

Notes on higher dimensional groups and groupoids and related topics

Christopher D Wensley

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Preface

The aim of these notes is to collect together, using a common notation and right actions, information about crossed modules; cat1-groups; crossed squares; and related structures.

Inspiration has come from Ronnie Brown; Tim Porter; postgraduate students; and the many visitors to the department in Bangor.

Help in the preparation of these notes has initially been provided by Murat Alp and Robert Rodrigues. A seminar course for Gareth Evans and Richard Lewis during 2002/2003 resulted in many corrections and additions.

This is work in progress, so there are many comments in the text of the form “[Such-and-such needs doing.]”.

The *hyperref* package is used from June 2009 to enable readers to skip around these notes.

The main sources used are:

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- K.J. Norrie, *Crossed modules and analogues of group theorems*, Ph.D. Thesis, King’s College, London, January 1987.
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- (more to be added)

1 Crossed Modules and Cat^1 -Groups

In this section we describe four equivalent categories:

- **XMod**, the category of crossed modules and their morphisms;
- **Cat1**, the category of cat^1 -groups and their morphisms;
- **GpGpd**, a category of sets with both a group structure and a groupoid structure; and
- **2-Gp**, a subcategory of **2-Cat**.

We also describe functors between these categories which exhibit the various equivalences.

1.1 Precrossed and Crossed Modules

Let S and R be groups acting upon themselves by conjugation:

$$s_0^s = s^{-1}s_0s, \quad r_0^r = r^{-1}r_0r.$$

A *precrossed module* $\mathcal{X} = (\delta : S \rightarrow R)$ consists of a group homomorphism δ , called the *boundary* of \mathcal{X} , together with an action $\alpha : R \rightarrow \text{Aut}(S)$ such that δ is an R -morphism. So, for all $s \in S$ and $r \in R$,

$$\mathbf{X1:} \quad \delta(s^r) = (\delta s)^r.$$

An alternative terminology is to say that (S, δ) is a precrossed R -module. The groups S, R are called the *source* and *range* of \mathcal{X} respectively.

The precrossed module \mathcal{X} is a *crossed module* if it also satisfies, for all $s_0, s \in S$,

$$\mathbf{X2:} \quad s_0^{\delta s} = s_0^s.$$

A *morphism of precrossed modules* $\alpha : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ is a pair $\alpha = (\ddot{\alpha}, \dot{\alpha})$, where $\ddot{\alpha} : S_1 \rightarrow S_2$ and $\dot{\alpha} : R_1 \rightarrow R_2$ are homomorphisms satisfying

$$\delta_2 \circ \ddot{\alpha} = \dot{\alpha} \circ \delta_1, \quad \ddot{\alpha}(s^r) = (\ddot{\alpha}s)^{\dot{\alpha}r},$$

making the following diagram commute:

$$\begin{array}{ccc} S_1 & \xrightarrow{\ddot{\alpha}} & S_2 \\ \delta_1 \downarrow & & \downarrow \delta_2 \\ R_1 & \xrightarrow{\dot{\alpha}} & R_2 \end{array} \quad (1)$$

When $\mathcal{X}_1, \mathcal{X}_2$ are both crossed modules then α is a *morphism of crossed modules* without any further condition. We thus obtain the category **PreXMod** of precrossed modules and their morphisms, and the category **XMod** of crossed modules and their morphisms. Furthermore, **XMod** is a full subcategory of **PreXMod**.

When $\mathcal{X}_2 = \mathcal{X}_1$ and $\ddot{\alpha}, \dot{\alpha}$ are automorphisms then α is an automorphism of \mathcal{X}_1 . The group of automorphisms is denoted by $\text{Aut}(\mathcal{X}_1)$.

We have to be careful about a notational problem which arises because, although we are using right actions, we are still writing functions of the left. Thus the group of automorphisms of R has multiplication $\beta_1 * \beta_2$ given by

$$(\beta_1 * \beta_2)r = \beta_2(\beta_1(r)) \quad \text{or, more conveniently,} \quad \beta_2\beta_1r,$$

where β_1 is applied *first* to r , and β_2 *second*.

1.2 Examples of Crossed Modules

Standard constructions for crossed modules include the following:

1. A *conjugation crossed module* ($\text{inc} : S \rightarrow R$) is an inclusion of a normal subgroup $S \trianglelefteq R$, where R acts on S by conjugation. The example takes the alternating subgroup a_4 of the symmetric group s_4 .

```
gap> s4 := Group( (1,2,3,4), (3,4) );; SetName( s4, "s4" );
gap> a4 := Subgroup( s4, [(1,2,3), (2,3,4)] );; SetName( a4, "a4" );
gap> X4 := XModByNormalSubgroup( s4, a4 );;
gap> Display( X4 );
Crossed module [a4->s4] :-
: Source group has generators:
  [ (1,2,3), (2,3,4) ]
: Range group s4 has generators:
  [ (1,2,3,4), (3,4) ]
: Boundary homomorphism maps source generators to:
  [ (1,2,3), (2,3,4) ]
: Action homomorphism maps range generators to automorphisms:
  (1,2,3,4) --> { source gens --> [ (2,3,4), (1,3,4) ] }
  (3,4) --> { source gens --> [ (1,2,4), (2,4,3) ] }
  These 2 automorphisms generate the group of automorphisms.
```

2. An *automorphism crossed module* ($\text{inn} : R \rightarrow S$) has as range a subgroup R of the automorphism group $\text{Aut}(S)$ of S which contains the inner automorphism group $\text{Inn}(S)$ of S . The boundary maps $s \in S$ to the inner automorphism of S by s . In the example the quaternion group q_8 has automorphism group isomorphic to s_4 . A permutation representation of degree 6 for this automorphism group is chosen automatically.

```
gap> q8 := Group( (1,2,3,4)(5,8,7,6), (1,5,3,7)(2,6,4,8) );;
gap> SetName( q8, "q8" );
gap> X8 := XModByAutomorphismGroup( q8 );;
gap> Display( X8 );
Crossed module [q8->PAut(q8)] :-
: Source group q8 has generators:
  [ (1,2,3,4)(5,8,7,6), (1,5,3,7)(2,6,4,8) ]
: Range group PAut(q8) has generators:
  [ (3,6,5,4), (1,6,3)(2,4,5), (3,5)(4,6), (1,2)(4,6) ]
: Boundary homomorphism maps source generators to:
  [ (3,5)(4,6), (1,2)(4,6) ]
: Action homomorphism maps range generators to automorphisms:
  (3,6,5,4) --> { source gens --> [ (1,2,3,4)(5,8,7,6), (1,8,3,6)(2,5,4,7) ] }
  (1,6,3)(2,4,5) --> { source gens -->
[ (1,8,3,6)(2,5,4,7), (1,2,3,4)(5,8,7,6) ] }
  (3,5)(4,6) --> { source gens --> [ (1,2,3,4)(5,8,7,6), (1,7,3,5)(2,8,4,6)
] }
  (1,2)(4,6) --> { source gens --> [ (1,4,3,2)(5,6,7,8), (1,5,3,7)(2,6,4,8)
] }
  These 4 automorphisms generate the group of automorphisms.
```

3. A *zero boundary crossed module* ($0 : M \rightarrow R$) has an R -module M as source and $\partial = 0$.
4. Any homomorphism $\partial : S \rightarrow R$, with S abelian and $\text{im } \partial$ in the centre of R , provides a crossed module with R acting trivially on S .

5. A *central extension crossed module* has as boundary a surjection $\partial : S \rightarrow R$ with central kernel, where $r \in R$ acts on S by conjugation with $\partial^{-1}r$.
6. The *direct product* of $\mathcal{X}_1 = (\partial_1 : S_1 \rightarrow R_1)$ and $\mathcal{X}_2 = (\partial_2 : S_2 \rightarrow R_2)$ is $\mathcal{X}_1 \times \mathcal{X}_2 = (\partial_1 \times \partial_2 : S_1 \times S_2 \rightarrow R_1 \times R_2)$ with R_1, R_2 acting trivially on S_2, S_1 respectively.

Here is a verification for the fifth example. Suppose $s_1, s_2 \in \partial^{-1}r$. Then $s_2 = s_1k$ for some $k \in \ker \partial$, so $s^{s_2} = s^{s_1k} = s_1^{-1}(k^{-1}sk)s_1 = s^{s_1}$, and the action is well-defined. The two axioms are then easily verified.

1.3 Properties of Crossed Modules

Lemma 1.1

- (i) The kernel K of ∂ is central in S , and so is abelian.
- (ii) The image $J = \text{im } \partial$ is normal in R , and so we may define $C = \text{coker } \partial = R/J$, with natural map $\nu : R \rightarrow C$, and hence obtain an exact sequence of groups

$$1 \longrightarrow J \xrightarrow{\iota} R \xrightarrow{\nu} C \longrightarrow 1.$$

- (iii) The group J acts trivially on the centre ZS of S , and so trivially on K . Hence K inherits an action of C , making K a C -module with $k^{Jr} := k^r$, and giving a crossed module $(\nu\partial|_K : K \rightarrow C)$.
- (iv) If $\partial' : S \rightarrow J$ is the restriction of ∂ , there is an exact sequence of R -groups

$$1 \longrightarrow K \longrightarrow S \xrightarrow{\partial'} J \longrightarrow 1.$$

- (v) The image J acts trivially on the abelianisation $S^{\text{ab}} = S/[S, S]$ of S , and so S^{ab} is also a C -module, with action

$$([S, S]s)^{(Jr)} = [S, S](s^r). \quad (2)$$

Proof:

- (i) If $k \in K$ and $s \in S$ then, by **X2**,

$$s^1 = s^{\partial k} = s^k = k^{-1}sk \quad \text{and so} \quad ks = sk.$$

- (ii) The conjugate of ∂s by $r \in R$ is $r^{-1}(\partial s)r = (\partial s)^r = \partial(s^r) \in J$.

- (iii) If $z \in ZS$ then, by **X2**, $z^{\partial s} = s^{-1}zs = z$. Then $\nu(\partial k) = J(\partial k)$ and

$$\nu\partial(k^{Jr}) = J(\partial(k^r)) = J((\partial k)^r) = (Jr^{-1})(J(\partial k))(Jr) = (\nu\partial k)^{Jr}.$$

- (iv) The kernel of ∂' is K and ∂' is surjective.

- (v)

$$[S, S]1 = [S, S][s_1, s_2^{-1}] = [S, S]s_2^{\partial s_1} s_2^{-1} \Rightarrow [S, S]s_2 = [S, S]s_2^{\partial s_1}.$$

Hence J acts trivially on S^{ab} , and it is easy to check that (2) defines an action.

□

1.4 Sub-crossed modules

Definition 1.2 A crossed module $\mathcal{X}_1 = (\partial_1 : S_1 \rightarrow R_1)$ is a sub-crossed module of $\mathcal{X} = (\partial : S \rightarrow R)$, written $\mathcal{X}_1 \leq \mathcal{X}$, if

- S_1, R_1 are subgroups of S, R respectively,
- ∂_1 is the restriction of ∂ to S_1 ,
- the action of R_1 on S_1 is induced by the action of R on S .

$$\begin{array}{ccc} S_1 & \xrightarrow{i_{S_1}} & S \\ \partial_1 \downarrow & & \downarrow \partial \\ R_1 & \xrightarrow{i_{R_1}} & R \end{array}$$

Definition 1.3 The inclusion morphism $i = (i_{S_1}, i_{R_1}) : \mathcal{X}_1 \rightarrow \mathcal{X}$ consists of the two subgroup inclusions $i_{S_1} : S_1 \rightarrow S$ and $i_{R_1} : R_1 \rightarrow R$.

Definition 1.4 The sub-crossed module \mathcal{X}_1 is a normal sub-crossed module of \mathcal{X} , written $\mathcal{X}_1 \trianglelefteq \mathcal{X}$, if

- (a) $R_1 \trianglelefteq R$,
- (b) $s_1^r \in S_1$ for all $r \in R$, $s_1 \in S_1$,
- (c) $s^{-1}s^{r_1} \in S_1$ for all $r_1 \in R_1$, $s \in S$.

Note that these conditions imply that $S_1 \trianglelefteq S$, for $s_1^s = s_1^{\partial s}$ by **X2**: and belongs to S_1 by (b).

Proposition 1.5 Given two normal sub-crossed modules $\mathcal{X}_1, \mathcal{X}_2$ of \mathcal{X} , there is a third normal sub-crossed module of \mathcal{X} called the commutator sub-crossed module $[\mathcal{X}_1, \mathcal{X}_2]$, having

- source group $[S_1, S_2]$,
- range group $[R_1, R_2]$,
- the restriction ∂' of ∂ to $[S_1, S_2]$ as boundary map.

Proof: First note that $\partial'[s_1, s_2] = [\partial s_1, \partial s_2]$, and that $[s_1, s_2]^{r_1} = [s_1^{r_1}, s_2^{r_1}]$, so $[\mathcal{X}_1, \mathcal{X}_2]$ is a sub-crossed module of \mathcal{X} . To show normality we check that

- (a) $[R_1, R_2] \trianglelefteq R$, since $r^{-1}[r_1, r_2]r = [r^{-1}r_1r, r^{-1}r_2r]$,
- (b) $[s_1, s_2]^r = [s_1^r, s_2^r] \in [S_1, S_2]$,
- (c) The product $s^{-1}s^{[r_1, r_2]}$ may be expanded as

$$\left[s^{-1}s^{r_1^{-1}} \right] \left[\left(s^{r_1^{-1}} \right)^{-1} \left(s^{r_1^{-1}} \right)^{r_2^{-1}} \right] \left[\left(s^{r_1^{-1}r_2^{-1}} \right)^{-1} \left(s^{r_1^{-1}r_2^{-1}} \right)^{r_1} \right] \left[\left(s^{r_1^{-1}r_2^{-1}r_1} \right)^{-1} \left(s^{r_1^{-1}r_2^{-1}r_1} \right)^{r_2} \right],$$

and each of the four pairs of terms has the form $s^{-1}s^{r_0}$, and so is in S_1 .

□

1.5 Properties of morphisms of crossed modules

Let $\mu = (\ddot{\mu}, \dot{\mu}) : \mathcal{X} \rightarrow \mathcal{X}'$ be a morphism of crossed modules. The kernel of μ is the sub-crossed module

$$\ker \mu = (\partial|_{\ker \ddot{\mu}} : \ker \ddot{\mu} \rightarrow \ker \dot{\mu})$$

of \mathcal{X} , as shown in the following diagram.

$$\begin{array}{ccccc} \ker \ddot{\mu} & \xrightarrow{i} & S & \xrightarrow{\ddot{\mu}} & S' \\ \partial|_{\ker \ddot{\mu}} \downarrow & & \downarrow \partial & & \downarrow \partial' \\ \ker \dot{\mu} & \xrightarrow{i} & R & \xrightarrow{\dot{\mu}} & R' \end{array}$$

This statement is justified by the following:

$$s \in \ker \ddot{\mu} \Leftrightarrow \ddot{\mu}s = 1 \Leftrightarrow \partial' \ddot{\mu}s = 1 \Leftrightarrow \dot{\mu}\partial s = 1 \Leftrightarrow \partial s \in \ker \dot{\mu}.$$

Lemma 1.6 *The kernel of μ is a normal sub-crossed module of \mathcal{X} .*

Proof:

- (a) $\ker \dot{\mu} \trianglelefteq R$,
- (b) if $\ddot{\mu}s_1 = 1$ then $\ddot{\mu}(s_1^r) = (\ddot{\mu}s_1)^{\dot{\mu}r} = 1^{\dot{\mu}r} = 1$, so $s_1^r \in \ker \ddot{\mu}$,
- (c) if $\dot{\mu}r_1 = 1$ then $\ddot{\mu}((s^{-1})^{r_1}s) = (\dot{\mu}s^{-1})^{\dot{\mu}r_1}(\dot{\mu}s) = 1$, so $(s^{-1})^{r_1}s \in \ker \ddot{\mu}$.

□

Theorem 1.7 *Given a normal sub-crossed module \mathcal{X}_1 of a crossed module \mathcal{X} , there is a quotient crossed module*

$$\mathcal{X}/\mathcal{X}_1 = (\delta : S/S_1 \rightarrow R/R_1)$$

where the action is defined by

$$(S_1s)^{R_1r} := S_1(s^r)$$

and the boundary map is given by

$$\delta(S_1s) := R_1(\partial s).$$

Proof: We first check that we do have an action:

$$\begin{aligned} (S_1s)^{R_1r}(S_1s')^{R_1r} &= (S_1(s^r))(S_1(s'^r)) = S_1(s^r)(s'^r) = S_1(ss')^r = (S_1(ss'))^{R_1r}, \\ ((S_1s)^{R_1q})^{R_1r} &= (S_1(s^q))^{R_1r} = S_1(s^q)^r = S_1(s^{qr}) = (S_1s)^{R_1(qr)}. \end{aligned}$$

Then we check the crossed module axioms:

$$\mathbf{M1:} \quad \delta((S_1s)^{R_1r}) = \delta(S_1s^r) = R_1\partial(s^r) = R_1r^{-1}(\partial s)r = (R_1r^{-1})(R_1\partial s)(R_1r) = (\delta(S_1s))^{R_1r},$$

$$\mathbf{M2:} \quad (S_1s')^{\delta(S_1s)} = (S_1s')^{R_1(\partial s)} = S_1(s')^{\partial s} = S_1s^{-1}s's = (S_1s^{-1})(S_1s')(S_1s).$$

□

Theorem 1.8 [from Tim Porter's Topology paper [51]]

Every crossed module is a quotient of normal inclusion crossed modules.

Proof: Given $\mathcal{X} = (\partial : S \rightarrow R)$, consider the diagram

$$\begin{array}{ccccc}
 1 & \xrightarrow{0} & S & \xrightarrow{1} & S \\
 \downarrow 0 & & \downarrow \epsilon & & \downarrow \partial \\
 S & \xrightarrow{\zeta} & R \ltimes S & \xrightarrow{h} & R
 \end{array}
 \tag{3}$$

$\Gamma_1 \mathcal{X} \qquad \qquad \Gamma_0 \mathcal{X} \qquad \qquad \mathcal{X}$

where the crossed modules $\Gamma_0 \mathcal{X}$ and $\Gamma_1 \mathcal{X}$ are given by

- $\epsilon s = (1, s), \quad \zeta s = (\partial s, s^{-1}), \quad s_1^{(r,s)} = s^{-1} s_1^r s = s_1^{r(\partial s)},$
- $h(r, s) = r(\partial s), \quad (r_1, s)^r = (r^{-1} r_1 r, s^r).$

The proof requires four verifications. Recall that in the semidirect product $R \ltimes S$ we have

$$(r, s)(r', s') = (rr', s^{r'} s') \quad \text{and} \quad (r, s)^{-1} = (r^{-1}, (s^{-1})^{r^{-1}}). \tag{4}$$

(a) Verification that $(\epsilon : S \rightarrow R \ltimes S)$ is a crossed module.

X1: $(\epsilon s_1)^{(r,s)} = (r^{-1}, (s^{-1})^{r^{-1}})(1, s_1)(r, s) = (1, s^{-1} s_1^r s) = \left(1, s_1^{(r,s)}\right) = \epsilon \left(s_1^{(r,s)}\right).$

X2: $s_1^{\epsilon s} = s_1^{(1,s)} = s^{-1} s_1 s.$

(b) Verification that $(1, h)$ and (ϵ, δ) are morphisms of crossed modules.

(i) $h\epsilon s = h(1, s) = \partial s,$

(ii) $\begin{cases} 1(s_1^{(r,s)}) = s_1^{(r,s)} = s^{-1} s_1^r s, \\ (1s_1)^{h(r,s)} = s_1^{r(\partial s)} = s^{-1} s_1^r s. \end{cases}$

(ii') $\begin{cases} \epsilon(s_1^s) = (1, s_1^s) = (1, s^{-1} s_1 s), \\ (\epsilon s_1)^{\partial s} = (1, s_1)^{\partial s} = (1, s_1)^{(1, \partial s)} = (1, s_1^{\partial s}) = (1, s^{-1} s_1 s). \end{cases}$

(c) Verification that $(0, \zeta)$ is the inclusion of a normal sub-crossed module.

We only need to show that $s_1^{-1} s_1^{\zeta s} = 1.$

$$s_1^{-1} s_1^{\zeta s} = s_1^{-1} s_1^{(\partial s, s^{-1})} = s_1^{-1} s s_1^{\partial s} s^{-1} = 1.$$

(d) Verification that $h\zeta = 0.$

$$h\zeta s = h(\partial s, s^{-1}) = (\partial s)(\partial s^{-1}) = 1.$$

□

Corollary 1.9 *In the previous result $\Gamma_0, \Gamma_1 : \mathbf{XMod} \rightarrow \mathbf{XMod}$ are functors and $(0, \zeta) : \Gamma_1 \rightarrow \Gamma_0$ is a natural transformation.*

Note that, in diagram (3), $(h : R \ltimes S \rightarrow R)$ is *not* in general a crossed module, so the right-hand square is not a crossed square (see Chapter 8).

We may extend the above Theorem to morphisms of crossed modules. In the diagram (5) below, $(\sigma, \rho) : \mathcal{X}_1 = (\partial : S \rightarrow R) \rightarrow \mathcal{X}_2 = (\delta : Q \rightarrow P)$ is a morphism of crossed modules, and both of these crossed modules are expressed as quotients of normal inclusion crossed modules. There are morphisms $(\sigma, \gamma) : \Gamma_0 \mathcal{X}_1 = (\epsilon : S \rightarrow R \ltimes S) \rightarrow \Gamma_0 \mathcal{X}_2 = (\eta : Q \rightarrow P \ltimes Q)$ and $(1, \sigma) : \Gamma_1 \mathcal{X}_1 \rightarrow \Gamma_1 \mathcal{X}_2$ where $\gamma(r, s) = (\rho r, \sigma s)$ and $\eta \sigma s = (1, \sigma s) = \gamma(1, s) = \gamma \epsilon s$.

$$\begin{array}{ccccc}
 & & 1 & \xrightarrow{0} & Q & \xrightarrow{1} & Q \\
 & & \downarrow 0 & & \downarrow \eta & & \downarrow \delta \\
 & & Q & \xrightarrow{\theta} & P \ltimes Q & \xrightarrow{k} & P \\
 & & \uparrow \sigma & & \uparrow \sigma & & \uparrow \sigma \\
 1 & \xrightarrow{0} & S & \xrightarrow{1} & S & & \\
 \downarrow 0 & & \downarrow \epsilon & & \downarrow \partial & & \\
 S & \xrightarrow{\zeta} & R \ltimes S & \xrightarrow{h} & R & &
 \end{array}
 \tag{5}$$

1.6 Peiffer subgroup of a precrossed module

We shall construct from any precrossed module $\mathcal{Q} = (\delta : Q \rightarrow R)$ a crossed module $\mathcal{X} = (\partial : Q/P \rightarrow R)$ whose source is a suitable quotient of Q .

Given $\mathcal{Q} = (\delta : Q \rightarrow R)$ the *Peiffer commutators* of \mathcal{Q} are elements of the form

$$\langle q_1, q_2 \rangle = (q_2^{-1})^{q_1} q_2^{\delta q_1}, \quad \text{where } q_1, q_2 \in Q,$$

so that

$$\langle q_1, q_2 \rangle = 1 \Leftrightarrow q_2^{\delta q_1} = q_1^{-1} q_2 q_1$$

The subgroup P of Q generated by the Peiffer commutators is known as the *Peiffer subgroup* of \mathcal{Q} .

Lemma 1.10

- (a) $\langle q_1, 1 \rangle = 1 = \langle 1, q_2 \rangle$;
- (b) $\delta \langle q_1, q_2 \rangle = 1_R$;
- (c) $\langle q_1 q_3, q_2 \rangle = \langle q_1, q_2 \rangle^{q_3} \langle q_3, q_2^{\delta q_1} \rangle$;
- (d) $\langle q_1, q_2 q_3 \rangle = \langle q_1, q_3 \rangle \langle q_1, q_2 \rangle^{q_3^{\delta q_1}}$;

- (e) $\langle q_1, q_2 \rangle^r = \langle q_1^r, q_2^r \rangle$;
(f) $\langle q_1, q_2 \rangle^{-1} = \langle q_1^{-1}, q_2^{\delta q_1} \rangle^{q_1}$.

Proof: Part (a) is immediate, and part (b) follows from $\delta(q_2^{\delta q_1}) = (\delta q_1)^{-1}(\delta q_2)(\delta q_1)$. The remaining parts follow on expanding the Peiffer elements, and are closely related to identities for commutators and crossed pairings (see Subsection 7.2):

$$\begin{aligned} \langle q_1 q_3, q_2 \rangle &= \left[q_3^{-1} q_1^{-1} q_2^{-1} q_1 q_2^{\delta q_1} q_3 \right] \left[q_3^{-1} (q_2^{\delta q_1})^{-1} q_3 q_2^{\delta(q_1 q_3)} \right] = \langle q_1, q_2 \rangle^{q_3} \langle q_3, q_2^{\delta q_1} \rangle , \\ \langle q_1, q_2 q_3 \rangle &= \left[q_1^{-1} q_3^{-1} q_1 q_3^{\delta q_1} \right] \left[(q_3^{\delta q_1})^{-1} q_1^{-1} q_2^{-1} q_1 (q_2 q_3)^{\delta q_1} \right] = \langle q_1, q_3 \rangle \langle q_1, q_2 \rangle^{q_3^{\delta q_1}} , \\ \langle q_1^r, q_2^r \rangle &= (q_1^{-1})^r (q_2^{-1})^r q_1^r (q_2^r)^{(r^{-1}(\delta q_1)r)} = (q_1^{-1} q_2^{-1} q_1 q_2^{\delta q_1})^r , \\ \langle q_1, q_2 \rangle^{-1} &= \left(q_2^{\delta q_1} \right)^{-1} q_1^{-1} q_2 q_1 = q_1^{-1} \left[q_1 \left(q_2^{\delta q_1} \right)^{-1} q_1^{-1} \left(q_2^{\delta q_1} \right)^{\delta q_1^{-1}} \right] q_1 = \langle q_1^{-1}, q_2^{\delta q_1} \rangle^{q_1} . \end{aligned}$$

□

The following result follows immediately from these identities.

Corollary 1.11 *Let $\mathcal{Q} = (\delta : Q \rightarrow R)$ be a precrossed module. Then the set P of Peiffer commutators in \mathcal{Q} is a subgroup of $\ker \delta$; is normal in Q ; and is R -invariant.*

Proposition 1.12 *If $(\ddot{\alpha}, \dot{\alpha}) : \mathcal{Q}_1 \rightarrow \mathcal{Q}_2$ is a morphism of precrossed modules, then Peiffer commutators in \mathcal{Q}_1 are mapped to Peiffer commutators in \mathcal{Q}_2 .*

Proof: The source map $\ddot{\alpha}$ is compatible with the Peiffer pairing:

$$\begin{aligned} \ddot{\alpha} \langle q_1, q_2 \rangle &= (\ddot{\alpha} q_1)^{-1} (\ddot{\alpha} q_2)^{-1} (\ddot{\alpha} q_1) (\ddot{\alpha} (q_2^{\delta_1 q_1})) \\ &= (\ddot{\alpha} q_1)^{-1} (\ddot{\alpha} q_2)^{-1} (\ddot{\alpha} q_1) (\ddot{\alpha} q_2)^{\dot{\alpha} \delta_1 q_1} \\ &= (\ddot{\alpha} q_1)^{-1} (\ddot{\alpha} q_2)^{-1} (\ddot{\alpha} q_1) (\ddot{\alpha} q_2)^{\delta_2(\ddot{\alpha} q_1)} \\ &= \langle \ddot{\alpha} q_1, \ddot{\alpha} q_2 \rangle . \end{aligned}$$

□

Proposition 1.13 *Let $\mathcal{Q} = (\delta : Q \rightarrow R)$ be a precrossed module. Then there is a crossed module $\mathcal{X} = (\partial : S \rightarrow R)$ and a morphism of precrossed modules $(\text{nat}, 1) : \mathcal{Q} \rightarrow \mathcal{X}$ such that $(\text{nat}, 1)$ is universal for morphisms from \mathcal{Q} to crossed modules over R .*

Proof: Let P be the Peiffer group of \mathcal{Q} . Then the quotient group $S = Q/P$ is well-defined, $\text{nat} : Q \rightarrow S$ is the natural quotient map, and S inherits an R -action and an R -morphism:

$$(Pq)^r = P(q^r) , \quad \partial : S \rightarrow R, (Pq) \mapsto \delta q .$$

So $\mathcal{X} = (\partial : S \rightarrow R)$ is a precrossed module. By definition of P , we have $s_1^{-1} s_2 s_1 = s_2^{\partial s_1}$ for all $s_1, s_2 \in S$, and so \mathcal{X} is a crossed module. The quotient morphism $(\text{nat}, 1)$ is clearly a morphism of precrossed modules.

$$\begin{array}{ccccc}
& & \theta & & \\
& \swarrow & & \searrow & \\
S' & \xleftarrow{n'} & Q & \xrightarrow{\text{nat}} & S = Q/P \\
\downarrow \partial' & & \downarrow \delta & & \downarrow \partial \\
R & \xleftarrow{1} & R & \xrightarrow{1} & R
\end{array}$$

If $\mathcal{X}' = (\partial' : S' \rightarrow R)$ is a crossed module and if $(n', 1) : \mathcal{Q} \rightarrow \mathcal{X}'$ is a precrossed module morphism, then there is a unique crossed module morphism $(\theta, 1) : \mathcal{X} \rightarrow \mathcal{X}'$ such that $(\theta, 1) \circ (\text{nat}, 1) = (n', 1)$ where $\theta(Pq) = n'q$. \square

Example 1.14 In the following GAP run the Peiffer subgroup is cyclic of order 4.

```

gap> b1 := (11,12,13,14,15,16,17,18);; b2 := (12,18)(13,17)(14,16);;
gap> d16 := Group( b1, b2 );;
gap> sk4 := Subgroup( d16, [ b1^4, b2 ] );;
gap> SetName( d16, "d16" ); SetName( sk4, "sk4" );
gap> bdy16 := GroupHomomorphismByImages( d16, sk4, [b1,b2], [b1^4*b2,b2] );;
gap> aut1 := GroupHomomorphismByImages( d16, d16, [b1,b2], [b1^5,b2] );;
gap> aut2 := GroupHomomorphismByImages( d16, d16, [b1,b2], [b1,b1^4*b2] );;
gap> aut16 := Group( [ aut1, aut2 ] );;
gap> act16 := GroupHomomorphismByImages( sk4, aut16, [b1^4,b2], [aut1,aut2] );;
gap> P16 := PreXModByBoundaryAndAction( bdy16, act16 );
gap> P := PeifferSubgroup( P16 );
Group([ (11,15)(12,16)(13,17)(14,18), (11,13,15,17)(12,14,16,18) ])
gap> X16 := XModByPeifferQuotient( P16 );;
Peiffer([d16->sk4])
gap> Display( X16 );
Crossed module Peiffer([d16->sk4]) :-
: Source group has generators:
[ f1, f2 ]
: Range group has generators:
[ (11,15)(12,16)(13,17)(14,18), (12,18)(13,17)(14,16) ]
: Boundary homomorphism maps source generators to:
[ (12,18)(13,17)(14,16), (11,15)(12,14)(16,18) ]
The automorphism group is trivial

```

The following result, which gives a normal generating set for the Peiffer group, is Proposition 3 of [15]. The method of proof was suggested by Philip Higgins.

Proposition 1.15 *Given the following ingredients:*

- $\mathcal{Q} = (\delta : Q \rightarrow R)$, a precrossed module;
- Γ , a generating set for Q , closed under the action of R ;
- P , the Peiffer group of \mathcal{Q} ;

- E , the set of Peiffer elements $\{\langle a, b \rangle \mid a, b \in \Gamma\}$;

then P is the normal closure of E in Γ .

Proof: Let P' be the normal closure of E in Q , so that

$$P' \trianglelefteq P \trianglelefteq \ker \delta \trianglelefteq Q .$$

If $P \neq P'$ then there is some $z = \langle x, y \rangle \in P \setminus P'$ such that $P'z \neq P'$ in Q/P' . Since

$$P'z \neq P' \Rightarrow zP' \neq P' \Rightarrow y^{\delta x} P' \neq (x^{-1}yx)P' \Rightarrow P'y^{\delta x} \neq P'(x^{-1}yx) ,$$

we need to show that $P'y^{\delta x} = P'(x^{-1}yx)$ for all $\langle x, y \rangle \in P'$.

Since Γ is R -invariant, part (d) of Lemma 1.10 shows that E is R -invariant, and hence P' is R -invariant. Since $P' \trianglelefteq Q$, form $S' = Q/P'$ with R -action $(P'q)^r = P'(q^r)$. The homomorphism δ induces $\partial' : S' \rightarrow P'$, $P'q \mapsto \delta q$, and $\mathcal{X}' = (\partial' : S' \rightarrow R)$ is a precrossed module since

$$\partial'((P'q)^r) = \partial'(P'(q^r)) = \delta q^r = r^{-1}(\delta q)r = r^{-1}(\partial'(P'q))r .$$

Since Γ generates Q as a group, the set of cosets $C' = \{P'a \mid a \in \Gamma\}$ generates S' and, for all $P'a, P'b \in C'$,

$$(P'a)^{\partial'(P'b)} = (P'a)^{\delta b} = P'(a^{\delta b}) = P'(b^{-1}ab) = (P'b)^{-1}(P'a)(P'b) . \quad (6)$$

For fixed b , the set of $P'a$ satisfying (6) is a subgroup of S' :

$$((P'a)(P'c))^{\partial'(P'b)} = (P'a)^{\partial'(P'b)}(P'c)^{\partial'(P'b)} = (P'b)^{-1}(P'a)(P'c)(P'b) .$$

So (6) is true for all $(P'a) \in S'$ and all $(P'b) \in C'$.

Also, the set of $(P'b) \in C'$ satisfying (6) is closed under multiplication and inversion since

- $(P'a)^{\partial'(P'(bc))} = ((P'a)^{\delta(b)})^{\delta c} = (P'(bc))^{-1}(P'a)(P'(bc)) ;$
- $(P'a)^{\partial'(P'b)^{-1}} = P'c \Rightarrow P'a = (P'b)^{-1}(P'c)(P'b) \Rightarrow (P'b)(P'a)(P'b)^{-1} = P'c .$

Thus (6) holds for all $P'a, P'b \in C'$. □

1.7 Free Crossed Modules

We first recall a property of free groups which we wish to generalise. Let Ω be a set. The *free group on Ω* is a group F and a function $\nu : \Omega \rightarrow F$ such that if G is a group and $\nu' : \Omega \rightarrow G$ a function, then there exists a unique group homomorphism $\theta : F \rightarrow G$ such that $\theta \circ \nu = \nu'$.

$$\begin{array}{ccc} & \theta & \\ \downarrow & \text{---} & \downarrow \\ G & \xleftarrow{\nu'} \Omega \xrightarrow{\nu} & F \end{array}$$

To construct a particular model for $F = F(\Omega)$ we take an alphabet consisting of all the elements of Ω together with their formal inverses, and take for F the set of all reduced words in this alphabet with concatenation as the group product and the empty word as the identity. The details of this construction should be familiar to the reader.

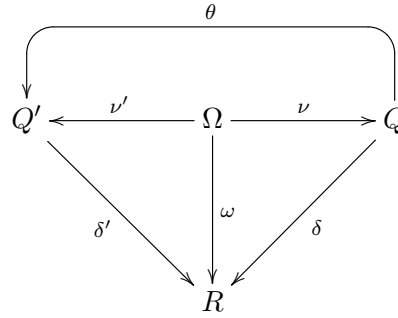
We now define a *free precrossed module* in an analogous manner. The ingredients for the construction are

- a set Ω ,
- a group R ,
- a function $\omega : \Omega \rightarrow R$.

The resulting construction consists of

- a precrossed module $\mathcal{Q} = (\delta : Q \rightarrow R)$,
- a function $\nu : \Omega \rightarrow Q$ such that $\delta \circ \nu = \omega$.

The universal property required of this construction is that if $\mathcal{Q}' = (\delta' : Q' \rightarrow R)$ is another precrossed module, and if $\nu' : \Omega \rightarrow Q'$ satisfies $\delta' \circ \nu' = \omega$, then there exists a unique morphism of precrossed modules $(\theta, 1) : \mathcal{Q} \rightarrow \mathcal{Q}'$ such that $(\theta, 1) \circ (\nu, 1) = (\nu', 1)$.



A particular model is obtained as follows:

- the source group Q is the free group $F(\Omega \times R)$,
- the boundary map is defined on generators by $\delta(\rho, r) = (\omega\rho)^r = r^{-1}(\omega\rho)r$,
- the action is given by $(\rho, r)^{r'} = (\rho, rr')$,
- the function is given by $\nu(\rho) = (\rho, 1)$.

We observe that

$$\delta \circ \nu(\rho) = \delta(\rho, 1) = \omega\rho \quad \text{for all } \rho \in \Omega,$$

and verify **X1**: as follows:

$$\delta((\rho, r)^{r'}) = \delta(\rho, rr') = r'^{-1}(r^{-1}(\omega\rho)r)r' = r'^{-1}\delta(\rho, r)r'.$$

To check the universal property we need to define $\theta : Q \rightarrow Q'$ in such a way that $\theta(\rho, 1) = \nu'\rho$. Since θ is to preserve the R -action, we are forced to define

$$\theta(\rho, r) = \theta((\rho, 1)^r) = (\nu'\rho)^r.$$

This defines θ on the whole of Q , and we verify that $(\theta, 1)$ is a morphism of precrossed modules:

- $\delta'\theta(\rho, r) = \delta'((\nu'\rho)^r) = r^{-1}(\delta'\nu'\rho)r = r^{-1}(\omega\rho)r = \delta(\rho, r),$
- $\theta((\rho, r)^{r'}) = \theta(\rho, rr') = (\nu'\rho)^{rr'} = (\theta(\rho, r))^{r'}.$

We are now in a position to construct the free crossed module associated to a free precrossed module, simply by factoring out the Peiffer commutators in $Q = F(\Omega \times R)$:

$$\begin{aligned} \langle (\rho_1, r_1), (\rho_2, r_2) \rangle &= (\rho_1, r_1)^{-1} (\rho_2, r_2)^{-1} (\rho_1, r_1) (\rho_2, r_2)^{\delta(\rho_1, r_1)} \\ &= (\rho_1, r_1)^{-1} (\rho_2, r_2)^{-1} (\rho_1, r_1) (\rho_2, r_2)^{r_1^{-1}(\omega\rho_1)r_1} \\ &= (\rho_1, r_1)^{-1} (\rho_2, r_2)^{-1} (\rho_1, r_1) (\rho_2, r_2 r_1^{-1}(\omega\rho_1)r_1) . \end{aligned}$$

We thus obtain a crossed module with source $C(\omega) = Q/P$:

$$\begin{aligned} \mathcal{F}_\rho &= (\partial : C(\omega) \rightarrow R), \\ \partial(P(\rho, r)) &= \delta(\rho, r) = r^{-1}(\omega\rho)r, \\ (P(\rho, r))^{r'} &= P(\rho, rr'). \end{aligned}$$

It is often convenient to use an alternative notation for elements in Q . Since $(\rho, r) = (\rho, 1)^r$ we may drop the “, 1” and write $(\rho)^r$ for (ρ, r) , and the inverse element by $(\rho^{-1})^r$:

$$\{(\rho)^r\}^{-1} = (\rho, r)^{-1} = \{(\rho, 1)^r\}^{-1} = \{(\rho, 1)^{-1}\}^r = (\rho^{-1})^r .$$

The action is then given by

$$((\rho^\epsilon)^r)^{r'} = (\rho^\epsilon)^{(rr')} \quad \text{where } \epsilon = \pm 1 .$$

The Peiffer commutators on generators in this notation are

$$\langle (\rho_1^{\epsilon_1})^{r_1}, (\rho_2^{\epsilon_2})^{r_2} \rangle = (\rho_1^{-\epsilon_1})^{r_1} (\rho_2^{-\epsilon_2})^{r_2} (\rho_1^{\epsilon_1})^{r_1} (\rho_2^{\epsilon_2})^{r_2 r_1^{-1}(\omega\rho_1)r_1} .$$

1.8 The monoid version of free crossed modules

Given $Y = \Omega \times R$, define

$$\bar{Y} = \{y^+ : y \in Y\} \sqcup \{y^- : y \in Y\} .$$

It is convenient to use an alternative notation, as above:

$$(\rho^+)^r := (\rho, r)^+, \quad (\rho^-)^r := (\rho, r)^- .$$

Then $H = \bar{Y}^*$ is the free monoid on \bar{Y} with empty word λ and elements

$$(\rho_1^{\epsilon_1})^{r_1} (\rho_2^{\epsilon_2})^{r_2} \cdots (\rho_n^{\epsilon_n})^{r_n}, \quad \rho_i \in \Omega, \epsilon_i \in \{+, -\}, r_i \in R .$$

The boundary map is the monoid morphism

$$\bar{\delta} : \bar{Y}^* \rightarrow R, \quad (\rho^+)^r \mapsto r^{-1}(\omega\rho)r, \quad (\rho^-)^r \mapsto r^{-1}(\omega\rho)^{-1}r .$$

Then $F(\omega)$ is the quotient monoid \bar{Y}^*/\equiv where \equiv is the congruence generated by

- inverse pairs $(y^\epsilon y^{-\epsilon}, \lambda)$,
- Peiffer pairs $(y^{-\epsilon} z^\eta y^\epsilon, (z^\eta)^{\bar{\delta} y^\epsilon})$, $\epsilon, \eta \in \{+, -\}$, $-(-) = +$, etc.

In the special case of a group presentation

$$\mathcal{P} = \text{grp}(X, \omega : \Omega \rightarrow F(X)),$$

$\omega\rho$ is a relator and so a word in $F(X)$. Then $Y = \Omega \times F(X)$ and $H = \bar{Y}^*$ has elements of the form

$$(\rho_1^{\epsilon_1})^{u_1} (\rho_2^{\epsilon_2})^{u_2} \dots (\rho_n^{\epsilon_n})^{u_n},$$

and $\ker \partial = \Pi_2(\mathcal{P})$ is the $\mathbb{Z}G$ -module of identities among the relators of \mathcal{P} .

Example 1.16 Consider the following presentation for the quaternion group of size 8.

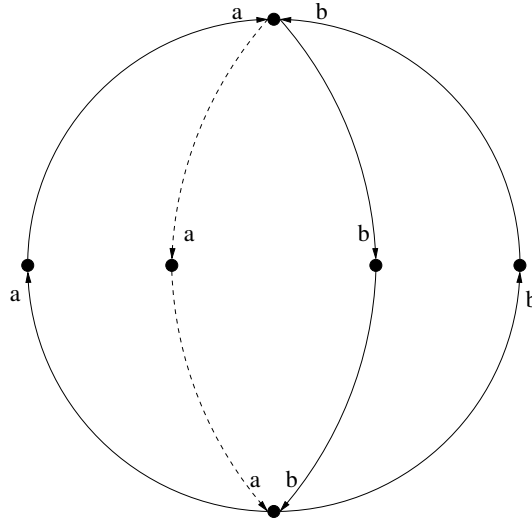
$$X = \{a, b\}, \quad \Omega = \{\rho_1, \rho_2, \rho_3, \rho_4\}, \quad \omega : \rho_1 \mapsto a^4, \rho_2 \mapsto b^4, \rho_3 \mapsto abab^{-1}, \rho_4 \mapsto a^2b^2.$$

The identity

$$\iota = (\rho_4^-) (\rho_1^+)^{a^2} (\rho_4^-)^{a^2} (\rho_2^+)$$

maps by ∂ to

$$(b^{-2}a^{-2}).a^{-2}(a^4)a^2.a^{-2}(b^{-2}a^{-2})a^2.(b^4) = \lambda.$$



In the van Kampen diagram, relators ρ_1, ρ_2 and ρ_4 (twice) tile a sphere as shown above. Tracing out $\partial\iota$ we walk around the boundaries of these four tiles, in the order back-right; back-left; front-left; front-right; in such a way that every edge cancels out with its inverse.

1.9 A Geometric Example of a Crossed Module

The major geometric example of a crossed module can be expressed in two ways. Let (X, A, a) be a based pair of spaces, with $a \in A \subseteq X$. Let $I = [0, 1]$ be the unit interval, I^2 the unit square with boundary \dot{I}^2 , and let $J^1 = (\{0, 1\} \times I) \cup (I \times \{1\}) \subset \dot{I}^2$ be three quarters of the boundary. The *second relative homotopy group* $\pi_2(X, A, a)$ consists of homotopy classes rel J^1 of continuous maps

$$\alpha : (I^2, \dot{I}^2, J^1) \rightarrow (X, A, a)$$

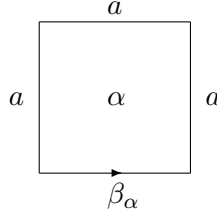


Figure 1: An element $\alpha \in \pi_2(X, A, a)$

Each such α is a map from I^2 to the space X mapping the left, top, and right sides of the square to the point a and the bottom side to a loop β_α at a . We may represent such a map by the diagram in Figure 1.

Recall that the fundamental group $\pi_1(A, a)$ consists of maps $\gamma : I \rightarrow A$, $\gamma(0) = \gamma(1) = a$. With such a γ it is easy to construct maps $I^2 \rightarrow A$ which map two sides of the square to a and two sides to γ . Five of these are shown in Figure 2.

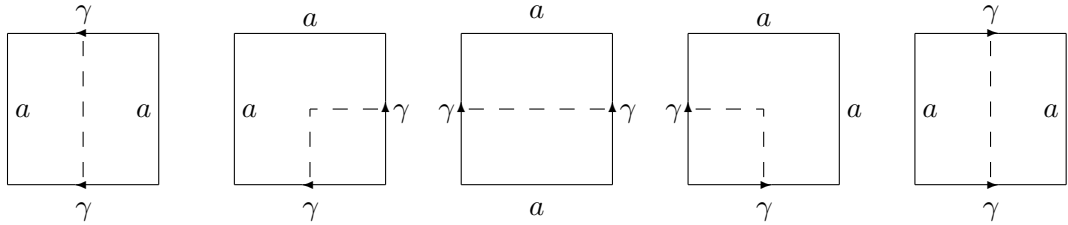


Figure 2: Five maps $I^2 \rightarrow A$ derived from $\gamma \in \pi_1(A, a)$

Whitehead showed in [54] that there is a crossed module $\Pi_2(X, A, a)$ with boundary map

$$\partial : \pi_2(X, A, a) \rightarrow \pi_1(A, a), \quad \alpha \mapsto \beta_\alpha = \alpha(I \times \{0\}) .$$

The image of $\alpha \in \pi_2(X, A, a)$ under the action of $\gamma \in \pi_1(A, a)$ is illustrated in Figure 3, surrounding α with the five maps in Figure 2. Note that the boundary loop is the conjugate $\gamma^{-1}\beta_\alpha\gamma$.

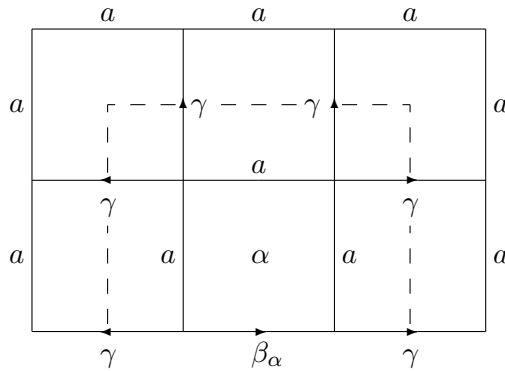


Figure 3: Action of γ on α

The meaning of this composite square is as follows. Squares may be joined along an edge when

the values agree on that edge. If a composite is then p units across by q units high, scaling factors $1/p$ horizontally and $1/q$ vertically are used to obtain a new map from I^2 to X .

Figure 4 gives an outline verification of the second crossed module axiom for $\Pi_2(X, A, a)$, where a square marked a represents the constant map $I^2 \rightarrow \{a\}$.

$$\begin{array}{c}
 \begin{array}{|c|c|c|} \hline a & a & a \\ \hline \alpha_2^{-1} & \alpha_1 & \alpha_2 \\ \hline \end{array} \quad \sim \quad \begin{array}{|c|c|c|} \hline a & a & a \\ \hline \alpha_2^{-1} & a & \alpha_2 \\ \hline \beta_2 & \alpha_1 & \beta_2 \\ \hline \end{array} \quad \sim \quad \begin{array}{|c|c|c|} \hline a & a & a \\ \hline \beta_2 & \alpha_2^{-1} \alpha_2 & \beta_2 \\ \hline \beta_2 & \alpha_1 & \beta_2 \\ \hline \end{array}
 \end{array}$$

$\alpha_1^{\alpha_2} = \alpha_2^{-1} \alpha_1 \alpha_2 \qquad \qquad \qquad \alpha_1^{\beta_2} = \alpha_1^{\partial \alpha_2}$

Figure 4: Verification of **X2**: for $\Pi_2(X, A, a)$.

Whitehead's main result in [53, 54, 56] was the following.

Theorem 1.17 (Whitehead) *If X is obtained from A by attaching 2-cells, then $\pi_2(X, A, a)$ is isomorphic to the free crossed $\pi_1(A, a)$ -module on the attaching maps of the 2-cells.*

[More here?]

1.10 Semidirect Products

We include here some basic results on semidirect products which will be needed in later sections.

Proposition 1.18

(a) *If a set X has a right G -action $x \mapsto x^g$ then X has an associated left G -action:*

$${}^g x \quad := \quad x^{g^{-1}} .$$

(b) *The semidirect products $R \ltimes S$ and $S \rtimes R$ have multiplication rules*

$$\begin{aligned}
 (r, s)(q, t) &= (rq, s^q t) \quad \text{in } R \ltimes S , \\
 (s, r)(t, q) &= (s^r t, rq) \quad \text{in } S \rtimes R .
 \end{aligned}$$

(c) *There is an isomorphism between these two groups:*

$$\psi : R \ltimes S \rightarrow S \rtimes R, \quad (r, s) \mapsto ({}^r s, r) ,$$

with inverse

$$\psi^{-1} : S \rtimes R \rightarrow R \ltimes S, \quad (s, r) \mapsto (r, s^r) .$$

1.11 Cat¹-groups and their morphisms

In [42] Loday reformulated the notion of a crossed module as a cat¹-group $(G; t, h)$, namely a group G with a pair of endomorphisms $t, h : G \rightarrow G$ having a common image R and satisfying certain axioms. We call these *traditional* cat¹-groups, to distinguish them from the more general description which follows.

Definition 1.19 A cat¹-group \mathcal{C} comprises a group G and two endomorphisms $\tau, \theta : G \rightarrow G$ as shown in the following diagram:

$$G \xrightleftharpoons[\theta]{\tau} G$$

These homomorphisms are required to satisfy the following axioms:

$$\begin{aligned} \mathbf{C1:} \quad & \tau \circ \theta = \theta \quad \text{and} \quad \theta \circ \tau = \tau, \\ \mathbf{C2:} \quad & [\ker \tau, \ker \theta] = \{1_G\}. \end{aligned}$$

The maps τ, θ are usually referred to as the *source* and *target* maps, but we choose to call them the *tail* and *head* of \mathcal{C} , because *source* is the GAP term for the domain of a function. It follows immediately from axiom **C1**: that:

$$\tau^2 = \tau, \quad \theta^2 = \theta, \quad \text{im } \tau = \text{im } \theta = Q \text{ (say),}$$

and that both τ and θ are the identity when restricted to Q .

Alternatively (and this is convenient for the implementation in XMod) we may define a cat¹-group as follows.

Definition 1.20 A cat¹-group $\mathcal{C} = (e; t, h : G \rightarrow R)$ has source group G , range group R , and three homomorphisms: two surjections $t, h : G \rightarrow R$ and an embedding $e : R \rightarrow G$ as shown in the following diagram:

$$G \xrightleftharpoons[t]{h} R \xleftarrow[e]{e} G$$

These homomorphisms are required to satisfy the following axioms:

$$\begin{aligned} \mathbf{C1:} \quad & (t \circ e) \text{ and } (h \circ e) \text{ are the identity mapping on } R, \\ \mathbf{C2:} \quad & [\ker t, \ker h] = \{1_G\}. \end{aligned}$$

It follows immediately from axiom **C1**: that:

$$t \circ e \circ t = t, \quad h \circ e \circ h = h, \quad t \circ e \circ h = h, \quad h \circ e \circ t = t.$$

Since e is an embedding, R acts on G by conjugation: $g^r = (er)^{-1}g(er)$.

Given a cat¹-group according to the second definition, we may convert it to a cat¹-group according to the first by setting

$$\tau = e \circ t, \quad \theta = e \circ h, \quad Q = eR,$$

so that $\ker \tau = \ker t$, $\ker \theta = \ker h$ and $\tau \circ \theta = e \circ t \circ e \circ h = e \circ h = \theta$, etc.

Example 1.21 In this example the group G is isomorphic to $C_2 \times (C_3 \times C_3)$.

```
gap> g18gens := [ (1,2,3), (4,5,6), (2,3)(5,6) ];;
gap> s3agens := [ (7,8,9), (8,9) ];;
gap> g18 := Group( g18gens );; SetName( g18, "g18" );
gap> s3a := Group( s3agens );; SetName( s3a, "s3a" );
gap> t1 := GroupHomomorphismByImages(g18,s3a,g18gens,[(7,8,9),(),(8,9)]);;
gap> h1 := GroupHomomorphismByImages(g18,s3a,g18gens,[(7,8,9),(7,8,9),(8,9)]);;
gap> e1 := GroupHomomorphismByImages(s3a,g18,s3agens,[(1,2,3),(2,3)(5,6)]);;
gap> C18 := PreCat1GroupByTailHeadEmbedding( t1, h1, e1 );;
gap> Display( C18 );
Cat1-group [g18=>s3a] :-
: Source group g18 has generators:
  [ (1,2,3), (4,5,6), (2,3)(5,6) ]
: Range group s3a has generators:
  [ (7,8,9), (8,9) ]
: tail homomorphism maps source generators to:
  [ (7,8,9), (), (8,9) ]
: head homomorphism maps source generators to:
  [ (7,8,9), (7,8,9), (8,9) ]
: range embedding maps range generators to:
  [ (1,2,3), (2,3)(5,6) ]
: kernel has generators:
  [ (4,5,6) ]
: boundary homomorphism maps generators of kernel to:
  [ (7,8,9) ]
: kernel embedding maps generators of kernel to:
  [ (4,5,6) ]
```

A morphism $\mathcal{C}_1 \rightarrow \mathcal{C}_2$ of cat^1 -groups is a pair (γ, ρ) where $\gamma : G_1 \rightarrow G_2$ and $\rho : R_1 \rightarrow R_2$ are homomorphisms satisfying

$$t_2 \circ \gamma = \rho \circ t_1, \quad h_2 \circ \gamma = \rho \circ h_1, \quad e_2 \circ \rho = \gamma \circ e_1. \quad (7)$$

The situation is displayed in the following diagram.

$$\begin{array}{ccc} G_1 & \xrightarrow{\gamma} & G_2 \\ \uparrow e_1 & \parallel t_1, h_1 & \uparrow e_2 \\ R_1 & \xrightarrow{\rho} & R_2 \end{array}$$

The morphism (id, e) is the isomorphism from a cat^1 -group $(e; t, h : G \rightarrow R)$ to the traditional cat^1 -group with endomorphisms $(\text{inc}_Q; e \circ t, e \circ h : G \rightarrow Q)$ with $Q = eR$, as described above.

Verification of the properties in the following lemma is routine.

