SIR model with Diffusion

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1 Motivation

Since the outbreak of Covid in 2019, China implements strict epidemic prevention policies, including reducing international flights and forcing localized lockdown for local outbreaks. On Dec.7th, China lifts the epidemic prevention guidelines and calls an end to localized lockdowns[1]. Since then, Beijing has witnessed a surge of infections caused by imported cases. Given the strong transmissibility of the new virus, it won't take long for the disease to spread over China. Thus it is important to give an estimate for the arrival time of disease for each city. In fact, some researchers has already given a rough estimate of the spreading speed by tracking the "search index" of the word "fever" on localized network. What we want to do is to come up with a more scientific method to model the spread speed of this outbreak.

2 Overview

SIR model has been widely used in modeling the local development of infectious disease since it was proposed by Kermack and McKendrick in 1927[2]. Based on SIR model, I choose to consider adding diffusion terms into the SIR model in order to investigate how infectious diseases spread geographically. In paper "Geographic and temporal development of plagues", JV Noble used SIR with diffusion to model the geographic spread of Black Death in middle ages, which gives a theoretical estimation close to the real data in spreading speed[3]. In this project, I will go through this paper and make theoretical estimation for the spreading speed of the recent outbreak of COVID-19 in Beijing after Chinese Government relaxes the epidemic prevention guidelines. Then I will deduce the estimated time needed for the disease to spread from Beijing to other cities based on the distance between them.

3 Rationale behind the choice of Model

Why is it proper to model the geographic spread of COVID in China with SIR+diffusion? Since China has been stick to "zero-policy" until recently, currently the majority of reported infections are caused by imported cases. Given that there is a strict regulation on entry administration during the epidemic, only airports of some major cities accept international flights. Moreover, plus the relaxation of the movement of persons within the national territory, the spread of COVID in China is just like adding a drop of ink to a glass of water, during which the spread of color is caused by the random walk of particles. Based on these reasons, I choose to use the classic SIR model + diffusion.

4 Main Formula

In the classic SIR model, the unit of S,I,R are all people. While in SIR plus diffusion, the unit of S,I,R are now susceptible/infectious/removed people per area, which is density. Noble creates SIR with diffusion model with terms as follows:

$$\frac{\partial I}{\partial t} = KIS - \mu I + D(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})I$$
$$\frac{\partial S}{\partial t} = -KIS + D(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})S$$

where K is the transmissibility coefficient, μ is the recovery rate, D is the diffusion constant[3].

5 Physical Idea of each term

$\frac{\partial I}{\partial t}$ is the rate of change of number of infectives per area. Noble models it to be the rate of transitions from susceptible per area (+ KIS), minus the rate of recovery per area ($-\mu I$), add the rate of diffusive outflow of infectives per area($+D(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})I$).[3] In short, for any place other than the source, the infection density is also related to random walk of population from the source, which is translated into math language as the diffusion term.

Similarly, $\frac{\partial S}{\partial t}$ is the rate of change of number of susceptible people per area. Noble models it to be minus the rate of transitions from susceptible per area (- KIS) plus rate of diffusive outflow of the susceptible per area($+D(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})S$).[3] In short, for any place other than the source, the susceptible density is also related to random walk of population from the source, which is translated into math language as the diffusion term.

6 Dimension Analysis

- $[I] = [S] = [\frac{people}{m^2 * s}]$
- $[\frac{\partial I}{\partial t}] = [\frac{\partial S}{\partial t}] = [\frac{people}{m^2 * s}]$

- $[(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})I]=[(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})S]=[\frac{people}{m^4}]$

Thus, we deduce that

- $[K]=[\frac{m^2}{people*s}]$, which represents the area of infection swept out by an infective per time.
- $[\mu]=[\frac{1}{s}]$
- $[D]=[\frac{m^2}{s}]$

7 Dimensionless

We assume that $S(\overrightarrow{x},0)=$ U, which is a uniform distribution. Based on the dimension analysis, let

- $I^*=\frac{I}{U}$
- $S^*=\frac{S}{U}$
- $x^*=(\frac{D}{KU})^{-\frac{1}{2}}\cdot x$
- $t^*=(KU)t$

