How to solve an enumeration problem

1. Describe a *set* S and a *weight function* w on S such that the answer to the give problem is "the number of elements of S of weight n"
2. Find the generating series ($[x^n]\Phi_S(x)$) for S with respect to w
   - definition
   - product lemma
   - sum lemma
3. Find the coefficient $[x^n]\Phi_S(x)$ This is the answer to the problem!

Example 1

Let n and k be non-negative integers. How many non-negative integer solutions are there to:

$$t_1 + t_2 + ... + t_k = n$$

Let us use the provided framework:

1. A non-negative integer solution is a k-tuple of non-negative integers.

   Choose $S = \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} ... \times \mathbb{Z}_{\geq 0} = (\mathbb{Z}_{\geq 0})^k\, 4$

   Choose weight function w on S defined by $w(t_1, t_2, ..., t_k) = t_1 + t_2 + ... + t_k$
   Note that we have only finitely many ways to have a sum of size n with non-negative integers, therefore this will be finite.

2. We know that $\Phi_{\mathbb{Z}_{\geq 0}}(x) = 1 + x + x^2 + ... = \frac{1}{1-x}$ with respect to the weight function of $w(\sigma) = \sigma$

   Note we meet the conditions of product lemma since $w(t_1, ..., t_k) = w_1(t_1) + ... + w_k(t_k)$

   Therefore:

   $$\Phi_S(x) = \left(\Phi_{\mathbb{Z}_{\geq 0}}(x)\right)^k = \left(\frac{1}{1-x}\right)^k$$

3. Find $[x^n](1-x)^{-k} = \binom{k+n-1}{k-1}$ by Negative Binomial Theorem

Example 2

How many ways can we choose a dozen donuts if the available flavours are chocolate, maple, lemon, and plain (and at least 12 of each are in stock)

i.e. How many non-negative interger solutions to $t_c + t_m + t_l + t_p = 12$?

From last example, we solve:

$$\binom{12+4-1}{4-1} = \binom{15}{3} = 455$$

Example 3
Same question but there are only 3 chocolate and 5 maple (still more than 12 of the others).

$$S = \{0,1,2,3\} \times \{0,1,2,3,4,5\} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$$

weight function, same as before $w(t_c, t_m, t_l, t_p) = t_c + t_m + t_l + t_p$.

Find $\Phi_S(x)$:

With respect to $w(\sigma) = \sigma$ we have:

$$\Phi_{\mathbb{Z}_{\geq 0}} = \frac{1}{1-x}$$

and

$$\Phi_{\{0,1,2,3\}} = \frac{1-x^4}{1-x}$$

and

$$\Phi_{\{0,1,2,3,4,5\}} = \frac{1-x^6}{1-x}$$

Therefore:

$$\begin{aligned}\Phi_S(x) &= \left(\frac{1-x^4}{1-x}\right)\left(\frac{1-x^6}{1-x}\right)\left(\frac{1}{1-x}\right)^2 \\ &= \frac{1-x^4-x^6+x^{10}}{(1-x)^4}\end{aligned}$$

Verify that $[x^12]\left(\frac{1-x^4-x^6+x^{10}}{(1-x)^4}\right) = 216$

Note: why not choose:

$$S = \{0,1,2,3\} \times \{0,1,2,3,4,5\} \times \{0,1,2,3,...,12\} \times \{0,1,2,3,...,12\}$$

We could've actually done this. Would just get different generating series, but wouldn't be as nice:

$$\Phi_S(x) = \frac{(1-x)^4(1-x^6)(1-x^{12})^2}{(1-x)^4} = \frac{(1-x^4-x^6+x^{10})(1-2x^{13}+x^{26})}{(1-x)^4}$$

But has the same coefficient on the x^{12}.

Composition of an Integer

definition: A *composition* of an integer with k parts is a k-tuple $(t_1, t_2, ..., t_k)$ of positive integers such that $t_1 + t_2 + ...t_k = n$

How many k-part compositions of n are there?

1. $S = s\mathbb{Z}_{\geq 1} \times s\mathbb{Z}_{\geq 1} \times ... \times \mathbb{Z}_{\geq 1} = (\mathbb{Z}_{\geq 1})^k$

   Choose $w(t_1, ..., t_k) = t_1 + ... + t_k$

2. $\Phi_{\mathbb{Z}_{\geq 1}}(x) = x + x^2 + x^3 + ... = \frac{x}{1-x}$
   with respect to $w(\sigma) = \sigma$

   By product lemma,

$$\Phi_S(x) = \left(\frac{x}{1-x}\right)^k = x^k(1-x)^{-k}$$

3. Find

$$\begin{aligned} &[x^n]\left(x^k(1-x)^{-k}\right) \\ &= [x^{n-k}](1-x)^{-k} \\ &= [x^{n-k}]\left(\sum_{i\geq 0}\binom{i+k-1}{k-1}x^i\right) \\ &= \binom{(n-k)+k-1}{k-1} \\ &= \binom{n-1}{k-1} \end{aligned}$$

Example 4

How many compositions of n are there with k parts, where the i^{th} part t_i is an even number at least $2i$?

1. $S = \mathbb{Z}^{even}_{\geq 2} \times \mathbb{Z}^{even}_{\geq 4} \times ... \times \mathbb{Z}^{even}_{\geq 2k}$

(continue next class)