

第四章 積分

4.1 不定積分

定義 (反導函數). 紿定 $F(x)$, 若 $\frac{d}{dx}F(x) = f(x)$, 則稱 $F(x)$ 為 $f(x)$ 的反導函數 (antiderivative) .

性質. 若 $F(x), G(x)$ 分別為 $f(x), g(x)$ 的反導函數, $c \in \mathbb{R}$. 則

- $F(x) + c$ 為 $f(x)$ 的反導函數.
- $c F(x)$ 為 $c f(x)$ 的反導函數.
- $F(x) + G(x)$ 為 $f(x) + g(x)$ 的反導函數.

結論.

- $\frac{d}{dx}F(x) = f(x) \implies dF(x) = f(x) \cdot dx \implies F(x) = \int f(x) \cdot dx = \int f(x) dx$
- $F(x)$ 為 $f(x)$ 的反導函數 $\iff f(x)$ 的反導函數為 $F(x) \iff F(x)$ 的導函數為 $f(x) \iff F(x)$ (對 x) 的微分為 $f(x) \iff f(x)$ (對 x) 的 (不定) 積分為 $F(x)$
- $f(x)$ 的反導函數 $\equiv f(x)$ (對 x) 的 (不定) 積分
- 基礎積分集: 以下 $\alpha \neq -1, a \neq 0$.

$f(x)$	x^α	$\frac{1}{x}$	e^{ax}	$\sin ax$	$\cos ax$	$\frac{1}{\sqrt{a^2 - x^2}}$	$\frac{1}{a^2 + x^2}$
$\int f(x) dx$	$\frac{1}{\alpha+1} x^{\alpha+1}$	$\ln x $	$\frac{1}{a} e^{ax}$	$-\frac{1}{a} \cos ax$	$\frac{1}{a} \sin ax$	$\sin^{-1} \frac{x}{ a }$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$

- (Liouville) $e^{-x^2}, \frac{e^x}{x}, \frac{1}{\ln x}, \sin(x^2), \cos(x^2), \frac{\sin x}{x}, \frac{\cos x}{x}, x^x$ 無 (初等函數形式之) 反導函數!

- 例.**
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|---|--|---|
| 1. $\int \sqrt{x} dx = \frac{2}{3} x^{\frac{3}{2}} + c$ | 4. $\int \cos xy dx = \frac{1}{y} \sin xy + c$ | 7. $\int \frac{1}{e+u^2} du = \frac{1}{\sqrt{e}} \tan^{-1} \frac{u}{\sqrt{e}} + c$ |
| 2. $\int x^\pi dx = \frac{1}{\pi+1} x^{\pi+1} + c$ | 5. $\int e^{-2x} dx = -\frac{1}{2} e^{-2x} + c$ | 8. $\int \left(\frac{\pi}{x} - e^{\pi x} \right) du = \pi \ln x - \frac{e^{\pi x}}{\pi} + c$ |
| 3. $\int \sin \pi x dx = -\frac{1}{\pi} \cos \pi x + c$ | 6. $\int \frac{1}{\sqrt{\pi-x^2}} dx = \sin^{-1} \frac{x}{\sqrt{\pi}} + c$ | 9. $\int \frac{3+x^2}{1+x^2} du = x + 2 \tan^{-1} x + c$ |

習題. 求下列不定積分.

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|--|---|
| 1. $\int \frac{x^3 - 1}{x^3} dx = x + \frac{1}{2x^3} + c$ | 5. $\int x \sqrt{3x} dx = \frac{2\sqrt{3}}{5} x^{\frac{5}{2}} + c$ |
| 2. $\int 5 - \frac{1}{\sqrt{x}} dx = 5x - 2\sqrt{x} + c$ | 6. $\int \frac{1}{x^3} - \frac{1}{x^5} dx = \frac{-1}{2x^2} + \frac{1}{4x^4} + c$ |
| 3. $\int (t-1)(t+1) dt = \frac{t^3}{3} - t + c$ | 7. $\int \sec^2 x - \sec x \tan x dx = \tan x - \sec x + c$ |
| 4. $\int (\sqrt{x} + 1)^2 dx = \frac{x^2}{2} + x + \frac{4x^{\frac{3}{2}}}{3} + c$ | 8. $\int \frac{e^{3x} + 1}{e^x + 1} dx = \frac{e^{2x}}{2} - e^x + x + c$ |

變數變換法

結論. $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x) \implies df(g(x)) = f'(g(x)) \cdot g'(x) dx \implies f(g(x)) = \int f'(g(x)) \cdot g'(x) dx.$
 令 $u = g(x)$, 則 $\frac{du}{dx} = g'(x) \implies du = g'(x) dx$; 故 $\int f'(g(x)) \cdot g'(x) dx = \int f'(u) du = f(u) + c = f(g(x)) + c.$

例. 求 $\int \frac{x}{\sqrt{x+1}} dx.$

解.

- (解一) 令 $u = x + 1$, 則 $x = u - 1$, $du = dx$. 故 $\int \frac{x}{\sqrt{x+1}} dx = \int \frac{u-1}{\sqrt{u}} du = \int u^{\frac{1}{2}} - u^{-\frac{1}{2}} du = \frac{2}{3}u^{\frac{3}{2}} - 2\sqrt{u} + c = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c.$
- (解二) $\int \frac{x}{\sqrt{x+1}} dx = \int \frac{x+1-1}{\sqrt{x+1}} dx = \int \sqrt{x+1} - \frac{1}{\sqrt{x+1}} dx$. 令 $u = x + 1$, 則 $du = dx$. 故 $\int \sqrt{x+1} - \frac{1}{\sqrt{x+1}} dx = \int u^{\frac{1}{2}} - u^{-\frac{1}{2}} du = \frac{2}{3}u^{\frac{3}{2}} - 2\sqrt{u} + c = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c.$
- (解三) 令 $u = \sqrt{x+1}$, 則 $x = u^2 - 1$, $du = \frac{1}{2\sqrt{x+1}} dx \implies \frac{1}{\sqrt{x+1}} dx = 2du$. 故 $\int \frac{x}{\sqrt{x+1}} dx = \int x \cdot \frac{1}{\sqrt{x+1}} dx = \int (u^2 - 1) \cdot 2 du = \frac{2}{3}u^3 - 2u + c = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c.$

例. 求 $\int \frac{x}{x^2+1} dx.$

解. 令 $u = x^2 + 1$, 則 $du = 2x dx \implies x dx = \frac{1}{2} du$. 故 $\int \frac{x}{x^2+1} dx = \int \frac{1}{x^2+1} \cdot x dx = \int \frac{1}{u} \cdot \frac{1}{2} du = \frac{1}{2} \ln|u| + c = \frac{1}{2} \ln(x^2 + 1) + c.$

例. 求 $\int \frac{\sin(3 \ln x)}{x} dx.$

解. 令 $u = \ln x$, 則 $du = \frac{1}{x} dx$. 故 $\int \frac{\sin(3 \ln x)}{x} dx = \int \sin(3 \ln x) \cdot \frac{1}{x} dx = \int \sin 3u \cdot du = -\frac{1}{3} \cos 3u + c = -\frac{1}{3} \cos(3 \ln x) + c.$

例. 求 $\int e^x \sqrt{1+e^x} dx.$

解. 令 $u = 1+e^x$, 則 $du = e^x dx$. 故 $\int e^x \sqrt{1+e^x} dx = \int \sqrt{1+e^x} \cdot e^x dx = \int \sqrt{u} \cdot du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}(1+e^x)^{\frac{3}{2}} + c$

例. 求 $\int \frac{e^x}{\sqrt{2-e^{2x}}} dx.$

解. 令 $u = e^x$, 則 $du = e^x dx$. 故 $\int \frac{e^x}{\sqrt{2-e^{2x}}} dx = \int \frac{1}{\sqrt{2-u^2}} du = \sin^{-1} \frac{u}{\sqrt{2}} + c = \sin^{-1} \frac{e^x}{\sqrt{2}} + c.$

例. 求 $\int \frac{1}{\sqrt{e^{2x}-1}} dx.$

解. $\int \frac{1}{\sqrt{e^{2x}-1}} dx = \int \frac{1}{e^x \sqrt{1-e^{-2x}}} dx = \int \frac{e^{-x}}{\sqrt{1-e^{-2x}}} dx$. 令 $u = e^{-x}$, 則 $du = -e^{-x} dx \Rightarrow e^{-x} dx = -du$;
故 $\int \frac{e^{-x}}{\sqrt{1-e^{-2x}}} dx = \int \frac{1}{\sqrt{1-e^{-2x}}} \cdot e^{-x} dx = - \int \frac{1}{\sqrt{1-u^2}} du = -\sin^{-1} u + c = -\sin^{-1} e^{-x} + c$.

例. 求 $\int \frac{1}{x^2+4x+5} dx$.

解. $\int \frac{1}{x^2+4x+5} dx = \int \frac{1}{(x+2)^2+1} dx$. 令 $u = x+2$, 則 $du = dx$; 故 $\int \frac{1}{(x+2)^2+1} dx = \int \frac{1}{u^2+1} du = \tan^{-1} u + c = \tan^{-1}(x+2) + c$.

例. 求 $\int \frac{1}{\sqrt{4+2x-x^2}} dx$

解. $\int \frac{1}{\sqrt{4+2x-x^2}} dx = \int \frac{1}{\sqrt{5-(x-1)^2}} dx$. 令 $u = x-1$, 則 $du = dx$; 故 $\int \frac{1}{\sqrt{5-(x-1)^2}} dx = \int \frac{1}{\sqrt{5-u^2}} du = \sin^{-1} \frac{u}{\sqrt{5}} + c = \sin^{-1} \frac{x-1}{\sqrt{5}} + c$.

例. 求 $\int \tan x dx$.

解. $\int \tan x dx = \int \frac{\sin x}{\cos x} dx$. 令 $u = \cos x$, 則 $du = -\sin x dx$; 故 $\int \frac{\sin x}{\cos x} dx = \int \frac{-1}{u} du = -\ln|u| + c = -\ln|\cos x| + c = \ln|\sec x| + c$

例. 求 $\int \cos^5 ax dx$, $a \neq 0$.

解. $\int \cos^5 ax dx = \int (1-\sin^2 ax)^2 \cos ax dx$. 令 $u = \sin ax$, 則 $du = a \cos ax dx$; 故 $\int (1-\sin^2 ax)^2 \cos ax dx = \frac{1}{a} \int (1-u^2)^2 du = \frac{1}{a} \int (1-2u^2+u^4) du = \frac{1}{a} \left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5 \right) + c = \frac{1}{a} \left(\sin ax - \frac{2}{3}\sin^3 ax + \frac{1}{5}\sin^5 ax \right) + c$.

例. 求 $\int \cos^4 ax dx$, $a \neq 0$.

解. $\int \cos^4 ax dx = \int \left(\frac{1+\cos 2ax}{2} \right)^2 dx = \frac{1}{4} \int (1+2\cos 2ax+\cos^2 2ax) dx = \frac{1}{4} \int \left(1+2\cos 2ax+\frac{1+\cos 4ax}{2} \right) dx = \int \left(\frac{3}{8} + \frac{1}{2}\cos 2ax + \frac{1}{8}\cos 4ax \right) dx = \frac{3}{8}x + \frac{1}{4a}\sin 2ax + \frac{1}{32a}\sin 4ax + c$

例. 求 $\int \sec x dx$.

解. $\int \sec x dx = \int \frac{\sec x \cdot (\sec x + \tan x)}{\sec x + \tan x} dx$. 令 $u = \sec x + \tan x$, 則 $du = (\sec^2 x + \sec x \tan x) dx$; 故 $\int \frac{\sec x \cdot (\sec x + \tan x)}{\sec x + \tan x} dx = \int \frac{1}{u} du = \ln|u| + c = \ln|\sec x + \tan x| + c$

例. 令 $T_1 = \int \frac{\sin x}{a \cos x + b \sin x} dx$, $T_2 = \int \frac{\cos x}{a \cos x + b \sin x} dx$, $a, b \neq 0$, 求 T_1, T_2 .

解.

$$(a) bT_1 + aT_2 = \int \frac{b \sin x}{a \cos x + b \sin x} dx + \int \frac{a \cos x}{a \cos x + b \sin x} dx = \int \frac{b \sin x + a \cos x}{a \cos x + b \sin x} dx = \int 1 dx = x.$$

$$(b) -aT_1 + bT_2 = \int \frac{-a \sin x}{a \cos x + b \sin x} dx + \int \frac{b \cos x}{a \cos x + b \sin x} dx = \int \frac{-a \sin x + b \cos x}{a \cos x + b \sin x} dx = \int \frac{du}{u} = \ln u = \ln |a \cos x + b \sin x| \quad (\text{令 } u = a \cos x + b \sin x, \text{ 則 } du = (-a \sin x + b \cos x) dx).$$

$$\text{解 } T_1, T_2 \text{ 方程式 (a), (b) 得 } T_1 = \frac{bx - a \ln |a \cos x + b \sin x|}{a^2 + b^2}, T_2 = \frac{ax + b \ln |a \cos x + b \sin x|}{a^2 + b^2}.$$

習題. 以變數變換法求下列不定積分. 注意: 可能會因為常數項而跟此處答案不同.

$$1. \int \frac{1}{\sqrt{2x-1}} dx$$

$$4. \int e^{\pi x-1} dx$$

$$7. \int \frac{\cos 3x}{\sin^2 3x} dx$$

$$2. \int \sqrt{7x+4} dx$$

$$5. \int (x^2 - 2x + 1)^{\frac{1}{3}} dx$$

$$8. \int \frac{x}{\sqrt{1+2x^2}} dx$$

$$3. \int \sin(3x-1) dx$$

$$6. \int \sin 3x \cos 3x dx$$

$$9. \int x^2 \sqrt{1-x} dx$$

解.

$$1. \text{令 } u = 2x-1, \text{ 則 } du = 2 dx \implies dx = \frac{1}{2} du. \text{ 故 } \int \frac{1}{\sqrt{2x-1}} dx = \int \frac{1}{\sqrt{u}} \cdot \frac{1}{2} du = \sqrt{u} + c = \sqrt{2x-1} + c.$$

$$2. \text{令 } u = 7x+4, \text{ 則 } du = 7 dx \implies dx = \frac{1}{7} du. \text{ 故 } \int \sqrt{7x+4} dx = \int \sqrt{u} \cdot \frac{1}{7} du = \frac{1}{7} \cdot \frac{2}{3} u^{\frac{3}{2}} + c = \frac{2}{21} (7x+4)^{\frac{3}{2}} + c.$$

$$3. \text{令 } u = 3x-1, \text{ 則 } du = 3 dx \implies dx = \frac{1}{3} du. \text{ 故 } \int \sin(3x-1) dx = \int \sin u \cdot \frac{1}{3} du = -\frac{1}{3} \cos u + c = -\frac{\cos(3x-1)}{3} + c.$$

$$4. \text{令 } u = \pi x-1, \text{ 則 } du = \pi dx \implies dx = \frac{1}{\pi} du. \text{ 故 } \int e^{\pi x-1} dx = \int e^u \cdot \frac{1}{\pi} du = \frac{1}{\pi} e^u + c = \frac{e^{\pi x-1}}{\pi} + c.$$

$$5. \text{令 } u = x-1, \text{ 則 } du = dx. \text{ 故 } \int (x^2 - 2x + 1)^{\frac{1}{3}} dx = \int (x-1)^{\frac{2}{3}} dx = \int u^{\frac{2}{3}} du = \frac{3}{5} u^{\frac{5}{3}} + c = \frac{3}{5} (x-1)^{\frac{5}{3}} + c.$$

$$6. \text{令 } u = \sin 3x, \text{ 則 } du = 3 \cos 3x dx \implies \cos 3x dx = \frac{1}{3} du. \text{ 故 } \int \sin 3x \cos 3x dx = \int u \cdot \frac{1}{3} du = \frac{1}{3} \cdot \frac{1}{2} u^2 + c = \frac{\sin^2 3x}{6} + c.$$

$$7. \text{令 } u = \sin 3x, \text{ 則 } du = 3 \cos 3x dx \implies \cos 3x dx = \frac{1}{3} du. \text{ 故 } \int \frac{\cos 3x}{\sin^2 3x} dx = \int \frac{1}{u^2} \cdot \frac{1}{3} du = -\frac{1}{3} \cdot \frac{1}{u} + c = -\frac{1}{3 \sin 3x} + c.$$

$$8. \text{令 } u = 1 + 2x^2, \text{ 則 } du = 4x dx \implies x dx = \frac{1}{4} du. \text{ 故 } \int \frac{x}{\sqrt{1+2x^2}} dx = \int \frac{1}{\sqrt{u}} \cdot \frac{1}{4} du = \frac{1}{4} \cdot 2 u^{\frac{1}{2}} + c = \frac{\sqrt{1+2x^2}}{2} + c.$$

$$9. \text{令 } u = \sqrt{1-x}, \text{ 則 } u^2 = 1-x \implies x = 1-u^2, dx = -2u du. \text{ 故 } \int x^2 \sqrt{1-x} dx = \int (1-u^2)^2 \cdot u \cdot (-2u) du = -2 \int (1-u^2)^2 \cdot u^2 du = -2 \int (u^2 - 2u^4 + u^6) du = -\frac{2u^3}{3} + \frac{4u^5}{5} - \frac{2u^7}{7} + c = -\frac{2(1-x)^{\frac{3}{2}}}{3} + \frac{4(1-x)^{\frac{5}{2}}}{5} - \frac{2(1-x)^{\frac{7}{2}}}{7}.$$

習題. 以變數變換法求下列不定積分, 其中 $a \neq b \neq 0$. 注意: 可能會因為常數項而跟此處答案不同.

