Properties of Normalization for a math based intermediate representation

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Abstract

The Normalization transformation plays a key rôle in the compilation of Diderot programs. The transformations are complicated and it would be easy for a bug to go undetected. To increase our confidence in normalization part of the compiler we provide a formal analysis on the rewriting system. We proof that the rewrite system is type preserving, value preserving (for tensor-valued expressions), and terminating.

1 Introduction

The Diderot language is a domain-specific language for scientific visualization and image analysis [3,4].Algorithms in this domain are used to visually explore data and compute features and properties. The language supports a high-level model of computation based on continuous tensor fields. The users rely on a high level of expressivity to implement visualization techniques.

Internally, we represent these computations with a a concise intermediate representation, called EIN [1,2]. Inside the compiler, we generate, compose, normalize, and optimize EIN operators. Unfortunately, the IR can quite large, dense, and impossible to read. It can be difficult to validate the correctness of computations represented in this IR.

To address the correctness of our work, we provide the following formal analysis. We define a type system for EIN operators and provide evaluation rules. We show that the rewriting system is type preserving and value preserving for the tensor valued rules. We define a size metric on the structure on an EIN expression. The rewriting system always decrease the size of an expression. We define a subset of the EIN expressions to be *normal form*. We show that termination implies normal form and that normal form implies termination. For any expression we can apply rewrites until termination, at which point we will have reached a normal form expression.

The paper is organized as follows.. We prove that the rewrite system is type preserving in Section 2. In Section 3 we show that for tensor-valued expressions the rewrite system is value preserving. Lastly, we show that the rewriting system is terminating in Section 4. We present the full proofs in the appendix.

2 Type Preservation

2.1 Typing EIN Operators

At the level of the SSA representation, we have types $\theta \in \textsc{Type}$ that correspond to the surface-level types:

$$
\begin{array}{rcll}
\theta & ::= & \mathbf{Ten}[d_1, \ldots, d_n] & \text{tensors} \\
 & | & \mathbf{Fld}(d)[d_1, \ldots, d_n] & \text{fields} \\
 & | & \mathbf{Img}(d)[d_1, \ldots, d_n] & \text{images} \\
 & | & \mathbf{Krn} & \text{kernels}
\end{array}
$$

An EIN operator $\lambda\,\bar{x}\langle e\rangle_\sigma$ can then be given a function type $(\theta_1 \times \cdots \times \theta_n) \rightarrow \theta$, where θ is either $\mathbf{Ten}[d_1, \ldots, d_n]$ or $\mathbf{Fld}(d)[d_1, \ldots, d_n]$ and σ is $1 < i_1 < d_1, \ldots, 1 < i_n < d_n$. The EIN expression (e) is the body of the operator, cannot be given a type θ, however since it represents a computation indexed by σ. Thus the type system for EIN expressions must track the index space as part of the context.

We define the syntax of indexed EIN-expression types as

$$
\begin{array}{c}
\tau_0 ::= \mathcal{T} \mid \mathcal{F}^d \\
\tau ::= (\sigma)\tau_0
\end{array}
$$

$$[\text{TYJUD}_1]\frac{\Gamma(T) = \textbf{Ten}[d_1,\,\ldots,\,d_n] \qquad |\alpha| = n \qquad \sigma \vdash \alpha < [d_1, \ldots d_n]}{\Gamma, \sigma \vdash T_\alpha : (\sigma)\mathcal{T}}$$

$$\frac{\Gamma(F) = \textbf{Fld}(d)[d_1,\,\ldots,\,d_n] \qquad |\alpha| = n \qquad \sigma \vdash \alpha < [d_1, \ldots d_n]}{\Gamma, \sigma \vdash F_\alpha : (\sigma)\mathcal{F}^d}$$

$$[\text{TYJUD}_2]\frac{\begin{array}{c}\Gamma(V) = \textbf{Img}(d)[d_1,\,\ldots,\,d_n] \qquad \Gamma(H) = \textbf{Krn} \\ |\alpha\beta| = n \qquad \sigma \vdash \alpha\beta < [d_1, \ldots d_n]\end{array}}{\Gamma, \sigma \vdash V_\alpha \circledast H^\beta : (\sigma)\mathcal{F}^d}$$

$$[\text{TYJUD}_3]\frac{i \notin \mathrm{dom}(\sigma) \qquad \sigma' = \sigma[i \mapsto (1, n)] \quad \Gamma, \sigma' \vdash e : (\sigma')\tau_0}{\Gamma, \sigma \vdash \sum_{i=1}^{n} e : (\sigma)\tau_0}$$

$$[\text{TYJUD}_4]\frac{\sigma(i) = d \qquad \sigma' = \sigma \setminus i \qquad \Gamma, \sigma' \vdash e : (\sigma')\mathcal{F}^d}{\Gamma, \sigma \vdash \frac{\partial}{\partial x_i} e : (\sigma)\mathcal{F}^d}$$

$$[\text{TYJUD}_5] \quad \frac{i, j \in dom(\sigma)}{\Gamma, \sigma \vdash \delta_{ij} : (\sigma)\mathcal{T}} \qquad\qquad \frac{\Gamma, \sigma \vdash ok}{\Gamma, \sigma \vdash \delta.\delta. : (\sigma)\mathcal{T}}$$

$$[\text{TYJUD}_5] \quad \frac{\sigma' = \sigma[j \mapsto (1, d)]/\ i \qquad \Gamma, \sigma' \vdash e : (\sigma')\tau_0}{\Gamma, \sigma \vdash (\delta_{ij} * e) : (\sigma)\tau_0}$$

$$[\text{TYJUD}_6] \quad \frac{\forall i \in \alpha . i \in dom(\sigma)}{\Gamma, \sigma \vdash \mathcal{E}_\alpha : (\sigma)\mathcal{T}} \qquad\qquad \Gamma, \sigma \vdash \mathcal{E}_{ijk}\mathcal{E}_{ilm} : (\sigma)\mathcal{T}$$

$$\frac{\Gamma, \sigma \vdash e : \tau}{\Gamma, \sigma \vdash (\mathcal{E}_\alpha * e) : \tau}$$

Figure 1: Typing Rules for each EIN expression.

$$[\text{TYJUD}_7]\frac{\begin{array}{c}\Gamma, \sigma \vdash \delta_{ij} : \tau \\ \Gamma, \sigma \vdash x : \textbf{Ten}[d]\end{array}}{\Gamma, \sigma \vdash \delta_{ij}@x : \tau} \qquad \frac{\begin{array}{c}\Gamma, \sigma \vdash \mathcal{E}_\alpha : \tau \\ \Gamma, \sigma \vdash x : \textbf{Ten}[d]\end{array}}{\Gamma, \sigma \vdash \mathcal{E}_\alpha@x : \tau} \qquad \frac{\begin{array}{c}\Gamma, \sigma \vdash e : (\sigma)\mathcal{F}^d \\ \Gamma, \sigma \vdash x : \textbf{Ten}[d]\end{array}}{\Gamma, \sigma \vdash e@x : (\sigma)\mathcal{T}}$$

$$[\text{TYJUD}_8] \quad \frac{\Gamma, \sigma \vdash e : (\sigma)\mathcal{T}}{\Gamma, \sigma \vdash \textbf{lift}_d(e) : (\sigma)\mathcal{F}^d}$$

$$[\text{TYJUD}_9]\frac{\Gamma, \sigma \vdash e : ()\tau_0 \;\; \odot_1 \in \{\sqrt{}, -, \kappa, \exp, (\cdot)^n\}}{\Gamma, \sigma \vdash \odot_1(e) : ()\tau_0}$$

$$[\text{TYJUD}_{10}]\frac{\Gamma, \sigma \vdash e_1 : \tau \;\; \Gamma, \sigma \vdash e_2 : \tau \;\; \odot_2 \in \{+, -\}}{\Gamma, \sigma \vdash (e_1 \odot_2 e_2) : \tau} \qquad \frac{\Gamma, \sigma \vdash e : \tau}{\Gamma, \sigma \vdash -e : \tau}$$

$$[\text{TYJUD}_{11}]\frac{\Gamma, \sigma \vdash e_1 : \tau \;\; \Gamma, \sigma \vdash e_2 : \tau}{\Gamma, \sigma \vdash (e_1 * e_2) : \tau}$$

$$[\text{TYJUD}_{12}]\frac{\Gamma, \sigma \vdash e_1 : (\sigma)\tau_0 \;\; \Gamma, \sigma \vdash e_2 : ()\tau_0}{\Gamma, \sigma \vdash \frac{e_1}{e_2} : (\sigma)\tau_0}$$

Figure 2: Typing Rules for each EIN expression.

where $(\sigma)\mathcal{T}$ is the type of indexed tensors and $(\sigma)\mathcal{F}^d$ is the type of indexed d-dimensional fields. We define our typing contexts as $\Gamma, \sigma \in (\textsc{Var} \overset{\text{fin}}{\to} \tau)^* \times (\textsc{IndexVar} \overset{\text{fin}}{\to} (\mathbb{Z} \times \mathbb{Z}))^*$. The typing context Γ, σ includes both the index map and an assignment of types to non-index variables.

With Γ we key the map with a variable. The notation

$$\Gamma(V) = \mathbf{Img}(d)[d_1, \ldots, d_n]$$

indicates that we can look up parameter id V in Γ and find the resulting type.

We key the map with an index $\sigma \in (\textsc{IndexVar} \overset{\text{fin}}{\to} (\mathbb{Z} \times \mathbb{Z}))^*$. To recall, the notation $i : n$ represents the upper boundary $1 < i < n$. We use notation

$$\sigma(i) = n$$

to indicate that we can look up variable (i) in σ and the upper bound of the variable is n. It is helpful to view σ as defining a finite map from index variables to the size of their range. To indicate the addition of a binding we use "$\sigma = \sigma'[i \mapsto (1, n)]$". The domain of σ is a sequence, which has to be disjoint ($\text{dom}(\sigma) = \{i_1, \ldots, i_n\}$). We use $i \notin \text{dom}(\sigma)$ to show that i is not in σ. We use "$\sigma = \sigma' \setminus i$" to indicate that i is not in σ' but it is in σ.

We state $\vdash \Gamma, \sigma$ **ok** to show that the environment is okay and the following apply

- with σ we key the map with an index and index variables do not repeat $\in \text{dom}(\sigma)$.
- in Γ we key the map with a unique variable parameter.

We define judgement form $\Gamma, \sigma \vdash e : \tau$ to mean that if the environment is okay then EIN expression e has type τ.

We define the judgement $\sigma \vdash \alpha < [d_1, \ldots d_n]$ as a shorthand for the following judgement.

$$\frac{\forall \mu_i \in \alpha, \text{ either } \mu_i \in \mathbb{N} \text{ and } 1 \leq \mu_i \leq d_i \text{ or } \sigma(\mu_i) = d_i}{\sigma \vdash \alpha < [d_1, \ldots d_n]}$$

Recall that an EIN index μ is either a constant ($\mu \in \mathbb{N}$) or a variable index $\mu \in \text{dom}(\sigma)$

We present a few typing rules next and refer the reader to Figure 1 and Figure 2 for a complete list of the rules. First consider the base case of a tensor variable T_α; the typing rule is

$$\frac{\Gamma, \sigma(T_\alpha) = \mathbf{Ten}[d_1, \ldots, d_n] \qquad |\alpha| = n \qquad \sigma \vdash \alpha < [d_1, \ldots d_n]}{\Gamma, \sigma \vdash T_\alpha : (\sigma)\mathcal{T}}$$

The antecedents of this rule state that T_α has a type that is compatible with both the multi-index α and the index map σ. A similar rule applies for field variables. The rule for convolution yields an indexed field type.

$$\frac{\begin{array}{c}\Gamma(V) = \mathbf{Img}(d)[d_1, \ldots, d_n] \qquad \Gamma(H) = \mathbf{Krn} \\ |\alpha\beta| = n \qquad \sigma \vdash \alpha\beta < [d_1, \ldots d_n]\end{array}}{\Gamma, \sigma \vdash V_\alpha \circledast H^\beta : (\sigma)\mathcal{F}^d}$$

Note that the index space covers both the shape of the image's range and the differentiation indices. Consider the following typing judgement for the EIN summation form:

$$\frac{i \notin \text{dom}(\sigma) \qquad \sigma' = \sigma[i \mapsto (1, n)] \qquad \Gamma, \sigma' \vdash e : (\sigma')\mathcal{T}}{\Gamma, \sigma \vdash \sum_{i=1}^{n} e : (\sigma)\mathcal{T}}$$

Here we extend the index map with $i : n$ when checking the body of the summation e. This rule reflects the fact that summation contracts the expression. We use a similar rule for differentiation.

$$\frac{\sigma(i) = d \qquad \sigma' = \sigma \setminus i \qquad \Gamma, \sigma' \vdash e : (\sigma')\mathcal{F}^d}{\Gamma, \sigma \vdash \frac{\partial}{\partial x_i} e : (\sigma)\mathcal{F}^d}$$

We can look up index i in σ with $\sigma(i) = d$ which indicates $1 \leq i \leq d$. The term $\sigma' = \sigma \setminus i$ indicates that the index map σ' has all the same index bindings as σ except i.

The term δ_{ij} does not change the context.

$$\frac{i, j \in \text{dom}(\sigma)}{\Gamma, \sigma \vdash \delta_{ij} : (\sigma)\mathcal{T}} \qquad \frac{\Gamma, \sigma \vdash ok}{\Gamma, \sigma \vdash \delta.\delta. : (\sigma)\mathcal{T}}$$

The application of a Kronecker delta function δ_{ij} adds index j to the context and removes index i.

$$\frac{\sigma' = \sigma[j \mapsto (1, d)]/\ i \qquad \Gamma, \sigma' \vdash e : (\sigma')\tau_0}{\Gamma, \sigma \vdash (\delta_{ij} * e) : (\sigma)\tau_0}$$

$$\Gamma, \sigma \vdash T_\alpha : \tau \quad \mapsto \tau = (\sigma)\mathcal{T}$$
$$\Gamma, \sigma \vdash F_\alpha : \tau \quad \mapsto \tau = (\sigma)\mathcal{F}^d$$
$$\ldots$$

Figure 3: The inversion lemma makes inferences based on a structural type judgements. Given a conclusion (left), we can infer something about the type τ (right).

Similarly, the $\mathcal{E}$ term by itself does not change the context.

$$\frac{\forall i \in \alpha. \quad i \in dom(\sigma)}{\Gamma, \sigma \vdash \mathcal{E}_\alpha : (\sigma)\mathcal{T}} \qquad \Gamma, \sigma \vdash \mathcal{E}_{ijk}\mathcal{E}_{ilm} : (\sigma)\mathcal{T}$$

When applying $\mathcal{E}$ to another term we preserve that term's type.

$$\frac{\Gamma, \sigma \vdash e : \tau}{\Gamma, \sigma \vdash (\mathcal{E}_\alpha * e) : \tau}$$

The Probe operation probes an expression and a tensor $\mathbf{Ten}[d]$.

$$\frac{\begin{array}{c}\Gamma, \sigma \vdash \delta_{ij} : \tau \\ \Gamma, \sigma \vdash x : \mathbf{Ten}[d]\end{array}}{\Gamma, \sigma \vdash \delta_{ij}@x : \tau} \qquad \frac{\begin{array}{c}\Gamma, \sigma \vdash \mathcal{E}_\alpha : \tau \\ \Gamma, \sigma \vdash x : \mathbf{Ten}[d]\end{array}}{\Gamma, \sigma \vdash \mathcal{E}_\alpha@x : \tau} \qquad \frac{\begin{array}{c}\Gamma, \sigma \vdash e : (\sigma)\mathcal{F}^d \\ \Gamma, \sigma \vdash x : \mathbf{Ten}[d]\end{array}}{\Gamma, \sigma \vdash e@x : (\sigma)\mathcal{T}}$$

Consider lifting a tensor term to the field level:

$$\frac{\Gamma, \sigma \vdash e : (\sigma)\mathcal{T}}{\Gamma, \sigma \vdash \mathbf{lift}_d(e) : (\sigma)\mathcal{F}^d}$$

The sub-term e has a tensor type $(\sigma)\mathcal{T}$ but the lifted term $\mathbf{lift}_d(e)$ has a field type $(\sigma)\mathcal{F}^d$. The rest of the judgements are quite straightforward. Some unary operators $\{\sqrt{}, -, \kappa, \exp, (\cdot)^n\}$ can only be applied to scalar valued terms such as reals and scalar fields.

$$\frac{\Gamma, \sigma \vdash e : ()\tau_0 \;\; \odot_1 \in \{\sqrt{}, -, \kappa, \exp, (\cdot)^n\}}{\Gamma, \sigma \vdash \odot_1(e) : ()\tau_0}$$

The subexpressions in an addition or subtraction expression have the same type as the result.

$$[\text{TYJUD}_{10}]\frac{\Gamma, \sigma \vdash e_1 : \tau \qquad \Gamma, \sigma \vdash e_2 : \tau \qquad \odot_2 \in \{+, -\}}{\Gamma, \sigma \vdash (e_1 \odot_2 e_2) : \tau}$$

The full set of typing judgements and corresponding inversion lemmas are contained in Figure 1, Figure 2, and Figure 3, respectively.

$$\frac{\sigma = i_1 : d_1, \ldots, i_m : d_m (\sigma, \{x_i \mapsto \theta_i \mid 1 \le i \le n\}) \vdash e : (\sigma)\mathcal{T}}{\vdash \lambda\,(x_1 : \theta_1, \ldots, x_n : \theta_1)\langle e\rangle_\sigma : (\theta_1 \times \cdots \times \theta_n) \to \mathbf{Ten}[d_1, \ldots, d_m]}$$

2.2 Type preservation Theorem

Given the type system for EIN expressions presented above, we prove that types are preserved by normalization.

Theorem 2.1 (Type preservation)**.** *If* $\vdash \Gamma, \sigma$ **ok**, $\Gamma, \sigma \vdash e : \tau$, *and* $e \xrightarrow[rule]{} e'$, *then* $\Gamma, \sigma \vdash e' : \tau$

Given a derivation d of the form $e \xrightarrow[rule]{} e'$ we state T(d) as a shorthand for the claim that the derivation preserves the type of the expression e. For each rewrite rule ($e \xrightarrow[rule]{} e'$), the structure of the left-hand-side (LHS) term determines the last typing rule(s) that apply in the derivation of $\Gamma, \sigma \vdash e : \tau$. We then apply a standard inversion lemma and derive the type of the right-hand-side (RHS) of the rewrite. Provided below are key cases of the proof (Section A).

R4 The rewrite rule (R4) has the form $(\sum_{i=1}^{n} e_1)@x \xrightarrow[rule]{} \sum_{i=1}^{n}(e_1@x)$.

The left hand side of the rewrite rule is a tensor type because it is the result of a probe operation. The LHS has the following type.

$$\Gamma, \sigma \vdash (\sum_{i=1}^{n} e_1)@x : (\sigma)\mathcal{T}$$

We want to show that the RHS has the same type.

$\Gamma, \sigma \vdash \sum_{i=1}^{n}(e_1 @ x) : (\sigma)\mathcal{T}$.

The type derivation for the LHS is the following structure.

$$\dfrac{\dfrac{\Gamma, \sigma[i \mapsto (1,n)] \vdash e_1 : (\sigma[i \mapsto (1,n)])\mathcal{F}^d [\text{TYINV}_3]}{\Gamma, \sigma \vdash (\sum_{i=1}^{n}(e_1)) : (\sigma)\mathcal{F}^d [\text{TYINV}_7]} \qquad \Gamma, \sigma \vdash x : \mathbf{Ten}[d]}{\Gamma, \sigma \vdash (\sum_{i=1}^{n}(e_1)) @ x : (\sigma)\mathcal{T}}$$

From that we can make the RHS derivations.

Given that $\Gamma, \sigma \vdash e_1 : (\sigma[i \mapsto (1,n)])\mathcal{F}^d$

then $\Gamma, \sigma \vdash e_1 @ x : (\sigma[i \mapsto (1,n)])\mathcal{T}$ by [TYJUD_7]

and $\Gamma, \sigma \vdash \sum_{i=1}^{n}(e_1 @ x) : (\sigma)\mathcal{T}$ by [TYJUD_3]

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T(R4) OK
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R6 The rewrite rule (R6) has the form $\frac{\partial}{\partial x_i} \diamond (e_1 * e_2) \xrightarrow[rule]{} e_1(\frac{\partial}{\partial x_i} \diamond e_2) + e_2(\frac{\partial}{\partial x_i} \diamond e_1)$.
The left hand side of the rewrite rule is a field type because it is the result of a field operation.
The LHS has the following type.

$\Gamma, \sigma \vdash \frac{\partial}{\partial x_i} \diamond (e_1 * e_2) : (\sigma)\mathcal{F}^d$

We want to show that the RHS has the same type.

$\Gamma, \sigma \vdash e_1 \frac{\partial}{\partial x_i} \diamond e_2 + e_2 \frac{\partial}{\partial x_i} \diamond e_1 : (\sigma)\mathcal{F}^d$.

The type derivation for the LHS is the following structure.
We use inversion to find the type for subexpressions e_1 and e_2.

$$\dfrac{\dfrac{\Gamma, \sigma \setminus i \vdash e_1 \qquad e_2 : (\sigma \setminus i)\mathcal{F}^d, [\text{TYINV}_{11}]}{(\Gamma, \sigma \setminus i \vdash e1 * e2 : (\sigma \setminus i)\mathcal{F}^d [\text{TYINV}_4]}}{\Gamma, \sigma \vdash \dfrac{\partial}{\partial x_{i:d}} \diamond (e1 * e2) : (\sigma)\mathcal{F}^d}$$

From that we can make the RHS derivations.

