

# 第四章 積分

## 4.1 不定積分

**定義** (反導函數). 紿定  $F(x)$ , 若  $\frac{d}{dx}F(x) = f(x)$ , 則稱  $F(x)$  為  $f(x)$  的反導函數 (antiderivative) .

**性質.** 若  $F(x), G(x)$  分別為  $f(x), g(x)$  的反導函數,  $c \in \mathbb{R}$ . 則

- $F(x) + c$  為  $f(x)$  的反導函數.
- $c F(x)$  為  $c f(x)$  的反導函數.
- $F(x) + G(x)$  為  $f(x) + g(x)$  的反導函數.

**結論.**

- $\frac{d}{dx}F(x) = f(x) \implies dF(x) = f(x) \cdot dx \implies F(x) = \int f(x) \cdot dx = \int f(x) dx$
- $F(x)$  為  $f(x)$  的反導函數  $\iff f(x)$  的反導函數為  $F(x) \iff F(x)$  的導函數為  $f(x) \iff F(x)$  (對  $x$ ) 的微分為  $f(x) \iff f(x)$  (對  $x$ ) 的 (不定) 積分為  $F(x)$
- $f(x)$  的反導函數  $\equiv f(x)$  (對  $x$ ) 的 (不定) 積分
- 基礎積分集: 以下  $\alpha \neq -1, a \neq 0$ .

$f(x)$	$x^\alpha$	$\frac{1}{x}$	$e^{ax}$	$\sin ax$	$\cos ax$	$\frac{1}{\sqrt{a^2 - x^2}}$	$\frac{1}{a^2 + x^2}$
$\int f(x) dx$	$\frac{1}{\alpha+1} x^{\alpha+1}$	$\ln x $	$\frac{1}{a} e^{ax}$	$-\frac{1}{a} \cos ax$	$\frac{1}{a} \sin ax$	$\sin^{-1} \frac{x}{ a }$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$

- (Liouville)  $e^{-x^2}, \frac{e^x}{x}, \frac{1}{\ln x}, \sin(x^2), \cos(x^2), \frac{\sin x}{x}, \frac{\cos x}{x}, x^x$  無 (初等函數形式之) 反導函數!

- 例.**
- |   |  |   |
|---|--|---|
| 1. $\int \sqrt{x} dx = \frac{2}{3} x^{\frac{3}{2}} + c$ | 4. $\int \cos xy dx = \frac{1}{y} \sin xy + c$                             | 7. $\int \frac{1}{e+u^2} du = \frac{1}{\sqrt{e}} \tan^{-1} \frac{u}{\sqrt{e}} + c$            |
| 2. $\int x^\pi dx = \frac{1}{\pi+1} x^{\pi+1} + c$      | 5. $\int e^{-2x} dx = -\frac{1}{2} e^{-2x} + c$                            | 8. $\int \left( \frac{\pi}{x} - e^{\pi x} \right) du = \pi \ln x - \frac{e^{\pi x}}{\pi} + c$ |
| 3. $\int \sin \pi x dx = -\frac{1}{\pi} \cos \pi x + c$ | 6. $\int \frac{1}{\sqrt{\pi-x^2}} dx = \sin^{-1} \frac{x}{\sqrt{\pi}} + c$ | 9. $\int \frac{3+x^2}{1+x^2} du = x + 2 \tan^{-1} x + c$                                      |

**習題.** 求下列不定積分.

- |  |   |
|--|---|
| 1. $\int \frac{x^3 - 1}{x^3} dx = x + \frac{1}{2x^3} + c$                          | 5. $\int x \sqrt{3x} dx = \frac{2\sqrt{3}}{5} x^{\frac{5}{2}} + c$                |
| 2. $\int 5 - \frac{1}{\sqrt{x}} dx = 5x - 2\sqrt{x} + c$                           | 6. $\int \frac{1}{x^3} - \frac{1}{x^5} dx = \frac{-1}{2x^2} + \frac{1}{4x^4} + c$ |
| 3. $\int (t-1)(t+1) dt = \frac{t^3}{3} - t + c$                                    | 7. $\int \sec^2 x - \sec x \tan x dx = \tan x - \sec x + c$                       |
| 4. $\int (\sqrt{x} + 1)^2 dx = \frac{x^2}{2} + x + \frac{4x^{\frac{3}{2}}}{3} + c$ | 8. $\int \frac{e^{3x} + 1}{e^x + 1} dx = \frac{e^{2x}}{2} - e^x + x + c$          |

## 變數變換法

**結論.**  $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x) \implies df(g(x)) = f'(g(x)) \cdot g'(x) dx \implies f(g(x)) = \int f'(g(x)) \cdot g'(x) dx.$   
 令  $u = g(x)$ , 則  $\frac{du}{dx} = g'(x) \implies du = g'(x) dx$ ; 故  $\int f'(g(x)) \cdot g'(x) dx = \int f'(u) du = f(u) + c = f(g(x)) + c.$

**例.** 求  $\int \frac{x}{\sqrt{x+1}} dx.$

**解.**

- (解一) 令  $u = x + 1$ , 則  $x = u - 1$ ,  $du = dx$ . 故  $\int \frac{x}{\sqrt{x+1}} dx = \int \frac{u-1}{\sqrt{u}} du = \int u^{\frac{1}{2}} - u^{-\frac{1}{2}} du = \frac{2}{3}u^{\frac{3}{2}} - 2\sqrt{u} + c = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c.$
- (解二)  $\int \frac{x}{\sqrt{x+1}} dx = \int \frac{x+1-1}{\sqrt{x+1}} dx = \int \sqrt{x+1} - \frac{1}{\sqrt{x+1}} dx$ . 令  $u = x + 1$ , 則  $du = dx$ . 故  $\int \sqrt{x+1} - \frac{1}{\sqrt{x+1}} dx = \int u^{\frac{1}{2}} - u^{-\frac{1}{2}} du = \frac{2}{3}u^{\frac{3}{2}} - 2\sqrt{u} + c = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c.$
- (解三) 令  $u = \sqrt{x+1}$ , 則  $x = u^2 - 1$ ,  $du = \frac{1}{2\sqrt{x+1}} dx \implies \frac{1}{\sqrt{x+1}} dx = 2du$ . 故  $\int \frac{x}{\sqrt{x+1}} dx = \int x \cdot \frac{1}{\sqrt{x+1}} dx = \int (u^2 - 1) \cdot 2 du = \frac{2}{3}u^3 - 2u + c = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c.$

**例.** 求  $\int \frac{x}{x^2+1} dx.$

**解.** 令  $u = x^2 + 1$ , 則  $du = 2x dx \implies x dx = \frac{1}{2} du$ . 故  $\int \frac{x}{x^2+1} dx = \int \frac{1}{x^2+1} \cdot x dx = \int \frac{1}{u} \cdot \frac{1}{2} du = \frac{1}{2} \ln|u| + c = \frac{1}{2} \ln(x^2 + 1) + c.$

**例.** 求  $\int \frac{\sin(3 \ln x)}{x} dx.$

**解.** 令  $u = \ln x$ , 則  $du = \frac{1}{x} dx$ . 故  $\int \frac{\sin(3 \ln x)}{x} dx = \int \sin(3 \ln x) \cdot \frac{1}{x} dx = \int \sin 3u \cdot du = -\frac{1}{3} \cos 3u + c = -\frac{1}{3} \cos(3 \ln x) + c.$

**例.** 求  $\int e^x \sqrt{1+e^x} dx.$

**解.** 令  $u = 1+e^x$ , 則  $du = e^x dx$ . 故  $\int e^x \sqrt{1+e^x} dx = \int \sqrt{1+e^x} \cdot e^x dx = \int \sqrt{u} \cdot du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}(1+e^x)^{\frac{3}{2}} + c$

**例.** 求  $\int \frac{e^x}{\sqrt{2-e^{2x}}} dx.$

**解.** 令  $u = e^x$ , 則  $du = e^x dx$ . 故  $\int \frac{e^x}{\sqrt{2-e^{2x}}} dx = \int \frac{1}{\sqrt{2-u^2}} du = \sin^{-1} \frac{u}{\sqrt{2}} + c = \sin^{-1} \frac{e^x}{\sqrt{2}} + c.$

**例.** 求  $\int \frac{1}{\sqrt{e^{2x}-1}} dx.$

解.  $\int \frac{1}{\sqrt{e^{2x}-1}} dx = \int \frac{1}{e^x \sqrt{1-e^{-2x}}} dx = \int \frac{e^{-x}}{\sqrt{1-e^{-2x}}} dx$ . 令  $u = e^{-x}$ , 則  $du = -e^{-x} dx \Rightarrow e^{-x} dx = -du$ ;  
故  $\int \frac{e^{-x}}{\sqrt{1-e^{-2x}}} dx = \int \frac{1}{\sqrt{1-e^{-2x}}} \cdot e^{-x} dx = - \int \frac{1}{\sqrt{1-u^2}} du = -\sin^{-1} u + c = -\sin^{-1} e^{-x} + c$ .

例. 求  $\int \frac{1}{x^2+4x+5} dx$ .

解.  $\int \frac{1}{x^2+4x+5} dx = \int \frac{1}{(x+2)^2+1} dx$ . 令  $u = x+2$ , 則  $du = dx$ ; 故  $\int \frac{1}{(x+2)^2+1} dx = \int \frac{1}{u^2+1} du = \tan^{-1} u + c = \tan^{-1}(x+2) + c$ .

例. 求  $\int \frac{1}{\sqrt{4+2x-x^2}} dx$

解.  $\int \frac{1}{\sqrt{4+2x-x^2}} dx = \int \frac{1}{\sqrt{5-(x-1)^2}} dx$ . 令  $u = x-1$ , 則  $du = dx$ ; 故  $\int \frac{1}{\sqrt{5-(x-1)^2}} dx = \int \frac{1}{\sqrt{5-u^2}} du = \sin^{-1} \frac{u}{\sqrt{5}} + c = \sin^{-1} \frac{x-1}{\sqrt{5}} + c$ .

例. 求  $\int \tan x dx$ .

解.  $\int \tan x dx = \int \frac{\sin x}{\cos x} dx$ . 令  $u = \cos x$ , 則  $du = -\sin x dx$ ; 故  $\int \frac{\sin x}{\cos x} dx = \int \frac{-1}{u} du = -\ln|u| + c = -\ln|\cos x| + c = \ln|\sec x| + c$

例. 求  $\int \cos^5 ax dx$ ,  $a \neq 0$ .

解.  $\int \cos^5 ax dx = \int (1-\sin^2 ax)^2 \cos ax dx$ . 令  $u = \sin ax$ , 則  $du = a \cos ax dx$ ; 故  $\int (1-\sin^2 ax)^2 \cos ax dx = \frac{1}{a} \int (1-u^2)^2 du = \frac{1}{a} \int (1-2u^2+u^4) du = \frac{1}{a} \left( u - \frac{2}{3}u^3 + \frac{1}{5}u^5 \right) + c = \frac{1}{a} \left( \sin ax - \frac{2}{3}\sin^3 ax + \frac{1}{5}\sin^5 ax \right) + c$ .

例. 求  $\int \cos^4 ax dx$ ,  $a \neq 0$ .

解.  $\int \cos^4 ax dx = \int \left( \frac{1+\cos 2ax}{2} \right)^2 dx = \frac{1}{4} \int (1+2\cos 2ax+\cos^2 2ax) dx = \frac{1}{4} \int \left( 1+2\cos 2ax+\frac{1+\cos 4ax}{2} \right) dx = \int \left( \frac{3}{8} + \frac{1}{2}\cos 2ax + \frac{1}{8}\cos 4ax \right) dx = \frac{3}{8}x + \frac{1}{4a}\sin 2ax + \frac{1}{32a}\sin 4ax + c$

例. 求  $\int \sec x dx$ .

解.  $\int \sec x dx = \int \frac{\sec x \cdot (\sec x + \tan x)}{\sec x + \tan x} dx$ . 令  $u = \sec x + \tan x$ , 則  $du = (\sec^2 x + \sec x \tan x) dx$ ; 故  $\int \frac{\sec x \cdot (\sec x + \tan x)}{\sec x + \tan x} dx = \int \frac{1}{u} du = \ln|u| + c = \ln|\sec x + \tan x| + c$

例. 令  $T_1 = \int \frac{\sin x}{a \cos x + b \sin x} dx$ ,  $T_2 = \int \frac{\cos x}{a \cos x + b \sin x} dx$ ,  $a, b \neq 0$ , 求  $T_1, T_2$ .

解.

$$(a) bT_1 + aT_2 = \int \frac{b \sin x}{a \cos x + b \sin x} dx + \int \frac{a \cos x}{a \cos x + b \sin x} dx = \int \frac{b \sin x + a \cos x}{a \cos x + b \sin x} dx = \int 1 dx = x.$$

$$(b) -aT_1 + bT_2 = \int \frac{-a \sin x}{a \cos x + b \sin x} dx + \int \frac{b \cos x}{a \cos x + b \sin x} dx = \int \frac{-a \sin x + b \cos x}{a \cos x + b \sin x} dx = \int \frac{du}{u} = \ln u = \ln |a \cos x + b \sin x| \quad (\text{令 } u = a \cos x + b \sin x, \text{ 則 } du = (-a \sin x + b \cos x) dx).$$

$$\text{解 } T_1, T_2 \text{ 方程式 (a), (b) 得 } T_1 = \frac{bx - a \ln |a \cos x + b \sin x|}{a^2 + b^2}, T_2 = \frac{ax + b \ln |a \cos x + b \sin x|}{a^2 + b^2}.$$

**習題.** 以變數變換法求下列不定積分. 注意: 可能會因為常數項而跟此處答案不同.

1. $\int \frac{1}{\sqrt{2x-1}} dx$	4. $\int e^{\pi x-1} dx$	7. $\int \frac{\cos 3x}{\sin^2 3x} dx$
2. $\int \sqrt{7x+4} dx$	5. $\int (x^2 - 2x + 1)^{\frac{1}{3}} dx$	8. $\int \frac{x}{\sqrt{1+2x^2}} dx$
3. $\int \sin(3x-1) dx$	6. $\int \sin 3x \cos 3x dx$	9. $\int x^2 \sqrt{1-x} dx$

**解.**

1. 令  $u = 2x-1$ , 則  $du = 2 dx \Rightarrow dx = \frac{1}{2} du$ . 故  $\int \frac{1}{\sqrt{2x-1}} dx = \int \frac{1}{\sqrt{u}} \cdot \frac{1}{2} du = \sqrt{u} + c = \sqrt{2x-1} + c$ .
2. 令  $u = 7x+4$ , 則  $du = 7 dx \Rightarrow dx = \frac{1}{7} du$ . 故  $\int \sqrt{7x+4} dx = \int \sqrt{u} \cdot \frac{1}{7} du = \frac{1}{7} \cdot \frac{2}{3} u^{\frac{3}{2}} + c = \frac{2}{21} (7x+4)^{\frac{3}{2}} + c$ .
3. 令  $u = 3x-1$ , 則  $du = 3 dx \Rightarrow dx = \frac{1}{3} du$ . 故  $\int \sin(3x-1) dx = \int \sin u \cdot \frac{1}{3} du = -\frac{1}{3} \cos u + c = -\frac{\cos(3x-1)}{3} + c$ .
4. 令  $u = \pi x - 1$ , 則  $du = \pi dx \Rightarrow dx = \frac{1}{\pi} du$ . 故  $\int e^{\pi x-1} dx = \int e^u \cdot \frac{1}{\pi} du = \frac{1}{\pi} e^u + c = \frac{e^{\pi x-1}}{\pi} + c$ .
5. 令  $u = x-1$ , 則  $du = dx$ . 故  $\int (x^2 - 2x + 1)^{\frac{1}{3}} dx = \int (x-1)^{\frac{2}{3}} dx = \int u^{\frac{2}{3}} du = \frac{3}{5} u^{\frac{5}{3}} + c = \frac{3}{5} (x-1)^{\frac{5}{3}} + c$ .
6. 令  $u = \sin 3x$ , 則  $du = 3 \cos 3x dx \Rightarrow \cos 3x dx = \frac{1}{3} du$ . 故  $\int \sin 3x \cos 3x dx = \int u \cdot \frac{1}{3} du = \frac{1}{3} \cdot \frac{1}{2} u^2 + c = \frac{\sin^2 3x}{6} + c$ .
7. 令  $u = \sin 3x$ , 則  $du = 3 \cos 3x dx \Rightarrow \cos 3x dx = \frac{1}{3} du$ . 故  $\int \frac{\cos 3x}{\sin^2 3x} dx = \int \frac{1}{u^2} \cdot \frac{1}{3} du = -\frac{1}{3} \cdot \frac{1}{u} + c = -\frac{1}{3 \sin 3x} + c$ .
8. 令  $u = 1 + 2x^2$ , 則  $du = 4x dx \Rightarrow x dx = \frac{1}{4} du$ . 故  $\int \frac{x}{\sqrt{1+2x^2}} dx = \int \frac{1}{\sqrt{u}} \cdot \frac{1}{4} du = \int \frac{1}{\sqrt{u}} \cdot \frac{1}{4} du = \frac{1}{4} \cdot 2 u^{\frac{1}{2}} + c = \frac{\sqrt{1+2x^2}}{2} + c$ .
9. 令  $u = \sqrt{1-x}$ , 則  $u^2 = 1-x \Rightarrow x = 1-u^2$ ,  $dx = -2u du$ . 故  $\int x^2 \sqrt{1-x} dx = \int (1-u^2)^2 \cdot u \cdot (-2u) du = -2 \int (1-u^2)^2 \cdot u^2 du = -2 \int (u^2 - 2u^4 + u^6) du = -\frac{2u^3}{3} + \frac{4u^5}{5} - \frac{2u^7}{7} + c = -\frac{2(1-x)^{\frac{3}{2}}}{3} + \frac{4(1-x)^{\frac{5}{2}}}{5} - \frac{2(1-x)^{\frac{7}{2}}}{7}$ .

**習題.** 以變數變換法求下列不定積分, 其中  $a \neq b \neq 0$ . 注意: 可能會因為常數項而跟此處答案不同.

1. $\int e^{2x} \sin e^{2x} dx$	7. $\int \frac{x^2}{2+x^6} dx$	13. $\int \sin^{-\frac{2}{3}} x \cos^3 x dx$
2. $\int xe^{-\frac{x^2}{2}} dx$	8. $\int \frac{1}{e^x + e^{-x}} dx$	14. $\int \frac{\sin^3(\ln x) \cos^3(\ln x)}{x} dx$
3. $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$	9. $\int \frac{x+1}{\sqrt{1-x^2}} dx$	15. $\int \frac{\sin^3 x}{\cos^4 x} dx$
4. $\int x^2 2^{x^3+1} dx$	10. $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}} dx$	16. $\int \cos ax \cos bx dx$
5. $\int \frac{\ln x}{x} dx$	11. $\int \sin^4 x \cos^5 x dx$	17. $\int \sin ax \sin bx dx$
6. $\int \frac{x+1}{\sqrt{x^2+2x+3}} dx$	12. $\int \sin^2 x \cos^2 x dx$	18. $\int \sin ax \cos bx dx$

解.