1. $\int e^{2x} \sin e^{2x} dx$	7. $\int \frac{x^2}{2+x^6} dx$	13. $\int \sin^{-\frac{2}{3}} x \cos^3 x dx$
2. $\int xe^{-\frac{x^2}{2}} dx$	8. $\int \frac{1}{e^x + e^{-x}} dx$	14. $\int \frac{\sin^3(\ln x) \cos^3(\ln x)}{x} dx$
3. $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$	9. $\int \frac{x+1}{\sqrt{1-x^2}} dx$	15. $\int \frac{\sin^3 x}{\cos^4 x} dx$
4. $\int x^2 2^{x^3+1} dx$	10. $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}} dx$	16. $\int \cos ax \cos bx dx$
5. $\int \frac{\ln x}{x} dx$	11. $\int \sin^4 x \cos^5 x dx$	17. $\int \sin ax \sin bx dx$
6. $\int \frac{x+1}{\sqrt{x^2+2x+3}} dx$	12. $\int \sin^2 x \cos^2 x dx$	18. $\int \sin ax \cos bx dx$

解.

1. 令 $u = e^{2x}$, 則 $du = 2e^{2x} dx \Rightarrow e^{2x} dx = \frac{1}{2} du$. 故 $\int e^{2x} \sin e^{2x} dx = \int \sin e^{2x} \cdot e^{2x} dx = \int \sin u \cdot \frac{1}{2} du = -\frac{1}{2} \cos u + c = -\frac{1}{2} \cos e^{2x} + c$.
2. 令 $u = \frac{x^2}{2}$, 則 $du = x dx$, 故 $\int xe^{-\frac{x^2}{2}} dx = \int e^{-\frac{x^2}{2}} \cdot x dx = \int e^{-u} du = -e^{-u} + c = -e^{-\frac{x^2}{2}} + c$
3. 令 $u = \sqrt{x}$, 則 $du = \frac{1}{2\sqrt{x}} dx \Rightarrow \frac{1}{\sqrt{x}} dx = 2 du$, 故 $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx = \int \sin \sqrt{x} \cdot \frac{1}{\sqrt{x}} dx = \int \sin u \cdot 2 du = -2 \cos u + c = -2 \cos \sqrt{x} + c$
4. 令 $u = x^3 + 1$, 則 $du = 3x^2 dx \Rightarrow x^2 dx = \frac{1}{3} du$, 故 $\int x^2 2^{x^3+1} dx = \int 2^{x^3+1} \cdot x^2 dx = \int 2^u \cdot \frac{1}{3} du = \frac{1}{3} \int e^{u \ln 2} du = \frac{1}{3 \ln 2} e^{u \ln 2} + c = \frac{2^{x^3+1}}{3 \ln 2} + c$
5. 令 $u = \ln x$, 則 $du = \frac{1}{x} dx$, 故 $\int \frac{\ln x}{x} dx = \int \ln x \cdot \frac{1}{x} dx = \int u du = \frac{1}{2} u^2 + c = \frac{1}{2} (\ln x)^2 + c$
6. 令 $u = x^2 + 2x + 3$, 則 $du = (2x+2) dx \Rightarrow (x+1) dx = \frac{1}{2} du$, 故 $\int \frac{x+1}{\sqrt{x^2+2x+3}} dx = \int \frac{1}{\sqrt{x^2+2x+3}} \cdot (x+1) dx = \int \frac{1}{\sqrt{u}} \cdot \frac{1}{2} du = \sqrt{u} + c = \sqrt{x^2+2x+3} + c$
7. 令 $u = x^3$, 則 $du = 3x^2 dx \Rightarrow x^2 dx = \frac{1}{3} du$, 故 $\int \frac{x^2}{2+x^6} dx = \int \frac{1}{2+x^6} \cdot x^2 dx = \int \frac{1}{2+u^2} \cdot \frac{1}{3} du = \frac{1}{3\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} + c = \frac{1}{3\sqrt{2}} \tan^{-1} \frac{x^3}{\sqrt{2}} + c$
8. $\int \frac{1}{e^x + e^{-x}} dx = \int \frac{e^x}{e^{2x} + 1} dx$. 令 $u = e^x$, 則 $du = e^x dx$, 故 $\int \frac{e^x}{e^{2x} + 1} dx = \int \frac{1}{e^{2x} + 1} \cdot e^x dx = \int \frac{1}{u^2 + 1} du = \tan^{-1} u + c = \tan^{-1} e^x + c$.
9. $\int \frac{x+1}{\sqrt{1-x^2}} dx = \int \frac{x}{\sqrt{1-x^2}} dx + \int \frac{1}{\sqrt{1-x^2}} dx$. 令 $u = \sqrt{1-x^2}$, 則 $du = \frac{-x}{\sqrt{1-x^2}} dx \Rightarrow \frac{x}{\sqrt{1-x^2}} dx = -du$, 故 $\int \frac{x}{\sqrt{1-x^2}} dx = \int -du = -u + c = -\sqrt{1-x^2}$, $\int \frac{1}{\sqrt{1-x^2}} dx = \int \frac{x+1}{\sqrt{1-x^2}} dx + \int \frac{1}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} + \sin^{-1} x + c$.

10. 因 $u = \tan x$, 則 $du = \sec^2 x dx$, 故 $\int \frac{\sec^2 x}{\sqrt{1 - \tan^2 x}} dx = \int \frac{1}{\sqrt{1 - \tan^2 x}} \cdot \sec^2 x dx = \int \frac{1}{\sqrt{1 - u^2}} du = \sin^{-1} u + c = \sin^{-1}(\tan x) + c$
11. $\int \sin^4 x \cos^5 x dx = \int \sin^4 x \cdot \cos^4 x \cdot \cos x dx = \int \sin^4 x \cdot (1 - \sin^2 x)^2 \cdot \cos x dx = \int (\sin^4 x - 2\sin^6 x + \sin^8 x) \cdot \cos x dx$. 因 $u = \sin x$, 則 $du = \cos x dx$, 故 $\int (\sin^4 x - 2\sin^6 x + \sin^8 x) \cdot \cos x dx = \int (u^4 - 2u^6 + u^8) du = \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + c = \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + c$
12. $\int \sin^2 x \cos^2 x dx = \int \frac{1 - \cos 2x}{2} \cdot \frac{1 + \cos 2x}{2} dx = \frac{1}{4} \int (1 - \cos^2 2x) dx = \frac{1}{4} \int \left(1 - \frac{1 + \cos 4x}{2}\right) dx = \frac{1}{4} \int \left(\frac{1}{2} + \frac{\cos 4x}{2}\right) dx = \frac{x}{8} - \frac{\sin 4x}{32} + c$
13. $\int \sin^{-\frac{2}{3}} x \cos^3 x dx = \int \sin^{-\frac{2}{3}} x \cdot \cos^2 x \cdot \cos x dx = \int \sin^{-\frac{2}{3}} x \cdot (1 - \sin^2 x) \cdot \cos x dx = \int (\sin^{-\frac{2}{3}} x - \sin^{\frac{4}{3}} x) \cdot \cos x dx$. 因 $u = \sin x$, 則 $du = \cos x dx$, 故 $\int (\sin^{-\frac{2}{3}} x - \sin^{\frac{4}{3}} x) \cdot \cos x dx = \int (u^{-\frac{2}{3}} - u^{\frac{4}{3}}) du = 3u^{\frac{1}{3}} - \frac{3}{7}u^{\frac{7}{3}} + c = 3\sin^{\frac{1}{3}} x - \frac{3}{7}\sin^{\frac{7}{3}} x + c$
14. 因 $u = \ln x$, 則 $du = \frac{1}{x} dx$, 故 $\int \frac{\sin^3(\ln x) \cos^3(\ln x)}{x} dx = \int \sin^3(\ln x) \cos^3(\ln x) \cdot \frac{1}{x} dx = \int \sin^3 u \cos^3 u du = \int \sin^3 u \cdot \cos^2 u \cdot \cos u du = \int \sin^3 u (1 - \sin^2 u) \cdot \cos u du = \int (\sin^3 u - \sin^5 u) \cdot \cos u du$. 因 $w = \sin u$, 則 $dw = \cos u du$, 故 $\int (\sin^3 u - \sin^5 u) \cdot \cos u du = \int (w^3 - w^5) dw = \frac{w^4}{4} - \frac{w^6}{6} + c = \frac{\sin^4(\ln x)}{4} - \frac{\sin^6(\ln x)}{6} + c$
15. $\int \frac{\sin^3 x}{\cos^4 x} dx = \int \frac{\sin^2 x \cdot \sin x}{\cos^4 x} dx = \int \frac{\sin^2 x}{\cos^4 x} \cdot \sin x dx = \int \frac{1 - \cos^2 x}{\cos^4 x} \cdot \sin x dx = \int \left(\frac{1}{\cos^4 x} - \frac{1}{\cos^2 x}\right) \cdot \sin x dx$. 因 $u = \cos x$, 則 $du = -\sin x dx \Rightarrow \sin x dx = -du$, 故 $\int \left(\frac{1}{\cos^4 x} - \frac{1}{\cos^2 x}\right) \cdot \sin x dx = \int \left(\frac{1}{u^4} - \frac{1}{u^2}\right)(-) du = \int \left(\frac{1}{u^2} - \frac{1}{u^4}\right) du = -\frac{1}{u} + \frac{1}{3u^3} + c = -\frac{1}{\cos x} + \frac{1}{3\cos^3 x} + c$
16. 由積化和差公式 $\cos \alpha \cos \beta = \frac{1}{2} (\cos(\alpha - \beta) + \cos(\alpha + \beta))$, $\int \cos ax \cos bx dx = \int \frac{1}{2} (\cos(ax - bx) + \cos(ax + bx)) dx = \int \frac{1}{2} (\cos(a - b)x + \cos(a + b)x) dx = \frac{\sin(a - b)x}{2(a - b)} + \frac{\sin(a + b)x}{2(a + b)} + c$.
17. 由積化和差公式 $\sin \alpha \sin \beta = \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta))$, $\int \sin ax \sin bx dx = \int \frac{1}{2} (\cos(ax - bx) - \cos(ax + bx)) dx = \int \frac{1}{2} (\cos(a - b)x - \cos(a + b)x) dx = \frac{\sin(a - b)x}{2(a - b)} - \frac{\sin(a + b)x}{2(a + b)} + c$.
18. 由積化和差公式 $\sin \alpha \cos \beta = \frac{1}{2} (\sin(\alpha - \beta) + \sin(\alpha + \beta))$, $\int \sin ax \cos bx dx = \int \frac{1}{2} (\sin(ax - bx) + \sin(ax + bx)) dx = \int \frac{1}{2} (\sin(a - b)x + \sin(a + b)x) dx = -\frac{\cos(a - b)x}{2(a - b)} - \frac{\cos(a + b)x}{2(a + b)} + c$.

部份積分法

結論. $\frac{d}{dx}(u(x)v(x)) = u(x)\frac{dv(x)}{dx} + v(x)\frac{du(x)}{dx} \Rightarrow u(x)v(x) = \int u(x)dv(x) + \int v(x)du(x)$
 $\Rightarrow \int u dv = uv - \int v du$.

例. 求 $\int xe^x dx$.

解. 令 $u = x$, 則 $du = dx$. 令 $dv = e^x dx$, 則 $v = e^x$. 故 $\int xe^x dx = x \cdot e^x - \int e^x dx = xe^x - e^x + c$.

例. 求 $\int \ln x dx$.

解. 令 $u = \ln x$, 則 $du = \frac{1}{x} dx$. 令 $dv = dx$, 則 $v = x$. 故 $\int \ln x dx = \ln x \cdot x - \int x \cdot \frac{1}{x} dx = x \ln x - x + c$.

例. 求 $\int x \tan^{-1} x dx$.

解. 令 $u = \tan^{-1} x$, 則 $du = \frac{1}{1+x^2} dx$. 令 $dv = x dx$, 則 $v = \frac{x^2}{2}$. 故 $\int x \tan^{-1} x dx = \tan^{-1} x \cdot \frac{x^2}{2} - \frac{1}{2} \int \frac{x^2}{1+x^2} dx = \tan^{-1} x \cdot \frac{x^2}{2} - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx = \tan^{-1} x \cdot \frac{x^2}{2} - \frac{1}{2}(x - \tan^{-1} x) + c$.

例. 求 $\int x^2 \sin x dx$.

解. 令 $u = x^2$, 則 $du = 2x dx$; 令 $dv = \sin x dx$, 則 $v = -\cos x$. 故 $\int x^2 \sin x dx = x^2 \cdot (-\cos x) + 2 \int x \cos x dx$.

令 $u = x$, 則 $du = dx$; 令 $dv = \cos x dx$, 則 $v = \sin x$. 故 $\int x \cos x dx = x \cdot \sin x - \int \sin x dx = x \cdot \sin x + \cos x$.

由上, $\int x^2 \sin x dx = -x^2 \cos x + 2(x \sin x + \cos x) + c$.

例. 求 $\int \sin^{-1} \sqrt{1-x^2} dx$, $x > 0$.

解. 令 $u = \sin^{-1} \sqrt{1-x^2}$, 則 $du = \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \frac{-x}{\sqrt{1-x^2}} dx = \frac{1}{\sqrt{x^2}} \cdot \frac{-x}{\sqrt{1-x^2}} dx = \frac{-1}{\sqrt{1-x^2}} dx$; 令 $dv = dx$, 則 $v = x$. 故 $\int \sin^{-1} \sqrt{1-x^2} dx = x \sin^{-1} \sqrt{1-x^2} + \int \frac{x}{\sqrt{1-x^2}} dx = x \sin^{-1} \sqrt{1-x^2} - \sqrt{1-x^2}$.

例. 求 $\int e^{ax} \cos bx dx$ 與 $\int e^{ax} \sin bx dx$, $a, b \neq 0$.

解.

(a) 考慮 $\int e^{ax} \cos bx dx$: 令 $u = \cos bx$, 則 $du = -b \sin bx dx$; 令 $dv = e^{ax} dx$, 則 $v = \frac{1}{a} e^{ax}$, 故 $\int e^{ax} \cos bx dx = \cos bx \cdot \frac{1}{a} e^{ax} + \frac{b}{a} \int e^{ax} \cdot \sin bx dx$.

(b) 考慮 $\int e^{ax} \sin bx dx$: 令 $u = \sin bx$, 則 $du = b \cos bx dx$; 令 $dv = e^{ax} dx$, 則 $v = \frac{1}{a} e^{ax}$, 故 $\int e^{ax} \sin bx dx = \sin bx \cdot \frac{1}{a} e^{ax} - \frac{b}{a} \int e^{ax} \cdot \cos bx dx$.

令 $X = \int e^{ax} \cos bx dx$, $Y = \int e^{ax} \sin bx dx$, 由 (a)(b) $X = \cos bx \cdot \frac{1}{a} e^{ax} + \frac{b}{a} Y$, $Y = \sin bx \cdot \frac{1}{a} e^{ax} - \frac{b}{a} X$. 解 X, Y 得 $X = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$, $Y = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$.

部份積分法：列表

結論.

- 將要微分函數寫左邊，積分函數寫右邊；左邊連續微分，右邊連續積分
- 依序左上連右下斜線函數相乘，最底部水平兩邊函數相乘並積分，符號正負相間
- 將上式所得項全部加總即為所求積分

例. 求 $\int (x+3) e^{2x} dx$.

Diff	Int
$x+3$	e^{2x}
1	$\frac{1}{2}e^{2x}$
0	$\frac{1}{4}e^{2x}$

$$\int (x+3) e^{2x} dx = (x+3) \frac{1}{2}e^{2x} - \frac{1}{4}e^{2x}$$

例. 求 $\int (x^2 - 2x) e^{kx} dx$.

Diff	Int
$x^2 - 2x$	e^{kx}
2(x-1)	$\frac{1}{k}e^{kx}$
2	$\frac{1}{k^2}e^{kx}$
0	$\frac{1}{k^3}e^{kx}$

$$\int (x^2 - 2x) e^{kx} dx = (x^2 - 2x) \frac{1}{k}e^{kx} - 2(x-1) \frac{1}{k^2}e^{kx} + 2 \frac{1}{k^3}e^{kx}$$

例. 求 $\int x^4 \sin 2x dx$.

Diff	Int
x^4	$\sin 2x$
4x ³	$-\frac{1}{2} \cos 2x$
12x ²	$-\frac{1}{4} \sin 2x$
24x	$\frac{1}{8} \cos 2x$
24	$\frac{1}{16} \sin 2x$
0	$-\frac{1}{32} \cos 2x$

$$\begin{aligned} & \int x^4 \sin 2x dx \\ &= -x^4 \frac{1}{2} \cos 2x + 4x^3 \frac{1}{4} \sin 2x + 12x^2 \frac{1}{8} \cos 2x - 24x \frac{1}{16} \sin 2x - 24 \frac{1}{32} \cos 2x \\ &= \left(-\frac{x^4}{2} + \frac{3x^2}{2} - \frac{3}{4} \right) \cos 2x + \left(x^3 - \frac{3x}{2} \right) \sin 2x \end{aligned}$$

例. 求 $\int x^5 e^{ax} dx$.

Diff	Int
x^5	e^{ax}
5x ⁴	$\frac{1}{a}e^{ax}$
20x ³	$\frac{1}{a^2}e^{ax}$
60x ²	$\frac{1}{a^3}e^{ax}$
120x	$\frac{1}{a^4}e^{ax}$
120	$\frac{1}{a^5}e^{ax}$
0	$\frac{1}{a^6}e^{ax}$

$$\begin{aligned} & \int x^5 e^{ax} dx \\ &= x^5 \frac{1}{a}e^{ax} - 5x^4 \frac{1}{a^2}e^{ax} + 20x^3 \frac{1}{a^3}e^{ax} - 60x^2 \frac{1}{a^4}e^{ax} + 120x \frac{1}{a^5}e^{ax} - 120 \frac{1}{a^6}e^{ax} \\ &= \left(\frac{x^5}{a} - \frac{5x^4}{a^2} + \frac{20x^3}{a^3} - \frac{60x^2}{a^4} + \frac{120x}{a^5} - \frac{120}{a^6} \right) e^{ax} \end{aligned}$$

例. 求 $\int (\sin^{-1} x)^2 dx$.

