P.Haxell MC 6308
OH M 2-3pm
W 3-4pm

1. Enumeration (5 weeks)
2. Graph Theory (7 weeks)

Assignments almost every week - Thursdays

Typical Question: "How many elements of a set S of "combinatorial objects" have a given property?"

We will usually answer such problems by encoding the properties of S into an algebraic expression: the generating series for S

E.g.: How many ways can we choose a dozen donuts if the available flavours are chocolate, maple, lemon, and plain (and at least 12 of each are in stock)
Ans: the coefficient of x^{12} in the series of $\frac{1}{(1-x)^4}$

e,g: Same question, but there are only 3 chocolate and 5 maple left (and at least 12 of the other two).
Ans: the coefficient of x^{12} in $\frac{(1-x^4)(1-x^6)}{(1-x)^4}$

Standard framework for enumeration problems:
we define:

- a set S
- a weight function w that assigns to each element σ of S a non-negative integer $w(\sigma)$

Ask: How many elements of S have weight n?

e.g.
in Q1 - S is the set of all collections of donuts form the 4 flavours

The weight function is the number of donuts in σ

Ask: How many elements of S have weight 12?

Usually our set S is a **Cartesian product** or a **disjoint union** of Cartesian products.

Recall that for sets A and B, the Cartesian product is

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Recall that for sets A and B, the union is

$$A \cup B = \{x : x \in A \vee x \in B\}$$

Recall that for sets A and B, the intersection is

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

Recall that for sets A and B, AuB is a disjoint union if

$$A \cap B = \emptyset$$

Recall that for finite sets A and B, the size of $A \times B$, $A \cup B$ is given by

$$|A \times B| = |A||B|$$
$$|A \cup B| = |A| + |B| - |A \cap B|$$

So, if $A \cup B$ is a disjoint union we get:

$$|A \cup B| = |A| + |B|$$

(All these come out later with a "Sum Lemma" and "Product Lemma")

Recall that for set A, the k^{th} Cartesian Power of A is the set of all ordered k-tuples of elements of A

$$A^k = \{(a_1, a_2, ..., a_k) : a_i \in A\}$$

Therefore the size of the k^{th} Cartesian Power of A (if A is a finite set) is

$$|A|^k$$

e.g. Suppose the donut shop has in stock:

- 3 chocolate
- 5 maple
- 19 lemon
- 17 plain

Describe all possible collections of donuts (of any size) as a **Cartesian product**.

$$\{chocolate\} \times \{maple\} \times \{lemon\} \times \{plain\}$$

$$= \{0, 1, 2, 3\} \times \{0, 1, 2, 3, 4, 5\} \times \{0, .., 19\} \times \{0, .., 17\}$$

$$= 4 * 6 * 20 * 18$$

Describe the set of all possible choices that do not include both a chocolate and a maple donut as a *disjoint union* of **Cartesian products**

$$\{chocolate\}\times\{NOmaple\}\times\{lemon\}\times\{plain\}\cup\{NOchocolate\}\times\{maple\}\times\{lemon\}\times\{plain\}$$

note - don't double count the NO maple + NO chocolate set!

$$= \{0, 1, 2, 3\} \times \{0\} \times \{0, .., 19\} \times \{0, .., 17\} + \{0, 1, 2, 3\} \times \{1, 2, 3, 4, 5\} \times \{0, .., 19\} \times \{0, .., 17\}$$

E.g Fix a positive integer m.
Let S be the set of all subsets of $\{1, ..., m\}$
i.e

$$S = \{\emptyset, \{1\}, \{2\}, ..., \{1, 2, 3\}\}$$

Let the weight function w be defined on S by

$$w(\sigma) = |\sigma|$$

Ask: How many elements of S have weight k (for any k, $0 \leq k \leq m$)?

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e.g: when $m = 2$, $k = 2$, answer is 3

We will see that the answer is:

$$\binom{m}{k}$$

$$= \frac{m(m-1)(m-2)(m-k+1)}{k!}$$

This is the number of subsets of $\{1, 2, ...m\}$ of size k.
we say a subset of size k is a k-subset.