Lemma 1.22 *The mapping $u : G \rightarrow \ker t$, $g \mapsto (etg^{-1})g$ has the following properties.*

- (i) $u^2 = u$,
- (ii) $tug = 1_R$, $hug = (tg^{-1})(hg)$, $uer = 1_G$,

- (iii) $u(g_1g_2) = (ug_2)(ug_1)^{g_2},$
- (iv) $(ug)^{-1} = g^{-1}(etg) = (u(g^{-1}))^g.$

Proposition 1.23 *An arbitrary cat^1 -group $\mathcal{C} = (e; t, h : G \rightarrow R)$ is isomorphic to the cat^1 -group $\mathcal{C}' = (e'; t', h' : R \ltimes S \rightarrow R)$, the semidirect form of \mathcal{C} , where $S = \ker t$; the homomorphisms t', h', e' are defined by*

$$t'(r, s) = r, \quad h'(r, s) = r(hs), \quad e'r = (r, 1); \quad (8)$$

and the action of R on S is given by $(r_0, s_0)^r = (r^{-1}r_0r, s_0^r).$

Proof: The isomorphism from G to $R \ltimes S$, and its inverse, are given by

$$\begin{aligned} \phi : G &\rightarrow R \ltimes S, & g &\mapsto (tg, ug), & \text{where } ug = (etg^{-1})g \in \ker t, \\ \phi^{-1} : R \ltimes S &\rightarrow G, & (r, s) &\mapsto (er)s. \end{aligned} \quad (9)$$

This ϕ is a homomorphism since

$$\begin{aligned} (\phi g_1)(\phi g_2) &= (tg_1, ug_1)(tg_2, ug_2) = ((tg_1)(tg_2), (ug_1)^{tg_2}(ug_2)) \\ &= (t(g_1g_2), (etg_2)^{-1}(ug_1)(etg_2)(ug_2)) \\ &= (t(g_1g_2), (etg_2)^{-1}g_2g_2^{-1}(ug_1)(etg_2)(etg_2)^{-1}g_2) \\ &= (t(g_1g_2), (ug_2)(ug_1)^{g_2}) = \phi(g_1g_2). \end{aligned}$$

The specified ϕ^{-1} is the inverse of ϕ since

$$\begin{aligned} \phi^{-1}(tg, (etg^{-1})g) &= (etg)(etg^{-1}g) = g, \\ \phi((er)s) &= ((ter)(ts), (ets^{-1})(eter^{-1})(er)s) = (r, s). \end{aligned}$$

The required isomorphism is then $(\phi, \text{id}) : \mathcal{C} \rightarrow \mathcal{C}'$. Axioms (7) are easily verified:

$$\begin{aligned} t'(tg, ug) &= tg, \\ h'(tg, ug) &= (tg)(hetg^{-1})(hg) = hg, \\ \phi(er) &= (ter, (eter^{-1})(er)) = (r, 1). \end{aligned}$$

□

The crossed module $\mathcal{X} = (\partial : S \rightarrow R)$ associated to \mathcal{C} and \mathcal{C}' has boundary $\partial = h|_S$ and action $s^r := s^{er}$. The cat^1 -group $\mathcal{C} = \mathcal{C}'$ associated to $\mathcal{X} = (\partial : S \rightarrow R)$ has $G = R \ltimes S$, where the action is that in \mathcal{X} , and homomorphisms given by:

$$t(r, s) = r, \quad h(r, s) = r(\partial s), \quad er = (r, 1). \quad (10)$$

We denote by ϵ the inclusion of S in G , so that $\partial = h\epsilon$.

Example 1.24 Here we convert X4 and C18, constructed in Subsection 1.2 and Example 1.21.

```
gap> CX4 := Cat1GroupOfXMod( X4 );
[s4 |X a4=>Group( [ (), (), (1,2), (2,3), (3,4) ] )]
gap> TailMap( CX4 ); HeadMap( CX4 );
[ (5,6,7), (6,7,8), (1,2)(5,6), (2,3)(6,7), (3,4)(7,8) ] ->
[ (), (), (1,2), (2,3), (3,4) ]
[ (5,6,7), (6,7,8), (1,2)(5,6), (2,3)(6,7), (3,4)(7,8) ] ->
[ (1,2,3), (2,3,4), (1,2), (2,3), (3,4) ]
gap> XC18 := XModOfCat1Group( C18 );
[Group( [ (4,5,6) ] )->s3a]
gap> Boundary( XC18 );
[ (4,5,6) ] -> [ (7,8,9) ]
```

Given a morphism $(\sigma, \rho) : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ of crossed modules, the associated morphism of cat^1 -groups is $(\gamma, \rho) : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ where $\gamma(r_1, s_1) = (\rho r_1, \sigma s_1)$. Similarly, given a morphism $(\gamma, \rho) : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ of cat^1 -groups, the associated morphism of crossed modules is $(\sigma, \rho) : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ where $\sigma s = \gamma(1, s)$.

George Janelidze has noted the following variant of the second cat^1 -group axiom:

$$\mathbf{C2'}: \quad [ug_1, ug_2] = 1_G \quad \text{for all } g_1, g_2 \in G.$$

It follows that cat^1 -groups form an equational variety.

Lemma 1.25 *If $\mathcal{C} = (e; t, h : G \rightarrow R)$ is a cat^1 -group then there is a group homomorphism*

$$(t, h) : G \rightarrow R \times R, \quad g \mapsto (tg, hg).$$

Proposition 1.26 (Comment by Tim Porter during a seminar on 18/10/02.)

A congruence \equiv (in the sense of congruence on a monoid) on a group R gives rise to a cat^1 -group.

Proof: The set of equivalent pairs,

$$G = \{(r_1, r_2) \in R \times R \mid r_1 \equiv r_2\}$$

is a subgroup of $R \times R$. The required $\mathcal{C} = (e; t, h : G \rightarrow R)$ has homomorphisms given by:

$$t(r_1, r_2) = r_1, \quad h(r_1, r_2) = r_2, \quad e(r) = (r, r).$$

The associated crossed module has source

$$\ker t = \{(r, 1) \mid r \equiv 1\}$$

which shows that the elements equivalent to 1_R in the congruence form a normal subgroup. \square

1.12 Pre- cat^1 -groups and sub- cat^1 -groups

When axioms **X2:** and **C2:** are *not* satisfied by $\mathcal{Q} = (\delta : Q \rightarrow R)$ and $\mathcal{B} = (e; t, h : R \ltimes Q \rightarrow R)$, the corresponding structures are known as *precrossed modules* and *pre- cat^1 -groups*. In this case recall from Subsection 1.6 that the *Peiffer subgroup* P of Q is the subgroup of $\ker \delta$ generated by *Peiffer commutators*

$$\langle q_1, q_2 \rangle = q_1^{-1} q_2^{-1} q_1 q_2^{\partial q_1}.$$

Then $\mathcal{P} = (0 : P \rightarrow \{1_R\})$ is a normal sub-precrossed module of \mathcal{Q} and $\mathcal{X} = \mathcal{Q}/\mathcal{P} = (\partial : S = Q/P \rightarrow R)$ is a crossed module. The restriction of $\epsilon : Q \rightarrow R \ltimes Q$ to P is given by

$$\epsilon \langle q_1, q_2 \rangle = [(\delta q_1^{-1}, q_1), (1_R, q_2^{\delta q_1})] \in [\ker h, \ker t].$$

The image ϵP is the Peiffer subgroup $[\ker h, \ker t]$ of $R \ltimes Q$ and, if ι is the inclusion $\{1_R\} \rightarrow R$, then $\mathcal{C}/(\epsilon, \iota)\mathcal{P} = (e; t, h : (R \ltimes Q)/\epsilon P \rightarrow R)$ is the cat^1 -group corresponding to $\mathcal{X} = \mathcal{Q}/\mathcal{P}$.

We now include definitions of sub- cat^1 -groups, normal sub- cat^1 -groups, and the kernel of a cat^1 -morphism.

A *sub- cat^1 -group* of $\mathcal{C} = (e; t, h : G \rightarrow R)$ is a cat^1 -group $\mathcal{C}' = (e'; t', h' : G' \rightarrow R')$ where G' is a subgroup of G ; R' is a subgroup of R ; and e', t', h' are the restrictions of e, t, h . Such a \mathcal{C}' is a *normal sub- cat^1 -group* of \mathcal{C} when $G' \trianglelefteq G$ and $R' \trianglelefteq R$.

For the kernel of a cat^1 -morphism $\mu : \mathcal{C} \rightarrow \mathcal{C}'$, consider the following diagram, where the left-hand morphism is a sub- cat^1 -group inclusion and the leftmost tail, head and embedding maps are the restrictions of t, h, e .

$$\begin{array}{ccccc}
 \ker \bar{\mu} & \xrightarrow{\quad} & G & \xrightarrow{\bar{\mu}} & G' \\
 \uparrow \Downarrow & & \uparrow \Downarrow & & \uparrow \Downarrow \\
 & & e & & e' \\
 & & \downarrow t, h & & \downarrow t', h' \\
 \ker \dot{\mu} & \xrightarrow{\quad} & R & \xrightarrow{\dot{\mu}} & R'
 \end{array}$$

The left-hand cat^1 -group is the kernel of μ since

$$\begin{aligned}
 g \in \ker \bar{\mu} &\Rightarrow t' \bar{\mu} g = 1, h' \bar{\mu} g = 1 \Rightarrow \dot{\mu} t g = 1, \dot{\mu} h g = 1 \Rightarrow t g \in \ker \dot{\mu}, h g \in \ker \dot{\mu}, \\
 r \in \ker \dot{\mu} &\Rightarrow e' \dot{\mu} r = 1 \Rightarrow \bar{\mu} e r = 1 \Rightarrow e r \in \ker \bar{\mu}.
 \end{aligned}$$

1.13 Group Groupoids

Cat^1 -groups may also be thought of as group-groupoids. A *group groupoid* is a set which has both a group structure and a groupoid structure (see subsection 4.1). From a categorical viewpoint, it is both a group object in the category of groupoids and a groupoid object in the category of groups. (For basic notions of groupoids see Section 4.)

The underlying groupoid \mathcal{G} of a cat^1 -group \mathcal{C} has the group R as object set G_0 and the group G as the set of arrows G_1 . The identity arrow at r is $1_r = er$. For each arrow g the tail (source) is tg and the head (target) is hg . Arrows g_1, g_2 are composable only when $hg_1 = tg_2 = r_2$ (say), in which case the composite arrow is

$$g_1 * g_2 = g_1(er_2^{-1})g_2 \quad \text{where} \quad t(g_1 * g_2) = tg_1 = r_1, \quad h(g_1 * g_2) = hg_2 = r_3. \quad (11)$$

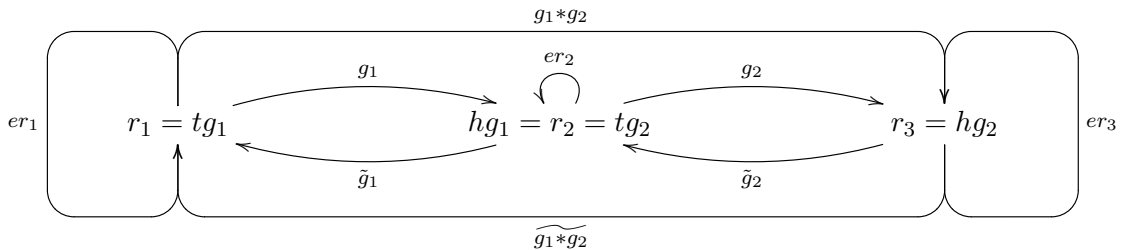
This composition is, of course, associative:

$$g_1 * g_2 * g_3 = g_1(er_2^{-1})g_2(er_3^{-1})g_3.$$

The groupoid inverse \tilde{g} of g for this composition is given by

$$\tilde{g} = (ehg)g^{-1}(etg) \quad \text{with} \quad t\tilde{g} = hg, h\tilde{g} = tg, g * \tilde{g} = etg \quad \text{and} \quad \tilde{g} * g = ehg.$$

This subset of \mathcal{G} is illustrated in the following diagram.



The composites of one element with the groupoid inverse of another, when defined, are given by

$$\tilde{g}_1 * g_3 = (ehg_1)g_1^{-1}g_3 \quad \text{and} \quad g_4 * \tilde{g}_2 = g_4g_2^{-1}(etg_2). \quad (12)$$

The equivalent formulae for composition and inverse when $R \ltimes S$ replaces G are:

$$(r, s) * (r(\partial s), s') = (r, ss') \quad \text{and} \quad \widetilde{(r, s)} = (r(\partial s), s^{-1}) .$$

Since $g^{-1}(etg) \in \ker t$ and $(ehg)g^{-1} \in \ker h$, the map $g \mapsto \tilde{g}$ is an automorphism of \mathcal{G} which restricts to the identity map on eR and provides a cat^1 -isomorphism from \mathcal{C} to the *reverse* cat^1 -group $\tilde{\mathcal{C}} = (e; h, t : G \rightarrow R)$ of \mathcal{C} . The set of arrows *out* from 1_R is $\ker t$ while the set of arrows *in* to 1_R is $\ker h$, so $\ker \partial$ is the set of loops at 1_R . The set of objects in the component of \mathcal{G} connected to 1_R is the image of ∂ , so \mathcal{G} is discrete when $\partial = 0$.

Alternatively, starting with a group groupoid $\mathcal{G} = (G, t, h)$, define

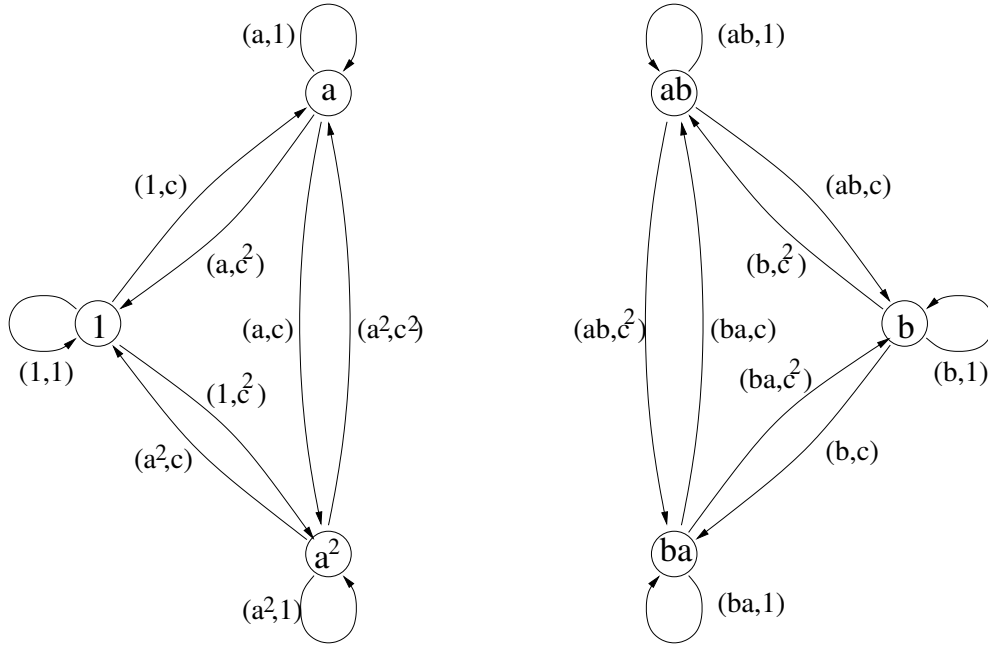
$$\begin{aligned} R &= \text{im } t = \text{im } h, \\ S &= \{g \mid tg = 1\} = \ker t = \text{arrows out from } 1_R, \\ s^r &= (er)^{-1}s(er), \quad \text{where } er \text{ is the identity loop at } r. \end{aligned}$$

See Subsection 2.3 for the group-groupoid equivalent of derivations and sections.

Example 1.27 *The normal inclusion crossed module $X_3 = (1 : C_3 \rightarrow S_3)$ of the cyclic group $C_3 = \langle c \mid c^3 \rangle$ in the symmetric group $S_3 = \langle a, b \mid a^3, b^2, (ab)^2 \rangle$, with conjugation action $c^a = c, c^b = c^2$, has associated cat^1 -group $(e; t, h : S_3 \ltimes C_3 \rightarrow S_3)$. The images of the tail and head functions are given in the following table:*

g	tg	hg	g	tg	hg
$(1, 1)$	1	1	$(b, 1)$	b	b
$(1, c)$	1	a	(b, c)	b	ba
$(1, c^2)$	1	a^2	(b, c^2)	b	ab
$(a, 1)$	a	a	$(ab, 1)$	ab	ab
(a, c)	a	a^2	(ab, c)	ab	b
(a, c^2)	a	1	(ab, c^2)	ab	ba
$(a^2, 1)$	a^2	a^2	$(ba, 1)$	ba	ba
(a^2, c)	a^2	1	(ba, c)	ba	ab
(a^2, c^2)	a^2	a	(ba, c^2)	ba	b

The corresponding group-groupoid has 6 objects, 18 morphisms, 2 connected components, and may be pictured as:



We may compare the group multiplication with the groupoid multiplication by calculating, for example,

$$\begin{aligned} (a,c)(a^2,c) &= (1,c^{a^2}c) = (1,c^2), \\ (a,c) * (a^2,c) &= (a,c)(a^2,1)^{-1}(a^2,c) = (a^4,c^{a^3}c) = (a,c^2). \end{aligned}$$

1.14 Regular Groupoids

Let $\mathcal{X} = (\partial : S \rightarrow R)$ be a precrossed module and let $\mathcal{C} = (e; t, h : G \rightarrow R)$ be the associated precat¹-group where $G = R \ltimes S$ has multiplication $(r_1, s_1)(r_2, s_2) = (r_1 r_2, s_1^{r_2} s_2)$ and inverse $(r, s)^{-1} = (r^{-1}, (s^{-1})^{r^{-1}})$. The homomorphisms e, t, h are given by $er = (r, 1)$, $t(r, s) = r$, $h(r, s) = r(\partial s)$.

The associated group-groupoid \mathcal{G} has vertex set R and arrows G with the tail (source) and head (target) maps given by t and h . Composition $*$ in \mathcal{G} is given by

$$(r_1, s_1) * (r_2, s_2) = (r_1, s_1)(r_2^{-1}, 1)(r_2, s_2) = (r_1, s_1 s_2)$$

and is defined when $r_1(\partial s_1) = r_2$. The identity element in the object group at r is $er = (r, 1)$.

Definition 1.28 Let \mathcal{G} be a groupoid with objects R and arrows G .

(i) Maps $\triangleright : R \times G \rightarrow G$ and $\triangleleft : G \times R \rightarrow G$ are respectively left and right actions of R on G if for all $q, r \in R$ and $g, h \in G$

- $(qr) \triangleright g = q \triangleright (r \triangleright g)$, $g \triangleleft (qr) = (g \triangleleft q) \triangleleft r$;
- $q \triangleright (g \triangleleft r) = (q \triangleright g) \triangleleft r$;
- $1 \triangleright g = g = g \triangleleft 1$;
- $t(r \triangleright g) = r(tg)$, $t(g \triangleleft r) = (tg)r$, $h(r \triangleright g) = r(hg)$, $h(g \triangleleft r) = (hg)r$;
- $r \triangleright (g * g') = (r \triangleright g) * (r \triangleright g')$, $(g * g') \triangleleft r = (g \triangleleft r) * (g' \triangleleft r)$, whenever $g * g'$ is defined;
- $q \triangleright er = e(qr) = eq \triangleleft r$.

(ii) \mathcal{G} is semiregular if R is a monoid and if \mathcal{G} has left and right actions as in (i).

(iii) A semiregular \mathcal{G} is regular if R is a group.

Lemma 1.29 When \mathcal{G} is the group groupoid associated to a precat¹-group \mathcal{C} , left and right actions are given by left and right multiplication:

$$r \triangleleft g := (er)g, \quad g \triangleright r := g(er).$$

Proof: We only verify the axioms for \triangleleft since those for \triangleright follow similarly.

$$\begin{aligned} g \triangleleft (qr) &= g \triangleleft (eq)(er) = (g \triangleleft q)(er) = (g \triangleleft q) \triangleleft r; \\ q \triangleright (g \triangleleft r) &= q \triangleright (g(er)) = (eq)g(er) = ((eq)g) \triangleleft r = (q \triangleright g) \triangleleft r; \\ g \triangleleft 1 &= g(e1) = g; \\ t(g \triangleleft r) &= t(g(er)) = (tg)(ter) = (tg)r; \\ h(g \triangleleft r) &= h(g(er)) = (hg)(her) = (hg)r; \\ (g \triangleleft r) * (g' \triangleleft r) &= (g(er))(eh(g(er)))^{-1} g'(er) = g(er)(er)^{-1} (ehg)^{-1} g'(er) = (g * g') \triangleleft r; \\ eq \triangleleft r &= (eq)(er) = e(qr). \end{aligned}$$

□

Proposition 1.30 Let \mathcal{G} be a semiregular groupoid. Then there are two everywhere defined binary operations on G given by:

$$\begin{aligned} g \odot g' &= (g \triangleleft tg') * (hg \triangleright g'), \\ g \otimes g' &= (tg \triangleright g') * (g \triangleleft hg'). \end{aligned}$$

When \mathcal{G} is a regular groupoid both \odot and \otimes make G into a group with identity $e1$.

Following on from the example above, when \mathcal{G} is a group groupoid, we find that \odot is just the multiplication in G .

$$\begin{aligned}
g \odot g' &= (g \triangleleft tg') * (hg \triangleright g') \\
&= (g(etg')) * ((ehg)g') \\
&= g(etg')(eh(g(etg')))^{-1}(ehg)g' \\
&= g(etg')((ehg)(etg'))^{-1}(ehg)g' \\
&= gg'.
\end{aligned}$$

The situation with \otimes is a little more complicated:

$$\begin{aligned}
g \otimes g' &= (tg \triangleright g') * (g \triangleleft hg') \\
&= ((etg)g')(eh((etg)g'))^{-1}(g(ehg')) \\
&= (etg)g'((etg)(ehg'))^{-1}g(ehg') \\
&= (etg)g'(ehg')^{-1}(etg)^{-1}g(ehg').
\end{aligned}$$

Now $g'(ehg')^{-1} \in \ker h$ and $(etg)^{-1}g \in \ker t$. When \mathcal{G} is a cat^1 -group these two products commute, so the expression reduces to gg' and \otimes is also just multiplication in G .

1.15 2-groups

Finally, we think of such a structure as a special case of a 2-category, which has objects, morphisms, and 2-cells. We follow the presentation in Subsection 1.2.3 of Forrester-Barker's thesis [33]. For an introduction to 2-groupoids, see Kamps and Porter [41]. (Note that a 2-group is *not* a special case of the group theorist's p -group, with $p = 2$, but is a 2-category with one object having all morphisms and 2-cells invertible.)

The 2-group \mathcal{H} associated to $\mathcal{X} = (\partial : S \rightarrow R)$ has

- a single object \bullet ,
- morphisms $r \in R$,
- 2-cells $(r, s) \in R \times S$ with tail r and head $r(\partial s)$.

$$\Downarrow(r, s) = \begin{array}{ccc} & r & \\ \bullet & \begin{array}{c} \Downarrow(r, s) \end{array} & \bullet \\ & r(\partial s) & \end{array}$$

Horizontal composition $(r_1, s_1) \#_0 (r_2, s_2)$ of 2-cells is given by

$$\begin{array}{ccc} \bullet & \begin{array}{c} r_1 \\ \Downarrow(r_1, s_1) \\ r_1(\partial s_1) \end{array} & \bullet \\ & \begin{array}{c} r_2 \\ \Downarrow(r_2, s_2) \\ r_2(\partial s_2) \end{array} & \bullet \end{array} = \begin{array}{ccc} & r_1 r_2 & \\ \bullet & \begin{array}{c} \Downarrow(r_1 r_2, s_1 r_2 s_2) \end{array} & \bullet \\ & r_1(\partial s_1) r_2(\partial s_2) & \end{array}$$

There is a unique *horizontal identity* 2-cell

$$\Downarrow(1, 1) = \begin{array}{ccc} & 1 & \\ \bullet & \begin{array}{c} \Downarrow(1, 1) \end{array} & \bullet \\ & 1 & \end{array}$$

Similarly, when $r_1(\partial s_1) = r_3$, vertical composition $(r_1, s_1) \#_1 (r_3, s_3)$ of 2-cells is given by

$$\begin{array}{ccc} & r_1 & \\ \bullet & \begin{array}{c} \Downarrow(r_1, s_1) \\ r_1(\partial s_1) \\ \Downarrow(r_1(\partial s_1), s_3) \\ r_1(\partial s_1)(\partial s_3) \end{array} & \bullet \end{array} = \begin{array}{ccc} & r_1 & \\ \bullet & \begin{array}{c} \Downarrow(r_1, s_1 s_3) \\ r_1(\partial(s_1 s_3)) \end{array} & \bullet \end{array}$$

For each $r \in R$ there is a *vertical identity* 2-cell

$$\Downarrow(r, 1) = \begin{array}{ccc} & r & \\ \bullet & \curvearrowright & \bullet \\ & r & \end{array} \quad \Downarrow(r, 1)$$

such that

$$\Downarrow(r, 1) \#_1 \Downarrow(r, s) \#_1 \Downarrow(r(\partial s), 1) = \Downarrow(r, s).$$

The *horizontal inverse* and the *vertical right inverse* of $\Downarrow(r, s)$ are $\Downarrow(r^{-1}, (s^{-1})^{r^{-1}})$ and $\Downarrow(r(\partial s), s^{-1})$ respectively.

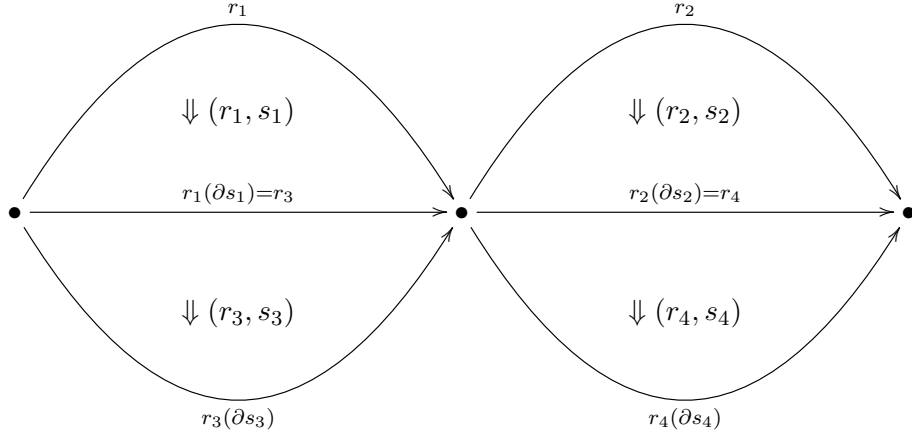
Horizontal composition with vertical identities is called *whiskering*. In diagrams it is often convenient to shrink $\Downarrow(q, 1)$ to a single arc, labelled q , as in the whiskering formulae:

$$q_1 \#_0 \Downarrow(r, s) \#_0 q_2 = \Downarrow(q_1 r q_2, s^{q_2}).$$

The Peiffer condition for cat^1 -groups establishes an *interchange law* for \mathcal{H} ,

$$((r_1, s_1) \#_0 (r_2, s_2)) \#_1 ((r_3, s_3) \#_0 (r_4, s_4)) = ((r_1, s_1) \#_1 (r_3, s_3)) \#_0 ((r_2, s_2) \#_1 (r_4, s_4))$$

for the well-defined composite when $r_1(\partial s_1) = r_3$ and $r_2(\partial s_2) = r_4$,



When this composite is defined,

$$s_2 s_3^{r_4} = s_2 s_3^{r_2(\partial s_2)} = s_3^{r_2} s_2$$

and the composite 2-cell is

$$\begin{array}{ccc} & r_1 r_2 & \\ \bullet & \curvearrowright & \bullet \\ & r_3(\partial s_3) r_4(\partial s_4) & \end{array} \quad \Downarrow(r_1 r_2, (s_1 s_3)^{r_2} s_2 s_4)$$

2 Derivations and Sections

2.1 Derivations

The Whitehead monoid $\text{Der}(\mathcal{X})$ of $\mathcal{X} = (\partial : S \rightarrow R)$ was defined in [55] to be the monoid of all *derivations* from R to S , that is the set of all maps $R \rightarrow S$, with composition \star , satisfying

$$\begin{aligned} \mathbf{D1:} \quad \chi(qr) &= (\chi q)^r (\chi r) \\ \mathbf{D2:} \quad (\chi_1 \star \chi_2)(r) &= (\chi_2 r)(\chi_1 r)(\chi_2 \partial \chi_1 r). \end{aligned}$$

The definition of Whitehead multiplication used here differs from that in [3] in that it is now defined as multiplication on the right rather than on the left, which is why we are using ' \star ' in place of ' \circ '. Invertible elements in the monoid are called *regular*. The Whitehead group $W = W(\mathcal{X})$ is the group of the monoid.

In Brown and Gilbert [8] the notion of derivation was extended to that of γ -derivation, as in the following definition. Since ordinary derivations may be obtained from γ -derivations by setting γ to be the identity automorphism of \mathcal{X} , we shall give properties in terms of the more general case.

Definition 2.1 *If $\gamma = (\ddot{\gamma}, \dot{\gamma})$ is an automorphism of \mathcal{X} , the Whitehead monoid $\text{Der}_\gamma(\mathcal{X})$ of \mathcal{X} is the monoid of all gamma-derivations from R to S , that is the set of all maps $R \rightarrow S$, with composition written \star_γ , satisfying*

$$\begin{aligned} \mathbf{D1:} \quad \chi(qr) &= (\chi q)^{\dot{\gamma}r} (\chi r) . \\ \mathbf{D2:} \quad (\chi_1 \star_\gamma \chi_2)(r) &= (\chi_2 r)(\chi_1 r)(\chi_2 \dot{\gamma}^{-1} \partial \chi_1 r). \end{aligned}$$

The following Lemma verifies that $\text{Der}_\gamma(\mathcal{X})$ is a monoid.

Lemma 2.2

- (a) $\chi 1 = 1$,
- (b) $(\chi r)^{-1} = (\chi r^{-1})^{\dot{\gamma}r}$,
- (c) the zero map is a derivation and an identity for the Whitehead multiplication,
- (d) the Whitehead multiplication is associative.

Proof:

- (a) This follows from $\chi(r1) = (\chi r)^1 (\chi 1)$.
- (b) This follows from $1 = \chi(r^{-1}r) = (\chi r^{-1})^{\dot{\gamma}r} (\chi r)$.
- (c) It is clear that $0 : R \rightarrow S$, $r \mapsto 1$ is a derivation, and that

$$(\chi \star_\gamma 0)r = 1(\chi r)1 = \chi r = (\chi r)11 = (0 \star_\gamma \chi)r .$$

- (d) Expansion by **D2:** using either bracketing (though one requires more work!) gives:

$$(\chi_1 \star \chi_2 \star \chi_3)r = (\chi_3 r)(\chi_2 r)(\chi_3 \dot{\gamma}^{-1} \partial \chi_2 r)(\chi_1 r)(\chi_3 \dot{\gamma}^{-1} \partial \chi_1 r)(\chi_2 \dot{\gamma}^{-1} \partial \chi_1 r)(\chi_3 \dot{\gamma}^{-1} \partial \chi_2 \dot{\gamma}^{-1} \partial \chi_1 r) .$$

□

For χ a γ -derivation, define $\psi = \psi_\chi : R \rightarrow S$ by $\psi r = \chi \dot{\gamma}^{-1} r$ or, equivalently, $\psi \dot{\gamma} r = \chi r$.

$$\begin{array}{ccc}
 S & \xrightarrow{\ddot{\gamma}} & S \\
 \downarrow \partial & \nearrow \chi & \downarrow \psi \quad \downarrow \partial \\
 R & \xrightarrow{\dot{\gamma}} & R
 \end{array} \tag{13}$$

Then ψ is a (identity-) derivation since

$$\psi(qr) = \chi((\dot{\gamma}^{-1}q)(\dot{\gamma}^{-1}r)) = (\chi \dot{\gamma}^{-1}q)^{\dot{\gamma}(\dot{\gamma}^{-1}r)}(\chi \dot{\gamma}^{-1}r) = (\psi q)^r(\psi r).$$

Lemma 2.3 *The map $\text{Der}_\gamma(\mathcal{X}) \rightarrow \text{Der}(\mathcal{X})$, $\chi \mapsto \psi_\chi$, is a monoid homomorphism.*

Proof: If ψ_1, ψ_2 are the derivations corresponding to γ -derivations χ_1, χ_2 , then

$$\begin{aligned}
 (\psi_1 \star \psi_2)(\dot{\gamma} r) &= (\psi_2 \dot{\gamma} r)(\psi_1 \dot{\gamma} r)(\psi_2 \dot{\gamma} \dot{\gamma}^{-1} \partial \psi_1 \dot{\gamma} r) \\
 &= (\chi_2 r)(\chi_1 r)(\chi_2 \dot{\gamma}^{-1} \partial \chi_1 r) \\
 &= (\chi_1 \star_\gamma \chi_2) r.
 \end{aligned}$$

□

Lemma 2.4 *Given a γ -derivation χ of \mathcal{X} there is an endomorphism $\beta_\chi = (\ddot{\beta}_\chi, \dot{\beta}_\chi)$ of \mathcal{X} where*

$$\ddot{\beta}_\chi : S \rightarrow S, \quad s \mapsto (\dot{\gamma} s)(\chi \partial s), \quad \dot{\beta}_\chi : R \rightarrow R, \quad r \mapsto (\dot{\gamma} r)(\partial \chi r)$$

such that

- (a) $\ddot{\beta}_\chi(s^r) = (\ddot{\beta}_\chi s)^{\dot{\beta}_\chi r} = (\chi r)^{-1} (\ddot{\beta}_\chi s)^{\dot{\gamma} r} (\chi r)$,
- (b) $\dot{\beta}_\chi(q^r) = (\dot{\beta}_\chi q)^{\dot{\beta}_\chi r} = ((\dot{\gamma} r)(\partial \chi r))^{-1} (\dot{\gamma} q)(\partial \chi q)((\dot{\gamma} r)(\partial \chi r))$,
- (c) $(\chi_1 \star_\gamma \chi_2) r = (\chi_2 r)(\ddot{\beta}_{\chi_2} \dot{\gamma}^{-1} \chi_1 r) = (\chi_1 r)(\chi_2 \dot{\gamma}^{-1} \dot{\beta}_{\chi_1} r)$,
- (d) $\chi * \dot{\gamma}^{-1} * \ddot{\beta}_\chi = \dot{\beta}_\chi * \dot{\gamma}^{-1} * \chi : R \rightarrow S, \quad r \mapsto (\chi r)(\chi \dot{\gamma}^{-1} \partial \chi r)$, so that the following diagram commutes:

$$\begin{array}{ccc}
 S & \xrightarrow{\dot{\gamma}^{-1} * \ddot{\beta}_\chi} & S \\
 \uparrow \chi & & \uparrow \chi \\
 R & \xrightarrow{\dot{\beta}_\chi * \dot{\gamma}^{-1}} & R
 \end{array} \tag{14}$$

- (e) *The endomorphism $\ddot{\beta}_\chi * \dot{\gamma}^{-1}$ commutes with $\partial * \chi * \dot{\gamma}^{-1}$ while $\dot{\gamma}^{-1} * \dot{\beta}_\chi$ commutes with $\dot{\gamma}^{-1} * \chi * \partial$.*
- (f) *When $\psi = \psi_\chi$ as in (13), then $\dot{\beta}_\chi r = \dot{\beta}_\psi(\dot{\gamma} r)$ and $\ddot{\beta}_\chi s = \ddot{\beta}_\psi(\dot{\gamma} s)$.*

Proof: We first check that $\dot{\beta}_\chi$ and $\ddot{\beta}_\chi$ are homomorphisms.

$$\begin{aligned}\dot{\beta}_\chi(r_1 r_2) &= \dot{\gamma}(r_1 r_2) \partial((\chi r_1)^{\dot{\gamma} r_2} (\chi r_2)) = (\dot{\gamma} r_1) (\partial \chi r_1) (\dot{\gamma} r_2) (\partial \chi r_2) = (\dot{\beta}_\chi r_1) (\dot{\beta}_\chi r_2), \\ \ddot{\beta}_\chi(s_1 s_2) &= \ddot{\gamma}(s_1 s_2) (\chi ((\partial s_1) (\partial s_2))) = (\ddot{\gamma} s_1) (\ddot{\gamma} s_2) (\chi \partial s_1)^{\partial \ddot{\gamma} s_2} (\chi \partial s_2) = (\ddot{\beta}_\chi s_1) (\ddot{\beta}_\chi s_2).\end{aligned}$$

We now verify the six properties.

- (a) $\ddot{\beta}_\chi(s^r) = (\ddot{\gamma} s^r) (\chi \partial(s^r)) = (\ddot{\gamma} s)^{\dot{\gamma} r} (\chi(r^{-1}(\partial s) r)) = (\ddot{\gamma} s)^{\dot{\gamma} r} (\chi(r^{-1}))^{(\partial \ddot{\gamma} s)(\dot{\gamma} r)} (\chi \partial s)^{\dot{\gamma} r} (\chi r)$
 $= \{(\chi(r^{-1}))(\ddot{\gamma} s)(\chi \partial s)\}^{\dot{\gamma} r} (\chi r) = (\chi r)^{-1} (\ddot{\beta}_\chi s)^{\dot{\gamma} r} (\chi r) = (\ddot{\beta}_\chi s)^{(\dot{\gamma} r)(\partial \chi r)} = (\ddot{\beta}_\chi s)^{\dot{\beta}_{\chi r}}.$
- (b) $\dot{\beta}_\chi(q^r) = (\dot{\gamma}(q^r)) (\partial \chi(r^{-1} q r)) = (\dot{\gamma}(q^r)) \partial \{(\chi(r^{-1}))^{\dot{\gamma}(q r)} (\chi q)^{\dot{\gamma} r} (\chi r)\}$
 $= (\dot{\gamma}(q^r)) (\partial \chi(r^{-1}))^{(\dot{\gamma} r)(\dot{\gamma} q r)} (\partial \chi q)^{\dot{\gamma} r} (\partial \chi r) = (\partial((\chi r)^{-1}) (\dot{\gamma}(q^r)) (\partial \chi q)^{\dot{\gamma} r} (\partial \chi r)$
 $= (\partial \chi r)^{-1} \{(\dot{\gamma} q) (\partial \chi q)\}^{\dot{\gamma} r} (\partial \chi r) = (\dot{\beta}_\chi q)^{\dot{\beta}_{\chi r}}.$
- (c) $(\chi_1 \star_\gamma \chi_2) r = (\chi_2 r) \{(\chi_1 r) (\chi_2 \partial(\ddot{\gamma}^{-1} \chi_1 r))\} = (\chi_2 r) (\ddot{\beta}_{\chi_2} \ddot{\gamma}^{-1} \chi_1 r),$
 $(\chi_2 \dot{\gamma}^{-1} \dot{\beta}_{\chi_1}) r = \chi_2 (r (\dot{\gamma}^{-1} \partial \chi_1 r)) = (\chi_2 r)^{\partial \chi_1 r} (\chi_2 \dot{\gamma}^{-1} \partial \chi_1 r) = (\chi_1 r)^{-1} (\chi_2 r) (\chi_1 r) (\chi_2 \dot{\gamma}^{-1} \partial \chi_1 r).$
- (d) $\ddot{\beta}_\chi(\ddot{\gamma}^{-1} \chi r) = \ddot{\gamma}(\ddot{\gamma}^{-1} \chi r) (\chi \partial \ddot{\gamma}^{-1} \chi r) = (\chi r) (\chi \dot{\gamma}^{-1} \partial \chi r),$ and
 $(\chi \dot{\gamma}^{-1}) (\dot{\beta}_\chi r) = \chi (r (\dot{\gamma}^{-1} \partial \chi r)) = (\chi r)^{\partial \chi r} (\chi \dot{\gamma}^{-1} \partial \chi r) = (\chi r) (\chi \dot{\gamma}^{-1} \partial \chi r).$
- (e) By (d), $(\ddot{\beta}_\chi * \ddot{\gamma}^{-1}) * (\partial * \chi * \ddot{\gamma}^{-1}) = \partial * (\dot{\beta}_\chi * \dot{\gamma}^{-1} * \chi) * \ddot{\gamma}^{-1} = (\partial * \chi * \ddot{\gamma}^{-1}) * (\ddot{\beta}_\chi * \ddot{\gamma}^{-1}),$
 $(\dot{\gamma}^{-1} * \dot{\beta}_\chi) * (\dot{\gamma}^{-1} * \chi * \partial) = \dot{\gamma}^{-1} * (\chi * \ddot{\gamma}^{-1} * \ddot{\beta}_\chi) * \partial = (\dot{\gamma}^{-1} * \chi * \partial) * (\dot{\gamma}^{-1} * \dot{\beta}_\chi).$

(f) This relationship between β_χ and β_ψ is immediate. \square

Using Lemma 2.4 and the first crossed module axiom, the identity **D1**: for derivations generalises as follows.

Lemma 2.5

- (a) $\chi(r_1 r_2 \dots r_k) = (\chi r_1)^{\dot{\gamma}(r_2 \dots r_k)} (\chi r_2)^{\dot{\gamma}(r_3 \dots r_k)} \dots (\chi r_{k-1})^{\dot{\gamma} r_k} (\chi r_k),$
- (b) $\partial \chi(r_1 r_2 \dots r_k) = (\dot{\gamma}(r_1 r_2 \dots r_k))^{-1} (\dot{\beta}_\chi r_1) (\dot{\beta}_\chi r_2) \dots (\dot{\beta}_\chi r_k),$
- (c) $\chi \partial(s_1 s_2 \dots s_k) = (\ddot{\gamma}(s_1 s_2 \dots s_k))^{-1} (\ddot{\beta}_\chi s_1) (\ddot{\beta}_\chi s_2) \dots (\ddot{\beta}_\chi s_k).$

It is straightforward to verify that for g an invertible element in a monoid M , the set $M_g = (M, *_g)$ with elements M and multiplication $*_g$ defined in terms of the usual multiplication by

$$m *_g n := m g^{-1} n, \quad (15)$$

is a monoid with identity g . If $m \in M$ is invertible in M then m has $*_g$ -inverse $\overline{m} := g m^{-1} g$. The resulting monoids are isomorphic, either by $\theta_g : M \rightarrow M_g, m \mapsto m g$ or by $\theta'_g : M \rightarrow M_g, m \mapsto g m$. When M is a group the g -conjugation automorphisms are the mappings

$$\wedge_g m : G \rightarrow G, n \mapsto \overline{m} *_g n *_g m = g m^{-1} n g^{-1} m. \quad (16)$$

This notion generalises to categories and to crossed modules, but the application we require here is to the monoid of endomorphisms $\text{End}_\gamma(\mathcal{X})$, where $\gamma = (\ddot{\gamma}, \dot{\gamma})$ is an automorphism of \mathcal{X} , with multiplication

$$\alpha *_\gamma \beta := (\ddot{\alpha} *_\ddot{\gamma} \ddot{\beta}, \dot{\alpha} *_\dot{\gamma} \dot{\beta}). \quad (17)$$

Theorem 2.6 *There is a monoid homomorphism $\Delta_\gamma : \text{Der}_\gamma(\mathcal{X}) \rightarrow \text{End}_\gamma(\mathcal{X})$, $\chi \mapsto \beta_\chi = (\ddot{\beta}_\chi, \dot{\beta}_\chi)$.*

Proof: Since

$$\begin{aligned}
(\ddot{\beta}_{\chi_1} *_\gamma \ddot{\beta}_{\chi_2})s &= (\ddot{\beta}_{\chi_1} * \ddot{\gamma}^{-1} * \ddot{\beta}_{\chi_2})s \\
&= \ddot{\beta}_{\chi_2}(s(\ddot{\gamma}^{-1}\chi_1\partial s)) \\
&= (\ddot{\gamma}s)(\chi_1\partial s)\chi_2((\partial s)(\dot{\gamma}^{-1}\partial\chi_1\partial s)) \\
&= (\ddot{\gamma}s)(\chi_1\partial s)(\chi_2\partial s)^{\partial\chi_1\partial s}(\chi_2\dot{\gamma}^{-1}\partial\chi_1\partial s) \\
&= (\ddot{\gamma}s)(\chi_2\partial s)(\chi_1\partial s)(\chi_2\dot{\gamma}^{-1}\partial\chi_1\partial s) \\
&= \ddot{\beta}_{\chi_1 \star_\gamma \chi_2} s, \\
(\dot{\beta}_{\chi_1} *_\gamma \dot{\beta}_{\chi_2})r &= (\dot{\beta}_{\chi_1} * \dot{\gamma}^{-1} * \dot{\beta}_{\chi_2})r \\
&= \dot{\beta}_{\chi_2}(r(\dot{\gamma}^{-1}\partial\chi_1r)) \\
&= (\dot{\gamma}r)(\partial\chi_1r)\partial((\chi_2r)^{\partial\chi_1r}(\chi_2\dot{\gamma}^{-1}\partial\chi_1r)) \\
&= (\dot{\gamma}r)\partial((\chi_2r)(\chi_1r)(\chi_2\dot{\gamma}^{-1}\partial\chi_1r)) \\
&= \dot{\beta}_{\chi_1 \star_\gamma \chi_2} r,
\end{aligned}$$

it follows that $(\Delta_\gamma\chi_1) *_\gamma (\Delta_\gamma\chi_2) = \Delta_\gamma(\chi_1 \star_\gamma \chi_2)$. □

We shall see later that there is a homomorphism from S to $\text{Der}(\mathcal{X})$ mapping s to the principal derivation η_s .

Lemma 2.7 *For each $s \in S$ the function η_s (which we may also write as $\eta_{\gamma,s}$ when required)*

$$\eta_s : R \rightarrow S, \quad r \mapsto (s^{-1})^{\dot{\gamma}r} s$$

is a γ -derivation, called a principal γ -derivation, satisfying

$$\eta_s(\partial s_0) = [\ddot{\gamma}s_0, s], \quad \text{and} \quad \partial(\eta_s r) = [\dot{\gamma}r, \partial s].$$

Proof:

$$\begin{aligned}
(\eta_s q)^{\dot{\gamma}r} (\eta_s r) &= ((s^{-1})^{\dot{\gamma}q} s)^{\dot{\gamma}r} (s^{-1})^{\dot{\gamma}r} s = (s^{-1})^{\dot{\gamma}(qr)} s = \eta_s(qr), \\
\eta_s \partial s_0 &= (s^{-1})^{\dot{\gamma}\partial s_0} s = (s^{-1})^{\partial\ddot{\gamma}s_0} s = (\ddot{\gamma}s_0)^{-1} s^{-1} (\ddot{\gamma}s_0) s = [\ddot{\gamma}s_0, s], \\
\partial(\eta_s r) &= (\partial s^{-1})^{\dot{\gamma}r} (\partial s) = [\dot{\gamma}r, \partial s].
\end{aligned}$$

□

Lemma 2.8 (Properties of principal derivations)

- (a) η_1 is the zero map,
- (b) $\ddot{\beta}_{\eta_s} s_0 = (\ddot{\gamma}s_0)^s$ and $\dot{\beta}_{\eta_s} r = (\dot{\gamma}r)^{\partial s}$,
- (c) $\eta_{s_1} \star_\gamma \eta_{s_2} = \eta_{s_1 s_2}$ and $\overline{\eta_s} = \eta_{s^{-1}}$.

Proof:

- (a) $\eta_1 r = 1^r 1 = 1$,
- (b) $\ddot{\beta}_{\eta_s} s_0 = (\ddot{\gamma}s_0)(\eta_s(\partial s_0)) = (\ddot{\gamma}s_0)[\ddot{\gamma}s_0, s] = (\ddot{\gamma}s_0)^s$
and $\dot{\beta}_{\eta_s} r = (\dot{\gamma}r)(\partial\eta_s r) = (\dot{\gamma}r)[\dot{\gamma}r, \partial s] = (\dot{\gamma}r)^{\partial s}$,
- (c) $(\eta_{s_2} r)(\eta_{s_1} r)(\eta_{s_2} \partial\ddot{\gamma}^{-1}\eta_{s_1} r) = (\eta_{s_2} r)(\eta_{s_1} r)[\eta_{s_1} r, s_2] = (\eta_{s_2} r)s_2^{-1}(\eta_{s_1} r)s_2 = (s_2^{-1})^{\dot{\gamma}r}(s_1^{-1})^{\dot{\gamma}r} s_1 s_2$.

□

Lemma 2.9 *The following statements are equivalent.*

- (i) χ has a Whitehead γ -inverse $\bar{\chi}$;
- (ii) $\ddot{\beta}_\chi \in \text{Aut}(S)$, where $\ddot{\beta}_\chi(s) = (\dot{\gamma}s)(\chi\partial s)$;
- (iii) $\dot{\beta}_\chi \in \text{Aut}(R)$, where $\dot{\beta}_\chi(r) = (\dot{\gamma}r)(\partial\chi r)$;
- (iv) $\beta = (\ddot{\beta}, \dot{\beta}) \in \text{Aut}_\gamma(\mathcal{X})$.

When these conditions are satisfied,

$$\bar{\chi}r = (\dot{\gamma}\overline{\dot{\beta}_\chi}\chi r)^{-1} = (\chi\overline{\dot{\beta}_\chi}\dot{\gamma}r)^{-1}, \quad (\chi r)(\bar{\chi}r) = (\chi\dot{\gamma}^{-1}\partial\bar{\chi}r)^{-1}, \quad \text{and} \quad (\bar{\chi}r)(\chi r) = (\bar{\chi}\dot{\gamma}^{-1}\partial\chi r)^{-1}.$$

Proof: Theorem 2.6 shows that, when χ is a regular derivation, both $\ddot{\beta}_\chi$ and $\dot{\beta}_\chi$ are automorphisms, so (i) implies (ii) and (iii), and hence (iv).

Now suppose that $\ddot{\beta}_\chi$ has γ -inverse $\overline{\ddot{\beta}_\chi}$. We first show that χ^\S is a derivation where $\chi^\S r = (\dot{\gamma}\overline{\dot{\beta}_\chi}\chi r)^{-1}$. Using the equivalent formula, $\ddot{\beta}_\chi\dot{\gamma}^{-1}\chi^\S r = (\chi r)^{-1}$,

$$\begin{aligned} \ddot{\beta}_\chi\dot{\gamma}^{-1}((\chi^\S q)^{\dot{\gamma}r}(\chi^\S r)) &= (\ddot{\beta}_\chi\dot{\gamma}^{-1}\chi^\S q)^{\dot{\beta}_\chi r}(\ddot{\beta}_\chi\dot{\gamma}^{-1}\chi^\S r) \quad \text{by Theorem 2.4 (a)} \\ &= ((\chi q)^{-1})^{(\dot{\gamma}r)(\partial\chi r)}(\chi r)^{-1} \quad \text{by definition of } \chi^\S \\ &= (\chi r)^{-1}((\chi q)^{-1})^{\dot{\gamma}r} = ((\chi q)^{\dot{\gamma}r}(\chi r))^{-1} = (\chi(qr))^{-1} = \ddot{\beta}_\chi\dot{\gamma}^{-1}\chi^\S(qr). \end{aligned}$$

We now show that χ^\S is the Whitehead γ -inverse $\bar{\chi}$ of χ , using Lemma 2.4 (c), (d) :

$$\begin{aligned} (\chi^\S \star_\gamma \chi)r &= (\chi r)(\ddot{\beta}_\chi\dot{\gamma}^{-1}\chi^\S r) = (\chi r)(\chi r)^{-1}, \\ (\chi \star_\gamma \chi^\S)r &= (\chi r)(\chi^\S\dot{\gamma}^{-1}\dot{\beta}_\chi r) = (\chi r)(\dot{\gamma}\overline{\dot{\beta}_\chi}\chi\dot{\gamma}^{-1}\dot{\beta}_\chi r)^{-1} = (\chi r)(\chi r)^{-1}. \end{aligned}$$

Thus (ii) implies (i), (iii) and (iv).

Similarly, suppose that $\dot{\beta}_\chi$ has an inverse $\dot{\beta}_\chi^{-1}$. We show that $\chi^\#$ is a derivation where $\chi^\# r = (\chi\overline{\dot{\beta}_\chi}\dot{\gamma}r)^{-1}$. Define $r' = \overline{\dot{\beta}_\chi}\dot{\gamma}r$ so that $\chi^\# r = (\chi r')^{-1}$, and similarly for q' . Then

$$\begin{aligned} (\chi^\# q)^{\dot{\gamma}r}(\chi^\# r) &= ((\chi q')^{-1})^{\dot{\gamma}r}(\chi r')^{-1} = \left((\chi r')(\chi q')^{\dot{\beta}_\chi r'}\right)^{-1} = \left((\chi r')(\chi q')^{(\dot{\gamma}r')(\partial\chi r')}\right)^{-1} \\ &= \left((\chi q')^{\dot{\gamma}r'}(\chi r')\right)^{-1} = (\chi(qr'))^{-1} = \chi^\#(qr). \end{aligned}$$

This $\chi^\#$ is another form of $\bar{\chi}$ since, again using Lemma 2.4 (c), (d) :

$$\begin{aligned} (\chi^\# \star_\gamma \chi)r &= (\chi r)(\ddot{\beta}_\chi\dot{\gamma}^{-1}\chi^\# r) = (\chi r)(\ddot{\beta}_\chi\dot{\gamma}^{-1}\chi\overline{\dot{\beta}_\chi}\dot{\gamma}r)^{-1} = (\chi r)(\chi r)^{-1}, \\ (\chi \star_\gamma \chi^\#)r &= (\chi r)(\chi^\#\dot{\gamma}^{-1}\dot{\beta}_\chi r) = (\chi r)(\chi r)^{-1}. \end{aligned}$$

Thus (iii) implies (i), (ii) and (iv).

Finally, (iv) implies (ii) and (iii), and hence (i).

The expressions for $(\chi r)(\bar{\chi}r)$ and $(\bar{\chi}r)(\chi r)$ are obtained by expanding $(\bar{\chi} \star_\gamma \chi)r$ and $(\chi \star_\gamma \bar{\chi})r$. \square

We shall see in Subsection 3.2 that $W_\gamma(\mathcal{X})$ is the source group in the γ -actor of \mathcal{X} ,

$$\text{Act}_\gamma(\mathcal{X}) = (\Delta_\gamma : W_\gamma(\mathcal{X}) \rightarrow \text{Aut}_\gamma(\mathcal{X})).$$

Lue and Norrie, in [44, 43, 48, 49], showed that $\text{Act}(\mathcal{X})$ is the automorphism object of \mathcal{X} in the category **XMod**. Gilbert, in [35], has discussed a connection between derivations and group extensions.

2.2 Sections

The construction for a cat^1 -group $\mathcal{C} = (e; t, h : G \rightarrow R)$ equivalent to the γ -derivation of the corresponding crossed module is the γ -section, namely a group monomorphism $\xi : R \rightarrow G$ satisfying:

$$\mathbf{S1:} \quad t\xi(r) = \dot{\gamma}r \text{ for all } r \in R.$$

The equations

$$\xi r = (e\dot{\gamma}r)(\epsilon\chi r) = (\dot{\gamma}r, \chi r), \quad \chi r = (e\dot{\gamma}r)^{-1}(\xi r) \quad (18)$$

define a section ξ of \mathcal{C} in terms of a derivation χ of \mathcal{X} , and conversely. The automorphism $\gamma = (\ddot{\gamma}, \dot{\gamma})$ of $\mathcal{X} = (\partial : S \rightarrow R)$ determines an automorphism $\bar{\gamma}$ of $R \ltimes S$, and hence an automorphism $(\bar{\gamma}, \dot{\gamma})$ of the corresponding cat^1 -group.

