THE STONE – WEIERSTRAß THEOREM

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Abstract. We.

Contents

1. Introduction

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2. Preliminaries

In the presentation of the material we mostly follow [2, 14, 10].

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Definition 1 (Topological space)**.** Let X be a set. A *topology* on X is a family $\mathcal{T} \subset 2^X$ that holds

- $\forall \{A_i\}_{i=1}^n \subset \mathcal{T} \implies \bigcap\limits_{i=1}^{n} A_i \in \mathcal{T}$.
- $\forall \{A_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{T} \implies \bigcup\limits_{\lambda \in \Lambda} A_\lambda \in \mathcal{T}$.

If $\mathcal{T}$ is a topology on X, then $(X, \mathcal{T})$ is a *topological space*. The sets in $\mathcal{T}$ are *open sets*.

Definition 2 (Open cover)**.** Let $(X, \mathcal{T})$ be a topological space. $\mathcal{C} \subset \mathcal{T}$ is an *open cover* of X iff $X \subseteq \bigcup\limits_{A \in \mathcal{C}} A$.

Definition 3 (Hausdorff topological space or T_2)**.** A topological space $(X, \mathcal{T})$ is *Hausdorff* iff $\forall\, x, y \in X, x \neq y$: $\exists\, U, V \in \mathcal{T}$ such that $\forall\, x \in U, y \in V : U \cap V = \emptyset$.

Definition 4 (Compact topological space)**.** A topological space $(X, \mathcal{T})$ is *compact* iff each open cover of X has a finite subcover.

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Definition 5 (σ-algebra)**.** Let X be a set. A *σ-algebra* on X is a family $\mathcal{F} \subset 2^X$ that holds

- $\forall\, A \in \mathcal{F} \implies X \setminus A \in \mathcal{F}$.
- $\forall\, \{A_i\}_{i\in\mathbb{N}} \subset \mathcal{F} \implies \bigcup\limits_{i\in\mathbb{N}} A_i \in \mathcal{F}$.

Definition 6 (Measure)**.** Let X be a set and $\mathcal{F}$ a σ-algebra on X. A *measure* on $(X, \mathcal{F})$ is a function $\mu\colon \mathcal{F} \to [0, \infty]$ that holds

- $\mu(\emptyset) = 0$.
- $\forall\, \{A_i\}_{i\in\mathbb{N}} \subset \mathcal{F} : \forall\, i \neq j : A_i \cap A_j = \emptyset \implies \mu\left(\bigcup\limits_{i\in\mathbb{N}} A_i\right) = \sum\limits_{i\in\mathbb{N}} \mu(A_i)$.

Definition 7 (Outer measure)**.** Let X be a set. An *outer measure* on X is a function $\mu^*\colon 2^X \to [0, \infty]$ that holds

- $\mu^*(\emptyset) = 0$.
- $\forall\, A, B \in 2^X : A \subset B \implies \mu^*(A) \leq \mu^*(B)$.
- $\forall\, \{A_i\}_{i\in\mathbb{N}} \subset 2^X \implies \mu^*\left(\bigcup\limits_{i\in\mathbb{N}} A_i\right) = \sum\limits_{i\in\mathbb{N}} \mu^*(A_i)$.

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Definition 8 (σ-algebra generated)**.** Let X be a set and $\mathcal{G} \subset 2^X$. The *σ-algebra generated* by $\mathcal{G}$ is the smallest σ-algebra on X which contains $\mathcal{G}$.

$$\sigma(\mathcal{G}) \coloneqq \bigcap_{\mathcal{A}\in\mathcal{F}(\mathcal{G})} \mathcal{A}, \quad \mathcal{F}(\mathcal{G}) = \left\{\mathcal{A} \subset 2^X \mid \mathcal{G} \subset \mathcal{A}, \mathcal{A} \text{ is a } \sigma\text{-algebra on } X\right\}.$$

Definition 9 (Borel σ-algebra)**.** Let $(X, \mathcal{T})$ be a topological space. The *Borel σ-algebra* on X is $\sigma(\mathcal{T})$. The sets in $\sigma(\mathcal{T})$ are *Borel sets*.

Definition 10 (Lebesgue measure)**.** The *Lebesgue measure* is a measure on $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra of subsets of $\mathbb{R}$, which assigns each Borel set its outer measure.

Definition 11 (Lebesgue space $\mathcal{L}^1(\mu)$)**.** Let $(X, \mathcal{F}, \mu)$ be a measure space. If $f\colon X \to [-\infty, \infty]$ is $\mathcal{F}$-measurable, then the *$\mathcal{L}^1$-norm* of f is

$$\|f\|_1 \coloneqq \int |f| \,\mathrm{d}\mu.$$

The *Lebesgue space* $\mathcal{L}^1(\mu)$ is $\mathcal{L}^1(\mu) \coloneqq \{f\colon X \to \mathbb{R} \mid f \text{ is a function } \mathcal{F}\text{-measurable and } \|f\|_1 < \infty\}$.

Definition 12 ($\|f\|_p$, essential supremum)**.** Let $(X, \mathcal{F}, \mu)$ be a measure space and $0 < p < \infty$. If $f\colon X \to \mathbb{C}$ is $\mathcal{F}$-measurable, then the *p-norm* of f is

$$\|f\|_p \coloneqq \left(\int |f|^p \mathrm{d}\mu\right)^{\frac{1}{p}}.$$

Also, the *essential supremum* of f is $\|f\|_\infty = \inf\{t > 0 : \mu(\{x \in X : |f(x)| > t\}) = 0\}$.

Theorem 13. *Let $(X, \mathcal{F}, \mu)$ be a measure space and $0 < p < \infty$. Then, $\mathcal{L}^p(\mu)$ is a vector space and it is holds:*

- $\forall\, f, g \in \mathcal{L}^p(\mu) : \|f + g\|_p^p \leq 2^p \left(\|f\|_p^p + \|g\|_p^p\right)$.
- $\forall\, f \in \mathcal{L}^p(\mu) : \forall\, \alpha \in \mathbb{C} : \|\alpha f\|_p = |\alpha| \, \|f\|_p$.

Proof. Let $f, g \in \mathcal{L}^p(\mu)$, $0 < p < \infty$, $x \in X$ and $\alpha \in \mathbb{C}$.

- Then, $|f(x)+g(x)|^p \leq (|f(x)|+|g(x)|)^p \leq (2\max\{|f(x)|,|g(x)|\})^p \leq 2^p(|f(x)|^p+|g(x)|^p)$.
  Integrating both sides of the inequality with respect to μ: $\|f+g\|_p^p \leq 2^p\left(\|f\|_p^p+\|g\|_p^p\right)$.
  I.e. if $\|f\|_p<\infty$ and $\|g\|_p<\infty$, then $\|f+g\|_p<\infty$.
- $\|\alpha f\|_p = \left(\int |\alpha f|^p \mathrm{d}\mu\right)^{\frac{1}{p}} = \left(\int |\alpha|^p |f|^p \mathrm{d}\mu\right)^{\frac{1}{p}} = |\alpha|^{\frac{p}{p}}\left(\int |f|^p \mathrm{d}\mu\right)^{\frac{1}{p}} = |\alpha|\,\|f\|_p$.

Since $0 \in \mathcal{L}^p(\mu)$, $\mathcal{L}^p(\mu) < \mathbb{C}^X$ is closed under addition and scalar multiplication.
$\therefore$ *$\mathcal{L}^p(\mu)$ is a vector space.* □

Remark [9]
Let $(X,\mathcal{F},\mu)$ be a measure space. The function

$$\mathcal{L}^2(\mu) \to \mathbb{R}$$
$$f \mapsto \left(\int_X |f|^2 \mathrm{d}\mu\right)^{\frac{1}{2}}$$

is not a norm on $\mathcal{L}^2(\mu)$ because $\exists f \in \mathcal{L}^2(\mu)$ non-zero such that $\displaystyle\int_X |f|^2\mathrm{d}\mu = 0 \in \mathbb{R}$.

Definition 14 ($\mathcal{Z}(\mu)$, $\widetilde{f}$)**.** Let $(X,\mathcal{F},\mu)$ be a measure space and $0<p\leq\infty$. We define

- $\mathcal{Z}(\mu) \coloneqq \{f\colon X\to\mathbb{C} \mid f$ is a function $\mathcal{F}$-measurable and $\mu(\{x\in X: f(x)\neq 0\})=0\}$.
- $\forall\, f\in\mathcal{L}^p(\mu): \widetilde{f} = \{f+z : z\in\mathcal{Z}(\mu)\} < \mathcal{L}^p(\mu)$.

