Unambiguous expressions for $\{0,1\}^\star/$

- 1 - decompositions $0^\star(10^\star)^\star$
- Prefix decomposition $(00 \cup 01 \cup 10 \cup 11)$ or $(0 \cup 1)^\star$
  First is the set of binary strings of even length

  Consider the set of all binary strings:

  $$(00 \cup 01 \cup 10 \cup 11)^\star(\epsilon \cup 0 \cup 1)$$

  This leads to the generating series of

  $$\frac{1}{1-(4x^2)}(1+x+x) = \frac{1+2x}{1-4x^2} = \frac{1}{1-2x}$$

- Block decompositions:
  - $0^\star(1^\star 10^\star 0)^\star 1^\star$
  - $1^\star(0^\star 01^\star 1)^\star 0^\star$

  (Note you can prove these are unambiguous through induction.. though this is messy)

Example Binary strings in which all blocks of 1s have length 1 (mod 3)

Adapt the 0-decomposition $1^{\star}(01^{\star})^{\star}$

String of 1s that is either empty or length 1 (mod 3). - $\epsilon \cup 1(111)^{\star}$. Since this is an adaptation constraining $0^{\star}$ it follows that this produces an umambiguous expression of

$$(\epsilon \cup 1(111)^{\star})(0(\epsilon \cup 1(111)^{\star}))^{\star}$$

we find the generating series for this expression is:

$$\begin{aligned}
\Phi_A(x) &= \left(1 + \frac{x}{1-x^3}\right) \frac{1}{1 - \left(x\left(1 + \frac{x}{1-x^3}\right)\right)} \\
&= \frac{1 + x - x^3}{1 - x^3 - (x(1-x^3) + x^2)} \\
&= \frac{1 + x - x^3}{1 - x - x^2 - x^3 + x^4} \\
&= \sum_{n=0} c_n x^n
\end{aligned}$$

By **Rational Function** (section 3.1)

Let $c_n = 0$ for $n < 0$ by convention. Then for all $n \geq 0$

$$c_n - c_{n-1} - c_{n-2} - c_{n-3} + c_{n-4} =$$

- 1 — $n = 0$
- 1 — $n = 1$
- 0 — $n = 2$
- -1 — $n = 3$
- 0 — $n \geq 4$

Above is from the denominator and the piecewise below is from the numerator.

- $n = 0$: $c_0 = 1$
- $n = 1$: $c_1 - c_0 = 1$ so $c_1 = 2$
- $n = 2$: $c_2 - c_1 - c_0 = 0$ so $c_2 = 3$
- $n = 3$: $c_3 - c_2 - c_1 - c_0 = -1$ so $c_3 = 5$
- $n \geq 4$: $c_n = c_{n-1} + c_{n-2} + c_{n-3} - c_{n-4}$

Example

Binary strings in which each block of 1s is followed by a block of 0s with the same pairity. (In particular such a string does not end with a 1.)

Start from the block decomposition of

$$0^\star(1^\star 10^\star 0)^\star 1^\star$$

turns into

$$0^\star((11)^\star 1(00)^\star 0 \cup (11)^\star 11(00)^\star 00)^\star \epsilon$$
$$0^\star((11)^\star(10 \cup 1100)(00)^\star)^\star$$

And the generating series is:

$$\Phi(x) = \frac{1}{1-x}\frac{1}{1-\frac{x^2+x^4}{(1-x^2)^2}} = \frac{(1-x^2)^2}{(1-x)(1-x^2)^2-x^2-x^4}\cdots$$