A TRACTABLE ALTERNATIVE TO COBB-DOUGLAS UTILITY FOR IMPERFECT COMPETITION

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This paper proposes a tractable alternative to Cobb-Douglas utility to resolve the problems of lack of reservation price and income effects in demand functions derived from Cobb-Douglas utility or quasilinear utility. Another advantage of this althernative is that it provides a closed-form general equilibrium in the case where some industries are monopoly while others are imperfectly competitive.

I. Introduction

Cobb-Douglas utility is one of most widely applied utility functions in economics. Its simple and elegant functional form enables economists to obtain closed-form solutions in a wide range of economic analysis. However, the simplicity does not come without costs. A major drawback is the lack of reservation price; that is, consumers with Cobb-Douglas utility spend a fixed proportion of their income on a commodity or a service, independent of the price of the commodity. Thus, they always purchase a positive amount of the commodity at whatever high price, although the quantity purchased tends to zero when the price tends to infinity. This obviously contradicts commonly observed consumer behaviour. An immediate result from this drawback is that the demand function drawn from a Cobb-Douglas utility function is not suitable for the analysis of monopoly, because the fixed expenditure makes a profit-maximising monopolist set an infinitely high price and produce an infinitesimal amount of output. Applying the Cobb-Douglas utility to a multi-industry general equilibrium model, monopoly or cartelisation must be excluded in all industries.

To overcome this flaw in reservation price, economists usually turn to linear demand functions in dealing with monopoly. However, a linear demand structure can only be derived from quasilinear utility (see Singh and Vives (1984)). Since a quasilinear utility function results in a demand function without income effects, linear demand functions cannot be used to study any issues relevant to consumer income and micro-macroeconomic interaction, and consequently are not suitable for general equilibrium analysis.

After Dixit and Stiglitz's (1977) seminal contribution, analysts become more delighted to use CES utility functions in general equilibrium analysis, especially in new growth

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models.[1] Since the consumption of each commodity enters the utility function symmetrically, it yields constant elasticity of substitution, as the name of the utility function suggested. The CES utility brings tractable analytical form on the one hand but it excludes more sophisticated industry structure and asymmetry on the other hand. Usually, it is assumed that each commodity is supplied by one firm and all firms in the economy are identical and engage in monopolistic competition. Of course, it is possible to consider that each commodity is supplied by an industry and industry structures across industries are different. However, such a setting will result in a model hardly tractable because the closed-form equilibrium with CES utility requires all industries are symmetric.

It is clear that these most-widely used utility functions have their advantages. But they fail to provide a tractable framework for general equilibrium analysis with asymmetric industries such as a monopoly industry along with other imperfectly competitive industries. This paper suggests an alternate utility to Cobb-Douglas to resolve the problem. At a partial equilibrium level, it produces a demand structure accommodating reservation price as well as income effects. For a general equilibrium analysis, it provides a tractable framework containing different industry structures.

II. An Alternative to Cobb-Douglas Utility

The suggested utility function is in the form[2]

$$u = \prod_i (Q_i + a_i)^{b_i}, \qquad a_i > 0,\ b_i > 0,\ \sum_i b_i = 1. \tag{1}$$

In this utility function, the subscript i indexes the category of commodities, and Q_i is the consumption quantity of commodity i. The number of commodities can be either finite or infinite.[3] This utility collapse to the conventional Cobb-Douglas utility function when $a_i = 0$. On the other hand, it becomes the Stone-Geary utility function if we assume that $a_i < 0$ (see Jehle (1991), p. 211). From demand function (4) below, it is not hard to find that the indirect utility derived from (1) leads to a special case of the Gorman form (see Varian (1992), p.153).[4] Taking partial derivative of utility (1) with respect to Q_i to obtain marginal utility,

$$MU_i = ub_i(Q_i + a_i)^{-1}.$$

The marginal utility is finite when a consumer consumes an infinitesimal amount of a commodity, which contradicts to the Cobb-Douglas utility. It is the finiteness of marginal utility that makes a consumer have a finite reservation price. Setting Q_i in the above equation equal to zero to obtain

$$b_i/a_i = MU_i/u, \tag{2}$$

which means that the ratio of two utility parameters, b_i and a_i, is equal to the ratio of marginal utility to utility.

Assume each consumer has income I so that they face the budget constraint

[1] For instance, Grossman and Helpman (1991), Romer (1990).
[2] The assumption that $\sum_i b_i = 1$ is not necessary but it can simplify notations.
[3] For the case of infinite categories, it is better to consider that i is distributed on [0, 1] and use the utility form $u = \int_0^1 b_i \log(q_i + a_i)\, di$.
[4] The author would like to thank an anonymous referee for indicating this.

$$\sum_i P_i Q_i = I, \tag{3}$$

where P_i is the market price of commodity i.

