第四章 積分

4.1 不定積分

定義 (反導函數). 給定 $F(x)$, 若 $\dfrac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$, 則稱 $F(x)$ 為 $f(x)$ 的反導函數 (antiderivative) .

性質. 若 $F(x)$, $G(x)$ 分別為 $f(x)$, $g(x)$ 的反導函數, $c \in \mathbb{R}$. 則

- $F(x) + c$ 為 $f(x)$ 的反導函數.
- $c\,F(x)$ 為 $c\,f(x)$ 的反導函數.
- $F(x) + G(x)$ 為 $f(x) + g(x)$ 的反導函數.

結論.

- $\dfrac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x) \implies \mathrm{d}F(x) = f(x)\cdot\mathrm{d}x \implies F(x) = \displaystyle\int f(x)\cdot\mathrm{d}x = \int f(x)\,\mathrm{d}x$
- $F(x)$ 為 $f(x)$ 的反導函數 $\iff$ $f(x)$ 的反導函數為 $F(x)$ $\iff$ $F(x)$ 的導函數為 $f(x)$ $\iff$ $F(x)$ (對 x) 的微分為 $f(x)$ $\iff$ $f(x)$ (對 x) 的 (不定) 積分為 $F(x)$
- $f(x)$ 的反導函數 $\equiv$ $f(x)$ (對 x) 的 (不定) 積分
- 基礎積分集: 以下 $\alpha \neq -1$, $a \neq 0$.

$f(x)$	x^α	$\dfrac{1}{x}$	e^{ax}	$\sin ax$	$\cos ax$	$\dfrac{1}{\sqrt{a^2-x^2}}$	$\dfrac{1}{a^2+x^2}$
$\displaystyle\int f(x)\,\mathrm{d}x$	$\dfrac{1}{\alpha+1}\,x^{\alpha+1}$	$\ln\lvert x\rvert$	$\dfrac{1}{a}\,e^{ax}$	$-\dfrac{1}{a}\cos ax$	$\dfrac{1}{a}\sin ax$	$\sin^{-1}\dfrac{x}{\lvert a\rvert}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$

- (Liouville) e^{-x^2}, $\dfrac{e^x}{x}$, $\dfrac{1}{\ln x}$, $\sin(x^2)$, $\cos(x^2)$, $\dfrac{\sin x}{x}$, $\dfrac{\cos x}{x}$, x^x 無 (初等函數形式之) 反導函數!

例.

1. $\displaystyle\int \sqrt{x}\,\mathrm{d}x = \frac{2}{3}\,x^{\frac{3}{2}} + c$
2. $\displaystyle\int x^\pi\,\mathrm{d}x = \frac{1}{\pi+1}\,x^{\pi+1} + c$
3. $\displaystyle\int \sin\pi x\,\mathrm{d}x = -\frac{1}{\pi}\cos\pi x + c$
4. $\displaystyle\int \cos xy\,\mathrm{d}x = \frac{1}{y}\sin xy + c$
5. $\displaystyle\int e^{-2x}\,\mathrm{d}x = -\frac{1}{2}\,e^{-2x} + c$
6. $\displaystyle\int \frac{1}{\sqrt{\pi - x^2}}\,\mathrm{d}x = \sin^{-1}\frac{x}{\sqrt{\pi}} + c$
7. $\displaystyle\int \frac{1}{e+u^2}\,\mathrm{d}u = \frac{1}{\sqrt{e}}\,\tan^{-1}\frac{u}{\sqrt{e}} + c$
8. $\displaystyle\int \left(\frac{\pi}{x} - e^{\pi x}\right)\mathrm{d}u = \pi\ln x - \frac{e^{\pi x}}{\pi} + c$
9. $\displaystyle\int \frac{3+x^2}{1+x^2}\,\mathrm{d}u = x + 2\,\tan^{-1}x + c$

習題. 求下列不定積分.

1. $\displaystyle\int \frac{x^3-1}{x^3}\,\mathrm{d}x = x + \frac{1}{2x^3} + c$
2. $\displaystyle\int 5 - \frac{1}{\sqrt{x}}\,\mathrm{d}x = 5x - 2\sqrt{x} + c$
3. $\displaystyle\int (t-1)(t+1)\,\mathrm{d}t = \frac{t^3}{3} - t + c$
4. $\displaystyle\int (\sqrt{x}+1)^2\,\mathrm{d}x = \frac{x^2}{2} + x + \frac{4x^{\frac{3}{2}}}{3} + c$
5. $\displaystyle\int x\sqrt{3x}\,\mathrm{d}x = \frac{2\sqrt{3}}{5}\,x^{\frac{5}{2}} + c$
6. $\displaystyle\int \frac{1}{x^3} - \frac{1}{x^5}\,\mathrm{d}x = \frac{-1}{2x^2} + \frac{1}{4x^4} + c$
7. $\displaystyle\int \sec^2 x - \sec x\tan x\,\mathrm{d}x = \tan x - \sec x + c$
8. $\displaystyle\int \frac{e^{3x}+1}{e^x+1}\,\mathrm{d}x = \frac{e^{2x}}{2} - e^x + x + c$

變數變換法

結論. $\dfrac{\mathrm{d}}{\mathrm{d}x}f(g(x)) = f'(g(x))\cdot g'(x) \implies \mathrm{d}f(g(x)) = f'(g(x))\cdot g'(x)\,\mathrm{d}x \implies f(g(x)) = \displaystyle\int f'(g(x))\cdot g'(x)\,\mathrm{d}x.$
令 $u = g(x)$, 則 $\dfrac{\mathrm{d}u}{\mathrm{d}x} = g'(x) \implies \mathrm{d}u = g'(x)\,\mathrm{d}x$; 故 $\displaystyle\int f'(g(x))\cdot g'(x)\,\mathrm{d}x = \int f'(u)\,\mathrm{d}u = f(u)+c = f(g(x))+c.$

例. 求 $\displaystyle\int \frac{x}{\sqrt{x+1}}\,\mathrm{d}x.$

解.

- (解一) 令 $u = x+1$, 則 $x = u-1$, $\mathrm{d}u = \mathrm{d}x$. 故 $\displaystyle\int \frac{x}{\sqrt{x+1}}\,\mathrm{d}x = \int \frac{u-1}{\sqrt{u}}\,\mathrm{d}u = \int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,\mathrm{d}u = \frac{2}{3}u^{\frac{3}{2}} - 2\sqrt{u} + c = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c.$
- (解二) $\displaystyle\int \frac{x}{\sqrt{x+1}}\,\mathrm{d}x = \int \frac{x+1-1}{\sqrt{x+1}}\,\mathrm{d}x = \int \sqrt{x+1} - \frac{1}{\sqrt{x+1}}\,\mathrm{d}x.$ 令 $u = x+1$, 則 $\mathrm{d}u = \mathrm{d}x$. 故 $\displaystyle\int \sqrt{x+1} - \frac{1}{\sqrt{x+1}}\,\mathrm{d}x = \int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,\mathrm{d}u = \frac{2}{3}u^{\frac{3}{2}} - 2\sqrt{u} + c = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c.$
- (解三) 令 $u = \sqrt{x+1}$, 則 $x = u^2-1$, $\mathrm{d}u = \dfrac{1}{2\sqrt{x+1}}\,\mathrm{d}x \implies \dfrac{1}{\sqrt{x+1}}\,\mathrm{d}x = 2\,\mathrm{d}u$. 故 $\displaystyle\int \frac{x}{\sqrt{x+1}}\,\mathrm{d}x = \int x\cdot\frac{1}{\sqrt{x+1}}\,\mathrm{d}x = \int (u^2-1)\cdot 2\,\mathrm{d}u = \frac{2}{3}u^3 - 2u + c = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c.$

例. 求 $\displaystyle\int \frac{x}{x^2+1}\,\mathrm{d}x.$

解. 令 $u = x^2+1$, 則 $\mathrm{d}u = 2\,x\,\mathrm{d}x \implies x\,\mathrm{d}x = \dfrac{1}{2}\,\mathrm{d}u$. 故 $\displaystyle\int \frac{x}{x^2+1}\,\mathrm{d}x = \int \frac{1}{x^2+1}\cdot x\,\mathrm{d}x = \int \frac{1}{u}\cdot\frac{1}{2}\,\mathrm{d}u = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln(x^2+1) + c.$

例. 求 $\displaystyle\int \frac{\sin(3\ln x)}{x}\,\mathrm{d}x.$

解. 令 $u = \ln x$, 則 $\mathrm{d}u = \dfrac{1}{x}\,\mathrm{d}x$. 故 $\displaystyle\int \frac{\sin(3\ln x)}{x}\,\mathrm{d}x = \int \sin(3\ln x)\cdot\frac{1}{x}\,\mathrm{d}x = \int \sin 3u\cdot\mathrm{d}u = -\frac{1}{3}\cos 3u + c = -\frac{1}{3}\cos(3\ln x) + c.$

例. 求 $\displaystyle\int e^x\sqrt{1+e^x}\,\mathrm{d}x.$

解. 令 $u = 1+e^x$, 則 $\mathrm{d}u = e^x\,\mathrm{d}x$. 故 $\displaystyle\int e^x\sqrt{1+e^x}\,\mathrm{d}x = \int \sqrt{1+e^x}\cdot e^x\,\mathrm{d}x = \int \sqrt{u}\cdot\mathrm{d}u = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}(1+e^x)^{\frac{3}{2}} + c$

例. 求 $\displaystyle\int \frac{e^x}{\sqrt{2-e^{2x}}}\,\mathrm{d}x.$

解. 令 $u = e^x$, 則 $\mathrm{d}u = e^x\,\mathrm{d}x$. 故 $\displaystyle\int \frac{e^x}{\sqrt{2-e^{2x}}}\,\mathrm{d}x = \int \frac{1}{\sqrt{2-u^2}}\,\mathrm{d}u = \sin^{-1}\frac{u}{\sqrt{2}} + c = \sin^{-1}\frac{e^x}{\sqrt{2}} + c.$

例. 求 $\displaystyle\int \frac{1}{\sqrt{e^{2x}-1}}\,\mathrm{d}x.$

解. $\displaystyle\int\frac{1}{\sqrt{e^{2x}-1}}\,\mathrm{d}x=\int\frac{1}{e^x\sqrt{1-e^{-2x}}}\,\mathrm{d}x=\int\frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,\mathrm{d}x$. 令 $u=e^{-x}$, 則 $\mathrm{d}u=-e^{-x}\,\mathrm{d}x \implies e^{-x}\,\mathrm{d}x=-\mathrm{d}u$; 故 $\displaystyle\int\frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,\mathrm{d}x=\int\frac{1}{\sqrt{1-e^{-2x}}}\cdot e^{-x}\,\mathrm{d}x=-\int\frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u=-\sin^{-1}u+c=-\sin^{-1}e^{-x}+c$.

例. 求 $\displaystyle\int\frac{1}{x^2+4x+5}\,\mathrm{d}x$.

解. $\displaystyle\int\frac{1}{x^2+4x+5}\,\mathrm{d}x=\int\frac{1}{(x+2)^2+1}\,\mathrm{d}x$. 令 $u=x+2$, 則 $\mathrm{d}u=\mathrm{d}x$; 故 $\displaystyle\int\frac{1}{(x+2)^2+1}\,\mathrm{d}x=\int\frac{1}{u^2+1}\,\mathrm{d}u=\tan^{-1}u+c=\tan^{-1}(x+2)+c$.

例. 求 $\displaystyle\int\frac{1}{\sqrt{4+2x-x^2}}\,\mathrm{d}x$

解. $\displaystyle\int\frac{1}{\sqrt{4+2x-x^2}}\,\mathrm{d}x=\int\frac{1}{\sqrt{5-(x-1)^2}}\,\mathrm{d}x$. 令 $u=x-1$, 則 $\mathrm{d}u=\mathrm{d}x$; 故 $\displaystyle\int\frac{1}{\sqrt{5-(x-1)^2}}\,\mathrm{d}x=\int\frac{1}{\sqrt{5-u^2}}\,\mathrm{d}u=\sin^{-1}\frac{u}{\sqrt5}+c=\sin^{-1}\frac{x-1}{\sqrt5}+c$.

例. 求 $\displaystyle\int\tan x\,\mathrm{d}x$.

解. $\displaystyle\int\tan x\,\mathrm{d}x=\int\frac{\sin x}{\cos x}\,\mathrm{d}x$. 令 $u=\cos x$, 則 $\mathrm{d}u=-\sin x\,\mathrm{d}x$; 故 $\displaystyle\int\frac{\sin x}{\cos x}\,\mathrm{d}x=\int\frac{-1}{u}\,\mathrm{d}u=-\ln|u|+c=-\ln|\cos x|+c=\ln|\sec x|+c$

例. 求 $\displaystyle\int\cos^5 ax\,\mathrm{d}x$, $a\neq0$.

解. $\displaystyle\int\cos^5 ax\,\mathrm{d}x=\int(1-\sin^2 ax)^2\cos ax\,\mathrm{d}x$. 令 $u=\sin ax$, 則 $\mathrm{d}u=a\cos ax\,\mathrm{d}x$; 故 $\displaystyle\int(1-\sin^2 ax)^2\cos ax\,\mathrm{d}x=\frac1a\int(1-u^2)^2\,\mathrm{d}u=\frac1a\int(1-2u^2+u^4)\,\mathrm{d}u=\frac1a\left(u-\frac23u^3+\frac15u^5\right)+c=\frac1a\left(\sin ax-\frac23\sin^3 ax+\frac15\sin^5 ax\right)+c$.

例. 求 $\displaystyle\int\cos^4 ax\,\mathrm{d}x$, $a\neq0$.

解. $\displaystyle\int\cos^4 ax\,\mathrm{d}x=\int\left(\frac{1+\cos 2ax}{2}\right)^2\mathrm{d}x=\frac14\int\left(1+2\cos 2ax+\cos^2 2ax\right)\mathrm{d}x=\frac14\int\left(1+2\cos 2ax+\frac{1+\cos 4ax}{2}\right)\mathrm{d}x$ $\displaystyle=\int\left(\frac38+\frac12\cos 2ax+\frac18\cos 4ax\right)\mathrm{d}x=\frac38x+\frac1{4a}\sin 2ax+\frac1{32a}\sin 4ax+c$

例. 求 $\displaystyle\int\sec x\,\mathrm{d}x$.

解. $\displaystyle\int\sec x\,\mathrm{d}x=\int\frac{\sec x\cdot(\sec x+\tan x)}{\sec x+\tan x}\,\mathrm{d}x$. 令 $u=\sec x+\tan x$, 則 $\mathrm{d}u=(\sec^2 x+\sec x\tan x)\,\mathrm{d}x$; 故 $\displaystyle\int\frac{\sec x\cdot(\sec x+\tan x)}{\sec x+\tan x}\,\mathrm{d}x=\int\frac1u\,\mathrm{d}u=\ln|u|+c=\ln|\sec x+\tan x|+c$

例. 令 $\displaystyle T_1=\int\frac{\sin x}{a\cos x+b\sin x}\,\mathrm{d}x$, $\displaystyle T_2=\int\frac{\cos x}{a\cos x+b\sin x}\,\mathrm{d}x$, a, $b\neq0$, 求 T_1, T_2.

解.

(a) $\displaystyle bT_1+aT_2=\int\frac{b\sin x}{a\cos x+b\sin x}\,\mathrm{d}x+\int\frac{a\cos x}{a\cos x+b\sin x}\,\mathrm{d}x=\int\frac{b\sin x+a\cos x}{a\cos x+b\sin x}\,\mathrm{d}x=\int 1\,\mathrm{d}x=x$.

(b) $-aT_1+bT_2=\displaystyle\int\frac{-a\sin x}{a\cos x+b\sin x}\,\mathrm{d}x+\int\frac{b\cos x}{a\cos x+b\sin x}\,\mathrm{d}x=\int\frac{-a\sin x+b\cos x}{a\cos x+b\sin x}\,\mathrm{d}x=\int\frac{\mathrm{d}u}{u}=\ln u=$ $\ln|a\cos x+b\sin x|$ (令 $u=a\cos x+b\sin x$, 則 $\mathrm{d}u=(-a\sin x+b\cos x)\,\mathrm{d}x$) .

解 T_1, T_2 方程式 (a), (b) 得 $T_1=\dfrac{bx-a\ln|a\cos x+b\sin x|}{a^2+b^2}$, $T_2=\dfrac{ax+b\ln|a\cos x+b\sin x|}{a^2+b^2}$.

習題. 以變數變換法求下列不定積分. 注意: 可能會因為常數項而跟此處答案不同.

1. $\displaystyle\int\frac{1}{\sqrt{2x-1}}\,\mathrm{d}x$
2. $\displaystyle\int\sqrt{7x+4}\,\mathrm{d}x$
3. $\displaystyle\int\sin(3x-1)\,\mathrm{d}x$
4. $\displaystyle\int e^{\pi x-1}\,\mathrm{d}x$
5. $\displaystyle\int(x^2-2x+1)^{\frac{1}{3}}\,\mathrm{d}x$
6. $\displaystyle\int\sin 3x\cos 3x\,\mathrm{d}x$
7. $\displaystyle\int\frac{\cos 3x}{\sin^2 3x}\,\mathrm{d}x$
8. $\displaystyle\int\frac{x}{\sqrt{1+2x^2}}\,\mathrm{d}x$
9. $\displaystyle\int x^2\sqrt{1-x}\,\mathrm{d}x$

解.

