Block Decomposition:

$$\{0,1\}^* = \{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*$$

and RHS is unambiguous.

Example Find the generating series with respect to length for binary strings S in which all blocks of 0s have even length and all blocks of 1s have odd length.

Block Decomposition

$$\{0,1\}^* = \{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*$$

1. $$S = (\{\epsilon\} \cup \{1\}\{11\}^*)\,(\{00\}\{00\}^*\{1\}\{11\}^*)^*$$

   This is unambiguous since it is a restriction of the block decomposition. Weight is length.

2. Find generating series:

$$\begin{aligned}\Phi_{\{1\}\{11\}^*}(x) &= x + x^3 + x^5 + ...\\ &= x(1 + x^2 + x^4 + ...)\\ &= \frac{x}{1-x^2}\end{aligned}$$

$$\begin{aligned}\Phi_{\{00\}\{00\}^*}(x) &= x^2 + x^4 + x^6 + ...\\ &= x^2(1 + x^2 + x^4 + ...)\\ &= \frac{x^2}{1-x^2}\end{aligned}$$

$$\begin{aligned}\Phi_{\{00\}^*}(x) &= 1 + x^2 + x^4 + x^6 + ..\\ &= \frac{1}{1-x^2}\end{aligned}$$

$$\Phi_{\{\epsilon\}\cup\{1\}\{11\}^*}(x) = 1 + \frac{x}{1-x^2}$$

By Product and Sum Lemma

$$\Phi_S(x) = \left(1 + \frac{x}{1-x^2}\right) \frac{1}{1 - \left(\frac{x^2}{1-x^2}\right)\left(\frac{x}{1-x^2}\right)} \frac{1}{1-x^2}$$

$$\left(\frac{1-x^2+x}{1-x^2}\right) \frac{(1-x^2)^2}{(1-x^2)^2 - x^3} \frac{1}{(1-x^2)}$$

$$= \frac{1+x-x^2}{1-2x^2-x^3+x^4}$$

Simplified rational expression.

Example 2 Find the generating series with respect to length for binary strings in which each even block of 0s is followed by a block of exactly two 1s.

1.
$$\{0,1\}^* = \{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*$$

$$S = \{1\}^* \left(\{00\}\{00\}^*\{11\} \cup \{0\}\{00\}^*\{1\}\{1\}^*\right)^* \left(\{\epsilon\} \cup \{0\}\{00\}^*\right)$$

This is a restriction of Block Decomposition, hence unambiguous

2.
$$\Phi_{\{00\}\{00\}^*}(x) = \frac{x^2}{1-x^2}$$

$$\Phi_{\{00\}\{00\}^*\{11\}}(x) = \frac{x^2}{1-x^2}(x^2)$$

$$\Phi_{\{00\}\{00\}^*\{1\}\{1\}*}(x) = \left(\frac{x}{1-x^2}\right)\frac{x}{1-x}$$

$$\Phi_{\{\epsilon\}\cup\{00\}\{00\}^*}(x) = 1 + \frac{x}{1-x^2}$$

So by Product Lemma, *-Lemma, Sum Lemma

$$\Phi_S(x) = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-\left(\frac{x^4}{1-x^2} + \frac{x^2}{(1-x)(1-x^2)}\right)}\right)\left(1 + \frac{x}{1-x^2}\right)$$

$$= \frac{1+x-x^2}{1-x-2x^2+x^3-x^4+x^5}$$

Example 3 Find the generating series with respect to length for binary strings that do not contain the substring 11100.

1. $$\{0,1\}^* = \{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*$$

   $$S = \{0\}^* \left(\{1\}\{1\}^* \cup \{0\}\{0\}^* \setminus \{111\}\{1\}^*\{00\}\{0\}^*\right)^* \{1\}^*$$

   This is a restriction of Block Decomposition, hence unambiguous

2. By Sum Lemma:

   $$\Phi_{\{1\}\{1\}^*\{0\}\{0\}^*}(x) = \Phi_M(x) + \Phi_A(x)$$
   $$\Phi_M(x) = \left(\frac{x}{1-x}\right)^2 - \left(\frac{x^3}{1-x}\right)\left(\frac{x^2}{1-x}\right)$$

   By Product Lemma (above)

   By Product Lemma and *-Lemma:

   $$\Phi_S(x) = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-\Phi_M(x)}\right)\left(\frac{1}{1-x}\right)$$
   $$= \frac{1}{1-2x+x^5}$$

How to tell which decomposition to use?

Example 4 find the generating series for strings with no substring 11 using Block Decomposition

1. $$\{0,1\}^* = \{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*$$

   $$S = \{\epsilon, 1\} * (\{0\}\{0\}^*\{1\}^*)^*\{0\}^*$$

   Unambiguous as before

2. $$\begin{aligned}\Phi_S(x) &= (1+x)\frac{1}{1-\frac{x}{1-x}x}\left(\frac{1}{1-x}\right)\\ &= (1+x)\frac{1-x}{1-x-x^2}\frac{1}{1-x}\\ &= \frac{1+x}{1-x-x^2}\end{aligned}$$

   Same as before using 0-decomposition! Any composition that works, works.