# Production Functions with Constant Elasticities of Substitution [1]

I. In the present note, we are concerned with characterizing the class of production functions for which elasticities of substitution are all constant regardless of factor prices.

II. In the case of two factors of production, the characterization of such production functions was discussed by Arrow, Chenery, Minhas, and Solow [2].

Let $f(x_1, x_2)$ be a production function where $x_1$ and $x_2$ resepctively represent the amounts of factors 1 and 2 employed. Production is assumed to be subject to constant returns to scale and to diminishing marginal rates of substitution. As is discussed in Hicks [3], pp. 241-46, the elasticity of substitution $\sigma$ may be defined by:

$$\sigma = \frac{\dfrac{\partial f}{\partial x_1}\,\dfrac{\partial f}{\partial x_2}}{f(x_1,x_2)\,\dfrac{\partial^2 f}{\partial x_1\,\partial x_2}} \tag{1}$$

It has been shown in [2] that the elasticity of substitution $\sigma$ is constant regardless of factor inputs $x_1$ and $x_2$ if and only if the production function $f(x_1,x_2)$ is of the following form:

$$f(x_1,x_2) = (\alpha_1 x_1{}^{-\beta} + \alpha_2 x_2{}^{-\beta})^{-1/\beta}, \tag{2}$$

where $\alpha_1$ and $\alpha_2$ are positive constants and

$$\beta = \frac{1}{\sigma} - 1. \tag{3}$$

The production function of the form (2) was first introduced by Solow [8] as an example to illustrate his model of economic growth.

III. The elasticity of substitution may be in several ways generalized to the case in which more than two factors of production are involved.[2] In what follows, we shall adapt the definition of partial elasticities of substitution as introduced in Allen [1], pp. 503-9.

Let the number of factors of production be $n$ and $f(x) = f(x_1, \ldots, x_n)$ a production function; $x_1, \ldots, x_n$ represent the amounts of factors of production $1, \ldots, n$ employed.

[1] This work in part was supported by the Office of Naval Research under Contract NR-047-004 at Stanford University and was written while the author was a Fellow at the Center for Advanced Study in the Behavioral Sciences. He is very much indebted to Dan McFadden and Marc Nerlove for valuable comments and suggestions.

[2] For the definition of the elsaticity of substitution, see, e.g., Hicks [3], p. 117; Lerner [4]; Robinson [5], pp. 256-7; Allen [1], pp. 340-3, pp. 503-9.

It is again assumed that production is subject to constant returns to scale and to diminishing marginal rates of substitution. Allen's *partial elasticity of substitution* $\sigma_{ij}$ between two factors of production, say factors $i$ and $j$ ($i \neq j$), is defined by:

$$\sigma_{ij} = \frac{x_1 f_1 + \ldots + x_n f_n}{x_i x_j} \frac{F_{ij}}{F}, \tag{4}$$

where

$$f_i = \frac{\partial f}{\partial x_i}, \; f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j},$$

$$F = \det \begin{bmatrix} 0, & f_1, \ldots, & f_n \\ f_1, & f_{11}, \ldots, & f_{1n} \\ \ldots & \ldots & \ldots \\ f_n, & f_{n1}, \ldots, & f_{nn} \end{bmatrix}$$

and $F_{ij}$ is the co-factor of the element $f_{ij}$ in the determinant $F$.

By definition (4), $\sigma_{ij}$ are symmetric; i.e.,

$$\sigma_{ij} = \sigma_{ji}, \qquad \text{for all } i \neq j. \tag{5}$$

IV. Before we investigate the implications of constancy of partial elasticities of substitution $\sigma_{ij}$, we shall first transform the definition (4) into one in terms of the unit cost function.

Let $p = (p_1, \ldots, p_n)$ be a vector of prices of factors of production; $p_i$ are assumed all positive. The vector $x = (x_1, \ldots, x_n)$ of factor inputs that minimizes the unit cost

$$\sum_{i=1}^{n} p_i x_i$$

subject to

$$f(x_1, \ldots, x_n) = 1$$

is uniquely determined. We may write

$$x_1 = x_1(p), \ldots, x_n = x_n(p),$$

$$\lambda = \lambda(p) = \sum_{i=1}^{n} p_i x_i(p).$$

The function $\lambda(p)$ will be referred to as the unit cost function associated with the production function $f(x)$. The factor input functions $x_i(p)$ are all homogeneous of degree zero, while the unit cost function $\lambda(p)$ is homogeneous of degree one.