Note that if $f,F\in\mathcal{L}^p(\mu)$, then $\widetilde{f}=\widetilde{F}$ iff $\mu(\{x\in X: f(x)\neq F(x)\})=0$.

Definition 15 ($L^p(\mu)$ space)**.** Let μ is a measure and $0<p\leq\infty$. The set $L^p(\mu)$ are the *equivalence classes of functions* on $\mathcal{L}^p(\mu)$, where two functions are equivalent iff they are equal almost everywhere.

- $L^p(\mu) \coloneqq \left\{\widetilde{f} : f\in\mathcal{L}^p(\mu)\right\} = \mathcal{L}^p(\mu)/\mathcal{Z}(\mu)$.
- $\forall\, \widetilde{f},\widetilde{g}\in L^p(\mu): \forall\,\alpha\in\mathbb{C}: \widetilde{f}+\widetilde{g} \coloneqq (f+g)\widetilde{\ }, \quad \alpha\widetilde{f} \coloneqq (\alpha f)\widetilde{\ }$.

Definition 16 ($\|\cdot\|_p$ on $L^p(\mu)$)**.** Let μ be a measure and $0<p\leq\infty$. We define $\forall\, f\in\mathcal{L}^p(\mu): \left\|\widetilde{f}\right\|_p = \|f\|_p$.

Note that if $f,F\in\mathcal{L}^p(\mu)$ and $\widetilde{f}=\widetilde{F}$, then $\|f\|_p=\|F\|_p$.

Theorem 17. *Let μ be a measure and $p\leq 1\leq\infty$. Then, $L^p(\mu)$ is a vector space and $\|\cdot\|_p$ is a norm on $L^p(\mu)$.*

A proof soon. Let $\widetilde{f},\widetilde{g}\in L^p(\mu)$ and $\alpha\in\mathbb{C}$.

□

Definition 18 (Convergent sequence)**.** Let $(X,\|\cdot\|)$ be a normed $\mathbb{C}$-vector space. A sequence $\{f_n\}_{n\in\mathbb{N}}\subset X$ is a *convergent sequence* iff $\exists\, f\in X$ such that $\forall\,\varepsilon>0:$ $\exists\, N\in\mathbb{N}$ such that $\forall\, n\geq N: \|f-f_n\|<\varepsilon$.

Definition 19 (Cauchy sequence)**.** Let $(X,\|\cdot\|)$ be a normed $\mathbb{C}$-vector space. A sequence $\{f_n\}_{n\in\mathbb{N}}\subset X$ is a *Cauchy sequence* iff $\forall\,\varepsilon>0: \exists\, N\in\mathbb{N}$ such that $\forall\, m,n\geq N: \|f_m-f_n\|<\varepsilon$. $(X,\|\cdot\|)$ is *complete* iff each Cauchy sequence in X is convergent in X.

Theorem 20. *Let $(X, \|\cdot\|)$ be a normed $\mathbb{C}$-vector space and $\{f_n\}_{n\in\mathbb{N}} \subset X$ a sequence. If $\{f_n\}_{n\in\mathbb{N}}$ is convergent in X, then $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in X. Also, if $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in X and has a convergent subsequence in X, then $\{f_n\}_{n\in\mathbb{N}}$ converges in X.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}} \subset X$ a convergent sequence. I.e. $\exists\, f \in X$ such that $\forall\, \varepsilon > 0 : \exists\, N \in \mathbb{N}$ such that $\forall\, n \geq N : \|f - f_n\| < \frac{\varepsilon}{2}$.

Hence, $\forall\, \varepsilon > 0 : \exists\, N \in \mathbb{N}$ such that $\forall\, m, n \geq N : \|f_m - f + f - f_n\| \leq \|f_m - f\| + \|f - f_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Let $\{f_n\}_{n\in\mathbb{N}} \subset X$ a Cauchy sequence that has a convergent subsequence $\{f_{n_k}\}_{k\in\mathbb{N}} \subset X$. Since $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy, i.e. $\exists N \in \mathbb{N}$ such that $\forall\, m, n \geq N : \|f_m - f_n\| < \frac{\varepsilon}{2}$. Also $\{f_{n_k}\}_{k\in\mathbb{N}}$ is convergent, i.e. $\exists\, n_k > N$ such that $\|f - f_{n_k}\| < \frac{\varepsilon}{2}$.

Therefore, $\forall\, \varepsilon > 0 : \exists\, N \in \mathbb{N}$ such that $\forall\, n \geq N : \|f - f_{n_k} + f_{n_k} - f_n\| \leq \|f - f_{n_k}\| + \|f_{n_k} - f_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. □

Theorem 21 (Riesz - Fischer theorem)**.** *Let $(X, \mathcal{F}, \mu)$ be a measure space and $1 \leq p \leq \infty$. Then, $L^p(\mu)$ is a Banach space.*

A proof soon. Let $\left\{\widetilde{f}_n\right\}_{n\in\mathbb{N}} \subset L^p(\mu)$ a Cauchy sequence, i.e. $\forall\, \varepsilon > 0 : \exists\, N \in \mathbb{N}$ such that $\forall\, m, n \geq N : \left\|\widetilde{f}_m - \widetilde{f}_n\right\|_p < \frac{\varepsilon}{2}$.

□

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Definition 22 (Antilinear function)**.** Let $(V, \mathbb{C})$ be a complex vector space. A function $\ell\colon V \to \mathbb{C}$ is *antilinear* iff

- $\forall\, x, y \in V : \ell(x + y) = \ell(x) + \ell(y)$.
- $\forall\, x \in V : \forall\, \lambda \in \mathbb{C} : \ell(\lambda x) = \overline{\lambda}\ell(x)$.

Definition 23 (Sesquilinear form)**.** Let $(V, \mathbb{C})$ be a complex vector space. A *sesquilinear form* on V is a function $V \times V \to \mathbb{C}$ such that $\forall\, x \in V : y \mapsto \langle x, y\rangle$ is linear and $y \mapsto \langle y, x\rangle$ is antilinear.

Definition 24 (Pre-Hilbert space)**.** A *pre-Hilbert space* is a complex vector space $(V, \mathbb{C})$, with a sesquilinear form that holds

- $\forall\, x \in V, x \neq 0 : \langle x, x\rangle \in \mathbb{R}$ y $\langle x, x\rangle > 0$.
- $\forall\, x, y \in V : \langle x, y\rangle = \overline{\langle y, x\rangle}$.

Definition 25 (Orthonormal set)**.** Let $(V, \langle\cdot,\cdot\rangle)$ be a pre-Hilbert space. $\{x_\lambda\}_{\lambda\in\Lambda} \subset V$ is *orthonormal* iff $\langle x_\alpha, x_\beta\rangle = \delta_{\alpha,\beta}$, i.e. $\forall\, \alpha, \beta \in \Lambda : \alpha \neq \beta \implies \langle x_\alpha, x_\beta\rangle = 0$ and $\forall\, \alpha \in \Lambda : \|x_\alpha\| = 1$.

Theorem 26 (Parallelogram law)**.** *Let $(V, \|\cdot\|)$ be a normed $\mathbb{C}$-vector space. Then, $\forall\, x, y \in V : \|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$.*

Theorem 27 (Jordan-von Neumann theorem [8])**.** *Let $(V, \|\cdot\|)$ be a normed $\mathbb{C}$-vector space. $\|\cdot\|$ is induced by an inner product iff $\|\cdot\|$ holds the parallelogram law.*

Theorem 28. *Let $1 \leq p < \infty$. The L^p-norm only holds the parallelogram law for $p = 2$.*

Proof. Let $(X, \mathcal{F}, \mu)$ be a measure space. Then, $\forall\, E \in \mathcal{F}$:

$$\|\chi_E\|_p = \left(\int_X |\chi_E|^p \mathrm{d}\mu\right)^{\frac{1}{p}} = \left(\int_E {\chi_E}^p \mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int_{E^C} {\chi_E}^p \mathrm{d}\mu\right)^{\frac{1}{p}} = \left(\int_E 1\,\mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int_{E^C} 0\,\mathrm{d}\mu\right)^{\frac{1}{p}} = \mu(E)^{\frac{1}{p}} + 0 = \mu$$

If $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$, $0 < \mu(A) < \infty$ y $0 < \mu(B) < \infty$, then $\chi_A + \chi_B = |\chi_A - \chi_B| = \chi_{A \uplus B}$.

$$\|\chi_A + \chi_B\|_p^2 + \|\chi_A - \chi_B\|_p^2 = \left(\int_X |\chi_A + \chi_B|^p \mathrm{d}\mu\right)^{\frac{2}{p}} + \left(\int_X |\chi_A - \chi_B|^p \mathrm{d}\mu\right)^{\frac{2}{p}} = 2\left(\int_X |\chi_{A \uplus B}|^p \mathrm{d}\mu\right)^{\frac{2}{p}} = 2(\mu\,(A$$

$$2\left(\|\chi_A\|_p^2 + \|\chi_B\|_p^2\right) = 2\left((\mu(A))^{\frac{2}{p}} + (\mu(B))^{\frac{2}{p}}\right).$$

Hence, L^p-norm only holds the *parallelogram law* for $p = 2$. □

Definition 29 (Hilbert space)**.** A *Hilbert space* $(H, \langle\cdot,\cdot\rangle)$ is a pre-Hilbert space that is complete with respect to the norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$.

