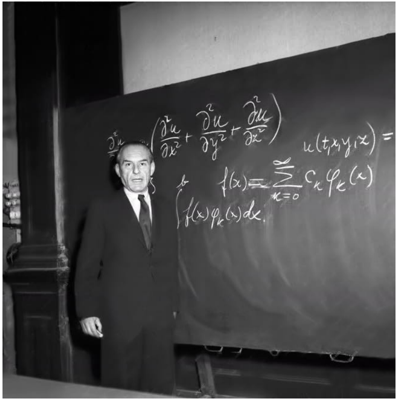
The Stone – Weierstraß Theorem



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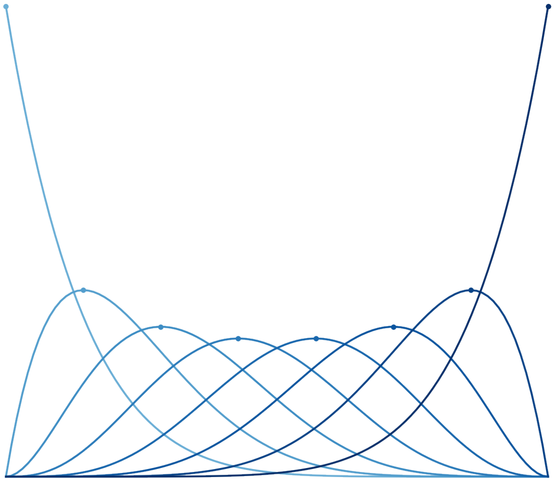


Objectives

37. Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen.

Von K. Weierstrass.

(Vorgetragen am 30. Juli; — gedruckt im Bericht vom gleichen Tage [St. XXXVIII]: — ausgegeben am 27. August.)

Zweite Mittheilung.

Es bedeute $f(x)$, wie in der am 9. Juli d. J. in der Akademie gelesenen Mittheilung, eine für jeden reellen Werth der Veränderlichen x eindeutig definirte, reelle und stetige Function, deren absoluter Betrag eine endliche obere Grenze (G) hat. Dagegen sei $\psi(x)$ eine transcendente ganze Function, von der zunächst nur angenommen wird, dass sie reell sei für reelle Werthe von x, und der Bedingung $\psi(-x) = \psi(x)$ genüge. Ferner seien u, v reelle, von einander unabhängige Veränderliche, und es werde

$$\sqrt{\psi(u+vi)\,\psi(u-vi)} = \psi(u, v)$$

gesetzt, wo der Quadratwurzel ihr positiver Werth beizulegen ist. Dann ist der absolute Betrag von $\frac{\psi(u+vi)}{\psi(u, v)}$ gleich 1, und man hat daher, wenn a, b reelle Grössen sind,

$$\int_a^b f(u)\,\psi(u+vi)\,du = \int_a^b f(u)\,\frac{\psi(u+vi)}{\psi(u, v)}\cdot\psi(u, v)\,du = \varepsilon G\int_a^b \psi(u, v)\,du,$$

wo ε eine complexe Grösse, deren absoluter Betrag kleiner als 1 ist, bezeichnet. Angenommen nun, es sei $\psi(x)$ so beschaffen, dass das Integral

$$\int_0^{+\infty} \psi(u, v)\,du$$

für jeden Werth von v einen endlichen Werth hat, so erhalten, wenn a_1, a_2, b_1, b_2 positive Grössen sind, $b_1 > a_1$, $b_2 > a_2$, die Integrale

$$\int_{a_2}^{b_2} \psi(u, v)\,du, \quad \int_{-b_1}^{-a_1} \psi(u, v)\,du,$$

- Weierstraß proved the approximation theorem at the age of 70. He used the Weierstraß transform [15, 13].

$$\boxed{\forall f \in C(\mathbb{R}, \mathbb{R}) : F(\theta) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} f(y) \exp\left(-\frac{(\theta - y)^2}{4}\right) \mathrm{d}y.}$$

- In 1912, Bernstein made a direct proof with the Bernstein polynomial [12].
- Proof the Fejér's theorem [5].
- Proof the Stone – Weierstraß theorem in the real, complex, quaternion and locally compact versions.
- Proof the Bishop's theorem [11].

Preliminaries

Definition (Topological space)

Let X be a set. A topology on X is a family $\mathcal{T} \subset 2^X$ that holds

- $\forall \{A_i\}_{i=1}^n \subset \mathcal{T} \implies \bigcap_{i=1}^n A_i \in \mathcal{T}$.
- $\forall \{A_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{T} \implies \bigcup_{\lambda \in \Lambda} A_\lambda \in \mathcal{T}$.

If $\mathcal{T}$ is a topology on X, then $(X, \mathcal{T})$ is a topological space. The sets in $\mathcal{T}$ are open sets.

Definition (Open cover)

Let $(X, \mathcal{T})$ be a topological space. $\mathcal{C} \subset \mathcal{T}$ is an open cover of X iff $X \subseteq \bigcup_{A \in \mathcal{C}} A$.

Definition (Hausdorff topological space or T_2)

A topological space $(X, \mathcal{T})$ is Hausdorff iff $\forall\, x, y \in X, x \neq y$: $\exists\, U, V \in \mathcal{T}$ such that $\forall\, x \in U, y \in V : U \cap V = \emptyset$.

Definition (Compact topological space)

A topological space $(X, \mathcal{T})$ is compact iff each open cover of X has a finite subcover.

Preliminaries

Definition (σ-algebra)

Let X be a set. A σ-algebra on X is a family $\mathcal{F} \subset 2^X$ that holds

- $\forall\, A \in \mathcal{F} \implies X \setminus A \in \mathcal{F}$.
- $\forall\, \{A_i\}_{i\in\mathbb{N}} \subset \mathcal{F} \implies \bigcup_{i\in\mathbb{N}} A_i \in \mathcal{F}$.

Definition (Measure)

Let X be a set and $\mathcal{F}$ a σ-algebra on X. A measure on $(X, \mathcal{F})$ is a function $\mu\colon \mathcal{F} \to [0, \infty]$ that holds

- $\mu(\emptyset) = 0$.
- $\forall\, \{A_i\}_{i\in\mathbb{N}} \subset \mathcal{F} : \forall\, i \neq j : A_i \cap A_j = \emptyset \implies \mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i)$.

Definition (Outer measure)

Let X be a set. An outer measure on X is a function $\mu^*\colon 2^X \to [0, \infty]$ that holds

- $\mu^*(\emptyset) = 0$.
- $\forall\, A, B \in 2^X : A \subset B \implies \mu^*(A) \leq \mu^*(B)$.
- $\forall\, \{A_i\}_{i\in\mathbb{N}} \subset 2^X \implies \mu^*\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu^*(A_i)$.

Preliminaries

Definition (σ-algebra generated)

Let X be a set and $\mathcal{G} \subset 2^X$. The σ-algebra generated by $\mathcal{G}$ is the smallest σ-algebra on X which contains $\mathcal{G}$.

$$\sigma(\mathcal{G}) := \bigcap_{\mathcal{A} \in \mathcal{F}(\mathcal{G})} \mathcal{A}, \quad \mathcal{F}(\mathcal{G}) = \left\{ \mathcal{A} \subset 2^X \mid \mathcal{G} \subset \mathcal{A}, \mathcal{A} \text{ is a } \sigma\text{-algebra on } X \right\}.$$

Definition (Borel σ-algebra)

Let $(X, \mathcal{T})$ be a topological space. The Borel σ-algebra on X is $\sigma(\mathcal{T})$. The sets in $\sigma(\mathcal{T})$ are Borel sets.

Definition (Lebesgue measure)

The Lebesgue measure is a measure on $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra of subsets of $\mathbb{R}$, which assigns each Borel set its outer measure.

Definition (Lebesgue space $\mathcal{L}^1(\mu)$)

Let $(X, \mathcal{F}, \mu)$ be a measure space. If $f\colon X \to [-\infty, \infty]$ is $\mathcal{F}$-measurable, then the $\mathcal{L}^1$-norm of f is

$$\|f\|_1 := \int |f| \, \mathrm{d}\mu.$$

The Lebesgue space $\mathcal{L}^1(\mu)$ is $\mathcal{L}^1(\mu) := \left\{ f\colon X \to \mathbb{R} \mid f \text{ is a function } \mathcal{F}\text{-measurable and } \|f\|_1 < \infty \right\}.$

Definition ($\|f\|_p$, essential supremum)

Let $(X, \mathcal{F}, \mu)$ be a measure space and $0 < p < \infty$. If $f\colon X \to \mathbb{C}$ is $\mathcal{F}$-measurable, then the p-norm of f is

$$\|f\|_p := \left(\int |f|^p \mathrm{d}\mu\right)^{\frac{1}{p}}.$$

Also, the essential supremum of f is $\|f\|_\infty = \inf\{t > 0 : \mu(\{x \in X : |f(x)| > t\}) = 0\}$.

