UM–SJTU JOINT INSTITUTE
PHYSICS LABORATORY
(VP141)

LABORATORY REPORT

EXERCISE 5

DAMPED AND DRIVEN OSCILLATIONS.
MECHANICAL RESONANCE

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1 INTRODUCTION

In our daily life, oscillation occurs everywhere. From the resonator of the string instruments, to the construction of hyperelastic membrane are both examples of oscillation in our world. To further study the problem, we assume that a force changing periodically with time is applied to a damped harmonic oscillator, i.e.

$$F_{dr} = F_0 cos\omega t$$

where F_{dr} is the external force called driving force, F_0 is the amplitude and ω is its angular frequency. After a period of time the motion will tend to be in equilibrium. The steady state is a simple harmonic oscillation with an angular frequency equal to that of the driving force. The amplitude depends on the driving force as well as the damping coefficient. In brief, such phenomenon is often called mechanical resonance.

Interestingly, after the oscillation reaches its equilibrium state, there will be a phase lag between the displacement from equilibrium position and the driving force. Particularly, the value equals to $\frac{\pi}{2}$ when the frequency of driving force is equal to the natural frequency.

In our experiment, we will focus on forced oscillation of a balance wheel. Electromagnets provide us with a damping force. Since in our experiment, our system is rotating, the quantities are all expressed in terms of angular value. For example, the amplitude is expressed in terms of its angular equivalent.

Then, we try to find out its numerical value. Assume that the wheel is exposed to a period driving torque $\tau_{dr} = \tau_0 cos\omega t$ and a damping torque $\tau_d = -b\frac{d\theta}{dt}$, with a restoring torque $\tau = -k\theta$. After that, we can write the equation of motion of the system:

$$I\frac{d^2\theta}{dt^2} = -k\theta - b\frac{d\theta}{dt} + \tau_0 cos\omega t \tag{1}$$

where I refers to the moment of inertia, and τ_0 as well as ω are the amplitude and frequency of τ_{dr} respectively. Then, we change its form, Eq.(1) turns out to be:

$$\frac{d^2\theta}{dt^2} + \frac{b}{I}\frac{d\theta}{dt} + \frac{k}{I}\theta = \frac{\tau_0}{I}cos(\omega t)$$

To further simplify its form, we replace some of parameters with the following symbols:

$$\omega_0^2 = \frac{k}{I},\ 2\beta = \frac{b}{I},\ \mu = \frac{\tau_0}{I}$$

Then, Eq.(1) can be expressed as:

$$\frac{d^2\theta}{dt^2} + 2\beta\frac{d\theta}{dt} + \omega_0^2\theta = \mu cos(\omega t) \tag{2}$$

Now, Eq(2) is an inhomogeneous ODE. We first consider the simplest case when $\mu = 0$. Then, it's an equation of motion for a damped harmonic oscillator. For a damped harmonic oscillator, the following possibilities exist: underdamped regime, overdamped regime and critical damping, and each of them has different $x-t$ curves. Furthermore, if in this case additionally $\beta = 0$ always holds, then the motion becomes a simple harmonic oscillation.

However, if $\mu \neq 0$, it's a little bit complicated. We solve the equation by first assuming the solution has the form:

$$\widetilde{\theta}(t) = Ae^{i(\omega_{dr}t+\varphi)}$$

Then, we obtain

$$\widetilde{\dot{\theta}(t)} = iA\omega_{dr}e^{i(\omega_{dr}t+\varphi)} = i\omega_{dr}\widetilde{\theta}(t)$$

$$\widetilde{\ddot{\theta}(t)} = -A\omega_{dr}^2 e^{i(\omega_{dr}t+\varphi)} = -\omega_{dr}^2\widetilde{\theta}(t)$$

Plug them in Eq.(2), and we will obtain

$$[(\omega_0^2 - \omega_{dr}^2) + 2i\beta\omega_{dr}]Ae^{i(\omega_{dr}t+\varphi)} = \mu cos\omega t \tag{3}$$

