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A FUNCTIONAL TREATMENT OF ASYMMETRIC COPULAS

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Abstract. The concept of asymmetric copulas is revisited and is made more precise. We give a rigorous topological argument for opportunity to define asymmetry measures defined recently by K.F Siburg [6] through exhibiting at least three ordered classes of copulas according to a suitable equivalence relation. We define a process of ordering subcopulas which makes clearer the degree of asymmetry. As illustration, we treat the asymmetric Cobb-Douglas utility model.

Introduction

Recently copulas are becoming an emerging concept in probability and statistic models. The copula permits in particular to model dependence structure between random variables. There are various and important applications of copulas in finance, actuarial and risk management. For a given data set, the practical and real problem is to determine a copula (or copulas) which fits for describing the dependence structure. Our summarized note on copulas is based on the edifying paper of A. Sklar and the theorem that bears his name[1] and the unavoidable book of R. Nelson [2]. The copula is a notion which allows a normalization of random variables law. Assume for example that one has a random vector (X, Y) where X and Y act on different probability spaces and eventually with different laws described by their distribution functions F^X and F^Y respectively. Let H denote the joint distribution of X and Y. In order to highlight properties of the couple (X, Y) concerning dependence, regression or symmetry it will be convenient to project X and Y in a common space where their projections share the same law and where arithmetic operations are rather possible. It is a classical result that $F^X(X)$ and $F^Y(Y)$ give a nice answer mainly when the margin distributions F^X and F^Y are continuous. In fact, $(F^X(X), F^Y(Y))$ has a uniform distribution (U, V) on $\mathbf{I}^2 = [0, 1] \times [O, 1]$ and a copula of the (X, Y) is exactly such as $C_{X,Y}(u, v) = H(x, y)$ as soon as $(u, v) = (F^X(x), F^Y(v))$. This technique of normalization consisting on researching of common projection space becomes recently a sufficient tool in many fields where heterogeneous data are used. We refer for example to [3, III,A] concerning the projection of heterogeneous multimedia data into a common low-dimensional Hamming space; and to [4] for similar procedure using *Krylov method* to reduce power systems by projecting data in smaller finite dimensional space.

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In order to give more autonomy to this work, we recall the essential results on copulas. Without loss of generality, we restrict ourself to the bivariate case. To this end we consider the unit closed square $I^2 = [0,1] \times [0,1]$.

Definition 0.1. *A copula C is a function on I^2 into $I = [0,1]$ which satisfies the following conditions for all $(u,v) \in I^2$:*
1. $C(0,v) = C(u,0) = 0$.
2. $C(1,v) = v$ *and* $C(u,1) = u$.
3. *the 2-increasing property:* $C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$.

When we deal with copulas, it is almost impossible to overcome Sklar's theorem which is the first bridge between marginal statistical distributions and their joint multivariate distribution. Let us give a restitution of this famous theorem in our context of bivariate case

Theorem 0.2 (Sklar's theorem)**.** *Let H be two-dimensional distribution function on a probability space (Ω, p) with marginal distribution functions F and G. Then there exists a copula C such that*

$$H(x,y) = C(F(x), G(y)). \tag{0.1}$$

If F and G are continuous then the copula C is unique. Conversely, for any copula C and any distribution functions F and G the equation (0.1) defines a two-dimensional distribution function H with marginal distribution functions F and G.

Mathematically speaking, it is not difficult to establish that the three conditions above (in definition 0.1) imply that a copula enjoys certain analytical properties such continuity and monotonicity on each argument. Moreover using condition (3) one may prove easily that a copula as defined in definition (0.1) is Lipschitz-continuous and then almost everywhere differentiable. By a classical integration theorem, it is so possible to rebuild the copula from one of its partial derivative. Algebraically speaking, the well known Fréchet-Hoeffding result states very important and canonical estimations for any given copula as formulated in the following proposition