1. 令 $u=2x-1$, 則 $\mathrm{d}u=2\,\mathrm{d}x \implies \mathrm{d}x=\frac{1}{2}\mathrm{d}u$. 故 $\displaystyle\int\frac{1}{\sqrt{2x-1}}\,\mathrm{d}x=\int\frac{1}{\sqrt{u}}\cdot\frac{1}{2}\,\mathrm{d}u=\sqrt{u}+c=\sqrt{2x-1}+c.$
2. 令 $u=7x+4$, 則 $\mathrm{d}u=7\,\mathrm{d}x \implies \mathrm{d}x=\frac{1}{7}\mathrm{d}u$. 故 $\displaystyle\int\sqrt{7x+4}\,\mathrm{d}x=\int\sqrt{u}\cdot\frac{1}{7}\,\mathrm{d}u=\frac{1}{7}\cdot\frac{2}{3}u^{\frac{3}{2}}+c=\frac{2}{21}(7x+4)^{\frac{3}{2}}+c.$
3. 令 $u=3x-1$, 則 $\mathrm{d}u=3\,\mathrm{d}x \implies \mathrm{d}x=\frac{1}{3}\mathrm{d}u$. 故 $\displaystyle\int\sin(3x-1)\,\mathrm{d}x=\int\sin u\cdot\frac{1}{3}\,\mathrm{d}u=-\frac{1}{3}\cos u+c=\frac{-\cos(3x-1)}{3}+c.$
4. 令 $u=\pi x-1$, 則 $\mathrm{d}u=\pi\,\mathrm{d}x \implies \mathrm{d}x=\frac{1}{\pi}\mathrm{d}u$. 故 $\displaystyle\int e^{\pi x-1}\,\mathrm{d}x=\int e^u\cdot\frac{1}{\pi}\,\mathrm{d}u=\frac{1}{\pi}e^u+c=\frac{e^{\pi x-1}}{\pi}+c.$
5. 令 $u=x-1$, 則 $\mathrm{d}u=\mathrm{d}x$. 故 $\displaystyle\int(x^2-2x+1)^{\frac{1}{3}}\,\mathrm{d}x=\int(x-1)^{\frac{2}{3}}\,\mathrm{d}x=\int u^{\frac{2}{3}}\,\mathrm{d}u=\frac{3}{5}u^{\frac{5}{3}}+c=\frac{3}{5}(x-1)^{\frac{5}{3}}+c.$
6. 令 $u=\sin 3x$, 則 $\mathrm{d}u=3\cos 3x\,\mathrm{d}x \implies \cos 3x\,\mathrm{d}x=\frac{1}{3}\mathrm{d}u$. 故 $\displaystyle\int\sin 3x\cos 3x\,\mathrm{d}x=\int u\cdot\frac{1}{3}\,\mathrm{d}u=\frac{1}{3}\cdot\frac{1}{2}u^2+c=\frac{\sin^2 3x}{6}+c.$
7. 令 $u=\sin 3x$, 則 $\mathrm{d}u=3\cos 3x\,\mathrm{d}x \implies \cos 3x\,\mathrm{d}x=\frac{1}{3}\mathrm{d}u$. 故 $\displaystyle\int\frac{\cos 3x}{\sin^2 3x}\,\mathrm{d}x=\int\frac{1}{u^2}\cdot\frac{1}{3}\,\mathrm{d}u=-\frac{1}{3}\cdot\frac{1}{u}+c=-\frac{1}{3\sin 3x}+c.$
8. 令 $u=1+2x^2$, 則 $\mathrm{d}u=4x\,\mathrm{d}x \implies x\,\mathrm{d}x=\frac{1}{4}\mathrm{d}u$. 故 $\displaystyle\int\frac{x}{\sqrt{1+2x^2}}\,\mathrm{d}x=\int\frac{1}{\sqrt{1+2x^2}}\cdot x\,\mathrm{d}x=\int\frac{1}{\sqrt{u}}\cdot\frac{1}{4}\,\mathrm{d}u=\frac{1}{4}\cdot 2u^{\frac{1}{2}}+c=\frac{\sqrt{1+2x^2}}{2}+c.$
9. 令 $u=\sqrt{1-x}$, 則 $u^2=1-x \implies x=1-u^2$, $\mathrm{d}x=-2u\,\mathrm{d}u$. 故 $\displaystyle\int x^2\sqrt{1-x}\,\mathrm{d}x=\int(1-u^2)^2\cdot u\cdot(-2)u\,\mathrm{d}u=-2\int(1-u^2)^2\cdot u^2\,\mathrm{d}u=-2\int\left(u^2-2u^4+u^6\right)\mathrm{d}u=-\frac{2u^3}{3}+\frac{4u^5}{5}-\frac{2u^7}{7}+c=-\frac{2(1-x)^{\frac{3}{2}}}{3}+\frac{4(1-x)^{\frac{5}{2}}}{5}-\frac{2(1-x)^{\frac{7}{2}}}{7}.$

習題. 以變數變換法求下列不定積分, 其中 $a\neq b\neq 0$. 注意: 可能會因為常數項而跟此處答案不同.

1. $\int e^{2x}\sin e^{2x}\,\mathrm{d}x$
2. $\int xe^{-\frac{x^2}{2}}\,\mathrm{d}x$
3. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x$
4. $\int x^2 2^{x^3+1}\,\mathrm{d}x$
5. $\int \frac{\ln x}{x}\,\mathrm{d}x$
6. $\int \frac{x+1}{\sqrt{x^2+2x+3}}\,\mathrm{d}x$
7. $\int \frac{x^2}{2+x^6}\,\mathrm{d}x$
8. $\int \frac{1}{e^x+e^{-x}}\,\mathrm{d}x$
9. $\int \frac{x+1}{\sqrt{1-x^2}}\,\mathrm{d}x$
10. $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,\mathrm{d}x$
11. $\int \sin^4 x\cos^5 x\,\mathrm{d}x$
12. $\int \sin^2 x\cos^2 x\,\mathrm{d}x$
13. $\int \sin^{-\frac{2}{3}} x\cos^3 x\,\mathrm{d}x$
14. $\int \frac{\sin^3(\ln x)\cos^3(\ln x)}{x}\,\mathrm{d}x$
15. $\int \frac{\sin^3 x}{\cos^4 x}\,\mathrm{d}x$
16. $\int \cos ax\cos bx\,\mathrm{d}x$
17. $\int \sin ax\sin bx\,\mathrm{d}x$
18. $\int \sin ax\cos bx\,\mathrm{d}x$

解.

1. 令 $u=e^{2x}$, 則 $\mathrm{d}u=2e^{2x}\,\mathrm{d}x \implies e^{2x}\,\mathrm{d}x=\frac{1}{2}\,\mathrm{d}u$. 故 $\int e^{2x}\sin e^{2x}\,\mathrm{d}x=\int \sin e^{2x}\cdot e^{2x}\,\mathrm{d}x=\int\sin u\cdot\frac{1}{2}\,\mathrm{d}u=-\frac{1}{2}\cos u+c=-\frac{1}{2}\cos e^{2x}+c$.
2. 令 $u=\frac{x^2}{2}$, 則 $\mathrm{d}u=x\,\mathrm{d}x$, 故 $\int xe^{-\frac{x^2}{2}}\,\mathrm{d}x=\int e^{-\frac{x^2}{2}}\cdot x\,\mathrm{d}x=\int e^{-u}\,\mathrm{d}u=-e^{-u}+c=-e^{-\frac{x^2}{2}}+c$
3. 令 $u=\sqrt{x}$, 則 $\mathrm{d}u=\frac{1}{2\sqrt{x}}\,\mathrm{d}x \implies \frac{1}{\sqrt{x}}\,\mathrm{d}x=2\,\mathrm{d}u$, 故 $\int\frac{\sin\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x=\int\sin\sqrt{x}\cdot\frac{1}{\sqrt{x}}\,\mathrm{d}x=\int\sin u\cdot 2\,\mathrm{d}u=-2\cos u+c=-2\cos\sqrt{x}+c$
4. 令 $u=x^3+1$, 則 $\mathrm{d}u=3x^2\,\mathrm{d}x \implies x^2\,\mathrm{d}x=\frac{1}{3}\,\mathrm{d}u$, 故 $\int x^2 2^{x^3+1}\,\mathrm{d}x=\int 2^{x^3+1}\cdot x^2\,\mathrm{d}x=\int 2^u\cdot\frac{1}{3}\,\mathrm{d}u=\frac{1}{3}\int e^{u\ln 2}\,\mathrm{d}u=\frac{1}{3\ln 2}e^{u\ln 2}+c=\frac{2^{x^3+1}}{3\ln 2}+c$
5. 令 $u=\ln x$, 則 $\mathrm{d}u=\frac{1}{x}\,\mathrm{d}x$, 故 $\int\frac{\ln x}{x}\,\mathrm{d}x=\int\ln x\cdot\frac{1}{x}\,\mathrm{d}x=\int u\,\mathrm{d}u=\frac{1}{2}u^2+c=\frac{1}{2}(\ln x)^2+c$
6. 令 $u=x^2+2x+3$, 則 $\mathrm{d}u=(2x+2)\,\mathrm{d}x \implies (x+1)\,\mathrm{d}x=\frac{1}{2}\,\mathrm{d}u$, 故 $\int\frac{x+1}{\sqrt{x^2+2x+3}}\,\mathrm{d}x=\int\frac{1}{\sqrt{x^2+2x+3}}\cdot(x+1)\,\mathrm{d}x=\int\frac{1}{\sqrt{u}}\cdot\frac{1}{2}\,\mathrm{d}u=\sqrt{u}+c=\sqrt{x^2+2x+3}+c$
7. 令 $u=x^3$, 則 $\mathrm{d}u=3x^2\,\mathrm{d}x \implies x^2\,\mathrm{d}x=\frac{1}{3}\,\mathrm{d}u$, 故 $\int\frac{x^2}{2+x^6}\,\mathrm{d}x=\int\frac{1}{2+x^6}\cdot x^2\,\mathrm{d}x=\int\frac{1}{2+u^2}\cdot\frac{1}{3}\,\mathrm{d}u=\frac{1}{3\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}}+c=\frac{1}{3\sqrt{2}}\tan^{-1}\frac{x^3}{\sqrt{2}}+c$
8. $\int\frac{1}{e^x+e^{-x}}\,\mathrm{d}x=\int\frac{e^x}{e^{2x}+1}\,\mathrm{d}x$. 令 $u=e^x$, 則 $\mathrm{d}u=e^x\,\mathrm{d}x$, 故 $\int\frac{e^x}{e^{2x}+1}\,\mathrm{d}x=\int\frac{1}{e^{2x}+1}\cdot e^x\,\mathrm{d}x=\int\frac{1}{u^2+1}\,\mathrm{d}u=\tan^{-1}u+c=\tan^{-1}e^x+c$.
9. $\int\frac{x+1}{\sqrt{1-x^2}}\,\mathrm{d}x=\int\frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x+\int\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$. 令 $u=\sqrt{1-x^2}$, 則 $\mathrm{d}u=\frac{-x}{\sqrt{1-x^2}}\,\mathrm{d}x \implies \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x=-\mathrm{d}u$, 故 $\int\frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x=\int -\mathrm{d}u=-u+c=-\sqrt{1-x^2}$, $\int\frac{x+1}{\sqrt{1-x^2}}\,\mathrm{d}x=\int\frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x+\int\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x=-\sqrt{1-x^2}+\sin^{-1}x+c$.

10. 令 $u=\tan x$, 則 $\mathrm{d}u=\sec^2 x\,\mathrm{d}x$, 故 $\displaystyle\int\frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,\mathrm{d}x=\int\frac{1}{\sqrt{1-\tan^2 x}}\cdot\sec^2 x\,\mathrm{d}x=\int\frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u=\sin^{-1}u+c=\sin^{-1}(\tan x)+c$

11. $\displaystyle\int\sin^4 x\cos^5 x\,\mathrm{d}x=\int\sin^4 x\cdot\cos^4 x\cdot\cos x\,\mathrm{d}x=\int\sin^4 x\cdot(1-\sin^2 x)^2\cdot\cos x\,\mathrm{d}x=\int\big(\sin^4 x-2\sin^6 x+\sin^8 x\big)\cdot\cos x\,\mathrm{d}x$. 令 $u=\sin x$, 則 $\mathrm{d}u=\cos x\,\mathrm{d}x$, 故 $\displaystyle\int\big(\sin^4 x-2\sin^6 x+\sin^8 x\big)\cdot\cos x\,\mathrm{d}x=\int\big(u^4-2u^6+u^8\big)\,\mathrm{d}u=\frac{u^5}{5}-\frac{2u^7}{7}+\frac{u^9}{9}+c=\frac{\sin^5 x}{5}-\frac{2\sin^7 x}{7}+\frac{\sin^9 x}{9}+c$

12. $\displaystyle\int\sin^2 x\cos^2 x\,\mathrm{d}x=\int\frac{1-\cos 2x}{2}\cdot\frac{1+\cos 2x}{2}\,\mathrm{d}x=\frac{1}{4}\int\big(1-\cos^2 2x\big)\,\mathrm{d}x=\frac{1}{4}\int\Big(1-\frac{1+\cos 4x}{2}\Big)\,\mathrm{d}x=\frac{1}{4}\int\Big(\frac{1}{2}+\frac{\cos 4x}{2}\Big)\,\mathrm{d}x=\frac{x}{8}-\frac{\sin 4x}{32}+c$

13. $\displaystyle\int\sin^{-\frac{2}{3}}x\cos^3 x\,\mathrm{d}x=\int\sin^{-\frac{2}{3}}x\cdot\cos^2 x\cdot\cos x\,\mathrm{d}x=\int\sin^{-\frac{2}{3}}x\cdot(1-\sin^2 x)\cdot\cos x\,\mathrm{d}x=\int\big(\sin^{-\frac{2}{3}}x-\sin^{\frac{4}{3}}x\big)\cdot\cos x\,\mathrm{d}x$. 令 $u=\sin x$, 則 $\mathrm{d}u=\cos x\,\mathrm{d}x$, 故 $\displaystyle\int\big(\sin^{-\frac{2}{3}}x-\sin^{\frac{4}{3}}x\big)\cdot\cos x\,\mathrm{d}x=\int\big(u^{-\frac{2}{3}}-u^{\frac{4}{3}}\big)\,\mathrm{d}u=3u^{\frac{1}{3}}-\frac{3}{7}u^{\frac{7}{3}}+c=3\sin^{\frac{1}{3}}x-\frac{3}{7}\sin^{\frac{7}{3}}x+c$

14. 令 $u=\ln x$, 則 $\mathrm{d}u=\dfrac{1}{x}\,\mathrm{d}x$, 故 $\displaystyle\int\frac{\sin^3(\ln x)\cos^3(\ln x)}{x}\,\mathrm{d}x=\int\sin^3(\ln x)\cos^3(\ln x)\cdot\frac{1}{x}\,\mathrm{d}x=\int\sin^3 u\cos^3 u\,\mathrm{d}u=\int\sin^3 u\cdot\cos^2 u\cdot\cos u\,\mathrm{d}u=\int\sin^3 u\big(1-\sin^2 u\big)\cdot\cos u\,\mathrm{d}u=\int\big(\sin^3 u-\sin^5 u\big)\cdot\cos u\,\mathrm{d}u$. 令 $w=\sin u$, 則 $\mathrm{d}w=\cos u\,\mathrm{d}u$, 故 $\displaystyle\int\big(\sin^3 u-\sin^5 u\big)\cdot\cos u\,\mathrm{d}u=\int\big(w^3-w^5\big)\,\mathrm{d}w=\frac{w^4}{4}-\frac{w^6}{6}+c=\frac{\sin^4(\ln x)}{4}-\frac{\sin^6(\ln x)}{6}+c$

15. $\displaystyle\int\frac{\sin^3 x}{\cos^4 x}\,\mathrm{d}x=\int\frac{\sin^2 x\cdot\sin x}{\cos^4 x}\,\mathrm{d}x=\int\frac{\sin^2 x}{\cos^4 x}\cdot\sin x\,\mathrm{d}x=\int\frac{1-\cos^2 x}{\cos^4 x}\cdot\sin x\,\mathrm{d}x=\int\Big(\frac{1}{\cos^4 x}-\frac{1}{\cos^2 x}\Big)\cdot\sin x\,\mathrm{d}x$. 令 $u=\cos x$, 則 $\mathrm{d}u=-\sin x\,\mathrm{d}x\implies\sin x\,\mathrm{d}x=-\mathrm{d}u$, 故 $\displaystyle\int\Big(\frac{1}{\cos^4 x}-\frac{1}{\cos^2 x}\Big)\cdot\sin x\,\mathrm{d}x=\int\Big(\frac{1}{u^4}-\frac{1}{u^2}\Big)(-)\,\mathrm{d}u=\int\Big(\frac{1}{u^2}-\frac{1}{u^4}\Big)\,\mathrm{d}u=-\frac{1}{u}+\frac{1}{3u^3}+c=-\frac{1}{\cos x}+\frac{1}{3\cos^3 x}+c$

16. 由積化和差公式 $\cos\alpha\cos\beta=\dfrac{1}{2}\big(\cos(\alpha-\beta)+\cos(\alpha+\beta)\big)$, $\displaystyle\int\cos ax\cos bx\,\mathrm{d}x=\int\frac{1}{2}\big(\cos(ax-bx)+\cos(ax+bx)\big)\,\mathrm{d}x=\int\frac{1}{2}\big(\cos(a-b)x+\cos(a+b)x\big)\,\mathrm{d}x=\frac{\sin(a-b)x}{2(a-b)}+\frac{\sin(a+b)x}{2(a+b)}+c.$

17. 由積化和差公式 $\sin\alpha\sin\beta=\dfrac{1}{2}\big(\cos(\alpha-\beta)-\cos(\alpha+\beta)\big)$, $\displaystyle\int\sin ax\sin bx\,\mathrm{d}x=\int\frac{1}{2}\big(\cos(ax-bx)-\cos(ax+bx)\big)\,\mathrm{d}x=\int\frac{1}{2}\big(\cos(a-b)x-\cos(a+b)x\big)\,\mathrm{d}x=\frac{\sin(a-b)x}{2(a-b)}-\frac{\sin(a+b)x}{2(a+b)}+c.$

18. 由積化和差公式 $\sin\alpha\cos\beta=\dfrac{1}{2}\big(\sin(\alpha-\beta)+\sin(\alpha+\beta)\big)$, $\displaystyle\int\sin ax\cos bx\,\mathrm{d}x=\int\frac{1}{2}\big(\sin(ax-bx)+\sin(ax+bx)\big)\,\mathrm{d}x=\int\frac{1}{2}\big(\sin(a-b)x+\sin(a+b)x\big)\,\mathrm{d}x=-\frac{\cos(a-b)x}{2(a-b)}-\frac{\cos(a+b)x}{2(a+b)}+c.$

部份積分法

結論. $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\big(u(x)\,v(x)\big)=u(x)\,\frac{\mathrm{d}v(x)}{\mathrm{d}x}+v(x)\,\frac{\mathrm{d}u(x)}{\mathrm{d}x}\implies u(x)\,v(x)=\int u(x)\,\mathrm{d}v(x)+\int v(x)\,\mathrm{d}u(x)$
$\displaystyle\implies\int u\,\mathrm{d}v=u\,v-\int v\,\mathrm{d}u.$

例. 求 $\displaystyle\int xe^x\,\mathrm{d}x$.

解. 令 $u=x$, 則 $\mathrm{d}u=\mathrm{d}x$. 令 $\mathrm{d}v=e^x\,\mathrm{d}x$, 則 $v=e^x$. 故 $\displaystyle\int xe^x\,\mathrm{d}x=x\cdot e^x-\int e^x\,\mathrm{d}x=xe^x-e^x+c$.

例. 求 $\displaystyle\int \ln x\,\mathrm{d}x$.

解. 令 $u=\ln x$, 則 $\mathrm{d}u=\dfrac{1}{x}\mathrm{d}x$. 令 $\mathrm{d}v=\mathrm{d}x$, 則 $v=x$. 故 $\displaystyle\int \ln x\,\mathrm{d}x=\ln x\cdot x-\int x\cdot\frac{1}{x}\,\mathrm{d}x=x\ln x-x+c$.

例. 求 $\displaystyle\int x\tan^{-1}x\,\mathrm{d}x$.

解. 令 $u=\tan^{-1}x$, 則 $\mathrm{d}u=\dfrac{1}{1+x^2}\mathrm{d}x$. 令 $\mathrm{d}v=x\,\mathrm{d}x$, 則 $v=\dfrac{x^2}{2}$. 故 $\displaystyle\int x\tan^{-1}x\,\mathrm{d}x=\tan^{-1}x\cdot\frac{x^2}{2}-\frac{1}{2}\int\frac{x^2}{1+x^2}\,\mathrm{d}x=\tan^{-1}x\cdot\frac{x^2}{2}-\frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)\mathrm{d}x=\tan^{-1}x\cdot\frac{x^2}{2}-\frac{1}{2}\big(x-\tan^{-1}x\big)+c$.