Since we have

$$e^{i(\omega t+\varphi)} = cos(\omega t + \varphi) + isin(\omega t + \varphi)$$

we further obtain that

$$Re\ e^{i(\omega t+\varphi)} = cos(\omega t + \varphi)$$

Plug it back to Eq.(3), and because $\omega_{dr} = \omega$, we have

$$[(\omega_0^2 - \omega^2) + 2i\beta\omega]Ae^{i\varphi} = \mu \tag{4}$$

For $(\omega_0^2 - \omega^2) + 2i\beta\omega$ from the left hand side, we transform it into polar coordinates, and we will obtain:

$$(\omega_0^2 - \omega^2) + 2i\beta\omega = \sqrt{(\omega_0^2 - \omega^2) + 4\beta^2\omega^2}\ e^{i\varphi'}$$

where $\varphi' = arctan(\frac{2\beta\omega}{\omega_0^2-\omega^2})$. Then, we plug it in Eq.(4), and we acquire immediately

$$\sqrt{(\omega_0^2 - \omega^2) + 4\beta^2\omega^2}Ae^{i\varphi'}e^{i\varphi} = \mu \tag{5}$$

In order that Eq.(5) holds, $\varphi' = -\varphi$. Hence, we finally obtain the value of A and φ:

$$\begin{cases} A = \dfrac{\mu}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \\ \varphi = arctan(\dfrac{2\beta\omega}{\omega^2 - \omega_0^2}) \end{cases}$$

Therefore, we are able to calculate the solution to Eq.(2) in a general form, i.e.

$$\theta(t) = \theta_{tr}(t) + \theta_{st}cos(\omega t + \varphi) \tag{6}$$

where θ_{tr} refers to the transient state which vanishes exponentially with the increase of t, and $\theta_{st}cos(\omega t + \varphi)$ refers to the steady-state situation, with an amplitude of the value:

$$\theta_{st} = \frac{\mu}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

and the phase shift

$$\varphi = arctan(\frac{2\beta\omega}{\omega^2 - \omega_0^2})$$

If we want to find out the maximum value of θ_{st}, we can obtain the resonance angular frequency and its amplitude as follows:

$$\omega_{max} = \omega_{res} = \sqrt{\omega_0^2 - 2\beta^2}$$

$$\theta_{max} = \theta_{res} = \frac{\mu}{2\beta\sqrt{\omega_0^2 - \beta^2}}$$

Specifically, if the damping coefficient is small, then the resonance angular frequency has a close value to the natural one. In Figure 2, the dependence of amplitude and phase shift on driving frequency are shown respectively [1].

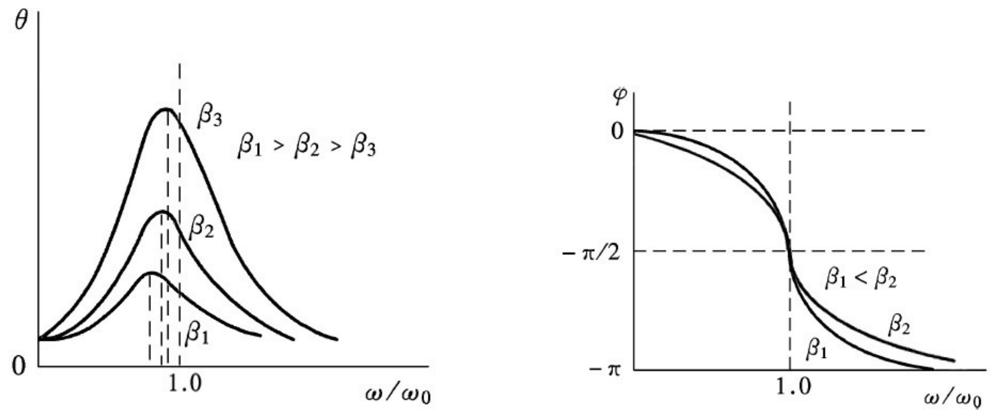


Figure 1: The dependence of amplitude and phase shift on driving frequency [1]

