Theorem Let G be a graph and let T be a spanning tree in G. let $e \in E(G)$ such that $e \notin E(T)$.

- the graph $T+e$ obtained by adding the edge e to T contains exactly one cycle C, which contains e
- for each $f \in E(C) \setminus \{e\}$ the graph $(T+e)-f$ is also a spanning tree of G.

Proof

Since T has no cycles, all cycles in $T+e$ contain e. Let $e = xy$. By an earlier lema, there is exactly one path p in T from x to y. Hence $C = P + e$ is the only cycle in $T+e$.

By our earlier theorem, ssince f is in the cycle C in $T+e$ it is not a bridge of $T+e$. Thus $(T+e)-f$ is connected.

But, $|V((T+e)-f)| = |V(T)|$ and $|E((T+e)-f)| = |E(T)| = |V(T)| - 1$

Thus, by yet another earlier theorem, $T+e-f$ is a tree, hence a spanning tree of G.

Theorem Let G be a graph and let Y be a spanning tree in G. Let $e \in E(T)$. say $e = xy$. Then:

- $T-e$ has exactly two components, T_x containing x and T_y containing y
- for each edge f in the cut S in G induced by T_x, $(T-e)+f$ is a spanning tree of G.

Proof

- Follows from our earlier lemma since e is a bridge of T (its removal leaves exactly components)
- Let $f = uv$ be an edge of $S \setminus \{e\}$ where $u \in T_x$ and $v \in T_y$. Since $v \in T_y$, all vertices in T_y are joined by a path in T_y to v.

  Each vertex in T_x is joined by a path P to v in $(T-e)+f$, where the last edge of P is uv.

  Hence $(T-e)+f$ is connected and spanning.

  But $|E(T-e+f)| = |E(T)|$ so $T-e+f$ is a spanning tree of G.

Minimum Spanning Tree

Let G be a graph and let $w : E(G) \to \mathbb{R}$ be a weight function on $E(G)$.

Definition A *minimum spanning tree* (MST) of G is a spanning tree T such that $w(T) = \sum_{e \in E(T)} w(e)$ is as small as possible (among all spanning trees).

Prim's Algorithm for MST
INPUT: Graph G and vertex $x \in V(G)$
OUTPUT: A MST of the component G_x containing x.

1. Set $V(D) = x$ and $E(D) = \varnothing$
2. If the cut S in G induced by $V(D)$ in nonempty:
   - let $e = uv$ be an edge in S of smallest weight where $u \notin V(D)$ and $v \in V(D)$.
   - Set $V(D) = V(D) \cup \{u\}$
   - Set $E(D) = E(D) \cup \{e\}$
3. If $S = \varnothing$ STOP and OUTPUT D.

Theorem Prim's Algorithm finds a MST of G_x

Proof Since this is a special case of our spanning tree algorithm, we know it finds a spanning tree D of G_x.

Let D_i denote the current subgraph D at step i of the algorithm. So D_1 has $V(D_1) = \{x\}$ and $E(D_1) = \varnothing$

We will prove the following claim by induction on i.

Claim D_i is contained in some MST of G.

This will prove the theorem, since the final output (say D_k, where k is teh total number of steps taken) is contained in a MST of G, but it is a spanning tree of G and therefore must be an MST itself.