例. 求 $\displaystyle\int x^2\sin x\,\mathrm{d}x$.

解. 令 $u=x^2$, 則 $\mathrm{d}u=2x\,\mathrm{d}x$; 令 $\mathrm{d}v=\sin x\,\mathrm{d}x$, 則 $v=-\cos x$. 故 $\displaystyle\int x^2\,\sin x\,\mathrm{d}x=x^2\cdot(-\cos x)+2\int x\cos x\,\mathrm{d}x$. 令 $u=x$, 則 $\mathrm{d}u=\mathrm{d}x$; 令 $\mathrm{d}v=\cos x\,\mathrm{d}x$, 則 $v=\sin x$. 故 $\displaystyle\int x\cos x\,\mathrm{d}x=x\cdot\sin x-\int\sin x\,\mathrm{d}x=x\cdot\sin x+\cos x$. 由上, $\displaystyle\int x^2\sin x\,\mathrm{d}x=-x^2\cos x+2(x\sin x+\cos x)+c$.

例. 求 $\displaystyle\int\sin^{-1}\sqrt{1-x^2}\,\mathrm{d}x$, $x>0$.

解. 令 $u=\sin^{-1}\sqrt{1-x^2}$, 則 $\mathrm{d}u=\dfrac{1}{\sqrt{1-\big(\sqrt{1-x^2}\big)^2}}\cdot\dfrac{-x}{\sqrt{1-x^2}}\,\mathrm{d}x=\dfrac{1}{\sqrt{x^2}}\cdot\dfrac{-x}{\sqrt{1-x^2}}\,\mathrm{d}x=\dfrac{-1}{\sqrt{1-x^2}}\,\mathrm{d}x$; 令 $\mathrm{d}v=\mathrm{d}x$, 則 $v=x$. 故 $\displaystyle\int\sin^{-1}\sqrt{1-x^2}\,\mathrm{d}x=x\,\sin^{-1}\sqrt{1-x^2}+\int\frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x=x\,\sin^{-1}\sqrt{1-x^2}-\sqrt{1-x^2}$.

例. 求 $\displaystyle\int e^{ax}\cos bx\,\mathrm{d}x$ 與 $\displaystyle\int e^{ax}\sin bx\,\mathrm{d}x$, a, $b\neq 0$.

解.

(a) 考慮 $\displaystyle\int e^{ax}\cos bx\,\mathrm{d}x$: 令 $u=\cos bx$, 則 $\mathrm{d}u=-b\sin bx\,\mathrm{d}x$; 令 $\mathrm{d}v=e^{ax}\,\mathrm{d}x$, 則 $v=\dfrac{1}{a}\,e^{ax}$, 故 $\displaystyle\int e^{ax}\cos bx\,\mathrm{d}x=\cos bx\cdot\frac{1}{a}e^{ax}+\frac{b}{a}\int e^{ax}\cdot\sin bx\,\mathrm{d}x$.

(b) 考慮 $\displaystyle\int e^{ax}\sin bx\,\mathrm{d}x$: 令 $u=\sin bx$, 則 $\mathrm{d}u=b\cos bx\,\mathrm{d}x$; 令 $\mathrm{d}v=e^{ax}\,\mathrm{d}x$, 則 $v=\dfrac{1}{a}\,e^{ax}$, 故 $\displaystyle\int e^{ax}\sin bx\,\mathrm{d}x=\sin bx\cdot\frac{1}{a}\,e^{ax}-\frac{b}{a}\int e^{ax}\cdot\cos bx\,\mathrm{d}x$.

令 $\displaystyle X=\int e^{ax}\cos bx\,\mathrm{d}x$, $\displaystyle Y=\int e^{ax}\sin bx\,\mathrm{d}x$, 由 (a)(b) $X=\cos bx\cdot\dfrac{1}{a}e^{ax}+\dfrac{b}{a}Y$, $Y=\sin bx\cdot\dfrac{1}{a}e^{ax}-\dfrac{b}{a}X$. 解 X, Y 得 $X=\dfrac{e^{ax}(a\cos bx+b\sin bx)}{a^2+b^2}$, $Y=\dfrac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}$.

部份積分法：列表

結論.

1. 將要微分函數寫左邊, 積分函數寫右邊; 左邊連續微分, 右邊連續積分
2. 依序左上連右下斜線函數相乘, 最底部水平兩邊函數相乘並積分, 符號正負相間
3. 將上式所得項全部加總即為所求積分

例. 求 $\displaystyle\int (x+3)\, e^{2x}\, \mathrm{d}x.$

解.

Diff		Int
$x+3$	$+$	e^{2x}
1	$-$	$\frac{1}{2}e^{2x}$
0		$\frac{1}{4}e^{2x}$

$$\int (x+3)\, e^{2x}\, \mathrm{d}x = (x+3)\,\frac{1}{2}e^{2x} - \frac{1}{4}e^{2x}$$

例. 求 $\displaystyle\int (x^2-2x)\, e^{kx}\, \mathrm{d}x.$

解.

Diff		Int
x^2-2x	$+$	e^{kx}
$2(x-1)$	$-$	$\frac{1}{k}e^{kx}$
2	$+$	$\frac{1}{k^2}e^{kx}$
0		$\frac{1}{k^3}e^{kx}$

$$\int (x^2-2x)\, e^{kx}\, \mathrm{d}x = (x^2-2x)\,\frac{1}{k}e^{kx} - 2(x-1)\frac{1}{k^2}e^{kx} + 2\frac{1}{k^3}e^{kx}$$

例. 求 $\displaystyle\int x^4\, \sin 2x\, \mathrm{d}x.$

解.

Diff		Int
x^4	$+$	$\sin 2x$
$4x^3$	$-$	$-\frac{1}{2}\cos 2x$
$12x^2$	$+$	$-\frac{1}{4}\sin 2x$
$24x$	$-$	$\frac{1}{8}\cos 2x$
24	$+$	$\frac{1}{16}\sin 2x$
0		$-\frac{1}{32}\cos 2x$

$$\begin{aligned}&\int x^4\, \sin 2x\, \mathrm{d}x\\ &= -x^4\,\frac{1}{2}\cos 2x + 4x^3\,\frac{1}{4}\sin 2x + 12x^2\,\frac{1}{8}\cos 2x - 24x\,\frac{1}{16}\sin 2x - 24\,\frac{1}{32}\cos 2x\\ &= \left(-\frac{x^4}{2}+\frac{3x^2}{2}-\frac{3}{4}\right)\cos 2x \,+\, \left(x^3-\frac{3x}{2}\right)\sin 2x\end{aligned}$$

例. 求 $\displaystyle\int x^5\, e^{ax}\, \mathrm{d}x.$

解.

Diff		Int
x^5	$+$	e^{ax}
$5x^4$	$-$	$\frac{1}{a}e^{ax}$
$20x^3$	$+$	$\frac{1}{a^2}e^{ax}$
$60x^2$	$-$	$\frac{1}{a^3}e^{ax}$
$120x$	$+$	$\frac{1}{a^4}e^{ax}$
120	$-$	$\frac{1}{a^5}e^{ax}$
0		$\frac{1}{a^6}e^{ax}$

$$\begin{aligned}&\int x^5\, e^{ax}\, \mathrm{d}x\\ &= x^5\,\frac{1}{a}e^{ax} - 5x^4\,\frac{1}{a^2}e^{ax} + 20x^3\,\frac{1}{a^3}e^{ax} - 60x^2\,\frac{1}{a^4}e^{ax} + 120x\,\frac{1}{a^5}e^{ax} - 120\,\frac{1}{a^6}e^{ax}\\ &= \left(\frac{x^5}{a}-\frac{5x^4}{a^2}+\frac{20x^3}{a^3}-\frac{60x^2}{a^4}+\frac{120x}{a^5}-\frac{120}{a^6}\right)e^{ax}\end{aligned}$$

例. 求 $\displaystyle\int (\sin^{-1} x)^2 \,\mathrm{d}x$.

解.

Diff		Int
$(\sin^{-1} x)^2$	$+$	1
$\frac{2\sin^{-1} x}{\sqrt{1-x^2}}$	$-$	x

Diff		Int
$2\sin^{-1} x$	$+$	$\frac{x}{\sqrt{1-x^2}}$
$\frac{2}{\sqrt{1-x^2}}$	$-$	$-\sqrt{1-x^2}$

$$\begin{aligned}
\int (\sin^{-1} x)^2 \,\mathrm{d}x &= (\sin^{-1} x)^2 \cdot x - \int \frac{2\sin^{-1} x}{\sqrt{1-x^2}} \cdot x \,\mathrm{d}x \\
&= (\sin^{-1} x)^2 \cdot x - \int 2\sin^{-1} x \cdot \frac{x}{\sqrt{1-x^2}} \,\mathrm{d}x \\
&= (\sin^{-1} x)^2 \cdot x - \left(-2\sin^{-1} x \cdot \sqrt{1-x^2} + \int \frac{2}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2} \,\mathrm{d}x\right) \\
&= (\sin^{-1} x)^2 \cdot x + 2\sin^{-1} x \cdot \sqrt{1-x^2} - 2x
\end{aligned}$$

例. 求 $\displaystyle\int e^{ax} \cos bx \,\mathrm{d}x$, a, $b \neq 0$.

解.

Diff		Int
$\cos bx$	$+$	e^{ax}
$-b\sin bx$	$-$	$\frac{1}{a}e^{ax}$
$-b^2 \cos bx$	$+$	$\frac{1}{a^2}e^{ax}$

$$\int e^{ax} \cos bx \,\mathrm{d}x = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} \int e^{ax} \cos bx \,\mathrm{d}x$$

$$\Longrightarrow \int e^{ax} \cos bx \,\mathrm{d}x = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2}$$

例. 求 $\displaystyle\int e^{ax} \sin bx \,\mathrm{d}x$, a, $b \neq 0$.

解.

Diff		Int
$\sin bx$	$+$	e^{ax}
$b\cos bx$	$-$	$\frac{1}{a}e^{ax}$
$-b^2 \sin bx$	$+$	$\frac{1}{a^2}e^{ax}$

$$\int e^{ax} \sin bx \,\mathrm{d}x = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} \int e^{ax} \sin bx \,\mathrm{d}x$$

$$\Longrightarrow \int e^{ax} \sin bx \,\mathrm{d}x = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2}$$

遞迴式

例. 令 $I_n = \displaystyle\int \frac{1}{(x^2+a^2)^n} \,\mathrm{d}x$, $n \in \mathbb{N}$, 則 $I_{n+1} = \dfrac{x}{2na^2(x^2+a^2)^n} + \dfrac{2n-1}{2na^2} I_n$.

解. 令 $u = \dfrac{1}{(x^2+a^2)^n}$, 則 $\mathrm{d}u = -2n \dfrac{x}{(x^2+a^2)^{n+1}} \,\mathrm{d}x$; $\mathrm{d}v = \mathrm{d}x$, 則 $v = x$. 故 $I_n = \displaystyle\int \frac{1}{(x^2+a^2)^n} \,\mathrm{d}x = \frac{x}{(x^2+a^2)^n} + 2n \int \frac{x^2}{(x^2+a^2)^{n+1}} \,\mathrm{d}x = \frac{x}{(x^2+a^2)^n} + 2n \int \frac{(x^2+a^2-a^2)}{(x^2+a^2)^{n+1}} \,\mathrm{d}x = \frac{x}{(x^2+a^2)^n} + 2nI_n - 2na^2 I_{n+1} \Longrightarrow$ $2na^2 I_{n+1} = \dfrac{x}{(x^2+a^2)^n} + (2n-1)I_n \Longrightarrow I_{n+1} = \dfrac{x}{2na^2(x^2+a^2)^n} + \dfrac{2n-1}{2na^2} I_n$.

註 (使用例). $I_1 = \dfrac{1}{a} \tan^{-1} \dfrac{x}{a}$, $I_2 = \dfrac{1}{2a^2} I_1 + \dfrac{x}{2a^2(x^2+a^2)} = \dfrac{1}{2a^3} \tan^{-1} \dfrac{x}{a} + \dfrac{x}{2a^2(x^2+a^2)}$, $I_3 = \dfrac{3}{4a^2} I_2 + \dfrac{x}{4a^2(x^2+a^2)^2} = \dfrac{3}{4a^2} \left(\dfrac{1}{2a^3} \tan^{-1} \dfrac{x}{a} + \dfrac{x}{2a^2(x^2+a^2)}\right) + \dfrac{x}{4a^2(x^2+a^2)^2} = \dfrac{3}{8a^5} \tan^{-1} \dfrac{x}{a} + \dfrac{3x}{8a^4(x^2+a^2)} + \dfrac{x}{4a^2(x^2+a^2)^2}$.

例. 令 $J_n = \displaystyle\int \sin^n x \,\mathrm{d}x$, $n \in \mathbb{N}$, $n \geqslant 2$, 則 $J_n = \dfrac{-\sin^{n-1} x \cos x}{n} + \dfrac{n-1}{n} J_{n-2}$.

解. 令 $u=\sin^{n-1}x$, 則 $\mathrm{d}u=(n-1)\sin^{n-2}x\cos x\,\mathrm{d}x$; $\mathrm{d}v=\sin x\,\mathrm{d}x$, 則 $v=-\cos x$. 故 $J_n=\displaystyle\int\sin^n x\,\mathrm{d}x=-\sin^{n-1}x\cdot\cos x+(n-1)\int\cos x\cdot\sin^{n-2}x\cos x\,\mathrm{d}x=-\sin^{n-1}x\cos x+(n-1)\int\sin^{n-2}x\cos^2 x\,\mathrm{d}x=-\sin^{n-1}x\cos x+(n-1)\int\sin^{n-2}x\cdot(1-\sin^2 x)\,\mathrm{d}x=-\sin^{n-1}x\cos x+(n-1)\int\sin^{n-2}x\,\mathrm{d}x-(n-1)\int\sin^n x\,\mathrm{d}x=-\sin^{n-1}x\cos x+(n-1)J_{n-2}+(1-n)J_n\implies nJ_n=-\sin^{n-1}x\cos x+(n-1)J_{n-2}\implies J_n=\dfrac{-\sin^{n-1}x\cos x}{n}+\dfrac{n-1}{n}\,J_{n-2}$.

註 (使用例)**.** $J_2=\dfrac{-\sin x\cos x}{2}+\dfrac{1}{2}\,J_0=\dfrac{-\sin x\cos x}{2}+\dfrac{x}{2}$, $J_3=\dfrac{-\sin^2 x\cos x}{3}+\dfrac{2}{3}\,J_1=\dfrac{-\sin^2 x\cos x}{3}-\dfrac{2\cos x}{3}$, $J_4=\dfrac{-\sin^3 x\cos x}{4}+\dfrac{3}{4}\,J_2=\dfrac{-\sin^3 x\cos x}{4}+\dfrac{3\sin x\cos x}{8}-\dfrac{3x}{8}$.

例. 令 $K_n=\displaystyle\int\sec^{2n+1}\theta\,\mathrm{d}\theta$, $n\in\mathbb{N}$, $n\geqslant 1$, 則 $K_n=\dfrac{\sec^{2n-1}\theta\tan\theta}{2n}+\dfrac{2n-1}{2n}\,K_{n-1}$.

解. 令 $u=\sec^{2n-1}\theta$, 則 $\mathrm{d}u=(2n-1)\sec^{2n-2}\theta\cdot\sec\theta\tan\theta\,\mathrm{d}\theta=(2n-1)\sec^{2n-1}\theta\tan\theta\,\mathrm{d}\theta$; 令 $\mathrm{d}v=\sec^2\theta\,\mathrm{d}\theta$, 則 $v=\tan\theta$. 故 $K_n=\displaystyle\int\sec^{2n+1}\theta\,\mathrm{d}\theta=\sec^{2n-1}\theta\cdot\tan\theta-\int\tan\theta\cdot(2n-1)\sec^{2n-1}\theta\tan\theta\,\mathrm{d}\theta=\sec^{2n-1}\theta\tan\theta-(2n-1)\int\tan^2\theta\cdot\sec^{2n-1}\theta\,\mathrm{d}\theta=\sec^{2n-1}\theta\tan\theta-(2n-1)\int(\sec^2\theta-1)\cdot\sec^{2n-1}\theta\,\mathrm{d}\theta=\sec^{2n-1}\theta\tan\theta-(2n-1)\int\sec^{2n+1}\theta\,\mathrm{d}\theta+(2n-1)\int\sec^{2n-1}\theta\,\mathrm{d}\theta\implies K_n=\sec^{2n-1}\theta\tan\theta-(2n-1)K_n+(2n-1)K_{n-1}\implies K_n=\dfrac{\sec^{2n-1}\theta\tan\theta}{2n}+\dfrac{2n-1}{2n}K_{n-1}$.

註 (使用例)**.** $K_0=\displaystyle\int\sec\theta\,\mathrm{d}\theta=\ln|\sec\theta+\tan\theta|$, $\displaystyle\int\sec^3\theta\,\mathrm{d}\theta=K_1=\dfrac{\sec\theta\tan\theta}{2}+\dfrac{1}{2}\,K_0=\dfrac{1}{2}\big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\big)$, $\displaystyle\int\sec^5\theta\,\mathrm{d}\theta=K_2=\dfrac{\sec^3\theta\tan\theta}{4}+\dfrac{3}{4}\,K_1=\dfrac{\sec^3\theta\tan\theta}{4}+\dfrac{3}{8}\big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\big)$.

習題. 以部份積分法求下列不定積分. 注意: 可能會因為常數項而跟此處答案不同.

1. $\displaystyle\int\frac{\sin^{-1}x}{x^2}\,\mathrm{d}x$
2. $\displaystyle\int\ln(x+\sqrt{1+x^2})\,\mathrm{d}x$
3. $\displaystyle\int x^3\ln x\,\mathrm{d}x$
4. $\displaystyle\int x(\ln x)^3\,\mathrm{d}x$
5. $\displaystyle\int x^2\tan^{-1}x\,\mathrm{d}x$
6. $\displaystyle\int\frac{xe^x}{(x+1)^2}\,\mathrm{d}x$
7. $\displaystyle\int x^5e^{-x^2}\,\mathrm{d}x$
8. $\displaystyle\int xe^{\sqrt{x}}\,\mathrm{d}x$
9. $\displaystyle\int(2x^2+1)e^{x^2}\,\mathrm{d}x$
10. $\displaystyle\int\sin(\ln x)\,\mathrm{d}x$
11. $\displaystyle\int x^2\ln\frac{1-x}{1+x}\,\mathrm{d}x$
12. $\displaystyle\int\frac{\ln x}{\sqrt{1+x}}\,\mathrm{d}x$

解.