In what follows it will be assumed that the factor input functions $x_i(p)$ are positive for all $p > 0$ and have continuous partial derivatives of the third order.

The definition (4) may be written as (see Allen [1], p. 508)

(6) $$\sigma_{ij} = \frac{\lambda \dfrac{\partial x_i}{\partial p_j}}{x_i x_j}, \qquad \text{for } i \neq j.$$

Since we have the following relations (see, e.g., Samuelson [6], p. 68):

(7) $$x_i = \frac{\partial \lambda}{\partial p_i} \qquad i = 1, \ldots, n,$$

and

(8) $$\frac{\partial x_i}{\partial p_j} = \frac{\partial^2 \lambda}{\partial p_i\, \partial p_j} \qquad i,j = 1, \ldots, n,$$

the elasticity of substitution $\sigma_{ij}$ may be written:

(9) $$\sigma_{ij} = \frac{\lambda \dfrac{\partial^2 \lambda}{\partial p_i\, \partial p_j}}{\dfrac{\partial \lambda}{\partial p_i} \dfrac{\partial \lambda}{\partial p_j}} \qquad \text{for } i \neq j,$$

where $\lambda = \lambda(p)$ is the unit cost function.

Let

(10) $$\Lambda = \Lambda(p) = \log \lambda(p);$$

then, the relation (9) is equivalent to the following:

$$\sigma_{ij} - 1 = \frac{\dfrac{\partial^2 \Lambda}{\partial p_i\, \partial p_j}}{\dfrac{\partial \Lambda}{\partial p_i} \dfrac{\partial \Lambda}{\partial p_j}}, \qquad \text{for } i \neq j,$$

or

(11) $$\frac{\partial^2 \Lambda}{\partial p_i\, \partial p_j} = (\sigma_{ij} - 1) \frac{\partial \Lambda}{\partial p_i} \frac{\partial \Lambda}{\partial p_j}, \text{ for } i \neq j.$$

V. The production function which extends the Arrow-Chenery-Minhas-Solow function to the $n$-factor case may be the following type:

(12) $$f(x_1, \ldots, x_n) = (\alpha_1 x^{-\beta} + \ldots + \alpha_n x^{-\beta})^{-1/\beta},$$

where $\alpha_1, \ldots, \alpha_n$ are positive constants and $\beta$ a number greater than $-1$.

The function $f(x_1, \ldots, x_n)$ is homogeneous of degree one, strictly quasi-concave, and has partial derivatives of any order. The unit cost function associated with the production function $f(x_1, \ldots, x_n)$ is given by:

(13) $$\lambda(p_1, \ldots, p_n) = (\alpha_1^{\sigma} p_1^{1-\sigma} + \ldots + \alpha_n^{\sigma} p_n^{1-\sigma})^{1/1-\sigma},$$

where

$$\sigma = \frac{1}{1+\beta}. \tag{14}$$

Hence,

$$\sigma_{ij} = \sigma, \qquad \text{for all } i \neq j. \tag{15}$$

*Therefore, if the production function* $f(x_1, \ldots, x_n)$ *is of the form* (12) *then the partial elasticities of substitution* $\sigma_{ij}$ *are independent of factor prices and are identical for all pairs of two factors of production.*

*On the other hand, if partial elasticities of substitution* $\sigma_{ij}$ *are all constant and identical for different pairs of factors, then the production function* $f(x_1, \ldots, x_n)$ *is of the form* (12).

In order to prove the latter statement, it suffices to show that if elasticities of substitution $\sigma_{ij}$ are constant and identical, say equal to $\sigma$, then the unit cost function $\lambda(p_1, \ldots, p_n)$ must be of the form (13), since by Shepherd's duality theorem ([7], pp. 17-22) the unit cost function uniquely determines the production function (in the case of constant returns to scale). In the case where $\sigma = 1$, it is easily shown from (11) that $\Lambda = \log \lambda$ is additive. But the unit cost function $\lambda = \lambda(p_1, \ldots, p_n)$ is homogeneous of degree one; hence it must be the limit of the form (13) as $\sigma$ tends to one:

$$\lambda(p_1, \ldots, p_n) = p_1^{\alpha'_1} \ldots p_n^{\alpha'_n}, \text{ with } \alpha'_1, \ldots, \alpha'_n > 0.$$

In the case where $\sigma \neq 1$, consider the following transformation:

$$z = \lambda^{1-\sigma}, \quad u_i = p_i^{1-\sigma}, \qquad i = 1, \ldots, n.$$

Then from (9) and (15) we have

$$\frac{\partial^2 z}{\partial u_i \, \partial u_j} = 0, \qquad \text{for } i \neq j.$$

But, the function $z = z(u_1, \ldots, u_n)$ is homogeneous of degree one with respect to $u_1, \ldots, u_n$; hence, $z$ is a linear function of $u_1, \ldots, u_n$. Therefore, the unit cost function $\lambda(p_1, \ldots, p_n)$ is of the form (13).

