第七章 重積分

7.1 二重積分

概念與基本性質

二重積分 $\approx$ (帶符號) 體積: z 軸上方為正, 下方為負.

定義. 給定 $\Omega = [a, b] \times [c, d]$, $b \geqslant a$, $d \geqslant c$, $f : \Omega \to \mathbb{R}$.

- Ω 分割 $\mathbb{P} : a = x_0 < x_1 < x_2 < \cdots < x_n = b$, $c = y_0 < y_1 < y_2 < \cdots < y_m = d$
- $\Delta x_k = x_k - x_{k-1}$, $\Delta y_l = y_l - y_{l-1}$, $k = 1, 2, \ldots, n$; $l = 1, 2, \ldots, m$
- $\|\mathbb{P}\| = \max \{\Delta x_k \Delta y_l \mid 1 \leqslant k \leqslant n, 1 \leqslant l \leqslant m\}$
- 樣本點 (ξ_k, ζ_l): $x_{k-1} \leqslant \xi_k \leqslant x_k$, $y_{l-1} \leqslant \zeta_l \leqslant y_l$, $k = 1, 2, \ldots, n$; $l = 1, 2, \ldots, m$.
- $u_{k,l} = \max \{f(x, y) \mid x_{k-1} \leqslant x \leqslant x_k, y_{l-1} \leqslant y \leqslant y_l\}$, $\ell_{k,l} = \min \{f(x, y) \mid x_{k-1} \leqslant x \leqslant x_k, y_{l-1} \leqslant y \leqslant y_l\}$, $k = 1, 2, \ldots, n$; $l = 1, 2, \ldots, m$.
- $R(f, \mathbb{P}) = \sum_{l=1}^{m} \sum_{k=1}^{n} f(\xi_k, \zeta_l) \, \Delta x_k \Delta y_l$, $U(f, \mathbb{P}) = \sum_{l=1}^{m} \sum_{k=1}^{n} u_{k,l} \, \Delta x_k \Delta y_l$, $L(f, \mathbb{P}) = \sum_{l=1}^{m} \sum_{k=1}^{n} \ell_{k,l} \, \Delta x_k \Delta y_l$; 顯然 $L(f, \mathbb{P}) \leqslant R(f, \mathbb{P}) \leqslant U(f, \mathbb{P})$.
- 求 $\lim_{\|\mathbb{P}\| \to 0} R(f, \mathbb{P})$. 若對不同分割與樣本點選取此極限均存在且相等, 稱 f 在 Ω 可積 (分); $f(x, y)$ 在 Ω 的定積分為 $\int_{\Omega} f(x, y) \, \mathrm{d}A \equiv \lim_{\|\mathbb{P}\| \to 0} R(f, \mathbb{P})$.

性質.

- 若 f 在 Ω 有界, 且除了 Ω 中有限個平滑曲線外 f 在 Ω 連續, 則 f 在 Ω 可積分.
- 若 $\Omega = \Omega_1 \cap \Omega_2$ 且 Ω_1, Ω_2 均為矩形, f 在 Ω_1, Ω_2 均可積, 則 $\int_{\Omega} f \, \mathrm{d}A = \int_{\Omega_1} f \, \mathrm{d}A + \int_{\Omega_2} f \, \mathrm{d}A$.
- 若 f_1, f_2 均在 Ω 可積且 $f_1(x, y) \leqslant f_2(x, y) \; \forall (x, y) \in \Omega$, 則 $\int_{\Omega} f_1 \, \mathrm{d}A \leqslant \int_{\Omega} f_2 \, \mathrm{d}A$.
- 若 $\alpha, \beta \in \mathbb{R}$, f_1, f_2 均在 Ω 可積, 則 $\int_{\Omega} (\alpha f_1 + \beta f_2) \, \mathrm{d}A = \alpha \int_{\Omega} f_1 \, \mathrm{d}A + \beta \int_{\Omega} f_2 \, \mathrm{d}A$.

逐次積分

矩形積分區域

定理 (Fubini). 若 $\Omega = [a, b] \times [c, d]$, $f : \Omega \to \mathbb{R}$ 為可積, 則

$$\int_{\Omega} f(x, y) \, \mathrm{d}A = \int_{c}^{d} \left\{ \int_{a}^{b} f(x, y) \, \mathrm{d}x \right\} \mathrm{d}y = \int_{a}^{b} \left\{ \int_{c}^{d} f(x, y) \, \mathrm{d}y \right\} \mathrm{d}x$$

例. 令 $\Omega = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$, 則 $\int_{\Omega} \sqrt{1 - x^2} \, \mathrm{d}A = \int_{-2}^{2} \left\{ \int_{-1}^{1} \sqrt{1 - x^2} \, \mathrm{d}x \right\} \mathrm{d}y = \frac{\pi}{2} \times 4 = 2\pi$.

例. 若 $\Omega = [0, 2] \times [1, 3]$, 求 $\int_{\Omega} x^2 y \, \mathrm{d}A$.

解.

- 由 Fubini 定理 $\int_{\Omega} x^2 y \, \mathrm{d}A = \int_{0}^{2} \int_{1}^{3} x^2 y \, \mathrm{d}y \, \mathrm{d}x = \int_{1}^{3} \int_{0}^{2} x^2 y \, \mathrm{d}x \, \mathrm{d}y$.

- $\displaystyle\int_0^2\int_1^3 x^2 y\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 x^2\Big(\frac{y^2}{2}\,\Big|_1^3\Big)\,\mathrm{d}x = \int_0^2 x^2\Big(\frac{9}{2}-\frac{1}{2}\Big)\,\mathrm{d}x = \frac{4}{3}x^3\,\Big|_0^2 = \frac{32}{3}.$

- $\displaystyle\int_1^3\int_0^2 x^2 y\,\mathrm{d}x\,\mathrm{d}y = \int_1^3 y\Big(\frac{x^3}{3}\,\Big|_0^2\Big)\,\mathrm{d}y = \int_1^3 y\,\frac{8}{3}\,\mathrm{d}x = \frac{4}{3}y^2\,\Big|_1^3 = \frac{36}{3}-\frac{4}{3} = \frac{32}{3}.$

例. 若 $\Omega = [0,\,1]\times[0,\,3]$, 求 $\displaystyle\int_\Omega e^{2x+y}\,\mathrm{d}A$.

解.

- 由 Fubini 定理 $\displaystyle\int_\Omega e^{2x+y}\,\mathrm{d}A = \int_0^3\int_0^1 e^{2x+y}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\int_0^3 e^{2x+y}\,\mathrm{d}y\,\mathrm{d}x.$

- $\displaystyle\int_0^3\int_0^1 e^{2x+y}\,\mathrm{d}x\,\mathrm{d}y = \int_0^3 e^y\Big(\int_0^1 e^{2x}\,\mathrm{d}x\Big)\,\mathrm{d}y = \int_0^3 e^y\Big(\frac{e^{2x}}{2}\,\Big|_0^1\Big)\,\mathrm{d}y = \frac{(e^2-1)(e^3-1)}{2}.$

- $\displaystyle\int_0^1\int_0^3 e^{2x+y}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 e^{2x}\Big(\int_0^3 e^y\,\mathrm{d}y\Big)\,\mathrm{d}x = \int_0^1 e^{2x}\Big(\frac{e^{2x}}{2}\,\Big|_0^3\Big)\,\mathrm{d}y = \frac{(e^2-1)(e^3-1)}{2}.$

例. 若 $\Omega = [0,\,\pi]\times[0,\,2\pi]$, 求 $\displaystyle\int_\Omega \sin(x+y)\,\mathrm{d}A$.

解.

- 由 Fubini 定理 $\displaystyle\int_\Omega \sin(x+y)\,\mathrm{d}A = \int_0^\pi\int_0^{2\pi}\sin(x+y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^{2\pi}\int_0^\pi\sin(x+y)\,\mathrm{d}x\,\mathrm{d}y.$

- $\displaystyle\int_0^\pi\int_0^{2\pi}\sin(x+y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^\pi\Big(-\cos(x+y)\,\Big|_0^{2\pi}\Big)\,\mathrm{d}x = -\int_0^\pi(\cos(x+2\pi)-\cos x)\,\mathrm{d}x = 0.$

- $\displaystyle\int_0^{2\pi}\int_0^\pi\sin(x+y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\Big(-\cos(x+y)\,\Big|_0^\pi\Big)\,\mathrm{d}y = -\int_0^{2\pi}(\cos(\pi+y)-\cos y)\,\mathrm{d}y = -\int_0^{2\pi}(\cos\pi\cos y -$
$\sin\pi\sin y - \cos y)\,\mathrm{d}y = 2\int_0^{2\pi}\cos y\,\mathrm{d}y = 0.$

例. 若 $\Omega = [1,\,2]\times[0,\,1]$, 求 $\displaystyle\int_\Omega y e^{xy}\,\mathrm{d}A$.

解.

- 由 Fubini 定理 $\displaystyle\int_\Omega y e^{xy}\,\mathrm{d}A = \int_0^1\int_1^2 y e^{xy}\,\mathrm{d}x\,\mathrm{d}y = \int_1^2\int_0^1 y e^{xy}\,\mathrm{d}y\,\mathrm{d}x.$

- $\displaystyle\int_0^1\int_1^2 y e^{xy}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 y\Big(\frac{e^{xy}}{y}\,\Big|_1^2\Big)\,\mathrm{d}y = \int_0^1(e^{2y}-e^y)\,\mathrm{d}y = \frac{e^2}{2}-e+\frac{1}{2}.$

- $\displaystyle\int_1^2\int_0^1 y e^{xy}\,\mathrm{d}y\,\mathrm{d}x = \int_1^2\Big(y\,\frac{e^{xy}}{x}-\frac{e^{xy}}{x^2}\,\Big|_0^1\Big)\,\mathrm{d}x = \int_1^2\Big(\underbrace{\frac{e^x}{x}-\frac{e^x}{x^2}}_{(\frac{e^x}{x})'=\frac{e^x}{x}-\frac{e^x}{x^2}}+\frac{1}{x^2}\Big)\,\mathrm{d}x = \Big(\frac{e^x}{x}-\frac{1}{x}\Big)\,\Big|_1^2 = \frac{e^2}{2}-e+\frac{1}{2}.$

習題. 求下列重積分.

1. $\displaystyle\int_2^3\int_1^5(x+2y)\,\mathrm{d}x\,\mathrm{d}y$

2. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\cos y}e^x\sin y\,\mathrm{d}x\,\mathrm{d}y$

3. $\displaystyle\int_{-\pi}^\pi\int_0^2 r\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta$

4. $\displaystyle\int_0^{\ln 4}\int_0^{\ln 3}e^{x+y}\,\mathrm{d}x\,\mathrm{d}y$

5. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta}\rho^2\sin\theta\,\mathrm{d}\rho\,\mathrm{d}\theta$

6. $\displaystyle\int_0^1\int_{x^2}^{x^3}(x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x$

7. $\displaystyle\int_1^2\int_0^y x\sqrt{y^2-x^2}\,\mathrm{d}x\,\mathrm{d}y$

8. $\displaystyle\int_0^1\int_{y^4}^{y^2}\sqrt{\frac{y}{x}}\,\mathrm{d}x\,\mathrm{d}y$

9. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^a\frac{r}{\sqrt{a^2-r^2\cos^2\theta}}\,\mathrm{d}r\,\mathrm{d}\theta$

解.

1. $\displaystyle\int_2^3\int_1^5 (x+2y)\,\mathrm{d}x\,\mathrm{d}y = \int_2^3 \Big(\frac{x^2}{2}+2yx\Big)\Big|_1^5\,\mathrm{d}y = \int_2^3 (12+8y)\,\mathrm{d}y = (12y+4y^2)\Big|_2^3 = 32$

2. $\displaystyle\int_{-\pi}^{\pi}\int_0^2 r\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta = \Big(\int_{-\pi}^{\pi}\sin\theta\,\mathrm{d}\theta\Big)\cdot\Big(\int_0^2 r\,\mathrm{d}r\Big) = 0$

3. $\displaystyle\int_0^{\ln 4}\int_0^{\ln 3} e^{x+y}\,\mathrm{d}x\,\mathrm{d}y = \Big(\int_0^{\ln 4} e^y\,\mathrm{d}y\Big)\cdot\Big(\int_0^{\ln 3} e^x\,\mathrm{d}x\Big) = (4-1)(3-1) = 6$

4. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\cos y} e^x\sin y\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{2}}\sin y\Big(\int_0^{\cos y} e^x\,\mathrm{d}x\Big)\,\mathrm{d}y = \int_0^{\frac{\pi}{2}}\sin y\,(e^{\cos y}-1)\,\mathrm{d}y = \big(-e^{\cos y}+\cos y\big)\Big|_0^{\frac{\pi}{2}} = e-2$

5. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta} \rho^2\sin\theta\,\mathrm{d}\rho\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}}\sin\theta\Big(\int_0^{\cos\theta}\rho^2\,\mathrm{d}\rho\Big)\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}}\sin\theta\,\frac{\cos^3\theta}{3}\,\mathrm{d}\theta = -\frac{\cos^4\theta}{12}\Big|_0^{\frac{\pi}{2}} = \frac{1}{12}$

6. $\displaystyle\int_0^1\int_{x^2}^{x^3} (x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\Big(x^2 y+\frac{y^3}{3}\Big)\Big|_{y=x^2}^{y=x^3}\,\mathrm{d}x = \int_0^1\Big(x^2\cdot x^3+\frac{x^9}{3}-x^2\cdot x^2-\frac{x^6}{3}\Big)\,\mathrm{d}x = -\frac{1}{21}$

7. $\displaystyle\int_0^y x\sqrt{y^2-x^2}\,\mathrm{d}x$ 中, 令 $u=y^2-x^2$, 則 $\mathrm{d}u=-2x\,\mathrm{d}x \implies x\,\mathrm{d}x = \frac{-1}{2}\,\mathrm{d}u$. 積分範圍 x 由 0 至 y, 則變數變換 $u=y^2-x^2$ 由 $y^2-0^2=y^2$ 至 $y^2-y^2=0$. 則 $\displaystyle\int_0^y x\sqrt{y^2-x^2}\,\mathrm{d}x = \int_{y^2}^0 \sqrt{u}\,\frac{-1}{2}\,\mathrm{d}u = \frac{1}{2}\int_0^{y^2}\sqrt{u}\,\mathrm{d}u = \frac{1}{2}\frac{2}{3}u^{\frac{3}{2}}\Big|_0^{y^2} = \frac{y^3}{3}$. 故 $\displaystyle\int_1^2\int_0^y x\sqrt{y^2-x^2}\,\mathrm{d}x\,\mathrm{d}y = \int_1^2\frac{y^3}{3}\,\mathrm{d}y = \frac{y^4}{12}\Big|_1^2 = \frac{5}{4}$.

