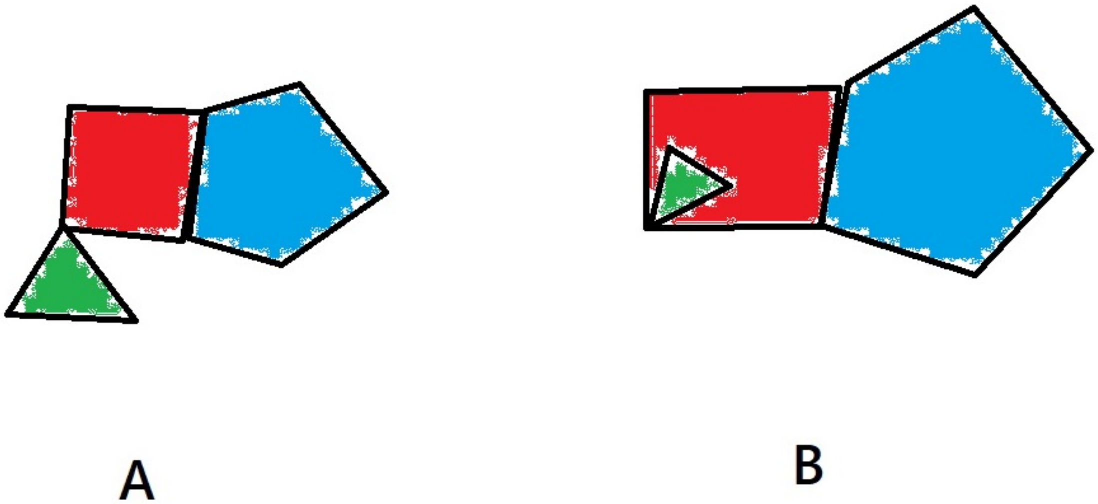

Figure 1: Multiple planar drawings

A planar graph can have several planar drawings (see Figure 1).

Graph A has face degrees 3, 4, 5, 10.
Graph B has face degrees 3, 5, 7, 7

Platonic Graphs

A connected graph G is called *platonic* if it has a planar drawing in which every vertex has the same degree $d \geq 3$, and every face has the same degree $d^* \geq 3$

(See Figures 2 and 3)

We will see that there are exactly 5 Platonic graphs.

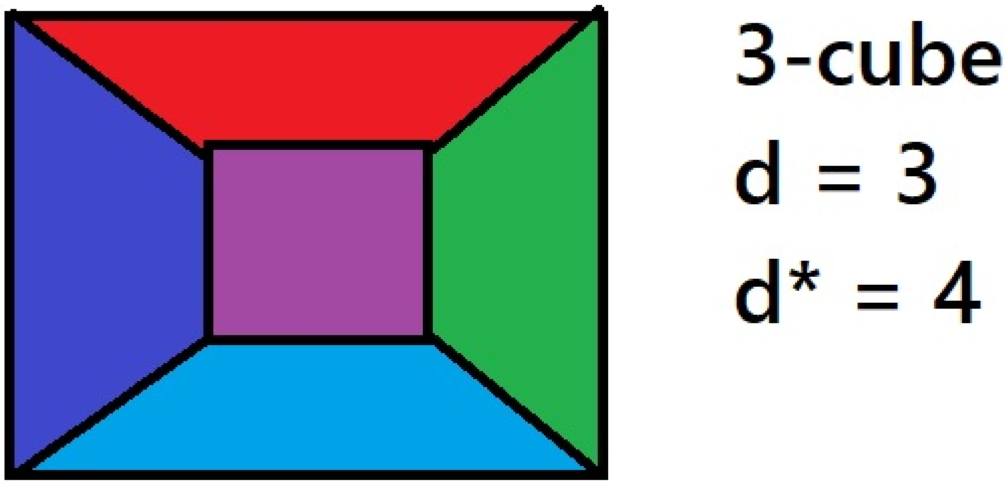

Figure 2: 3-cube is platonic

Lemma Let $\tilde{G}$ be a planar drawing of a connected planar graph G, where each vertex of G has degree $d \geq 3$, and each face of $\tilde{G}$ has degree $d^* \geq 3$. Let $p = |V(G)|$, $q = |E(G)|$, and $s = |F(\tilde{G})|$. Then