Definition 30 (Orthonormal basis)**.** Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. An *orthonormal basis* of H is a countable maximal orthonormal subset $\{e_n\}_{n\in\mathbb{N}}$ of H.

Theorem 31. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\{e_n\}_{n\in\mathbb{N}}$ an orthonormal basis on H. Then, we have convergence of the Fourier-Bessel series:*

$$\forall\, u \in H : \lim_{n\to\infty} \sum_{k=1}^{n} \langle u, e_k\rangle\, e_k = \sum_{n=1}^{\infty} \langle u, e_n\rangle\, e_n = u.$$

Theorem 32. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. If H has an orthonormal basis, then H is separable.*

Theorem 33. *Let $1 \le p < \infty$. The $L^p(\mu)$ is a Hilbert space iff $p = 2$.*

A proof soon. □

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Definition 34 (Fourier series of f relative)**.** Let $f \in L^2(I)$ and $\{\varphi_k\}_{k\in\mathbb{N}}$ an orthonormal sequence on I. The *Fourier series of f relative* of $\{\varphi_k\}_{k\in\mathbb{N}}$ is $\sum_{k\in\mathbb{N}} c_k \varphi_k(\theta)$, where $\forall\, k \in \mathbb{N} : c_k := \langle f, \varphi_k\rangle = \int_I f(\theta)\,\overline{\varphi_k(\theta)}$ are the *Fourier coefficients of f relative* of $\{\varphi_k\}_{k\in\mathbb{N}}$.

Example

If $I = [0, 2\pi]$ and two orthonormal sequences of trigonometric functions $\{\varphi_k\}_{k\in\mathbb{N}}, \{\phi_k\}_{k\in\mathbb{Z}}$:

real: $\varphi_0(\theta) = \dfrac{1}{\sqrt{2\pi}}, \quad \varphi_{2k-1}(\theta) = \dfrac{\cos(k\theta)}{\sqrt{\pi}}, \quad \varphi_{2k}(\theta) = \dfrac{\sin(k\theta)}{\sqrt{\pi}}.$

complex: $\phi_k(\theta) = \dfrac{e^{ik\theta}}{\sqrt{2\pi}} = \dfrac{\cos(k\theta) + i\sin(k\theta)}{\sqrt{2\pi}}.$

Then, the Fourier series of f relative of $\{\varphi_k\}_{k\in\mathbb{N}}$ and $\{\phi_k\}_{k\in\mathbb{N}}$ are

real: $\dfrac{a_0}{2} + \sum_{k\in\mathbb{N}} a_k \cos(k\theta) + b_k \sin(k\theta).$

complex: $\sum_{k\in\mathbb{Z}} \alpha_k e^{ik\theta}, \quad \alpha_k = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(\theta)\, e^{-ik\theta}\mathrm{d}\theta.$

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Remark [7]

The subset of functions $\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos(m\theta)}{\sqrt{\pi}}, \frac{\sin(n\theta)}{\sqrt{\pi}}\right\}_{m,n\in\mathbb{N}} \subset L^2([0, 2\pi])$ is an orthonormal subset of $L^2([0, 2\pi])$.

Indeed, $\forall\, n, m \in \mathbb{N}$:

- $\int_0^{2\pi} \left(\frac{1}{\sqrt{2\pi}}\right)^2 \mathrm{d}\theta = \int_0^{2\pi} \frac{1}{2\pi} \mathrm{d}\theta = \frac{1}{2\pi}\theta\Big|_0^{2\pi} = 1.$
- $\int_0^{2\pi} \left(\frac{\cos(m\theta)}{\sqrt{\pi}}\right)^2 \mathrm{d}\theta = \int_0^{2\pi} \frac{\cos^2(m\theta)}{\pi} \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} 1+\cos(2m\theta)\,\mathrm{d}\theta = \frac{1}{2\pi}\left(\theta + \frac{\sin(4m\theta)}{4m}\right)\Big|_0^{2\pi} = 1.$
- $\int_0^{2\pi} \left(\frac{\sin(n\theta)}{\sqrt{\pi}}\right)^2 \mathrm{d}\theta = \int_0^{2\pi} \frac{\sin^2(n\theta)}{\pi} \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} 1-\cos(2m\theta)\,\mathrm{d}\theta = \frac{1}{2\pi}\left(\theta - \frac{\sin(4m\theta)}{4m}\right)\Big|_0^{2\pi} = 1.$
- $\int_0^{2\pi} \frac{1}{\sqrt{2\pi}} \frac{\cos(m\theta)}{\sqrt{\pi}} \mathrm{d}\theta = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \cos(m\theta)\,\mathrm{d}\theta = 0.$
- $\int_0^{2\pi} \frac{1}{\sqrt{2\pi}} \frac{\sin(n\theta)}{\sqrt{\pi}} \mathrm{d}\theta = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \sin(n\theta)\,\mathrm{d}\theta = 0.$
- $\int_0^{2\pi} \frac{\cos(m\theta)}{\sqrt{\pi}} \frac{\sin(n\theta)}{\sqrt{\pi}} \mathrm{d}\theta = \frac{1}{\pi} \int_0^{2\pi} \sin(n\theta)\cos(m\theta)\,\mathrm{d}\theta = \frac{1}{\pi}\int_0^{2\pi} \frac{\sin((n+m)\theta) - \sin((n-m)\theta)}{2} \mathrm{d}\theta = 0.$

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Definition 35 (Fourier series generated by f)**.** Let $f \in L^2([0, 2\pi])$. The *Fourier coefficients* of f are given by

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta, \quad a_k = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(k\theta)\,\mathrm{d}\theta, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(k\theta)\,\mathrm{d}\theta.$$

and the *n-th partial Fourier sum* is

$$\boxed{s_n f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(k\theta) + b_k \sin(k\theta).}$$

Indeed, from the equalities $\forall\, k \in \mathbb{N}$:

- $\int_0^{2\pi} \frac{a_0}{2} \mathrm{d}\theta = \frac{a_0}{2}\theta\Big|_0^{2\pi} = \pi a_0.$
- $\int_0^{2\pi} \cos(k\theta)\,\mathrm{d}\theta = \frac{\sin(k\theta)}{k}\Big|_0^{2\pi} = 0.$
- $\int_0^{2\pi} \sin(k\theta)\,\mathrm{d}\theta = \frac{-\cos(k\theta)}{k}\Big|_0^{2\pi} = 0.$

If we integrate the Fourier series term by term

$$\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta = \int_0^{2\pi} \frac{a_0}{2}\mathrm{d}\theta + \int_0^{2\pi} \left(\sum_{k=1}^{\infty} a_k \cos(k\theta) + b_k \sin(k\theta)\right) \mathrm{d}\theta.$$

Then,

$$\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta = \frac{a_0}{2}\int_0^{2\pi} \mathrm{d}\theta + \sum_{k=1}^{\infty}\left(a_k \int_0^{2\pi} \cos(k\theta)\,\mathrm{d}\theta + b_k \int_0^{2\pi} \sin(k\theta)\,\mathrm{d}\theta\right).$$

$$\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta = \pi a_0 + \sum_{k=1}^{\infty}(a_k \cdot 0 + b_k \cdot 0). \quad \Longrightarrow \boxed{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta.}$$