VI. The problem naturally would arise if it were possible to find a production function with more than two factors of production for which elasticities of substitution are all constant but may differ for different pairs of factors of production.

Let $\{N_1, \ldots, N_S\}$ be a partition of the set $\{1, \ldots, n\}$ of $n$ factors of production; namely

$$N_1 \cup \ldots \cup N_S = \{1, \ldots, n\},$$
$$N_s \cap N_t = \text{empty}, \qquad \text{for all } s \neq t.$$

The vector $x = (x_1, \ldots, x_n)$ may be correspondingly partitioned into a set of subvectors:

$$x = (x^{(1)}, \ldots, x^{(S)}),$$

where $x^{(s)}$ is the subvector of $x$ whose components are $x_i$, $i \in N_s$. Similarly for price vector $p = (p_1, \ldots, p_n)$:

$$p = (p^{(1)}, \ldots, p^{(S)}).$$

Consider now a production function $f(x)$ defined by:

(16) $$f(x) = \prod_{s=1}^{S} f^{(s)}(x^{(s)})^{\rho_s},$$

where

(17) $$f^{(s)}(x^{(s)}) = \Big(\sum_{i \in N_s} \alpha_i x_i^{-\beta_s}\Big)^{-\frac{1}{\beta_s}},$$

$$\alpha_i > 0,$$
$$-1 < \beta_s < \infty, \qquad \beta_s \neq 0.$$
$$\rho_s > 0, \qquad \sum_{s=1}^{S} \rho_s = 1.$$

The production function $f(x)$ defined by (16) is homogeneous of degree one, strictly quasi-concave, and has partial derivatives of any order.

The unit cost function $\lambda(p)$ is easily derived ; namely,

(18) $$\lambda(p) = a \prod_{s=1}^{s} [\lambda^{(s)}(p^{(s)})]^{\rho_s},$$

where $a$ is a positive constant and

(19) $$\lambda^{(s)}(p^{(s)}) = \Big(\sum_{i \in N_s} \alpha_i^{\sigma_s} p_i^{1-\sigma_s}\Big)^{\frac{1}{1-\sigma_s}},$$

with

$$\sigma_s = \frac{1}{1+\beta_s},$$

is the unit cost function associated with the production function $f^{(s)}(x^{(s)})$, $s = 1, \ldots, S$.

Hence,

(20) $$\Lambda(p) = \log \lambda(p) = A + \sum_{s=1}^{S} \rho_s \Lambda^{(s)}(p^{(s)}),$$

where $A$ is a constant and

(21) $$\Lambda^{(s)}(p^{(s)}) = \frac{1}{1-\sigma_s} \log \Big(\sum_{i \in N_s} \alpha_i^{\sigma_s} p_i^{1-\sigma_s}\Big), \qquad s = 1, \ldots, S.$$

Partial elasticities of substitution $\sigma_{ij}$ are calculated from the formula (11) ; we have

(22) $$\sigma_{ij} = \begin{cases} 1, & \text{if } i \in N_s,\ j \in N_t,\ s \neq t, \\ \sigma_s, & \text{if } i, j \in N_s. \end{cases}$$

The foregoing analysis may be summarized by the following:

*Theorem* 1. *Let a production function $f(x)$ be of the form*:

$$(16) \qquad f(x) = \prod_{s=1}^{S} f^{(s)}(x^{(s)})^{\rho_s}$$

*where*

$$\rho_s > 0, \quad \sum_{s=1}^{S} \rho_s = 1,$$

*and*

$$(17) \qquad f^{(s)}(x^{(s)}) = \left(\sum_{i \varepsilon N_s} \alpha_i x_i^{-\beta_s}\right)^{-\frac{1}{\beta_s}},$$

$$\alpha_i > 0, \quad i = 1, \ldots, n,$$
$$-1 < \beta_s < \infty, \quad \beta_s \neq 0, \quad s = 1, \ldots, n.$$

*Then partial elasticities of substitution $\sigma_{ij}$ are all constant and*

$$(22) \qquad \sigma_{ij} = \begin{cases} 1, & \text{if } i \in N_s, \ j \in N_t, \ s \neq t, \\ \sigma_s, & \text{if } i, \ j \in N_s, \end{cases}$$

