Macquarie University

Grokking MATH135

A very unofficial textbook.

SID	Name	Initials	Email
4196 3539	CARMICHAEL, Adam Justin	AJC	adam.carmichael@ieee.org
xxxx xxxx	HUNT, Nathaniel K	NHK	
yyyy yyyy	RACE, Alison	AR	

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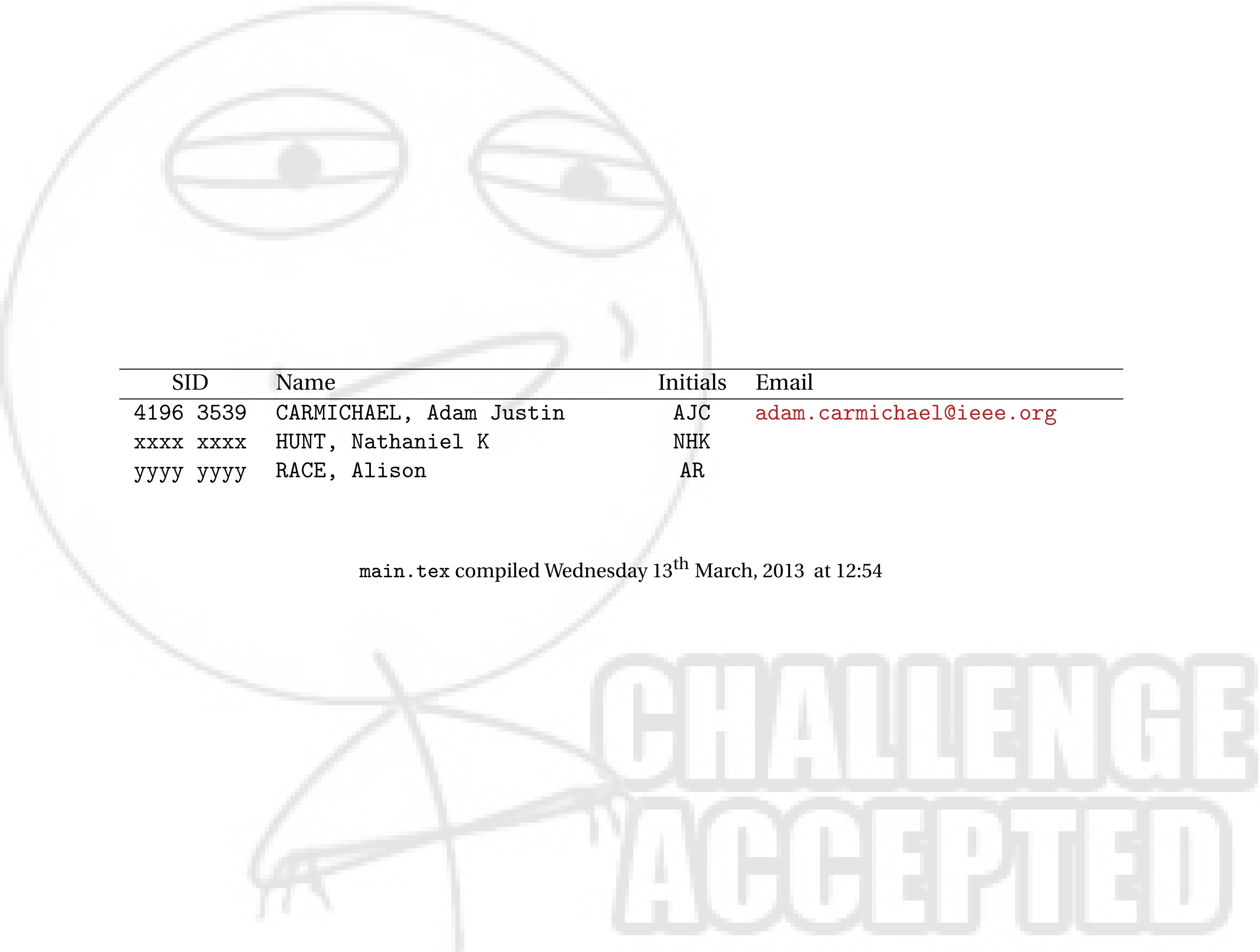

Contents

CHAPTER 1

Trigonometry Review

Further reading: `Chen & Duong ‘‘mtfym01.pdf’’`, `http://goo.gl/wyojl`

Big Ideas:

- Radians and arc length are interrelated. There are 2π radians in a circle.
- An angle subtended by θ will have arc length θ.
- $\cos(\theta) = y$-ordinate, $\sin(\theta) = x$-ordinate
- $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{y}{x}$
- Pythagorean Identities (page 7) are worth learning!

1.1 Radian & Arc Length

Further reading: `Chen & Duong ‘‘mtfym01.pdf’’ pp 1-2`, `http://goo.gl/wyojl`

Radians and arc length are interrelated. There are 2π radians in a circle, that is a semi-circle has about 3.14 radians (π radians to be exact).

In areas of maths, science and engineering radians are used as the main unit of measurement of angles rather than degrees, as such it can be handy to know how to convert between the two units.

$$\text{degrees} = \text{radians} \times \frac{180}{\pi} \tag{1.1}$$

$$\text{radians} = \text{degrees} \times \frac{\pi}{180} \tag{1.2}$$

Aside from the conversion formula, it is also useful to remember the following special values:

$$\frac{\pi}{6} = 30^\circ \tag{1.3}$$

$$\frac{\pi}{4} = 45^\circ \tag{1.4}$$

$$\frac{\pi}{3} = 60^\circ \tag{1.5}$$

$$\frac{\pi}{2} = 30^\circ \tag{1.6}$$

$$\pi = 180^\circ \tag{1.7}$$

$$2\pi = 360^\circ \tag{1.8}$$

1.2 The Unit Circle

In high school we learned that trigonometry was about angles of triangles, and while this remains true, we look at it in the context of circles. Consider a special circle called *the unit circle*. See (ref fig 1.2 page 7). If we run a vertical line from each point in the unit circle to the line $y = 0$, the triangle is formed.

1.3 Trigonometric Functions

Further reading: `Chen & Duong ‘‘mtfym01.pdf’’ pp 2-4`, `http://goo.gl/wyojl`

Looking at the unit circle, an angle is defined by the two points, the first at the origin $(0,0)$, the second is $(\cos\theta), \sin\theta)$. That is to say

$$x = \cos\theta \quad \text{and} \qquad y = \sin\theta \tag{1.9}$$

Additionally, we can define $\tan\theta$:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} \tag{1.10}$$

Finally there are the inverse trigonometric functions:

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{x} \quad \text{and} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{1}{y} \tag{1.11}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{x}{y} \tag{1.12}$$

Note: $\tan\theta$ and $\sec\theta$ are defined only when $\cos\theta \neq 0$ and $\cot\theta$ and $\csc\theta$ are defined only when $\sin\theta \neq 0$.