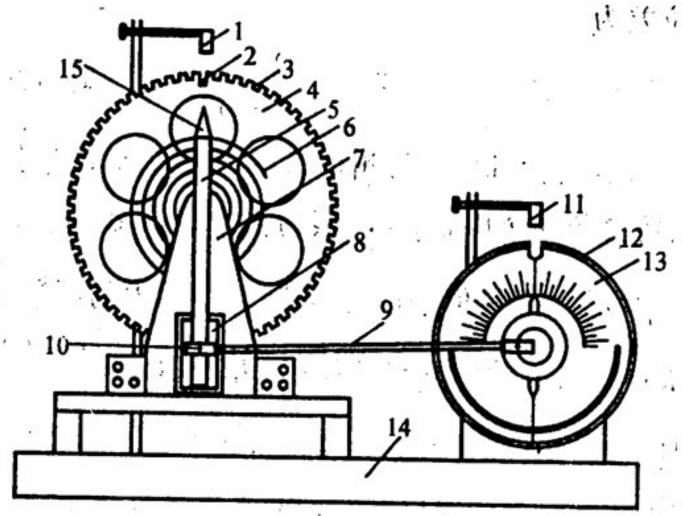
2 Apparatus and Experimental Setup

The main apparatus we use in experiment is the BG-2 Pohl resonator, which is shown in Figure 1. BG-2 Pohl Resonator consists of two parts: a vibrometer and a control box.

The most noticeable part of the resonator is its balance wheel. The wheel is placed on a supporting nod, with its axis attached to a scroll spring, providing the wheel with restoring force to make it rotate periodically around its equilibrium position.

The edge of the balance wheel consists of many grooves. A deep notch with a photodetector is arranged above one of them. The detector can be used to detect amplitudes and oscillation periods.

To equip the apparatus with the capability to create damping force, a pair of coils is placed at the bottom of the wheel. Due to some complicated electromagnetic actions, a damping force will be created so that we can simulate the cases of forced oscillations.



1. Photoelectric detector (光电门)
2. Deep notch (深凹槽)
3. Shallow notch (浅凹槽)
4. Copper balance wheel (铜质摆轮)
5. Swinging rod (摇杆)
6. Scroll spring (蜗卷弹簧)
7. Supporting frame (支撑架)
8. Damping coil (阻尼线圈)
9. Connecting rod (连杆)
10. Adjustment screw (调节螺钉)
11. Photoelectric detector
12. Angle scale (角度盘)
13. Glass turntable (玻璃转盘)
14. Pedestal (底座)
15. Locking screw (夹持螺钉)

Figure 2: The BG-2 Pohl resonator [1]

As for how to use the resonator, there are a few knobs on it. For **Period Selection** switch, we can choose how to measure the period: 1-period mode and 10-period mode. For **Period of Driving Force** knob, the frequency and amplitude of the driving force can be changed accurately. However, the particular scale on it is inaccurate. Thus only the trend is controllable in our experiment.

In terms of **Damping Selection** nob, there are six options, i.e. ”0”, ”1”, ”2”, ”3”, ”4”,

”5”, ”6” respectively. The current increase from 0 to 0.6 A with the increase of option numbers. In this experiment, option ”2”, ”3”, ”4” are used as damping mode.

There is also a **Strobe** bottom on our resonator. By turning on this bottom a flash light will be emitted so that we are able to read out the phase difference. However, in order to protect the strobe, only when measuring the phase difference will we turn on the bottom.

Also, there is a **Motor Switch** bottom which is used to control motor. When measuring natural angular frequency and damping coefficient, the bottom ought to be turned off.

For the detail position of these bottoms, see the following Figure 3.