Diff	Int
$(\sin^{-1} x)^2$	+
$\frac{2 \sin^{-1} x}{\sqrt{1-x^2}}$	-
x	

Diff	Int
$2 \sin^{-1} x$	+
$\frac{2}{\sqrt{1-x^2}}$	-

解.

$$\begin{aligned}\int (\sin^{-1} x)^2 dx &= (\sin^{-1} x)^2 \cdot x - \int \frac{2 \sin^{-1} x}{\sqrt{1-x^2}} \cdot x dx \\ &= (\sin^{-1} x)^2 \cdot x - \int 2 \sin^{-1} x \cdot \frac{x}{\sqrt{1-x^2}} dx \\ &= (\sin^{-1} x)^2 \cdot x - \left(-2 \sin^{-1} x \cdot \sqrt{1-x^2} + \int \frac{2}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2} dx \right) \\ &= (\sin^{-1} x)^2 \cdot x + 2 \sin^{-1} x \cdot \sqrt{1-x^2} - 2x\end{aligned}$$

例. 求 $\int e^{ax} \cos bx dx, a, b \neq 0$.

Diff	Int
$\cos bx$	+
$-b \sin bx$	-
$\frac{1}{a} e^{ax}$	

Diff	Int
$-b^2 \cos bx$	+
$\frac{1}{a^2} e^{ax}$	

解.

$$\begin{aligned}\int e^{ax} \cos bx dx &= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} \int e^{ax} \cos bx dx \\ \Rightarrow \int e^{ax} \cos bx dx &= \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}\end{aligned}$$

例. 求 $\int e^{ax} \sin bx dx, a, b \neq 0$.

Diff	Int
$\sin bx$	+
$b \cos bx$	-
$\frac{1}{a} e^{ax}$	

Diff	Int
$-b^2 \sin bx$	+
$\frac{1}{a^2} e^{ax}$	

解.

$$\begin{aligned}\int e^{ax} \sin bx dx &= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} \int e^{ax} \sin bx dx \\ \Rightarrow \int e^{ax} \sin bx dx &= \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}\end{aligned}$$

遞迴式

例. 令 $I_n = \int \frac{1}{(x^2 + a^2)^n} dx, n \in \mathbb{N}$, 則 $I_{n+1} = \frac{x}{2na^2(x^2 + a^2)^n} + \frac{2n-1}{2na^2} I_n$.

解. 令 $u = \frac{1}{(x^2 + a^2)^n}$, 則 $du = -2n \frac{x}{(x^2 + a^2)^{n+1}} dx$; $dv = dx$, 則 $v = x$. 故 $I_n = \int \frac{1}{(x^2 + a^2)^n} dx = \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{x^2}{(x^2 + a^2)^{n+1}} dx = \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{(x^2 + a^2 - a^2)}{(x^2 + a^2)^{n+1}} dx = \frac{x}{(x^2 + a^2)^n} + 2nI_n - 2na^2 I_{n+1} \Rightarrow 2na^2 I_{n+1} = \frac{x}{(x^2 + a^2)^n} + (2n-1)I_n \Rightarrow I_{n+1} = \frac{x}{2na^2(x^2 + a^2)^n} + \frac{2n-1}{2na^2} I_n$.

註 (使用例). $I_1 = \frac{1}{a} \tan^{-1} \frac{x}{a}, I_2 = \frac{1}{2a^2} I_1 + \frac{x}{2a^2(x^2 + a^2)} = \frac{1}{2a^3} \tan^{-1} \frac{x}{a} + \frac{x}{2a^2(x^2 + a^2)}, I_3 = \frac{3}{4a^2} I_2 + \frac{x}{4a^2(x^2 + a^2)^2} = \frac{3}{4a^2} \left(\frac{1}{2a^3} \tan^{-1} \frac{x}{a} + \frac{x}{2a^2(x^2 + a^2)} \right) + \frac{x}{4a^2(x^2 + a^2)^2} = \frac{3}{8a^5} \tan^{-1} \frac{x}{a} + \frac{3x}{8a^4(x^2 + a^2)} + \frac{x}{4a^2(x^2 + a^2)^2}$.

例. 令 $J_n = \int \sin^n x dx, n \in \mathbb{N}, n \geq 2$, 則 $J_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} J_{n-2}$.

解. 令 $u = \sin^{n-1}x$, 則 $du = (n-1) \sin^{n-2}x \cos x dx$; $dv = \sin x dx$, 則 $v = -\cos x$. 故 $J_n = \int \sin^n x dx = -\sin^{n-1}x \cdot \cos x + (n-1) \int \cos x \cdot \sin^{n-2}x \cos x dx = -\sin^{n-1}x \cos x + (n-1) \int \sin^{n-2}x \cos^2 x dx = -\sin^{n-1}x \cos x + (n-1) \int \sin^{n-2}x \cdot (1-\sin^2 x) dx = -\sin^{n-1}x \cos x + (n-1) \int \sin^{n-2}x dx - (n-1) \int \sin^n x dx = -\sin^{n-1}x \cos x + (n-1)J_{n-2} + (1-n)J_n \implies nJ_n = -\sin^{n-1}x \cos x + (n-1)J_{n-2} \implies J_n = \frac{-\sin^{n-1}x \cos x}{n} + \frac{n-1}{n} J_{n-2}$.

註 (使用例). $J_2 = \frac{-\sin x \cos x}{2} + \frac{1}{2} J_0 = \frac{-\sin x \cos x}{2} + \frac{x}{2}$, $J_3 = \frac{-\sin^2 x \cos x}{3} + \frac{2}{3} J_1 = \frac{-\sin^2 x \cos x}{3} - \frac{2 \cos x}{3}$,
 $J_4 = \frac{-\sin^3 x \cos x}{4} + \frac{3}{4} J_2 = \frac{-\sin^3 x \cos x}{4} + \frac{3 \sin x \cos x}{8} - \frac{3x}{8}$.

例. 令 $K_n = \int \sec^{2n+1} \theta d\theta$, $n \in \mathbb{N}$, $n \geq 1$, 則 $K_n = \frac{\sec^{2n-1} \theta \tan \theta}{2n} + \frac{2n-1}{2n} K_{n-1}$.

解. 令 $u = \sec^{2n-1} \theta$, 則 $du = (2n-1) \sec^{2n-2} \theta \cdot \sec \theta \tan \theta d\theta = (2n-1) \sec^{2n-1} \theta \tan \theta d\theta$; 令 $dv = \sec^2 \theta d\theta$, 則 $v = \tan \theta$. 故 $K_n = \int \sec^{2n+1} \theta d\theta = \sec^{2n-1} \theta \cdot \tan \theta - \int \tan \theta \cdot (2n-1) \sec^{2n-1} \theta \tan \theta d\theta = \sec^{2n-1} \theta \tan \theta - (2n-1) \int \tan^2 \theta \cdot \sec^{2n-1} \theta d\theta = \sec^{2n-1} \theta \tan \theta - (2n-1) \int (\sec^2 \theta - 1) \cdot \sec^{2n-1} \theta d\theta = \sec^{2n-1} \theta \tan \theta - (2n-1) \int \sec^{2n+1} \theta d\theta + (2n-1) \int \sec^{2n-1} \theta d\theta \implies K_n = \sec^{2n-1} \theta \tan \theta - (2n-1)K_n + (2n-1)K_{n-1} \implies K_n = \frac{\sec^{2n-1} \theta \tan \theta}{2n} + \frac{2n-1}{2n} K_{n-1}$.

註 (使用例). $K_0 = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta|$, $\int \sec^3 \theta d\theta = K_1 = \frac{\sec \theta \tan \theta}{2} + \frac{1}{2} K_0 = \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|)$, $\int \sec^5 \theta d\theta = K_2 = \frac{\sec^3 \theta \tan \theta}{4} + \frac{3}{4} K_1 = \frac{\sec^3 \theta \tan \theta}{4} + \frac{3}{8} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|)$.

習題. 以部份積分法求下列不定積分. 注意: 可能會因為常數項而跟此處答案不同.

1. $\int \frac{\sin^{-1} x}{x^2} dx$	5. $\int x^2 \tan^{-1} x dx$	9. $\int (2x^2 + 1)e^{x^2} dx$
2. $\int \ln(x + \sqrt{1+x^2}) dx$	6. $\int \frac{xe^x}{(x+1)^2} dx$	10. $\int \sin(\ln x) dx$
3. $\int x^3 \ln x dx$	7. $\int x^5 e^{-x^2} dx$	11. $\int x^2 \ln \frac{1-x}{1+x} dx$
4. $\int x(\ln x)^3 dx$	8. $\int xe^{\sqrt{x}} dx$	12. $\int \frac{\ln x}{\sqrt{1+x}} dx$

解.

1. 令 $u = \sin^{-1} x$, 則 $du = \frac{1}{\sqrt{1-x^2}} dx$; $dv = \frac{1}{x^2} dx$, 則 $v = -\frac{1}{x}$. 故 $\int \frac{\sin^{-1} x}{x^2} dx = \sin^{-1} x \cdot \frac{-1}{x} - \int \frac{-1}{x} \cdot \frac{1}{\sqrt{1-x^2}} dx = -\frac{\sin^{-1} x}{x} + \int \frac{1}{x\sqrt{1-x^2}} dx$. 令 $w = \sqrt{1-x^2}$, 則 $-x^2 = w^2 - 1$, $dw = \frac{-x}{\sqrt{1-x^2}} dx$. 故 $\int \frac{1}{x\sqrt{1-x^2}} dx = \int \frac{1}{-x^2} \cdot \frac{-x}{\sqrt{1-x^2}} dx = \int \frac{1}{w^2-1} dw = \frac{1}{2} \int \left(\frac{1}{w-1} - \frac{1}{w+1} \right) dw = \frac{1}{2} (\ln |w-1| - \ln |w+1|) = \frac{1}{2} \ln \left| \frac{w-1}{w+1} \right| = \frac{1}{2} \ln \left| \frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1} \right| = \frac{1}{2} \ln \left| \frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} \right| = \frac{1}{2} \ln \left| \frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} \right|$.

$$\left| \frac{1 - \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}} \right| = \frac{1}{2} \ln \left| \frac{(1 - \sqrt{1 - x^2})^2}{x^2} \right| = \ln \left| \frac{1 - \sqrt{1 - x^2}}{x} \right|.$$

以上, $\int \frac{\sin^{-1} x}{x^2} dx = -\frac{\sin^{-1} x}{x} + \int \frac{1}{x\sqrt{1-x^2}} dx = -\frac{\sin^{-1} x}{x} + \ln \left| \frac{1 - \sqrt{1 - x^2}}{x} \right| + c.$

2. 令 $u = \ln(x + \sqrt{1 + x^2})$, 則 $du = \frac{1}{x + \sqrt{1 + x^2}} \cdot (x + \sqrt{1 + x^2})' dx = \frac{1}{x + \sqrt{1 + x^2}} \cdot \left(1 + \frac{x}{\sqrt{1 + x^2}}\right) dx = \frac{1}{\sqrt{1 + x^2}} dx$; $dv = dx$, 則 $v = x$. 故 $\int \ln(x + \sqrt{1 + x^2}) dx = \ln(x + \sqrt{1 + x^2}) \cdot x - \int x \cdot \frac{1}{\sqrt{1 + x^2}} dx = \ln(x + \sqrt{1 + x^2}) x - \sqrt{1 + x^2} + c.$

3. 令 $u = \ln x$, 則 $du = \frac{1}{x} dx$; $dv = x^3 dx$, 則 $v = \frac{x^4}{4}$. 故 $\int x^3 \ln x dx = \ln x \cdot \frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{x} dx = \ln x \cdot \frac{x^4}{4} - \int \frac{x^3}{4} dx = \ln x \cdot \frac{x^4}{4} - \frac{x^4}{16} + c.$

4. 令 $u = (\ln x)^3$, 則 $du = 3(\ln x)^2 \cdot \frac{1}{x} dx$; $dv = x dx$, 則 $v = \frac{x^2}{2}$. 故 $\int x(\ln x)^3 dx = (\ln x)^3 \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot 3(\ln x)^2 \cdot \frac{1}{x} dx = (\ln x)^3 \cdot \frac{x^2}{2} - \frac{3}{2} \int x(\ln x)^2 dx$. 令 $u = (\ln x)^2$, 則 $du = 2 \ln x \cdot \frac{1}{x} dx$; $dv = x dx$, 則 $v = \frac{x^2}{2}$. 故 $\int x(\ln x)^2 dx = (\ln x)^2 \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot 2 \ln x \cdot \frac{1}{x} dx = (\ln x)^2 \cdot \frac{x^2}{2} - \int x \ln x dx$. 令 $u = \ln x$, 則 $du = \frac{1}{x} dx$; $dv = x dx$, 則 $v = \frac{x^2}{2}$. 故 $\int x \ln x dx = \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx = \ln x \cdot \frac{x^2}{2} - \frac{1}{2} \int x dx = \ln x \cdot \frac{x^2}{2} - \frac{x^2}{4}$. 以上, $\int x(\ln x)^3 dx = (\ln x)^3 \cdot \frac{x^2}{2} - \frac{3}{2} \left((\ln x)^2 \cdot \frac{x^2}{2} - \left(\ln x \cdot \frac{x^2}{2} - \frac{x^2}{4} \right) \right) = \frac{x^2}{2} \left((\ln x)^3 - \frac{3(\ln x)^2}{2} + \frac{3 \ln x}{2} - \frac{3}{4} \right) + c$

5. 令 $u = \tan^{-1} x$, 則 $du = \frac{1}{1+x^2} dx$; $dv = x^2 dx$, 則 $v = \frac{x^3}{3}$. 故 $\int x^2 \tan^{-1} x dx = \tan^{-1} x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{1+x^2} dx$. 又 $\int \frac{x^3}{3} \cdot \frac{1}{x^2+1} dx = \frac{1}{3} \int \frac{(x^3+x)-x}{x^2+1} dx = \frac{1}{3} \int \frac{x(x^2+1)-x}{x^2+1} dx = \frac{1}{3} \int \left(x - \frac{x}{x^2+1}\right) dx$. 令 $w = x^2 + 1$, 則 $dw = 2x dx \Rightarrow x dx = \frac{1}{2} dw$, 故 $\int \frac{x}{x^2+1} dx = \int \frac{1}{x^2+1} \cdot x dx = \int \frac{1}{w} \cdot \frac{1}{2} dw = \frac{1}{2} \ln w + c = \frac{1}{2} \ln(x^2 + 1) + c$. 以上, $\int x^2 \tan^{-1} x dx = \tan^{-1} x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x^2+1} dx = \tan^{-1} x \cdot \frac{x^3}{3} - \frac{x^2}{6} + \frac{\ln(x^2+1)}{6} + c$.

6. $\int \frac{xe^x}{(x+1)^2} dx = \int \frac{(x+1-1)e^x}{(x+1)^2} dx = \int \frac{e^x}{x+1} dx - \int \frac{e^x}{(x+1)^2} dx$. 令 $u = e^x$, 則 $du = e^x dx$; $dv = \frac{1}{(x+1)^2} dx$, 則 $v = \frac{-1}{x+1}$. 故 $\int \frac{e^x}{(x+1)^2} dx = e^x \cdot \frac{-1}{x+1} + \int \frac{1}{x+1} \cdot e^x dx = \frac{-e^x}{x+1} + \int \frac{e^x}{x+1} dx$; 原式 $\int \frac{xe^x}{(x+1)^2} dx = \int \frac{e^x}{x+1} dx - \int \frac{e^x}{(x+1)^2} dx = \int \frac{e^x}{x+1} dx - \left(\frac{-e^x}{x+1} + \int \frac{e^x}{x+1} dx \right) = \frac{e^x}{x+1}$.

7. 令 $w = x^2$, 則 $dw = 2x dx \Rightarrow x dx = \frac{1}{2} dw$, 故 $\int x^5 e^{-x^2} dx = \int e^{-x^2} \cdot (x^2)^2 \cdot x dx = \int e^{-w} \cdot w^2 \cdot \frac{1}{2} dw = \frac{1}{2} \int w^2 e^{-w} dw = -\frac{1}{2} e^{-w} (w^2 + 2w + 2) = -\frac{1}{2} e^{-x^2} (x^4 + 2x^2 + 2) + c$.