Given that $\Gamma, \sigma \vdash e_1, e_2 : (\sigma \setminus i)\mathcal{F}^d$

then $\Gamma, \sigma \vdash \frac{\partial}{\partial x_i} \diamond (e_1), \frac{\partial}{\partial x_i} \diamond (e_2) : (\sigma)\mathcal{F}^d$ by [TYJUD_4],

$\Gamma, \sigma \vdash e_1 * \frac{\partial}{\partial x_i} \diamond (e_2), e_2 * \frac{\partial}{\partial x_i} \diamond (e_1) : (\sigma)\mathcal{F}^d$ by [TYJUD_{11}] ,

and $\Gamma, \sigma \vdash e_1 * \frac{\partial}{\partial x_i} \diamond (e_2) + e_2 * \frac{\partial}{\partial x_i} \diamond (e_1) : (\sigma)\mathcal{F}^d$ by [TYJUD_{10}].

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T( R6) OK
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R7 The rewrite rule (R7) has the form $\frac{\partial}{\partial x_i} \diamond (\frac{e_1}{e_2}) \xrightarrow[rule]{} \frac{(\frac{\partial}{\partial x_i} \diamond e_1)e_2 - e_1(\frac{\partial}{\partial x_i} \diamond e_2)}{e_2^2}$. The left hand side of the rewrite rule is a field type because it is the result of a field operation. The LHS has the following type.

$\Gamma, \sigma \vdash \frac{\partial}{\partial x_i} \diamond (\frac{e_1}{e_2}) : (\sigma)\mathcal{F}^d$

We want to show that the RHS has the same type.

$\Gamma, \sigma \vdash \frac{(\frac{\partial}{\partial x_i} \diamond e_1)e_2 - e_1(\frac{\partial}{\partial x_i} \diamond e_2)}{e_2^2} : (\sigma)\mathcal{F}^d$.

The type derivation for the LHS is the following structure.
We use inversion to find the type for subexpressions e_1 and e_2.

$$\dfrac{\dfrac{\Gamma, \sigma \setminus i \vdash e_1 : (\sigma \setminus i)\mathcal{F}^d \qquad \Gamma, \sigma \vdash e_2 : ()\mathcal{F}^d, [\text{TYINV}_{12}]}{(\Gamma, \sigma \setminus i \vdash \dfrac{e1}{e2} : (\sigma \setminus i)\mathcal{F}^d [\text{TYINV}_4]}}{\Gamma, \sigma \vdash \dfrac{\partial}{\partial x_{i:d}} \diamond (\dfrac{e1}{e2}) : (\sigma)\mathcal{F}^d}$$

From that we can make the RHS derivations.
We use a type judgement to get the type of the subexpressions $(e_2 * e_2)$ in the right hand side of the rewrite rule.
Given that $\Gamma, \sigma \vdash e_2 : ()\mathcal{F}^d$ then $\Gamma, \sigma \vdash e_2 * e_2 : ()\mathcal{F}^d$ by [TYJUD_{11}]
We use a type judgement to get the type of the subexpressions $(\frac{\partial}{\partial x_{i:d}} \diamond e_2)$ in the right hand side of the rewrite rule.
Given that $\Gamma, \sigma \vdash e_2 : ()\mathcal{F}^d$ then $\Gamma, \sigma[i \mapsto (1,d)] \vdash \frac{\partial}{\partial x_{i:d}} \diamond e_2 : (i)\mathcal{F}^d$ by [TYJUD_4]
Next, we use a type judgement to get the type of the subexpressions $(e_1 * \frac{\partial}{\partial x_{i:d}} \diamond e_2)$ in the right hand side of the rewrite rule.
Given that $\Gamma, \sigma[i \mapsto (1,d)] \vdash \frac{\partial}{\partial x_{i:d}} \diamond e_2 : (i)\mathcal{F}^d$

and $\Gamma, \sigma \vdash e_1 : (\sigma \setminus i)\mathcal{F}^d$

then $\Gamma, \sigma \vdash e_1 \frac{\partial}{\partial x_{i:d}} \diamond e_2 : (\sigma)\mathcal{F}^d$ by [TYJUD_{11}]

The same is done to find $\Gamma, \sigma \vdash e_2 \frac{\partial}{\partial x_{i:d}} \diamond e_1 : (\sigma)\mathcal{F}^d$

Given that $\Gamma, \sigma \vdash ((\frac{\partial}{\partial x_i} \diamond e_1) * e_2), (e_1 * \frac{\partial}{\partial x_i} \diamond e_2) : (\sigma)\mathcal{F}^d$
and $\Gamma, \sigma \vdash e_2 * e_2 : ()\mathcal{F}^d$
then $\Gamma, \sigma \vdash ((\frac{\partial}{\partial x_i} \diamond e_1) * e_2) - (e_1 * \frac{\partial}{\partial x_i} \diamond e_2) : (\sigma)\mathcal{F}^d$ by [TYJUD_{10}]
and $\Gamma, \sigma \vdash \frac{(\frac{\partial}{\partial x_i} \diamond e_1)e_2 - e_1(\frac{\partial}{\partial x_i} \diamond e_2)}{e_2^2} : (\sigma)\mathcal{F}^d$ by [TYJUD_{12}]
`T( R7) OK`

R10 The rewrite rule (R10) has the form $\frac{\partial}{\partial x_i} \diamond (\mathbf{sine}(e_1)) \xrightarrow[rule]{} (\mathbf{cosine}(e_1)) * (\frac{\partial}{\partial x_i} \diamond e_1)$.
The left hand side of the rewrite rule is a field type because it is the result of a field operation.
The LHS has the following type.
$\Gamma, \sigma \vdash \frac{\partial}{\partial x_i} \diamond (\mathbf{sine}(e_1)) :(i)\mathcal{F}^d$
We want to show that the RHS has the same type.
$\Gamma, \sigma \vdash (\mathbf{cosine}(e_1)) * (\frac{\partial}{\partial x_i} \diamond e_1):(i)\mathcal{F}^d$.
The type derivation for the LHS is the following structure.
We use inversion to find the type for subexpression e_1.

$$\cfrac{\cfrac{\Gamma, \sigma \vdash e_1 : ()\mathcal{F}^d[\text{TYINV}_9]}{\Gamma, \sigma \vdash \mathbf{sine}(e_1) : ()\mathcal{F}^d}}{\Gamma, \sigma[i \mapsto (1, d)] \vdash \dfrac{\partial}{\partial x_i} \diamond (\mathbf{sine}(e_1)) : (i)\mathcal{F}^d}$$

From that we can make the RHS derivations.
Given that $\Gamma, \sigma \vdash e_1 : ()\mathcal{F}^d$
then $\Gamma, \sigma[i \mapsto (1, d)] \vdash \frac{\partial}{\partial x_i} \diamond e_1 : (i)\mathcal{F}^d$ by [TYJUD_4],
$\Gamma, \sigma \vdash \mathbf{cosine}(e_1) : ()\mathcal{F}^d$ by [TYJUD_9],
and $\Gamma, \sigma[i \mapsto (1, d)] \vdash (\mathbf{cosine}(e_1)) * (\frac{\partial}{\partial x_i} \diamond e_1) : (i)\mathcal{F}^d$ by [TYJUD_{11}].
`T(R10) OK`

R27 The rewrite rule (R27) has the form $\frac{\frac{e_1}{e_2}}{e_3} \xrightarrow[rule]{} \frac{e_1}{e_2 e_3}$.
We use inversion to find the type for subexpression e_1, e_2, e_3.
The LHS has the following type.
$\Gamma, \sigma \vdash \frac{\frac{e_1}{e_2}}{e_3} :(\sigma)\tau_0$
We want to show that the RHS has the same type.
$\Gamma, \sigma \vdash \frac{e_1}{e_2 e_3}:(\sigma)\tau_0$.
The type derivation for the LHS is the following structure.

$$\cfrac{\cfrac{\Gamma, \sigma \vdash e_1 : (\sigma)\tau_0, \Gamma, \sigma \vdash e_2 : ()\tau_0[\text{TYINV}_{12}]}{\Gamma, \sigma \vdash \dfrac{e_1}{e_2} : (\sigma)\tau_0} \qquad e_3 : ()\tau_0[\text{TYINV}_{12}]}{\Gamma, \sigma \vdash \dfrac{\frac{e_1}{e_2}}{e_3} : (\sigma)\tau_0}$$

From that we can make the RHS derivations.
Given that $\Gamma, \sigma \vdash e_1 : (\sigma)\mathcal{T}, \Gamma, \sigma \vdash e_2, e_3 : ()\mathcal{T}$
then $\Gamma, \sigma \vdash e_2 * e_3 : ()\mathcal{T}$ by [TYJUD_{11}],
and $\Gamma, \sigma \vdash \frac{e_1}{e_2 e_3} : (\sigma)\mathcal{T}$ by [TYJUD_{12}].
T(R27 for $\tau = (\sigma)\mathcal{T}$)
`T(R27) OK`

R40 The rewrite rule (R40) has the form $\delta_{ij} \frac{\partial}{\partial x_j} \diamond e_1 \xrightarrow[rule]{} \frac{\partial}{\partial x_i} \diamond (e_1)$.
We define a few variables $\sigma_2 = \sigma'/\ ij$, $\sigma_j = \sigma' j/\ i$, and $\sigma_i = \sigma' i/\ j$
We claim the type for the subexpression $(e_1).\Gamma, \sigma_2 \vdash e_1 : (\sigma_2)\mathcal{F}^d$
We use a type judgement to get the type of the subexpression $(\frac{\partial}{\partial x_j} \diamond e_1)$.
Given that $\Gamma, \sigma_2 \vdash e_1 : (\sigma_2)\mathcal{F}^d$ then $\Gamma, \sigma_j \vdash \frac{\partial}{\partial x_j} \diamond e_1 : (\sigma_j)\mathcal{F}^d$ by [TYJUD_4]
We switch the indices when applying the δ.
so that $\Gamma, \sigma_i \vdash \delta_{ij}(\frac{\partial}{\partial x_j} \diamond e_1) : (\sigma_i)\mathcal{F}^d$ by [TYJUD_5]
From that we can make the RHS derivations.
Given that $\Gamma, \sigma_2 \vdash e_1 : (\sigma_1)\mathcal{F}^d$ then $\Gamma, \sigma_i \vdash \frac{\partial}{\partial x_i} \diamond e_1 : (\sigma_i)\mathcal{F}^d$ by [TYJUD_4]
`T(R40) OK`

R41 The rewrite rule (R41) has the form $\sum(se_1) \xrightarrow[rule]{} s \sum e_1$.
We use inversion to find the type for subexpression s and e.
The LHS has the following type.
$\Gamma, \sigma \vdash \sum(se_1) :(\sigma)\tau_0$
We want to show that the RHS has the same type.
$\Gamma, \sigma \vdash s \sum e_1:(\sigma)\tau_0$.
The type derivation for the LHS is the following structure.

$$\dfrac{\sigma' = \sigma[i \mapsto (1,n)] \qquad \dfrac{\Gamma, \sigma' \vdash s : ()\tau_0, [\text{TYINV}_{11}] \qquad \Gamma, \sigma' \vdash e_1 : (\sigma')\tau_0}{\Gamma, \sigma' \vdash s * e_1 : (\sigma')\tau_0 [\text{TYINV}_3]}}{\Gamma, \sigma \vdash (\sum_{i=1}^{n} (s * e)) : (\sigma)\tau_0}$$

From that we can make the RHS derivations.
Given that $\Gamma, \sigma \vdash e_1 : (\sigma[i \mapsto (1,n)])\tau_0$ and $\Gamma, \sigma \vdash s : ()\tau_0$

then $\Gamma, \sigma \vdash \sum_{i=1}^{n} (e_1) : (\sigma)\tau_0$ by $[\text{TYJUD}_3]$

and $\Gamma, \sigma \vdash s * \sum_{i=1}^{n} (e_1) : (\sigma)\tau_0$ by $[\text{TYJUD}_{11}]$.

```
T( R41) OK
```

3 Value Preservation

3.1 Math background

In this section, we describe some additional mathematical concepts used by Diderot. We define some specific operators and their properties. These concepts are used in the following description about tensor fields and in other parts of the dissertation.

The permutation tensor or Levi-Civita tensor is represented in EIN with $\mathcal{E}_{ij}$ and $\mathcal{E}_{ijk}$ for the 2-d and 3-d case, respectively.

$$\mathcal{E}_{ij} = \begin{cases} +1 & ij \text{ is } (0,1) \\ -1 & ij \text{ is } (1,0) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathcal{E}_{ijk} = \begin{cases} +1 & ijk \text{ is cyclic } (0,1,2) \\ -1 & ijk \text{ is anti-cyclic } (2,1,0) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The kronecker delta function is δ_{ij}.

$$\delta_{ij} \quad = \quad \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The Krnocker delta value has the following property when two deltas share an index:

$$\delta_{ik}\delta_{kj} = \delta_{ij} \tag{3}$$

and the following when the indices are equal:

$$\delta_{ii} = 3 \tag{4}$$

We reflect on the following properties that hold in an orthonormal basis [5].Let us define an orthonormal basis β with unit basis vectors as $b_i, b_j, \ldots$. Each basis vector is linearly independent and normalized such that

$$\delta_{ij} \quad = \quad b_i \cdot b_j \quad = \quad \begin{cases} 1 & \text{i=j} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Any vector $\mathbf{u}$ can be defined by a linear combination of these basis vectors.

$$\mathbf{u} = \sum_i u_i b_i$$

A component of a tensor can be expressed in the following way

$$u_j = \mathbf{u} \cdot b_j \tag{6}$$

3.2 Value Definition

To show that the rewriting system preserves the semantics of the program, we must give a dynamic semantics to EIN expressions. We assume a set of values ($v \in \textsc{Value}$) that include reals, permutation tensor, Kronecker delta functions, and tensors. Rather than define the meaning of an expression to be a function from indices to values, we include a mapping ρ from index variables to indices as part of the dynamic environment. We define a dynamic environment to be $\Psi, \rho \in (\textsc{IndexVar} \stackrel{\text{fin}}{\to} \mathbb{Z}) \times (\textsc{Var} \stackrel{\text{fin}}{\to} \textsc{Value})$, where $\textsc{Value}$ is the domain of computational values (*e.g.*, tensors, *etc.*). We define the meaning of an EIN expression (for a subset of EIN expressions) using a big-step semantics $\Psi, \rho \vdash e \Downarrow v$, where v is a value. We describe values next and present evaluation rules Figure 5.

$$\begin{array}{lcll} \mathbf{v} & ::= & \mathit{Real}(n) & n \in \mathcal{R} \\ & | & \mathit{Tensor}[p \cdot b_1 \ldots b_n] & \text{index tensor argument p using basis values } b \\ & | & E_\alpha & \text{Reduces Levi-Civita tensor} \\ & | & K_{ij} & \text{Reduces Kronecker delta function} \end{array}$$

Figure 4: Value definitions ($\mathbf{v}$) for a subset of EIN expression

We assume an orthonormal basis function. Inspired by Equation 6, we use b_i to represent a basis vector inside a value expression. The value of a vector is defined as

$$\Psi, \rho \vdash T_i \Downarrow \mathit{Tensor}[T \cdot b_i]$$

A term b_i is created for each variable index i in the EIN expressions. The full tensor judgement

$$\Psi, \rho \vdash T_\alpha \Downarrow \mathit{Tensor}[T \cdot b_{\alpha 1} \ldots b_{\alpha n}]$$

is used to represent an arbitrary sized tensor. The lift operation is used to lift a tensor to a field. The value of a lifted term is the value of that term.

$$\frac{\Psi, \rho \vdash e \Downarrow v}{\Psi, \rho \vdash \mathbf{lift}_d(e) \Downarrow v}$$

We support arithmetic operations on and between u. The summation expression can be evaluated with the following judgement:

$$\frac{\Psi, \rho \vdash e \Downarrow v}{\Psi, \rho \vdash \sum_{i=1}^{n} e \Downarrow \Sigma_{i=1}^{n} v}$$

The summation operator is applied to the u. Generally, the judgement for unary operators ($\odot_1 \in \{\Sigma \mid \surd \mid - \mid \kappa \mid \exp \mid (\cdot)^n\}$) is as follows:

$$\frac{\Psi, \rho \vdash e_1 \Downarrow \mathit{Real}(r1)}{\Psi, \rho \vdash \odot_1 e_1 \Downarrow \mathit{Real}(\odot_1 r1)}$$

$$\frac{\Psi, \rho \vdash e_1 \Downarrow \mathit{Tensor}[e_1 \cdot b1]}{\Psi, \rho \vdash \odot_1 e_1 \Downarrow \odot_1(\mathit{Tensor}[e_1 \cdot b1])}$$

The binary operators ($\odot_2 = + \mid - \mid * \mid /$) can be applied between u. .

$$\frac{\Psi, \rho \vdash e_1 \Downarrow \mathit{Real}(r_1) \quad \Psi, \rho \vdash e_2 \Downarrow \mathit{Real}(r_2)}{\Psi, \rho \vdash (e_1 \odot_2 e_2) \Downarrow \mathit{Real}(r_1 \odot_2 r_2)}$$

$$\frac{\Psi, \rho \vdash e_1 \Downarrow \mathit{Tensor}[e_1 \cdot b1] \qquad \Psi, \rho \vdash e_2 \Downarrow \mathit{Tensor}[e_2 \cdot b2]}{\Psi, \rho \vdash (e_1 \odot_2 e_2) \Downarrow \mathit{Tensor}[e_1 \cdot b1] \odot_2 \mathit{Tensor}[e_2 \cdot b2]}$$

The epsilon and Kronecker delta functions are each reduced to a distinct permutation value (E_α or K_{ij}).

$$\Psi, \rho \vdash \mathcal{E}_{ijk} \Downarrow E_{ijk} \quad \Psi, \rho \vdash \delta_{ij} \Downarrow K_{ij}$$

The value for $\mathcal{E}_{ijk}$ is subject to Equation 1. The value for δ_{ij} is subject to Equation 2, Equation 3, and Equation 4.

We use notation $v_1 \mapsto v_2$ to indicate a value that is reduced or rewritten. We combine permutation values with tensor values as

$$K_{ij} * \mathit{Tensor}[T \cdot \beta] \mapsto \mathit{Tensor}[T \cdot b_i \cdot b_j \cdot \beta]. \tag{7}$$

The full set of evaluation rules are given in Figure 5.

3.3 Value Preservation Theorem

Our correctness theorem states the rewrite rules do not change the value of an expression with respect to a dynamic environment, assuming that the expression and dynamic environment are both type-able in the same static environment and their value is defined.

Theorem 3.1 (Value Preservation)**.** *If* $\vdash \Gamma, \sigma$ ***ok****,* $\Gamma, \sigma \vdash e : \tau$*,* $\Gamma, \sigma \vdash \Psi, \rho$ ***ok****,* $e \xrightarrow[rule]{} e'$*, and* $\Psi, \rho \vdash e \Downarrow v$*, then* $\Psi, \rho \vdash e' \Downarrow v$

Assume $\Psi, \rho \vdash e \Downarrow v$ and $e \xrightarrow[rule]{} e'$, then the proof proceeds by case analysis of the rewrite rules. Does not include rules that involve fields terms (values for fields are not defined). We show the full proof in Section B and select a few key examples below.

R24 The rewrite rule (R24) has the form $e_1 - 0 \xrightarrow[rule]{} e_1$.