Theorem

Let $(X, \mathcal{F}, \mu)$ be a measure space and $0 < p < \infty$. Then, $\mathcal{L}^p(\mu)$ is a vector space and it is holds:

- $\forall f, g \in \mathcal{L}^p(\mu) : \|f + g\|_p^p \leq 2^p \left(\|f\|_p^p + \|g\|_p^p\right)$.
- $\forall f \in \mathcal{L}^p(\mu) : \forall \alpha \in \mathbb{C} : \|\alpha f\|_p = |\alpha| \, \|f\|_p$.

Proof.

Let $f, g \in \mathcal{L}^p(\mu)$, $0 < p < \infty$, $x \in X$ and $\alpha \in \mathbb{C}$.

- Then, $|f(x) + g(x)|^p \leq (|f(x)| + |g(x)|)^p \leq (2 \max\{|f(x)|, |g(x)|\})^p \leq 2^p (|f(x)|^p + |g(x)|^p)$.
  Integrating both sides of the inequality with respect to μ: $\|f + g\|_p^p \leq 2^p \left(\|f\|_p^p + \|g\|_p^p\right)$.
  I.e. if $\|f\|_p < \infty$ and $\|g\|_p < \infty$, then $\|f + g\|_p < \infty$.
- $\|\alpha f\|_p = \left(\int |\alpha f|^p \mathrm{d}\mu\right)^{\frac{1}{p}} = \left(\int |\alpha|^p |f|^p \mathrm{d}\mu\right)^{\frac{1}{p}} = |\alpha|^{\frac{p}{p}} \left(\int |f|^p \mathrm{d}\mu\right)^{\frac{1}{p}} = |\alpha| \, \|f\|_p$.

Since $0 \in \mathcal{L}^p(\mu)$, $\mathcal{L}^p(\mu) < \mathbb{C}^X$ is closed under addition and scalar multiplication. $\therefore$ $\mathcal{L}^p(\mu)$ is a vector space. $\square$

Remark [9]

Let $(X, \mathcal{F}, \mu)$ be a measure space. The function

$$\begin{aligned} \mathcal{L}^2(\mu) &\to \mathbb{R} \\ f &\mapsto \left(\int_X |f|^2 \mathrm{d}\mu\right)^{\frac{1}{2}} \end{aligned}$$

is not a norm on $\mathcal{L}^2(\mu)$ because $\exists\, f \in \mathcal{L}^2(\mu)$ non-zero such that $\displaystyle\int_X |f|^2 \mathrm{d}\mu = 0 \in \mathbb{R}$.

Definition ($\mathcal{Z}(\mu)$, $\widetilde{f}$)

Let $(X, \mathcal{F}, \mu)$ be a measure space and $0 < p \leq \infty$. We define

- $\mathcal{Z}(\mu) := \{f \colon X \to \mathbb{C} \mid f \text{ is a function } \mathcal{F}\text{-measurable and } \mu(\{x \in X : f(x) \neq 0\}) = 0\}$.
- $\forall\, f \in \mathcal{L}^p(\mu) : \widetilde{f} = \{f + z : z \in \mathcal{Z}(\mu)\} < \mathcal{L}^p(\mu)$.

Note that if $f, F \in \mathcal{L}^p(\mu)$, then $\widetilde{f} = \widetilde{F}$ iff $\mu(\{x \in X : f(x) \neq F(x)\}) = 0$.

Definition ($L^p(\mu)$ space)

Let μ is a measure and $0 < p \leq \infty$. The set $L^p(\mu)$ are the equivalence classes of functions on $\mathcal{L}^p(\mu)$, where two functions are equivalent iff they are equal almost everywhere.

- $L^p(\mu) := \left\{\widetilde{f} : f \in \mathcal{L}^p(\mu)\right\} = \mathcal{L}^p(\mu) / \mathcal{Z}(\mu)$.
- $\forall\, \widetilde{f}, \widetilde{g} \in L^p(\mu) : \forall\, \alpha \in \mathbb{C} : \widetilde{f} + \widetilde{g} := (f+g)\widetilde{\ }, \quad \alpha\widetilde{f} := (\alpha f)\widetilde{\ }$.

Definition ($\|\cdot\|_p$ on $L^p(\mu)$)

Let μ be a measure and $0 < p \leq \infty$. We define $\forall f \in \mathcal{L}^p(\mu) : \left\|\widetilde{f}\right\|_p = \|f\|_p$.

Note that if $f, F \in \mathcal{L}^p(\mu)$ and $\widetilde{f} = \widetilde{F}$, then $\|f\|_p = \|F\|_p$.

Theorem

Let μ be a measure and $p \leq 1 \leq \infty$. Then, $L^p(\mu)$ is a vector space and $\|\cdot\|_p$ is a norm on $L^p(\mu)$.

A proof soon.

Let $\widetilde{f}, \widetilde{g} \in L^p(\mu)$ and $\alpha \in \mathbb{C}$.

$\square$

Definition (Convergent sequence)

Let $(X, \|\cdot\|)$ be a normed $\mathbb{C}$-vector space. A sequence $\{f_n\}_{n\in\mathbb{N}} \subset X$ is a convergent sequence iff $\exists\, f \in X$ such that $\forall\, \varepsilon > 0 : \exists\, N \in \mathbb{N}$ such that $\forall\, n \geq N : \|f - f_n\| < \varepsilon$.

Definition (Cauchy sequence)

Let $(X, \|\cdot\|)$ be a normed $\mathbb{C}$-vector space. A sequence $\{f_n\}_{n\in\mathbb{N}} \subset X$ is a Cauchy sequence iff $\forall\, \varepsilon > 0 : \exists\, N \in \mathbb{N}$ such that $\forall\, m, n \geq N : \|f_m - f_n\| < \varepsilon$. $(X, \|\cdot\|)$ is complete iff each Cauchy sequence in X is convergent in X.

Theorem

Let $(X, \|\cdot\|)$ be a normed $\mathbb{C}$-vector space and $\{f_n\}_{n\in\mathbb{N}} \subset X$ a sequence. If $\{f_n\}_{n\in\mathbb{N}}$ is convergent in X, then $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in X. Also, if $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in X and has a convergent subsequence in X, then $\{f_n\}_{n\in\mathbb{N}}$ converges in X.

Proof.

Let $\{f_n\}_{n\in\mathbb{N}} \subset X$ a convergent sequence. I.e. $\exists\, f \in X$ such that $\forall\, \varepsilon > 0 : \exists\, N \in \mathbb{N}$ such that $\forall\, n \geq N : \|f - f_n\| < \frac{\varepsilon}{2}$. Hence, $\forall\, \varepsilon > 0 : \exists\, N \in \mathbb{N}$ such that $\forall\, m, n \geq N : \|f_m - f + f - f_n\| \leq \|f_m - f\| + \|f - f_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Let $\{f_n\}_{n\in\mathbb{N}} \subset X$ a Cauchy sequence that has a convergent subsequence $\{f_{n_k}\}_{k\in\mathbb{N}} \subset X$. Since $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy, i.e. $\exists\, N \in \mathbb{N}$ such that $\forall\, m, n \geq N : \|f_m - f_n\| < \frac{\varepsilon}{2}$. Also $\{f_{n_k}\}_{k\in\mathbb{N}}$ is convergent, i.e. $\exists\, n_k > N$ such that $\left\|f - f_{n_k}\right\| < \frac{\varepsilon}{2}$. Therefore, $\forall\, \varepsilon > 0 : \exists\, N \in \mathbb{N}$ such that $\forall\, n \geq N : \left\|f - f_{n_k} + f_{n_k} - f_n\right\| \leq \left\|f - f_{n_k}\right\| + \left\|f_{n_k} - f_n\right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\square$

Theorem (Riesz - Fischer theorem)

Let $(X, \mathcal{F}, \mu)$ be a measure space and $1 \leq p \leq \infty$. Then, $L^p(\mu)$ is a Banach space.

A proof soon.

Let $\left\{\widetilde{f}_n\right\}_{n\in\mathbb{N}} \subset L^p(\mu)$ a Cauchy sequence, i.e. $\forall\, \varepsilon > 0 : \exists\, N \in \mathbb{N}$ such that $\forall\, m, n \geq N : \left\|\widetilde{f}_m - \widetilde{f}_n\right\|_p < \frac{\varepsilon}{2}$.

□

Preliminaries

Definition (Antilinear function)

Let $(V, \mathbb{C})$ be a complex vector space. A function $\ell\colon V \to \mathbb{C}$ is antilinear iff

- $\forall\, x, y \in V : \ell\,(x + y) = \ell\,(x) + \ell\,(y)$.
- $\forall\, x \in V : \forall\, \lambda \in \mathbb{C} : \ell\,(\lambda x) = \overline{\lambda}\ell\,(x)$.

Definition (Sesquilinear form)

Let $(V, \mathbb{C})$ be a complex vector space. A sesquilinear form on V is a function $V \times V \to \mathbb{C}$ such that $\forall\, x \in V : y \mapsto \langle x, y\rangle$ is linear and $y \mapsto \langle y, x\rangle$ is antilinear.

