 0$, so we have

$$\begin{aligned}z &= 0 \div \left(\frac{x-2}{x}\right)\\ &= 0 \cdot \left(\frac{x}{x-2}\right)\\ &= 0.\end{aligned}$$

Lesson 25: Adding and Subtracting Rational Expressions

Student Outcomes

- Students perform addition and subtraction of rational expressions.

Lesson Notes

In this lesson, we review addition and subtraction of fractions using the familiar number line technique that students have seen in earlier grades. This leads to an algebraic explanation of how to add and subtract fractions and an opportunity to practice MP.7. We then move to the process for adding and subtracting rational expressions by converting to equivalent rational expressions with a common denominator. As in the past three lessons, we continue to draw parallels between arithmetic of rational numbers and arithmetic of rational expressions.

Classwork

The four basic arithmetic operations are addition, subtraction, multiplication, and division. The previous lesson showed how to multiply and divide rational expressions. This lesson tackles the remaining operations of addition and subtraction of rational expressions, which are skills needed to address **A-APR.C.6**. As discussed in the previous lesson, we operate with rational expressions in the same way we work with rational numbers expressed as fractions. First, we will review the theory behind addition and subtraction of rational numbers.

Exercise 1 (8 minutes)

First, remind students how to add fractions with the same denominator. Allow them to work through the following sum individually. The solution should be presented to the class either by the teacher or by a student because we are going to extend the process of adding fractions to the new process of adding rational expressions.

Scaffolding:

If students need practice adding and subtracting fractions with a common denominator, have them compute the following.

- $\frac{2}{5}+\frac{1}{5}$
- $\frac{5}{7}-\frac{3}{7}$
- $\frac{17}{24}-\frac{12}{24}$

Exercises 1–4

1. Calculate the following sum: $\frac{3}{10}+\frac{6}{10}$.

One approach to this calculation is to factor out $\frac{1}{10}$ from each term.

$$\begin{aligned}\frac{3}{10}+\frac{6}{10} &= 3\cdot\frac{1}{10}+6\cdot\frac{1}{10}\\ &= (3+6)\cdot\frac{1}{10}\\ &= \frac{9}{10}\end{aligned}$$

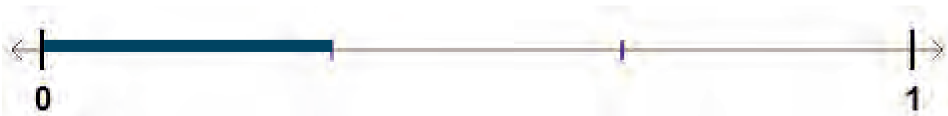
Ask students for help in stating the rule for adding and subtracting rational numbers with the same denominator.

> If a, b, and c are integers with $b \neq 0$, then
> $$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b} \quad \text{and} \quad \frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}.$$

The result in the box above is also valid for real numbers a, b, and c.

- But what if the fractions have different denominators? Let's examine a technique to add the fractions $\frac{2}{5}$ and $\frac{1}{3}$.
- Recall that when we first learned to add fractions, we represented them on a number line. Let's first look at $\frac{2}{5}$.

- And we want to add to this the fraction $\frac{1}{3}$.



- If we try placing these two segments next to each other, the exact location of the endpoint is difficult to identify.

- The units on the two original graphs do not match. We need to identify a common unit in order to identify the endpoint of the combined segments. We need to identify a number into which both denominators divide without remainder and write each fraction as an equivalent fraction with that number as denominator; such a number is known as a *common denominator*.
- Since 15 is a common denominator of $\frac{2}{5}$ and $\frac{1}{3}$, we divide the interval $[0, 1]$ into 15 parts of equal length. Now when we look at the segments of length $\frac{2}{5}$ and $\frac{1}{3}$ placed next to each other on the number line, we can see that the combined segment has length $\frac{11}{15}$.

- How can we do this without using the number line every time? The fraction $\frac{2}{5}$ is equivalent to $\frac{6}{15}$, and the fraction $\frac{1}{3}$ is equivalent to $\frac{5}{15}$. We then have

$$\begin{aligned}\frac{2}{5}+\frac{1}{3} &= \frac{6}{15}+\frac{5}{15}\\ &= \frac{11}{15}.\end{aligned}$$

- Thus, when adding rational numbers, we have to find a common multiple for the two denominators and write each rational number as an equivalent rational number with the new common denominator. Then we can add the numerators together.

Have students discuss how to rewrite the original fraction as an equivalent fraction with the chosen common denominator. Discuss how the identity property of multiplication allows you to multiply the top and the bottom by the same number so that product of the original denominator and the number gives the chosen common denominator.

- Generalizing, let's add together two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$. The first step is to rewrite both fractions as equivalent fractions with the same denominator. A simple common denominator that could be used is the product of the original two denominators:
$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}.$$
- Once we have a common denominator, we can add the two expressions together, using our previous rule for adding two expressions with the same denominator:
$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}.$$
- We could use the same approach to develop a process for subtracting rational numbers:
$$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}.$$
- Now that we know to find a common denominator before adding or subtracting, we can state the general rule for adding and subtracting rational numbers. Notice that one common denominator that will always work is the product of the two original denominators.

If a, b, c, and d are integers with $b \neq 0$ and $d \neq 0$, then
$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd} \quad \text{and} \quad \frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}.$$

As with our other rules developed in this and the previous lesson, the rule summarized in the box above is also valid for real numbers.

Exercises 2–4 (5 minutes)

Ask the students to work in groups to write what they have learned in their notebooks or journals. Check in to assess their understanding. Then, have students work in pairs to quickly work through the following review exercises. Allow them to think about how to approach Exercise 4, which involves adding three rational expressions. There are multiple ways to approach this problem. They could generalize the process for two rational expressions, rearrange terms using the commutative property to combine the terms with the same denominator, and then add using the above process, or they could group the addends using the associative property and perform addition twice.

2. $\frac{3}{20}-\frac{4}{15}$

$$\frac{3}{20}-\frac{4}{15}=\frac{9}{60}-\frac{16}{60}=-\frac{7}{60}$$

3. $\frac{\pi}{4}+\frac{\sqrt{2}}{5}$

$\frac{\pi}{4}+\frac{\sqrt{2}}{5}=\frac{5\pi}{20}+\frac{4\sqrt{2}}{20}=\frac{5\pi+4\sqrt{2}}{20}$

4. $\frac{a}{m}+\frac{b}{2m}-\frac{c}{m}$

$\frac{a}{m}+\frac{b}{2m}-\frac{c}{m}=\frac{2a}{2m}+\frac{b}{2m}-\frac{2c}{2m}=\frac{2a+b-2c}{2m}$

Discussion (2 minutes)

- Before we can add rational numbers or rational expressions, we need to convert to equivalent rational expressions with the same denominators. Finding such a denominator involves finding a common multiple of the original denominators. For example, 60 is a common multiple of 20 and 15. There are other common multiples, such as 120, 180, and 300, but smaller numbers are easier to work with.
- To add and subtract rational expressions, we follow the same procedure as when adding and subtracting rational numbers. First, we find a denominator that is a common multiple of the other denominators, and then we rewrite each expression as an equivalent rational expression with this new common denominator. We then apply the rule for adding or subtracting with the same denominator.

If a, b, and c are rational expressions with $b \neq 0$, then

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b} \quad \text{and} \quad \frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}.$$

Example 1 (10 minutes)

Work through these examples as a class, getting input from the students at each step.

Example 1

Perform the indicated operations below and simplify.

a. $\frac{a+b}{4}+\frac{2a-b}{5}$

A common multiple of 4 and 5 is 20, so we can write each expression as an equivalent rational expression with denominator 20. We have $\frac{a+b}{4}=\frac{5a+5b}{20}$ and $\frac{2a-b}{5}=\frac{8a-4b}{20}$, so that

$\frac{a+b}{4}+\frac{2a-b}{5}=\frac{5a+5b}{20}+\frac{8a-4b}{20}=\frac{13a+b}{20}$.

b. $\frac{4}{3x}-\frac{3}{5x^2}$

A common multiple of $3x$ and $5x^2$ is $15x^2$, so we can write each expression as an equivalent rational expression with denominator $15x^2$. We have $\frac{4}{3x}-\frac{3}{5x^2}=\frac{20x}{15x^2}-\frac{9}{15x^2}=\frac{20x-9}{15x^2}$.

c. $\frac{3}{2x^2+2x}+\frac{5}{x^2-3x-4}$

Since $2x^2+2x=2x(x+1)$ and $x^2-3x-4=(x-4)(x+1)$, a common multiple of $2x^2+2x$ and x^2-3x-4 is $2x(x+1)(x-4)$. Then we have $\frac{3}{2x^2+2x}+\frac{5}{x^2-3x-4}=\frac{3(x-4)}{2x(x+1)(x-4)}+\frac{5\cdot 2x}{2x(x+1)(x-4)}=\frac{13x-12}{2x(x+1)(x-4)}$.

Exercises 5–8 (8 minutes)

Have students work on these exercises in pairs or small groups.

Exercises 5–8

Perform the indicated operations for each problem below.

5. $\frac{5}{x-2}+\frac{3x}{4x-8}$

A common multiple is $4(x-2)$.

$$\frac{5}{x-2}+\frac{3x}{4x-8}=\frac{20}{4(x-2)}+\frac{3x}{4(x-2)}=\frac{3x+20}{4(x-2)}$$

6. $\frac{7m}{m-3}+\frac{5m}{3-m}$

Notice that $(3-m)=-(m-3)$.

A common multiple is $(m-3)$.

$$\frac{7m}{m-3}+\frac{5m}{3-m}=\frac{7m}{m-3}+\frac{-5m}{m-3}=\frac{7m}{m-3}-\frac{5m}{m-3}=\frac{2m}{m-3}$$

7. $\frac{b^2}{b^2-2bc+c^2}-\frac{b}{b-c}$

A common multiple is $(b-c)(b-c)$.

$$\frac{b^2}{b^2-2bc+c^2}-\frac{b}{b-c}=\frac{b^2}{(b-c)(b-c)}-\frac{b^2-bc}{(b-c)(b-c)}=\frac{bc}{(b-c)^2}$$

8. $\frac{x}{x^2-1}-\frac{2x}{x^2+x-2}$

A common multiple is $(x-1)(x+1)(x+2)$.

$$\frac{x}{x^2-1}-\frac{2x}{x^2+x-2}=\frac{x}{(x-1)(x+1)}-\frac{2x}{(x-1)(x+2)}=\frac{x(x+2)}{(x-1)(x+1)(x+2)}-\frac{2x(x+1)}{(x-1)(x+1)(x+2)}=\frac{-x^2}{(x-1)(x+1)(x+2)}$$

Example 2 (5 minutes)

Complex fractions were introduced in the previous lesson with multiplication and division of rational expressions, but these examples require performing addition and subtraction operations prior to doing the division. Remind students that when rewriting a complex fraction as division of rational expressions, they should add parentheses to the expressions both in the numerator and denominator. Then they should work inside the parentheses first following the standard order of operations.

MP.7

Example 2

Simplify the following expression.

$$\frac{\frac{b^2+b-1}{2b-1}-1}{4-\frac{8}{(b+1)}}$$

First, we can rewrite the complex fraction as a division problem, remembering to add parentheses.

$$\frac{\frac{b^2+b-1}{2b-1}-1}{4-\frac{8}{(b+1)}}=\left(\frac{b^2+b-1}{2b-1}-1\right)\div\left(4-\frac{8}{(b+1)}\right)$$

Remember that to divide rational expressions, we multiply by the reciprocal of the quotient. However, we first need to write each rational expression in an equivalent $\frac{P}{Q}$ form. For this, we need to find common denominators.

$$\begin{aligned}\frac{b^2+b-1}{2b-1}-1&=\frac{b^2+b-1}{2b-1}-\frac{2b-1}{2b-1}\\&=\frac{b^2-b}{2b-1}\end{aligned}$$

$$\begin{aligned}4-\frac{8}{(b+1)}&=\frac{4(b+1)}{b+1}-\frac{8}{(b+1)}\\&=\frac{4b-4}{(b+1)}\\&=\frac{4(b-1)}{b+1}\end{aligned}$$

Now, we can substitute these equivalent expressions into our calculation above and continue to perform the division as we did in Lesson 24.

$$\begin{aligned}\frac{\frac{b^2+b-1}{2b-1}-1}{4-\frac{8}{(b+1)}}&=\left(\frac{b^2+b-1}{2b-1}-1\right)\div\left(4-\frac{8}{(b+1)}\right)\\&=\left(\frac{b^2-b}{2b-1}\right)\div\left(\frac{4(b-1)}{b+1}\right)\\&=\left(\frac{b(b-1)}{2b-1}\right)\cdot\left(\frac{(b+1)}{4(b-1)}\right)\\&=\frac{b(b+1)}{4(2b-1)}\end{aligned}$$

Closing (2 minutes)

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this opportunity to informally assess their understanding of the lesson. In particular, ask students to verbally or symbolically articulate the processes for adding and subtracting rational expressions.

Lesson Summary

In this lesson, we extended addition and subtraction of rational numbers to addition and subtraction of rational expressions. The process for adding or subtracting rational expressions can be summarized as follows:

- **Find a common multiple of the denominators to use as a common denominator.**
- **Find equivalent rational expressions for each expression using the common denominator.**
- **Add or subtract the numerators as indicated and simplify if needed.**

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 25: Adding and Subtracting Rational Expressions

Exit Ticket

Perform the indicated operation.

1. $\frac{3}{a+2}+\frac{4}{a-5}$

2. $\frac{4r}{r+3}-\frac{5}{r}$

Exit Ticket Sample Solutions

Perform the indicated operation.

1. $\frac{3}{a+2}+\frac{4}{a-5}$

$$\frac{3}{a+2}+\frac{4}{a-5}=\frac{3a-15}{(a+2)(a-5)}+\frac{4a+8}{(a+2)(a-5)}$$
$$=\frac{7a-7}{(a+2)(a-5)}$$

2. $\frac{4r}{r+3}-\frac{5}{r}$

$$\frac{4r}{r+3}-\frac{5}{r}=\frac{4r^2}{r(r+3)}-\frac{5r+15}{r(r+3)}$$
$$=\frac{4r^2-5r-15}{r(r+3)}$$

Problem Set Sample Solutions

1. **Write each sum or difference as a single rational expression.**

   a. $\frac{7}{8}-\frac{\sqrt{3}}{5}$

   $\frac{35-8\sqrt{3}}{40}$

   b. $\frac{\sqrt{5}}{10}+\frac{\sqrt{2}}{6}+2$

   $\frac{3\sqrt{5}+5\sqrt{2}+60}{30}$

   c. $\frac{4}{x}+\frac{3}{2x}$

   $\frac{11}{2x}$

2. **Write as a single rational expression.**

   a. $\frac{1}{x}-\frac{1}{x-1}$

   $-\frac{1}{x(x-1)}$

   b. $\frac{3x}{2y}-\frac{5x}{6y}+\frac{x}{3y}$

   $\frac{x}{y}$

   c. $\frac{a-b}{a^2}+\frac{1}{a}$

   $\frac{2a-b}{a^2}$

   d. $\frac{1}{p-2}-\frac{1}{p+2}$

   $\frac{4}{(p-2)(p+2)}$

   e. $\frac{1}{p-2}+\frac{1}{2-p}$

   0

   f. $\frac{1}{b+1}-\frac{b}{1+b}$

   $\frac{1-b}{b+1}$

g. $1 - \frac{1}{1+p}$

$\frac{p}{1+p}$

h. $\frac{p+q}{p-q} - 2$

$\frac{3q-p}{p-q}$

i. $\frac{r}{s-r} + \frac{s}{r+s}$

$\frac{r^2+s^2}{(s-r)(r+s)}$

j. $\frac{3}{x-4} + \frac{2}{4-x}$

$\frac{1}{x-4}$

k. $\frac{3n}{n-2} + \frac{3}{2-n}$

$\frac{3n-3}{n-2}$

l. $\frac{8x}{3y-2x} + \frac{12y}{2x-3y}$

-4

m. $\frac{1}{2m-4n} - \frac{1}{2m+4n} - \frac{m}{m^2-4n^2}$

$-\frac{1}{m+2n}$

n. $\frac{1}{(2a-b)(a-c)} + \frac{1}{(b-c)(b-2a)}$

$\frac{b-a}{(a-c)(b-c)(2a-b)}$

o. $\frac{b^2+1}{b^2-4} + \frac{1}{b+2} + \frac{1}{b-2}$

$\frac{b^2+2b+1}{(b-2)(b+2)}$

3. Simplify the following expressions.

a. $\dfrac{\frac{1}{a} - \frac{1}{2a}}{\frac{4}{a}}$

$\frac{1}{8}$

b. $\dfrac{\frac{5x}{2} + 1}{\frac{5x}{4} - \frac{1}{5x}}$

$\frac{10x}{5x-2}$

c. $\dfrac{1 + \frac{4x+3}{x^2+1}}{1 - \frac{x+7}{x^2+1}}$

$\frac{x+2}{x-3}$

EXTENSION:

4. Suppose that $x \neq 0$ and $y \neq 0$. We know from our work in this section that $\frac{1}{x} \cdot \frac{1}{y}$ is equivalent to $\frac{1}{xy}$. Is it also true that $\frac{1}{x} + \frac{1}{y}$ is equivalent to $\frac{1}{x+y}$? Provide evidence to support your answer.

No, the rational expressions $\frac{1}{x} + \frac{1}{y}$ and $\frac{1}{x+y}$ are not equivalent. Consider $x = 2$ and $y = 1$. Then $\frac{1}{x+y} = \frac{1}{2+1} = \frac{1}{3}$, but $\frac{1}{x} + \frac{1}{y} = \frac{1}{2} + 1 = \frac{3}{2}$. Since $\frac{1}{3} \neq \frac{3}{2}$, the expressions $\frac{1}{x} + \frac{1}{y}$ and $\frac{1}{x+y}$ are not equivalent.

5. Suppose that $x = \frac{2t}{1+t^2}$ and $y = \frac{1-t^2}{1+t^2}$. Show that the value of $x^2 + y^2$ does not depend on the value of t.

$$\begin{aligned} x^2 + y^2 &= \left(\frac{2t}{1+t^2}\right)^2 + \left(\frac{1-t^2}{1+t^2}\right)^2 \\ &= \frac{4t^2}{(1+t^2)^2} + \frac{(1-t^2)^2}{(1+t^2)^2} \\ &= \frac{4t^2 + (1-2t^2+t^4)}{(1+t^2)^2} \\ &= \frac{1+2t^2+t^4}{1+2t^2+t^4} \\ &= 1 \end{aligned}$$

Since $x^2 + y^2 = 1$, the value of $x^2 + y^2$ does not depend on the value of t.

6. Show that for any real numbers a and b, and any integers x and y so that $x \neq 0$, $y \neq 0$, $x \neq y$, and $x \neq -y$,

$$\left(\frac{y}{x}-\frac{x}{y}\right)\left(\frac{ax+by}{x+y}-\frac{ax-by}{x-y}\right)=2(a-b).$$

$$\begin{aligned}
\left(\frac{y}{x}-\frac{x}{y}\right)\left(\frac{ax+by}{x+y}-\frac{ax-by}{x-y}\right) &= \left(\frac{y^2}{xy}-\frac{x^2}{xy}\right)\left(\frac{(ax+by)(x-y)}{(x+y)(x-y)}-\frac{(ax-by)(x+y)}{(x-y)(x+y)}\right)\\
&= \left(\frac{y^2-x^2}{xy}\right)\left(\frac{ax^2-axy+bxy-by^2}{x^2-y^2}-\frac{(ax^2+axy-bxy-by^2}{x^2-y^2}\right)\\
&= -\left(\frac{x^2-y^2}{xy}\right)\left(\frac{-2axy+2bxy}{x^2-y^2}\right)\\
&= -\left(\frac{1}{xy}\right)\left(\frac{-2xy(a-b)}{1}\right)\\
&= 2(a-b)
\end{aligned}$$

7. Suppose that n is a positive integer.

a. Simplify the expression $\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)$.

$$\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)=\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n+1}\right)=\left(\frac{n+2}{n}\right)$$

b. Simplify the expression $\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n+2}\right)$.

$$\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n+2}\right)=\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n+1}\right)\left(\frac{n+3}{n+2}\right)=\left(\frac{n+3}{n}\right)$$

c. Simplify the expression $\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n+2}\right)\left(1+\frac{1}{n+3}\right)$.

$$\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n+2}\right)\left(1+\frac{1}{n+3}\right)=\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n+1}\right)\left(\frac{n+3}{n+2}\right)\left(\frac{n+4}{n+3}\right)=\left(\frac{n+4}{n}\right)$$

MP.7

d. If this pattern continues, what is the product of n of these factors?

If we have n of these factors, then the product will be

$$\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\cdots\left(1+\frac{1}{n+(n-1)}\right)=\frac{n+n}{n}=2.$$

Lesson 26: Solving Rational Equations

Student Outcomes

- Students solve rational equations, monitoring for the creation of extraneous solutions.

Lesson Notes

In the preceding lessons, students learned to add, subtract, multiply, and divide rational expressions so that in this lesson we can solve equations involving rational expressions (**A-REI.A.2**). The skills developed in this lesson will be required to solve equations of the form $f(x) = c$ for a rational function f and constant c in Lesson 27 and later in Module 3 (**F-BF.B.4a**).

There is more than one approach to solving a rational equation, and we explore two such methods in this section. The first method is to multiply both sides by the common denominator to clear fractions. The second method is to find equivalent forms of all expressions with a common denominator, set the numerators equal to each other, and solve the resulting equation. Either approach requires that we keep an eye out for extraneous solutions; in other words, values that appear to be a solution to the equation but cause division by zero and are, thus, not valid. Throughout our work with rational expressions, students will be analyzing the structure of the expressions in order to decide on their next algebraic steps (MP.7). Encourage students to check their answers by substituting their solutions back into each side of the equation separately.

Classwork

Exercises 1–2 (8 minutes)

Allow students to solve this any way that they can and then discuss their answers. Focus on adding the fractions on the left and equating numerators or multiplying both sides by a common multiple. Indicate a practical technique of finding a common denominator. These first two exercises highlight MP.7, as the students must recognize the given expressions to be of the form $\frac{a}{b} + \frac{c}{b}$ or $\frac{a}{b} + \frac{c}{d}$; by expressing the equations in the simplified form $\frac{A}{B} = \frac{C}{B}$, they realize that we must have $A = C$.

Scaffolding:

Struggling students may benefit from first solving the equation $\frac{x}{5} - \frac{2}{5} = \frac{1}{5}$.

More advanced students may try to solve
$$\frac{x-1}{x+2} = \frac{3}{4}.$$

Exercises 1–2

Solve the following equations for x, and give evidence that your solutions are correct.

1. $\frac{x}{2} + \frac{1}{3} = \frac{5}{6}$

Combining the expressions on the left, we have $\frac{3x}{6} + \frac{2}{6} = \frac{5}{6}$, so $\frac{3x+2}{6} = \frac{5}{6}$; therefore, $3x + 2 = 5$. Then, $x = 1$.

Or, using another approach:

$6 \cdot \left(\frac{x}{2} + \frac{1}{3}\right) = 6 \cdot \left(\frac{5}{6}\right)$, so $3x + 2 = 5$; then, $x = 1$.

The solution to this equation is 1. To verify, we see that when $x = 1$, we have $\frac{x}{2} + \frac{1}{3} = \frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$, so 1 is a valid solution.

2. $\frac{2x}{9} + \frac{5}{9} = \frac{8}{9}$

Since the expressions already have a common denominator, we see that $\frac{2x}{9} + \frac{5}{9} = \frac{2x+5}{9}$, so we need to solve $\frac{2x+5}{9} = \frac{8}{9}$. It then follows that the numerators are equal, so $2x + 5 = 8$. Solving for x gives $x = \frac{3}{2}$. To verify, we see that when $x = \frac{3}{2}$, we have $\frac{2x}{9} + \frac{5}{9} = \frac{2\left(\frac{3}{2}\right)}{9} + \frac{5}{9} = \frac{3}{9} + \frac{5}{9} = \frac{8}{9}$; thus, $\frac{3}{2}$ is a valid solution.

Remind students that two rational expressions with the same denominator are equal if the numerators are equal.

Discussion (2 minutes)

Now that we know how to add, subtract, multiply, and divide rational expressions, we are going to use some of those basic operations to solve equations involving rational expressions. An equation involving rational expressions is called a *rational equation.* Keeping the previous exercise in mind, we will look at two different approaches to solving rational equations.

Example 1 (6 minutes)

MP.1

Ask students to try to solve this challenge problem on their own. Have them discuss and explain their methods in groups or with neighbors. The teacher should circulate and lead a discussion of both methods once students have had a chance to try solving on their own.

Example 1

Solve the following equation: $\frac{x+3}{12} = \frac{5}{6}$.

Equating Numerators Method: *Obtain expressions on both sides with the same denominator and equate numerators.*

$$\frac{x+3}{12} = \frac{5}{6} \cdot \frac{2}{2}$$
$$\frac{x+3}{12} = \frac{10}{12}$$

Thus, $x + 3 = 10$, and $x = 7$; therefore, 7 is the solution to our original equation.

Clearing Fractions Method: *Multiply both sides by a common multiple of the denominators to clear the fractions, and then solve the resulting equation.*

$$12 \cdot \left(\frac{x+3}{12}\right) = 12 \cdot \left(\frac{5}{6}\right)$$
$$x + 3 = 10$$

We can see, once again, that the solution is 7.

Discussion (3 minutes)

Ask students to discuss both methods used in the previous example. Which method do they prefer, and why? Does one method seem to be more efficient than the other? Have a few groups report their opinions to the class. At no time should students be required to use a particular method; just be sure they understand both approaches, and allow them to use whichever method seems more natural.

Exercise 3 (6 minutes)

Remind students that when we say a "solution" to an equation, we are talking about a value of the variable, usually x, that will result in a true number sentence. In Lesson 22, students learned that there are some values of the variable that are not allowed in order to avoid division by zero. Before students start working on the following exercise, ask them to identify the values of x that must be excluded. Wait for students to respond that we must have $x \neq 0$ and $x \neq 2$ before having them work with a partner on the following exercise.

MP.7

Exercises 3–7

3. **Solve the following equation: $\frac{3}{x} = \frac{8}{x-2}$.**

***Method 1:** Convert both expressions to equivalent expressions with a common denominator. The common denominator is $x(x-2)$, so we use the identity property of multiplication to multiply the left side by $\frac{x-2}{x-2}$ and the right side by $\frac{x}{x}$. This does not change the value of the expression on either side of the equation.*

$$\left(\frac{x-2}{x-2}\right)\cdot\left(\frac{3}{x}\right) = \left(\frac{x}{x}\right)\cdot\left(\frac{8}{x-2}\right)$$
$$\frac{3x-6}{x(x-2)} = \frac{8x}{x(x-2)}$$

Since the denominators are equal, we can see that the numerators must be equal; thus, $3x - 6 = 8x$. Solving for x gives a solution of $-\frac{6}{5}$. At the outset of this example, we noted that x cannot take on the value of 0 or 2, but there is nothing preventing x from taking on the value $-\frac{6}{5}$. Thus, we have found a solution. We can check our work. Substituting $-\frac{6}{5}$ into $\frac{3}{x}$ gives us $\frac{3}{\left(-6/5\right)} = -\frac{5}{2}$, and substituting $-\frac{6}{5}$ into $\frac{8}{x-2}$ gives us $\frac{8}{\left(-6/5\right)-2} = -\frac{5}{2}$. Thus, when $x = -\frac{6}{5}$, we have $\frac{3}{x} = \frac{8}{x-2}$; therefore, $-\frac{6}{5}$ is indeed a solution.

***Method 2:** Multiply both sides of the equation by the common denominator $x(x-2)$, and solve the resulting equation.*

$$x(x-2)\left(\frac{3}{x}\right) = x(x-2)\left(\frac{8}{x-2}\right)$$
$$3(x-2) = 8x$$
$$3x - 6 = 8x$$

From this point, we follow the same steps as we did in Method 1, and we get the same solution: $-\frac{6}{5}$.

Exercise 4 (6 minutes)

Have students continue to work with partners to solve the following equation. Walk around the room and observe student progress; if necessary, offer the following hints and reminders:

- Reminder: Ask students to identify excluded values of a. Suggest that they factor the denominator $a^2 - 4$. They should discover that we must specify $a \neq 2$ and $a \neq -2$.
- Hint 1: Ask students to identify a common denominator of the three expressions in the equation. They should respond with $(a-2)(a+2)$, or equivalently, $a^2 - 4$.
- Hint 2: What do we need to do with this common denominator? They should determine that they need to find equivalent rational expressions for each of the terms with denominator $(a-2)(a+2)$.

4. **Solve the following equation for a: $\frac{1}{a+2} + \frac{1}{a-2} = \frac{4}{a^2-4}$.**

First, we notice that we must have $a \neq 2$ and $a \neq -2$. Then, we apply Method 1:

$$\left(\frac{a-2}{a-2}\right)\cdot\left(\frac{1}{a+2}\right) + \left(\frac{a+2}{a+2}\right)\cdot\left(\frac{1}{a-2}\right) = \frac{4}{(a-2)(a+2)}$$

$$\frac{a-2}{(a-2)(a+2)} + \frac{a+2}{(a-2)(a+2)} = \frac{4}{(a-2)(a+2)}$$

$$\frac{2a}{(a-2)(a+2)} = \frac{4}{(a-2)(a+2)}.$$

Since the denominators are equal, we know that the numerators are equal; thus, we have $2a = 4$, which means that $a = 2$. Thus, the only solution to this equation is 2. However, a is not allowed to be 2 because if $a = 2$, then $\frac{1}{a-2}$ is not defined. This means that the original equation, $\frac{1}{a+2} + \frac{1}{a-2} = \frac{4}{a^2-4}$, has no solution.

Scaffolding:

Let students know that the word *extraneous* has meaning outside of the mathematics classroom; ask them to guess its definition, and then provide the actual definition.

> **Extraneous**: Irrelevant or unrelated to the subject being dealt with.

Students may benefit from choral repetition, as well as a visual representation of this word.

Introduce the term *extraneous solution*. An invalid solution that may arise when we manipulate a rational expression is called an extraneous solution. An extraneous solution is a value that satisfies a transformed equation but does not satisfy the original equation.

Exercises 5–7 (8 minutes)

Give students a few minutes to discuss extraneous solutions with a partner. When do they occur, and how do you know when you have one? Extraneous solutions occur when one of the solutions found does not make a true number sentence when substituted into the original equation. The only way to know you have one is to note the values of the variable that will cause a part of the equation to be undefined. In this lesson, we are concerned with division by zero; in later lessons, we exclude values of the variable that would cause the square root of a negative number. Make sure that all students have an understanding of extraneous solutions before proceeding. Then, have them work in pairs on the following exercises.

5. Solve the following equation. Remember to check for extraneous solutions.

$$\frac{4}{3x}+\frac{5}{4}=\frac{3}{x}$$

First, note that we must have $x \neq 0$.

Equating numerators: $\frac{16}{12x}+\frac{15x}{12x}=\frac{36}{12x}$ *Then, we have $16+15x=36$, and the solution is $x=\frac{4}{3}$.*

Clearing fractions: $12x\left(\frac{4}{3x}+\frac{5}{4}\right)=12x\left(\frac{3}{x}\right)$ *Then, we have $16+15x=36$, and the solution is $x=\frac{4}{3}$.*

The solution $\frac{4}{3}$ is valid since the only excluded value is 0.

6. Solve the following equation. Remember to check for extraneous solutions.

$$\frac{7}{b+3}+\frac{5}{b-3}=\frac{10b-2}{b^2-9}$$

First, note that we must have $x \neq 3$ and $x \neq -3$.

Equating numerators: $\frac{7(b-3)}{(b-3)(b+3)}+\frac{5(b+3)}{(b-3)(b+3)}=\frac{10b-2}{(b-3)(b+3)}$

Matching numerators, we have $7b-21+5b+15=10b-2$, which leads to $2b=4$; therefore, $b=2$.

Clearing fractions: $(b-3)(b+3)\left(\frac{7}{b+3}+\frac{5}{b-3}\right)=(b-3)(b+3)\left(\frac{10b-2}{b^2-9}\right)$

We have $7(b-3)+5(b+3)=10b-2$, which leads to $2b=4$; therefore, $b=2$.

The solution 2 is valid since the only excluded values are 3 and -3.

7. Solve the following equation. Remember to check for extraneous solutions.

$$\frac{1}{x-6}+\frac{x}{x-2}=\frac{4}{x^2-8x+12}$$

First, note that we must have $x \neq 6$ and $x \neq 2$.

Equating numerators:

$$\frac{x-2}{(x-6)(x-2)}+\frac{x^2-6x}{(x-6)(x-2)}=\frac{4}{(x-6)(x-2)}$$
$$x^2-5x-2=4$$
$$x^2-5x-6=0$$
$$(x-6)(x+1)=0$$

The solutions are 6 and -1.

Clearing fractions:

$$\left(\frac{1}{x-6}+\frac{x}{x-2}\right)(x-6)(x-2)=\left(\frac{4}{(x-6)(x-2)}\right)(x-6)(x-2)$$
$$(x-2)+x(x-6)=4$$
$$x^2-6x+x-2=4$$
$$x^2-5x-6=0$$
$$(x-6)(x+1)=0$$

The solutions are 6 and -1.

Because x is not allowed to be 6 in order to avoid division by zero, the solution 6 is extraneous; thus, -1 is the only solution to the given rational equation.

Closing (2 minutes)

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. In particular, ask students to explain how we identify extraneous solutions and why they arise when solving rational equations.

> **Lesson Summary**
>
> **In this lesson, we applied what we have learned in the past two lessons about addition, subtraction, multiplication, and division of rational expressions to solve rational equations. An extraneous solution is a solution to a transformed equation that is not a solution to the original equation. For rational functions, extraneous solutions come from the excluded values of the variable.**
>
> **Rational equations can be solved one of two ways:**
>
> 1. **Write each side of the equation as an equivalent rational expression with the same denominator and equate the numerators. Solve the resulting polynomial equation, and check for extraneous solutions.**
> 2. **Multiply both sides of the equation by an expression that is the common denominator of all terms in the equation. Solve the resulting polynomial equation, and check for extraneous solutions.**

Exit Ticket (4 minutes)

Name ______________________________________ Date ____________________

Lesson 26: Solving Rational Equations

Exit Ticket

Find all solutions to the following equation. If there are any extraneous solutions, identify them and explain why they are extraneous.

$$\frac{7}{b+3}+\frac{5}{b-3}=\frac{10b}{b^2-9}$$

Exit Ticket Sample Solutions

Find all solutions to the following equation. If there are any extraneous solutions, identify them and explain why they are extraneous.

$$\frac{7}{b+3}+\frac{5}{b-3}=\frac{10b}{b^2-9}$$

First, note that we must have $x \neq 3$ and $x \neq -3$.

Using the equating numerators method: $\frac{7(b-3)}{(b-3)(b+3)}+\frac{5(b+3)}{(b-3)(b+3)}=\frac{10b}{(b-3)(b+3)}$

Matching numerators, we have $7b-21+5b+15=10b$, which leads to $12b-6=10b$; therefore, $b=3$.

However, since the excluded values are 3 and -3, the solution 3 is an extraneous solution, and there is no solution to $\frac{7}{b+3}+\frac{5}{b-3}=\frac{10b}{b^2-9}$.

Problem Set Sample Solutions

1. **Solve the following equations and check for extraneous solutions.**

a. $\frac{x-8}{x-4}=2$
0

b. $\frac{4x-8}{x-2}=4$
All real numbers except 2

c. $\frac{x-4}{x-3}=1$
No solution

d. $\frac{4x-8}{x-2}=3$
No solution

e. $\frac{1}{2a}-\frac{2}{2a-3}=0$
$-\frac{3}{2}$

f. $\frac{3}{2x+1}=\frac{5}{4x+3}$
-2

g. $\frac{4}{x-5}-\frac{2}{5+x}=\frac{2}{x}$
$-\frac{5}{3}$

h. $\frac{y+2}{3y-2}+\frac{y}{y-1}=\frac{2}{3}$
$\frac{5}{6}, -2$

i. $\frac{3}{x+1}-\frac{2}{1-x}=1$
$0, 5$

j. $\frac{4}{x-1}+\frac{3}{x}-3=0$
$\frac{1}{3}, 3$

k. $\frac{x+1}{x+3}-\frac{x-5}{x+2}=\frac{17}{6}$
$0, -\frac{55}{17}$

l. $\frac{x+7}{4}-\frac{x+1}{2}=\frac{5-x}{3x-14}$
$5, 6$

m. $\frac{b^2-b-6}{b^2}-\frac{2b+12}{b}=\frac{b-39}{2b}$
$3, \frac{4}{3}$

n. $\frac{1}{p(p-4)}+1=\frac{p-6}{p}$
$\frac{23}{6}$

o. $\frac{1}{h+3}=\frac{h+4}{h-2}+\frac{6}{h-2}$
$-8, -4$

p. $\frac{m+5}{m^2+m}=\frac{1}{m^2+m}-\frac{m-6}{m+1}$
$4, 1$

2. **Create and solve a rational equation that has 0 as an extraneous solution.**

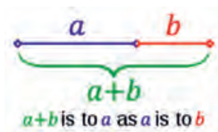
One such equation is $\frac{1}{x-1}+\frac{1}{x}=\frac{1}{x-x^2}$.

3. **Create and solve a rational equation that has 2 as an extraneous solution.**

One such equation is $\frac{1}{x-2}+\frac{1}{x+2}=\frac{4}{x^2-4}$.

EXTENSION:

4. **Two lengths a and b, where $a > b$, are in *golden ratio* if the ratio of $a+b$ is to a is the same as a is to b. Symbolically, this is expressed as $\frac{a}{b}=\frac{a+b}{a}$. We denote this common ratio by the Greek letter *phi (pronounced "fee")* with symbol φ, so that if a and b are in common ratio, then $\varphi=\frac{a}{b}=\frac{a+b}{a}$. By setting $b=1$, we find that $\varphi=a$ and φ is the positive number that satisfies the equation $\varphi=\frac{\varphi+1}{\varphi}$. Solve this equation to find the numerical value for φ.**



We can apply either method from the previous lesson to solve this equation.

$$\varphi=\frac{\varphi+1}{\varphi}$$
$$\varphi^2=\varphi+1$$
$$\varphi^2-\varphi-1=0$$

Applying the quadratic formula, we have two solutions:

$$\varphi=\frac{1+\sqrt{5}}{2} \quad \textit{or} \quad \varphi=\frac{1-\sqrt{5}}{2}.$$

Since φ is a positive number, and $\frac{1-\sqrt{5}}{2}<0$, *we have* $\varphi=\frac{1+\sqrt{5}}{2}$.

5. **Remember that if we use x to represent an integer, then the next integer can be represented by $x+1$.**

   a. **Does there exist a pair of consecutive integers whose reciprocals sum to $\frac{5}{6}$? Explain how you know.**

   Yes, 2 and 3 because $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$.

   b. **Does there exist a pair of consecutive integers whose reciprocals sum to $\frac{3}{4}$? Explain how you know.**

   If x represents the first integer, then $x+1$ represents the next integer. Suppose $\frac{1}{x}+\frac{1}{x+1}=\frac{3}{4}$. *Then,*

$$\frac{1}{x}+\frac{1}{x+1}=\frac{3}{4}$$
$$\frac{4(x+1)+4x}{4x(x+1)}=\frac{3x(x+1)}{4x(x+1)}$$
$$8x+4=3x^2+3x$$
$$3x^2-5x-4=0.$$

   The solutions to this quadratic equation are $\frac{5+\sqrt{73}}{6}$ *and* $\frac{5-\sqrt{73}}{6}$, *so there are no integers that solve this equation. Thus, there are no pairs of consecutive integers whose reciprocals sum to* $\frac{3}{4}$.

c. Does there exist a pair of consecutive *even* integers whose reciprocals sum to $\frac{3}{4}$? Explain how you know.

If x represents the first integer, then $x+2$ represents the next even integer. Suppose $\frac{1}{x}+\frac{1}{x+2}=\frac{3}{4}$. Then,

$$\frac{1}{x}+\frac{1}{x+2}=\frac{3}{4}$$
$$\frac{4(x+2)+4x}{4x(x+2)}=\frac{3x(x+2)}{4x(x+2)}$$
$$8x+8=3x^2+6x$$
$$3x^2-2x-8=0.$$

The solutions to this quadratic equation are $-\frac{4}{3}$ and 2; therefore, the only even integer x that solves the equation is 2. Then, 2 and 4 are consecutive even integers whose reciprocals sum to $\frac{3}{4}$.

d. Does there exist a pair of consecutive *even* integers whose reciprocals sum to $\frac{5}{6}$? Explain how you know.

If x represents the first integer, then $x+2$ represents the next even integer. Suppose $\frac{1}{x}+\frac{1}{x+2}=\frac{5}{6}$. Then,

$$\frac{1}{x}+\frac{1}{x+2}=\frac{5}{6}$$
$$\frac{6(x+2)+6x}{6x(x+2)}=\frac{5x(x+2)}{6x(x+2)}$$
$$12x+12=5x^2+10x$$
$$5x^2-2x-12=0.$$

The solutions to this quadratic equation are $\frac{1+\sqrt{61}}{5}$ and $\frac{1-\sqrt{61}}{5}$, so there are no integers that solve this equation. Thus, there are no pairs of consecutive even integers whose reciprocals sum to $\frac{5}{6}$.

Lesson 27: Word Problems Leading to Rational Equations

Student Outcomes

- Students solve word problems using models that involve rational expressions.

Lesson Notes

In the preceding lessons, students learned to add, subtract, multiply, and divide rational expressions and solve rational equations in order to develop the tools needed for solving application problems involving rational equations in this lesson (**A-REI.A.2**). Students will develop their problem-solving and modeling abilities by carefully reading the problem description and converting information into equations (MP.1), thus creating a mathematical model of the problem (MP.4).

Classwork

Exercise 1 (13 minutes)

We now turn to some applied problems that we can model with rational equations, strengthening students' problem-solving and modeling experience in alignment with standards MP.1 and MP.4. We can solve these equations using the skills developed in previous lessons. Have students work in small groups to answer this set of four questions. At the end of the work time, ask different groups to present their solutions to the class. Suggest to students that they should: (a) read the problem aloud, (b) paraphrase and summarize the problem in their own words, (c) find an equation that models the situation, and (d) say how it represents the quantities involved. Check to make sure that students understand the problem before they begin trying to solve it.

You may want to encourage students, especially with this first problem, by suggesting that they use m for the unknown quantity and asking if they can arrive at an equation that relates this unknown quantity to the known quantities.

Scaffolding:

Students may benefit from having the problem read aloud and summarized. They should be encouraged to restate the problem in their own words to a partner.

If students are struggling, present the equation $\frac{12+m}{20+m} = 0.75$, and ask students how this models the situation.

Students who may be above grade level could be challenged to write their own word problems that result in rational equations.

MP.1 & MP.4

Exercise 1

1. **Anne and Maria play tennis almost every weekend. So far, Anne has won 12 out of 20 matches.**

   a. **How many matches will Anne have to win in a row to improve her winning percentage to 75%?**

   Suppose that Anne has already won 12 of 20 matches, and let m represent the number of additional matches she must win to raise her winning percentage to 75%. After playing and winning all of those additional m matches, she has won $12+m$ matches out of a total of $20+m$ matches played. Her winning percentage is then $\frac{12+m}{20+m}$, and we want to find the value of m that solves the equation

$$\frac{12+m}{20+m} = 0.75.$$

MP.1 & MP.4

Multiply both sides by $20+m$.

$$12+m=0.75(20+m)$$
$$12+m=15+0.75m$$

Solving for m:

$$0.25m=3$$
$$m=12$$

So, Anne would need to win 12 matches in a row in order to improve her winning percentage to 75%.

b. How many matches will Anne have to win in a row to improve her winning percentage to 90%?

This situation is similar to that for part (a), except that we want a winning percentage of 0.90, instead of 0.75. Again, we let m represent the number of matches Anne must win consecutively to bring her winning percentage up to 90%.

$$\frac{12+m}{20+m}=0.90$$

Solving for m:

$$12+m=0.90(20+m)$$
$$12+m=18+0.90m$$
$$0.10m=6$$
$$m=60$$

In order for Anne to bring her winning percentage up to 90%, she would need to win the next 60 consecutive matches.

MP.2

c. Can Anne reach a winning percentage of 100%?

Allow students to come to the conclusion that Anne will never reach a winning percentage of 100% because she has already lost 8 matches.

d. After Anne has reached a winning percentage of 90% by winning consecutive matches as in part (b), how many matches can she now lose in a row to have a winning percentage of 50%?

Recall from part (b) that she had won 72 matches out of 80 to reach a winning percentage of 90%. We will now assume that she loses the next k matches in a row. Then, she will have won 72 matches out of $80+k$ matches, and we want to know the value of k that makes this a 50% win rate.

$$\frac{72}{80+k}=0.50$$

Solving the equation:

$$72=0.50(80+k)$$
$$72=40+0.50k$$
$$32=0.50k$$
$$64=k$$

Thus, after reaching a 90% winning percentage in 80 matches, Anne can lose 64 matches in a row to drop to a 50% winning percentage.

Example 1 (5 minutes)

Work this problem at the front of the room, but allow the class to provide input and steer the discussion. Depending on how students did with the first exercise, the teacher may lead a discussion of this problem as a class, ask students to work in groups, or ask students to work independently while targeting instruction with a small group that struggled on the first example.

MP.4

Example 1

Working together, it takes Sam, Jenna, and Francisco two hours to paint one room. When Sam is working alone, he can paint one room in 6 hours. When Jenna works alone, she can paint one room in 4 hours. Determine how long it would take Francisco to paint one room on his own.

Consider how much can be accomplished in one hour. Sam, Jenna, and Francisco together can paint half a room in one hour. If Sam can paint one room in 6 hours on his own, then in one hour he can paint $\frac{1}{6}$ of the room. Similarly, Jenna can paint $\frac{1}{4}$ of the room in one hour. We do not yet know how much Francisco can paint in one hour, so we will say he can paint $\frac{1}{f}$ of the room. So, in one hour, Sam has painted $\frac{1}{6}$ of the room, Jenna has painted $\frac{1}{4}$ of the room, and all three together can paint $\frac{1}{2}$ the room, leading to the following equation for how much can be painted in one hour:

$$\frac{1}{6}+\frac{1}{4}+\frac{1}{f}=\frac{1}{2}.$$

A common multiple of the denominators is $12f$. Multiplying both sides by $12f$ gives us:

$$\frac{12f}{6}+\frac{12f}{4}+\frac{12f}{f}=\frac{12f}{2}$$
$$2f+3f+12=6f,$$

which leads us to the value of f:

$$f=12.$$

So, Francisco can paint the room in 12 hours on his own.

Exercise 2 (5 minutes)

Remind students that distance equals rate times time ($d=r\cdot t$) before having them work on this exercise in pairs or small groups. Be sure to have groups share their results before continuing to the next exercise.

MP.1 & MP.4

Exercises 2–4

2. **Melissa walks 3 miles to the house of a friend and returns home on a bike. She averages 4 miles per hour faster when cycling than when walking, and the total time for both trips is two hours. Find her walking speed.**

Using the relationship $d = r \cdot t$, we have $t = \frac{d}{r}$. The time it takes for Melissa to walk to her friend's house is $\frac{3}{r}$, and the time to cycle back is $\frac{3}{r+4}$. Thus, we can write an equation that describes the combined time for both trips:

$$\frac{3}{r} + \frac{3}{r+4} = 2.$$

A common multiple of the denominators is $r(r+4)$, so we multiply both sides of the equation by $r(r+4)$.

$$3(r+4) + 3r = 2r(r+4)$$
$$3r + 12 + 3r = 2r^2 + 8r$$
$$2r^2 + 2r - 12 = 0$$
$$2(r-2)(r+3) = 0$$

Thus, $r = -3$ or $r = 2$. Since r represents Melissa's speed, it does not make sense for r to be negative. So, the only solution is 2, which means that Melissa's walking speed is 2 miles per hour.

Exercise 3 (10 minutes)

3. **You have 10 liters of a juice blend that is 60% juice.**

   a. **How many liters of pure juice need to be added in order to make a blend that is 75% juice?**

   We start off with 10 liters of a blend containing 60% juice. Then, this blend contains $0.60(10) = 6$ liters of juice in the 10 liters of mixture. If we add A liters of pure juice, then the concentration of juice in the blend is $\frac{6+A}{10+A}$. We want to know which value of A makes this blend 75% juice.

$$\frac{6+A}{10+A} = 0.75$$
$$6 + A = 0.75(10 + A)$$
$$6 + A = 7.5 + 0.75A$$
$$0.25A = 1.5$$
$$A = 6$$

   Thus, if we add 6 liters of pure juice, we have 16 liters of a blend that contains 12 liters of juice, meaning that the concentration of juice in this blend is 75%.

   b. **How many liters of pure juice need to be added in order to make a blend that is 90% juice?**

$$\frac{6+A}{10+A} = 0.90$$
$$6 + A = 0.9(10 + A)$$
$$6 + A = 9 + 0.9A$$
$$3 = 0.1A$$
$$A = 30$$

   Thus, if we add 30 liters of pure juice, we will have 40 liters of a blend that contains 36 liters of pure juice, meaning that the concentration of juice in this blend is 90%.

MP.4

c. Write a rational equation that relates the desired percentage p to the amount A of pure juice that needs to be added to make a blend that is $p\%$ juice, where $0 < p < 100$. What is a reasonable restriction on the set of possible values of p? Explain your answer.

$$\frac{6+A}{10+A} = \frac{p}{100}$$

We need to have $60 < p < 100$ for the problem to make sense. We already have 60% juice; the percentage cannot decrease by adding more juice, and we can never have a mixture that is more than 100 percent juice.

d. Suppose that you have added 15 liters of juice to the original 10 liters. What is the percentage of juice in this blend?

$$\frac{p}{100} = \frac{6+15}{10+15} = 0.84$$

So, the new blend contains 84% pure juice.

MP.2

e. Solve your equation in part (c) for the amount A. Are there any excluded values of the variable p? Does this make sense in the context of the problem?

$$\frac{6+A}{10+A} = \frac{p}{100}$$
$$100(6+A) = p(10+A)$$
$$600 + 100A = 10p + pA$$
$$100A - pA = 10p - 600$$
$$A(100-p) = 10p - 600$$
$$A = \frac{10p-600}{100-p}$$

We see from the equation for A that $p \neq 100$. This makes sense because we can never make a 100% juice solution since we started with a diluted amount of juice.

Exercise 4 (5 minutes)

Allow students to work together in pairs or small groups for this exercise. This exercise is a bit different from the previous example in that the amount of acid comes from a diluted solution, not a pure solution. Be sure that students get the numerator set up correctly. (If there is not enough time to do the entire problem, have students set up the equations in class and finish solving them for homework.)

4. You have a solution containing 10% acid and a solution containing 30% acid.

a. How much of the 30% solution must you add to 1 liter of the 10% solution to create a mixture that is 22% acid?

If we add A liters of the 30% solution, then the new mixture is $1 + A$ liters of solution that contains $0.1 + 0.3A$ liters of acid. We want the final mixture to be 22% acid, so we need to solve the equation:

$$\frac{0.1+0.3A}{1+A} = 0.22.$$

Solving this gives

$$0.1 + 0.3A = 0.22(1+A)$$
$$0.1 + 0.3A = 0.22 + 0.22A$$
$$0.08A = 0.12$$
$$A = 1.5.$$

Thus, if we add 1.5 liters of 30% acid solution to 1 liter of 10% acid solution, the result is 2.5 liters of 22% acid solution.

MP.4

b. Write a rational equation that relates the desired percentage p to the amount A of 30% acid solution that needs to be added to 1 liter of 10% acid solution to make a blend that is $p\%$ acid, where $0 < p < 100$. What is a reasonable restriction on the set of possible values of p? Explain your answer.

$$\frac{0.1 + 0.3A}{1 + A} = \frac{p}{100}$$

We must have $10 < p < 30$ because if we blend a 10% acid solution and a 30% acid solution, the blend will contain an acid percentage between 10% and 30%.

MP.2

c. Solve your equation in part (b) for A. Are there any excluded values of p? Does this make sense in the context of the problem?

$$A = \frac{10 - p}{p - 30}$$

We need to exclude 30 from the possible range of values of p, which makes sense in context because we can never reach a 30% acid solution since we started with a solution that was 10% acid.

d. If you have added some 30% acid solution to 1 liter of 10% acid solution to make a 26% acid solution, how much of the stronger acid did you add?

The formula in part (c) gives $A = \frac{10-26}{26-30}$; therefore, $A = 4$. We added 4 liters of the 30% acid solution to the 1 liter of 10% acid solution to make a 26% acid mixture.

Closing (2 minutes)

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson.

Lesson Summary

In this lesson, we developed the students' problem solving skills by asking them to carefully read a problem, rephrase it in a form comfortable for their own understanding, and convert fact sentences about unknown quantities into algebraic equations. Specifically, they used rational equations to model and solve some application problems and further developed their skills in working with rational expressions.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 27: Word Problems Leading to Rational Equations

Exit Ticket

Bob can paint a fence in 5 hours, and working with Jen, the two of them painted the fence in 2 hours. How long would it have taken Jen to paint the fence alone?

Exit Ticket Sample Solutions

Bob can paint a fence in 5 hours, and working with Jen, the two of them painted the fence in 2 hours. How long would it have taken Jen to paint the fence alone?

Let x represent the time it would take Jen to paint the fence alone. Then, Bob can paint the entire fence in 5 hours; therefore, in one hour he can paint $\frac{1}{5}$ of the fence. Similarly, Jen can paint $\frac{1}{x}$ of the fence in one hour. We know that it took them two hours to complete the job, and together they can paint $\frac{1}{2}$ of the fence in one hour. We then have to solve the equation:

$$\frac{1}{5}+\frac{1}{x}=\frac{1}{2}$$
$$\frac{2x}{10x}+\frac{10}{10x}=\frac{5x}{10x}$$
$$2x+10=5x$$
$$x=\frac{10}{3}.$$

Thus, it would have taken Jen 3 hours and 20 minutes to paint the fence alone.

Problem Set Sample Solutions

1. **If 2 inlet pipes can fill a pool in one hour and 30 minutes, and one pipe can fill the pool in two hours and 30 minutes on its own, how long would the other pipe take to fill the pool on its own?**

$$\frac{1}{2.5}+\frac{1}{x}=\frac{1}{1.5}$$

We find that $x=3.75$; therefore, it takes 3 hours and 45 minutes for the second pipe to fill the pool by itself.

2. **If one inlet pipe can fill the pool in 2 hours with the outlet drain closed, and the same inlet pipe can fill the pool in 2.5 hours with the drain open, how long does it take the drain to empty the pool if there is no water entering the pool?**

$$\frac{1}{2}-\frac{1}{x}=\frac{1}{2.5}$$

We find that $x=10$; therefore, it takes 10 hours for the drain to empty the pool by itself.

3. **It takes 36 minutes less time to travel 120 miles by car at night than by day because the lack of traffic allows the average speed at night to be 10 miles per hour faster than in the daytime. Find the average speed in the daytime.**

$$\frac{120}{t-36}=\frac{120}{t}+\frac{1}{6}$$

We find that $t=180$ minutes, which is 3 hours; therefore, the average speed in the daytime is $\frac{120}{3}=40$ miles per hour.

4. The difference in the average speed of two trains is 16 miles per hour. The slower train takes 2 hours longer to travel 170 miles than the faster train takes to travel 150 miles. Find the speed of the slower train.

$$\frac{150}{t} - \frac{170}{t+2} = 16$$

We find that $t = 3$ hours. The average speed of the slower train is 34 miles per hour.

5. A school library spends $\$80$ a month on magazines. The average price for magazines bought in January was 70 cents more than the average price in December. Because of the price increase, the school library was forced to subscribe to 7 fewer magazines. How many magazines did the school library subscribe to in December?

$$\frac{80}{x + 0.70} = \frac{80}{x} - 7$$

The average price in December is $x = \$2.50$, so the school subscribed to 32 magazines in December.

6. An investor bought a number of shares of stock for $\$1,600$. After the price dropped by $\$10$ per share, the investor sold all but 4 of her shares for $\$1,120$. How many shares did she originally buy?

$$\frac{1600}{x} = \frac{1120}{x-4} + 10$$

Two possible answers: $x = 32$ shares or $x = 20$ shares

7. Newton's law of universal gravitation, $F = \frac{Gm_1m_2}{r^2}$, measures the force of gravity between two masses m_1 and m_2, where r is the distance between the centers of the masses, and G is the universal gravitational constant. Solve this equation for G.

$$G = \frac{Fr^2}{m_1m_2}$$

8. Suppose that $= \frac{x+y}{1-xy}$.

   a. Show that when $x = \frac{1}{a}$ and $y = \frac{2a-1}{a+2}$, the value of t does not depend on the value of a.

   When simplified, we find that $t = 2$; therefore, the value of t does not depend on the value of a.

   b. For which values of a do these relationships have no meaning?

   If a is 0, then x has no meaning. If $a = -2$, then y has no meaning.

