Econometrica, Vol. 43, No. 5–6 (September–November, 1975)

THE MOST GENERAL CLASS OF CES FUNCTIONS

BY RYUZO SATO[1]

1. INTRODUCTION

IT IS NO EXAGGERATION to say that in economic analysis the Cobb-Douglas and CES functions are the most frequently used nonlinear type of special functions. They have been construed as the simplest type of meaningful functions in both theoretical and empirical studies of production functions [**1** and **10**] and have received privileged attention in the analysis of economic growth [**1**, **6**, and **10**]. In the theory of consumer behavior, this type of function has served theoretical and empirical needs as the Bergson family of utility index functions [**3** and **8**].

The purpose of this paper is to derive the general class of CES functions which are nonhomothetic and nonseparable, but include the ordinary CES or Cobb-Douglas functions as a special case, and to study the properties of the general CES functions in conjunction with production and utility analyses.[2] We first give economic justifications as to why such CES functions may be useful in economic analysis.

Consider a typical estimation problem of production functions under competitive markets and under the homogeneity or homotheticity assumption. If the underlying production function is of the CES type together with the Hicksian neutral type of technical progress, then the marginal rate of substitution function is equal to $\log \omega = \log a + (1/\sigma) \log k$, where ω is the ratio of the prices of labor and capital, k is the capital-labor ratio, and σ is the elasticity of substitution. Because of the homotheticity assumption, the marginal rate of substitution is *independent* of the level of production and of the neutral type of technical progress. Empirical data in many cases, however, suggest that the factor price ratio varies even at a constant input ratio. The well-known technique for approaching this type of situation is the introduction of the device of biased technical progress, in particular, the so-called factor-augmenting technical progress. But this device often fails due to the impossibility (theorem) of identification of the bias and substitution effect [**10**]. This problem will be solved if we relax the homogeneity or homotheticity assumption (hereafter referred to only as "homotheticity" assumption), so that *the level of output and the degree of the neutral technical progress will explicitly have effects on factor combinations.* The statistically convenient form of the marginal rate of substitution is again, $\log \omega = \log a + (1/\sigma) \log k + b \log Y + c \log T(t)$, where Y is the level of output and T is the index of biased technical progress. The production function which is consistent with the above relationship is, in fact, the most general class of CES functions that we propose to study.

Another justification stemming from the analysis of demand functions is that under the ordinary or homothetic CES utility index function, the expansion path is a straight line through the origin, implying that all commodities are always normal goods, while under the nonhomothetic CES function, meaningful income effects can be studied and the distinction between normal and inferior goods can be made within a statistically convenient framework.

2. DERIVATION OF THE GENERAL CLASS OF CES FUNCTIONS

The elasticity of substitution between commodities (or factors) x_1 and x_2 is defined as:

$$(1) \qquad \sigma = \frac{x_1 f_1 + x_2 f_2}{x_1 x_2\left(-\dfrac{f_1 f_{22}}{f_2} + 2f_{12} - \dfrac{f_2 f_{11}}{f_1}\right)} = \frac{x_1 + x_2\omega}{x_1 x_2\left(\dfrac{\partial \omega}{\partial x_1} - \dfrac{\partial \omega}{\partial x_2}\omega^{-1}\right)}$$

[1] This work was supported by the National Science Foundation Grant (No. GS-41436), and a Guggenheim Fellowship. The author wishes to acknowledge helpful comments from a referee.

