Technische Universität München
Fakultät für Informatik
Andreas Langbein, Adnane Jadid, David Plecher

Summer Term 2018
Example solutions for exercise 2
May 2rd, 2018

Introduction to Augmented Reality

Exercise 4 Finding Rectangles (example solution)

See file `Exercise-2.cpp` on Moodle.

Exercise 5 Rotation matrices (example solution)

(a) A simple calculation gives the point:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R_\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

As rotations in two-dimensional space always have the same axis (the origin), a combination of rotations results in:

$$R_\alpha R_\beta = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \cos\beta\sin\alpha \\ \cos\beta\sin\alpha + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix}$$

According to the rules for trigonometric functions, this results in: Fixme

$$R_\alpha R_\beta = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} = R_{\alpha+\beta} = R_\beta R_\alpha$$

As addition is commutative, the conjecture follows. However, even in two-dimensional space, this is not true anymore if rotations and translations are combined.

(b) This is a rotation by $-90°$ around the z axis. The rotation matrix is therefore:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$$

(c) Every matrix results in a linear mapping of space. However, as rotation matrices are orthogonal, such a transformation can't be a simple rotation. The only linear mappings which are possible while still keeping an orthogonal coordinate system are scaling operations along a single axis:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ \lambda \cdot z \end{pmatrix}$$

If orthogonality isn't required, shearings are also possible:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ a \cdot x + b \cdot y + c \cdot z \end{pmatrix}$$

(d) Proof by contradiction:

$$R(\omega,0,0)\cdot R(0,\phi,0)\cdot R(0,0,\kappa) = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ \sin\omega\sin\phi & \cos\omega & \sin\omega\cos\phi \\ \cos\omega\sin\phi & -\sin\omega & \cos\omega\cos\phi \end{pmatrix}\cdot R(0,0,\kappa)$$

$$= \begin{pmatrix} \cos\phi\cos\kappa & \cos\phi\sin\kappa & -\sin\phi \\ \sin\omega\sin\phi\cos\kappa-\cos\omega\sin\kappa & \sin\omega\sin\phi\sin\kappa+\cos\omega\cos\kappa & \sin\omega\cos\phi \\ \cos\omega\sin\phi\cos\kappa+\sin\omega\sin\kappa & \cos\omega\sin\phi\sin\kappa-\sin\omega\cos\kappa & \cos\omega\cos\phi \end{pmatrix}$$

$$R(0,0,\kappa)\cdot R(0,\phi,0)\cdot R(\omega,0,0) = \begin{pmatrix} \cos\phi\cos\kappa & \sin\kappa & -\sin\phi\cos\kappa \\ -\cos\phi\sin\kappa & \cos\kappa & \sin\phi\sin\kappa \\ \sin\phi & 0 & \cos\phi \end{pmatrix}\cdot R(\omega,0,0)$$

$$= \begin{pmatrix} \cos\phi\cos\kappa & \cos\omega\sin\kappa+\sin\omega\sin\phi\cos\kappa & \sin\omega\sin\kappa-\cos\omega\sin\phi\cos\kappa \\ -\cos\phi\sin\kappa & \cos\omega\cos\kappa-\sin\omega\sin\phi\sin\kappa & \sin\omega\cos\kappa+\cos\omega\sin\phi\sin\kappa \\ \sin\phi & -\sin\omega\cos\phi & \cos\omega\cos\phi \end{pmatrix}$$

The order of rotations is not arbitrary because the first rotation already influences the axis of the second rotation.

Exercise 6 Homogeneous matrices (example solution)

(a) This transformation can be composed from the rotation from exercise 1 and an additional translation, resulting in

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) The last row generates the fourth value w of the resulting homogeneous vector. If this value is not 1 (guaranteed when the last row is $(0,0,0,1)$), the whole vector is scaled by a factor of $\frac{1}{w}$. Therefore, if the last row contains (a,b,c,d), the result is a scaling of the vector by $1/(ax+by+cz+dw)$. This can be used, for example, for perspective transformations (as opposed to affine transformations, which require the last row to be $(0,0,0,1)$).

Note: In order to scale the vector by constant factors f_x, f_y, f_z along the coordinate axes, a matrix as follows can be used:

$$\begin{pmatrix} f_x & 0 & 0 & 0 \\ 0 & f_y & 0 & 0 \\ 0 & 0 & f_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$