Diff	Int
w^2	e^{-w}
$2w$	$-e^{-w}$
2	e^{-w}
0	$-e^{-w}$

$$\int w^2 e^{-w} dw = -w^2 e^{-w} - 2w e^{-w} - 2 e^{-w} = -e^{-w}(w^2 + 2w + 2)$$

8. 令 $w = \sqrt{x}$, 則 $w^2 = x$, $dx = 2w dw$, 故 $\int xe^{\sqrt{x}} dx = \int w^2 e^w \cdot 2w dw = 2 \int w^3 e^w dw = 2 e^w (w^3 - 3w^2 + 6w - 6) = 2 e^{\sqrt{x}} (x\sqrt{x} - 3x + 6\sqrt{x} - 6) + c$

Diff	Int
w^3	e^w
$3w^2$	$-e^w$
$6w$	e^w
6	$-e^w$
0	e^w

$$\int w^3 e^w dw = w^3 e^w - 3w^2 e^w + 6w e^w - 6 e^w = e^w (w^3 - 3w^2 + 6w - 6)$$

9. $\int (2x^2 + 1) e^{x^2} dx = \int 2x^2 e^{x^2} dx + \int e^{x^2} dx$. 令 $u = x$, 則 $du = dx$; $dv = 2x e^{x^2} dx$, 則 $v = e^{x^2}$. 故 $\int 2x^2 e^{x^2} dx = x \cdot e^{x^2} - \int e^{x^2} dx$; 原式 $\int 2x^2 e^{x^2} dx + \int e^{x^2} dx = x \cdot e^{x^2} - \int e^{x^2} dx + \int e^{x^2} dx = xe^{x^2}$

10. 令 $w = \ln x$, 則 $e^w = x$, $dx = e^w dw$, 故 $\int \sin(\ln x) dx = \int \sin w \cdot e^w dw = \int e^w \sin w dw = \frac{e^w(\sin w - \cos w)}{2} = \frac{x(\sin(\ln x) - \cos(\ln x))}{2} + c$

Diff	Int
$\sin w$	e^w
$\cos w$	$-e^w$
$-\sin w$	e^w

$$\begin{aligned} \int e^w \sin w dw &= e^w \sin w - e^w \cos w - \int e^w \sin w dw \\ \Rightarrow \int e^w \sin w dw &= \frac{e^w(\sin w - \cos w)}{2} \end{aligned}$$

11. 令 $u = \ln \frac{1-x}{1+x}$, 則 $du = \frac{1+x}{1-x} \cdot \left(\frac{1-x}{1+x}\right)' dx = \frac{1+x}{1-x} \cdot \frac{(1+x)(-1)-(1-x)}{(1+x)^2} dx = \frac{2}{x^2-1} dx$; $dv = x^2 dx$, 則 $v = \frac{x^3}{3}$. 故 $\int x^2 \ln \frac{1-x}{1+x} dx = \ln \frac{1-x}{1+x} \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{2}{x^2-1} dx$. 又 $\int \frac{x^3}{3} \cdot \frac{2}{x^2-1} dx = \frac{2}{3} \int \frac{x^3}{x^2-1} dx = \frac{2}{3} \int \left(x + \frac{x}{x^2-1}\right) dx = \frac{x^2}{3} + \frac{2}{3} \int \frac{x}{x^2-1} dx$. 令 $w = x^2 - 1$, 則 $dw = 2x dx \Rightarrow x dx = \frac{1}{2} dw$, 故 $\int \frac{x}{x^2-1} dx = \int \frac{1}{x^2-1} \cdot x dx = \int \frac{1}{w} \cdot \frac{1}{2} dw = \frac{1}{2} \ln w + c = \frac{1}{2} \ln(x^2 - 1) + c$. 以上, $\int x^2 \ln \frac{1-x}{1+x} dx = \frac{1+x}{1-x} \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x^2-1} dx = \ln \frac{1-x}{1+x} \cdot \frac{x^3}{3} - \frac{x^2 + \ln(x^2 - 1)}{3} + c$.

12. 令 $u = \ln x$, 則 $du = \frac{1}{x} dx$; $dv = \frac{1}{\sqrt{1+x}} dx$, 則 $v = 2\sqrt{1+x}$. 故 $\int \frac{\ln x}{\sqrt{1+x}} dx = \ln x \cdot 2\sqrt{1+x} - 2 \int \frac{\sqrt{1+x}}{x} dx$. 令 $w = \sqrt{1+x}$, 則 $w^2 = 1+x \Rightarrow x = w^2 - 1$, $2w dw = dx$, 故 $\int \frac{\sqrt{1+x}}{x} dx = \int \frac{w}{w^2-1} \cdot 2w dw = 2 \int \frac{w^2}{w^2-1} dw = 2 \int \frac{(w^2-1)+1}{w^2-1} dw = 2w + 2 \int \frac{1}{w^2-1} dw = 2\sqrt{1+x} +$

$$2 \int \frac{1}{w^2 - 1} dw. \text{ 由 } \frac{1}{w^2 - 1} = \frac{1}{2} \left(\frac{1}{w-1} - \frac{1}{w+1} \right), \int \frac{1}{w^2 - 1} dw = \frac{1}{2} \int \left(\frac{1}{w-1} - \frac{1}{w+1} \right) dw = \frac{1}{2} (\ln|w-1| - \ln|w+1|) = \frac{1}{2} (\ln|\sqrt{1+x}-1| - \ln|\sqrt{1+x}+1|). \text{ 以上, } \int \frac{\ln x}{\sqrt{1+x}} dx = \ln x \cdot 2\sqrt{1+x} - 2(2\sqrt{1+x} + 2(\frac{1}{2}(\ln|\sqrt{1+x}-1| - \ln|\sqrt{1+x}+1|))) = \ln x \cdot 2\sqrt{1+x} - 4\sqrt{1+x} - 2\ln|\sqrt{1+x}-1| + 2\ln|\sqrt{1+x}+1| + c.$$

4.2 定積分

定積分 \approx (帶符號) 面積: x 軸上方為正, 下方為負.

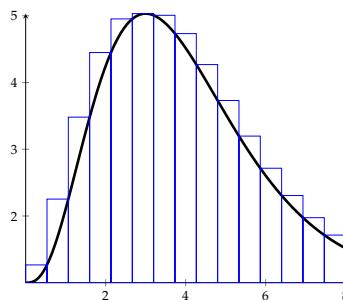
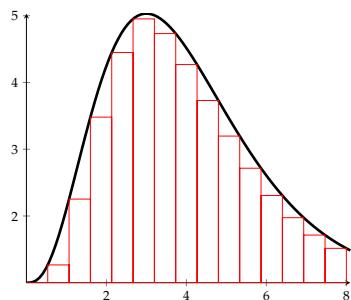
定義. 給定 $f : [a, b] \rightarrow \mathbb{R}$.

- $[a, b]$ 分割 $\mathbb{P} : a = x_0 < x_1 < x_2 < \dots < x_n = b$
- $\Delta x_k = x_k - x_{k-1}, k = 1, 2, \dots, n; \|\mathbb{P}\| = \max\{|\Delta x_k| \mid 1 \leq k \leq n\}$
- 樣本點 $\xi_k : x_{k-1} \leq \xi_k \leq x_k, k = 1, 2, \dots, n$
- $u_k = \sup\{f(x) \mid x_{k-1} \leq x \leq x_k\}, l_k = \inf\{f(x) \mid x_{k-1} \leq x \leq x_k\}, k = 1, 2, \dots, n$
- $R(f, \mathbb{P}) = \sum_{k=1}^n f(\xi_k) \Delta x_k, U(f, \mathbb{P}) = \sum_{k=1}^n u_k \Delta x_k, L(f, \mathbb{P}) = \sum_{k=1}^n l_k \Delta x_k;$
顯然 $L(f, \mathbb{P}) \leq R(f, \mathbb{P}) \leq U(f, \mathbb{P})$.
- 求 $\lim_{\|\mathbb{P}\| \rightarrow 0} R(f, \mathbb{P})$. 若對不同分割與樣本點選取此極限均存在且相等, 稱 f 在 $[a, b]$ 可積 (分); $f(x)$ 在 $[a, b]$ 的定積分 $\int_a^b f(x) dx \equiv \lim_{\|\mathbb{P}\| \rightarrow 0} R(f, \mathbb{P})$

註.

- 在 $\int_a^b f(x) dx$ 中, a 為積分下限 (lower limit of integration), b 為積分上限 (upper limit of integration), $f(x)$ 為被積分式 (integrand), x 為積分變數 (variable of integration).
- $\int_a^b f(x) dx = \int_a^b f(t) dt = \int_a^b f(u) du$ (定積分數值與積分變數無關)

結論. 若 f 在 $[a, b]$ 連續, 則 f 在 $[a, b]$ 可積.



性質. 令 f, g 在包含 a, b, c 之區間為可積, $\alpha, \beta \in \mathbb{R}$. 則

1. $\int_a^a f(x) dx = 0$

$$2. \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$3. \int_a^b (\alpha f(x) + \beta g(x)) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx$$

$$4. \int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$

$$5. \int_a^b f(x) dx \leq \int_a^b g(x) dx, \text{ 若 } f(x) \leq g(x) \forall x \in [a, b], a \leq b$$

$$6. \left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx, \quad a \leq b$$

$$7. f(x) \text{ 為奇函數: } \int_{-a}^a f(x) dx = 0$$

$$8. f(x) \text{ 為偶函數: } \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

例.

$$1. \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-x^2} dx = \pi \quad (\text{半徑 } \sqrt{2} \text{ 之半圓面積})$$

$$2. \text{定義 } \operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}, \text{ 則 } \int_{-1}^2 \operatorname{sgn}(x) dx = 2 \cdot 1 - 1 \cdot 1 = 1$$

$$3. \int_{-\pi}^{\pi} \sin(x^7 - 5x^3) dx = 0 \quad (\sin(x^7 - 5x^3) \text{ 為奇函數})$$

$$4. \int_{-6}^6 e^{-x^4} \sin(\sin x) dx = 0 \quad (e^{-x^4} \sin(\sin x) \text{ 為奇函數})$$

$$5. \int_{-4}^4 (e^x - e^{-x}) dx = 0 \quad (e^x - e^{-x} \text{ 為奇函數})$$

$$6. \int_{-2025}^{2025} (e^{9x^5-2x^7} - e^{-9x^5+2x^7}) dx = 0 \quad (e^{9x^5-2x^7} - e^{-9x^5+2x^7} \text{ 為奇函數})$$

$$7. \int_{-a}^a |x| dx = 2 \int_0^a |x| dx = a^2 \quad (|x| \text{ 為偶函數; 兩 } a \times a \text{ 等腰直角三角形面積})$$

$$8. \text{定義 } \int_1^x \frac{1}{\tau} d\tau = \ln x, \text{ 則 } \int_{\frac{1}{4}}^3 \frac{1}{x} dx = \int_{\frac{1}{4}}^1 \frac{1}{x} dx + \int_1^3 \frac{1}{x} dx = - \int_1^{\frac{1}{4}} \frac{1}{x} dx + \int_1^3 \frac{1}{x} dx = \ln 12$$

$$9. \int_0^2 \sqrt{4-x^2} \cdot \operatorname{sgn}(1-x) dx = \int_0^1 \sqrt{4-x^2} dx - \int_1^2 \sqrt{4-x^2} dx = \left(\frac{4\pi}{12} + \frac{\sqrt{3}}{2} \right) - \left(\frac{4\pi}{6} - \frac{\sqrt{3}}{2} \right) = \sqrt{3} - \frac{\pi}{3}$$

$$10. \forall n \in \mathbb{N}, \int_n^{n+1} \lfloor x \rfloor dx = n$$

以極限定義求定積分

$$\text{結論. } \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

證. 由 $k(k+1) = \frac{k(k+1)(k+2) - (k-1)k(k+1)}{3}$, $\sum_{k=1}^n k(k+1) = \frac{n(n+1)(n+2)}{3}$. 又 $k^2 = k(k+1) - k$,

$$\sum_{k=1}^n k^2 = \sum_{k=1}^n (k(k+1) - k) = \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2} = \frac{n(n+1)(2n+1)}{6}.$$

例. 求 $\int_0^1 x^2 dx$.

解. 建立 $[0, 1]$ 分割 $\mathbb{P} : \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\right\}$, 則 $\Delta x_k = \frac{1}{n} \forall k = 1, 2, \dots, n$ 且 $\|P\| \rightarrow 0$ 當 $n \rightarrow \infty$. $f(x) = x^2$ 且 f

在 $[0, 1]$ 嚴格遞增, $U(f, \mathbb{P}) = \sum_{k=1}^n u_k \Delta x_k = \sum_{k=1}^n \frac{k^2}{n^2} \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^n k^2$, $L(f, \mathbb{P}) = \sum_{k=1}^n l_k \Delta x_k = \sum_{k=1}^n \frac{(k-1)^2}{n^2} \cdot \frac{1}{n} =$

$\frac{1}{n^3} \sum_{k=1}^{n-1} k^2$, $L(f, \mathbb{P}) \leq R(f, \mathbb{P}) \leq U(f, \mathbb{P})$. 由 $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$, $\lim_{n \rightarrow \infty} U(f, \mathbb{P}) = \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{k=1}^n k^2 =$

$\lim_{n \rightarrow \infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}$, $\lim_{n \rightarrow \infty} L(f, \mathbb{P}) = \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{k=1}^{n-1} k^2 = \lim_{n \rightarrow \infty} \frac{(n-1)n(2n-1)}{6n^3} = \frac{1}{3}$, 故由夾擊定理

$$\lim_{n \rightarrow \infty} R(f, \mathbb{P}) = \frac{1}{3}.$$

結論. 若 $r \in \mathbb{R}$, $\sum_{k=0}^n r^k = \frac{r^{n+1} - 1}{r - 1}$.

證. 令 $s = \sum_{k=0}^n r^k$, 則 $rs = \sum_{k=1}^{n+1} r^k$; $rs - s = r^{n+1} - 1 \implies s = \frac{r^{n+1} - 1}{r - 1}$.

例. 求 $\int_0^1 e^x dx$.

解. 建立 $[0, 1]$ 分割 $\mathbb{P} : \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\right\}$, 則 $\Delta x_k = \frac{1}{n} \forall k = 1, 2, \dots, n$ 且 $\|P\| \rightarrow 0$ 當 $n \rightarrow \infty$. $f(x) = e^x$

且 f 在 $[0, 1]$ 嚴格遞增, $U(f, \mathbb{P}) = \sum_{k=1}^n u_k \Delta x_k = \sum_{k=1}^n e^{\frac{k}{n}} \cdot \frac{1}{n} = \frac{1}{n} \frac{e^{\frac{1}{n}}(e^{\frac{n}{n}} - 1)}{e^{\frac{1}{n}} - 1} = \frac{e-1}{n} \frac{e^{\frac{1}{n}}}{e^{\frac{1}{n}} - 1}$, $L(f, \mathbb{P}) =$

$\sum_{k=1}^n l_k \Delta x_k = \sum_{k=1}^n e^{\frac{k-1}{n}} \cdot \frac{1}{n} = \frac{1}{n} \frac{e^0(e^{\frac{n}{n}} - 1)}{e^{\frac{1}{n}} - 1} = \frac{e-1}{n} \frac{1}{e^{\frac{1}{n}} - 1}$, $L(f, \mathbb{P}) \leq R(f, \mathbb{P}) \leq U(f, \mathbb{P})$. $\lim_{n \rightarrow \infty} U(f, \mathbb{P}) =$

$\lim_{n \rightarrow \infty} \frac{e-1}{n} \frac{e^{\frac{1}{n}}}{e^{\frac{1}{n}} - 1} = (e-1) \lim_{x \rightarrow 0^+} \frac{xe^x}{e^x - 1} = (e-1) \lim_{x \rightarrow 0^+} \frac{e^x + xe^x}{e^x} = e-1$, $\lim_{n \rightarrow \infty} L(f, \mathbb{P}) = \lim_{n \rightarrow \infty} \frac{e-1}{n} \frac{1}{e^{\frac{1}{n}} - 1} =$

$(e-1) \lim_{x \rightarrow 0^+} \frac{x}{e^x - 1} = (e-1) \lim_{x \rightarrow 0^+} \frac{1}{e^x} = e-1$, 故由夾擊定理 $\lim_{n \rightarrow \infty} R(f, \mathbb{P}) = e-1$.

結論. $\sum_{k=1}^n \cos kx = \frac{1}{2} \left(\frac{\sin(n + \frac{1}{2})x}{\sin \frac{x}{2}} - 1 \right)$

證. 考慮 $\sum_{k=1}^n e^{ikx} = \frac{e^{ix}(1 - e^{inx})}{1 - e^{ix}}$, 則 $\sum_{k=1}^n \cos kx = \Re \left\{ \sum_{k=1}^n e^{ikx} \right\} = \Re \left\{ \frac{e^{ix}(1 - e^{inx})}{1 - e^{ix}} \right\} = \Re \left\{ \frac{e^{ix} e^{\frac{inx}{2}} (e^{-\frac{inx}{2}} - e^{\frac{inx}{2}})}{e^{\frac{ix}{2}} (e^{-\frac{ix}{2}} - e^{\frac{ix}{2}})} \right\} =$

$\frac{\cos \frac{(n+1)x}{2} \sin \frac{nx}{2}}{\sin \frac{x}{2}} = \frac{\sin \left(\frac{nx}{2} + \frac{(n+1)x}{2} \right) + \sin \left(\frac{nx}{2} + \frac{(n+1)x}{2} \right)}{2 \sin \frac{x}{2}} = \frac{\sin \left(n + \frac{1}{2} \right)x - \sin \frac{x}{2}}{2 \sin \frac{x}{2}} = \frac{1}{2} \left(\frac{\sin \left(n + \frac{1}{2} \right)x}{\sin \frac{x}{2}} - 1 \right)$.

例. 求 $\int_0^{\frac{\pi}{2}} \cos x dx$.