Definition (Pre-Hilbert space)

A pre-Hilbert space is a complex vector space $(V, \mathbb{C})$, with a sesquilinear form that holds

- $\forall\, x \in V, x \neq 0 : \langle x, x\rangle \in \mathbb{R}$ y $\langle x, x\rangle > 0$.
- $\forall\, x, y \in V : \langle x, y\rangle = \overline{\langle y, x\rangle}$.

Definition (Orthonormal set)

Let $(V, \langle\cdot,\cdot\rangle)$ be a pre-Hilbert space. $\{x_\lambda\}_{\lambda\in\Lambda} \subset V$ is orthonormal iff $\left\langle x_\alpha, x_\beta\right\rangle = \delta_{\alpha,\beta}$, i.e. $\forall\, \alpha, \beta \in \Lambda : \alpha \neq \beta \implies \left\langle x_\alpha, x_\beta\right\rangle = 0$ and $\forall\, \alpha \in \Lambda : \|x_\alpha\| = 1$.

Theorem (Parallelogram law)

Let $(V, \|\cdot\|)$ be a normed $\mathbb{C}$-vector space. Then, $\forall\, x, y \in V : \|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$.

Theorem (Jordan-von Neumann theorem [8])

Let $(V, \|\cdot\|)$ be a normed $\mathbb{C}$-vector space. $\|\cdot\|$ is induced by an inner product iff $\|\cdot\|$ holds the parallelogram law.

Theorem

Let $1 \leq p < \infty$. The L^p-norm only holds the parallelogram law for $p = 2$.

Proof.

Let $(X, \mathcal{F}, \mu)$ be a measure space. Then, $\forall\, E \in \mathcal{F}$:

$$\|\chi_E\|_p = \left(\int_X |\chi_E|^p \mathrm{d}\mu\right)^{\frac{1}{p}} = \left(\int_E {\chi_E}^p \mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int_{E^C} {\chi_E}^p \mathrm{d}\mu\right)^{\frac{1}{p}} = \left(\int_E 1\, \mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int_{E^C} 0\, \mathrm{d}\mu\right)^{\frac{1}{p}} = \mu(E)^{\frac{1}{p}} + 0 = \mu(E)^{\frac{1}{p}}.$$

If $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$, $0 < \mu(A) < \infty$ y $0 < \mu(B) < \infty$, then $\chi_A + \chi_B = |\chi_A - \chi_B| = \chi_{A \uplus B}$.

$$\|\chi_A + \chi_B\|_p^2 + \|\chi_A - \chi_B\|_p^2 = \left(\int_X |\chi_A + \chi_B|^p \mathrm{d}\mu\right)^{\frac{2}{p}} + \left(\int_X |\chi_A - \chi_B|^p \mathrm{d}\mu\right)^{\frac{2}{p}} = 2\left(\int_X |\chi_{A\uplus B}|^p \mathrm{d}\mu\right)^{\frac{2}{p}} = 2(\mu(A \uplus B))^{\frac{2}{p}}.$$

$$2\left(\|\chi_A\|_p^2 + \|\chi_B\|_p^2\right) = 2\left((\mu(A))^{\frac{2}{p}} + (\mu(B))^{\frac{2}{p}}\right).$$

Hence, L^p-norm only holds the parallelogram law for $p = 2$.

Definition (Hilbert space)

A Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is a pre-Hilbert space that is complete with respect to the norm $\|\cdot\| = \langle \cdot, \cdot \rangle^{\frac{1}{2}}$.

Definition (Orthonormal basis)

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. An orthonormal basis of H is a countable maximal orthonormal subset $\{e_n\}_{n \in \mathbb{N}}$ of H.

Theorem

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\{e_n\}_{n \in \mathbb{N}}$ an orthonormal basis on H. Then, we have convergence of the Fourier-Bessel series:

$$\forall\, u \in H : \lim_{n \to \infty} \sum_{k=1}^{n} \langle u, e_k \rangle\, e_k = \sum_{n=1}^{\infty} \langle u, e_n \rangle\, e_n = u.$$

Theorem

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. If H has an orthonormal basis, then H is separable.

Theorem

Let $1 \leq p < \infty$. The $L^p(\mu)$ is a Hilbert space iff $p = 2$.

A proof soon.

Preliminaries

Definition (Fourier series of f relative)

Let $f \in L^2(I)$ and $\{\varphi_k\}_{k\in\mathbb{N}}$ an orthonormal sequence on I. The Fourier series of f relative of $\{\varphi_k\}_{k\in\mathbb{N}}$ is $\sum_{k\in\mathbb{N}} c_k\varphi_k(\theta)$, where $\forall\, k \in \mathbb{N} : c_k := \langle f, \varphi_k\rangle = \int_I f(\theta)\,\overline{\varphi_k(\theta)}$ are the Fourier coefficients of f relative of $\{\varphi_k\}_{k\in\mathbb{N}}$.

Example

If $I = [0, 2\pi]$ and two orthonormal sequences of trigonometric functions $\{\varphi_k\}_{k\in\mathbb{N}}, \{\phi_k\}_{k\in\mathbb{Z}}$:

real $\varphi_0(\theta) = \dfrac{1}{\sqrt{2\pi}}, \quad \varphi_{2k-1}(\theta) = \dfrac{\cos(k\theta)}{\sqrt{\pi}}, \quad \varphi_{2k}(\theta) = \dfrac{\sin(k\theta)}{\sqrt{\pi}}.$

complex $\phi_k(\theta) = \dfrac{e^{ik\theta}}{\sqrt{2\pi}} = \dfrac{\cos(k\theta) + i\sin(k\theta)}{\sqrt{2\pi}}.$

Then, the Fourier series of f relative of $\{\varphi_k\}_{k\in\mathbb{N}}$ and $\{\phi_k\}_{k\in\mathbb{N}}$ are

real $\dfrac{a_0}{2} + \sum_{k\in\mathbb{N}} a_k\cos(k\theta) + b_k\sin(k\theta).$

complex $\sum_{k\in\mathbb{Z}} \alpha_k e^{ik\theta}, \quad \alpha_k = \dfrac{1}{2\pi}\int_0^{2\pi} f(\theta)\, e^{-ik\theta}\mathrm{d}\theta.$

Remark [7]

The subset of functions $\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos(m\theta)}{\sqrt{\pi}}, \frac{\sin(n\theta)}{\sqrt{\pi}}\right\}_{m,n\in\mathbb{N}} \subset L^2([0,2\pi])$ is an orthonormal subset of $L^2([0,2\pi])$.

Indeed, $\forall\, n, m \in \mathbb{N}$:

- $$\int_0^{2\pi}\left(\frac{1}{\sqrt{2\pi}}\right)^2 \mathrm{d}\theta = \int_0^{2\pi}\frac{1}{2\pi}\mathrm{d}\theta = \frac{1}{2\pi}\theta\Big|_0^{2\pi} = 1.$$
- $$\int_0^{2\pi}\left(\frac{\cos(m\theta)}{\sqrt{\pi}}\right)^2 \mathrm{d}\theta = \int_0^{2\pi}\frac{\cos^2(m\theta)}{\pi}\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} 1 + \cos(2m\theta)\,\mathrm{d}\theta = \frac{1}{2\pi}\left(\theta + \frac{\sin(4m\theta)}{4m}\right)\Bigg|_0^{2\pi} = 1.$$
- $$\int_0^{2\pi}\left(\frac{\sin(n\theta)}{\sqrt{\pi}}\right)^2 \mathrm{d}\theta = \int_0^{2\pi}\frac{\sin^2(n\theta)}{\pi}\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} 1 - \cos(2m\theta)\,\mathrm{d}\theta = \frac{1}{2\pi}\left(\theta - \frac{\sin(4m\theta)}{4m}\right)\Bigg|_0^{2\pi} = 1.$$

- $$\int_0^{2\pi}\frac{1}{\sqrt{2\pi}}\frac{\cos(m\theta)}{\sqrt{\pi}}\mathrm{d}\theta = \frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\cos(m\theta)\,\mathrm{d}\theta = 0.$$
- $$\int_0^{2\pi}\frac{1}{\sqrt{2\pi}}\frac{\sin(n\theta)}{\sqrt{\pi}}\mathrm{d}\theta = \frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\sin(n\theta)\,\mathrm{d}\theta = 0.$$

- $$\int_0^{2\pi}\frac{\cos(m\theta)}{\sqrt{\pi}}\frac{\sin(n\theta)}{\sqrt{\pi}}\mathrm{d}\theta = \frac{1}{\pi}\int_0^{2\pi}\sin(n\theta)\cos(m\theta)\,\mathrm{d}\theta = \frac{1}{\pi}\int_0^{2\pi}\frac{\sin((n+m)\theta) - \sin((n-m)\theta)}{2}\mathrm{d}\theta = 0.$$

Preliminaries

Definition (Fourier series generated by f)

