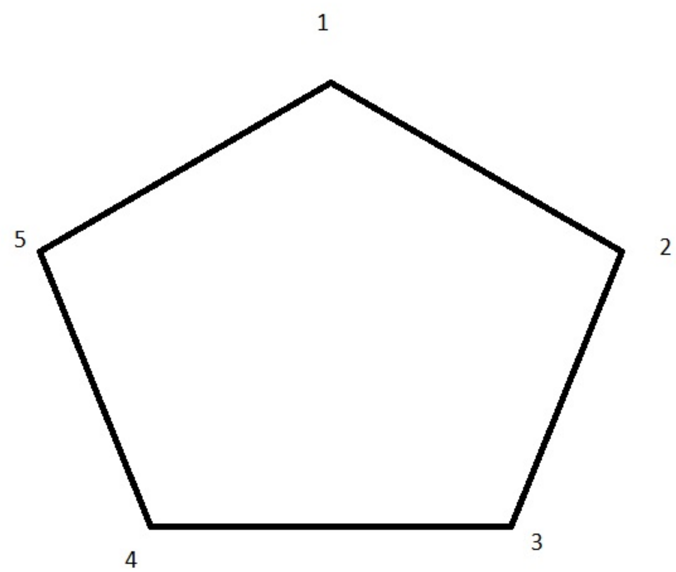

Figure 1: Odd length cycle

An *odd* cycle is a cycle of odd length. (as you would expect: triangle, pentagon, etc.)(See Figure 1)

Lemma If a graph G contains an odd cycle then G is *not* bipartite.

Proof Supposle on the contrary that $V(G) = A \cup B$ is a *bipartition* of G (i.e. every edge of G has a vertex in A and a vertex of B.) Let $C = v_1 v_2 ... v_k$ be an odd length cycle in G so k is odd.

Without loss of generality, $v_1 \in A$ then $v_2 \in B$ and in general

- $v_i \in A$ if (i is odd)
- $v_i \in B$ if (i is even)

But then the edge $v_1 v_k$ has both ends in A, which is a contradiction □

Definition A subgraph H of a graph G is *spanning* if $V(H) = V(G)$.

(take all of its vertices but only some of its edges)

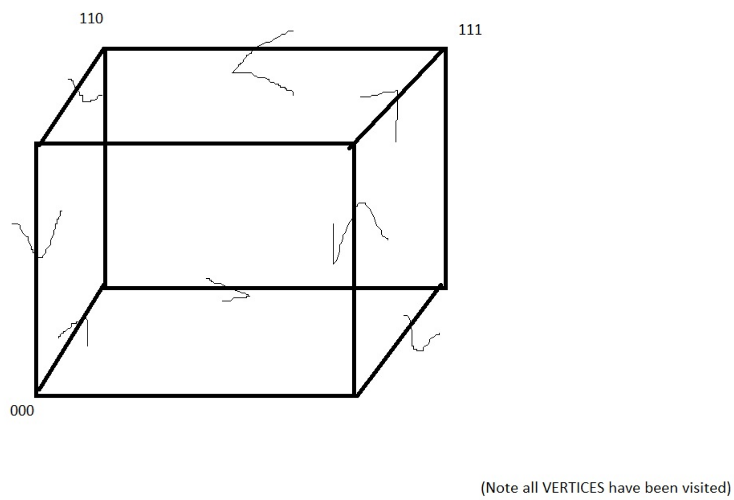

Figure 2: 3-cube is Hamiltonian

Definition A *Hamiliton* cycle in a graph G is a spanning cycle

Note if G has a Hamilton cycle then G is said to be a *Hamiltonian* graph.

Note a 3-cube is Hamiltonian (See Figure 2)

Note (Figure 3) cannot be Hamiltonian because of the odd-degree centre

(finding if a graph is Hamiltonian is **NP-Hard**)

Definition a graph G is said to be *conected* if there is a path in G from x to y, for **all** pairs of vertices x and y.

Note that (Figure 3) **is** connected by definition.

Note that if a graph is not connected it is called *disconnected*. (So a disconnected graph must have at least 2 components)