1. 令  $u = e^{2x}$ , 則  $du = 2e^{2x} dx \Rightarrow e^{2x} dx = \frac{1}{2} du$ . 故  $\int e^{2x} \sin e^{2x} dx = \int \sin e^{2x} \cdot e^{2x} dx = \int \sin u \cdot \frac{1}{2} du = -\frac{1}{2} \cos u + c = -\frac{1}{2} \cos e^{2x} + c$ .
2. 令  $u = \frac{x^2}{2}$ , 則  $du = x dx$ , 故  $\int xe^{-\frac{x^2}{2}} dx = \int e^{-\frac{x^2}{2}} \cdot x dx = \int e^{-u} du = -e^{-u} + c = -e^{-\frac{x^2}{2}} + c$
3. 令  $u = \sqrt{x}$ , 則  $du = \frac{1}{2\sqrt{x}} dx \Rightarrow \frac{1}{\sqrt{x}} dx = 2 du$ , 故  $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx = \int \sin \sqrt{x} \cdot \frac{1}{\sqrt{x}} dx = \int \sin u \cdot 2 du = -2 \cos u + c = -2 \cos \sqrt{x} + c$
4. 令  $u = x^3 + 1$ , 則  $du = 3x^2 dx \Rightarrow x^2 dx = \frac{1}{3} du$ , 故  $\int x^2 2^{x^3+1} dx = \int 2^{x^3+1} \cdot x^2 dx = \int 2^u \cdot \frac{1}{3} du = \frac{1}{3} \int e^{u \ln 2} du = \frac{1}{3 \ln 2} e^{u \ln 2} + c = \frac{2^{x^3+1}}{3 \ln 2} + c$
5. 令  $u = \ln x$ , 則  $du = \frac{1}{x} dx$ , 故  $\int \frac{\ln x}{x} dx = \int \ln x \cdot \frac{1}{x} dx = \int u du = \frac{1}{2} u^2 + c = \frac{1}{2} (\ln x)^2 + c$
6. 令  $u = x^2 + 2x + 3$ , 則  $du = (2x+2) dx \Rightarrow (x+1) dx = \frac{1}{2} du$ , 故  $\int \frac{x+1}{\sqrt{x^2+2x+3}} dx = \int \frac{1}{\sqrt{x^2+2x+3}} \cdot (x+1) dx = \int \frac{1}{\sqrt{u}} \cdot \frac{1}{2} du = \sqrt{u} + c = \sqrt{x^2+2x+3} + c$
7. 令  $u = x^3$ , 則  $du = 3x^2 dx \Rightarrow x^2 dx = \frac{1}{3} du$ , 故  $\int \frac{x^2}{2+x^6} dx = \int \frac{1}{2+x^6} \cdot x^2 dx = \int \frac{1}{2+u^2} \cdot \frac{1}{3} du = \frac{1}{3\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} + c = \frac{1}{3\sqrt{2}} \tan^{-1} \frac{x^3}{\sqrt{2}} + c$
8.  $\int \frac{1}{e^x + e^{-x}} dx = \int \frac{e^x}{e^{2x} + 1} dx$ . 令  $u = e^x$ , 則  $du = e^x dx$ , 故  $\int \frac{e^x}{e^{2x} + 1} dx = \int \frac{1}{e^{2x} + 1} \cdot e^x dx = \int \frac{1}{u^2 + 1} du = \tan^{-1} u + c = \tan^{-1} e^x + c$ .
9.  $\int \frac{x+1}{\sqrt{1-x^2}} dx = \int \frac{x}{\sqrt{1-x^2}} dx + \int \frac{1}{\sqrt{1-x^2}} dx$ . 令  $u = \sqrt{1-x^2}$ , 則  $du = \frac{-x}{\sqrt{1-x^2}} dx \Rightarrow \frac{x}{\sqrt{1-x^2}} dx = -du$ , 故  $\int \frac{x}{\sqrt{1-x^2}} dx = \int -du = -u + c = -\sqrt{1-x^2}$ ,  $\int \frac{1}{\sqrt{1-x^2}} dx = \int \frac{x+1}{\sqrt{1-x^2}} dx + \int \frac{1}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} + \sin^{-1} x + c$ .

10. 因  $u = \tan x$ , 則  $du = \sec^2 x dx$ , 故  $\int \frac{\sec^2 x}{\sqrt{1 - \tan^2 x}} dx = \int \frac{1}{\sqrt{1 - \tan^2 x}} \cdot \sec^2 x dx = \int \frac{1}{\sqrt{1 - u^2}} du = \sin^{-1} u + c = \sin^{-1}(\tan x) + c$
11.  $\int \sin^4 x \cos^5 x dx = \int \sin^4 x \cdot \cos^4 x \cdot \cos x dx = \int \sin^4 x \cdot (1 - \sin^2 x)^2 \cdot \cos x dx = \int (\sin^4 x - 2\sin^6 x + \sin^8 x) \cdot \cos x dx$ . 因  $u = \sin x$ , 則  $du = \cos x dx$ , 故  $\int (\sin^4 x - 2\sin^6 x + \sin^8 x) \cdot \cos x dx = \int (u^4 - 2u^6 + u^8) du = \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + c = \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + c$
12.  $\int \sin^2 x \cos^2 x dx = \int \frac{1 - \cos 2x}{2} \cdot \frac{1 + \cos 2x}{2} dx = \frac{1}{4} \int (1 - \cos^2 2x) dx = \frac{1}{4} \int \left(1 - \frac{1 + \cos 4x}{2}\right) dx = \frac{1}{4} \int \left(\frac{1}{2} + \frac{\cos 4x}{2}\right) dx = \frac{x}{8} - \frac{\sin 4x}{32} + c$
13.  $\int \sin^{-\frac{2}{3}} x \cos^3 x dx = \int \sin^{-\frac{2}{3}} x \cdot \cos^2 x \cdot \cos x dx = \int \sin^{-\frac{2}{3}} x \cdot (1 - \sin^2 x) \cdot \cos x dx = \int (\sin^{-\frac{2}{3}} x - \sin^{\frac{4}{3}} x) \cdot \cos x dx$ . 因  $u = \sin x$ , 則  $du = \cos x dx$ , 故  $\int (\sin^{-\frac{2}{3}} x - \sin^{\frac{4}{3}} x) \cdot \cos x dx = \int (u^{-\frac{2}{3}} - u^{\frac{4}{3}}) du = 3u^{\frac{1}{3}} - \frac{3}{7}u^{\frac{7}{3}} + c = 3\sin^{\frac{1}{3}} x - \frac{3}{7}\sin^{\frac{7}{3}} x + c$
14. 因  $u = \ln x$ , 則  $du = \frac{1}{x} dx$ , 故  $\int \frac{\sin^3(\ln x) \cos^3(\ln x)}{x} dx = \int \sin^3(\ln x) \cos^3(\ln x) \cdot \frac{1}{x} dx = \int \sin^3 u \cos^3 u du = \int \sin^3 u \cdot \cos^2 u \cdot \cos u du = \int \sin^3 u (1 - \sin^2 u) \cdot \cos u du = \int (\sin^3 u - \sin^5 u) \cdot \cos u du$ . 因  $w = \sin u$ , 則  $dw = \cos u du$ , 故  $\int (\sin^3 u - \sin^5 u) \cdot \cos u du = \int (w^3 - w^5) dw = \frac{w^4}{4} - \frac{w^6}{6} + c = \frac{\sin^4(\ln x)}{4} - \frac{\sin^6(\ln x)}{6} + c$
15.  $\int \frac{\sin^3 x}{\cos^4 x} dx = \int \frac{\sin^2 x \cdot \sin x}{\cos^4 x} dx = \int \frac{\sin^2 x}{\cos^4 x} \cdot \sin x dx = \int \frac{1 - \cos^2 x}{\cos^4 x} \cdot \sin x dx = \int \left(\frac{1}{\cos^4 x} - \frac{1}{\cos^2 x}\right) \cdot \sin x dx$ . 因  $u = \cos x$ , 則  $du = -\sin x dx \Rightarrow \sin x dx = -du$ , 故  $\int \left(\frac{1}{\cos^4 x} - \frac{1}{\cos^2 x}\right) \cdot \sin x dx = \int \left(\frac{1}{u^4} - \frac{1}{u^2}\right)(-) du = \int \left(\frac{1}{u^2} - \frac{1}{u^4}\right) du = -\frac{1}{u} + \frac{1}{3u^3} + c = -\frac{1}{\cos x} + \frac{1}{3\cos^3 x} + c$
16. 由積化和差公式  $\cos \alpha \cos \beta = \frac{1}{2} (\cos(\alpha - \beta) + \cos(\alpha + \beta))$ ,  $\int \cos ax \cos bx dx = \int \frac{1}{2} (\cos(ax - bx) + \cos(ax + bx)) dx = \int \frac{1}{2} (\cos(a - b)x + \cos(a + b)x) dx = \frac{\sin(a - b)x}{2(a - b)} + \frac{\sin(a + b)x}{2(a + b)} + c$ .
17. 由積化和差公式  $\sin \alpha \sin \beta = \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta))$ ,  $\int \sin ax \sin bx dx = \int \frac{1}{2} (\cos(ax - bx) - \cos(ax + bx)) dx = \int \frac{1}{2} (\cos(a - b)x - \cos(a + b)x) dx = \frac{\sin(a - b)x}{2(a - b)} - \frac{\sin(a + b)x}{2(a + b)} + c$ .
18. 由積化和差公式  $\sin \alpha \cos \beta = \frac{1}{2} (\sin(\alpha - \beta) + \sin(\alpha + \beta))$ ,  $\int \sin ax \cos bx dx = \int \frac{1}{2} (\sin(ax - bx) + \sin(ax + bx)) dx = \int \frac{1}{2} (\sin(a - b)x + \sin(a + b)x) dx = -\frac{\cos(a - b)x}{2(a - b)} - \frac{\cos(a + b)x}{2(a + b)} + c$ .

## 部份積分法

結論.  $\frac{d}{dx}(u(x)v(x)) = u(x)\frac{dv(x)}{dx} + v(x)\frac{du(x)}{dx} \Rightarrow u(x)v(x) = \int u(x)dv(x) + \int v(x)du(x)$   
 $\Rightarrow \int u dv = uv - \int v du$ .

例. 求  $\int xe^x dx$ .

解. 令  $u = x$ , 則  $du = dx$ . 令  $dv = e^x dx$ , 則  $v = e^x$ . 故  $\int xe^x dx = x \cdot e^x - \int e^x dx = xe^x - e^x + c$ .

例. 求  $\int \ln x dx$ .

解. 令  $u = \ln x$ , 則  $du = \frac{1}{x} dx$ . 令  $dv = dx$ , 則  $v = x$ . 故  $\int \ln x dx = \ln x \cdot x - \int x \cdot \frac{1}{x} dx = x \ln x - x + c$ .

例. 求  $\int x \tan^{-1} x dx$ .

解. 令  $u = \tan^{-1} x$ , 則  $du = \frac{1}{1+x^2} dx$ . 令  $dv = x dx$ , 則  $v = \frac{x^2}{2}$ . 故  $\int x \tan^{-1} x dx = \tan^{-1} x \cdot \frac{x^2}{2} - \frac{1}{2} \int \frac{x^2}{1+x^2} dx = \tan^{-1} x \cdot \frac{x^2}{2} - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx = \tan^{-1} x \cdot \frac{x^2}{2} - \frac{1}{2}(x - \tan^{-1} x) + c$ .

例. 求  $\int x^2 \sin x dx$ .

解. 令  $u = x^2$ , 則  $du = 2x dx$ ; 令  $dv = \sin x dx$ , 則  $v = -\cos x$ . 故  $\int x^2 \sin x dx = x^2 \cdot (-\cos x) + 2 \int x \cos x dx$ .

令  $u = x$ , 則  $du = dx$ ; 令  $dv = \cos x dx$ , 則  $v = \sin x$ . 故  $\int x \cos x dx = x \cdot \sin x - \int \sin x dx = x \cdot \sin x + \cos x$ .

由上,  $\int x^2 \sin x dx = -x^2 \cos x + 2(x \sin x + \cos x) + c$ .

例. 求  $\int \sin^{-1} \sqrt{1-x^2} dx$ ,  $x > 0$ .

解. 令  $u = \sin^{-1} \sqrt{1-x^2}$ , 則  $du = \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \frac{-x}{\sqrt{1-x^2}} dx = \frac{1}{\sqrt{x^2}} \cdot \frac{-x}{\sqrt{1-x^2}} dx = \frac{-1}{\sqrt{1-x^2}} dx$ ; 令  $dv = dx$ , 則  $v = x$ . 故  $\int \sin^{-1} \sqrt{1-x^2} dx = x \sin^{-1} \sqrt{1-x^2} + \int \frac{x}{\sqrt{1-x^2}} dx = x \sin^{-1} \sqrt{1-x^2} - \sqrt{1-x^2}$ .

例. 求  $\int e^{ax} \cos bx dx$  與  $\int e^{ax} \sin bx dx$ ,  $a, b \neq 0$ .

解.

(a) 考慮  $\int e^{ax} \cos bx dx$ : 令  $u = \cos bx$ , 則  $du = -b \sin bx dx$ ; 令  $dv = e^{ax} dx$ , 則  $v = \frac{1}{a} e^{ax}$ , 故  $\int e^{ax} \cos bx dx = \cos bx \cdot \frac{1}{a} e^{ax} + \frac{b}{a} \int e^{ax} \cdot \sin bx dx$ .

(b) 考慮  $\int e^{ax} \sin bx dx$ : 令  $u = \sin bx$ , 則  $du = b \cos bx dx$ ; 令  $dv = e^{ax} dx$ , 則  $v = \frac{1}{a} e^{ax}$ , 故  $\int e^{ax} \sin bx dx = \sin bx \cdot \frac{1}{a} e^{ax} - \frac{b}{a} \int e^{ax} \cdot \cos bx dx$ .

令  $X = \int e^{ax} \cos bx dx$ ,  $Y = \int e^{ax} \sin bx dx$ , 由 (a)(b)  $X = \cos bx \cdot \frac{1}{a} e^{ax} + \frac{b}{a} Y$ ,  $Y = \sin bx \cdot \frac{1}{a} e^{ax} - \frac{b}{a} X$ . 解  $X, Y$  得  $X = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$ ,  $Y = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$ .

## 部份積分法：列表

### 結論.

- 將要微分函數寫左邊，積分函數寫右邊；左邊連續微分，右邊連續積分
- 依序左上連右下斜線函數相乘，最底部水平兩邊函數相乘並積分，符號正負相間
- 將上式所得項全部加總即為所求積分

例. 求  $\int (x+3) e^{2x} dx$ .

Diff	Int
$x+3$	$e^{2x}$
1	$\frac{1}{2}e^{2x}$
0	$\frac{1}{4}e^{2x}$

$$\int (x+3) e^{2x} dx = (x+3) \frac{1}{2}e^{2x} - \frac{1}{4}e^{2x}$$

例. 求  $\int (x^2 - 2x) e^{kx} dx$ .

Diff	Int
$x^2 - 2x$	$e^{kx}$
2(x-1)	$\frac{1}{k}e^{kx}$
2	$\frac{1}{k^2}e^{kx}$
0	$\frac{1}{k^3}e^{kx}$

$$\int (x^2 - 2x) e^{kx} dx = (x^2 - 2x) \frac{1}{k}e^{kx} - 2(x-1) \frac{1}{k^2}e^{kx} + 2 \frac{1}{k^3}e^{kx}$$

例. 求  $\int x^4 \sin 2x dx$ .

Diff	Int
$x^4$	$\sin 2x$
4x <sup>3</sup>	$-\frac{1}{2} \cos 2x$
12x <sup>2</sup>	$-\frac{1}{4} \sin 2x$
24x	$\frac{1}{8} \cos 2x$
24	$\frac{1}{16} \sin 2x$
0	$-\frac{1}{32} \cos 2x$

$$\begin{aligned} & \int x^4 \sin 2x dx \\ &= -x^4 \frac{1}{2} \cos 2x + 4x^3 \frac{1}{4} \sin 2x + 12x^2 \frac{1}{8} \cos 2x - 24x \frac{1}{16} \sin 2x - 24 \frac{1}{32} \cos 2x \\ &= \left( -\frac{x^4}{2} + \frac{3x^2}{2} - \frac{3}{4} \right) \cos 2x + \left( x^3 - \frac{3x}{2} \right) \sin 2x \end{aligned}$$

例. 求  $\int x^5 e^{ax} dx$ .

Diff	Int
$x^5$	$e^{ax}$
5x <sup>4</sup>	$\frac{1}{a}e^{ax}$
20x <sup>3</sup>	$\frac{1}{a^2}e^{ax}$
60x <sup>2</sup>	$\frac{1}{a^3}e^{ax}$
120x	$\frac{1}{a^4}e^{ax}$
120	$\frac{1}{a^5}e^{ax}$
0	$\frac{1}{a^6}e^{ax}$

$$\begin{aligned} & \int x^5 e^{ax} dx \\ &= x^5 \frac{1}{a}e^{ax} - 5x^4 \frac{1}{a^2}e^{ax} + 20x^3 \frac{1}{a^3}e^{ax} - 60x^2 \frac{1}{a^4}e^{ax} + 120x \frac{1}{a^5}e^{ax} - 120 \frac{1}{a^6}e^{ax} \\ &= \left( \frac{x^5}{a} - \frac{5x^4}{a^2} + \frac{20x^3}{a^3} - \frac{60x^2}{a^4} + \frac{120x}{a^5} - \frac{120}{a^6} \right) e^{ax} \end{aligned}$$

例. 求  $\int (\sin^{-1} x)^2 dx$ .

Diff	Int
$(\sin^{-1} x)^2$	+
$\frac{2 \sin^{-1} x}{\sqrt{1-x^2}}$	-
$x$	

Diff	Int
$2 \sin^{-1} x$	+
$\frac{2}{\sqrt{1-x^2}}$	-

解.

$$\begin{aligned}\int (\sin^{-1} x)^2 dx &= (\sin^{-1} x)^2 \cdot x - \int \frac{2 \sin^{-1} x}{\sqrt{1-x^2}} \cdot x dx \\ &= (\sin^{-1} x)^2 \cdot x - \int 2 \sin^{-1} x \cdot \frac{x}{\sqrt{1-x^2}} dx \\ &= (\sin^{-1} x)^2 \cdot x - \left( -2 \sin^{-1} x \cdot \sqrt{1-x^2} + \int \frac{2}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2} dx \right) \\ &= (\sin^{-1} x)^2 \cdot x + 2 \sin^{-1} x \cdot \sqrt{1-x^2} - 2x\end{aligned}$$

例. 求  $\int e^{ax} \cos bx dx, a, b \neq 0$ .

Diff	Int
$\cos bx$	+
$-b \sin bx$	-
$\frac{1}{a} e^{ax}$	

Diff	Int
$-b^2 \cos bx$	+
$\frac{1}{a^2} e^{ax}$	

解.

$$\begin{aligned}\int e^{ax} \cos bx dx &= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} \int e^{ax} \cos bx dx \\ \Rightarrow \int e^{ax} \cos bx dx &= \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}\end{aligned}$$

例. 求  $\int e^{ax} \sin bx dx, a, b \neq 0$ .

Diff	Int
$\sin bx$	+
$b \cos bx$	-
$\frac{1}{a} e^{ax}$	

Diff	Int
$-b^2 \sin bx$	+
$\frac{1}{a^2} e^{ax}$	

解.

$$\begin{aligned}\int e^{ax} \sin bx dx &= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} \int e^{ax} \sin bx dx \\ \Rightarrow \int e^{ax} \sin bx dx &= \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}\end{aligned}$$

## 遞迴式

例. 令  $I_n = \int \frac{1}{(x^2 + a^2)^n} dx, n \in \mathbb{N}$ , 則  $I_{n+1} = \frac{x}{2na^2(x^2 + a^2)^n} + \frac{2n-1}{2na^2} I_n$ .

解. 令  $u = \frac{1}{(x^2 + a^2)^n}$ , 則  $du = -2n \frac{x}{(x^2 + a^2)^{n+1}} dx$ ;  $dv = dx$ , 則  $v = x$ . 故  $I_n = \int \frac{1}{(x^2 + a^2)^n} dx = \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{x^2}{(x^2 + a^2)^{n+1}} dx = \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{(x^2 + a^2 - a^2)}{(x^2 + a^2)^{n+1}} dx = \frac{x}{(x^2 + a^2)^n} + 2nI_n - 2na^2 I_{n+1} \Rightarrow 2na^2 I_{n+1} = \frac{x}{(x^2 + a^2)^n} + (2n-1)I_n \Rightarrow I_{n+1} = \frac{x}{2na^2(x^2 + a^2)^n} + \frac{2n-1}{2na^2} I_n$ .

註 (使用例).  $I_1 = \frac{1}{a} \tan^{-1} \frac{x}{a}, I_2 = \frac{1}{2a^2} I_1 + \frac{x}{2a^2(x^2 + a^2)} = \frac{1}{2a^3} \tan^{-1} \frac{x}{a} + \frac{x}{2a^2(x^2 + a^2)}, I_3 = \frac{3}{4a^2} I_2 + \frac{x}{4a^2(x^2 + a^2)^2} = \frac{3}{4a^2} \left( \frac{1}{2a^3} \tan^{-1} \frac{x}{a} + \frac{x}{2a^2(x^2 + a^2)} \right) + \frac{x}{4a^2(x^2 + a^2)^2} = \frac{3}{8a^5} \tan^{-1} \frac{x}{a} + \frac{3x}{8a^4(x^2 + a^2)} + \frac{x}{4a^2(x^2 + a^2)^2}$ .