Multiplying the Fourier series by $\cos(m\theta)$, $m \in \mathbb{N}$ and integrating term by term:

$$\int_0^{2\pi} \cos(m\theta) f(\theta)\,\mathrm{d}\theta = \int_0^{2\pi} \cos(m\theta)\frac{a_0}{2}\mathrm{d}\theta + \int_0^{2\pi} \cos(m\theta)\left(\sum_{k=1}^{\infty} a_k \cos(k\theta) + b_k \sin(k\theta)\right)\mathrm{d}\theta.$$

$$\int_0^{2\pi} f(\theta)\cos(m\theta)\,\mathrm{d}\theta = 0 + \sum_{k=1}^{\infty}\left(a_k \int_0^{2\pi} \cos(k\theta)\cos(m\theta)\,\mathrm{d}\theta + b_k \int_0^{2\pi} \sin(k\theta)\cos(m\theta)\,\mathrm{d}\theta\right).$$

$$\int_0^{2\pi} f(\theta)\cos(m\theta)\,\mathrm{d}\theta = \sum_{k=1}^{\infty}\left(\frac{a_k}{2}\int_0^{2\pi} \cos((m+k)\theta) + \cos((m-k)\theta)\,\mathrm{d}\theta + \frac{b_k}{2}\int_0^{2\pi} \sin((m+k)\theta) + \sin((m-k)\theta)\,\mathrm{d}\right.$$

When $m \neq k$ *both integrals* vanish, thus the infinite sum reduces to m-th addend.

$$\int_0^{2\pi} f(\theta)\cos(m\theta)\,\mathrm{d}\theta = a_m \int_0^{2\pi} \cos^2(m\theta)\,\mathrm{d}\theta + b_m \int_0^{2\pi} \sin(m\theta)\cos(m\theta)\,\mathrm{d}\theta.$$

$$\int_0^{2\pi} f(\theta)\cos(m\theta)\,\mathrm{d}\theta = \frac{a_m}{2}\int_0^{2\pi} 1 + \cos(2m\theta)\,\mathrm{d}\theta + b_m \cdot 0.$$

$$\int_0^{2\pi} f(\theta)\cos(m\theta)\,\mathrm{d}\theta = a_m \pi. \quad \Longrightarrow \boxed{a_m = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(m\theta)\,\mathrm{d}\theta.}$$

Multiplying the Fourier series by $\sin(m\theta)$, $m \in \mathbb{N}$ and integrating term by term:

$$\int_0^{2\pi} \sin(m\theta) f(\theta)\,\mathrm{d}\theta = \int_0^{2\pi} \sin(m\theta)\frac{a_0}{2}\mathrm{d}\theta + \int_0^{2\pi} \sin(m\theta)\left(\sum_{k=1}^{\infty} a_k \cos(k\theta) + b_k \sin(k\theta)\right)\mathrm{d}\theta.$$

$$\int_0^{2\pi} f(\theta)\sin(m\theta)\,\mathrm{d}\theta = 0 + \sum_{k=1}^{\infty}\left(a_k \int_0^{2\pi} \cos(k\theta)\sin(m\theta)\,\mathrm{d}\theta + b_k \int_0^{2\pi} \sin(k\theta)\sin(m\theta)\,\mathrm{d}\theta\right).$$

$$\int_0^{2\pi} f(\theta)\sin(m\theta)\,\mathrm{d}\theta = \sum_{k=1}^{\infty}\left(\frac{a_k}{2}\int_0^{2\pi} \sin((m+k)\theta) + \sin((m-k)\theta)\,\mathrm{d}\theta + \frac{b_k}{2}\int_0^{2\pi} \cos((m-k)\theta) - \cos((m+k)\theta)\,\mathrm{d}\right.$$

When $m \neq k$ *both integrals* vanish, thus the infinite sum reduces to m-th addend.

$$\int_0^{2\pi} f(\theta)\sin(m\theta)\,\mathrm{d}\theta = a_m \int_0^{2\pi} \cos(m\theta)\sin(m\theta)\,\mathrm{d}\theta + b_m \int_0^{2\pi} \sin^2(m\theta)\,\mathrm{d}\theta.$$

$$\int_0^{2\pi} f(\theta)\sin(m\theta)\,\mathrm{d}\theta = a_m \cdot 0 + \frac{b_m}{2} \int_0^{2\pi} 1 - \cos(2m\theta)\,\mathrm{d}\theta.$$

$$\int_0^{2\pi} f(\theta)\sin(m\theta)\,\mathrm{d}\theta = b_m \pi. \qquad \Longrightarrow \boxed{b_m = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(m\theta)\,\mathrm{d}\theta.}$$

.

Theorem 36. *If* $\theta \in \mathbb{R}$, *then*

$$\operatorname{Re}\left(\sum_{k=1}^{n} e^{ik\theta}\right) = \sum_{k=1}^{n} \operatorname{Re}\left(e^{ik\theta}\right) = \sum_{k=1}^{n} \cos(k\theta) = \begin{cases} \dfrac{\sin\left((2n+1)\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} - \dfrac{1}{2}, & \exists\, m \in \mathbb{Z} \textit{ such that } \theta \neq 2m\pi. \\ n, & \textit{otherwise.} \end{cases}$$

Proof. From the geometric sum of $e^{i\theta}$ and $2i\sin(\theta) = e^{i\theta} - e^{-i\theta}$:

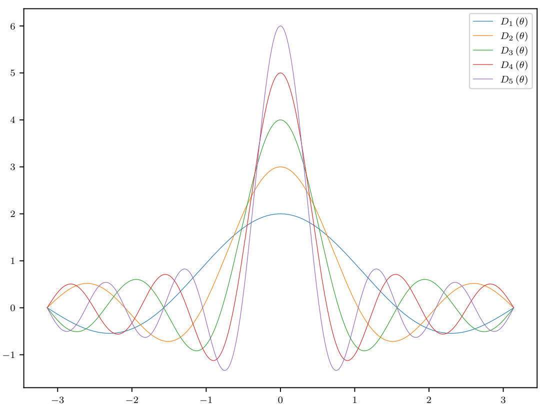
$$\sum_{k=1}^{n} \left(e^{i\theta}\right)^k = \frac{\left(e^{i\theta}\right)^{(n+1)} - e^{i\theta}}{e^{i\theta} - 1} = e^{i\theta}\frac{e^{in\theta} - 1}{e^{i\theta} - 1} = e^{i\theta}\frac{e^{in\frac{\theta}{2}}\left(e^{in\frac{\theta}{2}} - e^{-in\frac{\theta}{2}}\right)}{e^{i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}\right)} = e^{i(n+1)\frac{\theta}{2}}\frac{2i\sin\left(n\frac{\theta}{2}\right)}{2i\sin\left(\frac{\theta}{2}\right)} = e^{i(n+1)\frac{\theta}{2}}\frac{\sin\left(n\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

Taking the *real part* on the opposite sides of the equality and $\cos(\theta_1)\sin(\theta_2) = \frac{1}{2}\left(\sin(\theta_1+\theta_2) - \sin(\theta_1-\theta_2)\right)$:

$$\cos\left((n+1)\frac{\theta}{2}\right)\frac{\sin\left(n\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} = \frac{\sin\left((n+1)\frac{\theta}{2} + n\frac{\theta}{2}\right) - \sin\left((n+1)\frac{\theta}{2} - n\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} = \frac{\sin\left((2n+1)\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)}.$$

$\square$

.



Definition 37 (Dirichlet kernel)**.** The *Dirichlet kernel* D_n of n-order is

$$\boxed{D_n(\theta) := \frac{1}{2} + \sum_{k=1}^{n} \cos(k\theta)}$$

2π-periodic and even, i.e. $\forall\, \theta \in \mathbb{R} : D(-\theta) = D(\theta)$.

Definition 38 (Periodic function [3])**.** A function $f\colon \mathbb{R} \to \mathbb{C}$ is *T-periodic* iff $\exists\, T \in \mathbb{R}\setminus\{0\}$ such that $\forall\, x \in \mathbb{R} : f(x+T) = f(x)$.

