ON THE STABILITY OF MULTISECTORAL GROWTH EQUILIBRIUM

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1. Introduction

The existence and the relative stability of unique balanced growth for multisectoral models were established by Solow and Samuelson [8] under the assumption of *constant returns to scale*. They studied two types of equation systems: the *difference* equation system and the *differential* equation system. Later Muth [7] and Suits [9] studied the former type of system under the assumption of *decreasing returns to scale*. The first objective of this paper is to study some differential equation systems under assumptions weaker than those imposed by Solow and Samuelson but retaining the assumption of *constant returns* to scale. The second objective is to investigate certain differential equation systems under the assumption of *decreasing returns to scale*.

2. Constant returns to scale—General case

Our system is expressed by the following equations:

$$\dot{X}_i = H^i(X_1, \cdots, X_n), \qquad (i=1, \cdots, n). \tag{2.1}$$

The above system is the Solow-Samuelson balanced growth model. The H^i's are defined for any $(X_1, \cdots, X_n) \geqq 0$ and are assumed to be continuous with respect to every variable and to be positively homogeneous of degree one. Through this paper the X_i's are restricted to non-negative values. Hence the functions are defined only for non-negative values. It is assumed that

(2.2) $\qquad H^i$ is nondecreasing in all variables, except X^i,

and that

(2.2)′ $\qquad \{H^1, \cdots, H^n\}$ is indecomposable.

Here indecomposability is defined as in Morishima [6]. That is, for any set of indices $R=\{i_1, \cdots, i_r\}$, the relations $X_i=X_i'$ for $i \in R$ and $X_l<X_l'$ for $l \notin R$ imply that there exists at least one $i \in R$ such that $H^i(X_1, \cdots, X_n)<H^i(X_1', \cdots, X_n')$. We require that H^i is nondecreasing in X_j, for $j\neq i$, with no restriction on the dependence of H^i on X_i. By contrast Solow and Samuelson assumed that the H^i is increasing in all X_j.

Given their assumptions and the homogeneity of H^i $(i=1, \cdots, n)$, it follows that $H^i \geqq 0$ $(i=1, \cdots, n)$ for $X_j \geqq 0$ $(j=1, \cdots, n)$, and that, $H^i=0$ for all i, if and only if $X_j=0$ for all j. In our case, however, H^i is not necessarily increasing in X_i. Thus, we cannot obtain the above mentioned properties. So, we assume them. That is, *we assume that*

$$H^i \geqq 0 \ (i=1, \cdots, n) \qquad \text{for } X_j \geqq 0 \ (j=1, \cdots, n)\,. \tag{2.3}$$

Then, from the indecomposability and homogeneity of H^i, $H^i=0$ for all i, if and only if $X_j=0$ for all j. Our aim is to prove the following theorem.

THEOREM 1. *For the differential equation system* (2.1), *under the assumptions* (2.2), (2.2)$'$ *and* (2.3), *there exists a uniquely determined positive eigenvalue, a unique strictly positive normalized eigenvector and hence a unique balanced growth path. Moreover, any solution path of the system relatively approaches the balanced growth path.*

PROOF. We can show by a procedure similar to Solow and Samuelson's *the existence of a positive eigenvalue* λ and a *non-negative and non-zero eigenvector* $V=(V_1, V_2, \cdots, V_n)$ such that

$$\begin{aligned}\lambda V_1 &= H^1(V_1, \cdots, V_n)\,,\\ &\ \vdots\\ \lambda V_n &= H^n(V_1, \cdots, V_n)\,.\end{aligned}$$

