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DOUBLE MILLING IN SELF-PROPELLED SWARMS FROM KINETIC THEORY

J. A. Carrillo

ICREA (Institució Catalana de Recerca i Estudis Avan**c**cats)
and Departament de Matemàtiques
Universitat Autònoma de Barcelona, E-08193 Bellaterra, Spain

M. R. D'Orsogna

Department of Mathematics, California State University
Northridge, CA 91330-8313, USA

V. Panferov

Department of Mathematics, California State University
Northridge, CA 91330-8313, USA

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Abstract. We present a kinetic theory for swarming systems of interacting, self-propelled discrete particles. Starting from the Liouville equation for the many-body problem we derive a kinetic equation for the single particle probability distribution function and the related macroscopic hydrodynamic equations. General solutions include flocks of constant density and fixed velocity and other non-trivial morphologies such as compactly supported rotating mills. The kinetic theory approach leads us to the identification of macroscopic structures otherwise not recognized as solutions of the hydrodynamic equations, such as double mills of two superimposed flows. We find the conditions allowing for the existence of such solutions and compare to the case of single mills.

1. **Introduction.** Viewed as a collective, interacting individuals often flow into spectacular coherent patterns [6]. Systems that self-organize can be observed in nature across a wide variety of spatio-temporal scales: schools of fish, flocks of birds and swarms of insects among animal species; morphogenetic and bacterial growth at the cellular and subcellular levels. While each of these groups follows specific physical laws, all are able to organize in the absence of a leader, allowing order to arise even when starting from disordered configurations [27].

The ubiquity of the self-organizing phenomenon has lead to the development of several minimal models to describe a collection of interacting agents, both as discrete particles [35, 18, 10] or as a continuous density [32, 33, 34]. In particular, models of individuals driven by self-propelling forces and pairwise attractive and repulsive interactions have been shown to self-organize in various morphologies. Translationally invariant flocks, rotating mills, rings and clumps have all been observed and classified, so that specific interaction and propulsion values can be associated to specific collective configurations [22, 24, 16]. Aggregation patterns have also been

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identified in discrete models where averaging in direction or velocity is performed [11, 12] or where different zones of interaction and averaging are considered [1, 2].

However, as the number of particles grows, it becomes increasingly difficult to follow the dynamics of each individual agent. Indeed, to find the time evolution of a configuration of N individuals there are about N^2 interactions to compute, which in the limit of large N may yield to prohibitive calculations. A more compact, continuum approach where particles are represented by a density field, becomes thus desirable. While several continuum models based on heuristic derivations have been presented in the literature, few attempts have been made at deriving such models starting from principles of dynamics of individual agents [5, 9, 20, 13, 14].

Furthermore, even when parallels between microscopic and macroscopic descriptions exist, there might be cases in which the fullness of a microscopic solution is not immediately captured by the corresponding macroscopic one. For example, in simulations of rotating mills, discrete particle systems show the possibility of *two* compactly supported structures of roughly the same number of particles circulating in opposite directions. The corresponding macroscopic solution would be a "boring" stationary density since the two mills average out their velocities to zero. Due to the non-linearity of the problem, the trivial superposition of two rotating mills which solve the macroscopic problem, is not necessarily a solution itself.

This paper aims to bridge general microscopic descriptions of self-propelled interacting swarming systems to their macroscopic counterparts, using kinetic theory [7, 23] as middle ground. Here, the exact location and velocity of particles are considered irrelevant, but not to the extent that average velocities can be computed *tout court* at every position and every time step. Rather, several velocities may be possible, so that the focus is on determining the probability density function $f(\mathbf{x}, \mathbf{v}, t)$ that at time t a particle is at position $\mathbf{x}$ with velocity $\mathbf{v}$. Starting from a set of discrete swarming equations we shall thus derive the kinetic equation for $f(\mathbf{x}, \mathbf{v}, t)$ and hence present the corresponding hydrodynamic description. Solutions will be matched to the discrete case and most importantly, our kinetic model will allow us to identify the presence of a new class of solutions, those of double mills, which elude a strictly macroscopic derivation.

2. **Discrete model.**

We consider N interacting, self-propelled particles with Rayleigh friction in $\mathbb{R}^d$, governed by the following equations of motion [22, 16, 8, 9]

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = (\alpha - \beta\,|\mathbf{v}_i|^2)\mathbf{v}_i - \nabla_{\mathbf{x}_i} \displaystyle\sum_{j \neq i} U(|\mathbf{x}_i - \mathbf{x}_j|). \end{cases}$$

Here U is a pairwise interaction potential and $\alpha, \beta > 0$ are effective values for propulsion and friction forces. A common choice for U is the Morse potential composed of attractive and repulsive components

$$U(r) = -C_a e^{-r/\ell_a} + C_r e^{-r/\ell_r}, \tag{1}$$

with C_a, C_r denoting attractive and repulsive strengths and ℓ_a, ℓ_r their respective length scales. While the Morse potential is a common choice for interacting swarming systems, in this formulation we keep U general. To analyze the limit of large number of particles N, we scale the amplitude of the potential through an effective mass normalization. For simplicity, we assume identical particles of mass m with total mass fixed at $M = Nm$. The "weak coupling scaling" assumption for

the mean-field limit, see [30, 31] and [17, Part I], assumes that the potential is modulated by a factor M/N, i.e.,

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = (\alpha - \beta\,|\mathbf{v}_i|^2)\mathbf{v}_i - \dfrac{M}{N}\nabla_{\mathbf{x}_i}\displaystyle\sum_{j\neq i} U(|\mathbf{x}_i - \mathbf{x}_j|), \end{cases} \tag{2}$$

The above scaling of the potential allows the total kinetic and potential energy to bear the same N dependence since, in the unnormalized case, the total kinetic energy is a sum of N terms and the total pairwise potential energy scales as N^2. The weak coupling limit assumption can be justified as representing a scenario where a particle located at x_j imparts an interaction potential on the position x, proportional to its mass $m = M/N$, as in Coulomb or gravitational interactions.

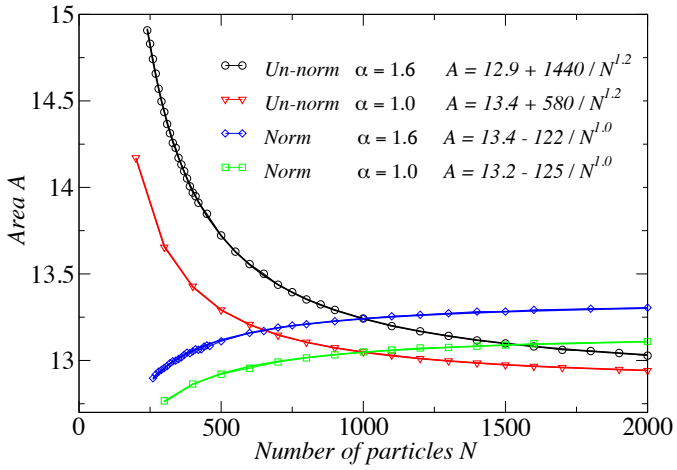

FIGURE 2.1. Area of unidirectional milling structures as a function of particle number N. The upper curve represents the unnormalized potentials, the lower one the potential normalized so that the total mass is kept at $M = Nm = 1000$. Milling parameters are chosen as $C_a = 0.5$, $C_r = 1.0$, $\ell_a = 3$, $\ell_r = 0.5$ while the self-propulsion terms are set at $\alpha = 1.6$, $\beta = 0.5$ and at $\alpha = 1.0$, $\beta = 0.5$. The normalized and unnormalized curves match at $N = 1000$, which is the reference point for the unnormalized potential. Curves can be fitted as $A \sim A_0 + B/N^{\xi}$, where $\xi = 1.2$ or $\xi = 1.0$ for the unnormalized and normalized potentials, respectively.