Remark
Let $x \in \mathbb{R}$. If $f \in L([0,T])$ is T-periodic, then with the change of variable $y \leftarrow \theta + x - \frac{T}{2}$:

$$\int_0^T f(\theta)\,\mathrm{d}\theta = \int_0^{\frac{T}{2}} f(\theta)\,\mathrm{d}\theta + \int_{\frac{T}{2}}^{T} f(\theta)\,\mathrm{d}\theta = \int_{x-\frac{T}{2}}^{x} f(y)\,\mathrm{d}y + \int_{x}^{x+\frac{T}{2}} f(y)\,\mathrm{d}y = \int_{x-\frac{T}{2}}^{x+\frac{T}{2}} f(y)\,\mathrm{d}y.$$

Lemma 39. *If $f \in L([0,2\pi])$ is 2π-periodic, then the sequence of partial sum $\{s_n f(\theta)\}_{n\in\mathbb{N}}$ of trigonometric Fourier series generated by f has the integral representation*

$$\boxed{s_n f(\theta) = \frac{2}{\pi}\int_0^{\pi} \frac{f(\theta+\xi)+f(\theta-\xi)}{2} D_n(\xi)\,\mathrm{d}\xi.}$$

Proof.

$$s_n f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k\cos(k\theta) + b_k\sin(k\theta)\right).$$

$$s_n f(\theta) = \frac{1}{2}\cdot\frac{1}{\pi}\int_0^{2\pi} f(\xi)\,\mathrm{d}\xi + \sum_{k=1}^{n}\left(\frac{1}{\pi}\int_0^{2\pi} f(\xi)\cos(k\xi)\,\mathrm{d}\xi\cos(k\theta) + \frac{1}{\pi}\int_0^{2\pi} f(\xi)\sin(k\xi)\,\mathrm{d}\xi\sin(k\theta)\right).$$

$$s_n f(\theta) = \frac{1}{\pi}\int_0^{2\pi} f(\xi)\left(\frac{1}{2} + \sum_{k=1}^{n}\cos(k\xi)\cos(k\theta) + \sin(k\xi)\sin(k\theta)\right)\mathrm{d}\xi = \frac{1}{\pi}\int_0^{2\pi} f(\xi)\left(\frac{1}{2} + \sum_{k=1}^{n}\cos(k\xi - k\theta)\right)\mathrm{d}\xi.$$

$$s_n f(\theta) = \frac{1}{\pi}\int_0^{2\pi} f(\xi)\,D_n(k(\xi-\theta))\,\mathrm{d}\xi.$$

The period of the product of two periodic functions f and D_n is the least common multiple of its periods, i.e. $\operatorname{lcm}(2\pi, 2\pi) = 2\pi$ and plugging the u-substitution $u = \xi - \theta$.

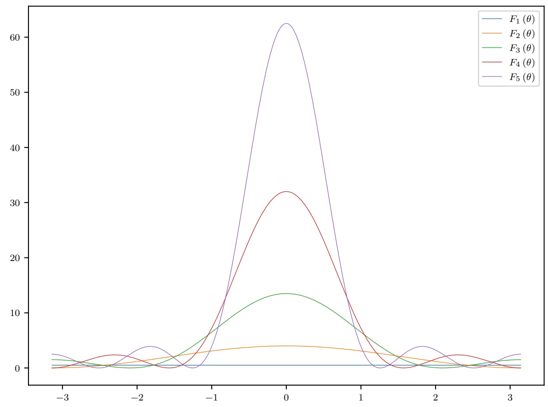
$$s_n f(\theta) = \frac{1}{\pi}\int_{\theta-\pi}^{\theta+\pi} f(\xi)\,D_n(k(\xi-\theta))\,\mathrm{d}\xi = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta+u)\,D_n(u)\,\mathrm{d}u.$$

$$s_n f(\theta) = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(\theta+u)\,D_n(u)\,\mathrm{d}u + \int_0^{\pi} f(\theta+u)\,D_n(u)\,\mathrm{d}u\right).$$

$$s_n f(\theta) = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(\theta+u)\,(D_n(-u))\,\mathrm{d}u + \int_0^{\pi} f(\theta+u)\,D_n(u)\,\mathrm{d}u\right).$$

$$s_n f(\theta) = \frac{1}{\pi}\left(\int_0^{\pi} f(\theta-u)\,D_n(u)\,\mathrm{d}u + \int_0^{\pi} f(\theta+u)\,D_n(u)\,\mathrm{d}u\right) = \frac{2}{\pi}\int_0^{\pi}\frac{f(\theta+u)+f(\theta-u)}{2}D_n(u)\,\mathrm{d}u.$$

$\square$



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Theorem 40. *If $\theta \in \mathbb{R}$, then*

$$\operatorname{Im}\left(\sum_{k=1}^{n} e^{i(2k-1)\theta}\right) = \sum_{k=1}^{n} \operatorname{Im}\left(e^{i(2k-1)\theta}\right) = \sum_{k=1}^{n} \sin\left((2k-1)\theta\right) = \begin{cases} \dfrac{\sin^2(n\theta)}{\sin(\theta)}, & \exists\, m \in \mathbb{Z} \text{ such that } \theta \neq 2m\pi. \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since $\sum\limits_{k=1}^{n} \left(e^{i\theta}\right)^k = e^{i(n+1)\frac{\theta}{2}} \frac{\sin\left(n\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$:

$$\sum_{k=1}^{n} \left(e^{i\theta}\right)^{2k-1} = e^{-i\theta} \sum_{k=1}^{n} \left(e^{i2\theta}\right)^k = e^{-i\theta} e^{i(n+1)\theta} \frac{\sin(n\theta)}{\sin(\theta)} = e^{in\theta} \frac{\sin(n\theta)}{\sin(\theta)}.$$

Taking the *imaginary part* on the opposite sides of the equality:

$$\operatorname{Im}\left(\sum_{k=1}^{n} e^{i(2k-1)\theta}\right) = \sin(n\theta) \frac{\sin(n\theta)}{\sin(\theta)} = \frac{\sin^2(n\theta)}{\sin(\theta)}.$$

□

.

Definition 41 (Fejér kernel)**.** The *Fejér kernel* K_n of n-order is

$$\boxed{K_n(\theta) := \frac{1}{n} \sum_{k=1}^{n} D_{k-1}(\theta)}$$

i.e. is the n-th Cesàro-Fourier means of the Dirichlet kernel.

Definition 42 (n-th Cesàro-Fourier means)**.** Let $f \in L^2([0, 2\pi])$. The *n-th Cesàro-Fourier means* of f is

$$\boxed{\sigma_n f(\theta) = \frac{1}{n} \sum_{k=1}^{n} s_{k-1} f(\theta).}$$

.

Lemma 43. *If $f \in L([0, 2\pi])$ is 2π-periodic and $\{s_n f(\theta)\}_{n \in \mathbb{N}}$ is the sequence of partial sum of the trigonometric Fourier series generated by f. Then, the sequence $\sigma_n f(\theta)$ has the integral representation*

$$\boxed{\sigma_n f(\theta) = \frac{2}{\pi} \int_0^{\pi} \frac{f(\theta + \xi) + f(\theta - \xi)}{2} K_n(\xi)\, \mathrm{d}\xi.}$$

Proof. If $s_n f(\theta) = \frac{2}{\pi} \int\limits_0^{\pi} \frac{f(\theta+\xi)+f(\theta-\xi)}{2} D_n(\xi)\,\mathrm{d}\xi$, then

$$\sigma_n f(\theta) = \frac{1}{n}\sum_{k=1}^{n} s_{k-1} f(\theta) = \frac{1}{n}\sum_{k=1}^{n} \frac{2}{\pi}\int\limits_0^{\pi} \frac{f(\theta+\xi)+f(\theta-\xi)}{2} D_{k-1}(\xi)\,\mathrm{d}\xi.$$

$$\sigma_n f(\theta) = \frac{2}{\pi}\int\limits_0^{\pi} \frac{f(\theta+\xi)+f(\theta-\xi)}{2}\frac{1}{n}\sum_{k=1}^{n} D_{k-1}(\xi)\,\mathrm{d}\xi = \frac{2}{\pi}\int\limits_0^{\pi} \frac{f(\theta+\xi)+f(\theta-\xi)}{2} K_n(\xi)\,\mathrm{d}\xi.$$

□

Theorem 44. *Let $\theta \in \mathbb{R}$. $\forall\, n \in \mathbb{N}$:*

- $\int\limits_0^{\pi} K_n(\theta) = \frac{\pi}{2}$.
- $K_n(\theta) = \frac{1}{2n}\frac{\sin^2\left(n\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)} \geq 0$.
- $\forall\, \delta \in (0,\pi) : \forall\, \delta \leq |\theta| \leq \pi : K_n(\theta) \leq \frac{1}{2n\sin^2\left(\frac{\delta}{2}\right)}$.

Proof.