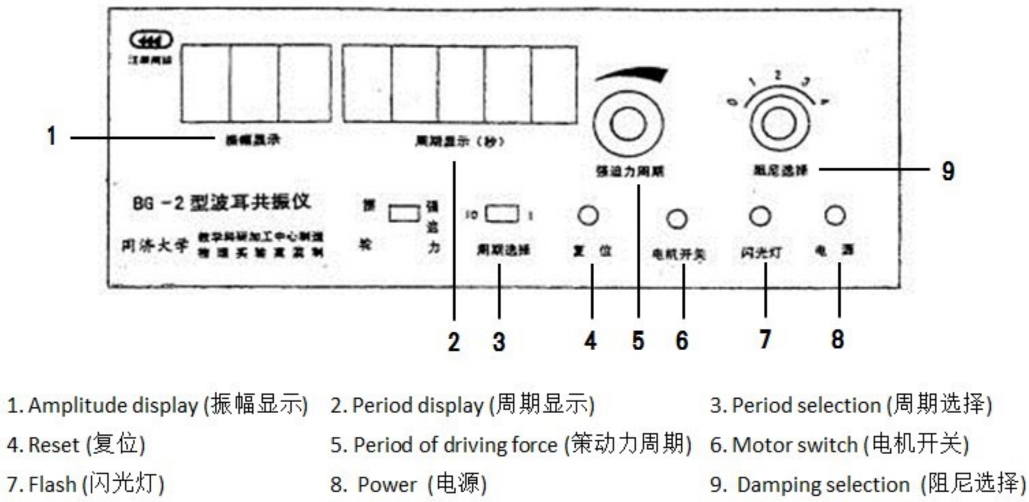


Figure 3: The panel of control box [1]

3 Measurements

3.1 Procedures [1]

3.1.1 Natural Angular Frequency

(1) Switch the **Damping Selection** bottom to ”0”.

(2) Rotate the balance wheel to approximately 150° and then release it. Select the **Period Selection** switch to ”10”, record the period.

(3) Repeat four times and then obtain the natural angular frequency.

3.1.2 Damping coefficient

(1) Switch the **Damping Selection** bottom to ”2”.

(2) Rotate the balance wheel to approximately $150°$ and then release it. Record each amplitude (omit the first one) and the corresponding 10-periods.

(3) Calculate the damping coefficient through the following format:

$$ln\frac{\theta_i}{\theta_j} = (j - i)\beta t$$

(4) In our experiment, calculate T as the average period, then plug it in the equation:

$$\beta = \frac{1}{5T}ln\frac{\theta_i}{\theta_{i+5}}$$

3.1.3 The $\theta_{st} - \omega$ AND $\varphi - \omega$ CHARACTERISTICS OF FORCED OSCILLATIONS

(1) Switch the **Damping Selection** bottom to ”2”. Set the speed of motor to one end, and after it reaches the steady state, record the amplitude θ_{st}, the period T, and the phase shift φ.

(2) Change the speed of motor slowly, then record the previous three values when it become steady. Collect at least 15 data points.

(3) Switch the **Damping Selection** bottom to ”1” or ”3” and repeat step (1) and (2).

(4) Plot $\theta_{st}(\omega) - \omega/\omega_0$ and $\varphi(\omega) - \omega/\omega_0$ characteristics respectively.

3.2 COMMENTS/OBSERVATIONS REGARDING THE MEASUREMENTS

In this experiment, there are some tips that we should take care of:

(1) When calculating the natural angular frequency in Procedure 3.1.1, be careful with its uncertainty.

(2) In Procedure 3.1.2, omit the first amplitude before released because it may have a large deviation.

(3) In Procedure 3.1.2, if some data is missed in the step (2), the whole step should be done again.

(4) In Procedure 3.1.3, remember to wait a few seconds so that you determine the wheel has already been in steady state. Otherwise there may be some errors take place.

