

# A Collection of Integrals

**Problem 1.**  $\int_{-2}^2 |x^2 - 4x| dx$

**Solution.**  $\int_{-2}^2 |x^2 - 4x| dx = \int_{-2}^2 |x(x-4)| dx = \int_{-2}^0 |x(x-4)| dx + \int_0^2 |x(x-4)| dx = \int_{-2}^0 x(x-4) dx - \int_0^2 x(x-4) dx = \int_{-2}^0 (x^2 - 4x) dx + \int_0^2 (4x - x^2) dx = \int_{-2}^0 (x^2 - 4x) dx + \int_0^2 (4x - x^2) dx = \left( \frac{x^3}{3} - 2x^2 \right) \Big|_{-2}^0 + \left( 2x^2 - \frac{x^3}{3} \right) \Big|_0^2 = 0 - \left( \frac{(-2)^3}{3} - 2(-2)^2 \right) + \left( 2 \cdot 2^2 - \frac{2^3}{3} \right) = 16$

**Problem 2.**  $\int_0^5 \frac{3x-1}{x+2} dx$

**Solution.**  $\int_0^5 \frac{3x-1}{x+2} dx = \int_0^5 \frac{3(x+2)-7}{x+2} dx = 3 \cdot 5 - 7 \int_0^5 \frac{1}{x+2} dx = 15 - 7 \ln|x+2| \Big|_0^5 = 15 - 7 \ln \frac{7}{2}$

**Problem 3.**  $\int_0^4 \frac{x-1}{x^2-4x-5} dx$

**Solution.**  $\int_0^4 \frac{x-1}{x^2-4x-5} dx = \int_0^4 \frac{(x+1)-2}{(x-5)(x+1)} dx = \int_0^4 \frac{1}{x-5} dx - 2 \int_0^4 \frac{1}{(x-5)(x+1)} dx = \int_0^4 \frac{1}{x-5} dx - \frac{1}{3} \int_0^4 \left( \frac{1}{x-5} - \frac{1}{x+1} \right) dx = \frac{2}{3} \int_0^4 \frac{1}{x-5} dx + \frac{1}{3} \int_0^4 \frac{1}{x+1} dx = \frac{2}{3} \ln|x-5| \Big|_0^4 + \frac{1}{3} \ln|x+1| \Big|_0^4 = -\frac{2}{3} \ln 5 + \frac{1}{3} \ln 5 = -\frac{1}{3} \ln 5.$

**Problem 4.**  $\int \frac{x-1}{x^2-4x+5} dx$

**Solution.**  $\int \frac{x-1}{x^2-4x+5} dx = \int \frac{(x-2)+1}{(x-2)^2+1} dx = \underbrace{\int \frac{x-2}{(x-2)^2+1} dx}_{\text{Let } u=(x-2)^2+1} + \int \frac{1}{(x-2)^2+1} dx = \frac{1}{2} \ln((x-2)^2+1) +$

$$\tan^{-1}(x-2) = \frac{1}{2} \ln(x^2 - 4x + 5) + \tan^{-1}(x-2)$$

**Problem 5.**  $\int \frac{x-1}{x^2-2x+5} dx$

**Solution.**  $\underbrace{\int \frac{x-1}{x^2-2x+5} dx}_{\text{Let } u=x^2-2x+5} = \frac{1}{2} \ln(x^2 - 2x + 5)$

**Problem 6.**  $\int_0^2 \frac{2x}{(x-3)^2} dx$

**Solution.**  $\int_0^2 \frac{2x}{(x-3)^2} dx = \int_0^2 \left( \frac{2}{x-3} + \frac{6}{(x-3)^2} \right) dx = 2 \ln|x-3| \Big|_0^2 - \frac{6}{x-3} \Big|_0^2 = -2 \ln 3 - 6 \left( -1 + \frac{1}{3} \right) = 4 - 2 \ln 3$

**Problem 7.**  $\int \frac{3x^2-2}{x^3-2x-8} dx$

**Solution.**  $\underbrace{\int \frac{3x^2-2}{x^3-2x-8} dx}_{\text{Let } u=x^3-2x-8} = \ln|x^3 - 2x - 8|$

**Problem 8.**  $\int_2^3 \frac{x^3+1}{x^3-x^2} dx$

**Solution.**  $\int_2^3 \frac{x^3+1}{x^3-x^2} dx = \int_2^3 \frac{(x^3-x^2)+x^2+1}{x^3-x^2} dx = \int_2^3 \left( 1 + \frac{x^2}{x^3-x^2} + \frac{1}{x^3-x^2} \right) dx = 1 + \int_2^3 \frac{1}{x-1} dx + \int_2^3 \left( -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1} \right) dx = 1 + \ln|x-1| \Big|_2^3 - \ln|x| \Big|_2^3 + \frac{1}{x} \Big|_2^3 + \ln|x-1| \Big|_2^3 = 1 + \ln 2 - \frac{1}{6} - \ln \frac{3}{2} + \ln 2 = \frac{5}{6} + 3 \ln 2 - \ln 3$

**Problem 9.**  $\int \frac{1}{(x-2)(x^2+4)} dx$

**Solution.**  $\int \frac{1}{(x-2)(x^2+4)} dx = \frac{1}{8} \int \left( \frac{1}{x-2} - \frac{x+2}{x^2+4} \right) dx = \frac{1}{8} \ln|x-2| - \frac{1}{16} \ln(x^2+4) - \frac{1}{8} \tan^{-1} \frac{x}{2}$

**Problem 10.**  $\int \frac{x}{x^4-a^4} dx$

**Solution.**  $\underbrace{\int \frac{x}{x^4-a^4} dx}_{\text{Let } u=x^2} = \frac{1}{2} \int \frac{1}{u^2-a^4} du = \frac{1}{4a^2} \int \left( \frac{1}{u-a^2} - \frac{1}{u+a^2} \right) du = \frac{1}{4a^2} \ln \left| \frac{u-a^2}{u+a^2} \right| = \frac{1}{4a^2} \ln \left| \frac{x^2-a^2}{x^2+a^2} \right|$

**Problem 11.**  $\int \frac{x}{x^4+x^2+1} dx$

**Solution.**  $\underbrace{\int \frac{x}{x^4+x^2+1} dx}_{\text{Let } u=x^2} = \frac{1}{2} \int \frac{1}{u^2+u+1} du = \frac{1}{2} \int \frac{1}{(u+\frac{1}{2})^2 + \frac{3}{4}} du = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1} \frac{2u+1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x^2+1}{\sqrt{3}}$

**Problem 12.**  $\int \frac{1}{x(x^4+1)} dx$

**Solution.**  $\underbrace{\int \frac{1}{x(x^4+1)} dx}_{\text{Let } u=x^2} = \frac{1}{2} \int \frac{1}{u(u^2+1)} du = \frac{1}{2} \int \left( \frac{1}{u} - \frac{u}{u^2+1} \right) du = \frac{1}{2} \ln|u| - \frac{1}{4} \ln(u^2+1) = \frac{1}{4} \ln \frac{x^4}{x^4+1}$

**Problem 13.**  $\int_1^\infty \frac{x^2-3}{x^4+2x^2+9} dx$

**Solution.** Note that  $x^4+2x^2+9 = (x^2+3)^2 - (2x)^2 = (x^2-2x+3)(x^2+2x+3)$ ,  $\int_1^\infty \frac{x^2-3}{x^4+2x^2+9} dx = \frac{1}{2} \int_1^\infty \left( \frac{x-1}{x^2-2x+3} - \frac{x+1}{x^2+2x+3} \right) dx = \frac{1}{4} \ln \left| \frac{x^2-2x+3}{x^2+2x+3} \right| \Big|_1^\infty = \frac{\ln 3}{4}$

**Problem 14.**  $\int \frac{1}{x^6-1} dx$

**Solution.**  $\frac{1}{x^6-1} = \frac{1}{2} \left( \frac{1}{x^3-1} - \frac{1}{x^3+1} \right) \cdot \frac{1}{x^3-1} = \frac{a}{x-1} + \frac{bx+C}{x^2+x+1} \implies 1 = a(x^2+x+1) + (bx+C)(x-1)$   
 $\implies a = \frac{1}{3}, b = -\frac{1}{3}, c = -\frac{2}{3}; \frac{1}{x^3+1} = \frac{a}{x+1} + \frac{bx+C}{x^2-x+1} \implies 1 = a(x^2-x+1) + (bx+C)(x+1)$   
 $\implies a = \frac{1}{3}, b = -\frac{1}{3}, c = \frac{2}{3}$ . So  $\frac{1}{x^6-1} = \frac{1}{6} \left( \frac{1}{x-1} - \frac{x+2}{x^2+x+1} - \frac{1}{x+1} + \frac{x-2}{x^2-x+1} \right)$ ,  $\int \frac{1}{x^6-1} dx = \frac{1}{6} \int \left( \frac{1}{x-1} - \frac{\frac{1}{2}(2x+1)+\frac{3}{2}}{x^2+x+1} - \frac{1}{x+1} + \frac{\frac{1}{2}(2x-1)-\frac{3}{2}}{x^2-x+1} \right) dx = \frac{1}{6} \ln \left| \frac{x-1}{x+1} \right| + \frac{1}{12} \ln \left| \frac{x^2-x+1}{x^2+x+1} \right| - \frac{1}{2\sqrt{3}} \left( \tan^{-1} \frac{2x+1}{\sqrt{3}} + \tan^{-1} \frac{2x-1}{\sqrt{3}} \right)$

