Corollary The complete graph K_5 with 5 vertices is not planar.

Proof $|E(K_5)| = 10$ but $p = 5$ and so $3p - 6 = 9$
So by our previous theorem K_5 is NOT planar $\square$

Definition The *girth* of a graph G is the length of the shortest cycle in G. (Undefined or infinite if G does not have a cycle.)

Corollary Let G be a graph with at least one cycle and girth at least k. If G is planar then

$$q \leq \frac{k}{k-2}(p-2)$$

Proof Follow the proof of our $q \leq 3p - 6$ theorem to get:

$$2q = \sum_{v \in F(\tilde{G})} deg(v) \geq ks$$

by using $s = 2 - p + q$

from Euler's Formula given:

$$2q \geq k(2 - p + q)$$

then rearrange:

$$q \leq \frac{k}{k-2}(p-2)$$

Corollary $K_{3,3}$ is not planar

Proof Since $K_{3,3}$ is bipartite, it has no odd cycles.

So its girth is $k \geq 4$

Hence since $|E(K_{3,3})| = 9$ but $\frac{k}{k-2}(p-2) = 8$

Therefore $K_{3,3}$ is NOT planar $\square$.

Definition Let H be a graph. An (edge) *subdivision* of H is any graph J obtained by replacing each edge $e = xy$ of H by a path P_e such that each vertex in

$$\bigcup_{e=xy\in E(H)} (V(P_e) \setminus \{x, y\})$$

has degree 2 in J.

Note that H is planar if and only if every subdivision of H is planar.

Theorem(Kuratowski's Theorem) A graph G is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Proof ($\Longrightarrow$) As we just showed since K_5 and $K_{3,3}$ are not planar.

How to tell if a graph is planar?

1. Try easy tests ($q \leq 3p - 6$ and $q \leq \frac{k}{k-2}(p-2)$ if girth ≥ 3)
2. Can you find a planar drawing?
3. Can you find a subdivision of K_5
4. Can you find a subdividion of $K_{3,3}$