9. Consider the rational equation $\frac{1}{R} = \frac{1}{x} + \frac{1}{y}$.

   a. Find the value of R when $x = \frac{2}{5}$ and $y = \frac{3}{4}$.

$$\frac{1}{R} = \frac{1}{2/5} + \frac{1}{3/4}$$

   So $R = \frac{6}{23}$.

b. Solve this equation for R and simplify.

There are two approaches to solve this equation for R.

The first way is to perform the addition on the right:

$$\frac{1}{R} = \frac{1}{x} + \frac{1}{y} = \frac{y}{xy} + \frac{x}{xy} = \frac{x+y}{xy}.$$

The second way is to take reciprocals of both sides, and then simplify:

$$R = \frac{1}{1/x + 1/y} = \frac{1}{y/xy + x/xy} = \frac{1}{(x+y)/xy}.$$

In either case, we find that $R = \frac{xy}{x+y}$.

10. Consider an ecosystem of rabbits in a park that starts with 10 rabbits and can sustain up to 60 rabbits. An equation that roughly models this scenario is

$$P = \frac{60}{1 + \frac{5}{t+1}},$$

where P represents the rabbit population in year t of the study.

a. What is the rabbit population in year 10? Round your answer to the nearest whole rabbit.

If $t = 10$, then $P = 41.25$; therefore, there are 41 rabbits in the park.

b. Solve this equation for t. Describe what this equation represents in the context of this problem.

$$t = \frac{60 - 6P}{P - 60}$$

This equation represents the relationship between the number of rabbits, P, and the year, t. If we know how many rabbits we have, $10 < P < 60$, we will know how long it took for the rabbit population to grow that much. If the population is 10, then this equation says we are in year 0 of the study, which fits with the given scenario.

c. At what time does the population reach 50 rabbits?

If $P = 50$, then $t = \frac{60-300}{50-60} = -\frac{240}{-10} = 24$; therefore, the rabbit population is 50 in year 24 of the study.

Extension:

11. Suppose that Huck Finn can paint a fence in 5 hours. If Tom Sawyer helps him paint the fence, they can do it in 3 hours. How long would it take for Tom to paint the fence by himself?

Huck paints the fence in 5 hours, so his rate of fence painting is $\frac{1}{5}$ fence per hour. Let T denote the percentage of the fence Tom can paint in an hour. Then

$$1 \text{ fence} = \left(\left(\frac{1}{5} + T\right)\text{fence per hour}\right) \cdot (3 \text{ hours}).$$

$$3 = \frac{1}{\frac{1}{5} + T} = \frac{1}{\frac{1}{5} + \frac{5T}{5}} = \frac{5}{1 + 5T}$$

$$3(1 + 5T) = 5$$

$$15T = 2$$

$$T = \frac{2}{15}$$

So, Tom can paint $\frac{2}{15}$ of the fence in an hour. Thus, Tom would take $\frac{15}{2} = 7.5$ hours to paint the fence by himself.

12. Huck Finn can paint a fence in 5 hours. After some practice, Tom Sawyer can now paint the fence in 6 hours.

a. How long would it take Huck and Tom to paint the fence together?

The amount of fence that Huck can paint per hour is $\frac{1}{5}$, and the amount that Tom can paint per hour is $\frac{1}{6}$. So, together they can paint $\frac{1}{5}+\frac{1}{6}$ of the fence per hour. Suppose the entire job of painting the fence takes h hours. Then, the amount of the fence that is painted is $h\left(\frac{1}{5}+\frac{1}{6}\right)$. Since the entire fence is painted, we need to solve the equation $h\left(\frac{1}{5}+\frac{1}{6}\right)=1$.

$$h\left(\frac{1}{5}+\frac{1}{6}\right)=1$$
$$h=\frac{1}{\frac{1}{5}+\frac{1}{6}}=\frac{30}{6+5}=\frac{30}{11}$$

So, together they can paint the fence in $\frac{30}{11}$ hours, which is 2 hours and 44 minutes.

b. Tom demands a half hour break while Huck continues to paint, and they finish the job together. How long does it take them to paint the fence?

Suppose the entire job of painting the fence takes h hours. Then, Huck paints at a rate of $\frac{1}{5}$ of the fence per hour for h hours, so he paints $\frac{h}{5}$ of the fence. Tom paints at a rate of $\frac{1}{6}$ of the fence per hour for $h-\frac{1}{2}$ hours, so he paints $\frac{1}{6}\left(h-\frac{1}{2}\right)$ of the fence. Together, they paint the whole fence; so, we need to solve the following equation for h:

$$\frac{1}{5}h+\frac{1}{6}\left(h-\frac{1}{2}\right)=1$$
$$\frac{1}{5}h+\frac{1}{6}h-\frac{1}{12}=1$$
$$\frac{1}{5}h+\frac{1}{6}h=\frac{13}{12}$$
$$60\left(\frac{1}{5}h+\frac{1}{6}h\right)=60\cdot\frac{13}{12}$$
$$12h+10h=65$$
$$h=\frac{65}{22}.$$

Thus, it takes $\frac{65}{22}$ hours, which is 2 hours 57 minutes, to paint the fence with Tom taking a $\frac{1}{2}$ hour break.

c. Suppose that they have to finish the fence in $3\frac{1}{2}$ hours. What's the longest break that Tom can take?

Suppose the entire job of painting the fence takes $\frac{7}{2}$ hours, and Tom stops painting for b hours for his break. Then, Huck paints at a rate of $\frac{1}{5}$ of the fence per hour for $\frac{7}{2}$ hours, so he paints $\frac{7}{10}$ of the fence. Tom paints at a rate of $\frac{1}{6}$ of the fence per hour for $\left(\frac{7}{2}-b\right)$ hours, so he paints $\frac{1}{6}\left(\frac{7}{2}-b\right)$ of the fence. Together, they paint the whole fence; so, we need to solve the following equation for b:

$$\frac{7}{10}+\frac{1}{6}\left(\frac{7}{2}-b\right)=1$$
$$\frac{7}{10}+\frac{7}{12}-\frac{b}{6}=1$$
$$60\left(\frac{7}{10}+\frac{7}{12}-\frac{b}{6}\right)=60$$
$$42+35-10b=60$$
$$42+35-60=10b$$
$$b=\frac{17}{10}.$$

Thus, if Tom takes a break for $\frac{17}{10}$ hours, which is 1 hour and 42 minutes, the fence will be painted in $3\frac{1}{2}$ hours.

Lesson 28: A Focus on Square Roots

Student Outcomes

- Students solve simple radical equations and understand the possibility of extraneous solutions. They understand that care must be taken with the role of square roots so as to avoid apparent paradoxes.
- Students explain and justify the steps taken in solving simple radical equations.

Lesson Notes

In the next two lessons, students work with radical expressions and equations. They extend their understanding of the idea that not all operations are invertible, which was explored in Algebra I and continued in the previous lessons on solving rational equations. Squaring both sides of an equation in some cases produces an extraneous solution. They also continue to work with fractional expressions and equations as seen in the previous lessons, but those expressions now contain radicals. This lesson highlights standards **A-REI.A.1**, which calls for students to be able to explain each step required to solve an equation, and **A-REI.A.2**, which calls for students to solve a radical equation and show how extraneous solutions might arise. It also addresses the standard MP.3 by building and analyzing arguments used in solving radical equations. In Example 2, students consider the difference between working with an *expression*, whose value should be preserved, and working with an *equation*, whose sides can be changed in value as long as the equality is preserved. This difference addresses the standard MP.7 because students are stepping back to get an overview of expressions and equations as objects subject to different rules.

Classwork

Opening (1 minute)

Recall that working with radicals can be tricky, especially when negative numbers are involved. When solving a radical equation, one must always check the answers found to verify that they are indeed valid solutions to the equation. In some cases, extraneous solutions appear and must be eliminated. Recall that an *extraneous solution* is one that satisfies a transformed equation but not the original one.

Exercises 1–4 (7 minutes)

Give students a few minutes to work through the first four exercises, and then discuss the results as a whole class. Circulate the room to assess the students' understanding.

Scaffolding:

- If students are struggling, show a few simpler examples such as solving $\sqrt{x} = 5$ or $\sqrt{x} = -5$.
- Another option would be to provide the following alternative model for students to complete.

Fill in the blanks to fully show and explain the solution process.

$\sqrt{x} - 6 = 4$

$\sqrt{x} = 10$ — **Added 6 to both sides to isolate the radical**

$\sqrt{x}^2 = 10^2$ — ***Square both sides***

$x = 100$

Exercises 1–4

For Exercises 1–4, describe each step taken to solve the equation. Then, check the solution to see if it is valid. If it is not a valid solution, explain why.

1. $\sqrt{x} - 6 = 4$

   $\sqrt{x} = 10$ *Add 6 to both sides*

   $x = 100$ *Square both sides*

   Check: $\sqrt{100} - 6 = 10 - 6 = 4$

   So 100 *is a valid solution.*

2. $\sqrt[3]{x} - 6 = 4$

   $\sqrt[3]{x} = 10$ *Add 6 to both sides*

   $x = 1000$ *Cube both sides*

   Check: $\sqrt[3]{1000} - 6 = 10 - 6 = 4$

   So 1000 *is a valid solution.*

3. $\sqrt{x} + 6 = 4$

   $\sqrt{x} = -2$

   $x = 4$

   Check: $\sqrt{4} + 6 = 2 + 6 = 8$*, and* $8 \neq 4$*, so* 4 *is not a valid solution.*

4. $\sqrt[3]{x} + 6 = 4$

   $\sqrt[3]{x} = -2$

   $x = -8$

   Check: $\sqrt[3]{-8} + 6 = -2 + 6 = 4$*, so* -8 *is a valid solution.*

Discussion

Consider each of the following questions, one at a time.

- What was the first step taken in solving the radical equations in Exercises 1 and 2?
  - *The radical was isolated.*
- What was the second step taken?
  - *Both sides were squared or cubed to eliminate the radical.*
- What happened in Exercise 3?
  - *The same steps were used to solve the equation as were used in Exercise 1, but this time the solution found did not work. There is no solution to the equation.*
- Why did that happen?

MP.3

This is one of the focal points of the lesson. Ask students to answer in writing or discuss with a partner before sharing their answers with the rest of the class. In the discussion, emphasize that 4 is an *extraneous solution*; it is the solution to $x = 4$ but not to the original equation.

  - *For* 4 *to be a solution,* $\sqrt{4}$ *would need to equal* -2*. Even though* $(-2)^2 = 4$*, we define* $\sqrt{4} = 2$ *so that* $f(x) = \sqrt{x}$ *takes on only one value for* $x \geq 0$ *and is thus a function. As a result, the square root of a positive number is only the positive value. Therefore,* 4 *is an extraneous solution.*
- What other types of equations sometimes have extraneous solutions?
  - *Rational equations can have extraneous solutions that create zero in the denominator.*
- Why did the solution process work in Exercise 4?
  - *The cube root of a negative number is negative, so a cube root equation does not have the same issues with the negative numbers as a square root does.*

Example 1 (5 minutes)

Work through the example to solidify the steps in solving a radical equation. Depending on how the students did with the first four exercises, you may want them to continue working with a partner, or you may want to work through this example with the whole class at once. Be sure the students can explain and justify the steps they are taking.

Example 1

Solve the radical equation. Be sure to check your solutions.

$$\sqrt{3x+5}-2=-1$$

Solution: $\sqrt{3x+5}=1$

$3x+5=1$

$3x=-4$

$x=-\frac{4}{3}$

Check: $\sqrt{3\left(-\frac{4}{3}\right)+5}-2=\sqrt{-4+5}-2=\sqrt{1}-2=-1$*, so* $-\frac{4}{3}$ *is a valid solution.*

Discussion

- What was the first step you took?
  - *Isolated the radical*
- Why did you do that first?
  - *Isolating the radical allows it to be eliminated by squaring or cubing both sides of the equation.*
- What was the next step?
  - *Squared both sides*
- Why did you do that?
  - *The purpose was to eliminate the radical from the equation.*
- Even though we are solving a new type of equation, does this feel like a familiar process?
  - *Yes. When solving an equation, we work on undoing any operation by doing the inverse. To undo a square root we use the inverse, so we square the expression.*
- How do the steps we are following relate to your previous experiences with solving other types of equations?
  - *We are still following the basic process to solve an equation, which is to undo any operation on the same side as the variable by using the inverse operation.*
- Why is it important to check the solution?
  - *Sometimes extraneous solutions appear because the square root of a positive number or zero is never negative.*

Summarize (in writing or with a partner) what you have learned about solving radical equations. Be sure that you explain what to do when you get an invalid solution.

Exercises 5–15 (15 minutes)

Allow students time to work the problems individually and then check with a partner. Circulate around the room. Make sure students are checking for extraneous solutions.

Exercises 5–15

Solve each radical equation. Be sure to check your solutions.

5. $\sqrt{2x-3} = 11$

   62

6. $\sqrt[3]{6-x} = -3$

   33

7. $\sqrt{x+5} - 9 = -12$

   No solution

8. $\sqrt{4x-7} = \sqrt{3x+9}$

   16

9. $-12\sqrt{x-6} = 18$

   No solution

10. $3\sqrt[3]{x+2} = 12$

    62

11. $\sqrt{x^2-5} = 2$

    3 *and* -3

12. $\sqrt{x^2+8x} = 3$

    -9 *and* 1

- Which exercises produced extraneous solutions?
  - *Exercises 7 and 9*
- Which exercises produced more than one solution? Why?
  - *Exercises 11 and 12 because after eliminating the radical, the equation became a quadratic equation. Both solutions were valid when checked.*

MP.3

- Write an example of a radical equation that has an extraneous solution. Exchange with a partner and confirm that the example does in fact have an extraneous solution.

Multiply each expression.

13. $(\sqrt{x}+2)(\sqrt{x}-2)$

    $x-4$

14. $(\sqrt{x}+4)(\sqrt{x}+4)$

    $x+8\sqrt{x}+16$

15. $(\sqrt{x}-5)(\sqrt{x}-5)$

    $x-5$

In the next example and exercises, we are working with rational expressions and equations that contain radicals. The purpose of these problems is to continue to build fluency working with radicals, to build on the work done in the previous lessons on rational expressions and equations, and to highlight MP.7, which calls for students to recognize and make use of structure in an expression.

Example 2 (5 minutes)

Work through the two examples as a class, making sure students are clear on the differences between working with an *expression* and working with an *equation*.

Example 2

Rationalize the denominator in each expression. That is, rewrite each expression so that the fraction has a rational expression in the denominator.

a. $\frac{x-9}{\sqrt{x-9}}$

$$\frac{x-9}{\sqrt{x-9}} \cdot \frac{\sqrt{x-9}}{\sqrt{x-9}}$$

$$\frac{(x-9)\sqrt{x-9}}{x-9}$$

$$\sqrt{x-9}$$

b. $\frac{x-9}{\sqrt{x}+3}$

$$\frac{x-9}{\sqrt{x}+3} \cdot \frac{\sqrt{x}-3}{\sqrt{x}-3}$$

$$\frac{(x-9)(\sqrt{x}-3)}{x-9}$$

$$\sqrt{x}-3$$

- What do the directions mean by "rationalize the denominator"?
  - *To remove the radical from the denominator so that the denominator will be a rational expression.*
- How can we accomplish this goal in part (a)?
  - *Multiply the numerator and denominator by* $\sqrt{x-9}$.
- Why not just square the expression?
  - *We are working with an expression, not an equation. You cannot square the expression because you would be changing its value. You can multiply the numerator and denominator by* $\sqrt{x-9}$ *because that is equivalent to multiplying by* 1. *It does not change the value of the expression.*
- Can we take the same approach in part (b)?
  - *No, multiplying by* $\sqrt{x}+3$ *would not remove the radical from the denominator.*
- Based on Exercise 13, what number should we multiply numerator and denominator by in part (b) in order to make the denominator rational?
  - $\sqrt{x}-3$
- In these examples, what was accomplished by rationalizing the denominator?
  - *It allowed us to create an equivalent expression that is simpler.*
- Why would that be advantageous?
  - *It would be easier to work with if we were evaluating it for a particular value of* x.

MP.7

Exercises 16–18 (5 minutes)

Allow students time to work on the three problems and then debrief. Students may have taken different approaches on Exercise 17, such as squaring both sides first or rationalizing the denominator. Share a few different approaches and compare.

Exercises 16–18

16. **Rewrite $\frac{1}{\sqrt{x}-5}$ in an equivalent form with a rational expression in the denominator.**

 $\frac{\sqrt{x}+5}{x-25}$

17. **Solve the radical equation $\frac{3}{\sqrt{x+3}} = 1$. Be sure to check for extraneous solutions.**

 $x = 6$

18. **Without solving the radical equation $\sqrt{x+5} + 9 = 0$, how could you tell that it has no real solution?**

 The radical expression $\sqrt{x+5}$ is positive or zero. In either case, adding 9 to it cannot give zero.

Scaffolding:

- If students are struggling with Exercise 17, have them approach the equation logically first rather than algebraically. If the output must equal 1 and the numerator is 3, what must the denominator equal?

Closing (2 minutes)

Ask students to respond to these questions in writing or with a partner. Use this as an opportunity to informally assess the students' understanding.

- Explain to your neighbor how to solve a radical equation. What steps do you take and why?
  - *Isolate the radical and then eliminate it by raising both sides to an exponent. The radical is isolated so that both sides can be squared or cubed as a means of eliminating the radical.*
- How is solving a radical equation similar to solving other types of equations we have solved?
  - *We are isolating the variable by undoing any operation on the same side.*
- Why is it important to check the solutions?
  - *Remember that the square root of a number takes on only the positive value. When solving a radical equation involving a square root, squaring both sides of the equation in the process of solving may make the negative "disappear" and may create an extraneous solution.*

Exit Ticket (5 minutes)

Name ______________________________ Date ________________

Lesson 28: A Focus on Square Roots

Exit Ticket

Consider the radical equation $3\sqrt{6-x}+4=-8$.

1. Solve the equation. Next to each step, write a description of what is being done.

2. Check the solution.

3. Explain why the calculation in Problem 1 does not produce the solution to the equation.

Exit Ticket Sample Solutions

Consider the radical equation $3\sqrt{6-x}+4=-8$

1. Solve the equation. Next to each step, write a description of what is being done.

$3\sqrt{6-x}=-12$	*Subtract 4 from both sides.*
$\sqrt{6-x}=-4$	*Divide both sides by 3 in order to isolate the radical.*
$6-x=16$	*Square both sides to eliminate the radical.*
$x=-10$	*Subtract 6 from both sides and divide by -1.*

2. Check the solution.

   $3\sqrt{6-(-10)}+4=3\sqrt{16}+4=3(4)+4=16$, and $16\neq -8$, *so -10 is not a valid solution.*

3. Explain why the calculation in Problem 1 does not produce the solution to the equation.

   Because the square root of a positive number is positive, $3\sqrt{6-x}$ will be positive. A positive number added to 4 cannot be -8.

Problem Set Sample Solutions

1. a. If $\sqrt{x}=9$, then what is the value of x?

   $x=81$

   b. If $x^2=9$, then what is the value of x?

   $x=3$ or $x=-3$

   c. Is there a value of x such that $\sqrt{x+5}=0$? If yes, what is the value? If no, explain why not.

   Yes, $x=-5$

   d. Is there a value of x such that $\sqrt{x}+5=0$? If yes, what is the value? If no, explain why not.

   No, $\sqrt{x}$ will be a positive value or zero for any value of x, so the sum cannot equal 0. If $x=25$, then $\sqrt{25}+5=10$.

2. a. Is the statement $\sqrt{x^2}=x$ true for all x-values? Explain.

   No, this statement is only true for $x\geq 0$. If $x<0$, it is not true. For example, if $x=-5$, $\sqrt{(-5)^2}=\sqrt{25}=5$, then $\sqrt{(-5)^2}\neq -5$.

   b. Is the statement $\sqrt[3]{x^3}=x$ true for all x-values? Explain.

   Yes, this statement is true for all x-values. For example, if $x=2$, then $\sqrt[3]{2^3}=2$. If $x=-2$, then $\sqrt[3]{(-2)^3}=-2$. Since the cube root of a positive number is positive, and the cube root of a negative number is negative, this statement is true for any value of x.

Rationalize the denominator in each expression.

3. $\frac{4-x}{2+\sqrt{x}}$

   $2-\sqrt{x}$

4. $\frac{2}{\sqrt{x-12}}$

   $\frac{2\sqrt{x-12}}{x-12}$

5. $\frac{1}{\sqrt{x+3}-\sqrt{x}}$

   $\frac{\sqrt{x+3}+\sqrt{x}}{3}$

Solve each equation and check the solutions.

6. $\sqrt{x+6}=3$

   $x=3$

7. $2\sqrt{x+3}=6$

   $x=6$

8. $\sqrt{x+3}+6=3$

   No solution

9. $\sqrt{x+3}-6=3$

   $x=78$

10. $16=8+\sqrt{x}$

    $x=64$

11. $\sqrt{3x-5}=7$

    $x=18$

12. $\sqrt{2x-3}=\sqrt{10-x}$

    $x=\frac{13}{3}$

13. $3\sqrt{x+2}+\sqrt{x-4}=0$

    No solution

14. $\frac{\sqrt{x+9}}{4}=3$

    $x=135$

15. $\frac{12}{\sqrt{x+9}}=3$

    $x=7$

16. $\sqrt{x^2+9}=5$

    $x=4$ *or* $x=-4$

17. $\sqrt{x^2-6x}=4$

    $x=8$ *or* $x=-2$

18. $\frac{5}{\sqrt{x-2}}=5$

    $x=3$

19. $\frac{5}{\sqrt{x}-2}=5$

    $x=9$

20. $\sqrt[3]{5x-3}+8=6$

    $x=-1$

21. $\sqrt[3]{9-x}=6$

    $x=-207$

22. Consider the inequality $\sqrt{x^2+4x}>0$. Determine whether each x-value is a solution to the inequality.

    a. $x=-10$ *Yes*

    b. $x=-4$ *No*

    c. $x=10$ *Yes*

    d. $x=4$ *Yes*

23. Show that $\frac{a-b}{\sqrt{a}-\sqrt{b}}=\sqrt{a}+\sqrt{b}$ for all values of a and b such that $a>0$ and $b>0$ and $a\neq b$.

    If we multiply the numerator and denominator of $\frac{a-b}{\sqrt{a}-\sqrt{b}}$ *by* $\sqrt{a}+\sqrt{b}$ *to rationalize the denominator, then we have*

$$\frac{a-b}{\sqrt{a}-\sqrt{b}}=\frac{a-b}{\sqrt{a}-\sqrt{b}}\cdot\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}=\frac{(a-b)(\sqrt{a}+\sqrt{b})}{a-b}=\sqrt{a}+\sqrt{b}.$$

24. Without actually solving the equation, explain why the equation $\sqrt{x+1}+2=0$ has no solution.

    The value of $\sqrt{x+1}$ *must be positive which is then added to* 2. *The sum of two positive numbers is positive; therefore, the sum cannot equal* 0.

Lesson 29: Solving Radical Equations

Student Outcomes

- Students develop facility in solving radical equations.

Lesson Notes

In the previous lesson, students were introduced to the notion of solving radical equations and checking for extraneous solutions (**A-REI.A.2**). Students continue this work by looking at radical equations that contain variables on both sides. The main point to stress to students is that radical equations become polynomial equations through exponentiation. So we really have not left the notion of polynomials that we have been studying throughout this module. This lesson also provides opportunities to emphasize MP.7 (look for and make use of structure).

Classwork

Discussion (5 minutes)

Before beginning the lesson, remind students of past experiences by providing the following scenario, which illustrates when an operation performed to both sides of an equation has changed the set of solutions.

Carlos and Andrea were solving the equation $x^2 + 2x = 0$. Andrea says that there are two solutions, 0 and -2. Carlos says the only solution is -2 because he divided both sides by x and got $x + 2 = 0$. Who is correct and why?

- Do both 0 and -2 satisfy the original equation?
  - *Yes. If we replace x with either 0 or -2, the answer is 0.*
- What happened when Carlos divided both sides of the equation by x?
  - *He changed the solutions from 0 and -2 to simply -2. He lost one solution to the equation.*
- What does this say about the solution of equations after we have performed algebraic operations on both sides?
  - *Performing algebraic steps may alter the set of solutions to the original equation.*

Now, Carlos and Andrea are solving the equation $\sqrt{x} = -3$. Andrea says the solution is 9 because she squared both sides and got $x = 9$. Carlos says there is no solution. Who is correct? Why?

MP.3

- Was Andrea correct to square both sides?
  - *Yes. To eliminate a radical from an equation, we raise both sides to an exponent.*
- Is she correct that the solution is 9?
  - *No. Carlos is correct. If we let $x = 9$, then we get $\sqrt{9} = 3$, and $3 \neq -3$, so 9 is not a solution.*

Scaffolding

- Use several examples to illustrate that if $a > 0$, then an equation of the form $\sqrt{x} = -a$ will not have a solution (e.g., $\sqrt{x} = -4$, $\sqrt{x} = -5$).
- Extension: Write an equation that has an extraneous solution of $x = 50$.

- What is the danger in squaring both sides of an equation?
  - *It sometimes produces an equation whose solution set is not equivalent to that of the original equation. If both sides of* $\sqrt{x} = -3$ *are squared, the equation* $x = 9$ *is produced, but* 9 *is not a solution to the original equation. The original equation has no solution.*
- Because of this danger, what is the final essential step of solving a radical equation?
  - *Checking the solution or solutions to ensure that an extraneous solution was not produced by the step of squaring both sides.*
- How could we have predicted that the equation would have no solution?
  - *The square root of a number is never equal to a negative value, so there is no* x*-value so that* $\sqrt{x} = -3$.

Example 1 (5 minutes)

MP.1 While this problem is difficult, students should attempt to solve it on their own first, by applying their understandings of radicals. Students should be asked to verify the solution they come up with and describe their solution method. Discuss Example 1 as a class once they have worked on it individually.

Example 1

Solve the equation $6 = x + \sqrt{x}$**.**

$$6 - x = \sqrt{x}$$
$$(6 - x)^2 = \sqrt{x}^2$$
$$36 - 12x + x^2 = x$$
$$x^2 - 13x + 36 = 0$$
$$(x - 9)(x - 4) = 0$$

The solutions are 9 ***and*** 4***.***

Check $x = 9$***:***

$$9 + \sqrt{9} = 9 + 3 = 12$$
$$6 \neq 12$$

So, 9 ***is not a solution.***

Check $x = 4$***:***

$$4 + \sqrt{4} = 4 + 2 = 6$$

The only solution is 4***.***

- How does this equation differ from the ones from yesterday's lesson?
  - *There are two* x*'s; one inside and one outside of the radical.*
- Explain how you were able to determine the solution to the equation above.
  - *Isolate the radical and square both sides. Solve the resulting equation.*
- Did that change the way in which the equation was solved?
  - *Not really. We still eliminated the radical by squaring both sides.*
- What type of equation were we left with after squaring both sides?
  - *A quadratic polynomial equation*
- Why did 9 fail to work as a solution?
  - *Because the square root of* 9 *takes the positive value of* 3

Exercises 1–4 (13 minutes)

Allow students time to work the problems independently and then pair up to compare solutions. Use this time to informally assess student understanding by examining their work. Display student responses, making sure that students checked for extraneous solutions.

Exercises 1–4

Solve.

1. $3x = 1 + 2\sqrt{x}$

   The only solution is 1.

   Note that $\frac{1}{9}$ is an extraneous solution.

2. $3 = 4\sqrt{x} - x$

   The two solutions are 9 and 1.

3. $\sqrt{x+5} = x - 1$

   The only solution is 4.

   Note that -1 is an extraneous solution.

4. $\sqrt{3x+7} + 2\sqrt{x-8} = 0$

   There are no solutions.

- When solving Exercise 1, what solutions did you find? What happened when you checked these solutions?
  - *The solutions found were $\frac{1}{9}$ and 1. Only 1 satisfies the original equation, so $\frac{1}{9}$ is an extraneous solution.*
- Did Exercise 2 have any extraneous solutions?
  - *No. Both solutions satisfied the original equation.*

MP.7

- Looking at Exercise 4, could we have predicted that there would be no solution?
  - *Yes. The only way the two square roots could add to zero would be if both of them produced a zero, meaning that $3x + 7 = 0$ and $x - 8 = 0$. Since x cannot be both $-\frac{7}{3}$ and 8, both radicals cannot be simultaneously zero. Thus, at least one of the square roots will be positive, and they cannot sum to zero.*

Example 2 (5 minutes)

MP.7

What do we do when there is no way to isolate the radical? What is going to be the easiest way to square both sides? Give students time to work on Example 2 independently. Point out that even though we had to square both sides twice we were still able to rewrite the equation as a polynomial.

Scaffolding:

- What if we had squared both sides of the equation as it was presented? Have early finishers work out the solution this way and share with the class.

Example 2

Solve the equation $\sqrt{x} + \sqrt{x+3} = 3$.

$$\begin{aligned} \sqrt{x+3} &= 3 - \sqrt{x} \\ \left(\sqrt{x+3}\right)^2 &= (3 - \sqrt{x})^2 \\ x + 3 &= 9 - 6\sqrt{x} + x \\ 1 &= \sqrt{x} \\ 1 &= x \end{aligned}$$

Check:

$\sqrt{1} + \sqrt{1+3} = 1 + 2 = 3$

So the solution is 1.

Exercises 5–6 (7 minutes)

Allow students time to work the problems independently and then pair up to compare solutions. Circulate to assess understanding. Consider targeted instruction with a small group of students while others are working independently. Display student responses, making sure that students check for extraneous solutions.

Exercises 5–6

Solve the following equations.

5. $\sqrt{x-3}+\sqrt{x+5}=4$

 4

6. $3+\sqrt{x}=\sqrt{x+81}$

 144

Closing (5 minutes)

Ask students to respond to these questions in writing or with a partner. Use these responses to informally assess their understanding of the lesson.

- How did these equations differ from the equations seen in yesterday's lesson?
  - *Most of them contained variables on both sides of the equation or a variable outside of the radical.*
- How were they similar to the equations from yesterday's lesson?
  - *They were solved using the same process of squaring both sides. Even though they were more complicated, the equations could still be rewritten as a polynomial equation and solved using the same process seen throughout this module.*
- Ask the students to summarize the lesson. Give an example where $a^n = b^n$ but $a \neq b$.
  - $(-3)^2 = 3^2$ *but* $-3 \neq 3$.

Lesson Summary

If $a=b$ and n is an integer, then $a^n=b^n$. However, the converse is not necessarily true. The statement $a^n=b^n$ does not imply that $a=b$. Therefore, it is necessary to check for extraneous solutions when both sides of an equation are raised to an exponent.

Exit Ticket (5 minutes)

Name ______________________________ Date ____________________

Lesson 29: Solving Radical Equations

Exit Ticket

1. Solve $\sqrt{2x+15} = x + 6$. Verify the solution(s).

2. Explain why it is necessary to check the solutions to a radical equation.

Exit Ticket Sample Solutions

1. Solve $\sqrt{2x+15} = x+6$. Verify the solution(s).

$$2x + 15 = x^2 + 12x + 36$$
$$0 = x^2 + 10x + 21$$
$$0 = (x+3)(x+7)$$

The solutions are -3 and -7.

Check $x = -3$:

$$\sqrt{2(-3)+15} = \sqrt{9} = 3$$
$$-3 + 6 = 3$$

So, -3 is a valid solution.

Check $x = -7$:

$$\sqrt{2(-7)+15} = \sqrt{1} = 1$$
$$-7 + 6 = -1$$

Since $-1 \neq 1$, we see that -1 is an extraneous solution.

Therefore, the only solution to the original equation is -3.

2. Explain why it is necessary to check the solutions to a radical equation.

Squaring both sides in some cases produces an equation whose solution set is not equivalent to that of the original equation. In the problem above, $x = -7$ does not satisfy the equation.

Problem Set Sample Solutions

Solve.

1. $\sqrt{2x-5} - \sqrt{x+6} = 0$

   11

2. $\sqrt{2x-5} + \sqrt{x+6} = 0$

   No solution

3. $\sqrt{x-5} - \sqrt{x+6} = 2$

   No solution

4. $\sqrt{2x-5} - \sqrt{x+6} = 2$

   43

5. $\sqrt{x+4} = 3 - \sqrt{x}$

   $\frac{25}{36}$

6. $\sqrt{x+4} = 3 + \sqrt{x}$

   No solution

7. $\sqrt{x+3} = \sqrt{5x+6} - 3$

   6

8. $\sqrt{2x+1} = x - 1$

   4

9. $\sqrt{x+12} + \sqrt{x} = 6$

   4

10. $2\sqrt{x} = 1 - \sqrt{4x-1}$

    $\frac{1}{4}$

11. $2x = \sqrt{4x-1}$

    $\frac{1}{2}$

12. $\sqrt{4x-1} = 2 - 2x$

    $\frac{1}{2}$

13. $x + 2 = 4\sqrt{x - 2}$

6

14. $\sqrt{2x - 8} + \sqrt{3x - 12} = 0$

4

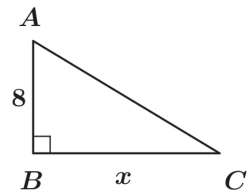
15. $x = 2\sqrt{x - 4} + 4$

$4, 8$

16. $x - 2 = \sqrt{9x - 36}$

$5, 8$

17. Consider the right triangle ABC shown to the right, with $AB = 8$ and $BC = x$.



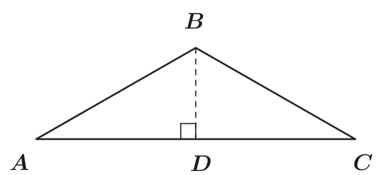
a. Write an expression for the length of the hypotenuse in terms of x.

$AC = \sqrt{64 + x^2}$

b. Find the value of x for which $AC - AB = 9$.

The solutions to the mathematical equation $\sqrt{64 + x^2} - 8 = 9$ are -15 and 15. Since lengths must be positive, -15 is an extraneous solution, and $x = 15$.

18. Consider the triangle ABC shown to the right where $AD = DC$ and $\overline{BD}$ is the altitude of the triangle.



a. If the length of $\overline{BD}$ is x cm and the length of $\overline{AC}$ is 18 cm, write an expression for the lengths of $\overline{AB}$ and $\overline{BC}$ in terms of x.

$AB = BC = \sqrt{81 + x^2}$ cm

b. Write an expression for the perimeter of ΔABC in terms of x.

$\left(2\sqrt{81 + x^2} + 18\right)$ cm

c. Find the value of x for which the perimeter of ΔABC is equal to 38 cm.

$\sqrt{19}$ cm

Lesson 30: Linear Systems in Three Variables

Student Outcomes

- Students solve linear systems in three variables algebraically.

Lesson Notes

Students solved systems of linear equations in two variables using substitution and elimination in Grade 8 and then encountered the topic again in Algebra I when talking about solving systems of linear equalities and inequalities. In this lesson, we begin with a quick review of elimination to solve a linear system in two variables along with one application problem before moving onto solving a system of equations in three variables using algebraic techniques.

Classwork

Opening (2 minutes)

This lesson transitions from solving linear 2-by-2 equations as in Algebra I to solving systems of equations involving linear and nonlinear equations in two variables in the next two lessons. These nonlinear systems will be solved algebraically using substitution or by graphing each equation and finding points of intersection, if any. This lesson helps remind students how to solve linear systems of equations and introduces them to 3-by-3 systems of linear equations (which will be later analyzed using matrices in Precalculus).

Exercises 1–3 (8 minutes)

Exercises 1–3

Determine the value of x and y in the following systems of equations.

1. $$\begin{aligned} 2x + 3y &= 7 \\ 2x + y &= 3 \end{aligned}$$

   $x = \frac{1}{2}, y = 2$

2. $$\begin{aligned} 5x - 2y &= 4 \\ -2x + y &= 2 \end{aligned}$$

   $x = 8, y = 18$

After this review of using elimination to solve a system, guide students through the set-up of the following problem, and then let them solve using the techniques reviewed in Exercises 1 and 2.

3. A scientist wants to create 120 ml of a solution that is 30% acidic. To create this solution, she has access to a 20% solution and a 45% solution. How many milliliters of each solution should she combine to create the 30% solution?

Solve this problem using a system of two equations in two variables.

Solution:

Milliliters of 20% solution: x ml
Milliliters of 45% solution: y ml

Write one equation to represent the total amounts of each solution needed:

$$x + y = 120$$

Since 30% of 120 ml is 36, we can write one equation to model the acidic portion:

$$0.20x + 0.45y = 36$$

Writing these two equations as a system:

$$x + y = 120$$
$$0.20x + 0.45y = 36$$

To solve, multiply both sides of the top equation by either 0.20 to eliminate x or 0.45 to eliminate y. The following work is for eliminating x:

$$0.20(x + y) = 0.20(120)$$
$$0.20x + 0.45y = 40$$

which gives

$$0.20x + 0.20y = 24$$
$$0.20x + 0.45y = 36.$$

Replacing the top equation with the difference between the bottom equation and top equation results in a new system with the same solutions:

$$0.25y = 12$$
$$0.20x + 0.45y = 36$$

The top equation can quickly be solved for y,

$$y = 48,$$

and substituting $y = 48$ back into the original first equation allows us to find x:

$$x + 48 = 120$$
$$x = 72$$

Thus, we need 48 ml of the 45% solution and 72 ml of the 20% solution.

Discussion (5 minutes)

MP.1

- In the previous examples we see how to solve a system of linear equations in two variables using elimination methods. However, what if we have three variables? For example, what are the solutions to the following system of equations?

$$2x + 3y - z = 5$$
$$4x - y - z = -1$$

Allow students time to work together and struggle with this system and realize that they cannot find a unique solution. Include the following third equation and ask students if they can solve it now.

$$x + 4y + z = 12$$

Scaffolding:

To help students, ask them if they can eliminate two of the variables from either equation (they cannot). Have a discussion around what that means (the graph of the solution set is a line, not a

Give students an opportunity to consider solutions or other ideas on how to begin the process of solving this system. After considering their suggestions and providing feedback, guide them through the process in the example below.

Example 1 (9 minutes)

Example 1

Determine the values for x, y, and z in the following system:

$\mathbf{2x + 3y - z = 5}$ **(1)**

$\mathbf{4x - y - z = -1}$ **(2)**

$\mathbf{x + 4y + z = 12}$ **(3)**

Suggest numbering the equations as shown above to help organize the process.

- Eliminate z from equations (1) and (2) by subtraction:

  $2x + 3y - z = 5$

  $\underline{4x - \ \ y - z = -1}$

  $-2x + 4y = 6$

- Our goal is to find two equations in two unknowns. Thus, we will also eliminate z from equations (2) and (3) by adding as follows:

  $4x - y - z = -1$

  $\underline{x + 4y + z = 12}$

  $5x + 3y = 11$

- Our new system of three equations in three variables has two equations with only two variables in them:

  $-2x + 4y = 6$

  $4x - y - z = -1$

  $5x + 3y = 11$

- These two equations now give us two equations in two variables, which we reviewed how to solve in Exercises 1–2.

$$-2x + 4y = 6$$
$$5x + 3y = 11$$

At this point, you can let students solve this individually or with partners, or guide them through the process if necessary.

- To get matching coefficients, we need to multiply both equations by a constant:

$$5(-2x + 4y) = 5(6) \quad \rightarrow \quad -10x + 20y = 30$$
$$2(5x + 3y) = 2(11) \quad \rightarrow \quad 10x + 6y = 22$$

- Replacing the top equation with the sum of the top and bottom equations together gives the following:

$$26y = 52$$
$$10x + 6y = 22$$

- The new top equation can be solved for y:

$$y = 2$$

- Replace $y = 2$ in one of the equations to find x:

$$5x + 3(2) = 11$$
$$5x + 6 = 11$$
$$5x = 5$$
$$x = 1$$

- Replace $x = 1$ and $y = 2$ in any of the original equations to find z:

$$2(1) + 3(2) - z = 5$$
$$2 + 6 - z = 5$$
$$8 - z = 5$$
$$z = 3$$

- The solution, $x = 1$, $y = 2$, and $z = 3$, can be written compactly as an ordered triple of numbers $(1, 2, 3)$.

You might want to point out to your students that the point $(1, 2, 3)$ can be thought of as a point in a three-dimensional coordinate plane, and that it is, like a two-by-two system of equations, the intersection point in three-space of the three planes given by the graphs of each equation. These concepts are not the point of this lesson, so addressing them is optional.

Point out that a linear system involving three variables requires three equations in order for the solution to possibly be a single point.

The following problems provide examples of situations that require solving systems of equations in three variables.

Exercise 4 (8 minutes)

Exercise 4

Given the system below, determine the values of r, s, and u that satisfy all three equations.

$$r + 2s - u = 8$$
$$s + u = 4$$
$$r - s - u = 2$$

Adding the second and third equation together produces the equation $r = 6$. Substituting this into the first equation and adding it to the second gives $6 + 3s = 12$, so that $s = 2$. Replacing s with 2 in the second equation gives $u = 2$. The solution to this system of equations is $(6, 2, 2)$.

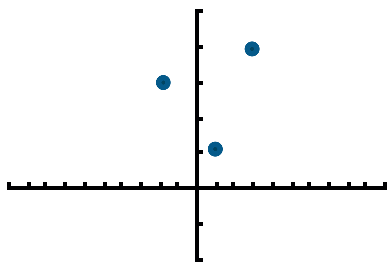
Exercise 5 (6 minutes)

Exercise 5

Find the equation of the form $y = ax^2 + bx + c$ that satisfies the points $(1, 6)$, $(3, 20)$, and $(-2, 15)$.

$a = 2$, $b = -1$, $c = 5$; therefore, the quadratic equation is $y = 2x^2 - x + 5$.

Students may need help setting this up. A graph of the points may help.



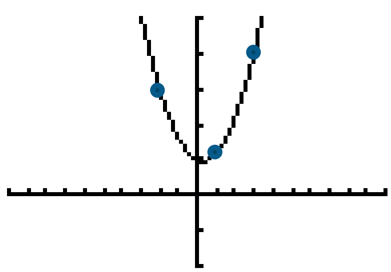
MP.7

- Since we know three ordered pairs, we can create three equations.

$$6 = a + b + c$$
$$20 = 9a + 3b + c$$
$$15 = 4a - 2b + c$$

Ask students to explain where the three equations came from. Then have them use the technique from Example 1 to solve this system.

Have students use a graphing utility to graph the equation using the coefficient solutions to confirm the answer.



Closing (2 minutes)

- Having solved systems of two linear equations, we see in the lesson that in order to solve a linear system in three variables, we need three equations. How many equations might we need to solve a system with four variables? Five?

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 30: Linear Systems in Three Variables

Exit Ticket

For the following system, determine the values of p, q, and r that satisfy all three equations:

$$\begin{aligned} 2p + q - r &= 8 \\ q + r &= 4 \\ p - q \quad &= 2 \end{aligned}$$

Exit Ticket Sample Solutions

For the following system, determine the values of p, q, and r that satisfy all three equations:

$$\begin{aligned} 2p + q - r &= 8 \\ q + r &= 4 \\ p - q \quad &= 2 \end{aligned}$$

$p = 4$, $q = 2$, $r = 2$, or equivalently $(4, 2, 2)$

Problem Set Sample Solutions

Solve the following systems.

1. $$\begin{aligned} x + y &= 3 \\ y + z &= 6 \\ x + z &= 5 \end{aligned}$$

   $x = 1$, $y = 2$, $z = 4$ or $(1, 2, 4)$

2. $$\begin{aligned} r &= 2(s - t) \\ 2t &= 3(s - r) \\ r + t &= 2s - 3 \end{aligned}$$

   $r = 2$, $s = 4$, $t = 3$, or $(2, 4, 3)$

3. $$\begin{aligned} 2a + 4b + c &= 5 \\ a - 4b &= -6 \\ 2b + c &= 7 \end{aligned}$$

   $a = -2$, $b = 1$, $c = 5$ or $(-2, 1, 5)$

4. $$\begin{aligned} 2x + y - z &= -5 \\ 4x - 2y + z &= 10 \\ 2x + 3y + 2z &= 3 \end{aligned}$$

   $x = \frac{1}{2}$, $y = -2$, $z = 4$ or $\left(\frac{1}{2}, -2, 4\right)$

5. $$\begin{aligned} r + 3s + t &= 3 \\ 2r - 3s + 2t &= 3 \\ -r + 3s - 3t &= 1 \end{aligned}$$

   $r = 3$, $s = \frac{1}{3}$, $t = -1$ or $\left(3, \frac{1}{3}, -1\right)$

6. $$\begin{aligned} x - y &= 1 \\ 2y + z &= -4 \\ x - 2z &= -6 \end{aligned}$$

   $x = -2$, $y = -3$, $z = 2$ or $(-2, -3, 2)$

7. $$\begin{aligned} x &= 3(y - z) \\ y &= 5(z - x) \\ x + y &= z + 4 \end{aligned}$$

   $x = 3$, $y = 5$, $z = 4$ or $(3, 5, 4)$

8. $$\begin{aligned} p + q + 3r &= 4 \\ 2q + 3r &= 7 \\ p - q - r &= -2 \end{aligned}$$

   $p = 2$, $q = 5$, $r = -1$ or $(2, 5, -1)$

9. $$\begin{aligned} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= 5 \\ \frac{1}{x} + \frac{1}{y} &= 2 \\ \frac{1}{x} - \frac{1}{z} &= -2 \end{aligned}$$

   $x = 1$, $y = 1$, $z = \frac{1}{3}$ or $\left(1, 1, \frac{1}{3}\right)$

10. $$\begin{aligned} \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &= 6 \\ \frac{1}{b} + \frac{1}{c} &= 5 \\ \frac{1}{a} - \frac{1}{b} &= -1 \end{aligned}$$

    $a = 1$, $b = \frac{1}{2}$, $c = \frac{1}{3}$ or $\left(1, \frac{1}{2}, \frac{1}{3}\right)$

11. Find the equation of the form $y = ax^2 + bx + c$ whose graph passes through the points $(1, -1)$, $(3, 23)$, and $(-1, 7)$.

    $y = 4x^2 - 4x - 1$

12. Show that for any number t, the values $x = t + 2$, $y = 1 - t$, and $z = t + 1$ are solutions to the system of equations below.

$$x + y = 3$$
$$y + z = 2$$

    (in this situation, we say that t *parameterizes the solution set of the system.*)

$$x + y = (t + 2) + (1 - t) = 3$$
$$y + z = (1 - t) + (t + 1) = 2$$

13. Some rational expressions can be written as the sum of two or more rational expressions whose denominators are the factors of its denominator (called a *partial fraction decomposition*). Find the partial fraction decomposition for the following example by filling in the blank to make the equation true for all n except 0 and -1.

$$\frac{1}{n(n+1)} = \frac{\_\_\_\_\_}{n} - \frac{1}{n+1}$$

    Adding $\frac{1}{n+1}$ *to both sides of the equations, we have* $\frac{1}{n(n+1)} + \frac{1}{(n+1)} = \frac{1}{n(n+1)} + \frac{n}{n(n+1)} = \frac{(n+1)}{n(n+1)} = \frac{1}{n}$*, so* $\frac{1}{n(n+1)} + \frac{1}{(n+1)} = \frac{1}{n}$*, and equivalently we have* $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{(n+1)}$*. Thus, the blank should contain a* 1*.*

14. A chemist needs to make 40 ml of a 15% acid solution. He has a 5% acid solution and a 30% acid solution on hand. If he uses the 5% and 30% solutions to create the 15% solution, how many ml of each will he need?

    He will need 24 *ml of the* 5% *solution and* 16 *ml of the* 30% *solution.*

15. An airplane makes a 400 mile trip against a head wind in 4 hours. The return trip takes 2.5 hours, the wind now being a tail wind. If the plane maintains a constant speed with respect to still air, and the speed of the wind is also constant and does not vary, find the still-air speed of the plane and the speed of the wind.

    The speed of the plane in still wind is 130 *mph, and the speed of the wind is* 30 *mph.*

16. A restaurant owner estimates that she needs in small change the same number of dimes as pennies and nickels together and the same number of pennies as nickels. If she gets $\$26$ worth of pennies, nickels, and dimes, how should they be distributed?

    She will need 200 *dimes (*$\$20$ *worth),* 100 *nickels (*$\$5$ *worth), and* 100 *pennies (*$\$1$ *worth) for a total of* $\$26$*.*

Lesson 31: Systems of Equations

Student Outcomes

- Students solve systems of linear equations in two variables and systems of a linear and a quadratic equation in two variables.
- Students understand that the points at which the two graphs of the equations intersect correspond to the solutions of the system.

Lesson Notes

Students review the solution of systems of linear equations, move on to systems of equations that represent a line and a circle and systems that represent a line and a parabola, and make conjectures as to how many points of intersection there can be in a given system of equations. They sketch graphs of a circle and a line to visualize the solution to a system of equations, solve the system algebraically, and note the correspondence between the solution and the intersection. Then they do the same for graphs of a parabola and a line.

The principal standards addressed in this lesson are **A-REI.C.6** (solve systems of linear equations exactly and approximately, e.g., with graphs, focusing on pairs of linear equations in two variables) and **A-REI.C.7** (solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically). The standards MP.5 (use appropriate tools strategically) and MP.8 (look for and express regularity in repeated reasoning) are also addressed.

Materials

Graph paper, straightedge, compass, and a tool for displaying graphs (e.g., projector, smart board, white board, chalk board, or squared poster paper)

Classwork

Exploratory Challenge 1 (8 minutes)

In this exercise, the students review ideas about systems of linear equations from Module 4 in Grade 8 (**A-REI.C.6**). Consider distributing graph paper for students to use throughout this lesson. Begin by posing the following problem for students to work on individually:

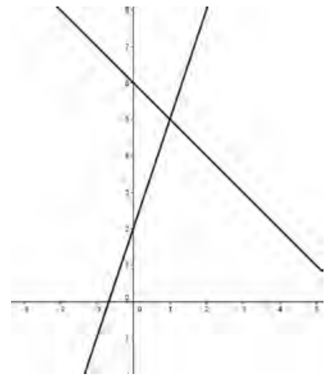
> *Scaffolding:*
> Circulate to identify students who might be asked to display their sketches and solutions.

Exploratory Challenge 1

a. Sketch the lines given by $x + y = 6$ and $-3x + y = 2$ on the same set of axes, and then solve the pair of equations algebraically to verify your graphical solution.

Once the students have made a sketch, ask one of them to use the display tool and draw the two graphs for the rest of the class to see. While the student is doing that, ask the other students how many points are shared (one) and what the coordinates of that point are.

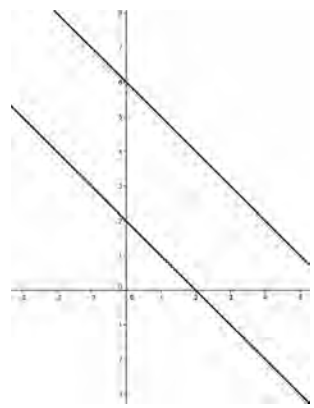
The point $(1, 5)$ should be easily identifiable from the sketch. See the graph to the right.



Point out that in this case, there is one solution. Now change the problem as follows. Then discuss the question as a class, and ask one or two students to show their sketches using the display tool.

b. Suppose the second line is replaced by the line with equation $x + y = 2$. Plot the two lines on the same set of axes, and solve the pair of equations algebraically to verify your graphical solution.

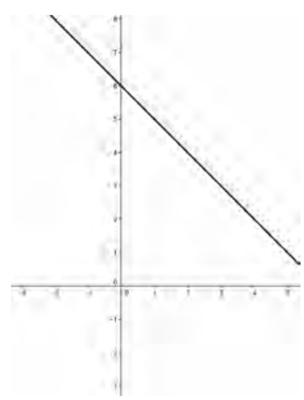
The lines are parallel, and there is no point in common. See the graph to the right.



Point out that in this case, there is no solution. Now change the problem again as follows, and again discuss the question as a class. Then ask one or two students to show their sketches using the display tool.

c. Suppose the second line is replaced by the line with equation $2x = 12 - 2y$. Plot the lines on the same set of axes, and solve the pair of equations algebraically to verify your graphical solution.

The lines coincide, and they have all points in common. See the graph to the right.



Point out that in this third case, there is an infinite number of solutions. Discuss the following problem as a class.

> **d. We have seen that a pair of lines can intersect in 1, 0, or an infinite number of points. Are there any other possibilities?**
>
> ***No. Students should convince themselves and each other that these three options exhaust the possibilities for the intersection of two lines.***

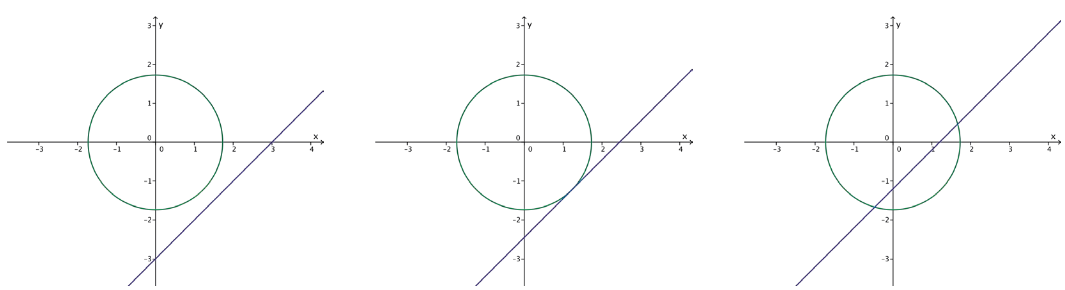
Exploratory Challenge 2 (12 minutes)

In this exercise, students move on to a system of a linear and quadratic equations (**A-REI.C.6**). Begin by asking students to work in pairs to sketch graphs and develop conjectures about the following item:

> *Scaffolding:*
> - Circulate to assist pairs of students who might be having trouble coming up with all three possibilities.
> - For students who are ready, ask them to write equations for the graphs they have sketched.

> **Exploratory Challenge 2**
>
> **a. Suppose that instead of equations for a pair of lines, you were given an equation for a circle and an equation for a line. What possibilities are there for the two figures to intersect? Sketch a graph for each possibility.**

Once the students have made their sketches, ask one pair to use the display tool and draw the graphs for the rest of the class to see.

> ***They can intersect in 0, 1, or 2 points as shown below.***
>
> 

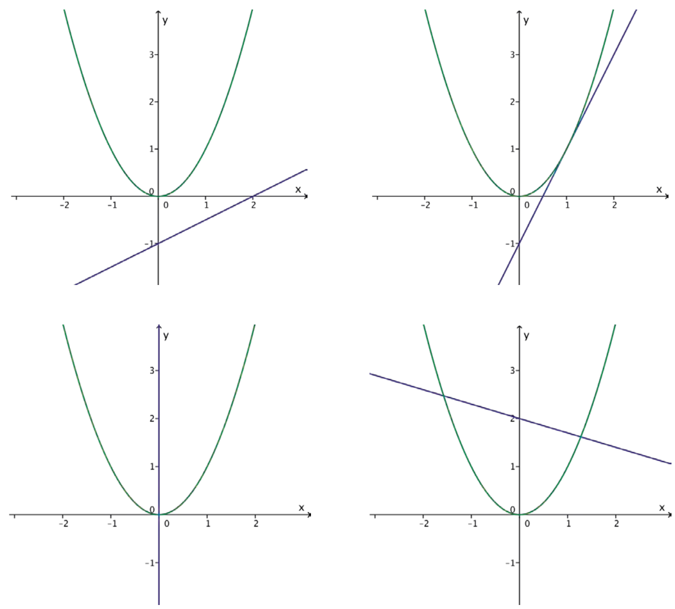
Next, the students should continue to work in pairs to sketch graphs and develop conjectures about the following item (**A-REI.C.6**):

> **b. Graph the parabola with equation $y = x^2$. What possibilities are there for a line to intersect the parabola? Sketch each possibility.**

> *Scaffolding:*
> - Again circulate to assist pairs of students who might be having trouble coming up with all three possibilities.

Once the students have made their sketches, ask one pair to use the display tool and draw the graphs for the rest of the class to see.

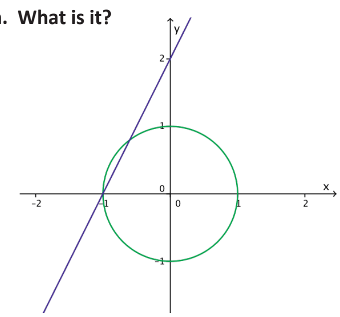
The parabola and line can intersect in 0, 1, or 2 points as shown below. Note that, in contrast to the circle, where all the lines intersecting the circle in one point are tangent to it, lines intersecting the parabola in one point are either tangent to it or are parallel to the parabola's axis of symmetry—in this case, the y-axis.



Next, ask the students to work on the following problem individually (**A-REI.C.7**):

c. Sketch the circle given by $x^2 + y^2 = 1$ and the line given by $y = 2x + 2$ on the same set of axes. One solution to the pair of equations has a value of y that is easily identifiable from the sketch. What is it?

The point $(-1, 0)$ should be easily identifiable from the sketch, but the other point is not.



Once the students have made a sketch, ask one of them to use the display tool and draw the two graphs for the rest of the class to see. While the student is doing that, ask the other students how many points are shared (two) and what the coordinates of those points are.