- $$K_n(\theta) = \frac{1}{n}\sum_{k=1}^{n} D_{k-1}(\theta) = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}+\sum_{m=1}^{k-1}\cos(m\theta)\right) = \frac{1}{n}\left(\sum_{k=1}^{n}\frac{1}{2}+\sum_{k=1}^{n}\sum_{m=1}^{k-1}\cos(m\theta)\right).$$

$$\int\limits_0^{\pi} K_n(\theta) = \int\limits_0^{\pi}\frac{1}{n}\left(\sum_{k=1}^{n}\frac{1}{2}+\sum_{k=1}^{n}\sum_{m=1}^{k-1}\cos(m\theta)\right)\mathrm{d}\theta = \frac{1}{n}\left(\frac{n}{2}\int\limits_0^{\pi}\mathrm{d}\theta+\sum_{k=1}^{n}\sum_{m=1}^{k-1}\int\limits_0^{\pi}\cos(m\theta)\,\mathrm{d}\theta\right) = \frac{1}{n}\left(\frac{n\pi}{2}+0\right) = \frac{\pi}{2}$$

- $$K_n(\theta) = \frac{1}{n}\left(\sum_{k=1}^{n}\frac{1}{2}+\sum_{k=1}^{n}\sum_{m=1}^{k-1}\cos(m\theta)\right) = \frac{1}{n}\left(\frac{n}{2}+\sum_{k=1}^{n}\left(\frac{\sin\left(\left(2\left(k-1\right)+1\right)\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)}-\frac{1}{2}\right)\right)$$

$$K_n(\theta) = \frac{1}{n}\left(\frac{n}{2}+\frac{\sum\limits_{k=1}^{n}\sin\left(\left(2\left(k-1\right)+1\right)\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)}-\sum_{k=1}^{n}\frac{1}{2}\right) = \frac{1}{n}\left(\frac{n}{2}+\frac{\sum\limits_{k=1}^{n}\sin\left(\left(2k-1\right)\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)}-\frac{n}{2}\right)$$

$$K_n(\theta) = \frac{1}{2n}\frac{\frac{\sin^2\left(n\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}}{\sin\left(\frac{\theta}{2}\right)} = \frac{1}{2n}\frac{\sin^2\left(n\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)} \geq 0.$$

- $\forall\, \delta \in (0,\pi) : \sin^2\left(\frac{\delta}{2}\right)$ is *even* and increasing. Then, $\forall\, \delta < |\theta| < \pi$:

$$\frac{\delta}{2} < \frac{|\theta|}{2} \implies \sin^2\left(\frac{\delta}{2}\right) < \sin^2\left(\frac{|\theta|}{2}\right) \implies \sin^2\left(\frac{\delta}{2}\right) < \sin^2\left(\frac{\theta}{2}\right) \implies \frac{1}{\sin^2\left(\frac{\theta}{2}\right)} < \frac{1}{\sin^2\left(\frac{\delta}{2}\right)}.$$

$$K_n(\theta) = \frac{1}{2n}\frac{\sin^2\left(n\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)} = \frac{\sin^2\left(n\frac{\theta}{2}\right)}{2n\sin^2\left(\frac{\theta}{2}\right)} \leq \frac{1}{2n\sin^2\left(\frac{\delta}{2}\right)}.$$

□

.

Remark

Applying the last lemma for $f \equiv 1 \in L([0,2\pi])$ which is 2π-periodic, then $\forall n \in \mathbb{N}$:

$$s_n f(\theta) = \frac{2}{\pi}\int\limits_0^\pi \frac{f(\theta+\xi)+f(\theta-\xi)}{2} D_n(\xi)\,\mathrm{d}\xi = \frac{2}{\pi}\int\limits_0^\pi \frac{1+1}{2}\left(\frac{1}{2}+\sum_{k=1}^{n}\cos(k\xi)\right)\mathrm{d}\xi = \frac{2}{\pi}\int\limits_0^\pi \left(\frac{1}{2}+\sum_{k=1}^{n}\cos(k\xi)\right)$$

$$s_n f(\theta) = \frac{2}{\pi}\int\limits_0^\pi \frac{\mathrm{d}\xi}{2} + \frac{2}{\pi}\int\limits_0^\pi \sum_{k=1}^{n}\cos(k\xi)\,\mathrm{d}\xi = 1 + \frac{2}{\pi}\sum_{k=1}^{n}\int\limits_0^\pi \cos(k\xi)\,\mathrm{d}\xi = 1 + \frac{2}{\pi}\sum_{k=1}^{n}\left(\frac{\sin(k\xi)}{k}\right)\Bigg|_0^\pi = 1 + \frac{2}{\pi}\sum_{k=1}^{n} 0 =$$

$$\sigma_n f(\theta) = \frac{2}{\pi}\int\limits_0^\pi \frac{f(\theta+\xi)+f(\theta-\xi)}{2} K_n(\xi)\,\mathrm{d}\xi = \frac{2}{\pi}\int\limits_0^\pi \frac{1+1}{2} K_n(\xi)\,\mathrm{d}\xi = \frac{2}{\pi}\int\limits_0^\pi K_n(\xi)\,\mathrm{d}\xi = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1.$$

We will see if $sf(\theta) := \lim\limits_{\xi\to 0^+} \dfrac{f(\theta+\xi)+f(\theta-\xi)}{2} \in \mathbb{R}$, then $\{\sigma_n f(\theta) - sf(\theta)\}_{n\in\mathbb{N}}$ converges to $0 \in L([0,2\pi])$.

$$\sigma_n f(\theta) - sf(\theta)\cdot 1 = \frac{2}{\pi}\int\limits_0^\pi \frac{f(\theta+\xi)+f(\theta-\xi)}{2} K_n(\xi)\,\mathrm{d}\xi - sf(\theta)\,\frac{2}{\pi}\int_0^\pi K_n(\xi)\,\mathrm{d}\xi.$$

$$(\bigstar) \qquad \boxed{\sigma_n f(\theta) - sf(\theta) = \frac{2}{\pi}\int\limits_0^\pi \left(\frac{f(\theta+\xi)+f(\theta-\xi)}{2} - sf(\theta)\right) K_n(\xi)\,\mathrm{d}\xi.}$$

3. Fejér's theorem

Theorem 45 (). *If $f \in L([0,2\pi])$ is 2π-periodic and $sf(\theta) \in \mathbb{R}$, then $\forall \theta \in \mathrm{dom}(sf) : \{\sigma_n f(\theta)\}_{n\in\mathbb{N}}$ is Cesàro summable. I.e.*

$$\lim_{n\to\infty} \sigma_n f(\theta) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} s_n f(\theta) = sf(\theta).$$

If f is continuous on $[0,2\pi]$, then $\{\sigma_n f\}_{n\in\mathbb{N}}$ converges uniformily to f on $[0,2\pi]$.

Proof. Suppose that $f \in L([0,2\pi])$ is 2π-periodic and $\theta \in \mathrm{dom}(sf)$. We define

$$\begin{aligned} g_\theta \colon\ & [0,2\pi] \to \mathbb{R} \\ & \xi \mapsto \frac{f(\theta+\xi)+f(\theta-\xi)}{2} - sf(\theta). \end{aligned}$$

Then,

$$\lim_{\xi\to 0^+} g_\theta(\xi) = \lim_{\xi\to 0^+}\left(\frac{f(\theta+\xi)+f(\theta-\xi)}{2} - sf(\theta)\right) = \lim_{\xi\to 0^+}\frac{f(\theta+\xi)+f(\theta-\xi)}{2} - sf(\theta) = sf(\theta) - sf(\theta) = 0.$$

I.e. $\forall\, \varepsilon > 0 : \exists\, 0 < \delta_\xi < \pi$ such that $\forall\, 0 < \xi < \delta_\xi : \boxed{|g_\theta(\xi)| < \frac{\varepsilon}{2}}$.

