The Singular Value Decomposition and the Pseudoinverse

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1 Singular Value Decomposition — SVD

The singular value decomposition is the appropriate tool for analyzing a mapping from one vector space into another vector space, possibly with a different dimension. Most systems of simultaneous linear equations fall into this second category.

Any m by n matrix $\mathbf{A}$ can be factored into:

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{1}$$

where $\mathbf{U}$ is orthogonal[1] m by m matrix and the columns of the $\mathbf{U}$ are the eigenvectors of $\mathbf{A}\mathbf{A}^T$. Likewise, $\mathbf{V}$ is orthogonal n by n matrix and the columns of the $\mathbf{V}$ are the eigenvectors[2] of $\mathbf{A}^T\mathbf{A}$. The matrix $\mathbf{S}$ is diagonal and it is the same size as $\mathbf{A}$. Its diagonal entries, also called sigma, $\sigma_1, \ldots, \sigma_r$, are the square roots of the nonzero eigenvalues of both $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$. They are the ***singular values*** of matrix $\mathbf{A}$ and they fill the first r places on the main diagonal of $\mathbf{S}$. r is the rank of $\mathbf{A}$.

The connections with $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$ must hold if the equation 1 is correct. It can be seen:

$$\mathrm{A}\mathrm{A}^T = \left(\mathbf{U}\mathbf{S}\mathbf{V}^T\right)\left(\mathbf{V}\mathbf{S}^T\mathbf{U}^T\right) = \mathbf{U}\mathbf{S}\mathbf{S}^T\mathbf{U}^T \tag{2}$$

and similarly

$$\mathrm{A}^T\mathrm{A} = \mathbf{V}\mathbf{S}^T\mathbf{S}\mathbf{V}^T \tag{3}$$

From eq. 2, $\mathbf{U}$ *must be the eigenvector matrix for* $\mathbf{A}\mathbf{A}^T$. The eigenvalue matrix in the middle is $\mathbf{S}\mathbf{S}^T$ — which is m by m with the eigenvalues $\lambda_1 = \sigma_1^2, \ldots, \lambda_r = \sigma_r^2$ on the diagonal. From eq. 3, $\mathbf{V}$ *must be the eigenvector matrix for* $\mathbf{A}^T\mathbf{A}$. The diagonal matrix $\mathbf{S}^T\mathbf{S}$ has the same $\lambda_1 = \sigma_1^2, \ldots, \lambda_r = \sigma_r^2$, but it is n by n.

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[1] An orthogonal matrix is a square matrix with columns built out of the orthonormal vectors. The vectors are orthonormal when their lengths are all 1 and their dot products are zero. If the matrix $\mathbf{Q}$ is orthogonal then $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ and $\mathbf{Q}^T = \mathbf{Q}^{-1}$, ***the transpose is the inverse.***

If the matrix $\mathbf{Q}$ *is not* square and $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ then $\mathbf{Q}$ is called orthonormal matrix.

If the columns of the $\mathbf{Q}$ are orthogonal vectors, (heir dot products are zero, but their lengths are *not* all 1) then $\mathbf{Q}^T\mathbf{Q}$ is a diagonal matrix, *not* the identity matrix.

[2] The number λ is an eigenvalue of matrix $\mathbf{M}$ if and only if: $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$. This is the characteristic equation, and each solution λ has a corresponding eigenvector $\underline{x}$: $(\mathbf{M} - \lambda\mathbf{I})\,\underline{x} = 0$ or $\mathbf{M}\underline{x} = \lambda\underline{x}$.

Example 1: Find the singular value decomposition of $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$.