*where*

$$(23) \qquad \sigma_s = \frac{1}{1 + \beta_s} \neq 1.$$

VII. In what follows, we shall show that the class of the production functions of the form (16) exhausts all possible linear and homogeneous production functions with constant partial elasticities of substitution; namely, we have:

*Theorem 2. Let a production function $f(x)$, $x = (x, \ldots, x_n)$, be homogeneous of degree one, strictly quasi-concave, and possess continuous partial derivatives of third order. If partial elasticities of substitution $\sigma_{ij}$ are constant for all pairs of factors of production, i and j, then there exists a partition $\{N_1, \ldots, N_S\}$ of the set $\{1, \ldots, n\}$ of n factors of production such that for partial elasticities of substitution $\sigma_{ij}$ the relations* (22) *hold and the production function $f(x)$ is of the form* (16) *with*

$$(24) \qquad \beta_s = \frac{1}{\sigma_s} - 1, \quad s = 1, \ldots, S.$$

*Proof.* Let us first prove that if $\sigma_{ij}$ are constant for all pairs of factors of production, there exists a partition $\{N_1, \ldots, N_S\}$ of the set $\{1, \ldots, n\}$ such that the realtions (22) hold.

Differentiating both sides of the relation (11) with respect to $p_k$, we get

$$\frac{\partial^3 \Lambda}{\partial p_k \, \partial p_i \, \partial p_j} = (\sigma_{ij} - 1)\left\{\frac{\partial^2 \Lambda}{\partial p_k \, \partial p_i} \frac{\partial \Lambda}{\partial p_j} + \frac{\partial^2 \Lambda}{\partial p_k \, \partial p_j} \frac{\partial \Lambda}{\partial p_i}\right\},$$

which, in view of (11), implies that

$$(25) \qquad \frac{\partial^3 \Lambda}{\partial p_k \, \partial p_i \, \partial p_j} = (\sigma_{ij} - 1)\left\{(\sigma_{ik} - 1) - (\sigma_{jk} - 1)\right\} \frac{\partial \Lambda}{\partial p_i} \frac{\partial \Lambda}{\partial p_j} \frac{\partial \Lambda}{\partial p_k}$$

for all distinct $i$, $j$, and $k$.

Interchanging $i$ and $k$ in (25), we get

(26) $$\frac{\partial^3 \Lambda}{\partial p_i \, \partial p_k \, \partial p_j} = (\sigma_{kj} - 1) \{ (\sigma_{ki} - 1) - (\sigma_{ji} - 1) \} \frac{\partial \Lambda}{\partial p_k} \frac{\partial \Lambda}{\partial p_j} \frac{\partial \Lambda}{\partial p_i}.$$

Since $\sigma_{ij}$ are symmetric, $x_i = \frac{\partial \lambda}{\partial p_i}$ positive, and $\frac{\partial^3 \Lambda}{\partial p_i \, \partial p_j \, \partial p_k}$ independent of the order of differentiation, we have from (25) and (26) that

(27) $$(\sigma_{ik} - 1)\,(\sigma_{ij} - \sigma_{kj}) = 0,$$
for all distinct $i$, $j$ and $k$.

Let us now define a binary relation $\sim$ between two factors of production $i$ and $k$ by:

(28) $$i \sim k \text{ if and only if } i = k \text{ or } \sigma_{ik} \neq 1.$$

We shall prove that the realtion $\sim$ is an equivalence relation ; namely,

(29) $$i \sim i,$$

(30) $$i \sim j \text{ implies } j \sim i,$$
and

(31) $$i \sim j \text{ and } j \sim k \text{ imply } i \sim k.$$

Since (29) and (30) are trivial, we prove only the transitivity relation (31), which may be implied by the following :

(32) If, for distinct $i$, $j$, and $k$,
$$i \sim j \text{ and } j \sim k, \text{ then } \sigma_{ij} = \sigma_{jk} = \sigma_{ik}.$$

Let, for distinct $i$, $j$, and $k$,

(33) $$\sigma_{ij} \neq 1 \text{ and } \sigma_{jk} \neq 1.$$

From (27) we have

(34) $$(\sigma_{ij} - 1)\,(\sigma_{ik} - \sigma_{jk}) = 0,$$

(35) $$(\sigma_{jk} - 1)\,(\sigma_{ij} - \sigma_{ik}) = 0.$$

The relations (33), (34) and (35) together imply that

$$\sigma_{ik} = \sigma_{ij} = \sigma_{jk} \neq 1.$$

Since the relation $\sim$ is an equivalence relation on the set $\{1, \ldots, n\}$ there exists a partition $\{N_1, \ldots, N_S\}$ of the set $\{1, \ldots, n\}$ such that

$$(36) \qquad i \sim j, \text{ if and only if } i, j \in N_s \text{ for some } s.$$