Proposition 1. With utility function (1), a consumer facing budget constraint (3) has the demand function below,

$$Q_i = (P + I)b_i P_i^{-1} - a_i = (P_{-i} + I)b_i P_i^{-1} - (1 - b_i)a_i, \tag{4}$$

where the price indexes are defined as

$$P \equiv \sum_i a_i P_i, \qquad P_{-1} \equiv \sum_{k \neq i} a_k P_k. \tag{5}$$

Proof. Define Lagrange function

$$\mathscr{J} = \prod_i (Q_i + a_i)^{b_i} - \lambda \left(\sum_i P_i Q_i - I \right).$$

Its partial derivative with respect to Q_i provides the first-order condition that

$$\prod_k (Q_k + a_k)^{b_k} b_i (Q_i + a_i)^{-1} = \lambda P_i. \tag{6}$$

Multiplying both sides of equation (6) by Q_i and summing up across i yield

$$\prod_k (Q_k + a_k)^{b_k} \sum_i b_i Q_i (Q_i + a_i)^{-1} = \lambda \sum_i P_i Q_i = \lambda I. \tag{7}$$

Multiplying both sides of equation (6) by a_i and summing up across i yield

$$\prod_k (Q_k + a_k)^{b_k} \sum_i a_i b_i (Q_i + a_i)^{-1} = \lambda \sum_i a_i P_i = \lambda P. \tag{8}$$

Substituting (7) into (6) to eliminate λ leads to

$$b_i (Q_i + a_i)^{-1} = I^{-1} P_i \sum_k b_k Q_k (Q_k + a_k)^{-1} = I^{-1} P_i \left(1 - \sum_k a_k b_k (Q_k + a_k)^{-1} \right). \tag{9}$$

Using (8) to eliminate λ in (6) we get

$$\sum_k a_k b_k (Q_k + a_k)^{-1} = P P_i^{-1} b_i (Q_i + a_i)^{-1}.$$

Substituting it into (9), (4) is immediate after routine calculation. ■

Given the prices of other commodities, demand function (4) shows that the consumer has a reservation price of good i that

$$P_{Ri} = (P_{-i} + I) a_i^{-1} b_i (1 - b_i)^{-1}. \tag{10}$$

When the market price of a commodity is higher than that, the demand for it shrinks to zero. In the determination of reservation price the taste parameter a_i is critical. The smaller is a_i, the higher is the reservation price. When a_i tends to zero, the reservation price tends to infinity, which is a characteristic of a Cobb-Douglas model. Recalling the definition of P_{-i} and P, the reservation can also be written as

$$P_{Ri} = (P + I)a_i^{-1}b_i. \tag{10'}$$

Noting $P \equiv \sum_i a_i P_i$, i.e., P is a combination of the price vector with a consumer's preferences, the equation implies that a consumer's reservation price is determined by their income, preferences and market prices. From (2) and (10′), we find that $P_{Ri}/(P + I) = MU_i/u$ when $Q_i = 0$.

The own price elasticity, cross price elasticity and income elasticity of demand (4) are, respectively, equal to

$$e_i = -(P_{-i} + I)b_i(P_iQ_i)^{-1} < -1, \qquad e_{ik} = a_k b_i P_k (Q_i P_i)^{-1}, \qquad e_{iI} = b_i I (Q_i P_i)^{-1}.$$

The absolute values of these elasticities tend to infinity when the consumption tends to zero.

The aggregate demand function can be easily obtained through integral when the distribution of consumer tastes (i.e., the distributions of a_i and b_i) and the distribution of income are known. Here we consider homogenous consumers only; that is, they have identical preferences and income. Moreover, the population is normalised to unity so that the demand function (4) is also the aggregate demand function.

For a monopolistic firm that is the only supplier of commodity i, it maximises profits

$$\pi = P_i Q_i - C(Q_i),$$

where $C(Q_i)$ is the cost of production. The conventional first-order condition of marginal revenue equal to marginal costs gives

$$P_i(1 + e_i^{-1}) = MC(Q_i). \tag{11}$$

Thus, the profit margin is equal to

$$[P_i - MC(Q_i)]P_i^{-1} = -e_i^{-1} = (P_{-i} + I)^{-1} b_i^{-1} P_i Q_i < 1. \tag{12}$$

For constant marginal cost; i.e., $MC(Q_i) \equiv c_i$, (11) provides a closed-form equilibrium.

