UM–SJTU JOINT INSTITUTE
PHYSICS LABORATORY
(VP141)

LABORATORY REPORT

EXERCISE 3

SIMPLE HARMONIC MOTION:
OSCILLATIONS IN MECHANICAL SYSTEMS

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Contents

1 INTRODUCTION

1.1 OBJECTIVES

In lab 3, our objective is to study the simple harmonic oscillation. First, we measure the spring constant and the effective mass of a spring. Then, we use air track to minimize the friction (external force). Finally, we study the relationship between oscillation period and the mass of the object. Also we find out whether the period is related to the amplitude of oscillation, and the dependence of maximum velocity on the amplitude of oscillation respectively.

1.2 THEORETICAL BACKGROUND

One of the simplest and commonest motions in our nature is the simple harmonic motion, whose restoring force is in proportional to the displacement from the equilibrium position, and its direction is often in the opposite direction to its displacement. After analysing its free-body diagram and obtain the equation of motion, we can get the solution (the relationship between displacement x and time t) is the sine or cosine function. Studying the simple harmonic oscillation is the foundation of learning more complicated situations.

1.2.1 HOOKE'S LAW

If the spring is in its elastic limit of deformation, then the force F_x needed to compress or stretch the spring to the certain distance x is in proportional to the very distance. The relation is called *Hooke's Law*, namely:

$$F_x = -kx \tag{1}$$

where k refers to the spring constant, which describe the difficulty for it to deform the spring. To be more specific, the larger the spring constant k is, the more difficult for the spring to be compressed or stretched. Also, according to the formula and Newton's third law of dynamics, we note that the direction of the force is always in the opposite direction to the displacement, trying to put the system back into its original state, which is also the reason why the force is called restoring force. In this lab, we use *Jolly balance* to measure the spring constant.

1.2.2 EQUATION OF MOTION OF THE SIMPLE HARMONIC OSCILLATOR

We start our analysis from the classical system "mass-spring system" shown in Figure 2. In the figure, an object with mass M is connected by two springs with spring constant k_1 and k_2 respectively, which are measured by the Jolly balance. The object is placed on an air track, which emit gases continuously to minimize the friction force between the surface of the track and object. The origin ($x = 0$) is set to be at the equilibrium position of the mass M.

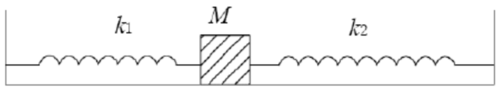

Figure 1: The mass–spring system. [1]

To begin with, we assume the mass of the springs can be neglected, and the damping of the system is ignored. As a result, the forces from the springs are the only forces exerted on the mass M. In this way, according to Newton's second law of dynamics, the equation of motion can be written in a simple form as follows:

$$M\ddot{x} + (k_1 + k_2)x = 0 \tag{2}$$

whose general solution is:

$$x(t) = Acos(\omega_0 t + \varphi_0) \tag{3}$$

where A is the amplitude of oscillation, $\omega_0 = \sqrt{(k_1 + k_2)/M}$ is the natural angular frequency of the system, and φ_0 is the initial phase. From Eq.(3), we know that the natural frequency only depends on M and k, and has nothing to do with the initial condition, while the initial phase depends on the initial conditions. Now, we are able to calculate the period of oscillation by:

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{M}{k_1 + k_2}} \tag{4}$$

In lab 2, we will focus on the relationship between the mass M of the object and its period of oscillation.

1.2.3 MASS OF THE SPRING

Our above discussion is only an ideal situation where the mass of spring is assumed to be massless. However, in the real state, the mass of the spring cannot be neglected. In this case, we ought to take into consideration the effective mass of the spring. The effective mass of the spring m_0 is $1/3$ of the actual mass m, i.e.

$$m_0 = \frac{1}{3}m \tag{5}$$

Then, the overall mass of the system is the sum of the object's mass and the effective mass of the springs, namely, the mass of the system is $M + m_0$. Hence, the angular frequency of the system can be expressed as follows:

$$\omega_0 = \sqrt{\frac{k_1 + k_2}{M + m_0}} \tag{6}$$

and thus the corresponding oscillation period is

$$T = 2\pi\sqrt{\frac{M + m_0}{k_1 + k_2}} \tag{7}$$

1.2.4 Mechanical Energy in Harmonic Motion

We know that the potential energy of a spring is:

$$U = \frac{1}{2}kx^2 \tag{8}$$

and the kinetic energy of the mass is:

$$K = \frac{1}{2}Mv^2 \tag{9}$$

Because in the system, there is no non-conservative force. Therefore we are able to apply energy conservation law, i.e.

$$U + K = const \tag{10}$$

Then we analyse the system from two extreme situation. First, when the mass reaches its equilibrium position ($x = 0$), the deformation x of the spring equals to 0, therefore the potential energy $U = 0$. Also, the velocity of the mass reaches its maximum, i.e. $v_1 = v_{max}$. Hence the kinetic energy $K = K_{max}$. On the other hand, when the mass reaches its maximum displacement, i.e. $x = \pm A$, then the velocity of the mass is $v_2 = 0$ and the corresponding kinetic energy $K = 0$. Also, the potential energy of the springs reaches its maximum, namely $K = K_{max}$. Taking into account Eq.(10), we are able to conclude that:

$$U_{max} = K_{max} \tag{11}$$

After plugging Eq.(8) and Eq.(9) into Eq.(11), we further obtain:

$$k = \frac{mv_{max}^2}{A^2} \tag{12}$$

2 Apparatus and Experimental Setup

In the first part of our experiment, we use the Jolly balance to measure the spring constant. The details are shown in Figure 2.

To use the Jolly balance to measure the spring constant, first we place a small mirror in the tube D and make three lines coincide with each other: the line on the mirror, the line on the tube, and the reflection in the mirror. In this way, we are able to make sure the height is accurate such that the value on the calliper is exactly the value we need.

To begin our experiment, first we don't add any weight on the spring, and make sure that the spring is parallel with the Jolly balance from two orthogonal directions. Then, we adjust knob G such that the three lines coincide. After that, read the value L_1.

Second, we add some mass m on the spring, then the spring will be stretched. By adjusting knob G again, we make the three lines coincide, and record the corresponding value L_2. Now, we are able to find out the spring constant from the following formula:

$$k = \frac{mg}{L_2 - L_1} \tag{13}$$

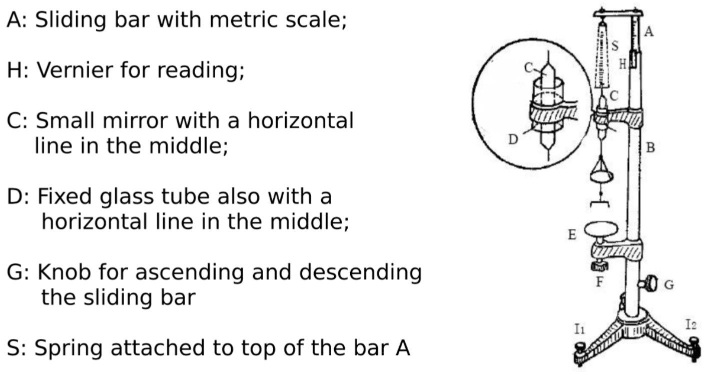

Figure 2: Jolly balance. [1]

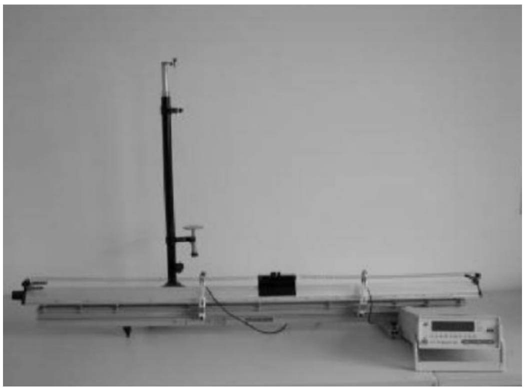

Figure 3: The experimental setup. [1]

In this lab, however, we change the value of the mass m and find out the spring constant by linear fit using the least squares method[1].