8. $\displaystyle\int_0^1\int_{y^4}^{y^2}\sqrt{\frac{y}{x}}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\sqrt{y}\Big(\int_{y^4}^{y^2}\frac{1}{\sqrt{x}}\,\mathrm{d}x\Big)\,\mathrm{d}y = \int_0^1\sqrt{y}\,(2\sqrt{x})\Big|_{y^4}^{y^2}\,\mathrm{d}y = 2\int_0^1\big(y^{\frac{3}{2}}-y^{\frac{5}{2}}\big)\,\mathrm{d}y = \frac{8}{35}$

9. $\displaystyle\int_0^a \frac{r}{\sqrt{a^2-r^2\cos^2\theta}}\,\mathrm{d}r$ 中, 令 $u=a^2-r^2\cos^2\theta$, 則 $\mathrm{d}u=-2\cos^2\theta\,r\,\mathrm{d}r \implies r\,\mathrm{d}r = \frac{-1}{2\cos^2\theta}\,\mathrm{d}u$. 積分範圍 r 由 0 至 a, 則變數變換 $u=a^2-r^2\cos^2\theta$ 由 $a^2-0^2\cos^2\theta=a^2$ 至 $a^2-a^2\cos^2\theta=a^2\sin^2\theta$. 則 $\displaystyle\int_0^a\frac{r}{\sqrt{a^2-r^2\cos^2\theta}}\,\mathrm{d}r = \int_{a^2}^{a^2\sin^2\theta}\frac{1}{\sqrt{u}}\frac{-1}{2\cos^2\theta}\,\mathrm{d}u = \int_{a^2\sin^2\theta}^{a^2}\frac{1}{2\cos^2\theta\sqrt{u}}\,\mathrm{d}u = \frac{\sqrt{u}}{\cos^2\theta}\Big|_{u=a^2\sin^2\theta}^{u=a^2} = a\frac{1-\sin\theta}{\cos^2\theta}$. 故 $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^a\frac{r}{\sqrt{a^2-r^2\cos^2\theta}}\,\mathrm{d}r\,\mathrm{d}\theta = a\int_0^{\frac{\pi}{2}}\frac{1-\sin\theta}{\cos^2\theta}\,\mathrm{d}\theta = a\frac{\sin\theta-1}{\cos\theta}\Big|_0^{\frac{\pi}{2}} = a\Big(\lim_{\theta\to\frac{\pi}{2}}\frac{\sin\theta-1}{\cos\theta}+1\Big) = a.$

一般積分區域

結論 (基本區域積分)**.**
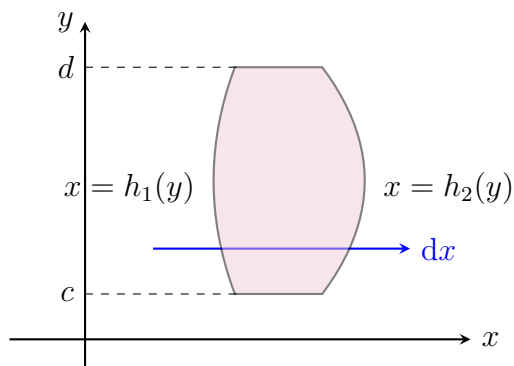
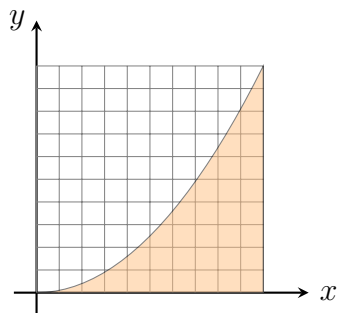1. 選擇初始積分方向 v: 延 x 方向為 $\mathrm{d}x$, 延 y 方向為 $\mathrm{d}y$.

2. 想像積分方向 v 為一射線, 進入區域之邊界函數為積分下界, 離開區域之邊界函數為積分上界; 邊界函數本身不含變數 v.

3. 次第區域為原始區域延 v 之投影 (令 $v=$ (常數) 之方程式, 或邊界的交集); 若此投影區域維度 > 1, 選擇積分方向.
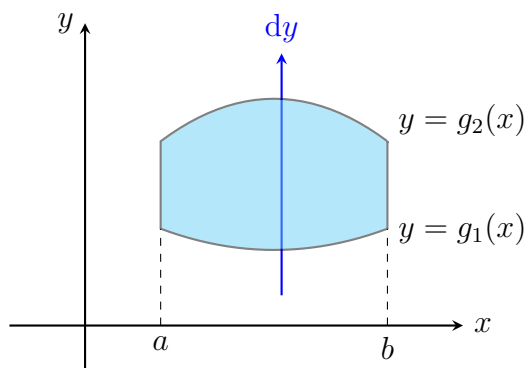
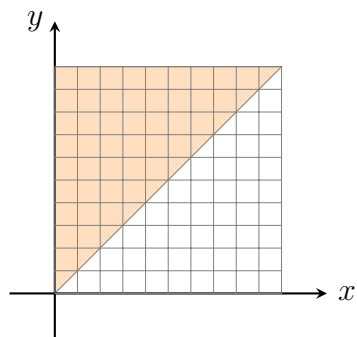$$\int_{\Omega} f(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

$$\int_{\Omega} f(x,y)\,\mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

例. 求 $\displaystyle\int_{\Omega} x\cos y\,\mathrm{d}A$, Ω 為 $y=0$, $x=1$, 與 $y=x^2$ 圍成之區域.

解. $\displaystyle\int_{\Omega} x\cos y\,\mathrm{d}A = \int_0^1 \int_0^{x^2} x\cos y\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 x\left(\sin y\,\Big|_0^{x^2}\right)\mathrm{d}x$
$\displaystyle = \int_0^1 x\sin x^2\,\mathrm{d}x = \frac{1-\cos 1}{2}$

例. 求 $\displaystyle\int_{\Omega} e^{-y^2}\,\mathrm{d}A$, $\Omega = \{(x,y)\mid 0\leqslant y\leqslant 1,\ 0\leqslant x\leqslant y\}$.

解. $\displaystyle\int_{\Omega} e^{-y^2}\,\mathrm{d}A = \int_0^1 \int_0^y e^{-y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 y e^{-y^2}\,\mathrm{d}y = \frac{1-e^{-1}}{2}$
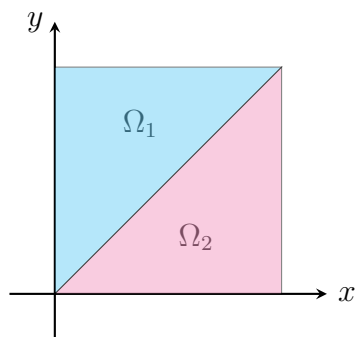
例. 求 $\displaystyle\int_0^1 \int_0^1 e^{\max\{x^2,y^2\}}\,\mathrm{d}x\,\mathrm{d}y$.

解. $\displaystyle\int_0^1 \int_0^1 e^{\max\{x^2,y^2\}}\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega_1} e^{y^2}\,\mathrm{d}A + \int_{\Omega_2} e^{x^2}\,\mathrm{d}A$
$\displaystyle = \int_0^1 \int_0^y e^{y^2}\,\mathrm{d}x\,\mathrm{d}y + \int_0^1 \int_0^x e^{x^2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 y e^{y^2}\,\mathrm{d}y + \int_0^1 x e^{x^2}\,\mathrm{d}x = e-1.$

積分順序交換
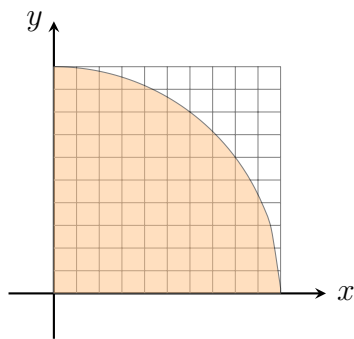
例. 求 $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{1-y^2}\,\mathrm{d}y\,\mathrm{d}x$.

解. $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{1-y^2}\,\mathrm{d}y\,\mathrm{d}x=\int_0^1\int_0^{\sqrt{1-y^2}}\sqrt{1-y^2}\,\mathrm{d}x\,\mathrm{d}y=\int_0^1(1-y^2)\,\mathrm{d}y=\frac{2}{3}$.

$\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{1-y^2}\,\mathrm{d}y\,\mathrm{d}x=\frac{1}{2}\int_0^1(\sin^{-1}\sqrt{1-x^2}+x\sqrt{1-x^2})\,\mathrm{d}x=\frac{1}{2}\left(1+\frac{1}{3}\right)=\frac{2}{3}$.

例. 將下列積分以不同積分順序寫出.

1. $\displaystyle\int_0^2\int_{x^2}^{2x}f(x,y)\,\mathrm{d}y\,\mathrm{d}x$

2. $\displaystyle\int_0^1\int_{\sin^{-1}y}^{\frac{\pi}{2}}f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

3. $\displaystyle\int_{-1}^2\int_{x^2}^{x+2}f(x,y)\,\mathrm{d}y\,\mathrm{d}x$

4. $\displaystyle\int_0^1\int_0^{2y}f(x,y)\,\mathrm{d}x\,\mathrm{d}y+\int_1^3\int_0^{3-y}f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

解.

1. $\displaystyle\int_0^2\int_{x^2}^{2x}f(x,y)\,\mathrm{d}y\,\mathrm{d}x=\int_0^4\int_{\sqrt{y}}^{\frac{y}{2}}f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

2. $\displaystyle\int_0^1\int_{\sin^{-1}y}^{\frac{\pi}{2}}f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_0^{\frac{\pi}{2}}\int_0^{\sin x}f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

3. $\displaystyle\int_{-1}^2\int_{x^2}^{x+2}f(x,y)\,\mathrm{d}y\,\mathrm{d}x=\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}}f(x,y)\,\mathrm{d}x\,\mathrm{d}y+\int_1^4\int_{y-2}^{\sqrt{y}}f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

4. $\displaystyle\int_0^1\int_0^{2y}f(x,y)\,\mathrm{d}x\,\mathrm{d}y+\int_1^3\int_0^{3-y}f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_0^2\int_{\frac{x}{2}}^{3-x}f(x,y)\,\mathrm{d}y\,\mathrm{d}x$

例. 證明 $\displaystyle\int_0^x\int_0^t F(u)\,\mathrm{d}u\,\mathrm{d}t=\int_0^x(x-u)F(u)\,\mathrm{d}u$.

解. $\displaystyle\int_0^x\int_0^t F(u)\,\mathrm{d}u\,\mathrm{d}t=\int_0^x\int_u^x F(u)\,\mathrm{d}t\,\mathrm{d}u=\int_0^x(x-u)F(u)\,\mathrm{d}u$.

例. 求 $\displaystyle\int_0^1\int_x^1 e^{-y^2}\,\mathrm{d}y\,\mathrm{d}x$.

解. $\displaystyle\int_0^1\int_x^1 e^{-y^2}\,\mathrm{d}y\,\mathrm{d}x=\int_0^1\int_0^y e^{-y^2}\,\mathrm{d}x\,\mathrm{d}y=\int_0^1 ye^{-y^2}\,\mathrm{d}y=\frac{1-e^{-1}}{2}$

例. 求 $\displaystyle\int_0^1\int_y^1\frac{\sin x}{x}\,\mathrm{d}x\,\mathrm{d}y$.

解. $\displaystyle\int_0^1\int_y^1\frac{\sin x}{x}\,\mathrm{d}x\,\mathrm{d}y=\int_0^1\int_0^x\frac{\sin x}{x}\,\mathrm{d}y\,\mathrm{d}x=\int_0^1\sin x\,\mathrm{d}x=1-\cos 1$.

例. 求 $\displaystyle\int_0^1\int_{\sin^{-1}y}^{\frac{\pi}{2}}\cos x\sqrt{1+\cos^2 x}\,\mathrm{d}x\,\mathrm{d}y$.

解.