例. 令  $J_n = \int \sin^n x dx, n \in \mathbb{N}, n \geq 2$ , 則  $J_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} J_{n-2}$ .

**解.** 令  $u = \sin^{n-1}x$ , 則  $du = (n-1) \sin^{n-2}x \cos x dx$ ;  $dv = \sin x dx$ , 則  $v = -\cos x$ . 故  $J_n = \int \sin^n x dx = -\sin^{n-1}x \cdot \cos x + (n-1) \int \cos x \cdot \sin^{n-2}x \cos x dx = -\sin^{n-1}x \cos x + (n-1) \int \sin^{n-2}x \cos^2 x dx = -\sin^{n-1}x \cos x + (n-1) \int \sin^{n-2}x \cdot (1-\sin^2 x) dx = -\sin^{n-1}x \cos x + (n-1) \int \sin^{n-2}x dx - (n-1) \int \sin^n x dx = -\sin^{n-1}x \cos x + (n-1)J_{n-2} + (1-n)J_n \implies nJ_n = -\sin^{n-1}x \cos x + (n-1)J_{n-2} \implies J_n = \frac{-\sin^{n-1}x \cos x}{n} + \frac{n-1}{n} J_{n-2}$ .

**註 (使用例).**  $J_2 = \frac{-\sin x \cos x}{2} + \frac{1}{2} J_0 = \frac{-\sin x \cos x}{2} + \frac{x}{2}$ ,  $J_3 = \frac{-\sin^2 x \cos x}{3} + \frac{2}{3} J_1 = \frac{-\sin^2 x \cos x}{3} - \frac{2 \cos x}{3}$ ,  
 $J_4 = \frac{-\sin^3 x \cos x}{4} + \frac{3}{4} J_2 = \frac{-\sin^3 x \cos x}{4} + \frac{3 \sin x \cos x}{8} - \frac{3x}{8}$ .

**例.** 令  $K_n = \int \sec^{2n+1} \theta d\theta$ ,  $n \in \mathbb{N}$ ,  $n \geq 1$ , 則  $K_n = \frac{\sec^{2n-1} \theta \tan \theta}{2n} + \frac{2n-1}{2n} K_{n-1}$ .

**解.** 令  $u = \sec^{2n-1} \theta$ , 則  $du = (2n-1) \sec^{2n-2} \theta \cdot \sec \theta \tan \theta d\theta = (2n-1) \sec^{2n-1} \theta \tan \theta d\theta$ ; 令  $dv = \sec^2 \theta d\theta$ , 則  $v = \tan \theta$ . 故  $K_n = \int \sec^{2n+1} \theta d\theta = \sec^{2n-1} \theta \cdot \tan \theta - \int \tan \theta \cdot (2n-1) \sec^{2n-1} \theta \tan \theta d\theta = \sec^{2n-1} \theta \tan \theta - (2n-1) \int \tan^2 \theta \cdot \sec^{2n-1} \theta d\theta = \sec^{2n-1} \theta \tan \theta - (2n-1) \int (\sec^2 \theta - 1) \cdot \sec^{2n-1} \theta d\theta = \sec^{2n-1} \theta \tan \theta - (2n-1) \int \sec^{2n+1} \theta d\theta + (2n-1) \int \sec^{2n-1} \theta d\theta \implies K_n = \sec^{2n-1} \theta \tan \theta - (2n-1)K_n + (2n-1)K_{n-1} \implies K_n = \frac{\sec^{2n-1} \theta \tan \theta}{2n} + \frac{2n-1}{2n} K_{n-1}$ .

**註 (使用例).**  $K_0 = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta|$ ,  $\int \sec^3 \theta d\theta = K_1 = \frac{\sec \theta \tan \theta}{2} + \frac{1}{2} K_0 = \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|)$ ,  $\int \sec^5 \theta d\theta = K_2 = \frac{\sec^3 \theta \tan \theta}{4} + \frac{3}{4} K_1 = \frac{\sec^3 \theta \tan \theta}{4} + \frac{3}{8} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|)$ .

**習題.** 以部份積分法求下列不定積分. 注意: 可能會因為常數項而跟此處答案不同.

1. $\int \frac{\sin^{-1} x}{x^2} dx$	5. $\int x^2 \tan^{-1} x dx$	9. $\int (2x^2 + 1)e^{x^2} dx$
2. $\int \ln(x + \sqrt{1+x^2}) dx$	6. $\int \frac{xe^x}{(x+1)^2} dx$	10. $\int \sin(\ln x) dx$
3. $\int x^3 \ln x dx$	7. $\int x^5 e^{-x^2} dx$	11. $\int x^2 \ln \frac{1-x}{1+x} dx$
4. $\int x(\ln x)^3 dx$	8. $\int xe^{\sqrt{x}} dx$	12. $\int \frac{\ln x}{\sqrt{1+x}} dx$

**解.**

1. 令  $u = \sin^{-1} x$ , 則  $du = \frac{1}{\sqrt{1-x^2}} dx$ ;  $dv = \frac{1}{x^2} dx$ , 則  $v = -\frac{1}{x}$ . 故  $\int \frac{\sin^{-1} x}{x^2} dx = \sin^{-1} x \cdot \frac{-1}{x} - \int \frac{-1}{x} \cdot \frac{1}{\sqrt{1-x^2}} dx = -\frac{\sin^{-1} x}{x} + \int \frac{1}{x\sqrt{1-x^2}} dx$ . 令  $w = \sqrt{1-x^2}$ , 則  $-x^2 = w^2 - 1$ ,  $dw = \frac{-x}{\sqrt{1-x^2}} dx$ . 故  $\int \frac{1}{x\sqrt{1-x^2}} dx = \int \frac{1}{-x^2} \cdot \frac{-x}{\sqrt{1-x^2}} dx = \int \frac{1}{w^2-1} dw = \frac{1}{2} \int \left( \frac{1}{w-1} - \frac{1}{w+1} \right) dw = \frac{1}{2} (\ln |w-1| - \ln |w+1|) = \frac{1}{2} \ln \left| \frac{w-1}{w+1} \right| = \frac{1}{2} \ln \left| \frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1} \right| = \frac{1}{2} \ln \left| \frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} \right| = \frac{1}{2} \ln \left| \frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} \right|$ .

$$\left| \frac{1 - \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}} \right| = \frac{1}{2} \ln \left| \frac{(1 - \sqrt{1 - x^2})^2}{x^2} \right| = \ln \left| \frac{1 - \sqrt{1 - x^2}}{x} \right|.$$

以上,  $\int \frac{\sin^{-1} x}{x^2} dx = -\frac{\sin^{-1} x}{x} + \int \frac{1}{x\sqrt{1-x^2}} dx = -\frac{\sin^{-1} x}{x} + \ln \left| \frac{1 - \sqrt{1 - x^2}}{x} \right| + c.$

2. 令  $u = \ln(x + \sqrt{1 + x^2})$ , 則  $du = \frac{1}{x + \sqrt{1 + x^2}} \cdot (x + \sqrt{1 + x^2})' dx = \frac{1}{x + \sqrt{1 + x^2}} \cdot \left(1 + \frac{x}{\sqrt{1 + x^2}}\right) dx = \frac{1}{\sqrt{1 + x^2}} dx$ ;  $dv = dx$ , 則  $v = x$ . 故  $\int \ln(x + \sqrt{1 + x^2}) dx = \ln(x + \sqrt{1 + x^2}) \cdot x - \int x \cdot \frac{1}{\sqrt{1 + x^2}} dx = \ln(x + \sqrt{1 + x^2}) x - \sqrt{1 + x^2} + c.$

3. 令  $u = \ln x$ , 則  $du = \frac{1}{x} dx$ ;  $dv = x^3 dx$ , 則  $v = \frac{x^4}{4}$ . 故  $\int x^3 \ln x dx = \ln x \cdot \frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{x} dx = \ln x \cdot \frac{x^4}{4} - \int \frac{x^3}{4} dx = \ln x \cdot \frac{x^4}{4} - \frac{x^4}{16} + c.$

4. 令  $u = (\ln x)^3$ , 則  $du = 3(\ln x)^2 \cdot \frac{1}{x} dx$ ;  $dv = x dx$ , 則  $v = \frac{x^2}{2}$ . 故  $\int x(\ln x)^3 dx = (\ln x)^3 \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot 3(\ln x)^2 \cdot \frac{1}{x} dx = (\ln x)^3 \cdot \frac{x^2}{2} - \frac{3}{2} \int x(\ln x)^2 dx$ . 令  $u = (\ln x)^2$ , 則  $du = 2 \ln x \cdot \frac{1}{x} dx$ ;  $dv = x dx$ , 則  $v = \frac{x^2}{2}$ . 故  $\int x(\ln x)^2 dx = (\ln x)^2 \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot 2 \ln x \cdot \frac{1}{x} dx = (\ln x)^2 \cdot \frac{x^2}{2} - \int x \ln x dx$ . 令  $u = \ln x$ , 則  $du = \frac{1}{x} dx$ ;  $dv = x dx$ , 則  $v = \frac{x^2}{2}$ . 故  $\int x \ln x dx = \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx = \ln x \cdot \frac{x^2}{2} - \frac{1}{2} \int x dx = \ln x \cdot \frac{x^2}{2} - \frac{x^2}{4}$ . 以上,  $\int x(\ln x)^3 dx = (\ln x)^3 \cdot \frac{x^2}{2} - \frac{3}{2} \left( (\ln x)^2 \cdot \frac{x^2}{2} - \left( \ln x \cdot \frac{x^2}{2} - \frac{x^2}{4} \right) \right) = \frac{x^2}{2} \left( (\ln x)^3 - \frac{3(\ln x)^2}{2} + \frac{3 \ln x}{2} - \frac{3}{4} \right) + c$

5. 令  $u = \tan^{-1} x$ , 則  $du = \frac{1}{1+x^2} dx$ ;  $dv = x^2 dx$ , 則  $v = \frac{x^3}{3}$ . 故  $\int x^2 \tan^{-1} x dx = \tan^{-1} x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{1+x^2} dx$ . 又  $\int \frac{x^3}{3} \cdot \frac{1}{x^2+1} dx = \frac{1}{3} \int \frac{(x^3+x)-x}{x^2+1} dx = \frac{1}{3} \int \frac{x(x^2+1)-x}{x^2+1} dx = \frac{1}{3} \int \left(x - \frac{x}{x^2+1}\right) dx$ . 令  $w = x^2 + 1$ , 則  $dw = 2x dx \Rightarrow x dx = \frac{1}{2} dw$ , 故  $\int \frac{x}{x^2+1} dx = \int \frac{1}{x^2+1} \cdot x dx = \int \frac{1}{w} \cdot \frac{1}{2} dw = \frac{1}{2} \ln w + c = \frac{1}{2} \ln(x^2 + 1) + c$ . 以上,  $\int x^2 \tan^{-1} x dx = \tan^{-1} x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x^2+1} dx = \tan^{-1} x \cdot \frac{x^3}{3} - \frac{x^2}{6} + \frac{\ln(x^2+1)}{6} + c$ .

6.  $\int \frac{xe^x}{(x+1)^2} dx = \int \frac{(x+1-1)e^x}{(x+1)^2} dx = \int \frac{e^x}{x+1} dx - \int \frac{e^x}{(x+1)^2} dx$ . 令  $u = e^x$ , 則  $du = e^x dx$ ;  $dv = \frac{1}{(x+1)^2} dx$ , 則  $v = \frac{-1}{x+1}$ . 故  $\int \frac{e^x}{(x+1)^2} dx = e^x \cdot \frac{-1}{x+1} + \int \frac{1}{x+1} \cdot e^x dx = \frac{-e^x}{x+1} + \int \frac{e^x}{x+1} dx$ ; 原式  $\int \frac{xe^x}{(x+1)^2} dx = \int \frac{e^x}{x+1} dx - \int \frac{e^x}{(x+1)^2} dx = \int \frac{e^x}{x+1} dx - \left( \frac{-e^x}{x+1} + \int \frac{e^x}{x+1} dx \right) = \frac{e^x}{x+1}$ .

7. 令  $w = x^2$ , 則  $dw = 2x dx \Rightarrow x dx = \frac{1}{2} dw$ , 故  $\int x^5 e^{-x^2} dx = \int e^{-x^2} \cdot (x^2)^2 \cdot x dx = \int e^{-w} \cdot w^2 \cdot \frac{1}{2} dw = \frac{1}{2} \int w^2 e^{-w} dw = -\frac{1}{2} e^{-w} (w^2 + 2w + 2) = -\frac{1}{2} e^{-x^2} (x^4 + 2x^2 + 2) + c$ .

Diff	Int
$w^2$	$e^{-w}$
$2w$	$-e^{-w}$
$2$	$e^{-w}$
$0$	$-e^{-w}$

$$\int w^2 e^{-w} dw = -w^2 e^{-w} - 2w e^{-w} - 2 e^{-w} = -e^{-w}(w^2 + 2w + 2)$$

8. 令  $w = \sqrt{x}$ , 則  $w^2 = x$ ,  $dx = 2w dw$ , 故  $\int xe^{\sqrt{x}} dx = \int w^2 e^w \cdot 2w dw = 2 \int w^3 e^w dw = 2 e^w (w^3 - 3w^2 + 6w - 6) = 2 e^{\sqrt{x}} (x\sqrt{x} - 3x + 6\sqrt{x} - 6) + c$

Diff	Int
$w^3$	$e^w$
$3w^2$	$-e^w$
$6w$	$e^w$
$6$	$-e^w$
$0$	$e^w$

$$\int w^3 e^w dw = w^3 e^w - 3w^2 e^w + 6w e^w - 6 e^w = e^w (w^3 - 3w^2 + 6w - 6)$$

9.  $\int (2x^2 + 1) e^{x^2} dx = \int 2x^2 e^{x^2} dx + \int e^{x^2} dx$ . 令  $u = x$ , 則  $du = dx$ ;  $dv = 2x e^{x^2} dx$ , 則  $v = e^{x^2}$ . 故  $\int 2x^2 e^{x^2} dx = x \cdot e^{x^2} - \int e^{x^2} dx$ ; 原式  $\int 2x^2 e^{x^2} dx + \int e^{x^2} dx = x \cdot e^{x^2} - \int e^{x^2} dx + \int e^{x^2} dx = xe^{x^2}$

10. 令  $w = \ln x$ , 則  $e^w = x$ ,  $dx = e^w dw$ , 故  $\int \sin(\ln x) dx = \int \sin w \cdot e^w dw = \int e^w \sin w dw = \frac{e^w(\sin w - \cos w)}{2} = \frac{x(\sin(\ln x) - \cos(\ln x))}{2} + c$

Diff	Int
$\sin w$	$e^w$
$\cos w$	$-e^w$
$-\sin w$	$e^w$

$$\begin{aligned} \int e^w \sin w dw &= e^w \sin w - e^w \cos w - \int e^w \sin w dw \\ \Rightarrow \int e^w \sin w dw &= \frac{e^w(\sin w - \cos w)}{2} \end{aligned}$$

11. 令  $u = \ln \frac{1-x}{1+x}$ , 則  $du = \frac{1+x}{1-x} \cdot \left(\frac{1-x}{1+x}\right)' dx = \frac{1+x}{1-x} \cdot \frac{(1+x)(-1)-(1-x)}{(1+x)^2} dx = \frac{2}{x^2-1} dx$ ;  $dv = x^2 dx$ , 則  $v = \frac{x^3}{3}$ . 故  $\int x^2 \ln \frac{1-x}{1+x} dx = \ln \frac{1-x}{1+x} \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{2}{x^2-1} dx$ . 又  $\int \frac{x^3}{3} \cdot \frac{2}{x^2-1} dx = \frac{2}{3} \int \frac{x^3}{x^2-1} dx = \frac{2}{3} \int \left(x + \frac{x}{x^2-1}\right) dx = \frac{x^2}{3} + \frac{2}{3} \int \frac{x}{x^2-1} dx$ . 令  $w = x^2 - 1$ , 則  $dw = 2x dx \Rightarrow x dx = \frac{1}{2} dw$ , 故  $\int \frac{x}{x^2-1} dx = \int \frac{1}{x^2-1} \cdot x dx = \int \frac{1}{w} \cdot \frac{1}{2} dw = \frac{1}{2} \ln w + c = \frac{1}{2} \ln(x^2 - 1) + c$ . 以上,  $\int x^2 \ln \frac{1-x}{1+x} dx = \frac{1+x}{1-x} \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x^2-1} dx = \ln \frac{1-x}{1+x} \cdot \frac{x^3}{3} - \frac{x^2 + \ln(x^2 - 1)}{3} + c$ .

12. 令  $u = \ln x$ , 則  $du = \frac{1}{x} dx$ ;  $dv = \frac{1}{\sqrt{1+x}} dx$ , 則  $v = 2\sqrt{1+x}$ . 故  $\int \frac{\ln x}{\sqrt{1+x}} dx = \ln x \cdot 2\sqrt{1+x} - 2 \int \frac{\sqrt{1+x}}{x} dx$ . 令  $w = \sqrt{1+x}$ , 則  $w^2 = 1+x \Rightarrow x = w^2 - 1$ ,  $2w dw = dx$ , 故  $\int \frac{\sqrt{1+x}}{x} dx = \int \frac{w}{w^2-1} \cdot 2w dw = 2 \int \frac{w^2}{w^2-1} dw = 2 \int \frac{(w^2-1)+1}{w^2-1} dw = 2w + 2 \int \frac{1}{w^2-1} dw = 2\sqrt{1+x} +$

$$2 \int \frac{1}{w^2 - 1} dw. \text{ 由 } \frac{1}{w^2 - 1} = \frac{1}{2} \left( \frac{1}{w-1} - \frac{1}{w+1} \right), \int \frac{1}{w^2 - 1} dw = \frac{1}{2} \int \left( \frac{1}{w-1} - \frac{1}{w+1} \right) dw = \frac{1}{2} (\ln |w-1| - \ln |w+1|) = \frac{1}{2} (\ln |\sqrt{1+x}-1| - \ln |\sqrt{1+x}+1|). \text{ 以上, } \int \frac{\ln x}{\sqrt{1+x}} dx = \ln x \cdot 2\sqrt{1+x} - 2(2\sqrt{1+x} + 2(\frac{1}{2}(\ln |\sqrt{1+x}-1| - \ln |\sqrt{1+x}+1|))) = \ln x \cdot 2\sqrt{1+x} - 4\sqrt{1+x} - 2\ln |\sqrt{1+x}-1| + 2\ln |\sqrt{1+x}+1| + c.$$

## 4.2 定積分

定積分  $\approx$  (帶符號) 面積:  $x$  軸上方為正, 下方為負.

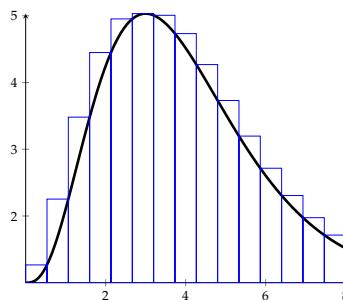
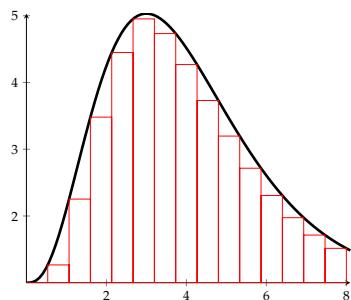
**定義.** 給定  $f : [a, b] \rightarrow \mathbb{R}$ .