The students should see that they can substitute the value for y in the second equation into the first equation. In other words, they need to solve the following quadratic equation (**A-REI.B.4**).

d. Solve $x^2 + (2x+2)^2 = 1$

Factoring or using the quadratic formula, the students should find that the solutions to the quadratic equation are -1 and $-\frac{3}{5}$.

If $x = -1$, then $y = 0$, as the sketch shows, so $(-1, 0)$ is a solution. If $x = -\frac{3}{5}$, then $y = 2\left(-\frac{3}{5}\right) + 2 = \frac{4}{5}$, so $\left(-\frac{3}{5}, \frac{4}{5}\right)$ is another solution.

Note that the problem above does not explicitly tell students to look for intersection points. Thus, the exercise assesses not only whether they can solve the system but also whether they understand that the intersection points of the graphs correspond to solutions of the system.

Students should understand that to solve the system of equations, we look for points that lie on the line and the circle. The points that lie on the circle are precisely those that satisfy $x^2 + y^2 = 1$, and the points that lie on the line are those that satisfy $y = 2x + 2$. So points on both are the intersection.

Exercise 1 (8 minutes)

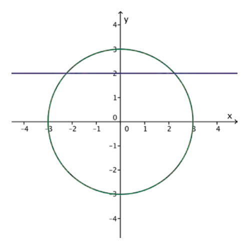
Pose the following three-part problem for students to work on individually and then discuss as a class.

Exercise 1

1. Draw a graph of the circle with equation $x^2 + y^2 = 9$.

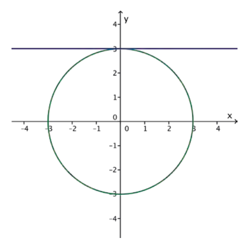
   a. What are the solutions to the system of circle and line when the circle is given by $x^2 + y^2 = 9$ and the line is given by $y = 2$?

   Substituting $y = 2$ in the equation of the circle yields $x^2 + 4 = 9$, so $x^2 = 5$, and $x = \sqrt{5}$ or $x = -\sqrt{5}$. The solutions are $(-\sqrt{5}, 2)$ and $(\sqrt{5}, 2)$.



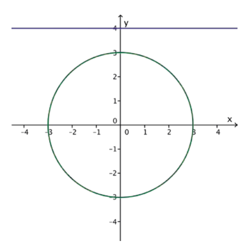
   b. What happens when the line is given by $y = 3$?

   Substituting $y = 3$ in the equation of the circle yields $x^2 + 9 = 9$, so $x^2 = 0$. The line is tangent to the circle, and the solution is $(0, 3)$.



   c. What happens when the line is given by $y = 4$?

   Substituting $y = 4$ in the equation of the circle yields $x^2 + 16 = 9$, so $x^2 = -7$. Since there are no real numbers that satisfy $x^2 = -7$, there is no solution to this equation. This indicates that the line and circle do not intersect.



Exercises 2–6 (8 minutes)

Students will need graph paper for this portion of the lesson. Complete Exercise 2 in groups so that students can check answers with each other. Then they can do Exercises 3 to 6 individually or in groups as they choose. Assist with the exercises if students have trouble understanding what it means to "verify your results both algebraically and graphically."

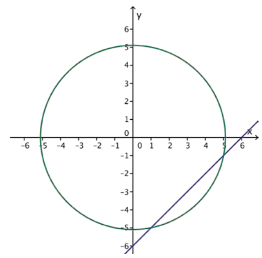
Scaffolding (for advanced learners):

- Create two different systems of one linear equation and one quadratic equation that have one solution at $(0, 2)$.

Exercises 2–6

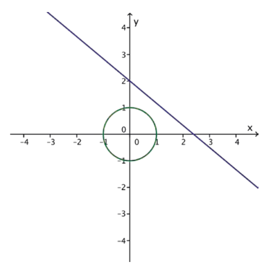
2. **By solving the equations as a system, find the points common to the line with equation $x - y = 6$ and the circle with equation $x^2 + y^2 = 26$. Graph the line and the circle to show those points.**

   $(5, -1)$ and $(1, -5)$. See picture to the right.



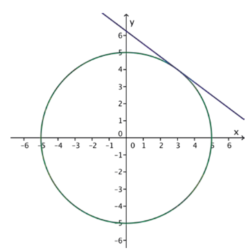
3. **Graph the line given by $5x + 6y = 12$ and the circle given by $x^2 + y^2 = 1$. Find all solutions to the system of equations.**

   There is no real solution; the line and circle do not intersect. See picture to the right.



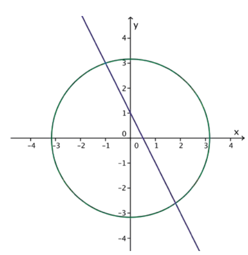
4. **Graph the line given by $3x + 4y = 25$ and the circle given by $x^2 + y^2 = 25$. Find all solutions to the system of equations. Verify your result both algebraically and graphically.**

   The line is tangent to the circle at $(3, 4)$, which is the only solution. See picture to the right.



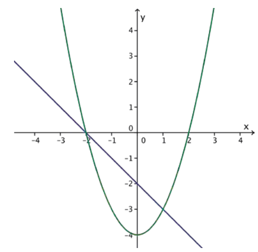
5. **Graph the line given by $2x + y = 1$ and the circle given by $x^2 + y^2 = 10$. Find all solutions to the system of equations. Verify your result both algebraically and graphically.**

   The line and circle intersect at $(-1, 3)$ and $\left(\frac{9}{5}, -\frac{13}{5}\right)$, which are the two solutions. See picture to the right.



6. **Graph the line given by $x + y = -2$ and the quadratic curve given by $y = x^2 - 4$. Find all solutions to the system of equations. Verify your result both algebraically and graphically.**

The line and the parabola intersect at $(1, -3)$ and $(-2, 0)$, which are the two solutions. See picture to the right.



Closing (4 minutes)

Ask students to respond to these questions with a partner or in writing. Share their responses as a class.

MP.1

- How does graphing a line and a quadratic curve help you solve a system consisting of a linear and a quadratic equation?
- What are the possibilities for the intersection of a line and a quadratic curve, and how are they related to the number of solutions of a system of linear and quadratic equations?

Present and discuss the Lesson Summary.

Scaffolding:

- Perhaps create a chart with the summary that can serve as a reminder to the students.

Lesson Summary

Here are some steps to consider when solving systems of equations that represent a line and a quadratic curve.

1. **Solve the linear equation for y in terms of x. This is equivalent to rewriting the equation in slope-intercept form. Note that working with the quadratic equation first would likely be more difficult and might cause the loss of a solution.**
2. **Replace y in the quadratic equation with the expression involving x from the slope-intercept form of the linear equation. That will yield an equation in one variable.**
3. **Solve the quadratic equation for x.**
4. **Substitute x into the linear equation to find the corresponding value of y.**
5. **Sketch a graph of the system to check your solution.**

Be sure to note that in the case of the circle, the reverse process of solving the equation for the circle first—for either x or y—and then substituting in the linear equation would have yielded an equation with a complicated radical expression and might have led students to miss part of the solution by considering only the positive square root.

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 31: Systems of Equations

Exit Ticket

Make and explain a prediction about the nature of the solution to the following system of equations and then solve it.

$$x^2 + y^2 = 25$$
$$4x + 3y = 0$$

Illustrate with a graph. Verify your solution and compare it with your initial prediction.

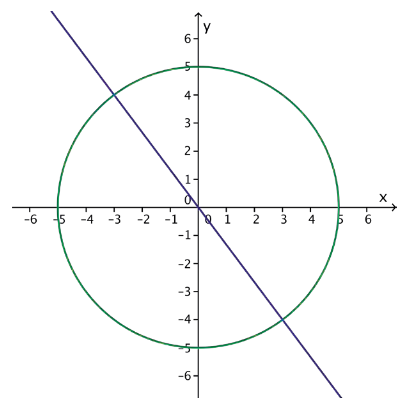
Exit Ticket Sample Solutions

Make and explain a prediction about the nature of the solution to the following system of equations and then solve it.

$$x^2 + y^2 = 25$$
$$4x + 3y = 0$$

Illustrate with a graph. Verify your solution and compare it with your initial prediction.

Prediction: By inspection of the equations, students should conclude that the circle is centered at the origin and that the line goes through the origin. So, the solution should consist of two points.



Solution: Solve the linear equation for one of the variables: $y = -\frac{4x}{3}$.

Substitute that variable in the quadratic equation: $x^2 + \left(-\frac{4x}{3}\right)^2 = 25$.

Remove parentheses and combine like terms: $25x^2 - 25 \cdot 9 = 0$, *so* $x^2 - 9 = 0$.

Solve the quadratic equation in x: $(x+3)(x-3) = 0$, *which gives the roots* 3 *and* -3.

Substitute into the linear equation: If $x = 3$, *then* $y = -4$; *if* $x = -3$, *then* $y = 4$.

As the graph shows, the solution is the two points of intersection of the circle and the line: $(3, -4)$ *and* $(-3, 4)$.

An alternative solution would be to solve the linear equation for x *instead of* y, *getting the quadratic equation* $(y+4)(y-4) = 0$, *which gives the roots* 4 *and* -4 *and the same points of intersection.*

As noted before, solving the quadratic equation for x *or* y *first is not a good procedure. It can lead to a complicated radical expression and loss of part of the solution.*

Problem Set Sample Solutions

Problem 4 yields a system with no real solution, and the graph shows that the circle and line have no point of intersection in the coordinate plane. In Problems 5 and 6, the curve is a parabola. In Problem 5, the line intersects the parabola in two points, whereas in Problem 6, the line is tangent to the parabola, and there is only one point of intersection. Note that there would also have been only one point of intersection if the line had been the line of symmetry of the parabola.

1. **Where do the lines given by $y = x + b$ and $y = 2x + 1$ intersect?**

   Since we do not know the value of b, *we cannot solve this problem by graphing, and we will have to approach it algebraically. Eliminating* y *gives the equation*

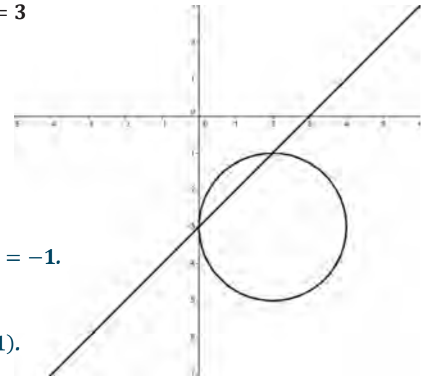
   $$x + b = 2x + 1$$
   $$x = b - 1$$

   Since $x = b - 1$, *we have* $y = x + b = (b-1) + b = 2b - 1$. *Thus, the lines intersect at the point* $(b - 1, 2b - 1)$.

2. Find all solutions to the following system of equations.

$$(x-2)^2 + (y+3)^2 = 4$$
$$x - y = 3$$

Illustrate with a graph.



Solve the linear equation for one of the variables: $x = y + 3$.

Substitute that variable in the quadratic equation: $(y+3-2)^2 + (y+3)^2 = 4$.

Rewrite the equation in standard form: $2y^2 + 8y + 6 = 0$.

Solve the quadratic equation: $2(y+3)(y+1) = 0$*, so* $y = -3$ *or* $y = -1$.

If $y = -3$*, then* $x = 0$*. If* $y = -1$*, then* $x = 2$.

As the graph shows, the solution is the two points $(0, -3)$ *and* $(2, -1)$.

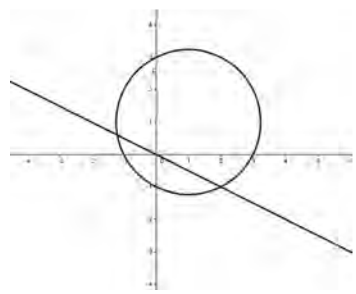
3. Find all solutions to the following system of equations.

$$x + 2y = 0$$
$$x^2 - 2x + y^2 - 2y - 3 = 0$$

Illustrate with a graph.

Solve the linear equation for one of the variables: $x = -2y$.

Substitute that variable in the quadratic equation: $(-2y)^2 - 2(-2y) + y^2 - 2y - 3 = 0$.



Rewrite the equation in standard form: $5y^2 + 2y - 3 = 0$.

Solve the quadratic equation: $(5y-3)(y+1) = 0$*, so* $y = \frac{3}{5}$ *or* $y = -1$.

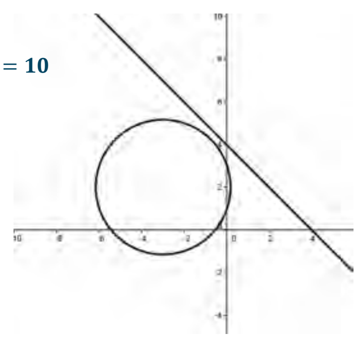
If $y = \frac{3}{5}$*, then* $x = -\frac{6}{5}$*. If* $y = -1$*, then* $x = 2$.

As the graph shows, the solutions are the two points: $\left(-\frac{6}{5}, \frac{3}{5}\right)$ *and* $(2, -1)$.

4. Find all solutions to the following system of equations.

$$x + y = 4$$
$$(x+3)^2 + (y-2)^2 = 10$$

Illustrate with a graph.



Solve the linear equation for one of the variables: $x = 4 - y$

Substitute that variable in the quadratic equation: $(4 - y + 3)^2 + (y-2)^2 = 10$

Rewrite the equation in standard form: $2y^2 - 18y + 43 = 0$

Solve the equation using the quadratic formula:

$$y = \frac{18 + \sqrt{324 - 344}}{4} \quad \text{or} \quad y = \frac{18 - \sqrt{324 - 344}}{4}$$

So we have $y = \frac{1}{2}\left(9 + \sqrt{-5}\right)$ *or* $y = \frac{1}{2}\left(9 - \sqrt{-5}\right)$

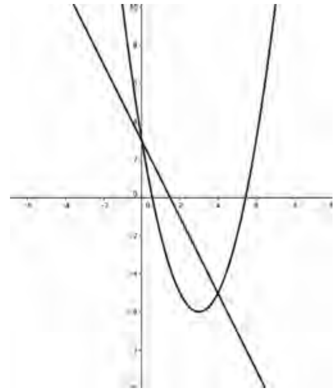
Therefore, there is no real solution to the system.

As the graph shows, the line and circle do not intersect.

5. **Find all solutions to the following system of equations.**

$$y = -2x + 3$$
$$y = x^2 - 6x + 3$$

Illustrate with a graph.



The linear equation is already solved for one of the variables: $y = -2x + 3$.

Substitute that variable in the quadratic equation: $-2x + 3 = x^2 - 6x + 3$.

Rewrite the equation in standard form: $x^2 - 4x = 0$.

Solve the quadratic equation: $x(x - 4) = 0$.

So, $x = 0$ *or* $x = 4$.

If $x = 0$*, then* $y = 3$. *If* $x = 4$*, then* $y = -5$.

As the graph shows, the solutions are the two points $(0, 3)$ *and* $(4, -5)$.

6. **Find all solutions to the following system of equations.**

$$-y^2 + 6y + x - 9 = 0$$
$$6y = x + 27$$

Illustrate with a graph.

Solve the second equation for x: $x = 6y - 27$.

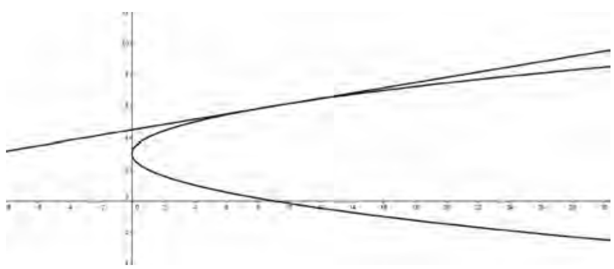
Substitute in the first equation: $-y^2 + 6y + 6y - 27 - 9 = 0$.

Combine like terms: $-y^2 + 12y - 36 = 0$.

Rewrite the equation in standard form and factor: $-(y - 6)^2 = 0$.

Therefore, $y = 6$. *Then* $x = 6y - 27$*, so* $x = 9$.

There is only one solution $(9, 6)$*, and as the graph shows, the line is tangent to the parabola.*



An alternative solution would be to solve the linear equation for y *instead of* x*, getting the quadratic equation* $(x - 9)(x - 9) = 0$*, which gives the repeated root* $x = 9$ *and the same point of tangency* $(9, 6)$.

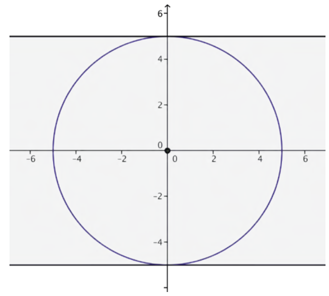
Another alternative solution would be to solve the quadratic equation for x*, so that* $x = y^2 - 6y + 9$. *Substituting in the linear equation would yield* $6y = y^2 - 6y + 9 + 27$. *Converting that to standard form would give* $y^2 - 12y + 36 = 0$*, which gives the repeated root* $y = 6$*, as in the first solution. Note that in this case, unlike when the graph of the quadratic equation is a circle, the quadratic equation can be solved for* x *in terms of* y *without getting a radical expression.*

7. If the following system of equations has two solutions, what is the value of k?

$$x^2 + y^2 = 25$$

$$y = k$$

Illustrate with a graph.



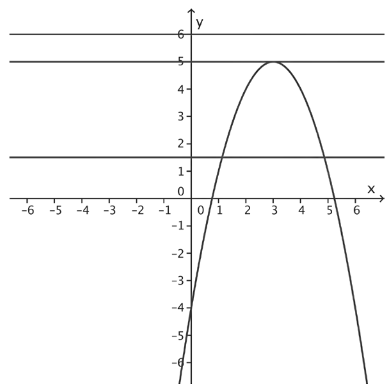
The center of the circle is the origin, and the line is parallel to the x-axis. Therefore, as the graph shows, there are two solutions only when $-5 < k < 5$.

8. If the following system of equations has exactly one solution, what is the value of k?

$$y = 5 - (x - 3)^2$$

$$y = k$$

Illustrate with a graph.



The parabola opens down, and its axis of symmetry is the vertical line $x = 3$. The line $y = k$ is a horizontal line and will intersect the parabola in either two, one, or no points. It intersects the parabola in one point only if it passes through the vertex of the parabola, which is $k = 5$.

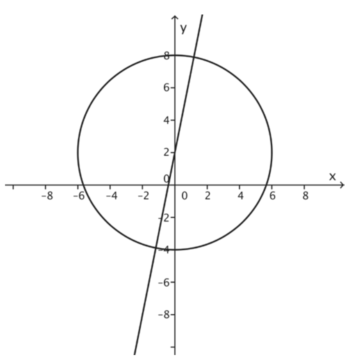
9. If the following system of equations has no solutions, what is the value of k?

$$x^2 + (y - k)^2 = 36$$

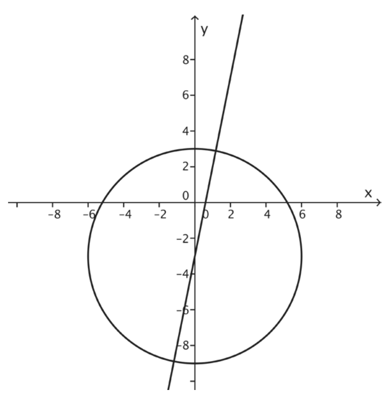
$$y = 5x + k$$

Illustrate with a graph.

The circle has radius 6 and center $(0, k)$. The line has slope 5 and crosses the y-axis at $(0, k)$. Since for any value of k the line passes through the center of the circle, the line intersects the circle twice. (In the figure on the left below $k = 2$, and in the one on the right below, $k = -3$.) There is no value of k for which there is no solution.



$k = 2$



$k = -3$

Lesson 32: Graphing Systems of Equations

Student Outcomes

- Students develop facility with graphical interpretations of systems of equations and the meaning of their solutions on those graphs. For example, they can use the distance formula to find the distance between the centers of two circles and thereby determine whether the circles intersect in 0, 1, or 2 points.
- By completing the squares, students can convert the equation of a circle in general form to the center-radius form and, thus, find the radius and center. They can also convert the center-radius form to the general form by removing parentheses and combining like terms.
- Students understand how to solve and graph a system consisting of two quadratic equations in two variables.

Lesson Notes

This lesson is an extension that goes beyond what is required in the standards. In particular, the standard **A-REI.C.7** (solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically) does not extend to a system of two quadratic equations, which is a natural culmination of the types of systems formed by linear and quadratic equations. The lesson also addresses standard MP.8 (look for and express regularity in repeated reasoning).

The lesson begins with a brief review of the distance formula and its connection both to the Pythagorean Theorem and to the center-radius equation of a circle. The distance formula will be used extensively in the next few lessons, so be sure to review it with students. Students also briefly review how to solve and graph a system of a linear equation and an equation of a circle. They then move to the main focus of the lesson, which is graphing and solving systems of pairs of quadratic equations whose graphs include parabolas as well as circles

Materials

This lesson requires use of graphing calculators or computer software, such as the Wolfram Alpha engine, the GeoGebra package, or the Geometer's Sketchpad software for graphing geometric figures, plus a tool for displaying graphs, such as a projector, smart board, white board, chalk board, or squared poster paper.

Classwork

Opening (1 minute)

Begin with questions that should remind students of the distance formula and how it is connected to the Pythagorean Theorem.

- Suppose you have a point A with coordinates $(1, 3)$. Find the distance AB if B has coordinates:

1. $(4, 2)$

   Answer: $AB = \sqrt{10}$

2. $(-3, 1)$

   Answer: $AB = 2\sqrt{5}$

3. (x, y)

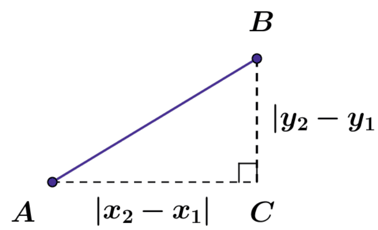
   Answer: $AB = \sqrt{(x-1)^2 + (y-3)^2}$

If the students cannot recall the distance formula (in the coordinate plane), they may need to be reminded of it.

<u>**The Distance Formula**</u>: Given two points (x_1, y_1) and (x_2, y_2), the distance d between these points is given by the formula

$$\boldsymbol{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}.$$

If A and B are points with coordinates (x_1, y_1) and (x_2, y_2), then the distance between them is the length AB. Draw horizontal and vertical lines through A and B to intersect in point C and form right triangle ΔABC. The length of the horizontal side is the difference in the x-coordinates $|x_2 - x_1|$, and the length of the vertical side is the difference in the y-coordinates $|y_2 - y_1|$. The Pythagorean Theorem gives the length of the hypotenuse as $(AB)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$. Taking the square root gives the distance formula.



Opening Exercise (3 minutes)

Make sure the students all have access to, and familiarity with, some technology (calculator or computer software) for graphing lines and circles in the coordinate plane. Have them work individually on the following exercise.

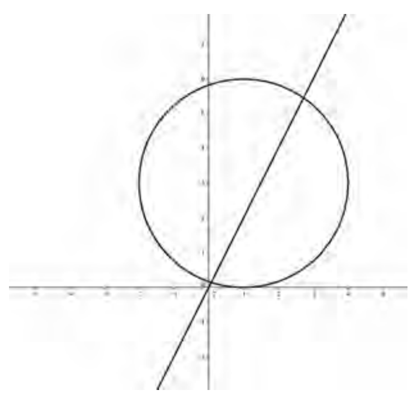
Opening Exercise

Given the line $y = 2x$, is there a point on the line at a distance 3 from $(1, 3)$? Explain how you know.

Yes, there are two such points. They are the intersection of the line $y = 2x$ and the circle $(x-1)^2 + (y-3)^2 = 9$. (The intersection points are roughly $(0.07, 0.15)$ and $(2.73, 5.45)$.)

Draw a graph showing where the point is.

There are actually two such points. See the graph to the right.



Scaffolding:

- Circulate to identify and help students who might have trouble managing the graphing tool.

Students should compare the graph they have drawn with that of a neighbor.

Exercise 1 (5 minutes)

This exercise reviews the solution of a simple system consisting of a linear equation and the equation of a circle from the perspective of the defining property of a circle (**A-REI.C.7**).

Exercise 1

Solve the system $(x-1)^2+(y-2)^2=2^2$ and $y=2x+2$.

Substituting $2x+2$ for y in the quadratic equation allows us to find the x-coordinates.

$$\begin{aligned}(x-1)^2+\left((2x+2)-2\right)^2&=4\\(x^2-2x+1)+4x^2&=4\\5x^2-2x-3&=0\\(x-1)(5x+3)&=0\end{aligned}$$

So, $x=1$ or $x=-\frac{3}{5}$, and the intersection points are $\left(-\frac{3}{5},\frac{4}{5}\right)$ and $(1,4)$.

What are the coordinates of the center of the circle?

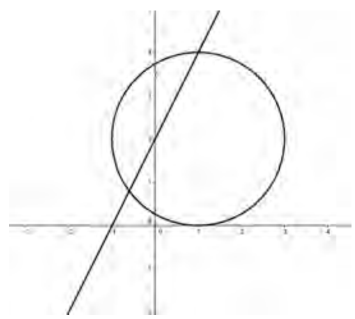
$(1,2)$

What can you say about the distance from the intersection points to the center of the circle?

Because they are points on the circle and the radius of the circle is 2, the intersection points are 2 units away from the center. This can be verified by the distance formula.

Using your graphing tool, graph the line and the circle.

See the graph at the right.



Example 1 (5 minutes)

It is important to keep in mind that not all quadratic equations in two variables represent circles.

Example 1

Rewrite $x^2+y^2-4x+2y=-1$ by completing the square in both x and y. Describe the circle represented by this equation.

Rearranging terms gives $x^2-4x+y^2+2y=-1$.

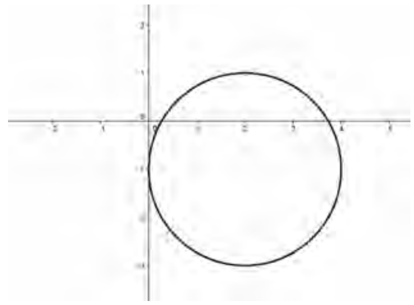
Then, completing the square in both x and y, we have

$$\begin{gathered}(x^2-4x+4)+(y^2+2y+1)=-1+4+1\\(x-2)^2+(y+1)^2=4.\end{gathered}$$

This is the equation of a circle with center $(2,-1)$ and radius 2.

Using your graphing tool, graph the circle.

See the graph to the right.



In contrast, consider the following equation: $x^2 + y^2 - 2x - 8y = -19$.

Rearranging terms gives $x^2 - 2x + y^2 - 8y = -19$.

Then, completing the square in both x *and* y*, we have*

$$(x^2 - 2x + 1) + (y^2 - 8y + 16) = -19 + 1 + 16$$
$$(x-1)^2 + (y-4)^2 = -2,$$

which is not a circle, because then the radius would be $\sqrt{-2}$.

What happens when you use your graphing tool with this equation?

The tool cannot draw the graph. There are no points in the plane that satisfy this equation, so the graph is empty.

Exercise 2 (5 minutes)

Allow students time to think these questions over, draw some pictures, and discuss with a partner before discussing as a class.

Exercise 2

Consider a circle with radius 5 and another circle with radius 3. Let d represent the distance between the two centers. We want to know how many intersections there are of these two circles for different values of d. Draw figures for each case.

a. **What happens if $d = 8$?**

If the distance is 8, then the circles will touch at only one point. We say that the circles are externally tangent.

b. **What happens if $d = 10$?**

If the distance is 10, the circles do not intersect, and one circle is outside of the other.

c. **What happens if $d = 1$?**

If the distance is 1, the circles do not intersect, but one circle lies inside the other.

d. **What happens if $d = 2$?**

If the distance is 2, the circles will touch at only one point, with one circle inside the other. We say that the circles are internally tangent.

e. **For which values of d do the circles intersect in exactly one point? Generalize this result to circles of any radius.**

If $d = 8$ or $d = 2$, the circles will be tangent. In general, if d is either the sum or the difference of the radii, then the circles will be tangent.

f. **For which values of d do the circles intersect in two points? Generalize this result to circles of any radius.**

If $2 < d < 8$, the circles will intersect in two points. In general, if d is between the sum and the difference of the radii then the circles will be tangent.

g. **For which values of d do the circles not intersect? Generalize this result to circles of any radius.**

The circles do not intersect if $d < 2$ or $d > 8$. In general, if d is smaller than the difference of the radii or larger than the sum of the radii, then the circles will not intersect.

Example 2 (5 minutes)

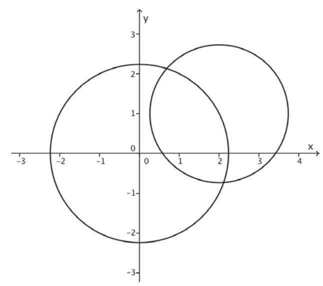
Example 2

Find the distance between the centers of the two circles with equations below, and use that distance to determine in how many points these circles intersect.

$$\boldsymbol{x^2 + y^2 = 5}$$
$$\boldsymbol{(x-2)^2 + (y-1)^2 = 3}$$

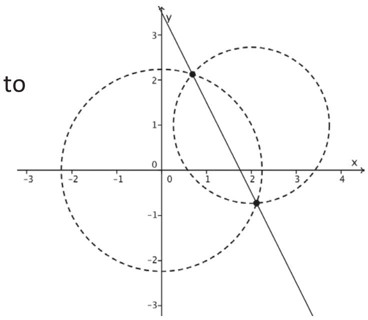
The first circle has center $(0,0)$, and the second circle has center $(2,1)$. Using the distance formula, the distance between the centers of these circles is $d = \sqrt{(2-0)^2 + (1-0)^2} = \sqrt{5}$.

Since the distance between the centers is between the sum and the difference of the two radii, that is, $\sqrt{5} - \sqrt{3} < \sqrt{5} < \sqrt{5} + \sqrt{3}$, we know that the circles must intersect in two distinct points.



- Find the coordinates of the intersection points of the circles.
  - *Multiplying out the terms in the second equation gives*:
  $$x^2 - 4x + 4 + y^2 - 2y + 1 = 3.$$
  - *We subtract the first equation: $x^2 + y^2 = 5$.*
- The reason for subtracting is that we are removing repeated information in the two equations.
  - *We get $-4x - 2y = -7$, which is the equation of the line through the two intersection points of the circles.*
- To find the intersection points, we find the intersection of the line $-4x - 2y = -7$ and the circle $x^2 + y^2 = 5$.
- As with the other systems of quadratic curves and lines, we solve the linear equation for y and substitute it into the quadratic equation to find two solutions for x: $x = \frac{7}{5} - \frac{\sqrt{51}}{10}$ and $x = \frac{7}{5} + \frac{\sqrt{51}}{10}$.

- The corresponding y-values are
$$y = \frac{7}{10} + \frac{\sqrt{51}}{5} \text{ and } y = \frac{7}{10} - \frac{\sqrt{51}}{5}.$$
- The graph of the circles and the line through the intersection points is shown to the right.



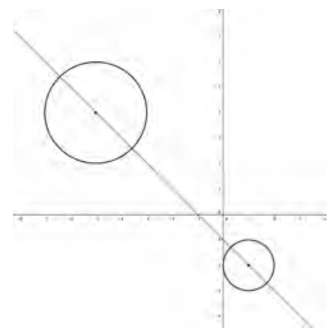
Exercise 3 (4 minutes)

This exercise concerns a system of equations that represents circles that do not intersect.

Exercise 3

Use the distance formula to show algebraically and graphically that the following two circles do not intersect.

$$(x-1)^2 + (y+2)^2 = 1$$
$$(x+5)^2 + (y-4)^2 = 4$$



The centers of the two circles are $(1, -2)$ and $(-5, 4)$, and the radii are 1 and 2. The distance between the two centers is $\sqrt{6^2 + 6^2} = 6\sqrt{2}$, which is greater than $1 + 2 = 3$. The graph to the right also shows that the circles do not intersect.

Example 3 (10 minutes)

Work through this example with the whole class, showing students how to find the tangent to a circle at a point and one way to determine how many points of intersection there are for a line and a circle.

Example 3

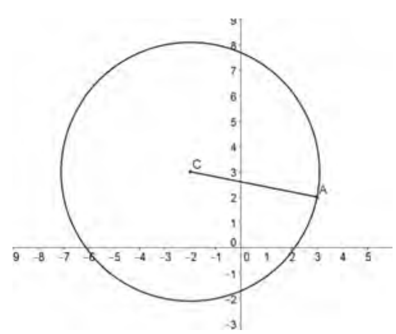
Point A $(3, 2)$ is on a circle whose center is C $(-2, 3)$. What is the radius of the circle?

The distance from A to C is given by $\sqrt{(3+2)^2 + (2-3)^2} = \sqrt{26}$, which is the length of the radius.

What is the equation of the circle? Graph it.

Given the center and the radius, we can write the equation of the circle as $(x+2)^2 + (y-3)^2 = 26$.

The graph is shown at the right.



Use the fact that the tangent at $A\ (3,2)$ is perpendicular to the radius at that point to find the equation of the tangent line. Then graph it.

The slope of the tangent line is the opposite reciprocal of the slope of $\overleftrightarrow{AC}$. The slope of $\overleftrightarrow{AC}$ is $\frac{3-2}{-2-3} = -\frac{1}{5}$, so the slope of the tangent line is 5. Using the point-slope form of the equation of a line with slope 5 and passing through point $(3,2)$ gives

$$\begin{aligned} y-2 &= 5(x-3) \\ y &= 5x-13. \end{aligned}$$

The equation of the tangent line is, therefore, $y = 5x - 13$.

Find the coordinates of point B, the second intersection of the line $\overleftrightarrow{AC}$ and the circle.

The system $(x+2)^2 + (y-3)^2 = 26$ and $5y = -x + 13$ can be solved by substituting $x = 13 - 5y$ into the equation of the circle, which yields $(13 - 5y + 2)^2 + (y-3)^2 = 26$. This gives $26(y-2)(y-4) = 0$. Thus, the y-coordinate is either 2 or 4. If $y = 2$, then $x = 13 - 5 \cdot 2 = 3$, and if $y = 4$, then $x = 13 - 5 \cdot 4 = -7$. Since A has coordinates $(3,2)$, it follows that B has coordinates $(-7,4)$.

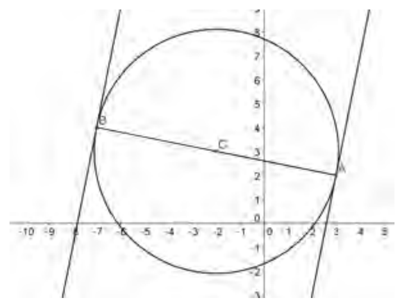
What is the equation of the tangent to the circle at $(-7,4)$? Graph it as a check.

Using the point-slope form of a line with slope 5 and point $(-7,4)$:

$$\begin{aligned} y-4 &= 5(x+7) \\ y &= 5x+39. \end{aligned}$$

The equation of the tangent line is, therefore, $y = 5x + 39$.

The graph is shown to the right.



The lines $y = 5x + b$ are parallel to the tangent lines to the circle at points A and B. How is the y-intercept b for these lines related to the number of times each line intersects the circle?

When $b = -13$ or $b = 39$, the line is tangent to the circle, intersecting in one point.

When $-13 < b < 39$, the line intersects the circle in two points.

When $b < -13$ or $b > 39$, the line and circle do not intersect.

Closing (2 minutes)

Ask students to summarize how to convert back and forth between the center-radius equation of a circle and the general quadratic equation of a circle.

Ask students to speculate about what might occur with respect to intersections if one or two of the quadratic equations in the system are not circles.

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 32: Graphing Systems of Equations

Exit Ticket

1. Find the intersection of the two circles

$$x^2 + y^2 - 2x + 4y - 11 = 0$$

and

$$x^2 + y^2 + 4x + 2y - 9 = 0.$$

2. The equations of the two circles in Question 1 can also be written as follows:

$$(x - 1)^2 + (y + 2)^2 = 16$$

and

$$(x + 2)^2 + (y + 1)^2 = 14.$$

Graph the circles and the line joining their points of intersection.

3. Find the distance between the centers of the circles in Questions 1 and 2.

Exit Ticket Sample Solutions

1. Find the intersection of the two circles

$$x^2 + y^2 - 2x + 4y - 11 = 0$$

and

$$x^2 + y^2 + 4x + 2y - 9 = 0.$$

Subtract the second equation from the first: $-6x + 2y - 2 = 0$.

Solve the equation for y*:* $y = 3x + 1$.

Substitute in the first equation: $x^2 + (3x+1)^2 - 2x + 4(3x+1) - 11 = 0$.

Remove parentheses and combine like terms: $5x^2 + 8x - 3 = 0$.

Substitute in the quadratic equation to find two values: $x = \frac{-4}{5} - \frac{\sqrt{31}}{5}$ *and* $x = \frac{-4}{5} + \frac{\sqrt{31}}{5}$.

The corresponding y*-values are the following:* $y = \frac{-7}{5} - \frac{3\sqrt{31}}{5}$ *and* $y = \frac{-7}{5} + \frac{3\sqrt{31}}{5}$.

2. The equations of the two circles in Question 1 can also be written as follows:

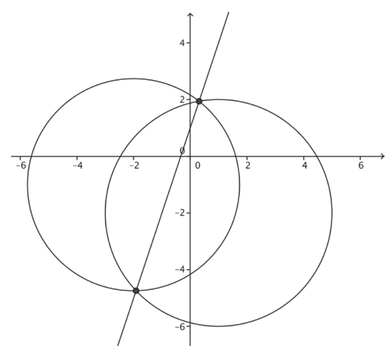
$$(x-1)^2 + (y+2)^2 = 16$$

and

$$(x+2)^2 + (y+1)^2 = 14.$$

Graph the circles and the line joining their points of intersection.

See the graph to the right.



3. Find the distance between the centers of the circles in Questions 1 and 2.

The center of the first circle is $(1,-2)$*, and the center of the second circle is* $(-2,-1)$*. We then have*

$d = \sqrt{(-2-1)^2 + (-1+2)^2} = \sqrt{9+1} = \sqrt{10}$.

Problem Set Sample Solutions

In this Problem Set, after solving some problems dealing with the distance formula, the students continue converting between forms of the equation of a circle and then move on to solving and graphing systems of quadratic equations, some of which represent circles and some of which do not.

1. Use the distance formula to find the distance between the points $(-1,-13)$ and $(3,-9)$.

Using the formula with $(-1,-13)$ *and* $(3,-9)$*:*

$$d = \sqrt{\left(3-(-1)\right)^2 + \left((-9)-(-13)\right)^2}$$
$$d = \sqrt{4^2 + 4^2} = \sqrt{16+16} = \sqrt{16}\cdot\sqrt{2} = 4\sqrt{2}.$$

Therefore, the distance is $4\sqrt{2}$.

2. Use the distance formula to find the length of the longer side of the rectangle whose vertices are $(1, 1)$, $(3, 1)$, $(3, 7)$, and $(1, 7)$.

Using the formula with $(1, 1)$ and $(1, 7)$:

$$d = \sqrt{(1-1)^2 + (7-1)^2}$$
$$d = \sqrt{(0)^2 + (6)^2} = \sqrt{36} = 6$$

Therefore, the length of the longer side is 6.

3. Use the distance formula to find the length of the diagonal of the square whose vertices are $(0, 0)$, $(0, 5)$, $(5, 5)$, and $(5, 0)$.

Using the formula with $(0, 0)$ and $(5, 5)$:

$$d = \sqrt{(5-0)^2 + (5-0)^2}$$
$$d = \sqrt{(5-0)^2 + (5-0)^2} = \sqrt{25 + 25} = 5\sqrt{2}$$

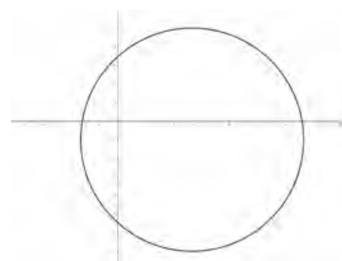
Therefore, the length of the diagonal is $5\sqrt{2}$.

Write an equation for the circles in Exercises 4–6 in the form $(x - h)^2 + (y - k)^2 = r^2$, where the center is (h, k) and the radius is r units. Then write the equation in the standard form $x^2 + ax + y^2 + by + c = 0$, and construct the graph of the equation.

4. A circle with center $(4, -1)$ and radius 6 units.

$(x-4)^2 + (y+1)^2 = 36$; standard form: $x^2 - 8x + y^2 + 2y - 19 = 0$

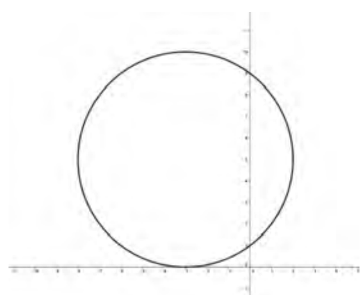
The graph is shown to the right.



5. A circle with center $(-3, 5)$ tangent to the x-axis.

$(x+3)^2 + (y-5)^2 = 25$; standard form: $x^2 + 6x + y^2 - 10y + 9 = 0$.

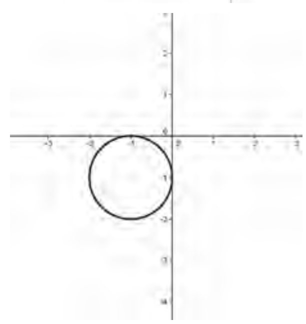
The graph is shown to the right.



6. A circle in the third quadrant, radius 1 unit, tangent to both axes.

$(x+1)^2 + (y+1)^2 = 1$; standard form: $x^2 + 2x + y^2 + 2y + 1 = 0$.

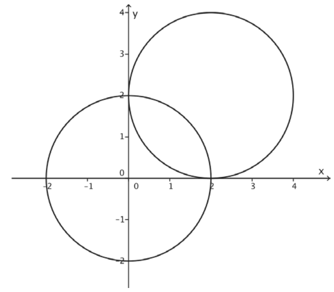
The graph is shown to the right.



7. By finding the radius of each circle and the distance between their centers, show that the circles $x^2 + y^2 = 4$ and $x^2 - 4x + y^2 - 4y + 4 = 0$ intersect. Illustrate graphically.

The second circle is $(x-2)^2 + (y-2)^2 = 4$. Each radius is 2, and the centers are at $(0,0)$ and $(2,2)$. The distance between the centers is $2\sqrt{2}$, which is less than 4, the sum of the radii.

The graph of the two circles is to the right.



8. Find the points of intersection of the circles $x^2 + y^2 - 15 = 0$ and $x^2 - 4x + y^2 + 2y - 5 = 0$. Check by graphing the equations.

Write the equations as

$$x^2 + y^2 = 15$$
$$x^2 + y^2 - 4x + 2y = 5$$

Subtracting the second equation from the first:

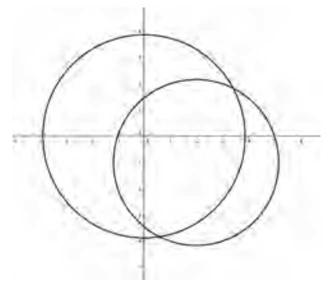
$$4x - 2y = 10,$$

which is equivalent to

$$2x - y = 5.$$

Solving the system $x^2 + y^2 = 15$ and $y = 2x - 5$ yields

$\left(2+\sqrt{2}, -1+2\sqrt{2}\right)$ and $\left(2-\sqrt{2}, -1-2\sqrt{2}\right)$. The graph is to the right.



9. Solve the system $y = x^2 - 2$ and $x^2 + y^2 = 4$. Illustrate graphically.

Substitute $x^2 = y + 2$ into the second equation:

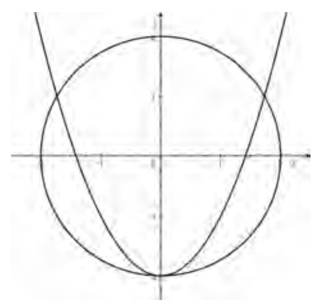
$$y + 2 + y^2 = 4$$
$$y^2 + y - 2 = 0$$
$$(y-1)(y+2) = 0$$

so $y = -2$ or $y = 1$.

If $y = -2$, then $x^2 = y + 2 = 0$ and thus $x = 0$.

If $y = 1$, then $x^2 = y + 2 = 3$, so $x = \sqrt{3}$ or $x = -\sqrt{3}$.

Thus, there are three solutions $(0,-2)$, $\left(\sqrt{3},1\right)$, and $\left(-\sqrt{3},1\right)$. The graph is to the right.



10. Solve the system $y = 2x - 13$ and $y = x^2 - 6x + 3$. Illustrate graphically.

Substitute $2x - 13$ for y in the second equation: $2x - 13 = x^2 - 6x + 3$.

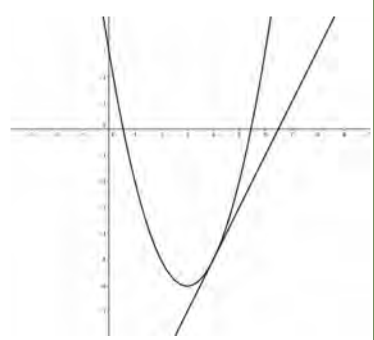
Rewrite the equation in standard form: $x^2 - 8x + 16 = 0$.

Solve for x: $(x-4)(x-4) = 0$.

The root is repeated, so there is only one solution $x = 4$.

The corresponding y-value is $y = -5$, and there is only one solution, $(4,-5)$.

As shown to the right, the line is tangent to the parabola.



Lesson 33: The Definition of a Parabola

Student Outcomes

- Students model the locus of points at equal distance between a point (focus) and a line (directrix). They construct a parabola and understand this geometric definition of the curve. They use algebraic techniques to derive the analytic equation of the parabola.

Lesson Notes

A Newtonian reflector telescope uses a parabolic mirror to reflect light to the focus of the parabola, bringing the image of a distant object closer to the eye. This lesson uses the Newtonian telescope to motivate the discussion of parabolas. The precise definitions of a parabola and the axis of symmetry of a parabola are given here. Figure 1 to the right depicts this definition of a parabola. In this diagram, $FP_1 = P_1Q_1$, $FP_2 = P_2Q_2$, $FP_3 = P_3Q_3$ illustrate that for any point P on the parabola, the distance between P and F is equal to the distance between P and the line L along a segment perpendicular to L.

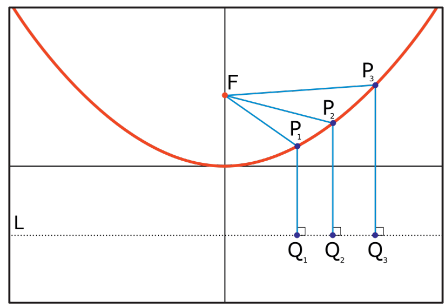


Figure 1

Parabola: G-GPE.A.2 A *parabola with directrix L and focus F* is the set of all points in the plane that are equidistant from the point F and line L.

Axis of Symmetry of a Parabola: G-GPE.A.2 The *axis of symmetry of a parabola* given by a focus point and a directrix is the perpendicular line to the directrix that passes through the focus.

Vertex of a Parabola: G-GPE.A.2 The *vertex of a parabola* is the point where the axis of symmetry intersects the parabola.

This lesson focuses on deriving the analytic equation for a parabola given the focus and directrix (**G.GPE.A.2**) and showing that it is a quadratic equation. In doing so, students are able to tie together many powerful ideas from geometry and algebra, including transformations, coordinate geometry, polynomial equations, the Pythagorean Theorem, and functions.

Parabolas all have the reflective property illustrated in Figure 2. Rays entering the parabola parallel to the axis of symmetry will reflect off the parabola and pass through the focus point F. A Newtonian telescope uses this property of parabolas.

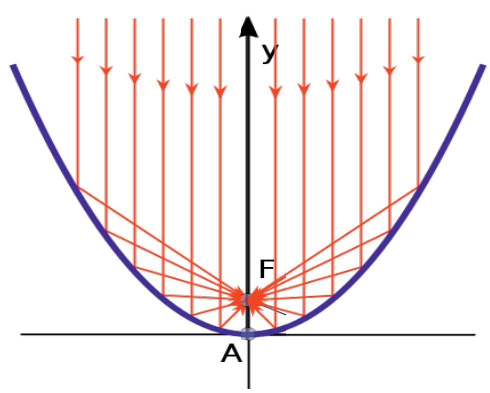


Figure 2

Parabolas have been studied by mathematicians since at least the 4th century B.C. James Gregory's *Optical Promata*, printed in 1663, contains the first known plans for a reflecting telescope using parabolic mirrors, though the idea itself was discussed earlier by many astronomers and mathematicians, including Galileo Galilei, as early as 1616. Isaac Newton, for whom the telescope is now named, needed such a telescope to prove his theory that white light is made up of a spectrum of colors. This theory explained why earlier telescopes that worked by refraction distorted the colors of objects in the sky. However, the technology did not exist at the time to accurately construct a parabolic mirror because of difficulties accurately engineering the curve of the parabola. In 1668, he built a reflecting telescope using a spherical mirror instead of a parabolic mirror, which distorted images but made the construction of the telescope possible. Even with the image distortion caused by the spherical mirror, Newton was able to see the moons of Jupiter without color distortion. Around 1721, John Hadley constructed the first reflecting telescope that used a parabolic mirror.

A Newtonian telescope reflects light back into the tube and requires a second mirror to direct the reflected image to the eyepiece. In a modern Newtonian telescope, the primary mirror is a paraboloid—the surface obtained by rotating a parabola around its axis of symmetry—and a second flat mirror positioned near the focus reflects the image directly to the eyepiece mounted along the side of the tube. A quick image search of the internet will show you simple diagrams of these types of telescopes. This type of telescope remains a popular design today and many amateur astronomers build their own Newtonian telescopes. The diagram shown in the student pages is adapted from this image which can be accessed at http://en.wikipedia.org/wiki/File:Newton01.png#filelinks.

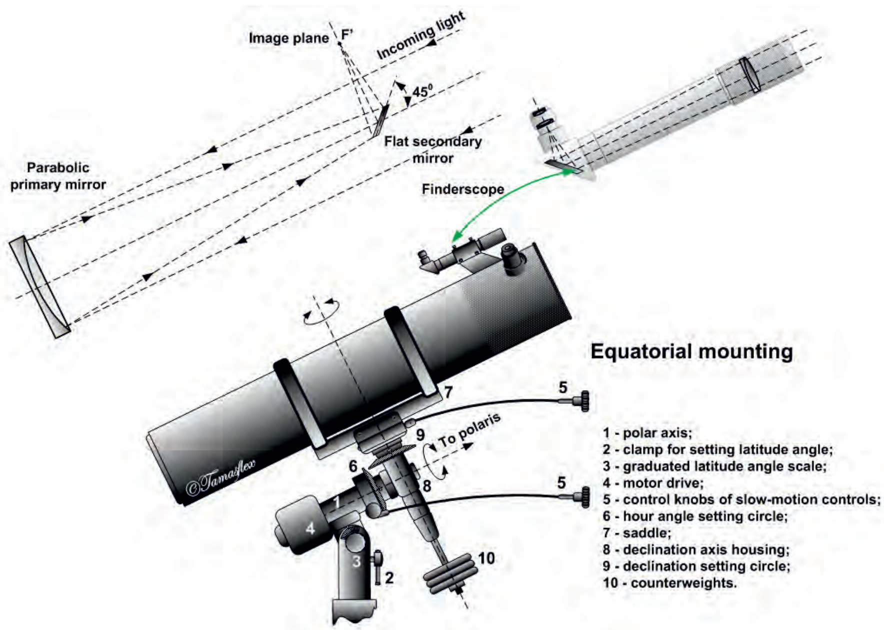


Image: Szöcs Tamás Tamasflex

Classwork

Opening (3 minutes)

The Opening Exercise below gets students thinking about reflections on different shaped lines and curves. From physics, we know that the measure of the angle of reflection of a ray of light is equal to the measure of the angle of incidence when it is bounced off a flat surface. For light reflecting on a curved surface, we measure the angles using the ray of light and the line tangent to the curve where the light ray touches the curve. As described below in the scaffolding box, we mainly want students to understand the shape of the mirror will result in different reflected images. After giving students a few minutes to work on this, ask for their ideas.

- How does each mirror reflect the light?
  - *Mirror 1 bounces light straight back at you.*
  - *Mirror 2 bounces light at a $90°$ angle across to the other side of the mirror, then it bounces at another $90°$ angle to away from the mirror.*
  - *Mirror 3 would bounce the light at different angles at different points because the surface is curved.*

Some background information that can help you process the Opening Exercise with students is summarized below.

- Semicircular mirrors do not send all rays of light to a single focus point; this fact can be seen by carefully drawing the path of three rays of light and noticing that they each intersect the other rays in different points after they reflect.
- From physics, the angle of reflection is congruent to the angle of incidence with a line tangent to the curve. On a curved mirror, the slope of the tangent line changes so the rays of light reflect at different angles.
- Remember when working with students to focus on the big ideas: These mirrors will not reflect light back to a single point. Simple student diagrams are acceptable.

After debriefing the opening with your class, introduce the idea of a telescope that uses mirrors to reflect light. We want a curved surface that focuses the incoming light to a single point in order to see reflected images from outer space. The question below sets the stage for this lesson. The rest of the lesson will define a parabola as a curve that meets the requirements of the telescope design.

- Is there a curved shape that will accomplish this goal?

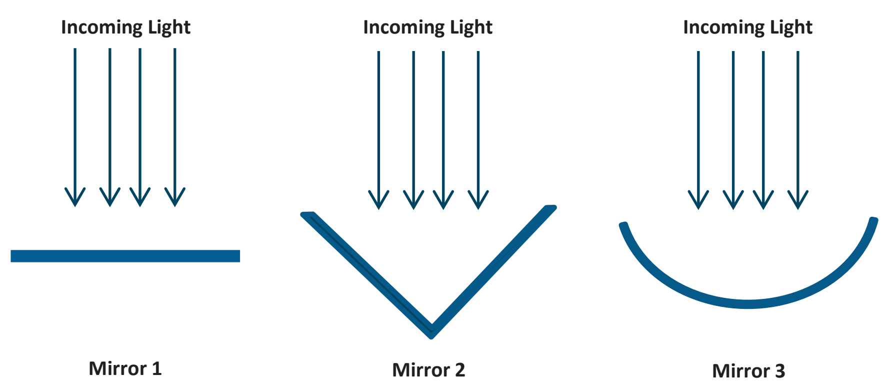
> *Scaffolding:*
>
> - Do not get sidetracked if students struggle to accurately draw the reflected light. Work through these as a class if necessary. Emphasize that the light will be reflected differently as the mirror's curvature changes.

Opening Exercise (2 minutes)

MP.1 & MP.4

Opening Exercise

Suppose you are viewing the cross-section of a mirror. Where would the incoming light be reflected in each type of design? Sketch your ideas below.



In Mirror 1, the light would reflect back onto the light rays. In Mirror 2, the light would reflect from one side of the mirror horizontally to the other side, and then reflect back upwards vertically. Incoming and outgoing rays would be parallel. In Mirror 3, the light would reflect back at different angles because the mirror is curved.

To transition from the Opening Exercise to the next discussion, tell students that telescopes work by reflecting light. If we want to see an image without distortion using a telescope, then we need the reflected light to focus on one point. Mirror 3 comes the closest to having this property but does not reflect the rays of light back to a single point. You can model this by showing a sample of a student solution or by providing a sketch that you created.

Discussion (15 minutes): Telescope Design

Lead a whole class discussion that ties together the definition of a parabola and its reflective property with Newton's telescope design requirements. A Newtonian telescope needs a mirror that will focus all the light on a single point to prevent a distorted image. A parabola by definition meets this requirement. During this discussion, you will share the definition of a parabola and how the focus and directrix give the graph its shape. If the distance between the focus and the directrix changes, the parabola's curvature changes. If we rotate a parabola $180°$ around the axis of symmetry, we get a curved surface called a *paraboloid*; this is the shape of a parabolic mirror. A Newtonian telescope requires a fairly flat mirror in order to see images of objects that are astronomically far away, so he built his mirror based on a parabola with a relatively large distance between its focus and directrix.

Discussion

When Newton designed his reflector telescope he understood two important ideas. Figure 1 shows a diagram of this type of telescope.

- **The curved mirror needs to focus all the light to a single point that we will call the focus. An angled flat mirror is placed near this point and reflects the light to the eyepiece of the telescope.**
- **The reflected light needs to arrive at the focus at the same time, otherwise the image is distorted.**

Figure 1

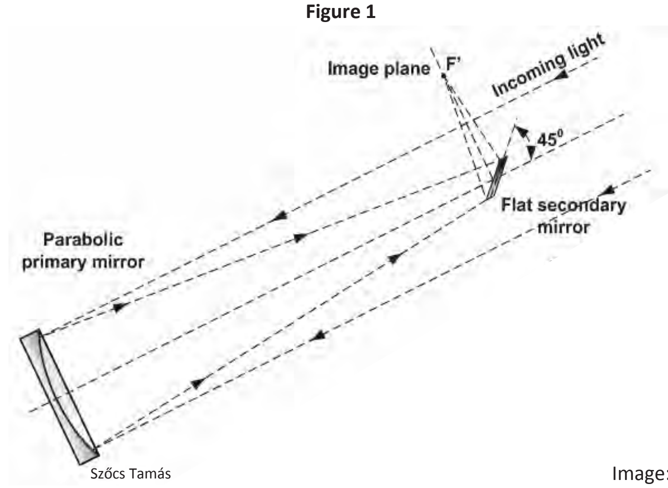


Image: Szöcs Tamás Tamasflex

In the diagram below, the dotted and solid lines show the incoming light. Model how to add these additional lines to the diagram. Make sure students annotate this on their student pages.