- $\int_0^1 \int_{\sin^{-1}y}^{\frac{\pi}{2}} \cos x \sqrt{1+\cos^2 x}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{2}} \int_0^{\sin x} \cos x \sqrt{1+\cos^2 x}\,\mathrm{d}y\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin x \cos x \sqrt{1+\cos^2 x}\,\mathrm{d}x$
  $= \int_0^1 u\sqrt{1+u^2}\,\mathrm{d}u = \int_1^2 \sqrt{v}\,\frac{1}{2}\,\mathrm{d}v = \frac{2\sqrt{2}-1}{3}$, 其中變數變換 $u = \cos x$ 與 $v = 1+u^2$.

- 令 $u = \sin x$, $\int_{\sin^{-1}y}^{\frac{\pi}{2}} \cos x \sqrt{1+\cos^2 x}\,\mathrm{d}x = \int_y^1 \sqrt{2-u^2}\,\mathrm{d}u$. 令 $u = \sqrt{2}\sin\theta$, 則 $\int_y^1 \sqrt{2-u^2}\,\mathrm{d}u =$
  $\int_{\sin^{-1}\frac{y}{\sqrt{2}}}^{\frac{\pi}{4}} 2\cos^2\theta\,\mathrm{d}\theta = \int_{\sin^{-1}\frac{y}{\sqrt{2}}}^{\frac{\pi}{4}} (1+\cos 2\theta)\,\mathrm{d}\theta = (\theta+\sin\theta\cos\theta)\Big|_{\sin^{-1}\frac{y}{\sqrt{2}}}^{\frac{\pi}{4}} = \frac{\pi}{4}+\frac{1}{2}-\sin^{-1}\frac{y}{\sqrt{2}}-\frac{y}{\sqrt{2}}\cdot\frac{\sqrt{2-y^2}}{\sqrt{2}} =$
  $\frac{\pi}{4}+\frac{1}{2}-\sin^{-1}\frac{y}{\sqrt{2}}-\frac{y\sqrt{2-y^2}}{2}$. 故 $\int_0^1 \int_{\sin^{-1}y}^{\frac{\pi}{2}} \cos x \sqrt{1+\cos^2 x}\,\mathrm{d}x\,\mathrm{d}y$
  $= \int_0^1 \left(\frac{\pi}{4}+\frac{1}{2}-\sin^{-1}\frac{y}{\sqrt{2}}-\frac{y\sqrt{2-y^2}}{2}\right)\mathrm{d}y = \frac{\pi}{4}+\frac{1}{2}-\left(\frac{\pi}{4}+1-\sqrt{2}\right)-\left(\frac{\sqrt{2}}{3}-\frac{1}{6}\right) = \frac{2\sqrt{2}-1}{3}$.

例. 求 $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,\mathrm{d}x$, $b > a > 0$.

解. 由 $\int_a^b e^{-xy}\,\mathrm{d}y = \frac{1}{-x}e^{-xy}\Big|_{y=a}^{y=b} = \frac{1}{-x}(e^{-xb}-e^{-xa}) = \frac{e^{-ax}-e^{-bx}}{x}$, $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,\mathrm{d}x = \int_0^\infty \left(\int_a^b e^{-xy}\,\mathrm{d}y\right)\mathrm{d}x =$
$\int_a^b \left(\int_0^\infty e^{-xy}\,\mathrm{d}x\right)\mathrm{d}y = \int_a^b \frac{1}{y}\,\mathrm{d}y = \ln\frac{b}{a}$.

例. 求 $\int_0^\infty \frac{\tan^{-1}\pi x - \tan^{-1}x}{x}\,\mathrm{d}x$.

解. 由 $\int_x^{\pi x} \frac{1}{1+y^2}\,\mathrm{d}y = \tan^{-1}\pi x - \tan^{-1}x$, $\int_0^\infty \frac{\tan^{-1}\pi x - \tan^{-1}x}{x}\,\mathrm{d}x = \int_0^\infty \frac{1}{x}\left(\int_x^{\pi x} \frac{1}{1+y^2}\,\mathrm{d}y\right)\mathrm{d}x$
$= \int_0^\infty \int_x^{\pi x} \frac{1}{x(1+y^2)}\,\mathrm{d}y\,\mathrm{d}x = \int_0^\infty \int_{\frac{y}{\pi}}^{y} \frac{1}{x(1+y^2)}\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty \frac{1}{1+y^2}\left(\int_{\frac{y}{\pi}}^y \frac{1}{x}\,\mathrm{d}x\right)\mathrm{d}y = \int_0^\infty \frac{1}{1+y^2}\left(\ln y - \ln\frac{y}{\pi}\right)\mathrm{d}y =$
$\int_0^\infty \frac{\ln\pi}{1+y^2}\,\mathrm{d}y = \frac{\pi}{2}\ln\pi$.

例. 求 $\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x$.

解. 由 $\int_0^\infty e^{-xy}\,\mathrm{d}y = \frac{1}{x}$, $x > 0$; 又 $\int e^{ax}\sin bx\,\mathrm{d}x = \frac{e^{ax}(a\cdot\sin bx - b\cdot\cos bx)}{a^2+b^2}$.
故 $\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \int_0^\infty \sin x\left(\int_0^\infty e^{-xy}\,\mathrm{d}y\right)\mathrm{d}x = \int_0^\infty\left(\int_0^\infty e^{-xy}\sin x\,\mathrm{d}y\right)\mathrm{d}x = \int_0^\infty\left(\int_0^\infty e^{-xy}\sin x\,\mathrm{d}x\right)\mathrm{d}y =$
$\int_0^\infty \frac{e^{-xy}((-y)\cdot\sin x - 1\cdot\cos x)}{(-y)^2+1^2}\Big|_{x=0}^{x=\infty}\,\mathrm{d}y = \int_0^\infty \frac{0-(-1)}{1+y^2}\,\mathrm{d}y = \int_0^\infty \frac{1}{1+y^2}\,\mathrm{d}y = \tan^{-1}y\Big|_0^\infty = \frac{\pi}{2}$.

變數變換

單變數積分 $\int_{[a,b]} f(x)\,\mathrm{d}x$ 中, 令 $x = x(u)$, 則 $\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u$; $\int_{[a,b]} f(x)\,\mathrm{d}x = \underbrace{\int_{x^{-1}[a,b]}}_{\text{由小到大}} f(x(u))\left|\frac{\mathrm{d}x}{\mathrm{d}u}\right|\mathrm{d}u$.

例.

- 求 $\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}}$: 令 $x = \sin u$, 則 $\mathrm{d}x = \cos u \cdot \mathrm{d}u$; x 由 0 至 1, 則 u 由 $\sin^{-1}0 = 0$ 至 $\sin^{-1}1 = \frac{\pi}{2}$. 故
  $\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int_0^{\frac{\pi}{2}} \frac{\cos u \cdot \mathrm{d}u}{\sqrt{1-\sin^2 u}} = \int_0^{\frac{\pi}{2}} \frac{\cos u \cdot \mathrm{d}u}{\cos u} = \int_0^{\frac{\pi}{2}} 1 \cdot \mathrm{d}u = \frac{\pi}{2}$.

- 求 $\int_{\frac{1}{2}}^{1} (1-2x)^2 \, \mathrm{d}x$: 令 $u = 1-2x$, 亦即 $x = \frac{1}{2} - \frac{u}{2}$, 則 $\mathrm{d}x = \frac{-1}{2} \, \mathrm{d}u$; x 由 $\frac{1}{2}$ 至 1, 則 u 由 $1 - 2 \cdot \frac{1}{2} = 0$ 至 $1 - 2 \cdot 1 = -1$. 故 $\int_{\frac{1}{2}}^{1} (1-2x)^2 \, \mathrm{d}x = \int_{0}^{-1} u^2 \left(\frac{-1}{2} \right) \mathrm{d}u = \int_{-1}^{0} u^2 \frac{1}{2} \, \mathrm{d}u = \frac{1}{6}$.

定理 (重積分變數變換). 給定 $\Omega_{\mathbf{x}}, \Omega_{\mathbf{u}} \subseteq \mathbb{R}^n$, $\mathbf{x} = \mathbf{x}(\mathbf{u}) : \Omega_{\mathbf{u}} \to \Omega_{\mathbf{x}}$ 且

- $\mathbf{x}$ 為嵌射 (亦即 $\mathbf{x}(\Omega_{\mathbf{u}}) = \Omega_{\mathbf{x}}$, $\mathbf{x}^{-1}(\Omega_{\mathbf{x}}) = \Omega_{\mathbf{u}}$)

- $\mathbf{x}$ 之各分量函數 x_j, $j = 1, 2 \ldots, n$ 為連續可偏微分

- $\forall \mathbf{u} \in \Omega_{\mathbf{u}}$, $\mathbf{x}$ 之 Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_n} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_n} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_n} & \cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix} (\mathbf{u}) \neq 0$

則對於可積函數 $f : \Omega_{\mathbf{x}} \to \mathbb{R}$, $\displaystyle\int_{\Omega_{\mathbf{x}}} f(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \int_{\mathbf{x}^{-1}(\Omega_{\mathbf{x}})} f(\mathbf{x}(\mathbf{u})) \left| J_{\mathbf{x}}(\mathbf{u}) \right| \mathrm{d}\mathbf{u} = \int_{\mathbf{x}^{-1}(\Omega_{\mathbf{x}})} f(\mathbf{x}(\mathbf{u})) \left| \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}} \right| \mathrm{d}\mathbf{u}$.

例. 利用變數變換 $x = u^2 - v^2$, $y = 2uv$ 求 $\displaystyle\int_{\Omega} y \, \mathrm{d}A$, Ω 為 $y \geqslant 0$, $y^2 = 4 - 4x$, $y^2 = 4 + 4x$ 所圍成之區域.

解. 由 $\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$, Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2 + v^2)$. 變數變換 $(x, y) \to$ (u, v) 後 Ω 由 $uv = 0$, $u^2 v^2 + u^2 - v^2 - 1 = (v^2 + 1)(u^2 - 1) = 0$, $u^2 v^2 - u^2 + v^2 - 1 = (u^2 + 1)(v^2 - 1) = 0$ 圍成, 亦即 $\Omega = \{0 \leqslant u \leqslant 1, \ 0 \leqslant v \leqslant 1\}$, 故 $\displaystyle\int_{\Omega} y \, \mathrm{d}A = \int_0^1 \int_0^1 2uv \left| 4(u^2 + v^2) \right| \mathrm{d}u \, \mathrm{d}v = 2$.

例. 求 $\displaystyle\int_{\Omega} (x + y)^2 \, \mathrm{d}A$, Ω 為 $x + y = 0$, $x + y = 1$, $2x - y = 0$, $2x - y = 3$ 所圍成之平行四邊形.

解. 令 $\begin{cases} u = x + y \\ v = 2x - y \end{cases}$, 則 $\begin{cases} x = \frac{1}{3}u + \frac{1}{3}v \\ y = \frac{2}{3}u - \frac{1}{3}v \end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{3}$. 變數變換 $(x, y) \to (u, v)$ 後 $\Omega = \{0 \leqslant u \leqslant 1, \ 0 \leqslant v \leqslant 3\}$, 故 $\displaystyle\int_{\Omega} (x + y)^2 \, \mathrm{d}A = \int_0^3 \int_0^1 u^2 \left| -\frac{1}{3} \right| \mathrm{d}u \, \mathrm{d}v = \frac{1}{3}$.

例. 求 $\displaystyle\int_{\Omega} \sqrt{x + y} \, \mathrm{d}A$, Ω 為頂點 $(0, 0)$, $(a, 0)$, $(0, a)$, $a > 0$ 之三角形.

解. 令 $\begin{cases} u = x + y \\ v = x - y \end{cases}$, 則 $\begin{cases} x = \frac{1}{2}u + \frac{1}{2}v \\ y = \frac{1}{2}u - \frac{1}{2}v \end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{vmatrix} = -\frac{1}{2}$. 變數變換前 Ω 由 $x + y = 0$, $x + y = a$, $x = 0$, $y = 0$ 所圍成, 變數變換 $(x, y) \to (u, v)$ 後 Ω 由 $u = 0$, $u = a$, $u + v = 0$, $u - v = 0$ 圍成, 故 $\displaystyle\int_{\Omega} \sqrt{x + y} \, \mathrm{d}A = \int_0^a \int_{-u}^u \sqrt{u} \left| -\frac{1}{2} \right| \mathrm{d}v \, \mathrm{d}u = \int_0^a u\sqrt{u} \, \mathrm{d}u = \frac{2 a^{\frac{5}{2}}}{5}$.

例. 求 $\displaystyle\int_{\Omega} (x + y) \, e^{x - y} \, \mathrm{d}A$, Ω 為頂點 $(4, 0)$, $(6, 2)$, $(4, 4)$, $(2, 2)$ 之四邊形.

解. 令 $\begin{cases} u = x + y \\ v = x - y \end{cases}$, 則 $\begin{cases} x = \frac{1}{2}u + \frac{1}{2}v \\ y = \frac{1}{2}u - \frac{1}{2}v \end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{vmatrix} = -\frac{1}{2}$. 變數變換前 Ω 由 $x - y = 0$, $x - y = 4$, $x + y = 4$, $x + y = 8$ 所圍成, 變數變換 $(x, y) \to (u, v)$ 後 Ω 由 $v = 0$, $v = 4$, $u = 4$, $u = 8$ 圍成, 故 $\displaystyle\int_{\Omega} (x + y) \, e^{x - y} \, \mathrm{d}A = \int_0^4 \int_4^8 u e^v \left| -\frac{1}{2} \right| \mathrm{d}u \, \mathrm{d}v = \frac{1}{2} \int_0^4 e^v \, \mathrm{d}v \int_4^8 u \, \mathrm{d}u = 12(e^4 - 1)$.