The mass normalization in (2) allows for the kinetic and potential energy to be the sum of terms that scale as N. While interaction amplitudes are now N dependent, the fundamental character of the resulting morphologies does not change since we simply introduce a multiplicative factor for U. As an example, in the case of the Morse potential, patterns of aggregation depend on the relative amplitudes $C = C_r/C_a$ and $\ell = \ell_r/\ell_a$, so that N dependencies do not affect the qualitative features of the observed patterns. The latter were classified in two dimensions in terms of C and ℓ in [16] using the concept of H-stability of potentials.

The most relevant set of parameters for biological applications concerns long-range attraction and short-range repulsion leading to $C > 1$ and $\ell < 1$. For these potentials, there exists a unique minimum of the pairwise potential and a typical distance minimizing the potential energy. However, the curve $C\ell^d = 1$ divides this parameter region into a catastrophic ($C\ell^d < 1$) and an H-stable regime ($C\ell^d > 1$). In the H-stable regime and for unnormalized potentials, particle simulations in two dimensions lead to crystalline-like patterns. Here, for sufficiently large N, agents find an optimal spacing and maintain a fixed relative distance from each other even as N is further increased. Conversely, in the catastrophic regime, particles settle at a typical rotational speed of magnitude $|v| = \sqrt{\alpha/\beta}$, and single and double mills are observed. The area of these rotating structures decreases as a function of N. We refer to [16] for full details.

Due to the fact that qualitative features depend solely on the ratios C and ℓ, both normalized and unnormalized potentials yield the same type of patterns, whether they be mills or flocks, however the way these structures scale with N is very different. In Figure 2.1 we show single mill areas of discrete particle systems in two dimensions as a function of N in the so called catastrophic regime, where, in the unnormalized case, rotating mills are expected to decrease in size, even to lose their inner radius, as the number of agents increases. As it can be seen, mill areas scale very differently in the unnormalized regime compared to the normalized case. We observe the tendency of mills to equilibrate to a finite density in a fixed annular region as the number of agents increases, as is typical of catastrophic potentials. The asymptotic value of the area as $N \to \infty$ depends on the intrinsic particle speed with higher values of $\sqrt{\alpha/\beta}$ yielding higher asymptotic limits. This is simply due to higher fluctuations in particle positions in the collapsed state.

While coherent flocks and single mill states are the most common patterns observed in biological swarms [27, 29], double-mill patterns are also reported in the biological literature, for instance *M. xanthus* cells show distinct cell subpopulations swarming in two opposite directions during part of their life cycle [21].

3. Collisionless kinetic model.

In this section, we briefly present the derivation of a mean-field kinetic model from the particle dynamics (2). This type of formal derivation is classical, see e. g. [3, 23] and also [4, 15, 28, 25, 26, 30, 31, 17] for applications to other physical systems and a rigorous analysis of related models.

Let us denote by $f^{(N)}(\{\mathbf{x}_i\}, \{\mathbf{v}_i\}, t)$ the N-particle probability density function, so that the probability of finding each of the N particles at position $\mathbf{x}_i$ and velocity $\mathbf{v}_i$ within a volume $d\mathbf{x}_i\, d\mathbf{v}_i$ in phase space is $f^{(N)}(\{\mathbf{x}_i\}, \{\mathbf{v}_i\}, t) \prod_i d\mathbf{x}_i\, d\mathbf{v}_i$. For simplicity and without restricting generality, we set in (2) the total mass $M = 1$. Conservation of mass allows us to write the time evolution of $f^{(N)}$ according to the following Liouville equation

$$\frac{\partial f^{(N)}}{\partial t} + \sum_{i=1}^{N} \left[\mathrm{div}_{\mathbf{x}_i}(\dot{\mathbf{x}}_i f^{(N)}) + \mathrm{div}_{\mathbf{v}_i}(\dot{\mathbf{v}}_i f^{(N)}) \right] = 0, \tag{3}$$

where $\dot{\mathbf{x}}_i$ and $\dot{\mathbf{v}}_i$ are expressed through (2). The one-particle distribution function $f^{(1)}(\mathbf{x}_1, \mathbf{v}_1, t)$ is defined as

$$f^{(1)}(\mathbf{x}_1, \mathbf{v}_1, t) = \int f^{(N)}\, d\mathbf{x}_2 \dots d\mathbf{x}_N d\mathbf{v}_2 \dots d\mathbf{v}_N. \tag{4}$$

To obtain an evolution equation for $f^{(1)}$ we will use two main assumptions: the propagation of chaos (5) and the particular $1/N$ scaling of the potential, as introduced in (2). We integrate the Liouville equation (3) to find the corresponding BBGKY hierarchy; restricting our analysis to the case of $f^{(1)}$ we find

$$\frac{\partial f^{(1)}}{\partial t} + \int \mathrm{div}_{\mathbf{x}_1}(\mathbf{v}_1 f^{(N)})\, d\mathbf{\Omega}_1 + \int \mathrm{div}_{\mathbf{v}_1}(\dot{\mathbf{v}}_1 f^{(N)})\, d\mathbf{\Omega}_1 = 0,$$

where $d\mathbf{\Omega}_1 = d\mathbf{x}_2 \dots d\mathbf{x}_N d\mathbf{v}_2 \dots d\mathbf{v}_N$ is the volume element. The spatial divergence term reduces to $\mathbf{v}_1 \cdot \nabla_{\mathbf{x}_1} f^{(1)}$, while for the momentum term we obtain the following two contributions

$$\mathrm{div}_{\mathbf{v}_1}[(\alpha - \beta|\mathbf{v}_1|^2)\mathbf{v}_1 f^{(1)}] \qquad \text{and} \qquad \mathrm{div}_{\mathbf{v}_1} \int (\nabla_{\mathbf{x}_1} \sum_{j\neq 1} U_{1,j})\, f^{(N)}\, d\mathbf{\Omega}_1,$$

where $U_{1,j}$ is a shorthand notation for $U(|\mathbf{x}_1 - \mathbf{x}_j|)$. Since particles are indistinguishable the last term can be recast as

$$(N-1)\int \nabla_{\mathbf{x}_1} U_{1,2}\, f^{(N)}\, d\mathbf{\Omega}_1 = (N-1)\int \nabla_{\mathbf{x}_1} U_{1,2}\, f^{(2)}\, d\mathbf{x}_2 d\mathbf{v}_2,$$

where $f^{(2)}$ is the two-particle probability density function

$$f^{(2)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2) = \int f^{(N)} d\mathbf{\Omega}_2,$$

with the notation $d\mathbf{\Omega_1} = d\mathbf{x}_2 d\mathbf{v}_2\, d\mathbf{\Omega}_2$. We furthermore assume the factorization

$$f^{(2)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2) = f^{(1)}(\mathbf{x}_1, \mathbf{v}_1)\, f^{(1)}(\mathbf{x}_2, \mathbf{v}_2) \tag{5}$$

which neglects the two-particle correlations. To simplify the notation we set $f^{(1)}(\mathbf{x}_1, \mathbf{v}_1) =: f(\mathbf{x}, \mathbf{v})$. The integral of f in the velocity variable is the macroscopic density of the system

$$\rho(\mathbf{x}, t) = \int f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}.$$