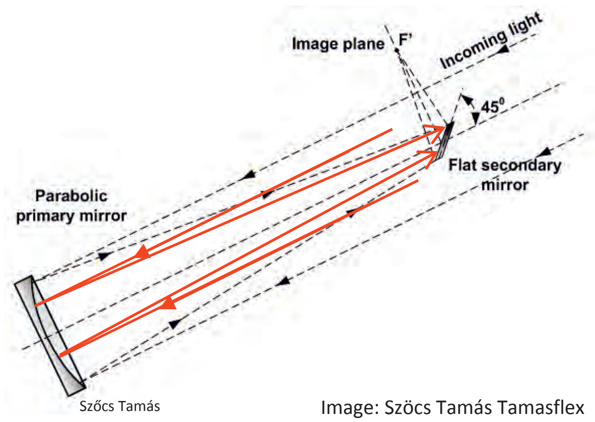


Image: Szöcs Tamás Tamasflex

Next, discuss the definition of parabola that appears in the student pages and the reflective property of parabolic curves. Take time to explain what the term equidistant means and how we define distance between a given point and a given line as the shortest distance, which will always be the length of the segment that lies on a line perpendicular to the given line whose endpoints are the given point and the intersection point of the given line and the perpendicular line. Ask students to recall the definition of a circle from Lessons 30–31 and use this to explore the definition of a parabola. Before reading through the definition, give students a ruler and ask them to measure the segments $\overline{FP_1}$, $\overline{Q_1P_1}$, $\overline{FP_2}$, $\overline{Q_2P_2}$, etc., in Figure 2. Then have them locate a few more points on the curve and measure the distance from the curve to point F and from the curve to the horizontal line L.

Definition: A *__parabola__ with __directrix__* L *and __focus point__* F is the set of all points in the plane that are equidistant from the point F and line L.

Figure 2 to the right illustrates this definition of a parabola. In this diagram, $FP_1 = P_1Q_1$, $FP_2 = P_2Q_2$, $FP_3 = P_3Q_3$ showing that for any point P on the parabola, the distance between P and F is equal to the distance between P and the line L.

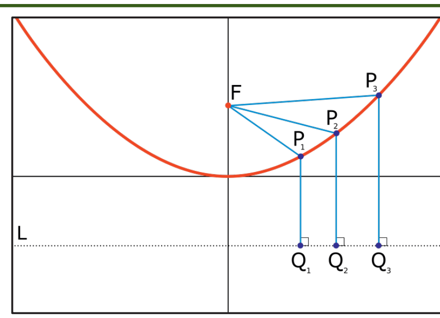


Figure 2

All parabolas have the reflective property illustrated in Figure 3 Rays parallel to the axis will reflect off the parabola and through the focus point, F.

Thus, a mirror shaped like a rotated parabola would satisfy Newton's requirements for his telescope design.

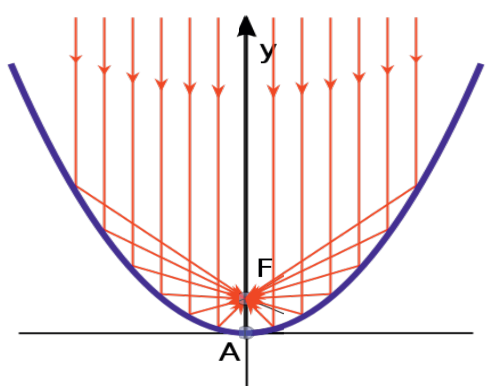


Figure 3

Then, move on to Figures 4 and 5. Here we transition back to thinking about the telescope and show how a mirror in the shape of a parabola (as opposed to say a semi-circle or other curve) will reflect light to the focus point. You could talk about fun house mirrors (modeled by some smartphone apps) that distort images as an example of how other curved surfaces reflect light differently.

- If we want the light to be reflected to the focus at exactly the same time, then what must be true about the distances between the focus and any point on the mirror and the distances between the directrix and any point on the mirror?
  - *Those distances must be equal.*

Scaffolding:

- To help students master the new vocabulary associated with parabolas, make a poster using the diagram shown below and label the parts. Adjust as needed for your classes but make sure to include the focus point and directrix in your poster along with marked congruent segments illustrating the definition.

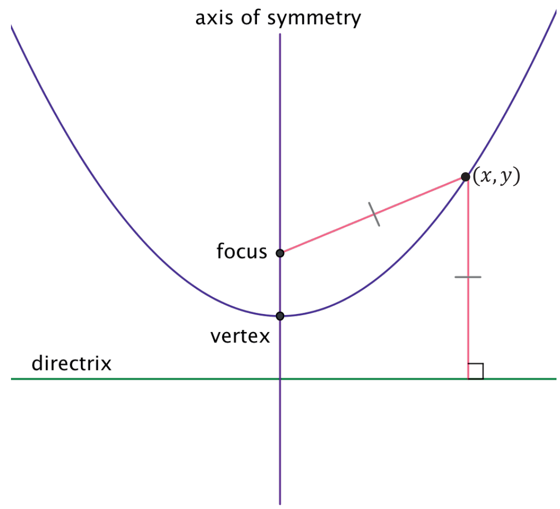


Figure 4 below shows several different line segments representing the reflected light with one endpoint on the curved mirror that is a parabola and the other endpoint at the focus. Anywhere the light hits this type of curved surface, it always reflects to the focus, F, at exactly the same time.

Figure 5 shows the same image with a directrix. Imagine for a minute that the mirror was not there. Then, the light would arrive at the directrix all at the same time. Since the distance from each point on the parabolic mirror to the directrix is the same as the distance from the point on the mirror to the focus, and the speed of light is constant, it takes the light the same amount of time to travel to the focus as it would have taken it to travel to the directrix. In the diagram, this means that $AF = AF_A$, $BF = BF_B$, and so on. Thus, the light rays arrive at the focus at the same time, and the image is not distorted.

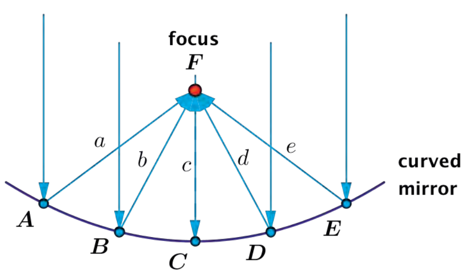


Figure 4

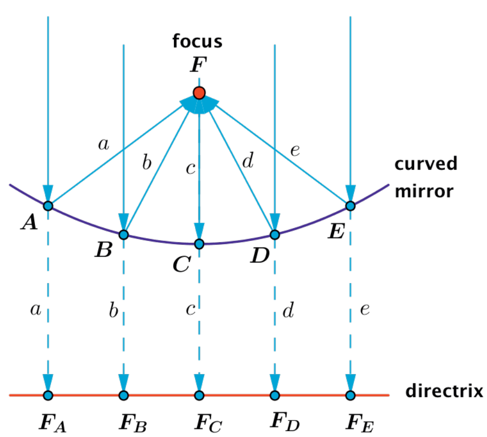


Figure 5

To further illustrate the definition of a parabola, ask students to mark on Figure 5 how the lengths AF, BF, CF, DF, and EF are equal to the lengths AF_A, BF_B, CF_C, DF_D, and EF_E, respectively.

- How does this definition fit the requirements for a Newtonian telescope?
  - *The definition states exactly what we need to make the incoming light hit the focus at the exact same time since the distance between any point on the curve to the directrix is equal to the distance between any point on the curve and the focus.*
- A parabola looks like the graph of what type of function?
  - *It looks like the graph of an even degree polynomial function.*

Transition to Example 1 by announcing that the prediction that an equation for a parabola would be a quadratic equation will be confirmed using a specific example.

Scaffolding:

- Provide some additional practice with using distance formula to find the length of a line segment for struggling learners.
- Post the distance formula on the wall in your classroom.
- Draw a diagram with the points $F(0,2)$, $F'(4,0)$, and $A(4,5)$ labeled. Have them find the lengths of FA and $F'A$.

Example 1 (13 minutes): Finding an Analytic Equation for a Parabola

This example derives an equation for a parabola given the focus and directrix. As you work through this example, give students time to record the steps along with you. Refer students back to the way the distance formula was used in the definition of a circle and explain we can use it here as well to find an analytic equation for this type of curve.

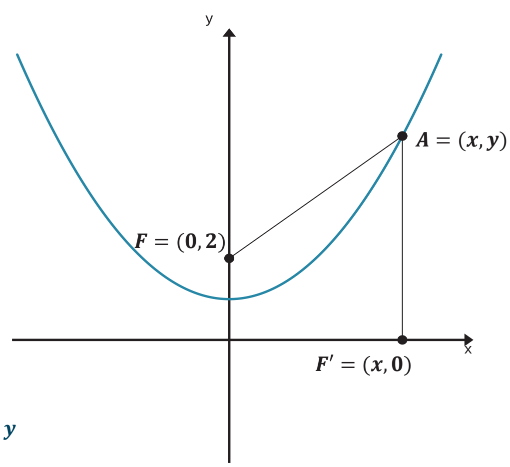
Example 1

Given a focus and a directrix, create an equation for a parabola.

Focus: $F = (0, 2)$

Directrix: x**-axis**

Parabola: $P = \{(x, y) \mid$
(x, y) **is equidistant to** F **and to the** x**-axis.**$\}$



Let A **be any point** (x, y) **on the parabola** P**. Let** F' **be a point on the directrix with the same** x**-coordinate as point** A**.**

What is the length of AF'**?**

$$AF' = y$$

Use the distance formula to create an expression that represents the length of AF**.**

$$AF = \sqrt{(x-0)^2 + (y-2)^2}$$

Create an equation that relates the two lengths and solve it for y.

Therefore, $P = \left\{(x,y) \mid \sqrt{(x-0)^2+(y-2)^2} = y\right\}$.

The two segments have equal lengths. $AF' = AF$

The length of each segment. $y = \sqrt{(x-0)^2+(y-2)^2}$

Square both sides of the equation. $y^2 = x^2 + (y-2)^2$

Expand the binomial. $y^2 = x^2 + y^2 - 4y + 4$

Solve for y. $4y = x^2 + 4$

$y = \frac{1}{4}x^2 + 1$

Replacing this equation in the definition of $P = \{(x,y) \mid (x,y)$ is equidistant to F and to the x-axis$\}$ gives the statement $P = \left\{(x,y) \mid y = \frac{1}{4}x^2 + 1\right\}$.

Thus, the parabola P is the graph of the equation $y = \frac{1}{4}x^2 + 1$.

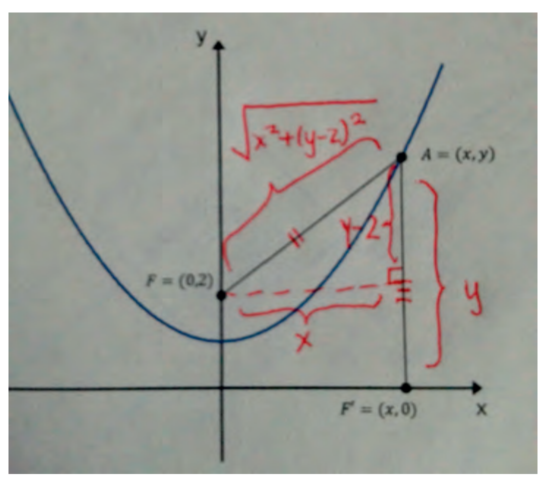
Verify that this equation appears to match the graph shown.

Consider the point where the y-axis intersects the parabola; let this point have coordinates $(0,b)$. From the graph, we see that $0 < b < 2$. The distance from the focus $(0,2)$ to $(0,b)$ is $2-b$ units, and the distance from the directrix to $(0,b)$ is b units. Since $(0,b)$ is on the parabola, we have $2-b=b$, so that $b=1$. From this perspective, we see that the point $(0,1)$ must be on the parabola. Does this point satisfy the equation we found? Let $x=0$. Then our equation gives $y = \frac{1}{4}x^2 + 1 = \frac{1}{4}(0)^2 + 1$, so $(0,1)$ satisfies the equation. This is the only point that we have determined to be on the parabola at this point, but it provides evidence that the equation matches the graph.

Use these questions below as you work through Example 1. Have students mark the congruent segments on their diagram and record the derivation of the equation as you work it out in front of the class. As you are working through this example, you can remind students that when we work with the distance formula we are applying the Pythagorean Theorem in the coordinate plane. This refers back to their work in both Grade 8 and high school Geometry.

- According to the definition of a parabola, which two line segments in the diagram must have equal measure? Mark them congruent on your diagram.
  - *AF must be equal to AF'.*
- How long is AF'? How do you know?
  - *It is y units long. The y-coordinate of point A is y.*
- Recall the distance formula, and use it to create an expression equal to the length of $\overline{AF}$.
  - *The distance formula is $D = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$, where (x_1,y_1) and (x_2,y_2) are two points in the Cartesian plane.*
- How can you tell if this equation represents a quadratic function?
  - *The degree of x will be 2 and the degree of y will be 1 and each x will correspond to exactly one y.*

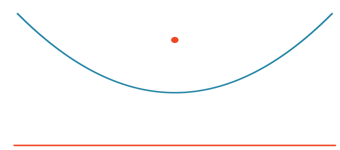
A marked up diagram is shown to the right.



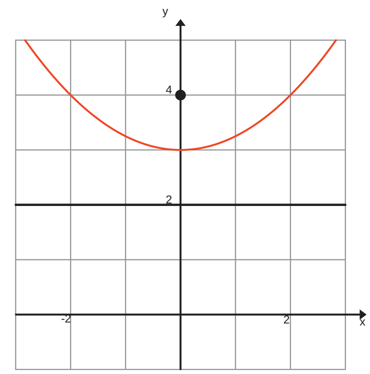
Exercises 1–2 (4 minutes)

Exercises 1–2

1. **Demonstrate your understanding of the definition of a parabola by drawing several pairs of congruent segments given the parabola, its focus, and directrix. Measure the segments that you drew to confirm the accuracy of your sketches in either centimeters or inches.**



2. **Derive the analytic equation of a parabola given the focus of $(0, 4)$ and the directrix $y = 2$. Use the diagram to help you work this problem.**



   a. **Label a point (x, y) anywhere on the parabola.**

   b. **Write an expression for the distance from the point (x, y) to the directrix.**

   $$y - 2$$

   c. **Write an expression for the distance from the point (x, y) to the focus.**

   $$\sqrt{(x-0)^2 + (y-4)^2}$$

   d. **Apply the definition of a parabola to create an equation in terms of x and y. Solve this equation for y.**

   $$y - 2 = \sqrt{(x-0)^2 + (y-4)^2}$$

   Solved for y, we find an equivalent equation is $y = \frac{1}{4}x^2 + 3$.

   e. **What is the translation that takes the graph of this parabola to the graph of the equation derived in Example 1?**

   A translation down two units will take this graph of this parabola to the one derived in Example 1.

Closing (3 minutes)

In this lesson, we limit the discussion to parabolas with a horizontal directrix. We will show in later lessons that all parabolas are similar and that the equations are quadratic regardless of the orientation of the parabola in the plane. Have students answer these questions individually in writing, then discuss their responses as a whole class.

- What is a parabola?
  - *A parabola is a geometric figure that represents the set of all points equidistant from a point called the focus and a line called the directrix.*
- Why are parabolic mirrors used in telescope designs?
  - *Parabolic mirrors are used in telescope designs because they focus reflected light to a single point.*
- What type of analytic equation can be used to model parabolas?
  - *The analytic equation of a parabola whose directrix is a horizontal line is quadratic in x.*

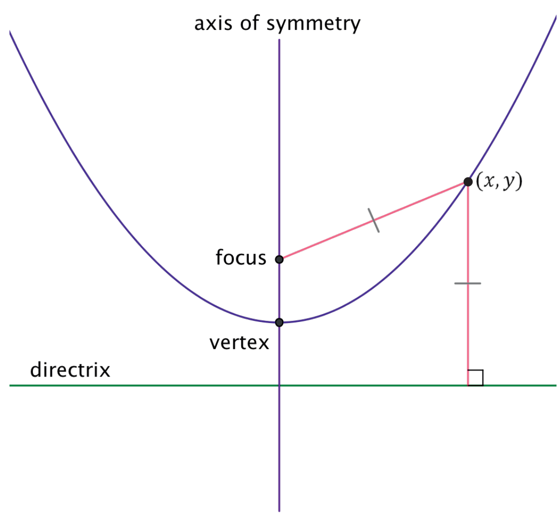
Lesson Summary

Parabola: A ***parabola*** with ***directrix line*** L and ***focus point*** F is the set of all points in the plane that are equidistant from the point F and line L.

Axis of symmetry: The ***axis of symmetry of a parabola*** given by a focus point and a directrix is the perpendicular line to the directrix that passes through the focus.

Vertex of a parabola: The ***vertex of a parabola*** is the point where the axis of symmetry intersects the parabola.

In the Cartesian plane, the distance formula can help us to derive an analytic equation for the parabola.



Exit Ticket (5 minutes)

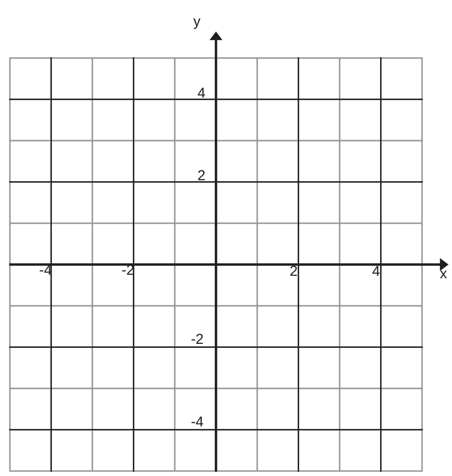
Name ______________________________ Date ______________

Lesson 33: The Definition of a Parabola

Exit Ticket

1. Derive an analytic equation for a parabola whose focus is $(0,4)$ and directrix is the x-axis. Explain how you got your answer.

2. Sketch the parabola from Question 1. Label the focus and directrix.



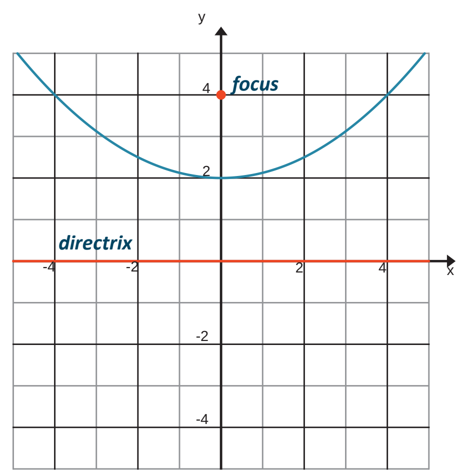
Exit Ticket Sample Solutions

1. **Derive an analytic equation for a parabola whose focus is $(0, 4)$ and directrix is the x-axis. Explain how you got your answer.**

Let (x, y) be a point on the parabola. Then, the distance between this point and the focus is given by $\sqrt{(x-0)^2 + (y-4)^2}$. The distance between the point (x, y) and the directrix is y. Then,

$$y = \sqrt{(x-0)^2 + (y-4)^2}$$
$$y^2 = x^2 + y^2 - 8y + 16$$
$$8y = x^2 + 16$$
$$y = \frac{1}{8}x^2 + 2$$

2. **Sketch the parabola from Question 1. Label the focus and directrix.**



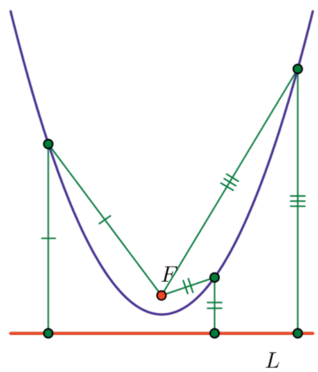
Problem Set Sample Solutions

These questions are designed to reinforce the ideas presented in this session. The first few questions focus on applying the definition of a parabola to sketch parabolas. Then the questions scaffold to creating an analytic equation for a parabola given its focus and directrix. Finally, questions near the end of the Problem Set help students to recall transformations of graphs of functions to prepare them for work in future lessons on proving when parabolas are congruent and that all parabolas are similar.

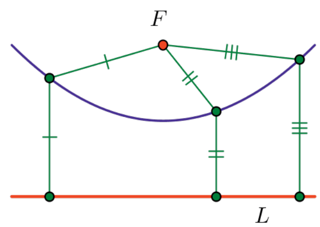
1. **Demonstrate your understanding of the definition of a parabola by drawing several pairs of congruent segments given each parabola, its focus, and directrix. Measure the segments that you drew in either inches or centimeters to confirm the accuracy of your sketches.**

   Measurements will depend on the location of the segments and the size of the printed document. Segments that should be congruent should be close to the same length.

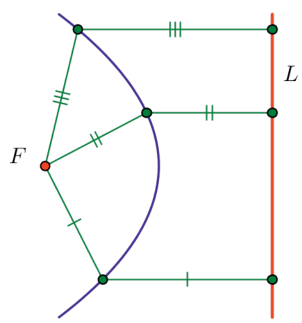
   a.



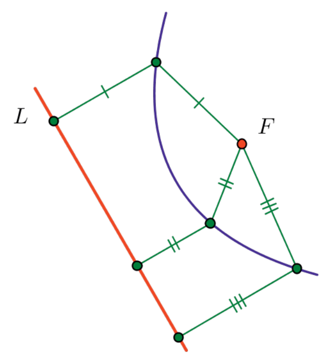
   b.



   c.



   d.



2. **Find the distance from the point $(4, 2)$ to the point $(0, 1)$.**

   The distance is $\sqrt{17}$ units.

3. **Find the distance from the point $(4, 2)$ to the line $y = -2$.**

   The distance is 4 units.

4. **Find the distance from the point $(-1, 3)$ to the point $(3, -4)$.**

   The distance is $\sqrt{65}$ units.

5. Find the distance from the point $(-1, 3)$ to the line $y = 5$.

The distance is 2 *units.*

6. Find the distance from the point $(x, 4)$ to the line $y = -1$.

The distance is 5 *units.*

7. Find the distance from the point $(x, -3)$ to the line $y = 2$.

The distance is 5 *units.*

8. Find the values of x for which the point $(x, 4)$ is equidistant from $(0, 1)$ and the line $y = -1$.

If $\sqrt{(x-0)^2 + (4-1)^2} = 5$*, then* $x = 4$ *or* $x = -4$.

9. Find the values of x for which the point $(x, -3)$ is equidistant from $(1, -2)$ and the line $y = 2$.

If $\sqrt{(x-1)^2 + \left(-3-(-2)\right)^2} = 5$*, then* $x = 1 + 2\sqrt{6}$ *or* $x = 1 - 2\sqrt{6}$.

10. Consider the equation $y = x^2$.

   a. Find the coordinates of the three points on the graph of $y = x^2$ whose x-values are 1, 2, and 3.

   The coordinates are $(1, 1)$, $(2, 4)$, *and* $(3, 9)$.

   b. Show that each of the three points in part (a) is equidistant from the point $\left(0, \frac{1}{4}\right)$ and the line $y = -\frac{1}{4}$.

   For $(1, 1)$*, show that*

$$\sqrt{(1-0)^2 + \left(1 - \frac{1}{4}\right)^2} = \sqrt{1 + \frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$$

   and

$$1 - \left(-\frac{1}{4}\right) = \frac{5}{4}.$$

   For $(2, 4)$*, show that*

$$\sqrt{(2-0)^2 + \left(4 - \frac{1}{4}\right)^2} = \sqrt{4 + \frac{225}{16}} = \sqrt{\frac{289}{16}} = \frac{17}{4}$$

   and

$$4 - \left(-\frac{1}{4}\right) = \frac{17}{4}.$$

   For $(3, 9)$*, show that*

$$\sqrt{(3-0)^2 + \left(9 - \frac{1}{4}\right)^2} = \sqrt{9 + \left(\frac{35}{4}\right)^2} = \sqrt{\frac{1369}{16}} = \frac{37}{4}$$

   and

$$9 - \left(-\frac{1}{4}\right) = \frac{37}{4}.$$

c. Show that if the point with coordinates (x, y) is equidistant from the point $\left(0, \frac{1}{4}\right)$, and the line $y = -\frac{1}{4}$, then $y = x^2$.

The distance from (x, y) to $\left(0, \frac{1}{4}\right)$ is $\sqrt{(x-0)^2 + \left(y - \frac{1}{4}\right)^2}$, and the distance from (x, y) to the line $y = -\frac{1}{4}$ is $y - \left(-\frac{1}{4}\right) = y + \frac{1}{4}$. Setting these distances equal gives

$$\sqrt{(x-0)^2 + \left(y - \frac{1}{4}\right)^2} = y + \frac{1}{4}$$

$$\sqrt{x^2 + y^2 - \frac{1}{2}y + \frac{1}{16}} = y + \frac{1}{4}$$

$$x^2 + y^2 - \frac{1}{2}y + \frac{1}{16} = \left(y + \frac{1}{4}\right)^2$$

$$x^2 - \frac{1}{2}y + \frac{1}{16} = \frac{1}{2}y + \frac{1}{16}$$

$$x^2 = y$$

Thus, if a point (x, y) is the same distance from the point $\left(0, \frac{1}{4}\right)$ and the line $y = -\frac{1}{4}$, then (x, y) lies on the parabola $y = x^2$.

11. Given the equation $y = \frac{1}{2}x^2 - 2x$,

a. Find the coordinates of the three points on the graph of $y = \frac{1}{2}x^2 - 2x$ whose x-values are -2, 0, and 4.

The coordinates are $(-2, 6)$, $(0, 0)$, $(4, 0)$.

b. Show that each of the three points in part (a) is equidistant from the point $\left(2, -\frac{3}{2}\right)$, and the line $y = -\frac{5}{2}$.

For $(-2, 6)$, show that

$$\sqrt{(-2-2)^2 + \left(6 - \left(-\frac{3}{2}\right)\right)^2} = \sqrt{16 + \frac{225}{4}} = \sqrt{\frac{289}{4}} = \frac{17}{2}$$

and

$$6 - \left(-\frac{5}{2}\right) = \frac{17}{2}.$$

For $(0, 0)$, show that

$$\sqrt{(0-2)^2 + \left(0 - \left(-\frac{3}{2}\right)\right)^2} = \sqrt{4 + \frac{9}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

and

$$0 - \left(-\frac{5}{2}\right) = \frac{5}{2}.$$

For $(4, 0)$, show that

$$\sqrt{(4-2)^2 + \left(0 - \left(-\frac{3}{2}\right)\right)^2} = \sqrt{4 + \frac{9}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

and

$$0 - \left(-\frac{5}{2}\right) = \frac{5}{2}.$$

c. Show that if the point with coordinates (x, y) is equidistant from the point $\left(2, -\frac{3}{2}\right)$ and the line $y = -\frac{5}{2}$ then $y = \frac{1}{2}x^2 - 2x$.

The distance from (x, y) to $\left(2, -\frac{3}{2}\right)$ is $\sqrt{(x-2)^2 + \left(y + \frac{3}{2}\right)^2}$, and the distance from (x, y) to the line $y = -\frac{5}{2}$ is $y - \left(-\frac{5}{2}\right) = y + \frac{5}{2}$. Setting these distances equal gives

$$\sqrt{(x-2)^2 + \left(y + \frac{3}{2}\right)^2} = y + \frac{5}{2}$$

$$\sqrt{x^2 - 4x + y^2 + 3y + \frac{25}{4}} = y + \frac{5}{2}$$

$$x^2 - 4x + y^2 + 3y + \frac{25}{4} = y^2 + 5y + \frac{25}{4}$$

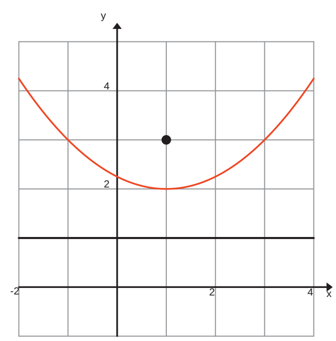
$$x^2 - 4x = 2y$$

$$\frac{1}{2}(x^2 - 2x) = y$$

Thus, if a point (x, y) is the same distance from the point $\left(2, -\frac{3}{2}\right)$, and the line $y = -\frac{5}{2}$, then (x, y) lies on the parabola $y = \frac{1}{2}(x^2 - 2x)$.

12. Derive the analytic equation of a parabola with focus $(1, 3)$ and directrix $y = 1$. Use the diagram to help you work this problem.

a. Label a point (x, y) anywhere on the parabola.



b. Write an expression for the distance from the point (x, y) to the directrix.

$$y - 1$$

c. Write an expression for the distance from the point (x, y) to the focus $(1, 3)$.

$$\sqrt{(x-1)^2 + (y-3)^2}$$

d. Apply the definition of a parabola to create an equation in terms of x and y. Solve this equation for y.

$$y - 1 = \sqrt{(x-1)^2 + (y-3)^2}$$

$$(y-1)^2 = (x-1)^2 + (y-3)^2$$

$$y^2 - 2y + 1 = (x-1)^2 + y^2 - 6y + 9$$

$$4y = (x-1)^2 + 8$$

$$y = \frac{1}{4}(x-1)^2 + 2$$

$$y = \frac{1}{4}x^2 - \frac{1}{2}x + \frac{9}{4}$$

e. Describe a sequence of transformations that would take this parabola to the parabola with equation $y = \frac{1}{4}x^2 + 1$ derived in Example 1.

A translation 1 unit to the left and 1 unit downward will take this parabola to the one derived in Example 1.

13. Consider a parabola with focus $(0, -2)$ and directrix on the x-axis.

a. Derive the analytic equation for this parabola.

$$y = -\frac{1}{4}x^2 - 1$$

b. Describe a sequence of transformations that would take the parabola with equation $y = \frac{1}{4}x^2 + 1$ derived in Example 1 to the graph of the parabola in part (a).

Reflect the graph in Example 1 across the x-axis to obtain this parabola.

14. Derive the analytic equation of a parabola with focus $(0, 10)$ and directrix on the x-axis.

$$y = \frac{1}{20}x^2 + 5$$

Lesson 34: Are All Parabolas Congruent?

Student Outcomes

- Students learn the vertex form of the equation of a parabola and how is arises from the definition of a parabola.
- Students perform geometric operations, such as rotations, reflections, and translations, on arbitrary parabolas to discover standard representations for their congruence classes. In doing so, they learn that all parabolas with the same distance p between the focus and the directrix are congruent to the graph of $y = \frac{1}{2p}x^2$.

Lesson Notes

This lesson builds upon the previous lesson and applies transformations to show that all parabolas with the same distance between their focus and directrix are congruent. Recall that two figures in the plane are congruent if there exists a finite sequence of rigid motions that maps one onto the other, so it makes sense for us to discuss congruency of parabolas. The lesson closes with a theorem and proof detailing the answer to the question posed in the lesson title. By using transformations in this lesson to determine the conditions under which two parabolas are congruent, this lesson builds coherence with the work students did in Geometry. This lesson specifically asks students to consider how we can use transformations to prove two figures are congruent. Additionally, the lesson reinforces the connections between geometric transformations and transformations of the graphs of functions.

There are many opportunities to provide scaffolding in this lesson for students who are not ready to move quickly to abstract representation. Use technology, patty paper or transparencies, and simple hand-drawn graphs as appropriate to support student learning throughout this lesson. Use the anchor poster created in Lesson 33, and keep key vocabulary words and formulas (e.g., the distance formula) displayed for student reference.

You might choose to break this lesson up into two days; on the first day, explore the definition of congruent parabolas, sketch parabolas given their focus and directrix, and explore the consequences of changing the distance between the focus and directrix, p. On the second day, derive the analytic equation for a parabola with a given focus and directrix and vertex at the origin, and prove the theorem on parabola congruence.

Classwork

Opening Exercise (7 minutes)

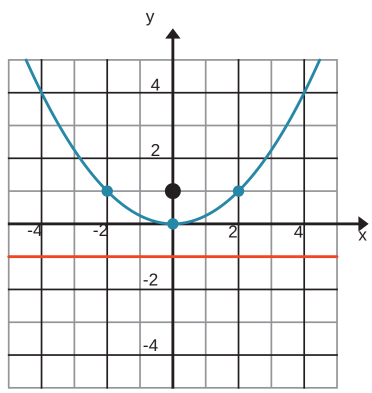
Allow students to discuss their approaches to this exercise with a partner or in small groups. Keep encouraging students to consider the definition of a parabola as they try to sketch the parabolas. Encourage students who draw a haphazard curve to consider how they could make sure their graph is the set of points equidistant from the focus and directrix. Throughout this lesson, provide students with access to graphing calculators or other computer graphing software that they can use to test and confirm conjectures. You can further model this and provide additional scaffolding by using an online applet located at the website http://www.intmath.com/plane-analytic-geometry/parabola-interactive.php. Within the applet, students can move either the focus or the directrix to change the value of p. They can also slide a point along the parabola noting the equal distances between that point and the focus and that point and the directrix. Depending on the level of your students, you can begin these exercises by moving directly to the applet or by having them start the exercises and use the applet later in this section.

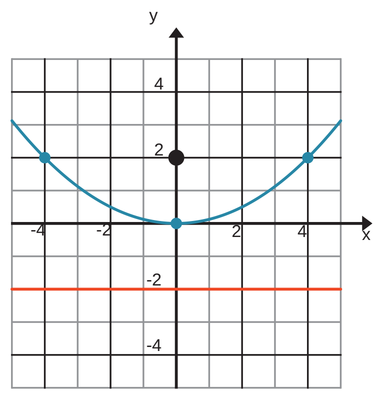
Opening Exercise

Are all parabolas congruent? Use the following questions to support your answer.

a. **Draw the parabola for each focus and directrix given below.**

The solution is shown below.





b. **What do we mean by congruent parabolas?**

Two parabolas would be congruent if we could find a sequence of rigid motions that takes one parabola onto the other. We could translate the vertex of the first one onto the vertex of the second one, then rotate the image of the first one so that the directrices are parallel and both parabolas open in the same direction. If the transformed first parabola coincides with the original second parabola, then the two original parabolas are congruent.

c. **Are the two parabolas from part (a) congruent? Explain how you know.**

These two parabolas are not congruent. They have the same vertex but different y-values for each x in the domain, except for the point $(0,0)$. If we translate the first one somewhere else, then the vertices will not align. If we reflect or rotate, then both parabolas will not open upward. There is no rigid transformation or set of transformations that takes the graph of one parabola onto the other.

MP.3

d. **Are all parabolas congruent?**

No. We just found two that are not congruent.

e. **Under what conditions might two parabolas be congruent? Explain your reasoning.**

Once we align the vertices and get the directrices parallel and parabolas opening in the same direction through rotation or reflection, the parabolas will have the same shape if the focus and directrix are aligned. Thus, two parabolas will be congruent if they have the same distance between the focus and directrix.

Scaffolding:

- If students struggle with vocabulary, refer them back to the anchor poster.
- As an extension, ask students to derive the equation for the parabola as was shown in Example 1 of the previous lesson.

Debrief this exercise using the following questions, which can also be used as scaffolds if students are struggling to begin this problem. During the debrief, record student thinking on chart paper to be used for reference at the end of this lesson as a means to confirm or refute their conjectures.

At some point in this discussion, students should recognize that it would be nice if we had a name for the distance between the focus and directrix of the parabola. When appropriate, let them know that this distance will be denoted by p in this lesson and lessons that follow. If they do not mention it here, it will be brought into the discussion in Exercise 5.

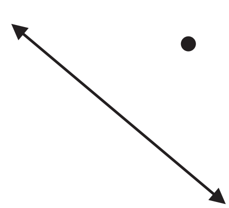
- How can you use the definition of a parabola to quickly locate at least three points on the graph of the parabola with a focus $(0,1)$ and directrix $y = -1$?
  - *Along the y-axis, the distance between the focus and the directrix is 2, so one point, the vertex, will be halfway between them at $(0,0)$. Since the distance between the focus and the directrix is $p = 2$ units, if we go 2 units to the left and right of the focus, we will be able to locate two more points on the parabola at $(2,1)$ and $(-2,1)$.*
- How can you use the definition of a parabola to quickly locate at least three points on the graph of the parabola with a focus $(0,2)$ and directrix $y = -2$?
  - *The distance between the focus and the directrix is $p = 4$ units, so the vertex is halfway between the focus and directrix at $(0,0)$. Moving right and left 4 units from the focus gives points $(4,2)$ and $(-4,2)$.*
- Generalize this process of finding three points on a parabola with given focus and directrix.
  - *Find the vertex halfway between the focus and directrix, and let p be the distance between the focus and directrix. Then, sketch a line through the focus parallel to the directrix. Locate the two points p units along that line in either direction from the focus. The vertex and these two points are all on the parabola.*
- How does this process help us determine if two parabolas are congruent?
  - *Two parabolas will be congruent if the two points found through this process are the same distance away from the focus. That is, two parabolas will be congruent if they have the same value of p, or the same distance between the focus and directrix.*

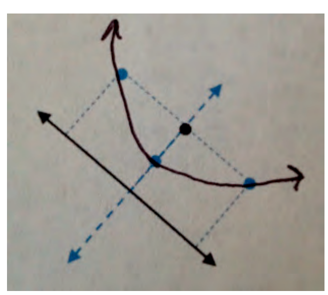
Exercises 1–5 (5 minutes)

Students practice drawing parabolas given a focus and a directrix. These parabolas are NOT all oriented with a horizontal directrix. Let students struggle with how to construct a sketch of the parabola. Remind them of their work in the Opening Exercise. You may choose to do one problem together to model the process. Draw a line perpendicular to the directrix through the focus. Locate the midpoint of the segment connecting the focus and directrix. Then, create a square on either side of this line with side length equal to the distance between the focus and the directrix. One vertex of each square that is not on the directrix or axis of symmetry is another point on the parabola. If you have dynamic geometry software, students could also model this construction using technology.

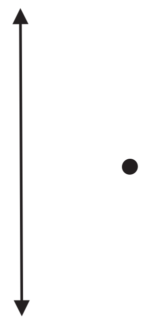
Exercises 1–5

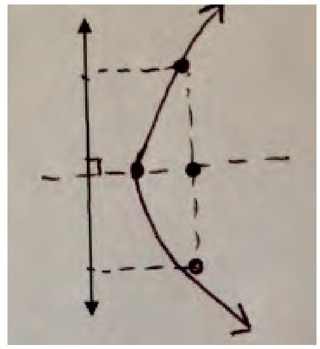
1. **Draw the parabola with the given focus and directrix.**



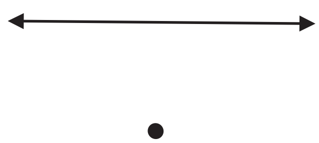


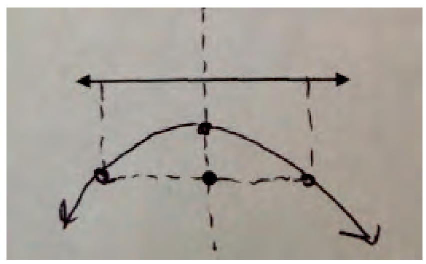
2. **Draw the parabola with the given focus and directrix.**





3. **Draw the parabola with the given focus and directrix.**





Scaffolding:

Use sentence frames for this exercise to support English Language Learners.

For example, in Exercise 5, use

"As the value of p grows, the parabola _______________."

Later in the lesson, use

"All parabolas with the same distance between the ____ and ____ of the parabola are congruent."

Give students time to work through these exercises alone or in small groups. Then, have a few students present their approaches on the board. Be sure to emphasize that by applying the definition, we can produce a fairly accurate sketch of a parabola.

- What two geometric objects determine the set of points that forms a parabola?
  - *A point called the focus and a line called the directrix.*

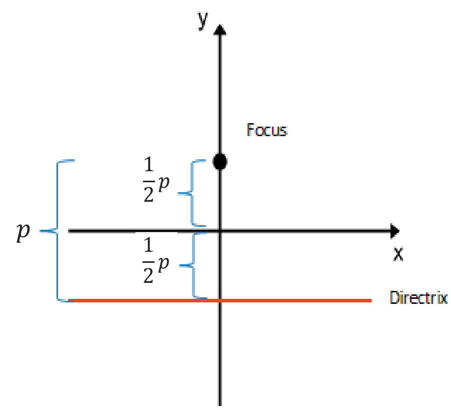
Direct students to compare and contrast Exercises 1–3 and discuss the implications within their small groups.

4. **What can you conclude about the relationship between the parabolas in Exercises 1–3?**

   The parabolas are all the same size and shape because the distance between the focus and the directrix stayed the same. These parabolas should be congruent.

Direct students attention to the diagram in Exercise 5. Have students respond individually and then discuss in small groups.

5. **Let p be the number of units between the focus and the directrix, as shown. As the value of p increases, what happens to the shape of the resulting parabola?**



As the value of p increases, the graph is dilated and shrinks vertically compared to graphs with a smaller value of p. It appears to get flatter.

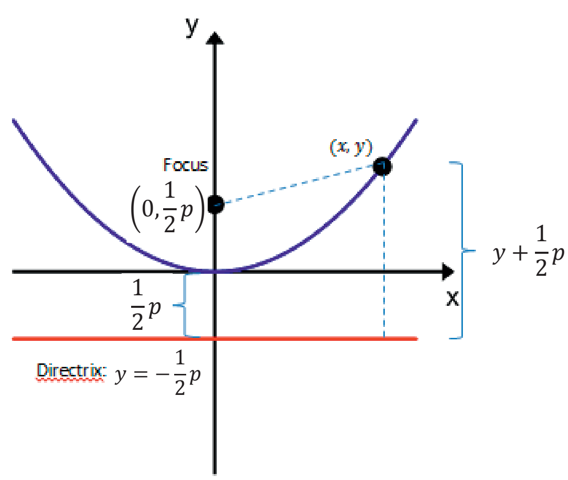
Example 1 (12 minutes): Derive an Equation for a Parabola

In this example, lead students through the process of creating an equation that represents a parabola with horizontal directrix, vertex at the origin, and the distance between the focus and directrix $p > 0$. This process will be similar to the work done in yesterday's lesson, except we are working with a general case instead of a specified value for p. Scaffolding may be necessary for students who are not ready to move to the abstract level. For those students, continued modeling with selected p values and use of the applet mentioned earlier will help bridge the gap between concrete and abstract. Also, recall that several of the exercises in the previous lesson worked through this process with specifc points. Remind students for whom this work is tedious that by deriving a general formula, we can simplify our work going forward, which is the heart of MP.7 and MP.8.

> *Scaffolding:*
> For more advanced students, derive the general vertex form of a parabola with vertex at (h, k), distance p between the focus and directrix, and horizontal directrix:
> $y = \pm\frac{1}{2p}(x-h)^2 + k$.

Example 1

Consider a parabola P with distance $p > 0$ between the focus with coordinates $\left(0, \frac{1}{2}p\right)$, and directrix $y = -\frac{1}{2}p$. What is the equation that represents this parabola?



- What are the coordinates of the vertex?
  - *The coordinates of the vertex are* $(0,0)$.
- Find a formula for the distance between the focus and the point (x, y).
  - $\sqrt{(x-0)^2 + \left(y - \frac{1}{2}p\right)^2}$
- Find a formula for the distance between (x, y) and the directrix.
  - $y + \frac{1}{2}p$
- By the definition of a parabola, these distances are equal. Create an equation.
  - $y + \frac{1}{2}p = \sqrt{(x-0)^2 + \left(y - \frac{1}{2}p\right)^2}$
- Solve the equation for y.
  - *Start by squaring both sides. When you expand the squared binomials, the* y^2 *and* $\frac{1}{4}p^2$ *terms drop out of the equation. You may need to provide some additional scaffolding for students who are still not fluent with expanding binomial expressions.*

$$\left(y+\frac{1}{2}p\right)^2 = x^2 + \left(y - \frac{1}{2}p\right)^2$$
$$\left(y+\frac{1}{2}p\right)^2 - \left(y - \frac{1}{2}p\right)^2 = x^2$$
$$y^2 + py + \frac{1}{4}p^2 - y^2 + py - \frac{1}{4}p = x^2$$
$$2py = x^2$$
$$y = \frac{1}{2p}x^2$$

  - *Therefore,* $P = \left\{(x, y) \mid y = \frac{1}{2p}x^2\right\}$.

Scaffolding:

As an alternative to Example 1, use graphing software to explore the relationship between the distance between the focus and directrix, and the coefficient of x^2. Organize the results in a table like the one shown below, and ask students to generalize the patterns they are seeing.

Equation	Focus	Directrix	Distance from Focus to Directrix
$y = \frac{1}{2}x^2$	$\left(0, \frac{1}{2}\right)$	$y = -\frac{1}{2}$	1
$y = \frac{1}{4}x^2$	$(0,1)$	$y = -1$	2
$y = \frac{1}{6}x^2$	$\left(0, \frac{3}{2}\right)$	$y = -\frac{3}{2}$	3
$y = \frac{1}{8}x^2$	$(0, 2)$	$y = -2$	4
$y = \frac{1}{2p}x^2$	$\left(0, \frac{1}{2}p\right)$	$y = -\frac{1}{2}p$	p

MP.7

- How does this result verify the conjecture that as p increases, the parabola gets flatter? (Note to teacher: Now is a good time to demonstrate with the applet http://www.intmath.com/plane-analytic-geometry/parabola-interactive.php.)
  - *We can confirm the conjecture that the graph of a parabola vertically shrinks as* p *increases because the expression* $\frac{1}{2p}$ *will get smaller as* p *gets larger; thus, the parabola appears flatter.*

Discussion (3 minutes)

The goal of this brief discussion is to introduce the vertex form of the equation of a parabola without going through the entire derivation of the formula and to make the connection between what we are doing now with parabolas and what was done in Algebra I, Module 2, Topic B with quadratic functions. Completing the square on any quadratic function $y = f(x)$ will give us the equation of a parabola in vertex form, which we could then quickly graph since we would know the coordinates of the vertex and distance p from focus to directrix.

Discussion

We have shown that any parabola with a distance $p > 0$ between the focus $\left(0, \frac{1}{2}p\right)$ and directrix $y = -\frac{1}{2}p$ has a vertex at the origin and is represented by a quadratic equation of the form $y = \frac{1}{2p}x^2$.

Suppose that the vertex of a parabola with a horizontal directrix that opens upward is (h, k), and the distance from the focus to directrix is $p > 0$. Then, the focus has coordinates $\left(h, k + \frac{1}{2}p\right)$, and the directrix has equation $y = k - \frac{1}{2}p$. If we go through the above derivation with focus $\left(h, k + \frac{1}{2}p\right)$ and directrix $y = k - \frac{1}{2}p$, we should not be surprised to get a quadratic equation. In fact, if we complete the square on that equation, we can write it in the form $y = \frac{1}{2p}(x - h)^2 + k$.

In Algebra I, Module 4, Topic B, we saw that any quadratic function can be put into vertex form: $f(x) = a(x - h)^2 + k$. Now we see that any parabola that opens upward can be described by a quadratic function in vertex form, where $a = \frac{1}{2p}$.

If the parabola opens downward, then the equation is $y = -\frac{1}{2p}(x - h)^2 + k$, and the graph of any quadratic equation of this form is a parabola with vertex at (h, k), distance p between focus and directrix, and opening downward. Likewise, we can derive analogous equations for parabolas that open to the left and right. This discussion is summarized in the box below.

<u>Vertex Form of a Parabola</u>

Given a parabola P with vertex (h, k), horizontal directrix, and distance $p > 0$ between focus and directrix, the analytic equation that describes the parabola P is

- **$y = \frac{1}{2p}(x - h)^2 + k$ if the parabola opens upward, and**
- **$y = -\frac{1}{2p}(x - h)^2 + k$ if the parabola opens downward.**

Conversely, if $p > 0$, then

- **The graph of the quadratic equation $y = \frac{1}{2p}(x - h)^2 + k$ is a parabola that opens upward with vertex at (h, k) and distance p from focus to directrix, and**
- **The graph of the quadratic equation $y = -\frac{1}{2p}(x - h)^2 + k$ is a parabola that opens downward with vertex at (h, k) and distance p from focus to directrix.**

Given a parabola P with vertex (h, k), vertical directrix, and distance $p > 0$ between focus and directrix, the analytic equation that describes the parabola P is:

- **$x = \frac{1}{2p}(y - k)^2 + h$ if the parabola opens to the right, and**
- **$x = -\frac{1}{2p}(y - k)^2 + h$ if the parabola opens to the left.**

Conversely, if $p > 0$, then

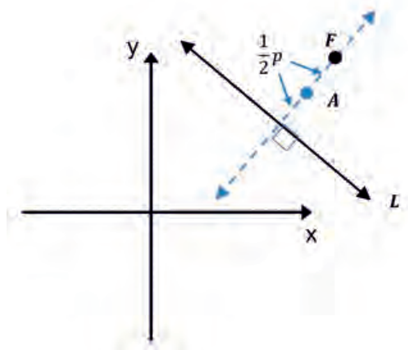
- **The graph of the quadratic equation $x = \frac{1}{2p}(y - k)^2 + h$ is a parabola that opens to the right with vertex at (h, k) and distance p from focus to directrix, and**
- **The graph of the quadratic equation $x = -\frac{1}{2p}(y - k)^2 + h$ is a parabola that opens to the left with vertex at (h, k) and distance p from focus to directrix.**

Example 2 (8 minutes)

The goal of this section is to present and prove the theorem that all parabolas that have the same distance from the focus to the directrix (that is, the same value of p) are congruent. Start by having students sketch the parabola defined by the focus and directrix on the diagram shown on the student pages.

Example 2

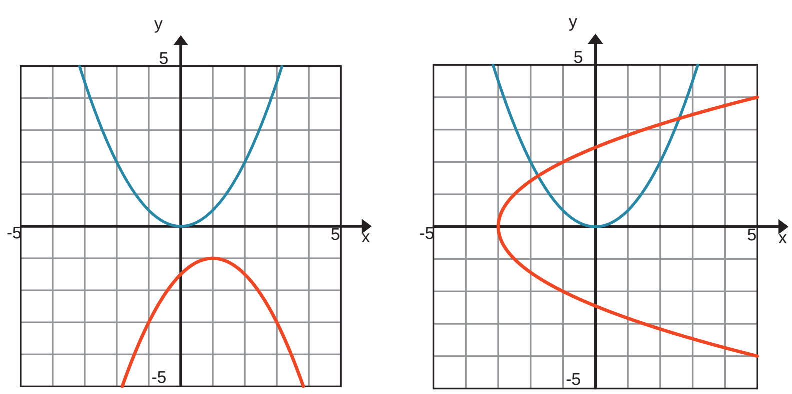
Theorem: Given a parabola P given by a directrix L and a focus F in the Cartesian plane, then P is congruent to the graph of $y = \frac{1}{2p}x^2$, where p is the distance from F to L.



A more concrete approach also appears below and could be used as an alternative approach to the formal proof or as a precursor to the formal proof. If you provide this additional scaffolding, this lesson may need to extend to an additional day to provide time to prove the theorem below rather than just demonstrate it by the examples provided in the scaffolds.

The following examples provide concrete evidence to support the proof provided on the next pages. Use this before or after the proof based on the needs of your students. Print graphs of these parabolas on paper. Then, print the graph of $y = \frac{1}{2}x^2$ on a transparency. Use this to help students understand the rigid transformations that will map the given parabola onto $y = \frac{1}{2}x^2$.

- The graphs of $y = \frac{1}{2}x^2$ and another parabola are shown in each coordinate plane. Describe a series of rigid transformations that will map the given parabola onto the graph of $y = \frac{1}{2}x^2$.



  - *There are many sequences of transformations that can take the red parabola to the blue one. For the graphs on the left, translate the red graph one unit vertically up and one unit to the left, then reflect the resulting graph about the x-axis. For the graphs on the right, rotate the graph $90°$ counter-clockwise about the point $(-3,0)$, then translate the graph 3 units to the right.*

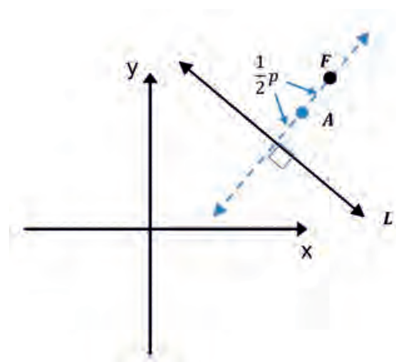
- Suppose we changed the distance between the focus and the directrix to 2 units instead of 1, and then mapped the given parabolas onto the graph of $y = \frac{1}{4}x^2$. Would the resulting graphs be the same? Why?
  - *The resulting graphs would be the same since the distance between the focus and the directrix for the parabola $y = \frac{1}{4}x^2$ is 2 units. The parabolas would map exactly onto the parabola whose vertex is $(0,0)$.*
- Under what conditions would two parabolas be congruent? How could you verify this using transformations that map one parabola onto the other?
  - *If the distance between the focus and the directrix is the same, then the two parabolas will be congruent. You could describe a series of rotations, reflections, and/or translations that will map one parabola onto the other. These rigid transformations preserve the size and shape of the graph and show that the two figures are congruent.*

Proof

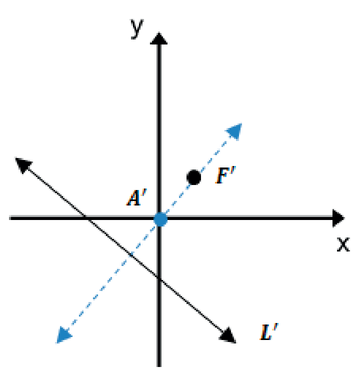
Let A be the point on the perpendicular line to L that passes through F, which is the midpoint of the line segment between F and L.

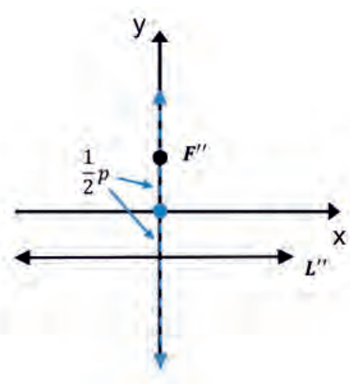
Translate A to the origin $A' = (0,0)$ using a translation. Then, F translates to F' and L translates to L'.

Next, rotate F' and L' about $(0,0)$ until $F'' = \left(0, \frac{1}{2}p\right)$ and $L'' = \left\{(x,y) \mid y = -\frac{1}{2}p\right\}$. We are guaranteed we can do this because A' is on the line perpendicular to L' that passes through F'.



The translations described above are shown below.





Since P is determined by F and L, the first translation takes P to a parabola P' such that $P \cong P'$. The rotation takes P' to a parabola P'' such that $P' \cong P''$. Therefore, $P \cong P''$ by transitivity.

Now, by Example 1 above,

$$P'' = \left\{(x,y) \mid y = \frac{1}{2p}x^2\right\};$$

that is, P'' is the graph of the equation $y = \frac{1}{2p}x^2$, which is what we wanted to prove.

Depending on the level of your students, you can begin with the scaffold examples using $y = \frac{1}{2}x^2$; you can begin with the proof of this theorem can be presented directly by you; or you can have students try to work through the proof with some scaffolding and support.

Consider these discussion questions to help students get started with thinking about a proof if you choose to have them work collaboratively in small groups to create a proof.

- How can we apply transformations to show that every parabola is congruent to $y = \frac{1}{2p}x^2$, where p is the distance between the focus and the directrix?
  - *The parabola's vertex is $(0,0)$. We could translate any parabola so that its vertex is also $(0,0)$.*
  - *Then we would need to rotate the parabolas so that the directrix is a horizontal line and the focus is a point along the y-axis.*

You may need to remind students that translations and rotations are rigid transformations and, therefore, guarantee that the parabolas determined by the focus and directrix as they are translated and rotated will remain the same shape and size. For more information, see Module 1 from high school Geometry.

Exercises 6–9 (4 minutes): Reflecting on the Theorem

Have students respond individually and then share within their groups. Post a few responses on the board for a whole class debrief and correct any misconceptions at that point.

Exercises 6–9

6. **Restate the results of the theorem from Example 2 in your own words.**

   All parabolas that have the same distance between the focus point and the directrix are congruent.

7. **Create the equation for a parabola that is congruent to $y = 2x^2$. Explain how you determined your answer.**

   $y = 2x^2 + 1$. As long as the coefficient of the x^2 term is the same, the parabolas will be congruent.

8. **Create an equation for a parabola that IS NOT congruent to $y = 2x^2$. Explain how you determined your answer.**

   $y = x^2$. As long as the coefficient of the x^2 term is different, the parabolas will not be congruent.

9. **Write the equation for two different parabolas that are congruent to the parabola with focus point $(0,3)$ and directrix line $y = -3$.**

   The distance between the focus and the directrix is 6 units. Therefore, any parabola with a coefficient of $\frac{1}{2p} = \frac{1}{2(6)} = \frac{1}{12}$ will be congruent to this parabola. Here are two options: $y = \frac{1}{12}x^2 + 1$ and $x = \frac{1}{12}y^2$.