4 RESULTS

4.1 NATURAL ANGULAR FREQUENCY

The natural angular frequency was measured by releasing the balance wheel from 150 ° and the corresponding 10 periods have been recorded in table 1 as

	$10T[s] \pm 0.001[s]$
1	15.812
2	15.810
3	15.810
4	15.811

Table 1: Data table for Natural Angular Frequency

Hence, we obtain the average period of the oscillator as follows:

$$\overline{10T} = \frac{1}{4}\sum_{i=1}^{4} 10T_i = 15.811[s] \pm 0.002[s]$$

$$\overline{T} = \frac{\overline{10T}}{10} = 1.5811[s] \pm 0.0002[s]$$

with a relative uncertainty $u_T = 0.01\%$.

Eventually, we can calculate the natural angular frequency

$$\omega_0 = \frac{2\pi}{T} = 3.9739[s^{-1}] \pm 0.0004[s^{-1}]$$

with a relative uncertainty $u_{\omega_0} = 0.01\%$. (The detailed calculations are shown in the Worksheet).

4.2 DAMPING COEFFICIENT

According to our measurements, our data is collected in Table 2. From the reading on the control box, our damping selection and period of damping are shown as follows:

Damping Selection: 2

$$T = \frac{10T}{T} = 1.5852[s] \pm 0.0001[s]$$

According to the formula we deduced in 3.1.2, we can calculate the damping coefficient accordingly.

$$\beta = \frac{1}{5T} ln \frac{\theta_i}{\theta_{i+5}} = 0.056[s^{-1}] \pm 0.004[s^{-1}]$$

with a relative uncertainty $u_\beta = 7\%$. (The detailed calculations are shown in the Worksheet).

$10T = 15.852[s] \pm 0.001[s]$ & Damping Selection: 2				
Amplitude$[^\circ] \pm 1[^\circ]$		Amplitude$[^\circ] \pm 1[^\circ]$		ln(θ_i/θ_{i+5})
θ_0	116	θ_5	73	0.4631
θ_1	105	θ_6	67	0.4493
θ_2	96	θ_7	62	0.4372
θ_3	87	θ_8	56	0.4406
θ_4	79	θ_9	51	0.4376
The average value of ln(θ_i/θ_{i+5})				0.4456

Table 2: Data table for Resonance Method

4.3 The $\theta_{st} - \omega$ and $\varphi - \omega$ Characteristics of Forced Oscillations

In this part, we use two different damping coefficients to measure the corresponding $\theta_{st} - \omega$ and $\varphi - \omega$ characteristics and then compare the following properties: the characteristics of the relationship between amplitude/phase lag and driving frequency, and their trends when changing the damping coefficients. We do two experiments choosing damping selection ”2” and ”3” respectively, and then record the data as follows (Table 3 and Table 4):

Then, using two sets of data, we are able to plot two curves. The figures are shown below (Figure 4 and Figure 5).

From Fig. 4, we are able to find out that the amplitude reaches maximum when ω/ω_0 reaches 1, and the plot shows a symmetric structure about the line $\omega/\omega_0 = 1$. We can also discover that with the increase of the damping force, the maximum value of amplitude decrease accordingly, and the symmetric line ω/ω_0 becomes slightly lower than 1, which moves leftwards.

From Fig.5, we are able to find out that the phase lag between the driving force and actual position φ is always negative, and with the increase of ω/ω_0, the value of phase lag increase accordingly. Also, we discover that when ω/ω_0 is close to 1, where the phase lag is close to $\frac{\pi}{2}$, the phase lag tends to increase at a large speed. Moreover, if we choose a larger damping force, the general slope of the figure will become more steady.