Since the interaction term is independent of $\mathbf{v}$ we can also write

$$\int \nabla_{\mathbf{x}_1} U_{1,2}\, f^{(2)}\, d\mathbf{x}_2 d\mathbf{v}_2 = (\nabla_{\mathbf{x}} U \star \rho)\, f,$$

where the $\star$ notation denotes the convolution in the $\mathbf{x}$ variable. Equation (4) now reduces to

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathrm{div}_{\mathbf{v}}[(\alpha - \beta|\mathbf{v}|^2)\mathbf{v}\, f] - \frac{N-1}{N}\, \mathrm{div}_{\mathbf{v}}\, [(\nabla_{\mathbf{x}} U \star \rho) f] = 0.$$

Taking the limit $N \to \infty$ yields *the Vlasov equation*

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathrm{div}_{\mathbf{v}}[(\alpha - \beta|\mathbf{v}|^2)\mathbf{v}\, f] - \mathrm{div}_{\mathbf{v}}\, [(\nabla_{\mathbf{x}} U \star \rho) f] = 0. \tag{6}$$

The mean-field limit $N \to \infty$ introduced above can be proved rigorously in the case of smooth potentials U by using the techniques in [25, 4, 15]. A review of these results can be found in [17, Part I]. More precisely, let us consider the empirical measure $\mu_N(t)$ associated with a solution $(\mathbf{x}_i(t), \mathbf{v}_i(t))$ of (2),

$$\mu_N(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{(\mathbf{x}_i(t), \mathbf{v}_i(t))},$$

where $\delta_{(\mathbf{x}_0, \mathbf{v}_0)}$ is the Dirac delta at $(\mathbf{x}_0, \mathbf{v}_0)$ in the phase space of one particle. Given an interaction potential $U(\mathbf{x})$ that is sufficiently well-behaved (e. g. of

class C^2, bounded with bounded derivatives and satisfying $\nabla_{\mathbf{x}} U(0) = 0$), then the empirical measure is always a solution of the kinetic equation (6) in the sense of distributions (see [17, Lemma I.2]). Moreover, a stability result for empirical measures as distributional solutions of the kinetic equation should be expected, although the proofs in the above references do not apply directly. More precisely, assume the initial data of the particle system (2) is chosen in such a way that $\mu_N(0) \to f_0$ weakly-* as measures, then all limit points of the empirical measure in the weak-* topology as measures should be distributional solutions of the kinetic equation (6) with initial data f_0. Such a result is in fact a proof of convergence of the particle method based on (2) for the kinetic equation (6) and will be treated elsewhere. A similar program has recently carried over in [19] for a kinetic model of flocking introduced in [20] based on the particle velocity-averaging interaction studied in [11, 12]. Let us remark that the typical Morse potential (1) is only Lipschitz-continuous for which the previous theory does not apply.

Finally, let us point out that the following estimate holds for the total energy

$$\mathcal{E}(t) = \frac{1}{2}\int f\,|\mathbf{v}|^2\,d\mathbf{x}\,d\mathbf{v} + \frac{1}{2}\int U(|\mathbf{x}-\mathbf{y}|)\,\rho(\mathbf{x},t)\rho(\mathbf{y},t)\,d\mathbf{x}\,d\mathbf{y}. \tag{7}$$

The function $\mathcal{E}(t)$ satisfies

$$\begin{aligned}\frac{d\mathcal{E}}{dt} &= \int f(\alpha-\beta|\mathbf{v}|^2)\,|\mathbf{v}|^2\,d\mathbf{x}\,d\mathbf{v} \\ &\le \alpha\int f|\mathbf{v}|^2\,d\mathbf{v} - \frac{\beta}{M}\Big(\int f|\mathbf{v}|^2\,d\mathbf{v}\Big)^2,\end{aligned} \tag{8}$$

since by Cauchy's inequality,

$$\Big(\int f|\mathbf{v}|^2\,d\mathbf{x}\,d\mathbf{v}\Big)^2 \le \int f\,d\mathbf{x}\,d\mathbf{v}\int f|\mathbf{v}|^4\,d\mathbf{x}\,d\mathbf{v} = M\int f|\mathbf{v}|^4\,d\mathbf{x}\,d\mathbf{v}.$$

Hence, we deduce

$$\frac{d\mathcal{E}}{dt} \le 0, \quad \text{if} \quad \int f|\mathbf{v}|^2\,d\mathbf{x}\,d\mathbf{v} \ge \frac{\alpha M}{\beta}.$$

Since the potential energy verifies

$$\frac{1}{2}\int U(|\mathbf{x}-\mathbf{y}|)\,\rho(\mathbf{x},t)\rho(\mathbf{y},t)\,d\mathbf{x}\,d\mathbf{y} \le CM^2,$$

where $C = \frac{1}{2}\sup|U|$, we obtain the following.

Proposition 3.1. *The energy $\mathcal{E}(t)$ satisfies*

$$\mathcal{E}(t) \le \max\big\{\mathcal{E}(0), CM^2 + \tfrac{\alpha M}{\beta}\big\}.$$

Remark 1. The same estimate holds for the discrete total energy

$$\frac{1}{2}\frac{M}{N}\sum_i |\mathbf{v}_i|^2 + \frac{1}{2}\Big(\frac{M}{N}\Big)^2\sum_{i\neq j} U(|\mathbf{x}_i-\mathbf{x}_j|),$$

defined on the solutions of the dynamical system (2).

Remark 2. We notice that the relation (8) implies the conservation of the energy (7) for solutions supported on the set $\{\beta|\mathbf{v}|^2 = \alpha\}$. For such solutions, since the self-propulsion and friction terms balance each other, and the model does not include any other dissipative mechanism, we cannot expect asymptotic equilibration of solutions as $t \to \infty$ towards a certain profile, however we might expect stability results for certain particular stationary solutions.

One of the objectives of the next sections is to show that all proposed continuum models for our swarming system (2) can be derived from the kinetic equation (6) and its variants. Moreover, all patterns observed in particle simulations can be considered as particular solutions of the kinetic equation (6) as will be shown below.

3.1. **Hydrodynamic equations.** As usual in kinetic theory, we obtain continuum-like equations by computing the evolution of macroscopic quantities starting from (6). These macroscopic quantities are the velocity moments of $f(\mathbf{x},\mathbf{v},t)$. Let us consider the coarse grained velocity field $\mathbf{u}(\mathbf{x},t)$ defined through

$$\rho\mathbf{u} = \int \mathbf{v} f(\mathbf{x},\mathbf{v})\, d\mathbf{v}$$

and the energy-flux $\mathbf{q}_K$, the pressure tensor $\hat{\sigma}_K$ and the temperature θ defined through fluctuation terms as

$$\mathbf{q}_K = \frac{1}{2}\int |\mathbf{v}-\mathbf{u}|^2(\mathbf{v}-\mathbf{u}) f\, d\mathbf{v}, \qquad \hat{\sigma}_K = \int (\mathbf{v}-\mathbf{u})\otimes(\mathbf{v}-\mathbf{u}) f\, d\mathbf{v},$$

and

$$\delta_K = \int |\mathbf{v}-\mathbf{u}|^2 f\, d\mathbf{v} = d\,\rho\,\theta.$$

Integrating (6) in $\mathbf{v}$ we obtain the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}_{\mathbf{x}}(\rho\mathbf{u}) = 0. \tag{9}$$

