Cycles

Continuing from last class...

Here is the result:

Theorem Let $G = (V, E)$ be a connected graph. G has an Eulerian Circuit if and only if every vertex in G has even degree.
Note must be even because "what goes in must come out" and must be done on separate edges.

Conceptually, we have Eulerian circuits attached to our cycle.

... (from last class)

However the graph with C removed might be disconnected. But all connected componets are connected graphs with all vertices of even degree.

So each component is either an isolated vertex (which trivially has an Eulerian circuit) or the inductive hypothesis applies.

Now pick a vertex v on C. Follow the Eulerian circuit of the component of $G - C$ contianing v. This returns to v now proceed along G at each vertex encountered along G follow the Eulerian circuit for the component of $G - C$ containing this vertex if it hasn't already been traversed. stop when we return to v along G.

This described an Eualrien circuit of G.

Theorem Let $G = (V, E)$ be a connected graph. G has an Eulerian Circuit if and only if every vertex in G has even degree.

Using this result we can determine when a graph has an Eulerian trail. (Recall: an *Eulerian trail* is a trail in a finite graph which visits every edge exactly once.)

Corollary Let G be a connected graph. G has an Eulerian trail that is not a circuit if and only if G has exactly two odd degree vertices.

Suppose G has exactly 2 vertices of odd degree let H be G with an edge between the two vertices of odd degree. The H has all vertices of even degree so H has an Eulerian circuit.

Choose an Eulerian circuit of H which starts at one of the vertices which has odd degree in G and where the last edge of the circuit is the added edge. This circuit without its last edge is an Eulerian trail of G.

(Note this actually uses the previous result for multigraphs in the case that there already was an edge between the odd degree vertices. But it is still the same proof. Alternately, as a separate case if the edge already exists make H by removing it.)