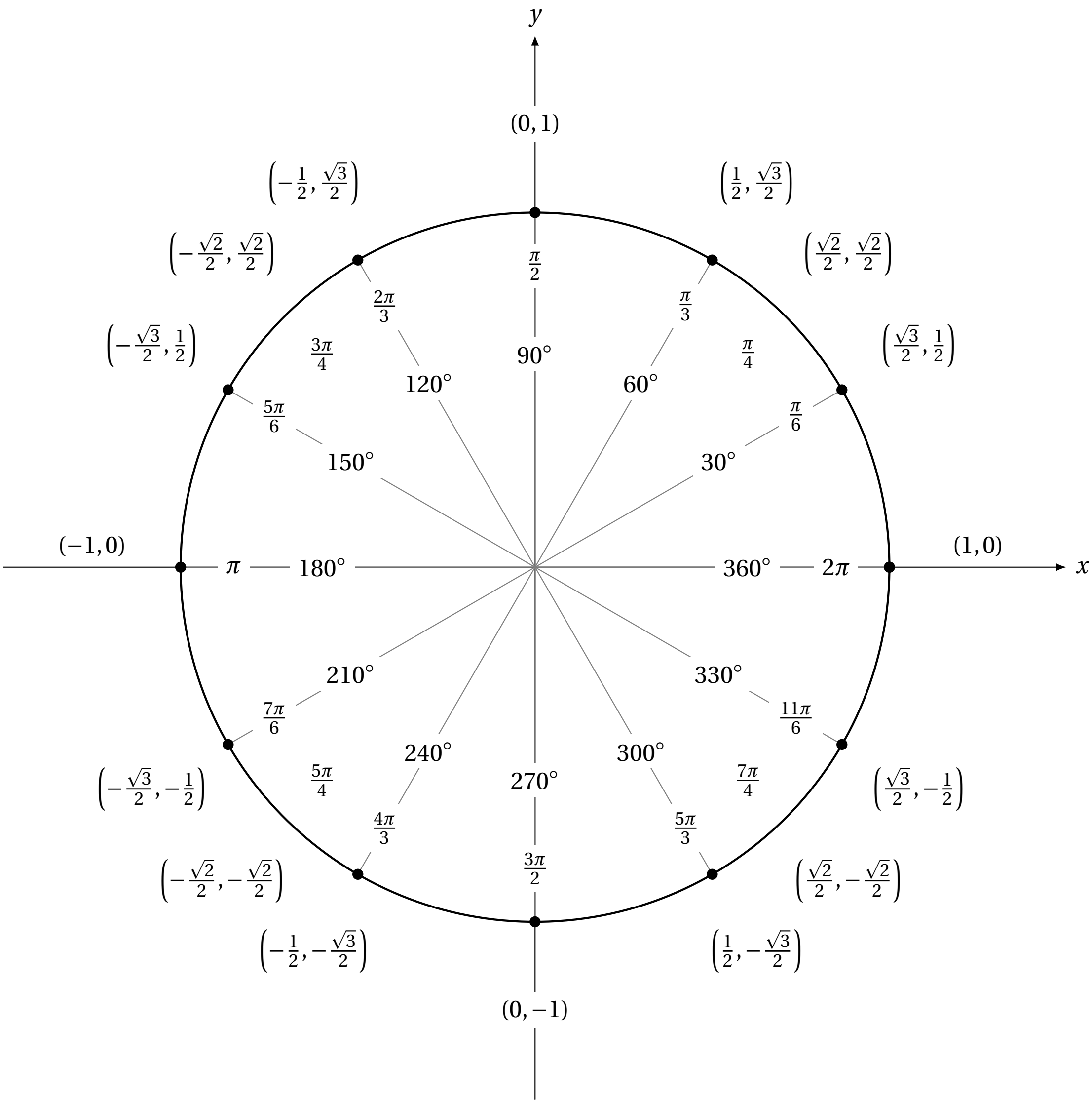

Figure 1.1: The Unit Circle, courtesy of Supreme Ayal, TiKZ Examples [**?**]

1.4 Pythagorean Identities

Further reading: `Chen & Duong ‘‘mtfym01.pdf’’ pp 4-5`, `http://goo.gl/wyojl`

$$\cos^2\theta + \sin^2\theta = 1 \tag{1.13}$$

By dividing 1.13 by $\cos^2\theta$ we get:

$$1 + \tan^2\theta = \sec^2\theta \tag{1.14}$$

By dividing 1.13 by $\sin^2\theta$ we get:

$$1 + \cot^2\theta = \csc^2\theta \tag{1.15}$$

1.5 Properties of Trigonometric Functions

Further reading: `Chen & Duong ‘‘mtfym01.pdf’’ pp 5-10`, `http://goo.gl/wyojl`

Properties of $\sin\theta$

$$\cos(\theta + 2\pi) = \cos\theta \tag{1.16}$$
$$\cos(-\theta) = -\cos\theta \tag{1.17}$$
$$\cos(\theta + \pi) = -\cos\theta \tag{1.18}$$
$$\cos(\pi - \theta) = \cos\theta \tag{1.19}$$

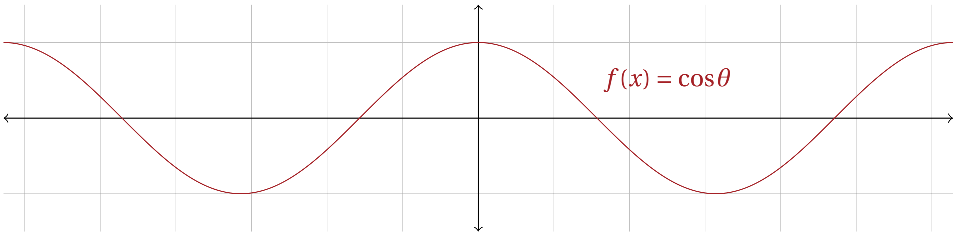

Figure 1.2: $f(x) = \cos\theta$

Properties of $\cos\theta$

$$\sin(\theta + 2\pi) = \sin\theta \tag{1.20}$$
$$\sin(-\theta) = -\sin\theta \tag{1.21}$$
$$\sin(\theta + \pi) = -\sin\theta \tag{1.22}$$
$$\sin(\pi - \theta) = \sin\theta \tag{1.23}$$

Properties of $\tan\theta$

$$\tan(\theta + \pi) = \tan\theta \tag{1.24}$$
$$\tan(-\theta) = -\tan\theta \tag{1.25}$$
$$\tan(\pi - \theta) = -\tan\theta \tag{1.26}$$

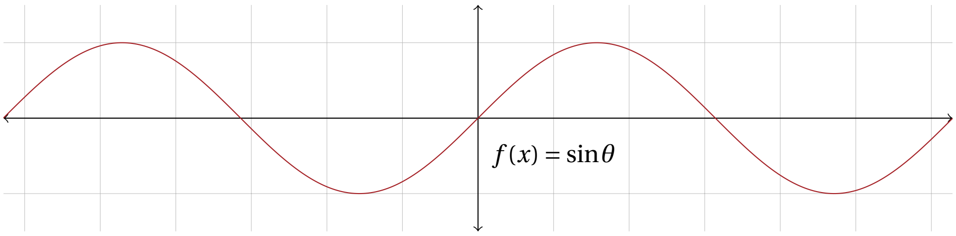

Figure 1.3: $f(x) = \sin\theta$

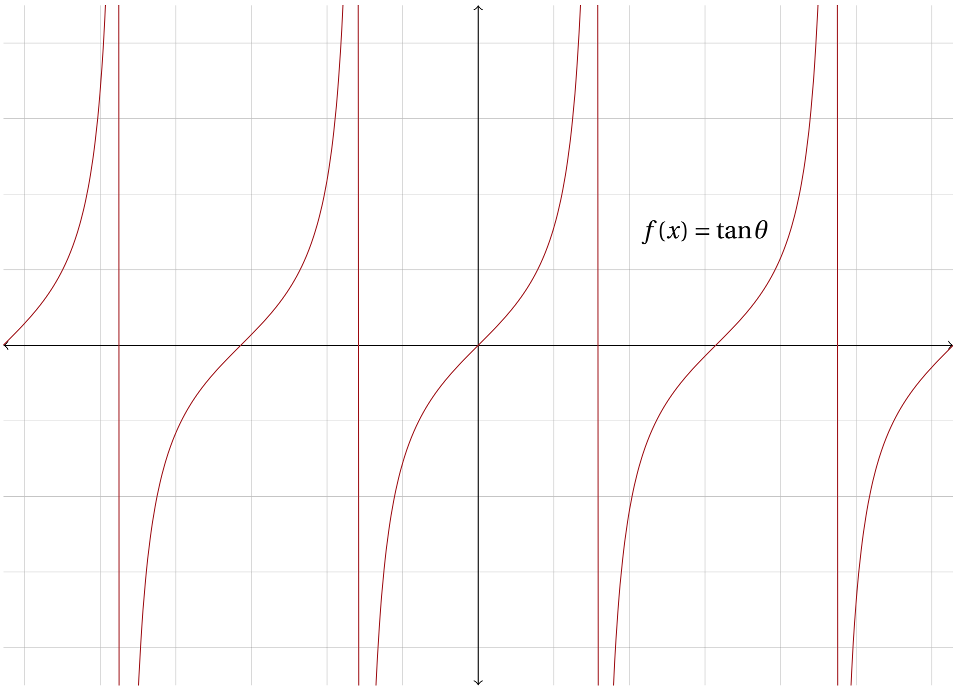

Figure 1.4: $f(x) = \tan\theta$

1.6 Trigonometric Identities

Further reading: `Chen & Duong ‘‘mtfym01.pdf’’ pp 11`, `http://goo.gl/wyojl`

Sine Rule

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} \tag{1.27}$$

Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\cos(A) \tag{1.28}$$

$$b^2 = a^2 + c^2 - 2ac\cos(B) \tag{1.29}$$

$$c^2 = a^2 + b^2 - 2ab\cos(C) \tag{1.30}$$

Angle Sum & Difference Identities

Further reading: `Chen & Duong ‘‘mtfym01.pdf’’ pp 12`, `http://goo.gl/wyojl`

$$\sin(A \pm B) = \sin(A)\cos(\mathrm{B}) \pm \cos(A)\sin(B) \tag{1.31}$$

$$\cos(A \pm B) = \cos(A)\cos(\mathrm{B}) \mp \sin(A)\sin(B) \tag{1.32}$$

Double Angle Identity

Further reading: `Chen & Duong ‘‘mtfym01.pdf’’ pp 12`, `http://goo.gl/wyojl`

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \tag{1.33}$$

$$\begin{aligned} \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) && (1.34)\\ &= 1 - 2\sin^(\theta) && (1.35)\\ &= 2c\cos^2(\theta) - 1 && (1.36) \end{aligned}$$

Half Angle Identity

Further reading: `Chen & Duong ‘‘mtfym01.pdf’’ pp 13-15`, `http://goo.gl/wyojl`

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos(\theta)}{2} \tag{1.37}$$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos(\theta)}{2} \tag{1.38}$$

CHAPTER 2

POLYNOMIALS

Further reading: `Chen & Duong ‘‘mtfym02.pdf’’`, `http://goo.gl/jHCXo`

Big Ideas:

- An expression of finite length constructed from variables and constants using only addition, subtraction, multiplication and non-negative integer powers.
- The highest power of a variable indicates the polynomials *degree*.
- Determining the *roots* of a poloynomial tells us where a polynomial equation intersects the x-axis.
- Polynomials of degree 2 are called *quadratic*.
- Polynomials of degree 3 are called *cubic*.
- A polynomial is often an approximation of other functions.

