Example: Finding Coefficients

Find

$$[x^8](x^5+5x^4)(1+3x)^6$$
$$= [x^8](x^5+5x^4)\sum_{k=0}^{6}\binom{6}{k}3^k x^k \text{B.T}$$
$$= [x^8](x^5\sum_{k=0}^{6}\binom{6}{k}3^k x^k + 5x^4\sum_{k=0}^{6}\binom{6}{k}3^k x^k)$$
$$= [x^3]\sum_{k=0}^{6}\binom{6}{k}3^k x^k + 5[x^4]\sum_{k=0}^{6}\binom{6}{k}3^k x^k)$$
$$= \binom{6}{3}3^3 + 5\binom{6}{k}3^4$$
$$= 7533$$

Example Finding Coefficients 2

Where $n, k \in \mathbb{Z}_{\geq 0}$

$$[x^n]((1-x^2)^{-k} + (1-7x^3)^{-k})$$
$$= [x^n](\sum_{i\geq 0}\binom{k+i-1}{k-1}x^{2i} + \sum_{j\geq 0}\binom{k+j-1}{k-1}x^{3j})$$

- 0 if $2 \not| n$ and $3 \not| n$
- $\binom{k+\frac{n}{2}-1}{k-1}$ if $2|n$ and $3 \not| n$
- $\binom{k+\frac{n}{3}-1}{k-1}$ if $2 \not| n$ and $3|n$
- $\binom{k+\frac{n}{2}-1}{k-1} + \binom{k+\frac{n}{3}-1}{k-1}$ if $2|n$ and $3|n$

Example Finding Coefficients 3

$$\begin{aligned}
&[x^9](1+x^2)^6(1-2x)^{-3}\\
&= [x^4]\left(\sum_{k=0}^{6}\binom{6}{k}x^{2k}\right)\left(\sum_{l\geq 0}\binom{3+l-1}{3-1}2^l x^l\right)\\
&= [x^4]\left(\sum_{k=0}^{6}\binom{6}{k}x^{2k}\right)\left(\sum_{l\geq 0}\binom{2+l}{2}2^l x^l\right)\\
&= \sum_{n\geq 0}\left(\sum_{(k,l):2k+l=n}\binom{6}{k}\binom{l+2}{2}2^l\right)x^n\\
&= \sum_{(k,l):2k+l=4}\binom{6}{k}\binom{l+2}{2}2^l
\end{aligned}$$

Note the pairs (k,l) where $2k+l=4$ are $(0,4),(1,2),(2,0)$

$$\binom{6}{0}\binom{6}{2}2^4+\binom{6}{1}\binom{4}{2}2^2+\binom{6}{2}\binom{2}{2}2^0$$
$$= 399$$

Example Finding Coefficients 4

Where $n, k \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned}
&[x^n](1 - x - x^2 + x^3)^{-k} \\
&= [x^n](1 - x^2)^{-k}(1 - x)^{-k} \\
&= [x^n]\left(\sum_{i \geq 0} \binom{k+i-1}{k-1} x^{2i}\right)\left(\sum_{j \geq 0} \binom{k+j-1}{k-1} x^j\right) \\
&= \sum_{m \geq 0} \left(\sum_{(i,j):2i+j=m} \binom{k+i-1}{k-1}\binom{k+j-1}{k-1}\right) x^m
\end{aligned}$$

Note that i cannot be bigger than $\frac{m}{2}$

$$\begin{aligned}
&= \sum_{(i,j):2i+j=n} \binom{k+i-1}{k-1}\binom{k+j-1}{k-1} \\
&= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{k+i-1}{k-1}\binom{k+n-2i-1}{k-1}
\end{aligned}$$

Recall that for a set S and weight funciton w on S, the generating series fo S with respect to w is :

$$\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{n \geq 0} a_n x^n$$

where $a_n =$ the number of elements of S of weight n

Sum Lemma:

If $S = A \cup B$ where $A \cap B = \emptyset$ then

$$\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$$

Proof

$$\begin{aligned} &\Phi_S(x) \\ &= \sum_{\sigma \in S} x^{w(\sigma)} \\ &= \sum_{\sigma \in A} x^{w(\sigma)} + \sum_{\sigma \in B} x^{w(\sigma)} \\ &= \Phi_A(x) + \Phi_B(x) \end{aligned}$$

Example

Recall that for $S = \mathbb{Z}_{\geq 0}$ with $w(\sigma) = \sigma$

$$\Phi_{\mathbb{Z}_{\geq 0}}(x) = 1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$$

Recall that for $S = \mathbb{Z}_{\geq 0}^{even}$ with $w(\sigma) = \sigma$

$$\Phi_{\mathbb{Z}_{\geq 0}^{even}}(x) = 1 + x^2 + x^4 + ... = \frac{1}{1-x^2}$$

But $\mathbb{Z}_{\geq 0} = \mathbb{Z}_{\geq 0}^{even} + \mathbb{Z}_{\geq 0}^{odd}$

$$\begin{aligned} \Phi_{\mathbb{Z}_{\geq 0}^{odd}} &= \Phi_{\mathbb{Z}_{\geq 0}} - \Phi_{\mathbb{Z}_{\geq 0}^{even}} \\ &= \frac{1}{1-x} - \frac{1}{1-x^2} \\ &= \frac{x}{1-x^2} \end{aligned}$$

Product Lemma:

Suppose A_1 and A_2 are sets with eight functions w_1 and w_2 respectively, and $S = A_1 \times A_2$ has weight function w where

$$(\star)\, w((a_1, a_2)) = w_1(a_1) + w_2(a_2)$$

then

$$\Phi_S(x) = \Phi_{A_1}(x)\Phi_{A_2}(x)$$

Proof

$$\Phi_S(x)$$
$$= \sum_{\sigma \in S} x^{w(\sigma)}$$
$$= \sum_{(a_1,a_2) \in S} x^{w((a_1,a_2))}$$

Make sure that the sum is finite...

$$= \sum_{(a_1,a_2) \in S} x^{w_1(a_1)+w_2(a_2)}$$
$$= \left(\sum_{a_1 \in A_1} x^{w_1(a_1)}\right)\left(\sum_{a_2 \in A_2} x^{w_1(a_2)}\right)$$
$$= \Phi_{A_1}(x)\Phi_{A_2}(x)$$