The second main apparatus we use in this experiment is the photoelectric measuring system. As shown in Figure 3, the system consists of two photoelectric gates and an electronic timer. Also, there is an air track which emit gases continuously to minimize the friction between contact surfaces. When an object passes through the gate, the light will be blocked. As a result, the receiver will receive a signal and the timer begin to record the time. For period measurements, we use the I-shape shutter.

However, when we measure the speed of an object, we use the U-shaped shutter, which is shown in Figure 4. In this way, the light will be blocked twice when the shutter passes through. Then, the time interval Δt will be recorded. Since the distance travelled can be estimated by $\Delta x = (x_{in} + x_{out})/2$, the velocity can be estimated as:

$$v = \frac{\Delta x}{\Delta t} = \frac{x_{in} + x_{out}}{2\Delta t} \tag{14}$$

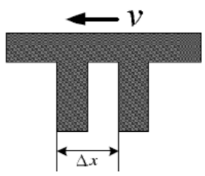

Figure 4: The U-shape shutter. [1]

The details for the apparatus are shown in Table 1.

Name of the instrument	Measured quantities	Precision
Timer (T mode)	Time	$\pm 0.0001[s]$
Timer (S_2 mode)	Time	$\pm 0.00001[s]$
Electronic scale	Mass of the objects	$\pm 0.01[g]$
Calliper (Jolly balance)	Length	$\pm 0.01[cm]$
Calliper	Inner and outer Distance of the U-shape shutter	$\pm 0.02[mm]$
Ruler (Air track)	Amplitude of oscillation	$\pm 0.1[cm]$

Table 1: Details and precisions of the apparatus.

3 Measurement Procedures

3.1 Spring Constant

To begin our measurement, we use the Jolly balance to find out the spring constant. First, we warm up and make some preparations for the measurement. We place the spring and mirror, and add a $20g$ preload. Then we move the knobs to make sure the mirror can be adjusted freely. After that, we check whether the spring is parallel to the balance by observing the system from two orthogonal directions. If they are not parallel, we make some adjustments on the knobs to let them coincide.

Second, we adjust knob G to make the three lines coincide with each other. Also, the height of the balance should be adjusted such that the initial position $L_0 = 5[cm] \sim 10[cm]$, and record the value L_1.

Third, we add mass m_1 and record length L_1, and after that, we keep adding another mass m_2 and record L_2. After repeating the process for six times, we collect six data. Note that the order of the masses should be recorded.

Fourth, the spring constant k_1 can be estimated by plotting a linear fit with least squares method.

Lastly, change another spring and use the same method to find out the spring constant k_2. And then attach spring 1 and spring 2 together with preload removed. We are able to compare k_3 with theories.

3.2 Relation Between the Oscillation Period T and the Mass of the Oscillator M

3.2.1 Early-stage adjustments

First, we check that the air track is horizontal. Then, we turn on the track and examine whether any hole is blocked. After that, we place the cart on the track, and by adjusting the single knob on the track, we make sure that the cart can move slowly back and forth. Note that before the air pump is turned on, no object is allowed to be placed on the track. Otherwise the track may deform, and the scales on it will be inaccurate.

3.2.2 Horizontal air track

In this part, we focus on the horizontal track. To begin with, we attach the springs to both sides of the track and use the I-shape shutter. Also, we should pay attention to the photoelectric gate so that it is in its equilibrium position.

Then, we begin our experiment. We place a mass m_1 on the cart and make it oscillate between the equilibrium position at an amplitude $A = 5[cm]$ with the help of a ruler. Then, we release the car and set the timer as the ”T” mode. The timer will automatically record the 10 periods of the oscillation. Record the mass and time period.

After that, we add some more weights on the cart, and repeat the procedure for 5 times. Finally, since other factors remain the same, we plot a linear fit to analyse the relationship between M and T.

3.2.3 Inclined air track

Actually, inclination of the track may also influence the oscillation period. To change the inclination of the air track, we put some plastic plates under it. In this lab, we place 3 and 6 plates respectively and repeat the steps in (3.2.2). Finally, we plot a linear fit to discuss the relationship between M and T.

3.3 Relation Between the Oscillation Period T and the Amplitude A

In this part, we study the relationship between period T and Amplitude A. We keep other factors unchanged and only change the amplitude. By choosing 6 different values (such as 5.0/ 10.0/ 15.0 [cm]...), we will obtain six groups of data.

Finally, we apply a linear fit to the data, and find out the dependence of period T on amplitude A with a correlation coefficient γ.

3.4 RELATION BETWEEN THE MAXIMUM SPEED AND THE AMPLITUDE

In this part, we measure the dependence of maximum speed on the amplitude. First, we change the shutter to the U-shaped one and select the "S_2" mode. Then the period interval Δt can be automatically recorded. Since the outer distance and the inner distance x_{in} and x_{out} can be measured by the calliper, the overall distance can be estimated as $\Delta x = (x_{in} + x_{out})/2$. Thus the maximum speed can be calculated accordingly.

After that, we change the amplitude of oscillation and we collect six data for different A (the recommended values are 5.0/ 10.0/ 15.0 [cm]...). Eventually, with the help of Eq.(12), we find out the v_{max} and compare the result.

3.5 MASS MEASUREMENT

In experiment 3, there are many masses we need to measure. When using the electronic scale, adjust it to be horizontal every time we use it. And when measuring the weight, we put them together. For instance, we measure the mass of the cart and the load together, which enables us to minimize the uncertainties which will otherwise occur when adding the masses together. Besides, note that we should begin our measurement before the circular symbol on the scale disappear.

4 RESULTS

4.1 MASS MEASUREMENT

In this lab, we obtain most of our results based on mass. Hence at first we analyse the mass involved. The data are recorded in Table 1 and Table 2 .

m [g] $\pm$ 0.01 [g]		
1	$\sum_{i=1}^{1} m_i$	4.70
2	$\sum_{i=1}^{2} m_i$	9.41
3	$\sum_{i=1}^{3} m_i$	14.17
4	$\sum_{i=1}^{4} m_i$	19.01
5	$\sum_{i=1}^{5} m_i$	23.81
6	$\sum_{i=1}^{6} m_i$	28.61

Table 2: The mass of loads.

where m_i refers to the i^{th} mass added to the cart.

m [g] $\pm$ 0.01 [g]	
Object with I-shape shutter m_I	177.22
Object with U-shape shutter m_U	185.37
Mass of Spring 1 & 2 $m_{spr1,2}$	21.31

Table 3: The mass of objects.

Based on Table 3, we can calculate the equivalent mass and their uncertainties, and the results are shown as follows (The detailed calculations are involved in Appendix A.1):

$$M_I = m_I + \frac{1}{3}m_{spr1,2} = 184.327[g] \pm 0.011[g] \qquad u_{rM_I} = 0.006\%$$

$$M_U = m_U + \frac{1}{3}m_{spr1,2} = 192.477[g] \pm 0.011[g] \qquad u_{rM_U} = 0.006\%$$

4.2 SPRING CONSTANT

We begin our experiment by using the Jolly balance to measure the spring constant. Then we record the different L corresponding to different loads. The data are recorded in Table 4.

spring 1 [cm] $\pm$ 0.01 [cm]		spring 2 [cm] $\pm$ 0.01 [cm]		series [cm] $\pm$ 0.01 [cm]	
L_0	6.03	L_0	5.69	L_0	9.84
L_1	8.08	L_1	7.55	L_1	13.58
L_2	10.09	L_2	9.45	L_2	17.58
L_3	12.30	L_3	11.37	L_3	21.54
L_4	14.50	L_4	13.25	L_4	25.65
L_5	16.69	L_5	15.18	L_5	29.66
L_6	18.83	L_6	17.10	L_6	33.81

Table 4: Measurement for single constant.

