Continuing from last class...

Proof We know that $\sum_{v \in V(G)} = 2|E(G)|$, hence $dp = 2q \implies p = \frac{2q}{d}$.

Also we know that $\sum_{f \in F(\tilde{G})} deg(f) = 2|E(G)|$ so $d^* s = 2q$ and hence $s = \frac{2q}{d^*}$

Euler's Formula says $p - q + s = 2$. So,

$$\frac{2q}{d} - q + \frac{2q}{d^*} = 2$$
$$\implies q(\frac{2}{d} - 1 + \frac{2}{d^*}) = 2$$
$$\implies q(2d^* - dd^* + 2d) = 2dd^*$$
$$\implies q = \frac{2dd^*}{2d^* - dd^* + 2d}$$

Here we see that

$$2d^* - dd^* + 2d > 0$$

$$\implies dd^* - 2d^* - 2d < 0$$

Add 4 to complete the square.

$$\implies dd^* - 2d^* - 2d + 4 < 4$$

$$(d-2)(d^*-2) < 4$$

But d, d^* are integers ≥ 3.

Only possible pairs are:

- $d = d^* = 3$
- $d = 4$, $d^* = 4$ (or vice versa)
- $d = 5$, $d^* = 3$ (or vice versa)

Note: For each of these pairs there is a *unique* platonic graph.

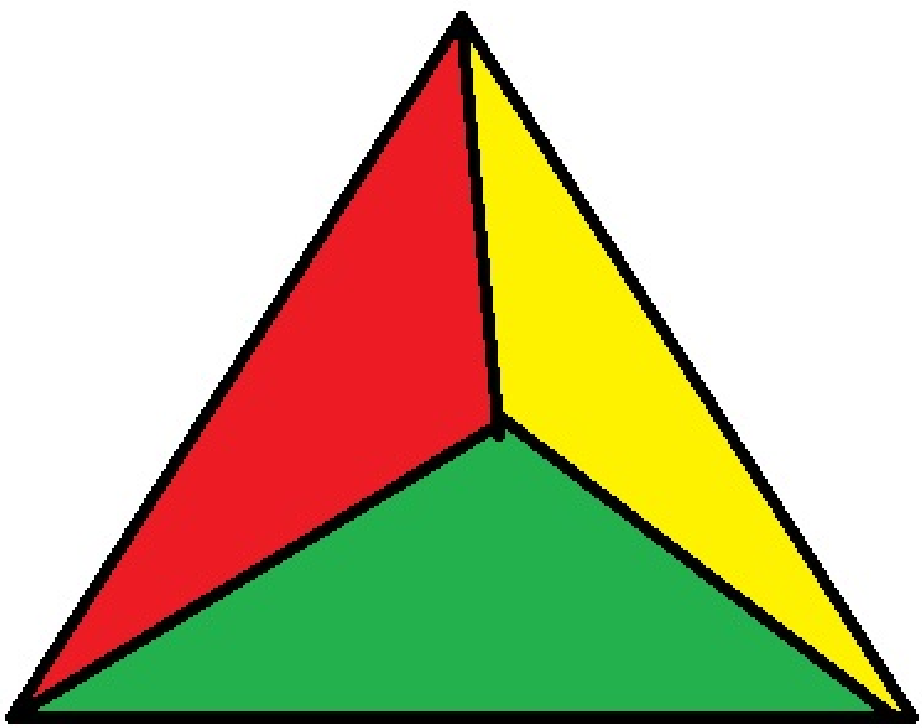

Figure 1: K_4

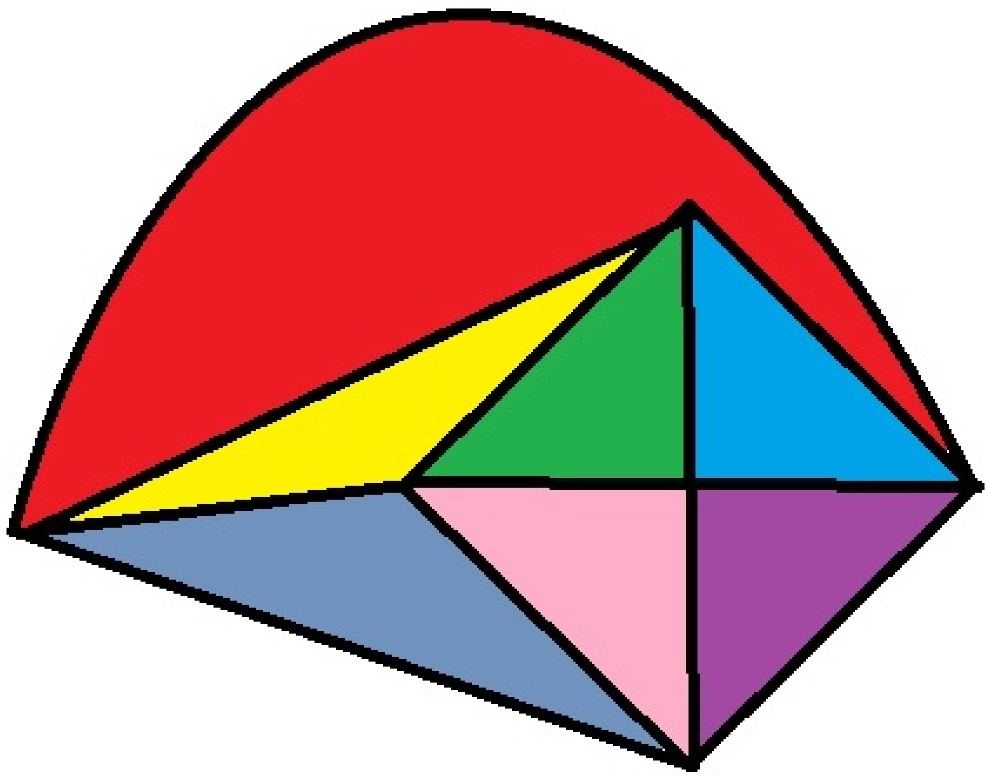

Figure 2: $d = 4, d^* = 3$ Platonic Graph

Try with $d = d^* = 3$:

Using the formula we get:

$$q = \frac{2dd^*}{2d^* - dd^* + 2d} = 6$$

then

$$p = \frac{2q}{d} = 4, s = 4$$

The number of graphs with 4 vertices and 6 edges is 1 (K_4). It is platonic. (See Figure 1)

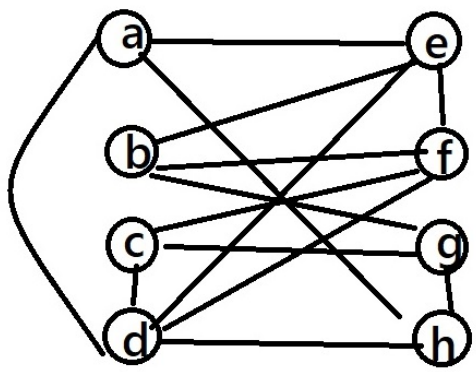

Figure 3: Step 0

Try with $d = 4, d^* = 3$:

Using the formula we get:

$$q = \frac{2dd^*}{2d^* - dd^* + 2d} = 12$$

then

$$p = \frac{2q}{d} = 6, s = \frac{2q}{d^*} = 8$$

So any graph G with these parameters has 6 vertices, and every vertex has degree 4.

So the vertices must come in 3 pairs (v, v') where $(v, v') \notin E(G)$ and $vw \in E(G)$ for all $w \in V(G) \setminus \{v, v'\}$

This is Platonic and unique.

(See Figure 2)

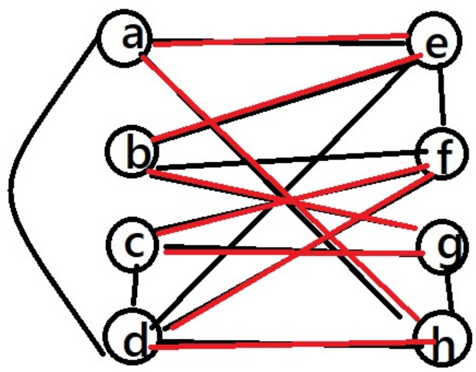

Figure 4: Step 1

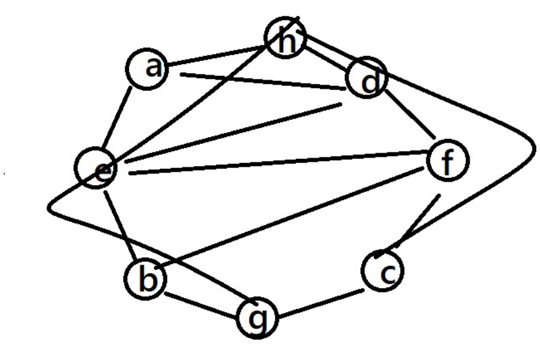

Figure 5: Step 2

Question How to show a graph is planar?
Answer Exhibit a planar embedding.

(See Figure 3)

Tip Look for a long cycle. Embed it. Try to complete, putting the remaining vertices / edges inside or outside.

(See Figure 4)
(See Figure 5)

Thus planar.

Question How to show a graph is not planar?
Answer Learn more

Lemma Let $\tilde{G}$ be a planar drawing of a connected planar grpah G. Let $f \in F(\tilde{G})$. If the boundary $B(f)$ of f doesn't contain a cycle then G is a tree.

Proof Since $B(f)$ is connected, and contains no cycles it is a tree.

But since G is connected, it cannot contain any vertices that are not in $B(f)$.

So $B(f) = G$ and G is a tree $\square$

Consequence Unless G is a tree, every face boundary contains a cycle.

Theorem Let G be a planar graph with $p \geq 3$ vertices and q edges. Then $q \leq 3p - 6$. (otherwise, would have too many edges to be planar).

Proof Note we can assume that G is connected since otherwise we may add edges between components of G to keep it planar, increase q but keep p the same. (Connected will only make things harder for ourselves.)

Important to have it connected to use Euler's Formula.

Let $\tilde{G}$ be a planar drawing of G.

If G does not contain a cycle then it is a tree so it has $p - 1$ edges. Then $p - 1 \leq 3p - 6$ because $p \geq 3$.

If G contains cycles then by the Lemma, every face of G contains a cycle and hence every face has degree ≥ 3. So,

$$2q = \sum_{f \in F(\tilde{G})} deg(f) \geq 3s$$

where $s = |F(\tilde{G})|$.

By Euler's Formula $s = 2 - p + q$

Thus,

$$2q \geq 3s = 3(2 - p + q)$$
$$2q \geq 6 - 3p + 3q$$
$$q \leq 3p - 6$$

□

Corollary Every planar graph has a vertex degree ≤ 5.

Proof Otherwise:

$$2q = \sum_{v \in V(G)} deg(v) \geq 6$$

$$q \geq 3p$$

a contradiction □