Moreover, integrating (6) against $\mathbf{v}d\mathbf{v}$ and using integration by parts, we find the momentum equation

$$\begin{aligned}\frac{\partial(\rho\mathbf{u})}{\partial t} + \operatorname{div}_{\mathbf{x}}(\rho\mathbf{u}\otimes\mathbf{u}) = {} & (\alpha-\beta|\mathbf{u}|^2)\rho\,\mathbf{u} - (\nabla_{\mathbf{x}}U\star\rho)\,\rho - \operatorname{div}_{\mathbf{x}}\hat{\sigma}_K \\ & - 2\,\beta\,\mathbf{q}_K - 2\,\beta\,\mathbf{u}\,\hat{\sigma}_K - \beta\,\delta_K\,\mathbf{u}.\end{aligned} \tag{10}$$

To close the moment system we assume that fluctuations are negligible and that the velocity distribution is monokinetic: $f(\mathbf{x},\mathbf{v},t) = \rho(\mathbf{x},t)\,\delta(\mathbf{v}-\mathbf{u}(\mathbf{x},t))$, where δ stands for the Dirac delta. The macroscopic system then reduces to

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}_{\mathbf{x}}(\rho\mathbf{u}) = 0, \\[2ex] \rho\,\dfrac{\partial \mathbf{u}}{\partial t} + \rho\,(\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} = \rho\,(\alpha-\beta|\mathbf{u}|^2)\mathbf{u} - \rho\,(\nabla_{\mathbf{x}}U\star\rho). \end{cases} \tag{11}$$

The system of equations (11) was already proposed in Ref. [9] based on computations of the empirical measure associated to N particles. Here, the same description is recovered from the monokinetic ansatz applied to the kinetic equation (6). In Ref. [9] the authors discussed the validity of this approximation based on numerical comparisons of the N-particle system and the hydrodynamic system (11). They concluded that the hydrodynamic system is a good approximation close to the steady state pattern situations and performed a linear stability analysis for the simple flocking solution around $\rho = \rho_0$ and $|u| = \sqrt{\alpha/\beta}$. Double mills however cannot be simply explained with this hydrodynamic approach due to the use of a single macroscopic velocity.

Based on the kinetic equation (6), we can also write an equation for the energy balance. The kinetic energy density ε_K and the potential energy density ε_U are

defined, respectively, as

$$\varepsilon_K = \frac{1}{2}\int f\,|\mathbf{v}|^2\,d\mathbf{v} = \rho\frac{|\mathbf{u}|^2}{2} + \frac{\delta_K}{2} \quad \text{and} \quad \varepsilon_U = \frac{1}{2}(U\star\rho)\,\rho,$$

so that the total energy density is given by $\varepsilon = \varepsilon_K + \varepsilon_U$. Upon integrating (6) against $\frac{|\mathbf{v}|^2}{2}\,d\mathbf{v}$, we obtain

$$\frac{\partial \varepsilon_K}{\partial t} + \operatorname{div}_{\mathbf{x}}\int f\,\mathbf{v}\tfrac{|\mathbf{v}|^2}{2}\,d\mathbf{v} - \int f\,|\mathbf{v}|^2(\alpha-\beta|\mathbf{v}|^2)\,d\mathbf{v} + (\nabla_{\mathbf{x}}U\star\rho)\,\rho\mathbf{u} = 0.$$

We can rewrite the integral terms as

$$\int f\,\mathbf{v}\tfrac{|\mathbf{v}|^2}{2}\,d\mathbf{v} = \varepsilon_K\mathbf{u} + \mathbf{q}_K + \mathbf{u}\cdot\hat{\sigma}_K,$$

and

$$\int f\,(\alpha-\beta|\mathbf{v}|^2)|\mathbf{v}|^2\,d\mathbf{v} = 2\alpha\,\varepsilon_K + \beta\int f\,|\mathbf{v}|^4\,d\mathbf{v}.$$

Using the chain rule and the continuity equation for ρ we find

$$\frac{\partial \varepsilon_U}{\partial t} = -\operatorname{div}_{\mathbf{x}}\left(\varepsilon_U\mathbf{u}\right) + \frac{1}{2}\left[(\nabla_{\mathbf{x}}U\star\rho\mathbf{u}) + (\nabla_{\mathbf{x}}U\star\rho)\,\mathbf{u}\right]\rho,$$

so that

$$\begin{aligned}\frac{\partial \varepsilon}{\partial t} &+ \operatorname{div}_{\mathbf{x}}(\varepsilon\mathbf{u}) + \operatorname{div}_{\mathbf{x}}(\mathbf{q}_K + \mathbf{u}\cdot\hat{\sigma}_K)\\ &= \frac{1}{2}\left[(\nabla_{\mathbf{x}}U\star\rho\mathbf{u}) - (\nabla_{\mathbf{x}}U\star\rho)\mathbf{u}\right]\rho + 2\alpha\,\varepsilon_K - \beta\int f|\mathbf{v}|^4\,d\mathbf{v}. \qquad (12)\end{aligned}$$

Finally, the last term can be expressed as

$$\int f|\mathbf{v}|^4\,d\mathbf{v} = |\mathbf{u}|^2\,(2\varepsilon_K + \delta_K) + 8\mathbf{u}\cdot\mathbf{q}_K + \tau_K + 4g_K,$$

where τ_K and g_K are defined as

$$\tau_K = \int f\,|\mathbf{v}-\mathbf{u}|^4\,d\mathbf{v} \quad \text{and} \quad g_K = \int f\,\left(\mathbf{u}\cdot(\mathbf{v}-\mathbf{u})\right)^2\,d\mathbf{v}.$$

This defines the evolution of the energy, where as usual the equations involve higher-order moments. In the limit of small fluctuations $\theta \simeq 0$, the above reduces to $2\varepsilon_K = \rho\,|\mathbf{u}|^2 + \delta_K \simeq \rho|\mathbf{u}|^2$, and the energy density balance equation with this closure assumption can be written as

$$\frac{\partial \varepsilon}{\partial t} + \operatorname{div}_{\mathbf{x}}(\varepsilon\,\mathbf{u}) = (\alpha-\beta|\mathbf{u}|^2)\rho|\mathbf{u}|^2 + \frac{1}{2}\left[(\nabla_{\mathbf{x}}U\star\rho\mathbf{u}) - (\nabla_{\mathbf{x}}U\star\rho)\mathbf{u}\right]\rho.$$

We remark that the above form can be derived directly from (9) and (10).

3.2. **Single-milling and flocking patterns: Monokinetic solutions.** We now try to find weak solutions to the kinetic equation (6) of the monokinetic form

$$f(\mathbf{x},\mathbf{v},t) = \rho(\mathbf{x},t)\,\delta(\mathbf{v}-\mathbf{u}(\mathbf{x},t)), \qquad (13)$$

where the constraints on ρ and $\mathbf{u}$ will be imposed by the weak formulation analysis. In this ansatz all fluctuating terms are strictly zero, since all microscopic velocities are identically set to $\mathbf{u}$, and we can find explicit weak solutions.