1. 令 $u=\sin^{-1}x$, 則 $\mathrm{d}u=\dfrac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$; $\mathrm{d}v=\dfrac{1}{x^2}\,\mathrm{d}x$, 則 $v=\dfrac{-1}{x}$. 故 $\displaystyle\int\frac{\sin^{-1}x}{x^2}\,\mathrm{d}x=\sin^{-1}x\cdot\frac{-1}{x}-\int\frac{-1}{x}\cdot\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x=-\frac{\sin^{-1}x}{x}+\int\frac{1}{x\sqrt{1-x^2}}\,\mathrm{d}x$. 令 $w=\sqrt{1-x^2}$, 則 $-x^2=w^2-1$, $\mathrm{d}w=\dfrac{-x}{\sqrt{1-x^2}}\,\mathrm{d}x$. 故 $\displaystyle\int\frac{1}{x\sqrt{1-x^2}}\,\mathrm{d}x=\int\frac{1}{-x^2}\cdot\frac{-x}{\sqrt{1-x^2}}\,\mathrm{d}x=\int\frac{1}{w^2-1}\,\mathrm{d}w=\frac{1}{2}\int\Big(\frac{1}{w-1}-\frac{1}{w+1}\Big)\,\mathrm{d}w=\frac{1}{2}\big(\ln|w-1|-\ln|w+1|\big)=\frac{1}{2}\ln\left|\frac{w-1}{w+1}\right|=\frac{1}{2}\ln\left|\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right|=\frac{1}{2}\ln\left|\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right|=\frac{1}{2}\ln\left|\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right|$.

$\dfrac{1-\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\Bigg| = \dfrac{1}{2}\ln\left|\dfrac{(1-\sqrt{1-x^2})^2}{x^2}\right| = \ln\left|\dfrac{1-\sqrt{1-x^2}}{x}\right|.$
以上, $\displaystyle\int \frac{\sin^{-1}x}{x^2}\,\mathrm{d}x = -\frac{\sin^{-1}x}{x} + \int \frac{1}{x\sqrt{1-x^2}}\,\mathrm{d}x = -\frac{\sin^{-1}x}{x} + \ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + c.$

2. 令 $u = \ln(x+\sqrt{1+x^2})$, 則 $\mathrm{d}u = \dfrac{1}{x+\sqrt{1+x^2}}\cdot\big(x+\sqrt{1+x^2}\big)'\,\mathrm{d}x = \dfrac{1}{x+\sqrt{1+x^2}}\cdot\Big(1+\dfrac{x}{\sqrt{1+x^2}}\Big)\,\mathrm{d}x = \dfrac{1}{\sqrt{1+x^2}}\,\mathrm{d}x$; $\mathrm{d}v = \mathrm{d}x$, 則 $v = x$. 故 $\displaystyle\int \ln(x+\sqrt{1+x^2})\,\mathrm{d}x = \ln(x+\sqrt{1+x^2})\cdot x - \int x\cdot\frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = \ln(x+\sqrt{1+x^2})\,x - \sqrt{1+x^2} + c.$

3. 令 $u = \ln x$, 則 $\mathrm{d}u = \dfrac{1}{x}\,\mathrm{d}x$; $\mathrm{d}v = x^3\,\mathrm{d}x$, 則 $v = \dfrac{x^4}{4}$. 故 $\displaystyle\int x^3\ln x\,\mathrm{d}x = \ln x\cdot\frac{x^4}{4} - \int\frac{x^4}{4}\cdot\frac{1}{x}\,\mathrm{d}x = \ln x\cdot\frac{x^4}{4} - \int\frac{x^3}{4}\,\mathrm{d}x = \ln x\cdot\frac{x^4}{4} - \frac{x^4}{16} + c.$

4. 令 $u = (\ln x)^3$, 則 $\mathrm{d}u = 3(\ln x)^2\cdot\dfrac{1}{x}\,\mathrm{d}x$; $\mathrm{d}v = x\,\mathrm{d}x$, 則 $v = \dfrac{x^2}{2}$. 故 $\displaystyle\int x(\ln x)^3\,\mathrm{d}x = (\ln x)^3\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot 3(\ln x)^2\cdot\frac{1}{x}\,\mathrm{d}x = (\ln x)^3\cdot\frac{x^2}{2} - \frac{3}{2}\int x(\ln x)^2\,\mathrm{d}x$. 令 $u = (\ln x)^2$, 則 $\mathrm{d}u = 2\,\ln x\cdot\dfrac{1}{x}\,\mathrm{d}x$; $\mathrm{d}v = x\,\mathrm{d}x$, 則 $v = \dfrac{x^2}{2}$. 故 $\displaystyle\int x(\ln x)^2\,\mathrm{d}x = (\ln x)^2\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot 2\ln x\cdot\frac{1}{x}\,\mathrm{d}x = (\ln x)^2\cdot\frac{x^2}{2} - \int x\ln x\,\mathrm{d}x$. 令 $u = \ln x$, 則 $\mathrm{d}u = \dfrac{1}{x}\,\mathrm{d}x$; $\mathrm{d}v = x\,\mathrm{d}x$, 則 $v = \dfrac{x^2}{2}$. 故 $\displaystyle\int x\ln x\,\mathrm{d}x = \ln x\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{1}{x}\,\mathrm{d}x = \ln x\cdot\frac{x^2}{2} - \frac{1}{2}\int x\,\mathrm{d}x = \ln x\cdot\frac{x^2}{2} - \frac{x^2}{4}$. 以上, $\displaystyle\int x(\ln x)^3\,\mathrm{d}x = (\ln x)^3\cdot\frac{x^2}{2} - \frac{3}{2}\Big((\ln x)^2\cdot\frac{x^2}{2} - \big(\ln x\cdot\frac{x^2}{2} - \frac{x^2}{4}\big)\Big) = \frac{x^2}{2}\big((\ln x)^3 - \frac{3(\ln x)^2}{2} + \frac{3\ln x}{2} - \frac{3}{4}\big) + c$

5. 令 $u = \tan^{-1}x$, 則 $\mathrm{d}u = \dfrac{1}{1+x^2}\,\mathrm{d}x$; $\mathrm{d}v = x^2\,\mathrm{d}x$, 則 $v = \dfrac{x^3}{3}$. 故 $\displaystyle\int x^2\tan^{-1}x\,\mathrm{d}x = \tan^{-1}x\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{1}{x^2+1}\,\mathrm{d}x$. 又 $\displaystyle\int\frac{x^3}{3}\cdot\frac{1}{x^2+1}\,\mathrm{d}x = \frac{1}{3}\int\frac{(x^3+x)-x}{x^2+1}\,\mathrm{d}x = \frac{1}{3}\int\frac{x(x^2+1)-x}{x^2+1}\,\mathrm{d}x = \frac{1}{3}\int\big(x - \frac{x}{x^2+1}\big)\,\mathrm{d}x.$ 令 $w = x^2+1$, 則 $\mathrm{d}w = 2x\,\mathrm{d}x \implies x\,\mathrm{d}x = \dfrac{1}{2}\,\mathrm{d}w$, 故 $\displaystyle\int\frac{x}{x^2+1}\,\mathrm{d}x = \int\frac{1}{x^2+1}\cdot x\,\mathrm{d}x = \int\frac{1}{w}\cdot\frac{1}{2}\,\mathrm{d}w = \frac{1}{2}\ln w + c = \frac{1}{2}\ln(x^2+1) + c$. 以上, $\displaystyle\int x^2\tan^{-1}x\,\mathrm{d}x = \tan^{-1}x\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{1}{x^2+1}\,\mathrm{d}x = \tan^{-1}x\cdot\frac{x^3}{3} - \frac{x^2}{6} + \frac{\ln(x^2+1)}{6} + c.$

6. $\displaystyle\int\frac{xe^x}{(x+1)^2}\,\mathrm{d}x = \int\frac{(x+1-1)e^x}{(x+1)^2}\,\mathrm{d}x = \int\frac{e^x}{x+1}\,\mathrm{d}x - \int\frac{e^x}{(x+1)^2}\,\mathrm{d}x.$ 令 $u = e^x$, 則 $\mathrm{d}u = e^x\,\mathrm{d}x$; $\mathrm{d}v = \dfrac{1}{(x+1)^2}\,\mathrm{d}x$, 則 $v = \dfrac{-1}{x+1}$. 故 $\displaystyle\int\frac{e^x}{(x+1)^2}\,\mathrm{d}x = e^x\cdot\frac{-1}{x+1} + \int\frac{1}{x+1}\cdot e^x\,\mathrm{d}x = \frac{-e^x}{x+1} + \int\frac{e^x}{x+1}\,\mathrm{d}x$; 原式 $\displaystyle\int\frac{xe^x}{(x+1)^2}\,\mathrm{d}x = \int\frac{e^x}{x+1}\,\mathrm{d}x - \int\frac{e^x}{(x+1)^2}\,\mathrm{d}x = \int\frac{e^x}{x+1}\,\mathrm{d}x - \Big(\frac{-e^x}{x+1} + \int\frac{e^x}{x+1}\,\mathrm{d}x\Big) = \frac{e^x}{x+1}.$

7. 令 $w = x^2$, 則 $\mathrm{d}w = 2x\,\mathrm{d}x \implies x\,\mathrm{d}x = \dfrac{1}{2}\,\mathrm{d}w$, 故 $\displaystyle\int x^5e^{-x^2}\,\mathrm{d}x = \int e^{-x^2}\cdot(x^2)^2\cdot x\,\mathrm{d}x = \int e^{-w}\cdot w^2\cdot\frac{1}{2}\,\mathrm{d}w = \frac{1}{2}\int w^2\,e^{-w}\,\mathrm{d}w = -\frac{1}{2}\,e^{-w}(w^2+2w+2) = -\frac{1}{2}\,e^{-x^2}(x^4+2x^2+2) + c.$

Diff		Int
w^2	$+$	e^{-w}
$2w$	$-$	$-e^{-w}$
2	$+$	e^{-w}
0		$-e^{-w}$

$$\int w^2 e^{-w}\,\mathrm{d}w = -w^2\,e^{-w} - 2w\,e^{-w} - 2\,e^{-w} = -e^{-w}(w^2+2w+2)$$

8. 令 $w = \sqrt{x}$, 則 $w^2 = x$, $\mathrm{d}x = 2w\,\mathrm{d}w$, 故 $\displaystyle\int xe^{\sqrt{x}}\,\mathrm{d}x = \int w^2e^w\cdot 2w\,\mathrm{d}w = 2\int w^3e^w\,\mathrm{d}w = 2\,e^w(w^3 - 3w^2+6w-6) = 2\,e^{\sqrt{x}}\,(x\sqrt{x}-3x+6\sqrt{x}-6)+c$

Diff		Int
w^3	$+$	e^w
$3w^2$	$-$	e^w
$6w$	$+$	e^w
6	$-$	e^w
0		e^w

$$\int w^3\,e^w\,\mathrm{d}w = w^3\,e^w - 3w^2\,e^w + 6w\,e^w - 6\,e^w = e^w(w^3-3w^2+6w-6)$$

9. $\displaystyle\int (2x^2+1)\,e^{x^2}\,\mathrm{d}x = \int 2x^2\,e^{x^2}\,\mathrm{d}x + \int e^{x^2}\,\mathrm{d}x$. 令 $u = x$, 則 $\mathrm{d}u = e^x\,\mathrm{d}x$; $\mathrm{d}v = 2x\,e^{x^2}\,\mathrm{d}x$, 則 $v = e^{x^2}$. 故 $\displaystyle\int 2x^2\,e^{x^2}\,\mathrm{d}x = x\cdot e^{x^2} - \int e^{x^2}\mathrm{d}x$; 原式 $\displaystyle\int 2x^2\,e^{x^2}\,\mathrm{d}x + \int e^{x^2}\,\mathrm{d}x = x\cdot e^{x^2} - \int e^{x^2}\mathrm{d}x + \int e^{x^2}\,\mathrm{d}x = xe^{x^2}$

10. 令 $w = \ln x$, 則 $e^w = x$, $\mathrm{d}x = e^w\,\mathrm{d}w$, 故 $\displaystyle\int \sin(\ln x)\,\mathrm{d}x = \int \sin w\cdot e^w\,\mathrm{d}w = \int e^w\,\sin w\,\mathrm{d}w = \frac{e^w(\sin w-\cos w)}{2} = \frac{x\,(\sin(\ln x)-\cos(\ln x))}{2}+c$

Diff		Int
$\sin w$	$+$	e^w
$\cos w$	$-$	e^w
$-\sin w$	$+$	e^w

$$\int e^w\sin w\,\mathrm{d}w = e^w\sin w - e^w\cos w - \int e^w\sin w\,\mathrm{d}w$$
$$\implies \int e^w\sin w\,\mathrm{d}w = \frac{e^w(\sin w-\cos w)}{2}$$

11. 令 $u = \ln\dfrac{1-x}{1+x}$, 則 $\mathrm{d}u = \dfrac{1+x}{1-x}\cdot\left(\dfrac{1-x}{1+x}\right)'\mathrm{d}x = \dfrac{1+x}{1-x}\cdot\dfrac{(1+x)\cdot(-1)-(1-x)}{(1+x)^2}\,\mathrm{d}x = \dfrac{2}{x^2-1}\,\mathrm{d}x$; $\mathrm{d}v = x^2\,\mathrm{d}x$, 則 $v = \dfrac{x^3}{3}$. 故 $\displaystyle\int x^2\ln\frac{1-x}{1+x}\,\mathrm{d}x = \ln\frac{1-x}{1+x}\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{2}{x^2-1}\,\mathrm{d}x$. 又 $\displaystyle\int\frac{x^3}{3}\cdot\frac{2}{x^2-1}\,\mathrm{d}x = \frac{2}{3}\int\frac{x^3}{x^2-1}\,\mathrm{d}x = \frac{2}{3}\int\left(x+\frac{x}{x^2-1}\right)\mathrm{d}x = \frac{x^2}{3}+\frac{2}{3}\int\frac{x}{x^2-1}\,\mathrm{d}x$. 令 $w = x^2-1$, 則 $\mathrm{d}w = 2x\,\mathrm{d}x \implies x\,\mathrm{d}x = \dfrac{1}{2}\,\mathrm{d}w$, 故 $\displaystyle\int\frac{x}{x^2-1}\,\mathrm{d}x = \int\frac{1}{x^2-1}\cdot x\,\mathrm{d}x = \int\frac{1}{w}\cdot\frac{1}{2}\,\mathrm{d}w = \frac{1}{2}\ln w + c = \frac{1}{2}\ln(x^2-1)+c$. 以上, $\displaystyle\int x^2\ln\frac{1-x}{1+x}\,\mathrm{d}x = \frac{1+x}{1-x}\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{1}{x^2-1}\,\mathrm{d}x = \ln\frac{1-x}{1+x}\cdot\frac{x^3}{3} - \frac{x^2+\ln(x^2-1)}{3}+c$.

12. 令 $u = \ln x$, 則 $\mathrm{d}u = \dfrac{1}{x}\,\mathrm{d}x$; $\mathrm{d}v = \dfrac{1}{\sqrt{1+x}}\,\mathrm{d}x$, 則 $v = 2\sqrt{1+x}$. 故 $\displaystyle\int\frac{\ln x}{\sqrt{1+x}}\,\mathrm{d}x = \ln x\cdot 2\sqrt{1+x} - 2\int\frac{\sqrt{1+x}}{x}\,\mathrm{d}x$. 令 $w = \sqrt{1+x}$, 則 $w^2 = 1+x \implies x = w^2-1$, $2w\,\mathrm{d}w = \mathrm{d}x$, 故 $\displaystyle\int\frac{\sqrt{1+x}}{x}\,\mathrm{d}x = \int\frac{w}{w^2-1}\cdot 2w\,\mathrm{d}w = 2\int\frac{w^2}{w^2-1}\,\mathrm{d}w = 2\int\frac{(w^2-1)+1}{w^2-1}\,\mathrm{d}w = 2w + 2\int\frac{1}{w^2-1}\,\mathrm{d}w = 2\sqrt{1+x} +$

$2\int \frac{1}{w^2-1}\,\mathrm{d}w$. 由 $\frac{1}{w^2-1} = \frac{1}{2}\left(\frac{1}{w-1} - \frac{1}{w+1}\right)$, $\int \frac{1}{w^2-1}\,\mathrm{d}w = \frac{1}{2}\int\left(\frac{1}{w-1} - \frac{1}{w+1}\right)\mathrm{d}w = \frac{1}{2}\left(\ln|w-1| - \ln|w+1|\right) = \frac{1}{2}\left(\ln|\sqrt{1+x}-1| - \ln|\sqrt{1+x}+1|\right)$. 以上, $\int \frac{\ln x}{\sqrt{1+x}}\,\mathrm{d}x = \ln x \cdot 2\sqrt{1+x} - 2\left(2\sqrt{1+x} + 2\left(\frac{1}{2}\left(\ln|\sqrt{1+x}-1| - \ln|\sqrt{1+x}+1|\right)\right)\right) = \ln x \cdot 2\sqrt{1+x} - 4\sqrt{1+x} - 2\ln|\sqrt{1+x}-1| + 2\ln|\sqrt{1+x}+1| + c$.

4.2 定積分

定積分 $\approx$ (帶符號) 面積: x 軸上方為正, 下方為負.

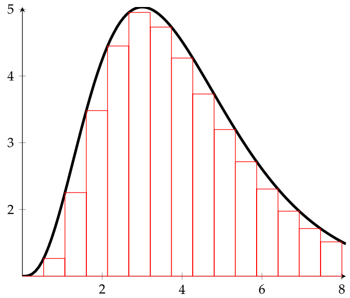
定義. 給定 $f : [a,\, b] \to \mathbb{R}$.

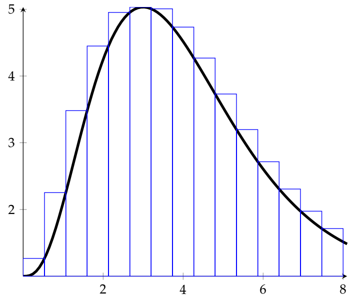
- $[a,\, b]$ 分割 $\mathbb{P} : a = x_0 < x_1 < x_2 < \cdots < x_n = b$
- $\Delta x_k = x_k - x_{k-1}$, $k = 1,\, 2,\, \ldots,\, n$; $\|\mathbb{P}\| = \max\{\,|\Delta x_k| \mid 1 \leqslant k \leqslant n\}$
- 樣本點 ξ_k: $x_{k-1} \leqslant \xi_k \leqslant x_k$, $k = 1,\, 2,\, \ldots,\, n$
- $u_k = \sup\{\, f(x) \mid x_{k-1} \leqslant x \leqslant x_k\}$, $l_k = \inf\{\, f(x) \mid x_{k-1} \leqslant x \leqslant x_k\}$, $k = 1,\, 2,\, \ldots,\, n$
- $R(f, \mathbb{P}) = \sum_{k=1}^{n} f(\xi_k)\,\Delta x_k$, $U(f, \mathbb{P}) = \sum_{k=1}^{n} u_k\,\Delta x_k$, $L(f, \mathbb{P}) = \sum_{k=1}^{n} l_k\,\Delta x_k$;
  顯然 $L(f, \mathbb{P}) \leqslant R(f, \mathbb{P}) \leqslant U(f, \mathbb{P})$.
- 求 $\lim_{\|\mathbb{P}\| \to 0} R(f, \mathbb{P})$. 若對不同分割與樣本點選取此極限均存在且相等, 稱 f 在 $[a, b]$ 可積 (分) ; $f(x)$ 在 $[a,\, b]$ 的定積分 $\int_a^b f(x)\,\mathrm{d}x \equiv \lim_{\|\mathbb{P}\| \to 0} R(f, \mathbb{P})$

註.