2.1 Parts of a Polynomial

A polynomial is a *sum of scaled non-negative powers of variable (x)*. The *sum of scaled* coefficient is called a *linear combination* of objects such as ☽, ♡, $\propto$ is an expression of form # + #☽, +# $\propto$.

A polynomial is of the form:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_p x + a_0 \tag{2.1}$$

where $a_n \neq 0$

$$\tag{2.2}$$

2.2 Approximation of sin function

$$
\begin{aligned}
\sin(x) &\approx \\
&\approx x - \frac{x^3}{3} && (2.3)\\
&\approx x - \frac{x^3}{3!} + \frac{x^5}{5!} && (2.4)\\
&\approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} && (2.5)\\
&\approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} && (2.6)\\
& && (2.7)
\end{aligned}
$$

Question: Why not simply show $\sin(x)$? Answer: How would a calculator using the sin button? A calculator uses simple arithmetic to add sums. These polynomials are nothing more than simple sums.

2.3 Polynomial Equations

Further reading: `Chen & Duong ‘‘mtfym02.pdf’’ pp 3-4.`

An equation in the form $ax + b = 0$ where $a, b \in \mathbb{F}$ and $a \neq 0$ is called a *linear equation* or *linear polynomial equation*. It has a degree of 1 (that is, the power of x is 1). We can determine the value of x:

$$ax + b = 0 \tag{2.8}$$

$$ax = -b \tag{2.9}$$

$$x = -\frac{b}{x} \tag{2.10}$$

An equation in the form $ax^2 + bx + c = 0$ where $a, b, c \in \mathbb{F}$ are constants and $a \neq 0$ is called a *quadratic equation*. To solve such an equation we use the *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2.11}$$

Where $b^2 - 4ac$ (called the *determinant*) is greater than or equal to 0. This will yield two solutions:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

However, equation 2.11 is the most compact and easiest to remember.

If the determinant < 0, then it indicates there are no roots exist in the real plane ($\mathbb{R}$) and that a complex plane ($\mathbb{C}$) solution exists.

For polynomials of degree 3 or more there is no general formula for a solution, however division of polynomials allows us to reduce polynomials of a higher degree to a factor and a lesser degree. By iteratively reducing polynomials we can eventually find all roots.

Cubics

Structure:

$$f(x) = ax^3 + bx^2 + cx + d$$

Standard / Basic cubic: $y = x^3$ Consider:

$$\begin{aligned} f(x) &= ax^3 + bx^2 + cx + d \\ f'(x) &= 3ax^2 + 2bx + c \end{aligned} \tag{2.12}$$

a quadratic!

$$\tag{2.13}$$

3 basic shapes of cubics: Like x^3 but with at most 2 “bumps”.
(show 0 bumps, 1 bump and 2 bumps)
Consider:

$$y_1 = x^3 \tag{2.14}$$

and

$$y_2 = x^3 + 1000x^2 + 1000 \tag{2.15}$$

For sufficiently large x the x^3 term dominates. They can be made to look identical.
Consider:

$$y = x^3 + 1000x^2 + 1000 \tag{2.16}$$

$$= x^3\left(1 + \frac{1000}{x} + \frac{1000}{x^3}\right) \tag{2.17}$$

$\frac{1000}{x}$ can be made as close to 0 as you like, similarly for $\frac{1000}{x^3}$.

2.4 Polynomial Division

Further reading: `Chen & Duong ‘‘mtfym02.pdf’’ pp 4-6.`

Remember long division from primary school? By the time we get to university most of us have forgotten. Checkout `http://goo.gl/TvDoI` for a Kahn Academy video on how to do it using just plain numbers. It is well worth the refresher even if you think you can remember how it works.

There is a simple recursive algorithm for polynomial long division, consisting of 3 steps which are repeated until either a divide by zero, or a remainder occurs.[1]

[1] I call this algorithm Jaye's Algorithm because Jaye, a 9 year old at the time, helped me memorise it a few weeks before writing it down.

Jaye's Algorithm:

1. Divide the next term numerator (the part under the long line) by the first term in the denominator. Write the result above the long line.

2. Multiply the result by the denominator and then by -1. Write the result underneath the term(s) operated on, underlining any new terms.

3. Perform a simple addition, carrying anything down below the line.

Rinse and repeat these 3 steps underneath your new long line.

The following example comes from Purple Math, http://goo.gl/GPi0z: Suppose we wish to divide $\frac{x^2-9x-10}{x+1}$:

1. First write the polynomial in the form:

$$\begin{array}{r|rrr}
\cline{2-4}
x+1\big) & x^2 & -9x & -10
\end{array}$$

We will look only at the leading x in the divisor and the leading x^2 in the numerator.

2. We want to divide the x^2 in the numerator by the x in the denominator. $\frac{x^2}{x} = x$ so we write the resulting x above the line.

$$\begin{array}{rrrr}
 & & x & \\ \cline{2-4}
x+1\big) & x^2 & -9x & -10
\end{array}$$

3. Now we must take care of the $+1$ in the denominator. Multiply the result above the line by -1 and multiply again by the denominator:

$$\begin{array}{rrrr}
 & & x & \\ \cline{2-4}
x+1\big) & x^2 & -9x & -10 \\
 & -x^2 & -x & \\ \cline{2-3}
\end{array}$$

4. Perform the subtraction as we would a normal sum carrying anything to the right down:

$$\begin{array}{rrrr}
 & & x & \\ \cline{2-4}
x+1\big) & x^2 & -9x & -10 \\
 & -x^2 & -x & \\ \cline{2-3}
 & & -10x & -10
\end{array}$$

5. Divide $-10x-10$ by x goes -10 times, so write -10 above the line

$$\begin{array}{rrrr}
 & & x & -10 \\ \cline{2-4}
x+1\big) & x^2 & -9x & -10 \\
 & -x^2 & -x & \\ \cline{2-3}
 & & -10x & -10
\end{array}$$

6. Multiply -10 by -1 and then again by $(x+1)$:

$$\begin{array}{rrrr}
 & & x & -10 \\ \cline{2-4}
x+1\big) & x^2 & -9x & -10 \\
 & -x^2 & -x & \\ \cline{2-3}
 & & -10x & -10 \\
 & & 10x & +10 \\ \cline{3-4}
 & & & 0
\end{array}$$

7. **The zero means there is no remainder, and we are done.**

Another example of polynomial long division, taken from the same site as above, to divide $\frac{x^2+9x+14}{x+7}$

1. Write out in the standard form:

$$
\begin{array}{r|r}
\cline{2-2}
x+7) & x^2+9x+14
\end{array}
$$

2. Next divide $\frac{x^2}{x} = x$ and write above the line.

$$
\begin{array}{r|r}
 & x \\ \cline{2-2}
x+7) & x^2+9x+14
\end{array}
$$

3. Multiply $-x$ by $x+7$ and write underneath:

$$
\begin{array}{r|r}
 & x \\ \cline{2-2}
x+7) & x^2+9x+14 \\
 & -x^2-7x \\ \cline{2-2}
\end{array}
$$

4. Perform the addition:

$$
\begin{array}{r|r}
 & x \\ \cline{2-2}
x+7) & x^2+9x+14 \\
 & -x^2-7x \\ \cline{2-2}
 & 2x+14
\end{array}
$$

5. Divide $\frac{2x}{x} = 2$

$$
\begin{array}{r|r}
 & x \quad +2 \\ \cline{2-2}
x+7) & x^2+9x+14 \\
 & -x^2-7x \\ \cline{2-2}
 & 2x+14
\end{array}
$$

6. Multiply -2 by $x+7$ and write underneath, and we can see there is no remainder.

$$
\begin{array}{r|r}
 & x \quad +2 \\ \cline{2-2}
x+7) & x^2+9x+14 \\
 & -x^2-7x \\ \cline{2-2}
 & 2x+14 \\
 & -2x-14 \\ \cline{2-2}
 & 0
\end{array}
$$

Sometimes there are tricky ones where you might have some terms with a coefficient of zero, such as $\frac{2x^3-9x^2+15}{2x-5}$. Here x^1 does not appear as a part of numerator, but we can rewrite it as $\frac{2x^3-9x^2+0x+15}{2x-5}$. With problems such as these, it is easiest to keep your writing neat and in columns. Where you might see $0x$ you can just as easily substitute a bit of whitespace.

$$
\begin{array}{rl}
 & \quad x^2 \quad -2x \quad -5 \\
\cline{2-2}
2x-5) & 2x^3 - 9x^2 \qquad\quad + 15 \\
 & \underline{-2x^3 + 5x^2} \\
 & \qquad -4x^2 \\
 & \qquad \underline{4x^2 - 10x} \\
 & \qquad\qquad -10x + 15 \\
 & \qquad\qquad \underline{10x - 25} \\
 & \qquad\qquad\qquad -10
\end{array}
$$

This example also shows a remainder of -10. The final answer could also be written as one of two ways, either:

1. $x^2 - 2x - 5 - \left(\frac{10}{2x-5}\right)$ – or –
2. $x^2 - 2x - 5$ r 10

Though the first method is probably better as it is less confusing sans the letter "r".

2.5 Polynomial Roots

Further reading: `Chen & Duong ''mtfym02.pdf'' pp 6-8.`

$x = \alpha$ is said to be a root of $f(x)$ if $f(\alpha) = 0$. If a polynomial is in factored form, the roots are trivial, eg
$f(x) = 6(x-3)(x-1)(x+4)$
clearly neat $f(x) = 0$ and the roots: $x = 3, 1, -4$.