Proposition 3.2. *Let $\rho(\mathbf{x},t)$ be a smooth function such that $U\star\rho$ is globally defined and smooth, and let $\mathbf{u}(\mathbf{x},t)$ be a smooth vector field. Then $f(\mathbf{x},\mathbf{v},t)$ given by (13) is a distributional solution of the kinetic equation (6) if and only if $(\rho,\mathbf{u})$ is a solution of (11).*

Proof. Imposing that (13) is a distributional solution of (6), then for any $\psi \in C_0^\infty$, $\psi(\mathbf{x},\mathbf{v},t)$,

$$\int f\,(-\frac{\partial}{\partial t} - \mathbf{v}\cdot\nabla_{\mathbf{x}} - \mathbf{F}\cdot\nabla_{\mathbf{v}})\,\psi\,d\mathbf{x}\,d\mathbf{v}\,dt$$
$$= \int \rho\,\big[(-\frac{\partial}{\partial t} - \mathbf{v}\cdot\nabla_{\mathbf{x}} - \mathbf{F}\cdot\nabla_{\mathbf{v}})\,\psi\big]_{\mathbf{v}=\mathbf{u}(\mathbf{x},t)}\,d\mathbf{x}\,dt = 0,$$

where $\mathbf{F}(\mathbf{x},\mathbf{v},t) = (\alpha-\beta|\mathbf{v}|^2)|\mathbf{v}|^2 - \nabla_{\mathbf{x}}U\star\rho$. Taking ψ in the form $\psi=\varphi(\mathbf{x},t)$ and $\psi=\varphi(\mathbf{x},t)\,\mathbf{v}_i$, $i=1,\dots,d$, where $\varphi\in C_0^\infty$ we find that equations (11) hold in the sense of distributions, and hence, due to the smoothness of ρ and $\mathbf{u}$, in the classical sense.

Conversely, if $(\rho,\mathbf{u})$ is a solution of (11), take $\psi\in C_0^\infty$, $\psi=\psi(\mathbf{x},\mathbf{v},t)$ and define $\varphi(\mathbf{x},t)=\psi(\mathbf{x},\mathbf{u}(\mathbf{x},t),t)$. Then

$$\big[\frac{\partial\psi}{\partial t}\big]_{\mathbf{v}=\mathbf{u}(\mathbf{x},t)} = \frac{\partial\varphi}{\partial t} - \frac{\partial\mathbf{u}}{\partial t}\cdot[\nabla_{\mathbf{v}}\psi]_{\mathbf{v}=\mathbf{u}(\mathbf{x},t)},$$
$$\big[\nabla_{\mathbf{x}}\psi\big]_{\mathbf{v}=\mathbf{u}(\mathbf{x},t)} = \nabla_{\mathbf{x}}\varphi - D_{\mathbf{x}}\mathbf{u}\,[\nabla_{\mathbf{v}}\psi]_{\mathbf{v}=\mathbf{u}(\mathbf{x},t)},$$

where $D_{\mathbf{x}}\mathbf{u}$ is the Jacobi matrix of $\mathbf{u}$, and we obtain

$$\int \rho\,\big[(-\frac{\partial}{\partial t} - \mathbf{v}\cdot\nabla_{\mathbf{x}} - \mathbf{F}\cdot\nabla_{\mathbf{v}})\,\psi\big]_{\mathbf{v}=\mathbf{u}(\mathbf{x},t)}\,d\mathbf{x}\,dt$$
$$= \int \rho\,\big(-\frac{\partial\varphi}{\partial t} - \mathbf{u}\cdot\nabla_{\mathbf{x}}\varphi\big)\,d\mathbf{x}\,dt$$
$$+ \int \rho\,\big(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} - [\mathbf{F}]_{\mathbf{v}=\mathbf{u}(\mathbf{x},t)}\big)\cdot[\nabla_{\mathbf{v}}\psi]_{\mathbf{v}=\mathbf{u}(\mathbf{x},t)}\,d\mathbf{x}\,dt.$$

Noticing that both integrals on the right-hand side are zero due to the mass conservation and the momentum balance, we obtain that f is a solution of (6) in the sense of distributions. □

We can now consider several types of particular solutions at the fluid level which are of either stationary or traveling wave type, and which, in the light of the previous result, correspond to monokinetic solutions of the Vlasov equation (6). If we impose that $\beta|\mathbf{u}(\mathbf{x},t)|^2=\alpha$, so that the self-propelling and frictional terms balance each other, then in the stationary case $\mathbf{u}=\mathbf{u}(\mathbf{x})$, $\rho=\rho(\mathbf{x})$ the hydrodynamic equations (11) reduce to

$$\begin{cases} \nabla_{\mathbf{x}}\cdot(\rho\,\mathbf{u}) = 0, \\ \rho\,(\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} = -\rho\,(\nabla_{\mathbf{x}}U\star\rho), \\ \beta|\mathbf{u}(\mathbf{x})|^2 = \alpha, \end{cases} \tag{14}$$

and in the traveling wave case, $\mathbf{u}=\text{const}$, $\rho(\mathbf{x},t)=\tilde\rho(\mathbf{x}-\mathbf{u}t)$, we obtain simply

$$\begin{cases} \tilde\rho\,(\nabla_{\mathbf{x}}U\star\tilde\rho) = 0, \\ \beta|\mathbf{u}|^2 = \alpha. \end{cases} \tag{15}$$

Obviously, $\rho=\text{const}$, $\mathbf{u}=\text{const}$, with $|\mathbf{u}|^2=\frac{\alpha}{\beta}$ is a solution of either of these cases, which can be thought of as a flocking solution of infinite extent. Other traveling wave solutions with compact support may be obtained by solving the integral equation

$$U\star\tilde\rho = C, \quad \tilde\rho\neq 0,$$

where C is a constant of integration, as verified numerically in [22].

Similarly, if we set $\mathbf{u}$ in a rotatory state,

$$\mathbf{u} = \pm\sqrt{\frac{\alpha}{\beta}}\,\frac{\mathbf{x}^\perp}{|\mathbf{x}|},$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{x}^\perp = (-x_2, x_1)$, and look for $\rho = \rho(|\mathbf{x}|)$ radial, then we see that $\nabla_{\mathbf{x}} \cdot \mathbf{u} = 0$, $\mathbf{u} \cdot \nabla_{\mathbf{x}} \rho = 0$, which implies the continuity equation, and furthermore,

$$\nabla_{\mathbf{x}} u_1 = \pm\sqrt{\frac{\alpha}{\beta}}\Big(-\frac{1}{|\mathbf{x}|}\,\mathbf{j} + \frac{x_2}{|\mathbf{x}|^3}\,\mathbf{x}\Big), \quad \nabla_{\mathbf{x}} u_2 = \pm\sqrt{\frac{\alpha}{\beta}}\Big(\frac{1}{|\mathbf{x}|}\,\mathbf{i} - \frac{x_1}{|\mathbf{x}|^3}\,\mathbf{x}\Big),$$

so that

$$(\mathbf{u} \cdot \nabla_{\mathbf{x}})\,\mathbf{u} = -\frac{\alpha}{\beta}\,\frac{\mathbf{x}}{|\mathbf{x}|^{\mathbf{2}}}.$$

Thus, (14) implies

$$U \star \rho = D + \frac{\alpha}{\beta}\ln|\mathbf{x}|, \quad \text{whenever} \quad \rho \neq 0, \tag{16}$$

where D is a constant of integration, which gives a linear integral equation that can be solved for ρ. As observed in [22] this equation has, possibly multiple, solutions $\rho(r)$ with compact support in $(0, \infty)$. Such solutions represent circular swarms in which all particles move with the same linear speed $\sqrt{\alpha/\beta}$. A special family of singular solutions to (16) are given by $\rho(r) = c\,\delta(r - r_0)$. They may be viewed as a swarm in which the thickness of the ring has collapsed to zero. Such behavior is indeed observed in the system for N large in the "catastrophic" regime, cf. [9]. Solutions with support filling an interval $[R_0, R_1]$ with $0 < R_0 < R_1$ were found numerically in Ref. [22] and matched to single mill patterns in Ref. [9].

