Math 504

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1 Question 1a

$$L_1\text{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 2/3 & 13/3 \end{bmatrix}$$

$$L_2(L_1\text{A}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 2/3 & 13/3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} = \text{U}$$

Now L $= (L_2L_1)^-1 = L_1^{-1}L_2^{-2}$ with

$$L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 1 \end{bmatrix}$$

hence

$$\text{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 1/3 & 1 \end{bmatrix}$$

$$\text{A} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

2 Question 1b

Because of the above factorization we can write the system in matrix form as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We then have Ly = b $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

and Ux = y $\begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

The first system is easily solved by forward substitution. From the first row, we can see that y_1 = -1; it follows that y_2 = 0. Finally, y_3 = 1 + $y_1/3$ -$y_2/3$ = 1 - 1/3 = 2/3

Now that we know y, the latter system becomes

$$\begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2/3 \end{bmatrix}$$

This is now solved by back-substitution. We can see that $4x_3$ = 2/3, so x_3 = 1/6

$2x_2$ + x_3 = 0, thus x_2 = -x_3 / 2 = -1/6 /2 = -1/12

$3x_1$ - x_2 + x_3 = -1, thus x_1 =(-1 - x_3 + x_2) / 3 = -5/12

3 Question 1c

We have that

$3x_1 - x_2 + x_3$ = -1, thus x_1 = (-1+$x_2 - x_3$)/3

$2x_2 + x_3$ = 0, thus $x_2 = -x_3/2$

$x_1 + x_2 + 4x_3$ = 1, thus x_3 = (1+$x_1 - x_2$)/4

With the first iteration, x_1 = (-1+1-0)/3 = 0

x_2 = -0 / 2 = 0

x_3 = (1-1-1) / 4 = -1/4

The second iteration, x_1 = (-1 + 0 - (-1/4)) / 3 = -1/4

x_2 = (1/4) / 2 = 1/8

x_3 = (1 - 0 - 0) / 4 = 1/4

The error in the first iteration, $||x^1 - x^*||_\infty = 5/12$

The error in the second iteration, $||x^2 - x^*||_\infty = 5/24$

4 Question 1d

We have that

$3x_1 - x_2 + x_3$ = -1, thus x_1 = (-1+$x_2 - x_3$)/3

$2x_2 + x_3$ = 0, thus $x_2 = -x_3/2$

$x_1 + x_2 + 4x_3$ = 1, thus x_3 = (1+$x_1 - x_2$)/4

With the first iteration, x_1 = (-1+1-0)/3 = 0

x_2 = -0 / 2 = 0

x_3 = (1+0-0) / 4 = 1/4

The second iteration, x_1 = (-1 + 0 - 1/4) / 3 = -5/12

x_2 = (1/4) / 2 = 1/8

x_3 = (1 - 5/12 -1/8) / 4 = 11/96

The error in the first iteration, $||x^1 - x^*||_\infty = 5/12$

The error in the second iteration, $||x^2 - x^*||_\infty = 5/24$

5 Question 1e

```python
def seidel(a, x ,b):
    #Finding length of a(3)
    n = len(a)
    # for loop for 3 times as to calculate x, y , z
    for j in range(0, n):
        # temp variable d to store b[j]
        d = b[j]

        # to calculate respective xi, yi, zi
        for i in range(0, n):
            if(j != i):
                d-=a[j][i] * x[i]
        # updating the value of our solution
        x[j] = d / a[j][j]
    # returning our updated solution
    return x
n = 3
a = []
b = []
# initial solution depending on n(here n=3)
x = [1, 1, 0]
a = [[3, -1, 1],[0, 2, 1],[1, 1, 4]]
b = [-1,0,1]
print(x)

#loop run for m times depending on m the error value
for i in range(0, 25):
    x = seidel(a, x, b)
    #print each time the updated solution
    print(x)
```

```
[1, 1, 0]
[0.0, 0.0, 0.25]
[-0.4166666666666667, -0.125, 0.3854166666666667]
[-0.5034722222222222, -0.19270833333333334, 0.4240451388888889]
[-0.5389178240740741, -0.21202256944444445, 0.4377350983796296]
[-0.5499192226080246, -0.2188675491898148, 0.44219669294945985]
[-0.5536880807130916, -0.22109834647472992, 0.44369660679695533]
[-0.5549316510905618, -0.22184830339847766, 0.44419498862225987]
[-0.5553477640069125, -0.22209749431112993, 0.44436131457951067]
[-0.5554862696302135, -0.22218065728975533, 0.4444167317299922]
[-0.5555324630065824, -0.2222083658649961, 0.4444352072178946]
[-0.5555478576942968, -0.2222176036089473, 0.444441365325811]
[-0.5555529896449195, -0.2222206826629055, 0.4444434180769562]
[-0.5555547002466206, -0.2222217090384781, 0.4444441023212747]
[-0.5555552704532509, -0.22222205116063734, 0.4444443304034721]
[-0.5555554605213698, -0.22222216520173604, 0.4444444064307765]
[-0.5555555238775042, -0.22222220321538824, 0.44444443177322307]
[-0.5555555449962037, -0.22222221588661153, 0.44444444022070384]
[-0.5555555520357718, -0.22222222011035192, 0.4444444430365309]
[-0.5555555543822942, -0.22222222151826546, 0.44444444397513994]
[-0.5555555551644685, -0.22222222198756997, 0.44444444428800967]
[-0.5555555554251933, -0.22222222214400483, 0.4444444443922996]
[-0.5555555555121016, -0.2222222221961498, 0.4444444444270629]
[-0.555555555541071, -0.22222222221353144, 0.4444444444386506]
[-0.5555555555507273, -0.222222222193253, 0.44444444444251313]
[-0.5555555555539461, -0.22222222222125657, 0.44444444444380066]
```