The number of k-subsets of $\{1, 2, ..., m\}$ is

$$\frac{m(m-1)(m-2)...(m-k+1)}{k!}$$

Proof:
Let L be the set of all ordered k-tuples of distinct elements of $\{1, 2, ..., m\}$
(e.g for $m = 3,\ k = 2,\ L = (1, 2)\,, ...\,(3, 2)\,,))$

Then $|L| = m(m-1)(m-2)...(m-k+1) = 6$

We could also count $|L|$ by listing all the k-subsets of $\{1, ..., m\}$ and ordered each in all $k!$ possilbe ways.

I.e

$$\{1, 2\} \rightarrow (1, 2)\, and\, (2, 1)\}$$

$$\{1, 3\} \rightarrow (1, 3)\, and\, (3, 1)\}$$

$$\{2, 3\} \rightarrow (2, 3)\, and\, (3, 2)\}$$

So $|L| = B(m, k) * k!$
Where $B(m, k)$ is the binomial coefficient!

We use notation as $\binom{m}{k}$

Note: This expression is defined for all non-negative integers k, and all m $\in \mathbb{R}$ or even ($\mathbb{C}$)!

- when $m \geq k$ is an integer

$$\binom{m}{k}$$

$$= \frac{m!}{(m-k)!k!}$$

$$\binom{m}{m-k}$$

- wheb $m \leq k$ is an integer $\binom{m}{k} = 0$
- 0! this is the *empty product* so

$$\binom{m}{0} = 1$$

Bijections
A bijection (or 1-1 correspondence) from a set S to a set T is a function

$$f : S \to T$$

this is

- 1-1 (means: if $f(s_1) = f(s_2) \implies s_1 = s_2$)
- onto (means: for each $t \in T$ there exists $s \in S$ with $f(s) = t$)

---

e.g.
There is a bijection from
$S =$ set of all subsets of $\{1, .., m\}$
to $T = \{0, 1\}^m$ (m^{th} cartesian power of $\{0, 1\}$)

(called the set of binary strings of length m) e.g. (0,1,0,1,0,0)

Must demonstrate that there is a corresponding element to each set!

We define $f : S \to T$ by $f(A) = (a_1, ..., a_m)$ where a_i is 1 if $i \in A$ and 0 otherwise

E.g. when $m = 3$

$$f(\{1, 3\}) = 101$$

The function f also gives a bijection from $S_k =$ set of all k-subsets of $\{1, ..., m\}$ to

$T_k =$ set o fall binary strings of length m that have exactly k 1s

Next we learn that a bijective function must have an inverse.

Then we will be able to conclude that the number of elements of S will be the same value as the number of elements of T. Since we will prove that if f is a bijection the $|S| = |T|$.

For functions $f : S \to T$ and $g : T \to U$ recall the *composition* functions

$$g \circ f : S \to U$$

is defined by

$$g \circ f(s) = g(f(s))$$

We say that $g : T \to S$ is the *inverse* of $f : S \to T$ if $g \circ f(s) = s$ for every $s \in S$. (i.e. $g \circ f$ is the *identity function*)

AND that $f \circ g(t) = t \ \forall t \in T$.

note that the inverse must be unique by its properties.

---

lemma: if $f : S \to T$ has an inverse then it is a bijection
(note that the converse is also true: if f is a bijection then it has an inverse)
Proof: let $g : T \to S$ be the inverse of f.

Check f is 1-1: if $f(s_1) = f(s_2)$ then $g(f(s_1)) = g(f(s_2))$ then $s_1 = s_2$.

Hence f is 1-1.

Check f is onto: let $t \in T$, let $s = g(t)$.
Then $f(s) = f(g(t)) = t$.

Hence f is onto.

Hence f is a bijection $\square$

The inverse of f (if it exists) is denoted f^{-1}

In our example:

The inverse of $f : S \to T$ is the function $g : T \to S$ by

$$g(a_1, a_2, ..., a_m) = \{i : a_i = 1\}$$

Clearly you get the same binary string you entered into it.

e.g.

A $= \{1, 3\}$ then $f(A) = (1, 0, 1)$
$g(1, 0, 1) = \{1, 3\}$

Conclusion: the number of subsets of $\{1, 2, ..., m\}$ is the same as $|\{0, 1\}^m| = 2^m$

Corollary:

$$\sum_{k=0}^{m} \binom{m}{k} = 2^m$$

We are writing the size of the set of all subsets of $\{1, ..., m\}$ in two different ways.

We have just demonstrated a *combinatorial proof* of an *identity*.