Let us point out that solutions to (14) have to satisfy that the velocity field $\mathbf{u}$ is orthogonal to the force in the momentum balance equation since the speed is constant. Milling solutions do satisfy this constraint, since the convolution of the radial potential $U(\mathbf{x})$ with a radial density ρ is a radial function. A more complete description of the possible steady states of this type can be achieved through a spectral analysis of the resulting integral equations and will be presented elsewhere. The stability of these particular steady solutions of the kinetic equation (6) or the hydrodynamic system (11) is certainly a challenging problem.

3.3. **Double milling patterns: Hydrodynamic superpositions at kinetic level.** In the previous section, we showed that particular solutions to the kinetic equation (6) are found by imposing the monokinetic ansatz. Here, we look for the conditions that must be met for a linear superposition of such monokinetic distributions to be a distributional solution of Eq. (6). For concreteness, we consider the case of two populations with densities ρ_1 and ρ_2 and with velocities $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively, so that

$$f = \rho_1(\mathbf{x}, t)\delta(\mathbf{v} - \mathbf{u}_1(\mathbf{x}, t)) + \rho_2(\mathbf{x}, t)\delta(\mathbf{v} - \mathbf{u}_2(\mathbf{x}, t)). \tag{17}$$

With this definition we find

$$\rho = \rho_1 + \rho_2, \tag{18}$$

$$\rho\mathbf{u} = \rho_1\mathbf{u}_1 + \rho_2\mathbf{u}_2. \tag{19}$$

We thus insert (17) into the weak form of the kinetic equation (6) with the test function $\psi \in C_0^\infty$, $\psi(x,v,t)$, to obtain

$$\begin{aligned}
&\int f\,(-\frac{\partial}{\partial t} - \mathbf{v}\cdot\nabla_{\mathbf{x}} - \mathbf{F}\cdot\nabla_{\mathbf{v}})\,\psi\,d\mathbf{x}\,d\mathbf{v}\,dt \\
&= \sum_{i=1}^{2}\int \rho_i\left[(-\frac{\partial}{\partial t} - \mathbf{v}\cdot\nabla_{\mathbf{x}} - \mathbf{F}\cdot\nabla_{\mathbf{v}})\,\psi\right]_{\mathbf{v}=\mathbf{u}_i(\mathbf{x},t)} d\mathbf{x}\,dt \\
&= \sum_{i=1}^{2}\int \rho_i\left(\frac{\partial \mathbf{u}_i}{\partial t} + (\mathbf{u}_i\cdot\nabla_{\mathbf{x}})\mathbf{u}_i - [\mathbf{F}]_{\mathbf{v}=\mathbf{u}_i(\mathbf{x},t)}\right)\cdot[\nabla_{\mathbf{v}}\psi]_{\mathbf{v}=\mathbf{u}_i(\mathbf{x},t)}\,d\mathbf{x}\,dt \\
&+ \sum_{i=1}^{2}\int \left(\frac{\partial \rho_i}{\partial t} + \mathrm{div}_{\mathbf{x}}(\rho_i\mathbf{u}_i)\right)[\psi]_{\mathbf{v}=\mathbf{u}_i(\mathbf{x},t)}\,d\mathbf{x}\,dt = 0. \qquad (20)
\end{aligned}$$

where as above, $\mathbf{F}(\mathbf{x},\mathbf{v},t) = (\alpha - \beta|\mathbf{v}|^2)\mathbf{v} - \nabla_{\mathbf{x}}U \star \rho$. As in the proof of Proposition 3.2, choosing ψ to be a function of t and $\mathbf{x}$ only leads to the continuity equation

$$\frac{\partial(\rho_1+\rho_2)}{\partial t} + \mathrm{div}_{\mathbf{x}}(\rho_1\mathbf{u}_1 + \rho_2\mathbf{u}_2) = 0. \qquad (21)$$

We can also choose $\psi = \varphi(\mathbf{x},t)\,\mathbf{v}_j$, $j = 1,\dots,d$, so that (20) gives

$$\sum_{i=1}^{2}\rho_i\left[\frac{\partial \mathbf{u}_i}{\partial t} + (\mathbf{u}_i\cdot\nabla_{\mathbf{x}})\mathbf{u}_i - (\alpha - \beta|\mathbf{u}_i|^2)\mathbf{u}_i\right] = -(\nabla_{\mathbf{x}}U\star\rho)\,\rho. \qquad (22)$$

Conversely, if (21) and (22) hold, then it follows from (20) that the two-mode superposition (17) is a solution of (6) in the sense of distributions. Summarizing, we obtain the following

Proposition 3.3. *Given $\rho_i(\mathbf{x},t)$ and $\mathbf{u}_i(\mathbf{x},t)$, $i = 1,2$, smooth functions, such that $U\star\rho_i$, $i=1,2$ are globally defined and smooth. Then f given by (17) is a distributional solution of the kinetic equation (6) if and only if $(\rho_i,\mathbf{u}_i)$ satisfy (21) and (22).*

We remark that the above result may be extended in a straightforward way to the case of arbitrary number of monokinetic modes using the ansatz

$$f(\mathbf{x},\mathbf{v},t) = \sum_{i=1}^{n}\rho_i(\mathbf{x},t)\,\delta(\mathbf{v}-\mathbf{u}_i(\mathbf{x},t)).$$

Based on the established result, we can look for steady state solutions satisfying $\beta|\mathbf{u}_i(\mathbf{x})|^2 = \alpha$, $i = 1,2$ which leads to

$$\begin{cases} \nabla_{\mathbf{x}}\cdot(\rho_1\mathbf{u}_1 + \rho_2\mathbf{u}_2) = 0, \\ \displaystyle\sum_{i=1}^{2}\rho_i\,(\mathbf{u}_i\cdot\nabla_{\mathbf{x}})\mathbf{u}_i = -\rho(\nabla_{\mathbf{x}}U\star\rho). \end{cases} \qquad (23)$$

Due to the nonlinear coupling term on the right-hand side, a simple linear superposition of monokinetic solutions is generally no longer a solution of (6). We can nonetheless combine monokinetic solutions in a slightly different way, by assuming $2\rho_1 = 2\rho_2 = \rho$ and $\mathbf{u}_1 = -\mathbf{u}_2 = \mathbf{u}$. In that case the second of the equations (23) becomes

$$\rho\,(\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} = -\rho\,(\nabla_{\mathbf{x}}U\star\rho),$$

and the system (23) then reduces to (14).

