UM-SJTU Joint Institute, Physics Laboratory I
Measurement Uncertainty Analysis Worksheet*
Exercise 5

WS-1 Natural Angular Frequency

The uncertainty for ten periods is found first. Then the result for the natural frequency is given along with its uncertainty.

The type-B uncertainty for T_{10} is $\Delta_{T_{10},B} = 0.001$ s. To find the type-A uncertainty, we first find the standard deviation

$$s_{T_{10}} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(T_{10,i} - \overline{T}_{10})^2} = \underline{0.00096} \quad [\underline{\;s\;}].$$

We have $n = \underline{4}$, so the type-A uncertainty $\Delta_{T_{10},A}$ is calculated as

$$\Delta_{T_{10},A} = \frac{t_{0.95}}{\sqrt{n}} s_{T_{10}} = \underline{1.59} \times \underline{0.00096} = \underline{0.0015} \quad [\underline{\;s\;}].$$

Hence the uncertainty for T_{10} is given by

$$u_{T_{10}} = \sqrt{\Delta^2_{T_{10},A} + \Delta^2_{T_{10},B}} = \underline{0.002} \quad [\underline{\;s\;}].$$

The period is found indirectly by measuring the ten periods. Therefore, its uncertainty u_T of a single period is found by applying the uncertainty propagation formula

$$\boxed{u_T} = \sqrt{\left(\frac{\partial T}{\partial T_{10}} u_{T_{10}}\right)^2} = \frac{u_{T_{10}}}{10} = \boxed{\underline{0.0002} \quad [\underline{\;s\;}]},$$

Hence the period is given by

$$\boxed{T = \underline{1.5811} \pm \underline{0.0002} \quad [\underline{\;s\;}]},$$

*Created by Peng Wenhao, edited by Fan Yixing, Ye Haojie, Mateusz Krzyzosiak [rev. 1.3] with relative uncertainty

$$\boxed{u_{rT}} = \frac{u_T}{T} \times 100\% = \boxed{\underline{0.01}}\%$$

The natural angular frequency ω_0 is found from the formula $\omega_0 = 2\pi/T$, so by the uncertainty propagation formula and the fact that

$$\frac{\partial\omega_0}{\partial T} = -\frac{2\pi}{T^2},$$

we obtain

$$\boxed{u_{\omega_0}} = \left|\frac{\partial\omega_0}{\partial T} u_T\right| = \boxed{\underline{0.0004} \quad [\underline{s^{-1}}]}$$

with the relative uncertainty

$$\boxed{u_{r,\omega_0}} = \frac{u_{\omega_0}}{\omega_0} \times 100\% = \boxed{\underline{0.01}\%}$$

WS-2 Damping Coefficient

The damping coefficient is found indirectly form measurements of the period T and the amplitude θ as $\beta = \frac{1}{5T}\ln(\theta_i/\theta_{i+5})$.

The uncertainty each single measurement of the amplitude is $u_\theta = \underline{1}$ °; so the uncertainty of the logarithm of the quotient of them $q_i = \ln(\theta_i/\theta_{i+5})$ is found from the uncertainty propagation formula

$$\Delta_{q_i,B} = \sqrt{\left(\frac{\partial(\ln(\theta_i/\theta_{i+5}))}{\partial\theta_i}\right)^2 u_\theta^2 + \left(\frac{\partial(\ln(\theta_i/\theta_{i+5}))}{\partial\theta_{i+5}}\right)^2 u_\theta^2} = \sqrt{\left(\frac{u_\theta}{\theta_{i+5}}\right)^2 + \left(\frac{u_\theta}{\theta_i}\right)^2}$$

For example, for $i = 1$,

$$\Delta_{q_1,B} = \sqrt{\left(\frac{u_\theta}{\theta_6}\right)^2 + \left(\frac{u_\theta}{\theta_1}\right)^2} = \underline{\overset{0.016}{\cancel{0.018}}}.$$

The results for all five sequences of measurements are given in Table WS-1.

i	$\Delta_{q_i,B}$
1	~~0.018~~ 0.016
2	0.018
3	0.019
4	0.021
5	0.023

Table WS-1: Type-B uncertainties for q_i.

