CS 395 Homework 4

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Grade: ____________________

PROBLEMS

1.

The basic definition for Θ-notation states:

$$\Theta(g(n)) = \{f(n) : \exists c_1, c_2, n_0; 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \forall n \geq n_0\} \quad \Theta\text{-notation Definition} \tag{1}$$
$$O(g(n)) = \{f(n) : f(n) \leq g(n)\} \quad O(n)\text{-notation Definition} \tag{2}$$
$$\Omega(g(n)) = \{f(n) : f(n) \geq g(n)\} \quad \Omega\text{-notation Definition} \tag{3}$$

Note that Theorem 3.1 states: For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n)$ and $f(n) = \Omega(g(n))$. Let's assume this is true:

$$\Omega(g(n)) = f(n) = O(g(n)) \tag{4}$$
$$f(n) \geq g(n) = f(n) = f(n) \leq g(n) \quad \text{Substitution} \tag{5}$$
$$f(n) = g(n) = f(n) = f(n) = g(n) \quad \text{Equivalence} \tag{6}$$

Step 6 above matches the conditions for Θ-notation described in step 1. But what about for when $f(n) \neq g(n)$? Assume an equation where this is true:

$$\Omega(g(n)) < f(n) < O(g(n)) \tag{7}$$
$$f(n) \geq g(n) < f(n) < f(n) \leq g(n) \quad \text{Substitution} \tag{8}$$
$$f(n) = g(n) \neq f(n) \neq f(n) = g(n) \quad \text{Contradiction} \tag{9}$$

Due to the contradiction assumption in step 7, the statement is simplified to a statement in step 9 that is a contradiction.

Therefore, by Proof by Contradiction, Theorem 3.1 is valid.

2

Problem two states that $f(n)$ and $g(n)$ are monotonically increasing functions. By definition, this means:

$$m \leq n \Rightarrow f(m) \leq f(n).$$

Figure 1: Definition for Monotonicity, Introduction to Algorithms Chapter 3.2

This means for both $f(n)$ and $g(n)$:

$$m \leq n \Rightarrow f(m) \leq f(n). \tag{10}$$
$$m \leq n \Rightarrow g(m) \leq g(n). \tag{11}$$

First, consider the function $h(n) = f(n) + g(n)$. Since both $f()$ and $g()$ are both monotonic, $f(m)+g(m) \leq f(n)+g(n)$. Therefore, $m \leq n \Rightarrow h(m) \leq h(n)$, and the function $h(n) = f(n)+g(n)$ by defintion is monotonic.

Second, consider the function $h(n) = f(g(n))$. Once can simply represent $g(m)$ as some variable m_1, and $g(n)$ as some variable n_1. Therefore, $h(n) = f(n_1)$ and $h(m) = f(m_1)$. Since $g()$ is monotonic, then $m_1 \leq n_1$.

Since both $f()$ and $g()$ are both monotonic, and $m_1 \leq n_1$, $f(m_1) \leq f(n_1)$. Therefore, $m \leq n \Rightarrow h(m) \leq h(n)$, and the function $h(n) = f(g(n))$ by defintion is monotonic.

Third, consider the equation $h(n) = f(n) * g(n) \exists f(n) + g(n) \geq 0$. If $f(n)$ and $g(n)$ are monotonic, then $f(m) \leq f(n)$ and $g(m) \leq g(n)$. Therefore, $f(m) * g(m) \leq f(n) * g(n)$. Substituting $(h()$, $h(m) \leq h(n)$, and by definition, is monotonic.

3

Proof that $a^{log_b(c)} = c^{log_b(a)}$:

$$a^{\frac{log_a(c)}{log_a(b)}} = c^{\frac{log_c(a)}{log_c(b)}} \quad \text{Changing bases rule} \tag{12}$$

$$a^{log_a(c)*log_b(a)} = c^{log_c(a)*log_b(c)} \quad \text{Logarithm property} \tag{13}$$

$$c * a^{log_b(a)} = a * c^{log_b(c)} \quad \text{Logarithm property} \tag{14}$$

This is as far as I got, did not find properties to continue solving this.