例. 求 $\displaystyle\int_\Omega e^{\frac{x+y}{x-y}}\,\mathrm{d}A$, Ω 為頂點 $(1, 0)$, $(2, 0)$, $(0, -2)$, $(0, -1)$ 之梯形.

解. 令 $\begin{cases} u = x+y \\ v = x-y \end{cases}$, 則 $\begin{cases} x = \frac{1}{2}u + \frac{1}{2}v \\ y = \frac{1}{2}u - \frac{1}{2}v \end{cases}$; Jacobian $J_\mathbf{x}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{vmatrix} = -\dfrac{1}{2}$. 變數變換前 Ω 由 $x - y = 1$, $x - y = 2$, $x = 0$, $y = 0$ 所圍成, 變數變換 $(x, y) \to (u, v)$ 後 Ω 由 $v = 1$, $v = 2$, $u + v = 0$, $u - v = 0$ 圍成, 故 $\displaystyle\int_\Omega e^{\frac{x+y}{x-y}}\,\mathrm{d}A = \int_1^2 \int_{-v}^{v} e^{\frac{u}{v}} \left| -\frac{1}{2} \right|\,\mathrm{d}u\,\mathrm{d}v = \frac{1}{2}\int_1^2 \left(v\,e^{\frac{u}{v}}\,\Big|_{u=-v}^{u=v} \right)\,\mathrm{d}v = \dfrac{3\left(e - e^{-1} \right)}{4}$.

例. 求 $y = x^2$, $y = 2x^2$, $x = y^2$, $x = 3y^2$ 所圍成之區域面積.

解. 令 $\begin{cases} u = \frac{y}{x^2} \\ v = \frac{x}{y^2} \end{cases}$, 則 $\begin{cases} x = (u^2 v)^{-\frac{1}{3}} \\ y = (uv^2)^{-\frac{1}{3}} \end{cases}$; Jacobian $J_\mathbf{x}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{3}(u^2 v)^{-\frac{4}{3}} \cdot 2uv & -\frac{1}{3}(u^2 v)^{-\frac{4}{3}} \cdot u^2 \\ -\frac{1}{3}(uv^2)^{-\frac{4}{3}} \cdot v^2 & -\frac{1}{3}(uv^2)^{-\frac{4}{3}} \cdot 2uv \end{vmatrix}$ $= \dfrac{1}{3u^2 v^2}$. 變數變換 $(x, y) \to (u, v)$ 後 Ω 由 $u = 1$, $u = 2$, $v = 1$, $v = 3$ 圍成, 亦即 $\Omega = \{1 \leqslant u \leqslant 2,\ 1 \leqslant v \leqslant 3\}$, 面積為 $\displaystyle\int_\Omega \mathrm{d}A = \int_1^3 \int_1^2 \left| \frac{1}{3u^2 v^2} \right|\,\mathrm{d}u\,\mathrm{d}v = \frac{1}{9}$.

例. 求 $\displaystyle\int_\Omega \frac{xy}{1 + x^2 y^2}\,\mathrm{d}A$, Ω 為 $xy = 1$, $xy = 5$, $x = 1$, $x = 6$ 所圍成之區域.

解. 令 $\begin{cases} u = xy \\ v = x \end{cases}$, 則 $\begin{cases} x = v \\ y = \frac{u}{v} \end{cases}$; Jacobian $J_\mathbf{x}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\dfrac{1}{v}$. 變數變換 $(x, y) \to (u, v)$ 後 Ω 由 $u = 1$, $u = 5$, $v = 1$, $v = 6$ 圍成, 亦即 $\Omega = \{1 \leqslant u \leqslant 5,\ 1 \leqslant v \leqslant 6\}$, 故 $\displaystyle\int_\Omega \frac{xy}{1 + x^2 y^2}\,\mathrm{d}A = \int_1^6 \int_1^5 \frac{u}{1+u^2} \left| -\frac{1}{v} \right|\,\mathrm{d}u\,\mathrm{d}v = \int_1^6 \frac{1}{v}\,\mathrm{d}v \int_1^5 \frac{u}{1+u^2}\,\mathrm{d}u = \dfrac{\ln 6 \cdot \ln 13}{2}$.

例. 求 $\displaystyle\int_\Omega \frac{y}{x}\,\mathrm{d}A$, Ω 為 $y = x$, $x^2 + 4y^2 = 4$, $y \geqslant 0$ 所圍成之區域.

解. 令 $\begin{cases} x = 2r\cos\theta \\ y = r\sin\theta \end{cases}$, Jacobian $J_\mathbf{x}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2\cos\theta & -2r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = 2r$. 變數變換 $(x, y) \to (r, \theta)$ 後 Ω 由 $r = 0$, $r = 1$, $\theta = 0$, $\theta = \tan^{-1} 2$ 圍成, 亦即 $\Omega = \{0 \leqslant r \leqslant 1,\ 0 \leqslant \theta \leqslant \tan^{-1} 2\}$, 故 $\displaystyle\int_\Omega \frac{y}{x}\,\mathrm{d}A = \int_0^{\tan^{-1} 2} \int_0^1 \frac{1}{2}\tan\theta \left| 2r \right|\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\tan^{-1} 2} \tan\theta\,\mathrm{d}\theta \int_0^1 r\,\mathrm{d}r = -\frac{1}{2}\ln|\cos\theta|\,\Big|_0^{\tan^{-1} 2} = \dfrac{\ln 5}{4}$.

例. 求 $\displaystyle\int_\Omega \sin(9x^2 + 4y^2)\,\mathrm{d}A$, Ω 為 $9x^2 + 4y^2 = 1$, $y \geqslant 0$, $x \geqslant 0$ 所圍成之區域.

解. 令 $\begin{cases} x = \frac{1}{3}r\cos\theta \\ y = \frac{1}{2}r\sin\theta \end{cases}$, Jacobian $J_\mathbf{x}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{1}{3}\cos\theta & -\frac{1}{3}r\sin\theta \\ \frac{1}{2}\sin\theta & \frac{1}{2}r\cos\theta \end{vmatrix} = \dfrac{1}{6}r$. 變數變換 $(x, y) \to (r, \theta)$ 後 Ω 由 $r = 0$, $r = 1$, $\theta = 0$, $\theta = \frac{\pi}{2}$ 圍成, 亦即 $\Omega = \{0 \leqslant r \leqslant 1,\ 0 \leqslant \theta \leqslant \frac{\pi}{2}\}$, 故 $\displaystyle\int_\Omega \sin(9x^2 + 4y^2)\,\mathrm{d}A = \int_0^{\frac{\pi}{2}} \int_0^1 \sin r^2 \left| \frac{1}{6}r \right|\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{6}\int_0^{\frac{\pi}{2}}\mathrm{d}\theta \int_0^1 r\sin r^2\,\mathrm{d}r = \dfrac{\pi}{24}(1 - \cos 1)$.

例. 求 $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y - 2x)^2\,\mathrm{d}y\,\mathrm{d}x$.

解. 令 $\begin{cases} u = x + y \\ v = y - 2x \end{cases}$, 則 $\begin{cases} x = \frac{1}{3}u - \frac{1}{3}v \\ y = \frac{2}{3}u + \frac{1}{3}v \end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{-1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \dfrac{1}{3}$. 變數變換前積分區域由 $x + y = 0$, $x + y = 1$, $x = 0$, $y = 0$ 所圍成, 變數變換 $(x, y) \to (u, v)$ 後積分區域由 $u = 0$, $u = 1$, $u - v = 0$, $2u + v = 0$ 圍成, 故 $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2 \,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_{-2u}^{u} \sqrt{u}\,v^2 \left|\frac{1}{3}\right| \mathrm{d}v\,\mathrm{d}u = \dfrac{1}{3}\int_0^1 \sqrt{u}\left(\dfrac{v^3}{3}\bigg|_{v=-2u}^{v=u}\right)\mathrm{d}u = \dfrac{2}{9}$.

例. 若 Ω 為頂點 $(0, 0)$, $(1, 0)$, $(0, 1)$ 之三角形, f 為可積, 證明 $\displaystyle\int_\Omega f(x+y)\,\mathrm{d}A = \int_0^1 u\,f(u)\,\mathrm{d}u$.

解. 令 $\begin{cases} u = x + y \\ v = x \end{cases}$, 則 $\begin{cases} x = v \\ y = u - v \end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$. 變數變換前 Ω 由 $x + y = 0$, $x + y = 1$, $x = 0$, $y = 0$ 所圍成, 變數變換 $(x, y) \to (u, v)$ 後 Ω 由 $u = 0$, $u = 1$, $v = 0$, $u - v = 0$ 圍成, 故 $\displaystyle\int_\Omega f(x+y)\,\mathrm{d}A = \int_0^1 \int_0^u f(u)\,|-1|\,\mathrm{d}v\,\mathrm{d}u = \int_0^1 u\,f(u)\,\mathrm{d}u$.

例. 1. 證明 $\displaystyle\int_0^1 \int_0^1 \dfrac{1}{1-xy}\,\mathrm{d}x\,\mathrm{d}y = \sum_{n=1}^\infty \dfrac{1}{n^2}$.   2. 以變數變換證明 $\displaystyle\int_0^1 \int_0^1 \dfrac{1}{1-xy}\,\mathrm{d}x\,\mathrm{d}y = \dfrac{\pi^2}{6}$.

解.

1. $|xy| < 1$ 時 $\dfrac{1}{1-xy} = \sum_{n=0}^\infty (xy)^n$ 且可逐項積分; 代入 $\displaystyle\int_0^1 \int_0^1 \dfrac{1}{1-xy}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^1 \sum_{n=0}^\infty (xy)^n \,\mathrm{d}x\,\mathrm{d}y = \sum_{n=0}^\infty \int_0^1 x^n \,\mathrm{d}x \int_0^1 y^n \,\mathrm{d}y = \sum_{n=0}^\infty \dfrac{1}{(n+1)^2} = \sum_{n=1}^\infty \dfrac{1}{n^2}$.

2. 令 $\begin{cases} x = \frac{u-v}{\sqrt{2}} \\ y = \frac{u+v}{\sqrt{2}} \end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = 1$. 變數變換前積分區域由 $x = 0$, $x = 1$, $y = 0$, $y = 1$ 所圍成, 變數變換 $(x, y) \to (u, v)$ 後積分區域由 $u - v = 0$, $u - v = \sqrt{2}$, $u + v = 0$, $u + v = \sqrt{2}$ 圍成, 故 $\displaystyle\int_0^1 \int_0^1 \dfrac{1}{1-xy}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\sqrt{2}}{2}} \int_{-u}^{u} \dfrac{\mathrm{d}v\,\mathrm{d}u}{1 - \frac{u-v}{\sqrt{2}}\frac{u+v}{\sqrt{2}}} + \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \int_{-\sqrt{2}+u}^{\sqrt{2}-u} \dfrac{\mathrm{d}v\,\mathrm{d}u}{1 - \frac{u-v}{\sqrt{2}}\frac{u+v}{\sqrt{2}}} = 2\left(\int_0^{\frac{\sqrt{2}}{2}} \int_{-u}^{u} \dfrac{\mathrm{d}v\,\mathrm{d}u}{2 - u^2 + v^2} + \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \int_{-\sqrt{2}+u}^{\sqrt{2}-u} \dfrac{\mathrm{d}v\,\mathrm{d}u}{2 - u^2 + v^2}\right) = 2\left(\int_0^{\frac{\sqrt{2}}{2}} \dfrac{1}{\sqrt{2-u^2}}\left(\tan^{-1}\dfrac{v}{\sqrt{2-u^2}}\bigg|_{v=-u}^{v=u}\right)\mathrm{d}u\right.$

    $\left. + \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \dfrac{1}{\sqrt{2-u^2}}\left(\tan^{-1}\dfrac{v}{\sqrt{2-u^2}}\bigg|_{v=-\sqrt{2}+u}^{v=\sqrt{2}-u}\right)\mathrm{d}u\right)$

    $= 4\left(\int_0^{\frac{\sqrt{2}}{2}} \dfrac{1}{\sqrt{2-u^2}}\tan^{-1}\dfrac{u}{\sqrt{2-u^2}}\,\mathrm{d}u + \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \dfrac{1}{\sqrt{2-u^2}}\tan^{-1}\dfrac{\sqrt{2}-u}{\sqrt{2-u^2}}\,\mathrm{d}u\right)$. 令 $u = \sqrt{2}\sin\theta$, 則 $\mathrm{d}u = \sqrt{2}\cos\theta\,\mathrm{d}\theta$; 化簡 $4\left(\int_0^{\frac{\sqrt{2}}{2}} \dfrac{1}{\sqrt{2-u^2}}\tan^{-1}\dfrac{u}{\sqrt{2-u^2}}\,\mathrm{d}u + \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \dfrac{1}{\sqrt{2-u^2}}\tan^{-1}\dfrac{\sqrt{2}-u}{\sqrt{2-u^2}}\,\mathrm{d}u\right)$

    $= 4\left(\int_0^{\frac{\pi}{6}} \tan^{-1}(\tan\theta)\,\mathrm{d}\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \tan^{-1}\left(\dfrac{1-\sin\theta}{\cos\theta}\right)\mathrm{d}\theta\right) = 4\left(\int_0^{\frac{\pi}{6}} \theta\,\mathrm{d}\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \dfrac{1}{2}\left(\dfrac{\pi}{2} - \theta\right)\mathrm{d}\theta\right) = \dfrac{\pi^2}{6}$; 其中 $\dfrac{1-\sin\theta}{\cos\theta} = \dfrac{1 - \cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)} = \dfrac{2\sin^2\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)}{2\sin\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\cos\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)} = \tan\dfrac{1}{2}\left(\dfrac{\pi}{2}-\theta\right)$.