**Problem 15.**  $\int \frac{x^3}{(x+1)^{10}} dx$

**Solution.**  $\underbrace{\int \frac{x^3}{(x+1)^{10}} dx}_{\text{Let } u=x+1} = \int \frac{(u-1)^3}{u^{10}} du = \int \frac{u^3-3u^2+3u-1}{u^{10}} du = \int (u^{-7}-3u^{-8}+3u^{-9}-u^{-10}) du = \frac{1}{-6}u^{-6} + 3\frac{1}{7}u^{-7} - 3\frac{1}{8}u^{-8} + \frac{1}{9}u^{-9} = -\frac{1}{6(x+1)^6} + \frac{3}{7(x+1)^7} - \frac{3}{8(x+1)^8} + \frac{1}{9(x+1)^9} = -\frac{84x^3+36x^2+9x+1}{504(x+1)^9}$

**Problem 16.**  $\int \frac{x^4}{x^{10}+16} dx$

**Solution.**  $\underbrace{\int \frac{x^4}{x^{10}+16} dx}_{\text{Let } u=x^5} = \frac{1}{5} \int \frac{1}{u^2+16} du = \frac{1}{20} \tan^{-1} \frac{u}{4} = \frac{1}{20} \tan^{-1} \frac{x^5}{4}$

**Problem 17.**  $\int_{-1}^{\infty} \left( \frac{x^4}{1+x^6} \right)^2 dx$

**Solution.**  $\int_{-1}^{\infty} \left( \frac{x^4}{1+x^6} \right)^2 dx = \underbrace{\int_{-1}^{\infty} \frac{x^8}{(1+x^6)^2} dx}_{\text{Let } w=x^3} = \frac{1}{3} \int_{-1}^{\infty} \frac{w^2}{(1+w^2)^2} dw$ . Note that by evaluating  $\int \frac{1}{1+w^2} dw$

through integration by parts: let  $u = \frac{1}{1+w^2}$ , then  $du = -\frac{2w}{(1+w^2)^2} dw$ ; let  $dv = dw$ , then  $v = w$ . So  $\int \frac{1}{1+w^2} dw = \frac{1}{1+w^2} \cdot w + \int w \cdot \frac{2w}{(1+w^2)^2} dw \Rightarrow \int \frac{w^2}{(1+w^2)^2} dw = \frac{1}{2} \int \frac{1}{1+w^2} dw - \frac{w}{2(1+w^2)} = \frac{1}{2} \left( \tan^{-1} w - \frac{w}{1+w^2} \right)$ ; Hence  $\frac{1}{3} \int_{-1}^{\infty} \frac{w^2}{(1+w^2)^2} dw = \frac{1}{6} \left( \tan^{-1} w - \frac{w}{1+w^2} \right) \Big|_{-1}^{\infty} = \frac{1}{6} \left( \left( \frac{\pi}{2} - 0 \right) - \left( -\frac{\pi}{4} + \frac{1}{2} \right) \right) = \frac{\pi}{8} - \frac{1}{12}$

**Problem 18.**  $\int \frac{1}{x^n - x} dx$

**Solution.**  $\int \frac{1}{x^n - x} dx = \int \frac{1}{x(x^{n-1} - 1)} dx = \int \left( \frac{x^{n-2}}{x^{n-1} - 1} - \frac{1}{x} \right) dx = \underbrace{\int \frac{x^{n-2}}{x^{n-1} - 1} dx}_{\text{Let } u = x^{n-1} - 1} - \int \frac{1}{x} dx = \frac{1}{n-1} \ln |x^{n-1} - 1| - \ln |x|$

**Problem 19.**  $\int_0^{\infty} \frac{x^n}{(1+x^2)^{1+\frac{n}{2}}} dx, \forall n \in \mathbb{N}$ .

**Solution.**  $\underbrace{\int_0^{\infty} \frac{x^n}{(1+x^2)^{1+\frac{n}{2}}} dx}_{\text{Let } x = \tan \theta} = \int_0^{\frac{\pi}{2}} \frac{\tan^n \theta}{\sec^2 \theta \cdot \sec^n \theta} \cdot \sec^2 \theta d\theta = \int_0^{\frac{\pi}{2}} \sin^n \theta d\theta = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} & n \text{ odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & n \text{ even} \end{cases}$

**Problem 20.**  $\int x \sqrt[3]{x+c} dx$

**Solution.**  $\underbrace{\int x \sqrt[3]{x+c} dx}_{\text{Let } u = \sqrt[3]{x+c}} = \int (u^3 - c) \cdot u \cdot 3u^2 du = 3 \int u^6 du - 3c \int u^3 du = \frac{3}{7} u^7 - \frac{3c}{4} u^4 = \frac{3}{7} (\sqrt[3]{x+c})^7 - \frac{3c}{4} (\sqrt[3]{x+c})^4$

**Problem 21.**  $\int_0^1 (1 + \sqrt{x})^8 dx$

**Solution.**  $\underbrace{\int_0^1 (1 + \sqrt{x})^8 dx}_{\text{Let } u = 1 + \sqrt{x}} = \int_1^2 u^8 \cdot 2(u-1) du = 2 \int_1^2 (u^9 - u^8) du = 2 \left( \frac{u^{10}}{10} - \frac{u^9}{9} \right) \Big|_1^2 = 2 \cdot \left( \left( \frac{2^{10}}{10} - \frac{2^9}{9} \right) - \left( \frac{1}{10} - \frac{1}{9} \right) \right) = \frac{4097}{45}$

**Problem 22.**  $\int \frac{\sqrt{x}}{1+x^3} dx$

**Solution.**  $\underbrace{\int \frac{\sqrt{x}}{1+x^3} dx}_{\text{Let } u = x^{\frac{3}{2}}} = \int \frac{\frac{2}{3} du}{1+u^2} = \frac{2}{3} \int \frac{du}{1+u^2} = \frac{2}{3} \tan^{-1} u = \frac{2}{3} \tan^{-1} x^{\frac{3}{2}}$

**Problem 23.**  $\int \frac{\sqrt{x}}{1+\sqrt[3]{x}} dx$

**Solution.**  $\underbrace{\int \frac{\sqrt{x}}{1+\sqrt[3]{x}} dx}_{\text{Let } u = x^{\frac{1}{6}}} = \int \frac{u^3}{1+u^2} \cdot 6u^5 du = 6 \int \left( u^6 - u^4 + u^2 - 1 + \frac{1}{1+u^2} \right) du = 6 \left( \frac{u^7}{7} - \frac{u^5}{5} + \frac{u^3}{3} - u + \tan^{-1} u \right) = 6 \left( \frac{x^{\frac{7}{6}}}{7} - \frac{x^{\frac{5}{6}}}{5} + \frac{x^{\frac{1}{2}}}{3} - x^{\frac{1}{6}} + \tan^{-1} x^{\frac{1}{6}} \right)$

**Problem 24.**  $\int \frac{1}{\sqrt{x+1} + \sqrt{x}} dx$

**Solution.**  $\int \frac{1}{\sqrt{x+1} + \sqrt{x}} dx = \int \frac{\sqrt{x+1} - \sqrt{x}}{\sqrt{x+1} - \sqrt{x}} \cdot \frac{1}{\sqrt{x+1} + \sqrt{x}} dx = \int \frac{\sqrt{x+1} - \sqrt{x}}{(\sqrt{x+1})^2 - (\sqrt{x})^2} dx = \int (\sqrt{x+1} - \sqrt{x}) dx = \frac{2}{3}((x+1)^{\frac{3}{2}} - x^{\frac{3}{2}})$

**Problem 25.**  $\int \frac{1}{x\sqrt{4x+1}} dx$

**Solution.**  $\underbrace{\int \frac{1}{x\sqrt{4x+1}} dx}_{\text{Let } u = \sqrt{4x+1}} = \int \frac{\frac{1}{2}u du}{\frac{u^2-1}{4} \cdot u} = \int \frac{2}{u^2-1} du = \int \left( \frac{1}{u-1} - \frac{1}{u+1} \right) du = \ln \left| \frac{u-1}{u+1} \right| = \ln \left| \frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1} \right|$

**Problem 26.**  $\int \frac{1}{x^2\sqrt{4x+1}} dx$

**Solution.**  $\underbrace{\int \frac{1}{x^2\sqrt{4x+1}} dx}_{\text{Let } u = \sqrt{4x+1}} = \int \frac{\frac{1}{2}u du}{\frac{(u^2-1)^2}{16} \cdot u} = \int \frac{8}{(u^2-1)^2} du = 2 \int \left( \frac{1}{u-1} - \frac{1}{u+1} \right)^2 du$   
 $= 2 \int \left( \frac{1}{(u-1)^2} + \frac{1}{(u+1)^2} - \frac{2}{(u-1)(u+1)} \right) du = 2 \int \left( \frac{1}{(u-1)^2} + \frac{1}{(u+1)^2} + \frac{1}{u+1} - \frac{1}{u-1} \right) du = 2 \left( -\frac{1}{u+1} - \frac{1}{u-1} + \ln \left| \frac{u+1}{u-1} \right| \right) = -\frac{\sqrt{4x+1}}{x} + 2 \ln \left| \frac{\sqrt{4x+1}+1}{\sqrt{4x+1}-1} \right|$

**Problem 27.**  $\int \frac{1}{x+4+4\sqrt{x+1}} dx$

**Solution.**  $\underbrace{\int \frac{1}{x+4+4\sqrt{x+1}} dx}_{\text{Let } u = \sqrt{x+1}} = \int \frac{2u du}{u^2+3+4u} = \int \frac{2u}{(u+3)(u+1)} du = \int \left( \frac{3}{u+3} - \frac{1}{u+1} \right)^2 du = \ln \left| \frac{(u+3)^3}{u+1} \right| = \ln \left| \frac{(\sqrt{x+1}+3)^3}{\sqrt{x+1}+1} \right|$

