Notes on higher dimensional groups and groupoids and related topics

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Contents

Preface

The aim of these notes is to collect together, using a common notation and right actions, information about crossed modules; cat1-groups; crossed squares; cat2-groups; and related structures.

Inspiration has come from Ronnie Brown; Tim Porter; postgraduate students; and the many visitors to the department in Bangor.

Help in the preparation of these notes has initially been provided by Murat Alp and Robert Rodrigues. A seminar course for Gareth Evans and Richard Lewis during 2002/2003 resulted in many corrections and additions. Many additions were made when Murat Alp visited Bangor in July and August 2003.

After that, some material was added up until 2008. The notes were returned to in April 2017 when work started with Ronnie Brown of induced crossed squares and with Alper Odabas on GAP code for cat2-groups. The notes were deposited on the arXiv in February 2018.

In parallel with these notes the GAP packages XMod and Groupoids have been developed to perform computations with these structures. These packages may be obtained from the main GAP package distribution at `https://www.gap-system.org/Packages/` or from the GitHub repositories at: `https://gap-packages.github.io/groupoids/` and `https://gap-packages.github.io/xmod/`.

If you have a question relating to either of these packages, encounter any problems, or have a suggestion for extending the packages in any way, please make an issue at: `https://github.com/gap-packages/groupoids/issues/new` or `https://github.com/gap-packages/xmod/issues/new`.

The main sources used are (up until 2008): Alp & Wensley [3] (2000); Brown & Gilbert [9] (1989); Brown & Huebschmann [16] (1982); Brown and İçen [17] (2003); Brown & Loday [19] (1987); Ellis [30] (1984); Ellis & Steiner [32] (1987); Kamps & Porter [42] (2002); Norrie [49] (1987) and Porter [52] (1993).

These notes are subject to intermittent development. Later sections of somer chapters contain lots of comments and suggestions of items which need to be corrected, added or improved.

Much of the later sections were to form part of a paper on the actor of a crossed square, but this was never completed.

1 Crossed Modules and Cat^1-Groups

In this section we describe four equivalent categories:

- **XMod**, the category of crossed modules and their morphisms;
- **Cat**1, the category of cat^1-groups and their morphisms;
- **GpGpd**, a category of sets with both a group structure and a groupoid structure; and
- **2-Gp**, a subcategory of **2-Cat**.

We also describe functors between these categories which exhibit the various equivalences.

1.1 Precrossed and Crossed Modules

Let S and R be groups acting upon themselves by conjugation:

$$ {s_0}^s = s^{-1}s_0s, \qquad {r_0}^r = r^{-1}r_0r. $$

A *precrossed module* $\mathcal{X} = (\delta : S \to R)$ consists of a group homomorphism δ, called the *boundary* of $\mathcal{X}$, together with an action $\alpha : R \to \mathrm{Aut}(S)$ such that δ is an R-morphism. So, for all $s \in S$ and $r \in R$,

$$ \textbf{X1:} \quad \delta(s^r) \quad = \quad (\delta s)^r \ . $$

An alternative terminology is to say that (S, δ) is a precrossed R-module. The groups S, R are called the *source* and *range* of $\mathcal{X}$ respectively.

The precrossed module $\mathcal{X}$ is a *crossed module* if it also satisfies, for all $s_0, s \in S$,

$$ \textbf{X2:} \quad {s_0}^{\delta s} \quad = \quad {s_0}^s \ . $$

A *morphism of precrossed modules* $\alpha : \mathcal{X}_1 \to \mathcal{X}_2$ is a pair $\alpha = (\ddot{\alpha}, \dot{\alpha})$, where $\ddot{\alpha} : S_1 \to S_2$ and $\dot{\alpha} : R_1 \to R_2$ are homomorphisms satisfying

$$ \delta_2 \circ \ddot{\alpha} = \dot{\alpha} \circ \delta_1, \quad \ddot{\alpha}(s^r) = (\ddot{\alpha}s)^{\dot{\alpha}r}, $$

making the following diagram commute:

$$\begin{tikzcd} S_1 \arrow[r, "\ddot{\alpha}"] \arrow[d, "\delta_1"'] & S_2 \arrow[d, "\delta_2"] \\ R_1 \arrow[r, "\dot{\alpha}"'] & R_2 \end{tikzcd} \tag{1}$$

When $\mathcal{X}_1, \mathcal{X}_2$ are both crossed modules then α is a *morphism of crossed modules* without any further condition. We thus obtain the category **PreXMod** of precrossed modules and their morphisms, and the category **XMod** of crossed modules and their morphisms. Furthermore, **XMod** is a full subcategory of **PreXMod**.

When $\mathcal{X}_2 = \mathcal{X}_1$ and $\ddot{\alpha}, \dot{\alpha}$ are automorphisms then α is an automorphism of $\mathcal{X}_1$. The group of automorphisms is denoted by $\mathrm{Aut}(\mathcal{X}_1)$.

We have to be careful about a notational problem which arises because, although we are using right actions, we are still writing functions of the left. Thus the group of automorphisms of R has multiplication $\beta_1 * \beta_2$ given by

$$ (\beta_1 * \beta_2)r \ = \ \beta_2(\beta_1(r)) \quad \text{or, more conveniently,} \quad \beta_2\beta_1 r \ , $$

where β_1 is applied *first* to r, and β_2 *second.*

1.2 Examples of Crossed Modules

Standard constructions for crossed modules include the following:

1. A *conjugation crossed module* (inc : $S \to R$) is an inclusion of a normal subgroup $S \trianglelefteq R$, where R acts on S by conjugation. The example takes the alternating subgroup a_4 of the symmetric group s_4.

```
gap> s4 := Group( (1,2,3,4), (3,4) );;  SetName( s4, "s4" );
gap> a4 := Subgroup( s4, [(1,2,3),(2,3,4)] );;  SetName( a4, "a4" );
gap> X4 := XModByNormalSubgroup( s4, a4 );;
gap> Display( X4 );
Crossed module [a4->s4] :-
: Source group has generators:
  [ (1,2,3), (2,3,4) ]
: Range group s4 has generators:
  [ (1,2,3,4), (3,4) ]
: Boundary homomorphism maps source generators to:
  [ (1,2,3), (2,3,4) ]
: Action homomorphism maps range generators to automorphisms:
  (1,2,3,4) --> { source gens --> [ (2,3,4), (1,3,4) ] }
  (3,4) --> { source gens --> [ (1,2,4), (2,4,3) ] }
  These 2 automorphisms generate the group of automorphisms.
```

2. An *automorphism crossed module* (inn : $R \to S$) has as range a subgroup R of the automorphism group Aut(S) of S which contains the inner automorphism group Inn(S) of S. The boundary maps $s \in S$ to the inner automorphism of S by s. In the example the quaternion group q_8 has automorphism group isomorphic to s_4. A permutation representation of degree 6 for this automorphism group is chosen automatically.

```
gap> q8 := Group( (1,2,3,4)(5,8,7,6), (1,5,3,7)(2,6,4,8) );;
gap> SetName( q8, "q8" );
gap> X8 := XModByAutomorphismGroup( q8 );;
gap> Display( X8 );
Crossed module [q8->PAut(q8)] :-
: Source group q8 has generators:
  [ (1,2,3,4)(5,8,7,6), (1,5,3,7)(2,6,4,8) ]
: Range group PAut(q8) has generators:
  [ (3,6,5,4), (1,6,3)(2,4,5), (3,5)(4,6), (1,2)(4,6) ]
: Boundary homomorphism maps source generators to:
  [ (3,5)(4,6), (1,2)(4,6) ]
: Action homomorphism maps range generators to automorphisms:
  (3,6,5,4) --> { source gens --> [ (1,2,3,4)(5,8,7,6), (1,8,3,6)(2,5,4,7) ] }
  (1,6,3)(2,4,5) --> { source gens -->
[ (1,8,3,6)(2,5,4,7), (1,2,3,4)(5,8,7,6) ] }
  (3,5)(4,6) --> { source gens --> [ (1,2,3,4)(5,8,7,6), (1,7,3,5)(2,8,4,6)
 ] }
  (1,2)(4,6) --> { source gens --> [ (1,4,3,2)(5,6,7,8), (1,5,3,7)(2,6,4,8)
 ] }
  These 4 automorphisms generate the group of automorphisms.
```

3. A *zero boundary crossed module* ($0 : M \to R$) has an R-module M as source and $\partial = 0$.

4. Any homomorphism $\partial : S \to R$, with S abelian and im ∂ in the centre of R, provides a crossed module with R acting trivially on S.

5. A *central extension crossed module* has as boundary a surjection $\partial : S \to R$ with central kernel, where $r \in R$ acts on S by conjugation with $\partial^{-1}r$.

6. The *direct product* of $\mathcal{X}_1 = (\partial_1 : S_1 \to R_1)$ and $\mathcal{X}_2 = (\partial_2 : S_2 \to R_2)$ is $\mathcal{X}_1 \times \mathcal{X}_2 = (\partial_1 \times \partial_2 : S_1 \times S_2 \to R_1 \times R_2)$ with R_1, R_2 acting trivially on $S_2,\ S_1$ respectively.

Here is a verification for the fifth example. Suppose $s_1, s_2 \in \partial^{-1}r$. Then $s_2 = s_1 k$ for some $k \in \ker\partial$, so $s^{s_2} = s^{s_1 k} = s_1^{-1}(k^{-1}sk)s_1 = s^{s_1}$, and the action is well-defined. The two axioms are then easily verified.

1.3 Properties of Crossed Modules

Lemma 1.1

(i) *The kernel K of ∂ is central in S, and so is abelian.*

(ii) *The image $J = \operatorname{im}\partial$ is normal in R, and so we may define $C = \operatorname{coker}\partial = R/J$, with natural map $\nu : R \to C$, and hence obtain an exact sequence of groups*

$$1 \longrightarrow J \xrightarrow{\iota} R \xrightarrow{\nu} C \longrightarrow 1\ .$$

(iii) *The group J acts trivially on the centre ZS of S, and so trivially on K. Hence K inherits an action of C, making K a C-module with $k^{Jr} := k^r$, and giving a crossed module $(\nu\partial|_K : K \to C)$.*

(iv) *If $\partial' : S \to J$ is the restriction of ∂, there is an exact sequence of R-groups*

$$1 \longrightarrow K \longrightarrow S \xrightarrow{\partial'} J \longrightarrow 1\ .$$

(v) *The image J acts trivially on the abelianisation $S^{\mathrm{ab}} = S/[S,S]$ of S, and so S^{ab} is also a C-module, with action*

$$([S,S]s)^{(Jr)} = [S,S](s^r)\ . \tag{2}$$

Proof:

(i) If $k \in K$ and $s \in S$ then, by **X2**,

$$s^1 = s^{\partial k} = s^k = k^{-1}sk \qquad \text{and so} \qquad ks = sk\,.$$

(ii) The conjugate of ∂s by $r \in R$ is $r^{-1}(\partial s)r = (\partial s)^r = \partial(s^r) \in J$.

(iii) If $z \in ZS$ then, by **X2**, $z^{\partial s} = s^{-1}zs = z$. Then $\nu(\partial k) = J(\partial k)$ and

$$\nu\partial(k^{Jr}) = J(\partial(k^r)) = J((\partial k)^r) = (Jr^{-1})(J(\partial k))(Jr) = (\nu\partial k)^{Jr}.$$

(iv) The kernel of ∂' is K and ∂' is surjective.

(v)

$$[S,S]1 = [S,S][s_1, s_2^{-1}] = [S,S]\,s_2^{\partial s_1}\,s_2^{-1} \qquad \Rightarrow \qquad [S,S]s_2 = [S,S]s_2^{\partial s_1}\ .$$

Hence J acts trivially on S^{ab}, and it is easy to check that (2) defines an action.

□

1.4 Sub-crossed modules

Definition 1.2 *A crossed module* $\mathcal{X}_1 = (\partial_1 : S_1 \to R_1)$ *is a sub-crossed module of* $\mathcal{X} = (\partial : S \to R)$*, written* $\mathcal{X}_1 \leqslant \mathcal{X}$*, if*

- S_1, R_1 *are subgroups of* S, R *respectively,*
- ∂_1 *is the restriction of* ∂ *to* S_1*,*
- *the action of* R_1 *on* S_1 *is induced by the action of* R *on* S*.*

$$\begin{tikzcd}
S_1 \arrow[r, "i_{S_1}"] \arrow[d, "\partial_1"'] & S \arrow[d, "\partial"] \\
R_1 \arrow[r, "i_{R_1}"'] & R
\end{tikzcd}$$

Definition 1.3 *The* inclusion morphism $i = (i_{S_1}, i_{R_1}) : \mathcal{X}_1 \to \mathcal{X}$ *consists of the two subgroup inclusions* $i_{S_1} : S_1 \to S$ *and* $i_{R_1} : R_1 \to R$*.*

Definition 1.4 *The sub-crossed module* $\mathcal{X}_1$ *is a* normal sub-crossed module *of* $\mathcal{X}$*, written* $\mathcal{X}_1 \trianglelefteq \mathcal{X}$*, if*

(a) $R_1 \trianglelefteq R$*,*

(b) ${s_1}^r \in S_1$ *for all* $r \in R,\ s_1 \in S_1$*,*

(c) $s^{-1}s^{r_1} \in S_1$ *for all* $r_1 \in R_1,\ s \in S$*.*

Note that these conditions imply that $S_1 \trianglelefteq S$, for ${s_1}^s = {s_1}^{\partial s}$ by **X2:** and belongs to S_1 by (b).

Proposition 1.5 *Given two normal sub-crossed modules* $\mathcal{X}_1, \mathcal{X}_2$ *of* $\mathcal{X}$*, there is a third normal sub-crossed module of* $\mathcal{X}$ *called the* commutator sub-crossed module $[\mathcal{X}_1, \mathcal{X}_2]$*, having*

- *source group* $[S_1, S_2]$*,*
- *range group* $[R_1, R_2]$*,*
- *the restriction* ∂' *of* ∂ *to* $[S_1, S_2]$ *as boundary map.*

Proof: First note that $\partial'[s_1, s_2] = [\partial s_1, \partial s_2]$, and that $[s_1, s_2]^{r_1} = [{s_1}^{r_1}, {s_2}^{r_1}]$, so $[\mathcal{X}_1, \mathcal{X}_2]$ is a sub-crossed module of $\mathcal{X}$. To show normality we check that

(a) $[R_1, R_2] \trianglelefteq R$, since $r^{-1}[r_1, r_2]r = [r^{-1}r_1r, r^{-1}r_2r]$,

(b) $[s_1, s_2]^r = [{s_1}^r, {s_2}^r] \in [S_1, S_2]$,

(c) The product $s^{-1}s^{[r_1,r_2]}$ may be expanded as

$$\left[s^{-1}s^{r_1^{-1}}\right]\left[\left(s^{r_1^{-1}}\right)^{-1}\left(s^{r_1^{-1}}\right)^{r_2^{-1}}\right]\left[\left(s^{r_1^{-1}r_2^{-1}}\right)^{-1}\left(s^{r_1^{-1}r_2^{-1}}\right)^{r_1}\right]\left[\left(s^{r_1^{-1}r_2^{-1}r_1}\right)^{-1}\left(s^{r_1^{-1}r_2^{-1}r_1}\right)^{r_2}\right],$$

and each of the four pairs of terms has the form $s^{-1}s^{r_0}$, and so is in S_1.

□

1.5 Properties of morphisms of crossed modules

Let $\mu = (\ddot{\mu}, \dot{\mu}) : \mathcal{X} \to \mathcal{X}'$ be a morphism of crossed modules. The kernel of μ is the sub-crossed module

$$\ker \mu \;=\; (\partial \mid_{\ker \ddot{\mu}} : \, \ker \ddot{\mu} \to \ker \dot{\mu})$$

of $\mathcal{X}$, as shown in the following diagram.

$$\begin{tikzcd}
\ker \ddot{\mu} \arrow[r, "i"] \arrow[d, "\partial|_{\ker \ddot{\mu}}"'] & S \arrow[r, "\ddot{\mu}"] \arrow[d, "\partial"] & S' \arrow[d, "\partial'"] \\
\ker \dot{\mu} \arrow[r, "i"'] & R \arrow[r, "\dot{\mu}"'] & R'
\end{tikzcd}$$

This statement is justified by the following:

$$s \in \ker \ddot{\mu} \;\Leftrightarrow\; \ddot{\mu}s = 1 \;\Leftrightarrow\; \partial'\ddot{\mu}s = 1 \;\Leftrightarrow\; \dot{\mu}\partial s = 1 \;\Leftrightarrow\; \partial s \in \ker \dot{\mu}.$$

Lemma 1.6 *The kernel of μ is a normal sub-crossed module of $\mathcal{X}$.*

Proof:

(a) $\ker \dot{\mu} \trianglelefteq R$,

(b) if $\ddot{\mu}s_1 = 1$ then $\ddot{\mu}(s_1^r) = (\ddot{\mu}s_1)^{\dot{\mu}r} = 1^{\dot{\mu}r} = 1$, so $s_1^r \in \ker \ddot{\mu}$,

(c) if $\dot{\mu}r_1 = 1$ then $\ddot{\mu}((s^{-1})^{r_1}s) = \left(\ddot{\mu}s^{-1}\right)^{\dot{\mu}r_1}(\ddot{\mu}s) = 1$, so $(s^{-1})^{r_1}s \in \ker \ddot{\mu}$.

□

Theorem 1.7 *Given a normal sub-crossed module $\mathcal{X}_1$ of a crossed module $\mathcal{X}$, there is a quotient crossed module*

$$\mathcal{X}/\mathcal{X}_1 \;=\; (\delta : S/S_1 \to R/R_1)$$

where the action is defined by

$$(S_1 s)^{R_1 r} \;:=\; S_1(s^r)$$

and the boundary map is given by

$$\delta(S_1 s) \;:=\; R_1(\partial s).$$

Proof: We first check that we do have an action:

$$(S_1 s)^{R_1 r}(S_1 s')^{R_1 r} \;=\; (S_1(s^r))(S_1(s'^r)) \;=\; S_1(s^r)(s'^r) \;=\; S_1(ss')^r \;=\; (S_1(ss'))^{R_1 r},$$

$$((S_1 s)^{R_1 q})^{R_1 r} \;=\; (S_1(s^q))^{R_1 r} \;=\; S_1(s^q)^r \;=\; S_1(s^{qr}) \;=\; (S_1 s)^{R_1(qr)}.$$

Then we check the crossed module axioms:

M1: $\delta((S_1 s)^{R_1 r}) \;=\; \delta(S_1 s^r) \;=\; R_1\partial(s^r) \;=\; R_1 r^{-1}(\partial s)r \;=\; (R_1 r^{-1})(R_1\partial s)(R_1 r) \;=\; (\delta(S_1 s))^{R_1 r}$,

M2: $(S_1 s')^{\delta(S_1 s)} \;=\; (S_1 s')^{R_1(\partial s)} \;=\; S_1(s')^{\partial s} \;=\; S_1 s^{-1}s's \;=\; (S_1 s^{-1})(S_1 s')(S_1 s)$.

□

Theorem 1.8 *[from Tim Porter's Topology paper [52]]*
Every crossed module is a quotient of normal inclusion crossed modules.

Proof: Given $\mathcal{X} = (\partial : S \to R)$, consider the diagram

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[d, "0"'] & S \arrow[r, "1"] \arrow[d, "\epsilon"] & S \arrow[d, "\partial"] \\
S \arrow[r, "\zeta"'] & R \ltimes S \arrow[r, "h"'] & R \\
\Gamma_1\mathcal{X} & \Gamma_0\mathcal{X} & \mathcal{X}
\end{tikzcd} \tag{3}$$

where the crossed modules $\Gamma_0\mathcal{X}$ and $\Gamma_1\mathcal{X}$ are given by

- $\epsilon s = (1, s), \quad \zeta s = (\partial s, s^{-1}), \quad {s_1}^{(r,s)} = s^{-1}{s_1}^r s = {s_1}^{r(\partial s)}$,
- $h(r, s) = r(\partial s), \quad (r_1, s)^r = (r^{-1}r_1 r, s^r)$.

The proof requires four verifications. Recall that in the semidirect product $R \ltimes S$ we have

$$(r, s)(r', s') = (rr', s^{r'}s') \qquad \text{and} \qquad (r, s)^{-1} = (r^{-1}, (s^{-1})^{r^{-1}}). \tag{4}$$

(a) Verification that $(\epsilon : S \to R \ltimes S)$ is a crossed module.

X1: $(\epsilon s_1)^{(r,s)} \;=\; (r^{-1}, (s^{-1})^{r^{-1}})(1, s_1)(r, s) \;=\; (1, s^{-1}s_1^r s) \;=\; \left(1, s_1^{(r,s)}\right) \;=\; \epsilon\left(s_1^{(r,s)}\right)$.

X2: ${s_1}^{\epsilon s} \;=\; {s_1}^{(1,s)} \;=\; s^{-1}s_1 s$.

(b) Verification that $(1, h)$ and (ϵ, δ) are morphisms of crossed modules.

(i) $h\epsilon s \;=\; h(1, s) \;=\; \partial s$,

(ii) $\left\{ \begin{array}{rcl} 1({s_1}^{(r,s)}) &=& {s_1}^{(r,s)} \;=\; s^{-1}{s_1}^r s, \\ (1s_1)^{h(r,s)} &=& {s_1}^{r(\partial s)} \;=\; s^{-1}{s_1}^r s. \end{array} \right.$

(ii′) $\left\{ \begin{array}{rcl} \epsilon({s_1}^s) &=& (1, {s_1}^s) \;=\; (1, s^{-1}s_1 s), \\ (\epsilon s_1)^{\partial s} &=& (1, s_1)^{\partial s} \;=\; (1, s_1)^{(1,\partial s)} \;=\; (1, {s_1}^{\partial s}) \;=\; (1, s^{-1}s_1 s). \end{array} \right.$

(c) Verification that $(0, \zeta)$ is the inclusion of a normal sub-crossed module.

We only need to show that ${s_1}^{-1}{s_1}^{\zeta s} = 1$.

$${s_1}^{-1}{s_1}^{\zeta s} \;=\; {s_1}^{-1}{s_1}^{(\partial s, s^{-1})} \;=\; {s_1}^{-1}s{s_1}^{\partial s}s^{-1} \;=\; 1.$$

(d) Verification that $h\zeta = 0$.

$$h\zeta s \;=\; h(\partial s, s^{-1}) \;=\; (\partial s)(\partial s^{-1}) \;=\; 1.$$

□

Corollary 1.9 *In the previous result* $\Gamma_0, \Gamma_1 : \mathsf{XMod} \to \mathsf{XMod}$ *are functors and* $(0,\zeta) : \Gamma_1 \to \Gamma_0$ *is a natural transformation.*

Note that, in diagram (3), $(h : R \ltimes S \to R)$ is *not* in general a crossed module, so the right-hand square is not a crossed square (see Chapter 8).

We may extend the above Theorem to morphisms of crossed modules. In the diagram (5) below, $(\sigma, \rho) : \mathcal{X}_1 = (\partial : S \to R) \to \mathcal{X}_2 = (\delta : Q \to P)$ is a morphism of crossed modules, and both of these crossed modules are expressed as quotients of normal inclusion crossed modules. There are morphisms $(\sigma, \gamma) : \Gamma_0\mathcal{X}_1 = (\epsilon : S \to R \ltimes S) \to \Gamma_0\mathcal{X}_2 = (\eta : Q \to P \ltimes Q)$ and $(1, \sigma) : \Gamma_1\mathcal{X}_1 \to \Gamma_1\mathcal{X}_2$ where $\gamma(r,s) = (\rho r, \sigma s)$ and $\eta\sigma s = (1, \sigma s) = \gamma(1,s) = \gamma\epsilon s$.

$$\begin{tikzcd}
 & 1 \arrow[rr, "0"] \arrow[d, "0"] & & Q \arrow[rr, "1"] \arrow[d, "\eta"] & & Q \arrow[d, "\delta"] \\
 & Q \arrow[rr, "\theta"] & & P \ltimes Q \arrow[rr, "k"] & & P \\
1 \arrow[rr, "0"] \arrow[d, "0"'] \arrow[uur, "1"] & & S \arrow[rr, "1"] \arrow[d, "\epsilon"'] \arrow[uur, "\sigma"] & & S \arrow[d, "\partial"'] \arrow[uur, "\sigma"] & \\
S \arrow[rr, "\zeta"'] \arrow[uur, "\sigma"] & & R \ltimes S \arrow[rr, "h"'] \arrow[uur, "\gamma"] & & R \arrow[uur, "\rho"'] &
\end{tikzcd} \tag{5}$$

1.6 Peiffer subgroup of a precrossed module

We shall construct from any precrossed module $\mathcal{Q} = (\delta : Q \to R)$ a crossed module $\mathcal{X} = (\partial : Q/P \to R)$ whose source is a suitable quotient of Q.

Given $\mathcal{Q} = (\delta : Q \to R)$ the *Peiffer commutators* of $\mathcal{Q}$ are elements of the form

$$\langle q_1, q_2 \rangle \quad = \quad (q_2^{-1})^{q_1}\, {q_2}^{\delta q_1}, \qquad \text{where} \quad q_1, q_2 \in Q,$$

so that

$$\langle q_1, q_2 \rangle = 1 \quad \Leftrightarrow \quad {q_2}^{\delta q_1} \; = \; q_1^{-1}\, q_2\, q_1$$

The subgroup P of Q generated by the Peiffer commutators is known as the *Peiffer subgroup* of $\mathcal{Q}$.

Lemma 1.10

(a) $\langle q_1, 1 \rangle \; = \; 1 \; = \; \langle 1, q_2 \rangle$;

(b) $\delta\langle q_1, q_2 \rangle \; = \; 1_R$;

(c) $\langle q_1 q_3, q_2 \rangle \; = \; \langle q_1, q_2 \rangle^{q_3}\, \langle q_3, {q_2}^{\delta q_1} \rangle$;

(d) $\langle q_1, q_2 q_3 \rangle \; = \; \langle q_1, q_3 \rangle\, \langle q_1, q_2 \rangle^{{q_3}^{\delta q_1}}$;

(e) $\langle q_1, q_2\rangle^r \;=\; \langle {q_1}^r, {q_2}^r\rangle$;

(f) $\langle q_1, q_2\rangle^{-1} \;=\; \langle q_1^{-1}, q_2^{\delta q_1}\rangle^{q_1}$.

Proof: Part (a) is immediate, and part (b) follows from $\delta({q_2}^{\delta q_1}) \;=\; (\delta q_1)^{-1}(\delta q_2)(\delta q_1)$. The remaining parts follow on expanding the Peiffer elements, and are closely related to identities for commutators and crossed pairings (see Subsection 7.2):

$$\begin{array}{rcl}
\langle q_1 q_3, q_2\rangle & = & \left[{q_3}^{-1}{q_1}^{-1}{q_2}^{-1}q_1{q_2}^{\delta q_1}q_3\right]\left[{q_3}^{-1}({q_2}^{\delta q_1})^{-1}q_3{q_2}^{\delta(q_1q_3)}\right] \;=\; \langle q_1, q_2\rangle^{q_3}\,\langle q_3, {q_2}^{\delta q_1}\rangle\ , \\
\langle q_1, q_2q_3\rangle & = & \left[{q_1}^{-1}{q_3}^{-1}q_1{q_3}^{\delta q_1}\right]\left[({q_3}^{\delta q_1})^{-1}{q_1}^{-1}{q_2}^{-1}q_1(q_2q_3)^{\delta q_1}\right] \;=\; \langle q_1, q_3\rangle\,\langle q_1, q_2\rangle^{{q_3}^{\delta q_1}}\ , \\
\langle {q_1}^r, {q_2}^r\rangle & = & ({q_1}^{-1})^r({q_2}^{-1})^r{q_1}^r({q_2}^r)^{(r^{-1}(\delta q_1)r)} \;=\; ({q_1}^{-1}{q_2}^{-1}q_1{q_2}^{\delta q_1})^r\ , \\
\langle q_1, q_2\rangle^{-1} & = & \left(q_2^{\delta q_1}\right)^{-1}q_1^{-1}q_2q_1 \;=\; q_1^{-1}\left[q_1\left(q_2^{\delta q_1}\right)^{-1}q_1^{-1}\left(q_2^{\delta q_1}\right)^{\delta q_1^{-1}}\right]q_1 \;=\; \langle q_1^{-1}, q_2^{\delta q_1}\rangle^{q_1}\ .
\end{array}$$

□

The following result follows immediately from these identities.

Corollary 1.11 *Let $\mathcal{Q} = (\delta : Q \to R)$ be a precrossed module. Then the set P of Peiffer commutators in $\mathcal{Q}$ is a subgroup of* $\ker\delta$*; is normal in Q; and is R-invariant.*

Proposition 1.12 *If $(\ddot{\alpha}, \dot{\alpha}) : \mathcal{Q}_1 \to \mathcal{Q}_2$ is a morphism of precrossed modules, then Peiffer commutators in Q_1 are mapped to Peiffer commutators in Q_2.*

Proof: The source map $\ddot{\alpha}$ is compatible with the Peiffer pairing:

$$\begin{array}{rcl}
\ddot{\alpha}\langle q_1, q_2\rangle & = & (\ddot{\alpha}q_1)^{-1}\,(\ddot{\alpha}q_2)^{-1}\,(\ddot{\alpha}q_1)\,(\ddot{\alpha}({q_2}^{\delta_1 q_1})) \\
& = & (\ddot{\alpha}q_1)^{-1}\,(\ddot{\alpha}q_2)^{-1}\,(\ddot{\alpha}q_1)\,(\ddot{\alpha}q_2)^{\dot{\alpha}\delta_1 q_1} \\
& = & (\ddot{\alpha}q_1)^{-1}\,(\ddot{\alpha}q_2)^{-1}\,(\ddot{\alpha}q_1)\,(\ddot{\alpha}q_2)^{\delta_2(\ddot{\alpha}q_1)} \\
& = & \langle\, \ddot{\alpha}q_1,\, \ddot{\alpha}q_2\,\rangle\ .
\end{array}$$

□

Proposition 1.13 *Let $\mathcal{Q} = (\delta : Q \to R)$ be a precrossed module. Then there is a crossed module $\mathcal{X} = (\partial : S \to R)$ and a morphism of precrossed modules* $(\mathrm{nat}, 1) : \mathcal{Q} \to \mathcal{X}$ *such that* $(\mathrm{nat}, 1)$ *is universal for morphisms from $\mathcal{Q}$ to crossed modules over R.*

Proof: Let P be the Peiffer group of $\mathcal{Q}$. Then the quotient group $S = Q/P$ is well-defined, $\mathrm{nat} : Q \to S$ is the natural quotient map, and S inherits an R-action and an R-morphism:

$$(Pq)^r = P(q^r)\ , \qquad\qquad \partial : S \to R,\ (Pq) \mapsto \delta q\ .$$

So $\mathcal{X} = (\partial : S \to R)$ is a precrossed module. By definition of P, we have $s_1^{-1}s_2s_1 = {s_2}^{\partial s_1}$ for all $s_1, s_2 \in S$, and so $\mathcal{X}$ is a crossed module. The quotient morphism $(\mathrm{nat}, 1)$ is clearly a morphism of precrossed modules.

$$\begin{tikzcd}
S' \arrow[d, "\partial'"'] & Q \arrow[l, "n'"'] \arrow[r, "\mathrm{nat}"] \arrow[d, "\delta"] & S = Q/P \arrow[d, "\partial"] \arrow[ll, bend right, "\theta"'] \\
R & R \arrow[l, "1"] \arrow[r, "1"'] & R
\end{tikzcd}$$

If $\mathcal{X}' = (\partial' : S' \to R)$ is a crossed module and if $(n', 1) : \mathcal{Q} \to \mathcal{X}'$ is a precrossed module morphism, then there is a unique crossed module morphism $(\theta, 1) : \mathcal{X} \to \mathcal{X}'$ such that $(\theta, 1) \circ (\mathrm{nat}, 1) = (n', 1)$ where $\theta(Pq) = n'q$. □

Example 1.14 In the following GAP run the Peiffer subgroup is cyclic of order 4.

```
gap> b1 := (11,12,13,14,15,16,17,18);;  b2 := (12,18)(13,17)(14,16);;
gap> d16 := Group( b1, b2 );;
gap> sk4 := Subgroup( d16, [ b1^4, b2 ] );;
gap> SetName( d16, "d16" );  SetName( sk4, "sk4" );
gap> bdy16 := GroupHomomorphismByImages( d16, sk4, [b1,b2], [b1^4*b2,b2] );;
gap> aut1 := GroupHomomorphismByImages( d16, d16, [b1,b2], [b1^5,b2] );;
gap> aut2 := GroupHomomorphismByImages( d16, d16, [b1,b2], [b1,b1^4*b2] );;
gap> aut16 := Group( [ aut1, aut2 ] );;
gap> act16 := GroupHomomorphismByImages( sk4, aut16, [b1^4,b2], [aut1,aut2] );;
gap> P16 := PreXModByBoundaryAndAction( bdy16, act16 );
gap> P := PeifferSubgroup( P16 );
Group([ (11,15)(12,16)(13,17)(14,18), (11,13,15,17)(12,14,16,18) ])
gap> X16 := XModByPeifferQuotient( P16 );;
Peiffer([d16->sk4])
gap> Display( X16 );
Crossed module Peiffer([d16->sk4]) :-
: Source group has generators:
  [ f1, f2 ]
: Range group has generators:
  [ (11,15)(12,16)(13,17)(14,18), (12,18)(13,17)(14,16) ]
: Boundary homomorphism maps source generators to:
  [ (12,18)(13,17)(14,16), (11,15)(12,14)(16,18) ]
  The automorphism group is trivial
```

The following result, which gives a normal generating set for the Peiffer group, is Proposition 3 of [16]. The method of proof was suggested by Philip Higgins.

Proposition 1.15 *Given the following ingredients:*

- $\mathcal{Q} = (\delta : Q \to R)$, *a precrossed module;*
- Γ, *a generating set for* Q, *closed under the action of* R*;*
- P, *the Peiffer group of* $\mathcal{Q}$*;*

- *E, the set of Peiffer elements $\{\langle a,b\rangle \mid a,b \in \Gamma\}$;*

then P is the normal closure of E in Γ.

Proof: Let P' be the normal closure of E in Q, so that

$$P' \trianglelefteq P \trianglelefteq \ker\delta \trianglelefteq Q \ .$$

If $P \neq P'$ then there is some $z = \langle x,y\rangle \in P \setminus P'$ such that $P'z \neq P'$ in Q/P' . Since

$$P'z \neq P' \quad\Rightarrow\quad zP' \neq P' \quad\Rightarrow\quad y^{\delta x}\,P' \neq (x^{-1}yx)P' \quad\Rightarrow\quad P'y^{\delta x} \neq P'(x^{-1}yx) \ ,$$

we need to show that $P'\,y^{\delta x} \ = \ P'(x^{-1}yx)$ for all $\langle x,y\rangle \in P'$.

Since Γ is R-invariant, part (d) of Lemma 1.10 shows that E is R-invariant, and hence P' is R-invariant. Since $P' \trianglelefteq Q$, form $S' = Q/P'$ with R-action $(P'q)^r = P'(q^r)$. The homomorphism δ induces $\partial' : S' \to P'$, $P'q \mapsto \delta q$, and $\mathcal{X}' = (\partial' : S' \to R)$ is a precrossed module since

$$\partial'((P'q)^r) \ = \ \partial'(P'(q^r)) \ = \ \delta q^r \ = \ r^{-1}(\delta q)r \ = \ r^{-1}(\partial'(P'q))r \ .$$

Since Γ generates Q as a group, the set of cosets $C' = \{P'a \mid a \in \Gamma\}$ generates S' and, for all $P'a, P'b \in C'$,

$$(P'a)^{\partial'(P'b)} \ = \ (P'a)^{\delta b} \ = \ P'(a^{\delta b}) \ = \ P'(b^{-1}ab) \ = \ (P'b)^{-1}(P'a)(P'b) \ . \tag{6}$$

For fixed b, the set of $P'a$ satisfying (6) is a subgroup of S' :

$$((P'a)(P'c))^{\partial'(P'b)} \ = \ (P'a)^{\partial'(P'b)}(P'c)^{\partial'(P'b)} \ = \ (P'b)^{-1}(P'a)(P'c)(P'b) \ .$$

So (6) is true for all $(P'a) \in S'$ and all $(P'b) \in C'$.

Also, the set of $(P'b) \in C'$ satisfying (6) is closed under multiplication and inversion since

- $(P'a)^{\partial'(P'(bc))} \ = \ ((P'a)^{\delta(b)})^{\delta c} \ = \ (P'(bc))^{-1}\,(P'a)\,(P'(bc))$;
- $(P'a)^{\partial'(P'b)^{-1}} = P'c \quad\Rightarrow\quad P'a = (P'b)^{-1}(P'c)(P'b) \quad\Rightarrow\quad (P'b))(P'a)(P'b)^{-1} = P'c$.

Thus (6) holds for *all* $P'a, P'b \in C'$. □

1.7 Free Crossed Modules

We first recall a property of free groups which we wish to generalise. Let Ω be a set. The *free group on* Ω is a group F and a function $\nu : \Omega \to F$ such that if G is a group and $\nu' : \Omega \to G$ a function, then there exists a unique group homomorphism $\theta : F \to G$ such that $\theta \circ \nu = \nu'$.

$$\begin{tikzcd} G & \Omega \arrow[l, "\nu'"'] \arrow[r, "\nu"'] & F \arrow[ll, bend right, "\theta"'] \end{tikzcd}$$

To construct a particular model for $F = F(\Omega)$ we take an alphabet consisting of all the elements of Ω together with their formal inverses, and take for F the set of all reduced words in this alphabet with concatenation as the group product and the empty word as the identity. The details of this construction should be familiar to the reader.

We now define a *free precrossed module* in an analogous manner. The ingredients for the construction are

- a set Ω,
- a group R,
- a function $\omega : \Omega \to R$.

The resulting construction consists of

- a precrossed module $\mathcal{Q} = (\delta : Q \to R)$,
- a function $\nu : \Omega \to Q$ such that $\delta \circ \nu = \omega$.

The universal property required of this construction is that if $\mathcal{Q}' = (\delta' : Q' \to R)$ is another precrossed module, and if $\nu' : \Omega \to Q'$ satisfies $\delta' \circ \nu' = \omega$, then there exists a unique morphism of precrossed modules $(\theta, 1) : \mathcal{Q} \to \mathcal{Q}'$ such that $(\theta, 1) \circ (\nu, 1) = (\nu', 1)$.

$$\begin{tikzcd}
Q' \arrow[dr, "\delta'"'] & \Omega \arrow[l, "\nu'"'] \arrow[r, "\nu"] \arrow[d, "\omega"] & Q \arrow[ll, bend right, "\theta"'] \arrow[dl, "\delta"] \\
& R &
\end{tikzcd}$$

A particular model is obtained as follows:

- the source group Q is the free group $F(\Omega \times R)$,
- the boundary map is defined on generators by $\delta(\rho, r) = (\omega\rho)^r = r^{-1}(\omega\rho)r$,
- the action is given by $(\rho, r)^{r'} = (\rho, rr')$,
- the function is given by $\nu(\rho) = (\rho, 1)$.

We observe that

$$\delta \circ \nu(\rho) \;=\; \delta(\rho, 1) \;=\; \omega\rho \quad \text{for all} \quad \rho \in \Omega,$$

and verify **X1:** as follows:

$$\delta((\rho, r)^{r'}) \;=\; \delta(\rho, rr') \;=\; r'^{-1}(r^{-1}(\omega\rho)r)r' \;=\; r'^{-1}\,\delta(\rho, r)\,r' \;.$$

To check the universal property we need to define $\theta : Q \to Q'$ in such a way that $\theta(\rho, 1) = \nu'\rho$. Since θ is to preserve the R-action, we are forced to define

$$\theta(\rho, r) \;=\; \theta((\rho, 1)^r) \;=\; (\nu'\rho)^r \;.$$

This defines θ on the whole of Q, and we verify that $(\theta, 1)$ is a morphism of precrossed modules:

- $\delta'\theta(\rho, r) \;=\; \delta'((\nu'\rho)^r) \;=\; r^{-1}(\delta'\nu'\rho)r \;=\; r^{-1}(\omega\rho)r \;=\; \delta(\rho, r)\;,$
- $\theta((\rho, r)^{r'}) \;=\; \theta(\rho, rr') \;=\; (\nu'\rho)^{rr'} \;=\; (\theta(\rho, r))^{r'}\;.$

We are now in a position to construct the free crossed module associated to a free precrossed module, simply by factoring out the Peiffer commutators in $Q = F(\Omega \times R)$:

$$\begin{aligned}\langle(\rho_1, r_1), (\rho_2, r_2)\rangle &= (\rho_1, r_1)^{-1}\,(\rho_2, r_2)^{-1}\,(\rho_1, r_1)\,(\rho_2, r_2)^{\delta(\rho_1, r_1)} \\ &= (\rho_1, r_1)^{-1}\,(\rho_2, r_2)^{-1}\,(\rho_1, r_1)\,(\rho_2, r_2)^{r_1^{-1}(\omega\rho_1)r_1} \\ &= (\rho_1, r_1)^{-1}\,(\rho_2, r_2)^{-1}\,(\rho_1, r_1)\,(\rho_2, r_2 r_1^{-1}(\omega\rho_1)r_1)\ .\end{aligned}$$

We thus obtain a crossed module with source $C(\omega) = Q/P$:

$$\begin{aligned}\mathcal{F}_\rho &= (\partial : C(\omega) \to R)\,, \\ \partial(P(\rho, r)) &= \delta(\rho, r) \ = \ r^{-1}(\omega\rho)r\,, \\ (P(\rho, r))^{r'} &= P(\rho, rr')\,.\end{aligned}$$

It is often convenient to use an alternative notation for elements in Q. Since $(\rho, r) = (\rho, 1)^r$ we may drop the "$, 1$" and write $(\rho)^r$ for (ρ, r), and the inverse element by $(\rho^{-1})^r$:

$$\{(\rho)^r\}^{-1} \ = \ (\rho, r)^{-1} \ = \ \{(\rho, 1)^r\}^{-1} \ = \ \{(\rho, 1)^{-1}\}^r \ = \ (\rho^{-1})^r\ .$$

The action is then given by

$$((\rho^\epsilon)^r)^{r'} \ = \ (\rho^\epsilon)^{(rr')} \quad \text{where} \quad \epsilon = \pm 1\ .$$

The Peiffer commutators on generators in this notation are

$$\langle\,({\rho_1}^{\epsilon_1})^{r_1},\, ({\rho_2}^{\epsilon_2})^{r_2}\,\rangle \ = \ ({\rho_1}^{-\epsilon_1})^{r_1}\,({\rho_2}^{-\epsilon_2})^{r_2}\,({\rho_1}^{\epsilon_1})^{r_1}\,({\rho_2}^{\epsilon_2})^{r_2 r_1^{-1}(\omega\rho_1)r_1}\ .$$

1.8 The monoid version of free crossed modules

Given $Y = \Omega \times R$, define

$$\bar{Y} \ = \ \{y^+ : y \in Y\} \sqcup \{y^- : y \in Y\}\,.$$

It is convenient to use an alternative notation, as above:

$$(\rho^+)^r \ := \ (\rho, r)^+\,, \qquad (\rho^-)^r \ := \ (\rho, r)^-\,.$$

Then $H = \bar{Y}^*$ is the free monoid on $\bar{Y}$ with empty word λ and elements

$$(\rho_1^{\epsilon_1})^{r_1}\,(\rho_2^{\epsilon_2})^{r_2}\cdots(\rho_n^{\epsilon_n})^{r_n}\,, \quad \rho_i \in \Omega,\ \epsilon_i \in \{+, -\},\ r_i \in R\,.$$

The boundary map is the monoid morphism

$$\bar{\delta}\ :\ \bar{Y}^* \to R, \quad (\rho^+)^r \mapsto r^{-1}(\omega\rho)r, \quad (\rho^-)^r \mapsto r^{-1}(\omega\rho)^{-1}r.$$

Then $F(\omega)$ is the quotient monoid $\bar{Y}^*/\equiv$ where $\equiv$ is the congruence generated by

- inverse pairs $(y^\epsilon y^{-\epsilon}, \lambda)$,
- Peiffer pairs $(y^{-\epsilon}z^\eta y^\epsilon, (z^\eta)^{\bar{\delta}y^\epsilon})$, $\quad \epsilon, \eta \in \{+, -\}, \quad -(-) = +, \quad$ etc.

In the special case of a group presentation

$$\mathcal{P} \;=\; \mathrm{grp}(X, \omega : \Omega \to F(X)),$$

$\omega\rho$ is a relator and so a word in $F(X)$. Then $Y = \Omega \times F(X)$ and $H = \bar{Y}^*$ has elements of the form

$$(\rho_1^{\epsilon_1})^{u_1}\,(\rho_2^{\epsilon_2})^{u_2}\,\cdots(\rho_n^{\epsilon_n})^{u_n}\,,$$

and $\ker\partial = \Pi_2(\mathcal{P})$ is the $\mathbb{Z}G$-module of identities among the relators of $\mathcal{P}$.

Example 1.16 *Consider the following presentation for the quaternion group of size* 8.

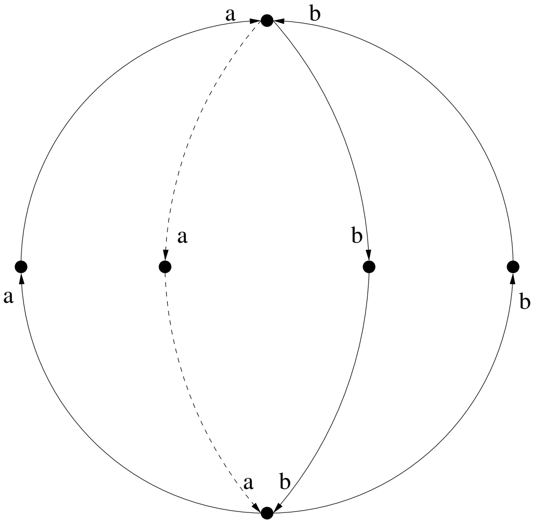
$$X = \{a,b\}, \quad \Omega = \{\rho_1,\rho_2,\rho_3,\rho_4\}, \quad \omega : \rho_1 \mapsto a^4,\ \rho_2 \mapsto b^4,\ \rho_3 \mapsto abab^{-1},\ \rho_4 \mapsto a^2b^2.$$

The identity

$$\iota \;=\; (\rho_4^-)\,(\rho_1^+)^{a^2}\,(\rho_4^-)^{a^2}\,(\rho_2^+)$$

maps by ∂ to

$$(b^{-2}a^{-2}).a^{-2}(a^4)a^2.a^{-2}(b^{-2}a^{-2})a^2.(b^4) \;=\; \lambda\,.$$



In the van Kampen diagram, relators ρ_1, ρ_2 and ρ_4 (twice) tile a sphere as shown above. Tracing out $\partial\iota$ we walk around the boundaries of these four tiles, in the order back-right; back-left; front-left; front-right; in such a way that every edge cancels out with its inverse.

1.9 A Geometric Example of a Crossed Module

The major geometric example of a crossed module can be expressed in two ways. Let (X, A, a) be a based pair of spaces, with $a \in A \subseteq X$. Let $I = [0,1]$ be the unit interval, I^2 the unit square with boundary $\dot{I}^2$, and let $J^1 = (\{0,1\} \times I) \cup (I \times \{1\}) \subset \dot{I}^2$ be three quarters of the boundary. The *second relative homotopy group* $\pi_2(X, A, a)$ consists of homotopy classes rel J^1 of continuous maps

$$\alpha \;:\; (I^2, \dot{I}^2, J^1) \;\to\; (X, A, a)$$

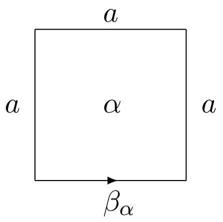


Figure 1: An element $\alpha \in \pi_2(X, A, a)$

Each such α is a map from I^2 to the space X mapping the left, top, and right sides of the square to the point a and the bottom side to a loop β_α at a. We may represent such a map by the diagram in Figure 1.

Recall that the fundamental group $\pi_1(A, a)$ consists of maps $\gamma : I \to A$, $\gamma(0) = \gamma(1) = a$. With such a γ it is easy to construct maps $I^2 \to A$ which map two sides of the square to a and two sides to γ. Five of these are shown in Figure 2.

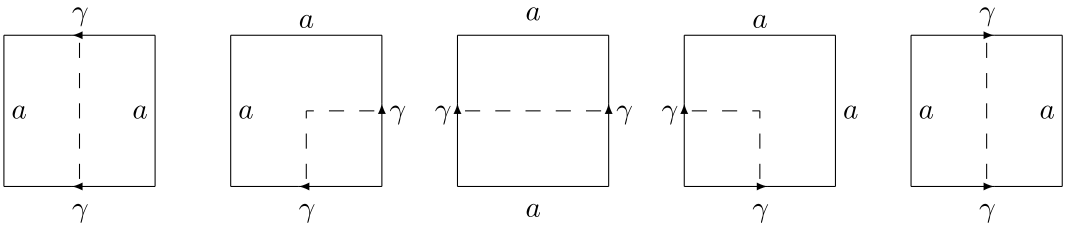


Figure 2: Five maps $I^2 \to A$ derived from $\gamma \in \pi_1(A, a)$

Whitehead showed in [55] that there is a crossed module $\Pi_2(X, A, a)$ with boundary map

$$\partial \ : \ \pi_2(X, A, a) \to \pi_1(A, a), \quad \alpha \mapsto \beta_\alpha = \alpha(I \times \{0\}) \ .$$

The image of $\alpha \in \pi_2(X, A, a)$ under the action of $\gamma \in \pi_1(A, a)$ is illustrated in Figure 3, surounding α with the five maps in Figure 2. Note that the boundary loop is the conjugate $\gamma^{-1}\beta_\alpha\gamma$.

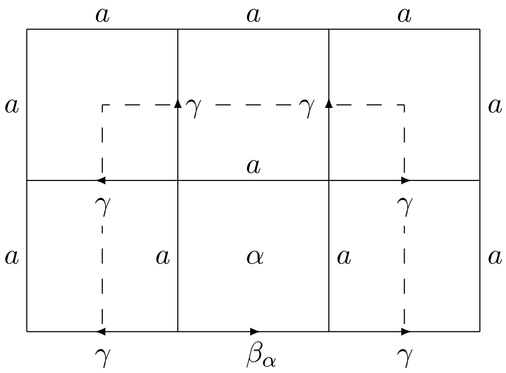
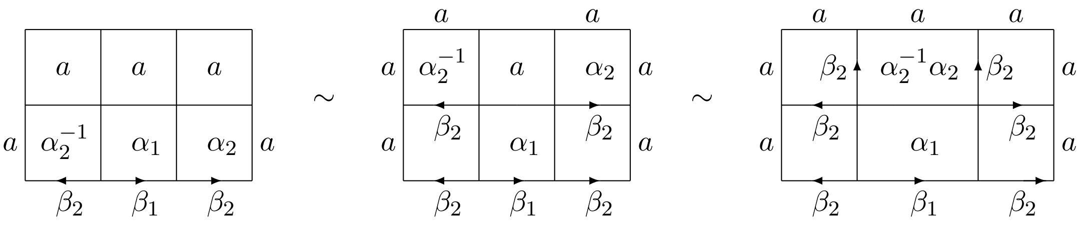


Figure 3: Action of γ on α

The meaning of this composite square is as follows. Squares may be joined along an edge when the values agree on that edge. If a composite is then p units across by q units high, scaling factors $1/p$ horizontally and $1/q$ vertically are used to obtain a new map from I^2 to X.

Figure 4 gives an outline verification of the second crossed module axiom for $\Pi_2(X, A, a)$, where a square marked a represents the constant map $I^2 \to \{a\}$.



$$\alpha_1^{\alpha_2} = \alpha_2^{-1}\alpha_1\alpha_2 \qquad\qquad {\alpha_1}^{\beta_2} = {\alpha_1}^{\partial\alpha_2}$$

Figure 4: Verification of **X2:** for $\Pi_2(X, A, a)$.

Whitehead's main result in [54, 55, 57] was the following.

Theorem 1.17 (Whitehead) *If X is obtained from A by attaching 2-cells, then $\pi_2(X, A, a)$ is isomorphic to the free crossed $\pi_1(A, a)$-module on the attaching maps of the 2-cells.*

[More here?]

1.10 Semidirect Products

We include here some basic results on semidirect products which will be needed in later sections.

Proposition 1.18

(a) *If a set X has a right G-action $x \mapsto x^g$ then X has an associated left G-action:*

$$ {}^g x \quad := \quad x^{g^{-1}} \; .$$

(b) *The semidirect products $R \ltimes S$ and $S \rtimes R$ have multiplication rules*

$$\begin{aligned} (r,s)(q,t) &= (rq,\, s^q\, t) \quad \text{in} \quad R \ltimes S \; , \\ (s,r)(t,q) &= (s\,{}^r t,\, rq) \quad \text{in} \quad S \rtimes R \; . \end{aligned}$$

(c) *There is an isomorphism between these two groups:*

$$\psi \;:\; R \ltimes S \to S \rtimes R\,, \quad (r,s) \mapsto ({}^r s,\, r) \; ,$$

with inverse

$$\psi^{-1} \;:\; S \rtimes R \to R \ltimes S\,, \quad (s,r) \mapsto (r,\, s^r) \; .$$

1.11 Cat^1-groups and their morphisms

In [43] Loday reformulated the notion of a crossed module as a cat^1-group $(G; t, h)$, namely a group G with a pair of endomorphisms $t, h : G \to G$ having a common image R and satisfying certain axioms. We call these *traditional* cat^1-groups, to distinguish them from the more general description which follows.

Definition 1.19 *A* cat^1-group $\mathcal{C}$ *comprises a group* G *and two endomorphisms* $\tau, \theta : G \to G$ *as shown in the following diagram:*

$$\begin{tikzcd} G \arrow[r, shift left, "{\tau,\ \theta}"] \arrow[r, shift right] & G \end{tikzcd}$$

These homomorphisms are required to satisfy the following axioms:

$$\begin{array}{ll} \textbf{\textit{C1:}} & \tau \circ \theta = \theta \quad \textit{and} \quad \theta \circ \tau = \tau, \\ \textbf{\textit{C2:}} & [\ker \tau, \ker \theta] = \{1_G\}. \end{array}$$

The maps τ, θ are usually referred to as the *source* and *target* maps, but we choose to call them the *tail* and *head* of $\mathcal{C}$, because *source* is the GAP term for the domain of a function. It follows immediately from axiom **C1:** that:

$$\tau^2 = \tau, \quad \theta^2 = \theta, \quad \operatorname{im} \tau = \operatorname{im} \theta = Q \text{ (say)},$$

and that both τ and θ are the identity when restricted to Q.

Alternatively (and this is convenient for the implementation in XMod) we may define a cat^1-group as follows.

Definition 1.20 *A* cat^1-group $\mathcal{C} = (e; t, h : G \to R)$ *has source group* G*, range group* R*, and three homomorphisms: two surjections* $t, h : G \to R$ *and an embedding* $e : R \to G$ *as shown in the following diagram:*

$$\begin{tikzcd} G \arrow[r, shift left=2, "{t,h}"] \arrow[r, shift left] & R \arrow[l, shift left, "e"] \end{tikzcd}$$

These homomorphisms are required to satisfy the following axioms:

$$\begin{array}{ll} \textbf{\textit{C1:}} & (t \circ e) \textit{ and } (h \circ e) \textit{ are the identity mapping on } R, \\ \textbf{\textit{C2:}} & [\ker t, \ker h] = \{1_G\}. \end{array}$$

It follows immediately from axiom **C1:** that:

$$t \circ e \circ t = t, \quad h \circ e \circ h = h, \quad t \circ e \circ h = h, \quad h \circ e \circ t = t. \tag{7}$$

Since e is an embedding, R acts on G by conjugation: $g^r = (er)^{-1} g(er)$.

Given a cat^1-group according to the second definition, we may convert it to a cat^1-group according to the first by setting

$$\tau = e \circ t, \quad \theta = e \circ h, \quad Q = eR,$$

so that $\ker \tau = \ker t$, $\ker \theta = \ker h$ and $\tau \circ \theta = e \circ t \circ e \circ h = e \circ h = \theta$, etc.

A cat^1-group is *symmetric* if the tail and head maps are equal. By (7) a sufficient condition for this is that $t \circ e \circ h \ = \ h \circ e \circ t$.

Example 1.21 In this example the group G is isomorphic to $C_2 \ltimes (C_3 \times C_3)$.

```
gap> g18gens := [ (1,2,3), (4,5,6), (2,3)(5,6) ];;
gap> s3agens := [ (7,8,9), (8,9) ];;
gap> g18 := Group( g18gens );;  SetName( g18, "g18" );
gap> s3a := Group( s3agens );;  SetName( s3a, "s3a" );
gap> t1 := GroupHomomorphismByImages(g18,s3a,g18gens,[(7,8,9),(),(8,9)]);;
gap> h1 := GroupHomomorphismByImages(g18,s3a,g18gens,[(7,8,9),(7,8,9),(8,9)]);;
gap> e1 := GroupHomomorphismByImages(s3a,g18,s3agens,[(1,2,3),(2,3)(5,6)]);;
gap> C18 := PreCat1GroupByTailHeadEmbedding( t1, h1, e1 );;
gap> Display{ C18 );
Cat1-group [g18=>s3a] :-
: Source group g18 has generators:
  [ (1,2,3), (4,5,6), (2,3)(5,6) ]
: Range group s3a has generators:
  [ (7,8,9), (8,9) ]
: tail homomorphism maps source generators to:
  [ (7,8,9), (), (8,9) ]
: head homomorphism maps source generators to:
  [ (7,8,9), (7,8,9), (8,9) ]
: range embedding maps range generators to:
  [ (1,2,3), (2,3)(5,6) ]
: kernel has generators:
  [ (4,5,6) ]
: boundary homomorphism maps generators of kernel to:
  [ (7,8,9) ]
: kernel embedding maps generators of kernel to:
  [ (4,5,6) ]
```

A morphism $\mathcal{C}_1 \to \mathcal{C}_2$ of cat^1-groups is a pair (γ, ρ) where $\gamma : G_1 \to G_2$ and $\rho : R_1 \to R_2$ are homomorphisms satisfying

$$t_2 \circ \gamma = \rho \circ t_1, \quad h_2 \circ \gamma = \rho \circ h_1, \quad e_2 \circ \rho = \gamma \circ e_1. \tag{8}$$

The situation is displayed in the following diagram.

$$\begin{tikzcd}
G_1 \arrow[r, "\gamma"] \arrow[d, shift left, "{t_1,h_1}"] \arrow[d, shift left=3] & G_2 \arrow[d, shift left, "{t_2,h_2}"] \arrow[d, shift left=3] \\
R_1 \arrow[u, shift left, "e_1"] \arrow[r, "\rho"'] & R_2 \arrow[u, shift left, "e_2"]
\end{tikzcd}$$

The morphism (id, e) is the isomorphism from a cat^1-group $(e; t, h : G \to R)$ to the traditional cat^1-group with endomorphisms $(\mathrm{inc}_Q; e \circ t, e \circ h : G \to Q)$ with $Q = eR$, as described above.

Verification of the properties in the following lemma is routine.

Lemma 1.22 *The mapping* $u : G \to \ker t,\ g \mapsto (etg^{-1})g$ *has the following properties.*

(i) $u^2 = u$,

(ii) $tug = 1_R, \quad hug = (tg^{-1})(hg), \quad uer = 1_G$,

(iii) $u(g_1g_2) = (ug_2)(ug_1)^{g_2}$,

(iv) $(ug)^{-1} = g^{-1}(etg) = (u(g^{-1}))^{g}$.

Proposition 1.23 *An arbitrary cat^1-group $\mathcal{C} = (e;t,h : G \to R)$ is isomorphic to the cat^1-group $\mathcal{C}' = (e';t',h' : R \ltimes S \to R)$, the* semidirect form *of $\mathcal{C}$, where $S = \ker t$; the homomorphisms t', h', e' are defined by*

$$t'(r,s) = r, \quad h'(r,s) = r(hs), \quad e'r = (r,1); \tag{9}$$

and the action of R on S is given by $(r_0,s_0)^r \;=\; (r^{-1}r_0r, s_0^r)$.

Proof: The isomorphism from G to $R \ltimes S$, and its inverse, are given by

$$\begin{array}{rcll} \phi & : & G \to R \ltimes S, \quad g \mapsto (tg, ug), & \quad \text{where} \;\; ug = (etg^{-1})g \in \ker t, \\ \phi^{-1} & : & R \ltimes S \to G, \quad (r,s) \mapsto (er)s. & \end{array} \tag{10}$$

This ϕ is a homomorphism since

$$\begin{aligned} (\phi g_1)(\phi g_2) &= (tg_1, ug_1)(tg_2, ug_2) \;=\; \big((tg_1)(tg_2), (ug_1)^{tg_2}(ug_2)\big) \\ &= \big(t(g_1g_2), (etg_2)^{-1}(ug_1)(etg_2)(ug_2)\big) \\ &= \big(t(g_1g_2), (etg_2)^{-1}g_2g_2^{-1}(ug_1)(etg_2)(etg_2)^{-1}g_2\big) \\ &= \big(t(g_1g_2), (ug_2)(ug_1)^{g_2}\big) \;=\; \phi(g_1g_2). \end{aligned}$$

The specified ϕ^{-1} *is* the inverse of ϕ since

$$\begin{aligned} \phi^{-1}(tg, (etg^{-1})g) &= (etg)(etg^{-1}g \;=\; g, \\ \phi((er)s) &= \big((ter)(ts), (ets^{-1})(eter^{-1})(er)s\big) \;=\; (r,s). \end{aligned}$$

The required isomorphism is then $(\phi, \mathrm{id}) : \mathcal{C} \to \mathcal{C}'$. Axioms (8) are easily verified:

$$\begin{aligned} t'(tg, ug) &= tg, \\ h'(tg, ug) &= (tg)(hetg^{-1})(hg) \;=\; hg, \\ \phi(er) &= (ter, (eter^{-1})(er)) \;=\; (r,1). \end{aligned}$$

□

The crossed module $\mathcal{X} = (\partial : S \to R)$ associated to $\mathcal{C}$ and $\mathcal{C}'$ has boundary $\partial = h|_S$ and action $s^r := s^{er}$. The cat^1-group $\mathcal{C} = \mathcal{C}'$ associated to $\mathcal{X} = (\partial : S \to R)$ has $G = R \ltimes S$, where the action is that in $\mathcal{X}$, and homomorphisms given by:

$$t(r,s) = r, \quad h(r,s) = r(\partial s), \quad er = (r,1). \tag{11}$$

We denote by ϵ the inclusion of S in G, so that $\partial = h\epsilon$.

Example 1.24 Here we convert `X4` and `C18`, constructed in Subsection 1.2 and Example 1.21.

```
gap> CX4 := Cat1GroupOfXMod( X4 );
[s4 |X a4=>Group( [ (), (), (1,2), (2,3), (3,4) ] )]
gap> TailMap( CX4 );  HeadMap( CX4 );
[ (5,6,7), (6,7,8), (1,2)(5,6), (2,3)(6,7), (3,4)(7,8) ] ->
[ (), (), (1,2), (2,3), (3,4) ]
[ (5,6,7), (6,7,8), (1,2)(5,6), (2,3)(6,7), (3,4)(7,8) ] ->
[ (1,2,3), (2,3,4), (1,2), (2,3), (3,4) ]
gap> XC18 := XModOfCat1Group( C18 );
[Group( [ (4,5,6) ] )->s3a]
gap> Boundary( XC18 );
[ (4,5,6) ] -> [ (7,8,9) ]
```

Given a morphism $(\sigma, \rho) : \mathcal{X}_1 \to \mathcal{X}_2$ of crossed modules, the associated morphism of cat1-groups is $(\gamma, \rho) : \mathcal{C}_1 \to \mathcal{C}_2$ where $\gamma(r_1, s_1) = (\rho r_1, \sigma s_1)$. Similarly, given a morphism $(\gamma, \rho) : \mathcal{C}_1 \to \mathcal{C}_2$ of cat^1-groups, the associated morphism of crossed modules is $(\sigma, \rho) : \mathcal{X}_1 \to \mathcal{X}_2$ where $\sigma s = \gamma(1, s)$.

George Janelidze has noted the following variant of the second cat^1-group axiom:

$$\mathbf{C2'}: \qquad [ug_1, ug_2] \;=\; 1_G \qquad \text{for all } \; g_1, g_2 \in G\;.$$

It follows that cat^1-groups form an equational variety.

Lemma 1.25 *If $\mathcal{C} = (e; t, h : G \to R)$ is a cat^1-group then there is a group homomorphism*

$$(t, h) \;:\; G \to R \times R, \;\; g \mapsto (tg, hg)\;.$$

Proposition 1.26 (Comment by Tim Porter during a seminar on 18/10/02.)
A congruence $\equiv$ *(in the sense of congruence on a monoid) on a group R gives rise to a cat^1-group.*

Proof: The set of equivalent pairs,

$$G \;=\; \{(r_1, r_2) \in R \times R \;\mid\; r_1 \equiv r_2\}$$

is a subgroup of $R \times R$. The required $\mathcal{C} = (e; t, h : G \to R)$ has homomorphisms given by:

$$t(r_1, r_2) = r_1, \qquad h(r_1, r_2) = r_2, \qquad e(r) = (r, r).$$

The associated crossed module has source

$$\ker t \;=\; \{(r, 1) \mid r \equiv 1\}$$

which shows that the elements equivalent to 1_R in the congruence form a normal subgroup. □

1.12 Pre-cat^1-groups and sub-cat^1-groups

When axioms **X2:** and **C2:** are *not* satisfied by $\mathcal{Q} = (\delta : Q \to R)$ and $\mathcal{B} = (e; t, h : R \ltimes Q \to R)$, the corresponding structures are known as *precrossed modules* and *pre-cat^1-groups*. In this case recall from Subsection 1.6 that the *Peiffer subgroup* P of Q is the subgroup of $\ker \delta$ generated by *Peiffer commutators*

$$\langle q_1, q_2 \rangle \;=\; q_1^{-1}\, q_2^{-1}\, q_1\, {q_2}^{\partial q_1}\;.$$

Then $\mathcal{P} = (0 : P \to \{1_R\})$ is a normal sub-precrossed module of $\mathcal{Q}$ and $\mathcal{X} = \mathcal{Q}/\mathcal{P} = (\partial : S = Q/P \to R)$ is a crossed module. The restriction of $\epsilon : Q \to R \ltimes Q$ to P is given by

$$\epsilon\langle q_1, q_2\rangle = [(\delta q_1^{-1}, q_1), (1_R, {q_2}^{\delta q_1})] \in [\ker h, \ker t].$$

The image ϵP is the Peiffer subgroup $[\ker h, \ker t]$ of $R \ltimes Q$ and, if ι is the inclusion $\{1_R\} \to R$, then $\mathcal{C}/(\epsilon, \iota)\mathcal{P} = (e; t, h : (R \ltimes Q)/\epsilon P \to R)$ is the cat^1-group corresponding to $\mathcal{X} = \mathcal{Q}/\mathcal{P}$.

We now include definitions of sub-cat^1-groups, normal sub-cat^1-groups, and the kernel of a cat^1-morphism.

A *sub-cat^1-group* of $\mathcal{C} = (e; t, h : G \to R)$ is a cat^1-group $\mathcal{C}' = (e'; t', h' : G' \to R')$ where G' is a subgroup of G; R' is a subgroup of R; and e', t', h' are the restrictions of e, t, h. Such a $\mathcal{C}'$ is a *normal sub-cat^1-group* of $\mathcal{C}$ when $G' \trianglelefteq G$ and $R' \trianglelefteq R$.

For the kernel of a cat^1-morphism $\mu : \mathcal{C} \to \mathcal{C}'$, consider the following diagram, where the left-hand morphism is a sub-cat^1-group inclusion and the leftmost tail, head and embedding maps are the restrictions of t, h, e.

$$\begin{tikzcd}
\ker\bar{\mu} \arrow[r] \arrow[d, shift left] \arrow[d, shift left=2] & G \arrow[r, "\bar{\mu}"] \arrow[d, shift left] \arrow[d, shift left=2, "t{,}h"] & G' \arrow[d, shift left] \arrow[d, shift left=2, "t'{,}h'"] \\
\ker\dot{\mu} \arrow[r] \arrow[u, shift left] & R \arrow[r, "\dot{\mu}"'] \arrow[u, shift left, "e"] & R' \arrow[u, shift left, "e'"]
\end{tikzcd}$$

The left-hand cat^1-group *is* the kernel of μ since

$$\begin{array}{llllllll}
g \in \ker\bar{\mu} & \Rightarrow & t'\bar{\mu}g = 1,\ h'\bar{\mu}g = 1 & \Rightarrow & \dot{\mu}tg = 1,\ \dot{\mu}hg = 1 & \Rightarrow & tg \in \ker\dot{\mu},\ hg \in \ker\dot{\mu}, \\
r \in \ker\dot{\mu} & \Rightarrow & e'\dot{\mu}r = 1 & \Rightarrow & \bar{\mu}er = 1 & \Rightarrow & er \in \ker\bar{\mu}.
\end{array}$$

1.13 Group Groupoids

Cat^1-groups may also be thought of as group-groupoids. A *group groupoid* is a set which has both a group structure and a groupoid structure (see subsection 4.1). From a categorical viewpoint, it is both a group object in the category of groupoids and a groupoid object in the category of groups. (For basic notions of groupoids see Section 4.)

The underlying groupoid $\mathcal{G}$ of a cat^1-group $\mathcal{C}$ has the group R as object set G_0 and the group G as the set of arrows G_1. The identity arrow at r is $1_r = er$. For each arrow g the tail (source) is tg and the head (target) is hg. Arrows g_1, g_2 are composable only when $hg_1 = tg_2 = r_2$ (say), in which case the composite arrow is

$$g_1 * g_2 \ = \ g_1({er_2}^{-1})g_2 \quad \text{where} \quad t(g_1 * g_2) = tg_1 = r_1, \quad h(g_1 * g_2) = hg_2 = r_3. \tag{12}$$

This composition is, of course, associative:

$$g_1 * g_2 * g_3 \ = \ g_1({er_2}^{-1})g_2({er_3}^{-1})g_3.$$

The groupoid inverse $\tilde{g}$ of g for this composition is given by

$$\tilde{g} \ = \ (ehg)g^{-1}(etg) \quad \text{with} \quad t\tilde{g} = hg,\ h\tilde{g} = tg,\ g * \tilde{g} = etg \ \text{ and } \ \tilde{g} * g = ehg\ .$$

This subset of $\mathcal{G}$ is illustrated in the following diagram.

$$\begin{tikzcd}
r_1 = tg_1 \arrow[loop left, "er_1"] \arrow[r, bend left, "g_1"] \arrow[rr, bend left=50, "g_1 * g_2"] & hg_1 = r_2 = tg_2 \arrow[loop above, "er_2"] \arrow[l, bend left, "\tilde{g}_1"] \arrow[r, bend left, "g_2"] & r_3 = hg_2 \arrow[loop right, "er_3"] \arrow[l, bend left, "\tilde{g}_2"] \arrow[ll, bend left=50, "\widetilde{g_1 * g_2}"]
\end{tikzcd}$$

The composites of one element with the groupoid inverse of another, when defined, are given by

$$\tilde{g}_1 * g_3 \ = \ (ehg_1)g_1^{-1}g_3 \qquad \text{and} \qquad g_4 * \tilde{g}_2 \ = \ g_4g_2^{-1}(etg_2). \tag{13}$$

The equivalent formulae for composition and inverse when $R \ltimes S$ replaces G are:

$$(r,s) * (r(\partial s), s') = (r, ss') \quad \text{and} \quad \widetilde{(r,s)} = (r(\partial s), s^{-1}) \ .$$

Since $g^{-1}(etg) \in \ker t$ and $(ehg)g^{-1} \in \ker h$, the map $g \mapsto \tilde{g}$ is an automorphism of $\mathcal{G}$ which restricts to the identity map on eR and provides a cat^1-isomorphism from $\mathcal{C}$ to the *reverse* cat^1-group $\tilde{\mathcal{C}} = (e; h, t : G \to R)$ of $\mathcal{C}$. The set of arrows *out* from 1_R is $\ker t$ while the set of arrows *in* to 1_R is $\ker h$, so $\ker \partial$ is the set of loops at 1_R. The set of objects in the component of $\mathcal{G}$ connected to 1_R is the image of ∂, so $\mathcal{G}$ is discrete when $\partial = 0$.

