$i \backslash j$	0	1	2	3	4		k
0	1						
1	1	1				$\emptyset$	
2	1	2	1				
3	1	3	3	1			
n 4	1	4	6	4	1		
	$\vdots$					$\ddots$	1
$n-1$	1						
n	1						

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\binom{n-1}{k-1} \quad \binom{n-1}{k}$$

$$\binom{n}{k}$$

$O(nk)$ if $k = O(n)$

$\downarrow$

$O(n^2)$

V_{ij} = value for the optimal knapsack involving elements $a_1 \dots a_i$ with a weight capacity j

Inputs

items: $a_1 \dots a_n$

wts: $w_1 \dots w_n$

values v_1 v_n

capacity: W

Final Problem Solution:

$$V_{\boxed{n}\,\boxed{W}}$$

$$V_{10,\,100} = ?$$

$n = 200$

$W = 1000$

$i = 0, 1, 2, \ldots n$

$j = 0 \ldots W$

Item	Weight	Value
a_1	5	10
a_2	2	5
a_3	4	8
a_4	2	7
a_5	3	7

$W = 7$ $\quad n = 5$

V_{ij} = value of the optimal knapsack involving elements $a_1 \dots a_i$ with a weight capacity j

take no elements

$i = 0, 1, 2, \dots n$

take everything

$j = 0, 1, 2, \dots W$

choose nothing

full capacity

Base Cases:

$i = 0$ value $= 0$

$j = 0$ value $= 0$

final optimal value:

$V_{n,W}$

V_{ij} = you are now considering including or excluding item a_i (wt: w_i value: v_i)

- is taking a_i even feasible

$j - w_i \geq 0$ feasible → take a_i / → don't take a_i

$j - w_i < 0$ infeasible → no choice, you <u>cannot</u> take a_i

$\underbrace{a_1 \; a_2 \; \cdots \; a_{i-1}} \; a_i$

$\uparrow$ don't take a_i

$\Rightarrow$ wt capacity is still j

$V_{i-1,j}$ = value of the optimal solution involving $a_1 \cdots a_{i-1}$, wt $= j$

take a_i:

$j \longrightarrow j - w_i$ (remaining capacity)