Proposition 2. If $MC(Q_i) \equiv c_i$ and $P_{Ri} > c_i$, there is a unique monopoly equilibrium

$$Q_i = [(P_{-i} + I)(1 - b_i)a_i b_i c_i^{-1}]^{1/2} - (1 - b_i)a_i, \tag{13}$$

$$P_i = [(P_{-i} + I)(1 - b_i)^{-1} a_i^{-1} b_i c_i]^{1/2}. \tag{14}$$

Proof. Substituting $MR(Q_i) = (P_{-i} + I)(1 - b_i)a_i b_i [Q_i + (1 - b_i)a_i]^{-2}$ into (11), we immediately obtain (13). Noting $MR(0) = P_{Ri} > c_i$, $\lim_{Q_i \to \infty} MR(Q_i) = 0$ and the monotonicity of $MR(Q_i)$, we find Q_i is the only solution to (11). Substituting (13) into (4) and solving for P_i yield (14). ■

Equations (12)–(14) show that as the marginal cost declines, not only equilibrium profits rise but also equilibrium revenue and profit margin increase. The monopoly output and price determined by (13) and (14) include the effects of aggregate income and price of other commodities on them. In a model of Cobb-Douglas utility or quasilinear utility, it is impossible to accommodate these effects simultaneously. Proposition 2 can be considered more dynamically. Suppose an entrepreneur brings a brand new product into a market through certain breakthroughs in product innovation. However, as far as the marginal production cost is higher than the reservation price, the market will not accept the product. Only when the cost is lower than the reservation price, would the innovation be economically successful. This characteristic contradicts a demand structure with infinitely high reservation price, where any new product can succeed in marketplace no matter how high the production cost is.

III. A Further Extension

Of course, the model in the previous section is not necessarily limited to monopoly. Consider a situation that each commodity is supplied by an industry and each industry accommodates more than one firm. Assuming all firms produce a homogenous product, it is straightforward to extend the previous model to oligopoly or other imperfect competition forms. A more interesting case is that each firm produces one differentiated variety of a commodity. Each variety enters the utility function through a CES composition that

$$Q_i = \left(n_i^{\theta_i - 1} \sum_{j=1}^{n_i} q_{ij}^{\theta_i} \right)^{1/\theta_i}, \qquad 0 < \theta_i < 1, \tag{15}$$

where n_i is the number of varieties in commodity i, q_{ij} is the consumption quantity of variety j and $1/(\theta_i - 1)$ is the elasticity of substitution between any pair of varieties in the same industry. Thus, (1) and (15) consist of a utility function of a range of commodities and each commodity has a number of close substitutes. Given the aggregate expenditure, I, and the price of each product, p_{ij}, the budget constraint becomes

$$\sum_{i,j} p_{ij} q_{ij} = I. \tag{16}$$

Proposition 3. A representative consumer with utility (1) and (15) and budget constraint (16) has a demand function

$$q_{ij} = (P_i p_{ij}^{-1})^{1/(1-\theta_i)} n_i^{-1} Q_i, \tag{17}$$

where industry price index P_i is defined as

$$P_i \equiv \left(n_i^{-1} \sum_{j=1}^{n_i} p_{ij}^{\theta_i/(\theta_i - 1)} \right)^{(\theta_i - 1)/\theta_i} \tag{18}$$

and industry demand Q_i is given by (4).

Proof. Define Lagrange function

$$\mathscr{F}_1 = \prod_i \left[\left(n_i^{\theta_i - 1} \sum_{j=1}^{n_i} q_{ij}^{\theta_i} \right)^{1/\theta_i} + a_i \right]^{b_i} - \lambda_1 \left(\sum_{i,j} p_{ij} q_{ij} - I \right).$$

Its partial derivative with respect to q_{ij} provides the first-order condition

$$\prod_k (Q_k + a_k)^{b_k} b_i (Q_i + a_i)^{-1} \left(n_i^{\theta_i - 1} \sum_k q_{ik}^{\theta_i} \right)^{1/\theta_i - 1} n_i^{\theta_i - 1} q_{ij}^{\theta_i - 1} = \lambda_1 p_{ij}. \tag{19}$$

Recalling (15), multiplying both sides of the equation by q_{ij} and summing up across i and j yield

$$\prod_k (Q_k + a_k)^{b_k} \sum_i b_i Q_i (Q_i + a_i)^{-1} = \lambda_1 \sum_{i,j} p_{ij} q_{ij} = \lambda_1 I. \tag{20}$$