Alternatively, starting with a group groupoid $\mathcal{G} = (G, t, h)$, define

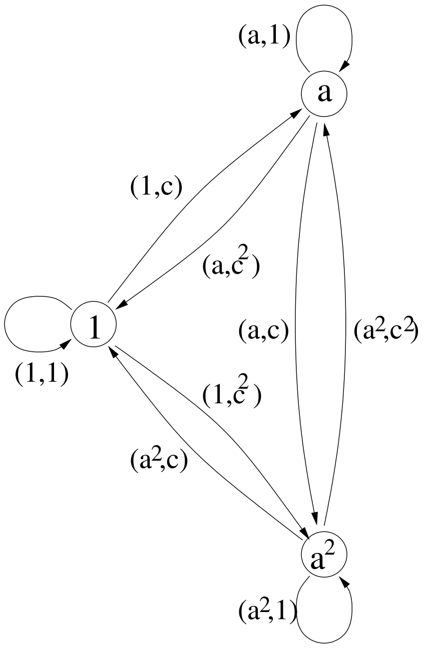
$$\begin{array}{rcl} R & = & \operatorname{im} t \; = \; \operatorname{im} h, \\ S & = & \{g \mid tg = 1\} \; = \; \ker t \; = \; \text{arrows out from } 1_R, \\ s^r & = & (er)^{-1} s(er), \quad \text{where } er \text{ is the identity loop at } r. \end{array}$$

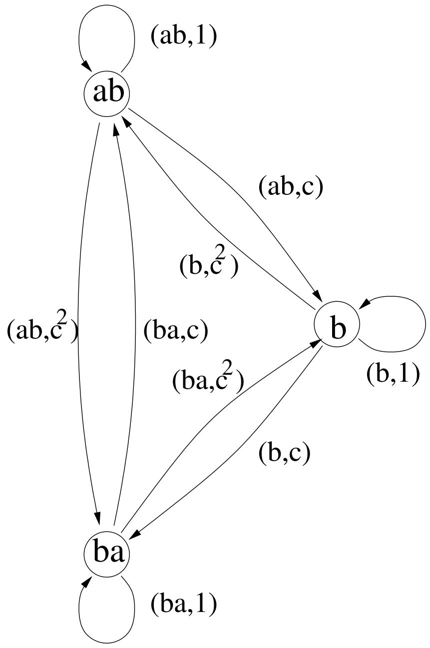
See Subsection 2.3 for the group-groupoid equivalent of derivations and sections.

Example 1.27 *The normal inclusion crossed module* $X_3 = (1 : C_3 \to S_3)$ *of the cyclic group* $C_3 = \langle c \mid c^3 \rangle$ *in the symmetric group* $S_3 = \langle a, b \mid a^3, b^2, (ab)^2 \rangle$, *with conjugation action* $c^a = c, c^b = c^2$, *has associated cat*^{1}*-group* $(e; t, h : S_3 \ltimes C_3 \to S_3)$. *The images of the tail and head functions are given in the following table:*

g	tg	hg	g	tg	hg
$(1,1)$	1	1	$(b,1)$	b	b
$(1,c)$	1	a	(b,c)	b	ba
$(1,c^2)$	1	a^2	(b,c^2)	b	ab
$(a,1)$	a	a	$(ab,1)$	ab	ab
(a,c)	a	a^2	(ab,c)	ab	b
(a,c^2)	a	1	(ab,c^2)	ab	ba
$(a^2,1)$	a^2	a^2	$(ba,1)$	ba	ba
(a^2,c)	a^2	1	(ba,c)	ba	ab
(a^2,c^2)	a^2	a	(ba,c^2)	ba	b

The corresponding group-groupoid has 6 *objects,* 18 *morphisms,* 2 *connected components, and may be pictured as:*





We may compare the group multiplication with the groupoid multiplication by calculating, for example,

$$\begin{aligned} (a,c)(a^2,c) &= (1,c^{a^2}c) \;=\; (1,c^2), \\ (a,c)*(a^2,c) &= (a,c)(a^2,1)^{-1}(a^2,c) \;=\; (a^4,c^{a^3}c) \;=\; (a,c^2). \end{aligned}$$

Example 1.28 This example may be investigated in GAP with the following corredpondence:

$$a \mapsto (7,8,9),\ b \mapsto (8,9),\ c \mapsto (2,3)(4,5)$$

```
gap> G18 := GroupGroupoid( C18);
groupoid with 2 pieces:
1:  single piece groupoid with rays: < Group( [ ()>-()->() ] ),
[ (), (7,8,9), (7,9,8) ], [ ()>-()->(), ()>-(4,6,5)->(7,9,8),
  ()>-(4,5,6)->(7,8,9) ] >
2:  single piece groupoid with rays: < Group( [ (8,9)>-(2,3)(5,6)->(8,9) ] ),
[ (8,9), (7,8), (7,9) ], [ (8,9)>-(2,3)(5,6)->(8,9), (8,9)>-(2,3)(4,5)->(7,8),
  (8,9)>-(2,3)(4,6)->(7,9) ] >
gap> piece2 := Pieces( G18 )[2];;
gap> obs2 := piece2!.objects;
[ (8,9), (7,8), (7,9) ]
gap> RaysOfGroupoid( piece2 );
[ (8,9)>-(2,3)(5,6)->(8,9), (8,9)>-(2,3)(4,5)->(7,8),
  (8,9)>-(2,3)(4,6)->(7,9) ]
gap> elts2 := ElementsOfGroupoid( piece2 );;
gap> x := elts2[3];
[(8,9)>-(2,3)(4,6)->(7,9) : (8,9) -> (7,9)]
gap> y := elts2[8];
[(7,9)>-(1,3)(4,6)->(7,8) : (7,9) -> (7,8)]
gap> x*y;
[(8,9)>-(2,3)(4,5)->(7,8) : (8,9) -> (7,8)]
```

1.14 Regular Groupoids

Let $\mathcal{X} = (\partial : S \to R)$ be a precrossed module and let $\mathcal{C} = (e; t, h : G \to R)$ be the associated precat^1-group where $G = R \ltimes S$ has multiplication $(r_1, s_1)(r_2, s_2) = (r_1 r_2, s_1^{r_2} s_2)$ and inverse $(r, s)^{-1} = (r^{-1}, (s^{-1})^{r^{-1}})$. The homomorphisms e, t, h are given by $er = (r, 1),\ t(r, s) = r,\ h(r, s) = r(\partial s)$.

The associated group-groupoid $\mathcal{G}$ has vertex set R and arrows G with the tail (source) and head (target) maps given by t and h. Composition $*$ in $\mathcal{G}$ is given by

$$(r_1, s_1) * (r_2, s_2) = (r_1, s_1)(r_2^{-1}, 1)(r_2, s_2) = (r_1, s_1 s_2)$$

and is defined when $r_1(\partial s_1) = r_2$. The identity element in the object group at r is $er = (r, 1)$.

Definition 1.29 *Let $\mathcal{G}$ be a groupoid with objects R and arrows G.*

(i) Maps $\rhd : R \times G \to G$ and $\lhd : G \times R \to G$ are respectively left *and* right actions *of R on G if for all $q, r \in R$ and $g, h \in G$*

- $(qr) \rhd g = q \rhd (r \rhd g),\ \ g \lhd (qr) = (g \lhd q) \lhd r;$
- $q \rhd (g \lhd r) = (q \rhd g) \lhd r;$
- $1 \rhd g = g = g \lhd 1;$
- $t(r \rhd g) = r(tg),\ \ t(g \lhd r) = (tg)r,\ \ h(r \rhd g) = r(hg),\ \ h(g \lhd r) = (hg)r;$
- $r \rhd (g * g') = (r \rhd g) * (r \rhd g'),\ \ (g * g') \lhd r = (g \lhd r) * (g' \lhd r)$, *whenever $g * g'$ is defined;*
- $q \rhd er = e(qr) = eq \lhd r.$

(ii) $\mathcal{G}$ is semiregular *if R is a monoid and if $\mathcal{G}$ has left and right actions as in (i).*

(iii) A semiregular $\mathcal{G}$ is regular *if R is a group.*

Lemma 1.30 *When $\mathcal{G}$ is the group groupoid associated to a precat^1-group $\mathcal{C}$, left and right actions are given by left and right multiplication:*

$$r \lhd g := (er)g, \qquad g \rhd r := g(er).$$

Proof: We only verify the axioms for $\lhd$ since those for $\rhd$ follow similarly.

$$\begin{aligned}
g \lhd (qr) &= g \lhd (eq)(er) = (g \lhd q)(er) = (g \lhd q) \lhd r;\\
q \rhd (g \lhd r) &= q \rhd (g(er)) = (eq)g(er) = ((eq)g) \lhd r = (q \rhd g) \lhd r;\\
g \lhd 1 &= g(e1) = g;\\
t(g \lhd r) &= t(g(er)) = (tg)(ter) = (tg)r;\\
h(g \lhd r) &= h(g(er)) = (hg)(her) = (hg)r;\\
(g \lhd r) * (g' \lhd r) &= (g(er))(eh(g(er)))^{-1} g'(er) = g(er)(er)^{-1}(ehg)^{-1} g'(er) = (g * g') \lhd r;\\
eq \lhd r &= (eq)(er) = e(qr).
\end{aligned}$$

□

Proposition 1.31 *Let $\mathcal{G}$ be a semiregular groupoid. Then there are two everywhere defined binary operations on G given by:*

$$\begin{aligned}
g \circledcirc g' &= (g \lhd tg') * (hg \rhd g'),\\
g \circledast g' &= (tg \rhd g') * (g \lhd hg').
\end{aligned}$$

When $\mathcal{G}$ is a regular groupoid both $\circledcirc$ and $\circledast$ make G into a group with identity $e1$.

Following on from the example above, when $\mathcal{G}$ is a group groupoid, we find that $\circledcirc$ is just the multiplication in G.

$$\begin{array}{rcl} g \circledcirc g' & = & (g \lhd tg') * (hg \rhd g') \\ & = & (g(etg')) * ((ehg)g') \\ & = & g(etg')(eh(g(etg')))^{-1}(ehg)g' \\ & = & g(etg')((ehg)(etg'))^{-1}(ehg)g' \\ & = & gg'. \end{array}$$

The situation with $\circledast$ is a little more complicated:

$$\begin{array}{rcl} g \circledast g' & = & (tg \rhd g') * (g \lhd hg') \\ & = & ((etg)g')(eh((etg)g'))^{-1}(g(ehg') \\ & = & (etg)g')((etg)(ehg'))^{-1}g(ehg') \\ & = & (etg)g'(ehg')^{-1}(etg)^{-1}g(ehg'). \end{array}$$

Now $g'(ehg')^{-1} \in \ker h$ and $(etg)^{-1}g \in \ker t$. When $\mathcal{G}$ is a cat^1-group these two products commute, so the expression reduces to gg' and $\circledast$ is also just multiplication in G.

1.15 2-groups

Finally, we think of such a structure as a special case of a 2-category, which has objects, morphisms, and 2-*cells*. We follow the presentation in Subsection 1.2.3 of Forrester-Barker's thesis [34]. For an introduction to 2-groupoids, see Kamps and Porter [42]. (Note that a 2-group is *not* a special case of the group theorist's p-group, with $p = 2$, but is a 2-category with one object having all morphisms and 2-cells invertible.)

The 2-group $\mathcal{H}$ associated to $\mathcal{X} = (\partial : S \to R)$ has

- a single object $\bullet$,
- morphisms $r \in R$,
- 2-cells $(r, s) \in R \ltimes S$ with tail r and head $r(\partial s)$.

$$\begin{tikzcd}
\Downarrow (r,s) \;\; = & \bullet \arrow[r, bend left=40, "r"] \arrow[r, bend right=40, "r(\partial s)"'] \arrow[r, phantom, "\Downarrow (r,s)"] & \bullet
\end{tikzcd}$$

Horizontal composition $(r_1, s_1) \sharp_0 (r_2, s_2)$ of 2-cells is given by

$$\begin{tikzcd}
\bullet \arrow[r, bend left=40, "r_1"] \arrow[r, bend right=40, "r_1(\partial s_1)"'] \arrow[r, phantom, "\Downarrow (r_1,s_1)"] & \bullet \arrow[r, bend left=40, "r_2"] \arrow[r, bend right=40, "r_2(\partial s_2)"'] \arrow[r, phantom, "\Downarrow (r_2,s_2)"] & \bullet & = & \bullet \arrow[r, bend left=30, "r_1r_2"] \arrow[r, bend right=30, "r_1(\partial s_1)r_2(\partial s_2)"'] \arrow[r, phantom, "\Downarrow (r_1r_2, s_1{}^{r_2}s_2)"] & \bullet
\end{tikzcd}$$

There is a unique *horizontal identity* 2-cell

$$\begin{tikzcd}
\Downarrow (1,1) \;\; = & \bullet \arrow[r, bend left=40, "1"] \arrow[r, bend right=40, "1"'] \arrow[r, phantom, "\Downarrow (1,1)"] & \bullet
\end{tikzcd}$$

Similarly, when $r_1(\partial s_1) = r_3$, vertical composition $(r_1, s_1) \sharp_1 (r_3, s_3)$ of 2-cells is given by

$$\begin{tikzcd}
\bullet \arrow[r, bend left=70, "r_1"] \arrow[r, "r_1(\partial s_1)"] \arrow[r, bend right=70, "r_1(\partial s_1)(\partial s_3)"'] \arrow[r, phantom, bend left=35, "\Downarrow (r_1, s_1)"] \arrow[r, phantom, bend right=35, "\Downarrow (r_1(\partial s_1), s_3)"] & \bullet & = & \bullet \arrow[r, bend left=40, "r_1"] \arrow[r, bend right=40, "r_1(\partial(s_1s_3))"'] \arrow[r, phantom, "\Downarrow (r_1, s_1s_3)"] & \bullet
\end{tikzcd}$$

For each $r \in R$ there is a *vertical identity* 2-cell

$$\begin{tikzcd} \Downarrow (r,1) \;=\; & \bullet \arrow[r, bend left=40, "r"] \arrow[r, bend right=40, "r"'] \arrow[r, phantom, "\Downarrow (r,1)"] & \bullet \end{tikzcd}$$

such that

$$\Downarrow (r,1) \;\sharp_1\; \Downarrow (r,s) \;\sharp_1\; \Downarrow (r(\partial s),1) \;=\; \Downarrow (r,s).$$

The *horizontal inverse* and the *vertical right inverse* of $\Downarrow (r,s)$ are $\Downarrow (r^{-1},(s^{-1})^{r^{-1}})$ and $\Downarrow (r(\partial s),s^{-1})$ respectively.

Horizontal composition with vertical identities is called *whiskering.* In diagrams it is often convenient to shrink $\Downarrow (q,1)$ to a single arc, labelled q, as in the whiskering formlae:

$$q_1 \sharp_0 \Downarrow (r,s) \;\sharp_0\; q_2 \;=\; \Downarrow (q_1 r q_2, s^{q_2}).$$

The Peiffer condition for cat^1-groups establishes an *interchange law* for $\mathcal{H}$,

$$((r_1,s_1)\,\sharp_0\,(r_2,s_2))\,\sharp_1\,((r_3,s_3)\,\sharp_0\,(r_4,s_4)) \quad = \quad ((r_1,s_1)\,\sharp_1\,(r_3,s_3))\,\sharp_0\,((r_2,s_2)\,\sharp_1\,(r_4,s_4))$$

for the well-defined composite when $r_1(\partial s_1) = r_3$ and $r_2(\partial s_2) = r_4$,

$$\begin{tikzcd}[column sep=huge, row sep=huge] \bullet \arrow[r, bend left=70, "r_1"] \arrow[r, "r_1(\partial s_1)=r_3"] \arrow[r, bend right=70, "r_3(\partial s_3)"'] \arrow[r, phantom, bend left=35, "\Downarrow (r_1,s_1)"] \arrow[r, phantom, bend right=35, "\Downarrow (r_3,s_3)"] & \bullet \arrow[r, bend left=70, "r_2"] \arrow[r, "r_2(\partial s_2)=r_4"] \arrow[r, bend right=70, "r_4(\partial s_4)"'] \arrow[r, phantom, bend left=35, "\Downarrow (r_2,s_2)"] \arrow[r, phantom, bend right=35, "\Downarrow (r_4,s_4)"] & \bullet \end{tikzcd}$$

When this composite is defined,

$$s_2 {s_3}^{r_4} \;=\; s_2 {s_3}^{r_2(\partial s_2)} \;=\; {s_3}^{r_2} s_2$$

and the composite 2-cell is

$$\begin{tikzcd}[column sep=huge] \bullet \arrow[r, bend left=25, "r_1 r_2"] \arrow[r, bend right=25, "r_3(\partial s_3) r_4(\partial s_4)"'] \arrow[r, phantom, "\Downarrow (r_1 r_2, (s_1 s_3)^{r_2} s_2 s_4)"] & \bullet \end{tikzcd}$$

2 Derivations and Sections

2.1 Derivations

The Whitehead monoid $\mathrm{Der}(\mathcal{X})$ of $\mathcal{X} = (\partial : S \to R)$ was defined in [56] to be the monoid of all *derivations* from R to S, that is the set of all maps $R \to S$, with composition $\star$, satisfying

$$\begin{array}{lrcl} \textbf{D1:} & \chi(qr) & = & (\chi q)^r\,(\chi r) \\ \textbf{D2:} & (\chi_1 \star \chi_2)(r) & = & (\chi_2 r)(\chi_1 r)(\chi_2 \partial \chi_1 r). \end{array}$$

The definition of Whitehead multiplication used here differs from that in [3] in that it is now defined as multiplication on the right rather than on the left, which is why we are using '$\star$' in place of '$\circ$'. Invertible elements in the monoid are called *regular*. The Whitehead group $W = W(\mathcal{X})$ is the group of the monoid.

In Brown and Gilbert [9] the notion of derivation was extended to that of γ-derivation, as in the following definition. Since ordinary derivations may be obtained from γ-derivations by setting γ to be the identity automorphism of $\mathcal{X}$, we shall give properties in terms of the more general case.

Definition 2.1 *If $\gamma = (\ddot{\gamma}, \dot{\gamma})$ is an automorphism of $\mathcal{X}$, the Whitehead monoid* $\mathrm{Der}_\gamma(\mathcal{X})$ *of $\mathcal{X}$ is the monoid of all γ-derivations from R to S, that is the set of all maps $R \to S$, with composition written $\star_\gamma$, satisfying*

$$\begin{array}{lrcl} \textbf{D1:} & \chi(qr) & = & (\chi q)^{\dot{\gamma} r}(\chi r)\ . \\ \textbf{D2:} & (\chi_1 \star_\gamma \chi_2)(r) & = & (\chi_2 r)(\chi_1 r)(\chi_2 \dot{\gamma}^{-1} \partial \chi_1 r). \end{array}$$

The following Lemma verifies that $\mathrm{Der}_\gamma(\mathcal{X})$ *is* a monoid.

Lemma 2.2

(a) $\chi 1 = 1$,

(b) $(\chi r)^{-1} \;=\; (\chi\, r^{-1})^{\dot{\gamma} r}$,

(c) *the zero map is a derivation and an identity for the Whitehead multiplication,*

(d) *the Whitehead multiplication is associative.*

Proof:

(a) This follows from $\chi(r1) = (\chi r)^1(\chi 1)$.

(b) This follows from $1 = \chi(r^{-1} r) = (\chi r^{-1})^{\dot{\gamma} r}(\chi r)$.

(c) It is clear that $0 : R \to S,\ r \mapsto 1$ is a derivation, and that

$$(\chi \star_\gamma 0)r = 1(\chi r)1 = \chi r = (\chi r)11 = (0 \star_\gamma \chi)r\ .$$

(d) Expansion by **D2:** using either bracketing (though one requires more work!) gives:

$$(\chi_1 \star \chi_2 \star \chi_3)r \;=\; (\chi_3 r)(\chi_2 r)(\chi_3 \dot{\gamma}^{-1} \partial \chi_2 r)(\chi_1 r)(\chi_3 \dot{\gamma}^{-1} \partial \chi_1 r)(\chi_2 \dot{\gamma}^{-1} \partial \chi_1 r)(\chi_3 \dot{\gamma}^{-1} \partial \chi_2 \dot{\gamma}^{-1} \partial \chi_1 r)\ .$$

□

For χ a γ-derivation, define $\psi = \psi_\chi : R \to S$ by $\psi r = \chi\dot{\gamma}^{-1}r$ or, equivalently, $\psi\dot{\gamma}r = \chi r$.

$$\begin{tikzcd}
S \arrow[r, "\ddot{\gamma}"] \arrow[d, "\partial"'] & S \arrow[d, shift left, "\partial"] \\
R \arrow[r, "\dot{\gamma}"'] \arrow[ur, "\chi"] & R \arrow[u, shift left, "\psi"]
\end{tikzcd} \tag{14}$$

Then ψ is a (identity-) derivation since

$$\psi(qr) \;=\; \chi((\dot{\gamma}^{-1}q)(\dot{\gamma}^{-1}r)) \;=\; (\chi\dot{\gamma}^{-1}q)^{\dot{\gamma}(\dot{\gamma}^{-1}r)}(\chi\dot{\gamma}^{-1}r) \;=\; (\psi q)^r(\psi r).$$

Lemma 2.3 *The map* $\ddot{\theta} : \mathrm{Der}_\gamma(\mathcal{X}) \to \mathrm{Der}(\mathcal{X}),\; \chi \mapsto \psi_\chi$, *is a monoid isomorphism.*

Proof: If ψ_1, ψ_2 are the derivations corresponding to γ-derivations χ_1, χ_2, then

$$\begin{aligned}
\Big((\ddot{\theta}\chi_1) \star (\ddot{\theta}\chi_2)\Big)^r &= (\psi_1 \star \psi_2)r \\
&= (\psi_2 r)(\psi_1 r)(\psi_2\partial\psi_1 r) \\
&= \big(\chi_2(\dot{\gamma}^{-1}r)\big)\big(\chi_1(\dot{\gamma}^{-1}r)\big)\big(\chi_2\dot{\gamma}^{-1}\partial\chi_1(\dot{\gamma}^{-1}r)\big) \\
&= (\chi_1 \star_\gamma \chi_2)(\dot{\gamma}^{-1}r) \\
&= \ddot{\theta}(\chi_1 \star_\gamma \chi_2)r.
\end{aligned}$$

So $\ddot{\theta}$ is a homomorphism, and it is invertible since

$$\psi_1 = \psi_2 \quad\Rightarrow\quad \chi_1\dot{\gamma}^{-1}r = \chi_2\dot{\gamma}^{-1}r \;\;(\forall r \in R) \quad\Rightarrow\quad \chi_1 = \chi_2.$$

□

Lemma 2.4 *Given a* γ*-derivation* χ *of* $\mathcal{X}$ *there is an endomorphism* $\beta_\chi = (\ddot{\beta}_\chi, \dot{\beta}_\chi)$ *of* $\mathcal{X}$ *where*

$$\ddot{\beta}_\chi : S \to S,\;\; s \mapsto (\ddot{\gamma}s)(\chi\partial s), \quad \dot{\beta}_\chi : R \to R,\;\; r \mapsto (\dot{\gamma}r)(\partial\chi r)$$

such that

(a) $\ddot{\beta}_\chi(s^r) \;=\; (\ddot{\beta}_\chi s)^{\dot{\beta}_\chi r} \;=\; (\chi r)^{-1}\,(\ddot{\beta}_\chi s)^{\dot{\gamma}r}\,(\chi r)$,

(b) $\dot{\beta}_\chi(q^r) \;=\; (\dot{\beta}_\chi q)^{\dot{\beta}_\chi r} \;=\; ((\dot{\gamma}r)(\partial\chi r))^{-1}(\dot{\gamma}q)(\partial\chi q)((\dot{\gamma}r)(\partial\chi r))$,

(c) $(\chi_1 \star_\gamma \chi_2)r \;=\; (\chi_2 r)(\ddot{\beta}_{\chi_2}\ddot{\gamma}^{-1}\chi_1 r) \;=\; (\chi_1 r)(\chi_2\dot{\gamma}^{-1}\dot{\beta}_{\chi_1} r)$,

(d) $\chi * \ddot{\gamma}^{-1} * \ddot{\beta}_\chi \;=\; \dot{\beta}_\chi * \dot{\gamma}^{-1} * \chi \;:\; R \to S, \quad r \mapsto (\chi r)(\chi\dot{\gamma}^{-1}\partial\chi r)$, so that the following diagram commutes:

$$\begin{tikzcd}
S \arrow[r, "\ddot{\gamma}^{-1} * \ddot{\beta}_\chi"] & S \\
R \arrow[u, "\chi"] \arrow[r, "\dot{\beta}_\chi * \dot{\gamma}^{-1}"'] & R \arrow[u, "\chi"']
\end{tikzcd} \tag{15}$$

(e) *The endomorphism* $\ddot\beta_\chi * \ddot\gamma^{-1}$ *commutes with* $\partial * \chi * \ddot\gamma^{-1}$ *while* $\dot\gamma^{-1} * \dot\beta_\chi$ *commutes with* $\dot\gamma^{-1} * \chi * \partial$.

(f) *When* $\psi = \psi_\chi$ *as in* (14), *then* $\dot\beta_\chi r \;=\; \dot\beta_\psi(\dot\gamma r)$ *and* $\ddot\beta_\chi s \;=\; \ddot\beta_\psi(\ddot\gamma s)$, *so that* $\beta_\chi \;=\; \beta_\psi \circ \gamma$.

Proof: We first check that $\dot\beta_\chi$ and $\ddot\beta_\chi$ are homomorphisms.

$$\begin{array}{rcl}
\dot\beta_\chi(r_1r_2) & = & \dot\gamma(r_1r_2)\partial((\chi r_1)^{\dot\gamma r_2}(\chi r_2)) \;=\; (\dot\gamma r_1)(\partial\chi r_1)(\dot\gamma r_2)(\partial\chi r_2) \;=\; (\dot\beta_\chi r_1)(\dot\beta_\chi r_2), \\
\ddot\beta_\chi(s_1s_2) & = & \ddot\gamma(s_1s_2)(\chi((\partial s_1)(\partial s_2)) \;=\; (\ddot\gamma s_1)(\ddot\gamma s_2)(\chi\partial s_1)^{\partial\ddot\gamma s_2}(\chi\partial s_2) \;=\; (\ddot\beta s_1)(\ddot\beta s_2).
\end{array}$$

We now verify the six properties.

(a)
$$\begin{array}{rcl}
\ddot\beta_\chi(s^r) & = & (\ddot\gamma s^r)(\chi\partial(s^r)) \;=\; (\ddot\gamma s)^{\dot\gamma r}(\chi(r^{-1}(\partial s)r)) \;=\; (\ddot\gamma s)^{\dot\gamma r}(\chi(r^{-1}))^{(\partial\ddot\gamma s)(\dot\gamma r)}(\chi\partial s)^{\dot\gamma r}(\chi r) \\
 & = & \{(\chi(r^{-1}))(\ddot\gamma s)(\chi\partial s)\}^{\dot\gamma r}(\chi r) \;=\; (\chi r)^{-1}(\ddot\beta_\chi s)^{\dot\gamma r}(\chi r) \;=\; (\ddot\beta_\chi s)^{(\dot\gamma r)(\partial\chi r)} \;=\; (\ddot\beta_\chi s)^{\dot\beta_\chi r} \,.
\end{array}$$

(b)
$$\begin{array}{rcl}
\dot\beta_\chi(q^r) & = & (\dot\gamma(q^r))(\partial\chi(r^{-1}qr)) \;=\; (\dot\gamma(q^r))\,\partial\{(\chi(r^{-1}))^{\dot\gamma(qr)}(\chi q)^{\dot\gamma r}(\chi r)\} \\
 & = & (\dot\gamma(q^r))(\partial\chi(r^{-1}))^{(\dot\gamma r)(\dot\gamma q^r)}(\partial\chi q)^{\dot\gamma r}(\partial\chi r) \;=\; (\partial((\chi r)^{-1})(\dot\gamma(q^r))(\partial\chi q)^{\dot\gamma r}(\partial\chi r) \\
 & = & (\partial\chi r)^{-1}\{(\dot\gamma q)(\partial\chi q)\}^{\dot\gamma r}(\partial\chi r) \;=\; (\dot\beta_\chi q)^{\dot\beta_\chi r} \,.
\end{array}$$

(c)
$$\begin{array}{rcl}
(\chi_1 \star_\gamma \chi_2)r & = & (\chi_2 r)\{(\chi_1 r)(\chi_2\partial(\ddot\gamma^{-1}\chi_1 r))\} \;=\; (\chi_2 r)(\ddot\beta_{\chi_2}\ddot\gamma^{-1}\chi_1 r) \,, \\
(\chi_2\dot\gamma^{-1}\dot\beta_{\chi_1})r & = & \chi_2(r(\dot\gamma^{-1}\partial\chi_1 r)) \;=\; (\chi_2 r)^{\partial\chi_1 r}\,(\chi_2\dot\gamma^{-1}\partial\chi_1 r) \;=\; (\chi_1 r)^{-1}(\chi_2 r)(\chi_1 r)(\chi_2\dot\gamma^{-1}\partial\chi_1 r) \,.
\end{array}$$

(d)
$$\begin{array}{rcl}
\ddot\beta_\chi(\ddot\gamma^{-1}\chi r) & = & \ddot\gamma(\ddot\gamma^{-1}\chi r)(\chi\partial\ddot\gamma^{-1}\chi r) \;=\; (\chi r)(\chi\dot\gamma^{-1}\partial\chi r) \,, \quad \text{and} \\
(\chi\dot\gamma^{-1})(\dot\beta_\chi r) & = & \chi(r(\dot\gamma^{-1}\partial\chi r)) \;=\; (\chi r)^{\partial\chi r}(\chi\dot\gamma^{-1}\partial\chi r) \;=\; (\chi r)(\chi\dot\gamma^{-1}\partial\chi r) \,.
\end{array}$$

(e) By (d),
$$\begin{array}{rcl}
(\ddot\beta_\chi * \ddot\gamma^{-1}) * (\partial * \chi * \ddot\gamma^{-1}) & = & \partial * (\dot\beta_\chi * \dot\gamma^{-1} * \chi) * \ddot\gamma^{-1} \;=\; (\partial * \chi * \ddot\gamma^{-1}) * (\ddot\beta_\chi * \ddot\gamma^{-1}), \\
(\dot\gamma^{-1} * \dot\beta_\chi) * (\dot\gamma^{-1} * \chi * \partial) & = & \dot\gamma^{-1} * (\chi * \ddot\gamma^{-1} * \ddot\beta_\chi) * \partial \;=\; (\dot\gamma^{-1} * \chi * \partial) * (\dot\gamma^{-1} * \dot\beta_\chi).
\end{array}$$

(f) This relationship between β_χ and β_ψ is immediate. □

Using Lemma 2.4 and the first crossed module axiom, the identity **D1:** for derivations generalises as follows.

Lemma 2.5

(a) $\chi(r_1r_2\ldots r_k) \;=\; (\chi r_1)^{\dot\gamma(r_2\ldots r_k)}(\chi r_2)^{\dot\gamma(r_3\ldots r_k)}\,\ldots\,(\chi r_{k-1})^{\dot\gamma r_k}(\chi r_k)$,

(b) $\partial\chi(r_1r_2\ldots r_k) \;=\; (\dot\gamma(r_1r_2\ldots r_k))^{-1}\,(\dot\beta_\chi r_1)(\dot\beta_\chi r_2)\ldots(\dot\beta_\chi r_k)$,

(c) $\chi\partial(s_1s_2\ldots s_k) \;=\; (\ddot\gamma(s_1s_2\ldots s_k))^{-1}\,(\ddot\beta_\chi s_1)(\ddot\beta_\chi s_2)\ldots(\ddot\beta_\chi s_k)$.

It is straightforward to verify that for g an invertible element in a monoid M, the set $M_g = (M, *_g)$ with elements M and multiplication $*_g$ defined in terms of the usual multiplication by

$$m *_g n \;:=\; mg^{-1}n, \tag{16}$$

is a monoid with identity g. If $m \in M$ is invertible in M then m has $*_g$-inverse $\overline{m} := gm^{-1}g$. The resulting monoids are isomorphic, either by $\theta_g : M \to M_g, m \mapsto mg$ or by $\theta'_g : M \to M_g, m \mapsto gm$. When M is a group the g-conjugation automorphisms are the mappings

$$\wedge_g m \;:\; G \to G,\; n \;\mapsto\; \overline{m} *_g n *_g m = gm^{-1}ng^{-1}m\,. \tag{17}$$

This notion generalises to categories and to crossed modules, but the application we require here is to the monoid of endomorphisms $\mathrm{End}_\gamma(\mathcal{X})$, where $\gamma = (\ddot\gamma, \dot\gamma)$ is an automorphism of $\mathcal{X}$, with multiplication

$$\alpha *_\gamma \beta \;:=\; (\ddot\alpha *_{\ddot\gamma} \ddot\beta,\; \dot\alpha *_{\dot\gamma} \dot\beta). \tag{18}$$

Theorem 2.6 *There is a monoid homomorphism* $\Delta_\gamma : \mathrm{Der}_\gamma(\mathcal{X}) \to \mathrm{End}_\gamma(\mathcal{X})$, $\chi \mapsto \beta_\chi = (\ddot{\beta}_\chi, \dot{\beta}_\chi)$.

Proof: Since

$$\begin{array}{rcl}
(\ddot{\beta}_{\chi_1} *_\gamma \ddot{\beta}_{\chi_2})s & = & (\ddot{\beta}_{\chi_1} * \ddot{\gamma}^{-1} * \ddot{\beta}_{\chi_2})s \\
 & = & \ddot{\beta}_{\chi_2}(s(\ddot{\gamma}^{-1}\chi_1\partial s) \\
 & = & (\ddot{\gamma}s)(\chi_1\partial s)\chi_2((\partial s)(\dot{\gamma}^{-1}\partial\chi_1\partial s) \\
 & = & (\ddot{\gamma}s)(\chi_1\partial s)(\chi_2\partial s)^{\partial\chi_1\partial s}(\chi_2\dot{\gamma}^{-1}\partial\chi_1\partial s) \\
 & = & (\ddot{\gamma}s)(\chi_2\partial s)(\chi_1\partial s)(\chi_2\dot{\gamma}^{-1}\partial\chi_1\partial s) \\
 & = & \ddot{\beta}_{\chi_1 \star_\gamma \chi_2}\ s, \\
(\dot{\beta}_{\chi_1} *_\gamma \dot{\beta}_{\chi_2})r & = & (\dot{\beta}_{\chi_1} * \dot{\gamma}^{-1} * \dot{\beta}_{\chi_2})r \\
 & = & \dot{\beta}_{\chi_2}(r(\dot{\gamma}^{-1}\partial\chi_1 r)) \\
 & = & (\dot{\gamma}r)(\partial\chi_1 r)\partial\left((\chi_2 r)^{\partial\chi_1 r}(\chi_2\dot{\gamma}^{-1}\partial\chi_1 r)\right) \\
 & = & (\dot{\gamma}r)\partial\left((\chi_2 r)(\chi_1 r)(\chi_2\dot{\gamma}^{-1}\partial\chi_1 r)\right) \\
 & = & \dot{\beta}_{\chi_1 \star \chi_2}\ r,
\end{array}$$

it follows that $(\Delta_\gamma\chi_1) *_\gamma (\Delta_\gamma\chi_2) = \Delta_\gamma(\chi_1 \star_\gamma \chi_2)$. □

Lemma 2.7 *For each* $s \in S$ *the function* η_s

$$\eta_s : R \to S, \quad r \mapsto (s^{-1})^r s$$

is a derivation, called a principal derivation, *satisfying*

$$\eta_s(\partial s_0) = [s_0, s], \qquad \textit{and} \qquad \partial(\eta_s r) = [r, \partial s].$$

Similarly

$$\eta_{(s,\gamma)} := \ddot{\theta}^{-1}\eta_s : r \mapsto (s^{-1})^{\dot{\gamma}r}\, s$$

is the corresponding principal γ-derivation *satisfying*

$$\eta_{(s,\gamma)}(\partial s_0) = [\ddot{\gamma}s_0, s], \qquad \textit{and} \qquad \partial(\eta_{(s,\gamma)}r) = [\dot{\gamma}r, \partial s].$$

Proof:

$$\eta_s(qr) = (s^{-1})^{qr}(s^r(s^{-1})^r)s = ((s^{-1})^q s)^r(s^{-1})^r s = (\eta_s g)^r(\eta_s r) \qquad \text{satisfying D1},$$

$$\eta_s\partial s_0 = (s^{-1})^{\partial s_0}\, s = (s_0)^{-1}\, s^{-1}\, s_0\, s = [s_0, s]\ ,$$

$$\partial(\eta_s r) = (\partial s^{-1})^r(\partial s) = [r, \partial s]\ .$$

The corresponding properties of $\eta_{(s,\gamma)}$ are easily verified in the same way. □

We shall see later that there is a homomorphism from S to $\mathrm{Der}(\mathcal{X})$ mapping s to the principal derivation η_s.

Lemma 2.8 (More properties of principal γ-derivations)

(a) $\eta_{(1,\gamma)}$ *is the zero map,*

(b) $\ddot{\beta}_{\eta_{(s,\gamma)}} s_0 = (\ddot{\gamma}s_0)^s$ *and* $\dot{\beta}_{\eta_{(s,\gamma)}} r = (\dot{\gamma}r)^{\partial s}$,

(c) $\eta_{(s_1,\gamma)} \star_\gamma \eta_{(s_2,\gamma)} = \eta_{(s_1 s_2,\gamma)}$,

(d) *the zero map is the identity in* $\mathrm{End}_\gamma(\mathcal{X})$ *and* $\overline{\eta_{(s,\gamma)}} = \eta_{(s^{-1},\gamma)}$.

Proof:

(a) $\eta_{(1,\gamma)}r \;=\; 1^{\dot\gamma r}\,1 \;=\; 1\;,$

(b) $\ddot\beta_{\eta_{(s,\gamma)}}s_0 \;=\; (\ddot\gamma s_0)(\eta_{(s,\gamma)}(\partial s_0)) \;=\; (\ddot\gamma s_0)[\ddot\gamma s_0, s] \;=\; (\ddot\gamma s_0)^s$

and $\dot\beta_{\eta_{(s,\gamma)}}r \;=\; (\dot\gamma r)(\partial\eta_s r) \;=\; (\dot\gamma r)[\dot\gamma r, \partial s] \;=\; (\dot\gamma r)^{\partial s}\;,$

(c) $(\eta_{(s_1,\gamma)} \star_\gamma \eta_{(s_2,\gamma)})r \;=\; (\eta_{(s_2,\gamma)}r)(\eta_{(s_1,\gamma)}r)(\eta_{(s_2,\gamma)}\partial\ddot\gamma^{-1}\eta_{(s_1,\gamma)}r)) \;=\; (\eta_{(s_2,\gamma)}r)(\eta_{(s_1,\gamma)}r)[\eta_{(s_1,\gamma)}r, s_2]$
$\;=\; (\eta_{(s_2,\gamma)}r)s_2^{-1}(\eta_{(s_1,\gamma)}r)s_2 \;=\; (s_2^{-1})^{\dot\gamma r}(s_1^{-1})^{\dot\gamma r}s_1s_2 \;=\; \eta_{(s_1s_2,\gamma)}r.$

(d) It follows from (a) and (c) that $\eta_{(s,\gamma)} \star_\gamma \eta(1,\gamma) \;=\; \eta_{(s,\gamma)} \;=\; \eta_{(1,\gamma)} \star_\gamma \eta(s,\gamma)$ and also that $\eta_{(s,\gamma)} \star_\gamma \eta(s^{-1},\gamma) = \eta_{(1,\gamma)}$.

□

Lemma 2.9 *The following statements are equivalent.*

(i) χ *has a Whitehead* γ*-inverse* $\overline{\chi}$*;*

(ii) $\ddot\beta_\chi \in \mathrm{Aut}(S)$*, where* $\ddot\beta_\chi(s) = (\ddot\gamma s)(\chi\partial s)$*;*

(iii) $\dot\beta_\chi \in \mathrm{Aut}(R)$*, where* $\dot\beta_\chi(r) = (\dot\gamma r)(\partial\chi r)$*;*

(iv) $\beta = (\ddot\beta, \dot\beta) \in \mathrm{Aut}_\gamma(\mathcal{X})$.

When these conditions are satisfied,

$$\overline{\chi}r \;=\; (\ddot\gamma\overline{\ddot\beta_\chi}\,\chi\,r)^{-1} \;=\; (\chi\,\overline{\dot\beta_\chi}\dot\gamma\,r)^{-1},\quad (\chi r)(\overline{\chi}r) \;=\; (\chi\dot\gamma^{-1}\partial\overline{\chi}r)^{-1},\quad \textit{and}\quad (\overline{\chi}r)(\chi r) \;=\; (\overline{\chi}\dot\gamma^{-1}\partial\chi r)^{-1}\,.$$

Proof: Theorem 2.6 shows that, when χ is a regular derivation, both $\ddot\beta_\chi$ and $\dot\beta_\chi$ are automorphisms, so (i) implies (ii) and (iii), and hence (iv).

Now suppose that $\ddot\beta_\chi$ has γ-inverse $\overline{\ddot\beta_\chi}$. We first show that $\chi^\$$ is a derivation where $\chi^\$r = (\ddot\gamma\overline{\ddot\beta_\chi}\chi r)^{-1}$. Using the equivalent formula, $\ddot\beta_\chi\ddot\gamma^{-1}\chi^\$r = (\chi r)^{-1}$,

$$\begin{aligned}
\ddot\beta_\chi\ddot\gamma^{-1}(\,(\chi^\$q)^{\dot\gamma r}\,(\chi^\$r)\,) \;&=\; (\ddot\beta_\chi\ddot\gamma^{-1}\chi^\$q)^{\dot\beta_\chi r}(\ddot\beta_\chi\ddot\gamma^{-1}\chi^\$r) \quad \text{by Theorem 2.4 (a)}\\
&=\; ((\chi q)^{-1})^{(\dot\gamma r)(\partial\chi r)}(\chi r)^{-1} \qquad \text{by definition of} \quad \chi^\$\\
&=\; (\chi r)^{-1}(\,(\chi q)^{-1}\,)^{\dot\gamma r} \;=\; ((\chi q)^{\dot\gamma r}\,(\chi r))^{-1} \;=\; (\chi(qr))^{-1} \;=\; \ddot\beta_\chi\ddot\gamma^{-1}\chi^\$(qr)\;.
\end{aligned}$$

We now show that $\chi^\$$ is the Whitehead γ-inverse $\overline{\chi}$ of χ, using Lemma 2.4 (c), (d) :

$$\begin{aligned}
(\chi^\$ \star_\gamma \chi)r \;&=\; (\chi r)(\ddot\beta_\chi\ddot\gamma^{-1}\chi^\$r) \;=\; (\chi r)(\chi r)^{-1}\;,\\
(\chi \star_\gamma \chi^\$)r \;&=\; (\chi r)(\chi^\$\dot\gamma^{-1}\dot\beta_\chi r) \;=\; (\chi r)(\ddot\gamma\overline{\ddot\beta_\chi}\chi\dot\gamma^{-1}\dot\beta_\chi r)^{-1} \;=\; (\chi r)(\chi r)^{-1}\;.
\end{aligned}$$

Thus (ii) implies (i), (iii) and (iv).

Similarly, suppose that $\dot\beta_\chi$ has an inverse $\overline{\dot\beta_\chi}$. We show that $\chi^\#$ is a derivation where $\chi^\#r = (\chi\overline{\dot\beta_\chi}\dot\gamma r)^{-1}$. Define $r' = \overline{\dot\beta_\chi}\dot\gamma r$ so that $\chi^\#r = (\chi r')^{-1}$, and similarly for q'. Then

$$\begin{aligned}
(\chi^\#q)^{\dot\gamma r}\,(\chi^\#r) \;&=\; ((\chi q')^{-1})^{\dot\gamma r}\,(\chi r')^{-1} \;=\; \Big((\chi r')(\chi q')^{\dot\beta_\chi r'}\Big)^{-1} \;=\; \Big((\chi r')(\chi q')^{(\dot\gamma r')(\partial\chi r')}\Big)^{-1}\\
&=\; \Big((\chi q')^{\dot\gamma r'}(\chi r')\Big)^{-1} \;=\; (\chi(qr)')^{-1} \;=\; \chi^\#(qr)\;.
\end{aligned}$$

This $\chi^{\#}$ is another form of $\overline{\chi}$ since, again using Lemma 2.4 (c), (d) :

$$\begin{array}{rcl}(\chi^{\#} \star_{\gamma} \chi)r & = & (\chi r)(\ddot{\beta}_{\chi}\ddot{\gamma}^{-1}\chi^{\#}r) \;=\; (\chi r)(\ddot{\beta}_{\chi}\ddot{\gamma}^{-1}\chi\overline{\dot{\beta}_{\chi}}\dot{\gamma}r)^{-1} \;=\; (\chi r)(\chi r)^{-1}, \\ (\chi \star_{\gamma} \chi^{\#})r & = & (\chi r)(\chi^{\#}\dot{\gamma}^{-1}\dot{\beta}_{\chi}r) \;=\; (\chi r)(\chi r)^{-1} \ .\end{array}$$

Thus (iii) implies (i), (ii) and (iv).

Finally, (iv) implies (ii) and (iii), and hence (i).

The expressions for $(\chi r)(\overline{\chi}r)$ and $(\overline{\chi}r)(\chi r)$ are obtained by expanding $(\overline{\chi} \star_{\gamma} \chi)r$ and $(\chi \star_{\gamma} \overline{\chi})r$. □

We shall see in Subsection 3.2 that $W_{\gamma}(\mathcal{X})$ is the source group in the γ-*actor* of $\mathcal{X}$,

$$\mathrm{Act}_{\gamma}(\mathcal{X}) \;=\; (\Delta_{\gamma} : W_{\gamma}(\mathcal{X}) \to \mathrm{Aut}_{\gamma}(\mathcal{X})) \ .$$

Lue and Norrie, in [45, 44, 49, 50], showed that $\mathrm{Act}(\mathcal{X})$ is the automorphism object of $\mathcal{X}$ in the category **XMod**. Gilbert, in [36], has discussed a connection between derivations and group extensions.

2.2 Sections

The construction for a cat^1-group $\mathcal{C} = (e; t, h : G \to R)$ equivalent to the γ-derivation of the corresponding crossed module is the *γ-section*, namely a group monomorphism $\xi : R \to G$ satisfying:

$$\textbf{S1:} \quad t\xi(r) = \dot{\gamma}r \ \text{ for all } \ r \in R.$$

The equations

$$\xi r = (e\dot{\gamma}r)(\epsilon\chi r) = (\dot{\gamma}r, \chi r)\,, \qquad \chi r = (e\dot{\gamma}r)^{-1}(\xi r) \tag{19}$$

define a section ξ of $\mathcal{C}$ in terms of a derivation χ of $\mathcal{X}$, and conversely. The automorphism $\gamma = (\ddot{\gamma}, \dot{\gamma})$ of $\mathcal{X} = (\partial : S \to R)$ determines an automorphism $\bar{\gamma}$ of $R \ltimes S$, and hence an automorphism $(\bar{\gamma}, \dot{\gamma})$ of the corresponding cat^1-group.

The *principal section* $\kappa_s, \ s \in \ker t,$ and the corresponding principal derivation η_s are given by

$$\eta_s r \ = \ (s^{-1})^{\dot{\gamma}r} s \qquad \kappa_s r \ = \ (e\dot{\gamma}r)^s \ = \ s^{-1}(e\dot{\gamma}r)s\,.$$

In the semidirect product notation we have

$$\kappa_s r \ = \ (\dot{\gamma}r, \eta_s r) \ = \ (\dot{\gamma}r, (s^{-1})^{\dot{\gamma}r}s) \ = \ (1, s^{-1})(\dot{\gamma}r, 1)(1, s) \ = \ (\dot{\gamma}r, 1)^{(1,s)}\,.$$

Since $(ehg^{-1})(\xi\dot{\gamma}^{-1}hg) \in \ker t$ and $(ehg^{-1})g \in \ker h$ we have, in the group groupoid,

$$g * \xi\dot{\gamma}^{-1}hg \ = \ g(ehg^{-1})(\xi\dot{\gamma}^{-1}hg) \ = \ (ehg)((ehg^{-1})g)((ehg^{-1})(\xi\dot{\gamma}^{-1}hg)) \ = \ (\xi\dot{\gamma}^{-1}hg)(ehg^{-1})g. \tag{20}$$

These sections form the monoid Sect($\mathcal{C}$) of $\mathcal{C}$, whose composition rule we determine from the rule **D2:** for Der($\mathcal{X}$) by evaluating:

$$\begin{aligned}
(\xi_1 \star_\gamma \xi_2)r \ &= \ (e\dot{\gamma}r)(\epsilon(\chi_1 \star \chi_2)r) \\
&= \ (e\dot{\gamma}r)(\epsilon\chi_2 r)(\epsilon\chi_1 r)(\epsilon\chi_2\dot{\gamma}^{-1}h\epsilon\chi_1 r) \\
&= \ (\xi_2 r)(e\dot{\gamma}r^{-1})(\xi_1 r)(eh(\epsilon\chi_1 r)^{-1})(\xi_2\dot{\gamma}^{-1}h\epsilon\chi_1 r) \\
&= \ (\xi_2 r)(e\dot{\gamma}r^{-1})(\xi_1 r)(eh((\xi_1 r)^{-1}(e\dot{\gamma}r)))(\xi_2\dot{\gamma}^{-1}h((e\dot{\gamma}r^{-1})(\xi_1 r))) \\
&= \ ((e\dot{\gamma}r)(\xi_2 r^{-1}))^{-1}((\xi_1 r)(eh\xi_1 r^{-1}))((e\dot{\gamma}r)(\xi_2 r^{-1}))(\xi_2\dot{\gamma}^{-1}h\xi_1 r).
\end{aligned}$$

Since $(e\dot{\gamma}r)(\xi_2 r^{-1}) \in \ker t$ while $(\xi_1 r)(eh\xi_1 r^{-1}) \in \ker h$, we obtain, using (20),

$$\textbf{S2:} \quad (\xi_1 \star_\gamma \xi_2)r \ = \ (\xi_1 r)(eh\xi_1 r^{-1})(\xi_2\dot{\gamma}^{-1}h\xi_1 r) \ = \ (\xi_2\dot{\gamma}^{-1}h\xi_1 r)(eh\xi_1 r^{-1})(\xi_1 r). \tag{21}$$

(Note that this axiom also differs from that in [3] in that it is converted to a multiplication on the right.)

The section $\dot{\gamma} * e$ is the identity for this composition, and equation (19) determines a monoid isomorphism Der($\mathcal{X}$) $\cong$ Sect($\mathcal{C}$). A section is *regular* when $h\xi$ is an automorphism of R, and the group of regular sections is isomorphic to the Whitehead group.

Each χ and its associated ξ determine endomorphisms of $R, S, G, \mathcal{X}$ and $\mathcal{C}$, namely

$$\begin{array}{rclcll}
\dot{\beta}_\chi & = & \dot{\beta}_\xi & : & R \to R, & r \mapsto (\dot{\gamma}r)(\partial\chi r) = h\xi r, \\
\ddot{\beta}_\chi & = & \ddot{\beta}_\xi & : & S \to S, & s \mapsto (\ddot{\gamma}s)(\chi\partial s) \ = \ (\ddot{\gamma}s)(e\partial\ddot{\gamma}s^{-1})(\xi\partial s) \ = \ (\xi\partial s)(e\partial\ddot{\gamma}s^{-1})(\ddot{\gamma}s), \\
\bar{\beta}_\chi & = & \bar{\beta}_\xi & : & G \to G, & g \mapsto (eh\xi tg)(\xi tg^{-1})(\bar{\gamma}g)(eh\bar{\gamma}g^{-1})(\xi hg), \\
(\ddot{\beta}_\chi, \dot{\beta}_\chi) & = & (\ddot{\beta}_\xi, \dot{\beta}_\xi) & : & \mathcal{X} \to \mathcal{X}, & \\
(\bar{\beta}_\chi, \dot{\beta}_\chi) & = & (\bar{\beta}_\xi, \dot{\beta}_\xi) & : & \mathcal{C} \to \mathcal{C}, &
\end{array} \tag{22}$$

and these assignments determine group homomorphisms from the Whitehead group to these five endomorphism groups. The accompanying diagram shows the relationship between the various groups and homomorphisms.

$$\begin{tikzcd}
\mathrm{Aut}(S) \arrow[r, dotted, "\curvearrowright"] & S \arrow[r, "\epsilon"] \arrow[d, "\partial"] \arrow[loop above, "\ddot{\beta}_\chi, \ddot{\gamma}"] & G \arrow[d, shift right, "t"'] \arrow[d, shift left, "h"] \arrow[loop above, "\bar{\beta}_\chi, \bar{\gamma}"] \\
& R \arrow[ul, bend left, "\alpha"] \arrow[u, bend left=60, "\chi"] \arrow[r, "\mathrm{id}_R"'] & R \arrow[u, bend left=60, "e"] \arrow[u, bend right=60, "\xi"'] \arrow[loop below, "\dot{\beta}_\chi, \dot{\gamma}"]
\end{tikzcd}$$

2.3 The group-groupoid equivalent of derivations and sections

(This Subsection (for now) covers only identity derivations and sections.)

The cat^1-formula (21) for Whitehead composition of sections is

$$\mathbf{S2:} \qquad (\xi_1 \star \xi_2)r \;=\; (\xi_1 r)(eh\xi_1 r^{-1})(\xi_2 h\xi_1 r) \;=\; (\xi_2 h\xi_1 r)(eh\xi_1 r^{-1})(\xi_1 r)\ ,$$

which is rather obscure. Considering the group-groupoid $\mathcal{G}$ associated to the cat^1-group $\mathcal{C}$, as discussed in Subsection 1.13, we see that sections of $\mathcal{C}$ are associated to automorphisms of $\mathcal{G}$.

A section ξ of $\mathcal{C}$ defines a groupoid endomorphism $\lambda = \lambda_\xi : \mathcal{G} \to \mathcal{G}$ as follows. Consider the diagram

$$\begin{tikzcd}
h\xi tg \arrow[r, "\lambda g"] & h\xi hg \\
tg \arrow[u, "\xi tg"] \arrow[r, "g"'] & hg \arrow[u, "\xi hg"']
\end{tikzcd} \tag{23}$$

where $t\lambda g = h\xi tg$ and $h\lambda g = h\xi hg$. The morphism λ is defined on objects and arrows by

$$\lambda r \;=\; h\xi r, \qquad \lambda g \;=\; (\widetilde{\xi tg}) * g * \xi hg \;=\; (eh\xi tg)(\xi tg^{-1})g(ehg^{-1})(\xi hg). \tag{24}$$

The product of the first four terms is in $\ker h$, while the product of the last four terms is in $\ker t$. It is easily verified that λ is a groupoid morphism. If $r_0 = tg_1,\ r_1 = hg_1 = tg_2$ and $r_2 = hg_2$, then

$$\begin{aligned}
(\lambda g_1) * (\lambda g_2) &= (eh\xi r_0)(\xi r_0^{-1})g_1(er_1^{-1})(\xi r_1).(eh\xi r_1^{-1}).(eh\xi r_1)(\xi r_1^{-1})g_2(er_2^{-1})(\xi r_2) \\
&= (eh\xi r_0)(\xi r_0^{-1})(g_1 * g_2)(er_2^{-1})(\xi r_2) \\
&= \lambda(g_1 * g_2).
\end{aligned}$$

When we consider ξ_1 followed by ξ_2 we get

$$\begin{tikzcd}
h\xi_2h\xi_1tg \arrow[r, "\lambda_2\lambda_1 g"] & h\xi_2h\xi_1hg \\
h\xi_1tg \arrow[u, "\xi_2h\xi_1tg"] \arrow[r, "\lambda_1 g"] \arrow[loop left, "eh\xi_1tg"'] & h\xi_1hg \arrow[u, "\xi_2h\xi_1hg"'] \arrow[loop right, "eh\xi_1hg"] \\
tg \arrow[u, "\xi_1tg"] \arrow[r, "g"] & hg \arrow[u, "\xi_1hg"']
\end{tikzcd}$$

and the composite on the left-hand side is

$$(\xi_1tg) * (\xi_2h\xi_1tg) \;=\; (\xi_1tg)(eh\xi_1tg^{-1})(\xi_2h\xi_1tg)$$

in agreement with **S2:**, and similarly for the right-hand side. Thus $\lambda_{\xi_1\star\xi_2} = \lambda_{\xi_1} * \lambda_{\xi_2}$ and we have the following result.

Lemma 2.10 *There is a monoid homomorphism*

$$\mathrm{Sect}(\mathcal{C}) \to \mathrm{End}(\mathcal{G}), \qquad \xi \mapsto \lambda_\xi$$

which restricts to a homomorphism $W(\mathcal{C}) \to \mathrm{Aut}(\mathcal{G})$.

Associated to a principal derivation $\eta_s r = (s^{-1})^r s$, and the corresponding principal section $\kappa_s r = (er)^s$, there is a *principal endomorphism* λ_s of $\mathcal{G}$.

Proposition 2.11 *The principal endomorphism* $\lambda_s = \lambda_{\kappa_s} : \mathcal{G} \to \mathcal{G}$ *is given by*

$$\lambda_s r \;=\; r^{hs}, \qquad \lambda_s g \;=\; g^{hs}.$$

Proof: Applying the formulae in equation (24), and $[\ker t, \ker s] = 1$,

$$\begin{aligned}
\lambda_s r &= h\kappa_s r \;=\; (hs^{-1})r(hs) \;=\; r^{hs}, \\
\lambda_s g &= \widetilde{(\kappa_s tg)} * g * (\kappa_s hg) \\
&= \widetilde{((etg)^s)} * g * (ehg)^s \\
&= (eh(s^{-1}(etg)s)(s^{-1}(etg)s)^{-1}(et(s^{-1}(etg)s)(etg^{-1})g(ehg)^{-1}(ehg)^s \\
&= (ehs^{-1})(etg)(ehs)s^{-1}(etg^{-1})[s][g(ehg)^{-1}]s^{-1}(ehg)s \\
&= (ehs^{-1})(etg)[(ehs)s^{-1}][(etg^{-1})g]s \\
&= (ehs^{-1})g(ehs) \\
&= g^{hs}.
\end{aligned}$$

□

3 The Actor of a Crossed Module

This section is based on the material covered in pages 25-28 of Norrie's thesis [49]. We will, however, be extending her actor crossed module $\mathrm{Act}(\mathcal{X})$ to the more general $\mathrm{Act}_\gamma(\mathcal{X})$ where γ is an automorphism of $\mathcal{X}$. Here is a table giving Norrie's symbols and the ones used here.

section	type	old symbol	new symbol
3	xmod	$\partial : T \to G$	$\mathcal{X} = (\partial : S \to R)$
	Whitehead group	$\mathrm{Der}(G,T)$	$W(\mathcal{X})$
	xmod morphism	(σ,θ)	$\beta_\chi \;=\; (\ddot{\beta}_\chi, \dot{\beta}_\chi) \;=\; (\sigma,\rho)$
	principal derivations	$E(G,T)$	$E(\mathcal{X})$
3.6	xmod	$\mu : M \to P$	$\mathcal{X} = (\partial : S \to R)$
	xmod	$\nu : N \to V$	$\mathcal{Y} = (\delta : Q \to P)$
	xmod morphism	$< \epsilon, \rho >$	$\beta \;(= \;(\ddot{\beta}, \dot{\beta})\,)$
	derivation group	$D(P,M)$	$W = W(\mathcal{X})$
	automorphism group	$\mathrm{Aut}(M,P)$	$A = A(\mathcal{X})$
	semidirect product	$(M,P) \sqsupset_{<\epsilon,\rho>} (N,V)$	$\mathcal{Y} \ltimes \mathcal{X}$
	xmod	$\pi \;:\; M \sqsupset N \;\to\; P \sqsupset V$	$(\pi \;:\; Q \ltimes S \to P \ltimes R)$
	automorphism	$\rho(v) = (\rho_1(v), \rho_2(v))$	$\beta(p) = \beta_p : \mathcal{X} \to \mathcal{X}$
	elements	$m,\; n,\; p,\; v$	$s,\; q,\; r,\; p$
	Whitehead boundary	$\Delta(\chi) \;= < \theta_\chi, \ddot{\beta}_\chi >$	$\Delta(\chi) \;=\; \beta_\chi \;=\; (\ddot{\beta}_\chi, \dot{\beta}_\chi)$

When γ is an automorphism of $\mathcal{X}$, the group of automorphisms $\mathrm{Aut}_\gamma\mathcal{X}$, has composition (as in (18)) given by

$$\alpha_1 *_\gamma \alpha_2 \;:=\; (\ddot{\alpha}_1 *_{\ddot{\gamma}} \ddot{\alpha}_2,\, \dot{\alpha}_1 *_{\dot{\gamma}} \dot{\alpha}_2) \;=\; (\ddot{\alpha}_1 * \ddot{\gamma}^{-1} * \ddot{\alpha}_2,\, \dot{\alpha}_1 * \dot{\gamma}^{-1} * \dot{\alpha}_2)\,.$$

3.1 Lue and Norrie crossed modules

We generalise the automorphism crossed module $(\iota : R \to \mathrm{Aut}\,R)$, where ιr is conjugation by r, to the *Norrie crossed module* $\mathcal{N}_\gamma = (i_\gamma : R \to \mathrm{Aut}_\gamma\mathcal{X})$ where:

- the γ-conjugation map is given by $i_\gamma r := \beta_r$ where $\dot{\beta}_r q := (\dot{\gamma}q)^r,\; \ddot{\beta}_r s := (\ddot{\gamma}s)^r$, and
- $\mathrm{Aut}_\gamma\mathcal{X}$ has right actions on R and S given by $r^\alpha \;:=\; \dot{\alpha}\dot{\gamma}^{-1}r, \quad s^\alpha \;:=\; \ddot{\alpha}\ddot{\gamma}^{-1}s\,.$

(An alternative set of definitions is given by $\dot{\beta}_r q = (\dot{\gamma}q)^r,\; \ddot{\beta}_r s = (\ddot{\gamma}s)^r,\; r^\alpha = \dot{\alpha}\dot{\gamma}^{-1}r,\; s^\alpha = \ddot{\alpha}\ddot{\gamma}^{-1}s$, but these do not combine with the principal derivation map to give a morphism of crossed modules.)
Note that β_r^{-1} is given by $\dot{\beta}_r^{-1}q = \dot{\gamma}^{-1}(q^{r^{-1}}),\; \ddot{\beta}_r^{-1}s = \ddot{\gamma}^{-1}(s^{r^{-1}})$.

We now check the various axioms for $\mathcal{N}_\gamma$.
The map i_γ is a homomorphism:

$$\begin{aligned}
(\beta_{r_1} *_\gamma \beta_{r_2})\,q \;&=\; \dot{\beta}_{r_2}\dot{\gamma}^{-1}((\dot{\gamma}q)^{r_1}) \;=\; \dot{\beta}_{r_2}(q^{\dot{\gamma}^{-1}r_1}) \;=\; (\dot{\gamma}(q^{\dot{\gamma}^{-1}r_1}))^{r_2} \;=\; (\dot{\gamma}q)^{r_1r_2} \;=\; \dot{\beta}_{r_1r_2}q\,,\\
(\beta_{r_1} *_\gamma \beta_{r_2})\,s \;&=\; \ddot{\beta}_{r_2}\ddot{\gamma}^{-1}((\ddot{\gamma}s)^{r_1}) \;=\; \ddot{\beta}_{r_2}(s^{\dot{\gamma}^{-1}r_1}) \;=\; (\ddot{\gamma}(s^{\dot{\gamma}^{-1}r_1}))^{r_2} \;=\; (\ddot{\gamma}s)^{r_1r_2} \;=\; \ddot{\beta}_{r_1r_2}s\,.
\end{aligned}$$

The given formulae do specify an action of $\mathrm{Aut}_\gamma\mathcal{X}$ on $\mathcal{X}$:

$$\begin{aligned}
(r^{\alpha_1})^{\alpha_2} \;&=\; \dot{\alpha}_2\dot{\gamma}^{-1}(\dot{\alpha}_1\dot{\gamma}^{-1}r) \;=\; (\dot{\alpha}_1 *_{\dot{\gamma}} \dot{\alpha}_2)\dot{\gamma}^{-1}r \;=\; r^{\alpha_1 *_\gamma \alpha_2}\,,\\
(s^{\alpha_1})^{\alpha_2} \;&=\; \ddot{\alpha}_2\ddot{\gamma}^{-1}(\ddot{\alpha}_1\ddot{\gamma}^{-1}s) \;=\; (\ddot{\alpha}_1 *_{\ddot{\gamma}} \ddot{\alpha}_2)\ddot{\gamma}^{-1}s \;=\; s^{\alpha_1 *_\gamma \dot{\alpha}_2}\,.
\end{aligned}$$

First crossed module axiom (using the γ-conjugation of (17)):

$$\begin{aligned}(\wedge_\gamma\alpha)\dot{\beta}_r q &= \dot{\alpha}\dot{\gamma}^{-1}\dot{\beta}_r(\dot{\alpha}^{-1}\dot{\gamma}q) = \dot{\alpha}\dot{\gamma}^{-1}((\dot{\gamma}\dot{\alpha}^{-1}\dot{\gamma}q)^r) = \dot{\alpha}((\dot{\alpha}^{-1}\dot{\gamma}q)^{\dot{\gamma}^{-1}r}) = (\dot{\gamma}q)^{r^\alpha} = \dot{\beta}_{r^\alpha}q\,,\\ (\wedge_\gamma\alpha)\ddot{\beta}_r s &= \ddot{\alpha}\ddot{\gamma}^{-1}\ddot{\beta}_r(\ddot{\alpha}^{-1}\ddot{\gamma}s) = \ddot{\alpha}\ddot{\gamma}^{-1}((\ddot{\gamma}\ddot{\alpha}^{-1}\ddot{\gamma}s)^r) = \ddot{\alpha}((\ddot{\alpha}^{-1}\ddot{\gamma}s)^{\dot{\gamma}^{-1}r}) = (\ddot{\gamma}s)^{r^\alpha} = \ddot{\beta}_{r^\alpha}s\,.\end{aligned}$$

Second crossed module axiom:

$$r^{i_\gamma r'} = \dot{\beta}_{r'}\dot{\gamma}^{-1}r = r^{r'}\,.$$

Similarly, for the *Lue crossed module* $\mathcal{L}_\gamma(\mathcal{X}) = (\partial * i_\gamma : S \to \mathrm{Aut}_\gamma\mathcal{X})$,

- the boundary maps $s \in S$ to $\beta_{\partial s}$ where $\dot{\beta}_{\partial s}q = (\dot{\gamma}q)^{\partial s}$, $\ddot{\beta}_{\partial s}s' = (\ddot{\gamma}s')^{\partial s}$, and
- the action of $\mathrm{Aut}_\gamma\mathcal{X}$ on R and S are as above.

The verification of the crossed module axioms for $\mathcal{L}_\gamma\mathcal{X}$ are similar to those for $\mathcal{N}_\gamma\mathcal{X}$.

3.2 The actor crossed module

The missing part of the structure of the actor crossed module $\mathrm{Act}_\gamma(\mathcal{X})$ is a γ-action of the automorphisms on the derivations.

Lemma 3.1 *There is an action of* $\mathrm{Aut}_\gamma(\mathcal{X})$ *on* $W_\gamma(\mathcal{X})$ *given by*

$$\chi^\alpha \;=\; \gamma * \alpha^{-1} * \chi * \gamma^{-1} * \alpha \;:\; R \to S\,, \quad r \mapsto \ddot{\alpha}\ddot{\gamma}^{-1}\chi\dot{\alpha}^{-1}\dot{\gamma}\,r\,,$$

such that $\beta_{\chi^\alpha} = (\wedge_\gamma\alpha)(\beta_\chi)$ *where* $\wedge_\gamma\alpha$ *is the* γ*-conjugation automorphism of (17).*

Proof: We first check the axiom for an action:

$$(\chi^{\alpha_1})^{\alpha_2} \;=\; \gamma*\alpha_2^{-1}*(\gamma*\alpha_1^{-1}*\chi*\gamma^{-1}*\alpha_1)*\gamma^{-1}*\alpha_2 \;=\; \gamma*(\alpha_1*_\gamma\alpha_2)^{-1}*\chi*\gamma^{-1}*(\alpha_1*_\gamma\alpha_2) \;=\; \chi^{(\alpha_1*_\gamma\alpha_2)}.$$

Secondly, we observe that $(\wedge_\gamma\alpha)(\beta_\chi) = \gamma * \alpha^{-1} * \beta_\chi * \gamma^{-1} * \alpha$. □

Definition 3.2 *For* γ *an automorphism of* $\mathcal{X} = (\partial : S \to R)$, *the* actor crossed module over γ of $\mathcal{X}$ *is* $\mathcal{A}_\gamma(\mathcal{X}) = (\Delta_\gamma : W_\gamma \to A_\gamma)$ *where*

- $W_\gamma = W_\gamma(\mathcal{X})$ *is the Whitehead group of invertible derivations*

  $$\chi : R \to S, \quad \text{such that} \quad \chi(qr) = (\chi q)^{\dot{\gamma}r}\,(\chi r) \quad \text{for all} \quad q, r \in R\,, \tag{25}$$

  and with Whitehead multiplication (on the right)

  $$\chi_1 \star_\gamma \chi_2 \;:\; R \to S, \;\; r \mapsto (\chi_2 r)(\chi_1 r)(\chi_2\dot{\gamma}^{-1}\partial\chi_1 r)\;; \tag{26}$$

- $A_\gamma = \mathrm{Aut}_\gamma(\mathcal{X})$ *is the group of automorphisms of* $\mathcal{X}$, *namely those invertible* $\alpha = (\ddot{\alpha}, \dot{\alpha}) : \mathcal{X} \to \mathcal{X}$ *such that*

  $$\dot{\alpha}\partial = \partial\ddot{\alpha}\;, \quad \ddot{\alpha}(s^r) = (\ddot{\alpha}s)^{\dot{\alpha}r} \quad \text{and} \quad \dot{\alpha}(q^r) = (\dot{\alpha}q)^{\dot{\alpha}r} \quad \text{for all } s \in S \text{ and } q, r \in R\,,$$

  with composition $\alpha_1 *_\gamma \alpha_2 := (\ddot{\alpha}_1 *_{\ddot{\gamma}} \ddot{\alpha}_2,\, \dot{\alpha}_1 *_{\dot{\gamma}} \dot{\alpha}_2)$, *and action given by Lemma 3.1.*

- *The boundary map is obtained by restricting the monoid homomorphism* $\Delta_\gamma : \mathrm{Der}_\gamma(\mathcal{X}) \to \mathrm{End}_\gamma(\mathcal{X})$ *of Theorem 2.6 to the regular derivations:*

$$\Delta_\gamma : W_\gamma \to A_\gamma, \quad \chi \mapsto \beta_\chi = (\ddot{\beta}_\chi, \dot{\beta}_\chi) \ ,$$

$$\textit{where} \quad \ddot{\beta}_\chi : S \to S, \quad s \mapsto (\ddot{\gamma}s)(\chi\partial s), \qquad \dot{\beta}_\chi : R \to R, \quad r \mapsto (\dot{\gamma}r)(\partial\chi r)\,. \tag{27}$$

When it is convenient not to distinguish the two group homomorphisms in the crossed module morphism, we write α for both $\ddot{\alpha}$ and $\dot{\alpha}$.

These groups and morphisms are exhibited in the following diagram (the inner morphism $\iota_\gamma = (\ddot{\imath}_\gamma, \dot{\imath}_\gamma)$ is defined in Subsection 3.3 below):

$$\begin{tikzcd}
S \arrow[d, "\partial"'] & S \arrow[l, "{\ddot{\alpha},\, \ddot{\beta}_r,\, \ddot{\beta}_\chi,\, \ddot{\gamma}}"'] \arrow[r, "\ddot{\imath}_\gamma"] \arrow[d, "\partial"'] & W_\gamma \arrow[d, "\Delta_\gamma"] \\
R & R \arrow[l, "{\dot{\alpha},\, \dot{\beta}_r,\, \dot{\beta}_\chi,\, \dot{\gamma}}"] \arrow[r, "\dot{\imath}_\gamma"'] & A_\gamma
\end{tikzcd} \tag{28}$$

Theorem 3.3 *With this action,* $\mathcal{A}_\gamma(\mathcal{X}) = (\Delta_\gamma : W_\gamma \to A_\gamma)$ *is a crossed module.*

Proof: We have already shown that Δ_γ is a group homomorphism.

