**Homogeneous examples** There are two basic things to keep in mind

1. the roots
2. whether they are repeated

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**Example 1**

$$a_n = 2a_{n-1} + 3a_{n-2}$$

with $n \geq 2$ and initial conditions $a_0 = 1$ and $a_1 = 2$

The characteristic polynomial is:

$$x^2 = 2x + 3$$

so $x^2 - 2x - 3 = (x - 3)(x + 1)$ so the roots are 3, $-1$ and they are distinct.

Between 3 and -1 these must have polynomials $c_i$ with degree less than the multiplicity of the root!
So $a_n = c_1 3^n + c_2(-1)^n$ is the general solution to the recurrence

Use the initial conditions to find the $c_1$ and $c_2$ which work for those initial conditions:

$$1 = a_0 = c_1 3^0 + c_2(-1)^0 = c_1 + c_2$$

and

$$2 = a_1 = c_1 3^1 + c_2(-1)^1 = 3c_1 - c_2$$

Now solve the system to arrive at $c_1 = \frac{3}{4}$ and $c_2 = \frac{1}{4}$

So the solution to the recurrence is:

$$a_n = \frac{3}{4}3^n + \frac{1}{4}(-1)^n = \frac{3^{n+1} + (-1)^n}{4}$$

**Example 2**

$$a_n = -2a_{n-1} - a_{n-2}$$

with initial conditions $a_0 = 1$ and $a_1 = 3$

The characteristic polynomial is $x^2 = -2x - 1$ so $x^2 + 2x + 1 = (x+1)^2$

So the root is $-1$ with multiplicity 2 thereofre the polynomial in the general solution has degree $< 2$ (generically deg $= 1$)

So $a_n = (c_1 + c_2)(-1)^n$ is the general solution. Use the initial conditions to solve for $c_1$ and $c_2$

$$1 = a_0 = (c_1 * 0 + c_2)(-1)^0 = c_2$$

and

$$3 = a_1 = (c_1 * 1 + c_2)(-1)^1 = -c_1 - c_2$$

so $c_2 = 1$ and $c_1 = 4$ thus $a_n = (-4n + 1)(-1)^n$

**Nonhomogeneous Recurrences**
In principal the sotry is the same!
Recurrence $\rightarrow$ generating series $\rightarrow$ rational function $\rightarrow$ extract coefficients with partial fractions.

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**Example 3** $a_n = a_{n-1} - 6a_{n-2} + 2^n$ for $n \geq 2$, $a_0 = 2$ and $a_1 = 4$

multiply both sides by $x^n$

$$a_n x^n = a_{n-1}x^n - 6a_{n-2}x^n + 2^n x^n$$

sum for $n \geq 2$ (i.e. where the recurrence is valid)

$$\begin{aligned}\sum_{n\geq 2} a_n x^n &= \sum_{n\geq 2} a_{n-1}x^n - \sum_{n\geq 2} 6a_{n-2}x^n + \sum_{n\geq 2} 2^n x^n \\ &= x\sum_{n\geq 2} a_{n-1}x^{n-1} - 6x^2\sum_{n\geq 2} a_{n-2}x^{n-2} + \frac{(2x)^2}{1-2x}\end{aligned}$$

rewrite in terms of $A(x) = \sum_{n\geq 0} a_n x^n$

$$A(x) - a_0 - a_1 x = x\,(A(x) - a_0) - 6x^2\,(A(x)) + \frac{(2x)^2}{1-2x}$$

Solve for $A(x)$

$$A(x)(1 - x + 6x^2) = a_0 + a_1 x - a_0 x + \frac{4x^2}{1-2x}$$

so

$$\begin{aligned}A(x) &= \frac{a_0 + a_1 x - a_0 x + \frac{4x^2}{1-2x}}{1 - x + 6x^2} \\ &= \frac{2 + 4x - 2x + \frac{4x^2}{1-2x}}{1 - x + 6x^2} \\ &= \frac{2 - 2x}{(1 - x + 6x^2)(1 - 2x)}\end{aligned}$$

What is missinng here is the shortcut theorems which got us directly from the recurrence and its characteristic polynomial to the general solution

There is something like this but it isn't as nice

**Proposition** Consider the recurrence

$$(\star)\, b_n + q_1 b_{n-1} + ... + q_k b_{n-k} = f(n)$$

for $n \geq k$.

Let $a_0, a_1, ...$ be a solution (ignoring initial conditions)

Let the general solution to the associated homogeneous recurrence (i.e $b_n + q_1 b_{n-1} + ... + q_k b_{n-k} = 0$) be $c_n = p_1(n)\theta_1^n + ... + p_j(n)\theta_j^n$

Then the general solution to $(\star)$ is $c_n + a_n$.

- $c_n$ - general solution to homogeneous recurrence
- $a_n$ - a single fixed particular solution to the homogeneous recurrence

If you have initial conditions and want to obtain a particular solution use the initial conditions with $c_n + a_n$ for $n = 0, ..., k-1$.

The variables you're solving for are in the $p_i(n)$

This might remind you of linear algebra

- eigenvalues - in both here and in eigenvalues we are interested in roots of charactersitic polynomials
- you have something like this last proposition where you add a particular solution to the general homogeneous solution... sort of like solving the vector space.. (elements in your vector space are sequences)

So the problem you still exists when you want a particular solution.. how do you find it?

So as long as you can find one solution to the non-homogeneous recurrence you have an algorithm to find all of them.

The problem is how to find that *one* solution

You *can* use the generating function (but not so nice, at least you get to choose initial condition so your numerator is nice)... but for this course, this is enough. (though there are heuristics based on what $f(n)$ is)