極座標二重積分

令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$.

例. 求 $\displaystyle\int_\Omega (3x+4y^2)\,\mathrm{d}A$, Ω 為 $x^2+y^2=1$, $x^2+y^2=4$ 與 $y\geqslant 0$ 所圍成之區域.

解. 變數變換 $(x,y)\to(r,\theta)$ 後 Ω 由 $r=1$, $r=2$, $\theta=0$, $\theta=\pi$ 圍成, 故 $\displaystyle\int_\Omega(3x+4y^2)\,\mathrm{d}A=\int_0^\pi\int_1^2(3r\cos\theta+4r^2\sin^2\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta=\int_0^\pi(7\cos\theta+15\sin^2\theta)\,\mathrm{d}\theta=\dfrac{15\,\pi}{2}$.

例. 求 $z=1-x^2-y^2$ 與 $z\geqslant 0$ 所圍成之體積.

解. 體積為 $\displaystyle\int_\Omega(1-x^2-y^2)\,\mathrm{d}A$, Ω 為單位圓 $x^2+y^2=1$. 變數變換 $(x,y)\to(r,\theta)$ 後 Ω 由 $r=0$, $r=1$, $\theta=0$, $\theta=2\pi$ 圍成, 故 $\displaystyle\int_\Omega(1-x^2-y^2)\,\mathrm{d}A=\int_0^{2\pi}\int_0^1(1-r^2)\,r\,\mathrm{d}r\,\mathrm{d}\theta=\dfrac{\pi}{2}$.

例. 求 $\displaystyle\int_\Omega\dfrac{y^2}{x^2}\,\mathrm{d}A$, Ω 為 $0<a^2\leqslant x^2+y^2\leqslant b^2$ 與 $x\geqslant y\geqslant 0$ 所圍成之區域.

解. 變數變換 $(x,y)\to(r,\theta)$ 後 Ω 由 $r=a$, $r=b$, $\theta=0$, $\theta=\dfrac{\pi}{4}$ 圍成, 故 $\displaystyle\int_\Omega\dfrac{y^2}{x^2}\,\mathrm{d}A=\int_0^{\frac{\pi}{4}}\int_a^b\tan^2\theta\cdot r\,\mathrm{d}r\,\mathrm{d}\theta=$ $\dfrac{b^2-a^2}{2}\displaystyle\int_0^{\frac{\pi}{4}}\tan^2\theta\,\mathrm{d}\theta=\dfrac{b^2-a^2}{2}\int_0^{\frac{\pi}{4}}(\sec^2\theta-1)\,\mathrm{d}\theta=\dfrac{b^2-a^2}{2}(\tan\theta-\theta)\Big|_0^{\frac{\pi}{4}}=\dfrac{b^2-a^2}{2}\Big(1-\dfrac{\pi}{4}\Big)$.

例. 反變數變換: $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^a\dfrac{r}{\sqrt{a^2-r^2\cos^2\theta}}\,\mathrm{d}r\,\mathrm{d}\theta=\int_0^a\int_0^{\sqrt{a^2-x^2}}\dfrac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}y\,\mathrm{d}x=\int_0^a\dfrac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\,\mathrm{d}x=a$.

例. 若 $a>0$, 求

1. $\displaystyle\int_0^\infty e^{-ax^2}\,\mathrm{d}x$.　　　　　　2. $\displaystyle\int_0^\infty x\,e^{-ax^2}\,\mathrm{d}x$.　　　　　　3. $\displaystyle\int_0^\infty x^2\,e^{-ax^2}\,\mathrm{d}x$.

解.

1. $\left(\displaystyle\int_{-\infty}^\infty e^{-x^2}\,\mathrm{d}x\right)^2=\left(\int_{-\infty}^\infty e^{-x^2}\,\mathrm{d}x\right)\left(\int_{-\infty}^\infty e^{-y^2}\,\mathrm{d}y\right)=\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y$. 變數變換 $(x,y)\to(r,\theta)$ 後得 $\displaystyle\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y=\int_0^{2\pi}\int_0^\infty e^{-r^2}r\,\mathrm{d}r\,\mathrm{d}\theta=2\pi\left(\dfrac{-e^{-r^2}}{2}\bigg|_0^\infty\right)=\pi$, 故 $\displaystyle\int_{-\infty}^\infty e^{-x^2}\,\mathrm{d}x=\sqrt{\pi}$; 由 e^{-x^2} 之對稱性, $\displaystyle\int_0^\infty e^{-x^2}\,\mathrm{d}x=\dfrac{\sqrt{\pi}}{2}$. 在 $\displaystyle\int_0^\infty e^{-ax^2}\,\mathrm{d}x$ 中, 令 $w=\sqrt{a}\,x\implies x=\dfrac{w}{\sqrt{a}}\implies\mathrm{d}x=\dfrac{1}{\sqrt{a}}\,\mathrm{d}w$; 積分範圍 x 由 0 至 ∞, 變數變換後 w 由 $\sqrt{a}\cdot 0=0$ 至 $\sqrt{a}\cdot\infty=\infty$, 則 $\displaystyle\int_0^\infty e^{-ax^2}\,\mathrm{d}x=\dfrac{1}{\sqrt{a}}\int_0^\infty e^{-w^2}\,\mathrm{d}w=\dfrac{\sqrt{\pi}}{2\sqrt{a}}$.

2. 令 $w=a\,x^2$, 則 $\mathrm{d}w=2\,a\,x\,\mathrm{d}x\implies x\,\mathrm{d}x=\dfrac{\mathrm{d}w}{2a}$. 積分範圍 x 由 0 至 ∞, 則變數變換後 w 由 $a\cdot 0^2=0$ 至 $a\cdot\infty^2=\infty$, 故 $\displaystyle\int_0^\infty x\,e^{-a\,x^2}\,\mathrm{d}x=\int_0^\infty e^{-w}\dfrac{\mathrm{d}w}{2a}=\dfrac{1}{2a}\int_0^\infty e^{-w}\,\mathrm{d}w=\dfrac{1}{2a}\left(-e^{-w}\bigg|_0^\infty\right)=\dfrac{1}{2a}$.

3. 令 $w=\sqrt{a}\,x\implies x=\dfrac{w}{\sqrt{a}}\implies\mathrm{d}x=\dfrac{1}{\sqrt{a}}\,\mathrm{d}w$; 積分範圍 x 由 0 至 ∞, 變數變換後 w 由 $\sqrt{a}\cdot 0=0$ 至 $\sqrt{a}\cdot\infty=\infty$, 故 $\displaystyle\int_0^\infty x^2\,e^{-ax^2}\,\mathrm{d}x=\int_0^\infty\dfrac{w^2}{a}\,e^{-w^2}\dfrac{1}{\sqrt{a}}\,\mathrm{d}w=\dfrac{1}{a^{\frac{3}{2}}}\int_0^\infty w^2\,e^{-w^2}\,\mathrm{d}w$. 令 $u=-\dfrac{1}{2}\,w$, 則 $\mathrm{d}u=-\dfrac{1}{2}\,\mathrm{d}w$. 令 $\mathrm{d}v=-2\,w\,e^{-w^2}\,\mathrm{d}w$, 則 $v=e^{-w^2}$. 故 $\dfrac{1}{a^{\frac{3}{2}}}\displaystyle\int_0^\infty w^2\,e^{-w^2}\,\mathrm{d}w=\dfrac{1}{a^{\frac{3}{2}}}\left(-\dfrac{1}{2}w\cdot e^{-w^2}\bigg|_0^\infty+\dfrac{1}{2}\int_0^\infty e^{-w^2}\,\mathrm{d}w\right)=\dfrac{1}{a^{\frac{3}{2}}}\cdot\dfrac{1}{2}\cdot\dfrac{\sqrt{\pi}}{2}=\dfrac{\sqrt{\pi}}{4a^{\frac{3}{2}}}$.

曲面表面積

結論. 若曲面 S 由 $z = f(x,y)$, $(x,y) \in \Omega$ 所定義, 則 S 的表面積為 $\int_{\Omega} \sqrt{1 + f_x^2 + f_y^2} \, dA$.

證 (sketch)**.** 以切平面近似曲面 S; S 參數式為 $\mathbf{r} = (x, y, f(x,y))$, x, y 方向切向量分別為 $\mathbf{r}_x = (1, 0, f_x)$, $\mathbf{r}_y = (0, 1, f_y)$; $|\mathbf{r}_x \times \mathbf{r}_y| = |(-f_x, -f_y, 1)| = \sqrt{1 + f_x^2 + f_y^2}$.

例. 求半球面 $z = \sqrt{a^2 - x^2 - y^2}$ 與圓柱體 $(x - \frac{a}{2})^2 + y^2 = (\frac{a}{2})^2$ 相交區域之表面積.

解. 由 $z = f(x,y) = \sqrt{a^2 - x^2 - y^2}$, $f_x = \dfrac{-x}{\sqrt{a^2 - x^2 - y^2}}$, $f_y = \dfrac{-y}{\sqrt{a^2 - x^2 - y^2}}$, $dS = \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy =$

$\sqrt{1 + \dfrac{x^2}{a^2 - x^2 - y^2} + \dfrac{y^2}{a^2 - x^2 - y^2}} \, dx \, dy = \sqrt{\dfrac{a^2}{a^2 - x^2 - y^2}} \, dx \, dy$. 所求表面積為 $\int_{\Omega} dS$, $\Omega = \big\{(x,y) \,\big|\, (x - $

$\frac{a}{2})^2 + y^2 \leqslant (\frac{a}{2})^2 \big\}$; $\int_{\Omega} dS = \int_{\Omega} \sqrt{\dfrac{a^2}{a^2 - x^2 - y^2}} \, dx \, dy = 2 \int_0^{\frac{\pi}{2}} \int_0^{a\cos\theta} \sqrt{\dfrac{a^2}{a^2 - r^2}} \, r \, dr \, d\theta$

$= 2a \int_0^{\frac{\pi}{2}} \Big(-\sqrt{a^2 - r^2} \,\Big|_{r=0}^{r=a\cos\theta} \Big) \, d\theta = 2a^2 \int_0^{\frac{\pi}{2}} (a - \sin\theta) \, d\theta = a^2(\pi - 2)$.

7.2 三重積分

一般區域積分

例. 若 E 為中心 0, 半徑 a 之球體, 寫出 $\int_E f(x,y,z) \, dV$.

解. $\int_E f(x,y,z) \, dV$

$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} f(x,y,z) \, dz \, dy \, dx = \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} f(x,y,z) \, dz \, dx \, dy$

$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-z^2}}^{\sqrt{a^2-x^2-z^2}} f(x,y,z) \, dy \, dz \, dx = \int_{-a}^{a} \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} \int_{-\sqrt{a^2-x^2-z^2}}^{\sqrt{a^2-x^2-z^2}} f(x,y,z) \, dy \, dx \, dz$

$= \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \int_{-\sqrt{a^2-y^2-z^2}}^{\sqrt{a^2-y^2-z^2}} f(x,y,z) \, dx \, dz \, dy = \int_{-a}^{a} \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} \int_{-\sqrt{a^2-y^2-z^2}}^{\sqrt{a^2-y^2-z^2}} f(x,y,z) \, dx \, dy \, dz$

例. 若 E 為第一卦限中 $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} \leqslant 1$ 圍成之區域, a, b, $c > 0$, 寫出 $\int_E f(x,y,z) \, dV$.

解. $\int_E f(x,y,z) \, dV$

$= \int_0^a \int_0^{b - \frac{bx}{a}} \int_0^{c - \frac{cx}{a} - \frac{cy}{b}} f(x,y,z) \, dz \, dy \, dx = \int_0^b \int_0^{a - \frac{ay}{b}} \int_0^{c - \frac{cx}{a} - \frac{cy}{b}} f(x,y,z) \, dz \, dx \, dy$

$= \int_0^a \int_0^{c - \frac{cx}{a}} \int_0^{b - \frac{bx}{a} - \frac{bz}{c}} f(x,y,z) \, dy \, dz \, dx = \int_0^c \int_0^{a - \frac{az}{c}} \int_0^{b - \frac{bx}{a} - \frac{bz}{c}} f(x,y,z) \, dy \, dx \, dz$

$= \int_0^b \int_0^{c - \frac{cy}{b}} \int_0^{a - \frac{ay}{b} - \frac{az}{c}} f(x,y,z) \, dx \, dz \, dy = \int_0^c \int_0^{b - \frac{bz}{c}} \int_0^{a - \frac{ay}{b} - \frac{az}{c}} f(x,y,z) \, dx \, dy \, dz$

例. 求 $x^2 + y^2 = a^2$ 與 $x^2 + z^2 = a^2$ 交集區域之表面積與體積.
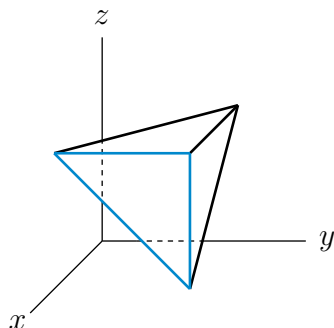
解.
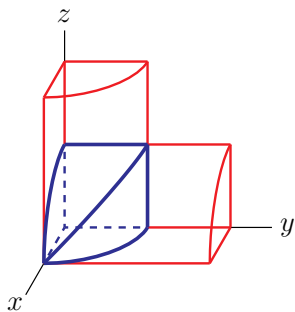
$$(\text{表面積}) = 16 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{1 + \left(\frac{-x}{\sqrt{a^2-x^2}}\right)^2} \, dy \, dx = 16 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{\frac{a^2}{a^2-x^2}} \, dy \, dx$$

$$= 16a \int_0^a \frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}} \, dx = 16a^2.$$

$$(\text{體積}) = 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2}} dz \, dy \, dx = 8 \int_0^a (a^2-x^2) \, dx = \frac{16a^3}{3}.$$

例. 若 E 為 $x=1$, $y=1$, $z=1$, $x+y+z=2$ 圍成之四面體, 求 $\int_E x \, dV$.