We verify the first crossed module axiom for $\mathcal{A}_\gamma(\mathcal{X})$ as follows.

$$\textbf{X1:} \qquad \Delta_\gamma(\chi^\alpha) \ = \ (\wedge_\gamma\alpha)(\Delta_\gamma\chi) \ = \ \gamma * \alpha^{-1} * \beta_\chi * \gamma^{-1} * \alpha\,.$$

Now $\Delta_\gamma(\chi^\alpha) \ = \ \beta_{\chi^\alpha}$ where

$$\begin{aligned}
\ddot{\beta}_{\chi^\alpha}s \ = \ (\ddot{\gamma}s)(\chi^\alpha\partial s) \ &= \ (\ddot{\gamma}s)(\ddot{\alpha}\ddot{\gamma}^{-1}\chi\dot{\alpha}^{-1}\dot{\gamma}\partial s) \ = \ (\ddot{\gamma}s)(\ddot{\alpha}\ddot{\gamma}^{-1}\chi\partial\ddot{\alpha}^{-1}\ddot{\gamma}s) \ = \ \ddot{\alpha}\ddot{\gamma}^{-1}((\ddot{\gamma}\ddot{\alpha}^{-1}\ddot{\gamma}s)(\chi\partial\ddot{\alpha}^{-1}\ddot{\gamma}s)) \\
&= \ \ddot{\alpha}\ddot{\gamma}^{-1}\ddot{\beta}_\chi(\ddot{\alpha}^{-1}\ddot{\gamma}s) \ = \ (\gamma * \alpha^{-1} * \beta_\chi * \gamma^{-1} * \alpha)\,s\ , \\
\dot{\beta}_{\chi^\alpha}r \ = \ (\dot{\gamma}r)(\partial\chi^\alpha r) \ &= \ (\dot{\gamma}r)(\partial\ddot{\alpha}\ddot{\gamma}^{-1}\chi\dot{\alpha}^{-1}\dot{\gamma}r) \ = \ (\dot{\gamma}r)(\dot{\alpha}\dot{\gamma}^{-1}\partial\chi\dot{\alpha}^{-1}\dot{\gamma}r) \ = \ \dot{\alpha}\dot{\gamma}^{-1}(\dot{\gamma}\dot{\alpha}^{-1}\dot{\gamma}r)(\partial\chi\dot{\alpha}^{-1}\dot{\gamma}r)) \\
&= \ \dot{\alpha}\dot{\gamma}^{-1}\dot{\beta}_\chi(\dot{\alpha}^{-1}\dot{\gamma}r) \ = \ (\gamma * \alpha^{-1} * \beta_\chi * \gamma^{-1} * \alpha)\,r\ .
\end{aligned}$$

The second crossed module axiom for $\mathcal{A}_\gamma(\mathcal{X})$,

$$\textbf{X2:} \qquad {\chi_1}^{\Delta_\gamma\chi_2} \ = \ \overline{\chi_2} \star_\gamma \chi_1 \star_\gamma \chi_2\,,$$

is verified by showing that $\chi_2 \star_\gamma {\chi_1}^{\Delta_\gamma\chi_2} = \chi_1 \star_\gamma \chi_2$, using Lemma 2.4 (c),

$$(\chi_2 \star_\gamma {\chi_1}^{\Delta_\gamma\chi_2})\,r \ = \ (\chi_2 r)({\chi_1}^{\Delta_\gamma\chi_2}\dot{\gamma}^{-1}\dot{\beta}_{\chi_2}r) \ = \ (\chi_2 r)(\ddot{\beta}_{\chi_2}\ddot{\gamma}^{-1}\chi_1 r) \ = \ (\chi_1 \star_\gamma \chi_2)r\,.$$

□

3.3 The inner morphism

We next describe the morphism of crossed modules $\iota_\gamma = (\ddot{\imath}_\gamma, \dot{\imath}_\gamma) : \mathcal{X} \to \mathcal{A}_\gamma(\mathcal{X})$. The conditions in (1) for ι_γ to be a morphism are:

$$\ddot{\imath}_\gamma(s_1 s_2) \ = \ \ddot{\imath}_\gamma s_1 \star_\gamma \ddot{\imath}_\gamma s_2\,, \qquad \dot{\imath}_\gamma(r_1 r_2) \ = \ \dot{\imath}_\gamma r_1 *_\gamma \dot{\imath}_\gamma r_2\,. \tag{29}$$

The range part i_γ of ι_γ is given in Subsection 3.1 by:

$$i_\gamma \; : \; R \to A_\gamma, \quad r \mapsto \beta_r = (\ddot{\beta}_r, \dot{\beta}_r) : \mathcal{X} \to \mathcal{X}, \quad \ddot{\beta}_r s_0 = (\ddot{\gamma} s_0)^r, \quad \dot{\beta}_r r_0 = (\dot{\gamma} r_0)^r \,.$$

The source part $\ddot{\imath}_\gamma$ of ι_γ maps s to its principal derivation (see Lemmas 2.7, 2.8):

$$\ddot{\imath}_\gamma \; : \; S \to W_\gamma, \quad s \mapsto \eta_s : R \to S, \; r \mapsto (s^{-1})^{\dot{\gamma} r} \, s \,.$$

Theorem 3.4 *The pair of group homomorphisms* $\iota_\gamma = (\ddot{\imath}_\gamma, i_\gamma) \; : \; \mathcal{X} \to \mathcal{A}_\gamma(\mathcal{X})$ *is a morphism of crossed modules.*

Proof: The square commutes if $\Delta_\gamma \, \ddot{\imath}_\gamma = i_\gamma \, \partial$. To verify this we show that $(\Delta_\gamma \ddot{\imath}_\gamma)s = \Delta_\gamma \eta_s = \beta_{\eta_s}$ is the same automorphism of $\mathcal{X}$ as $(i\partial)s = \beta_{\partial s}$. By definition of $\beta_{\eta_s} = (\ddot{\beta}_{\eta_s}, \dot{\beta}_{\eta_s})$ we have:

$$\begin{array}{rcl} \ddot{\beta}_{\eta_s}(s_0) & = & (\ddot{\gamma} s_0)(\eta_s \partial s_0) \; = \; (\ddot{\gamma} s_0)(s^{-1})^{\partial \ddot{\gamma} s_0} s \; = \; s^{-1}(\ddot{\gamma} s_0) s \; = \; (\ddot{\gamma} s_0)^{\partial s} \; = \; \ddot{\beta}_{\partial s}(s_0) \,, \\ \dot{\beta}_{\eta_s}(r_0) & = & (\dot{\gamma} r_0)(\partial \eta_s r_0) \; = \; (\dot{\gamma} r_0) \partial((s^{-1})^{\dot{\gamma} r_0} s) \; = \; (\partial s)^{-1}(\dot{\gamma} r_0)(\partial s) \; = \; (\dot{\gamma} r_0)^{\partial s} \; = \; \dot{\beta}_{\partial s}(r_0) \,. \end{array}$$

Then we check that the action is preserved:

$$\begin{array}{rcl} (\ddot{\imath} s)^{ir}(q) & = & (\eta_s)^{\beta_r} q \; = \; \ddot{\beta}_r \ddot{\gamma}^{-1} \eta_s \dot{\beta}_r^{-1} \dot{\gamma} q \; = \; \ddot{\beta}_r \ddot{\gamma}^{-1} \eta_s \dot{\gamma}^{-1} \left((\dot{\gamma} q)^{r^{-1}} \right) \; = \; \left(\ddot{\gamma} \ddot{\gamma}^{-1} \eta_s (q^{\dot{\gamma}^{-1} r^{-1}}) \right)^r \\ & = & \left((s^{-1})^{r (\dot{\gamma} q) r^{-1}} s \right)^r \; = \; \left((s^r)^{-1} \right)^{\dot{\gamma} q} s^r \; = \; \eta_{(s^r)} q \; = \; \ddot{\imath}(s^r)(q) \,. \end{array}$$

□

The γ-version of the *inner actor crossed module* of $\mathcal{X}$ is the image $\iota_\gamma \mathcal{X}$. The source group consists of the principal γ-derivations, and the range group consists of the γ-conjugation automorphisms. For further details see Norrie's thesis [49].

3.4 The Whitehead crossed module

Lemma 3.5 *There is an action of the Whitehead group* W_γ *on* S *given by*

$$s^\chi \; := \; s^{\beta_\chi} \; = \; \ddot{\beta}_\chi \ddot{\gamma}^{-1} s \; = \; s(\chi \dot{\gamma}^{-1} \partial s)$$

which makes $\mathcal{W}_\gamma(\mathcal{X}) = (\ddot{\imath}_\gamma : S \to W_\gamma)$ *a crossed module.*

Proof: We first verify that this is an action:

$$\begin{array}{rcl} (s^{\chi_1})^{\chi_2} & = & \ddot{\beta}_{\chi_2} \ddot{\gamma}^{-1} \left(s(\chi_1 \dot{\gamma}^{-1} \partial s) \right) \; = \; s(\chi_1 \dot{\gamma}^{-1} \partial s) \, \chi_2 \partial \ddot{\gamma}^{-1} \left(s(\chi_1 \dot{\gamma}^{-1} \partial s) \right) \\ & = & s(\chi_1 \dot{\gamma}^{-1} \partial s) \, \chi_2 \left((\dot{\gamma}^{-1} \partial s)(\dot{\gamma}^{-1} \partial \chi_1 \dot{\gamma}^{-1} \partial s) \right) \\ & = & s(\chi_1 \dot{\gamma}^{-1} \partial s) \, (\chi_2 \dot{\gamma}^{-1} \partial s)^{\partial \chi_1 \dot{\gamma}^{-1} \partial s} (\chi_2 \dot{\gamma}^{-1} \partial \chi_1 \dot{\gamma}^{-1} \partial s) \\ & = & s(\chi_2 \dot{\gamma}^{-1} \partial s)(\chi_1 \dot{\gamma}^{-1} \partial s)(\chi_2 \dot{\gamma}^{-1} \partial \chi_1 \dot{\gamma}^{-1} \partial s) \\ & = & s \left((\chi_1 \star_\gamma \chi_2)(\dot{\gamma}^{-1} \partial s) \right) \; = \; s^{(\chi_1 \star_\gamma \chi_2)} \,. \end{array}$$

The first crossed module axiom $\eta_{(s^\chi)} = \overline{\chi} \star_\gamma \eta_s \star_\gamma \chi$ is verified by checking

$$\begin{array}{rcl} (\eta_s \star_\gamma \chi) \, r & = & (\chi r)(\ddot{\beta}_\chi \ddot{\gamma}^{-1} \eta_s r) \; = \; (\chi r) \left(\ddot{\beta}_\chi \ddot{\gamma}^{-1} ((s^{-1})^{\dot{\gamma} r} s) \right) \; = \; (\chi r)(\ddot{\beta}_\chi \ddot{\gamma}^{-1} s^{-1})^{\dot{\beta}_\chi r} (\ddot{\beta}_\chi \ddot{\gamma}^{-1} s) \\ & = & \left((s^\chi)^{-1} \right)^{\dot{\gamma} r} (\chi r)(s^\chi) \; = \; \left((s^\chi)^{-1} \right)^{\dot{\gamma} r} (s^\chi)(\chi r) \left((s^\chi)^{-1} \right)^{\dot{\gamma}(\dot{\gamma}^{-1} \partial \chi r)} (s^\chi) \\ & = & \left(\eta_{(s^\chi)} r \right) (\chi r) \left(\eta_{(s^\chi)} \dot{\gamma}^{-1} \partial \chi r \right) \; = \; \left(\chi \star_\gamma \eta_{(s^\chi)} \right) \, r \,. \end{array}$$

The second crossed module axiom is verified by:

$$s^{\ddot{\iota}s'} \;=\; s^{\eta_{s'}} \;=\; s^{\beta_{\eta_{s'}}} \;=\; \ddot{\beta}_{\eta_{s'}}\ddot{\gamma}^{-1}s \;=\; s(\eta_{s'}\partial\ddot{\gamma}^{-1}s) \;=\; s[s,s'] \;=\; s^{s'}\,.$$

$\square$

[We might show here that (∂,Δ) is a morphism.]

Lemma 3.6

$$(\eta_s)^{\alpha} \;=\; \eta_{(s^{\alpha})}\;.$$

Proof:

$$\begin{aligned}(\eta_s)^{\alpha}q \;&=\; \ddot{\alpha}\ddot{\gamma}^{-1}\eta_s\dot{\alpha}^{-1}\dot{\gamma}q \;=\; \ddot{\alpha}\ddot{\gamma}^{-1}\left((s^{-1})^{\dot{\gamma}\dot{\alpha}^{-1}\dot{\gamma}q}\,s\right) \;=\; \ddot{\alpha}\left((\ddot{\gamma}^{-1}s^{-1})^{\dot{\alpha}^{-1}\dot{\gamma}q}\,(\ddot{\gamma}^{-1}s)\right)\\ &=\; (\ddot{\alpha}\ddot{\gamma}^{-1}s^{-1})^{\dot{\gamma}q}\,(\ddot{\alpha}\ddot{\gamma}^{-1}s) \;=\; ((s^{\alpha})^{-1})^{\dot{\gamma}q}\,(s^{\alpha}) \;=\; \eta_{(s^{\alpha})}q\,.\end{aligned}$$

$\square$

The right-hand square of morphisms of crossed modules in (28) becomes a *crossed square* $\mathcal{S}_{\gamma}(\mathcal{X})$ (see Example 8.10 for the identity case) when the *crossed pairing* (see Section 7) $\boxtimes\,:\,R\times W_{\gamma}\to S,\;(r,\chi)\mapsto\chi\dot{\gamma}^{-1}r$, is added to the structure.

3.5 The actor of a cat^1-group

(This Subsection (for now) covers only identity derivations and sections.)

The diagram corresponding to equation (28) is

$$\begin{tikzcd} G \arrow[d, shift right=3, "t{,}h"'] \arrow[d, shift right=1] & G \arrow[l, "\bar{\beta}"'] \arrow[r, "\ddot{\kappa}"] \arrow[d, shift right=3, "t{,}h"'] \arrow[d, shift right=1] & A(\mathcal{C})\ltimes W(\mathcal{C}) \arrow[d, shift right=3, "T{,}H"'] \arrow[d, shift right=1] \\ R \arrow[u, shift right=1, "e"'] & R \arrow[l, "\dot{\beta}"] \arrow[r, "\dot{\kappa}"'] \arrow[u, shift right=1, "e"'] & A(\mathcal{C}) \arrow[u, shift right=1, "E"'] \end{tikzcd} \qquad (30)$$

where $W=W(\mathcal{C})$ and $A=A(\mathcal{C})$ are defined as follows:

- W is the group of sections of $\mathcal{C}$ with composition given by equation (21),
- $A=\mathrm{Aut}(\mathcal{C})$ is the group of automorphisms of $\mathcal{C}$,
- $\Delta_{\mathcal{C}}:W\to A,\;\xi\mapsto(\bar{\beta}_{\xi},\dot{\beta}_{\xi})$ (see equations (22)).

Note: T,H,E were previously written $\Delta_t,\Delta_h,\Delta_e$.

The homomorphisms $\ddot{\kappa},\dot{\kappa},T,H,E$ are given as follows.

- $\dot{\kappa}\,:\,R\to A,\quad r\;\mapsto\;\beta'_r=(\bar{\beta}_r,\dot{\beta}_r):\mathcal{C}\to\mathcal{C},\quad \bar{\beta}_rg_0=g_0^r\,,\quad \dot{\beta}_rr_0=r_0^r=r^{-1}r_0r\,,$ using the action of R on G in equation (**??**).
- $\ddot{\kappa}:G\to A\ltimes W,\;g\mapsto(\dot{\kappa}tg,\kappa_g:R\to G)\quad$ where $\;\kappa_g(r)=(er)^{(etg^{-1})g}\;.$
- $T(\beta,\xi)=\beta,\quad H(\beta,\xi)=\beta*\Delta_{\mathcal{C}}(\xi),\quad E(\beta)=(\beta,\mathrm{id}).$

[Add in here the associated cat2-group with groups $(A\ltimes W)\ltimes(R\ltimes S),\;A\ltimes W,\;A\ltimes R,\;A$.]

3.6 Actions of a Crossed Module

The material in the rest of this section is taken, in the main, from Norrie's thesis [49]. Recall that an action of a group H on a group G is a group homomorphism from H to the actor of G. The following definition is a straightforward generalisation.

Definition 3.7 *An action of a crossed module $\mathcal{Y} = (\delta : Q \to P)$ on a crossed module $\mathcal{X} = (\partial : S \to R)$ is a morphism of crossed modules*

$$\alpha = (\ddot{\alpha}, \dot{\alpha}) \quad : \quad \mathcal{Y} \to \mathcal{A}(\mathcal{X}) = \mathrm{Act}(\mathcal{X}) \ ,$$

from $\mathcal{Y}$ to the actor *of $\mathcal{X}$, as in the following diagram.*

$$\begin{tikzcd}
S \arrow[r, "\ddot{\imath}"] \arrow[d, "\partial"'] & W \arrow[d, "\Delta"] & Q \arrow[l, "\ddot{\alpha}"'] \arrow[d, "\delta"] \\
R \arrow[r, "\imath"'] & A & P \arrow[l, "\dot{\alpha}"] \\
\mathcal{X} & \mathcal{A}(\mathcal{X}) & \mathcal{Y}
\end{tikzcd} \tag{31}$$

Here

- $\mathcal{A}(\mathcal{X})$ is the crossed module $(\Delta : W \to A)$ of Subsection 3.2,
- $\ddot{\alpha}q \ = \ \chi_q : R \to S$, a derivation of $\mathcal{X}$;
- $\dot{\alpha}p \ = \ \beta_p = (\ddot{\beta}_p, \dot{\beta}_p)$, an automorphism of $\mathcal{X}$ giving actions of P on S and R :

$$s^p = \ddot{\beta}_p s \qquad \text{and} \qquad r^p = \dot{\beta}_p r \ .$$

We have seen in Theorem 3.4 that $\iota = (\ddot{\imath}, \imath)$ is the *inner action* of $\mathcal{X}$ on itself.

Here are five useful identities.

Lemma 3.8

(a) $\partial(s^p) \ = \ \partial\ddot{\beta}_p s \ = \ \dot{\beta}_p \partial s \ = \ (\partial s)^p$;

(b) $s^q \ = \ s^{\delta q} \ = \ \ddot{\beta}_{\delta q} s \ = \ \ddot{\beta}_{\chi_q} s \ = \ s(\chi_q \partial s)$;

(c) $\dot{\beta}_{\delta q} r \ = \ r^{\delta q} \ = \ r^{\dot{\alpha}\delta q} \ = \ r^{\Delta\chi_q} \ = \ \dot{\beta}_{\chi_q} r \ = \ r(\partial\chi_q r)$;

(d) $\chi_{q_1 q_2} r \ = \ (\chi_{q_2} r)(\chi_{q_1} r)^{q_2}$,

(e) $\chi_{q^p} \ = \ \ddot{\beta}_p \circ \chi_q \circ \dot{\beta}_p^{-1}$.

Proof:

(a) $\partial(s^p) \ = \ \partial\ddot{\beta}_p s \ = \ \dot{\beta}_p \partial s \ = \ (\partial s)^p$.

(b) The action of Q on S is via P:

$$s^q \;=\; s^{\delta q} \;=\; \ddot\beta_{\delta q} s \;=\; \ddot\beta_{\chi_q} s \;=\; s(\chi_q \partial s)\ ,$$

and we may check that

$$\begin{aligned}
(s^{q_1})^{q_2} &= (s(\chi_{q_1}\partial s))^{q_2} \\
&= s(\chi_{q_1}\partial s)\ \chi_{q_2}((\partial s)(\partial(\chi_{q_1}\partial s))) \\
&= s(\chi_{q_1}\partial s)(\chi_{q_2}\partial s)^{\partial \chi_{q_1}\partial s}(\chi_{q_2}\partial\chi_{q_1}\partial s) \\
&= s(\chi_{q_2}\partial s)(\chi_{q_1}\partial s)(\chi_{q_2}\partial\chi_{q_1}\partial s) \\
&= s(\chi_{q_1} * \chi_{q_2})(\partial s) \\
&= s^{q_1 q_2}\ .
\end{aligned}$$

(c) $\dot\beta_{\delta q} r \;=\; r^{\delta q} \;=\; r^{\dot\alpha\delta q} \;=\; r^{\Delta\chi_q} \;=\; \dot\beta_{\chi_q} r \;=\; r(\partial\chi_q r)\ .$

(d) $\chi_{q_1 q_2} r \;=\; (\chi_{q_1} \star \chi_{q_2}) r \;=\; (\chi_{q_2} r)(\ddot\beta_{\chi_{q_2}} \chi_{q_1} r) \;=\; (\chi_{q_2} r)(\chi_{q_1} r)^{q_2}\ .$

(e) $\chi_{q^p} \;=\; \ddot\alpha(q^p) \;=\; (\ddot\alpha q)^{\dot\alpha p} \;=\; (\chi_q)^{\beta_p} \;=\; \ddot\beta_p \circ \chi_q \circ \dot\beta_p^{-1}$ by (3.1).

$\square$

[Could do with some more examples of crossed module actions!]

3.7 Semidirect product of crossed modules

Just as a group action gives rise to a semidirect product group (see subsection 1.10), so a crossed module action gives a semidirect product crossed module.

Definition 3.9 The *crossed module semidirect product* with $\mathcal{Y}$ acting on $\mathcal{X}$ is

$$\mathcal{Y} \ltimes \mathcal{X} \;=\; (\,\pi \,:\, Q \ltimes S \longrightarrow P \ltimes R\,)$$

where

(i) $Q \ltimes S$ is the semidirect product with action $s^q \;=\; s^{\delta q} \;=\; \ddot\beta_{\delta q} s$ (see Lemma 3.8(b) below);

(ii) $P \ltimes R$ is the semidirect product with action $r^p = \dot\beta_p r$;

(iii) $\pi(q,s) \;=\; (\delta q,\, \partial s)$;

(iv) $(q,s)^{(p,r)} \;=\; (q^p,\, (\chi_{q^p}\, r)^{-1}\, s^{pr})$.

Note that the action specified in (iv) is a special case of Proposition 8.17.

Lemma 3.10 *π is a group homomorphism.*

Proof:

$$\begin{aligned}
\pi(\,(q_1,s_1)(q_2,s_2)\,) &= \pi(q_1q_2,\ {s_1}^{q_2}\, s_2) \\
&= (\delta(q_1q_2),\ \partial(\ddot\beta_{\delta q_2} s_1)(\partial s_2)) \quad \text{by (i)} \\
\pi(q_1,s_1)\ \pi(q_2,s_2) &= (\delta q_1, \partial s_1)\ (\delta q_2, \partial s_2) \\
&= (\,(\delta q_1)(\delta q_2),\ (\partial s_1)^{\delta q_2}\,(\partial s_2)\,) \\
&= (\delta(q_1q_2),\ (\dot\beta_{\delta q_2}\partial s_1)\,(\partial s_2)\,) \quad \text{by (ii).}
\end{aligned}$$

and the two right-hand sides are equal since $\partial\ddot\beta_p = \dot\beta_p\partial$ for all $p \in P$. $\square$

Theorem 3.11 *The crossed module $\mathcal{Y} \ltimes \mathcal{X}$ as defined* does *satisfy the two crossed modules axioms.*

Proof: Verification of the first axiom:

$$\begin{array}{rcll}
(p,r)^{-1}\,\pi(q,s)\,(p,r) & = & (p,r)^{-1}\,(\delta q, \partial s)\,(p,r) & \\
& = & (p^{-1},\,(r^{-1})^{p^{-1}})\,((\delta q)p,\,(\partial s)^{p}\,r\,) & \\
& = & (p^{-1}(\delta q)p,\,(r^{-1})^{p^{-1}(\delta q)p}\,(\partial s)^{p}\,r\,) & \\
& = & (\,\delta(q^{p}),\,(r^{\delta q^{p}})^{-1}\,(\partial s^{p})\,r\,) & \\
& = & (\,\delta(q^{p}),\,(r\,(\partial \chi_{q^{p}}\,r))^{-1}\,(\partial s^{p})\,r\,) & \text{(by 3.8(c))} \\
& = & (\,\delta(q^{p}),\,(\partial \chi_{q^{p}}\,r)^{-1}\,\partial(s^{pr}\,) & \\
& = & (\,\delta(q^{p}),\,\partial((\chi_{q^{p}}\,r)^{-1}\,s^{pr})\,) & \\
& = & \pi(\,(q,s)^{(p,r)}\,)\ . &
\end{array}$$

Verification of the second axiom:

$$\begin{array}{rcll}
(q,s)^{-1}\,(q_1,s_1)\,(q,s) & = & (q^{-1},\,((s^{-1})^{q^{-1}})\,(q_1q,\,{s_1}^{q}s\,) & \\
& = & (\,q^{-1}q_1q,\,(s^{-1})^{q^{-1}q_1q}\,{s_1}^{q}\,s\,) & \\
& = & (\,{q_1}^{\delta q},\,(s^{-1})^{{q_1}^{\delta q}}\,{s_1}^{q}\,s\,) & \\
& = & (\,{q_1}^{\delta q},\,s^{-1}\,(\chi_{({q_1}^{\delta q})}\,(\partial s^{-1}))\,s\,s^{-1}\,{s_1}^{(\delta q)}\,s\,) & \text{(by 3.8(b))} \\
& = & (\,{q_1}^{\delta q},\,(\chi_{({q_1}^{\delta q})}\,(\partial s^{-1}))^{\partial s}\,{s_1}^{(\delta q)(\partial s)}\,) & \\
& = & (\,{q_1}^{\delta q},\,(\chi_{({q_1}^{\delta q})}\,\partial s)^{-1}\,{s_1}^{(\delta q)(\partial s)}\,) & \text{(by 2.2(a))} \\
& = & (q_1,s_1)^{(\delta q,\,\partial s)} & \text{(by 3.9(iv))} \\
& = & (q_1,s_1)^{\pi(q,s)} & \text{(by 3.9(iii))}\ .
\end{array}$$

□

3.8 Actions of a cat^1-group

Definition 3.12 *To be added.*

4 Groupoids

4.1 Basic definitions

A *groupoid* is a category in which every arrow is invertible. Thus a groupoid $\mathbb{C} = (C_1, C_0)$ consists of the following:

- a set $\mathrm{Ob}(\mathbb{C}) = C_0$ of *objects*,
- a set $\mathrm{Arr}(\mathbb{C}) = C_1$ of *arrows*,
- source and target maps $s, t : C_1 \to C_0$, so that we write $(a : u \to v)$ whenever $sa = u$ and $ta = v$, and we denote by $\mathbb{C}(u, v)$ the set of arrows with source u and target v,
- an *identity arrow* 1_u at each object u, with $s1_u = t1_u = u$,
- an associative partial composition $\diamond : C_1 \times_0 C_1 \to C_1$, with $a \diamond b$ defined whenever $ta = sb$, such that $s(a \diamond b) = sa$ and $t(a \diamond b) = tb$, so that $\mathbb{C}(u) := \mathbb{C}(u, u)$ is a group, called the *object group* at u,
- for each arrow $(a : u \to v)$ an inverse arrow $(a^{-1} : v \to u)$ such that $a \diamond a^{-1} = 1_u$ and $a^{-1} \diamond a = 1_v$.

It will often be convenient to omit the symbol $\diamond$ and use simple juxtaposition to indicate composition. (In our GAP implementation source and target are called *tail* and *head*.)

A morphism of groupoids, as for general categories, is called a functor. Thus a *functor* $\phi = (\phi_1, \phi_0) : \mathbb{C} \to \mathbb{D}$ is a pair of maps $\phi_1 : C_1 \to D_1$ and $\phi_0 : C_0 \to D_0$ such that $\phi_1 1_u = 1_{\phi_0 u}$ and $\phi_1(a \diamond b) = (\phi_1 a) \diamond (\phi_1 b)$ whenever the composite arrow is defined. It is often convenient to omit the subscripts $0, 1$ since it should be clear from the context whether an object or an arrow is being mapped. A morphism ϕ is *injective* and/or *surjective* if both ϕ_0, ϕ_1 are.

Example 4.1 *A group is a groupoid with a single object (usually written* $*$*). This gives a functor* Groupoids *from* **Gp** *to* **Gpd**.

Example 4.2 *For* X *a set, the* trivial groupoid $\mathbb{O}(X)$ on X *has* $\mathrm{Ob}(\mathbb{O}) = X$ *and* $\mathrm{Arr}(\mathbb{O}) = \{1_u \mid u \in \mathrm{Ob}(\mathbb{O})\}$. *We denote* $\mathbb{O}(\{1, \ldots, n\})$ *by* $\mathbb{O}_n$.

Example 4.3 *The* unit groupoid $\mathbb{I}$ *has two objects* $0, 1$ *and four arrows. The two non-identity arrows are* $(\iota : 0 \to 1)$ *and its inverse* $(\iota^{-1} : 1 \to 0)$.

The *underlying digraph* of a groupoid is obtained by forgetting the composition, so the objects become vertices, the arrows become arcs, while the source and target maps keep their usual digraph meaning. A groupoid is *connected* if its underlying digraph is connected.

Example 4.4 *The* tree groupoid $\mathbb{I}_n$ *has* n *objects* $\{1, 2, \ldots, n\}$ *and* n^2 *arrows* $\{(p, q) \mid 0 \leqslant p, q \leqslant n\}$ *where* $s(p, q) = p$, $t(p, q) = q$, $(p, q) \diamond (q, r) = (p, r)$, *and* $(p, q)^{-1} = (q, p)$. *Note that* $\mathbb{I}_2 \cong \mathbb{I}$. *We also write* $\mathbb{I}(X)$ *for the tree groupoid on a set of objects* X. *The underlying digraph of* $\mathbb{I}_n$ *is complete.*

The name tree groupoid comes from the fact that a subset of arrows which form a spanning tree in the underlying digraph generate the whole groupoid using composition and inversion. For example, taking the subset $X_n = \{(1, p) \mid 2 \leqslant p \leqslant n\}$, *we have* $(q, r) = (q, 1)^{-1} \diamond (1, r)$.

The *product* $\mathbb{C} \times \mathbb{D}$ of groupoids $\mathbb{C}, \mathbb{D}$ has objects $C_0 \times D_0$, arrows $C_1 \times D_1$, and composition $(a_1, b_1) \diamond (a_2, b_2) = (a_1 \diamond a_2, b_1 \diamond b_2)$, so that $(a, b)^{-1} = (a^{-1}, b^{-1})$.

Example 4.5 *If $\mathbb{G}$ is a group, considered as a one-object groupoid, and $\mathbb{I}_n$ is a tree groupoid, then $\mathbb{C} = \mathbb{G} \times \mathbb{I}_n$ may be thought of as the groupoid with n objects $\{1, 2, \ldots, n\}$ and $n^2|G|$ arrows $\{(p, g, q) \mid g \in G, 1 \leqslant p, q \leqslant n\}$, with $t(p, g, q) = p$, $h(p, g, q) = q$, composition $(p, g, q) \diamond (q, h, r) = (p, gh, r)$, and inverses $(p, g, q)^{-1} = (q, g^{-1}, p)$. A generating set for $\mathbb{G}$ is given by $\{(1, g, 1) \mid g \in X_G\} \cup X_n$ where X_G is any generating set for $\mathbb{G}$. Every finite, connected groupoid is isomorphic to a direct product of a group and a tree groupoid in this way, and we call such a representation a* standard connected groupoid.

A groupoid $\mathbb{A}$ is *abelian* if and only if all its object groups are abelian.

We now describe the construction of a free groupoid on a graph. Let D be a digraph with vertices $V = V(D)$, arcs $A^+ = A(D)$, and source and target maps $s, t : A^+ \to V$. Let $A^- = \{a^- \mid a^+ \in A^+\}$ be a copy of A^+, and let $A = A^+ \cup A^-$. Extend s, t to A by defining $sa^- = ta^+$, $ta^- = sa^+$. Consider A as an alphabet with A^* the monoid of words in A under concatenation. A word $w = a_1^{\epsilon_1} a_2^{\epsilon_2} \ldots a_k^{\epsilon_k} \in A^*$, where $\epsilon_i \in \{+, -\}$, is *composable* if $ha_i^{\epsilon_i} = ta_{i+1}^{\epsilon_{i+1}}$ for all $1 \leqslant i < k$. The *free groupoid* $\mathbb{D}$ on D is defined by:

- the object set is $\mathrm{Ob}(\mathbb{D}) = V(D)$,
- $\mathrm{Arr}(\mathbb{D})$ is the set of all composable words in A^*,
- $s(a_1^{\epsilon_1} a_2^{\epsilon_2} \ldots a_k^{\epsilon_k}) = sa_1^{\epsilon_1}$, and $t(a_1^{\epsilon_1} a_2^{\epsilon_2} \ldots a_k^{\epsilon_k}) = ta_k^{\epsilon_k}$,
- $w_1 \diamond w_2$ is the concatenation of w_1 and w_2, defined if the result is a composable word.

A groupoid $\mathbb{D}$ is *free* if it is isomorphic to the free groupoid on some digraph D.

A *subgroupoid* $\mathbb{S} = (S_1, S_0)$ of $\mathbb{C} = (C_1, C_0)$ is a groupoid with $S_0 \subseteq C_0$, $S_1 \subseteq C_0$, having the same source, target and composition. A subgroupoid $\mathbb{S}$ is *full* if $\mathbb{S}(u, v) = \mathbb{C}(u, v)$ for all $u, v \in S_0$ and *wide* if $\mathrm{Ob}(\mathbb{S}) = \mathrm{Ob}(\mathbb{C})$. The *(connected) components* of $\mathbb{C}$ are its maximal connected subgroupoids, with one component $\mathbb{C}_i$ for each of the k connected components Γ_i of the underlying digraph. We write $\mathbb{C} = \mathbb{C}_1 \cup \cdots \cup \mathbb{C}_k$. A groupoid whose components all have a single object is a *union of groups*, and is said to be *totally disconnected.* We denote by $\mathrm{ids}(\mathbb{C})$ the wide trivial subgroupoid $\mathbb{O}(\mathrm{Ob}(\mathbb{C}))$.

Given a wide subgroupoid $\mathbb{S} \subseteq \mathbb{C}$, we may define a relation $\equiv_R$ on $\mathrm{Arr}(\mathbb{C})$ by $c \equiv_R c' \Leftrightarrow c = a \diamond c'$ for some $a \in \mathrm{Arr}(\mathbb{S})$. This is an equivalence relation since:

- $c = 1_{sc} \diamond c$, since $\mathbb{S}$ contains all the identity loops in $\mathbb{C}$,
- $c \equiv_R c' \Rightarrow c = a \diamond c' \Rightarrow a^{-1} \diamond c = c' \Rightarrow c' \equiv_R c$, so $\equiv_R$ is symmetric,
- $(c_1 \equiv_R c_2,\ c_2 \equiv_R c_3) \ \Rightarrow\ (c_1 = a_1 \diamond c_2,\ c_2 = a_2 \diamond c_3) \ \Rightarrow\ c_1 = a_1 \diamond a_2 \diamond c_3 \ \Rightarrow\ c_1 \equiv_R c_3$.

The equivalence classes $\mathbb{S}c$ for this relation are called the *right cosets of* $\mathbb{S}$ *in* $\mathbb{C}$.

For $u \in \mathrm{Ob}(\mathbb{C})$ the *star* $\mathrm{Star}(u)$ of u is the set $\{a \in \mathrm{Arr}(\mathbb{C}) \mid sa = u\}$, the set of all arrows with source u. Similarly the *costar* $\mathrm{Costar}(u)$ of u is the set $\{a \in \mathrm{Arr}(\mathbb{C}) \mid ta = u\}$, the set of all arrows with target u. Note that each right coset of $\mathbb{S}$ in $\mathbb{C}$ is a subset of a costar. We may define a second equivalence relation $\equiv_L$ on $\mathrm{Arr}(\mathbb{C})$ by $c \equiv_L c' \Rightarrow c = c'a$ for some $a \in \mathrm{Arr}(\mathbb{S})$. The equivalence classes $c\mathbb{S}$ for this relation are the *left cosets of* $\mathbb{S}$ *in* $\mathbb{C}$, and this time each class is a subset of some star.

Example 4.6 *Let $\mathbb{G} = \mathsf{Groupoids}(G = \langle a, b \mid a^3, b^2, (ab)^2\rangle)$, where we write e for the identity in G, and let $\mathbb{C} = \mathbb{G} \times \mathbb{I}_3$. Let $\mathbb{S}$ be the union of $\mathsf{Groupoids}(C_2) \times \mathbb{I}(\{1, 2\})$ with $\mathsf{Groupoids}(C_3)$ at object 3, where $C_2 = \{e, b\}$ and $C_3 = \{e, a, a^2\}$ are subgroups of the symmetric group G. The 54 arrows in $\mathbb{C}$ form 15 right cosets of $\mathbb{S}$ in $\mathbb{C}$, as shown in the following table. Note that $|\mathrm{Costar}(1)| = 4 = |\mathrm{Costar}(2)|$ while $|\mathrm{Costar}(3)| = 3$, so some cosets contain 4 arrows and some 3. Note also that the 11 arrows in $\mathbb{S}$ are partitioned into the first, second and fourteenth coset.*

$$\begin{array}{ll}
\mathbb{S}(1,e,1) = \{(1,e,1),(1,b,1),(2,e,1),(2,b,1)\} & \mathbb{S}(1,e,2) = \{(1,e,2),(1,b,2),(2,e,2),(2,b,2)\} \\
\mathbb{S}(1,a,1) = \{(1,a,1),(1,a^2b,1),(2,a,1),(2,a^2b,1)\} & \mathbb{S}(1,a,2) = \{(1,a,2),(1,a^2b,2),(2,a,2),(2,a^2b,2)\} \\
\mathbb{S}(1,a^2,1) = \{(1,a^2,1),(1,ab,1),(2,a^2,1),(2,ab,1)\} & \mathbb{S}(1,a^2,2) = \{(1,a^2,2),(1,ab,2),(2,a^2,2),(2,ab,2)\} \\
\mathbb{S}(1,e,3) = \{(1,e,3),(1,b,3),(2,e,3),(2,b,3)\} & \mathbb{S}(1,a,3) = \{(1,a,3),(1,a^2b,3),(2,a,3),(2,a^2b,3)\} \\
\mathbb{S}(1,a^2,3) = \{(1,a^2,3),(1,ab,3),(2,a^2,3),(2,ab,3)\} & \\
\mathbb{S}(3,e,1) = \{(3,e,1),(3,a,1),(3,a^2,1)\} & \mathbb{S}(3,b,1) = \{(3,b,1),(3,ab,1),(3,a^2b,1)\} \\
\mathbb{S}(3,e,2) = \{(3,e,2),(3,a,2),(3,a^2,2)\} & \mathbb{S}(3,b,2) = \{(3,b,2),(3,ab,2),(3,a^2b,2)\} \\
\mathbb{S}(3,e,3) = \{(3,e,3),(3,a,3),(3,a^2,3)\} & \mathbb{S}(3,b,3) = \{(3,b,3),(3,ab,3),(3,a^2b,3)\}
\end{array}$$

A subgroupoid $\mathbb{N}$ of $\mathbb{C}$ is *normal in* $\mathbb{C}$, written $\mathbb{N} \trianglelefteq \mathbb{C}$, if $\mathrm{Ob}(\mathbb{N}) = \mathrm{Ob}(\mathbb{C})$ and $a^{-1}\mathbb{N}(u)a \trianglelefteq \mathbb{C}(v)$ for all $(a : u \to v) \in \mathbb{C}(u, v)$.

Example 4.7 *Let $\mathbb{C} = \mathbb{G} \times \mathbb{I}_n$, as in Example 4.5 with $\mathbb{G} = \mathsf{Groupoids}(G)$.*
If $N \trianglelefteq G$, then we may construct:

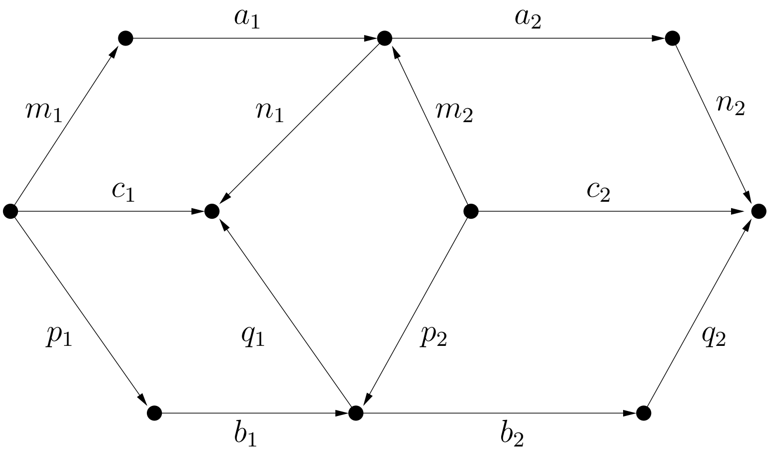
- *a normal subgroupoid $\mathsf{Groupoids}(N) = \mathbb{N} \trianglelefteq \mathbb{G}$,*
- *for each partition $\pi = \pi_1 \cup \cdots \cup \pi_k$ of $\{1, 2, \ldots, n\}$ into k parts, a normal subgroupoid $(\mathbb{N} \times \mathbb{I}(\pi_1)) \cup \cdots \cup (\mathbb{N} \times \mathbb{I}(\pi_k))$.*

The two extreme cases of the second construction are when π has n singleton parts, giving the totally disconnected normal subgroupoid $(\mathbb{N} \times \mathbb{O}_n) \trianglelefteq \mathbb{C}$, and when π has a single part, giving a connected normal subgroupoid $(\mathbb{N} \times \mathbb{I}_n) \trianglelefteq \mathbb{C}$.

When $\mathbb{N}$ is a normal subgroupoid of $\mathbb{C}$ it is *not* in general the case that left cosets coincide with right cosets. A different equivalence relation is therefore required on $\mathrm{Arr}(\mathbb{C})$ in order to be able to define a *quotient groupoid*. The following material is taken from Higgins [39]. Define a relation $\equiv$ on $\mathrm{Arr}(\mathbb{C})$ by $c \equiv c' \Leftrightarrow c = m \diamond c' \diamond n$ for some $m, n \in \mathrm{Arr}(\mathbb{N})$. This is an equivalence relation since

- $c = 1_{sc} \diamond c \diamond 1_{tc}$ since $\mathbb{N}$ contains all the identity loops in $\mathbb{C}$,
- $c \equiv c' \Rightarrow c = m \diamond c' \diamond n \Rightarrow m^{-1} \diamond c \diamond n^{-1} = c' \Rightarrow c' \equiv c$, so $\equiv$ is symmetric,
- $c_1 \equiv c_2, c_2 \equiv c_3 \;\Rightarrow\; c_1 = m_1 \diamond c_2 \diamond n_1, c_2 = m_2 \diamond c_3 \diamond n_2 \;\Rightarrow\; c_1 = m_1 \diamond m_2 \diamond c_3 \diamond n_2 \diamond n_1 \;\Rightarrow\; c_1 \equiv c_3$.

Note that equivalent arrows have sources in the same component of $\mathbb{N}$ and similarly for their targets, so we define an equivalence relation $\equiv_0$ on $\mathrm{Ob}(\mathbb{C})$ by $u \equiv_0 v$ if u, v are in the same component of $\mathbb{N}$. We denote the $\equiv_0$-class of $u \in \mathrm{Ob}(\mathbb{C})$ by $\overline{u}$ and the $\equiv$-class of $c \in \mathrm{Arr}(\mathbb{C})$ by $\overline{c}$. The quotient groupoid $\mathbb{Q} = \mathbb{C}/\mathbb{N}$ has $\mathrm{Ob}(\mathbb{Q}) = \mathrm{Ob}(\mathbb{C})/\equiv_0$ and $\mathrm{Arr}(\mathbb{Q}) = \mathrm{Arr}(\mathbb{C})/\equiv$. Source and target are given by $s\overline{c} = \overline{sc}$, $t\overline{c} = \overline{sc}$ (it is clear that s, t are well-defined). Arrows $\overline{c}_1, \overline{c}_2$ are defined to be composable if there exist $a_1 \equiv c_1$, $a_2 \equiv c_2$ with $a_1 \diamond a_2$ defined in $\mathbb{C}$, in which case $\overline{c_1} \diamond \overline{c_2}$ is defined to be $\overline{a_1 \diamond a_2}$. This composition is well-defined since, if there exist $b_1 \equiv c_1$, $b_2 \equiv c_2$ with $b_1 \diamond b_2$ defined in $\mathbb{C}$, then $b_1b_2 = (p_1^{-1}m_1)a_1(n_1q_1^{-1}p_2^{-1}m_2)a_2(n_2q_2^{-1})$ where $m_i, n_i, p_i, q_i \in \mathrm{Arr}(\mathbb{N})$, as shown in the diagram below. Since $\ell = n_1q_1^{-1}p_2^{-1}m_2 \in \mathbb{N}(sa_2)$, normality of $\mathbb{N}$ implies that $a_2^{-1}\ell a_2 \in \mathbb{N}(ta_2)$ so that $b_1b_2 = (p_1^{-1}m_1)a_1a_2(a_2^{-1}\ell a_2n_2q_2^{-1})$, and $b_1b_2 \equiv a_1a_2$.



Example 4.8 (a) *When* $N \trianglelefteq G$, $\mathbb{G} = \mathsf{Groupoids}(G) \times \mathbb{I}_n$ *and* $\mathbb{N} = \mathsf{Groupoids}(N) \times \mathbb{I}_n$, *the quotient groupoid is the single-object* $\mathsf{Groupoids}(G/N)$.

(b) *When* $\mathbb{G}$ *is as in (a), but* $\mathbb{N}$ *is totally disconnected, with* n *copies of* $\mathsf{Groupoids}(N)$, *the quotient groupoid is the connected* $\mathsf{Groupoids}(G/N) \times \mathbb{I}_n$.

(c) *The general case is when* $\mathbb{G}$ *is as above,* $\pi = \pi_1 \cup \cdots \cup \pi_k$ *is a partition of* $\{1, 2, \ldots, n\}$ *into* k *parts, and* $\mathbb{N} = (\mathsf{Groupoids}(N) \times \mathbb{I}(\pi_1)) \cup \cdots \cup (\mathsf{Groupoids}(N) \times \mathbb{I}(\pi_k))$. *Then the quotient groupoid is* $\mathsf{Groupoids}(G/N) \times \mathbb{I}_k$.

The *kernel* $\ker \phi$ of a groupoid morphism $\phi : \mathbb{C} \to \mathbb{D}$ is the set of arrows in C_1 which are mapped to one of the identity arrows in D_1. Since identities are always mapped to identities, this kernel is normal in $\mathbb{C}$.

Example 4.9 *Let* $S_3 = \langle a, b \mid a^3 = b^2 = (ab)^2 = e \rangle$ *be the symmetric group with normal subgroup* $C_3 = \langle a \rangle$, *and let* $C_2 = \langle c \mid c^2 = e \rangle$. *There is a groupoid morphism from* $\mathbb{C} = \mathsf{Groupoids}(S_3) \times \mathbb{I}\{u, v, w\}$ *to* $\mathbb{D} = \mathsf{Groupoids}(C_2) \times \mathbb{I}\{x, y\}$, *mapping* u, v *to* x *and* w *to* y, *and killing* C_3, *defined on generators by*

$$(u, e, v) \mapsto (x, e, x),\ (u, e, w) \mapsto (x, e, y),\ (u, a, u) \mapsto (x, e, x),\ (u, b, u) \mapsto (x, c, x).$$

The kernel $\mathsf{Groupoids}(C_3) \times \mathbb{I}\{u, v\} \cup \mathsf{Groupoids}(C_3) \times \mathbb{I}\{w\}$ *has two components, and* $\mathbb{C}/(\ker \phi) \cong \mathbb{D}$.

If $a \in \mathbb{C}(u)$ and $c \in \mathbb{C}(u, v)$ then the *conjugate* $a^c \in \mathbb{C}(v)$ is defined to be $c^{-1} \diamond a \diamond c$.

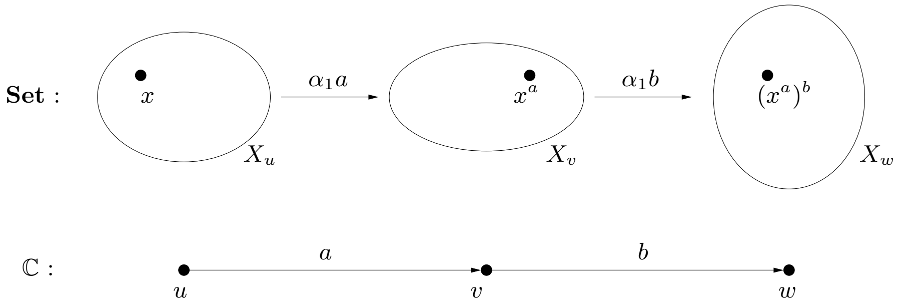
We now consider actions of a groupoid $\mathbb{C}$. We restrict to the case when $\mathbb{C}$ is connected since there is a clear extension to the general case.

For $\mathbb{C}$ a groupoid, a $\mathbb{C}$-set-system (or, by abuse of language, a $\mathbb{C}$-set) is a functor α from $\mathbb{C}$ to **Set**, mapping arrows to bijections. So, for $(a : u \to v) \in \mathrm{Arr}(\mathbb{C})$, there are sets $\alpha_0 u = X_u,\ \alpha_0 v = X_v$ and a bijection $\alpha_1 a : X_u \to X_v$. We also call α an *action of* $\mathbb{C}$ *on* $\bigsqcup_{u \in \mathrm{Ob}(\mathbb{C})} X_u$. If $(b : v \to w)$ is a second arrow in $\mathbb{C}$ and $\alpha_0 w = X_w$, then, since α preserves composition, we have

$$\alpha_1(a \diamond b) \ = \ (\alpha_1 a) * (\alpha_1 b) \ = \ (\alpha_1 b) \circ (\alpha_1 a) \ : \ X_u \to X_w.$$

For $x \in X_u$ we denote, in the usual way, $(\alpha_1 a)(x)$ by x^a, and then the condition becomes $(x^a)^b = x^{a \diamond b}$.

A similar notion applies to sets with structure. For example, $\mathbb{C}$-*graphs* are functors from $\mathbb{C}$ to the groupoid of (combinatorial) graphs and their isomorphisms.

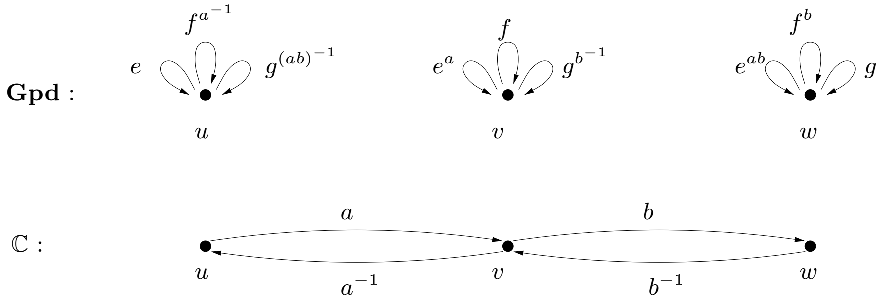


Example 4.10 *Let Γ be a connected graph with automorphism group $A = \mathrm{Aut}\Gamma$. Let Δ be the graph consisting of n copies of Γ, which we may consider as a* graph-system *with n components. The appropriate groupoid to consider is $\mathbb{A} = \mathsf{Groupoids}(A) \times \mathbb{I}_n$, which has an obvious action on Δ. It is reasonable to consider $\mathbb{A}$ to be the automorphism gadget of Δ, rather than its wreath product automorphism group $S_n \wr A$.*

A $\mathbb{C}$-*group-system* (or $\mathbb{C}$-*group*) provides, for each object u a group F_u and, for each $(a : u \to v)$, an isomorphism of groups $\alpha_1 a : F_u \to F_v$. As usual, we write f^a for $(\alpha_1 a)(f)$ when $f \in F$. The group structure has to be preserved so, as well as $(f^a)^b = f^{a \diamond b}$, we require $(e_u)^a = e_v$ and $(f_1 f_2)^a = (f_1^a)(f_2^a)$.

A $\mathbb{C}$-*module* is a $\mathbb{C}$-group in which the F_u are all abelian.

A $\mathbb{C}$-*groupoid-system* is a functor α from $\mathbb{C}$ to **Gpd**, where now there are groupoids $\alpha_0 u = \mathbb{B}_u$, $\alpha_o v = \mathbb{B}_v$ and an invertible functor $\alpha_1 a : \mathbb{B}_u \to \mathbb{B}_v$. As a simple case, note that a $\mathbb{C}$-group determines a $\mathbb{C}$-groupoid on replacing each F_u by $\mathbb{F}_u = \mathsf{Groupoids}(F_u)$, taking u as the single object. Thus a $\mathbb{C}$-module may be consided as an abelian $\mathbb{C}$-groupoid. In these cases f^a is defined when f is a loop at u and then f^a is a loop at v. Here is a picture showing part of the structure.



A particular example, when $\mathbb{C} = \mathsf{Groupoids}(G) \times \mathbb{I}_n$ and $N \trianglelefteq G$, is given by taking $F_u \cong N$ for all $u \in \mathrm{Ob}(\mathbb{C})$ and the action to be conjugation:

$$(p,h,p)^{(p,g,q)} \;=\; (q,g^{-1},p)(p,h,p)(p,g,q) \;=\; (q,g^{-1}hg,q). \tag{32}$$

This will provide our first example of a crossed module of groupoids.

4.2 Automorphisms of Groupoids

An automorphism of a category $\mathbb{C}$ is a functor $\alpha : \mathbb{C} \to \mathbb{C}$ which is an isomorphism. Let $\mathbb{C}$ be the connected groupoid with object set $U = \{u_1, \ldots, u_n\}$ and let $\{(a_p : u_1 \to u_p) \mid 2 \leqslant p \leqslant n\}$ be a generating set for a spanning tree in $\mathbb{C}$. If G_1 is the object group at u_1, an automorphism of $\mathbb{C}$ is obtained on choosing

- $\pi \in \mathrm{Symm}(U)$, permuting the objects in U,
- $\gamma \in \mathrm{Aut}\, G$, permuting the elements of G_1,
- $\{(b_p : u_1 \to u_p) \mid 2 \leqslant p \leqslant n\}$, replacing the a_p in the tree.

Thus there are in total $n! \times |\mathrm{Aut}\, G| \times |G|^{n-1}$ automorphisms of $\mathbb{C}$.

We now analyse the automorphisms of a standard connected groupoid. For G a group, $\mathbb{G} = \mathsf{Groupoids}(G)$, let $\mathbb{C} = \mathbb{G} \times \mathbb{I}_n$ with objects $\{1, \ldots, n\}$; arrows $\{(q, g, r) \mid g \in G,\ 1 \leqslant q, r \leqslant n\}$; composition $(p, h, q) \diamond (q, g, r) = (p, hg, r)$; identities (p, e, p) where e is the identity in G; and inverses $(q, g, r)^{-1} = (r, g^{-1}, q)$. If G has generating set $\Gamma_G = \{g_1, \ldots, g_\ell\}$ then the groupoid is generated by sets

$$\Gamma_p \;=\; \{(p, g_k, p) \mid g_k \in \Gamma_G\} \cup \{(p, e, q) \mid q \neq p\},$$

where the second set forms a spanning tree T_p in the underlying digraph of G. The remaining arrows are given as the composites:

$$\begin{array}{rcll} (p, g, p) & = & (p, g_{k_1}, p)(p, g_{k_2}, p) \ldots (p, g_{k_j}, p) & \text{when } g = g_{k_1} g_{k_2} \ldots g_{k_j} \in G,\ g_{k_i} \in \Gamma_G, \\ (q, g, r) & = & (q, e, p)^{-1}(p, g, p)(p, e, r). & \end{array}$$

An automorphism of $\mathbb{C}$ will be specified by giving the images of the arrows in one of the Γ_p.

There are three sets of automorphisms which generate the group $A = \mathrm{Aut}(\mathbb{C})$.

(1) For π a permutation in the symmetric group S_n we define an automorphism α_π by

$$\alpha_\pi(q, g, r) \;=\; (\pi q, g, \pi r).$$

(2) We may apply an automorphism κ of G to the loops at object 1, giving an automorphism α_κ of $\mathbb{C}$ which fixes the objects, where

$$\alpha_\kappa(1, g, 1) \;=\; (1, \kappa g, 1), \qquad \alpha_\kappa(1, e, q) \;=\; (1, e, q).$$

It follows that $\alpha_\kappa(q, g, r) = (q, \kappa g, r)$, so α_κ applies κ to all the hom-sets at once.

(3) The hom-set $\mathbb{C}(1, q)$ provides a regular representation for G with action $(1, g, q)^b = (1, gb, q)$. For each $q \neq p$ choose $b_q \in G$ and map the arrow $(p, e, q) \in T_p$ to (p, b_q, q). The n-tuple $\boldsymbol{b} = (b_1, \ldots, b_p = e, \ldots, b_n)$ determines an automorphism $\alpha_{p,\boldsymbol{b}}$ of $\mathbb{C}$, fixing the objects, where

$$\alpha_{p,\boldsymbol{b}}(p, g, p) \;=\; (p, g, p), \qquad \alpha_{p,\boldsymbol{b}}(p, e, r) \;=\; (p, b_r, r).$$

For $\boldsymbol{b} \in G^n$, the n-fold direct product of G with itself, we generalise the $\alpha_{p,\boldsymbol{b}}$ by defining

$$\alpha_{\boldsymbol{b}} \;:\; \mathbb{C} \to \mathbb{C}, \quad (q, g, r) \;\mapsto\; (q, b_q^{-1} g b_r, r).$$

This map is a homomorphism since

$$(p, b_p^{-1} h b_q, q)(q, b_q^{-1} g b_r, r) \;=\; (p, b_p^{-1}(hg) b_r, r).$$

Furthermore, there is a homomorphism $\theta : G^n \to \operatorname{Aut}\mathbb{C},\ \boldsymbol{b} \mapsto \alpha_{\boldsymbol{b}}$, since

$$(\alpha_{\boldsymbol{b}} * \alpha_{\boldsymbol{c}})(q, g, r) \;=\; (q, c_q^{-1} b_q^{-1} g b_r c_r, r) \;=\; \alpha_{\boldsymbol{bc}}(q, g, r).$$

For $\boldsymbol{z} \in \ker\theta$ we require $g z_r = z_q g$ for all $g \in G,\ 1 \leqslant q, r \leqslant n$. It follows that $\boldsymbol{z}$ is a constant vector $(z, z, \ldots, z)$ with $z \in Z(G)$, the centre of G. When $\boldsymbol{g} = (g, g, \ldots, g)$ is an arbitrary constant vector in G^n, we see that $\alpha_{\boldsymbol{g}}$ is the type (1) conjugation automorphism $\alpha_{(\wedge g)}$. We denote by $\hat{G}$ the diagonal subgroup in G^n, put $\hat{Z} = \ker\theta$, and define $Q = G^n/\hat{Z}$.

There are actions of both S_n and $\operatorname{Aut} G$ on G^n, where

$$\boldsymbol{b}^{\pi} \;=\; \pi\boldsymbol{b} \;=\; (b_{\pi^{-1}1}, \ldots, b_{\pi^{-1}ip}, \ldots, b_{\pi^{-1}n}), \qquad \boldsymbol{b}^{\kappa} \;=\; \kappa\boldsymbol{b} \;=\; (\kappa b_1, \ldots, \kappa b_p, \ldots, \kappa b_n), \tag{33}$$

and these actions commute, giving an action of $S_n \times \operatorname{Aut} G$ on G^n. We denote by G^n_p the subset $\{\boldsymbol{b} \in G^n \mid b_p = e\}$ and note that $\alpha_{\boldsymbol{b}} = \alpha_{p,\boldsymbol{b}}$ when $\boldsymbol{b} \in G^n_p$. Note also that G^n_p is closed under multiplication in G^n; that $G^n_p \cong G^{n-1}$; that the kernel of θ restricted to G^n_p is trivial; and that S_n and $\operatorname{Aut} G$ act trivially on $\hat{Z}$.

We now investigate conmposites of the set

$$\Gamma_A \;=\; \{\alpha_\pi \mid \pi \in S_n\} \cup \{\alpha_\kappa \mid \kappa \in \operatorname{Aut} G\} \cup \{\alpha_{\boldsymbol{b}} \mid \boldsymbol{b} \in G^n\}.$$

In keeping with the use of right actions, we write $\alpha * \beta$ for the composite mapping $\beta \circ \alpha$. It is straightforward to verify the following identities.

Lemma 4.11 *Pairs of automorphisms in Γ_A compose as follows.*

$$\begin{aligned}
(\alpha_\pi * \alpha_\rho)(q, g, r) \;=\; \alpha_{\pi*\rho}(q, g, r) &\;=\; ((\pi * \rho)q, g, (\pi * \rho)r)\,, \\
(\alpha_\kappa * \alpha_\lambda)(q, g, r) \;=\; \alpha_{\kappa*\lambda}(q, g, r) &\;=\; (q, (\kappa * \lambda)g, r), \\
(\alpha_{\boldsymbol{b}} * \alpha_{\boldsymbol{c}})(q, g, r) \;=\; \alpha_{\boldsymbol{bc}}(q, g, r) &\;=\; (q, (b_q c_q)^{-1} g (b_r c_r), r), \\
(\alpha_\pi * \alpha_\kappa)(q, g, r) \;=\; (\alpha_\kappa * \alpha_\pi)(q, g, r) &\;=\; (\pi q, \kappa g, \pi r), \\
(\alpha_{\boldsymbol{b}} * \alpha_\pi)(q, g, r) \;=\; (\alpha_\pi * \alpha_{\pi\boldsymbol{b}})(q, g, r) &\;=\; (\pi q, b_q^{-1} g b_r, \pi r), \\
(\alpha_\kappa * \alpha_{\boldsymbol{b}})(q, g, r) \;=\; (\alpha_{\kappa^{-1}\boldsymbol{b}} * \alpha_\kappa)(q, g, r) &\;=\; (q, b_q^{-1}(\kappa g) b_r, r).
\end{aligned}$$

Proof: The fifth isomorphism is the least obvious one. When $\pi q = q',\ \pi r = r'$ we obtain

$$\alpha_{\pi\boldsymbol{b}}(q', g, r') \;=\; \left(q', \left(b_{\pi^{-1}q'}\right)^{-1} g \left(b_{\pi^{-1}r'}\right), r'\right) \;=\; (\pi q, b_q^{-1} g b_r, \pi r).$$

□

It is clear that the group A_1 generated by the α_π is isomorphic to the symmetric group S_n; that the group A_2 generated by the α_κ is isomorphic to $\operatorname{Aut} G$; and that the group A_3 generated by the $\alpha_{1,\boldsymbol{b}}$ is isomorphic to G^{n-1}.

It is clear that the join $A_{1,2}$ of A_1 and A_2 in the automorphism group $\operatorname{Aut}\mathbb{C}$ of $\mathbb{C}$ is isomorphic to $A_1 \times A_2$. We denote by $A_{1,3}, A_{2,3}$ the joins of A_1, A_3 and A_2, A_3 respectively.

Proposition 4.12 *The groups $A_{1,3},\ A_{2,3},\ \operatorname{Aut}\mathbb{C}$ are isomorphic to the following semidirect products.*

(1) $A_{2,3} \;\cong\; \mathrm{Aut}\,G \ltimes G^{n-1}$*, where the action on* G_1^n *is defined in equation (33).*

(2) $A_{1,3} \;\cong\; S_n \ltimes Q$*, using the action in (33), so that* $(\kappa, \boldsymbol{b})^{\pi} = (\kappa, \pi\boldsymbol{b})$*.*

(3) $\mathrm{Aut}\,\mathbb{C} \;\cong\; (S_n \times \mathrm{Out}\,G) \ltimes Q$*, using the same action as in (2).*

Proof: The sixth identity in Lemma 4.11 shows that every element of $A_{2,3}$ has the form $\alpha_\kappa * \alpha_{1,\boldsymbol{b}}$. We define an isomorphism, $\theta_1 : A_{2,3} \to \mathrm{Aut}\,G \ltimes G^{n-1}$, by $\alpha_\kappa \mapsto (\kappa, 1),\;\; \alpha_{1,\boldsymbol{b}} \mapsto (1, \boldsymbol{b})$. Then we check that $\theta_1(\alpha_\kappa * \alpha_\lambda) = \theta_1(\alpha_{\kappa*\lambda}) = (\kappa * \lambda, 1)$, that $\theta_1(\alpha_{1,\boldsymbol{b}} * \alpha_{1,\boldsymbol{c}}) = \theta_1(\alpha_{\boldsymbol{bc}}) = (1, \boldsymbol{bc})$, and that

$$\theta_1(\alpha_{1,\kappa^{-1}\boldsymbol{b}} * \alpha_\kappa) \;=\; (1, \kappa^{-1}\boldsymbol{b})(\kappa, 1) \;=\; (\kappa, \kappa^{-1}\boldsymbol{b}^{\kappa}) \;=\; (\kappa, \boldsymbol{b}) \;=\; \theta_1(\alpha_\kappa * \alpha_{1,\boldsymbol{b}}).$$

For the second isomorphism we use Lemma 4.11 to obtain

$$\alpha_\pi * \alpha_{1,\boldsymbol{b}} * \alpha_\rho * \alpha_{1,\boldsymbol{c}} \;=\; \alpha_\pi * \alpha_\rho * \alpha_{\rho\boldsymbol{b}} * \alpha_{\boldsymbol{c}} \;=\; \alpha_{\pi*\rho} * \alpha_{(\rho\boldsymbol{b})\boldsymbol{c}}. \tag{34}$$

Iterating this procedure, we see that every word in the generators of $A_{1,3}$ has a normal form $\alpha_\pi * \alpha_{\boldsymbol{b}}$. We define $\theta'_2 : S_n \ltimes G^n \to A_{1,3},\;\; (\pi, \boldsymbol{b}) \mapsto \alpha_\pi * \alpha_{\boldsymbol{b}}$. This is a homomorphism since, by (34),

$$\theta'_2(\pi, \boldsymbol{b})\theta'_2(\rho, \boldsymbol{c}) \;=\; \alpha_{\pi*\rho} * \alpha_{(\rho\boldsymbol{b})\boldsymbol{c}} \;=\; \theta'_2(\pi * \rho, \boldsymbol{b}^{\rho}\boldsymbol{c}).$$

The kernel of θ'_2 is $\{((\;), \boldsymbol{z}) \mid \boldsymbol{z} \in \hat{Z}\}$, where $(\;)$ denotes the identity permutation, so there is an isomorphism $\theta_2 : A_{1,3} \to S_n \ltimes Q$ mapping generators $\alpha_\pi * \alpha_{1,\boldsymbol{b}}$ to $(\pi, \hat{Z}\boldsymbol{b})$.

To prove the third isomorphism we define $\theta'_3 : (S_n \times \mathrm{Aut}\,G) \ltimes G^n \to \mathrm{Aut}\,\mathbb{C}$ mapping $((\pi, \kappa), \boldsymbol{b})$ to $\alpha_\pi * \alpha_\kappa * \alpha_{\boldsymbol{b}}$. The formulae in Lemma 4.11 again show that every automorphism can be written in this form. The kernel of θ'_3 is generated by elements $(((\;), \mathrm{id}), \boldsymbol{z})$ for $z \in \hat{Z}$ and elements $(((\;), \wedge g), \boldsymbol{g}^{-1})$, so there is an isomorphism $\theta_3 : \mathrm{Aut}\,\mathbb{C} \to (S_n \times \mathrm{Out}\,G) \ltimes Q$ mapping $\alpha_\pi * \alpha_\kappa * \alpha_{1,\boldsymbol{b}}$ to $((\pi, (\mathrm{Inn}\,G)\overline{\kappa}), \hat{Z}\boldsymbol{gb})$ where $\kappa = (\wedge g)\overline{\kappa} \in \mathrm{Aut}\,G$. □

We conclude this subsection by observing that an automorphism $\alpha = (\alpha_1, \alpha_0) : \mathbb{C} \to \mathbb{C}$ is specified by giving

- the permutation α_0 on the objects;
- an automorphism, written $\overline{\alpha}$, of the object group G_1, so that $\alpha_1(1, g, 1) = \overline{\alpha}g$;
- images $\alpha_1(1, e, q) = (\alpha_0 1, \alpha_q, \alpha_0 q)$, with $\alpha_q \in G,\;\; 2 \leqslant j \leqslant n$, for the tree T_1. .

Then α acts on a typical arrow by

$$\alpha_1(q, g, r) \;=\; (\alpha_0 q, \alpha_q^{-1}(\alpha g)\alpha_r, \alpha_0 r).$$

It is clear how to replace object 1 by an arbitrary object p in these formulae.

The next type of groupoid to consider is the disjoint union of m copies of a connected groupoid. More generally, let X be some structure with an equivalence relation $\equiv$ which is preserved by endomorphisms. We argue that the the correct automorphism structure for X is not the group of automorphisms but rather the *groupoid of automorphisms* $\mathbb{A} = \mathbb{A}\mathrm{ut}\,X$ whose objects are the $\equiv$-classes $[x]$. The object group at class $[x]$ is the group of automorphsisms $\mathrm{Aut}\,[x]$ of the elements in $[x]$. The hom-set $\mathbb{A}([x], [y])$ comprises all isomorphisms from $[x]$ to $[y]$ (if there are any). A connected component of $\mathbb{A}$ has the form $\mathrm{Aut}\,[x] \times \mathbb{I}_m$, where m is the number of components of X isomorphic to $[x]$.

A simple example of this situation is a graph Γ with m connected components all isomorphic to Γ_0, having automorphism groupoid $\mathrm{Aut}\,\Gamma \cong \mathrm{Aut}\,\Gamma_0 \times \mathbb{I}_m$.

So let $\mathbb{B}$ be the disjoint union $\mathbb{C} \times \mathbb{O}_n$ of m copies of $\mathbb{C} = G \times \mathbb{I}_n$, and denote the i-th copy by $\mathbb{C}_i$ and its elements by $(q,g,r)_i$. The equivalence relation here is the connectedness of the underlying digraph. There should be no confusion if we take $\mathrm{Ob}(\mathbb{A}\mathrm{u}t\,\mathbb{B})$ to be $\{1,\ldots,m\}$. From our previous discussion we see that $\mathbb{A}\mathrm{u}t\,\mathbb{B} \cong \mathrm{Aut}\,\mathbb{C} \times \mathbb{I}_m \cong \mathsf{Groupoids}((S_n \times \mathrm{Out}\,G) \ltimes Q) \times \mathbb{I}_m$. Generators for this groupoid are provided by:

- $\{(1,\alpha_r,1)\}$, where the α_r are generators for $\mathrm{Aut}\,\mathbb{C}$;
- $\{(1,\epsilon_i,i) \mid 2 \leqslant i \leqslant m\}$, where $\epsilon_i : \mathbb{C}_1 \to \mathbb{C}_i,\ (q,g,r)_1 \mapsto (q,g,r)_i$.

An automorphism of $\mathbb{B}$ which does not interchange the components is obtained by choosing an automorphism for each component. These form a group isomorphic to $(\mathrm{Aut}\,\mathbb{C})^m$, and the automorphism group of $\mathbb{B}$ is the wreath product $S_m \wr \mathrm{Aut}\,G$ with action $(\kappa_1,\ldots,\kappa_m)^\pi = (\kappa_{(\pi^{-1}1)},\ldots,\kappa_{(\pi^{-1}m)})$.

Example 4.13 *We shall see groupoids of the form $\mathbb{B} = \mathbb{G} \times \mathbb{O}_m$, a disjoint union of groups, when we come to consider crossed modules. Clearly $\mathbb{A}\mathrm{u}t\,\mathbb{B} \cong (\mathrm{Aut}\,G) \times \mathbb{I}_m$ and $\mathrm{Aut}\,\mathbb{B} \cong S_m \wr \mathrm{Aut}\,G$.*

The final case to consider is that of an arbitrary groupoid $\mathbb{G}$, whose connected components $\mathbb{G}_j$ form m classes $[\mathbb{G}_i]$ of isomorphic connected groupoids with m_i components in class $[\mathbb{G}_i]$. The automorphism groupoid $\mathrm{Aut}\,\mathbb{G}$ has $\sum_{i=1}^m m_i$ objects and m connected components, with the i-th component being isomorphic to $\mathrm{Aut}\,\mathbb{G}_i \times \mathbb{I}_{m_i}$.