The *end behaviour* of f $6x^3$ because the x^3 term dominates the function.

Use the roots to plot a function, simply mark the roots on the x axis at $y = 0$ and put in the "bumps".

Double Roots

Consider:

$$f(x) = (x-1)^2 \cdot (x-4)$$

It has only two roots: $x = 1$ and $x = 4$, however, $x = 1$ is considered to be a "double root" because the "bump" touches the x axis at it's peak.

When $x \to 1$ the graph resembles a quadratic

$$f(x) = (x-1)^2 \cdot (x-4) \qquad\qquad (x-4) \equiv 1-4 = -3 f(x) \approx -3(x-1)^2$$

(only when $x = 1$)

Graphs to Formulae - Factor Theorem

So far we have considered going from a formula to a graph, but if we were to go the other way: Examples we have seen have the property that if $x = \alpha$ is a root, then $x = -\alpha$ is a factor of the polynomial. This result is called the *factor theorem.*[2].

Suppose we had some polynomial with the graph:

-5 is a root: $(x-(-5)) = (x+5)$ is a factor, so $f(x) = (x+5) \times$ `<some polynomial>`.

[2] For a proof see outline in tutorial 2

Such that:

$$f(x) = (x+5)(x-2)(x-6)^2(x-9) \times \texttt{<some polynomial>}$$

<some polynomial> is a constant. An appropriate choice for f(x):

$$a(x+5)(x-2)(x-6)^2(x-9) \tag{2.18}$$

All very well if we are given the roots. What about if we are given:

$$f(x) = 12x^3 - 16x^2 - 7x + 6 \tag{2.19}$$

$$\tag{2.20}$$

We can use the *XYZ theorem* which narrows down possible roots.[3]
Consider a known formula / factorization:

$$(3x-1)\cdot(2x+5) = 6x^2 + 13x - 5 \tag{2.21}$$

Roots are known:

$$\frac{1}{3}, \frac{-5}{2} \tag{2.22}$$

What is the relationship between the roots and coefficients:

$$6, 13, -5 \tag{2.23}$$

The result from tutorial 2 is this:

$$\text{If } \frac{p}{q} \text{ is a rational root in lowest form then} \tag{2.24}$$

$$p \text{ divides constant term} - 5 \tag{2.25}$$

$$q \text{ divides leading coefficient,} 6 \tag{2.26}$$

$$\text{Factors of } 5: 1, 5: \text{Factors of } 6{:}1, 2, 3, 6 \tag{2.27}$$

Let's apply this to:

$$f(x) = 12x^3 - 16x^2 - 7x + 6 \tag{2.28}$$

Possibly rational roots are

$$\pm\frac{\text{factors of } 6}{\text{factors} 12} \tag{2.29}$$

$$\pm\frac{1, 2, 3, 6}{1, 2, 3, 4, 6, 12} \tag{2.30}$$

$$\pm 1, 1/2, 1/3, 1/4, 1/6, 1/12 \tag{2.31}$$

$$\pm 2, 2/3, 3, 3/2, 3/4, 6 \tag{2.32}$$

[3] Proof in tutorial 2

Are any roots? Yes

$$3/2, -2/3, 1/2 \tag{2.33}$$

That is

$$f(x) = 12x^3 - 16x^2 - 7x + 6 \tag{2.34}$$
$$= (2x-3)\cdot q(x) \tag{2.35}$$

Now we have to construct q(x):

$$12x^3 - 16x^2 - 7x + 6 \tag{2.36}$$
$$= (2x-3)(6x^2 \ldots) \tag{2.37}$$
$$= (2x-3)(6x^2 \ldots -2) \tag{2.38}$$

We want -16x^2

$$. \tag{2.39}$$

We have $-18x^2$

$$. \tag{2.40}$$

What do we need to to make up the shortfall? $2x^2$

$$= (2x-3)(6x^2 + x - 2) \tag{2.41}$$
$$\tag{2.42}$$

Since $12x^3 - 16x^2 - 7x + 6 = (2x-3)(6x^2 + x - 2)$, all roots can be found using our standard methods for quadratics[4].

$f(x) = (2x-3)(2x-1)(3x+2)$

2.6 Fundamental Theorem of Algebra

Further reading: `Chen & Duong ‘‘mtfym02.pdf’’ pp 7.`

2.7 Roots of Real Polynomials

Further reading: `Chen & Duong ‘‘mtfym02.pdf’’ pp 7-8.`

2.8 More Polynomial Roots

Further reading: `Chen & Duong ‘‘mtfym02.pdf’’ pp 8.`

2.9 Rational Functions

Further reading: `Chen & Duong ‘‘mtfym02.pdf’’ pp 8-11.`

[4] eg completing the square, quadratic formula, etc…

2.10 Greatest Common Divisor

Further reading: `Chen & Duong ''mtfym02.pdf'' pp 12-13.`
Further reading: `Kenneth H ROSEN ''Discrete Mathematics and Its Applications 6e'', Chapter Primes and Greatest Common Divisors, pp 210-227.`

Definition: Let a and b be integers no both zero. The largest integer d such that $d|a$[a] and $d|b$ is called the greatest common divisor of a and b, denoted as $\gcd(a,b)$.

[a] read as d divides a

There are many ways to determine the gcd of two numbers, including a products of primes and Euclid's algorithm. Euclid's algorithm is by far the most efficient and is discussed in the next subsection.

Euclid's Algorithm

Further reading: `Chen & Duong ''mtfym02.pdf'' pp 12-13.`
Further reading: `Rosen ''Discrete Mathematics and Its Applications'', 6e, pp 227-229`

Euclid's algorithm is recursive, and consists of 2 steps:

Euclid's algorithm:

1. Divide the larger of the two numbers by the smaller of the two numbers, and add the remainder onto the end.
2. Find the gcd of the remainder and the smaller of the two original numbers.

In summary:

$$\text{Let } a = bq + r \quad | \quad a, b, q, r \in \mathbb{Z} \quad \text{Then } \gcd(a,b) = \gcd(b,r) \tag{2.43}$$

The following example comes from ROSEN p 229:

$$662 = 414 \cdot 1 + 248 \tag{2.44}$$
$$414 = 248 \cdot 1 + 166 \tag{2.45}$$
$$248 = 166 \cdot 1 + 82 \tag{2.46}$$
$$166 = 82 \cdot 2 + 2 \tag{2.47}$$
$$82 = 2 \cdot 41 \tag{2.48}$$

2 is the last non-zero remainder, therefore gcd(414,662) = 2.

CHAPTER 3

Complex Numbers

Further reading: Chen & Duong ''em09-cn.pdf'', http://goo.gl/C5rfq
Further reading: Chen & Duong ''fyc01.pdf'', http://goo.gl/e0E7y
Further reading: Chris Cooper ''LINALG03 Complex Numbers'', http://goo.gl/N85u0

Big Ideas:

- Complex numbers are created to solve what would otherwise be unsolveable equations because we cannot square any number in $\mathbb{R}$ to get a negative number.
- Complex numbers are made of two parts, a real part and an imaginary part (i).
- i exhibits ring-like behaviour when raised to powers.
- A complex number resembles the form:

$$z = x + y\mathrm{i} \quad | \; x, y \in \mathbb{R} \tag{3.1}$$

  and the set of complex numbers, $\mathbb{C}$ is defined as:

$$\mathbb{C} = a + ib \quad | \; a, b \in \mathbb{R}, \; i^2 = -1 \tag{3.2}$$

- Rules exist for adding, subtracting, multiplying and dividing complex numbers.
- Complex numbers have a "twin" called a *complex conjugate*, suppose we have:

$$z = (x + y\mathrm{i})$$

  then the conjugate is:

$$\overline{z} = (x - y\mathrm{i})$$

When we square any number in the real number plane, we *always* get a postive number. This begs the question, how do we square root a negative number? Previously, a problem such as

$$i^2 + 1 = 0 \tag{3.3}$$
$$i^2 = -1 \tag{3.4}$$

Would be left with "no real solution" as an answer in a paper. While this is true, we **can in fact find a solution**. It just happens to be outside of the real plane and is defined exactly as:

$$i = \sqrt{-1} \tag{3.5}$$

This gives rise to a very interesting property...What if we were to square i? We already know, the answer from equation 3.4

$$i^2 = -1 \tag{3.6}$$

What about if we cube i?

$$i^3 = i * i^2 \tag{3.7}$$
$$= i * -1 \tag{3.8}$$
$$= -i \tag{3.9}$$

What if we go again?

$$i^4 = i^2 * i^2 \tag{3.10}$$
$$= -1 * -1 \tag{3.11}$$
$$= 1 \tag{3.12}$$

And again?

$$i^5 = i * i^4 \tag{3.13}$$
$$= i * 1 \tag{3.14}$$
$$= i \tag{3.15}$$

This repetitive behaviour is called a *ring* and is cyclic for every 4^{th} power of i. Remember this fact, it will become useful in later trigonometry.