We will show that *all components of the eigenvector V are positive.* Suppose that some components of V are zeros. Without loss of generality, we may suppose that

$$V_i=0 \qquad \text{for } i \leqq r(<n)\,,$$

and

$$V_i>0 \qquad \text{for } n \geqq i>r\,.$$

Then,

$$\begin{aligned}0 &= H^1(0, \cdots, 0, V_{r+1}, \cdots, V_n)\,,\\ &\ \vdots\\ 0 &= H^r(0, \cdots, 0, V_{r+1}, \cdots, V_n)\,,\\ 0<\lambda V_{r+1} &= H^{r+1}(0, \cdots, 0, V_{r+1}, \cdots, V_n)\,,\\ &\ \vdots\\ 0<\lambda V_n &= H^n(0, \cdots, 0, V_{r+1}, \cdots, V_n)\,.\end{aligned}$$

But this contradicts the assumption of indecomposability, as is easily seen by letting

$$R \equiv \{1, \cdots, r\} ,$$

$$(X_1, \cdots, X_r, X_{r+1}, \cdots, X_n) \equiv (0, \cdots, 0, V_{r+1}, \cdots, V_n) ,$$

$$(X_1', \cdots, X_r', X_{r+1}', \cdots, X_n') \equiv (0, \cdots, 0, 2V_{r+1}, \cdots, 2V_n) .$$

Next, we shall show the uniqueness of the eigenvalue. Suppose that there is another *tuple* of positive eigenvalue and eigenvector (μ, U). Then, we get the following two sets of relations.

$$\lambda = H^1\left(1, \frac{V_2}{V_1}, \cdots, \frac{V_n}{V_1}\right), \tag{2.4.1}$$

$$\lambda = H^2\left(\frac{V_1}{V_2}, 1, \cdots, \frac{V_n}{V_2}\right), \tag{2.4.2}$$

$$\vdots$$

$$\lambda = H^n\left(\frac{V_1}{V_n}, \frac{V_2}{V_n}, \cdots, 1\right). \tag{2.4.n}$$

$$\mu = H^1\left(1, \frac{U_2}{U_1}, \cdots, \frac{U_n}{U_1}\right), \tag{2.5.1}$$

$$\mu = H^2\left(\frac{U_1}{U_2}, 1, \cdots, \frac{U_n}{U_2}\right), \tag{2.5.2}$$

$$\vdots$$

$$\mu = H^n\left(\frac{U_1}{U_n}, \frac{U_2}{U_n}, \cdots, 1\right). \tag{2.5.n}$$

Assume that $\lambda > \mu$. Compare (2.4.1) and (2.5.1). Then,

$$H^1\left(1, \frac{V_2}{V_1}, \cdots, \frac{V_n}{V_1}\right) > H^1\left(1, \frac{U_2}{U_1}, \cdots, \frac{U_n}{U_1}\right).$$

Since H^1 is nondecreasing in all arguments, except the first one, $V_i/V_1 > U_i/U_1$ must hold for at least one i. Without loss of generality, we can put $i=2$. That is,

$$\frac{V_2}{V_1} > \frac{U_2}{U_1} . \tag{2.6}$$

Compare (2.4.2) and (2.5.2). Then

$$H^2\left(\frac{V_1}{V_2}, 1, \cdots, \frac{V_n}{V_2}\right) > H^2\left(\frac{U_1}{U_2}, 1, \cdots, \frac{U_n}{U_2}\right).$$

Since $V_1/V_2<U_1/U_2$, and H^2 is nondecreasing in all arguments, except the second one, we must have, say,

$$\frac{V_3}{V_2}>\frac{U_3}{U_2}\,. \tag{2.7}$$

From (2.6) and (2.7), we get $V_1/V_3<U_1/U_3$ and $V_2/V_3<U_2/U_3$. Continuing this reasoning, we reach a contradiction for the last relations (2.4.n) and (2.5.n). Assuming that $\mu>\lambda$, we reach a contradiction also.

Since the diagonal arguments in the right-hand sides of both groups of relations are all one, we need not assume that the H^i is increasing in X_i. The above reasoning has already been used by Solow and Samuelson to show the uniqueness of the eigenvalues for the $n=2$ case. But they used different reasoning for the general case. In that reasoning, they used the property that the H^i is increasing in X_j.