4.3 Natural Transformations

Functors are related by *natural transformations.* If $\alpha, \beta : \mathbb{C} \to \mathbb{D}$ are functors, then a natural transformation $\tau : \alpha \to \beta$ is determined by a function $\tau : \mathrm{Ob}(\mathbb{C}) \to \mathrm{Arr}(\mathbb{D})$, $u \mapsto \tau_u$, such that for every arrow $(a : u \to v) \in \mathbb{C}$ the following diagram commutes.

$$\begin{tikzcd}
\alpha u \arrow[r, "\tau_u"] \arrow[d, "\alpha a"'] \arrow[dr, "\tau a"] & \beta u \arrow[d, "\beta a"] \\
\alpha v \arrow[r, "\tau_v"'] & \beta v
\end{tikzcd}$$

Notice that commutativity of the diagram enables us to extend τ to a function $\mathrm{Arr}(\mathbb{C}) \to \mathrm{Arr}(\mathbb{D})$, where τa is this diagonal arrow and $\tau 1_u = \tau_u$ for each object u. Notice also that $[s\tau_{u_1}, s\tau_{u_2}, \ldots, s\tau_{u_n}]$ is a permutation of the list of objects, otherwise one or more of these diagrams would be undefined.

Restricting to groupoids, so that arrows are invertible, we have $\tau_v = (\alpha a)^{-1} \diamond (\tau_u) \diamond (\beta a)$, so τ is defined if we are given, for each component of $\mathbb{C}$, the image of one object. Furthermore, when α, β are surjective, every transformation is invertible with $(\tau^{-1})_u = (\tau_u)^{-1}$,

$$\begin{tikzcd}
\beta u \arrow[r, "(\tau^{-1})_u = \tau_u^{-1}"] \arrow[d, "\beta a"'] & \alpha u \arrow[d, "\alpha a"] \\
\beta v \arrow[r, "(\tau^{-1})_v = \tau_v^{-1}"'] & \alpha v
\end{tikzcd}$$

In this case the list $[t\tau_{u_1}, t\tau_{u_2}, \ldots, t\tau_{u_n}]$ is a second permutation of the objects in $\mathbb{D}$.

When $\mathbb{C} = \mathbb{D}$ and α, β are isomorphisms, we obtain our first example of a *homotopy*, with τ being considered as a homotopy from α to β, as displayed in the following diagram. The significant feature of τ is that it lifts from one level to the next.

$$\begin{tikzcd}
C_1 \arrow[r, "\alpha_1,\, \beta_1"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & C_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
C_0 \arrow[r, "\alpha_0,\, \beta_0"'] \arrow[ur, "\tau"] & C_0
\end{tikzcd}$$

Natural transformations compose in the obvious way. If κ is a third functor from $\mathbb{C}$ to $\mathbb{D}$, and if $\sigma : \beta \to \kappa$ is a second natural transformation, then we obtain the diagrams:

$$\begin{tikzcd}
\alpha u \arrow[r, "\tau_u"] \arrow[d, "\alpha a"'] \arrow[dr, "\tau a"] & \beta u \arrow[r, "\sigma_u"] \arrow[d, "\beta a"'] \arrow[dr, "\sigma a"] & \kappa u \arrow[d, "\kappa a"] \\
\alpha v \arrow[r, "\tau_v"'] & \beta v \arrow[r, "\sigma_v"'] & \kappa v
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[r, "\alpha_1,\, \beta_1,\, \kappa_1"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & C_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
C_0 \arrow[r, "\alpha_0,\, \beta_0,\, \kappa_0"'] \arrow[ur, "\tau,\, \sigma"] & C_0
\end{tikzcd}$$

We obtain a composite natural transformation $\tau \diamond \sigma \ : \ \alpha \to \kappa$ where

$$\begin{aligned}(\tau \diamond \sigma)u &= \tau_u \diamond \sigma_u, \\ (\tau \diamond \sigma)a &= (\tau a) \diamond (\sigma_v) \ = \ (\tau_u) \diamond (\sigma a) \ = \ (\tau a) \diamond (\beta a)^{-1} \diamond (\sigma a).\end{aligned}$$

We thus obtain a groupoid whose objects are isomorphisms and whose arrows are natural transformations. When $\mathbb{C} = \mathbb{D}$ and we obtain the *automorphism groupoid* of $\mathbb{C}$.

Theorem 4.14 *Let* $\mathbb{C} = G \times \mathbb{I}_n$. *Then the automorphism groupoid* $\mathrm{Aut}\,\mathbb{C}$ *of* $\mathbb{C}$ *has*

- $n!.|\mathrm{Aut}\,G|.|G|^{n-1}$ *objects (automorphisms),*
- $(n!)^2.|\mathrm{Aut}\,G|.|G|^{2n-1}$ *arrows (natural transformations),*
- *degree* $|Z(G)| = |G|/|\mathrm{Inn}\,G|$,
- $\mathrm{Out}\,G$ *connected components, with* $n!.|\mathrm{Inn}\,G|.|G|^{n-1}$ *objects in each component.*

Proof: An automorphism is specified by choosing a permutation of the objects; an automorphism of the group G; and, for each arrow in the spanning tree, a choice of one of the G arrows between the appropriate vertices.

When specifying a natural transformation $\tau : \alpha \to \beta$, the sources of the τ_u determine one of the $n!$ permutations of the objects, and the targets determine a second permutation, and there are then $|G|$ choices for each τu_q. The automorphism α is specified on the object group by choosing an automorphism from $\mathrm{Aut}\,G$ and on each of the $n-1$ arrows $(1, e, q)$ in the tree by choosing one of the $|G|$ arrows $(\alpha_0 1, \alpha_q, \alpha_0 q)$.

The degree is determined by the number of loops at the identity automorphism id. If τ is such a loop and $\tau_1 = z$, then $z^{-1}az = a$ for every generator a of G, so $z \in Z(G)$. Each τ_q is then determined by $\tau_q = \alpha_q^{-1} \diamond z \diamond \alpha_q$.

The number of arrows whose source is id is $n!.|G|^n$ since there are $n!$ choices for the targets of the τ_q, and the $|G|$ choices for each τ_q. Dividing this number by the degree gives the number of objects in the component containing id. The automorphism group acts on the objects of the automorphism groupoid by right multiplication, permuting the components, so the components are isomorphic and their number is the obvious quotient. □

Corollary 4.15 *When* $\mathbb{C}$ *is a group* G *considered as a one-object groupoid, the automorphism groupoid has* $|\mathrm{Aut}\,G|$ *objects;* $|\mathrm{Aut}\,G|.|G|$ *natural transformations;* $|\mathrm{Out}\,G|$ *components;* $|\mathrm{Inn}\,G|$ *objects in each component; and degree* $|Z(G)|$.

4.4 Admissible and coadmissible sections

These two types of section are related to special cases of natural transformations between automorphisms of a groupoid.

For $\mathbb{C}$ a groupoid, an *admissible section* $H_0 : C_0 \to C_1$ is a section of the source map which composes with the target map to give a bijection on C_0,

$$H_0 * s \ = \ \mathrm{id}_{C_0}, \qquad h_0 \ := \ H_0 * t \ : \ C_0 \to C_0 \quad \text{is a bijection.}$$

Note that if $\tau : \mathrm{id}_{\mathbb{C}} \to h$ is a natural transformation, then τ is an admissible section.

The set of admissible sections $M(\mathbb{C})$ of $\mathbb{C}$ is a group with multiplication

$$(H_0 \star J_0)u \ := \ (H_0u)(J_0tH_0u) \ = \ ((H_0u)(J_0h_0u) : u \to (h_0 * j_0)u)$$

where $j_0 = J_0 * t$. It is straightforward to verify that this product is associative,

$$(H_0 \star J_0 \star K_0)u \ = \ ((H_0u)(J_0h_0u)(K_0j_0h_0u) : u \to (h_0 * j_0 * k_0)u).$$

Here is a sketch showing the situation:

$$\begin{tikzcd}
 & & j_0u & & \\
h_0^{-1}u & u \arrow[l, "H_0^{-1}u"'] \arrow[ur, "J_0u"] \arrow[r, "H_0u"] \arrow[dr, "K_0u"'] & h_0u \arrow[r, "J_0h_0u"] & j_0h_0u \arrow[r, "K_0j_0h_0u"] & k_0j_0h_0u \\
 & & k_0u & &
\end{tikzcd}$$

The identity admissible section is I_0 where $I_0u = 1_u$ for all $u \in C_0$. The inverse of H_0 is the admissible section where

$$H_0^{-1}u = (H_0h_0^{-1}u)^{-1}, \qquad \text{so} \quad H_0^{-1}h_0u = (H_0u)^{-1} \quad \text{and} \quad H_0^{-1} * t = h_0^{-1}.$$

Note that the map $M(\mathbb{C}) \to \mathrm{Symm}(C_0)$, $H_0 \mapsto H_0 * t$ is a homomorphism.

Similarly, a *coadmissible section* $H_0 : C_0 \to C_1$ is a section of the target map which composes with the source map to give a bijection of C_0. For a picture of this situation just reverse all the arrows in the diagram above. The multiplication is given by

$$(H_0 \star J_0)u \ := \ (J_0sH_0u)(H_0u) \ = \ ((J_0h_0u)(H_0u) : (h_0 * j_0)u \to u)$$

where $h_0 = H_0 * s$, $j_0 = J_0 * s$. Note that if $\tau : h \to \mathrm{id}_{\mathbb{C}}$ is a natural transformation, then τ is a coadmissible section.

We now generalise these notions. For g_0, h_0 a pair of permutations of the objects of a groupoid $\mathbb{C}$, a (g_0, h_0)*-section* $H_0 : C_0 \to C_1$ is a map which composes with the source and target maps to give g_0 and h_0 respectively:

$$g_0 \ = \ H_0 * s, \qquad h_0 \ = \ H_0 * t.$$

Note that if $\tau : g \to h$ is a natural transformation between automorphisms of $\mathbb{C}$, then τ is an admissible section. A (g_0, h_0)-section is also called an *admissible* g_0*-section* and a *coadmissible* h_0*-section.*

We have constructed a groupoid $\mathbb{S} = \mathbb{S}(\mathbb{C})$ having the automorphisms of $\mathbb{C}$ as objects and the (g_0, h_0)-sections as the elements of the hom-set $\mathbb{S}(g_0, h_0)$. Composition in $\mathbb{S}$ is defined by

$$(H_0 : g_0 \to h_0) \diamond (J_0 : h_0 \to j_0)u \;:=\; (H_0u : g_0u \to h_0u) \diamond (J_0u : h_0u \to J_0u).$$

We now define a multiplication on the set of admissible g_0-sections $M_g(\mathbb{C})$ of $\mathbb{C}$. Note that there is a multiplication on the permutations of C_0 given in terms of the standard composition by $h_0 \star j_0 := h_0 * g_0^{-1} * j_0$, such that g_0 is the identity and h_0 has inverse $g_0 * h_0^{-1} * g_0$. We define the product on $M_g(\mathbb{C})$ by

$$(H_0 \star J_0)u \;:=\; (H_0u)(J_0g_0^{-1}tH_0u) \;=\; ((H_0u)(J_0g_0^{-1}h_0u) : u \to (h_0 \star j_0)u)$$

where $j_0 = J_0 * t$. It is straightforward to verify that this product is associative,

$$(H_0 \star J_0 \star K_0)u \;=\; ((H_0u)(J_0g_0^{-1}h_0u)(K_0g_0^{-1}j_0g_0^{-1}h_0u) : u \to (h_0 \star j_0 \star k_0)u).$$

Here is a sketch showing the situation:

$$\begin{tikzcd}
 & j_0u & & \\
g_0u \arrow[ur, "J_0u"] \arrow[r, "H_0u"] \arrow[dr, "K_0u"'] & h_0u \arrow[r, "J_0g_0^{-1}tH_0u"] & j_0g_0^{-1}h_0u \arrow[r, "K_0g_0^{-1}tJ_0g_0^{-1}tH_0u"] & k_0g_0^{-1}j_0g_0^{-1}h_0u \\
 & k_0u & &
\end{tikzcd}$$

The identity admissible section is I_0 where $I_0u = 1_{g_0u}$ for all $u \in C_0$. The inverse of H_0 is the admissible section where

$$H_0^{-1}u = (H_0h_0^{-1}g_0u)^{-1}, \qquad \text{so} \quad H_0^{-1}g_0^{-1}h_0u = (H_0u)^{-1} \quad \text{and} \quad H_0^{-1} * t = g_0 * h_0^{-1} * g_0.$$

Note that the map from $M_g(\mathbb{C})$ to $\mathrm{Symm}(C_0)$ with the $\star$ product, mapping H_0 to $H_0 * t$ is a group homomorphism.

4.5 Groupoid Actions

An action of a groupoid $\mathbb{C}$ on a groupoid $\mathbb{B}$ is usually defined in the case where $\mathbb{B}$ is a union of groups and has the same objects as $\mathbb{C}$. Then, when $(c : w \to x) \in \mathbb{C}$ and $(b : w \to w) \in \mathbb{B}$, we have $(b^c : x \to x)$. So c does not act by permuting the arrows of $\mathbb{B}$, but by providing an isomorphism from $\mathbb{B}(w)$ to $\mathbb{B}(x)$. A particular case of this situation is when $\mathbb{B}$ is a totally discrete subgroupoid of $\mathbb{C}$ and the action is conjugation, $b^c = c^{-1}bc$. We now give an alternative definition of an action, using the automorphism groupoid $\mathbb{A}\mathrm{u}t\,\mathbb{B}$, which does not require $\mathbb{B}$ to be totally disconnected, and which *does* provide a permutation of the arrows.

Definition 4.16 *An* action *of a groupoid $\mathbb{C}$ on a groupoid $\mathbb{B}$ is a groupoid morphism $\alpha = (\alpha_0, \alpha_1) : \mathbb{C} \to \mathbb{A}\mathrm{u}t\,\mathbb{B}$.*

????? This seems to be saying that $\mathbb{C}$ has m objects while $\mathbb{B}$ has mn objects, so how can $/alpha{+}_0$ be a bijection?

This means that, when α_0 is a bijection and $\mathbb{C} = \mathbb{H} \times \mathbb{I}_m$ with $\mathbb{H} = \mathsf{Groupoids}(H)$, then $\mathbb{B}$ has m isomorphic components, $\mathbb{B}_i \cong \mathbb{G} \times \mathbb{I}_n$ say, and $\alpha_1(p, h, q) = (\alpha_0 p, \alpha_{p,q} h, \alpha_0 q)$ where $\alpha_{p,q} : \mathbb{B}_{\alpha_0 p} \to \mathbb{B}_{\alpha_0 q}$ is an isomorphism.

4.6 Conjugation in groupoids

Definition 4.17 *Let $(c : p \to q)$, where $p \neq q$, be an arrow in a connected groupoid $\mathbb{C} = \mathbb{G} \times \mathbb{I}_n$. Then* conjugation by c, *written $\wedge c$, is an automorphism of $\mathbb{C}$ where:*

- *p, q are interchanged, and the remaining objects are fixed;*
- *the loops at p are interchanged with those at q,*
$$(p, g, p) \mapsto (q, c^{-1}gc, q), \qquad (q, g, q) \mapsto (p, cgc^{-1}, p);$$
- *the hom-set $\mathbb{C}(p, q)$ is interchanged with $\mathbb{C}(q, p)$,*
$$(p, g, q) \mapsto (q, c^{-1}gc^{-1}, p), \qquad (q, g, p) \mapsto (p, cgc, q);$$
- *the rest of the costar at p is interchanged with that at q,*
$$(r, g, p) \mapsto (r, gc, q), \qquad (r, g, q) \mapsto (r, gc^{-1}, p);$$
- *the rest of the star at p is interchanged with that at q,*
$$(p, g, r) \mapsto (q, c^{-1}g, r), \qquad (q, g, r) \mapsto (p, cg, r);$$
- *the remaining arrows are unchanged.*

There are a number of cases to consider when checking that composition is preserved by this mapping, for example

$$(r, g, p)^c (p, h, q)^c \;=\; (r, gc, q)(q, c^{-1}hc^{-1}, p) \;=\; (r, (gh)c^{-1}, p) \;=\; (r, gh, q)^c.$$

We now express $\wedge c$ as a word in our standard sets of generators.

[to be continued]

[**Add the corresponding formula for the case $p = q$.**]

It is *not* the case that the map $\wedge : \mathbb{C} \to \mathbb{A}\mathrm{u}t, \mathbb{C}$ is a groupoid morphism. This is clear just by considering the images of the objects, where the symmetric group is acting.

Lemma 4.18

$$\wedge(cd) \;=\; (\wedge c) * (\wedge d) * (\wedge c) \;=\; (\wedge d) * (\wedge c) * (\wedge d)$$

Proof: To be added. □

5 Crossed modules of groupoids

5.1 Basic definitions

Let $\mathbb{C}_1 = (C_1, C_0)$ be a groupoid and $\mathbb{C}_2 = (C_2, C_0)$ a union of groups with the same object set, and let $\mathbb{C}_2, \mathbb{C}_1$ act upon themselves by conjugation:

$$a_1{}^a = a^{-1} a_1 a, \qquad c_1{}^c = c^{-1} \diamond c_1 \diamond c,$$

defined when a_1, a are loops in $\mathbb{C}_2$ at the same object and when $sc = sc_1 = tc_1$ in $\mathbb{C}_1$.
[But is this the correct notion of conjugacy?]

A *pre-crossed module of groupoids* $\mathcal{C} = (\gamma : \mathbb{C}_2 \to \mathbb{C}_1)$ consists of a morphism of groupoids $\gamma = (\gamma, \mathrm{id})$ (abusing notation), the *boundary* of $\mathcal{C}$, pictured as:

$$\mathbb{C}_2 \;=\; \begin{tikzcd} C_2 \arrow[r, "\gamma"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & C_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ C_0 \arrow[r, "\mathrm{id}"'] & C_0 \end{tikzcd} \qquad \text{or, more simply,} \qquad \begin{tikzcd} C_2 \arrow[r, "\gamma"] & C_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & C_0 \end{tikzcd},$$

together with an action of $\mathbb{C}_1$ on $\mathbb{C}_2$ such that γ is a $\mathbb{C}_1$-morphism. So $\gamma * s = s,\ \gamma * t = t$ and, for all $a \in \mathrm{Arr}(\mathbb{C}_2)$ and $c \in \mathrm{Arr}(\mathbb{C}_1)$,

$$\textbf{X1:} \qquad \gamma(a^c) \;=\; c^{-1}(\gamma a)c \quad \text{when} \quad sa = ta = sc.$$

The pre-crossed module $\mathcal{C}$ is a *crossed module of groupoids* if it also satisfies

$$\textbf{X2:} \qquad a_1{}^{\gamma a} \;=\; a_1{}^a \quad \text{for all} \quad a_1, a \in \mathbb{C}_2(u),\ u \in C_0.$$

Note that, when both axioms are satisfied, the restriction $(\gamma_u : \mathbb{C}_2(u) \to \mathbb{C}_1(u))$ is a crossed module of groups for all $u \in C_0$.

A *morphism of pre-crossed modules* $\alpha : \mathcal{C} \to \mathcal{D}$, where $\mathcal{D} = (\delta : \mathbb{D}_2 \to \mathbb{D}_1)$, is a triple $(\alpha_2, \alpha_1, \alpha_0)$, where $(\alpha_2, \alpha_0) : \mathbb{C}_2 \to \mathbb{D}_2$ and $(\alpha_1, \alpha_0) : \mathbb{C}_1 \to \mathbb{D}_1$ are morphisms of groupoids satisfying

$$\alpha_2 * \delta \;=\; \gamma * \alpha_1, \qquad \alpha_2(a^c) \;=\; (\alpha_2 a)^{\alpha_1 c},$$

making the following diagram commute:

$$\begin{tikzcd} C_2 \arrow[r, "\gamma"] \arrow[d, "\alpha_2"'] & C_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "\alpha_1"'] & C_0 \arrow[d, "\alpha_0"] \\ D_2 \arrow[r, "\delta"'] & D_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & D_0 \end{tikzcd}$$

When $\mathcal{C}, \mathcal{D}$ are crossed modules, α is a morphism of crossed modules.

Example 5.1 Let $N \trianglelefteq G$ and $\mathbb{G} = \mathsf{Groupoids}(G)$. Take $\mathbb{C}_1 = \mathbb{G} \times \mathbb{I}_n$ and let $\mathbb{C}_2 = \mathbb{N} \times \mathbb{O}_n$ be the totally disconnected subgroupoid consisting of n copies of $\mathbb{N} = \mathsf{Groupoids}(N)$. Then $\mathbb{C}_1$ acts on $\mathbb{C}_2$ by conjugation, and $\mathcal{C} = (\iota : \mathbb{C}_2 \to \mathbb{C}_1)$ is a *conjugation crossed module*, where ι is the inclusion map.

5.2 Homotopies of a crossed module of groupoids

This subsection is intended to cover section 2 of Brown and İçen [17].

Let $g, h : \mathcal{C} = (\gamma : \mathbb{C}_2 \to \mathbb{C}_1) \to \mathcal{D} = (\delta : \mathbb{D}_2 \to \mathbb{D}_1)$ be morphisms of crossed modules. A *(g,h)-homotopy* $H : g \simeq h$ is a pair of functions (H_0, H_1) such that

- $H_0 : C_0 \to C_1$ is a (g,h)-section: $H_0 * s = g_0, \quad H_0 * t = h_0$,
- $H_1 : C_1 \to C_2$ satisfies: $H_1 * t = h_1 * t, \quad H_1(cc') \;=\; (H_1c)^{h_1c'}(H_1c')$,
- for all $(c : u \to v) \in \mathbb{C}_1$ and $(a : v \to v) \in \mathbb{C}_2$, $\delta H_1 c$ and $H_1\gamma a$ measure the divergence from commutativity of the following squares (in the second square dashed lines denote arrows in $\mathbb{C}_2$),

$$\begin{tikzcd}
g_0u \arrow[r, "g_1c"] \arrow[d, "H_0u"'] & g_0v \arrow[d, "H_0v"] \\
h_0u \arrow[r, "h_1c"'] & h_0v
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
g_0v \arrow[r, "g_2a"] \arrow[d, dashed, "H_0v"'] & g_0v \arrow[d, dashed, "H_0v"] \\
h_0v \arrow[r, "h_2a"'] & h_0v
\end{tikzcd}$$

$$\delta H_1 c \;=\; (h_1c)^{-1}(H_0u)^{-1}(g_1c)(H_0v), \qquad H_1\gamma a \;=\; (h_2a)^{-1}(g_2a)^{H_0v}. \tag{35}$$

When H is a (g,h)-homotopy, we call H_1 a (g,h)-derivation. We shall usually be concerned with the case $\mathcal{C} = \mathcal{D}$, so that g, h are automorphisms. In the special case that $g = \mathrm{id}_{\mathcal{C}}$ we call H a *free homotopy* and H_1 a *free derivation*. In another special case, when $H_0u = 1_u$ for all $u \in C_0$ we call H a *homotopy over the identity* and H_1 a *derivation over the identity*. A free derivation over the identity is simply called a *derivation*.

Consider the case when $\mathcal{C} = (\gamma : \mathbb{C}_2 \to \mathbb{C}_1)$ is connected, so $\mathbb{C}_1 = \mathbb{G}_1 \ltimes \mathbb{I}_n$, $\mathbb{C}_2 = \mathbb{G}_2 \times \mathbb{O}_n$, $\mathbb{G}_1 = \mathsf{Groupoids}(G_1)$, $\mathbb{G}_2 = \mathsf{Groupoids}(G_2)$, and $\mathcal{X} = (\overline{\gamma} : G_2 \to G_1)$ is a crossed module of groups, where $\gamma(j,a,j) = (j, \overline{\gamma}a, j)$. Because of the multiplication rule for H_1, we may define an h-derivation by specifying the images of a generating set (just as we did for automorphisms). Thus an h-derivation of $\mathcal{C}$ is determined by

- a derivation $\chi : G_1 \to G_2$ for the crossed module $\mathcal{X}$, so $\chi(cc') = (\chi c)^{c'}(\chi c')$,
- a choice of images $H_1(1,e,q) = (h_0q, a_q, h_0q)$, $2 \leqslant q \leqslant n$ for arrows in the tree T_1.

The h-derivation for $\mathcal{X}$ associated to χ is given by $\psi c = \chi h_1 c$, so that $\psi(cc') = (\psi c)^{h_1c'}(\psi c')$, and we define $H_1(1,c,1) = (h_01, \chi h_1 c, h_01)$. Applying the multiplication rule, we find

$$\begin{aligned}
H_1(q,e,1) &= (H_1(1,e,q)^{-1})^{h(q,e,1)} \;=\; (h_01, a_q^{-1}, h_01), \\
H_1(1,c,q) &= (h_01, \chi h_1c, h_01)^{h(1,e,q)}(h_0q, a_q, h_0q) \;=\; (h_0q, (\chi h_1c)a_q, h_0q), \\
H_1(q,c,q) &= (h_01, a_q^{-1}, h_01)^{h(1,c,q)}(h_01, \chi h_1c, h_01)^{h(1,e,q)}(h_0q, a_q, h_0q) \;=\; (h_0q, (a_q^{-1})^{h_1c}(\chi h_1c)a_q, h_0q), \\
H_1(q,c,1) &= (h_0q, (a_q^{-1})^{h_1c}(\chi h_1c)a_q, h_0q)^{h(q,e,1)}(h_01, a_q^{-1}, h_01) \;=\; (h_01, (a_q^{-1})^{h_1c}(\chi h_1c), h_01), \\
H_1(q,c,r) &= (h_0r, (a_q^{-1})^{h_1c}(\chi h_1c)a_r, h_0r).
\end{aligned}$$

We may check the multiplication rule as follows:

$$\begin{aligned} & H_1\left((p,c,q)(q,c',r)\right) \\ = \ & (h_0q, (a_p^{-1})^{h_1c}(\chi h_1c)a_q, h_0q)^{h(q,c',r)}\ (h_0r, (a_q^{-1})^{h_1c'}(\chi h_1c')a_r, h_0r) \\ = \ & (h_0r, ((a_p^{-1})^{h_1c}(\chi h_1c)a_q)^{h_1c'}, h_0r)(h_0r, (a_q^{-1})^{h_1c'}(\chi h_1c')a_r, h_0r) \\ = \ & (h_0r, (a_p^{-1})^{(h_1c)(h_1c')}(\chi h_1c)^{h_1c'}{a_q}^{h_1c'}(a_q^{-1})^{h_1c'}(\chi h_1c')a_r, h_0r) \\ = \ & (h_0r, (a_p^{-1})^{h_1(cc')}(\chi h_1c)^{h_1c'}(\chi h_1c')a_r, h_0r) \\ = \ & (h_0r, (a_p^{-1})^{h_1(cc')}(\chi h_1(cc')a_r, h_0r) \\ = \ & H_1(p,cc',r). \end{aligned}$$

[It is the case that the multiplication rule holds for this H_1, but we probably also need to check the axioms in (35).]

When H, K are two (g,h)-homotopies determined by derivations χ, ζ of $\mathcal{X}$, by $H_1(1,e,q) = (h_0q, a_q, h_0q)$, and by $K_1(1,e,q) = (h_0q, b_q, h_0q)$, the Whitehead product $H_1 \star K_1$ is given, as usual, by the formula

$$(H_1 \star K_1)x \ = \ (K_1x)(H_1x)(K_1\gamma h_1^{-1}H_1x). \qquad \textbf{???}$$

[The following calculation is probably wrong, and needs to be checked!]

For the loops at 1 this gives

$$(H_1 \star K_1)(1,c,1) \ = \ (h_01, (\chi \star \zeta)h_1c, h_01),$$

while the image of $(1,e,q)$ is given by

$$\begin{aligned} (H_1 \star K_1)(1,e,q) \ &= \ (h_0q, b_q, h_0q)(h_0q, a_q, h_0q)(K_1\gamma(h_0q, a_q, h_0q)) \\ &= \ (h_0q, b_qa_q, h_0q)(K_1(h_0q, \gamma a_q, h_0q)) \\ &= \ (h_0q, b_qa_q, h_0q)(h_0q, (b_q^{-1})^{\gamma a_q}(\zeta\gamma a_q)b_q, h_0q) \\ &= \ (h_0q, a_q(\zeta\gamma a_q)b_q, h_0q). \end{aligned}$$

6 Crossed complexes

The main references for this section are the book [20] which covers much of the material in the papers [9, 12, 13, 14, 15].

Definition 6.1 *A (many object) crossed complex* $(\mathcal{C}, \chi)$ *is a sequence of morphisms of groupoids with common object set* C_0*,*

$$\begin{tikzcd}
\cdots \arrow[r, "\chi_{n+1}"] & C_n \arrow[r, "\chi_n"] \arrow[d, "t=h"'] & C_{n-1} \arrow[r, "\chi_{n-1}"] \arrow[d, "t=h"'] & \cdots \arrow[r, "\chi_3"] & C_2 \arrow[r, "\chi_2"] \arrow[d, "t=h"'] & C_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "h"] \\
 & C_0 & C_0 & & C_0 & C_0
\end{tikzcd} \tag{36}$$

where:

- $\mathbb{C}_n = (C_n, C_0)$ *is a groupoid for* $n \geqslant 1$ *and, for* $n \geqslant 2$, $\mathbb{C}_n$ *is totally disconnected,*
- *for* $n \geqslant 3$, $\mathbb{C}_n$ *is abelian, and is a* $\mathbb{C}_1$*-module such that the image of* χ_2 *acts trivially,*
- *for* $n \geqslant 2$, $((\chi_n, \mathrm{id}) : \mathbb{C}_n \to \mathbb{C}_{n-1})$ *is a groupoid morphism which preserves the action of* $\mathbb{C}_1$ *(in the case* $n = 2$ *the action of* $\mathbb{C}_1$ *on itself is conjugation) so, for* $a \in C_1, c_n \in C_n$*, we have* $\chi_n({c_n}^a) = (\chi c_n)^a$*,*
- $((\chi_2, \mathrm{id}) : \mathbb{C}_2 \to \mathbb{C}_1)$ *is a crossed module of groupoids, so* $\{a_2\}^{\chi_2 b_2} = b_2^{-1} a_2 b_2$*,*
- *for* $n \geqslant 3$ *the composite* $\chi_n * \chi_{n-1}$ *is the zero map.*

The repeated C_0 in equation (36) is rather untidy, so the diagram of a crossed complex is usually simplified, as shown for $\mathbb{C}$ and $\mathbb{D}$ in (37).

For convenience in the general formulae to be considered later, we specify source and target maps on objects by defining $s, t : C_0 \to C_0$ to be the identity map on the set of objects.

A *morphism of crossed complexes* $\phi : (\mathcal{C}, \chi) \to (\mathcal{D}, \delta)$ is a family of groupoid morphisms $\{(\phi_n, \phi_0) : \mathbb{C}_n \to \mathbb{D}_n\}_{n \geqslant 1}$

$$\begin{tikzcd}
\cdots \arrow[r, "\chi_{n+1}"] & C_n \arrow[r, "\chi_n"] \arrow[d, "\phi_n"'] & C_{n-1} \arrow[r, "\chi_{n-1}"] \arrow[d, "\phi_{n-1}"'] & \cdots \arrow[r, "\chi_3"] & C_2 \arrow[r, "\chi_2"] \arrow[d, "\phi_2"'] & C_1 \arrow[r, shift left, "t"] \arrow[r, shift right, "h"'] \arrow[d, "\phi_1"'] & C_0 \arrow[d, "\phi_0"] \\
\cdots \arrow[r, "\delta_{n+1}"'] & D_n \arrow[r, "\delta_n"'] & D_{n-1} \arrow[r, "\delta_{n-1}"'] & \cdots \arrow[r, "\delta_3"'] & D_2 \arrow[r, "\delta_2"'] & D_1 \arrow[r, shift left, "t'"] \arrow[r, shift right, "h'"'] & D_0
\end{tikzcd} \tag{37}$$

compatible with the morphisms and actions on $\mathcal{C}, \mathcal{D}$, so that

$$\phi_n * \delta_n \;=\; \chi_n * \phi_{n-1} \qquad \text{and} \qquad \phi_n({c_n}^a) \;=\; (\phi_n c_n)^{\phi_1 a} \qquad \forall\ c_n \in C_n,\ n \geqslant 2,\ a \in C_1.$$

The category **XComp** has crossed complexes as objects and their morphisms as arrows.

Example 6.2 *The* unit interval crossed complex $(\mathcal{I}, 1)$ *has for* $\mathbb{I}_1$ *the groupoid* $\mathbb{I}$ *of Example 4.3 and, for* $n \geqslant 2$, $\mathbb{I}_n$ *is the trivial groupoid* $\mathrm{ids}(\mathcal{I})$*. The morphisms are all inclusion morphisms.*

When, for $n > m$, $\mathbb{C}_n$ is the trivial groupoid $\mathrm{ids}(\mathcal{C})$ of Example 4.2, and χ_n is the inclusion morphism, we say that $\mathcal{C}$ is *m-truncated*. This means that the structures above level m may be ignored, so that a 1-truncated crossed complex is effectively a groupoid (for example $(\mathbb{I}, \iota)$), and a 2-truncated crossed complex is a crossed module of groupoids. We denote the m-truncated subcrossed complex of $\mathcal{C}$ by $\mathcal{C}^m$. (We could refer here to the m-th *skeleton functor* sk^m.)

6.1 Tensor Product of Crossed Complexes

In [20] the tensor product $\mathcal{C} \otimes \mathcal{D}$ of crossed complexes $\mathcal{C}, \mathcal{D}$ is defined using a universal bimorphism, but we will not discuss bimorphisms of crossed complexes here (not yet, anyhow).

Definition 6.3 *Let $(\mathcal{C}, \chi), (\mathcal{D}, \delta)$ be crossed complexes. Then $(\mathcal{C} \otimes \mathcal{D}, \partial)$ is the crossed complex generated by elements $c \otimes d$ in dimension $m+n$, where $c \in C_m, d \in D_n$, with the following defining relations (plus the laws for crossed complexes).*

Source and target:

$$s(c \otimes d) = sc \otimes sd, \qquad t(c \otimes d) = tc \otimes td.$$

Action axioms:

$$c \otimes d^{d_1} = (c \otimes d)^{(tc \otimes d_1)} \quad \textit{when } n \geqslant 2,\ d_1 \in D_1, \qquad c^{c_1} \otimes d = (c \otimes d)^{(c_1 \otimes td)} \quad \textit{when } m \geqslant 2,\ c_1 \in C_1.$$

Product axioms:

$$\begin{aligned} c \otimes (dd_1) &= \begin{cases} (c \otimes d)^{(tc \otimes d_1)}(c \otimes d_1) & \textit{if } n = 1,\ m \geqslant 1, \\ (c \otimes d)(c \otimes d_1) & \textit{otherwise}, \end{cases} \\ (cc_1) \otimes d &= \begin{cases} (c_1 \otimes d)(c \otimes d)^{(c_1 \otimes td)} & \textit{if } m = 1,\ n \geqslant 1, \\ (c \otimes d)(c_1 \otimes d) & \textit{otherwise}. \end{cases} \end{aligned}$$

Boundary map:

$$\partial_{m+n}(c \otimes d) = \begin{cases} (tc \otimes d)^{-1}(c \otimes sd)^{-1}(sc \otimes d)(c \otimes td) & \textit{if } m = n = 1, \\ c \otimes \delta_n d & \textit{if } m = 0,\ n \geqslant 2, \\ \chi_m c \otimes d & \textit{if } m \geqslant 2,\ n = 0, \\ (c \otimes \delta_n d)^{-1}(tc \otimes d)^{-1}(sc \otimes d)^{(c \otimes td)} & \textit{if } m = 1,\ n \geqslant 2, \\ (c \otimes td)^{(-1)^{m+1}} \left((c \otimes sd)^{(tc \otimes d)}\right)^{(-1)^m} (\chi_m c \otimes d) & \textit{if } n = 1,\ m \geqslant 2, \\ (\chi_m c \otimes d)(c \otimes \delta_n d)^{(-1)^m} & \textit{if } n \geqslant 2,\ m \geqslant 2. \end{cases}$$

Note that $(\mathcal{C} \otimes \mathcal{D})_0 = C_0 \times D_0$, so we may write $u \otimes x$ as (u, x) when $u \in C_0, x \in D_0$.

The groupoid $((\mathcal{C} \otimes \mathcal{D})_1, (\mathcal{C} \otimes \mathcal{D})_0)$ is isomorphic to $C_1 \# D_1$, the groupoid coproduct of $\mathcal{C}^1 \times \mathrm{ids}(\mathcal{D}^1)$ and $\mathrm{ids}(\mathcal{C}^1) \times \mathcal{D}^1$. Every element of $C_1 \# D_1$ is uniquely expressible in one of the following normal forms.

(i) An identity arrow $(1_u, 1_x)$.

(ii) A *generating arrow* $(c, 1_x)$ or $(1_u, d)$, where $c \in C_1,\ x \in D_0,\ u \in C_0,\ d \in D_1$ and c, d are not identities. We write $\mathrm{arr}(c, 1_x) = c,\ \mathrm{obj}(c, 1_x) = x$ and $\mathrm{arr}(1_u, d) = d,\ \mathrm{obj}(1_u, d) = u$.

(iii) A composite $k = k_1 k_2 \ldots k_n$ $(n \geqslant 2)$ of generating arrows in which the $\mathrm{arr}(k_i)$ lie alternately in C_1 and D_1 (or conversely), and the odd and even products $\mathrm{odd}\,(k) = \mathrm{arr}(k_1)\mathrm{arr}(k_3)\cdots$ and $\mathrm{even}\,(k) = \mathrm{arr}(k_2)\mathrm{arr}(k_4)\cdots$ are defined in C_1 or D_1. We define k_{C_1} to be $\mathrm{odd}\,(k)$ or $\mathrm{even}\,(k)$, whichever composite arrow is in C_1, and then k_{D_1} is the other composite arrow.

6.2 Tensor product of groupoids

When $\mathcal{C}, \mathcal{D}$ are groupoids, considered as 1-truncated crossed complexes, the tensor product $\mathcal{C} \otimes \mathcal{D}$ is 2-truncated. We have already described the groupoid at level 1, so it remains to specify $(\mathcal{C} \otimes \mathcal{D})_2$.

There is a canonical morphism $\sigma : C_1 \# D_1 \to C_1 \times D_1$ mapping k to (k_{C_1}, k_{C_2}). The kernel of σ is the *Cartesian subgroup* $C_1 \square D_1$ of $C_1 \# D_1$ consisting of all the identity arrows and all the words k such that k_{C_1} and k_{D_1} are identities. It is generated by "commutators" $[c, d]$ where, when $(c : u \to v) \in C_1$ and $(d : x \to y) \in D_1$ are not identities, $[c, d]$ is the loop at $(v, y) = (tc, td)$ given by

$$[c, d] \;:=\; (c^{-1}, 1_y)(1_u, d^{-1})(c, 1_x)(1_v, d),$$

as shown in the following diagram:

$$\begin{tikzcd}
(v,y) \arrow[d, "(c^{-1},1_y)"'] & (v,x) \arrow[l, "(1_v,d)"'] \\
(u,y) \arrow[r, "(1_u,d^{-1})"'] & (u,x) \arrow[u, "(c,1_x)"']
\end{tikzcd}$$

These commutators satisfy the usual commutator identities:

$$\begin{aligned}
[d, c] &= [c, d]^{-1}, \\
[dd_1, c] &= [d, c]^{d_1}[d_1, c], \\
[d, cc_1] &= [d, c_1][d, c]^{c_1},
\end{aligned}$$

whenever cc_1, dd_1 are defined. Note that the action is conjugation: for example, when $(d_1 : y \to z) \in D_1$ the loop $[d, c]^{d_1}$ at (v, z) is given by

$$\begin{aligned}
[d, c]^{d_1} &= (1_v, d_1^{-1})(1_v, d^{-1})(c^{-1}, 1_x)(1_u, d)(c, 1_y)(1_v, d_1) \\
&= (1_v, (dd_1)^{-1})(c^{-1}, 1_x)(1_u, dd_1)(c, 1_z).(c^{-1}, 1_z)(1_u, d_1^{-1})(c, 1_y)(1_v, d_1) \\
&= [dd_1, c][c, d_1],
\end{aligned}$$

and similarly $[d, c]^{c_1} = [c_1, d][d, cc_1]$.

It turns out that $(\mathcal{C} \otimes \mathcal{D})_2 = C_1 \square D_1$ with $c \otimes d := [d, c]$. We may check the product axioms as follows:

$$\begin{aligned}
cc_1 \otimes d &= [d, cc_1] = [d, c_1][d, c]^{c_1} = (c_1 \otimes d)(c \otimes d)^{(c_1 \otimes td)}, \\
c \otimes dd_1 &= [dd_1, c] = [d, c]^{d_1}[d_1, c] = (c \otimes d)^{(tc \otimes d_1)}(c \otimes d_1).
\end{aligned}$$

We have constructed a normal inclusion crossed module of groupoids

$$(C_1 \square D_1 \longrightarrow C_1 \# D_1 \rightrightarrows C_0 \times D_0).$$

The action on loops in dimension two is given by:

$$\begin{aligned}
(c \otimes d)^{(tc \otimes d_1)} &= (c \otimes dd_1)(c \otimes d_1)^{-1} \;, \\
(c \otimes d)^{(c_1 \otimes td)} &= (c_1 \otimes d)^{-1}(cc_1 \otimes d) \;.
\end{aligned}$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
(0,1) \arrow[loop above, "(1_0,1_1)"] \arrow[r, shift left, "(\iota,1_1)"] \arrow[d, shift left, "(1_0,\kappa^{-1})"] & (1,1) \arrow[loop above, "(1_1,1_1)"] \arrow[l, shift left, "(\iota^{-1},1_1)"] \arrow[d, shift right, "(1_1.\kappa^{-1})"'] \\
(0,0) \arrow[loop below, "(1_0,1_0)"] \arrow[u, shift left, "(1_0,\kappa)"] \arrow[r, shift right, "(\iota,1_0)"'] & (1,0) \arrow[loop below, "(1_1,1_0)"] \arrow[l, shift right, "(\iota^{-1},1_0)"'] \arrow[u, shift right, "(1_1,\kappa)"']
\end{tikzcd}$$

Example 6.4 *The tensor product* $\mathcal{I}^{\otimes 2} = \mathcal{I} \otimes \mathcal{I}$ *has four objects and eight generating arrows in dimension one, as shown in the following diagram. We relabel* ι *in the second factor as* κ *so that the "commutator" notation is not ambiguous.*

The generating commutators for $I_2^{\otimes 2}(0,0)$ *as:*

$$\begin{aligned}
k_{(0,0)} &= (\iota,1_0)(1_1,\kappa)(\iota^{-1},1_1)(1_0,\kappa^{-1}) &= [\iota^{-1},\kappa^{-1}], \\
{k_{(0,0)}}^{-1} &= (1_0,\kappa)(\iota,1_1)(1_1,\kappa^{-1})(\iota^{-1},1_0) &= [\kappa^{-1},\iota^{-1}].
\end{aligned}$$

So the vertex group $I_2^{\otimes 2}(0,0)$ *is free on one generator. It is easy to see that the vertex groups at* $(1,0),(1,1),(0,1)$ *are respectively generated by*

$$k_{(1,0)} = [\iota,\kappa^{-1}], \qquad k_{(1,1)} = [\iota,\kappa], \qquad k_{(0,1)} = [\iota^{-1},\kappa].$$

Example 6.5 *Consider* $\mathcal{D} = \mathcal{I} \otimes \mathcal{C}$, *where* $\mathcal{C}$ *is a crossed module of groups with* $C_0 = \{\bullet\}$. *The objects in* D_0 *are* $\{(0,\bullet),(1,\bullet)\}$, *but it is convenient to replace these by* $\{0,1\}$. *The generating arrows in* D_1 *are*

$$\{((\iota,1_\bullet):0\to 1),\ ((\iota^{-1},1_\bullet):1\to 0)\}\ \cup\ \{((1_0,g):0\to 0),\ ((1_1,g):1\to 1)\mid g\in C_1\},$$

where we may restrict g *to be a member of a generating set for* C_1 *since* $(1_0,g_1)(1_0,g_2) = (1_0,g_1g_2)$.

$$\begin{tikzcd}[column sep=huge]
(0,\bullet) \arrow[loop left, "(1_0,g)"] \arrow[r, shift right, "(\iota,1_\bullet)"'] & (1,\bullet) \arrow[loop right, "(1_1,g)"] \arrow[l, shift right, "(\iota^{-1},1_\bullet)"']
\end{tikzcd}$$

There should be no confusion if we simply write ι *for* $(\iota,1_\bullet)$ *and* g *for* $(1_0,g)$, *etc. A typical element of* D_1 *with source* 0 *is therefore* $k = g_1\iota g_2\iota^{-1}g_3\iota\dots$. *As before, we define*

$$\iota\otimes g \ = \ [g,\iota] \ = \ g^{-1}\iota^{-1}g\iota, \qquad \text{a loop at } 1.$$

In dimension 2 we also have a contribution from elements of C_2. *For* $c \in C_2$ *we write* c *for* $c\otimes 0$ *and* $c\otimes 1$ *when the object is clear. Then a typical loop at* 0 *in* D_2 *is*

$$c_1w_1c_2w_2c_3\dots \quad \text{where } w_i \text{ is a word in the } [g,\iota].$$

Example 6.6 *The tensor product $\mathcal{I}^{\otimes 3} = \mathcal{I} \otimes \mathcal{I} \otimes \mathcal{I}$ should, strictly speaking, be calculated as one of the isomorphic products $\mathcal{I}^{\otimes 2} \otimes \mathcal{I}$ or $\mathcal{I} \otimes \mathcal{I}^{\otimes 2}$. Thus we might consider, for example, the object $((0,0),0)$ and the generating arrows $(1_{(0,0)}, \iota)$ and $((1_0, \iota), 1_0)$. It is simpler to consider the 8 objects and 24 generating arrows in dimension one as triples, as shown in the following cubical diagram where, to avoid confusion, we relabel the three copies of ι as ι, κ, λ.*

$$\begin{tikzcd}
 & (0,1,1) \arrow[rr, shift right, "{(\iota,1_1,1_1)}"'] \arrow[dl, shift right, "{(1_0,1_1,\lambda^{-1})}"'] \arrow[dd, shift right, "{(1_0,\kappa^{-1},1_1)}"'] & & (1,1,1) \arrow[ll, shift right, "{(\iota^{-1},1_1,1_1)}"'] \arrow[dl, shift right, "{(1_1,1_1,\lambda^{-1})}"'] \arrow[dd, shift right, "{(1_1,\kappa^{-1},1_1)}"'] \\
(0,1,0) \arrow[ur, shift right, "{(1_0,1_1,\lambda)}"'] \arrow[rr, shift left, "{(\iota,1_1,1_0)}"] \arrow[dd, shift right, "{(1_0,\kappa^{-1},1_0)}"'] & & (1,1,0) \arrow[ur, shift right, "{(1_1,1_1,\lambda)}"'] \arrow[ll, shift left, "{(\iota^{-1},1_1,1_0)}"] \arrow[dd, shift right, "{(1_1,\kappa^{-1},1_0)}"'] & \\
 & (0,0,1) \arrow[uu, shift right, "{(1_0,\kappa,1_1)}"'] \arrow[rr, shift right, "{(\iota,1_0,1_1)}"'] \arrow[dl, shift right, "{(1_0,1_0,\lambda^{-1})}"'] & & (1,0,1) \arrow[uu, shift right, "{(1_1,\kappa,1_1)}"'] \arrow[ll, shift right, "{(\iota^{-1},1_0,1_1)}"'] \arrow[dl, shift right, "{(1_1,1_0,\lambda^{-1})}"'] \\
(0,0,0) \arrow[uu, shift right, "{(1_0,\kappa,1_0)}"'] \arrow[ur, shift right, "{(1_0,1_0,\lambda)}"'] \arrow[rr, shift right, "{(\iota,1_0,1_0)}"'] & & (1,0,0) \arrow[uu, shift right, "{(1_1,\kappa,1_0)}"'] \arrow[ur, shift right, "{(1_0,1_0,\lambda)}"'] \arrow[ll, shift right, "{(\iota^{-1},1_0,1_0)}"'] &
\end{tikzcd}$$

For dimension 2 loops at $(1,1,1)$ we have commutators from the back, top, and right faces:

$$[\iota, \kappa], \qquad [\iota, \lambda], \qquad [\kappa, \lambda]. \tag{38}$$

The other three faces also contribute loops at $(1,1,1)$. For example. the front face is traversed (in a clockwise direction) by

$$[\kappa, \iota]^{\lambda} \;=\; (1_1, 1_1, \lambda^{-1})(1_1, \kappa^{-1}, 1_0)(\iota^{-1}, 1_0, 1_0)(1_0, \kappa, 1_0)(\iota, 1_1, 1_0)(1_1, 1_1, \lambda).$$

A typical element in the vertex group $K_2((1,1,1))$ is a word in the three commutators (38) and their conjugates.

Finally, there are non-trivial vertex groups in K_3, associated to the whole cube. Each of the eight elements $(\iota^{\pm 1} \otimes \kappa^{\pm 1} \otimes \lambda^{\pm 1}) \in K_3$ generates an infinite cyclic group at $(h\iota^{\pm 1}, h\kappa^{\pm 1}, h\lambda^{\pm 1})$. The boundary map is given by a version of the Jacobi-Hall-Witt identity for commutators,

$$[x^y, [y,z]]\;[y^z, [z,x]]\;[z^x, [x,y]] \;=\; 1,$$

which is used by Ellis in [31] to give an identity among the relators for the free abelian group on three generators. (Another relevant reference is [11].) Thus, at $K_3((1,1,1))$, we define

$$\partial_3(\iota \otimes \kappa \otimes \lambda) \;=\; [\lambda, \kappa]^{\iota^{\kappa}}\;[\kappa, \lambda]\;[\iota, \lambda]^{\kappa^{\lambda}}\;[\lambda, \iota]\;[\kappa, \iota]^{\lambda^{\iota}}\;[\iota, \kappa]\;,$$

where

$$\begin{aligned}
\partial_2[\lambda, \kappa]^{\iota^{\kappa}} &= (1_1, \kappa^{-1}, 1_1)(\iota^{-1}, 1_0, 1_1)(1_0, \kappa, 1_1)(1_0, 1_1, \lambda^{-1})(1_0, \kappa^{-1}, 1_0)(1_0, 1_0, \lambda)(\iota, 1_0, 1_1)(1_1, \kappa, 1_1),\\
\partial_2[\kappa, \lambda] &= (1_1, \kappa^{-1}, 1_1)(1_1, 1_0, \lambda^{-1})(1_1, \kappa, 1_0)(1_1, 1_1, \lambda),\\
\partial_2[\iota, \lambda]^{\kappa^{\lambda}} &= (1_1, 1_1, \lambda^{-1})(1_1, \kappa^{-1}, 1_0)(1_1, 1_0, \lambda)(\iota^{-1}, 1_0, 1_1)(1_0, 1_0, \lambda^{-1})(\iota, 1_0, 1_0)(1_1, \kappa, 1_0)(1_1, 1_1, \lambda),\\
\partial_2[\lambda, \iota] &= (1_1, 1_1, \lambda^{-1})(\iota^{-1}, 1_1, 1_0)(1_0, 1_1, \lambda)(\iota, 1_1, 1_1),\\
\partial_2[\kappa, \iota]^{\lambda^{\iota}} &= (\iota^{-1}, 1_1, 1_1)(1_0, 1_1, \lambda^{-1})(\iota, 1_1, 1_0)(1_1, \kappa^{-1}, 1_0)(\iota^{-1}, 1_0, 1_0)(1_0, \kappa, 1_0)(1_0, 1_1, \lambda)(\iota, 1_1, 1_1),\\
\partial_2[\iota, \kappa] &= (\iota^{-1}, 1_1, 1_1)(1_0, \kappa^{-1}, 1_1)(\iota, 1_0, 1_1)(1_1, \kappa, 1_1).
\end{aligned}$$

The Jacobi-Hall-Witt identity ensures that $\partial^2(\iota \otimes \kappa \otimes \lambda) = 1_{(1,1,1)}$.

Here are the formulae for the boundary maps of the tensor product $\mathcal{C} \otimes \mathcal{D}$, where $(c_1 : u \to v) \in C_1$ $(d_1 : x \to y) \in D_1$, $(c : w \to w) \in C_m,\ m \geqslant 2$, and $(d : z \to z) \in D_n,\ n \geqslant 2$.

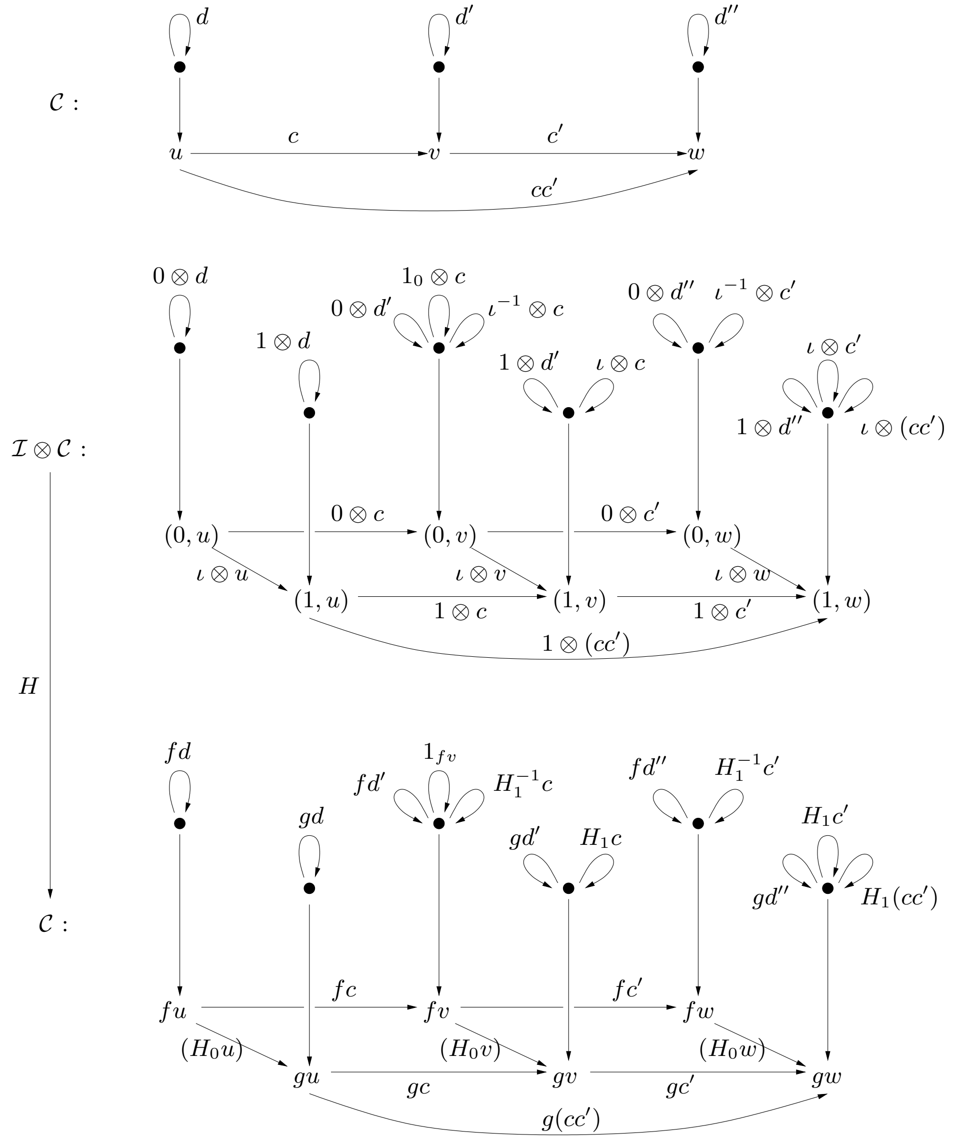
$$\begin{array}{rcl}
\partial_{0+n}(u \otimes d) & = & u \otimes \partial_n d, \\
\partial_{m+0}(c \otimes x) & = & \partial_m c \otimes x, \\
\partial_2(c_1 \otimes d_1) & = & (v \otimes d_1)^{-1}(c_1 \otimes x)^{-1}(u \otimes d_1)(c_1 \otimes y), \\
\partial_{1+n}(c_1 \otimes d) & = & (c_1 \otimes \partial_n d)^{-1}(v \otimes d)^{-1}(u \otimes d)^{(c_1 \otimes z)}, \\
\partial_{m+1}(c \otimes d_1) & = & (c \otimes y)^{(-1)^{m+1}} \left((c \otimes x)^{(w \otimes d_1)}\right)^{(-1)^m} (\partial_m c \otimes d_1), \\
\partial_{m+n}(c \otimes d) & = & (\partial_m c \otimes d) + (-1)^m (c \otimes \partial_n d), \quad m, n \geqslant 2.
\end{array}$$

6.3 Homotopies between morphisms of crossed complexes

For f, g two automorphisms of $\mathcal{C}$, a 1-*homotopy* $H : f \simeq g$ is a set of maps $H_n : C_n \to C_{n+1},\ n \geqslant 0$, satisfying various axioms. These axioms are most easily obtained by viewing the homotopy as a morphism $H : \mathcal{I} \otimes \mathcal{C} \to \mathcal{C}$, making the following diagram commute where $i_0 c = 0 \otimes c$ and $i_1 c = 1 \otimes c$.

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dl, "i_0"'] \arrow[dr, "f"] & \\
\mathcal{I} \otimes \mathcal{C} \arrow[rr, "H"] & & \mathcal{C} \\
 & \mathcal{C} \arrow[ul, "i_1"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Such an H comprises maps $H_{m,n} : I_m \times C_n \to C_{m+n},\ m, n \geqslant 0$. Part of such a morphism is shown in the following diagram.



We now consider the maps $H_{m,n}$ for small m, n.

$$\begin{array}{rl}
H_{0,0} : \{0,1\} \times C_0 \to C_0, & (0,u) \mapsto fu,\ (1,u) \mapsto gu, \\
H_{0,n} : \{0,1\} \times C_n \to C_n, & 0 \otimes c \mapsto fc,\ 1 \otimes c \mapsto gc, \\
H_{1,0} : \{1_0, 1_1, \iota, \iota^{-1}\} \times C_0 \to C_1, & 1_0 \otimes u \mapsto 1_{fu},\ 1_1 \otimes u \mapsto 1_{gu}, \\
 & \iota \otimes u \mapsto (H_0 u : fu \to gu), \\
 & \iota^{-1} \otimes u \mapsto (H_0 u)^{-1} : gu \to fu, \\
 & \text{(which defines the map } H_0 : C_0 \to C_1\text{)}, \\
H_{m,0} : \{1_0, 1_1\} \times C_0 \to C_m, & 1_0 \otimes u \mapsto 1_{fu},\ 1_1 \otimes u \mapsto 1_{gu},\ m \geqslant 2, \\
H_{1,1} : \{1_0, 1_1, \iota, \iota^{-1}\} \times C_1 \to C_2, & 1_0 \otimes c \mapsto 1_{fv},\ 1_1 \otimes c \mapsto 1_{gv}, \\
 & \iota \otimes c \mapsto H_1 c = [c, \iota] \in C_2(t\iota, tc) = C_2(1, v), \\
 & \text{(which defines the map } H_1 : C_1 \to C_2\text{)}.
\end{array}$$

We now derive some of the axioms. In $\mathcal{I} \otimes \mathcal{C}$ the arrow $\iota \otimes c$ decomposes as

$$(1_0 \otimes c)(\iota \otimes 1_v) \ = \ (\iota \otimes c) \ = \ (\iota \otimes 1_u)(1_1 \otimes c),$$

so this commuting square is mapped to a commuting square in $\mathcal{C}$, defining

$$H_0 : C_1 \to C_1, \quad c \mapsto (fc)(H_0 v) = (H_0 u)(gc).$$

The image under H_1 of a composite arrow is given by

$$H_1(cc') \ = \ \iota \otimes cc' \ = \ (\iota \otimes c)^{c'}(\iota \otimes c') \ = \ (H_1 c)^{(1_1 \otimes c')}(H_1 c') \ = \ (H_1 c)^{gc'}(H_1 c').$$

Similarly, applying the boundary maps,

$$\begin{aligned}
\partial(H_1 c) \ &= \ H\left((1_1, c^{-1})(\iota^{-1}, 1_u)(1_0, c)(\iota, 1_v)\right) \\
&= \ H_{0,1}(1, c^{-1}) \ H_{1,0}(\iota^{-1}, u) \ H_{0,1}(0, c) \ H_{1,0}(\iota, v) \\
&= \ (gc)^{-1} \ (H_0 u)^{-1} \ (fc) \ (H_0 v).
\end{aligned}$$

In the formulae for $H_{1,1}$ above the image of $\iota^{-1} \otimes c$ has not been defined. It may be determined on expanding $\iota\iota^{-1} \otimes c$ as follows:

$$\begin{array}{rrcl}
 & 1_0 \otimes c & = & \iota\iota^{-1} \otimes c \ = \ (\iota^{-1} \otimes c)(\iota \otimes c)^{(\iota \otimes v)^{-1}} \\
\Rightarrow & 1_{fv} & = & H(\iota^{-1} \otimes c)\,(H_1 c)^{(H_0 v)^{-1}} \\
\Rightarrow & H_{1,1}(\iota^{-1} \otimes c) & = & ((H_1 c)^{-1})^{(H_0 v)^{-1}}.
\end{array}$$

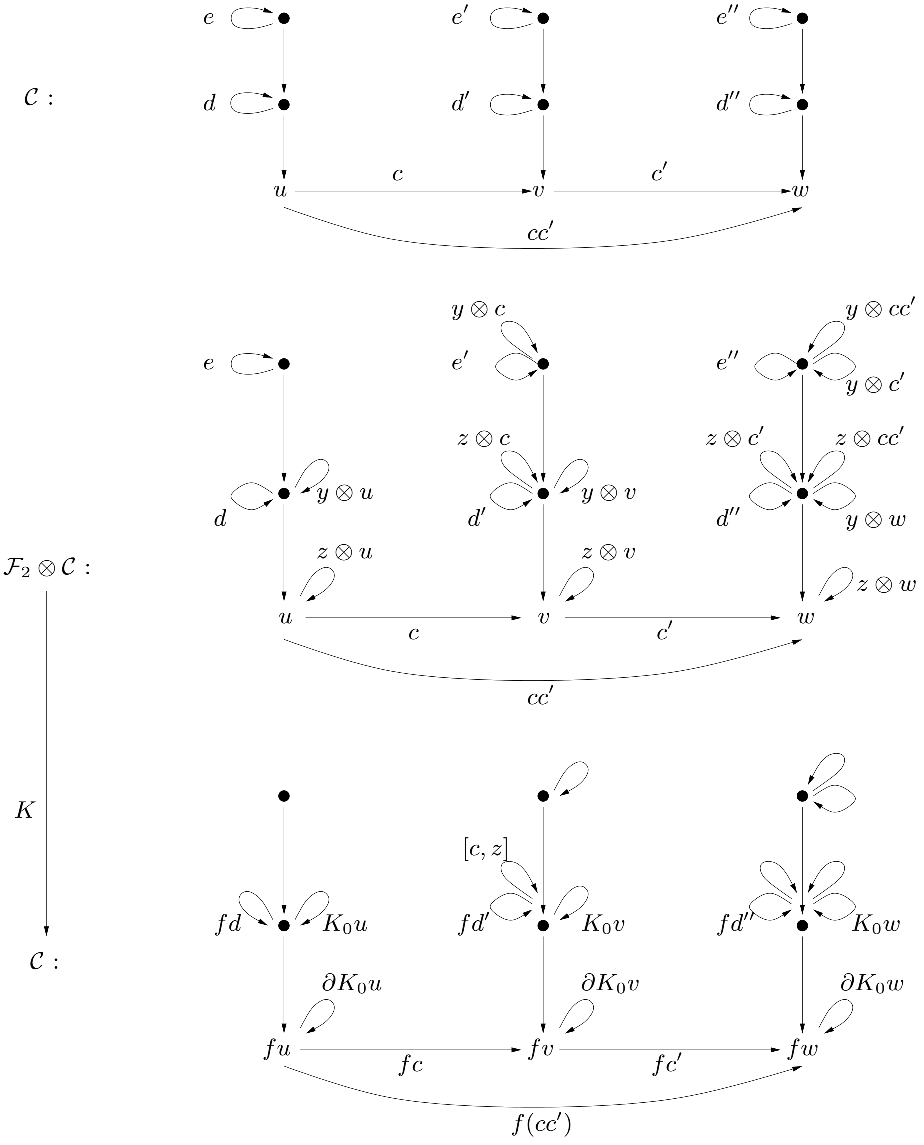
For f an automorphism of $\mathcal{C}$, a 2-*homotopy* K *over* f is a set of maps $K_n : C_n \to C_{n+2},\ n \geqslant 0$, satisfying various axioms. These axioms are most easily obtained by viewing the homotopy as a morphism $K : \mathcal{F}_2 \otimes \mathcal{C} \to \mathcal{C}$, making the following diagram commute where $\mathcal{F}_2$ is the free crossed complex

$$\begin{tikzcd}
\cdots \arrow[r] & 1 \arrow[r] & F_2 = \langle y \rangle \arrow[r, "\partial"] & F_1 = \langle z \rangle \arrow[r] & \{\bullet\}
\end{tikzcd} ,$$

the boundary map is given by $\partial y = z$, the action is trivial, $y^z = y$, and $ic = \bullet \otimes c$.

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dl, "i"'] \arrow[dr, "f"] & \\
\mathcal{F}_2 \otimes \mathcal{C} \arrow[rr, "K"'] & & \mathcal{C}
\end{tikzcd}$$

Such a K comprises maps $K_{m,n} : (F_2)_m \times C_n \to C_{m+n},\ m, n \geqslant 0$. Part of such a morphism is shown in the following diagram where, since there is only one object in $\mathcal{F}_2$, we write u for $\bullet \otimes u$.



We now consider the maps $K_{m,n}$ for small m, n.

$$\begin{array}{rl}
K_{0,0} : \{\bullet\} \times C_0 \to C_0, & u \mapsto fu, \\
K_{0,n} : \{\bullet\} \times C_n \to C_n, & c \mapsto fc, \\
K_{2,0} : \langle y \rangle \times C_0 \to C_2, & y \otimes u \mapsto K_0 u, \quad \text{where } \partial K_0 u = z \otimes u, \\
 & \text{(which defines the map } K_0 : C_0 \to C_2), \\
K_{1,0} : \langle z \rangle \times C_0 \to C_1, & z \otimes u \mapsto \partial(K_0 u), \\
K_{1,1} : \langle z \rangle \times C_1 \to C_2, & z \otimes c \mapsto [c, z] \in C_2(v) \quad \text{where} \\
 & \partial[c, z] = K\left((c^{-1}, 1_v)(1_u, z^{-1})(c, 1_u)(1_v, z)\right) \\
 & \qquad = (fc)^{-1}(\partial K_0 u)^{-1}(fc)(\partial K_0 v), \\
K_{2,1} : \langle y \rangle \times C_1 \to C_2, & y \otimes c \mapsto K_1 c \in C_3(v) \qquad \text{where } \partial K_1 c = ??? \\
 & \text{(which defines the map } K_1 : C_1 \to C_3).
\end{array}$$

We now derive some of the axioms or properties of these maps.

(a)

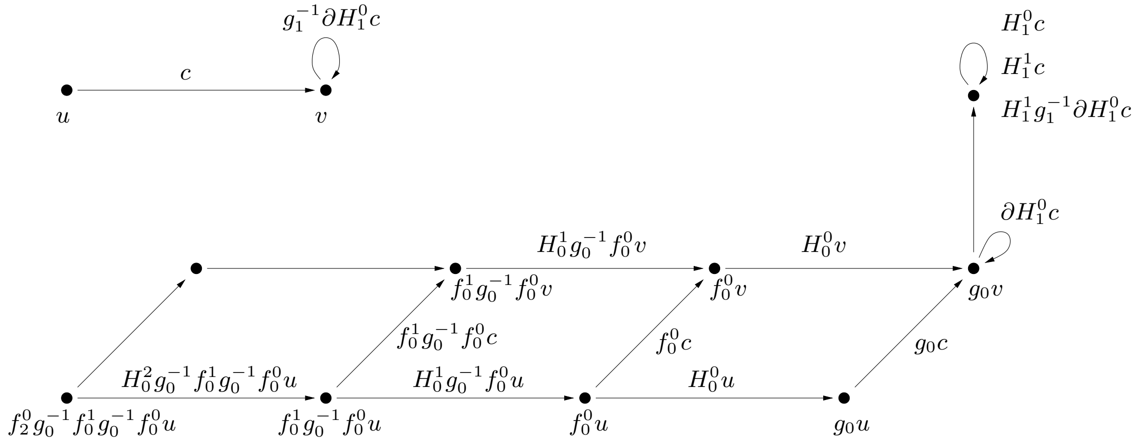
$$\begin{array}{rrcl}
 & \partial_3(y \otimes c) & = & (y \otimes v)^{-1}(y \otimes u)^c(z \otimes c) \\
\Rightarrow & \partial_3 K_1 c & = & (K_0 v)^{-1}(K_0 u)^c[c, z].
\end{array}$$

6.4 Whitehead product of 1-homotopies

We wish to define a monoid structure on the set of homotopies to g, so let $H^0 : f^0 \simeq g$ and $H^1 : f^1 \simeq g$ be two homotopies between automorphisms of $\mathcal{C}$. Then we define the *Whitehead product* of H^0, H^1 to be the homotopy $H^0 \star H^1 : f^0 * g^{-1} * f^1 \simeq g$ where

$$\begin{array}{rcl}
(H^0 \star H^1)_0(u) & = & (H^1_0 g_0^{-1} f^0_0 u)(H^0_0 u), \\
(H^0 \star H^1)_1(c) & = & (H^1_1 c)(H^0_1 c)(H^1_1 g_1^{-1} \partial H^0_1 c).
\end{array}$$

The arrows in these composite formulae are shown in the following diagram.



This product is associative: the homotopy is $H^0 \star H^1 : f^0 * g^{-1} * f^1 \simeq g$ where

$$\begin{aligned}
(H^0 \star H^1 \star H^2)_0(u) &= (H^2_0 g_0^{-1} f^1_0 g_0^{-1} f^0_0 u)(H^1_0 g_0^{-1} f^0_0 u)(H^0_0 u), \\
(H^0 \star H^1 \star H^2)_1(c) &= (H^2_1 c)(H^1_1 c)(H^2_1 g_1^{-1} \partial H^1_1 c)(H^0_1 c)(H^2_1 g_1^{-1} \partial H^0_1 c)(H^1_1 g_1^{-1} \partial H^0_1 c)(H^2_1 g_1^{-1} \partial H^1_1 g_1^{-1} \partial H^0_1 c).
\end{aligned}$$

7 Crossed Pairings and Nonabelian Tensor Products

The *nonabelian tensor product* was introduced by Brown and Loday in [19] and developed in Brown, Johnson, Robertson [18].

When G, H are both abelian, $G \otimes H$ is the usual tensor product.

Many computations of the *nonabelain tensor square* $G \otimes G$ of a group G have been made. Here is a small sample of known results:

symmetric group	S_3	C_6
alternating group	A_4	$Q_8 \times C_3$
dihedral groups	D_{2n}, n odd	$\mathbf{Z}_{2n}$
Heisenberg group	$\mathcal{H}$	$\mathbf{Z}^6$

The nonabelian tensor product is a special case of a crossed pairing.

7.1 Compatible Group Actions

Definition 7.1 *Let G and H be groups which act on themselves by conjugation, and also act on each other. These four actions are said to be* compatible *if*

$$g_1{}^{(h^g)} \;=\; ((g_1{}^{g^{-1}})^h)^g \;, \qquad h_1{}^{(g^h)} \;=\; ((h_1{}^{h^{-1}})^g)^h \;.$$

Example 7.2 *If G, H are normal subgroups of a group Γ, then each acts on the other by conjugation and the actions are compatible.*

Example 7.3 *Let $\mathcal{X} = (\partial : S \to R)$ be a crossed module. If r^s is defined to be $(\partial s^{-1})r(\partial s)$ then both R and S act on each other and on themselves. Compatibility is easily checked:*

- $s_1{}^{(r^s)} \;=\; s_1{}^{(\partial s^{-1})r(\partial s)} \;=\; s_1{}^{s^{-1}r(\partial s)} \;=\; s_1{}^{s^{-1}rs}\,,$
- $r_1{}^{(s^r)} \;=\; (\partial s^r)^{-1} r_1 (\partial s^r) \;=\; (r^{-1}(\partial s)r)^{-1} r_1 (r^{-1}(\partial s)r) \;=\; ((r_1{}^{r^{-1}})^s)^r\,.$

7.2 Crossed Pairings

There are two standard definitions of a *crossed pairing.* Here is the one which we shall use.
(There is a more general definition when the two actions are not compatible.)

