Definition A walk $W = v_0v_1...v_n$ in a graph G is called *closed* if $v_0 = v_n$.

Note that if w is a closed walk then for each i: $v_iv_{i+1}...v_nv_1...vi-1v_i$ is also a closed walk.

Definition An *Eulerian circuit* (or Euler tour) in a graph G is a closed walk that contains every edge of G exactly once.

Definition A graph G is called *even* if all its vertices have even degree.

Important Note If v is a vertex in G, then any closed walk w contains an even number of the edges indcident to v (half going in, half going out).

Thus, if G has an Eulerian cirtuit, it must be even.

Theorem Let G be a connected graph. Then G has an Eulerian circuit if and only if it is even.

Proof ($\Longrightarrow$) we just covered.

Proof ($\Longleftarrow$) Let G be an even connected graph. We use induction on $m = |E(G)|$

For $m = 0$ then G has one vertex v_0 which is a trivial Eulerian circuit.

Inductive Hypothesis Assume $m \geq 1$ and that every even connected grpah with fewer than m edges has an Eulerian circuit.

Consider G with m edges. Since G is connected and has some edges, every vertex has degree ≥ 2. Then by earlier lemma, if every vertex in a graph at a degree at least two then G must have a cycle (by earlier lemma).

If $C = v_1 ... v_k$ then note that $v_1 v_2 ... v_k v_1$ is a closed walk.

Let $G' = G - E(C)$ be the graph obtained from G by removing the edges of C.

Since G is even, every vertex has even degree. The degree of each $v \in V(G')$ is

- $deg_G(v)$ if $v \notin V(C)$
- $deg_G(v) - 2$ if $v \in V(C)$

Therefore G' is an even graph. We can't yet say that G' is connected... But we can say that every component of G' is a connected even graph with $< m$ edges.

Hence by **IH** each component G_i of G' has an Eulerian circuit W_i.

Since G is connected, each G_i contains a vertex a_i of C.

View each W_i as starting and ending of a_i. (See Figure 1).

Then by taking the closed walk around C and inserting at each a_i the Eulerian circuit w_i of G_i, we obtain an Eulerian circuit of G.

Hence by induction the theorem is proved $\square$.

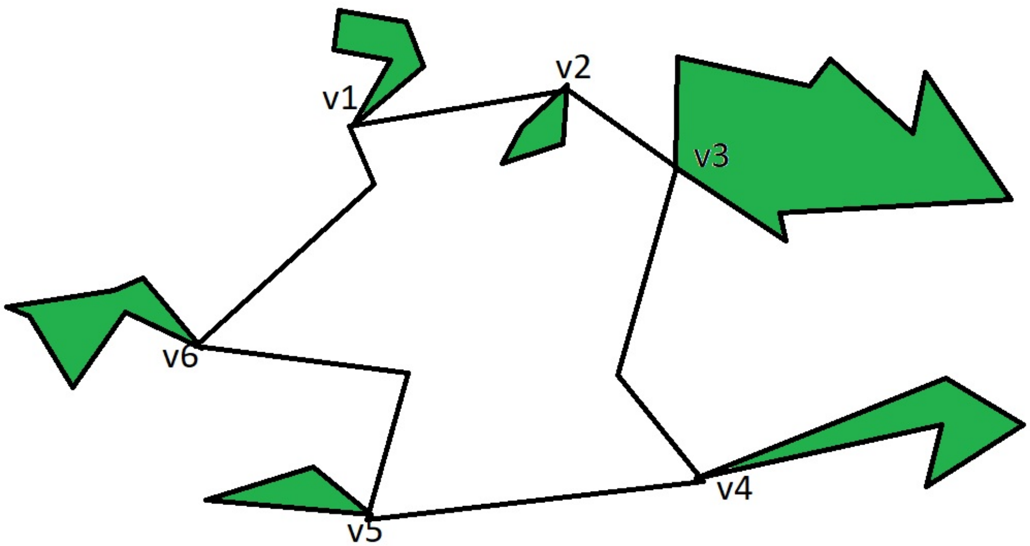

Figure 1: Visualization of Eulerian Proof Concept

Definition A *bridge* in a graph G is an edge w with the property that $G - e$ (the graph with vertex set $V(G)$ and the edge set $E(G) - \{e\}$) has more components than G.

In particular if G is connected than a bridge is an edge e such that $G - e$ is disconnected.