$$(d, d^*) \in \{(3,3), (3,4), (4,3), (3,5), (5,3)\}$$

Moreover

$$q = \frac{2dd^*}{2d + 2d^* - dd^*}$$

and

$$p = \frac{2q}{d}$$

and

$$s = \frac{2q}{d^*}$$

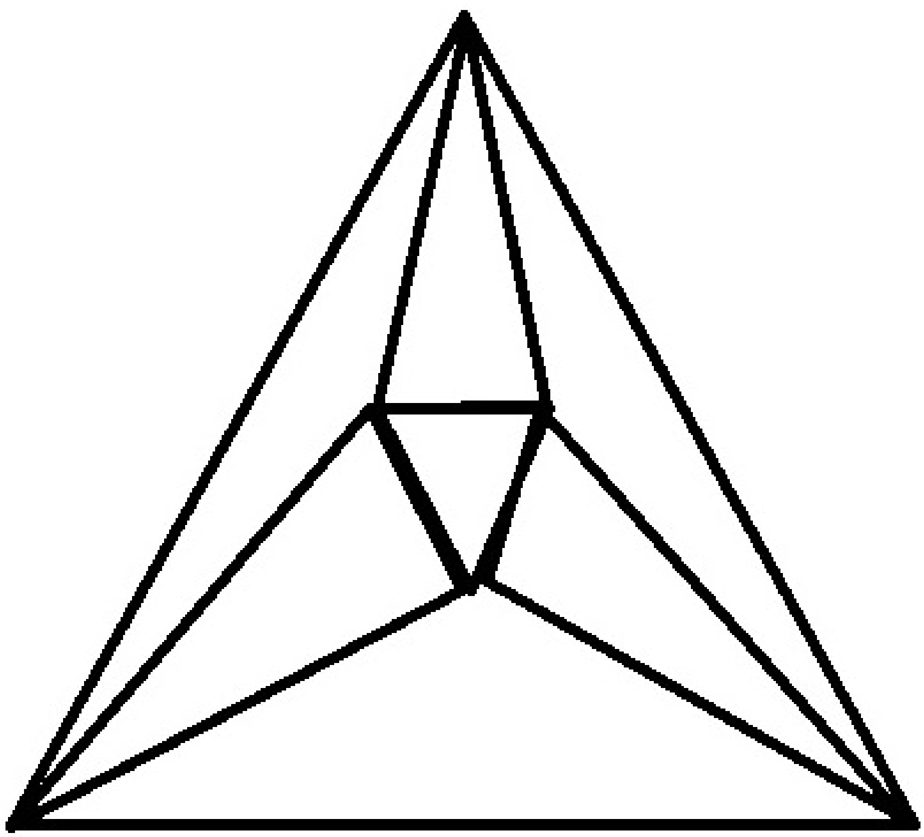

Figure 3: Platonic graph

Proof We know that $\sum_{v \in V(G)} = 2|E(G)|$, hence $dp = 2q \implies p = \frac{2q}{d}$.

Also we know that $\sum_{f \in F(\tilde{G})} deg(f) = 2|E(G)|$ so $d^* s = 2q$ and hence $s = \frac{2q}{d^*}$

Euler's Formula says $p - q + s = 2$. So,

$$\frac{2q}{d} - q + \frac{2q}{d^*} = 2$$
$$\implies q(\frac{2}{d} - 1 + \frac{2}{d^*}) = 2$$
$$\implies q(2d^* - dd^* + 2d) = 2dd^*$$
$$\implies q = \frac{2dd^*}{2d^* - dd^* + 2d}$$

Here we see that

$$2d^* - dd^* + 2d > 0$$

... interrupted by fire alarm ...