The *principal section* κ_s , $s \in \ker t$, and the corresponding principal derivation η_s are given by

$$\eta_s r = (s^{-1})^{\dot{\gamma}r} s \quad \kappa_s r = (e\dot{\gamma}r)^s = s^{-1}(e\dot{\gamma}r)s.$$

In the semidirect product notation we have

$$\kappa_s r = (\dot{\gamma}r, \eta_s r) = (\dot{\gamma}r, (s^{-1})^{\dot{\gamma}r} s) = (1, s^{-1})(\dot{\gamma}r, 1)(1, s) = (\dot{\gamma}r, 1)^{(1, s)}.$$

Since $(ehg^{-1})(\xi\dot{\gamma}^{-1}hg) \in \ker t$ and $(ehg^{-1})g \in \ker h$ we have, in the group groupoid,

$$g * \xi\dot{\gamma}^{-1}hg = g(ehg^{-1})(\xi\dot{\gamma}^{-1}hg) = (ehg)((ehg^{-1})g)((ehg^{-1})(\xi\dot{\gamma}^{-1}hg)) = (\xi\dot{\gamma}^{-1}hg)(ehg^{-1})g. \quad (19)$$

These sections form the monoid $\text{Sect}(\mathcal{C})$ of \mathcal{C} , whose composition rule we determine from the rule **D2**: for $\text{Der}(\mathcal{X})$ by evaluating:

$$\begin{aligned} (\xi_1 \star_\gamma \xi_2)r &= (e\dot{\gamma}r)(\epsilon(\chi_1 \star \chi_2)r) \\ &= (e\dot{\gamma}r)(\epsilon\chi_2 r)(\epsilon\chi_1 r)(\epsilon\chi_2 \dot{\gamma}^{-1}h\epsilon\chi_1 r) \\ &= (\xi_2 r)(e\dot{\gamma}r^{-1})(\xi_1 r)(eh(\epsilon\chi_1 r)^{-1})(\xi_2 \dot{\gamma}^{-1}h\epsilon\chi_1 r) \\ &= (\xi_2 r)(e\dot{\gamma}r^{-1})(\xi_1 r)(eh((\xi_1 r)^{-1}(e\dot{\gamma}r)))(\xi_2 \dot{\gamma}^{-1}h((e\dot{\gamma}r^{-1})(\xi_1 r))) \\ &= ((e\dot{\gamma}r)(\xi_2 r^{-1}))^{-1}((\xi_1 r)(eh\xi_1 r^{-1}))((e\dot{\gamma}r)(\xi_2 r^{-1}))(\xi_2 \dot{\gamma}^{-1}h\xi_1 r). \end{aligned}$$

Since $(e\dot{\gamma}r)(\xi_2 r^{-1}) \in \ker t$ while $(\xi_1 r)(eh\xi_1 r^{-1}) \in \ker h$, we obtain, using (19),

$$\mathbf{S2:} \quad (\xi_1 \star_\gamma \xi_2)r = (\xi_1 r)(eh\xi_1 r^{-1})(\xi_2 \dot{\gamma}^{-1}h\xi_1 r) = (\xi_2 \dot{\gamma}^{-1}h\xi_1 r)(eh\xi_1 r^{-1})(\xi_1 r). \quad (20)$$

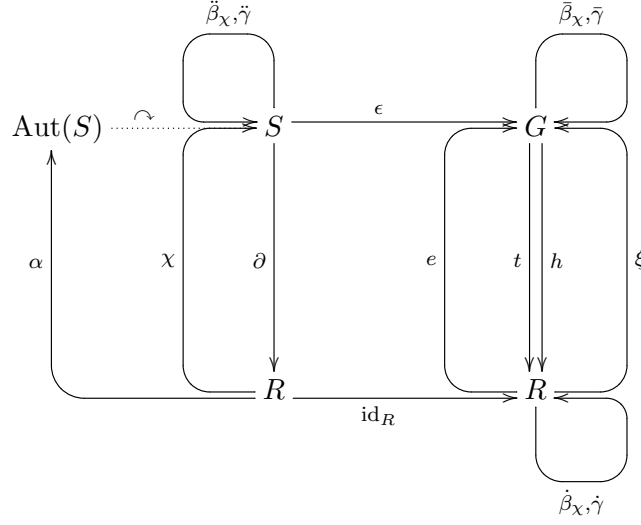
(Note that this axiom also differs from that in [3] in that it is converted to a multiplication on the right.)

The section $\dot{\gamma} * e$ is the identity for this composition, and equation (18) determines a monoid isomorphism $\text{Der}(\mathcal{X}) \cong \text{Sect}(\mathcal{C})$. A section is *regular* when $h\xi$ is an automorphism of R , and the group of regular sections is isomorphic to the Whitehead group.

Each χ and its associated ξ determine endomorphisms of R, S, G, \mathcal{X} and \mathcal{C} , namely

$$\begin{aligned} \dot{\beta}_\chi &= \dot{\beta}_\xi : R \rightarrow R, & r &\mapsto (\dot{\gamma}r)(\partial\chi r) = h\xi r, \\ \ddot{\beta}_\chi &= \ddot{\beta}_\xi : S \rightarrow S, & s &\mapsto (\ddot{\gamma}s)(\chi\partial s) = (\ddot{\gamma}s)(e\partial\ddot{\gamma}s^{-1})(\xi\partial s) = (\xi\partial s)(e\partial\ddot{\gamma}s^{-1})(\ddot{\gamma}s), \\ \bar{\beta}_\chi &= \bar{\beta}_\xi : G \rightarrow G, & g &\mapsto (eh\xi tg)(\xi tg^{-1})(\bar{\gamma}g)(eh\bar{\gamma}g^{-1})(\xi hg), \\ (\ddot{\beta}_\chi, \dot{\beta}_\chi) &= (\ddot{\beta}_\xi, \dot{\beta}_\xi) : \mathcal{X} \rightarrow \mathcal{X}, \\ (\bar{\beta}_\chi, \dot{\beta}_\chi) &= (\bar{\beta}_\xi, \dot{\beta}_\xi) : \mathcal{C} \rightarrow \mathcal{C}, \end{aligned} \quad (21)$$

and these assignments determine group homomorphisms from the Whitehead group to these five endomorphism groups. The accompanying diagram shows the relationship between the various groups and homomorphisms.



2.3 The group-groupoid equivalent of derivations and sections

(This Subsection (for now) covers only identity derivations and sections.)

The cat^1 -formula (20) for Whitehead composition of sections is

$$\mathbf{S2:} \quad (\xi_1 \star \xi_2)r = (\xi_1 r)(eh\xi_1 r^{-1})(\xi_2 h\xi_1 r) = (\xi_2 h\xi_1 r)(eh\xi_1 r^{-1})(\xi_1 r),$$

which is rather obscure. Considering the group-groupoid \mathcal{G} associated to the cat^1 -group \mathcal{C} , as discussed in Subsection 1.13, we see that sections of \mathcal{C} are associated to automorphisms of \mathcal{G} .

A section ξ of \mathcal{C} defines a groupoid endomorphism $\lambda = \lambda_\xi : \mathcal{G} \rightarrow \mathcal{G}$ as follows. Consider the diagram

$$\begin{array}{ccc} h\xi tg & \xrightarrow{\lambda g} & h\xi hg \\ \uparrow \xi tg & & \uparrow \xi hg \\ tg & \xrightarrow{g} & hg \end{array} \quad (22)$$

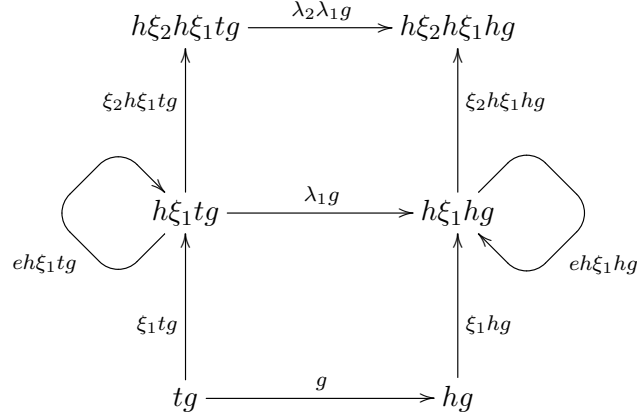
where $t\lambda g = h\xi tg$ and $h\lambda g = h\xi hg$. The morphism λ is defined on objects and arrows by

$$\lambda r = h\xi r, \quad \lambda g = (\widetilde{\xi tg}) * g * \xi hg = (eh\xi tg)(\xi tg^{-1})g(ehg^{-1})(\xi hg). \quad (23)$$

The product of the first four terms is in $\ker h$, while the product of the last four terms is in $\ker t$. It is easily verified that λ is a groupoid morphism. If $r_0 = tg_1$, $r_1 = hg_1 = tg_2$ and $r_2 = hg_2$, then

$$\begin{aligned} (\lambda g_1) * (\lambda g_2) &= (eh\xi r_0)(\xi r_0^{-1})g_1(er_1^{-1})(\xi r_1).(eh\xi r_1^{-1}).(eh\xi r_1)(\xi r_1^{-1})g_2(er_2^{-1})(\xi r_2) \\ &= (eh\xi r_0)(\xi r_0^{-1})(g_1 * g_2)(er_2^{-1})(\xi r_2) \\ &= \lambda(g_1 * g_2). \end{aligned}$$

When we consider ξ_1 followed by ξ_2 we get



and the composite on the left-hand side is

$$(\xi_1 tg) * (\xi_2 h\xi_1 tg) = (\xi_1 tg)(eh\xi_1 tg^{-1})(\xi_2 h\xi_1 tg)$$

in agreement with **S2**., and similarly for the right-hand side. Thus $\lambda_{\xi_1 * \xi_2} = \lambda_{\xi_1} * \lambda_{\xi_2}$ and we have the following result.

Lemma 2.10 *There is a monoid homomorphism*

$$\text{Sect}(\mathcal{C}) \rightarrow \text{End}(\mathcal{G}), \quad \xi \mapsto \lambda_\xi$$

which restricts to a homomorphism $W(\mathcal{C}) \rightarrow \text{Aut}(\mathcal{G})$.

Associated to a principal derivation $\eta_s r = (s^{-1})^r s$, and the corresponding principal section $\kappa_s r = (er)^s$, there is a *principal endomorphism* λ_s of \mathcal{G} .

Proposition 2.11 *The principal endomorphism $\lambda_s = \lambda_{\kappa_s} : \mathcal{G} \rightarrow \mathcal{G}$ is given by*

$$\lambda_s r = r^{hs}, \quad \lambda_s g = g^{hs}.$$

Proof: Applying the formulae in equation (23), and $[\ker t, \ker s] = 1$,

$$\begin{aligned} \lambda_s r &= h\kappa_s r = (hs^{-1})r(hs) = r^{hs}, \\ \lambda_s g &= (\widetilde{\kappa_s tg}) * g * (\kappa_s hg) \\ &= (\widetilde{(etg)^s}) * g * (ehg)^s \\ &= (eh(s^{-1}(etg)s)(s^{-1}(etg)s)^{-1}(et(s^{-1}(etg)s)(etg^{-1})g(ehg)^{-1}(ehg)^s) \\ &= (ehs^{-1})(etg)(ehs)s^{-1}(etg^{-1})[s][g(ehg)^{-1}]s^{-1}(ehg)s \\ &= (ehs^{-1})(etg)[(ehs)s^{-1}][(etg^{-1})g]s \\ &= (ehs^{-1})g(ehs) \\ &= g^{hs}. \end{aligned}$$

□

3 The Actor of a Crossed Module

This section is based on the material covered in pages 25-28 of Norrie's thesis [48]. We will, however, be extending her actor crossed module $\text{Act}(\mathcal{X})$ to the more general $\text{Act}_\gamma(\mathcal{X})$ where γ is an automorphism of \mathcal{X} . Here is a table giving Norrie's symbols and the ones used here.

section	type	old symbol	new symbol
3	xmod Whitehead group xmod morphism principal derivations	$\partial : T \rightarrow G$ $\text{Der}(G, T)$ (σ, θ) $E(G, T)$	$\mathcal{X} = (\partial : S \rightarrow R)$ $W(\mathcal{X})$ $\beta_\chi = (\ddot{\beta}_\chi, \dot{\beta}_\chi) = (\sigma, \rho)$ $E(\mathcal{X})$
??	xmod xmod xmod morphism derivation group automorphism group semidirect product xmod automorphism elements Whitehead boundary	$\mu : M \rightarrow P$ $\nu : N \rightarrow V$ $\langle \epsilon, \rho \rangle$ $D(P, M)$ $\text{Aut}(M, P)$ $(M, P) \sqsupset_{\langle \epsilon, \rho \rangle} (N, V)$ $\pi : M \sqsupset N \rightarrow P \sqsupset V$ $\rho(v) = (\rho_1(v), \rho_2(v))$ m, n, p, v $\Delta(\chi) = \langle \theta_\chi, \ddot{\beta}_\chi \rangle$	$\mathcal{X} = (\partial : S \rightarrow R)$ $\mathcal{Y} = (\delta : Q \rightarrow P)$ $\beta (= (\ddot{\beta}, \dot{\beta}))$ $W = W(\mathcal{X})$ $A = A(\mathcal{X})$ $\mathcal{Y} \ltimes \mathcal{X}$ $(\pi : Q \ltimes S \rightarrow P \ltimes R)$ $\beta(p) = \beta_p : \mathcal{X} \rightarrow \mathcal{X}$ s, q, r, p $\Delta(\chi) = \beta_\chi = (\ddot{\beta}_\chi, \dot{\beta}_\chi)$

When γ is an automorphism of \mathcal{X} , the group of automorphisms $\text{Aut}_\gamma \mathcal{X}$, has composition (as in (17)) given by

$$\alpha_1 *_\gamma \alpha_2 := (\ddot{\alpha}_1 *_{\dot{\gamma}} \ddot{\alpha}_2, \dot{\alpha}_1 *_{\dot{\gamma}} \dot{\alpha}_2) = (\ddot{\alpha}_1 * \ddot{\gamma}^{-1} * \ddot{\alpha}_2, \dot{\alpha}_1 * \dot{\gamma}^{-1} * \dot{\alpha}_2).$$

3.1 Lue and Norrie crossed modules

We generalise the automorphism crossed module $(\iota : R \rightarrow \text{Aut } R)$, where ιr is conjugation by r , to the *Norrie crossed module* $\mathcal{N}_\gamma = (i_\gamma : R \rightarrow \text{Aut}_\gamma \mathcal{X})$ where:

- the γ -conjugation map is given by $i_\gamma r := \beta_r$ where $\dot{\beta}_r q := (\dot{\gamma} q)^r$, $\ddot{\beta}_r s := (\ddot{\gamma} s)^r$, and
- $\text{Aut}_\gamma \mathcal{X}$ has right actions on R and S given by $r^\alpha := \dot{\alpha} \dot{\gamma}^{-1} r$, $s^\alpha := \ddot{\alpha} \ddot{\gamma}^{-1} s$.

(An alternative set of definitions is given by $\dot{\beta}_r q = (\dot{\gamma} q)^r$, $\ddot{\beta}_r s = (\ddot{\gamma} s)^r$, $r^\alpha = \dot{\alpha} \dot{\gamma}^{-1} r$, $s^\alpha = \ddot{\alpha} \ddot{\gamma}^{-1} s$, but these do not combine with the principal derivation map to give a morphism of crossed modules.)

Note that β_r^{-1} is given by $\dot{\beta}_r^{-1} q = \dot{\gamma}^{-1}(q^{r^{-1}})$, $\ddot{\beta}_r^{-1} s = \ddot{\gamma}^{-1}(s^{r^{-1}})$.

We now check the various axioms for \mathcal{N}_γ .

The map i_γ is a homomorphism:

$$\begin{aligned} (\beta_{r_1} *_\gamma \beta_{r_2}) q &= \dot{\beta}_{r_2} \dot{\gamma}^{-1} ((\dot{\gamma} q)^{r_1}) = \dot{\beta}_{r_2} (q^{\dot{\gamma}^{-1} r_1}) = (\dot{\gamma} (q^{\dot{\gamma}^{-1} r_1}))^{r_2} = (\dot{\gamma} q)^{r_1 r_2} = \dot{\beta}_{r_1 r_2} q, \\ (\beta_{r_1} *_\gamma \beta_{r_2}) s &= \ddot{\beta}_{r_2} \ddot{\gamma}^{-1} ((\ddot{\gamma} s)^{r_1}) = \ddot{\beta}_{r_2} (s^{\ddot{\gamma}^{-1} r_1}) = (\ddot{\gamma} (s^{\ddot{\gamma}^{-1} r_1}))^{r_2} = (\ddot{\gamma} s)^{r_1 r_2} = \ddot{\beta}_{r_1 r_2} s. \end{aligned}$$

The given formulae do specify an action of $\text{Aut}_\gamma \mathcal{X}$ on \mathcal{X} :

$$\begin{aligned} (r^{\alpha_1})^{\alpha_2} &= \dot{\alpha}_2 \dot{\gamma}^{-1} (\dot{\alpha}_1 \dot{\gamma}^{-1} r) = (\dot{\alpha}_1 *_\gamma \dot{\alpha}_2) \dot{\gamma}^{-1} r = r^{\alpha_1 *_\gamma \alpha_2}, \\ (s^{\alpha_1})^{\alpha_2} &= \ddot{\alpha}_2 \ddot{\gamma}^{-1} (\ddot{\alpha}_1 \ddot{\gamma}^{-1} s) = (\ddot{\alpha}_1 *_\gamma \ddot{\alpha}_2) \ddot{\gamma}^{-1} s = s^{\alpha_1 *_\gamma \alpha_2}. \end{aligned}$$

First crossed module axiom (using the γ -conjugation of (16)):

$$\begin{aligned} (\wedge_\gamma \alpha) \dot{\beta}_r q &= \dot{\alpha} \dot{\gamma}^{-1} \dot{\beta}_r (\dot{\alpha}^{-1} \dot{\gamma} q) = \dot{\alpha} \dot{\gamma}^{-1} ((\dot{\gamma} \dot{\alpha}^{-1} \dot{\gamma} q)^r) = \dot{\alpha} ((\dot{\alpha}^{-1} \dot{\gamma} q)^{\dot{\gamma}^{-1} r}) = (\dot{\gamma} q)^{r^\alpha} = \dot{\beta}_{r^\alpha} q, \\ (\wedge_\gamma \alpha) \ddot{\beta}_r s &= \ddot{\alpha} \ddot{\gamma}^{-1} \ddot{\beta}_r (\ddot{\alpha}^{-1} \ddot{\gamma} s) = \ddot{\alpha} \ddot{\gamma}^{-1} ((\ddot{\gamma} \ddot{\alpha}^{-1} \ddot{\gamma} s)^r) = \ddot{\alpha} ((\ddot{\alpha}^{-1} \ddot{\gamma} s)^{\ddot{\gamma}^{-1} r}) = (\ddot{\gamma} s)^{r^\alpha} = \ddot{\beta}_{r^\alpha} s. \end{aligned}$$

Second crossed module axiom:

$$r^{i_\gamma r'} = \dot{\beta}_{r'} \dot{\gamma}^{-1} r = r^{r'}.$$

Similarly, for the *Lue crossed module* $\mathcal{L}_\gamma(\mathcal{X}) = (\partial * i_\gamma : S \rightarrow \text{Aut}_\gamma \mathcal{X})$,

- the boundary maps $s \in S$ to $\beta_{\partial s}$ where $\dot{\beta}_{\partial s} q = (\dot{\gamma} q)^{\partial s}$, $\ddot{\beta}_{\partial s} s' = (\ddot{\gamma} s')^{\partial s}$, and
- the action of $\text{Aut}_\gamma \mathcal{X}$ on R and S are as above.

The verification of the crossed module axioms for $\mathcal{L}_\gamma \mathcal{X}$ are similar to those for $\mathcal{N}_\gamma \mathcal{X}$.

3.2 The actor crossed module

The missing part of the structure of the actor crossed module $\text{Act}_\gamma(\mathcal{X})$ is a γ -action of the automorphisms on the derivations.

Lemma 3.1 *There is an action of $\text{Aut}_\gamma(\mathcal{X})$ on $W_\gamma(\mathcal{X})$ given by*

$$\chi^\alpha = \gamma * \alpha^{-1} * \chi * \gamma^{-1} * \alpha : R \rightarrow S, \quad r \mapsto \ddot{\alpha} \ddot{\gamma}^{-1} \chi \dot{\alpha}^{-1} \dot{\gamma} r,$$

such that $\beta_{\chi^\alpha} = (\wedge_\gamma \alpha)(\beta_\chi)$ where $\wedge_\gamma \alpha$ is the γ -conjugation automorphism of (16).

Proof: We first check the axiom for an action:

$$(\chi^{\alpha_1})^{\alpha_2} = \gamma * \alpha_2^{-1} * (\gamma * \alpha_1^{-1} * \chi * \gamma^{-1} * \alpha_1) * \gamma^{-1} * \alpha_2 = \gamma * (\alpha_1 *_\gamma \alpha_2)^{-1} * \chi * \gamma^{-1} * (\alpha_1 *_\gamma \alpha_2) = \chi^{(\alpha_1 *_\gamma \alpha_2)}.$$

Secondly, we observe that $(\wedge_\gamma \alpha)(\beta_\chi) = \gamma * \alpha^{-1} * \beta_\chi * \gamma^{-1} * \alpha$. □

Definition 3.2 *For γ an automorphism of $\mathcal{X} = (\partial : S \rightarrow R)$, the actor crossed module over γ of \mathcal{X} is $\mathcal{A}_\gamma(\mathcal{X}) = (\Delta_\gamma : W_\gamma \rightarrow A_\gamma)$ where*

- $W_\gamma = W_\gamma(\mathcal{X})$ is the Whitehead group of invertible derivations

$$\chi : R \rightarrow S, \quad \text{such that} \quad \chi(qr) = (\chi q)^{\dot{\gamma} r} (\chi r) \quad \text{for all } q, r \in R, \quad (24)$$

and with Whitehead multiplication (on the right)

$$\chi_1 *_\gamma \chi_2 : R \rightarrow S, \quad r \mapsto (\chi_2 r)(\chi_1 r)(\chi_2 \dot{\gamma}^{-1} \partial \chi_1 r); \quad (25)$$

- $A_\gamma = \text{Aut}_\gamma(\mathcal{X})$ is the group of automorphisms of \mathcal{X} , namely those invertible $\alpha = (\ddot{\alpha}, \dot{\alpha}) : \mathcal{X} \rightarrow \mathcal{X}$ such that

$$\dot{\alpha} \partial = \partial \ddot{\alpha}, \quad \ddot{\alpha}(s^r) = (\ddot{\alpha} s)^{\dot{\alpha} r} \quad \text{and} \quad \dot{\alpha}(q^r) = (\dot{\alpha} q)^{\dot{\alpha} r} \quad \text{for all } s \in S \text{ and } q, r \in R,$$

with composition $\alpha_1 *_\gamma \alpha_2 := (\ddot{\alpha}_1 * \ddot{\alpha}_2, \dot{\alpha}_1 * \dot{\alpha}_2)$, and action given by Lemma 3.1.

- The boundary map is obtained by restricting the monoid homomorphism $\Delta_\gamma : \text{Der}_\gamma(\mathcal{X}) \rightarrow \text{End}_\gamma(\mathcal{X})$ of Theorem 2.6 to the regular derivations:

$$\Delta_\gamma : W_\gamma \rightarrow A_\gamma, \quad \chi \mapsto \beta_\chi = (\ddot{\beta}_\chi, \dot{\beta}_\chi),$$

where $\ddot{\beta}_\chi : S \rightarrow S, \quad s \mapsto (\ddot{\gamma}s)(\chi\partial s), \quad \dot{\beta}_\chi : R \rightarrow R, \quad r \mapsto (\dot{\gamma}r)(\partial\chi r).$ (26)

When it is convenient not to distinguish the two group homomorphisms in the crossed module morphism, we write α for both $\ddot{\alpha}$ and $\dot{\alpha}$.

These groups and morphisms are exhibited in the following diagram (the inner morphism $\iota_\gamma = (\ddot{\iota}_\gamma, \dot{\iota}_\gamma)$ is defined in Subsection 3.3 below):

$$\begin{array}{ccccc} S & \xleftarrow{\ddot{\alpha}, \ddot{\beta}_r, \ddot{\beta}_\chi, \ddot{\gamma}} & S & \xrightarrow{\ddot{\iota}_\gamma} & W_\gamma \\ \partial \downarrow & & \partial \downarrow & & \downarrow \Delta_\gamma \\ R & \xleftarrow{\dot{\alpha}, \dot{\beta}_r, \dot{\beta}_\chi, \dot{\gamma}} & R & \xrightarrow{\dot{\iota}_\gamma} & A_\gamma \end{array} \quad (27)$$

Theorem 3.3 With this action, $\mathcal{A}_\gamma(\mathcal{X}) = (\Delta_\gamma : W_\gamma \rightarrow A_\gamma)$ is a crossed module.

Proof: We have already shown that Δ_γ is a group homomorphism.

We verify the first crossed module axiom for $\mathcal{A}_\gamma(\mathcal{X})$ as follows.

$$\mathbf{X1:} \quad \Delta_\gamma(\chi^\alpha) = (\wedge_\gamma \alpha)(\Delta_\gamma \chi) = \gamma * \alpha^{-1} * \beta_\chi * \gamma^{-1} * \alpha.$$

Now $\Delta_\gamma(\chi^\alpha) = \beta_{\chi^\alpha}$ where

$$\begin{aligned} \ddot{\beta}_{\chi^\alpha} s &= (\ddot{\gamma}s)(\chi^\alpha \partial s) = (\ddot{\gamma}s)(\ddot{\alpha}\ddot{\gamma}^{-1}\chi\dot{\alpha}^{-1}\dot{\gamma}\partial s) = (\ddot{\gamma}s)(\ddot{\alpha}\ddot{\gamma}^{-1}\chi\partial\ddot{\alpha}^{-1}\ddot{\gamma}s) = \ddot{\alpha}\ddot{\gamma}^{-1}((\ddot{\gamma}\ddot{\alpha}^{-1}\ddot{\gamma}s)(\chi\partial\ddot{\alpha}^{-1}\ddot{\gamma}s)) \\ &= \ddot{\alpha}\ddot{\gamma}^{-1}\ddot{\beta}_\chi(\ddot{\alpha}^{-1}\ddot{\gamma}s) = (\gamma * \alpha^{-1} * \beta_\chi * \gamma^{-1} * \alpha) s, \\ \dot{\beta}_{\chi^\alpha} r &= (\dot{\gamma}r)(\partial\chi^\alpha r) = (\dot{\gamma}r)(\partial\ddot{\alpha}\ddot{\gamma}^{-1}\chi\dot{\alpha}^{-1}\dot{\gamma}r) = (\dot{\gamma}r)(\dot{\alpha}\dot{\gamma}^{-1}\partial\chi\dot{\alpha}^{-1}\dot{\gamma}r) = \dot{\alpha}\dot{\gamma}^{-1}(\dot{\gamma}\dot{\alpha}^{-1}\dot{\gamma}r)(\partial\chi\dot{\alpha}^{-1}\dot{\gamma}r) \\ &= \dot{\alpha}\dot{\gamma}^{-1}\dot{\beta}_\chi(\dot{\alpha}^{-1}\dot{\gamma}r) = (\gamma * \alpha^{-1} * \beta_\chi * \gamma^{-1} * \alpha) r. \end{aligned}$$

The second crossed module axiom for $\mathcal{A}_\gamma(\mathcal{X})$,

$$\mathbf{X2:} \quad \chi_1^{\Delta_\gamma \chi_2} = \overline{\chi_2} \star_\gamma \chi_1 \star_\gamma \chi_2,$$

is verified by showing that $\chi_2 \star_\gamma \chi_1^{\Delta_\gamma \chi_2} = \chi_1 \star_\gamma \chi_2$, using Lemma 2.4 (c),

$$(\chi_2 \star_\gamma \chi_1^{\Delta_\gamma \chi_2}) r = (\chi_2 r)(\chi_1^{\Delta_\gamma \chi_2} \dot{\gamma}^{-1} \dot{\beta}_{\chi_2} r) = (\chi_2 r)(\ddot{\beta}_{\chi_2} \ddot{\gamma}^{-1} \chi_1 r) = (\chi_1 \star_\gamma \chi_2) r.$$

□

3.3 The inner morphism

We next describe the morphism of crossed modules $\iota_\gamma = (\ddot{\iota}_\gamma, \dot{\iota}_\gamma) : \mathcal{X} \rightarrow \mathcal{A}_\gamma(\mathcal{X})$. The conditions in (1) for ι_γ to be a morphism are:

$$\ddot{\iota}_\gamma(s_1 s_2) = \ddot{\iota}_\gamma s_1 \star_\gamma \ddot{\iota}_\gamma s_2, \quad \dot{\iota}_\gamma(r_1 r_2) = \dot{\iota}_\gamma r_1 \star_\gamma \dot{\iota}_\gamma r_2. \quad (28)$$

The range part i_γ of ι_γ is given in Subsection 3.1 by:

$$i_\gamma : R \rightarrow A_\gamma, \quad r \mapsto \beta_r = (\ddot{\beta}_r, \dot{\beta}_r) : \mathcal{X} \rightarrow \mathcal{X}, \quad \ddot{\beta}_r s_0 = (\ddot{\gamma} s_0)^r, \quad \dot{\beta}_r r_0 = (\dot{\gamma} r_0)^r.$$

The source part \ddot{i}_γ of ι_γ maps s to its principal derivation (see Lemmas 2.7, 2.8):

$$\ddot{i}_\gamma : S \rightarrow W_\gamma, \quad s \mapsto \eta_s : R \rightarrow S, \quad r \mapsto (s^{-1})^{\dot{\gamma}r} s.$$

Theorem 3.4 *The pair of group homomorphisms $\iota_\gamma = (\ddot{i}_\gamma, i_\gamma) : \mathcal{X} \rightarrow \mathcal{A}_\gamma(\mathcal{X})$ is a morphism of crossed modules.*

Proof: The square commutes if $\Delta_\gamma \ddot{i}_\gamma = i_\gamma \partial$. To verify this we show that $(\Delta_\gamma \ddot{i}_\gamma)s = \Delta_\gamma \eta_s = \beta_{\eta_s}$ is the same automorphism of \mathcal{X} as $(i\partial)s = \beta_{\partial s}$. By definition of $\beta_{\eta_s} = (\ddot{\beta}_{\eta_s}, \dot{\beta}_{\eta_s})$ we have:

$$\begin{aligned} \ddot{\beta}_{\eta_s}(s_0) &= (\ddot{\gamma} s_0)(\eta_s \partial s_0) = (\ddot{\gamma} s_0)(s^{-1})^{\partial \dot{\gamma} s_0} s = s^{-1}(\ddot{\gamma} s_0)s = (\ddot{\gamma} s_0)^{\partial s} = \ddot{\beta}_{\partial s}(s_0), \\ \dot{\beta}_{\eta_s}(r_0) &= (\dot{\gamma} r_0)(\partial \eta_s r_0) = (\dot{\gamma} r_0)\partial((s^{-1})^{\dot{\gamma} r_0} s) = (\partial s)^{-1}(\dot{\gamma} r_0)(\partial s) = (\dot{\gamma} r_0)^{\partial s} = \dot{\beta}_{\partial s}(r_0). \end{aligned}$$

Then we check that the action is preserved:

$$\begin{aligned} (is)^{ir}(q) &= (\eta_s)^{\beta_r} q = \ddot{\beta}_r \ddot{\gamma}^{-1} \eta_s \dot{\beta}_r^{-1} \dot{\gamma} q = \ddot{\beta}_r \ddot{\gamma}^{-1} \eta_s \dot{\gamma}^{-1} ((\dot{\gamma} q)^{r^{-1}}) = (\ddot{\gamma} \ddot{\gamma}^{-1} \eta_s (q^{\dot{\gamma}^{-1} r^{-1}}))^r \\ &= ((s^{-1})^{r(\dot{\gamma} q)^{r^{-1}}} s)^r = ((s^r)^{-1})^{\dot{\gamma} q} s^r = \eta_{(s^r)} q = \ddot{i}(s^r)(q). \end{aligned}$$

□

The γ -version of the *inner actor crossed module* of \mathcal{X} is the image $\iota_\gamma \mathcal{X}$. The source group consists of the principal γ -derivations, and the range group consists of the γ -conjugation automorphisms. For further details see Norrie's thesis [48].

3.4 The Whitehead crossed module

Lemma 3.5 *There is an action of the Whitehead group W_γ on S given by*

$$s^\chi := s^{\beta_\chi} = \ddot{\beta}_\chi \ddot{\gamma}^{-1} s = s(\chi \dot{\gamma}^{-1} \partial s)$$

which makes $\mathcal{W}_\gamma(\mathcal{X}) = (\ddot{i}_\gamma : S \rightarrow W_\gamma)$ a crossed module.

Proof: We first verify that this is an action:

$$\begin{aligned} (s^{\chi_1})^{\chi_2} &= \ddot{\beta}_{\chi_2} \ddot{\gamma}^{-1} (s(\chi_1 \dot{\gamma}^{-1} \partial s)) = s(\chi_1 \dot{\gamma}^{-1} \partial s) \chi_2 \partial \ddot{\gamma}^{-1} (s(\chi_1 \dot{\gamma}^{-1} \partial s)) \\ &= s(\chi_1 \dot{\gamma}^{-1} \partial s) \chi_2 ((\dot{\gamma}^{-1} \partial s)(\dot{\gamma}^{-1} \partial \chi_1 \dot{\gamma}^{-1} \partial s)) \\ &= s(\chi_1 \dot{\gamma}^{-1} \partial s) (\chi_2 \dot{\gamma}^{-1} \partial s)^{\partial \chi_1 \dot{\gamma}^{-1} \partial s} (\chi_2 \dot{\gamma}^{-1} \partial \chi_1 \dot{\gamma}^{-1} \partial s) \\ &= s(\chi_2 \dot{\gamma}^{-1} \partial s) (\chi_1 \dot{\gamma}^{-1} \partial s) (\chi_2 \dot{\gamma}^{-1} \partial \chi_1 \dot{\gamma}^{-1} \partial s) \\ &= s((\chi_1 \star_\gamma \chi_2)(\dot{\gamma}^{-1} \partial s)) = s^{\chi_1 \star_\gamma \chi_2}. \end{aligned}$$

The first crossed module axiom $\eta_{(s^\chi)} = \bar{\chi} \star_\gamma \eta_s \star_\gamma \chi$ is verified by checking

$$\begin{aligned} (\eta_s \star_\gamma \chi) r &= (\chi r) (\ddot{\beta}_\chi \ddot{\gamma}^{-1} \eta_s r) = (\chi r) (\ddot{\beta}_\chi \ddot{\gamma}^{-1} ((s^{-1})^{\dot{\gamma} r} s)) = (\chi r) (\ddot{\beta}_\chi \ddot{\gamma}^{-1} s^{-1})^{\dot{\beta}_{\chi r}} (\ddot{\beta}_\chi \ddot{\gamma}^{-1} s) \\ &= ((s^\chi)^{-1})^{\dot{\gamma} r} (\chi r) (s^\chi) = ((s^\chi)^{-1})^{\dot{\gamma} r} (s^\chi) (\chi r) ((s^\chi)^{-1})^{\dot{\gamma} (\dot{\gamma}^{-1} \partial \chi r)} (s^\chi) \\ &= (\eta_{(s^\chi)} r) (\chi r) (\eta_{(s^\chi)} \dot{\gamma}^{-1} \partial \chi r) = (\chi \star_\gamma \eta_{(s^\chi)}) r. \end{aligned}$$

The second crossed module axiom is verified by:

$$s^{\ddot{s}'} = s^{\eta_{s'}} = s^{\beta_{\eta_{s'}}} = \ddot{\beta}_{\eta_{s'}} \ddot{\gamma}^{-1} s = s(\eta_{s'} \partial \ddot{\gamma}^{-1} s) = s[s, s'] = s^{s'}.$$

□

[We might show here that (∂, Δ) is a morphism.]

Lemma 3.6

$$(\eta_s)^\alpha = \eta_{(s^\alpha)}.$$

Proof:

$$\begin{aligned} (\eta_s)^\alpha q &= \ddot{\alpha} \ddot{\gamma}^{-1} \eta_s \dot{\alpha}^{-1} \dot{\gamma} q = \ddot{\alpha} \ddot{\gamma}^{-1} \left((s^{-1})^{\dot{\gamma} \dot{\alpha}^{-1} \dot{\gamma} q} s \right) = \ddot{\alpha} \left((\ddot{\gamma}^{-1} s^{-1})^{\dot{\alpha}^{-1} \dot{\gamma} q} (\ddot{\gamma}^{-1} s) \right) \\ &= (\ddot{\alpha} \ddot{\gamma}^{-1} s^{-1})^{\dot{\gamma} q} (\ddot{\alpha} \ddot{\gamma}^{-1} s) = ((s^\alpha)^{-1})^{\dot{\gamma} q} (s^\alpha) = \eta_{(s^\alpha)} q. \end{aligned}$$

□

The right-hand square of morphisms of crossed modules in (27) becomes a *crossed square* $\mathcal{S}_\gamma(\mathcal{X})$ (see Example 8.10 for the identity case) when the *crossed pairing* (see Section 7) $\boxtimes : R \times W_\gamma \rightarrow S$, $(r, \chi) \mapsto \chi \dot{\gamma}^{-1} r$, is added to the structure.

3.5 The actor of a cat^1 -group

(This Subsection (for now) covers only identity derivations and sections.)

The diagram corresponding to equation (27) is

$$\begin{array}{ccccc} G & \xleftarrow{\bar{\beta}} & G & \xrightarrow{\ddot{\kappa}} & A(\mathcal{C}) \ltimes W(\mathcal{C}) \\ \downarrow t, h & \uparrow e & \downarrow t, h & \uparrow e & \downarrow T, H \uparrow E \\ R & \xleftarrow{\dot{\beta}} & R & \xrightarrow{\dot{\kappa}} & A(\mathcal{C}) \end{array} \quad (29)$$

where $W = W(\mathcal{C})$ and $A = A(\mathcal{C})$ are defined as follows:

- W is the group of sections of \mathcal{C} with composition given by equation (20),
- $A = \text{Aut}(\mathcal{C})$ is the group of automorphisms of \mathcal{C} ,
- $\Delta_{\mathcal{C}} : W \rightarrow A$, $\xi \mapsto (\bar{\beta}_\xi, \dot{\beta}_\xi)$ (see equations (21)).

Note: T, H, E were previously written $\Delta_t, \Delta_h, \Delta_e$.

The homomorphisms $\ddot{\kappa}, \dot{\kappa}, T, H, E$ are given as follows.

- $\dot{\kappa} : R \rightarrow A$, $r \mapsto \beta'_r = (\bar{\beta}_r, \dot{\beta}_r) : \mathcal{C} \rightarrow \mathcal{C}$, $\bar{\beta}_r g_0 = g_0^r$, $\dot{\beta}_r r_0 = r_0^r = r^{-1} r_0 r$, using the action of R on G in equation (??).
- $\ddot{\kappa} : G \rightarrow A \ltimes W$, $g \mapsto (\dot{\kappa} g, \kappa_g : R \rightarrow G)$ where $\kappa_g(r) = (er)^{(etg^{-1})g}$.
- $T(\beta, \xi) = \beta$, $H(\beta, \xi) = \beta * \Delta_{\mathcal{C}}(\xi)$, $E(\beta) = (\beta, \text{id})$.

[Add in here the associated $\text{cat}2$ -group with groups $(A \ltimes W) \ltimes (R \ltimes S)$, $A \ltimes W$, $A \ltimes R$, A .]

4 Groupoids

4.1 Basic definitions

A *groupoid* is a category in which every arrow is invertible. Thus a groupoid $\mathbb{C} = (C_1, C_0)$ consists of the following:

- a set $\text{Ob}(\mathbb{C}) = C_0$ of *objects*,
- a set $\text{Arr}(\mathbb{C}) = C_1$ of *arrows*,
- source and target maps $s, t : C_1 \rightarrow C_0$, so that we write $(a : u \rightarrow v)$ whenever $sa = u$ and $ta = v$, and we denote by $\mathbb{C}(u, v)$ the set of arrows with source u and target v ,
- an *identity arrow* 1_u at each object u , with $s1_u = t1_u = u$,
- an associative partial composition $\diamond : C_1 \times_0 C_1 \rightarrow C_1$, with $a \diamond b$ defined whenever $ta = sb$, such that $s(a \diamond b) = sa$ and $t(a \diamond b) = tb$, so that $\mathbb{C}(u) := \mathbb{C}(u, u)$ is a group, called the *object group* at u ,
- for each arrow $(a : u \rightarrow v)$ an inverse arrow $(a^{-1} : v \rightarrow u)$ such that $a \diamond a^{-1} = 1_u$ and $a^{-1} \diamond a = 1_v$.

It will often be convenient to omit the symbol \diamond and use simple juxtaposition to indicate composition. (In our GAP implementation source and target are called *tail* and *head*.)

A morphism of groupoids, as for general categories, is called a functor. Thus a *functor* $\phi = (\phi_1, \phi_0) : \mathbb{C} \rightarrow \mathbb{D}$ is a pair of maps $\phi_1 : C_1 \rightarrow D_1$ and $\phi_0 : C_0 \rightarrow D_0$ such that $\phi_1 1_u = 1_{\phi_0 u}$ and $\phi_1(a \diamond b) = (\phi_1 a) \diamond (\phi_1 b)$ whenever the composite arrow is defined. It is often convenient to omit the subscripts 0, 1 since it should be clear from the context whether an object or an arrow is being mapped. A morphism ϕ is *injective* and/or *surjective* if both ϕ_0, ϕ_1 are.

Example 4.1 A group is a groupoid with a single object (usually written $*$). This gives a functor Groupoids from **Gp** to **Gpd**.

Example 4.2 For X a set, the trivial groupoid $\mathbb{O}(X)$ on X has $\text{Ob}(\mathbb{O}) = X$ and $\text{Arr}(\mathbb{O}) = \{1_u \mid u \in \text{Ob}(\mathbb{O})\}$. We denote $\mathbb{O}(\{1, \dots, n\})$ by \mathbb{O}_n .

Example 4.3 The unit groupoid \mathbb{I} has two objects 0, 1 and four arrows. The two non-identity arrows are $(\iota : 0 \rightarrow 1)$ and its inverse $(\iota^{-1} : 1 \rightarrow 0)$.

The *underlying digraph* of a groupoid is obtained by forgetting the composition, so the objects become vertices, the arrows become arcs, while the source and target maps keep their usual digraph meaning. A groupoid is *connected* if its underlying digraph is connected.

Example 4.4 The tree groupoid \mathbb{I}_n has n objects $\{1, 2, \dots, n\}$ and n^2 arrows $\{(p, q) \mid 0 \leq p, q \leq n\}$ where $s(p, q) = p$, $t(p, q) = q$, $(p, q) \diamond (q, r) = (p, r)$, and $(p, q)^{-1} = (q, p)$. Note that $\mathbb{I}_2 \cong \mathbb{I}$. We also write $\mathbb{I}(X)$ for the tree groupoid on a set of objects X . The underlying digraph of \mathbb{I}_n is complete.

The name *tree groupoid* comes from the fact that a subset of arrows which form a spanning tree in the underlying digraph generate the whole groupoid using composition and inversion. For example, taking the subset $X_n = \{(1, p) \mid 2 \leq p \leq n\}$, we have $(q, r) = (q, 1)^{-1} \diamond (1, r)$.

The *product* $\mathbb{C} \times \mathbb{D}$ of groupoids \mathbb{C}, \mathbb{D} has objects $C_0 \times D_0$, arrows $C_1 \times D_1$, and composition $(a_1, b_1) \diamond (a_2, b_2) = (a_1 \diamond a_2, b_1 \diamond b_2)$, so that $(a, b)^{-1} = (a^{-1}, b^{-1})$.

Example 4.5 If \mathbb{G} is a group, considered as a one-object groupoid, and \mathbb{I}_n is a tree groupoid, then $\mathbb{C} = \mathbb{G} \times \mathbb{I}_n$ may be thought of as the groupoid with n objects $\{1, 2, \dots, n\}$ and $n^2|G|$ arrows $\{(p, g, q) \mid g \in G, 1 \leq p, q \leq n\}$, with $t(p, g, q) = p$, $h(p, g, q) = q$, composition $(p, g, q) \diamond (q, h, r) = (p, gh, r)$, and inverses $(p, g, q)^{-1} = (q, g^{-1}, p)$. A generating set for \mathbb{G} is given by $\{(1, g, 1) \mid g \in X_G\} \cup X_n$ where X_G is any generating set for \mathbb{G} . Every finite, connected groupoid is isomorphic to a direct product of a group and a tree groupoid in this way, and we call such a representation a *standard connected groupoid*.

A groupoid \mathbb{A} is *abelian* if and only if all its object groups are abelian.

We now describe the construction of a free groupoid on a graph. Let D be a digraph with vertices $V = V(D)$, arcs $A^+ = A(D)$, and source and target maps $s, t : A^+ \rightarrow V$. Let $A^- = \{a^- \mid a^+ \in A^+\}$ be a copy of A^+ , and let $A = A^+ \cup A^-$. Extend s, t to A by defining $sa^- = ta^+$, $ta^- = sa^+$. Consider A as an alphabet with A^* the monoid of words in A under concatenation. A word $w = a_1^{\epsilon_1} a_2^{\epsilon_2} \dots a_k^{\epsilon_k} \in A^*$, where $\epsilon_i \in \{+, -\}$, is *composable* if $ha_i^{\epsilon_i} = ta_{i+1}^{\epsilon_{i+1}}$ for all $1 \leq i < k$. The *free groupoid* \mathbb{D} on D is defined by:

- the object set is $\text{Ob}(\mathbb{D}) = V(D)$,
- $\text{Arr}(\mathbb{D})$ is the set of all composable words in A^* ,
- $s(a_1^{\epsilon_1} a_2^{\epsilon_2} \dots a_k^{\epsilon_k}) = sa_1^{\epsilon_1}$, and $t(a_1^{\epsilon_1} a_2^{\epsilon_2} \dots a_k^{\epsilon_k}) = ta_k^{\epsilon_k}$,
- $w_1 \diamond w_2$ is the concatenation of w_1 and w_2 , defined if the result is a composable word.

A groupoid \mathbb{D} is *free* if it is isomorphic to the free groupoid on some digraph D .

A *subgroupoid* $\mathbb{S} = (S_1, S_0)$ of $\mathbb{C} = (C_1, C_0)$ is a groupoid with $S_0 \subseteq C_0$, $S_1 \subseteq C_1$, having the same source, target and composition. A subgroupoid \mathbb{S} is *full* if $\mathbb{S}(u, v) = \mathbb{C}(u, v)$ for all $u, v \in S_0$ and *wide* if $\text{Ob}(\mathbb{S}) = \text{Ob}(\mathbb{C})$. The (*connected*) *components* of \mathbb{C} are its maximal connected subgroupoids, with one component \mathbb{C}_i for each of the k connected components Γ_i of the underlying digraph. We write $\mathbb{C} = \mathbb{C}_1 \cup \dots \cup \mathbb{C}_k$. A groupoid whose components all have a single object is a *union of groups*, and is said to be *totally disconnected*. We denote by $\text{ids}(\mathbb{C})$ the wide trivial subgroupoid $\mathbb{O}(\text{Ob}(\mathbb{C}))$.

Given a wide subgroupoid $\mathbb{S} \subseteq \mathbb{C}$, we may define a relation \equiv_R on $\text{Arr}(\mathbb{C})$ by $c \equiv_R c' \Leftrightarrow c = a \diamond c'$ for some $a \in \text{Arr}(\mathbb{S})$. This is an equivalence relation since:

- $c = 1_{sc} \diamond c$, since \mathbb{S} contains all the identity loops in \mathbb{C} ,
- $c \equiv_R c' \Rightarrow c = a \diamond c' \Rightarrow a^{-1} \diamond c = c' \Rightarrow c' \equiv_R c$, so \equiv_R is symmetric,
- $(c_1 \equiv_R c_2, c_2 \equiv_R c_3) \Rightarrow (c_1 = a_1 \diamond c_2, c_2 = a_2 \diamond c_3) \Rightarrow c_1 = a_1 \diamond a_2 \diamond c_3 \Rightarrow c_1 \equiv_R c_3$.

The equivalence classes $\mathbb{S}c$ for this relation are called the *right cosets* of \mathbb{S} in \mathbb{C} .

For $u \in \text{Ob}(\mathbb{C})$ the *star* $\text{Star}(u)$ of u is the set $\{a \in \text{Arr}(\mathbb{C}) \mid sa = u\}$, the set of all arrows with source u . Similarly the *costar* $\text{Costar}(u)$ of u is the set $\{a \in \text{Arr}(\mathbb{C}) \mid ta = u\}$, the set of all arrows with target u . Note that each right coset of \mathbb{S} in \mathbb{C} is a subset of a costar. We may define a second equivalence relation \equiv_L on $\text{Arr}(\mathbb{C})$ by $c \equiv_L c' \Rightarrow c = c'a$ for some $a \in \text{Arr}(\mathbb{S})$. The equivalence classes $c\mathbb{S}$ for this relation are the *left cosets* of \mathbb{S} in \mathbb{C} , and this time each class is a subset of some star.

Example 4.6 Let $\mathbb{G} = \text{Groupoids}(G = \langle a, b \mid a^3, b^2, (ab)^2 \rangle)$, where we write e for the identity in G , and let $\mathbb{C} = \mathbb{G} \times \mathbb{I}_3$. Let \mathbb{S} be the union of $\text{Groupoids}(C_2) \times \mathbb{I}(\{1, 2\})$ with $\text{Groupoids}(C_3)$ at object 3, where $C_2 = \{e, b\}$ and $C_3 = \{e, a, a^2\}$ are subgroups of the symmetric group G . The 54 arrows in \mathbb{C} form 15 right cosets of \mathbb{S} in \mathbb{C} , as shown in the following table. Note that $|\text{Costar}(1)| = 4 = |\text{Costar}(2)|$ while $|\text{Costar}(3)| = 3$, so some cosets contain 4 arrows and some 3. Note also that the 11 arrows in \mathbb{S} are partitioned into the first, second and fourteenth coset.

$$\begin{array}{ll}
\mathbb{S}(1, e, 1) = \{(1, e, 1), (1, b, 1), (2, e, 1), (2, b, 1)\} & \mathbb{S}(1, e, 2) = \{(1, e, 2), (1, b, 2), (2, e, 2), (2, b, 2)\} \\
\mathbb{S}(1, a, 1) = \{(1, a, 1), (1, a^2b, 1), (2, a, 1), (2, a^2b, 1)\} & \mathbb{S}(1, a, 2) = \{(1, a, 2), (1, a^2b, 2), (2, a, 2), (2, a^2b, 2)\} \\
\mathbb{S}(1, a^2, 1) = \{(1, a^2, 1), (1, ab, 1), (2, a^2, 1), (2, ab, 1)\} & \mathbb{S}(1, a^2, 2) = \{(1, a^2, 2), (1, ab, 2), (2, a^2, 2), (2, ab, 2)\} \\
\mathbb{S}(1, e, 3) = \{(1, e, 3), (1, b, 3), (2, e, 3), (2, b, 3)\} & \mathbb{S}(1, a, 3) = \{(1, a, 3), (1, a^2b, 3), (2, a, 3), (2, a^2b, 3)\} \\
\mathbb{S}(1, a^2, 3) = \{(1, a^2, 3), (1, ab, 3), (2, a^2, 3), (2, ab, 3)\} & \\
\mathbb{S}(3, e, 1) = \{(3, e, 1), (3, a, 1), (3, a^2, 1)\} & \mathbb{S}(3, b, 1) = \{(3, b, 1), (3, ab, 1), (3, a^2b, 1)\} \\
\mathbb{S}(3, e, 2) = \{(3, e, 2), (3, a, 2), (3, a^2, 2)\} & \mathbb{S}(3, b, 2) = \{(3, b, 2), (3, ab, 2), (3, a^2b, 2)\} \\
\mathbb{S}(3, e, 3) = \{(3, e, 3), (3, a, 3), (3, a^2, 3)\} & \mathbb{S}(3, b, 3) = \{(3, b, 3), (3, ab, 3), (3, a^2b, 3)\}
\end{array}$$

A subgroupoid \mathbb{N} of \mathbb{C} is *normal* in \mathbb{C} , written $\mathbb{N} \trianglelefteq \mathbb{C}$, if $\text{Ob}(\mathbb{N}) = \text{Ob}(\mathbb{C})$ and $a^{-1}\mathbb{N}(u)a \trianglelefteq \mathbb{C}(v)$ for all $(a : u \rightarrow v) \in \mathbb{C}(u, v)$.

Example 4.7 Let $\mathbb{C} = \mathbb{G} \times \mathbb{I}_n$, as in Example 4.5 with $\mathbb{G} = \text{Groupoids}(G)$.

If $\mathbb{N} \trianglelefteq \mathbb{G}$, then we may construct:

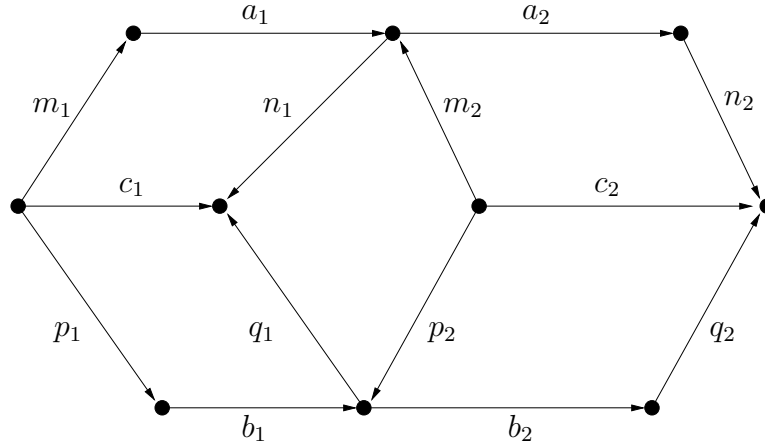
- a normal subgroupoid $\text{Groupoids}(\mathbb{N}) = \mathbb{N} \trianglelefteq \mathbb{G}$,
- for each partition $\pi = \pi_1 \cup \dots \cup \pi_k$ of $\{1, 2, \dots, n\}$ into k parts, a normal subgroupoid $(\mathbb{N} \times \mathbb{I}(\pi_1)) \cup \dots \cup (\mathbb{N} \times \mathbb{I}(\pi_k))$.

The two extreme cases of the second construction are when π has n singleton parts, giving the totally disconnected normal subgroupoid $(\mathbb{N} \times \mathbb{O}_n) \trianglelefteq \mathbb{C}$, and when π has a single part, giving a connected normal subgroupoid $(\mathbb{N} \times \mathbb{I}_n) \trianglelefteq \mathbb{C}$.

When \mathbb{N} is a normal subgroupoid of \mathbb{C} it is *not* in general the case that left cosets coincide with right cosets. A different equivalence relation is therefore required on $\text{Arr}(\mathbb{C})$ in order to be able to define a *quotient groupoid*. The following material is taken from Higgins [38]. Define a relation \equiv on $\text{Arr}(\mathbb{C})$ by $c \equiv c' \Leftrightarrow c = m \diamond c' \diamond n$ for some $m, n \in \text{Arr}(\mathbb{N})$. This is an equivalence relation since

- $c = 1_{sc} \diamond c \diamond 1_{tc}$ since \mathbb{N} contains all the identity loops in \mathbb{C} ,
- $c \equiv c' \Rightarrow c = m \diamond c' \diamond n \Rightarrow m^{-1} \diamond c \diamond n^{-1} = c' \Rightarrow c' \equiv c$, so \equiv is symmetric,
- $c_1 \equiv c_2, c_2 \equiv c_3 \Rightarrow c_1 = m_1 \diamond c_2 \diamond n_1, c_2 = m_2 \diamond c_3 \diamond n_2 \Rightarrow c_1 = m_1 \diamond m_2 \diamond c_3 \diamond n_2 \diamond n_1 \Rightarrow c_1 \equiv c_3$.