Closing (2 minutes)

Revisit the conjecture from the beginning of this lesson: Under what conditions will two parabolas be congruent? Give students time to reflect on this question in writing before reviewing the points listed below. Summarize the following key points as you wrap up this lesson. In the next lesson, we will consider whether or not all parabolas are similar. This lesson has established that given a distance p between the directrix and focus, all parabolas with equal values of p are congruent to the parabola that is the graph of the equation $y = \frac{1}{2p}x^2$.

- The points of a parabola are determined by the directrix and a focus.
- Every parabola is congruent to a parabola defined by a focus on the y-axis and a directrix that is parallel to the x-axis.
- All parabolas that have the same distance between the focus and the directrix are congruent.
- When the focus is at $\left(0, \frac{1}{2}p\right)$ and the directrix is given by the equation $y = -\frac{1}{2}p$, then the parabola is the graph of the equation $y = \frac{1}{2p}x^2$.
- When the vertex is at (h, k), and the distance from the focus to directrix is $p > 0$, then:
  - If it opens upward, the parabola is the graph of the equation $y = \frac{1}{2p}(x-h)^2 + k$;
  - If it opens downward, the parabola is the graph of the equation $y = -\frac{1}{2p}(x-h)^2 + k$;
  - If it opens to the right, the parabola is the graph of the equation $x = \frac{1}{2p}(y-k)^2 + h$;
  - If it opens to the left, the parabola is the graph of the equation $x = -\frac{1}{2p}(y-k)^2 + h$.

Exit Ticket (4 minutes)

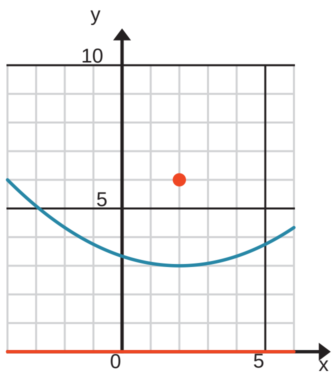
Name ______________________________ Date ______________

Lesson 34: Are All Parabolas Congruent?

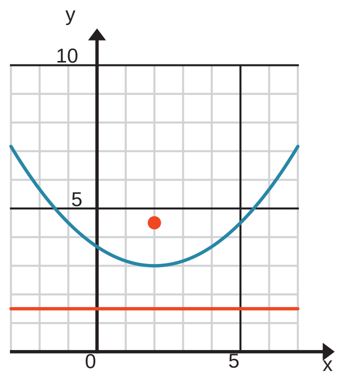
Exit Ticket

Which parabolas shown below are congruent to the parabola that is the graph of the equation $y = \frac{1}{12}x^2$? Explain how you know.

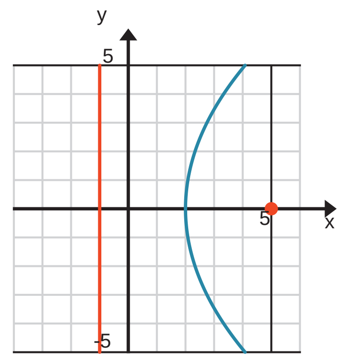
a.



b.



c.

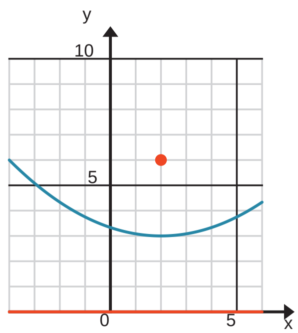


Exit Ticket Sample Solutions

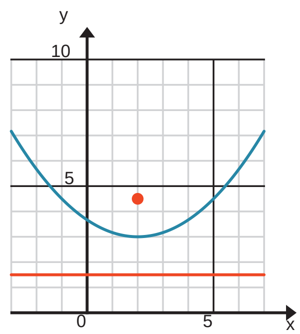
Which parabolas shown below are congruent to the parabola that is the graph of the equation $y = \frac{1}{12}x^2$? Explain how you know.

The p-value is 6. So, any parabola where the distance between the focus and the directrix is equal to 6 units will be congruent to the parabola that is the graph of the equation $y = \frac{1}{12}x^2$. Of the parabolas shown below, (a) and (c) meet this condition, but (b) does not.

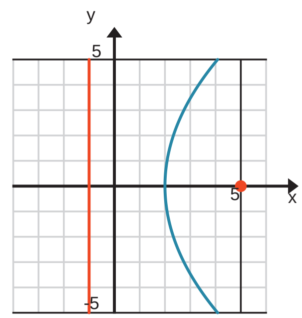
a.



b.



c.



Problem Set Sample Solutions

Problems 1–9 in this Problem Set review how to create the analytic equation of a parabola. Students may use the process from the previous lesson, or use the vertex form of the equation of a parabola included in this lesson. Starting with Problem 10, the focus of the Problem Set shifts to recognizing when parabolas are congruent.

1. **Show that if the point with coordinates (x, y) is equidistant from $(4, 3)$ and the line $y = 5$, then $y = -\frac{1}{4}x^2 + 2x$.**

   Students might start with the equation $\sqrt{(x-4)^2 + (y-3)^2} = 5 - y$ and solve for y as follows:

$$\begin{aligned}
\sqrt{(x-4)^2 + y^2 - 6y + 9} &= 5 - y \\
(x-4)^2 + y^2 - 6y + 9 &= 25 - 10y + y^2 \\
(x-4)^2 &= -4y + 16 \\
4y &= -(x-4)^2 + 16 \\
y &= -\frac{1}{4}(x^2 - 8x + 16) + 4 \\
y &= -\frac{1}{4}x^2 + 2x.
\end{aligned}$$

   Or, they might apply what we have learned about the vertex form of the equation of a parabola. Since the directrix is above the focus, we know the parabola opens downward, so p will be negative. Since the distance from the point $(4, 3)$ to the line $y = 5$ is 2 units, we know that $p = -2$. The vertex is halfway between the focus and directrix, so the coordinates of the vertex are $(4, 4)$. Then, the vertex form of the equation that represents the parabola is

$$\begin{aligned}
y &= -\frac{1}{4}(x-4)^2 + 4 \\
y &= -\frac{1}{4}x^2 + 2x.
\end{aligned}$$

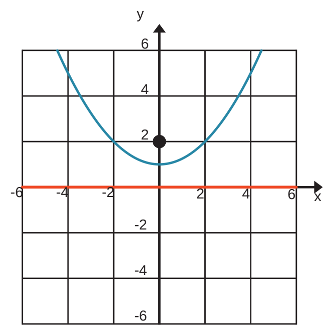
2. Show that if the point with coordinates (x, y) is equidistant from the point $(2, 0)$ and the line $y = -4$, then $y = \frac{1}{8}(x-2)^2 - 2$.

Students might start with the equation $\sqrt{(x-2)^2 + (y-0)^2} = y + 4$, and then solve it for y, or they might apply the vertex form of the equation of a parabola. Since the vertex is above the directrix, we know that the parabola opens upward, and $p > 0$. Since the distance from the point $(2, 0)$ to the line $y = -4$ is 4 units, we know that $p = 4$. The vertex is halfway between the focus and directrix, so the vertex is $(2, -2)$. Thus, the equation that represents the parabola is $y = \frac{1}{8}(x-2)^2 - 2$.

3. Find the equation of the set of points which are equidistant from $(0, 2)$ and the x-axis. Sketch this set of points.

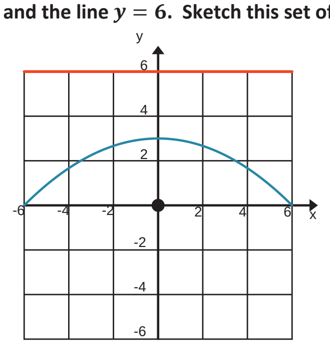
The focus is $(0, 2)$, and the directrix is the x-axis. Thus, the vertex is the point $(0, 1)$, which is halfway between the vertex and directrix. Since the parabola opens upward, $p > 0$, so $p = 2$. Then, the vertex form of the equation of the parabola is

$$y = \frac{1}{4}x^2 + 1.$$



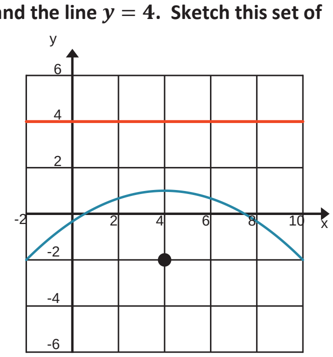
4. Find the equation of the set of points which are equidistant from the origin and the line $y = 6$. Sketch this set of points.

$$y = -\frac{1}{12}x^2 + 3$$



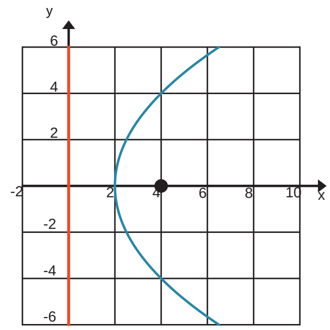
5. Find the equation of the set of points which are equidistant from $(4, -2)$ and the line $y = 4$. Sketch this set of points.

$$y = -\frac{1}{12}(x-4)^2 + 1$$



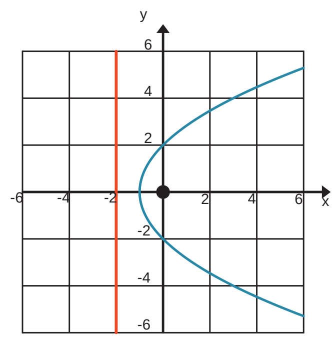
6. **Find the equation of the set of points which are equidistant from $(4, 0)$ and the y-axis. Sketch this set of points.**

$$x = \frac{1}{8}y^2 + 2$$



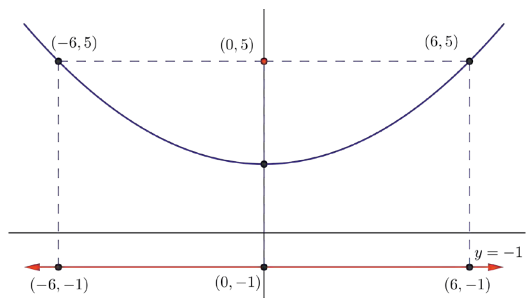
7. **Find the equation of the set of points which are equidistant from the origin and the line $x = -2$. Sketch this set of points.**

$$x = \frac{1}{4}y^2 - 1$$

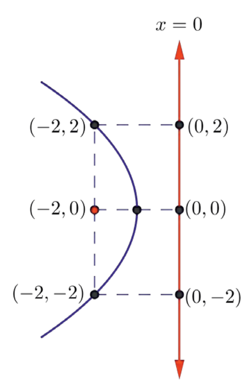


8. **Use the definition of a parabola to sketch the parabola defined by the given focus and directrix.**

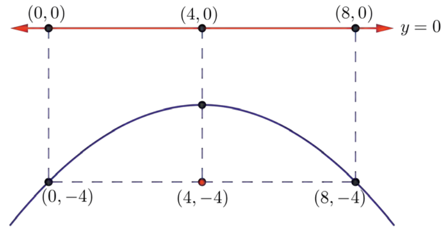
   a. **Focus: $(0, 5)$ Directrix: $y = -1$**



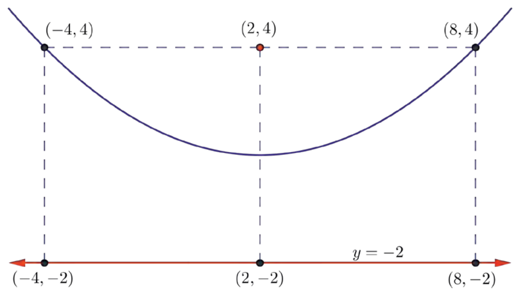
   b. **Focus: $(-2, 0)$ Directrix: y-axis**



c. Focus: $(4, -4)$ Directrix: x-axis



d. Focus: $(2, 4)$ Directrix: $y = -2$



9. Find an analytic equation for each parabola described in Problem 8.

   a. $P = \left\{(x, y) \mid y = \frac{1}{12}x^2 + 2\right\}$; *thus, P is the graph of the equation $y = \frac{1}{12}x^2 + 2$.*

   b. $P = \left\{(x, y) \mid x = -\frac{1}{4}y^2 - 1\right\}$; *thus, P is the graph of the equation $x = -\frac{1}{4}y^2 - 1$.*

   c. $P = \left\{(x, y) \mid y = -\frac{1}{8}(x-4)^2 - 2\right\}$; *thus, P is the graph of the equation $y = -\frac{1}{8}(x-4)^2 - 2$.*

   d. $P = \left\{(x, y) \mid y = \frac{1}{12}(x-2)^2 + 1\right\}$; *thus, P is the graph of the equation $y = \frac{1}{12}(x-2)^2 + 1$.*

10. Are any of the parabolas described in Problem 9 congruent? Explain your reasoning.

    (a) $p = 6$, (b) $p = 2$, (c) $p = 4$, and (d) $p = 6$; therefore, the parabolas in parts (a) and (d) are congruent because they have the same distance between the focus and directrix.

11. Sketch each parabola, labeling its focus and directrix.

    Each sketch should have the appropriate vertex, focus, and directrix and be fairly accurate. Sketches for (a) and (c) are shown.

    a. $y = \frac{1}{2}x^2 + 2$

       Distance between focus and directrix is 1 unit, vertex $(0, 2)$, focus $(0, 2.5)$, directrix $y = 1.5$

    b. $y = -\frac{1}{4}x^2 + 1$

       Distance between focus and directrix is 2 units, vertex $(0, 1)$, focus $(0, 0)$, directrix $y = 2$

    c. $x = \frac{1}{8}y^2$

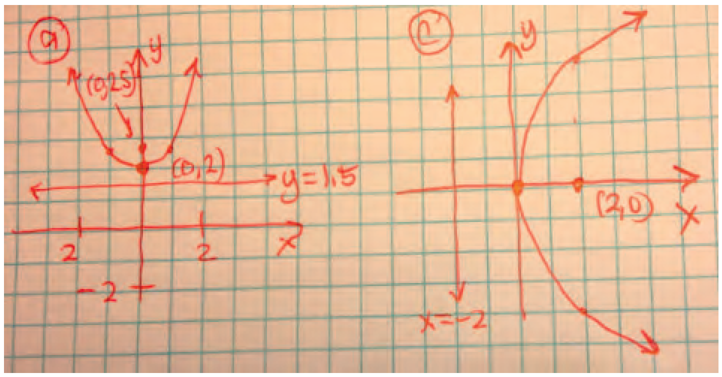
       Distance between focus and directrix is 4 units, vertex $(0, 0)$, focus $(2, 0)$, directrix $x = -2$

d. $x = \frac{1}{2}y^2 + 2$

Distance between focus and directrix is 1 unit, vertex $(2, 0)$, focus $(2.5, 0)$, directrix $x = 1.5$

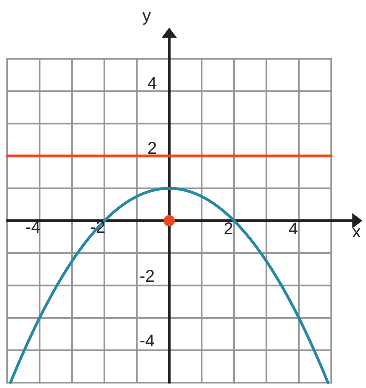
e. $y = \frac{1}{10}(x-1)^2 - 2$

Distance between focus and directrix is 5 units, vertex $(1, -2)$, focus $(1, 0.5)$, directrix $y = -4.5$

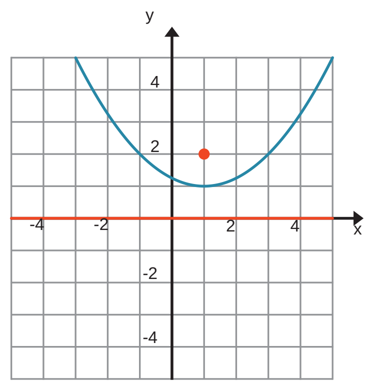


12. Determine which parabolas are congruent to the parabola that is the graph of the equation $y = -\frac{1}{4}x^2$.

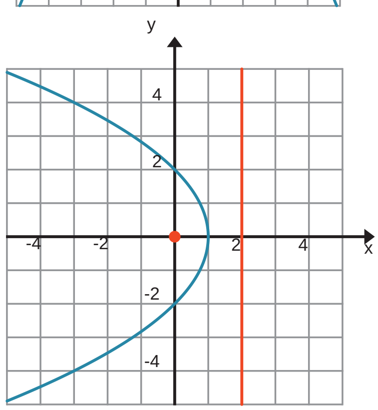
a.



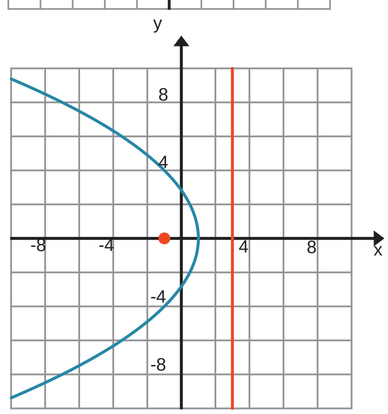
c.



b.



d.



Parabolas (a), (b), and (c) are congruent because all have $p = 2$. Parabola (d) has $p = 1$, so it is not congruent to the others.

13. Determine which equations represent the graph of a parabola that is congruent to the parabola shown to the right.

a. $y = \frac{1}{20}x^2$

b. $y = \frac{1}{10}x^2 + 3$

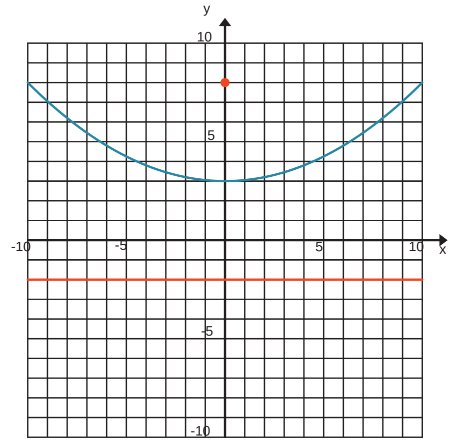
c. $y = -\frac{1}{20}x^2 + 8$

d. $y = \frac{1}{5}x^2 + 5$

e. $x = \frac{1}{10}y^2$

f. $x = \frac{1}{5}(y-3)^2$

g. $x = \frac{1}{20}y^2 + 1$



The parabolas in parts (a), (c), and (g) are congruent and are congruent to the parabola shown. They all have the same distance of 10 units between the focus and the directrix like the parabola shown.

MP.3

14. Jemma thinks that the parabola whose graph is the equation $y = \frac{1}{3}x^2$ is NOT congruent to the parabola whose graph is the equation $y = -\frac{1}{3}x^2 + 1$. Do you agree or disagree? Create a convincing argument to support your reasoning.

Jemma is wrong. These two parabolas are congruent. If you translate the graph of $y = -\frac{1}{3}x^2 + 1$ down one unit and then reflect the resulting graph about the x-axis, the resulting graph will have the equation $y = \frac{1}{3}x^2$. Alternately, the focus and directrix of each parabola are the same distance apart, 1.5 units.

15. Let P be the parabola with focus $(2, 6)$ and directrix $y = -2$.

a. Write an equation whose graph is a parabola congruent to P with a focus $(0, 4)$.

$y = \frac{1}{16}x^2$ is one option. The directrix for this parabola is $y = -4$. Another possible solution would be the parabola with focus $(0, 4)$ and directrix $y = 12$. The equation would be $y = -\frac{1}{16}x^2 + 8$.

b. Write an equation whose graph is a parabola congruent to P with a focus $(0, 0)$.

$$y = \frac{1}{16}x^2 - 4$$

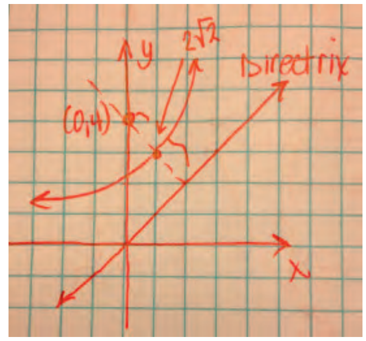
c. Write an equation whose graph is a parabola congruent to P with the same directrix, but different focus.

The focus would be a reflection of the original focus across the directrix, or $(2, -10)$. The equation would be $y = -\frac{1}{16}(x-2)^2 - 6$.

d. Write an equation whose graph is a parabola congruent to P with the same focus, but with a vertical directrix.

$x = \frac{1}{16}(y-6)^2 - 2$ *or* $x = -\frac{1}{16}(y-6)^2 + 8$

16. Let P be the parabola with focus $(0, 4)$ and directrix $y = x$.



a. Sketch this parabola.

Sketch is shown to the right.

b. By how many degrees would you have to rotate P about the focus to make the directrix line horizontal?

One possible answer is a clockwise rotation of $45°$.

c. Write an equation in the form $y = \frac{1}{2a}x^2$ whose graph is a parabola that is congruent to P.

The distance between the focus and the directrix is $2\sqrt{2}$. The equation is $y = \frac{1}{4\sqrt{2}}x^2$.

d. Write an equation whose graph is a parabola with a vertical directrix that is congruent to P.

Since the exact focus and directrix are not specified, there are infinitely many possible parabolas. A vertical directrix does require that the y-term be squared. Thus, $x = \frac{1}{4\sqrt{2}}y^2$ satisfies the conditions specified in the problem.

e. Write an equation whose graph is P', the parabola congruent to P that results after P is rotated clockwise $45°$ about the focus.

The directrix will be $y = 4 - 2\sqrt{2}$. The equation is $y = \frac{1}{4\sqrt{2}}x^2 + 4 - \sqrt{2}$.

f. Write an equation whose graph is P'', the parabola congruent to P that results after the directrix of P is rotated $45°$ about the origin.

The focus will be $(2\sqrt{2}, 2\sqrt{2})$, and the directrix will be the x-axis. The equation is $y = \frac{1}{4\sqrt{2}}\left(x - 2\sqrt{2}\right)^2 + \sqrt{2}$.

EXTENSION

17. Consider the function $f(x) = \frac{2x^2-8x+9}{-x^2+4x-5}$, where x is a real number.

a. Use polynomial division to rewrite f in the form $f(x) = q + \frac{r}{-x^2+4x-5}$ for some real numbers q and r.

Using polynomial division, $f(x) = -2 + \frac{-1}{-x^2+4x-5}$.

MP.2

b. Find the x-value where the maximum occurs for the function f, without using graphing technology. Explain how you know.

We can rewrite f as $f(x) = -2 + \frac{1}{x^2-4x+5}$. Since $x^2 - 4x + 5 = (x-2)^2 + 1$, the graph of $y = x^2 - 4x + 5$ is a parabola with vertex $(2, 1)$ that opens upward. Thus, the lowest point on the graph is $(2, 1)$. The function f will take on its maximum value when $\frac{1}{x^2-4x+5}$ is maximized; this happens when the value of $x^2 - 4x + 5$ is minimized. Since we have already seen that $x^2 - 4x + 5$ is minimized at $x = 2$, the expression $\frac{1}{x^2-4x+5}$ takes on its maximum value when $x = 2$, and, thus, the original function f takes on its maximum value when $x = 2$.

Lesson 35: Are All Parabolas Similar?

Student Outcomes

- Students apply the geometric transformation of dilation to show that all parabolas are similar.

Lesson Notes

In the previous lesson, students used transformations to prove that all parabolas with the same distance between the focus and directrix are congruent. In the process, they made a connection between geometry, coordinate geometry, transformations, equations, and functions. In this lesson, students explore how dilation can be applied to prove that all parabolas are similar.

Students may express disagreement with or confusion about the claim that all parabolas are similar because the various graphical representations of parabolas they have seen do not appear to have the "same shape." Because a parabola is an open figure as opposed to a closed figure, like a triangle or quadrilateral, it is not easy to see similarity among parabolas. Students must understand that we are strictly defining *similar* via similarity transformations; in other words, two parabolas are similar because there is a sequence of translations, rotations, reflections, and dilations that takes one parabola to the other. In the last lesson, we showed that every parabola is congruent to the graph of the equation $y = \frac{1}{2p}x^2$ for some $p > 0$; in this lesson, we need only consider dilations.

When students claim that two parabolas are not similar, they should be reminded that the parts of the parabolas they are looking at may well appear to be different in size or magnification, but the parabolas themselves are not different in shape. Remind students that similarity is established by dilation; in other words, by magnifying a figure in both the horizontal and vertical directions. By analogy, although circles with different radii have different curvature, every student should agree that any circle can be dilated to be the same size and shape as any other circle; thus, all circles are similar.

Quadratic curves such as parabolas belong to a family of curves known as *conic sections*. The technical term in mathematics for how much a conic section deviates from being circular is *eccentricity,* and two conic sections with the same eccentricity are similar. Circles have eccentricity 0, and parabolas have eccentricity 1. After this lesson, you might ask students to research and write a report on eccentricity.

Classwork

Provide graph paper for students as they work the first five exercises. They will first examine three congruent parabolas and then make a conjecture about whether or not all parabolas are similar. Finally, they will explore this conjecture by graphing parabolas of the form $y = ax^2$ that have different a-values.

Scaffolding:

- Allow students access to graphing calculators or software to focus on conceptual understanding if they are having difficulty sketching the graphs.
- Consider providing students with transparencies with a variety of parabolas drawn on them (as in prior lesson), such as $y = x^2$, $y = \frac{1}{2}x^2$, and $y = \frac{1}{4}x^2$ to help them illustrate these principles.

Exercises 1–5 (4 minutes)

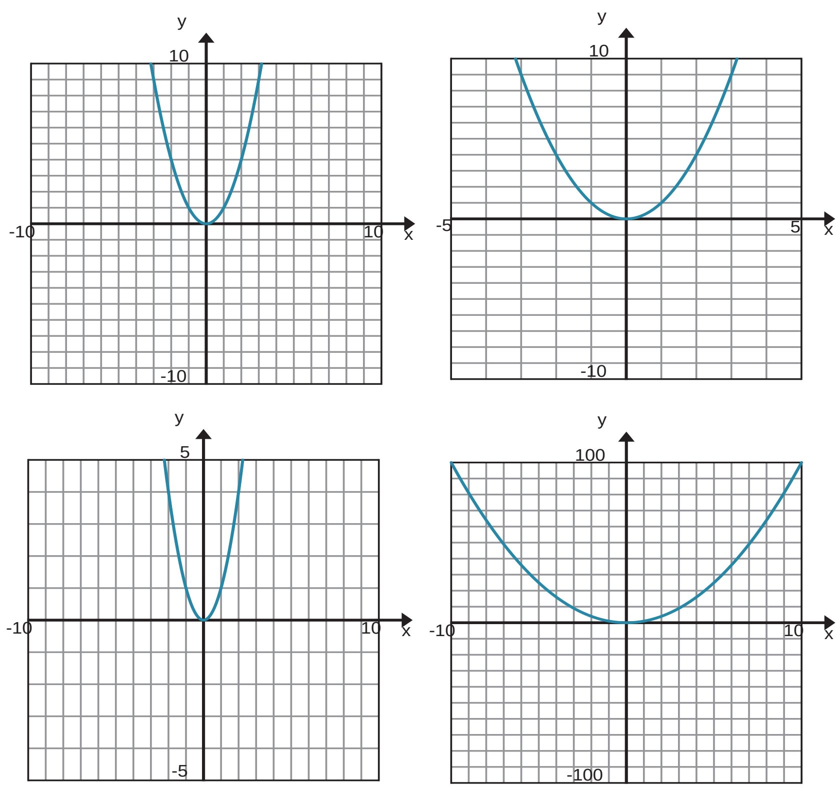
Exercises 1–6

1. **Write the equation of two parabolas that are congruent to $y = x^2$ and explain how you determined your equations.**
2. **Sketch the graph of $y = x^2$ and the two parabolas you created on the same coordinate axes.**
3. **Write the equation of two parabolas that are NOT congruent to $y = x^2$. Explain how you determined your equations.**
4. **Sketch the graph of $y = x^2$ and the two non-congruent parabolas you created on the same coordinate axes.**
5. **Use your work on Exercises 1–4 to answer the question posed in the lesson title: Are all parabolas similar? Explain your reasoning.**

MP.3

Discussion

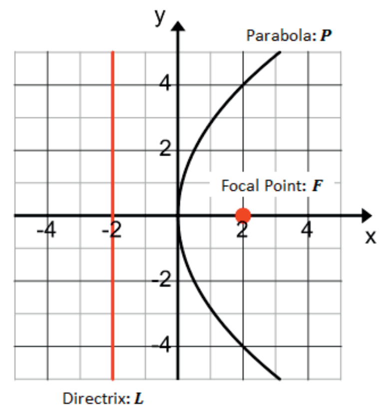
After students have examined the fact that when we change the a-value in the parabola equation, the resulting graph is is basically the same shape, you can further emphasize this point by exploring the graph of $y = x^2$ on a graphing calculator or graphing program on your computer. Use the same equation but different viewing windows so students can see that we can create an image of what appears to be a different parabola by transforming the dimensions of the viewing window. However, the images are just a dilation of the original that is created when we change the scale. See the images to the right. Each figure is a graph of the equation $y = x^2$ with different scales on the horizontal and vertical axes.



Exercise 6 (5 minutes)

In this exercise, students derive the analytic equation for a parabola given its graph, focus, and directrix. Students have worked briefly with parabolas with a vertical directrix in previous lessons, so this exercise will be an opportunity for the teacher to assess whether or not students are able to transfer and extend their thinking to a slightly different situation.

6. **The parabola at right is the graph of what equation?**



a. **Label a point (x, y) on the graph of P.**

b. **What does the definition of a parabola tell us about the distance between the point (x, y) and the directrix L, and the distance between the point (x, y) and the focus F?**

Let (x, y) be any point on the graph of P. Then, these distances are equal because $P = \{(x, y) | (x, y)$ is equidistant from F and $L\}$.

c. **Create an equation that relates these two distances.**

Distance from (x, y) to F: $\sqrt{(x-2)^2 + (y-0)^2}$

Distance from (x, y) to L: $x + 2$

Therefore, any point on the parabola has coordinates (x, y) that satisfy $\sqrt{(x-2)^2 + (y-0)^2} = x + 2$.

d. **Solve this equation for x.**

The equation can be solved as follows.

$$\begin{aligned} \sqrt{(x-2)^2 + (y-0)^2} &= x + 2 \\ (x-2)^2 + y^2 &= (x+2)^2 \\ x^2 - 4x + 4 + y^2 &= x^2 + 4x + 4 \\ y^2 &= 8x \\ x &= \frac{1}{8}y^2 \end{aligned}$$

Thus,
$P = \left\{(x, y) \mid x = \frac{1}{8}y^2\right\}$.

e. **Find two points on the parabola P, and show that they satisfy the equation found in part (d).**

By observation, $(2, 4)$ and $(2, -4)$ are points on the graph of P. Both points satisfy the equation that defines P.

$$(2,4): \quad \frac{1}{8}(4)^2 = \frac{16}{8} = 2$$

$$(2,-4): \quad \frac{1}{8}(-4)^2 = \frac{16}{8} = 2$$

Discussion (8 minutes)

After giving students time to work through the Exercises 1–5, ask the following questions to revisit concepts from Algebra I, Module 3.

- In the previous exercise, is P a function of x?
  - *No, because the x-value 2 corresponds to two y-values.*
- Is P a function of y?
  - *Yes. If you rotate the Cartesian plane by $90°$, you can see that it would be a function. Alternately, if we take y to be in the domain and x to be in the range, then each y-value on P corresponds to exactly one x-value, which is the definition of a function.*

These two questions remind students that just because we typically use the variable x to represent the domain element of an algebraic function, this does not mean that it must always represent the domain element.

Next, transition to summarizing what was learned in the last two lessons. We have defined a parabola and determined the conditions required for two parabolas to be congruent. Use the following questions to summarize these ideas.

- What have we learned about the definition of a parabola?
  - *The points on a parabola are equidistant from the directrix and the focus.*
- What transformations can be applied to a parabola to create a parabola congruent to the original one?
  - *If the directrix and the focus are transformed by a rigid motion (e.g., translation, rotation, or reflection), then the new parabola defined by the transformed directrix and focus will be congruent to the original.*

Essentially, every parabola that has a distance of p units between its focus and directrix is congruent to a parabola with focus $\left(0, \frac{1}{2}p\right)$ and directrix $y = -\frac{1}{2}p$. What is the equation of this parabola?

$$P = \left\{(x, y) \mid y = \frac{1}{2p}x^2\right\}$$

Thus, all parabolas that have the same distance between the focus and the directrix are congruent.

The family of graphs given by the equation $y = \frac{1}{2p}x^2$ for $p > 0$ describes the set of non-congruent parabolas, one for each value of p.

Ask students to consider the question from the lesson title. Chart responses to revisit at the end of this lesson to confirm or refute their claims.

MP.3

Discussion

How many of you think that all parabolas are similar? Explain why you think so.

In geometry, two figures were similar if a sequence of transformations would take one figure onto the other one.

What could we do to show that two parabolas are similar? How might you show this?

Since every parabola can be transformed into a congruent parabola by applying one or more rigid transformations, perhaps similar parabolas can be created by applying a dilation which is a non-rigid transformation.

To check to see if all parabolas are similar, we only need to show that any parabola that is the graph of $y = \frac{1}{2p}x^2$ for $p > 0$ is similar to the graph of $y = x^2$. This is done through a dilation by some scale factor $k > 0$ at the origin $(0,0)$. Note that a dilation of the graph of a function is the same as performing a horizontal scaling followed by a vertical scaling that students studied in Algebra I, Module 3.

Exercises 7–10 (8 minutes)

The following exercises review the function transformations studied in Algebra I that are required to define dilation at the origin. These exercises provide students with an opportunity to recall what they learned in a previous course so that they can apply it here. Students must read points on the graphs to determine that the vertical scaling is by a factor of 2 for the graphs on the left and by a factor of $\frac{1}{2}$ for the graphs on the right. In Algebra I, Module 3, we saw that the graph of a function can be transformed with a non-rigid transformation in two ways: *vertical* and *horizontal scaling*.

A *vertical scaling* of a graph by a scale factor $k > 0$ takes every y-value of points (x, y) on the graph of $y = f(x)$ to ky. The result of the transformation is given by the graph of $y = kf(x)$.

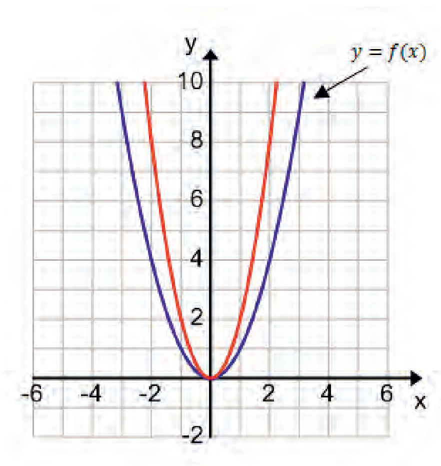
A *horizontal scaling* of a graph by a scale factor $k > 0$ takes every x-value of points (x, y) on the graph of $y = f(x)$ to kx. The result of the transformation is given by the graph of $y = f\left(\frac{1}{k}x\right)$.

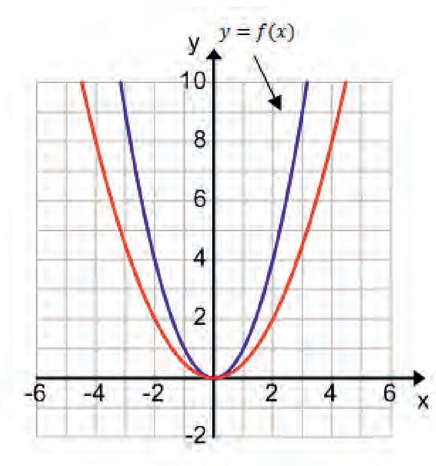
Scaffolding:

- Allow students access to graphing calculators or software to focus on conceptual understanding if they are having difficulty sketching the graphs.
- The graphs shown in Exercises 7 and 8 are $f(x) = x^2$, $g(x) = 2x^2$, and $h(x) = \frac{1}{2}x^2$. The graphs shown in Exercises 9 and 10 are $f(x) = x^2$, $g(x) = \left(\frac{1}{2}x\right)^2$, and $h(x) = (2x)^2$.

Exercises 7–10

Use the graphs below to answer Exercises 7 and 8.





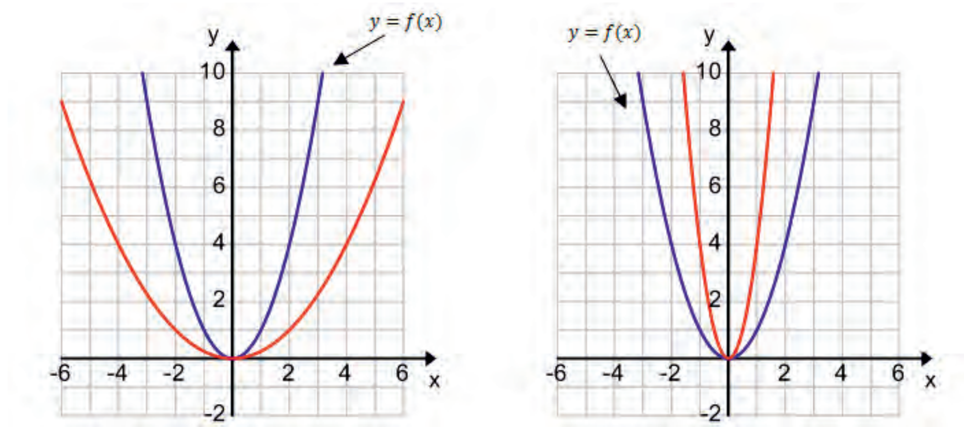
7. **Suppose the unnamed red graph on the left coordinate plane is the graph of the function g. Describe g as a vertical scaling of the graph of $y = f(x)$; that is, find a value of k so that $g(x) = kf(x)$. What is the value of k? Explain how you determined your answer.**

   The graph of g is a vertical scaling of the graph of f by a factor of 2. Thus, $g(x) = 2f(x)$. By comparing points on the graph of f to points on the graph of g, you can see that the y-values on g are all twice the y-values on f.

8. **Suppose the unnamed red graph on the right coordinate plane is the graph of the function h. Describe h as a vertical scaling of the graph of $y = f(x)$; that is, find a value of k so that $h(x) = kf(x)$. Explain how you determined your answer.**

   The graph of h is a vertical scaling of the graph of f by a factor of $\frac{1}{2}$. Thus, $h(x) = \frac{1}{2}f(x)$. By comparing points on the graph of f to points on the graph of h, you can see that the y-values on h are all half of the y-values on f.

Use the graphs below to answer Exercises 9–10.



9. **Suppose the unnamed function graphed in red on the left coordinate plane is g. Describe g as a horizontal scaling of the graph of $y = f(x)$. What is the value of k? Explain how you determined your answer.**

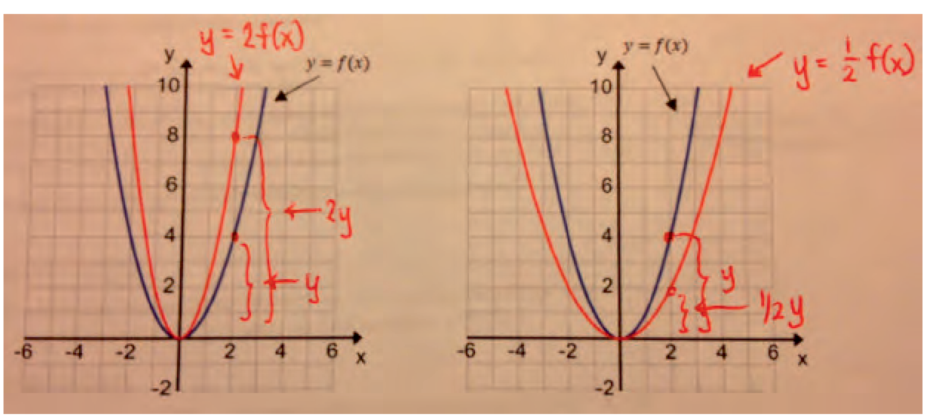
   The graph of g is a horizontal scaling of the graph of f by a factor of 2. Thus, $g(x) = f\left(\frac{1}{2}x\right)$. By comparing points on the graph of f to points on the graph of g, you can see that for the same y-values, the x-values on g are all twice the x-values on f.

10. **Suppose the unnamed function graphed in red on the right coordinate plane is h. Describe h as a horizontal scaling of the graph of $y = f(x)$. What is the value of k? Explain how you determined your answer.**

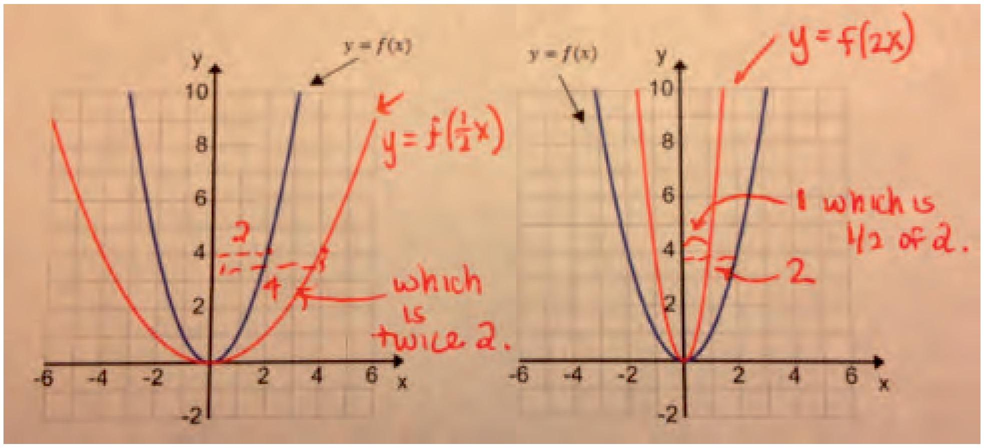
   The graph of h is a horizontal scaling of the graph of f by a factor of $\frac{1}{2}$. Thus, $h(x) = f(2x)$. By comparing points on the graph of f to points on the graph of h, you can see that for the same y-values, the x-values on h are all half of the x-values on f.

When you debrief these exercises, model marking up the diagrams to illustrate the vertical and horizontal scaling. A sample is provided below.

Marked up diagrams for vertical scaling in Exercises 7 and 8:



Marked up diagrams for horizontal scaling in Exercises 9 and 10:



After working through Exercises 7–10, pose the following discussion question.

- If a dilation by scale factor k involves both horizontal and vertical scaling by a factor of k, how could we express the dilation of the graph of $y = f(x)$?
  - *You could combine both types of scaling. Thus,* $y = kf\left(\frac{1}{k}x\right)$.

Explain the definition of dilation at the origin as a combination of a horizontal and then vertical scaling by the same factor. Exercises 1–3 in the Problem Set will address this idea further.

Definition: A *dilation at the origin* D_k is a horizontal scaling by $k > 0$ followed by a vertical scaling by the same factor k. In other words, this dilation of the graph of $y = f(x)$ is the graph of the equation $y = kf\left(\frac{1}{k}x\right)$.

It will be important for students to clearly understand that this dilation of the graph of $y = f(x)$ is the graph of the equation $y = kf\left(\frac{1}{k}x\right)$. Remind students of the following two facts that they studied in high school Geometry:

1. When one figure is a dilation of another figure, the two figures are similar.
2. A dilation at the origin is just a particular type of dilation transformation.

Thus, the graph of $y = f(x)$ is similar to the graph of $y = kf\left(\frac{1}{k}x\right)$. Students may realize here that their thinking about "stretching" the graph creating a similar parabola is not quite enough to prove that all parabolas are similar because we must consider both a horizontal and vertical dilation in order to connect back to the geometric definition of similar figures.

Example 1 (5 minutes): Dilation at the Origin

This example helps students gain a level of comfort with the notation and mathematics before moving on to proving that all parabolas are similar.

Example 1

Let $f(x) = x^2$ and let $k = 2$. Write a formula for the function g that results from dilating f at the origin by a factor of $\frac{1}{2}$.

The new function will have equation $g(x) = 2f\left(\frac{1}{2}x\right)$. Since $f(x) = x^2$, the new function will have equation $g(x) = 2\left(\frac{1}{2}x\right)^2$. That is, $g(x) = \frac{1}{2}x^2$.

What would the results be for $k = 3, 4$, or 5? What about $k = \frac{1}{2}$?

For $k = 3$, $g(x) = \frac{1}{3}x^2$.

For $k = 4$, $g(x) = \frac{1}{4}x^2$.

For $k = 5$, $g(x) = \frac{1}{5}x^2$.

For $k = \frac{1}{2}$, $g(x) = 2x^2$.

After working through this example, the following questions will help prepare students for the upcoming proof using a general parabola from the earlier discussion.

- Based on this example, what can you conclude about these parabolas?
  - *They are all similar to one another because they represent dilations of the graph at the origin of the original function.*
- Based on this example, what can you conclude about these parabolas?
- Is this enough information to prove ALL parabolas are similar?
  - *No, we have only proven that these specific parabolas are similar.*
- How could we prove that all parabolas are similar?
  - *We would have to use the patterns we observed here to make a generalization and algebraically show that it works in the same way.*

Scaffolding:

Some students might find this derivation easier if you use the parabola $y = ax^2$. Then, the proof would be as follows:

If $f(x) = ax^2$, then the graph of f is similar to the graph of the equation

$$y = k\left(a\left(\frac{1}{k}x\right)\right)^2.$$

Simplifying the right side gives:

$$y = \frac{a}{k}x^2.$$

We want this new parabola to be similar to $y = x^2$, which it will be if $\frac{a}{k} = 1$.

Therefore, let $a = k$. Thus, dilating the graph of $y = ax^2$ about the origin by a factor of a, we have shown this parabola is similar to $y = x^2$.

To further support students, supply written reasons, such as those provided, as you work through these steps on the board.

Discussion (8 minutes): Prove All Parabolas Are Similar

In this discussion, you will work through a dilation at the origin on a general parabola with equation $y = \frac{1}{2p}x^2$ to transform it to our basic parabola with equation $y = x^2$ by selecting the appropriate value of k. At that point, we can argue that all parabolas are similar. Walk through the outline below slowly, and ask the class for input at each step, but expect that much of this discussion will be teacher-centered. For students not ready to show this result at an abstract level, have them work in small groups to show that a few parabolas, such as

$$y = \frac{1}{2}x^2,\ y = 4x^2, \text{ and } y = \frac{1}{8}x^2,$$

are similar to $y = x^2$ by finding an appropriate dilation about the origin. Then, generalize from these examples in the following discussion.

- Recall from Lesson 34 that any parabola is congruent to an "upright" parabola of the form $y = \frac{1}{2p}x^2$, where p is the distance between the vertex and directrix. That is, given any parabola we can rotate, reflect and translate it so that it has its vertex at the origin and axis of symmetry along the y-axis. We now want to show that all parabolas of the form $y = \frac{1}{2p}x^2$are similar to the parabola $y = x^2$. To do this, we will apply a dilation at the origin to the parabola $y = \frac{1}{2p}x^2$. We just need to find the right value of k for the dilation.
- Recall that the graph of $y = f(x)$ is similar to the graph of $y = kf\left(\frac{1}{k}x\right)$.

If $f(x) = \frac{1}{2p}x^2$, then the graph of f is similar to the graph of the equation $y = kf\left(\frac{1}{k}x\right) = k\left(\frac{1}{2p}\left(\frac{1}{k}x\right)^2\right)$, which simplifies to $y = \frac{1}{2pk}x^2$.

We want to find the value of k that dilates the graph of $f(x) = \frac{1}{2p}x^2$ into $y = x^2$. That is, we need to choose the dilation factor k so that $y = \frac{1}{2p}x^2$ becomes $y = x^2$; therefore, we want $\frac{1}{2pk} = 1$. Solving this equation for k gives $k = \frac{1}{2p}$.

- Therefore, if we dilate the parabola $y = \frac{1}{2p}x^2$ about the origin by a factor of $\frac{1}{2p}$, we have

$$\begin{aligned} y &= kf\left(\frac{1}{k}x\right) \\ &= k\left(\frac{1}{2p}\left(\frac{1}{k}x\right)^2\right) \\ &= \frac{1}{2pk}x^2 \\ &= x^2. \end{aligned}$$

Thus, we have shown that the original parabola is similar to $y = x^2$.

- In the previous lesson, we showed that any parabola is congruent to a parabola $y = \frac{1}{2p}x^2$ for some value of p. Now, we have shown that every parabola of the form $y = \frac{1}{2p}x^2$ is similar to our basic parabola $y = x^2$. Then, any parabola in the plane is similar to the basic parabola $y = x^2$.
- Further, all parabolas are similar to each other because we have just shown that they are all similar to the same parabola.

Closing (3 minutes)

Revisit the title of this lesson by asking students to summarize what they learned about the reason why all parabolas are similar. Then, take time to bring closure to this cycle of three lessons. The work students have engaged in has drawn together three different domains: geometry, algebra, and functions. In working through these examples and exercises and engaging in the discussions presented here, students can gain an appreciation for how mathematics can model real-world scenarios. The past three lessons show the power of using algebra and functions to solve problems in geometry.

Solving geometric problems using algebra and functions is one of the most powerful techniques we have to solve science, engineering, and technology problems.

Lesson Summary

- **We started with a geometric figure of a parabola defined by geometric requirements and recognized that it involved the graph of an equation we studied in algebra.**
- **We used algebra to prove that all parabolas with the same distance between the focus and directrix are congruent to each other, and in particular, they are congruent to a parabola with vertex at the origin, axis of symmetry along the y-axis, and equation of the form $y = \frac{1}{2p}x^2$.**
- **Noting that the equation for a parabola with axis of symmetry along the y-axis is of the form $y = f(x)$ for a quadratic function f, we proved that all parabolas are similar using transformations of functions.**

Exit Ticket (4 minutes)

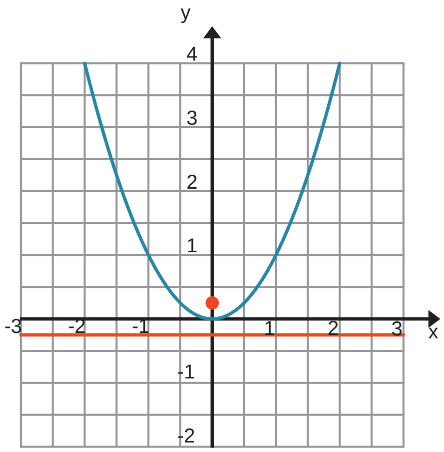
Name ____ Date ____________________

Lesson 35: Are All Parabolas Similar?

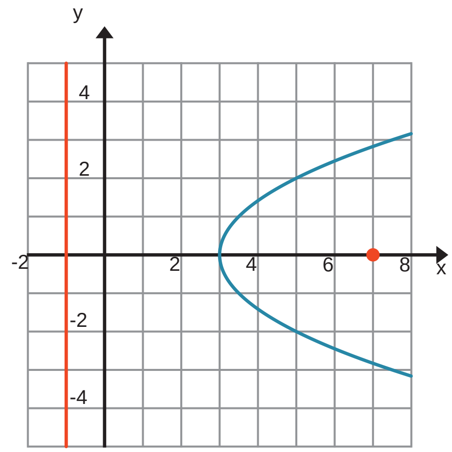
Exit Ticket

1. Describe the sequence of transformations that will transform the parabola P_x into the similar parabola P_y.

Graph of P_x



Graph of P_y



2. Are the two parabolas defined below similar or congruent or both? Justify your reasoning.

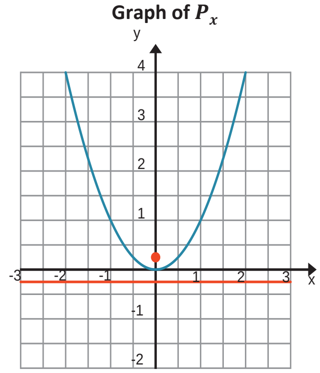
Parabola 1: The parabola with a focus of $(0,2)$ and a directrix line of $y = -4$

Parabola 2: The parabola that is the graph of the equation $y = \frac{1}{6}x^2$

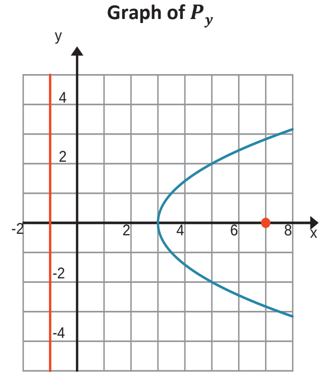
Exit Ticket Sample Solutions

1. Describe the sequence of transformations that would transform the parabola P_x into the similar parabola P_y.

Graph of P_x



Graph of P_y



Vertical scaling by a factor of $\frac{1}{2}$, vertical translation up 3 units, and a $90°$ rotation clockwise about the origin.

2. Are the two parabolas defined below similar or congruent or both?

MP.3

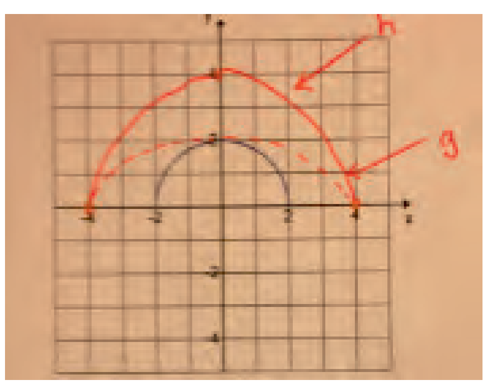
Parabola 1: The parabola with a focus of $(0, 2)$ and a directrix line of $y = -4$

Parabola 2: The parabola that is the graph of the equation $y = \frac{1}{6}x^2$

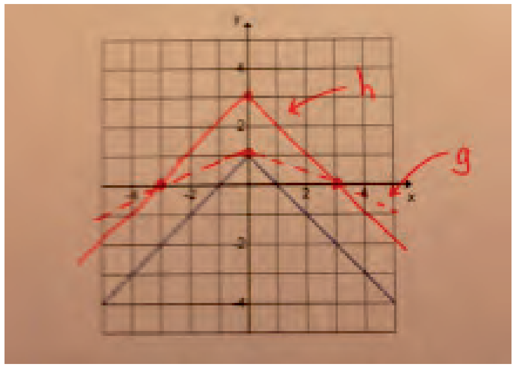
They are similar but not congruent because the distance between the focus and the directrix on Parabola 1 is 6 units, but on Parabola 2, it is only 3 units. Alternatively, students may describe that you cannot apply a series of rigid transformations that will map Parabola 1 onto Parabola 2. However, by using a dilation and a series of rigid transformations, the two parabolas can be shown to be similar since ALL parabolas are similar.

Problem Set Sample Solutions

1. Let $f(x) = \sqrt{4 - x^2}$. The graph of f is shown below. On the same axes, graph the function g, where $g(x) = f\left(\frac{1}{2}x\right)$. Then, graph the function h, where $h(x) = 2g(x)$.



2. Let $f(x) = -|x| + 1$. The graph of f is shown below. On the same axes, graph the function g, where $g(x) = f\left(\frac{1}{3}x\right)$. Then, graph the function h, where $h(x) = 3g(x)$.



3. Based on your work in Problems 1 and 2, describe the resulting function when the original function is transformed with a horizontal and then a vertical scaling by the same factor, k.

The resulting function is scaled by a factor of k in both directions. It is a dilation about the origin of the original figure and is similar to it.

4. Let $f(x) = x^2$.

   a. What are the focus and directrix of the parabola that is the graph of the function $f(x) = x^2$?

   Since $\frac{1}{2p} = 1$, we know $p = \frac{1}{2}$, and that is the distance between the focus and the directrix. The point $(0, 0)$ is the vertex of the parabola and the midpoint of the segment connecting the focus and the directrix. Since the distance between the focus and vertex is $\frac{1}{2}p = \frac{1}{4}$, which is the same as the distance between the vertex and directrix; therefore, the focus has coordinates $\left(0, \frac{1}{4}\right)$, and the directrix is $y = -\frac{1}{4}$.

   b. Describe the sequence of transformations that would take the graph of f to each parabola described below.

   i. Focus: $\left(0, -\frac{1}{4}\right)$, directrix: $y = \frac{1}{4}$

   This parabola is a reflection of the graph of f across the x-axis.

   ii. Focus: $\left(\frac{1}{4}, 0\right)$, directrix: $x = -\frac{1}{4}$

   This parabola is a $90°$ clockwise rotation of the graph of f.

   iii. Focus: $(0, 0)$, directrix: $y = -\frac{1}{2}$

   This parabola is a vertical translation of the graph of f down $\frac{1}{4}$ unit.

   iv. Focus: $\left(0, \frac{1}{4}\right)$, directrix: $y = -\frac{3}{4}$

   This parabola is a vertical scaling of the graph of f by a factor of $\frac{1}{2}$ and a vertical translation of the resulting image down $\frac{1}{4}$ unit.

v. Focus: $(0, 3)$, directrix: $y = -1$

This parabola is a vertical scaling of the graph of f by a factor of $\frac{1}{8}$ and a vertical translation of the resulting image up 1 unit.

c. Which parabolas are similar to the parabola that is the graph of f? Which are congruent to the parabola that is the graph of f?

