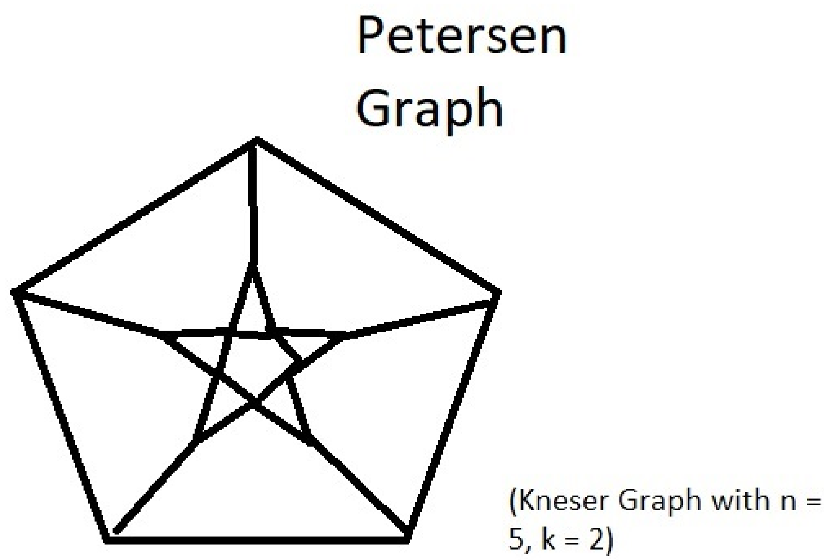

Figure 1: Sample Kneser Graph (Petersen Graph)

Example let $n \geq k$ be positive integers. The *Kneser graph* with parameters (n, k) has vertex set: $\{S \subseteq \{1, ..., n\} : |S| = k\}$ and edge set $\{ST : S \cap T = \varnothing\}$

In general the graph has $\binom{n}{k}$ vertices and it is regular of degree $\binom{n-k}{k}$

Hence it has $\frac{1}{2}\sum_{v \in V} deg(v) = \frac{1}{2}\binom{n}{k}\binom{n-k}{k}$ edges.

Note that if $n < 2k$ then this graph has 0 edges. Can't have two disjoint subsets if you're selecting more than half the elements in each subset.

See Figure 1 for the Petersen Graph - famous counter example.

Definition A *walk* in a graph G from a vertex x to a vertex y is a sequence $w = v_0 e_1 v_1 ... e_k v_k$ of vertices v_i and edges e_i such that $v_0 = x$ and $v_k = y$ and $e_i = v_{i-1}v_i$ for each i.

Here k is the *length* of w (the number of edges). Note $k = 0$ is possible.

(No restriction on revisitng edges and vertices).

Often we specify a walk just with the sequence $v_0 ... v_k$ of vertices since the edges are determined by the vertices.

Definition A *path* is a walk in which no vertex (and hence no edge) is repeated.

Definition A *subgraph* of a graph G is a graph H with vertex set $V(H) \subseteq V(G)$ and $E(H) \subseteq \{xy, \in E(G) : x, y \in V(H)\}$

Example Any path $v_0 v_1 ... v_k$ in a graph G determines a subgraph with vertex set $\{v_0, ..., v_k\}$ and edge set $\{v_{i-1} v_i : 1 \leq i \leq k\}$

Definition A *cycle* in a graph G is a subgraph with vertex set $\{v_1, ..., v_k\}$ (where $k \geq 3$) and edge set $\{v_{i-1} v_i : 1 \leq i \leq k\} \cup \{v_k v_1\}$.

Again k is the *length* of the cycle which is the number of edges and the number of vrtices. Write C_k for the cycle of length k.

Question How to show two graphs G and H are **NOT** isomorphic?

Things to try:

- Do they have the same number of vertices? (No $\rightarrow$ No. Yes $\rightarrow$ Maybe)
- Do they have the same number of edges? (No $\rightarrow$ No. Yes $\rightarrow$ Maybe)
- Do they have the same degree sequence? (No $\rightarrow$ No. Yes $\rightarrow$ Maybe)
- Do they have the same cycle lengths? (No $\rightarrow$ No. Yes $\rightarrow$ Maybe)
- Do they have the same (small) subgraphs? (No $\rightarrow$ No. Yes $\rightarrow$ Maybe)