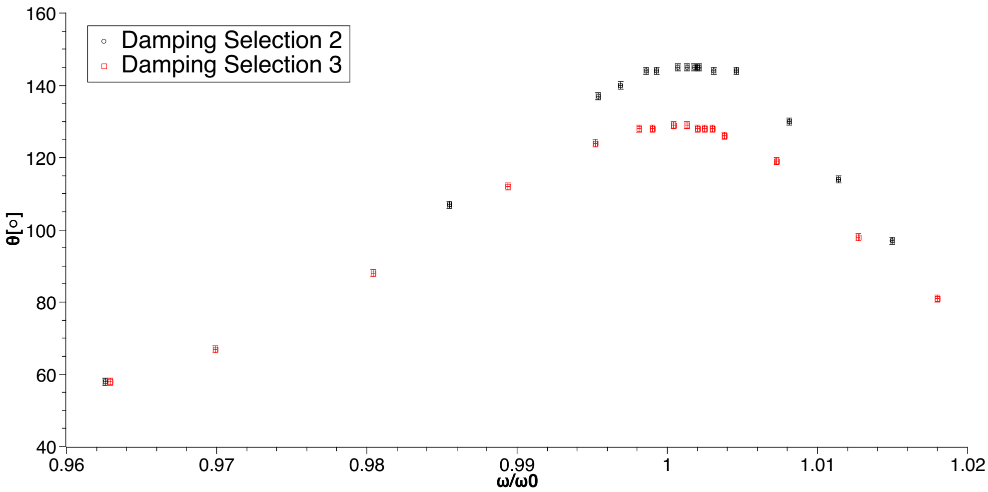


Figure 4: Characteristics of $\theta_{st} - \omega$

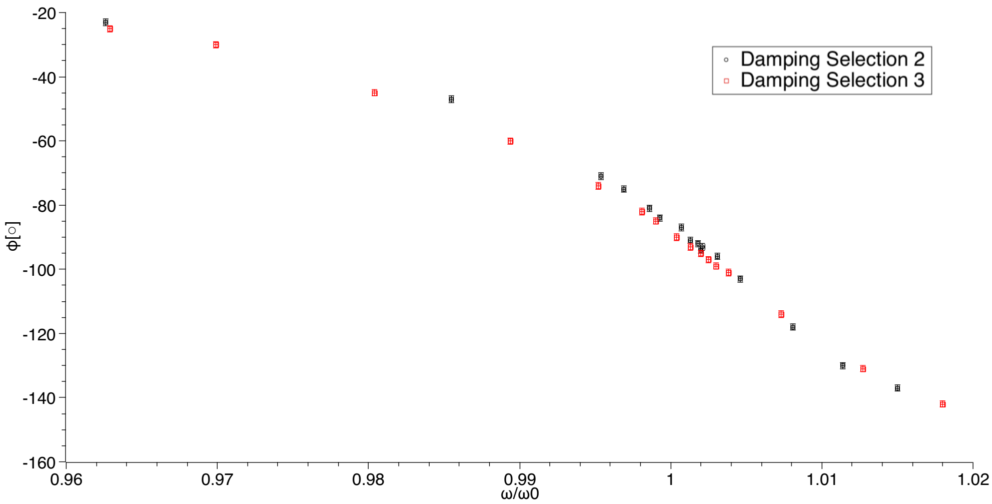


Figure 5: Characteristics of $\varphi - \omega$

Damping Selection: 2					
	$10T[s] \pm 0.001[s]$	$\varphi[^\circ] \pm 1[^\circ]$	$\theta[^\circ] \pm 1[^\circ]$	ω/ω_0	u_{ω/ω_0}
1	16.426	23	58	0.9626	1.4×10^{-4}
2	16.044	47	107	0.9855	1.4×10^{-4}
3	15.884	71	137	0.9954	1.4×10^{-4}
4	15.860	75	140	0.9969	1.4×10^{-4}
5	15.833	81	144	0.9986	1.4×10^{-4}
6	15.822	84	144	0.9993	1.4×10^{-4}
7	15.800	87	145	1.0007	1.4×10^{-4}
8	15.790	91	145	1.0013	1.4×10^{-4}
9	15.783	92	145	1.0018	1.4×10^{-4}
10	15.778	93	145	1.0021	1.4×10^{-4}
11	15.780	94	145	1.0020	1.4×10^{-4}
12	15.762	96	144	1.0031	1.4×10^{-4}
13	15.739	103	144	1.0046	1.4×10^{-4}
14	15.684	118	130	1.0081	1.4×10^{-4}
15	15.633	130	114	1.0114	1.4×10^{-4}
16	15.577	137	97	1.0150	1.4×10^{-4}