Ring behaviour of powers of i:

$$i^0 = 1 \tag{3.16}$$
$$i^1 = i \tag{3.17}$$
$$i^2 = -1 \tag{3.18}$$
$$i^3 = -i \tag{3.19}$$
$$i^4 = 1 \tag{3.20}$$
$$i^5 = i \tag{3.21}$$
$$i^6 = -1 \tag{3.22}$$
$$i^7 = -i \tag{3.23}$$
$$i^8 = 1 \tag{3.24}$$

Why does this work so neatly? Consider our basic index laws and apply to numbers between 0 and 3 inclusive:

$$i^{a+b} = i^a * i^b \quad | \quad a, b \in [0,3] \tag{3.25}$$

3.1 Arithmetic of Complex Numbers

Further reading: `Chen & Duong ''em09-cn.pdf'' pp 1-2.`
Further reading: `Chen & Duong ''fyc01.pdf'' pp 6-8.`

Addition Rule:

$$(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i} \tag{3.26}$$

Subtraction Rule:

$$(a + b\mathrm{i}) - (c + d\mathrm{i}) = (a - c) + (b - d)\mathrm{i} \tag{3.27}$$

Multiplication Rule:

$$(a + b\mathrm{i}) \cdot (c + d\mathrm{i}) = (ac - bd) + (ad + bc)\mathrm{i} \tag{3.28}$$

Division Rule:

$$\frac{(a + b\mathrm{i})}{(c + d\mathrm{i})} = x + y\mathrm{i} \quad | \quad x, y \in \mathbb{R} \tag{3.29}$$

such that

$$a + b\mathrm{i} = (c + d\mathrm{i})(x + y\mathrm{i}) = (cx - dy)(cy + dx)\mathrm{i} \tag{3.30}$$

It follows that

$$a = cx - dy \tag{3.31}$$

$$b = cy + dx \tag{3.32}$$

With the solution:

$$x = \frac{ac + bd}{c^2 + d^2} \quad \text{and} \quad y = \frac{bc - ad}{c^2 + d^2} \tag{3.33}$$

3.2 Complex Conjugates

Complex numbers have a "twin" called a *complex conjugate*, suppose we have:

$$z = (x + y\mathrm{i})$$

then the conjugate is:

$$\overline{z} = (x - y\mathrm{i})$$

Conjugates exhibit special properties, they are:

$$\overline{z+w} = \overline{z} + \overline{w} \tag{3.34}$$

and

$$\overline{z \cdot w} = \overline{z} \cdot \overline{w} \tag{3.35}$$

Proof of equation 3.34 is as follows:

$$\begin{aligned}
\text{Let: } z &= x + y\mathrm{i} \quad | \quad x, y \in \mathbb{R} && (3.36)\\
\text{Let: } w &= u + v\mathrm{i} \quad | \quad u, v \in \mathbb{R} && (3.37)\\
\overline{z+w} &= \overline{(x+u) + (y+v)\mathrm{i}} && (3.38)\\
&= (x+u) - (y+v)\mathrm{i} && (3.39)\\
&= (x - y\mathrm{i}) + (u - v\mathrm{i}) && (3.40)\\
&= \overline{z} + \overline{w} && (3.41)
\end{aligned}$$

□

Proof of equation 3.35 is as follows:

$$\begin{aligned}
\text{Let: } z &= x + y\mathrm{i} \quad | \quad x, y \in \mathbb{R} && (3.42)\\
\text{Let: } w &= u + v\mathrm{i} \quad | \quad u, v \in \mathbb{R} && (3.43)\\
\overline{z \cdot w} &= \overline{(x + y\mathrm{i})(u + v\mathrm{i})} && (3.44)\\
&= \overline{(xu - yv) + (xv + yu)\mathrm{i}} && (3.45)\\
&= (xu - yv) - (xv + yu)\mathrm{i} && (3.46)\\
&= (x - y\mathrm{i})(u - v\mathrm{i}) && (3.47)\\
&= \overline{z} \cdot \overline{w} && (3.48)
\end{aligned}$$

□

3.3 Polynomials with Complex Coefficients

Every polynomial with complex coefficients has a complex root. This result is called the *Fundamental Theorem of Algebra.*[1]

3.4 Equalities of Complex Numbers

In developing new numbers, a property is lost.
For example $\mathbb{N} \to \mathbb{Z}$, we lose the well ordering property. As we go from $\mathbb{R} \to \mathbb{C}$ we lose the property of equalities (such as $<$ and $>$).[2].

[1] Unfortunately the proof of this is rather complicated.
[2] cf Apostol, Mathematical Analysis

Suppose that < has usual properties and also suppose that $i > 0$. We need to choose one of $i > 0$, or $i < 0$, or $i = 0$. Since $i > 0$:

$$\begin{aligned} i &> 0 && (3.49)\\ i * i &> i * 0 && (3.50)\\ -1 &> 0 \quad \text{problem!} && (3.51)\\ -i &> 0 \quad \text{problem, because} && (3.52)\\ i &< 0 \quad \text{How can i be bigger and less than zero at the same time?} && (3.53) \end{aligned}$$

Recap: How does equality work?

Let

$$z = a + ib \tag{3.54}$$
$$w = c + id \tag{3.55}$$

then

$$z = w \quad \text{iff } a = c \text{and} b = d \tag{3.56}$$

We have don't go outside $\mathbb{C}$, let's consider, $\sqrt{i}$. Is this complex? Consider the question, "What polynomial is this the root of?".

Let

$$x = \sqrt{i} \tag{3.57}$$
$$x^2 = i \tag{3.58}$$
$$x^2 - i = 0 \tag{3.59}$$

such that

$$\sqrt{i} \quad \text{is a root of} \tag{3.60}$$
$$p(x) = x^2 - i \tag{3.61}$$

By the Fundamental Theorem of Algebra, we must be able to find a, b so that a,b $\in \mathbb{R}$

$$\tag{3.62}$$
$$a + ib = \sqrt{i} \tag{3.63}$$
$$\tag{3.64}$$

$$a + ib = \sqrt{i} \tag{3.65}$$
$$(a + ib)(a + ib) = i \tag{3.66}$$
$$a^2 + 2iab + i^2(b^2) = i \tag{3.67}$$
$$a^2 + 2iab - b^2 = i \tag{3.68}$$

i has great practical importance. Without i we can't do much engineering or physics without i. Mainly electronics engineers already have i reserved for the unit of current, so j is used in place.[3]

We have already seen how to draw i.

What of 2+i?

Identify a + ib with (a,b). $a + ib \in \mathbb{C} | (a, b) \in \mathbb{R}^2$

$a + ib \leftrightarrow (a, b)$

3.5 Scaling complex numbers

Approach from a numerical example:

Let

$$z = 3 + i \tag{3.69}$$

Consider

$$2z = 6 + 2i \tag{3.70}$$
$$3z = 9 + 3i \tag{3.71}$$
$$\frac{1}{2}z = \frac{3}{2} + \frac{1}{2}i \tag{3.72}$$
$$-z = -3 - i \tag{3.73}$$
$$\tag{3.74}$$

A very important concept in complex numbers is the distance away from 0. As for reals, we denote this by absolute value. This is also called the *modulus* or *magnitude*.

$$|z| = \text{is the distance from } z \text{ to } 0$$

[3]Because engineers are even less creative than mathematicians.

CHAPTER 4

Complex Plane

Further reading: `Chen & Duong ‘‘em09-cn.pdf’’ pp 3-4.`
Further reading: `Chen & Duong ‘‘fyc01.pdf’’ pp 8-11.`

Big Ideas:

- Complex numbers take the form $(a + b\mathrm{i})$.
- We can represent x and y ordinates in the real plane ($\mathbb{R}^2$) in the form $(x + y\mathrm{i})$.
- x is called the real part (represented by $\mathfrak{Re}[z]$)
- y is called the imaginary part (represented by $\mathfrak{Im}[z]$)
- Plotting these on an x, y axis is called an *Argand diagram*.
- Polar coordinates use two variables, (r,θ) to describe a point.
- θ is an angle measured in radians
- r is a length.
- Conversion between polar coordinates and Cartesian coordinates is done using Pythagoras theorem or trigonmetry.

4.1 Argand Diagrams

Complex numbers consist of two parts, such as $z = (x + y\mathrm{i})$. Suppose we were to plot both x and y ordinates on an x, y plane as seen in figure 4.1.

Following our addition rules in section 3.1 (page 23), we can add, two complex numbers as seen in figure 4.2.

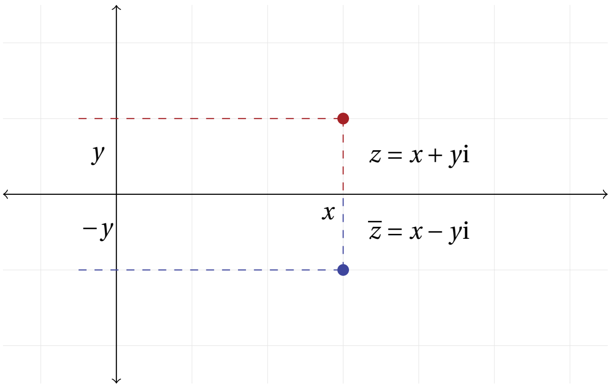

Figure 4.1: Argand Diagram, z, and complex conjugate $\overline{z}$.