Solution: Compute $\mathbf{AA}^T$, find its eigenvalues (it is generally preferred to put them into decreasing order) and then find corresponding unit eigenvectors:

$$\mathbf{AA}^T = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \det\left(\mathbf{AA}^T - \lambda\mathbf{I}\right) = \det\begin{bmatrix} 8-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} = 0$$

$$(8-\lambda)(2-\lambda) = 0 \Rightarrow \lambda_1 = 8, \lambda_2 = 2$$

Their corresponding unit eigenvectors are:

$$\mathbf{AA}^T\underline{u}_1 = \lambda_1\underline{u}_1 \Rightarrow \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = 8\begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} \Rightarrow \begin{array}{l} 8u_{11} = 8u_{11} \Rightarrow u_{11} = 1 \\ 2u_{12} = 8u_{12} \Rightarrow u_{12} = 0 \end{array} \Rightarrow \underline{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{AA}^T\underline{u}_2 = \lambda_2\underline{u}_2 \Rightarrow \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = 2\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} \Rightarrow \begin{array}{l} 8u_{21} = 2u_{21} \Rightarrow u_{21} = 0 \\ 2u_{22} = 2u_{22} \Rightarrow u_{22} = 1 \end{array} \Rightarrow \underline{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The matrix $\mathbf{U}$ is then:

$$\mathbf{U} = [\ \underline{u}_1 \quad \underline{u}_2\ ] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The eigenvalues of the $\mathbf{A}^T\mathbf{A}$ are the same as the eigenvalues of the $\mathbf{AA}^T$. The eigenvectors of the $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ are:

$$\mathbf{A}^T\mathbf{A}\underline{v}_1 = \lambda_1\underline{v}_1 \Rightarrow \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 8\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \Rightarrow \begin{array}{l} 5v_{11} + 3v_{12} = 8v_{11} \Rightarrow v_{11} = v_{12} \\ 3v_{11} + 5v_{12} = 8v_{12} \Rightarrow v_{12} = v_{11} \end{array}$$

Choice of v_{11} will define v_{12} and vice versa. In general v_{11} and v_{12} can be any numbers, but since vector $\underline{v}_1$ should have length of 1, the v_{11} and v_{12} are chosen as follows:

$$\|\underline{v}_1\| = 1 \Rightarrow \sqrt{v_{11}^2 + v_{12}^2} = 1 \Rightarrow v_{11} = v_{12} = \frac{1}{\sqrt{2}} \Rightarrow \underline{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Unit eigenvector $\underline{v}_2$ is:

$$\mathbf{A}^T\mathbf{A}\underline{v}_2 = \lambda_2\underline{v}_2 \Rightarrow \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 2\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} \Rightarrow \begin{array}{l} 5v_{21} + 3v_{22} = 2v_{21} \Rightarrow v_{21} = 0 \\ 3v_{21} + 5v_{22} = 2v_{22} \Rightarrow v_{22} = 1 \end{array}$$

$$v_{21} = -v_{22} \Rightarrow \underline{v}_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The matrix $\mathbf{V}$ is then:

$$\mathbf{V} = [\ \underline{v}_1 \quad \underline{v}_2\ ] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } \mathbf{V}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The matrix $\mathbf{S}$ is:

$$\mathbf{S} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}.$$

Finally the SVD of the $\mathbf{A}$ is:

$$\mathbf{A} = \mathbf{USV}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Example 2: Find the singular value decomposition of $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$.

Solution: The rank $r = 1$!

$$\mathrm{AA}^T = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \Rightarrow \lambda_1 = 10, \lambda_2 = 0 \Rightarrow \underline{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}.$$

For the second eigenvalue $\lambda_1 = 0$, equation $\mathrm{AA}^T\underline{u}_2 = \lambda_2\underline{u}_2$ is no use. Since the matrix $\mathbf{U}$ is usually square, another column (vector $\underline{u}_2$ in this case) is needed. Eigenvectors $\underline{u}_1$ and $\underline{u}_1$ must be orthogonal, their dot product is zero (eq. 4), and their length must be 1 (eq. 5).

$$u_{11}u_{12} + u_{21}u_{22} = 0 \tag{4}$$

$$\|\underline{u}_2\| = 1 \Rightarrow \sqrt{u_{12}^2 + u_{22}^2} = 1 \tag{5}$$

The eigenvector $\underline{u}_1$ is:

$$\left.\begin{array}{r} \frac{1}{\sqrt{5}}u_{12} + \frac{2}{\sqrt{5}}u_{22} = 0 \Rightarrow u_{22} = -2u_{12} \\ \sqrt{u_{12}^2 + u_{22}^2} = 1 \Rightarrow \sqrt{u_{12}^2 + (-2u_{12})^2} \Rightarrow u_{12} = \frac{1}{\sqrt{5}} \end{array}\right. \Rightarrow \underline{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{bmatrix}$$

and matrix $\mathbf{U}$ is:

$$\mathbf{U} = [\ \underline{u}_1 \quad \underline{u}_2\ ] = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix}.$$

$$\mathrm{A}^T\mathrm{A} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \Rightarrow \mathrm{A}^T\mathrm{A}\underline{v}_1 = \lambda_1\underline{v}_1 \Rightarrow \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 10 \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \Rightarrow \underline{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

