Theorem (5-colour theorem) Every planar graph is 5-colourable.

Proof The proof is by induction on the number of vertices p.

Base case For any planar graph on 1 vertex can be 5 coloured.

Induction Hypothesis Assume for all planar graphs on $p \leq k$ vertices is 5-colourable for $k \geq 1$.

Inductive Step

1. There is a vertex of degree at most 4. Then we can remove it - colour the graph - and place back the vertex with a colour not used by any neighbour. We obtain a 5-colouring of G (Exactly like the six colour theorem).

2. No vertex of G has degree at most 4 (i.e. all vertices have degrees higher than 4).

   By Corollary 7.5.5, G has a vertex v, of degree 5.

   Note that v has at least two non-adjacent neighbours a and b as otherwise G contains a subdivision of K_5, contradiction that G is planar, like in Figure 1.

   Contract the edge $\{a, v\}$ and call the new vertex v. Then contract the edge $\{v, b\}$ and call the new vertex v.

   (Note that contracting an edge on a planar graph must preserve planarity)

   This process produces a new graph H that contains $k - 1$ vertices. By our *Inductive Hypothesis* H is 5-colourable.

   Take such a 5-colouring and apply it to G, but not colouring a, b, v.

   We assign the vertices a and b the same colour that was assigned to v in the colouring of H. Note that this preserves the 5-colouring - since the colour of v was different from all its neighbours.

   Since a and b have the colour, at most 4 colours appear on the neighbours of v, and we can assign v the missing colour, completing the 5-colouring of G.

Therefore in all cases G is 5-colourable.
Therefore, by induction all planar graphs are 5 colourable □

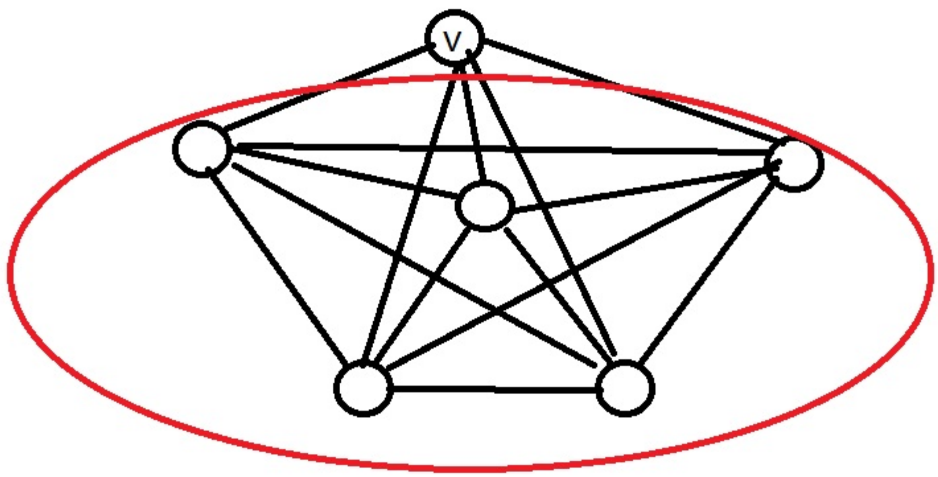

Figure 1: A graph with all vertices of 5 or more has subdivision of K_5

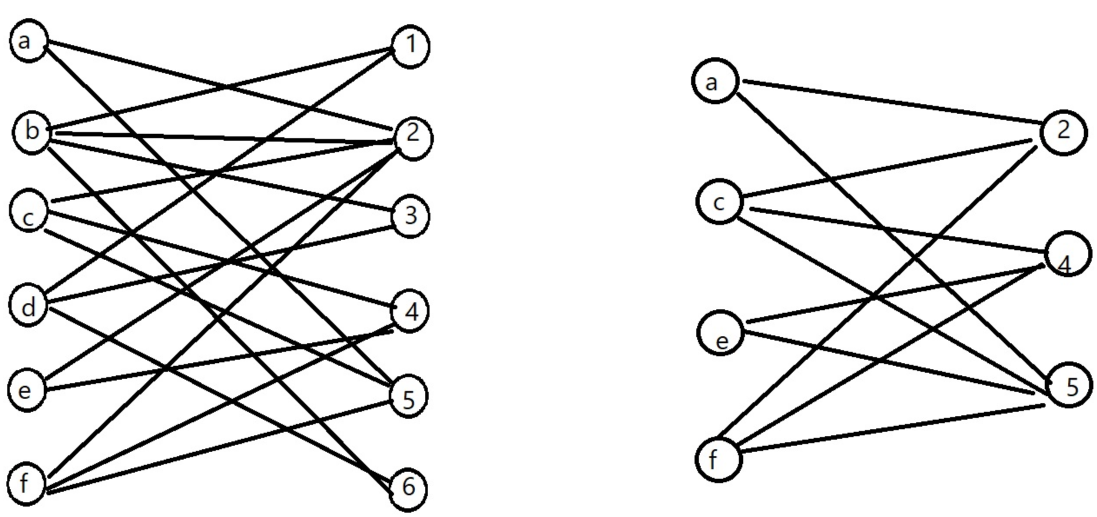

Figure 2: A graph that is not possible to create a perfect matching

Note: One condition for every student to be assigned a job is that any set of students collectively applies to at least as many jobs as the size of the set of students.

See Figure 2

Definition A *matching* in a graph is a set M of edges such that no two edges in M share a vertex.

For example, see Figure 3. M_1 is in red, M_2 is in green.

Definition A *maximum matching* is a matching in a graph of maximum size.

Definition A *perfect matching* is a matching M where each vertex in the graph is incident with an edge in M. (Note Figures 2 and 3 do not have a possible perfect matching).

Definition A vertex is *saturated* by a matching M if it is incident with an edge in M.