Definition 7.4 *Let G, H be groups which act compatibly on each other and on a group L.*
A map $\boxtimes : G \times H \to L$, $(g, h) \mapsto g \boxtimes h$, is a crossed pairing if

(a) $(g_1 g_2 \boxtimes h) \;=\; (g_1 \boxtimes h)^{g_2}\,(g_2 \boxtimes h)\;,$

(b) $(g \boxtimes h_1 h_2) \;=\; (g \boxtimes h_2)\,(g \boxtimes h_1)^{h_2}\;,$

(c) $(g \boxtimes h)^x \;=\; g^x \boxtimes h^x$ *for all* $x \in G \cup H\;.$

The alternative definition does not require actions on L and omits axiom (c). It is then observed that $\operatorname{im} \boxtimes \leq L$ inherits G- and H-actions given by (c).

Example 7.5 If $N \trianglelefteq G$ then a crossed pairing is provided by commutators:

$$\begin{array}{llll} (a) & \boxtimes \ : \ G \times N \to N, & & g \boxtimes n = [g,n] = (n^{-1})^g\, n, \\ (b) & \boxtimes \ : \ N \times G \to N, & & n \boxtimes g = [n,g] = n^{-1}\, n^g. \end{array}$$

Here are some standard properties of crossed pairings (see Proposition 3 of [18]).

Proposition 7.6 *The following relations hold for all* $g, g_1, g_2 \in G$ *and for all* $h, h_1, h_2 \in H$.

(a) $(g \boxtimes 1_H) \;=\; (1_G \boxtimes h) \;=\; 1_L$;

(b) $(g \boxtimes h)^{-1} \;=\; (g^{-1} \boxtimes h)^g \;=\; (g \boxtimes h^{-1})^h \;=\; (g^{-1} \boxtimes h^g) \;=\; (g^h \boxtimes h^{-1})$;

(c) $(g_1 \boxtimes h_1)^{h_2 g_2}\,(g_2 \boxtimes h_2) \;=\; (g_2 \boxtimes h_2)\,(g_1 \boxtimes h_1)^{g_2 h_2}$;

(d) $(g^h \boxtimes h_1) \;=\; (g \boxtimes h)^{-1}\,(g \boxtimes h_1)\,(g \boxtimes h)^{h_1}$ *and* $(g_1 \boxtimes h^g) \;=\; (g \boxtimes h)^{g_1}\,(g_1 \boxtimes h)\,(g \boxtimes h)^{-1}$;

(e) $(g_1 g_2 \boxtimes h) \;=\; (g_2 \boxtimes h^{g_1})(g_1 \boxtimes h)$ *and* $(g \boxtimes h_1 h_2) \;=\; (g \boxtimes h_1)(g^{h_1} \boxtimes h_2)$;

(f) $(g \boxtimes h)^{[g_2,h_2]} \;=\; (g_2 \boxtimes h_2)^{-1}\,(g \boxtimes h)\,(g_2 \boxtimes h_2)$;

(g) $(g^{-1}\, g^h \boxtimes h_1) \;=\; (g \boxtimes h)^{-1}\,(g \boxtimes h)^{h_1}$ *and* $(g_1 \boxtimes (h^{-1})^g\, h) \;=\; ((g \boxtimes h)^{-1})^{g_1}\,(g \boxtimes h)$;

(h) $[(g_1 \boxtimes h_1),\, (g_2 \boxtimes h_2)] \;=\; ((g_1^{-1}\, {g_1}^{h_1}) \boxtimes ((h_2^{-1})^{g_2}\, h_2))$.

Proof: Where there are two formulae, the proof of the second mirrors that of the first.

(a) $g \boxtimes h \;=\; g1 \boxtimes h \;=\; (g \boxtimes h)^1\,(1 \boxtimes h)$

(b) $1 \;=\; 1 \boxtimes h \;=\; g^{-1} g \boxtimes h \;=\; (g^{-1} \boxtimes h)^g\,(g \boxtimes h)$

(c)

$$\begin{array}{rrcl} & g_1 g_2 \boxtimes h_1 h_2 & = & (g_1 g_2 \boxtimes h_2)\,(g_1 g_2 \boxtimes h_1)^{h_2} \\ & & = & (g_1 \boxtimes h_2)^{g_2}\,(g_2 \boxtimes h_2)\,(g_1 \boxtimes h_1)^{g_2 h_2}\,(g_2 \boxtimes h_1)^{h_2} \\ \text{and} & g_1 g_2 \boxtimes h_1 h_2 & = & (g_1 \boxtimes h_1 h_2)^{g_2}\,(g_2 \boxtimes h_1 h_2) \\ & & = & (g_1 \boxtimes h_2)^{g_2}\,(g_1 \boxtimes h_1)^{h_2 g_2}\,(g_2 \boxtimes h_2)\,(g_2 \boxtimes h_1)^{h_2} \\ \Rightarrow & (g_1 \boxtimes h_1)^{h_2 g_2}\,(g_2 \boxtimes h_2) & = & (g_2 \boxtimes h_2)\,(g_1 \boxtimes h_1)^{g_2 h_2} \end{array}$$

(d)

$$\begin{array}{rcl} g^h \boxtimes h_1 & = & (g \boxtimes {h_1}^{h^{-1}})^h \;=\; (g \boxtimes h h_1 h^{-1})^h \\ & = & (g \boxtimes h^{-1})^h\,(g \boxtimes h_1)\,(g \boxtimes h)^{h_1} \\ & = & (g \boxtimes h)^{-1}\,(g \boxtimes h_1)\,(g \boxtimes h)^{h_1} \end{array}$$

(e) These alternative forms for 7.4 (a),(b) follow immediately from (d).

(f) Substitute $g = {g_1}^{h_2 g_2}$, $h = {h_1}^{h_2 g_2}$ in (c).

(g)

$$\begin{array}{rcll} (g^{-1} g^h) \boxtimes h_1 & = & (g^{-1} \boxtimes h_1)^{g^h}\,(g^h \boxtimes h_1) \;=\; (g^{-1} \boxtimes h_1^g)^{[g,h]}\,(g^h \boxtimes h_1) & \\ & = & (g \boxtimes h)^{-1}\,(g \boxtimes h_1)^{-1}\,(g \boxtimes h).(g \boxtimes h)^{-1}\,(g \boxtimes h_1)\,(g \boxtimes h)^{h_1} & \text{by (b),(d),(e)} \\ & = & (g \boxtimes h)^{-1}\,(g \boxtimes h)^{h_1} & \end{array}$$

(h)
$$\begin{aligned}
[g_1 \boxtimes h_1, g_2 \boxtimes h_2] &= (g_1 \boxtimes h_1)^{-1}\,(g_2 \boxtimes h_2)^{-1}\,(g_1 \boxtimes h_1)\,(g_2 \boxtimes h_2) \\
&= (g_1 \boxtimes h_1)^{-1}\,(g_1 \boxtimes h_1)^{[g_2,h_2]} \;=\; (g_1 \boxtimes h_1)^{-1}\,(g_1 \boxtimes h_1)^{(h_2^{-1})^{g_2} h_2} \quad \text{by (e)} \\
&= ((g_1^{-1}\, {g_1}^{h_1}) \boxtimes ((h_2^{-1})^{g_2}\, h_2)) \quad \text{by (f).}
\end{aligned}$$

□

Lemma 7.7 *The* principal crossed pairing *of a crossed module $\mathcal{X} = (\partial : S \to R)$ is given by*

$$\boxtimes : R \times S \to S,\ \ (r,s) \mapsto \eta_s(r) = (s^{-1})^r s\ .$$

Proof: We have seen in Example 7.3 that R and S have compatible actions. The three axioms are easily checked.

(a)
$$(r_1 \boxtimes s)^{r_2}(r_2 \boxtimes s) \;=\; ((s^{-1})^{r_1} s)^{r_2}((s^{-1})^{r_2} s) \;=\; (s^{-1})^{r_1 r_2} s \;=\; r_1 r_2 \boxtimes s$$

(b)
$$(r \boxtimes s_1)(r \boxtimes s_1)^{s_2} \;=\; ((s_2^{-1})^r s_2)(s_2^{-1}(s_1^{-1})^r s_1 s_2) \;=\; ((s_1 s_2)^{-1})^r (s_1 s_2) \;=\; r \boxtimes s_1 s_2$$

(c)
$$\begin{aligned}
r^{r_0} \boxtimes s^{r_0} &= ((s^{r_0})^{-1})^{r^{r_0}} s^{r_0} \;=\; (s^{-1})^{r r_0} s^{r_0} \;=\; (r \boxtimes s)^{r_0} \\
r^{\partial s_0} \boxtimes s^{s_0} &= ((s^{s_0})^{-1})^{s_0^{-1} r s_0} s^{s_0} \;=\; ((s^{-1})^r s)^{s_0} \;=\; (r \boxtimes s)^{s_0}
\end{aligned}$$

□

So we may write principal derivations as $\eta_s r = r \boxtimes s$ and principal sections as $\kappa_s r = (r, r \boxtimes s)$.

A standard result concerning crossed pairings shows that the nonabelian tensor product is the universal object for this construction.

Definition 7.8 *Given groups G and H which act compatibly on each other, the* nonabelian tensor product *$G \otimes H$ of G and H has generating set $\{g \otimes h \mid g \in G,\, h \in H\}$ subject to relations*

$$(g_1 g_2 \otimes h) \;=\; (g_1 \otimes h)^{g_2}\,(g_2 \otimes h)\ , \quad (g \otimes h_1 h_2) \;=\; (g \otimes h_2)\,(g \otimes h_1)^{h_2}\ , \tag{39}$$

where

$$(g \otimes h)^x \;=\; (g^x \otimes h^x) \quad \textit{for all } x \in G \cup H\ .$$

Theorem 7.9 *The nonabelian tensor product function*

$$\otimes \;:\; G \times H \to G \otimes H,\ \ (g,h) \mapsto g \otimes h$$

is a crossed pairing. Moreover, given any crossed pairing $\boxtimes : G \times H \to L$, there is a unique homomorphism $\boxtimes_\otimes : G \otimes H \to L$ satisfying $\boxtimes = \boxtimes_\otimes \circ \otimes$ so that the following diagram commutes:

$$\begin{tikzcd}
G \times H \arrow[d, "\otimes"'] \arrow[dr, "\boxtimes"] & \\
G \otimes H \arrow[r, "\boxtimes_\otimes"'] & L\ .
\end{tikzcd}$$

Checking that a potential crossed pairing satisfies the axioms of Definition 7.4 can be a tedious process. However we can convert this into checking that maps to certain semidirect products are homomorphisms.

Lemma 7.10 *Let $\boxtimes : G \times H \to L$ be a crossed pairing. Then*

(a) *given a fixed element $h \in H$, the map*

$$\theta_h \ : \ G \to G \ltimes L, \ \ g \mapsto (g, \, g \boxtimes h)$$

is a group homomorphism;

(b) *given a fixed element $g \in G$, the map*

$$\theta_g \ : \ H \to H \ltimes L, \ \ h \mapsto (h, (g \boxtimes h)^{-1})$$

is a group homomorphism.

Proof:

$$\theta_h(g_1 g_2) \ = \ (g_1 g_2, \, (g_1 \boxtimes h)^{g_2} \, (g_2 \boxtimes h)) \ = \ (g_1, \, g_1 \boxtimes h) \, (g_2, \, g_2 \boxtimes h) \ = \ (\theta_h g_1) \, (\theta_h g_2) \ .$$

$$\theta_g(h_1 h_2) \ = \ (h_1 h_2, \, \{(g \boxtimes h_2) \, ((g \boxtimes h_1))^{h_2}\}^{-1}) \ = \ (h_1, \, (g \boxtimes h_1)^{-1}) \, (h_2, \, (g \boxtimes h_2)^{-1}) \ = \ (\theta_g h_1) \, (\theta_g h_2) \ .$$

□

The converse proposition gives a way of checking that a given map is a crossed pairing.

Proposition 7.11 *Given a map $\odot : G \times H \to L$ and an action $(g \odot h)^x = g^x \odot h^x$ for all $x \in G \cup H$, such that for all $h \in H$ and $g \in G$*

- *$\theta_h : G \to G \ltimes L, \ g \mapsto (g, g \odot h)$ is a homomorphism, and*
- *$\theta_g : H \to H \ltimes L, \ h \mapsto (h, (g \odot h)^{-1})$ is a homomorphism,*

then $\odot$ is a crossed pairing.

Proof:

$$\begin{aligned} (g_1 g_2, \, g_1 g_2 \odot h) \ &= \ \theta_h(g_1 g_2) \ = \ (\theta_h g_1)(\theta_h g_2) \\ &= \ (g_1, \, g_1 \odot h)(g_2, \, g_2 \odot h) \\ &= \ (g_1 g_2, \, (g_1 \odot h)^{g_2} \, (g_2 \odot h)) \ . \end{aligned}$$

A similar argument shows that (7.4)(b) is also satisfied, so $\odot$ is a crossed pairing. □

[The remaining sections in this chapter are experimental.]

7.3 Interchange Laws

The crossed pairing rules of Definition 7.4:

$$(g_1 g_2 \boxtimes h) \;=\; (g_1 \boxtimes h)^{g_2}(g_2 \boxtimes h)\ , \qquad (g \boxtimes h_1 h_2) \;=\; (g \boxtimes h_2)(g \boxtimes h_1)^{h_2}\ ,$$

generalise to

$$\begin{aligned} (g_1 \dots g_k \boxtimes h) &= (g_1 \boxtimes h)^{g_2 \dots g_k}(g_2 \boxtimes h)^{g_3 \dots g_k} \dots (g_{k-1} \boxtimes h)^{g_k}(g_k \boxtimes h)\ , && (40)\\ \text{and} \quad (g \boxtimes h_1 \dots h_l) &= (g \boxtimes h_l)(g \boxtimes h_{l-1})^{h_l} \dots (g \boxtimes h_2)^{h_3 \dots h_l}(g \boxtimes h_1)^{h_2 \dots h_l}\ . && (41) \end{aligned}$$

The element $(g_1 g_2 \dots g_k \boxtimes h_1 h_2 \dots h_l)$ can be expanded in many ways, but by its *GH-expansion* we mean applying (40) first, and then applying (41) to each of the k terms which result. Similarly the *HG-expansion* is obtained by applying (41) first, and then (40) to the l terms.

In Proposition 7.6(c) we equated the two expansions of $(g_1 g_2 \boxtimes h_1 h_2)$ to obtain the basic *interchange law*

$$(g_1 \boxtimes h_1)^{h_2 g_2}(g_2 \boxtimes h_2) \;=\; (g_2 \boxtimes h_2)(g_1 \boxtimes h_1)^{g_2 h_2}\ . \qquad (42)$$

We now consider the cases $k = 2, l = 3$ and $k = l = 3$.

The GH-expansion of $(g_1 g_2 \boxtimes h_1 h_2 h_3)$ is:

$$\begin{aligned} g_1 g_2 \boxtimes h_1 h_2 h_3 &= (g_1 \boxtimes h_1 h_2 h_3)^{g_2}(g_2 \boxtimes h_1 h_2 h_3)\\ &= (g_1 \boxtimes h_3)^{g_2}(g_1 \boxtimes h_2)^{h_3 g_2}(g_1 \boxtimes h_1)^{h_2 h_3 g_2}(g_2 \boxtimes h_3)(g_2 \boxtimes h_2)^{h_3}(g_2 \boxtimes h_1)^{h_2 h_3} \end{aligned}$$

The HG-expansion of $(g_1 g_2 \boxtimes h_1 h_2 h_3)$ is:

$$\begin{aligned} g_1 g_2 \boxtimes h_1 h_2 h_3 &= (g_1 g_2 \boxtimes h_3)(g_1 g_2 \boxtimes h_2)^{h_3}(g_1 g_2 \boxtimes h_1)^{h_2 h_3}\\ &= (g_1 \boxtimes h_3)^{g_2}(g_2 \boxtimes h_3)(g_1 \boxtimes h_2)^{g_2 h_3}(g_2 \boxtimes h_2)^{h_3}(g_1 \boxtimes h_1)^{g_2 h_2 h_3}(g_2 \boxtimes h_1)^{h_2 h_3} \end{aligned}$$

By applying a series of interchanges to the second expansion of $(g_1 g_2 \boxtimes h_1 h_2 h_3)$, moving each term involving g_2 to the right, it is clear that these two expansions are equal:

$$\begin{aligned} & (g_1 \boxtimes h_3)^{g_2}\underbrace{(g_2 \boxtimes h_3)(g_1 \boxtimes h_2)^{g_2 h_3}}\,\underbrace{(g_2 \boxtimes h_2)^{h_3}(g_1 \boxtimes h_1)^{g_2 h_2 h_3}}(g_2 \boxtimes h_1)^{h_2 h_3}\\ = \;& (g_1 \boxtimes h_3)^{g_2}(g_1 \boxtimes h_2)^{h_3 g_2}\underbrace{(g_2 \boxtimes h_3)(g_1 \boxtimes h_1)^{h_2 g_2 h_3}}(g_2 \boxtimes h_2)^{h_3}(g_2 \boxtimes h_1)^{h_2 h_3}\\ = \;& (g_1 \boxtimes h_3)^{g_2}(g_1 \boxtimes h_2)^{h_3 g_2}(g_1 \boxtimes h_1)^{h_2 h_3 g_2}(g_2 \boxtimes h_3)(g_2 \boxtimes h_2)^{h_3}(g_2 \boxtimes h_1)^{h_2 h_3}\ . \end{aligned}$$

Similarly, by applying a series of interchanges to the HG-expansion of $(g_1 g_2 g_3 \boxtimes h_1 h_2 h_3)$, moving terms involving g_3 to the right, we obtain:

$$\begin{aligned}
& (g_1 \boxtimes h_3)^{g_2g_3}(g_2 \boxtimes h_3)^{g_3}\underbrace{(g_3 \boxtimes h_3)(g_1 \boxtimes h_2)^{g_2g_3h_3}}(g_2 \boxtimes h_2)^{g_3h_3} \\
& \qquad \underbrace{(g_3 \boxtimes h_2)^{h_3}(g_1 \boxtimes h_1)^{g_2g_3h_2h_3}}(g_2 \boxtimes h_1)^{g_3h_2h_3}(g_3 \boxtimes h_1)^{h_2h_3} \\
= \; & (g_1 \boxtimes h_3)^{g_2g_3}(g_2 \boxtimes h_3)^{g_3}(g_1 \boxtimes h_2)^{g_2h_3g_3}\underbrace{(g_3 \boxtimes h_3)(g_2 \boxtimes h_2)^{g_3h_3}} \\
& \qquad (g_1 \boxtimes h_1)^{g_2h_2g_3h_3}\underbrace{(g_3 \boxtimes h_2)^{h_3}(g_2 \boxtimes h_1)^{g_3h_2h_3}}(g_3 \boxtimes h_1)^{h_2h_3} \\
= \; & (g_1 \boxtimes h_3)^{g_2g_3}(g_2 \boxtimes h_3)^{g_3}(g_1 \boxtimes h_2)^{g_2h_3g_3}(g_2 \boxtimes h_2)^{h_3g_3} \\
& \qquad \underbrace{(g_3 \boxtimes h_3)(g_1 \boxtimes h_1)^{g_2h_2g_3h_3}}(g_2 \boxtimes h_1)^{h_2g_3h_3}(g_3 \boxtimes h_2)^{h_3}(g_3 \boxtimes h_1)^{h_2h_3} \\
= \; & (g_1 \boxtimes h_3)^{g_2g_3}(g_2 \boxtimes h_3)^{g_3}(g_1 \boxtimes h_2)^{g_2h_3g_3}(g_2 \boxtimes h_2)^{h_3g_3} \\
& \qquad (g_1 \boxtimes h_1)^{g_2h_2h_3g_3}\underbrace{(g_3 \boxtimes h_3)(g_2 \boxtimes h_1)^{h_2g_3h_3}}(g_3 \boxtimes h_2)^{h_3}(g_3 \boxtimes h_1)^{h_2h_3} \\
= \; & (g_1 \boxtimes h_3)^{g_2g_3}(g_2 \boxtimes h_3)^{g_3}(g_1 \boxtimes h_2)^{g_2h_3g_3}(g_2 \boxtimes h_2)^{h_3g_3} \\
& \qquad (g_1 \boxtimes h_1)^{g_2h_2h_3g_3}(g_2 \boxtimes h_1)^{h_2h_3g_3}(g_3 \boxtimes h_3)(g_3 \boxtimes h_2)^{h_3}(g_3 \boxtimes h_1)^{h_2h_3} \\
= \; & \Big(HG - \text{expansion of} \quad (g_1g_2 \boxtimes h_1h_2h_3)\Big)^{g_3}\Big(g_3 \boxtimes h_1h_2h_3\Big) \\
= \; & \Big(GH - \text{expansion of} \quad (g_1g_2 \boxtimes h_1h_2h_3)\Big)^{g_3}\Big(g_3 \boxtimes h_1h_2h_3\Big) \\
= \; & \Big(GH - \text{expansion of} \quad (g_1g_2g_3 \boxtimes h_1h_2h_3)\Big) \; .
\end{aligned}$$

These two examples indicate how to develop an inductive argument to prove the following result.

Proposition 7.12 *The GH-expansion and HG-expansion of* $(g_1g_2 \ldots g_k \boxtimes h_1h_2 \ldots h_l)$ *are equal.*

Proof: Assume the result to be true for $(g_1g_2 \ldots g_{k-1} \boxtimes h_1h_2 \ldots h_l)$.
The first $l-1$ terms involving g_k in the HG-expansion can each be interchanged with the term on the right:

$$\{(g_k \boxtimes h_j)(g_1 \boxtimes h_{j+1})^{g_2 \ldots g_k h_j}\}^{h_{j+1} \ldots h_l} \quad \longrightarrow \quad \{(g_1 \boxtimes h_{j+1})^{g_2 \ldots g_{k-1} h_j g_k}(g_k \boxtimes h_j)\}^{h_{j+1} \ldots h_l} \; ,$$

and then with the next term on the right:

$$\{(g_k \boxtimes h_j)(g_2 \boxtimes h_{j+1})^{g_3 \ldots g_k h_j}\}^{h_{j+1} \ldots h_l} \quad \longrightarrow \quad \{(g_2 \boxtimes h_{j+1})^{g_3 \ldots g_{k-1} h_j g_k}(g_k \boxtimes h_j)\}^{h_{j+1} \ldots h_l} \; ,$$

and so on. After $k-1$ sets of interchanges the last two terms in the expansion are $(g_k \boxtimes h_2)^{h_3 \ldots h_l}\,(g_k \boxtimes h_1)^{h_2 \ldots h_l}$, and there remain $l-2$ terms involving g_k still to be moved to the right. After a total of $\frac{1}{2}(k-1)(l-1)l$ interchanges, we obtain

$$\begin{aligned}
& \Big(HG - \text{expansion of} \quad (g_1 \ldots g_k \boxtimes h_1 \ldots h_l)\Big) \\
= \; & \Big(HG - \text{expansion of} \quad (g_1 \ldots g_{k-1} \boxtimes h_1 \ldots h_l)\Big)^{g_k}\Big(g_k \boxtimes h_1 \ldots h_l\Big) \\
= \; & \Big(GH - \text{expansion of} \quad (g_1 \ldots g_{k-1} \boxtimes h_1 \ldots h_l)\Big)^{g_k}\Big(g_k \boxtimes h_1 \ldots h_l\Big) \qquad \text{(by induction)} \\
= \; & \Big(GH - \text{expansion of} \quad (g_1 \ldots g_k \boxtimes h_1 \ldots h_l\Big) \; .
\end{aligned}$$

A similar argument can be used when starting with the GH-expansion. □

7.4 An alternative construction for the tensor product

This describes work of Rodrigues and Wensley.

Let G and H be groups which act on themselves and each other in a compatible way.

A presentation $< C_G \,|\, R_G >$ for G is constructed as follows.

- A generating set C_G for G is chosen, which is closed under the actions of both G and H.
- A list of elements E_G for G is initialised as $[1_G]$ and R_G is initialised as an empty list.
- A fixed word in the generators for each element of G is chosen by the following iteration:
  - take the next element g from E_G ;
  - for each $c \in C_G$ let $g' = gc$;
  - if $g' \notin E_G$ then add g' to E_G, else add $gc = g'$ to R_G ;

A similar presentation $< C_H \,|\, R_H >$ for H is also constructed. We use the notation

$$C_G = \{c_1, c_2, \dots\}, \quad E_G = \{g_1, g_2, \dots\}, \quad C_H = \{d_1, d_2, \dots\}, \quad E_H = \{h_1, h_2, \dots\} \ .$$

Define $G \odot H =< C_{(G \odot H)} \,|\, R_{(G \odot H)} >$ to be the group where

- $C_{(G \odot H)}$ is the set of symbols $\{c \odot d \,|\, c \in C_G,\, d \in C_H\}$ with actions $(c \odot d)^x \ = \ (c^x \odot h^x)$ for all $x \in G \cup H$,
- $R_{(G \odot H)}$ is the relator set consisting of:

(1) for each $gc = g' \in R_G$ where $g = c_1 c_2 \dots c_k,\ g' = c'_1 c'_2 \dots c'_{k'}$, and for each $d_j \in C_H$,

$$(c_1 \odot d_j)^{c_2 \dots c_k c} \dots (c_k \odot d_j)^{c} \, (c \odot d_j) \quad = \quad (c'_1 \odot d_j)^{c'_2 \dots c'_{k'}} \dots (c'_{k'} \odot d_j) \ ;$$

(2) for each $hd = h' \in R_H$ where $h = d_1 d_2 \dots d_k,\ h' = d'_1 d'_2 \dots d'_{k'}$, and for each $c_i \in C_G$,

$$(c_i \odot d) \, (c_i \odot d_k)^{d} \dots (c_i \odot d_1)^{d_2 \dots d_k d} \quad = \quad (c'_i \odot d'_{k'}) \dots (c'_i \odot d'_1)^{d'_2 \dots d'_{k'}} \ ;$$

(3) for each $c_i, c_{i'} \in C_G$ and for each $d_j, d_{j'} \in C_H$ the interchange laws

$$(c_{i'} \odot d_{j'})^{d_j c_i} \, (c_i \odot d_j) \quad = \quad (c_i \odot d_j) \, (c_{i'} \odot d_{j'})^{c_i d_j} \ .$$

(**Note:** we hope to restrict to a subset of these in due course.)

Define a map, which we shall show to be a crossed pairing, $\odot : G \times H \to G \odot H,\ (g,h) \mapsto g \odot h$, extending the obvious map $C_G \times C_H \to C_{(G \odot H)}$ to the whole of $G \times H$ by

(a) if $g_i \ = \ c_1 c_2 \dots c_k$ then $g_i \odot d_j \ = \ (c_1 \odot d_j)^{c_2 \dots c_k} \dots (c_{k-1} \odot d_j)^{c_k} \, (c_k \odot d_j)$,

(b) if $h_j \ = \ d_1 d_2 \dots d_l$ then $c_i \odot h_j \ = \ (c_i \odot d_l) \, (c_i \odot d_{l-1})^{d_l} \dots (c_i \odot d_1)^{d_2 \dots d_l}$,

(c) $g_k \odot h_l \ = \ GH$-expansion of $c_1 c_2 \dots c_k \odot d_1 d_2 \dots d_l$.

Recall the *Substitution Test* (see Johnson, p.29), a corollary of von Dyck's Theorem, which we may use to verify that a given map is a homomorphism of finitely presented groups.

Theorem 7.13 *Suppose we are given a presentation* $\Gamma =< C\,|\,R >$*, a group* Δ *and a mapping* $\theta' : C \to \Delta$ *. Then* θ' *extends to a homomorphism* $\theta : \Gamma \to \Delta$ *if and only if, for all* $(u_i = v_i) \in R$*, the result of substituting* $\theta' c$ *for each letter* c *in* u_i *and* v_i *yeilds a relation for* Δ *.*

Lemma 7.14 *The map*

$$\theta'_{d_j} : C_G \to G \ltimes G \odot H, \quad c_i \mapsto (c_i,\, c_i \odot d_j)$$

determines a homomorphism $\theta_{d_j} : G \to G \ltimes (G \odot H)$ *.*

Proof: Let $(gc = g') \in R_G$ be a deduction for G. Then, by definition of θ_{d_j} and by the semidirect product multiplication rule,

$$\begin{aligned} \theta_{d_j}(gc) \;&=\; (\theta_{d_j}c_1)\dots(\theta_{d_j}c_k)(\theta_{d_j}c) \;&=&\; (c_1, c_1 \odot d_j)(c_2, c_2 \odot d_j)\dots(c_k, c_k \odot d_j)(c, c \odot d_j) \\ & &=&\; (c_1 \dots c_k c,\, (c_1 \odot d_j)^{c_2 \dots c_k c} \dots (c_k \odot d_j)^{c}\,(c, c \odot d_j)\,) \;, \\ \text{and} \qquad \theta_{d_j}(g') \;&=\; (\theta_{d_j}c'_1)\dots(\theta_{d_j}c'_{k'}) \;&=&\; (c'_1 \dots c'_{k'},\, (c'_1 \odot d_j)^{c'_2 \dots c'_{k'}} \dots (c'_{k'} \odot d_j)\,) \;. \end{aligned}$$

The two $(G \odot H)$-components in these expansions are equal by a relator of type (1) in $R_{(G \odot H)}$. □

Lemma 7.15 *The map*

$$\theta'_{d_{j_1} \dots d_{j_l}} : C_G \to G \ltimes (G \odot H), \quad c_i \mapsto (c_i,\, c_i \odot d_{j_1} \dots d_{j_l})$$

determines a homomorphism $\theta_{d_{j_1} \dots d_{j_l}} : G \to G \ltimes (G \odot H)$ *.*

Proof: We again use the Substitution Test to verify this result.

The proof of Proposition 7.12 requires only the basic interchange laws for elements of $G \otimes H$. Since the interchange laws for the generating elements of $G \odot H$ have been imposed in $R_{(G \odot H)}$, it is also true that the GH-expansion and HG-expansion of $(c_1 c_2 \dots c_k \odot d_1 d_2 \dots d_l)$ are equal.

Let $(gc = g') \in R_G$ be a deduction for G. Then, by definition of $\theta_{d_{j_1} \dots d_{j_l}}$,

$$\theta_{d_{j_1} \dots d_{j_l}}(c_1 c_2 \dots c_k c) \;=\; (c_1, c_1 \odot d_{j_1} \dots d_{j_l}) \dots (c_k, c_k \odot d_{j_1} \dots d_{j_l})(c, c \odot d_{j_1} \dots d_{j_l}) \;.$$

Part (b) of the definition of $\odot$ gives,

$$(c',\, c' \odot d_{j_1} \dots d_{j_l}) \;=\; (c',\, c' \odot d_{j_l})\,(c',\, c' \odot d_{j_{l-1}})^{d_{j_l}} \dots (c',\, c' \odot d_{j_1})^{d_{j_2} \dots d_{j_l}} \quad \text{for all} \quad c' \in C_G \;.$$

Hence

$$\begin{aligned} \theta_{d_{j_1} \dots d_{j_l}}(gc) \;&=\; GH - \text{expansion of} \quad gc \odot d_{j_1} \dots d_{j_l} \\ &=\; HG - \text{expansion of} \quad gc \odot d_{j_1} \dots d_{j_l} \\ &=\; (\theta_{d_{j_1}}(gc))\,(\theta_{d_{j_2}}(gc)) \dots (\theta_{d_{j_l}}(gc)) \\ &=\; (\theta_{d_{j_1}}(g'))\,(\theta_{d_{j_2}}(g')) \dots (\theta_{d_{j_l}}(g')) \qquad\qquad \text{by the previous Lemma,} \\ &=\; HG - \text{expansion of} \quad g' \odot d_{j_1} \dots d_{j_l} \\ &=\; GH - \text{expansion of} \quad g' \odot d_{j_1} \dots d_{j_l} \\ &=\; \theta_{d_{j_1} \dots d_{j_l}}(g') \;. \end{aligned}$$

□

Theorem 7.16 *The group* $G \odot H$ *is isomorphic to* $G \otimes H$ *.*

Proof: We have shown in Lemma 7.15 that $\theta_h : G \to G \ltimes (G \odot H)$ is a homomorphism, and it can be shown in the same way that $\theta_g : H \to H \ltimes (G \odot H)$ is also a homomorphism. It follows from Proposition 7.11 that $\odot$ is a crossed pairing. The unique homomorphism $\odot_\otimes : G \otimes H \to G \odot H$ given by the universal property of Theorem 7.9 maps $g \otimes h$ to $g \odot h$.

In order to show that $G \odot H$ is a universal crossed pairing we require homomorphisms $\boxtimes_\odot$ and $\otimes_\odot$ making the following diagram commute:

$$\begin{tikzcd}
& G \times H \arrow[dl, "\otimes"'] \arrow[d, "\odot"] \arrow[dr, "\boxtimes"] & \\
G \otimes H \arrow[r, shift left, "\odot_\otimes"] \arrow[rr, bend right, "\boxtimes_\otimes"'] & G \odot H \arrow[l, shift left, "\otimes_\odot"] \arrow[r, "\boxtimes_\odot"'] & L \, .
\end{tikzcd}$$

Commutativity forces the images of the generators to be given by maps

$$\boxtimes'_\odot : C_{(G \odot H)} \to L \, , \; c \odot d \mapsto c \boxtimes d \, , \qquad \otimes'_\odot : C_{(G \odot H)} \to G \otimes H \, , \; c \odot d \mapsto c \otimes d \, .$$

It is easy to see that these two maps determine the required homomorphisms and that these are unique. Since $G \odot H$ satisfies the universal property of the tensor product, $G \odot H \cong G \otimes H$ and $\odot_\otimes$, $\otimes_\odot$ are inverse isomorphisms. □

7.5 The Heisenberg group

This describes work of Bacon and Kappe.

Nilpotent groups generalise abelian groups in that multiple commutators of a certain length are all trivial. A nilpotent group G of class 2 is such that its commutator subgroup G' is a subgroup of the centre of G.

The Heisenberg group $\mathcal{H}_2$ is the free nilpotent group of class 2 with presentation

$$\mathcal{H}_2 \quad = \quad < \, a, b, c \mid [a, b] = c, \; [a, c] = 1, \; [b, c] = 1 \, > \quad .$$

Proposition 7.17

$$\mathcal{H}_2 \, \otimes \, \mathcal{H}_2 \quad \cong \quad \mathbf{Z}^6 \; .$$

Proof: Here is a brief outline of the proof.

- show that the tensor square is abelian;
- show that every element of $\mathcal{H}_2$ can be written in the form $a^m \, b^n \, c^l$;
- show that every element of the tensor square is a sum of multiples of

$$a \otimes a, \quad b \otimes b, \quad a \otimes b, \quad b \otimes a, \quad a \otimes c, \quad \text{and} \quad b \otimes c \, ;$$

- it follows that the tensor square is a quotient of $\mathbf{Z}^6$;
- construct a surjective crossed pairing $\phi : \mathcal{H}_2 \times \mathcal{H}_2 \to \mathbf{Z}^6$;
- it follows that there is a unique homomorphism from the tensor square onto $\mathbf{Z}^6$ and hence that the tensor square is isomorphic to $\mathbf{Z}^6$.

□

8 Crossed Squares and Cat^2-groups

Crossed squares were introduced by Guin-Waléry and Loday (see, for example, [38, 43, 19]) as fundamental crossed squares of commutative squares of spaces, but are also of purely algebraic interest.

Definition 8.1 *A crossed square consists of the following:*

(i) *a commutative diagram of group homomorphisms*

$$\mathcal{S} \quad = \quad \begin{tikzcd} L \arrow[r, "\kappa"] \arrow[d, "\lambda"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd} \quad ; \tag{43}$$

(ii) *actions of P on N, M and L which determine actions of N on M and L via ν and actions of M on N and L via μ ;*

(iii) *a function* $\boxtimes : N \times M \to L$.

The following axioms must be satisfied for all $l \in L,\ m, m_1, m_2 \in M,\ n, n_1, n_2 \in N,\ p \in P$:

(a) *the homomorphisms κ, λ preserve the action of P ;*

(b) *each of $\mathcal{K} = (\kappa : L \to M),\ \mathcal{L} = (\lambda : L \to N),\ \mathcal{M} = (\mu : M \to P),\ \mathcal{N} = (\nu : N \to P)$ and the diagonal $\mathcal{D} = (\delta = \nu \circ \lambda = \mu \circ \kappa : L \to P)$ are crossed modules;*

(c) $\boxtimes$ *is a* crossed pairing*:*

$$\begin{aligned} &\text{(i)} \quad (n_1 n_2 \boxtimes m) \ = \ (n_1 \boxtimes m)^{n_2}\,(n_2 \boxtimes m)\ , \\ &\text{(ii)} \quad (n \boxtimes m_1 m_2) \ = \ (n \boxtimes m_2)\,(n \boxtimes m_1)^{m_2}\ , \\ &\text{(iii)} \quad (n \boxtimes m)^p \ = \ (n^p \boxtimes m^p)\ ; \end{aligned}$$

(d) $\quad \lambda(n \boxtimes m) \ = \ n^{-1}\, n^m \quad$ *and* $\quad \kappa(n \boxtimes m) \ = \ (m^{-1})^n\, m\ ,$

(e) $\quad (\lambda l \boxtimes m) \ = \ l^{-1}\, l^m \quad$ *and* $\quad (n \boxtimes \kappa l) \ = \ (l^{-1})^n\, l\ .$

Note that the actions of N on M and M on N via P are compatible since

$$m_1{}^{(n^m)} \ = \ m_1{}^{\nu(n^m)} \ = \ m_1{}^{m^{-1}(\nu n)m} \ = \ ((m_1{}^{m^{-1}})^n)^m\ .$$

Note also that identities c(i) and c(ii) are based on the commutator identities

$$[n, m] \ = \ n^{-1}m^{-1}nm, \qquad [n_1 n_2, m] \ = \ [n_1, m]^{n_2}[n_2, m], \qquad [n, m_1 m_2] \ = \ [n, m_2][n, m_1]^{m_2}.$$

Lemma 8.2

$$1 = (1 \boxtimes m) = (n \boxtimes 1) \qquad \textit{and} \qquad (n \boxtimes m)^{-1} = (n^{-1} \boxtimes m)^n = (n \boxtimes m^{-1})^m.$$

Proof: These follow immediately on expanding $(n.1 \boxtimes m),\ (n \boxtimes m.1),\ (n^{-1}n \boxtimes m)$ and $(n \boxtimes m^{-1}m)$.
□

Lemma 8.3 *Pairs* $(\kappa,\nu):\mathcal{L}\to\mathcal{M}$ *and* $(\lambda,\mu)=\mathcal{K}\to\mathcal{N}$ *are morphisms of crossed modules, and so also are* $(\mathrm{id}_L,\nu):\mathcal{L}\to\mathcal{D}$, $(\mathrm{id}_L,\mu):\mathcal{K}\to\mathcal{D}$, $(\lambda,\mathrm{id}_P):\mathcal{D}\to\mathcal{N}$ *and* $(\kappa,\mathrm{id}_P):\mathcal{D}\to\mathcal{M}$.

Proof: For (κ,ν) we note that (43) commutes, that $\kappa(\ell^m)=(\kappa l)^m$ by (a), and that $(\kappa\ell)^n=(\kappa\ell)^{\nu n}$ by (ii). The arguments for the other five morphisms are similar. □

Note in particular that

$$\kappa(\ell^p)=(\kappa\ell)^p \qquad \text{and} \qquad \lambda(\ell^p)=(\lambda\ell)^p.$$

Lemma 8.4 *In the crossed square* $\mathcal{S}$ *above:*

(a) $$\ell^{(n\boxtimes m)} \;=\; \ell^{[n,m]}\,,$$

(b) $$\ell^{mn}\,(n\boxtimes m) \;=\; (n\boxtimes m)\,\ell^{nm} \qquad \textit{and} \qquad \ell^{(m^n)}\,(n\boxtimes m) \;=\; (n\boxtimes m)\,\ell^{m}\,,$$

(c) $$m^{\lambda\ell}=m^{\kappa\ell} \qquad \textit{and} \qquad n^{\lambda\ell}=n^{\kappa\ell}\,,$$

(d) $$(n\boxtimes m)^{m'n'}(n'\boxtimes m') \;=\; (n'\boxtimes m')(n\boxtimes m)^{n'm'}\,.$$

Proof:

(a) By the crossed module identity **(X2)**,

$$\ell^{(n\boxtimes m)} \;=\; \ell^{\nu\lambda(n\boxtimes m)} \;=\; \ell^{\nu(n^{-1}n^m)} \;=\; \ell^{\nu(n^{-1}n^{\mu m})} \;=\; \ell^{[\nu n,\mu m]} \;=\; \ell^{[n,m]}\,.$$

(b) The first identity is given by

$$(n\boxtimes m)^{-1}\,\ell^{mn}\,(n\boxtimes m) \;=\; \ell^{mn\lambda(n\boxtimes m)} \;=\; \ell^{mn(n^{-1})n^m} \;=\; \ell^{nm}\,.$$

Then replace ℓ by $\ell^{n^{-1}}$ to get the second.

(c)
$$m^{\lambda\ell} \;=\; m^{\nu\lambda\ell} \;=\; m^{\mu\kappa\ell} \;=\; m^{\kappa\ell}\,, \qquad \text{and similarly for } n.$$

(d) Expand $(nn'\boxtimes mm')$ in two different ways.

□

8.1 Examples of crossed squares

Example 8.5 If M,N are normal subgroups of the group P then the diagram of inclusions

$$\begin{tikzcd}
M\cap N \arrow[r, "i_1"] \arrow[d, "i_2"'] \arrow[dr, "i_{12}"] & M \arrow[d, "i_M"] \\
N \arrow[r, "i_N"'] & P
\end{tikzcd}$$

together with the actions of P on M,N and $M\cap N$ given by conjugation and the function

$$\boxtimes \;:\; N\times M\to M\cap N, \quad (n,m)\mapsto[n,m] \;=\; n^{-1}m^{-1}nm$$

is a crossed square. We may check the axioms as follows:

(a) The identity maps preserve P-actions.

(b) The five crossed modules are all conjugation crossed modules.

(c) We have already noted that (i) and (ii) are analogues of commutator identities. For (iii),

$$(n^p \boxtimes m^p) \;=\; [n^p, m^p] \;=\; \{p^{-1}np\}^{-1}\,\{p^{-1}mp\}^{-1}\,\{p^{-1}np\}\{p^{-1}mp\} \;=\; p^{-1}[n,m]p \;=\; [n,m]^p\,.$$

(d) $\quad \ddot{\imath}_1(n \boxtimes m) \;=\; n^{-1}\,(m^{-1}nm) \;=\; n^{-1}\,n^m \quad \textit{and} \quad \ddot{\imath}_2(n \boxtimes m) \;=\; (n^{-1}m^{-1}n)m \;=\; m^{-1^n}\,m\;.$

(e) $\quad (n \boxtimes i_1 l) \;=\; (n^{-1}l^{-1}n)l \;=\; (l^{-1})^n\,l \quad \textit{and} \quad (i_2 l \boxtimes m) \;=\; l^{-1}\,(m^{-1}lm) \;=\; l^{-1}\,l^m\;.$

Example 8.6 If M, N are ordinary $P-$modules and A is an arbitrary abelian group on which P is assumed to act trivially, then there is a crossed square

$$\begin{tikzcd}
A \arrow[r, "0"] \arrow[d, "0"'] \arrow[dr, "0"] & M \arrow[d, "0"] \\
N \arrow[r, "0"'] & P
\end{tikzcd}$$

Note that M acts trivially on N, and conversely, and that $n \boxtimes m = 1_A$.

Example 8.7 The diagram

$$\begin{tikzcd}
M \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & \mathrm{Inn}M \arrow[d, "\iota"] \\
\mathrm{Inn}M \arrow[r, "\iota"'] & \mathrm{Aut}M
\end{tikzcd}$$

is a crossed square, where α maps $m \in M$ to the inner automorphism $\beta_m : M \to M,\ m' \mapsto m^{-1}m'm$; where ι is the inclusion of Inn$M$ in Aut$M$; the actions are standard; and the crossed pairing is

$$\boxtimes \;:\; \mathrm{Inn}M \times \mathrm{Inn}M \to M, \quad (\beta_m, \beta_{m'}) \;\mapsto\; [m, m']\;.$$

Example 8.8 If U, V are subspaces of a space X with a point x_0 in common, then the diagram of boundary maps

$$\begin{tikzcd}
\pi_3(X; U, V, x_0) \arrow[r] \arrow[d] & \pi_2(V, U \cap V, x_0) \arrow[d] \\
\pi_2(U, U \cap V, x_0) \arrow[r] & \pi_1(U \cap V, x_0)
\end{tikzcd}$$

in which $\pi_3(X; U, V, x_0)$ is the triad homotopy group, together with the standard actions and the triad Whitehead product

$$\boxtimes \;:\; \pi_2(U, U \cap V, x_0) \times \pi_2(V, U \cap V, x_0) \;\to\; \pi_3(X; U, V, x_0)$$

is a crossed square.

Lemma 8.9 *The* transpose

$$\tilde{\mathcal{S}} \quad = \quad \begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\kappa"'] & N \arrow[d, "\nu"] \\ M \arrow[r, "\mu"'] & P \end{tikzcd} \qquad \textit{of} \qquad \mathcal{S} \quad = \quad \begin{tikzcd} L \arrow[r, "\kappa"] \arrow[d, "\lambda"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

is a crossed square with crossed pairing

$$\tilde{\boxtimes} \ : \ M \times N \to L, \quad (m,n) \ \mapsto \ m \ \tilde{\boxtimes} \ n := (n \boxtimes m)^{-1} \ . \tag{44}$$

Proof:

$$\begin{aligned} m_1 m_2 \ \tilde{\boxtimes} \ n \ &= \ (n \boxtimes m_1 m_2)^{-1} \ = \ ((n \boxtimes m_2)\,(n \boxtimes m_1)^{m_2})^{-1} \\ &= \ ((n \boxtimes m_1)^{m_2})^{-1}\,(n \boxtimes m_2)^{-1} \ = \ (m_1 \ \tilde{\boxtimes} \ n)^{m_2}\,(m_2 \ \tilde{\boxtimes} \ n) \ ; \\ m \ \tilde{\boxtimes} \ n_1 n_2 \ &= \ (n_1 n_2 \boxtimes m)^{-1} \ = \ ((n_1 \boxtimes m)^{n_2}\,(n_2 \boxtimes m))^{-1} \\ &= \ ((n_2 \boxtimes m)^{-1}\,((n_1 \boxtimes m)^{n_2})^{-1} \ = \ (m \ \tilde{\boxtimes} \ n_2)\,(m \ \tilde{\boxtimes} \ n_1)^{n_2} \ ; \\ (m \ \tilde{\boxtimes} \ n)^p \ &= \ ((n \boxtimes m)^{-1})^p \ = \ ((n \boxtimes m)^p)^{-1} \ = \ (n^p \boxtimes m^p)^{-1} \ = \ (m^p \ \tilde{\boxtimes} \ n^p) \ . \end{aligned}$$

□

Example 8.10 The actor crossed module $\mathcal{A}(\mathcal{X})$ of a crossed module $\mathcal{X}$ (see subsection 3.2)

$$\mathcal{A} \quad = \quad \begin{tikzcd} S \arrow[r, "\ddot{\imath}"] \arrow[d, "\partial"'] & W \arrow[d, "\Delta"] \\ R \arrow[r, "\imath"'] & A \end{tikzcd} \tag{45}$$

is a crossed square with crossed pairing

$$\boxtimes \ : \ R \times W \ \to \ S, \quad (r,\chi) \ \mapsto \ \chi r \ .$$

We already know that the square $\mathcal{A}$ contains 5 crossed modules, but we still need to check the axioms (c), (d), and (e) which involve the crossed pairing:

(c) (i) $(qr \boxtimes \chi) \ = \ \chi(qr) \ = \ (\chi q)^r\,(\chi r) \ = \ (q \boxtimes \chi)^r\,(r \boxtimes \chi)$.

(ii) $(r \boxtimes \chi_1 \star \chi_2) \ = \ (\chi_2 r)(\chi_1 r)(\chi_2 \partial \chi_1 r) \ = \ (\chi_2 r)\,(\chi_1 r)^{\chi_2} \ = \ (r \boxtimes \chi_2)\,(r \boxtimes \chi_1)^{\chi_2}$,

using the action of W on S given in Lemma 3.5.

(iii) $(r^\beta \boxtimes \chi^\beta) \ = \ \chi^\beta(r^\beta) \ = \ (\ddot{\beta}\chi\dot{\beta}^{-1})(\dot{\beta}r) \ = \ \ddot{\beta}(\chi r) \ = \ (r \boxtimes \chi)^\beta$.

(d) The first formula follows by

$\partial(r \boxtimes \chi) \ = \ \partial\chi(r) \ = \ r^{-1}r(\partial\chi r) \ = \ r^{-1}(\dot{\beta}_\chi r) \ = \ r^{-1}r^\chi$.

For the second formula, since $\ddot{\imath}(r \boxtimes \chi) = \ddot{\imath}(\chi r) = \eta_{\chi r}$, we wish to prove that

$$\eta_{\chi r} = (\chi^{-1})^r \star \chi \quad \text{or, equivalently,} \quad \chi^r = \chi \star (\eta_{\chi r})^{-1} = \chi \star \eta_{(\chi r)^{-1}} \ .$$

Starting with the right-hand side,

$$\begin{array}{rcll} (\chi \star \eta_{(\chi r)^{-1}})q & = & (\eta_{(\chi r)^{-1}}q)(\ddot{\beta}_{\eta_{(\chi r)^{-1}}}\chi q) & \text{by Lemma 2.4 (c)} \\ & = & (\chi r)^q (\chi r)^{-1} (\chi q)^{(\chi r)^{-1}} & \text{by Lemma 2.8 (b)} \\ & = & (\chi r)^q (\chi q)(\chi r)^{-1} \;=\; (\chi r)^q (\chi q)(\chi r^{-1})^r & \text{by Lemma 2.2 (b)} \\ & = & \ddot{\beta}_r((\chi r)^{qr^{-1}} (\chi q)^{r^{-1}} (\chi r^{-1})) \;=\; \ddot{\beta}_r \chi (rqr^{-1}) \;=\; (\ddot{\beta}_r \chi \dot{\beta}_r^{-1})q & \text{by Lemma 2.4 (c)} \\ & = & \chi^{ir} q \;=\; \chi^r q\,. & \end{array}$$

(e) $$\begin{array}{ll} (\partial s \boxtimes \chi) \;=\; \chi(\partial s) \;=\; s^{-1}s(\chi \partial s) \;=\; s^{-1}(\ddot{\beta}_\chi s) \;=\; s^{-1}s^\chi \;, & \\ (r \boxtimes \ddot{\imath}(s)) \;=\; \eta_s(r) \;=\; (s^{-1})^r s & \text{by Lemma 2.7.} \end{array}$$

8.2 Morphisms of crossed squares

A morphism $\theta : \mathcal{S}_1 \to \mathcal{S}_2$ of crossed squares is a 4-tuple of group homomorphisms which commute with the morphisms in $\mathcal{S}_1$ and $\mathcal{S}_2$ and preserve all the actions and the crossed pairings.

Definition 8.11 *A morphism $\theta : \mathcal{S}_1 \to \mathcal{S}_2$ of crossed squares consists of four group homomorphisms*

$$\theta_L : L_1 \to L_2, \quad \theta_M : M_1 \to M_2, \quad \theta_N : N_1 \to N_2, \quad \theta_P : P_1 \to P_2,$$

forming a commutative cube with the morphisms $\kappa_1, \lambda_1, \mu_1, \nu_1$ of $\mathcal{S}_1$ and $\kappa_2, \lambda_2, \mu_2, \nu_2$ of $\mathcal{S}_2$, which pair off in appropriate ways to form crossed module morphisms

$$(\theta_L, \theta_M) : \mathcal{K}_1 \to \mathcal{K}_2, \quad (\theta_L, \theta_N) : \mathcal{L}_1 \to \mathcal{L}_2, \quad (\theta_M, \theta_P) : \mathcal{M}_1 \to \mathcal{M}_2, \quad (\theta_N, \theta_P) : \mathcal{N}_1 \to \mathcal{N}_2,$$

and which preserve the crossed pairing:

$$\theta_L(n \boxtimes_1 m) \;=\; (\theta_N n) \boxtimes_2 (\theta_M m).$$

Definition 8.12 *The group* $\mathrm{Aut}(\mathcal{S})$ *of automorphisms of the crossed square $\mathcal{S}$ is*

$$\mathrm{Aut}(\mathcal{S}) \;=\; \{\alpha = (\alpha_L, \alpha_M, \alpha_N, \alpha_P) \;:\; \mathcal{S} \to \mathcal{S}\}$$

such that (α_L, α_M) is an automorphism of $\mathcal{K}$, (α_L, α_N) is an automorphism of $\mathcal{L}$, (α_M, α_P) is an automorphism of $\mathcal{M}$, (α_N, α_P) is an automorphism of $\mathcal{N}$, and $\alpha_L(n \boxtimes m) \;=\; (\alpha_N n) \boxtimes (\alpha_M m)$.

[Surely there is a crossed square version of Theorem 1.8 ?]

Theorem 8.13 *Every crossed square is a quotient of normal inclusion crossed squares.* **?**

(Note: notes 10/7/03 only go one way.)

8.3 Cat^2-groups

When defining a cat^2-group we may require all the homomorphisms to be endomorphisms, as in Definition 1.19, or we may take a more general view, as in Definition 1.20. For now, we consider only endomorphisms. When we come to define cat^n-groups we shall give a similar set of definitions. Firstly, we give the definition of a *traditional* cat^2-group adapted from Section 5 of Brown and Loday [19] and Ellis-Steiner [32].

Definition 8.14 A cat^2-group $\mathcal{C} = (G; \tau_1, \theta_1, \tau_2, \theta_2)$ comprises a group G and 4 endomorphisms, as shown in the following diagram, where $Q_1 = \operatorname{im} \tau_1 = \operatorname{im} \theta_1$, $Q_2 = \operatorname{im} \tau_2 = \operatorname{im} \theta_2$ and $Q_0 = Q_1 \cap Q_2$,

$$\mathcal{C} \quad = \quad \begin{tikzcd}[row sep=huge, column sep=huge]
G \arrow[r, shift left, "\tau_1{,}\theta_1"] \arrow[r, shift right] \arrow[d, shift right, "\tau_2{,}\theta_2"'] \arrow[d, shift left] \arrow[dr, shift left] \arrow[dr, shift right, "\tau_0{,}\theta_0"'] & Q_1 \arrow[d, shift left, "\tau_2{,}\theta_2"] \arrow[d, shift right] \\
Q_2 \arrow[r, shift left] \arrow[r, shift right, "\tau_1{,}\theta_1"'] & Q_0
\end{tikzcd} \tag{46}$$

subject to the following axioms:

(a) $\mathcal{C}_1 = (G; \tau_1, \theta_1)$ and $\mathcal{C}_2 = (G; \tau_2, \theta_2)$ are (traditional) cat^1-groups,

(b) $\tau_1 \circ \tau_2 = \tau_2 \circ \tau_1 = \tau_0, \quad \theta_1 \circ \theta_2 = \theta_2 \circ \theta_1 = \theta_0,$

(c) $\tau_1 \circ \theta_2 = \theta_2 \circ \tau_1, \quad \tau_2 \circ \theta_1 = \theta_1 \circ \tau_2.$

If $r \in Q_0$ then $\tau_0 r = \tau_1(\tau_2 r) = \tau_1 r = r$, and similarly for $\theta_0 r$, so $\operatorname{im} \tau_0 = \operatorname{im} \theta_0 = Q_0$. It is clear that $\mathcal{C}_3 = (Q_1; \tau_2, \theta_2)$ is a sub-cat^1-group of $\mathcal{C}_2$ and that $\mathcal{C}_4 = (Q_2; \tau_1, \theta_1)$ is a sub-cat^1-group of $\mathcal{C}_1$. It is easy to verify that $(G; \tau_0, \theta_0)$ is a cat^1-group since, for example,

$$\tau_0 \circ \theta_0 \;=\; \tau_1 \circ \tau_2 \circ \theta_1 \circ \theta_2 \;=\; \tau_1 \circ \theta_1 \circ \tau_2 \circ \theta_2 \;=\; \theta_1 \circ \theta_2 \;=\; \theta_0.$$

It also follows from these identities that, for example, (τ_1, τ_1) and $(\theta_1, \theta_1) : \mathcal{C}_2 \to \mathcal{C}_3$ are morphisms of cat^1-groups.

Secondly, we give a definition of a cat^2-group using homomorphisms, rather than endomorphisms.

Definition 8.15 A cat^2-group $\mathcal{C}$ is generated by three cat^1-groups $\mathcal{C}_i = (e_i; t_i, h_i : G \to R_i), 0 \leqslant i \leqslant 2$, as shown in the following diagram,

$$\mathcal{C} \quad = \quad \begin{tikzcd}[row sep=huge, column sep=huge]
G \arrow[r, shift left=2, "t_1{,}h_1"] \arrow[r, shift left=1] \arrow[d, shift right=2, "t_2{,}h_2"'] \arrow[d, shift right=1] \arrow[dr, shift right=1, "t_0{,}h_0"'] \arrow[dr, shift right=2] & R_1 \arrow[l, shift left=1, "e_1"] \arrow[d, shift left=2, "t_3{,}h_3"] \arrow[d, shift left=1] \\
R_2 \arrow[u, shift right=1, "e_2"'] \arrow[r, shift right=1] \arrow[r, shift right=2, "t_4{,}h_4"'] & R_0 \arrow[l, "e_4"'] \arrow[u, shift left=1, "e_3"] \arrow[ul, "e_0"']
\end{tikzcd} \tag{47}$$

The remaining homomorphisms are defined by

$$t_3 = t_0 \circ e_1, \quad h_3 = h_0 \circ e_1, \quad e_3 = e_0 \circ t_1,$$
$$t_4 = t_0 \circ e_2, \quad h_3 = h_0 \circ e_2, \quad e_3 = e_0 \circ t_2.$$

The following axioms must be satisfied:

(a) $(e_1 \circ t_1) \circ (e_2 \circ t_2) = (e_2 \circ t_2) \circ (e_1 \circ t_1) = e_0 \circ t_0, \quad (e_1 \circ h_1) \circ (e_2 \circ h_2) = (e_2 \circ h_2) \circ (e_1 \circ h_1) = e_0 \circ h_0,$

(b) $(e_1 \circ t_1) \circ (e_2 \circ h_2) = (e_2 \circ h_2) \circ (e_1 \circ t_1), \quad (e_2 \circ t_2) \circ (e_1 \circ h_1) = (e_1 \circ h_1) \circ (e_2 \circ t_2).$

(c) $(t_0 \circ e_1) \circ (t_1 \circ e_0) = 1, \quad (h_0 \circ e_1) \circ (t_1 \circ e_0) = 1, \quad (t_0 \circ e_2) \circ (t_2 \circ e_0) = 1, \quad (h_0 \circ e_2) \circ (t_2 \circ e_0) = 1.$

We now show that Definition 8.15 is equivalent to Definition 8.14. It is routine to check that $(e_3; t_3, h_3 : R_1 \to R_0)$ and $(e_4; t_4, h_4 : R_2 \to R_0)$ are cat^1-groups. In particular, $[\ker t_3, \ker h_3] = e_1^{-1}[\ker t_0, \ker h_0] = 1$.

We may convert the cat^2-group in Definition 8.15 into a traditional cat^2-group by defining:

$$\tau_1 = e_1 \circ t_1, \quad \theta_1 = e_1 \circ h_1, \quad \tau_2 = e_2 \circ t_2, \quad \theta_2 = e_2 \circ h_2, \quad \tau_0 = e_0 \circ t_0, \quad \theta_0 = e_0 \circ h_0,$$

and setting $Q_1 = e_1 R_1, \ Q_2 = e_2 R_2, \ Q_0 = e_0 R_0$.

Proposition 8.16 *A cat^2-group $\mathcal{C}$ is a cat^1-group of cat^1-groups.*

Proof: We have already seen that $(\tau_1, \tau_1) : \mathcal{C}_2 \to \mathcal{C}_3$ is a cat^1-group morphism – the boundary in this case. We also have to consider the action of $\mathcal{C}_3$ on $\mathcal{C}_2$. [To be continued.] □

Kernels of cat^1-mappings have been considered in subsection 1.12. The diagram for the situation here is

$$\begin{tikzcd}
\ker \tau_1, \ \ker \theta_1 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & G \arrow[r, shift left, "{\tau_1,\theta_1}"] \arrow[r, shift right] \arrow[d, shift left, "{\tau_2,\theta_2}"] \arrow[d, shift right] & Q_1 \arrow[d, shift left, "{\tau_2,\theta_2}"] \arrow[d, shift right] \\
\ker \tau_1, \ \ker \theta_1 \arrow[r] & Q_2 \arrow[r, shift left, "{\tau_1,\theta_1}"] \arrow[r, shift right] & Q_0
\end{tikzcd}$$

8.4 The cat^2-group associated to a crossed square

Given a crossed square

$$\mathcal{S} \quad = \quad \begin{tikzcd} L \arrow[r, "\kappa"] \arrow[d, "\lambda"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd} \tag{48}$$

with crossed pairing $\boxtimes : N \times M \to L$, we wish to construct an associated cat^2-group.

Proposition 8.17 *For $\mathcal{S}$ a crossed square (as in Definition 8.1) there are group actions of $P \ltimes M$ on $N \ltimes L$ and $P \ltimes N$ on $M \ltimes L$ given by*

$$\begin{aligned} (n,\ell)^{(p,m)} &= (n^p, (n^p \boxtimes m)\ell^{pm}) && (49)\\ (m,\ell)^{(p,n)} &= (m^p, (n \boxtimes m^p)^{-1}\, \ell^{pn})\ ,\ . && (50) \end{aligned}$$

Proof: There are two axioms to be checked for the first identity:

$$\begin{aligned} (n_1,\ell_1)^{(p,m)}(n_2,\ell_2)^{(p,m)} &= (n_1^p, (n_1^p \boxtimes m)\ell_1^{pm})(n_2^p, (n_2^p \boxtimes m)\ell_2^{pm}) \\ &= (n_1^p n_2^p, (n_1^p \boxtimes m)^{n_2^p}[\ell_1^{pmn_2^p}(n_2^p \boxtimes m)]\ell_2^{pm}) \\ &= (n_1^p n_2^p, (n_1^p \boxtimes n)^{n_2^p}[(n_2^p \boxtimes m)\ell_1^{pn_2^p m}]\ell_2^{pm}) \qquad \text{by Lemma 8.4(b)} \\ &= ((n_1n_2)^p, ((n_1n_2)^p \boxtimes m)(\ell_1^{n_2}\ell_2)^{pm}) \\ &= (n_1n_2, \ell_1^{n_2}\ell_2)^{(p,m)} \\ &= ((n_1,\ell_1)(n_2,\ell_2))^{(p,m)}\ , \end{aligned}$$

$$\begin{aligned} ((n,\ell)^{(p_1,m_1)})^{(p_2,m_2)} &= (n^{p_1}, (n^{p_1} \boxtimes m_1)\ell^{p_1m_1})^{(p_2,m_2)} \\ &= ((n^{p_1})^{p_2}, (n^{p_1p_2} \boxtimes m_2)((n^{p_1} \boxtimes m_1)\ell^{p_1m_1})^{p_2m_2}) \\ &= (n^{p_1p_2}, (n^{p_1p_2} \boxtimes m_2)(n^{p_1p_2} \boxtimes {m_1}^{p_2})^{m_2}\ell^{p_1m_1p_2m_2}) \\ &= (n^{p_1p_2}, (n^{p_1p_2} \boxtimes {m_1}^{p_2}m_2)\ell^{p_1p_2{m_1}^{p_2}m_2}) \\ &= (n,\ell)^{(p_1p_2,{m_1}^{p_2}m_2)} \\ &= (n,\ell)^{((p_1,m_1)(p_2,m_2))}\ . \end{aligned}$$

The second identity follows using the transpose crossed pairing (44). □

We saw in (11) that the cat^1-group associated to a crossed module $\mathcal{X}$ has homomorphisms

$$t, h : R \ltimes S \to S, \quad t(r,s) = r, \quad h(r,s) = r(\partial s)\ .$$

Applying this construction to $\mathcal{L}$ and $\mathcal{M}$ we obtain a *crossed module of cat^1-groups*:

$$\begin{tikzcd} N \ltimes L \arrow[r, "(\nu,\kappa)"] \arrow[d, shift right, "(t_2,h_2)"'] \arrow[d, shift left] & P \ltimes M \arrow[d, shift left, "(t_3,h_3)"] \arrow[d, shift right] \\ N \arrow[r, "\nu"'] & P \end{tikzcd} \tag{51}$$

Lemma 8.18 *The mapping*

$$(\nu,\kappa) : N \ltimes L \to P \ltimes M \ : \ (n,\ell) \mapsto (\nu n, \kappa l)$$

is a group homomorphism.

Proof:

$$\begin{aligned}
(\nu,\kappa)((n_1,\ell_1)(n_2,\ell_2)) &= (\nu,\kappa)(n_1 n_2, \ell_1^{n_2}\ell_2) \\
&= (\nu(n_1 n_2), \kappa(\ell_1^{n_2}\ell_2)) \\
&= ((\nu n_1)(\nu n_2), (\kappa \ell_1^{\nu n_2})(\kappa \ell_2)) \\
&= ((\nu n_1)(\nu n_2), (\kappa \ell_1)^{\nu n_2}(\kappa \ell_2)) \\
&= (\nu n_1, \kappa \ell_1)(\nu n_2, \kappa \ell_2) \\
&= (\nu,\kappa)(n_1,\ell_1).(\nu,\kappa)(n_2,\ell_2).
\end{aligned}$$

□

Lemma 8.19 *The action given in Proposition 8.17 makes* $((\nu,\kappa) : N \ltimes L \to P \ltimes M)$ *a crossed module.*

Proof: **X1:**

$$\begin{aligned}
(\nu,\kappa)((n,\ell)^{(p,m)}) &= (\nu,\kappa)(n^p, (n^p \boxtimes m)\ell^{pm}) \\
&= (\nu(n^p), \kappa(n^p \boxtimes m)\kappa(\ell^{pm})) \\
&= (\nu(n^p), (m^{-1})^{n^p} m\kappa(\ell^{pm})) \\
&= (\nu(n^p), (m^{-1})^{(\nu n^p)} m\kappa((\ell^p)^m)) \\
&= (\nu(n^p), (m^{-1})^{\nu(n^p)} (\kappa\ell)^p m) \\
&= (p^{-1}(\nu n)p, (m^{-1})^{p^{-1}(\nu n)p} (\kappa\ell)^p m) \\
&= (p^{-1}, (m^{-1})^{p^{-1}})(\nu n, \kappa\ell)(p,m) \\
&= (p,m)^{-1}\ ((\nu,\kappa)(n,\ell))(p,m)\ .
\end{aligned}$$

X2:

$$\begin{aligned}
(n_1,\ell_1)^{(\nu,\kappa)(n_2,\ell_2)} &= (n_1,\ell_1)^{(\nu n_2, \kappa\ell_2)} \\
&= ({n_1}^{\nu n_2}, ({n_1}^{\nu n_2} \boxtimes \kappa\ell_2){\ell_1}^{(\nu n_2)(\kappa\ell_2)}) \\
&= ({n_1}^{n_2}, (n_2^{-1} n_1 n_2 \boxtimes \kappa\ell_2){\ell_1}^{(\nu n_2)(\kappa\ell_2)}) \\
&= ({n_1}^{n_2}, (n_2^{-1} \boxtimes \kappa\ell_2)^{n_1 n_2}(n_1 \boxtimes \kappa\ell_2)^{n_2}(n_2 \boxtimes \kappa\ell_2){\ell_1}^{(\nu n_2)(\kappa\ell_2)}) \\
&= (n_2^{-1} n_1 n_2, [(\ell_2^{-1})^{n_2^{-1}}\ell_2]^{n_1 n_2})[(\ell_2^{-1})^{n_1}\ell_2]^{n_2})[(\ell_2^{-1})^{n_2}\ell_2])\ell_2^{-1}{\ell_1}^{(\nu n_2)}\ell_2 \\
&= (n_2^{-1} n_1 n_2, (\ell_2^{-1})^{n_2^{-1} n_1 n_2}\ell_1^{n_2}\ell_2) \\
&= \left(n_2^{-1}, (\ell_2^{-1})^{n_2^{-1}}\right)(n_1 n_2, \ell_1^{n_2}\ell_2) \\
&= (n_2,\ell_2)^{-1}(n_1,\ell_1)(n_2,\ell_2)
\end{aligned}$$

□

We may then construct a *cat*^{1}*-group of cat*^{1}*-groups* where the required homomorphisms are:

$$\begin{aligned}
t_1, h_1 \ : \ (P \ltimes M) \ltimes (N \ltimes L) &\to P \ltimes M \qquad (52)\\
t_1((p,m),(n,\ell)) &= (p,m) \\
h_1((p,m),(n,\ell)) &= (p,m)\,(\nu,\kappa)(n,\ell) \ = \ (p,m)(\nu n, \kappa\ell) \ = \ (p(\nu n),\, m^{\nu n}(\kappa\ell)) \\
\text{and} \qquad e_1 \ : \ P \ltimes M &\to (P \ltimes M) \ltimes (N \ltimes L) \qquad (53)\\
e_1(p,m) &= ((p,m),(1,1))\ .
\end{aligned}$$

We now check that $(e_1; t_1, h_1 : (P \ltimes M) \ltimes (N \ltimes L) \to P \ltimes M)$ is a cat^1-group. Note that

$$\begin{array}{rcl} \ker t_1 & = & \{\ ((1,1),(n,\ell))\ \} \\ \ker h_1 & = & \{\ ((p,m),(n,\ell))\ \} \quad \text{where} \quad p = (\nu n)^{-1} \ \text{ and } \ m = (\kappa \ell^{-1})^p , \end{array}$$

so that

$$\ell^{pn} \ = \ \ell^{(\nu n)^{-1}(\nu n)} \ = \ \ell \qquad \text{in} \quad \ker h_1 .$$

The formula for the action is given by equation (50):

$$(n,\ell)^{(p,m)} \ := \ (\ n^p ,\ (n^p \boxtimes m)\ell^{pm}) \ .$$

We now check the cat^1-group axioms.

C1:

$$\begin{array}{rcccl} t_1 \circ e_1(p,m) & = & t_1((p,m),(1,1)) & = & (p,m) \\ h_1 \circ e_1(p,m) & = & h_1((p,m),(1,1)) & = & (p,m) \end{array}$$

C2:

$$\begin{array}{rcl} ((1,1),(n_0,\ell_0))\,((p,m),(n,\ell)) & = & ((p,m),(n_0,\ell_0)^{(p,m)}(n,\ell)) \\ & = & ((p,m),({n_0}^p,({n_0}^p \boxtimes m){\ell_0}^{pm})(n,\ell)) \\ & = & ((p,m),({n_0}^{(\nu n)^{-1}}n,({n_0}^p \boxtimes m)^n {\ell_0}^{pmn}\ell)) \\ & = & ((p,m),(n n_0 n^{-1} n,(n_0 \boxtimes \kappa \ell^{-1})^{pn}\ {\ell_0}^{p\kappa\ell^{-1}n}\ell)) \\ & = & ((p,m),(n n_0,[\ell^{n_0}\ell^{-1}]^{pn}{\ell_0}^{\kappa\ell^{-1}pn}\ell)) \\ & = & ((p,m),(n n_0,\ell^{n_0}\ell^{-1}{\ell_0}^{\kappa\ell^{-1}pn}\ell)) \\ & = & ((p,m),(n n_0,\ell^{n_0}\ell^{-1}[\ell\ell_0\ell^{-1}]^{pn}\ell)) \\ & = & ((p,m),(n n_0,\ell^{n_0}(\ell^{-1}\ell)\ell_0^{p(\nu n)}(\ell^{-1}\ell))) \\ & = & ((p,m),(n,\ell)(n_0,\ell_0)) \\ & = & ((p,m),(n,\ell))\,((1,1),(n_0,\ell_0)) \end{array}$$

and so the two kernels commute.