[2] This is a further extension of an earlier work [**11**].

where $\omega = f_2/f_1$ is the marginal rate of substitution between x_1 and x_2. Rewriting (1) as $\sigma x_1 x_2(\partial\omega/\partial x_1) - \sigma x_1 x_2 \omega^{-1}(\partial\omega/\partial x_2) - x_1 - x_2\omega = 0$, and letting $\omega = e^u$, equation (1) may be expressed as:

$$(1') \qquad e^u \frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_2} = \frac{1}{\sigma x_2} + \frac{1}{\sigma x_1} e^u, \quad \sigma = \sigma(x_1, x_2, u).$$

This is a general class of the first-order quasi-linear partial differential equation involving x_1, x_2, and u. If σ is constant, $0 < \sigma < \infty$, equation (1′) reduces to a rather simple partial differential equation which can be solved for explicit solution (see Lax [**4**, pp. 54–57]). With the appropriate initial conditions such as $x_1(\tau, t) = \bar{x}_1(\tau, 0) = \tau$, $x_2(\tau, t) = \bar{x}_2(\tau, 0) = \tau$, and $u(\tau, t) = \bar{u}(\tau, 0) = \tau, 0 \leqslant \tau \leqslant 1$, the above quasi-linear partial differential equation has a unique solution derived from $dx_1/dt = e^u$, $dx_2/dt = -1$, and $du/dt = (1/\sigma x_2) + (1/\sigma x_1)e^u$, i.e., $u = (1/\sigma)(\log x_1 - \log x_2) + \log C(\bar{f}, \tau)$, or from $\omega = e^u$,

$$(2) \qquad \omega = \left(\frac{x_1}{x_2}\right)^{1/\sigma} C(f), \quad C > 0.$$

This is a general expression for the marginal rate of substitution of the class containing all CES (homothetic or nonhomothetic) functions.

In order to obtain the general class of CES functions, we have to solve (2) by setting $\omega = f_2/f_1$, i.e.,

$$(2') \qquad x_1^{1/\sigma} C(f) \cdot \frac{\partial f}{\partial x_1} - x_2^{1/\sigma} \cdot \frac{\partial f}{\partial x_2} = 0, \quad \sigma = \text{constant}.$$

Unless f is separable, (2′) *cannot* be reduced to an ordinary differential equation. Again the partial differential equation (2′) has a unique solution for appropriate initial conditions, i.e., $x_1 = \bar{x}_1 = \bar{x}_1(\tau)$, $x_2 = \bar{x}_2(\tau)$, and $f = \bar{f}(\tau)$, for $0 \leqslant \tau \leqslant 1$. From $dx_1/dt = C(f)x_1^{1/\sigma}$, $dx_2/dt = -x_2^{1/\sigma}$, and $df/dt = 0$, we immediately obtain, $(\sigma/(\sigma - 1))x_1^{(\sigma - 1/\sigma)} - (\sigma/(\sigma - 1)) \times \bar{x}_1(\tau)^{(\sigma - 1/\sigma)} = C(f) \cdot t$, $-(\sigma/(\sigma - 1))x_2^{(\sigma - 1/\sigma)} + (\sigma/(\sigma - 1))\bar{x}_2(\tau)^{(\sigma - 1/\sigma)} = t, f - \bar{f}(\tau) = 0, \sigma \neq 1$. If we assume that $\bar{f}$ is monotone for $0 \leqslant \tau \leqslant 1$ and $\bar{f}' \neq 0$, we can solve for τ as $\tau = \bar{f}^{-1}(f) = h(f)$. Substituting τ in $\bar{x}_1(\tau)$ and $\bar{x}_2(\tau)$ and eliminating t, we obtain an implicit formulation of the general class of CES ($\sigma \neq 1$) functions,

$$(3) \qquad C(f) = \frac{\rho^{-1}x_1^{-\rho} - \rho^{-1}\bar{x}_1(h(f))^{-\rho}}{-\rho^{-1}x_2^{-\rho} + \rho^{-1}\bar{x}_2(h(f))^{-\rho}}, \quad \rho = \frac{1-\sigma}{\sigma}, \quad \sigma \neq 1.$$

