For a set S, a *weight function* on S is

- a function $w : S \to \mathbb{Z}_{\geq 0}$
- for each $n \in \mathbb{Z}_{\geq 0}$, the number of elements $\sigma \in S$ with $w(\sigma) = n$ is finite

e.g.

$S =$ the set of all subsets of $\{1, 2, ..., m\}$ and $w : S \to \mathbb{Z}_{\geq 0}$ is defined by $w(\sigma) = |\sigma|$.

i.e

Gives every element of S some non-negative integer weight.

Definition: let S be a set and let w be a weight function on S. The *generating series* for S with respect to w, in the indeterminate x, is:

$$\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)}$$

We can collect like terms and write:

$$\Phi_S(x) = \sum_{k \geq 0} a_k x^k$$

where a_k is the number of elements of S of weight k

(Note that a_k is also a non-negative integer by the definition of weight function!)

E.g: $S =$ the set of all subsets of $\{1, 2, ..., m\}$ and $w : S \to \mathbb{Z}_{\geq 0}$ is defined by $w(\sigma) = |\sigma|$.

Then

$$\Phi_S(x) = \sum_{k \geq 0} a_k x^k = \sum_{k \geq 0} \binom{m}{k} x^k = \sum_{k \geq 0}^{m} \binom{m}{k} x^k v = (1 + x)^m$$

Note that any choose where $k > m$ is 0.
Note end conclusion from Binomial Theorem.

Q** If a set is infinite then something must be infinite - one way or another.

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e.g. $S = \{1, 2, 3, ..., 20\}$. Define $w(\sigma)$ to be the number of digits (in base 10) of σ.
Then,

$$\Phi_S(x) = 0x^0 + 9x^1 + 11x^2 + 0x^3 + ...$$

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e.g. $S = \mathbb{Z}_{\geq 0}$ where $w(\sigma) = \sigma$.

Then,

$$\Phi_S(x) = 1x^0 + 1x^1 + 1x^2 + 1x^3 + ... = \sum_{i=0} x^i$$

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e.g. $S = \mathbb{Z}_{\geq 0}^{even}$ where $w(\sigma) = \sigma$.

Then,

$$\Phi_S(x) = 1x^0 + 0x^1 + 1x^2 + 0x^3 + 1x^4 + ... = \sum_{i=0} x^{2i}$$

Note: $-x$ is an *indeterminate*, i.e. a symbol that satisfies:

$$x^0 = 1$$

and

$$x^a x^b = x^{a+b}$$

and

$$(x^a)^b = x^{ab}$$

Normally (unless S is finite), we do not substitute numbers for x.

If S is infinite, $\Phi_S(x)$ is an infinite sum. This will be a *formal power series* (more later)

Theorem: Let S be a finite set with weight function w. Then $\Phi_S(x)$ is a polynomial. Then,

- $\Phi_S(1) = |S|$
- $\frac{d}{dx}\Phi_S(1)$ is the total weight of all elements in S

Proof:

$$\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{\sigma \in S} 1^{w(\sigma)} = |S|$$

and

$$\frac{d}{dx}\Phi_S(x) = \sum_{\sigma \in S} w(\sigma) x^{w(\sigma)-1}$$

plugging in 1...

$$\frac{d}{dx}\Phi_S(1) = \sum_{\sigma \in S} w(\sigma) 1^{w(\sigma)-1} = \sum_{\sigma \in S} w(\sigma)$$

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e.g. When S = set of all subsets of $\{1, ..., m\}$ where $w(\sigma) = |\sigma|$.
Note that

$$\Phi_S(x) = \sum_{i \geq 0} a_i x^i = \sum_{i \geq 0} \binom{m}{i} x^i = (1+x)^m$$

Plugging in 1,

$$\Phi_S(1) = (1+1)^m = 2^m$$

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e.g. S is the set consisting of 5 nickels, 12 dimes, and 3 quarters. Let $w(\sigma)$ = value of σ in cents

$$\Phi_S(x) = 5x^5 + 12x^10 + 3x^25$$

$$\frac{d}{dx}\Phi_S(x) = 25x^4 + 120x^9 + 75x^24$$

Identify:
$\Phi_S(1) = 5 + 12 + 3 = 20$, the number of coins.

$\frac{d}{dx}\Phi_S(x) = 25 + 120 + 75 = 220$, the total value of the coins.

Formal Power Series

Def: a *formal power series* over the rational numbers (doesn't need to be the rationals), in the indeterminante x is an expression of the following form:

$$A(x) = \sum_{i \geq 0} a_i x^i$$

where $\forall i, a_i \in \mathbb{Q}$
Generalization of an infinite polynomial.
What can you do? Add, subtract, multiply and divide!

Addition of Formal Power Series
For $A(x) = \sum_{i \geq 0} a_i x^i$ and $B(x) = \sum_{i \geq 0} b_i x^i$:

$$A(x) \pm B(x) = \sum_{i \geq 0} (a_i \pm b_i) x^i$$

Multiplication We define $A(x)B(x)$ by:

$$A(x)B(x) = \sum_{i \geq 0} (\sum_{j \geq 0} a_j b_{i-j}) x^i$$

e.g. Must generalize behaviour! (also note symmetric)

$$(a_0 + a_1 x + a_2 x^2 + ...)(b_0 + b_1 x + b_2 x^2 + ...)$$
$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + ...$$

Equality: $A(x) = B(x)$ means $a_i = b_i, \forall i \geq 0$

Coefficients: a_i is called the *coefficient* of x^i in $A(x)$. We write $a_i = [x^i]A(x)$.

We say $B(x)$ is the *multiplicative inverse* of $A(x)$ if $A(x)B(x) = 1$.