All of the parabolas are similar. We have proven that all parabolas are similar. The congruent parabolas are (i), (ii), and (iii). These parabolas are the result of a rigid transformation of the original parabola that is the graph of f. They have the same distance between the focus and directrix line as the original parabola.

5. Derive the analytic equation for each parabola described in Problem 4(b) by applying your knowledge of transformations.

i. $y = -x^2$

ii. $x = y^2$

iii. $y = x^2 - \frac{1}{4}$

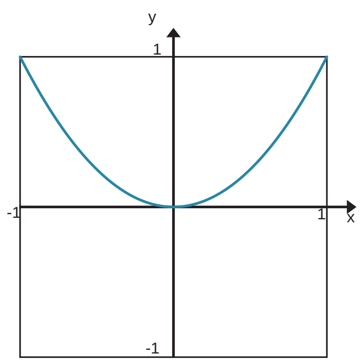
iv. $y = \frac{1}{2}x^2 - \frac{1}{4}$

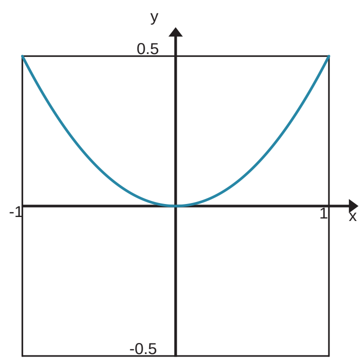
v. $y = \frac{1}{8}x^2 + 1$

6. Are all parabolas the graph of a function of x in the xy-plane? If so, explain why, and if not, provide an example (by giving a directrix and focus) of a parabola that is not.

No, they are not. Examples include the graph of the equation $x = y^2$, or a list stating a directrix and focus. For example, students may give the example of a directrix given by $x = -2$ and focus $(2, 0)$, or an even more interesting example, such as a directrix given by $y = x$ with focus $(-1, -1)$. Any line and any point not on that line define a parabola.

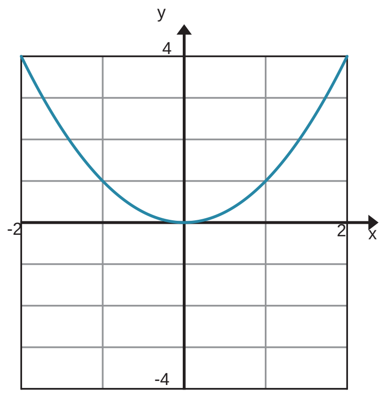
7. Are the following parabolas congruent? Explain your reasoning.

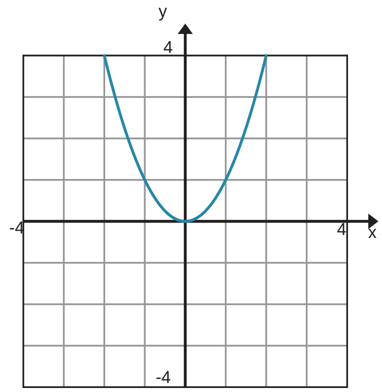




They are not congruent but they are similar. I can see that the parabola on the left appears to contain the point $(1, 1)$, while the parabola on the right appears to contain the point $\left(1, \frac{1}{2}\right)$. This implies that the graph of the parabola on the right is a dilation of the graph of the parabola on the left, so they are not congruent.

8. **Are the following parabolas congruent? Explain your reasoning.**





They are congruent. Both graphs contain the points $(0,0)$, $(1,1)$, and $(2,4)$ that satisfy the equation $y = x^2$. The scales are different on these graphs, making them appear non-congruent.

9. **Write the equation of a parabola congruent to $y = 2x^2$ that contains the point $(1,-2)$. Describe the transformations that would take this parabola to your new parabola.**

There are many solutions. Two possible solutions:

Reflect the graph about the x-axis to get $y = -2x^2$.

OR

Translate the graph down four units to get $y = 2x^2 - 4$.

10. **Write the equation of a parabola similar to $y = 2x^2$ that does NOT contain the point $(0,0)$, but does contain the point $(1,1)$.**

There are many solutions. One solution is $y = (x-1)^2 + 1$. This parabola is congruent to $y = x^2$ and, therefore, similar to the original parabola, but the graph has been translated horizontally and vertically to contain the point $(1,1)$, but not the point $(0,0)$.

Topic D:

A Surprise from Geometry—Complex Numbers Overcome All Obstacles

N-CN.A.1, N-CN.A.2, N-CN.C.7, A-REI.A.2, A-REI.B.4b, A-REI.C.7

Focus Standard:	N-CN.A.1	Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
	N-CN.A.2	Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
	N-CN.C.7	Solve quadratic equations with real coefficients that have complex solutions.
	A-REI.A.2	Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.
	A-REI.B.4	Solve quadratic equations in one variable. b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.
	A-REI.C.7	Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.*
Instructional Days:	5	
Lesson 36:	Overcoming a Third Obstacle to Factoring—What If There Are No Real Number Solutions? (P)[1]	
Lesson 37:	A Surprising Boost from Geometry (P)	
Lesson 38:	Complex Numbers as Solutions to Equations (P)	
Lesson 39:	Factoring Extended to the Complex Realm (P)	
Lesson 40:	Obstacles Resolved—A Surprising Result (S)	

[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

In Topic D, students extend their facility with finding zeros of polynomials to include complex zeros. Lesson 36 presents a third obstacle to using factors of polynomials to solve polynomial equations. Students begin by solving systems of linear and non-linear equations to which no real solutions exist, and then relate this to the possibility of quadratic equations with no real solutions. Lesson 37 introduces complex numbers through their relationship to geometric transformations. That is, students observe that scaling all numbers on a number line by a factor of -1 turns the number line out of its one-dimensionality and rotates it $180°$ through the plane. They then answer the question, “What scale factor could be used to create a rotation of $90°$?” In Lesson 38, students discover that complex numbers have real uses; in fact, they can be used in finding real solutions of polynomial equations. In Lesson 39, students develop facility with properties and operations of complex numbers and then apply that facility to factor polynomials with complex zeros. Lesson 40 brings the module to a close with the result that every polynomial can be rewritten as the product of linear factors, which is not possible without complex numbers. Even though standards **N-CN.C.8** and **N-CN.C.9** are not assessed at the Algebra II level, they are included instructionally to develop further conceptual understanding.

Lesson 36: Overcoming a Third Obstacle to Factoring—What If There Are No Real Number Solutions?

Student Outcomes

- Students understand the possibility that an equation—or a system of equations—has no real solutions. Students identify these situations and make the appropriate geometric connections.

Lesson Notes

Lessons 36–40 provide students with the necessary tools to find solutions to polynomial equations outside the realm of the real numbers. This lesson illustrates how to both analytically and graphically identify a system of equations that has no real number solution. In the next lesson, the imaginary unit i is defined, and students begin to work with complex numbers through the familiar geometric context of rotation. Students will realize that the set of complex numbers inherits the arithmetic and algebraic properties from the real numbers. The work with complex solutions to polynomial equations in these lessons culminates with the Fundamental Theorem of Algebra in Lesson 40, the last lesson in this module.

Classwork

Opening (1 minutes)

This lesson illustrates how to identify a system of equations that has no real number solution, both graphically and analytically. In this lesson, we explore systems of equations that have no real number solutions.

Opening Exercise 1 (5 minutes)

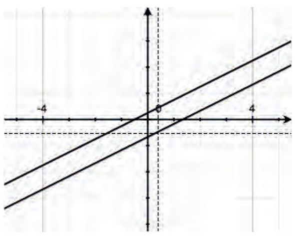
Instruct students to complete the following exercise individually and then to pair up with a partner after a few minutes to compare their answers. Allow students to search for solutions analytically or graphically as they choose. After a few minutes, ask students to share their answers and solution methods. Both an analytic and a graphical solution should be presented for each system, either by a student or by the teacher if all students used the same approach. Circulate while students are working, and take note of which students are approaching the question analytically and which are approaching the question graphically.

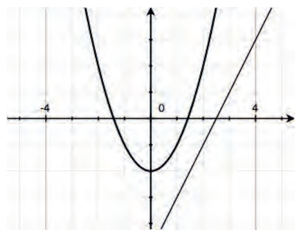
Opening Exercise

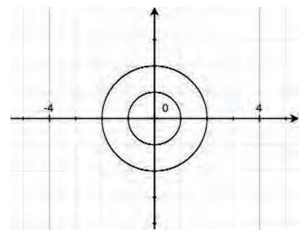
Find all solutions to each of the systems of equations below using any method.

$2x - 4y = -1$	$y = x^2 - 2$	$x^2 + y^2 = 1$
$3x - 6y = 4$	$y = 2x - 5$	$x^2 + y^2 = 4$

All three systems have no real number solutions, which is evident from the non-intersecting graphs in each. Instead of graphing the systems, students may have used an analytic approach such as the approach outlined in the Discussion below.







Discussion (10 minutes)

Ask students to explain their reasoning for each of the three systems in the Opening Exercise with both approaches shown for each part. This means that six students should have the opportunity to present their solutions to the class. It is important that you go through both the analytical and graphical approaches for each system so that students draw the connection between graphs that do not intersect and systems that have no analytic solution. Be sure that you have some way to display the graph of each system of equations as you lead students through this discussion.

Part (a):

- Looking at the graph of the first system $\begin{cases} 2x - 4y = -1 \\ 3x - 6y = 4 \end{cases}$, how can we tell that there is no solution?
  - *The two lines never intersect.*
  - *The two lines are parallel.*
- Using an algebraic approach, how can we tell that there is no solution?
  - *If we multiply both sides of the top equation by* 3 *and the bottom equation by* 2, *we see that an equivalent system can be written.*

    $$6x - 12y = -3$$
    $$6x - 12y = 8$$

    Subtracting the first equation from the second results in the false number sentence

    $$0 = 11.$$

    Thus, there are no real numbers x *and* y *that satisfy both equations.*

- The graphs of these equations are lines. What happens if we put them in slope-intercept form?
    - *Rewriting both linear equations in slope-intercept form, the system from part (a) can be written as*
    $$y = \frac{1}{2}x + \frac{1}{4}$$
    $$y = \frac{1}{2}x - \frac{2}{3}.$$
    From what we know about graphing lines, the lines associated to these equations have the same slope and different y-intercepts, so they will be parallel. Since parallel lines do not intersect, the lines have no points in common and, therefore, this system has no solution.

Part (b):

- Looking at the graph of the second system $\begin{cases} y = x^2 - 2 \\ y = 2x - 5 \end{cases}$, how can we tell that there is no solution?
    - *The line and the parabola never intersect.*
- Can we confirm, algebraically, that the system in part (b) has no real solution?
    - *Yes. Since $y = x^2 - 2$ and $y = 2x - 5$, we must have $x^2 - 2 = 2x - 5$, which is equivalent to the quadratic equation $x^2 - 2x + 3 = 0$. Solving for x using the quadratic formula, we get*
    $$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)} = 1 \pm \frac{\sqrt{-8}}{2}.$$
    Since the square root of a negative real number is not a real number, there is no real number x that satisfies the equation $x^2 - 2x + 3 = 0$; therefore, there is no point in the plane with coordinates (x, y) that satisfies both equations in the original system.

Part (c):

- Looking at the graph of the final system $\begin{cases} x^2 + y^2 = 1 \\ x^2 + y^2 = 4 \end{cases}$, how can we tell that there is no solution?
    - *The circles are concentric, meaning that they have the same center and different radii. Thus, they never intersect, and there are no points that lie on both circles.*
- Can we algebraically confirm that the system in part (c) has no solution?
    - *Yes. If we try to solve this system, we could subtract the first equation from the second, giving the false number sentence $0 = 3$. Since this statement is false, we know that there are no values of x and y that satisfy both equations simultaneously; thus, the system has no solution.*

At this point, ask students to summarize in writing or with a partner what they have learned so far. Use this brief exercise as an opportunity to check for understanding.

Exercise 1 (4 minutes)

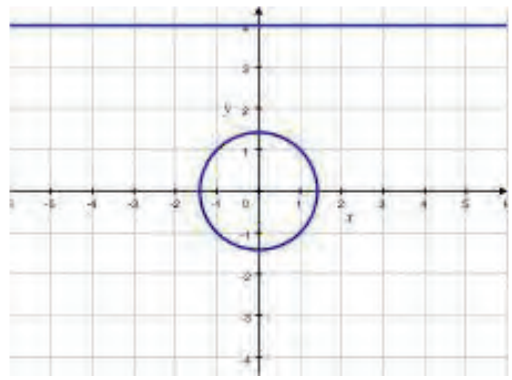
Have students work individually, and then check their answers with a partner. Make sure they write out their steps as done in the sample solutions. After a few minutes, invite students to share one or two solutions on the board.

Exercises 1–4

1. **Are there any real number solutions to the system $\begin{cases} y = 4 \\ x^2 + y^2 = 2 \end{cases}$? Support your findings both analytically and graphically.**

$\boldsymbol{x^2 + (4)^2 = 2}$
$\boldsymbol{x^2 + 16 = 2}$
$\boldsymbol{x^2 = -14}$

Since x^2 is non-negative for all real numbers x, there are no real numbers x so that $x^2 = -14$. Then, there is no pair of real numbers (x, y) that solves the system consisting of the line $y = 4$ and the circle $x^2 + y^2 = 2$. Thus, the line $y = 4$ does not intersect the circle $x^2 + y^2 = 2$ in the real plane. This is confirmed graphically as follows.



Discussion (7 minutes)

We are still withholding any mention of complex numbers or complex solutions; those will be introduced in the next lesson. Make sure your students understand that analytical findings can be confirmed graphically and vice-versa. We turn our focus to quadratic equations in one variable x without real solutions and to how the absence of any real solution x can be confirmed by graphing a system of equations with two variables x and y.

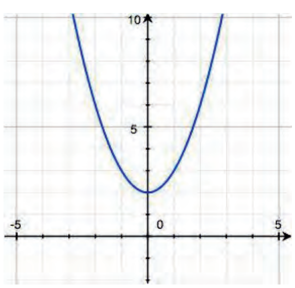
Scaffolding:

Feel free to assign an optional extension exercise, such as: "Which of these equations will have no solution? Explain how you know in terms of a graph."

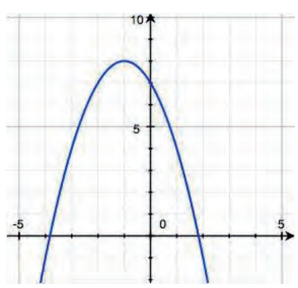
$x^2 + 5 = 0$
$x^2 - 4 = 0$
$x^2 + 1 = 0$
$x^2 - 10 = 0$

Solution: $x^2 + 5 = 0$ and $x^2 + 1 = 0$ will not have real solutions because the graphs of the equations $y = x^2 + 5$ and $y = x^2 + 1$ do not intersect the x-axis, the line given by $y = 0$.

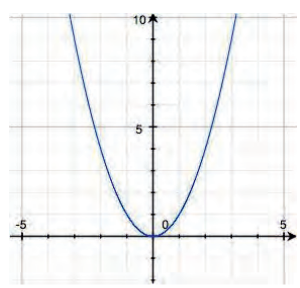
Parabola 1



Parabola 2



Parabola 3



Present students with the following graphs of parabolas:

- Remember that a parabola with a vertical axis of symmetry is the graph of an equation of the form $y = ax^2 + bx + c$ for some real number coefficients a, b, and c with $a \neq 0$. We can consider the solutions of the quadratic equation $ax^2 + bx + c = 0$ to be the x-coordinates of solutions to the system of equations $y = ax^2 + bx + c$ and $y = 0$. Thus, when we are investigating whether a quadratic equation $ax^2 + bx + c = 0$ has a solution, we can think of this as finding the x-intercepts of the graph of $y = ax^2 + bx + c$.
- Which of these three parabolas are represented by a quadratic equation $y = ax^2 + bx + c$ that has no solution to $ax^2 + bx + c = 0$? Explain how you know.
    - *Parabola 1 because its graph does not intersect the x-axis. No x-intercepts of the parabola means there are no solutions to the associated equation $ax^2 + bx + c = 0$.*

- Now, consider Parabola 2, which is the graph of the equation $y = 8 - (x+1)^2$. How many solutions are there to the equation $8 - (x+1)^2 = 0$? Explain how you know.
  - *Because Parabola 2 intersects the x-axis twice, the system consisting of $y = 8 - (x+1)^2$ and $y = 0$ has two real solutions. The graph suggests that the system will have one positive solution and one negative solution.*
- Now, consider Parabola 3, which is the graph of the equation $y = x^2$. How does the graph tell us how many solutions there are to the equation $x^2 = 0$? Explain how you know.
  - *Parabola 3 touches the x-axis only at $(0,0)$ so the parabola and the line with equation $y = 0$ intersect at only one point. Accordingly, the system has exactly one solution, and there is exactly one solution to the equation $x^2 = 0$.*

Pause, and ask students to again summarize what they have learned, either in writing or orally to a neighbor. Students should be making connections between the graph of the quadratic equation $y = ax^2 + bx + c$ (which is a parabola), the number of x-intercepts of the graph, and the number of solutions to the system consisting of $y = 0$ and $y = ax^2 + bx + c$.

Exercises 2–4 (12 minutes)

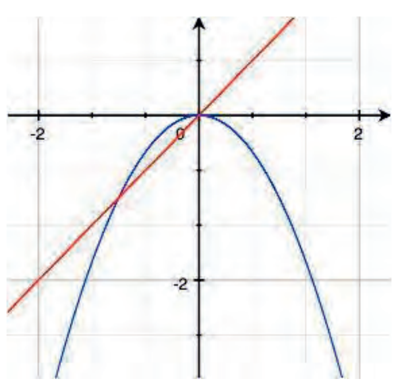
Students should work individually or in pairs on these exercises. To solve these problems analytically, they need to understand that they can determine the x-coordinates of the intersection points of the graphs of these geometric figures by solving an equation. Make sure your students are giving their answers to these questions as coordinate pairs. Encourage students to solve the problems analytically and verify the solutions graphically.

2. Does the line $y = x$ intersect the parabola $y = -x^2$? If so, how many times, and where? Draw graphs on the same set of axes.

$$\begin{aligned} x &= -x^2 \\ x + x^2 &= 0 \\ x(1+x) &= 0 \\ x &= 0 \text{ or } x = -1 \end{aligned}$$

If $x = 0$, then $y = -x^2 = 0$, and if $x = -1$, then $y = -x^2 = -(-1)^2 = -1$.

The line $y = x$ intersects the parabola $y = -x^2$ at two distinct points: $(0,0)$ and $(-1,-1)$.



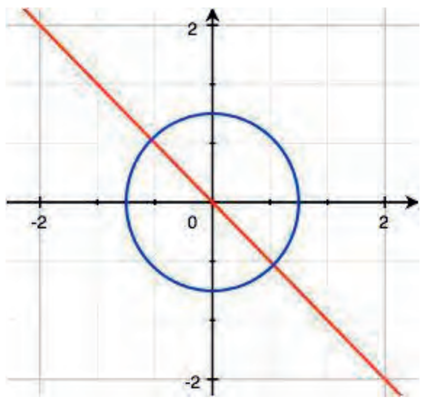
Scaffolding:

- Consider having students follow the instructor along using a graphing calculator to show that the graph of $y = x$ intersects the graph of $y = -x^2$ twice, at the points indicated.
- Consider tasking advanced students to generate a system that meets certain criteria. For example, ask them to write the equations of a circle and a parabola that intersect once at $(0,1)$. One appropriate answer is $x^2 + y^2 = 1$ and $y = x^2 + 1$.

3. **Does the line $y = -x$ intersect the circle $x^2 + y^2 = 1$? If so, how many times, and where? Draw graphs on the same set of axes.**

$$x^2 + (-x)^2 = 1$$
$$2x^2 = 1$$
$$x^2 = \frac{1}{2}$$
$$x = -\frac{\sqrt{2}}{2} \text{ or } x = \frac{\sqrt{2}}{2}$$

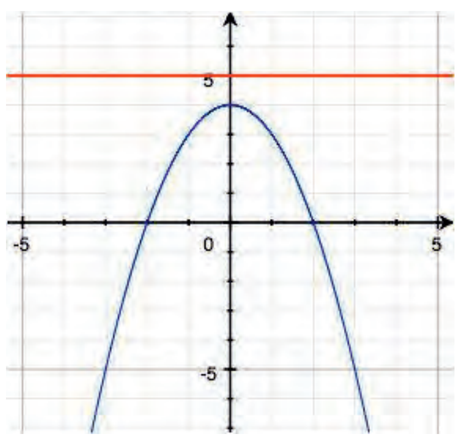
The line $y = -x$ intersects the circle $x^2 + y^2 = 1$ at two distinct points: $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.



4. **Does the line $y = 5$ intersect the parabola $y = 4 - x^2$? Why or why not? Draw the graphs on the same set of axes.**

$$5 = 4 - x^2$$
$$1 = -x^2$$
$$x^2 = -1$$

A squared real number cannot be negative, so the line $y = 5$ does not intersect the parabola $y = 4 - x^2$.



Before moving on, discuss these results as a whole class. Have students put both graphical and analytical solutions to each exercise on the board. Start to reinforce the connection that when the graphs intersect, the related system of equations has real solutions, and when the graphs do not intersect, there are no real solutions to the related system of equations.

Closing (2 minutes)

Have students discuss with their neighbors the key points from today's lesson. Encourage them to discuss the relationship between the solution(s) to a quadratic equation of the form $ax^2 + bx + c = 0$ and the system

$$y = ax^2 + bx + c$$
$$y = 0.$$

They should discuss an understanding of the relationship between any solution(s) to a system of two equations and the x-coordinate of any point(s) of intersection of the graphs of the equations in the system.

The Lesson Summary below contains key findings from today's lesson.

Lesson Summary

An equation or a system of equations may have one or more solutions in the real numbers, or it may have no real number solution.

Two graphs that do not intersect in the real plane describe a system of two equations without a real solution. If a system of two equations does not have a real solution, the graphs of the two equations do not intersect in the real plane.

A quadratic equation in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$, that has no real solution indicates that the graph of $y = ax^2 + bx + c$ does not intersect the x-axis.

Exit Ticket (4 minutes)

In this Exit Ticket, students will show that a particular system of two equations has no real solutions. They will demonstrate this both analytically and graphically.

Name ____ Date ____________________

Lesson 36: Overcoming a Third Obstacle—What If There Are No Real Number Solutions?

Exit Ticket

Solve the following system of equations or show that it does not have a real solution. Support your answer analytically and graphically.

$$y = x^2 - 4$$
$$y = -(x + 5)$$

Exit Ticket Sample Solutions

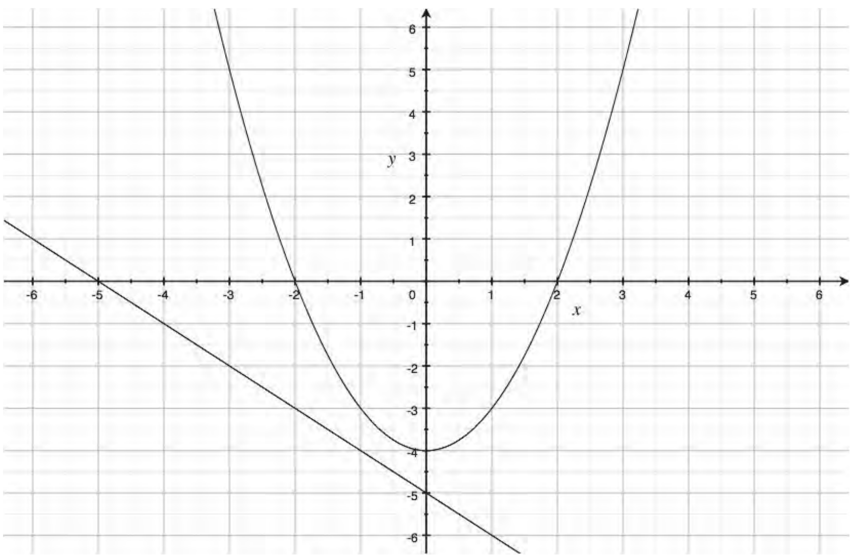
Solve the following system of equations or show that it does not have a real solution. Support your answer analytically and graphically.

$$y = x^2 - 4$$
$$y = -(x+5)$$

We distribute over the set of parentheses in the second equation and rewrite the system.

$$y = x^2 - 4$$
$$y = -(x+5)$$

The graph of the system shows a parabola and a line that do not intersect. As such, we know that the system does not have a real solution.



Algebraically,

$$x^2 - 4 = -x - 5$$
$$x^2 + x + 1 = 0.$$

Using the quadratic formula with $a = 1$, $b = 1$, and $c = 1$,

$$x = \frac{-1 + \sqrt{1^2 - 4(1)(1)}}{2(1)} \text{ or } x = \frac{-1 - \sqrt{1^2 - 4(1)(1)}}{2(1)},$$

which indicates that the solutions would be $\frac{-1+\sqrt{-3}}{2}$ and $\frac{-1-\sqrt{-3}}{2}$.

Since the square root of a negative number is not a real number, there is no real number x that solves this equation. Thus, the system has no solution (x, y) where x and y are real numbers.

Problem Set Sample Solutions

1. For each part, solve the system of linear equations, or show that no real solution exists. Graphically support your answer.

   a. $4x + 2y = 9$
   $x + y = 3$

   Multiply the bottom equation by 4.

   $$4x + 2y = 9$$
   $$4x + 4y = 12$$

   Subtract top from bottom.

   $$2y = 3$$

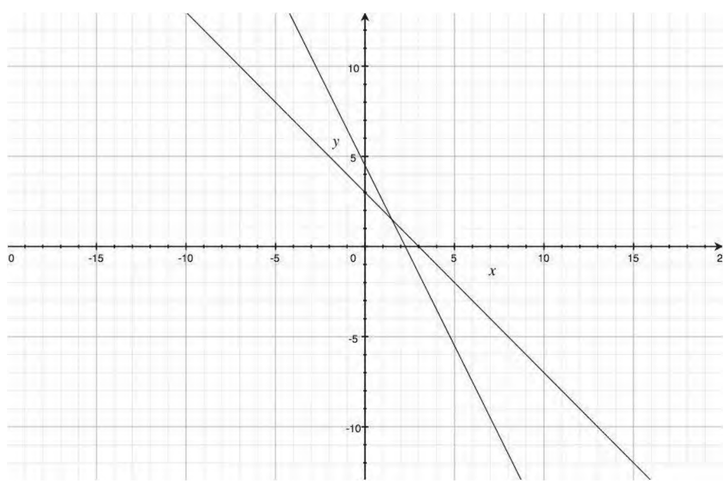
   We get $y = \frac{3}{2}$.

   Substitute $\frac{3}{2}$ *for* y *in the original bottom equation.*

   $$x + \frac{3}{2} = 3$$

   We get $x = \frac{3}{2}$.

   The lines from the system intersect at $\left(\frac{3}{2}, \frac{3}{2}\right)$.

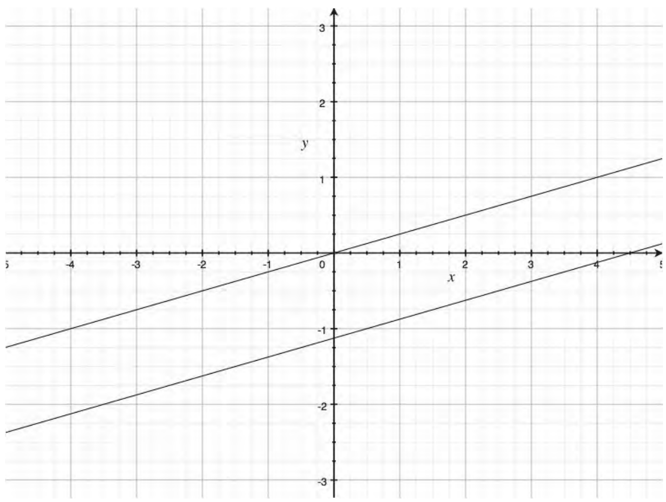


b. $2x - 8y = 9$
$3x - 12y = 0$

Multiply the top equation by 3 and the bottom equation by 2 on both sides.

$$6x - 24y = 27$$
$$6x - 24y = 0$$

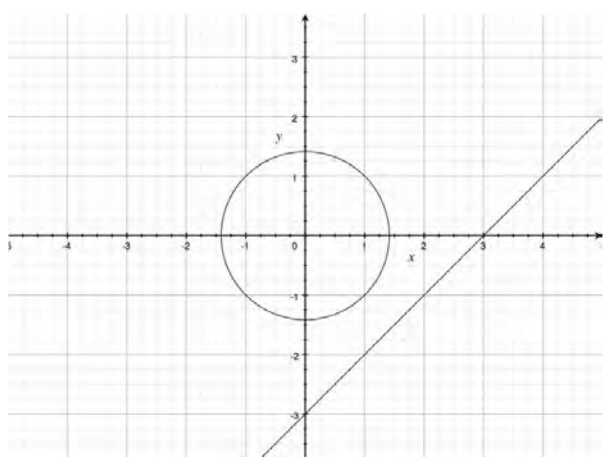
Subtracting the bottom equation from the top equation gives $27 = 0$, but $27 = 0$ is a false number sentence. Thus, there is no solution to the system. The graph of the system appropriately shows two parallel lines.



2. Solve the following system of equations, or show that no real solution exists. Graphically confirm your answer.

$$3x^2 + 3y^2 = 6$$
$$x - y = 3$$

We can factor out 3 from the top equation and isolate y in the bottom equation to give us a better idea of what the graphs of the equations in the system look like. The first equation represents a circle centered at the origin with radius $\sqrt{2}$, and the second equation represent the line $y = x - 3$, which is simply the $45°$ line through the origin, $y = x$, shifted down by 3 units.



Algebraically,

$$3x^2 + 3(x-3)^2 = 6$$
$$x^2 + (x-3)^2 = 2$$
$$x^2 + (x^2 - 6x + 9) = 2$$
$$2x^2 - 6x + 7 = 0$$

We solve for x using the quadratic formula:

$a = 2, \ b = -6, \ c = 7$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2 \cdot 7)}}{2 \cdot 2}$$

$$x = \frac{6 \pm \sqrt{36 - 56}}{4}.$$

The solutions would be $\frac{6+\sqrt{-20}}{4}$ and $\frac{6-\sqrt{-20}}{4}$.

Since both solutions for x contain a square root of a negative number, no real solution x exists; so, the system has no solution (x, y) where x and y are real numbers.

3. Find the value of k so that the graph of the following system of equations has no solution.

$$3x - 2y - 12 = 0$$
$$kx + 6y - 10 = 0$$

First, we rewrite the linear equations in the system in slope-intercept form.

$$y = \frac{3}{2}x - 6$$
$$y = -\frac{k}{6}x + \frac{10}{6}$$

There is no solution to this system when the lines are parallel. Two lines are parallel when they share the same slope and have different y-intercepts. Here, the first line has slope $\frac{3}{2}$ and y-intercept -6, and the second line has slope $-\frac{k}{6}$ and y-intercept $\frac{10}{6}$. The lines have different y-intercepts and will be parallel when $-\frac{k}{6} = \frac{3}{2}$.

$$\frac{3}{2} = -\frac{k}{6}$$
$$2k = -18$$
$$k = -9$$

Thus, there is no solution only when $k = -9$.

4. Offer a geometric explanation to why the equation $x^2 - 6x + 10 = 0$ has no real solutions.

The graph of $y = x^2 - 6x + 10$ opens upward (since the leading coefficient is positive) and takes on its lowest value at the vertex $(3, 1)$. Hence, it does not intersect the x-axis, and, therefore, the equation has no real solutions.

5. Without his pencil or calculator, Joey knows that $2x^3 + 3x^2 - 1 = 0$ has at least one real solution. How does he know?

The graph of every cubic polynomial function intersects the x-axis at least once because the end behaviors are opposite: one end goes up and the other goes down. This means that the graph of any cubic equation $y = ax^3 + bx^2 + cx + d$ must have at least one x-intercept. Thus, every cubic equation must have at least one real number solution.

MP.3

6. The graph of the quadratic equation $y = x^2 + 1$ has no x-intercepts. However, Gia claims that when the graph of $y = x^2 + 1$ is translated by a distance of 1 in a certain direction, the new (translated) graph would have exactly one x-intercept. Further, if $y = x^2 + 1$ is translated by a distance greater than 1 in the same direction, the new (translated) graph would have exactly two x-intercepts. Support or refute Gia's claim. If you agree with her, in which direction did she translate the original graph? Draw graphs to illustrate.

By translating the graph of $y = x^2 + 1$ DOWN by 1 unit, the new graph has equation $y = x^2$, which has one x-intercept at $x = 0$. When translating the original graph DOWN by more than 1 unit, the new graph will cross the x-axis exactly twice.

7. In the previous problem, we mentioned that the graph of $y = x^2 + 1$ has no x-intercepts. Suppose that $y = x^2 + 1$ is one of two equations in a system of equations and that the other equation is a line. Give an example of a linear equation such that this system has exactly one solution.

The line with equation $y = 1$ is tangent to $y = x^2 + 1$ only at $(0, 1)$; so, there would be exactly one real solution to the system.

$$y = x^2 + 1$$
$$y = 1$$

Another possibility is an equation of any vertical line, such as $x = -3$ or $x = 4$, or $x = a$ for any real number a.

8. **In prior problems, we mentioned that the graph of $y = x^2 + 1$ has no x-intercepts. Does the graph of $y = x^2 + 1$ intersect the graph of $y = x^3 + 1$?**

Setting these equations together, we can rearrange terms to get $x^3 - x^2 = 0$, which is an equation we can solve by factoring. We have $x^2(x - 1) = 0$, which has solutions at 0 and 1. Thus, the graphs of these equations intersect when $x = 0$ and when $x = 1$. When $x = 0$, $y = 1$, and when $x = 1$, $y = 2$. Thus, the two graphs intersect at the points $(0, 1)$ and $(1, 2)$.

The quick answer: The highest term in both equations has degree 3. The third degree term does not cancel when setting the two equations (in terms of x) equal to each other. All cubic equations have at least one real solution, so the two graphs intersect at least at one point.

Lesson 37: A Surprising Boost from Geometry

Student Outcomes

- Students define a complex number in the form $a + bi$, where a and b are real numbers and the imaginary unit i satisfies $i^2 = -1$. Students geometrically identify i as a multiplicand effecting a $90°$ counterclockwise rotation of the real number line. Students locate points corresponding to complex numbers in the complex plane.
- Students understand complex numbers as a superset of the real numbers; i.e., a complex number $a + bi$ is real when $b = 0$. Students learn that complex numbers share many similar properties of the real numbers: associative, commutative, distributive, addition/subtraction, multiplication, etc.

Lesson Notes

Students first receive an introduction to the imaginary unit i and develop an algebraic and geometric understanding of the complex numbers (**N-CN.A.1**). The lesson then underscores that complex numbers also satisfy the properties of operations that real numbers do (**N-CN.A.2**). Finally, students perform exercises to reinforce their understanding of and facility with complex numbers in an algebraic arena. This lesson ties into the work in the next lesson, which involves complex solutions to quadratic equations (**N-CN.C.7**).

Complex numbers are neither *imaginary*, as in make believe, nor *complex*, as in complicated. Students first encounter them when they classify equations such as $x^2 + 1 = 0$ as having no real number solutions. At that point, we do not introduce the possibility that a solution exists within a superset of the real numbers called the complex numbers. At the end of this module, we briefly introduce the idea that every polynomial P of degree n has n values r_i for which $P(r_i) = 0$, where n is a whole number and r_i is a real or complex number. Further, in preparation for students' work in Precalculus, we state (but do not expect students to know) that P can be written as the product of n linear factors, a result known as the Fundamental Theorem of Algebra. The usefulness of complex numbers as solutions to polynomial equations comes with a cost: While real numbers can be ordered (put in order from smallest to greatest), complex numbers cannot be compared; for example, the complex number $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ is not larger or smaller than $\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$. However, this is a small price to pay. Students will begin to see just how important complex numbers are to geometry and computer science in Modules 1 and 2 in Precalculus. In college level science and engineering courses, complex numbers are used in conjunction with differential equations to model circular motion and periodic phenomena in two dimensions.

Classwork

Opening (1 minute)

We introduce a geometric context for complex numbers by demonstrating the analogous relationship between rotations in the plane and multiplication. The intention is for students to develop a deep understanding of i through geometry.

- Today, we define a new number system that allows us to identify solutions to some equations that have no real number solutions. The complex numbers, as you will see, in fact share many properties with the real numbers with which you are familiar. We will be taking a geometric approach to introducing complex numbers.

Opening Exercise (5 minutes)

Have students work alone on this motivating Opening Exercise. This exercise provides the context and invites the necessity for defining an alternative number system, namely the complex numbers. Go over parts (a), (b), and (c) with the class; then, suggest that part (d) may be solvable using an alternative number system. Have students table this thought while beginning a geometrically-oriented discussion.

> *Scaffolding:*
> - There were times in the past when people would have said that an equation such as $x^2 = 2$ also had no solution.

Opening Exercise

Solve each equation for x.

a. $x - 1 = 0$ — 1

b. $x + 1 = 0$ — -1

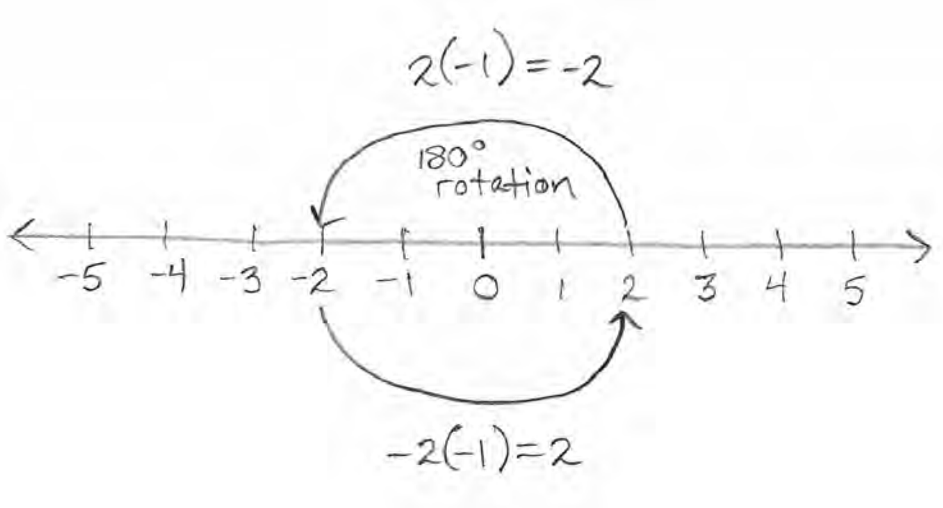
c. $x^2 - 1 = 0$ — $1, -1$

d. $x^2 + 1 = 0$ — *No real solution*

Discussion (20 minutes)

Before beginning, allow students to prepare graph paper for drawing images as the discussion unfolds. At the close of this discussion, have students work with partners to summarize at least one thing they learned; then, provide time for some teacher-guided note-taking to capture the definition of the imaginary unit and its connection to geometric rotation.

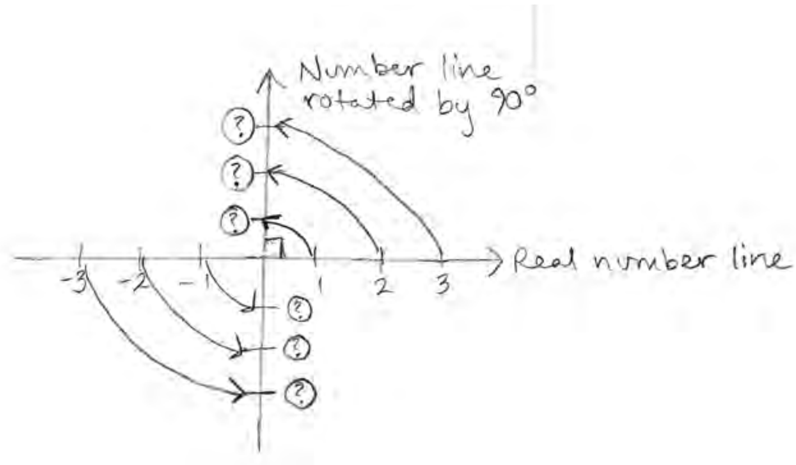
Recall that multiplying by -1 rotates the number line in the plane by $180°$ about the point 0.



> *Scaffolding:*
> - You can demonstrate the rotation concept by drawing the number line carefully on a piece of white paper, drawing an identical number line on a transparency, putting a pin at zero, and rotating the transparency to show that the number line is rotating. For example, go from 2 to -2. This, of course, is the same as multiplying by -1.

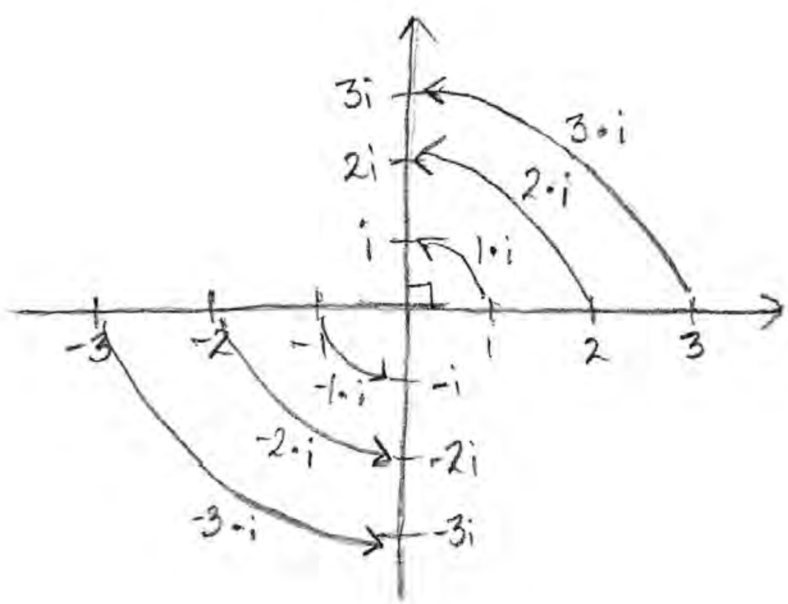
Pose this interesting thought question to students: Is there a number we can multiply by that corresponds to a $90°$ rotation?

Students may find that this is a strange question. First, such a number *does not* take the number line to itself, so we have to *imagine* another number line that is a $90°$ rotation of the original:

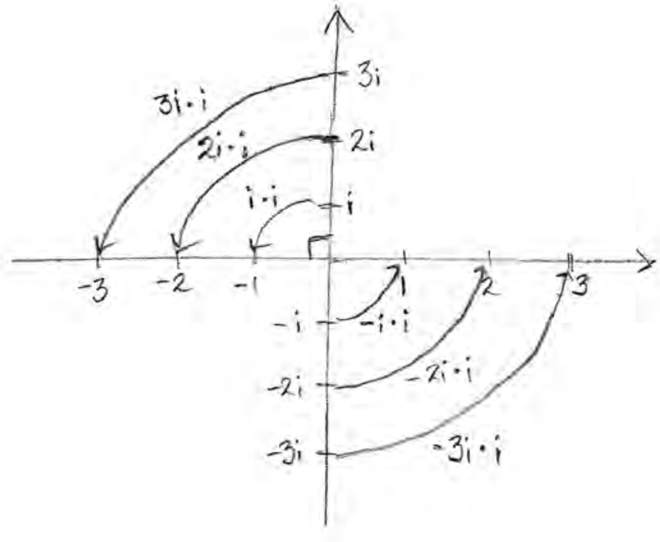


This is like the coordinate plane. However, how should we label the points on the vertical axis?

Well, since we *imagined* such a number existed, let's call it the imaginary axis and subdivide it into units of something called i. Then, the point 1 on the number line rotates to $1 \cdot i$ on the rotated number line and so on, as follows:



- What happens if we multiply a point on the vertical number line by i?
  - *We rotate that point by* $90°$ *counterclockwise:*



When we perform two $90°$ rotations, it is the same as performing a $180°$ rotation, so multiplying by i twice results in the same rotation as multiplying by -1. Since two rotations by $90°$ is the same as a single rotation by $180°$, two rotations by $90°$ is equivalent to multiplication by i twice, and one rotation by $180°$ is equivalent to multiplication by -1, we have

MP.2

$$i^2 \cdot x = -1 \cdot x$$

for any real number x; thus,

$$i^2 = -1.$$

- Why might this new number i be useful?
  - *Recall from the Opening Exercise that there are no real solutions to the equation*
    $$x^2 + 1 = 0.$$
    However, this new number i is a solution.

$$(i)^2 + 1 = -1 + 1 = 0$$

In fact, "solving" the equation $x^2 + 1 = 0$, we get

$$x^2 = -1$$
$$\sqrt{x^2} = \sqrt{-1}$$
$$x = \sqrt{-1} \text{ or } x = -\sqrt{-1}.$$

However, because we know from above that $i^2 = -1$, and $(-i)^2 = (-1)^2(i)^2 = -1$, we have two solutions to the quadratic equation $x^2 = -1$, which are i and $-i$.

These result suggests that "$i = \sqrt{-1}$." That seems a little weird, but this new imagined number i already appears to solve problems we could not solve before.

For example, in Algebra I, when we applied the quadratic formula to

$$x^2 + 2x + 5 = 0,$$

we found that

$$x = \frac{-2+\sqrt{2^2-4(1)(5)}}{2(1)} \text{ or } x = \frac{-2-\sqrt{2^2-4(1)(5)}}{2(1)}$$

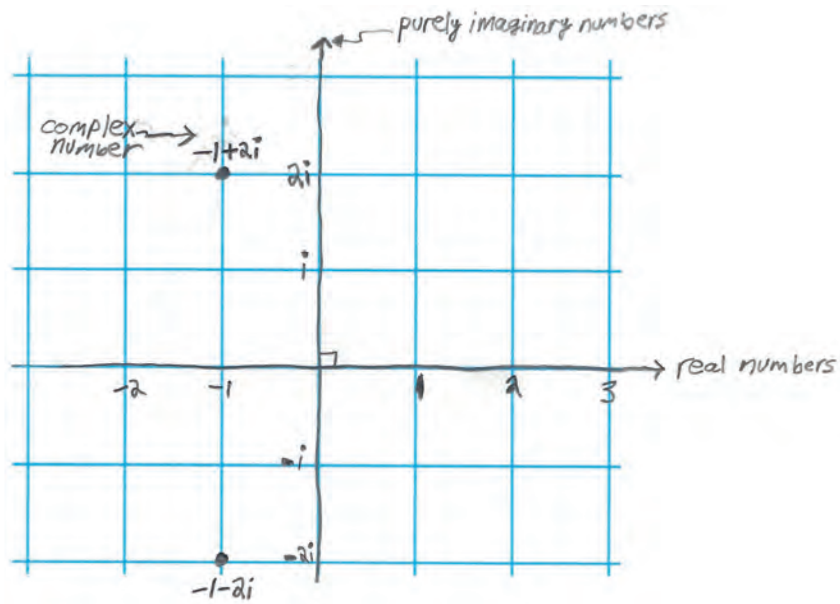
$$x = \frac{-2+\sqrt{-16}}{2} \text{ or } x = \frac{-2-\sqrt{-16}}{2}.$$

Recognizing the negative number under the square root, we reported that the equation $x^2 + 2x + 5 = 0$ has no real solutions. Now, however, we can write

$$\sqrt{-16} = \sqrt{16 \cdot -1} = \sqrt{16} \cdot \sqrt{-1} = 4i.$$

Therefore, $x = -1 + 2i$ or $x = -1 - 2i$, which means $-1 + 2i$ and $-1 - 2i$ are the solutions to $x^2 + 2x + 5 = 0$.

The solutions $-1 + 2i$ and $-1 - 2i$ are numbers called complex numbers, which we can locate in the complex plane.



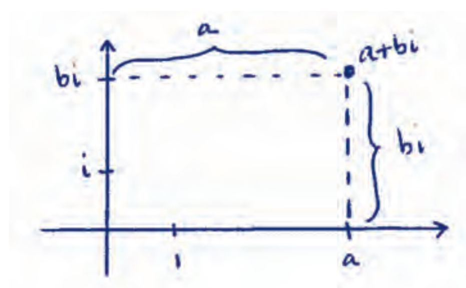
Scaffolding:

- Name a few complex numbers for students to plot on their graph paper. This will build an understanding of their locations in this coordinate system. For example, consider $-2i - 3$, $-i$, i, $i - 1$, and $\frac{3}{2}i + 2$. Make sure students are also cognizant of the fact that real numbers are also complex numbers, e.g., $-\frac{3}{2}$, 0, 1, π.

In fact, all complex numbers can be written in the form

$$a + bi,$$

where a and b are real numbers. Just as we can represent real numbers on the number line, we can represent complex numbers in the complex plane. Each complex number $a + bi$ can be located in the complex plane in the same way we locate the point (a, b) in the Cartesian plane. From the origin, translate a units horizontally along the real axis and b units vertically along the imaginary axis.



Since complex numbers are built from real numbers, we should be able to add, subtract, multiply, and divide them. They should also satisfy the commutative, associative, and distributive properties, just as real numbers do.

Let's check how some of these operations work for complex numbers.

Examples 1–2 (4 minutes): Addition and Subtraction with Complex Numbers

MP.7

Addition of variable expressions is a matter of re-arranging terms according to the properties of operations. Often, we call this "combining like terms." These properties of operations apply to complex numbers.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Example 1: Addition with Complex Numbers

Compute $(3 + 4i) + (7 - 20i)$.

$(3 + 4i) + (7 - 20i) = 3 + 4i + 7 - 20i = (3 + 7) + (4 - 20)i = 10 - 16i$

Example 2: Subtraction with Complex Numbers

Compute $(3 + 4i) - (7 - 20i)$.

$(3 + 4i) - (7 - 20i) = 3 + 4i - 7 + 20i = (3 - 7) + (4 + 20)i = -4 + 20i$

Scaffolding:

If necessary, further examples of addition and multiplication with complex numbers are as follows:

- $(6 - i) + (3 - 2i) = 9 - 3i$
- $(3 + 2i)(-3 + 2i) = -13$
- $(5 + 4i)(2 - i) = 14 + 3i$
- $(2 + \sqrt{3}\,i)(-2 + \sqrt{3}\,i) = -7$
- $(1 - 6i)^2 = 37 - 12i$
- $(-3 - i)((2 - 4i) + (1 + 3i)) = -10$

Examples 3-4 (6 minutes): Multiplication with Complex Numbers

MP.7

Multiplication uses the properties of operations and the fact that $i^2 = -1$. It is analogous to polynomial multiplication.

$$\begin{aligned}(a + bi) \cdot (c + di) &= ac + bci + adi + bdi^2 \\ &= (ac - bd) + (bc + ad)i\end{aligned}$$

Example 3: Multiplication with Complex Numbers

Compute $(1 + 2i)(1 - 2i)$.

$$\begin{aligned}(1 + 2i)(1 - 2i) &= 1 + 2i - 2i - 4i^2 \\ &= 1 + 0 - 4(-1) \\ &= 1 + 4 \\ &= 5\end{aligned}$$

Example 4: Multiplication with Complex Numbers

Verify that $-1 + 2i$ and $-1 - 2i$ are solutions to $x^2 + 2x + 5 = 0$.

$-1 + 2i$:

$$\begin{aligned}(-1 + 2i)^2 + 2(-1 + 2i) + 5 &= 1 - 4i + 4i^2 - 2 + 4i + 5 \\ &= 4i^2 - 4i + 4i + 1 - 2 + 5 \\ &= -4 + 0 + 4 \\ &= 0\end{aligned}$$

$-1 - 2i$:

$$\begin{aligned}(-1 - 2i)^2 + 2(-1 - 2i) + 5 &= 1 + 4i + 4i^2 - 2 - 4i + 5 \\ &= 4i^2 + 4i - 4i + 1 - 2 + 5 \\ &= -4 + 0 + 4 \\ &= 0\end{aligned}$$

So, both complex numbers $-1 - 2i$ and $-1 + 2i$ are solutions to the quadratic equation $x^2 + 2x + 5 = 0$.

Closing (4 minutes)

Close by asking students to write or discuss with a neighbor some reasons for defining the set of complex numbers in the first place. Have them explain the importance of complex numbers satisfying the arithmetic properties of real numbers. How does geometry help explain i?

The Lesson Summary box presents key findings from today's lesson.

Lesson Summary

Multiplying by i rotates every complex number in the complex plane by $90°$ about the origin.

Every complex number is in the form $a + bi$, where a is the real part and b is the imaginary part of the number. Real numbers are also complex numbers; the real number a can be written as the complex number $a + 0i$.

Adding two complex numbers is analogous to combining like terms in a polynomial expression.

Multiplying two complex numbers is like multiplying two binomials, except one can use $i^2 = -1$ to further write the expression in simpler form.

Complex numbers satisfy the associative, commutative, and distributive properties.

Complex numbers can now allow us to find solutions to equations that previously had no real number solutions.

Exit Ticket (5 minutes)

In this Exit Ticket, students reduce a complex expression into its $a + bi$ form and then locate the corresponding point on the complex plane.

Name ____ Date ____________________

Lesson 37: A Surprising Boost from Geometry

Exit Ticket

Express the quantities below in $a + bi$ form, and graph the corresponding points on the complex plane. If you use one set of axes, be sure to label each point appropriately.

$$(1 + i) - (1 - i)$$

$$(1 + i)(1 - i)$$

$$i(2 - i)(1 + 2i)$$

Exit Ticket Sample Solutions

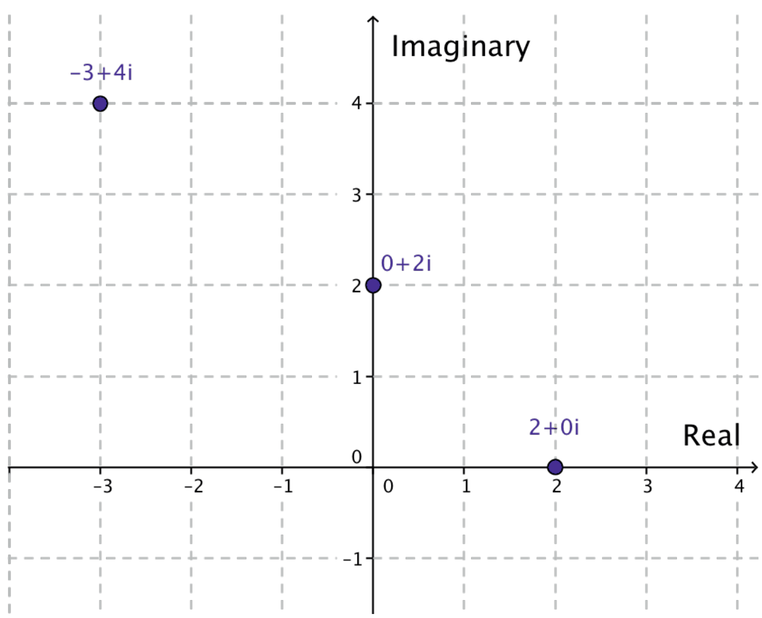
Express the quantities below in $a + bi$ form, and graph the corresponding points on the complex plane. If you use one set of axes, be sure to label each point appropriately.

$$(1+i)-(1-i)$$
$$(1+i)(1-i)$$
$$i(2-i)(1+2i)$$

$$\begin{aligned}(1+i)-(1-i) &= 0+2i\\ &= 2i\end{aligned}$$

$$\begin{aligned}(1+i)(1-i) &= 1+i-i-i^2\\ &= 1-i^2\\ &= 1+1\\ &= 2+0i\\ &= 2\end{aligned}$$

$$\begin{aligned}i(2-i)(1+2i) &= i(2+4i-i-2i^2)\\ &= i(2+3i-2(-1))\\ &= i(2+3i+2)\\ &= i(4+3i)\\ &= 4i+3i^2\\ &= -3+4i\end{aligned}$$

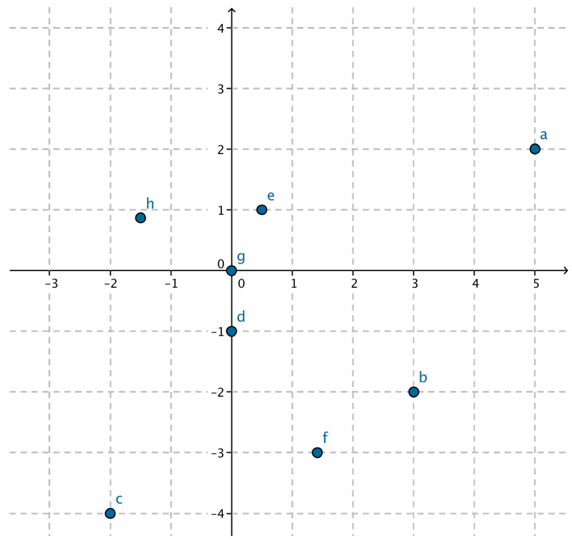


Problem Set Sample Solutions

This problem set offers your students an opportunity to practice and gain facility with complex numbers and complex number arithmetic.