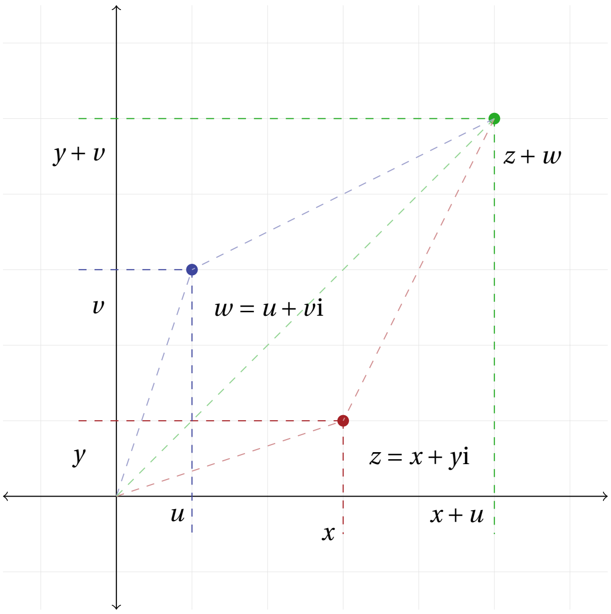

Figure 4.2: Addition of two complex numbers in an Argand Diagram

The working for 4.2 is as follows:

$$\text{Let: } z = x + y\mathrm{i} \tag{4.1}$$
$$\text{Let: } w = u + v\mathrm{i} \tag{4.2}$$

By the addition rule:

$$(x + y\mathrm{i}) + (u + v\mathrm{i}) = x + u + y\mathrm{i} + v\mathrm{i} \tag{4.3}$$
$$= (x + u) + (y + v)\mathrm{i} \tag{4.4}$$

4.2 Polar Coordinates

1. Polar coordinates use two variables, (r, θ) to describe a point.
2. r is a length.
3. θ is an angle

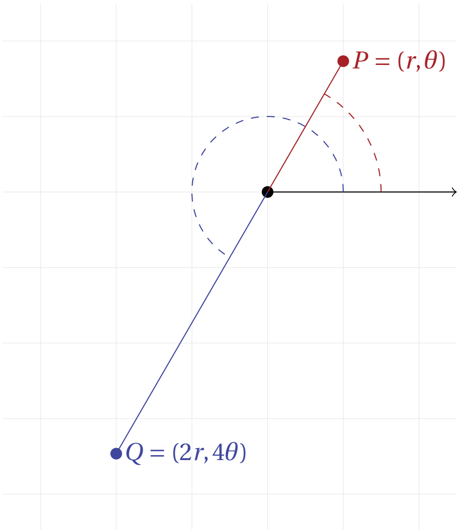

Figure 4.3: Two polar coordinates. Note how r, the radius, is specified first followed by the angle.

Converting between polar coordinates and cartesian coordinates is a matter of applying some trigonometry but is summarised as follows:

$$x = r\cos\theta \tag{4.5}$$

$$y = r\sin\theta \tag{4.6}$$

Iff $r \geq 0$ and $-\pi \leq \theta < \pi$ we can also convert to Cartesian coordinates as follows:

$$r = \sqrt{x^2 + y^2} \tag{4.7}$$

$$\theta = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \arctan\left(\frac{x}{y}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ 0 & \text{if } x = 0 \text{ and } y = 0 \end{cases} \tag{4.8}$$

4.3 Modulus

Consider diagram 4.4:

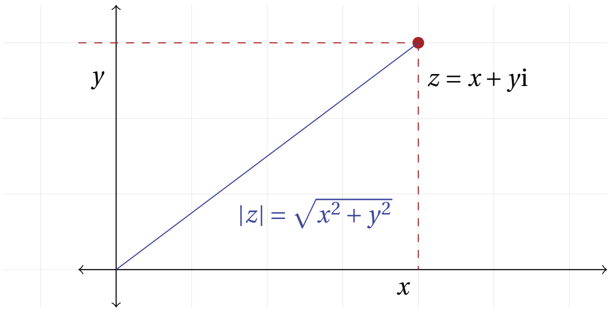

Figure 4.4: The modulus, $|z|$, is given by Pythagoras' theorem

4.4 Euler's Formula

Euler's Formula: Euler was this dude who had a formula:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{4.9}$$

In the context of the unit circle, projected onto an Argand diagram, it starts to form an important relationship shown in figure 4.5. This relationship is further extended in chapter 5 on de Moivre's Theorem (page 31).

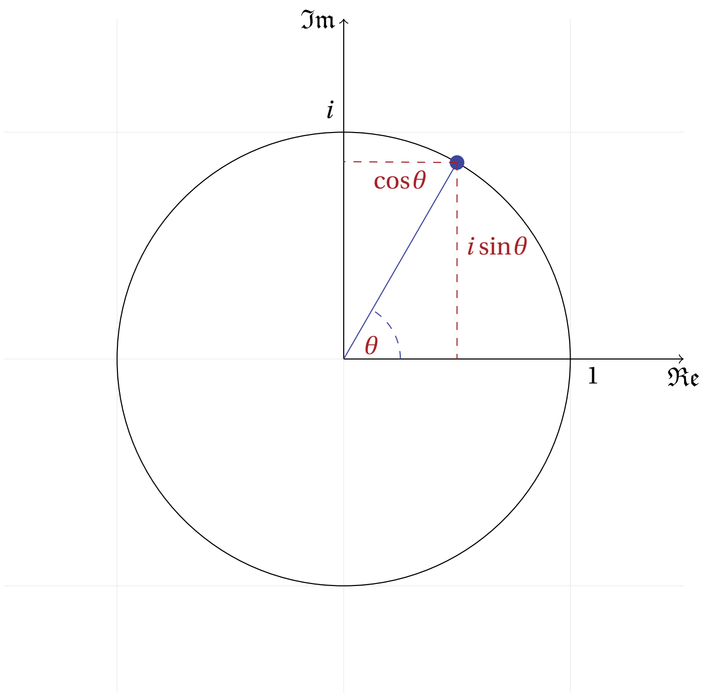

Figure 4.5: Euler's Formula plotted on an Argand diagram

CHAPTER 5

de Moivre's Theorem & Roots of Complex Numbers

Further reading: `Chen & Duong ''fyc01.pdf''`, `http://goo.gl/e0E7y`

Big Ideas:

- de Moivre was this dude who had a theorem named after him, though this specific theorem was never actually mentioned in his work.
- Derived from Euler's Formula (see section 4.4 p 30).

5.1 de Moivre's Theorem

Further reading: `Chen & Duong ''fyc01.pdf'' pp 12.`

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{5.1}$$

apply an index law:

$$\left(e^{i\theta}\right)^n = e^{in\theta} \quad \text{iff} \quad n \in \mathbb{Z} \tag{5.2}$$

then by Euler's formula:

$$e^{i(n\theta)} = \cos\theta + i\sin(n\theta) \tag{5.3}$$

Proof that this follows for all integers can be done through a proof by induction. The following proof comes courtesy of Wikipedia, `http://goo.gl/K8nRR`.

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \quad | \quad n \in \mathbb{Z} \tag{5.4}$$

For $n > 0$, we proceed by principle of mathematical induction. S(1) is true. For $S(k)$ is true where $k \in \mathbb{N}$. That is we assume:

$$(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta) \tag{5.5}$$

Consider $S(k+1)$:

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos(\theta) + i\sin(\theta))^k(\cos\theta + i\sin\theta) &(5.6)\\ &= [\cos(k\theta) + i\sin(k\theta)]\,(\cos\theta + i\sin\theta) &(5.7)\\ &= \cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta + i\,[\cos(k\theta)\sin(\theta) + \sin(k\theta)\cos\theta] &(5.8)\\ &= \cos[(k+1)\theta] + i\sin[(k+1)\theta] &(5.9)\end{aligned}$$

Because $S(k)$ implies $S(k+1)$, by POMI, it follows that the result is true for all $\mathbb{N}$. For negative integers, consider an index of $-n$ for $\mathbb{N}$:

$$\begin{aligned}(\cos\theta + i\sin\theta)^{-n} &= \left[(\cos\theta + i\sin\theta)^n\right]^{-1} &(5.10)\\ &= [\cos(n\theta) + i\sin(n\theta)]^{-1} &(5.11)\\ &= \cos(-n\theta) + i\sin(-n\theta) &(5.12)\end{aligned}$$

5.2 Finding Roots of Complex Numbers

Further reading: `Chen & Duong ‘‘fyc01.pdf’’ pp 12-14.`

5.3 Analytic Geometry

Further reading: `Chen & Duong ‘‘fyc01.pdf’’ pp 14-15.`

CHAPTER 6

LINEAR EQUATIONS

Further reading: `Chen & Duong ‘‘la01.pdf’’`, `http://goo.gl/N7S2s`

Big Ideas:

- Linear equations are often called a *system of linear equations* when equations are linked together.
- A system of equations can be represented in a *matrix*.