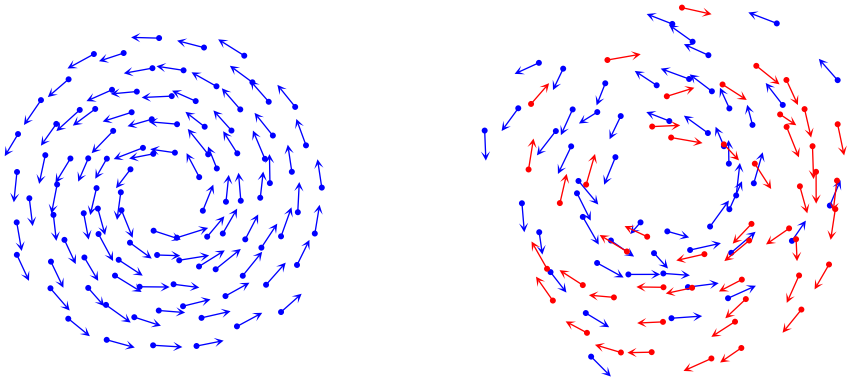

FIGURE 3.2. Single and double milling solutions from numerical simulations of (2) in the unnormalized case where $M = N$ and parameters are chosen as $N = 100$, $C_a = 0.5$ $C_r = 1.0$, $\ell_a = 3$, $\ell_r = 0.5$, $\alpha = 1.0$, $\beta = 0.5$. Random initial conditions were chosen for particles in the left panel, whereas $N/2$ particles were initiated rotating clockwise and $N/2$ counterclockwise in the right panel. General random initial conditions typically yield single rotational structures.

Summarizing, we conclude that the combination of two delta functions,

$$f(\mathbf{x},\mathbf{v}) = \tfrac{1}{2}\,\rho(\mathbf{x})\,\delta(\mathbf{v}-\mathbf{u}(\mathbf{x})) + \tfrac{1}{2}\,\rho(\mathbf{x})\,\delta(\mathbf{v}+\mathbf{u}(\mathbf{x}))$$

with $\beta|\mathbf{u}(\mathbf{x})|^2 = \alpha$ is a stationary solution to the kinetic equation (6) whenever $\rho(\mathbf{x})\,\delta(\mathbf{v}-\mathbf{u}(\mathbf{x}))$ is a solution. Thus, double milling solutions, where half the particles travel at the speed $\mathbf{u}$ and the rest at its opposite $-\mathbf{u}$, exist provided equations (14) are satisfied, and any single mill configuration thereby produces the corresponding double mill solution. The question of stability of the single and double-mill solutions depending on the parameters of the model then arises naturally, but its discussion is beyond the scope of the present work.

Note that in the case of double mills the average macroscopic velocity is zero and therefore, such solutions cannot be explained by a hydrodynamic model with a single macroscopic velocity. The kinetic theory approach provides a natural framework in which this type of solutions may be studied.

4. **Kinetic model for interacting particles with random noise.** Finally, let us study the case of the interacting particle system with random noise and with a linear Stokes friction term in addition to the self-propulsion and the Rayleigh type friction,

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \\ d\mathbf{v}_i = \left[(\alpha - \beta\,|\mathbf{v}_i|^2)\mathbf{v}_i - \dfrac{M}{N}\nabla_{\mathbf{x}_i}\displaystyle\sum_{j\neq i} U(|\mathbf{x}_i-\mathbf{x}_j|)\right]dt + \sqrt{2\sigma}\,d\Gamma_i(t), \end{cases}$$

where $\Gamma_i(t)$ are N independent copies of standard Wiener processes with values in $\mathbb{R}^d$ and $\sigma > 0$ is the noise strength. Here, $\alpha \in \mathbb{R}$ is the effective friction constant

coming from $\alpha = \alpha_1 - \alpha_0$ with $\alpha_0, \alpha_1 > 0$, and α_0 is the Stokes friction component and α_1 is the self-propulsion generated by the organisms.

Using Ito's formula to obtain a Fokker-Planck equation for the N-particle distribution and following the same procedure as in section 2, it is easy to derive the following kinetic Fokker-Planck equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \mathrm{div}_{\mathbf{v}}\big[(\alpha - \beta|\mathbf{v}|^2)\mathbf{v}\, f\big] - \mathrm{div}_{\mathbf{v}}\left[(\nabla_{\mathbf{x}} U \star \rho) f\right] = \sigma\Delta_{\mathbf{v}} f. \tag{24}$$

In order to derive some asymptotic limit equations starting from the kinetic equation (24), we introduce a dimensionless formulation of the equation. Let us rewrite the kinetic equation as

$$\begin{aligned}\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f \; &+ \mathrm{div}_{\mathbf{v}}[(\alpha_1 - \beta|\mathbf{v}|^2)\mathbf{v}\, f] \\ &- \mathrm{div}_{\mathbf{v}}\left[(\nabla_{\mathbf{x}} U \star \rho) f\right] = \mathrm{div}_{\mathbf{v}}[\alpha_0 \mathbf{v} f + \sigma\nabla_{\mathbf{v}} f].\end{aligned} \tag{25}$$

Let us first remark that $\mathcal{T}_{FP} = \frac{1}{\alpha_0}$ is the natural relaxation time for the Fokker-Planck operator $\mathrm{div}_{\mathbf{v}}[\alpha_0\mathbf{v}f + \sigma\nabla_{\mathbf{v}} f]$. As usual, let us consider $\mathcal{V}_{th} = \sqrt{\sigma/\alpha_0}$, the typical value of fluctuations in particle velocity, called the thermal speed. Let us introduce the time and length units T and $\mathrm{L} = \ell_a$ which are determined by the units of observation, and the typical speed $\mathrm{U} = \mathrm{L}/\mathrm{T}$. Then we can define dimensionless variables, denoted by primed quantities, as

$$t = \mathrm{T}\, t', \qquad \mathbf{x} = \mathrm{L}\,\mathbf{x}', \qquad \mathbf{v} = \mathcal{V}_{th}\mathbf{v}',$$

$$f'(t', \mathbf{x}', \mathbf{v}') = \mathrm{L}^d\ \mathcal{V}_{th}^d\ f(\mathrm{T}t', \mathrm{L}\mathbf{x}', \mathcal{V}_{th}\mathbf{v}'), \qquad \text{and} \qquad U'(\mathbf{x}') = C_a\ U(\mathrm{L}\mathbf{x}').$$

With this change of scales, we finally find the following dimensionless kinetic equation

$$\begin{aligned}\frac{\partial f}{\partial t} + \eta\mathbf{v}\cdot\nabla_{\mathbf{x}} f + \gamma_1 \mathrm{div}_{\mathbf{v}}[\mathbf{v}\, f] \; &- \gamma_2 \mathrm{div}_{\mathbf{v}}[|\mathbf{v}|^2\mathbf{v}\, f] \\ &- \chi\mathrm{div}_{\mathbf{v}}\left[(\nabla_{\mathbf{x}} U \star \rho) f\right] = \frac{1}{\epsilon} L f.\end{aligned} \tag{26}$$

where primes have been eliminated for notational simplicity and the operator L is defined as $Lf \equiv \mathrm{div}_{\mathbf{v}}[\mathbf{v}f + \nabla_{\mathbf{v}} f]$. Here, η, γ_1, γ_2 and χ are dimensionless parameters given by

$$\eta = \frac{\mathcal{V}_{th}}{U}, \qquad \gamma_1 = T\alpha_1, \qquad \gamma_2 = T\beta\mathcal{V}_{th}^2,$$

$$\chi = \frac{C_a}{U\mathcal{V}_{th}} \qquad \text{and} \qquad \epsilon = \frac{\mathcal{T}_{FP}}{T} = \frac{1}{T\alpha_0}.$$

With this dimensionless formulation, the potential becomes $U(r) = -e^{-r} + Ce^{-r/\ell}$ and we can find two different regimes in which the kinetic equation may be approximated by distinct macroscopic equations.