The vector $\underline{v}_2$ can be found using eq. 4 and eq. 5:

$$\underline{v}_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \text{ the matrix } \mathbf{V} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } \mathbf{V}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Matrix $\mathbf{S}$ is:

$$\mathbf{S} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}$$

and SVD of $\mathbf{A}$ can be written as:

$$\mathbf{A} = \mathbf{USV}^T = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The Matlab function `svd()` can be used to find the singular value decomposition. If the matrix **A** has many more rows than columns, the resulting **U** can be quite large, but most of its columns are multiplied by zeros in **A**. In this situation, the *economy* sized decomposition saves both time and storage by producing an m by n **U**, an n by n **S** and the same **V**.

Example 3: For the matrix **A**

```
A =
     2     0
     0    -3
     0     0
```

the full singular value decomposition is

```
[U,S,V]=svd(A)

U =
     0     1     0
    -1     0     0
     0     0     1

S =
     3     0
     0     2
     0     0

V =
     0     1
     1     0
```

For this small problem, the economy size decomposition is only slightly smaller:

```
[U,S,V]=svd(A,0)

U =
     0     1
    -1     0
     0     0

S =
     3     0
     0     2

V =
     0     1
     1     0
```

2 Application of the SVD

2.1 The effective rank

Rank of the matrix is the number of independent rows or equivalently the number of independent columns. In computations this can be hard to decide. Counting the pivots [2] is correct in exact arithmetic, but in real arithmetic the roundoff error makes life more complicated. More stable measure of the rank is to compute $\mathbf{A}^T\mathbf{A}$ or $\mathbf{A}\mathbf{A}^T$, which are symmetric but share the same rank as $\mathbf{A}$. Based on the accuracy of the data, we can set the tolerance, 10^{-6} for example, and count the singular values of $\mathbf{A}^T\mathbf{A}$ or $\mathbf{A}\mathbf{A}^T$ above it. The number of singular values above the specified tolerance is the ***effective rank***.

2.2 Least Squares and the Pseudoinverse

Vector $\underline{x}$ is called a ***least squares solution*** of linear system

$$\mathbf{A}\underline{x} = \underline{b} \tag{6}$$

when it minimizes

$$\left\| \mathbf{A}\underline{x} - \underline{b} \right\| . \tag{7}$$

For any linear system $\mathbf{A}\underline{x} = \underline{b}$, the associate normal system

$$\mathbf{A}^T\mathbf{A}\underline{x} = \mathbf{A}^T\underline{b} \tag{8}$$

is consistent, and all solutions of the normal system are least squares solutions of $\mathbf{A}\underline{x} = \underline{b}$. If $\mathbf{A}$ has linearly independent columns and if $\mathbf{A}^T\mathbf{A}$ is invertible then the system $\mathbf{A}\underline{x} = \underline{b}$ has a ***unique*** least squares solution:

$$\underline{x} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\underline{b} \tag{9}$$

If $\mathbf{A}$ does not have full rank, the least squares problem still has a solution, but it is no longer unique. There are many vectors $\underline{x}$ that minimize $\|\mathbf{A}\underline{x} - \underline{b}\|$. ***The optimal solution of the*** $\mathbf{A}\underline{x} = \underline{b}$ ***is the vector*** $\underline{x}$***, that has the minimum length***[3]. This optimal solution is called $\underline{x}^+$ and *the matrix which produces* $\underline{x}^+$ *from* $\underline{b}$ is called the ***pseudoinverse*** of $\mathbf{A}$. Pseudoinverse of $\mathbf{A}$ is denoted by $\mathbf{A}^+$, so we have:

$$\underline{x}^+ = \mathbf{A}^+\underline{b} \tag{10}$$

If the singular value decomposition of $\mathbf{A}$ is $\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T$ (eq.1), the pseudoinverse of $\mathbf{A}$ is then:

$$\mathbf{A}^+ = \mathbf{V}\mathbf{S}^+\mathbf{U}^T \tag{11}$$

The singular values $\sigma_1, \ldots, \sigma_r$ are on the diagonal of m by n matrix $\mathbf{S}$, and the reciprocals of the singular values $\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_r}$ are on the diagonal of n by m matrix $\mathbf{S}^+$. *The minimum length solution to* $\mathbf{A}\underline{x} = \underline{b}$ *is* $\underline{x}^+ = \mathbf{A}^+\underline{b} = \mathbf{V}\mathbf{S}^+\mathbf{U}^T\underline{b}$. The pseudoinverse of $\mathbf{A}^+$ is $\mathbf{A}^{++} = \mathbf{A}$. If $\mathbf{A}^{-1}$ exists, then $\mathbf{A}^+ = \mathbf{A}^{-1}$ and the system 6 has a unique least squares solution 9.

Example 4: Find the pseudoinverse of $\mathbf{A} = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}$.
Solution: The SVD of $\mathbf{A}$ is:

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

[3] Also known as the ***Minimal Norm*** Least Squares solution.

The pseudoinverse of $\mathbf{A}$ is:

$$\mathbf{A}^{+} = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}^{+} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \end{bmatrix} [1] = \begin{bmatrix} -\frac{1}{9} \\ \frac{2}{9} \\ \frac{2}{9} \end{bmatrix}$$

The Matlab function `pinv()` can be used to find the pseudoinverse:

```
A=pinv([-1 2 2])

A =
   -0.1111
    0.2222
    0.2222
```

2.3 Ill–conditioned Least Squares Problem

The Least Squares procedure will fail, when $\mathbf{A}$ is rank deficient. The best we can do is to find $\underline{x}^{+}$. When $\mathbf{A}$ is nearly rank deficient, small changes[4] in the vector $\underline{b}$ will produce wildly different solution vectors. $\underline{x}$. In this case the pseudoinverse gives more stable solution.

Example 5: Consider system given in eq. 6, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ 4 & 8.0001 \end{bmatrix} \text{ and } \underline{b} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}. \tag{12}$$

The second column of $\mathbf{A}$ is almost first column multiplied by 2, $\mathbf{A}$ is close to being singular. The Least Squares solution is:

$$\underline{x} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\underline{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \tag{13}$$

If the third element in vector $\underline{b}$ changes from 6 to 5.9999 and 6.0001, Least Squares gives the following solutions:

$$\underline{x} = \begin{bmatrix} 0.2857 \\ 0.8571 \end{bmatrix} \text{ for } \underline{b}(3) = 5.9999 \text{ and } \underline{x} = \begin{bmatrix} 3.7143 \\ -0.8571 \end{bmatrix} \text{ for } \underline{b}(3) = 6.0001.$$

It can be seen that solution vector changes from $\underline{x} = \begin{bmatrix} 0.2857 \\ 0.8571 \end{bmatrix}$ to $\underline{x} = \begin{bmatrix} 3.7143 \\ -0.8571 \end{bmatrix}$ given this very small change in the vector $\underline{b}$. The pseudoinverse[5] provides the following solution of the system given in eq. 12:

$$\underline{x}^{+} = \mathbf{A}^{+}\underline{b} = \begin{bmatrix} 0.4000 \\ 0.8000 \end{bmatrix}. \tag{14}$$

Changes in the $\underline{b}$ would no longer change the result significantly:

$$\underline{x} = \begin{bmatrix} 0.4000 \\ 0.8000 \end{bmatrix} \text{ for } \underline{b}(3) = 5.9999 \text{ and } \underline{x} = \begin{bmatrix} 0.4000 \\ 0.8000 \end{bmatrix} \text{ for } \underline{b}(3) = 6.0001$$

This example illustrates that the use of a pseudoinverse can enhance the stability of our calculations.

[4] Vector $\underline{b}$ is the data vector and it is usuary corrupted by the noise.

[5] Matlab command `pinv(A)*[2 4 6 8]'` will not give the same result as 14. See section 2.3.1.