解. 建立 $[0, \frac{\pi}{2}]$ 分割 $\mathbb{P} : \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \dots, \frac{n\pi}{2n}\right\}$, 則 $\Delta x_k = \frac{\pi}{2n} \forall k = 1, 2, \dots, n$ 且 $\|P\| \rightarrow 0$ 當 $n \rightarrow \infty$, 則積分為 $\lim_{n \rightarrow \infty} R(f, \mathbb{P}) = \sum_{k=1}^n f(\xi_k) \Delta x_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n \cos \frac{k\pi}{2n} \cdot \frac{\pi}{2n} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \cos \left(k \left(\frac{\pi}{2n}\right)\right) \cdot \frac{\pi}{2n} = \lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{\sin((n + \frac{1}{2}) \cdot \frac{\pi}{2n})}{\sin(\frac{1}{2} \cdot \frac{\pi}{2n})} - 1 \right) \cdot \frac{\pi}{2n} = \lim_{n \rightarrow \infty} \frac{\sin((n + \frac{1}{2}) \frac{\pi}{2n}) \frac{1}{2} \cdot \frac{\pi}{2n}}{\sin(\frac{1}{2} \cdot \frac{\pi}{2n})} = \lim_{n \rightarrow \infty} \sin\left(\frac{1}{2} + \frac{1}{4n}\right) \pi \cdot \lim_{n \rightarrow \infty} \frac{\frac{1}{2} \cdot \frac{\pi}{2n}}{\sin(\frac{1}{2} \cdot \frac{\pi}{2n})} = \sin\frac{\pi}{2} \cdot 1 = 1.$

微積分基本定理

複習：連續函數性質

定理 (中間值定理). 若 f 在 $[a, b]$ 連續, 則對任意介於 $f(a)$ 與 $f(b)$ 之間的數 d , 存在 $c \in [a, b]$ 使得 $f(c) = d$.

定理 (最大最小值定理). 若函數在定義域為有限閉區間, 或有限閉區間的有限聯集連續, 則函數在定義域上有最大值及最小值.

積分均值定理

定理 (積分均值定理). 設 f 在 $[a, b]$ 連續, 則存在 $c \in (a, b)$ 使得 $\int_a^b f(x) dx = f(c) \cdot (b - a)$.

證. 第一步: 建立上下界. 因 f 在 $[a, b]$ 連續, 根據最大最小值定理, f 在 $[a, b]$ 有最大值 M 及最小值 m , 亦即 $\exists x_m, x_M$ 使 $m = f(x_m) \leq f(x) \leq f(x_M) = M, \forall x \in [a, b]$.

第二步: 積分的夾擠. 對不等式 $m \leq f(x) \leq M$ 在 $[a, b]$ 上積分, 得 $m(b - a) \leq \int_a^b f(x) dx \leq M(b - a)$; 除以 $(b - a) > 0$ 得 $m \leq \frac{1}{b - a} \int_a^b f(x) dx \leq M$.

第三步: 應用中間值定理. 令 $\mu = \frac{1}{b - a} \int_a^b f(x) dx$: 我們已證明 $m \leq \mu \leq M$. 因為 f 連續且 $f(x_m) = m$, $f(x_M) = M$, 由中間值定理, 存在 c 介於 x_m 與 x_M 間 (故 $c \in [a, b]$), 使 $f(c) = \mu = \frac{1}{b - a} \int_a^b f(x) dx \implies \int_a^b f(x) dx = f(c) \cdot (b - a)$.

定理 (微積分基本定理 I (積分的微分)). 設 f 在 $[a, b]$ 上連續. 定義 $F(x) = \int_a^x f(t) dt, x \in [a, b]$, 則 F 在 (a, b) 可微, 且 $F'(x) = f(x)$.

證. 對於 $x \in (a, b)$ 及充分小的 $h \neq 0$, 考慮 $\frac{F(x + h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t) dt$. 因 f 在 $[a, b]$ 連續, 由積分均值定理, 存在 c_h 介於 x 與 $x + h$ 間, 使 $\int_x^{x+h} f(t) dt = f(c_h) \cdot h$, 故 $\frac{F(x + h) - F(x)}{h} = f(c_h)$. 當 $h \rightarrow 0$, $c_h \rightarrow x$. 由 f 的連續性, $f(c_h) \rightarrow f(x)$. 故 $F'(x) = \lim_{h \rightarrow 0} \frac{F(x + h) - F(x)}{h} = f(x)$.

定理 (微積分基本定理 II (牛頓-萊布尼茲公式)). 設 f 在 $[a, b]$ 連續, 且 G 是 f 的任一反導函數 (即 $G'(x) = f(x)$), 則 $\int_a^b f(x) dx = G(b) - G(a)$.

證. 由微積分基本定理 I, $F(x) = \int_a^x f(t) dt$ 滿足 $F'(x) = f(x)$. 因為 G 也是 f 的反導函數, F 與 G 相差一個常數: $F(x) = G(x) + C$. 由 $F(a) = \int_a^a f(t) dt = 0$ 得 $0 = G(a) + C \implies C = -G(a)$. 故 $\int_a^b f(x) dx = F(b) = G(b) + C = G(b) - G(a)$.

例 (以 FTC 求定積分).

$$1. x^2 \text{ 之反導函數為 } \frac{x^3}{3}, \text{ 故 } \int_0^1 x^2 dx = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

$$2. e^x \text{ 之反導函數為 } e^x, \text{ 故 } \int_0^1 e^x dx = e^1 - e^0 = e - 1.$$

$$3. \cos x \text{ 之反導函數為 } \sin x, \text{ 故 } \int_0^{\frac{\pi}{2}} \cos x dx = \sin \frac{\pi}{2} - \sin 0 = 1.$$

結論 (定積分變數變換).

- 求反導函數後代入: $\int_a^b f'(g(x)) g'(x) dx = f(g(x)) \Big|_{x=a}^{x=b} = f(g(b)) - f(g(a))$
- 變數變換並改變積分範圍: $\int_a^b f'(g(x)) g'(x) dx = \int_a^b f'(g(x)) dg(x) = \int_{g(a)}^{g(b)} f'(u) du = f(u) \Big|_{u=g(a)}^{u=g(b)} = f(g(b)) - f(g(a))$

例. 求 $\int_0^1 x^3(1+x^4)^3 dx$.

解.

- 求反導函數後代入: 令 $u = 1 + x^4$, 則 $du = 4x^3 dx \implies x^3 dx = \frac{du}{4}$, 故 $\int x^3(1+x^4)^3 dx = \int (1+x^4)^3 x^3 dx = \int u^3 \frac{du}{4} = \frac{u^4}{16} + c = \frac{(1+x^4)^4}{16} + c$. 故 $\int_0^1 x^3(1+x^4)^3 dx = \frac{(1+x^4)^4}{16} \Big|_{x=0}^{x=1} = \frac{(1+1^4)^4 - (1+0^4)^4}{16} = \frac{15}{16}$.

- 變數變換並改變積分範圍: 令 $u = 1 + x^4$, 則 $du = 4x^3 dx \implies x^3 dx = \frac{du}{4}$. 積分範圍 x 由 0 至 1, 則 變數變換後 u 由 $1+0^4 = 1$ 至 $1+1^4 = 2$, 故 $\int_0^1 x^3(1+x^4)^3 dx = \int_0^1 (1+x^4)^3 x^3 dx = \int_1^2 u^3 \frac{du}{4} = \frac{1}{4} \int_1^2 u^3 du = \frac{1}{16} u^4 \Big|_{u=1}^{u=2} = \frac{2^4 - 1^4}{16} = \frac{15}{16}$.

例. 求 $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{1+x^2}} dx$.

解.

- 求反導函數後代入: 令 $u = 1 + x^2$, 則 $du = 2x dx \implies 4x dx = 2 du$, 故 $\int \frac{4x}{\sqrt{1+x^2}} dx = \int \frac{2}{\sqrt{u}} du = 4\sqrt{u} + c = 4\sqrt{1+x^2} + c$. 故 $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{1+x^2}} dx = 4\sqrt{1+x^2} \Big|_{x=0}^{x=\sqrt{3}} = 4\sqrt{1+3} - 4\sqrt{1} = 4$.
- 變數變換並改變積分範圍: 令 $u = 1 + x^2$, 則 $du = 2x dx \implies 4x dx = 2 du$. 積分範圍 x 由 0 至 $\sqrt{3}$, 則變數變換後 u 由 $1+0^2 = 1$ 至 $1+(\sqrt{3})^2 = 4$, 故 $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{1+x^2}} dx = \int_1^4 \frac{2}{\sqrt{u}} du = 4\sqrt{u} \Big|_{u=1}^{u=4} = 4(\sqrt{4} - \sqrt{1}) = 4$.

例. 求 $\int_0^\pi 3 \cos^2 x \sin x \, dx$.

解.

- 求反導函數後代入: 令 $u = \cos x$, 則 $du = -\sin x \, dx \implies \sin x \, dx = -du$, 故 $\int 3 \cos^2 x \sin x \, dx = -3 \int u^2 \, du = -u^3 + c = -\cos^3 x + c$. 故 $\int_0^\pi 3 \cos^2 x \sin x \, dx = -\cos^3 x \Big|_{x=0}^{x=\pi} = -(\cos^3 \pi - \cos^3 0) = -((-1)^3 - 1^3) = 2$.
- 變數變換並改變積分範圍: 令 $u = \cos x$, 則 $du = -\sin x \, dx \implies \sin x \, dx = -du$. 積分範圍 x 由 0 至 π , 則變數變換後 u 由 $\cos 0 = 1$ 至 $\cos \pi = -1$, 故 $\int_0^\pi 3 \cos^2 x \sin x \, dx = - \int_1^{-1} 3u^2 \, du = -u^3 \Big|_{u=1}^{u=-1} = -((-1)^3 - 1^3) = 2$.

例. 若 f 在 $[a, b]$ 二次可微且 $f(a) = f(b) = 0$, 證明 $\int_a^b (x-a)(b-x) f''(x) \, dx = -2 \int_a^b f(x) \, dx$.

解. $\int_a^b (x-a)(b-x) f''(x) \, dx = ((x-a)(b-x) f'(x) - (a+b-2x) f(x)) \Big|_a^b - 2 \int_a^b f(x) \, dx = ((b-a)(b-b) f'(b) - (a+b-2b) f(b)) - ((a-a)(b-a) f'(a) - (a+b-2a) f(a)) - 2 \int_a^b f(x) \, dx = -2 \int_a^b f(x) \, dx$.

Diff	Int	
$(x-a)(b-x)$	$f''(x)$	$\int (x-a)(b-x) f''(x) \, dx$
$a+b-2x$	$f'(x)$	$= ((x-a)(b-x) f'(x) - (a+b-2x) f(x)) - 2 \int f(x) \, dx$
-2	$f(x)$	
0	$\int f(x) \, dx$	

性質. 令 $F(x) = \int_{v(x)}^{u(x)} f(\tau) \, d\tau$, 則 $F'(x) = f(u(x)) \cdot u'(x) - f(v(x)) \cdot v'(x)$.

證. 令 $a \in \mathbb{R}$, $F(x) = \int_{v(x)}^{u(x)} f(\tau) \, d\tau = \int_a^{u(x)} f(\tau) \, d\tau - \int_a^{v(x)} f(\tau) \, d\tau$. 令 $G(x) \equiv \int_a^x f(\tau) \, d\tau$, 則 $G'(x) = f(x)$, $F(x) = \int_a^{u(x)} f(\tau) \, d\tau - \int_a^{v(x)} f(\tau) \, d\tau = G(u(x)) - G(v(x))$; 故 $F'(x) = (G(u(x)) - G(v(x)))' = G'(u(x)) \cdot u'(x) - G'(v(x)) \cdot v'(x) = f(u(x)) \cdot u'(x) - f(v(x)) \cdot v'(x)$.

例.

1. $F(x) = \int_1^x \frac{1}{1+\tau^4} \, d\tau \implies F'(x) = \frac{1}{1+x^4}$
2. $F(x) = \int_2^{\sqrt{x}} \sin \tau \, d\tau \implies F'(x) = \sin \sqrt{x} \cdot \frac{1}{2\sqrt{x}}$
3. $F(x) = \int_x^{2x} \tau^3 \, d\tau \implies F'(x) = (2x)^3 \cdot 2 - x^3 \cdot 1 = 15x^3$
4. $F(x) = \int_{\sin x}^{\tan^{-1} x} e^{\tau^2} \, d\tau \implies F'(x) = e^{(\tan^{-1} x)^2} \cdot \frac{1}{1+x^2} - e^{\sin^2 x} \cdot \cos x$

例. 若 $g(x) = \int_0^{\cos x} (1 + \sin(t^2)) \, dt$, $f(x) = \int_0^{g(x)} \frac{1}{\sqrt{1+t^3}} \, dt$, 求 $f'\left(\frac{\pi}{2}\right)$.

解. $f'(x) = \frac{1}{\sqrt{1+g(x)^3}} \cdot g'(x)$, $g'(x) = (1+\sin(\cos^2 x)) \cdot (-\sin x)$. 代入 $x = \frac{\pi}{2}$, $g\left(\frac{\pi}{2}\right) = 0$, $g'\left(\frac{\pi}{2}\right) = -1$, 故 $f'\left(\frac{\pi}{2}\right) = -1$.

例. 若 $\int_0^{x^2} f(t) dt = x \sin \pi x$, 求 $f'(9)$.

解. $\int_0^{x^2} f(t) dt = x \sin \pi x$ 兩邊對 x 微分得 $f(x^2) \cdot 2x = \sin \pi x + x \cdot \pi \cos \pi x$. 兩邊再對 x 微分得 $(f'(x^2) \cdot 2x) \cdot 2x + f(x^2) \cdot 2 = \pi \cos \pi x + \pi \cos \pi x - x \cdot \pi^2 \sin \pi x$. 代入 $x = 3$, 則 $(f'(9) \cdot 2 \cdot 3) \cdot 2 \cdot 3 + f(9) \cdot 2 = \pi \cos 3\pi + \pi \cos 3\pi - 3 \cdot \pi^2 \sin 3\pi \Rightarrow f'(9) \cdot 36 + f(9) \cdot 2 = -2\pi$. 又 $f(3^2) \cdot (2 \cdot 3) = \sin 3\pi + 3 \cdot \pi \cos 3\pi \Rightarrow f(9) = -\frac{\pi}{2}$, 故 $f'(9) = -\frac{\pi}{36}$.

例. 求函數 f 與 $a \in \mathbb{R}$ 使 $6 + \int_a^x \frac{f(t)}{t^2} dt = 2\sqrt{x}$, $\forall x > 0$.

解. $6 + \int_a^x \frac{f(t)}{t^2} dt = 2\sqrt{x}$ 兩邊對 x 微分得 $\frac{f(x)}{x^2} = \frac{1}{\sqrt{x}} \Rightarrow f(x) = x^{\frac{3}{2}}$. 代入原式得 $6 + \int_a^x \frac{t^{\frac{3}{2}}}{t^2} dt = 6 + \int_a^x \frac{1}{\sqrt{t}} dt = 6 + 2\sqrt{t} \Big|_a^x = 6 + 2\sqrt{x} - 2\sqrt{a} = 2\sqrt{x} \Rightarrow a = 9$.

例. 求下列極限.

$$1. \lim_{x \rightarrow 0} \frac{\int_0^x (\sec t - 1) dt}{x^3}$$

$$2. \lim_{x \rightarrow \infty} \frac{\int_{x^2}^0 e^{t-x^2} (2t^2 + 1) dt}{x^4}$$

$$3. \lim_{x \rightarrow 0} \frac{1}{x} \int_0^{\tan x} f(u)(\sin x - \cos u) du, f \text{ 為連續函數}$$

$$4. \lim_{x \rightarrow 0} \frac{1}{x} \int_0^x (1 - \tan 2t)^{\frac{1}{t}} dt$$

解.

$$1. \lim_{x \rightarrow 0} \frac{\int_0^x (\sec t - 1) dt}{x^3} = \lim_{x \rightarrow 0} \frac{\sec x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{\sec x \tan x}{6x} = \lim_{x \rightarrow 0} \frac{\sec x \tan x \cdot \tan x + \sec x \cdot \sec^2 x}{6} = \frac{1}{6}.$$

$$2. \lim_{x \rightarrow \infty} \frac{\int_{x^2}^0 e^{t-x^2} (2t^2 + 1) dt}{x^4} = \lim_{x \rightarrow \infty} \frac{\int_{x^2}^0 e^t (2t^2 + 1) dt}{e^{x^2} x^4} = \lim_{x \rightarrow \infty} \frac{-e^{x^2} (2(x^2)^2 + 1) \cdot 2x}{e^{x^2} 2x \cdot x^4 + e^{x^2} \cdot 4x^3} = \lim_{x \rightarrow \infty} \frac{-(4x^5 + 2x)}{2x^5 + 4x^3} = -2.$$

$$3. \lim_{x \rightarrow 0} \frac{1}{x} \int_0^{\tan x} f(u)(\sin x - \cos u) du = \lim_{x \rightarrow 0} \frac{\int_0^{\tan x} f(u)(\sin x - \cos u) du}{x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x \int_0^{\tan x} f(u) du - \int_0^{\tan x} f(u) \cos u du}{x}$$