Conveniently rewriting this we have the implicit formulation of the general CES functions ($\sigma \neq 1$) as:

$$(3') \qquad F[-(\beta_1 x_1^{-\rho} + \theta_1) + (\beta_2 x_2^{-\rho} + \theta_2)C(f) + g_2(f)C(f) + g_1(f)] = 0,$$

where β_i and θ_i ($i = 1, 2$) are appropriate constants and $-\rho = (\sigma - 1)/\sigma, \sigma \neq 1$. When $\sigma = 1$ (nonhomothetic Cobb-Douglas) we replace $x_1^{-\rho}$ and $x_2^{-\rho}$ by $\log x_1$ and $\log x_2$ respectively. Combining terms and omitting transformations, equation (3′) may be simply expressed as:

$$(4) \qquad F(X_1, X_2, f) = C_1(f)X_1 + C_2(f)X_2 + H(f) = 0$$

or

$$(4a) \qquad G(X_1, X_2, f) = C_1(f)X_1 + C_2(f)X_2 = 1,$$

where X_i either equals $\beta_i x_i^{-\rho} + \theta_i$ for $\sigma \neq 1$ or equals $\beta_i \log x_i + \theta_i$ for $\sigma = 1, i = 1, 2$.

3. DIFFERENT TYPES OF CES FUNCTIONS

There are basically two types of CES functions: one, homothetic (or ordinary) CES including Cobb-Douglas and the other, nonhomothetic CES functions, depending upon the relationships among $C_1(f)$, $C_2(f)$, and $H(f)$.

(i) HOMOTHETIC CES: When $dC_i/df \equiv 0$ ($i = 1, 2$) and $dH/df \neq 0$, equation (4) represents the class of ordinary CES functions, $\sigma \neq 1$ and $\sigma = 1$ (Cobb-Douglas) (See [**2**, **5**, **7**, and **13**].)

(ii) NONHOMOTHETIC CES: When $dC_i/df \neq 0$ ($i = 1, 2$), $dH/df \neq 0$, and $C_1 \neq aC_2$ (a being a constant), equation (4) represents nonhomothetic CES functions. If $\sigma = 1$, it may be called the class of *nonhomothetic Cobb-Douglas functions*. All homothetic CES functions are separable, but there are both separable and nonseparable nonhomothetic CES functions.

(a) *Separable Nonhomothetic CES*: When (4) is written as $-X_1 + C(f)X_2 = 0$, $X_i =$ either $\beta_i x_i^{-\rho} + \theta_i$ or $\beta_i \log x_i + \theta_i, \theta_i = (a - \beta_i)\rho - \beta_i$ ($i = 1, 2$), we get the class of *separable* nonhomothetic CES functions (Sato [**11**]):

$$(5) \qquad f = V\left(\frac{X_1}{X_2}\right), \quad \rho\beta_1 < 0, \quad \rho\beta_2 > 0,$$

where the β_i are *distribution parameters*, $1/\alpha$ is the *nonhomotheticity parameter*, and ρ is the *substitution parameter*. (For a more detailed analysis of separable CES functions, see Sato [**11**].)

(b) *Nonseparable CES*: Another convenient form of (4) is the following:

$$(4') \qquad F = -X_1 + C(f)X_2 + H(f) = 0.$$

Nonseparable CES functions are the *nonlinear* solutions of (4′) in terms of f or $C(f)$. (Separable CES functions are the *linear* solutions.) Examples of nonseparable CES are: if $C(f) = af$ and $H(f) = bf^2$, then $f = (-aX_2 \pm \sqrt{(aX_2)^2 + 4bX_1})/2b > 0$; if $C(f) = af^2$ and $H(f) = bf$, then $f = (-b \pm \sqrt{b^2 + 4aX_1X_2})/2aX_2 > 0$.