Theorem 8.20 *The homomorphisms in (52) and (53) give a cat^2-group:*

$$\mathcal{C}(\mathcal{R}) \quad = \quad \begin{tikzcd}[column sep=huge, row sep=huge] (P \ltimes N) \ltimes (M \ltimes L) \arrow[r, shift left=2, "t_1{,}h_1"] \arrow[r, shift left=1] \arrow[d, shift right=2, "t_2{,}h_2"'] \arrow[d, shift right=1] \arrow[dr, shift left=1] \arrow[dr, shift right=1, "t_0{,}h_0"'] & P \ltimes M \arrow[l, shift left=1, "e_1"] \arrow[d, shift left=1, "t_3{,}h_3"] \arrow[d, shift left=2] \arrow[dl, shift right=2, "e_0"'] \\ P \ltimes N \arrow[u, shift right=1, "e_2"'] \arrow[r, shift right=1] \arrow[r, shift right=2, "t_4{,}h_4"'] & P \arrow[l, shift right=1, "e_4"'] \arrow[u, shift left=1, "e_3"] \end{tikzcd} \qquad (54)$$

Proof: To be added. (Is material required from Section 7.2 ?) □

8.5 The other cat^2-structure

Now the underlying diagram (43) of the crossed square $\mathcal{R}$, together with the crossed pairing

$$\tilde{\boxtimes}\ : M \times N \to L,\ (m \mathbin{\tilde{\boxtimes}} n) = (n \boxtimes m)^{-1}$$

forms a second crossed square $\tilde{\mathcal{S}}$. Thus we can form a second cat^2-group $\mathcal{C}(\tilde{\mathcal{S}})$ with

$$\tilde{t}_1, \tilde{h}_1 : (P \ltimes M) \ltimes (N \ltimes L) \to (P \ltimes M)\ .$$

Let

$$G = (P \ltimes N) \ltimes (M \ltimes L), \qquad \tilde{G} = (P \ltimes M) \ltimes (N \ltimes L)\ .$$

Proposition 8.21 *There is an isomorphism between these two semidirect products:*

$$\begin{array}{rclll} \tau & : & G \to \tilde{G}, & ((p,n),(m,\ell)) \mapsto ((p,m),(n,(n \boxtimes m)\ell))\ , & (55) \\ \tilde{\tau}\ :=\ \tau^{-1} & : & \tilde{G} \to G, & ((p,m),(n,\ell)) \mapsto ((p,n),(m,(m \mathbin{\tilde{\boxtimes}} n)\ell))\ . & \end{array}$$

Proof:

$$\begin{array}{rl} & \tau((p_1,n_1),(m_1,\ell_1))((p_2,n_2),(m_2,\ell_2)) \\ = & \tau((p_1,n_1)(p_2,n_2),(m_1,\ell_1)^{(p_2,n_2)}(m_2,\ell_2)) \\ = & \tau((p_1p_2,{n_1}^{p_2}n_2),({m_1}^{p_2},(n_2 \boxtimes {m_1}^{p_2})^{-1}{\ell_1}^{p_2n_2})(m_2,\ell_2)) \\ = & \tau((p_1p_2,{n_1}^{p_2}n_2),({m_1}^{p_2}m_2,({n_2}^{m_2} \boxtimes {m_1}^{p_2m_2})^{-1}{\ell_1}^{p_2n_2m_2}\ell_2)) \\ = & ((p_1p_2,{m_1}^{p_2}m_2),({n_1}^{p_2}n_2,({n_1}^{p_2}n_2 \boxtimes {m_1}^{p_2}m_2)({n_2}^{m_2} \boxtimes {m_1}^{p_2m_2})^{-1}{\ell_1}^{p_2n_2m_2}\ell_2)) \\ = & ((p_1p_2,{m_1}^{p_2}m_2),({n_1}^{p_2}n_2,({n_1}^{p_2} \boxtimes m_2)^{n_2}({n_1}^{p_2} \boxtimes {m_1}^{p_2})^{m_2n_2}(n_2 \boxtimes m_2){\ell_1}^{p_2n_2m_2}\ell_2)) \\ = & ((p_1p_2,{m_1}^{p_2}m_2),({n_1}^{p_2}n_2,({n_1}^{p_2} \boxtimes {m_1}^{p_2}m_2)^{n_2}(n_2 \boxtimes m_2)\ell_1^{p_2n_2m_2}\ell_2)) \\ = & ((p_1p_2,{m_1}^{p_2}m_2),({n_1}^{p_2}n_2,({n_1}^{p_2} \boxtimes {m_1}^{p_2}m_2)^{n_2}\ell_1^{p_2m_2n_2}(n_2 \boxtimes m_2)\ell_2)) \\ = & ((p_1p_2,{m_1}^{p_2}m_2),({n_1}^{p_2},({n_1}^{p_2} \boxtimes {m_1}^{p_2}m_2)\ell_1^{p_2m_2})(n_2,(n_2 \boxtimes m_2)\ell_2)) \\ = & ((p_1p_2,{m_1}^{p_2}m_2),({n_1}^{p_2},({n_1}^{p_2} \boxtimes m_2)({n_1}^{p_2} \boxtimes {m_1}^{p_2})^{m_2}\ell_1^{p_2m_2})(n_2,(n_2 \boxtimes m_2)\ell_2)) \\ = & ((p_1,m_1)(p_2,m_2),(n_1,(n_1 \boxtimes m_1)\ell_1))^{(p_2,m_2)}(n_2,(n_2 \boxtimes m_2)\ell_2)) \\ = & ((p_1,m_1),(n_1,(n_1 \boxtimes m_1)\ell_1))\ ((p_2,m_2),(n_2,(n_2 \boxtimes m_2)\ell_2)) \\ = & \tau((p_1,n_1),(m_1,\ell_1))\ \tau((p_2,n_2),(m_2,\ell_2)) \end{array}$$

□

Note that the subgroup $(1 \ltimes N) \ltimes (M \ltimes 1)$ does *not* in general get mapped by τ to the subgroup $(1 \ltimes M) \ltimes (N \ltimes 1)$.

[The final section in this chapter is experimental.]

These used to be Lemmas 9.22, 9.23 following the Proposition for τ.

The isomorphism τ provides a neat formula for the inverse of a general element in $(P \ltimes N) \ltimes (M \ltimes L)$.

Lemma 8.22

$$((p,n),(m,\ell))^{-1} \;=\; \tau((p^{-1},(m^{-1})^{p^{-1}}),\ ((n^{-1})^{p^{-1}},(\ell^{-1})^{m^{-1}n^{-1}p^{-1}}))$$

Proof:

$$\begin{array}{rcl} ((p,n),(m,\ell))^{-1} & = & ((p^{-1},(n^{-1})^{p^{-1}}),\ (m^{-1},(\ell^{-1})^{m^{-1}})^{(p^{-1},(n^{-1})^{p^{-1}})})) \\ & = & ((p^{-1},(n^{-1})^{p^{-1}}),\ ((m^{-1})^{p^{-1}},((m^{-1})^{p^{-1}} \boxtimes (n^{-1})^{p^{-1}})(\ell^{-1})^{m^{-1}n^{-1}p^{-1}})) \\ & = & \tau((p^{-1},(m^{-1})^{p^{-1}}),\ ((n^{-1})^{p^{-1}},(\ell^{-1})^{m^{-1}n^{-1}p^{-1}}))) \end{array}$$

□

Lemma 8.23 *The commutator of elements in $\ddot{e}_1(P \ltimes N)$ and $\ddot{e}_2(P \ltimes M)$ is*

$$[((p,n),(1,1)),\ ((q,1),(m,1))] \;=\; (([p,q],n^{-1}n^q),((m^{-1})^{q^{-1}pq}m,((m^{-1})^{q^{-1}pq} \boxtimes n^q)^m))$$

Proof:

$$\begin{array}{rl} & [((p,n),(1,1)),\ ((q,1),(m,1))] \\ = & ((p^{-1},(n^{-1})^{p^{-1}}),(1,1))\ ((q^{-1},1),((m^{-1})^{q^{-1}},1))\ ((p,n),(1,1))\ ((q,1),(m,1)) \\ = & ((p^{-1}q^{-1},(n^{-1})^{p^{-1}q^{-1}}),((m^{-1})^{q^{-1}},1))\ ((pq,n^q),(m,1)) \\ = & ((p^{-1}q^{-1}pq,n^{-1}n^q),((m^{-1})^{q^{-1}pq},((m^{-1})^{q^{-1}pq} \boxtimes n^q))(m,1)) \\ = & (([p,q],n^{-1}n^q),((m^{-1})^{q^{-1}pq}m,((m^{-1})^{q^{-1}pq} \boxtimes n^q)^m)) \end{array}$$

□

8.6 Semidirect Factorisation of a cat^2-group

Starting with the cat^2-group

$$\mathcal{G} \quad = \quad \begin{tikzcd}[column sep=huge, row sep=huge] G_{[2]} \arrow[r, shift left=3, "\ddot{t}_1{,}\ddot{h}_1"] \arrow[r, shift left=1] \arrow[d, shift right=3, "\ddot{t}_2{,}\ddot{h}_2"'] \arrow[d, shift right=1] & G_{\{2\}} \arrow[l, shift left=1, "\ddot{e}_1"] \arrow[d, shift left=1] \arrow[d, shift left=3, "\dot{t}_2{,}\dot{h}_2"] \\ G_{\{1\}} \arrow[u, shift right=1, "\ddot{e}_2"'] \arrow[r, shift right=1] \arrow[r, shift right=3, "\dot{t}_1{,}\dot{h}_1"'] & G_{\emptyset} \arrow[u, shift left=1, "\dot{e}_2"] \arrow[l, shift right=1, "\dot{e}_1"'] \end{tikzcd}$$

we wish to generalise the semidirect factorisation of Subsection 1.11 equation (10). Put

$$\ddot{S}_1 = \ker \ddot{t}_1, \quad \ddot{S}_2 = \ker \ddot{t}_2, \quad \dot{S}_1 = \ker \dot{t}_1, \quad \dot{S}_2 = \ker \dot{t}_2 \quad \text{and} \quad S_{[2]} = \ker t_{[2]},$$

giving isomorphisms

$$\begin{array}{llllll}\dot\phi_1 & : & G_{\{1\}} & \cong & G_\emptyset \ltimes \dot S_1, & b \mapsto (\dot t_1 b, \dot u_1 b), \qquad \ddot\phi_1 \;:\; G_{[2]} \;\cong\; G_{\{2\}} \ltimes \ddot S_1, \quad g \mapsto (\ddot t_1 g, \ddot u_1 g), \\ \dot\phi_2 & : & G_{\{2\}} & \cong & G_\emptyset \ltimes \dot S_2, & c \mapsto (\dot t_2 c, \dot u_2 c), \qquad \ddot\phi_2 \;:\; G_{[2]} \;\cong\; G_{\{1\}} \ltimes \ddot S_2, \quad g \mapsto (\ddot t_2 g, \ddot u_2 g),\end{array}$$

where, for example, $\dot u_1 b = (\dot e_1 \dot t_1 b^{-1})b$.

Note that $S_{[2]} = (\ker \ddot t_1) \cap (\ker \ddot t_2)$.

Where possible we keep to the convention that

$$g \in G_{[2]}, \quad b \in G_{\{1\}}, \quad c \in G_{\{2\}}, \quad p \in G_\emptyset, \quad j \in \ddot S_1, \quad k \in \ddot S_2, \quad m \in \dot S_1, \quad n \in \dot S_2, \quad \ell \in S_{[2]}.$$

The following identities are easily verified:

$$\dot u_1 \ddot t_2 = \ddot t_2 \ddot u_1, \quad \dot u_1 \ddot h_2 = \ddot h_2 \ddot u_1, \quad \ddot u_1 \ddot e_2 = \ddot e_2 \dot u_1, \quad \dot u_2 \ddot t_1 = \ddot t_1 \ddot u_2, \quad \dot u_2 \ddot h_1 = \ddot h_1 \ddot u_2, \quad \ddot u_2 \ddot e_1 = \ddot e_1 \dot u_2. \tag{56}$$

We thus obtain the following result.

Lemma 8.24 *There is a cat*^{1}*-group isomorphism* $\phi_1 = (\ddot\phi_1, \dot\phi_1) : \ddot{\mathcal{G}}_2 \to \bar{\mathcal{G}}_2$,

$$\begin{tikzcd}
G_{[2]} \arrow[r, "\ddot\phi_1"] \arrow[d, shift right=2, "\ddot t_2{,}\ddot h_2"'] \arrow[d, shift right=4] & G_{\{2\}} \ltimes \ddot S_1 \arrow[d, shift left=2, "\bar t_2{,}\bar h_2"] \arrow[d, shift left=4] \\
G_{\{1\}} \arrow[r, "\dot\phi_1"'] \arrow[u, shift right=1, "\ddot e_2"'] & G_\emptyset \ltimes \dot S_1 \arrow[u, shift left=1, "\bar e_2"]
\end{tikzcd}$$

with structure maps on $\bar{\mathcal{G}}_2$ *given by*

$$\bar t_2(c,j) = (\dot t_2 c, \ddot t_2 j), \quad \bar h_2(c,j) = (\dot h_2 c, \ddot h_2 j), \quad \bar e_2(p,m) = (\dot e_2 p, \ddot e_2 m).$$

Proof: There are three equations to be verified.

$$\bar t_2(c,j) \;=\; \dot\phi_1 \ddot t_2 \ddot\phi_1^{-1}(c,j) \;=\; \dot\phi_1 \ddot t_2 (\ddot e_1 c) j \;=\; \dot\phi_1((\dot e_1 \dot t_2 c)(\ddot t_2 j)) \;=\; (\dot t_1((\dot e_1 \dot t_2 c)(\ddot t_2 j)), \dot u_1((\dot e_1 \dot t_2 c)(\ddot t_2 j))).$$

The first of these terms is $(\dot t_2 c)(\dot t_2 \ddot t_1 j) \;=\; \dot t_2 c$ since $\ddot t_1 j = 1$.

The second term is $(\dot u_1(\ddot t_2 j))(\dot u_1(\dot e_1 \dot t_2 c))^{\ddot t_2 j} \;=\; (\dot e_1 \dot t_1 \ddot t_2 j^{-1})(\ddot t_2 j) \;=\; \dot u_1 \ddot t_2 j \;=\; \ddot t_2 j$ since $\ddot u_1 j = j$.

[Parts 2,3 added 01/05/07, and need to be checked.]

Similarly

$$\bar h_2(c,j) \;=\; \dot\phi_1 \ddot h_2 \ddot\phi_1^{-1}(c,j) \;=\; \dot\phi_1 \ddot h_2 (\ddot e_1 c) j \;=\; \dot\phi_1((\dot e_1 \dot h_2 c)(\ddot h_2 j)) \;=\; (\dot t_1((\dot e_1 \dot h_2 c)(\ddot h_2 j)), \dot u_1((\dot e_1 \dot h_2 c)(\ddot h_2 j))).$$

The first of these terms is $(\dot h_2 c)(\dot t_1 \ddot h_2 j) \;=\; (\dot h_2 c)(\dot h_2 \ddot t_1 j) \;=\; \dot h_2(c(\ddot t_1 j))$.

The second term is $(\dot u_1(\ddot h_2 j))(\dot u_1(\dot e_1 \dot h_2 c))^{\ddot h_2 j} \;=\; (\dot e_1 \dot t_1 \ddot h_2 j^{-1})(\ddot h_2 j) \;=\; \dot u_1 \ddot h_2 j \;=\; \ddot h_2(\ddot u_1 j)$. Hence

$$\bar h_2(c,j) \;=\; (\dot h_2(c(\ddot t_1 j)), \ddot h_2(\ddot u_1 j)) \;=\; (\dot h_2 c, \ddot h_2 j).$$

Thirdly,

$$\bar{e}_2(p,m) \;=\; \ddot{\phi}_1\ddot{e}_2\ddot{\phi}_1^{-1}(p,m) \;=\; \ddot{\phi}_1\ddot{e}_2(\dot{e}_1p)m \;=\; \ddot{\phi}_1(\ddot{e}_1\dot{e}_2p)(\ddot{e}_2m) \;=\; (\ddot{t}_1((\ddot{e}_1\dot{e}_2p)(\ddot{e}_2m)),\ddot{u}_1((\ddot{e}_1\dot{e}_2p)(\ddot{e}_2m))).$$

The first of these terms is $(\dot{e}_2p)(\dot{e}_2\dot{t}_1m) \;=\; \dot{e}_2(p(\dot{t}_1m)$.

The second term is $(\ddot{u}_1(\ddot{e}_2m))(\ddot{u}_1(\ddot{e}_1\dot{e}_2p))^{\ddot{e}_2m} \;=\; (\ddot{e}_1\ddot{t}_1\ddot{e}_2m^{-1})(\ddot{e}_2m) \;=\; (\ddot{e}_2\dot{e}_1\dot{t}_1m^{-1})(\ddot{e}_2m) \;=\; \ddot{e}_2(\dot{u}_1m)$.
Hence

$$\bar{e}_2(p,m) \;=\; (\dot{e}_2(p(\ddot{t}_1m)),\ddot{e}_2(\dot{u}_1m)) \;=\; (\dot{e}_2p,\ddot{e}_2m).$$

□

Now $\ker\bar{t}_2 = \ddot{\phi}_1(\ker\ddot{t}_2) = \ddot{\phi}_1\ddot{S}_2$. **[Really?]**

A typical element in $\ddot{S}_2$ is $\ddot{u}_2g$, and

$$\ddot{\phi}_1\ddot{u}_2g = (\ddot{t}_1\ddot{u}_2g,\ddot{u}_1\ddot{u}_2g) \in \dot{S}_2 \ltimes S_{[2]} \qquad \text{so} \qquad \ker\bar{t}_2 \;=\; \dot{S}_2 \ltimes S_{[2]} \;\trianglelefteq\; G_{\{2\}} \ltimes \ddot{S}_1. \tag{57}$$

Lemma 8.25 *There is a cat*^{1}*-group isomorphism* $\phi_1{}' = (\ddot{\phi}_1{}',1) : \bar{\mathcal{G}}_2 \to \ddot{\mathcal{G}}_2{}'$

$$\begin{tikzcd}
G_{\{2\}} \ltimes \ddot{S}_1 \arrow[r, "\ddot{\phi}_1{}'"] \arrow[d, shift right=3, "\bar{t}_2{,}\bar{h}_2"'] \arrow[d, shift right=1] & (G_\emptyset \ltimes \dot{S}_1) \ltimes (\dot{S}_2 \ltimes S_{[2]}) \arrow[d, shift left=1] \arrow[d, shift left=3, "\ddot{t}_2{}'{,}\ddot{h}_2{}'"] \\
G_\emptyset \ltimes \dot{S}_1 \arrow[r, "1"'] \arrow[u, shift right=1, "\bar{e}_2"'] & G_\emptyset \ltimes \dot{S}_1 \arrow[u, shift left=1, "\ddot{e}_2{}'"]
\end{tikzcd}$$

where $\ddot{\phi}_1{}'(c,j) \quad = \quad (\bar{t}_2(c,j),\bar{u}_2(c,j))$, *and the structure maps on* $\ddot{\mathcal{G}}'$ *are given by*

$$\begin{aligned}
\ddot{t}_2{}'((p,m),(n,\ell)) \;&=\; (p(\dot{t}_2n),m^{\dot{t}_2n}(\ddot{t}_2\ell)),\\
\ddot{h}_2{}'((p,m),(n,\ell)) \;&=\; (p(\dot{h}_2n),m^{\dot{h}_2n}(\ddot{h}_2\ell)),\\
\ddot{e}_2{}'(p,m) \;&=\; ((p,m),(1,1)).
\end{aligned}$$

Proof: Since

$$\bar{u}_2(c,j) \;=\; (\bar{e}_2\bar{t}_2(c^{-1},(j^{-1})^{c^{-1}}))(c,j) \;=\; (\dot{e}_2\dot{t}_2c^{-1},\ddot{e}_2\ddot{t}_2(j^{-1})^{c^{-1}})(c,j) \;=\; (\dot{u}_2c,\ddot{u}_2j),$$

it follows that

$$\ddot{\phi}_1{}'(c,j) \quad = \quad (\,(\dot{t}_2c,\ \ddot{t}_2j),\ (\dot{u}_2c,\ \ddot{u}_2j)\,).$$

We verify the first of the structure map formulae – the rest follow similarly:

$$\begin{aligned}
\ddot{t}_2{}'((p,m),(n,\ell)) \;&=\; \bar{t}_2\ddot{\phi}_1{}'^{-1}((p,m),(n,\ell))\\
&=\; \bar{t}_2(\bar{e}_2(p,m)(n,\ell))\\
&=\; (p,m)\bar{t}_2(n,\ell)\\
&=\; (p,m)(\dot{t}_2n,\ddot{t}_2\ell)\\
&=\; (p,m)(1,1) \;=\; (p,m).
\end{aligned}$$

□

The composite source group isomorphism $\phi_{[2]} = \ddot{\phi}_1{}' \circ \ddot{\phi}_1$ is therefore

$$\phi_{[2]} \;:\; G_{[2]} \to (G_\emptyset \ltimes \dot{S}_1) \ltimes (\dot{S}_2 \ltimes S_{[2]}), \quad g \mapsto (\,(t_{[2]}g,\dot{u}_1\ddot{t}_2g),\ (\dot{u}_2\ddot{t}_1g,\ \ddot{u}_2\ddot{u}_1g)\,).$$

Combining ϕ_1 with $\phi_1{}'$ we obtain the following result.

Theorem 8.26 *There is an isomorphism of cat*^{2}*-groups*

$$\phi \;=\; (\phi_{[2]}, \dot{\phi}_2, \dot{\phi}_1, 1) \;:\; \mathcal{G} \to \mathcal{G}' \ .$$

$$\begin{tikzcd}
 & G_{[2]} \arrow[r, shift left=3, "{\ddot{t}_1,\ddot{h}_1}"] \arrow[r, shift left=1] \arrow[dd, shift right=3, "{\ddot{t}_2,\ddot{h}_2}"'] \arrow[dd, shift right=1] & G_{\{2\}} \arrow[l, shift left=1, "\ddot{e}_1"] \arrow[dd, shift left=1] \arrow[dd, shift left=3, "{\dot{t}_2,\dot{h}_2}"] & & (G_\emptyset \ltimes \dot{S}_1) \ltimes (\dot{S}_2 \ltimes S_{[2]}) \arrow[r, shift left=3, "{\ddot{t}_1{}',\ddot{h}_1{}'}"] \arrow[r, shift left=1] \arrow[dd, shift left=1] \arrow[dd, shift left=3, "{\ddot{t}_2{}',\ddot{h}_2{}'}"] & G_\emptyset \ltimes \dot{S}_2 \arrow[l, shift left=1, "\ddot{e}_1{}'"] \arrow[dd, shift left=1] \arrow[dd, shift left=3, "{\dot{t}_2{}',\dot{h}_2{}'}"] \\
\phi \ : & & & \to & & \\
 & G_{\{1\}} \arrow[uu, shift right=1, "\ddot{e}_2"'] \arrow[r, shift right=1] \arrow[r, shift right=3, "{\dot{t}_1,\dot{h}_1}"'] & G_\emptyset \arrow[l, shift right=1, "\dot{e}_1"'] \arrow[uu, shift left=1, "\dot{e}_2"] & & G_\emptyset \ltimes \dot{S}_1 \arrow[uu, shift left=1, "\ddot{e}_2{}'"] \arrow[r, shift right=1] \arrow[r, shift right=3, "{\dot{t}_1{}',\dot{h}_1{}'}"'] & G_\emptyset \arrow[l, shift right=1, "\ddot{e}_1{}'"'] \arrow[uu, shift left=1, "\dot{e}_2{}'"]
\end{tikzcd}$$

where

$$\begin{aligned}
\phi_{[2]}(g) &= (\,(t_{[2]}g, \dot{u}_1\ddot{t}_2 g),\ (\dot{u}_2\ddot{t}_1 g,\ \ddot{u}_2\ddot{u}_1 g)\,),\\
\dot{\phi}_2(c) &= (\dot{t}_2 c, \dot{u}_2 c),\\
\dot{\phi}_1(b) &= (\dot{t}_1 b, \dot{u}_1 b),
\end{aligned}$$

and the 12 *homomorphisms are given by*

$$\begin{aligned}
\ddot{t}_2{}'((p,m),(n,\ell)) &= (p(\dot{t}_2 n), m^{\dot{t}_2 n}(\ddot{t}_2\ell)),\\
\ddot{h}_2{}'((p,m),(n,\ell)) &= (p(\dot{h}_2 n), m^{\dot{h}_2 n}(\ddot{h}_2\ell)),\\
\ddot{e}_2{}'(p,m) &= ((p,m),(1,1)),\\
\ddot{t}_1{}'((p,m),(n,\ell)) &= (p(\dot{t}_1 m)(\dot{t}_2 n), (\dot{u}_2 n)(\dot{u}_2\ddot{t}_1\ell)),\\
\ddot{h}_1{}'((p,m),(n,\ell)) &= ((?,?),(?,?)),\\
\ddot{e}_1{}'(p,n) &= ((p,1),(n,1)),\\
\dot{t}_2(p,n) &= p(\dot{t}_2 n),\\
\dot{h}_2(p,n) &= p(\dot{h}_2 n),\\
\dot{e}_2(p) &= (p,1)\\
\dot{t}_1(p,m) &= p(\dot{t}_1 m),\\
\dot{h}_1(p,m) &= p(\dot{h}_1 m),\\
\dot{e}_1(p) &= (p,1).
\end{aligned}$$

Proof: To be added. □

9 Crossed squares and 2-fold crossed modules

We now show how the action of one crossed module on another leads to a crossed square. As a first step, we consider alternative formulations of the crossed module axioms in order to find one which can most easily be generalised.

9.1 An alternative formulation for crossed modules

Let $\mathcal{X} = (\partial : S \to R)$ be a crossed module. Define maps ι_S (similarly ι_R) and $\bar{\alpha}$ by

$$\bar{\iota}_S : S \times S \to S,\ (s_1, s) \mapsto {s_1}^s = s^{-1} s_1 s, \qquad \bar{\alpha} : S \times R \to S,\ (s, r) \mapsto s^r.$$

From these we may define

$$\begin{aligned} \iota_s \ &= \ \bar{\iota}_S(-, s) \ : \ S \to S,\ s_1 \mapsto s^{-1} s_1 s \quad \text{and} \quad \iota_S \ : \ S \to \mathrm{Aut}(S),\ s \mapsto \iota_s, \\ \alpha_r \ &= \ \bar{\alpha}(-, r) \ : \ S \to S,\ s \mapsto s^r \quad \text{and} \quad \alpha \ : \ R \to \mathrm{Aut}(S),\ r \mapsto \alpha_r, \end{aligned}$$

The first crossed module axiom says that the following diagram commutes:

$$\begin{tikzcd} S \times R \arrow[r, "1_S \times \alpha"] \arrow[d, "\partial \times \iota_R"'] & S \times \mathrm{Aut}S \arrow[r, "\mathrm{eval}"] & S \arrow[d, "\partial"] \\ R \times \mathrm{Aut}R \arrow[rr, "\mathrm{eval}"'] & & R \end{tikzcd} \tag{58}$$

mapping (s, r) to $\alpha_r(\partial s) = r^{-1}(\partial s) r$ and to $\partial(s^r)$.

The corresponding diagram for 2-fold crossed modules, where $\mathcal{Y} = (\delta : Q \to P)$, is

$$\begin{tikzcd} \mathcal{X} \times \mathcal{Y} \arrow[r, "1_{\mathcal{X}} \times \alpha"] \arrow[d, "\mu \times \iota_{\mathcal{Y}}"'] & \mathcal{X} \times \mathrm{Act}\mathcal{X} \arrow[r, "\mathrm{eval}"] & \mathcal{X} \arrow[d, "\mu"] \\ \mathcal{Y} \times \mathrm{Act}\mathcal{Y} \arrow[rr, "\mathrm{eval}"'] & & \mathcal{Y} \end{tikzcd} \tag{59}$$

Applying these maps to $s \in S,\ r \in R,\ q \in Q,\ p \in P$ clockwise around the rectangle we obtain:

$$\begin{pmatrix} s \\ r \end{pmatrix}, \begin{pmatrix} q \\ p \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} s \\ r \end{pmatrix}, \begin{pmatrix} \chi_q \\ \alpha_p \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \chi_q r \\ \begin{pmatrix} \ddot{\alpha}_p s \\ \dot{\alpha}_p r \end{pmatrix} \end{pmatrix}, \begin{pmatrix} \ddot{\alpha} \ddot{\mu} s \\ \dot{\alpha} \dot{\mu} r \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \ddot{\mu}(\chi_q r) \\ \begin{pmatrix} \ddot{\mu}(s^p) \\ \dot{\mu}(r^p) \end{pmatrix} \end{pmatrix}.$$

In the other direction,

$$\begin{pmatrix} s \\ r \end{pmatrix}, \begin{pmatrix} q \\ p \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \ddot{\mu} s \\ \dot{\mu} r \end{pmatrix}, \begin{pmatrix} \eta_q \\ \beta_p \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \eta_q(\dot{\mu} r) \\ \begin{pmatrix} \ddot{\beta}_p(\ddot{\mu} s) \\ \dot{\beta}_p(\dot{\mu} r) \end{pmatrix} \end{pmatrix}.$$

Hence we obtain

$$\begin{array}{rcccl} \eta_q(\dot{\mu}r) & = & (q^{-1})^{\dot{\mu}r}q & = & \ddot{\mu}(\chi_q r) \\ \ddot{\beta}_p(\ddot{\mu}s) & = & (\ddot{\mu}s)^p & = & \ddot{\mu}(s^p) \\ \dot{\beta}_p(\dot{\mu}r) & = & p^{-1}(\dot{\mu}r)p & = & \dot{\mu}(r^p) \end{array} \tag{60}$$

From this we may deduce (for example)

$$\ddot{\mu}(s^q) = \ddot{\mu}(s^{\delta q}) = (\ddot{\mu}s)^{\delta q} = q^{-1}(\ddot{\mu}s)q \ . \tag{61}$$

Now we have to do a similar development for the second crossed module axiom. Corresponding to the diagram

$$\begin{tikzcd} S \times S \arrow[r, "1_S \times \partial"] \arrow[d, "1_S \times \iota_S"'] & S \times R \arrow[r, "1_S \times \alpha"] & S \times \mathrm{Aut}S \arrow[d, "\mathrm{eval}"] \\ S \times \mathrm{Aut}S \arrow[rr, "\mathrm{eval}"'] & & S \end{tikzcd} \tag{62}$$

mapping (s_0, s) to ${s_0}^s$ and to ${s_0}^{\partial s}$, the diagram for 2-fold crossed modules is :

$$\begin{tikzcd} \mathcal{X} \times \mathcal{X} \arrow[r, "1_{\mathcal{X}} \times \mu"] \arrow[d, "1_{\mathcal{X}} \times \iota_{\mathcal{X}}"'] & \mathcal{X} \times \mathcal{Y} \arrow[r, "1_{\mathcal{X}} \times \alpha"] & \mathcal{X} \times \mathrm{Act}\mathcal{X} \arrow[d, "\mathrm{eval}"] \\ \mathcal{X} \times \mathrm{Act}\mathcal{X} \arrow[rr, "\mathrm{eval}"'] & & \mathcal{X} \end{tikzcd} \tag{63}$$

Applying these maps to $s, s_0 \in S,\ r, r_0 \in R$ clockwise around the rectangle we have:

$$\begin{pmatrix} s_0 \\ r_0 \end{pmatrix}, \begin{pmatrix} s \\ r \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} s_0 \\ r_0 \end{pmatrix}, \begin{pmatrix} \ddot{\mu}s \\ \dot{\mu}r \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} s_0 \\ r_0 \end{pmatrix}, \begin{pmatrix} \ddot{\alpha}\ddot{\mu}s \\ \dot{\alpha}\dot{\mu}r \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \chi_{\ddot{\mu}s} r_0 \\ {s_0}^{\dot{\mu}r} \\ {r_0}^{\dot{\mu}r} \end{pmatrix}.$$

In the other direction,

$$\begin{pmatrix} s_0 \\ r_0 \end{pmatrix}, \begin{pmatrix} s \\ r \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} s_0 \\ r_0 \end{pmatrix}, \begin{pmatrix} \eta_s \\ \beta_r \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \eta_s r_0 \\ {s_0}^r \\ {r_0}^r \end{pmatrix}.$$

Thus we obtain

$$\begin{array}{rcl} \chi_{\ddot{\mu}s} r_0 & = & (s^{-1})^{r_0} s \ , \\ {s_0}^{\dot{\mu}r} & = & {s_0}^r \ , \\ {r_0}^{\dot{\mu}r} & = & r^{-1} r_0 r \ . \end{array} \tag{64}$$

9.2 Crossed Modules of Crossed Modules

Using the generalisation of the crossed module axioms given in the previous section we obtain the following definition.

Definition 9.1 *A* crossed module of crossed modules *or* 2-fold crossed module *has the form* $\mathcal{M} = (\mu : \mathcal{X} \to \mathcal{Y})$ *where* $\mathcal{X} = (\partial : S \to R)$, $\mathcal{Y} = (\delta : Q \to P)$, *and* $\mu = (\ddot{\mu}, \dot{\mu}) : \mathcal{X} \to \mathcal{Y}$ *is a morphism of crossed modules, together with an action of* $\mathcal{Y}$ *on* $\mathcal{X}$, *as in Subsection 3.6, satisfying:*

$$\textbf{CC1:} \quad \ddot{\mu}(s^p) \;=\; (\ddot{\mu}s)^p\,, \quad \dot{\mu}(r^p) \;=\; p^{-1}(\dot{\mu}r)p\,, \quad \ddot{\mu} \circ \chi_q \;=\; \eta_q \circ \dot{\mu}\,,$$

$$\textbf{CC2:} \quad {s_0}^{\dot{\mu}r} \;=\; {s_0}^{r}\,, \qquad {r_0}^{\dot{\mu}r} \;=\; r^{-1}\,r_0\,r\,, \qquad \chi_{\ddot{\mu}s} \;=\; \eta_s\,.$$

$$\begin{tikzcd}
S \arrow[r, "\ddot{\iota}"] \arrow[d, "\partial"'] \arrow[rr, bend left, "\ddot{\mu}"'] & W(\mathcal{X}) \arrow[d, "\Delta"] & Q \arrow[l, "\ddot{\alpha}"'] \arrow[d, "\delta"] \\
R \arrow[r, "\iota"'] \arrow[rr, bend right, "\dot{\mu}"] & A(\mathcal{X}) & P \arrow[l, "\dot{\alpha}"]
\end{tikzcd} \tag{65}$$

Axioms **CC1:** and **CC2:** are sufficient to make the diagram commute.

Lemma 9.2

$$\alpha\mu \;=\; \iota\,.$$

Proof:

$$\ddot{\alpha}\ddot{\mu}s \;=\; \chi_{\ddot{\mu}s} \;=\; \eta_s \;=\; \ddot{\iota}s \qquad \text{and} \qquad \dot{\alpha}\dot{\mu}r \;=\; \beta_{\dot{\mu}r} \;=\; \beta_r \;=\; \iota r\;.$$

□

In order to show that $\mathcal{X}$ and $\mathcal{Y}$ form a crossed square we have to define a crossed pairing, and there is one obvious choice.

Proposition 9.3 *The* 2*-fold crossed module* $\mathcal{M}$ *is a crossed square with crossed pairing*

$$\boxtimes \;:\; R \times Q \to S, \quad (r,q) \mapsto r \boxtimes q \;=\; (\ddot{\alpha}q)r \;=\; \chi_q r.$$

Proof: Axiom (a) for crossed squares requires $\ddot{\mu}(s^p) = (\ddot{\mu}s)^p$ and $\partial(s^p) = (\partial s)^p$. The first of these is given by **CC1:** while $(\partial s)^p = \dot{\beta}_p \partial s = \partial \ddot{\beta}_p s = \partial(s^p)$.

For (b) we require that $(\ddot{\mu} : S \to Q)$ and $(\dot{\mu} : R \to P)$ are crossed modules:

$$\ddot{\mu}(s^q) = q^{-1}(\ddot{\mu}s)q \quad \text{by (61)} \quad \text{and} \quad {s_0}^{\ddot{\mu}s} = {s_0}^{\delta\ddot{\mu}s} = {s_0}^{\dot{\mu}\partial s} = {s_0}^{\partial s} = s^{-1}s_0 s\;,$$

$$\dot{\mu}(r^p) = p^{-1}(\dot{\mu}r)p \quad \text{by } \textbf{CC1:} \quad \text{and} \quad {r_0}^{\dot{\mu}r} = r^{-1}r_0 r \quad \text{by } \textbf{CC2:}\;.$$

We then verify that $\boxtimes$ is a crossed pairing.

$$\begin{array}{llllllll}
\text{(i)} & (r_1r_2)\boxtimes q & = & (\ddot{\alpha}q)(r_1r_2) & = & ((\ddot{\alpha}q)(r_1))^{r_2}\,((\ddot{\alpha}q)(r_2)) & = & (r_1\boxtimes q)^{r_2}\,(r_2\boxtimes q) \\
\text{(ii)} & r\boxtimes(q_1q_2) & = & \ddot{\alpha}(q_1q_2)(r) & = & (\ddot{\alpha}q_1\star\ddot{\alpha}q_2)(r) & = & (\chi_{q_2}r)(\chi_{q_1}r)(\chi_{q_2}\partial\chi_{q_1}r) \\
 & & = & (\chi_{q_2}r)\,(\chi_{q_1}r)^{q_2} & & \text{(using Lemma 3.8 (b))} & = & (r\boxtimes q_2)\,(r\boxtimes q_1)^{q_2}\,.
\end{array}$$

$$\text{(iii)} \quad r^p\boxtimes q^p \;=\; (\ddot{\alpha}q^p)\,(r^p) \;=\; (\ddot{\alpha}q)^{\dot{\alpha}p}\,(r^p) \;=\; (\chi_q)^{\beta_p}\,(r^p) \;=\; \ddot{\beta}_p\chi_q\dot{\beta}_p^{-1}(r^p) \;=\; \ddot{\beta}_p\chi_q r \;=\; (r\boxtimes q)^p\,.$$

To verify axiom (d) for a crossed square, we check:

$$\ddot{\mu}(r\boxtimes q) = \ddot{\mu}(\chi_q r) = (q^{-1})^{\dot{\mu}r}q = (q^{-1})^r q\;,$$

$$r\partial(r\boxtimes q) = r(\partial\chi_q r) = \dot{\beta}_{\chi_q}r = \Delta\chi_q r = (\Delta\ddot{\alpha}q)r = (\dot{\alpha}\delta q)r = \dot{\beta}_{\delta q}r = r^{\delta q} = r^q\;.$$

To verify axiom (e) we check the following:

$$(r\boxtimes\ddot{\mu}s) = \chi_{\ddot{\mu}s}r = \eta_s r = (s^{-1})^r s\;,$$

$$s(\partial s\boxtimes q) = s(\chi_q\partial s) = \ddot{\beta}_{\chi_q}s = \Delta\chi_q s = (\Delta\ddot{\alpha}q)s = (\dot{\alpha}\delta q)s = \ddot{\beta}_{\delta q}s = s^{\delta q} = s^q\;.$$

□

Definition 9.4 *A* morphism of crossed module of crossed modules

$$\theta \quad : \quad \mathcal{M}_1 = (\mu_1 : \mathcal{X}_1 \to \mathcal{Y}_1) \;\to\; \mathcal{M}_2 = (\mu_2 : \mathcal{X}_2 \to \mathcal{Y}_2)$$

is a pair of crossed module morphisms

$$\ddot{\theta} : \mathcal{X}_1 \to \mathcal{X}_2, \qquad \dot{\theta} : \mathcal{Y}_1 \to \mathcal{Y}_2,$$

plus some extra condition(s) which say that the action is preserved.
[Complete this definition.]

9.3 A 2-fold crossed module from a crossed square

In order to show that the category XXMod of 2-fold crossed modules and their morphisms is equivalent to the category XSq of crossed squares and their morphisms we require a construction in the reverse direction.

Theorem 9.5 *Let $\mathcal{R}$ be a crossed square with crossed pairing $\boxtimes : R_{\{1\}} \times R_{\{2\}} \to R_{[2]}$,*

$$\mathcal{R} \quad = \quad \begin{tikzcd} R_{[2]} \arrow[r, "\ddot{\partial}_1"] \arrow[d, "\ddot{\partial}_2"'] & R_{\{2\}} \arrow[d, "\dot{\partial}_2"] \\ R_{\{1\}} \arrow[r, "\dot{\partial}_1"'] & R_{\emptyset} \end{tikzcd}$$

so that $\ddot{\mathcal{R}}_2 = (\ddot{\partial}_2 : R_{[2]} \to R_{\{1\}})$ and $\dot{\mathcal{R}}_2 = (\dot{\partial}_2 : R_{\{2\}} \to R_{\emptyset})$ are crossed modules, $(\ddot{\partial}_1, \dot{\partial}_1) : \ddot{\mathcal{R}}_2 \to \dot{\mathcal{R}}_2$ is a crossed module morphism, and $R_{\{1\}}$ and $R_{\{2\}}$ act on $R_{[2]}$ via $R_{\emptyset}$. Then there is an action of $\dot{\mathcal{R}}_2$ on $\ddot{\mathcal{R}}_2$ which makes $(\partial_1 : \ddot{\mathcal{R}}_2 \to \dot{\mathcal{R}}_2)$ a crossed module of crossed modules.

Proof: We first demonstrate an action of $\dot{\mathcal{R}}_2$ on $\ddot{\mathcal{R}}_2$ by showing that there exists a morphism of crossed modules $\alpha : \dot{\mathcal{R}}_2 \to \mathcal{A}(\ddot{\mathcal{R}}_2)$ as in the following diagram:

$$\begin{tikzcd}
R_{[2]} \arrow[loop left, "\ddot{\beta}_m{,}\ddot{\beta}_p"] \arrow[r, "\ddot{\imath}"] \arrow[d, shift left, "\ddot{\partial}_2"] \arrow[rr, bend left, "\ddot{\partial}_1"] & W(\ddot{\mathcal{R}}_2) \arrow[d, "\Delta"] & R_{\{2\}} \arrow[l, "\ddot{\alpha}"'] \arrow[d, "\dot{\partial}_2"] \\
R_{\{1\}} \arrow[loop left, "\dot{\beta}_m{,}\dot{\beta}_p"] \arrow[u, shift left, "\chi_\ell{,}\chi_n"] \arrow[r, "i"'] \arrow[rr, bend right, "\dot{\partial}_1"'] & A(\ddot{\mathcal{R}}_2) & R_\emptyset \arrow[l, "\dot{\alpha}"] \\
\ddot{\mathcal{R}}_2 & \mathcal{A}(\ddot{\mathcal{R}}_2) & \dot{\mathcal{R}}_2
\end{tikzcd} \tag{66}$$

Define

$$\dot{\alpha} : R_\emptyset \to \mathcal{A}(\ddot{\mathcal{R}}_2), \quad p \mapsto \beta_p : \ddot{\mathcal{R}}_2 \to \ddot{\mathcal{R}}_2 \quad \text{for all} \quad p \in R_\emptyset \ ,$$

where

$$\ddot{\beta}_p \ell = \ell^p \quad \text{for all} \quad \ell \in R_{[2]} \qquad \text{and} \qquad \dot{\beta}_p m = m^p \quad \text{for all} \quad m \in R_{\{1\}} \ .$$

We see that $\dot{\alpha}p$ is a crossed module morphism since $\ddot{\beta}$ and $\dot{\beta}$ are group homomorphisms and

$$(\ddot{\beta}_p \ell)^{\dot{\beta}_p m} \ = \ (\ell^p)^{m^p} \ = \ \ell^{pp^{-1}mp} \ = \ \ell^{mp} \ = \ \ddot{\beta}_p(\ell^m) \ .$$

The map $\ddot{\alpha} : R_{\{2\}} \to W(\ddot{\mathcal{R}}_2)$ is defined by

$$\ddot{\alpha} n \ = \ \chi_n : R_{\{1\}} \to R_{[2]}, \quad \chi_n m = m \boxtimes n, \quad \text{for all} \ \ m \in R_{\{1\}} \ . \tag{67}$$

Each χ_n is a derivation of $\ddot{\mathcal{R}}_1$ since

$$\chi_n(m_1 m_2) \ = \ (m_1 m_2 \boxtimes n) \ = \ (m_1 \boxtimes n)^{m_2}\,(m_2 \boxtimes n) \ = \ (\chi_n m_1)^{m_2}\,(\chi_n m_2) \ .$$

That $\ddot{\alpha}$ is a homomorphism is verified by

$$\begin{aligned}
(\chi_{n_1} \star \chi_{n_2})\, m \ &= \ (\chi_{n_2} m)\,(\chi_{n_1} m)\,\chi_{n_2}\ddot{\partial}_2(m \boxtimes n_1) \\
&= \ (m \boxtimes n_2)\,(m \boxtimes n_1)\,\chi_{n_2}(m^{-1}\,m^{n_1}) \\
&= \ (m \boxtimes n_2)\,(m \boxtimes n_1)\,(\chi_{n_2} m^{-1})^{m^{n_1}}\,(\chi_{n_2} m^{n_1}) \\
&= \ (m \boxtimes n_2)\,(m \boxtimes n_1)\,(m^{-1} \boxtimes n_2)^{n_1^{-1} m n_1}\,(m^{n_1} \boxtimes n_2) \\
&= \ (m \boxtimes n_2)\,(m \boxtimes n_1)\,((m \boxtimes n_2)^{[m,n_1]})^{-1}\,(m \boxtimes n_1)^{-1}\,(m \boxtimes n_2)\,(m \boxtimes n_1)^{n_2} \\
&= \ (m \boxtimes n_1)\,\{(m \boxtimes n_1)^{-1}\,(m \boxtimes n_2)\}\,(m \boxtimes n_1)^{n_2} \\
&= \ (m \boxtimes n_2)\,(m \boxtimes n_1)^{n_2} \ = \ (m \boxtimes n_1 n_2) \ = \ \chi_{n_1 n_2}\, m \ .
\end{aligned}$$

To show that α preserves actions, we calculate

$$\begin{aligned} \ddot\alpha(n^p) \;&=\; \chi_{n^p} \quad : \quad m \mapsto (m \boxtimes n^p) \ , \\ (\ddot\alpha n)^{\dot\alpha p} \;&=\; \beta_p^{-1} * \chi_n * \beta_p \quad : \quad m \mapsto \ddot\beta_p \chi_n \dot\beta_p^{-1} m \;=\; \ddot\beta_p \chi_n (m^{p^{-1}}) \;=\; (m^{p^{-1}} \boxtimes n)^p \;=\; (m \boxtimes n^p) \ . \end{aligned}$$

To show that the right-hand square in diagram (65) commutes, $\dot\alpha\dot\partial_2 \;=\; \Delta\ddot\alpha$, note that

$$\Delta\ddot\alpha\, n \;=\; \Delta\chi_n \;=\; (\ddot\beta_{\chi_n}, \dot\beta_{\chi_n})$$

where

$$\dot\beta_{\chi_n} m \;=\; m(\ddot\partial_2 \chi_n m) \;=\; m(\ddot\partial_2(m \boxtimes n)) \;=\; m(m^{-1}m^n) \;=\; m^n \;=\; \dot\beta_{\dot\partial_2 n} m \ ,$$

and

$$\ddot\beta_{\chi_n} \ell \;=\; \ell(\chi_n \ddot\partial_2 \ell) \;=\; \ell(\ddot\partial_2 \ell \boxtimes n) \;=\; \ell^n \;=\; \ddot\beta_{\dot\partial_2 n} \ell \ .$$

So

$$\Delta\ddot\alpha n \;=\; (\ddot\beta_{\dot\partial_2 n}, \dot\beta_{\dot\partial_2 n}) \;=\; \beta_{\dot\partial_2 n} \;=\; \dot\alpha\dot\partial_2 n \ .$$

This completes the proof that $\alpha : \dot{\mathcal{R}}_2 \to \mathcal{A}(\ddot{\mathcal{R}}_2)$ is a morphism of crossed modules.

Thus the crossed square $\mathcal{R}$ gives rise to a semidirect crossed module $\dot{\mathcal{R}}_2 \ltimes \ddot{\mathcal{R}}_2$.

Of course we could obtain a second semidirect product crossed module $\dot{\mathcal{R}}_1 \ltimes \ddot{\mathcal{R}}_1$ for the transpose of $\mathcal{R}$ by reversing the roles of $R_{\{1\}}$ and $R_{\{2\}}$ and using the crossed pairing $\tilde\boxtimes$.

The morphism $\iota = (\dot\imath, \ddot\imath)$ in diagram (66) is given (see Subsection 3.3) by:

$$\dot\imath \;:\; R_{\{1\}} \to A, \quad \dot\imath m = \beta_m = (\ddot\beta_m, \dot\beta_m) : \ddot{\mathcal{R}}_2 \to \ddot{\mathcal{R}}_2, \quad \text{for all} \quad m \in R_{\{1\}} \ ,$$

where $\ddot\beta_m \ell = \ell^m$ for all $\ell \in R_{[2]}$ and $\dot\beta_m m_0 = {m_0}^m = m^{-1} m_0 m$ for all $m_0 \in R_{\{1\}}$, and

$$\ddot\imath \;:\; R_{[2]} \to W, \quad \ell \mapsto \eta_\ell : R_{\{1\}} \to R_{[2]},\ m \mapsto (\ell^{-1})^m\, \ell \quad \text{for all} \quad m \in R_{\{1\}} \ .$$

The 2-fold crossed module axioms for $(\partial_1 : \ddot{\mathcal{R}}_2 \to \dot{\mathcal{R}}_2)$ are easily verified.

CC1: (i) $\ddot\partial_1(\ell^p) = (\ddot\partial_1 \ell)^p$ by (a),
(ii) $\dot\partial_1(m^p) = p^{-1}(\dot\partial_1 m)p$ by **M1:** for $\dot{\mathcal{R}}_1$,
(iii) $\ddot\partial_1(\chi_n m) = \ddot\partial_1(m \boxtimes n) = (n^{-1})^m n = (n^{-1})^{\dot\partial_1 m} n = \eta_n(\dot\partial_1 m)$.

CC2: (i) $\ell^{\dot\partial_1 m} = \ell^m$ by definition of the action,
(ii) ${m_0}^{\dot\partial_1 m} = m^{-1} m_0 m$ by **M2:** for $\dot{\mathcal{R}}_1$,
(iii) $\chi_{\ddot\partial_1 \ell} m = m \boxtimes \ddot\partial_1 \ell = (\ell^{-1})^m \ell = \eta_\ell m.$

□

9.4 Derivations of a 2-fold crossed module

The following definition (b) of an $\mathcal{R}_1$-derivation *appears* to be correct, but we shall see that an extra axiom is appropriate when we consider the corresponding version of a section. The missing information is an expansion for $\ddot{\chi}(n^p)$. A revised version will be given as Definition 9.18.

Definition 9.6 *Let* $\mathcal{R}_1$ *be the usual crossed square* $\mathcal{R}$ *of (43) considered as a crossed module of crossed modules* $(\partial_1 = (\ddot{\partial}_1, \dot{\partial}_1) \ : \ \ddot{\mathcal{R}}_2 \to \dot{\mathcal{R}}_2)$.

$$\mathcal{R}_1 \qquad = \qquad \begin{tikzcd} R_{[2]} \arrow[r, shift left, "\ddot{\partial}_1"] \arrow[d, "\ddot{\partial}_2"'] & R_{\{2\}} \arrow[l, shift left, "{\ddot{\phi}, \ddot{\chi}}"] \arrow[d, "\dot{\partial}_2"] \\ R_{\{1\}} \arrow[r, shift right, "\dot{\partial}_1"'] & R_{\emptyset} \arrow[l, shift right, "{\dot{\phi}, \dot{\chi}}"'] \end{tikzcd} \tag{68}$$

(a) *An* $\mathcal{R}_1$*-map is a pair of maps* $\phi = (\ddot{\phi}, \dot{\phi}) : \dot{\mathcal{R}}_2 \to \ddot{\mathcal{R}}_2$ *which commute with the two boundaries:*

$$\ddot{\phi} * \ddot{\partial}_2 = \dot{\partial}_2 * \dot{\phi}$$

(b) *An* $\mathcal{R}_1$*-derivation is an* $\mathcal{R}$*-map* $\chi = (\ddot{\chi}, \dot{\chi})$ *such that* $\ddot{\chi}$ *is a derivation of* $\ddot{\mathcal{R}}_1$ *and* $\dot{\chi}$ *is a derivation of* $\dot{\mathcal{R}}_1$*.*

Similarly, let $\mathcal{R}_2$ *be* $\mathcal{R}$ *considered as* $(\partial_2 = (\ddot{\partial}_2, \dot{\partial}_2) : \ddot{\mathcal{R}}_1 \to \dot{\mathcal{R}}_1)$*, giving notions of* $\mathcal{R}_2$*-map and* $\mathcal{R}_2$*-derivation.*

Lemma 9.7 *To check that a pair of derivations* $\chi = (\ddot{\chi}, \dot{\chi})$ *is an* $\mathcal{R}_1$*-derivation it is sufficient to check that* $\ddot{\chi} * \ddot{\partial}_2 = \dot{\partial}_2 * \dot{\chi}$ *on a generating set of* $R_{\{2\}}$*.*

Proof: If $\ddot{\partial}_2\ddot{\chi}n_1 = \dot{\chi}\dot{\partial}_2 n_1$ and $\ddot{\partial}_2\ddot{\chi}n_2 = \dot{\chi}\dot{\partial}_2 n_2$ then

$$\ddot{\partial}_2\ddot{\chi}(n_1n_2) = \ddot{\partial}_2((\ddot{\chi}n_1)^{n_2}(\ddot{\chi}n_2)) = (\ddot{\partial}_2\ddot{\chi}n_1)^{\dot{\partial}_2 n_2}(\ddot{\partial}_2\ddot{\chi}n_2) = (\dot{\chi}\dot{\partial}_2 n_1)^{n_2}(\dot{\chi}\dot{\partial}_2 n_2) = \dot{\chi}\dot{\partial}_2(n_1n_2).$$

□

Lemma 9.8 *The set of* $\mathcal{R}_1$*-derivations has a Whitehead multiplication*

$$\chi_1 \star \chi_2 = (\ddot{\chi}_1 \star \ddot{\chi}_2,\ \dot{\chi}_1 \star \dot{\chi}_2).$$

Proof: Since $\ddot{\chi}_1 \star \ddot{\chi}_2$ and $\dot{\chi}_1 \star \dot{\chi}_2$ are derivations of $\ddot{\mathcal{R}}_1$ and $\dot{\mathcal{R}}_1$ respectively, we only have to show that $\chi_1 \star \chi_2$ is an $\mathcal{R}_1$-map.

$$\begin{aligned} \ddot{\partial}_2(\ddot{\chi}_1 \star \ddot{\chi}_2)(n) &= (\ddot{\partial}_2\ddot{\chi}_2 n)(\ddot{\partial}_2\ddot{\chi}_1 n)(\ddot{\partial}_2\ddot{\chi}_2\ddot{\partial}_1\ddot{\chi}_1 n) \\ &= (\dot{\chi}_2\dot{\partial}_2 n)(\dot{\chi}_1\dot{\partial}_2 n)(\dot{\chi}_2\dot{\partial}_2\ddot{\partial}_1\ddot{\chi}_1 n) \\ &= (\dot{\chi}_2\dot{\partial}_2 n)(\dot{\chi}_1\dot{\partial}_2 n)(\dot{\chi}_2\dot{\partial}_1\ddot{\partial}_2\ddot{\chi}_1 n) \\ &= (\dot{\chi}_2\dot{\partial}_2 n)(\dot{\chi}_1\dot{\partial}_2 n)(\dot{\chi}_2\dot{\partial}_1\dot{\chi}_1\dot{\partial}_2 n) \\ &= (\dot{\chi}_2 \star \dot{\chi}_1)(\dot{\partial}_2 n)\ . \end{aligned}$$

□

[Not sure if the following Lemma is true!]

Lemma 9.9 *A pair of principal derivations $\eta = (\ddot\eta_\ell, \dot\eta_m)$ is an $\mathcal{R}_1$-derivation if and only if $(\ddot\partial_2\ell)m^{-1}$ is fixed by the action of $\dot\partial_2 R_{\{2\}}$.*

Proof:

$$\ddot\partial_2\ddot\eta_\ell n \;=\; \dot\eta_m\dot\partial_2 n \;\Leftrightarrow\; (\ddot\partial_2\ell^{-1})^{\dot\partial_2 n}(\ddot\partial_2\ell) \;=\; (m^{-1})^{\dot\partial_2 n}m \;\Leftrightarrow\; (\ddot\partial_2\ell)m^{-1} \;=\; ((\ddot\partial_2\ell)m^{-1})^{\dot\partial_2 n}\;.$$

[But what about equation (70)?] □

Definition 9.10 *A* principal $\mathcal{R}_1$-derivation *is an $\mathcal{R}_1$-derivation $\eta = (\ddot\eta_\ell, \dot\eta_m)$ such that $\ddot\eta_\ell, \dot\eta_m$ are principal derivations.*

Given an $\mathcal{R}_1$-derivation $\chi = (\ddot\chi, \dot\chi)$ we have automorphisms $\beta_{\ddot\chi} = (\ddot\beta_{\ddot\chi}, \dot\beta_{\ddot\chi})$ of $\ddot{\mathcal{R}}_1$ and $\beta_{\dot\chi} = (\ddot\beta_{\dot\chi}, \dot\beta_{\dot\chi})$ of $\dot{\mathcal{R}}_1$ given by

$$\ddot\beta_{\ddot\chi}\ell = \ell(\ddot\chi\ddot\partial_1\ell), \qquad \dot\beta_{\ddot\chi}n = n(\ddot\partial_1\ddot\chi n), \qquad \ddot\beta_{\dot\chi}m = m(\dot\chi\dot\partial_1 m), \qquad \dot\beta_{\dot\chi}p = p(\dot\partial_1\dot\chi p),$$

such that

$$\ddot\beta_{\ddot\chi}\ddot\chi \;=\; \ddot\chi\dot\beta_{\ddot\chi},\; n \mapsto (\ddot\chi n)(\ddot\chi\ddot\partial_1\ddot\chi n) \qquad \text{and} \qquad \ddot\beta_{\dot\chi}\dot\chi \;=\; \dot\chi\dot\beta_{\dot\chi},\; p \mapsto (\dot\chi p)(\dot\chi\dot\partial_1\dot\chi p).$$

These automorphisms will be important for our construction of the actor crossed square.

Since these maps are automorphisms, we know that

$$\ddot\beta_{\ddot\chi}(\ell^n) \;=\; (\ddot\beta_{\ddot\chi}\ell)^{\dot\beta_{\ddot\chi}n} \qquad \text{and} \qquad \ddot\beta_{\dot\chi}(m^p) \;=\; (\ddot\beta_{\dot\chi}m)^{\dot\beta_{\dot\chi}p}.$$

Lemma 9.11 *These four group automorphisms combine to give automorphisms*

$$\beta_{\ddot\chi} = (\ddot\beta_{\ddot\chi}, \ddot\beta_{\dot\chi}) \text{ of } \ddot{\mathcal{R}}_2 \qquad \text{and} \qquad \beta_{\dot\chi} = (\dot\beta_{\ddot\chi}, \dot\beta_{\dot\chi}) \text{ of } \dot{\mathcal{R}}_2.$$

Proof: We first check commutation:

$$\begin{aligned}
\ddot\partial_2\ddot\beta_{\ddot\chi}\ell \;&=\; (\ddot\partial_2\ell)(\ddot\partial_2\ddot\chi\ddot\partial_1\ell) \;=\; (\ddot\partial_2\ell)(\dot\chi\dot\partial_2\ddot\partial_1\ell) \;=\; (\ddot\partial_2\ell)(\dot\chi\dot\partial_1(\ddot\partial_2\ell)) \;=\; \ddot\beta_{\dot\chi}\ddot\partial_2\ell,\\
\dot\partial_2\dot\beta_{\ddot\chi}n \;&=\; (\dot\partial_2 n)(\dot\partial_2\ddot\partial_1\ddot\chi n) \;=\; (\dot\partial_2 n)(\dot\partial_1\ddot\partial_2\ddot\chi n) \;=\; (\dot\partial_2 n)(\dot\partial_1\dot\chi(\dot\partial_2 n)) \;=\; \dot\beta_{\dot\chi}\dot\partial_2 n.
\end{aligned}$$

We now require to prove

$$\text{(a)}\;\; \ddot\beta_{\ddot\chi}(\ell^m) \;=\; (\ddot\beta_{\ddot\chi}\ell)^{\ddot\beta_{\dot\chi}m} \qquad \text{and} \qquad \text{(b)}\;\; \dot\beta_{\ddot\chi}(n^p) \;=\; (\dot\beta_{\ddot\chi}n)^{\dot\beta_{\dot\chi}p}.$$

We prove (b) as follows.

$$\begin{aligned}
\dot\beta_{\ddot\chi}(n^p) \;&=\; n^p(\ddot\partial_1\ddot\chi(n^p))\\
&=\; n^p\;\ddot\partial_1[(\dot\chi p \boxtimes n^p)^{-1}(\ddot\chi n)^{p(\dot\chi p)}]\\
&=\; n^p\;(n^p)^{-1}\;(n^p)^{\dot\chi p}\;(\ddot\partial_1\ddot\chi n)^{p(\dot\partial_1\dot\chi p)}\\
&=\; (n(\ddot\partial_1\ddot\chi n))^{p(\dot\partial_1\dot\chi p)}\\
&=\; (\dot\beta_{\ddot\chi}n)^{\dot\beta_{\dot\chi}p}.
\end{aligned}$$

[We still need to check (a) !] □

Lemma 9.12 *The endomorphisms* $\ddot{\beta}_{\ddot{\chi}}, \dot{\beta}_{\ddot{\chi}}, \ddot{\beta}_{\dot{\chi}}, \dot{\beta}_{\dot{\chi}}$ *commute with* $\ddot{\chi}\ddot{\partial}_1, \ddot{\partial}_1\ddot{\chi}, \dot{\chi}\dot{\partial}_1, \dot{\partial}_1\dot{\chi}$ *respectively.*

Proof: This follows immediately from Lemma 2.4(d). □

Lemma 9.13 *Given a composite* $\mathcal{R}_1$*-derivation* $\chi = \chi_1 * \chi_2$*,*

$$\ddot{\beta}_{\ddot{\chi}} \;=\; \ddot{\beta}_{\ddot{\chi}_1} * \ddot{\beta}_{\ddot{\chi}_2} \qquad \textit{and} \qquad \dot{\beta}_{\dot{\chi}} \;=\; \dot{\beta}_{\dot{\chi}_1} * \dot{\beta}_{\dot{\chi}_2}.$$

Proof: This is immediate from Corollary 2.3.

[Result on derivations changed to γ-derivations – what happens here?] □

9.5 Sections of a crossed module of cat^1-groups

Replacing the crossed modules $\ddot{\mathcal{R}}_1, \dot{\mathcal{R}}_1$ by the corresponding cat^1-groups, we obtain the following *crossed module of cat*^{1}*-groups*, (equivalently a *cat*^{1}*-group of crossed modules* as in Section 8.4), where $\bar{\partial}_2 : R_{\{2\}} \ltimes R_{[2]} \to R_{\emptyset} \ltimes R_{\{1\}}$, $(n,\ell) \mapsto (\dot{\partial}_2 n, \ddot{\partial}_2 \ell)$.

$$\mathcal{C}_1 \quad = \quad \begin{tikzcd}
R_{\{2\}} \ltimes R_{[2]} \arrow[r, shift left=3, "{\ddot{t}_1,\ddot{h}_1}"] \arrow[r, shift left=1] \arrow[d, "\bar{\partial}_2"'] & R_{\{2\}} \arrow[l, shift left=1] \arrow[l, shift left=3, "{\ddot{e}_1,\ddot{\xi}}"] \arrow[d, "\dot{\partial}_2"] \\
R_{\emptyset} \ltimes R_{\{1\}} \arrow[r, shift right=1] \arrow[r, shift right=3, "{\dot{t}_1,\dot{h}_1}"'] & R_{\emptyset} \arrow[l, shift right=3, "{\dot{e}_1,\dot{\xi}}"'] \arrow[l, shift right=1]
\end{tikzcd} \tag{69}$$

Lemma 9.14 *The pair* $(\bar{\partial}_2, \dot{\partial}_2)$ *is a morphism of cat*^{1}*-groups.*

Proof: We first show that $\bar{\partial}_2$ is a group homomorphism.

$$\begin{aligned}
\bar{\partial}_2((n_1,\ell_1)(n_1,\ell_2)) \;&=\; \bar{\partial}_2(n_1n_2, \ell_1^{n_2}\ell_2) \\
&=\; (\dot{\partial}_2(n_1n_2), \ddot{\partial}_2(\ell_1^{n_2}\ell_2)) \\
&=\; ((\dot{\partial}_2n_1)(\dot{\partial}_2n_2), (\ddot{\partial}_2\ell_1)^{\dot{\partial}_2n_2}\ddot{\partial}_2\ell_2) \\
&=\; (\dot{\partial}_2n_1, \ddot{\partial}_2\ell_1)(\dot{\partial}_2n_2, \ddot{\partial}_2\ell_2)
\end{aligned}$$

Next we check that $(\bar{\partial}_2, \dot{\partial}_2)$ commutes with $t_1 = (\ddot{t}_1, \dot{t}_1)$, $h_1 = (\ddot{h}_1, \dot{h}_1)$ and $e_1 = (\ddot{e}_1, \dot{e}_1)$.

$$(\bar{\partial}_2 * \dot{t}_1)(n,\ell) \;=\; \dot{t}_1(\dot{\partial}_2n, \ddot{\partial}_2\ell) \;=\; \dot{\partial}_2n \;=\; (\ddot{t}_1 * \dot{\partial}_2)(n,\ell)$$

$$(\bar{\partial}_2 * \dot{h}_1)(n,\ell) \;=\; \dot{h}_1(\dot{\partial}_2n, \ddot{\partial}_2\ell) \;=\; (\dot{\partial}_2n)(\dot{\partial}_1\ddot{\partial}_2\ell) \;=\; (\dot{\partial}_2n)(\dot{\partial}_2\ddot{\partial}_1\ell) \;=\; \dot{\partial}_2(n(\ddot{\partial}_1\ell)) \;=\; (\ddot{h}_1 * \dot{\partial}_2)(n,\ell)$$

$$(\dot{e}_1 * \bar{\partial}_2)n \;=\; \bar{\partial}_2(n,1) \;=\; (\dot{\partial}_2n, 1) \;=\; \dot{e}_1(\dot{\partial}_2n) \;=\; (\dot{\partial}_2 * \dot{e}_1)n$$

□

Definition 9.15

(a) *A* $\mathcal{C}_1$-map *is a pair of maps* $\phi = (\ddot{\phi}, \dot{\phi}) : \dot{\mathcal{R}}_2 \to \dot{\mathcal{R}}_2 \ltimes \ddot{\mathcal{R}}_2$ *which commute with the two boundaries.*

(b) *A* $\mathcal{C}_1$-section *is a pair* $\xi = (\ddot{\xi}, \dot{\xi})$ *such that*

- ξ *is a* $\mathcal{C}_1$*-map:* $\bar{\partial}_2 \circ \ddot{\xi} \;=\; \dot{\xi} \circ \dot{\partial}_2$,
- $\ddot{\xi}$ *is a section of* $\ddot{\mathcal{R}}_1$ *and* $\dot{\xi}$ *is a section of* $\dot{\mathcal{R}}_1$,
- $t \circ \xi = 1_{\dot{\mathcal{R}}_2}$,
- ξ *is a crossed module morphism:* $\ddot{\xi}(n^p) = (\ddot{\xi}n)^{\dot{\xi}p}$.

Lemma 9.16 *If* ξ *is an* $\mathcal{C}_1$*-section, then there is an* $\mathcal{R}_1$*-derivation* χ *such that* $\dot{\xi}p = (p, \dot{\chi}p)$ *and* $\ddot{\xi}n = (n, \ddot{\chi}n)$.

Proof: The sections $\ddot{\xi}, \dot{\xi}$ determine derivations $\ddot{\chi} : R_{\{2\}} \to R_{\emptyset}$ and $\dot{\chi} : R_{\emptyset} \to R_{\{1\}}$, so we have to verify that $\chi = (\ddot{\chi}, \dot{\chi})$ is an $\mathcal{R}_1$-map. Since

$$(\ddot{\xi} * \bar{\partial}_2)n \;=\; \bar{\partial}_2(n, \ddot{\chi}n) \;=\; (\dot{\partial}_2 n, \ddot{\partial}_2\ddot{\chi}n) \qquad \text{and} \qquad (\dot{\partial}_2 * \dot{\xi})n \;=\; \dot{\xi}(\dot{\partial}_2 n) \;=\; (\dot{\partial}_2 n, \dot{\chi}\dot{\partial}_2 n),$$

commuting sections imply commuting derivations. □

Definition 9.17 *A Whitehead multiplication for* $\mathcal{R}_1$*-sections is defined as follows*

$$\xi_1 \star \xi_2 \;=\; (\ddot{\xi}_1 \star \ddot{\xi}_2,\, \dot{\xi}_1 \star \dot{\xi}_2).$$

Let us investigate whether the requirement that ξ is a morphism of crossed modules gives further information about χ. Since

$$\begin{aligned} \ddot{\xi}(n^p) &\;=\; (n^p, \ddot{\chi}(n^p))\ , \\ (\ddot{\xi}n)^{\dot{\xi}p} &\;=\; (n, \ddot{\chi}n)^{(p,\dot{\chi}p)} \;=\; (n^p,\ (\dot{\chi}p \boxtimes n^p)^{-1}\ (\ddot{\chi}n)^{p(\dot{\chi}p)})\ , \end{aligned}$$

we obtain the additional identity

$$\ddot{\chi}(n^p) \;=\; (\dot{\chi}p \boxtimes n^p)^{-1}\ (\ddot{\chi}n)^{p(\dot{\chi}p)})\ . \tag{70}$$

We can now provide the modified version of Definition 9.6 as promised earlier.

Definition 9.18 *An* $\mathcal{R}_1$*-derivation is an* $\mathcal{R}$*-map* $\chi = (\ddot{\chi}, \dot{\chi})$ *such that* $\ddot{\chi}$ *is a derivation of* $\ddot{\mathcal{R}}_1$*,* $\dot{\chi}$ *is a derivation of* $\dot{\mathcal{R}}_1$*, and equation (70) is satisfied.*

The hope is that this extra information (70) will prove useful in simplifying formulae to found later. On the other hand, the right-hand side is rather a messy expression, so perhaps we should stick to sections?

Here is a tricky task: prove that a principal derivation $\eta = (\ddot{\eta}_\ell, \dot{\eta}_m)$ satisfies (70). Perhaps it is easier to do this for principal sections?

9.6 2-fold derivations of a crossed square

The maps $\theta : R_\emptyset \to R_{[2]}$ which we shall use to form the group of 2-derivations can be considered as the final part of a map

$$C_\emptyset \to C_{[2]}, \quad p \mapsto (\ (p, \chi p),\ (\phi p, \theta p)\ )\,,$$

where $C_\emptyset = R_\emptyset$ and $C_{[2]} = (R_\emptyset \ltimes R_{\{1\}}) \ltimes (R_{\{2\}} \ltimes R_{[2]})$ as in diagram (54).