4

Consider the equation $x^2 = x + 1$. The equation can be reduced as follows:

$$0 = x^2 - x - 1 \tag{15}$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4*(1)*(-1)}}{2*1} \quad \text{Quadratic Formula} \tag{16}$$

$$x = \frac{1 \pm \sqrt{5}}{2} \tag{17}$$

$$x = \{\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\} \tag{18}$$

The definitions for ϕ and $\wedge\phi$ are:

$$\phi = \frac{1+\sqrt{5}}{2} \tag{19}$$

$$\wedge\phi = \frac{1-\sqrt{5}}{2} \tag{20}$$

The set $\{\phi, \wedge\phi\} = \{\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\}$ = the solution set for $x^2 = x+1$ as shown in step 19. Therefore, ϕ and $\wedge\phi$ satisfy $x^2 = x + 1$.

5

For this problem, it will be shown by Proof by Induction that the ith Fibonacci umber satisfies the equality:

$$F_i = \frac{\phi^i - \wedge\phi^i}{\sqrt{5}} \tag{21}$$

Step 1: Base step

Let $i = 2$. It can then be shown:

$$F_2 = \frac{\phi^2 - \wedge\phi^2}{\sqrt{5}} \tag{22}$$

$$F_2 = \frac{(\frac{1+\sqrt{5}}{2})^2 - (\frac{1-\sqrt{5}}{2})^2}{\sqrt{5}} \quad \text{Definitions from steps 20 and 21} \tag{23}$$

$$F_2 = \frac{(\frac{1+\sqrt{5}}{2})^2 - (\frac{1-\sqrt{5}}{2})^2}{\sqrt{5}} \tag{24}$$

$$F_2 = \frac{(\frac{1+\sqrt{5}}{2} * \frac{1+\sqrt{5}}{2}) - (\frac{1-\sqrt{5}}{2} * \frac{1-\sqrt{5}}{2})}{\sqrt{5}} \tag{25}$$

$$F_2 = \frac{\frac{1+2\sqrt{5}+5}{4} - \frac{1-2\sqrt{5}+5}{4}}{\sqrt{5}} \tag{26}$$

$$F_2 = \frac{\frac{1+2\sqrt{5}+5-1+2\sqrt{5}-5}{4}}{\sqrt{5}} \tag{27}$$

$$F_2 = \frac{\frac{4\sqrt{5}}{4}}{\sqrt{5}} \tag{28}$$

$$F_2 = \frac{\sqrt{5}}{\sqrt{5}} \tag{29}$$

$$F_2 = 1 \tag{30}$$

Note that the **Fibonacci numbers** are defined by the following recurrence:

$$F_0 = 0, \tag{31}$$

$$F_1 = 1, \tag{32}$$

$$F_i = F_{i-1} + F_{i-2} \tag{33}$$

Per the definition of the **Fibonacci numbers**, the 2nd number in the sequence is 1, so the Base Step holds.

Induction Assumption

For the Induction Assumption, it is assumed that $F_i = \frac{\phi^i - \wedge\phi^i}{\sqrt{5}}$ for i.

Induction Step

For the Induction Step, it will be shown that $F_k = \frac{\phi^k - \wedge\phi^k}{\sqrt{5}}$ for $k = i + 1$.

First, we will show that $F_k = \frac{\phi^k - \wedge\phi^k}{\sqrt{5}}$ for $k = i$.

$$F_i = F_{i+1} + F_i \quad \text{Definition from step 34} \tag{34}$$

$$= \frac{\phi^{i-1} - \wedge\phi^{i-1}}{\sqrt{5}} + \frac{\phi^{i-2} - \wedge\phi^{i-2}}{\sqrt{5}} \quad \text{The Induction Assumption} \tag{35}$$

$$= \frac{\phi^{i-1} - \wedge\phi^{i-1} + \phi^{i-2} - \wedge\phi^{i-2}}{\sqrt{5}} \tag{36}$$

$$= \frac{\phi^{i-2}(\phi+1)-\phi^{i-2}(\wedge\phi^{i}+)}{\sqrt{5}} \tag{37}$$

$$= \frac{\phi^{i}-\wedge\phi^{i}}{\sqrt{5}} \tag{38}$$

$$\tag{39}$$

Therefore, as shown above leading to step 39, the Inductions Step holds, showing proof by Induction.