Note that equivalent arrows have sources in the same component of \mathbb{N} and similarly for their targets, so we define an equivalence relation \equiv_0 on $\text{Ob}(\mathbb{C})$ by $u \equiv_0 v$ if u, v are in the same component of \mathbb{N} . We denote the \equiv_0 -class of $u \in \text{Ob}(\mathbb{C})$ by \bar{u} and the \equiv -class of $c \in \text{Arr}(\mathbb{C})$ by \bar{c} . The quotient groupoid $\mathbb{Q} = \mathbb{C}/\mathbb{N}$ has $\text{Ob}(\mathbb{Q}) = \text{Ob}(\mathbb{C})/\equiv_0$ and $\text{Arr}(\mathbb{Q}) = \text{Arr}(\mathbb{C})/\equiv$. Source and target are given by $s\bar{c} = \overline{sc}$, $t\bar{c} = \overline{tc}$ (it is clear that s, t are well-defined). Arrows \bar{c}_1, \bar{c}_2 are defined to be composable if there exist $a_1 \equiv c_1, a_2 \equiv c_2$ with $a_1 \diamond a_2$ defined in \mathbb{C} , in which case $\bar{c}_1 \diamond \bar{c}_2$ is defined to be $\overline{a_1 \diamond a_2}$. This composition is well-defined since, if there exist $b_1 \equiv c_1, b_2 \equiv c_2$ with $b_1 \diamond b_2$ defined in \mathbb{C} , then $b_1 b_2 = (p_1^{-1} m_1) a_1 (n_1 q_1^{-1} p_2^{-1} m_2) a_2 (n_2 q_2^{-1})$ where $m_i, n_i, p_i, q_i \in \text{Arr}(\mathbb{N})$, as shown in the diagram below. Since $\ell = n_1 q_1^{-1} p_2^{-1} m_2 \in \mathbb{N}(sa_2)$, normality of \mathbb{N} implies that $a_2^{-1} \ell a_2 \in \mathbb{N}(ta_2)$ so that $b_1 b_2 = (p_1^{-1} m_1) a_1 a_2 (a_2^{-1} \ell a_2 n_2 q_2^{-1})$, and $b_1 b_2 \equiv a_1 a_2$.



- Example 4.8** (a) When $N \trianglelefteq G$, $\mathbb{G} = \text{Groupoids}(G) \times \mathbb{I}_n$ and $\mathbb{N} = \text{Groupoids}(N) \times \mathbb{I}_n$, the quotient groupoid is the single-object $\text{Groupoids}(G/N)$.
- (b) When \mathbb{G} is as in (a), but \mathbb{N} is totally disconnected, with n copies of $\text{Groupoids}(N)$, the quotient groupoid is the connected $\text{Groupoids}(G/N) \times \mathbb{I}_n$.
- (c) The general case is when \mathbb{G} is as above, $\pi = \pi_1 \cup \dots \cup \pi_k$ is a partition of $\{1, 2, \dots, n\}$ into k parts, and $\mathbb{N} = (\text{Groupoids}(N) \times \mathbb{I}(\pi_1)) \cup \dots \cup (\text{Groupoids}(N) \times \mathbb{I}(\pi_k))$. Then the quotient groupoid is $\text{Groupoids}(G/N) \times \mathbb{I}_k$.

The kernel $\ker \phi$ of a groupoid morphism $\phi : \mathbb{C} \rightarrow \mathbb{D}$ is the set of arrows in C_1 which are mapped to one of the identity arrows in D_1 . Since identities are always mapped to identities, this kernel is normal in \mathbb{C} .

Example 4.9 Let $S_3 = \langle a, b \mid a^3 = b^2 = (ab)^2 = e \rangle$ be the symmetric group with normal subgroup $C_3 = \langle a \rangle$, and let $C_2 = \langle c \mid c^2 = e \rangle$. There is a groupoid morphism from $\mathbb{C} = \text{Groupoids}(S_3) \times \mathbb{I}\{u, v, w\}$ to $\mathbb{D} = \text{Groupoids}(C_2) \times \mathbb{I}\{x, y\}$, mapping u, v to x and w to y , and killing C_3 , defined on generators by

$$(u, e, v) \mapsto (x, e, x), \quad (u, e, w) \mapsto (x, e, y), \quad (u, a, u) \mapsto (x, e, x), \quad (u, b, u) \mapsto (x, c, x).$$

The kernel $\text{Groupoids}(C_3) \times \mathbb{I}\{u, v\} \cup \text{Groupoids}(C_3) \times \mathbb{I}\{w\}$ has two components, and $\mathbb{C}/(\ker \phi) \cong \mathbb{D}$.

If $a \in \mathbb{C}(u)$ and $c \in \mathbb{C}(u, v)$ then the conjugate $a^c \in \mathbb{C}(v)$ is defined to be $c^{-1} \diamond a \diamond c$.

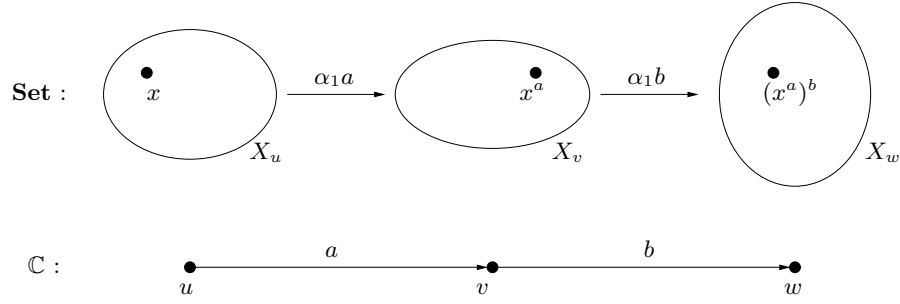
We now consider actions of a groupoid \mathbb{C} . We restrict to the case when \mathbb{C} is connected since there is a clear extension to the general case.

For \mathbb{C} a groupoid, a \mathbb{C} -set-system (or, by abuse of language, a \mathbb{C} -set) is a functor α from \mathbb{C} to **Set**, mapping arrows to bijections. So, for $(a : u \rightarrow v) \in \text{Arr}(\mathbb{C})$, there are sets $\alpha_0 u = X_u$, $\alpha_0 v = X_v$ and a bijection $\alpha_1 a : X_u \rightarrow X_v$. We also call α an *action* of \mathbb{C} on $\bigsqcup_{u \in \text{Ob}(\mathbb{C})} X_u$. If $(b : v \rightarrow w)$ is a second arrow in \mathbb{C} and $\alpha_0 w = X_w$, then, since α preserves composition, we have

$$\alpha_1(a \diamond b) = (\alpha_1 a) * (\alpha_1 b) = (\alpha_1 b) \circ (\alpha_1 a) : X_u \rightarrow X_w.$$

For $x \in X_u$ we denote, in the usual way, $(\alpha_1 a)(x)$ by x^a , and then the condition becomes $(x^a)^b = x^{a \diamond b}$.

A similar notion applies to sets with structure. For example, \mathbb{C} -graphs are functors from \mathbb{C} to the groupoid of (combinatorial) graphs and their isomorphisms.

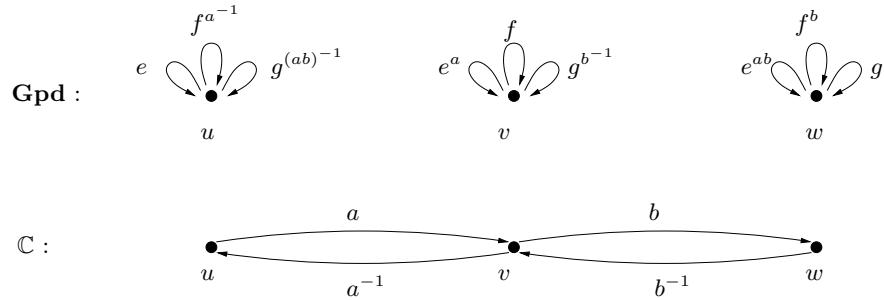


Example 4.10 Let Γ be a connected graph with automorphism group $A = \text{Aut}\Gamma$. Let Δ be the graph consisting of n copies of Γ , which we may consider as a graph-system with n components. The appropriate groupoid to consider is $\mathbb{A} = \mathbf{Groupoids}(A) \times \mathbb{I}_n$, which has an obvious action on Δ . It is reasonable to consider \mathbb{A} to be the automorphism gadget of Δ , rather than its wreath product automorphism group $S_n \wr A$.

A \mathbb{C} -group-system (or \mathbb{C} -group) provides, for each object u a group F_u and, for each $(a : u \rightarrow v)$, an isomorphism of groups $\alpha_1 a : F_u \rightarrow F_v$. As usual, we write f^a for $(\alpha_1 a)(f)$ when $f \in F$. The group structure has to be preserved so, as well as $(f^a)^b = f^{a \circ b}$, we require $(e_u)^a = e_v$ and $(f_1 f_2)^a = (f_1^a)(f_2^a)$.

A \mathbb{C} -module is a \mathbb{C} -group in which the F_u are all abelian.

A \mathbb{C} -groupoid-system is a functor α from \mathbb{C} to \mathbf{Gpd} , where now there are groupoids $\alpha_0 u = \mathbb{B}_u$, $\alpha_0 v = \mathbb{B}_v$ and an invertible functor $\alpha_1 a : \mathbb{B}_u \rightarrow \mathbb{B}_v$. As a simple case, note that a \mathbb{C} -group determines a \mathbb{C} -groupoid on replacing each F_u by $\mathbb{F}_u = \mathbf{Groupoids}(F_u)$, taking u as the single object. Thus a \mathbb{C} -module may be considered as an abelian \mathbb{C} -groupoid. In these cases f^a is defined when f is a loop at u and then f^a is a loop at v . Here is a picture showing part of the structure.



A particular example, when $\mathbb{C} = \mathbf{Groupoids}(G) \times \mathbb{I}_n$ and $N \trianglelefteq G$, is given by taking $F_u \cong N$ for all $u \in \text{Ob}(\mathbb{C})$ and the action to be conjugation:

$$(p, h, p)^{(p, g, q)} = (q, g^{-1}, p)(p, h, p)(p, g, q) = (q, g^{-1} h g, q). \quad (30)$$

This will provide our first example of a crossed module of groupoids.

4.2 Automorphisms of Groupoids

An automorphism of a category \mathbb{C} is a functor $\alpha : \mathbb{C} \rightarrow \mathbb{C}$ which is an isomorphism. Let \mathbb{C} be the connected groupoid with object set $U = \{u_1, \dots, u_n\}$ and let $\{(a_p : u_1 \rightarrow u_p) \mid 2 \leq p \leq n\}$ be a generating set for a spanning tree in \mathbb{C} . If G_1 is the object group at u_1 , an automorphism of \mathbb{C} is obtained on choosing

- $\pi \in \text{Symm}(U)$, permuting the objects in U ,
- $\gamma \in \text{Aut } G$, permuting the elements of G_1 ,
- $\{(b_p : u_1 \rightarrow u_p) \mid 2 \leq p \leq n\}$, replacing the a_p in the tree.

Thus there are in total $n! \times |\text{Aut } G| \times |G|^{n-1}$ automorphisms of \mathbb{C} .

We now analyse the automorphisms of a standard connected groupoid. For G a group, $\mathbb{G} = \text{Groupoids}(G)$, let $\mathbb{C} = \mathbb{G} \times \mathbb{I}_n$ with objects $\{1, \dots, n\}$; arrows $\{(q, g, r) \mid g \in G, 1 \leq q, r \leq n\}$; composition $(p, h, q) \diamond (q, g, r) = (p, hg, r)$; identities (p, e, p) where e is the identity in G ; and inverses $(q, g, r)^{-1} = (r, g^{-1}, q)$. If G has generating set $\Gamma_G = \{g_1, \dots, g_\ell\}$ then the groupoid is generated by sets

$$\Gamma_p = \{(p, g_k, p) \mid g_k \in \Gamma_G\} \cup \{(p, e, q) \mid q \neq p\},$$

where the second set forms a spanning tree T_p in the underlying digraph of G . The remaining arrows are given as the composites:

$$\begin{aligned} (p, g, p) &= (p, g_{k_1}, p)(p, g_{k_2}, p) \dots (p, g_{k_j}, p) \quad \text{when } g = g_{k_1}g_{k_2} \dots g_{k_j} \in G, \quad g_{k_i} \in \Gamma_G, \\ (q, g, r) &= (q, e, p)^{-1}(p, g, p)(p, e, r). \end{aligned}$$

An automorphism of \mathbb{C} will be specified by giving the images of the arrows in one of the Γ_p .

There are three sets of automorphisms which generate the group $A = \text{Aut}(\mathbb{C})$.

- (1) For π a permutation in the symmetric group S_n we define an automorphism α_π by

$$\alpha_\pi(q, g, r) = (\pi q, g, \pi r).$$

- (2) We may apply an automorphism κ of G to the loops at object 1, giving an automorphism α_κ of \mathbb{C} which fixes the objects, where

$$\alpha_\kappa(1, g, 1) = (1, \kappa g, 1), \quad \alpha_\kappa(1, e, q) = (1, e, q).$$

It follows that $\alpha_\kappa(q, g, r) = (q, \kappa g, r)$, so α_κ applies κ to all the hom-sets at once.

- (3) The hom-set $\mathbb{C}(1, q)$ provides a regular representation for G with action $(1, g, q)^b = (1, gb, q)$. For each $q \neq p$ choose $b_q \in G$ and map the arrow $(p, e, q) \in T_p$ to (p, b_q, q) . The n -tuple $\mathbf{b} = (b_1, \dots, b_p = e, \dots, b_n)$ determines an automorphism $\alpha_{p, \mathbf{b}}$ of \mathbb{C} , fixing the objects, where

$$\alpha_{p, \mathbf{b}}(p, g, p) = (p, g, p), \quad \alpha_{p, \mathbf{b}}(p, e, r) = (p, b_r, r).$$

For $\mathbf{b} \in G^n$, the n -fold direct product of G with itself, we generalise the $\alpha_{p, \mathbf{b}}$ by defining

$$\alpha_{\mathbf{b}} : \mathbb{C} \rightarrow \mathbb{C}, \quad (q, g, r) \mapsto (q, b_q^{-1}gb_r, r).$$

This map is a homomorphism since

$$(p, b_p^{-1} h b_q, q)(q, b_q^{-1} g b_r, r) = (p, b_p^{-1} (h g) b_r, r).$$

Furthermore, there is a homomorphism $\theta : G^n \rightarrow \text{Aut } \mathbb{C}$, $\mathbf{b} \mapsto \alpha_{\mathbf{b}}$, since

$$(\alpha_{\mathbf{b}} * \alpha_{\mathbf{c}})(q, g, r) = (q, c_q^{-1} b_q^{-1} g b_r c_r, r) = \alpha_{\mathbf{bc}}(q, g, r).$$

For $\mathbf{z} \in \ker \theta$ we require $g z_r = z_q g$ for all $g \in G$, $1 \leq q, r \leq n$. It follows that \mathbf{z} is a constant vector (z, z, \dots, z) with $z \in Z(G)$, the centre of G . When $\mathbf{g} = (g, g, \dots, g)$ is an arbitrary constant vector in G^n , we see that $\alpha_{\mathbf{g}}$ is the type (1) conjugation automorphism $\alpha_{(\wedge g)}$. We denote by \hat{G} the diagonal subgroup in G^n , put $\hat{Z} = \ker \theta$, and define $Q = G^n / \hat{Z}$.

There are actions of both S_n and $\text{Aut } G$ on G^n , where

$$\mathbf{b}^\pi = \pi \mathbf{b} = (b_{\pi^{-1}1}, \dots, b_{\pi^{-1}ip}, \dots, b_{\pi^{-1}n}), \quad \mathbf{b}^\kappa = \kappa \mathbf{b} = (\kappa b_1, \dots, \kappa b_p, \dots, \kappa b_n), \quad (31)$$

and these actions commute, giving an action of $S_n \times \text{Aut } G$ on G^n . We denote by G_p^n the subset $\{\mathbf{b} \in G^n \mid b_p = e\}$ and note that $\alpha_{\mathbf{b}} = \alpha_{p, \mathbf{b}}$ when $\mathbf{b} \in G_p^n$. Note also that G_p^n is closed under multiplication in G^n ; that $G_p^n \cong G^{n-1}$; that the kernel of θ restricted to G_p^n is trivial; and that S_n and $\text{Aut } G$ act trivially on \hat{Z} .

We now investigate composites of the set

$$\Gamma_A = \{\alpha_\pi \mid \pi \in S_n\} \cup \{\alpha_\kappa \mid \kappa \in \text{Aut } G\} \cup \{\alpha_{\mathbf{b}} \mid \mathbf{b} \in G^n\}.$$

In keeping with the use of right actions, we write $\alpha * \beta$ for the composite mapping $\beta \circ \alpha$. It is straightforward to verify the following identities.

Lemma 4.11 *Pairs of automorphisms in Γ_A compose as follows.*

$$\begin{aligned} (\alpha_\pi * \alpha_\rho)(q, g, r) &= \alpha_{\pi * \rho}(q, g, r) = ((\pi * \rho)q, g, (\pi * \rho)r), \\ (\alpha_\kappa * \alpha_\lambda)(q, g, r) &= \alpha_{\kappa * \lambda}(q, g, r) = (q, (\kappa * \lambda)g, r), \\ (\alpha_{\mathbf{b}} * \alpha_{\mathbf{c}})(q, g, r) &= \alpha_{\mathbf{bc}}(q, g, r) = (q, (b_q c_q)^{-1} g (b_r c_r), r), \\ (\alpha_\pi * \alpha_\kappa)(q, g, r) &= (\alpha_\kappa * \alpha_\pi)(q, g, r) = (\pi q, \kappa g, \pi r), \\ (\alpha_{\mathbf{b}} * \alpha_\pi)(q, g, r) &= (\alpha_\pi * \alpha_{\pi \mathbf{b}})(q, g, r) = (\pi q, b_q^{-1} g b_r, \pi r), \\ (\alpha_\kappa * \alpha_{\mathbf{b}})(q, g, r) &= (\alpha_{\kappa^{-1} \mathbf{b}} * \alpha_\kappa)(q, g, r) = (q, b_q^{-1} (\kappa g) b_r, r). \end{aligned}$$

Proof: The fifth isomorphism is the least obvious one. When $\pi q = q'$, $\pi r = r'$ we obtain

$$\alpha_{\pi \mathbf{b}}(q', g, r') = \left(q', (b_{\pi^{-1}q'})^{-1} g (b_{\pi^{-1}r'}), r' \right) = (\pi q, b_q^{-1} g b_r, \pi r).$$

□

It is clear that the group A_1 generated by the α_π is isomorphic to the symmetric group S_n ; that the group A_2 generated by the α_κ is isomorphic to $\text{Aut } G$; and that the group A_3 generated by the $\alpha_{\mathbf{b}}$ is isomorphic to G^{n-1} .

It is clear that the join $A_{1,2}$ of A_1 and A_2 in the automorphism group $\text{Aut } \mathbb{C}$ of \mathbb{C} is isomorphic to $A_1 \times A_2$. We denote by $A_{1,3}, A_{2,3}$ the joins of A_1, A_3 and A_2, A_3 respectively.

Proposition 4.12 *The groups $A_{1,3}, A_{2,3}, \text{Aut } \mathbb{C}$ are isomorphic to the following semidirect products.*

(1) $A_{2,3} \cong \text{Aut } G \ltimes G^{n-1}$, where the action on G_1^n is defined in equation (31).

(2) $A_{1,3} \cong S_n \ltimes Q$, using the action in (31), so that $(\kappa, \mathbf{b})^\pi = (\kappa, \pi \mathbf{b})$.

(3) $\text{Aut } \mathbb{C} \cong (S_n \times \text{Out } G) \ltimes Q$, using the same action as in (2).

Proof: The sixth identity in Lemma 4.11 shows that every element of $A_{2,3}$ has the form $\alpha_\kappa * \alpha_{1,\mathbf{b}}$. We define an isomorphism, $\theta_1 : A_{2,3} \rightarrow \text{Aut } G \ltimes G^{n-1}$, by $\alpha_\kappa \mapsto (\kappa, 1)$, $\alpha_{1,\mathbf{b}} \mapsto (1, \mathbf{b})$. Then we check that $\theta_1(\alpha_\kappa * \alpha_\lambda) = \theta_1(\alpha_{\kappa * \lambda}) = (\kappa * \lambda, 1)$, that $\theta_1(\alpha_{1,\mathbf{b}} * \alpha_{1,\mathbf{c}}) = \theta_1(\alpha_{\mathbf{bc}}) = (1, \mathbf{bc})$, and that

$$\theta_1(\alpha_{1,\kappa^{-1}\mathbf{b}} * \alpha_\kappa) = (1, \kappa^{-1}\mathbf{b})(\kappa, 1) = (\kappa, \kappa^{-1}\mathbf{b}^\kappa) = (\kappa, \mathbf{b}) = \theta_1(\alpha_\kappa * \alpha_{1,\mathbf{b}}).$$

For the second isomorphism we use Lemma 4.11 to obtain

$$\alpha_\pi * \alpha_{1,\mathbf{b}} * \alpha_\rho * \alpha_{1,\mathbf{c}} = \alpha_\pi * \alpha_\rho * \alpha_{\rho\mathbf{b}} * \alpha_{\mathbf{c}} = \alpha_{\pi * \rho} * \alpha_{(\rho\mathbf{b})\mathbf{c}}. \quad (32)$$

Iterating this procedure, we see that every word in the generators of $A_{1,3}$ has a normal form $\alpha_\pi * \alpha_{\mathbf{b}}$. We define $\theta'_2 : S_n \ltimes G^n \rightarrow A_{1,3}$, $(\pi, \mathbf{b}) \mapsto \alpha_\pi * \alpha_{\mathbf{b}}$. This is a homomorphism since, by (32),

$$\theta'_2(\pi, \mathbf{b})\theta'_2(\rho, \mathbf{c}) = \alpha_{\pi * \rho} * \alpha_{(\rho\mathbf{b})\mathbf{c}} = \theta'_2(\pi * \rho, \mathbf{b}^\rho \mathbf{c}).$$

The kernel of θ'_2 is $\{((\text{id}), \mathbf{z}) \mid \mathbf{z} \in \hat{Z}\}$, where (id) denotes the identity permutation, so there is an isomorphism $\theta_2 : A_{1,3} \rightarrow S_n \ltimes Q$ mapping generators $\alpha_\pi * \alpha_{1,\mathbf{b}}$ to $(\pi, \hat{Z}\mathbf{b})$.

To prove the third isomorphism we define $\theta'_3 : (S_n \times \text{Aut } G) \ltimes G^n \rightarrow \text{Aut } \mathbb{C}$ mapping $((\pi, \kappa), \mathbf{b})$ to $\alpha_\pi * \alpha_\kappa * \alpha_{\mathbf{b}}$. The formulae in Lemma 4.11 again show that every automorphism can be written in this form. The kernel of θ'_3 is generated by elements $(((\text{id}), \text{id}), \mathbf{z})$ for $\mathbf{z} \in \hat{Z}$ and elements $(((\text{id}), \wedge g), \mathbf{g}^{-1})$, so there is an isomorphism $\theta_3 : \text{Aut } \mathbb{C} \rightarrow (S_n \times \text{Out } G) \ltimes Q$ mapping $\alpha_\pi * \alpha_\kappa * \alpha_{1,\mathbf{b}}$ to $((\pi, (\text{Inn } G)\bar{\kappa}), \hat{Z}\mathbf{g}\mathbf{b})$ where $\kappa = (\wedge g)\bar{\kappa} \in \text{Aut } G$. \square

We conclude this subsection by observing that an automorphism $\alpha = (\alpha_1, \alpha_0) : \mathbb{C} \rightarrow \mathbb{C}$ is specified by giving

- the permutation α_0 on the objects;
- an automorphism, written $\bar{\alpha}$, of the object group G_1 , so that $\alpha_1(1, g, 1) = \bar{\alpha}g$;
- images $\alpha_1(1, e, q) = (\alpha_0 1, \alpha_q, \alpha_0 q)$, with $\alpha_q \in G$, $2 \leq q \leq n$, for the tree T_1 .

Then α acts on a typical arrow by

$$\alpha_1(q, g, r) = (\alpha_0 q, \alpha_q^{-1}(\alpha g)\alpha_r, \alpha_0 r).$$

It is clear how to replace object 1 by an arbitrary object p in these formulae.

The next type of groupoid to consider is the disjoint union of m copies of a connected groupoid. More generally, let X be some structure with an equivalence relation \equiv which is preserved by endomorphisms. We argue that the correct automorphism structure for X is not the group of automorphisms but rather the *groupoid of automorphisms* $\mathbb{A} = \text{Aut } X$ whose objects are the \equiv -classes $[x]$. The object group at class $[x]$ is the group of automorphisms $\text{Aut } [x]$ of the elements in $[x]$. The hom-set $\mathbb{A}([x], [y])$ comprises all isomorphisms from $[x]$ to $[y]$ (if there are any). A connected component of \mathbb{A} has the form $\text{Aut } [x] \times \mathbb{I}_m$, where m is the number of components of X isomorphic to $[x]$.

A simple example of this situation is a graph Γ with m connected components all isomorphic to Γ_0 , having automorphism groupoid $\text{Aut } \Gamma \cong \text{Aut } \Gamma_0 \times \mathbb{I}_m$.

So let \mathbb{B} be the disjoint union $\mathbb{C} \times \mathbb{O}_n$ of m copies of $\mathbb{C} = G \times \mathbb{I}_n$, and denote the i -th copy by \mathbb{C}_i and its elements by $(q, g, r)_i$. The equivalence relation here is the connectedness of the underlying digraph. There should be no confusion if we take $\text{Ob}(\text{Aut } \mathbb{B})$ to be $\{1, \dots, m\}$. From our previous discussion we see that $\text{Aut } \mathbb{B} \cong \text{Aut } \mathbb{C} \times \mathbb{I}_m \cong \text{Groupoids}((S_n \times \text{Out } G) \ltimes Q) \times \mathbb{I}_m$. Generators for this groupoid are provided by:

- $\{(1, \alpha_r, 1)\}$, where the α_r are generators for $\text{Aut } \mathbb{C}$;
- $\{(1, \epsilon_i, i) \mid 2 \leq i \leq m\}$, where $\epsilon_i : \mathbb{C}_1 \rightarrow \mathbb{C}_i$, $(q, g, r)_1 \mapsto (q, g, r)_i$.

An automorphism of \mathbb{B} which does not interchange the components is obtained by choosing an automorphism for each component. These form a group isomorphic to $(\text{Aut } \mathbb{C})^m$, and the automorphism group of \mathbb{B} is the wreath product $S_m \wr \text{Aut } G$ with action $(\kappa_1, \dots, \kappa_m)^\pi = (\kappa_{(\pi^{-1}1)}, \dots, \kappa_{(\pi^{-1}m)})$.

Example 4.13 *We shall see groupoids of the form $\mathbb{B} = \mathbb{G} \times \mathbb{O}_m$, a disjoint union of groups, when we come to consider crossed modules. Clearly $\text{Aut } \mathbb{B} \cong (\text{Aut } G) \times \mathbb{I}_m$ and $\text{Aut } \mathbb{B} \cong S_m \wr \text{Aut } G$.*

The final case to consider is that of an arbitrary groupoid \mathbb{G} , whose connected components \mathbb{G}_j form m classes $[\mathbb{G}_i]$ of isomorphic connected groupoids with m_i components in class $[\mathbb{G}_i]$. The automorphism groupoid $\text{Aut } \mathbb{G}$ has $\sum_{i=1}^m m_i$ objects and m connected components, with the i -th component being isomorphic to $\text{Aut } \mathbb{G}_i \times \mathbb{I}_{m_i}$.

4.3 Natural Transformations

Functors are related by *natural transformations*. If $\alpha, \beta : \mathbb{C} \rightarrow \mathbb{D}$ are functors, then a natural transformation $\tau : \alpha \rightarrow \beta$ is determined by a function $\tau : \text{Ob}(\mathbb{C}) \rightarrow \text{Arr}(\mathbb{D})$, $u \mapsto \tau_u$, such that for every arrow $(a : u \rightarrow v) \in \mathbb{C}$ the following diagram commutes.

$$\begin{array}{ccc} \alpha u & \xrightarrow{\tau_u} & \beta u \\ \alpha a \downarrow & \searrow \tau a & \downarrow \beta a \\ \alpha v & \xrightarrow{\tau_v} & \beta v \end{array}$$

Notice that commutativity of the diagram enables us to extend τ to a function $\text{Arr}(\mathbb{C}) \rightarrow \text{Arr}(\mathbb{D})$, where τa is this diagonal arrow and $\tau 1_u = \tau_u$ for each object u . Notice also that $[s\tau_{u_1}, s\tau_{u_2}, \dots, s\tau_{u_n}]$ is a permutation of the list of objects, otherwise one or more of these diagrams would be undefined.

Restricting to groupoids, so that arrows are invertible, we have $\tau_v = (\alpha a)^{-1} \diamond (\tau_u) \diamond (\beta a)$, so τ is defined if we are given, for each component of \mathbb{C} , the image of one object. Furthermore, when α, β are surjective, every transformation is invertible with $(\tau^{-1})_u = (\tau_u)^{-1}$,

$$\begin{array}{ccc} \beta u & \xrightarrow{(\tau^{-1})_u = \tau_u^{-1}} & \alpha u \\ \beta a \downarrow & & \downarrow \alpha a \\ \beta v & \xrightarrow{(\tau^{-1})_v = \tau_v^{-1}} & \alpha v \end{array}$$

In this case the list $[t\tau_{u_1}, t\tau_{u_2}, \dots, t\tau_{u_n}]$ is a second permutation of the objects in \mathbb{D} .

When $\mathbb{C} = \mathbb{D}$ and α, β are isomorphisms, we obtain our first example of a *homotopy*, with τ being considered as a homotopy from α to β , as displayed in the following diagram. The significant feature of τ is that it lifts from one level to the next.

$$\begin{array}{ccc} C_1 & \xrightarrow{\alpha_1, \beta_1} & C_1 \\ s \downarrow & \nearrow \tau & \downarrow t \\ C_0 & \xrightarrow{\alpha_0, \beta_0} & C_0 \end{array}$$

Natural transformations compose in the obvious way. If κ is a third functor from \mathbb{C} to \mathbb{D} , and if $\sigma : \beta \rightarrow \kappa$ is a second natural transformation, then we obtain the diagrams:

$$\begin{array}{ccccc} \alpha u & \xrightarrow{\tau_u} & \beta u & \xrightarrow{\sigma_u} & \kappa u \\ \alpha a \downarrow & \searrow \tau a & \downarrow \beta a & \searrow \sigma a & \downarrow \kappa a \\ \alpha v & \xrightarrow{\tau_v} & \beta v & \xrightarrow{\sigma_v} & \kappa v \end{array} \qquad \begin{array}{ccc} C_1 & \xrightarrow{\alpha_1, \beta_1, \kappa_1} & C_1 \\ s \downarrow & \nearrow \tau, \sigma & \downarrow t \\ C_0 & \xrightarrow{\alpha_0, \beta_0, \kappa_0} & C_0 \end{array}$$

We obtain a composite natural transformation $\tau \diamond \sigma : \alpha \rightarrow \kappa$ where

$$\begin{aligned} (\tau \diamond \sigma)u &= \tau_u \diamond \sigma_u, \\ (\tau \diamond \sigma)a &= (\tau a) \diamond (\sigma v) = (\tau_u) \diamond (\sigma a) = (\tau a) \diamond (\beta a)^{-1} \diamond (\sigma a). \end{aligned}$$

We thus obtain a groupoid whose objects are isomorphisms and whose arrows are natural transformations. When $\mathbb{C} = \mathbb{D}$ and we obtain the *automorphism groupoid* of \mathbb{C} .

Theorem 4.14 *Let $\mathbb{C} = G \times \mathbb{I}_n$. Then the automorphism groupoid $\text{Aut } \mathbb{C}$ of \mathbb{C} has*

- $n!.|\text{Aut } G|.|G|^{n-1}$ objects (automorphisms),
- $(n!)^2.|\text{Aut } G|.|G|^{2n-1}$ arrows (natural transformations),
- degree $|Z(G)| = |G|/|\text{Inn } G|$,
- $\text{Out } G$ connected components, with $n!.|\text{Inn } G|.|G|^{n-1}$ objects in each component.

Proof: An automorphism is specified by choosing a permutation of the objects; an automorphism of the group G ; and, for each arrow in the spanning tree, a choice of one of the G arrows between the appropriate vertices.

When specifying a natural transformation $\tau : \alpha \rightarrow \beta$, the sources of the τ_u determine one of the $n!$ permutations of the objects, and the targets determine a second permutation, and there are then $|G|$ choices for each τ_{u_q} . The automorphism α is specified on the object group by choosing an automorphism from $\text{Aut } G$ and on each of the $n - 1$ arrows $(1, e, q)$ in the tree by choosing one of the $|G|$ arrows $(\alpha_0 1, \alpha_q, \alpha_0 q)$.

The degree is determined by the number of loops at the identity automorphism id . If τ is such a loop and $\tau_1 = z$, then $z^{-1}az = a$ for every generator a of G , so $z \in Z(G)$. Each τ_q is then determined by $\tau_q = \alpha_q^{-1} \diamond z \diamond \alpha_q$.

The number of arrows whose source is id is $n!.|G|^n$ since there are $n!$ choices for the targets of the τ_q , and the $|G|$ choices for each τ_q . Dividing this number by the degree gives the number of objects in the component containing id . The automorphism group acts on the objects of the automorphism groupoid by right multiplication, permuting the components, so the components are isomorphic and their number is the obvious quotient. \square

Corollary 4.15 *When \mathbb{C} is a group G considered as a one-object groupoid, the automorphism groupoid has $|\text{Aut } G|$ objects; $|\text{Aut } G|.|G|$ natural transformations; $|\text{Out } G|$ components; $|\text{Inn } G|$ objects in each component; and degree $|Z(G)|$.*

4.4 Admissible and coadmissible sections

These two types of section are related to special cases of natural transformations between automorphisms of a groupoid.

For \mathbb{C} a groupoid, an *admissible section* $H_0 : C_0 \rightarrow C_1$ is a section of the source map which composes with the target map to give a bijection on C_0 ,

$$H_0 * s = \text{id}_{C_0}, \quad h_0 := H_0 * t : C_0 \rightarrow C_0 \text{ is a bijection.}$$

Note that if $\tau : \text{id}_{\mathbb{C}} \rightarrow h$ is a natural transformation, then τ is an admissible section.

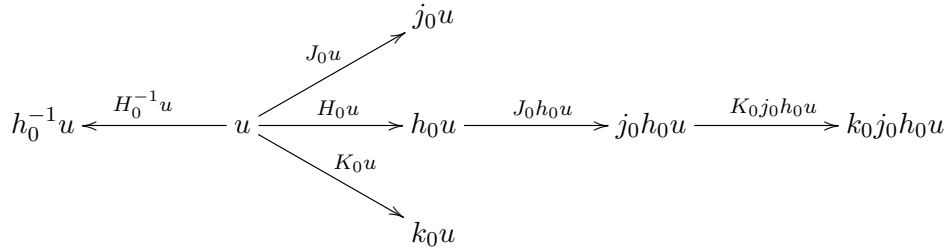
The set of admissible sections $M(\mathbb{C})$ of \mathbb{C} is a group with multiplication

$$(H_0 \star J_0)u := (H_0u)(J_0tH_0u) = ((H_0u)(J_0h_0u) : u \rightarrow (h_0 * j_0)u)$$

where $j_0 = J_0 * t$. It is straightforward to verify that this product is associative,

$$(H_0 \star J_0 \star K_0)u = ((H_0u)(J_0h_0u)(K_0j_0h_0u) : u \rightarrow (h_0 * j_0 * k_0)u).$$

Here is a sketch showing the situation:



The identity admissible section is I_0 where $I_0u = 1_u$ for all $u \in C_0$. The inverse of H_0 is the admissible section where

$$H_0^{-1}u = (H_0h_0^{-1}u)^{-1}, \quad \text{so} \quad H_0^{-1}h_0u = (H_0u)^{-1} \quad \text{and} \quad H_0^{-1} * t = h_0^{-1}.$$

Note that the map $M(\mathbb{C}) \rightarrow \text{Symm}(C_0)$, $H_0 \mapsto H_0 * t$ is a homomorphism.

Similarly, a *coadmissible section* $H_0 : C_0 \rightarrow C_1$ is a section of the target map which composes with the source map to give a bijection of C_0 . For a picture of this situation just reverse all the arrows in the diagram above. The multiplication is given by

$$(H_0 \star J_0)u := (J_0sH_0u)(H_0u) = ((J_0h_0u)(H_0u) : (h_0 * j_0)u \rightarrow u)$$

where $h_0 = H_0 * s$, $j_0 = J_0 * s$. Note that if $\tau : h \rightarrow \text{id}_{\mathbb{C}}$ is a natural transformation, then τ is a coadmissible section.

We now generalise these notions. For g_0, h_0 a pair of permutations of the objects of a groupoid \mathbb{C} , a (g_0, h_0) -section $H_0 : C_0 \rightarrow C_1$ is a map which composes with the source and target maps to give g_0 and h_0 respectively:

$$g_0 = H_0 * s, \quad h_0 = H_0 * t.$$

Note that if $\tau : g \rightarrow h$ is a natural transformation between automorphisms of \mathbb{C} , then τ is an admissible section. A (g_0, h_0) -section is also called an *admissible g_0 -section* and a *coadmissible h_0 -section*.

We have constructed a groupoid $\mathbb{S} = \mathbb{S}(\mathbb{C})$ having the automorphisms of \mathbb{C} as objects and the (g_0, h_0) -sections as the elements of the hom-set $\mathbb{S}(g_0, h_0)$. Composition in \mathbb{S} is defined by

$$(H_0 : g_0 \rightarrow h_0) \diamond (J_0 : h_0 \rightarrow j_0)u := (H_0u : g_0u \rightarrow h_0u) \diamond (J_0u : h_0u \rightarrow J_0u).$$

We now define a multiplication on the set of admissible g_0 -sections $M_g(\mathbb{C})$ of \mathbb{C} . Note that there is a multiplication on the permutations of C_0 given in terms of the standard composition by $h_0 \star j_0 := h_0 * g_0^{-1} * j_0$, such that g_0 is the identity and h_0 has inverse $g_0 * h_0^{-1} * g_0$. We define the product on $M_g(\mathbb{C})$ by

$$(H_0 \star J_0)u := (H_0u)(J_0g_0^{-1}tH_0u) = ((H_0u)(J_0g_0^{-1}h_0u) : u \rightarrow (h_0 \star j_0)u)$$

where $j_0 = J_0 * t$. It is straightforward to verify that this product is associative,

$$(H_0 \star J_0 \star K_0)u = ((H_0u)(J_0g_0^{-1}h_0u)(K_0g_0^{-1}j_0g_0^{-1}h_0u) : u \rightarrow (h_0 \star j_0 \star k_0)u).$$

Here is a sketch showing the situation:

$$\begin{array}{ccccccc} & & & j_0u & & & \\ & & \nearrow^{J_0u} & & \searrow_{K_0u} & & \\ g_0u & \xrightarrow{H_0u} & h_0u & \xrightarrow{J_0g_0^{-1}tH_0u} & j_0g_0^{-1}h_0u & \xrightarrow{K_0g_0^{-1}tJ_0g_0^{-1}tH_0u} & k_0g_0^{-1}j_0g_0^{-1}h_0u \\ & & \searrow_{K_0u} & & & & \\ & & & k_0u & & & \end{array}$$

The identity admissible section is I_0 where $I_0u = 1_{g_0u}$ for all $u \in C_0$. The inverse of H_0 is the admissible section where

$$H_0^{-1}u = (H_0h_0^{-1}g_0u)^{-1}, \quad \text{so} \quad H_0^{-1}g_0^{-1}h_0u = (H_0u)^{-1} \quad \text{and} \quad H_0^{-1} * t = g_0 * h_0^{-1} * g_0.$$

Note that the map from $M_g(\mathbb{C})$ to $\text{Symm}(C_0)$ with the \star product, mapping H_0 to $H_0 * t$ is a group homomorphism.

4.5 Groupoid Actions

An action of a groupoid \mathbb{C} on a groupoid \mathbb{B} is usually defined in the case where \mathbb{B} is a union of groups and has the same objects as \mathbb{C} . Then, when $(c : w \rightarrow x) \in \mathbb{C}$ and $(b : w \rightarrow w) \in \mathbb{B}$, we have $(b^c : x \rightarrow x)$. So c does not act by permuting the arrows of \mathbb{B} , but by providing an isomorphism from $\mathbb{B}(w)$ to $\mathbb{B}(x)$. A particular case of this situation is when \mathbb{B} is a totally discrete subgroupoid of \mathbb{C} and the action is conjugation, $b^c = c^{-1}bc$. We now give an alternative definition of an action, using the automorphism groupoid $\text{Aut } \mathbb{B}$, which does not require \mathbb{B} to be totally disconnected, and which *does* provide a permutation of the arrows.

Definition 4.16 *An action of a groupoid \mathbb{C} on a groupoid \mathbb{B} is a groupoid morphism $\alpha = (\alpha_0, \alpha_1) : \mathbb{C} \rightarrow \text{Aut } \mathbb{B}$.*

????? This seems to be saying that \mathbb{C} has m objects while \mathbb{B} has mn objects, so how can α_0 be a bijection?

This means that, when α_0 is a bijection and $\mathbb{C} = \mathbb{H} \times \mathbb{I}_m$ with $\mathbb{H} = \text{Groupoids}(H)$, then \mathbb{B} has m isomorphic components, $\mathbb{B}_i \cong \mathbb{G} \times \mathbb{I}_n$ say, and $\alpha_1(p, h, q) = (\alpha_0 p, \alpha_{p,q} h, \alpha_0 q)$ where $\alpha_{p,q} : \mathbb{B}_{\alpha_0 p} \rightarrow \mathbb{B}_{\alpha_0 q}$ is an isomorphism.

4.6 Conjugation in groupoids

Definition 4.17 *Let $(c : p \rightarrow q)$, where $p \neq q$, be an arrow in a connected groupoid $\mathbb{C} = \mathbb{G} \times \mathbb{I}_n$. Then conjugation by c , written $\wedge c$, is an automorphism of \mathbb{C} where:*

- p, q are interchanged, and the remaining objects are fixed;
- the loops at p are interchanged with those at q ,

$$(p, g, p) \mapsto (q, c^{-1}gc, q), \quad (q, g, q) \mapsto (p, cgc^{-1}, p);$$

- the hom-set $\mathbb{C}(p, q)$ is interchanged with $\mathbb{C}(q, p)$,

$$(p, g, q) \mapsto (q, c^{-1}gc^{-1}, p), \quad (q, g, p) \mapsto (p, cgc, q);$$

- the rest of the costar at p is interchanged with that at q ,

$$(r, g, p) \mapsto (r, gc, q), \quad (r, g, q) \mapsto (r, gc^{-1}, p);$$

- the rest of the star at p is interchanged with that at q ,

$$(p, g, r) \mapsto (q, c^{-1}gr, r), \quad (q, g, r) \mapsto (p, cgr, r);$$

- the remaining arrows are unchanged.

There are a number of cases to consider when checking that composition is preserved by this mapping, for example

$$(r, g, p)^c (p, h, q)^c = (r, gc, q)(q, c^{-1}hc^{-1}, p) = (r, (gh)c^{-1}, p) = (r, gh, q)^c.$$

We now express $\wedge c$ as a word in our standard sets of generators.

[to be continued]

[Add the corresponding formula for the case $p = q$.]

It is *not* the case that the map $\wedge : \mathbb{C} \rightarrow \text{Aut}, \mathbb{C}$ is a groupoid morphism. This is clear just by considering the images of the objects, where the symmetric group is acting.

Lemma 4.18

$$\wedge(cd) = (\wedge c) * (\wedge d) * (\wedge c) = (\wedge d) * (\wedge c) * (\wedge d)$$

Proof: To be added.

□

5 Crossed modules of groupoids

5.1 Basic definitions

Let $\mathbb{C}_1 = (C_1, C_0)$ be a groupoid and $\mathbb{C}_2 = (C_2, C_0)$ a union of groups with the same object set, and let $\mathbb{C}_2, \mathbb{C}_1$ act upon themselves by conjugation:

$$a_1^a = a^{-1}a_1a, \quad c_1^c = c^{-1} \diamond c_1 \diamond c,$$

defined when a_1, a are loops in \mathbb{C}_2 at the same object and when $sc = sc_1 = tc_1$ in \mathbb{C}_1 .

[But is this the correct notion of conjugacy?]

A *pre-crossed module of groupoids* $\mathcal{C} = (\gamma : \mathbb{C}_2 \rightarrow \mathbb{C}_1)$ consists of a morphism of groupoids $\gamma = (\gamma, \text{id})$ (abusing notation), the *boundary* of \mathcal{C} , pictured as:

$$\mathbb{C}_2 = \begin{array}{ccc} C_2 & \xrightarrow{\gamma} & C_1 \\ s \downarrow & & \downarrow s \\ C_0 & \xrightarrow{\text{id}} & C_0 \end{array} \quad \text{or, more simply,} \quad C_2 \xrightarrow{\gamma} C_1 \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{t} \end{array} C_0,$$

together with an action of \mathbb{C}_1 on \mathbb{C}_2 such that γ is a \mathbb{C}_1 -morphism. So $\gamma * s = s$, $\gamma * t = t$ and, for all $a \in \text{Arr}(\mathbb{C}_2)$ and $c \in \text{Arr}(\mathbb{C}_1)$,

$$\mathbf{X1:} \quad \gamma(a^c) = c^{-1}(\gamma a)c \quad \text{when} \quad sa = ta = sc.$$

The pre-crossed module \mathcal{C} is a *crossed module of groupoids* if it also satisfies

$$\mathbf{X2:} \quad a_1^{\gamma a} = a_1^a \quad \text{for all} \quad a_1, a \in \mathbb{C}_2(u), \quad u \in C_0.$$

Note that, when both axioms are satisfied, the restriction $(\gamma_u : \mathbb{C}_2(u) \rightarrow \mathbb{C}_1(u))$ is a crossed module of groups for all $u \in C_0$.

A *morphism of pre-crossed modules* $\alpha : \mathcal{C} \rightarrow \mathcal{D}$, where $\mathcal{D} = (\delta : \mathbb{D}_2 \rightarrow \mathbb{D}_1)$, is a triple $(\alpha_2, \alpha_1, \alpha_0)$, where $(\alpha_2, \alpha_0) : \mathbb{C}_2 \rightarrow \mathbb{D}_2$ and $(\alpha_1, \alpha_0) : \mathbb{C}_1 \rightarrow \mathbb{D}_1$ are morphisms of groupoids satisfying

$$\alpha_2 * \delta = \gamma * \alpha_1, \quad \alpha_2(a^c) = (\alpha_2 a)^{\alpha_1 c},$$

making the following diagram commute:

$$\begin{array}{ccccc} C_2 & \xrightarrow{\gamma} & C_1 & \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{t} \end{array} & C_0 \\ \alpha_2 \downarrow & & \alpha_1 \downarrow & & \downarrow \alpha_0 \\ D_2 & \xrightarrow{\delta} & D_1 & \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{t} \end{array} & D_0 \end{array}$$

When \mathcal{C}, \mathcal{D} are crossed modules, α is a morphism of crossed modules.

Example 5.1 Let $N \trianglelefteq G$ and $\mathbb{G} = \mathbf{Groupoids}(G)$. Take $\mathbb{C}_1 = \mathbb{G} \times \mathbb{I}_n$ and let $\mathbb{C}_2 = \mathbb{N} \times \mathbb{O}_n$ be the totally disconnected subgroupoid consisting of n copies of $\mathbb{N} = \mathbf{Groupoids}(N)$. Then \mathbb{C}_1 acts on \mathbb{C}_2 by conjugation, and $\mathcal{C} = (\iota : \mathbb{C}_2 \rightarrow \mathbb{C}_1)$ is a *conjugation crossed module*, where ι is the inclusion map.

5.2 Homotopies of a crossed module of groupoids

This subsection is intended to cover section 2 of Brown and İçen [16].

Let $g, h : \mathcal{C} = (\gamma : \mathbb{C}_2 \rightarrow \mathbb{C}_1) \rightarrow \mathcal{D} = (\delta : \mathbb{D}_2 \rightarrow \mathbb{D}_1)$ be morphisms of crossed modules. A (g, h) -homotopy $H : g \simeq h$ is a pair of functions (H_0, H_1) such that

- $H_0 : C_0 \rightarrow C_1$ is a (g, h) -section: $H_0 * s = g_0$, $H_0 * t = h_0$,
- $H_1 : C_1 \rightarrow C_2$ satisfies: $H_1 * t = h_1 * t$, $H_1(cc') = (H_1c)^{h_1c'}(H_1c')$,
- for all $(c : u \rightarrow v) \in \mathbb{C}_1$ and $(a : v \rightarrow v) \in \mathbb{C}_2$, δH_1c and $H_1\gamma a$ measure the divergence from commutativity of the following squares (in the second square dashed lines denote arrows in \mathbb{C}_2),

$$\begin{array}{ccc}
 g_0u & \xrightarrow{g_1c} & g_0v \\
 \downarrow H_0u & & \downarrow H_0v \\
 h_0u & \xrightarrow{h_1c} & h_0v
 \end{array}
 \qquad
 \begin{array}{ccc}
 g_0v & \xrightarrow{g_2a} & g_0v \\
 \vdots H_0v & & \vdots H_0v \\
 h_0v & \xrightarrow{h_2a} & h_0v
 \end{array}$$

$$\delta H_1c = (h_1c)^{-1}(H_0u)^{-1}(g_1c)(H_0v), \quad H_1\gamma a = (h_2a)^{-1}(g_2a)^{H_0v}. \quad (33)$$

When H is a (g, h) -homotopy, we call H_1 a (g, h) -derivation. We shall usually be concerned with the case $\mathcal{C} = \mathcal{D}$, so that g, h are automorphisms. In the special case that $g = \text{id}_{\mathcal{C}}$ we call H a *free homotopy* and H_1 a *free derivation*. In another special case, when $H_0u = 1_u$ for all $u \in C_0$ we call H a *homotopy over the identity* and H_1 a *derivation over the identity*. A free derivation over the identity is simply called a *derivation*.

Consider the case when $\mathcal{C} = (\gamma : \mathbb{C}_2 \rightarrow \mathbb{C}_1)$ is connected, so $\mathbb{C}_1 = \mathbb{G}_1 \times \mathbb{I}_n$, $\mathbb{C}_2 = \mathbb{G}_2 \times \mathbb{O}_n$, $\mathbb{G}_1 = \text{Groupoids}(G_1)$, $\mathbb{G}_2 = \text{Groupoids}(G_2)$, and $\mathcal{X} = (\bar{\gamma} : G_2 \rightarrow G_1)$ is a crossed module of groups, where $\gamma(j, a, j) = (j, \bar{\gamma}a, j)$. Because of the multiplication rule for H_1 , we may define an h -derivation by specifying the images of a generating set (just as we did for automorphisms). Thus an h -derivation of \mathcal{C} is determined by

- a derivation $\chi : G_1 \rightarrow G_2$ for the crossed module \mathcal{X} , so $\chi(cc') = (\chi c)^{c'}(\chi c')$,
- a choice of images $H_1(1, e, q) = (h_0q, a_q, h_0q)$, $2 \leq q \leq n$ for arrows in the tree T_1 .

The h -derivation for \mathcal{X} associated to χ is given by $\psi c = \chi h_1c$, so that $\psi(cc') = (\psi c)^{h_1c'}(\psi c')$, and we define $H_1(1, c, 1) = (h_01, \chi h_1c, h_01)$. Applying the multiplication rule, we find

$$\begin{aligned}
 H_1(q, e, 1) &= (H_1(1, e, q)^{-1})^{h(q, e, 1)} = (h_01, a_q^{-1}, h_01), \\
 H_1(1, c, q) &= (h_01, \chi h_1c, h_01)^{h(1, e, q)}(h_0q, a_q, h_0q) = (h_0q, (\chi h_1c)a_q, h_0q), \\
 H_1(q, c, q) &= (h_01, a_q^{-1}, h_01)^{h(1, c, q)}(h_01, \chi h_1c, h_01)^{h(1, e, q)}(h_0q, a_q, h_0q) = (h_0q, (a_q^{-1})^{h_1c}(\chi h_1c)a_q, h_0q), \\
 H_1(q, c, 1) &= (h_0q, (a_q^{-1})^{h_1c}(\chi h_1c)a_q, h_0q)^{h(q, e, 1)}(h_01, a_q^{-1}, h_01) = (h_01, (a_q^{-1})^{h_1c}(\chi h_1c), h_01), \\
 H_1(q, c, r) &= (h_0r, (a_q^{-1})^{h_1c}(\chi h_1c)a_r, h_0r).
 \end{aligned}$$

We may check the multiplication rule as follows:

$$\begin{aligned}
& H_1((p, c, q)(q, c', r)) \\
&= (h_0q, (a_p^{-1})^{h_1c}(\chi h_1c)a_q, h_0q)^{h(q, c', r)} (h_0r, (a_q^{-1})^{h_1c'}(\chi h_1c')a_r, h_0r) \\
&= (h_0r, ((a_p^{-1})^{h_1c}(\chi h_1c)a_q)^{h_1c'}, h_0r)(h_0r, (a_q^{-1})^{h_1c'}(\chi h_1c')a_r, h_0r) \\
&= (h_0r, (a_p^{-1})^{(h_1c)(h_1c')}(\chi h_1c)^{h_1c'} a_q^{h_1c'} (a_q^{-1})^{h_1c'} (\chi h_1c')a_r, h_0r) \\
&= (h_0r, (a_p^{-1})^{h_1(cc')}(\chi h_1c)^{h_1c'} (\chi h_1c')a_r, h_0r) \\
&= (h_0r, (a_p^{-1})^{h_1(cc')}(\chi h_1(cc')a_r, h_0r) \\
&= H_1(p, cc', r).
\end{aligned}$$

[It is the case that the multiplication rule holds for this H_1 , but we probably also need to check the axioms in (33).]

When H, K are two (g, h) -homotopies determined by derivations χ, ζ of \mathcal{X} , by $H_1(1, e, q) = (h_0q, a_q, h_0q)$, and by $K_1(1, e, q) = (h_0q, b_q, h_0q)$, the Whitehead product $H_1 \star K_1$ is given, as usual, by the formula

$$(H_1 \star K_1)x = (K_1x)(H_1x)(K_1\gamma h_1^{-1}H_1x). \quad ???$$

[The following calculation is probably wrong, and needs to be checked!]

For the loops at 1 this gives

$$(H_1 \star K_1)(1, c, 1) = (h_01, (\chi \star \zeta)h_1c, h_01),$$

while the image of $(1, e, q)$ is given by

$$\begin{aligned}
(H_1 \star K_1)(1, e, q) &= (h_0q, b_q, h_0q)(h_0q, a_q, h_0q)(K_1\gamma(h_0q, a_q, h_0q)) \\
&= (h_0q, b_q a_q, h_0q)(K_1(h_0q, \gamma a_q, h_0q)) \\
&= (h_0q, b_q a_q, h_0q)(h_0q, (b_q^{-1})^{\gamma a_q}(\zeta \gamma a_q)b_q, h_0q) \\
&= (h_0q, a_q(\zeta \gamma a_q)b_q, h_0q).
\end{aligned}$$

6 Crossed complexes

The main references for this section are the book [19] which covers much of the material in the papers [8, 11, 12, 13, 14].

Definition 6.1 A (many object) crossed complex (\mathcal{C}, χ) is a sequence of morphisms of groupoids with common object set C_0 ,

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\chi_{n+1}} & C_n & \xrightarrow{\chi_n} & C_{n-1} & \xrightarrow{\chi_{n-1}} & \cdots \xrightarrow{\chi_3} C_2 \xrightarrow{\chi_2} C_1 \\ & & \downarrow t=h & & \downarrow t=h & & \downarrow t=h & \downarrow t=h \\ & & C_0 & & C_0 & & C_0 & C_0 \end{array} \quad (34)$$

where:

- $\mathbb{C}_n = (C_n, C_0)$ is a groupoid for $n \geq 1$ and, for $n \geq 2$, \mathbb{C}_n is totally disconnected,
- for $n \geq 3$, \mathbb{C}_n is abelian, and is a \mathbb{C}_1 -module such that the image of χ_2 acts trivially,
- for $n \geq 2$, $((\chi_n, \text{id}) : \mathbb{C}_n \rightarrow \mathbb{C}_{n-1})$ is a groupoid morphism which preserves the action of \mathbb{C}_1 (in the case $n = 2$ the action of \mathbb{C}_1 on itself is conjugation) so, for $a \in C_1, c_n \in C_n$, we have $\chi_n(c_n^a) = (\chi c_n)^a$,
- $((\chi_2, \text{id}) : \mathbb{C}_2 \rightarrow \mathbb{C}_1)$ is a crossed module of groupoids, so $\{a_2\}^{\chi_2 b_2} = b_2^{-1} a_2 b_2$,
- for $n \geq 3$ the composite $\chi_n * \chi_{n-1}$ is the zero map.