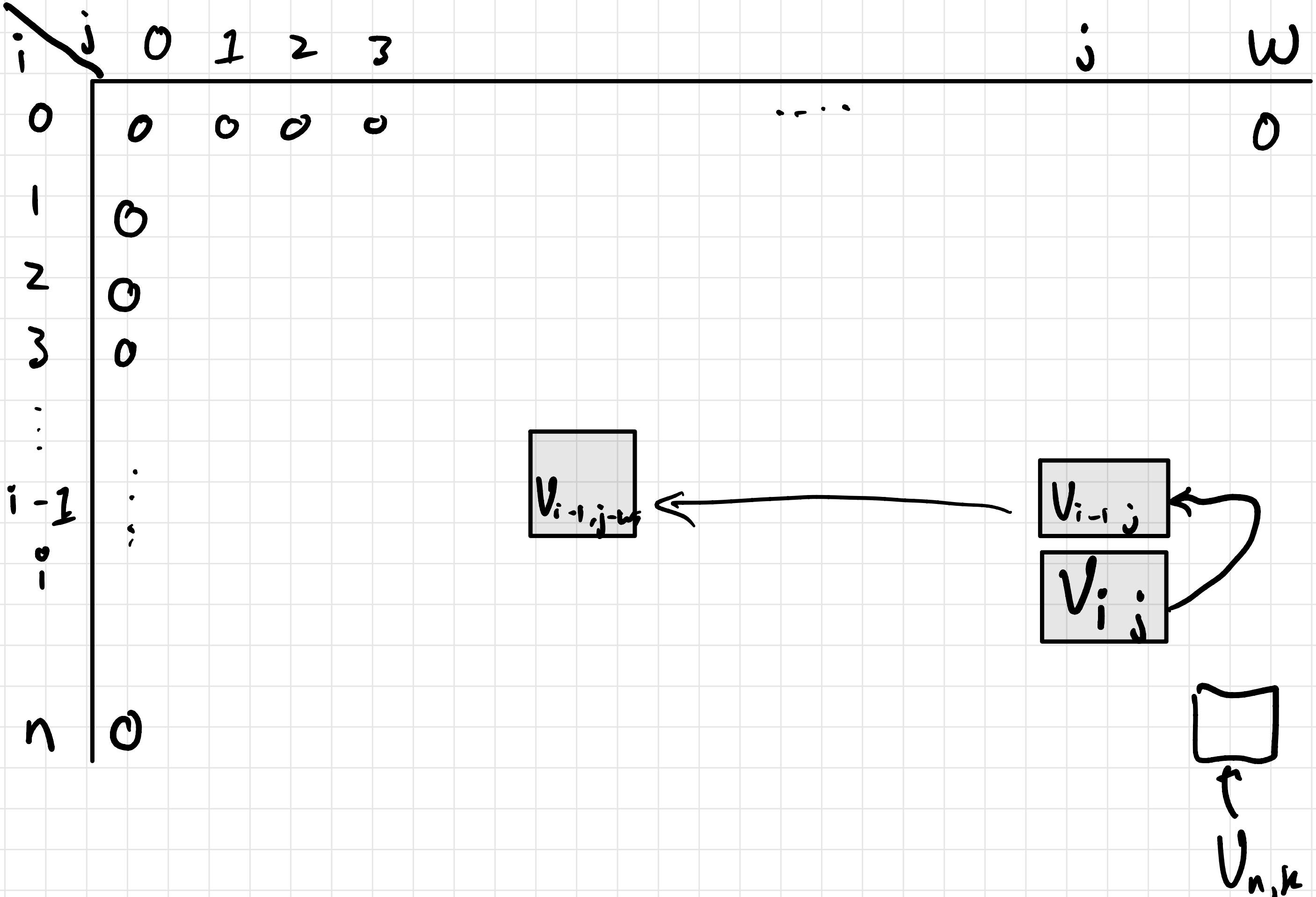
$a_1 a_2 \cdots a_{i-1} \boxed{a_i}$

$V_{i-1,\, j-w_i} + v_i$

get the value of a_i

$$V_{ij} = \begin{cases} \max\left\{ \underbrace{V_{i-1,j} \quad V_{i-1,j-w_i}}_{\text{choice}} \right\} & \text{if } j-w_i \geq 0 \\ V_{i-1,j} & \text{if } j-w_i < 0 \end{cases}$$

no choice (infeasible)



i
j
0
1
2
3
j
W
0
1
2
3
i-1
i
n
0
0
0
0
0
0
0
0
0
0
$V_{i-1,j-w}$
$V_{i-1,j}$
$V_{i,j}$
$V_{n,k}$

for $i = 0 \ldots n$

for $j = 0 \ldots W$

if $i = 0$ or $j = 0$

$V_{ij} \leftarrow 0$

else if $j - w_i < 0$

$V_{ij} \leftarrow V_{i-1,j}$

value of the a_i item

else if $V_{i-1,j} > V_{i-1,j-w_i} + v_i$

$V_{ij} \leftarrow V_{i-1,j}$ Don't take a_i

else

$V_{ij} \leftarrow V_{i-1,j-w_i} + v_i$

$i \backslash j$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10
2	0	0	5	5	5	10	10	15
3	0	0	5	5	8	10	13	15
4	0	0	7	7	12	12	15	17
5	0	0	7	7	12	14	15	19

Item	Weight	Value
a_1	5	10
a_2	2	5
a_3	4	8
a_4	2	7
a_5	3	7

$V_{4,5} =$ 1) feasible? $j - w_i = 5 - 2 \geq 0$ feasible

max: take it: $V_{i-1, j-w_i} + v_i$

$= V_{3\,3} + v_4 = 5 + 7 = 12$

or omit

$V_{i-1,j} = V_{3,5} = 10$

$i \backslash j$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10
2	0	0	5	5	5	10	10	15
3	0	0	5	5	$\boxed{8}$	10	13	15
4	0	0	7	7	$\boxed{12}$	12	15	17
5	0	0	7	7	12	14	15	$\boxed{19}$

Item	Weight	Value
a_1	5	10
$\boxed{a_2}$	$\boxed{2}$	5
~~a_3~~	4	8
$\boxed{a_4}$	$\boxed{2}$	7
$\boxed{a_5}$	$\boxed{3}$	7

≠ ⇒ take a_5

→ wt ⇒ 4

⇒ take a_4

= ⇒ did not take a_3

⇒ take a_2

Solution:

$\{a_2\ a_4\ a_5\}$

Val: 19 wt: 7

Input: Tableau V of 0-1 knapsack

Output: Optimal knapsack $S \subseteq A = \{a_1 \cdots a_n\}$

$S \leftarrow \emptyset$

$i \leftarrow n$ // start at the bottom

$j \leftarrow W$ // start at the last column.

while $i \geq 0$ and $j \geq 0$ // stay in bounds of the table

while $i \geq 1$ and $V_{ij} = V_{i-1,j}$:

i--

$S \leftarrow S \cup \{a_i\}$

$j = j - w_i$

$i = i - 1$

Output S