2.3.1 Truncation, Precision and Tolerance

To examine the problem of precision and tolerance it is necessary to look at more than four digits after the decimal point.

The SVD of **A** in example 5 is:

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T$$

where:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ 4 & 8.0001 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 0.18257321210769 & -0.19518092543763 & -0.48445018521904 & -0.83299426565087 \\ 0.36514642421537 & -0.39036185084826 & -0.64472592855133 & 0.54645602873302 \\ 0.54771963632306 & -0.58554277626589 & 0.59130068078336 & -0.08663926392867 \\ 0.73030015136897 & 0.68312640770977 & -0.00000000000709 & -0.00000000000729 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 12.24751403383832 & 0 \\ 0 & 0.00003055034170 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\mathbf{V} = \begin{bmatrix} 0.44721121036078 & -0.89442838356553 \\ 0.89442838356553 & 0.44721121036078 \end{bmatrix}.$$

In practice, the singular values in **S** which are close to zero are usually set to zero[6]. The singular value $\sigma_2 = 0.00003055034170$ is pretty close to zero. There are two possible decisions:

I.) If σ_2 **is not set to zero**, the following truncation[7] of the matrices **U**, **S** and **V** can be made and the Minimal Norm Least Squares solution of the system 12 can be found:

$$\underline{x}^+ = \mathbf{A}^+\underline{b} = \mathbf{V}\mathbf{S}^+\mathbf{U}^T\underline{b} = \begin{bmatrix} 2.00000000006548 \\ -0.00000000002910 \end{bmatrix}$$

where:

$$\mathbf{V} = \begin{bmatrix} 0.44721121036078 & -0.89442838356553 \\ 0.89442838356553 & 0.44721121036078 \end{bmatrix}$$

$$\mathbf{S}^+ = \begin{bmatrix} 0.08164922262895 & 0 \\ 0 & 3.273285810744304e+004 \end{bmatrix}$$

$$\mathbf{U}^T = \begin{bmatrix} 0.18257321210769 & -0.19518092543763 \\ 0.36514642421537 & -0.39036185084826 \\ 0.54771963632306 & -0.58554277626589 \\ 0.73030015136897 & 0.68312640770977 \end{bmatrix}^T$$

and

$$\underline{b} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}.$$

[6] Answer to the question "what is close to 0" can be difficult task.

[7] `svd(A,0)` can be used in Matlab. It produces the "economy size" decomposition.

This is the same solution as would be obtained from the Matlab function `pinv()`, which uses the SVD and sets all singular values that are within machine precision to zero. As can be seen, this solution is close to The Least Squares solution 13. It was shown in example 5, that solution vector $\underline{x}$ found this way, is very sensitive to the noise in the vector $\underline{b}$.

II.) If σ_2 **is set to zero**, the Minimal Norm Least Squares solution[8] of the system 12 is:

$$\underline{x}^{+} = \mathbf{A}^{+}\underline{b} = \mathbf{V}\mathbf{S}^{+}\mathbf{U}^{T}\underline{b} = \begin{bmatrix} 0.39999573334471 \\ 0.79999679999076 \end{bmatrix} \tag{15}$$

where:

$$\mathbf{V} = \begin{bmatrix} 0.44721121036078 \\ 0.89442838356553 \end{bmatrix}$$

$$\mathbf{S}^{+} = [0.08164922262895]$$

$$\mathbf{U}^{T} = \begin{bmatrix} 0.18257321210769 \\ 0.36514642421537 \\ 0.54771963632306 \\ 0.73030015136897 \end{bmatrix}^{T}$$

and

$$\underline{b} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}.$$

Tolerance in the `pinv()` can be specified manually[9]. If the tolerance is loose enough[10], the `pinv()` will give the same result as 15.

```
x=pinv(A,0.0001)*b

x =
   0.39999573334471
   0.79999679999076
```

It was shown in example 5, that this solution is less sensitive to the noise in the vector $\underline{b}$.

[8]The Matlab function `svds(A,k)` can be used here. `k` is the number of k largest singular values of `A`. See `help svds`.

[9]See `help pinv`

[10]Same problem as remark 6.

References

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