4.1. **Weak-interaction/Strong-noise regime.** We choose the following relation between the dimensionless parameters: $\eta \simeq \gamma_1 \simeq \gamma_2 \simeq \chi \simeq \epsilon^{-1/2}$. In this regime, the dominant mechanisms are the noise and the linear Stokes friction. In order to obtain macroscopic equations, we use the standard Hilbert expansion method. Inserting the following Hilbert expansion

$$f_\epsilon = f^{(0)} + \sqrt{\epsilon} f^{(1)} + \epsilon f^{(2)} + \ldots \qquad \text{and} \qquad \rho_\epsilon = \rho + \sqrt{\epsilon}\rho^{(1)} + \ldots \tag{27}$$

into (26) and identifying terms with equal power of $\sqrt{\epsilon}$, we get:

- ϵ^{-1} terms: $Lf^{(0)} = 0$ which implies that $f^{(0)}(t,\mathbf{x},\mathbf{v}) = \rho(t,\mathbf{x})\ M(\mathbf{v})$, where $M(\mathbf{v})$ is the Maxwellian distribution with unit temperature.
- $\epsilon^{-1/2}$ terms:

$$\begin{aligned} Lf^{(1)} &= \mathbf{v}\cdot\nabla_{\mathbf{x}} f^{(0)} + \mathrm{div}_{\mathbf{v}}[(1-|\mathbf{v}|^2)\mathbf{v}\, f^{(0)}] - \mathrm{div}_{\mathbf{v}}\left[(\nabla_{\mathbf{x}} U \star \rho) f^{(0)}\right] \\ &= \mathbf{v}\cdot\left[\nabla_{\mathbf{x}}\rho + (\nabla_{\mathbf{x}} U\star\rho)\rho\right]\ M(\mathbf{v}) + \rho\left[d-(d+3)|\mathbf{v}|^2+|\mathbf{v}|^4\right] M(\mathbf{v}). \end{aligned}$$

  This equation can be readily inverted using that $L(\mathbf{v}M(\mathbf{v})) = -\mathbf{v}M(\mathbf{v})$ and

$$L\left[(\frac{1}{2}|\mathbf{v}|^2 - \frac{1}{4}|\mathbf{v}|^4)M(\mathbf{v})\right] = \left[d-(d+3)|\mathbf{v}|^2+|\mathbf{v}|^4\right] M(\mathbf{v})$$

  to obtain

$$f^{(1)}(t,x,\mathbf{v}) = -\mathbf{v}\cdot\left[\nabla_{\mathbf{x}}\rho + (\nabla_{\mathbf{x}} U\star\rho)\rho\right]\ M(\mathbf{v}) + \rho\left[\frac{1}{2}|\mathbf{v}|^2 - \frac{1}{4}|\mathbf{v}|^4\right] M(\mathbf{v})$$

- ϵ^0 terms:

$$\begin{aligned} Lf^{(2)} =& \partial_t f^{(1)} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f^{(1)} + \mathrm{div}_{\mathbf{v}}[(1-|\mathbf{v}|^2)\mathbf{v}\, f^{(1)}] \\ &- \mathrm{div}_{\mathbf{v}}\left[(\nabla_{\mathbf{x}} U\star\rho) f^{(1)}\right] - \mathrm{div}_{\mathbf{v}}\left[(\nabla_{\mathbf{x}} U\star\rho^{(1)}) f^{(0)}\right], \end{aligned}$$

  with $\rho^{(1)} = \int f^{(1)}\, d\mathbf{v}$. However, since $\int h\, d\xi = 0$ is a necessary condition for the equation $L(f) = h$ to admit a solution, we conclude

$$\begin{aligned} \partial_t\Big(\int_{\mathbb{R}^d} f^{(0)}\, d\mathbf{v}\Big) + \mathrm{div}_{\mathbf{x}}\Big(\int_{\mathbb{R}^d} \mathbf{v} f^{(1)}\, d\mathbf{v}\Big) \\ = \partial_t\rho - \nabla_{\mathbf{x}}\cdot\big(\nabla_{\mathbf{x}}\rho + (\nabla_{\mathbf{x}} U\star\rho)\rho\big) = 0. \end{aligned}$$

Therefore, in the $\epsilon\to 0$ limit regime we expect the macroscopic density to be well approximated by the solution to the equation

$$\partial_t\rho = \nabla_{\mathbf{x}}\cdot\big((\nabla_{\mathbf{x}} U\star\rho)\rho\big) + \Delta_{\mathbf{x}}\rho. \tag{28}$$

4.2. **Strong-interaction/Strong-noise regime.** We choose the following relation between the dimensionless parameters: $\eta \simeq \gamma_1 \simeq \gamma_2 \simeq 1$ and $\chi \simeq \epsilon^{-1}$. In this regime, the dominant mechanisms are the noise, the linear Stokes friction and the interaction term. We use again the Hilbert expansion

$$f_\epsilon = f^{(0)} + \epsilon f^{(1)} + \dots \qquad \text{and} \qquad \rho_\epsilon = \rho + \epsilon\rho^{(1)} + \dots$$

in (26) and collect terms with equal powers of ϵ:

- ϵ^{-1} terms:

$$Lf^{(0)} = -\mathrm{div}_{\mathbf{v}}\left[(\nabla_{\mathbf{x}} U\star\rho) f^{(0)}\right]$$

  which implies that $f^{(0)}(t,\mathbf{x},\mathbf{v}) = \rho(t,\mathbf{x})\ M(\mathbf{v} - (\nabla_{\mathbf{x}} U\star\rho))$.
- ϵ^0 terms:

$$\begin{aligned} &L(f^{(1)}) + \mathrm{div}_{\mathbf{v}}\left[(\nabla_{\mathbf{x}} U\star\rho) f^{(1)}\right] \\ &= \partial_t f^{(0)} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f^{(0)} + \mathrm{div}_{\mathbf{v}}[(1-|\mathbf{v}|^2)\mathbf{v}\, f^{(0)}], \end{aligned}$$

  and integration with respect to $\mathbf{v}$ yields the mass conservation

$$\partial_t\rho = \nabla_{\mathbf{x}}\cdot\big((\nabla_{\mathbf{x}} U\star\rho)\rho\big). \tag{29}$$

Let us point out again that both equations (28) and (29) were proposed in Refs. [33, 34] as continuum models for swarming and are here recovered through the presented kinetic theory.

5. **Conclusions.** We have developed a kinetic theory approach aimed at describing self-propelling swarming systems driven by general pairwise interactions. Our analysis leads to a new class of macroscopic solutions to swarming systems consisting of double, superimposed rotating mills. These are indeed observed in discrete simulations but they cannot be identified from hydrodynamic equations with a single macroscopic velocity since the inherent dual velocity distributions yield a macroscopic average of zero. We find the conditions under which double mills can coexist. Due to the nonlinearity of the problem these solutions are not trivial since in general the superposition of two existing solutions does not necessarily satisfy the governing equations of motion. We have also extended the kinetic theory to the interacting particle system with random noise effects. The resulting kinetic Fokker-Planck equations yield, under suitable scalings, macroscopic equations for the density of organisms already proposed in the literature. We have demonstrated that the kinetic theory approach leads to a unified hierarchy of swarming models bridging the particle description to all the hydrodynamic and continuum descriptions available in the literature. Future development includes a full numerical solution of the kinetic equations, both in one and two dimensions.

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E-mail address: carrillo@mat.uab.es
E-mail address: dorsogna@csun.edu
E-mail address: vladislav.panferov@csun.edu