Example 1

How many compositions of n are there with k parts, where the i^{th} part t_i is an even number at least $2i$?

1. $S = \mathbb{Z}_{\geq 2}^{even} \times \mathbb{Z}_{\geq 4}^{even} \times ... \times \mathbb{Z}_{\geq 2k}^{even}$

   with weight function of $w(t_1, ..., t_k) = t_1 + ... + t_k$

2. Find $\Phi_S(x)$ using product lemma:

$$\Phi_{\mathbb{Z}_{\geq 2i}^{even}}(x) = x^{2i} + x^{2i+2} + x^{2i+4} + ...$$

$$= x^{2i}\left(1 + x^2 + x^4 + ...\right)$$
$$= \frac{x^{2i}}{1 - x^2}$$

   Note by choice of w and Product Lemma:

$$\Phi_S(x) = \left(\frac{x^2}{1-x^2}\right)\left(\frac{x^4}{1-x^2}\right)\left(\frac{x^6}{1-x^2}\right)...$$

$$= \frac{x^{2(1+2+3+...+k)}}{(1-x^2)^k}$$

$$= \frac{x^{k(k+1)}}{(1-x^2)^k}$$

3. Find $[x^n]\left(\frac{x^{k(k+1)}}{(1-x^2)^k}\right)$

   This is:

   - 0 if n is odd or $n < k(k+1)$
   - $\binom{\frac{k}{2} - \frac{k^2}{2} + \frac{n}{2} - 1}{k-1}$ otherwise

Example 2

How many compositions of n with k parts are there in which each part is ≥ 2 and ≤ 8?

1. $S = \{2, 3, ..., 8\} \times ... \times \{2, ..., 8\} = \{2, ...8\}^k$

   With weight function of $w(t_1, t_2, ..., t_k) = t_1 + t_2 + ... + t_k$

2. Find $\Phi_S(x)$:

$$\Phi_{\{2,...,8\}} = x^2 + x^3 + ... + x^8 = x^2(1 + x + ... + x^6)$$

   By Product Lemma we have that:

$$\Phi_S(x) = \left(\frac{x^2(1 - x^7)}{1 - x}\right)^k$$

3.

$$[x^n]\Phi_S(x) = \sum_{i=0}^{\lfloor \frac{n-2k}{7} \rfloor} \binom{k}{i}(-1)^i \binom{n - k - 7i - 1}{k - 1}$$

Example 3

How may compositions of n are there? (i.e the number of parts is *not* fixed!)

Remember parts in a composition **must be positive** (i.e. no zeroes!)

We consider the *empty* composition (with 0 parts) as the composition of 0.

1.
$$S = \{\emptyset\} \cup \mathbb{Z}_{\geq 1} \cup (\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}) \cup ... \cup (\mathbb{Z}_{\geq 1})^k \cup ...$$

$$= \bigcup_{k \geq 0} (\mathbb{Z}_{\geq 1})^k$$

Note that this is a infinite disjoint union, since each set to be unioned has different number of components from any other set

choose $w(t_1, ...t_k) = t_1 + ... + t_k$

2. Use Product Lemma and Sum Lemma to find $\Phi_S(x)$:

We know that by Product Lemma:

$$\Phi_{\mathbb{Z}^k_{\geq 1}}(x) = \left(\frac{x}{1-x}\right)^k$$

By Sum Lemma

$$\Phi_S(x) = \sum_{k \geq 0} \left(\frac{x}{1-x}\right)^k$$
$$= \frac{1}{1 - \frac{x}{1-x}}$$
$$= \frac{1-x}{1-2x}$$

3.
$$[x^n](1-x)(1-2x)^{-1}$$
$$= [x^n](1-x) \sum_{i \geq 0} 2^i x^i$$
$$= [x^n] \sum_{i \geq 0} 2^i x^i - \sum_{i \geq 0} 2^i x^{i+1}$$

$$= [x^n] \sum_{i \geq 0} 2^i x^i - \sum_{i \geq 1} 2^{i-1} x^i$$

(Note it is useful to have both xs to the same power)
So $[x^n]\frac{1-x}{1-2x}$:

- 1 if n = 0
- $2^n - 2^{n-1} = 2^{n-1}$ if $n \geq 1$

Example 4

Find the generating series for compositions of n with an odd number of parts, each of which is odd.

1.
$$S = \bigcup_{k\geq 0} \left(\mathbb{Z}_{\geq 1}^{odd}\right)^{2k+1}$$

with $w(t_1, ..., t_k) = t_1 + ... + t_k$

2.
$$\Phi_{\mathbb{Z}_{\geq 1}^{odd}} = x + x^3 + x^5 + ... = \frac{x}{1-x^2}$$

So by Sum Lemma and Product Lemma:

$$\Phi_S(x) = \sum_{k\geq 0} \left(\frac{x}{1-x^2}\right)^{2k+1}$$
$$= \left(\frac{x}{1-x^2}\right) \sum_{k\geq 0} \left(\frac{x}{1-x^2}\right)^{2k}$$
$$= \frac{x}{1-x^2} \frac{1}{1-\frac{x^2}{(1-x^2)^2}}$$

(Can substitute since $\frac{x^2}{(1-x^2)^2)}$ has constant coefficient 0)

$$= \frac{x}{1-x^2} \frac{(1-x^2)^2}{1-2x^2+x^4-x^2}$$
$$= \frac{x(1-x^2)}{1-3x^2+x^4}$$

Let's find a *recurrence relation* for the coefficients $a_n = [x^n]\Phi_S(x)$

$$(1-3x^2+x^4)\sum_{n\geq 0} a_n x^n = x - x^3$$

$$\sum_{n\geq 0} a_n x^n - 3x^2 \sum_{n\geq 0} a_n x^n + x^4 \sum_{n\geq 0} a_n x^n = x - x^3$$

$$\sum_{n\geq 0} a_n x^n - 3\sum_{n\geq 0} a_n x^{n+2} + \sum_{n\geq 0} a_n x^{n+4} = x - x^3$$

$$\sum_{n\geq 0} a_n x^n - 3\sum_{n\geq 2} a_{n-2} x^{n} + \sum_{n\geq 4} a_{n-4} x^{n} = x - x^3$$

Note that $[x^n]$ must be the same on left-hand side and right-hand side sfor every n

- n = 0: $a_0 = 0$
- n = 1: $a_1 = 1$
- n = 2: $a_2 - 3a_0 = 0$
- n = 3: $a_3 - 3a_1 = -1$
- $n \geq 4$: $a_n - 3a_{n-2} + a_{n-4} = 0$