Claim $e_1 - 0$ evaluates to v.
We need to define v.
Assume that $e_1 \Downarrow v'$
then $\Psi, \rho \vdash e_1 - 0 \Downarrow v' - \mathit{Real}(0)$ by [VALJUD_1], [VALJUD_5].
The value of v is $v' - \mathit{Real}(0)$.
By using algebraic reasoning: $v' - \mathit{Real}(0) = v'$.
Since $e_1 - 0 \Downarrow v$ and $e_1 - 0 \Downarrow v'$ then $v = v'$
The last step leads to $e_1 \Downarrow v$
`V(R24) OK`

$$[\text{VALJUD}_1] \quad \Psi, \rho \vdash c \Downarrow \mathit{Real}(c)$$

$$[\text{VALJUD}_2] \quad \Psi, \rho \vdash T_\alpha \Downarrow \mathit{Tensor}[T \cdot b_{\alpha 1} \dots b_{\alpha n}]$$

$$[\text{VALJUD}_3] \quad \frac{\Psi, \rho \vdash e \Downarrow v}{\Psi, \rho \vdash \mathbf{lift}_d(e) \Downarrow v}$$

$$[\text{VALJUD}_4] \quad \odot_1 \in \{\sum \mid \sqrt{} \mid - \mid \kappa \mid \exp \mid (\cdot)^n\}$$
$$\frac{\Psi, \rho \vdash e_1 \Downarrow \mathit{Real}(r1)}{\Psi, \rho \vdash \odot_1 e_1 \Downarrow \mathit{Real}(\odot_1 r1)} \qquad \frac{\Psi, \rho \vdash e_1 \Downarrow \mathit{Tensor}[e_1 \cdot b1]}{\Psi, \rho \vdash \odot_1 e_1 \Downarrow \odot_1 \mathit{Tensor}[e_1 \cdot b1]}$$

$$[\text{VALJUD}_5] \quad \odot_2 = + \mid - \mid * \mid /$$
$$\frac{\Psi, \rho \vdash e_1 \Downarrow \mathit{Real}(r1) \qquad \Psi, \rho \vdash e_2 \Downarrow \mathit{Real}(r2)}{\Psi, \rho \vdash (e_1 \odot_2 e_2) \Downarrow \mathit{Real}(r1 \odot_2 r2)}$$
$$\frac{\Psi, \rho \vdash e_1 \Downarrow \mathit{Tensor}[e_1 \cdot b1] \qquad \Psi, \rho \vdash e_2 \Downarrow \mathit{Tensor}[e_2 \cdot b2]}{\Psi, \rho \vdash (e_1 \odot_2 e_2) \Downarrow \mathit{Tensor}[e_1 \cdot b1] \odot_2 \mathit{Tensor}[e_2 \cdot b2]}$$

$$\frac{\Psi, \rho \vdash e_1 \Downarrow v_1 \quad \Psi, \rho \vdash e_2 \Downarrow v_2 \quad \odot_2 = + \mid - \mid * \mid /}{\Psi, \rho \vdash (e_1 \odot_2 e_2)@x \Downarrow \mathit{Probe}(v_1)[x] \odot_2 \mathit{Probe}(v_2)[x]}$$

$$[\text{VALJUD}_6] \quad \frac{\Psi, \rho \vdash e \Downarrow v}{\Psi, \rho \vdash \mathbf{lift}_d(e)@e \Downarrow v} \qquad \frac{\Psi, \rho \vdash \delta_{ij} \Downarrow v}{\Psi, \rho \vdash \delta_{ij}@e \Downarrow v} \qquad \frac{\Psi, \rho \vdash \mathcal{E}_\alpha \Downarrow v}{\Psi, \rho \vdash \mathcal{E}_\alpha@e \Downarrow v}$$

$$[\text{VALJUD}_7] \quad \Psi, \rho \vdash \delta_{ij} \Downarrow K_{ij} \qquad \Psi, \rho \vdash \mathcal{E}_\alpha \Downarrow E_\alpha$$

Figure 5: Value Judgements for each EIN expression.

R32 The rewrite rule (R32) has the form $\sqrt{(e_1)} * \sqrt{(e_1)} \xrightarrow[rule]{} e_1$.

Claim $\sqrt{(e_1)} * \sqrt{(e_1)}$ evaluates to v.
We need to define v.
Assume that $e_1 \Downarrow v'$
then $\Psi, \rho \vdash \sqrt{e_1} \Downarrow \sqrt{(v')}$ by [VALJUD$_4$],
and $\Psi, \rho \vdash \sqrt{e_1}\sqrt{e_1} \Downarrow \sqrt{v'}\sqrt{v'}$ by [VALJUD$_5$]
The value of v is $\sqrt{v'} * \sqrt{v'}$
By using algebraic reasoning to analyze v
$v = \sqrt{v'} * \sqrt{v'} = v'$ by reduction
The last step leads to $e_1 \Downarrow v$
`V(R32) OK`

R35 The rewrite rule (R35) has the form $\mathcal{E}_{ijk}\mathcal{E}_{ilm} \xrightarrow[rule]{} \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.

Claim $\mathcal{E}_{ijk}\mathcal{E}_{ilm}$ evaluates to v.
We need to define v.
Given that $\mathcal{E}_{ijk} \Downarrow E_{ijk}$ and $\mathcal{E}_{pqr} \Downarrow E_{pqr}$ then $\mathcal{E}_{ijk}\mathcal{E}_{pqr} \Downarrow E_{ijk}E_{pqr}$.
The value of v is $E_{ijk}E_{pqr}$.
Consider the product of two E expressions as

$$E_{ijk}E_{pqr} \longrightarrow \begin{vmatrix} K_{ip} & K_{iq} & K_{ir} \\ K_{jp} & K_{jq} & K_{jr} \\ K_{kp} & K_{kq} & K_{kr} \end{vmatrix}$$

$\longrightarrow K_{ip}(K_{jq}K_{kr} - K_{jr}K_{kq}) + K_{iq}(K_{jr}K_{kp} - K_{jp}K_{kr}) + K_{ir}(K_{jp}K_{kq} - K_{jq}K_{kp})$
Rewriting so that there is a shared index ($p = i$):
$\longrightarrow K_{ii}K_{jq}K_{kr} - K_{ii}K_{jr}K_{kq} + K_{iq}K_{jr}K_{ki} - K_{iq}K_{ji}K_{kr} + K_{ir}K_{ji}K_{kq} - K_{ir}K_{jq}K_{ki}$
Applying Equation 4:
$\longrightarrow 3K_{jq}K_{kr} - 3K_{jr}K_{kq} + K_{iq}K_{jr}K_{ki} - K_{iq}K_{ji}K_{kr} + K_{ir}K_{ji}K_{kq} - K_{ir}K_{jq}K_{ki}$
Applying Equation 3:
$\longrightarrow 3K_{jq}K_{kr} - 3K_{jr}K_{kq} + K_{kq}K_{jr} - K_{jq}K_{kr} + K_{jr}K_{kq} - K_{kr}K_{jq}$
Reduces to:
$\longrightarrow K_{jq}K_{kr} - K_{jr}K_{kq}$

Match indices to rule ($q \longrightarrow l$ and $r \longrightarrow m$)
$\longrightarrow K_{jl}K_{km} - K_{jm}K_{kl}$

We need to show that $\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ evaluates to v.

Given that $\Psi, \rho \vdash \delta_{jl} \Downarrow K_{jl}\ \ \delta_{km} \Downarrow K_{km}\ \ \delta_{jm} \Downarrow K_{jm}\ \ \delta_{kl} \Downarrow K_{kl}$ by [VALJUD$_7$]
then $\Psi, \rho \vdash \delta_{jl}\delta_{km} \Downarrow K_{jl}K_{km}\quad \delta_{jm}\delta_{kl} \Downarrow K_{jm}K_{kl}$ by [VALJUD$_5$]
and $\Psi, \rho \vdash \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \Downarrow K_{jl}K_{km} - K_{jm}K_{kl}$ by [VALJUD$_5$]

The last step leads to $\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \Downarrow v$

`V( R35) OK`

R36 The rewrite rule (R36) has the form $\delta_{ij}T_j \xrightarrow[rule]{} T_i$.

Claim $\delta_{ij}T_j$ evaluates to v.
We need to define v.
Given that $\Psi, \rho \vdash T_j \Downarrow \mathit{Tensor}[T \cdot b_j]$ by [VALJUD$_2$]
and $\Psi, \rho \vdash \delta_{ij} \Downarrow K_{ij}$ by [VALJUD$_7$]
then $\Psi, \rho \vdash \delta_{ij}T_j \Downarrow \mathit{Tensor}[T \cdot b_j \cdot b_i \cdot b_j]$ by *Equation* 7
The value of v is $\mathit{Tensor}[T \cdot b_j \cdot b_i \cdot b_j]$
By using algebraic reasoning to analyze v
$v = \mathit{Tensor}[T \cdot b_i]$ by reducing value $b_j \cdot b_j$ using *Equation* 5

We need to show that T_i evaluates to v.
Lastly, $\Psi, \rho \vdash T_i \Downarrow \mathit{Tensor}[T \cdot b_i]$ by [VALJUD$_2$]
The last step leads to $T_i \Downarrow v$

`V( R36) OK`

4 Termination

In this section we make the following claims:

1. Rewriting terminates
2. if e $\xrightarrow[rule]{}^{*}$ e' and $\nexists$ e" such that e' $\xrightarrow[rule]{}$ e", then e' $\in \mathcal{N}$

We prove that the normalization rewriting will terminate and that the resulting term will be in normal form.

Our approach uses the standard technique of defining a well-founded size metric $[\![e]\!]$ to show that the rewrite rules always decrease the size of an expression. The size metric guarantees that the normalization process terminates (Section 4.1). We also want to guarantee that normalization actually produces a normal-form. We define a subset of the EIN expressions that are in *normal form* by a grammar Section 4.2. We then define the *terminal* expressions as $\mathcal{T} = \{e \mid \nexists e' \text{ such that } e \xrightarrow[rule]{} e'\}$. The last section (Section 4.3) relates normal form expressions and terminal expressions. We show that termination implies normal form (Lemma 4.2) and that normal form implies termination (Lemma 4.3). For any expression we can apply rewrites until termination, at which point we will have reached a normal form expression (Theorem 4.4).

Table 3: We define a size metric $[\![\bullet]\!] : e \to \mathbb{N}$ inductively on the structure of the grammar of EIN in [2].

EIN expression (e)	Size metric $[\![[\![e]\!]$
$c,\ T_\alpha,\ F_\alpha,\ (v_\beta \circledast h^\mu),\ \delta_{ij}$	1
$\mathcal{E}_\alpha$	4
$\mathbf{lift}_d(e),\ \sqrt{e},\ -e,\ \exp(e),\ e^n,\ \kappa(e)$	$1 + [\![e]\!]$
$e_1 + e_2\ ,e_1 - e_2,\ e_1 * e_2$	$1 + [\![e_1]\!] + [\![e_2]\!]$
$\frac{a}{b}$	$2 + [\![e_1]\!] + [\![e_2]\!]$
$\sum e$	$2 + 2[\![e]\!]$
$\frac{\partial}{\partial x_\nu} \diamond \diamond e$	$5^{[\![e]\!]}[\![e]\!]$
$e(x)$	$2[\![e]\!]$

4.1 Size Metric

We define a size metric $[\![e]\!]$ for EIN expressions in Table 3 and use it to show that rewrites always decrease the size of the EIN expression.

Lemma 4.1. *If* $e \xrightarrow[rule]{} e'$ *then* $[\![e]\!] > [\![e']\!]$

Our proof does a case analysis on the rewrite rules ($e \xrightarrow[rule]{} e'$) and compares the size (Table 3) of each side of the rule. Provided below are key cases of the proof (Section C.1).

R1 The rewrite rule (R1) has the form $(e_1 \odot_n e_2)@x \xrightarrow[rule]{} (e_1@x) \odot_n (e_2@x)$.

case analysis on the operator $\odot_n$

if $\odot_n = *$

$$\begin{aligned} [\![(e_1 * e_2)@x]\!] &= 2 + 2[\![e_1]\!] + 2[\![e_2]\!] \\ &> 1 + 2[\![e_1]\!] + 2\mathcal{S} \\ &= [\![[\![(e_1@x) * (e_2@x)]\!]\!] \end{aligned}$$

if $\odot_n = \bullet/\bullet$

$$\begin{aligned} [\![[\![(\tfrac{e_1}{e_2})@x]\!]\!] &= 4 + 2[\![e_1]\!] + 2[\![e_2]\!] \\ &> 2 + 2[\![e_1]\!] + 2[\![e_2]\!] \\ &= [\![[\![\tfrac{e_1@x}{e_2@x}]\!]\!] \end{aligned}$$

P(d)

R9 The rewrite rule (R9) has the form $\frac{\partial}{\partial x_i} \diamond (\mathbf{cosine}(e_1)) \xrightarrow[rule]{} (-\mathbf{sine}(e_1)) * (\frac{\partial}{\partial x_i} \diamond e_1)$.

$$\begin{aligned} [\![\tfrac{\partial}{\partial x_i} \diamond (\mathbf{cosine}(e_1))]\!] &= (1 + [\![e_1]\!])5^{(1+[\![e_1]\!])} \\ &> [\![e_1]\!] * (1 + 5^{[\![e_1]\!]}) + 3 \\ &= [\![(-\mathbf{sine}(e_1)) * (\tfrac{\partial}{\partial x_i} \diamond e_1)]\!] \end{aligned}$$

P(d)

R17 The rewrite rule (R17) has the form $\frac{\partial}{\partial x_i} \diamond (e_1 \odot e_2) \xrightarrow[rule]{} (\frac{\partial}{\partial x_i} \diamond e_1) \odot (\frac{\partial}{\partial x_i} \diamond e_2)$.

$$\begin{array}{rcl} [\![\frac{\partial}{\partial x_i} \diamond (e_1 \odot e_2)]\!] & = & (1 + [\![e_1]\!] + [\![e_2]\!])5^{(1+[\![e_1]\!]+[\![e_2]\!])} \\ & > & [\![e_1]\!]5^{([\![e_1]\!])} + [\![e_2]\!]5^{([\![e_2]\!])} + 1 \\ & = & [\![(\frac{\partial}{\partial x_i} \diamond e_1) \odot (\frac{\partial}{\partial x_i} \diamond e_2)]\!] \end{array}$$

P(d)

R27 The rewrite rule (R27) has the form $\frac{\frac{e_1}{e_2}}{e_3} \xrightarrow[rule]{} \frac{e_1}{e_2 e_3}$.

$$\begin{array}{rcl} [\![\frac{\frac{e_1}{e_2}}{e_3}]\!] & = & 4 + [\![e_1]\!] + [\![e_2]\!] + [\![e_3]\!] \\ & > & 3 + [\![e_1]\!] + [\![e_2]\!] + [\![e_3]\!] \\ & = & [\![\frac{e_1}{e_2 e_3}]\!] \end{array}$$

P(d)

4.2 Normal Form

An EIN expression is in normal form if it can not be reduced. The normal form is defined as the subset $\mathcal{N}$ of EIN expressions. In the following, we describe the normal form with the following examples. Some tensors, constants, and permutation terms that are in normal form include:

$$T_\alpha, c \neq 0, \delta_{ij}, \mathcal{E}_{ij}, \text{ and } \mathcal{E}_{ijk}$$

The field forms $\mathcal{F}$ include:

$$F_\alpha, V \circledast H, \frac{\partial}{\partial x_i} \diamond F_\alpha$$

All differentiation is applied (via product rule or otherwise) so in normal form the differentiation is only applied to a field term:

$$\frac{\partial}{\partial x_i} \diamond F_\alpha$$

until it is pushed down to the convolution kernel:

$$V \circledast \frac{\partial}{\partial x_i} \diamond H$$

The only probed terms are field forms $\mathcal{F}$:

$$F_\alpha @T, (V \circledast H)@x, \text{ and } (\frac{\partial}{\partial x_i} \diamond F)@x$$

Some unary operations are in normal form, as long as their sub-term e_1 is in normal form:

$$\text{sine}(e_1), \mathbf{lift}_d(e_1), \sqrt{e_1}, \exp(e_1)$$

Other arithmetic operations cannot have a zero constant sub-term [2]

$$-e_1, e_1 + e_2, e_1 - e_2, e_1 * e_2, \frac{e_1}{e_2}$$

The division structure is subject to algebraic rewrites [2]. The normal form of the product and summation structure is more restricted in part because of index-based rewrites. Normal form is presented more formally next:

Normal Form The following grammar specifies the subset $\mathcal{N}$ of EIN expressions that are in *normal form*:

$$\begin{array}{rcl} \mathcal{N} & ::= & \mathcal{A} \mid c \\ \mathcal{A} & ::= & \mathcal{D} \mid \mathcal{G} \\ \mathcal{D} & ::= & \mathcal{B} \mid -\mathcal{G} \\ \mathcal{G} & ::= & \mathcal{B} \mid \frac{\mathcal{D}}{\mathcal{D}} \\ \mathcal{B} & ::= & T_\alpha \mid \mathcal{F} \mid \mathcal{F}@T_\alpha \mid c \neq 0 \mid \delta_{ij} \mid \mathcal{E}_{ij} \mid \mathcal{E}_{ijk} \\ & \mid & \mathcal{A} + \mathcal{A} \mid \mathcal{A} - \mathcal{A} \mid \sqrt{\mathcal{N}} \\ & \mid & \mathbf{lift}_d(\mathcal{N}) \mid \exp(\mathcal{N}) \mid \mathcal{N}^c \mid \kappa(\mathcal{N}) \\ & \mid & (\mathcal{A} * \mathcal{A})^{1,2,3,4} \\ & \mid & (\sum \mathcal{N})^5 \\ \mathcal{F} & ::= & F_\alpha \mid v \circledast h \mid \frac{\partial}{\partial x_i} \diamond F_\alpha \end{array}$$

subject to the following additional restrictions (noted in the syntax with an upper index):

1. If a term has the form $\mathcal{E}_{ijk} * \mathcal{E}_{i'j'k'}$ then the indices ijk must be disjoint from $i'j'k'$.

2. If a term contains the form $\mathcal{E}_{ijk} * \mathcal{A}$ and $\mathcal{A}$ has a differentiation component then no two of the indices i, j, and k may occur in the differentiation component of $\mathcal{A}$. For example, $\mathcal{E}_{ijk} * \frac{\partial}{\partial x_{jk}} \diamond e$ is not in normal form and can be rewritten as $\mathcal{E}_{ijk} * \frac{\partial}{\partial x_{jk}} \diamond e \xrightarrow[rule]{}^* 0$.
3. If a term has the form $\delta_{ij} * \mathcal{A}$ then j may not occur in $\mathcal{A}$. For example, the expression $\delta_{ij} * T_j$ is not in normal form, and thus $\delta_{ij} * T_j$ can be rewritten to T_i.
4. If a term has the form $\sqrt{e_1} * \sqrt{e_2}$ then $e_1 \neq e_2$.
5. If a term is of the form $\sum(e_1 * e_2)$ then e_1 can not be a scalar s, scalar field φ, or constant c. For example, terms $\sum(s * e_2)$ or $\sum(\varphi * e_2)$ are not in normal form and can be rewritten as $s \sum e_2$ and $\varphi \sum e_2$, respectively.

4.3 Termination and Normal form

The following two lemmas relate the set of normal forms expressions to the terminal expressions. The first shows that termination implies normal form.

Lemma 4.2. *If $e \in \mathcal{T}$, then $e \in \mathcal{N}$*

The proof is by examination of the EIN syntax in [2].For any syntactic construct, we show that either the term is in normal form, or there is a rewrite rule that applies. We define $Q(e_x) \equiv \nexists e'_x$ such that $e_x \xrightarrow[rule]{} e'_x$ and $e_x \in \mathcal{N}$. The following is a sample of a proof by contradiction (full proof is available Section C.2).

case on structure e_x

If $e_x = c$ then $Q(e_x)$ because e_x is in normal form.
If $e_x = T_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_x = F_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_x = V_\alpha \circledast H$ then $Q(e_x)$ because e_x is in normal form.
If $e_x = \delta_{ij}$ then $Q(e_x)$ because e_x is in normal form.
If $e_x = \mathcal{E}_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_x = \mathbf{lift}_d(e_1)$
Prove Q(e) by contradiction.
If $e_1 = c$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = T_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = F_\alpha$ then $Q(e_x)$ because e_x is not a supported type.
If $e_1 = e \circledast e$ then $Q(e_x)$ because e_x is not a supported type.
If $e_1 = \delta_{ij}$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = \mathcal{E}_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = \mathbf{lift}_d(e)$ then $Q(e_x)$ because e_x is not a supported type.
If $e_1 = M(e_1)$ and assuming $Q(e)$ then $Q(e_x)$
Given $M(e) = \sqrt{e} \mid exp(e) \mid e_1^n \mid \kappa(e)$
If $e_1 = -e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = \frac{\partial}{\partial x_\alpha} \diamond e$ then $Q(e_x)$ because e_x is not a supported type.
If $e_1 = \sum e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = e_3 + e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = e_3 - e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = e_3 * e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e = \frac{e_3}{e_4}$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = e_3 @ e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
$Q(e_x)$

The next lemma demonstrates that normal form implies termination.

Lemma 4.3. *If $e \in \mathcal{N}$, then $e \in \mathcal{T}$*

We state M(e) as a shorthand for the claim that if e is in normal form then it has terminated. The following is a proof by contradiction. CM(e): There exists an expression e that has not terminated and is in normal form. More precisely, given a derivation d of the form $e \xrightarrow[rule]{} e'$, there exists an expression that is the source term e of derivation d therefore not-terminated, and is in normal form. Below are cases of the proof (Section C.3).

Case R1.$(e_1 \odot_n e_2)@x \xrightarrow[rule]{} (e_1@x) \odot_n (e_2@x)$

Let y= $(e_1 \odot_n e_2)@x$ and since y is not in normal form then `M(R1) OK`

Case R2.$(e_0 \odot_2 e_1)@x \xrightarrow[rule]{} (e_0@x) \odot_2 (e_1@x)$

Let y= $(e_0 \odot_2 e_1)@x$ and since y is not in normal form then `M(R2) OK`

Theorem 4.4 (Normalization)**.** *For any closed EIN expression e the following two properties hold:*

1. *there exists an EIN expression $e' \in \mathcal{N}$, such that $e \xrightarrow[rule]{}^{*} e'$, and*
2. *there is no infinite sequence of rewrites starting with e.*

In other words, for any expression e we can apply rewrites until termination, at which point we will have reached a normal form expression e'.

The theorem follows from Lemmas 4.1, 4.2, and 4.3 described in Section C.

5 Discussion

The properties that we have described demonstrate the correctness of the normalization transformations for EIN. Unfortunately, the rewriting system is not confluent (because different pairings of $\mathcal{E}_{ijk}$ can be rewritten and produce different normal forms). In our system, we apply rules in a standard order, but there may be opportunities for improving performance by tuning the order of rewrites.

While there are still many opportunities for compiler bugs, normalization is the most critical part of compiling tensor-field expressions down to executable code, so these results increase our confidence in the correctness of the compiler. There are other parts of the compiler pipeline for which we hope to prove correctness in the future.