**Problem 28.**  $\int \sqrt{\frac{1+x}{1-x}} dx$

**Solution.**  $\int \sqrt{\frac{1+x}{1-x}} dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}} dx = \int \frac{1+x}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-x^2}} dx + \int \frac{x}{\sqrt{1-x^2}} dx = \sin^{-1} x - \sqrt{1-x^2}$

**Problem 29.**  $\int \frac{\sqrt{2-x}}{\sqrt{x}} dx$

**Solution.**  $\underbrace{\int \frac{\sqrt{2-x}}{\sqrt{x}} dx}_{\text{Let } u = 1-x} = - \int \frac{\sqrt{1+u}}{\sqrt{1-u}} du = \sqrt{1-u^2} - \sin^{-1} u = \sqrt{x(2-x)} - \sin^{-1}(1-x)$

**Problem 30.**  $\int_0^\infty \frac{dx}{\sqrt{x+x^2}}$

**Solution.**  $\underbrace{\int_0^\infty \frac{dx}{\sqrt{x+x^2}}}_{\text{Let } u = \sqrt{x}} = \int_0^\infty \frac{2u du}{u+u^4} = 2 \int_0^\infty \frac{du}{1+u^3}$ . From  $\frac{1}{u^3+1} = \frac{\frac{1}{3}}{u+1} + \frac{-\frac{1}{3}u+\frac{2}{3}}{u^2-u+1} = \frac{\frac{1}{3}}{u+1} + \frac{-\frac{1}{3}(u-\frac{1}{2})+\frac{1}{2}}{u^2-u+1} = \frac{\frac{1}{3}}{u+1} + \frac{-\frac{1}{6}(2u-1)+\frac{1}{2}}{u^2-u+1} + \frac{\frac{1}{2}}{(u-\frac{1}{2})^2+\frac{3}{4}}$ ,  $\int \frac{du}{u^3+1} = \frac{1}{3} \ln |u+1| - \frac{1}{6} \ln(u^2-u+1) + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2u-1}{\sqrt{3}}$ , so  $\int_0^\infty \frac{dx}{x^2+\sqrt{x}} = 2 \int_0^\infty \frac{du}{u^3+1} = \frac{4\pi}{3\sqrt{3}}$  by  $\lim_{u \rightarrow \infty} \frac{(u+1)^2}{u^2-u+1} = 1$ ,  $\lim_{u \rightarrow 0} \frac{(u+1)^2}{u^2-u+1} = 1$ ,  $\lim_{u \rightarrow \infty} \tan^{-1} \frac{2u-1}{\sqrt{3}} = \frac{\pi}{2}$ , and  $\lim_{u \rightarrow 0} \tan^{-1} \frac{2u-1}{\sqrt{3}} = -\frac{\pi}{6}$ .

**Problem 31.**  $\int \sqrt{3-2x-x^2} dx$

**Solution.**  $\int \sqrt{3 - 2x - x^2} dx = \int \sqrt{4 - 1 - 2x - x^2} dx = \underbrace{\int \sqrt{4 - (x+1)^2} dx}_{\text{Let } u = x+1} = \underbrace{\int \sqrt{4 - u^2} du}_{\text{Let } u = 2\sin\theta}$

$$= \int 2\cos\theta \cdot 2\cos\theta d\theta = 2 \int (\cos 2\theta + 1) d\theta = \sin 2\theta + 2\theta = 2\sin\theta\cos\theta + 2\sin^{-1}\frac{u}{2} = 2 \cdot \frac{u}{2} \cdot \frac{\sqrt{4-u^2}}{2} + 2\sin^{-1}\frac{u}{2} =$$

$$\frac{(x+1)\sqrt{3-2x-x^2}}{2} + 2\sin^{-1}\frac{x+1}{2}$$

**Problem 32.**  $\int \frac{dx}{(2x-1)\sqrt{x^2-x}}$

**Solution.**  $\underbrace{\int \frac{dx}{(2x-1)\sqrt{x^2-x}}}_{\text{Let } u = \sqrt{x^2-x}} = \int \frac{2u du}{(4u^2+1)u} = \tan^{-1}(2\sqrt{x^2-x})$

**Problem 33.**  $\int \frac{2x^2 + \sqrt{1-x^2}}{x\sqrt{1-x^2}} dx$

**Solution.**  $\int \frac{2x^2 + \sqrt{1-x^2}}{x\sqrt{1-x^2}} dx = \int \frac{2x}{\sqrt{1-x^2}} dx + \int \frac{dx}{x} = -2\sqrt{1-x^2} + \ln|x|$

**Problem 34.**  $\int \frac{dx}{(1-x^2)^{\frac{3}{2}}}$

**Solution.**  $\underbrace{\int \frac{dx}{(1-x^2)^{\frac{3}{2}}}}_{\text{Let } x = \sin\theta} = \int \frac{\cos\theta d\theta}{\cos^3\theta} = \int \sec^2\theta d\theta = \tan\theta = \frac{x}{\sqrt{1-x^2}}$

**Problem 35.**  $\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx$

**Solution.**  $\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} \Big|_0^{\frac{1}{2}} = -\left(\frac{\sqrt{3}}{2} - 1\right) = 1 - \frac{\sqrt{3}}{2}$

**Problem 36.**  $\int_0^{\frac{\sqrt{2}}{2}} \frac{x^2}{\sqrt{1-x^2}} dx$

**Solution.**  $\int_0^{\frac{\sqrt{2}}{2}} \frac{x^2}{\sqrt{1-x^2}} dx = \int_0^{\frac{\sqrt{2}}{2}} \frac{(x^2-1)+1}{\sqrt{1-x^2}} dx = -\int_0^{\frac{\sqrt{2}}{2}} \sqrt{1-x^2} dx + \sin^{-1}x \Big|_0^{\frac{\sqrt{2}}{2}} = -\left(\frac{\pi}{8} + \frac{1}{4}\right) + \frac{\pi}{4} = \frac{\pi}{8} - \frac{1}{4}$

**Problem 37.**  $\int \frac{x}{1-x^2+\sqrt{1-x^2}} dx$

**Solution.**  $\underbrace{\int \frac{x}{1-x^2+\sqrt{1-x^2}} dx}_{\text{Let } u = \sqrt{1-x^2}} = \int \frac{-u du}{u^2+u} = -\ln|u+1| = -\ln|\sqrt{1-x^2}+1|$

**Problem 38.**  $\int \frac{1}{\sqrt{4x^2-4x-3}} dx$

**Solution.**  $\int \frac{1}{\sqrt{4x^2-4x-3}} dx = \int \frac{1}{\sqrt{4x^2-4x+1-4}} dx = \underbrace{\int \frac{1}{\sqrt{(2x-1)^2-2^2}} dx}_{\text{Let } 2x-1 = 2\sec\theta} = \int \frac{\sec\theta \tan\theta d\theta}{2\tan\theta}$ 

$$= \frac{1}{2} \int \sec\theta d\theta = \frac{1}{2} \ln|\sec\theta + \tan\theta| = \frac{1}{2} \ln\left|x - \frac{1}{2} + \sqrt{x^2 - x - \frac{3}{4}}\right|$$

**Problem 39.**  $\int \frac{1}{x-\sqrt{1-x^2}} dx$

**Solution.**  $\int \underbrace{\frac{1}{x - \sqrt{1-x^2}} dx}_{\text{Let } x = \sin \theta} = \int \frac{\cos \theta}{\sin \theta - \cos \theta} d\theta = \frac{1}{2} \int \frac{2 \cos \theta}{\sin \theta - \cos \theta} d\theta = \frac{1}{2} \int \frac{(\cos \theta - \sin \theta) + (\cos \theta + \sin \theta)}{\sin \theta - \cos \theta} d\theta$

$$-\frac{1}{2}\theta + \frac{1}{2} \int \underbrace{\frac{\cos \theta + \sin \theta}{\sin \theta - \cos \theta} d\theta}_{\text{Let } u = \sin \theta - \cos \theta} = \frac{1}{2}(-\theta + \ln |\sin \theta - \cos \theta|) = \frac{1}{2}(-\sin^{-1} x + \ln |x - \sqrt{1-x^2}|)$$

**Problem 40.**  $\int \frac{x}{\sqrt{3-x^4}} dx$

**Solution.**  $\int \underbrace{\frac{x}{\sqrt{3-x^4}} dx}_{\text{Let } u = x^2} = \int \frac{\frac{du}{2}}{\sqrt{3-u^2}} = \frac{1}{2} \int \frac{du}{\sqrt{3-u^2}} = \frac{1}{2} \sin^{-1} \frac{u}{\sqrt{3}} = \frac{1}{2} \sin^{-1} \frac{x^2}{\sqrt{3}}$

**Problem 41.**  $\int \frac{dx}{\sqrt[3]{1+x^2}}$

**Solution.**  $\int \underbrace{\frac{dx}{\sqrt[3]{1+x^2}}}_{\text{Let } u = \sqrt[3]{1+x^2}} = \int \frac{(u^2-1)^2 6u du}{u} = 6 \int (u^4 - 2u^2 + 1) du = \frac{6}{5}u^5 - 4u^3 + 6u = \frac{6}{5}(\sqrt[3]{1+\sqrt[3]{x}})^5 -$

$$4(\sqrt[3]{1+\sqrt[3]{x}})^3 + 6\sqrt[3]{1+\sqrt[3]{x}} = \frac{2}{5}\sqrt[3]{1+\sqrt[3]{x}}(8 - 4\sqrt[3]{x} + 6\sqrt[3]{x^2})$$

**Problem 42.**  $\int_0^1 (\sqrt[3]{1-x^7} - \sqrt[7]{1-x^3}) dx$

**Solution.** Let  $u = \sqrt[3]{1-x^7} \implies x = \sqrt[7]{1-u^3}$ . Integration by parts yield  $\int_0^1 \sqrt[3]{1-x^7} dx = \int_1^0 u d(\sqrt[7]{1-u^3}) = u \cdot \sqrt[7]{1-u^3} \Big|_1^0 - \int_1^0 \sqrt[7]{1-u^3} du = \int_0^1 \sqrt[7]{1-u^3} du$ . So  $\int_0^1 (\sqrt[3]{1-x^7} - \sqrt[7]{1-x^3}) dx = 0$ .

