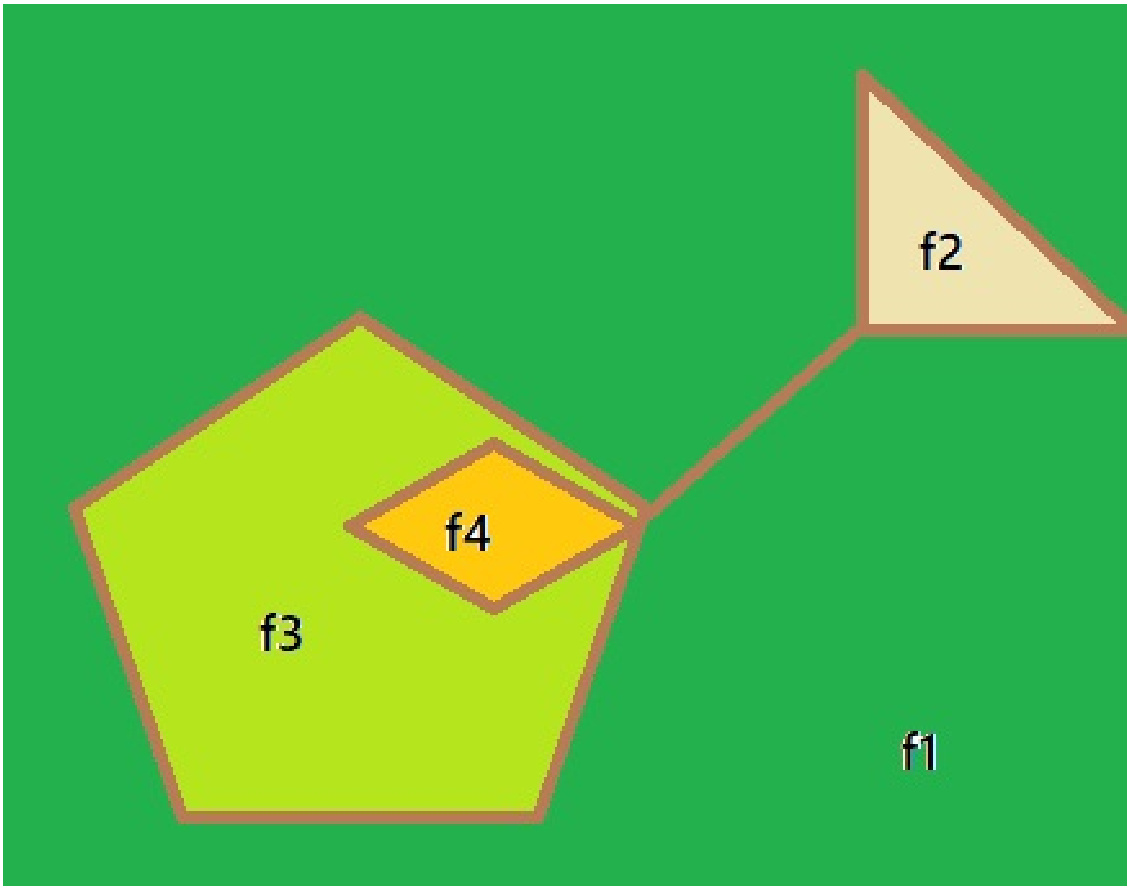

Figure 1: Example of faces

With Figure 1 as an example of faces.

For a face f of a planar drawing $\tilde{G}$ of a planar graph G:
Definition List

- The set of vertices and edges on the perimeter of f form the *boundary* of f (also said to be *incident* to f)
- Faces f and f' are said to be *adjacent* if they are incident to a common edge
- with a face f and a vertex v_0 incident to f, a *boundary walk* $w(f)$ of f is a sequence $v_0 e_1 v_1 e_2 v_2 ... e_k v_k$ (where $v_k = v_0$) of all the vertices and edges of the boundary of f. These are listed in the order that they are encountered as one moves around the whole perimeter of f.

  Note: Imagine this like you're trailing your hand along one side of the fence. You can't just stop when you've re-encountered a vertex, but you have to be in the position that you'd be on the "starting side" to the fence.

- the degree of f is the length of $w(f)$. (not doubling up on the last vertex)

  Note that edges e in the boundary of f contribute 1 to $deg(f)$ uf e is not a bridge and 2 if e is a bridge. This is beacuase if C is a cycle then in any planar drawing, each edge of C is incident to 2 faces.

If G is a tree, then any planr drawing of G has 1 face (the outer face), which has degree 2(p - 1) where $p = |V(G)|$ since you'll be counting every edge twice.

Lemma Let $\tilde{G}$ be a planar drawing of a connected planar graph. Let $F(\tilde{G})$ denote the set of faces of $\tilde{G}$ then

$$\sum_{f \in F(\tilde{G})} = 2|E(G)|$$

(since every edge is counted twice - either separating two faces from each other, or a bridge)

Proof Each edge e of G contributes 1 to the 2 distinct faces in the sum to the Left Hand Side. If e is not a brdige, and 2 to the same face if e is a bridge. So we get $2|E(G)|$.

(sometimes called "Face Shake Lemma")

Theorem (Euler's Formula) Let $\tilde{G}$ be the planar drawing of a connected planar graph G. Let $p = |V(G)|$, $q = |E(G)|$, and $s = |F(\tilde{G})|$. Then

$$p - q + s = 2$$

(basically an invariant of the plane. Special number that goes along with $\mathbb{R}^2$)

Proof We fix p and use induction on q

Base Case $q = p - 1$ smallest it can be is minimally connected, and therefore a tree by our earlier lemma. Then $s = 1$. So $p - q + s = p - (p - 1) + 1 = 2$

Inductive Hypothesis Assume that $q \geq p$ and that for all $q' - q$, and all planar drawings of connected planar graphs G' with p vertices and q' edges and s' faces, $p - q' + s' = 2$.

Let $\tilde{G}$ have p vertices, q edges and s faces.

Then G contains a cycle, hence we can choose an edge e of G that is not a bridge of G.

Then e is incident to 2 distinct faces of $\tilde{G}$

Then $G - e$ is connected (because e is not a bridge), and the drawing $\tilde{G}$ has p vertices, $q - 1$ edges, and $s - 1$ faces.

Hence by the inductive hypothesis, $\tilde{G} - e$ has $p - (q - 1) + (s - 1) = 2$ and therefore $p - q + s = 2$ since when e was removed we removed one face.

Therefore by induction the hteorem holds □