Proposition 0.3. *The functions C_l and C_u defined for all $(u,v) \in I^2$ by setting $C_l(u,v) = \min(u,v)$ and $C_u(u,v)) = \max(u+v-1,0)$ are copulas. Furthermore, for every copula C, one has*

$$\forall (u,v) \in I^2 \qquad C_l(u,v) \leq C(u,v) \leq C_u(u,v).$$

In the litterature, C_l and C_u are said respectively *the Fréchet-Hoeffding lower bound* and the *the Fréchet-Hoeffding upper bound.*
It is worth to mention that these bounds depend on the natural order defined on the set of all copulas as follows

$$C_1 \leq C_2 \iff \forall (u,v) \in I^2 : \ C_1(u,v) \leq C_2(u,v).$$

This natural order is far from being total (it is just a preorder). For example, according to [2, Example 2.18], the copulas $\frac{C_l + C_u}{2}$ and $\Pi : (u,v) \mapsto \pi(u,v) = uv$ are not comparable. Other orders are possible. For example, in [5], a general concept of a regression dependence order (RDO, as denoted by the authors) was introduced. Recently, to characterize the symmetry degree of a given copula compared to another K.F. Siburg et al. [6] defined a partial order on copulas set by introducing the difference between a copula C and its transpose C^T defined by

$$\forall(u,v)\in I^2 \;\; C^\top(u,v)=C(v,u).$$

Let us recall the order adopted there (in[6]). We say that a copula C_1 is *more symmetric* than another one C_2 and we write $C_1 \preceq C_2$ if and only if, roughly speaking, the difference between C_1 and its transpose is smaller than the difference between C_2 and $C_2^\top$. To be more precise, we state:

$$C_1 \preceq C_2 \Longleftrightarrow \forall(u,v)I^2 : |C_1(u,v)-C_1^\top(u,v)| \leq |C_2(u,v)-C_2^\top(u,v)|. \tag{0.2}$$

For the partial order "$\preceq$" the lower bound is manifestly the product copula, said also *independence copula* P given by $\forall(u,v)\in I^2 \; P(u,v)=uv$. Moreover P is the smallest element of the partially ordered set $(\mathcal{CP},\preceq))$ where $\mathcal{CP}$ denotes the set of all copulas. Unfortunately as proved in [8], this order has not an upper bound and so a fortiori has not a greatest element nor an upper bound.
The main result of this paper is to establish that the order " $\preceq$ " is not trivial in the following sense: the classes of copulas with the same degree of symmetry are not reduced to one. Thus for a given measure of symmetry (or asymmetry), it will be convenient to search the *most symmetric* among a parameterized copulas. Topological arguments are given to prove our purposes.

In the first section, we recall and reshuffle the most important results on asymmetric copulas. As the mathematical ingredients are needed to make clear and precise, we introduce, at the second section, all mathematical tools which will be used in the remaining of this paper. The third section is devoted to the main results and their proofs. At last we give a way to define a new measure of asymmetry baptized "*local asymmetry measure*."

1. Asymmetric copulas

A large and most well known class of copulas is the Archimedean ones. A copula C is said Archimedean if there exists a decreasing and convex function $\phi:\ [0,1]\mapsto\mathbb{R}^+$ with $\phi(1)=0$ such that for all $(u,v)\in I^2$ we have $C(u,v)=\phi^{[-1]}(\phi(u)+\phi(v))$. The function ϕ is said *a generator* of C. Here $\phi^{[-1]}$ denotes the general inverse of ϕ. A lot of researches were axed on generalizing this class of copulas. For example, in [9] the authors defined a general copula as been every function C_g that we may write $C_g(u,v)=\phi^{[-1]}(\phi(\max(u,v))+\psi(\min(u,v))$ where ϕ is a strictly decreasing continuous function, ψ is assumed to be just decreasing continuous function and $\psi-\phi$ is increasing. Further steps have been taken in this direction. Among others,