We also note that the above reasoning is used by Solow and Samuelson to show the uniqueness of the normalized eigenvector and *that procedure is applicable with a slight modification in our case also.* So, we skip the proof of $V=\alpha U$. Here, α is a proportionate constant.

Our next target is to *show the relative stability of the dynamic path.* We define new variables,

$$y_i=\frac{X_i}{V_i e^{\lambda t}}\,,\qquad (i=1,\cdots,n)\,. \tag{2.8}$$

Then

$$y_i V_i e^{\lambda t}=X_i\,.$$

Differentiating both sides of this relation, we get

$$\dot{y}_i V_i e^{\lambda t}+\lambda y_i V_i e^{\lambda t}=\dot{X}_i\,,\qquad (i=1,\cdots,n)\,. \tag{2.9}$$

Substituting relations (2.8) and (2.9) into the original system, we get

$$\dot{y}_i=H^i\left(\frac{V_1}{V_i}y_1,\cdots,\frac{V_n}{V_i}y_n\right)-\lambda y_i\,,\qquad (i=1,\cdots,n)\,. \tag{2.10}$$

Put

$$\operatorname{Min}\{y_i(t)\}=m(t)=y_{k_1}(t)=\cdots=y_{k_r}(t)\,,$$

and suppose that

$$y_l(t)>m(t)\qquad \text{for } l\neq k_j\,.$$

Then,

$$\dot{y}_{k_j}(t)\geqq 0\qquad \text{for all } j\leqq r$$

and

$$\dot{y}_{k_j}(t)>0 \qquad \text{for at least one } j\leqq r\,.$$

This is shown as follows.

$$\begin{aligned}\dot{y}_{k_j}&=H^{k_j}\left(\frac{V_1}{V_{k_j}}y_1,\ \cdots,\ \frac{V_n}{V_{k_j}}y_n\right)-\lambda y_{k_j}\\ &\geqq H^{k_j}\left(\frac{V_1}{V_{k_j}}m(t),\ \cdots,\ \frac{V_n}{V_{k_j}}m(t)\right)-\lambda m(t)\\ &=m(t)H^{k_j}\left(\frac{V_1}{V_{k_j}},\ \cdots,\ \frac{V_n}{V_{k_j}}\right)-\lambda m(t)=0\,,\\ &\qquad\qquad\qquad\qquad \text{for } j=1,\ \cdots,\ r\,.\end{aligned}$$

But the inequality holds for at least one k_j. This follows from the assumption of the indecomposability if we put

$$R\equiv\{k_1,\ \cdots,\ k_r\}\,,$$

$$(X_1,\ \cdots,\ X_n)=(V_1m(t),\ \cdots,\ V_nm(t))\,,$$

and

$$(X_1',\ \cdots,\ X_n')=(V_1y_1,\ \cdots,\ V_ny_n)\,.$$

From this property, we infer that the minimum value of $y_i(t)$ cannot stay constant for ever. For, at each moment of time, the number of minimum $y_k(t)$'s is decreasing. Eventually there is just one minimum $y_k(t)$. Henceforth the minimum itself must increase. Since the time elapses continuously in our case, $m(t)$ always increases over time, provided that $y_l(t)>m(t)$ for at least one l. It is possible that

$$\frac{dm(t)}{dt}=0\,,$$

at a certain time point. But $m(t)$ stays constant only for an infinitesimally short period. It does not remain stationary for a finite period. Fig. 1 shows the situation. Put

$$\operatorname*{Max}_i\,\{y_i(t)\}=M(t)\,.$$

Then we can show that $M(t)$ decreases, provided that $Y_l(t)<M(t)$ for at least one l.

