Math 504

Cuong Ly

February 2022

1 Question 1

$$det(A - I\lambda) = \begin{bmatrix} 3-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{bmatrix}.$$

$= -\lambda^3 + 8\lambda^2 - 19\lambda + 12$
Solve the equation $-\lambda^3 + 8\lambda^2 - 19\lambda + 12 = 0$, we obtain the roots are $\lambda_1 = 1$, $\lambda_2 = 3$ and $\lambda_3 = 4$
These are the eigenvalues
We now proceed to find eigenvectors

$*\lambda_1 = 1$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

We need to solve

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In order to solve for x_1, x_2, x_3, we need to solve $\begin{bmatrix} 2 & 1 & 0 & | & 0 \\ 1 & 1 & 1 & | & 0 \\ 0 & 1 & 2 & | & 0 \end{bmatrix}$

The null space and also the eigenvectors are $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$*\lambda_2 = 3$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

We need to solve

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In order to solve for x_1, x_2, x_3, we need to solve $\begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 1 & -1 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \end{bmatrix}$

The null space and also the eigenvectors are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$*\lambda_2 = 4$
$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$
We need to solve
$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
In order to solve for x_1, x_2, x_3, we need to solve $\begin{bmatrix} -1 & 1 & 0 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \end{bmatrix}$

The null space and also the eigenvectors are $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

The inverse of the matrix is
$$\begin{bmatrix} 1 & -1 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ -1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$
Eigenvalue decomposition
A = U$\Lambda U^{-}1$

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ -1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

2 Question 2

f(0,0,0) = 1

∇f(x_1, x_2, x_3) = $\begin{bmatrix} 2\mathrm{x}_1\mathrm{exp}(x_1^2, x_2^2, x_3^2) \\ 2\mathrm{x}_2\mathrm{exp}(x_1^2, x_2^2, x_3^2) \\ 2\mathrm{x}_3\mathrm{exp}(x_1^2, x_2^2, x_3^2) \end{bmatrix}$

∇f(0,0,0) = $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

∇^2f(x_1, x_2, x_3) = $\begin{bmatrix} 2(2\mathrm{x}_1^2+1)\mathrm{exp}(x_1^2, x_2^2, x_3^2) & 4\mathrm{x}_1 x_2 exp(\mathrm{x}_1^2, x_2^2, x_3^2) & 4\mathrm{x}_1 x_3 exp(\mathrm{x}_1^2, x_2^2, x_3^2) \\ 4\mathrm{x}_1 x_2 exp(\mathrm{x}_1^2, x_2^2, x_3^2) & 2(2\mathrm{x}_2^2+1)\mathrm{exp}(x_1^2, x_2^2, x_3^2) & 4\mathrm{x}_2 x_3 exp(\mathrm{x}_1^2, x_2^2, x_3^2) \\ 4\mathrm{x}_1 x_3 exp(\mathrm{x}_1^2, x_2^2, x_3^2) & 4\mathrm{x}_2 x_3 exp(\mathrm{x}_1^2, x_2^2, x_3^2) & 2(2\mathrm{x}_3^2+1)\mathrm{exp}(x_1^2, x_2^2, x_3^2) \end{bmatrix}$

∇^2f(0,0,0) = $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The quadratic approximation is

$$1 + \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 1$$

3 Question 3

A = $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$

The eigenvalues are 1 and 0

Since A is not negative or positive definite, the quadratic function has a saddle point

4 Question 4

αx^Tx + (Ax)$^T(Ax) - 2b^T Ax + b^T b$
αx^Tx + $x^T A^T$Ax - 2b$^T Ax + b^T b$
$x^T(\alpha$IA$^T A) - 2b^T Ax + b^T b$
$B = \alpha$IA$^T A$
c = (-2b$^T A)^T = -2A^T b$
$d = b^T b = ||b||^2$

The gradient is ∇f(x) = 2Bx + c = 2(αI + A$^T A)x - 2A^T b$

The Hessian matrix is ∇f$^2(x) = 2B = 2(\alpha$I + A$^T A)$

Check if the Hessian matrix is positive definite

2x$^T Bx > 0$
$x^T 2(\alpha$I + A$^T A)x = 2(\alpha$x$^T Ix + x^T A^T Ax) = 2(\alpha$x$^T x + x^T A^T Ax) = 2(\alpha||x||^2 + ||Ax||^2$
Norm is always positive, thus the Hessian matrix is positive definite