On the problem proposed by paper *Singularity Computation for Rational Parametric Surfaces Using Moving Planes. ACM Transactions on Graphics, 2022.*

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1. Introduction

The note is to answer the following problem propsed by the paper *Xiaohong Jia, Falai Chen and Shanshan Yao; Singularity Computation for Rational Parametric Surfaces Using Moving Planes. ACM Transactions on Graphics, 2022.*

Problem 1. *Let K be a field of characteristic 0, C a rational surface in $\mathbb{A}^3_K$. Let $\varphi : \mathbb{A}^2_K \dashrightarrow C \subseteq \mathbb{A}^3_K, (u,v) \mapsto (\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$ be a birational map, where $p_i(u,v)$, $q_j(u,v)$ are polynomials, $i,j = 1,2,3$. If $P \in C$ is an isolated singularity, and $\varphi^{-1}(P)$ is nonempty, then $\varphi^{-1}(P)$ has infinitely many points.*

Remark 1.1.
1) *The condition of isolated singularity is necessary. For example,*
$$\begin{array}{rccc} \varphi : & \mathbb{A}^2 & \to & \mathbb{A}^3 \\ & (u,v) & \mapsto & (u^2, u^3, v) \end{array}$$
The singularities of the nonnormal surface $C = Im\ \varphi$, which is $y^2 - x^3 = 0 \in \mathbb{A}^3$ are the line $x = y = 0$. Let $P = (0,0,v)$ be one singularity of C, then $\varphi^{-1}(P)$ is a single point $(0,v)$.

   Isolated singularity for a hypersurface C implies that C is locally normal around P by Serre S_2 normality criterion (Prop 2.6). So the above won't happen in our problem.
2) *The condition φ birational is necessary. For example, let $Y \subseteq \mathbb{A}^3$ be a cone over a conic, defined by $uv - w^2$. Let $\varphi : \mathbb{A}^2 \to Y \subseteq \mathbb{A}^3, (x,y) \mapsto (x^2, y^2, xy)$ be a finite morphism of degree 2. Notice that $P = (0,0,0) \in Y$ is the unique singularity of Y, and $\varphi^{-1}(P)$ consist of one point $(0,0)$.*

2. Preliminaries

Definition 2.1 ([Ha77] page 84)**.** *A morphism $f : X \to Y$ is a finite morphism if there exists a covering of Y by open affine subsets $V_i = Spec\ B_i$, such that for each i, $f^{-1}(V_i)$ is affine, equal to $Spec\ A_i$, where A_i is a B_i-algebra which is a finitely generated B_i-module.*

Proposition 2.2.
1) *A morphism $f : X \to Y$ is finite if and only if for every open affine subset $V = Spec\ B$ of Y, $f^{-1}(V)$ is affine, equal to $Spec\ A$, where A is a finite B-module. (*[Ha77] *Exercise 3.4, page 91)*
2) *A morphism $f : X \to Y$ is quasi-finite if for every point $y \in Y$, $f^{-1}(y)$ is a finite set. (*[Ha77] *Exercise 3.5 (a), page 91)*
3) *A quasi-finite projective morphism is a finite morphism. (*[Ha77] *Exercise 11.2, page 280)*

Definition 2.3 ([Ha77] page 103)**.** *A morphism $f : X \to Y$ of schemes is projective if it factors into a closed immersion of $i : X \to \mathbb{P}^n_Y$ for some n, followed by the projection $\mathbb{P}^n_Y \to Y$.*

Proposition 2.4 ([Liu02] Cor 3.32 page 108)**.** *The following properties are true.*

a) *Closed immersion are projective morphisms.*
b) *The composition of two projective morphisms is a projective morphism.*
c) *Projective morphisms are stable under base change.*
d) *Let $X \to S$, $Y \to S$ be projective morphisms, then $X \times_S Y \to S$ is projective.*
e) *If the composition of $X \to Y$, $Y \to Z$ is projective, and if $Y \to Z$ is separated, then $X \to Y$ is projective.*

For our problem, to show any morphism of projective varieties is a projective morphism, we apply Proposition 2.4 e) to letting $Z = \text{Spec } k$, X, Y projective varieties over k. Moreover, if $f : X \to Y$ is a projective morphism, and $V \subseteq Y$ is an open subset, then $f : f^{-1}(V) \to V$ is projective by Proposition 2.4 c), since $X \times_Y V = f^{-1}(V)$ and $Y \times_Y V = V$.

Proposition 2.5 ([Liu02] Prop 2.7 page 89)**.** *Let X be an algebraic variety over k, and let K/k be an algebraic extension. The following properties are true.*
a) We have $\dim X_K = \dim X$, where $X_K = X \times_k K$.

By Proposition 2.5, dimension is stable under the base change of algebraic field extension. So we can assume the base field K is an algebraically closed field of characteristic 0 in the problem. If $\dim \varphi^{-1}(P) = 1$, then $\varphi^{-1}(P)$ contains infinitely many points.

Proposition 2.6 (Serre S_2 normality criterion, [Ha77] page 186)**.** *Let Y be a locally complete intersection subscheme of a nonsingular variety X over k. Then Y is normal if and only if it is regular in codimension 1.*

Any codimension 1 algebraic set in $\mathbb{A}^3$ or $\mathbb{P}^3$ is a hypersurface (see [Ha77] Chap 1, Prop. 1.13, and Chap 1 Exercise 2.8), which is apparently complete intersection. Isolated singularity implies that the surface C is regular in codimension 1 near the isolated singularity P. Apply Prop 2.6, we know that the surface C is normal in an open neighborhood of P.