Then, we are able to obtain the change of length ΔL by the following formula:

$$\Delta L_i = L_i - L_0$$

where $i = 1, 2, 3, 4, 5, 6$. Now we are able to obtain the ΔL for each set of data, and their uncertainties can be calculated as follows (The detailed calculations are shown in Appendix A.2). We take spring 1 as an example:

$$\Delta L_1 = L_1 - L_0 = 2.050[cm] \pm 0.014[cm] \qquad u_{\Delta L_1} = 0.7\%$$

The rest of the data can be calculated in exact the same way, and the overall data with their uncertainties are recorded in Table 5, Table 6, and Table 7.

spring 1	ΔL [cm]	$u_{\Delta L_i}$ [cm]	$u_{r\Delta L_i}$
ΔL_1	2.050	0.014	0.7%
ΔL_2	4.060	0.014	0.3%
ΔL_3	6.270	0.014	0.2%
ΔL_4	8.470	0.014	0.17%
ΔL_5	10.660	0.014	0.13%
ΔL_6	12.800	0.014	0.11%

Table 5: Uncertainties for spring 1.

spring 1	ΔL [cm]	$u_{\Delta L_i}$ [cm]	$u_{r\Delta L_i}$
ΔL_1	1.860	0.014	0.8%
ΔL_2	3.760	0.014	0.4%
ΔL_3	5.880	0.014	0.2%
ΔL_4	7.560	0.014	0.19%
ΔL_5	9.490	0.014	0.15%
ΔL_6	11.410	0.014	0.12%

Table 6: Uncertainties for spring 2.

From the mass we have calculated in 4.1, we have:

$$u_m = 0.010[g]$$

Therefore, we are able to obtain that (The detailed calculations are shown in Appendix A.2):

$$u_{mg} = u_G = 0.00010[N]$$

Now, we are able to use *Origin* to plot the linear fit for them. The results are shown in Figure 5, Figure 6, and Figure 7.

It can be seem that from Eq.(1), we have $mg = k\Delta l$. Therefore, we obtain the spring constants from the following graphs. The uncertainties are the CI Half-Width, which are shown in the last row of the attached table of each graph. Based on the figures, we obtain the spring constants as follows:

$$k_1 = 2.17[N/m] \pm 0.03[N/m] \qquad u_{k1} = 1.4\%$$

$$k_2 = 2.47[N/m] \pm 0.09[N/m] \qquad u_{k2} = 4\%$$

$$k_s = 1.162[N/m] \pm 0.008[N/m] \qquad u_{ks} = 0.7\%$$

spring 1	ΔL [cm]	$u_{\Delta L_i}$ [cm]	$u_{r\Delta L_i}$
ΔL_1	3.740	0.014	0.4%
ΔL_2	7.740	0.014	0.18%
ΔL_3	11.700	0.014	0.12%
ΔL_4	15.810	0.014	0.09%
ΔL_5	19.820	0.014	0.07%
ΔL_6	23.970	0.014	0.06%

Table 7: Uncertainties for spring series.

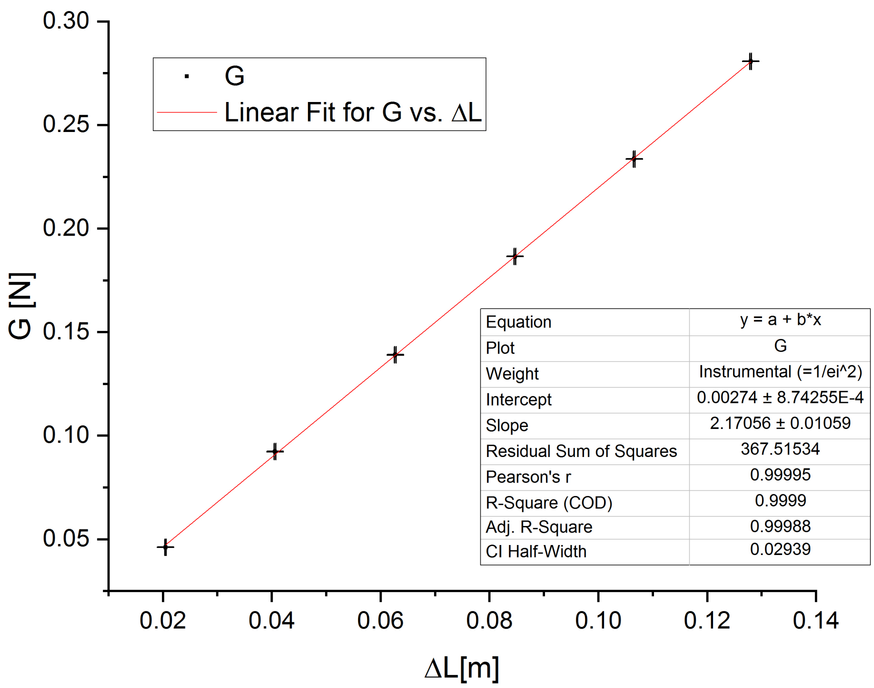

Figure 5: Linear Fit for spring 1.

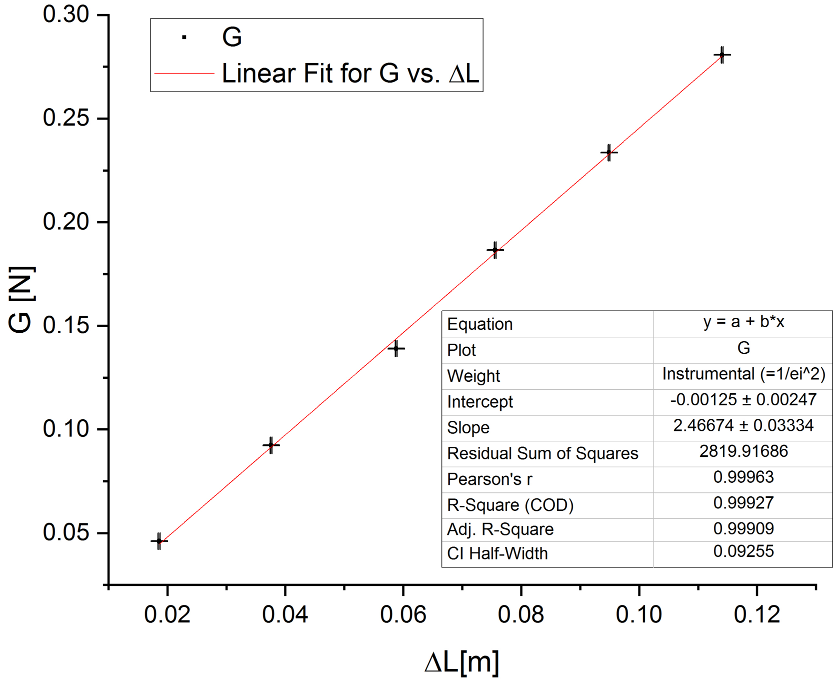

Figure 6: Linear Fit for spring 2.

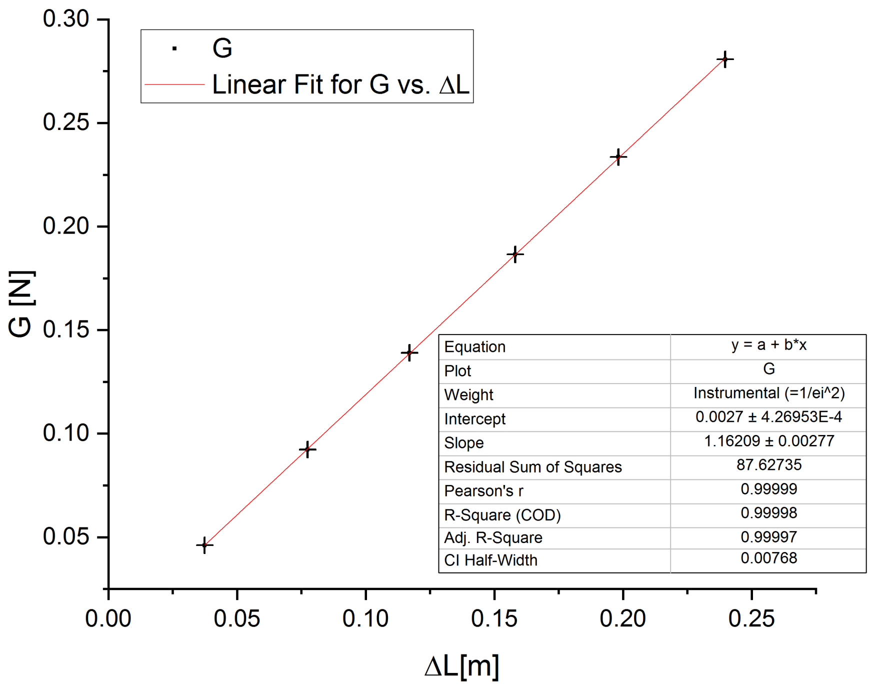

Figure 7: Linear Fit for spring series.