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A Type Preservation Proof

The following is a proof for Theorem 2.1

Given a derivation d of the form $e \xrightarrow[rule]{} e'$ we state T(d) as a shorthand for the claim that the derivation preserves the type of the expression e. For each rule, the structure of the left-hand-side term determines the last typing rule(s) that apply in the derivation of $\Gamma, \sigma \vdash e : \tau$. We then apply a standard inversion lemma and derive the type of the right-hand-side of the rewrite . The proof demonstrates that $\forall d.T(d)$.

Case on structure of d

Case R1.$(e_1 \odot_n e_2)@x \xrightarrow[rule]{} (e_1@x) \odot_n (e_2@x)$

We will do a case analysis on the structure on the left-hand-side

where $\odot_n = \{*|/\}$.

First we will prove T(d) for $\odot_n =*$ then $\odot_n =/$.

if $\odot_n = *$

Find $\Gamma, \sigma \vdash ((e_1 * e_2)@x)$

This type of structure inside a probe operation results in a tensor type.

The LHS has the following type.

$\Gamma, \sigma \vdash (e_1 \odot_n e_2)@x : (\sigma)\mathcal{T}$

We want to show that the RHS has the same type.

$\Gamma, \sigma \vdash (e_1@x) \odot_n (e_2@x) : (\sigma)\mathcal{T}$.

The type derivation for the LHS is the following structure.

$$\dfrac{\dfrac{\Gamma, \sigma \vdash e_1 : (\sigma)\mathcal{F}^d \qquad \Gamma, \sigma \vdash e_2 : (\sigma)\mathcal{F}^d[\text{TYINV}_{11}]}{\Gamma, \sigma \vdash e1 * e2 : (\sigma)\mathcal{F}^d[\text{TYINV}_7]} \qquad \Gamma, \sigma \vdash x : \textbf{Ten}[d]}{\Gamma, \sigma \vdash (e1 * e2)@x : (\sigma)\mathcal{T}}$$

From that we can make the RHS derivations.

Find $\Gamma, \sigma \vdash ((e_1@x) * (e_2@x))$
Given that $\Gamma, \sigma \vdash e_1, e_2 : (\sigma)\mathcal{F}^d$,
then $\Gamma, \sigma \vdash e_1@x, e_2@x : (\sigma)\mathcal{T}$ by [TYJUD$_7$],
and $\Gamma, \sigma \vdash e_1@x * e_2@x : (\sigma)\mathcal{T}$ by [TYJUD$_{11}$]
T(R1 for $\odot_n = *$)
if $\odot_n = /$
Find $\Gamma, \sigma \vdash ((\frac{e_1}{e_2})@x)$
This type of structure inside a probe operation results in a tensor type.
$\Gamma, \sigma \vdash (e_1 \odot_n e_2)@x : (\sigma)\mathcal{T}$ ([TYINV$_7$])
Find $\Gamma, \sigma \vdash (e_1$ and $e_2)$

$$\frac{\dfrac{\Gamma, \sigma \vdash e_1 : (\sigma)\mathcal{F}^d, \qquad \Gamma, \sigma \vdash e_2 : ()\mathcal{F}^d[\text{TYINV}_{12}]}{\Gamma, \sigma \vdash (\dfrac{e1}{e2}) : (\sigma)\mathcal{F}^d[\text{TYINV}_7]} \qquad \Gamma, \sigma \vdash x : \mathbf{Ten}[d]}{\Gamma, \sigma \vdash (\dfrac{e1}{e2})@x : (\sigma)\mathcal{T}}$$

Find $\Gamma, \sigma \vdash (\frac{(e_1@x)}{(e_2@x)})$
Given that $\Gamma, \sigma \vdash e_1 : (\sigma)\mathcal{F}^d, \Gamma, \sigma \vdash e_2 : ()\mathcal{F}^d$
then $\Gamma, \sigma \vdash e_1@x : (\sigma)\mathcal{T}$ by [TYJUD$_7$],
$\Gamma, \sigma \vdash e_2@x : ()\mathcal{T}$ by [TYJUD$_7$],
and $\Gamma, \sigma \vdash \frac{e_1@x}{e_2@x} : (\sigma)\mathcal{T}$ by [TYJUD$_{12}$].
T(R1 for $\odot_n = /$)
`T(R1) OK`
Case R2.$(e_0 \odot_2 e_1)@x \xrightarrow[rule]{} (e_0@x) \odot_2 (e_1@x)$
$\odot_2 = + \mid -$
Find $\Gamma, \sigma \vdash ((e_1 \odot_2 e_2)@x)$
This type of structure inside a probe operation results in a tensor type.
The LHS has the following type.
$\Gamma, \sigma \vdash (e_0 \odot_2 e_1)@x :(\sigma)\mathcal{T}$
We want to show that the RHS has the same type.
$\Gamma, \sigma \vdash (e_0@x) \odot_2 (e_1@x):(\sigma)\mathcal{T}$.
The type derivation for the LHS is the following structure.

$$\frac{\dfrac{\Gamma, \sigma \vdash e_1, e_2 : (\sigma)\mathcal{F}^d[\text{TYINV}_{10}]}{\Gamma, \sigma \vdash e_1 \odot_2 e2 : (\sigma)\mathcal{F}^d[\text{TYINV}_7]} \qquad \Gamma, \sigma \vdash x : \mathbf{Ten}[d]}{\Gamma, \sigma \vdash (e1 \odot_2 e2)@x : (\sigma)\mathcal{T}}$$

From that we can make the RHS derivations.
Given that $\Gamma, \sigma \vdash e_1, e_2 : (\sigma)\mathcal{F}^d$
then $\Gamma, \sigma \vdash e_1@x, e_2@x : (\sigma)\mathcal{T}$ by [TYJUD$_7$]
and $\Gamma, \sigma \vdash e_1@x \odot_2 e_2@x : (\sigma)\mathcal{T}$ by [TYJUD$_{10}$]
Case R3.$(\odot_1 e_1)@x \xrightarrow[rule]{} \odot_1 (e_1@x)$
We will do a case analysis on the structure on the left-hand-side
where $\odot_1 =\{-|M(.)\}$.
First we will prove T(d) for $\odot_1 =-$ then $\odot_1 =M(.)$.
if $\odot_1 = -$,
Find $\Gamma, \sigma \vdash ((-e_1)@x)$
This type of structure inside a probe operation results in a tensor type.
The LHS has the following type.
$\Gamma, \sigma \vdash (\odot_1 e_1)@x :(\sigma)\mathcal{T}$
We want to show that the RHS has the same type.
$\Gamma, \sigma \vdash \odot_1(e_1@x):(\sigma)\mathcal{T}$.
The type derivation for the LHS is the following structure.

$$\frac{\dfrac{\Gamma, \sigma \vdash e_1 : (\sigma)\mathcal{F}^d[\text{TYINV}_{10}]}{\Gamma, \sigma \vdash -e_1 : (\sigma)\mathcal{F}^d[\text{TYINV}_7]} \qquad \Gamma, \sigma \vdash x : \mathbf{Ten}[d]}{\Gamma, \sigma \vdash (-e_1)@x : (\sigma)\mathcal{T}}$$

From that we can make the RHS derivations.
Find $\Gamma, \sigma \vdash (-(e_1@x))$
Given that $\Gamma, \sigma \vdash e_1 : (\sigma)\mathcal{F}^d$
then $\Gamma, \sigma \vdash e_1@x : (\sigma)\mathcal{T}$ by [TYJUD$_7$]
and $\Gamma, \sigma \vdash -e_1@x : (\sigma)\mathcal{T}$ by [TYJUD$_{10}$]
T(R3 for $\odot_1 = -$)
if $\odot_1 = M(e_1)$
Note: $M(e_1) = \sqrt{e_1} \mid \kappa(e_1) \mid \exp(e_1) \mid e^n$

Find $\Gamma,\sigma \vdash ((M(e_1))@x)$
This type of structure inside a probe operation results in a tensor type.
The LHS has the following type.
$\Gamma,\sigma \vdash (\odot_1 e_1)@x :(\sigma)\mathcal{T}$
We want to show that the RHS has the same type.
$\Gamma,\sigma \vdash \odot_1(e_1@x):(\sigma)\mathcal{T}$.
The type derivation for the LHS is the following structure.

$$\dfrac{\dfrac{\Gamma,\sigma \vdash e_1 : (\sigma)\mathcal{F}^d([\text{TYINV}_9])}{\Gamma,\sigma \vdash M(e_1) : (\sigma)\mathcal{F}^d[\text{TYINV}_7]} \qquad \Gamma,\sigma \vdash x : \mathbf{Ten}[d]}{\Gamma,\sigma \vdash M(e_1)@x : (\sigma)\mathcal{T}}$$

From that we can make the RHS derivations.
Given that $\Gamma,\sigma \vdash e_1 : (\sigma)\mathcal{F}^d$
then $\Gamma,\sigma \vdash e_1@x : (\sigma)\mathcal{T}$ by [TYJUD$_7$]
and $\Gamma,\sigma \vdash M(e_1@x) : (\sigma)\mathcal{T}$ by [TYJUD$_9$]
T(R3 for $\odot_1 = M$)

```
T(R3) OK
```

Case R4.$(\sum_{i=1}^{n} e_1)@x \xrightarrow[rule]{} \sum_{i=1}^{n}(e_1@x)$. Included in the earlier prose.
Case R5.$(\chi)@x \xrightarrow[rule]{} \chi$
We will do a case analysis on the structure on the left-hand-side
where χ=$\{\mathbf{lift}_d(e_1)|\delta_{ij} \mid \mathcal{E}_\alpha\}$.
First we will prove T(d) for χ=$\mathbf{lift}_d(e_1)$ then χ=$\delta_{ij} \mid \mathcal{E}_\alpha$.
case $\chi = \mathbf{lift}_d(e_1)$
Find $\Gamma,\sigma \vdash ((\chi(e_1))@x)$
This type of structure inside a probe operation results in a tensor type.
The LHS has the following type.
$\Gamma,\sigma \vdash (\chi)@x :(\sigma)\mathcal{T}$
We want to show that the RHS has the same type.
$\Gamma,\sigma \vdash \chi:(\sigma)\mathcal{T}$.
The type derivation for the LHS is the following structure.

$$\dfrac{\dfrac{\Gamma,\sigma \vdash e_1 : (\sigma)\mathcal{F}^d([\text{TYINV}_8])}{\Gamma,\sigma \vdash (\mathbf{lift}_d(e_1)) : (\sigma)\mathcal{F}^d[\text{TYINV}_7]} \qquad \Gamma,\sigma \vdash x : \mathbf{Ten}[d]}{\Gamma,\sigma \vdash (\mathbf{lift}_d(e_1))@x : (\sigma)\mathcal{T}}$$

From that we can make the RHS derivations.
Given that $\Gamma,\sigma \vdash e_1 : (\sigma)\mathcal{F}^d$
then $\Gamma,\sigma \vdash e_1@x : (\sigma)\mathcal{T}$ by [TYJUD$_7$]
and $\Gamma,\sigma \vdash \mathbf{lift}_d(e_1@x) : (\sigma)\mathcal{T}$ by [TYJUD$_8$]
T(R5 where $\chi = \mathbf{lift}_d(e_1)$)
For the case $\chi = \delta_{ij} \mid \mathcal{E}_\alpha$
Given that $\Gamma,\sigma \vdash \chi : \tau$ then $\Gamma,\sigma \vdash \chi@x : \tau$ by [TYJUD$_7$]
T(R5 where $\chi = \delta_{ij} \mid \mathcal{E}_\alpha$)

```
T(R5) OK
```

Case R6.$\frac{\partial}{\partial x_i} \diamond (e_1 * e_2) \xrightarrow[rule]{} e_1(\frac{\partial}{\partial x_i} \diamond e_2) + e_2(\frac{\partial}{\partial x_i} \diamond e_1)$. Included in the earlier prose.
Case R7.$\frac{\partial}{\partial x_i} \diamond (\frac{e_1}{e_2}) \xrightarrow[rule]{} \frac{(\frac{\partial}{\partial x_i} \diamond e_1)e_2 - e_1(\frac{\partial}{\partial x_i} \diamond e_2)}{e_2^2}$. Included in the earlier prose.
Case R8.$\frac{\partial}{\partial x_i} \diamond (\sqrt{e_1}) \xrightarrow[rule]{} \mathbf{lift}_d(1/2) * \frac{\frac{\partial}{\partial x_i} \diamond e_1}{\sqrt{e_1}}$
Find $\Gamma,\sigma \vdash (\frac{\partial}{\partial x_i} \diamond (\sqrt{e_1}))$
This type of structure inside a derivative operation results in a field type
and the $\sqrt{e_1}$ term results in a scalar.
Claim: $\Gamma \vdash \sqrt{e_1} : ()\mathcal{F}^d$ then $\Gamma_i \vdash \nabla_i \diamond (\sqrt{e_1}) : (i)\mathcal{F}^d$ by [TYJUD$_4$]
The LHS has the following type.
$\Gamma,\sigma \vdash \frac{\partial}{\partial x_i} \diamond (\sqrt{e_1}) :(i)\mathcal{F}^d$
We want to show that the RHS has the same type.
$\Gamma,\sigma \vdash \mathbf{lift}_d(1/2) * \frac{\frac{\partial}{\partial x_i} \diamond e}{\sqrt{e_1}}:(i)\mathcal{F}^d$.
The type derivation for the LHS is the following structure.

$$\dfrac{\dfrac{\Gamma,\sigma \vdash e_1 : ()\mathcal{F}^d[\text{TYINV}_9]}{\Gamma,\sigma \vdash \sqrt{e_1} : ()\mathcal{F}^d[\text{TYINV}_4]}}{\Gamma,\sigma \vdash \dfrac{\partial}{\partial x_i} \diamond (\sqrt{e_1}) : (\sigma)\mathcal{F}^d \text{ and } \sigma = \{i : d\}(\text{Claim})}$$

From that we can make the RHS derivations.

Given that $\Gamma, \sigma \vdash e_1 : ()\mathcal{F}^d$

then $\Gamma, \sigma[i \mapsto (1,d)] \vdash \frac{\partial}{\partial x_{i:d}} \diamond e_1 : (i)\mathcal{F}^d$([TYJUD$_4$])

and $\Gamma, \sigma \vdash \sqrt{e_1} : ()\mathcal{F}^d$([TYJUD$_9$])

Additionally, $\Gamma, \sigma \vdash \mathbf{lift}_d(-) : (\sigma)\mathcal{F}^d$ by [TYJUD$_8$]

Given that $\Gamma, \sigma \vdash \sqrt{e_1} : ()\mathcal{F}^d$ and $\Gamma, \sigma[i \mapsto (1,d)] \vdash \frac{\partial}{\partial x_{i:d}} \diamond e_1 : (i)\mathcal{F}^d$

then $\Gamma, \sigma[i \mapsto (1,d)] \vdash \frac{\frac{\partial}{\partial x_{i:d}} \diamond e_1}{\sqrt{e_1}} : (i)\mathcal{F}^d$ by [TYJUD$_{12}$]

and $\Gamma, \sigma[i \mapsto (1,d)] \vdash \mathbf{lift}_d(1/2) * \frac{\frac{\partial}{\partial x_i} \diamond e}{\sqrt{e_1}} : (i)\mathcal{F}^d$ by [TYJUD$_{11}$]

Case R9. $\frac{\partial}{\partial x_i} \diamond (\mathbf{cosine}(e_1)) \xrightarrow[rule]{} (-\mathbf{sine}(e_1)) * (\frac{\partial}{\partial x_i} \diamond e_1)$

Find $\Gamma, \sigma \vdash (\frac{\partial}{\partial x_i} \diamond (\mathbf{cosine}(e_1)))$

This type of structure inside a derivative operation results in a field type

and the $\mathbf{cosine}(e_1)$ term results in a scalar.

Claim: $\Gamma \vdash \mathbf{cosine}(e_1) : ()\mathcal{F}^d$ then $\Gamma_i \vdash \nabla_i \diamond (\mathbf{cosine}(e_1)) : (i)\mathcal{F}^d$ by [TYJUD$_4$]

The LHS has the following type.

$\Gamma, \sigma \vdash \frac{\partial}{\partial x_i} \diamond (\mathbf{cosine}(e_1))$:$(i)\mathcal{F}^d$

We want to show that the RHS has the same type.

$\Gamma, \sigma \vdash (-\mathbf{sine}(e_1)) * (\frac{\partial}{\partial x_i} \diamond e_1)$:$(i)\mathcal{F}^d$.

The type derivation for the LHS is the following structure.

$$\frac{\dfrac{\Gamma, \sigma \vdash e_1 : ()\mathcal{F}^d[\text{TYINV}_9]}{\Gamma, \sigma \vdash \mathbf{cosine}(e_1) : ()\mathcal{F}^d}}{\Gamma, \sigma[i \mapsto (1,d)] \vdash \dfrac{\partial}{\partial x_i} \diamond (\mathbf{cosine}(e_1)) : (i)\mathcal{F}^d}$$

From that we can make the RHS derivations.

Given that $\Gamma, \sigma \vdash e_1 : ()\mathcal{F}^d$

then $\Gamma, \sigma[i \mapsto (1,d)] \vdash \frac{\partial}{\partial x_i} \diamond e_1 : (i)\mathcal{F}^d$ by [TYJUD$_4$],

$\Gamma, \sigma \vdash \mathbf{sine}(e_1) : ()\mathcal{F}^d$ by [TYJUD$_9$],

$\Gamma, \sigma \vdash -\mathbf{sine}(e_1) : ()\mathcal{F}^d$ by [TYJUD$_{10}$],

and $\Gamma, \sigma[i \mapsto (1,d)] \vdash (-\mathbf{sine}(e_1)) * (\frac{\partial}{\partial x_i} \diamond e_1) : (i)\mathcal{F}^d$ by [TYJUD$_{11}$]

`T(R9) OK`

Case R10. $\frac{\partial}{\partial x_i} \diamond (\mathbf{sine}(e_1)) \xrightarrow[rule]{} (\mathbf{cosine}(e_1)) * (\frac{\partial}{\partial x_i} \diamond e_1)$. Included in the earlier prose. `T(R10) OK`

Case R11. $\frac{\partial}{\partial x_i} \diamond (\mathbf{tangent}(e_1)) \xrightarrow[rule]{} \frac{\frac{\partial}{\partial x_i} \diamond e}{\mathbf{cosine}(e_1) * \mathbf{cosine}(e_1)}$

This type of structure inside a derivative operation results in a field type

and the $\mathbf{tangent}(e_1)$ term results in a scalar.

Claim: $\Gamma \vdash \mathbf{tangent}(e_1) : ()\mathcal{F}^d$ then $\Gamma_i \vdash \nabla_i \diamond (\mathbf{tangent}(e_1)) : (i)\mathcal{F}^d$ by [TYJUD$_4$]

The LHS has the following type.

$\Gamma, \sigma \vdash \frac{\partial}{\partial x_i} \diamond (\mathbf{tangent}(e_1))$:$(i)\mathcal{F}^d$

We want to show that the RHS has the same type.

$\Gamma, \sigma \vdash \frac{\frac{\partial}{\partial x_i} \diamond e}{\mathbf{cosine}(e_1) * \mathbf{cosine}(e_1)}$:$(i)\mathcal{F}^d$.

The type derivation for the LHS is the following structure.

$$\frac{\dfrac{\Gamma, \sigma \vdash e_1 : ()\mathcal{F}^d[\text{TYINV}_9]}{\Gamma, \sigma \vdash \mathbf{tangent}(e_1) : ()\mathcal{F}^d}}{\Gamma, \sigma[i \mapsto (1,d)] \vdash \dfrac{\partial}{\partial x_i} \diamond (\mathbf{tangent}(e_1)) : (i)\mathcal{F}^d}$$

From that we can make the RHS derivations.

Given that $\Gamma, \sigma \vdash e_1 : ()\mathcal{F}^d$

then $\Gamma, \sigma[i \mapsto (1,d)] \vdash \frac{\partial}{\partial x_i} \diamond e_1 : (i)\mathcal{F}^d$ by [TYJUD$_4$],

$\Gamma, \sigma \vdash \mathbf{cosine}(e_1) * \mathbf{cosine}(e_1) : ()\mathcal{F}^d$ by [TYJUD$_9$], [TYJUD$_{11}$],

and $\Gamma, \sigma \vdash \frac{\frac{\partial}{\partial x_i} \diamond e}{\mathbf{cosine}(e_1) * \mathbf{cosine}(e_1)} : ()\mathcal{F}^d$ by [TYJUD$_{12}$]

`T( R11) OK`

Case R12. $\frac{\partial}{\partial x_i} \diamond (\mathbf{arccosine}(e_1)) \xrightarrow[rule]{} (\frac{-\mathbf{lift}_d(1)}{\sqrt{(\mathbf{lift}_d(1) - (e * e)}}) * (\frac{\partial}{\partial x_i} \diamond e_1)$

Similar approach to R13 `T(R12) OK`

Case R13. $\frac{\partial}{\partial x_i} \diamond (\mathbf{arcsine}(e_1)) \xrightarrow[rule]{} (\frac{\mathbf{lift}_d(1)}{\sqrt{(\mathbf{lift}_d(1) - (e * e)}}) * (\frac{\partial}{\partial x_i} \diamond e_1)$

Find $\Gamma, \sigma \vdash (\frac{\partial}{\partial x_i} \diamond (\mathbf{arcsine}(e_1)))$

This type of structure inside a derivative operation results in a field type

and the $\mathbf{arcsine}(e_1)$ term results in a scalar.