The repeated C_0 in equation (34) is rather untidy, so the diagram of a crossed complex is usually simplified, as shown for \mathbb{C} and \mathbb{D} in (35).

For convenience in the general formulae to be considered later, we specify source and target maps on objects by defining $s, t : C_0 \rightarrow C_0$ to be the identity map on the set of objects.

A morphism of crossed complexes $\phi : (\mathcal{C}, \chi) \rightarrow (\mathcal{D}, \delta)$ is a family of groupoid morphisms $\{(\phi_n, \phi_0) : \mathbb{C}_n \rightarrow \mathbb{D}_n\}_{n \geq 1}$

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\chi_{n+1}} & C_n & \xrightarrow{\chi_n} & C_{n-1} & \xrightarrow{\chi_{n-1}} & \cdots \xrightarrow{\chi_3} C_2 \xrightarrow{\chi_2} C_1 \xrightarrow[t]{h} C_0 \\ & & \downarrow \phi_n & & \downarrow \phi_{n-1} & & \downarrow \phi_2 & \downarrow \phi_1 & \downarrow \phi_0 \\ \cdots & \xrightarrow{\delta_{n+1}} & D_n & \xrightarrow{\delta_n} & D_{n-1} & \xrightarrow{\delta_{n-1}} & \cdots \xrightarrow{\delta_3} D_2 \xrightarrow{\delta_2} D_1 \xrightarrow[t']{h'} D_0 \end{array} \quad (35)$$

compatible with the morphisms and actions on \mathcal{C}, \mathcal{D} , so that

$$\phi_n * \delta_n = \chi_n * \phi_{n-1} \quad \text{and} \quad \phi_n(c_n^a) = (\phi_n c_n)^{\phi_1 a} \quad \forall c_n \in C_n, n \geq 2, a \in C_1.$$

The category **XComp** has crossed complexes as objects and their morphisms as arrows.

Example 6.2 The unit interval crossed complex $(\mathcal{I}, 1)$ has for \mathbb{I}_1 the groupoid \mathbb{I} of Example 4.3 and, for $n \geq 2$, \mathbb{I}_n is the trivial groupoid $\text{ids}(\mathcal{I})$. The morphisms are all inclusion morphisms.

When, for $n > m$, \mathbb{C}_n is the trivial groupoid $\text{ids}(\mathcal{C})$ of Example 4.2, and χ_n is the inclusion morphism, we say that \mathcal{C} is *m-truncated*. This means that the structures above level m may be ignored, so that a 1-truncated crossed complex is effectively a groupoid (for example (\mathbb{I}, ι)), and a 2-truncated crossed complex is a crossed module of groupoids. We denote the *m*-truncated subcrossed complex of \mathcal{C} by \mathcal{C}^m . (We could refer here to the *m*-th *skeleton functor* sk^m .)

6.1 Tensor Product of Crossed Complexes

In [19] the tensor product $\mathcal{C} \otimes \mathcal{D}$ of crossed complexes \mathcal{C}, \mathcal{D} is defined using a universal bimorphism, but we will not discuss bimorphisms of crossed complexes here (not yet, anyhow).

Definition 6.3 *Let $(\mathcal{C}, \chi), (\mathcal{D}, \delta)$ be crossed complexes. Then $(\mathcal{C} \otimes \mathcal{D}, \partial)$ is the crossed complex generated by elements $c \otimes d$ in dimension $m+n$, where $c \in C_m, d \in D_n$, with the following defining relations (plus the laws for crossed complexes).*

Source and target:

$$s(c \otimes d) = sc \otimes sd, \quad t(c \otimes d) = tc \otimes td.$$

Action axioms:

$$c \otimes d^{d_1} = (c \otimes d)^{(tc \otimes d_1)} \quad \text{when } n \geq 2, d_1 \in D_1, \quad c^{c_1} \otimes d = (c \otimes d)^{(c_1 \otimes td)} \quad \text{when } m \geq 2, c_1 \in C_1.$$

Product axioms:

$$\begin{aligned} c \otimes (dd_1) &= \begin{cases} (c \otimes d)^{(tc \otimes d_1)}(c \otimes d_1) & \text{if } n = 1, m \geq 1, \\ (c \otimes d)(c \otimes d_1) & \text{otherwise,} \end{cases} \\ (cc_1) \otimes d &= \begin{cases} (c_1 \otimes d)(c \otimes d)^{(c_1 \otimes td)} & \text{if } m = 1, n \geq 1, \\ (c \otimes d)(c_1 \otimes d) & \text{otherwise.} \end{cases} \end{aligned}$$

Boundary map:

$$\partial_{m+n}(c \otimes d) = \begin{cases} (tc \otimes d)^{-1}(c \otimes sd)^{-1}(sc \otimes d)(c \otimes td) & \text{if } m = n = 1, \\ c \otimes \delta_n d & \text{if } m = 0, n \geq 2, \\ \chi_m c \otimes d & \text{if } m \geq 2, n = 0, \\ (c \otimes \delta_n d)^{-1}(tc \otimes d)^{-1}(sc \otimes d)(c \otimes td) & \text{if } m = 1, n \geq 2, \\ (c \otimes td)^{(-1)^{m+1}}((c \otimes sd)^{(tc \otimes d)})^{(-1)^m}(\chi_m c \otimes d) & \text{if } n = 1, m \geq 2, \\ (\chi_m c \otimes d)(c \otimes \delta_n d)^{(-1)^m} & \text{if } n \geq 2, m \geq 2. \end{cases}$$

Note that $(\mathcal{C} \otimes \mathcal{D})_0 = C_0 \times D_0$, so we may write $u \otimes x$ as (u, x) when $u \in C_0, x \in D_0$.

The groupoid $((\mathcal{C} \otimes \mathcal{D})_1, (\mathcal{C} \otimes \mathcal{D})_0)$ is isomorphic to $C_1 \# D_1$, the groupoid coproduct of $\mathcal{C}^1 \times \text{ids}(\mathcal{D}^1)$ and $\text{ids}(\mathcal{C}^1) \times \mathcal{D}^1$. Every element of $C_1 \# D_1$ is uniquely expressible in one of the following normal forms.

- (i) An identity arrow $(1_u, 1_x)$.
- (ii) A *generating arrow* $(c, 1_x)$ or $(1_u, d)$, where $c \in C_1, x \in D_0, u \in C_0, d \in D_1$ and c, d are not identities. We write $\text{arr}(c, 1_x) = c, \text{obj}(c, 1_x) = x$ and $\text{arr}(1_u, d) = d, \text{obj}(1_u, d) = u$.
- (iii) A composite $k = k_1 k_2 \dots k_n$ ($n \geq 2$) of generating arrows in which the $\text{arr}(k_i)$ lie alternately in C_1 and D_1 (or conversely), and the odd and even products $\text{odd}(k) = \text{arr}(k_1)\text{arr}(k_3)\dots$ and $\text{even}(k) = \text{arr}(k_2)\text{arr}(k_4)\dots$ are defined in C_1 or D_1 . We define k_{C_1} to be $\text{odd}(k)$ or $\text{even}(k)$, whichever composite arrow is in C_1 , and then k_{D_1} is the other composite arrow.

6.2 Tensor product of groupoids

When \mathcal{C}, \mathcal{D} are groupoids, considered as 1-truncated crossed complexes, the tensor product $\mathcal{C} \otimes \mathcal{D}$ is 2-truncated. We have already described the groupoid at level 1, so it remains to specify $(\mathcal{C} \otimes \mathcal{D})_2$.

There is a canonical morphism $\sigma : C_1 \# D_1 \rightarrow C_1 \times D_1$ mapping k to (k_{C_1}, k_{D_1}) . The kernel of σ is the *Cartesian subgroup* $C_1 \sqcap D_1$ of $C_1 \# D_1$ consisting of all the identity arrows and all the words k such that k_{C_1} and k_{D_1} are identities. It is generated by “commutators” $[c, d]$ where, when $(c : u \rightarrow v) \in C_1$ and $(d : x \rightarrow y) \in D_1$ are not identities, $[c, d]$ is the loop at $(v, y) = (tc, td)$ given by

$$[c, d] := (c^{-1}, 1_y)(1_u, d^{-1})(c, 1_x)(1_v, d),$$

as shown in the following diagram:

$$\begin{array}{ccc} (v, y) & \xleftarrow{(1_v, d)} & (v, x) \\ (c^{-1}, 1_y) \downarrow & & \uparrow (c, 1_x) \\ (u, y) & \xrightarrow{(1_u, d^{-1})} & (u, x) \end{array}$$

These commutators satisfy the usual commutator identities:

$$\begin{aligned} [d, c] &= [c, d]^{-1}, \\ [dd_1, c] &= [d, c]^{d_1} [d_1, c], \\ [d, cc_1] &= [d, c_1] [d, c]^{c_1}, \end{aligned}$$

whenever cc_1, dd_1 are defined. Note that the action is conjugation: for example, when $(d_1 : y \rightarrow z) \in D_1$ the loop $[d, c]^{d_1}$ at (v, z) is given by

$$\begin{aligned} [d, c]^{d_1} &= (1_v, d_1^{-1})(1_v, d^{-1})(c^{-1}, 1_x)(1_u, d)(c, 1_y)(1_v, d_1) \\ &= (1_v, (dd_1)^{-1})(c^{-1}, 1_x)(1_u, dd_1)(c, 1_z) \cdot (c^{-1}, 1_z)(1_u, d_1^{-1})(c, 1_y)(1_v, d_1) \\ &= [dd_1, c][c, d_1], \end{aligned}$$

and similarly $[d, c]^{c_1} = [c_1, d][d, cc_1]$.

It turns out that $(\mathcal{C} \otimes \mathcal{D})_2 = C_1 \sqcap D_1$ with $c \otimes d := [d, c]$. We may check the product axioms as follows:

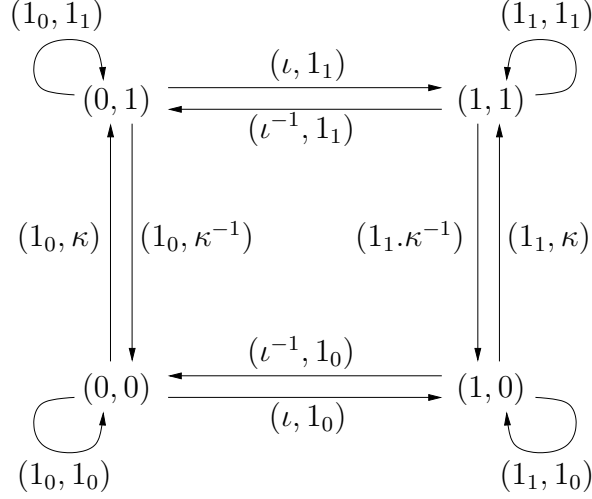
$$\begin{aligned} cc_1 \otimes d &= [d, cc_1] = [d, c_1][d, c]^{c_1} = (c_1 \otimes d)(c \otimes d)^{(c_1 \otimes td)}, \\ c \otimes dd_1 &= [dd_1, c] = [d, c]^{d_1} [d_1, c] = (c \otimes d)^{(tc \otimes d_1)}(c \otimes d_1). \end{aligned}$$

We have constructed a normal inclusion crossed module of groupoids

$$(C_1 \sqcap D_1 \longrightarrow C_1 \# D_1 \rightrightarrows C_0 \times D_0).$$

The action on loops in dimension two is given by:

$$\begin{aligned} (c \otimes d)^{(tc \otimes d_1)} &= (c \otimes dd_1)(c \otimes d_1)^{-1}, \\ (c \otimes d)^{(c_1 \otimes td)} &= (c_1 \otimes d)^{-1}(cc_1 \otimes d). \end{aligned}$$



Example 6.4 The tensor product $\mathcal{I}^{\otimes 2} = \mathcal{I} \otimes \mathcal{I}$ has four objects and eight generating arrows in dimension one, as shown in the following diagram. We relabel ι in the second factor as κ so that the “commutator” notation is not ambiguous.

The generating commutators for $I_2^{\otimes 2}(0,0)$ as:

$$\begin{aligned} k_{(0,0)} &= (\iota, 1_0)(1_1, \kappa)(\iota^{-1}, 1_1)(1_0, \kappa^{-1}) = [\iota^{-1}, \kappa^{-1}], \\ k_{(0,0)}^{-1} &= (1_0, \kappa)(\iota, 1_1)(1_1, \kappa^{-1})(\iota^{-1}, 1_0) = [\kappa^{-1}, \iota^{-1}]. \end{aligned}$$

So the vertex group $I_2^{\otimes 2}(0,0)$ is free on one generator. It is easy to see that the vertex groups at $(1,0), (1,1), (0,1)$ are respectively generated by

$$k_{(1,0)} = [\iota, \kappa^{-1}], \quad k_{(1,1)} = [\iota, \kappa], \quad k_{(0,1)} = [\iota^{-1}, \kappa].$$

Example 6.5 Consider $\mathcal{D} = \mathcal{I} \otimes \mathcal{C}$, where \mathcal{C} is a crossed module of groups with $C_0 = \{\bullet\}$. The objects in D_0 are $\{(0, \bullet), (1, \bullet)\}$, but it is convenient to replace these by $\{0, 1\}$. The generating arrows in D_1 are

$$\{((\iota, 1_\bullet) : 0 \rightarrow 1), ((\iota^{-1}, 1_\bullet) : 1 \rightarrow 0)\} \cup \{((1_0, g) : 0 \rightarrow 0), ((1_1, g) : 1 \rightarrow 1) \mid g \in C_1\},$$

where we may restrict g to be a member of a generating set for C_1 since $(1_0, g_1)(1_0, g_2) = (1_0, g_1 g_2)$.

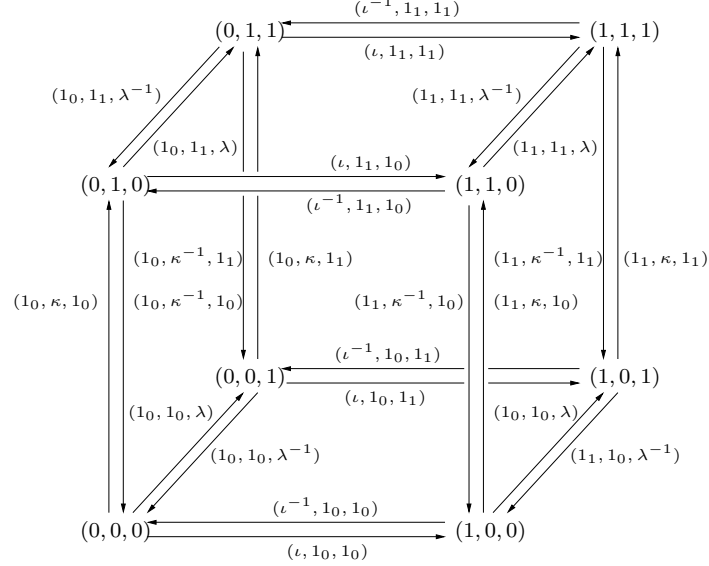
There should be no confusion if we simply write ι for $(\iota, 1_\bullet)$ and g for $(1_0, g)$, etc. A typical element of D_1 with source 0 is therefore $k = g_1 \iota g_2 \iota^{-1} g_3 \iota \dots$. As before, we define

$$\iota \otimes g = [g, \iota] = g^{-1} \iota^{-1} g \iota, \quad \text{a loop at 1.}$$

In dimension 2 we also have a contribution from elements of C_2 . For $c \in C_2$ we write c for $c \otimes 0$ and $c \otimes 1$ when the object is clear. Then a typical loop at 0 in D_2 is

$$c_1 w_1 c_2 w_2 c_3 \dots \quad \text{where } w_i \text{ is a word in the } [g, \iota].$$

Example 6.6 The tensor product $\mathcal{I}^{\otimes 3} = \mathcal{I} \otimes \mathcal{I} \otimes \mathcal{I}$ should, strictly speaking, be calculated as one of the isomorphic products $\mathcal{I}^{\otimes 2} \otimes \mathcal{I}$ or $\mathcal{I} \otimes \mathcal{I}^{\otimes 2}$. Thus we might consider, for example, the object $((0,0),0)$ and the generating arrows $(1_{(0,0)}, \iota)$ and $((1_0, \iota), 1_0)$. It is simpler to consider the 8 objects and 24 generating arrows in dimension one as triples, as shown in the following cubical diagram where, to avoid confusion, we relabel the three copies of ι as ι, κ, λ .



For dimension 2 loops at $(1,1,1)$ we have commutators from the back, top, and right faces:

$$[\iota, \kappa], \quad [\iota, \lambda], \quad [\kappa, \lambda]. \quad (36)$$

The other three faces also contribute loops at $(1,1,1)$. For example. the front face is traversed (in a clockwise direction) by

$$[\kappa, \iota]^\lambda = (1_1, 1_1, \lambda^{-1})(1_1, \kappa^{-1}, 1_0)(\iota^{-1}, 1_0, 1_0)(1_0, \kappa, 1_0)(\iota, 1_1, 1_0)(1_1, 1_1, \lambda).$$

A typical element in the vertex group $K_2((1,1,1))$ is a word in the three commutators (36) and their conjugates.

Finally, there are non-trivial vertex groups in K_3 , associated to the whole cube. Each of the eight elements $(\iota^{\pm 1} \otimes \kappa^{\pm 1} \otimes \lambda^{\pm 1}) \in K_3$ generates an infinite cyclic group at $(h\iota^{\pm 1}, h\kappa^{\pm 1}, h\lambda^{\pm 1})$. The boundary map is given by a version of the Jacobi-Hall-Witt identity for commutators,

$$[x^y, [y, z]] [y^z, [z, x]] [z^x, [x, y]] = 1,$$

which is used by Ellis in [30] to give an identity among the relators for the free abelian group on three generators. (Another relevant reference is [10].) Thus, at $K_3((1,1,1))$, we define

$$\partial_3(\iota \otimes \kappa \otimes \lambda) = [\lambda, \kappa]^{\iota^\kappa} [\kappa, \lambda] [\iota, \lambda]^{\kappa^\lambda} [\lambda, \iota] [\kappa, \iota]^{\lambda^\iota} [\iota, \kappa],$$

where

$$\begin{aligned} \partial_2[\lambda, \kappa]^{\iota^\kappa} &= (1_1, \kappa^{-1}, 1_1)(\iota^{-1}, 1_0, 1_1)(1_0, \kappa, 1_1)(1_0, 1_1, \lambda^{-1})(1_0, \kappa^{-1}, 1_0)(1_0, 1_0, \lambda)(\iota, 1_0, 1_1)(1_1, \kappa, 1_1), \\ \partial_2[\kappa, \lambda] &= (1_1, \kappa^{-1}, 1_1)(1_1, 1_0, \lambda^{-1})(1_1, \kappa, 1_0)(1_1, 1_1, \lambda), \\ \partial_2[\iota, \lambda]^{\kappa^\lambda} &= (1_1, 1_1, \lambda^{-1})(1_1, \kappa^{-1}, 1_0)(1_1, 1_0, \lambda)(\iota^{-1}, 1_0, 1_1)(1_0, 1_0, \lambda^{-1})(\iota, 1_0, 1_0)(1_1, \kappa, 1_0)(1_1, 1_1, \lambda), \\ \partial_2[\lambda, \iota] &= (1_1, 1_1, \lambda^{-1})(\iota^{-1}, 1_1, 1_0)(1_0, 1_1, \lambda)(\iota, 1_1, 1_1), \\ \partial_2[\kappa, \iota]^{\lambda^\iota} &= (\iota^{-1}, 1_1, 1_1)(1_0, 1_1, \lambda^{-1})(\iota, 1_1, 1_0)(1_1, \kappa^{-1}, 1_0)(\iota^{-1}, 1_0, 1_0)(1_0, \kappa, 1_0)(1_0, 1_1, \lambda)(\iota, 1_1, 1_1), \\ \partial_2[\iota, \kappa] &= (\iota^{-1}, 1_1, 1_1)(1_0, \kappa^{-1}, 1_1)(\iota, 1_0, 1_1)(1_1, \kappa, 1_1). \end{aligned}$$

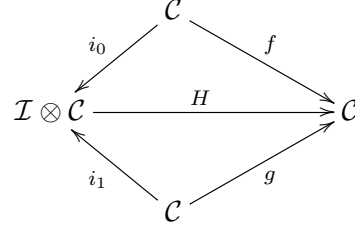
The Jacobi-Hall-Witt identity ensures that $\partial^2(\iota \otimes \kappa \otimes \lambda) = 1_{(1,1,1)}$.

Here are the formulae for the boundary maps of the tensor product $\mathcal{C} \otimes \mathcal{D}$, where $(c_1 : u \rightarrow v) \in C_1$, $(d_1 : x \rightarrow y) \in D_1$, $(c : w \rightarrow w) \in C_m$, $m \geq 2$, and $(d : z \rightarrow z) \in D_n$, $n \geq 2$.

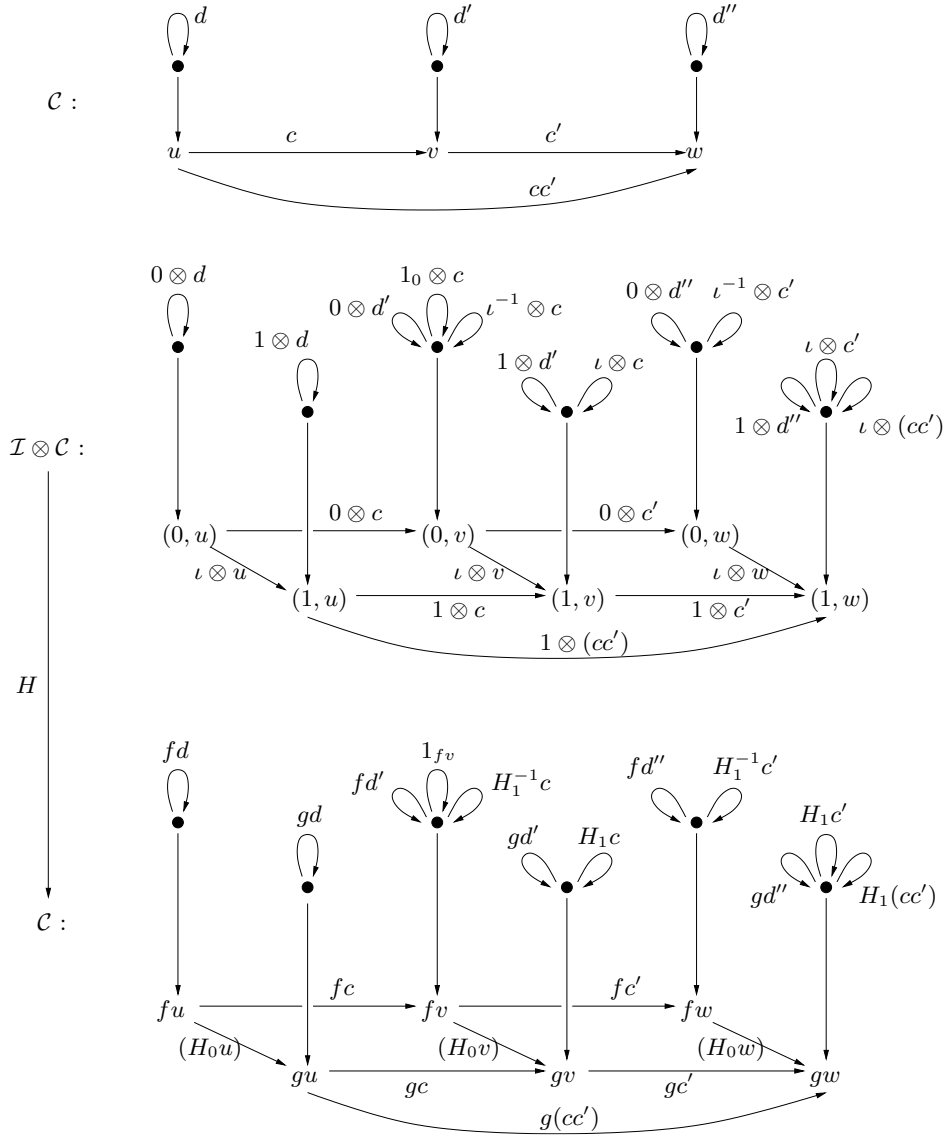
$$\begin{aligned}
\partial_{0+n}(u \otimes d) &= u \otimes \partial_n d, \\
\partial_{m+0}(c \otimes x) &= \partial_m c \otimes x, \\
\partial_2(c_1 \otimes d_1) &= (v \otimes d_1)^{-1}(c_1 \otimes x)^{-1}(u \otimes d_1)(c_1 \otimes y), \\
\partial_{1+n}(c_1 \otimes d) &= (c_1 \otimes \partial_n d)^{-1}(v \otimes d)^{-1}(u \otimes d)^{(c_1 \otimes z)}, \\
\partial_{m+1}(c \otimes d_1) &= (c \otimes y)^{(-1)^{m+1}} \left((c \otimes x)^{(w \otimes d_1)} \right)^{(-1)^m} (\partial_m c \otimes d_1), \\
\partial_{m+n}(c \otimes d) &= (\partial_m c \otimes d) + (-1)^m (c \otimes \partial_n d), \quad m, n \geq 2.
\end{aligned}$$

6.3 Homotopies between morphisms of crossed complexes

For f, g two automorphisms of \mathcal{C} , a 1-homotopy $H : f \simeq g$ is a set of maps $H_n : C_n \rightarrow C_{n+1}$, $n \geq 0$, satisfying various axioms. These axioms are most easily obtained by viewing the homotopy as a morphism $H : \mathcal{I} \otimes \mathcal{C} \rightarrow \mathcal{C}$, making the following diagram commute where $i_0 c = 0 \otimes c$ and $i_1 c = 1 \otimes c$.



Such an H comprises maps $H_{m,n} : I_m \times C_n \rightarrow C_{m+n}$, $m, n \geq 0$. Part of such a morphism is shown in the following diagram.



We now consider the maps $H_{m,n}$ for small m, n .

$$\begin{aligned}
H_{0,0} : \{0, 1\} \times C_0 &\rightarrow C_0, & (0, u) &\mapsto fu, \quad (1, u) \mapsto gu, \\
H_{0,n} : \{0, 1\} \times C_n &\rightarrow C_n, & 0 \otimes c &\mapsto fc, \quad 1 \otimes c \mapsto gc, \\
H_{1,0} : \{1_0, 1_1, \iota, \iota^{-1}\} \times C_0 &\rightarrow C_1, & 1_0 \otimes u &\mapsto 1_{fu}, \quad 1_1 \otimes u \mapsto 1_{gu}, \\
&& \iota \otimes u &\mapsto (H_0 u : fu \rightarrow gu), \\
&& \iota^{-1} \otimes u &\mapsto (H_0 u)^{-1} : gu \rightarrow fu, \\
&& & \text{(which defines the map } H_0 : C_0 \rightarrow C_1), \\
H_{m,0} : \{1_0, 1_1\} \times C_0 &\rightarrow C_m, & 1_0 \otimes u &\mapsto 1_{fu}, \quad 1_1 \otimes u \mapsto 1_{gu}, \quad m \geq 2, \\
H_{1,1} : \{1_0, 1_1, \iota, \iota^{-1}\} \times C_1 &\rightarrow C_2, & 1_0 \otimes c &\mapsto 1_{fv}, \quad 1_1 \otimes c \mapsto 1_{gv}, \\
&& \iota \otimes c &\mapsto H_1 c = [c, \iota] \in C_2(t\iota, tc) = C_2(1, v), \\
&& & \text{(which defines the map } H_1 : C_1 \rightarrow C_2).
\end{aligned}$$

We now derive some of the axioms. In $\mathcal{I} \otimes \mathcal{C}$ the arrow $\iota \otimes c$ decomposes as

$$(1_0 \otimes c)(\iota \otimes 1_v) = (\iota \otimes c) = (\iota \otimes 1_u)(1_1 \otimes c),$$

so this commuting square is mapped to a commuting square in \mathcal{C} , defining

$$H_0 : C_1 \rightarrow C_1, \quad c \mapsto (fc)(H_0 v) = (H_0 u)(gc).$$

The image under H_1 of a composite arrow is given by

$$H_1(cc') = \iota \otimes cc' = (\iota \otimes c)^{c'}(\iota \otimes c') = (H_1 c)^{(1_1 \otimes c')}(H_1 c') = (H_1 c)^{gc'}(H_1 c').$$

Similarly, applying the boundary maps,

$$\begin{aligned}
\partial(H_1 c) &= H((1_1, c^{-1})(\iota^{-1}, 1_u)(1_0, c)(\iota, 1_v)) \\
&= H_{0,1}(1, c^{-1}) H_{1,0}(\iota^{-1}, u) H_{0,1}(0, c) H_{1,0}(\iota, v) \\
&= (gc)^{-1} (H_0 u)^{-1} (fc) (H_0 v).
\end{aligned}$$

In the formulae for $H_{1,1}$ above the image of $\iota^{-1} \otimes c$ has not been defined. It may be determined on expanding $\iota^{-1} \otimes c$ as follows:

$$\begin{aligned}
1_0 \otimes c &= \iota^{-1} \otimes c = (\iota^{-1} \otimes c)(\iota \otimes c)^{(\iota \otimes v)^{-1}} \\
\Rightarrow 1_{fv} &= H(\iota^{-1} \otimes c) (H_1 c)^{(H_0 v)^{-1}} \\
\Rightarrow H_{1,1}(\iota^{-1} \otimes c) &= ((H_1 c)^{-1})^{(H_0 v)^{-1}}.
\end{aligned}$$

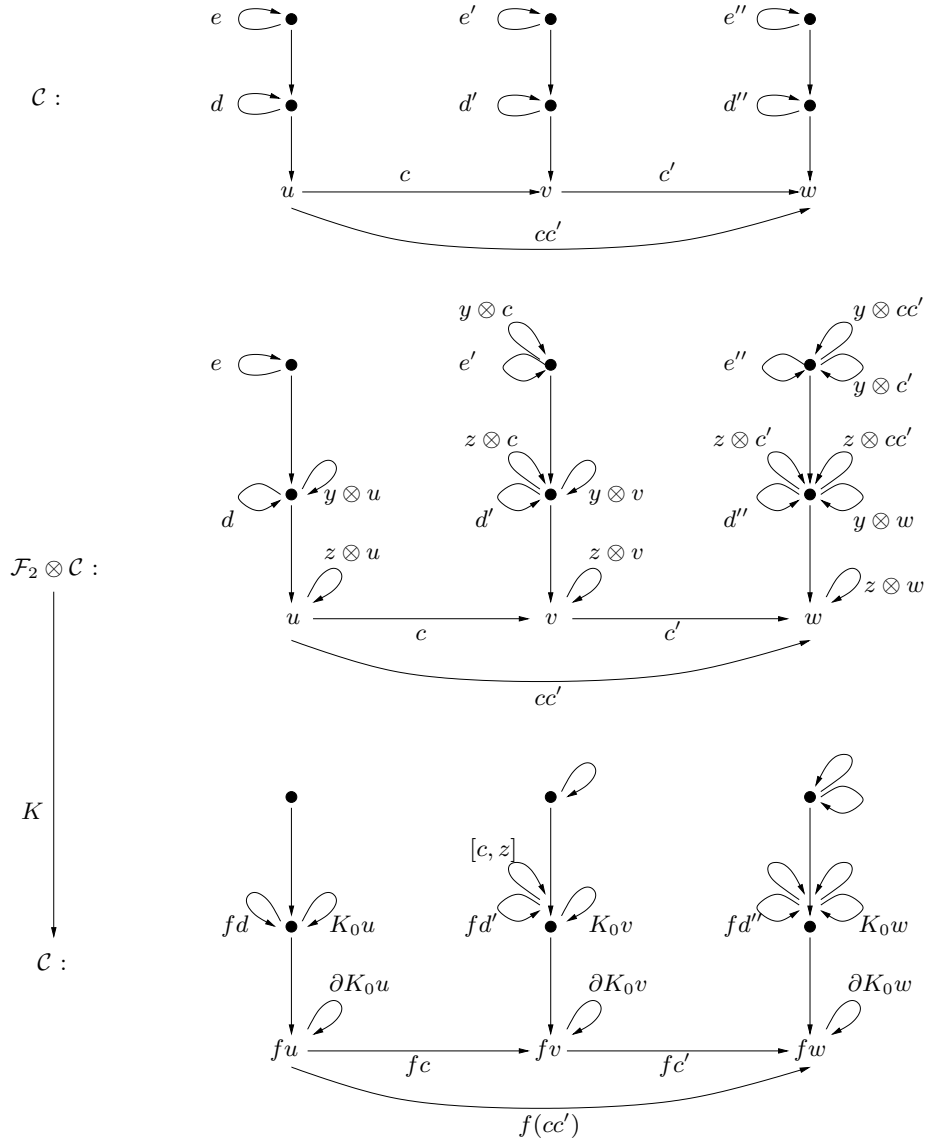
For f an automorphism of \mathcal{C} , a 2-homotopy K over f is a set of maps $K_n : C_n \rightarrow C_{n+2}$, $n \geq 0$, satisfying various axioms. These axioms are most easily obtained by viewing the homotopy as a morphism $K : \mathcal{F}_2 \otimes \mathcal{C} \rightarrow \mathcal{C}$, making the following diagram commute where \mathcal{F}_2 is the free crossed complex

$$\cdots \longrightarrow 1 \longrightarrow F_2 = \langle y \rangle \xrightarrow{\partial} F_1 = \langle z \rangle \longrightarrow \{\bullet\},$$

the boundary map is given by $\partial y = z$, the action is trivial, $y^z = y$, and $ic = \bullet \otimes c$.

$$\begin{array}{ccc}
& \mathcal{C} & \\
i \swarrow & & \searrow f \\
\mathcal{F}_2 \otimes \mathcal{C} & \xrightarrow{K} & \mathcal{C}
\end{array}$$

Such a K comprises maps $K_{m,n} : (F_2)_m \times C_n \rightarrow C_{m+n}$, $m, n \geq 0$. Part of such a morphism is shown in the following diagram where, since there is only one object in \mathcal{F}_2 , we write u for $\bullet \otimes u$.



We now consider the maps $K_{m,n}$ for small m, n .

$$\begin{aligned}
K_{0,0} : \{\bullet\} \times C_0 &\rightarrow C_0, & u &\mapsto fu, \\
K_{0,n} : \{\bullet\} \times C_n &\rightarrow C_n, & c &\mapsto fc, \\
K_{2,0} : \langle y \rangle \times C_0 &\rightarrow C_2, & y \otimes u &\mapsto K_0 u, \quad \text{where } \partial K_0 u = z \otimes u, \\
&& & \text{(which defines the map } K_0 : C_0 \rightarrow C_2), \\
K_{1,0} : \langle z \rangle \times C_0 &\rightarrow C_1, & z \otimes u &\mapsto \partial(K_0 u), \\
K_{1,1} : \langle z \rangle \times C_1 &\rightarrow C_2, & z \otimes c &\mapsto [c, z] \in C_2(v) \quad \text{where} \\
&& \partial[c, z] &= K((c^{-1}, 1_v)(1_u, z^{-1})(c, 1_u)(1_v, z)) \\
&& &= (fc)^{-1}(\partial K_0 u)^{-1}(fc)(\partial K_0 v), \\
K_{2,1} : \langle y \rangle \times C_1 &\rightarrow C_2, & y \otimes c &\mapsto K_1 c \in C_3(v) \quad \text{where } \partial K_1 c = ??? \\
&& & \text{(which defines the map } K_1 : C_1 \rightarrow C_3).
\end{aligned}$$

We now derive some of the axioms or properties of these maps.

(a)

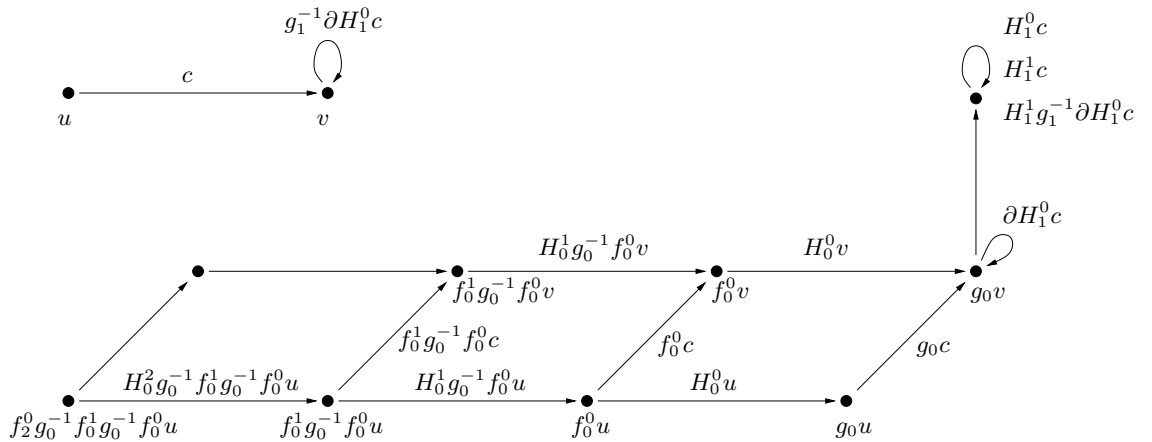
$$\begin{aligned}
\partial_3(y \otimes c) &= (y \otimes v)^{-1}(y \otimes u)^c(z \otimes c) \\
\Rightarrow \partial_3 K_1 c &= (K_0 v)^{-1}(K_0 u)^c[c, z].
\end{aligned}$$

6.4 Whitehead product of 1-homotopies

We wish to define a monoid structure on the set of homotopies to g , so let $H^0 : f^0 \simeq g$ and $H^1 : f^1 \simeq g$ be two homotopies between automorphisms of \mathcal{C} . Then we define the *Whitehead product* of H^0, H^1 to be the homotopy $H^0 \star H^1 : f^0 * g^{-1} * f^1 \simeq g$ where

$$\begin{aligned}
(H^0 \star H^1)_0(u) &= (H_0^1 g_0^{-1} f_0^0 u)(H_0^0 u), \\
(H^0 \star H^1)_1(c) &= (H_1^1 c)(H_1^0 c)(H_1^1 g_1^{-1} \partial H_1^0 c).
\end{aligned}$$

The arrows in these composite formulae are shown in the following diagram.



This product is associative: the homotopy is $H^0 \star H^1 : f^0 * g^{-1} * f^1 \simeq g$ where

$$\begin{aligned} (H^0 \star H^1 \star H^2)_0(u) &= (H_0^2 g_0^{-1} f_0^1 g_0^{-1} f_0^0 u)(H_0^1 g_0^{-1} f_0^0 u)(H_0^0 u), \\ (H^0 \star H^1 \star H^2)_1(c) &= (H_1^2 c)(H_1^1 c)(H_1^2 g_1^{-1} \partial H_1^1 c)(H_1^0 c)(H_1^2 g_1^{-1} \partial H_1^0 c)(H_1^1 g_1^{-1} \partial H_1^0 c)(H_1^2 g_1^{-1} \partial H_1^1 g_1^{-1} \partial H_1^0 c). \end{aligned}$$

7 Crossed Pairings and Nonabelian Tensor Products

The *nonabelian tensor product* was introduced by Brown and Loday in [18] and developed in Brown, Johnson, Robertson [17].

When G, H are both abelian, $G \otimes H$ is the usual tensor product.

Many computations of the *nonabelian tensor square* $G \otimes G$ of a group G have been made. Here is a small sample of known results:

symmetric group	S_3	C_6
alternating group	A_4	$Q_8 \times C_3$
dihedral groups	$D_{2n}, n \text{ odd}$	\mathbf{Z}_{2n}
Heisenberg group	\mathcal{H}	\mathbf{Z}^6

The nonabelian tensor product is a special case of a crossed pairing.

7.1 Compatible Group Actions

Definition 7.1 Let G and H be groups which act on themselves by conjugation, and also act on each other. These four actions are said to be compatible if

$$g_1^{(h^g)} = ((g_1^{g^{-1}})^h)^g, \quad h_1^{(g^h)} = ((h_1^{h^{-1}})^g)^h.$$

Example 7.2 If G, H are normal subgroups of a group Γ , then each acts on the other by conjugation and the actions are compatible.

Example 7.3 Let $\mathcal{X} = (\partial : S \rightarrow R)$ be a crossed module. If r^s is defined to be $(\partial s^{-1})r(\partial s)$ then both R and S act on each other and on themselves. Compatibility is easily checked:

- $s_1^{(r^s)} = s_1^{(\partial s^{-1})r(\partial s)} = s_1^{s^{-1}r(\partial s)} = s_1^{s^{-1}rs},$
- $r_1^{(s^r)} = (\partial s^r)^{-1}r_1(\partial s^r) = (r^{-1}(\partial s)r)^{-1}r_1(r^{-1}(\partial s)r) = ((r_1^{r^{-1}})^s)^r.$

7.2 Crossed Pairings

There are two standard definitions of a *crossed pairing*. Here is the one which we shall use. (There is a more general definition when the two actions are not compatible.)

Definition 7.4 Let G, H be groups which act compatibly on each other and on a group L . A map $\boxtimes : G \times H \rightarrow L$, $(g, h) \mapsto g \boxtimes h$, is a *crossed pairing* if

- (a) $(g_1 g_2 \boxtimes h) = (g_1 \boxtimes h)^{g_2} (g_2 \boxtimes h),$
- (b) $(g \boxtimes h_1 h_2) = (g \boxtimes h_2) (g \boxtimes h_1)^{h_2},$
- (c) $(g \boxtimes h)^x = g^x \boxtimes h^x \text{ for all } x \in G \cup H.$

The alternative definition does not require actions on L and omits axiom (c). It is then observed that $\text{im } \boxtimes \leq L$ inherits G - and H -actions given by (c).

Example 7.5 If $N \trianglelefteq G$ then a crossed pairing is provided by commutators:

$$\begin{aligned} (a) \quad \boxtimes & : G \times N \rightarrow N, & g \boxtimes n &= [g, n] = (n^{-1})^g n, \\ (b) \quad \boxtimes & : N \times G \rightarrow N, & n \boxtimes g &= [n, g] = n^{-1} n^g. \end{aligned}$$

Here are some standard properties of crossed pairings (see Proposition 3 of [17]).

Proposition 7.6 *The following relations hold for all $g, g_1, g_2 \in G$ and for all $h, h_1, h_2 \in H$.*

- (a) $(g \boxtimes 1_H) = (1_G \boxtimes h) = 1_L$;
- (b) $(g \boxtimes h)^{-1} = (g^{-1} \boxtimes h)^g = (g \boxtimes h^{-1})^h = (g^{-1} \boxtimes h^g) = (g^h \boxtimes h^{-1})$;
- (c) $(g_1 \boxtimes h_1)^{h_2 g_2} (g_2 \boxtimes h_2) = (g_2 \boxtimes h_2) (g_1 \boxtimes h_1)^{g_2 h_2}$;
- (d) $(g^h \boxtimes h_1) = (g \boxtimes h)^{-1} (g \boxtimes h_1) (g \boxtimes h)^{h_1}$ and $(g_1 \boxtimes h^g) = (g \boxtimes h)^{g_1} (g_1 \boxtimes h) (g \boxtimes h)^{-1}$;
- (e) $(g_1 g_2 \boxtimes h) = (g_2 \boxtimes h^{g_1}) (g_1 \boxtimes h)$ and $(g \boxtimes h_1 h_2) = (g \boxtimes h_1) (g^{h_1} \boxtimes h_2)$;
- (f) $(g \boxtimes h)^{[g_2, h_2]} = (g_2 \boxtimes h_2)^{-1} (g \boxtimes h) (g_2 \boxtimes h_2)$;
- (g) $(g^{-1} g^h \boxtimes h_1) = (g \boxtimes h)^{-1} (g \boxtimes h)^{h_1}$ and $(g_1 \boxtimes (h^{-1})^g h) = ((g \boxtimes h)^{-1})^{g_1} (g \boxtimes h)$;
- (h) $[(g_1 \boxtimes h_1), (g_2 \boxtimes h_2)] = ((g_1^{-1} g_1^{h_1}) \boxtimes ((h_2^{-1})^{g_2} h_2))$.

Proof: Where there are two formulae, the proof of the second mirrors that of the first.

- (a) $g \boxtimes h = g 1 \boxtimes h = (g \boxtimes h)^1 (1 \boxtimes h)$
- (b) $1 = 1 \boxtimes h = g^{-1} g \boxtimes h = (g^{-1} \boxtimes h)^g (g \boxtimes h)$
- (c)
$$\begin{aligned} g_1 g_2 \boxtimes h_1 h_2 &= (g_1 g_2 \boxtimes h_2) (g_1 g_2 \boxtimes h_1)^{h_2} \\ &= (g_1 \boxtimes h_2)^{g_2} (g_2 \boxtimes h_2) (g_1 \boxtimes h_1)^{g_2 h_2} (g_2 \boxtimes h_1)^{h_2} \\ \text{and } g_1 g_2 \boxtimes h_1 h_2 &= (g_1 \boxtimes h_1 h_2)^{g_2} (g_2 \boxtimes h_1 h_2) \\ &= (g_1 \boxtimes h_2)^{g_2} (g_1 \boxtimes h_1)^{h_2 g_2} (g_2 \boxtimes h_2) (g_2 \boxtimes h_1)^{h_2} \\ \Rightarrow (g_1 \boxtimes h_1)^{h_2 g_2} (g_2 \boxtimes h_2) &= (g_2 \boxtimes h_2) (g_1 \boxtimes h_1)^{g_2 h_2} \end{aligned}$$
- (d)
$$\begin{aligned} g^h \boxtimes h_1 &= (g \boxtimes h_1^{h^{-1}})^h = (g \boxtimes h h_1 h^{-1})^h \\ &= (g \boxtimes h^{-1})^h (g \boxtimes h_1) (g \boxtimes h)^{h_1} \\ &= (g \boxtimes h)^{-1} (g \boxtimes h_1) (g \boxtimes h)^{h_1} \end{aligned}$$
- (e) These alternative forms for 7.4 (a),(b) follow immediately from (d).
- (f) Substitute $g = g_1^{h_2 g_2}$, $h = h_1^{h_2 g_2}$ in (c).
- (g)
$$\begin{aligned} (g^{-1} g^h) \boxtimes h_1 &= (g^{-1} \boxtimes h_1)^{g^h} (g^h \boxtimes h_1) = (g^{-1} \boxtimes h_1^{g^h})^{[g, h]} (g^h \boxtimes h_1) \\ &= (g \boxtimes h)^{-1} (g \boxtimes h_1)^{-1} (g \boxtimes h) \cdot (g \boxtimes h)^{-1} (g \boxtimes h_1) (g \boxtimes h)^{h_1} \quad \text{by (b),(d),(e)} \\ &= (g \boxtimes h)^{-1} (g \boxtimes h)^{h_1} \end{aligned}$$

$$\begin{aligned}
(h) \quad [g_1 \boxtimes h_1, g_2 \boxtimes h_2] &= (g_1 \boxtimes h_1)^{-1} (g_2 \boxtimes h_2)^{-1} (g_1 \boxtimes h_1) (g_2 \boxtimes h_2) \\
&= (g_1 \boxtimes h_1)^{-1} (g_1 \boxtimes h_1)^{[g_2, h_2]} = (g_1 \boxtimes h_1)^{-1} (g_1 \boxtimes h_1)^{(h_2^{-1})^{g_2} h_2} \quad \text{by (e)} \\
&= ((g_1^{-1} g_1^{h_1}) \boxtimes ((h_2^{-1})^{g_2} h_2)) \quad \text{by (f)}.
\end{aligned}$$

□

Lemma 7.7 *The principal crossed pairing of a crossed module $\mathcal{X} = (\partial : S \rightarrow R)$ is given by*

$$\boxtimes : R \times S \rightarrow S, \quad (r, s) \mapsto \eta_s(r) = (s^{-1})^r s.$$

Proof: We have seen in Example 7.3 that R and S have compatible actions. The three axioms are easily checked.

$$\begin{aligned}
(a) \quad (r_1 \boxtimes s)^{r_2} (r_2 \boxtimes s) &= ((s^{-1})^{r_1} s)^{r_2} ((s^{-1})^{r_2} s) = (s^{-1})^{r_1 r_2} s = r_1 r_2 \boxtimes s \\
(b) \quad (r \boxtimes s_1) (r \boxtimes s_1)^{s_2} &= ((s_2^{-1})^r s_2) (s_2^{-1} (s_1^{-1})^r s_1 s_2) = ((s_1 s_2)^{-1})^r (s_1 s_2) = r \boxtimes s_1 s_2 \\
(c) \quad r^{r_0} \boxtimes s^{r_0} &= ((s^{r_0})^{-1})^{r^{r_0}} s^{r_0} = (s^{-1})^{r r_0} s^{r_0} = (r \boxtimes s)^{r_0} \\
r^{\partial s_0} \boxtimes s^{s_0} &= ((s^{s_0})^{-1})^{s_0^{-1} r s_0} s^{s_0} = ((s^{-1})^r s)^{s_0} = (r \boxtimes s)^{s_0}
\end{aligned}$$

□

So we may write principal derivations as $\eta_s r = r \boxtimes s$ and principal sections as $\kappa_s r = (r, r \boxtimes s)$.

A standard result concerning crossed pairings shows that the nonabelian tensor product is the universal object for this construction.

Definition 7.8 *Given groups G and H which act compatibly on each other, the nonabelian tensor product $G \otimes H$ of G and H has generating set $\{g \otimes h \mid g \in G, h \in H\}$ subject to relations*

$$(g_1 g_2 \otimes h) = (g_1 \otimes h)^{g_2} (g_2 \otimes h), \quad (g \otimes h_1 h_2) = (g \otimes h_2) (g \otimes h_1)^{h_2}, \quad (37)$$

where

$$(g \otimes h)^x = (g^x \otimes h^x) \quad \text{for all } x \in G \cup H.$$

Theorem 7.9 *The nonabelian tensor product function*

$$\otimes : G \times H \rightarrow G \otimes H, \quad (g, h) \mapsto g \otimes h$$

is a crossed pairing. Moreover, given any crossed pairing $\boxtimes : G \times H \rightarrow L$, there is a unique homomorphism $\boxtimes_{\otimes} : G \otimes H \rightarrow L$ satisfying $\boxtimes = \boxtimes_{\otimes} \circ \otimes$ so that the following diagram commutes:

$$\begin{array}{ccc}
G \times H & & \\
\downarrow \otimes & \searrow \boxtimes & \\
G \otimes H & \xrightarrow{\boxtimes_{\otimes}} & L.
\end{array}$$

Checking that a potential crossed pairing satisfies the axioms of Definition 7.4 can be a tedious process. However we can convert this into checking that maps to certain semidirect products are homomorphisms.

Lemma 7.10 *Let $\boxtimes : G \times H \rightarrow L$ be a crossed pairing. Then*

(a) *given a fixed element $h \in H$, the map*

$$\theta_h : G \rightarrow G \ltimes L, \quad g \mapsto (g, g \boxtimes h)$$

is a group homomorphism;

(b) *given a fixed element $g \in G$, the map*

$$\theta_g : H \rightarrow H \ltimes L, \quad h \mapsto (h, (g \boxtimes h)^{-1})$$

is a group homomorphism.

Proof:

$$\begin{aligned} \theta_h(g_1 g_2) &= (g_1 g_2, (g_1 \boxtimes h)^{g_2} (g_2 \boxtimes h)) = (g_1, g_1 \boxtimes h) (g_2, g_2 \boxtimes h) = (\theta_h g_1) (\theta_h g_2) . \\ \theta_g(h_1 h_2) &= (h_1 h_2, \{(g \boxtimes h_2) ((g \boxtimes h_1)^{h_2})^{-1}\}) = (h_1, (g \boxtimes h_1)^{-1}) (h_2, (g \boxtimes h_2)^{-1}) = (\theta_g h_1) (\theta_g h_2) . \end{aligned}$$

□

The converse proposition gives a way of checking that a given map is a crossed pairing.

Proposition 7.11 *Given a map $\odot : G \times H \rightarrow L$ and an action $(g \odot h)^x = g^x \odot h^x$ for all $x \in G \cup H$, such that for all $h \in H$ and $g \in G$*

- $\theta_h : G \rightarrow G \ltimes L, \quad g \mapsto (g, g \odot h)$ *is a homomorphism, and*
- $\theta_g : H \rightarrow H \ltimes L, \quad h \mapsto (h, (g \odot h)^{-1})$ *is a homomorphism,*

then \odot is a crossed pairing.

Proof:

$$\begin{aligned} (g_1 g_2, g_1 g_2 \odot h) &= \theta_h(g_1 g_2) = (\theta_h g_1)(\theta_h g_2) \\ &= (g_1, g_1 \odot h)(g_2, g_2 \odot h) \\ &= (g_1 g_2, (g_1 \odot h)^{g_2} (g_2 \odot h)) . \end{aligned}$$

A similar argument shows that (7.4)(b) is also satisfied, so \odot is a crossed pairing. □

8 Crossed Squares and Cat^2 -groups

Crossed squares were introduced by Guin-Waléry and Loday (see, for example, [37, 42, 18]) as fundamental crossed squares of commutative squares of spaces, but are also of purely algebraic interest.

Definition 8.1 *A crossed square consists of the following:*

- (i) *a commutative diagram of group homomorphisms*

$$\mathcal{S} = \begin{array}{ccc} L & \xrightarrow{\kappa} & M \\ \lambda \downarrow & & \downarrow \mu \\ N & \xrightarrow{\nu} & P \end{array} ; \quad (38)$$

- (ii) *actions of P on N, M and L which determine actions of N on M and L via ν and actions of M on N and L via μ ;*
- (iii) *a function $\boxtimes : N \times M \rightarrow L$.*

The following axioms must be satisfied for all $l \in L$, $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, $p \in P$:

- (a) *the homomorphisms κ, λ preserve the action of P ;*
- (b) *each of $\mathcal{K} = (\kappa : L \rightarrow M)$, $\mathcal{L} = (\lambda : L \rightarrow N)$, $\mathcal{M} = (\mu : M \rightarrow P)$, $\mathcal{N} = (\nu : N \rightarrow P)$ and the diagonal $\mathcal{D} = (\delta = \nu \circ \lambda = \mu \circ \kappa : L \rightarrow P)$ are crossed modules;*
- (c) *\boxtimes is a crossed pairing:*
- (i) $(n_1 n_2 \boxtimes m) = (n_1 \boxtimes m)^{n_2} (n_2 \boxtimes m)$,
 - (ii) $(n \boxtimes m_1 m_2) = (n \boxtimes m_2) (n \boxtimes m_1)^{m_2}$,
 - (iii) $(n \boxtimes m)^p = (n^p \boxtimes m^p)$;
- (d) $\lambda(n \boxtimes m) = n^{-1} n^m$ and $\kappa(n \boxtimes m) = (m^{-1})^n m$,
- (e) $(\lambda l \boxtimes m) = l^{-1} l^m$ and $(n \boxtimes \kappa l) = (l^{-1})^n l$.

