Math 504

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1 Question 1

Let $z_1 = (x^1, y^1) \in \hat{C}$. $||x^1||^2 \leq y^1$
Let $z_2 = (x^2, y^2) \in \hat{C}$. $||x^2||^2 \leq y^2$ is convex
$\iff$ αz_1 + (1- α) $z_2 \in \hat{C}$
$\iff$ αx^1 + (1-α)(x^2), α y^1 + (1-α)(y^2)
$||\alpha$x^1 + (1-α)(x^2)$||^2 \leq$ (α $||x^1||$ + (1 - α)$||x^2||$)^2
$||\alpha$x^1 + (1-α)(x^2)$||_2^2 \leq \alpha||y^1||$ + (1-α)(y^2)
= α^2 $||x^1||^2$ + (1-α)$^2||x^2||^2$ + 2α(1-α)$||x^1||||x^2||$
= α^2 $||x^1||^2$ + (1-α)$^2||x^2||^2$ + α(1-α)$||x^1||^2$ + α(1-α)$||x^2||^2 \leq \alpha||x^1||^2$ + (-α)$||x^2||^2 \leq \alpha$y^1 + (1-α)y^2 $\hat{C}$ is convex

2 Question 2

Let z_1 = (x_1,t_1) $\in$ C, $||x_1||_2 \leq t_1$
Let z_2 = (x_2,t_2) $\in$ C, $||x_2||_2 \leq t_2$
z = αz_1 + (1-α)z_2 = (x_1t)
z = α(x_1,t_1) + (1-α)(x_2,t_2)
= (αx_1, αt_1) + ((1-α)x_2, (1-α)t_2)
= (αx_1 + (1-α)x_2, (αt_1 + (1-α)t_2
Thus,
x = (αx_1 + (1-α)x_2
t = (αt_1 + (1-α)t_2
Finally, $||x||_2$ = $||$(αx_1 + (1-α)$x_2||_2 \leq ||\alpha x_1||$ + $||(1-\alpha)$x$_2||_2$
$\leq \alpha||x_1||$ + (1-α)$||x_2||$
$\leq \alpha$ t_1 + (1-α)t$_2$ = t
$||x||_2 \leq$ t
C is a convex set

3 Question 3

3.1 a

From f(x) = h(g(x)), we have f(αx + (1-α)y = h(g(αx + (1-α)y)) $\leq$ h(αg(x) + (1-α)g(y) $\leq$ α h(g(x) + (1-α)h(g(y))

Thus, f(αx + (1-α)y $\leq$ αf(x) + (1-α)f(y)

Thus, h is convex and non-decreasing

3.2 b

f(αx + (1-α)y = h(g(αx + (1-α)y)) $\geq$ h(αg(x) + (1-α)g(y) $\leq$ α h(g(x) + (1-α)h(g(y))
Thus, f(αx + (1-α)y $\leq$ αf(x) + (1-α)f(y)

4 Question 4

f(x) = -x_1^2 - 4x_2^2
∇f = (-2$x_1, -8$x$_2)^T$
$\bar{x} = [\sqrt{3}, 1/2]^T$ and $\tilde{x} = [2, 0]^T$
-x_1^2 - 4x_2^2 = -2^2 -4 x 0^2 and -x_1^2 - 4x_2^2 = $-(\sqrt{3}^2$ - 4 x $(1/2)^2$
-x_1^2 - 4x_2^2 = $4 and -$x$_1^2$ - 4x_2^2 = 4

Define the parametric curve as x(t) = (2Cos(t) + Sin(t))^T, then $\tilde{x}$ corresponds to $\tilde{t}$ = 0 and $\bar{x}$ corresponds to $\bar{t} = \pi/6$
x$'$(t) = (-2Sin(t),Cos(t)^T
∇f(x(t)) = (-2 . 2Cos(t), -8Sin(t))^T = (-4Cos(t),-8Sin(t))^T
∇f($\tilde{x}$)^T = (-4Cos(0), -8Sin(0)) = (-4,0)
∇f($\bar{x}$)^T = (-4Cos(π/6),-8Sin(π/6) = (-4. $\sqrt{3}$/2,-8 . 1/2) = -2$\sqrt{3}$,-4)
x$'$($\tilde{t}$ = (-2Sin(0),Cos(0)) = (0,1)^T
and x$'$($\bar{t}$ = (-2Sin(π/6), Cos(π/6) = (-2/2,$\sqrt{3}$/2) = (-1,$\sqrt{3}$/2)^T

∇f($\tilde{x}$)^{T}x$'$($\tilde{t}$ = (-4,0) (0,1)^T = (-4) x 0 + 0 x 1 = 0

∇f($\bar{x}$)^{T}x$'$($\bar{t}$ = (-2$\sqrt{3}$,-4) ((-1,$\sqrt{3}$/2)^T) = 0

Thus, we prove that ∇f($\tilde{x}$)^{T}x$'$($\tilde{t}$ = ∇f($\bar{x}$)^{T}x$'$($\bar{t}$ = 0