# Notes on turbulence and mixing

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This are notes I've been writing for my self study as part of the class "Turbulence and Mixing" (MAE 214A), taught by Prof. Sutanu Sarkar in Winter 2015. The class covers essentially homogeneous three-dimensional turbulence but I'm also self-studying two-dimensional and quasi-two-dimensional turbulence.

I claim no originality to the content of these notes. In particular, I'm loosely following Sutanu's class notes, and the following books and notes

- *Turbulent flows* by Pope;

- *A first course in turbulence* by Tennekes and Lumley;

- *Turbulence* by Frisch;

- *The theory of homogeneous turbulence* by Batchelor;

- *Notes on turbulence* by McWilliams.

## 1  Introduction

### 1.1  What is turbulence?

Turbulence is certainly a field in which the maxims "To define it is to limit it" is valid. Indeed, we must avoid the temptation of formally defining turbulence. Instead, we shall collect some characteristics common to turbulent flows:

1. **Large Reynolds number**

    We will define the Reynolds number (Re), and contrast different interpretations. Here it suffices the loose scaling definition as the ration of the non-linear advective terms to viscous term. This readily suggests that turbulent flows are a consequence of non-linearities of the equations of motion. In the opposite limit, that of of low Re, the equations of motions become asymptotically linear, and one can typically make progress analytically. In contrasts, we rarely can solve for turbulent flows analytically, and studies on turbulence almost always resort to experiments both in the laboratory or on computers. An important implication of non-linearity is a remarkable sensitivity to initial and boundary conditions.

2. **Irregularity**

    Turbulent flows are spatially irregular and evolve chaotically. This characteristic is frequently observed in natural phenomena. For example, measurements of velocity the turbulent planetary boundary layer at a fixed point display irregular temporal behavior. The same is true for chimney flows that often one can see; the regular flow eventually breaks into a chaotic motion.

3. **Three-dimensional vorticity**

   Turbulent flows are typically three-dimensionally rotational. The absence of vorticity can certainly be used to characterize non-turbulent flows. For example, quasi-linear surface gravity waves in the ocean can combine to give rise to patterns that are irregular. But these flows are definitely not turbulent (they lack vorticity).

4. **Wide range of scales**

   It is really hard to find particular scales consistent with turbulent motions. Indeed, turbulent flows span a wide range of temporal and spatial scales: from the very large to tiny dissipative spatial scales; from very short to significantly long time scales.

5. **Transport and mixing**

   Turbulent flows effectively transport and mixes mass, momentum, and scalars. Transport and mixing in turbulent flows are typically orders of magnitude larger than purely molecular processes.

6. **Coherent structures**

   There is some order in the midst of disorder. For that reason, statistical theories are not always successful.

## 1.2   Methods of study

1. Statistical theory of idealized turbulence (isotropic, homogeneous).

2. Phenomenological theory of turbulence. This is a semi-empirical approach also known as one-point modeling.

3. Characterization of coherent structures.

4. Two-point statistical models (correlations and spectra).

5. Larger eddy simulations. These are computational simulations in which only the largest scales of the turbulent flow are explicitly resolved.

6. Direct numerical simulations (DNS). This is the gold standard of study of turbulence, in which computational simulations are used to resolved all scales of motions explicitly. Needless to say, DNSs can be very computationally expensive.

## 1.3   Application

Applications of turbulence studies are every where. Here are some examples

1. Plane design: Estimation of drag on a wing;

2. Mixing rates in a chemical reactor;

3. Parameterization of small-scale motions in climate models.

# 2   The Navier-Stokes Equations

We will typically use index notation since its compact and shrinks the algebra. We will eventually rewrite some terms in vector notation to facilitate physical interpretation. Throughout we use summation convention: when two repeated indices appear together, summation over that index in implied unless otherwise stated.

The momentum equation is

$$\partial_t u_i + u_j \partial_{x_j} u_i = \rho^{-1} \partial_{x_j} \sigma_{ij} - \rho^{-1} \partial_{x_3} \Psi \,, \tag{1}$$

where the stress tensor is given by

$$\sigma_{ij} = -p \delta_{ij} + 2\mu \, s_{ij} \,, \tag{2}$$

with $p$ the thermodynamic pressure, and the stain rate

$$s_{ij} = \frac{1}{2} \left( \partial_{x_j} u_i + \partial_{x_i} u_j \right) \,. \tag{3}$$