**Problem 43.**  $\int_0^1 3(x-1)^2 \left( \int_0^x \sqrt{1+(t-1)^4} dt \right) dx$

**Solution.**  $\int_0^1 3(x-1)^2 \left( \int_0^x \sqrt{1+(t-1)^4} dt \right) dx$

Let  $u = \int_0^x \sqrt{1+(t-1)^4} dt$ ,  $dv = 3(x-1)^2 dx$

$$= \left( \int_0^x \sqrt{1+(t-1)^4} dt \right) \cdot (x-1)^3 \Big|_0^1 - \underbrace{\int_0^1 (x-1)^3 \cdot \sqrt{1+(x-1)^4} dx}_{\text{Let } u = 1+(x-1)^4} = \frac{1}{4} \int_1^2 \sqrt{u} du = \frac{2\sqrt{2}-1}{6}$$

**Problem 44.**  $\int \sin^3 x \cos^5 x dx$

**Solution.**  $\int \sin^3 x \cos^5 x dx = \int \sin^3 x \cos^4 x \cos x dx = \underbrace{\int \sin^3 x (1-\sin^2 x)^2 \cos x dx}_{\text{Let } u = \sin x} = \int u^3 (1-u^2)^2 du = \int (u^3 -$

$$2u^5 + u^7) du = \frac{u^4}{4} - \frac{u^6}{3} + \frac{u^8}{8} = \frac{\sin^4 x}{4} - \frac{\sin^6 x}{3} + \frac{\sin^8 x}{8}$$

**Problem 45.**  $\int \sin x \cos(\cos x) dx$

**Solution.**  $\int \underbrace{\sin x \cos(\cos x) dx}_{\text{Let } u = \cos x} = - \int \cos u du = -\sin u = -\sin(\cos x)$

**Problem 46.**  $\int \tan^3 x dx$

**Solution.**  $\int \tan^3 x \, dx = \int (\sec^2 x - 1) \tan x \, dx = \int \sec^2 x \tan x \, dx - \int \tan x \, dx = \underbrace{\int \sec x \cdot \sec x \tan x \, dx}_{\text{Let } u = \sec x} - \underbrace{\int \frac{\sin x}{\cos x} \, dx}_{\text{Let } w = \cos x} = \frac{\sec^2 x}{2} + \ln |\cos x|$

**Problem 47.**  $\int \tan^5 x \sec^4 x \, dx$

**Solution.**  $\int \tan^5 x \sec^4 x \, dx = \int \tan^5 x \sec^2 x \sec^2 x \, dx = \underbrace{\int \tan^5 x (\tan^2 x + 1) \sec^2 x \, dx}_{\text{Let } u = \tan x} = \int (u^7 + u^5) \, du = \frac{u^8}{8} + \frac{u^6}{6} = \frac{\tan^8 x}{8} + \frac{\tan^6 x}{6}$

**Problem 48.**  $\int \tan^5 x \sec^7 x \, dx$

**Solution.**  $\int \tan^5 x \sec^7 x \, dx = \int \tan^4 x \sec^6 x \cdot \tan x \sec x \, dx = \underbrace{\int (\sec^2 x - 1)^2 \sec^6 x \cdot \tan x \sec x \, dx}_{\text{Let } u = \sec x} = \int (u^2 - 1)^2 u^6 \, du = \int (u^{10} - 2u^8 + u^6) \, du = \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} = \frac{\sec^{11} x}{11} - \frac{2\sec^9 x}{9} + \frac{\sec^7 x}{7}$

**Problem 49.**  $\int_0^{\frac{\pi}{4}} \cos^2 x \tan^2 x \, dx$

**Solution.**  $\int_0^{\frac{\pi}{4}} \cos^2 x \tan^2 x \, dx = \int_0^{\frac{\pi}{4}} \sin^2 x \, dx = \int_0^{\frac{\pi}{4}} \frac{1 - \cos 2x}{2} \, dx = \frac{\pi}{8} - \frac{1}{4} \sin 2x \Big|_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{1}{4}$

**Problem 50.**  $\int (\sin x + \cos x)^2 \, dx$

**Solution.**  $\int (\sin x + \cos x)^2 \, dx = \int (\sin^2 x + 2 \sin x \cos x + \cos^2 x) \, dx = \int (1 + \sin 2x) \, dx = x - \frac{\cos 2x}{2}$

**Problem 51.**  $\int \sin 4x \cos 3x \, dx$

**Solution.**  $\int \sin 4x \cos 3x \, dx = \frac{1}{2} \int (\sin(4x + 3x) + \sin(4x - 3x)) \, dx = -\frac{1}{2} \left( \frac{\cos 7x}{7} + \cos x \right)$

**Problem 52.**  $\int \sin x \sin 2x \sin 3x \, dx$

**Solution.**  $\int \sin x \sin 2x \sin 3x \, dx = \frac{1}{2} \int (\cos(x - 2x) - \cos(x + 2x)) \sin 3x \, dx = \frac{1}{2} \int (\cos x \sin 3x - \cos 3x \sin 3x) \, dx = \frac{1}{4} \int (\sin(3x - x) + \sin(3x + x) - \sin 6x) \, dx = \frac{1}{4} \int (\sin 2x + \sin 4x - \sin 6x) \, dx = -\frac{\cos 2x}{8} - \frac{\cos 4x}{16} + \frac{\cos 6x}{24}$

**Problem 53.**  $\int \sin \sqrt{ax} \, dx$

**Solution.**  $\underbrace{\int \sin \sqrt{ax} \, dx}_{\text{Let } u = \sqrt{ax}} = \int \sin u \cdot \frac{2u}{a} \, du = \frac{2}{a} \int u \sin u \, du = \frac{2}{a} (\sin u - u \cos u) = \frac{2}{a} (\sin \sqrt{ax} - \sqrt{ax} \cos \sqrt{ax})$

**Problem 54.**  $\int \frac{\cos^5 x}{\sqrt{\sin x}} \, dx$

**Solution.**  $\int \frac{\cos^5 x}{\sqrt{\sin x}} \, dx = \int \frac{\cos^4 x \cos x}{\sqrt{\sin x}} \, dx = \underbrace{\int \frac{(1 - \sin^2 x)^2 \cos x}{\sqrt{\sin x}} \, dx}_{\text{Let } u = \sin x} = \int \frac{(1 - u^2)^2}{\sqrt{u}} \, du = \int (u^{-\frac{1}{2}} - 2u^{\frac{3}{2}} + u^{\frac{7}{2}}) \, du = 2u^{\frac{1}{2}} - \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{9}u^{\frac{9}{2}} = 2\sin^{\frac{1}{2}} x - \frac{4}{5}\sin^{\frac{5}{2}} x + \frac{2}{9}\sin^{\frac{9}{2}} x$

**Problem 55.**  $\int \frac{\cos^6 x}{\sin^4 x} dx$

**Solution.**  $\int \frac{\cos^6 x}{\sin^4 x} dx = \int \frac{(1 - \sin^2 x)^3}{\sin^4 x} dx = \int \frac{1 - 3 \sin^2 x + 3 \sin^4 x - \sin^6 x}{\sin^4 x} dx = \int (\csc^4 x - 3 \csc^2 x + 3 - \sin^2 x) dx = \left( -\frac{1}{3} \cot^3 x - \cot x \right) + 3 \cot x + 3x - \left( \frac{x}{2} - \frac{\sin 2x}{4} \right) = -\frac{1}{3} \cot^3 x + 2 \cot x + \frac{5x}{2} + \frac{\sin 2x}{4}$

**Problem 56.**  $\int \frac{\tan x}{\sec^2 x} dx$

**Solution.**  $\int \frac{\tan x}{\sec^2 x} dx = \int \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos^2 x}} dx = \int \sin x \cos x dx = \frac{1}{2} \int \sin 2x dx = -\frac{1}{4} \cos 2x$

**Problem 57.**  $\int \frac{\tan^3 x}{\cos^3 x} dx$

**Solution.**  $\int \frac{\tan^3 x}{\cos^3 x} dx = \int \tan^2 x \sec^2 x \cdot \tan x \sec x dx = \underbrace{\int (\sec^2 x - 1) \sec^2 x \cdot \tan x \sec x dx}_{\text{Let } u = \sec x} = \int (u^2 - 1) u^2 du = \int (u^4 - u^2) du = \frac{u^5}{5} - \frac{u^3}{3} = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3}$

**Problem 58.**  $\int \frac{1}{1 - \cos x} dx$

**Solution.**  $\int \frac{1}{1 - \cos x} dx = \int \frac{1 + \cos x}{1 - \cos^2 x} dx = \int \frac{1 + \cos x}{\sin^2 x} dx = \int (\csc^2 x + \csc x \cot x) dx = -\cot x - \csc x$