Then, we check the theoretical value k_{th} of the spring constant in series based on k_1 and k_2, and compared k_{th} with our measured k_s. The theoretical value k_{th} can be given by the following formula (The detailed calculations are involved in Appendix A.2):

$$k_{th} = \frac{1}{1/k_1 + 1/k_2} = \frac{k_1 k_2}{k_1 + k_2} = 1.16[N/m] \pm 0.02[N/m]$$

with a relative uncertainty:

$$u_{rk_{th}} = 2\%$$

Now, we are able to calculate the deviation of the experimental value between the measured value as follows:

$$u_r = \frac{|k_{th} - k_s|}{k_{th}} \times 100\% = 0.17\%$$

4.3 Relation between the oscillation period T and the mass of the oscillator M

In this lab, we measure the dependence of period T on the mass m, and change the inclination of the air track. The data with three different inclination are recorded in Table 8.

ten periods [s] $\pm$ 0.0001[s]					
horizontal		incline 1 (3 disks)		incline 2 (6 disks)	
m_1	12.6481	m_1	12.5973	m_1	12.6160
m_2	12.7636	m_2	12.7657	m_2	12.7725
m_3	12.9203	m_3	12.9330	m_3	12.9112
m_4	13.0617	m_4	13.0699	m_4	13.0861
m_5	13.2077	m_5	13.2258	m_5	13.2481
m_6	13.3534	m_6	13.3807	m_6	13.3875

Table 8: Measurement data for the T vs. M relation.

Since in this lab, we plot a linear fit in terms of M vs. T^2, and we already have the data of m calculated in 4.1, what we need to do is to find out the values of T^2 and their corresponding uncertainties. The general formula for obtaining T^2 can be written as follows:

$$T^2 = (\frac{T_{0,i}}{10})^2 = \frac{T_{0,i}^2}{100}$$

Then, we are able to calculate its value and corresponding uncertainty (The detailed calculations are shown in Appendix A.3). The data are shown in Table 8, Table 9, and Table 10.

Besides, in order to plot the M vs. T^2 linear fit, the uncertainties of masses are also needed. Based on the calculations in A.3, our uncertainties of mass is:

Horizontal	$T_i^2[s^2]$	$u_{T_i^2}[s^2]$	$u_{rT_i^2}$
m_1	1.59974	0.00003	0.0018%
m_2	1.62909	0.00003	0.0018%
m_3	1.66934	0.00003	0.0018%
m_4	1.70608	0.00003	0.0018%
m_5	1.74443	0.00003	0.0017%
m_6	1.78313	0.00003	0.0017%

Table 9: Uncertainties for horizontal track.

Incline 1	$T_i^2[s^2]$	$u_{T_i^2}[s^2]$	$u_{rT_i^2}$
m_1	1.58692	0.00003	0.0019%
m_2	1.62963	0.00003	0.0018%
m_3	1.67262	0.00003	0.0018%
m_4	1.70822	0.00003	0.0018%
m_5	1.74922	0.00003	0.0017%
m_6	1.79043	0.00003	0.0017%

Table 10: Uncertainties for track with incline 1.

$$u_{M_i} = 0.015[g] = 0.000015[kg]$$

Now, with all the uncertainties involved, we are able to use *Origin* to plot a M vs. T^2 linear fit. The results are shown in Figure 8, Figure 9, and Figure 10.

From Eq.(7), we obtain that

$$M = \frac{k}{4\pi^2}T^2$$

where k is our spring constant.

From the three graphs, we obtain their slopes and corresponding relative uncertainties as follows:

$$k_{slope,h} = 0.129[N/m] \pm 0.006[N/m] \qquad u_h = 5\%$$

$$k_{slope,i1} = 0.119[N/m] \pm 0.005[N/m] \qquad u_h = 4\%$$

$$k_{slope,i2} = 0.118[N/m] \pm 0.005[N/m] \qquad u_h = 4\%$$

Because $k = 4\pi^2 k_{slope}$, we can further find out the spring constants respectively:

$$k_h = 5.1[N/m] \pm 0.2[N/m] \qquad u_h = 5\%$$

$$k_{i1} = 4.68[N/m] \pm 0.18[N/m] \qquad u_h = 4\%$$

Incline 2	$T_i^2[s^2]$	$u_{T_i^2}[s^2]$	$u_{rT_i^2}$
m_1	1.59163	0.00003	0.0019%
m_2	1.63137	0.00003	0.0018%
m_3	1.66699	0.00003	0.0018%
m_4	1.71246	0.00003	0.0018%
m_5	1.75512	0.00003	0.0017%
m_6	1.79225	0.00003	0.0017%

Table 11: Uncertainties for track with incline 2.

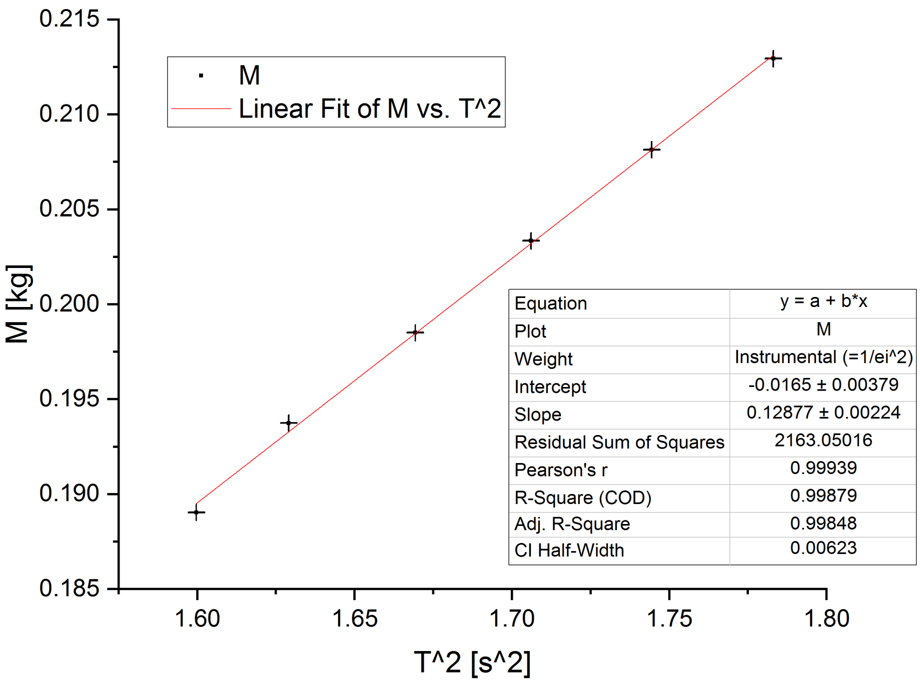

Figure 8: Linear Fit of M vs. T^2 (horizontal).

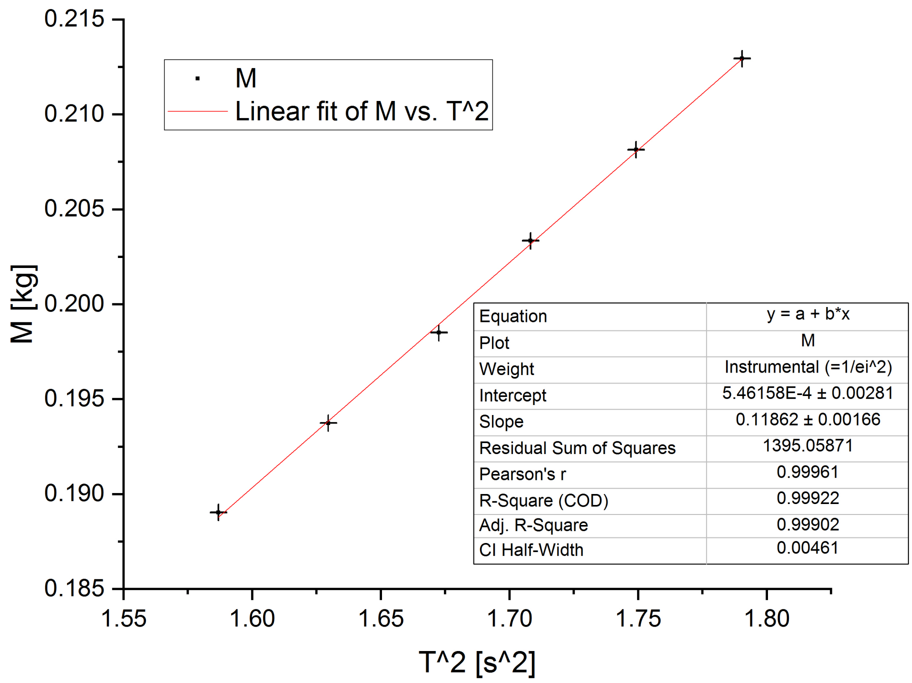

Figure 9: Linear Fit of M vs. T^2 (incline 1).