Thus,

$$
\begin{aligned}
x^2 &= \frac{D}{KU}x^{*2}\\
\frac{\partial^2 I}{\partial x^2} &= \frac{\partial^2(I^*U)}{\partial(\frac{D}{KU}\cdot x^{*2})}\\
&= \frac{KU^2}{D}\cdot\frac{\partial^2 I^*}{\partial(x^{*2}}\\
\frac{\partial I^*}{\partial t^*} &= \frac{\partial I^*}{\partial I}\cdot\frac{\partial I}{\partial t}\cdot\frac{\partial t}{\partial t^*}\\
&= \frac{1}{U}\cdot\frac{\partial I}{\partial t}\cdot\frac{1}{KU}\\
&= \frac{KIS-\mu I+D(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})I}{KU^2}\\
&= I^*S^*-\frac{\mu}{KU}I^*+\frac{D}{KU^2}\cdot\frac{KU^2}{D}\cdot[\frac{\partial^2 I^*}{\partial x^{*2}}+\frac{\partial^2 I^*}{\partial y^{*2}}]\\
&=: I^*S^*-\lambda I^*+[\frac{\partial^2}{\partial x^{*2}}+\frac{\partial^2}{\partial y^{*2}}]I^*
\end{aligned}
$$

where $\lambda=\frac{\mu}{KU}$ Similarly, we can get that

$$\frac{\partial S^*}{\partial t^*}=I^*S^*+[\frac{\partial^2}{\partial x^{*2}}+\frac{\partial^2}{\partial y^{*2}}]S^*$$

8 Solve PDE as ODE

We consider that the solution of $S^*(\overrightarrow{x},t), I^*(\overrightarrow{x},t)$ be given as a function of a new variable: $x - vt$. That is to say, let $I^*(\overrightarrow{x},t) = I(x-vt)$ and $S^*(\overrightarrow{x},t) = S(x-vt)$ Then, we obtain two ODEs:

$$-vI' = IS - \lambda I + I''$$
$$-vS' = -IS + S''$$

Noble is interested in for what value λ does the wave-like solution exist under the condition that

- $I(-\infty) = I(\infty) = 0$
- $0 \leq S(-\infty) < S(\infty) = 1$
- $I, S, v > 0$

Adding these two equations, after some integration, he deduced that we need $\lambda = \frac{\mu}{KU} < 1$.[2]

9 Parameters

Using the dimension analysis, we deduce that $[\sqrt{KUD}] = [\frac{m}{s}]$, which suggests that the spreading speed of the disease is proportional to $\sqrt{KUD}$. Now we list our assumptions to estimate the spreading speed of COVID in Beijing.

1. $v = \sqrt{KUD}$
2. $U = 1000\frac{people}{km^2}$,which is the average population density of China.
3. Take the same as Nobel's assumption on page 727 that $K = 0.4\frac{mile^2}{yr}$, which equals $1\frac{km^2}{yr}$.
4. For the diffusive coefficient D, I assume it to be proportional to the population density U. For Nobel's estimation for the middle age Europe, he takes $U_E = 50\frac{people}{mile^2} = 20\frac{people}{km^2}$, and deduces $D_E = 10^4\frac{mile^2}{yr} = 25600\frac{km^2}{yr}$ (page 727). Thus for the case of Beijing, where $U = 1000\frac{people}{km^2}$, we take $D = 1280000\frac{km^2}{yr}$.

10 Estimation for spreading speed

$v = \sqrt{KUD} = \sqrt{1\frac{km^2}{yr} \cdot 1000\frac{people}{km^2} \cdot 1280000\frac{km^2}{yr}} = 35777\frac{km}{yr} = 98\frac{km}{day}$

Since the epidemic peak in Beijing is arrived on Dec.13th, given that the distance between Shanghai and Beijing is 1200 km, the epidemic peak in Shanghai is arrived after $\frac{1200}{98} = 12$ days, which is approximately Dec.25th. For Xinjiang, which is the most distant city from Beijing, the distance between them is approximately 3000 km, then its epidemic peak arrives on Jan.13th. Plus the time needed for recovery, the epidemic shall go to a relative peaceful stage before February.

11 Is zero-policy sustainable?

Noble stated that once $\lambda = \frac{\mu}{KU} < 1$, the wave-like solution will exist. Since it takes approximately 10 days to recover from COVID, we take the recovery rate $\mu = 36(yr^{-1})$. Thus,$\lambda = \frac{\mu}{KU} = \frac{36}{1*1000} << 1$. Thus, the zero-policy is not sustainable and people should learn to co-exist with the virus.

12 Citations

[1]https://www.nature.com/articles/d41586-022-04382-0
[2]Kermack, William Ogilvy, and Anderson G. McKendrick. "A contribution to the mathematical theory of epidemics." Proceedings of the royal society of london. Series A, Containing papers of a mathematical and physical character 115.772 (1927): 700-721.
[3]Noble, Julia V. "Geographic and temporal development of plagues." Nature 250.5469 (1974): 726-729.