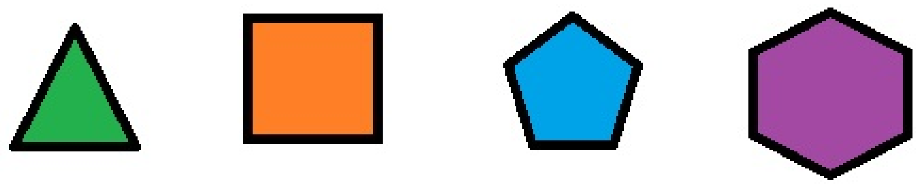

Figure 1: Graph Cycles

Cycles

For $n \geq 3$ let C_n be the graph with vertices $V(C_n) = \{0, ..., n-1\}$ and edges $E(C_n) = \{\{i-1, i\} : \forall i, 1 \leq i \leq n-1\} \cup \{\{0, n-1\}\}$.

An *n-cycle* is any graph isomorphic with C_n.

See Figure 1 - imagine vertices are labelled numerically.

Bipartite

A graph $G = (V, E)$ is *bipartite* if one can partition the vertices $V = A \cup B$ such that:

- $A \cap B = \varnothing$
- every edge e has one end in A and one end in B

See Figure 2.

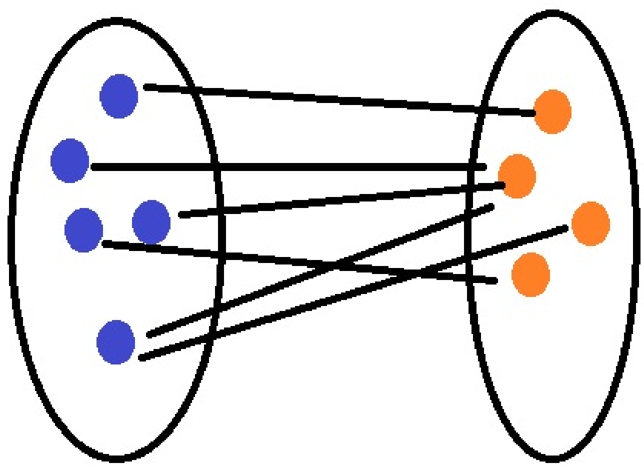

Figure 2: Bipartite Graph

Complete Bipartite Graphs

For $a, b \geq 1$, $K_{a,b}$ is the graph with $V = A \cup B$

- $A = \{v_1, v_2, ..., v_a\}$
- $B = \{w_1, w_2, ..., w_b\}$

and edges $E = \{\{v_i, w_j\} : \forall i, j, 1 \leq i \leq a, 1 \leq j \leq b\}$ or anything isomorphic to this.

Claim $K_{a,b} \cong K_{b,a}$

By $\phi(v_i) = w_i$ and $\phi(w_j) = v_j$ this is isomorphic.

If $a \leq 2$ (or $b \leq 2$)) then $K_{a,b}$ can be drawn without crossing edges.

No matter how you draw $K_{3,3}$ you must have a pair of crossing edges.
(See Figure 3)

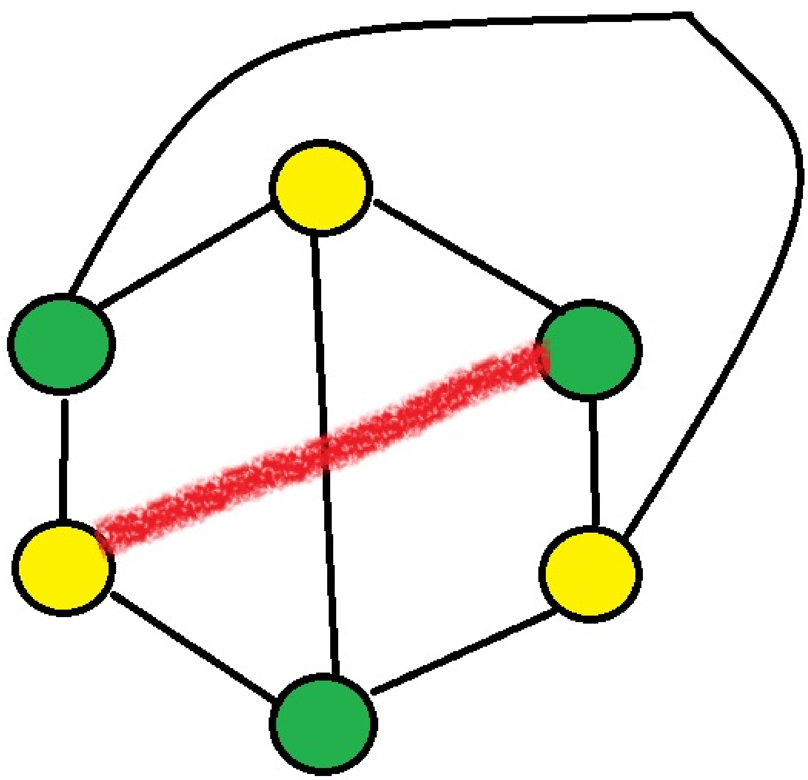

Figure 3: $K_{3,3}$ Graph

Q Which cycles are bipartite?

Proposition For $n \geq 3$ C_n is bipartite if and only if n is even.

Proof

Assume (A, B) is a bipartition of C_n.

Without loss of generality may assume that 0 is in A. For $1 \leq i \leq n-1$, if $i-1 \in A$ then $i \in B$, and if $i-1 \in B$ then $i \in A$ since (A, B) is a bipartition.

This means $A = \{0, 2, 4, ...\}$ (is the set of even numbers) in $V = \{0, ..., n-1\}$ and therefoer $B = \{1, 3, 5, ...\}$ is the set of odd numbers in V.

So if n is even then $n-1$ is odd, so $n-1 \in B$ and $\{0, n-1\}$ has one end in A and one end in B.

So (A, B) is a bipartition if n is even. ☺

However, if n is odd then $n-1$ is even so $n-1 \in A$ and $\{0, n-1\}$ has both ends in A is not a bipartition. This is contradiction shows that there is no bipartition for an odd cycle. ☹

Lemma If G is bipartite and H is a subgraph of G, then H is bipartite.
Corollary Any graph that contains an odd cycle is not bipartite.