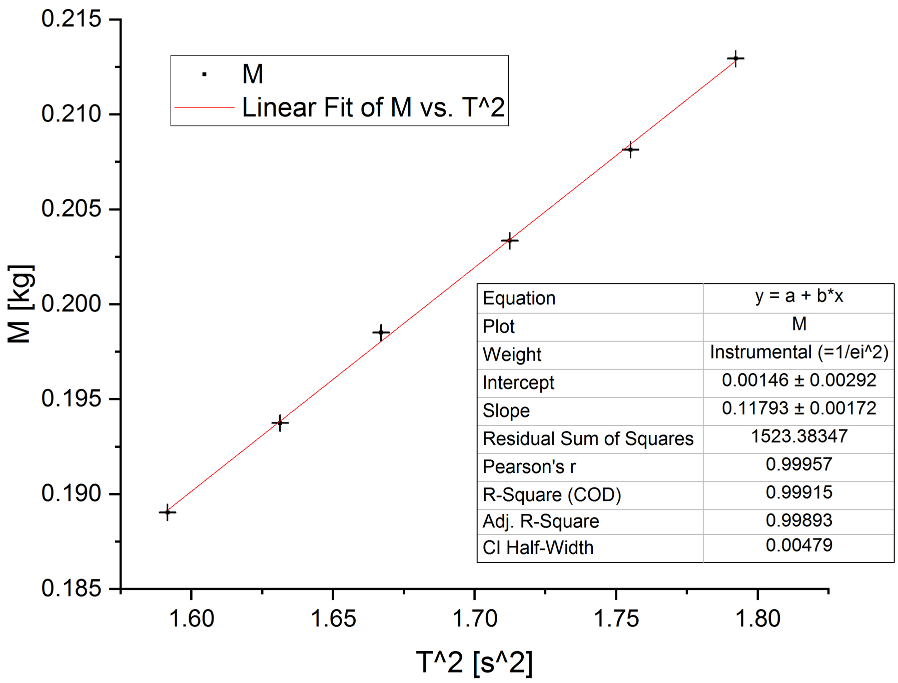

Figure 10: Linear Fit of M vs. T^2 (incline 2).

$$k_{i2} = 4.66[N/m] \pm 0.19[N/m] \qquad u_h = 4\%$$

In conclusion, it can be find the following formula is satisfied

$$k = k_1 + k_2$$

This will be discussed later in Discussion.

4.4 Relation Between the Oscillation Period T and the Amplitude A

In this section, we focus on the relationship between oscillation period T and amplitude A. The data are recorded in Table 12.

A[cm] $\pm$ 0.1[cm]		ten periods [s] $\pm$ 0.0001 [s]
1	5.0	13.3788
2	10.0	13.3800
3	15.0	13.3884
4	20.0	13.3930
5	25.0	13.3929
6	30.0	13.3940

Table 12: Data for T vs. A relation.

In this section, we plan to plot a T vs. A linear fit. Hence we ought to find out these two parameters' uncertainties. First, we focus on the period T. Since the formula is given by

$$T_i = \frac{T_{10,i}}{10}$$

we are able to find out its uncertainty (The detailed calculations are shown in A.4):

$$u_{T_i} = \sqrt{(\frac{\partial T_i}{\partial T_{10,i}})^2 (u_{T_{10,i}})^2} = 0.000010[s]$$

Hence, all data and their uncertainties are shown in Table 13.

Based on the data in Table 13, we are able to use *Origin* to plot a linear fit. The results are shown below (Figure 11).

From the linear fit plot, we obtain the correlation coefficient γ and its uncertainty as follows:

$$\gamma = 0.007 \pm 0.004 \qquad u_{r\gamma} = 57\%$$

In this linear fit, the Pearson's r is 0.9. According to the linear dependence, since the Pearson' r is larger than 0.75, then there possibly exist a linear relation between T and A. However, theoretically speaking, it is obvious that there is no relationship between T and A. What leads to the contradiction will be discussed later.

A[m]	u_A [m]	u_{rA}	T [s]	u_T [s]	u_{rT}
0.050	0.001	2%	1.337880	0.000010	0.0007%
0.100	0.001	1%	1.338000	0.000010	0.0007%
0.150	0.001	0.7%	1.338840	0.000010	0.0007%
0.200	0.001	0.5%	1.339300	0.000010	0.0007%
0.250	0.001	0.4%	1.339290	0.000010	0.0007%
0.300	0.001	0.3%	1.339400	0.000010	0.0007%

Table 13: Uncertainties of T vs. A relation.

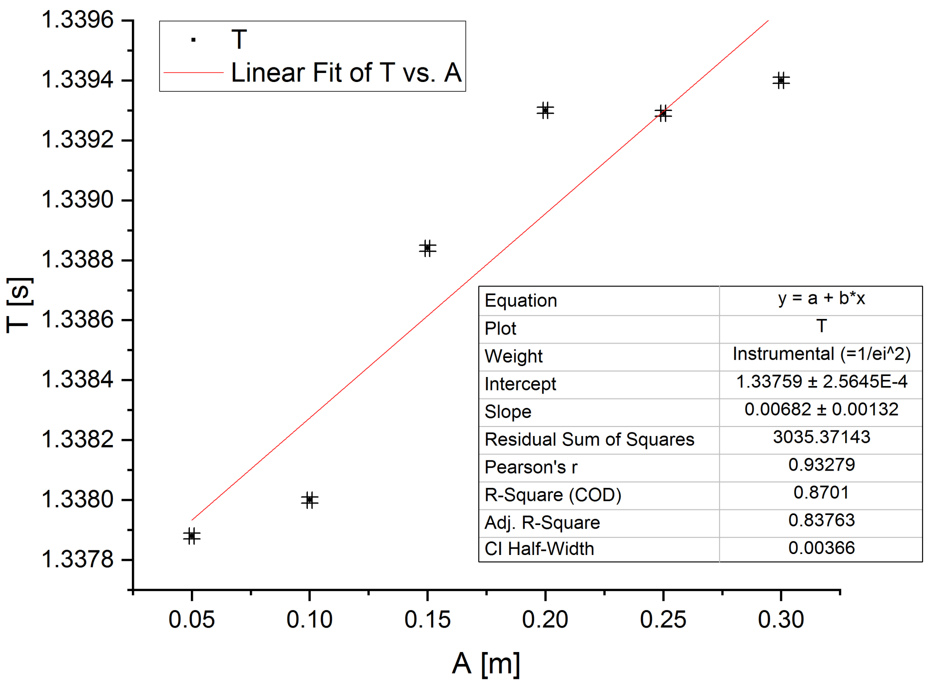

Figure 11: Linear Fit of T vs. A.

4.5 RELATION BETWEEN THE MAXIMUM SPEED AND THE AMPLITUDE

In this section, we study the relation between the maximum speed and amplitude. First, the measurement data are recorded in Table 14 and Table 15.

A[cm] $\pm$ 0.1[cm]		Δt [s] $\pm$ 0.00001 [s]
1	5.0	0.04244
2	10.0	0.02154
3	15.0	0.01411
4	20.0	0.01058
5	25.0	0.00848
6	30.0	0.00707

Table 14: Data for the v_{max}^2 vs. A^2 relation.

x_{in} [mm] $\pm$ 0.02 [mm]	x_{out} [mm] $\pm$ 0.02 [mm]
4.50	15.42
4.50	15.42
4.50	15.42

Table 15: Data for the v_{max}^2 vs. A^2 relation.