4. PROPERTIES OF GENERAL CES FUNCTIONS

(i) POSITIVITY OF f: We first assume that X_i in (4′) is positive for positive inputs x_i. Then sufficient conditions for $f > 0$ in (4′) are either:

$$(6a) \qquad \begin{cases} F(x_1, x_2, 0) < 0, \\ \dfrac{\partial F}{\partial f} > 0, \\ F(x_1, x_2, \infty) > 0, \end{cases}$$

or

$$(6b) \qquad \begin{cases} F(x_1, x_2, 0) > 0, \\ \dfrac{\partial F}{\partial f} < 0, \\ F(x_1, x_2, \infty) < 0, \end{cases}$$

for $x_i > 0$ ($i = 1, 2$).

(ii) POSITIVITY OF $f_i = (\partial f/\partial x_i)$ AND THE LAW OF DIMINISHING MARGINAL RATE OF SUBSTITUTION: Marginal values of f are $f_1 = -\rho\beta_1 x_1^{-(1+\rho)}/(X_2C'(f) + H'(f))$ and $f_2 = \rho\beta_2 x_2^{-(1+\rho)} \cdot C(f)/(X_2C'(f) + H'(f))$. As long as $f_i > 0$ and $x_i > 0$, the bordered Hessian will be positive for $\sigma > 0$. Positivity of f_i and ω requires that sign β_1 be equal to $-$sign β_2.

(iii) IMPLICIT CES FUNCTIONS AS UTILITY INDEX FUNCTIONS: As the implicit CES function is *nonhomothetic*, it may be of some interest to calculate the income effects of the Slutsky equations. Expressing them in elasticity form, we have,

$$\frac{\partial x_i}{\partial p_j} \cdot \frac{p_j}{x_i} = \kappa_j\sigma + x_jp_j \frac{-\dfrac{C'(f)}{C(f)}x_jf_j + \sigma^{-1}}{x_1f_1 + x_2f_2}, \quad i = 1, 2,$$

where p_i is the price of the ith good, κ_i is the relative cost of the ith good $= x_i p_i / I$, and I is income ($i = 1, 2$). Then we have the following theorem:

THEOREM: *Under the general class of CES utility index functions: (i) x_1 and x_2 are both normal goods if and only if $-(1/\delta_1\sigma) < \xi < (1/\delta_2\sigma)$; (ii) x_i is a normal, but x_j is an inferior good if and only if $\xi < -(1/\delta_i\sigma)$ $(i \neq j)$; (iii) in the case of homothetic CES, i.e., $\xi \equiv 0$, x_1 and x_2 are both normal goods, where $\xi = (dC/df)\cdot(f/C)$ and $\delta_i = x_i f_i / f$ $(i = 1, 2)$.*

It is shown in [**11**] that the system of demand functions associated with separable nonhomothetic CES functions when $\theta_1 = 0$ are *self-dual* and the own *price elasticity of demand is identically equal to the elasticity of substitution* (see [**12**]).

(iv) NONHOMOTHETIC CES PRODUCTION FUNCTIONS: The minimum cost condition under competitive markets requires

$$\frac{p_2}{p_1} = \omega = \left(\frac{x_1}{x_2}\right)^{1/\sigma} C(Y), \quad p_i = \text{price of } x_i, \quad Y = \text{output}. \tag{7}$$

If, for instance, $C(Y) = aY^bTc$, T being the technical progress factor, equation (7) implies:

$$\log\left(\frac{p_2}{p_1}\right) = \log\omega = \log a + \frac{1}{\sigma}\log\left(\frac{x_1}{x_2}\right) + b\log Y + c\log T. \tag{7'}$$

This is the equation mentioned in Section 1. Given the data for the marginal rate of substitution ω, the capital-labor ratio x_1/x_2, and output series Y, we can completely identify all the parameters for nonhomothetic CES functions, including the nonhomothetic parameter b. Empirical applications will be later reported in a separate paper.

