Formal Power Series

Def: a *formal power series* over the rational numbers (doesn't need to be the rationals), in the indeterminante x is an expression of the following form:

$$A(x) = \sum_{i \geq 0} a_i x^i$$

where $\forall i, a_i \in \mathbb{Q}$
Generalization of an infinite polynomial.
What can you do? Add, subtract, multiply and divide!

Addition of Formal Power Series
For $A(x) = \sum_{i\geq 0} a_i x^i$ and $B(x) = \sum_{i \geq 0} b_i x^i$:

$$A(x) \pm B(x) = \sum_{i\geq 0} (a_i \pm b_i) x^i$$

Multiplication We define $A(x)B(x)$ by:

$$A(x)B(x) = \sum_{i\geq 0} (\sum_{j \geq 0} a_j b_{i-j}) x^i$$

e.g. Must generalize behaviour! (also note symmetric)

$$(a_0 + a_1 x + a_2 x^2 + ...)(b_0 + b_1 x + b_2 x^2 + ...)$$
$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + ...$$

Equality: $A(x) = B(x)$ means $a_i = b_i, \forall i \geq 0$

Coefficients: a_i is called the *coefficient* of x^i in $A(x)$. We write $a_i = [x^i]A(x)$.

We say $B(x)$ is the *multiplicative inverse* of $A(x)$ if $A(x)B(x) = 1$.

We write $B(x) = A^{-1}(x)$ or $B(x) = \frac{1}{A(x)}$

Note that not every formal power series has a multiplicative inverse.

example of inverse
Let $A(x) = 1 + x + x^2 + x^3 + ... = \sum_{i \geq 0} x^i$. We prove that the inverse is $B(x) = 1 - x$.

Check:

$$\begin{aligned}
&(1-x)(1+x+x^2+x^3+...)\\
&= 1 + x - x + x^2 - x^2 + x^3 - x^3 ...\\
&= 1
\end{aligned}$$

We often write $1 + x + x^2 + ... = \frac{1}{1-x}$

Composition (or Substitution) of Formal Power Series:
If $B(x)$ has constant coefficient $b_0 = 0$ then $A(B(x))$ is a formal power series.

Check:

We can write $B(x) = xC(x)$ where $C(x) = b_1 + b_2 x + b_3 x^2 + ...$

Then $A(B(x)) = a_0 + a_1 x C(x) + a_2 x^2 C^2(x) + ...$

We need to verify that for every n the coefficient $[x^n]A(B(x))$ of x^n is a **finite number**!

But $[x^n]A(B(x)) = [x^n](a_0 + a_1 x C(x) + ... + a_n x^n C^n(x))$ (why though?)

Why can we cut off a potentially infinite series after the nth term?

Note all the powers after that point are all strictly larger than n so they don't need to be considered. (Won't influence the coefficient of x^n further)

This *is* a finite number since we have a **finite** sum of products of formal power series. Hence, $A(B(x))$ is a formal power series.

What goes wrong when $b_0 \neq 0$?

$$A(B(x)) = a_0 + a_1 B(x) + a_2 B^2(x) + a_3 B^3(x) + ...$$

Note that we are forced to use $B(x)$ each time, since there is no simplification for the formal power series. It is possible for $B(x)$ to be an infinite series. We can't "cut off" the series like last time because $B^k(x)$ still has some x^n terms regardless of k because of the definition of multiplication.

Theorem: A formal power series $A(x)$ has an inverse if and only if the constant coefficient $a_0 \neq 0$.

Proof: Suppose $C(x)$ is the inverse of $A(x)$. Then $A(x)C(x) = 1$. This implies that the constant coefficient of $A(x)C(x)$ is $a_0c_0 = 1$ Hence $a_0 \neq 1$. $\square$

Suppose $a_0 \neq 0$. Then we can write:

$$A(x) = a_0(1 - xB(x))$$

where

$$B(x) = \frac{-a_1}{a_0} + \frac{-a_2}{a_0}x + ...$$

So $1 - xB(x)$ is $xB(x)$ substituted into $1 - x$, and we know $1 - x$ has an inverse:

$$\sum_{i \geq 0} x^i$$

Hence $1 - xB(x)$ has inverse:

$$\sum_{i \geq 0} (xB(x))^i$$

So $A(x)$ has an inverse:

$$\frac{1}{a_0} \sum_{i \geq 0} (xB(x))^i$$

Example

$$1 + x^2 + x^4 + ... = \frac{1}{1 - x^2}$$

Exercise: show if a formal power series has an inverse it is unique

Theorem (Negative Binomial Theorem) Let m be a positive integer. Then

$$(1-x)^{-m} = \sum_{k\geq 0} \binom{m+k-1}{m-1} x^k$$

Proof By induction on m.

When $m = 1$, left hand side $= (x-1)^{-1}$.

When $m = 1$, right hand side $= \binom{m+k-1}{m-1} = \binom{k}{0} = 1$
So RHS $= \sum_{k\geq 0} x^k = (1-x)^{-1}$ as required.(*Recall our earlier proof on the inverse being a formal power series*)

Inductive Hypothesis Assume $m \geq 1$ and that $(1-x)^{-m} = \sum_{k\geq 0} \binom{m+k-1}{m-1} x^k$

Consider $m+1$.

$$(1-x)^{-(m+1)} = (1-x)^{-1}(1-x)^{-m}$$

from the inductive hypothesis...

$$= \left(\sum_{k\geq 0} x^k\right)\left(\sum_{k\geq 0} \binom{m+k-1}{m-1} x^k\right)$$

from the definition of multiplication of formal power series...

$$= \sum_{k\geq 0} \left(\sum_{j\geq 0}^{k} \binom{m+j-1}{m-1}\right) x^k$$

Last week we proved

$$\sum_{j\geq 0}^{k} \binom{m+j-1}{m-1} = \binom{m+k}{m}$$

(*Recall the bijection of the largest element being $m+j$ for each subset*)

So

$$(1-x)^{-(m+1)} = \sum_{k\geq 0} \binom{m+k}{m} x^k$$

as required.

So by induction the statement holders. $\square$

Theorem (Finite Geometric Series) Let k be a non-negative integer. Then

$$1 + x + x^2 + ... + x^k = \frac{1 - x^{k+1}}{1 - x}$$

Proof

$$\begin{aligned}&(1-x)(1 + x + x^2 + ... + x^k)\\ &= 1 - x + x - x^2 + x^2 - .... - x^k + x^k - x^{k+1}\\ &= \frac{1 - x^{k+1}}{1 - x}\end{aligned}$$

As required. $\square$

Your Toolbox for finding coefficients in formal power series

- $(1-x)^{-m} = \sum_{k \geq 0} \binom{m+k-1}{m-1} x^k$
- $(1-x)^{-1} = \sum_{k \geq 0} x^k$
- Finite Geometric Series
- Binomial Theorem