The overall type-B uncertainty for the quotient can be estimated as the maximum of uncertainties listed in in Table WS-1

$$\Delta_{q,B} = \underline{0.023}.$$

To estimate the type-A uncertainty of q, the standard deviation of q is calculated as

$$s_q = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(q_i - \bar{q})^2} = \underline{0.011}$$

Hence the type-A uncertainty for $n = 5$ is calculated as

$$\Delta_{q,A} = \frac{t_{0.95}}{\sqrt{n}} s_q = \underline{1.204} \times \underline{0.011} = \underline{0.013},$$

and the combined uncertainty

$$u_q = \sqrt{\Delta_{q,B}^2 + \Delta_{q,A}^2} = \sqrt{\underline{0.013^2 + \cancel{0.016^2}\ 0.023^2}} = \underline{0.03}.$$

A single measurement for ten periods is recorded as $T_{10} = \underline{15.852} \pm \underline{0.001}$ [S]. Hence $T = \underline{1.5852} \pm \underline{0.0001}$ [S].

Then the uncertainty propagation equation is used to calculate the uncertainty for the damping coefficient $\beta = \frac{1}{5T}q$ as

$$\boxed{u_\beta} = \sqrt{\left(\frac{\partial\beta}{\partial T}\right)^2 u_T^2 + \left(\frac{\partial\beta}{\partial q}\right)^2 u_q^2} = \sqrt{\left(-\frac{q}{5T^2}\right)^2 u_T^2 + \left(\frac{1}{5T}\right)^2 u_q^2}$$

$$= \sqrt{\left(-\frac{0.4456}{5\cdot 1.5852^2}\right)^2 \cdot 0.0001^2 + \left(\frac{1}{5\cdot 1.5852}\right)^2 \cdot 0.03^2} = \boxed{0.004\ [\ s^{-1}\ ]}$$

with relative uncertainty

$$\boxed{u_{r,\beta}} = \frac{u_\beta}{\beta} \times 100\% = \boxed{7\ \%}$$

WS-3 The θ_{st}-ω and φ-ω Characteristics of Forced Oscillations

On the graphs included in the report, the uncertainty is shown in the form of error bars.[1] In both the φ vs. (ω/ω_0) graph and the θ_{st} vs. (ω/ω_0) graph, the

[1] Please follow this part to find the uncertainties and mark them on the graphs of the phase shift φ vs. (ω/ω_0) graph and the amplitude of steady-state oscillations θ_{st} vs. (ω/ω_0).

measurements of φ and θ_{st} are single measurements with uncertainty 1 °, determined by the resolution of our equipment. However, to find the uncertainty of (ω/ω_0) we need to derive it from the uncertainty propagation formula. Let us introduce symbols $Q = \frac{\omega}{\omega_0}$, $T_{10,\text{natural}} = N$ and $T_{10,\text{driven}} = D$, where the uncertainty of D is again the minimum scale (resolution) of the equipment used. Since these are single measurements, we have

$$Q = \frac{\omega}{\omega_0} = \frac{T_{10,\text{natural}}}{T_{10,\text{driven}}} = \frac{N}{D}$$

and the uncertainty of the ratio Q, found from the uncertainty propagation formula, is

$$u_Q = \sqrt{\left(\frac{\partial Q}{\partial N}u_N\right)^2 + \left(\frac{\partial Q}{\partial D}u_D\right)^2} = \sqrt{\left(\frac{u_N}{D}\right)^2 + \left(\frac{N u_D}{D^2}\right)^2}$$

In particular, with $N =$ 15.811 [S], $u_N =$ 0.002 [S], and $u_D =$ 0.001 [S], so with every set of N and D a unique uncertainty is generated. For instance,[2] for $D =$ 16.426 [S], we can calculate Q as

$$Q = \frac{N}{D} = \frac{15.811}{16.426} = 0.9626$$

with uncertainty u_Q calculated as

$$\boxed{u_Q} = \sqrt{\left(\frac{0.002}{16.426}\right)^2 + \left(\frac{15.811 \cdot 0.001}{16.426^2}\right)^2} = \boxed{1.4 \times 10^{-4}},$$

and

$$\boxed{u_\varphi = 1° = 0.017 \text{ rad}} \quad \boxed{u_{\theta_{st}} = 1° = 0.017 \text{ rad}}.$$

[2] Here, based on your measurement data, give one sample calculation for a chosen value of ω/ω_0. All values of the calculated uncertainties u_Q that you have used to plot error bars, should be given in the *Results* section, where tables with the data for the plots φ vs. (ω/ω_0) and θ_{st} vs. (ω/ω_0) is included.