1. Locate the point on the complex plane corresponding to the complex number given in parts (a)–(h). On one set of axes, label each point by its identifying letter. For example, the point corresponding to $5 + 2i$ should be labeled "a."

   a. $5 + 2i$
   b. $3 - 2i$
   c. $-2 - 4i$
   d. $-i$
   e. $\frac{1}{2} + i$
   f. $\sqrt{2} - 3i$
   g. 0
   h. $-\frac{3}{2} + \frac{\sqrt{3}}{2}i$



2. Express each of the following in $a + bi$ form.

   a. $(13 + 4i) + (7 + 5i)$

   $(13 + 7) + (4 + 5)i = 20 + 9i$

   b. $(5 - i) - 2(1 - 3i)$

   $5 - i - 2 + 6i = 3 + 5i$

   c. $\left((5 - i) - 2(1 - 3i)\right)^2$

   $$\begin{aligned}(3 + 5i)^2 &= 9 + 30i + 25i^2\\ &= 9 + 30i + (-25)\\ &= -16 + 30i\end{aligned}$$

   d. $(3 - i)(4 + 7i)$

   $$\begin{aligned}12 - 4i + 21i - 7i^2 &= 12 + 17i - (-7)\\ &= 19 + 17i\end{aligned}$$

   e. $(3 - i)(4 + 7i) - \left((5 - i) - 2(1 - 3i)\right)$

   $$\begin{aligned}(19 + 17i) - (3 + 5i) &= (19 - 3) + (17 - 5)i\\ &= 16 + 12i\end{aligned}$$

3. Express each of the following in $a + bi$ form.

a. $(2+5i)+(4+3i)$

$$\begin{aligned}(2+5i)+(4+3i) &= (2+4)+(5+3)i \\ &= 6+8i\end{aligned}$$

b. $(-1+2i)-(4-3i)$

$$\begin{aligned}(-1+2i)-(4-3i) &= -1+2i-4+3i \\ &= -5+5i\end{aligned}$$

c. $(4+i)+(2-i)-(1-i)$

$$\begin{aligned}(4+i)+(2-i)-(1-i) &= 4+i+2-i-1+i \\ &= 5+i\end{aligned}$$

d. $(5+3\mathrm{i})(3+5\mathrm{i})$

$$\begin{aligned}(5+3i)(3+5i) &= 5\cdot 3+3\cdot 3i+5\cdot 5i+3i\cdot 5i \\ &= 15+9i+25i+15i^2 \\ &= 15+34i-15 \\ &= 0+34i \\ &= 34i\end{aligned}$$

e. $-i(2-i)(5+6i)$

$$\begin{aligned}-i(2-i)(5+6i) &= -i(10-5i+12i-6i^2) \\ &= -i(10+7i+6) \\ &= -i(16+7i) \\ &= -16i-7i^2 \\ &= -16i+7 \\ &= 7-16i\end{aligned}$$

f. $(1+i)(2-3i)+3i(1-i)-i$

$$\begin{aligned}(1+i)(2-3i)+3i(1-i)-i &= (2+2i-3i-3i^2)+3i-3i^2-i \\ &= 2+2i-3i+3+3i+3-i \\ &= 8+i\end{aligned}$$

MP.7

4. Find the real values of x and y in each of the following equations using the fact that if $a+bi=c+di$, then $a=c$ and $b=d$.

a. $5x+3yi=20+9i$

$$5x = 20 \qquad\qquad 3yi = 9i$$
$$x = 4 \qquad\qquad y = 3$$

MP.7

b. $2(5x+9) = (10-3y)i$

$2(5x+9) + 0i = 0 + (10-3y)i$

$$2(5x+9) = 0$$
$$x = -\frac{9}{5}$$

$$0i = (10-3y)i$$
$$10 - 3y = 0$$
$$y = \frac{10}{3}$$

c. $3(7-2x) - 5(4y-3)i = x - 2(1+y)i$

$$3(7-2x) = x$$
$$21 - 6x = x$$
$$21 = 7x$$
$$x = 3$$

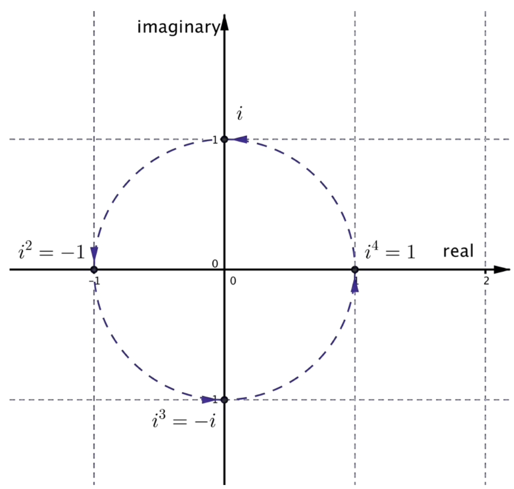
$$-5(4y-3)i = -2(1+y)i$$
$$-5(4y-3) = -2(1+y)$$
$$-20y + 15 = -2 - 2y$$
$$17 = 18y$$
$$y = \frac{17}{18}$$

5. Since $i^2 = -1$, we see that

$$i^3 = i^2 \cdot i = -1 \cdot i = -i$$
$$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1.$$

Plot i, i^2, i^3, and i^4 on the complex plane and describe how multiplication by each rotates points in the complex plane.

Multiplying by i rotates points by $90°$counterclockwise around $(0,0)$. Multiplying by $i^2 = -1$ rotates points by $180°$ about $(0,0)$. Multiplying by $i^3 = -i$ rotates points counterclockwise by $270°$ about the origin, which is equivalent to rotation by $90°$ clockwise about the origin. Multiplying by i^4 rotates points counterclockwise by $360°$, which is equivalent to not rotating at all. The points i, i^2, i^3, and i^4 are plotted below on the complex plane.



6. Express each of the following in $a + bi$ form.

a. i^5 $\quad 0 + i$

b. i^6 $\quad -1 + 0i$

c. i^7 $\quad 0 - i$

d. i^8 $\quad 1 + 0i$

e. i^{102} $\quad -1 + 0i$

MP.8

A simple approach is to notice that every 4 multiplications by i result in four $90°$ rotations, which takes i^4 back to 1. Therefore, divide 102 by 4, which is 25 with a remainder 2. So, 102 $90°$ rotations will take i^{102} onto -1.

7. Express each of the following in $a + bi$ form.

$(1+i)^2$

$$\begin{aligned}(1+i)(1+i) &= 1+i+i+i^2\\ &= 1+2i-1\\ &= 2i\end{aligned}$$

$(1+i)^4$

$$\begin{aligned}(1+i)^4 &= ((1+i)^2)^2\\ &= (2i)^2\\ &= 4i^2\\ &= -4\end{aligned}$$

$(1+i)^6$

$$\begin{aligned}(1+i)^6 &= (1+i)^2(1+i)^4\\ &= (2i)(-4)\\ &= -8i\end{aligned}$$

8. Evaluate $x^2 - 6x$ when $x = 3 - i$.

-10

9. Evaluate $4x^2 - 12x$ when $x = \frac{3}{2} - \frac{i}{2}$.

-10

10. Show by substitution that $\frac{5-i\sqrt{5}}{5}$ is a solution to $5x^2 - 10x + 6 = 0$.

$$\begin{aligned}5\left(\frac{5-i\sqrt{5}}{5}\right)^2 - 10\left(\frac{5-i\sqrt{5}}{5}\right) + 6 &= \frac{1}{5}(5-i\sqrt{5})(5-i\sqrt{5}) - 2(5-i\sqrt{5}) + 6\\ &= \frac{1}{5}(25-10i\sqrt{5}+5i^2) - 2(5-i\sqrt{5}) + 6\\ &= \frac{1}{5}(25-10i\sqrt{5}-5) - 2(5-i\sqrt{5}) + 6\\ &= 5 - 2i\sqrt{5} - 1 - 10 + 2i\sqrt{5} + 6\\ &= 0\end{aligned}$$

MP.7

11. a. Evaluate the four products below.

Evaluate $\sqrt{9}\cdot\sqrt{4}$. $3 \cdot 2 = 6$

Evaluate $\sqrt{9}\cdot\sqrt{-4}$. $3 \cdot 2\mathrm{i} = 6\mathrm{i}$

Evaluate $\sqrt{-9}\cdot\sqrt{4}$. $3\mathrm{i} \cdot 2 = 6\mathrm{i}$

Evaluate $\sqrt{-9}\cdot\sqrt{-4}$. $3\mathrm{i} \cdot 2\mathrm{i} = 6\mathrm{i}^2 = -6$

b. Suppose a and b are positive real numbers. Determine whether the following quantities are equal or not equal.

$\sqrt{a}\cdot\sqrt{b}$ and $\sqrt{-a}\cdot\sqrt{-b}$ *not equal*

$\sqrt{-a}\cdot\sqrt{b}$ and $\sqrt{a}\cdot\sqrt{-b}$ *equal*

Lesson 38: Complex Numbers as Solutions to Equations

Student Outcomes

- Students solve quadratic equations with real coefficients that have complex solutions (**N-CN.C.7**). They recognize when the quadratic formula gives complex solutions and write them as $a + bi$ for real numbers a and b. (**A-REI.B.4b**)

Lesson Notes

This lesson models how to solve quadratic equations over the set of complex numbers. Students relate the sign of the discriminant to the nature of the solution set for a quadratic equation. Continue to encourage students to make connections between the graphs of a quadratic equation, $y = ax^2 + bx + c$, and the number and type of solutions to the equation $ax^2 + bx + c = 0$.

Classwork

Opening (2 minutes)

In Algebra I, students learned that when the quadratic formula resulted in an expression that contained a negative number in the radicand, the equation would have no real solution. Now, we have defined the imaginary unit as $i = \sqrt{-1}$. That allows us to solve quadratic equations over the set of complex numbers and see that every quadratic equation has at least one solution.

> *Scaffolding:*
>
> Advanced students may be able to handle a more abstract framing of—in essence—the same exercise. The exercise below offers the advanced student an opportunity to discover the discriminant and its significance on his or her own.
>
> "Recall that a quadratic equation can have exactly two distinct real solutions, exactly one distinct real solution, or exactly two distinct complex solutions. What is the quadratic formula that we can use to solve an equation in the form $ax^2 + bx + x = 0$, where a, b, and c are real numbers and $a \neq 0$. Analyze this formula to decide when the equation will have two, one, or no real solutions."
>
> Solution:
>
> $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
>
> The "type" of solutions to a quadratic equation hinges on the expression under the radical in the quadratic formula, namely, $b^2 - 4ac$. When $b^2 - 4ac < 0$, both solutions will have imaginary parts. When $b^2 - 4ac > 0$, the quadratic equation has two distinct real solutions. When $b^2 - 4ac = 0$, the quadratic formula simplifies as $x = -\frac{b}{2a}$. In this case, there is only one real solution, which we call a zero of multiplicity two.

Opening Exercises (5 minutes)

Have students work on this opening exercise alone or in pairs. In this exercise, students apply the quadratic formula to three different relatively simple quadratic equations: one with two real roots, one with one real repeated root, and one with two imaginary roots. Students are then asked to explain the results in terms of the discriminant. Afterward, go over the answers with the class.

Review the quadratic formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ before beginning this exercise, and define the *discriminant* as the number under the radical; that is, the discriminant is the quantity $b^2 - 4ac$.

Opening Exercises

1. **Use the quadratic formula to solve the following quadratic equations. Calculate the discriminant for each equation.**

   a. $x^2 - 9 = 0$

   The equation $x^2 - 9 = 0$ has two real solutions: $x = 3$ and $x = -3$. The discriminant of $x^2 - 9 = 0$ is 36.

   b. $x^2 - 6x + 9 = 0$

   The equation $x^2 - 6x + 9 = 0$ has one real solution: $x = 3$. The discriminant of $x^2 - 6x + 9 = 0$ is 0.

   c. $x^2 + 9 = 0$

   The equation $x^2 + 9 = 0$ has two complex solutions: $x = 3i$ and $x = -3i$. The discriminant of $x^2 + 9 = 0$ is -36.

2. **How does the value of the discriminant for each equation relate the number of solutions you found?**

   If the discriminant is negative, we get complex solutions. If the discriminant is zero, we get one real solution. If the discriminant is positive, we get two real solutions.

Discussion (8 minutes)

The expression under the radical in the quadratic formula $b^2 - 4ac$ is called the discriminant.

- Why do you think we call it the discriminant?
  - *In English, a discriminant is a characteristic that allows "something" (e.g., an object, a person, a function) among a group of other "somethings" to be distinguished.*
  - *In this case, the discriminant distinguishes a quadratic equation by its number and type of solutions: one real solution (repeated), two real solutions, or two complex solutions.*

Scaffolding:

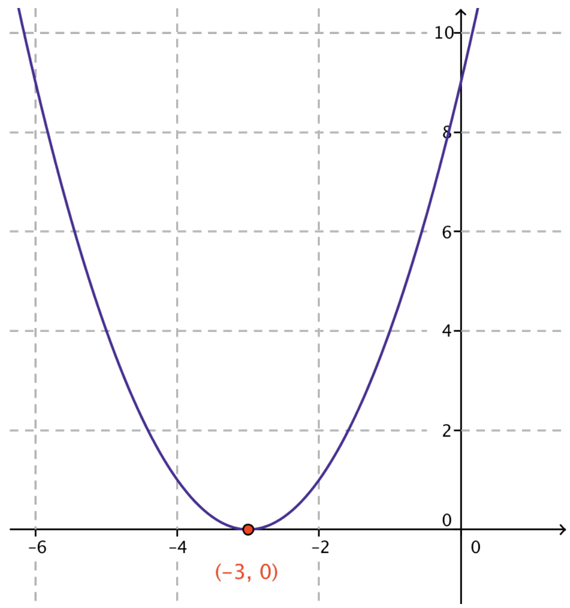
- English Language Learners may benefit from a Frayer diagram or other vocabulary exercise for the word "discriminant."

- Let's examine the situation when the discriminant is zero. Why does a quadratic equation with discriminant zero have only one real solution?
  - *When the discriminant is zero, the quadratic formula gives the single solution $-\frac{b \pm 0}{2a} = -\frac{b}{2a}$.*
- Why is the solution when $b^2 - 4ac = 0$ a repeated zero?
  - *If $b^2 - 4ac = 0$, then $c = \frac{b^2}{4a}$, and we can factor the quadratic expression $ax^2 + bx + c$ as follows:*

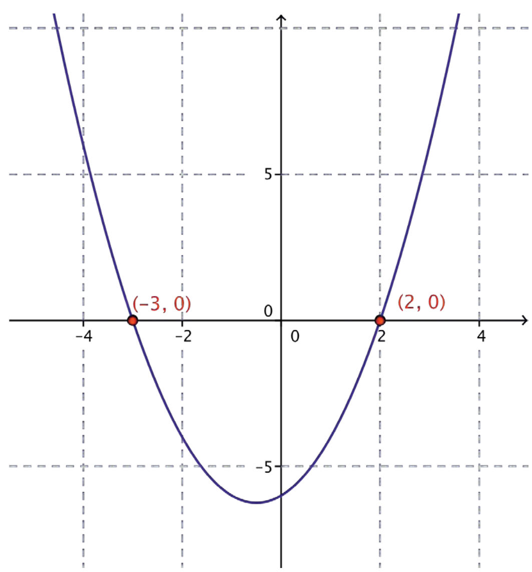
    $$ax^2 + bx + c = ax^2 + bx + \frac{b^2}{4a} = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) = a\left(x + \frac{b}{2a}\right)^2.$$

    From what we know of factoring quadratic expressions from Lesson 11, $-\frac{b}{2a}$ is a repeated zero.
  - *Analytically, the solutions can be thought of as $-\frac{b+0}{2a}$ and $-\frac{b-0}{2a}$, which are both $-\frac{b}{2a}$. So, there are two solutions that are the same number.*
  - *Geometrically, we can write the equation of the parabola as $y = a\left(x + \frac{b}{2a}\right)^2$, so the vertex of this parabola is $\left(-\frac{b}{2a}, 0\right)$, meaning the vertex of the parabola lies on the x-axis. Thus, the parabola is tangent to the x-axis and intersects the x-axis only at the point $\left(-\frac{b}{2a}, 0\right)$.*

- For example, the graph of $y = x^2 + 6x + 9$ intersects the x-axis only at $(-3, 0)$, as follows.

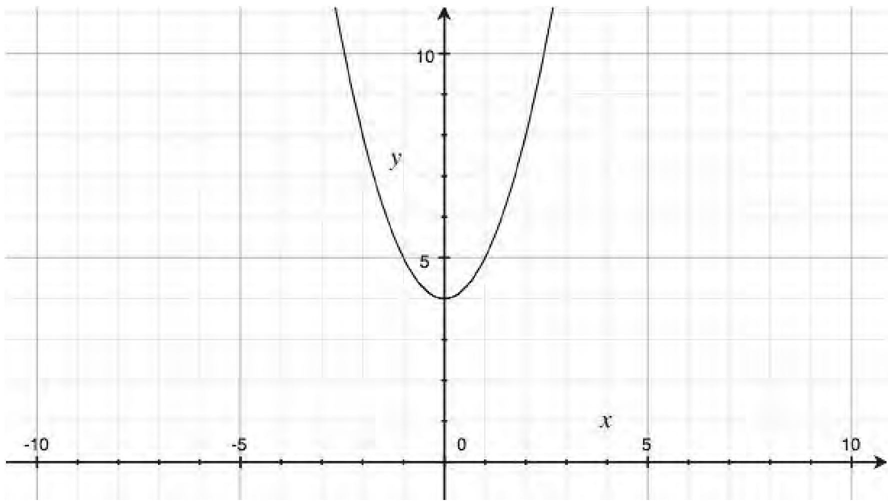


- Describe the graph of a quadratic equation with positive discriminant.
    - *If the discriminant is positive, then the quadratic formula gives two different real solutions.*
    - *Two real solutions mean the graph intersects the x-axis at two distinct real points.*
- For example, the graph of $y = x^2 + x - 6$ intersects the x-axis at $(-3, 0)$ and $(2, 0)$, as follows.



- Describe the graph of a quadratic equation with negative discriminant.
    - *Since the discriminant is negative, the quadratic formula will give two different complex solutions.*
    - *Since there are no real solutions, the graph does not cross or touch the x-axis in the real plane.*

- For example, the graph of $y = x^2 + 4$, shown below, does not intersect the x-axis.



Example 1 (5 minutes)

Consider the equation $3x + x^2 = -7$.

- What does the value of the discriminant tell us about number of solutions to this equation?
  - The equation in standard form is $x^2 + 3x + 7 = 0$.
  - $a = 1, b = 3, c = 7$
  - The discriminant is $3^2 - 4(1)(7) = -19$. The negative discriminant indicates that no real solutions exist. There are two complex solutions.
- Solve the equation. Does the number of solutions match the information provided by the discriminant? Explain.
  - *Using the quadratic formula,* $x = \frac{-3+\sqrt{-19}}{2}$ *or* $x = \frac{-3-\sqrt{-19}}{2}$.
  - *The solutions, in* $a + bi$ *form, are* $-\frac{3}{2} + \frac{\sqrt{19}}{2}i$ *and* $-\frac{3}{2} - \frac{\sqrt{19}}{2}i$.
  - *The two complex solutions are consistent with the rule for a negative discriminant.*

Exercise (15 minutes)

MP.3

Have students work individually on this exercise; then, have them work with a partner or in a small group to check their solutions. You could also conduct this exercise using personal white boards and have your students show their answers to each question after a few minutes. When many students are stuck, invite them to exchange papers with a partner to check for errors. Having students identify errors in their work or the work of others will help them to build fluency when working with these complicated expressions. Debrief this exercise by showing the related graph of the equation in the coordinate plane and verify that the number of solutions corresponds to the number of x-intercepts.

Exercise

Compute the value of the discriminant of the quadratic equation in each part. Use the value of the discriminant to predict the number and type of solutions. Find all real and complex solutions.

a. $x^2 + 2x + 1 = 0$

We have $a = 1$, $b = 2$, and $c = 1$. Then

$$b^2 - 4ac = 2^2 - 4(1)(1) = 0.$$

Note that the discriminant is zero, so this equation has exactly one real solution.

$$x = \frac{-(2) \pm \sqrt{0}}{2(1)} = -1$$

Thus, the only solution is -1.

b. $x^2 + 4 = 0$

We have $a = 1$, $b = 0$, and $c = 4$. Then

$$b^2 - 4ac = -16.$$

Note that the discriminant is negative, so this equation has two complex solutions.

$$x = \frac{-0 \pm \sqrt{-16}}{2(1)}$$

Thus, the two complex solutions are $2i$ and $-2i$.

c. $9x^2 - 4x - 14 = 0$

We have $a = 9$, $b = -4$, and $c = -14$. Then

$$\begin{aligned} b^2 - 4ac &= (-4)^2 - 4(9)(-14) \\ &= 16 + 504 \\ &= 520. \end{aligned}$$

Note that the discriminant is positive, so this equation has two distinct real solutions.

Using the quadratic formula,

$$x = \frac{-(-4) \pm 2\sqrt{130}}{2(9)}.$$

So, the two real solutions are $\frac{2+\sqrt{130}}{9}$ and $\frac{2-\sqrt{130}}{9}$.

d. $3x^2 + 4x + 2 = 0$

We have $a = 3$, $b = 4$, and $c = 2$. Then

$$\begin{aligned} b^2 - 4ac &= 4^2 - 4(3)(2) \\ &= 16 - 24 \\ &= -8. \end{aligned}$$

The discriminant is negative, so there will be two complex solutions. Using the quadratic formula,

$$x = \frac{-4 \pm \sqrt{-8}}{2(3)}.$$

So, the two complex solutions are $-\frac{2}{3} + \frac{\sqrt{2}}{3}i$ and $-\frac{2}{3} - \frac{\sqrt{2}}{3}i$.

e. $x = 2x^2 + 5$

We can rewrite this equation in standard form with $a = 2$, $b = -1$, and $c = 5$:

$$2x^2 - x + 5 = 0.$$

Then

$$\begin{aligned} b^2 - 4ac &= (-1)^2 - 4(2)(5) \\ &= 1 - 40 \\ &= -39. \end{aligned}$$

The discriminant is negative, so there will be two complex solutions. Using the quadratic formula,

$$x = \frac{-(-1) \pm \sqrt{-39}}{2(2)}$$
$$x = \frac{1 \pm i\sqrt{39}}{4}.$$

The two solutions are $\frac{1}{4} + \frac{\sqrt{39}}{4}i$ and $\frac{1}{4} - \frac{\sqrt{39}}{4}i$.

f. $8x^2 + 4x + 32 = 0$

We can factor 4 from the left side of this equation to obtain $4(2x^2 + x + 8) = 0$, and we know that a product is zero when one of the factors are zero. Since $4 \neq 0$, we must have $2x^2 + x + 8 = 0$. This is a quadratic equation with $a = 2$, $b = 1$, and $c = 8$. Then

$$b^2 - 4ac = 1^2 - 4(2)(8) = -63.$$

The discriminant is negative, so there will be two complex solutions. Using the quadratic formula,

$$x = \frac{-1 \pm \sqrt{-63}}{2(2)}$$
$$x = \frac{-1 \pm 3i\sqrt{7}}{4}.$$

The complex solutions are $-\frac{1}{4} + \frac{3\sqrt{7}}{4}i$ and $-\frac{1}{4} - \frac{3\sqrt{7}}{4}i$.

Scaffolding:

You may assign advanced students to create quadratic equations that have specific solutions. For example, request a quadratic equation that has only the solution -5.

Answer: $x^2 - 10x + 25 = 0$. This follows from the expansion of the left side of $(x + 5)^2 = 0$. You may also request a quadratic equation with solution set $3 + i$ and $3 - i$. The answer is $x^2 - 6x + 10 = 0$.

Closing (5 minutes)

As you summarize this lesson, ask your students to create a graphic organizer that allows them to compare and contrast the nature of the discriminant, the number and types of solutions to $ax^2 + bx + c = 0$, and the graphs of the equation $y = ax^2 + bx + c$. Have them record a problem of each type from the previous exercise as an example in their graphic organizer.

Lesson Summary

- **A quadratic equation with real coefficients and a real constant may have real or complex solutions.**
- **Given a quadratic equation $ax^2 + bx + c = 0$, the discriminant $b^2 - 4ac$ indicates whether the equation has two distinct real solutions, one real solution, or two complex solutions.**
  - **If $b^2 - 4ac > 0$, there are two real solutions to $ax^2 + bx + c = 0$.**
  - **If $b^2 - 4ac = 0$, there is one real solution to $ax^2 + bx + c = 0$.**
  - **If $b^2 - 4ac < 0$, there are two complex solutions to $ax^2 + bx + c = 0$.**

Exit Ticket (5 minutes)

The Exit Ticket gives students the opportunity to demonstrate their mastery of this lesson's content.

Name ___________________________ Date ________________

Lesson 38: Complex Numbers as Solutions to Equations

Exit Ticket

Use the discriminant to predict the nature of the solutions to the equation $4x - 3x^2 = 10$. Then, solve the equation.

Exit Ticket Sample Solutions

Use the discriminant to predict the nature of the solutions to the equation $4x - 3x^2 = 10$. Then, solve the equation.

$$3x^2 - 4x + 10 = 0$$

We have $a = 3$, $b = -4$, and $c = 10$. Then

$$\begin{aligned} b^2 - 4ac &= (-4)^2 - 4(3)(10) \\ &= 16 - 120 \\ &= -104. \end{aligned}$$

The value of the discriminant is negative, indicating that there are two complex solutions.

$$x = \frac{-(-4) \pm \sqrt{-104}}{2(3)}$$

$$x = \frac{4 \pm 2i\sqrt{26}}{6}$$

Thus, the two solutions are $\frac{2}{3} + \frac{\sqrt{26}}{3}i$ and $\frac{2}{3} - \frac{\sqrt{26}}{3}i$.

Problem Set Sample Solutions

The Problem Set offers students more practice solving quadratic equations with complex solutions.

1. **Give an example of a quadratic equation in standard form that has…**

   a. **Exactly two distinct real solutions.**

      Since $(x+1)(x-1) = x^2 - 1$, the equation $x^2 - 1 = 0$ has two distinct real solutions, 1 and -1.

   b. **Exactly one distinct real solution.**

      Since $(x+1)^2 = x^2 + 2x + 1$, the equation $x^2 + 2x + 1 = 0$ has only one real solution, 1.

   c. **Exactly two complex (non-real) solutions.**

      Since $x^2 + 1 = 0$ has no solutions in the real numbers, this equation must have two complex solutions. They are i and $-i$.

2. **Suppose we have a quadratic equation $ax^2 + bx + c = 0$ so that $a + c = 0$. Does the quadratic equation have one solution or two distinct solutions? Are they real or complex? Explain how you know.**

   If $a + c = 0$, then either $a = c = 0$, $a > 0$ and $c < 0$, or $a < 0$ and $c > 0$.

   The definition of a quadratic polynomial requires that $a \neq 0$, so either $a > 0$ and $c < 0$ or $a < 0$ and $c > 0$.

   In either case, $ac < 0$. Because b^2 is positive and $4ac$ is positive, we know $b^2 - 4ac > 0$.

   Therefore, a quadratic equation $ax^2 + bx + c = 0$ always has two distinct real solutions when $a + c = 0$.

3. Solve the equation $5x^2 - 4x + 3 = 0$.

We have a quadratic equation with $a = 5$, $b = -4$, and $c = 3$.

$$x = \frac{-(-4) \pm 2\sqrt{-11}}{2(5)}$$

So, the solutions are $\frac{2+i\sqrt{11}}{5}$ *and* $\frac{2-i\sqrt{11}}{5}$.

4. Solve the equation $2x^2 + 8x = -9$.

In standard form, this is the quadratic equation $2x^2 + 8x + 9 = 0$ *with* $a = 2$, $b = 8$, *and* $c = 9$.

$$x = \frac{-8 \pm 2\sqrt{-2}}{2(2)} = \frac{-4 \pm i\sqrt{2}}{2}$$

Thus, the solutions are $2 + \frac{i\sqrt{2}}{2}$ *and* $2 - \frac{i\sqrt{2}}{2}$.

5. Solve the equation $9x - 9x^2 = 3 + x + x^2$.

In standard form, this is the quadratic equation $10x^2 - 8x + 3 = 0$ *with* $a = 10$, $b = -8$, *and* $c = 3$.

$$x = -\frac{-(-8) \pm 2\sqrt{-14}}{2(10)} = \frac{8 \pm 2i\sqrt{14}}{20}$$

Thus, the solutions are $\frac{4+i\sqrt{14}}{10}$ *and* $\frac{4-i\sqrt{14}}{10}$.

6. Solve the equation $3x^2 - x + 1 = 0$.

This is a quadratic equation with $a = 3$, $b = -1$, *and* $c = 1$.

$$x = -\frac{-(-1) \pm \sqrt{-11}}{2(3)} = \frac{1 \pm i\sqrt{11}}{6}$$

Thus, the solutions are $\frac{1+i\sqrt{11}}{6}$ *and* $\frac{1-i\sqrt{11}}{6}$.

7. Solve the equation $6x^4 + 4x^2 - 3x + 2 = 2x^2(3x^2 - 1)$.

When expanded, this is a quadratic equation with $a = 6$, $b = -3$, *and* $c = 2$.

$$6x^4 + 4x^2 - 3x + 2 = 6x^4 - 2x^2$$
$$6x^2 - 3x + 2 = 0$$
$$x = \frac{-(-3) \pm \sqrt{\{-39\}}}{2(6)}$$

So, the solutions are $x = \frac{3+i\sqrt{39}}{12}$ *and* $x = \frac{3-i\sqrt{39}}{12}$.

8. Solve the equation $25x^2 + 100x + 200 = 0$.

We can factor 25 from the left side of this equation to obtain $25(x^2 + 4x + 8) = 0$, and we know that a product is zero when one of the factors is zero. Since $25 \neq 0$, we must have $x^2 + 4x + 8 = 0$. This is a quadratic equation with $a = 1$, $b = 4$, and $c = 8$. Then

$$x = \frac{-4 \pm 4\sqrt{-1}}{2},$$

and the solutions are $-2 + 2i$ and $-2 - 2i$.

9. Write a quadratic equation in standard form such that -5 is its only solution.

$$(x+5)^2 = 0$$
$$x^2 + 10x + 25 = 0$$

MP.2

10. Is it possible that the quadratic equation $ax^2 + bx + c = 0$ has a positive real solution if a, b, and c are all positive real numbers?

No. The solutions are $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$. If b is positive, the second one of these will be negative. So, we need to think about whether or not the first one can be positive. If $-b + \sqrt{b^2 - 4ac} > 0$, then $\sqrt{b^2 - 4ac} > b$; so, $b^2 - 4ac > b^2$, and $-4ac > 0$. This means that either a or c must be negative. So, if all three coefficients are positive, then there cannot be a positive solution to $ax^2 + bx + c = 0$.

11. Is it possible that the quadratic equation $ax^2 + bx + c = 0$ has a positive real solution if a, b, and c are all negative real numbers?

No. If a, b, and c are all negative, then $-a$, $-b$, and $-c$ are all positive. The solutions of $ax^2 + bx + c = 0$ are the same as the solutions to $-ax^2 - bx - c = 0$, and by Problem 10, this equation has no positive real solution since it has all positive coefficients.

Extension:

12. Show that if $k > 3.2$, the solutions of $5x^2 - 8x + k = 0$ are not real numbers.

We have $a = 5$, $b = -8$, and $c = k$, then

$$\begin{aligned} b^2 - 4ac &= (-8)^2 - 4 \cdot 5 \cdot k \\ &= 64 - 20k. \end{aligned}$$

When the discriminant is negative, the solutions of the quadratic function are not real numbers.

$$\begin{aligned} 0 &> 64 - 20k \\ 20k &> 64 \\ k &> \frac{64}{20} \\ k &> 3.2 \end{aligned}$$

Thus, if $k > 3.2$, then the discriminant is negative and the solutions of $5x^2 - 8x + k = 0$ are not real numbers.

13. Let k be a real number, and consider the quadratic equation $(k+1)x^2 + 4kx + 2 = 0$.

a. Show that the discriminant of $(k+1)x^2 + 4kx + 2 = 0$ defines a quadratic function of k.

The discriminant of a quadratic equation written in the form $ax^2 + bx + c = 0$ is $b^2 - 4ac$.

Here, $a = k+1$, $b = 4k$, and $c = 2$. We get

$$\begin{aligned} b^2 - 4ac &= (4k)^2 - 4 \cdot (k+1) \cdot 2 \\ &= 16k^2 - 8(k+1) \\ &= 16k^2 - 8k - 8. \end{aligned}$$

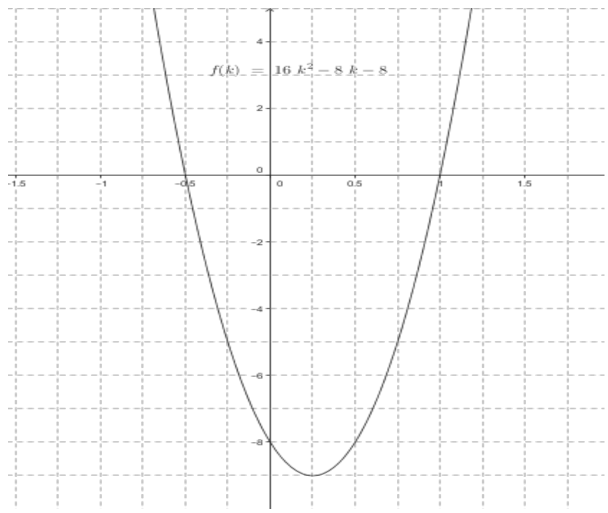
With k unknown, we can write $f(k) = 16k^2 - 8k - 8$, which is a quadratic function of k.

b. Find the zeros of the function in part (a) and make a sketch of its graph.

If $f(k) = 0$, then we have

$$\begin{aligned} 0 &= 16k^2 - 8k - 8 \\ &= 2k^2 - k - 1 \\ &= 2k^2 - 2k + k - 1 \\ &= 2k(k-1) + 1(k-1) \\ &= (k-1)(2k+1). \end{aligned}$$

Then, $k - 1 = 0$ or $2k + 1 = 0$. So, $k = 1$ or $k = -\frac{1}{2}$.



c. For what value of k are there two distinct real solutions to the given quadratic equation?

The original quadratic equation has two distinct real solutions when the discriminant given by $f(k)$ is positive. This occurs for all real numbers k such that $k < -\frac{1}{2}$ or $k > 1$.

d. For what value of k are there two complex solutions to the given quadratic equation?

There are two complex solutions when $f(k) < 0$. This occurs for all real numbers k such that $-\frac{1}{2} < k < 1$.

e. For what value of k is there one solution to the given quadratic equation?

There is one solution when $f(k) = 0$. This occurs at $k = -\frac{1}{2}$ and $k = 1$.

14. **We can develop two formulas that can help us find errors in calculated solutions of quadratic equations.**

a. **Find a formula for the sum S of the solutions of the quadratic equation $ax^2 + bx + c = 0$.**

The zeros of the quadratic equation are given by $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. *Then*

$$\begin{aligned} S &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b + \sqrt{b^2 - 4ac} + -b - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b + -b + \sqrt{b^2 - 4ac} - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2b}{2a} \\ &= -\frac{b}{a}. \end{aligned}$$

Thus, $S = -\frac{b}{a}$.

b. **Find a formula for the product R of the solutions of the quadratic equation $ax^2 + bx + c = 0$.**

$$R = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Note that the numerators differ only in that one is a sum, and one is a difference. The formula $(m + n) \cdot (m - n) = m^2 - n^2$ *applies where* $m = -b$ *and* $n = \sqrt{b^2 - 4ac}$. *We get*

$$\begin{aligned} R &= \frac{(-b)^2 - \left(\sqrt{b^2 - 4ac}\right)^2}{2a \cdot 2a} \\ &= \frac{b^2 - b^2 + 4ac}{4a^2} \\ &= \frac{4ac}{4a^2} \\ &= \frac{c}{a}. \end{aligned}$$

So, the product is $R = \frac{c}{a}$.

c. **June calculated the solutions 7 and -1 to the quadratic equation $x^2 - 6x + 7 = 0$. Do the formulas from parts (a) and (b) detect an error in her solutions? If not, determine if her solution is correct.**

The sum formula agrees with June's calculations. From June's zeros,

$$7 + -1 = 6,$$

and from the formula,

$$S = \frac{6}{1} = 6.$$

However, the product formula does not agree with her calculations. From June's zeros,

$$7 \cdot -1 = -7,$$

and from the formula,

$$R = \frac{7}{1} = 7.$$

June's solutions are not correct: $(7)^2 - 6(7) + 7 = 49 - 42 + 7 = 14$; *so,* 7 *is not a solution to this quadratic equation. Likewise,* $1 - 6 + 7 = 2$, *so* 1 *is also not a solution to this equation. Thus, the formulas caught her error.*

d. Paul calculated the solutions $3 - i\sqrt{2}$ and $3 + i\sqrt{2}$ to the quadratic equation $x^2 - 6x + 7 = 0$. Do the formulas from parts (a) and (b) detect an error in his solutions? If not, determine if his solutions are correct.

In part (c), we calculated that $R = 7$ and $S = 6$. From Paul's zeros,

$$S = 3 + i\sqrt{2} + 3 - i\sqrt{2} = 6,$$

and for the product,

$$\begin{aligned} R &= \left(3 + i\sqrt{2}\right) \cdot \left(3 - i\sqrt{2}\right) \\ &= 3^2 - \left(i\sqrt{2}\right)^2 \\ &= 9 — 1 \cdot 2 \\ &= 11. \end{aligned}$$

This disagrees with the calculated version of R. So, the formulas do find that he made an error.

e. Joy calculated the solutions $3 - \sqrt{2}$ and $3 + \sqrt{2}$ to the quadratic equation $x^2 - 6x + 7 = 0$. Do the formulas from parts (a) and (b) detect an error in her solutions? If not, determine if her solutions are correct.

Joy's zeros will have the same sum as Paul's, so $S = 6$, which agrees with the sum from the formula. For the product of her zeros we get

$$\begin{aligned} R &= \left(3 - \sqrt{2}\right)\left(3 + \sqrt{2}\right) \\ &= 9 - 2 \\ &= 7, \end{aligned}$$

which agrees with the formulas.

Checking her solutions in the original equation, we find

$$\begin{aligned} \left(3 - \sqrt{2}\right)^2 - 6\left(3 - \sqrt{2}\right) + 7 &= \left(9 - 6\sqrt{2} + 2\right) - 18 + 6\sqrt{2} + 7 \\ &= 0, \\ \left(3 + \sqrt{2}\right)^2 - 6\left(3 + \sqrt{2}\right) + 7 &= \left(9 + 6\sqrt{2} + 2\right) - 18 - 6\sqrt{2} + 7 \\ &= 0. \end{aligned}$$

Thus, Joy has correctly found the solutions of this quadratic equation.

MP.2

f. If you find solutions to a quadratic equations that match the results from parts (a) and (b), does that mean your solutions are correct?

Not necessarily. We only know that if the sum and product of the solutions do not match S and R, then we have not found a solution. Evidence suggests that if the sum and product of the solutions do match S and R, then we have found the correct solutions, but we do not know for sure until we check.

g. Summarize the results of this exercise.

For a quadratic equation of the form $ax^2 + bx + c = 0$, the sum of the solutions is given by $S = -\frac{b}{a}$, and the product of the solutions is given by $R = \frac{c}{a}$. So, multiplying and adding the calculated solutions will identify if we have made an error. Passing these checks, however, does not guarantee that the numbers we found are the correct solutions.

Lesson 39: Factoring Extended to the Complex Realm

Student Outcomes

- Students solve quadratic equations with real coefficients that have complex solutions. Students extend polynomial identities to the complex numbers.
- Students note the difference between solutions to the equation and the x-intercepts of the graph of said equation.

Lesson Notes

This lesson extends the factoring concepts and techniques covered in Topic B of this module to the complex number system and specifically addresses **N-CN.C.7**. Students will learn how to solve and express the solutions to any quadratic equation. Students observe that complex solutions to polynomial equations with real coefficients occur in conjugate pairs, and that only real solutions to polynomial equations are also the x-intercepts of the graph of the related polynomial function. In essence, this is the transition lesson to the next lesson on factoring all polynomials into linear factors.

Classwork

Opening (1 minute)

Since your introduction to the complex number system, you have hopefully recognized its theme of sharing arithmetic and algebraic properties with the real numbers. Today, we extend factoring polynomial expressions and finding solutions to polynomial equations to the complex realm.

Opening Exercise (8 minutes)

Have students individually complete this opening exercise. Students will eventually identify the expressions in this exercise as cases of the identity $(x + ai)(x - ai) = x^2 + a^2$. But, for now, allow them the experience of the algebra and to confirm for themselves that the imaginary terms combine to 0 in each example, resulting in polynomials in standard form with real coefficients. Invite students to the board to write their solutions, and let the class have the first opportunity to correct any mistakes, should it be necessary.

Opening Exercise

Rewrite each expression as a polynomial in standard form.

a. $(x+i)(x-i)$

$$\begin{aligned}(x+i)(x-i) &= x^2 + ix - ix - i^2 \\ &= x^2 - i^2 \\ &= x^2 - (-1) \\ &= x^2 + 1\end{aligned}$$

b. $(x+5i)(x-5i)$

$$\begin{aligned}(x+5i)(x-5i) &= x^2+5ix-5ix-25i^2\\ &= x^2-25i^2\\ &= x^2-25(-1)\\ &= x^2+25\end{aligned}$$

c. $(x-(2+i))(x-(2-i))$

$$\begin{aligned}\big(x-(2+i)\big)\big(x-(2-i)\big) &= x^2-(2+i)x-(2-i)x+[(2+i)(2-i)]\\ &= x^2-2x-ix-2x+ix+[4-i^2]\\ &= x^2-4x+[4-(-1)]\\ &= x^2-4x+5\end{aligned}$$

Discussion (5 minutes)

Here we begin a dialogue that discusses patterns and regularity observed in the Opening Exercise. As you pose each question, give students time to discuss them with a partner or in their small groups. To encourage students to be accountable for responding to questions during discussion, you can have them write answers on personal whiteboards, show a thumbs-up when they have an idea, whisper their idea to a partner before asking for a response with the whole group, or show their agreement or disagreement to a question by showing a thumbs up/down.

- Do you observe any patterns among parts (a)–(c) in the opening exercise?
  - *After each expression is expanded and like terms are collected, we have quadratic polynomials with real coefficients. The imaginary terms were opposites and combined to* 0.

MP.7

- How could you generalize the patterns into a rule (or identity)?
  - *Parts (a) and (b) are instances of the identity*
    $(x+ai)(x-ai)=x^2+a^2$.
- What about part (c)? Do you notice an instance of the same identity?
  - *Yes.*
    $\big(x-(2+i)\big)\big(x-(2-i)\big)=\big((x-2)-i\big)\big((x-2)+i\big)$
- Where have we seen a similar identity to $(x+ai)(x-ai)=x^2+a^2$?
  - *Recall the polynomial identity* $(x+a)(x-a)=x^2-a^2$ *from Lesson 6.*
- Recall the quick mental arithmetic we learned in Lesson 7. Can you compute $(3+2i)(3-2i)$ really quickly?
  - *Yes.* $(3+2i)(3-2i)=3^2+2^2=9+4=13$
- How about $(9+4i)(9-4i)$?
  - $(9+4i)(9-4i)=9^2+4^2=81+16=97$

Exercises 1–2 (5 minutes)

Students understand that the expansion of $(x+ai)(x-ai)$ is a polynomial with real coefficients; the imaginary terms disappear when working through the algebra. Now, students are expected to understand this process in reverse; in other words, they factor polynomials with real coefficients but complex factors.

Exercises 1–4

Completely factor the following polynomial expressions.

1. $x^2 + 9$

$$x^2 + 9 = (x + 3i)(x - 3i)$$

2. $x^2 + 5$

$$x^2 + 5 = \left(x + i\sqrt{5}\right)\left(x - i\sqrt{5}\right)$$

Discussion (6 minutes)

This discussion is the introduction to conjugate pairs in the context of complex numbers. Relate this idea back to the idea of conjugate pairs for radical expressions from Lesson 29.

- In Lesson 29, we saw that the conjugate of an expression such as $x + \sqrt{5}$ is the expression $x - \sqrt{5}$, and if we multiply a radical expression by its conjugate, the result is a rational expression—the radical part disappears.
$$\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right) = x^2 - x\sqrt{5} + x\sqrt{5} - 5 = x^2 - 5$$
- Analogously, complex numbers $a + bi$ have conjugates. The conjugate of $a + bi$ is $a - bi$. Then, we see that
$$(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2.$$
- We have observed, for real values of x and a, that the expression $(x + ai)(x - ai)$ is a real number. The factors $(x + ai)$ and $(x - ai)$ form a conjugate pair.

Quadratic expressions with real coefficients, as we have seen, can be decomposed into real factors or non-real complex factors. However, non-real factors must be members of a conjugate pair; hence, a quadratic expression with real coefficients cannot have exactly one complex factor.

> *Scaffolding:*
> - Show conjugate pairs graphically by graphing (as in the previous lessons) a parabola with 0, 1, and 2 solutions and cubic curves with a various number of solutions. Let student determine visually what is possible.
> - Additionally, consider having students complete a Frayer diagram for the term "conjugate."
> - As an extension, ask students to generate conjugate pairs.

- Similarly, quadratic equations can have real or non-real complex solutions. If there are complex solutions, they will be conjugates of each other.
- Can a polynomial equation with real number coefficients have just one complex solution?
  - *No. If there is a complex solution, then the conjugate is also a solution. Complex solutions come in pairs.*
- Now, can a polynomial equation have real and non-real solutions?
  - *Yes, as long as all non-real complex solutions occur in conjugate pairs.*
  - *For example, the polynomial equation $(x^2 + 1)(x^2 - 1) = 0$ has two real solutions, 1 and -1, and two complex solutions. The complex solutions, i and $-i$, form a conjugate pair.*
- If you know that $2i$ is a solution to the polynomial equation $P(x) = 0$, can you tell me another solution?
  - *Complex solutions come in conjugate pairs, so if $2i$ is a solution to the equation, then its conjugate, $-2i$, is also a solution.*

At this point, have students write down or discuss with their neighbors what they have learned so far. The teacher should walk around the room and check for understanding.

Exercise 3 (6 minutes)

Students should work in groups of 2–4 on this exercise. Invite students to the board to present their solutions.

Scaffolding:

- Consider having groups work Exercise 3 with different polynomials:

  $4x^3 - x$
  $x^3 - 4x^2 + 29x$
  $x^3 - 6x^2 + 25x$
  $x^4 - 1$
  $x^4 + 3x^2 - 4$
  $x^4 - 4x^3 + 5x^2$
  $x^4 - 4x^3 + 13x^2$
  $x^4 - 2x^3 - 10x^2$
  $x^4 + 13x^2 + 36$

3. Consider the polynomial $P(x) = x^4 - 3x^2 - 4$.

a. What are the solutions to $x^4 - 3x^2 - 4 = 0$?

$$x^4 - 3x^2 - 4 = 0$$
$$(x^2)^2 - 3x^2 - 4 = 0$$
$$(x^2 + 1)(x^2 - 4) = 0$$
$$(x + i)(x - i)(x + 2)(x - 2) = 0$$

The solutions are $-i$, i, -2, and 2.

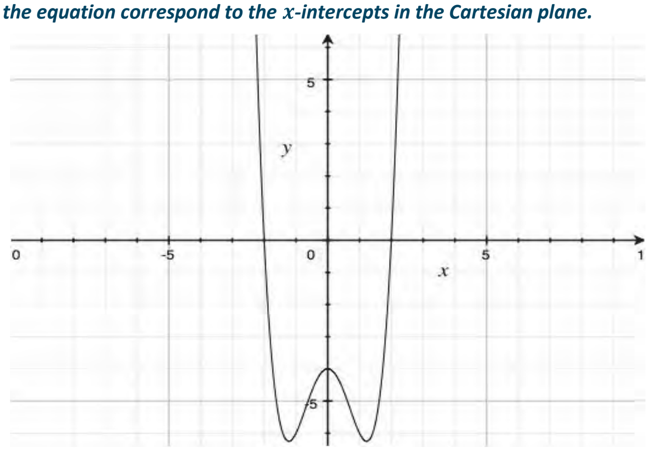
b. How many x-intercepts does the graph of the equation $y = x^4 - 3x^2 - 4$ have? What are the coordinates of the x-intercepts?

The graph of $y = x^4 - 3x^2 - 4$ has two x-intercepts: $(-2, 0)$ and $(2, 0)$.

c. Are solutions to the polynomial equation $P(x) = 0$ the same as the x-intercepts of the graph of $y = P(x)$? Justify your reasoning.

MP.3

No. Only the real solutions to the equation are x-intercepts of the graph. By comparing the graph of the polynomial in part (b) to the equation's solutions from part (c), you can see that only the real number solutions to the equation correspond to the x-intercepts in the Cartesian plane.



Exercise 4 (5 minutes)

Transition students to the next exercise by announcing that we now want to reverse our thinking. In the previous problem, we solved an equation to find the solutions. Now, pose the question: Can we construct an equation if we know its solutions? You may want to remind students that when a polynomial equation is written in factored form $(x - r_1)(x - r_2)\ldots(x - r_n) = 0$, the solutions to the equation are $r_1, r_2, \ldots, r_n$. Students will apply what they learned in the previous exercise to create a polynomial equation given its solutions. The problems scaffold from easier to more difficult. Students are encouraged to rewrite the factored form to show the polynomial in standard form for additional practice with complex numbers, but you may choose to have them leave the polynomial in factored form if time is a concern. Have students work with a partner on this exercise.

4. Write a polynomial P with the lowest possible degree that has the given solutions. Explain how you generated each answer.

a. $-2, 3, -4i, 4i$

The polynomial P has two real zeroes and two complex zeros. Since the two complex zeros are members of a conjugate pair, P may have as few as four total factors. Therefore, P has degree at least 4.

$$\begin{aligned} P(x) &= (x+2)(x-3)(x+4i)(x-4i) \\ &= (x^2-x-6)(x^2-16i^2) \\ &= (x^2-x-6)(x^2+16) \\ &= x^4-x^3-6x^2+16x^2-16x-96 \\ &= x^4-x^3+10x^2-16x-96 \end{aligned}$$

b. $-1, 3i$

The polynomial P has one real zero and two complex zeros because complex zeros come in pairs. Since $3i$ and $-3i$ form a conjugate pair, P has at least three total factors. Therefore, P has degree at least 3.

$$\begin{aligned} P(x) &= (x+1)(x-3i)(x+3i) \\ &= (x+1)(x^2-9i^2) \\ &= (x+1)(x^2+9) \\ &= x^3+x^2+9x+9 \end{aligned}$$

c. $0, 2, 1+i, 1-i$

Since $1+i$ and $1-i$ are complex conjugates, P is at least a 4th degree polynomial.

$$\begin{aligned} P(x) &= x(x-2)\big(x-(1+i)\big)\big(x-(1-i)\big) \\ &= x(x-2)[(x-1)-i][(x-1)+i] \\ &= x(x-2)[(x-1)^2-i^2] \\ &= x(x-2)[(x^2-2x+1)+1 \\ &= x(x-2)(x^2-2x+2) \\ &= x(x^3-2x^2+2x-2x^2+4x-4) \\ &= x(x^3-4x^2+6x-4) \\ &= x^4-4x^3+6x^2-4x \end{aligned}$$

d. $\sqrt{2}, -\sqrt{2}, 3, 1+2i$

Since$1+2i$ is a complex solution to $P(x)=0$, its conjugate, $1-2i$, must also be a complex solution. Thus, P is at least a fifth-degree polynomial.

$$\begin{aligned} P(x) &= \big(x-\sqrt{2}\big)\big(x+\sqrt{2}\big)(x-3)\big(x-(1+2i)\big)\big(x-(1-2i)\big) \\ &= (x^2-2)(x-3)[(x-1)-2i][(x-1)+2i] \\ &= (x^2-2)(x-3)[(x-1)^2-4i^2] \\ &= (x^2-2)(x-3)[(x^2-2x+1)+4] \\ &= (x^2-2)(x-3)(x^2-2x+5) \\ &= (x^3-3x^2-2x+6)(x^2-2x+5) \\ &= x^5-5x^4+9x^3-5x^2-22x+30 \end{aligned}$$

e. $2i, 3 - i$

The complex conjugates of $2i$ and $3 - i$ are $-2i$ and $3 + i$, respectively. So, P is at least a fourth-degree polynomial.

$$\begin{aligned}
P(x) &= (x - 2i)(x + 2i)\big(x - (3 - i)\big)\big(x - (3 + i)\big) \\
&= (x^2 - 4i^2)[(x - 3) + i][(x - 3) - i] \\
&= (x^2 + 4)[(x - 3)^2 - i^2] \\
&= (x^2 + 4)[(x^2 - 6x + 9) + 1] \\
&= (x^2 + 4)(x^2 - 6x + 10) \\
&= x^4 - 6x^3 + 14x^2 - 24x + 40
\end{aligned}$$

Closing (3 minutes)

Have students break into small groups to discuss what they learned today. Today's lesson is summarized in the box below.

Lesson Summary

- **Polynomial equations with real coefficients can have real or complex solutions or they can have both.**
- **Complex solutions to polynomial equations with real coefficients are always members of a conjugate pair.**
- **Real solutions to polynomial equations are the same as x-intercepts of the associated graph, but complex solutions are not.**

Exit Ticket (6 minutes)

In this Exit Ticket, students solve quadratic equations with real and complex solutions.

Name ______________________________ Date ______________

Lesson 39: Factoring Extended to the Complex Realm

Exit Ticket

1. Solve the quadratic equation $x^2 + 9 = 0$. What are the x-intercepts of the graph of the function $f(x) = x^2 + 9$?

2. Find the solutions to $2x^5 - 5x^3 - 3x = 0$. What are the x-intercepts of the graph of the function $f(x) = 2x^5 - 5x^3 - 3x$?

Exit Ticket Sample Solutions

1. Solve the quadratic equation $x^2 + 9 = 0$. What are the x-intercepts of the graph of the function $f(x) = x^2 + 9$?

$$x^2 + 9 = 0$$
$$x^2 = -9$$

$$x = \sqrt{-9} \text{ or } x = -\sqrt{-9}$$
$$x = 3\sqrt{-1} \text{ or } x = -3\sqrt{-1}$$
$$x = 3i \text{ or } x = -3i$$

The x-intercepts of the graph of the function $f(x) = x^2 + 9$ are any real solutions to the equation $x^2 + 9 = 0$. However, since both solutions to $x^2 + 9 = 0$ are not real, the function $f(x) = x^2 + 9$ does not have any x-intercepts.

2. Find the solutions to $2x^5 - 5x^3 - 3x = 0$. What are the x-intercepts of the graph of the function $f(x) = 2x^5 - 5x^3 - 3x$?

$$(2x^4 - 5x^2 - 3) = 0$$
$$x(x^2 - 3)(2x^2 + 1) = 0$$
$$x(x + \sqrt{3})(x - \sqrt{3})(2x^2 + 1) = 0$$
$$x(x + \sqrt{3})(x - \sqrt{3})\left(x + \frac{i\sqrt{2}}{2}\right)\left(x - \frac{i\sqrt{2}}{2}\right) = 0$$

Thus, $x = 0$, $x = -\sqrt{3}$, $x = \sqrt{3}$, $x = -\frac{i\sqrt{2}}{2}$, *or* $x = \frac{i\sqrt{2}}{2}$.

The solutions are 0, $\sqrt{3}$, $-\sqrt{3}$, $\frac{i\sqrt{2}}{2}$, *and* $-\frac{i\sqrt{2}}{2}$.

The x-intercepts of the graph of the function $f(x) = 2x^5 - 5x^3 - 3x$ are the real solutions to the equation $2x^5 - 5x^3 - 3x = 0$, so the x-intercepts are 0, $\sqrt{3}$, and $-\sqrt{3}$.

Problem Set Sample Solutions

1. Rewrite each expression in standard form.

   a. $(x + 3i)(x - 3i)$

   $x^2 + 3^2 = x^2 + 9$

   b. $(x - a + bi)(x - (a + bi))$

$$\begin{aligned}(x - a + bi)(x - (a + bi)) &= ((x - a) + bi)((x - a) - bi)\\ &= (x - a)^2 + b^2\\ &= x^2 - 2ax + a^2 + b^2\end{aligned}$$

   c. $(x + 2i)(x - i)(x + i)(x - 2i)$

$$\begin{aligned}(x + 2i)(x - 2i)(x + i)(x - i) &= (x^2 + 2^2)(x^2 + 1^2)\\ &= (x^2 + 4)(x^2 + 1)\\ &= x^4 + 5x^2 + 4\end{aligned}$$

d. $(x+i)^2 \cdot (x-i)^2$

$$(x+i)(x-i)\cdot(x+i)(x-i) = (x^2+1)(x^2+1)$$
$$= x^4+2x^2+1$$

2. **Suppose in Problem 1 that you had no access to paper, writing utensils, or technology. How do you know that the expressions in parts (a)–(d) are polynomials with real coefficients?**

In part (a), the identity $(x+ai)(x-ai) = x^2+a^2$ can be applied. Since the number a is real, the resulting polynomial will have real coefficients. The remaining three expressions can all be rearranged to take advantage of the conjugate pairs identity. In parts (c) and (d), regrouping terms will produce products of polynomial expressions with real coefficients, which will again have real coefficients.

