Notes on
Two-dimensional turbulence above topography, JFM 1976
by Bretherton and Haidvogel

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1 Phenomenology of two-dimensional turbulence

The two invariants of the two-dimensional system are (kinetic) energy

$$E = \tfrac{1}{2}\iint |\nabla\psi|^2 \mathrm{d}x\,\mathrm{d}y\,. \tag{1}$$

and enstrophy

$$Q = \tfrac{1}{2}\iint q^2 \mathrm{d}x\,\mathrm{d}y\,, \tag{2}$$

where

$$q = \nabla^2\psi + h\,. \tag{3}$$

2 A minimum enstrophy principle

Bretherton and Haidvogel start by asking: "what is the flow pattern which minimizes Q for a given E?". That is, one minimizes the functional Q given the constrain E to obtain (see Appendix A)

$$\delta Q + k_0^2\delta E = \iint \left[q\nabla^2\sigma + k_0^2(\nabla\psi\cdot\nabla\sigma)\right]\,\mathrm{d}x\,\mathrm{d}y \tag{4}$$

$$= \iint \nabla^2\left(\nabla^2\psi + h - k_0^2\psi\right)\sigma\,\mathrm{d}x\,\mathrm{d}y = 0\,, \tag{5}$$

where λ^{-2} is a Lagrange multiplier and σ is an arbitrary function. To obtain (5) from (4) we use integration by parts and harmless boundary conditions (doubly periodic or no flux), e.g.,

$$\iint \nabla\psi\cdot\nabla\sigma \mathrm{d}x\mathrm{d}y = \underbrace{\iint \nabla\cdot(\sigma\nabla\psi)\,\mathrm{d}x\,\mathrm{d}y}_{=0} - \iint \sigma\nabla^2\psi \mathrm{d}x\,\mathrm{d}y\,. \tag{6}$$

Given the arbitrariness of σ and assuming a doubly periodic domain, (5) reduces to the elliptic problem

$$\left(k_0^2 - \nabla^2\right)\psi = h\,. \tag{7}$$

In Fourier space, (7) reduces to

$$\hat{\psi} = \frac{\hat{h}}{k_0^2 + k^2 + l^2}\,. \tag{8}$$

A Algebra leading to (4)

Let

$$\psi \to \psi + \alpha\sigma\,, \tag{9}$$

so that

$$q \to q + \alpha\nabla^2\sigma\,, \tag{10}$$

where α is a small parameter and σ is an arbitrary solution for the streamfunction. We now minimize the functional

$$S = Q + k_0^2 E\,, \tag{11}$$

with respect to α. That is, we require

$$\frac{\mathrm{d}S}{\mathrm{d}\alpha} = 0\,, \text{for all } \sigma\,. \tag{12}$$

We obtain

$$\iint \left[q\nabla^2\sigma + k_0^2(\nabla\psi\cdot\nabla\sigma)\right]\,\mathrm{d}x\,\mathrm{d}y + \alpha\iint\left[(\nabla^2\sigma)^2 + |\nabla\sigma|^2\right]\mathrm{d}x\,\mathrm{d}y = 0\,. \tag{13}$$

Thus, the leading order (in α) constrain is

$$\delta S = \delta(Q + k_0^2 E) = \iint\left[q\nabla^2\sigma + k_0^2(\nabla\psi\cdot\nabla\sigma)\right]\,\mathrm{d}x\,\mathrm{d}y = 0\,. \tag{14}$$