- $[a, b]$  分割  $\mathbb{P} : a = x_0 < x_1 < x_2 < \dots < x_n = b$
- $\Delta x_k = x_k - x_{k-1}, k = 1, 2, \dots, n; \|\mathbb{P}\| = \max\{|\Delta x_k| \mid 1 \leq k \leq n\}$
- 樣本點  $\xi_k : x_{k-1} \leq \xi_k \leq x_k, k = 1, 2, \dots, n$
- $u_k = \sup \{f(x) \mid x_{k-1} \leq x \leq x_k\}, l_k = \inf \{f(x) \mid x_{k-1} \leq x \leq x_k\}, k = 1, 2, \dots, n$
- $R(f, \mathbb{P}) = \sum_{k=1}^n f(\xi_k) \Delta x_k, U(f, \mathbb{P}) = \sum_{k=1}^n u_k \Delta x_k, L(f, \mathbb{P}) = \sum_{k=1}^n l_k \Delta x_k;$   
顯然  $L(f, \mathbb{P}) \leq R(f, \mathbb{P}) \leq U(f, \mathbb{P})$ .
- 求  $\lim_{\|\mathbb{P}\| \rightarrow 0} R(f, \mathbb{P})$ . 若對不同分割與樣本點選取此極限均存在且相等, 稱  $f$  在  $[a, b]$  可積 (分);  $f(x)$  在  $[a, b]$  的定積分  $\int_a^b f(x) dx \equiv \lim_{\|\mathbb{P}\| \rightarrow 0} R(f, \mathbb{P})$

**註.**

- 在  $\int_a^b f(x) dx$  中,  $a$  為積分下限 (lower limit of integration),  $b$  為積分上限 (upper limit of integration),  $f(x)$  為被積分式 (integrand),  $x$  為積分變數 (variable of integration).
- $\int_a^b f(x) dx = \int_a^b f(t) dt = \int_a^b f(u) du$  (定積分數值與積分變數無關)

**結論.** 若  $f$  在  $[a, b]$  連續, 則  $f$  在  $[a, b]$  可積.



**性質.** 令  $f, g$  在包含  $a, b, c$  之區間為可積,  $\alpha, \beta \in \mathbb{R}$ . 則

1.  $\int_a^a f(x) dx = 0$

2.  $\int_a^b f(x) dx = - \int_b^a f(x) dx$

3.  $\int_a^b (\alpha f(x) + \beta g(x)) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx$

4.  $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$

5.  $\int_a^b f(x) dx \leq \int_a^b g(x) dx$ , 若  $f(x) \leq g(x) \forall x \in [a, b], a \leq b$

6.  $\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx, a \leq b$

7.  $f(x)$  為奇函數:  $\int_{-a}^a f(x) dx = 0$

8.  $f(x)$  為偶函數:  $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

例.

1.  $\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-x^2} dx = \pi$  (半徑  $\sqrt{2}$  之半圓面積)

2. 定義  $\text{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$ , 則  $\int_{-1}^2 \text{sgn}(x) dx = 2 \cdot 1 - 1 \cdot 1 = 1$

3.  $\int_{-\pi}^{\pi} \sin(x^7 - 5x^3) dx = 0$  ( $\sin(x^7 - 5x^3)$  為奇函數)

4.  $\int_{-6}^6 e^{-x^4} \sin(\sin x) dx = 0$  ( $e^{-x^4} \sin(\sin x)$  為奇函數)

5.  $\int_{-4}^4 (e^x - e^{-x}) dx = 0$  ( $e^x - e^{-x}$  為奇函數)

6.  $\int_{-2025}^{2025} (e^{9x^5-2x^7} - e^{-9x^5+2x^7}) dx = 0$  ( $e^{9x^5-2x^7} - e^{-9x^5+2x^7}$  為奇函數)

7.  $\int_{-a}^a |x| dx = 2 \int_0^a |x| dx = a^2$  ( $|x|$  為偶函數; 兩  $a \times a$  等腰直角三角形面積)

8. 定義  $\int_1^x \frac{1}{\tau} d\tau = \ln x$ , 則  $\int_{\frac{1}{4}}^3 \frac{1}{x} dx = \int_{\frac{1}{4}}^1 \frac{1}{x} dx + \int_1^3 \frac{1}{x} dx = - \int_1^{\frac{1}{4}} \frac{1}{x} dx + \int_1^3 \frac{1}{x} dx = \ln 12$

9.  $\int_0^2 \sqrt{4-x^2} \cdot \text{sgn}(1-x) dx = \int_0^1 \sqrt{4-x^2} dx - \int_1^2 \sqrt{4-x^2} dx = \left( \frac{4\pi}{12} + \frac{\sqrt{3}}{2} \right) - \left( \frac{4\pi}{6} - \frac{\sqrt{3}}{2} \right) = \sqrt{3} - \frac{\pi}{3}$

10.  $\forall n \in \mathbb{N}, \int_n^{n+1} \lfloor x \rfloor dx = n$

## 以極限定義求定積分

結論.  $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$ .

**證.** 由  $k(k+1) = \frac{k(k+1)(k+2) - (k-1)k(k+1)}{3}$ ,  $\sum_{k=1}^n k(k+1) = \frac{n(n+1)(n+2)}{3}$ . 又  $k^2 = k(k+1) - k$ ,

$$\sum_{k=1}^n k^2 = \sum_{k=1}^n (k(k+1) - k) = \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2} = \frac{n(n+1)(2n+1)}{6}.$$

**例.** 求  $\int_0^1 x^2 dx$ .

**解.** 建立  $[0, 1]$  分割  $\mathbb{P} : \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\right\}$ , 則  $\Delta x_k = \frac{1}{n} \forall k = 1, 2, \dots, n$  且  $\|P\| \rightarrow 0$  當  $n \rightarrow \infty$ .  $f(x) = x^2$  且  $f$

在  $[0, 1]$  嚴格遞增,  $U(f, \mathbb{P}) = \sum_{k=1}^n u_k \Delta x_k = \sum_{k=1}^n \frac{k^2}{n^2} \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^n k^2$ ,  $L(f, \mathbb{P}) = \sum_{k=1}^n l_k \Delta x_k = \sum_{k=1}^n \frac{(k-1)^2}{n^2} \cdot \frac{1}{n} =$

$\frac{1}{n^3} \sum_{k=1}^{n-1} k^2$ ,  $L(f, \mathbb{P}) \leq R(f, \mathbb{P}) \leq U(f, \mathbb{P})$ . 由  $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$ ,  $\lim_{n \rightarrow \infty} U(f, \mathbb{P}) = \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{k=1}^n k^2 =$

$\lim_{n \rightarrow \infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}$ ,  $\lim_{n \rightarrow \infty} L(f, \mathbb{P}) = \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{k=1}^{n-1} k^2 = \lim_{n \rightarrow \infty} \frac{(n-1)n(2n-1)}{6n^3} = \frac{1}{3}$ , 故由夾擊定理

$$\lim_{n \rightarrow \infty} R(f, \mathbb{P}) = \frac{1}{3}.$$

**結論.** 若  $r \in \mathbb{R}$ ,  $\sum_{k=0}^n r^k = \frac{r^{n+1} - 1}{r - 1}$ .

**證.** 令  $s = \sum_{k=0}^n r^k$ , 則  $rs = \sum_{k=1}^{n+1} r^k$ ;  $rs - s = r^{n+1} - 1 \implies s = \frac{r^{n+1} - 1}{r - 1}$ .

**例.** 求  $\int_0^1 e^x dx$ .

**解.** 建立  $[0, 1]$  分割  $\mathbb{P} : \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\right\}$ , 則  $\Delta x_k = \frac{1}{n} \forall k = 1, 2, \dots, n$  且  $\|P\| \rightarrow 0$  當  $n \rightarrow \infty$ .  $f(x) = e^x$

且  $f$  在  $[0, 1]$  嚴格遞增,  $U(f, \mathbb{P}) = \sum_{k=1}^n u_k \Delta x_k = \sum_{k=1}^n e^{\frac{k}{n}} \cdot \frac{1}{n} = \frac{1}{n} \frac{e^{\frac{1}{n}}(e^{\frac{n}{n}} - 1)}{e^{\frac{1}{n}} - 1} = \frac{e-1}{n} \frac{e^{\frac{1}{n}}}{e^{\frac{1}{n}} - 1}$ ,  $L(f, \mathbb{P}) =$

$\sum_{k=1}^n l_k \Delta x_k = \sum_{k=1}^n e^{\frac{k-1}{n}} \cdot \frac{1}{n} = \frac{1}{n} \frac{e^0(e^{\frac{n}{n}} - 1)}{e^{\frac{1}{n}} - 1} = \frac{e-1}{n} \frac{1}{e^{\frac{1}{n}} - 1}$ ,  $L(f, \mathbb{P}) \leq R(f, \mathbb{P}) \leq U(f, \mathbb{P})$ .  $\lim_{n \rightarrow \infty} U(f, \mathbb{P}) =$

$\lim_{n \rightarrow \infty} \frac{e-1}{n} \frac{e^{\frac{1}{n}}}{e^{\frac{1}{n}} - 1} = (e-1) \lim_{x \rightarrow 0^+} \frac{xe^x}{e^x - 1} = (e-1) \lim_{x \rightarrow 0^+} \frac{e^x + xe^x}{e^x} = e-1$ ,  $\lim_{n \rightarrow \infty} L(f, \mathbb{P}) = \lim_{n \rightarrow \infty} \frac{e-1}{n} \frac{1}{e^{\frac{1}{n}} - 1} =$

$(e-1) \lim_{x \rightarrow 0^+} \frac{x}{e^x - 1} = (e-1) \lim_{x \rightarrow 0^+} \frac{1}{e^x} = e-1$ , 故由夾擊定理  $\lim_{n \rightarrow \infty} R(f, \mathbb{P}) = e-1$ .

**結論.**  $\sum_{k=1}^n \cos kx = \frac{1}{2} \left( \frac{\sin(n + \frac{1}{2})x}{\sin \frac{x}{2}} - 1 \right)$

**證.** 考慮  $\sum_{k=1}^n e^{ikx} = \frac{e^{ix}(1 - e^{inx})}{1 - e^{ix}}$ , 則  $\sum_{k=1}^n \cos kx = \Re \left\{ \sum_{k=1}^n e^{ikx} \right\} = \Re \left\{ \frac{e^{ix}(1 - e^{inx})}{1 - e^{ix}} \right\} = \Re \left\{ \frac{e^{ix} e^{\frac{inx}{2}} (e^{-\frac{inx}{2}} - e^{\frac{inx}{2}})}{e^{\frac{ix}{2}} (e^{-\frac{ix}{2}} - e^{\frac{ix}{2}})} \right\} =$

$\frac{\cos \frac{(n+1)x}{2} \sin \frac{nx}{2}}{\sin \frac{x}{2}} = \frac{\sin \left( \frac{nx}{2} + \frac{(n+1)x}{2} \right) + \sin \left( \frac{nx}{2} + \frac{(n+1)x}{2} \right)}{2 \sin \frac{x}{2}} = \frac{\sin \left( n + \frac{1}{2} \right)x - \sin \frac{x}{2}}{2 \sin \frac{x}{2}} = \frac{1}{2} \left( \frac{\sin \left( n + \frac{1}{2} \right)x}{\sin \frac{x}{2}} - 1 \right)$ .

**例.** 求  $\int_0^{\frac{\pi}{2}} \cos x dx$ .

解. 建立  $[0, \frac{\pi}{2}]$  分割  $\mathbb{P} : \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \dots, \frac{n\pi}{2n}\right\}$ , 則  $\Delta x_k = \frac{\pi}{2n} \forall k = 1, 2, \dots, n$  且  $\|P\| \rightarrow 0$  當  $n \rightarrow \infty$ , 則積分為  $\lim_{n \rightarrow \infty} R(f, \mathbb{P}) = \sum_{k=1}^n f(\xi_k) \Delta x_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n \cos \frac{k\pi}{2n} \cdot \frac{\pi}{2n} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \cos \left(k \left(\frac{\pi}{2n}\right)\right) \cdot \frac{\pi}{2n} = \lim_{n \rightarrow \infty} \frac{1}{2} \left( \frac{\sin((n + \frac{1}{2}) \cdot \frac{\pi}{2n})}{\sin(\frac{1}{2} \cdot \frac{\pi}{2n})} - 1 \right) \cdot \frac{\pi}{2n} = \lim_{n \rightarrow \infty} \frac{\sin((n + \frac{1}{2}) \frac{\pi}{2n}) \frac{1}{2} \cdot \frac{\pi}{2n}}{\sin(\frac{1}{2} \cdot \frac{\pi}{2n})} = \lim_{n \rightarrow \infty} \sin\left(\frac{1}{2} + \frac{1}{4n}\right) \pi \cdot \lim_{n \rightarrow \infty} \frac{\frac{1}{2} \cdot \frac{\pi}{2n}}{\sin(\frac{1}{2} \cdot \frac{\pi}{2n})} = \sin\frac{\pi}{2} \cdot 1 = 1.$

## 微積分基本定理

### 複習：連續函數性質

**定理** (中間值定理). 若  $f$  在  $[a, b]$  連續, 則對任意介於  $f(a)$  與  $f(b)$  之間的數  $d$ , 存在  $c \in [a, b]$  使得  $f(c) = d$ .

**定理** (最大最小值定理). 若函數在定義域為有限閉區間, 或有限閉區間的有限聯集連續, 則函數在定義域上有最大值及最小值.

### 積分均值定理

**定理** (積分均值定理). 設  $f$  在  $[a, b]$  連續, 則存在  $c \in (a, b)$  使得  $\int_a^b f(x) dx = f(c) \cdot (b - a)$ .

**證. 第一步: 建立上下界.** 因  $f$  在  $[a, b]$  連續, 根據最大最小值定理,  $f$  在  $[a, b]$  有最大值  $M$  及最小值  $m$ , 亦即  $\exists x_m, x_M$  使  $m = f(x_m) \leq f(x) \leq f(x_M) = M, \forall x \in [a, b]$ .

**第二步: 積分的夾擠.** 對不等式  $m \leq f(x) \leq M$  在  $[a, b]$  上積分, 得  $m(b - a) \leq \int_a^b f(x) dx \leq M(b - a)$ ; 除以  $(b - a) > 0$  得  $m \leq \frac{1}{b - a} \int_a^b f(x) dx \leq M$ .

**第三步: 應用中間值定理.** 令  $\mu = \frac{1}{b - a} \int_a^b f(x) dx$ : 我們已證明  $m \leq \mu \leq M$ . 因為  $f$  連續且  $f(x_m) = m$ ,  $f(x_M) = M$ , 由中間值定理, 存在  $c$  介於  $x_m$  與  $x_M$  間 (故  $c \in [a, b]$ ), 使  $f(c) = \mu = \frac{1}{b - a} \int_a^b f(x) dx \implies \int_a^b f(x) dx = f(c) \cdot (b - a)$ .

**定理** (微積分基本定理 I (積分的微分)). 設  $f$  在  $[a, b]$  上連續. 定義  $F(x) = \int_a^x f(t) dt, x \in [a, b]$ , 則  $F$  在  $(a, b)$  可微, 且  $F'(x) = f(x)$ .

**證.** 對於  $x \in (a, b)$  及充分小的  $h \neq 0$ , 考慮  $\frac{F(x + h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t) dt$ . 因  $f$  在  $[a, b]$  連續, 由積分均值定理, 存在  $c_h$  介於  $x$  與  $x + h$  間, 使  $\int_x^{x+h} f(t) dt = f(c_h) \cdot h$ , 故  $\frac{F(x + h) - F(x)}{h} = f(c_h)$ . 當  $h \rightarrow 0$ ,  $c_h \rightarrow x$ . 由  $f$  的連續性,  $f(c_h) \rightarrow f(x)$ . 故  $F'(x) = \lim_{h \rightarrow 0} \frac{F(x + h) - F(x)}{h} = f(x)$ .

**定理** (微積分基本定理 II (牛頓-萊布尼茲公式)). 設  $f$  在  $[a, b]$  連續, 且  $G$  是  $f$  的任一反導函數 (即  $G'(x) = f(x)$ ), 則  $\int_a^b f(x) dx = G(b) - G(a)$ .

**證.** 由微積分基本定理 I,  $F(x) = \int_a^x f(t) dt$  滿足  $F'(x) = f(x)$ . 因為  $G$  也是  $f$  的反導函數,  $F$  與  $G$  相差一個常數:  $F(x) = G(x) + C$ . 由  $F(a) = \int_a^a f(t) dt = 0$  得  $0 = G(a) + C \implies C = -G(a)$ . 故  $\int_a^b f(x) dx = F(b) = G(b) + C = G(b) - G(a)$ .

**例** (以 FTC 求定積分).

$$1. x^2 \text{ 之反導函數為 } \frac{x^3}{3}, \text{ 故 } \int_0^1 x^2 dx = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

$$2. e^x \text{ 之反導函數為 } e^x, \text{ 故 } \int_0^1 e^x dx = e^1 - e^0 = e - 1.$$

$$3. \cos x \text{ 之反導函數為 } \sin x, \text{ 故 } \int_0^{\frac{\pi}{2}} \cos x dx = \sin \frac{\pi}{2} - \sin 0 = 1.$$

**結論** (定積分變數變換).

- 求反導函數後代入:  $\int_a^b f'(g(x)) g'(x) dx = f(g(x)) \Big|_{x=a}^{x=b} = f(g(b)) - f(g(a))$
- 變數變換並改變積分範圍:  $\int_a^b f'(g(x)) g'(x) dx = \int_a^b f'(g(x)) dg(x) = \int_{g(a)}^{g(b)} f'(u) du = f(u) \Big|_{u=g(a)}^{u=g(b)} = f(g(b)) - f(g(a))$

**例.** 求  $\int_0^1 x^3(1+x^4)^3 dx$ .

**解.**

- 求反導函數後代入: 令  $u = 1 + x^4$ , 則  $du = 4x^3 dx \implies x^3 dx = \frac{du}{4}$ , 故  $\int x^3(1+x^4)^3 dx = \int (1+x^4)^3 x^3 dx = \int u^3 \frac{du}{4} = \frac{u^4}{16} + c = \frac{(1+x^4)^4}{16} + c$ . 故  $\int_0^1 x^3(1+x^4)^3 dx = \frac{(1+x^4)^4}{16} \Big|_{x=0}^{x=1} = \frac{(1+1^4)^4 - (1+0^4)^4}{16} = \frac{15}{16}$ .

- 變數變換並改變積分範圍: 令  $u = 1 + x^4$ , 則  $du = 4x^3 dx \implies x^3 dx = \frac{du}{4}$ . 積分範圍  $x$  由 0 至 1, 則 變數變換後  $u$  由  $1+0^4 = 1$  至  $1+1^4 = 2$ , 故  $\int_0^1 x^3(1+x^4)^3 dx = \int_0^1 (1+x^4)^3 x^3 dx = \int_1^2 u^3 \frac{du}{4} = \frac{1}{4} \int_1^2 u^3 du = \frac{1}{16} u^4 \Big|_{u=1}^{u=2} = \frac{2^4 - 1^4}{16} = \frac{15}{16}$ .

**例.** 求  $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{1+x^2}} dx$ .

**解.**

- 求反導函數後代入: 令  $u = 1 + x^2$ , 則  $du = 2x dx \implies 4x dx = 2 du$ , 故  $\int \frac{4x}{\sqrt{1+x^2}} dx = \int \frac{2}{\sqrt{u}} du = 4\sqrt{u} + c = 4\sqrt{1+x^2} + c$ . 故  $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{1+x^2}} dx = 4\sqrt{1+x^2} \Big|_{x=0}^{x=\sqrt{3}} = 4\sqrt{1+3} - 4\sqrt{1} = 4$ .
- 變數變換並改變積分範圍: 令  $u = 1 + x^2$ , 則  $du = 2x dx \implies 4x dx = 2 du$ . 積分範圍  $x$  由 0 至  $\sqrt{3}$ , 則變數變換後  $u$  由  $1+0^2 = 1$  至  $1+(\sqrt{3})^2 = 4$ , 故  $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{1+x^2}} dx = \int_1^4 \frac{2}{\sqrt{u}} du = 4\sqrt{u} \Big|_{u=1}^{u=4} = 4(\sqrt{4} - \sqrt{1}) = 4$ .

例. 求  $\int_0^\pi 3 \cos^2 x \sin x \, dx$ .

解.