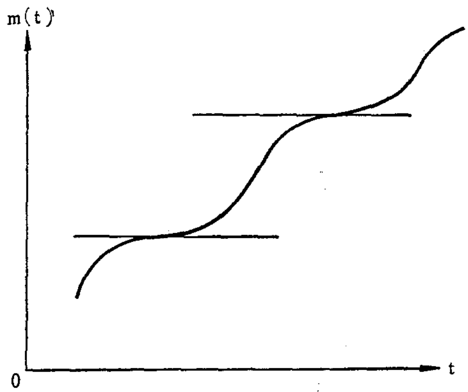

Fig. 1

Thus, $m(t)$ increases and converges to a certain positive value m^* and

$M(t)$ decreases and converges to a certain positive value M^*. That is;

$$\lim_{t\to\infty} m(t)=m^* .$$

$$\lim_{t\to\infty} M(t)=M^* .$$

Then,

$$m^* \leqq M^* .$$

We seek to prove that

$$m^*=M^* .$$

Suppose that $m^*<M^*$. Consider a set of vectors in an n-dimensional space such that

$$S \equiv \{y \equiv (y_1, \cdots, y_n):\ \underset{i}{\mathrm{Min}}\, y_i=m^* \text{ and } \underset{i}{\mathrm{Max}}\, y_i=M^*\} .$$

This set is compact[1]. Consider a dynamic path which starts from a point in this set. Then, by the same reasoning as above, the minimum value of the y_i's increases and the maximum value of the y_i's decreases. To make explicit their dependence on the initial point of y in S, we write, respectively,

$$m^*(t; y) \quad \text{and} \quad M^*(t; y) .$$

Then,

$$m^*(\tau; y)>m^*(0; y)=m^* \quad \text{and} \quad M^*(\tau; y)<M^*(0; y)=M^* .$$

Here, τ is an arbitrarily chosen positive value. Put

[1] In the two dimensional case, this set consists of two points (m^*, M^*) and (M^*, m^*), as is shown in Fig. 2. In the three dimensional case, this set is shown by the set of real lines in Fig. 3.

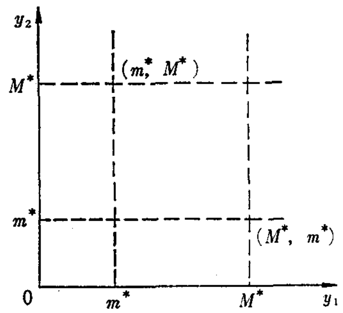

Fig. 2

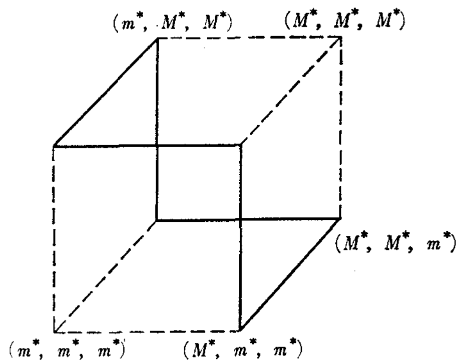

Fig. 3

$$\operatorname*{Inf}_{y \in S} \{m^*(\tau; y) - m^*(0; y)\} = \varepsilon ,$$

and

$$\operatorname*{Inf}_{y \in S} \{M^*(0; y) - M^*(\tau; y)\} = \delta .$$

Since S is compact, both ε and δ are positive.

Now, back to the original dynamic path. As shown above, $\operatorname*{Min}_i y_i(t) = m(t)$ and $\operatorname*{Max}_i y_i(t) = M(t)$, respectively, are sufficiently close to m^* and M^* for any $t \geqq T$, provided T is taken sufficiently large. Then, every point on the dynamic path is sufficiently close to a point in S. From the continuity of the H^i's,[2]

$$m(t+\tau) - m(t) > \frac{\varepsilon}{2} > 0$$

and

$$M(t) - M(t+\tau) > \frac{\delta}{2} > 0$$

for $t \geqq T$, provided T is sufficiently large. But this contradicts

$$\lim_{t \to \infty} m(t) = m^* \quad \text{and} \quad \lim_{t \to \infty} M(t) = M^* .$$

Thus

$$m^* = M^* .$$

This is the desired result. It is noted here that if all components of the initial point $X(0)$ are non-negative and at least one of them is positive, then this property holds for any point $X(t)$ for all $t \geqq 0$.