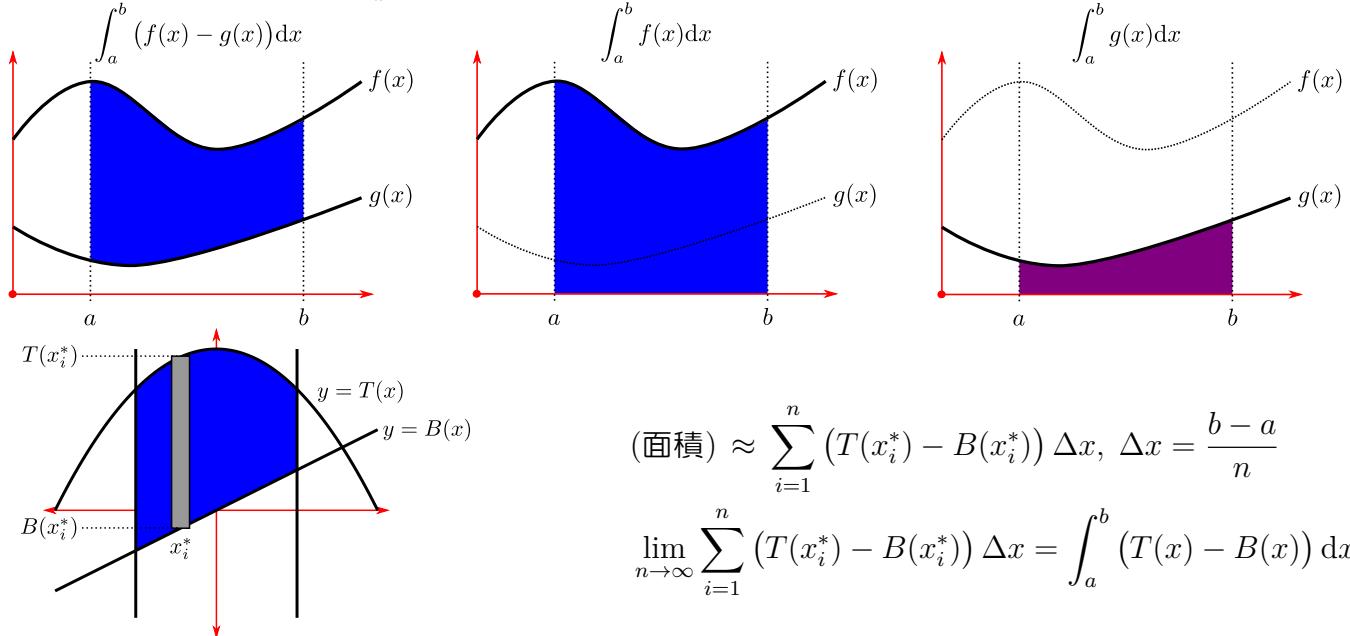
$$= \lim_{x \rightarrow 0} \frac{\cos x \int_0^{\tan x} f(u) du + \sin x \cdot f(\tan x) \cdot \sec^2 x - f(\tan x) \cos(\tan x) \cdot \sec^2 x}{1} = -f(0)$$

$$4. \lim_{x \rightarrow 0} \frac{\int_0^x (1 - \tan 2t)^{\frac{1}{t}} dt}{x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{(1 - \tan 2x)^{\frac{1}{x}}}{1} = \exp \left(\lim_{x \rightarrow 0} \frac{\ln(1 - \tan 2x)}{x} \right) \stackrel{H}{=} \exp \left(\lim_{x \rightarrow 0} \frac{-2 \sec^2 2x}{1 - \tan 2x} \right) = \exp \left(\frac{-2 \cdot 1^2}{1 - 0} \right) = e^{-2}$$

4.3 面積與體積

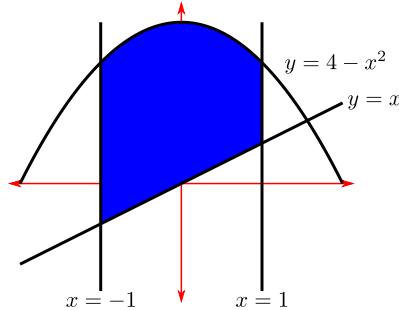
面積

結論. 當 $f(x) \geq 0 \forall x \in [a, b]$, $\int_a^b f(x) dx$ 為 $y = f(x)$, x 軸, $x = a$, 與 $x = b$ 所圍成之面積.



例. 求以 $y = 4 - x^2$, $y = x$, $x = -1$, 與 $x = 1$ 圍成之區域面積.

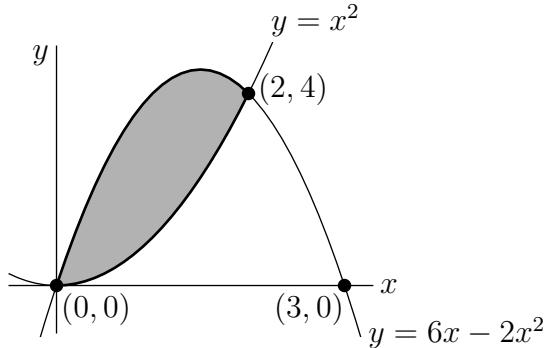
解.



$$(面積) = \int_{-1}^1 ((4 - x^2) - x) dx = \frac{22}{3}$$

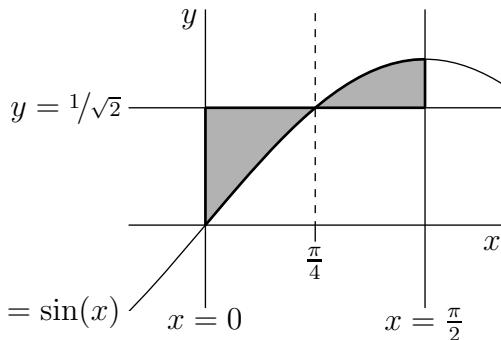
例. 求 $y = x^2$ 與 $y = 6x - 2x^2$ 圍成之區域面積.

解.



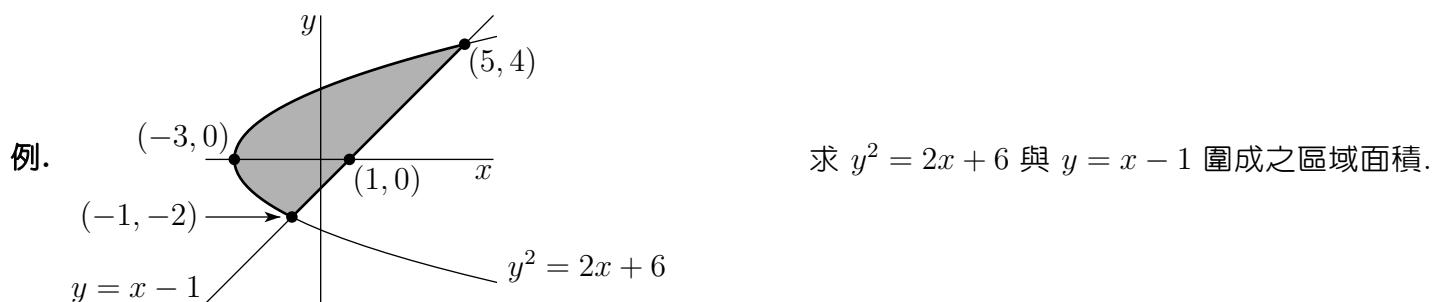
$$(面積) = \int_0^2 ((6x - 2x^2) - x^2) dx = 4$$

例. 求 $y = \frac{1}{\sqrt{2}}$ 與 $y = \sin x$ 在 x 從 0 至 $\frac{\pi}{2}$ 範圍內圍成之區域面積.



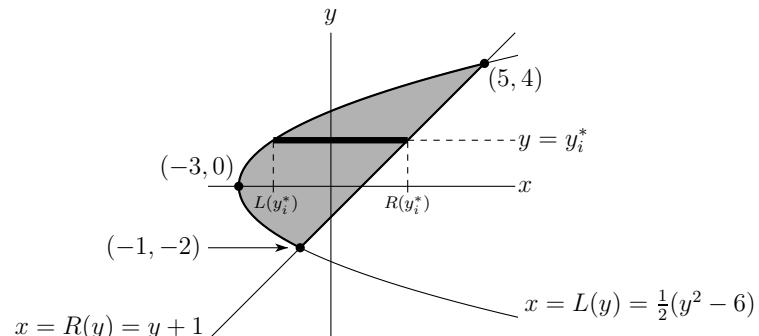
$$(面積) = \int_0^{\frac{\pi}{4}} \left(\frac{1}{\sqrt{2}} - \sin x \right) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\sin x - \frac{1}{\sqrt{2}} \right) dx = \sqrt{2} - 1$$

解.

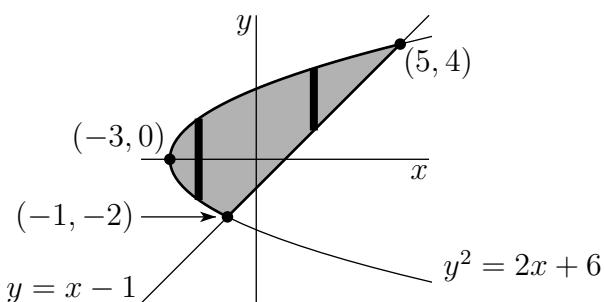


求 $y^2 = 2x + 6$ 與 $y = x - 1$ 圍成之區域面積.

解.

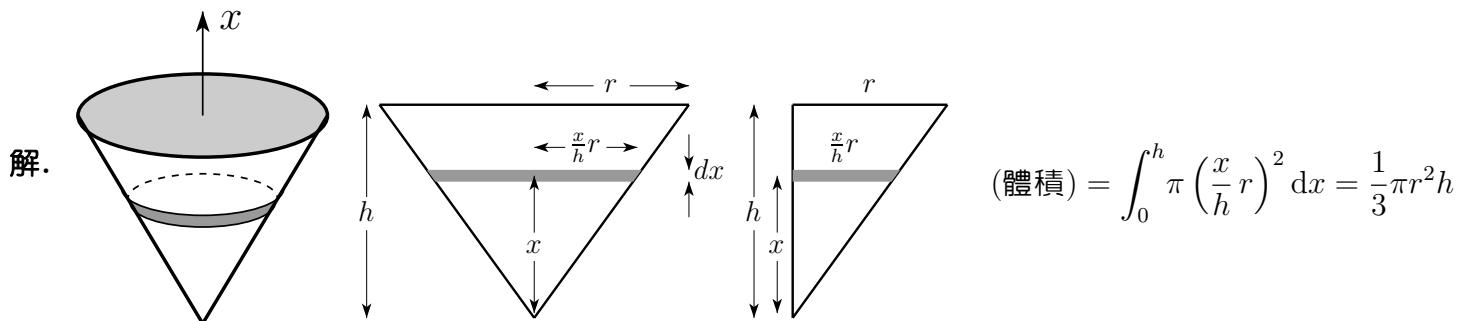


$$(面積) = \int_{-2}^4 \left((y+1) - \frac{1}{2}(y^2 - 6) \right) dy = 18$$



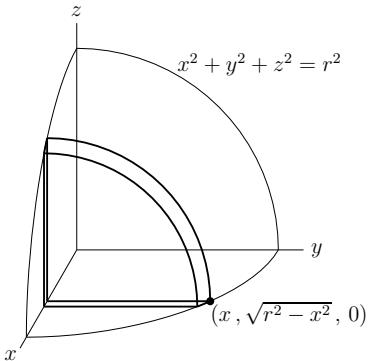
$$(面積) = \int_{-3}^{-1} 2\sqrt{2x+6} dx + \int_{-1}^5 (\sqrt{2x+6} - x + 1) dx = 18$$

例. 求高 h 與底半徑 r 之圓錐體體積.



$$(體積) = \int_0^h \pi \left(\frac{x}{h} r \right)^2 dx = \frac{1}{3} \pi r^2 h$$

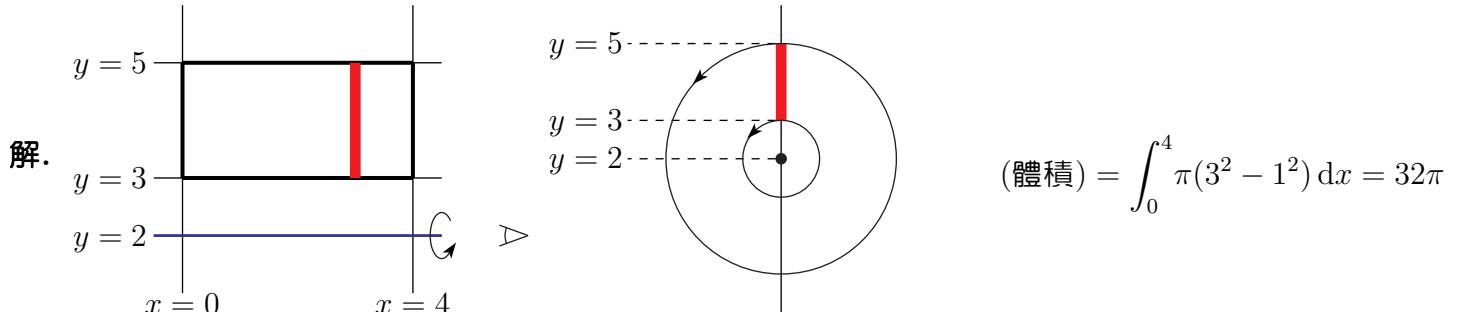
例. 求半徑 r 之三維球體體積.



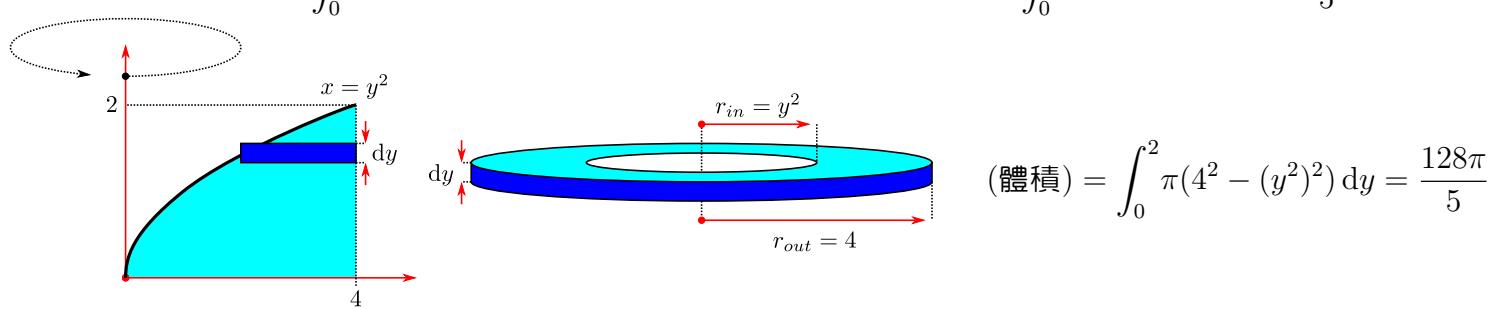
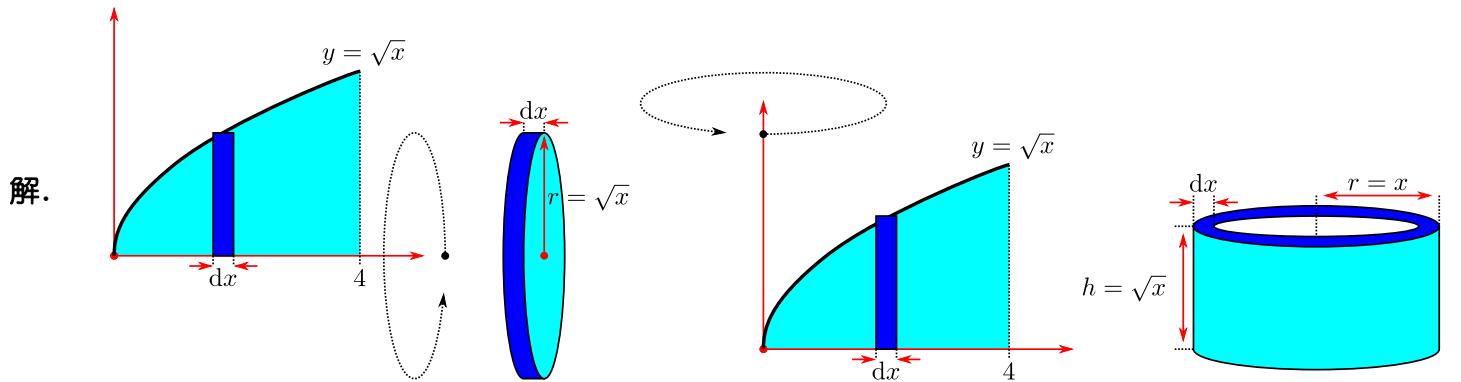
解.

$$(\text{體積}) = 8 \cdot \int_0^r \frac{\pi}{4} (\sqrt{r^2 - x^2})^2 dx = \frac{4}{3} \pi r^3$$

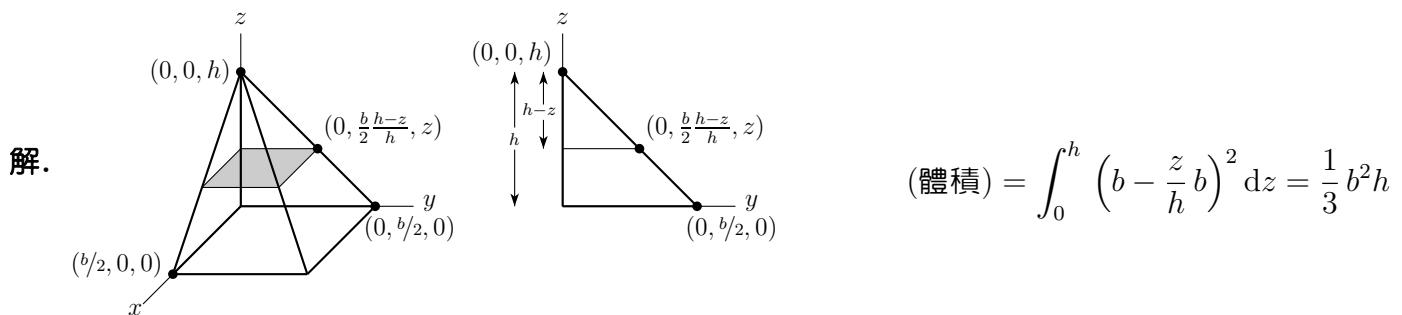
例. 求以 $y = 3, y = 5, x = 0$ 與 $x = 4$ 圍成之區域繞 $y = 2$ 旋轉而成之旋轉體體積.



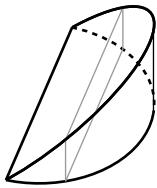
例. 求以 $y = \sqrt{x}, y = 0, x = 0$ 與 $x = 4$ 圍成之區域繞 (i) $y = 0$ (ii) $x = 0$ 旋轉而成之旋轉體體積.



例. 求高為 h , 底面為邊長 b 正方形之錐體體積.

