(recall from last class) D_i denotes the subgraph D of Step i of Prim's Algorithm.

Claim For each $i \geq 1$ there exists a Minimum Spanning Tree of G_x that contains D_i.

Proof By induction on i.

Base Case $i = 1$ $D_1 = \{x\}$. Then every spanning tree of G_x contains D_1, so in particular every Minimum Spanning Tree contains D_i.

Inductive Hypothesis Assume $i \geq 2$ and there exists a Minimum Spanning Tree T that contains D_{i-1}.

Consider D_i.

Let e be the edge we add to D_{i-1} in the algorithm to get D_i.

Cases:

1. T contains e. Then T is a Minimum Spanning Tree of G_x that contains D_i as required.

2. T does not contain e. By our earlier lemma, $T + e$ contains exactly one cycle C which contains e. Then C must contain another edge $e' \neq e$ that is also in the cut of G_x induced by $V(D_{i-1})$.

   But e is an edge of this cut of minimum weight, so $w(e) \leq w(e')$.

   By our (other) spanning tree lemma, we know $T' = T + e - e'$ is also a spanning tree of G_x.

   But $w(T') = w(T) + w(e) - w(e') \leq w(T)$.

   Hence, because T is a Minimum Spanning Tree of G_x, we find T' is also a Minimum Spanning Tree (and $w(e) = w(e')$). Thus T' contains D_i as required.

   Hence by the induction the claim is true, which proves that Prim's ALgorithm finds a MST. □

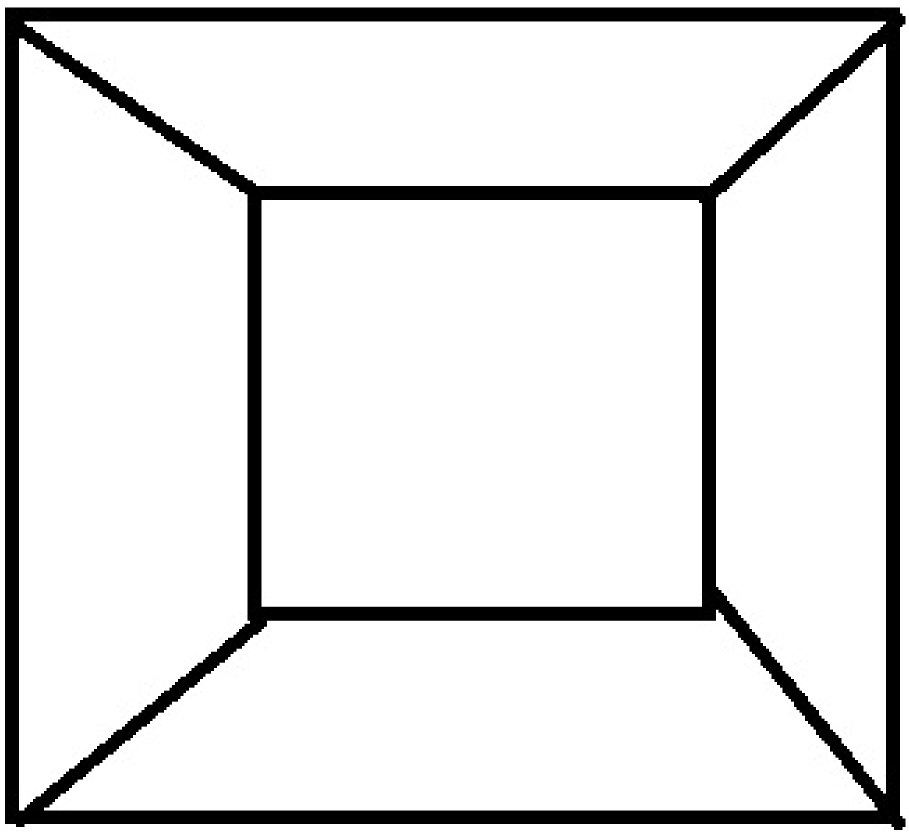

Figure 1: Planar drawing of 3-cube

Planar Graphs

Definition A graph G is said to be *planar* if it has a drawing in the plane (i.e. $\mathbb{R}^2$) so that:

- no two edges intersect (except at their common end vertices, if any)
- no two vertices coincide

Note A drawing that verifies G is planar is called a *planar drawing* or a *planar embedding* of G.

Furthermore, a graph is planar if and only if all its components are plnara. So often we will consider only *connected* planar graphs.