**Problem 59.**  $\int \frac{dx}{3 \sin x - 4 \cos x}$

**Solution.**  $3 \sin x - 4 \cos x = -5 \left( \frac{4}{5} \cos x - \frac{3}{5} \sin x \right) = -5 \cos(x + \alpha)$ , where  $\cos \alpha = \frac{4}{5}$ . So  $\int \frac{dx}{3 \sin x - 4 \cos x} = -\frac{1}{5} \int \frac{dx}{\cos(x + \alpha)} = -\frac{1}{5} \int \sec(x + \alpha) dx = -\frac{1}{5} \ln |\sec(x + \alpha) + \tan(x + \alpha)|$

**Problem 60.**  $\int \frac{dx}{1 + \sin x - \cos x}$

**Solution.**  $\int \frac{dx}{1 + \sin x - \cos x} = \int \frac{1 + \sin x + \cos x}{(1 + \sin x)^2 - \cos^2 x} dx = \int \frac{(1 + \sin x) + \cos x}{2 \sin x (1 + \sin x)} dx = \frac{1}{2} \int \frac{1}{\sin x} dx + \frac{1}{2} \int \frac{\cos x}{\sin x (1 + \sin x)} dx = -\frac{1}{2} \ln |\csc x + \cot x| + \frac{1}{2} \int \frac{du}{u(1+u)} = -\frac{1}{2} \ln |\csc x + \cot x| + \frac{1}{2} \int \left( \frac{1}{u} - \frac{1}{u+1} \right) du = -\frac{1}{2} \ln |\csc x + \cot x| + \frac{1}{2} \ln \left| \frac{\sin x}{\sin x + 1} \right|$   
Let  $u = \sin x$

**Problem 61.**  $\int \frac{\cos x + \sin x}{\sin 2x} dx$

**Solution.**  $\int \frac{\cos x + \sin x}{\sin 2x} dx = \int \frac{\cos x + \sin x}{2 \sin x \cos x} dx = \frac{1}{2} \int \left( \frac{1}{\sin x} + \frac{1}{\cos x} \right) dx = \frac{1}{2} \int (\csc x + \sec x) dx = \frac{1}{2} (-\ln |\csc x + \cot x| + \ln |\sec x + \tan x|) = \frac{1}{2} \ln \left| \frac{\sec x + \tan x}{\csc x + \cot x} \right| = \frac{1}{2} \ln \left| \frac{(1 + \sin x) \sin x}{(1 + \cos x) \cos x} \right|$

**Problem 62.**  $\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$

**Solution.**  $\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx = \int \frac{\frac{1}{2} \sin 2x}{(\frac{1-\cos 2x}{2})^2 + (\frac{1+\cos 2x}{2})^2} dx = \int \frac{\sin 2x}{1 + \cos^2 2x} dx = -\frac{1}{2} \int \frac{du}{1 + u^2} = -\frac{1}{2} \tan^{-1} u = -\frac{1}{2} \tan^{-1}(\cos 2x)$   
Let  $u = \cos 2x$

**Problem 63.**  $\int \frac{\cos x}{\sqrt{\cos^2 x + 2}} dx$

**Solution.**  $\int \frac{\cos x}{\sqrt{\cos^2 x + 2}} dx = \underbrace{\int \frac{\cos x}{\sqrt{3 - \sin^2 x}} dx}_{\text{Let } u = \sin x} = \int \frac{du}{\sqrt{3 - u^2}} = \sin^{-1} \frac{u}{\sqrt{3}} = \sin^{-1} \left( \frac{\sin x}{\sqrt{3}} \right)$

**Problem 64.**  $\int \frac{\sec x \cos 2x}{\sin x + \sec x} dx$

**Solution.**  $\int \frac{\sec x \cos 2x}{\sin x + \sec x} dx = \int \frac{\cos 2x}{\sin x \cos x + 1} dx = \underbrace{\int \frac{\cos 2x}{\frac{1}{2} \sin 2x + 1} dx}_{\text{Let } u = \frac{1}{2} \sin 2x + 1} = \int \frac{du}{u} = \ln \left| \frac{1}{2} \sin 2x + 1 \right|$

**Problem 65.**  $\int \frac{\tan x}{\tan x + \sec x} dx$

**Solution.**  $\int \frac{\tan x}{\tan x + \sec x} dx = \int \frac{\frac{\sin x}{\cos x}}{\frac{\sin x}{\cos x} + \frac{1}{\cos x}} dx = \int \frac{\sin x}{1 + \sin x} dx = \int \frac{(1 - \sin x) \sin x}{(1 - \sin x)(1 + \sin x)} dx = \int \frac{\sin x - \sin^2 x}{\cos^2 x} dx$   
 $= \int \frac{\sin x}{\cos^2 x} dx - \int \frac{\sin^2 x}{\cos^2 x} dx = \sec x - \int \tan^2 x dx = \sec x - \int (\sec^2 x - 1) dx = \sec x - \tan x + x$

**Problem 66.**  $\int \frac{1}{1 - \tan^2 x} dx$

**Solution.**  $\int \frac{1}{1 - \tan^2 x} dx = \int \frac{1 + \tan^2 x}{1 - \tan^4 x} dx = \underbrace{\int \frac{\sec^2 x}{1 - \tan^4 x} dx}_{\text{Let } u = \tan x} = \int \frac{du}{1 - u^4} = \frac{1}{2} \int \left( \frac{1}{1 + u^2} + \frac{1}{1 - u^2} \right) du =$   
 $\frac{1}{2} \int \left( \frac{1}{1 + u^2} + \frac{1}{2} \left( \frac{1}{1 - u} + \frac{1}{1 + u} \right) \right) du = \frac{1}{2} \tan^{-1} u + \frac{1}{4} \ln \left| \frac{1+u}{1-u} \right| = \frac{x}{2} + \frac{1}{4} \ln \left| \frac{1+\tan x}{1-\tan x} \right|$

**Problem 67.**  $\int \frac{3 + \sec^2 x + \sin x}{\tan x} dx$

**Solution.**  $\int \frac{3 + \sec^2 x + \sin x}{\tan x} dx = \int (3 \cot x + 2 \csc 2x + \cos x) dx = 3 \ln |\sin x| - \ln |\csc 2x + \cot 2x| + \sin x =$   
 $4 \ln |\sin x| - \ln |\cos x| + \sin x$

**Problem 68.**  $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1 + 4 \cot x}{4 - \cot x} dx$

**Solution.**  $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1 + 4 \cot x}{4 - \cot x} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1 + 4 \frac{\cos x}{\sin x}}{4 - \frac{\cos x}{\sin x}} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\frac{\sin x + 4 \cos x}{\sin x}}{\frac{4 \sin x - \cos x}{\sin x}} dx = \underbrace{\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x + 4 \cos x}{4 \sin x - \cos x} dx}_{\text{Let } u = 4 \sin x - \cos x} = \int_{\frac{3}{\sqrt{2}}}^4 \frac{du}{u} =$

$\frac{5}{2} \ln 2 - \ln 3$

**Problem 69.**  $\int \sqrt{\tan x} dx$

**Solution.**  $\int \sqrt{\tan x} dx = \int \frac{2u^2}{1 + u^4} du$ . By  $1 + u^4 = 1 + 2u^2 + u^4 - 2u^2 = (1 + u^2)^2 - (\sqrt{2}u)^2 = (1 + \sqrt{2}u + u^2)(1 -$   
 $\sqrt{2}u + u^2)$ ,  $\frac{2u^2}{1 + u^4} = \frac{au + b}{1 + \sqrt{2}u + u^2} + \frac{cu + d}{1 - \sqrt{2}u + u^2} \Rightarrow (au + b)(1 - \sqrt{2}u + u^2) + (cu + d)(1 + \sqrt{2}u + u^2) = 2u^2$

$\Rightarrow (a + c)u^3 + (-\sqrt{2}a + b + \sqrt{2}c + d)u^2 + (a - \sqrt{2}b + c + \sqrt{2}d)u + (b + d) = 2u^2 \Rightarrow a + c = 0,$   
 $-\sqrt{2}a + b + \sqrt{2}c + d = 2, a - \sqrt{2}b + c + \sqrt{2}d = 0, b + d = 0 \Rightarrow a = -\frac{\sqrt{2}}{2}, c = \frac{\sqrt{2}}{2}, b = d = 0$

$\Rightarrow \frac{2u^2}{1 + u^4} = \frac{-\frac{1}{\sqrt{2}}u}{1 + \sqrt{2}u + u^2} + \frac{\frac{1}{\sqrt{2}}u}{1 - \sqrt{2}u + u^2} = \frac{-\frac{1}{2\sqrt{2}}(2u + \sqrt{2}) + \frac{1}{2}}{1 + \sqrt{2}u + u^2} + \frac{\frac{1}{2\sqrt{2}}(2u - \sqrt{2}) + \frac{1}{2}}{1 - \sqrt{2}u + u^2}$ . So  $\int \frac{2u^2}{1 + u^4} du =$   
 $\frac{1}{2\sqrt{2}} \ln \left| \frac{1 - \sqrt{2}u + u^2}{1 + \sqrt{2}u + u^2} \right| + \frac{1}{2} \int \left( \frac{1}{(u + \frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2} + \frac{1}{(u - \frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2} \right) du = \frac{1}{2\sqrt{2}} \ln \left| \frac{1 - \sqrt{2}u + u^2}{1 + \sqrt{2}u + u^2} \right|$

$$+\frac{1}{\sqrt{2}}(\tan^{-1}(\sqrt{2}u+1)+\tan^{-1}(\sqrt{2}u-1)) = \frac{1}{2\sqrt{2}}\ln\left|\frac{1-\sqrt{2\tan x}+\tan x}{1+\sqrt{2\tan x}+\tan x}\right| + \frac{1}{\sqrt{2}}(\tan^{-1}(\sqrt{2\tan x}+1)+\tan^{-1}(\sqrt{2\tan x}-1))$$