- Let $n \in \mathbb{N}$ and $\xi \in [0,\delta]$.

$$|\sigma_n f(\theta) - sf(\theta)| = \left|\frac{2}{\pi}\int\limits_0^\delta \left(\frac{f(\theta+\xi)+f(\theta-\xi)}{2} - sf(\theta)\right) K_n(\xi)\,\mathrm{d}\xi\right| = \left|\frac{2}{\pi}\int\limits_0^\delta g_\theta(\xi) K_n(\xi)\,\mathrm{d}\xi\right| \le \frac{2}{\pi}\int\limits_0^\delta |g_\theta(\xi)|\,|$$

$$|\sigma_n f(\theta) - sf(\theta)| < \frac{2}{\pi}\int\limits_0^\delta \frac{\varepsilon}{2}|K_n(\xi)|\,\mathrm{d}\xi = \frac{\varepsilon}{\pi}\int\limits_0^\delta K_n(\xi)\,\mathrm{d}\xi < \frac{\varepsilon}{\pi}\int\limits_0^\pi K_n(\xi)\,\mathrm{d}\xi = \frac{\varepsilon}{\pi}\cdot\frac{\pi}{2} = \frac{\varepsilon}{2}.$$

- Let $n \in \mathbb{N}$ and $\xi \in [\delta, \pi]$. Since $g_\theta \in L([0,2\pi])$, then $M = \int\limits_\delta^\pi |g_\theta(\xi)| \, \mathrm{d}\xi \leq \int\limits_0^{2\pi} |g_\theta(\xi)| \, \mathrm{d}\xi < \infty$.

$$|\sigma_n f(\theta) - sf(\theta)| \leq \frac{2}{\pi} \int\limits_\delta^\pi |g_\theta(\xi)| \, |K_n(\xi)| \, \mathrm{d}\xi = \frac{2}{\pi} \int\limits_\delta^\pi |g_\theta(\xi)| \, K_n(\xi) \, \mathrm{d}\xi \leq \frac{2}{\pi} \int\limits_\delta^\pi |g_\theta(\xi)| \frac{1}{2n\sin^2\left(\frac{\delta}{2}\right)} \mathrm{d}\xi = \int\limits_\delta^\pi \frac{|g_\theta(\xi)}{n\pi\sin^2 (}$$

Since $\mathbb{R}$ is an *archimedean* ordered field, satisfies the *archimedean property*, i.e.

$$\forall \, \varepsilon > 0 : \forall \, L \in \mathbb{R} : \exists \, n_0 \in \mathbb{N} \text{ such that } \frac{1}{n_0} L < \frac{\varepsilon}{2}.$$

□

Proof. Let $L = \dfrac{M}{\pi\sin^2\left(\frac{\delta}{2}\right)}$. Next, $\forall \, \varepsilon > 0 : \forall \, L \in \mathbb{R} : \exists n_0 \in \mathbb{N}$ such that $\dfrac{1}{n_0} \dfrac{\int_\delta^\pi |g_\theta(\xi)| \, \mathrm{d}\xi}{\pi\sin^2\left(\frac{\delta}{2}\right)} < \dfrac{\varepsilon}{2}$. Then, $\forall \, n > n_0$:

$$\frac{1}{n} < \frac{1}{n_0}.$$
$$\frac{1}{n} \frac{|g_\theta(\xi)|}{\pi\sin^2\left(\frac{\delta}{2}\right)} < \frac{1}{n_0} \frac{|g_\theta(\xi)|}{\pi\sin^2\left(\frac{\delta}{2}\right)}.$$

$$|\sigma_n f(\theta) - sf(\theta)| \leq \int\limits_\delta^\pi \frac{1}{n} \frac{|g_\theta(\xi)|}{\pi\sin^2\left(\frac{\delta}{2}\right)} \mathrm{d}\xi < \int\limits_\delta^\pi \frac{1}{n_0} \frac{|g_\theta(\xi)|}{\pi\sin^2\left(\frac{\delta}{2}\right)} \mathrm{d}\xi = \frac{1}{n_0} \frac{\int_\delta^\pi |g_\theta(\xi)| \, \mathrm{d}\xi}{\pi\sin^2\left(\frac{\delta}{2}\right)} < \frac{\varepsilon}{2}.$$

I.e. $\forall \, \varepsilon > 0 : \exists \, n_0 \in \mathbb{N}$ such that $\forall \, n \geq n_0$:

$$|\sigma_n f(\theta) - sf(\theta)| \leq \frac{2}{\pi} \int\limits_0^\pi |g_\theta(\xi)| \, K_n(\xi) \, \mathrm{d}\xi = \frac{2}{\pi} \int\limits_0^\delta |g_\theta(\xi)| \, K_n(\xi) \, \mathrm{d}\xi + \frac{2}{\pi} \int\limits_\delta^\pi |g_\theta(\xi)| \, K_n(\xi) \, \mathrm{d}\xi < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Proof. Suppose that f is continuous in $[0, 2\pi]$. We define

$$\begin{aligned} h_\theta \colon \; & [0, 2\pi] \to \mathbb{R} \\ & \xi \mapsto \frac{f(\theta + \xi) + f(\theta - \xi)}{2} - f(\theta). \end{aligned}$$

Since f is continuous in $[0, 2\pi]$, h_θ is uniformly continuous in $[0, 2\pi]$. I.e. $\forall \, \varepsilon > 0$: $\exists \, 0 < \delta < \pi$ such that $\forall \, \xi_1, \xi_2 \in [0, 2\pi]$:

$$|\xi_1 - \xi_2| < \delta \implies |h_\theta(\xi_1) - h_\theta(\xi_2)| < \frac{\varepsilon}{2}.$$

Hence, for $\xi_1 = \xi$ and $\xi_2 = 0$: $|h_\theta(\xi) - h_\theta(0)| = \left| h_\theta(\xi) - \left(\frac{f(\theta+0)+f(\theta-0)}{2} - f(\theta) \right) \right| = \boxed{|h_\theta(\xi)| < \frac{\varepsilon}{2}}$.

In other hand,

$$|\sigma_n f(\theta) - f(\theta)\cdot 1| = \left|\frac{2}{\pi}\int_0^\pi \frac{f(\theta+\xi)+f(\theta-\xi)}{2} K_n(\xi)\,\mathrm{d}\xi - f(\theta)\frac{2}{\pi}\int_0^\pi K_n(\xi)\,\mathrm{d}\xi\right| = \left|\frac{2}{\pi}\int_0^\pi h_\theta(\xi) K_n(\xi)\,\mathrm{d}\xi\right|$$
$$\leq \frac{2}{\pi}\int_0^\pi |h_\theta(\xi)|\,|K_n(\xi)|\,\mathrm{d}\xi = \frac{2}{\pi}\int_0^\pi |h_\theta(\xi)|\,K_n(\xi)\,\mathrm{d}\xi.$$

- Let $n \in \mathbb{N}$ and $\xi \in [0,\delta]$.

$$|\sigma_n f(\theta) - f(\theta)| \leq \frac{2}{\pi}\int_0^\delta |h_\theta(\xi)|\,K_n(\xi)\,\mathrm{d}\xi < \frac{2}{\pi}\int_0^\delta \frac{\varepsilon}{2} K_n(\xi)\,\mathrm{d}\xi < \frac{\varepsilon}{\pi}\int_0^\pi K_n(\xi)\,\mathrm{d}\xi = \frac{\varepsilon}{\pi}\cdot\frac{\pi}{2} = \frac{\varepsilon}{2}.$$

- Let $n \in \mathbb{N}$ and $\xi \in [\delta,\pi]$. Since h_θ is bounded on $[\delta,\pi]$, attains the maximum $M \coloneqq \max\limits_{\theta\in[\delta,\pi]} |h_\theta|$.

$$|\sigma_n f(\theta) - f(\theta)| \leq \frac{2}{\pi}\int_\delta^\pi |h_\theta(\xi)|\,K_n(\xi)\,\mathrm{d}\xi \leq \frac{2}{\pi}\int_\delta^\pi M K_n(\xi)\,\mathrm{d}\xi \leq \frac{2}{\pi}\int_\delta^\pi \frac{M}{2n\sin^2\left(\frac{\delta}{2}\right)}\mathrm{d}\xi = \int_\delta^\pi \frac{M}{n\pi\sin^2\left(\frac{\delta}{2}\right)}\mathrm{d}\xi.$$

Since $\mathbb{R}$ is an *archimedean* ordered field, satisfies the *archimedean property*, i.e.

$$\forall\,\varepsilon > 0 : \forall\,L \in \mathbb{R} : \exists\,n_0 \in \mathbb{N} \text{ such that } \frac{1}{n_0}L < \frac{\varepsilon}{2}.$$

Let $L = \dfrac{M}{\pi\sin^2\left(\frac{\delta}{2}\right)}$. Next, $\forall\,\varepsilon > 0 : \forall\,L \in \mathbb{R} : \exists n_0 \in \mathbb{N}$ such that $\dfrac{1}{n_0}\dfrac{M}{\pi\sin^2\left(\frac{\delta}{2}\right)} < \dfrac{\varepsilon}{2}$. Then, $\forall\,n > n_0$:

$$\frac{1}{n} < \frac{1}{n_0}.$$
$$\frac{1}{n}\frac{M}{\pi\sin^2\left(\frac{\delta}{2}\right)} < \frac{1}{n_0}\frac{M}{\pi\sin^2\left(\frac{\delta}{2}\right)}.$$
$$|\sigma_n f(\theta) - f(\theta)| \leq \int_\delta^\pi \frac{1}{n}\frac{M}{\pi\sin^2\left(\frac{\delta}{2}\right)}\mathrm{d}\xi < \int_\delta^\pi \frac{1}{n_0}\frac{M}{\pi\sin^2\left(\frac{\delta}{2}\right)}\mathrm{d}\xi = \frac{1}{n_0}\frac{M}{\pi\sin^2\left(\frac{\delta}{2}\right)} < \frac{\varepsilon}{2}.$$

I.e. $\forall\,\varepsilon > 0 : \exists\,n_0 \in \mathbb{N}$ such that $\forall\,n \geq n_0 : \forall\,\theta \in [0,2\pi]$:

$$|\sigma_n f(\theta) - f(\theta)| \leq \frac{2}{\pi}\int_0^\pi |h_\theta(\xi)|\,K_n(\xi)\,\mathrm{d}\xi = \frac{2}{\pi}\int_0^\delta |h_\theta(\xi)|\,K_n(\xi)\,\mathrm{d}\xi + \frac{2}{\pi}\int_\delta^\pi |h_\theta(\xi)|\,K_n(\xi)\,\mathrm{d}\xi < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

□

4. Weierstrass approximation theorem

Let is remember from the course of Complex analysis.