3. **Write a polynomial equation of degree 4 in standard form that has the solutions $i, -i, 1, -1$.**

The first step is writing the equation in factored form:

$$(x+i)(x-i)(x+1)(x-1) = 0.$$

Then, use the commutative property to rearrange terms and apply the difference of squares formula:

$$(x+i)(x-i)(x+1)(x-1) = (x^2+1)(x^2-1)$$
$$= x^4-1.$$

So, the standard form of the equation is

$$x^4-1=0.$$

4. **Explain the difference between x-intercepts and solutions to an equation. Give an example of a polynomial with real coefficients that has twice as many solutions as x-intercepts. Write it in standard form.**

The x-intercepts are the real solutions to a polynomial equation with real coefficients. The solutions to an equation can be real or not real. The previous problem is an example of a polynomial with twice as many solutions than x-intercepts. Or, we could consider the equation $x^4-6x^3+13x^2-12x+4=0$, which has zeros of multiplicity 2 at both 1 and 2.

5. **Find the solutions to $x^4-5x^2-36=0$ and the x-intercepts of the graph of $y=x^4-5x^2-36$.**

$$(x^2+4)(x^2-9)=0$$
$$(x+2i)(x-2i)(x+3)(x-3)=0$$

Since the solutions are $2i$, $-2i$, 3, and -3, and only real solutions to the equation are x-intercepts of the graph, the x-intercepts are 3 and -3.

6. **Find the solutions to $2x^4-24x^2+40=0$ and the x-intercepts of the graph of $y=2x^4-24x^2+40$.**

$$2(x^4-12x^2+20)=0$$
$$2(x^2-10)(x^2-2)=0$$

Since all of the solutions $\sqrt{10}$, $-\sqrt{10}$, $\sqrt{2}$ and $-\sqrt{2}$ are real numbers, the x-intercepts of the graph are $\sqrt{10}$, $-\sqrt{10}$, $\sqrt{2}$ and $-\sqrt{2}$.

7. **Find the solutions to $x^4-64=0$ and the x-intercepts of the graph of $y=x^4-64$.**

$$(x^2+8)(x^2-8)=0$$
$$\left(x+\sqrt{8}i\right)\left(x-\sqrt{8}i\right)\left(x+\sqrt{8}\right)\left(x-\sqrt{8}\right)=0$$

The x-intercepts are $2\sqrt{2}$ and $2-\sqrt{2}$.

8. **Use the fact that $x^4 + 64 = (x^2 - 4x + 8)(x^2 + 4x + 8)$ to explain how you know that the graph of $y = x^4 + 64$ has no x-intercepts. You need not find the solutions.**

The x-intercepts of $y = x^4 + 64$ are solutions to $(x^2 - 4x + 8)(x^2 + 4x + 8) = 0$. Both $x^2 - 4x + 8 = 0$ and $x^2 + 4x + 8 = 0$ have negative discriminant values of -16, so the equations $x^2 - 4x + 8 = 0$ and $x^2 + 4x + 8 = 0$ have no real solutions. Thus, the equation $x^4 + 64 = 0$ has no real solutions, and the graph of $y = x^4 + 64$ has no x-intercepts.

Since $x^4 + 64 = 0$ has no real solutions, the graph of $y = x^4 + 64$ has no x-intercepts.

Lesson 40: Obstacles Resolved—A Surprising Result

Student Outcomes

- Students understand the Fundamental Theorem of Algebra; that all polynomial expressions factor into linear terms in the realm of complex numbers. Consequences, in particular, for quadratic and cubic equations are understood.

Lesson Notes

There is no real consensus in the literature about what exactly constitutes the Fundamental Theorem of Algebra; it is stated differently in different texts. The two-part theorem stated in this lesson encapsulates the main ideas of the theorem and its corollaries while remaining accessible to students. The first part of what is stated here as the Fundamental Theorem of Algebra is the one that we are not mathematically equipped to prove or justify at this level; this part states that every polynomial equation has at least one solution in the complex numbers and will need to be accepted without proof. The consequence of this first part is what we find really interesting–that every polynomial expression factors into the same number of linear factors as its degree. Justification for this second part of the Fundamental Theorem of Algebra is accessible for students as long as we can accept the first part without needing proof. Since every polynomial of degree $n \geq 1$ will factor into n linear factors, we know that any polynomial function of degree n will have n zeros (including repeated zeros).

Classwork

Opening Exercise (5 minutes)

At the beginning of the lesson, we focus on the most familiar of polynomial expressions, the quadratic equations. Ensure that students understand the link provided by the Remainder Theorem between solutions of polynomial equations and factors of the associated polynomial expression. Allow students time to work on the Opening Exercise, and then debrief.

Opening Exercise

Write each of the following quadratic expressions as a product of linear factors. Verify that the factored form is equivalent.

1. $x^2 + 12x + 27$ $\quad$ $(x+3)(x+9)$
2. $x^2 - 16$ $\quad$ $(x+4)(x-4)$
3. $x^2 + 16$ $\quad$ $(x+4i)(x-4i)$
4. $x^2 + 4x + 5$ $\quad$ $(x+2+i)(x+2-i)$

Discussion (7 minutes)

Remind students about the Remainder Theorem, studied earlier in the module. The Remainder Theorem states that if P is a polynomial function, and $P(a) = 0$ for some value of a, then $x - a$ is a factor of P. The Remainder Theorem plays an important role in the development of this lesson, linking the solutions of a polynomial equation to the factors of the associated polynomial expression.

- With a partner, describe any patterns you see in the Opening Exercise.
- Can every quadratic polynomial be written in terms of linear factors? If so, how many linear factors?
  - *Yes, two.*
- How do you know?
  - *Every quadratic equation has two solutions that can be found using the quadratic formula. These solutions of the equation lead to linear factors of the quadratic polynomial.*
- What types of solutions can a quadratic equation have? What does this mean about the graph of the corresponding function?
  - *The equation has either two real solutions, one real solution, or two complex solutions. These situations correspond to the graph having two x-intercepts, one x-intercept, or no x-intercepts.*

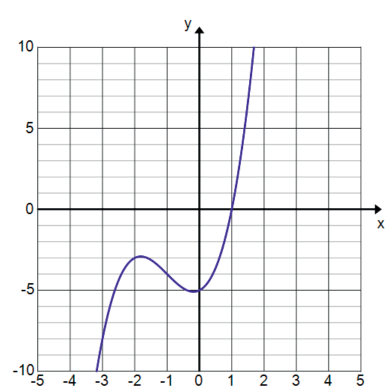
Be sure that students realize that real numbers are also complex numbers; if a is real, then we can write it as $a + 0i$.

Example 1 (8 minutes)

The purpose of this example is to help students move fluently between the concepts of x-intercepts of the graph of a polynomial equation $y = P(x)$, the solutions of the polynomial equation $P(x) = 0$, and the factors in the factored form of the associated polynomial P. Talk the students through parts (a)–(e), and then allow them time to work alone or in pairs on part (f) before completing the discussion.

Example 1

Consider the polynomial $P(x) = x^3 + 3x^2 + x - 5$ whose graph is shown to the right.



a. Looking at the graph, how do we know that there is only one real solution?

The graph has only one x-intercept.

b. Is it possible for a cubic polynomial function to have no zeros?

No. Since the opposite ends of the graph of a cubic function go in opposite directions, the graph must cross the x-axis at some point. Since the graph must have an x-intercept, the function must have a zero.

c. From the graph, what appears to be one solution to the equation $x^3 + 3x^2 + x - 5 = 0$?

The only real solution appears to be 1.

d. **How can we verify that this is a solution?**

Evaluate the function at 1; that is, verify that $P(1) = 0$.

$$P(1) = (1)^3 + 3(1)^2 + 1 - 5 = 1 + 2 + 1 - 5 = 0$$

e. **According to the Remainder Theorem, what is one factor of the cubic expression $x^3 + 3x^2 + x - 5$?**

$(x - 1)$

f. **Factor out the expression you found in part (e) from $x^3 + 3x^2 + x - 5$.**

Using polynomial division, we see that $x^3 + 3x^2 + x - 5 = (x - 1)(x^2 + 4x + 5)$.

g. **What are all of the solutions to $x^3 + 3x^2 + x - 5 = 0$?**

The quadratic equation $x^2 + 4x + 5 = 0$ has solutions $2 - i$ and $2 + i$ by the quadratic formula, so the original equation has solutions 1, $2 - i$, and $2 + i$.

h. **Write the expression $x^3 + 3x^2 + x - 5$ in terms of linear factors.**

The factored form of the cubic expression is $x^3 + 3x^2 + x - 5 = (x - 1)(x - (2 - i))(x - (2 + i))$.

Scaffolding:

- For students who are struggling with part (g), point out that the remaining quadratic polynomial is the same as one of the problems from the Opening Exercise.
- As an extension, ask students to create a polynomial equation that has three real and two complex solutions.

- We established earlier in the lesson that all quadratic expressions can be written in terms of two linear factors. How many factors did our cubic expression have?
  - *Three.*
- Is it true that every cubic expression can be factored into three linear factors?
  - *Yes, because a cubic equation will always have at least one real solution that corresponds to a linear factor of the expression. What is left over will be a quadratic expression, which can be written in terms of two linear factors.*

If students don't seem ready to answer the last question or are unsure of the answer, let them work through Exercise 1 and then re-address it.

Exercises 1–2 (6 minutes)

Give students time to work through the two exercises and then lead the discussion that follows.

Exercises 1–2

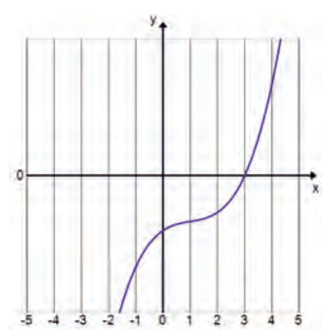
Write each polynomial in terms of linear factors. The graph of $y = x^3 - 3x^2 + 4x - 12$ is provided for Exercise 2.

1. $f(x) = x^3 + 5x$

   $f(x) = x(x + i\sqrt{5})(x - i\sqrt{5})$

2. $g(x) = x^3 - 3x^2 + 4x - 12$

   $g(x) = (x - 3)(x + 2i)(x - 2i)$



Discussion (3 minutes)

- Do your results from Exercises 1 and 2 agree with our conclusions from Example 1?
  - *Yes, each cubic function could be written as a product of three linear factors.*

MP.3

- Make a conjecture about what might happen if we factored a degree 4 polynomial. What about a degree 5 polynomial? Explain your reasoning.
  - *A degree 4 polynomial should have 4 linear factors. Based on the previous examples, it seems that a polynomial has as many linear factors as its degree. Similarly, a degree 5 polynomial should be able to be written as a product of 5 linear factors.*
- Our major conclusion in this lesson is a two-part theorem known as the Fundamental Theorem of Algebra (FTA). Part 1 of the Fundamental Theorem of Algebra says that every polynomial equation has at least one solution in the complex numbers. Does that agree with our experience?
  - *Yes.*

> *Scaffolding:*
>
> To meet a variety of student needs, ask students to:
>
> - Restate the FTA in their own words (either in writing or verbally).
> - Illustrate the FTA with an example. For instance, show that the FTA is true for some polynomial function.
> - Apply the FTA to some examples. For instance, how many linear factors would be in the factored form of $x^5 - 3x + 1$?

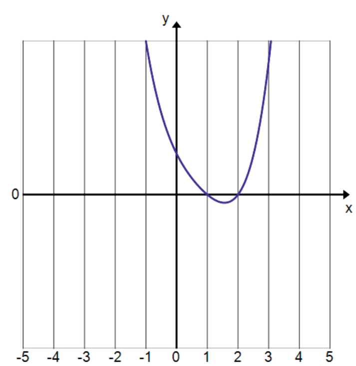
- Think about how we factor a polynomial expression P: We find one solution a to $P(x) = 0$, then we factor out the term $(x - a)$. We are left with a new polynomial expression of one degree lower than P, so we look for another solution, and repeat until we have factored everything into linear parts.
- Consider the polynomial $P(x) = x^4 - 3x^3 + 6x^2 - 12x + 8$ in the next example.

Example 2 (8 minutes)

While we do not have the mathematical tools or experience needed to prove the Fundamental Theorem of Algebra (either part), this example illustrates how the logic of the second part of the FTA works. Lead students through this example, allowing time for factoring and discussion at each step.

Example 2

Consider the polynomial function $P(x) = x^4 - 3x^3 + 6x^2 - 12x + 8$, whose corresponding graph $y = x^4 - 3x^3 + 6x^2 - 12x + 8$ is shown to the right. How many zeros does P have?



a. Part 1 of the Fundamental Theorem of Algebra says that this equation will have at least one solution in the complex numbers. How does this align with what we can see in the graph to the right?

Since the graph has 2 x-intercepts, there appear to be 2 zeros to the function. We were guaranteed one zero, but we know there are at least two.

b. Identify one zero from the graph.

One zero is 1. (The other is 2.)

c. Use polynomial division to factor out one linear term from the expression $x^4 - 3x^3 + 6x^2 - 12x + 8$.

$x^4 - 3x^3 + 6x^2 - 12x + 8 = (x-1)(x^3 - 2x^2 + 4x - 8)$

d. **Now we have a cubic polynomial to factor. We know by part 1 of the Fundamental Theorem of Algebra that a polynomial function will have at least one real zero. What is that zero in this case?**

The original polynomial function had real zeros at 1 and 2, so the cubic function $P(x) = x^3 - 2x^2 + 4x - 8$ has a zero at 2.

e. **Use polynomial division to factor out another linear term of $x^4 - 3x^3 + 6x^2 - 12x + 8$.**

$x^4 - 3x^3 + 6x^2 - 12x + 8 = (x-1)(x^3 - 2x^2 + 4x - 8) = (x-1)(x-2)(x^2+4)$

f. **Are we done? Can we factor this polynomial any further?**

At this point, we can see that $x^2 + 4 = (x+2i)(x-2i)$, so

$$x^4 - 3x^3 + 6x^2 - 12x + 8 = (x-1)(x-2)(x+2i)(x-2i).$$

g. **Now that the polynomial is in factored form, we can quickly see how many solutions there are to the original equation $x^4 - 3x^3 + 6x^2 - 12x + 8 = 0$.**

If $x^4 - 3x^3 + 6x^2 - 12x + 8 = 0$, then $(x-1)(x-2)(x+2i)(x-2i) = 0$, so the solutions are 1, 2, $2i$ and $-2i$. So, the polynomial P has 4 zeros; 2 are real numbers, and 2 are complex numbers.

MP.8

h. **What if we had started with a polynomial function of degree 8?**

We would find the first zero, and factor out a linear term, leaving a polynomial of degree 7. We would then find another zero, factor out a linear term, leaving a polynomial of degree 6. We would repeat this process until we had a quadratic polynomial remaining; then, we would factor that with the help of the quadratic formula. We would have 8 linear factors at the end of the process that correspond to the 8 zeros of the original function.

The logic we just followed leads to part 2 of the Fundamental Theorem of Algebra, which is the result that we have already conjectured: A polynomial of degree $N \geq 1$ will factor into N linear factors with complex coefficients. Collectively, these two results are often just referred to as the Fundamental Theorem of Algebra. Although we have only worked with polynomials with real coefficients, the FTA applies to polynomial functions with real coefficients, such as $P(x) = x^3 + 2x^2 - 4$ as well as to polynomial functions with non-real coefficients, such as $P(x) = x^3 + 3ix^2 + 4 - 2i$. We have not attempted to justify the first part, but the students should be able to justify the second part of the theorem.

Fundamental Theorem of Algebra

1. Every polynomial function of degree $N \geq 1$ with real or complex coefficients has at least one real or complex zero.
2. Every polynomial of degree $N \geq 1$ with real or complex coefficients factors into N linear terms with real or complex coefficients.

- Why is the Fundamental Theorem of Algebra so "fundamental" to mathematics?
  - *The Fundamental Theorem says that the complex number system contains every zero of every polynomial function. We do not need to look anywhere else to find zeros to these types of functions.*

- Notice that the Fundamental Theorem just tells us that the factorization of the polynomial exists; it does not help us actually find it. If we had been given a polynomial function that did not have any real zeros, it would have been very hard to start the factorization process.

Closing (3 minutes)

- With a partner, summarize the key points of this lesson.
- What does the Fundamental Theorem of Algebra guarantee?
  - *A polynomial of degree $N \geq 1$ will factor into N linear factors, and the associated function will have N zeros, some of which may be repeated.*
- Why is this important?
  - *The Fundamental Theorem of Algebra ensures that there are as many zeros as we'd expect for a polynomial function, and that factoring will always (in theory) work to find solutions to polynomial equations.*
- Illustrate the Fundamental Theorem of Algebra with an example.

Lesson Summary

Every polynomial function of degree n, for $n \geq 1$, has n roots over the complex numbers, counted with multiplicity. Therefore, such polynomials can always be factored into n linear factors, and the obstacles to factoring we saw before have all disappeared in the larger context of allowing solutions to be complex numbers.

The Fundamental Theorem of Algebra:

1. **If P is a polynomial function of degree $n \geq 1$, with real or complex coefficients, then there exists at least one number r (real or complex) such that $P(r) = 0$.**
2. **If P is a polynomial function of degree $n \geq 1$, given by $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with real or complex coefficients a_i, then P has exactly n zeros $r_1, r_2, \ldots, r_n$ (not all necessarily distinct), such that $P(x) = a(x - r_1)(x - r_2)\cdots(x - r_n)$.**

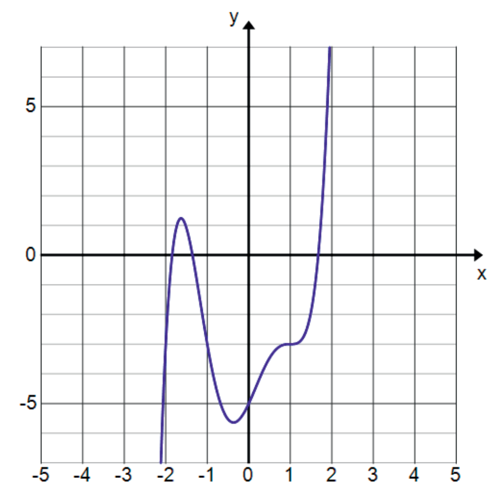
Exit Ticket (5 minutes)

Name ______________________________ Date ________________

Lesson 40: Obstacles Resolved—A Surprising Result

Exit Ticket

Consider the degree 5 polynomial function $P(x) = x^5 - 4x^3 + 2x^2 + 3x - 5$, whose graph is shown below. You do not need to factor this polynomial to answer the questions below.



1. How many linear factors is P guaranteed to have? Explain.

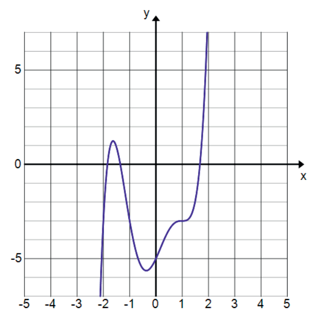
2. How many zeros does P have? Explain.

3. How many real zeros does P have? Explain.

4. How many complex zeros does P have? Explain.

Exit Ticket Sample Solutions

Consider the degree 5 polynomial function $P(x) = x^5 - 4x^3 + 2x^2 + 3x - 5$ whose graph is shown below. You do not need to factor this polynomial to answer the questions below.



1. How many linear factors is P guaranteed to have? Explain.

   The polynomial expression must have 5 linear factors. The Fundamental Theorem of Algebra guarantees that a polynomial function can be written in terms of linear factors and must have the same number of linear factors as its degree.

2. How many zeros does P have? Explain.

   Since P can be written in terms of 5 linear factors, the equation P must have 5 zeros (counted with multiplicity).

3. How many real zeros does P have? Explain.

   The graph crosses the x-axis 3 times, which means that three of the zeros are real numbers.

4. How many complex zeros does P have? Explain.

   Since P must have 5 total zeros and only 3 of them are real, there must be 2 complex zeros.

Problem Set Sample Solutions

1. Write each quadratic function below in terms of linear factors.

   a. $f(x) = x^2 - 25$

   $f(x) = (x+5)(x-5)$

   b. $f(x) = x^2 + 25$

   $f(x) = (x+5i)(x-5i)$

   c. $f(x) = 4x^2 + 25$

   $f(x) = (2x+5i)(2x-5i)$

   d. $f(x) = x^2 - 2x + 1$

   $f(x) = (x-1)(x-1)$

   e. $f(x) = x^2 - 2x + 4$

   $f(x) = (x-1+i\sqrt{3})(x-1-i\sqrt{3})$

2. Consider the polynomial function $P(x) = (x^2+4)(x^2+1)(2x+3)(3x-4)$.

   a. Express P in terms of linear factors.

   $P(x) = (x+2i)(x-2i)(x+i)(x-i)(2x+3)(3x-4)$

   b. Fill in the blanks of the following sentence.

   The polynomial P has degree ________ and can, therefore, be written in terms of ________ linear factors. The function P has ________ zeros. There are _______ real zeros and ________ complex zeros. The graph of $y = P(x)$ has ________ x-intercepts.

   The polynomial P has degree 6 and can, therefore, be written in terms of 6 linear factors. The function P has 6 solutions. There are 2 real zeros and 4 complex zeros. The graph of $y = P(x)$ has 2 x-intercepts.

3. Express each cubic function below in terms of linear factors.

   a. $f(x) = x^3 - 6x^2 - 27x$

   $f(x) = x(x-9)(x+3)$

   b. $f(x) = x^3 - 16x^2$

   $f(x) = x^2(x-16)$

   c. $f(x) = x^3 + 16x$

   $f(x) = x(x+4i)(x-4i)$

4. For each cubic function below, one of the zeros is given. Express each cubic function in terms of linear factors.

   a. $f(x) = 2x^3 - 9x^2 - 53x - 24; f(8) = 0$

   $f(x) = (x-8)(2x+1)(x+3)$

   b. $f(x) = x^3 + x^2 + 6x + 6; f(-1) = 0$

   $f(x) = (x+1)(x+i\sqrt{6})(x-i\sqrt{6})$

5. Determine if each statement is always true or sometimes false. If it is sometimes false, explain why it is not always true.

   a. A degree 2 polynomial function will have two linear factors.

   Always true.

   b. The graph of a degree 2 polynomial function will have two x-intercepts.

   False. It is possible for the solutions to a degree 2 polynomial to be complex, in which case the graph would not cross the x-axis. It is also possible for the graph to have only one x-intercept if the vertex lies on the x-axis.

   c. The graph of a degree 3 polynomial function might not cross the x-axis.

   False. A degree 3 polynomial must cross the x-axis at least one time.

   d. A polynomial function of degree n can be written in terms of n linear factors.

   Always true.

6. Consider the polynomial function $f(x) = x^6 - 9x^3 + 8$.

   a. How many linear factors does $x^6 - 9x^3 + 8$ have? Explain.

   Since the degree is 6, the polynomial must have 6 linear factors.

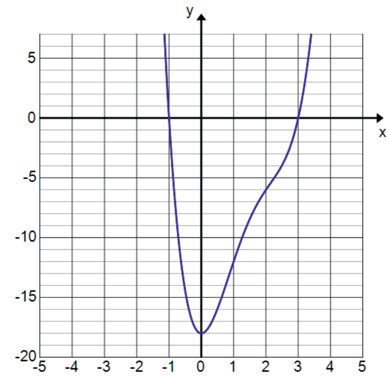
   b. How is this information useful for finding the zeros of f?

   We know that the function has 6 zeros since there are 6 linear factors. Each factor corresponds to a zero of the function.

   c. Find the zeros of f. (Hint: Let $Q = x^3$. Rewrite the equation in terms of Q to factor.)

   $1, 2, -1+i\sqrt{3}, -1-i\sqrt{3}, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$

7. Consider the polynomial function $P(x) = x^4 - 6x^3 + 11x^2 - 18$.



a. Use the graph to find the real zeros of P

The real zeros are -1 and 3.

b. Confirm that the zeros are correct by evaluating the function P at those values.

$P(-1) = 0$ *and* $P(3) = 0$

c. Express P in terms of linear factors.

$P(x) = (x+1)(x-3)\left(x-\left(2+i\sqrt{2}\right)\right)\left(x-\left(2-i\sqrt{2}\right)\right)$

d. Find all zeros of P.

$-1, 3, 2 - i\sqrt{2}, 2 + i\sqrt{2}$

MP.3

8. Penny says that the equation $x^3 - 8 = 0$ has only one solution, $x = 2$. Use the Fundamental Theorem of Algebra to explain to her why she is incorrect.

Because $x^3 - 8$ is a degree 3 polynomial, the Fundamental Theorem of Algebra guarantees that $x^3 - 8$ can be written as the product of three linear factors; therefore, the corresponding equation has 3 solutions. One of the 3 solutions is 2. We know that 2 cannot be the only solution because $(x-2)(x-2)(x-2) \neq x^3 - 8$.

9. Roger says that the equation $x^2 - 12x + 36 = 0$ has only one solution, 6. Regina says Roger is wrong and that the Fundamental Theorem of Algebra guarantees that a quadratic equation must have two solutions. Who is correct and why?

Roger is correct. While the Fundamental Theorem of Algebra guarantees that a quadratic polynomial can be written in terms of two linear factors, the factors are not necessarily distinct. We know that $x^2 - 12x + 36 = (x-6)(x-6)$, so the equation $x^2 - 12x + 36 = 0$ has only one solution, which is 6.

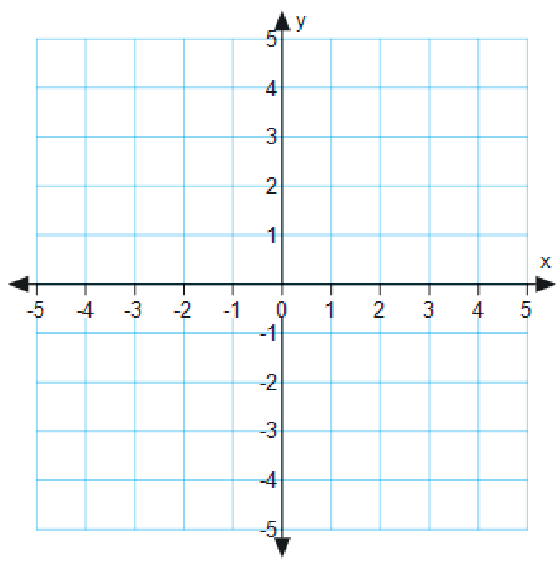
Name ____ Date ______________________

1. A *parabola* is defined as the set of points in the plane that are equidistant from a fixed point (called the *focus* of the parabola) and a fixed line (called the *directrix* of the parabola).

   Consider the parabola with focus point $(1, 1)$ and directrix the horizontal line $y = -3$.

   a. What are the coordinates of the vertex of the parabola?

   b. Plot the focus and draw the directrix on the graph below. Then draw a rough sketch of the parabola.



c. Find the equation of the parabola with this focus and directrix.

d. What is the y-intercept of this parabola?

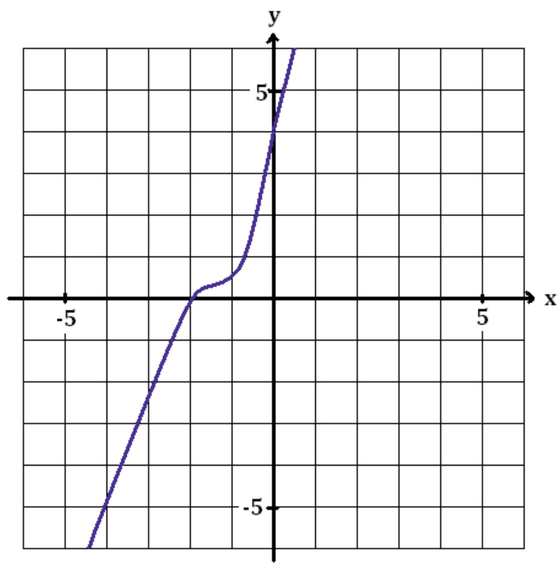
e. Demonstrate that your answer from part (d) is correct by showing that the y-intercept you identified is indeed equidistant from the focus and the directrix.

f. Is the parabola in this question (with focus point $(1, 1)$ and directrix $y = -3$) congruent to a parabola with focus $(2, 3)$ and directrix $y = -1$? Explain.

g. Is the parabola in this question (with focus point $(1, 1)$ and directrix $y = -3$) congruent to the parabola with equation given by $y = x^2$? Explain.

h. Are the two parabolas from part (g) similar? Why or why not?

2. The graph of the polynomial function $f(x) = x^3 + 4x^2 + 6x + 4$ is shown below.



a. Based on the appearance of the graph, what does the real solution to the equation $x^3 + 4x^2 + 6x + 4 = 0$ appear to be? Jiju does not trust the accuracy of the graph. Prove to her algebraically that your answer is in fact a zero of $y = f(x)$.

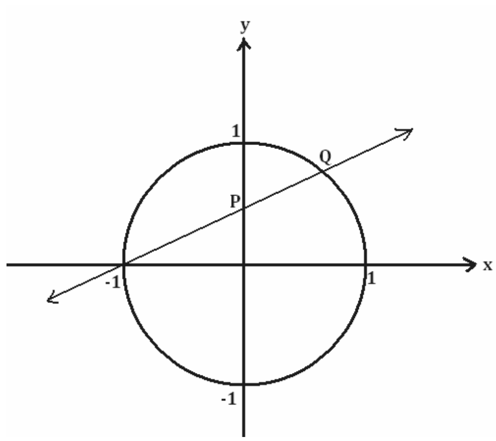
b. Write f as a product of a linear factor and a quadratic factor, each with real-number coefficients.

c. What is the value of $f(10)$? Explain how knowing the linear factor of f establishes that $f(10)$ is a multiple of 12.

d. Find the two complex-number zeros of $y = f(x)$.

e. Write f as a product of three linear factors.

3. A line passes through the points $(-1, 0)$ and $P = (0, t)$ for some real number t and intersects the circle $x^2 + y^2 = 1$ at a point Q different from $(-1, 0)$.



a. If $t = \frac{1}{2}$, so that the point P has coordinates $\left(0, \frac{1}{2}\right)$, find the coordinates of the point Q.

A Pythagorean triple is a set of three positive integers a, b, and c satisfying $a^2 + b^2 = c^2$. For example, setting $a = 3$, $b = 4$, and $c = 5$ gives a Pythagorean triple.

b. Suppose that $\left(\frac{a}{c}, \frac{b}{c}\right)$ is a point with rational-number coordinates lying on the circle $x^2 + y^2 = 1$. Explain why then a, b, and c form a Pythagorean triple.

c. Which Pythagorean triple is associated with the point $Q = \left(\frac{5}{13}, \frac{12}{13}\right)$ on the circle?

d. If $Q = \left(\frac{5}{13}, \frac{12}{13}\right)$, what is the value of t so that the point P has coordinates $(0, t)$?

e. Suppose we set $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$, for a real number t. Show that (x, y) is then a point on the circle $x^2 + y^2 = 1$.

f. Set $t = \frac{3}{4}$ in the formulas $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$. Which point on the circle $x^2 + y^2 = 1$ does this give? What is the associated Pythagorean triple?

g. Suppose t is a value greater than 1, $P = (0, t)$, and Q is the point in the second quadrant (different from $(-1, 0)$) at which the line through $(-1, 0)$ and P intersects the circle $x^2 + y^2 = 1$. Find the coordinates of the point Q in terms of t.

4.

a. Write a system of two equations in two variables where one equation is quadratic and the other is linear such that the system has no solution. Explain, using graphs, algebra, and/or words, why the system has no solution.

b. Prove that $x = \sqrt{-5x - 6}$ has no solution.

c. Does the following system of equations have a solution? If so, find one. If not, explain why not.

$$\begin{aligned} 2x + y + z &= 4 \\ x - y + 3z &= -2 \\ -x + y + z &= -2 \end{aligned}$$

A Progression Toward Mastery					
Assessment Task Item		**STEP 1** Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.	**STEP 2** Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.	**STEP 3** A correct answer with some evidence of reasoning or application of mathematics to solve the problem <u>or</u> an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.	**STEP 4** A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
1	**a–c** N-Q.A.2 F-IF.C.7c G-GPE.A.2	(a) Student provides incorrect vertex coordinates. (b) Student sketches a parabola that does not open up or a parabola that is horizontal. (c) Student provides an equation that is not in the form of a vertical parabola.	(a) Student provides either an incorrect x- or y-coordinate. (b) Student provides a sketch of a parabola that opens up but with little or no scale or labels. (c) Student provides an incorrect equation using the vertex from part (a); a-value is incorrect due to conceptual errors.	(a) Student provides the correct vertex. (b) Student provides a sketch of a parabola that opens up with correct vertex. The sketch may be incomplete or lack sufficient labels or scale. (c) Student provides a parabola equation with correct vertex. Work showing a-value calculation may contain minor errors.	(a) Student provides the correct vertex. (b) Student provides a well-labeled and accurate sketch of a parabola that opens up and includes the focus, directrix, and vertex. (c) Student provides the correct parabola equation in vertex or standard form with or without work showing how $a = \frac{1}{8}$.
	d–e N-Q.A.2 F-IF.C.7c G-GPE.A.2	(d) Student provides incorrect y-intercept. No work is shown or a conceptual error is made. (e) Student makes no attempt or provides two incorrect distances.	(d) Student provides incorrect y-intercept. No work is shown or a conceptual error is made (e.g., $y = 0$ instead of $x = 0$). (e) Student provides one correct distance but not both, using student's incorrect y-intercept and the given focus and directrix. <u>OR</u> Student provides the correct y-intercept in part (d), but the student is unable to compute one or both distances	(d) Student substitutes $x = 0$ to determine the y-intercept, but may make a minor calculation error. (e) Student provides the correct distance to the directrix using student's y-intercept. Student provides the correct distance between focus and y-intercept using student's y-intercept. Note that if these are not equal, the student solution should indicate that they should be based on the definition	(d) Student correctly identifies the y-intercept. (e) Student correctly identifies the distance to directrix and applies the distance formula to calculate the distance from focus and y-intercept. Both are equal to $\frac{17}{8}$.

			between the y-intercept and the given focus and directrix.	of a parabola.	
	f–h **N-Q.A.2** **F-IF.C.7c** **G-GPE.A.2**	Student incorrectly answers two or more parts with no justification in all three parts. OR Student incorrectly answers all three parts with faulty or no justification.	Student incorrectly answers two or more parts. Minimal justification is provided that includes a reference to the a-value.	Student correctly answers all three parts with no justification. OR Student correctly answers two out of three parts with correct justification.	Student correctly answers all three parts. Justification states that parabolas with equal a-values are congruent, but all parabolas are similar.
2	**a–b** **A-SSE.A.2** **A-APR.A.1** **A-APR.B.2** **A-APR.B.3** **A-REI.A.1** **A-REI.B.4b** **F-IF.C.7c**	(a) Student concludes that $x = -2$ is NOT a zero due to conceptual or major calculation errors (e.g., incorrect application of division algorithm) OR Student shows no work at all. (b) Student does not provide factored form or it is incorrect.	(a) Student concludes that $x = -2$ is NOT a zero due to minor calculation errors and provides limited justification for the solution. (b) Student does not provide factored form or it is incorrect.	(a) Student concludes that $x = -2$ is a zero, but may not support this mathematically or verbally. (b) Student correctly identifies both factors, but the polynomial may not be written in factored form $\left((x+2)(x^2+2x+2)\right)$.	(a) Student concludes that $x = -2$ is a zero by showing $f(-2) = 0$ or using division and getting a remainder equal to 0. Work or written explanation supports conclusion. (b) Student writes f in correct factored form and provides work to support the solution. Note that the work may be done in part (a).
	c **A-SSE.A.2** **A-APR.B.2** **A-APR.B.3** **A-REI.A.1** **A-REI.B.4b** **F-IF.C.7c**	Student provides an incorrect value of $f(10)$ and a conclusion regarding 12 being a factor is missing or unsupported by any mathematical work or explanation.	Student provides an incorrect value of $f(10)$ and an incorrect factored form of f. The student does not attempt to find the numerical factors of $f(10)$ or divide $f(10)$ by 12 to see if the remainder is 0.	Student provides a complete solution, but the solution may contain minor calculation errors on the value of $f(10)$. OR Student concludes that 12 is NOT a factor of $f(10)$ because the student used an incorrectly factored form of f in the first place or an incorrect value for $f(10)$.	Student correctly identifies the solution as $f(10) = 1464$. The explanation clearly communicates that 12 is a factor of $f(10)$, i.e., when $x = 10$, $(x+2)$ is 12.
	d–e **A-SSE.A.2** **A-APR.A.1** **A-APR.B.2** **A-APR.B.3** **A-REI.A.1**	(d) Student does not use quadratic formula or uses incorrect formula. (e) Student provides incorrect complex roots and the solution is not a cubic equivalent to	(d) Student makes minor errors in the quadratic formula and provides incorrect roots. (e) Student uses the incorrect roots from (d), but the solution is a cubic equivalent to $(x+2)(x-r_1)(x-r_2)$, where	(d) Student provides the correct complex roots using the quadratic formula (does not have to be in simplest form). (e) Student provides a cubic polynomial using -2 and complex roots from (d). May contain	(d) Student provides the correct complex roots expressed as $(-1 \pm i)$. (e) Student provides a cubic polynomial equivalent to $(x+2)\left(x-(1+i)\right)\left(x-(1-i)\right)$. It is acceptable to leave it in factored

	A-REI.B.4b F-IF.C.7c	$(x+2)(x-r_1)(x-r_2)$, where r_1 and r_2 are complex conjugates.	r_1 and r_2 are the student solutions to (d).	minor errors (e.g., leaving out parentheses on $(x-(1+i))$ or a multiplication error when writing the polynomial in standard form.	form.
3	**a** A-APR.C.4 A-APR.D.6 A-REI.A.2 A-REI.C.6 A-REI.C.7	Student provides an incorrect equation of the line and makes major mathematical errors in attempting to solve a system of a linear and non-linear equation.	Student provides an incorrect equation of the line, but the solution shows substitution of the student's linear equation into the circle equation. The solution to the system may also contain minor calculation errors. OR Student provides a correct equation of the line, but the student is unable to solve the system due to major mathematical errors.	Student provides the correct equation of the line. The solution to the system may contain minor calculation errors. The correct solution is not expressed as an ordered pair or the solution only includes a correct x- or y-value for point Q.	Student provides the correct equation of the line and the correct solution to the system of equations. The solution is expressed as an ordered pair, $Q\left(\frac{3}{5},\frac{4}{5}\right)$.
	b–c A-APR.C.4 A-APR.D.6 A-REI.A.2 A-REI.C.6 A-REI.C.7	(b) Student does not provide an answer or the answer is incorrect showing limited understanding of they were asked to do. (c) Student does not provide a triple or it is incorrect.	(b) Student substitutes $\left(\frac{a}{c},\frac{b}{c}\right)$ into the equation of the circle, but fails to show that this equation is equivalent to $a^2+b^2=c^2$. (c) Student provides an incorrect triple.	(b) Student provides an almost-complete solution (i.e., student substitutes $\left(\frac{a}{c},\frac{b}{c}\right)$ into the equation of the circle and states that the point satisfies the Pythagorean Triple condition but doesn't show why). The solution may contain minor algebra mistakes. (c) Student identifies 5, 12, 13 as the triple.	(b) Student provides a correct solution showing substitution of $\left(\frac{a}{c},\frac{b}{c}\right)$ into the equation of a circle. The work clearly demonstrates this equation is equivalent to $a^2+b^2=c^2$. (c) Student identifies 5, 12, 13 as the triple.
	d–f A-APR.C.4 A-APR.D.6 A-REI.A.2 A-REI.C.6 A-REI.C.7	Student does not provide a solution or provides an incomplete solution to (d), (e), and (f) with major mathematical errors.	(d) Student provides correct slope of the line but fails to identify correct value of t. (e) Student substitutes coordinates into $x^2+y^2=1$, but makes major errors in attempt to show they satisfy the equation. (f) Student substitutes $\frac{3}{4}$ for t, but the solution is incorrect.	(d) Student provides the correct slope and equation of the line but fails to identify the correct value of t. (e) Student substitutes coordinates into $x^2+y^2=1$ and simplifies to show they satisfy the equation. (f) Student identifies point Q and the triple correctly.	(d) Student provides the correct slope and equation of the line and correctly identifies the t-value. (e) Student substitutes coordinates into $x^2+y^2=1$ and simplifies to show they satisfy the equation. (f) Student identifies point Q and the triple correctly.

				Note that one or more parts may contain minor calculation errors.	Note that all solutions use proper mathematical notation and clearly demonstrate student understanding.
	g **A-APR.C.4** **A-APR.D.6** **A-REI.A.2** **A-REI.C.6** **A-REI.C.7**	Student does not provide a solution or provides an incomplete solution with major mathematical errors.	The solution may include an accurate sketch and the equation of the line $y = tx + t$ but little additional work.	Student attempts to solve the system by substituting $y = tx + t$ into the circle equation and recognizes the need to apply the quadratic equation to solve for x. May contain algebraic errors.	Student provides a complete and correct solution, showing sufficient work and calculation of both the x- and y-coordinate of the point.
4	**a** **A-REI.A.2** **A-REI.B.4b** **A-REI.C.6** **A-REI.C.7**	Student does not provide work or it is incomplete. System does not include a linear and quadratic equation.	Student provides a system that has a solution, but student work indicates understanding that the graphs of the equations should not intersect or that algebraically the system has no real number solutions.	Student provides a system that does not have a solution, but the justification may reveal minor errors in student's thought process. If a graphical justification is the only one provided the graph must be scaled sufficiently to provide a convincing argument that the two equations do not intersect.	Student provides a system that does not have a solution. Justification includes a graphical, verbal explanation, or algebraic explanation that clearly demonstrates student thinking.
	b–c **A-REI.A.2** **A-REI.B.4b** **A-REI.C.6** **A-REI.C.7**	Student provides incorrect solutions with little or no supporting work shown.	Student provides incorrect solutions to parts (b) and (c). Solutions are limited and reveal major mathematical errors in the solution process.	Student provides incorrect solutions to part (b) or part (c). Solutions show considerable understanding of the processes, but may contain minor errors.	Student provides correct solutions with sufficient work shown. AND Mathematical work or verbal explanation show why -2 and -3 are NOT solutions to part (b).

Name ______________________________ Date ____________________

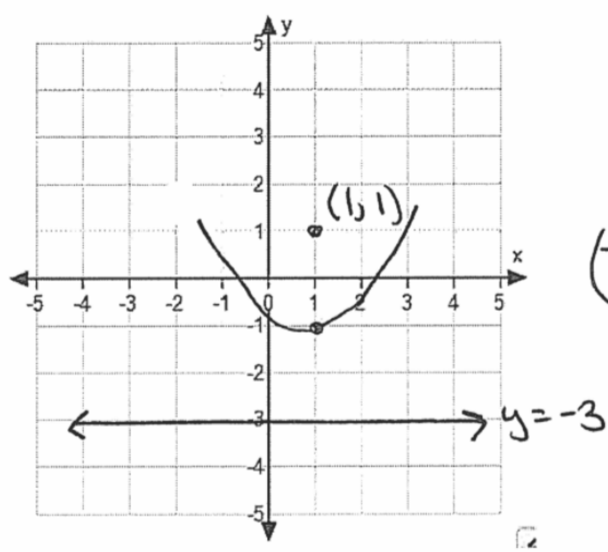
1. A *parabola* is defined as the set of points in the plane that are equidistant from a fixed point (called the *focus* of the parabola) and a fixed line (called the *directrix* of the parabola).

   Consider the parabola with focus point $(1, 1)$ and directrix the horizontal line $y = -3$.

   a. What are the coordinates of the vertex of the parabola?

   (1,-1)

   b. Plot the focus and draw the directrix on the graph below. Then draw a rough sketch of the parabola.



(This is very rough!)

c. Find the equation of the parabola with this focus and directrix.

A point (x,y) on the parabola is equidistant from the directrix and the focus:

$$y+3=\sqrt{(x-1)^2+(y-1)^2}$$
$$(y+3)^2=(x-1)^2+(y-1)^2$$
$$y^2+6y+9=(x-1)^2+y^2-2y+1$$
$$8y=(x-1)^2-8$$
$$y=\frac{1}{8}(x-1)^2-1$$

d. What is the y-intercept of this parabola?

At the y-intercept, $x=0$, so $y=\frac{1}{8}(-1)^2-1=-\frac{7}{8}$

$-\frac{7}{8}$ is the y intercept.

e. Demonstrate that your answer from part (d) is correct by showing that the y-intercept you identified is indeed equidistant from the focus and the directrix.

The distance of $\left(0,-\frac{7}{8}\right)$ from the focus is:

$$\sqrt{(0-1)^2+\left(-\frac{7}{8}-1\right)^2}=\sqrt{1+\left(\frac{15}{8}\right)^2}=\sqrt{\frac{289}{64}}=\frac{17}{8}.$$

The distance of $\left(0,-\frac{7}{8}\right)$ from the line $y=3$ is:

$$\left|\left(-\frac{7}{8}\right)-(-3)\right|=2\frac{1}{8}.$$

These are the same!

f. Is the parabola in this question (with focus point $(1, 1)$ and directrix $y = -3$) congruent to a parabola with focus $(2, 3)$ and directrix $y = -1$? Explain.

The parabola with focus $(2,3)$ and directrix $y=-1$ is $y+1=\sqrt{(x-2)^2+(y-3)^2}$. Solving for y, $(y+1)^2=(x+2)^2+(y-3)^2$,

$y^2+2y+1=(x-2)^2+y^2-6y+9$

$8y=(x-2)^2+8$

$y=\frac{1}{8}(x-2)^2+1$

This parabola is congruent to the parabola with focus point $(1,1)$ and directrix $y=-3$ because the leading coefficients are the same.

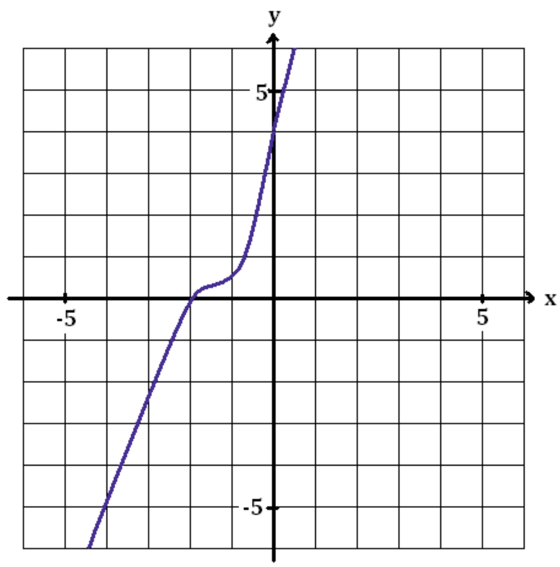
g. Is the parabola in this question (with focus point $(1, 1)$ and directrix $y = -3$) congruent to the parabola with equation given by $y = x^2$? Explain.

No, $y=x^2$ and $y=\frac{1}{8}(x-1)^2+1$ do not have the same leading coefficient so they are not congruent. ($y=x^2$ has a leading coefficient 1, and $y=\frac{1}{8}(x-1)^2+1$ had a leading coefficient $\frac{1}{8}$.

h. Are the two parabolas from part (g) similar? Why or why not?

Yes! Because all parabolas are similar.

2. The graph of the polynomial function $f(x) = x^3 + 4x^2 + 6x + 4$ is shown below.



a. Based on the appearance of the graph, what does the real solution to the equation $x^3 + 4x^2 + 6x + 4 = 0$ appear to be? Jiju does not trust the accuracy of the graph. Prove to her algebraically that your answer is in fact a zero of $y = f(x)$.

The real zero appears to be $x = -2$.

$f(-2) = (-2)^3 + 4(-2)^2 + 6(-2) + 4$
$= -8 + 16 - 12 + 4$
$= 0$

b. Write f as a product of a linear factor and a quadratic factor, each with real-number coefficients.

Since $x = -2$ is a zero, $x+2$ must be a factor.
Dividing $f(x)$ by $(x+2)$ gives $(x^2 + 2x + 2)$

So,
$f(x) = (x+2)(x^2 + 2x + 2)$

	x^2	$2x$	2
x	x^2	$2x^2$	$2x$
2	$2x^2$	$4x$	4

x^3, $4x^2$, $6x$, 4

c. What is the value of $f(10)$? Explain how knowing the linear factor of f establishes that $f(10)$ is a multiple of 12.

$$f(10) = (10+2)(100+20+2)$$
$$= 12(122)$$
$$= 1220 + 244$$
$$= 1464$$

f(10) is a multiple of 12 because f(x) has a linear factor of x+2, and x+2 = 12, when x = 10.

d. Find the two complex-number zeros of $y = f(x)$.

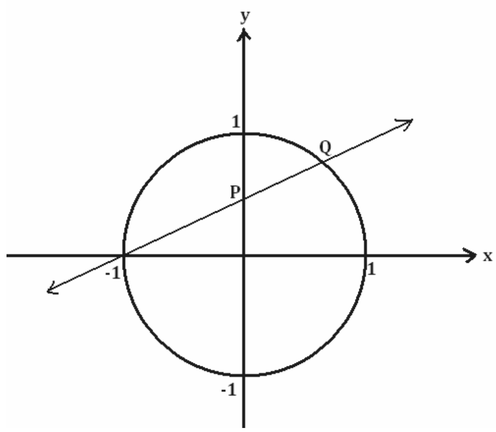
We need to solve $x^2 + 2x + 2 = 0$.

$$x^2 + 2x + 1 = -1$$
$$(x+1)^2 = -1$$
$$x + 1 = \pm i$$
$$x = -1 \pm i$$

e. Write f as a product of three linear factors.

$$f(x) = (x+2)(x-(-1+i))(x-(-1-i))$$

3. A line passes through the points $(-1, 0)$ and $P = (0, t)$ for some real number t and intersects the circle $x^2 + y^2 = 1$ at a point Q different from $(-1, 0)$.



a. If $t = \frac{1}{2}$, so that the point P has coordinates $\left(0, \frac{1}{2}\right)$, find the coordinates of the point Q.

The slope of $\overleftrightarrow{PQ} = t = \frac{1}{2}$

Line $\overleftrightarrow{PQ}$ has equation $y = \frac{1}{2}(x+1)$

Point Q lies on the line $y = \frac{1}{2}(x+1)$ and the circle $x^2+y^2=1$.

So, $x^2 + \left(\frac{1}{2}(x+1)\right)^2 = 1$

$x^2 + \frac{1}{4}(x^2+2x+1) = 1$

$4x^2 + x^2 + 2x + 1 = 4$

$5x^2 + 2x + 1 = 4$

$25x^2 + 10x + 5 = 20$

$25x^2 + 10x + 1 = 16$

$(5x+1)^2 = 16$

$5x+1 = 4$ or $5x+1 = -4$

$5x = 3$ or -5

$x = \frac{3}{5}$ or -1

Since Q is in the first quadrant, choose $x = \frac{3}{5}$.

Then $y = \frac{1}{2}\left(\frac{3}{5}+1\right) = \frac{4}{5}$.

The point Q is $\left(\frac{3}{5}, \frac{4}{5}\right)$.

A Pythagorean triple is a set of three positive integers a, b, and c satisfying $a^2 + b^2 = c^2$. For example, setting $a = 3$, $b = 4$, and $c = 5$ gives a Pythagorean triple.

b. Suppose that $\left(\frac{a}{c}, \frac{b}{c}\right)$ is a point with rational-number coordinates lying on the circle $x^2 + y^2 = 1$. Explain why then a, b, and c form a Pythagorean triple.

We have $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$.
That is $\frac{a^2+b^2}{c^2} = 1$. Thus $a^2 + b^2 = c^2$.
If a, b, and c are integers, this is a Pythagorean triple.

c. Which Pythagorean triple is associated with the point $Q = \left(\frac{5}{13}, \frac{12}{13}\right)$ on the circle?

5, 12, 13

d. If $Q = \left(\frac{5}{13}, \frac{12}{13}\right)$, what is the value of t so that the point P has coordinates $(0, t)$?

Slope $\overline{PQ} = \frac{t-0}{0-(-1)} = t$, using points $(-1,0)$ and $(0,t)$.

Using points $\left(\frac{5}{13}, \frac{12}{13}\right)$ and $(-1,0)$, slope $\overline{PQ} = \frac{\frac{12}{13} - 0}{\frac{5}{13} + 1}$

$= \frac{\frac{12}{13}}{\frac{18}{13}} = \frac{2}{3}$.

Thus, $t = \frac{2}{3}$.

e. Suppose we set $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$, for a real number t. Show that (x, y) is then a point on the circle $x^2 + y^2 = 1$.

We need to show that $\left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2$ equals 1.

$$\left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2 = \frac{1-2t^2+t^2+4t^2}{(1+t^2)^2}$$

$$= \frac{t^4+2t^2+1}{(1+t^2)^2}$$

$$= \frac{(t^2+1)^2}{(t^2+1)^2} = 1$$

We're good!

f. Set $t = \frac{3}{4}$ in the formulas $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$. Which point on the circle $x^2 + y^2 = 1$ does this give? What is the associated Pythagorean triple?

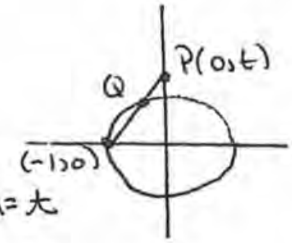
For $t = \frac{3}{4}$, $x = \dfrac{1-\frac{9}{16}}{1+\frac{9}{16}} = \dfrac{\frac{7}{16}}{\frac{25}{16}} = \dfrac{7}{25}$

and $y = \dfrac{2\left(\frac{3}{4}\right)}{1+\frac{9}{16}} = \dfrac{\frac{6}{4}}{\frac{25}{16}} = \dfrac{24}{25}$

So (x, y) is $\left(\frac{7}{25}, \frac{24}{25}\right)$

and the Pythagorean triple is 7, 24, 25.

g. Suppose t is a value greater than 1, $P = (0, t)$, and Q is the point in the second quadrant (different from $(-1, 0)$) at which the line through $(-1, 0)$ and P intersects the circle $x^2 + y^2 = 1$. Find the coordinates of the point Q in terms of t.



$m = t$

Line $\overline{PQ}$ has equation $y = t(x+1)$.

Point Q lies on the line $y = t(x+1)$ and the circle $x^2+y^2=1$.

So, $x^2 + t^2(x+1)^2 = 1$. Solving for x:

$x^2 + t^2(x^2+2x+1) = 1$

$(1+t^2)x^2 + 2t^2x + t^2 - 1 = 0$

$$x = \frac{-2t^2 \pm \sqrt{4t^4 - 4(1+t^2)(t^2-1)}}{2(1+t^2)} = \frac{-t^2 \pm \sqrt{t^4-(t^4-1)}}{(1+t^2)} = \frac{-t^2 \pm 1}{1+t^2}$$

$x = \frac{1-t^2}{1+t^2}$ or $x = -1$. Since we are looking for a point different than P, we choose $x = \frac{1-t^2}{1+t^2}$

Substituting back into the equation of line $\overline{PQ}$,

$$y = t\left(\frac{1-t^2}{1+t^2} + 1\right)$$

$$= t\left(\frac{1-t^2+1+t^2}{1+t^2}\right)$$

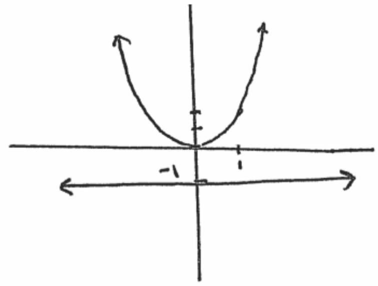
$$= \frac{2t}{1+t^2}$$

The point Q is $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$.

4.

a. Write a system of two equations in two variables where one equation is quadratic and the other is linear such that the system has no solution. Explain, using graphs, algebra, and/or words, why the system has no solution.

$y = x^2$
$y = -1$



From the graph, these two curves do not intersect, and so there is no solution to this system of equations.

b. Prove that $x = \sqrt{-5x - 6}$ has no solution.

If $x = \sqrt{-5x-6}$ holds for some number x, then $x^2 = -5x - 6$ would hold for that number, too.

That is, $x^2 + 5x + 6 = 0$

$(x+3)(x+2) = 0$

$x = -3$ or $x = -2$

But, $x = -3$ does not work: $-3 \neq \sqrt{15-6}$

and $x = -2$ does not work: $-2 \neq \sqrt{10-6}$.

So there is no solution after all.

c. Does the following system of equations have a solution? If so, find one. If not, explain why not.

1. $2x + y + z = 4$
2. $x - y + 3z = -2$
3. $-x + y + z = -2$

1. − 3. $\Rightarrow 3x = 6$

$x = 2$

1. $2(2) + y + z = 4 \Rightarrow y + z = 0$
2. $2 - y + 3z = -2 \Rightarrow -y + 3z = -4$
3. $-2 + y + z = -2 \Rightarrow y + z = 0$

1. + 2. $\Rightarrow 4z = -4$

$z = -1 \Rightarrow y = 1$

Check: $4 + 1 - 1 = 4$ ✓

$2 - 1 - 3 = -2$ ✓

$-2 + 1 - 1 = -2$ ✓

The solution is $(2, 1, -1)$.