- 求反導函數後代入: 令  $u = \cos x$ , 則  $du = -\sin x \, dx \implies \sin x \, dx = -du$ , 故  $\int 3 \cos^2 x \sin x \, dx = -3 \int u^2 \, du = -u^3 + c = -\cos^3 x + c$ . 故  $\int_0^\pi 3 \cos^2 x \sin x \, dx = -\cos^3 x \Big|_{x=0}^{x=\pi} = -(\cos^3 \pi - \cos^3 0) = -((-1)^3 - 1^3) = 2$ .
- 變數變換並改變積分範圍: 令  $u = \cos x$ , 則  $du = -\sin x \, dx \implies \sin x \, dx = -du$ . 積分範圍  $x$  由 0 至  $\pi$ , 則變數變換後  $u$  由  $\cos 0 = 1$  至  $\cos \pi = -1$ , 故  $\int_0^\pi 3 \cos^2 x \sin x \, dx = - \int_1^{-1} 3u^2 \, du = -u^3 \Big|_{u=1}^{u=-1} = -((-1)^3 - 1^3) = 2$ .

例. 若  $f$  在  $[a, b]$  二次可微且  $f(a) = f(b) = 0$ , 證明  $\int_a^b (x-a)(b-x) f''(x) \, dx = -2 \int_a^b f(x) \, dx$ .

解.  $\int_a^b (x-a)(b-x) f''(x) \, dx = ((x-a)(b-x) f'(x) - (a+b-2x) f(x)) \Big|_a^b - 2 \int_a^b f(x) \, dx = ((b-a)(b-b) f'(b) - (a+b-2b) f(b)) - ((a-a)(b-a) f'(a) - (a+b-2a) f(a)) - 2 \int_a^b f(x) \, dx = -2 \int_a^b f(x) \, dx$ .

Diff	Int	
$(x-a)(b-x)$	$f''(x)$	$\int (x-a)(b-x) f''(x) \, dx$
$a+b-2x$	$f'(x)$	$= ((x-a)(b-x) f'(x) - (a+b-2x) f(x)) - 2 \int f(x) \, dx$
$-2$	$f(x)$	
0	$\int f(x) \, dx$	

性質. 令  $F(x) = \int_{v(x)}^{u(x)} f(\tau) \, d\tau$ , 則  $F'(x) = f(u(x)) \cdot u'(x) - f(v(x)) \cdot v'(x)$ .

證. 令  $a \in \mathbb{R}$ ,  $F(x) = \int_{v(x)}^{u(x)} f(\tau) \, d\tau = \int_a^{u(x)} f(\tau) \, d\tau - \int_a^{v(x)} f(\tau) \, d\tau$ . 令  $G(x) \equiv \int_a^x f(\tau) \, d\tau$ , 則  $G'(x) = f(x)$ ,  $F(x) = \int_a^{u(x)} f(\tau) \, d\tau - \int_a^{v(x)} f(\tau) \, d\tau = G(u(x)) - G(v(x))$ ; 故  $F'(x) = (G(u(x)) - G(v(x)))' = G'(u(x)) \cdot u'(x) - G'(v(x)) \cdot v'(x) = f(u(x)) \cdot u'(x) - f(v(x)) \cdot v'(x)$ .

例.

- $F(x) = \int_1^x \frac{1}{1+\tau^4} \, d\tau \implies F'(x) = \frac{1}{1+x^4}$

- $F(x) = \int_2^{\sqrt{x}} \sin \tau \, d\tau \implies F'(x) = \sin \sqrt{x} \cdot \frac{1}{2\sqrt{x}}$

- $F(x) = \int_x^{2x} \tau^3 \, d\tau \implies F'(x) = (2x)^3 \cdot 2 - x^3 \cdot 1 = 15x^3$

- $F(x) = \int_{\sin x}^{\tan^{-1} x} e^{\tau^2} \, d\tau \implies F'(x) = e^{(\tan^{-1} x)^2} \cdot \frac{1}{1+x^2} - e^{\sin^2 x} \cdot \cos x$

例. 若  $g(x) = \int_0^{\cos x} (1 + \sin(t^2)) \, dt$ ,  $f(x) = \int_0^{g(x)} \frac{1}{\sqrt{1+t^3}} \, dt$ , 求  $f'\left(\frac{\pi}{2}\right)$ .

解.  $f'(x) = \frac{1}{\sqrt{1+g(x)^3}} \cdot g'(x)$ ,  $g'(x) = (1+\sin(\cos^2 x)) \cdot (-\sin x)$ . 代入  $x = \frac{\pi}{2}$ ,  $g\left(\frac{\pi}{2}\right) = 0$ ,  $g'\left(\frac{\pi}{2}\right) = -1$ , 故  $f'\left(\frac{\pi}{2}\right) = -1$ .

例. 若  $\int_0^{x^2} f(t) dt = x \sin \pi x$ , 求  $f'(9)$ .

解.  $\int_0^{x^2} f(t) dt = x \sin \pi x$  兩邊對  $x$  微分得  $f(x^2) \cdot 2x = \sin \pi x + x \cdot \pi \cos \pi x$ . 兩邊再對  $x$  微分得  $(f'(x^2) \cdot 2x) \cdot 2x + f(x^2) \cdot 2 = \pi \cos \pi x + \pi \cos \pi x - x \cdot \pi^2 \sin \pi x$ . 代入  $x = 3$ , 則  $(f'(9) \cdot 2 \cdot 3) \cdot 2 \cdot 3 + f(9) \cdot 2 = \pi \cos 3\pi + \pi \cos 3\pi - 3 \cdot \pi^2 \sin 3\pi \Rightarrow f'(9) \cdot 36 + f(9) \cdot 2 = -2\pi$ . 又  $f(3^2) \cdot (2 \cdot 3) = \sin 3\pi + 3 \cdot \pi \cos 3\pi \Rightarrow f(9) = -\frac{\pi}{2}$ , 故  $f'(9) = -\frac{\pi}{36}$ .

例. 求函數  $f$  與  $a \in \mathbb{R}$  使  $6 + \int_a^x \frac{f(t)}{t^2} dt = 2\sqrt{x}$ ,  $\forall x > 0$ .

解.  $6 + \int_a^x \frac{f(t)}{t^2} dt = 2\sqrt{x}$  兩邊對  $x$  微分得  $\frac{f(x)}{x^2} = \frac{1}{\sqrt{x}} \Rightarrow f(x) = x^{\frac{3}{2}}$ . 代入原式得  $6 + \int_a^x \frac{t^{\frac{3}{2}}}{t^2} dt = 6 + \int_a^x \frac{1}{\sqrt{t}} dt = 6 + 2\sqrt{t} \Big|_a^x = 6 + 2\sqrt{x} - 2\sqrt{a} = 2\sqrt{x} \Rightarrow a = 9$ .

例. 求下列極限.

$$1. \lim_{x \rightarrow 0} \frac{\int_0^x (\sec t - 1) dt}{x^3}$$

$$2. \lim_{x \rightarrow \infty} \frac{\int_{x^2}^0 e^{t-x^2} (2t^2 + 1) dt}{x^4}$$

$$3. \lim_{x \rightarrow 0} \frac{1}{x} \int_0^{\tan x} f(u)(\sin x - \cos u) du, f \text{ 為連續函數}$$

$$4. \lim_{x \rightarrow 0} \frac{1}{x} \int_0^x (1 - \tan 2t)^{\frac{1}{t}} dt$$

解.

$$1. \lim_{x \rightarrow 0} \frac{\int_0^x (\sec t - 1) dt}{x^3} = \lim_{x \rightarrow 0} \frac{\sec x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{\sec x \tan x}{6x} = \lim_{x \rightarrow 0} \frac{\sec x \tan x \cdot \tan x + \sec x \cdot \sec^2 x}{6} = \frac{1}{6}.$$

$$2. \lim_{x \rightarrow \infty} \frac{\int_{x^2}^0 e^{t-x^2} (2t^2 + 1) dt}{x^4} = \lim_{x \rightarrow \infty} \frac{\int_{x^2}^0 e^t (2t^2 + 1) dt}{e^{x^2} x^4} = \lim_{x \rightarrow \infty} \frac{-e^{x^2} (2(x^2)^2 + 1) \cdot 2x}{e^{x^2} 2x \cdot x^4 + e^{x^2} \cdot 4x^3} = \lim_{x \rightarrow \infty} \frac{-(4x^5 + 2x)}{2x^5 + 4x^3} = -2.$$

$$3. \lim_{x \rightarrow 0} \frac{1}{x} \int_0^{\tan x} f(u)(\sin x - \cos u) du = \lim_{x \rightarrow 0} \frac{\int_0^{\tan x} f(u)(\sin x - \cos u) du}{x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x \int_0^{\tan x} f(u) du - \int_0^{\tan x} f(u) \cos u du}{x}$$

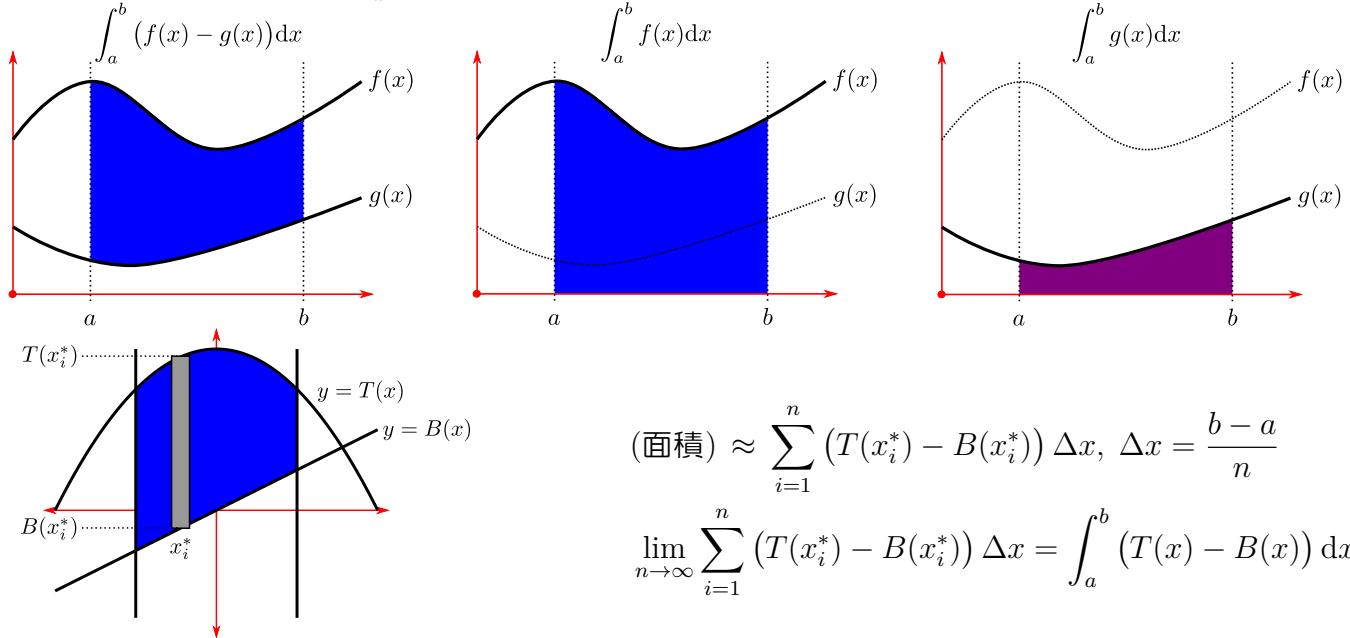
$$= \lim_{x \rightarrow 0} \frac{\cos x \int_0^{\tan x} f(u) du + \sin x \cdot f(\tan x) \cdot \sec^2 x - f(\tan x) \cos(\tan x) \cdot \sec^2 x}{1} = -f(0)$$

$$4. \lim_{x \rightarrow 0} \frac{\int_0^x (1 - \tan 2t)^{\frac{1}{t}} dt}{x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{(1 - \tan 2x)^{\frac{1}{x}}}{1} = \exp \left( \lim_{x \rightarrow 0} \frac{\ln(1 - \tan 2x)}{x} \right) \stackrel{H}{=} \exp \left( \lim_{x \rightarrow 0} \frac{-2 \sec^2 2x}{1 - \tan 2x} \right) = \exp \left( \frac{-2 \cdot 1^2}{1 - 0} \right) = e^{-2}$$

## 4.3 面積與體積

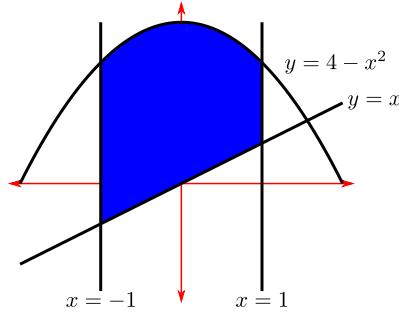
### 面積

結論. 當  $f(x) \geq 0 \forall x \in [a, b]$ ,  $\int_a^b f(x) dx$  為  $y = f(x)$ ,  $x$  軸,  $x = a$ , 與  $x = b$  所圍成之面積.



例. 求以  $y = 4 - x^2$ ,  $y = x$ ,  $x = -1$ , 與  $x = 1$  圍成之區域面積.

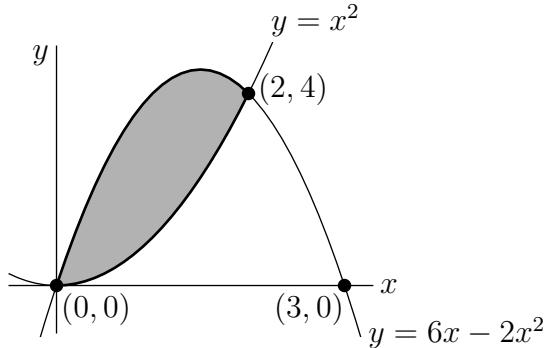
解.



$$(面積) = \int_{-1}^1 ((4 - x^2) - x) dx = \frac{22}{3}$$

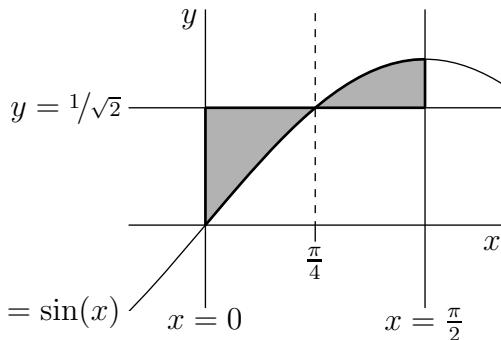
例. 求  $y = x^2$  與  $y = 6x - 2x^2$  圍成之區域面積.

解.



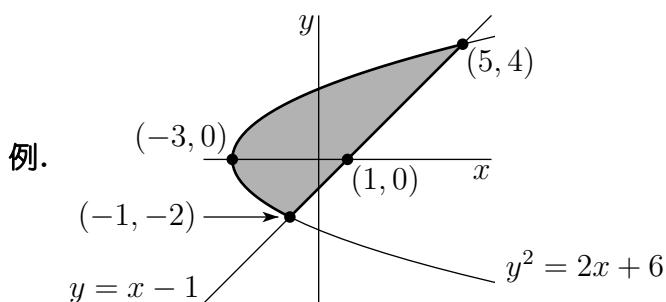
$$(面積) = \int_0^2 ((6x - 2x^2) - x^2) dx = 4$$

例. 求  $y = \frac{1}{\sqrt{2}}$  與  $y = \sin x$  在  $x$  從 0 至  $\frac{\pi}{2}$  範圍內圍成之區域面積.

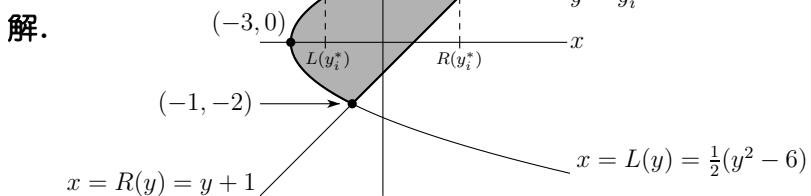


解.

$$(面積) = \int_0^{\frac{\pi}{4}} \left( \frac{1}{\sqrt{2}} - \sin x \right) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left( \sin x - \frac{1}{\sqrt{2}} \right) dx = \sqrt{2} - 1$$

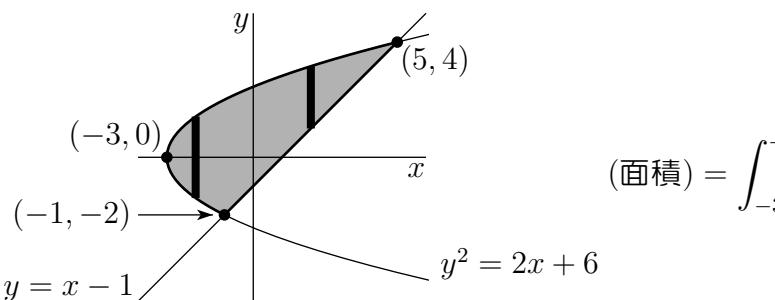


求  $y^2 = 2x + 6$  與  $y = x - 1$  圍成之區域面積.



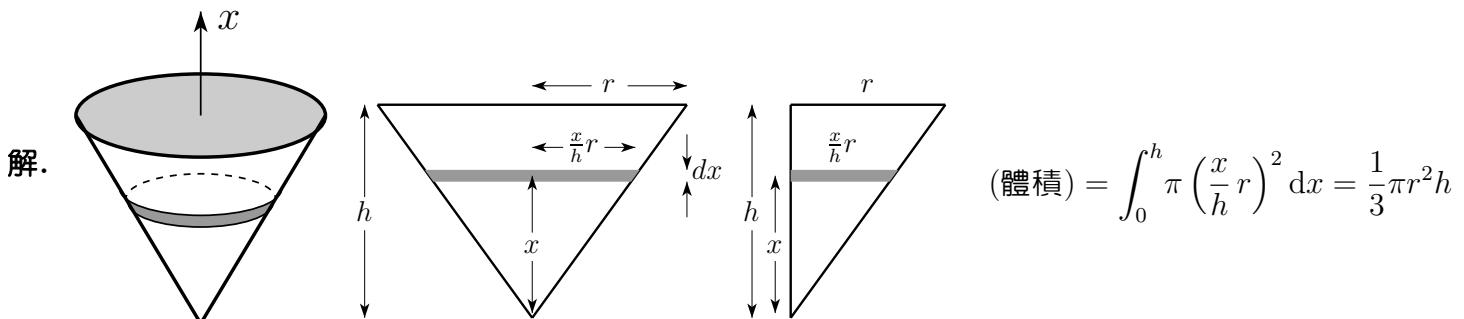
解.

$$(面積) = \int_{-2}^4 \left( (y+1) - \frac{1}{2}(y^2 - 6) \right) dy = 18$$



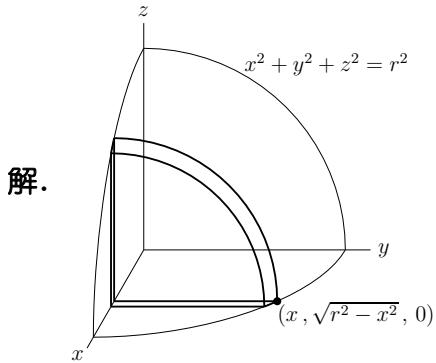
$$(面積) = \int_{-3}^{-1} 2\sqrt{2x+6} dx + \int_{-1}^5 (\sqrt{2x+6} - x + 1) dx = 18$$

例. 求高  $h$  與底半徑  $r$  之圓錐體體積.



$$(體積) = \int_0^h \pi \left( \frac{x}{h} r \right)^2 dx = \frac{1}{3} \pi r^2 h$$

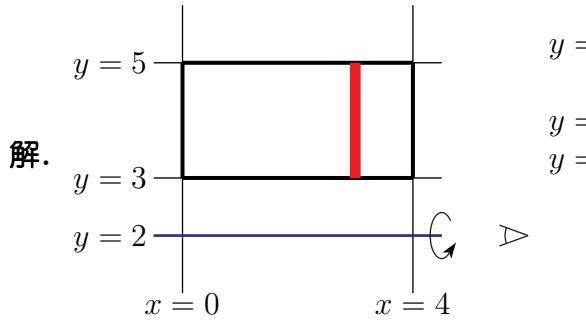
例. 求半徑  $r$  之三維球體體積.



