

# CSE512 Fall 2018 Machine Learning - Homework 1

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# 1 Question 1

1. The expectation:  $E(X) = \frac{N^2-1}{6N}$

The problem is discrete. There are  $N^2$  cases with equal probability, since  $X_1$  and  $X_2$  are independent and both are uniformly distributed.

When  $X_2 \leq X_1$ ,  $X = 0$ , so only need to consider cases all cases where  $X_2 > X_1$ .

$$P(X = 1) = \frac{N-1}{N^2}$$

$$P(X = 2) = \frac{N-2}{N^2}$$

$$P(X = i) = \frac{N-i}{N^2}$$

$$E(X) = \sum_{i=1}^N iP(i) = \sum_{i=1}^N i \cdot \frac{N-i}{N^2} = \frac{N^2-1}{6N}$$

2. The variance:  $\text{Var}(X) = \frac{2N^4-N^2-1}{36N^2}$

$$\text{Var}(X) = E(X^2) - E(X)^2$$

We already derived  $E(X)$ , so we only need to derive  $E(X^2)$ . We just need to slightly alter the process of computing  $E(X)$

$$P(X = 1^2) = \frac{N-1}{N^2}$$

$$P(X = 2^2) = \frac{N-2}{N^2}$$

$$P(X = i^2) = \frac{N-i}{N^2}$$

$$E(X^2) = \sum_{i=1}^N i^2 P(i) = \sum_{i=1}^N i^2 \cdot \frac{N-i}{N^2} = \frac{N^2-1}{12}$$

$$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{2N^4-N^2-1}{36N^2}$$

3. The covariance:  $\text{Cov}(X, X_1) = \frac{-N^2+1}{24}$

$$\text{Cov}(X, X_1) = E[(X - E(X)) \cdot (X_1 - E(X_1))] = E(X \cdot X_1) - E(X)E(X_1)$$

We already derived  $E(X)$ , and  $E(X_1) = \frac{N+1}{2}$ . So we need to derive  $E(X \cdot X_1)$

$$\begin{aligned} E(X \cdot X_1) &= \frac{1}{N^2} \sum_{X_1=1}^N \sum_{X_2=1}^N [\max(X_2 - X_1, 0) \cdot X_1] \\ &= \frac{1}{N^2} \sum_{X_1=1}^N \sum_{X_2=X_1+1}^N [(X_2 - X_1) \cdot X_1] + \frac{1}{N^2} \sum_{X_1=1}^N \sum_{X_2=1}^{X_1} 0 \\ &= \frac{1}{N^2} \sum_{X_1=1}^N \left[ \frac{(X_1 + 1 + N)(N - X_1)}{2} - X_1(N - X_1) \right] X_1 \\ &= \frac{1}{N^2} \sum_{X_1=1}^N [X_1^3 - (2N + 1)X_1^2 + (N^2 + N)X_1] \\ &= \frac{N^3 + 2N^2 - N - 2}{24N} \end{aligned}$$

$$\text{Cov}(X, X_1) = \text{Cov}(X, X_1) - E(X)E(X_1) = \frac{N^3+2N^2-N-2}{24N} - \frac{N^2-1}{6N} \cdot \frac{N+1}{2} = \frac{-N^2+1}{24}$$