Note that the actions of N on M and M on N via P are compatible since

$$m_1^{(n^m)} = m_1^{\nu(n^m)} = m_1^{m^{-1}(\nu n)m} = ((m_1^{m^{-1}})^n)^m .$$

Note also that identities c(i) and c(ii) are based on the commutator identities

$$[n, m] = n^{-1} m^{-1} n m, \quad [n_1 n_2, m] = [n_1, m]^{n_2} [n_2, m], \quad [n, m_1 m_2] = [n, m_2] [n, m_1]^{m_2} .$$

Lemma 8.2

$$1 = (1 \boxtimes n) = (m \boxtimes 1) \quad \text{and} \quad (m \boxtimes n)^{-1} = (m^{-1} \boxtimes n)^m = (m \boxtimes n^{-1})^n .$$

Proof: These follow immediately on expanding $(m.1 \boxtimes n)$, $(m \boxtimes n.1)$, $(m^{-1}m \boxtimes n)$ and $(m \boxtimes n^{-1}n)$.
□

Lemma 8.3 Pairs $(\kappa, \nu) : \mathcal{L} \rightarrow \mathcal{M}$ and $(\lambda, \mu) : \mathcal{K} \rightarrow \mathcal{N}$ are morphisms of crossed modules, and so also are $(\text{id}_L, \nu) : \mathcal{L} \rightarrow \mathcal{D}$, $(\text{id}_L, \mu) : \mathcal{K} \rightarrow \mathcal{D}$, $(\lambda, \text{id}_P) : \mathcal{D} \rightarrow \mathcal{N}$ and $(\kappa, 1_P) : \mathcal{D} \rightarrow \mathcal{M}$.

Proof: For (κ, ν) we note that (38) commutes, that $\kappa(\ell^m) = (\kappa\ell)^m$ by (a), and that $(\kappa\ell)^n = (\kappa\ell)^{\nu n}$ by (ii). The arguments for the other five morphisms are similar. \square

Note in particular that

$$\kappa(\ell^p) = (\kappa\ell)^p \quad \text{and} \quad \lambda(\ell^p) = (\lambda\ell)^p.$$

Lemma 8.4 In the crossed square \mathcal{S} above:

- (a) $\ell^{(n \boxtimes m)} = \ell^{[n, m]},$
- (b) $\ell^{mn} (n \boxtimes m) = (n \boxtimes m) \ell^{nm} \quad \text{and} \quad \ell^{(m^n)} (n \boxtimes m) = (n \boxtimes m) \ell^m,$
- (c) $m^{\lambda\ell} = m^{\kappa\ell} \quad \text{and} \quad n^{\lambda\ell} = n^{\kappa\ell},$
- (d) $(n \boxtimes m)^{m'n'} (n' \boxtimes m') = (n' \boxtimes m') (n \boxtimes m)^{n'm'}.$

Proof:

- (a) By the crossed module identity (X2),

$$\ell^{(n \boxtimes m)} = \ell^{\nu\lambda(n \boxtimes m)} = \ell^{\nu(n^{-1}n^m)} = \ell^{\nu(n^{-1}n^{\mu m})} = \ell^{[\nu n, \mu m]} = \ell^{[n, m]}.$$

- (b) The first identity is given by

$$(n \boxtimes m)^{-1} \ell^{mn} (n \boxtimes m) = \ell^{mn\lambda(n \boxtimes m)} = \ell^{mn(n^{-1}n^m)} = \ell^{nm}.$$

Then replace ℓ by $\ell^{n^{-1}}$ to get the second.

- (c)

$$m^{\lambda\ell} = m^{\nu\lambda\ell} = m^{\mu\kappa\ell} = m^{\kappa\ell}, \quad \text{and similarly for } n.$$

- (d) Expand $(nn' \boxtimes mm')$ in two different ways.

\square

8.1 Examples of crossed squares

Example 8.5 If M, N are normal subgroups of the group P then the diagram of inclusions

$$\begin{array}{ccc} M \cap N & \xrightarrow{i_1} & M \\ \downarrow i_2 & \searrow i_{12} & \downarrow i_M \\ N & \xrightarrow{i_N} & P \end{array}$$

together with the actions of P on M, N and $M \cap N$ given by conjugation and the function

$$\boxtimes : N \times M \rightarrow M \cap N, \quad (n, m) \mapsto [n, m] = n^{-1}m^{-1}nm$$

is a crossed square. We may check the axioms as follows:

- (a) The identity maps preserve P -actions.
(b) The five crossed modules are all conjugation crossed modules.
(c) We have already noted that (i) and (ii) are analogues of commutator identities. For (iii),

$$(n^p \boxtimes m^p) = [n^p, m^p] = \{p^{-1}np\}^{-1} \{p^{-1}mp\}^{-1} \{p^{-1}np\} \{p^{-1}mp\} = p^{-1}[n, m]p = [n, m]^p.$$

- (d) $\ddot{i}_1(n \boxtimes m) = n^{-1}(m^{-1}nm) = n^{-1}n^m$ and $\ddot{i}_2(n \boxtimes m) = (n^{-1}m^{-1}n)m = m^{-1}n^m$.
(e) $(n \boxtimes i_1l) = (n^{-1}l^{-1}n)l = (l^{-1})^n l$ and $(i_2l \boxtimes m) = l^{-1}(m^{-1}lm) = l^{-1}l^m$.

Example 8.6 If M, N are ordinary P -modules and A is an arbitrary abelian group on which P is assumed to act trivially, then there is a crossed square

$$\begin{array}{ccc} A & \xrightarrow{0} & M \\ \downarrow 0 & \searrow 0 & \downarrow 0 \\ N & \xrightarrow{0} & P \end{array}$$

Note that M acts trivially on N , and conversely, and that $n \boxtimes m = 1_A$.

Example 8.7 The diagram

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & \text{Inn}M \\ \downarrow \alpha & & \downarrow \iota \\ \text{Inn}M & \xrightarrow{\iota} & \text{Aut}M \end{array}$$

is a crossed square, where α maps $m \in M$ to the inner automorphism $\beta_m : M \rightarrow M$, $m' \mapsto m^{-1}m'm$; where ι is the inclusion of $\text{Inn}M$ in $\text{Aut}M$; the actions are standard; and the crossed pairing is

$$\boxtimes : \text{Inn}M \times \text{Inn}M \rightarrow M, \quad (\beta_m, \beta_{m'}) \mapsto [m, m'].$$

Example 8.8 If U, V are subspaces of a space X with a point x_0 in common, then the diagram of boundary maps

$$\begin{array}{ccc} \pi_3(X; U, V, x_0) & \longrightarrow & \pi_2(V, U \cap V, x_0) \\ \downarrow & & \downarrow \\ \pi_2(U, U \cap V, x_0) & \longrightarrow & \pi_1(U \cap V, x_0) \end{array}$$

in which $\pi_3(X; U, V, x_0)$ is the triad homotopy group, together with the standard actions and the triad Whitehead product

$$\boxtimes : \pi_2(U, U \cap V, x_0) \times \pi_2(V, U \cap V, x_0) \rightarrow \pi_3(X; U, V, x_0)$$

is a crossed square.

Lemma 8.9 *The transpose*

$$\tilde{\mathcal{S}} = \begin{array}{ccc} L & \xrightarrow{\lambda} & N \\ \kappa \downarrow & & \downarrow \nu \\ M & \xrightarrow{\mu} & P \end{array} \quad \text{of} \quad \mathcal{S} = \begin{array}{ccc} L & \xrightarrow{\kappa} & M \\ \lambda \downarrow & & \downarrow \mu \\ N & \xrightarrow{\nu} & P \end{array}$$

is a crossed square with crossed pairing

$$\tilde{\boxtimes} : M \times N \rightarrow L, \quad (m, n) \mapsto m \tilde{\boxtimes} n := (n \boxtimes m)^{-1}. \quad (39)$$

Proof:

$$\begin{aligned} m_1 m_2 \tilde{\boxtimes} n &= (n \boxtimes m_1 m_2)^{-1} = ((n \boxtimes m_2) (n \boxtimes m_1)^{m_2})^{-1} \\ &= ((n \boxtimes m_1)^{m_2})^{-1} (n \boxtimes m_2)^{-1} = (m_1 \tilde{\boxtimes} n)^{m_2} (m_2 \tilde{\boxtimes} n); \\ m \tilde{\boxtimes} n_1 n_2 &= (n_1 n_2 \boxtimes m)^{-1} = ((n_1 \boxtimes m)^{n_2} (n_2 \boxtimes m))^{-1} \\ &= ((n_2 \boxtimes m)^{-1} ((n_1 \boxtimes m)^{n_2})^{-1} = (m \tilde{\boxtimes} n_2) (m \tilde{\boxtimes} n_1)^{n_2}; \\ (m \tilde{\boxtimes} n)^p &= ((n \boxtimes m)^{-1})^p = ((n \boxtimes m)^p)^{-1} = (n^p \boxtimes m^p)^{-1} = (m^p \tilde{\boxtimes} n^p). \end{aligned}$$

□

Example 8.10 The actor crossed module $\mathcal{A}(\mathcal{X})$ of a crossed module \mathcal{X} (see subsection 3.2)

$$\mathcal{A} = \begin{array}{ccc} S & \xrightarrow{i} & W \\ \partial \downarrow & & \downarrow \Delta \\ R & \xrightarrow{i} & A \end{array} \quad (40)$$

is a crossed square with crossed pairing

$$\boxtimes : R \times W \rightarrow S, \quad (r, \chi) \mapsto \chi r.$$

We already know that the square \mathcal{A} contains 5 crossed modules, but we still need to check the axioms (c), (d), and (e) which involve the crossed pairing:

- (c) (i) $(qr \boxtimes \chi) = \chi(qr) = (\chi q)^r (\chi r) = (q \boxtimes \chi)^r (r \boxtimes \chi).$
(ii) $(r \boxtimes \chi_1 \star \chi_2) = (\chi_2 r)(\chi_1 r)(\chi_2 \partial \chi_1 r) = (\chi_2 r)(\chi_1 r)^{\chi_2} = (r \boxtimes \chi_2)(r \boxtimes \chi_1)^{\chi_2},$
using the action of W on S given in Lemma 3.5.
(iii) $(r^\beta \boxtimes \chi^\beta) = \chi^\beta(r^\beta) = (\ddot{\beta} \chi \dot{\beta}^{-1})(\dot{\beta} r) = \ddot{\beta}(\chi r) = (r \boxtimes \chi)^\beta.$

(d) The first formula follows by

$$\partial(r \boxtimes \chi) = \partial \chi(r) = r^{-1} r (\partial \chi r) = r^{-1} (\dot{\beta} \chi r) = r^{-1} r^\chi.$$

For the second formula, since $i(r \boxtimes \chi) = i(\chi r) = \eta_{\chi r}$, we wish to prove that

$$\eta_{\chi r} = (\chi^{-1})^r \star \chi \quad \text{or, equivalently,} \quad \chi^r = \chi \star (\eta_{\chi r})^{-1} = \chi \star \eta_{(\chi r)^{-1}}.$$

Starting with the right-hand side,

$$\begin{aligned}
(\chi \star \eta_{(\chi r)^{-1}})q &= (\eta_{(\chi r)^{-1}}q)(\ddot{\beta}_{\eta_{(\chi r)^{-1}}}\chi q) && \text{by Lemma 2.4 (c)} \\
&= (\chi r)^q(\chi r)^{-1}(\chi q)^{(\chi r)^{-1}} && \text{by Lemma 2.8 (b)} \\
&= (\chi r)^q(\chi q)(\chi r)^{-1} = (\chi r)^q(\chi q)(\chi r^{-1})^r && \text{by Lemma 2.2 (b)} \\
&= \ddot{\beta}_r((\chi r)^{qr^{-1}}(\chi q)^{r^{-1}}(\chi r^{-1})) = \ddot{\beta}_r\chi(rqr^{-1}) = (\ddot{\beta}_r\chi\dot{\beta}_r^{-1})q && \text{by Lemma 2.4 (c)} \\
&= \chi^{ir}q = \chi^r q.
\end{aligned}$$

$$\begin{aligned}
\text{(e)} \quad (\partial s \boxtimes \chi) &= \chi(\partial s) = s^{-1}s(\chi\partial s) = s^{-1}(\ddot{\beta}_\chi s) = s^{-1}s^\chi, \\
(r \boxtimes \ddot{i}(s)) &= \eta_s(r) = (s^{-1})^r s && \text{by Lemma 2.7.}
\end{aligned}$$

8.2 Morphisms of crossed squares

A morphism $\theta : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ of crossed squares is a 4-tuple of group homomorphisms which commute with the morphisms in \mathcal{S}_1 and \mathcal{S}_2 and preserve all the actions and the crossed pairings.

Definition 8.11 A morphism $\theta : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ of crossed squares consists of four group homomorphisms

$$\theta_L : L_1 \rightarrow L_2, \quad \theta_M : M_1 \rightarrow M_2, \quad \theta_N : N_1 \rightarrow N_2, \quad \theta_P : P_1 \rightarrow P_2,$$

forming a commutative cube with the morphisms $\kappa_1, \lambda_1, \mu_1, \nu_1$ of \mathcal{S}_1 and $\kappa_2, \lambda_2, \mu_2, \nu_2$ of \mathcal{S}_2 , which pair off in appropriate ways to form crossed module morphisms

$$(\theta_L, \theta_M) : \mathcal{K}_1 \rightarrow \mathcal{K}_2, \quad (\theta_L, \theta_N) : \mathcal{L}_1 \rightarrow \mathcal{L}_2, \quad (\theta_M, \theta_P) : \mathcal{M}_1 \rightarrow \mathcal{M}_2, \quad (\theta_N, \theta_P) : \mathcal{N}_1 \rightarrow \mathcal{N}_2,$$

and which preserve the crossed pairing:

$$\theta_L(n \boxtimes_1 m) = (\theta_N n) \boxtimes_2 (\theta_M m).$$

Definition 8.12 The group $\text{Aut}(\mathcal{S})$ of automorphisms of the crossed square \mathcal{S} is

$$\text{Aut}(\mathcal{S}) = \{ \alpha = (\alpha_L, \alpha_M, \alpha_N, \alpha_P) : \mathcal{S} \rightarrow \mathcal{S} \}$$

such that (α_L, α_M) is an automorphism of \mathcal{K} , (α_L, α_N) is an automorphism of \mathcal{L} , (α_M, α_P) is an automorphism of \mathcal{M} , (α_N, α_P) is an automorphism of \mathcal{N} , and $\alpha_L(n \boxtimes m) = (\alpha_N n) \boxtimes (\alpha_M m)$.

[Surely there is a crossed square version of Theorem 1.8 ?]

Theorem 8.13 Every crossed square is a quotient of normal inclusion crossed squares. ?

(Note: notes 10/7/03 only go one way.)

8.3 Cat²-groups

When defining a cat²-group we may require all the homomorphisms to be endomorphisms, as in Definition 1.19, or we may take a more general view, as in Definition 1.20. For now, we consider only endomorphisms. When we come to define catⁿ-groups we shall give a similar set of definitions. Firstly, we give the definition of a *traditional* cat²-group adapted from Section 5 of Brown and Loday [18] and Ellis-Steiner [31].

Definition 8.14 A cat²-group $\mathcal{C} = (G; \tau_1, \theta_1, \tau_2, \theta_2)$ comprises a group G and 4 endomorphisms, as shown in the following diagram, where $Q_1 = \text{im } \tau_1 = \text{im } \theta_1$, $Q_2 = \text{im } \tau_2 = \text{im } \theta_2$ and $Q_0 = Q_1 \cap Q_2$,

$$\mathcal{C} = \begin{array}{ccc} G & \xrightleftharpoons{\tau_1, \theta_1} & Q_1 \\ \tau_2, \theta_2 \downarrow & \searrow \tau_0, \theta_0 & \downarrow \tau_2, \theta_2 \\ Q_2 & \xrightleftharpoons{\tau_1, \theta_1} & Q_0 \end{array} \quad (41)$$

subject to the following axioms:

- (a) $\mathcal{C}_1 = (G; \tau_1, \theta_1)$ and $\mathcal{C}_2 = (G; \tau_2, \theta_2)$ are (traditional) cat¹-groups,
- (b) $\tau_1 \circ \tau_2 = \tau_2 \circ \tau_1 = \tau_0$, $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1 = \theta_0$,
- (c) $\tau_1 \circ \theta_2 = \theta_2 \circ \tau_1$, $\tau_2 \circ \theta_1 = \theta_1 \circ \tau_2$.

If $r \in Q_0$ then $\tau_0 r = \tau_1(\tau_2 r) = \tau_1 r = r$, and similarly for $\theta_0 r$, so $\text{im } \tau_0 = \text{im } \theta_0 = Q_0$. It is clear that $\mathcal{C}_3 = (Q_1; \tau_2, \theta_2)$ is a sub-cat¹-group of \mathcal{C}_2 and that $\mathcal{C}_4 = (Q_2; \tau_1, \theta_1)$ is a sub-cat¹-group of \mathcal{C}_1 . It is easy to verify that $(G; \tau_0, \theta_0)$ is a cat¹-group since, for example,

$$\tau_0 \circ \theta_0 = \tau_1 \circ \tau_2 \circ \theta_1 \circ \theta_2 = \tau_1 \circ \theta_1 \circ \tau_2 \circ \theta_2 = \theta_1 \circ \theta_2 = \theta_0.$$

It also follows from these identities that, for example, (τ_1, τ_1) and $(\theta_1, \theta_1) : \mathcal{C}_2 \rightarrow \mathcal{C}_3$ are morphisms of cat¹-groups.

Secondly, we give a definition of a cat²-group using homomorphisms, rather than endomorphisms.

Definition 8.15 A cat²-group \mathcal{C} is generated by three cat¹-groups $\mathcal{C}_i = (e_i; t_i, h_i : G \rightarrow R_i), 0 \leq i \leq 2$, as shown in the following diagram,

$$\mathcal{C} = \begin{array}{ccccc} & & \xrightleftharpoons{t_1, h_1} & & \\ & & \xrightleftharpoons{e_1} & & \\ G & \xleftarrow{e_2} & & R_1 & \xrightarrow{e_3} \\ \tau_2, h_2 \downarrow & \searrow e_0 & & \downarrow \tau_3, h_3 & \\ & \xrightarrow{t_0, h_0} & & & \\ R_2 & \xleftarrow{e_4} & & R_0 & \xrightarrow{t_4, h_4} \end{array} \quad (42)$$

The remaining homomorphisms are defined by

$$\begin{aligned} t_3 &= t_0 \circ e_1, & h_3 &= h_0 \circ e_1, & e_3 &= e_0 \circ t_1, \\ t_4 &= t_0 \circ e_2, & h_3 &= h_0 \circ e_2, & e_3 &= e_0 \circ t_2. \end{aligned}$$

The following axioms must be satisfied:

- (a) $(e_1 \circ t_1) \circ (e_2 \circ t_2) = (e_2 \circ t_2) \circ (e_1 \circ t_1) = e_0 \circ t_0$, $(e_1 \circ h_1) \circ (e_2 \circ h_2) = (e_2 \circ h_2) \circ (e_1 \circ h_1) = e_0 \circ h_0$,
- (b) $(e_1 \circ t_1) \circ (e_2 \circ h_2) = (e_2 \circ h_2) \circ (e_1 \circ t_1)$, $(e_2 \circ t_2) \circ (e_1 \circ h_1) = (e_1 \circ h_1) \circ (e_2 \circ t_2)$.
- (c) $(t_0 \circ e_1) \circ (t_1 \circ e_0) = 1$, $(h_0 \circ e_1) \circ (t_1 \circ e_0) = 1$, $(t_0 \circ e_2) \circ (t_2 \circ e_0) = 1$, $(h_0 \circ e_2) \circ (t_2 \circ e_0) = 1$.

We now show that Definition 8.15 is equivalent to Definition 8.14. It is routine to check that $(e_3; t_3, h_3 : R_1 \rightarrow R_0)$ and $(e_4; t_4, h_4 : R_2 \rightarrow R_0)$ are cat^1 -groups. In particular, $[\ker t_3, \ker h_3] = e_1^{-1}[\ker t_0, \ker h_0] = 1$.

We may convert the cat^2 -group in Definition 8.15 into a traditional cat^2 -group by defining:

$$\tau_1 = e_1 \circ t_1, \quad \theta_1 = e_1 \circ h_1, \quad \tau_2 = e_2 \circ t_2, \quad \theta_2 = e_2 \circ h_2, \quad \tau_0 = e_0 \circ t_0, \quad \theta_0 = e_0 \circ h_0,$$

and setting $Q_1 = e_1 R_1$, $Q_2 = e_2 R_2$, $Q_0 = e_0 R_0$.

Proposition 8.16 *A cat^2 -group \mathcal{C} is a cat^1 -group of cat^1 -groups.*

Proof: We have already seen that $(\tau_1, \tau_1) : \mathcal{C}_2 \rightarrow \mathcal{C}_3$ is a cat^1 -group morphism – the boundary in this case. We also have to consider the action of \mathcal{C}_3 on \mathcal{C}_2 . [To be continued.] \square

Kernels of cat^1 -mappings have been considered in subsection 1.12. The diagram for the situation here is

$$\begin{array}{ccccc} \ker \tau_1, \ker \theta_1 & \longrightarrow & G & \xrightarrow{\tau_1, \theta_1} & Q_1 \\ \Downarrow & & \Downarrow & & \Downarrow \\ \ker \tau_1, \ker \theta_1 & \longrightarrow & Q_2 & \xrightarrow{\tau_1, \theta_1} & Q_0 \end{array}$$

$\begin{array}{ccc} & \tau_2, \theta_2 & \\ & \Downarrow & \\ & \tau_2, \theta_2 & \end{array}$

8.4 The cat^2 -group associated to a crossed square

Given a crossed square

$$\mathcal{S} = \begin{array}{ccc} L & \xrightarrow{\kappa} & M \\ \lambda \downarrow & & \downarrow \mu \\ N & \xrightarrow{\nu} & P \end{array} \quad (43)$$

with crossed pairing $\boxtimes : N \times M \rightarrow L$, we wish to construct an associated cat^2 -group.

Proposition 8.17 *For \mathcal{S} a crossed square (as in Definition 8.1) there are group actions of $P \ltimes M$ on $N \ltimes L$ and $P \ltimes N$ on $M \ltimes L$ given by*

$$(n, \ell)^{(p,m)} = (n^p, (n^p \boxtimes m) \ell^{pm}) \quad (44)$$

$$(m, \ell)^{(p,n)} = (m^p, (n \boxtimes m^p)^{-1} \ell^{pn}) , \quad (45)$$

Proof: There are two axioms to be checked for the first identity:

$$\begin{aligned} (n_1, \ell_1)^{(p,m)} (n_2, \ell_2)^{(p,m)} &= (n_1^p, (n_1^p \boxtimes m) \ell_1^{pm}) (n_2^p, (n_2^p \boxtimes m) \ell_2^{pm}) \\ &= (n_1^p n_2^p, (n_1^p \boxtimes m) n_2^p [\ell_1^{pm} n_2^p (n_2^p \boxtimes m)] \ell_2^{pm}) \\ &= (n_1^p n_2^p, (n_1^p \boxtimes n) n_2^p [(n_2^p \boxtimes m) \ell_1^{pm} n_2^p] \ell_2^{pm}) \quad \text{by Lemma 8.4(b)} \\ &= ((n_1 n_2)^p, ((n_1 n_2)^p \boxtimes m) (\ell_1^{n_2} \ell_2)^{pm}) \\ &= (n_1 n_2, \ell_1^{n_2} \ell_2)^{(p,m)} \\ &= ((n_1, \ell_1)(n_2, \ell_2))^{(p,m)} , \end{aligned}$$

$$\begin{aligned} ((n, \ell)^{(p_1, m_1)})^{(p_2, m_2)} &= (n^{p_1}, (n^{p_1} \boxtimes m_1) \ell^{p_1 m_1})^{(p_2, m_2)} \\ &= ((n^{p_1})^{p_2}, (n^{p_1 p_2} \boxtimes m_2) ((n^{p_1} \boxtimes m_1) \ell^{p_1 m_1})^{p_2 m_2}) \\ &= (n^{p_1 p_2}, (n^{p_1 p_2} \boxtimes m_2) (n^{p_1 p_2} \boxtimes m_1^{p_2}) \ell^{p_1 m_1 p_2 m_2}) \\ &= (n^{p_1 p_2}, (n^{p_1 p_2} \boxtimes m_1^{p_2} m_2) \ell^{p_1 p_2 m_1 p_2 m_2}) \\ &= (n, \ell)^{(p_1 p_2, m_1^{p_2} m_2)} \\ &= (n, \ell)^{((p_1, m_1)(p_2, m_2))} . \end{aligned}$$

The second identity follows using the transpose crossed pairing (39). □

We saw in (10) that the cat^1 -group associated to a crossed module \mathcal{X} has homomorphisms

$$t, h : R \ltimes S \rightarrow S, \quad t(r, s) = r, \quad h(r, s) = r(\partial s) .$$

Applying this construction to \mathcal{L} and \mathcal{M} we obtain a *crossed module of cat^1 -groups*:

$$\begin{array}{ccc} N \ltimes L & \xrightarrow{(\nu, \kappa)} & P \ltimes M \\ \Downarrow (t_2, h_2) & & \Downarrow (t_3, h_3) \\ N & \xrightarrow{\nu} & P \end{array} \quad (46)$$

Lemma 8.18 *The mapping*

$$(\nu, \kappa) : N \ltimes L \rightarrow P \ltimes M : (n, \ell) \mapsto (\nu n, \kappa \ell)$$

is a group homomorphism.

Proof:

$$\begin{aligned} (\nu, \kappa)((n_1, \ell_1)(n_2, \ell_2)) &= (\nu, \kappa)(n_1 n_2, \ell_1^{n_2} \ell_2) \\ &= (\nu(n_1 n_2), \kappa(\ell_1^{n_2} \ell_2)) \\ &= ((\nu n_1)(\nu n_2), (\kappa \ell_1^{\nu n_2})(\kappa \ell_2)) \\ &= ((\nu n_1)(\nu n_2), (\kappa \ell_1)^{\nu n_2}(\kappa \ell_2)) \\ &= (\nu n_1, \kappa \ell_1)(\nu n_2, \kappa \ell_2) \\ &= (\nu, \kappa)(n_1, \ell_1) \cdot (\nu, \kappa)(n_2, \ell_2). \end{aligned}$$

□

Lemma 8.19 *The action given in Proposition 8.17 makes $((\nu, \kappa) : N \ltimes L \rightarrow P \ltimes M)$ a crossed module.*

Proof: X1:

$$\begin{aligned} (\nu, \kappa)((n, \ell)^{(p, m)}) &= (\nu, \kappa)(n^p, (n^p \boxtimes m) \ell^{pm}) \\ &= (\nu(n^p), \kappa(n^p \boxtimes m) \kappa(\ell^{pm})) \\ &= (\nu(n^p), (m^{-1})^{n^p} m \kappa(\ell^{pm})) \\ &= (\nu(n^p), (m^{-1})^{(\nu n^p)} m \kappa((\ell^p)^m)) \\ &= (\nu(n^p), (m^{-1})^{\nu(n^p)} (\kappa \ell)^p m) \\ &= (p^{-1}(\nu n) p, (m^{-1})^{p^{-1}(\nu n) p} (\kappa \ell)^p m) \\ &= (p^{-1}, (m^{-1})^{p^{-1}})(\nu n, \kappa \ell)(p, m) \\ &= (p, m)^{-1} ((\nu, \kappa)(n, \ell))(p, m). \end{aligned}$$

X2:

$$\begin{aligned} (n_1, \ell_1)^{(\nu, \kappa)(n_2, \ell_2)} &= (n_1, \ell_1)^{(\nu n_2, \kappa \ell_2)} \\ &= (n_1^{\nu n_2}, (n_1^{\nu n_2} \boxtimes \kappa \ell_2) \ell_1^{(\nu n_2)(\kappa \ell_2)}) \\ &= (n_1^{n_2}, (n_2^{-1} n_1 n_2 \boxtimes \kappa \ell_2) \ell_1^{(\nu n_2)(\kappa \ell_2)}) \\ &= (n_1^{n_2}, (n_2^{-1} \boxtimes \kappa \ell_2)^{n_1 n_2} (n_1 \boxtimes \kappa \ell_2)^{n_2} (n_2 \boxtimes \kappa \ell_2) \ell_1^{(\nu n_2)(\kappa \ell_2)}) \\ &= (n_2^{-1} n_1 n_2, [(\ell_2^{-1})^{n_2^{-1}} \ell_2]^{n_1 n_2} [(\ell_2^{-1})^{n_1} \ell_2]^{n_2} [(\ell_2^{-1})^{n_2} \ell_2] \ell_2^{-1} \ell_1^{(\nu n_2)} \ell_2) \\ &= (n_2^{-1} n_1 n_2, (\ell_2^{-1})^{n_2^{-1} n_1 n_2} \ell_1^{n_2} \ell_2) \\ &= \left(n_2^{-1}, (\ell_2^{-1})^{n_2^{-1}} \right) (n_1 n_2, \ell_1^{n_2} \ell_2) \\ &= (n_2, \ell_2)^{-1} (n_1, \ell_1)(n_2, \ell_2) \end{aligned}$$

□

We may then construct a cat^1 -group of cat^1 -groups where the required homomorphisms are:

$$t_1, h_1 : (P \ltimes M) \ltimes (N \ltimes L) \rightarrow P \ltimes M \quad (47)$$

$$t_1((p, m), (n, \ell)) = (p, m)$$

$$h_1((p, m), (n, \ell)) = (p, m)(\nu, \kappa)(n, \ell) = (p, m)(\nu n, \kappa \ell) = (p(\nu n), m^{\nu n}(\kappa \ell))$$

$$\text{and } e_1 : P \ltimes M \rightarrow (P \ltimes M) \ltimes (N \ltimes L) \quad (48)$$

$$e_1(p, m) = ((p, m), (1, 1)).$$

We now check that $(e_1; t_1, h_1 : (P \ltimes M) \ltimes (N \ltimes L) \rightarrow P \ltimes M)$ is a cat^1 -group. Note that

$$\begin{aligned} \ker t_1 &= \{ ((1, 1), (n, \ell)) \} \\ \ker h_1 &= \{ ((p, m), (n, \ell)) \} \quad \text{where } p = (\nu n)^{-1} \text{ and } m = (\kappa \ell^{-1})^p, \end{aligned}$$

so that

$$\ell^{pn} = \ell^{(\nu n)^{-1}(\nu n)} = \ell \quad \text{in } \ker h_1.$$

The formula for the action is given by equation (45):

$$(n, \ell)^{(p, m)} := (n^p, (n^p \boxtimes m) \ell^{pm}).$$

We now check the cat^1 -group axioms.

$$\begin{aligned} \mathbf{C1:} \quad t_1 \circ e_1(p, m) &= t_1((p, m), (1, 1)) = (p, m) \\ h_1 \circ e_1(p, m) &= h_1((p, m), (1, 1)) = (p, m) \end{aligned}$$

C2:

$$\begin{aligned} ((1, 1), (n_0, \ell_0))((p, m), (n, \ell)) &= ((p, m), (n_0, \ell_0)^{(p, m)}(n, \ell)) \\ &= ((p, m), (n_0^p, (n_0^p \boxtimes m) \ell_0^{pm})(n, \ell)) \\ &= ((p, m), (n_0^{(\nu n)^{-1}} n, (n_0^p \boxtimes m)^n \ell_0^{pmn} \ell)) \\ &= ((p, m), (nn_0 n^{-1} n, (n_0 \boxtimes \kappa \ell^{-1})^{pn} \ell_0^{p\kappa \ell^{-1} n} \ell)) \\ &= ((p, m), (nn_0, [\ell^{n_0} \ell^{-1}]^{pn} \ell_0^{\kappa \ell^{-1} pn} \ell)) \\ &= ((p, m), (nn_0, \ell^{n_0} \ell^{-1} \ell_0^{\kappa \ell^{-1} pn} \ell)) \\ &= ((p, m), (nn_0, \ell^{n_0} \ell^{-1} [\ell \ell_0 \ell^{-1}]^{pn} \ell)) \\ &= ((p, m), (nn_0, \ell^{n_0} (\ell^{-1} \ell) \ell_0^{p(\nu n)} (\ell^{-1} \ell))) \\ &= ((p, m), (n, \ell)(n_0, \ell_0)) \\ &= ((p, m), (n, \ell))((1, 1), (n_0, \ell_0)) \end{aligned}$$

and so the two kernels commute.

Theorem 8.20 *The homomorphisms in (47) and (48) give a cat^2 -group:*

$$\mathcal{C}(\mathcal{R}) = \begin{array}{ccc} (P \ltimes N) \ltimes (M \ltimes L) & \begin{array}{c} \xrightarrow{t_1, h_1} \\ \xleftarrow{e_1} \end{array} & P \ltimes M \\ \begin{array}{c} \uparrow \scriptstyle t_2, h_2 \\ \downarrow \scriptstyle e_2 \end{array} & \begin{array}{c} \swarrow \scriptstyle e_0 \\ \searrow \scriptstyle t_0, h_0 \end{array} & \begin{array}{c} \uparrow \scriptstyle e_3 \\ \downarrow \scriptstyle t_3, h_3 \end{array} \\ P \ltimes N & \begin{array}{c} \xleftarrow{e_4} \\ \xrightarrow{t_4, h_4} \end{array} & P \end{array} \quad (49)$$

Proof: To be added. (Is material required from Section 7.2 ?)

□

8.5 The other cat^2 -structure

Now the underlying diagram (38) of the crossed square \mathcal{R} , together with the crossed pairing

$$\tilde{\boxtimes} : M \times N \rightarrow L, \quad (m \tilde{\boxtimes} n) = (n \boxtimes m)^{-1}$$

forms a second crossed square $\tilde{\mathcal{S}}$. Thus we can form a second cat^2 -group $\mathcal{C}(\tilde{\mathcal{S}})$ with

$$\tilde{t}_1, \tilde{h}_1 : (P \ltimes M) \ltimes (N \ltimes L) \rightarrow (P \ltimes M) .$$

Let

$$G = (P \ltimes N) \ltimes (M \ltimes L), \quad \tilde{G} = (P \ltimes M) \ltimes (N \ltimes L) .$$

Proposition 8.21 *There is an isomorphism between these two semidirect products:*

$$\begin{aligned} \tau : G &\rightarrow \tilde{G}, & ((p, n), (m, \ell)) &\mapsto ((p, m), (n, (n \boxtimes m)\ell)) , \\ \tilde{\tau} := \tau^{-1} : \tilde{G} &\rightarrow G, & ((p, m), (n, \ell)) &\mapsto ((p, n), (m, (m \tilde{\boxtimes} n)\ell)) . \end{aligned} \tag{50}$$

Proof:

$$\begin{aligned} &\tau((p_1, n_1), (m_1, \ell_1))((p_2, n_2), (m_2, \ell_2)) \\ &= \tau((p_1, n_1)(p_2, n_2), (m_1, \ell_1)^{(p_2, n_2)}(m_2, \ell_2)) \\ &= \tau((p_1 p_2, n_1^{p_2} n_2), (m_1^{p_2}, (n_2 \boxtimes m_1^{p_2})^{-1} \ell_1^{p_2 n_2})(m_2, \ell_2)) \\ &= \tau((p_1 p_2, n_1^{p_2} n_2), (m_1^{p_2} m_2, (n_2^{m_2} \boxtimes m_1^{p_2 m_2})^{-1} \ell_1^{p_2 n_2 m_2} \ell_2)) \\ &= ((p_1 p_2, m_1^{p_2} m_2), (n_1^{p_2} n_2, (n_1^{p_2} n_2 \boxtimes m_1^{p_2} m_2)(n_2^{m_2} \boxtimes m_1^{p_2 m_2})^{-1} \ell_1^{p_2 n_2 m_2} \ell_2)) \\ &= ((p_1 p_2, m_1^{p_2} m_2), (n_1^{p_2} n_2, (n_1^{p_2} \boxtimes m_2)^{n_2} (n_1^{p_2} \boxtimes m_1^{p_2})^{m_2 n_2} (n_2 \boxtimes m_2) \ell_1^{p_2 n_2 m_2} \ell_2)) \\ &= ((p_1 p_2, m_1^{p_2} m_2), (n_1^{p_2} n_2, (n_1^{p_2} \boxtimes m_1^{p_2} m_2)^{n_2} (n_2 \boxtimes m_2) \ell_1^{p_2 n_2 m_2} \ell_2)) \\ &= ((p_1 p_2, m_1^{p_2} m_2), (n_1^{p_2} n_2, (n_1^{p_2} \boxtimes m_1^{p_2} m_2)^{n_2} \ell_1^{p_2 m_2 n_2} (n_2 \boxtimes m_2) \ell_2)) \\ &= ((p_1 p_2, m_1^{p_2} m_2), (n_1^{p_2}, (n_1^{p_2} \boxtimes m_1^{p_2} m_2) \ell_1^{p_2 m_2})(n_2, (n_2 \boxtimes m_2) \ell_2)) \\ &= ((p_1 p_2, m_1^{p_2} m_2), (n_1^{p_2}, (n_1^{p_2} \boxtimes m_2)(n_1^{p_2} \boxtimes m_1^{p_2})^{m_2} \ell_1^{p_2 m_2})(n_2, (n_2 \boxtimes m_2) \ell_2)) \\ &= ((p_1, m_1)(p_2, m_2), (n_1, (n_1 \boxtimes m_1) \ell_1))^{(p_2, m_2)}(n_2, (n_2 \boxtimes m_2) \ell_2) \\ &= ((p_1, m_1), (n_1, (n_1 \boxtimes m_1) \ell_1)) ((p_2, m_2), (n_2, (n_2 \boxtimes m_2) \ell_2)) \\ &= \tau((p_1, n_1), (m_1, \ell_1)) \tau((p_2, n_2), (m_2, \ell_2)) \end{aligned}$$

□

Note that the subgroup $(1 \ltimes N) \ltimes (M \ltimes 1)$ does *not* in general get mapped by τ to the subgroup $(1 \ltimes M) \ltimes (N \ltimes 1)$.

9 Crossed squares and 2-fold crossed modules

We now show how the action of one crossed module on another leads to a crossed square. As a first step, we consider alternative formulations of the crossed module axioms in order to find one which can most easily be generalised.

9.1 An alternative formulation for crossed modules

Let $\mathcal{X} = (\partial : S \rightarrow R)$ be a crossed module. Define maps ι_S (similarly ι_R) and $\bar{\alpha}$ by

$$\bar{\iota}_S : S \times S \rightarrow S, (s_1, s) \mapsto s_1^s = s^{-1}s_1s, \quad \bar{\alpha} : S \times R \rightarrow S, (s, r) \mapsto s^r.$$

From these we may define

$$\begin{aligned} \iota_s &= \bar{\iota}_S(-, s) : S \rightarrow S, s_1 \mapsto s^{-1}s_1s & \text{and} & & \iota_S : S \rightarrow \text{Aut}(S), s \mapsto \iota_s, \\ \alpha_r &= \bar{\alpha}(-, r) : S \rightarrow S, s \mapsto s^r & \text{and} & & \alpha : R \rightarrow \text{Aut}(S), r \mapsto \alpha_r, \end{aligned}$$

The first crossed module axiom says that the following diagram commutes:

$$\begin{array}{ccccc} S \times R & \xrightarrow{1_S \times \alpha} & S \times \text{Aut}S & \xrightarrow{\text{eval}} & S \\ \downarrow \partial \times \iota_R & & & & \downarrow \partial \\ R \times \text{Aut}R & \xrightarrow{\quad \quad \quad} & & \xrightarrow{\text{eval}} & R \end{array} \quad (51)$$

mapping (s, r) to $\alpha_r(\partial s) = r^{-1}(\partial s)r$ and to $\partial(s^r)$.

The corresponding diagram for 2-fold crossed modules, where $\mathcal{Y} = (\delta : Q \rightarrow P)$, is

$$\begin{array}{ccccc} \mathcal{X} \times \mathcal{Y} & \xrightarrow{1_{\mathcal{X}} \times \alpha} & \mathcal{X} \times \text{Act}\mathcal{X} & \xrightarrow{\text{eval}} & \mathcal{X} \\ \downarrow \mu \times \iota_{\mathcal{Y}} & & & & \downarrow \mu \\ \mathcal{Y} \times \text{Act}\mathcal{Y} & \xrightarrow{\quad \quad \quad} & & \xrightarrow{\text{eval}} & \mathcal{Y} \end{array} \quad (52)$$

Applying these maps to $s \in S$, $r \in R$, $q \in Q$, $p \in P$ clockwise around the rectangle we obtain:

$$\begin{pmatrix} s \\ r \end{pmatrix}, \begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} s \\ r \end{pmatrix}, \begin{pmatrix} \chi_q \\ \alpha_p \end{pmatrix} \mapsto \left(\begin{pmatrix} \chi_q r \\ \ddot{\alpha}_p s \\ \dot{\alpha}_p r \end{pmatrix} \right), \begin{pmatrix} \ddot{\alpha} \dot{\mu} s \\ \dot{\alpha} \dot{\mu} r \end{pmatrix} \mapsto \left(\begin{pmatrix} \ddot{\mu}(\chi_q r) \\ \ddot{\mu}(s^p) \\ \dot{\mu}(r^p) \end{pmatrix} \right).$$

In the other direction,

$$\begin{pmatrix} s \\ r \end{pmatrix}, \begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} \ddot{\mu} s \\ \dot{\mu} r \end{pmatrix}, \begin{pmatrix} \eta_q \\ \beta_p \end{pmatrix} \mapsto \left(\begin{pmatrix} \eta_q(\dot{\mu} r) \\ \dot{\beta}_p(\ddot{\mu} s) \\ \dot{\beta}_p(\dot{\mu} r) \end{pmatrix} \right).$$

Hence we obtain

$$\begin{aligned}
\eta_q(\dot{\mu}r) &= (q^{-1})^{\dot{\mu}r} q = \ddot{\mu}(\chi_q r) \\
\beta_p(\ddot{\mu}s) &= (\ddot{\mu}s)^p = \ddot{\mu}(s^p) \\
\beta_p(\dot{\mu}r) &= p^{-1}(\dot{\mu}r)p = \dot{\mu}(r^p)
\end{aligned} \tag{53}$$

From this we may deduce (for example)

$$\ddot{\mu}(s^q) = \ddot{\mu}(s^{\delta q}) = (\ddot{\mu}s)^{\delta q} = q^{-1}(\ddot{\mu}s)q . \tag{54}$$

Now we have to do a similar development for the second crossed module axiom. Corresponding to the diagram

$$\begin{array}{ccccc}
S \times S & \xrightarrow{1_S \times \partial} & S \times R & \xrightarrow{1_S \times \alpha} & S \times \text{Aut} S \\
\downarrow 1_S \times \iota_S & & & & \downarrow \text{eval} \\
S \times \text{Aut} S & \xrightarrow{\text{eval}} & & & S
\end{array} \tag{55}$$

mapping (s_0, s) to s_0^s and to $s_0^{\partial s}$, the diagram for 2-fold crossed modules is :

$$\begin{array}{ccccc}
\mathcal{X} \times \mathcal{X} & \xrightarrow{1_{\mathcal{X}} \times \mu} & \mathcal{X} \times \mathcal{Y} & \xrightarrow{1_{\mathcal{X}} \times \alpha} & \mathcal{X} \times \text{Act} \mathcal{X} \\
\downarrow 1_{\mathcal{X}} \times \iota_{\mathcal{X}} & & & & \downarrow \text{eval} \\
\mathcal{X} \times \text{Act} \mathcal{X} & \xrightarrow{\text{eval}} & & & \mathcal{X}
\end{array} \tag{56}$$

Applying these maps to $s, s_0 \in S$, $r, r_0 \in R$ clockwise around the rectangle we have:

$$\begin{pmatrix} s_0 \\ r_0 \end{pmatrix}, \begin{pmatrix} s \\ r \end{pmatrix} \mapsto \begin{pmatrix} s_0 \\ r_0 \end{pmatrix}, \begin{pmatrix} \ddot{\mu}s \\ \dot{\mu}r \end{pmatrix} \mapsto \begin{pmatrix} s_0 \\ r_0 \end{pmatrix}, \begin{pmatrix} \ddot{\alpha}\ddot{\mu}s \\ \dot{\alpha}\dot{\mu}r \end{pmatrix} \mapsto \begin{pmatrix} \chi_{\ddot{\mu}s} r_0 \\ s_0^{\dot{\mu}r} \\ r_0^{\dot{\mu}r} \end{pmatrix} .$$

In the other direction,

$$\begin{pmatrix} s_0 \\ r_0 \end{pmatrix}, \begin{pmatrix} s \\ r \end{pmatrix} \mapsto \begin{pmatrix} s_0 \\ r_0 \end{pmatrix}, \begin{pmatrix} \eta_s \\ \beta_r \end{pmatrix} \mapsto \begin{pmatrix} \eta_s r_0 \\ s_0^r \\ r_0^r \end{pmatrix} .$$

Thus we obtain

$$\begin{aligned}
\chi_{\ddot{\mu}s} r_0 &= (s^{-1})^{r_0} s , \\
s_0^{\dot{\mu}r} &= s_0^r , \\
r_0^{\dot{\mu}r} &= r^{-1} r_0 r .
\end{aligned} \tag{57}$$

9.2 Crossed Modules of Crossed Modules

Using the generalisation of the crossed module axioms given in the previous section we obtain the following definition.

Definition 9.1 A crossed module of crossed modules or 2-fold crossed module has the form $\mathcal{M} = (\mu : \mathcal{X} \rightarrow \mathcal{Y})$ where $\mathcal{X} = (\partial : S \rightarrow R)$, $\mathcal{Y} = (\delta : Q \rightarrow P)$, and $\mu = (\ddot{\mu}, \dot{\mu}) : \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of crossed modules, together with an action of \mathcal{Y} on \mathcal{X} , as in Subsection ??, satisfying:

$$\mathbf{CC1:} \quad \ddot{\mu}(s^p) = (\ddot{\mu}s)^p, \quad \dot{\mu}(r^p) = p^{-1}(\dot{\mu}r)p, \quad \ddot{\mu} \circ \chi_q = \eta_q \circ \dot{\mu},$$

$$\mathbf{CC2:} \quad s_0 \dot{\mu} r = s_0^r, \quad r_0 \dot{\mu} r = r^{-1} r_0 r, \quad \chi_{\ddot{\mu}s} = \eta_s.$$

$$\begin{array}{ccccc}
 & & \ddot{\mu} & & \\
 & \text{---} & & \text{---} & \\
 S & \xrightarrow{\ddot{i}} & W(\mathcal{X}) & \xleftarrow{\ddot{\alpha}} & Q \\
 \downarrow \partial & & \downarrow \Delta & & \downarrow \delta \\
 R & \xrightarrow{i} & A(\mathcal{X}) & \xleftarrow{\dot{\alpha}} & P \\
 & \text{---} & & \text{---} & \\
 & & \dot{\mu} & &
 \end{array} \tag{58}$$

Axioms **CC1:** and **CC2:** are sufficient to make the diagram commute.

Lemma 9.2

$$\alpha\mu = \iota.$$

Proof:

$$\ddot{\alpha}\ddot{\mu}s = \chi_{\ddot{\mu}s} = \eta_s = \ddot{i}s \quad \text{and} \quad \dot{\alpha}\dot{\mu}r = \beta_{\dot{\mu}r} = \beta_r = ir.$$

□

In order to show that \mathcal{X} and \mathcal{Y} form a crossed square we have to define a crossed pairing, and there is one obvious choice.

Proposition 9.3 The 2-fold crossed module \mathcal{M} is a crossed square with crossed pairing

$$\boxtimes : R \times Q \rightarrow S, \quad (r, q) \mapsto r \boxtimes q = (\ddot{\alpha}q)r = \chi_q r.$$

Proof: Axiom (a) for crossed squares requires $\ddot{\mu}(s^p) = (\ddot{\mu}s)^p$ and $\partial(s^p) = (\partial s)^p$. The first of these is given by **CC1:** while $(\partial s)^p = \dot{\beta}_p \partial s = \partial \dot{\beta}_p s = \partial(s^p)$.

For (b) we require that $(\ddot{\mu} : S \rightarrow Q)$ and $(\dot{\mu} : R \rightarrow P)$ are crossed modules:

$$\ddot{\mu}(s^q) = q^{-1}(\ddot{\mu}s)q \quad \text{by (54)} \quad \text{and} \quad s_0 \ddot{\mu} s = s_0^{\delta \ddot{\mu} s} = s_0^{\dot{\mu} \partial s} = s_0^{\partial s} = s^{-1} s_0 s,$$

$$\dot{\mu}(r^p) = p^{-1}(\dot{\mu}r)p \quad \text{by CC1:} \quad \text{and} \quad r_0 \dot{\mu} r = r^{-1} r_0 r \quad \text{by CC2:}.$$

We then verify that \boxtimes is a crossed pairing.

- (i) $(r_1 r_2) \boxtimes q = (\ddot{\alpha} q)(r_1 r_2) = ((\ddot{\alpha} q)(r_1))^{r_2} ((\ddot{\alpha} q)(r_2)) = (r_1 \boxtimes q)^{r_2} (r_2 \boxtimes q)$
- (ii) $r \boxtimes (q_1 q_2) = \ddot{\alpha}(q_1 q_2)(r) = (\ddot{\alpha} q_1 \star \ddot{\alpha} q_2)(r) = (\chi_{q_2} r)(\chi_{q_1} r)(\chi_{q_2} \partial \chi_{q_1} r)$
 $= (\chi_{q_2} r)(\chi_{q_1} r)^{q_2} \text{ (using Lemma ?? (b))} = (r \boxtimes q_2)(r \boxtimes q_1)^{q_2}.$
- (iii) $r^p \boxtimes q^p = (\ddot{\alpha} q^p)(r^p) = (\ddot{\alpha} q)^{\dot{\alpha} p}(r^p) = (\chi_q)^{\beta_p}(r^p) = \ddot{\beta}_p \chi_q \dot{\beta}_p^{-1}(r^p) = \ddot{\beta}_p \chi_q r = (r \boxtimes q)^p.$

To verify axiom (d) for a crossed square, we check:

$$\begin{aligned} \ddot{\mu}(r \boxtimes q) &= \ddot{\mu}(\chi_q r) = (q^{-1})^{\dot{\mu} r} q = (q^{-1})^r q, \\ r \partial(r \boxtimes q) &= r(\partial \chi_q r) = \dot{\beta}_{\chi_q} r = \Delta \chi_q r = (\Delta \ddot{\alpha} q) r = (\dot{\alpha} \delta q) r = \dot{\beta}_{\delta q} r = r^{\delta q} = r^q. \end{aligned}$$

To verify axiom (e) we check the following:

$$\begin{aligned} (r \boxtimes \ddot{\mu} s) &= \chi_{\ddot{\mu} s} r = \eta_s r = (s^{-1})^r s, \\ s(\partial s \boxtimes q) &= s(\chi_q \partial s) = \ddot{\beta}_{\chi_q} s = \Delta \chi_q s = (\Delta \ddot{\alpha} q) s = (\dot{\alpha} \delta q) s = \ddot{\beta}_{\delta q} s = s^{\delta q} = s^q. \end{aligned}$$

□

Definition 9.4 A morphism of crossed module of crossed modules

$$\theta : \mathcal{M}_1 = (\mu_1 : \mathcal{X}_1 \rightarrow \mathcal{Y}_1) \rightarrow \mathcal{M}_2 = (\mu_2 : \mathcal{X}_2 \rightarrow \mathcal{Y}_2)$$

is a pair of crossed module morphisms

$$\ddot{\theta} : \mathcal{X}_1 \rightarrow \mathcal{X}_2, \quad \dot{\theta} : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2,$$

plus some extra condition(s) which say that the action is preserved.

[Complete this definition.]

9.3 A 2-fold crossed module from a crossed square

In order to show that the category \mathbf{XXMod} of 2-fold crossed modules and their morphisms is equivalent to the category \mathbf{XSq} of crossed squares and their morphisms we require a construction in the reverse direction.

Theorem 9.5 Let \mathcal{R} be a crossed square with crossed pairing $\boxtimes : R_{\{1\}} \times R_{\{2\}} \rightarrow R_{[2]}$,

$$\mathcal{R} = \begin{array}{ccc} R_{[2]} & \xrightarrow{\ddot{\partial}_1} & R_{\{2\}} \\ \ddot{\partial}_2 \downarrow & & \downarrow \dot{\partial}_2 \\ R_{\{1\}} & \xrightarrow{\dot{\partial}_1} & R_{\emptyset} \end{array}$$

so that $\ddot{\mathcal{R}}_2 = (\ddot{\partial}_2 : R_{[2]} \rightarrow R_{\{1\}})$ and $\dot{\mathcal{R}}_2 = (\dot{\partial}_2 : R_{\{2\}} \rightarrow R_{\emptyset})$ are crossed modules, $(\ddot{\partial}_1, \dot{\partial}_1) : \ddot{\mathcal{R}}_2 \rightarrow \dot{\mathcal{R}}_2$ is a crossed module morphism, and $R_{\{1\}}$ and $R_{\{2\}}$ act on $R_{[2]}$ via R_{\emptyset} . Then there is an action of $\dot{\mathcal{R}}_2$ on $\ddot{\mathcal{R}}_2$ which makes $(\partial_1 : \ddot{\mathcal{R}}_2 \rightarrow \dot{\mathcal{R}}_2)$ a crossed module of crossed modules.