Also in (2) $\mu$ is the dynamic viscosity, and the gravity force $\Psi = \rho g x_3$. Thus, the more usual form of the NS equations is

$$\partial_t u_i + u_j \partial_{x_j} u_i = \rho^{-1} \partial_{x_j} p + \nu \partial_{x_j} \partial_{x_j} u_i \,, \tag{4}$$

where $\nu \overset{\text{def}}{=} \mu/\rho$ is the kinematic viscosity. We will deal with incompressible fluids. Thus the mass conservation reduces to the solenoidal condition

$$\partial_{x_j} u_j = 0 \,. \tag{5}$$

With (5) we note that then nonlinear advective term can be rewritten as

$$u_j \partial_{x_j} u_i = \partial_{x_j} (u_i u_j) \,. \tag{6}$$

It is convenient define the modified pressure

$$p = -p \delta_{ij} + \Psi \,. \tag{7}$$

## Reynolds decomposition

We introduce the decomposition

$$u_i = U_i + u_i' \,, \tag{8}$$

where the mean velocity is

$$U_i = \, < u_i > \overset{\text{def}}{=} \frac{1}{T} \int_0^T u_i \mathrm{d}t \,, \tag{9}$$

with the arbitrary period $T$. Defining a good average is an art. Sometimes it is better to think in terms of ensemble mean. Notice that by definition $< u_i' > = 0$

We introduce similar decompositions for the pressure. The idea is to introduce the decomposition (8) into the NS equations, and average over time. The linear terms are easy. For instance, mass conservations gives

$$\partial_{x_j} U_j = 0 \,, \qquad \text{and} \qquad \partial_{x_j} u_j = 0 \,. \tag{10}$$

The average of the nonlinear term is

$$< (U_i + u_i')(U_j + u_j') > \; = \; < U_i U_j > + < U_i u_j' > + < U_j u_i' > + < u_i' u_j' >$$
$$= U_i U_j + < u_i' u_j' > \,. \tag{11}$$

Thus, the average momentum equation is

$$\partial_t U_i + U_j \partial_{x_j} U_i = \rho^{-1} \partial_{x_j} P + \nu \partial_{x_j} \partial_{x_j} U_i - \partial_{x_j} < u_i' u_j' > \,. \tag{12}$$

The last term in represents the mean eddy transport of fluctuating momentum. It is generally nonzero because the velocity fluctuations are typically correlated. The velocity correlation is termed the Reynolds stress

$$\sigma'_{ij} \stackrel{\text{def}}{=} -\rho < u'_i u'_j > .\tag{13}$$

This decomposition reveals a problem that is in the heart of turbulence. Because of the nonlinearity of the advective term in (4) one cannot write a closed equation for average fields. Indeed, this is termed the closure problem. If one tries to write equations for $< u'_i u'_j >$, one finds that it depends on higher order terms, say $< u'_i u'_j u'_k >$.

We can form an equation for the turbulent velocity $u'_i$ by subtracting the equation for the total flow (4) from the average momentum equation (12)

$$\partial_t u'_i + u_j \partial_{x_j} u'_i = -\rho^{-1} \partial_{x_i} p' + \nu \partial_{x_j} \partial_{x_j} u'_i - u'_j \partial_{x_j} U_i + \partial_{x_j} < u'_i u'_j > .\tag{14}$$

## Eddy viscosity

To make progress, we will have to close the system. The most used method is to write the Reynolds stresses as a function of $U_i$. This is inspired by the apparent similarity between the viscous and Reynolds stresses. It is important to realize that this is a crude representation, and that $\tau'_{ij}$ depend fundamentally on the properties of the flows, whereas viscous stresses do not. The parameterize expression is

$$\tau'_{ij} = K \partial_{x_i} U_j ,\tag{15}$$

and is called eddy viscosity. The eddy viscous coefficient $K$ is typically orders of magnitude larger than the molecular viscosity $\nu$.

## Passive scalar

We can also write an equation for the passive scale $\phi$

$$\partial_t \phi + u_j \partial_{x_j} \phi = \gamma \partial^2_{x_j x_j} \phi ,\tag{16}$$

where $\gamma$ is the scaler diffusivity. Similarly to the procedure used above in the equations of motions, we can introduce a Reynolds-like decomposition, to obtain the mean passive scalar equation

$$\partial_t \Phi + U_j \partial_{x_j} \Phi = \gamma \partial^2_{x_j x_j} \Phi - \partial_{x_j} < u'_j \phi' > .\tag{17}$$

Again, assumptions have to be made to close the system.

