Partial Fractions

We have many examples where we can get the generating series of some interesting combinatorial objects as a rational function (ration of polynomials). But so far we don't have a good way to extract coefficients and we really want coefficients not generating series.

That is we care about $[x^n]\Phi(x)$

Example Suppose we found that $\Phi(s) = \frac{1}{(1-x)(1-2x)}$

then use partial fractions to rewrite $\Phi(x)$.

$$\frac{1}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x}$$

for some A, B. Now solve for A, B.

$$= \frac{A(1-2x)}{(1-x)(1-2x)} + \frac{B(1-x)}{(1-x)(1-2x)}$$

$$= \frac{x(-2A-B)+(A+B)}{(1-x)(1-2x)}$$

And so: $-2A - B = 0$ and $A + B = 1$.

we can solve A and B based off the earlier value.

Theorem

Suppose f and g are polynomials with degree $f <$ degree g and

$$g(x) = (1-\theta_1 x)^{m_1}(1-\theta_2 x)^{m_2}...(1-\theta_k x)^{m_k}$$

with the θ_k distinct (possibly complex scalars). then there exist polynomials $p_1, ..., p_k$ with degree $p_i < m_i$ such that

$$[x^n]\frac{f(x)}{g(x)} = p_1(n)\theta_1^n + ... + p_k(n)\theta_k^n$$

The general result is:

Proof

Claim if $g(x) = g_1(x)g_2(x)$ with g_1, g_2 coprime then there exist polynomials f_1, f_2 with degree $f_i <$ degree g_i such that

$$\frac{f(x)}{g(x)} = \frac{f_1(x)}{g_1(x)} + \frac{f_2(x)}{g_2(x)}$$

The key to proving the claim is the extended Euclidean algorithm. By the extended Euclidean algorithm we have a_1, a_2 polynomials with degree $a_1 <$ degree g_1 and degree $a_2 <$ degree g_2 such that

$$a_1(x)g_1(x) + a_2(x)g_2(x) = 1$$

since g_1, g_2 coprime. So:

$$f(x)a_1(x)g_1(x) + f(x)a_2(x)g_2(x) = f(x)$$

If I divide by g then in the first term I'm faced with $\frac{f(x)a_1(x)}{g_2(x)}$

Let f_2 be the remainder of $f(x)a_1(x)$ after dividing by $g_2(x)$, similarly for the second term let f_1 be the remainder of $f(x)a_2(x)$ after dividing by $g_1(x)$.

Then $f_1(x)g_2(x) + f_2(x)g_1(x) \equiv f(x)$ modulo $g(x) = g_1(x)g_2(x)$

Note that both sides have degree less than g, so

$$f_1(x)g_2(x) + f_2(x)g_1(x) = f(x)$$

dividing everything by g once again provides:

$$\frac{f_1(x)}{g_1(x)} + \frac{f_2(x)}{g_2(x)} = \frac{f(x)}{g(x)}$$

which proves the claim $\square$

—Applying the claim multiple times we get

$$[x^n]\frac{f(x)}{g(x)} = [x^n]\frac{f_1(x)}{(1-\theta_1 x)^{m_1}} + ... + [x^n]\frac{f_k(x)}{(1-\theta_k x)^{m_k}}$$

with degree $f_i \leq$ degree $(1-\theta_i x)^{m_i} = m_i$.

Note this f_1, f_2 are not necessarily same as in claim.

Now apply the binomial theorem on each term...

Recurrences

Suppose you have a sequence $a_0, a_1, ...$ and you havle an expresion for a_n in terms of some or all of $a_0, ..., a_{n-1}$

This expression is called a recurrence.

E.g. Fibonnacii numbers of $f_n = f_{n-1} + f_{n-2}$ with $f_0 = f_1 = 1$.

We will be interested in recurrenced which are:

1) linear. That is no products of the a_is occur so we have this the form

$$a_n = c_1 a_{n-1} + ...c_k a_{n-k} + F$$

(where k is ther order of the recurrence) for $n \geq k$

E.g. the fibonacci is linear. You also saw them on $A3$ when solving for a multiplicative inverse of a series.

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2) constant coefficient that is : c_i are independent of n.

Note F is allowed to depend on n write $F(n)$ to emphasize this. (where F is the constant).

Note: If $F(n) = 0$ for all n, we say the recurrence is homogeneous, otherwise it is nonhomogenous!

Example

$$c_n = c_{n-1} + c_{n-2}$$

for $n \geq 2$.

Multiply by x^n.

$$c_n x^n = c_{n-1} x^n + c_{n-2} x^n$$

now sum over the range where the recurrence holds:

$$\sum_{n\geq 2} c_n x^n = \sum_{n\geq 2} c_{n-2} x^n + \sum_{n\geq 2} c_{n-1} x^n$$

Let $\Phi_C(x) = \sum_{n\geq 0} c_n x^n$. Let's adjust to be written in terms of Φ_c

$$\begin{aligned}
&\Phi_c(x) - c_0 - c_1 x \\
&= x^2 \left(\sum_{n\geq 2} c_{n-2} x^{n-2} \right) + x \left(\sum_{n\geq 2} c_{n-1} x^{n-1} \right) \\
&= x^2 \Phi_C(x) + x(\Phi_C(x) - c_0)
\end{aligned}$$

Now solve for $\Phi_C(x) = \frac{c_0 + c_1 x + c_0 x}{1 - x - x^2}$

Now to finish we need to extract coefficients

$$c_n = [x^n] \frac{c_0 + c_1 x + c_0 x}{1 - x - x^2}$$

Note I need initil conditions to tell me c_0, c_1.

We can now do this coefficient extraction by today's first theorem. The point is that this always works.

Theorem

Let

$$c_n + q_1 c_{n-1} + ... + q_k c_{n-k} = 0$$

for $n \geq k$ (the recurrence)

Let $g(x) = 1 + q_1 x + ... + q_k x^k$

Then there exists a polynomial f with degree $f < k$ such that the formal power series $C(x) = \sum_{n \geq 0} c_n x^n$ is $C(x) = \frac{f(x)}{g(x)}$

and so

$$c_n = p_1(n)\theta_1^n + p_2(n)\theta_2^n + ... + p_j(n)\theta_j^n$$

by the previous theorem

Where $g(x) = (1 - \theta_1 x)^{m_1} + ... + (1 - \theta_j x)^{m_j}$
degree $p_i < m_i$ and the p_i can be determined by $c_0, ..., c_{k-1}$

Proof calculate as in the example to obtain $c(x) = \frac{f(x)}{g(x)}$ then apply the previous theorem.

Note the polynomial $x^k g(\frac{1}{x})$ is called the characteristic polynomial of the recurrence. The θ_i are the roots of the characteristic polynomial. You never actually need $g(x)$ or $c(x)$ themeslves. The characteristic polynomial and its roots are all you need.