Proof: We first demonstrate an action of \mathcal{R}_2 on \mathcal{R}_2 by showing that there exists a morphism of crossed modules $\alpha : \dot{\mathcal{R}}_2 \rightarrow \mathcal{A}(\ddot{\mathcal{R}}_2)$ as in the following diagram:

$$\begin{array}{ccccc}
\ddot{\beta}_m, \ddot{\beta}_p & \begin{array}{c} \curvearrowright \\ \rightarrow R_{[2]} \end{array} & \xrightarrow{\ddot{i}} & W(\ddot{\mathcal{R}}_2) & \xleftarrow{\ddot{\alpha}} R_{\{2\}} \\
& \uparrow \chi^\ell, \chi^n & & \downarrow \Delta & \downarrow \dot{\partial}_2 \\
& \begin{array}{c} \leftarrow R_{\{1\}} \\ \curvearrowright \end{array} & \xrightarrow{i} & A(\ddot{\mathcal{R}}_2) & \xleftarrow{\dot{\alpha}} R_\emptyset \\
& \downarrow \ddot{\partial}_2 & & \uparrow \dot{\partial}_1 & \\
& \begin{array}{c} \leftarrow R_{[2]} \\ \curvearrowright \end{array} & & & \\
\dot{\beta}_m, \dot{\beta}_p & \begin{array}{c} \curvearrowright \\ \rightarrow R_{\{1\}} \end{array} & & & \\
\ddot{\mathcal{R}}_2 & & \ddot{\mathcal{A}}(\ddot{\mathcal{R}}_2) & & \dot{\mathcal{R}}_2
\end{array}
\tag{59}$$

Define

$$\dot{\alpha} : R_\emptyset \rightarrow \mathcal{A}(\ddot{\mathcal{R}}_2), \quad p \mapsto \beta_p : \ddot{\mathcal{R}}_2 \rightarrow \ddot{\mathcal{R}}_2 \quad \text{for all } p \in R_\emptyset ,$$

where

$$\ddot{\beta}_p \ell = \ell^p \quad \text{for all } \ell \in R_{[2]} \quad \text{and} \quad \dot{\beta}_p m = m^p \quad \text{for all } m \in R_{\{1\}}.$$

We see that $\dot{\alpha}p$ is a crossed module morphism since $\ddot{\beta}$ and $\dot{\beta}$ are group homomorphisms and

$$(\ddot{\beta}_p \ell)^{\dot{\beta}_p m} = (\ell^p)^{m^p} = \ell^{pp^{-1}mp} = \ell^{mp} = \ddot{\beta}_p(\ell^m) .$$

The map $\ddot{\alpha} : R_{\{2\}} \rightarrow W(\ddot{\mathcal{R}}_2)$ is defined by

$$\ddot{\alpha}n = \chi_n: R_{\{1\}} \rightarrow R_{[2]}, \quad \chi_n m = m \boxtimes n, \quad \text{for all } m \in R_{\{1\}}. \quad (60)$$

Each χ_n is a derivation of \mathcal{R}_1 since

$$\chi_n(m_1 m_2) = (m_1 m_2 \boxtimes n) = (m_1 \boxtimes n)^{m_2} (m_2 \boxtimes n) = (\chi_n m_1)^{m_2} (\chi_n m_2).$$

That $\ddot{\alpha}$ is a homomorphism is verified by

$$\begin{aligned}
(\chi_{n_1} \star \chi_{n_2}) m &= (\chi_{n_2} m) (\chi_{n_1} m) \chi_{n_2} \ddot{\partial}_2 (m \boxtimes n_1) \\
&= (m \boxtimes n_2) (m \boxtimes n_1) \chi_{n_2} (m^{-1} m^{n_1}) \\
&= (m \boxtimes n_2) (m \boxtimes n_1) (\chi_{n_2} m^{-1})^{m^{n_1}} (\chi_{n_2} m^{n_1}) \\
&= (m \boxtimes n_2) (m \boxtimes n_1) (m^{-1} \boxtimes n_2)^{n_1^{-1} m n_1} (m^{n_1} \boxtimes n_2) \\
&= (m \boxtimes n_2) (m \boxtimes n_1) ((m \boxtimes n_2)^{[m, n_1]})^{-1} (m \boxtimes n_1)^{-1} (m \boxtimes n_2) (m \boxtimes n_1)^{n_2} \\
&= (m \boxtimes n_1) \{ (m \boxtimes n_1)^{-1} (m \boxtimes n_2) \} (m \boxtimes n_1)^{n_2} \\
&= (m \boxtimes n_2) (m \boxtimes n_1)^{n_2} = (m \boxtimes n_1 n_2) = \chi_{n_1 n_2} m .
\end{aligned}$$

To show that α preserves actions, we calculate

$$\begin{aligned}\ddot{\alpha}(n^p) &= \chi_{n^p} : m \mapsto (m \boxtimes n^p), \\ (\ddot{\alpha}n)^{\dot{\alpha}p} &= \beta_p^{-1} * \chi_n * \beta_p : m \mapsto \ddot{\beta}_p \chi_n \dot{\beta}_p^{-1} m = \ddot{\beta}_p \chi_n (m^{p^{-1}}) = (m^{p^{-1}} \boxtimes n)^p = (m \boxtimes n^p).\end{aligned}$$

To show that the right-hand square in diagram (58) commutes, $\dot{\alpha}\dot{\partial}_2 = \Delta\ddot{\alpha}$, note that

$$\Delta\ddot{\alpha}n = \Delta\chi_n = (\ddot{\beta}_{\chi_n}, \dot{\beta}_{\chi_n})$$

where

$$\dot{\beta}_{\chi_n} m = m(\ddot{\partial}_2 \chi_n m) = m(\ddot{\partial}_2(m \boxtimes n)) = m(m^{-1}m^n) = m^n = \dot{\beta}_{\dot{\partial}_2 n} m,$$

and

$$\ddot{\beta}_{\chi_n} \ell = \ell(\chi_n \ddot{\partial}_2 \ell) = \ell(\ddot{\partial}_2 \ell \boxtimes n) = \ell^n = \ddot{\beta}_{\dot{\partial}_2 n} \ell.$$

So

$$\Delta\ddot{\alpha}n = (\ddot{\beta}_{\dot{\partial}_2 n}, \dot{\beta}_{\dot{\partial}_2 n}) = \beta_{\dot{\partial}_2 n} = \dot{\alpha}\dot{\partial}_2 n.$$

This completes the proof that $\alpha : \dot{\mathcal{R}}_2 \rightarrow \mathcal{A}(\ddot{\mathcal{R}}_2)$ is a morphism of crossed modules.

Thus the crossed square \mathcal{R} gives rise to a semidirect crossed module $\dot{\mathcal{R}}_2 \ltimes \ddot{\mathcal{R}}_2$.

Of course we could obtain a second semidirect product crossed module $\dot{\mathcal{R}}_1 \ltimes \ddot{\mathcal{R}}_1$ for the transpose of \mathcal{R} by reversing the roles of $R_{\{1\}}$ and $R_{\{2\}}$ and using the crossed pairing $\tilde{\boxtimes}$.

The morphism $\iota = (i, \ddot{i})$ in diagram (59) is given (see Subsection 3.3) by:

$$i : R_{\{1\}} \rightarrow A, \quad im = \beta_m = (\ddot{\beta}_m, \dot{\beta}_m) : \ddot{\mathcal{R}}_2 \rightarrow \ddot{\mathcal{R}}_2, \quad \text{for all } m \in R_{\{1\}},$$

where $\ddot{\beta}_m \ell = \ell^m$ for all $\ell \in R_{[2]}$ and $\dot{\beta}_m m_0 = m_0^m = m^{-1}m_0m$ for all $m_0 \in R_{\{1\}}$, and

$$\ddot{i} : R_{[2]} \rightarrow W, \quad \ell \mapsto \eta_\ell : R_{\{1\}} \rightarrow R_{[2]}, \quad m \mapsto (\ell^{-1})^m \ell \quad \text{for all } m \in R_{\{1\}}.$$

The 2-fold crossed module axioms for $(\partial_1 : \ddot{\mathcal{R}}_2 \rightarrow \dot{\mathcal{R}}_2)$ are easily verified.

- CC1:** (i) $\ddot{\partial}_1(\ell^p) = (\ddot{\partial}_1 \ell)^p$ by (a),
(ii) $\dot{\partial}_1(m^p) = p^{-1}(\dot{\partial}_1 m)p$ by **M1:** for $\dot{\mathcal{R}}_1$,
(iii) $\ddot{\partial}_1(\chi_n m) = \ddot{\partial}_1(m \boxtimes n) = (n^{-1})^m n = (n^{-1})^{\dot{\partial}_1 m} n = \eta_n(\dot{\partial}_1 m)$.
- CC2:** (i) $\ell^{\dot{\partial}_1 m} = \ell^m$ by definition of the action,
(ii) $m_0^{\dot{\partial}_1 m} = m^{-1}m_0m$ by **M2:** for $\dot{\mathcal{R}}_1$,
(iii) $\chi_{\ddot{\partial}_1 \ell} m = m \boxtimes \ddot{\partial}_1 \ell = (\ell^{-1})^m \ell = \eta_\ell m$.

□

9.4 Derivations of a 2-fold crossed module

The following definition (b) of an \mathcal{R}_1 -derivation *appears* to be correct, but we shall see that an extra axiom is appropriate when we consider the corresponding version of a section. The missing information is an expansion for $\ddot{\chi}(n^p)$. A revised version will be given as Definition 9.18.

Definition 9.6 Let \mathcal{R}_1 be the usual crossed square \mathcal{R} of (38) considered as a crossed module of crossed modules $(\partial_1 = (\ddot{\partial}_1, \dot{\partial}_1) : \ddot{\mathcal{R}}_2 \rightarrow \dot{\mathcal{R}}_2)$.

$$\mathcal{R}_1 = \begin{array}{ccc} R_{[2]} & \begin{array}{c} \xrightarrow{\ddot{\partial}_1} \\ \xleftarrow{\ddot{\phi}, \ddot{\chi}} \end{array} & R_{\{2\}} \\ \downarrow \ddot{\partial}_2 & & \downarrow \dot{\partial}_2 \\ R_{\{1\}} & \begin{array}{c} \xleftarrow{\dot{\phi}, \dot{\chi}} \\ \xrightarrow{\dot{\partial}_1} \end{array} & R_{\emptyset} \end{array} \quad (61)$$

(a) An \mathcal{R}_1 -map is a pair of maps $\phi = (\ddot{\phi}, \dot{\phi}) : \dot{\mathcal{R}}_2 \rightarrow \ddot{\mathcal{R}}_2$ which commute with the two boundaries:

$$\ddot{\phi} * \ddot{\partial}_2 = \dot{\partial}_2 * \dot{\phi}$$

(b) An \mathcal{R}_1 -derivation is an \mathcal{R} -map $\chi = (\ddot{\chi}, \dot{\chi})$ such that $\ddot{\chi}$ is a derivation of $\ddot{\mathcal{R}}_1$ and $\dot{\chi}$ is a derivation of $\dot{\mathcal{R}}_1$.

Similarly, let \mathcal{R}_2 be \mathcal{R} considered as $(\partial_2 = (\ddot{\partial}_2, \dot{\partial}_2) : \ddot{\mathcal{R}}_1 \rightarrow \dot{\mathcal{R}}_1)$, giving notions of \mathcal{R}_2 -map and \mathcal{R}_2 -derivation.

Lemma 9.7 To check that a pair of derivations $\chi = (\ddot{\chi}, \dot{\chi})$ is an \mathcal{R}_1 -derivation it is sufficient to check that $\ddot{\chi} * \ddot{\partial}_2 = \dot{\partial}_2 * \dot{\chi}$ on a generating set of $R_{\{2\}}$.

Proof: If $\ddot{\partial}_2 \ddot{\chi} n_1 = \dot{\chi} \dot{\partial}_2 n_1$ and $\ddot{\partial}_2 \ddot{\chi} n_2 = \dot{\chi} \dot{\partial}_2 n_2$ then

$$\ddot{\partial}_2 \ddot{\chi}(n_1 n_2) = \ddot{\partial}_2((\ddot{\chi} n_1)^{n_2}(\ddot{\chi} n_2)) = (\ddot{\partial}_2 \ddot{\chi} n_1)^{\dot{\partial}_2 n_2}(\ddot{\partial}_2 \ddot{\chi} n_2) = (\dot{\chi} \dot{\partial}_2 n_1)^{n_2}(\dot{\chi} \dot{\partial}_2 n_2) = \dot{\chi} \dot{\partial}_2(n_1 n_2).$$

□

Lemma 9.8 The set of \mathcal{R}_1 -derivations has a Whitehead multiplication

$$\chi_1 \star \chi_2 = (\ddot{\chi}_1 \star \ddot{\chi}_2, \dot{\chi}_1 \star \dot{\chi}_2).$$

Proof: Since $\ddot{\chi}_1 \star \ddot{\chi}_2$ and $\dot{\chi}_1 \star \dot{\chi}_2$ are derivations of $\ddot{\mathcal{R}}_1$ and $\dot{\mathcal{R}}_1$ respectively, we only have to show that $\chi_1 \star \chi_2$ is an \mathcal{R}_1 -map.

$$\begin{aligned} \ddot{\partial}_2(\ddot{\chi}_1 \star \ddot{\chi}_2)(n) &= (\ddot{\partial}_2 \ddot{\chi}_2 n)(\ddot{\partial}_2 \ddot{\chi}_1 n)(\ddot{\partial}_2 \ddot{\chi}_2 \dot{\partial}_1 \ddot{\chi}_1 n) \\ &= (\dot{\chi}_2 \dot{\partial}_2 n)(\dot{\chi}_1 \dot{\partial}_2 n)(\dot{\chi}_2 \dot{\partial}_2 \dot{\partial}_1 \ddot{\chi}_1 n) \\ &= (\dot{\chi}_2 \dot{\partial}_2 n)(\dot{\chi}_1 \dot{\partial}_2 n)(\dot{\chi}_2 \dot{\partial}_1 \ddot{\partial}_2 \ddot{\chi}_1 n) \\ &= (\dot{\chi}_2 \dot{\partial}_2 n)(\dot{\chi}_1 \dot{\partial}_2 n)(\dot{\chi}_2 \dot{\partial}_1 \dot{\chi}_1 \dot{\partial}_2 n) \\ &= (\dot{\chi}_2 \star \dot{\chi}_1)(\dot{\partial}_2 n). \end{aligned}$$

□

[Not sure if the following Lemma is true!]

Lemma 9.9 *A pair of principal derivations $\eta = (\ddot{\eta}_\ell, \dot{\eta}_m)$ is an \mathcal{R}_1 -derivation if and only if $(\ddot{\partial}_2 \ell) m^{-1}$ is fixed by the action of $\dot{\partial}_2 R_{\{2\}}$.*

Proof:

$$\ddot{\partial}_2 \ddot{\eta}_\ell n = \dot{\eta}_m \dot{\partial}_2 n \Leftrightarrow (\ddot{\partial}_2 \ell^{-1})^{\dot{\partial}_2 n} (\ddot{\partial}_2 \ell) = (m^{-1})^{\dot{\partial}_2 n} m \Leftrightarrow (\ddot{\partial}_2 \ell) m^{-1} = ((\ddot{\partial}_2 \ell) m^{-1})^{\dot{\partial}_2 n}.$$

[But what about equation (63)?] □

Definition 9.10 *A principal \mathcal{R}_1 -derivation is an \mathcal{R}_1 -derivation $\eta = (\ddot{\eta}_\ell, \dot{\eta}_m)$ such that $\ddot{\eta}_\ell, \dot{\eta}_m$ are principal derivations.*

Given an \mathcal{R}_1 -derivation $\chi = (\ddot{\chi}, \dot{\chi})$ we have automorphisms $\beta_{\ddot{\chi}} = (\ddot{\beta}_{\ddot{\chi}}, \dot{\beta}_{\ddot{\chi}})$ of $\ddot{\mathcal{R}}_1$ and $\beta_{\dot{\chi}} = (\dot{\beta}_{\dot{\chi}}, \ddot{\beta}_{\dot{\chi}})$ of $\dot{\mathcal{R}}_1$ given by

$$\ddot{\beta}_{\ddot{\chi}} \ell = \ell(\ddot{\chi} \ddot{\partial}_1 \ell), \quad \dot{\beta}_{\dot{\chi}} n = n(\dot{\chi} \dot{\partial}_1 n), \quad \ddot{\beta}_{\ddot{\chi}} m = m(\dot{\chi} \dot{\partial}_1 m), \quad \dot{\beta}_{\dot{\chi}} p = p(\dot{\chi} \dot{\partial}_1 p),$$

such that

$$\ddot{\beta}_{\ddot{\chi}} \ddot{\chi} = \ddot{\chi} \dot{\beta}_{\dot{\chi}}, \quad n \mapsto (\ddot{\chi} n)(\ddot{\chi} \ddot{\partial}_1 \ddot{\chi} n) \quad \text{and} \quad \dot{\beta}_{\dot{\chi}} \dot{\chi} = \dot{\chi} \ddot{\beta}_{\dot{\chi}}, \quad p \mapsto (\dot{\chi} p)(\dot{\chi} \dot{\partial}_1 \dot{\chi} p).$$

These automorphisms will be important for our construction of the actor crossed square.

Since these maps are automorphisms, we know that

$$\ddot{\beta}_{\ddot{\chi}}(\ell^m) = (\ddot{\beta}_{\ddot{\chi}} \ell)^{\dot{\beta}_{\dot{\chi}} n} \quad \text{and} \quad \dot{\beta}_{\dot{\chi}}(m^p) = (\dot{\beta}_{\dot{\chi}} m)^{\ddot{\beta}_{\dot{\chi}} p}.$$

Lemma 9.11 *These four group automorphisms combine to give automorphisms*

$$\beta_{\ddot{\chi}} = (\ddot{\beta}_{\ddot{\chi}}, \dot{\beta}_{\dot{\chi}}) \text{ of } \ddot{\mathcal{R}}_2 \quad \text{and} \quad \beta_{\dot{\chi}} = (\dot{\beta}_{\dot{\chi}}, \ddot{\beta}_{\dot{\chi}}) \text{ of } \dot{\mathcal{R}}_2.$$

Proof: We first check commutation:

$$\begin{aligned} \ddot{\partial}_2 \ddot{\beta}_{\ddot{\chi}} \ell &= (\ddot{\partial}_2 \ell)(\ddot{\partial}_2 \ddot{\chi} \ddot{\partial}_1 \ell) = (\ddot{\partial}_2 \ell)(\dot{\chi} \dot{\partial}_1 \ddot{\partial}_1 \ell) = (\ddot{\partial}_2 \ell)(\dot{\chi} \dot{\partial}_1 (\ddot{\partial}_2 \ell)) = \ddot{\beta}_{\dot{\chi}} \ddot{\partial}_2 \ell, \\ \dot{\partial}_2 \dot{\beta}_{\dot{\chi}} n &= (\dot{\partial}_2 n)(\dot{\partial}_2 \dot{\chi} \dot{\partial}_1 n) = (\dot{\partial}_2 n)(\dot{\partial}_1 \ddot{\partial}_2 \dot{\chi} n) = (\dot{\partial}_2 n)(\dot{\partial}_1 \dot{\chi} (\dot{\partial}_2 n)) = \dot{\beta}_{\ddot{\chi}} \dot{\partial}_2 n. \end{aligned}$$

We now require to prove

$$(a) \quad \ddot{\beta}_{\ddot{\chi}}(\ell^m) = (\ddot{\beta}_{\ddot{\chi}} \ell)^{\ddot{\beta}_{\dot{\chi}} m} \quad \text{and} \quad (b) \quad \dot{\beta}_{\dot{\chi}}(n^p) = (\dot{\beta}_{\dot{\chi}} n)^{\ddot{\beta}_{\dot{\chi}} p}.$$

We prove (b) as follows.

$$\begin{aligned} \dot{\beta}_{\dot{\chi}}(n^p) &= n^p(\dot{\partial}_1 \dot{\chi}(n^p)) \\ &= n^p \dot{\partial}_1[(\dot{\chi} p \boxtimes n^p)^{-1}(\dot{\chi} n)^{p(\dot{\chi} p)}] \\ &= n^p (n^p)^{-1} (n^p)^{\dot{\chi} p} (\dot{\partial}_1 \dot{\chi} n)^{p(\dot{\partial}_1 \dot{\chi} p)} \\ &= (n(\dot{\partial}_1 \dot{\chi} n))^{p(\dot{\partial}_1 \dot{\chi} p)} \\ &= (\dot{\beta}_{\dot{\chi}} n)^{\ddot{\beta}_{\dot{\chi}} p}. \end{aligned}$$

[We still need to check (a) !] □

Lemma 9.12 *The endomorphisms $\ddot{\beta}_{\ddot{\chi}}, \dot{\beta}_{\ddot{\chi}}, \ddot{\beta}_{\dot{\chi}}, \dot{\beta}_{\dot{\chi}}$ commute with $\ddot{\chi}\ddot{\partial}_1, \ddot{\partial}_1\ddot{\chi}, \dot{\chi}\dot{\partial}_1, \dot{\partial}_1\dot{\chi}$ respectively.*

Proof: This follows immediately from Lemma 2.4(d). \square

Lemma 9.13 *Given a composite \mathcal{R}_1 -derivation $\chi = \chi_1 * \chi_2$,*

$$\ddot{\beta}_{\ddot{\chi}} = \ddot{\beta}_{\ddot{\chi}_1} * \ddot{\beta}_{\ddot{\chi}_2} \quad \text{and} \quad \dot{\beta}_{\dot{\chi}} = \dot{\beta}_{\dot{\chi}_1} * \dot{\beta}_{\dot{\chi}_2}.$$

Proof: This is immediate from Corollary 2.3.

[Result on derivations changed to γ -derivations – what happens here?] \square

9.5 Sections of a crossed module of cat^1 -groups

Replacing the crossed modules $\ddot{\mathcal{R}}_1, \dot{\mathcal{R}}_1$ by the corresponding cat^1 -groups, we obtain the following *crossed module of cat^1 -groups*, (equivalently a *cat^1 -group of crossed modules* as in Section 8.4), where $\bar{\partial}_2 : R_{\{2\}} \ltimes R_{[2]} \rightarrow R_\emptyset \ltimes R_{\{1\}}$, $(n, \ell) \mapsto (\dot{\partial}_2 n, \ddot{\partial}_2 \ell)$.

$$\mathcal{C}_1 = \begin{array}{ccc} R_{\{2\}} \ltimes R_{[2]} & \begin{array}{c} \xrightarrow{\ddot{t}_1, \ddot{h}_1} \\ \xleftarrow{\ddot{e}_1, \ddot{\xi}} \end{array} & R_{\{2\}} \\ \downarrow \bar{\partial}_2 & & \downarrow \dot{\partial}_2 \\ R_\emptyset \ltimes R_{\{1\}} & \begin{array}{c} \xleftarrow{\dot{e}_1, \dot{\xi}} \\ \xrightarrow{\dot{t}_1, \dot{h}_1} \end{array} & R_\emptyset \end{array} \quad (62)$$

Lemma 9.14 *The pair $(\bar{\partial}_2, \dot{\partial}_2)$ is a morphism of cat^1 -groups.*

Proof: We first show that $\bar{\partial}_2$ is a group homomorphism.

$$\begin{aligned} \bar{\partial}_2((n_1, \ell_1)(n_1, \ell_2)) &= \bar{\partial}_2(n_1 n_2, \ell_1^{n_2} \ell_2) \\ &= (\dot{\partial}_2(n_1 n_2), \ddot{\partial}_2(\ell_1^{n_2} \ell_2)) \\ &= ((\dot{\partial}_2 n_1)(\dot{\partial}_2 n_2), (\ddot{\partial}_2 \ell_1)^{\dot{\partial}_2 n_2} \ddot{\partial}_2 \ell_2) \\ &= (\dot{\partial}_2 n_1, \ddot{\partial}_2 \ell_1)(\dot{\partial}_2 n_2, \ddot{\partial}_2 \ell_2) \end{aligned}$$

Next we check that $(\bar{\partial}_2, \dot{\partial}_2)$ commutes with $t_1 = (\ddot{t}_1, \dot{t}_1)$, $h_1 = (\ddot{h}_1, \dot{h}_1)$ and $e_1 = (\ddot{e}_1, \dot{e}_1)$.

$$\begin{aligned} (\bar{\partial}_2 * \dot{t}_1)(n, \ell) &= \dot{t}_1(\dot{\partial}_2 n, \ddot{\partial}_2 \ell) = \dot{\partial}_2 n = (\ddot{t}_1 * \dot{\partial}_2)(n, \ell) \\ (\bar{\partial}_2 * \dot{h}_1)(n, \ell) &= \dot{h}_1(\dot{\partial}_2 n, \ddot{\partial}_2 \ell) = (\dot{\partial}_2 n)(\dot{\partial}_1 \ddot{\partial}_2 \ell) = (\dot{\partial}_2 n)(\dot{\partial}_2 \ddot{\partial}_1 \ell) = \dot{\partial}_2(n(\ddot{\partial}_1 \ell)) = (\ddot{h}_1 * \dot{\partial}_2)(n, \ell) \\ (\dot{e}_1 * \bar{\partial}_2)n &= \bar{\partial}_2(n, 1) = (\dot{\partial}_2 n, 1) = \dot{e}_1(\dot{\partial}_2 n) = (\dot{\partial}_2 * \dot{e}_1)n \end{aligned}$$

Definition 9.15 \square

(a) A \mathcal{C}_1 -map is a pair of maps $\phi = (\ddot{\phi}, \dot{\phi}) : \dot{\mathcal{R}}_2 \rightarrow \ddot{\mathcal{R}}_2 \ltimes \dot{\mathcal{R}}_2$ which commute with the two boundaries.

(b) A \mathcal{C}_1 -section is a pair $\xi = (\ddot{\xi}, \dot{\xi})$ such that

- ξ is a \mathcal{C}_1 -map: $\bar{\partial}_2 \circ \ddot{\xi} = \dot{\xi} \circ \partial_2$,
- $\ddot{\xi}$ is a section of $\ddot{\mathcal{R}}_1$ and $\dot{\xi}$ is a section of $\dot{\mathcal{R}}_1$,
- $t \circ \xi = 1_{\dot{\mathcal{R}}_2}$,
- ξ is a crossed module morphism: $\ddot{\xi}(n^p) = (\ddot{\xi}n)^{\dot{\xi}p}$.

Lemma 9.16 *If ξ is an \mathcal{C}_1 -section, then there is an \mathcal{R}_1 -derivation χ such that $\dot{\xi}p = (p, \dot{\chi}p)$ and $\ddot{\xi}n = (n, \ddot{\chi}n)$.*

Proof: The sections $\ddot{\xi}, \dot{\xi}$ determine derivations $\ddot{\chi} : R_{\{2\}} \rightarrow R_\emptyset$ and $\dot{\chi} : R_\emptyset \rightarrow R_{\{1\}}$, so we have to verify that $\chi = (\ddot{\chi}, \dot{\chi})$ is an \mathcal{R}_1 -map. Since

$$(\ddot{\xi} * \bar{\partial}_2)n = \bar{\partial}_2(n, \ddot{\chi}n) = (\partial_2 n, \ddot{\partial}_2 \ddot{\chi}n) \quad \text{and} \quad (\dot{\partial}_2 * \dot{\xi})n = \dot{\xi}(\partial_2 n) = (\partial_2 n, \dot{\chi}\partial_2 n),$$

commuting sections imply commuting derivations. \square

Definition 9.17 *A Whitehead multiplication for \mathcal{R}_1 -sections is defined as follows*

$$\xi_1 \star \xi_2 = (\ddot{\xi}_1 \star \ddot{\xi}_2, \dot{\xi}_1 \star \dot{\xi}_2).$$

Let us investigate whether the requirement that ξ is a morphism of crossed modules gives further information about χ . Since

$$\begin{aligned} \ddot{\xi}(n^p) &= (n^p, \ddot{\chi}(n^p)) , \\ (\ddot{\xi}n)^{\dot{\xi}p} &= (n, \ddot{\chi}n)^{(p, \dot{\chi}p)} = (n^p, (\dot{\chi}p \boxtimes n^p)^{-1} (\ddot{\chi}n)^{p(\dot{\chi}p)}) , \end{aligned}$$

we obtain the additional identity

$$\ddot{\chi}(n^p) = (\dot{\chi}p \boxtimes n^p)^{-1} (\ddot{\chi}n)^{p(\dot{\chi}p)} . \quad (63)$$

We can now provide the modified version of Definition 9.6 as promised earlier.

Definition 9.18 *An \mathcal{R}_1 -derivation is an \mathcal{R} -map $\chi = (\ddot{\chi}, \dot{\chi})$ such that $\ddot{\chi}$ is a derivation of $\ddot{\mathcal{R}}_1$, $\dot{\chi}$ is a derivation of $\dot{\mathcal{R}}_1$, and equation (63) is satisfied.*

The hope is that this extra information (63) will prove useful in simplifying formulae to found later. On the other hand, the right-hand side is rather a messy expression, so perhaps we should stick to sections?

Here is a tricky task: prove that a principal derivation $\eta = (\ddot{\eta}_\ell, \dot{\eta}_m)$ satisfies (63). Perhaps it is easier to do this for principal sections?

9.6 2-fold derivations of a crossed square

The maps $\theta : R_\emptyset \rightarrow R_{[2]}$ which we shall use to form the group of 2-derivations can be considered as the final part of a map

$$C_\emptyset \rightarrow C_{[2]}, \quad p \mapsto ((p, \chi p), (\phi p, \theta p)),$$

where $C_\emptyset = R_\emptyset$ and $C_{[2]} = (R_\emptyset \times R_{\{1\}}) \times (R_{\{2\}} \times R_{[2]})$ as in diagram (49).

Applying the formula for semidirect product multiplication (Proposition 46(b) ???) and the semidirect product action (45), we obtain

$$\begin{aligned} & ((p, \chi p), (\phi p, \theta p))((q, \chi q), (\phi q, \theta q)) \\ &= ((p, \chi p)(q, \chi q), (\phi p, \theta p)^{(q, \chi q)}(\phi q, \theta q)) \\ &= ((pq, (\chi p)^q \chi q), ((\phi p)^q, (\chi q \boxtimes (\phi p)^q)^{-1}(\theta p)^{q(\chi q)})(\phi q, \theta q)) \\ &= ((pq, (\chi p)^q \chi q), ((\phi p)^q, ((\phi p)^q \boxtimes \chi q)(\theta p)^{q(\chi q)})(\phi q, \theta q)) \\ &= ((pq, (\chi p)^q \chi q), ((\phi p)^q \phi q, ((\phi p)^q \boxtimes \chi q)^{\phi q}(\theta p)^{[q(\chi q)(\phi q)]}(\theta q))) \end{aligned}$$

9.7 Sections of a cat^2 -group

Recall from Section 8.3 that a cat^2 -group \mathcal{C} comprises 4 groups and 15 homomorphisms, as shown in the following diagram,

$$\mathcal{C} = \begin{array}{ccc} & \begin{array}{c} \xrightarrow{\check{t}_1, \check{h}_1} \\ \xleftarrow{\check{e}_1, \check{\xi}_1} \end{array} & C_{\{2\}} \\ \begin{array}{c} \uparrow \check{t}_2, \check{h}_2 \\ \downarrow \check{e}_2, \check{\xi}_2 \end{array} & \begin{array}{c} C_{[2]} \\ \xrightarrow{e_{[2]}, \xi_{[2]}} \\ \xleftarrow{t_{[2]}, h_{[2]}} \end{array} & \begin{array}{c} \uparrow \check{e}_2, \check{\xi}_2 \\ \downarrow \check{t}_2, \check{h}_2 \end{array} \\ & \begin{array}{c} \xleftarrow{\check{e}_1, \check{\xi}_1} \\ \xrightarrow{\check{t}_1, \check{h}_1} \end{array} & C_\emptyset \\ & C_{\{1\}} & \end{array}$$

where the four sides of the square are all cat^1 -groups, and

$$\left\{ \begin{array}{l} \check{t}_1 \check{h}_2 = \check{h}_2 \check{t}_1, \quad \check{t}_2 \check{h}_1 = \check{h}_1 \check{t}_2, \quad \check{e}_1 \check{t}_2 = \check{t}_2 \check{e}_1, \quad \check{e}_2 \check{t}_1 = \check{t}_1 \check{e}_2, \quad \check{e}_1 \check{h}_2 = \check{h}_2 \check{e}_1, \quad \check{e}_2 \check{h}_1 = \check{h}_1 \check{e}_2, \\ \check{t}_1 \check{t}_2 = \check{t}_2 \check{t}_1 = t_{[2]}, \quad \check{h}_1 \check{h}_2 = \check{h}_2 \check{h}_1 = h_{[2]}, \quad \check{e}_1 \check{e}_2 = \check{e}_2 \check{e}_1 = e_{[2]}, \end{array} \right. \quad (64)$$

while the diagonal is only a pre- cat^1 -group.

We name the four cat^1 -groups as $\check{C}_1, \dot{C}_1, \check{C}_2, \dot{C}_2$ and the diagonal as $\mathcal{C}_{[2]}$, so that $t_1 = (\check{t}_1, \dot{t}_1)$, $h_1 = (\check{h}_1, \dot{h}_1) : \check{C}_1 \rightarrow \dot{C}_1$ and $e_1 = (\check{e}_1, \dot{e}_1) : \dot{C}_1 \rightarrow \check{C}_1$ are horizontal morphisms of cat^1 -groups, while $t_2 = (\check{t}_2, \dot{t}_2)$, $h_2 = (\check{h}_2, \dot{h}_2) : \check{C}_2 \rightarrow \dot{C}_2$ and $e_2 = (\check{e}_2, \dot{e}_2) : \dot{C}_2 \rightarrow \check{C}_2$ are vertical morphisms. Pairs such as $(\check{t}_1, \text{id}_{C_\emptyset})$ are pre- cat^1 -morphisms.

Lemma 9.19 *For $\check{\xi}_1$ a section of \check{C}_1 , the map $\dot{\xi}_1 = \check{t}_2 \check{\xi}_1 \dot{e}_2$ is a section of \dot{C}_1 .*

Proof:

$$t_1 \dot{\xi}_1 = t_1 \ddot{t}_2 \ddot{\xi}_1 \dot{e}_2 = t_2 \ddot{t}_1 \ddot{\xi}_1 \dot{e}_2 = t_2 \dot{e}_2 = \text{id}_{C_0}.$$

□

We now consider $\mathcal{C}_1 = (e_1; t_1, h_1 : \check{\mathcal{C}}_1 \rightarrow \dot{\mathcal{C}}_1)$ as a cat^1 -group of cat^1 -groups and define the notion of section in this situation.

Definition 9.20 A section of \mathcal{C}_1 is a pair of maps $\xi_1 = (\ddot{\xi}_1, \dot{\xi}_1)$ where $\ddot{\xi}_1$ is a section of $\check{\mathcal{C}}_1$ and

$$\ddot{t}_2 \ddot{\xi}_1 = \dot{\xi}_1 \dot{t}_2, \quad \ddot{h}_2 \ddot{\xi}_1 = \dot{\xi}_1 \dot{h}_2 \quad \text{and} \quad \ddot{e}_2 \dot{\xi}_1 = \ddot{\xi}_1 \dot{e}_2. \quad (65)$$

By Lemma 9.19, ξ_1 a morphism of cat^1 -groups. Since both $\ddot{\xi}_1$ and $\dot{\xi}_1$ are sections, the usual formula **S1**: $t_1 \xi_1 = \text{id}_{\dot{\mathcal{C}}_2}$ holds.

Recall that the Whitehead multiplication for sections of a cat^1 -group applied to two sections of $\check{\mathcal{C}}_1$ gives

$$(\ddot{\xi}_1 \star \ddot{\zeta}_1)k = (\ddot{\xi}_1 k)(\ddot{e}_1 \ddot{h}_1 \ddot{\xi}_1 k^{-1})(\ddot{\zeta}_1 \ddot{h}_1 \ddot{\xi}_1 k) = (\ddot{\zeta}_1 \ddot{h}_1 \ddot{\xi}_1 k)(\ddot{e}_1 \ddot{h}_1 \ddot{\xi}_1 k^{-1})(\ddot{\xi}_1 k).$$

Lemma 9.21 The section of $\dot{\mathcal{C}}_1$ associated to $\ddot{\xi}_1 \star \ddot{\zeta}_1$ is $\dot{\xi}_1 \star \dot{\zeta}_1$.

Proof: The associated section is $\ddot{t}_2(\ddot{\xi}_1 \star \ddot{\zeta}_1)\dot{e}_2$ which maps $p \in C_0$ to

$$\begin{aligned} (\ddot{t}_2 \ddot{\xi}_1 \dot{e}_2 p) (\ddot{t}_2 \ddot{e}_1 \ddot{h}_1 \ddot{\xi}_1 \dot{e}_2 p^{-1}) (\ddot{t}_2 \ddot{\zeta}_1 \ddot{h}_1 \ddot{\xi}_1 \dot{e}_2 p) &= (\dot{\xi}_1 \dot{t}_2 \dot{e}_2 p) (\dot{e}_1 \dot{h}_1 \dot{\xi}_1 \dot{t}_2 \dot{e}_2 p^{-1}) (\dot{\zeta}_1 \dot{h}_1 \dot{\xi}_1 \dot{t}_2 \dot{e}_2 p) \\ &= (\dot{\xi}_1 p) (\dot{e}_1 \dot{h}_1 \dot{\xi}_1 p^{-1}) (\dot{\zeta}_1 \dot{h}_1 \dot{\xi}_1 p) = (\dot{\xi}_1 \star \dot{\zeta}_1) p, \end{aligned}$$

where identities (64) and (65) are frequently used. □

Recall that a section ξ of a cat^1 -group $\mathcal{C} = (e; t, h : C \rightarrow R)$ determines a cat^1 -group endomorphism (γ, ρ) of \mathcal{C} where

$$\begin{aligned} \rho : R &\rightarrow R, & r &\mapsto h \xi r, \\ \gamma : C &\rightarrow C, & g &\mapsto (e h \xi t g)(\xi t g^{-1}) g (e h g^{-1})(\xi h g). \end{aligned}$$

The equivalent definitions for our $\dot{\mathcal{C}}_1$ are

$$\begin{aligned} \rho_1 &= (\ddot{\rho}_1, \dot{\rho}_1) : \dot{\mathcal{C}}_2 \rightarrow \dot{\mathcal{C}}_2, & k &\mapsto \ddot{h}_1 \ddot{\xi}_1 k, & p &\mapsto \dot{h}_1 \dot{\xi}_1 p, \\ \gamma_1 &= (\ddot{\gamma}_1, \dot{\gamma}_1) : \check{\mathcal{C}}_2 \rightarrow \check{\mathcal{C}}_2, & j &\mapsto (\dot{e}_1 \dot{h}_1 \dot{\xi}_1 \dot{t}_1 j)(\dot{\xi}_1 \dot{t}_1 j^{-1}) j (\dot{e}_1 \dot{h}_1 j^{-1})(\dot{\xi}_1 \dot{h}_1 j), \\ & & g &\mapsto (\ddot{e}_1 \ddot{h}_1 \ddot{\xi}_1 \ddot{t}_1 g)(\ddot{\xi}_1 \ddot{t}_1 g^{-1}) g (\ddot{e}_1 \ddot{h}_1 g^{-1})(\ddot{\xi}_1 \ddot{h}_1 g). \end{aligned}$$

It is easy to check that ρ_1 and γ_1 are endomorphisms of cat^1 -groups. For example,

$$t_2 \ddot{\rho}_1 k = t_2 \ddot{h}_1 \ddot{\xi}_1 k = \dot{h}_1 \ddot{t}_2 \ddot{\xi}_1 k = \dot{h}_1 \dot{\xi}_1 \dot{t}_2 k = \dot{\rho}_1 \dot{t}_2 k.$$

Of course we may also consider $\mathcal{C}_2 = (e_2; t_2, h_2 : \check{\mathcal{C}}_2 \rightarrow \dot{\mathcal{C}}_2)$ as a second cat^1 -group of cat^1 -groups, and define sections $\xi_2 = (\ddot{\xi}_2, \dot{\xi}_2) : \check{\mathcal{C}}_2 \rightarrow \dot{\mathcal{C}}_2$.

9.8 More on sections of a cat^2 -group

A first attempt at defining a section of \mathcal{C} might be as a homomorphism $\xi_{[2]} : C_\emptyset \rightarrow C_{[2]}$ which composes with the tail maps in various ways to give identities

$$t_{[2]}\xi_{[2]} = \text{id}_{C_\emptyset}, \quad \dot{t}_1\xi_{[2]}\dot{t}_2 = \text{id}_{C_{\{2\}}}, \quad \dot{t}_2\xi_{[2]}\dot{t}_1 = \text{id}_{C_{\{1\}}}.$$

This will not do because, when $\dot{t}_1 = 0$ (say) the last identity puts too many restrictions on $\xi_{[2]}$. Since $t_{[2]} = \dot{t}_1\dot{t}_2 = \dot{t}_2\dot{t}_1$, the first identity gives

$$\dot{t}_1(\ddot{t}_2\xi_{[2]}) = \text{id}_{C_\emptyset} \quad \text{and} \quad \dot{t}_2(\ddot{t}_1\xi_{[2]}) = \text{id}_{C_\emptyset}$$

so that $\ddot{t}_2\xi_{[2]}$ is a section of $\dot{\mathcal{C}}_1$ and $\ddot{t}_1\xi_{[2]}$ is a section of $\dot{\mathcal{C}}_2$.

An alternative approach is to consider commuting pairs of sections.

Definition 9.22 A cat^2 -group section of \mathcal{C} is a pair $\xi = (\xi_1, \xi_2)$ where $\xi_1 = (\ddot{\xi}_1, \dot{\xi}_1)$ is a section of \mathcal{C}_1 , and $\xi_2 = (\ddot{\xi}_2, \dot{\xi}_2)$ is a section of \mathcal{C}_2 , such that

$$\ddot{\xi}_2\dot{\xi}_1 = \ddot{\xi}_1\dot{\xi}_2 : C_\emptyset \rightarrow C_{[2]}.$$

We denote the common composite by $\xi_{[2]} : C_\emptyset \rightarrow C_{[2]}$ and, since

$$t_{[2]}\xi_{[2]} = \dot{t}_1\ddot{t}_2\ddot{\xi}_1\dot{\xi}_2 = \dot{t}_1\dot{\xi}_1\dot{t}_2\dot{\xi}_2 = 1,$$

the map $\xi_{[2]}$ is a section of $\mathcal{C}_{[2]}$.

We expect Whitehead multiplication of cat^2 -sections to be defined by

$$\xi \star \zeta = (\ddot{\xi}_1 \star \ddot{\zeta}_1)(\dot{\xi}_2 \star \dot{\zeta}_2) : C_\emptyset \rightarrow C_{[2]},$$

but for this to make sense we must first show that a product of commuting sections commutes.

Lemma 9.23

$$(\ddot{\xi}_1 \star \ddot{\zeta}_1)(\dot{\xi}_2 \star \dot{\zeta}_2) = (\ddot{\xi}_2 \star \ddot{\zeta}_2)(\dot{\xi}_1 \star \dot{\zeta}_1).$$

Proof:

$$\begin{aligned} & (\ddot{\xi}_1 \star \ddot{\zeta}_1)(\dot{\xi}_2 \star \dot{\zeta}_2)p \\ = & (\ddot{\xi}_1\dot{\xi}_2p)(\ddot{\xi}_1\dot{\xi}_2\dot{h}_2\dot{\xi}_2p^{-1})(\ddot{\zeta}_1\dot{\zeta}_2\dot{h}_2\dot{\zeta}_2p^{-1})(\ddot{e}_1\ddot{h}_1\ddot{\xi}_1\dot{\zeta}_2\dot{h}_2\dot{\xi}_2p^{-1})(\ddot{e}_1\ddot{h}_1\ddot{\xi}_1\dot{\xi}_2\dot{h}_2\dot{\xi}_2p^{-1})(\ddot{e}_1\ddot{h}_1\ddot{\xi}_1\dot{\xi}_2p^{-1}) \\ & (\ddot{\zeta}_1\ddot{h}_1\ddot{\xi}_1\dot{\xi}_2p)(\ddot{\zeta}_1\ddot{h}_1\ddot{\xi}_1\dot{\xi}_2\dot{h}_2\dot{\xi}_2p^{-1})(\ddot{\zeta}_1\ddot{h}_1\ddot{\xi}_1\dot{\zeta}_2\dot{h}_2\dot{\xi}_2p) \\ = & (\ddot{\xi}_2\dot{\xi}_1p)(\ddot{e}_2\ddot{h}_2\ddot{\xi}_2\dot{\xi}_1p^{-1})(\ddot{\xi}_1\dot{\xi}_2\dot{h}_2\dot{\xi}_2p)(\ddot{e}_1\ddot{h}_1\ddot{\xi}_1\dot{\zeta}_2\dot{h}_2\dot{\xi}_2p^{-1})(\ddot{e}_1\ddot{h}_1\ddot{\xi}_1\dot{\xi}_2\dot{h}_2\dot{\xi}_2p^{-1})(\ddot{e}_1\ddot{h}_1\ddot{\xi}_1\dot{\xi}_2p^{-1}) \\ & (\ddot{\zeta}_1\ddot{h}_1\ddot{\xi}_1\dot{\xi}_2p)(\ddot{\zeta}_1\ddot{h}_1\ddot{\xi}_1\dot{\xi}_2\dot{h}_2\dot{\xi}_2p^{-1})(\ddot{\zeta}_1\ddot{h}_1\ddot{\xi}_1\dot{\zeta}_2\dot{h}_2\dot{\xi}_2p) \\ = & \text{?????} \end{aligned}$$

This seems to be going nowhere fast!

Perhaps the group-groupoid or double groupoid approaches are better (see Subsection 2.3 of the Notes) – and the next Section. \square

What needs doing next?

- Prove that the definition of Whitehead multiplication of sections makes sense.
- Investigate the corresponding (γ, ρ) pairs.
- **Eventually:** convert to crossed squares and see what the corresponding formulae are for 2-fold derivations.

10 2-crossed modules

The reader is referred to Brown–Gilbert [8], Conduché [26],[27], and Mutlu–Porter [47] for background information on 2-crossed modules.

Definition 10.1 *A 2-crossed module is comprised of the following:*

- a 2-complex of groups

$$\mathcal{Z} = (T \xrightarrow{\delta_2} U \xrightarrow{\delta_1} V)$$

(so that $\delta_1 \circ \delta_2 = 0 : T \rightarrow V$);

- an action of V on T and on U , and on itself by conjugation, such that δ_1, δ_2 are morphisms which preserve the actions;
- a function, called the Peiffer lifting,

$$\{ , \} : U \times U \rightarrow T,$$

making $(\delta_2 : T \rightarrow U)$ a crossed module with action

$$t^u := t \{ \delta_2 t, u \}. \quad (66)$$

The following axioms are also required:

$$(2X1) \quad \delta_2 \{ u_1, u_2 \} = \langle u_1, u_2 \rangle = u_1^{-1} u_2^{-1} u_1 u_2^{\delta_1 u_1} \quad (\text{a Peiffer commutator}),$$

$$(2X2) \quad \{ u, \delta_2 t \} = (t^{-1})^u t^{\delta_1 u},$$

$$(2X3) \quad \{ u_1 u_2, u_3 \} = \{ u_1, u_3 \}^{u_2} \{ u_2, u_3^{\delta_1 u_1} \},$$

$$(2X4) \quad \{ u_1, u_2 u_3 \} = \{ u_1, u_3 \} \{ u_1, u_2 \}^{u_3^{\delta_1 u_1}},$$

$$(2X5) \quad \{ u_1, u_2 \}^v = \{ u_1^v, u_2^v \}.$$

An additional axiom, $\{ \delta_2 t, u \} = t^{-1} t^u$ is often specified, but we have used this identity in (66) to define the action. Note that δ_2 maps (2X3), (2X4) and (2X5) to identities (b), (c) and (d) in Lemma 1.10 for Peiffer commutators. Compare also (2X3), (2X4) with identities (a), (b) for crossed pairings in Definition 7.4.

We will check that the crossed module action $t^u = t \{ \delta_2 t, u \}$ given by formula (66) is well defined:

$$\begin{aligned} (t_1 t_2)^u &= t_1 t_2 \{ (\delta_2 t_1) (\delta_2 t_2), u \} \\ &= t_1 t_2 \{ \delta_2 t_1, u \}^{\delta_2 t_2} \{ \delta_2 t_2, u^{\delta_1 \delta_2 t_1} \} \\ &= t_1 t_2 t_2^{-1} \{ \delta_2 t_1, u \} t_2 \{ \delta_2 t_2, u \} \\ &= t_1^u t_2^u, \\ t^{(u_1 u_2)} &= t \{ \delta_2 t, u_1 u_2 \} \\ &= t \{ \delta_2 t, u_2 \} \{ \delta_2 t, u_1 \}^{u_2^{\delta_1 \delta_2 t}} \\ &= t^{u_2} \{ \delta_2 t, u_1 \}^{u_2} \\ &= (t^{u_1})^{u_2}. \end{aligned}$$

Lemma 10.2 (a) $\{\delta_2 t_1, \delta_2 t_2\} = [t_1, t_2]$.

Proof:

(a) $\{\delta_2 t_1, \delta_2 t_2\} = t_1^{-1} t_1^{\delta_2 t_2} = [t_1, t_2]$, by definition of the crossed module action.

□

10.1 Morphisms and Homotopies of 2-crossed modules

Definition 10.3 A morphism of 2-crossed modules is a triple of group homomorphisms

$$f_{\bullet} \equiv (f_2, f_1, f_0) : \mathcal{Z} \rightarrow \mathcal{Z}'$$

such that

$$f_1 \delta_2 = \delta'_2 f_2, \quad f_0 \delta_1 = \delta'_1 f_1, \quad f_2(t^v) = (f_2 t)^{f_0 v}, \quad f_1(u^v) = (f_1 u)^{f_0 v}, \quad f_2\{u_1, u_2\} = \{f_1 u_1, f_1 u_2\}.$$

$$\begin{array}{ccccc} \mathcal{Z} : & T & \xrightarrow{\delta_2} & U & \xrightarrow{\delta_1} & V \\ & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 \\ f_{\bullet} : & & & & & \\ & T' & \xrightarrow{\delta'_2} & U' & \xrightarrow{\delta'_1} & V' \\ \mathcal{Z}' : & & & & & \end{array}$$

An automorphism of \mathcal{Z} is an endomorphism with inverse $f_{\bullet}^{-1} = (f_2^{-1}, f_1^{-1}, f_0^{-1})$.

Note that

$$f_2(t^u) = (f_2 t)\{f_1 \delta_2 t, f_1 u\} = (f_2 t)\{\delta'_2 f_2 t, f_1 u\} = (f_2 t)^{f_1 u}$$

so $(f_2, f_1) : (\delta_2 : T \rightarrow U) \rightarrow (\delta'_2 : T' \rightarrow U')$ is a morphism of crossed modules.

Definition 10.4 A homotopy of the 2-crossed module \mathcal{Z} is a pair of homomorphisms $\phi_{\bullet} = (\phi_1, \phi_0)$ such that

$$\delta_2 \phi_1 = \phi_0 \delta_1 \quad [\text{Is that all ???}]$$

$$\begin{array}{ccccc} T & \xrightarrow{\delta_2} & U & \xrightarrow{\delta_1} & V \\ & \swarrow \phi_1 & \downarrow f_1 & \swarrow \phi_0 & \downarrow f_0 \\ f_2 \downarrow & & & & \\ T & \xrightarrow{\delta_2} & U & \xrightarrow{\delta_1} & V \end{array}$$

10.2 2-crossed modules of groupoids

[To be added.]

[The rest of this section really belongs in a crossed squares chapter.]

10.3 The 2-crossed module associated to a crossed square

We follow Brown–Gilbert [8] in defining the appropriate Peiffer lifting to be $\{(m_1, n_1), (m_2, n_2)\} = m_2^{m_1} \boxtimes n_1$.

Proposition 10.5 *Given a crossed square \mathcal{R} there is an associated 2-crossed module \mathcal{Z} , as shown in the following diagram:*

$$\mathcal{R} : \begin{array}{ccc} R_{[2]} & \xrightarrow{\ddot{\partial}_1} & R_{\{2\}} \\ \ddot{\partial}_2 \downarrow & & \downarrow \dot{\partial}_2 \\ R_{\{1\}} & \xrightarrow{\dot{\partial}_1} & R_\emptyset \end{array} \quad \mathcal{Z} : \quad (R_{[2]} \xrightarrow{\delta_2} R_{\{1\}} \times R_{\{2\}} \xrightarrow{\delta_1} R_\emptyset)$$

where

$$\delta_2 \ell = (\ddot{\partial}_2 \ell, \ddot{\partial}_1 \ell^{-1}) \quad \text{and} \quad \delta_1(m, n) = (\dot{\partial}_1 m)(\dot{\partial}_2 n),$$

R_\emptyset acts diagonally on $R_{\{1\}} \times R_{\{2\}}$, and the Peiffer lifting is given by

$$\{(m_1, n_1), (m_2, n_2)\} = m_2^{m_1} \boxtimes n_1.$$

Proof: We first check that δ_1 and δ_2 are homomorphisms preserving the R_\emptyset -actions, and that $\delta_1 \delta_2 = 0$:

$$\begin{aligned} \delta_1((m_1, n_1), (m_2, n_2)) &= \delta_1(m_1 m_2, n_1^{m_2} n_2) = \dot{\partial}_1(m_1 m_2) \dot{\partial}_2(n_1^{\dot{\partial}_1 m_2} n_2) \\ &= (\dot{\partial}_1 m_1)(\dot{\partial}_1 m_2)(\dot{\partial}_2 n_1)^{\dot{\partial}_1 m_2} (\dot{\partial}_2 n_2) = \delta_1(m_1, n_1) \delta_1(m_2, n_2). \\ \delta_1((m, n)^p) &= \delta_1(m^p, n^p) = \dot{\partial}_1(m^p) \dot{\partial}_2(n^p) = (\dot{\partial}_1 m)^p (\dot{\partial}_2 n)^p = (\delta_1(m, n))^p. \\ (\delta_2 \ell_1)(\delta_2 \ell_2) &= (\ddot{\partial}_2 \ell_1, \ddot{\partial}_1 \ell_1^{-1})(\ddot{\partial}_2 \ell_2, \ddot{\partial}_1 \ell_2^{-1}) \\ &= ((\ddot{\partial}_2 \ell_1)(\ddot{\partial}_2 \ell_2), (\ddot{\partial}_1 \ell_1^{-1})^{\dot{\partial}_1 \ddot{\partial}_2 \ell_2} (\ddot{\partial}_1 \ell_2^{-1})) \\ &= (\ddot{\partial}_2(\ell_1 \ell_2), (\ddot{\partial}_1 \ell_1^{-1})^{\dot{\partial}_1 \ddot{\partial}_1 \ell_2} (\ddot{\partial}_1 \ell_2^{-1})) \\ &= (\ddot{\partial}_2(\ell_1 \ell_2), (\ddot{\partial}_1 \ell_2^{-1})(\ddot{\partial}_1 \ell_1^{-1})) \quad \text{by } \mathbf{X2:} \text{ for } \dot{\mathcal{R}}_2 \\ &= \delta_2(\ell_1 \ell_2). \\ \delta_2(\ell^p) &= (\ddot{\partial}_2(\ell^p), \ddot{\partial}_1((\ell^p)^{-1})) = ((\ddot{\partial}_2 \ell)^p, (\ddot{\partial}_1 \ell^{-1})^p) = (\ddot{\partial}_2 \ell, \ddot{\partial}_1 \ell^{-1})^p = (\delta_2 \ell)^p. \\ \delta_1 \delta_2 \ell &= \delta_1(\ddot{\partial}_2 \ell, \ddot{\partial}_1 \ell^{-1}) = (\dot{\partial}_1 \ddot{\partial}_2 \ell)(\dot{\partial}_2 \ddot{\partial}_1 \ell^{-1}) = 1. \end{aligned}$$

Secondly we identify the crossed module action in \mathcal{Z} in this case to be $\ell^{(m, n)} = \ell^m$.

$$\begin{aligned} \ell^{(m, n)} &= \ell \{(\ddot{\partial}_2 \ell, \ddot{\partial}_1 \ell^{-1}), (m, n)\} \\ &= \ell(m^{\ddot{\partial}_2 \ell} \boxtimes \ddot{\partial}_1 \ell^{-1}) \\ &= \ell \ell^{(\ddot{\partial}_2 \ell^{-1})m(\ddot{\partial}_2 \ell)} \ell^{-1} \quad \text{by Definition 8.1 (e)} \\ &= \ell(\ell^m)^{(\ddot{\partial}_2 \ell)} \ell^{-1} = \ell^m \quad \text{by } \mathbf{X2:} \text{ (twice).} \end{aligned}$$

It is clear that this *is* an action, so we verify the two crossed module axioms:

$$\begin{aligned}
\mathbf{X1}: \quad (\delta_2 \ell)^{(m,n)} &= (m^{-1}, (n^{-1})^{m^{-1}})(\ddot{\partial}_2 \ell, \ddot{\partial}_1 \ell^{-1})(m, n) \\
&= (m^{-1}(\ddot{\partial}_2 \ell)m, (n^{-1})^{\dot{\partial}_1 \ddot{\partial}_2 \ell^m}(\ddot{\partial}_1 \ell^{-1})^m n) \\
&= ((\ddot{\partial}_2 \ell)^m, (n^{-1})^{\dot{\partial}_2(\ddot{\partial}_1 \ell)^m}(\ddot{\partial}_1 \ell^m)^{-1} n) \\
&= ((\ddot{\partial}_2 \ell)^m, (\ddot{\partial}_1 \ell^m)^{-1} n^{-1} n) \\
&= \delta_2(\ell^m) = \delta_2(\ell^{(m,n)}), \\
\mathbf{X2}: \quad \ell_0^{\delta_2 \ell} &= \ell_0^{(\ddot{\partial}_2 \ell, \ddot{\partial}_1 \ell^{-1})} = \ell_0^{\ddot{\partial}_2 \ell} = \ell_0^\ell.
\end{aligned}$$

Thirdly, we verify the five axioms.