## The role of pressure

Taking the divergence of the momentum equation (4) we obtain a Poisson problem for pressure

$$\partial_{x_j} \partial_{x_j} p = -\rho \partial_{x_j} u_i \partial_{x_i} u_j .\tag{18}$$

Hence the pressure can be diagnosed from the flow $u_i$, provided boundary conditions are prescribed.

## The vorticity equation

The vorticity is defined as the curl of the velocity

$$\omega_i = \varepsilon_{ijk} \partial_{x_j} u_k .\tag{19}$$

We can form an equation for $\omega_i$ by taking the curl of the momentum equation (4). The final result is

$$\partial_t \omega_i + u_j \partial_{x_j} \omega_i = \omega_j \partial_{x_j} u_i + \nu \partial_{x_j x_j} \omega_i .\tag{20}$$

We can form an enstrophy equation by dotting the equation above with $\omega_i$, to obtain

$$\partial_t \omega_i^2 + u_j \partial_{x_j} \omega_i^2 = 2\omega_i \omega_j \partial_{x_j} u_i + \nu \partial_{x_j x_j} \omega_i^2 - 2\nu \left( \partial_{x_j} \omega_i \right)^2 . \tag{21}$$

The last term is negative definite, and it represents the viscous dissipation of enstrophy. The second last term is the divergence of a viscous flux, which redistribute enstrophy across the domain. Because the tensor $\omega_i \omega_j$ is symmetric, we can rewrite the first term as

$$\omega_i \omega_j \partial_{x_j} u_i = \omega_i \omega_j s_{ij} , \tag{22}$$

where $s_{ij}$ is the rate of strain tensor defined in (3). Thus this term represents the generation of enstrophy through straining by the flow.

# 3 Kinetic Energy

Kinetic energy is a diagnostic of fluid flows. Thus, a descriptions of its transfers and dissipation plays a fundamental role in the study of turbulence. The kinetic energy per unit mass is defined as

$$E(x_i, t) \stackrel{\text{def}}{=} \frac{u_i u_i}{2} , \tag{23}$$

and its average can be decomposed in two parts

$$< E(x_i, t) > = \bar{E}(x_i, t) + e(x_i, t) , \tag{24}$$

where the kinetic energy of the average flow is

$$\bar{E}(x_i, t) \stackrel{\text{def}}{=} \frac{< U_i U_i >}{2} , \tag{25}$$

and the turbulence kinetic energy is

$$e(x_i, t) \stackrel{\text{def}}{=} \frac{< u_i' u_i' >}{2} . \tag{26}$$

The decomposition (24) is easily verifiable: just recall the Reynolds decomposition $u_i \stackrel{\text{def}}{=} U_i + u_i'$. Also, the turbulent kinetic energy $e(x_i, t)$ is determined by the isotropic part of the Reynolds stress tensor. The anisotropic part of that tensor also scales with $e(x_i, t)$; experiments show that $|< u, v >| \sim 0.27e$. Mathematically, the turbulent kinetic energy is a bound on the anisotropic part of the Reynolds stress tensor is

$$|< uv >| \leq e . \tag{27}$$

**The instantaneous kinetic energy equation**

We begin by noticing that for incompressible flows

$$u_i \frac{Du_i}{Dt} = \frac{DE}{Dt} = \partial_t E + \partial_{x_j} (u_j E) . \tag{28}$$

Hence, multiplying the momentum equation (4) by $u_i$, and using the definition of the rate of strain tensor (3), we obtain the instantaneous kinetic energy equation

$$\partial_t E + \partial_{x_j} (u_j E) + \partial_{x_j} T_j = -2\nu s_{ij} s_{ij} , \tag{29}$$

where the kinetic energy flux is

$$T_j \stackrel{\text{def}}{=} u_j p / \rho - 2\nu u_k s_{ik} . \tag{30}$$

Integrating the energy equation (29) over a fixed control volume $V$ with surface areas $S$ we obtain the instantaneous energy budget

$$\frac{\mathrm{d}}{\mathrm{d}t} \int E \, \mathrm{d}V + \int (u_j E + T_j)_k \, \hat{n}_k \mathrm{d}S = -2 \int \nu s_{ij} s_{ij} \, \mathrm{d}V \,, \tag{31}$$

where $\hat{n}_k$ is a unit vector normal to $S$. The surface term in (31) accounts for the surface energy flux into the volume $V$ and the work done by the pressure and viscous stress on the surface $S$. The term on the right of (31) is negative definite and it represents the kinetic energy viscous dissipation, i.e., the conversion of kinetic energy into internal energy.