Then, we first try to figure out v. To achieve this, we calculate the distance between two blanks on the U-shape shutter, i.e., we have (The detailed calculations are shown in Appendix A.5.)

$$x_{in} = \frac{\sum_{i=1}^{3} x_{in,i}}{3} = 0.00450[m] \pm 0.00002[m] \qquad u_{rin} = 0.4\%$$

$$x_{out} = \frac{\sum_{i=1}^{3} x_{out,i}}{3} = 0.01542[m] \pm 0.00002[m] \qquad u_{rout} = 0.13\%$$

After that, we can have

$$\Delta x = \frac{x_{in} + x_{out}}{2} = 9.960[mm] \pm 0.014[mm] = 0.009960[m] \pm 0.000014[m]$$

with a relative uncertainty

$$u_{r\Delta_x} = 0.14\%$$

Now, we would like to find out Mv_{max}^2, which is given by

$$Mv_{max}^2 = M(\frac{\Delta x}{\Delta t})^2$$

And A^2 similarly.

The detailed calculation are shown in Appendix A.5, and the final data with uncertainties are shown in Table 16.

Mv_{max}^2 [J]	$u_{Mv_{max}^2}$ [J]	$u_{rMv_{max}^2}$	$A^2[m^2]$	u_{A^2}	u_{rA^2}
0.01086	0.00003	0.3%	0.00250	0.00010	4%
0.04216	0.00013	0.3%	0.0100	0.0002	2%
0.0982	0.0003	0.3%	0.0225	0.0003	1.3%
0.1747	0.0006	0.3%	0.0400	0.0004	1%
0.2720	0.0011	0.4%	0.0625	0.0005	0.8%
0.3913	0.0018	0.5%	0.0900	0.0006	0.7%

Table 16: Uncertainties for the v_{max}^2 vs. A^2 relation.

Now, we can use *Origin* to plot a linear fit. The figure is shown in Figure 12.

From Eq.(12), we know that

$$kA^2 = mv_{max}^2$$

Because the system is connected in parallel, the spring constant k is supposed to be the sum of the two springs $k_1 + k_2$. Namely, we have

$$Mv_{max}^2 = (k_1 + k_2)A^2$$

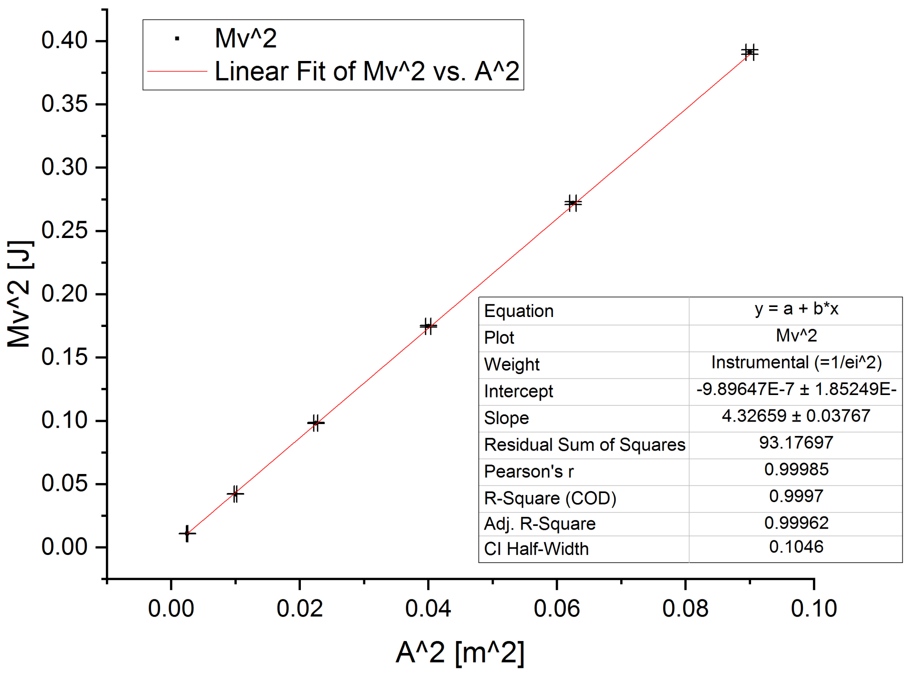

Figure 12: Linear Fit of M^2 vs. A^2.

where $k_1 + k_2$ is the slope of the linear fit.

Hence, from Figure 12, we obtain the slope and uncertainty of the linear fit is:

$$k_v = 4.33[N/m] \pm 0.10[N/m] \qquad u_{rk_v} = 2\%$$

In 4.2 we have already figured out the spring constant $k_1 = 2.17[N/m]$ and $k_2 = 2.47[N/m]$. Therefore, the theoretical value of the slope is

$$k_p = k_1 + k_2 = 4.64[N/m] \pm 0.12[N/m] \qquad u_{rk_p} = 3\%$$

We compare the theoretical value with the measured one, and then calculate the deviation from the theoretical value:

$$d = \frac{|k_v - k_p|}{k_p} \times 100\% = 7\%$$

5 DISCUSSION

5.1 THE SPRING CONSTANT

5.1.1 SPRINGS IN SERIES

In this lab, we use the Jolly balance to measure the spring constant of the spring and verify the theorem of springs in series. The value we obtain in our experiment are recorded below:

$$k_1 = 2.17[N/m] \pm 0.03[N/m] \qquad u_{k1} = 1.4\%$$

$$k_2 = 2.47[N/m] \pm 0.09[N/m] \qquad u_{k2} = 4\%$$

$$k_s = 1.162[N/m] \pm 0.008[N/m] \qquad u_{ks} = 0.7\%$$

Then, we calculate the theoretical value of the springs in series and compare it with our measured value. The result has been shown below:

$$k_{th} = \frac{k_1 k_2}{k_1 + k_2} = 1.16[N/m] \pm 0.02[N/m] \qquad u_{rk_{th}} = 2\%$$

Now, we are able to calculate the deviation of the experimental value between the measured value as follows:

$$u_r = \frac{|k_{th} - k_s|}{k_{th}} \times 100\% = 0.17\%$$

In this way, we can conclude that our result is reliable and our experiment is successful in general. The satisfactory result may comes from:

- The high precision of the Jolly balance.
- The load is not too large so that the springs are in their elastic limit.
- The accurate performance of the procedure.

5.1.2 SPRINGS IN PARALLEL

In this lab, we also verify the situation where the springs are connected parallel. This is realized by studying the cart on an air track. The technical details have been shown in 4.5, and our result are shown below:

$$k_v = 4.33[N/m] \pm 0.10[N/m] \qquad u_{rk_v} = 2\%$$

Then, we find out the theoretical value of the slope:

$$k_p = k_1 + k_2 = 4.64[N/m] \pm 0.12[N/m] \qquad u_{rk_p} = 3\%$$

We compare the theoretical value with the measured one, and then calculate the deviation from the theoretical value:

$$d = \frac{|k_v - k_p|}{k_p} \times 100\% = 7\%$$

Unfortunately, the deviation reaches 7%, which is relatively high compared with the 0.17% in 5.1.1. There may be a variety of reasons which may lead to this large difference:

- Although we have used the air track to minimize the friction, there still exist some small friction between the contact surfaces, whose influence can not be neglected with the increase of oscillation amplitude. For instance, when $A = \pm 30[cm]$, the influence of friction can't be easily neglected.
- In the lab, there also exists air drag, which can not be ignored when the velocity of the cart becomes large. To be more specific, when doing the experiment, we found that when the amplitude is set to be $A = \pm 30[cm]$, it will decrease to approximately $25[cm]$ after 10 periods of oscillation. This means that the existence of air drag and friction can't be ignored in this case.
- Because in this lab, we stretch the springs to as long as $30[cm]$ to obtain our data, chances are that the deformation exceeds its elastic limit, and the irreversible deformation of springs is sure to make some deviations.
- The air track may not be completely horizontal, and the small inclination may lead to some subtle difference on environment, which will influence the result.