different ratios of correlation and dependence like τ of Kendall and ρ of Spearman were calculated in terms of the generator ϕ and its cogenerator ψ of generalized copula.
unfortunately all of these copulas are symmetric (i.e $C(u,v) = C(v,u), \ \forall (u,v) \in I^2$.). In practice, this kind of copulas does not suffice to model phenomena which do not present asymmetric data. Awareness of the interest of this type of copulas began with the publication of the Klement and Mesiar paper [8]. Just a year after, appeared the article of Nelson [10] where the concept of asymmetric copula is dealt with, in some way, to the equivalent notion of non exchangeability of two random variables. As a consequence, many tests of parametric and nonparametric symmetry tests are implemented. As a revealing example, we refer to [6] and the important paper of A. Erdely and J. M. Gonzàlez-Barrios [11] and in another and close context [13]. We mention also the attempt of dependence coefficient *symmetrization* as initiated by Cifarelli et al.[7].

Recently Siburg et al. [6, Definition 2.2] define via (0.2) an order on the set of all copulas. This is in fact the starting point of some important remarks and ideas that we will mention or which are already mentioned in [6].
First, the order $\preceq$ does not distinguish between elements of a large class of copulas. To see this, it suffices to remember that it is possible, for suitable copulas, to have simultaneously $C_1 \preceq C_2$ and $C_2 \preceq C_1$ but they do not turn equal. This is clear for every Archimedean copulas or generally for every symmetric ones. For more general case, one may choose $C_2 = C_1^\top$ as given in [6, Remark 2.5]. A more serious drawback is the incompleteness problem: A large class of copulas are not comparable. To overcome these inconsistencies, the asymmetry measures are introduced as a complement of the order $\preceq$. A large catalog of asymmetry and symmetry measures are proposed in many works mainly, among others, [12]. So it will be interesting to know how many classes of copulas that have not the same measure. We will take advantage of the growth of measures and some elementary topological results to prove that there are at least three classes of copulas without exhibiting them explicitly when we deal with quotient topological space. All these ingredients will be developed in the next section. In the fourth section, we define a new and general measure of asymmetry based on Zygmond-Calderon theorem [16] for general copulas and give a bridge with the known Arrow's impossibility theorem " *Social choice and individual values*" published in 1951. More information on this important result in political economy and economic science may be found in [14] or in the last research on this topic [15]. The generality of our measure consists on its ability to compare subcopulas (see [2] and/or section 3 below) instead of copulas.

2. Mathematical ingredients

Topological results presented in this section are known and may be consulted in any elementary course of topology and functional analysis. We suggest as a typical

one the famous Choquet's book [18] and the irreplaceable monograph of H. Brezis [19] or the recent and English version [20].
Let X be a toplogical space. An equivalence denoted $\sim$ on X is a relation which responds to three well known properties

(1) Reflexivity : $x \sim x, \forall x \in X$.
(2) Symmetry: $x \sim y \iff y \sim x$.
(3) Transitivity: $x \sim y$ and $y \sim z \implies x \sim z$.

For a given x in X the class (or orbit) $\dot{x}$ of x is the set of all elements y which are related with x (i.e $\dot{x} = \{y \in X,\ x \sim y.\}$) The equivalence classes give a partition of X and the mapping $x \in X \mapsto \dot{x} \in X/\sim$ is said the *canonical projection* of X into the set $X/\sim$ of a all orbits. It is a classical question to ask what adequate topology one may define on $X/\sim$. The following proposition gives an answer

