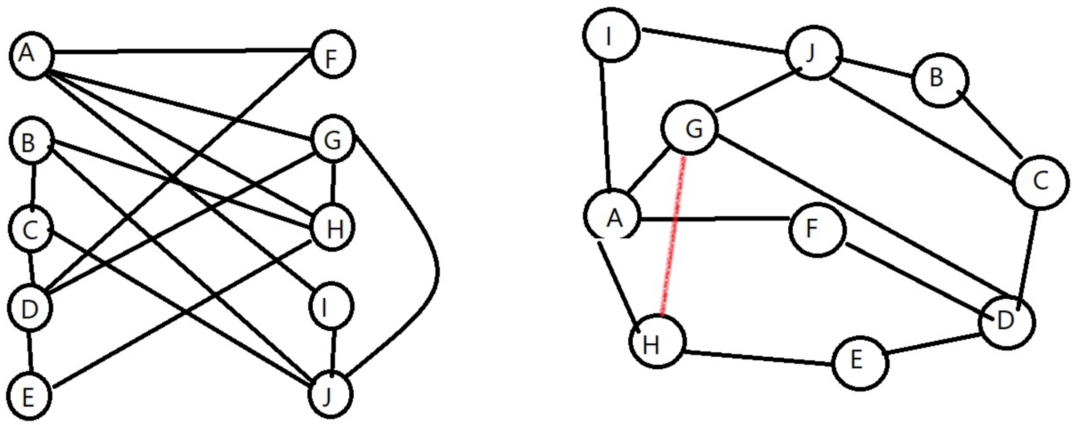

Figure 1: Planar Graph Attempt

... black out so some of this was hard to see ...

Start with the graph on the left of Figure 1.
We try to create a planar embedding but fail (see the right of Figure 1)

We now try to find a subdivision of $K_{5,5}$ - which must have 5 vertices of degree 4.
See Figure 2 (incomplete)

Definition A *colouring* of a graph G is a function $f : G \to \{1, 2, ...\}$ such that for each edge $xy \in E(G)$ we have $f(x) \neq f(y)$.

We say that f is a *k-colouring* if $f : G \to \{1, 2, ..., k\}$ and that G is *k-colourable* if there exists a k-colouring of G.

See Figure 3 for a 3-colourable graph.

Note that if G has p vertices then it is p-colourable.

Note that if G is 1-colourable then $E(G) = \varnothing$.

Note that if G is 2-colourable then G is bipartite.

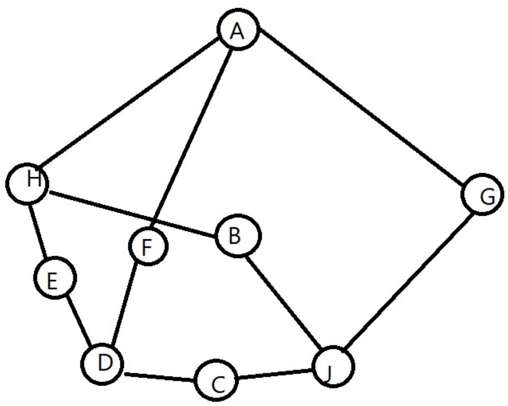

Figure 2: $K_{5,5}$ subdivision of the graph (incomplete)