6.1 System of Linear Equations as Matrices

Further reading: `Chen & Duong ‘‘la01.pdf’’ pp 1-3.`

6.2 Elementary Row Operations

Further reading: `Chen & Duong ‘‘la01.pdf’’ pp 3-6.`

6.3 Row Echelon Form

Further reading: `Chen & Duong ‘‘la01.pdf’’ pp 6-10.`

6.4 Reduced Row Echelon Form

Further reading: `Chen & Duong ‘‘la01.pdf’’ pp 11-12.`

6.5 Solving a System of Linear Equations

Further reading: `Chen & Duong ‘‘la01.pdf’’ pp 12-15.`

6.6 Homogeneous Systems

Further reading: `Chen & Duong ‘‘la01.pdf’’ pp 15-16.`

CHAPTER 7

Matrices & Linear Equations

Big Ideas:

-

CHAPTER 8

APPLICATION OF LINEAR EQUATIONS

Big Ideas:

-

8.1 Application to Network Flow

Further reading: `Chen & Duong ‘‘la01.pdf’’ pp 16-18.`

8.2 Application to Electric Circuits

Further reading: `Chen & Duong ‘‘la01.pdf’’ pp 18-21.`

8.3 Application to Economics

Further reading: `Chen & Duong ‘‘la01.pdf’’ pp 21-22.`

8.4 Application to Chemistry

Further reading: `Chen & Duong ‘‘la01.pdf’’ pp 22-23.`

8.5 Application to Mechanics

Further reading: `Chen & Duong ‘‘la01.pdf’’ pp 23-25.`

CHAPTER 9

Arithmetic of Matrices

Big Ideas:

-

CHAPTER 10

Determinant of a Matrix

Big Ideas:

-

Chapter 11

Vectors

Required reading: Complex numbers.

Big Ideas:

-

Chapter 12

Vectors & Geometry

Big Ideas:

-

CHAPTER 13

COMBINATORICS & BINOMIAL THEOREM

Further reading: Chris Cooper ''ALG03 The Binomial Theorem'', http://goo.gl/bqTQH

Big Ideas:

-

CHAPTER 14

REAL NUMBERS

Big Ideas:

-

CHAPTER 15

INDUCTION

Big Ideas:

- POMI establishes that a given statement is true for all $\mathbb{N}$.
- Start by proving (or assuming if given) that the first statement is true. This statement is called the **basis** or **base case**
- Next prove that **any one** statement then so is the next statement. This is known as the **inductive step**.

15.1 Set Notation

- $\mathbb{N}$ Natural numbers, $\{1,2,3,4,\ldots\}$
- $\mathbb{Z}$ Integers (from the German word for "Number"), $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$
- $\mathbb{Q}$ Quotients (fractions), $\left\{\frac{a}{b}|a\in\mathbb{Z},b\in\mathbb{Z},b\neq 0\right\}$, rationals.
- $\mathbb{R}$ Real numbers $\mathbb{Q}\cup\{\text{irrationals}\}$.

If a number, A is a set of numbers and the number x is a member of the set A, then we write $x\in A$. If x is not a member of A then we write $x\notin A$.

15.2 Intervals

- $[a,b] = x\in\mathbb{R}|a\le x\le b)$ closed interval
- $(a,b) = x\in\mathbb{R}|a<x<b)$ open interval
- $(a,b] = x\in\mathbb{R}|a<x\le b)$ partly open interval

Supose that A and B are two sets, we say that A is a subset of B if $x \in A$ implies that $x \in B$ and is denoted by $A \subset B$. If A is a subset of B then we also say that B contains the set A.

$$A \subset B \text{if no element of A can be found which is not in B}$$

15.3 Well Ordering

Less than or equal ($\leq$) to provide an ordering on $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$. Given an order on a set (any set, not just those 4), we call the set *well-ordered* if every non-empty subset has a least element.

Example

Are any of the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ well-ordered? Ask ourselves, is $\mathbb{Z}$ well ordered? No because $\mathbb{Z} \subset \mathbb{Z}$

Is $\mathbb{N}$ well ordered? Yes, because every non-empty subset has a least element. This is an axiomatic property of $\mathbb{N}$[1].

15.4 Propositions

A proposition is a statement that is either true ($\top$) xor false ($\perp$).

15.5 The Principle of Mathematical Induction

Mathematical induction is a technique for proving if propositions index by natural numbers are true for all natural numbers. For example, consider the propositions P_n, where P_n is the satement "3^n - 1" disivible by 2.

Supposed that $\{P_n\}$ is a collation of propositions such at

1. P_b is true for some fixed $b \in \mathbb{N}$.

2. P_n is true, then P_{n+1} is true.

3. The P_n is true for all $n \in \mathbb{N}$.

To use mathematical induction to prove sets of propositions like these, you should follow the following four step algorithm:

1. Prove that a proposition is $\top$ for some fixed natural number (usually P_0 or P_1). called the *base case.*

2. Write down your assumption about the truth of the proposition P_k for some fixed arbitrary natural number k.

3. Write down the proposition P_{k+1}. Using the assumption that P_k is true, prove that P_{k+1} is true.

4. State that you have proved the propositions using POMI [2].

[1] That is, we define $\mathbb{N}$ to start at 1 (or maybe 0).

[2] Principle of Mathematical Induction.

Example

Show that the following formula is true for all natural numbers, n:

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Firstly we must prove it for the base case

$$\text{Let } n = 1 \text{ such that}$$
$$P_1 = 1^2 = \frac{1 \cdot (1+1)(2 \cdot 1 + 1)}{6} \tag{15.1}$$

Assume P_k is true:

$$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6} \tag{15.2}$$

Substitute P_{k+1}

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \ldots + (k+1)^2 &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} && (15.3)\\
&= \frac{(k+1)(k+2)(2(k+1)+1)}{6} && (15.4)\\
&= \frac{(k+1)(k+2)(2k+3)}{6} && (15.5)\\
&= \frac{(k^2+3k+2)(2k+3)}{6} && (15.6)\\
&= \frac{(2k^3+3k^2+6k^2+9k+4k+6)}{6} && (15.7)\\
&= \frac{(2k^3+9k^2+13k)}{6} + 1 && (15.8)\\
&= \frac{k^3}{3} + \frac{3k^2}{2} + \frac{13k}{6} + 1 && (15.9)\\
1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 &= && (15.10)\\
1^2 + 2^2 + 3^2 + \ldots + k^2 + k^2 + 2k + 1 &= && (15.11)\\
1^2 + 2^2 + 3^2 + \ldots + 2k^2 + 2k + 1 &= && (15.12)
\end{aligned}$$

subtract 1 from both sides:

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \ldots + 2k^2 + 2k &= \frac{k^3}{3} + \frac{3k^2}{2} + \frac{13k}{6} && (15.13)\\
1^2 + 2^2 + 3^2 + \ldots + 2k(k+1) &= && (15.14)
\end{aligned}$$

now i give up

$$\tag{15.15}$$

Proposition P_n:

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

Assume P_k:

$$1^2+2^2+3^2+\ldots+k^2=\frac{k(k+1)(2k+1)}{6}$$

Assume P_{k+1}

$$1^2+2^2+3^2+\ldots+(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \tag{15.16}$$

Add $(k+1)^2$ to both sides

$$\begin{aligned}1^2+2^2+3^2+\ldots+(k+1)^2+(k+1)^2&=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}+(k+1)^2 &(15.17)\\&=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}+\frac{6(k+1)^2}{6} &(15.18)\\& &(15.19)\end{aligned}$$

Foo

$$1^2+2^2+\ldots+(k+1)^2=\frac{(k+1)((k+1)(+1))(2k+3)}{6} \tag{15.20}$$

So since P_1 is true and if P_k is true, then P_{k+1} is true so by the POMI P_n is true for $n\in\mathbb{N}$.

Example

Show that 3^n-1 is divisible by 2 for all natural numbers n.

Step 1: prove the base case

$$\begin{aligned}P_1\text{says } 3^1-1 \text{ is divisble by 2 which is true} &&(15.21)\\P_2\text{says } 3^2-1 \text{ is divisible by 2 which is true} &&(15.22)\end{aligned}$$

Assume P_k

$$\begin{aligned}3^k-1\text{is divisible by 2} &&(15.23)\\\leftrightarrow 3^k-1=2s|s\in\mathbb{Z} &&(15.24)\end{aligned}$$

Step 3 P_{k+1} says:

$$\begin{aligned}3^{k+1}-1 \text{ is divisible by 2} &&(15.25)\\\leftrightarrow 3^{k+1}-1=2s'|s'\in\mathbb{Z} &&(15.26)\end{aligned}$$

Example

Show $2^k > k^3$ for natural numbers where $n \geq 10$.

Prove base case

Assume P_k

$$2^k > k^3 \tag{15.27}$$

P_{k+1} says

$$2^{k+1} > (k+1)^3 \tag{15.28}$$

Assume

$$\begin{align}
2^k &> k^3 \tag{15.29}\\
2^{k+1} &> 2k^3 \tag{15.30}\\
&> k^3 + k^3 \tag{15.31}\\
&\geq k^3 + 10k^2 \tag{15.32}\\
&\geq k^3 + 3k^2 + 3k^2 + 4k^2 \tag{15.33}\\
&> k^3 + 3k^2 + 3k + 1 \tag{15.34}\\
2^{k+1} &> (k+1)^3 \tag{15.35}
\end{align}$$

Which is P_{k+1}

15.6 Some remarks on terminology

Inductive arguments (ordinary English) - An argument where we use evidence from specific statements to conclude that something is probably. Example: All life forms we know of depend on water to exist. Therefore, if we discover a new life form, it will probably depend on water to exist. **Deductive Aruments (ordinary English)** - An argument where we use general statements to conclude that something is certain. Example: All champagne is made in France. Dom Perignon is a champagne. Therefore Dom Perignon is made in France. **Mathematical Induction** is badly named, because in ordinary English it is really a form of deduction not induction. This is because we use priniciples to guarantee the truth of the given propositions.