- 在 $\int_a^b f(x)\,\mathrm{d}x$ 中, a 為*積分下限* (lower limit of integration) , b 為*積分上限* (upper limit of integration) , $f(x)$ 為*被積分式* (integrand) , x 為*積分變數* (variable of integration) .
- $\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(t)\,\mathrm{d}t = \int_a^b f(u)\,\mathrm{d}u$ (定積分數值與積分變數無關)

結論. 若 f 在 $[a,\, b]$ 連續, 則 f 在 $[a,\, b]$ 可積.





性質. 令 f, g 在包含 a, b, c 之區間為可積, α, $\beta \in \mathbb{R}$. 則

1. $\int_a^a f(x)\,\mathrm{d}x = 0$

2. $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$

3. $\displaystyle\int_a^b \big(\alpha\, f(x) + \beta\, g(x)\big)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x + \beta \int_a^b g(x)\,\mathrm{d}x$

4. $\displaystyle\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x$

5. $\displaystyle\int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b g(x)\,\mathrm{d}x$, 若 $f(x) \leqslant g(x)\ \forall\, x \in [a,b],\ a \leqslant b$

6. $\displaystyle\left|\int_a^b f(x)\,\mathrm{d}x\right| \leqslant \int_a^b |\,f(x)\,|\ \mathrm{d}x, \quad a \leqslant b$

7. $f(x)$ 為奇函數: $\displaystyle\int_{-a}^a f(x)\,\mathrm{d}x = 0$

8. $f(x)$ 為偶函數: $\displaystyle\int_{-a}^a f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x$

例.

1. $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-x^2}\,\mathrm{d}x = \pi$ (半徑 $\sqrt{2}$ 之半圓面積)

2. 定義 $\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$, 則 $\displaystyle\int_{-1}^2 \operatorname{sgn}(x)\,\mathrm{d}x = 2\cdot 1 - 1\cdot 1 = 1$

3. $\displaystyle\int_{-\pi}^{\pi} \sin(x^7 - 5x^3)\,\mathrm{d}x = 0$ ($\sin(x^7 - 5x^3)$ 為奇函數)

4. $\displaystyle\int_{-6}^{6} e^{-x^4}\sin(\sin x)\,\mathrm{d}x = 0$ ($e^{-x^4}\sin(\sin x)$ 為奇函數)

5. $\displaystyle\int_{-4}^{4} \big(e^x - e^{-x}\big)\,\mathrm{d}x = 0$ ($e^x - e^{-x}$ 為奇函數)

6. $\displaystyle\int_{-2024}^{2024} \big(e^{9x^5-2x^7} - e^{-9x^5+2x^7}\big)\,\mathrm{d}x = 0$ ($e^{9x^5-2x^7} - e^{-9x^5+2x^7}$ 為奇函數)

7. $\displaystyle\int_{-a}^{a} |x|\,\mathrm{d}x = 2\int_0^a |x|\,\mathrm{d}x = a^2$ ($|x|$ 為偶函數; 兩 $a \times a$ 等腰直角三角形面積)

8. 定義 $\displaystyle\int_1^x \frac{1}{\tau}\,\mathrm{d}\tau = \ln x$, 則 $\displaystyle\int_{\frac{1}{4}}^3 \frac{1}{x}\,\mathrm{d}x = \int_{\frac{1}{4}}^1 \frac{1}{x}\,\mathrm{d}x + \int_1^3 \frac{1}{x}\,\mathrm{d}x = -\int_1^{\frac{1}{4}} \frac{1}{x}\,\mathrm{d}x + \int_1^3 \frac{1}{x}\,\mathrm{d}x = \ln 12$

9. $\displaystyle\int_0^2 \sqrt{4-x^2}\cdot\operatorname{sgn}(1-x)\,\mathrm{d}x = \int_0^1 \sqrt{4-x^2}\,\mathrm{d}x - \int_1^2 \sqrt{4-x^2}\,\mathrm{d}x = \left(\frac{4\pi}{12} + \frac{\sqrt{3}}{2}\right) - \left(\frac{4\pi}{6} - \frac{\sqrt{3}}{2}\right) = \sqrt{3} - \frac{\pi}{3}$

10. $\forall\, n \in \mathbb{N}$, $\displaystyle\int_n^{n+1} \lfloor x \rfloor\,\mathrm{d}x = n$

以極限定義求定積分

結論. $\displaystyle\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$

證. 由 $k(k+1)=\dfrac{k(k+1)(k+2)-(k-1)k(k+1)}{3}$, $\displaystyle\sum_{k=1}^{n}k(k+1)=\frac{n(n+1)(n+2)}{3}$. 又 $k^2=k(k+1)-k$, $\displaystyle\sum_{k=1}^{n}k^2=\sum_{k=1}^{n}\big(k(k+1)-k\big)=\frac{n(n+1)(n+2)}{3}-\frac{n(n+1)}{2}=\frac{n(n+1)(2n+1)}{6}$.

例. 求 $\displaystyle\int_0^1 x^2\,\mathrm{d}x$.

解. 建立 $[0,\,1]$ 分割 $\mathbb{P}:\Big\{0,\dfrac{1}{n},\dfrac{2}{n},\dots,\dfrac{n}{n}\Big\}$, 則 $\Delta x_k=\dfrac{1}{n}\;\forall\,k=1,2,\dots,n$ 且 $\|P\|\to 0$ 當 $n\to\infty$. $f(x)=x^2$ 且 f 在 $[0,\,1]$ 嚴格遞增, $\displaystyle U(f,\mathbb{P})=\sum_{k=1}^{n}u_k\,\Delta x_k=\sum_{k=1}^{n}\frac{k^2}{n^2}\cdot\frac{1}{n}=\frac{1}{n^3}\sum_{k=1}^{n}k^2$, $\displaystyle L(f,\mathbb{P})=\sum_{k=1}^{n}l_k\,\Delta x_k=\sum_{k=1}^{n}\frac{(k-1)^2}{n^2}\cdot\frac{1}{n}=\frac{1}{n^3}\sum_{k=1}^{n-1}k^2$, $L(f,\mathbb{P})\leqslant R(f,\mathbb{P})\leqslant U(f,\mathbb{P})$. 由 $\displaystyle\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$, $\displaystyle\lim_{n\to\infty}U(f,\mathbb{P})=\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}k^2=\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}=\frac{1}{3}$, $\displaystyle\lim_{n\to\infty}L(f,\mathbb{P})=\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n-1}k^2=\lim_{n\to\infty}\frac{(n-1)n(2n-1)}{6n^3}=\frac{1}{3}$, 故由夾擊定理 $\displaystyle\lim_{n\to\infty}R(f,\mathbb{P})=\frac{1}{3}$.

結論. 若 $r\in\mathbb{R}$, $\displaystyle\sum_{k=0}^{n}r^k=\frac{r^{n+1}-1}{r-1}$.

證. 令 $\displaystyle s=\sum_{k=0}^{n}r^k$, 則 $\displaystyle r\,s=\sum_{k=1}^{n+1}r^k$; $r\,s-s=r^{n+1}-1\implies s=\dfrac{r^{n+1}-1}{r-1}$.

例. 求 $\displaystyle\int_0^1 e^x\,\mathrm{d}x$.

解. 建立 $[0,\,1]$ 分割 $\mathbb{P}:\Big\{0,\dfrac{1}{n},\dfrac{2}{n},\dots,\dfrac{n}{n}\Big\}$, 則 $\Delta x_k=\dfrac{1}{n}\;\forall\,k=1,2,\dots,n$ 且 $\|P\|\to 0$ 當 $n\to\infty$. $f(x)=e^x$ 且 f 在 $[0,\,1]$ 嚴格遞增, $\displaystyle U(f,\mathbb{P})=\sum_{k=1}^{n}u_k\,\Delta x_k=\sum_{k=1}^{n}e^{\frac{k}{n}}\cdot\frac{1}{n}=\frac{1}{n}\frac{e^{\frac{1}{n}}\big(e^{\frac{n}{n}}-1\big)}{e^{\frac{1}{n}}-1}=\frac{e-1}{n}\frac{e^{\frac{1}{n}}}{e^{\frac{1}{n}}-1}$, $\displaystyle L(f,\mathbb{P})=\sum_{k=1}^{n}l_k\,\Delta x_k=\sum_{k=1}^{n}e^{\frac{k-1}{n}}\cdot\frac{1}{n}=\frac{1}{n}\frac{e^0\big(e^{\frac{n}{n}}-1\big)}{e^{\frac{1}{n}}-1}=\frac{e-1}{n}\frac{1}{e^{\frac{1}{n}}-1}$, $L(f,\mathbb{P})\leqslant R(f,\mathbb{P})\leqslant U(f,\mathbb{P})$. $\displaystyle\lim_{n\to\infty}U(f,\mathbb{P})=\lim_{n\to\infty}\frac{e-1}{n}\frac{e^{\frac{1}{n}}}{e^{\frac{1}{n}}-1}=(e-1)\lim_{x\to0+}\frac{xe^x}{e^x-1}=(e-1)\lim_{x\to0+}\frac{e^x+xe^x}{e^x}=e-1$, $\displaystyle\lim_{n\to\infty}L(f,\mathbb{P})=\lim_{n\to\infty}\frac{e-1}{n}\frac{1}{e^{\frac{1}{n}}-1}=(e-1)\lim_{x\to0+}\frac{x}{e^x-1}=(e-1)\lim_{x\to0+}\frac{1}{e^x}=e-1$, 故由夾擊定理 $\displaystyle\lim_{n\to\infty}R(f,\mathbb{P})=e-1$.

結論. $\displaystyle\sum_{k=1}^{n}\cos kx=\frac{1}{2}\left(\frac{\sin\big(n+\frac{1}{2}\big)x}{\sin\frac{x}{2}}-1\right)$

證. 考慮 $\displaystyle\sum_{k=1}^{n}e^{ikx}=\frac{e^{ix}(1-e^{inx})}{1-e^{ix}}$, 則 $\displaystyle\sum_{k=1}^{n}\cos kx=\Re\Big\{\sum_{k=1}^{n}e^{ikx}\Big\}=\Re\Big\{\frac{e^{ix}(1-e^{inx})}{1-e^{ix}}\Big\}=\Re\Bigg\{\frac{e^{ix}\,e^{\frac{inx}{2}}\big(e^{-\frac{inx}{2}}-e^{\frac{inx}{2}}\big)}{e^{\frac{ix}{2}}\big(e^{-\frac{ix}{2}}-e^{\frac{ix}{2}}\big)}\Bigg\}=\frac{\cos\frac{(n+1)x}{2}\,\sin\frac{nx}{2}}{\sin\frac{x}{2}}=\frac{\sin\big(\frac{nx}{2}+\frac{(n+1)x}{2}\big)+\sin\big(\frac{nx}{2}+\frac{(n+1)x}{2}\big)}{2\,\sin\frac{x}{2}}=\frac{\sin\big(n+\frac{1}{2}\big)x-\sin\frac{x}{2}}{2\,\sin\frac{x}{2}}=\frac{1}{2}\left(\frac{\sin\big(n+\frac{1}{2}\big)x}{\sin\frac{x}{2}}-1\right)$.

例. 求 $\displaystyle\int_0^{\frac{\pi}{2}}\cos x\,\mathrm{d}x$.

解. 建立 $\left[0,\,\frac{\pi}{2}\right]$ 分割 $\mathbb{P}:\left\{0,\frac{\pi}{2n},\frac{2\pi}{2n},\ldots,\frac{n\pi}{2n}\right\}$, 則 $\Delta x_k=\frac{\pi}{2n}\ \forall\, k=1,2,\ldots,n$ 且 $\|P\|\to 0$ 當 $n\to\infty$, 則積分為 $\lim\limits_{n\to\infty}R(f,\mathbb{P})=\sum\limits_{k=1}^{n}f(\xi_k)\,\Delta x_k=\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\cos\frac{k\pi}{2n}\cdot\frac{\pi}{2n}=\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\cos\left(k\left(\frac{\pi}{2n}\right)\right)\cdot\frac{\pi}{2n}=$ $\lim\limits_{n\to\infty}\frac{1}{2}\left(\frac{\sin\left(\left(n+\frac12\right)\cdot\frac{\pi}{2n}\right)}{\sin\left(\frac12\cdot\frac{\pi}{2n}\right)}-1\right)\cdot\frac{\pi}{2n}=\lim\limits_{n\to\infty}\frac{\sin\left(\left(n+\frac12\right)\frac{\pi}{2n}\right)\frac12\cdot\frac{\pi}{2n}}{\sin\left(\frac12\cdot\frac{\pi}{2n}\right)}=\lim\limits_{n\to\infty}\sin\left(\frac12+\frac{1}{4n}\right)\pi\cdot\lim\limits_{n\to\infty}\frac{\frac12\cdot\frac{\pi}{2n}}{\sin\left(\frac12\cdot\frac{\pi}{2n}\right)}=$ $\sin\frac{\pi}{2}\cdot 1=1$.

微積分基本定理

定理 (微積分基本定理 (Fundamental Theorem of Calculus, FTC))**.**

1. 若 f 在 $[a,\,b]$ 連續, 令 $F(x)=\int_a^x f(\tau)\,\mathrm{d}\tau$ 且 $a\leqslant x\leqslant b$, 則 $F'(x)=f(x)\ \forall\, x\in[a,\,b]$.
2. 若 $G'(x)=f(x)\ \forall\, x\in[a,\,b]$, 則 $\int_a^b f(x)\,\mathrm{d}x=G(b)-G(a)\equiv G(x)\Big|_a^b$.

註. 由 FTC, $\int_a^b f(x)\,\mathrm{d}x$ 可由 f 的反導函數 (不定積分) 得出, 不需繁複極限計算!

證.

1. WLOG 令 $h>0$. 則 $\left|\frac{F(x+h)-F(x)}{h}-f(x)\right|=\left|\frac{1}{h}\int_x^{x+h}(f(\tau)-f(x))\,\mathrm{d}\tau\right|\leqslant\frac{1}{h}\int_x^{x+h}\big|f(\tau)-f(x)\big|\,\mathrm{d}\tau\leqslant\max\limits_{x\leqslant\tau\leqslant x+h}\big|f(\tau)-f(x)\big|\to 0$, 當 $h\to 0$ 且 f 為連續.
2. $(G(x)-F(x))'=f(x)-f(x)=0\ \forall\, x\in[a,b]$, 故 $G(x)-F(x)$ 在 $[a,b]$ 為常數 $\implies G(a)-F(a)=G(b)-F(b)\implies G(b)-G(a)=F(b)-F(a)=\int_a^b f(x)\,\mathrm{d}x$.

例 (以 FTC 求定積分)**.**

1. x^2 之反導函數為 $\frac{x^3}{3}$, 故 $\int_0^1 x^2\,\mathrm{d}x=\frac{1^3}{3}-\frac{0^3}{3}=\frac{1}{3}$.
2. e^x 之反導函數為 e^x, 故 $\int_0^1 e^x\,\mathrm{d}x=e^1-e^0=e-1$.
3. $\cos x$ 之反導函數為 $\sin x$, 故 $\int_0^{\frac{\pi}{2}}\cos x\,\mathrm{d}x=\sin\frac{\pi}{2}-\sin 0=1$.

結論 (定積分變數變換)**.**

- 求反導函數後代入: $\int_a^b f'(g(x))\,g'(x)\,\mathrm{d}x=f(g(x))\Big|_{x=a}^{x=b}=f(g(b))-f(g(a))$
- 變數變換並改變積分範圍: $\int_a^b f'(g(x))\,g'(x)\,\mathrm{d}x=\int_a^b f'(g(x))\,\mathrm{d}g(x)=\int_{g(a)}^{g(b)}f'(u)\,\mathrm{d}u=f(u)\Big|_{u=g(a)}^{u=g(b)}=f(g(b))-f(g(a))$

例. 求 $\int_0^1 x^3(1+x^4)^3\,\mathrm{d}x$.

解.

- 求反導函數後代入: 令 $u=1+x^4$, 則 $\mathrm{d}u=4x^3\,\mathrm{d}x \implies x^3\,\mathrm{d}x=\frac{\mathrm{d}u}{4}$, 故 $\int x^3(1+x^4)^3\,\mathrm{d}x=\int(1+x^4)^3\,x^3\,\mathrm{d}x=\int u^3\,\frac{\mathrm{d}u}{4}=\frac{u^4}{16}+c=\frac{(1+x^4)^4}{16}+c$. 故 $\int_0^1 x^3(1+x^4)^3\,\mathrm{d}x=\frac{(1+x^4)^4}{16}\Big|_{x=0}^{x=1}=\frac{(1+1^4)^4-(1+0^4)^4}{16}=\frac{15}{16}$.
- 變數變換並改變積分範圍: 令 $u=1+x^4$, 則 $\mathrm{d}u=4x^3\,\mathrm{d}x \implies x^3\,\mathrm{d}x=\frac{\mathrm{d}u}{4}$. 積分範圍 x 由 0 至 1, 則變數變換後 u 由 $1+0^4=1$ 至 $1+1^4=2$, 故 $\int_0^1 x^3(1+x^4)^3\,\mathrm{d}x=\int_0^1(1+x^4)^3\,x^3\,\mathrm{d}x=\int_1^2 u^3\,\frac{\mathrm{d}u}{4}=\frac{1}{4}\int_1^2 u^3\,\mathrm{d}u=\frac{1}{16}u^4\Big|_{u=1}^{u=2}=\frac{2^4-1^4}{16}=\frac{15}{16}$.

例. 求 $\int_0^{\sqrt{3}}\frac{4x}{\sqrt{1+x^2}}\,\mathrm{d}x$.

解.

- 求反導函數後代入: 令 $u=1+x^2$, 則 $\mathrm{d}u=2x\,\mathrm{d}x \implies 4x\,\mathrm{d}x=2\,\mathrm{d}u$, 故 $\int\frac{4x}{\sqrt{1+x^2}}\,\mathrm{d}x=\int\frac{2}{\sqrt{u}}\,\mathrm{d}u=4\sqrt{u}+c=4\sqrt{1+x^2}+c$. 故 $\int_0^{\sqrt{3}}\frac{4x}{\sqrt{1+x^2}}\,\mathrm{d}x=4\sqrt{1+x^2}\Big|_{x=0}^{x=\sqrt{3}}=4\sqrt{1+3}-4\sqrt{1}=4$.
- 變數變換並改變積分範圍: 令 $u=1+x^2$, 則 $\mathrm{d}u=2x\,\mathrm{d}x \implies 4x\,\mathrm{d}x=2\,\mathrm{d}u$. 積分範圍 x 由 0 至 $\sqrt{3}$, 則變數變換後 u 由 $1+0^2=1$ 至 $1+(\sqrt{3})^2=4$, 故 $\int_0^{\sqrt{3}}\frac{4x}{\sqrt{1+x^2}}\,\mathrm{d}x=\int_1^4\frac{2}{\sqrt{u}}\,\mathrm{d}u=4\sqrt{u}\Big|_{u=1}^{u=4}=4\,(\sqrt{4}-\sqrt{1})=4$.