解.

$$\int_E x \, dV = \int_0^1 \int_{1-x}^1 \int_{2-x-y}^1 x \, dz \, dy \, dx = \int_0^1 \int_{1-x}^1 x \,(x+y-1) \, dy \, dx$$

$$= \int_0^1 \int_{1-x}^1 (x^2 - x + xy) \, dy \, dx = \int_0^1 \left((x^2-x)\,x + x\,\frac{1-(1-x)^2}{2} \right) dx$$

$$= \int_0^1 \left(x^3 - x^2 + x\,\frac{2x-x^2}{2} \right) dx = \int_0^1 \frac{x^3}{2} \, dx = \frac{1}{8}$$

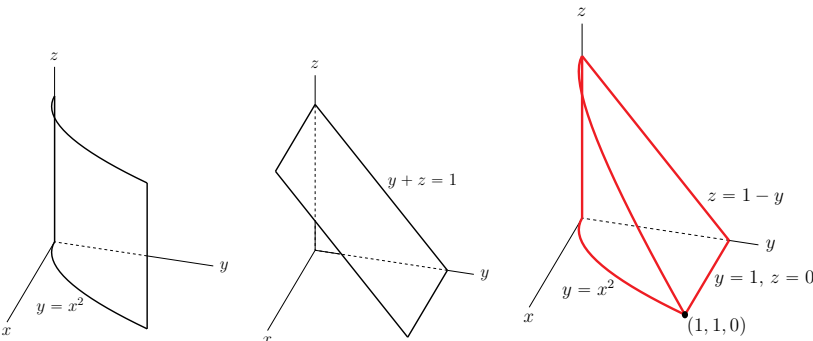**例.** 若 E 為第一卦限中 $y^2 + z^2 = 1$, $x+y=2$, $y=0$, $z=0$ 圍成之區域, 求 $\int_E z \, dV$.

解.
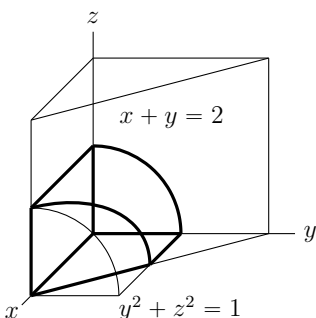
$$\int_E z \, dV = \int_0^1 \int_0^{2-y} \int_0^{\sqrt{1-y^2}} z \, dz \, dx \, dy = \int_0^1 \frac{1}{2}(1-y^2)(2-y) \, dy$$

$$= \frac{1}{2} \int_0^1 (2 - y - 2y^2 + y^3) \, dy = \frac{1}{2} \left(2 - \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{1}{2} \cdot \frac{13}{12} = \frac{13}{24}$$

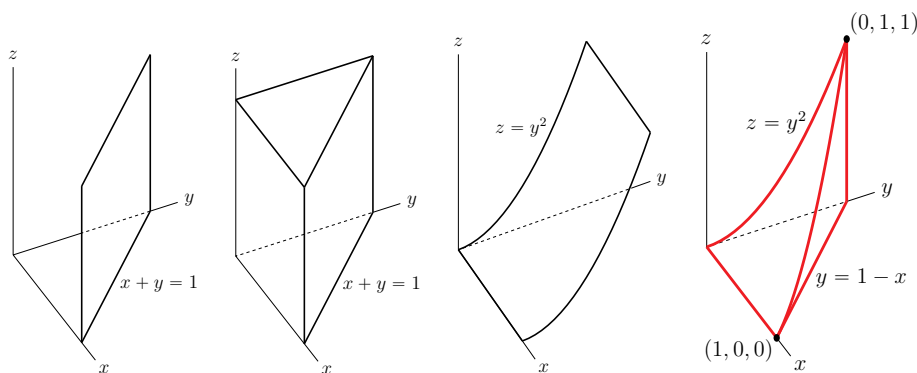**例.** 若 E 為第一卦限中 $y = x^2$, $y + z = 1$ 圍成之區域, 求 $\int_E x \, dV$.

解.

$$\int_E x \, dV = \int_0^1 \int_{x^2}^1 \int_0^{1-y} x \, dz \, dy \, dx$$

$$= \int_0^1 \int_{x^2}^1 x\,(1-y) \, dy \, dx$$

$$= \int_0^1 \left(x(1-x^2) - x \left(\frac{y^2}{2} \Big|_{y=x^2}^{y=1} \right) \right) dx$$

$$= \int_0^1 \left(x - x^3 + \frac{x^5}{2} - \frac{x}{2} \right) dx = \frac{1}{12}$$

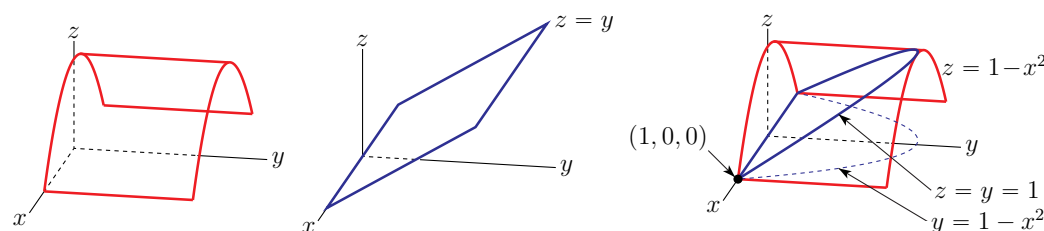**例.** 若 E 為第一卦限中 $z = y^2$, $x+y=1$ 圍成之區域, 求 $\int_E z \, dV$.

解.

$$\int_E z \, dV = \int_0^1 \int_0^{1-x} \int_0^{y^2} z \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{y^4}{2} \, dy \, dx$$

$$= \int_0^1 \frac{(1-x)^5}{10} \, dx = \frac{1}{60}$$

例. 若 E 為 $z = 1 - x^2$, $y = z$, $y = 0$, $z = 0$ 圍成之區域, 求 $\displaystyle\int_E dV$.

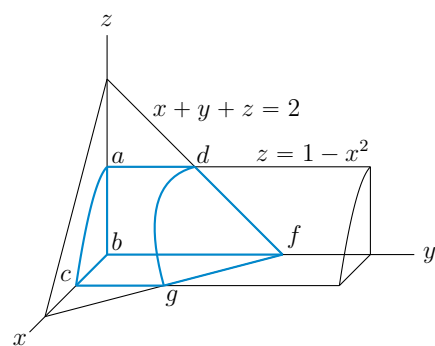**解.**

$$\int_E dV$$

$$= \int_{-1}^1 \int_0^{1-x^2} \int_y^{1-x^2} dz \, dy \, dx$$

$$= \int_{-1}^1 \frac{(1-x^2)^2}{2} \, dx = \frac{8}{15}$$

例. 若 E 為 $x = 0$, $y = 0$, $z = 0$, $x + y + z = 2$, $x^2 + z = 1$ 圍成之區域, 求 $\displaystyle\int_E x \, dV$.

解.

$$\int_E x \, dV = \int_0^1 \int_0^{1-x^2} \int_0^{2-x-z} x \, dy \, dz \, dx = \int_0^1 \int_0^{1-x^2} (2-x-z) \, x \, dz \, dx$$

$$= \int_0^1 \left((2-x) \, x \, (1-x^2) - x \, \frac{(1-x^2)^2}{2} \right) dx$$

$$= \frac{1}{2} \int_0^1 (-x^5 + 2x^4 - 2x^3 - 2x^2 + 3x) \, dx$$

$$= \frac{1}{2} \left(-\frac{1}{6} + \frac{2}{5} - \frac{1}{2} - \frac{2}{3} + \frac{3}{2} \right) = \frac{17}{60}$$

例. 若 E 為 $x + y + z = 1$, $x = 0$, $y = 0$, $z = 0$ 圍成之區域, 求 $\displaystyle\int_E z \, dV$.

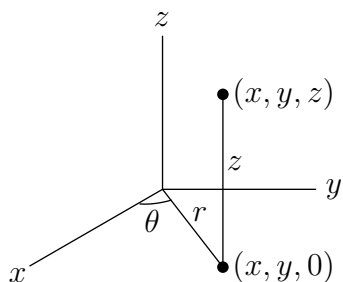**解.** $\displaystyle\int_E z \, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} \frac{(1-x-y)^2}{2} \, dy \, dx = \int_0^1 -\frac{(1-x-y)^3}{6} \Big|_{y=0}^{y=1-x} dx$

$= \displaystyle\int_0^1 \frac{1}{6}(1-x)^3 \, dx = -\frac{1}{24}(1-x)^4 \Big|_0^1 = \frac{1}{24}.$

例. 若 E 為 $x + 2y + z = 2$, $x = 2y$, $x = 0$, $z = 0$ 圍成之區域, 求 $\displaystyle\int_E y \, dV$.

解. $\displaystyle\int_E y \, dV = \int_0^1 \int_{\frac{x}{2}}^{\frac{2-x}{2}} \int_0^{2-x-2y} y \, dz \, dy \, dx = \int_0^1 \int_{\frac{x}{2}}^{\frac{2-x}{2}} (2-x-2y) \, y \, dy \, dx = \int_0^1 \left(\frac{(2-x)y^2}{2} - \frac{2y^3}{3} \right) \Big|_{\frac{x}{2}}^{\frac{2-x}{2}} dx =$

$\displaystyle\int_0^1 \left(\frac{2-x}{2} \left(\left(\frac{2-x}{2} \right)^2 - \left(\frac{x}{2} \right)^2 \right) - \frac{2}{3} \left(\left(\frac{2-x}{2} \right)^3 - \left(\frac{x}{2} \right)^3 \right) \right) dx = \int_0^1 \left(\frac{2-x}{2}(1-x) - \frac{2}{3}(1-x) \left(\left(\frac{2-x}{2} \right)^2 + \right.$

$\left. \frac{2-x}{2} \cdot \frac{x}{2} + \left(\frac{x}{2} \right)^2 \right) \, dx = \frac{1}{6} \int_0^1 (x^3 - 3x + 2) \, dx = \frac{1}{6} \left(\frac{1}{4} - \frac{3}{2} + 2 \right) = \frac{1}{8}.$

變數變換

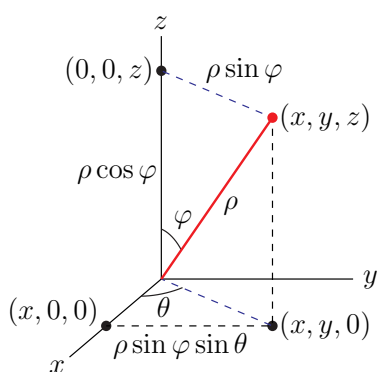### 柱面座標

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \iff \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\frac{y}{x} \\ z = z \end{cases}$$

$$\text{Jacobian } J_{\mathbf{x}}(\mathbf{u}) = \frac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$
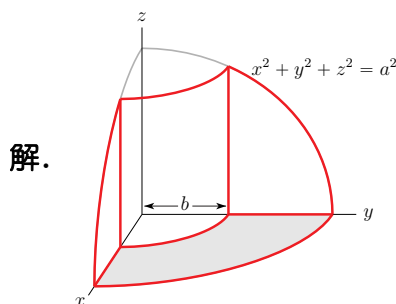
球面座標

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases} \iff \begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\frac{y}{x} \\ \varphi = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} \end{cases}$$

$$\text{Jacobian } J_{\mathbf{x}}(\mathbf{u}) = \frac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix} = -\rho^2\sin\varphi$$
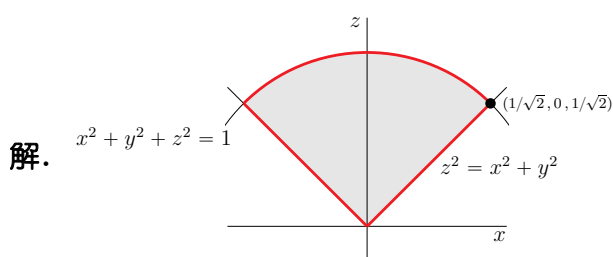
例. 半徑為 a 之球中心對稱鑽半徑為 b 之圓孔, $a > b > 0$, 求球剩下的體積.

解.

