Binary Trees

We regard the empty tree ϵ as a binary tree.

A *binary tree* with at least one vertex is a tree with a distinguished vertex, called the *root*, such that every vertex has a *left branch* and a *right branch*, each of which is itself a (possibly empty) binary tree.

There is a natural way to draw a binary tree in the plane. (Left, Root, Right)

Binary trees they many be different even if they are isomorphic as graphs

We define a weight function on the set $\mathbb{T}$ of all binary trees by $n(T)$ = number of vertices of T.

If $T \in \mathbb{T}$ is nonempty, and has left branch $T_1 \in \mathbb{T}$ and right branch $T_2 \in \mathbb{T}$ then $n(T) = 1 + n(T_1) + n(T_2)$.

Question How many binary trees are there of weight n?

There is a natural bijection from $\mathbb{T} \setminus \{\epsilon\}$ to $\{o\} \times \mathbb{T} \times \mathbb{T}$

Where $\{o\}$ is the single vertex binary tree and the two copies of $\mathbb{T}$ correspond to the left and right branches.

The weight function n satifies the conditions for the Product Lemma, so:

$$\Phi_{\mathbb{T} \setminus \{\epsilon\}}(x) = \Phi_{\{o\}}(x) \left(\Phi_{\mathbb{T}}(x)\right)^2$$

$$\Phi_{\mathbb{T}}(x) = 1 + x \left(\Phi_{\mathbb{T}}(x)\right)^2$$

$$x \left(\Phi_{\mathbb{T}}(x)\right)^2 - \Phi_{\mathbb{T}}(x) + 1 = 0$$

Let us complete the square

$$4x^2 \left(\Phi_{\mathbb{T}}(x)\right)^2 - 4x\Phi_{\mathbb{T}}(x) + 4x + 1 = 0 + 1$$

$$4x^2 \left(\Phi_{\mathbb{T}}(x)\right)^2 - 4x\Phi_{\mathbb{T}}(x) + 1 = 1 - 4x$$
$$\left(2x\Phi_{\mathbb{T}}(x) - 1\right)^2 = 1 - 4x$$

So

$$2x\Phi_{\mathbb{T}}(x) - 1 = \pm \left(1 - 4x\right)^{\frac{1}{2}}$$

Recall the Binomial Theorem

$$(1 + x)^n = \sum_{k \geq 0} \binom{n}{k} x^k$$

Where $\binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}$.
What is this when $n = \frac{1}{2}$.

First assume $k \geq 1$

$$\binom{\frac{1}{2}}{k} = \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\ldots(\frac{1}{2} - k + 1)}{k!}$$

$$= \frac{(-1)^{k-1}(1)(3)(5)\ldots(2k-3)}{2^k k!} \times \frac{(2)(4)\ldots(2k-2)}{(2)(4)\ldots(2k-2)}$$

$$= \frac{(-1)^{k-1}(2k-2)!}{2^k k! 2^{k-1}(k-1)!}$$

$$= \frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}k(k-1)!(k-1)!}$$

$$= \frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}k} \times \binom{2k-2}{k-1}$$

If $k = 0$ then $\binom{\frac{1}{2}}{k} = 1$

So the Binomial theorem says:

$$(1+x)^{\frac{1}{2}} = 1 + \sum_{k \geq 1} \frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}k} \times \binom{2k-2}{k-1} x^k$$

then

$$(1-4x)^{\frac{1}{2}} == 1 + \sum_{k \geq 1} \frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}k} \times \binom{2k-2}{k-1} 4^k x^k (-1)^k$$

$$= 1 - 2 \sum_{k \geq 1} \frac{1}{k} \binom{2k-2}{k-1} x^k$$

So we get

$$2x\Phi_{\mathbb{T}}(x) - 1 = \pm \left(1 - 2 \sum_{k \geq 1} \frac{x^k}{k} \binom{2k-2}{k-1} \right)$$

Note the left hand has a constant coefficient of -1, so we want to take the negative to get a matching constant coefficient.

$$2x\Phi_{\mathbb{T}}(x) = 2 \sum_{k \geq 1} \frac{x^k}{k} \binom{2k-2}{k-1}$$

$$2x\Phi_{\mathbb{T}}(x) = 2 \sum_{n \geq 0} \frac{x^n}{n+1} \binom{2n}{n}$$

$$\Phi_{\mathbb{T}}(x) = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n$$

So the number of binary trees with n vertices is $\frac{1}{n+1}\binom{2n}{n}$ (Called the n^{th} Catalan number).

These count many different combinatoiral structures.