Math 504

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1 Question 1

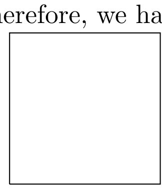
From the definition of norm, any function is called a norm if it satisfies three properties: positivity, scaling and triangle inequality.
Firstly, we will prove that the dual norm is greater than or equal to 0
If $||x||_* = 0$, we must have x = 0. Thus, if x $\neq$ 0, $||x||_* \neq 0$
$||x||_* = max_{||z|| \leq 1} z^T x$ and taking z = $\frac{x}{|x|}$, we have $||x||_* \geq \frac{||x||_2^2}{||x||} > 0$
Thus, the dual norm is greater than 0, the first property is satisfied.

Secondly, the scaling $||x\alpha||_* = max_{||z|| \leq 1}|z^T x\alpha| = max_{||z|| \leq 1}|\alpha||z^T x| = |\alpha| max_{||z|| \leq 1}|z^T x| = |\alpha|||x||_*$
Thus, the second property is satisfied.

Thirdly, triangle inequality From the Cauchy-Schawartz Inequality definition, $z^T x \leq ||z||_2.||x||_2$. Thus, $z^T x \leq ||z||.||x||_*$. The third property holds. Dual norm is indeed a norm

2 Question 2

$B_1(1) = \{a \in \mathrm{R}^2 \; |||x||_\infty \leq 1\}$
=max$\{|a_1|,|a_2|\} = 1$
Divide this to two cases, max$\{|a_1|\} = 1$ and max$\{,|a_2|\} = 1$
In the first case, max$\{|a_1|\} = 1$, so a_1 = 1,-1
Likewise, max$\{|a_2|\} = 1$, so a_2 = 1,-1
Therefore, we have a square at 4 points (1,1),(1,-1),(-1,1),(-1,-1)



3 Question 3

$||x+y||_2^2 = < x+y, x+y > = < x,x > + < x,y > + < x,y > + < y,y >$
$= ||x||_2^2 + 2 < x,y > + ||y||_2^2 \leq ||x||_2^2 + 2||x||_2||y||_2 + ||y||_2$ (using Cauchy-Schwarz inequality)
$= (||x||_2 + ||y||_2)^2$
Square root both sides, we have $||x+y||_2 \leq ||x||_2 + ||y||_2$

4 Question 4

$(A^TA)^T = A^T(A^T)^T = A^TA$ By the definition of symmetry, A^TA is symmetric.