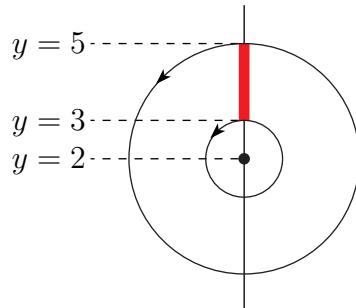
解.

$$(\text{體積}) = 8 \cdot \int_0^r \frac{\pi}{4} (\sqrt{r^2 - x^2})^2 dx = \frac{4}{3} \pi r^3$$

例. 求以  $y = 3, y = 5, x = 0$  與  $x = 4$  圍成之區域繞  $y = 2$  旋轉而成之旋轉體體積.

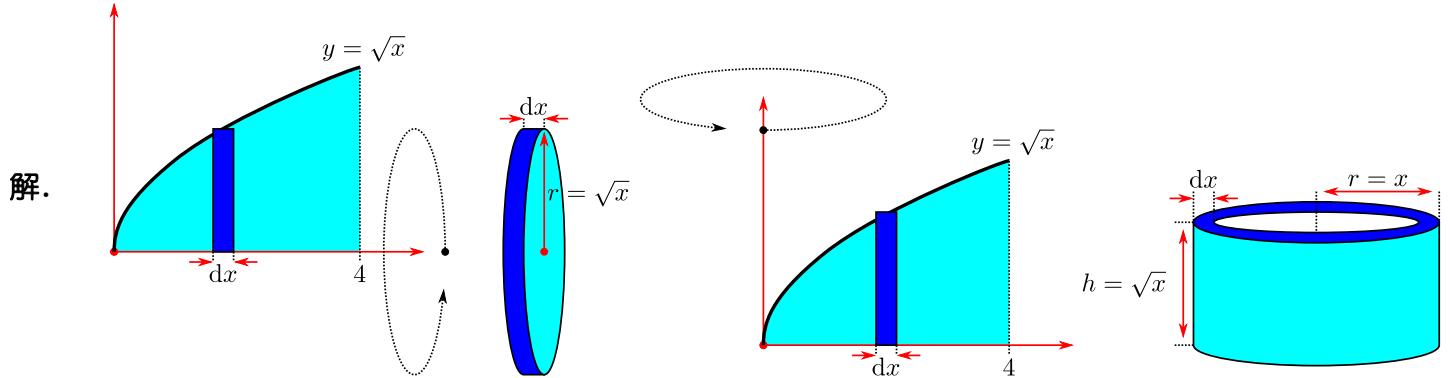


解.



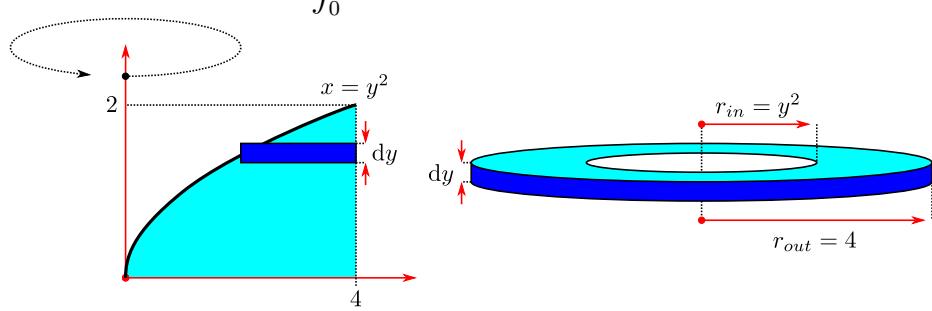
$$(\text{體積}) = \int_0^4 \pi (3^2 - 1^2) dx = 32\pi$$

例. 求以  $y = \sqrt{x}, y = 0, x = 0$  與  $x = 4$  圍成之區域繞 (i)  $y = 0$  (ii)  $x = 0$  旋轉而成之旋轉體體積.



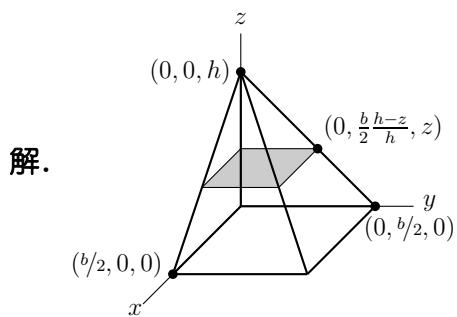
$$(\text{體積}) = \int_0^4 \pi (\sqrt{x})^2 dx = 8\pi$$

$$(\text{體積}) = \int_0^4 2\pi \cdot x \cdot \sqrt{x} dx = \frac{128\pi}{5}$$

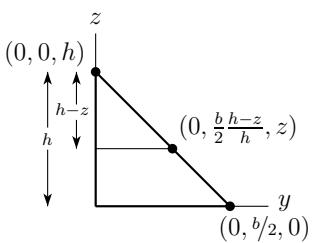


$$(\text{體積}) = \int_0^2 \pi (4^2 - (y^2)^2) dy = \frac{128\pi}{5}$$

例. 求高為  $h$ , 底面為邊長  $b$  正方形之錐體體積.

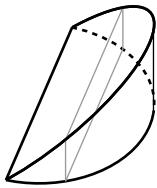


解.



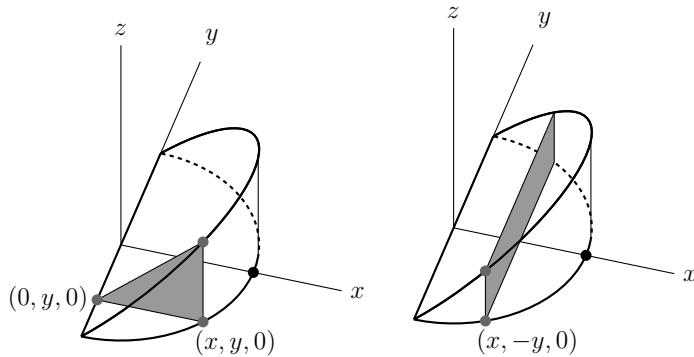
$$(\text{體積}) = \int_0^h \left(b - \frac{z}{h} b\right)^2 dz = \frac{1}{3} b^2 h$$

例.



將一半徑為  $a$  之圓柱體水平橫切, 再對其底面圓心  $45^\circ$  角斜切, 求如圖所示結果體積.

解.



$$(\text{體積}) = \int_{-a}^a \frac{1}{2} (\sqrt{a^2 - y^2})^2 dy = \frac{2}{3} a^3$$

$$(\text{體積}) = \int_0^a x \cdot 2\sqrt{a^2 - x^2} dx = \frac{2}{3} a^3$$

## 4.4 積分技巧

### 部份分式

例 (動機). 若  $a \neq 0$ , 求  $\int \frac{1}{x^2 - a^2} dx$ .

解. 由  $\frac{1}{x^2 - a^2} = \frac{1}{2a} \left( \frac{1}{x-a} - \frac{1}{x+a} \right)$ ,  $\int \frac{1}{x^2 - a^2} dx = \int \frac{1}{2a} \left( \frac{1}{x-a} - \frac{1}{x+a} \right) dx = \frac{1}{2a} (\ln|x-a| - \ln|x+a|) = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right|$

結論.

- 實係數多項式可分解成不可約的一次及二次因式的乘積.
- 有理式可寫成多項式與真分式之和.
- 若  $\frac{p(x)}{q(x)}$  為一真分式,  $q(x) = (x+a_1)^{m_1}(x+a_2)^{m_2} \cdots (x+a_k)^{m_k} \cdot (x^2+b_1x+c_1)^{n_1}(x^2+b_2x+c_2)^{n_2} \cdots (x^2+b_lx+c_l)^{n_l}$ , 其中  $(x+a_i)$ ,  $(x^2+b_ix+c_i)$  均相異,  $(x^2+b_ix+c_i)$  為不可分解之二次式 ( $b_i^2 - 4c_i < 0$ ), 則

$$\begin{aligned} \frac{p(x)}{q(x)} &= \frac{\alpha_{11}}{x+a_1} + \frac{\alpha_{12}}{(x+a_1)^2} + \cdots + \frac{\alpha_{1m_1}}{(x+a_1)^{m_1}} + \\ &\quad \frac{\alpha_{21}}{x+a_2} + \frac{\alpha_{22}}{(x+a_2)^2} + \cdots + \frac{\alpha_{2m_2}}{(x+a_2)^{m_2}} + \cdots + \\ &\quad \frac{\alpha_{k1}}{x+a_k} + \frac{\alpha_{k2}}{(x+a_k)^2} + \cdots + \frac{\alpha_{km_k}}{(x+a_k)^{m_k}} + \\ &\quad \frac{\beta_{11}x + \gamma_{11}}{x^2 + b_1x + c_1} + \frac{\beta_{12}x + \gamma_{12}}{(x^2 + b_1x + c_1)^2} + \cdots + \frac{\beta_{1n_1}x + \gamma_{1n_1}}{(x^2 + b_1x + c_1)^{n_1}} + \\ &\quad \frac{\beta_{21}x + \gamma_{21}}{x^2 + b_2x + c_2} + \frac{\beta_{22}x + \gamma_{22}}{(x^2 + b_2x + c_2)^2} + \cdots + \frac{\beta_{2n_2}x + \gamma_{2n_2}}{(x^2 + b_2x + c_2)^{n_2}} + \cdots + \\ &\quad \frac{\beta_{l1}x + \gamma_{l1}}{x^2 + b_lx + c_l} + \frac{\beta_{l2}x + \gamma_{l2}}{(x^2 + b_lx + c_l)^2} + \cdots + \frac{\beta_{ln_l}x + \gamma_{ln_l}}{(x^2 + b_lx + c_l)^{n_l}} \end{aligned}$$

其中  $\alpha_{ij}, \beta_{ij}, \gamma_{ij} \in \mathbb{R}$ .

註. 任一有理函數之積分可分解為多項式積分與以下兩型積分:

$$\bullet \int \frac{1}{(x+a)^n} dx, n \in \mathbb{N}$$

$$\bullet \int \frac{f(x)}{(x^2+bx+c)^n} dx, f(x) = 1 \text{ 或 } f(x) = x; b^2 - 4c < 0, n \in \mathbb{N}$$

例. 求  $\int \frac{x}{x^2 - 5x + 6} dx$ .

解.  $\frac{x}{x^2 - 5x + 6} = \frac{x}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} \Rightarrow x = A(x-2) + B(x-3)$ . 代入  $x=3 \Rightarrow 3 = A$ ; 代入  $x=2 \Rightarrow 2 = B(2-3) \Rightarrow B = -2$ , 故  $\frac{x}{x^2 - 5x + 6} = \frac{3}{x-3} - \frac{2}{x-2}$ ,  $\int \frac{x}{x^2 - 5x + 6} dx = \int \left( \frac{3}{x-3} - \frac{2}{x-2} \right) dx = 3 \ln|x-3| - 2 \ln|x-2|$

例. 求  $\int \frac{1}{x^3 + 1} dx$ .

解.  $\frac{1}{x^3 + 1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \Rightarrow 1 = A(x^2-x+1) + (Bx+C)(x+1)$ . 代入  $x=-1 \Rightarrow 1 = A((-1)^2 - (-1) + 1) = 3A \Rightarrow A = \frac{1}{3}$ ; 代入  $x=0 \Rightarrow 1 = A + C \Rightarrow C = \frac{2}{3}$ ; 代入  $x=1 \Rightarrow 1 = A + (B+C) \cdot 2 \Rightarrow 1 = \frac{1}{3} + \left(B + \frac{2}{3}\right) \cdot 2 \Rightarrow B = -\frac{1}{3}$ ; 故  $\frac{1}{x^3 + 1} = \frac{1}{3} \left(\frac{1}{x+1} - \frac{x-2}{x^2-x+1}\right)$ ,  $\int \frac{1}{x^3 + 1} dx = \int \frac{1}{3} \left(\frac{1}{x+1} - \frac{x-2}{x^2-x+1}\right) dx = \int \frac{1}{3} \left(\frac{1}{x+1} - \frac{\frac{1}{2}(2x-1) - \frac{3}{2}}{x^2-x+1}\right) dx = \frac{1}{3} \int \frac{1}{x+1} dx - \frac{1}{6} \int \frac{2x-1}{x^2-x+1} dx - \frac{1}{2} \int \frac{1}{x^2-x+1} dx = \frac{1}{3} \ln|x+1| - \frac{1}{6} \ln(x^2-x+1) - \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{3}}$

例. 求  $\int \frac{1}{x^4 + 4} dx$ .

解. 由  $x^4 + 4 = (x^2 + 2)^2 - (2x)^2 = (x^2 - 2x + 2)(x^2 + 2x + 2)$ ,  $\frac{1}{x^4 + 4} = \frac{Ax+B}{x^2 - 2x + 2} + \frac{Cx+D}{x^2 + 2x + 2} \Rightarrow (Ax+B)(x^2+2x+2) + (Cx+D)(x^2-2x+2) = 1 \Rightarrow (A+C)x^3 + (2A+B-2C+D)x^2 + (2A+2B+2C-2D)x + (2B+2D) = 1 \Rightarrow A+C = 0, 2A+B-2C+D = 0, 2A+2B+2C-2D = 0, 2B+2D = 1 \Rightarrow A = -\frac{1}{8}, B = \frac{1}{4}, C = \frac{1}{8}, D = \frac{1}{4}$ . 故  $\frac{1}{x^4 + 4} = \frac{1}{8} \left( \frac{x+2}{x^2+2x+2} - \frac{x-2}{x^2-2x+2} \right)$ ,  $\int \frac{1}{x^4 + 4} dx = \frac{1}{8} \int \left( \frac{x+2}{x^2+2x+2} - \frac{x-2}{x^2-2x+2} \right) dx = \frac{1}{8} \int \left( \frac{(x+1)+1}{(x+1)^2+1} - \frac{(x-1)-1}{(x-1)^2+1} \right) dx = \frac{1}{16} \ln \frac{x^2+2x+2}{x^2-2x+2} + \frac{1}{8} (\tan^{-1}(x+1) - \tan^{-1}(x-1))$

例. 求  $\int \frac{1}{\cos^3 x} dx$ .

解.  $\int \frac{1}{\cos^3 x} dx = \int \frac{\cos x}{\cos^4 x} dx = \int \frac{\cos x}{(1-\sin^2 x)^2} dx$ . 令  $u = \sin x$ , 则  $du = \cos x dx$ , 则  $\int \frac{\cos x}{(1-\sin^2 x)^2} dx = \int \frac{1}{(1-u^2)^2} du$ .  $\frac{1}{(1-u^2)^2} = \frac{1}{(u-1)^2(u+1)^2} = \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{C}{u+1} + \frac{D}{(u+1)^2} \Rightarrow 1 = A(u-1)(u+1)^2 + B(u+1)^2 + C(u-1)^2(u+1) + D(u-1)^2$ . 代入  $u=1 \Rightarrow 1 = 4B \Rightarrow B = \frac{1}{4}$ ; 代入  $u=-1 \Rightarrow 1 = 4D \Rightarrow D = \frac{1}{4}$ ; 代入  $u=0 \Rightarrow 1 = -A + B + C + D \Rightarrow 1 = -A + \frac{1}{4} + C + \frac{1}{4} \Rightarrow \frac{1}{2} = -A + C$ ; 代入  $u=2 \Rightarrow 1 = 9A + 9B + 3C + D \Rightarrow 1 = 9A + \frac{9}{4} + 3C + \frac{1}{4} \Rightarrow A = -\frac{1}{4}, C = \frac{1}{4}$ ; 则  $\frac{1}{(1-u^2)^2} = \frac{1}{4} \left( \frac{-1}{u-1} + \frac{1}{(u-1)^2} + \frac{1}{u+1} + \frac{1}{(u+1)^2} \right)$ ,  $\int \frac{1}{(1-u^2)^2} du = \int \frac{1}{4} \left( \frac{-1}{u-1} + \frac{1}{(u-1)^2} + \frac{1}{u+1} + \frac{1}{(u+1)^2} \right) dx = \frac{1}{4} \left( -\ln|u-1| - \frac{1}{u-1} + \ln|u+1| - \frac{1}{u+1} \right) = \frac{1}{4} \left( -\ln|\sin x - 1| - \frac{1}{\sin x - 1} + \ln|\sin x + 1| - \frac{1}{\sin x + 1} \right)$ .

$$\text{部份分式另解: } \frac{1}{(1-u^2)^2} = \left(\frac{1}{u^2-1}\right)^2 = \left(\frac{1}{(u-1)(u+1)}\right)^2 = \frac{1}{4} \left(\frac{1}{u-1} - \frac{1}{u+1}\right)^2 = \frac{1}{4} \left(\frac{1}{(u-1)^2} - \frac{2}{(u-1)(u+1)} + \frac{1}{(u+1)^2}\right) = \frac{1}{4} \left(\frac{1}{(u-1)^2} - \left(\frac{1}{u-1} - \frac{1}{u+1}\right) + \frac{1}{(u+1)^2}\right) = \frac{1}{4} \left(\frac{-1}{u-1} + \frac{1}{(u-1)^2} + \frac{1}{u+1} + \frac{1}{(u+1)^2}\right).$$

## 三角函數代換

結論.

- 遇  $\sqrt{a^2 - x^2}$ , 考慮  $x = a \sin \theta \implies \theta = \sin^{-1} \frac{x}{a}$ ,  $dx = a \cos \theta d\theta$
- 遇  $\sqrt{a^2 + x^2}$ , 考慮  $x = a \tan \theta \implies \theta = \tan^{-1} \frac{x}{a}$ ,  $dx = a \sec^2 \theta d\theta$
- 遇  $\sqrt{x^2 - a^2}$ , 考慮  $x = a \sec \theta \implies \theta = \sec^{-1} \frac{x}{a}$ ,  $dx = a \sec \theta \tan \theta d\theta$
- 遇  $\sin x, \cos x$  之有理式, 考慮  $u = \tan \frac{x}{2}$ , 由以下化為  $u$  之有理式:

$$\begin{aligned} - \sin x &= 2 \sin \frac{x}{2} \cos \frac{x}{2} = 2 \frac{u}{\sqrt{1+u^2}} \frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2} \\ - \cos x &= 2 \cos^2 \frac{x}{2} - 1 = 2 \cdot \frac{1}{1+u^2} - 1 = \frac{1-u^2}{1+u^2} \\ - du &= \frac{1}{2} \sec^2 \frac{x}{2} dx \implies dx = \frac{2}{1+u^2} du \end{aligned}$$

例. 若  $a \neq 0$ , 求下列不定積分.

1. $\int \sqrt{a^2 - x^2} dx$	4. $\int \frac{1}{\sqrt{x^2 + a^2}} dx$	7. $\int \frac{1}{\sqrt{x^2 - a^2}} dx$
2. $\int \sqrt{x^2 + a^2} dx$	5. $\int \frac{1}{x^2 \sqrt{x^2 + a^2}} dx$	8. $\int \frac{1}{\tan x + \sin x} dx$
3. $\int x^2 \sqrt{x^2 + a^2} dx$	6. $\int \sqrt{x^2 - a^2} dx$	9. $\int \frac{1}{a + \sin x} dx, a > 1$

解.

