Matchings Recap

A matching M saturates the vertices that are in the matching. A matching of maximum size is a *maximum matching* and a matching where every vertex is saturated by it is a *perfect matching*.

Note: a graph needs to have an even number of vertices for a perfect matching to exist.

Definition Let M be a matching in G. An *alternating path* with respect to M is a path in G which has every second edge in M

Definition An *augmenting path* is an alternating path that begins and ends with an edge not in M.

Note that switching along an augmenting path makes the matching larger. For augmenting path P and matching M, "switching" means $\{E(M) \setminus \{E(M) \cap E(P)\}\} \cup \{E(P) \setminus E(M)\}$.

If our augmenting path starts and ends at unsaturated vertices, this switching produces a new, larger matching.

Note that augmenting paths always have odd length.

Theorem Let G be a graph and M a matching in G. If G has an augmenting path with respect to M, then M is not a maximum matching.

Proof Suppose G has an augmenting path P. Then P contains more edges not in M than in M.

Let M' be the set removing the edges in $E(M) \cap E(P)$ from M and adding the edges of $E(P) \setminus E(M)$ to M.

Since the endpoints of P are unsaturated, no vertex in $V(M')$ is incident to more than one edge in M'. Hence M' is a matching

Since $|E(P) \setminus E(M)| > |E(M) \cap E(P)|$, $|M'| > |M|$. Thus M is not a maximum matching $\square$

Definition A *cover* in a graph G is a set of vertices C such that every edge in G is incident to at least one vertex in C.

A cover in a bipartite graph is either the characteristic set A or B.

Lemma Let G be agraph, M a matching and C a cover. Then

$$|M| \leq |C|$$

Proof Every edge in M is incident to at least one vertex in C.

Also, no two edges are incident to the same vertex in C. Thus $|M| \leq |C|$, $\square$.

Lemma Let G be a graph, M a matching, and C a cover. If $|M| = |C|$ then M is a maximum matching and C is a minimum cover.

Proof Let M' be a maximum matching. Then $|M'| \leq |C|$ by previous lemma. As $|C| = |M|$, $|M'| \leq |C| = |M|$, so M is a maximum matching.

Let C' be a minimum cover.Then $|C'| \geq |M|$ by previous lemma. Since $|M| = |C|$, $|C'| \geq |M| = |C|$, so C is a minimum cover. $\square$.

However, not there are graphs with $|C| > |M|$ - more common when a perfect matching doesn't exist.

Setting up definitions for next class
For a bipartite graph (A, B).

Let $X_0 = \{v \in A :$ v is not saturated by M$\}$.

Let $X = \{v \in A :$ there is an alternating path from v to a vertex in $X_0\}$

Let $Y = \{v \in B :$ there is an alternating path from v to a vertex in $X_0\}$