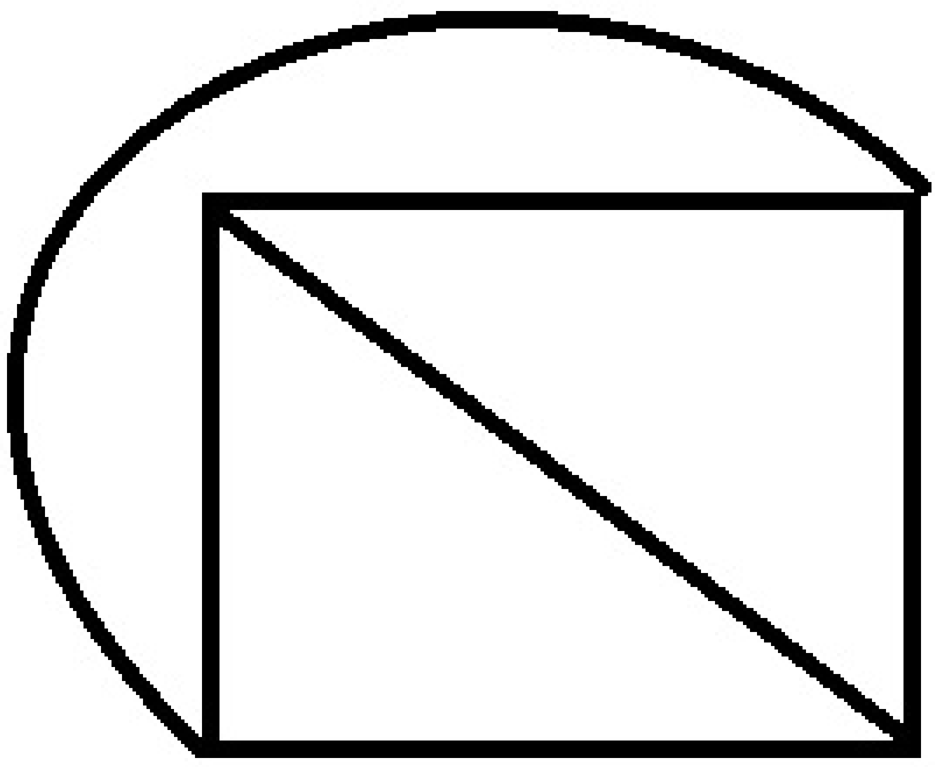

Figure 2: Planar drawing of k_4

Example 1
The 3-cube can be drawn in a planar way. (See Figure 1)

Example 2
The k_4 can be drawn in a planar way. (See Figure 2)

Definition Let $\tilde{G}$ be a planar drawing of a connected planar graph G. Then $\tilde{G}$ partitions the plane into connected regions of the plane, these are called the *faces* of G.

Definition: The *outer face* is a *face* that is unbounded and exists for every planar drawing.

See Figure 3 for a visulization of faces.

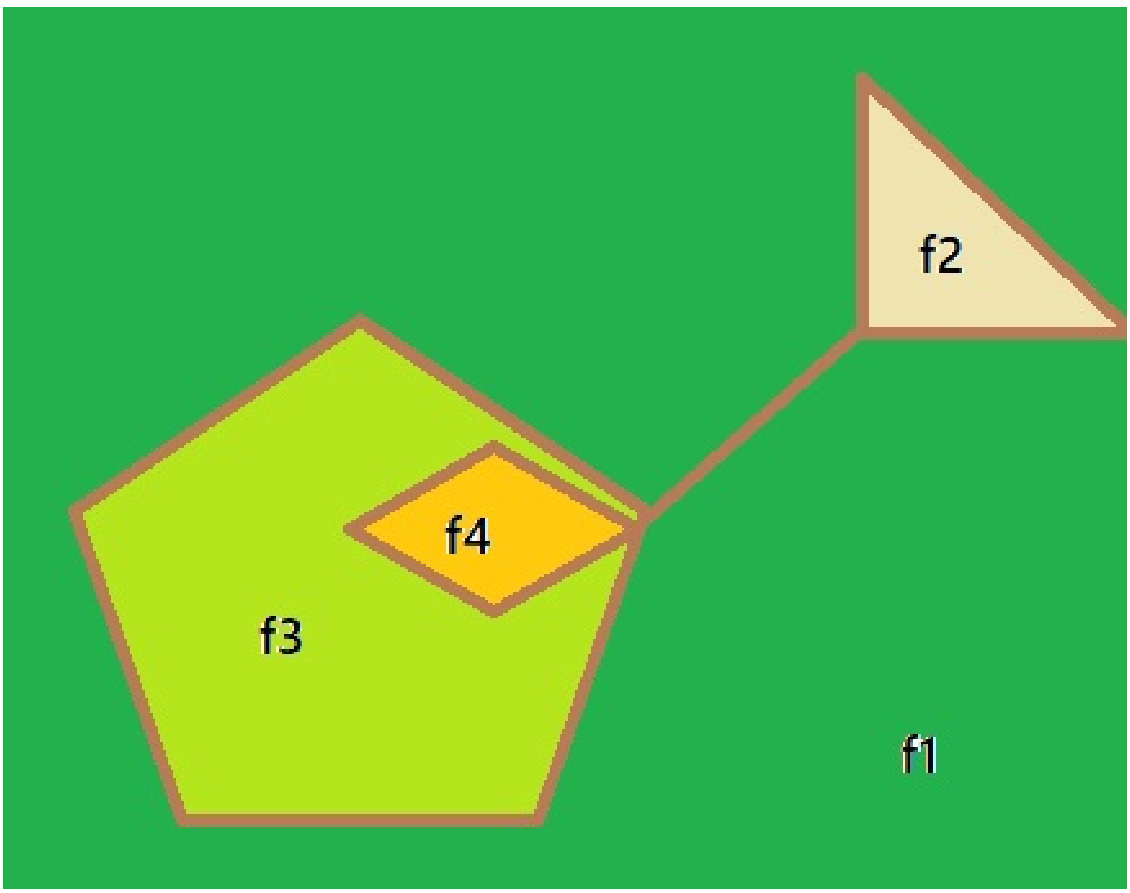

Figure 3: Example of faces