- 柱面座標: $\displaystyle\int_E \mathrm{d}V = 8\int_0^{\frac{\pi}{2}}\int_b^a\int_0^{\sqrt{a^2-r^2}} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta = \frac{4\pi}{3}(a^2-b^2)^{\frac{3}{2}}$

- 球面座標: $\displaystyle\int_E \mathrm{d}V = 2\int_{\sin^{-1}\frac{b}{a}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_{\frac{b}{\sin\varphi}}^a \rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi = \frac{4\pi}{3}(a^2-b^2)^{\frac{3}{2}}$

例. 若 E 為 $x^2+y^2 \leqslant z^2$, $x^2+y^2+z^2 \leqslant 1$ 與 $z \geqslant 0$ 圍成之區域, 求 $\displaystyle\int_E \sqrt{x^2+y^2+z^2}\,\mathrm{d}V$.

解.

- 柱面座標: $\displaystyle\int_E \sqrt{x^2+y^2+z^2}\,\mathrm{d}V$

$$= \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\int_r^{\sqrt{1-r^2}} \sqrt{r^2+z^2}\cdot r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

- 球面座標: $\displaystyle\int_E \sqrt{x^2+y^2+z^2}\,\mathrm{d}V$

$$= \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^1 \rho\cdot\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi = \frac{\pi}{2}\left(1-\frac{1}{\sqrt{2}}\right)$$

例. 若 E 為 $0 \leqslant z \leqslant \sqrt{x^2 + y^2}$ 與 $x^2 + y^2 \leqslant 1$ 圍成之區域, 求 $\displaystyle\int_E z\sqrt{x^2 + y^2 + z^2}\,\mathrm{d}V$.

解.
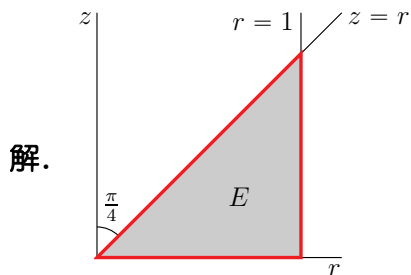
- 柱面座標: $\displaystyle\int_E z\sqrt{x^2 + y^2 + z^2}\,\mathrm{d}V = \int_0^{2\pi}\int_0^1\int_0^r z\sqrt{r^2 + z^2}\cdot r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$

$$= 2\pi\int_0^1 \frac{r}{3}(r^2 + z^2)^{\frac{3}{2}}\Big|_{z=0}^{z=r}\mathrm{d}r = \frac{2\pi}{3}\int_0^1 r\cdot\left(2^{\frac{3}{2}}-1\right)r^3\,\mathrm{d}r = \frac{2\pi\left(2^{\frac{3}{2}}-1\right)}{15}$$

- 球面座標: $\displaystyle\int_E z\sqrt{x^2+y^2+z^2}\,\mathrm{d}V = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{\frac{1}{\sin\varphi}}\rho\cos\varphi\cdot\rho\cdot\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$

例. 若 E 為 $z = x^2 + y^2$ 與 $z \leqslant \sqrt{x^2 + y^2}$ 圍成之區域, 求 $\displaystyle\int_E z\left(x^2 + y^2 + z^2\right)\mathrm{d}V$.

解.
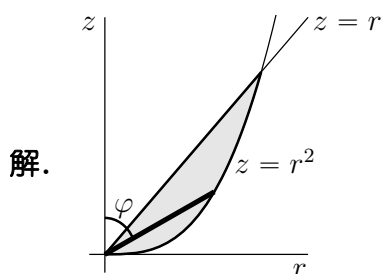
- 柱面座標: $\displaystyle\int_E z\left(x^2 + y^2 + z^2\right)\mathrm{d}V = \int_0^{2\pi}\int_0^1\int_{r^2}^r z\left(r^2 + z^2\right)\cdot r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$

$$= 2\pi\int_0^1\int_{r^2}^r\left(r^3 z + r z^3\right)\mathrm{d}z\,\mathrm{d}r = 2\pi\int_0^1\left(\frac{r^3}{2}(r^2 - r^4) + \frac{r}{4}(r^4 - r^8)\right)\mathrm{d}r = \frac{3\pi}{40}$$

- 球面座標: $z = x^2 + y^2 \implies \rho\cos\varphi = \rho^2\sin^2\varphi \implies \rho = \dfrac{\cos\varphi}{\sin^2\varphi}$;

$z = \sqrt{x^2 + y^2} \implies \rho\cos\varphi = \rho\sin\varphi \implies \tan\varphi = 1$;

$$\int_E z\left(x^2 + y^2 + z^2\right)\mathrm{d}V = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{\frac{\cos\varphi}{\sin^2\varphi}}\rho\cos\varphi\cdot\rho^2\cdot\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$$
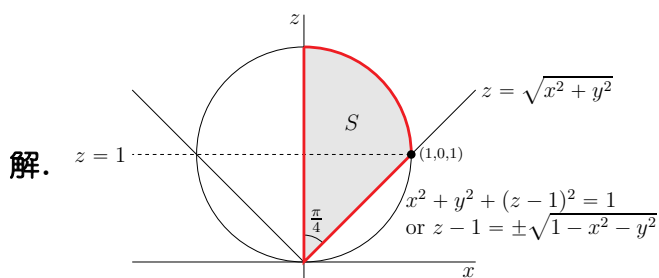
例. 若 E 為 $y = x^2 + z^2$ 與 $y = 4$ 圍成之區域, 求 $\displaystyle\int_E\sqrt{x^2 + z^2}\,\mathrm{d}V$.

解. 令 $\begin{cases} x = r\cos\theta \\ z = r\sin\theta \end{cases}$, 投影之 Ω 為 $x^2 + z^2 \leqslant 4$, 則 $\displaystyle\int_E\sqrt{x^2 + z^2}\,\mathrm{d}V = \int_\Omega\int_{x^2+z^2}^4\sqrt{x^2+z^2}\,\mathrm{d}y\,\mathrm{d}A$

$$= \int_\Omega\left(4 - (x^2 + z^2)\right)\sqrt{x^2 + z^2}\,\mathrm{d}A = \int_0^{2\pi}\int_0^2(4 - r^2)\,r\cdot r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi\int_0^2(4r^2 - r^4)\,\mathrm{d}r = 2\pi\cdot\left(\frac{4\cdot 2^3}{3} - \frac{2^5}{5}\right) = \frac{128\pi}{15}.$$

例. 若 $E = \{(x, y, z)\,|\,x^2 + y^2 + (z-1)^2 \leqslant 1\}$, 求 $\displaystyle\int_E\left(x^2 + y^2 + z^2\right)^{\frac{5}{2}}\mathrm{d}V$.

解. 代入球面座標於 $x^2 + y^2 + (z - 1)^2 \leqslant 1 \implies (\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2 + (\rho\cos\varphi - 1)^2 \leqslant 1 \implies$
$\rho^2\sin^2\varphi + \rho^2\cos^2\varphi - 2\rho\cos\varphi + 1 \leqslant 1 \implies \rho^2 \leqslant 2\rho\cos\varphi \implies \rho \leqslant 2\cos\varphi$, 故 $\displaystyle\int_E\left(x^2 + y^2 + z^2\right)^{\frac{5}{2}}\mathrm{d}V =$

$$\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{2\cos\varphi}\rho^5\cdot\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi = 2\pi\int_0^{\frac{\pi}{2}}\frac{(2\cos\varphi)^8}{8}\sin\varphi\,\mathrm{d}\varphi = \frac{64\pi}{9}(-\cos^9\varphi)\Big|_0^{\frac{\pi}{2}} = \frac{64\pi}{9}.$$

例. 若 E 為 $z = \sqrt{x^2 + y^2}$ 與 $x^2 + y^2 + (z-1)^2 = 1$ 於第一卦限圍成之區域, 求其體積.

解.

- 柱面座標: $\displaystyle\int_E\mathrm{d}V = \int_0^{\frac{\pi}{2}}\int_0^1\int_r^{1+\sqrt{1-r^2}} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta = \frac{\pi}{4}$

- 球面座標: $\displaystyle\int_E\mathrm{d}V = \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}}\int_0^{2\cos\varphi}\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi = \frac{\pi}{4}$

7.3 綜合問題

定理 (積分下取微分). 若 $f(x,y)$ 與 $\dfrac{\partial f}{\partial y}(x,y)$ 在 $[a,b] \times [c,d]$ 連續, 設 $F(y) = \displaystyle\int_a^b f(x,y)\,\mathrm{d}x$, 則 $F'(y) = \displaystyle\int_a^b \dfrac{\partial f}{\partial y}(x,y)\,\mathrm{d}x$.

證. 若 $y_0 \in (c,d)$ 且 $y \neq y_0$, $\dfrac{F(y) - F(y_0)}{y - y_0} = \displaystyle\int_a^b \dfrac{f(x,y) - f(x,y_0)}{y - y_0}\,\mathrm{d}x \to \int_a^b \dfrac{\partial f}{\partial y}(x,y_0)\,\mathrm{d}x$.

例. 證明 $\displaystyle\int_0^\infty e^{-x^2}\,\mathrm{d}x = \dfrac{\sqrt{\pi}}{2}$.

證. 令 $F(t) = \displaystyle\int_0^\infty \dfrac{e^{-t^2(1+x^2)}}{1+x^2}\,\mathrm{d}x$, $t \geqslant 0$; $F(\infty) = 0$, $F(0) = \displaystyle\int_0^\infty \dfrac{\mathrm{d}x}{1+x^2} = \tan^{-1} x\,\Big|_0^\infty = \tan^{-1}\infty - \tan^{-1} 0 = \dfrac{\pi}{2}$.

則 $F'(t) = \displaystyle\int_0^\infty \dfrac{-2\,t\,(1+x^2)\,e^{-t^2(1+x^2)}}{1+x^2}\,\mathrm{d}x = \underbrace{-2\,t\,e^{-t^2}\int_0^\infty e^{-t^2 x^2}\,\mathrm{d}x}_{\text{令 } u = t\,x,\ \text{則 } \mathrm{d}u = t\,\mathrm{d}x} = -2\,e^{-t^2}\underbrace{\int_0^\infty e^{-u^2}\,\mathrm{d}u}_{= I} = -2\,e^{-t^2} I \implies$

$\displaystyle\int_0^b F'(t)\,\mathrm{d}t = -2I\int_0^b e^{-t^2}\,\mathrm{d}t \implies F(b) - F(0) = -2I\int_0^b e^{-t^2}\,\mathrm{d}t$. 令 $b \to \infty$, 則 $F(\infty) - F(0) = -2I^2 \implies$

$0 - \dfrac{\pi}{2} = -2I^2 \implies I = \dfrac{\sqrt{\pi}}{2}$.

例. 若 $a > 0$, 求

1. $\displaystyle\int_{-\infty}^\infty x^2 e^{-ax^2}\,\mathrm{d}x$. 　　2. $\displaystyle\int_{-\infty}^\infty x^4 e^{-ax^2}\,\mathrm{d}x$. 　　3. $\displaystyle\int_{-\infty}^\infty x^6 e^{-ax^2}\,\mathrm{d}x$.

解. 由 $F(a) = \displaystyle\int_{-\infty}^\infty e^{-ax^2}\,\mathrm{d}x = \sqrt{\pi}\,a^{-\frac{1}{2}}$,

1. $F'(a) = -\displaystyle\int_{-\infty}^\infty x^2 e^{-ax^2}\,\mathrm{d}x = \dfrac{\mathrm{d}}{\mathrm{d}a}(\sqrt{\pi}\,a^{-\frac{1}{2}}) = \sqrt{\pi} \cdot \dfrac{-1}{2}\,a^{-\frac{3}{2}} \implies \int_{-\infty}^\infty x^2 e^{-ax^2}\,\mathrm{d}x = \dfrac{\sqrt{\pi}}{2\,a^{\frac{3}{2}}}$.

2. $F''(a) = \displaystyle\int_{-\infty}^\infty x^4 e^{-ax^2}\,\mathrm{d}x = \dfrac{\mathrm{d}^2}{\mathrm{d}a^2}(\sqrt{\pi}\,a^{-\frac{1}{2}}) = \sqrt{\pi} \cdot \dfrac{3}{4}\,a^{-\frac{5}{2}} \implies \int_{-\infty}^\infty x^4 e^{-ax^2}\,\mathrm{d}x = \dfrac{3\sqrt{\pi}}{4\,a^{\frac{5}{2}}}$.

3. $F'''(a) = -\displaystyle\int_{-\infty}^\infty x^6 e^{-ax^2}\,\mathrm{d}x = \dfrac{\mathrm{d}^3}{\mathrm{d}a^3}(\sqrt{\pi}\,a^{-\frac{1}{2}}) = \sqrt{\pi} \cdot \dfrac{-15}{8}\,a^{-\frac{7}{2}} \implies \int_{-\infty}^\infty x^6 e^{-ax^2}\,\mathrm{d}x = \dfrac{15\sqrt{\pi}}{8\,a^{\frac{7}{2}}}$.

例. 證明 $\displaystyle\int_0^\infty e^{-tx}\dfrac{\sin x}{x}\,\mathrm{d}x = \dfrac{\pi}{2} - \tan^{-1} t$, $\forall\, t > 0$.