**Problem 70.**  $\int \frac{\sin 2x}{1+\tan x} dx$

**Solution.**  $\int \frac{\sin 2x}{1+\tan x} dx = \int \frac{2\sin x \cos x}{1+\frac{\sin x}{\cos x}} dx = \int \frac{2\sin x \cos^2 x}{\cos x + \sin x} dx = \int \frac{2\sin x \cos^2 x}{\cos x + \sin x} \cdot \frac{\cos x - \sin x}{\cos x - \sin x} dx$   
 $= \int \frac{\cos^2 x \sin 2x - \frac{1}{2} \sin^2 2x}{\cos 2x} dx = \int \frac{\frac{1+\cos 2x}{2} \cdot \sin 2x - \frac{1}{2}(1-\cos^2 2x)}{\cos 2x} dx = \frac{1}{2} \int \frac{(1+\cos 2x) \cdot \sin 2x - (1-\cos^2 2x)}{\cos 2x} dx =$   
 $\frac{1}{2} \int \left( \frac{\sin 2x}{\cos 2x} + \sin 2x - \sec 2x + \cos 2x \right) dx = \frac{1}{4} \left( \ln |\sec 2x| - \cos 2x - \ln |\sec 2x + \tan 2x| + \sin 2x \right)$

**Problem 71.**  $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{\frac{\tan x}{\sin 2x}} dx$

**Solution.**  $\int \sqrt{\frac{\tan x}{\sin 2x}} dx = \int \sqrt{\frac{\frac{\sin x}{\cos x}}{2\sin x \cos x}} dx = \frac{1}{\sqrt{2}} \int \sec x dx = \frac{1}{\sqrt{2}} \ln |\sec x + \tan x|, \text{ so } \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{\frac{\tan x}{\sin 2x}} dx =$   
 $\frac{1}{\sqrt{2}} \ln |\sec x + \tan x| \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{1}{\sqrt{2}} \ln \frac{2+\sqrt{3}}{\sqrt{2}+1}$

**Problem 72.**  $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\sin 2x} dx$

**Solution.** By  $(\sqrt{\tan x})' = \frac{1}{2\sqrt{\tan x}} \cdot \sec^2 x = \frac{\sec^2 x \sqrt{\tan x}}{2\tan x} = \frac{\sqrt{\tan x}}{\sin 2x}$ ,  $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\sin 2x} dx = \sqrt{\tan x} \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \sqrt[4]{3} - 1$

**Problem 73.**  $\int \tan^5 x \sqrt[3]{\cos x} dx$

**Solution.**  $\int \tan^5 x \sqrt[3]{\cos x} dx = \int \frac{\sin^5 x}{\cos^5 x} \sqrt[3]{\cos x} dx = \underbrace{\int (1-\cos^2 x)^2 \cos^{\frac{1}{3}-5} x \cdot \sin x dx}_{\text{Let } u=\cos x} = - \int (u^{-\frac{14}{3}} - 2u^{-\frac{8}{3}} +$   
 $u^{-\frac{2}{3}}) du = \frac{3}{11}u^{-\frac{11}{3}} - \frac{6}{5}u^{-\frac{5}{3}} - 3u^{\frac{1}{3}} = \frac{3}{11}\cos^{-\frac{11}{3}}x - \frac{6}{5}\cos^{-\frac{5}{3}}x - 3\cos^{\frac{1}{3}}x$

**Problem 74.**  $\int x \sin^2 x dx$

**Solution.**  $\int x \sin^2 x dx = \frac{1}{2} \int (x - x \cos 2x) dx = \frac{1}{2} \left( x \cdot \frac{1}{2} \sin 2x + \frac{1}{4} \cos 2x \right) = \frac{x \sin 2x}{4} + \frac{\cos 2x}{8}$

**Problem 75.**  $\int (x + \sin x)^2 dx$

**Solution.**  $\int (x + \sin x)^2 dx = \int (x^2 + 2x \sin x + \sin^2 x) dx = \frac{x^3}{6} + 2 \int x \sin x dx + \frac{1}{2} \int (1 - \cos 2x) dx = \frac{x^3}{6} +$   
 $2(\sin x - x \cos x) + \frac{x}{2} - \frac{1}{4} \sin 2x$

**Problem 76.**  $\int x \sin^2 x \cos x dx$

**Solution.**  $\int x \sin^2 x \cos x dx = \int x \cdot \frac{1-\cos 2x}{2} \cdot \cos x dx = \frac{1}{2} \int (x \cos x - x \cos 2x \cos x) dx = \frac{1}{2} \int (x \cos x -$   
 $x \cdot \frac{1}{2} (\cos(2x-x) + \cos(2x+x)) dx = \frac{1}{4} \int (2x \cos x - x \cos x - x \cos 3x) dx = \frac{1}{4} \int x (\cos x - \cos 3x) dx =$   
 $\frac{1}{4} \left( x \left( \sin x - \frac{1}{3} \sin 3x \right) + \cos x - \frac{1}{9} \cos 3x \right)$

**Problem 77.**  $\int x \tan^2 x dx$

**Solution.**  $\int x \tan^2 x \, dx = \int x(\sec^2 x - 1) \, dx = \underbrace{\int x \sec^2 x \, dx}_{\text{Let } u=x, dv=\sec^2 x \, dx} - \frac{x^2}{2} = x \tan x + \ln |\cos x| - \frac{x^2}{2}$

**Problem 78.**  $\int_{-1}^1 x^8 \sin x \, dx$

**Solution.**  $\int_{-1}^1 x^8 \sin x \, dx = 0$  for  $x^8 \sin x$  is odd.

**Problem 79.**  $\int x \sin^{-1} x \, dx$

**Solution.**  $\underbrace{\int x \sin^{-1} x \, dx}_{\text{Let } u=\sin^{-1} x, dv=dx} = \sin^{-1} x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{\sqrt{1-x^2}} \, dx = \frac{x^2 \sin^{-1} x}{2} - \frac{1}{2} \int \frac{(x^2-1)+1}{\sqrt{1-x^2}} \, dx = \frac{x^2 \sin^{-1} x}{2} +$   
 $\frac{1}{2} \int \sqrt{1-x^2} \, dx - \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} \, dx = \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left( \frac{\theta}{2} + \frac{1}{4} \sin 2\theta \right) - \frac{1}{2} \sin^{-1} x = \frac{x^2 \sin^{-1} x}{2} + \frac{1}{4} \left( \sin^{-1} x + x \sqrt{1-x^2} \right) -$   
 $\frac{1}{2} \sin^{-1} x = \frac{x^2 \sin^{-1} x}{2} + \frac{x \sqrt{1-x^2}}{4} - \frac{\sin^{-1} x}{4}$

**Problem 80.**  $\int x^{-2} \tan^{-1} x \, dx$

**Solution.**  $\underbrace{\int x^{-2} \tan^{-1} x \, dx}_{\text{Let } u=\tan^{-1} x, dv=x^{-2} dx} = \tan^{-1} x \cdot \frac{-1}{x} + \int \frac{1}{x} \cdot \frac{1}{1+x^2} \, dx = -\frac{\tan^{-1} x}{x} + \int \left( \frac{1}{x} - \frac{x}{1+x^2} \right) \, dx = -\frac{\tan^{-1} x}{x} +$   
 $\ln|x| - \underbrace{\int \frac{x}{1+x^2} \, dx}_{\text{Let } u=1+x^2} = -\frac{\tan^{-1} x}{x} + \ln|x| - \int \frac{\frac{du}{2}}{u} = -\frac{\tan^{-1} x}{x} + \ln|x| - \frac{1}{2} \ln(1+x^2)$

**Problem 81.**  $\int_1^3 \frac{\tan^{-1} \sqrt{x}}{\sqrt{x}} \, dx$

**Solution.**  $\underbrace{\int_1^3 \frac{\tan^{-1} \sqrt{x}}{\sqrt{x}} \, dx}_{\text{Let } u=\sqrt{x}} = \int_1^{\sqrt{3}} \frac{\tan^{-1} u}{u} \cdot 2u \, du = 2 \int_1^{\sqrt{3}} \tan^{-1} u \, du.$  From  $\underbrace{\int \tan^{-1} w \, dw}_{\text{Let } u=\tan^{-1} w, dv=du} = \tan^{-1} w \cdot w -$   
 $\int w \cdot \frac{1}{1+w^2} \, dw = w \tan^{-1} w - \frac{1}{2} \ln(1+w^2), 2 \int_1^{\sqrt{3}} \tan^{-1} u \, du = (2u \tan^{-1} u - \ln(1+u^2)) \Big|_1^{\sqrt{3}} = \frac{(4\sqrt{3}-3)\pi}{6} - \ln 2$