Applying the formula for semidirect product multiplication (Proposition 51(b) ???) and the semidirect product action (50), we obtain

$$\begin{aligned}
& ((p,\chi p),(\phi p,\theta p))((q,\chi q),(\phi q,\theta q)) \\
= \ & ((p,\chi p)(q,\chi q),(\phi p,\theta p)^{(q,\chi q)}(\phi q,\theta q)) \\
= \ & ((pq,(\chi p)^q\chi q),((\phi p)^q,(\chi q \boxtimes (\phi p)^q)^{-1}(\theta p)^{q(\chi q)})(\phi q,\theta q)) \\
= \ & ((pq,(\chi p)^q\chi q),((\phi p)^q,((\phi p)^q \boxtimes \chi q)(\theta p)^{q(\chi q)})(\phi q,\theta q)) \\
= \ & ((pq,(\chi p)^q\chi q),((\phi p)^q\phi q,((\phi p)^q \boxtimes \chi q)^{\phi q}(\theta p)^{[q(\chi q)(\phi q)]}(\theta q)))
\end{aligned}$$

9.7 Sections of a cat^2-group

Recall from Section 8.3 that a cat^2-group $\mathcal{C}$ comprises 4 groups and 15 homomorphisms, as shown in the following diagram,

$$\mathcal{C} \quad = \quad \begin{tikzcd}[row sep=huge, column sep=huge]
C_{[2]} \arrow[r, shift left=2] \arrow[r, shift left=1, "{\ddot t_1,\ddot h_1}"] \arrow[d, shift right=2] \arrow[d, shift right=1, "{\ddot t_2,\ddot h_2}"'] \arrow[dr, shift right=1] \arrow[dr, "{t_{[2]},h_{[2]}}"'] & C_{\{2\}} \arrow[l, shift left=1, "{\ddot e_1,\ddot\xi_1}"] \arrow[d, shift left=1] \arrow[d, shift left=2, "{\dot t_2,\dot h_2}"] \\
C_{\{1\}} \arrow[u, shift right=1, "{\ddot e_2,\ddot\xi_2}"'] \arrow[r, shift right=1] \arrow[r, shift right=2, "{\dot t_1,\dot h_1}"'] & C_\emptyset \arrow[l, shift right=1, "{\dot e_1,\dot\xi_1}"'] \arrow[u, shift left=1, "{\dot e_2,\dot\xi_2}"] \arrow[ul, shift right=1, "{e_{[2]},\xi_{[2]}}"']
\end{tikzcd}$$

where the four sides of the square are all cat^1-groups, and

$$\left\{ \begin{array}{l}
\dot t_1\ddot h_2 = \dot h_2\ddot t_1, \quad \dot t_2\ddot h_1 = \dot h_1\ddot t_2, \quad \dot e_1\dot t_2 = \ddot t_2\ddot e_1, \quad \dot e_2\dot t_1 = \ddot t_1\ddot e_2, \quad \dot e_1\dot h_2 = \ddot h_2\ddot e_1, \quad \dot e_2\dot h_1 = \ddot h_1\ddot e_2, \\
\dot t_1\ddot t_2 = \dot t_2\ddot t_1 = t_{[2]}, \quad \dot h_1\ddot h_2 = \dot h_2\ddot h_1 = h_{[2]}, \quad \dot e_1\ddot e_2 = \dot e_2\ddot e_1 = e_{[2]},
\end{array} \right. \tag{71}$$

while the diagonal is only a pre-cat^1-group.

We name the four cat^1-groups as $\ddot{\mathcal{C}}_1$, $\dot{\mathcal{C}}_1$, $\ddot{\mathcal{C}}_2$, $\dot{\mathcal{C}}_2$ and the diagonal as $\mathcal{C}_{[2]}$, so that $t_1 = (\ddot t_1, \dot t_1)$, $h_1 = (\ddot h_1, \dot h_1) : \ddot{\mathcal{C}}_1 \to \dot{\mathcal{C}}_1$ and $e_1 = (\ddot e_1, \dot e_1) : \dot{\mathcal{C}}_1 \to \ddot{\mathcal{C}}_1$ are horizontal morphisms of cat^1-groups, while $t_2 = (\ddot t_2, \dot t_2)$, $h_2 = (\ddot h_2, \dot h_2) : \ddot{\mathcal{C}}_2 \to \dot{\mathcal{C}}_2$ and $e_2 = (\ddot e_2, \dot e_2) : \dot{\mathcal{C}}_2 \to \ddot{\mathcal{C}}_2$ are vertical morphisms. Pairs such as $(\ddot t_1, \mathrm{id}_{C_\emptyset})$ are pre-cat^1-morphisms.

Lemma 9.19 *For $\ddot\xi_1$ a section of $\ddot{\mathcal{C}}_1$, the map $\dot\xi_1 = \ddot t_2\ddot\xi_1\dot e_2$ is a section of $\dot{\mathcal{C}}_1$.*

Proof:

$$\dot t_1\dot\xi_1 \;=\; \dot t_1\ddot t_2\ddot\xi_1\dot e_2 \;=\; \dot t_2\ddot t_1\ddot\xi_1\dot e_2 \;=\; \dot t_2\dot e_2 \;=\; \mathrm{id}_{C_\emptyset}.$$

□

We now consider $\mathcal{C}_1 = (e_1; t_1, h_1 : \ddot{\mathcal{C}}_1 \to \dot{\mathcal{C}}_1)$ as a *cat*^{1}*-group of cat*^{1}*-groups* and define the notion of section in this situation.

Definition 9.20 *A section of* $\mathcal{C}_1$ *is a pair of maps* $\xi_1 = (\ddot\xi_1, \dot\xi_1)$ *where* $\ddot\xi_1$ *is a section of* $\ddot{\mathcal{C}}_1$ *and*

$$\ddot t_2\ddot\xi_1 = \dot\xi_1\dot t_2, \quad \ddot h_2\ddot\xi_1 = \dot\xi_1\dot h_2 \quad \textit{and} \quad \ddot e_2\dot\xi_1 = \ddot\xi_1\dot e_2. \tag{72}$$

By Lemma 9.19, ξ_1 a morphism of cat^1-groups. Since both $\ddot\xi_1$ and $\dot\xi_1$ are sections, the usual formula **S1:** $t_1\xi_1 = \mathrm{id}_{\dot{\mathcal{C}}_2}$ holds.

Recall that the Whitehead multiplication for sections of a cat^1-group applied to two sections of $\ddot{\mathcal{C}}_1$ gives

$$(\ddot\xi_1 \star \ddot\zeta_1)k \;=\; (\ddot\xi_1 k)(\ddot e_1\ddot h_1\ddot\xi_1 k^{-1})(\ddot\zeta_1\ddot h_1\ddot\xi_1 k) \;=\; (\ddot\zeta_1\ddot h_1\ddot\xi_1 k)(\ddot e_1\ddot h_1\ddot\xi_1 k^{-1})(\ddot\xi_1 k)\ .$$

Lemma 9.21 *The section of* $\dot{\mathcal{C}}_1$ *associated to* $\ddot\xi_1 \star \ddot\zeta_1$ *is* $\dot\xi_1 \star \dot\zeta_1$.

Proof: The associated section is $\ddot t_2(\ddot\xi_1 \star \ddot\zeta_1)\dot e_2$ which maps $p \in C_\emptyset$ to

$$\begin{aligned}
(\ddot t_2\ddot\xi_1\dot e_2 p)\ (\ddot t_2\ddot e_1\ddot h_1\ddot\xi_1\dot e_2 p^{-1})\ (\ddot t_2\ddot\zeta_1\ddot h_1\ddot\xi_1\dot e_2 p) \;&=\; (\dot\xi_1\dot t_2\dot e_2 p)\ (\dot e_1\dot h_1\dot\xi_1\dot t_2\dot e_2 p^{-1})\ (\dot\zeta_1\dot h_1\dot\xi_1\dot t_2\dot e_2 p) \\
&=\; (\dot\xi_1 p)\ (\dot e_1\dot h_1\dot\xi_1 p^{-1})\ (\dot\zeta_1\dot h_1\dot\xi_1 p) \;=\; (\dot\xi_1 \star \dot\zeta_1)p\ ,
\end{aligned}$$

where identities (71) and (72) are frequently used. □

Recall that a section ξ of a cat^1-group $\mathcal{C} = (e; t, h : C \to R)$ determines a cat^1-group endomorphism (γ, ρ) of $\mathcal{C}$ where

$$\begin{aligned}
&\rho : R \to R, && r \mapsto h\xi r, \\
&\gamma : C \to C, && g \mapsto (eh\xi tg)(\xi tg^{-1})g(ehg^{-1})(\xi hg).
\end{aligned}$$

The equivalent definitions for our $\dot{\mathcal{C}}_1$ are

$$\begin{aligned}
&\rho_1 = (\ddot\rho_1, \dot\rho_1) : \dot{\mathcal{C}}_2 \to \dot{\mathcal{C}}_2, && k \mapsto \ddot h_1\ddot\xi_1 k, \quad p \mapsto \dot h_1\dot\xi_1 p, \\
&\gamma_1 = (\ddot\gamma_1, \dot\gamma_1) : \ddot{\mathcal{C}}_2 \to \ddot{\mathcal{C}}_2, && j \mapsto (\dot e_1\dot h_1\dot\xi_1\dot t_1 j)(\dot\xi_1\dot t_1 j^{-1})j(\dot e_1\dot h_1 j^{-1})(\dot\xi_1\dot h_1 j), \\
& && g \mapsto (\ddot e_1\ddot h_1\ddot\xi_1\ddot t_1 g)(\ddot\xi_1\ddot t_1 g^{-1})g(\ddot e_1\ddot h_1 g^{-1})(\ddot\xi_1\ddot h_1 g).
\end{aligned}$$

It is easy to check that ρ_1 and γ_1 are endomorphisms of cat^1-groups. For example,

$$\dot t_2\ddot\rho_1 k \;=\; \dot t_2\ddot h_1\ddot\xi_1 k \;=\; \dot h_1\ddot t_2\ddot\xi_1 k \;=\; \dot h_1\dot\xi_1\dot t_2 k \;=\; \dot\rho_1\dot t_2 k\ .$$

Of course we may also consider $\mathcal{C}_2 = (e_2; t_2, h_2 : \ddot{\mathcal{C}}_2 \to \dot{\mathcal{C}}_2)$ as a second cat^1-group of cat^1-groups, and define sections $\xi_2 = (\ddot\xi_2, \dot\xi_2) : \dot{\mathcal{C}}_2 \to \ddot{\mathcal{C}}_2$.

9.8 More on sections of a cat^2-group

A first attempt at defining a section of $\mathcal{C}$ might be as a homomorphism $\xi_{[2]} : C_\emptyset \to C_{[2]}$ which composes with the tail maps in various ways to give identities

$$t_{[2]}\xi_{[2]} \;=\; \mathrm{id}_{C_\emptyset}, \qquad \ddot{t}_1\xi_{[2]}\dot{t}_2 \;=\; \mathrm{id}_{C_{\{2\}}}, \qquad \ddot{t}_2\xi_{[2]}\dot{t}_1 \;=\; \mathrm{id}_{C_{\{1\}}}.$$

This will not do because, when $\dot{t}_1 = 0$ (say) the last identity puts to many restrictions of $\xi_{[2]}$.
Since $t_{[2]} = \dot{t}_1\ddot{t}_2 = \dot{t}_2\ddot{t}_1$, the first identity gives

$$\dot{t}_1(\ddot{t}_2\xi_{[2]}) \;=\; \mathrm{id}_{C_\emptyset} \quad \text{and} \quad \dot{t}_2(\ddot{t}_1\xi_{[2]}) \;=\; \mathrm{id}_{C_\emptyset}$$

so that $\ddot{t}_2\xi_{[2]}$ is a section of $\dot{\mathcal{C}}_1$ and $\ddot{t}_1\xi_{[2]}$ is a section of $\dot{\mathcal{C}}_2$.

An alternative approach is to consider commuting pairs of sections.

Definition 9.22 *A* cat^2-group section *of $\mathcal{C}$ is a pair $\xi = (\xi_1, \xi_2)$ where $\xi_1 = (\ddot{\xi}_1, \dot{\xi}_1)$ is a section of $\mathcal{C}_1$, and $\xi_2 = (\ddot{\xi}_2, \dot{\xi}_2)$ is a section of $\mathcal{C}_2$, such that*

$$\ddot{\xi}_2\dot{\xi}_1 \;=\; \ddot{\xi}_1\dot{\xi}_2 \;:\; C_\emptyset \to C_{[2]}.$$

We denote the common composite by $\xi_{[2]} : C_\emptyset \to C_{[2]}$ and, since

$$t_{[2]}\xi_{[2]} \;=\; \dot{t}_1\ddot{t}_2\ddot{\xi}_1\dot{\xi}_2 \;=\; \dot{t}_1\dot{\xi}_1\dot{t}_2\dot{\xi}_2 \;=\; 1,$$

the map $\xi_{[2]}$ is a section of $\mathcal{C}_{[2]}$.

We expect Whitehead multiplication of cat^2-sections to be defined by

$$\xi \star \zeta \;=\; (\ddot{\xi}_1 \star \ddot{\zeta}_1)(\dot{\xi}_2 \star \dot{\zeta}_2) \;:\; C_\emptyset \to C_{[2]},$$

but for this to make sense we must first show that a product of commuting sections commutes.

Lemma 9.23

$$(\ddot{\xi}_1 \star \ddot{\zeta}_1)(\dot{\xi}_2 \star \dot{\zeta}_2) \;=\; (\ddot{\xi}_2 \star \ddot{\zeta}_2)(\dot{\xi}_1 \star \dot{\zeta}_1).$$

Proof:

$$\begin{array}{cl}
 & (\ddot{\xi}_1 \star \ddot{\zeta}_1)(\dot{\xi}_2 \star \dot{\zeta}_2)p \\
= & (\ddot{\xi}_1\dot{\xi}_2p)(\ddot{\xi}_1\dot{e}_2\dot{h}_2\dot{\xi}_2p^{-1})(\ddot{\xi}_1\dot{\zeta}_2\dot{h}_2\dot{\xi}_2p)(\ddot{e}_1\ddot{h}_1\ddot{\xi}_1\dot{\zeta}_2\dot{h}_2\dot{\xi}_2p^{-1})(\ddot{e}_1\ddot{h}_1\ddot{\xi}_1\dot{e}_2\dot{h}_2\dot{\xi}_2p)(\ddot{e}_1\ddot{h}_1\ddot{\xi}_1\dot{\xi}_2p^{-1}) \\
 & \quad (\ddot{\zeta}_1\ddot{h}_1\ddot{\xi}_1\dot{\xi}_2p)(\ddot{\zeta}_1\ddot{h}_1\ddot{\xi}_1\dot{e}_2\dot{h}_2\dot{\xi}_2p^{-1})(\ddot{\zeta}_1\ddot{h}_1\ddot{\xi}_1\dot{\zeta}_2\dot{h}_2\dot{\xi}_2p) \\
= & (\ddot{\xi}_2\dot{\xi}_1p)(\ddot{e}_2\ddot{h}_2\ddot{\xi}_2\dot{\xi}_1p^{-1})(\ddot{\xi}_1\dot{\zeta}_2\dot{h}_2\dot{\xi}_2p)(\ddot{e}_1\ddot{h}_1\ddot{\xi}_1\dot{\zeta}_2\dot{h}_2\dot{\xi}_2p^{-1})(\ddot{e}_1\ddot{h}_1\ddot{e}_2\ddot{h}_2\ddot{\xi}_2\dot{\xi}_1p)(\ddot{e}_1\ddot{h}_1\ddot{\xi}_2\dot{\xi}_1p^{-1}) \\
 & \quad (\ddot{\zeta}_1\ddot{h}_1\ddot{\xi}_1\dot{\xi}_2p)(\ddot{\zeta}_1\ddot{h}_1\ddot{e}_2\ddot{h}_2\ddot{\xi}_2\dot{\xi}_1p^{-1})(\ddot{\zeta}_1\ddot{h}_1\ddot{\xi}_1\dot{\zeta}_2\dot{h}_2\dot{\xi}_2p) \\
= & ?????
\end{array}$$

This seems to be going nowhere fast!
Perhaps the group-groupoid or double groupoid approaches are better (see Subsection 2.3 of the Notes) – and the next Section. □

What needs doing next?

- Prove that the definition of Whitehead multiplication of sections makes sense.
- Investigate the corresponding (γ, ρ) pairs.
- **Eventually:** convert to crossed squares and see what the corresponding formulae are for 2-fold derivations.

10 2-crossed modules

The reader is refered to Brown–Gilbert [9], Conduché [27],[28], and Mutlu–Porter [48] for background information on 2-crossed modules.

Definition 10.1 *A* 2*-crossed module is comprised of the following:*

- *a* 2*-complex of groups*

$$\mathcal{Z} \quad = \quad (\, T \xrightarrow{\delta_2} U \xrightarrow{\delta_1} V \,)$$

  (so that $\delta_1 \circ \delta_2 = 0 : T \to V$*);*

- *an action of* V *on* T *and on* U*, and on itself by conjugation, such that* δ_1, δ_2 *are morphisms which preserve the actions;*

- *a function, called the* Peiffer lifting*,*

$$\{\ ,\ \} \ : \ U \times U \to T,$$

  making $(\delta_2 : T \to U)$ *a crossed module with action*

$$t^u \ := \ t\,\{\delta_2 t, u\}\ . \tag{73}$$

The following axioms are also required:

(2X1) $\delta_2\{u_1, u_2\} = \langle u_1, u_2\rangle = u_1^{-1}u_2^{-1}u_1 {u_2}^{\delta_1 u_1}$ *(a Peiffer commutator),*

(2X2) $\{u, \delta_2 t\} = (t^{-1})^u \, t^{\delta_1 u}$,

(2X3) $\{u_1 u_2, u_3\} = \{u_1, u_3\}^{u_2}\{u_2, {u_3}^{\delta_1 u_1}\}$,

(2X4) $\{u_1, u_2 u_3\} = \{u_1, u_3\}\{u_1, u_2\}^{{u_3}^{\delta_1 u_1}}$,

(2X5) $\{u_1, u_2\}^v = \{{u_1}^v, {u_2}^v\}$.

An additional axiom, $\{\delta_2 t, u\} = t^{-1}t^u$ id often specified, but we have used this identity in (73) to define the action. Note that δ_2 maps (2X3), (2X4) and (2X5) to identities (b), (c) and (d) in Lemma 1.10 for Peiffer commutators. Compare also (2X3), (2X4) with identities (a), (b) for crossed pairings in Definition 7.4.

We will check that the crossed module action $t^u = t\,\{\delta_2 t, u\}$ given by formula (73) is well defined:

$$\begin{aligned}
(t_1 t_2)^u &= t_1 t_2\{(\delta_2 t_1)(\delta_2 t_2), u\} \\
&= t_1 t_2\{\delta_2 t_1, u\}^{\delta_2 t_2}\{\delta_2 t_2, u^{\delta_1 \delta_2 t_1}\} \\
&= t_1 t_2 t_2^{-1}\{\delta_2 t_1, u\} t_2\{\delta_2 t_2, u\} \\
&= {t_1}^u \, {t_2}^u, \\
t^{(u_1 u_2)} &= t\{\delta_2 t, u_1 u_2\} \\
&= t\{\delta_2 t, u_2\}\{\delta_2 t, u_1\}^{{u_2}^{\delta_1 \delta_2 t}} \\
&= t^{u_2}\{\delta_2 t, u_1\}^{u_2} \\
&= (t^{u_1})^{u_2}.
\end{aligned}$$

Lemma 10.2 (a) $\{\delta_2 t_1, \delta_2 t_2\} = [t_1, t_2]$.

Proof:

(a) $\{\delta_2 t_1, \delta_2 t_2\} = t_1^{-1}\, {t_1}^{\delta_2 t_2} = [t_1, t_2]$, by definition of the crossed module action.

$\square$

10.1 Morphisms and Homotopies of 2-crossed modules

Definition 10.3 *A morphism of 2-crossed modules is a triple of group homomorphisms*

$$f_\bullet \;\equiv\; (f_2, f_1, f_0) \;:\; \mathcal{Z} \to \mathcal{Z}'$$

such that

$$f_1\delta_2 = \delta_2' f_2, \quad f_0\delta_1 = \delta_1' f_1, \quad f_2(t^v) = (f_2 t)^{f_0 v}, \quad f_1(u^v) = (f_1 u)^{f_0 v}, \quad f_2\{u_1, u_2\} = \{f_1 u_1, f_1 u_2\}.$$

$$\begin{tikzcd}
\mathcal{Z} & : & T \arrow[r, "\delta_2"] \arrow[dd, "f_2"'] & U \arrow[r, "\delta_1"] \arrow[dd, "f_1"'] & V \arrow[dd, "f_0"'] \\
f_\bullet & : & & & \\
\mathcal{Z}' & : & T' \arrow[r, "\delta_2'"] & U' \arrow[r, "\delta_1'"] & V'
\end{tikzcd}$$

An automorphism of $\mathcal{Z}$ is an endomorphism with inverse ${f_\bullet}^{-1} = (f_2^{-1}, f_1^{-1}, f_0^{-1})$.

Note that

$$f_2(t^u) \;=\; (f_2 t)\{f_1\delta_2 t, f_1 u\} \;=\; (f_2 t)\{\delta_2' f_2 t, f_1 u\} \;=\; (f_2 t)^{f_1 u}$$

so $(f_2, f_1) : (\delta_2 : T \to U) \to (\delta_2' : T' \to U')$ is a morphism of crossed modules.

Definition 10.4 *A homotopy of the 2-crossed module $\mathcal{Z}$ is a pair of homomorphisms $\phi_\bullet = (\phi_1, \phi_0)$ such that*

$$\delta_2\phi_1 \;=\; \phi_0\delta_1 \qquad \textbf{[Is that all ???]}$$

$$\begin{tikzcd}
T \arrow[r, "\delta_2"] \arrow[d, "f_2"'] & U \arrow[r, "\delta_1"] \arrow[d, "f_1"'] & V \arrow[d, "f_0"] \\
T \arrow[r, "\delta_2"] & U \arrow[r, "\delta_1"] \arrow[ul, "\phi_1"] & V \arrow[ul, "\phi_0"]
\end{tikzcd}$$

10.2 2-crossed modules of groupoids

[To be added.]

[The rest of this section really belongs in a crossed squares chapter.]

10.3 The 2-crossed module associated to a crossed square

We follow Brown–Gilbert [9] in defining the appropriate Peiffer lifting to be $\{(m_1,n_1),(m_2,n_2)\} = {m_2}^{m_1} \boxtimes n_1$.

Proposition 10.5 *Given a crossed square $\mathcal{R}$ there is an associated 2-crossed module $\mathcal{Z}$, as shown in the following diagram:*

$$\mathcal{R} \;:\; \begin{tikzcd} R_{[2]} \arrow[r, "\ddot{\partial}_1"] \arrow[d, "\ddot{\partial}_2"'] & R_{\{2\}} \arrow[d, "\dot{\partial}_2"] \\ R_{\{1\}} \arrow[r, "\dot{\partial}_1"'] & R_{\emptyset} \end{tikzcd} \qquad\qquad \mathcal{Z} \;:\; \left(\begin{tikzcd} R_{[2]} \arrow[r, "\delta_2"] & R_{\{1\}} \ltimes R_{\{2\}} \arrow[r, "\delta_1"] & R_{\emptyset} \end{tikzcd} \right)$$

where

$$\delta_2\ell = (\ddot{\partial}_2\ell, \ddot{\partial}_1\ell^{-1}) \qquad \textit{and} \qquad \delta_1(m,n) = (\dot{\partial}_1 m)(\dot{\partial}_2 n)\ ,$$

$R_{\emptyset}$ acts diagonally on $R_{\{1\}} \ltimes R_{\{2\}}$, and the Peiffer lifting is given by

$$\{(m_1,n_1),(m_2,n_2)\} \;=\; {m_2}^{m_1} \boxtimes n_1\ .$$

Proof: We first check that δ_1 and δ_2 are homomorphisms preserving the $R_{\emptyset}$-actions, and that $\delta_1\delta_2 = 0$:

$$\begin{aligned}
\delta_1((m_1,n_1),(m_2,n_2)) &= \delta_1(m_1m_2, {n_1}^{m_2}n_2) \;=\; \dot{\partial}_1(m_1m_2)\,\dot{\partial}_2({n_1}^{\dot{\partial}_1 m_2}n_2) \\
&= (\dot{\partial}_1 m_1)(\dot{\partial}_1 m_2)(\dot{\partial}_2 n_1)^{\dot{\partial}_1 m_2}(\dot{\partial}_2 n_2) \;=\; \delta_1(m_1,n_1)\,\delta_1(m_2,n_2). \\
\delta_1((m,n)^p) &= \delta_1(m^p,n^p) \;=\; \dot{\partial}_1(m^p)\,\dot{\partial}_2(n^p) \;=\; (\dot{\partial}_1 m)^p\,(\dot{\partial}_2 m)^p \;=\; (\delta_1(m,n))^p\ . \\
(\delta_2\ell_1)(\delta_2\ell_2) &= (\ddot{\partial}_2\ell_1, \ddot{\partial}_1\ell_1^{-1})(\ddot{\partial}_2\ell_2, \ddot{\partial}_1\ell_2^{-1}) \\
&= ((\ddot{\partial}_2\ell_1)(\ddot{\partial}_2\ell_2), (\ddot{\partial}_1\ell_1^{-1})^{\dot{\partial}_1\ddot{\partial}_2\ell_2}(\ddot{\partial}_1\ell_2^{-1})) \\
&= (\ddot{\partial}_2(\ell_1\ell_2), (\ddot{\partial}_1\ell_1^{-1})^{\dot{\partial}_2\ddot{\partial}_1\ell_2}(\ddot{\partial}_1\ell_2^{-1})) \\
&= (\ddot{\partial}_2(\ell_1\ell_2), (\ddot{\partial}_1\ell_2^{-1})(\ddot{\partial}_1\ell_1^{-1})) && \text{by } \textbf{X2:} \text{ for } \dot{\mathcal{R}}_2 \\
&= \delta_2(\ell_1\ell_2). \\
\delta_2(\ell^p) &= (\ddot{\partial}_2(\ell^p), \ddot{\partial}_1((\ell^p)^{-1})) \;=\; ((\ddot{\partial}_2\ell)^p, (\ddot{\partial}_1\ell^{-1})^p) \;=\; (\ddot{\partial}_2\ell, \ddot{\partial}_1\ell^{-1})^p \;=\; (\delta_2\ell)^p\ . \\
\delta_1\delta_2\ell &= \delta_1(\ddot{\partial}_2\ell, \ddot{\partial}_1\ell^{-1}) \;=\; (\dot{\partial}_1\ddot{\partial}_2\ell)(\dot{\partial}_2\ddot{\partial}_1\ell^{-1}) \;=\; 1\ .
\end{aligned}$$

Secondly we identify the crossed module action in $\mathcal{Z}$ in this case to be $\ell^{(m,n)} = \ell^m$.

$$\begin{aligned}
\ell^{(m,n)} &= \ell\,\{(\ddot{\partial}_2\ell, \ddot{\partial}_1\ell^{-1}),(m,n)\} \\
&= \ell\,(m^{\ddot{\partial}_2\ell} \boxtimes \ddot{\partial}_1\ell^{-1}) \\
&= \ell\,\ell^{(\ddot{\partial}_2\ell^{-1})m(\ddot{\partial}_2\ell)}\,\ell^{-1} && \text{by Definition 8.1 (e)} \\
&= \ell\,(\ell^m)^{(\ddot{\partial}_2\ell)}\,\ell^{-1} \;=\; \ell^m && \text{by } \textbf{X2:} \text{ (twice).}
\end{aligned}$$

It is clear that this *is* an action, so we verify the two crossed module axioms:

$$\begin{array}{rrcl}
\mathbf{X1}: & (\delta_2\ell)^{(m,n)} & = & (m^{-1},(n^{-1})^{m^{-1}})(\ddot{\partial}_2\ell,\ddot{\partial}_1\ell^{-1})(m,n) \\
 & & = & (m^{-1}(\ddot{\partial}_2\ell)m,\ (n^{-1})^{\dot{\partial}_1\ddot{\partial}_2\ell^m}(\ddot{\partial}_1\ell^{-1})^m n) \\
 & & = & ((\ddot{\partial}_2\ell)^m,\ (n^{-1})^{\dot{\partial}_2(\ddot{\partial}_1\ell)^m}(\ddot{\partial}_1\ell^m)^{-1}n) \\
 & & = & ((\ddot{\partial}_2\ell)^m,\ (\ddot{\partial}_1\ell^m)^{-1}n^{-1}n) \\
 & & = & \delta_2(\ell^m) \ = \ \delta_2(\ell^{(m,n)})\,, \\
\mathbf{X2}: & {\ell_0}^{\delta_2\ell} & = & {\ell_0}^{(\ddot{\partial}_2\ell,\ddot{\partial}_1\ell^{-1})} \ = \ {\ell_0}^{\ddot{\partial}_2\ell} \ = \ {\ell_0}^{\ell}\,.
\end{array}$$

Thirdly, we verify the five axioms.

2X1:

$$\begin{array}{rl}
 & (m_1,n_1)^{-1}\,(m_2,n_2)^{-1}\,(m_1,n_1)\,(m_2,n_2)^{\delta_1(m_1,n_1)} \\
= & ({m_1}^{-1},({n_1}^{-1})^{{m_1}^{-1}})\,({m_2}^{-1},({n_2}^{-1})^{{m_2}^{-1}})\,(m_1,n_1)\,(m_2,n_2)^{(\dot{\partial}_1 m_1)(\dot{\partial}_2 n_1)} \\
= & (m_1^{-1}m_2^{-1}m_1{m_2}^{m_1n_1},(n_1^{-1})^{m_1^{-1}m_2^{-1}m_1{m_2}^{m_1n_1}}(n_2^{-1})^{m_2^{-1}m_1{m_2}^{m_1n_1}}{n_1}^{{m_2}^{m_1n_1}}{n_2}^{m_1n_1})
\end{array}$$

The left hand element is

$$({m_2}^{m_1})^{-1}\,({m_2}^{m_1})^{n_1} \ = \ \ddot{\partial}_2({m_2}^{m_1}\boxtimes n_1)$$

It follows that the right hand element is

$$\begin{array}{rll}
 & (n_1^{-1}(n_2^{-1})^{m_1})^{\ddot{\partial}_2({m_2}^{m_1}\boxtimes n_1)}\,{n_1}^{{m_2}^{m_1n_1}}\,{n_2}^{m_1n_1} & \\
= & (n_1^{-1}({n_2}^{m_1})^{-1})^{(n_1^{-1})^{{m_2}^{m_1}}n_1}\,({n_1}^{{m_2}^{m_1}})^{n_1}\,({n_2}^{m_1})^{n_1} & \text{by Lemmas 2.7(c) and 2.2(d),} \\
= & (n_1^{-1})({n_1}^{{m_2}^{m_1}})(n_1^{-1})({n_2}^{m_1})^{-1}({n_1}^{{m_2}^{m_1}})^{-1}(n_1)\,(n_1^{-1})({n_1}^{{m_2}^{m_1}})(n_1)\,(n_1^{-1})({n_2}^{m_1})(n_1) & \\
= & \ddot{\partial}_1({m_2}^{m_1}\boxtimes n_1)^{-1} &
\end{array}$$

So the pair of elements is $\delta_2({m_2}^{m_1}\boxtimes n_1) \ = \ \delta_2\{(m_1,n_1),(m_2,n_2)\}$.

2X2:

$$\{(m,n),\delta_2\ell\} \ = \ \{(m,n),(\ddot{\partial}_2\ell,\ddot{\partial}_1\ell^{-1})\} \ = \ \ddot{\partial}_2(\ell^m)\boxtimes n \ = \ (\ell^m)^{-1}(\ell^m)^n \ = \ (\ell^{-1})^{(m,n)}\ell^{\delta_1(m,n)}\,.$$

2X3:

$$\begin{array}{rll}
 & \{(m_1,n_1),(m_3,n_3)\}^{(m_2,n_2)}\,\{(m_2,n_2),(m_3,n_3)^{m_1n_1}\} & \\
= & ({m_3}^{m_1}\boxtimes n_1)^{m_2}\,({m_3}^{m_1n_1m_2}\boxtimes n_2) & \\
= & (m_0\boxtimes n_0)\,({m_0}^{n_0}\boxtimes n_2) & \text{where } m_0 = {m_3}^{m_1m_2},\ n_0 = {n_1}^{m_2} \\
= & (m_0\boxtimes n_2)\,(m_0\boxtimes n_0)^{n_2} & \text{by Proposition 7.6(d)} \\
= & m_0\boxtimes n_0n_2 & \\
= & {m_3}^{m_1m_2}\boxtimes {n_1}^{m_2}n_2 & \\
= & \{(m_1m_2,{n_1}^{m_2}n_2),(m_3,n_3)\} & \\
= & \{(m_1,n_1)(m_2,n_2),(m_3,n_3)\} &
\end{array}$$

2X4:

$$\begin{array}{rll} & \{(m_1,n_1),(m_2m_3,{n_2}^{m_3}n_3)\} & \\ = & (m_2m_3)^{m_1}\boxtimes n_1 & \\ = & ({m_2}^{m_1}\boxtimes n_1)^{{m_3}^{m_1}}\,({m_3}^{m_1}\boxtimes n_1) & \\ = & (m_4\boxtimes n_1)^{n_1m_5}\,(m_5\boxtimes n_1) & \text{where } m_4={m_2}^{m_1n_1^{-1}},\ m_5={m_3}^{m_1} \\ = & (m_5\boxtimes n_1)(m_4\boxtimes n_1)^{m_5n_1} & \text{by Proposition 7.6(c)} \\ = & ({m_3}^{m_1}\boxtimes n_1)({m_2}^{m_1}\boxtimes n_1)^{n_1^{-1}{m_3}^{m_1}n_1} & \\ = & \{(m_1,n_1),(m_3,n_3)\}\,\{(m_1,n_1),(m_2,n_2)\}^{(m_3,n_3)^{\delta_1(m_1,n_1)}} & \end{array}$$

2X5:

$$\{({m_1}^p,{n_1}^p),({m_2}^p,{n_2}^p)\} \;=\; ({m_2}^p)^{{m_1}^p}\boxtimes {n_1}^p \;=\; ({m_2}^{m_1}\boxtimes n_1)^p \;=\; \{(m_1,n_1),(m_2,n_2)\}^p$$

$\square$

10.4 The crossed square associated to a 2-crossed module

Maybe there is no exact construction?

10.5 Homotopies of the actor 2-crossed module

[This needs significant revision.]

Definition 10.6 *A homotopy of the* 2*-crossed module* $\mathcal{Z}$ *is a pair of homomorphisms* $\phi_\bullet = (\phi_1, \phi_0)$ *such that* $\phi_0 : R_\emptyset \to R_{\{1\}} \ltimes R_{\{2\}},\ p \mapsto (\phi_0^N p, \phi_0^M p)$ *where*

$$\begin{array}{rcl} \phi_0(p_1p_2) & = & (\phi_0^N(p_1p_2), \phi_0^M(p_1p_2)) \\ & = & (\phi_0^N p_1, \phi_0^M p_1)(\phi_0^N p_2, \phi_0^M p_2) \\ & = & (\phi_0^N p_1 \phi_0^N p_2, (\phi_0^M p_1)^{\phi_0^N p_2}\phi_0^M p_2) \end{array}$$

So ϕ_0^M is a ϕ_0^N-derivation. $\phi_1 : R_{\{1\}} \ltimes R_{\{2\}} \to R_{[2]},\ (n,m) \mapsto (\phi_1^N n)(\phi_1^M m)$

$$\begin{array}{rcl} \phi_1(n,m) & = & \phi_1((n,1)(1,m)) \\ & = & \phi_1(n,1)\phi_1(1,m) \\ & = & (\phi_1^N n)(\phi_1^M m) \end{array}$$

[Moved what was Lemma 7.6 to cat2-group section.]

Lemma 10.7 *There is an action of* $R_\emptyset \ltimes R_{\{2\}}$ *on* $R_{\{1\}} \ltimes R_{[2]}$ *defined on the image of some derivation* χ *by*

$$\chi(p,m)^{(p_1,m_1)} = \chi(pp_1, m^{p_1}m_1)\chi(p_1,m_1)^{-1}$$

which is well defined **(should use the usual action!)**

Proof:

$$\begin{aligned}
\chi((p,m)^{(p_1,m_1)})^{(p_2,m_2)} &= (\chi(pp_1, m^{p_1}m_1)\chi(p_1,m_1)^{-1})^{(p_2,m_2)} \\
&= \chi(pp_1, m^{p_1}m_1)^{(p_2,m_2)}(\chi(p_1,m_1)^{(p_2,m_2)})^{-1} \\
&= \chi(pp_1p_2, m^{p_1p_2}m_1^{p_2}m_2)(\chi(p_2,m_2)^{-1}(\chi(p_1p_2, m_1^{p_2}m_2)\chi(p_2,m_2)^{-1})^{-1} \\
&= \chi(pp_1p_2, m^{p_1p_2}m_1^{p_2}m_2)\chi(p_1p_2, m_1^{p_2}m_2)^{-1} \\
\chi(p,m)^{(p_1,m_1)(p_2,m_2)} &= \chi(p,m)^{(p_1p_2, m_1^{p_2}m_2)} \\
&= \chi(pp_1p_2, m^{p_1p_2}m_1^{p_2}m_2)\chi(p_1p_2, m_1^{p_2}m_2)^{-1}
\end{aligned}$$

□

Lemma 10.8 *Consider a derivation* $\chi : R_\emptyset \ltimes R_{\{2\}} \to R_{\{1\}} \ltimes R_{[2]}$ *such that* $\chi(p,m) = (\chi^L(p,m), \chi^N(p,m))$. *Then the rules for* χ^L *and* χ^N *are as follows: ? ? ?*

Proof: we will show that above derivation is satisfies the derivation rule

$$\begin{aligned}
\chi((p_1,m_1)(p_2,m_2)) &= \chi(p_1p_2, m_1^{p_2}m_2) \\
\chi((p_1,m_1)(p_2,m_2)) &= \chi(p_1,m_1)^{(p_2,m_2)}\chi(p_2,m_2) \\
&= \chi(p_1p_2, m_1^{p_2}m_2)\chi(p_2,m_2)^{-1}\chi(p_2,m_2) \\
&= \chi(p_1p_2, m_1^{p_2}m_2)
\end{aligned}$$

□

Definition 10.9 *Now we can define a derivation which is depends on the derivation* χ^L

$$\begin{aligned}
\theta_p &= \chi^L(p,1) \\
\ddot{\chi}m &= \chi^L(1,m)
\end{aligned}$$

$$\begin{aligned}
\theta(p_1p_2) &= \chi^L(p_1p_2,1) \\
&= \chi^L((p_1,1),(p_1,1)) \\
&= (\chi^L(p_1,1))^{(p_2,1)}(\chi^L(p_2,1)) \\
&= (\theta p_1)^{p_2}(\theta p_2)
\end{aligned}$$

Lemma 10.10 $\chi^L : R_\emptyset \ltimes R_{\{2\}} \to R_{[2]}$ *is a derivation.*

Proof:

$$\begin{aligned}
\chi^L(p,m) &= \chi^L((p,1)(1,m)) \\
&= (\chi^L(p,1))^{(1,m)}(\chi^L(1,m)) \\
&= (\theta p)^m(\ddot{\chi}m)
\end{aligned}$$

□

11 Braidings

Brown and Gilbert, in [9], have shown that 2-crossed modules are equivalent to regular braided crossed modules. This construction has also been discussed in [17]. Our aim here is review this material, using right actions consistently.

11.1 Crossed module bimorphisms

Let $\mathcal{A} = ((\partial, \mathrm{id}) : \mathbb{A}_2 \to \mathbb{A}_1)$ be a crossed module of groupoids. A *bimorphism* $b : (\mathcal{A}, \mathcal{A}) \to \mathcal{A}$ consists of a family of maps

$$b_{ij} \ : \ A_i \times A_j \to A_{i+j}, \qquad 0 \leqslant i+j \leqslant 2,$$

satisfying the following axioms.

(a) The map b_{00} provides a monoid structure on A_0, with identity e, written $b_{00}(u,v) = uv$. As usual, A_0 acts on itself on the left and the right using this multiplication.

(b) The maps b_{10}, b_{20} give left actions of A_0 on A_1, A_2 respectively, while b_{01}, b_{02} provide right actions. These left actions commute with the right actions. We write these actions using '$\cdot$' to avoid confusion with the crossed module action of A_1 on A_2. So, for $u,v \in A_0$, $a \in A_1$ and $\ell \in A_2$, we have

$$(u \cdot a) \cdot v \ = \ u \cdot (a \cdot v), \qquad (u \cdot \ell) \cdot v \ = \ u \cdot (\ell \cdot v).$$

(c) These actions are compatible with the groupoid structure: for $p,q \in A_1$ or $p,q \in A_2$,

$$\begin{array}{rclrcl} s(u \cdot p) & = & u(sp), & t(u \cdot p) & = & u(tp), \\ s(p \cdot v) & = & (sp)v, & t(p \cdot v) & = & (tp)v, \\ u \cdot (pq) & = & (u \cdot p)(u \cdot q), & (pq) \cdot v & = & (p \cdot v)(q \cdot v), \quad \text{provided } tp = sq. \end{array}$$

Hence $u \cdot 1_v = 1_{uv} = 1_u \cdot v$ and $(u \cdot b)^{-1} = u \cdot b^{-1}$, $(b \cdot v)^{-1} = b^{-1} \cdot v$.

(d) The actions are compatible with the crossed module action: when $\ell \in A_2(x)$, $a \in A_1(x,y)$, and $\ell^a \in A_2(y)$,

$$u \cdot (\ell^a) \ = \ (u \cdot \ell)^{u \cdot a} \in A_2(uy), \qquad (\ell^a) \cdot v \ = \ (\ell \cdot v)^{a \cdot v} \in A_2(yv).$$

(e) The boundary morphism $\partial : A_2 \to A_1$ is equivariant with respect to these actions:

$$\partial(u \cdot \ell) \ = \ u \cdot (\partial \ell), \qquad \partial(\ell \cdot v) \ = \ (\partial \ell) \cdot v.$$

(f) We write the images of the map $b_{11} : A_1 \times A_1 \to A_2$ as a bracing:

$$b_{11}(a,b) \ = \ \{a,b\}, \quad \text{a loop at } (ta)(tb).$$

This bracing interacts with composition in A_1 according to:

$$\{1_e, b\} = 1_{tb}, \qquad \{a, 1_e\} = 1_{ta}, \qquad \{aa', b\} = \{a', b\}\{a,b\}^{a' \cdot tb}, \qquad \{a, bb'\} = \{a,b\}^{ta \cdot b'}\{a, b'\}.$$

(g) Given $a, b \in A_1$, we may act with the source or target of one on the other in four ways, forming $sa \cdot b,\ ta \cdot b,\ a \cdot sb,\ a \cdot tb$. These arrows fit together to form the square

$$\begin{tikzcd}
(sa)(sb) \arrow[r, "sa\cdot b"] \arrow[d, "a\cdot sb"'] & (sa)(tb) \arrow[d, "a\cdot tb"] \\
(ta)(sb) \arrow[r, "ta\cdot b"'] & (ta)(tb) \arrow[loop right, "\partial\{a,b\}"]
\end{tikzcd}$$

and the loop $\partial\{a, b\}$ measures the lack of commutativity:

$$\partial\{a, b\} \ = \ (ta \cdot b)^{-1}(a \cdot sb)^{-1}(sa \cdot b)(a \cdot tb).$$

(h) The bracing interacts with the boundary map as follows:

$$\{a, \partial\ell\} \ = \ (sa \cdot \ell\}^{a \cdot t\ell}\,(ta \cdot \ell), \qquad \{\partial\ell, b\} \ = \ (\ell^{-1} \cdot tb)\,(\ell \cdot sb)^{t\ell \cdot b}.$$

(i) The actions of A_0 on the bracing are given by:

$$u \cdot \{a, b\} \ = \ \{u \cdot a.b\}, \qquad \{a, b\} \cdot v \ = \ \{a, b \cdot v\}, \qquad \{a \cdot u, b\} \ = \ \{a, u \cdot b\}.$$

[Now do some checks?]

12 Double Categories and Double Groupoids

Our interest here is in double groupoids and their connection with crossed squares and cat^2-groups.

12.1 Double Categories

A *double category* $\mathcal{D}$ consists of four sets and four category structures, and satisfies an interchange law:

- $D_{[2]}$, a set of *squares*,
- $D_{\{2\}}$, a set of *vertical* or *up-down arrows*,
- $D_{\{1\}}$, a set of *horizontal* or *left-right arrows*,
- $D_{\emptyset}$, a set of *objects*,
- maps $\ddot{t}_1, \ddot{h}_1 : D_{[2]} \to D_{\{2\}}$ and $\ddot{e}_1 : D_{\{2\}} \to D_{[2]}$ and a composition $\ddot{*}_1$ giving a category structure $\ddot{\mathcal{D}}_1$ on squares displayed horizontally,
- maps $\ddot{t}_2, \ddot{h}_2 : D_{[2]} \to D_{\{1\}}$ and $\ddot{e}_2 : D_{\{1\}} \to D_{[2]}$ and a composition $\ddot{*}_2$ giving a category structure $\ddot{\mathcal{D}}_2$ on squares displayed vertically,
- maps $\dot{t}_1, \dot{h}_1 : D_{\{1\}} \to D_{\emptyset}$, and $\dot{e}_1 : D_{\emptyset} \to D_{\{1\}}$ and a composition $\dot{*}_1$ giving a category structure $\dot{\mathcal{D}}_1$ on horizontal arrows,
- maps $\dot{t}_2, \dot{h}_2 : D_{\{2\}} \to D_{\emptyset}$, and $\dot{e}_2 : D_{\emptyset} \to D_{\{2\}}$ and a composition $\dot{*}_2$ giving a category structure $\dot{\mathcal{D}}_2$ on vertical arrows,
- the tail and head maps commute as follows:

$$\dot{t}_2\ddot{t}_1 = \dot{t}_1\ddot{t}_2, \quad \dot{t}_2\ddot{h}_1 = \dot{h}_1\ddot{t}_2, \quad \dot{h}_2\ddot{t}_1 = \dot{t}_1\ddot{h}_2, \quad \dot{h}_2\ddot{h}_1 = \dot{h}_1\ddot{h}_2,$$

- for all squares A, B, C, D such that the compositions are defined,

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "A"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "B"] & \cdot \arrow[d] \\
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "C"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "D"] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

$$(A \ddot{*}_1 B) \ddot{*}_2 (C \ddot{*}_1 D) \; = \; (A \ddot{*}_2 C) \ddot{*}_1 (B \ddot{*}_2 D).$$

For a square A the four arrows and the four objects are displayed as follows.

$$\begin{tikzcd}
\dot{t}_1\ddot{t}_2A \arrow[r, "\ddot{t}_2A"] \arrow[d, "\ddot{t}_1A"'] \arrow[dr, phantom, "A"] & \dot{h}_1\ddot{t}_2A \arrow[d, "\ddot{h}_1A"] \\
\dot{t}_1\ddot{h}_2A \arrow[r, "\ddot{h}_2A"'] & \dot{h}_1\ddot{h}_2A
\end{tikzcd}$$

The structure of a double category is shown in the following diagram:

$$\mathcal{D} \quad = \quad \begin{tikzcd}[column sep=large, row sep=large]
D_{[2]} \arrow[r, shift left=3, "\ddot{t}_1{,}\ddot{h}_1"] \arrow[r, shift left=1.5] \arrow[d, shift right=3, "\ddot{t}_2{,}\ddot{h}_2"'] \arrow[d, shift right=1.5] & D_{\{2\}} \arrow[l, "\ddot{e}_1"] \arrow[d, shift left=1.5] \arrow[d, shift left=3, "\dot{t}_2{,}\dot{h}_2"] \\
D_{\{1\}} \arrow[u, "\ddot{e}_2"'] \arrow[r, shift right=1.5] \arrow[r, shift right=3, "\dot{t}_1{,}\dot{h}_1"'] & D_{\emptyset} \arrow[l, "\dot{e}_1"'] \arrow[u, "\dot{e}_2"]
\end{tikzcd} \tag{74}$$

Horizontal, vertical and double identity squares ($e_{[2]} = \ddot{e}_2\dot{e}_1 = \ddot{e}_1\dot{e}_2$) are represented by

$$\begin{tikzcd}
\cdot \arrow[r, "\dot{e}_1\dot{t}_1 a"] \arrow[d, "a"'] \arrow[dr, phantom, "\ddot{e}_1 a"] & \cdot \arrow[d, "a"] \\
\cdot \arrow[r, "\dot{e}_1\dot{h}_1 a"'] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "c"] \arrow[d, "\dot{e}_2\dot{t}_2 c"'] \arrow[dr, phantom, "\ddot{e}_2 c"] & \cdot \arrow[d, "\dot{e}_2\dot{h}_2 c"] \\
\cdot \arrow[r, "c"'] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
r \arrow[r, "\dot{e}_1 r"] \arrow[d, "\dot{e}_2 r"'] \arrow[dr, phantom, "e_{[2]} r"] & r \arrow[d, "\dot{e}_2 r"] \\
r \arrow[r, "\dot{e}_1 r"'] & r
\end{tikzcd}$$

Definition 12.1 *A morphism of double categories is a* double functor $\lambda : \mathcal{D} \to \mathcal{E}$*, given by a quadruple* $\lambda = (\lambda_{[2]}, \lambda_{\{2\}}, \lambda_{\{1\}}, \lambda_{\emptyset})$ *of maps*

$$\lambda_{[2]} : D_{[2]} \to E_{[2]}, \quad \lambda_{\{2\}} : D_{\{2\}} \to E_{\{2\}}, \quad \lambda_{\{1\}} : D_{\{1\}} \to E_{\{1\}}, \quad \lambda_{\emptyset} : D_{\emptyset} \to E_{\emptyset},$$

which combine to give functors from $\ddot{\mathcal{D}}_1, \ddot{\mathcal{D}}_2, \dot{\mathcal{D}}_1, \dot{\mathcal{D}}_2$ *to* $\ddot{\mathcal{E}}_1, \ddot{\mathcal{E}}_2, \dot{\mathcal{E}}_1, \dot{\mathcal{E}}_2$ *respectively, and which preserve all compositions.*

[Expand on this?]

12.2 Double Groupoids and Group – Double Groupoids

A *double groupoid* is a double category in which the four category structures are all groupoids. When the two sets of arrows coincide, $D_{\{1\}} = D_{\{2\}}$, we have an *arrow-symmetric double groupoid*. These are the subject of Chapter 6 of Brown and Sivera [20].

By analogy with the equivalence between cat^1-groups and group-groupoids, we attempt to describe an equivalence between cat^2-groups and group-double groupoids.

Definition 12.2 *A* group-double groupoid *is a double groupoid* $\mathcal{D}$ *where each of* $\mathcal{D}_{[2]}, \mathcal{D}_{\{2\}}, \mathcal{D}_{\{1\}}, \mathcal{D}_{\emptyset}$ *are groups and the twelve structure maps in (74) are group homomorphisms, so that the four category structures* $\ddot{\mathcal{D}}_1, \dot{\mathcal{D}}_1, \ddot{\mathcal{D}}_2, \dot{\mathcal{D}}_2$ *are all group-groupoids.*

[Is the diagonal a group-groupoid?]

12.3 Horizontal, Vertical and Double Sections

We saw in Subsections 1.13 and 2.3 that a section of a group-groupoid is a group monomorphism $\xi : G_0 \to G_1$ such that $t\xi = 1_{G_0}$. In order to generalise this to a horizontal section of a group-double groupoid $\mathcal{D}$, we require compatible monomorphisms from up-down arrows to squares and from points to left-right arrows. In order to see what 'compatible' means in this context we note that, in the following diagrams, the left-right boundaries of $\ddot\xi_1 g$ should be the images of the points of g under $\dot\xi_1$, while $\ddot\xi_1\dot e_2 r$ should be the vertical identity square for $\dot\xi_1 r$. We show up-down arrows as dashed in the diagrams in this Subsection.

$$\begin{tikzcd}
\dot t_2 g \arrow[r, "\dot\xi_1\dot t_2 g=\ddot t_2\ddot\xi_1 g"] \arrow[d, dashed, "g"'] \arrow[dr, phantom, "\ddot\xi_1 g"] & \dot t_2\ddot h_1\ddot\xi_1 g \arrow[d, dashed, "\ddot h_1\ddot\xi_1 g"] \\
\dot h_2 g \arrow[r, "\dot\xi_1\dot h_2 g=\ddot h_2\ddot\xi_1 g"'] & \dot h_2\ddot h_1\ddot\xi_1 g
\end{tikzcd}
\qquad
\begin{tikzcd}
r \arrow[r, "\dot\xi_1 r"] \arrow[d, dashed, "\dot e_2 r"'] \arrow[dr, phantom, "\ddot\xi_1\dot e_2 r = \ddot e_2\dot\xi_1 r"] & \dot h_1\dot\xi_1 r \arrow[d, dashed, "\ddot h_1\ddot\xi_1\dot e_2 r"] \\
r \arrow[r, "\dot\xi_1 r"'] & \dot h_1\dot\xi_1 r
\end{tikzcd}$$

Definition 12.3
(a) *A* horizontal section *of a group-double groupoid $\mathcal{D}$ is a pair $\xi_1 = (\ddot\xi_1, \dot\xi_1)$ of monomorphisms $\ddot\xi_1 : D_1 \to D_{[2]}$ and $\dot\xi_1 : D_\emptyset \to D_{\{2\}}$ such that*

$$\ddot t_1\ddot\xi_1 = 1_{D_{\{1\}}}, \quad \dot t_1\dot\xi_1 = 1_{D_\emptyset}, \quad \ddot t_2\ddot\xi_1 g = \dot\xi_1\dot t_2 g, \quad \ddot h_2\ddot\xi_1 g = \dot\xi_1\dot h_2 g, \quad \ddot\xi_1\dot e_2 r = \ddot e_2\dot\xi_1 r.$$

(b) *A* vertical section *of a group-double groupoid $\mathcal{D}$ is a pair $\xi_2 = (\ddot\xi_2, \dot\xi_2)$ of monomorphisms $\ddot\xi_2 : D_2 \to D_{[2]}$ and $\dot\xi_2 : D_\emptyset \to D_{\{1\}}$ such that*

$$\ddot t_2\ddot\xi_2 = 1_{D_{\{2\}}}, \quad \dot t_2\dot\xi_2 = 1_{D_\emptyset}, \quad \ddot t_1\ddot\xi_2 g = \dot\xi_2\dot t_1 g, \quad \ddot h_1\ddot\xi_2 g = \dot\xi_2\dot h_1 g, \quad \ddot\xi_2\dot e_1 r = \ddot e_1\dot\xi_2 r.$$

[Maybe we should swap 1 **and** 2 **in** ξ_1, ξ_2 **?]**

Given a section $\xi_1 = (\ddot\xi_1, \dot\xi_1)$ of $\ddot{\mathcal{D}}_1$ we may apply the construction in Subsection 2.3 to obtain a groupoid automorphism λ_1 of $\ddot{\mathcal{D}}_1$, which extends to a double groupoid automorphisms $\lambda_1 : \mathcal{D} \to \mathcal{D}$.

$$\begin{tikzcd}
\dot h_1\dot\xi_1\dot t_1\ddot t_2 A \arrow[r, "\dot\lambda_1\ddot t_2 A"] \arrow[d, dashed, "\ddot h_1\ddot\xi_1\ddot t_1 A"'] \arrow[dr, phantom, "\ddot\lambda_1 A"] & \dot h_1\dot\xi_1\dot h_1\ddot t_2 A \arrow[d, dashed, "\ddot h_1\ddot\xi_1\ddot h_1 A"] \\
\dot h_1\dot\xi_1\dot t_1\ddot h_2 A \arrow[r, "\dot\lambda_1\ddot h_2 A"'] & \dot h_1\dot\xi_1\dot h_1\ddot h_2 A
\end{tikzcd}
=
\begin{tikzcd}
\dot h_1\dot\xi_1\dot t_1\ddot t_2 A \arrow[d, dashed, "\ddot h_1\ddot\xi_1\ddot t_1 A"'] \arrow[dr, phantom, "\ddot\xi_1\ddot t_1 A"] & \dot t_1\ddot t_2 A \arrow[l, "\dot\xi_1\dot t_1\ddot t_2 A"'] \arrow[r, "\ddot t_2 A"] \arrow[d, dashed, "\ddot t_1 A"] \arrow[dr, phantom, "A"] & \dot h_1\ddot t_2 A \arrow[r, "\dot\xi_1\dot h_1\ddot t_2 A"] \arrow[d, dashed, "\ddot h_1 A"'] \arrow[dr, phantom, "\ddot\xi_1\ddot h_1 A"] & \dot h_1\dot\xi_1\dot h_1\ddot t_2 A \arrow[d, dashed, "\ddot h_1\ddot\xi_1\ddot h_1 A"] \\
\dot h_1\dot\xi_1\dot t_1\ddot h_2 A & \dot t_1\ddot h_2 A \arrow[l, "\dot\xi_1\dot t_1\ddot h_2 A"] \arrow[r, "\ddot h_2 A"'] & \dot h_1\ddot h_2 A \arrow[r, "\dot\xi_1\dot h_1\ddot h_2 A"'] & \dot h_1\dot\xi_1\dot h_1\ddot h_2 A
\end{tikzcd}$$

The defining equations for λ_1 are:

$$\begin{aligned}
(\lambda_1)_{[2]}\, A &= \widetilde{(\ddot\xi_1\ddot t_1 A)} \mathbin{\ddot *_1} A \mathbin{\ddot *_1} (\ddot\xi_1\ddot h_1 A)\,, \qquad (75)\\
(\lambda_1)_{\{2\}}\, a &= \ddot h_1\ddot\xi_1\, a\,,\\
(\lambda_1)_{\{1\}}\, c &= \widetilde{(\dot\xi_1\dot t_1 c)} \mathbin{\dot *_1} c \mathbin{\dot *_1} (\dot\xi_1\dot h_1 c),\\
(\lambda_1)_\emptyset\, p &= \dot h_1\dot\xi_1\, p\,.
\end{aligned}$$

Similarly, given a section $\xi_2 = (\ddot\xi_2, \dot\xi_2)$ of $\ddot{\mathcal{D}}_2$, applying the same construction we obtain

$$\begin{tikzcd}
\dot h_2\dot\xi_2\dot t_1\ddot t_2 A \arrow[r, "\ddot h_2\ddot\xi_2\ddot t_2 A"] \arrow[d, dashed, "\dot\lambda_2\ddot t_1 A"'] \arrow[dr, phantom, "\ddot\lambda_2 A"] & \dot h_2\dot\xi_2\dot h_1\ddot t_2 A \arrow[d, dashed, "\dot\lambda_2\ddot h_1 A"] \\
\dot h_2\dot\xi_2\dot t_1\ddot h_2 A \arrow[r, "\ddot h_2\ddot\xi_2\ddot h_2 A"'] & \dot h_2\dot\xi_2\dot h_1\ddot h_2 A
\end{tikzcd}
=
\begin{tikzcd}
\dot h_2\dot\xi_2\dot t_1\ddot t_2 A \arrow[r, "\ddot h_2\ddot\xi_2\ddot t_2 A"] & \dot h_2\dot\xi_2\dot h_1\ddot t_2 A \\
\dot t_1\ddot t_2 A \arrow[u, dashed, "\dot\xi_2\dot t_1\ddot t_2 A"] \arrow[r, "\ddot t_2 A"'] \arrow[d, dashed, "\ddot t_1 A"'] \arrow[ur, phantom, "\ddot\xi_2\ddot t_2 A"] \arrow[dr, phantom, "A"] & \dot h_1\ddot t_2 A \arrow[u, dashed, "\dot\xi_2\dot h_1\ddot t_2 A"'] \arrow[d, dashed, "\ddot h_1 A"] \\
\dot t_1\ddot h_2 A \arrow[r, "\ddot h_2 A"] \arrow[d, dashed, "\dot\xi_2\dot t_1\ddot h_2 A"'] \arrow[dr, phantom, "\ddot\xi_2\ddot h_2 A"] & \dot h_1\ddot h_2 A \arrow[d, dashed, "\dot\xi_2\dot h_1\ddot h_2 A"] \\
\dot h_2\dot\xi_2\dot t_1\ddot h_2 A \arrow[r, "\ddot h_2\ddot\xi_2\ddot h_2 A"'] & \dot h_2\dot\xi_2\dot h_1\ddot h_2 A
\end{tikzcd}$$

This determines a double groupoid automorphism λ_2 of $\mathcal{D}$ where

$$\begin{aligned}
(\lambda_2)_{[2]}\, A &= \widetilde{(\ddot\xi_2\ddot t_2 A)}\ \ddot{*}_2\ A\ \ddot{*}_2\ (\ddot\xi_2\ddot h_2 A)\,, \\
(\lambda_2)_{\{2\}}\, a &= \widetilde{(\dot\xi_2\dot t_2 a)}\ \dot{*}_2\ a\ \dot{*}_2\ (\dot\xi_2\dot h_2 a)\,, \\
(\lambda_2)_{\{1\}}\, c &= \ddot h_2\ddot\xi_2\, c\,, \\
(\lambda_2)_{\emptyset}\, p &= \dot h_2\dot\xi_2\, p\,.
\end{aligned} \tag{76}$$

Given a pair of sections $\xi = (\xi_1, \xi_2)$, one horizontal and one vertical, if we apply both constructions we obtain an automorphism $\lambda = \lambda_\xi$ of $\mathcal{D}$ provided $\lambda_1 A = \lambda_2 A$. We call ξ a *double section* of $\mathcal{D}$. The requirement $\lambda_1 A = \lambda_2 A$ implies four equations at the four levels of $\mathcal{D}$,

$$\begin{aligned}
&& \dot h_1\dot\xi_1\, p &= \dot h_2\dot\xi_2\, p\,, \\
&& \ddot h_1\ddot\xi_1\, a &= \widetilde{(\dot\xi_2\dot t_2 a)}\ \dot{*}_2\ a\ \dot{*}_2\ (\dot\xi_2\dot h_2 a) \ = \ \dot\lambda_2\, a\,, \\
\dot\lambda_1\, c &= \widetilde{(\dot\xi_1\dot t_1 c)}\ \dot{*}_1\ c\ \dot{*}_1\ (\dot\xi_1\dot h_1 c) &&= \ddot h_2\ddot\xi_2\, c\,, \\
\ddot\lambda_1\, A &= \widetilde{(\ddot\xi_1\ddot t_1 A)}\ \ddot{*}_1\ A\ \ddot{*}_1\ (\ddot\xi_1\ddot h_1 A) &&= \widetilde{(\ddot\xi_2\ddot t_2 A)}\ \ddot{*}_2\ A\ \ddot{*}_2\ (\ddot\xi_2\ddot h_2 A) \ = \ \ddot\lambda_2\, A\,.
\end{aligned}$$

The previous two diagrams may be combined into the cube-like figure of equation (77), where the six square faces are denoted F=front, L=left, R=right, U=up, D=down and B=back, and

$$F = A, \quad L = \ddot\xi_1\ddot t_1 A, \quad R = \ddot\xi_1\ddot h_1 A, \quad U = \ddot\xi_2\ddot t_2 A, \quad D = \ddot\xi_2\ddot h_2 A, \quad B = \lambda A.$$

$$\begin{tikzcd}
 & \dot h_1\dot\xi_1\dot t_1\ddot t_2A \arrow[rr, "\ddot h_2\ddot\xi_2\ddot t_2A\,=\,\dot\lambda_1\ddot t_2A"] \arrow[dd, dashed, "\ddot h_1\ddot\xi_1\ddot t_1A \\ = \\ \dot\lambda_2\ddot t_1A" {align=left}] & & \dot h_1\dot\xi_1\dot h_1\ddot t_2A \arrow[dd, dashed, "\ddot h_1\ddot\xi_1\ddot h_1A \\ = \\ \dot\lambda_2\ddot h_1A" {align=left}] \\
\dot t_1\ddot t_2A \arrow[ur, bend left, "\dot\xi_1\dot t_1\ddot t_2A"] \arrow[ur, dashed, bend right, "\dot\xi_2\dot t_1\ddot t_2A"'] \arrow[rr, "\ddot t_2A"'] \arrow[dd, dashed, "\ddot t_1A"'] & & \dot h_1\ddot t_2A \arrow[ur, bend right, "\dot\xi_1\dot h_1\ddot t_2A"'] \arrow[ur, dashed, bend left, "\dot\xi_2\dot h_1\ddot t_2A"] \arrow[dd, dashed, "\ddot h_1A"'] & \\
 & \dot h_1\dot\xi_1\dot t_1\ddot h_2A \arrow[rr, "\ddot h_2\ddot\xi_2\ddot h_2A\,=\,\dot\lambda_1\ddot h_2A"] & & \dot h_1\dot\xi_1\dot h_1\ddot h_2A \\
\dot t_1\ddot h_2A \arrow[ur, bend left, "\dot\xi_1\dot t_1\ddot h_2A"] \arrow[ur, dashed, bend right, "\dot\xi_2\dot t_1\ddot h_2A"'] \arrow[rr, "\ddot h_2A"'] & & \dot h_1\ddot h_2A \arrow[ur, bend right, "\dot\xi_1\dot h_1\ddot h_2A"'] \arrow[ur, dashed, bend left, "\dot\xi_2\dot h_1\ddot h_2A"] &
\end{tikzcd} \tag{77}$$

[The remaining sections in this chapter are experimental.]

12.4 From actor crossed square to group-double groupoid

[Why start with the actor? – Because we know it works!]

The correspondence from actor crossed square to cat^2-group is

$$\begin{tikzcd}
S \arrow[r, "\eta"] \arrow[d, "\partial"'] & W \arrow[d, "\Delta"] \\
R \arrow[r, "\iota"'] & A
\end{tikzcd}
\quad \longrightarrow \quad
\begin{tikzcd}
(A \ltimes R) \ltimes (W \ltimes S) \arrow[r, shift left=2, "\ddot{t}_1{,}\ddot{h}_1"] \arrow[r, shift left=0.5] \arrow[d, shift right=2, "\ddot{t}_2{,}\ddot{h}_2"'] \arrow[d, shift right=0.5] & A \ltimes W \arrow[l, shift left=1.5, "\ddot{e}_1"] \arrow[d, shift left=0.5] \arrow[d, shift left=2, "\dot{t}_2{,}\dot{h}_2"] \\
A \ltimes R \arrow[u, shift right=1.5, "\ddot{e}_2"'] \arrow[r, shift right=0.5] \arrow[r, shift right=2, "\dot{t}_1{,}\dot{h}_1"'] & R \arrow[l, shift right=1.5, "\dot{e}_1"'] \arrow[u, shift left=1.5, "\dot{e}_2"]
\end{tikzcd}$$

We saw in Subsection 1.11 (9) that the maps on $(e; t, h : R \ltimes S \to R)$ are $t(r,s) = r, h(r,s) = r(\partial s)$ and $er = (r,1)$. These may be used immediately to define $\ddot{t}_2, \ddot{h}_2$ and $\ddot{e}_2$. They may also be used to define $\ddot{t}_1, \ddot{h}_1$ and $\ddot{e}_1$ after applying the isomorphism τ between $G = (A \ltimes R) \ltimes (W \ltimes S)$ and $\tilde{G} = (A \ltimes W) \ltimes (R \ltimes S)$ as shown in the following diagram

$$\begin{tikzcd}
G \arrow[rr, shift left=1, "\tau"] \arrow[dr, shift right=2, "\ddot{t}_1{,}\ddot{h}_1"'] \arrow[dr, shift right=0.5] & & \tilde{G} \arrow[ll, shift left=1, "\tilde{\tau}"] \arrow[dl, shift left=0.5] \arrow[dl, shift left=2, "\tilde{t}_1{,}\tilde{h}_1"] \\
& A \ltimes W \arrow[ul, shift right=1, "\ddot{e}_1"'] \arrow[ur, shift left=1, "\tilde{e}_1"] &
\end{tikzcd}$$

Recall from Proposition 8.21 that the formulae for τ and its inverse $\tilde{\tau}$ are

$$\begin{aligned}
\tau((\alpha, r), (\chi, s)) &= ((\alpha, \chi), (r, (r \boxtimes \chi)s)), \\
\tilde{\tau}((\alpha, \chi), (r, s)) &= ((\alpha, r), (\chi, (r \boxtimes \chi)^{-1}s)).
\end{aligned}$$

Applying these formulae, using $r \boxtimes \chi = \chi r$, we obtain

$$\begin{array}{rcl}
g &=& ((\alpha, r), (\chi, s)), \\
\tau g &=& ((\alpha, \chi), (r, (\chi r)s)), \\
\ddot{t}_1 g \;=\; \tilde{t}_1 \tau g &=& (\alpha, \chi), \\
\tilde{e}_1 \tilde{t}_1 \tau g &=& ((\alpha, \chi), (1, 1)), \\
\ddot{e}_1 \ddot{t}_1 g \;=\; \tilde{\tau} \tilde{e}_1 \tilde{t}_1 \tau g &=& ((\alpha, 1), (\chi, 1)), \\
\ddot{h}_1 g \;=\; \tilde{h}_1 \tau g &=& (\alpha, \chi)(\iota r, \eta((\chi r)s)) \\
&=& (\alpha, \chi)(\beta_r, \eta_{\chi r} \star \eta_s) \\
&=& (\alpha * \beta_r, \chi^{\beta_r} \star \eta_{\chi r} \star \eta_s), \\
\tilde{e}_1 \tilde{h}_1 \tau g &=& ((\alpha * \beta_r, \chi^{\beta_r} \star \eta_{\chi r} \star \eta_s), (1, 1)), \\
\ddot{e}_1 \ddot{h}_1 g \;=\; \tilde{\tau} \tilde{e}_1 \tilde{h}_1 \tau g &=& ((\alpha * \beta_r, 1), (\chi^{\beta_r} \star \eta_{\chi r} \star \eta_s, 1)).
\end{array}$$

Thus the images of the various maps in the cat^2-group are given by

$$\begin{aligned}
\ddot{t}_1((\alpha,r),(\chi,s)) &= (\alpha,\chi),\\
\ddot{h}_1((\alpha,r),(\chi,s)) &= (\alpha*\beta_r,\ \chi^{\beta_r}\star\eta_{\chi r}\star\eta_s),\\
\ddot{t}_2((\alpha,r),(\chi,s)) &= (\alpha,r),\\
\ddot{h}_2((\alpha,r),(\chi,s)) &= (\alpha*\beta_\chi,\ r(\partial\chi r)(\partial s)),\\
\dot{t}_1(\alpha,r) &= \alpha,\\
\dot{h}_1(\alpha,r) &= \alpha*\beta_r,\\
\dot{t}_2(\alpha,\chi) &= \alpha,\\
\dot{h}_2(\alpha,\chi) &= \alpha*\beta_\chi.
\end{aligned}$$

From crossed square to group-double groupoid:

Here is the same calculation, but starting with a general crossed square rather than an actor crossed square.

$$\begin{tikzcd}
 & (P\ltimes M)\ltimes(N\ltimes L) \arrow[dl, shift left, "\ddot{t}_2{,}\ddot{h}_2"] \arrow[dl, shift left=2] \arrow[dr, shift left, "\ddot{t}_1{,}\ \ddot{h}_1"'] \arrow[dr, shift left=2] \arrow[rr, shift left, "\tau"] & & (P\ltimes N)\ltimes(M\ltimes L) \arrow[ll, shift left, "\tilde{\tau}"] \arrow[dl, shift left, "\tilde{t}_1{,}\ \tilde{h}_1"] \arrow[dl, shift left=2] \\
P\ltimes M \arrow[ur, shift left=3, "\ddot{e}_2"] \arrow[dr, shift left, "\dot{t}_1{,}\ \dot{h}_1"'] \arrow[dr, shift left=2] & & P\ltimes N \arrow[ul, shift left=3, "\ddot{e}_1"'] \arrow[ur, shift left=3, "\tilde{e}_1"] \arrow[dl, shift left, "\dot{t}_2{,}\dot{h}_2"] \arrow[dl, shift left=2] & \\
 & P \arrow[ul, shift left=3, "\dot{e}_1"'] \arrow[ur, shift left=3, "\dot{e}_2"] & &
\end{tikzcd}$$

Again we evaluate some images. The formulae for $\ddot{t}_1,\ddot{h}_1$ are obtained by first applying $\tilde{\tau}$ and then using formulae equivalent to $\ddot{t}_2,\ddot{h}_2$.

$$\begin{aligned}
g &= ((p,m),(n,\ell)),\\
\ddot{t}_2 g &= (p,m),\\
\ddot{h}_2 g &= (p,m)(\dot{\partial}_2 n,\ddot{\partial}_2\ell)\\
&= (p(\dot{\partial}_2 n), m^n(\ddot{\partial}_2\ell)), && \textbf{(**)}\\
\tau g &= ((p,n),(m,(m\boxtimes n)\ell)),\\
\ddot{t}_1 g \ =\ \tilde{t}_1\tau g &= (p,n),\\
\ddot{h}_1 g \ =\ \tilde{h}_1\tau g &= (p,n)(\dot{\partial}_1 m,\ddot{\partial}_1((m\boxtimes n)\ell))\\
&= (p(\dot{\partial}_1 m), n^m((n^{-1})^m n)(\ddot{\partial}_1\ell))\\
&= (p(\dot{\partial}_1 m), n(\ddot{\partial}_1\ell)) && \textbf{(***)}\\
\dot{t}_1(p,m) &= p,\\
\dot{h}_1(p,m) &= p(\dot{\partial}_1 m),\\
\dot{t}_2(p,n) &= p,\\
\dot{h}_2(p,n) &= p(\dot{\partial}_2 n).
\end{aligned}$$

[Note the minor difference betwen (**) and (***), which will be very important!]