Let $f \in L^2([0, 2\pi])$. The Fourier coefficients of f are given by

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta, \quad a_k = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(k\theta)\,\mathrm{d}\theta, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(k\theta)\,\mathrm{d}\theta.$$

and the n-th partial Fourier sum is

$$\boxed{s_n f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(k\theta) + b_k \sin(k\theta).}$$

Indeed, from the equalities $\forall\, k \in \mathbb{N}$:

- $\displaystyle\int_0^{2\pi} \frac{a_0}{2}\,\mathrm{d}\theta = \frac{a_0}{2}\theta\Big|_0^{2\pi} = \pi a_0.$
- $\displaystyle\int_0^{2\pi} \cos(k\theta)\,\mathrm{d}\theta = \frac{\sin(k\theta)}{k}\Big|_0^{2\pi} = 0.$
- $\displaystyle\int_0^{2\pi} \sin(k\theta)\,\mathrm{d}\theta = \frac{-\cos(k\theta)}{k}\Big|_0^{2\pi} = 0.$

If we integrate the Fourier series term by term

$$\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta = \int_0^{2\pi} \frac{a_0}{2}\mathrm{d}\theta + \int_0^{2\pi} \left(\sum_{k=1}^{\infty} a_k \cos(k\theta) + b_k \sin(k\theta)\right) \mathrm{d}\theta.$$

Then,

$$\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta = \frac{a_0}{2}\int_0^{2\pi} \mathrm{d}\theta + \sum_{k=1}^{\infty} \left(a_k \int_0^{2\pi} \cos(k\theta)\,\mathrm{d}\theta + b_k \int_0^{2\pi} \sin(k\theta)\,\mathrm{d}\theta\right).$$

$$\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta = \pi a_0 + \sum_{k=1}^{\infty} (a_k \cdot 0 + b_k \cdot 0). \qquad \Longrightarrow \boxed{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta.}$$

Multiplying the Fourier series by $\cos(m\theta)$, $m \in \mathbb{N}$ and integrating term by term:

$$\int_0^{2\pi} \cos(m\theta) f(\theta)\,\mathrm{d}\theta = \int_0^{2\pi} \cos(m\theta)\frac{a_0}{2}\mathrm{d}\theta + \int_0^{2\pi} \cos(m\theta)\left(\sum_{k=1}^{\infty} a_k \cos(k\theta) + b_k \sin(k\theta)\right)\mathrm{d}\theta.$$

$$\int_0^{2\pi} f(\theta)\cos(m\theta)\,\mathrm{d}\theta = 0 + \sum_{k=1}^{\infty}\left(a_k \int_0^{2\pi} \cos(k\theta)\cos(m\theta)\,\mathrm{d}\theta + b_k \int_0^{2\pi} \sin(k\theta)\cos(m\theta)\,\mathrm{d}\theta\right).$$

$$\int_0^{2\pi} f(\theta)\cos(m\theta)\,\mathrm{d}\theta = \sum_{k=1}^{\infty}\left(\frac{a_k}{2}\int_0^{2\pi} \cos((m+k)\theta) + \cos((m-k)\theta)\,\mathrm{d}\theta + \frac{b_k}{2}\int_0^{2\pi} \sin((m+k)\theta) + \sin((m-k)\theta)\,\mathrm{d}\theta\right).$$

When $m \neq k$ both integrals vanish, thus the infinite sum reduces to m-th addend.

$$\int_0^{2\pi} f(\theta)\cos(m\theta)\,\mathrm{d}\theta = a_m \int_0^{2\pi} \cos^2(m\theta)\,\mathrm{d}\theta + b_m \int_0^{2\pi} \sin(m\theta)\cos(m\theta)\,\mathrm{d}\theta.$$

$$\int_0^{2\pi} f(\theta)\cos(m\theta)\,\mathrm{d}\theta = \frac{a_m}{2}\int_0^{2\pi} 1 + \cos(2m\theta)\,\mathrm{d}\theta + b_m \cdot 0.$$

$$\int_0^{2\pi} f(\theta)\cos(m\theta)\,\mathrm{d}\theta = a_m \pi. \qquad \Longrightarrow \qquad \boxed{a_m = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(m\theta)\,\mathrm{d}\theta.}$$

Multiplying the Fourier series by $\sin(m\theta)$, $m \in \mathbb{N}$ and integrating term by term:

$$\int_0^{2\pi} \sin(m\theta) f(\theta)\,\mathrm{d}\theta = \int_0^{2\pi} \sin(m\theta)\frac{a_0}{2}\mathrm{d}\theta + \int_0^{2\pi} \sin(m\theta)\left(\sum_{k=1}^{\infty} a_k \cos(k\theta) + b_k \sin(k\theta)\right)\mathrm{d}\theta.$$

$$\int_0^{2\pi} f(\theta)\sin(m\theta)\,\mathrm{d}\theta = 0 + \sum_{k=1}^{\infty}\left(a_k \int_0^{2\pi} \cos(k\theta)\sin(m\theta)\,\mathrm{d}\theta + b_k \int_0^{2\pi} \sin(k\theta)\sin(m\theta)\,\mathrm{d}\theta\right).$$

$$\int_0^{2\pi} f(\theta)\sin(m\theta)\,\mathrm{d}\theta = \sum_{k=1}^{\infty}\left(\frac{a_k}{2}\int_0^{2\pi} \sin((m+k)\theta) + \sin((m-k)\theta)\,\mathrm{d}\theta + \frac{b_k}{2}\int_0^{2\pi} \cos((m-k)\theta) - \cos((m+k)\theta)\,\mathrm{d}\theta\right).$$

When $m \neq k$ both integrals vanish, thus the infinite sum reduces to m-th addend.

$$\int_0^{2\pi} f(\theta)\sin(m\theta)\,\mathrm{d}\theta = a_m \int_0^{2\pi} \cos(m\theta)\sin(m\theta)\,\mathrm{d}\theta + b_m \int_0^{2\pi} \sin^2(m\theta)\,\mathrm{d}\theta.$$

$$\int_0^{2\pi} f(\theta)\sin(m\theta)\,\mathrm{d}\theta = a_m \cdot 0 + \frac{b_m}{2}\int_0^{2\pi} 1 - \cos(2m\theta)\,\mathrm{d}\theta.$$

$$\int_0^{2\pi} f(\theta)\sin(m\theta)\,\mathrm{d}\theta = b_m \pi. \quad \Longrightarrow \quad \boxed{b_m = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(m\theta)\,\mathrm{d}\theta.}$$

Preliminaries

Theorem

If $\theta \in \mathbb{R}$, then

$$\operatorname{Re}\left(\sum_{k=1}^{n} e^{ik\theta}\right) = \sum_{k=1}^{n} \operatorname{Re}\left(e^{ik\theta}\right) = \sum_{k=1}^{n} \cos(k\theta) = \begin{cases} \dfrac{\sin\left((2n+1)\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} - \dfrac{1}{2}, & \exists\, m \in \mathbb{Z} \textit{ such that } \theta \neq 2m\pi. \\ n, & \textit{otherwise.} \end{cases}$$

Proof.

From the geometric sum of $e^{i\theta}$ and $2i\sin(\theta) = e^{i\theta} - e^{-i\theta}$:

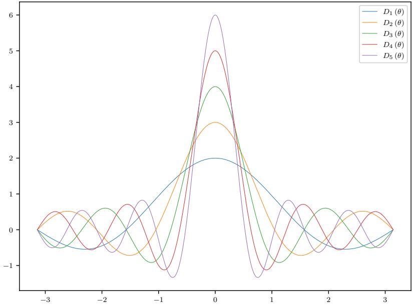
$$\sum_{k=1}^{n}\left(e^{i\theta}\right)^{k} = \frac{\left(e^{i\theta}\right)^{(n+1)} - e^{i\theta}}{e^{i\theta} - 1} = e^{i\theta}\,\frac{e^{in\theta} - 1}{e^{i\theta} - 1} = e^{i\theta}\,\frac{e^{in\frac{\theta}{2}}\left(e^{in\frac{\theta}{2}} - e^{-in\frac{\theta}{2}}\right)}{e^{i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}\right)} = e^{i(n+1)\frac{\theta}{2}}\,\frac{2i\sin\left(n\frac{\theta}{2}\right)}{2i\sin\left(\frac{\theta}{2}\right)} = e^{i(n+1)\frac{\theta}{2}}\,\frac{\sin\left(n\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}.$$

Taking the real part on the opposite sides of the equality and $\cos(\theta_1)\sin(\theta_2) = \frac{1}{2}\left(\sin(\theta_1+\theta_2) - \sin(\theta_1-\theta_2)\right)$:

$$\cos\left((n+1)\frac{\theta}{2}\right)\frac{\sin\left(n\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} = \frac{\sin\left((n+1)\frac{\theta}{2} + n\frac{\theta}{2}\right) - \sin\left((n+1)\frac{\theta}{2} - n\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} = \frac{\sin\left((2n+1)\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)}.$$

$\square$

Preliminaries



Definition (Dirichlet kernel)

The Dirichlet kernel D_n of n-order is

$$\boxed{D_n(\theta) := \frac{1}{2} + \sum_{k=1}^{n} \cos(k\theta)}$$

2π-periodic and even, i.e. $\forall\, \theta \in \mathbb{R} : D(-\theta) = D(\theta)$.