**Problem 82.**  $\int \sqrt{1-x^2} \sin^{-1} x \, dx$

**Solution.** First note that  $\underbrace{\int \sqrt{1-x^2} \, dx}_{\text{Let } u=\sin \theta} = \int \cos \theta \cdot \cos \theta \, d\theta = \frac{1}{2} \int (1+\cos 2\theta) \, d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} = \frac{1}{2} (\theta + \sin \theta \cos \theta) =$   
 $\frac{1}{2} (\sin^{-1} x + x \sqrt{1-x^2}),$  then  $\underbrace{\int \sqrt{1-x^2} \sin^{-1} x \, dx}_{\text{Let } u=\sin^{-1} x, dv=\sqrt{1-x^2} \, dx} = \sin^{-1} x \cdot \frac{1}{2} (\sin^{-1} x + x \sqrt{1-x^2}) - \frac{1}{2} \int (\sin^{-1} x + x \sqrt{1-x^2}) \cdot$

$$\frac{1}{\sqrt{1-x^2}} \, dx = \frac{1}{2} \sin^{-1} x (\sin^{-1} x + x \sqrt{1-x^2}) - \frac{1}{2} \int \left( \frac{\sin^{-1} x}{\sqrt{1-x^2}} + x \right) \, dx = \frac{1}{2} \sin^{-1} x (\sin^{-1} x + x \sqrt{1-x^2}) - \frac{x^2}{4} -$$

$$\frac{1}{2} \int \frac{\sin^{-1} x}{\sqrt{1-x^2}} \, dx = \frac{1}{2} \sin^{-1} x (\sin^{-1} x + x \sqrt{1-x^2}) - \frac{x^2}{4} - \frac{1}{4} (\sin^{-1} x)^2 = \frac{1}{4} (\sin^{-1} x)^2 + \frac{1}{2} x \sqrt{1-x^2} \sin^{-1} x - \frac{x^2}{4}$$

Let  $u = \sin^{-1} x$

**Problem 83.**  $\int_0^1 \frac{\sin^{-1} \sqrt{x}}{\sqrt{x(1-x)}} \, dx$

**Solution.** By  $(\sin^{-1}\sqrt{x})' = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(1-x)}}$ ,  $\int_0^1 \frac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}} dx = 2 \int_0^1 \sin^{-1}\sqrt{x} d(\sin^{-1}\sqrt{x})$   
 $= (\sin^{-1}\sqrt{x})^2 \Big|_0^1 = \frac{\pi^2}{4}$

**Problem 84.**  $\int e^{x+e^x} dx$

**Solution.**  $\int e^{x+e^x} dx = \underbrace{\int e^{e^x} \cdot e^x dx}_{\text{Let } u=e^x} = \int e^u du = e^u = e^{e^x}$

**Problem 85.**  $\int \frac{1}{e^{3x} - e^x} dx$

**Solution.**  $\int \frac{1}{e^{3x} - e^x} dx = \underbrace{\int \frac{e^{-3x}}{1 - e^{-2x}} dx}_{\text{Let } u=e^{-x}} = \int \frac{u^2}{u^2 - 1} du = \int \left(1 + \frac{1}{2} \left(\frac{1}{u-1} - \frac{1}{u+1}\right)\right) du = u + \frac{1}{2} \ln \left|\frac{u-1}{u+1}\right| = e^{-x} + \frac{1}{2} \ln \left|\frac{e^{-x}-1}{e^{-x}+1}\right|$

**Problem 86.**  $\int \frac{e^{2x}}{1+e^x} dx$

**Solution.**  $\underbrace{\int \frac{e^{2x}}{1+e^x} dx}_{\text{Let } u=e^x} = \int \frac{u}{1+u} du = \int \left(1 - \frac{1}{1+u}\right) du = u - \ln|1+u| = e^x - \ln(1+e^x)$

**Problem 87.**  $\int \frac{1}{1+2e^x - e^{-x}} dx$

**Solution.**  $\int \frac{1}{1+2e^x - e^{-x}} dx = \underbrace{\int \frac{e^x}{e^x + 2e^{2x} - 1} dx}_{\text{Let } u=e^x} = \int \frac{1}{u+2u^2-1} du = \int \left(\frac{-\frac{1}{3}}{u+1} + \frac{\frac{2}{3}}{2u-1}\right) du = \frac{1}{3} \ln \left|\frac{2u-1}{u+1}\right| = \frac{1}{3} \ln \left|\frac{2e^x-1}{e^x+1}\right|$

**Problem 88.**  $\int \frac{e^{2x}}{1+e^{4x}} dx$

**Solution.**  $\underbrace{\int \frac{e^{2x}}{1+e^{4x}} dx}_{\text{Let } u=e^{2x}} = \frac{1}{2} \int \frac{du}{1+u^2} = \frac{1}{2} \tan^{-1} u = \frac{1}{2} \tan^{-1} e^{2x}$

**Problem 89.**  $\int e^x \sqrt{1+e^x} dx$

**Solution.**  $\underbrace{\int e^x \sqrt{1+e^x} dx}_{\text{Let } u=1+e^x} = \int \sqrt{u} du = \frac{2}{3} u^{\frac{3}{2}} = \frac{2}{3} (1+e^x)^{\frac{3}{2}}$

**Problem 90.**  $\int \sqrt{1+e^x} dx$

**Solution.**  $\underbrace{\int \sqrt{1+e^x} dx}_{\text{Let } u=\sqrt{1+e^x}} = \int u \cdot \frac{2u}{u^2-1} du = \int \frac{2(u^2-1)+2}{u^2-1} du = 2u + \int \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du = 2u + \ln \left|\frac{u-1}{u+1}\right| = 2\sqrt{1+e^x} + \ln \left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right|$

**Problem 91.**  $\int \frac{2e^{2x} - e^x}{\sqrt{3e^{2x} - 6e^x - 1}} dx$

**Solution.**  $\int \frac{2e^{2x} - e^x}{\sqrt{3e^{2x} - 6e^x - 1}} dx = \int \frac{(2e^{2x} - 2e^x) + e^x}{\sqrt{3e^{2x} - 6e^x - 1}} dx = \frac{1}{3} \underbrace{\int \frac{6e^{2x} - 6e^x}{\sqrt{3e^{2x} - 6e^x - 1}} dx}_{\text{Let } u = 3e^{2x} - 6e^x - 1} + \underbrace{\int \frac{e^x}{\sqrt{3e^{2x} - 6e^x - 1}} dx}_{\text{Let } w = e^x} =$

$$\frac{1}{3} \int \frac{du}{\sqrt{u}} + \int \frac{dw}{\sqrt{3w^2 - 6w - 1}} = \frac{2}{3} \sqrt{3e^{2x} - 6e^x - 1} + \int \frac{dw}{\sqrt{3(w-1)^2 - 4}} = \frac{2}{3} \sqrt{3e^{2x} - 6e^x - 1}$$

$$+ \frac{1}{\sqrt{3}} \int \frac{dw}{\sqrt{(w-1)^2 - (\frac{2}{\sqrt{3}})^2}} = \frac{2}{3} \sqrt{3e^{2x} - 6e^x - 1} + \frac{1}{\sqrt{3}} \ln \left| \frac{e^x - 1}{\frac{2}{\sqrt{3}}} + \frac{\sqrt{(e^x - 1)^2 - (\frac{2}{\sqrt{3}})^2}}{\frac{2}{\sqrt{3}}} \right| = \frac{2}{3} \sqrt{3e^{2x} - 6e^x - 1} +$$

$$\frac{1}{\sqrt{3}} \ln \left| \frac{\sqrt{3}(e^x - 1)}{2} + \frac{\sqrt{3(e^x - 1)^2 - 4}}{2} \right| \text{ by } \int \frac{1}{\sqrt{x^2 - a^2}} dx = \int \frac{a \sec \theta \tan \theta}{a \tan \theta} d\theta = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| =$$

$$\ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right|$$

**Problem 92.**  $\int e^{\sqrt[3]{x}} dx$

**Solution.**  $\underbrace{\int e^{\sqrt[3]{x}} dx}_{\text{Let } u = \sqrt[3]{x}} = \int e^u \cdot 3u^2 du = (3u^2 - 6u + 6)e^u = (3(\sqrt[3]{x})^2 - 6\sqrt[3]{x} + 6)e^{\sqrt[3]{x}}$

**Problem 93.**  $\int \sqrt{x} e^{\sqrt{x}} dx$

**Solution.**  $\underbrace{\int \sqrt{x} e^{\sqrt{x}} dx}_{\text{Let } u = \sqrt{x}} = \int u \cdot e^u \cdot 2u du = \int 2u^2 e^u du = (2u^2 - 4u + 4)e^u = (2x - 4\sqrt{x} + 4)e^{\sqrt{x}}$

**Problem 94.**  $\int x^5 e^{-x^3} dx$

**Solution.**  $\underbrace{\int x^5 e^{-x^3} dx}_{\text{Let } u = x^3} = \frac{1}{3} \int u \cdot e^{-u} du = -\frac{u+1}{3} e^u = -\frac{x^3+1}{3} e^{-x^3}$

**Problem 95.**  $\int x^3 e^{-2x} dx$

**Solution.**  $\int x^3 e^{-2x} dx = -\left(\frac{x^3}{2} + \frac{3x^2}{4} + \frac{3x}{4} + \frac{3}{8}\right) e^{-2x}$

**Problem 96.**  $\int 27^{3x+1} dx$

**Solution.**  $\int 27^{3x+1} dx = \int e^{(3x+1)\ln 27} dx = 27 \int e^{3\ln 27 \cdot x} dx = \frac{27}{3 \ln 27} e^{3\ln 27 \cdot x} = \frac{27^{3x+1}}{3 \ln 27}$

**Problem 97.**  $\int_{-1}^1 \frac{e^{\tan^{-1} x}}{1+x^2} dx$

**Solution.**  $\underbrace{\int_{-1}^1 \frac{e^{\tan^{-1} x}}{1+x^2} dx}_{\text{Let } u = \tan^{-1} x} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^u du = e^{\frac{\pi}{4}} - e^{-\frac{\pi}{4}}$

**Problem 98.**  $\int e^x \sin(ax - 3) dx$

**Solution.**  $\int e^x \sin(ax - 3) dx = -\frac{1}{a} e^x \cos(ax - 3) + \frac{1}{a^2} e^x \sin(ax - 3) - \frac{1}{a^2} \int e^x \sin(ax - 3) dx$ 

$$\Rightarrow \int e^x \sin(ax - 3) dx = \frac{e^x (\sin(ax - 3) - a \cos(ax - 3))}{a^2 + 1}$$