Example

In the parlour game Nim, there are two players and two piles of matches. At each turn a player removes some (non-zero) number of matches from one of the piles. The player who removes the last match wins.

Prove that if the two piles contain the second number of matches at the start of the game, then the second player can always win.

Winning strategy

Player 2 takes from the opposite pile the same number of pens that player removed.

Induction on the number of pens, n. This means that player 2 *must* win.

CHAPTER 16

FUNCTIONS

Big Ideas:

- Many naturally occurring quantities that vary with time can be modelled using *functions*.

16.1 Example

The volume of water stored in Canberra's dams is a variable that depends on time. For any particular time t, we could denot the corresponding volume by $f(t)$. In this situation

- f is a function,
- if t is an input for the function, f, then $f(t)$ is the corresponding output.

Notation:

$$t \xrightarrow{f} f(t)$$

16.2 Specifying a function

A function f has three parts to its definition:

- The rule which explains how to get outputs form inputs,
- the specification of the function's *domain* (ie set of inputs)
- the specification of the function's *co-domain* (ie the set of where the outputs lie)

The second part of the definition is often not explicitly stated, but it is very important. In MATH135, the third part is almost always $\mathbb{R}$.

16.3 Functions as maps - a pictorial view

Function acts on *every* element of the domain. Example: $f(x) = x^2$ For example, $f(-4) = f(4) = 16$. This is OK, and is called a surjective function.

16.4 Example and terminology

A function f with domain $[0,\infty)$ is given by the rule:

$$f(x) = x^2 \quad \forall x \in [0,\infty]$$

.

- $[0,\infty)$ is the set of input values, the domain of f
- Write `Dom(`$[0,\infty)$`)`.
- $\forall$ means for *all*.
- We say that f is defined on $[0,\infty)$
- We say that f is real-valued because the outputs are real numbers (the codomain is $\mathbb{R}$).

  Important note: Technically, it would be incorrect to write "Consider the function $f(x) = x^2$ although this is a common abuse of notation.
- f is the function; $f(x)$ is the value of the function at x.
- The domain of f is not specified.

$$f : [0,\infty) \to \mathbb{R}$$

. If we want to include the rule as well:

$$f : [0,\infty) \to \mathbb{R} : x \mapsto x^2$$

If we see $f : A \to B$, it means f is a rule that acts on every element of A and to each element of A assigns a definite element of B.

16.5 The Maximal Domain

16.6 Examples

- $f(x) = \frac{1}{x-5}$, the maximal domain: $\mathbb{R}$ 5.
- $g(x) = \sqrt{x+2}$, $x \geq -2$

16.7 The range of a function

Suppose that f is a function. The *range* of f, denoted by $\texttt{Range}(f)$, is defined by: $\texttt{Range}(f) = \{f(x) \in B : x \in \texttt{Dom}(f)\}$.

Note that $\texttt{Range}(f)$ **is the set of all output values for** f.

Suppose that f is defined by

$$\texttt{Dom}(f) = \mathbb{R}, f(x) = \sin(x), \forall x \in \mathbb{R}$$

Choose codomain $\mathbb{R}$

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto \sin(x)$$

Range? Is 3 an output? No. Rnage is [-1,1]

16.8 Absolute Values

Definition, if $x \in \mathbb{R}$ then $|x|$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

16.9 More piecewise definitions

Piecewise functions are very important in applications. Absolute value is an example of a piecewise function. Example:

$$f(x) = \begin{cases} 2 & \text{if } x < 0 \\ x^2 - 1 & \text{if } 0 \leq x < 3 \\ 8 - 2x \text{if } x \geq 3 \end{cases}$$

16.10 Combining functions

If two functions f and g have the same domain A, we can construct new functions $f + g$, $f - g$ and $f \cdot g$ each with the domain A. These are defined pointwise by te following formuale:

$$(f + g)(x) := f(x) + g(x) \tag{16.1}$$

$$(f - g)(x) := f(x) - g(x) \tag{16.2}$$

$$(f \cdot g)(x) := f(x) \cdot g(x) \tag{16.3}$$

We can also define $\frac{f}{g}$ by the formula:

$$\frac{f}{g}(x) := \frac{f(x)}{g(x)} \qquad \text{provided that } g(x) \neq 0 \tag{16.4}$$

The domain of $\frac{f}{g}$ is $\{x \in A : g(x) \neq 0\}$.

16.11 Composition

If Range(f) is a subset of Dom(g) then we can define a new function, $g \circ f$ on A by the rule:

CHAPTER 17

LIMITS

Big Ideas:

-

CHAPTER 18

CONTINUITY

Big Ideas:

-

Chapter 19

Differentiability

Big Ideas:

-

CHAPTER 20

STATIONARY POINTS

Big Ideas:

-

CHAPTER 21

CURVE SKETCHING

Big Ideas:

-

CHAPTER 22

Applications of the Derivative

Big Ideas:

-

CHAPTER 23

ANTIDERIVATIVES

Big Ideas:

-

CHAPTER 24

TECHNIQUES OF INTEGRATION

Big Ideas:

-

CHAPTER 25

Separable Differentiable Equations

Big Ideas:

-

CHAPTER 26

AREA UNDER A CURVE

Big Ideas:

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CHAPTER 27

VOLUME OF SOLIDS OF REVOLUTION

Big Ideas:

-

CHAPTER 28

More Applications of Integration

Big Ideas:

-

CHAPTER 29

GLOSSARY

Symbol	Name	LaTeX
$\mathbb{A}$	Blackboard A	\mathbb{A}
$\mathbb{B}$	Blackboard B	\mathbb{B}
$\mathbb{C}$	Blackboard C	\mathbb{C}
$\mathbb{D}$	Blackboard D	\mathbb{D}
$\mathbb{E}$	Blackboard E	\mathbb{E}
$\mathbb{F}$	Blackboard F	\mathbb{F}
$\mathbb{G}$	Blackboard G	\mathbb{G}
$\mathbb{H}$	Blackboard H	\mathbb{H}
$\mathbb{I}$	Blackboard I	\mathbb{I}
$\mathbb{J}$	Blackboard J	\mathbb{J}
$\mathbb{K}$	Blackboard K	\mathbb{K}
$\mathbb{L}$	Blackboard L	\mathbb{L}
$\mathbb{M}$	Blackboard M	\mathbb{M}
$\mathbb{N}$	Blackboard N	\mathbb{N}
$\mathbb{O}$	Blackboard O	\mathbb{O}
$\mathbb{P}$	Blackboard P	\mathbb{P}
$\mathbb{Q}$	Blackboard Q	\mathbb{Q}
$\mathbb{R}$	Blackboard R	\mathbb{R}
$\mathbb{S}$	Blackboard S	\mathbb{S}
$\mathbb{T}$	Blackboard T	\mathbb{T}
$\mathbb{U}$	Blackboard U	\mathbb{U}
$\mathbb{V}$	Blackboard V	\mathbb{V}
$\mathbb{W}$	Blackboard W	\mathbb{W}
$\mathbb{X}$	Blackboard X	\mathbb{X}
$\mathbb{Y}$	Blackboard Y	\mathbb{Y}
$\mathbb{Z}$	Blackboard Z	\mathbb{Z}

Table 29.1: Blackboard letters. Blackboard notation must be inside math mode.

Greek Uppercase	Lowercase	Name	LaTeX(Upper)	LaTeX(Lower)
A	α	Alpha	A	\alpha
B	β	Beta	B	\beta
Γ	γ	Gamma	\Gamma	\gamma
Δ	δ	Delta	\Delta	\delta
E	ϵ	Epsilon	E	\epsilon
	ε	Epsilon Variant		\varepsilon
Z	ζ	Zeta	Z	\zeta
H	η	Eta	H	\eta
Θ	θ	Theta	\Theta	\theta
	ϑ	Theta Variant		\vartheta
I	ι	Iota	I	\iota
K	κ	Kappa	K	\kappa
	κ	Kappa Variant		\varkappa
Λ	λ	Lambda	\Lambda	\lambda
M	μ	Mu	M	\mu
N	ν	Nu	N	\nu
Ξ	ξ	Xi	\Xi	\xi
O	o	Omicron	O	o
Π	π	Pi	\Pi	\pi
	ϖ	Pi Variant		\varpi
P	ρ	Rho	P	\rho
	ϱ	Rho Variant		\varrho
Σ	σ	Sigma	\Sigma	\sigma
	ς	Sigma Variant		\varsigma
T	τ	Tau	T	\tau
Υ	υ	Upsilon	\Upsilon	\upsilon
Φ	ϕ	Phi	\Phi	\phi
	φ	Phi Variant		\varphi
X	χ	Chi	X	\chi
Ψ	ψ	Psi	\Psi	\psi
Ω	ω	Omega	\Omega	\omega

Table 29.2: The Greek alphabet.

Symbol	My LaTeX	LaTeX	Description
$\Re$	`$\Re$`	`$\operatorname{\mathfrak{Re}}$`	Real part of a number
$\Im$	`$\Im$`	`$\operatorname{\mathfrak{Im}}$`	Imaginary part of a number

Table 29.3: Real and Imaginary symbols