Definition (Periodic function [3])

A function $f : \mathbb{R} \to \mathbb{C}$ is T-periodic iff $\exists\, T \in \mathbb{R} \setminus \{0\}$ such that $\forall\, x \in \mathbb{R} : f(x+T) = f(x)$.

Remark

Let $x \in \mathbb{R}$. If $f \in L([0, T])$ is T-periodic, then with the change of variable $y \leftarrow \theta + x - \frac{T}{2}$:

$$\int_0^T f(\theta)\,\mathrm{d}\theta = \int_0^{\frac{T}{2}} f(\theta)\,\mathrm{d}\theta + \int_{\frac{T}{2}}^{T} f(\theta)\,\mathrm{d}\theta = \int_{x-\frac{T}{2}}^{x} f(y)\,\mathrm{d}y + \int_{x}^{x+\frac{T}{2}} f(y)\,\mathrm{d}y = \int_{x-\frac{T}{2}}^{x+\frac{T}{2}} f(y)\,\mathrm{d}y.$$

Lemma

If $f \in L([0,2\pi])$ is 2π-periodic, then the sequence of partial sum $\{s_n f(\theta)\}_{n\in\mathbb{N}}$ of trigonometric Fourier series generated by f has the integral representation

$$\boxed{s_n f(\theta) = \frac{2}{\pi}\int_0^{\pi} \frac{f(\theta+\xi)+f(\theta-\xi)}{2} D_n(\xi)\,\mathrm{d}\xi.}$$

Proof.

$$s_n f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos(k\theta) + b_k \sin(k\theta)\right).$$

$$s_n f(\theta) = \frac{1}{2}\cdot\frac{1}{\pi}\int_0^{2\pi} f(\xi)\,\mathrm{d}\xi + \sum_{k=1}^{n}\left(\frac{1}{\pi}\int_0^{2\pi} f(\xi)\cos(k\xi)\,\mathrm{d}\xi \cos(k\theta) + \frac{1}{\pi}\int_0^{2\pi} f(\xi)\sin(k\xi)\,\mathrm{d}\xi \sin(k\theta)\right).$$

$$s_n f(\theta) = \frac{1}{\pi}\int_0^{2\pi} f(\xi)\left(\frac{1}{2} + \sum_{k=1}^{n} \cos(k\xi)\cos(k\theta) + \sin(k\xi)\sin(k\theta)\right)\mathrm{d}\xi = \frac{1}{\pi}\int_0^{2\pi} f(\xi)\left(\frac{1}{2} + \sum_{k=1}^{n}\cos(k\xi - k\theta)\right)\mathrm{d}\xi.$$

Proof.

$$s_n f(\theta) = \frac{1}{\pi} \int_{0}^{2\pi} f(\xi)\, D_n(k(\xi - \theta))\, \mathrm{d}\xi.$$

The period of the product of two periodic functions f and D_n is the least common multiple of its periods, i.e. $\operatorname{lcm}(2\pi, 2\pi) = 2\pi$ and plugging the u-substitution $u = \xi - \theta$.

$$s_n f(\theta) = \frac{1}{\pi} \int_{\theta-\pi}^{\theta+\pi} f(\xi)\, D_n(k(\xi - \theta))\, \mathrm{d}\xi = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta + u)\, D_n(u)\, \mathrm{d}u.$$

$$s_n f(\theta) = \frac{1}{\pi} \left(\int_{-\pi}^{0} f(\theta + u)\, D_n(u)\, \mathrm{d}u + \int_{0}^{\pi} f(\theta + u)\, D_n(u)\, \mathrm{d}u \right).$$

$$s_n f(\theta) = \frac{1}{\pi} \left(\int_{-\pi}^{0} f(\theta + u)\, (D_n(-u))\, \mathrm{d}u + \int_{0}^{\pi} f(\theta + u)\, D_n(u)\, \mathrm{d}u \right).$$

$$s_n f(\theta) = \frac{1}{\pi} \left(\int_{0}^{\pi} f(\theta - u)\, D_n(u)\, \mathrm{d}u + \int_{0}^{\pi} f(\theta + u)\, D_n(u)\, \mathrm{d}u \right) = \frac{2}{\pi} \int_{0}^{\pi} \frac{f(\theta + u) + f(\theta - u)}{2} D_n(u)\, \mathrm{d}u.$$

□

Preliminaries

Theorem

If $\theta \in \mathbb{R}$, then

$$\operatorname{Im}\left(\sum_{k=1}^{n} e^{i(2k-1)\theta}\right) = \sum_{k=1}^{n} \operatorname{Im}\left(e^{i(2k-1)\theta}\right) = \sum_{k=1}^{n} \sin\left((2k-1)\,\theta\right) = \begin{cases} \dfrac{\sin^2(n\theta)}{\sin(\theta)}, & \exists\, m \in \mathbb{Z} \text{ such that } \theta \neq 2m\pi. \\ 0, & \text{otherwise.} \end{cases}$$

Proof.

Since $\sum_{k=1}^{n} \left(e^{i\theta}\right)^k = e^{i(n+1)\frac{\theta}{2}} \frac{\sin\left(n\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$:

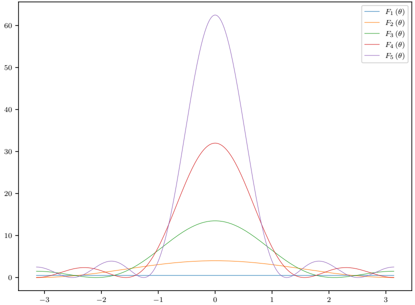
$$\sum_{k=1}^{n} \left(e^{i\theta}\right)^{2k-1} = e^{-i\theta} \sum_{k=1}^{n} \left(e^{i2\theta}\right)^k = e^{-i\theta} e^{i(n+1)\theta} \frac{\sin(n\theta)}{\sin(\theta)} = e^{in\theta} \frac{\sin(n\theta)}{\sin(\theta)}.$$

Taking the imaginary part on the opposite sides of the equality:

$$\operatorname{Im}\left(\sum_{k=1}^{n} e^{i(2k-1)\theta}\right) = \sin(n\theta) \frac{\sin(n\theta)}{\sin(\theta)} = \frac{\sin^2(n\theta)}{\sin(\theta)}.$$

$\square$

Preliminaries



Definition (Fejér kernel)

The Fejér kernel K_n of n-order is

$$\boxed{K_n(\theta) := \frac{1}{n}\sum_{k=1}^{n} D_{k-1}(\theta)}$$

i.e. is the n-th Cesàro-Fourier means of the Dirichlet kernel.

Definition (n-th Cesàro-Fourier means)

Let $f \in L^2([0, 2\pi])$. The n-th Cesàro-Fourier means of f is

$$\boxed{\sigma_n f(\theta) = \frac{1}{n}\sum_{k=1}^{n} s_{k-1} f(\theta).}$$

Preliminaries

Lemma

If $f \in L([0,2\pi])$ is 2π-periodic and $\{s_n f(\theta)\}_{n\in\mathbb{N}}$ is the sequence of partial sum of the trigonometric Fourier series generated by f. Then, the sequence $\sigma_n f(\theta)$ has the integral representation

$$\boxed{\sigma_n f(\theta) = \frac{2}{\pi}\int_0^{\pi} \frac{f(\theta+\xi)+f(\theta-\xi)}{2} K_n(\xi)\,\mathrm{d}\xi.}$$

Proof.

If $s_n f(\theta) = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \frac{f(\theta+\xi)+f(\theta-\xi)}{2} D_n(\xi)\,\mathrm{d}\xi$, then

$$\sigma_n f(\theta) = \frac{1}{n}\sum_{k=1}^{n} s_{k-1} f(\theta) = \frac{1}{n}\sum_{k=1}^{n} \frac{2}{\pi}\int_0^{\pi} \frac{f(\theta+\xi)+f(\theta-\xi)}{2} D_{k-1}(\xi)\,\mathrm{d}\xi.$$

$$\sigma_n f(\theta) = \frac{2}{\pi}\int_0^{\pi} \frac{f(\theta+\xi)+f(\theta-\xi)}{2} \frac{1}{n}\sum_{k=1}^{n} D_{k-1}(\xi)\,\mathrm{d}\xi = \frac{2}{\pi}\int_0^{\pi} \frac{f(\theta+\xi)+f(\theta-\xi)}{2} K_n(\xi)\,\mathrm{d}\xi.$$

□

Theorem

Let $\theta \in \mathbb{R}$*.* $\forall\, n \in \mathbb{N}$*:*

- $\int\limits_0^\pi K_n(\theta) = \frac{\pi}{2}$.
- $K_n(\theta) = \frac{1}{2n} \frac{\sin^2\left(n\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)} \geq 0$.
- $\forall\, \delta \in (0, \pi) : \forall\, \delta \leq |\theta| \leq \pi : K_n(\theta) \leq \frac{1}{2n\sin^2\left(\frac{\delta}{2}\right)}$.

