**Theorem** Every tree with $p \geq 1$ vertices has exactly $p - 1$ edges.

**Proof** By induction on $p$.

**Base Case** $p = 1$: The only tree (in fact graph) on 1 vertex is a single vertex.... which has no edges i.e. $p - 1 = 0$.

**Inductive Hypothesis** Assume $p \geq 2$ and that every tree with $1 \leq p' < p$ vertices has exactly $p' - 1$ edges.

Let $T$ be a tree with $p$ vertices. Since $p \geq 2$ and $T$ is connected, it has some edge $e = xy$.

Then since $T$ has no cycles, the edge $e$ is a bridge of $T$ then by our earlier lemma $T - e$ has exactly two components. $T_x$ continas $x$ and $T_y$ contains $y$.

Since $T_x$ and $T_y$ are components, both are connencted. Neither $T_x$ nor $T_y$ contains a cycle, since other wise $T$ would have a cycle.

Let $p_x$ be the number of vertices of $T_x$ and $p_y$ be the number of vertices of $T_y$.

Then $1 \leq p_x < p$ since $x \in V(T_x)$ and $y \notin V(T_y)$.

Similarly, $1 \leq p_y < p$.

Hence by Inductive Hypothesis, we find that $T_x$ has exactly $p_x - 1$ edges and $T_y$ has exactly $p_y - 1$ edges.

So $|E(T)| = (p_x - 1) + (p_y - 1) + 1 = p_x + p_y - 1 = p - 1$

Therefore by induction the theorem is proved $\square$ (☺)

$\star\star\star$ ***NOTE*** $\star\star\star$ When using induction in graph theory, in the induction step you must consider a **general** graph (with the given properties) and do something to obtain a **smaller** graph in which you apply the inductive hypothesis!

**Definition** A vertex $x$ of degree 1 in a tree $T$ is called a *leaf*.

Then every tree with at least two vertices has a leaf. Since:

- $\sum_{v \in T} deg(v) = 2|E(T)|$
- we proved that every vertex has degree $\geq 2$ then the graph contains a cycle!

**Theorem** Suppose $p \geq 2$. Let $T$ be a tree with $p$ vertices, and let $n_i$ denote the number of vertices of $T$ of degree $i$ for $1 \leq i \leq p$. Then the number of the leaves of $T$ satisfies:

$$n_1 = 2 + n_3 + 2n_4 + ... + (p-2)n_p$$

**Proof** We know that $|E(T)| = \frac{1}{2}\sum_{v \in V(T)} deg(v) = p - 1$ from the earlier previous lemma and the Handshake (degree-sum) lemma.

But by grouping the vertices of $T$ into groups according to their degree we find:

$$\sum_{v \in V(T)} deg(v) = |n_1 + 2n_2 + 3n_3 + ... + pn_p|$$

But $n_1 + ... + n_p = p$ (1)
So $2p = 2n_1 + 2n_2 + 2n_3 + ...2n_p$ (2) and $2p - 2 = n_1 + 2n_2 + 3n_3 + ... + pn_p$ (3)

Then (2) - (3) $\implies 2 = n_1 - n_3 - 2n_4 ... - (p-2)n_p$ and rearranging we get:

$n_1 = 2 + n_3 + 2n_4 + ... + (p-2)n_p$ as required □

**Example** What is th esmallest possible number of leaves in a tree with 4 vertices of degree 3, 2 vertices of degree 4, and 2 vertices of degree 5?

By formula:

$$n_1 = 2 + 4 + 2(2) + ... \geq 16$$

**Theorem** Every tree is bipartite.

**Proof** By induction, we prove that every tree with $p$ vertices is bipartite.

**Base case** $p = 1$. This is bipartite. Label the one vertex to be of set $A$ and we're done.

**Inductive Hypothesis** Assume $p \geq 2$ and every tree with $p-1$ vertices is bipartite.

Let $T$ be a tree on $p$ vertices. Let $x$ be a leaf of $T$ (by our lemma, such $x$ exists). Then the graph $T - x$ obtained by removing $x$ is connected, since each vertex is joined by a path to $y$ where $xy$ is the unique edge incident to $x$. So $T - x$ is a tree, which by Inductive Hypothesis is bipartite with vertex classes $A$ and $B$

Say without loss of generality $y \in A$ then put $x \in B$ to show $T$ is bipartite □.