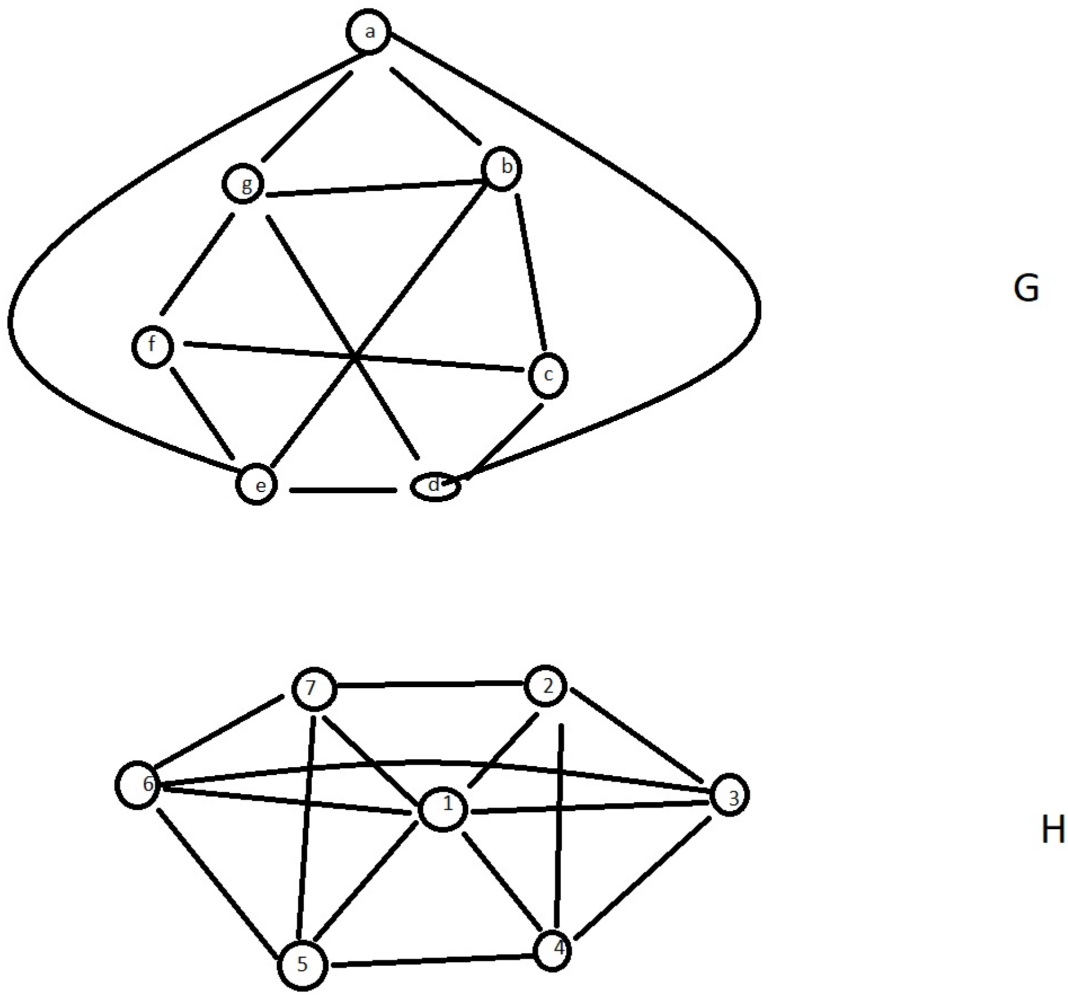

Figure 2: Sample Graph Comparison for Isomorphism

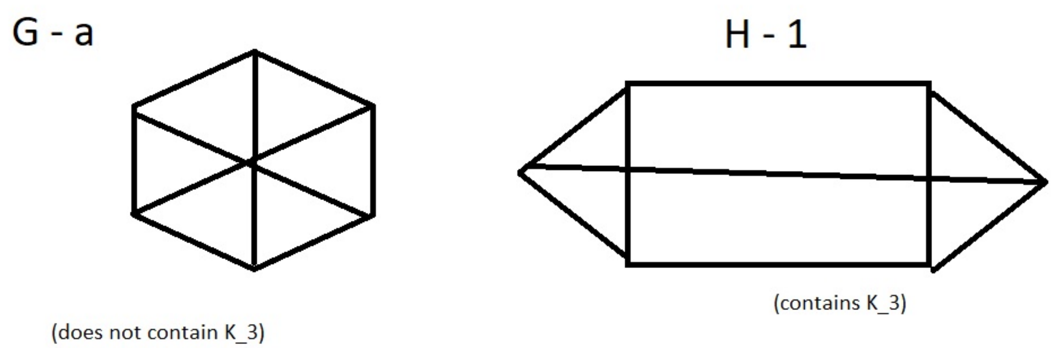

Figure 3: Sample Subgraph Comparison for Isomorphism

Question How to show two graphs G and H are **NOT** isomorphic? (See Figure 2)

Things to try:

- Do they have the same number of vertices? Yes
- Do they have the same number of edges? Yes
- Do they have the same degree sequence?
  - G: 6, 4, 4, 4, 4, 4, 4
  - H: 6, 4, 4, 4, 4, 4, 4

  Yes
- Do they have the same cycle lengths?
  - G: 3, 4, 5, 6, 7
  - H: 3, 4, 5, 6, 7

  Yes
- Do they have the same (small) subgraphs? No! H contains K_4 as a subgraph, in fact G does not! so they are not isomorphic!
- Another Criterion: IF we had an isomorphism from G to H, it would have to map a to 1. Then G with the vertex a removed would have to be isomorphic to H with the vertex 1 removed!

  Formally: $G - a$ (obtained by removing a and all its incident edges from G) would be isomporhic to $H - 1$. ($H - 1$ contains K_3, $G - a$ does not. See Figure 3).