Table 3: Data table for $\theta_{st} - \omega$ and $\varphi - \omega$ characteristics

Damping Selection: 3					
	$10T[s] \pm 0.001[s]$	$\varphi[°] \pm 1[°]$	$\theta[°] \pm 1[°]$	ω/ω_0	u_{ω/ω_0}
1	16.420	25	58	0.9629	1.4×10^{-4}
2	16.301	30	67	0.9699	1.4×10^{-4}
3	16.126	45	88	0.9804	1.4×10^{-4}
4	15.981	60	112	0.9894	1.4×10^{-4}
5	15.888	74	124	0.9952	1.4×10^{-4}
6	15.841	82	128	0.9981	1.4×10^{-4}
7	15.827	85	128	0.9990	1.4×10^{-4}
8	15.804	90	129	1.0004	1.4×10^{-4}
9	15.790	93	129	1.0013	1.4×10^{-4}
10	15.780	95	128	1.0020	1.4×10^{-4}
11	15.772	97	128	1.0025	1.4×10^{-4}
12	15.764	99	128	1.0030	1.4×10^{-4}
13	15.751	101	126	1.0038	1.4×10^{-4}
14	15.697	114	119	1.0073	1.4×10^{-4}
15	15.612	131	98	1.0127	1.4×10^{-4}
16	15.531	142	81	1.0180	1.4×10^{-4}

Table 4: Data table for $\theta_{st} - \omega$ and $\varphi - \omega$ characteristics

5 Conclusion and Discussion

The experiment is divided into three parts. In the first part, we measured the natural frequency of the balance wheel. By turning the "Period Selection" to "10", we measured the 10 periods of the oscillation. After repeating four times, we finally obtained 4 data. Then we use the formula to calculate out our natural frequency:

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\overline{10T}/10}$$

Then we got our natural frequency as $\omega_0 = 3.9739[s^{-1}] \pm 0.0004[s^{-1}]$, with a relative uncertainty 0.01%.

Generally speaking, the uncertainty is satisfactory for us. However, we still do some improvements about it: Although the certainty is not large, we will use the natural frequency throughout the experiment. Hence even a minor improvement will be beneficial to our overall precision of experiment later. Since the uncertainty mainly comes from Type A and Type B, and the Type A uncertainty occurs due to the limitations of the apparatuses, we can try our best to decrease the Type B uncertainty. In our experiment, we measured 4 data for the period. If we are able to collect more data. For example, we repeat the experiment for 10 times, then chances are that our uncertainty of natural frequency become lower.

In the next part, we measured the damping coefficient. We recorded the changing amplitudes successively, and then find out the damping coefficient through the following formula:

$$\beta = \frac{1}{5T} ln \frac{\theta_i}{\theta_{i+5}}$$

Then we obtained our results: $\beta = 0.056[s^{-1}] \pm 0.004[s^{-1}]$, with a relative uncertainty about 7%.

In this experiment, we have a relative large deviation compared with the previous errors. The reasons are complicated. From one aspect, the measurements becomes more complex, which makes the uncertainty more likely to increase. From another aspect, In this process, since we should set the balance wheel at 150 degrees at rest before letting it rotate, the uncertainty may comes from our misplacing of the wheel. In other words, it hard for us to place the wheel at accurate 150 degrees simply by eyes. Hence large uncertainty may occur here. Moreover, if we take into consideration the influence of air drag or the instability of the resonator, the uncertainty will be even larger.