5.2 Relation Between the Oscillation Period T and the Mass of the Oscillator M

In this section, we apply an M vs. T^2 linear fit, and we find out that our Pearson's r is close to 1:

$$r = 0.99939 \approx 1$$

Since the r is larger than 0.75, it is possibly that there exist a linear relation between M and T^2. From our theoretical analysis, the slope is:

$$k_{slope} = \frac{k}{4\pi^2}$$

Thus, we have

$$k_h = 5.1[N/m] \pm 0.2[N/m] \qquad u_h = 5\%$$

$$k_{i1} = 4.68[N/m] \pm 0.18[N/m] \qquad u_h = 4\%$$

$$k_{i2} = 4.66[N/m] \pm 0.19[N/m] \qquad u_h = 4\%$$

In conclusion, it can be find the following formula is satisfied

$$k = k_1 + k_2$$

It can be noticed that our first result has a large difference from the other two ones. And some reason may lead to such difference:

- The I-shape shutter contact the sensor when moving back and through, which leads to error in time recording.
- The air track may not be that horizontal, which may lead to errors.

5.3 Relation Between the Oscillation Period T and the Amplitude A

In this section, we apply a T vs. A linear fit, and try to find out the relationship between them. The Pearson's r of the linear fit is:

$$r = 0.93279 \approx 0.9$$

Since r is larger than 0.75, the result yields a possible relationship between T and A. Nevertheless, theoretically speaking, such relationship doesn't exist. Hence we list some possible factors contributing to the contradiction:

- This are just some accidental errors. It can be observed from the figure that the points are actually kind of scatter, and chances are that they coincide to have a "linear relation". In other words, it is the accidental errors which lead to the large value of Pearson's r.
- The friction between the contact surface and the air drag. These factors also influence the result. Since we have discussed them before, we do not discuss then in detail here.

5.4 Relation Between the Maximum Speed and the Amplitude

In this section, we focus on the relation between the maximum speed and the amplitude, and plot a Mv_{max}^2 vs. A^2 linear fit. The Pearson's r is:

$$r = 0.9985 \approx 1$$

Since r is larger than 0.75, we can hopefully conclude that there exist a linear relation between Mv_{max}^2 and A^2. In fact, from our analysis, the slope is just the spring constant.

5.5 Some possible improvement to the experiment

Here are some improvement which may increase the accuracy of this experiment:

- In this lab, the amplitude of the oscillation is hard to control. It's difficult for us to control the amplitude accurately simply by hands or ruler. Hence some improvements may exist here.
- The number of periods should be carefully chosen. It can neither be too large nor be too small. Too large number of periods may lead to the obvious influence of friction and air drag, while too small number of periods may lead to the instability of data. In our lab, we just choose the number of periods to be 10. However, if the speed of the cart is large, we may decrease the number of periods we measure so that the accuracy is increased.
- The I-shaped and U-shaped shutter may be unstable in our experiment. To be more specific, it may hit the sensor, or contact the sensor, both of which lead to errors. Hence it would be better if the shutters are much more stable.

6 Conclusion

In conclusion, in this lab, we have learned a lot:

- How to use the Jolly balance.
- How to use *Origin* to plot a linear fit.
- How to analyse the data involved in a linear fit in terms of Pearson's r.
- The factors which may influence the simple harmonic oscillation.

Also, we verify some important conclusions:

- When springs are connected in series, the spring constant satisfies $k = \dfrac{k_1 k_2}{k_1 + k_2}$.
- When springs are connected in parallel, the spring constant satisfies $k = k_1 + k_2$.
- Though two quantities do not have a linear relation theoretically, a linear fit may still yield an $r > 0.75$ such that a relation may exist. However, this may simply result from accidental errors.
- The energy conservation in simple harmonic motion.

In all, in lab 3 I really learned a lot, and my understanding towards simple harmonic motion becomes deeper and deeper. Although some errors may still exist, the factors have been detected and hopefully we have successfully accomplished this experiment.

A Measurement uncertainty analysis

A.1 Uncertainty of mass measurement

First, we calculate the uncertainty of mass. For other masses except the equivalent mass, since it is a single measurement, the type A uncertainties equal to zero, and the type B uncertainty is 0.01 [g] which is due to the instruments. Hence the overall uncertainty is:

$$u = \sqrt{\Delta_A^2 + \Delta_B^2} = 0.001[g]$$

For the equivalent mass M, however, the uncertainty propagation should be considered. Since the formula are expressed as follows:

$$M_I = m_I + \frac{1}{3} m_{spr1,2}$$

$$M_U = m_U + \frac{1}{3} m_{spr1,2}$$

The uncertainty can be calculated as follows:

$$\frac{\partial M_I}{\partial m_I} = 1$$

$$\frac{\partial M_I}{\partial m_{spr1,2}} = \frac{1}{3}$$

Therefore, we obtain:

$$u_{M_I} = \sqrt{(\frac{\partial M_I}{\partial m_I})^2 (u_{M_I})^2 + (\frac{\partial M_I}{\partial m_{spr1,2}})^2 (u_{m_{spr1,2}})^2} = 0.011[g]$$

Similarly, we have:

$$u_{M_U} = \sqrt{(\frac{\partial M_U}{\partial m_U})^2 (u_{M_U})^2 + (\frac{\partial M_U}{\partial m_{spr1,2}})^2 (u_{m_{spr1,2}})^2} = 0.011[g]$$

Hence, the equivalent mass with their relative uncertainties can be obtained as follows:

$$M_I = m_I + \frac{1}{3} m_{spr1,2} = 184.327[g] \pm 0.011[g] \qquad u_{rM_I} = 0.006\%$$

$$M_U = m_U + \frac{1}{3} m_{spr1,2} = 192.477[g] \pm 0.011[g] \qquad u_{rM_U} = 0.006\%$$

A.2 UNCERTAINTY OF SPRING CONSTANT

To find out the spring constant, first we use the Jolly balance, and the uncertainties of the ΔL are what we should obtain first. Since we only measure the value once, the type A uncertainty for length measurement always equals to 0. Thus the overall uncertainty is simply the type B uncertainty, namely, $u_{L_i} = 0.01[cm] = 0.0001[m]$. Furthermore, because the measurement is indirect, the propagation of uncertainty should be considered, and we calculate the partial derivatives first.

$$\Delta L_i = L_i - L_0$$

$$\frac{\partial \Delta L_i}{\partial L_i} = 1$$

$$\frac{\partial \Delta L_i}{\partial L_0} = -1$$

$$u_{\Delta L_i} = \sqrt{(\frac{\partial \Delta L_i}{\partial L_i})^2 (u_{L_i})^2 + (\frac{\partial \Delta L_i}{\partial L_0})^2 (u_{L_0})^2} = 0.014[cm] = 0.00014[m]$$

Also, in this section, the weight of the loads and their uncertainties are also needed. Since we have:

$$G = mg$$

we can further obtain that:

$$u_G = \sqrt{(\frac{\partial G}{\partial m})^2 (u_m)^2} = 0.00010[N]$$

Then, we obtain the uncertainties of the spring constant of 1, 2, and in series with the help of *Origin*, and the value are listed below:

$$u_{k1} = 0.03[N/m]$$

$$u_{k2} = 0.09[N/m]$$

$$u_{ks} = 0.008[N/m]$$

Now, we calculate the theoretical value of the spring in series based on k_1 and k_2. The formula to calculate k_{th} can be expressed as:

$$k_{th} = \frac{k_1 k_2}{k_1 + k_2} = 1.16[N/m] \pm 0.02[N/m]$$

The uncertainty of k_{th} can be calculate accordingly. First, we write its partial derivatives:

$$\frac{\partial k_{th}}{\partial k_1} = \frac{k_2^2}{(k_1 + k_2)^2}$$

$$\frac{\partial k_{th}}{\partial k_2} = \frac{k_1^2}{(k_1 + k_2)^2}$$

Therefore, the overall uncertainty is:

$$u_{k_{th}} = \sqrt{(\frac{\partial k_{th}}{\partial k_1})^2 (u_{k1})^2 + (\frac{\partial k_{th}}{\partial k_2})^2 (u_{k2})^2} = 0.02[N/m]$$

with a relative uncertainty

$$u_{rk_{th}} = \frac{u_{k_{th}}}{k_{th}} = 1.7\%$$

A.3 Relation between the oscillation period T and the mass of the oscillator M

In this section, we first need to find out the value and uncertainty of T^2. Because we only measure the time once, the overall uncertainty is simply the type B uncertainty, i.e.