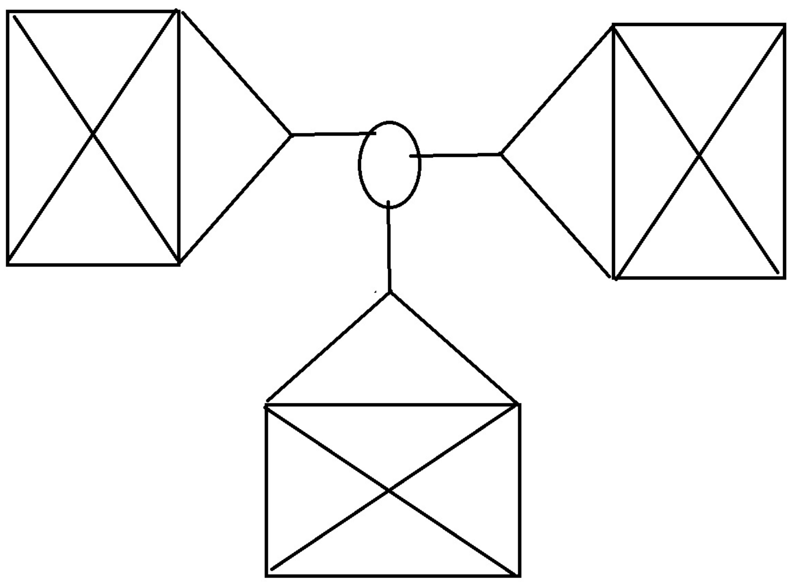

Figure 3: Not Hamiltonian

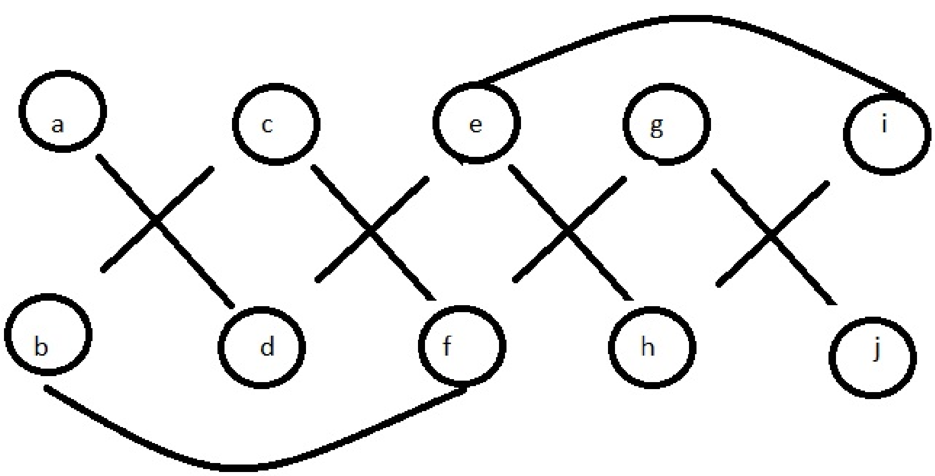

Figure 4: Example Graph

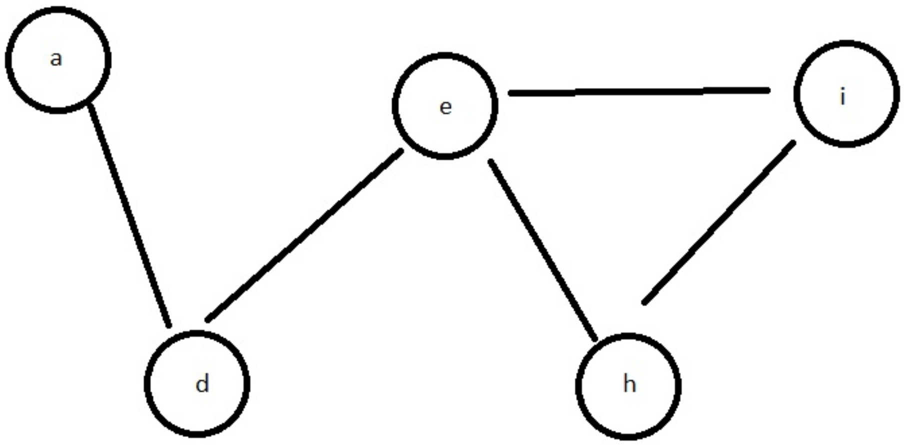

Figure 5: Example Component of Figure 4

Definition A *component* of a graph G is a maximal connected subgraph of G.

If H is a component of G:

- H is a connected subgraph of G
- H is not a subgraph of any larger connected subgraph of G

Example See Figure 4.

Identify that we can find one component of (Figure 4) to be (Figure 5).

Components are essentially the connected pieces of the graph.

(Note that there are only two components.)

Note that if G is a connected graph then the number of components is 1.

Lemma Suppose that we have two vertices x and y and there is a walk in the graph G from x to y, then there is a path in G from x to y.

Proof (Remove all cycles from your walk. Now you have a path ☺)
Let $W = v_0v_1...v_n$ be a walk from x to y (so $x = v_0$ and $y = v_1$) of shortest length (i.e. n is minimal).

If all v_i are distinct then W is a path from x to y. We are done.

Otherwise, have $v_i = v_j$ where $i < j$ then

$$W' = v_0v_1...v_iv_{j+1}...v_n$$

Then W' is a shorter walk from x to y contradicting our choice of W.

Thus W is a path as required □

Proposition Let G be a graph in which every vertex has degree ≥ 2 then G contains a cycle.

Proof Let $P = v_0v_1...v_{k-1}v_k$ be a longest path in G.

(What are vertices are adjacent to v_k ?)

Then since v_k has degree at least two, there is a vertex $w \neq v_{k-1}$ such that $v_kw \in E(G)$ (w is a neighbour of v_k).

(Note that w must be along the path. Otherwise we could extend our path to include w and have a new longer path, a contradiction of our assumption.)

Then $w = v_i$ for some $i < k - 1$ since otherwise $P\{w\}$ is a path longer than P.

So $v_iv_{i+1}...v_k$ is a cycle in G, as required □