- 令  $x = a \sin \theta$ , 則  $\int \sqrt{a^2 - x^2} dx = \int \sqrt{a^2 - a^2 \sin^2 \theta} a \cos \theta d\theta = a^2 \int \cos^2 \theta d\theta = a^2 \int \frac{1 + \cos 2\theta}{2} d\theta = \frac{a^2}{2} \theta + \frac{a^2}{4} \sin 2\theta = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{a^2}{2} \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a} = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2}.$
- 令  $x = a \tan \theta$ , 則  $\int \sqrt{x^2 + a^2} dx = \int a \sec \theta \cdot a \sec^2 \theta d\theta = a^2 \int \sec^3 \theta d\theta = \frac{a^2}{2} (\sec \theta \cdot \tan \theta + \ln |\sec \theta + \tan \theta|) = \frac{a^2}{2} \left( \frac{x}{a} \cdot \frac{\sqrt{x^2 + a^2}}{a} + \ln \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| \right) = \frac{x \sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \ln \left| \sqrt{x^2 + a^2} + x \right| - \frac{a^2}{2} \ln |a|$
- 令  $x = a \tan \theta$ , 則  $\int x^2 \sqrt{x^2 + a^2} dx = \int a \sec^2 \theta \cdot a^2 \tan^2 \theta \cdot a \sec \theta d\theta = a^4 \int \sec^3 \theta \tan^2 \theta d\theta = a^4 \int \sec^3 \theta (\sec^2 \theta - 1) d\theta = a^4 \int (\sec^5 \theta - \sec^3 \theta) d\theta = \frac{a^4}{4} \left( \sec^3 \theta \tan \theta - \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \right) = \frac{a^4}{4} \left( \sec^3 \theta \tan \theta - \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \right) = \frac{x(x^2 + a^2)^{\frac{3}{2}}}{4} - \frac{a^2 x \sqrt{x^2 + a^2}}{8} - \frac{a^4 \ln |\sqrt{x^2 + a^2} + x|}{8} + \frac{a^4 \ln |a|}{8}.$

$$4. \text{ 令 } x = a \tan \theta, \text{ 則 } \int \frac{1}{\sqrt{x^2 + a^2}} dx = \int \frac{1}{a \sec \theta} \cdot a \sec^2 \theta d\theta = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| \\ = \ln \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| = \ln |\sqrt{x^2 + a^2} + x| - \ln |a|.$$

$$5. \text{ 令 } x = a \tan \theta, \text{ 則 } \int \frac{1}{x^2 \sqrt{x^2 + a^2}} dx = \int \frac{a \sec^2 \theta}{a^2 \tan^2 \theta \cdot a \sec \theta} d\theta = \int \frac{\sec \theta}{a^2 \tan^2 \theta} d\theta = \int \frac{\cos \theta}{a^2 \sin^2 \theta} d\theta \\ = -\frac{1}{a^2 \sin \theta} = -\frac{\sqrt{x^2 + a^2}}{a^2 x}.$$

$$6. \text{ 令 } x = a \sec \theta, \text{ 則 } \int \sqrt{x^2 - a^2} dx = \int a \tan \theta \cdot a \sec \theta \tan \theta d\theta = a^2 \int \sec \theta \tan^2 \theta d\theta = a^2 \int \sec \theta (\sec^2 \theta - 1) d\theta \\ = a^2 \left( \int \sec^3 \theta d\theta - \int \sec \theta d\theta \right) = \frac{a^2}{2} \left( \sec \theta \cdot \tan \theta + \int \sec \theta d\theta - 2 \int \sec \theta d\theta \right) = \frac{a^2}{2} \left( \sec \theta \cdot \tan \theta - \int \sec \theta d\theta \right) = \frac{a^2}{2} (\sec \theta \cdot \tan \theta - \ln |\sec \theta + \tan \theta|) = \frac{a^2}{2} \left( \frac{x}{a} \cdot \frac{\sqrt{x^2 - a^2}}{a} - \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| \right) = \\ \frac{x \sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| = \frac{x \sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \ln |\sqrt{x^2 - a^2} + x| + \frac{a^2}{2} \ln |a|.$$

$$7. \text{ 令 } x = a \sec \theta, \text{ 則 } \int \frac{1}{\sqrt{x^2 - a^2}} dx = \int \frac{a \sec \theta \tan \theta}{a \tan \theta} d\theta = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| \\ = \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| = \ln |\sqrt{x^2 - a^2} + x| - \ln |a|.$$

$$8. \text{ 令 } u = \tan \frac{x}{2}, \text{ 則 } \int \frac{1}{\tan x + \sin x} dx = \int \frac{1}{\frac{2u}{1-u^2} + \frac{2u}{1+u^2}} \frac{2}{1+u^2} du = \int \frac{1-u^2}{2u} du = \frac{\ln |u|}{2} - \frac{u^2}{4} = \\ \frac{\ln |\tan \frac{x}{2}|}{2} - \frac{\tan^2 \frac{x}{2}}{4}$$

$$9. \text{ 令 } u = \tan \frac{x}{2}, \text{ 則 } \int \frac{1}{a + \sin x} dx = \int \frac{1}{a + \frac{2u}{1+u^2}} \frac{2}{1+u^2} du = 2 \int \frac{1}{au^2 + 2u + a} du = \frac{2}{a \sqrt{\frac{a^2-1}{a^2}}} \tan^{-1} \frac{u + \frac{1}{a}}{\sqrt{\frac{a^2-1}{a^2}}} \\ = \frac{2}{\sqrt{a^2-1}} \tan^{-1} \frac{au+1}{\sqrt{a^2-1}} = \frac{2}{\sqrt{a^2-1}} \tan^{-1} \frac{a \tan \frac{x}{2} + 1}{\sqrt{a^2-1}}.$$

## 4.5 瑕積分

**定義** (瑕積分 (improper integral) ).

- 無限區間 (第一型) 瑕積分

- 若  $f(x)$  在  $[a, \infty)$  連續, 則  $\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx.$
- 若  $f(x)$  在  $(-\infty, b]$  連續, 則  $\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx.$
- 若  $f(x)$  在  $(-\infty, \infty)$  連續, 則任取  $c \in \mathbb{R}$ ,  $\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^\infty f(x) dx.$

- 不連續點 (第二型) 瑕積分

- 若  $f(x)$  在  $(a, b]$  連續, 則  $\int_a^b f(x) dx = \lim_{c \rightarrow a+} \int_c^b f(x) dx.$
- 若  $f(x)$  在  $[a, b)$  連續, 則  $\int_a^b f(x) dx = \lim_{c \rightarrow b-} \int_a^c f(x) dx.$

- 令  $c \in (a, b)$ . 若  $f(x)$  在  $[a, c) \cup (c, b]$  連續且在  $x = c$  不連續, 則  $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$ .

例. 1.  $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx.$       2.  $\int_1^{\infty} \frac{1}{x^2} dx.$       3.  $\int_0^1 \frac{1}{\sqrt{x}} dx.$

解.

$$1. \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = 2 \int_0^{\infty} \frac{1}{1+x^2} dx = 2 \lim_{b \rightarrow \infty} \int_0^b \frac{1}{1+x^2} dx = 2 \lim_{b \rightarrow \infty} \tan^{-1} b = 2 \cdot \frac{\pi}{2} = \pi.$$

$$2. \int_1^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \left( -\frac{1}{b} + 1 \right) = 1$$

$$3. \int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{c \rightarrow 0+} \int_c^1 \frac{1}{\sqrt{x}} dx = \lim_{c \rightarrow 0+} 2\sqrt{x} \Big|_c^1 = \lim_{c \rightarrow 0+} (2 - 2\sqrt{c}) = 2$$

例. 證明  $\forall n \in \mathbb{N}, \int_0^1 (\ln x)^n dx = (-1)^n n!$

解. 使用數學歸納法:  $n = 1$  時  $\int_0^1 \ln x dx = (x \ln x - x) \Big|_0^1 = -1 - \lim_{x \rightarrow 0+} x \ln x = -1 + \lim_{x \rightarrow 0+} \frac{\ln \frac{1}{x}}{\frac{1}{x}} = -1 + \lim_{y \rightarrow \infty} \frac{\ln y}{y} = -1 = (-1)^1 1!$ . 令等式在  $n - 1$  成立:  $\int_0^1 (\ln x)^{n-1} dx = (-1)^{n-1} (n-1)!$ , 則  $\int_0^1 (\ln x)^n dx = x (\ln x)^n \Big|_0^1 - n \int_0^1 x \cdot (\ln x)^{n-1} \cdot \frac{1}{x} dx = 0 - \lim_{x \rightarrow 0+} x (\ln x)^n + (-1) \cdot n \cdot (-1)^{n-1} (n-1)! = - \lim_{x \rightarrow 0+} x (\ln x)^n + (-1)^n n!$

反覆使用 L'Hôpital 法則得  $\lim_{x \rightarrow 0+} x (\ln x)^n = (-1)^n \cdot \lim_{x \rightarrow 0+} \frac{(\ln \frac{1}{x})^n}{\frac{1}{x}} = (-1)^n \lim_{y \rightarrow \infty} \frac{(\ln y)^n}{y} = (-1)^n \cdot n \lim_{y \rightarrow \infty} \frac{(\ln y)^{n-1}}{y} = (-1)^n \cdot n(n-1) \lim_{y \rightarrow \infty} \frac{(\ln y)^{n-2}}{y} = \dots = (-1)^n n! \lim_{y \rightarrow \infty} \frac{1}{y} = 0$ , 故  $\int_0^1 (\ln x)^n dx = (-1)^n n!$  成立.

定理. 給定  $0 < a < \infty$ .

- $\int_a^{\infty} \frac{1}{x^p} dx$  當  $p > 1$  收斂至  $\frac{a^{1-p}}{p-1}$ , 當  $p \leq 1$  發散至  $\infty$ .

- $\int_0^a \frac{1}{x^p} dx$  當  $p < 1$  收斂至  $\frac{a^{1-p}}{1-p}$ , 當  $p \geq 1$  發散至  $\infty$ .

定理. 令  $-\infty \leq a < b \leq \infty$ ,  $f, g$  在  $(a, b)$  連續, 且  $0 \leq f(x) \leq g(x) \forall x$ .

- 若  $\int_a^b g(x) dx$  收斂,  $\int_a^b f(x) dx$  收斂.
- 若  $\int_a^b f(x) dx$  發散,  $\int_a^b g(x) dx$  發散.

定理. 令  $f, g$  在  $[a, \infty)$ ,  $a \in \mathbb{R}$  連續, 均為正值, 且  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$  存在, 則  $\int_a^{\infty} f(x) dx$  與  $\int_a^{\infty} g(x) dx$  同斂散.

例. 證明  $\int_0^{\infty} e^{-x^2} dx$  收斂.

解. 由  $e^{-x^2} \leq 1 \forall 0 \leq x < 1$  及  $e^{-x^2} \leq e^{-x} \forall x \geq 1$ ,  $\int_0^{\infty} e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^{\infty} e^{-x^2} dx \leq \int_0^1 1 dx + \int_1^{\infty} e^{-x} dx = 1 + \frac{1}{e}$ , 故  $\int_0^{\infty} e^{-x^2} dx$  收斂.

例. 定義  $\Gamma$  函數  $\Gamma(x) \equiv \int_0^{\infty} t^{x-1} e^{-t} dt$ ; 已知  $\int_0^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$ .

1. 證明  $\Gamma(x)$  收斂,  $\forall x > 0$ .
2. 證明  $\Gamma(x+1) = x\Gamma(x)$ ,  $\forall x > 0$ .

3. 證明  $\Gamma(n+1) = n!$ ,  $\forall n \in \mathbb{N}$ .
4. 求  $\Gamma\left(\frac{1}{2}\right)$  與  $\Gamma\left(\frac{3}{2}\right)$ .

解.

1.  $\int_0^\infty t^{x-1}e^{-t} dt = \int_0^1 t^{x-1}e^{-t} dt + \int_1^\infty t^{x-1}e^{-t} dt$ .  $\int_1^\infty t^{x-1}e^{-t} dt$  收斂, 因為

- $\int_0^1 t^{x-1}e^{-t} dt \leq \int_0^1 t^{x-1} dt = \frac{t^x}{x} \Big|_{t=0}^{t=1} = \frac{1}{x}$ ,  $\int_0^1 t^{x-1}e^{-t} dt$  收斂.
- $\int_1^\infty \frac{1}{t^2} dt$  收斂,  $\lim_{t \rightarrow \infty} \frac{t^{x-1}e^{-t}}{\frac{1}{t^2}} = \lim_{t \rightarrow \infty} \frac{t^{x+1}}{e^t} = 0$ ,  $\int_1^\infty t^{x-1}e^{-t} dt$  收斂.

2.  $\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = \lim_{\substack{a \rightarrow 0+ \\ b \rightarrow \infty}} \int_a^b t^x e^{-t} dt$ . 令  $u = t^x$ , 則  $du = xt^{x-1} dt$ . 令  $dv = e^{-t} dt$ , 則  $v = -e^{-t}$ .  
故  $\lim_{\substack{a \rightarrow 0+ \\ b \rightarrow \infty}} \int_a^b t^x e^{-t} dt = \lim_{\substack{a \rightarrow 0+ \\ b \rightarrow \infty}} \left( -t^x e^{-t} \Big|_a^b + \int_a^b e^{-t} \cdot xt^{x-1} dt \right) = 0 + x \int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x)$ .

3. 由上  $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-2) = \dots = n(n-1)\dots 2\Gamma(1)$ , 又  $\Gamma(1) = \lim_{b \rightarrow \infty} \int_0^b e^{-t} dt = \lim_{b \rightarrow \infty} -e^{-t} \Big|_0^b = \lim_{b \rightarrow \infty} 1 - e^{-b} = 1$ , 得證.

4. (a)  $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt = \int_0^\infty \frac{e^{-t}}{\sqrt{t}} dt$ . 令  $u = \sqrt{t}$ , 則  $t = u^2$ ,  $du = \frac{1}{2\sqrt{t}} dt \implies \frac{1}{\sqrt{t}} dt = 2 du$ . 積分範圍  $t$  由 0 至  $\infty$ , 則變數變換後  $u$  由  $\sqrt{0} = 0$  至  $\sqrt{\infty} = \infty$ , 故  $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}} dt = 2 \int_0^\infty e^{-u^2} du = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$ .

- (b) 由  $\Gamma(x+1) = x\Gamma(x)$ ,  $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2} + 1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$ .

## 4.6 Leibniz 積分法則: 積分號下取微分

**定理** (Leibniz 積分法則). 設  $(X, \mathcal{A}, \mu)$  為測度空間,  $U \subseteq \mathbb{R}$  為開區間, 函數  $f : X \times U \rightarrow \mathbb{R}$  (或  $\mathbb{C}$ ) 滿足

- (i) **可測性:** 對每個  $t \in U$ , 函數  $x \mapsto f(x, t)$  是  $\mathcal{A}$ -可測的.
- (ii) **可微性:** 對  $\mu$ -幾乎所有  $x \in X$ , 函數  $t \mapsto f(x, t)$  在  $U$  上可微.
- (iii) **控制條件:** 存在函數  $g \in L^1(\mu)$  使得對所有  $t \in U$  及  $\mu$ -幾乎所有  $x \in X$ ,  $\left| \frac{\partial f}{\partial t}(x, t) \right| \leq g(x)$ .

則

- (a) 對每個  $t \in U$ , 函數  $x \mapsto f(x, t)$  屬於  $L^1(\mu)$ .
- (b) 函數  $F(t) := \int_X f(x, t) d\mu(x)$  在  $U$  上可微.
- (c) 微分與積分可交換次序:  $\frac{dF}{dt}(t) = \int_X \frac{\partial f}{\partial t}(x, t) d\mu(x)$ .

## 重要特例

**系理** (Lebesgue 測度下的經典形式). 設  $U, V \subseteq \mathbb{R}$  為開區間,  $f : U \times V \rightarrow \mathbb{R}$  滿足

- (i)  $f$  與  $\frac{\partial f}{\partial t}$  在  $U \times V$  上連續;
- (ii) 存在可積函數  $g : U \rightarrow [0, \infty)$  使得  $\left| \frac{\partial f}{\partial t}(x, t) \right| \leq g(x)$  對所有  $(x, t) \in U \times V$  成立.

則  $F(t) = \int_U f(x, t) dx$  在  $V$  上可微, 且  $F'(t) = \int_U \frac{\partial f}{\partial t}(x, t) dx$ .

**系理** (緊支撐或有界區間情形). 設  $[a, b] \subset \mathbb{R}$  為有界閉區間,  $V \subseteq \mathbb{R}$  為開區間,  $f : [a, b] \times V \rightarrow \mathbb{R}$  滿足

- (i)  $f$  對  $x$  可積, 對  $t$  可微; (ii)  $\frac{\partial f}{\partial t}$  在  $[a, b] \times V$  上連續.

則  $\frac{d}{dt} \int_a^b f(x, t) dx = \int_a^b \frac{\partial f}{\partial t}(x, t) dx$ .

**證.** 連續函數在緊集  $[a, b] \times K$  ( $K \subset V$  為任意緊子區間) 上有界, 故控制條件自動滿足.

## 應用範例

**例.**  $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ .

**證.** 令  $F(t) = \int_0^\infty \frac{e^{-t^2(1+x^2)}}{1+x^2} dx, t \geq 0$ ;  $F(\infty) = 0, F(0) = \int_0^\infty \frac{dx}{1+x^2} = \tan^{-1} x \Big|_0^\infty = \tan^{-1} \infty - \tan^{-1} 0 = \frac{\pi}{2}$ .  
 則  $F'(t) = \int_0^\infty \frac{-2t(1+x^2)e^{-t^2(1+x^2)}}{1+x^2} dx = \underbrace{-2te^{-t^2} \int_0^\infty e^{-t^2x^2} dx}_{\text{令 } u=tx, \text{ 則 } du=t dx} = -2e^{-t^2} \int_0^\infty e^{-u^2} du = -2e^{-t^2} I \Rightarrow$

$\int_0^b F'(t) dt = -2I \int_0^b e^{-t^2} dt \Rightarrow F(b) - F(0) = -2I \int_0^b e^{-t^2} dt$ . 令  $b \rightarrow \infty$ , 則  $F(\infty) - F(0) = -2I^2 \Rightarrow 0 - \frac{\pi}{2} = -2I^2 \Rightarrow I = \frac{\sqrt{\pi}}{2}$ .

**例.**  $\int_0^\infty x^{2n} e^{-ax^2} dx = \frac{(2n-1)!!}{2^{n+1}} \sqrt{\frac{\pi}{a^{2n+1}}} = \frac{(2n)!}{n! 4^n} \sqrt{\frac{\pi}{a^{2n+1}}}, \forall a > 0, n \in \mathbb{N}$ , 其中  $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$ .

**證.** 令  $F(a) = \int_0^\infty e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}}$ , 則  $F'(a) = - \int_0^\infty x^2 e^{-ax^2} dx = -\frac{1}{2} \sqrt{\pi} \cdot (-\frac{1}{2}) a^{-\frac{3}{2}} = \frac{\sqrt{\pi}}{4a^{\frac{3}{2}}} \Rightarrow$   
 $\int_0^\infty x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{4a^{\frac{3}{2}}}, F''(a) = \int_0^\infty x^4 e^{-ax^2} dx = \frac{3\sqrt{\pi}}{8a^{\frac{5}{2}}}$ ; 可歸納出通式為  $(-1)^n F^{(n)}(a) = \int_0^\infty x^{2n} e^{-ax^2} dx$ . 由  
 $\frac{d^n}{da^n} (a^{-\frac{1}{2}}) = (-1)^n \frac{(2n-1)!!}{2^n} a^{-\frac{2n+1}{2}}$ , 得證.

**例.**  $\int_0^\infty e^{-ax^2} \cos bx dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}}, \forall a > 0$ .

**證.** 令  $F(t) = \int_0^\infty e^{-ax^2} \cos tx dx$ , 則  $F(0) = \frac{1}{2} \sqrt{\frac{\pi}{a}}$ ;  $F'(t) = - \int_0^\infty x e^{-ax^2} \sin tx dx$ . 令  $u = \sin tx$ , 則  $du = t \cos tx dx$ ;  $dv = -xe^{-ax^2} dx$ , 則  $v = \frac{1}{2a} e^{-ax^2}$ ; 故  $F'(t) = \sin tx \cdot \frac{1}{2a} e^{-ax^2} \Big|_{x=0}^{x=\infty} - \frac{t}{2a} \int_0^\infty e^{-ax^2} \cos tx dx = -\frac{t}{2a} F(t) \Rightarrow \int_0^b \frac{F'(t)}{F(t)} dt = -\frac{1}{2a} \int_0^b t dt \Rightarrow \ln F(t) \Big|_0^b = -\frac{b^2}{4a} \Rightarrow F(b) = F(0) e^{-\frac{b^2}{4a}} = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}}$ .