It is also noted here that the reasoning developed above is not valid for the difference equation system

$$X_i(t+1) = H^i(X_1(t), \cdots, X_n(t)) , \qquad (i = 1, \cdots, n) .$$

That is, if H^i is decreasing in X_i, we can construct an example in which the difference equation system is unstable. Morishima [6] has shown the relative stability of the difference equation system under the assumption that the H^i's are nondecreasing in all X_j's, and $(H^1, \cdots, H^n)$

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[2] We are assuming the existence of the unique solution path passing through any point $(X_1, \cdots, X_n) \geqq 0$. Then, the solution is continuous with respect to the initial point. See Theorem 4.4, Chapter 2 in [2].

is indecomposable and primitive[3], that is, the assumptions of nondecreasingness of the H^i in X^i and of primitivity are additionally required.

Stability is shown in our case even without the assumption of primitivity. Indecomposability is sufficient. But here again stability is not obtained for the difference equation system without the assumption of primitivity, that is, we can construct an example in which instability is shown with indecomposability but without primitivity. Summarizing the results, stability is shown for the differential equation system without the assumptions of nondecreasingness of the H^i in X^i and of primitivity.

The reason why we can relax these assumptions for the differential equation system but not for the difference equation system will be explained in the next section.

3. Constant returns to scale—Matrix case

Our system in this case is

$$\dot{X}=AX. \tag{3.1}$$

Here, X is a vector of which the components are the X_i's. *A is an* $n\times n$ *indecomposable matrix of which the off-diagonal elements are assumed to be all non-negative.* That is, A in a Metzler Matrix. The following theorem is to be proved in this section.

THEOREM 2. *For the differential equation system* (3.1), *under the assumption that all off-diagonal elements of A are non-negative, and A*

[3] The primitivity of non-negative indecomposable matrix is defined in [3] as follows. A matrix A is called primitive if there exists no permutation matrix P such that

$$PAP^{-1}=\begin{bmatrix} 0 & A_{12} & 0 & \cdots 0 \\ 0 & 0 & A_{23} & \cdots 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots A_{k-1k} \\ A_{k1} & 0 & 0 & \cdots 0 \end{bmatrix}$$

with square submatrices (all zero matrices) on the diagonal and $k>1$. Suppose the dynamic system is

$$X(t+1)=AX(t)$$

and A is a non-negative, indecomposable and primitive matrix. Then $X_i(t)$ becomes positive for $t\geqq n$, and for $i=1,\cdots,n$, provided $X_j(0)$ is positive for at least one j. Morishima [6] defined the primitivity of the system

$$X(t+1)=H(x(t))$$

by generalizing above property.

is indecomposable, there exists a unique path of balanced growth or decay, and any solution path approaches relatively it.

Note that the 'growth' rate can be negative.

PROOF OF THEOREM 2. Let α be a positive number which is larger than the absolute value of any diagonal element of Matrix A. Put

$$B \equiv A + \alpha I .$$

Here, I is the identity matrix. Then, all elements of B are non-negative and B is indecomposable. Then B has a unique simple positive eigenvalue μ_1 and a unique normalized positive eigenvector $\bar{X}^{(1)}$ associated with it such that μ_1 is not less than the absolute values of other eigenvalues μ_i's $(i=2, \cdots, n)$ of matrix B. Now, it is easily seen that the $\mu_i - \alpha$ $(\equiv \lambda_i)$ are eigenvalues of A. For