Theorem 2.7 (Zariski Main Theorem, [Ha77] page 280)**.** *Let $f : X \to Y$ be a birational peojective morphism of noetherian integral schemes, and assume that Y is normal. Then for very $y \in Y$, $f^{-1}(y)$ is connected.*

Lemma 2.8 ([Ha77] Lemma 5.1, page 410)**.** *If $T : X \to Y$ is a birational transformation of projective varieties, and if X is normal, then the fundamental points of T form a closed subset of codimension ≥ 2.*

The above lemma shows that for a rational map of surfaces $T : X \to Y$, if X is normal, then the points where T is NOT defined form a codimension ≥ 2 subset of X. In other words, T is not defined for only finitely many points.

Theorem 2.9 (Elimination of points of indeterminacy, [Liu02] Theorem 2.7 page 396, or [Ha77] Example 7.17.3)**.** *Let $\varphi : X \dashrightarrow \mathbb{P}^n$ be a rational map and defined over an open subset $U \subseteq X$, then there is a blow up $\pi : \tilde{X} \to X$ to make the following diagram commutative*

$$\begin{tikzcd} \tilde{X} \arrow[d, "\pi"'] \arrow[drr, "\tilde{\varphi}"] & & \\ X & U \arrow[l] \arrow[r, "\varphi"] & \mathbb{P}^n \end{tikzcd}$$

By the construction of elimination of points of indeterminacy, we keep blowing up the points in $X \setminus U$, which is a set of finitely many points for a smooth projective surface X by Lemma 2.8. After finitely many blow ups, we shall get a morphism $\tilde{\varphi}$ extending the rational map φ. Notice that $\pi : \pi^{-1}(U) \to U$ is an isomorphism since we blow up points outside U.

By Hironaka's work, *Resolution of singularties* works over a field of characteristic 0. Moreover, Resolution of singularties for surfaces works over any field. Hence, elimination for indeterminacies works for our case for any field.

3. Proof

Most of the results we quoted are for projective varieties. So we reduce the problem to the projective varieties and over an algebraically closed field of characteristic 0.

Let $\bar{\varphi} : \mathbb{P}^2_K \dashrightarrow \mathbb{P}^3_K$ be the projective closure of $\varphi : \mathbb{A}^2_K \dashrightarrow \mathbb{A}^3_K$. Furthermore, the line of infinity $\mathbb{P}^2 \setminus \mathbb{A}^2$ is mapped into the plane of inifinity $\mathbb{P}^3 \setminus \mathbb{A}^3$. Then $\varphi^{-1}(P)$ is finite if and only if $\bar{\varphi}^{-1}(P)$ is finite. Let $U \subseteq \mathbb{P}^2_K$ be the largest subset where $\bar{\varphi}$ is defined. Then by assumption $\bar{\varphi}^{-1}(P)$, $\bar{\varphi}$ is defined on $\bar{\varphi}^{-1}(P)$. By elimination of indeterminancy, there exists a blow up $\pi : \tilde{X} \to \mathbb{P}^2$ and a morphism $\tilde{\varphi} : \tilde{X} \to C$, such that $\tilde{\varphi}$ extends $\bar{\varphi}$, where C is the projective closure of the affine rational surface.

Since $P \in C$ is an isolated singularity, there exists an open neighborhood V of P, such that V is normal by Proposition 2.6.

The morphism $\tilde{\varphi}$ is a birational projective (proper) morphism over K. By Zariski Main Theorem, $\tilde{\varphi}^{-1}(P)$ is connected. To prove the result, we only need to show that $\dim \tilde{\varphi}^{-1}(P) > 0$. Indeed, $\tilde{\varphi}^{-1}(P) = \pi^{-1}\bar{\varphi}^{-1}(P)$. If $\dim \tilde{\varphi}^{-1}(P) > 0$ and $\dim \bar{\varphi}^{-1}(P) = 0$, then all curves in $\tilde{\varphi}^{-1}(P)$ are contracted by π. Since π is an isomorphism on $\pi^{-1}(U)$ and $\bar{\varphi}^{-1}(P) \cap U \neq \emptyset$, it is a contradiction!

Therefore, to prove $\dim \tilde{\varphi}^{-1}(P) > 0$, we can assume that K is an algebraically closed field by Proposition 2.5.

By Proposition 2.4 c), we can assume, by shrinking V if necessary, that $\varphi_V : \varphi^{-1}(V) \to V$ is a projective birational morphism. By Zariski's Main Theorem, $\varphi^{-1}(P)$ is connected. If $\dim \varphi^{-1}(P) = 0$, then $\varphi|_{\varphi^{-1}(V)}$ is a quasi-finite projective morphism. Hence it is a finite morphism by Proposition 2.2 c). So there is an affine neighborhood W =Spec A of P, such that $\varphi^{-1}W =$ Spec A is affine and A is a finite B-module. Since φ is birational and W is normal, A and B have same frational field. Moreover B is integrally closed. So $A = B$ and thus $\varphi|_{\varphi^{-1}(W)}$ is an isomorphism. It is a contradiction since $P \in W$ is a singularity but $\tilde{X}$ is smooth.

References

[Ha77] R. Hartshorne; *Algebraic Geometry*, GTM52.

[Liu02] Q. Liu; *Algebraic geometry and arithmetic curves.*