Proof.

-
$$K_n(\theta) = \frac{1}{n}\sum_{k=1}^{n} D_{k-1}(\theta) = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2} + \sum_{m=1}^{k-1}\cos(m\theta)\right) = \frac{1}{n}\left(\sum_{k=1}^{n}\frac{1}{2} + \sum_{k=1}^{n}\sum_{m=1}^{k-1}\cos(m\theta)\right).$$
$$\int\limits_0^\pi K_n(\theta) = \int\limits_0^\pi \frac{1}{n}\left(\sum_{k=1}^{n}\frac{1}{2} + \sum_{k=1}^{n}\sum_{m=1}^{k-1}\cos(m\theta)\right)\mathrm{d}\theta = \frac{1}{n}\left(\frac{n}{2}\int\limits_0^\pi \mathrm{d}\theta + \sum_{k=1}^{n}\sum_{m=1}^{k-1}\int\limits_0^\pi \cos(m\theta)\,\mathrm{d}\theta\right) = \frac{1}{n}\left(\frac{n\pi}{2} + 0\right) = \frac{\pi}{2}.$$

Proof.

- $$K_n(\theta) = \frac{1}{n}\left(\sum_{k=1}^{n}\frac{1}{2} + \sum_{k=1}^{n}\sum_{m=1}^{k-1}\cos(m\theta)\right) = \frac{1}{n}\left(\frac{n}{2} + \sum_{k=1}^{n}\left(\frac{\sin\left(\left(2(k-1)+1\right)\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} - \frac{1}{2}\right)\right)$$

  $$K_n(\theta) = \frac{1}{n}\left(\frac{n}{2} + \frac{\sum_{k=1}^{n}\sin\left(\left(2(k-1)+1\right)\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} - \sum_{k=1}^{n}\frac{1}{2}\right) = \frac{1}{n}\left(\frac{n}{2} + \frac{\sum_{k=1}^{n}\sin\left(\left(2k-1\right)\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} - \frac{n}{2}\right)$$

  $$K_n(\theta) = \frac{1}{2n}\frac{\frac{\sin^2\left(n\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}}{\sin\left(\frac{\theta}{2}\right)} = \frac{1}{2n}\frac{\sin^2\left(n\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)} \geq 0.$$

- $\forall\,\delta \in (0,\pi) : \sin^2\left(\frac{\delta}{2}\right)$ is even and increasing. Then, $\forall\,\delta < |\theta| < \pi$:

  $$\frac{\delta}{2} < \frac{|\theta|}{2} \implies \sin^2\left(\frac{\delta}{2}\right) < \sin^2\left(\frac{|\theta|}{2}\right) \implies \sin^2\left(\frac{\delta}{2}\right) < \sin^2\left(\frac{\theta}{2}\right) \implies \frac{1}{\sin^2\left(\frac{\theta}{2}\right)} < \frac{1}{\sin^2\left(\frac{\delta}{2}\right)}.$$

  $$K_n(\theta) = \frac{1}{2n}\frac{\sin^2\left(n\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)} = \frac{\sin^2\left(n\frac{\theta}{2}\right)}{2n\sin^2\left(\frac{\theta}{2}\right)} \leq \frac{1}{2n\sin^2\left(\frac{\delta}{2}\right)}.$$

□

Remark

Applying the last lemma for $f \equiv 1 \in L([0, 2\pi])$ which is 2π-periodic, then $\forall\, n \in \mathbb{N}$:

$$s_n f(\theta) = \frac{2}{\pi}\int_0^\pi \frac{f(\theta+\xi)+f(\theta-\xi)}{2} D_n(\xi)\,\mathrm{d}\xi = \frac{2}{\pi}\int_0^\pi \frac{1+1}{2}\left(\frac{1}{2}+\sum_{k=1}^{n}\cos(k\xi)\right)\mathrm{d}\xi = \frac{2}{\pi}\int_0^\pi \left(\frac{1}{2}+\sum_{k=1}^{n}\cos(k\xi)\right)\mathrm{d}\xi$$

$$s_n f(\theta) = \frac{2}{\pi}\int_0^\pi \frac{\mathrm{d}\xi}{2} + \frac{2}{\pi}\int_0^\pi \sum_{k=1}^{n}\cos(k\xi)\,\mathrm{d}\xi = 1 + \frac{2}{\pi}\sum_{k=1}^{n}\int_0^\pi \cos(k\xi)\,\mathrm{d}\xi = 1 + \frac{2}{\pi}\sum_{k=1}^{n}\left(\frac{\sin(k\xi)}{k}\right)\Bigg|_0^\pi = 1 + \frac{2}{\pi}\sum_{k=1}^{n} 0 = 1.$$

$$\sigma_n f(\theta) = \frac{2}{\pi}\int_0^\pi \frac{f(\theta+\xi)+f(\theta-\xi)}{2} K_n(\xi)\,\mathrm{d}\xi = \frac{2}{\pi}\int_0^\pi \frac{1+1}{2} K_n(\xi)\,\mathrm{d}\xi = \frac{2}{\pi}\int_0^\pi K_n(\xi)\,\mathrm{d}\xi = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1.$$

We will see if $sf(\theta) := \lim\limits_{\xi\to 0^+} \dfrac{f(\theta+\xi)+f(\theta-\xi)}{2} \in \mathbb{R}$, then $\{\sigma_n f(\theta) - sf(\theta)\}_{n\in\mathbb{N}}$ converges to $0 \in L([0, 2\pi])$.

$$\sigma_n f(\theta) - sf(\theta)\cdot 1 = \frac{2}{\pi}\int_0^\pi \frac{f(\theta+\xi)+f(\theta-\xi)}{2} K_n(\xi)\,\mathrm{d}\xi - sf(\theta)\,\frac{2}{\pi}\int_0^\pi K_n(\xi)\,\mathrm{d}\xi.$$

$$\boxed{\sigma_n f(\theta) - sf(\theta) = \frac{2}{\pi}\int_0^\pi \left(\frac{f(\theta+\xi)+f(\theta-\xi)}{2} - sf(\theta)\right) K_n(\xi)\,\mathrm{d}\xi.} \qquad (\star)$$

Theorem (Fejér's theorem)

If $f \in L([0,2\pi])$ is 2π-periodic and $sf(\theta) \in \mathbb{R}$, then $\forall\, \theta \in \operatorname{dom}(sf) : \{\sigma_n f(\theta)\}_{n\in\mathbb{N}}$ is Cesàro summable. I.e.

$$\lim_{n\to\infty} \sigma_n f(\theta) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} s_n f(\theta) = sf(\theta).$$

If f is continuous on $[0,2\pi]$, then $\{\sigma_n f\}_{n\in\mathbb{N}}$ converges uniformly to f on $[0,2\pi]$.

Proof.

Suppose that $f \in L([0,2\pi])$ is 2π-periodic and $\theta \in \operatorname{dom}(sf)$. We define

$$\begin{aligned} g_\theta \colon\ & [0,2\pi] \to \mathbb{R} \\ & \xi \mapsto \frac{f(\theta+\xi)+f(\theta-\xi)}{2} - sf(\theta). \end{aligned}$$

Then,

$$\lim_{\xi\to 0^+} g_\theta(\xi) = \lim_{\xi\to 0^+} \left(\frac{f(\theta+\xi)+f(\theta-\xi)}{2} - sf(\theta) \right) = \lim_{\xi\to 0^+} \frac{f(\theta+\xi)+f(\theta-\xi)}{2} - sf(\theta) = sf(\theta) - sf(\theta) = 0.$$

I.e. $\forall\, \varepsilon > 0 : \exists\, 0 < \delta_\xi < \pi$ such that $\forall\, 0 < \xi < \delta_\xi : \boxed{|g_\theta(\xi)| < \frac{\varepsilon}{2}}$.

Proof.