$$u_{T_{0,i}} = \Delta_{T,B} = 0.0001[s]$$

Since we already have the formula:

$$T^2 = \frac{T_{0,i}^2}{100}$$

we are able to figure out its partial derivatives first:

$$\frac{\partial T^2}{\partial T_{0,i}} = \frac{T_{0,i}}{50}$$

Hence, the overall uncertainty can be written as:

$$u_{T^2} = \sqrt{(\frac{\partial T^2}{\partial T_{0,i}})^2 (u_{T_{0,i}})^2} = \frac{T_{0,i} u_{T_{0,i}}}{50}$$

which varies from different $T_{0,i}$. Thus we only take some examples such as $T_{0,1}$. This can be calculated as follows:

$$u_{T_1{}^2} = 0.00003[s]$$

with a relatively uncertainty:

$$u_{rT_1{}^2} = \frac{u_{T_1{}^2}}{T_1^2} \times 100\% = 0.0018\%$$

Horizontal	$T_i^2[s^2]$	$u_{T_i^2}[s^2]$	$u_{rT_i^2}$
m_1	1.59974	0.00003	0.0018%
m_2	1.62909	0.00003	0.0018%
m_3	1.66934	0.00003	0.0018%
m_4	1.70608	0.00003	0.0018%
m_5	1.74443	0.00003	0.0017%
m_6	1.78313	0.00003	0.0017%

Table 17: Uncertainties for horizontal track.

Incline 1	$T_i^2[s^2]$	$u_{T_i^2}[s^2]$	$u_{rT_i^2}$
m_1	1.58692	0.00003	0.0019%
m_2	1.62963	0.00003	0.0018%
m_3	1.67262	0.00003	0.0018%
m_4	1.70822	0.00003	0.0018%
m_5	1.74922	0.00003	0.0017%
m_6	1.79043	0.00003	0.0017%

Table 18: Uncertainties for track with incline 1.

Similarly, other T^2s can be obtained in exactly the same way, and the uncertainties are shown in the following Tables.

Now, we still need to find out the uncertainty of the mass measurements. Since the mass is given by:

$$M_i = M_{obj} + \tfrac{1}{3}m_{spr1,2} + m_i$$

it is an indirect measurement. Therefore we first calculate its partial derivative as follows:

$$\frac{\partial M_i}{\partial M_{obj}} = 1$$

$$\frac{\partial M_i}{\partial m_{spr1,2}} = \frac{1}{3}$$

$$\frac{\partial M_i}{\partial m_i} = 1$$

Then, the overall uncertainty can be calculated as:

$$u_{M_i} = \sqrt{(\frac{\partial M_i}{\partial M_{obj}})^2(u_{M_{obj}})^2 + (\frac{\partial M_i}{\partial m_{spr1,2}})^2(u_{m_{spr1,2}})^2 + (\frac{\partial M_i}{\partial m_i})^2(u_{m_i})^2} = 0.015[g]$$

Incline 2	$T_i^2[s^2]$	$u_{T_i^2}[s^2]$	$u_{rT_i^2}$
m_1	1.59163	0.00003	0.0019%
m_2	1.63137	0.00003	0.0018%
m_3	1.66699	0.00003	0.0018%
m_4	1.71246	0.00003	0.0018%
m_5	1.75512	0.00003	0.0017%
m_6	1.79225	0.00003	0.0017%

Table 19: Uncertainties for track with incline 2.

A.4 Relation Between the Oscillation Period T and the Amplitude A

In this section, we first find out the uncertainty of T. Since the formula is given by

$$T_i = \frac{T_{10,i}}{10}$$

it is an indirect measurement, and the uncertainty is calculated by the following procedure:

$$\frac{\partial T_i}{\partial T_{10,i}} = \frac{1}{10}$$

Hence the overall uncertainty is

$$u_{T_i} = \sqrt{(\frac{\partial T_i}{\partial T_{10,i}})^2 (u_{T_{10,i}})^2} = \frac{u_{T_{10,i}}}{10} = 0.000010[s]$$

Similarly, the overall uncertainty of amplitude A can be calculated as follows:

$$u_A = \Delta_{B,A} = 0.1[cm] = 0.001[m]$$

A.5 Relation Between the Maximum Speed and the Amplitude

In this part, we first calculate the value of maximum v. To achieve this, we find out the Δx first. For x_{in} and x_{out}, we have:

$$x_{in} = \frac{\sum_{i=1}^{3} x_{in,i}}{3}$$

$$x_{out} = \frac{\sum_{i=1}^{3} x_{out,i}}{3}$$

which is a multiple measurement. Hence, we calculate its std. deviation first:

$$s_{x_{in}} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{3}(x_{in,i} - \bar{x_{in}})^2} = 0$$

$$s_{x_{out}} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{3}(x_{out,i} - \bar{x_{out}})^2} = 0$$

Hence, the type A uncertainty is:

$$\Delta_{A1} = \frac{t_{0.95}}{\sqrt{n}} s_{x_{in}} = 0$$

$$\Delta_{A2} = \frac{t_{0.95}}{\sqrt{n}} s_{x_{out}} = 0$$

Eventually, the overall uncertainty is simply the type B uncertainty, namely:

$$u_{x_{in}} = \Delta_{B1} = 0.02[mm] = 0.00002[m]$$

$$u_{x_{out}} = \Delta_{B2} = 0.02[mm] = 0.00002[m]$$

We still have an indirect measurement, namely, we have

$$\Delta x = \frac{x_{in} + x_{out}}{2}$$

We have:

$$\frac{\partial \Delta x}{\partial x_{in}} = \frac{1}{2}$$

$$\frac{\partial \Delta x}{\partial x_{out}} = \frac{1}{2}$$

Thus, we have

$$u_{\Delta x} = \sqrt{(\frac{\partial \Delta x}{\partial x_{in}})^2 (u_{x_{in}})^2 + (\frac{\partial \Delta x}{\partial x_{out}})^2 (u_{x_{out}})^2} = 0.000014[m]$$

To calculate $u_{Mv_{max}^2}$, since we have

$$Mv_{max}^2 = M(\frac{\Delta x}{\Delta t})^2$$

we find their partial derivatives:

$$\frac{\partial Mv_{max}^2}{\partial M} = (\frac{\Delta x}{\Delta t})^2$$

$$\frac{\partial Mv_{max}^2}{\partial \Delta x} = \frac{2M\Delta x}{\Delta t^2}$$

$$\frac{\partial Mv_{max}^2}{\partial \Delta t} = \frac{-2M\Delta x^2}{\Delta t^3}$$

We have:

$$u_{Mv_{max}^2} = \sqrt{(\frac{\partial Mv_{max}^2}{\partial M})^2(u_M)^2 + (\frac{\partial Mv_{max}^2}{\partial \Delta x})^2(u_{\Delta x})^2 + (\frac{\partial Mv_{max}^2}{\partial \Delta t})^2(u_{\Delta t})^2}$$

Besides, for A^2, we have

$$\frac{\partial A^2}{\partial A} = 2A$$

$$u_{A^2} = \sqrt{(\frac{\partial A^2}{\partial A})^2(u_A)^2} = 2Au_A$$

The final results are shown in the Table below:

Mv_{max}^2 [J]	$u_{Mv_{max}^2}$ [J]	$u_{rMv_{max}^2}$	$A^2[m^2]$	u_{A^2}	u_{rA^2}
0.01086	0.00003	0.3%	0.00250	0.00010	4%
0.04216	0.00013	0.3%	0.0100	0.0002	2%
0.0982	0.0003	0.3%	0.0225	0.0003	1.3%
0.1747	0.0006	0.3%	0.0400	0.0004	1%
0.2720	0.0011	0.4%	0.0625	0.0005	0.8%
0.3913	0.0018	0.5%	0.0900	0.0006	0.7%

Table 20: Uncertainties for the v_{max}^2 vs. A^2 relation.

B DATASHEET

See the last page.

References

[1] Krzyzosiak, M. & VP141 TA Groups. *Exercise 3 - lab manual [rev 5.2].pdf*. 2019.