Proposition 2.4. *Let X be a topological space and $\sim$ an equivalence on X. Consider the canonical projection $p : X \to X/\sim$. The quotient topology is the family*

$$\tau = \{U \subset X/\sim,\ p^{-1}(U) \text{ is open set in } X\}. \tag{2.3}$$

The topology on $X/\sim$ given by (2.3) is the optimal one which makes the canonical projection continuous. Henceforward, $X/\sim$ will be endowed with this canonical topology.
A most important property of a topology on X is the *separation* which means that if x and y are different elements of X, one can always find two respective neighbors O_x and O_y of x and y such that $O_x \cap O_y = \emptyset$. Unfortunately, the quotient topology is rather pathologic because the separation property is not always warranted. We mention without developing it the classical and illustrative example:

$$X = \mathbb{R} \text{ and } x \sim y \iff x - y \in \mathbb{Q}.$$

One proves easily that the quotient space $\mathbb{R}/\sim$ is not separate. As a consequence, among quotient topological spaces it will be interesting to characterize those which are separate. To this end, we give a short answer that we will use as a recipe to establish our first main result on copulas in the next section

Theorem 2.5. *Let $\sim$ be an equivalence relation on a topological space X. Assume that one of the following conditions is satisfied*

a: *The relation $\sim$ is open.*
b: *The space X is compact and separate.*

Then the quotient space $X/\sim$ is separate.

The relation $\sim$ is said open if the canonical projection is so. We will not give more details because we will just use the second item in theorem (2.5). This latter condition implies that the compactness of the set of all copulas is needed. In functional analysis, the natural tool that we have at disposal is

Theorem 2.6 (Ascoli-Arzela)**.** *Let X be a compact Hausdorff space and $\mathbf{F} \subset C(X, \mathbb{R})$. Then $\mathbf{F}$ is compact if and only if it fulfills simultaneously the two conditions*

i: *The set $\mathbf{F}$ is equicontinuous, i.e for every $x \in X$ and every $\varepsilon > 0$, x has a neighborhood O_x such that*

$$\forall y \in O_x, \forall f \in \mathbf{F} : \ |f(y) - f(x)| < \varepsilon$$

ii: *The set $\mathbf{F}$ is pointwise relatively compact and closed which means that for each $x \in$, the set $F_x = \{f(x) : f \in \mathbf{F}\}$ is relatively compact in $\mathbb{R}$.*

More general versions of this important theorem exist. In particular, it is possible to replace $\mathbb{R}$ in theorem (2.6) with any metric space Y. But for our purposes, the real case ($Y = \mathbb{R}$) is largely enough.
Now, to make easier the introduction of our new asymmetry measure, the key will be a fundamental result in integration theory. It is a kind of suitable decomposition of element of $L^1([0, 1]) = \{f : I \to \mathbb{R}^+$ measurable such that $\int_0^1 |f(x)|dx < \infty\}$. For such functions, the values $f(x)$ may be greater as one may want, but Calderon-Zygmund theorem ensures that, *on average* or *on expectation*, the mean values of f in some disjointed subsets can be controlled. On the complement of those subsets, the function f is entirely dominated. For more precise, we have (see [16])

Theorem 2.7 (Calderon-Zygmund Lemma)**.** *Consider $f \in L^1(I^2)$ and $t > 0$. There exist at most countably many closed intervals $Q_i = [a_i, b_i], i \in \mathbb{N}$ with disjoint interiors such that*

$$\frac{1}{b_i - a_i} \int_{I^2} |f(x, y)|dxdy = t \tag{2.4}$$

and

$$\|f\mathbf{1}_{\{I^2 \setminus \cup_i Q_i\}}\|_\infty < t. \tag{2.5}$$

The intervals $(Q_i)_i$ are such that $Vol(\underset{i\geq 1}{\cup} Q_i) = \sum_{i=1}^{\infty} b_i - a_i \leq \frac{1}{t}\|f\|_{L^1}$.

In the literature, this result is known to be the Calderon-Zygmund Lemma. The theorem that bears this name states that every function $f \in L^1$ may be decomposed as follows: $f = g + b$ where g reveals "good function" and b connotes "bad" one in the following sense

Proposition 2.8 (Calderon-Zygmund Theorem)**.** *Under the same assumptions of theorem (2.7), the function $f \in L^2(I^2)$ may be written $f = g + b$ where g and b are two functions on I^2 satisfying*

- *$g(x) \leq 2t$.*
- *for every $1 \leq p \leq \infty$, one has $\|g\|_{L^p} \leq K_p \|f\|_{L^1}^{\frac{1}{p}}$ for some constant K_p depending uniquely on p.*
- *$\int_{Q_i} b(x, y)dxdy = 0$.*

- $\|b\|_{L^1} \leq 3\|f\|_{L^1}$.

