Binomial Theorem: For every $m \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, ...\}$ and every x

$$(1+x)^m = \sum_{k=0}^{m} \binom{m}{k} x^k$$

Proof:

$$\begin{aligned}(1+x)^m &= (1+x)(1+x)....(1+x) \text{ - (m factors)} \\ &= (x^0+x^1)(x^0+x^1)...(x^0+x^1) \\ &= x^{0+0+0+0....} + x^{1+0+0....} + x^{0+1+0....} + ... + x^{1+1+1+...}\end{aligned}$$

The set of exponents is:

$$\{e_1 + e_2 + ... + e_m : (e_1, e_2, ..., e_m) \in \{0,1\}^m\}$$

Our bijection f from last class maps the set of all subsets of $\{1, ..., m\}$ to $T = \{(e_1, e_2, ..., e_m) \in \{0,1\}^m\}$

The number of terms in which the exponent adds up to exactly k is the number of elements of T with exactly k 1s. By our bijection, we know that this is the number of k-subsets of $\{1, ..., m\}$. So it is $\binom{m}{k}$

So when we collect like terms in $(1+x)^m$, the coefficient of x^k is $\binom{m}{k}$

Hence

$$(1+x)^m = \sum_{k=0}^{m} \binom{m}{k} x^k$$

□

Corollary:

$$2^m = (1+1)^m = \sum_{k=0}^{m} \binom{m}{k}$$

This was the idea of a *combinatorial proof* - describing numbers as sizes of sets in different ways.

Give a *combinatorial proof* that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

for all integers $1 \leq k \leq n-1$.

Proof: We have $\binom{n}{k}$ is the size of the set S of all k-subsets of $\{1, 2, ..., n\}$

We can write $S = S_0 \cup S_1$ where $S_0 \cap S_1 = \emptyset$ where:

- $S_0 =$ the set of all k-subsets of $\{1, ..., n\}$ that do NOT contain the last element n.
- $S_1 =$ the set of all k-subsets of $\{1, ..., n\}$ that DO contain the last element n.

Then S_0 is the set of k-subsets of $\{1, 2, ..., n-1\}$, therefore

$$|S_0| = \binom{n-1}{k}$$

There is a bijection from S, to the set of all (—k-1—-subsets) of $\{1, ..., n-1\}$ obtained by removing the element n. Thus

$$|S_1| = \binom{n-1}{k-1}$$

Hence $\binom{n}{k} = |S| = |S_0| + |S_1| = \binom{n-1}{k} + \binom{n-1}{k-1}$ $\square$

Theorem: For $n, k \in \mathbb{Z}_{\geq 0}$ we have:

$$\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$$

Proof: Let S be the set of all n-subsets of $\{1, 2, ..., n+k\}$. Then $|S| = \binom{n+k}{n}$

Let S_i be the set of all n-subsets of $\{1, ..., n+k\}$ whose largest element is $n+i$. Do this for $0 \leq i \leq k$.

Then $S = S_0 \cup S_1 \cup ... \cup S_k$ and this is a disjoint union (since any n-ubset has a unique largest element).

For each i each element of S_i is of the form $A \cup \{n+i\}$. Where A is a (n-1)-subset of $\{1, ..., n+i-1\}$, since $n+i$ is the *largest* element in σ.

Conversely, every (n - 1)-subset A of $\{1, 2, ..., n+i-1\}$ together with $n+i$ gives an element of S_i. So we get a bijection from S_i to the set of (n - 1)-subsets of $\{1, ..., n+i-1\}$ obtained by removing $n+i$.

Hence

$$|S_i| = \binom{n+i-1}{n-1}$$

Thus

$$\binom{n+k}{n} = |S| = \sum_{i=0}^{k} |S_i| = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$$

$\square$

For a set S, a *weight function* on S is

- a function $w : S \to \mathbb{Z}_{\geq 0}$
- for each $n \in \mathbb{Z}_{\geq 0}$, the number of elements $\sigma \in S$ with $w(\sigma) = n$ is finite

e.g.

$S =$ the set of all subsets of $\{1, 2, ..., m\}$ and $w : S \to \mathbb{Z}_{\geq 0}$ is defined by $w(\sigma) = |\sigma|$.