例. 求 $\int_0^{\pi}3\cos^2 x\sin x\,\mathrm{d}x$.

解.

- 求反導函數後代入: 令 $u=\cos x$, 則 $\mathrm{d}u=-\sin x\,\mathrm{d}x \implies \sin x\,\mathrm{d}x=-\mathrm{d}u$, 故 $\int 3\cos^2 x\sin x\,\mathrm{d}x=-3\int u^2\,\mathrm{d}u=-u^3+c=-\cos^3 x+c$. 故 $\int_0^{\pi}3\cos^2 x\sin x\,\mathrm{d}x=-\cos^3 x\Big|_{x=0}^{x=\pi}=-(\cos^3\pi-\cos^3 0)=-((-1)^3-1^3)=2$.
- 變數變換並改變積分範圍: 令 $u=\cos x$, 則 $\mathrm{d}u=-\sin x\,\mathrm{d}x \implies \sin x\,\mathrm{d}x=-\mathrm{d}u$. 積分範圍 x 由 0 至 π, 則變數變換後 u 由 $\cos 0=1$ 至 $\cos\pi=-1$, 故 $\int_0^{\pi}3\cos^2 x\sin x\,\mathrm{d}x=-\int_1^{-1}3u^2\,\mathrm{d}u=-u^3\Big|_{u=1}^{u=-1}=-((-1)^3-1^3)=2$.

例. 若 f 在 $[a,b]$ 二次可微且 $f(a)=f(b)=0$, 證明 $\int_a^b(x-a)(b-x)\,f''(x)\,\mathrm{d}x=-2\int_a^b f(x)\,\mathrm{d}x$.

解. $\int_a^b(x-a)(b-x)\,f''(x)\,\mathrm{d}x=\big((x-a)(b-x)\,f'(x)-(a+b-2x)\,f(x)\big)\Big|_a^b-2\int_a^b f(x)\,\mathrm{d}x=\big((b-a)(b-b)\,f'(b)-(a+b-2b)\,f(b)\big)-\big((a-a)(b-a)\,f'(a)-(a+b-2a)\,f(a)\big)-2\int_a^b f(x)\,\mathrm{d}x=-2\int_a^b f(x)\,\mathrm{d}x$.

Diff		Int
$(x-a)(b-x)$	$+$	$f''(x)$
$a+b-2x$	$-$	$f'(x)$
-2	$+$	$f(x)$
0		$\int f(x)\,\mathrm{d}x$

$$\begin{aligned}&\int (x-a)(b-x)\,f''(x)\,\mathrm{d}x\\ &=\big((x-a)(b-x)\,f'(x)-(a+b-2x)\,f(x)\big)-2\int f(x)\,\mathrm{d}x\end{aligned}$$

性質. 令 $F(x)=\displaystyle\int_{v(x)}^{u(x)} f(\tau)\,\mathrm{d}\tau$, 則 $F'(x)=f(u(x))\cdot u'(x)-f(v(x))\cdot v'(x)$.

證. 令 $a\in\mathbb{R}$, $F(x)=\displaystyle\int_{v(x)}^{u(x)} f(\tau)\,\mathrm{d}\tau=\int_a^{u(x)} f(\tau)\,\mathrm{d}\tau-\int_a^{v(x)} f(\tau)\,\mathrm{d}\tau$. 令 $G(x)\equiv\displaystyle\int_a^{x} f(\tau)\,\mathrm{d}\tau$, 則 $G'(x)=f(x)$, $F(x)=\displaystyle\int_a^{u(x)} f(\tau)\,\mathrm{d}\tau-\int_a^{v(x)} f(\tau)\,\mathrm{d}\tau=G(u(x))-G(v(x))$; 故 $F'(x)=(G(u(x))-G(v(x)))'=G'(u(x))\cdot u'(x)-G'(v(x))\cdot v'(x)=f(u(x))\cdot u'(x)-f(v(x))\cdot v'(x)$.

例.

1. $F(x)=\displaystyle\int_1^x \frac{1}{1+\tau^4}\,\mathrm{d}\tau \implies F'(x)=\frac{1}{1+x^4}$
2. $F(x)=\displaystyle\int_2^{\sqrt{x}} \sin\tau\,\mathrm{d}\tau \implies F'(x)=\sin\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$
3. $F(x)=\displaystyle\int_x^{2x} \tau^3\,\mathrm{d}\tau \implies F'(x)=(2x)^3\cdot 2-x^3\cdot 1=15x^3$
4. $F(x)=\displaystyle\int_{\sin x}^{\tan^{-1}x} e^{\tau^2}\,\mathrm{d}\tau \implies F'(x)=e^{(\tan^{-1}x)^2}\cdot\frac{1}{1+x^2}-e^{\sin^2 x}\cdot\cos x$

例. 若 $g(x)=\displaystyle\int_0^{\cos x}(1+\sin(t^2))\,\mathrm{d}t$, $f(x)=\displaystyle\int_0^{g(x)}\frac{1}{\sqrt{1+t^3}}\,\mathrm{d}t$, 求 $f'\left(\dfrac{\pi}{2}\right)$.

解. $f'(x)=\dfrac{1}{\sqrt{1+g(x)^3}}\cdot g'(x)$, $g'(x)=(1+\sin(\cos^2 x))\cdot(-\sin x)$. 代入 $x=\dfrac{\pi}{2}$, $g\left(\dfrac{\pi}{2}\right)=0$, $g'\left(\dfrac{\pi}{2}\right)=-1$, 故 $f'\left(\dfrac{\pi}{2}\right)=-1$.

例. 若 $\displaystyle\int_0^{x^2} f(t)\,\mathrm{d}t=x\sin\pi x$, 求 $f'(9)$.

解. $\displaystyle\int_0^{x^2} f(t)\,\mathrm{d}t=x\sin\pi x$ 兩邊對 x 微分得 $f(x^2)\cdot 2x=\sin\pi x+x\cdot\pi\cos\pi x$. 兩邊再對 x 微分得 $(f'(x^2)\cdot 2x)\cdot 2x+f(x^2)\cdot 2=\pi\cos\pi x+\pi\cos\pi x-x\cdot\pi^2\sin\pi x$. 代入 $x=3$, 則 $(f'(9)\cdot 2\cdot 3)\cdot 2\cdot 3+f(9)\cdot 2=\pi\cos 3\pi+\pi\cos 3\pi-3\cdot\pi^2\sin 3\pi \implies f'(9)\cdot 36+f(9)\cdot 2=-2\pi$. 又 $f(3^2)\cdot(2\cdot 3)=\sin 3\pi+3\cdot\pi\cos 3\pi \implies f(9)=-\dfrac{\pi}{2}$, 故 $f'(9)=-\dfrac{\pi}{36}$.

例. 求函數 f 與 $a\in\mathbb{R}$ 使 $6+\displaystyle\int_a^x \frac{f(t)}{t^2}\,\mathrm{d}t=2\sqrt{x}$, $\forall x>0$.

解. $6+\displaystyle\int_a^x \frac{f(t)}{t^2}\,\mathrm{d}t=2\sqrt{x}$ 兩邊對 x 微分得 $\dfrac{f(x)}{x^2}=\dfrac{1}{\sqrt{x}} \implies f(x)=x^{\frac{3}{2}}$. 代入原式得 $6+\displaystyle\int_a^x \frac{t^{\frac{3}{2}}}{t^2}\,\mathrm{d}t=6+\int_a^x \frac{1}{\sqrt{t}}\,\mathrm{d}t=6+2\sqrt{t}\Big|_a^x=6+2\sqrt{x}-2\sqrt{a}=2\sqrt{x} \implies a=9$.

例. 求下列極限.

1. $\lim_{x\to 0} \dfrac{\int_0^x (\sec t - 1)\,\mathrm{d}t}{x^3}$

2. $\lim_{x\to \infty} \dfrac{\int_{x^2}^0 e^{t-x^2}(2t^2+1)\,\mathrm{d}t}{x^4}$

3. $\lim_{x\to 0} \dfrac{1}{x}\displaystyle\int_0^{\tan x} f(u)(\sin x - \cos u)\,\mathrm{d}u$, f 為連續函數

解.

1. $\lim_{x\to 0} \dfrac{\int_0^x (\sec t - 1)\,\mathrm{d}t}{x^3} = \lim_{x\to 0} \dfrac{\sec x - 1}{3x^2} = \lim_{x\to 0} \dfrac{\sec x \tan x}{6x} = \lim_{x\to 0} \dfrac{\sec x \tan x \cdot \tan x + \sec x \cdot \sec^2 x}{6} = \dfrac{1}{6}.$

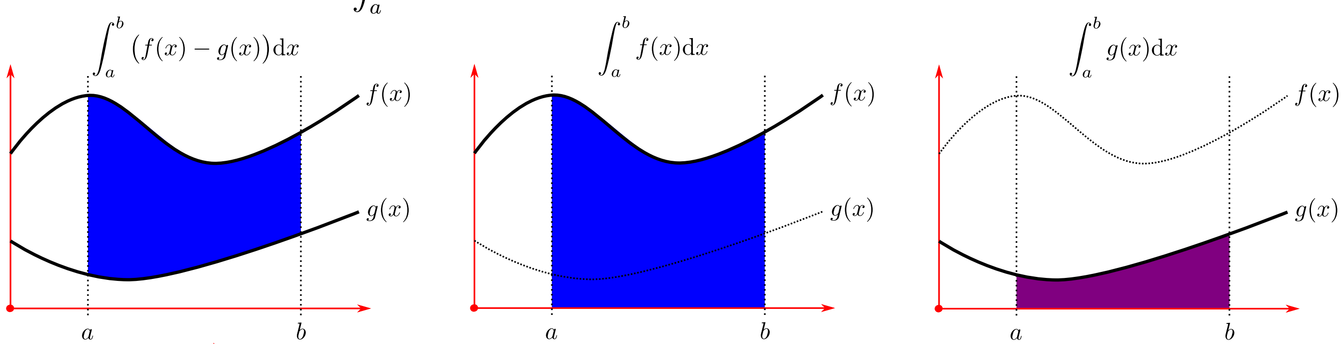
2. $\lim_{x\to \infty} \dfrac{\int_{x^2}^0 e^{t-x^2}(2t^2+1)\,\mathrm{d}t}{x^4} = \lim_{x\to \infty} \dfrac{\int_{x^2}^0 e^{t}(2t^2+1)\,\mathrm{d}t}{e^{x^2}x^4} = \lim_{x\to\infty} \dfrac{-e^{x^2}(2(x^2)^2+1)\cdot 2x}{e^{x^2}\,2x\cdot x^4 + e^{x^2}\cdot 4x^3} = \lim_{x\to\infty} \dfrac{-(4x^5+2x)}{2x^5+4x^3} = -2.$

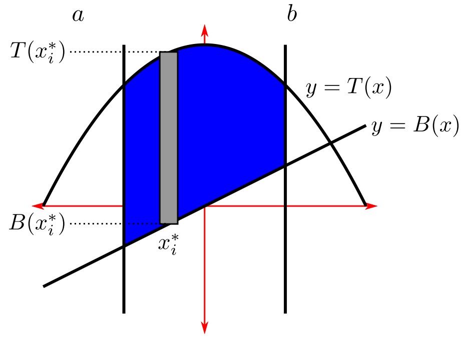
3. $$\lim_{x\to 0} \frac{1}{x}\int_0^{\tan x} f(u)(\sin x - \cos u)\,\mathrm{d}u = \lim_{x\to 0} \frac{\int_0^{\tan x} f(u)(\sin x - \cos u)\,\mathrm{d}u}{x}$$
$$= \lim_{x\to 0} \frac{\sin x \int_0^{\tan x} f(u)\,\mathrm{d}u - \int_0^{\tan x} f(u)\cos u\,\mathrm{d}u}{x}$$
$$= \lim_{x\to 0} \frac{\cos x \int_0^{\tan x} f(u)\,\mathrm{d}u + \sin x \cdot f(\tan x)\cdot \sec^2 x - f(\tan x)\cos(\tan x)\cdot \sec^2 x}{1} = -f(0)$$

4.3 面積與體積

面積

結論. 當 $f(x) \geqslant 0\ \forall x \in [a, b]$, $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ 為 $y = f(x)$, x 軸, $x = a$, 與 $x = b$ 所圍成之面積.



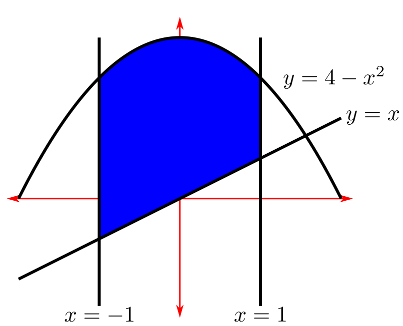


$$(\text{面積}) \approx \sum_{i=1}^{n} \big(T(x_i^*) - B(x_i^*)\big)\,\Delta x,\ \ \Delta x = \frac{b-a}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \big(T(x_i^*) - B(x_i^*)\big)\,\Delta x = \int_a^b \big(T(x) - B(x)\big)\,\mathrm{d}x$$

例. 求以 $y = 4 - x^2$, $y = x$, $x = -1$, 與 $x = 1$ 圍成之區域面積.

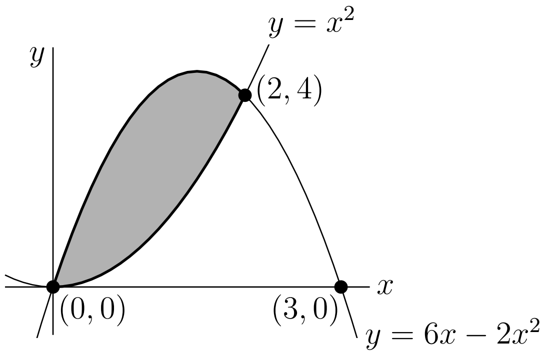
解.



$$(\text{面積}) = \int_{-1}^{1} \big((4 - x^2) - x\big)\,\mathrm{d}x = \frac{22}{3}$$

例. 求 $y = x^2$ 與 $y = 6x - 2x^2$ 圍成之區域面積.

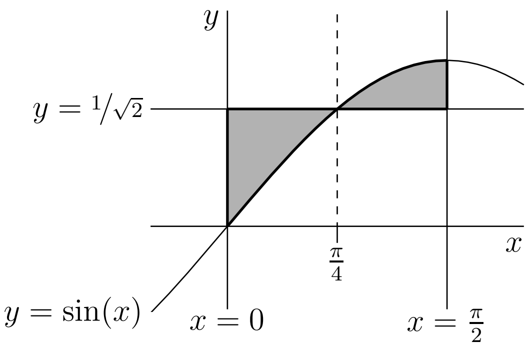
解.



$$(\text{面積}) = \int_0^2 \big((6x - 2x^2) - x^2\big)\,\mathrm{d}x = 4$$

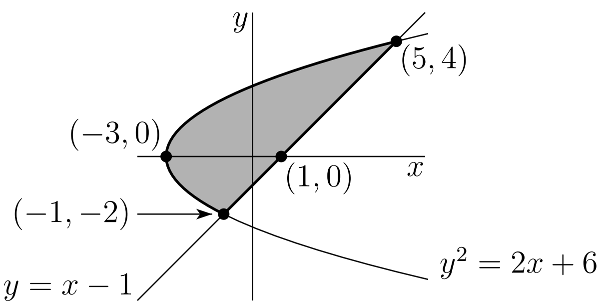
例. 求 $y = \dfrac{1}{\sqrt{2}}$ 與 $y = \sin x$ 在 x 從 0 至 $\dfrac{\pi}{2}$ 範圍內圍成之區域面積.

解.



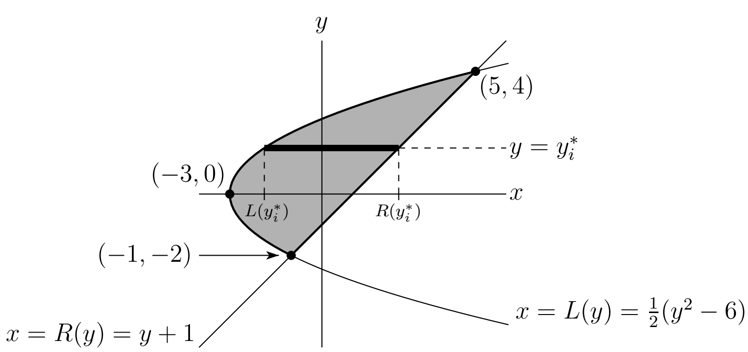
$$(\text{面積}) = \int_0^{\frac{\pi}{4}} \left(\frac{1}{\sqrt{2}} - \sin x\right)\mathrm{d}x + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\sin x - \frac{1}{\sqrt{2}}\right)\mathrm{d}x = \sqrt{2} - 1$$

例.

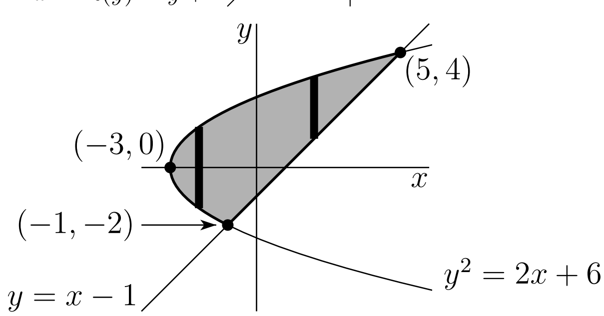


求 $y^2 = 2x + 6$ 與 $y = x - 1$ 圍成之區域面積.

解.



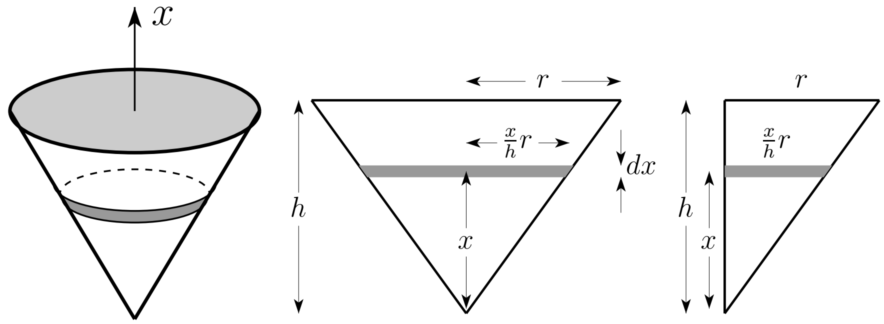
$$(\text{面積}) = \int_{-2}^{4} \left((y+1) - \frac{1}{2}\left(y^2 - 6\right)\right)\mathrm{d}y = 18$$



$$(\text{面積}) = \int_{-3}^{-1} 2\sqrt{2x+6}\,\mathrm{d}x + \int_{-1}^{5} \left(\sqrt{2x+6} - x + 1\right)\mathrm{d}x = 18$$

例. 求高 h 與底半徑 r 之圓錐體體積.