The two results above will allow us to define a suitable asymmetry measure when we deal with subcopulas. Before developing the sequel, a remark on functional analysis results exposed above deserves to be mentioned: all formulations are adapted to the context and to the needs. In [16], one find more general statements and many applications.

3. CLASSES OF COPULAS

In the sequel we will denote by $\mathcal{CP}$, the set of all copulas. We state first that according to definition (0.1), $\mathcal{CP}$ is a convex subset of the Banach space $(C(I^2), \|.\|_\infty)$ of all continuous functions on I^2 endowed with its natural uniform convergence norm $\|f\|_\infty = \max_{(x,y)\in I^2} |f(x,y)|$. This is an immediate consequence of the proposition below. Convexity of $\mathcal{CP}$ is easy to establish since conditions (1), (2) and (3) in definition (0.1) are verified for every convex combination $tC_1 + (1-t)C_2,\ 0 \leq t \leq 1$ of two copulas C_1 and C_2.

Proposition 3.9. *Let C be a copula as given by definition (0.1). For all (u_1, u_2), (v_1, v_2) in I^2 we have*

$$|C(u_2, v_2) - C(u_1, v_1)| \leq |u_2 - u_1| + |v_2 - v_1| \tag{3.6}$$

The estimation (3.6) implies that the mapping $(x, y) \mapsto C(x, y)$ is Lipschitz function and thus continuous on I^2. Moreover, as already mentioned at the introduction, all partial derivative of this mapping exist almost everywhere (exactly in Lebesgue point) and in particular for almost all $(x, y) \in I^2$, we have $C(x, y) = \int_0^x \frac{\partial C(s,y)}{\partial s} ds$. We refer for such results to [17].
We are now able to state the important results concerning the compactness of $\mathcal{CP}$

Theorem 3.10. *The set $\mathcal{CP}$ of all copulas on I^2 is convex and compact subset of $C(I^2)$.*

Proof. The convexity is already discussed above and according to theorem (2.6) it suffices to verify that $\mathcal{CP}$ is equicontinuous and relatively compact. The first property is easy to check from (3.6) since for a given point $(x_0, y_0) \in I^2 \in$, and all $\varepsilon > 0$ one has

$$\forall (x, y) \in I^2 \text{ and } \forall C \in \mathcal{CP}\ \|(x, y) - (x_0, y_0)\|_1 < \varepsilon \implies |C(x, y) - C(x_0, y_0)| < \varepsilon.$$

The pointwise relative compactness is obvious because for all $f \in \mathcal{CP}$ the set $\overline{\{f(x, y),\ (x, y) \in I^2\}}$ is compact as a closed subset of the compact set I.

□

Remark 3.11. *The proof of theorem (3.10) is made easy thanks to boundedness of all copulas which take their values in I (they are joint distribution in some sense). For general subsets of equicontinuous functions on I^2, the pointwise relative compactness is more subtle.*

Consider now the topological space $\mathcal{CP}$ equipped with the topology arising from the $C(I^2)$ norm. Indeed it is a metric space with the natural distance associated with the norm $\|.\|_\infty$. It is then obviously separate.
On the separate topological space $\mathcal{CP}$, we consider the binary relation given by

$$\forall (C_1, C_2) \in \mathcal{CP}^2 : \ C_1 \sim C_2 \iff |C_1 - C_1^\top| = |C_2 - C_2^\top|. \tag{3.7}$$

Without any difficulty one checks that $\sim$ is an equivalence on $\mathcal{CP}$. A natural following step is to investigate the quotient set $\mathcal{CP}/\sim$. The objective is to establish that this latter set is not trivial. This will give sense first for the order defined in [6] and second for the opportunity to search and develop asymmetry measures. The following theorem gives an affirmative answer