Claim: $\Gamma \vdash \mathbf{arcsine}(e_1) : ()\mathcal{F}^d$ then $\Gamma_i \vdash \nabla_i \diamond (\mathbf{arcsine}(e_1)) : (i)\mathcal{F}^d$ by [TYJUD$_4$]

The LHS has the following type.

$\Gamma, \sigma \vdash \frac{\partial}{\partial x_i} \diamond (\mathbf{arcsine}(e_1))$:$(i)\mathcal{F}^d$

We want to show that the RHS has the same type.

$\Gamma, \sigma \vdash (\frac{\mathbf{lift}_d(1)}{\sqrt{(\mathbf{lift}_d(1)-(e*e)}}) * (\frac{\partial}{\partial x_i} \diamond e_1)$:$(i)\mathcal{F}^d$.

The type derivation for the LHS is the following structure.

$$\frac{\dfrac{\Gamma, \sigma \vdash e_1 : ()\mathcal{F}^d([\mathrm{TYINV}_9])}{\Gamma, \sigma \vdash \mathbf{arcsine}(e_1) : ()\mathcal{F}^d}}{\Gamma, \sigma[i \mapsto (1,d)] \vdash \dfrac{\partial}{\partial x_i} \diamond (\mathbf{arcsine}(e_1)) : (i)\mathcal{F}^d}$$

Since $\Gamma, \sigma \vdash e_1 : ()\mathcal{F}^d$ then $\Gamma, \sigma[i \mapsto (1,d)] \vdash \frac{\partial}{\partial x_i} \diamond e_1 : (i)\mathcal{F}^d$ by [TYJUD_4]

Find $\Gamma, \sigma \vdash (\mathbf{lift}_d(1))$

$\Gamma, \sigma \vdash \mathbf{lift}_d(1) : (\sigma)\mathcal{F}^d([\mathrm{TYJUD}_8])$

From that we can make the RHS derivations.

Given that $\Gamma, \sigma \vdash e_1 : ()\mathcal{F}^d$

then $\Gamma, \sigma \vdash e_1 * e_1 : ()\mathcal{F}^d$ by [TYJUD_{11}],

$\Gamma, \sigma \vdash \mathbf{lift}_d(1) - (e_1 * e_1) : ()\mathcal{F}^d$ by [TYJUD_{10}],

$\Gamma, \sigma \vdash \sqrt{\mathbf{lift}_d(1) - (e_1 * e_1)} : ()\mathcal{F}^d$ by [TYJUD_9],

$\Gamma, \sigma \vdash \frac{\mathbf{lift}_d(1)}{\sqrt{\mathbf{lift}_d(1)-(e_1*e_1)}} : ()\mathcal{F}^d$ by [TYJUD_{12}],

and $\Gamma, \sigma[i \mapsto (1,d)] \vdash (\frac{\mathbf{lift}_d(1)}{\sqrt{(\mathbf{lift}_d(1)-(e*e)}}) * (\frac{\partial}{\partial x_i} \diamond e_1) : (i)\mathcal{F}^d$ by [TYJUD_{11}]

`T( R13) OK`

Case R14. $\frac{\partial}{\partial x_i} \diamond (\mathbf{arctangent}(e_1)) \xrightarrow[rule]{} \frac{\mathbf{lift}_d(1)}{\mathbf{lift}_d(1)+(e_1*e_1)} * (\frac{\partial}{\partial x_i} \diamond e_1)$

Similar approach to R13 `T( R14) OK`

Case R15. $\frac{\partial}{\partial x_i} \diamond (\mathbf{exp}(e_1)) \xrightarrow[rule]{} \mathbf{exp}(e_1) * (\frac{\partial}{\partial x_i} \diamond e_1)$

Find $\Gamma, \sigma \vdash (\frac{\partial}{\partial x_i} \diamond (\mathbf{exp}(e_1)))$

This type of structure inside a derivative operation results in a field type and the $\exp(e_1)$ term results in a scalar.

Claim: $\Gamma \vdash \exp(e_1) : ()\mathcal{F}^d$ then $\Gamma_i \vdash \nabla_i \diamond (\exp(e_1)) : (i)\mathcal{F}^d$ by [TYJUD_4]

The LHS has the following type.

$\Gamma, \sigma \vdash \frac{\partial}{\partial x_i} \diamond (\mathbf{exp}(e_1))$:$(i)\mathcal{F}^d$

We want to show that the RHS has the same type.

$\Gamma, \sigma \vdash \mathbf{exp}(e_1) * (\frac{\partial}{\partial x_i} \diamond e_1)$:$(i)\mathcal{F}^d$.

The type derivation for the LHS is the following structure.

$$\frac{\Gamma, \sigma \vdash e_1 : ()\mathcal{F}^d([\mathrm{TYINV}_9])}{\Gamma, \sigma \vdash \mathbf{exp}(e_1) : ()\mathcal{F}^d}$$

From that we can make the RHS derivations.

Given that $\Gamma, \sigma \vdash e_1 : ()\mathcal{F}^d$

then $\Gamma, \sigma[i \mapsto (1,d)] \vdash \frac{\partial}{\partial x_i} \diamond e_1 : (i)\mathcal{F}^d$ by [TYJUD_4],

$\Gamma, \sigma \vdash \mathbf{exp}(e_1) : ()\mathcal{F}^d$ by [TYJUD_9],

and $\Gamma, \sigma[i \mapsto (1,d)] \vdash \mathbf{exp}(e_1) * (\frac{\partial}{\partial x_i} \diamond e_1) : (i)\mathcal{F}^d$ by [TYJUD_{11}]

`T(R15) OK`

Case R16. $\frac{\partial}{\partial x_i} \diamond (e_1^n) \xrightarrow[rule]{} \mathbf{lift}_d(n) * e_1^{n-1} * (\frac{\partial}{\partial x_i} \diamond e_1)$

This type of structure inside a derivative operation results in a field type and the e_1^n term results in a scalar.

Claim: $\Gamma \vdash e_1^n : ()\mathcal{F}^d$ then $\Gamma_i \vdash \nabla_i \diamond (e_1^n) : (i)\mathcal{F}^d$ by [TYJUD_4]

The LHS has the following type.

$\Gamma, \sigma \vdash \frac{\partial}{\partial x_i} \diamond (e_1^n)$:$(i)\mathcal{F}^d$

We want to show that the RHS has the same type.

$\Gamma, \sigma \vdash \mathbf{lift}_d(n) * e_1^{n-1} * (\frac{\partial}{\partial x_i} \diamond e_1)$:$(i)\mathcal{F}^d$.

The type derivation for the LHS is the following structure.

$$\frac{\dfrac{\Gamma, \sigma \vdash e_1 : ()\mathcal{F}^d, \Gamma, \sigma \vdash n : ()\mathcal{T} \text{ and } \sigma = \{i : d\}([\mathrm{TYINV}_9])}{\Gamma, \sigma \setminus i \vdash (e^n) : (\sigma \setminus i)\mathcal{F}^d[\mathrm{TYINV}_4]}}{\Gamma, \sigma \vdash \dfrac{\partial}{\partial x_{i:d}} \diamond (e^n) : (i)\mathcal{F}^d}$$

From that we can make the RHS derivations.

Given that $\Gamma, \sigma \vdash e_1 : ()\mathcal{F}^d$ then $\Gamma, \sigma[i \mapsto (1,d)] \vdash \frac{\partial}{\partial x_i} \diamond e_1 : (i)\mathcal{F}^d$ by [TYJUD_4].

Given that $\Gamma, \sigma \vdash e_1 : ()\mathcal{F}^d, \Gamma, \sigma \vdash n : ()\mathcal{T}$

then $\Gamma, \sigma \vdash \mathbf{lift}_d(n) : ()\mathcal{F}^d$ by [TYJUD_8] and $\Gamma, \sigma \vdash e^{n-1} : ()\mathcal{F}^d$ by [TYJUD_9].

Given that $\Gamma, \sigma \vdash e^{n-1} : ()\mathcal{F}^d$ and $\Gamma, \sigma[i \mapsto (1,d)] \vdash \frac{\partial}{\partial x_i} \diamond e_1 : (i)\mathcal{F}^d$

then $\Gamma, \sigma[i \mapsto (1,d)] \vdash \mathbf{lift}_d(n) * e_1^{n-1} * (\frac{\partial}{\partial x_i} \diamond e_1) : (i)\mathcal{F}^d$ by [TYJUD_{11}].

`T(R16) OK`

Case R17. $\frac{\partial}{\partial x_i} \diamond (e_1 \odot e_2) \underset{rule}{\longrightarrow} (\frac{\partial}{\partial x_i} \diamond e_1) \odot (\frac{\partial}{\partial x_i} \diamond e_2)$
Find $\Gamma, \sigma \vdash (\frac{\partial}{\partial x_i} \diamond (e_1 \odot e_2))$
This type of structure inside a derivative operation results in a field type.
Given the subterm: $\Gamma, \sigma/\ i \vdash e_1 \odot e_2 : (\sigma/\ i)\mathcal{F}^d$
then by [TYJUD$_4$] we know it's derivative $\Gamma, \sigma \vdash \nabla_i \diamond (e_1 \odot e_2) : (\sigma)\mathcal{F}^d$
The LHS has the following type.
$\Gamma, \sigma \vdash \frac{\partial}{\partial x_i} \diamond (e_1 \odot e_2) :(\sigma)\mathcal{F}^d$
We want to show that the RHS has the same type.
$\Gamma, \sigma \vdash (\frac{\partial}{\partial x_i} \diamond e_1) \odot (\frac{\partial}{\partial x_i} \diamond e_2):(\sigma)\mathcal{F}^d$.
The type derivation for the LHS is the following structure.
Find $\Gamma, \sigma \vdash (\tau(e_1)$ and $\tau(e_2))$

$$\frac{\dfrac{\Gamma, \sigma \setminus i \vdash e_1, e_2 : (\sigma \setminus i)\mathcal{F}^d[\text{TYINV}_{10}]}{\Gamma, \sigma \setminus i \vdash e_1 \odot e_2 : (\sigma \setminus i)\mathcal{F}^d[\text{TYINV}_4]}}{\Gamma, \sigma \vdash \dfrac{\partial}{\partial x_i} \diamond (e_1 \odot e_2) : (\sigma)\mathcal{F}^d}$$

From that we can make the RHS derivations.
Given that $\Gamma, \sigma \vdash e_1, e_2 : (\sigma \setminus i)\mathcal{F}^d$
then $\Gamma, \sigma \vdash \frac{\partial}{\partial x_i} \diamond (e_1) : (\sigma)\mathcal{F}^d$ by [TYJUD$_4$]
and $\Gamma, \sigma \vdash (\frac{\partial}{\partial x_i} \diamond e_1) \odot (\frac{\partial}{\partial x_i} \diamond e_2) : (\sigma)\mathcal{F}^d$ by [TYINV$_{10}$].

```
T( R17) OK
```

Case R18. $\frac{\partial}{\partial x_i} \diamond (-e_1) \underset{rule}{\longrightarrow} -(\frac{\partial}{\partial x_i} \diamond e_1)$
Find $\Gamma, \sigma \vdash (\frac{\partial}{\partial x_i} \diamond (-e_1))$
This type of structure inside a derivative operation results in a field type.
Given the subterm: $\Gamma, \sigma/\ i \vdash -e_1 : (\sigma/\ i)\mathcal{F}^d$
then by [TYJUD$_4$] we know it's derivative $\Gamma, \sigma \vdash \nabla_i \diamond (-e_1) : (\sigma)\mathcal{F}^d$
The LHS has the following type.
$\Gamma, \sigma \vdash \frac{\partial}{\partial x_i} \diamond (-e_1) :(\sigma)\mathcal{F}^d$
We want to show that the RHS has the same type.
$\Gamma, \sigma \vdash -(\frac{\partial}{\partial x_i} \diamond e_1):(\sigma)\mathcal{F}^d$.
The type derivation for the LHS is the following structure.

$$\frac{\dfrac{\Gamma, \sigma \setminus i \vdash e_1 : (\sigma \setminus i)\mathcal{F}^d[\text{TYINV}_{10}]}{\Gamma, \sigma \setminus i \vdash -e_1 : (\sigma \setminus i)\mathcal{F}^d[\text{TYINV}_4]}}{\Gamma, \sigma \vdash \dfrac{\partial}{\partial x_i} \diamond (-e_1) : (\sigma)\mathcal{F}^d}$$

From that we can make the RHS derivations.
Given that $\Gamma, \sigma \vdash e_1 : (\sigma \setminus i)\mathcal{F}^d$
then $\Gamma, \sigma \vdash \frac{\partial}{\partial x_i} \diamond (e_1) : (\sigma)\mathcal{F}^d$ by [TYJUD$_4$]
and $\Gamma, \sigma \vdash -(\frac{\partial}{\partial x_i} \diamond e_1) : (\sigma)\mathcal{F}^d$ by [TYINV$_{10}$]

```
T( R18) OK
```

Case R19. $\frac{\partial}{\partial x_i} \sum_{v=1}^{n} e_1 \underset{rule}{\longrightarrow} \sum_{v=1}^{n} (\frac{\partial}{\partial x_i} e_1)$
This type of structure inside a derivative operation results in a field type.
Given the subterm: $\Gamma, \sigma/\ i \vdash \sum_{v=1}^{n} : (\sigma/\ i)\mathcal{F}^d$
then by [TYJUD$_4$] we know it's derivative $\Gamma, \sigma \vdash \nabla_i \diamond (\sum_{v=1}^{n}) : (\sigma)\mathcal{F}^d$
The LHS has the following type.
$\Gamma, \sigma \vdash \frac{\partial}{\partial x_i} \sum_{v=1}^{n} e_1 :(\sigma)\mathcal{F}^d$
We want to show that the RHS has the same type.
$\Gamma, \sigma \vdash \sum_{v=1}^{n} (\frac{\partial}{\partial x_i} e_1):(\sigma)\mathcal{F}^d$.
The type derivation for the LHS is the following structure.

$$\frac{\dfrac{\Gamma, \sigma \setminus i, v : n \vdash e_1 : (\sigma \setminus i, v : n)\mathcal{F}^d([\text{TYINV}_3])}{\Gamma, \sigma \setminus i \vdash (\displaystyle\sum_{v=1}^{n} e_1) : (\sigma \setminus i)\mathcal{F}^d[\text{TYINV}_4]}}{\Gamma, \sigma \vdash \dfrac{\partial}{\partial x_{i:d}} \diamond (\displaystyle\sum_{v=1}^{n} e_1) : (\sigma)\mathcal{F}^d}$$

From that we can make the RHS derivations.
Given that $\Gamma, \sigma \vdash e_1 : (\sigma \setminus i, v : n)\mathcal{F}^d$
then $\Gamma, \sigma \vdash \frac{\partial}{\partial x_{i:d}} \diamond (e_1) : (\sigma, v : n)\mathcal{F}^d$ by [TYJUD$_4$]

and $\Gamma, \sigma \vdash \sum_{v=1}^{n} (\frac{\partial}{\partial x_{i:d}} \diamond (e_1)) : (\sigma)\mathcal{F}^d$ by ([TYJUD$_3$])

`T(R19) OK`

Case R20.$\frac{\partial}{\partial x_i}\chi \xrightarrow[rule]{} 0$

This type of structure inside a derivative operation results in a field type.

Given the subterm: $\Gamma, \sigma/\ i \vdash \nabla\chi : (\sigma/\ i)\mathcal{F}^d$

then by [TYJUD$_4$] we know it's derivative $\Gamma, \sigma \vdash \nabla_i \diamond (\nabla\chi) : (\sigma)\mathcal{F}^d$

Lastly,$\Gamma, \sigma \vdash 0 : (\sigma)\mathcal{F}^d$ by [TYJUD$_8$]. `T( R20) OK`

Case R21.$\frac{\partial}{\partial x_i} \diamond (V_\alpha \circledast H^\nu) \xrightarrow[rule]{} (V_\alpha \circledast h^{i\nu})$

Given$\Gamma, \sigma \vdash V_\alpha \circledast H^v : (\sigma/\ i)\mathcal{F}^d$ by [TYJUD$_2$]

then $\Gamma, \sigma \vdash \frac{\partial}{\partial x_i} \diamond (V_\alpha \circledast H^\nu) : (\sigma)\mathcal{F}^d$ by [TYJUD$_4$].

Lastly, $\Gamma, \sigma \vdash (V_\alpha \circledast H^{i\nu}) : (\sigma)\mathcal{F}^d$ by [TYJUD$_2$].

`T(R21) OK`

Case R22.$- - e_1 \xrightarrow[rule]{} e_1$

Find $\Gamma, \sigma \vdash (- - e_1)$

Assign generic type $\Gamma, \sigma \vdash e_1 : \tau$

$$\cfrac{\cfrac{\Gamma, \sigma \vdash - - e_1 : \tau[\text{TYINV}_{10}]}{\Gamma, \sigma \vdash -e_1 : \tau[\text{TYINV}_{10}]}}{\Gamma, \sigma \vdash e_1 : \tau}$$

From that we can make the RHS derivations.

`T(R22) OK`

Case R23.$-0 \xrightarrow[rule]{} 0$

Find $\Gamma, \sigma \vdash (-0)$

Assign generic type $\Gamma, \sigma \vdash -0 : \tau$

Find $\Gamma, \sigma \vdash (0)$

The LHS has the following type.

$\Gamma, \sigma \vdash -0 :\tau$

We want to show that the RHS has the same type.

$\Gamma, \sigma \vdash 0:\tau$.

The type derivation for the LHS is the following structure.

$$\frac{\Gamma, \sigma \vdash 0 : \tau[\text{TYINV}_{10}]}{\Gamma, \sigma \vdash -0 : \tau}$$

From that we can make the RHS derivations.

`T(R23) OK`

Case R24.$e_1 - 0 \xrightarrow[rule]{} e_1$

Find $\Gamma, \sigma \vdash (e_1 - 0)$

Assign generic type $\Gamma, \sigma \vdash e_1 - 0 : \tau$

$\Gamma, \sigma \vdash e - 0 : (\sigma)\tau_0$

$\Gamma, \sigma \vdash 0 : (\sigma)\tau_0$ by [TYJUD$_1$]

T(R24)

`T(R24) OK`

Case R25.$0 - e_1 \xrightarrow[rule]{} - e_1$

Similar approach to R24 `T(R25) OK`

Case R26.$\frac{0}{e_1} \xrightarrow[rule]{} 0$

Similar approach to R24 `T(R26) OK`

Case R27.$\frac{\frac{e_1}{e_2}}{e_3} \xrightarrow[rule]{} \frac{e_1}{e_2 e_3}$. Included in the earlier prose.

Case R28.$\frac{e_1}{\frac{e_2}{e_3}} \xrightarrow[rule]{} \frac{e_1 e_3}{e_2}$

Similar approach to R27 `T(R28) OK`

Case R29.$\frac{\frac{e_1}{e_2}}{\frac{e_3}{e_4}} \xrightarrow[rule]{} \frac{e_1 e_4}{e_2 e_3}$

The LHS has the following type.

$\Gamma, \sigma \vdash \frac{\frac{e_1}{e_2}}{\frac{e_3}{e_4}} :(\sigma)\tau_0$

We want to show that the RHS has the same type.

$\Gamma, \sigma \vdash \frac{e_1 e_4}{e_2 e_3}:(\sigma)\tau_0$.