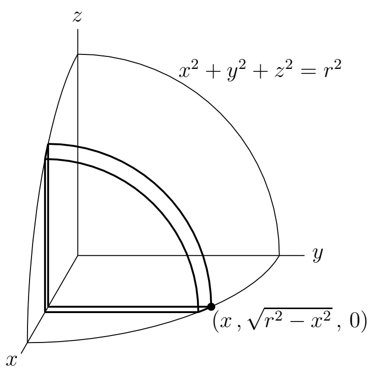
解.



$$(\text{體積}) = \int_0^h \pi\left(\frac{x}{h}r\right)^2 \mathrm{d}x = \frac{1}{3}\pi r^2 h$$

例. 求半徑 r 之三維球體體積.

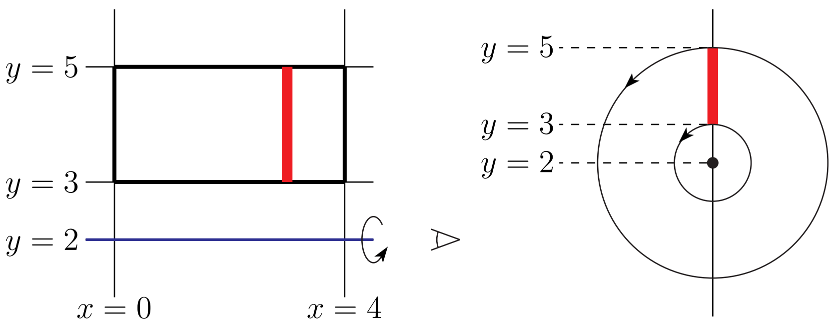
解.



$$(\text{體積}) = 8\cdot\int_0^r \frac{\pi}{4}\left(\sqrt{r^2-x^2}\right)^2 \mathrm{d}x = \frac{4}{3}\pi r^3$$

例. 求以 $y=3$, $y=5$, $x=0$ 與 $x=4$ 圍成之區域繞 $y=2$ 旋轉而成之旋轉體體積.

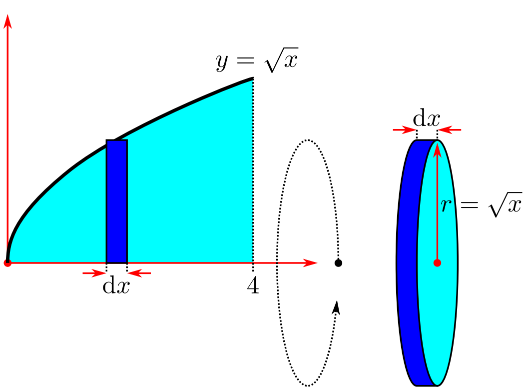
解.

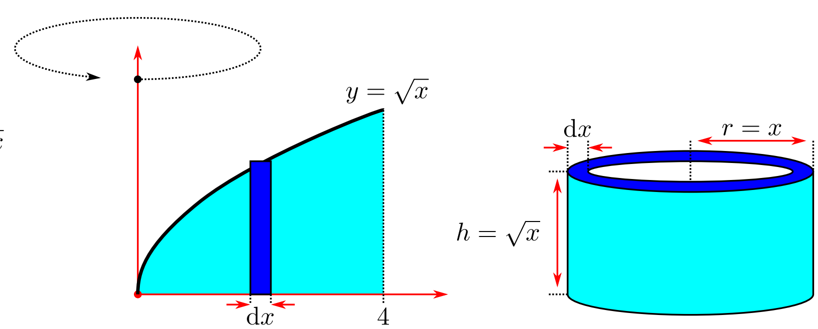


$$(\text{體積}) = \int_0^4 \pi(3^2-1^2)\,\mathrm{d}x = 32\pi$$

例. 求以 $y=\sqrt{x}$, $y=0$, $x=0$ 與 $x=4$ 圍成之區域繞 (i) $y=0$ (ii) $x=0$ 旋轉而成之旋轉體體積.

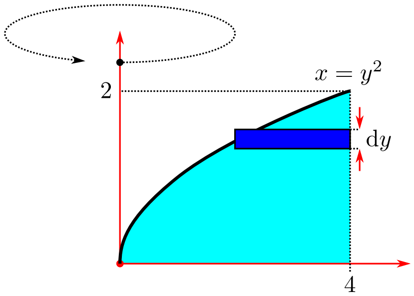
解.

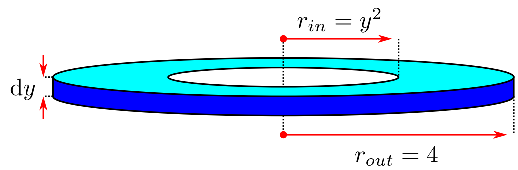




$$(\text{體積}) = \int_0^4 \pi\left(\sqrt{x}\right)^2 \mathrm{d}x = 8\pi$$

$$(\text{體積}) = \int_0^4 2\pi\cdot x\cdot\sqrt{x}\,\mathrm{d}x = \frac{128\pi}{5}$$

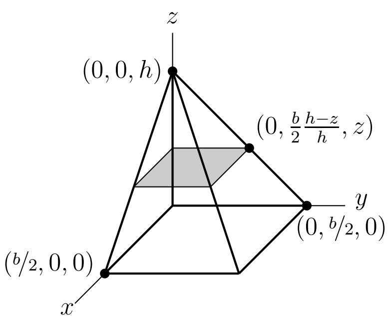


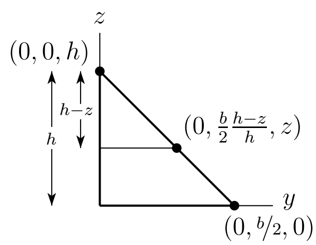


$$(\text{體積}) = \int_0^2 \pi(4^2 - (y^2)^2)\,\mathrm{d}y = \frac{128\pi}{5}$$

例. 求高為 h, 底面為邊長 b 正方形之錐體體積.

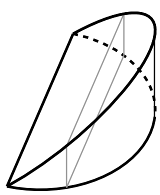
解.





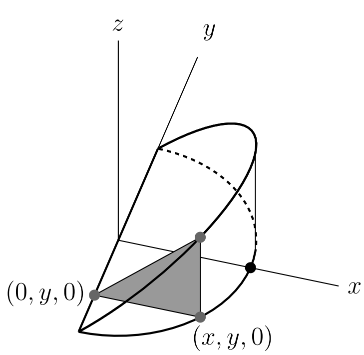
$$(\text{體積}) = \int_0^h \left(b - \frac{z}{h}\,b\right)^2 \mathrm{d}z = \frac{1}{3}\,b^2 h$$

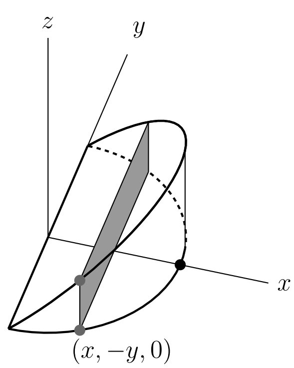
例.



將一半徑為 a 之圓柱體水平橫切, 再對其底面圓心 45° 角斜切, 求如圖所示結果體積。

解.





$$(\text{體積}) = \int_{-a}^{a} \frac{1}{2}\left(\sqrt{a^2 - y^2}\right)^2 \mathrm{d}y = \frac{2}{3}\,a^3$$

$$(\text{體積}) = \int_0^a x \cdot 2\sqrt{a^2 - x^2}\,\mathrm{d}x = \frac{2}{3}\,a^3$$

4.4 積分技巧

部份分式

例 (動機). 若 $a \neq 0$, 求 $\displaystyle\int \frac{1}{x^2 - a^2}\,\mathrm{d}x$.

解. 由 $\displaystyle\frac{1}{x^2-a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$, $\displaystyle\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \int \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)\mathrm{d}x = \frac{1}{2a}\big(\ln|x-a| - \ln|x+a|\big) = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

結論.

- 實係數多項式可分解成不可約的一次及二次因式的乘積.
- 有理式可寫成多項式與真分式之和.

- 若 $\dfrac{p(x)}{q(x)}$ 為一真分式, $q(x)=(x+a_1)^{m_1}(x+a_2)^{m_2}\cdots(x+a_k)^{m_k}\cdot(x^2+b_1x+c_1)^{n_1}(x^2+b_2x+c_2)^{n_2}\cdots(x^2+b_lx+c_l)^{n_l}$, 其中 $(x+a_i)$, $(x^2+b_ix+c_i)$ 均相異, $(x^2+b_ix+c_i)$ 為不可分解之二次式 $(b_i^2-4c_i<0)$, 則

$$\begin{aligned}\frac{p(x)}{q(x)} &= \frac{\alpha_{11}}{x+a_1}+\frac{\alpha_{12}}{(x+a_1)^2}+\cdots+\frac{\alpha_{1m_1}}{(x+a_1)^{m_1}}+\\ &\frac{\alpha_{21}}{x+a_2}+\frac{\alpha_{22}}{(x+a_2)^2}+\cdots+\frac{\alpha_{2m_2}}{(x+a_2)^{m_2}}+\cdots+\\ &\frac{\alpha_{k1}}{x+a_k}+\frac{\alpha_{k2}}{(x+a_k)^2}+\cdots+\frac{\alpha_{km_k}}{(x+a_k)^{m_k}}+\\ &\frac{\beta_{11}x+\gamma_{11}}{x^2+b_1x+c_1}+\frac{\beta_{12}x+\gamma_{12}}{(x^2+b_1x+c_1)^2}+\cdots+\frac{\beta_{1n_1}x+\gamma_{1n_1}}{(x^2+b_1x+c_1)^{n_1}}+\\ &\frac{\beta_{21}x+\gamma_{21}}{x^2+b_2x+c_2}+\frac{\beta_{22}x+\gamma_{22}}{(x^2+b_2x+c_2)^2}+\cdots+\frac{\beta_{2n_2}x+\gamma_{2n_2}}{(x^2+b_1x+c_1)^{n_2}}+\cdots+\\ &\frac{\beta_{l1}x+\gamma_{l1}}{x^2+b_lx+c_l}+\frac{\beta_{l2}x+\gamma_{l2}}{(x^2+b_lx+c_l)^2}+\cdots+\frac{\beta_{ln_l}x+\gamma_{ln_l}}{(x^2+b_lx+c_l)^{n_l}}\end{aligned}$$

  其中 α_{ij}, β_{ij}, $\gamma_{ij}\in\mathbb{R}$.

註. 任一有理函數之積分可分解為多項式積分與以下兩型積分:

- $\displaystyle\int\frac{1}{(x+a)^n}\,\mathrm{d}x$, $n\in\mathbb{N}$
- $\displaystyle\int\frac{f(x)}{(x^2+bx+c)^n}\,\mathrm{d}x$, $f(x)=1$ 或 $f(x)=x$; $b^2-4c<0$, $n\in\mathbb{N}$

例. 求 $\displaystyle\int\frac{x}{x^2-5x+6}\,\mathrm{d}x$.

解. $\dfrac{x}{x^2-5x+6}=\dfrac{x}{(x-3)(x-2)}=\dfrac{A}{x-3}+\dfrac{B}{x-2}\implies x=A\,(x-2)+B\,(x-3)$. 代入 $x=3\implies 3=A$; 代入 $x=2\implies 2=B(2-3)\implies B=-2$, 故 $\dfrac{x}{x^2-5x+6}=\dfrac{3}{x-3}-\dfrac{2}{x-2}$, $\displaystyle\int\frac{x}{x^2-5x+6}\,\mathrm{d}x=\int\left(\frac{3}{x-3}-\frac{2}{x-2}\right)\mathrm{d}x=3\ln|x-3|-2\ln|x-2|$

例. 求 $\displaystyle\int\frac{1}{x^3+1}\,\mathrm{d}x$.

解. $\dfrac{1}{x^3+1}=\dfrac{1}{(x+1)(x^2-x+1)}=\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2-x+1}\implies 1=A\,(x^2-x+1)+(Bx+C)\,(x+1)$. 代入 $x=-1\implies 1=A((-1)^2-(-1)+1)=3A\implies A=\dfrac{1}{3}$; 代入 $x=0\implies 1=A+C\implies C=\dfrac{2}{3}$; 代入 $x=1\implies 1=A+(B+C)\cdot 2\implies 1=\dfrac{1}{3}+\left(B+\dfrac{2}{3}\right)\cdot 2\implies B=-\dfrac{1}{3}$; 故 $\dfrac{1}{x^3+1}=\dfrac{1}{3}\left(\dfrac{1}{x+1}-\dfrac{x-2}{x^2-x+1}\right)$, $\displaystyle\int\frac{1}{x^3+1}\,\mathrm{d}x=\int\frac{1}{3}\left(\frac{1}{x+1}-\frac{x-2}{x^2-x+1}\right)\mathrm{d}x=\int\frac{1}{3}\left(\frac{1}{x+1}-\frac{\frac{1}{2}(2x-1)-\frac{3}{2}}{x^2-x+1}\right)\mathrm{d}x=\frac{1}{3}\int\frac{1}{x+1}\,\mathrm{d}x-\frac{1}{6}\int\frac{2x-1}{x^2-x+1}\,\mathrm{d}x-\frac{1}{2}\int\frac{1}{x^2-x+1}\,\mathrm{d}x=\frac{1}{3}\ln|x+1|-\frac{1}{6}\ln(x^2-x+1)-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}$

例. 求 $\displaystyle\int\frac{1}{x^4+4}\,\mathrm{d}x$.

解. 由 $x^4+4=(x^2+2)^2-(2x)^2=(x^2-2x+2)(x^2+2x+2)$, $\dfrac{1}{x^4+4}=\dfrac{Ax+B}{x^2-2x+2}+\dfrac{Cx+D}{x^2+2x+2}\implies (Ax+B)(x^2+2x+2)+(Cx+D)(x^2-2x+2)=1\implies (A+C)x^3+(2A+B-2C+D)x^2+(2A+2B+2C-2D)x+(2B+2D)=1\implies A+C=0,\ 2A+B-2C+D=0,\ 2A+2B+2C-2D=0,\ 2B+2D=1\implies A=-\dfrac{1}{8}$,

$B=\frac{1}{4}$, $C=\frac{1}{8}$, $D=\frac{1}{4}$. 故 $\frac{1}{x^4+4}=\frac{1}{8}\left(\frac{x+2}{x^2+2x+2}-\frac{x-2}{x^2-2x+2}\right)$, $\int\frac{1}{x^4+4}\,\mathrm{d}x=\frac{1}{8}\int\left(\frac{x+2}{x^2+2x+2}-\frac{x-2}{x^2-2x+2}\right)\mathrm{d}x=\frac{1}{8}\int\left(\frac{(x+1)+1}{(x+1)^2+1}-\frac{(x-1)-1}{(x-1)^2+1}\right)\mathrm{d}x=\frac{1}{16}\ln\frac{x^2+2x+2}{x^2-2x+2}+\frac{1}{8}\big(\tan^{-1}(x+1)-\tan^{-1}(x-1)\big)$

例. 求 $\int\frac{1}{\cos^3 x}\,\mathrm{d}x$.

解. $\int\frac{1}{\cos^3 x}\,\mathrm{d}x=\int\frac{\cos x}{\cos^4 x}\,\mathrm{d}x=\int\frac{\cos x}{(1-\sin^2 x)^2}\,\mathrm{d}x$. 令 $u=\sin x$, 則 $\mathrm{d}u=\cos x\,\mathrm{d}x$, 則 $\int\frac{\cos x}{(1-\sin^2 x)^2}\,\mathrm{d}x=\int\frac{1}{(1-u^2)^2}\,\mathrm{d}u$. $\frac{1}{(1-u^2)^2}=\frac{1}{(u-1)^2(u+1)^2}=\frac{A}{u-1}+\frac{B}{(u-1)^2}+\frac{C}{u+1}+\frac{D}{(u+1)^2}\implies 1=A\,(u-1)(u+1)^2+B\,(u+1)^2+C\,(u-1)^2(u+1)+D\,(u-1)^2$. 代入 $u=1\implies 1=4B\implies B=\frac{1}{4}$; 代入 $u=-1\implies 1=4D\implies D=\frac{1}{4}$; 代入 $u=0\implies 1=-A+B+C+D\implies 1=-A+\frac{1}{4}+C+\frac{1}{4}\implies\frac{1}{2}=-A+C$; 代入 $u=2\implies 1=9A+9B+3C+D\implies 1=9A+\frac{9}{4}+3C+\frac{1}{4}\implies A=-\frac{1}{4}$, $C=\frac{1}{4}$; 則 $\frac{1}{(1-u^2)^2}=\frac{1}{4}\left(\frac{-1}{u-1}+\frac{1}{(u-1)^2}+\frac{1}{u+1}+\frac{1}{(u+1)^2}\right)$, $\int\frac{1}{(1-u^2)^2}\,\mathrm{d}u=\int\frac{1}{4}\left(\frac{-1}{u-1}+\frac{1}{(u-1)^2}+\frac{1}{u+1}+\frac{1}{(u+1)^2}\right)\mathrm{d}x=\frac{1}{4}\left(-\ln|u-1|-\frac{1}{u-1}+\ln|u+1|-\frac{1}{u+1}\right)=\frac{1}{4}\left(-\ln|\sin x-1|-\frac{1}{\sin x-1}+\ln|\sin x+1|-\frac{1}{\sin x+1}\right)$.
部份分式另解: $\frac{1}{(1-u^2)^2}=\left(\frac{1}{u^2-1}\right)^2=\left(\frac{1}{(u-1)(u+1)}\right)^2=\frac{1}{4}\left(\frac{1}{u-1}-\frac{1}{u+1}\right)^2=\frac{1}{4}\left(\frac{1}{(u-1)^2}-\frac{2}{(u-1)(u+1)}+\frac{1}{(u+1)^2}\right)=\frac{1}{4}\left(\frac{1}{(u-1)^2}-\left(\frac{1}{u-1}-\frac{1}{u+1}\right)+\frac{1}{(u+1)^2}\right)=\frac{1}{4}\left(\frac{-1}{u-1}+\frac{1}{(u-1)^2}+\frac{1}{u+1}+\frac{1}{(u+1)^2}\right)$.

三角函數代換

結論.

- 遇 $\sqrt{a^2-x^2}$, 考慮 $x=a\sin\theta\implies\theta=\sin^{-1}\frac{x}{a}$, $\mathrm{d}x=a\cos\theta\,\mathrm{d}\theta$
- 遇 $\sqrt{a^2+x^2}$, 考慮 $x=a\tan\theta\implies\theta=\tan^{-1}\frac{x}{a}$, $\mathrm{d}x=a\sec^2\theta\,\mathrm{d}\theta$
- 遇 $\sqrt{x^2-a^2}$, 考慮 $x=a\sec\theta\implies\theta=\sec^{-1}\frac{x}{a}$, $\mathrm{d}x=a\sec\theta\,\tan\theta\,\mathrm{d}\theta$
- 遇 $\sin x$, $\cos x$ 之有理式, 考慮 $u=\tan\frac{x}{2}$, 由以下化為 u 之有理式:
  - $\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}=2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}}=\frac{2u}{1+u^2}$
  - $\cos x=2\cos^2\frac{x}{2}-1=2\cdot\frac{1}{1+u^2}-1=\frac{1-u^2}{1+u^2}$
  - $\mathrm{d}u=\frac{1}{2}\sec^2\frac{x}{2}\,\mathrm{d}x\implies\mathrm{d}x=\frac{2}{1+u^2}\,\mathrm{d}u$

例. 若 $a\neq 0$, 求下列不定積分.