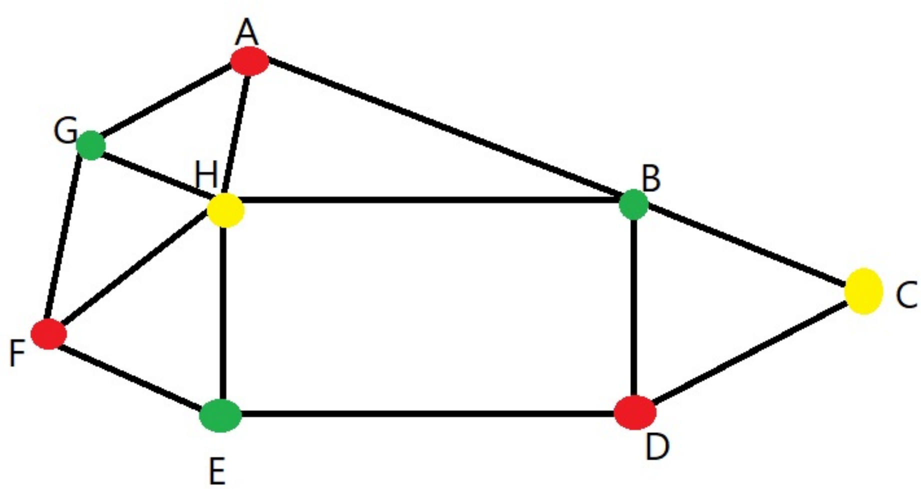

Figure 3: $3 - \mathit{colouring}$ graph

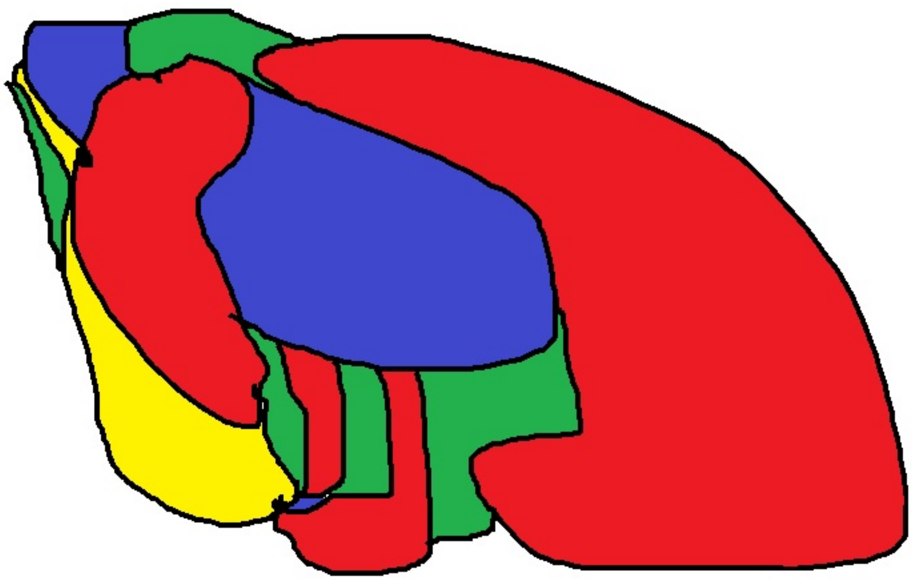

Figure 4: $4-colouring$ graph

Theorem Every planar graph is 6-colourable.

Proof Let G be a planar graph with p-vertices. We use induction on p.

Base case: $p \leq 6$. Since $|V(G)| \leq 6$ we can use a different colour for each vertex, and use ≤ 6 colours.

Inductive Hypothesis Assume $p \geq 7$ and that every planar graph with fewer than p vertices is $6-colourable$.

Inductive Step Let G be a planar graph with p vertices. We proved that every planar graph has a vertex v of degree ≤ 5.

Let $G' = G - v$ be obtained by removing v.

G' would still be planar, now with $p-1$ vertices. Then by *Induction Hypothesis* G' has a 6-colouring f.

Then on the neighbours of v, at most 5 colours out of $\{1, 2, 3, 4, 5, 6\}$ are used by f. Hence we can extend f to a colouring of G by giving v a colouring not appearing on any neighbour.

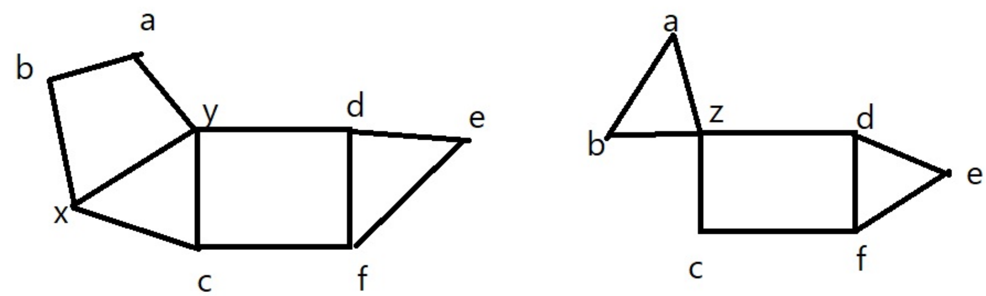

Figure 5: Visualization of contracting

Note: every planar graph is $4-colourable$ - "4-colour theorem"

Otherwise known as - every geographical map can be coloured with 4 colours such that no two countries sharing a border get the same colour.

See Figure 4.

Proof of 4-colouring theorem is beyond the scope of this course, but we will cover the 5-colourable theorem.

Definition Let G be a graph and let $e = xy$ be an edge of G. The graph G/e formed by *contracting* e has vertex set $V(G) \setminus \{x, y\} \cup \{z\}$ and the edge set $\{uv \in E(G) : \{u, v\} \cap \{x, y\} = \varnothing\} \cup \{uz : u \notin \{x, y, z\} \wedge ux \in E(G) \vee uy \in E(G)\}$
See Figure 5