**Problem 99.**  $\int \frac{x e^x}{\sqrt{1+e^x}} dx$

**Solution.**  $\int \underbrace{\frac{xe^x}{\sqrt{1+e^x}} dx} = x \cdot 2\sqrt{1+e^x} - 2 \int \sqrt{1+e^x} dx = 2x\sqrt{1+e^x} - 2 \int \sqrt{1+e^x} dx = 2x\sqrt{1+e^x} -$   
 Let  $u = x, dv = \frac{e^x}{\sqrt{1+e^x}} dx$   
 $4\sqrt{1+e^x} - 2 \ln \left| \frac{\sqrt{1+e^x} - 1}{\sqrt{1+e^x} + 1} \right|$

**Problem 100.**  $\int_0^\infty xe^{-x} \sin x dx$

**Solution.** From  $\int e^{ax} \sin bx dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$ ,  $\int e^{ax} \cos bx dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$ ,  $\int e^{-x} \sin x dx = -\frac{e^{-x}(\sin x + \cos x)}{2}$ ,  $\int e^{-x} \cos x dx = \frac{e^{-x}(\sin x - \cos x)}{2}$ , by tabulation  $\int xe^{-x} \sin x dx = x \cdot -\frac{e^{-x}(\sin x + \cos x)}{2} + \frac{1}{2} \left( -\frac{e^{-x}(\sin x + \cos x)}{2} + \frac{e^{-x}(\sin x - \cos x)}{2} \right) = -\frac{e^{-x}}{2}(x \sin x + (x+1) \cos x)$ , so  $\int_0^\infty xe^{-x} \sin x dx = \frac{1}{2}$ .

**Problem 101.**  $\int x^2 \ln(1+x) dx$

**Solution.**  $\int x^2 \ln(1+x) dx = \underbrace{\ln(x+1) \cdot \frac{x^3}{3}} - \frac{1}{3} \int x^3 \cdot \frac{1}{1+x} dx = \ln(x+1) \cdot \frac{x^3}{3} - \frac{1}{3} \int \left( x^2 - x + 1 - \frac{1}{x+1} \right) dx =$   
 Let  $u = \ln(x+1), dv = x^2 dx$   
 $\ln(x+1) \cdot \frac{x^3}{3} - \frac{x^3}{9} + \frac{x^2}{6} - \frac{x}{3} + \frac{\ln(x+1)}{3}$

**Problem 102.**  $\int \ln \sqrt{x-1} dx$

**Solution.**  $\int \underbrace{\ln \sqrt{x-1} dx} = \int u \cdot 2e^{2u} du = \left( u - \frac{1}{2} \right) e^{2u} = \left( \ln \sqrt{x-1} - \frac{1}{2} \right) (x-1)$   
 Let  $u = \ln \sqrt{x-1}$

**Problem 103.**  $\int \ln(\sqrt{x} + \sqrt{1+x}) dx$

**Solution.**  $\int \underbrace{\ln(\sqrt{x} + \sqrt{1+x}) dx} = \ln(\sqrt{x} + \sqrt{1+x}) \cdot x - \int x \cdot \frac{1}{2\sqrt{x(1+x)}} dx = x \ln(\sqrt{x} + \sqrt{1+x}) - \frac{1}{2} \int \underbrace{\sqrt{\frac{x}{1+x}} dx}_{\text{Let } u = \sqrt{1+x}}$   
 Let  $u = \ln(\sqrt{x} + \sqrt{1+x}), dv = dx$   
 $x \ln(\sqrt{x} + \sqrt{1+x}) - \frac{1}{2} \int \frac{\sqrt{u^2-1}}{u} \cdot 2u du = x \ln(\sqrt{x} + \sqrt{1+x}) - \underbrace{\int \sqrt{u^2-1} du}_{\text{Let } u = \sec \theta} = x \ln(\sqrt{x} + \sqrt{1+x}) - \frac{1}{2} (u \sqrt{u^2-1} - \ln |u + \sqrt{u^2-1}|) = \left( x + \frac{1}{2} \right) \ln(\sqrt{x} + \sqrt{1+x}) - \frac{1}{2} \sqrt{x(1+x)}$

**Problem 104.**  $\int \frac{\sqrt{1+\ln x}}{x \ln x} dx$

**Solution.**  $\int \underbrace{\frac{\sqrt{1+\ln x}}{x \ln x} dx}_{\text{Let } u = \ln x} = \int \underbrace{\frac{\sqrt{1+u}}{u} du}_{\text{Let } v = \sqrt{1+u}} = \int \frac{v}{v^2-1} 2v dv = \int \frac{2v^2}{v^2-1} dv = \int \left( \frac{2v^2-2}{v^2-1} + \frac{2}{v^2-1} \right) dv = 2v + \int \left( \frac{1}{v-1} - \frac{1}{v+1} \right) dv = 2v + \ln \left| \frac{v-1}{v+1} \right| = 2\sqrt{1+\ln x} + \ln \left| \frac{\sqrt{1+\ln x}-1}{\sqrt{1+\ln x}+1} \right|$

**Problem 105.**  $\int \frac{dx}{x \sqrt{\ln x}}$

**Solution.**  $\int \underbrace{\frac{dx}{x \sqrt{\ln x}}}_{\text{Let } u = \ln x} = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} = 2\sqrt{\ln x}$

**Problem 106.**  $\int \frac{\ln(x+1)}{x^2} dx$

**Solution.** 
$$\int \underbrace{\frac{\ln(x+1)}{x^2} dx}_{\text{Let } u = \ln(x+1), dv = \frac{1}{x^2} dx} = \ln(x+1) \cdot \frac{-1}{x} + \int \frac{1}{x} \cdot \frac{1}{x+1} dx = \frac{-\ln(x+1)}{x} + \int \left( \frac{1}{x} - \frac{1}{x+1} \right) dx$$

$$= \frac{-\ln(x+1)}{x} + \ln \frac{x}{x+1}$$

**Problem 107.**  $\int \frac{x \ln x}{\sqrt{x^2 - 1}} dx$

**Solution.** 
$$\int \underbrace{\frac{x \ln x}{\sqrt{x^2 - 1}} dx}_{\text{Let } u = \ln x, dv = \frac{x}{\sqrt{x^2 - 1}} dx} = \ln x \cdot \sqrt{x^2 - 1} - \int \sqrt{x^2 - 1} \cdot \frac{1}{x} dx = \sqrt{x^2 - 1} \ln x - \int \underbrace{\frac{\sqrt{x^2 - 1}}{x} dx}_{\text{Let } u = \sqrt{x^2 - 1}}$$

$$\int \frac{u^2}{u^2 + 1} du = \sqrt{x^2 - 1} \ln x - \int \left( 1 - \frac{1}{u^2 + 1} \right) du = \sqrt{x^2 - 1} \ln x - u + \tan^{-1} u = \sqrt{x^2 - 1} \ln x - \sqrt{x^2 - 1} + \tan^{-1} \sqrt{x^2 - 1}$$

**Problem 108.**  $\int \frac{dx}{x(1 + \ln x)\sqrt{(\ln x)(2 + \ln x)}}$

**Solution.** 
$$\int \underbrace{\frac{dx}{x(1 + \ln x)\sqrt{(\ln x)(2 + \ln x)}}}_{\text{Let } u = 1 + \ln x} = \int \underbrace{\frac{du}{u\sqrt{u^2 - 1}}}_{\text{Let } w = \ln u} = \int w dw = \frac{w^2}{2} = \frac{(\ln \sin x)^2}{2}$$

**Problem 109.**  $\int \cot x \ln(\sin x) dx$

**Solution.** 
$$\int \cot x \ln \sin x dx = \int \underbrace{\frac{\cos x}{\sin x} \ln \sin x dx}_{\text{Let } u = \sin x} = \int \underbrace{\frac{\ln u}{u} du}_{\text{Let } w = \ln u} = \int w dw = \frac{w^2}{2} = \frac{(\ln \sin x)^2}{2}$$

**Problem 110.**  $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\sin x \cos x} dx$

**Solution.** Note that  $(\ln(\tan x))' = \frac{1}{\sin x \cos x}$ , so  $\int \frac{\ln(\tan x)}{\sin x \cos x} dx = \int \ln(\tan x) d(\ln(\tan x)) = \frac{1}{2}(\ln(\tan x))^2$ ;  

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\sin x \cos x} dx = \frac{1}{2}(\ln(\tan x))^2 \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{(\ln 3)^2}{8}.$$

**Problem 111.**  $\int \frac{\cot x}{\ln \sin x} dx$

**Solution.** 
$$\int \frac{\cot x}{\ln \sin x} dx = \int \underbrace{\frac{\cos x}{\sin x \ln \sin x} dx}_{\text{Let } u = \sin x} = \int \underbrace{\frac{du}{u \ln u}}_{\text{Let } w = \ln u} = \int \frac{dw}{w} = \ln |w| = \ln |\ln(\sin x)|$$

**Problem 112.**  $\int (1 + \ln x) \sqrt{1 + (x \ln x)^2} dx$

**Solution.** 
$$\int \underbrace{(1 + \ln x) \sqrt{1 + (x \ln x)^2} dx}_{\text{Let } u = x \ln x} = \int \sqrt{1 + u^2} du = \frac{u\sqrt{u^2 + 1} + \ln|\sqrt{u^2 + 1} + u|}{2}$$

$$= \frac{x \ln x \sqrt{(x \ln x)^2 + 1} + \ln|\sqrt{(x \ln x)^2 + 1} + x \ln x|}{2}$$