Theorem 3.12. *The quotient $\mathcal{CP}/\sim$ contains more than three elements and thus is not trivial.*

Proof. As it will be showed below, it suffices to prove that the quotient space $\mathcal{CP}/\sim$ equipped with the quotient topology is separate. In fact, by theorem (2.5) it is sufficient to prove that $\mathcal{CP}$ is compact. But this compactness is stated in theorem (3.10).
Let us make clear why a separate space is not trivial:

- **a:** If $\mathcal{CP}/\sim$ is reduced to a singleton then all copulas are equivalent, which is a contradiction.
- **b:** If $\mathcal{CP}/\sim = \{\dot{C}_1, \dot{C}_2\}$ with $C_1 \neq C_2$ then its topology may not be separate unless it is discrete. But in this case $\mathcal{CP}$ is not connected (thus not convex) space since $\mathcal{CP} = P^{-1}\{C_1\} \cup P^{-1}\{C_1\}$ which splits $\mathcal{CP}$ on two disjoint open sets. This contradicts theorem (3.10).

Finally, the set $\mathcal{CP}$ contains more than three elements. This result yields a beforehand possibility to construct at least three copulas with different measure values according to the order $\preceq$. In other words, the order proposed by Sigurov et.al in [6] is consistent.

□

4. Generalization of asymmetry measure of copulas

In [2], a subcopula is defined as a generalization of copula. It is precisely a kind of " copula with domain." To make clearer the concept we recap the definition as written therein (adapted to bivariate case)

Definition 4.13. *A 2-dimensional subcopula (or 2-subcopula, or bivariate one) is a function C which satisfies*

1. *The domain $Dom(C)$ of C is such as $Dom(C) = S_1 \times S_2$, where each S_k is a measurable subset of I containing 0 and 1.*
2. *C is grounded and 2-increasing, which means that C satisfies the property (3) in definition (0.1).*

(3) *C has (one-dimensional) margins C_1 and C_2 which satisfy $C_k(t) = t$ for all $t \in S_k$.*

The regularity of a given subcopula is warranted only on the domain $S_1 \times S_2$ where it is, thanks to Theorem 3.6, Lipschitz-continuous. On the complement of $S_1 \times S_2$, its behavior may be pathologic.(See shuffle of M) It yields that subcopulas deserve another treatment concerning asymmetry measure. Here we will recall and then adapt the μ_p-measures defined in [6] to the context of subcopulas. Our approach is based on Theorem 2.7 and Proposition 2.8 of section 2.
For a given subcopula C we consider its *bracket* defined by $C_s = |C - C^\top|$. The behavior of C_s may be arbitrarily impredictable on I^2. So we split it in accordance with Proposition 2.8: $C_s = g + b$.
In order to compare the asymmetry of two subcopulas C_1 and C_2 and since the respective bad parts b_1 and b_2 of brackets $C_{s,1}$ and $C_{s,2}$ are *in average* null (equal zero), it will be more convenient to compare good parts g_1 and g_2. This yields that the following definition makes sense

Definition 4.14. *Let us fix $t \in]0, 1[$ and let C_1 and C_2 two subcopulas with brackets $C_{s,2}$ and $C_{s,1}$ We will say that C_1 is more symmetric than C_2 with tolerance t and write $C_2 \prec_t C_1$ if and only if $\|g_1\|_1 \geq \|g_2\|_1$, where g_1 and g_2 are the good parts associated with C_1 and C_2 respectively as given in proposition (2.8).*

For instance, we consider the measure μ_1 which quantifies $\mu_1(f) = \mu(f) = \|g\|_1$ for any $f \in L^1(I^2)$ with good part g. One may follow the same way for generalizing to $p-$measures $\|g\|_p$.
The parameter t or *tolerance* which appears in Definition (4) is an a priori asymmetry accepted (ou tolerated) degree. For convenience and as we are dealing with copulas, it is natural to choose $t = 1$ since a copula is joint distribution function.
To accredit definition (4), we illustrate it with the following example which gives simultaneously an application and illustration of above purposes