- Let $n \in \mathbb{N}$ and $\xi \in [0, \delta]$.

$$|\sigma_n f(\theta) - sf(\theta)| = \left| \frac{2}{\pi} \int_0^\delta \left(\frac{f(\theta+\xi) + f(\theta-\xi)}{2} - sf(\theta) \right) K_n(\xi)\, \mathrm{d}\xi \right| = \left| \frac{2}{\pi} \int_0^\delta g_\theta(\xi) K_n(\xi)\, \mathrm{d}\xi \right| \leq \frac{2}{\pi} \int_0^\delta |g_\theta(\xi)| \, |K_n(\xi)|\, \mathrm{d}\xi.$$

$$|\sigma_n f(\theta) - sf(\theta)| < \frac{2}{\pi} \int_0^\delta \frac{\varepsilon}{2} |K_n(\xi)|\, \mathrm{d}\xi = \frac{\varepsilon}{\pi} \int_0^\delta K_n(\xi)\, \mathrm{d}\xi < \frac{\varepsilon}{\pi} \int_0^\pi K_n(\xi)\, \mathrm{d}\xi = \frac{\varepsilon}{\pi} \cdot \frac{\pi}{2} = \frac{\varepsilon}{2}.$$

- Let $n \in \mathbb{N}$ and $\xi \in [\delta, \pi]$. Since $g_\theta \in L([0, 2\pi])$, then $M = \int_\delta^\pi |g_\theta(\xi)|\, \mathrm{d}\xi \leq \int_0^{2\pi} |g_\theta(\xi)|\, \mathrm{d}\xi < \infty$.

$$|\sigma_n f(\theta) - sf(\theta)| \leq \frac{2}{\pi} \int_\delta^\pi |g_\theta(\xi)| \, |K_n(\xi)|\, \mathrm{d}\xi = \frac{2}{\pi} \int_\delta^\pi |g_\theta(\xi)| \, K_n(\xi)\, \mathrm{d}\xi \leq \frac{2}{\pi} \int_\delta^\pi |g_\theta(\xi)| \frac{1}{2n \sin^2\left(\frac{\delta}{2}\right)} \mathrm{d}\xi = \int_\delta^\pi \frac{|g_\theta(\xi)|}{n\pi \sin^2\left(\frac{\delta}{2}\right)} \mathrm{d}\xi.$$

Since $\mathbb{R}$ is an archimedean ordered field, satisfies the archimedean property, i.e.

$$\forall\, \varepsilon > 0 : \forall\, L \in \mathbb{R} : \exists\, n_0 \in \mathbb{N} \text{ such that } \frac{1}{n_0} L < \frac{\varepsilon}{2}.$$

□

Proof.

Let $L = \dfrac{M}{\pi \sin^2\left(\frac{\delta}{2}\right)}$. Next, $\forall\, \varepsilon > 0 : \forall\, L \in \mathbb{R} : \exists\, n_0 \in \mathbb{N}$ such that $\dfrac{1}{n_0} \dfrac{\int\limits_{\delta}^{\pi} |g_\theta(\xi)|\, \mathrm{d}\xi}{\pi \sin^2\left(\frac{\delta}{2}\right)} < \dfrac{\varepsilon}{2}$. Then, $\forall\, n > n_0$:

$$\frac{1}{n} < \frac{1}{n_0}.$$

$$\frac{1}{n} \frac{|g_\theta(\xi)|}{\pi \sin^2\left(\frac{\delta}{2}\right)} < \frac{1}{n_0} \frac{|g_\theta(\xi)|}{\pi \sin^2\left(\frac{\delta}{2}\right)}.$$

$$|\sigma_n f(\theta) - sf(\theta)| \le \int\limits_{\delta}^{\pi} \frac{1}{n} \frac{|g_\theta(\xi)|}{\pi \sin^2\left(\frac{\delta}{2}\right)} \mathrm{d}\xi < \int\limits_{\delta}^{\pi} \frac{1}{n_0} \frac{|g_\theta(\xi)|}{\pi \sin^2\left(\frac{\delta}{2}\right)} \mathrm{d}\xi = \frac{1}{n_0} \frac{\int\limits_{\delta}^{\pi} |g_\theta(\xi)|\, \mathrm{d}\xi}{\pi \sin^2\left(\frac{\delta}{2}\right)} < \frac{\varepsilon}{2}.$$

I.e. $\forall\, \varepsilon > 0 : \exists\, n_0 \in \mathbb{N}$ such that $\forall\, n \ge n_0$:

$$|\sigma_n f(\theta) - sf(\theta)| \le \frac{2}{\pi} \int\limits_{0}^{\pi} |g_\theta(\xi)|\, K_n(\xi)\, \mathrm{d}\xi = \frac{2}{\pi} \int\limits_{0}^{\delta} |g_\theta(\xi)|\, K_n(\xi)\, \mathrm{d}\xi + \frac{2}{\pi} \int\limits_{\delta}^{\pi} |g_\theta(\xi)|\, K_n(\xi)\, \mathrm{d}\xi < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\square$

Proof.

Suppose that f is continuous in $[0, 2\pi]$. We define

$$\begin{aligned} h_\theta :\ & [0, 2\pi] \to \mathbb{R} \\ & \xi \mapsto \frac{f(\theta+\xi) + f(\theta-\xi)}{2} - f(\theta). \end{aligned}$$

Since f is continuous in $[0, 2\pi]$, h_θ is uniformily continuous in $[0, 2\pi]$. I.e. $\forall\, \varepsilon > 0 : \exists\, 0 < \delta < \pi$ such that $\forall\, \xi_1, \xi_2 \in [0, 2\pi]$:

$$|\xi_1 - \xi_2| < \delta \implies |h_\theta(\xi_1) - h_\theta(\xi_2)| < \frac{\varepsilon}{2}.$$

Hence, for $\xi_1 = \xi$ and $\xi_2 = 0$: $|h_\theta(\xi) - h_\theta(0)| = \left|h_\theta(\xi) - \left(\frac{f(\theta+0)+f(\theta-0)}{2} - f(\theta)\right)\right| = \boxed{|h_\theta(\xi)| < \frac{\varepsilon}{2}}$.

In other hand,

$$|\sigma_n f(\theta) - f(\theta)\cdot 1| = \left|\frac{2}{\pi}\int_0^\pi \frac{f(\theta+\xi)+f(\theta-\xi)}{2} K_n(\xi)\,\mathrm{d}\xi - f(\theta)\,\frac{2}{\pi}\int_0^\pi K_n(\xi)\,\mathrm{d}\xi\right| = \left|\frac{2}{\pi}\int_0^\pi h_\theta(\xi)\,K_n(\xi)\,\mathrm{d}\xi\right|$$

$$\leq \frac{2}{\pi}\int_0^\pi |h_\theta(\xi)|\,|K_n(\xi)|\,\mathrm{d}\xi = \frac{2}{\pi}\int_0^\pi |h_\theta(\xi)|\,K_n(\xi)\,\mathrm{d}\xi.$$

Proof.

- Let $n \in \mathbb{N}$ and $\xi \in [0, \delta]$.

$$|\sigma_n f(\theta) - f(\theta)| \leq \frac{2}{\pi} \int_0^\delta |h_\theta(\xi)| K_n(\xi) \, \mathrm{d}\xi < \frac{2}{\pi} \int_0^\delta \frac{\varepsilon}{2} K_n(\xi) \, \mathrm{d}\xi < \frac{\varepsilon}{\pi} \int_0^\pi K_n(\xi) \, \mathrm{d}\xi = \frac{\varepsilon}{\pi} \cdot \frac{\pi}{2} = \frac{\varepsilon}{2}.$$

- Let $n \in \mathbb{N}$ and $\xi \in [\delta, \pi]$. Since h_θ is bounded on $[\delta, \pi]$, attains the maximum $M := \max_{\theta \in [\delta, \pi]} |h_\theta|$.

$$|\sigma_n f(\theta) - f(\theta)| \leq \frac{2}{\pi} \int_\delta^\pi |h_\theta(\xi)| K_n(\xi) \, \mathrm{d}\xi \leq \frac{2}{\pi} \int_\delta^\pi M K_n(\xi) \, \mathrm{d}\xi \leq \frac{2}{\pi} \int_\delta^\pi \frac{M}{2n \sin^2\left(\frac{\delta}{2}\right)} \mathrm{d}\xi = \int_\delta^\pi \frac{M}{n\pi \sin^2\left(\frac{\delta}{2}\right)} \mathrm{d}\xi.$$

Since $\mathbb{R}$ is an archimedean ordered field, satisfies the archimedean property, i.e.

$$\forall \varepsilon > 0 : \forall L \in \mathbb{R} : \exists n_0 \in \mathbb{N} \text{ such that } \frac{1}{n_0} L < \frac{\varepsilon}{2}.$$

Let $L = \dfrac{M}{\pi \sin^2\left(\frac{\delta}{2}\right)}$. Next, $\forall \varepsilon > 0 : \forall L \in \mathbb{R} : \exists n_0 \in \mathbb{N}$ such that $\dfrac{1}{n_0} \dfrac{M}{\pi \sin^2\left(\frac{\delta}{2}\right)} < \dfrac{\varepsilon}{2}$. Then, $\forall n > n_0$:

Proof.

$$\frac{1}{n} < \frac{1}{n_0}.$$

$$\frac{1}{n}\frac{M}{\pi\sin^2\left(\frac{\delta}{2}\right)} < \frac{1}{n_0}\frac{M}{\pi\sin^2\left(\frac{\delta}{2}\right)}.$$

$$\left|\sigma_n f(\theta) - f(\theta)\right| \leq \int_{\delta}^{\pi} \frac{1}{n}\frac{M}{\pi\sin^2\left(\frac{\delta}{2}\right)}\mathrm{d}\xi < \int_{\delta}^{\pi} \frac{1}{n_0}\frac{M}{\pi\sin^2\left(\frac{\delta}{2}\right)}\mathrm{d}\xi = \frac{1}{n_0}\frac{M}{\pi\sin^2\left(\frac{\delta}{2}\right)} < \frac{\varepsilon}{2}.$$

I.e. $\forall\,\varepsilon > 0 : \exists\, n_0 \in \mathbb{N}$ such that $\forall\, n \geq n_0 : \forall\,\theta \in [0, 2\pi]$:

$$\left|\sigma_n f(\theta) - f(\theta)\right| \leq \frac{2}{\pi}\int_{0}^{\pi} \left|h_\theta(\xi)\right| K_n(\xi)\,\mathrm{d}\xi = \frac{2}{\pi}\int_{0}^{\delta} \left|h_\theta(\xi)\right| K_n(\xi)\,\mathrm{d}\xi + \frac{2}{\pi}\int_{\delta}^{\pi} \left|h_\theta(\xi)\right| K_n(\xi)\,\mathrm{d}\xi < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\square$

Let is remember from the course of Complex analysis.