2X1:

$$\begin{aligned}
&(m_1, n_1)^{-1} (m_2, n_2)^{-1} (m_1, n_1) (m_2, n_2)^{\delta_1(m_1, n_1)} \\
&= (m_1^{-1}, (n_1^{-1})^{m_1^{-1}}) (m_2^{-1}, (n_2^{-1})^{m_2^{-1}}) (m_1, n_1) (m_2, n_2)^{(\dot{\partial}_1 m_1)(\dot{\partial}_2 n_1)} \\
&= (m_1^{-1} m_2^{-1} m_1 m_2^{m_1 n_1}, (n_1^{-1})^{m_1^{-1} m_2^{-1} m_1 m_2^{m_1 n_1}} (n_2^{-1})^{m_2^{-1} m_1 m_2^{m_1 n_1}} n_1^{m_2^{m_1 n_1}} n_2^{m_1 n_1})
\end{aligned}$$

The left hand element is

$$(m_2^{m_1})^{-1} (m_2^{m_1})^{n_1} = \ddot{\partial}_2(m_2^{m_1} \boxtimes n_1)$$

It follows that the right hand element is

$$\begin{aligned}
&(n_1^{-1} (n_2^{-1})^{m_1})^{\ddot{\partial}_2(m_2^{m_1} \boxtimes n_1)} n_1^{m_2^{m_1 n_1}} n_2^{m_1 n_1} \\
&= (n_1^{-1} (n_2^{m_1})^{-1})^{(n_1^{-1})^{m_2^{m_1}} n_1} (n_1^{m_2^{m_1}})^{n_1} (n_2^{m_1})^{n_1} \quad \text{by Lemmas 2.7(c) and 2.2(d),} \\
&= (n_1^{-1})^{(n_1^{m_2^{m_1}})(n_1^{-1})(n_2^{m_1})^{-1}(n_1^{m_2^{m_1}})^{-1}(n_1)(n_1^{-1})(n_1^{m_2^{m_1}})(n_1)(n_1^{-1})(n_2^{m_1})(n_1))} \\
&= \ddot{\partial}_1(m_2^{m_1} \boxtimes n_1)^{-1}
\end{aligned}$$

So the pair of elements is $\delta_2(m_2^{m_1} \boxtimes n_1) = \delta_2\{(m_1, n_1), (m_2, n_2)\}$.

2X2:

$$\{(m, n), \delta_2 \ell\} = \{(m, n), (\ddot{\partial}_2 \ell, \ddot{\partial}_1 \ell^{-1})\} = \ddot{\partial}_2(\ell^m) \boxtimes n = (\ell^m)^{-1}(\ell^m)^n = (\ell^{-1})^{(m,n)} \ell^{\delta_1(m,n)}.$$

2X3:

$$\begin{aligned}
&\{(m_1, n_1), (m_3, n_3)\}^{(m_2, n_2)} \{(m_2, n_2), (m_3, n_3)\}^{m_1 n_1} \\
&= (m_3^{m_1} \boxtimes n_1)^{m_2} (m_3^{m_1 n_1 m_2} \boxtimes n_2) \\
&= (m_0 \boxtimes n_0) (m_0^{n_0} \boxtimes n_2) \quad \text{where } m_0 = m_3^{m_1 m_2}, \quad n_0 = n_1^{m_2} \\
&= (m_0 \boxtimes n_2) (m_0 \boxtimes n_0)^{n_2} \quad \text{by Proposition 7.6(d)} \\
&= m_0 \boxtimes n_0 n_2 \\
&= m_3^{m_1 m_2} \boxtimes n_1^{m_2} n_2 \\
&= \{(m_1 m_2, n_1^{m_2} n_2), (m_3, n_3)\} \\
&= \{(m_1, n_1)(m_2, n_2), (m_3, n_3)\}
\end{aligned}$$

2X4:

$$\begin{aligned}
& \{(m_1, n_1), (m_2 m_3, n_2^{m_3} n_3)\} \\
&= (m_2 m_3)^{m_1} \boxtimes n_1 \\
&= (m_2^{m_1} \boxtimes n_1)^{m_3^{m_1}} (m_3^{m_1} \boxtimes n_1) \\
&= (m_4 \boxtimes n_1)^{n_1 m_5} (m_5 \boxtimes n_1) \quad \text{where } m_4 = m_2^{m_1 n_1^{-1}}, m_5 = m_3^{m_1} \\
&= (m_5 \boxtimes n_1)(m_4 \boxtimes n_1)^{m_5 n_1} \quad \text{by Proposition 7.6(c)} \\
&= (m_3^{m_1} \boxtimes n_1)(m_2^{m_1} \boxtimes n_1)^{n_1^{-1} m_3^{m_1} n_1} \\
&= \{(m_1, n_1), (m_3, n_3)\} \{(m_1, n_1), (m_2, n_2)\}^{(m_3, n_3)^{\delta_1(m_1, n_1)}}
\end{aligned}$$

2X5:

$$\{(m_1^p, n_1^p), (m_2^p, n_2^p)\} = (m_2^p)^{m_1^p} \boxtimes n_1^p = (m_2^{m_1} \boxtimes n_1)^p = \{(m_1, n_1), (m_2, n_2)\}^p$$

□

10.4 The crossed square associated to a 2-crossed module

Maybe there is no exact construction?

10.5 Homotopies of the actor 2-crossed module

[This needs significant revision.]

Definition 10.6 A homotopy of the 2-crossed module \mathcal{Z} is a pair of homomorphisms $\phi_\bullet = (\phi_1, \phi_0)$ such that $\phi_0 : R_\emptyset \rightarrow R_{\{1\}} \times R_{\{2\}}$, $p \mapsto (\phi_0^N p, \phi_0^M p)$ where

$$\begin{aligned}
\phi_0(p_1 p_2) &= (\phi_0^N(p_1 p_2), \phi_0^M(p_1 p_2)) \\
&= (\phi_0^N p_1, \phi_0^M p_1)(\phi_0^N p_2, \phi_0^M p_2) \\
&= (\phi_0^N p_1 \phi_0^N p_2, (\phi_0^M p_1)^{\phi_0^N p_2} \phi_0^M p_2)
\end{aligned}$$

So ϕ_0^M is a ϕ_0^N -derivation. $\phi_1 : R_{\{1\}} \times R_{\{2\}} \rightarrow R_{[2]}$, $(n, m) \mapsto (\phi_1^N n)(\phi_1^M m)$

$$\begin{aligned}
\phi_1(n, m) &= \phi_1((n, 1)(1, m)) \\
&= \phi_1(n, 1)\phi_1(1, m) \\
&= (\phi_1^N n)(\phi_1^M m)
\end{aligned}$$

[Moved what was Lemma 7.6 to cat2-group section.]

Lemma 10.7 There is an action of $R_\emptyset \times R_{\{2\}}$ on $R_{\{1\}} \times R_{[2]}$ defined on the image of some derivation χ by

$$\chi(p, m)^{(p_1, m_1)} = \chi(pp_1, m^{p_1} m_1) \chi(p_1, m_1)^{-1}$$

which is well defined (should use the usual action!)

Proof:

$$\begin{aligned}
\chi((p, m)^{(p_1, m_1)}(p_2, m_2)) &= (\chi(pp_1, m^{p_1}m_1)\chi(p_1, m_1)^{-1})^{(p_2, m_2)} \\
&= \chi(pp_1, m^{p_1}m_1)^{(p_2, m_2)}(\chi(p_1, m_1)^{(p_2, m_2)})^{-1} \\
&= \chi(pp_1p_2, m^{p_1p_2}m_1^{p_2}m_2)(\chi(p_2, m_2)^{-1}(\chi(p_1p_2, m_1^{p_2}m_2)\chi(p_2, m_2)^{-1})^{-1})^{-1} \\
&= \chi(pp_1p_2, m^{p_1p_2}m_1^{p_2}m_2)\chi(p_1p_2, m_1^{p_2}m_2)^{-1} \\
\chi(p, m)^{(p_1, m_1)}(p_2, m_2) &= \chi(p, m)^{(p_1p_2, m_1^{p_2}m_2)} \\
&= \chi(pp_1p_2, m^{p_1p_2}m_1^{p_2}m_2)\chi(p_1p_2, m_1^{p_2}m_2)^{-1}
\end{aligned}$$

□

Lemma 10.8 Consider a derivation $\chi : R_\emptyset \ltimes R_{\{2\}} \rightarrow R_{\{1\}} \ltimes R_{[2]}$ such that $\chi(p, m) = (\chi^L(p, m), \chi^N(p, m))$. Then the rules for χ^L and χ^N are as follows: ? ? ?

Proof: we will show that above derivation is satisfies the derivation rule

$$\begin{aligned}
\chi((p_1, m_1)(p_2, m_2)) &= \chi(p_1p_2, m_1^{p_2}m_2) \\
\chi((p_1, m_1)(p_2, m_2)) &= \chi(p_1, m_1)^{(p_2, m_2)}\chi(p_2, m_2) \\
&= \chi(p_1p_2, m_1^{p_2}m_2)\chi(p_2, m_2)^{-1}\chi(p_2, m_2) \\
&= \chi(p_1p_2, m_1^{p_2}m_2)
\end{aligned}$$

□

Definition 10.9 Now we can define a derivation which is depends on the derivation χ^L

$$\begin{aligned}
\theta_p &= \chi^L(p, 1) \\
\ddot{\chi}m &= \chi^L(1, m)
\end{aligned}$$

$$\begin{aligned}
\theta(p_1p_2) &= \chi^L(p_1p_2, 1) \\
&= \chi^L((p_1, 1), (p_2, 1)) \\
&= (\chi^L(p_1, 1))^{(p_2, 1)}(\chi^L(p_2, 1)) \\
&= (\theta_{p_1})^{p_2}(\theta_{p_2})
\end{aligned}$$

Lemma 10.10 $\chi^L : R_\emptyset \ltimes R_{\{2\}} \rightarrow R_{[2]}$ is a derivation.

Proof:

$$\begin{aligned}
\chi^L(p, m) &= \chi^L((p, 1)(1, m)) \\
&= (\chi^L(p, 1))^{(1, m)}(\chi^L(1, m)) \\
&= (\theta_p)^m(\ddot{\chi}m)
\end{aligned}$$

□

11 Braiding

Brown and Gilbert, in [8], have shown that 2-crossed modules are equivalent to regular braided crossed modules. This construction has also been discussed in [16]. Our aim here is review this material, using right actions consistently.

11.1 Crossed module bimorphisms

Let $\mathcal{A} = ((\partial, \text{id}) : \mathbb{A}_2 \rightarrow \mathbb{A}_1)$ be a crossed module of groupoids. A *bimorphism* $b : (\mathcal{A}, \mathcal{A}) \rightarrow \mathcal{A}$ consists of a family of maps

$$b_{ij} : A_i \times A_j \rightarrow A_{i+j}, \quad 0 \leq i + j \leq 2,$$

satisfying the following axioms.

- (a) The map b_{00} provides a monoid structure on A_0 , with identity e , written $b_{00}(u, v) = uv$. As usual, A_0 acts on itself on the left and the right using this multiplication.
- (b) The maps b_{10}, b_{20} give left actions of A_0 on A_1, A_2 respectively, while b_{01}, b_{02} provide right actions. These left actions commute with the right actions. We write these actions using \cdot to avoid confusion with the crossed module action of A_1 on A_2 . So, for $u, v \in A_0$, $a \in A_1$ and $\ell \in A_2$, we have

$$(u \cdot a) \cdot v = u \cdot (a \cdot v), \quad (u \cdot \ell) \cdot v = u \cdot (\ell \cdot v).$$

- (c) These actions are compatible with the groupoid structure: for $p, q \in A_1$ or $p, q \in A_2$,

$$\begin{aligned} s(u \cdot p) &= u(sp), & t(u \cdot p) &= u(tp), \\ s(p \cdot v) &= (sp)v, & t(p \cdot v) &= (tp)v, \\ u \cdot (pq) &= (u \cdot p)(u \cdot q), & (pq) \cdot v &= (p \cdot v)(q \cdot v), \quad \text{provided } tp = sq. \end{aligned}$$

Hence $u \cdot 1_v = 1_{uv} = 1_u \cdot v$ and $(u \cdot b)^{-1} = u \cdot b^{-1}$, $(b \cdot v)^{-1} = b^{-1} \cdot v$.

- (d) The actions are compatible with the crossed module action: when $\ell \in A_2(x)$, $a \in A_1(x, y)$, and $\ell^a \in A_2(y)$,

$$u \cdot (\ell^a) = (u \cdot \ell)^{u \cdot a} \in A_2(uy), \quad (\ell^a) \cdot v = (\ell \cdot v)^{a \cdot v} \in A_2(yv).$$

- (e) The boundary morphism $\partial : A_2 \rightarrow A_1$ is equivariant with respect to these actions:

$$\partial(u \cdot \ell) = u \cdot (\partial \ell), \quad \partial(\ell \cdot v) = (\partial \ell) \cdot v.$$

- (f) We write the images of the map $b_{11} : A_1 \times A_1 \rightarrow A_2$ as a bracing:

$$b_{11}(a, b) = \{a, b\}, \quad \text{a loop at } (ta)(tb).$$

This bracing interacts with composition in A_1 according to:

$$\{1_e, b\} = 1_{tb}, \quad \{a, 1_e\} = 1_{ta}, \quad \{aa', b\} = \{a', b\}\{a, b\}^{a' \cdot tb}, \quad \{a, bb'\} = \{a, b\}^{ta \cdot b'}\{a, b'\}.$$

- (g) Given $a, b \in A_1$, we may act with the source or target of one on the other in four ways, forming $sa \cdot b$, $ta \cdot b$, $a \cdot sb$, $a \cdot tb$. These arrows fit together to form the square

$$\begin{array}{ccc}
 (sa)(sb) & \xrightarrow{sa \cdot b} & (sa)(tb) \\
 \downarrow a \cdot sb & & \downarrow a \cdot tb \\
 (ta)(sb) & \xrightarrow{ta \cdot b} & (ta)(tb)
 \end{array}$$

$\partial\{a, b\}$

and the loop $\partial\{a, b\}$ measures the lack of commutativity:

$$\partial\{a, b\} = (ta \cdot b)^{-1}(a \cdot sb)^{-1}(sa \cdot b)(a \cdot tb).$$

- (h) The bracing interacts with the boundary map as follows:

$$\{a, \partial\ell\} = (sa \cdot \ell)^{a \cdot t\ell} (ta \cdot \ell), \quad \{\partial\ell, b\} = (\ell^{-1} \cdot tb) (\ell \cdot sb)^{t\ell \cdot b}.$$

- (i) The actions of A_0 on the bracing are given by:

$$u \cdot \{a, b\} = \{u \cdot a, b\}, \quad \{a, b\} \cdot v = \{a, b \cdot v\}, \quad \{a \cdot u, b\} = \{a, u \cdot b\}.$$

[Now do some checks?]

12 Double Categories and Double Groupoids

Our interest here is in double groupoids and their connection with crossed squares and cat^2 -groups.

12.1 Double Categories

A *double category* \mathcal{D} consists of four sets and four category structures, and satisfies an interchange law:

- $D_{[2]}$, a set of *squares*,
- $D_{\{2\}}$, a set of *vertical* or *up-down arrows*,
- $D_{\{1\}}$, a set of *horizontal* or *left-right arrows*,
- D_\emptyset , a set of *objects*,
- maps $\ddot{t}_1, \ddot{h}_1 : D_{[2]} \rightarrow D_{\{2\}}$ and $\ddot{e}_1 : D_{\{2\}} \rightarrow D_{[2]}$ and a composition $\ddot{*}_1$ giving a category structure $\ddot{\mathcal{D}}_1$ on squares displayed horizontally,
- maps $\ddot{t}_2, \ddot{h}_2 : D_{[2]} \rightarrow D_{\{1\}}$ and $\ddot{e}_2 : D_{\{1\}} \rightarrow D_{[2]}$ and a composition $\ddot{*}_2$ giving a category structure $\ddot{\mathcal{D}}_2$ on squares displayed vertically,
- maps $\dot{t}_1, \dot{h}_1 : D_{\{1\}} \rightarrow D_\emptyset$, and $\dot{e}_1 : D_\emptyset \rightarrow D_{\{1\}}$ and a composition $\dot{*}_1$ giving a category structure $\dot{\mathcal{D}}_1$ on horizontal arrows,
- maps $\dot{t}_2, \dot{h}_2 : D_{\{2\}} \rightarrow D_\emptyset$, and $\dot{e}_2 : D_\emptyset \rightarrow D_{\{2\}}$ and a composition $\dot{*}_2$ giving a category structure $\dot{\mathcal{D}}_2$ on vertical arrows,
- the tail and head maps commute as follows:

$$\dot{t}_2 \ddot{t}_1 = \dot{t}_1 \ddot{t}_2, \quad \dot{t}_2 \ddot{h}_1 = \dot{h}_1 \ddot{t}_2, \quad \dot{h}_2 \ddot{t}_1 = \dot{t}_1 \ddot{h}_2, \quad \dot{h}_2 \ddot{h}_1 = \dot{h}_1 \ddot{h}_2,$$

- for all squares A, B, C, D such that the compositions are defined,

$$\begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \begin{array}{cc} A & B \end{array} & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \begin{array}{cc} C & D \end{array} & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \quad (A \ddot{*}_1 B) \ddot{*}_2 (C \ddot{*}_1 D) = (A \ddot{*}_2 C) \ddot{*}_1 (B \ddot{*}_2 D).$$

For a square A the four arrows and the four objects are displayed as follows.

$$\begin{array}{ccc} \dot{t}_1 \ddot{t}_2 A & \xrightarrow{\ddot{t}_2 A} & \dot{h}_1 \ddot{t}_2 A \\ \dot{t}_1 A \downarrow & A & \downarrow \dot{h}_1 A \\ \dot{t}_1 \ddot{h}_2 A & \xrightarrow{\ddot{h}_2 A} & \dot{h}_1 \ddot{h}_2 A \end{array}$$

The structure of a double category is shown in the following diagram:

$$\mathcal{D} = \begin{array}{ccc} & \begin{array}{c} \xrightarrow{\check{t}_1, \check{h}_1} \\ \xleftarrow{\check{e}_1} \end{array} & \\ \begin{array}{c} \uparrow \check{e}_2 \\ \downarrow \check{t}_2, \check{h}_2 \end{array} & \begin{array}{c} D_{[2]} \\ \\ D_{\{1\}} \end{array} & \begin{array}{c} \uparrow \check{e}_2 \\ \downarrow \check{t}_2, \check{h}_2 \end{array} \\ & \begin{array}{c} \xleftarrow{\check{e}_1} \\ \xrightarrow{\check{t}_1, \check{h}_1} \end{array} & \begin{array}{c} D_{\{2\}} \\ \\ D_{\emptyset} \end{array} \end{array} \quad (67)$$

Horizontal, vertical and double identity squares ($e_{[2]} = \check{e}_2 \check{e}_1 = \check{e}_1 \check{e}_2$) are represented by

$$\begin{array}{ccc} \begin{array}{ccc} \cdot & \xrightarrow{\check{e}_1 \check{t}_1 a} & \cdot \\ \downarrow a & \check{e}_1 a & \downarrow a \\ \cdot & \xrightarrow{\check{e}_1 \check{h}_1 a} & \cdot \end{array} & \begin{array}{ccc} \cdot & \xrightarrow{c} & \cdot \\ \downarrow \check{e}_2 \check{t}_2 c & \check{e}_2 c & \downarrow \check{e}_2 \check{h}_2 c \\ \cdot & \xrightarrow{c} & \cdot \end{array} & \begin{array}{ccc} r & \xrightarrow{\check{e}_1 r} & r \\ \downarrow \check{e}_2 r & e_{[2]} r & \downarrow \check{e}_2 r \\ r & \xrightarrow{\check{e}_1 r} & r \end{array} \end{array}$$

Definition 12.1 A morphism of double categories is a double functor $\lambda : \mathcal{D} \rightarrow \mathcal{E}$, given by a quadruple $\lambda = (\lambda_{[2]}, \lambda_{\{2\}}, \lambda_{\{1\}}, \lambda_{\emptyset})$ of maps

$$\lambda_{[2]} : D_{[2]} \rightarrow E_{[2]}, \quad \lambda_{\{2\}} : D_{\{2\}} \rightarrow E_{\{2\}}, \quad \lambda_{\{1\}} : D_{\{1\}} \rightarrow E_{\{1\}}, \quad \lambda_{\emptyset} : D_{\emptyset} \rightarrow E_{\emptyset},$$

which combine to give functors from $\check{D}_1, \check{D}_2, \dot{D}_1, \dot{D}_2$ to $\check{E}_1, \check{E}_2, \dot{E}_1, \dot{E}_2$ respectively, and which preserve all compositions.

[Expand on this?]

12.2 Double Groupoids and Group – Double Groupoids

A *double groupoid* is a double category in which the four category structures are all groupoids. When the two sets of arrows coincide, $D_{\{1\}} = D_{\{2\}}$, we have an *arrow-symmetric double groupoid*. These are the subject of Chapter 6 of Brown and Sivera [19].

By analogy with the equivalence between cat^1 -groups and group-groupoids, we attempt to describe an equivalence between cat^2 -groups and group-double groupoids.

Definition 12.2 A group-double groupoid is a double groupoid \mathcal{D} where each of $\mathcal{D}_{[2]}, \mathcal{D}_{\{2\}}, \mathcal{D}_{\{1\}}, \mathcal{D}_{\emptyset}$ are groups and the twelve structure maps in (67) are group homomorphisms, so that the four category structures $\check{D}_1, \dot{D}_1, \check{D}_2, \dot{D}_2$ are all group-groupoids.

[Is the diagonal a group-groupoid?]

12.3 Horizontal, Vertical and Double Sections

We saw in Subsections 1.13 and 2.3 that a section of a group-groupoid is a group monomorphism $\xi : G_0 \rightarrow G_1$ such that $t\xi = 1_{G_0}$. In order to generalise this to a horizontal section of a group-double groupoid \mathcal{D} , we require compatible monomorphisms from up-down arrows to squares and from points to left-right arrows. In order to see what ‘compatible’ means in this context we note that, in the following diagrams, the left-right boundaries of $\ddot{\xi}_1 g$ should be the images of the points of g under $\dot{\xi}_1$, while $\ddot{\xi}_1 \dot{e}_2 r$ should be the vertical identity square for $\dot{\xi}_1 r$. We show up-down arrows as dashed in the diagrams in this Subsection.

$$\begin{array}{ccc}
 t_2 g & \xrightarrow{\dot{\xi}_1 t_2 g = \ddot{t}_2 \ddot{\xi}_1 g} & t_2 \ddot{h}_1 \ddot{\xi}_1 g \\
 \downarrow g & & \downarrow \ddot{h}_1 \dot{\xi}_1 g \\
 \dot{h}_2 g & \xrightarrow{\dot{\xi}_1 \dot{h}_2 g = \ddot{h}_2 \ddot{\xi}_1 g} & \dot{h}_2 \ddot{h}_1 \ddot{\xi}_1 g
 \end{array}
 \qquad
 \begin{array}{ccc}
 r & \xrightarrow{\dot{\xi}_1 r} & \dot{h}_1 \dot{\xi}_1 r \\
 \downarrow \dot{e}_2 r & & \downarrow \ddot{h}_1 \ddot{\xi}_1 \dot{e}_2 r \\
 r & \xrightarrow{\dot{\xi}_1 r} & \dot{h}_1 \dot{\xi}_1 r
 \end{array}$$

Definition 12.3

(a) A horizontal section of a group-double groupoid \mathcal{D} is a pair $\xi_1 = (\ddot{\xi}_1, \dot{\xi}_1)$ of monomorphisms $\ddot{\xi}_1 : D_1 \rightarrow D_{[2]}$ and $\dot{\xi}_1 : D_\emptyset \rightarrow D_{\{2\}}$ such that

$$\ddot{t}_1 \ddot{\xi}_1 = 1_{D_{\{1\}}}, \quad \dot{t}_1 \dot{\xi}_1 = 1_{D_\emptyset}, \quad \ddot{t}_2 \ddot{\xi}_1 g = \dot{\xi}_1 \dot{t}_2 g, \quad \ddot{h}_2 \ddot{\xi}_1 g = \dot{\xi}_1 \dot{h}_2 g, \quad \ddot{\xi}_1 \dot{e}_2 r = \dot{e}_2 \dot{\xi}_1 r.$$

(b) A vertical section of a group-double groupoid \mathcal{D} is a pair $\xi_2 = (\ddot{\xi}_2, \dot{\xi}_2)$ of monomorphisms $\ddot{\xi}_2 : D_2 \rightarrow D_{[2]}$ and $\dot{\xi}_2 : D_\emptyset \rightarrow D_{\{1\}}$ such that

$$\ddot{t}_2 \ddot{\xi}_2 = 1_{D_{\{2\}}}, \quad \dot{t}_2 \dot{\xi}_2 = 1_{D_\emptyset}, \quad \ddot{t}_1 \ddot{\xi}_2 g = \dot{\xi}_2 \dot{t}_1 g, \quad \ddot{h}_1 \ddot{\xi}_2 g = \dot{\xi}_2 \dot{h}_1 g, \quad \ddot{\xi}_2 \dot{e}_1 r = \dot{e}_1 \dot{\xi}_2 r.$$

[Maybe we should swap 1 and 2 in ξ_1, ξ_2 ?]

Given a section $\xi_1 = (\ddot{\xi}_1, \dot{\xi}_1)$ of \ddot{D}_1 we may apply the construction in Subsection 2.3 to obtain a groupoid automorphism λ_1 of \ddot{D}_1 , which extends to a double groupoid automorphisms $\lambda_1 : \mathcal{D} \rightarrow \mathcal{D}$.

$$\begin{array}{ccc}
 \dot{h}_1 \dot{\xi}_1 \dot{t}_1 \ddot{t}_2 A & \xrightarrow{\dot{\lambda}_1 \dot{t}_2 A} & \dot{h}_1 \dot{\xi}_1 \dot{h}_1 \ddot{t}_2 A \\
 \downarrow \ddot{h}_1 \ddot{\xi}_1 \dot{t}_1 A & & \downarrow \ddot{h}_1 \ddot{\xi}_1 \dot{h}_1 A \\
 \dot{h}_1 \dot{\xi}_1 \dot{t}_1 \ddot{h}_2 A & \xrightarrow{\dot{\lambda}_1 \ddot{h}_2 A} & \dot{h}_1 \dot{\xi}_1 \dot{h}_1 \ddot{h}_2 A
 \end{array}
 \qquad
 \begin{array}{ccccccc}
 \dot{h}_1 \dot{\xi}_1 \dot{t}_1 \ddot{t}_2 A & \xleftarrow{\dot{\xi}_1 \dot{t}_1 \ddot{t}_2 A} & \dot{t}_1 \ddot{t}_2 A & \xrightarrow{\ddot{t}_2 A} & \dot{h}_1 \ddot{t}_2 A & \xrightarrow{\dot{\xi}_1 \dot{h}_1 \ddot{t}_2 A} & \dot{h}_1 \dot{\xi}_1 \dot{h}_1 \ddot{t}_2 A \\
 \downarrow \ddot{h}_1 \ddot{\xi}_1 \dot{t}_1 A & & \downarrow \dot{t}_1 A & & \downarrow \ddot{h}_1 A & & \downarrow \ddot{h}_1 \ddot{\xi}_1 \dot{h}_1 A \\
 \dot{h}_1 \dot{\xi}_1 \dot{t}_1 \ddot{h}_2 A & \xleftarrow{\dot{\xi}_1 \dot{t}_1 \ddot{h}_2 A} & \dot{t}_1 \ddot{h}_2 A & \xrightarrow{\ddot{h}_2 A} & \dot{h}_1 \ddot{h}_2 A & \xrightarrow{\dot{\xi}_1 \dot{h}_1 \ddot{h}_2 A} & \dot{h}_1 \dot{\xi}_1 \dot{h}_1 \ddot{h}_2 A
 \end{array}$$

The defining equations for λ_1 are:

$$\begin{aligned}
 (\lambda_1)_{[2]} A &= \widetilde{(\ddot{\xi}_1 \dot{t}_1 A)} \ast_1 A \ast_1 (\ddot{\xi}_1 \ddot{h}_1 A), \\
 (\lambda_1)_{\{2\}} a &= \ddot{h}_1 \ddot{\xi}_1 a, \\
 (\lambda_1)_{\{1\}} c &= \widetilde{(\dot{\xi}_1 \dot{t}_1 c)} \ast_1 c \ast_1 (\dot{\xi}_1 \dot{h}_1 c), \\
 (\lambda_1)_\emptyset p &= \dot{h}_1 \dot{\xi}_1 p.
 \end{aligned} \tag{68}$$

Similarly, given a section $\xi_2 = (\ddot{\xi}_2, \dot{\xi}_2)$ of $\tilde{\mathcal{D}}_2$, applying the same construction we obtain

$$\begin{array}{ccc}
\begin{array}{ccc}
\dot{h}_2 \dot{\xi}_2 \dot{t}_1 \ddot{t}_2 A & \xrightarrow{\ddot{h}_2 \ddot{\xi}_2 \ddot{t}_2 A} & \dot{h}_2 \dot{\xi}_2 \dot{h}_1 \ddot{t}_2 A \\
\downarrow \lambda_2 \ddot{t}_1 A & & \downarrow \lambda_2 \ddot{h}_1 A \\
\dot{h}_2 \dot{\xi}_2 \dot{t}_1 \ddot{h}_2 A & \xrightarrow{\ddot{h}_2 \ddot{\xi}_2 \ddot{h}_2 A} & \dot{h}_2 \dot{\xi}_2 \dot{h}_1 \ddot{h}_2 A
\end{array} & = &
\begin{array}{ccc}
\begin{array}{ccc}
\dot{h}_2 \dot{\xi}_2 \dot{t}_1 \ddot{t}_2 A & \xrightarrow{\ddot{h}_2 \ddot{\xi}_2 \ddot{t}_2 A} & \dot{h}_2 \dot{\xi}_2 \dot{h}_1 \ddot{t}_2 A \\
\downarrow \xi_2 \dot{t}_1 \ddot{t}_2 A & & \downarrow \xi_2 \dot{h}_1 \ddot{t}_2 A \\
\dot{t}_1 \ddot{t}_2 A & \xrightarrow{\ddot{t}_2 A} & \dot{h}_1 \ddot{t}_2 A \\
\downarrow \ddot{t}_1 A & & \downarrow \ddot{h}_1 A \\
\dot{t}_1 \ddot{h}_2 A & \xrightarrow{\ddot{h}_2 A} & \dot{h}_1 \ddot{h}_2 A \\
\downarrow \xi_2 \dot{t}_1 \ddot{h}_2 A & & \downarrow \xi_2 \dot{h}_1 \ddot{h}_2 A \\
\dot{h}_2 \dot{\xi}_2 \dot{t}_1 \ddot{h}_2 A & \xrightarrow{\ddot{h}_2 \ddot{\xi}_2 \ddot{h}_2 A} & \dot{h}_2 \dot{\xi}_2 \dot{h}_1 \ddot{h}_2 A
\end{array}
\end{array}
\end{array}$$

This determines a double groupoid automorphism λ_2 of \mathcal{D} where

$$\begin{aligned}
(\lambda_2)_{[2]} A &= \widetilde{(\ddot{\xi}_2 \ddot{t}_2 A)} *_2 A *_2 (\ddot{\xi}_2 \ddot{h}_2 A), \\
(\lambda_2)_{\{2\}} a &= \widetilde{(\dot{\xi}_2 \dot{t}_2 a)} *_2 a *_2 (\dot{\xi}_2 \dot{h}_2 a), \\
(\lambda_2)_{\{1\}} c &= \ddot{h}_2 \ddot{\xi}_2 c, \\
(\lambda_2)_{\emptyset} p &= \dot{h}_2 \dot{\xi}_2 p.
\end{aligned} \tag{69}$$

Given a pair of sections $\xi = (\xi_1, \xi_2)$, one horizontal and one vertical, if we apply both constructions we obtain an automorphism $\lambda = \lambda_\xi$ of \mathcal{D} provided $\lambda_1 A = \lambda_2 A$. We call ξ a *double section* of \mathcal{D} . The requirement $\lambda_1 A = \lambda_2 A$ implies four equations at the four levels of \mathcal{D} ,

$$\begin{aligned}
\dot{h}_1 \dot{\xi}_1 p &= \dot{h}_2 \dot{\xi}_2 p, \\
\ddot{h}_1 \ddot{\xi}_1 a &= \widetilde{(\dot{\xi}_2 \dot{t}_2 a)} *_2 a *_2 (\dot{\xi}_2 \dot{h}_2 a) = \dot{\lambda}_2 a, \\
\dot{\lambda}_1 c &= \widetilde{(\dot{\xi}_1 \dot{t}_1 c)} *_1 c *_1 (\dot{\xi}_1 \dot{h}_1 c) = \ddot{h}_2 \ddot{\xi}_2 c, \\
\ddot{\lambda}_1 A &= \widetilde{(\ddot{\xi}_1 \ddot{t}_1 A)} *_1 A *_1 (\ddot{\xi}_1 \ddot{h}_1 A) = \widetilde{(\ddot{\xi}_2 \ddot{t}_2 A)} *_2 A *_2 (\ddot{\xi}_2 \ddot{h}_2 A) = \ddot{\lambda}_2 A.
\end{aligned}$$

The previous two diagrams may be combined into the cube-like figure of equation (70), where the six square faces are denoted F =front, L =left, R =right, U =up, D =down and B =back, and

$$F = A, \quad L = \ddot{\xi}_1 \ddot{t}_1 A, \quad R = \ddot{\xi}_1 \ddot{h}_1 A, \quad U = \ddot{\xi}_2 \ddot{t}_2 A, \quad D = \ddot{\xi}_2 \ddot{h}_2 A, \quad B = \lambda A.$$

The figure consists of two commutative diagrams, one above the other, illustrating the relationship between the Lie algebra of the adjoint action and the Lie algebra of the adjoint action of the adjoint action.

Top Diagram:

- Nodes:**
 - Top-left: $\dot{h}_1 \dot{\xi}_1 \dot{t}_1 \ddot{t}_2 A$
 - Top-right: $\dot{h}_1 \dot{\xi}_1 \dot{h}_1 \ddot{t}_2 A$
 - Middle-left: $\dot{t}_1 \ddot{t}_2 A$
 - Middle-right: $\dot{h}_1 \ddot{t}_2 A$
 - Bottom-left: $\dot{t}_1 \ddot{h}_2 A$
 - Bottom-right: $\dot{h}_1 \ddot{h}_2 A$
- Arrows:**
 - Horizontal arrows: $\dot{t}_2 A$ (from middle-left to middle-right) and $\dot{h}_2 \ddot{\xi}_2 \ddot{t}_2 A = \dot{\lambda}_1 \ddot{t}_2 A$ (from top-left to top-right).
 - Vertical arrows: $\dot{t}_1 A$ (from middle-left to bottom-left), $\dot{h}_1 A$ (from middle-right to bottom-right), and $\ddot{h}_1 \ddot{\xi}_1 \ddot{h}_1 A = \ddot{\lambda}_2 \ddot{h}_1 A$ (from top-right to bottom-right).
 - Curved arrows: $\dot{\xi}_1 \dot{t}_1 \ddot{t}_2 A$ (from middle-left to top-left), $\dot{\xi}_1 \dot{h}_1 \ddot{t}_2 A$ (from middle-right to top-right), $\dot{\xi}_1 \dot{t}_1 \ddot{h}_2 A$ (from bottom-left to top-left), and $\dot{\xi}_1 \dot{h}_1 \ddot{h}_2 A$ (from bottom-right to top-right).
 - Dashed arrows: $\dot{\xi}_2 \dot{t}_1 \ddot{t}_2 A$ (from middle-left to top-left), $\dot{\xi}_2 \dot{h}_1 \ddot{t}_2 A$ (from middle-right to top-right), $\dot{\xi}_2 \dot{t}_1 \ddot{h}_2 A$ (from bottom-left to top-left), and $\dot{\xi}_2 \dot{h}_1 \ddot{h}_2 A$ (from bottom-right to top-right).

Bottom Diagram:

- Nodes:**
 - Top-left: $\dot{h}_1 \dot{\xi}_1 \dot{t}_1 \ddot{h}_2 A$
 - Top-right: $\dot{h}_1 \dot{\xi}_1 \dot{h}_1 \ddot{h}_2 A$
 - Middle-left: $\dot{t}_1 \ddot{h}_2 A$
 - Middle-right: $\dot{h}_1 \ddot{h}_2 A$
 - Bottom-left: $\dot{t}_1 \ddot{t}_2 A$
 - Bottom-right: $\dot{h}_1 \ddot{t}_2 A$
- Arrows:**
 - Horizontal arrows: $\dot{h}_2 \ddot{\xi}_2 \ddot{h}_2 A = \dot{\lambda}_1 \ddot{h}_2 A$ (from top-left to top-right) and $\dot{h}_2 A$ (from middle-left to middle-right).
 - Vertical arrows: $\dot{t}_1 A$ (from middle-left to bottom-left), $\dot{h}_1 A$ (from middle-right to bottom-right), and $\ddot{h}_1 \ddot{\xi}_1 \ddot{h}_1 A = \ddot{\lambda}_2 \ddot{h}_1 A$ (from top-right to bottom-right).
 - Curved arrows: $\dot{\xi}_1 \dot{t}_1 \ddot{h}_2 A$ (from middle-left to top-left), $\dot{\xi}_1 \dot{h}_1 \ddot{h}_2 A$ (from middle-right to top-right), $\dot{\xi}_1 \dot{t}_1 \ddot{t}_2 A$ (from bottom-left to top-left), and $\dot{\xi}_1 \dot{h}_1 \ddot{t}_2 A$ (from bottom-right to top-right).
 - Dashed arrows: $\dot{\xi}_2 \dot{t}_1 \ddot{h}_2 A$ (from middle-left to top-left), $\dot{\xi}_2 \dot{h}_1 \ddot{h}_2 A$ (from middle-right to top-right), $\dot{\xi}_2 \dot{t}_1 \ddot{t}_2 A$ (from bottom-left to top-left), and $\dot{\xi}_2 \dot{h}_1 \ddot{t}_2 A$ (from bottom-right to top-right).

13 Simplicial Groups

This material is taken mainly from seminars given by Tim Porter, designed to explain the four functors in the diagram

$$\text{Spaces} \begin{array}{c} \xleftarrow{||} \\ \xrightarrow{\text{Sing}} \end{array} \text{Simplicial Sets} \begin{array}{c} \xleftarrow{W} \\ \xrightarrow{G} \end{array} \text{Simplicial Groupoids}$$

13.1 Simplicial Sets and Simplicial Complexes

A *simplicial set* S comprises

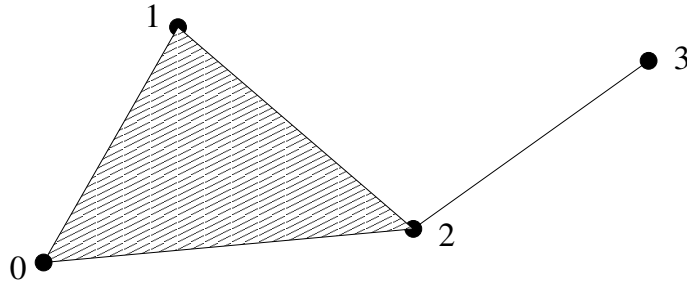
- a sequence $\{S_n\}_{n \geq 0}$ of sets, where S_n is the *set of simplices of dimension n* , linked by
- *face maps* $d_k : S_n \rightarrow S_{n-1}$, $0 \leq k \leq n$, where we think of $d_k(S_n)$ as the “face opposite S_n ”, and
- *degeneracy maps* $s_k : S_n \rightarrow S_{n+1}$, $0 \leq k \leq n$.

A *simplicial complex* K is a set $V(K)$ of vertices, and a family $S \subset (2^{V(K)} \setminus \emptyset)$ closed under \subseteq .

To get from a simplicial complex K to a simplicial set S_K

- pick a total order on $V(K)$,
- let S_K be the set of all finite subsets of $V(K)$,
- set $(S_K)_n = \{[a_0, \dots, a_n] \mid \{a_0, \dots, a_n\} \in S, a_0 \leq \dots \leq a_n\}$.
- choose face maps $d_i[a_0, \dots, a_n] = [a_0, \dots, \hat{a}_i, \dots, a_n]$, omitting the i -th entry, starting at 0,
- choose degeneracy maps $s_i[a_0, \dots, a_n] = [a_0, \dots, a_i, a_i, \dots, a_n]$, duplicating the i -th entry.

Example 13.1 Here is a simplicial complex K with 4 vertices; 4 edges; and 1 triangle.



$$V(K) = \{0, 1, 2, 3\},$$

$$S = \{ \{0, 1, 2\}, \{2, 3\}, \{0, 1\}, \{1, 2\}, \{0, 2\}, \{0\}, \{1\}, \{2\}, \{3\} \}$$

$$\text{order} = 2 < 1 < 3 < 0 \quad (\text{say}),$$

$$(S_K)_0 = \{[2], [1], [3], [0]\},$$

$$(S_K)_1 = \{[2, 2] = s_0[2], [1, 1] = s_0[1], [3, 3] = s_0[3], [0, 0] = s_0[0], [2, 1], [2, 3], [2, 0], [1, 0]\},$$

$$(S_K)_2 = \{[2, 2, 2] = s_0^2[2], [1, 1, 1] = s_0^2[1], [3, 3, 3] = s_0^2[3], [0, 0, 0] = s_0^2[0], [2, 2, 1] = s_0[2, 1], \\ [2, 2, 3] = s_0[2, 3], [2, 2, 0] = s_0[2, 0], [2, 1, 1] = s_1[2, 1], [2, 3, 3] = s_1[2, 3], \\ [2, 0, 0] = s_1[2, 0], [1, 1, 0] = s_0[1, 0], [1, 0, 0] = s_1[1, 0], [2, 1, 0]\}.$$

The face and degeneracy maps satisfy the following identities. **[need to check these!]**

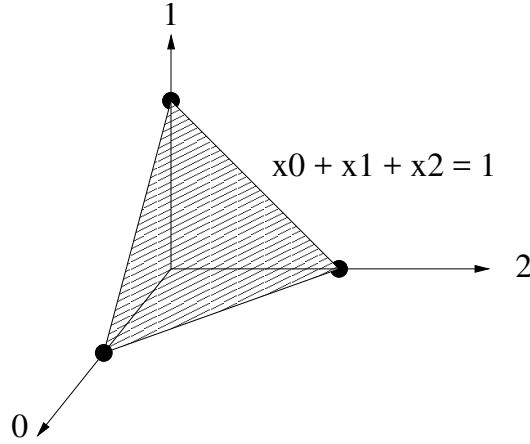
The second column gives the common image of $[a_0, \dots, a_n]$.

$$\begin{array}{ll}
 d_{j-1} \circ d_i = d_i \circ d_j & (0 \leq i < j \leq n) & [a_0, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_n], \\
 d_{j+1} \circ s_i = s_i \circ d_j & (0 \leq i < j \leq n) & [a_0, \dots, a_i, a_i, \dots, \hat{a}_j, \dots, a_n], \\
 d_{i+1} \circ s_i = d_i \circ s_i = \text{id} & (0 \leq i \leq n) & [a_0, \dots, a_n], \\
 d_i \circ s_j = s_{j-1} \circ d_i & (0 \leq i < j \leq n) & [a_0, \dots, \hat{a}_i, \dots, a_j, a_j, \dots, a_n], \\
 s_{j+1} \circ s_i = s_i \circ s_j & (0 \leq i < j \leq n) & [a_0, \dots, a_i, a_i, \dots, a_j, a_j, \dots, a_n], \\
 s_{i+1} \circ s_i = s_i^2 & (0 \leq i \leq n) & [a_0, \dots, a_i, a_i, a_i, \dots, a_n].
 \end{array} \tag{71}$$

The standard n -simplex, Δ^n

(See Ehlers [28] for some of this material.)

Let $B_{n+1} = \{e_0, \dots, e_n\}$ be the standard basis for \mathbb{R}^{n+1} , and let Δ^n be the convex hull of the points B_{n+1} , a subset of the hyperplane $x_0 + x_1 + \dots + x_n = 1$. The picture below shows $\Delta^2 \subseteq \mathbb{R}^3$.



Any $x \in \Delta^n$ may be expressed as

$$x = \sum_{i=0}^n x_i e_i, \quad \sum_{i=0}^n x_i = 1, \quad x_i \geq 0.$$

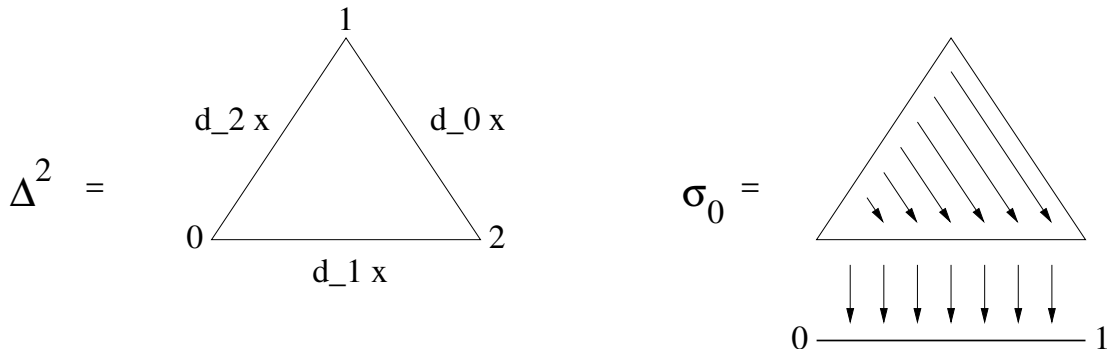
We then define the *linear* maps

$$\delta_k : \Delta^{n-1} \rightarrow \Delta^n, \quad (x_0, \dots, x_{n-1}) \mapsto (x_0, \dots, x_k, 0, x_{k+1}, \dots, x_{n-1}), \quad 0 \leq k \leq n.$$

Similarly, we define

$$\sigma_k : \Delta^n \rightarrow \Delta^{n-1}, \quad (x_0, \dots, x_n) \mapsto (x_0, \dots, x_{k-1}, x_k + x_{k+1}, x_{k+2}, \dots, x_n), \quad 0 \leq k \leq n-1.$$

The picture for Δ^2 is

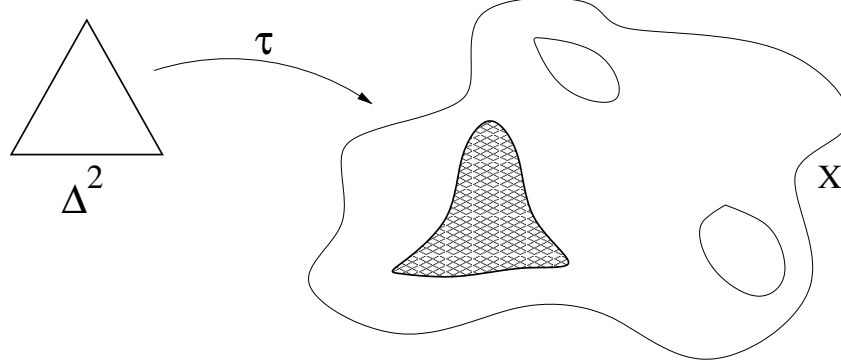


The Singular Complex $\text{Sing}(X)$ of a space X .

For X a space we define the *singular complex* of X to be the set of maps

$$\text{Sing}(X) = \text{Top}(\Delta^n, X).$$

The diagram below shows such a map τ to a space X with two holes.



The Nerve of a Category

Let \mathbb{C} be a small category. The *nerve* of \mathbb{C} is defined to be the simplicial set $\text{Ner } \mathbb{C}$ where

- $(\text{Ner } \mathbb{C})_0 = \text{Ob}(\mathbb{C}),$
- $(\text{Ner } \mathbb{C})_1 = \text{Arr}(\mathbb{C}),$
- for $x \in (\text{Ner } \mathbb{C})_1, d_0x = tx$ and $d_1x = hx,$
- for $u \in (\text{Ner } \mathbb{C})_0, s_0u = 1_u,$
- $(\text{Ner } \mathbb{C})_n = \{\text{composable words } (x_1, \dots, x_n) \mid x_i \in \text{Arr}(\mathbb{C})\},$
- $d_0(x_1, x_2, \dots, x_n) = (x_2, \dots, x_n),$
- $d_i(x_1, \dots, x_n) = (x_1, \dots, x_i x_{i+1}, \dots, x_n), \quad (0 < i < n),$
- $d_n(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}),$
- $s_0(x_1, \dots, x_n) = (1_{tx_0}, x_1, \dots, x_n),$
- $s_i(x_1, \dots, x_n) = (x_1, \dots, x_i, 1_{hx_i}, x_{i+1}, \dots, x_n), \quad (0 < i < n),$
- $s_n(x_1, \dots, x_n) = (x_1, \dots, x_n, 1_{hx_n}).$

This may be described more compactly by saying that $(\text{Ner } \mathbb{C})_n = \text{Cat}([n], \mathbb{C})$. This construction extends to a functor from Cat to SimpSet in the obvious way.

14 Crossed n -cubes of groups and cat^n -groups

Here we include the basic ideas about crossed n -cubes (of groups) taken from Chapter 1 of Ellis' thesis [29], Ellis-Steiner [31], and Brown-Loday [18], and the associated cat^n -groups.

14.1 Crossed n -cubes of groups

Let $[n] = \{1, 2, \dots, n\}$, and let A, B, C, \dots be subsets of $[n]$.

A crossed n -cube consists of the following.

- (i) Groups R_A for each subset A of $[n]$, where we write R for R_\emptyset .
- (ii) Group homomorphisms

$$\partial_{A,i} : R_A \rightarrow R_{A \setminus \{i\}} \quad \text{for all } A \subseteq [n], i \in A,$$

such that $\partial_{A \setminus \{j\},i} \circ \partial_{A,j} = \partial_{A \setminus \{i\},j} \circ \partial_{A,i}$ for $i \neq j \in A$.

Since the $\partial_{A,i}$ commute, composite homomorphisms $\partial_{A,B} : R_A \rightarrow R_{A \setminus B}$ are well defined and $\partial_{A,B} = \partial_{A,(B \setminus C)} \circ \partial_{A,C}$.

- (iii) For all $B \subseteq A$ an action of $R_{A \setminus B}$ on R_A making $\mathcal{R}_{A,A \setminus B} = (\partial_B : R_A \rightarrow R_{A \setminus B})$ a crossed module. For each $j \in [n]$ the maps

$$(1, \partial_{A \setminus \{j\}}) : \mathcal{R}_{A,A \setminus \{j\}} \rightarrow \mathcal{R}_{A,A \setminus \{j\}} \quad \text{and} \quad (\partial_{A,j}, 1) : \mathcal{R}_{A,A \setminus \{j\}} \rightarrow \mathcal{R}_{A \setminus \{j\}, A \setminus \{j\}}$$

are crossed module homomorphisms.

It follows that all the actions act via R :

$$a^b = a^{\partial_{B,B} b} \quad \text{for } a \in R_A, b \in R_B, \text{ and } B \subseteq A.$$

- (iv) For all $A, B \subseteq [n]$ a crossed pairing $\boxtimes_{A,B} : R_A \times R_B \rightarrow R_{A \cup B}$, such that $(b \boxtimes_{B,A} a) = (a \boxtimes_{A,B} b)^{-1}$ and, when $B \subseteq A$, $\boxtimes_{A,B}$ is the principal crossed pairing for $\mathcal{R}_{A,B}$ given by $a \boxtimes b = a^{-1} a^b$ and $b \boxtimes a = (a^{-1})^b a$.
- (v) Various axioms relating the homomorphisms, actions, and crossed pairings, for example

- $\partial_{A \cup B, i}(a \boxtimes_{A,B} b) = (\partial_{A,i} a) \boxtimes_{A \setminus \{i\}, B \setminus \{i\}} (\partial_{B,i} b)$.
- when $i \in A \cap B$, so that $A \cup B = (A \setminus \{i\}) \cup B = A \cup (B \setminus \{i\})$,

$$a \boxtimes_{A,B} b = \partial_{A,i} a \boxtimes_{A \setminus \{i\}, B \setminus \{i\}} b = a \boxtimes_{A \setminus \{i\}, B} \partial_{B,i} b.$$

(This means that we need only define $\boxtimes_{A,B}$ when $A \cup B = \emptyset$.)

- $(a \boxtimes b)^c = a^c \boxtimes b^c$ when $C \subseteq A$ and $C \subseteq B$. **[Is this correct?]**

We might now wish to define an n -derivation, which would seem to be a set of maps

$$\chi_{B,A} : R_B \rightarrow R_A \quad \text{for all } B \subseteq A$$

satisfying suitable axioms:

- (i) perhaps $\chi_{B,A}(bb') = (\chi_{B,A}b)^{b'}(\chi_{B,A}b')$?
- (ii) closure: $\chi_{B,A} \circ \chi_{C,B} = \chi_{C,A}$?
- (iii) ???

Exercise 14.1 Derive the crossed square axioms from those of a crossed 2-cube.

[These are just some thoughts to be worked on!]

14.2 Catⁿ-groups

As with cat²-groups (see 8.3), we will give three definitions.

Definition 14.2 A catⁿ-group consists of the following.

- (i) 2ⁿ groups G_A , one for each subset A of $[n]$, the *vertices* of an n -cube.
- (ii) Group homomorphisms forming $n2^{n-1}$ commuting cat¹-groups,

$$\mathcal{C}_{A,i} = (e_{A,i}; t_{A,i}, h_{A,i} : G_A \rightarrow G_{A \setminus \{i\}}), \quad \text{for all } A \subseteq [n], i \in A,$$

the *edges* of the cube.

- (iii) These cat¹-groups combine (in sets of 4) to form $n(n-1)2^{n-3}$ cat²-groups $\mathcal{C}_{A,\{i,j\}}$ for all $\{i,j\} \subseteq A \subseteq [n]$, $i \neq j$, the *faces* of the cube.

Note that, since the $t_{A,i}, h_{A,i}$ and $e_{A,i}$ commute, composite homomorphisms $t_{A,B}, h_{A,B} : G_A \rightarrow G_{A \setminus B}$ and $e_{A,B} : G_{A \setminus B} \rightarrow G_A$ are well defined for all $B \subseteq A \subseteq [n]$.

Secondly, we give the simplest of the three definitions, again adapted from Ellis-Steiner [31].

Definition 14.3 A catⁿ-group \mathcal{C} consists of 2ⁿ groups G_A , one for each subset A of $[n]$, and $3n$ homomorphisms $t_{[n],i}, h_{[n],i} : G_{[n]} \rightarrow G_{[n] \setminus \{i\}}$, $e_{[n],i} : G_{[n] \setminus \{i\}} \rightarrow G_{[n]}$, satisfying the following axioms for all $1 \leq i \leq n$,

- (a) the $\mathcal{C}_{[n],i} = (e_{[n],i}; t_{[n],i}, h_{[n],i} : G_{[n]} \rightarrow G_{[n] \setminus \{i\}})$ are *commuting* cat¹-groups, so that:
- (b) $(e_1 \circ t_1) \circ (e_2 \circ t_2) = (e_2 \circ t_2) \circ (e_1 \circ t_1)$, $(e_1 \circ h_1) \circ (e_2 \circ h_2) = (e_2 \circ h_2) \circ (e_1 \circ h_1)$,
- (c) $(e_1 \circ t_1) \circ (e_2 \circ h_2) = (e_2 \circ h_2) \circ (e_1 \circ t_1)$, $(e_2 \circ t_2) \circ (e_1 \circ h_1) = (e_1 \circ h_1) \circ (e_2 \circ t_2)$.

Our third definition defines a catⁿ-group as a "cat¹-group of cat⁽ⁿ⁻¹⁾-groups".

Definition 14.4 A catⁿ-group \mathcal{C} consists of two cat⁽ⁿ⁻¹⁾-groups:

- \mathcal{A} with groups $G_A, A \subseteq [n-1]$, and homomorphisms $\ddot{t}_{A,i}, \ddot{h}_{A,i}, \ddot{e}_{A,i}$,
- \mathcal{B} with groups $H_B, B \subseteq [n-1]$, and homomorphisms $\dot{t}_{B,i}, \dot{h}_{B,i}, \dot{e}_{B,i}$,
- and cat⁽ⁿ⁻¹⁾-morphisms $t, h : \mathcal{A} \rightarrow \mathcal{B}$ and $e : \mathcal{B} \rightarrow \mathcal{A}$ subject to the following conditions:

- C1:** $(t \circ e)$ and $(h \circ e)$ are the identity mapping on \mathcal{B} ,
- C2:** $[\ker t, \ker h] = \{1_{\mathcal{A}}\}$.

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