Example 4.15. *We propose asymmetric Cobb-Douglas model of cardinal utility. Let us recall essential notions about this important model in economical studies. Cobb-Douglas functions are used for both production functions*

$$Q(K, L) = K^\alpha L^\beta, \ \alpha + \beta = 1$$

where Q is the production quantity, K denotes the capital and L is the labor. The coefficients α and β explain the contribution of each production factor.
The same functional form is also used for the utility function of two different goods X and Y. We often define it as

$$U(X, Y) = X^\alpha Y^{1-\alpha}.$$

Note that the hypothesis $\alpha + \beta = 1$ is not a restriction since it is assumed just to simplify the computation of some characteristics such as the slope *or* the marginal rate of substitution *given by $\frac{dY}{dX} = \frac{-\beta Y}{(1-\beta)X}$. So one may consider the more general*

product $U = X^\alpha Y^\beta$ without any other conditions on coefficients α and β except their positivity.
Let $\mathcal{Q} = (q_n)_{n\geq 1}$ and consider Cobb-Douglas subcopula C *defined by*

$$C(u,v) = \frac{2}{3}u^\alpha v\mathbf{1}_{\overline{\mathcal{Q}^2}} + \frac{1}{3}q_n q_m \mathbf{1}_{\mathcal{Q}^2}(q_n,q_m),\ 0<\alpha<1.$$

It is easy to check that C is not symmetric and that the asymmetric part is $\frac{2}{3}u^\alpha v\mathbf{1}_{\overline{\mathcal{Q}^2}}$. So, the bracket C_s is reduced to $C_s = |\frac{2}{3}u^\alpha v\mathbf{1}_{\overline{\mathcal{Q}^2}} - \frac{2}{3}v^\alpha u\mathbf{1}_{\overline{\mathcal{Q}^2}}|$ which will measure asymmetry degree of the copula C. In other words, the bad part in C_s bracket is null.

Let us know consider the copula

$$D(u,v) = \frac{2}{3}uv\mathbf{1}_{\overline{\mathcal{Q}^2}} + \frac{1}{3}q_n^\alpha q_m \mathbf{1}_{\mathcal{Q}^2}(q_n,q_m),\ 0<\alpha<1.$$

The copula D is clearly non symmetric. In order to be convinced, it suffices to see example 2.11 in [21] *where it was shown that D and C suggested in both of two last examples are asymmetric copulas. To measure the asymmetry of D, one may first compute its bracket:*

$$D_s(u,v) = |\frac{1}{3}q_n^\alpha q_m - \frac{1}{3}q_n q_m^\alpha|\mathbf{1}_{\mathcal{Q}}(u,v) \text{ when } u=q_n \text{ and } v=q_m.$$

Let us discuss more copulas C and D. Since they are not symmetric, the two inequalities $K \preceq C$ and $K \preceq D$ hold for any symmetric copula K because the bracket of K satisfies $K_s = |K - K^\top| = 0$. It is easy to check that C and D are not comparable with respect to preoder $\preceq$. On the other hand one may use the measure μ to see that $\mu(C) > \mu(D)$ since $D_s = 0$ almost everywhere. The measure μ gives more than the simple ranking, it allows a comparison of magnitudes which means the contribution margin of each production factor (K and L) or in general for each good (X and Y).