The type derivation for the LHS is the following structure.

$$\dfrac{\dfrac{\Gamma,\sigma\vdash e_1:(\sigma)\tau_0 \qquad \Gamma,\sigma\vdash e_2:()\tau_0[\text{TYINV}_{12}]}{\Gamma,\sigma\vdash(\dfrac{e_1}{e_2}):(\sigma)\tau_0} \qquad \dfrac{\Gamma,\sigma\vdash e_3,e_4:()\tau_0[\text{TYINV}_{12}]}{\Gamma,\sigma\vdash(\dfrac{e_3}{e_4}):()\tau_0[\text{TYINV}_{12}]}}{\Gamma,\sigma\vdash\frac{\frac{e_1}{e_2}}{\frac{e_3}{e_4}}:(\sigma)\tau_0}$$

From that we can make the RHS derivations.
Find $\Gamma,\sigma\vdash(\frac{e_1e_4}{e_2e_3})$
Given that $\Gamma,\sigma\vdash e_1:(\sigma)\tau_0$ and $\Gamma,\sigma\vdash e_2,e_3,e_4:()\tau_0$
then $\Gamma,\sigma\vdash e_1*e_4:(\sigma)\tau_0$ by [TYJUD_{11}],
$\Gamma,\sigma\vdash e_2*e_3:()\tau_0$ by [TYJUD_{11}],
and $\Gamma,\sigma\vdash\frac{e_1e_4}{e_2e_3}:(\sigma)\tau_0$ by [TYJUD_{12}].
`T(R29) OK`

Case R30.$0+e_1,e_1+0\xrightarrow[rule]{}e_1$
Similar approach to R24 `T(R30) OK`

Case R31.$0e,e0\xrightarrow[rule]{}0$
Similar approach to R24 `T(R31) OK`

Case R32.$\sqrt{(e_1)}*\sqrt{(e_1)}\xrightarrow[rule]{}e_1$
Assign generic type $\Gamma,\sigma\vdash\sqrt{(e_1)}*\sqrt{(e_1)}:\tau$
Find $\Gamma,\sigma\vdash(e_1)$

$$\dfrac{\dfrac{\Gamma,\sigma\vdash e_1:\tau([\text{TYINV}_9])}{\Gamma,\sigma\vdash\sqrt{e_1}:\tau[\text{TYINV}_{11}]}}{\Gamma,\sigma\vdash\sqrt{e_1}*\sqrt{e_1}:\tau}$$

`T(R32) OK`

Case R33.$\mathcal{E}_{ijk}\frac{\partial}{\partial x_{i,j}}\diamond e_1\xrightarrow[rule]{}\textbf{lift}_d(0)$
Similar approach to R34 `T(R33) OK`

Case R34.$\mathcal{E}_{ijk}(V_\alpha\circledast h^{jk})\xrightarrow[rule]{}\textbf{lift}_d(0)$
Given $\Gamma,\sigma\vdash V_\alpha\circledast h^{jk}:(\sigma)\mathcal{F}^d$ by [TYJUD_2]
then $\Gamma,\sigma\vdash\epsilon_{ijk}V_\alpha\circledast h^{jk}:(\sigma)\mathcal{F}^d$ by [TYJUD_6].
Lastly, $\Gamma,\sigma\vdash\textbf{lift}_d(0):(\sigma)\mathcal{F}^d$ by [TYJUD_8]
`T(R34) OK`

Case R35.$\mathcal{E}_{ijk}\mathcal{E}_{ilm}\xrightarrow[rule]{}\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}$
We know $\Gamma,\sigma\vdash\mathcal{E}_{ijk}\mathcal{E}_{ilm}:(\sigma)\mathcal{T}$ by [TYJUD_6].
Given $\Gamma,\sigma\vdash\delta_{jl}\delta_{km}:(\sigma)\mathcal{T}$ by [TYJUD_5]
then $\Gamma,\sigma\vdash\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}:(\sigma)\mathcal{T}$ by [TYJUD_{10}].
`T(R35) OK`

Case R36.$\delta_{ij}T_j\xrightarrow[rule]{}T_i$
Find $\Gamma,\sigma\vdash(\delta_{ij}T_j)$
Given $\Gamma,\sigma\vdash T_j:(\sigma)\mathcal{T}$ and $\sigma=\{j\}$ by [TYJUD_1]
then $\Gamma,\sigma\vdash\delta_{ij}(T_j):(\sigma)\mathcal{T}$ by [TYJUD_5]
and $\sigma=\{i\}$[TYJUD_5]
Find $\Gamma,\sigma\vdash(T_i)$
$\Gamma,\sigma\vdash T_i:(\sigma)\mathcal{F}^d$ and $\sigma=\{i\}$[TYJUD_1]
`T(R36) OK`

Case R37.$\delta_{ij}F_j\xrightarrow[rule]{}F_i$
Similar approach to R36 `T(R37) OK`

Case R38.$\delta_{ij}V\circledast H^{\delta_{cj}}\xrightarrow[rule]{}V\circledast H^{\delta_{ci}}$
Given $\Gamma,\sigma\vdash V\circledast H^{\delta_{cj}}:(\sigma)\mathcal{F}^d$ and $\sigma=\{j\}$ by [TYJUD_2]
then $\Gamma,\sigma\vdash\delta_{ij}(V\circledast H^{\delta_{cj}}):(\sigma)\mathcal{F}^d$ and $\sigma=\{i\}$ by [TYJUD_5]
$\Gamma,\sigma\vdash V\circledast H^{\delta_{ci}}:(\sigma)\mathcal{F}^d$ and $\sigma=\{i\}$[TYJUD_2]
`T(R38) OK`

Case R39.$\delta_{ij}V\circledast H^{\delta_{cj}}(x)\xrightarrow[rule]{}V\circledast H^{\delta_{ci}}(x)$
Similar approach to R38 `T(R39) OK`

Case R40.$\delta_{ij}\frac{\partial}{\partial x_j}\diamond e_1\xrightarrow[rule]{}\frac{\partial}{\partial x_i}\diamond(e_1)$. Included in the earlier prose.

Case R41.$\sum(se_1)\xrightarrow[rule]{}s\sum e_1$. Included in the earlier prose.

Case R42.$\frac{\partial}{\partial x_\alpha}\diamond\frac{\partial}{\partial x_\beta}\diamond e_1\xrightarrow[rule]{}\frac{\partial}{\partial x_{\beta\alpha}}\diamond e_1$
This type of structure inside a derivative operation results in a field type.
Claim: $\Gamma,\sigma/\ \alpha\beta\vdash e_1:(\sigma/\ \alpha\beta)\mathcal{F}^d$
The LHS has the following type.

$\Gamma, \sigma \vdash \frac{\partial}{\partial x_\alpha} \diamond \frac{\partial}{\partial x_\beta} \diamond e_1 :(\sigma)\mathcal{F}^d$

We want to show that the RHS has the same type.

$\Gamma, \sigma \vdash \frac{\partial}{\partial x_{\beta\alpha}} \diamond e_1{:}(\sigma)\mathcal{F}^d$.

The type derivation for the LHS is the following structure.

$$\cfrac{\cfrac{\Gamma, \sigma \vdash e_1 : (\sigma/\ \alpha\beta)\mathcal{F}^d[\text{TYJUD}_4]}{\Gamma, \sigma \vdash (\dfrac{\partial}{\partial x_\beta} \diamond e_1) : (\sigma/\ \alpha)\mathcal{F}^d[\text{TYJUD}_4]}}{\Gamma, \sigma \vdash (\dfrac{\partial}{\partial x_\alpha} \diamond \dfrac{\partial}{\partial x_\beta} \diamond e_1) : (\sigma)\mathcal{F}^d}$$

From that we can make the RHS derivations.

Given that $\Gamma, \sigma \vdash e : \sigma/\ \alpha\beta$ then $\Gamma, \sigma \vdash \frac{\partial}{\partial x_{\beta\alpha}} \diamond e : (\sigma)\mathcal{F}^d$ by [TYJUD_4]

`T( R42) OK` T(d) Lemma 2.1

B Value Preservation Proof

The following is a proof for Theorem 3.1 Given a derivation d of the form $e \longrightarrow e'$ we state V(d) as a shorthand for the claim that the derivation preserves the value of the expression e. The proof demonstrates that $\forall d.V(d)$.

Case on structure of d

Case Rules R1-R5 use the probe operator.

Value representation of the probe operator is not supported.

Case Rules R6-R21 use the differentiation operator.

Value representation of the differentiation operator is not supported.

Case R22.$--e_1 \underset{rule}{\longrightarrow} e_1$

Claim $--e_1$ evaluates to v.

We need to define v.

Assume that $e_1 \Downarrow v'$

then $\Psi, \rho \vdash -e_1 \Downarrow -v'$ by [VALJUD_4],

and $\Psi, \rho \vdash --e_1 \Downarrow --v'$ by [VALJUD_4]

The value of v is $--v'$.

By using algebraic reasoning: $--v' = v'$. Since $--e_1 \Downarrow v$ and $--e_1 \Downarrow v'$ then $v = v'$.

The last step leads to $e_1 \Downarrow v$

`V(R22) OK`

Case R23.$-0 \underset{rule}{\longrightarrow} 0$

Claim -0 evaluates to v.

We need to define v.

then $\Psi, \rho \vdash 0 \Downarrow \textit{Real}()(0)$ by [VALJUD_1] , and $\Psi, \rho \vdash -0 \Downarrow \textit{Real}()(-0)$ by [VALJUD_4]

The value of v is $\textit{Real}()(-0)$

By using algebraic reasoning: $\textit{Real}()(-0) = \textit{Real}()(0)$

The last step leads to $0 \Downarrow v$

`V(R23) OK`

Case R24.$e_1 - 0 \underset{rule}{\longrightarrow} e_1$

Included in the earlier prose.

Case R25.$0 - e_1 \underset{rule}{\longrightarrow} -e_1$

Claim $0 - e_1$ evaluates to v.

We need to define v.

Assume that $-e_1 \Downarrow v'$

then $\Psi, \rho \vdash 0 - e_1 \Downarrow \textit{Real}()(0) + v'$ by ([VALJUD_1], [VALJUD_5]).

The value of v is $\textit{Real}()(0) + v'$. By using algebraic reasoning: $\textit{Real}()(0) + v' = v'$.

Since $0 - e_1 \Downarrow v$ and $0 - e_1 \Downarrow v'$ then $v = v'$

The last step leads to $-e_1 \Downarrow v$

`V(R25) OK`

Case R26.$\frac{0}{e_1} \underset{rule}{\longrightarrow} 0$

Assume that $e_1 \Downarrow \textit{Real}()(v2)$ then $\Psi, \rho \vdash \frac{0}{e_1} \Downarrow \textit{Real}()(\frac{0}{v2})$ by ([VALJUD_1], [VALJUD_5]).

The value of v is $\textit{Real}()(\frac{0}{v2})$. By using algebraic reasoning: $\textit{Real}()(\frac{0}{v2}) = \textit{Real}()(0)$

Lastly, $\Psi, \rho \vdash 0 \Downarrow \textit{Real}()(0)$ by ([VALJUD_1])

The last step leads to $0 \Downarrow v$

`V(R26) OK`

Case R27.$\frac{\frac{e_1}{e_2}}{e_3} \underset{rule}{\longrightarrow} \frac{e_1}{e_2 e_3}$

Claim $\frac{\frac{e_1}{e_2}}{e_3}$ evaluates to v.
We need to define v.
Assume that $\frac{e_1}{e_2 e_3} \Downarrow v', e_1 \Downarrow v1, e_2 \Downarrow v2, e_3 \Downarrow v3$.
then $\Psi, \rho \vdash \frac{e_1}{e_2} \Downarrow \frac{v1}{v2}$ by [VALJUD$_5$] and $\Psi, \rho \vdash \frac{\frac{e1}{e_2}}{e_3} \Downarrow \frac{\frac{v1}{v2}}{v3}$ by [VALJUD$_5$].
Given that $e_1 \Downarrow v1$ $e_2 \Downarrow v2$ $e_3 \Downarrow v3$
then $\Psi, \rho \vdash e_2 e_3 \Downarrow v2 * v3$ by [VALJUD$_5$] and $\Psi, \rho \vdash \frac{e1}{e_2 e_3} \Downarrow \frac{v1}{v2*v3}$ by [VALJUD$_5$].
The value of v is $\frac{v1}{v2*v3}$. By using algebraic reasoning: $v' = \frac{v1}{v2*v3} = \frac{\frac{v1}{v2}}{v3} = v$.
The last step leads to $\frac{e_1}{e_2 e_3} \Downarrow v$
V(R27) OK
Case R28.$\frac{\frac{e_1}{e_2}}{e_3} \xrightarrow[rule]{} \frac{e_1 e_3}{e_2}$
Similar approach to R27 V(R28) OK
Case R29.$\frac{\frac{e_1}{e_2}}{\frac{e_3}{e_4}} \xrightarrow[rule]{} \frac{e_1 e_4}{e_2 e_3}$
Similar approach to R27 V(R29) OK
Case R30.$0 + e_1, e_1 + 0 \xrightarrow[rule]{} e_1$ Claim $0 + e_1, e_1 + 0$ evaluates to v.
We need to define v.
Assume that $e_1 \Downarrow v'$ then $\Psi, \rho \vdash e_1 + 0 \Downarrow v' + \mathit{Real}()(0)$ by ([VALJUD$_1$], [VALJUD$_5$]).
By using algebraic reasoning $v' + \mathit{Real}()(0) = v'$
The last step leads to $e_1 \Downarrow v$
V(R30) OK
Case R31.$0e, e0 \xrightarrow[rule]{} 0$
Similar approach to R26 V(R31) OK
Case R32.$\sqrt{(e_1)} * \sqrt{(e_1)} \xrightarrow[rule]{} e_1$
Included in the earlier prose.
Case R33.$\mathcal{E}_{ijk} \frac{\partial}{\partial x_{i,j}} \diamond e_1 \xrightarrow[rule]{} \mathbf{lift}_d(0)$
Value representation not supported
Case R34.$\mathcal{E}_{ijk}(V_\alpha \circledast h^{jk}) \xrightarrow[rule]{} \mathbf{lift}_d(0)$
Value representation not supported
Case R35.$\mathcal{E}_{ijk}\mathcal{E}_{ilm} \xrightarrow[rule]{} \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$
Included in the earlier prose.
Case R36.$\delta_{ij} T_j \xrightarrow[rule]{} T_i$
Included in the earlier prose.
Case Rules R37-R40 uses field terms
Value representation of the field terms is not supported.

Case R41.$\sum(se_1) \xrightarrow[rule]{} s \sum e_1$
Claim $\sum(se_1)$ evaluates to v.
We need to define v.
Assume that $s \Downarrow v_s$ and $e_1 \Downarrow v_e$
then $\Psi, \rho \vdash s * e_1 \Downarrow v_s * v_e$ by ([VALJUD$_5$])
and $\Psi, \rho \vdash \sum(se_1) \Downarrow \sum(v_s * v_e)$ by [VALJUD$_4$]
The value of v is $\sum(v_s * v_e)$
$v = v_s * \sum(v_e)$ by moving scalar outside summation
We need to show that $s \sum e_1$ evaluates to v.
Given that $s \Downarrow v_s$ and $e \Downarrow v_e$
then $\Psi, \rho \vdash \sum e \Downarrow \sum v_e$ by ([VALJUD$_4$]) and $\Psi, \rho \vdash s \sum e_1 \Downarrow v_s * \sum v_e$ by ([VALJUD$_5$])
The last step leads to $s \sum e_1 \Downarrow v$
V(R41) OK
Case R42.$\frac{\partial}{\partial x_\alpha} \diamond \frac{\partial}{\partial x_\beta} \diamond e_1 \xrightarrow[rule]{} \frac{\partial}{\partial x_{\beta\alpha}} \diamond e_1$
Value representation not supported

C Termination

C.1 Size reduction

If e $\Longrightarrow$ e' then $\mathcal{S}(e) > \mathcal{S}(e') \geq 0$ (Lemma 4.1). The following are a few helpful lemmas that will be referred to in the proof.

Lemma C.1. $5^{(1+x)} > (16 + 5^x)$

$$\begin{array}{ll} 5^x > 4. & \textit{Given } x >= 1 \\ 4 * 5^x > 16 & \textit{Multiply by 4} \\ 5 * 5^x - 5^x > 16 & \textit{Refactor left side} \\ 5 * 5^x > (16 + 5^x) & \textit{Add } 5^x \\ 5^{(1+x)} > (16 + 5^x) & \textit{Rewritten} \end{array}$$

Lemma C.2. $5^{([\![e_1]\!]+[\![e_2]\!])} > 5^{([\![e_1]\!])} > 4.$

Lemma C.3. $(1 + [\![e_1]\!])5^{(1+[\![e_1]\!])} > [\![e_1]\!](16 + 5^{[\![e_1]\!]}) + 20$

$$\begin{array}{ll} 5^{(1+[\![e_1]\!])} > 16 + 5^{[\![e_1]\!]} & \textit{Lemma C.1} \\ [\![e_1]\!]5^{(1+[\![e_1]\!])} > [\![e_1]\!](16 + 5^{[\![e_1]\!]}) & \textit{Multiply by } [\![e_1]\!] \\ [\![e_1]\!]5^{(1+[\![e_1]\!])} + 5^{(1+[\![e_1]\!])} > [\![e_1]\!](16 + 5^{[\![e_1]\!]}) + 5^{(1+[\![e_1]\!])} & \textit{Add } 5^{(1+[\![e_1]\!])} \\ (1 + [\![e_1]\!])5^{(1+[\![e_1]\!])} > [\![e_1]\!](16 + 5^{[\![e_1]\!]}) + 5 * 5^{[\![e_1]\!]} > [\![e_1]\!](16 + 5^{[\![e_1]\!]}) + 20 & (\textit{Lemma C.2}) \end{array}$$

The following is a proof for Lemma 4.1 Given a derivation d of the form $e \longrightarrow e'$ we state P(d) as a shorthand for the claim that the derivation reduces the size of the expression e. By case analysis and comparing the size metric provided. This proof does a case analysis to show $\forall d \in \mathit{Deriv}.P(d)$. Case on structure of d

Case R1.$(e_1 \odot_n e_2)@x \xrightarrow[rule]{} (e_1@x) \odot_n (e_2@x)$. Included in the earlier prose.

Case R2.$(e_0 \odot_2 e_1)@x \xrightarrow[rule]{} (e_0@x) \odot_2 (e_1@x)$

$$\begin{aligned} [\![(e_0 \odot_2 e_1)@x]\!] &= 2 + 2[\![e_1]\!] + 2[\![e_2]\!] \\ &> 1 + 2[\![e_1]\!] + 2[\![e_2]\!] \\ &= [\![(e_0@x) \odot_2 (e_1@x)]\!] \end{aligned}$$

P(d)

Case R3.$(\odot_1 e_1)@x \xrightarrow[rule]{} \odot_1 (e_1@x)$

$$\begin{aligned} [\![(\odot_1 e_1)@x]\!] &= 2 + 2[\![e_1]\!] \\ &> 1 + 2[\![e_1]\!] \quad = [\![\odot_1(e_1@x)]\!] \end{aligned}$$

P(d)

Case R4.$(\sum_{i=1}^{n} e_1)@x \xrightarrow[rule]{} \sum_{i=1}^{n}(e_1@x)$

$$\begin{aligned} [\![(\sum_{i=1}^{n} e_1)@x]\!] &= 4 + 4[\![e_1]\!] \\ &> 2 + 4[\![e_1]\!] \\ &= [\![\sum_{i=1}^{n}(e_1@x)]\!] \end{aligned}$$

P(d)

Case R5.$(\chi)@x \xrightarrow[rule]{} \chi$

$$\begin{aligned} [\![(\chi)@x]\!] &= 2\mathcal{S}(\chi) \\ &> \mathcal{S}(\chi) \quad = [\![\chi]\!] \end{aligned}$$

Case R6.$\frac{\partial}{\partial x_i} \diamond (e_1 * e_2) \xrightarrow[rule]{} e_1(\frac{\partial}{\partial x_i} \diamond e_2) + e_2(\frac{\partial}{\partial x_i} \diamond e_1)$

We define $[\![(\frac{\partial}{\partial x_i} \diamond (e_1 * e_2))]\!]$=$s_1 + s_2 + s_3$

where $s_1 = [\![e_1]\!] * 5^{1+[\![e_1]\!]+[\![e_2]\!]}$, $s_2 = [\![e_2]\!] * 5^{1+[\![e_1]\!]+[\![e_2]\!]}$, and $s_3 = 5^{1+[\![e_1]\!]+[\![e_2]\!]}$,

We define $[\![(e_1 \frac{\partial}{\partial x_i} \diamond e_2 + e_2 \frac{\partial}{\partial x_i} \diamond e_1)]\!]$=$t_1 + t_2 + t_3$

where $t_1 = [\![e_1]\!](5^{[\![e_1]\!]} + 1)$, $t_2 = [\![e_2]\!](5^{[\![e_1]\!]} + 1)$, and $t_3 = 3$

Given $4 * 5^{1+[\![e_1]\!]} > 1$ then

$\longrightarrow 5 * 5^{[\![e_1]\!]} > 5^{[\![e_1]\!]} + 1$ by adding $5^{[\![e_1]\!]}$

$\longrightarrow 5^{1+[\![e_1]\!]+[\![e_2]\!]} > 5^{[\![e_1]\!]} + 1$ by refactoring

$\longrightarrow [\![e_1]\!] * 5^{1+[\![e_1]\!]+[\![e_2]\!]} > [\![e_1]\!](5^{[\![e_1]\!]} + 1)$ by multiplying by $[\![e_1]\!]$

$\longrightarrow [\![e_2]\!] * 5^{1+[\![e_1]\!]+[\![e_2]\!]} > [\![e_2]\!](5^{[\![e_1]\!]} + 1)$ by multiplying by $[\![e_2]\!]$

where and so $s_1 > t_1$,$s_2 > t_2$

where Lastly, $5^{1+[\![e_1]\!]+[\![e_2]\!]} > 3$ (Lm C.2) and so $s_3 > t_3$

Finally, $[\![\frac{\partial}{\partial x_i} \diamond (e_1 * e_2)]\!] > [\![e_1 \frac{\partial}{\partial x_i} \diamond e_2 + e_2 \frac{\partial}{\partial x_i} \diamond e_1]\!]$

P(d)

Case R7.$\frac{\partial}{\partial x_i} \diamond (\frac{e_1}{e_2}) \xrightarrow[rule]{} \frac{(\frac{\partial}{\partial x_i} \diamond e_1)e_2 - e_1(\frac{\partial}{\partial x_i} \diamond e_2)}{e_2^2}$

We define $[\![(\frac{\partial}{\partial x_i} \diamond (\frac{e_1}{e_2}))]\!]$=$s_1 + s_2 + s_3$

where $s_1 = [\![e_1]\!]5^{2+[\![e_1]\!]+[\![e_2]\!]}$, $s_2 = [\![e_2]\!]5^{2+[\![e_1]\!]+[\![e_2]\!]}$, and $s_3 = 2 * 5^{2+[\![e_1]\!]+[\![e_2]\!]}$

We define $[\![(\frac{(\frac{\partial}{\partial x_i} \diamond e_1)e_2 - e_1(\frac{\partial}{\partial x_i} \diamond e_2)}{e_2^2})]\!]$=$t_1 + t_2 + t_3$

where $t_1 = [\![e_1]\!](1 + 5^{[\![e_1]\!]})$, $t_2 = [\![e_2]\!](3 + 5^{[\![e_2]\!]})$, and $t_3 = 6$

Given $5^{2+[\![e_1]\!]+[\![e_2]\!]} > (1 + 5^{[\![e_1]\!]})$(Lm C.1)

where then $[\![e_1]\!]5^{2+[\![e_1]\!]+[\![e_2]\!]} > [\![e_1]\!](1+5^{[\![e_1]\!]})$ by multiplying by $[\![e_1]\!]$
where so $s_1 > t_1$,$s_2 > t_2$
Given $5^{1+[\![e_1]\!]+[\![e_2]\!]} > 5^{[\![e_2]\!]} + 3$ (Lm C.1)
where then $2 * 5^{1+[\![e_1]\!]+[\![e_2]\!]} > 2 * 5^{[\![e_2]\!]} + 6$ by multiplying by 2
where so $s_3 > t_3$
$[\![\text{source}(d)]\!] > [\![\text{target}(d)]\!]$
P(d)

Case R8. $\frac{\partial}{\partial x_i} \diamond (\sqrt{e_1}) \xrightarrow[rule]{} \textbf{lift}_d(1/2) * \frac{\frac{\partial}{\partial x_i} \diamond e_1}{\sqrt{e_1}}$

$$\begin{aligned} [\![\tfrac{\partial}{\partial x_i} \diamond (\sqrt{e_1})]\!] &= (1+[\![e_1]\!])5^{(1+[\![e_1]\!])} \\ &> [\![e_1]\!](1+5^{[\![e_1]\!]}) + 6 \\ &= [\![\textbf{lift}_d(1/2) * \tfrac{\frac{\partial}{\partial x_i} \diamond e}{\sqrt{e_1}}]\!] \end{aligned}$$

P(d)

Case R9. $\frac{\partial}{\partial x_i} \diamond (\textbf{cosine}(e_1)) \xrightarrow[rule]{} (-\textbf{sine}(e_1)) * (\frac{\partial}{\partial x_i} \diamond e_1)$. Included in the earlier prose.