$$\mu_i \bar{X}^{(i)} = B\bar{X}^{(i)} = (A+\alpha I)\bar{X}^{(i)} ,$$

and hence

$$\lambda_i \bar{X}^{(i)} = (\mu_i - \alpha)\bar{X}^{(i)} = A\bar{X}^{(i)} .$$

Here $\bar{X}^{(i)}$ is the eigenvector associated with μ_i and $\bar{X}^{(i)} \not> 0$ for $i \neq 1$. From above, we also see that $\bar{X}^{(i)}$ is the eigenvector associated with λ_i, and that A has the unique normalized *positive* eigenvector $\bar{X}^{(1)}$. The solution of (3.1) is written explicitly in the following way:

$$X(t) = \sum_{i=1}^{n} c_i \bar{X}^{(i)} e^{\lambda_i t} . \tag{3.2}$$

(Here, it is assumed that every eigenvalue of matrix A is a single root of the characteristic equation

$$|A - \lambda I| = 0 ,$$

but this assumption is not essential to the following discussion). Now consider eigenvalues of $A+\alpha I$. The absolute value of μ_i attains a maximum when $i=1$. Recalling that μ_1 is simple, real and positive, we see that the real part of μ_i also attains a maximum when and only when $i=1$. Since

$$\lambda_i = \mu_i - \alpha , \qquad (i=1, \cdots, n)$$

we see that the real part of λ_i also attains a maximum when and only when $i=1$. Then we note from the expression (3.2) that the solution of (3.1) is dominated by the first term $c_1 \bar{X}^{(1)} e^{\lambda_1 t}$ in the summation when

t tends to infinity. Since $\bar{X}^{(1)}$ is strictly positive, the relative stability of the balanced growth path $c_1\bar{X}^{(1)}e^{\lambda_1 t}$ is proved.

However we have to show that the values of the $X_i(t)$'s stay non-negative provided the initial values of the $X_i(t)$'s are chosen so. This is easily seen as follows. Suppose that $X_1(t)=0$. Then

$$\begin{aligned}\dot{X}_1(t)&=a_{11}X_1(t)+a_{12}X_2(t)+\cdots+a_{1n}X_n(t)\\&=a_{12}X_2(t)+\cdots+a_{1n}X_n(t)\geqq 0\ .\end{aligned}$$

Thus the solution of the system does not go into the economically meaningless region where some components of X become negative. The theorem is, therefore, proved.

This is almost the same procedure as that used to show that system (3.1) is absolutely (not relatively) stable if and only if the eigenvalue of the Metzler matrix with the largest real part is negative. In that sense, our theorem is only a trivial extension of this property. We have cited the theorem, however, to explain why the reasoning employed to prove this theorem is not applicable to the difference equation system. That is, the system

$$X(t+1)=AX(t)$$

is not necessarily relatively stable if A is a Metzler Matrix.

The absolute values of eigenvalues are relevant to stability for the difference equation case. In the procedure we have followed to relate the eigenvalues of $A+\alpha I$ to those of A, we see all eigenvalues slide to the left by α. As far as the real parts are concerned, the relative positions of the eigenvalues are kept the same. But of course the absolute values do change. This explains why the relaxation of the assumption of non-negativity of the diagonal elements of A is possible for the differential equation system but not for the difference equation system. The non-matrix case discussed in the preceding section also reflects this fact.

The reason why the assumption of the primitivity is necessary in the case of the difference equation system but not in the case of the differential equation system is the same. That is, the absolute values of the eigenvalues are relevant to stability in the former case, whereas their real parts are relevant to stability in the latter case.[4]

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[4] If a nonnegative indecomposable matrix is primitive, the absolute values of the eigenvalues other than unique positive eigenvalue are less than the latter. See [3]. Thus, if the primitivity is assumed for A, the system $X(t+1)=AX(t)$ is relatively stable. But if A is imprimitive, there are some complex eigenvalues of which absolute value is equal to the unique positive eigenvalue. Thus the system is not necessarily stable.