Comments

1. Further steps to modeling more complicated utility functions which vary with respect the time will be investigated in our future work. The idea consists on gluing such functions following the same way of frozen coefficients as in [22].
2. Using any scientific computing software, one may compute the measure $\mu(h)$ of a given copula $h : (u,v) \mapsto ku^\alpha v,\ k,\alpha \in]0,1[$. It suffices to implement the following (e.g with Pyton):
from scipy.integrate import dblquad
def Integrale(alpha,k): area=(dblquad(k u,v: u**alpha*v, 0, 1, k u:0, k u=1))
return(k*area[0])

```
I=Integrale(1,1)
print(I)
```

References

[1] Sklar; *Fonctions de répartition à n dimensions et leurs marges.* Publ. Inst. Statist. Univ. Paris. 8, 229-231. (1959)

[2] Roger B. Nelsen. *An Introduction to Copulas.* Springer Series in Statistics. Springer Science+Business Media, Inc., New York, 2^{nd} edition, (2006).

[3] J. Tang, K. Wang, L.Shao. *Supervised Matrix Factorization Hashing for Cross-Modal Retrieval.* Transactions on image Processing, Vol. 25, NO. 7, July (2016).

[4] *Model Reduction in Power Systems Using Krylov Subspace Methods* IEEE transaction on power systems, Vol. 20, NO. 2, may 2005

[5] H.Dette, K.F Siburg and P. Stoimenov *A copula based nonparametric measure of regression dependence.* (Preprint) Submitted to *The Annals of Statistics.*(2010).

[6] Siburg et al. *An order of asymmetry in copulas, and implications for risk management.* Insurance: Mathematics and Economics. Volume 68, 247-251. (2016)

[7] Cifarelli, D.M.; Conti, P.L.; Regazzini, E. *On the asymptotic distribution of a general measure of monotone dependence.* Ann. Statist. 24, 1386-1399. (1996)

[8] Klement, E., Mesiar, R., *.How non-symmetric can a copula be?* Comment. Math. Univ. Carolin. 47, 141148. (2006)

[9] Ayumi Ida, Naoyuki Ishimuraa, MasaAki Nakamura. *Note on the measures of dependence in terms of copulas.* Procedia Economics and Finance 14 . 273-279. (2014)

[10] Nelsen, R.B., 2007. *Extremes of nonexchangeability.* Statist. Papers 48, 329-336.

[11] Arturo Erdely José M. Gonzàlez-Barrios. *A nonparametric symmetry test for absolutely continuous bivariate copulas* Stat Methods Appl (2010) 19:541565. DOI 10.1007/s10260-010-0147-7

[12] P.N. Patil, P.P. Patil, and Bagkavos. *A measure of asymmetry* D. Stat Papers (2012) 53: 971. https://doi.org/10.1007/s00362-011-0401-6

[13] Iman Marvian, and Robert W. Spekkens *Extending Noether's theorem by quantifying the asymmetry of quantum states.* arXiv:1404.3236v1 [quant-ph] 11 Apr 2014.

[14] D. Black. *On Arrow's Impossibility Theorem.* The Journal of Law and Economics Vol. 12, No. 2, pp. 227-248 (Oct., 1969).

[15] Z. Yuan. *A New Explanation of K. J. Arrows Impossibility Theorem: On Conditions of Social Welfare Functions.* Open Journal of Political Science, 5, 26-39, 2015.

[16] N. V Krylov *On the Calderon-Zygmund theorem and application to parabolic equations.* St Petersburg math. J 13 , 509-525. (2002).

[17] L.C Evans, *Partial differential equations.* American Mathematical Society. Providence, RI.(1998)

[18] G. Choquet and Amiel Feinstein (Translator). *Topology.* Pure and Applied Mathematics. Academic Press, 1^{st} edition, Volume 19.(April 11, 1966)

[19] H. Brezis. *Analyse fonctionnelle : Théorie et applications .* (Mathematiques Appliquées pour la Maitrise.) Paris : Masson, (1983).

[20] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer. (2011)

[21] C. Genest, J. Quessy. Neslehovà,. *Tests of symmetry for bivariate copulas.* Ann. Inst. Statist. Math. 64, 811834. (2012).

[22] A. Sani and H. Laasri *Evolution equations governed by Lipschitz continuous nonautonomous* forms. Czechoslovak Mathematical Journal 65(2) 475-491. (2015)

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