Case R10. $\frac{\partial}{\partial x_i} \diamond (\textbf{sine}(e_1)) \xrightarrow[rule]{} (\textbf{cosine}(e_1)) * (\frac{\partial}{\partial x_i} \diamond e_1)$

$$\begin{aligned} [\![\tfrac{\partial}{\partial x_i} \diamond (\textbf{sine}(e_1))]\!] &= (1+[\![e_1]\!])5^{(1+[\![e_1]\!])} \\ &> [\![e_1]\!](1+5^{[\![e_1]\!]}) + 2 \\ &= [\![(\textbf{cosine}(e_1)) * (\tfrac{\partial}{\partial x_i} \diamond e_1)]\!] \end{aligned}$$

P(d)

Case R11. $\frac{\partial}{\partial x_i} \diamond (\textbf{tangent}(e_1)) \xrightarrow[rule]{} \frac{\frac{\partial}{\partial x_i} \diamond e}{\textbf{cosine}(e_1) * \textbf{cosine}(e_1)}$

$$\begin{aligned} [\![\tfrac{\partial}{\partial x_i} \diamond (\textbf{tangent}(e_1))]\!] &= (1+[\![e_1]\!])5^{(1+[\![e_1]\!])} \\ &> [\![e_1]\!](5^{[\![e_1]\!]} + 2) + 5 \\ &= [\![\tfrac{\frac{\partial}{\partial x_i} \diamond e}{\textbf{cosine}(e_1) * \textbf{cosine}(e_1)}]\!] \end{aligned}$$

P(d)

Case R12. $\frac{\partial}{\partial x_i} \diamond (\textbf{arccosine}(e_1)) \xrightarrow[rule]{} (\frac{-\textbf{lift}_d(1)}{\sqrt{(\textbf{lift}_d(1)-(e*e)}}) * (\frac{\partial}{\partial x_i} \diamond e_1)$

$$\begin{aligned} [\![\tfrac{\partial}{\partial x_i} \diamond (\textbf{arccosine}(e_1))]\!] &= (1+[\![e_1]\!])5^{(1+[\![e_1]\!])} \\ &> [\![e_1]\!](2+5^{[\![e_1]\!]}) + 11 \\ &= [\![(\tfrac{-\textbf{lift}_d(1)}{\sqrt{(\textbf{lift}_d(1)-(e*e)}}) * (\tfrac{\partial}{\partial x_i} \diamond e_1)]\!] \end{aligned}$$

P(d)

Case R13. $\frac{\partial}{\partial x_i} \diamond (\textbf{arcsine}(e_1)) \xrightarrow[rule]{} (\frac{\textbf{lift}_d(1)}{\sqrt{(\textbf{lift}_d(1)-(e*e)}}) * (\frac{\partial}{\partial x_i} \diamond e_1)$

$$\begin{aligned} [\![\tfrac{\partial}{\partial x_i} \diamond (\textbf{arcsine}(e_1))]\!] &= (1+[\![e_1]\!])5^{(1+[\![e_1]\!])} \\ &> [\![e_1]\!](2+5^{[\![e_1]\!]}) + 10 \\ &= [\![(\tfrac{\textbf{lift}_d(1)}{\sqrt{(\textbf{lift}_d(1)-(e*e)}}) * (\tfrac{\partial}{\partial x_i} \diamond e_1)]\!] \end{aligned}$$

P(d)

Case R14. $\frac{\partial}{\partial x_i} \diamond (\textbf{arctangent}(e_1)) \xrightarrow[rule]{} \frac{\textbf{lift}_d(1)}{\textbf{lift}_d(1)+(e_1*e_1)} * (\frac{\partial}{\partial x_i} \diamond e_1)$

$$\begin{aligned} [\![\tfrac{\partial}{\partial x_i} \diamond (\textbf{arctangent}(e_1))]\!] &= (1+[\![e_1]\!])5^{(1+[\![e_1]\!])} \\ &> [\![e_1]\!](2+5^{[\![e_1]\!]}) + 9 \\ &= [\![\tfrac{1}{1+(e*e)} * (\tfrac{\partial}{\partial x_i} \diamond e_1)]\!] \end{aligned}$$

P(d)

Case R15. $\frac{\partial}{\partial x_i} \diamond (\textbf{exp}(e_1)) \xrightarrow[rule]{} \textbf{exp}(e_1) * (\frac{\partial}{\partial x_i} \diamond e_1)$

$$\begin{aligned} [\![\tfrac{\partial}{\partial x_i} \diamond (\textbf{exp}(e_1))]\!] &= (1+[\![e_1]\!])5^{(1+[\![e_1]\!])} \\ &> [\![e_1]\!](1+5^{[\![e_1]\!]}) + 2 \\ &= [\![\textbf{exp}(e_1) * (\tfrac{\partial}{\partial x_i} \diamond e_1)]\!] \end{aligned}$$

P(d)

Case R16. $\frac{\partial}{\partial x_i} \diamond (e_1^n) \xrightarrow[rule]{} \textbf{lift}_d(n) * e_1^{n-1} * (\frac{\partial}{\partial x_i} \diamond e_1)$

$$\begin{aligned} [\![\tfrac{\partial}{\partial x_i} \diamond (e_1^n)]\!] &= (1+[\![e_1]\!])5^{(1+[\![e_1]\!])} \\ &> 5 + [\![e_1]\!](1+5^{[\![e_1]\!]}) \\ &= [\![\textbf{lift}_d(n) * e_1^{n-1} * (\tfrac{\partial}{\partial x_i} \diamond e_1)]\!] \end{aligned}$$

P(d)

Case R17. $\frac{\partial}{\partial x_i} \diamond (e_1 \odot e_2) \xrightarrow[rule]{} (\frac{\partial}{\partial x_i} \diamond e_1) \odot (\frac{\partial}{\partial x_i} \diamond e_2)$Included in the earlier prose.

Case R18. $\frac{\partial}{\partial x_i} \diamond (-e_1) \xrightarrow[rule]{} -(\frac{\partial}{\partial x_i} \diamond e_1)$

$$\begin{aligned} [\![\tfrac{\partial}{\partial x_i} \diamond (-e_1)]\!] &= 5^{1+[\![e_1]\!]}(1+[\![e_1]\!]) \\ &> 1 + [\![e_1]\!]5^{[}e_1]\!] \\ &= [\![-(\tfrac{\partial}{\partial x_i} \diamond e_1)]\!] \end{aligned}$$

P(d)

Case R19. $\frac{\partial}{\partial x_i}\sum_{v=1}^{n} e_1 \xrightarrow[rule]{} \sum_{v=1}^{n}(\frac{\partial}{\partial x_i}e_1)$

$$\begin{aligned}[\![\frac{\partial}{\partial x_i}\sum_{v=1}^{n} e_1]\!] &= (2+2[\![e_1]\!])*5^{2+2[\![e_1]\!]} \\ &> 2+2[\![e_1]\!]5^{[\![e_1]\!]} \\ &= [\![\sum_{v=1}^{n}(\frac{\partial}{\partial x_i}e_1)]\!]\end{aligned}$$

P(d)

Case R20. $\frac{\partial}{\partial x_i}\chi \xrightarrow[rule]{} 0$

$[\![\frac{\partial}{\partial x_i}]\!] \quad = \mathcal{S}\chi 5^{\mathcal{S}\chi}$

$> 2 \qquad = [\![0]\!]$

Case R21. $\frac{\partial}{\partial x_i}\diamond(V_\alpha \circledast H^\nu) \xrightarrow[rule]{} (V_\alpha \circledast h^{i\nu})$

$[\![\frac{\partial}{\partial x_i}\diamond(V_\alpha \circledast H^\nu)]\!] \quad = 5$

$> 1 \quad = [\![(V_\alpha \circledast H^{i\nu})]\!]$

Case R22. $--e_1 \xrightarrow[rule]{} e_1$

$[\![--e_1]\!] \quad = 2+[\![e_1]\!]$

$> [\![e_1]\!] \qquad = [\![e_1]\!]$

Case R23. $-0 \xrightarrow[rule]{} 0$

$[\![-0]\!] \quad = 2$

$> 1 \quad = [\![0]\!]$

Case R24. $e_1 - 0 \xrightarrow[rule]{} e_1$

$[\![e_1-0]\!] \quad = 2+[\![e_1]\!]$

$> [\![e_1]\!] \qquad = [\![e_1]\!]$

Case R25. $0 - e_1 \xrightarrow[rule]{} -e_1$

Similar approach to R24 `P(R25) OK`

Case R26. $\frac{0}{e_1} \xrightarrow[rule]{} 0$

$[\![\frac{0}{e_1}]\!] \quad = 3+[\![e_1]\!]$

$> 1 \qquad = [\![0]\!]$

Case R27. $\frac{\frac{e_1}{e_2}}{e_3} \xrightarrow[rule]{} \frac{e_1}{e_2e_3}$ Included in the earlier prose.

Case R28. $\frac{e_1}{\frac{e_2}{e_3}} \xrightarrow[rule]{} \frac{e_1e_3}{e_2}$

Similar approach to R27 `P(R28) OK`

Case R29. $\frac{\frac{e_1}{e_2}}{\frac{e_3}{e_4}} \xrightarrow[rule]{} \frac{e_1e_4}{e_2e_3}$

$[\![\frac{\frac{e_1}{e_2}}{\frac{e_3}{e_4}}]\!] \quad = 6+[\![e_1]\!]+[\![e_2]\!]+[\![e_3]\!]$

$> 4+[\![e_1]\!]+[\![e_2]\!]+[\![e_3]\!] \quad = [\![\frac{e_1e_4}{e_2e_3}]\!]$

Case R30. $0+e_1, e_1+0 \xrightarrow[rule]{} e_1$

$[\![0+e_1, e_1+0]\!] \quad = 2+[\![e_1]\!]$

$> [\![e_1]\!] \qquad = [\![e_1]\!]$

Case R31. $0e, e0 \xrightarrow[rule]{} 0$

Similar approach to R30 `P(R31) OK`

Case R32. $\sqrt{(e_1)}*\sqrt{(e_1)} \xrightarrow[rule]{} e_1$

$[\![\sqrt{(e_1)}*\sqrt{(e_1)}]\!] \quad = 3+2[\![e_1]\!]$

$> [\![e_1]\!] \qquad = [\![e_1]\!]$

Case R33. $\mathcal{E}_{ijk}\frac{\partial}{\partial x_{i,j}}\diamond e_1 \xrightarrow[rule]{} \mathbf{lift}_d(0)$

$[\![\mathcal{E}_{ijk}\frac{\partial}{\partial x_{i,j}}\diamond e_1]\!] \quad = 5+[\![e_1]\!]5^{[}e_1]\!]$

$> 2 \qquad = [\![\mathbf{lift}_d(0)]\!]$

Case R34. $\mathcal{E}_{ijk}(V_\alpha \circledast h^{jk}) \xrightarrow[rule]{} \mathbf{lift}_d(0)$

$[\![\mathcal{E}_{ijk}(V_\alpha \circledast h^{jk})]\!] \quad = 6$

$> 2 \quad = [\![\mathbf{lift}_d(0)]\!]$

Case R35. $\mathcal{E}_{ijk}\mathcal{E}_{ilm} \xrightarrow[rule]{} \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$

$[\![\mathcal{E}_{ijk}\mathcal{E}_{ilm}]\!] \quad = 9$

$> 7 \quad = [\![\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}]\!]$

Case R36. $\delta_{ij}T_j \xrightarrow[rule]{} T_i$

$[\![\delta_{ij}T_j]\!] \quad = 3$

$> 1 \quad = [\![T_i]\!]$

Case R37.$\delta_{ij}F_j \xrightarrow[rule]{} F_i$

Similar approach to R36 `P(R37) OK`

Case R38.$\delta_{ij}V \circledast H^{\delta_{cj}} \xrightarrow[rule]{} V \circledast H^{\delta_{ci}}$

Similar approach to R36 `P(R38) OK`

Case R39.$\delta_{ij}V \circledast H^{\delta_{cj}}(x) \xrightarrow[rule]{} V \circledast H^{\delta_{ci}}(x)$

$$\begin{aligned} [\![\delta_{ij}V \circledast H^{\delta_{cj}}(x)]\!] &= 4 \\ &> 2 \quad = [\![V \circledast H^{\delta_{ci}}(x)]\!] \end{aligned}$$

Case R40.$\delta_{ij}\frac{\partial}{\partial x_j} \diamond e_1 \xrightarrow[rule]{} \frac{\partial}{\partial x_i} \diamond (e_1)$

$$\begin{aligned} [\![\delta_{ij}\frac{\partial}{\partial x_j} \diamond (e_1)]\!] &= 2 + [\![e_1]\!]5^{[\![e_1]\!]} \\ &> [\![e_1]\!]5^{[\![e_1]\!]} \qquad = [\![\frac{\partial}{\partial x_i} \diamond (e_1)]\!] \end{aligned}$$

Case R41.$\sum(se_1) \xrightarrow[rule]{} s\sum e_1$

$$\begin{aligned} [\![\sum(se_1)]\!] &= 6 + 2[\![e_1]\!] \\ &> 4 + 2[\![e_1]\!] \quad = [\![s\sum e_1]\!] \end{aligned}$$

P(d) Lemma 4.1

C.2 Termination implies Normal Form

Termination implies normal form (Lemma 4.2). The proof is by examination of the EIN syntax in [2].For any syntactic construct, we show that either the term is in normal form, or there is a rewrite rule that applies (Section C.2). We state $Q(e_x)$ as a shorthand for the claim that if x has terminated and is normal form. Additionally we state $CQ(e_x)$ if there exists an expression that is not in normal form and has terminated. The following is a proof by contradiction.

Define the following shorthand: M(e_1) = $\sqrt{e_1} \mid exp(e_1) \mid e_1^n \mid \kappa(e_1)$

case on structure e_x

If $e_x = c$ then $Q(e_x)$ because e_x is in normal form.
If $e_x = T_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_x = F_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_x = V_\alpha \circledast H$ then $Q(e_x)$ because e_x is in normal form.
If $e_x = \delta_{ij}$ then $Q(e_x)$ because e_x is in normal form.
If $e_x = \mathcal{E}_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_x = \mathbf{lift}_d(e_1)$

Prove Q(e_x) by contradiction.

case on structure e_1

If $e_1 = c$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = T_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = F_\alpha$ then $Q(e_x)$ because e_x is not a supported type.
If $e_1 = e \circledast e$ then $Q(e_x)$ because e_x is not a supported type.
If $e_1 = \delta_{ij}$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = \mathcal{E}_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = \mathbf{lift}_d(e)$ then $Q(e_x)$ because e_x is not a supported type.
If $e_1 = M(e)$ and assuming $Q(e)$ then $Q(e_x)$
Given $M(e_3) = \sqrt{e_3} \mid exp(e_3) \mid e_3^n \mid \kappa(e_3)$
If $e_1 = -e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = \frac{\partial}{\partial x_\alpha} \diamond e$ then $Q(e_x)$ because e_x is not a supported type.
If $e_1 = \sum e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = e_3 + e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = e_3 - e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = e_3 * e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = \frac{e_3}{e_4}$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = e_3 @ e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
$Q(e_x)$

e_x=M(e_1)

Show Q(x) with proof by contradiction. Assume CQ(Q_x)

case on structure e_1

Note. M(e_1) = $\sqrt{e_3} \mid exp(e_3) \mid e_3^n \mid \kappa(e_3)$

If $e_1 = c$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = T_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = F_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = V_\alpha \circledast H$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = \delta_{ij}$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = \mathcal{E}_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = \mathbf{lift}_d(e)$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = M(e)$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = -e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = \frac{\partial}{\partial x_\alpha} e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = \sum e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = e_3 + e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = e_3 - e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = e_3 * e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = \frac{e_3}{e_4}$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = e_3 @ e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
$Q(e_x)$

$e_x = -e_1$
Show Q(x) with proof by contradiction. Assume CQ(Q_x)
case on structure e_1
If $e_1 = 0$ then $Q(e_x)$ because we can apply rule$R23$
If $e_1 = c$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = T_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = F_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = V_\alpha \circledast H$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = \delta_{ij}$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = \mathcal{E}_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = \mathbf{lift}_d(e)$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = M(e)$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = -e$ then $Q(e_x)$ because we can apply rule$R22$
If $e_1 = \frac{\partial}{\partial x_\alpha} e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = \sum e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = e_3 + e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = e_3 - e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = e_3 * e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = \frac{e_3}{e_4}$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = e_3 @ e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
$Q(e_x)$

$e_x = e_1 + e_2$
Prove Q(x)
case on structure e_1
If $e_x = 0$ then $Q(e_x)$ because we can apply rule $R30$
If $e_x = c$ then $Q(e_x)$ because e_x is in normal form.
If $e_x = T_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_x = F_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_x = V_\alpha \circledast H$ then $Q(e_x)$ because e_x is in normal form.
If $e_x = \delta_{ij}$ then $Q(e_x)$ because e_x is in normal form.
If $e_x = \mathcal{E}_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_x = \mathbf{lift}_d(e)$ and assuming $Q(e)$ then $Q(e_x)$
If $e_x = M(e)$ and assuming $Q(e)$ then $Q(e_x)$
If $e_x = -e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_x = \frac{\partial}{\partial x_\alpha} e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = \sum e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = e_3 + e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = e_3 - e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = e_3 * e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = \frac{e_3}{e_4}$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = e_3 @ e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
$Q(e_x)$
case on structure e_2
Proof same as above Q(x)

$e_x = e_1 - e_2$
Show Q(x) with proof by contradiction. Assume CQ(Q_x)

case on structure e_1

If $e_1 = 0$ then $Q(e_x)$ because we can apply rule $R25$

If $e_1 = c$ then $Q(e_x)$ because e_x is in normal form.

If $e_1 = T_\alpha$ then $Q(e_x)$ because e_x is in normal form.

If $e_1 = F_\alpha$ then $Q(e_x)$ because e_x is in normal form.

If $e_1 = V_\alpha \circledast H$ then $Q(e_x)$ because e_x is in normal form.

If $e_1 = \delta_{ij}$ then $Q(e_x)$ because e_x is in normal form.

If $e_1 = \mathcal{E}_\alpha$ then $Q(e_x)$ because e_x is in normal form.

If $e_1 = \mathbf{lift}_d(e)$ and assuming $Q(e)$ then $Q(e_x)$

If $e_1 = M(e)$ and assuming $Q(e)$ then $Q(e_x)$

If $e_1 = -e$ and assuming $Q(e)$ then $Q(e_x)$

If $e_1 = \frac{\partial}{\partial x_\alpha} e$ and assuming $Q(e)$ then $Q(e_x)$

If $e_1 = \sum e$ and assuming $Q(e)$ then $Q(e_x)$

If $e_1 = e_3 + e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$

If $e_1 = e_3 - e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$

If $e_1 = e_3 * e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$

If $e_1 = \frac{e_3}{e_4}$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$

If $e_1 = e_3 @ e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$

$Q(e_x)$

case on structure e_2

If $e_x = 0$ then $Q(e_x)$ because we can apply rule $R24$

Proof same as above

$Q(x)$

e$_x$=$e_1 * e_2$

Show Q(x) with proof by contradiction. Assume CQ(Q$_x$)

case on structure e_1

If $e_1 = 0$ then $Q(e_x)$ because we can apply rule $R31$

If $e_1 = c$ then $Q(e_x)$ because e_x is in normal form.

If $e_1 = T_\alpha$ then $Q(e_x)$ because e_x is in normal form.

If $e_1 = F_\alpha$ then $Q(e_x)$ because e_x is in normal form.

If $e_1 = V_\alpha \circledast H$ then $Q(e_x)$ because e_x is in normal form.