Taking a $\theta_i/(\theta_i - 1)$-root of both sides of (19) and summing up across j, then taking a $(\theta_i - 1)/\theta_i$-root and using (18) to obtain

$$\prod_k (Q_k + a_k)^{b_k} b_i (Q_i + a_i)^{-1} = \lambda_1 P_i. \tag{21}$$

It is clear that (20) and (21) are identical to (6) and (7) so that they determine an industry demand function (4). Substituting (21) into (19) to eliminate λ_1, we obtain (17). ■

An interesting feature of demand function (17) is that the demand for a variety is positive as far as the industry demand is positive. However, the industry demand has a reservation price of (10). By the definition, the industry price index is determined by the price of each individual variety. If variety prices are very high, so is the industry price index and in turn the demand for each individual variety becomes zero.

Demand function (17) provides a flexible structure for either partial or general equilibrium analysis. We start with partial equilibrium of a particular industry i, where the commodity prices of other industries and aggregate income are exogenous variables. Firms within the industry compete with each other to maximise their own profits

$$\pi_{ij} = p_{ij} q_{ij} - C_{ij}(q_{ij}) - f_{ij},$$

where $C_{ij}(q_{ij})$ is the variable costs of firm j in industry i and f_{ij} is the fixed costs. Although it is hard to figure out equilibrium in a closed-form with a general functional form of variable costs, a symmetric equilibrium with identical constant marginal cost across all firms within an industry can be plotted out in a closed-form.[5]

Proposition 4. When firms in an industry have a common constant marginal cost $c_i < P_{Ri}$, there exists a unique symmetric partial equilibrium

$$P_i = \tfrac{1}{2}[-\xi_{i2} - (\xi_{i2}^2 - 4\xi_{i1}\xi_{i3})^{1/2}]\xi_{i1}^{-1}, \qquad \text{when } \xi_{i1} \neq 0, \tag{22}$$

$$P_i = -\xi_{i2}^{-1}\xi_{i3}, \qquad \text{when } \xi_{i1} = 0, \tag{23}$$

where

$$\xi_{i1} \equiv (1 - b_i) a_i (\theta_i - n_i^{-1}),$$

$$\xi_{i2} \equiv [(P_{-i} + I) b_i \theta_i + (1 - b_i) a_i c_i](n_i^{-1} - 1),$$

$$\xi_{i3} \equiv (P_{-i} + I) b_i c_i (1 - \theta_i n_i^{-1}).$$

Proof. From (4) and (17), we have

$$\pi_{ij} = n_i^{-1}(p_{ij} - c_i) p_{ij}^{-1/(1-\theta_i)} [(P_{-i} + I) b_i P_i^{\theta_i/(1-\theta_i)} - (1 - b_i) a_i P_i^{1/(1-\theta_i)}] - f_i.$$

[5] Symmetric equilibrium means that firms in an industry charge the same price for their products. However, the prices across industries can be different.

Recalling (18), $\partial\pi_{ij}/\partial p_{ij} = 0 \rightarrow$

$$q_{ij} - n_i^{-1}(p_{ij} - c_i)(1-\theta_i)^{-1} p_{ij}^{-1/(1-\theta_i)-1}[(P_{-i} + I)b_i P_i^{\theta_i/(1-\theta_i)} - (1-b_i)a_i P_i^{1/(1-\theta_i)}]$$
$$+\ n_i^{-1}(p_{ij} - c_i)p_{ij}^{-1/(1-\theta_i)}[-(P_{-i} + I)b_i P_{ij}^{2\theta_i/(1-\theta_i)}\theta_i(\theta_i - 1)^{-1} n_i^{-1} p_{ij}^{\theta_i/(\theta_i-1)-1}$$
$$+\ (1-b_i)a_i P_i^{(1+\theta_i)/(1-\theta_i)}(\theta_i - 1)^{-1} n_i^{-1} p_{ij}^{\theta_i/(\theta_i-1)-1}] =\ 0.$$

In symmetric equilibrium, $p_{ij} = P_i$, $q_{ij} = q_i = n_i^{-1}Q_i = n_i^{-1}[(P_{-1} + I)b_i P_i^{-1} - (1-b_i)a_i]$. Substituting them into the last equation and multiplying both sides of the equation by $(1-\theta_i)n_i P_i^2$ yield

$$\xi_{i1}P_i^2 + \xi_{i2}P_i + \xi_{i3} = 0. \tag{24}$$

The solutions to the equation are

$$P_i = \tfrac{1}{2}[-\xi_{i2} + (\xi_{i2}^2 - 4\xi_{i1}\xi_{i3})^{1/2}]\xi_{i1}^{-1} \text{ and } P_i = \tfrac{1}{2}[-\xi_{i2} - (\xi_{i2}^2 - 4\xi_{i1}\xi_{i3})^{1/2}]\xi_{i1}^{-1}. \tag{25}$$