例.  $\int_0^\infty xe^{-ax^2} \sin bx dx = \frac{b}{4a} \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}}, \forall a > 0.$

證. 令  $F(b) = \int_0^\infty e^{-ax^2} \cos bx dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}}$ , 則  $F'(b) = - \int_0^\infty xe^{-ax^2} \sin bx dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \cdot e^{-\frac{b^2}{4a}} \cdot \left( -\frac{b}{2a} \right) = -\frac{b}{4a} \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}}.$

例.  $\int_0^\infty e^{-ax^2 - \frac{b}{x^2}} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}, \forall a, b > 0.$

證. 令  $F(t) = \int_0^\infty e^{-ax^2 - \frac{t}{x^2}} dx, t > 0$ , 則  $F(0) = \frac{1}{2} \sqrt{\frac{\pi}{a}}$ ;  $F'(t) = - \int_0^\infty \frac{e^{-ax^2 - \frac{t}{x^2}}}{x^2} dx$ . 變數變換  $u = \sqrt{\frac{t}{a}} \frac{1}{x}$ , 則  $x = \sqrt{\frac{t}{a}} \frac{1}{u}, -\sqrt{\frac{a}{t}} du = \frac{dx}{x^2}$ , 積分範圍  $x$  從  $0+$  到  $\infty$ , 則  $u = \sqrt{\frac{t}{a}} \frac{1}{x}$  從  $\infty$  到  $0$ ,  $F'(t) = - \int_0^\infty \frac{e^{-ax^2 - \frac{t}{x^2}}}{x^2} dx = - \int_0^0 e^{-a \cdot \frac{t}{a} \cdot \frac{1}{u^2} - t \cdot \frac{a}{t} \cdot u^2} (-) \sqrt{\frac{a}{t}} du = -\sqrt{\frac{a}{t}} \int_0^\infty e^{-\frac{t}{u^2} - au^2} du = -\sqrt{\frac{a}{t}} F(t) \Rightarrow \int_0^b \frac{F'(t)}{F(t)} dt = -\sqrt{a} \int_0^b \frac{1}{\sqrt{t}} dt \Rightarrow \ln F(t) \Big|_0^b = -2\sqrt{ab} \Rightarrow F(b) = F(0) e^{-2\sqrt{ab}} = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}.$

定理 (Glasser's Master Theorem (GMT)). 若  $f$  在  $(-\infty, \infty)$  可積,  $\forall c > 0$ ,  $\int_{-\infty}^\infty f(x - \frac{c}{x}) dx = \int_{-\infty}^\infty f(x) dx$ .

證. 變數變換  $u = x - \frac{c}{x}$ , 則  $x^2 - ux - c = 0 \Rightarrow x = \frac{u + \sqrt{u^2 + 4c}}{2} > 0$  或  $x = \frac{u - \sqrt{u^2 + 4c}}{2} < 0$ , 分別對應  $dx = \left(\frac{1}{2} + \frac{u}{\sqrt{u^2 + 4c}}\right) du$  或  $dx = \left(\frac{1}{2} - \frac{u}{\sqrt{u^2 + 4c}}\right) du$ ; 積分範圍  $x$  分別由  $-\infty$  至  $0-$ ,  $0+$  至  $\infty$ ,  $u$  則分別由  $-\infty$  至  $\infty$ ,  $-\infty$  至  $\infty$ . 故  $\int_{-\infty}^\infty f(x - \frac{c}{x}) dx = \int_{-\infty}^{0-} f(x - \frac{c}{x}) dx + \int_{0+}^\infty f(x - \frac{c}{x}) dx = \int_{-\infty}^\infty f(u) \left(\frac{1}{2} + \frac{u}{\sqrt{u^2 + 4c}}\right) du + \int_{-\infty}^\infty f(u) \left(\frac{1}{2} - \frac{u}{\sqrt{u^2 + 4c}}\right) du = \int_{-\infty}^\infty f(u) du = \int_{-\infty}^\infty f(x) dx$ .

例 (GMT 範例). •  $\int_0^\infty e^{-ax^2 - \frac{b}{x^2}} dx = \underbrace{\int_0^\infty e^{-(\sqrt{ax} - \frac{\sqrt{b}}{x})^2 - 2\sqrt{ab}} dx}_{\text{令 } u = \sqrt{ax}; \frac{du}{\sqrt{a}} = dx} = \frac{e^{-2\sqrt{ab}}}{\sqrt{a}} \int_0^\infty e^{-(u - \frac{\sqrt{ab}}{u})^2} du = \underbrace{\frac{e^{-2\sqrt{ab}}}{2\sqrt{a}} \int_{-\infty}^\infty e^{-u^2} du}_{\text{偶函數, } \sqrt{ab} > 0; \text{符合 GMT}} = \frac{e^{-2\sqrt{ab}}}{2\sqrt{a}} \int_{-\infty}^\infty e^{-u^2} du = \frac{e^{-2\sqrt{ab}}}{2\sqrt{a}} \cdot \sqrt{\pi} = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}.$

•  $\int_{-\infty}^\infty \frac{x^2}{x^4 - 3x^2 + 4} dx = \int_{-\infty}^\infty \frac{1}{(x - \frac{2}{x})^2 + 1} dx = \int_{-\infty}^\infty \frac{1}{x^2 + 1} dx = \pi$

例. 若  $a > 0$ , 求  $\int_0^\infty \frac{dx}{(x^2 + a^2)^2}, \int_0^\infty \frac{dx}{(x^2 + a^2)^3}$  並推導  $\int_0^\infty \frac{dx}{(x^2 + a^2)^n}, n \in \mathbb{N}$ .

解. 由  $F(a) = \int_0^\infty \frac{dx}{x^2 + a^2} = \frac{\pi}{2a} = \frac{\pi}{2} a^{-1}$ ,  $F'(a) = \frac{d}{da} \left( \frac{\pi}{2} a^{-1} \right) = \frac{\pi}{2} (-1) a^{-2} = (-1) 2a \int_0^\infty \frac{dx}{(x^2 + a^2)^2} \Rightarrow \int_0^\infty \frac{dx}{(x^2 + a^2)^2} = \frac{\pi}{4a^3}$ ; 等式兩邊再對  $a$  微分  $\Rightarrow (-2)(2a) \int_0^\infty \frac{dx}{(x^2 + a^2)^3} = (-3) \frac{\pi}{4a^4} \Rightarrow \int_0^\infty \frac{dx}{(x^2 + a^2)^3} = \frac{3\pi}{16a^5}$ . 由數學歸納法可得  $\int_0^\infty \frac{dx}{(x^2 + a^2)^n} = \frac{\pi}{(2a)^{2n-1}} \binom{2(n-1)}{n-1}, n \in \mathbb{N}$ .

例.  $\int_0^\infty e^{-tx} \frac{\sin x}{x} dx = \frac{\pi}{2} - \tan^{-1} t, \forall t > 0$

證. 令  $F(t) = \int_0^\infty e^{-tx} \frac{\sin x}{x} dx$ , 則  $F'(t) = - \int_0^\infty e^{-tx} \sin x dx = - \frac{e^{-tx}(-t \sin x - \cos x)}{1+t^2} \Big|_{x=0}^{x=\infty} = - \frac{1}{1+t^2} \Rightarrow F(t) = -\tan^{-1} t + c$ . 令  $t \rightarrow \infty$ ,  $F(\infty) = 0 = -\tan^{-1} \infty + c = -\frac{\pi}{2} + c \Rightarrow c = \frac{\pi}{2}$ .

註. 上式令  $t \rightarrow 0+$  得  $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ ; 由變數變換  $u = ax$  得  $\int_0^\infty \frac{\sin ax}{x} dx = \frac{\pi}{2} \operatorname{sgn}(a), \forall a \neq 0$ .

例.  $\int_0^\infty e^{-ax} \frac{\sin bx}{x} dx = \tan^{-1} \frac{b}{a}, \forall a, b > 0$ .

證. 令  $F(t) = \int_0^\infty e^{-ax} \frac{\sin tx}{x} dx$ , 則  $F(0) = 0$ ;  $F'(t) = \int_0^\infty e^{-ax} \cos tx dx = \frac{a}{t^2 + a^2} \Rightarrow \int_0^b F'(t) dt = a \int_0^b \frac{dt}{t^2 + a^2} \Rightarrow F(b) - F(0) = \tan^{-1} \frac{t}{a} \Big|_{t=0}^{t=b} \Rightarrow F(b) = \tan^{-1} \frac{b}{a}$ .

例.  $\int_0^\infty \frac{\sin ax \sin bx}{x^2} dx = \frac{\pi}{2} \min(a, b), \forall a, b > 0$ .

證. 由  $\sin ax \sin bx = \frac{\cos(a-b)x - \cos(a+b)x}{2}$ ,  $\int_0^\infty \frac{\sin ax \sin bx}{x^2} dx = \frac{1}{2} \int_0^\infty \frac{\cos(a-b)x - \cos(a+b)x}{x^2} dx$ .

設  $F(t) = \int_0^\infty \frac{1 - \cos tx}{x^2} dx$ , 則  $F(0) = 0$ .  $F'(t) = \int_0^\infty \frac{\sin tx}{x} dx = \frac{\pi}{2} \operatorname{sgn}(t) \Rightarrow F(t) = \frac{\pi|t|}{2}$ .

$\int_0^\infty \frac{\cos(a-b)x - \cos(a+b)x}{x^2} dx = F(a+b) - F(a-b) = \frac{\pi(a+b)}{2} - \frac{\pi|a-b|}{2} = \frac{\pi}{2} \cdot 2 \min(a, b) = \pi \min(a, b)$ ,

故  $\int_0^\infty \frac{\sin ax \sin bx}{x^2} dx = \frac{1}{2} \cdot \pi \min(a, b) = \frac{\pi}{2} \min(a, b)$ .

例.  $\int_0^\infty \frac{\sin^2 ax}{x^2} dx = \frac{\pi a}{2}, \forall a > 0$ .

證. 令  $F(t) = \int_0^\infty \frac{\sin^2 tx}{x^2} dx, t > 0$ , 則  $F(0) = 0$ ;  $F'(t) = \int_0^\infty \frac{\partial}{\partial t} \left( \frac{\sin^2 tx}{x^2} \right) dx = \int_0^\infty \frac{2 \sin tx \cos tx \cdot x}{x^2} dx = \int_0^\infty \frac{\sin 2tx}{x} dx = \frac{\pi}{2} \Rightarrow \int_0^a F'(t) dt = \frac{\pi}{2} \int_0^a dt \Rightarrow F(a) - F(0) = \frac{\pi a}{2} \Rightarrow F(a) = \frac{\pi a}{2}$ .

例.  $\int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx = \frac{\pi(b-a)}{2} \quad \forall a, b > 0$ .

證. 設  $F(t) = \int_0^\infty \frac{1 - \cos tx}{x^2} dx, t > 0$ , 則  $F(0) = 0$ ;  $F'(t) = \int_0^\infty \frac{\sin tx}{x} dx = \frac{\pi}{2} \Rightarrow \int_0^t F'(\tau) d\tau = \frac{\pi}{2} \int_0^t d\tau \Rightarrow F(t) - F(0) = \frac{\pi t}{2} \Rightarrow F(t) = \frac{\pi t}{2}$ ;  $\int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx = F(b) - F(a) = \frac{\pi(b-a)}{2}$ .

定理 (Frullani 公式). 若  $f$  在  $(0, \infty)$  連續, 且  $f(0+)$  和  $f(\infty) = \lim_{x \rightarrow \infty} f(x)$  都存在且有限, 則  $\forall a, b > 0$   
 $\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = (f(\infty) - f(0+)) \ln \frac{a}{b}$ .

證. 令  $F(t) = \int_0^\infty \frac{f(tx) - f(bx)}{x} dx$ , 則  $F(b) = 0$ ; 變數變換  $u = tx, F'(t) = \int_0^\infty f'(tx) dx = \frac{1}{t} \int_0^\infty f'(u) du = \frac{1}{t} (f(\infty) - f(0+))$ . 則  $\int_b^a F'(t) dt = (f(\infty) - f(0+)) \int_b^a \frac{1}{t} dt \Rightarrow F(a) = (f(\infty) - f(0+)) \ln \frac{a}{b}$ .

**註.**  $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \ln \frac{b}{a}$ ,  $\int_0^\infty \frac{\tan^{-1} ax - \tan^{-1} bx}{x} dx = \frac{\pi}{2} \ln \frac{a}{b}$  可分別由 Frullani 公式求得: 分別取  $f(x) = e^{-x}$ ;  $f(0+) = 1$ ,  $f(\infty) = 0$  及取  $f(x) = \tan^{-1} x$ ;  $f(0+) = 0$ ,  $f(\infty) = \frac{\pi}{2}$ .

**例.**  $\int_0^{\frac{\pi}{2}} \ln(a^2 \cos^2 \theta + b^2 \sin^2 \theta) d\theta = \pi \ln \frac{a+b}{2}$ ,  $\forall a, b > 0$ .

**證.** 令  $F(a, b) = \int_0^{\frac{\pi}{2}} \ln(a^2 \cos^2 \theta + b^2 \sin^2 \theta) d\theta$ ,  $\frac{\partial F}{\partial a} = \int_0^{\frac{\pi}{2}} \frac{2a \cos^2 \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta$ ,  $\frac{\partial F}{\partial b} = \int_0^{\frac{\pi}{2}} \frac{2b \sin^2 \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta$ .  
 令  $K = \int_0^{\frac{\pi}{2}} \frac{\cos^2 \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta$ ,  $L = \int_0^{\frac{\pi}{2}} \frac{\sin^2 \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta$ . 變數變換  $t = \tan \theta$ , 則  $\cos^2 \theta = \frac{1}{1+t^2}$ ,  $\sin^2 \theta = \frac{t^2}{1+t^2}$ ,  $d\theta = \frac{dt}{1+t^2}$ ,  $t$  範圍從 0 至  $\infty$ ;  $K+L = \int_0^{\frac{\pi}{2}} \frac{1}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta = \int_0^\infty \frac{1}{a^2 + b^2 t^2} dt = \frac{\pi}{2ab}$ . 又  $a^2 K + b^2 L = \int_0^{\frac{\pi}{2}} \frac{a^2 \cos^2 \theta + b^2 \sin^2 \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta = \frac{\pi}{2}$ , 解得  $K = \frac{\pi}{2a(a+b)}$ ,  $L = \frac{\pi}{2b(a+b)}$ ; 代入得  $\frac{\partial F}{\partial a} = \frac{\partial F}{\partial b} = \frac{\pi}{a+b}$ ,  $F(a, b) = \pi \ln(a+b) + C$ . 令  $a = b$ ,  $F(a, a) = \pi \ln a = \pi \ln(2a) + C \Rightarrow C = -\pi \ln 2$ , 故  $F(a, b) = \pi \ln(a+b) - \pi \ln 2 = \pi \ln \frac{a+b}{2}$

**例.**  $\int_0^{\frac{\pi}{2}} \ln(\sin \theta) d\theta = \int_0^{\frac{\pi}{2}} \ln(\cos \theta) d\theta = -\frac{\pi}{2} \ln 2$ ,  $\int_0^\pi \ln(2 \pm 2 \cos \theta) d\theta = 0$ .

**證.** 變數變換  $u = \frac{\pi}{2} - \theta$ ,  $\int_0^{\frac{\pi}{2}} \ln(\sin \theta) d\theta = \int_{\frac{\pi}{2}}^0 \ln(\sin(\frac{\pi}{2} - u))(-1) du = \int_0^{\frac{\pi}{2}} \ln(\cos u) du = \int_0^{\frac{\pi}{2}} \ln(\cos \theta) d\theta$ ;  
 接連變數變換  $x = 2\theta$ ,  $u = \frac{\pi}{2} - x$ ,  $\int_0^{\frac{\pi}{2}} \ln(\sin 2\theta) d\theta = \frac{1}{2} \int_0^\pi \ln(\sin x) dx = \frac{1}{2} \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \ln(\cos u)(-1) du = \int_0^{\frac{\pi}{2}} \ln(\cos u) du = \int_0^{\frac{\pi}{2}} \ln(\cos \theta) d\theta$ ;  $\int_0^{\frac{\pi}{2}} \ln(\sin \theta) d\theta = \int_0^{\frac{\pi}{2}} \ln(\sin 2\theta) d\theta = \int_0^{\frac{\pi}{2}} \ln(\sin \theta) d\theta + \int_0^{\frac{\pi}{2}} \ln(\cos \theta) d\theta + \frac{\pi}{2} \ln 2 = \int_0^{\frac{\pi}{2}} \ln(\cos \theta) d\theta \Rightarrow \int_0^{\frac{\pi}{2}} \ln(\sin \theta) d\theta = -\frac{\pi}{2} \ln 2$ .  $\int_0^\pi \ln(2 + 2 \cos \theta) d\theta = \int_0^\pi \ln\left(4 \cos^2 \frac{\theta}{2}\right) d\theta = 2\pi \ln 2 + 2 \int_0^\pi \ln\left(\cos \frac{\theta}{2}\right) d\theta = 2\pi \ln 2 + 4 \int_0^{\frac{\pi}{2}} \ln(\cos \theta) d\theta = 0$ ;  $\int_0^\pi \ln(2 - 2 \cos \theta) d\theta = \int_0^\pi \ln\left(4 \sin^2 \frac{\theta}{2}\right) d\theta = 2\pi \ln 2 + 2 \int_0^\pi \ln\left(\sin \frac{\theta}{2}\right) d\theta = 2\pi \ln 2 + 4 \int_0^{\frac{\pi}{2}} \ln(\sin \theta) d\theta = 0$ ;

**例 (Poisson 積分).**  $\int_0^\pi \ln(1 - 2r \cos \theta + r^2) d\theta = \begin{cases} 0 & |r| \leq 1 \\ 2\pi \ln|r| & |r| > 1 \end{cases}$

**證.** 設  $F(r) = \int_0^\pi \ln(1 - 2r \cos \theta + r^2) d\theta$ ,  $\forall |r| \neq 1$ ,  $F(0) = \int_0^\pi \ln 1 d\theta = 0$ . 令  $u = \tan \frac{\theta}{2}$ , 則  $\cos \theta = \frac{1-u^2}{1+u^2}$ ,  $d\theta = \frac{2}{1+u^2} du$ ,  $u : 0 \rightarrow \infty$ . 由  $1 - 2r \cos \theta + r^2 = 1 - 2r \cdot \frac{1-u^2}{1+u^2} + r^2 = \frac{(1+u^2) - 2r(1-u^2) + r^2(1+u^2)}{1+u^2} = \frac{(1-r)^2 + (1+r)^2 u^2}{1+u^2}$ ,  $-2 \cos \theta + 2r = -2 \cdot \frac{1-u^2}{1+u^2} + 2r = \frac{2(r-1) + 2(1+r)u^2}{1+u^2}$ ,  $F'(r) = \int_0^\pi \frac{-2 \cos \theta + 2r}{1 - 2r \cos \theta + r^2} d\theta = \int_0^\infty \frac{2(r-1) + 2(1+r)u^2}{(1-r)^2 + (1+r)^2 u^2} \cdot \frac{2}{1+u^2} du = 4 \int_0^\infty \frac{(r-1) + (1+r)u^2}{((1-r)^2 + (1+r)^2 u^2)(1+u^2)} du$ . 求部份分式:  
 $\frac{(r-1) + (1+r)u^2}{((1-r)^2 + (1+r)^2 u^2)(1+u^2)} = \frac{A}{(1-r)^2 + (1+r)^2 u^2} + \frac{B}{1+u^2}$ , 則  $A = \frac{r^2-1}{2r}$ ,  $B = \frac{1}{2r}$ .  
 $\int_0^\infty \frac{du}{(1-r)^2 + (1+r)^2 u^2} = \frac{1}{(1+r)^2} \int_0^\infty \frac{du}{\frac{(1-r)^2}{(1+r)^2} + u^2} = \frac{1}{(1+r)^2} \frac{|1+r|}{|1-r|} \cdot \frac{\pi}{2} = \frac{1}{|1+r| \cdot |1-r|} \cdot \frac{\pi}{2} = \frac{\pi}{2} \cdot \frac{1}{|1-r^2|}$ ,

故  $F'(r) = 4 \int_0^\infty \frac{(r-1)+(1+r)u^2}{((1-r)^2+(1+r)^2u^2)(1+u^2)} du = 4 \cdot \frac{\pi}{2} \left( \frac{A}{|1-r^2|} + B \right) = \frac{\pi}{r} \left( \frac{r^2-1}{|r^2-1|} + 1 \right)$ . 故  $F'(r) = \begin{cases} 0 & |r| < 1 \\ \frac{2\pi}{r} & |r| > 1 \end{cases}$ ,  $F(r) = \begin{cases} 0 & |r| < 1 \\ 2\pi \ln|r| & |r| > 1 \end{cases}$ ;  $|r|=1$  情形由  $\int_0^\pi \ln(2 \pm 2 \cos \theta) d\theta = 0$  含括.