Definition A *bridge* in a graph G is an edge e such that $G-e$ has more components than G.

Lemma Let G be a connected graph. Let $e = xy$ be a bridge of G.

Then $G-e$ has exactly two components C_x containing x and C_y containing y.

Proof Since e is a bridge, by definition $C-e$ has at least two components.

Let C_x be the component of $G-e$ containing x.

Let z be a vertex of G that is not in C_x.

Since G was connected there was a path P in G from x to z. Therefore $e = xy$ is an edge of P, say $P = xyy_1y_2y_3...y_kz$. Then $y_1y_2y_3...y_kz$ is a path from y to z in $G-e$, hence z is in the component C_y of $G-e$ containing y. Hence C_y is the only other component of $G-e$. □

Theorem Let g be a graph and let e be an edge of G. Then e is a bridge of G if and only if e is not an edge of any cycle in G.

Proof We'll prove that e is in a cycle of G if and only if it is not a bridge.

Note: that if G is not connected we can just consider the component of G that contains e. So we may assume that G is connected.

- ($\implies$) Suppose $e = xy$ is in a cycle C of G, say $xy_1y_2...y_k$ is a cycle in G. Then in the graph $G-e$, we have the path $yy_1y_2...y_kx$ (since it was a cycle) hence x and y are in the same component of $G-e$.

  However, by earlier lemma if e was a bridge we would have two components, namely one containing x the other containing y. Therefore e is not a bridge (since $C_x = C_y$).

- ($\impliedby$) Conversely, suppose e is not a bridge. Then by definition $G-e$ is connected, hence there exists a path $P = xy_1y_2...y_ky$ from x to y in $G-e$.

  Then P together with $e = xy$ forms a cycle containing e in G. □

Corollary Suppose G is a connected graph with no cycles. Then for every pair of vertices x and y in G, there is a unique path from x to y.

Proof Fix x and y. Since G is connected there exists a path P_1 from x to y. Suppose on the contrary there exists a path $P_2 \neq P_1$, from x to y in G.

Since $P_1 \neq P_2$ there exists som edge that in P_1 and not in P_2.

Let $P_1 = xy_1 ... y_k y$ and suppose $y_i y_{i+1}$ is not an edge of P_2.

We claim that $e = y_i y_{i+1}$ is not a bridge of G, that is, it lies in a cycle of G.

But $y_i y_{i-1} ... y_1 x P_2 y y_k y_{k-1} ... y_{i+1}$ is a walk from y_i to y_{i+1} in $G - e$. Hence by an earlier lemma ther is a path from y_i to y_{i+1} in $G - e$.

Hence e is not a bridge i.e. it is in a cycle of G.

This contradiction shows that P_1 is unique $\square$

Trees

Definition a *tree* is a connected graph with no cycles.

- So we just showed any pair of vertices in a tree are joined by a *unique* path.
- Every edge in a tree is a bridge
- There are $n - 1$ edges for a tree of n vertices (proof for next time)