Definition 46 (Power series)**.** An infinite series

$$a_0 + \sum_{n=1}^{\infty} a_n (z-z_0)^n$$

is a *power series* in $z - z_0$.

Theorem 47. *Let* $a_0 + \sum_{n=1}^{\infty} a_n(z-z_0)^n$ *be a power series.*

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

Then, the series converges absolutely if $|z-z_0| < R$ *and diverges if* $|z-z_0| > R$. *Also, the series converges uniformily on every compact subset interior to the disk of convergence.*

Definition 48 (Power series expansion)**.** The *power series expansion* of a function f about a given point z_0 is uniquely determined by

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

Theorem 49 ([1])**.** *If* $f\colon [a,b]\to\mathbb{R}$ *is continuous, then* $\forall\,\varepsilon>0 : \exists\, p_\varepsilon\colon [a,b]\to\mathbb{R}$ *such that* $\forall\,\theta\in[a,b] : |f(x) - p_\varepsilon(\theta)| < \varepsilon$.

Proof. Suppose that $f\colon [a,b]\to\mathbb{R}$ is continuous. We define the 2π-periodic extension of f as

$$\begin{aligned} g\colon \mathbb{R} &\to \mathbb{R} \\ \theta &\mapsto \begin{cases} f\left(a+\theta\frac{(b-a)}{\pi}\right), & \theta\in[0,\pi). \\ f\left(a+\theta\frac{(2\pi-\theta)(b-a)}{\pi}\right), & \theta\in[\pi,2\pi]. \\ g(\theta-2m\pi), & \exists\, m\in\mathbb{Z}\setminus\{0\} \text{ such that } \theta\in[2m\pi, 2(m+1)\pi]. \end{cases} \end{aligned}$$

Since $g\in L([0,2\pi])$ is 2π-periodic. By the *Fejér's theorem*, $\forall\,\theta\in\operatorname{dom}(sg)$: $\{\sigma_n g(\theta)\}_{n\in\mathbb{N}}$ is Cesàro summable. I.e. $\forall\,\varepsilon>0 : \exists\, n_0\in\mathbb{N}$ such that $\forall\, n\geq n_0$:

$$|\sigma_n g(\theta) - sg(\theta)| < \frac{\varepsilon}{2}, \qquad sg(\theta) = a_0 + \sum_{k=1}^{n}(a_k\cos(k\theta) + b_k\sin(k\theta)).$$

Also, the power series defined as 1

Let $\forall\,\varepsilon>0$. $\forall\,\theta\in[0,2\pi] : |p_m(\theta) - g(\theta)| < \varepsilon$. By the triangular inequality.

$$|p_m(\theta)-g(\theta)| = |p_m(\theta)-\sigma(\theta)+\sigma(\theta)-g(\theta)| \leq |p_m(\theta)-\sigma(\theta)|+|\sigma(\theta)-g(\theta)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

We define the polynomial as

$$\begin{aligned} p_\varepsilon\colon [a,b] &\to \mathbb{R} \\ \theta &\mapsto p_m\left(\pi\frac{\theta-a}{b-a}\right) \end{aligned}$$

$$|f(\theta) - p_\varepsilon(\theta)| < \varepsilon.$$

$t\mapsto a+(b-a)\,t$ is a continuous bijection from $[0,1]$ to $[a,b]$. □

.

Definition 50 (Poset)**.** Sea $\mathcal{L}\neq\emptyset$ un conjunto. Una *relación de orden parcial* $\leq$ en $\mathcal{L}$ es una relación binaria en $\mathcal{L}$ que cumple la

reflexiva: $\forall\, a\in\mathcal{L} : a\leq a$.
antisimétrica: $\forall\, a,b\in\mathcal{L} : a\leq b$ y $b\leq a \implies a=b$.
transitiva: $\forall\, a,b,c\in\mathcal{L} : a\leq b$ y $b\leq c \implies a\leq c$.

Si $\leq$ es una relación de orden parcial en $\mathcal{L}$, entonces $(\mathcal{L},\leq)$ es un *conjunto parcialemente ordenado*.

Definition 51 (Látice)**.** Un conjunto parcialmente ordenado $(\mathcal{L}, \leq)$ es *látice* sii $\forall\, a, b \in \mathcal{L}$ tiene un supremo, $a \wedge b$ y tiene un ínfimo $a \vee b$.

Definition 52 (Látice vectorial o Espacio de Riesz)**.** Un *látice vectorial* V es un $\mathbb{R}$-espacio vectorial, que tiene un orden en cual este es un látice, con las propiedades

$$a \leq b \implies x + a \leq x + b, \quad \lambda \in [0, \infty), a \leq b \implies \lambda a \leq \lambda b \vee \wedge.$$

.

Definition 53 (Espacio las funciones continuas)**.** Sea $(X, \mathcal{T})$ un espacio topológico compacto y Hausdorff. Definimos

$$C(X, \mathbb{R}) \coloneqq \{\text{todas las funciones continuas } f \colon X \to \mathbb{R}\}.$$
$$C(X, \mathbb{C}) \coloneqq \{\text{todas las funciones continuas } f \colon X \to \mathbb{C}\}.$$

Definition 54 (Separa puntos)**.** Un conjunto de funciones $S \subset C(X, \mathbb{R})$ *separa puntos* sii $\forall\, x, y \in X : x \neq y \implies \exists\, f \in S$ tal que $f(x) \neq f(y)$. Además, S *separa puntos fuertemente* sii $\forall\, x, y \in X, x \neq y : \{(f(x), f(y))\}_{f \in S} = \mathbb{R}^2$.

Theorem 55. *Si $S \subset C(X, \mathbb{R})$ es un $\mathbb{R}$-espacio vectorial, separa puntos y $\mathbb{1} \in S$, entonces S separa puntos fuertemente.*

Proof. Sean $x, y \in X$ distintos. □

5. Stone – Weierstrass theorem (real case)

Theorem 56 ()**.** *Sean $(X, \mathcal{T})$ un espacio topológico compacto y Hausdorff. Si $S \subset C(X, \mathbb{R})$ cumple*

- **Subálgebra:** *$\forall\, f, g \in S : \forall\, \lambda \in \mathbb{R} : \implies f + g, fg, \lambda f \in S$.*
- **Separa puntos fuertemente:** *Para cualquier $x, y \in X$ y $\alpha, \beta \in \mathbb{R}$, existe $f \in S$ con $f(x) = \alpha, g(x) = \beta$.*

Entonces, S es denso (en norma $\|\cdot\|_\infty$) en $C(X, \mathbb{R})$.

A proof soon. . □

6. Stone – Weierstrass theorem (complex case)

Theorem 57 ()**.** *Sean (X, d) un espacio topológico Hausdorff compacto*

A proof soon. . □

Theorem 58. *El espacio $L^2([0,1])$ es separable.*

A proof soon [6]. Es decir, $\exists\, S \subset L^2([0,1])$ denso y numerable.

$$\mathbb{P}[0,1] = \{p \colon [0,1] \to \mathbb{R}\}.$$
$$C([0,1], \mathbb{R}) = \{\}.$$
$$L^2([0,1]) = \{\}.$$

□

7. Generalizations

Existen diversas *generalizaciones* del clásico teorema de Stone – Weierstraß que amplia la clase de funciones continuas escalares o vectoriales que se van a aproximar. Una de ellas es debida a Errett Bishop.

Theorem 59 (Bishop's theorem)**.** *Sean $(X, \mathcal{T})$ un espacio topológico compacto y Hausdorff, $C(X, \mathbb{C})$.*

A proof soon [11]. . □

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