Definition (Power series)

An infinite series

$$a_0 + \sum_{n=1}^{\infty} a_n (z - z_0)^n$$

is a power series in $z - z_0$.

Theorem

Let $a_0 + \sum_{n=1}^{\infty} a_n (z - z_0)^n$ *be a power series.*

$$\frac{1}{R} = \limsup_{n \to \infty} \sqrt[n]{|a_n|}.$$

Then, the series converges absolutely if $|z - z_0| < R$ *and diverges if* $|z - z_0| > R$*. Also, the series converges uniformly on every compact subset interior to the disk of convergence.*

Definition (Power series expansion)

The power series expansion of a function f about a given point z_0 is uniquely determined by

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

Theorem (Weierstraß approximation theorem [1])

If $f\colon\ [a,b] \to \mathbb{R}$ *is continuous, then* $\forall\, \varepsilon > 0 : \exists\, p_\varepsilon\colon\ [a,b] \to \mathbb{R}$ *such that* $\forall\, \theta \in [a,b] : |f(x) - p_\varepsilon(\theta)| < \varepsilon$.

Proof.

Suppose that $f\colon\ [a,b] \to \mathbb{R}$ is continuous. We define the 2π-periodic extension of f as

$$
\begin{aligned}
g\colon \mathbb{R} &\to \mathbb{R} \\
\theta &\mapsto \begin{cases} f\left(a + \theta\frac{(b-a)}{\pi}\right), & \theta \in [0,\pi)\,. \\ f\left(a + \theta\frac{(2\pi-\theta)(b-a)}{\pi}\right), & \theta \in [\pi, 2\pi]\,. \\ g(\theta - 2m\pi)\,, & \exists\, m \in \mathbb{Z}\setminus\{0\} \text{ such that } \theta \in [2m\pi, 2(m+1)\pi]\,. \end{cases}
\end{aligned}
$$

Since $g \in L([0,2\pi])$ is 2π-periodic. By the Fejér's theorem, $\forall\, \theta \in \operatorname{dom}(sg) : \{\sigma_n g(\theta)\}_{n\in\mathbb{N}}$ is Cesàro summable. I.e. $\forall\, \varepsilon > 0 : \exists\, n_0 \in \mathbb{N}$ such that $\forall\, n \geq n_0$:

$$
|\sigma_n g(\theta) - sg(\theta)| < \frac{\varepsilon}{2}, \qquad sg(\theta) = a_0 + \sum_{k=1}^{n} \left(a_k \cos(k\theta) + b_k \sin(k\theta)\right).
$$

Also, the power series defined as 1

Proof.

Let $\forall\, \varepsilon > 0$. $\forall\, \theta \in [0, 2\pi] : |p_m(\theta) - g(\theta)| < \varepsilon$. By the triangular inequality.

$$|p_m(\theta) - g(\theta)| = |p_m(\theta) - \sigma(\theta) + \sigma(\theta) - g(\theta)| \leq |p_m(\theta) - \sigma(\theta)| + |\sigma(\theta) - g(\theta)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We define the polynomial as

$$p_\varepsilon : \; [a, b] \to \mathbb{R}$$

$$\theta \mapsto p_m\left(\pi \frac{\theta - a}{b - a}\right)$$

$$|f(\theta) - p_\varepsilon(\theta)| < \varepsilon.$$

$t \mapsto a + (b - a)\, t$ is a continuous bijection from $[0, 1]$ to $[a, b]$. □

Definition (Poset)

Sea $\mathcal{L} \neq \emptyset$ un conjunto. Una relación de orden parcial $\leq$ en $\mathcal{L}$ es una relación binaria en $\mathcal{L}$ que cumple la

- reflexiva $\forall\, a \in \mathcal{L} : a \leq a$.
- antisimétrica $\forall\, a, b \in \mathcal{L} : a \leq b$ y $b \leq a \implies a = b$.
- transitiva $\forall\, a, b, c \in \mathcal{L} : a \leq b$ y $b \leq c \implies a \leq c$.

Si $\leq$ es una relación de orden parcial en $\mathcal{L}$, entonces $(\mathcal{L}, \leq)$ es un conjunto parcialmente ordenado.

Definition (Látice)

Un conjunto parcialmente ordenado $(\mathcal{L}, \leq)$ es látice sii $\forall\, a, b \in \mathcal{L}$ tiene un supremo, $a \wedge b$ y tiene un ínfimo $a \vee b$.

Definition (Látice vectorial o Espacio de Riesz)

Un látice vectorial V es un $\mathbb{R}$-espacio vectorial, que tiene un orden en cual este es un látice, con las propiedades

$$a \leq b \implies x + a \leq x + b, \quad \lambda \in [0, \infty)\,, a \leq b \implies \lambda a \leq \lambda b \vee \wedge.$$

Weierstraß approximation theorem

Definition (Espacio las funciones continuas)

Sea $(X, \mathcal{T})$ un espacio topológico compacto y Hausdorff. Definimos

$$C(X, \mathbb{R}) \coloneqq \{\text{todas las funciones continuas } f\colon X \to \mathbb{R}\}.$$

$$C(X, \mathbb{C}) \coloneqq \{\text{todas las funciones continuas } f\colon X \to \mathbb{C}\}.$$

Definition (Separa puntos)

Un conjunto de funciones $S \subset C(X, \mathbb{R})$ separa puntos sii $\forall\, x, y \in X : x \neq y \implies \exists\, f \in S$ tal que $f(x) \neq f(y)$. Además, S separa puntos fuertemente sii $\forall\, x, y \in X, x \neq y : \{(f(x), f(y))\}_{f \in S} = \mathbb{R}^2$.

Theorem

Si $S \subset C(X, \mathbb{R})$ es un $\mathbb{R}$-espacio vectorial, separa puntos y $\mathbb{1} \in S$, entonces S separa puntos fuertemente.

Proof.

Sean $x, y \in X$ distintos. $\square$

Theorem (Stone – Weierstraß theorem (real case))

Sean $(X, \mathcal{T})$ un espacio topológico compacto y Hausdorff. Si $S \subset C(X, \mathbb{R})$ cumple

Subálgebra $\forall f, g \in S : \forall \lambda \in \mathbb{R} : \Longrightarrow f + g, fg, \lambda f \in S$.

Separa puntos fuertemente *Para cualquier* $x, y \in X$ *y* $\alpha, \beta \in \mathbb{R}$, *existe* $f \in S$ *con* $f(x) = \alpha, g(x) = \beta$.

Entonces, S *es denso (en norma* $\|\cdot\|_\infty$*) en* $C(X, \mathbb{R})$.

A proof soon.

. $\square$

Theorem (Stone – Weierstraß theorem (complex case))

Sean (X, d) un espacio topológico Hausdorff compacto

A proof soon.

. $\square$

Theorem

El espacio $L^2([0,1])$ *es separable.*

A proof soon [6].

Es decir, $\exists\, S \subset L^2([0,1])$ denso y numerable.

$$
\begin{aligned}
\mathbb{P}[0,1] &= \{p\colon\ [0,1] \to \mathbb{R}\}\,. \\
C([0,1],\mathbb{R}) &= \{\}\,. \\
L^2([0,1]) &= \{\}\,.
\end{aligned}
$$

□

Existen diversas generalizaciones del clásico teorema de Stone – Weierstraß que amplia la clase de funciones continuas escalares o vectoriales que se van a aproximar. Una de ellas es debida a Errett Bishop.

Theorem (Bishop's theorem)

Sean $(X, \mathcal{T})$ un espacio topológico compacto y Hausdorff, $C(X, \mathbb{C})$.

A proof soon [11].

. $\square$

„Ein Mathematiker, der nicht irgendwie ein Dichter ist, wird nie ein vollkommener Mathematiker sein.“

– Karl Theodor Wilhelm Weierstraß (1815 - 1897)

“Science is reasoning; reasoning is mathematics; and, therefore, science is mathematics.”

– Marshall Harvey Stone (1903 - 1989)

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– Marshall Harvey Stone (1903 - 1989)

Cuando escribo un artículo, “Tengo que volver a derivar por mí mismo las reglas de derivación y, a veces, incluso la ley conmutativa de la multiplicación.”

– Lipót Fejér (1880 - 1959)

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