Notice that

$$\partial_{x_j} \partial_{x_j} \tfrac{1}{2} u_i u_i = u_i \partial_{x_j} \partial_{x_j} u_i + \partial_{x_j} u_i \partial_{x_j} u_i \,. \tag{32}$$

Hence an alternate form for the instantaneous kinetic energy equation is

$$\partial_t E + \partial_{x_j} (u_j E) + \partial_{x_j} \tilde{T}_j = -\nu \partial_{x_j} u_i \partial_{x_j} u_i \,, \tag{33}$$

where the energy flux is

$$\tilde{T}_j = u_i p / \rho - \nu \partial_{x_j} E \,. \tag{34}$$

## The average kinetic energy equation

Taking the average of the instantaneous kinetic energy equation (29) we obtain

$$\partial_t < E > + \partial_{x_j} (U_j < E >) + \partial_{x_j} (< u_j E > + < T_j >) = -\varepsilon - \bar{\varepsilon} \,, \tag{35}$$

where $< u_j E >$ is the turbulence transport of kinetic energy, and the terms on the right of (35) are the viscous disspation of mean kinetic energy

$$\bar{\varepsilon} \stackrel{\mathrm{def}}{=} 2\nu S_{ij} S_{ij} \,, \tag{36}$$

and the viscous dissipation of turbulent kinetic energy

$$\varepsilon \stackrel{\mathrm{def}}{=} 2\nu s'_{ij} s'_{ij} \,, \tag{37}$$

with mean and turbulent rate of strain tensors

$$S_{ij} = < s_{ij} > \stackrel{\mathrm{def}}{=} \frac{1}{2} \left( \partial_{x_j} U_i + \partial_{x_i} U_j \right) \,, \tag{38}$$

and

$$s_{ij} = s_{ij} - S_{ij} \stackrel{\mathrm{def}}{=} \frac{1}{2} \left( \partial_{x_j} u_i + \partial_{x_i} u_j \right) \,. \tag{39}$$

Pope note that the dissipation associated with the mean flow scales as $\mathrm{Re}^{-1}$, and is typically negligible.

## The equation for the kinetic energy of the mean flow

We can form an equation for the kinetic energy of the mean flow by multiplying (12) by $U_i$

$$\partial_t \bar{E} + \partial_{x_j} U_j \bar{E} + \partial_{x_j} < T_j > = -\mathcal{P} - \bar{\varepsilon} \,, \tag{40}$$

where the average kinetic energy transport is

$$< T_j > = U_j < u'_i u'_j > + U_j P / \rho - 2\nu U_j S_{ij} \,. \tag{41}$$

The production term $\mathcal{P}$ is

$$\mathcal{P} \stackrel{\mathrm{def}}{=} - < u'_i u'_j > \partial_{x_j} U_i \,. \tag{42}$$

## The equation for the kinetic energy of the turbulent flow

We form an equation for the kinetic energy of the turbulent flow by multiplying (14) by $u_i'$ and averaging

$$\partial_t e + \partial_{x_j} U_j e + \partial_{x_j} T_j' = \mathcal{P} - \varepsilon \,, \tag{43}$$

where the flux of turbulent kinetic energy is

$$T_i' \overset{\text{def}}{=} <u_i'p'>/\rho - 2\nu <u_i's_{ij}'> + \tfrac{1}{2} <u_i'u_i'u_j'> \,. \tag{44}$$

Notice that the production $\mathcal{P}$ term appears in both mean and turbulent kinetic energy equations with signs changed. Thus this term represents a conversion between mean and turbulent flows. This term is associated with the interaction of the Reynolds stress with the background velocity shear. Because this term is typically positive, the kinetic energy is transferred from mean to turbulent flow. Pope note some important properties of the production term

1. Because $<u_i'u_j'>$ is symmetric, only the symmetric part of the velocity gradient tensor affects the production term

$$\mathcal{P} = - <u_i'u_j'> S_{ij}; \tag{45}$$

2. Only the anisotropic part of the Reynolds stress tensor affects the production term

$$\mathcal{P} = -a_{ij}S_{ij} \,, \tag{46}$$

$a_{ij} = <u_i'u_j'> - \frac{2e}{3}\delta_{ij};$

3. Assuming the eddy viscosity hypothesis $a_{ij} = -2\nu_T S_{ij}$ the production term has the form

$$\mathcal{P} = 2\nu_T S_{ij}S_{ij} \geq 0 \,. \tag{47}$$

Hence $\mathcal{P}$ assumes the form of the dissipation of mean kinetic energy $\bar{\varepsilon}$ with the eddy viscosity $\nu_T$ in place of the molecular viscosity $\nu$.