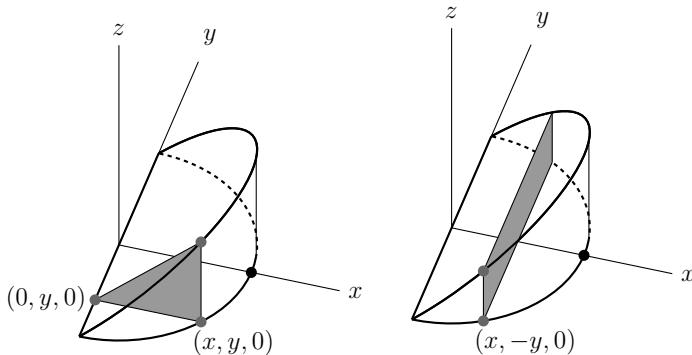


例.



將一半徑為 a 之圓柱體水平橫切, 再對其底面圓心 45° 角斜切, 求如圖所示結果體積.

解.



$$(\text{體積}) = \int_{-a}^a \frac{1}{2} (\sqrt{a^2 - y^2})^2 dy = \frac{2}{3} a^3$$

$$(\text{體積}) = \int_0^a x \cdot 2\sqrt{a^2 - x^2} dx = \frac{2}{3} a^3$$

4.4 積分技巧

部份分式

例 (動機). 若 $a \neq 0$, 求 $\int \frac{1}{x^2 - a^2} dx$.

解. 由 $\frac{1}{x^2 - a^2} = \frac{1}{2a} \left(\frac{1}{x-a} - \frac{1}{x+a} \right)$, $\int \frac{1}{x^2 - a^2} dx = \int \frac{1}{2a} \left(\frac{1}{x-a} - \frac{1}{x+a} \right) dx = \frac{1}{2a} (\ln|x-a| - \ln|x+a|) = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right|$

結論.

- 實係數多項式可分解成不可約的一次及二次因式的乘積.
- 有理式可寫成多項式與真分式之和.
- 若 $\frac{p(x)}{q(x)}$ 為一真分式, $q(x) = (x+a_1)^{m_1}(x+a_2)^{m_2} \cdots (x+a_k)^{m_k} \cdot (x^2+b_1x+c_1)^{n_1}(x^2+b_2x+c_2)^{n_2} \cdots (x^2+b_lx+c_l)^{n_l}$, 其中 $(x+a_i)$, $(x^2+b_ix+c_i)$ 均相異, $(x^2+b_ix+c_i)$ 為不可分解之二次式 ($b_i^2 - 4c_i < 0$), 則

$$\begin{aligned} \frac{p(x)}{q(x)} &= \frac{\alpha_{11}}{x+a_1} + \frac{\alpha_{12}}{(x+a_1)^2} + \cdots + \frac{\alpha_{1m_1}}{(x+a_1)^{m_1}} + \\ &\quad \frac{\alpha_{21}}{x+a_2} + \frac{\alpha_{22}}{(x+a_2)^2} + \cdots + \frac{\alpha_{2m_2}}{(x+a_2)^{m_2}} + \cdots + \\ &\quad \frac{\alpha_{k1}}{x+a_k} + \frac{\alpha_{k2}}{(x+a_k)^2} + \cdots + \frac{\alpha_{km_k}}{(x+a_k)^{m_k}} + \\ &\quad \frac{\beta_{11}x + \gamma_{11}}{x^2 + b_1x + c_1} + \frac{\beta_{12}x + \gamma_{12}}{(x^2 + b_1x + c_1)^2} + \cdots + \frac{\beta_{1n_1}x + \gamma_{1n_1}}{(x^2 + b_1x + c_1)^{n_1}} + \\ &\quad \frac{\beta_{21}x + \gamma_{21}}{x^2 + b_2x + c_2} + \frac{\beta_{22}x + \gamma_{22}}{(x^2 + b_2x + c_2)^2} + \cdots + \frac{\beta_{2n_2}x + \gamma_{2n_2}}{(x^2 + b_2x + c_2)^{n_2}} + \cdots + \\ &\quad \frac{\beta_{l1}x + \gamma_{l1}}{x^2 + b_lx + c_l} + \frac{\beta_{l2}x + \gamma_{l2}}{(x^2 + b_lx + c_l)^2} + \cdots + \frac{\beta_{ln_l}x + \gamma_{ln_l}}{(x^2 + b_lx + c_l)^{n_l}} \end{aligned}$$

其中 $\alpha_{ij}, \beta_{ij}, \gamma_{ij} \in \mathbb{R}$.

註. 任一有理函數之積分可分解為多項式積分與以下兩型積分:

$$\bullet \int \frac{1}{(x+a)^n} dx, n \in \mathbb{N} \quad \bullet \int \frac{f(x)}{(x^2+bx+c)^n} dx, f(x) = 1 \text{ 或 } f(x) = x; b^2 - 4c < 0, n \in \mathbb{N}$$

例. 求 $\int \frac{x}{x^2 - 5x + 6} dx$.

解. $\frac{x}{x^2 - 5x + 6} = \frac{x}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} \Rightarrow x = A(x-2) + B(x-3)$. 代入 $x=3 \Rightarrow 3=A$; 代入 $x=2 \Rightarrow 2=B(2-3) \Rightarrow B=-2$, 故 $\frac{x}{x^2 - 5x + 6} = \frac{3}{x-3} - \frac{2}{x-2}$, $\int \frac{x}{x^2 - 5x + 6} dx = \int \left(\frac{3}{x-3} - \frac{2}{x-2} \right) dx = 3 \ln|x-3| - 2 \ln|x-2|$

例. 求 $\int \frac{1}{x^3 + 1} dx$.

解. $\frac{1}{x^3 + 1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \Rightarrow 1 = A(x^2-x+1) + (Bx+C)(x+1)$. 代入 $x=-1 \Rightarrow 1 = A((-1)^2 - (-1) + 1) = 3A \Rightarrow A = \frac{1}{3}$; 代入 $x=0 \Rightarrow 1 = A + C \Rightarrow C = \frac{2}{3}$; 代入 $x=1 \Rightarrow 1 = A + (B+C) \cdot 2 \Rightarrow 1 = \frac{1}{3} + \left(B + \frac{2}{3}\right) \cdot 2 \Rightarrow B = -\frac{1}{3}$; 故 $\frac{1}{x^3 + 1} = \frac{1}{3} \left(\frac{1}{x+1} - \frac{x-2}{x^2-x+1}\right)$, $\int \frac{1}{x^3 + 1} dx = \int \frac{1}{3} \left(\frac{1}{x+1} - \frac{x-2}{x^2-x+1}\right) dx = \int \frac{1}{3} \left(\frac{1}{x+1} - \frac{\frac{1}{2}(2x-1) - \frac{3}{2}}{x^2-x+1}\right) dx = \frac{1}{3} \int \frac{1}{x+1} dx - \frac{1}{6} \int \frac{2x-1}{x^2-x+1} dx - \frac{1}{2} \int \frac{1}{x^2-x+1} dx = \frac{1}{3} \ln|x+1| - \frac{1}{6} \ln(x^2-x+1) - \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{3}}$

例. 求 $\int \frac{1}{x^4 + 4} dx$.

解. 由 $x^4 + 4 = (x^2 + 2)^2 - (2x)^2 = (x^2 - 2x + 2)(x^2 + 2x + 2)$, $\frac{1}{x^4 + 4} = \frac{Ax+B}{x^2 - 2x + 2} + \frac{Cx+D}{x^2 + 2x + 2} \Rightarrow (Ax+B)(x^2+2x+2)+(Cx+D)(x^2-2x+2)=1 \Rightarrow (A+C)x^3+(2A+B-2C+D)x^2+(2A+2B+2C-2D)x+(2B+2D)=1 \Rightarrow A+C=0, 2A+B-2C+D=0, 2A+2B+2C-2D=0, 2B+2D=1 \Rightarrow A=-\frac{1}{8}$, $B=\frac{1}{4}$, $C=\frac{1}{8}$, $D=\frac{1}{4}$. 故 $\frac{1}{x^4 + 4} = \frac{1}{8} \left(\frac{x+2}{x^2+2x+2} - \frac{x-2}{x^2-2x+2} \right)$, $\int \frac{1}{x^4 + 4} dx = \frac{1}{8} \int \left(\frac{x+2}{x^2+2x+2} - \frac{x-2}{x^2-2x+2} \right) dx = \frac{1}{8} \int \left(\frac{(x+1)+1}{(x+1)^2+1} - \frac{(x-1)-1}{(x-1)^2+1} \right) dx = \frac{1}{16} \ln \frac{x^2+2x+2}{x^2-2x+2} + \frac{1}{8} (\tan^{-1}(x+1) - \tan^{-1}(x-1))$

例. 求 $\int \frac{1}{\cos^3 x} dx$.

解. $\int \frac{1}{\cos^3 x} dx = \int \frac{\cos x}{\cos^4 x} dx = \int \frac{\cos x}{(1-\sin^2 x)^2} dx$. 令 $u = \sin x$, 则 $du = \cos x dx$, 则 $\int \frac{\cos x}{(1-\sin^2 x)^2} dx = \int \frac{1}{(1-u^2)^2} du$. $\frac{1}{(1-u^2)^2} = \frac{1}{(u-1)^2(u+1)^2} = \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{C}{u+1} + \frac{D}{(u+1)^2} \Rightarrow 1 = A(u-1)(u+1)^2 + B(u+1)^2 + C(u-1)^2(u+1) + D(u-1)^2$. 代入 $u=1 \Rightarrow 1=4B \Rightarrow B=\frac{1}{4}$; 代入 $u=-1 \Rightarrow 1=4D \Rightarrow D=\frac{1}{4}$; 代入 $u=0 \Rightarrow 1=-A+B+C+D \Rightarrow 1=-A+\frac{1}{4}+\frac{1}{4} \Rightarrow \frac{1}{2}=-A+C$; 代入 $u=2 \Rightarrow 1=9A+9B+3C+D \Rightarrow 1=9A+\frac{9}{4}+3C+\frac{1}{4} \Rightarrow A=-\frac{1}{4}, C=\frac{1}{4}$; 则 $\frac{1}{(1-u^2)^2} = \frac{1}{4} \left(\frac{-1}{u-1} + \frac{1}{(u-1)^2} + \frac{1}{u+1} + \frac{1}{(u+1)^2} \right)$, $\int \frac{1}{(1-u^2)^2} du = \int \frac{1}{4} \left(\frac{-1}{u-1} + \frac{1}{(u-1)^2} + \frac{1}{u+1} + \frac{1}{(u+1)^2} \right) dx = \frac{1}{4} \left(-\ln|u-1| - \frac{1}{u-1} + \ln|u+1| - \frac{1}{u+1} \right) = \frac{1}{4} \left(-\ln|\sin x-1| - \frac{1}{\sin x-1} + \ln|\sin x+1| - \frac{1}{\sin x+1} \right)$.

$$\text{部份分式另解: } \frac{1}{(1-u^2)^2} = \left(\frac{1}{u^2-1}\right)^2 = \left(\frac{1}{(u-1)(u+1)}\right)^2 = \frac{1}{4} \left(\frac{1}{u-1} - \frac{1}{u+1}\right)^2 = \frac{1}{4} \left(\frac{1}{(u-1)^2} - \frac{2}{(u-1)(u+1)} + \frac{1}{(u+1)^2}\right) = \frac{1}{4} \left(\frac{1}{(u-1)^2} - \left(\frac{1}{u-1} - \frac{1}{u+1}\right) + \frac{1}{(u+1)^2}\right) = \frac{1}{4} \left(\frac{-1}{u-1} + \frac{1}{(u-1)^2} + \frac{1}{u+1} + \frac{1}{(u+1)^2}\right).$$

三角函數代換

結論.

- 遇 $\sqrt{a^2 - x^2}$, 考慮 $x = a \sin \theta \implies \theta = \sin^{-1} \frac{x}{a}$, $dx = a \cos \theta d\theta$
- 遇 $\sqrt{a^2 + x^2}$, 考慮 $x = a \tan \theta \implies \theta = \tan^{-1} \frac{x}{a}$, $dx = a \sec^2 \theta d\theta$
- 遇 $\sqrt{x^2 - a^2}$, 考慮 $x = a \sec \theta \implies \theta = \sec^{-1} \frac{x}{a}$, $dx = a \sec \theta \tan \theta d\theta$
- 遇 $\sin x, \cos x$ 之有理式, 考慮 $u = \tan \frac{x}{2}$, 由以下化為 u 之有理式:

$$\begin{aligned} - \sin x &= 2 \sin \frac{x}{2} \cos \frac{x}{2} = 2 \frac{u}{\sqrt{1+u^2}} \frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2} \\ - \cos x &= 2 \cos^2 \frac{x}{2} - 1 = 2 \cdot \frac{1}{1+u^2} - 1 = \frac{1-u^2}{1+u^2} \\ - du &= \frac{1}{2} \sec^2 \frac{x}{2} dx \implies dx = \frac{2}{1+u^2} du \end{aligned}$$

例. 若 $a \neq 0$, 求下列不定積分.

1. $\int \sqrt{a^2 - x^2} dx$	4. $\int \frac{1}{\sqrt{x^2 + a^2}} dx$	7. $\int \frac{1}{\sqrt{x^2 - a^2}} dx$
2. $\int \sqrt{x^2 + a^2} dx$	5. $\int \frac{1}{x^2 \sqrt{x^2 + a^2}} dx$	8. $\int \frac{1}{\tan x + \sin x} dx$
3. $\int x^2 \sqrt{x^2 + a^2} dx$	6. $\int \sqrt{x^2 - a^2} dx$	9. $\int \frac{1}{a + \sin x} dx, a > 1$

解.

- 令 $x = a \sin \theta$, 則 $\int \sqrt{a^2 - x^2} dx = \int \sqrt{a^2 - a^2 \sin^2 \theta} a \cos \theta d\theta = a^2 \int \cos^2 \theta d\theta = a^2 \int \frac{1 + \cos 2\theta}{2} d\theta = \frac{a^2}{2} \theta + \frac{a^2}{4} \sin 2\theta = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{a^2}{2} \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a} = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2}.$
- 令 $x = a \tan \theta$, 則 $\int \sqrt{x^2 + a^2} dx = \int a \sec \theta \cdot a \sec^2 \theta d\theta = a^2 \int \sec^3 \theta d\theta = \frac{a^2}{2} (\sec \theta \cdot \tan \theta + \ln |\sec \theta + \tan \theta|) = \frac{a^2}{2} \left(\frac{x}{a} \cdot \frac{\sqrt{x^2 + a^2}}{a} + \ln \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| \right) = \frac{x \sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \ln \left| \sqrt{x^2 + a^2} + x \right| - \frac{a^2}{2} \ln |a|$
- 令 $x = a \tan \theta$, 則 $\int x^2 \sqrt{x^2 + a^2} dx = \int a \sec^2 \theta \cdot a^2 \tan^2 \theta \cdot a \sec \theta d\theta = a^4 \int \sec^3 \theta \tan^2 \theta d\theta = a^4 \int \sec^3 \theta (\sec^2 \theta - 1) d\theta = a^4 \int (\sec^5 \theta - \sec^3 \theta) d\theta = \frac{a^4}{4} \left(\sec^3 \theta \tan \theta - \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \right) = \frac{a^4}{4} \left(\sec^3 \theta \tan \theta - \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \right) = \frac{x(x^2 + a^2)^{\frac{3}{2}}}{4} - \frac{a^2 x \sqrt{x^2 + a^2}}{8} - \frac{a^4 \ln |\sqrt{x^2 + a^2} + x|}{8} + \frac{a^4 \ln |a|}{8}.$

$$4. \text{ 令 } x = a \tan \theta, \text{ 則 } \int \frac{1}{\sqrt{x^2 + a^2}} dx = \int \frac{1}{a \sec \theta} \cdot a \sec^2 \theta d\theta = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| \\ = \ln \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| = \ln |\sqrt{x^2 + a^2} + x| - \ln |a|.$$

$$5. \text{ 令 } x = a \tan \theta, \text{ 則 } \int \frac{1}{x^2 \sqrt{x^2 + a^2}} dx = \int \frac{a \sec^2 \theta}{a^2 \tan^2 \theta \cdot a \sec \theta} d\theta = \int \frac{\sec \theta}{a^2 \tan^2 \theta} d\theta = \int \frac{\cos \theta}{a^2 \sin^2 \theta} d\theta \\ = -\frac{1}{a^2 \sin \theta} = -\frac{\sqrt{x^2 + a^2}}{a^2 x}.$$

$$6. \text{ 令 } x = a \sec \theta, \text{ 則 } \int \sqrt{x^2 - a^2} dx = \int a \tan \theta \cdot a \sec \theta \tan \theta d\theta = a^2 \int \sec \theta \tan^2 \theta d\theta = a^2 \int \sec \theta (\sec^2 \theta - 1) d\theta \\ = a^2 \left(\int \sec^3 \theta d\theta - \int \sec \theta d\theta \right) = \frac{a^2}{2} \left(\sec \theta \cdot \tan \theta + \int \sec \theta d\theta - 2 \int \sec \theta d\theta \right) = \frac{a^2}{2} \left(\sec \theta \cdot \tan \theta - \int \sec \theta d\theta \right) = \frac{a^2}{2} (\sec \theta \cdot \tan \theta - \ln |\sec \theta + \tan \theta|) = \frac{a^2}{2} \left(\frac{x}{a} \cdot \frac{\sqrt{x^2 - a^2}}{a} - \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| \right) = \\ \frac{x \sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| = \frac{x \sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \ln |\sqrt{x^2 - a^2} + x| + \frac{a^2}{2} \ln |a|.$$

$$7. \text{ 令 } x = a \sec \theta, \text{ 則 } \int \frac{1}{\sqrt{x^2 - a^2}} dx = \int \frac{a \sec \theta \tan \theta}{a \tan \theta} d\theta = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| \\ = \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| = \ln |\sqrt{x^2 - a^2} + x| - \ln |a|.$$

$$8. \text{ 令 } u = \tan \frac{x}{2}, \text{ 則 } \int \frac{1}{\tan x + \sin x} dx = \int \frac{1}{\frac{2u}{1-u^2} + \frac{2u}{1+u^2}} \frac{2}{1+u^2} du = \int \frac{1-u^2}{2u} du = \frac{\ln |u|}{2} - \frac{u^2}{4} = \\ \frac{\ln |\tan \frac{x}{2}|}{2} - \frac{\tan^2 \frac{x}{2}}{4}$$

$$9. \text{ 令 } u = \tan \frac{x}{2}, \text{ 則 } \int \frac{1}{a + \sin x} dx = \int \frac{1}{a + \frac{2u}{1+u^2}} \frac{2}{1+u^2} du = 2 \int \frac{1}{au^2 + 2u + a} du = \frac{2}{a \sqrt{\frac{a^2-1}{a^2}}} \tan^{-1} \frac{u + \frac{1}{a}}{\sqrt{\frac{a^2-1}{a^2}}} \\ = \frac{2}{\sqrt{a^2-1}} \tan^{-1} \frac{au+1}{\sqrt{a^2-1}} = \frac{2}{\sqrt{a^2-1}} \tan^{-1} \frac{a \tan \frac{x}{2} + 1}{\sqrt{a^2-1}}.$$

4.5 瑕積分

定義 (瑕積分 (improper integral)).