## 2 Question 2

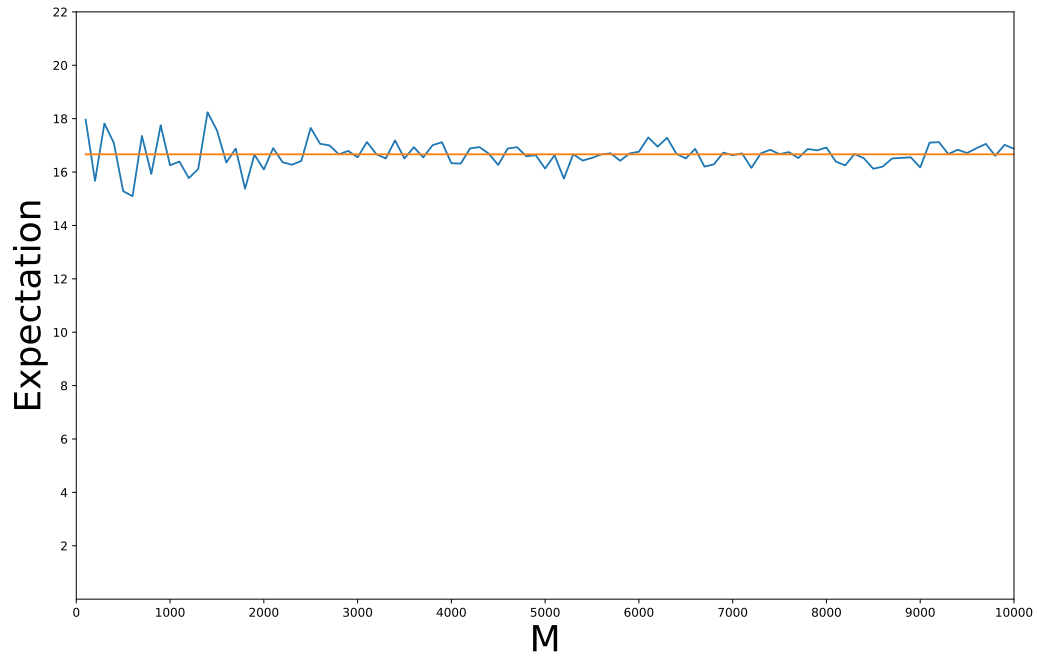


Figure 1: Expectation as a function of  $M$ . Theoretic  $E(X) = 16.665$

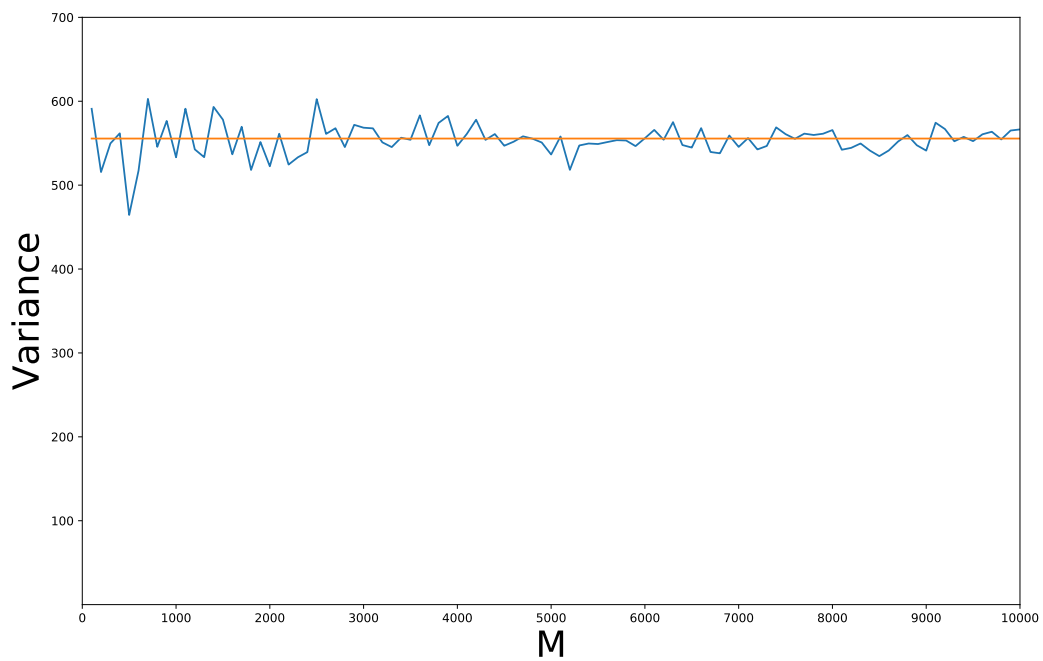


Figure 2: Variance as a function of  $M$ . Theoretic  $\text{Var}(X) = 555.527775$