Note, $\xi_{i2} \leqslant 0$, $\xi_{i3} > 0$ but the sign of ξ_{i1} is ambiguous. If $\xi_{i1} > 0$, the profit increases when p_{ij} is small since $\lim_{p_{ij}\rightarrow 0} d\pi_{ij}/dp_{ij} \rightarrow +\infty$ so that the second solution in (25) determines the equilibrium price. If $\xi_{i1} < 0$, only the second solution is positive. If $\xi_{i1} = 0$, (24) leads to equilibrium price $P_i = -\xi_{i2}^{-1}\xi_{i3}$. ■

It is interesting to check two special cases for the equilibrium determined by (22). For the monopoly case, $n_i = 1$, there are

$$\xi_{i1} = (1-b_i)a_i(\theta_i - 1), \qquad \xi_{i2} = 0, \qquad \xi_{i3} = (P_{-i} + I)b_i c_i(1-\theta_i).$$

Thus, equation (22) collapse to (14). On the other hand, monopolistic competition is usually presumed to have many competitors in an industry so that $n_i \gg 1$. Neglecting terms of order n_i^{-1} in the definitions of ξ_{i1}, ξ_{i2} and ξ_{i3}, we have

$$\xi_{i1} = (1-b_i)a_i\theta_i, \qquad \xi_{i2} = -[(P_{-i} + I)b_i\theta_i + (1-b_i)a_i c_i], \qquad \xi_{i3} = (P_{-i} + I)b_i c_i.$$

Substituting them into (22) yields

$$P_i = c_i/\theta_i. \tag{26}$$

In Proposition 4, the number of firms in an industry is exogenously given. To endogenise it and industry structure, we require a zero profit condition determining the number of firms in equilibrium,

$$n_i = (1-\theta_i)P_i Q_i/f_i = (1-\theta_i)[(P_{-i} + I)b_i - (1-b_i)a_i P_i]/f_i. \tag{27}$$

In this condition, P_i is given by (22) or (23).

Turning to general equilibrium, the aggregate income can be normalised to unity; i.e., $I \equiv 1$ (see chapter 2 of Grossman and Helpman (1991)) but all prices are endogenous variables. Since the price index of industry i in (22) or (23) is a function of price indexes of other industries, we need to pool equation (22) or (23) of each industry together and simultaneously solve them to obtain general equilibrium. Because (22) and (23) are nonlinear, it is impossible to calculate the closed-form equilibrium for a general setting of the model. However, for many interesting specifications, the model is analytically tractable. For instance, the model can be applied to study the interaction between a giant monopolist like *Microsoft* and the whole economy. In this case, we can consider one industry is a monopoly and the others are monopolistically competitive. The general equilibrium then is given in the proposition below.

Proposition 5. If industry 1 is a monopoly and other industries are monopolistically competitive, there exists a unique quasi-symmetric general equilibrium[6]

$$P_1 = \left[\left(\sum_{i\neq 1} a_i P_i + 1\right)(1-b_1)^{-1} a_1^{-1} b_1 c_1\right]^{1/2}, \tag{28}$$

$$P_i = c_i/\theta_i, \qquad i \neq 1 \tag{29}$$

and equilibrium output for each industry can be obtained by substituting (28) and (29) into (4).

Proof. (29) is immediate from (26). (28) is derived from (14) by substituting $I = 1$ and $P_{-1} = \sum_{i\neq 1} a_i P_i$. ■

IV. Concluding Remarks

This paper proposes a tractable alternative to Cobb-Douglas utility. The demand function derived from this utility function can resolve the problems of missing reservation price or income effects in other commonly applied demand functions. It has a particular advantage in the study of the interaction between an industry and the whole economy. The model presented in Proposition 5 is only a simple example illustrating such an advantage in general equilibrium analysis. The great potential of the utility function remains to be explored in further research. For instance, the model of Proposition 5 can be easily extended to investigate the dynamics of economic development, where a new monopolist emerges through the invention of a drastically new product. When the technology has matured and/or the patent expires, more firms enter the industry and it becomes monopolistically competitive. Then another new technology breakthrough creates a newer industry... The economy develops through the creation of new industry and the evolution of industries. In comparison with existing growth models, such a model can accommodate more sophisticated microeconomic specification and provide more room for the analysis of economic evolution.

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[6] Quasi-symmetry is in the sense that firms within an industry set the same equilibrium price, but the prices index for each industry can differ.