5. EXTENSIONS

An obvious extension is that σ itself depends on the level of f, i.e.,

$$\omega = \left(\frac{x_1}{x_2}\right)^{1/\sigma(f)} \cdot C(f), \quad \sigma(f) > 0, \quad C(f) > 0. \tag{8}$$

Thus the elasticity of substitution is *constant* at a given isoquant (or indifference curve), but *varies* as the level of output (or utility) varies. Integrating (8) again we get

$$F = -X_1 + X_2C(f) + H(f) = 0 \quad \text{and} \tag{9}$$

$$X_i = \beta_i x_i^{-\rho} + \theta_i, \quad \rho = \frac{1-\sigma(f)}{\sigma(f)} \qquad (i = 1, 2).$$

Equation (9) includes all the cases of CES functions. For instance, one isoquant (or indifference curve) can take Cobb-Douglas or ordinary CES form, while the next isoquant takes a nonhomothetic CES form.

We can derive n-variable nonhomothetic CES functions from

$$\sigma_{ij} = \sigma = \frac{\partial \log\left(\dfrac{x_i}{x_j}\right)}{\partial \log \omega_{ij}} = \text{constant} \qquad (i \neq j, 1 \leqslant i, j \leqslant n). \tag{10}$$

Solving the system of partial differential equations once to obtain

$$\omega_{ij} = \left(\frac{x_i}{x_j}\right)^{1/\sigma} C_{ij}(f), \quad C_{ij} > 0, \tag{11}$$

and again solving the above partial differential equations, we have

$$F = \sum_{i=1}^{n} C_i(f)X_i + H(f) = 0 \tag{12}$$

where $X_i = \beta_i x_i^{-\rho} + \theta_i$ for $\sigma \neq 1$, or $X_i = \beta_i \log x_i + \theta_i$ for $\sigma = 1$.[3] Again, σ in the above may still be a function of f so that $\sigma = \sigma(f) =$ constant at a given f. Also, a composite commodity extension of implicit nonhomothetic CES functions may be obtained from the usual tree properties of utility functions, but this will be left to the interested reader in order to save space.

Brown University

Manuscript received May, 1973; revision received August, 1974.

REFERENCES

[1] ARROW, K., ET AL.: "Capital-Labor Substitution and Economic Efficiency," *Review of Economics and Statistics*, 43 (1961), 225–250.

[2] COURANT, R.: *Differential and Integral Calculus*, Vol. 1, Second Edition. London and Glasgow: 1937.

[3] HOUTHAKKER, H.: "A Note on Self-Dual Preferences," *Econometrica*, 33 (1965), 797–801.

[4] LAX, P. D.: *Partial Differential Equations*. New York: 1951.

[5] MCELROY, F. W.: "Note on the CES Production Function," *Econometrica*, 35 (1967), 154–156.

[6] MCFADDEN, D.: "Constant Elasticity of Substitution Production Functions," *Review of Economic Studies*, 30 (1963), 73–83.

[7] PAROUSH, J.: "A Note on the CES Production Function," *Econometrica*, 32 (1964), 213–214.

[8] SAMUELSON, P. A.: "Using Full Duality to Show that Simultaneously Additive Direct and Indirect Utilities Implies Unitary Price Elasticity of Demand," *Econometrica*, 33 (1965), 781–796.

[9] ———: "Two Generalizations of the Elasticity of Substitution," in *Value, Capital and Growth. Papers in Honor of Sir John Hicks*, Edinburgh: Edinburgh University Press, 1968.

[10] SATO, R.: "The Estimation of Biased Technical Progress and the Production Function," *International Economic Review*, 11 (1970), 179–208.

[11] ———: "On the Class of Separable Non-Homothetic CES Functions," *Economic Studies Quarterly*, 15 (1974), 42–55.

[12] ———: "On Self-Dual Preferences," Mimeograph, Brown University, December, 1972. Revised, March, 1974.

[13] YASUI, T.: "The CES Production Function: A Note," *Econometrica*, 33 (1965), 646–648.

[3] When σ is constant, it is equal to the Samuelson elasticity [9].