We now check the axioms in (71) for a cat^2-group, where we put $g = ((p,m),(n,\ell))$,

$$\begin{array}{rcl}
\dot{t}_1\ddot{h}_2 g & = & \dot{t}_1(p(\dot{\partial}_2 n), m^n(\ddot{\partial}_2\ell)) \;=\; p(\dot{\partial}_2 n) \;=\; \dot{h}_2(p,n) \;=\; \dot{h}_2\ddot{t}_1 g, \\
\dot{t}_2\ddot{h}_1 g & = & \dot{t}_2(p(\dot{\partial}_1 m), n(\ddot{\partial}_1\ell)) \;=\; p(\dot{\partial}_1 m) \;=\; \dot{h}_1(p,m) \;=\; \dot{h}_1\ddot{t}_2 g, \\
\ddot{t}_2\ddot{e}_1(p,n) & = & \ddot{t}_2((p,1),(n,1)) \;=\; (p,1) \;=\; \dot{e}_1 p \;=\; \dot{e}_1\dot{t}_2(p,n), \\
\ddot{t}_1\ddot{e}_2(p,m) & = & \ddot{t}_1((p,m),(1,1)) \;=\; (p,1) \;=\; \dot{e}_2 p \;=\; \dot{e}_2\dot{t}_1(p,m), \\
\ddot{h}_2\ddot{e}_1(p,n) & = & \ddot{h}_2((p,1),(n,1)) \;=\; (p(\dot{\partial}_2 n),1) \;=\; \dot{e}_1(p(\dot{\partial}_2 n)) \;=\; \dot{e}_1\dot{h}_2(p,n), \\
\ddot{h}_1\ddot{e}_2(p,m) & = & \ddot{h}_1((p,m),(1,1)) \;=\; (p(\dot{\partial}_1 m),1) \;=\; \dot{e}_2(p(\dot{\partial}_1 m)) \;=\; \dot{e}_2\dot{h}_1(p,m). \\
\dot{t}_2\ddot{t}_1 g & = & \dot{t}_2(p,n) \;=\; p \;=\; \dot{t}_1(p,m) \;=\; \dot{t}_1\ddot{t}_2 g, \\
\dot{h}_1\ddot{h}_2 g & = & \dot{h}_1(p(\dot{\partial}_2 n), m^n(\ddot{\partial}_2\ell)) \;=\; p(\dot{\partial}_2 n)(\dot{\partial}_1 m^n)(\bar{\partial}\ell) \;=\; p(\dot{\partial}_1 m)(\dot{\partial}_2 n)(\bar{\partial}\ell) \qquad \textbf{(****)} \\
 & = & \dot{h}_2(p(\dot{\partial}_1 m), n(\ddot{\partial}_1\ell)) \;=\; \dot{h}_2\ddot{h}_1 g, \\
\ddot{e}_1\dot{e}_2 p & = & \ddot{e}_1(p,1) \;=\; ((p,1),(1,1)) \;=\; \ddot{e}_2(p,1) \;=\; \ddot{e}_2\dot{e}_1 p,
\end{array}$$

[Note that the third equality in (****) is the crucial step!]

Vertical composition is easy to evaluate.

Lemma 12.4 *If we let* $(p',m') = (p(\dot{\partial}_2 n), m^n(\ddot{\partial}_2\ell))$ *then a vertical composite square evaluates as*

$$((p,m),(n,\ell)) \;\ddot{*}_2\; ((p',m'),(n',\ell')) \;=\; ((p,m),(nn',\ell^{n'}\ell')).$$

Proof: Applying formula (12) for the composite,

$$\begin{array}{rl}
 & ((p,m),(n,\ell)) \;\ddot{*}_1\; ((p',m'),(n',\ell')) \\
= & ((p,m),(n,\ell))((p',m')^{-1},(1,1))((p',m'),(n',\ell')) \\
= & ((p,m),(n,\ell)(n',\ell')).
\end{array}$$

□

Horizontal composition is more complicated.

Lemma 12.5 *If we let* $(p'',n'') = (p(\dot{\partial}_1 m), n(\ddot{\partial}_1\ell))$ *then a horizontal composite square evaluates as*

$$((p,m),(n,\ell)) \;\ddot{*}_1\; ((p'',m''),(n'',\ell'')) \;=\; ((p,mm''),(n,\ell\ell'')).$$

Proof: Applying formula (12) again,

$$\begin{array}{rll}
 & ((p,m),(n,\ell))\;((p''^{-1},1),((n''^{-1})^{p''^{-1}},1))\;((p'',m''),(n'',\ell'')) & \\
= & ((p,m)(1,m''),\;\; (n,\ell)^{(1,m'')}((n''^{-1})^{p''^{-1}},1)^{(p'',m'')}(n'',\ell'')) & \\
= & ((p,mm''),(n,(m''\boxtimes n)^{-1}\ell^{m''})(n''^{-1},(m''\boxtimes n''^{-1})^{-1})(n'',\ell'')) & \\
= & ((p,mm''),(n,(m''\boxtimes n)^{-1}\ell^{m''}((m''\boxtimes n''^{-1})^{-1})^{n''}\ell'')) & \\
= & ((p,mm''),(n,(m''\boxtimes n)^{-1}\ell^{m''}(m''\boxtimes n(\ddot{\partial}_1\ell))\ell'')) & \text{(by Proposition 7.6(b))} \\
= & ((p,mm''),(n,(m''\boxtimes n)^{-1}\ell^{m''}(m''\boxtimes\ddot{\partial}_1\ell)(m''\boxtimes n)^{\ddot{\partial}_1\ell}\ell'')) & \\
= & ((p,mm''),(n,(m''\boxtimes n)^{-1}\ell^{m''}((\ell^{-1})^{m''}\ell)(\ell^{-1}(m''\boxtimes n)\ell)\ell'')) & \\
= & ((p,mm''),(n,\ell\ell''). &
\end{array}$$

□

[The difference between these two composites should be very significant!]

Now consider the two composites of four squares with elements as shown:

$$\begin{tikzcd}
\cdot \arrow[r, "{(p_1,m_1)}"] \arrow[d, "{(p_1,n_1)}"'] \arrow[dr, phantom, "{((p_1,m_1),(n_1,\ell_1))}"] & \cdot \arrow[r, "{(p_2,m_2)}"] \arrow[d] \arrow[dr, phantom, "{((p_2,m_2),(n_2,\ell_2))}"] & \cdot \arrow[d, "{(p_2(\dot{\partial}_1 m_2),n_2(\ddot{\partial}_1 \ell_2))}"] \\
\cdot \arrow[r] \arrow[d, "{(p_3,n_3)}"'] \arrow[dr, phantom, "{((p_3,m_3),(n_3,\ell_3))}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "{((p_4,m_4),(n_4,\ell_4))}"] & \cdot \arrow[d, "{(p_4(\dot{\partial}_1 m_4),n_4(\ddot{\partial}_1 \ell_4))}"] \\
\cdot \arrow[r, "{(p_3(\dot{\partial}_2 n_3),m_3^{n_3}(\ddot{\partial}_2 \ell_3))}"'] & \cdot \arrow[r, "{(p_4(\dot{\partial}_2 n_4),m_4^{n_4}(\ddot{\partial}_2 \ell_4))}"'] & \cdot
\end{tikzcd}$$

For the horizontal composites to be defined we require

$$p_2 = p_1(\dot{\partial}_1 m_1), \quad n_2 = n_1(\ddot{\partial}_1 \ell_1), \quad p_4 = p_3(\dot{\partial}_1 m_3), \quad n_4 = n_3(\ddot{\partial}_1 \ell_3).$$

Similarly, for the vertical composites, we require

$$p_3 = p_1(\dot{\partial}_2 n_1), \quad m_3 = {m_1}^{n_1}(\ddot{\partial}_2 \ell_1), \quad p_4 = p_2(\dot{\partial}_2 n_2), \quad m_4 = {m_2}^{n_2}(\ddot{\partial}_2 \ell_2).$$

The two requirements on p_4 are consistent since

$$p_4 \;=\; p_2(\dot{\partial}_2 n_2) \;=\; p_1(\dot{\partial}_1 m_1)(\dot{\partial}_2 n_1)(\dot{\partial}_2 \ddot{\partial}_1 \ell_1) \;=\; p_1(\dot{\partial}_2 n_1)(\dot{\partial}_1({m_1}^{n_1}))(\dot{\partial}_1 \ddot{\partial}_2 \ell_1) \;=\; p_3(\dot{\partial}_1 m_3).$$

Composing horizontally, we obtain

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "{((p_1,m_1m_2),(n_1,\ell_1\ell_2))}"] & \cdot \arrow[d] \\
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "{((p_3,m_3m_4),(n_3,\ell_3\ell_4))}"] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}$$

The vertical composite is then

$$((p_1, m_1m_2), (n_1, \ell_1\ell_2)(n_3, \ell_3\ell_4)) \;=\; ((p_1, m_1m_2), (n_1n_3, (\ell_1\ell_2)^{n_3}(\ell_3\ell_4)).$$

When the four squares are composed vertically then horizontally, we obtain

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "{((p_1,m_1),(n_1n_3,{\ell_1}^{n_3}\ell_3))}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "{((p_2,m_2),(n_2n_4,{\ell_2}^{n_4}\ell_4))}"] & {} \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & {}
\end{tikzcd}$$

and the horizontal composite is

$$((p_1, m_1m_2), (n_1n_3, {\ell_1}^{n_3}\ell_3{\ell_2}^{n_3(\ddot{\partial}_1 \ell_3)}\ell_4)) \;=\; ((p_1, m_1m_2), (n_1n_3, {\ell_1}^{n_3}{\ell_2}^{n_3}\ell_3\ell_4))$$

as before. We have therefore verified the interchange law for this double groupoid.

12.5 From cat^2-group to group-double groupoid (1)

Recall the cat^1-group semidirect factorisation of Subsection 1.11 (10)

$$(\phi', \mathrm{id}_R) \ : \ \mathcal{C} = (e; t, h : G \to R) \ \to \ \mathcal{C}' = (e'; t', h' : R \ltimes S \to R), \quad g \mapsto (tg, ug), \quad ug = (etg^{-1})g,$$

and the properties of the projection $u : G \to \ker t$ in Lemma 1.22,

$$tug = 1_R, \quad hug = (tg^{-1})(hg), \quad uer = 1_G, \quad u(g_1g_2) = (ug_2)(ug_1)^{g_2}, \quad (ug)^{-1} = g^{-1}(etg) = (u(g^{-1}))^g.$$

Generalising this isomorphism to cat^2-groups, the relevant diagram is:

$$\begin{tikzcd}
G \arrow[r, shift left=3, "\ddot t_1{,}\ddot h_1"] \arrow[r, shift left=1] \arrow[d, shift right=3, "\ddot t_2{,}\ddot h_2"'] \arrow[d, shift right=1] & K \arrow[l, shift left=1, "\ddot e_1"] \arrow[d, shift left=3, "\dot t_2{,}\dot h_2"] \arrow[d, shift left=1] \\
J \arrow[u, shift right=1, "\ddot e_2"'] \arrow[r, shift right=1] \arrow[r, shift right=3, "\dot t_1{,}\dot h_1"'] & P \arrow[l, shift right=1, "\dot e_1"'] \arrow[u, shift left=1, "\dot e_2"]
\end{tikzcd}
\quad \xrightarrow{\phi} \quad
\begin{tikzcd}
(J \ltimes F) \arrow[r, shift left=3, "\ddot t_1'{,}\ddot h_1'"] \arrow[r, shift left=1] \arrow[d, shift right=3, "\ddot t_2'{,}\ddot h_2'"'] \arrow[d, shift right=1] & P \ltimes N \arrow[l, shift left=1, "\ddot e_1'"] \arrow[d, shift left=3, "\dot t_2'{,}\dot h_2'"] \arrow[d, shift left=1] \\
P \ltimes M \arrow[u, shift right=1, "\ddot e_2'"'] \arrow[r, shift right=1] \arrow[r, shift right=3, "\dot t_1'{,}\dot h_1'"'] & P \arrow[l, shift right=1, "\dot e_1'"'] \arrow[u, shift left=1, "\dot e_2'"]
\end{tikzcd}$$

where

$$J \cong P \ltimes M, \quad K \cong P \ltimes N, \quad E \cong M \ltimes L, \quad F \cong N \ltimes L.$$

The components of ϕ and the fourth semidirect product isomorphism are:

$$\begin{array}{rcll}
\phi_{[2]} & : & G \to K \ltimes E, & g \mapsto (\ddot t_1 g, (\ddot e_1 \ddot t_1 g^{-1})g) \ = \ (\ddot t_1 g, \ddot u_1 g), \\
\phi_{\{1\}} & : & J \to P \ltimes M, & j \mapsto (\dot t_1 j, (\dot e_1 \dot t_1 j^{-1})j) \ = \ (\dot t_1 j, \dot u_1 j), \\
\phi_{\{2\}} & : & K \to P \ltimes N, & k \mapsto (\dot t_2 k, (\dot e_2 \dot t_2 k^{-1})k) \ = \ (\dot t_2 k, \dot u_2 k), \\
\phi_{\emptyset} & : & P \to P, & p \mapsto p, \\
\phi_{[2']} & : & F \to N \ltimes L, & f \mapsto (\ddot t_2 f, (\ddot e_2 \ddot t_2 f^{-1})f) \ = \ (\ddot t_2 f, \ddot u_2 f), \\
\text{or} \ \ ? & : & G \to J \ltimes F, & g \mapsto (\ddot t_2 g, (\ddot e_2 \ddot t_2 g^{-1})g) \ = \ (\ddot t_2 g, \ddot u_2 g).
\end{array}$$

Recall the following identities from (56),

$$\dot u_1 \ddot t_2 = \ddot t_2 \ddot u_1, \quad \dot u_1 \ddot h_2 = \ddot h_2 \ddot u_1, \quad \ddot u_1 \ddot e_2 = \ddot e_2 \dot u_1, \quad \dot u_2 \ddot t_1 = \ddot t_1 \ddot u_2, \quad \dot u_2 \ddot h_1 = \ddot h_1 \ddot u_2, \quad \ddot u_2 \ddot e_1 = \ddot e_1 \dot u_2.$$

Combining these isomorphisms (as in Lemma 8.25) we obtain

$$\begin{array}{rcl}
\phi \ : \ G & \to & (P \ltimes M) \ltimes (N \ltimes L), \\
g & \mapsto & (\ (t_{[2]}g, \ \dot u_2 \ddot t_1 g), \ (\dot u_1 \ddot t_2 g, \ \ddot u_2 \ddot u_1 g)\ ), \\
\text{where} \quad \ddot u_2 \ddot u_1 g & = & (\ddot e_2 \ddot t_2 g^{-1})(e_{[2]} t_{[2]} g)(\ddot e_1 \ddot t_1 g^{-1})g.
\end{array}$$

For the alternative semidirect product decomposition of G we obtain:

$$\begin{array}{rcl}
\tilde\phi \ : \ G & \to & (P \ltimes N) \ltimes (M \ltimes L), \\
g & \mapsto & (\ (t_{[2]}g, \ \dot u_1 \ddot t_2 g), \ (\dot u_2 \ddot t_1 g, \ \ddot u_1 \ddot u_2 g)\ ).
\end{array}$$

The difference between the two fourth coordinates (corresponding to $m \boxtimes n$ in the crossed square version) is given by $(\ddot u_2 \ddot u_1 g)(\ddot u_1 \ddot u_2 g)^{-1}$. Thus the crossed pairing corresponds to a mapping

$$\begin{aligned} u_{[2]} \ : \ G \to G, \quad g \ \mapsto & \ (\ddot u_2 \ddot u_1 g)(\ddot u_1 \ddot u_2 g)^{-1} \\ = & \ (\ddot e_2 \ddot t_2 g^{-1})(e_{[2]} t_{[2]} g)(\ddot e_1 \ddot t_1 g^{-1})(\ddot e_2 \ddot t_2 g)(e_{[2]} t_{[2]} g^{-1})(\ddot e_1 \ddot t_1 g), \end{aligned} \tag{78}$$

which measures the difference between the two composite projections onto $\ker \ddot t_1 \cap \ker \ddot t_2$.

Lemma 12.6 $u_{[2]} g \ \in \ [\ker \ddot t_1, \ker \ddot t_2]$.

Proof: Put $y = (e_{[2]} t_{[2]} g^{-1})(\ddot e_2 \ddot t_2 g) \in \ker \ddot t_1$ and $z = (e_{[2]} t_{[2]} g^{-1})(\ddot e_1 \ddot t_1 g) \in \ker \ddot t_2$. Then, by (78), $u_{[2]} g = [y, z] \ \in \ [\ker \ddot t_1, \ker \ddot t_2]$. $\Box$

We are now ready to define the associated group-double groupoid. There is a square for each element $g \in G$, and the tail and head cat^2 homomorphisms give the two groupoid source and target functions.

$$\begin{tikzcd}
\dot t_1 \ddot t_2 g \arrow[r, "\ddot t_2 g"] \arrow[d, "\ddot t_1 g"'] & \dot t_1 \ddot h_2 g \arrow[d, "\ddot h_1 g"] \\
\dot h_1 \ddot t_2 g \arrow[r, "\ddot h_2 g"'] \arrow[ur, phantom, "g"] & \dot h_1 \ddot h_2 g
\end{tikzcd}$$

Vertical composition is given as in (12),

$$g_1 *_1 g_3 \ = \ g_1(\ddot e_2 \ddot h_2 g_1{}^{-1}) g_3 \qquad \text{when} \quad \ddot h_2 g_1 = \ddot t_2 g_3,$$

while horizontal composition first applies $\phi_{M,N}$ and then (12),

... ... **to be completed** ... ...

12.6 From cat^2-group to group-double groupoid (2)

Given a cat^2-group $\mathcal{G}$, define a douple groupoid $\mathcal{D} = \mathcal{D}(\mathcal{G})$ to have as squares the elements of $\mathcal{G}$, with the head and tail maps in $\mathcal{G}$ providing the boundary maps in $\mathcal{D}$.

Consider the following composite square of cat^1-group elements

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "a"] & \cdot \arrow[r] \arrow[d, "w"] \arrow[dr, phantom, "b"] & \cdot \arrow[d] \\
\cdot \arrow[r, "x"] \arrow[d] \arrow[dr, phantom, "c"] & p \arrow[r, "y"] \arrow[d, "z"] \arrow[dr, phantom, "d"] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

$$w = \ddot{h}_1 a = \ddot{t}_1 b, \quad z = \ddot{h}_1 c = \ddot{t}_1 d,$$

$$x = \ddot{h}_2 a = \ddot{t}_2 c, \quad y = \ddot{h}_2 b = \ddot{t}_2 d,$$

$$p = \dot{h}_1 \ddot{h}_2 a = \dot{h}_2 \ddot{t}_1 b = \dot{t}_2 \ddot{h}_1 c = \dot{t}_1 \ddot{t}_2 d.$$

Define horizontal and vertical composition as in

$$a \ \ddot{*}_2\ b \ = \ a(\ddot{e}_1 w^{-1})b, \qquad a \ \ddot{*}_1\ c \ = \ a(\ddot{e}_2 x^{-1})c,$$

[N.B. Either one of these is incorrect, or else some boundary maps should be changed!]

Note that squares such as $\ddot{e}_1 w$ and $\ddot{e}_2 x$ are identities for the groupoid structures $\ddot{*}_2$, $\ddot{*}_1$ respectively:

$$a \ \ddot{*}_2\ (\ddot{e}_1 w) \ = \ a(\ddot{e}_1 w^{-1})(\ddot{e}_1 w) \ = \ a, \qquad a \ \ddot{*}_1\ (\ddot{e}_2 x) \ = \ a(\ddot{e}_2 x^{-1})(\ddot{e}_2 x) \ = \ a,$$

while $e_{[2]} p = \dot{e}_1 \ddot{e}_2 p$ is an identity for both groupoid structures.

Note also that

$$\ddot{h}_2(a \ \ddot{*}_2\ b) \ = \ (\ddot{h}_2 a)(\ddot{h}_2 \ddot{e}_1 w^{-1})(\ddot{h}_2 b) \ = \ x(\dot{e}_1 p^{-1})y \ = \ \ddot{t}_2(c \ \ddot{*}_2\ d).$$

Here is an expanded view of the composite square using nine squares:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "a"] & \cdot \arrow[r] \arrow[d, "w"] \arrow[dr, phantom, "\ddot{e}_1 w"] & \cdot \arrow[r] \arrow[d, "w"] \arrow[dr, phantom, "b"] & \cdot \arrow[d] \\
\cdot \arrow[r, "x"] \arrow[d] \arrow[dr, phantom, "\ddot{e}_2 x"] & p \arrow[r, "\dot{e}_1 p"] \arrow[d, "\dot{e}_2 p"] \arrow[dr, phantom, "e_{[2]} p"] & p \arrow[r, "y"] \arrow[d, "\dot{e}_2 p"] \arrow[dr, phantom, "\ddot{e}_2 y"] & \cdot \arrow[d] \\
\cdot \arrow[r, "x"] \arrow[d] \arrow[dr, phantom, "c"] & p \arrow[r, "\dot{e}_1 p"] \arrow[d, "z"] \arrow[dr, phantom, "\ddot{e}_1 z"] & p \arrow[r, "y"] \arrow[d, "z"] \arrow[dr, phantom, "d"] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

Composing in two ways, we obtain

$$\begin{aligned}
(a \mathbin{\ddot{*}_2} b) \mathbin{\ddot{*}_1} (c \mathbin{\ddot{*}_2} d) &= (a(\ddot{e}_1 w^{-1})b) \mathbin{\ddot{*}_1} (c(\ddot{e}_1 z^{-1})d) \\
&= (a(\ddot{e}_1 w^{-1})b)\ (\ddot{e}_2(y^{-1}(\dot{e}_1 p)x^{-1}))\ (c(\ddot{e}_1 z^{-1})d) \\
&= a(\ddot{e}_1 w^{-1})b(\ddot{e}_2 y^{-1})(e_{[2]}p)(\ddot{e}_2 x^{-1})c(\ddot{e}_1 z^{-1})d, \\
(a \mathbin{\ddot{*}_1} c) \mathbin{\ddot{*}_2} (b \mathbin{\ddot{*}_1} d) &= (a(\ddot{e}_2 x^{-1})c) \mathbin{\ddot{*}_2} (b(\ddot{e}_2 y^{-1})d) \\
&= (a(\ddot{e}_2 x^{-1})c)\ (\ddot{e}_1(z^{-1}(\dot{e}_2 p)w^{-1}))\ (b(\ddot{e}_2 y^{-1})d) \\
&= a(\ddot{e}_2 x^{-1})c(\ddot{e}_1 z^{-1})(e_{[2]}p)(\ddot{e}_1 w^{-1})b(\ddot{e}_2 y^{-1})d,
\end{aligned}$$

For these to be equal we require

$$a(\ddot{e}_1 w^{-1})b(\ddot{e}_2 y^{-1})(e_{[2]}p)(\ddot{e}_2 x^{-1})c(\ddot{e}_1 z^{-1})d \quad = \quad a(\ddot{e}_2 x^{-1})c(\ddot{e}_1 z^{-1})(e_{[2]}p)(\ddot{e}_1 w^{-1})b(\ddot{e}_2 y^{-1})d$$

which we may rewrite as

$$a(\ddot{u}_1 b)(\ddot{u}_1 \ddot{e}_2 y)^{-1}(\ddot{u}_2 c)(\ddot{u}_1 d) \quad = \quad a(\ddot{u}_2 c)(\ddot{u}_2 \ddot{e}_1 z)^{-1}(\ddot{u}_1 b)(\ddot{u}_2 d).$$

This equation does not appear to be satisfied, so the interchange law is not satisfied. On the other hand, the group-double groupoid obtained earlier from a crossed square does satisfy the interchange law.

[This contradiction still needs to be sorted out!]

13 Simplicial Groups

This material is taken mainly from seminars given by Tim Porter, designed to explain the four functors in the diagram

$$\begin{tikzcd}
\text{Spaces} \arrow[r, shift right, "\text{Sing}"'] & \text{Simplicial Sets} \arrow[l, shift right, "|\ |"'] \arrow[r, shift right, "G"'] & \text{Simplicial Groupoids} \arrow[l, shift right, "W"']
\end{tikzcd}$$

13.1 Simplicial Sets and Simplicial Complexes

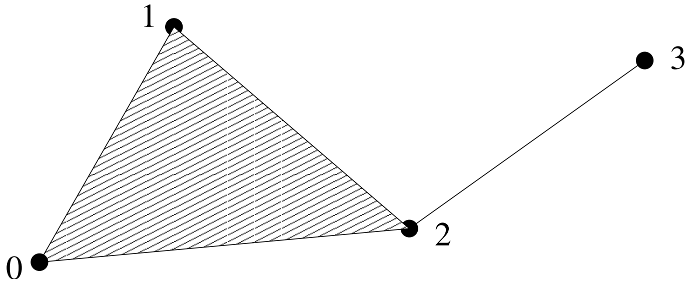
A *simplicial set* S comprises

- a sequence $\{S_n\}_{n\geqslant 0}$ of sets, where S_n is the *set of simplices of dimension* n, linked by
- *face maps* $d_k : S_n \to S_{n-1}, 0 \leqslant k \leqslant n$, where we think of $d_k(S_n)$ as the "face opposite S_n", and
- *degeneracy maps* $s_k : S_n \to S_{n+1}, 0 \leqslant k \leqslant n.$

A *simplicial complex* K is a set $V(K)$ of vertices, and a family $S \subset \left(2_{\text{fin}}^{V(K)} \setminus \emptyset\right)$ closed under $\subseteq$.
To get from a simplicial complex K to a simplicial set S_K

- pick a total order on $V(K)$,
- let S_K be the set of all finite subsets of $V(K)$,
- set $(S_K)_n = \{[a_0, \dots, a_n] \mid \{a_0, \dots, a_n\} \in S, a_0 \leqslant \cdots \leqslant a_n\}$.
- choose face maps $d_i[a_0, \dots, a_n] = [a_0, \dots, \hat{a_i}, \dots, a_n]$, omitting the i-th entry, starting at 0,
- choose degeneracy maps $s_i[a_0, \dots, a_n] = [a_0, \dots, a_i, a_i, \dots, a_n]$, duplicating the i-th entry.

Example 13.1 *Here is a simplicial complex K with 4 vertices; 4 edges; and 1 triangle.*



$$\begin{array}{rcl}
V(K) & = & \{0,1,2,3\}, \\
S & = & \{\ \{0,1,2\},\ \{2,3\},\ \{0,1\},\ \{1,2\},\ \{0,2\},\ \{0\},\ \{1\},\ \{2\},\ \{3\}\ \} \\
\textit{order} & = & 2 < 1 < 3 < 0 \quad \textit{(say)}, \\
(S_K)_0 & = & \{[2],\ [1],\ [3],\ [0]\}, \\
(S_K)_1 & = & \{[2,2] = s_0[2],\ [1,1] = s_0[1],\ [3,3] = s_0[3],\ [0,0] = s_0[0],\ [2,1],\ [2,3],\ [2,0],\ [1,0]\}, \\
(S_K)_2 & = & \{[2,2,2] = s_0^2[2],\ [1,1,1] = s_0^2[1],\ [3,3,3] = s_0^2[3],\ [0,0,0] = s_0^2[0],\ [2,2,1] = s_0[2,1], \\
 & & \quad [2,2,3] = s_0[2,3],\ [2,2,0] = s_0[2,0],\ [2,1,1] = s_1[2,1],\ [2,3,3] = s_1[2,3], \\
 & & \quad [2,0,0] = s_1[2,0],\ [1,1,0] = s_0[1,0],\ [1,0,0] = s_1[1,0],\ [2,1,0]\}.
\end{array}$$

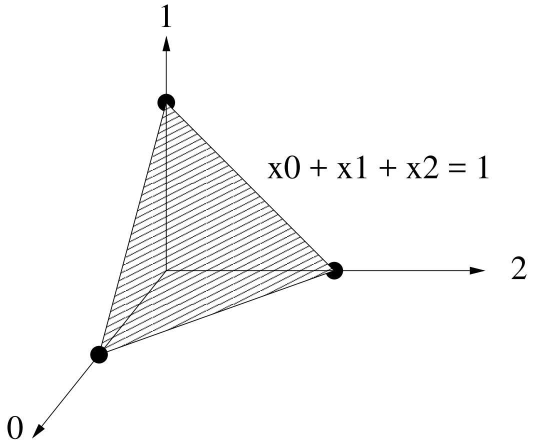
The face and degeneracy maps satisfy the following identities. **[need to check these!]** The second column gives the common image of $[a_0, \dots, a_n]$.

$$\begin{array}{llll}
d_{j-1} \circ d_i \;=\; d_i \circ d_j & (0 \leqslant i < j \leqslant n) & & [a_0, \dots, \hat{a_i}, \dots, \hat{a_j}, \dots, a_n], \\
d_{j+1} \circ s_i \;=\; s_i \circ d_j & (0 \leqslant i < j \leqslant n) & & [a_0, \dots, a_i, a_i, \dots, \hat{a_j}, \dots, a_n], \\
d_{i+1} \circ s_i \;=\; d_i \circ s_i \;=\; \mathrm{id} & (0 \leqslant i \leqslant n) & & [a_0, \dots, a_n], \\
d_i \circ s_j \;=\; s_{j-1} \circ & (0 \leqslant i < j \leqslant n) & & [a_0, \dots, \hat{a_i}, \dots, a_j, a_j, \dots, a_n], \\
s_{j+1} \circ s_i \;=\; s_i \circ s_j & (0 \leqslant i < j \leqslant n) & & [a_0, \dots, a_i, a_i, \dots, a_j, a_j, \dots, a_n], \\
s_{i+1} \circ s_i \;=\; s_i^2 & (0 \leqslant i \leqslant n) & & [a_0, \dots, a_i, a_i, a_i, \dots, a_n].
\end{array} \qquad (79)$$

The standard n-simplex, Δ^n

(See Ehlers [29] for some of this material.)

Let $B_{n+1} = \{\boldsymbol{e}_0, \dots, \boldsymbol{e}_n\}$ be the standard basis for $\mathbb{R}^{n+1}$, and let Δ^n be the convex hull of the points B_{n+1}, a subset of the hyperplane $x_0 + x_1 + \cdots + x_n = 1$. The picture below shows $\Delta^2 \subseteq \mathbb{R}^3$.



Any $\boldsymbol{x} \in \Delta^n$ may be expressed as

$$\boldsymbol{x} \;=\; \sum_{i=0}^{n} x_i\, \boldsymbol{e}_i, \qquad \sum_{i=0}^{n} x_i \;=\; 0, \qquad x_i \geqslant 0.$$

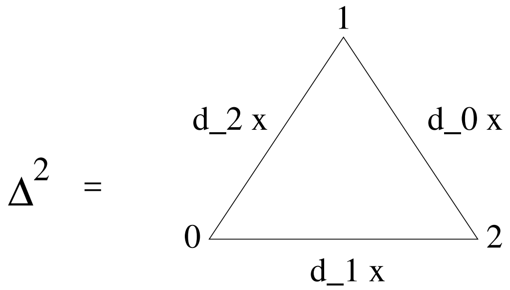
We then define the *linear* maps

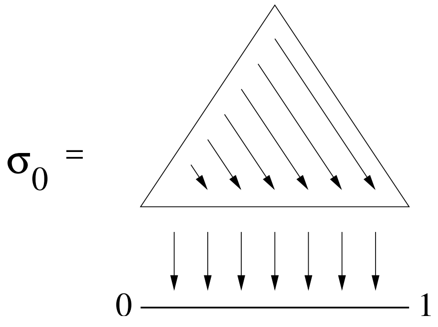
$$\delta_k \;:\; \Delta^{n-1} \to \Delta^n, \qquad (x_0, \dots, x_{n-1}) \mapsto (x_0, \dots, x_k, 0, x_{k+1}, \dots, x_{n-1}), \qquad 0 \leqslant k \leqslant n.$$

Similarly, we define

$$\sigma_k \;:\; \Delta^n \to \Delta^{n-1}, \qquad (x_0, \dots, x_n) \mapsto (x_0, \dots, x_{k-1}, x_k + x_{k+1}, x_{k+2}, \dots, x_n), \qquad 0 \leqslant k \leqslant n-1.$$

The picture for Δ^2 is



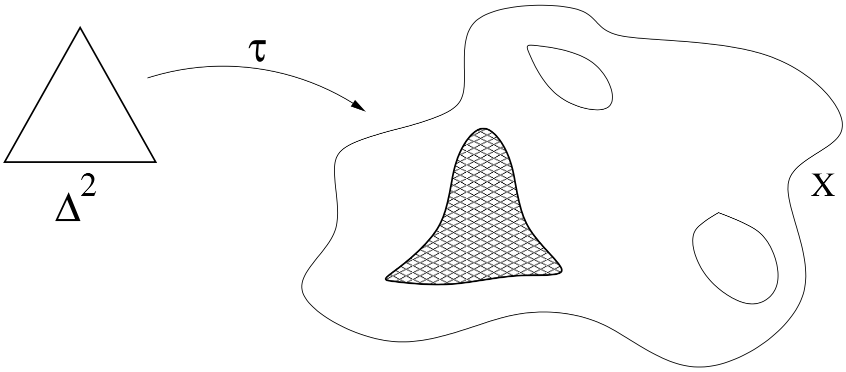


The Singular Complex $\mathrm{Sing}(X)$ **of a space** X**.**

For X a space we defing the *singular complex* of X to be the set of maps

$$\mathrm{Sing}(X) \;=\; \mathrm{Top}(\Delta^n, X).$$

The diagram below shows such a map τ to a space X with two holes.



The Nerve of a Category

/noindent Let $\mathbb{C}$ be a small category. The *nerve* of $\mathbb{C}$ is defined to be the simplicial set $\mathrm{Ner}\,\mathbb{C}$ where

- $(\mathrm{Ner}\,\mathbb{C})_0 = \mathrm{Ob}(\mathbb{C})$,
- $(\mathrm{Ner}\,\mathbb{C})_1 = \mathrm{Arr}(\mathbb{C})$,
- for $x \in (\mathrm{Ner}\,\mathbb{C})_1$, $d_0 x = tx$ and $d_1 x = hx$,
- for $u \in (\mathrm{Ner}\,\mathbb{C})_0$, $s_0 u = 1_u$,
- $(\mathrm{Ner}\,\mathbb{C})_n = \{\text{composable words } (x_1, \ldots, x_n) \mid x_i \in \mathrm{Arr}(\mathbb{C})\}$,
- $d_0(x_1, x_2, \ldots, x_n) \;=\; (x_2, \ldots, x_n)$,
- $d_i(x_1, \ldots, x_n) \;=\; (x_1, \ldots, x_i x_{i+1}, \ldots, x_n), \quad (0 < i < n)$,
- $d_n(x_1 \ldots, x_{n-1}, x_n) \;=\; (x_1, \ldots, x_{n-1})$,
- $s_0(x_1, \ldots, x_n) \;=\; (1_{tx_0}, x_1, \ldots, x_n)$,
- $s_i(x_1, \ldots, x_n) \;=\; (x_1, \ldots, x_i, 1_{hx_i}, x_{i+1}, \ldots, x_n), \quad (0 < i < n)$,
- $s_n(x_1, \ldots, x_n) \;=\; (x_1, \ldots, x_n, 1_{hx_n})$.

This may be described more compactly by saying that $(\mathrm{Ner}\,\mathbb{C})_n = \mathsf{Cat}([n], \mathbb{C})$. This construction extends to a functor from Cat to $\mathsf{SimpSet}$ in the obvious way.

14 Crossed n-cubes of groups and cat^n-groups

Here we include the basic ideas about crossed n-cubes (of groups) taken from Chapter 1 of Ellis' thesis [30], Ellis-Steiner [32], and Brown-Loday [19], and the associated cat^n-groups.

14.1 Crossed n-cubes of groups

Let $[n] = \{1, 2, ..., n\}$, and let $A, B, C, \ldots$ be subsets of $[n]$.

A crossed n-cube consists of the following.

(i) Groups R_A for each subset A of $[n]$, where we write R for $R_\emptyset$.

(ii) Group homomorphisms

$$\partial_{A,i} : R_A \to R_{A\setminus\{i\}} \quad \text{for all} \quad A \subseteq [n],\ i \in A,$$

such that $\partial_{A\setminus\{j\},i} \circ \partial_{A,j} \ = \ \partial_{A\setminus\{i\},j} \circ \partial_{A,i}$ for $i \neq j \in A$.

Since the $\partial_{A,i}$ commute, composite homomorphisms $\partial_{A,B} : R_A \to R_{A\setminus B}$ are well defined and $\partial_{A,B} = \partial_{A,(B\setminus C)} \circ \partial_{A,C}$.

(iii) For all $B \subseteq A$ an action of $R_{A\setminus B}$ on R_A making $\mathcal{R}_{A,A\setminus B} = (\partial_B : R_A \to R_{A\setminus B})$ a crossed module. For each $j \in [n]$ the maps

$$(1, \partial_{A\ setminus\{i\}}, j) : \mathcal{R}_{A,A\setminus\{i\}} \to \mathcal{R}_{A,A\setminus\{i,j\}} \quad \text{and} \quad (\partial_{A,j}, 1) : \mathcal{R}_{A,A\setminus\{i,j\}} \to \mathcal{R}_{A\setminus\{j\},A\setminus\{i,j\}}$$

are crossed module homomorphisms.

It follows that all the actions act via R :

$$a^b \ = \ a^{\partial_{B,B} b} \quad \text{for} \quad a \in R_A,\ b \in R_B, \quad \text{and} \quad B \subseteq A.$$

(iv) For all $A, B \subseteq [n]$ a crossed pairing $\boxtimes_{A,B} : R_A \times R_B \to R_{A\cup B}$, such that $(b \boxtimes_{B,A} a) = (a \boxtimes_{A,B} b)^{-1}$ and, when $B \subseteq A$, $\boxtimes_{A,B}$ is the principal crossed pairing for $\mathcal{R}_{A,B}$ given by $a \boxtimes b = a^{-1}a^b$ and $b \boxtimes a = (a^{-1})^b a$.

(v) Various axioms relating the homomorphisms, actions, and crossed pairings, for example

- $\partial_{A\cup B,i}(a \boxtimes_{A,B} b) = (\partial_{A,i}a) \boxtimes_{A\setminus\{i\},B\setminus\{i\}} (\partial_{B,i}b)$.
- when $i \in A \cap B$, so that $A \cup B = (A \setminus \{i\}) \cup B = A \cup (B \setminus \{i\})$,

  $$a \boxtimes_{A,B} b \ = \ \partial_{A,i}a \boxtimes_{A,B\setminus\{i\}} b \ = \ a \boxtimes_{A\setminus\{i\},B} \partial_{B,i}b.$$

  (This means that we need only define $\boxtimes_{A,B}$ when $A \cup B = \emptyset$.)
- $(a \boxtimes b)^c \ = \ a^c \boxtimes b^c$ when $C \subseteq A$ and $C \subseteq B$. **[Is this correct?]**

We might now wish to define an *n-derivation*, which would seem to be a set of maps

$$\chi_{B,A} : R_B \to R_A \quad \text{for all} \quad B \subseteq A$$

satifying suitable axioms:

(i) perhaps $\quad \chi_{B,A}(bb') = (\chi_{B,A}b)^{b'}(\chi_{B,A}b')$?

(ii) closure: $\quad \chi_{B,A} \circ \chi_{C,B} = \chi_{C,A}$?

(iii) ???

Exercise 14.1 Derive the crossed square axioms from those of a crossed 2-cube.

[These are just some thoughts to be worked on!]

14.2 Cat^n-groups

As with cat^2-groups (see 8.3), we will give three definitions.

Definition 14.2 A cat^n-group consists of the following.

(i) 2^n groups G_A, one for each subset A of $[n]$, the *vertices* of an n-cube.

(ii) Group homomorphisms forming $n2^{n-1}$ commuting cat^1-groups,

$$\mathcal{C}_{A,i} = (e_{A,i};\ t_{A,i},\ h_{A,i} : G_A \to G_{A\backslash\{i\}}), \quad \text{for all} \quad A \subseteq [n],\ i \in A,$$

the *edges* of the cube.

(iii) These cat^1-groups combine (in sets of 4) to form $n(n-1)2^{n-3}$ cat^2-groups $\mathcal{C}_{A,\{i,j\}}$ for all $\{i,j\} \subseteq A \subseteq [n],\ i \neq j$, the *faces* of the cube.

Note that, since the $t_{A,i}, h_{A,i}$ and $e_{A,i}$ commute, composite homomorphisms $t_{A,B}, h_{A,B} : G_A \to G_{A\backslash B}$ and $e_{A,B} : G_{A\backslash B} \to G_A$ are well defined for all $B \subseteq A \subseteq [n]$.

Secondly, we give the simplest of the three definitions, again adapted from Ellis-Steiner [32].

Definition 14.3 A cat^n-group $\mathcal{C}$ consists of 2^n groups G_A, one for each subset A of $[n]$, and $3n$ homomorphisms $t_{[n],i}, h_{[n],i} : G_{[n]} \to G_{n\backslash\{i\}},\ e_{[n],i} : G_{[n]\backslash\{i\}} \to G_{[n]}$, satisfying the following axioms for all $1 \leqslant i \leqslant n$,

(a) the $\mathcal{C}_{[n],i} = (e_{[n],i}; t_{[n],i}, h_{[n],i} : G_{[n]} \to G_{[n]\backslash\{i\}})$ are *commuting* cat^1-groups, so that:

(b) $(e_1 \circ t_1) \circ (e_2 \circ t_2) = (e_2 \circ t_2) \circ (e_1 \circ t_1), \quad (e_1 \circ h_1) \circ (e_2 \circ h_2) = (e_2 \circ h_2) \circ (e_1 \circ h_1),$

(c) $(e_1 \circ t_1) \circ (e_2 \circ h_2) = (e_2 \circ h_2) \circ (e_1 \circ t_1), \quad (e_2 \circ t_2) \circ (e_1 \circ h_1) = (e_1 \circ h_1) \circ (e_2 \circ t_2).$

Our third definition defines a cat^n-group as a "cat^1-group of cat$^{(n-1)}$-groups".

Definition 14.4 A cat^n-group $\mathcal{C}$ consists of two cat$^{(n-1)}$-groups:

- $\mathcal{A}$ with groups $G_A, A \subseteq [n-1]$, and homomorphisms $\ddot{t}_{A,i}, \ddot{h}_{A,i}, \ddot{e}_{A,i}$,
- $\mathcal{B}$ with groups $H_B, B \subseteq [n-1]$, and homomorphisms $\dot{t}_{B,i}, \dot{h}_{B,i}, \dot{e}_{B,i}$,
- and cat$^{(n-1)}$-morphisms $t, h : \mathcal{A} \to \mathcal{B}$ and $e : \mathcal{B} \to \mathcal{A}$ subject to the following conditions:

C1: $(t \circ e)$ and $(h \circ e)$ are the identity mapping on $\mathcal{B}$,
C2: $[\ker t, \ker h] = \{1_{\mathcal{A}}\}$.

15 Other types of Crossed Module

This chapter needs a lot of work.

Ellis, in [30], defined crossed modules, cat^1-groups, and associated structures in *categories of* Ω-*groups* which, as well as including ordinary groups, includes associative algebra and Lie algebras. In this section we describe the constructions in these two particular cases.

15.1 Crossed modules of associative algebras and cat^1-algebras

This material is adapted from Arvasi and Odabas [4].

15.2 Crossed modules of Lie algebras

This material is adapted from Casas and Ladra [25].

15.3 Representations of crossed modules

This material is adapted from Forrester-Barker [34].

16 2-groupoid enrichments

This chapter is experimental.

How does a group act on a set?

Let G be a group (thought of as a 1-object groupoid) and X a set (with 0 groupoid structures). Construct the (1-)groupoid Symm(X), whose arrows form the group of permutations of X. An action of G on X is then determined by a groupoid homomorphism $\theta : G \to \text{Symm}(X)$.

We now form a cat^0-group X_G (which is just a group) from G, by throwing away the single object, leaving a set with extra structure (the group multiplication). We then require the subgroupoid of Symm(X_G) whose arrows are the permutations which preserve the multiplication. This subgroupoid is of course Aut(G). The inner morphism ι, mapping $g \in G$ to conjugation by g, gives an action of G on itself. The actor crossed module of G is then $(\iota : G \to \text{Aut}(G))$, which can be thought of as a cat^1-group, or group-groupoid, or 2-groupoid.

How does a crossed module act on a groupoid?

Let $\mathcal{X}$ be a crossed module, thought of as a 2-groupoid $\mathcal{G}$ with one object, and let Γ be a groupoid. Construct the 2-groupoid Symm(Γ), with one object $\bullet$, the automorphisms of Γ as arrows, thought of as functors $\Gamma \to \Gamma$, and natural transformations between these functors as 2-cells. An action of $\mathcal{X}$ on Γ is then determined by a 2-groupoid homomorphism $\theta : \mathcal{G} \to \text{Symm}(\Gamma)$.

We now form a cat^1-group (group-groupoid) $\Gamma_{\mathcal{X}}$ from $\mathcal{X}$ by throwing away the single object, leaving a groupoid with extra structure (again group multiplication). We then require the sub-2-groupoid Act($\mathcal{X}$) of Symm($\Gamma_{\mathcal{X}}$) whoses arrows are the automorphims which preserve the cat^1-structure (and whose 2-cells preserve ???). The inner morphism $\iota : \mathcal{X} \to \text{Act}(\mathcal{X})$ then has to be determined. The actor crossd square of $\mathcal{X}$ is then $(\iota : \mathcal{X} \to \text{Act}(\mathcal{X}))$, which can be thought of as a cat^2-group, or 3-groupoid, or group-double groupoid (is this correct?), or whatever.

How does a crossed square act on a double groupoid?

Let $\mathcal{S}$ be a crossed square, thought of as a 3-groupoid, and let $\mathcal{D}$ be a double groupoid. Construct the 3-groupoid Symm($\mathcal{D}$) (or should this be a 2-double groupoid?) with one object, whose arrows are automorphisms of $\mathcal{D}$ (double functors), 2-cells are homotopies, or double natural transformations (?), and 3-cells are 2-homotopies of some sort? An action of $\mathcal{S}$ on $\mathcal{D}$ is then determined by a 3-groupoid homomorphism $\theta : \mathcal{S} \to \text{Symm}(\mathcal{D})$.

We now form a cat^2-group (group-double groupoid) $\mathcal{D}_{\mathcal{S}}$ from $\mathcal{S}$ by throwing away the single object, leaving a double groupoid with extra structure (again group multiplication). We then require the sub-3-groupoid Act($\mathcal{S}$) of Symm($\mathcal{D}_{\mathcal{S}}$) whoses arrows are the automorphims which preserve the cat^2-structure (and whose 2-cells and 3-cells preserve ???). The inner morphism $\iota : \mathcal{S} \to \text{Act}(\mathcal{S})$ then has to be determined. The actor crossd cube of $\mathcal{S}$ is then $(\iota : \mathcal{S} \to \text{Act}(\mathcal{S}))$, which can be thought of as a cat^3-group, or 4-groupoid, or group-triple groupoid (is this also correct?) and, no doubt, lots of other gadgets in the crossed menagerie!

16.1 The automorphism 2-category of a 2-group

This material is adapted from [42].

Let $\mathcal{G}$ be the “2-group associated to the crossed module $\mathcal{X} = (\partial : S \to R)$”, as described in Subsection 1.15, with 2-cells

$$\begin{tikzcd} \bullet \arrow[r, bend left, "r"] \arrow[r, bend right, "r(\partial s)"'] \arrow[r, phantom, "\Downarrow (r,s)"] & \bullet \end{tikzcd}$$

We now seek to describe $\mathcal{F} = \mathcal{F}(\mathcal{G}, \mathcal{G})$, the automorphism 2-category of $\mathcal{G}$.

objects of $\mathcal{F}$

These are strict 2-functors $\beta, \gamma, \ldots : \mathcal{G} \to \mathcal{G}$, where we write $\beta = (\mathrm{id}_\bullet, \dot\beta, \bar\beta = (\dot\beta, \ddot\beta))$, mapping $\bullet$ to itself, r to $\dot\beta r$, and (r,s) to $(\dot\beta r, \ddot\beta s)$. Of course $\dot\beta, \ddot\beta$ are automorphisms of R, S respectively. (Perhaps generalise to endomorphisms.) Since β is a 2-functor, tail and head maps are preserved, so $\partial\ddot\beta = \dot\beta\partial$.

1-arrows of $\mathcal{F}$

These have the form $\bar\chi : \beta \to \gamma$, and are 2-natural transformations which make the following assignments. To the object $\bullet$ we assign an element $q = q^\gamma_\beta = \bar\chi(\bullet) \in R$.
To $r \in R$ we assign a square 2-cell

$$\begin{tikzcd} \bullet \arrow[r, "\dot\beta r"] \arrow[dr, Rightarrow, "\bar\chi r"'] & \bullet & & \bullet \arrow[r, bend left, "q(\dot\beta r)"] \arrow[r, bend right, "(\dot\gamma r)q"'] \arrow[r, phantom, "\Downarrow \bar\chi r"] & \bullet \\ \bullet \arrow[u, "q"] \arrow[r, "\dot\gamma r"'] & \bullet \arrow[u, "q"'] & \equiv & & \end{tikzcd}$$

where $\bar\chi r = \bar\chi^\gamma_\beta r = (q(\dot\beta r), \ddot\chi r) \in R \ltimes S$, say. Since $\bar\chi r$ has head $(\dot\gamma r)q$, we have

$$(\dot\beta r)(\partial\ddot\chi r) \;=\; (\dot\gamma r)^q \quad \text{in } R, \tag{80}$$

so that $\partial\ddot\chi r$ measures the amount by which the square does not commute.

The squares $\bar\chi r$, for $r \in R$, are required to satisfy three axioms.

(i) For the identity in R we have $\bar\chi 1_R = \Downarrow (1, q)$, the vertical identity 2-cell at q.

(ii) Product in R is preserved: $\bar\chi(r_1 \sharp_0 r_2)$ corresponds to the vertical composite $\bar\chi r_1 \star_0 \bar\chi r_2$ shown in the following diagram.

$$\begin{tikzcd} \bullet \arrow[r, "\dot\beta r_1"] \arrow[dr, Rightarrow, "\bar\chi r_1"'] & \bullet \arrow[r, "\dot\beta r_2"] \arrow[dr, Rightarrow, "\bar\chi r_2"'] & \bullet & & & & & & \\ \bullet \arrow[u, "q"] \arrow[r, "\dot\gamma r_1"'] & \bullet \arrow[u, "q"'] \arrow[r, "\dot\gamma r_2"'] & \bullet \arrow[u, "q"'] & = & \bullet \arrow[r, bend left, "q(\dot\beta r_1)"] \arrow[r, bend right, "(\dot\gamma r_1)q"'] \arrow[r, phantom, "\Downarrow (q(\dot\beta r_1), \ddot\chi r_1)"] & \bullet \arrow[r, "q^{-1}"] & \bullet \arrow[r, bend left, "q(\dot\beta r_2)"] \arrow[r, bend right, "(\dot\gamma r_2)q"'] \arrow[r, phantom, "\Downarrow (q(\dot\beta r_2), \ddot\chi r_2)"] & \bullet \end{tikzcd}$$

The composite of these is

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "q(\dot\beta r_1)(\dot\beta r_2)"] \arrow[r, bend right, "(\dot\gamma r_1)(\dot\gamma r_2)q"'] \arrow[r, phantom, "\Downarrow (q(\dot\beta r_1)(\dot\beta r_2), (\ddot\chi r_1)^{\dot\beta r_2}(\ddot\chi r_2))"] & \bullet
\end{tikzcd}$$

This composite 2-cell is required to be be equal to

$$\begin{tikzcd}
\bullet \arrow[r, "\dot\beta(r_1r_2)"] \arrow[dr, phantom, "\bar\chi(r_1r_2)\ \Searrow"] & \bullet \\
\bullet \arrow[u, "q"] \arrow[r, "\dot\gamma(r_1r_2)"'] & \bullet \arrow[u, "q"']
\end{tikzcd}
\equiv
\begin{tikzcd}
\bullet \arrow[r, bend left, "q\dot\beta(r_1r_2)"] \arrow[r, bend right, "\dot\gamma(r_1r_2)q"'] \arrow[r, phantom, "\Downarrow (q\dot\beta(r_1r_2), \ddot\chi(r_1r_2))"] & \bullet
\end{tikzcd}$$

We deduce that $\ddot\chi$ is a β-derivation:

$$\ddot\chi(r_1r_2) \;=\; (\ddot\chi r_1)^{\dot\beta r_2}\,(\ddot\chi r_2).$$

(iii) If $(r,s) : r \to r(\partial s)$ then $(\bar\beta(r,s), \bar\gamma(r,s))$ forms a vertical homotopy from $\bar\chi r$ to $\bar\chi(r(\partial s))$. This means that the following composite 2-cells are equal.

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "\dot\beta r"] \arrow[dr, phantom, "\bar\chi r\ \Searrow"] & \bullet \\
\bullet \arrow[u, "q"] \arrow[r, bend left, "\dot\gamma r"] \arrow[r, bend right, "\dot\gamma(r(\partial s))"'] \arrow[r, phantom, "\Downarrow (\dot\gamma r, \ddot\gamma r)"] & \bullet \arrow[u, "q"']
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, bend left, "\dot\beta r"] \arrow[r, bend right, "\dot\beta(r(\partial s))"'] \arrow[r, phantom, "\Downarrow (\dot\beta r, \ddot\beta r)"] \arrow[dr, phantom, "\bar\chi(r(\partial s))\ \Searrow"] & \bullet \\
\bullet \arrow[u, "q"] \arrow[r, bend right, "\dot\gamma(r(\partial s))"'] & \bullet \arrow[u, "q"']
\end{tikzcd}$$

It follows that

$$\begin{aligned}
(\bar\chi r)\,\sharp_1\,((\dot\gamma r, \ddot\gamma s)\,\sharp_0\,q) &= (q\,\sharp_0\,(\dot\beta r, \ddot\beta s))\,\sharp_1\,\bar\chi(r(\partial s)),\\
(q(\dot\beta r), \ddot\chi r)\,\sharp_1\,((\dot\gamma r)q, (\ddot\gamma s)^q) &= (q(\dot\beta r), \ddot\beta s)\,\sharp_1\,(q\dot\beta(r(\partial s)), \ddot\chi(r(\partial s))),\\
(q(\dot\beta r), (\ddot\chi r)(\ddot\gamma s)^q) &= (q(\dot\beta r), (\ddot\beta s)(\ddot\chi r)^{\dot\beta\partial s}(\ddot\chi\partial s)),\\
(\ddot\chi r)(\ddot\gamma s)^q &= (\ddot\beta s)(\ddot\chi r)^{\partial\ddot\beta s}(\ddot\chi\partial s),\\
(\ddot\chi r)(\ddot\gamma s)^q &= (\ddot\chi r)(\ddot\beta s)(\ddot\chi\partial s),
\end{aligned}$$

which gives the identity (compare with equation (80))

$$(\ddot{\gamma}s)^q \;=\; (\ddot{\beta}s)(\ddot{\chi}\partial s), \tag{81}$$

which shows that $\ddot{\chi}\partial s$ measures the amount by which $(\ddot{\gamma}s)^q$ differs from $\ddot{\beta}s$.

The composite $\bar{\chi}^{\delta}_{\beta}$ of $\bar{\chi}^{\gamma}_{\beta} : \beta \to \gamma$ with $\bar{\chi}^{\delta}_{\gamma} : \gamma \to \delta$, is shown in the following diagram.

$$(\bar{\chi}^{\gamma}_{\beta}r) \star_1 (\bar{\chi}^{\delta}_{\gamma}r) \;=\; \begin{tikzcd}
\bullet \arrow[r, "\dot{\beta}r"] \arrow[dr, phantom, "\overset{\bar{\chi}^{\gamma}_{\beta}r}{\Downarrow}"] & \bullet \\
\bullet \arrow[u, "q^{\gamma}_{\beta}"] \arrow[r, "\dot{\gamma}r"] \arrow[dr, phantom, "\overset{\bar{\chi}^{\delta}_{\gamma}r}{\Downarrow}"] & \bullet \arrow[u, "q^{\gamma}_{\beta}"'] \\
\bullet \arrow[u, "q^{\delta}_{\gamma}"] \arrow[r, "\dot{\delta}r"'] & \bullet \arrow[u, "q^{\delta}_{\gamma}"']
\end{tikzcd} \;=\; (q^{\delta}_{\gamma} \sharp_0 (\bar{\chi}^{\gamma}_{\beta}r)) \sharp_1 ((\bar{\chi}^{\delta}_{\gamma}r) \sharp_0 (q^{\gamma}_{\beta})),$$

so that

$$\bar{\chi}^{\delta}_{\beta}(\bullet) \;=\; q^{\delta}_{\gamma}q^{\gamma}_{\beta} \qquad \text{and} \qquad \bar{\chi}^{\delta}_{\beta}(r) \;= \Downarrow (q^{\delta}_{\gamma}q^{\gamma}_{\beta}(\dot{\beta}r), (\chi^{\gamma}_{\beta}r)(\chi^{\delta}_{\gamma}r)^{q^{\gamma}_{\beta}}).$$

Suppose there exists $s \in S$ such that $\partial s = q$. Then, using the principal γ-derivation η_s, we may define the *principal* 2-*cell* $\bar{\eta}^{\gamma}_{\gamma}r = (q(\dot{\gamma}r), \eta_s r)$ with tail $q(\dot{\gamma}r)$ and head

$$q(\dot{\gamma}r)\partial((s^{-1})^{\dot{\gamma}r}s) \;=\; q(\dot{\gamma}r)(q^{-1})^{\dot{\gamma}r}q \;=\; (\dot{\gamma}r)q\ .$$

2-arrows of $\mathcal{F}$

To $\bar{\chi}, \bar{\chi}' \in \mathcal{F}$

17 Oddments

This chapter is a place to include experimental material.

17.1 Actor of a 2-fold crossed module

Definition 17.1 *The Whitehead group* $W_1(\mathcal{R})$ *is defined to be the group of the monoid* $\mathrm{Der}_1(\mathcal{R})$*. Elements of* $W_1(\mathcal{R})$ *are called regular derivations.*

The following result is a higher-dimensional version of Lemma 2.9.

Lemma 17.2 *The following statements are equivalent*

(a) $\chi = (\ddot{\chi}, \dot{\chi}) \in W_1(\mathcal{R})$

(b) $\sigma_\chi = (\ddot{\sigma}_\chi, \dot{\sigma}_\chi) \in \mathrm{Aut}(\ddot{\mathcal{R}}_2)$

(c) $\rho_\chi = (\ddot{\rho}_\chi, \dot{\rho}_\chi) \in \mathrm{Aut}(\dot{\mathcal{R}}_2)$

Proof: Here is an incomplete argument. Assume that σ_χ, ρ_χ are invertible. Then, by Lemma 2.9,

$$\ddot{\chi}^{-1}n \;=\; (\ddot{\sigma}_\chi^{-1}\ddot{\chi}n)^{-1} \;=\; (\ddot{\chi}\ddot{\rho}_\chi^{-1}n)^{-1} \qquad \text{and} \qquad \dot{\chi}^{-1}p \;=\; (\dot{\sigma}_\chi^{-1}\dot{\chi}p)^{-1} \;=\; (\dot{\chi}\dot{\rho}_\chi^{-1}p)^{-1},$$

and hence

$$\ddot{\partial}_2 \circ \ddot{\chi}^{-1}(n) \;=\; (\ddot{\partial}_2\ddot{\sigma}_\chi^{-1}\ddot{\chi}n)^{-1} \;=\; (\dot{\sigma}_\chi^{-1}\ddot{\partial}_2\ddot{\chi}n)^{-1} \;=\; (\dot{\sigma}_\chi^{-1}\dot{\chi}\dot{\partial}_2 n)^{-1} \;=\; \dot{\chi}^{-1} \circ \dot{\partial}_2(n).$$

So $\chi^{-1} = (\ddot{\chi}^{-1}, \dot{\chi}^{-1}) \in W_1(\mathcal{R})$. □

Definition 17.3 *The group* $\mathrm{Aut}(\mathcal{R})$ *of automorphisms of the crossed square* $\mathcal{R}$ *is*

$$\mathrm{Aut}(\mathcal{R}) \;=\; \{\alpha = (\alpha_{[2]}, \alpha_{\{1\}}, \alpha_{\{2\}}, \alpha_\emptyset)\}$$

such that $(\alpha_{[2]}, \alpha_{\{1\}})$ *is an automorphism of* $\ddot{\mathcal{R}}_2$*,* $(\alpha_{[2]}, \alpha_{\{2\}})$ *is an automorphism of* $\ddot{\mathcal{R}}_1$*,* $(\alpha_{\{2\}}, \alpha_\emptyset)$ *is an automorphism of* $\dot{\mathcal{R}}_2$*, and* $(\alpha_{\{1\}}, \alpha_\emptyset)$ *is an automorphism of* $\dot{\mathcal{R}}_1$*.*

Lemma 17.4 *If* $\chi \in \mathrm{Der}_1(\mathcal{R})$ *then* $\alpha_\chi = (\ddot{\sigma}, \dot{\sigma}, \ddot{\rho}, \dot{\rho}) \in \mathrm{Aut}(\mathcal{R})$ *where*

$$\ddot{\sigma}\ell = \ell(\ddot{\chi}\ddot{\partial}_1\ell), \quad \dot{\sigma}m = m(\dot{\chi}\dot{\partial}_1 m), \quad \ddot{\rho}n = n(\ddot{\partial}_1\ddot{\chi}n), \quad \dot{\rho}p = p(\dot{\partial}_1\dot{\chi}p).$$

Proof: This follows immediately from Lemma 17.2. □

Lemma 17.5 *There is a crossed module*

$$\mathcal{A}_1\mathcal{R} \;=\; (\dot{\Delta}_1 : W_1(\mathcal{R}) \to \mathrm{Aut}(\mathcal{R}),\ \chi \mapsto \alpha_\chi)$$

where the action of $\mathrm{Aut}(\mathcal{R})$ *on* $W_1(\mathcal{R})$ *is given by* $\chi^\alpha = \psi = (\ddot{\psi}, \dot{\psi})$ *where*

$$\ddot{\psi} : R_{\{2\}} \to R_{[2]} \;=\; \alpha_{\{2\}}^{-1} * \ddot{\chi} * \alpha_{[2]}, \quad \textit{and} \quad \dot{\psi} : R_\emptyset \to R_{\{1\}} \;=\; \alpha_\emptyset^{-1} * \dot{\chi} * \alpha_{\{1\}}.$$

Proof: We first show that the boundary map $\dot{\Delta}_1$ is a homomorphism. The proof is the same as the proof of Lemma 3.3 in the notes.

Secondly we must show that the action is well-defined.

$$\begin{aligned}
(\chi^{\alpha})^{\beta} &= ((\alpha_{\{2\}}^{-1} * \ddot{\chi} * \alpha_{[2]})^{\psi_2},\, (\alpha_{\emptyset}^{-1} * \dot{\chi} * \alpha_{\{1\}})^{\psi_2}) \\
&= (\beta_{\{2\}}^{-1} * (\alpha_{\{2\}}^{-1} * \ddot{\chi} * \alpha_{[2]}) * \beta_{[2]},\, \beta_{\emptyset}^{-1} * (\alpha_{\emptyset}^{-1} * \dot{\chi} * \alpha_{\{1\}}) * \beta_{\{1\}}) \\
&= ((\alpha_{\{2\}}\beta_{\{2\}})^{-1} * \ddot{\chi} * (\alpha_{[2]}\beta_{[2]}),\, (\alpha_{\emptyset}\beta_{\emptyset})^{-1} * \dot{\chi} * (\alpha_{\{1\}}\beta_{\{1\}})) \\
&= \chi^{\alpha*\beta}
\end{aligned}$$

Thirdly we show that $\psi = \chi^{\alpha}$ commutes with $\partial_1 = (\ddot{\partial}_1, \dot{\partial}_1)$, so that $\dot{\psi}\dot{\partial}_1 = \ddot{\partial}_1\ddot{\psi}$.

$$\begin{aligned}
(\dot{\psi}\dot{\partial}_1)(n) &= (\alpha_{\emptyset}^{-1} * \dot{\chi} * \alpha_{\{1\}})(\dot{\partial}_1 n) \\
&= (\alpha_{\{1\}}\dot{\chi}\alpha_{\emptyset}^{-1})(\dot{\partial}_1 n) \\
&= \alpha_{\{1\}}\dot{\chi}\dot{\partial}_1\alpha_{\{2\}}^{-1} n \\
&= \alpha_{\{1\}}\ddot{\partial}_1\ddot{\chi}\alpha_{\{2\}}^{-1} n \\
&= \ddot{\partial}_1\alpha_{[2]}\ddot{\chi}\alpha_{\{2\}}^{-1} n \\
&= \ddot{\partial}_1(\alpha_{\{2\}}^{-1} \star \ddot{\chi} \star \alpha_{[2]})(n) \\
&= \ddot{\partial}_1\ddot{\psi}(n)\ .
\end{aligned}$$

Now we verify the first crossed module axiom, $\dot{\Delta}_1(\chi^{\alpha}) = \alpha^{-1} * \dot{\Delta}_1\chi * \alpha$ where

$$\begin{aligned}
\alpha^{-1} * \dot{\Delta}_1\chi * \alpha &= (\alpha_{[2]}, \alpha_{\{1\}}, \alpha_{\{2\}}, \alpha_{\emptyset})^{-1} * (\ddot{\sigma}_{\chi}, \dot{\sigma}_{\chi}, \ddot{\rho}_{\chi}, \dot{\rho}_{\chi}) * (\alpha_{[2]}, \alpha_{\{1\}}, \alpha_{\{2\}}, \alpha_{\emptyset}) \\
&= (\alpha_{[2]}^{-1} * \ddot{\sigma}_{\chi} * \alpha_{[2]},\ \alpha_{\{1\}}^{-1} * \dot{\sigma}_{\chi} * \alpha_{\{1\}},\ \alpha_{\{2\}}^{-1} * \ddot{\rho}_{\chi} * \alpha_{\{1\}},\ \alpha_{\emptyset}^{-1} * \dot{\rho}_{\chi} * \alpha_{\emptyset})
\end{aligned}$$

We verify just the first and the last coordinate, since the other two are similar.

$$\begin{aligned}
\ddot{\sigma}_{\psi}\ell &= \ell(\ddot{\psi}\ddot{\partial}_1\ell) \\
&= \ell(\alpha_{\{2\}}^{-1} * \ddot{\chi} * \alpha_{[2]})(\ddot{\partial}_1\ell) \\
&= \ell(\alpha_{[2]}\ddot{\chi}\alpha_{\{2\}}^{-1}\ddot{\partial}_1\ell) \\
&= \ell(\alpha_{[2]}\ddot{\chi}\ddot{\partial}_1\alpha_{[2]}^{-1}\ell) \\
(\alpha_{[2]}^{-1} * \ddot{\sigma}_{\chi} * \alpha_{[2]})\ell &= \alpha_{[2]}\ddot{\sigma}_{\chi}(\alpha_{[2]}^{-1}\ell) \\
&= \alpha_{[2]}\{(\alpha_{[2]}^{-1}\ell)(\ddot{\chi}\ddot{\partial}_1\alpha_{[2]}^{-1}\ell)\} \\
&= \ell(\alpha_{[2]}\ddot{\chi}\ddot{\partial}_1\alpha_{[2]}^{-1}\ell)
\end{aligned}$$

$$\begin{aligned}
\dot{\rho}_{\dot{\psi}}(p) &= p(\dot{\partial}_1\dot{\psi}p) \\
&= p(\dot{\partial}_1(\alpha_{\emptyset}^{-1} * \dot{\chi} * \alpha_{\{1\}}))(p)) \\
&= p(\dot{\partial}_1(\alpha_{\{1\}}\dot{\chi}\alpha_{\emptyset}^{-1})(p)) \\
&= p(\alpha_{\{1\}}\dot{\partial}_1\dot{\chi}\alpha_{\{1\}}^{-1}p) \\
(\alpha_{\emptyset}^{-1} * \dot{\rho}_{\chi} * \alpha_{\emptyset}) &= \alpha_{\emptyset}\dot{\rho}_{\chi}(\alpha_{\emptyset}^{-1}p) \\
&= \alpha_{\emptyset}\{(\alpha_{\emptyset}^{-1}p)(\dot{\partial}_1\dot{\chi}\alpha_{\emptyset}^{-1}p)\} \\
&= p(\alpha_{\{1\}}\dot{\partial}_1\dot{\chi}\alpha_{\{1\}}^{-1}p)
\end{aligned}$$

Finally, we show that the second crossed module axiom holds:

$$\chi_1 \star \chi_2 \;=\; \chi_2 \star {\chi_1}^{\dot{\Delta}_1 \chi_2} \;=\; \chi_2 \star \psi.$$

We have $\dot{\Delta}_1 \chi_2 = (\ddot{\sigma}_2, \dot{\sigma}_2, \ddot{\rho}_2, \dot{\rho}_2)$ and $\ddot{\psi} = \ddot{\rho}_2^{-1} * \ddot{\chi}_1 * \ddot{\sigma}_2$ and $\dot{\psi} = \dot{\rho}_2^{-1} * \dot{\chi}_1 * \dot{\sigma}_2$, so

$$\begin{array}{rcccccl} (\ddot{\chi}_1 \star \ddot{\chi}_2)(n) & = & (\ddot{\chi}_2 n)(\ddot{\sigma}_2 \ddot{\chi}_1 n) & = & (\ddot{\chi}_2 n)(\ddot{\psi} n) & = & (\ddot{\chi}_2 \star \ddot{\psi})(n) \\ (\dot{\chi}_1 \star \dot{\chi}_2)(p) & = & (\dot{\chi}_2 p)(\dot{\sigma}_2 \dot{\chi}_1 p) & = & (\dot{\chi}_2 p)(\dot{\psi} p) & = & (\dot{\chi}_2 \star \dot{\psi})(p) \; . \end{array}$$

(where we have used Lemma 2.4 (c) – need an $\mathcal{R}$-version?). □

17.2 Searching for a derivation-like map

We are hoping to find a function

$$\xi \;:\; R_\emptyset \to (R_\emptyset \ltimes R_{\{1\}}) \ltimes (R_{\{2\}} \ltimes R_{[2]}), \quad p \mapsto ((p, \chi p), (\phi p, \theta p)),$$

such that $\xi(pq) = (\xi p)(\xi q)$. So we require

$$\begin{array}{rcl} ((pq, \chi(pq)), (\phi(pq), \theta(pq))) & = & ((p, \chi p), (\phi p, \theta p))((q, \chi q), (\phi q, \theta q)) \\ & = & ((p, \chi p)(q, \chi q), (\phi p, \theta p)^{(q, \chi q)}(\phi q, \theta q)) \\ & = & ((pq, (\chi p)^q \chi q), ((\phi p)^q, (\chi q \boxtimes (\phi p)^q)^{-1} (\theta p)^{q(\chi q)})(\phi q, \theta q)) \\ & = & ((pq, (\chi p)^q \chi q), ((\phi p)^q, ((\phi p)^q \boxtimes \chi q)(\theta p)^{q(\chi q)})(\phi q, \theta q)) \\ & = & ((pq, (\chi p)^q \chi q), ((\phi p)^q \phi q, ((\phi p)^q \boxtimes \chi q)^{\phi q} (\theta p)^{[q(\chi q)(\phi q)]} (\theta q))) \end{array}$$

Thus χ has to be a derivation $R_\emptyset \to R_{\{1\}}$, and ϕ has to be a derivation $R_\emptyset \to R_{\{2\}}$, while $\theta : R_\emptyset \to R_{[2]}$ must be such that $\theta(pq)$ depends upon $p, q, \chi q, \phi p, \theta p, \theta q$, and satisfies

$$\theta(pq) \;=\; ((\phi p)^q \boxtimes \chi q)^{\phi q} (\theta p)^{[q(\chi q)(\phi q)]} (\theta q).$$

[This needs checking/expanding.]

17.3 Further Oddments

No entries here so far.

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