1. $\int\sqrt{a^2-x^2}\,\mathrm{d}x$
2. $\int\sqrt{x^2+a^2}\,\mathrm{d}x$
3. $\int x^2\sqrt{x^2+a^2}\,\mathrm{d}x$

4. $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x$

5. $\displaystyle\int \frac{1}{x^2\sqrt{x^2+a^2}}\,\mathrm{d}x$

6. $\displaystyle\int \sqrt{x^2-a^2}\,\mathrm{d}x$

7. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x$

8. $\displaystyle\int \frac{1}{\tan x+\sin x}\,\mathrm{d}x$

9. $\displaystyle\int \frac{1}{a+\sin x}\,\mathrm{d}x,\ a>1$

解.

1. 令 $x=a\sin\theta$, 則 $\displaystyle\int \sqrt{a^2-x^2}\,\mathrm{d}x=\int\sqrt{a^2-a^2\sin^2\theta}\,a\cos\theta\,\mathrm{d}\theta=a^2\int\cos^2\theta\,\mathrm{d}\theta=a^2\int\frac{1+\cos 2\theta}{2}\,\mathrm{d}\theta$ $\displaystyle=\frac{a^2}{2}\theta+\frac{a^2}{4}\sin 2\theta=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{a^2}{2}\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}.$

2. 令 $x=a\tan\theta$, 則 $\displaystyle\int\sqrt{x^2+a^2}\,\mathrm{d}x=\int a\sec\theta\cdot a\sec^2\theta\,\mathrm{d}\theta=a^2\int\sec^3\theta\,\mathrm{d}\theta=\frac{a^2}{2}\big(\sec\theta\cdot\tan\theta+\ln|\sec\theta+$ $\displaystyle\tan\theta|\big)=\frac{a^2}{2}\left(\frac{x}{a}\cdot\frac{\sqrt{x^2+a^2}}{a}+\ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right|\right)=\frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\ln\left|\sqrt{x^2+a^2}+x\right|-\frac{a^2}{2}\ln|a|$

3. 令 $x=a\tan\theta$, 則 $\displaystyle\int x^2\sqrt{x^2+a^2}\,\mathrm{d}x=\int a\sec^2\theta\cdot a^2\tan^2\theta\cdot a\sec\theta\,\mathrm{d}\theta=a^4\int\sec^3\theta\tan^2\theta\,\mathrm{d}\theta$ $\displaystyle=a^4\int\sec^3\theta\,(\sec^2\theta-1)\,\mathrm{d}\theta=a^4\int(\sec^5\theta-\sec^3\theta)\,\mathrm{d}\theta=\frac{a^4}{4}\left(\sec^3\theta\tan\theta-\frac{1}{2}\big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\big)\right)$ $\displaystyle=\frac{a^4}{4}\left(\sec^3\theta\tan\theta-\frac{1}{2}\big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\big)\right)=\frac{x(x^2+a^2)^{\frac{3}{2}}}{4}-\frac{a^2x\sqrt{x^2+a^2}}{8}-\frac{a^4\ln|\sqrt{x^2+a^2}+x|}{8}+$ $\displaystyle\frac{a^4\ln|a|}{8}.$

4. 令 $x=a\tan\theta$, 則 $\displaystyle\int\frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x=\int\frac{1}{a\sec\theta}\cdot a\sec^2\theta\,\mathrm{d}\theta=\int\sec\theta\,\mathrm{d}\theta=\ln|\sec\theta+\tan\theta|$ $\displaystyle=\ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right|=\ln|\sqrt{x^2+a^2}+x|-\ln|a|.$

5. 令 $x=a\tan\theta$, 則 $\displaystyle\int\frac{1}{x^2\sqrt{x^2+a^2}}\,\mathrm{d}x=\int\frac{a\sec^2\theta}{a^2\tan^2\theta\cdot a\sec\theta}\,\mathrm{d}\theta=\int\frac{\sec\theta}{a^2\tan^2\theta}\,\mathrm{d}\theta=\int\frac{\cos\theta}{a^2\sin^2\theta}\,\mathrm{d}\theta$ $\displaystyle=-\frac{1}{a^2\sin\theta}=-\frac{\sqrt{x^2+a^2}}{a^2x}.$

6. 令 $x=a\sec\theta$, 則 $\displaystyle\int\sqrt{x^2-a^2}\,\mathrm{d}x=\int a\tan\theta\cdot a\sec\theta\tan\theta\,\mathrm{d}\theta=a^2\int\sec\theta\tan^2\theta\,\mathrm{d}\theta=a^2\int\sec\theta\,(\sec^2\theta-$ $\displaystyle1)\,\mathrm{d}\theta=a^2\left(\int\sec^3\theta\,\mathrm{d}\theta-\int\sec\theta\,\mathrm{d}\theta\right)=\frac{a^2}{2}\left(\sec\theta\cdot\tan\theta+\int\sec\theta\,\mathrm{d}\theta-2\int\sec\theta\,\mathrm{d}\theta\right)=\frac{a^2}{2}\Big(\sec\theta\cdot$ $\displaystyle\tan\theta-\int\sec\theta\,\mathrm{d}\theta\Big)=\frac{a^2}{2}\big(\sec\theta\cdot\tan\theta-\ln|\sec\theta+\tan\theta|\big)=\frac{a^2}{2}\left(\frac{x}{a}\cdot\frac{\sqrt{x^2-a^2}}{a}-\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|\right)=$ $\displaystyle\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\ln\left|\sqrt{x^2-a^2}+x\right|+\frac{a^2}{2}\ln|a|.$

7. 令 $x=a\sec\theta$, 則 $\displaystyle\int\frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x=\int\frac{a\sec\theta\tan\theta}{a\tan\theta}\,\mathrm{d}\theta=\int\sec\theta\,\mathrm{d}\theta=\ln|\sec\theta+\tan\theta|$ $\displaystyle=\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|=\ln|\sqrt{x^2-a^2}+x|-\ln|a|.$

8. 令 $u=\tan\dfrac{x}{2}$, 則 $\displaystyle\int\frac{1}{\tan x+\sin x}\,\mathrm{d}x=\int\frac{1}{\frac{2u}{1-u^2}+\frac{2u}{1+u^2}}\frac{2}{1+u^2}\,\mathrm{d}u=\int\frac{1-u^2}{2u}\,\mathrm{d}u=\frac{\ln|u|}{2}-\frac{u^2}{4}=$ $\displaystyle\frac{\ln|\tan\frac{x}{2}|}{2}-\frac{\tan^2\frac{x}{2}}{4}$

9. 令 $u=\tan\frac{x}{2}$, 則 $\int\frac{1}{a+\sin x}\,\mathrm{d}x=\int\frac{1}{a+\frac{2u}{1+u^2}}\,\frac{2}{1+u^2}\,\mathrm{d}u=2\int\frac{1}{au^2+2u+a}\,\mathrm{d}u=\frac{2}{a\sqrt{\frac{a^2-1}{a^2}}}\tan^{-1}\frac{u+\frac{1}{a}}{\sqrt{\frac{a^2-1}{a^2}}}$
$=\frac{2}{\sqrt{a^2-1}}\tan^{-1}\frac{au+1}{\sqrt{a^2-1}}=\frac{2}{\sqrt{a^2-1}}\tan^{-1}\frac{a\tan\frac{x}{2}+1}{\sqrt{a^2-1}}$.

4.5 瑕積分

定義 (瑕積分 (improper integral))**.**

- 無限區間 (第一型) 瑕積分
  - 若 $f(x)$ 在 $[a,\,\infty)$ 連續, 則 $\int_a^\infty f(x)\,\mathrm{d}x=\lim\limits_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x$.
  - 若 $f(x)$ 在 $(-\infty,\,b]$ 連續, 則 $\int_{-\infty}^b f(x)\,\mathrm{d}x=\lim\limits_{a\to-\infty}\int_a^b f(x)\,\mathrm{d}x$.
  - 若 $f(x)$ 在 $(-\infty,\,\infty)$ 連續, 則任取 $c\in\mathbb{R}$, $\int_{-\infty}^\infty f(x)\,\mathrm{d}x=\int_{-\infty}^c f(x)\,\mathrm{d}x+\int_c^\infty f(x)\,\mathrm{d}x$.
- 不連續點 (第二型) 瑕積分
  - 若 $f(x)$ 在 $(a,\,b]$ 連續, 則 $\int_a^b f(x)\,\mathrm{d}x=\lim\limits_{c\to a+}\int_c^b f(x)\,\mathrm{d}x$.
  - 若 $f(x)$ 在 $[a,\,b)$ 連續, 則 $\int_a^b f(x)\,\mathrm{d}x=\lim\limits_{c\to b-}\int_a^c f(x)\,\mathrm{d}x$.
  - 令 $c\in(a,\,b)$. 若 $f(x)$ 在 $[a,\,c)\cup(c,b]$ 連續且在 $x=c$ 不連續, 則 $\int_a^b f(x)\,\mathrm{d}x=\int_a^c f(x)\,\mathrm{d}x+\int_c^b f(x)\,\mathrm{d}x$.

例. 1. $\int_{-\infty}^\infty\frac{1}{1+x^2}\,\mathrm{d}x$. 2. $\int_1^\infty\frac{1}{x^2}\,\mathrm{d}x$. 3. $\int_0^1\frac{1}{\sqrt{x}}\,\mathrm{d}x$.

解.

1. $\int_{-\infty}^\infty\frac{1}{1+x^2}\,\mathrm{d}x=2\int_0^\infty\frac{1}{1+x^2}\,\mathrm{d}x=2\lim\limits_{b\to\infty}\int_0^b\frac{1}{1+x^2}\,\mathrm{d}x=2\lim\limits_{b\to\infty}\tan^{-1}b=2\cdot\frac{\pi}{2}=\pi.$
2. $\int_1^\infty\frac{1}{x^2}\,\mathrm{d}x=\lim\limits_{b\to\infty}\int_1^b\frac{1}{x^2}\,\mathrm{d}x=\lim\limits_{b\to\infty}\Big(-\frac{1}{b}+1\Big)=1$
3. $\int_0^1\frac{1}{\sqrt{x}}\,\mathrm{d}x=\lim\limits_{c\to0+}\int_c^1\frac{1}{\sqrt{x}}\,\mathrm{d}x=\lim\limits_{c\to0+}2\sqrt{x}\,\Big|_c^1=\lim\limits_{c\to0+}(2-2\sqrt{c})=2$

例. 證明 $\forall n\in\mathbb{N}$, $\int_0^1(\ln x)^n\,\mathrm{d}x=(-1)^n n!$.

解. 使用數學歸納法: $n=1$ 時 $\int_0^1\ln x\,\mathrm{d}x=(x\ln x-x)\Big|_0^1=-1-\lim\limits_{x\to0+}x\ln x=-1+\lim\limits_{x\to0+}\frac{\ln\frac{1}{x}}{\frac{1}{x}}=-1+\lim\limits_{y\to\infty}\frac{\ln y}{y}=-1=(-1)^1\,1!$. 令等式在 $n-1$ 成立: $\int_0^1(\ln x)^{n-1}\,\mathrm{d}x=(-1)^{n-1}(n-1)!$, 則 $\int_0^1(\ln x)^n\,\mathrm{d}x=x\,(\ln x)^n\Big|_0^1-n\int_0^1 x\cdot(\ln x)^{n-1}\cdot\frac{1}{x}\,\mathrm{d}x=0-\lim\limits_{x\to0+}x\,(\ln x)^n+(-1)\cdot n\cdot(-1)^{n-1}(n-1)!=-\lim\limits_{x\to0+}x\,(\ln x)^n+(-1)^n n!$.

反覆使用 L'Hôpital 法則得 $\lim\limits_{x\to 0+} x(\ln x)^n = (-1)^n \cdot \lim\limits_{x\to 0+} \dfrac{(\ln \frac{1}{x})^n}{\frac{1}{x}} = (-1)^n \lim\limits_{y\to\infty} \dfrac{(\ln y)^n}{y} = (-1)^n \cdot n \lim\limits_{y\to\infty} \dfrac{(\ln y)^{n-1}}{y} = (-1)^n \cdot n(n-1) \lim\limits_{y\to\infty} \dfrac{(\ln y)^{n-2}}{y} = \cdots = (-1)^n n! \lim\limits_{y\to\infty} \dfrac{1}{y} = 0$, 故 $\displaystyle\int_0^1 (\ln x)^n \,\mathrm{d}x = (-1)^n n!$ 成立.

定理. 給定 $0 < a < \infty$.

- $\displaystyle\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x$ 當 $p > 1$ 收斂至 $\dfrac{a^{1-p}}{p-1}$, 當 $p \leqslant 1$ 發散至 ∞.
- $\displaystyle\int_0^a \frac{1}{x^p}\,\mathrm{d}x$ 當 $p < 1$ 收斂至 $\dfrac{a^{1-p}}{1-p}$, 當 $p \geqslant 1$ 發散至 ∞.

定理. 令 $-\infty \leqslant a < b \leqslant \infty$, f, g 在 (a,b) 連續, 且 $0 \leqslant f(x) \leqslant g(x)\ \forall x$.

- 若 $\displaystyle\int_a^b g(x)\,\mathrm{d}x$ 收斂, $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ 收斂.
- 若 $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ 發散, $\displaystyle\int_a^b g(x)\,\mathrm{d}x$ 發散.

定理. 令 f, g 在 $[a, \infty)$, $a \in \mathbb{R}$ 連續, 均為正値, 且 $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)}$ 存在, 則 $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$ 與 $\displaystyle\int_a^\infty g(x)\,\mathrm{d}x$ 同斂散.

例. 證明 $\displaystyle\int_0^\infty e^{-x^2}\,\mathrm{d}x$ 收斂.

解. 由 $e^{-x^2} \leqslant 1\ \forall 0 \leqslant x < 1$ 及 $e^{-x^2} \leqslant e^{-x}\ \forall x \geqslant 1$, $\displaystyle\int_0^\infty e^{-x^2}\,\mathrm{d}x = \int_0^1 e^{-x^2}\,\mathrm{d}x + \int_1^\infty e^{-x^2}\,\mathrm{d}x \leqslant \int_0^1 1\,\mathrm{d}x + \int_1^\infty e^{-x}\,\mathrm{d}x = 1 + \frac{1}{e}$, 故 $\displaystyle\int_0^\infty e^{-x^2}\,\mathrm{d}x$ 收斂.

例. 定義 Γ 函數 $\Gamma(x) \equiv \displaystyle\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t$; 已知 $\displaystyle\int_0^\infty e^{-t^2}\,\mathrm{d}t = \frac{\sqrt{\pi}}{2}$.

1. 證明 $\Gamma(x)$ 收斂, $\forall x > 0$.
2. 證明 $\Gamma(x+1) = x\,\Gamma(x)$, $\forall x > 0$.
3. 證明 $\Gamma(n+1) = n!$, $\forall n \in \mathbb{N}$.
4. 求 $\Gamma\Big(\dfrac{1}{2}\Big)$ 與 $\Gamma\Big(\dfrac{3}{2}\Big)$.

解.

1. (證一) 由 $\lim\limits_{t\to\infty} t^{x-1}e^{-\frac{t}{2}} = 0$, $\exists T > 0$ 使 $t^{x-1}e^{-\frac{t}{2}} \leqslant 1\ \forall t \geqslant T$. $\displaystyle\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t = \int_0^T t^{x-1}e^{-t}\,\mathrm{d}t + \int_T^\infty t^{x-1}e^{-t}\,\mathrm{d}t \leqslant \int_0^T t^{x-1}\,\mathrm{d}t + \int_T^\infty 1\cdot e^{-\frac{t}{2}}\,\mathrm{d}t = \frac{T^x}{x} + 2\,e^{-\frac{T}{2}}$. 故 $\forall x > 0$, $\displaystyle\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t$ 收斂.

   (證二) $\displaystyle\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t = \int_0^1 t^{x-1}e^{-t}\,\mathrm{d}t + \int_1^\infty t^{x-1}e^{-t}\,\mathrm{d}t$. $\displaystyle\int_1^\infty t^{x-1}e^{-t}\,\mathrm{d}t$ 收斂, 因為

   - $\displaystyle\int_0^1 t^{x-1}e^{-t}\,\mathrm{d}t \leqslant \int_0^1 t^{x-1}\,\mathrm{d}t = \frac{t^x}{x}\bigg|_{x=0}^{x=1} = \frac{1}{x}$, $\displaystyle\int_0^1 t^{x-1}e^{-t}\,\mathrm{d}t$ 收斂.
   - $\displaystyle\int_1^\infty \frac{1}{t^2}\,\mathrm{d}t$ 收斂, $\lim\limits_{t\to\infty} \dfrac{t^{x-1}e^{-t}}{\frac{1}{t^2}} = \lim\limits_{t\to\infty} \dfrac{t^{x+1}}{e^t} = 0$, $\displaystyle\int_1^\infty t^{x-1}e^{-t}\,\mathrm{d}t$ 收斂.

2. $\Gamma(x+1) = \displaystyle\int_0^\infty t^x e^{-t}\,\mathrm{d}t = \lim_{\substack{a\to 0+\\ b\to\infty}} \int_a^b t^x e^{-t}\,\mathrm{d}t$. 令 $u = t^x$, 則 $\mathrm{d}u = xt^{x-1}\,\mathrm{d}t$. 令 $\mathrm{d}v = e^{-t}\,\mathrm{d}t$, 則 $v = -e^{-t}$.

   故 $\displaystyle\lim_{\substack{a\to 0+\\ b\to\infty}} \int_a^b t^x e^{-t}\,\mathrm{d}t = \lim_{\substack{a\to 0+\\ b\to\infty}} \Big(-t^x e^{-t}\Big|_a^b + \int_a^b e^{-t}\cdot xt^{x-1}\,\mathrm{d}t\Big) = 0 + x\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t = x\,\Gamma(x)$.

3. 由上 $\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\,\Gamma(n-2) = \cdots = n(n-1)\cdots 2\,\Gamma(1)$, 又 $\Gamma(1) = \lim_{b\to\infty}\int_0^b e^{-t}\,\mathrm{d}t = \lim_{b\to\infty} -e^{-t}\Big|_0^b = \lim_{b\to\infty} 1-e^{-b} = 1$, 得證.

4. (a) $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}}e^{-t}\,\mathrm{d}t = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,\mathrm{d}t$. 令 $u = \sqrt{t}$, 則 $t = u^2$, $\mathrm{d}u = \frac{1}{2\sqrt{t}}\,\mathrm{d}t \implies \frac{1}{\sqrt{t}}\,\mathrm{d}t = 2\,\mathrm{d}u$. 積分範圍 t 由 0 至 ∞, 則變數變換後 u 由 $\sqrt{0} = 0$ 至 $\sqrt{\infty} = \infty$, 故 $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,\mathrm{d}t = 2\int_0^\infty e^{-u^2}\,\mathrm{d}u = 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$.

   (b) 由 $\Gamma(x+1) = x\,\Gamma(x)$, $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$.