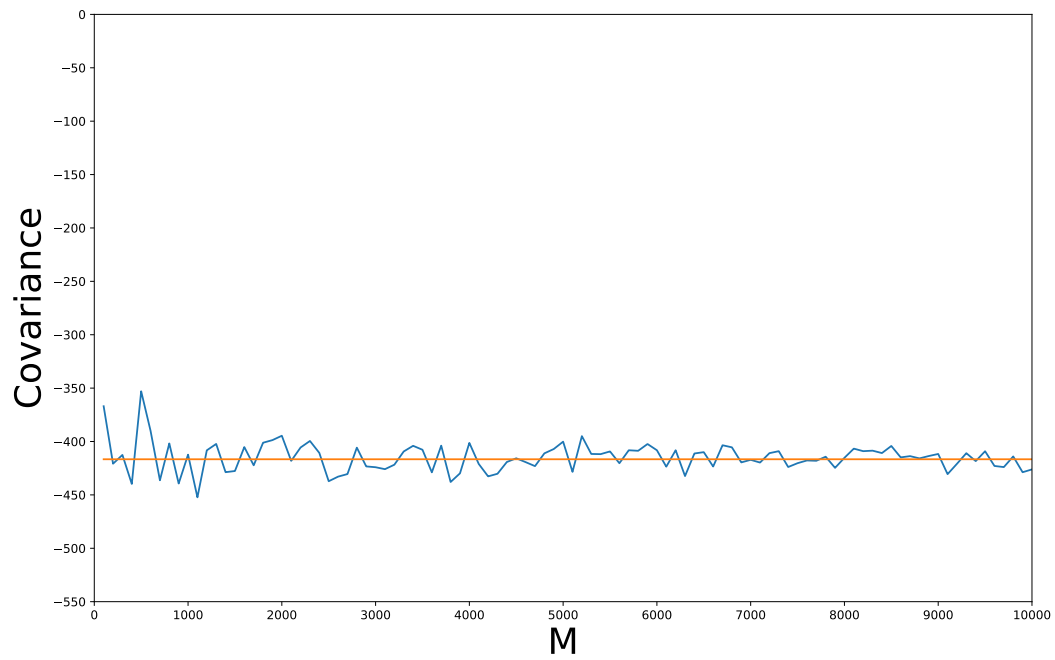


Figure 3: Variance as a function of M. Theoretic  $\text{Cov}(X) = -416.625$

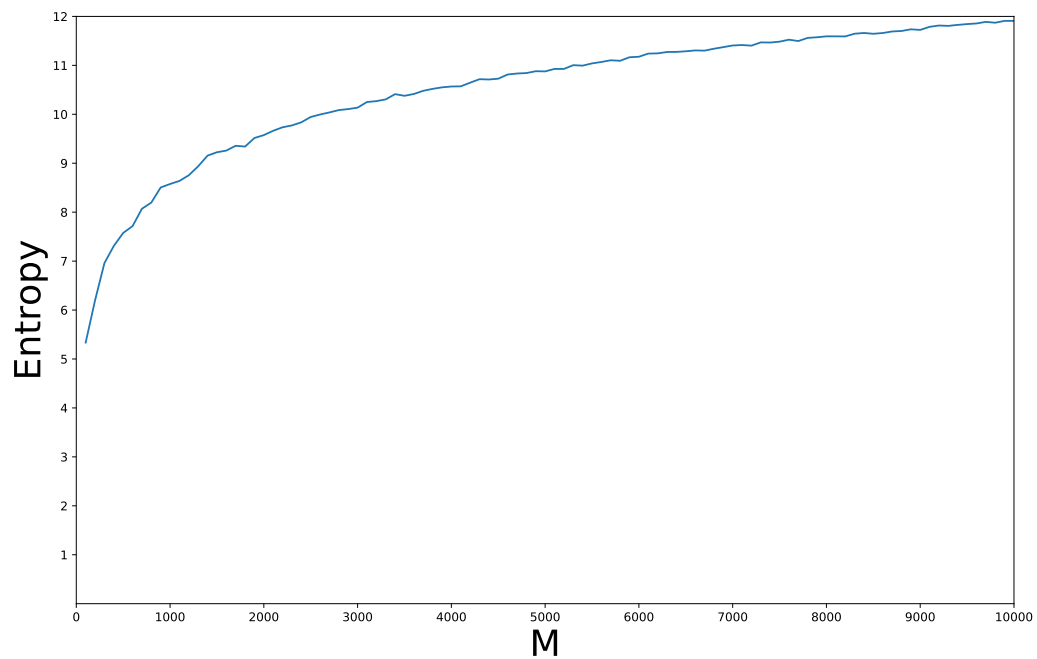


Figure 4: Entropy as a function of M.

Note: The python plotting script is in the jupyter notebook.