CS 395 Homework 7

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Grade: ____________________

PROBLEMS

1.

Considering the equation $T(n) = T(\frac{n}{2}) + n^2$. The cost for the recursive tree is:

$$
\begin{aligned}
T(n) \quad &= T(\tfrac{n}{2}) + n^2 \\
&= (T(\tfrac{n}{4}) + n^2) + n^2 \\
&= ((T(\tfrac{n}{8}) + n^2) + n^2) + n^2 \\
&= n^2 + (\tfrac{n}{2})^2 + (\tfrac{n}{4})^2 + (\tfrac{n}{8})^2 + \ldots + 1 \\
&= n^2 \tfrac{1-(\frac{1}{4})^{log2^2}}{1-\frac{1}{4}} \\
&\leq \tfrac{4}{3} n^2 \\
&= O(n^2)
\end{aligned}
$$

Using the **substitution method**, $T(n) = O(n^2)$ will be the upper bound. It will be shown that $T(n) < dn^2$ for some constant $d > 0$.

$$
\begin{aligned}
T(n) \quad &= T(\tfrac{n}{2}) + n^2 \leq d(\tfrac{n}{2})^2 + n^2 \\
&= d(\tfrac{n^2}{4}) + n^2 \leq dn^2 \\
&= (\tfrac{d}{4} + 1)n^2 \leq dn^2
\end{aligned}
$$

For $n > 0$ and $d = 4$, the condition holds. Therefore, proof by substitution method.

2.

Considering the equation $T(n) = T(n-1) + 1$. The cost for the recursive tree is:

$$
\begin{aligned}
T(n) \quad &= T(n-1) + 1 \\
&\leq T(n-1) + T(n-1) \\
&\leq 2(n-1) + 2^2(n-2) + \ldots + 2^{n-1} \\
&\leq -n + 2 + 2^2 + \dot{+} 2^{n-1} + 2^n \\
&\leq -n + 2\tfrac{1-2^n}{1-2} \\
&\leq 2 * 2^n \\
&= O(2^n)
\end{aligned}
$$

Using the **substitution method**, $T(n) = O(2^n)$ will be the upper bound. It will be shown that $T(n) < c2^n$ for some constant $c > 0$.

$$
\begin{aligned}
T(n) \quad &\leq c2^{n-1} + c2^{\frac{n}{2}} \\
&\leq c2^{n-1} + c2^{n-2} \\
&= c2^n
\end{aligned}
$$

For $n \geq 4$ and $c = 1$, the condition holds. Therefore, proof by substitution method.

3.

The recursive tree for $T(n) = T(\frac{n}{3}) + T(\frac{2n}{3}) + cn$ is as follows:

$$
\begin{array}{c|cccc}
 & & Cn & & \\
lg_{\frac{3}{2}} n & C(\frac{n}{3}) & & C(\frac{2n}{3}) & \\
 & C(n/9)\ C(2n/9) & & C(2n/9)\ C(4n/9) &
\end{array}
$$

The greatest cost in the tree is $T(n) = (\frac{2/3}{)}^2 n$, where $T(n) = (\frac{2}{3})^k n$. When $k = log_{\frac{3}{2}} n$, the depth of the tree is $log_{\frac{3}{2}} n$. $T(n)$ is at least $cnlog_3 n = \Omega(nlogn)$ where every node has 2 children.

4.

(a) $T(n) = 2T(\frac{n}{4}) + 1$

Consider $a = 2, b = 4, f(n) = 1$. Using the **master theorem**:

$$\begin{aligned} n^{log_b a} &= n^{log_4 2} \\ &= \Theta(n^{\frac{1}{2}}) \end{aligned}$$

Applying Case 3, $T(n) = \Theta(n)$.

(b) $T(n) = 2T(\frac{n}{4}) + \sqrt{n}$

Consider $a = 2, b = 4, and f(n) = \sqrt{n}$ Using the **master theorem**:

$$\begin{aligned} n^{log_b a} &= n^{log_4 2} \\ &= \Theta(n^{\frac{1}{2}}) \\ f(n) &= \Theta(n^{\frac{1}{2}} logn) \end{aligned}$$

Therefore, $T(n) = \Theta(n^{\frac{1}{2}} logn)$.

(c) $T(n) = 2T(\frac{n}{4}) + n$

Consider $a = 2, b = 4, f(n) = n$. Using the **master theorem**:

$$\begin{aligned} n^{log_b a} &= n^{log_4 2} \\ &= \Theta(n^{\frac{1}{2}}) \\ f(n) &= O(n^{log_3 4 - e}) \end{aligned}$$

Applying Case 1, $T(n) = \Theta(n)$.

(d) $T(n) = 2T(\frac{n}{4}) + n^2$

Consider $a = 2, b = 4, and f(n) = n^2$ Using the **master theorem**:

$$\begin{aligned} n^{log_b a} &= n^{log_4 2} \\ &= \Theta(n^{\frac{1}{2}}) \\ f(n) &= O(n^{log_3 4 - e}) \end{aligned}$$

Applying Case 3, $T(n) = \Theta(n^{\frac{1}{2}})$.

5.

Consider the equation $T(n) = T(\frac{n}{4}) + \Theta(n^2)$. With $a = 2, b = 4, and f(n) = \Theta(n^2)$, using the **master theorem**:

$$\begin{aligned} n^{log_b a} &= n^{log_4 2} + \Theta(n^2) \\ &= \Theta(n^{\frac{1}{2}}) + \Theta(n^2) \\ f(n) &= \Theta(n^{\frac{1}{2}} logn) + \Theta(n^2) \end{aligned}$$

Applying Case 2, $T(n) = \Theta(n^{\frac{1}{2}} logn) + \Theta(n^2) \geq \Theta(n^2)$. Which is equal or slower than Strassen's algorithm at Θn^2, so Caesar's algorithm is no better.

6

Consider the equation $T(n) = T(\frac{n}{2}) + \Theta(1)$. With $a = 1, b = 2, and f(n) = \Theta(1)$, using the **master theorem**:

$$\begin{aligned}
n^{log_b a} &= n^{log_2 1} \\
&= n^0 \\
&= 1 \\
f(n) &= \Theta(n^{log_b a}) \\
&= \Theta(n^{log_2 1}) \\
&= \Theta(n^0) \\
&= \Theta(1)
\end{aligned}$$

Applying Case 2:

$$\begin{aligned}
T(n) &= \Theta(n^{log_b a} logn) \\
&= \Theta(n^{log_2 1} lgn) \\
&= \Theta(n^0 lgn) \\
&= \Theta(1 lgn)
\end{aligned}$$

Therefore, $= \Theta(lgn)$