If $e_1 = \delta_{ij}$

case on structure e_2

If $e_2 = T_j$ then $Q(e_x)$ because we can apply rule $R36$

If $e_2 = F_j$ then $Q(e_x)$ because we can apply rule $R37$

If $e_2 = V_\alpha \circledast H$ then $Q(e_x)$ because we can apply rule $R38$

If $e_2 = V_\alpha \circledast H @ e$ then $Q(e_x)$ because we can apply rule $R39$

If $e_2 = \frac{\partial}{\partial x_\alpha} e$ then $Q(e_x)$ because we can apply rule $R40$

else $Q(e_x)$ because e_x is in normal form.

If $e_1 = \mathcal{E}_{ij}$

If $e_1 = \mathcal{E}_{ijk}$

case on structure e_2

If $e_2 = \frac{\partial}{\partial x_{ij}}(e)$ then $Q(e_x)$ because we can apply rule $R33$

If $e_2 = V \circledast H_{jk}$ then $Q(e_x)$ because we can apply rule $R34$

If $e_2 = \mathcal{E}_{ijk}$ then $Q(e_x)$ because we can apply rule $R35$

else $Q(e_x)$ because e_x is in normal form.

If $e_1 = \mathbf{lift}_d(e_1)$ and assuming $Q(e)$ then $Q(e_x)$

If $e_1 = \sqrt{e_3}$

If $e_2 = \sqrt{e_4}$ then $Q(e_x)$ because we can apply rule $R32$

otherwise $Q(e_x)$ because e_x is in normal form.

If $e_1 = -e$ and assuming $Q(e)$ then $Q(e_x)$

If $e_1 = \frac{\partial}{\partial x_\alpha} \diamond e$ then $Q(e_x)$ because e_x is not a supported type.

If $e_1 = \sum e$ and assuming $Q(e)$ then $Q(e_x)$

If $e_1 = e_3 + e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$

If $e_1 = e_3 - e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$

If $e_1 = e_3 * e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$

If $e_1 = \frac{e_3}{e_4}$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$

If $e_1 = e_3 @ e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$

$Q(e_x)$

e$_x$=$\frac{e_1}{e_2}$

Show Q(x) with proof by contradiction. Assume CQ(Q$_x$)

case on structure e_1

If $e_1 = \frac{e_3}{e_4}$

If $e_2 = \frac{e_5}{e_6}$ then $Q(e_x)$ because we can apply rule $R27$
otherwise $Q(e_x)$ because we can apply rule R29.

If $e_1 = 0$ then $Q(e_x)$ because we can apply rule $R26$
If $e_1 = c$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = T_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = F_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = V \circledast H$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = \delta_{ij}, \mathcal{E}_{ij}, \mathcal{E}_{ijk}$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = \frac{\partial}{\partial x_\alpha} e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = \sum e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = \mathbf{lift}_d(e)$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = M(e)$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = -e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = e + e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = e - e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = e * e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = e@e$ and assuming $Q(e)$ then $Q(e_x)$

case on structure e_2
If $e_2 = \frac{e_4}{e_5}$ then $Q(e_x)$ because we can apply rule $R28$
otherwise proof same as above
$Q(e_x)$

e$_x$=$e_1@e_2$
Show Q(x) with proof by contradiction. Assume CQ(Q$_x$)
case on structure e_1
If $e_1 = c$ then $Q(e_x)$ because e_x is not a supported type.
If $e_1 = T_\alpha$ then $Q(e_x)$ because e_x is not a supported type.
If $e_1 = F_\alpha$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = e \circledast e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = \delta_{ij}, \mathcal{E}_\alpha$ then $Q(e_x)$ because we can apply rule $R5$
If $e_1 = \mathbf{lift}_d(e)$ then $Q(e_x)$ because we can apply rule $R5$
If $e_1 = M(e)$ then $Q(e_x)$ because we can apply rule $R3$
If $e_1 = -e$ then $Q(e_x)$ because we can apply rule $R3$
If $e_x = \frac{\partial}{\partial x_\alpha} \diamond e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = \sum e$ then $Q(e_x)$ because we can apply rule $R4$
If $e_1 = e + e$ then $Q(e_x)$ because we can apply rule $R2$
If $e_1 = e - e$ then $Q(e_x)$ because we can apply rule $R2$
If $e_1 = e * e$ then $Q(e_x)$ because we can apply rule $R1$
If $e_1 = \frac{e}{e}$ then $Q(e_x)$ because we can apply rule $R1$
If $e_1 = e@e$ then $Q(e_x)$ because e_x is not a supported type.
$Q(e_x)$

e$_x$=$\frac{\partial}{\partial x_\alpha} e_1$
Show Q(x) with proof by contradiction. Assume CQ(Q$_x$)
case on structure e_1
If $e_1 = c$ then $Q(e_x)$ because e_x is not a supported type.
If $e_1 = T_\alpha$ then $Q(e_x)$ because e_x is not a supported type.
If $e_1 = F_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = e \circledast e$ then $Q(e_x)$ because we can apply rule $R21$
If $e_1 = \delta_{ij}, \mathcal{E}_\alpha$ then $Q(e_x)$ because we can apply rule $R20$
If $e_1 = \mathbf{lift}_d(e)$ then $Q(e_x)$ because we can apply rule $R20$
If $e_1 = M(e_2)$
case on structure e_2
If $e_2 = Cosine(e)$ then $Q(e_x)$ because we can apply rule $R9$
If $e_2 = Sine(e)$ then $Q(e_x)$ because we can apply rule $R10$
If $e_2 = Tangent(e)$ then $Q(e_x)$ because we can apply rule $R11$
If $e_2 = ArcCosine(e)$ then $Q(e_x)$ because we can apply rule $R12$
If $e_2 = ArcSine(e)$ then $Q(e_x)$ because we can apply rule $R13$
If $e_2 = ArcTangent(e)$ then $Q(e_x)$ because we can apply rule $R14$
If $e_2 = exp(e)$ then $Q(e_x)$ because we can apply rule $R15$
If $e_2 = e^n$ then $Q(e_x)$ because we can apply rule $R16$
If $e_2 = \sqrt{e}$ then $Q(e_x)$ because we can apply rule $R8$
$Q(e_x)$

If $e_1 = -e$ then $Q(e_x)$ because we can apply rule $R18$
If $e_1 = \frac{\partial}{\partial x_\alpha} \diamond e$ then $Q(e_x)$ because we can apply rule $R42$
If $e_1 = \sum e$ then $Q(e_x)$ because we can apply rule $R19$
If $e_1 = e + e$ then $Q(e_x)$ because we can apply rule $R17$
If $e_1 = e - e$ then $Q(e_x)$ because we can apply rule $R17$
If $e_1 = e * e$ then $Q(e_x)$ because we can apply rule $R6$
If $e_1 = \frac{e}{e}$ then $Q(e_x)$ because we can apply rule $R7$
If $e_1 = e@e$ then $Q(e_x)$ because e_x is not a supported type.
$Q(e_x)$

$e_x=\sum(e_1)$
Show Q(x) with proof by contradiction. Assume CQ(Q_x)
case on structure e_1
If $e_1 = c$ then $Q(e_x)$ because we can apply rule $R41$
If $e_1 = T$ then $Q(e_x)$ because we can apply rule $R41$
If $e_1 = T_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = F$ then $Q(e_x)$ because we can apply rule $R41$
If $e_1 = F_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = V_\alpha \circledast H$ then $Q(e_x)$ because we can apply rule $R41$
If $e_1 = \delta_{ij}, \mathcal{E}_\alpha$ then $Q(e_x)$ because e_x is in normal form.
If $e_1 = \mathbf{lift}_d(e)$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = M(e)$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = -e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = \frac{\partial}{\partial x_\alpha} e$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = \sum e_1$ and assuming $Q(e)$ then $Q(e_x)$
If $e_1 = e_3 + e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = e_3 - e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = e_3 * e_4$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = \frac{e_3}{e_4}$ and assuming $Q(e_3)$ and $Q(e_4)$ then $Q(e_x)$
If $e_1 = F@e$ then $Q(e_x)$ because we can apply rule $R41$
If $e_1 = V \circledast h@e$ then $Q(e_x)$ because we can apply rule $R41$
If $e_1 = e@e$ then $Q(e_x)$ because e_x is in normal form.
$Q(e_x)$

C.3 Normal Form implies Termination

The section offers a proof for Lemma 4.3.

Non-terminated A term has not terminated if it is the source term of a rewrite rule.

Normal form implies Termination. (Lemma 4.3).

Proof. We state M(e) as a shorthand for the claim that if e is in normal form then it has terminated. The following is a proof by contradiction. CM(e): There exists an expression e that has not terminated and is in normal form. More precisely, given a derivation d of the form $e \longrightarrow e'$, there exists an expression that is the source term e of derivation d therefore not-terminated, and is in normal form. □

Case analysis on the source of each rule

Case R1.$(e_1 \odot_n e_2)@x \underset{rule}{\longrightarrow} (e_1@x) \odot_n (e_2@x)$
Let y= $(e_1 \odot_n e_2)@x$ and since y is not in normal form then `M(R1) OK`

Case R2.$(e_0 \odot_2 e_1)@x \underset{rule}{\longrightarrow} (e_0@x) \odot_2 (e_1@x)$
Let y= $(e_0 \odot_2 e_1)@x$ and since y is not in normal form then `M(R2) OK`

Case R3.$(\odot_1 e_1)@x \underset{rule}{\longrightarrow} \odot_1 (e_1@x)$
Let y= $(\odot_1 e_1)@x$ and since y is not in normal form then `M(R3) OK`

Case R4.$(\sum_{i=1}^{n} e_1)@x \underset{rule}{\longrightarrow} \sum_{i=1}^{n}(e_1@x)$
Let y= $(\sum_{i=1}^{n} e_1)@x$ and since y is not in normal form then `M(R4) OK`

Case R5.$(\chi)@x \underset{rule}{\longrightarrow} \chi$
Let y= $(\chi)@x$ and since y is not in normal form then `M(R5) OK`

Case R6.$\frac{\partial}{\partial x_i} \diamond (e_1 * e_2) \underset{rule}{\longrightarrow} e_1(\frac{\partial}{\partial x_i} \diamond e_2) + e_2(\frac{\partial}{\partial x_i} \diamond e_1)$
Let y= $\frac{\partial}{\partial x_i} \diamond (e_1 * e_2)$ and since y is not in normal form then `M( R6) OK`

Case R7.$\frac{\partial}{\partial x_i} \diamond (\frac{e_1}{e_2}) \xrightarrow[rule]{} \frac{(\frac{\partial}{\partial x_i} \diamond e_1)e_2 - e_1(\frac{\partial}{\partial x_i} \diamond e_2)}{e_2^2}$

Let y= $\frac{\partial}{\partial x_i} \diamond (\frac{e_1}{e_2})$ and since y is not in normal form then `M( R7) OK`

Case R8.$\frac{\partial}{\partial x_i} \diamond (\sqrt{e_1}) \xrightarrow[rule]{} \mathbf{lift}_d(1/2) * \frac{\frac{\partial}{\partial x_i} \diamond e_1}{\sqrt{e_1}}$

Let y= $\frac{\partial}{\partial x_i} \diamond (\sqrt{e_1})$ and since y is not in normal form then `M( R8) OK`

Case R9.$\frac{\partial}{\partial x_i} \diamond (\mathbf{cosine}(e_1)) \xrightarrow[rule]{} (-\mathbf{sine}(e_1)) * (\frac{\partial}{\partial x_i} \diamond e_1)$

Let y= $\frac{\partial}{\partial x_i} \diamond (\mathbf{cosine}(e_1))$ and since y is not in normal form then `M(R9) OK`

Case R10.$\frac{\partial}{\partial x_i} \diamond (\mathbf{sine}(e_1)) \xrightarrow[rule]{} (\mathbf{cosine}(e_1)) * (\frac{\partial}{\partial x_i} \diamond e_1)$

Let y= $\frac{\partial}{\partial x_i} \diamond (\mathbf{sine}(e_1))$ and since y is not in normal form then `M(R10) OK`

Case R11.$\frac{\partial}{\partial x_i} \diamond (\mathbf{tangent}(e_1)) \xrightarrow[rule]{} \frac{\frac{\partial}{\partial x_i} \diamond e}{\mathbf{cosine}(e_1) * \mathbf{cosine}(e_1)}$

Let y= $\frac{\partial}{\partial x_i} \diamond (\mathbf{tangent}(e_1))$ and since y is not in normal form then `M( R11) OK`

Case R12.$\frac{\partial}{\partial x_i} \diamond (\mathbf{arccosine}(e_1)) \xrightarrow[rule]{} (\frac{-\mathbf{lift}_d(1)}{\sqrt{(\mathbf{lift}_d(1) - (e * e)}}) * (\frac{\partial}{\partial x_i} \diamond e_1)$

Let y= $\frac{\partial}{\partial x_i} \diamond (\mathbf{arccosine}(e_1))$ and since y is not in normal form then `M(R12) OK`

Case R13.$\frac{\partial}{\partial x_i} \diamond (\mathbf{arcsine}(e_1)) \xrightarrow[rule]{} (\frac{\mathbf{lift}_d(1)}{\sqrt{(\mathbf{lift}_d(1) - (e * e)}}) * (\frac{\partial}{\partial x_i} \diamond e_1)$

Let y= $\frac{\partial}{\partial x_i} \diamond (\mathbf{arcsine}(e_1))$ and since y is not in normal form then `M( R13) OK`

Case R14.$\frac{\partial}{\partial x_i} \diamond (\mathbf{arctangent}(e_1)) \xrightarrow[rule]{} \frac{\mathbf{lift}_d(1)}{\mathbf{lift}_d(1) + (e_1 * e_1)} * (\frac{\partial}{\partial x_i} \diamond e_1)$

Let y= $\frac{\partial}{\partial x_i} \diamond (\mathbf{arctangent}(e_1))$ and since y is not in normal form then `M( R14) OK`

Case R15.$\frac{\partial}{\partial x_i} \diamond (\mathbf{exp}(e_1)) \xrightarrow[rule]{} \mathbf{exp}(e_1) * (\frac{\partial}{\partial x_i} \diamond e_1)$

Let y= $\frac{\partial}{\partial x_i} \diamond (\mathbf{exp}(e_1))$ and since y is not in normal form then `M(R15) OK`

Case R16.$\frac{\partial}{\partial x_i} \diamond (e_1^n) \xrightarrow[rule]{} \mathbf{lift}_d(n) * e_1^{n-1} * (\frac{\partial}{\partial x_i} \diamond e_1)$

Let y= $\frac{\partial}{\partial x_i} \diamond (e_1^n)$ and since y is not in normal form then `M(R16) OK`

Case R17.$\frac{\partial}{\partial x_i} \diamond (e_1 \odot e_2) \xrightarrow[rule]{} (\frac{\partial}{\partial x_i} \diamond e_1) \odot (\frac{\partial}{\partial x_i} \diamond e_2)$

Let y= $\frac{\partial}{\partial x_i} \diamond (e_1 \odot e_2)$ and since y is not in normal form then `M( R17) OK`

Case R18.$\frac{\partial}{\partial x_i} \diamond (-e_1) \xrightarrow[rule]{} -(\frac{\partial}{\partial x_i} \diamond e_1)$

Let y= $\frac{\partial}{\partial x_i} \diamond (-e_1)$ and since y is not in normal form then `M( R18) OK`

Case R19.$\frac{\partial}{\partial x_i} \sum_{v=1}^{n} e_1 \xrightarrow[rule]{} \sum_{v=1}^{n} (\frac{\partial}{\partial x_i} e_1)$

Let y= $\frac{\partial}{\partial x_i} \sum_{v=1}^{n} e_1$ and since y is not in normal form then `M(R19) OK`

Case R20.$\frac{\partial}{\partial x_i} \mathbf{lift}_d(e_1) \xrightarrow[rule]{} 0$

Let y= $\frac{\partial}{\partial x_i}\mathrm{Lift}(e_1)$ and since y is not in normal form then `M( R20) OK`

Case R20.$\frac{\partial}{\partial x_i} \chi \xrightarrow[rule]{} 0$

Let y= $\frac{\partial}{\partial x_i}$ and since y is not in normal form then `M( R20) OK`

Case R21.$\frac{\partial}{\partial x_i} \diamond (V_\alpha \circledast H^\nu) \xrightarrow[rule]{} (V_\alpha \circledast h^{i\nu})$

Let y= $\frac{\partial}{\partial x_i} \diamond (V_\alpha \circledast H^\nu)$ and since y is not in normal form then `M(R21) OK`

Case R22.$- - e_1 \xrightarrow[rule]{} e_1$

Let y= $- - e_1$ and since y is not in normal form then `M(R22) OK`

Case R23.$-0 \xrightarrow[rule]{} 0$

Let y= -0 and since y is not in normal form then `M(R23) OK`

Case R24.$e_1 - 0 \xrightarrow[rule]{} e_1$

Let y= $e_1 - 0$ and since y is not in normal form then `M(R24) OK`

Case R25.$0 - e_1 \xrightarrow[rule]{} -e_1$

Let y= $0 - e_1$ and since y is not in normal form then `M(R25) OK`

Case R26.$\frac{0}{e_1} \xrightarrow[rule]{} 0$

Let y= $\frac{0}{e_1}$ and since y is not in normal form then `M(R26) OK`

Case R27.$\frac{\frac{e_1}{e_2}}{e_3} \xrightarrow[rule]{} \frac{e_1}{e_2 e_3}$

Let y= $\frac{\frac{e_1}{e_2}}{e_3}$ and since y is not in normal form then `M(R27) OK`

Case R28.$\frac{\frac{e_1}{e_2}}{e_3} \xrightarrow[rule]{} \frac{e_1 e_3}{e_2}$

Let y= $\frac{\frac{e_1}{e_2}}{e_3}$ and since y is not in normal form then `M(R28) OK`

Case R29.$\frac{\frac{e_1}{e_2}}{\frac{e_3}{e_4}} \xrightarrow[rule]{} \frac{e_1 e_4}{e_2 e_3}$

Let y= $\frac{\frac{e_1}{e_2}}{\frac{e_3}{e_4}}$ and since y is not in normal form then `M( R29) OK`

Case R30.$0 + e_1, e_1 + 0 \xrightarrow[rule]{} e_1$

Let y= $0 + e_1, e_1 + 0$ and since y is not in normal form then `M( R30) OK`

Case R31.$0e, e0 \xrightarrow[rule]{} 0$

Let y= $0e, e0$ and since y is not in normal form then `M( R31) OK`

Case R32.$\sqrt{(e_1)} * \sqrt{(e_1)} \xrightarrow[rule]{} e_1$

Let y= $\sqrt{(e_1)} * \sqrt{(e_1)}$ and since y is not in normal form then `M( R32) OK`

Case R33.$\mathcal{E}_{ijk} \frac{\partial}{\partial x_{i,j}} \diamond e_1 \xrightarrow[rule]{} \mathbf{lift}_d(0)$

Let y= $\mathcal{E}_{ijk} \frac{\partial}{\partial x_{i,j}} \diamond e_1$ and since y is not in normal form then `M( R33) OK`

Case R34.$\mathcal{E}_{ijk}(V_\alpha \circledast h^{jk}) \xrightarrow[rule]{} \mathbf{lift}_d(0)$

Let y= $\mathcal{E}_{ijk}(V_\alpha \circledast h^{jk})$ and since y is not in normal form then `M( R34) OK`

Case R35.$\mathcal{E}_{ijk}\mathcal{E}_{ilm} \xrightarrow[rule]{} \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$

Let y= $\mathcal{E}_{ijk}\mathcal{E}_{ilm}$ and since y is not in normal form then `M( R35) OK`

Case R36.$\delta_{ij} T_j \xrightarrow[rule]{} T_i$

Let y= $\delta_{ij} T_j$ and since y is not in normal form then `M( R36) OK`

Case R37.$\delta_{ij} F_j \xrightarrow[rule]{} F_i$

Let y= $\delta_{ij} F_j$ and since y is not in normal form then `M( R37) OK`

Case R38.$\delta_{ij} V \circledast H^{\delta_{cj}} \xrightarrow[rule]{} V \circledast H^{\delta_{ci}}$

Let y= $\delta_{ij} V \circledast H^{\delta_{cj}}$ and since y is not in normal form then `M( R38) OK`

Case R39.$\delta_{ij} V \circledast H^{\delta_{cj}}(x) \xrightarrow[rule]{} V \circledast H^{\delta_{ci}}(x)$

Let y= $\delta_{ij} V \circledast H^{\delta_{cj}}(x)$ and since y is not in normal form then `M( R39) OK`

Case R40.$\delta_{ij} \frac{\partial}{\partial x_j} \diamond e_1 \xrightarrow[rule]{} \frac{\partial}{\partial x_i} \diamond (e_1)$

Let y= $\delta_{ij} \frac{\partial}{\partial x_j} \diamond (e_1)$ and since y is not in normal form then `M( R40) OK`

Case R41.$\sum(se_1) \xrightarrow[rule]{} s \sum e_1$

Let y= $\sum(se_1)$ and since y is not in normal form then `M( R41) OK`

Case R42.$\frac{\partial}{\partial x_\alpha} \diamond \frac{\partial}{\partial x_\beta} \diamond e_1 \xrightarrow[rule]{} \frac{\partial}{\partial x_{\beta\alpha}} \diamond e_1$

Let y= $\frac{\partial}{\partial x_\alpha} \diamond \frac{\partial}{\partial x_\beta} \diamond e_1$ and since y is not in normal form then `M( R42) OK`

M(x) Lemma 4.3