4. Decreasing returns to scale

In this section, we shall study the following system.[5)]

$$(4.1) \qquad \dot{X}_i = H^i(X_1, \cdots, X_n) \equiv F^i(X_1, \cdots, X_n) - \delta_i X_i \qquad (i=1, \cdots, n).$$

Here, F^i is, for example, the gross output of capital good of type i, and δ_i is the instantaneous rate of depreciation of capital good of type i. *We assume that all F^i's are strictly positive for any strictly positive X, differentiable with respect to every variable and positively homogeneous of degree m_i which are less than one, and that*

$$(4.2) \qquad \frac{\partial H^i}{\partial X_j} \equiv \frac{\partial F^i}{\partial X_j} \geqq 0 \qquad \text{for } j \neq i.$$

Here, we need not assume that

$$\frac{\partial H^i}{\partial X_i} \equiv \frac{\partial F^i}{\partial X_2} - \delta_i > 0, \qquad (i=1, \cdots, n).$$

and the indecomposability of matrix (H^i_j). Since the main purpose is to show the stability of the system, *we assume from the outset the existence of the unique and strictly positive equilibrium* $(\bar{X}_1 \cdots, \bar{X}_n)$. It is noted here that even if we were to assume $\partial H^i/\partial X_i > 0$ for our system, our system could not be a special case of Muth's and Suit's. For, we assumed the homogeneity of F^i but not H^i. Moreover, even if $\delta_i = 0$ for all i, our system would not be a special case of theirs. For we have assumed that the degree of homogeneity in one sector can be different from those in other sectors. In Muth's case, they are all equal. In Suit's case, a more general form of homogeneity is introduced, but the degree of homogeneity is the same in each production sector. Now, we will prove the following theorem.

THEOREM 3. *Under the assumptions of* (4.2), *the degree-less-than-one homogeneity of all F^i's and the existence of unique and positive equilibrium, the solution of the differential equation system* (4.1) *approaches*

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5) We assume in this section that all X_i's are always positive, provided that the initial values of the X_i's are positive. This is so obtained if

$$\lim_{X_i \to 0} \frac{\partial F^i}{\partial X_i} = \infty \qquad (i=1, \cdots, n).$$

This condition is named in [4] the *derivative condition*.

the equilibrium.[6]

PROOF. From (4.1),

$$\frac{\dot{X}_i}{X_i}=\frac{1}{X_i}F^i(X_1, \cdots, X_n)-\delta_i , \qquad (i=1, \cdots, n) . \tag{4.3}$$

Put

$$\frac{1}{X_i}F^i(X_1, \cdots, X_n)\equiv G^i(X_1, \cdots, X_n) , \qquad (i=1, \cdots, n) . \tag{4.4}$$

Then, G^i is homogeneous of degree m_i-1 which is negative. Put

$$\log X_i=\xi_i , \qquad (i=1, \cdots, n) . \tag{4.5}$$

Then,

$$X_i=e^{\xi_i} \quad \text{and} \quad \dot{X}_i/X_i=\dot{\xi}_i , \qquad (i=1, \cdots, n) .$$

From (4.3) and (4.4),

$$\dot{\xi}_i=G^i(e^{\xi_1}, \cdots, e^{\xi_n})-\delta_i , \qquad (i=1, \cdots, n) .$$

Put

$$G^i(e^{\xi_1}, \cdots, e^{\xi_n})-\delta_i\equiv g^i(\xi_1, \cdots, \xi_n) , \qquad (i=1, \cdots, n) . \tag{4.6}$$

Then,

$$\dot{\xi}_i=g^i(\xi_1, \cdots, \xi_n) , \qquad (i=1, \cdots, n) . \tag{4.7}$$

Now, $G^i(X_1, \cdots, X_n)$ is homogeneous of degree m_i-1. Thus

$$(m_i-1)G^i=\sum_{j=1}^{n}\frac{\partial G^i}{\partial X_j}X_j , \qquad (i=1, \cdots, n) .$$

Since $m_i-1<0$ for all i, we get

$$\sum_{j=1}^{n}\frac{\partial G^i}{\partial X_j}X_j<0 \qquad \text{for all } i . \tag{4.8}$$

Now calculate $\partial g^i/\partial\xi_j$. From (4.5) and (4.6),

$$\frac{\partial g^i}{\partial \xi_j}=\frac{\partial G^i}{\partial X_j}\frac{\partial X_j}{\partial \xi_j}=\frac{\partial G^i}{\partial X_j}X_j . \tag{4.9}$$

We have assumed that

$$\frac{\partial F^i}{\partial X_j}\geqq 0 \qquad \text{for } j\neq i .$$

[6] This theorem is a slight modification of the result in [5].