證. 令 $F(t) = \displaystyle\int_0^\infty e^{-tx}\dfrac{\sin x}{x}\,\mathrm{d}x$, 則 $F'(t) = -\displaystyle\int_0^\infty e^{-tx}\sin x\,\mathrm{d}x = -\dfrac{e^{-tx}(-t\sin x - \cos x)}{1+t^2}\,\Big|_{x=0}^{x=\infty} = -\dfrac{1}{1+t^2} \implies$

$F(t) = -\tan^{-1} t + c$. 令 $t \to \infty$, $F(\infty) = 0 = -\tan^{-1}\infty + c = -\dfrac{\pi}{2} + c \implies c = \dfrac{\pi}{2}$.

例. 若 $a > 0$, 求 $\displaystyle\int_0^\infty \dfrac{\mathrm{d}x}{(x^2+a^2)^2}$, $\displaystyle\int_0^\infty \dfrac{\mathrm{d}x}{(x^2+a^2)^3}$ 並推導 $\displaystyle\int_0^\infty \dfrac{\mathrm{d}x}{(x^2+a^2)^n}$, $n \in \mathbb{N}$.

解. 由 $F(a) = \displaystyle\int_0^\infty \dfrac{\mathrm{d}x}{x^2+a^2} = \dfrac{\pi}{2a} = \dfrac{\pi}{2}\,a^{-1}$, $F'(a) = \dfrac{\mathrm{d}}{\mathrm{d}a}\Big(\dfrac{\pi}{2}\,a^{-1}\Big) = \dfrac{\pi}{2}(-1)\,a^{-2} = (-1)2a\displaystyle\int_0^\infty \dfrac{\mathrm{d}x}{(x^2+a^2)^2} \implies$

$\displaystyle\int_0^\infty \dfrac{\mathrm{d}x}{(x^2+a^2)^2} = \dfrac{\pi}{4\,a^3}$; 等式兩邊再對 a 微分 $\implies (-2)(2a)\displaystyle\int_0^\infty \dfrac{\mathrm{d}x}{(x^2+a^2)^3} = (-3)\dfrac{\pi}{4\,a^4} \implies \displaystyle\int_0^\infty \dfrac{\mathrm{d}x}{(x^2+a^2)^3} =$

$\dfrac{3\,\pi}{16\,a^5}$. 由數學歸納法可得 $\displaystyle\int_0^\infty \dfrac{\mathrm{d}x}{(x^2+a^2)^n} = \dfrac{\pi}{(2a)^{2n-1}}\binom{2(n-1)}{n-1}$, $n \in \mathbb{N}$.

例. 證明 $\displaystyle\int_0^\infty e^{-x^2}\cos tx\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}\,e^{\frac{-t^2}{4}}$, $\forall\, t \in \mathbb{R}$.

證. 令 $F(t) = \displaystyle\int_0^\infty e^{-x^2}\cos tx\,\mathrm{d}x$, 則 $F'(t) = -\displaystyle\int_0^\infty e^{-x^2}x\sin tx\,\mathrm{d}x$. 令 $u = \dfrac{1}{2}\sin tx$, 則 $\mathrm{d}u = \dfrac{t}{2}\cos tx$; 令 $\mathrm{d}v = -2xe^{-x^2}\,\mathrm{d}x$, 則 $v = e^{-x^2}$. 故 $F'(t) = -\displaystyle\int_0^\infty e^{-x^2}x\sin tx\,\mathrm{d}x = \dfrac{1}{2}\sin tx\cdot e^{-x^2}\,\Big|_{x=0}^{x=\infty} - \displaystyle\int_0^\infty e^{-x^2}\cdot\dfrac{t}{2}\cos tx\,\mathrm{d}x = -\dfrac{t}{2}F(t)$, 故 $F(t) = c\,e^{\frac{-t^2}{4}}$; 又 $F(0) = \dfrac{\sqrt{\pi}}{2} = c\cdot e^0 = c$.

例. 已知 $\forall\, 0 < \alpha < 1$, $\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x = \frac{\pi}{\sin\alpha\pi}$, 證明 $\Gamma(\alpha)\,\Gamma(1-\alpha) = \dfrac{\pi}{\sin\alpha\pi}$.

解. $\Gamma(\alpha)\,\Gamma(1-\alpha) = \displaystyle\int_0^\infty e^{-t}t^{\alpha-1}\,\mathrm{d}t\int_0^\infty e^{-s}s^{-\alpha}\,\mathrm{d}s = \int_0^\infty\int_0^\infty e^{-(t+s)}t^{\alpha-1}s^{-\alpha}\,\mathrm{d}s\,\mathrm{d}t$. 令 $\begin{cases} u = s+t \\ v = \frac{t}{s} \end{cases}$, 則 $\begin{cases} s = \frac{u}{1+v} \\ t = \frac{uv}{1+v} \end{cases}$;

Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{1+v} & \frac{-u}{(1+v)^2} \\ \frac{v}{1+v} & \frac{u}{(1+v)^2} \end{vmatrix} = \dfrac{u}{(1+v)^2}$. 變數變換 $(s,t) \to (u,v)$ 後積分

範圍仍為 $0 < u < \infty$, $0 < v < \infty$, 故 $\displaystyle\int_0^\infty\int_0^\infty e^{-(t+s)}t^{\alpha-1}s^{-\alpha}\,\mathrm{d}s\,\mathrm{d}t = \int_0^\infty\int_0^\infty e^{-(t+s)}\left(\dfrac{t}{s}\right)^\alpha t^{-1}\,\mathrm{d}s\,\mathrm{d}t =$

$\displaystyle\int_0^\infty\int_0^\infty e^{-u}v^\alpha\dfrac{1+v}{uv}\dfrac{u}{(1+v)^2}\,\mathrm{d}u\,\mathrm{d}v = \int_0^\infty\int_0^\infty e^{-u}\dfrac{v^{\alpha-1}}{1+v}\,\mathrm{d}u\,\mathrm{d}v = \int_0^\infty e^{-u}\,\mathrm{d}u\int_0^\infty\dfrac{v^{\alpha-1}}{1+v}\,\mathrm{d}v = \int_0^\infty\dfrac{v^{\alpha-1}}{1+v}\,\mathrm{d}v = \dfrac{\pi}{\sin\alpha\pi}$.

例. Beta 函數定義為 $B(m,n) = \displaystyle\int_0^1 t^{m-1}(1-t)^{n-1}\,\mathrm{d}t$ $\forall\, m,n > 0$. 證明 $B(m,n) = B(n,m) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

證. $\Gamma(x) = \displaystyle\int_0^\infty e^{-t}t^{x-1}\,\mathrm{d}t$; 令 $t = u^2$, 則 $\Gamma(x) = 2\displaystyle\int_0^\infty e^{-u^2}u^{2x-1}\,\mathrm{d}u$.

$\Gamma(m)\,\Gamma(n) = \left(2\displaystyle\int_0^\infty e^{-u^2}u^{2m-1}\,\mathrm{d}u\right)\left(2\displaystyle\int_0^\infty e^{-v^2}v^{2n-1}\,\mathrm{d}v\right) = 4\displaystyle\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}u^{2m-1}v^{2n-1}\,\mathrm{d}u\,\mathrm{d}v$

$= 4\displaystyle\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}(r\cos\theta)^{2m-1}(r\sin\theta)^{2n-1}r\,\mathrm{d}r\mathrm{d}\theta = \left(2\displaystyle\int_0^\infty e^{-r^2}r^{2(m+n)-1}\,\mathrm{d}r\right)\left(2\displaystyle\int_0^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta\,\mathrm{d}\theta\right)$

$= \Gamma(m+n)\cdot 2\displaystyle\int_0^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta\,\mathrm{d}\theta$. 令 $t = \cos^2\theta$, 則 $\mathrm{d}t = -2\cos\theta\sin\theta\,\mathrm{d}\theta$; 故 $2\displaystyle\int_0^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta\,\mathrm{d}\theta$

$= \displaystyle\int_0^{\frac{\pi}{2}}\cos^{2m-2}\theta\sin^{2n-2}\theta\big(-2\cos\theta\sin\theta\,\mathrm{d}\theta\big) = -\int_1^0 t^{m-1}(1-t)^{n-1}\,\mathrm{d}t = \int_0^1 t^{m-1}(1-t)^{n-1}\,\mathrm{d}t = B(m,n)$.

例. 設 $V_n(a)$ 為半徑 a 之 n 維球體積, $n \geqslant 1$, $a > 0$; 證明 $V_n(1) = \dfrac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

證. $V_n(a) = \displaystyle\int_{x_1^2+x_2^2+\cdots+x_n^2\leqslant a^2}\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n$. 變數變換 $x_i = au_i$ $\forall\, i = 1,2,\ldots,n$, 則 Jacobian 為 a^n, 積分範

圍變為 $u_1^2+u_2^2+\cdots+u_n^2\leqslant 1$, 故 $V_n(a) = a^n\,V_n(1)$. $V_n(1) = \displaystyle\int_{u_1^2+u_2^2+\cdots+u_n^2\leqslant 1}\mathrm{d}u_1\,\mathrm{d}u_2\cdots\mathrm{d}u_n$

$= \displaystyle\int_{u_n^2\leqslant 1}\left(\int_{u_1^2+u_2^2+\cdots+u_{n-1}^2\leqslant 1-u_n^2}\mathrm{d}u_1\,\mathrm{d}u_2\cdots\mathrm{d}u_{n-1}\right)\mathrm{d}u_n = \int_{-1}^1 V_{n-1}\big(\sqrt{1-u_n^2}\big)\,\mathrm{d}u_n = V_{n-1}(1)\cdot\int_{-1}^1(1-u_n^2)^{\frac{n-1}{2}}\,\mathrm{d}u_n$

$= V_{n-1}(1)\cdot 2\displaystyle\int_0^1(1-u_n^2)^{\frac{n-1}{2}}\,\mathrm{d}u_n$. 令 $t = u_n^2 \implies u_n = \sqrt{t} \implies \mathrm{d}u_n = \dfrac{\mathrm{d}t}{2\sqrt{t}}$, 則 $2\displaystyle\int_0^1(1-u_n^2)^{\frac{n-1}{2}}\,\mathrm{d}u_n =$

$\displaystyle\int_0^1(1-t)^{\frac{n-1}{2}}t^{-\frac{1}{2}}\,\mathrm{d}t = B\left(\dfrac{n+1}{2},\dfrac{1}{2}\right) = \dfrac{\Gamma(\frac{n+1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n+2}{2})}$. 故 $V_n(1) = \dfrac{\Gamma(\frac{n+1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n+2}{2})}V_{n-1}(1) = \dfrac{\Gamma(\frac{n+1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n+2}{2})}\cdot$

$\dfrac{\Gamma(\frac{n}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})}V_{n-2}(1) = \dfrac{\Gamma(\frac{n+1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n+2}{2})}\cdot\dfrac{\Gamma(\frac{n}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})}\cdot\dfrac{\Gamma(\frac{n-1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})}V_{n-3}(1) = \cdots\cdots\cdots = \dfrac{\Gamma(\frac{3}{2})\cdot\big(\Gamma(\frac{1}{2})\big)^{n-1}}{\Gamma(\frac{n+2}{2})}V_1(1) =$

$\dfrac{\frac{1}{2}\sqrt{\pi}\cdot(\sqrt{\pi})^{n-1}}{\Gamma(\frac{n}{2}+1)}\cdot 2 = \dfrac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

例. 設 $V_n(a)$ 為 n 維區域 $|x_1| + |x_2| + \cdots + |x_n| \leqslant a$ 之體積, $n \geqslant 1$, $a > 0$; 證明 $V_n(a) = a^n \dfrac{2^n}{n!}$.

證. $V_n(a) = \displaystyle\int_{|x_1|+|x_2|+\cdots+|x_n|\leqslant a} \mathrm{d}x_1 \, \mathrm{d}x_2 \cdots \mathrm{d}x_n$. 變數變換 $x_i = au_i \; \forall\, i = 1, 2, \ldots, n$, 則 Jacobian 為 a^n, 積分範圍變為 $|u_1| + |u_2| + \cdots + |u_n| \leqslant 1$, 故 $V_n(a) = a^n V_n(1)$. $V_n(1) = \displaystyle\int_{|u_1|+|u_2|+\cdots+|u_n|\leqslant 1} \mathrm{d}u_1 \, \mathrm{d}u_2 \cdots \mathrm{d}u_n$

$= \displaystyle\int_{|u_n|\leqslant 1} \left(\int_{|u_1|+|u_2|+\cdots+|u_{n-1}|\leqslant 1-|u_n|} \mathrm{d}u_1 \, \mathrm{d}u_2 \cdots \mathrm{d}u_{n-1} \right) \mathrm{d}u_n = \int_{-1}^{1} V_{n-1}(1-|u_n|) \, \mathrm{d}u_n = V_{n-1}(1) \int_{-1}^{1} (1-|u_n|)^{n-1} \, \mathrm{d}u_n$

$= V_{n-1}(1) \left(\displaystyle\int_{-1}^{0} (1+u_n)^{n-1} \, \mathrm{d}u_n + \int_{0}^{1} (1-u_n)^{n-1} \, \mathrm{d}u_n \right) = \dfrac{2}{n} V_{n-1}(1).$ 故 $V_n(1) = \dfrac{2^{n-1}}{n \cdot (n-1)\cdots 2} V_1(1) = \dfrac{2^{n-1}}{n!} \cdot 2 = \dfrac{2^n}{n!}$, $V_n(a) = a^n \dfrac{2^n}{n!}$.