To minimize our uncertainty, we can first release the balance in the exact angle of 150 degrees with the help of the angular ruler. Also, we can try to do the experiment in the place where the wind is relatively gentle, so that the disturbance will be less, and our results will become more accurate

Finally, in the last part, we focused on the situation where a periodically changing external force is added to our system. According to our deduction in Introduction part, the relationship between θ_{st}/φ and ω should be as follows:

$$\theta_{st} = \frac{\mu}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

$$\varphi = arctan(\frac{2\beta\omega}{\omega^2 - \omega_0^2})$$

Since in this experiment, the uncertainty can't be easily analysed numerically, we plot the theoretical curve for our experiment. With the help of Matlab, the theoretical plot for θ_{st} vs. ω are shown as follows (Fig.6 and Fig.7):

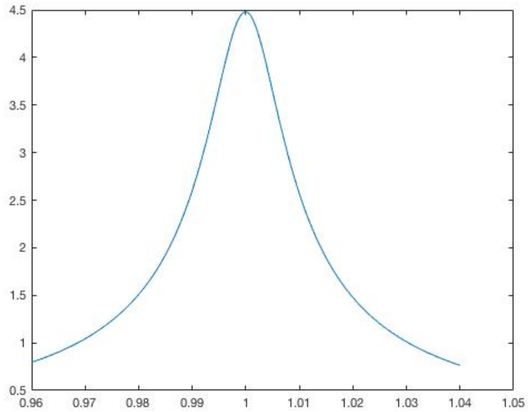


Figure 6: Theoretical characteristics of $\theta_{st} - \omega$

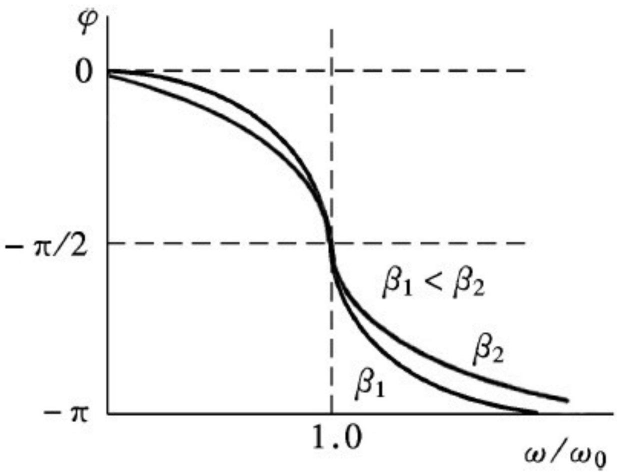


Figure 7: Theoretical characteristics of $\varphi - \omega$ [1]

From the comparison of our experimental plot and theoretical plot, we find out that there exists some uncertainties.

One significant difference is that our plot is not as smooth as the theoretical plot, which may be caused by the instability of our reading. To be more specific, we may not wait enough time before the system becomes completely steady. As a result, the data we collected may be those who are not at equilibrium, leading to the not-smooth curve. Besides, our curve tends to be a little bit "steep" compared with the theoretical one. This may comes from the existence of air drag, which makes the curve an irreversible deformation.

To minimize these issues, first, we can be patient while doing our experiment. We should wait a few seconds so that we make sure our balance wheel has reaches its equilibrium state. After that, the data collected can be accurate. Besides, in the process we begin to measure its period, during which the amplitude begins to change, it means we are not at the equilibrium. Hence in this case we should measure the period again.

Moreover, in terms of air drag, we can do this experiment at a wind-gentle environment with windows closed, and during the process of experiment, we shouldn't move or disturb the resonator. In this way the external disturbance can be made as less as possible, which results in the decrease of error.

Lastly, in our lab, there is another place which we can improve, i.e., when collecting the data, it's better for us to collect more data when ω/ω_0 is approaching 1, because at this time the curve tends to decrease largely. Thus more data points can indicate a more accurate result. Besides, when collecting the data, we 'd better control our phase lag so that φ is symmetric about the line $\omega/\omega_0 = 1$ so that we can better find out the general trend. To be more specific, in our data, if we could collect a data point where $\varphi \approx 160°$, our plot will be symmetric and closer to the theoretical one.

A MEASUREMENT UNCERTAINTY ANALYSIS

See the Worksheet.

B DATA SHEET

See the attached data sheet.

References

[1] Krzyzosiak, M. & VP141 TA Groups. *Exercise 5 - lab manual [rev 4.3].pdf*. 2019.