Then, from (4.4),

$$\frac{\partial G^i}{\partial X_j}=\frac{\partial}{\partial X_j}\left(\frac{1}{X_i}F^i\right)=\frac{1}{X_i}\frac{\partial F^i}{\partial X_j}\geqq 0 \qquad \text{for } j\neq i\,. \tag{4.10}$$

Thus, from (4.9),

$$\frac{\partial g^i}{\partial \xi_j}\geqq 0 \qquad \text{for } j\neq i\,. \tag{4.11}$$

From (4.8) and (4.10),

$$\frac{\partial G^i}{\partial X_i}X_i<-\sum_{j\neq i}\frac{\partial G^i}{\partial X_j}X_j\leqq 0\,. \tag{4.12}$$

Then, from (4.9),

$$\frac{\partial g^i}{\partial \xi_i}<0\,. \tag{4.13}$$

From (4.9), (4.11) (4.12) and (4.13), we get

$$\left|\frac{\partial g^i}{\partial \xi_i}\right|>\sum_{j\neq i}\left|\frac{\partial g^i}{\partial \xi_j}\right| \qquad \text{for all } i\,. \tag{4.14}$$

Relations (4.13) and (4.14) are sufficient for the stability of system (4.7) and hence, of system (4.1). Relations (4.13) and (4.14) are known as the Condition of Dominant Diagonal, and the stability of system satisfying this is shown by Arrow, Block and Hurwicz [1].

In the above we assumed the homogeneity of functions $F^i(X_1, \cdots, X_n)$, $(i=1, \cdots, n)$. But that assumption is not necessary for stability. If we can obtain relation (4.8), stability is obtained also. Consider the following alternative set of assumptions. Assume that the amounts of natural resources (including labour force) are given.[7] Let them be $Z_1, \cdots, Z_m$. Assume that the gross production functions of the sausage grinder type are

$$F^i(X_1, \cdots, X_n, Z_1, \cdots, Z_m)\,, \qquad (i=1, \cdots, n)\,.$$

Assume that the F^i are positively homogeneous of degree one in $X_1, \cdots, X_n, Z_1, \cdots, Z_m$. When we take into account all kinds of production factors, the assumption of first degree homogeneity is natural. Now G^i is defined in the same way as (4.4), so that G^i is homogeneous of degree zero in $X_1, \cdots, X_n, Z_1, \cdots, Z_m$. That is,

$$\sum_{j=1}^{n}\frac{\partial G^i}{\partial X_j}X_j+\sum_{k=1}^{m}\frac{\partial G^i}{\partial Z_k}Z_k=0\,, \qquad (i=1, \cdots, n)\,.$$

[7] The following is a slight generalization of the result in [5].

Assuming that

$$\frac{\partial G^i}{\partial Z_k} \geqq 0 \qquad \text{for all } i \text{ and } k .$$

and that

$$\frac{\partial G^i}{\partial Z_{k_i}} > 0 \qquad \text{for at least one } k = k_i, \ (i = 1, \cdots, n) ,$$

we get

$$\sum_{j=1}^{n} \frac{\partial G^i}{\partial X_j} X_j < 0 , \qquad (i = 1, \cdots, n) .$$

This is sufficient for stability of the following system,

$$\dot{X}_i = F^i(X_1, \cdots, X_n, Z_1, \cdots, Z_m) \qquad (i = 1, \cdots, n) .$$

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