Hence,

$$(37) \qquad \sigma_{ij} = 1 \qquad \text{if } i \in N_s,\ j \in N_t, \quad s \neq t.$$

The relation (32) implies the existence of $\sigma_1, \ldots, \sigma_S$ such that

$$(38) \qquad \sigma_{ij} = \sigma_s \neq 1, \qquad \text{for all } i, j \in N_s, \qquad i \neq j.$$

The relations (37) and (38) together imply (22).

From (11) and (22), we have

$$(39) \qquad \frac{\partial^2 \Lambda}{\partial p_i \, \partial p_j} = \begin{cases} 0, & \text{if } i \in N_s,\ j \in N_t,\ s \neq t \\ (\sigma_{ij} - 1) \dfrac{\partial \Lambda}{\partial p_i} \dfrac{\partial \Lambda}{\partial p_j}, & \text{if } i,\ j \in N_s,\ i \neq j. \end{cases}$$

Hence, there exist $S$ functions $\psi^{(1)}(p^{(1)}), \ldots, \psi^{(S)}(p^{(S)})$ such that

$$(40) \qquad \Lambda(p) = \sum_{s=1}^{S} \psi^{(s)}(p^{(s)}),$$

where

$$(41) \qquad \frac{\partial^2 \psi^{(s)}}{\partial p_i \, \partial p_j} = (\sigma_s - 1) \frac{\partial \psi^{(s)}}{\partial p_i} \frac{\partial \psi^{(s)}}{\partial p_j}, \qquad \text{for } i \neq j,\ i, f \in N_s.$$

Let

$$\varphi^{(s)}(p^{(s)}) = e^{\psi^{(s)}(p^{(s)})}, \qquad s = 1, \ldots, S.$$

Then

$$(42) \qquad \lambda(p) = \prod_{s=1}^{S} \varphi^{(s)}(p^{(s)}),$$

$$(43) \qquad \varphi^{(s)} \frac{\partial \varphi^{(s)2}}{\partial p_i \, \partial p_j} = \sigma_s \frac{\partial \varphi^{(s)}}{\partial p_i} \frac{\partial \varphi^{(s)}}{\partial p_j}, \quad \text{for } i,\ j \in N_s,\ i \neq j.$$

By considering the transformation

$$\varphi^{(s)} \longrightarrow \varphi^{(s)^{1-\sigma_s}}, \quad p_i \longrightarrow p_i^{1-\sigma_s} \quad (i \in N_s),$$

and taking the homogeneity of the unit cost function $\lambda(p)$ into account, we have

$$(44) \qquad \varphi^{(s)}(p^{(s)}) = \Big(\sum_{i \in N_s} \alpha_i p_i^{1-\sigma_s}\Big)^{\frac{1}{1-\sigma_s}}, \quad s = 1, \ldots, S, \text{ with } \alpha_i > 0 \ (i \in N_s).$$

Hence, the unit cost function $\lambda(p)$ is of the form (18). Therefore, by applying

Shepherd's duality theorem ([7], pp. 17-22), the production function $f(x)$ must be of the form (16).

Q.E.D.

*Stanford* HIROFUMI UZAWA.

## REFERENCES

[1] Allen, R. G. D. *Mathematical Analysis for Economists*, London: Macmillan, 1938.

[2] Arrow, K. J., H. B. Chenery, B. S. Minhas, and R. M. Solow. "Capital-Labor Substitution and Economic Efficiency," *Review of Economics and Statistics*, Vol. 63 (1961), pp. 225-250.

[3] Hicks, J. R. *The Theory of Wages*, London: Macmillan, 1932.

[4] Lerner, A. P. " Notes on the Elasticities of Substitution," *Review of Economic Studies*, Vol. 1 (1933-4), pp. 39-44 and pp. 68-71.

[5] Robinson, J. *The Economics of Imperfect Competition*, London: Macmillan, 1933.

[6] Samuelson, P. A. *Foundations of Economic Analysis*, Cambridge: Harvard University Press, 1947.

[7] Shepherd, R. W. *Cost and Production Functions*, Princeton: Princeton University Press, 1953.

[8] Solow, R. M. " A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, Vol. 70 (1956), pp. 65-94.