- 無限區間 (第一型) 瑕積分

- 若 $f(x)$ 在 $[a, \infty)$ 連續, 則 $\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$.
- 若 $f(x)$ 在 $(-\infty, b]$ 連續, 則 $\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$.
- 若 $f(x)$ 在 $(-\infty, \infty)$ 連續, 則任取 $c \in \mathbb{R}$, $\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^\infty f(x) dx$.

- 不連續點 (第二型) 瑕積分

- 若 $f(x)$ 在 $(a, b]$ 連續, 則 $\int_a^b f(x) dx = \lim_{c \rightarrow a+} \int_c^b f(x) dx$.
- 若 $f(x)$ 在 $[a, b)$ 連續, 則 $\int_a^b f(x) dx = \lim_{c \rightarrow b-} \int_a^c f(x) dx$.

- 令 $c \in (a, b)$. 若 $f(x)$ 在 $[a, c) \cup (c, b]$ 連續且在 $x = c$ 不連續, 則 $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$.

例. 1. $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx.$ 2. $\int_1^{\infty} \frac{1}{x^2} dx.$ 3. $\int_0^1 \frac{1}{\sqrt{x}} dx.$

解.

$$1. \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = 2 \int_0^{\infty} \frac{1}{1+x^2} dx = 2 \lim_{b \rightarrow \infty} \int_0^b \frac{1}{1+x^2} dx = 2 \lim_{b \rightarrow \infty} \tan^{-1} b = 2 \cdot \frac{\pi}{2} = \pi.$$

$$2. \int_1^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + 1 \right) = 1$$

$$3. \int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{c \rightarrow 0+} \int_c^1 \frac{1}{\sqrt{x}} dx = \lim_{c \rightarrow 0+} 2\sqrt{x} \Big|_c^1 = \lim_{c \rightarrow 0+} (2 - 2\sqrt{c}) = 2$$

例. 證明 $\forall n \in \mathbb{N}, \int_0^1 (\ln x)^n dx = (-1)^n n!$

解. 使用數學歸納法: $n = 1$ 時 $\int_0^1 \ln x dx = (x \ln x - x) \Big|_0^1 = -1 - \lim_{x \rightarrow 0+} x \ln x = -1 + \lim_{x \rightarrow 0+} \frac{\ln \frac{1}{x}}{\frac{1}{x}} = -1 + \lim_{y \rightarrow \infty} \frac{\ln y}{y} = -1 = (-1)^1 1!$. 令等式在 $n - 1$ 成立: $\int_0^1 (\ln x)^{n-1} dx = (-1)^{n-1} (n-1)!$, 則 $\int_0^1 (\ln x)^n dx = x (\ln x)^n \Big|_0^1 - n \int_0^1 x \cdot (\ln x)^{n-1} \cdot \frac{1}{x} dx = 0 - \lim_{x \rightarrow 0+} x (\ln x)^n + (-1) \cdot n \cdot (-1)^{n-1} (n-1)! = - \lim_{x \rightarrow 0+} x (\ln x)^n + (-1)^n n!$

反覆使用 L'Hôpital 法則得 $\lim_{x \rightarrow 0+} x (\ln x)^n = (-1)^n \cdot \lim_{x \rightarrow 0+} \frac{(\ln \frac{1}{x})^n}{\frac{1}{x}} = (-1)^n \lim_{y \rightarrow \infty} \frac{(\ln y)^n}{y} = (-1)^n \cdot n \lim_{y \rightarrow \infty} \frac{(\ln y)^{n-1}}{y} = (-1)^n \cdot n(n-1) \lim_{y \rightarrow \infty} \frac{(\ln y)^{n-2}}{y} = \dots = (-1)^n n! \lim_{y \rightarrow \infty} \frac{1}{y} = 0$, 故 $\int_0^1 (\ln x)^n dx = (-1)^n n!$ 成立.

定理. 給定 $0 < a < \infty$.

- $\int_a^{\infty} \frac{1}{x^p} dx$ 當 $p > 1$ 收斂至 $\frac{a^{1-p}}{p-1}$, 當 $p \leq 1$ 發散至 ∞ .

- $\int_0^a \frac{1}{x^p} dx$ 當 $p < 1$ 收斂至 $\frac{a^{1-p}}{1-p}$, 當 $p \geq 1$ 發散至 ∞ .

定理. 令 $-\infty \leq a < b \leq \infty$, f, g 在 (a, b) 連續, 且 $0 \leq f(x) \leq g(x) \forall x$.

- 若 $\int_a^b g(x) dx$ 收斂, $\int_a^b f(x) dx$ 收斂.
- 若 $\int_a^b f(x) dx$ 發散, $\int_a^b g(x) dx$ 發散.

定理. 令 f, g 在 $[a, \infty)$, $a \in \mathbb{R}$ 連續, 均為正值, 且 $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$ 存在, 則 $\int_a^{\infty} f(x) dx$ 與 $\int_a^{\infty} g(x) dx$ 同斂散.

例. 證明 $\int_0^{\infty} e^{-x^2} dx$ 收斂.

解. 由 $e^{-x^2} \leq 1 \forall 0 \leq x < 1$ 及 $e^{-x^2} \leq e^{-x} \forall x \geq 1$, $\int_0^{\infty} e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^{\infty} e^{-x^2} dx \leq \int_0^1 1 dx + \int_1^{\infty} e^{-x} dx = 1 + \frac{1}{e}$, 故 $\int_0^{\infty} e^{-x^2} dx$ 收斂.

例. 定義 Γ 函數 $\Gamma(x) \equiv \int_0^{\infty} t^{x-1} e^{-t} dt$; 已知 $\int_0^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$.

1. 證明 $\Gamma(x)$ 收斂, $\forall x > 0$.
2. 證明 $\Gamma(x+1) = x\Gamma(x)$, $\forall x > 0$.

3. 證明 $\Gamma(n+1) = n!$, $\forall n \in \mathbb{N}$.
4. 求 $\Gamma\left(\frac{1}{2}\right)$ 與 $\Gamma\left(\frac{3}{2}\right)$.

解.

1. $\int_0^\infty t^{x-1}e^{-t} dt = \int_0^1 t^{x-1}e^{-t} dt + \int_1^\infty t^{x-1}e^{-t} dt$. $\int_1^\infty t^{x-1}e^{-t} dt$ 收斂, 因為

- $\int_0^1 t^{x-1}e^{-t} dt \leq \int_0^1 t^{x-1} dt = \frac{t^x}{x} \Big|_{t=0}^{t=1} = \frac{1}{x}$, $\int_0^1 t^{x-1}e^{-t} dt$ 收斂.
- $\int_1^\infty \frac{1}{t^2} dt$ 收斂, $\lim_{t \rightarrow \infty} \frac{t^{x-1}e^{-t}}{\frac{1}{t^2}} = \lim_{t \rightarrow \infty} \frac{t^{x+1}}{e^t} = 0$, $\int_1^\infty t^{x-1}e^{-t} dt$ 收斂.

2. $\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = \lim_{\substack{a \rightarrow 0+ \\ b \rightarrow \infty}} \int_a^b t^x e^{-t} dt$. 令 $u = t^x$, 則 $du = xt^{x-1} dt$. 令 $dv = e^{-t} dt$, 則 $v = -e^{-t}$.
故 $\lim_{\substack{a \rightarrow 0+ \\ b \rightarrow \infty}} \int_a^b t^x e^{-t} dt = \lim_{\substack{a \rightarrow 0+ \\ b \rightarrow \infty}} \left(-t^x e^{-t} \Big|_a^b + \int_a^b e^{-t} \cdot xt^{x-1} dt \right) = 0 + x \int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x)$.

3. 由上 $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-2) = \dots = n(n-1)\dots 2\Gamma(1)$, 又 $\Gamma(1) = \lim_{b \rightarrow \infty} \int_0^b e^{-t} dt = \lim_{b \rightarrow \infty} -e^{-t} \Big|_0^b = \lim_{b \rightarrow \infty} 1 - e^{-b} = 1$, 得證.

4. (a) $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt = \int_0^\infty \frac{e^{-t}}{\sqrt{t}} dt$. 令 $u = \sqrt{t}$, 則 $t = u^2$, $du = \frac{1}{2\sqrt{t}} dt \implies \frac{1}{\sqrt{t}} dt = 2 du$. 積分範圍 t 由 0 至 ∞ , 則變數變換後 u 由 $\sqrt{0} = 0$ 至 $\sqrt{\infty} = \infty$, 故 $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}} dt = 2 \int_0^\infty e^{-u^2} du = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$.

- (b) 由 $\Gamma(x+1) = x\Gamma(x)$, $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2} + 1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$.

4.6 Leibniz 積分法則: 積分號下取微分

定理 (Leibniz 積分法則). 設 (X, \mathcal{A}, μ) 為測度空間, $U \subseteq \mathbb{R}$ 為開區間, 函數 $f : X \times U \rightarrow \mathbb{R}$ (或 \mathbb{C}) 滿足

- (i) **可測性:** 對每個 $t \in U$, 函數 $x \mapsto f(x, t)$ 是 \mathcal{A} -可測的.
- (ii) **可微性:** 對 μ -幾乎所有 $x \in X$, 函數 $t \mapsto f(x, t)$ 在 U 上可微.
- (iii) **控制條件:** 存在函數 $g \in L^1(\mu)$ 使得對所有 $t \in U$ 及 μ -幾乎所有 $x \in X$, $\left| \frac{\partial f}{\partial t}(x, t) \right| \leq g(x)$.

則

- (a) 對每個 $t \in U$, 函數 $x \mapsto f(x, t)$ 屬於 $L^1(\mu)$.
- (b) 函數 $F(t) := \int_X f(x, t) d\mu(x)$ 在 U 上可微.
- (c) 微分與積分可交換次序: $\frac{dF}{dt}(t) = \int_X \frac{\partial f}{\partial t}(x, t) d\mu(x)$.

重要特例

系理 (Lebesgue 測度下的經典形式). 設 $U, V \subseteq \mathbb{R}$ 為開區間, $f : U \times V \rightarrow \mathbb{R}$ 滿足

- (i) f 與 $\frac{\partial f}{\partial t}$ 在 $U \times V$ 上連續;
- (ii) 存在可積函數 $g : U \rightarrow [0, \infty)$ 使得 $\left| \frac{\partial f}{\partial t}(x, t) \right| \leq g(x)$ 對所有 $(x, t) \in U \times V$ 成立.

則 $F(t) = \int_U f(x, t) dx$ 在 V 上可微, 且 $F'(t) = \int_U \frac{\partial f}{\partial t}(x, t) dx$.

系理 (緊支撐或有界區間情形). 設 $[a, b] \subset \mathbb{R}$ 為有界閉區間, $V \subseteq \mathbb{R}$ 為開區間, $f : [a, b] \times V \rightarrow \mathbb{R}$ 滿足

- (i) f 對 x 可積, 對 t 可微; (ii) $\frac{\partial f}{\partial t}$ 在 $[a, b] \times V$ 上連續.

則 $\frac{d}{dt} \int_a^b f(x, t) dx = \int_a^b \frac{\partial f}{\partial t}(x, t) dx$.

證. 連續函數在緊集 $[a, b] \times K$ ($K \subset V$ 為任意緊子區間) 上有界, 故控制條件自動滿足.

應用範例

例. $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

證. 令 $F(t) = \int_0^\infty \frac{e^{-t^2(1+x^2)}}{1+x^2} dx, t \geq 0$; $F(\infty) = 0, F(0) = \int_0^\infty \frac{dx}{1+x^2} = \tan^{-1} x \Big|_0^\infty = \tan^{-1} \infty - \tan^{-1} 0 = \frac{\pi}{2}$.
 則 $F'(t) = \int_0^\infty \frac{-2t(1+x^2)e^{-t^2(1+x^2)}}{1+x^2} dx = \underbrace{-2te^{-t^2} \int_0^\infty e^{-t^2x^2} dx}_{\text{令 } u=tx, \text{ 則 } du=t dx} = -2e^{-t^2} \int_0^\infty e^{-u^2} du = -2e^{-t^2} I \Rightarrow$

$\int_0^b F'(t) dt = -2I \int_0^b e^{-t^2} dt \Rightarrow F(b) - F(0) = -2I \int_0^b e^{-t^2} dt$. 令 $b \rightarrow \infty$, 則 $F(\infty) - F(0) = -2I^2 \Rightarrow 0 - \frac{\pi}{2} = -2I^2 \Rightarrow I = \frac{\sqrt{\pi}}{2}$.

例. $\int_0^\infty x^{2n} e^{-ax^2} dx = \frac{(2n-1)!!}{2^{n+1}} \sqrt{\frac{\pi}{a^{2n+1}}} = \frac{(2n)!}{n! 4^n} \sqrt{\frac{\pi}{a^{2n+1}}}, \forall a > 0, n \in \mathbb{N}$, 其中 $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$.

證. 令 $F(a) = \int_0^\infty e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}}$, 則 $F'(a) = - \int_0^\infty x^2 e^{-ax^2} dx = -\frac{1}{2} \sqrt{\pi} \cdot (-\frac{1}{2}) a^{-\frac{3}{2}} = \frac{\sqrt{\pi}}{4a^{\frac{3}{2}}} \Rightarrow$
 $\int_0^\infty x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{4a^{\frac{3}{2}}}, F''(a) = \int_0^\infty x^4 e^{-ax^2} dx = \frac{3\sqrt{\pi}}{8a^{\frac{5}{2}}}$; 可歸納出通式為 $(-1)^n F^{(n)}(a) = \int_0^\infty x^{2n} e^{-ax^2} dx$. 由
 $\frac{d^n}{da^n} (a^{-\frac{1}{2}}) = (-1)^n \frac{(2n-1)!!}{2^n} a^{-\frac{2n+1}{2}}$, 得證.

例. $\int_0^\infty e^{-ax^2} \cos bx dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}}, \forall a > 0$.

證. 令 $F(t) = \int_0^\infty e^{-ax^2} \cos tx dx$, 則 $F(0) = \frac{1}{2} \sqrt{\frac{\pi}{a}}$; $F'(t) = - \int_0^\infty x e^{-ax^2} \sin tx dx$. 令 $u = \sin tx$, 則 $du = t \cos tx dx$; $dv = -xe^{-ax^2} dx$, 則 $v = \frac{1}{2a} e^{-ax^2}$; 故 $F'(t) = \sin tx \cdot \frac{1}{2a} e^{-ax^2} \Big|_{x=0}^{x=\infty} - \frac{t}{2a} \int_0^\infty e^{-ax^2} \cos tx dx = -\frac{t}{2a} F(t) \Rightarrow \int_0^b \frac{F'(t)}{F(t)} dt = -\frac{1}{2a} \int_0^b t dt \Rightarrow \ln F(t) \Big|_0^b = -\frac{b^2}{4a} \Rightarrow F(b) = F(0) e^{-\frac{b^2}{4a}} = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}}$.

例. $\int_0^\infty xe^{-ax^2} \sin bx dx = \frac{b}{4a} \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}}, \forall a > 0.$

證. 令 $F(b) = \int_0^\infty e^{-ax^2} \cos bx dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}}$, 則 $F'(b) = - \int_0^\infty xe^{-ax^2} \sin bx dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \cdot e^{-\frac{b^2}{4a}} \cdot \left(-\frac{b}{2a} \right) = -\frac{b}{4a} \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}}.$

例. $\int_0^\infty e^{-ax^2 - \frac{b}{x^2}} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}, \forall a, b > 0.$

證. 令 $F(t) = \int_0^\infty e^{-ax^2 - \frac{t}{x^2}} dx, t > 0$, 則 $F(0) = \frac{1}{2} \sqrt{\frac{\pi}{a}}$; $F'(t) = - \int_0^\infty \frac{e^{-ax^2 - \frac{t}{x^2}}}{x^2} dx$. 變數變換 $u = \sqrt{\frac{t}{a}} \frac{1}{x}$, 則 $x = \sqrt{\frac{t}{a}} \frac{1}{u}, -\sqrt{\frac{a}{t}} du = \frac{dx}{x^2}$, 積分範圍 x 從 $0+$ 到 ∞ , 則 $u = \sqrt{\frac{t}{a}} \frac{1}{x}$ 從 ∞ 到 0 , $F'(t) = - \int_0^\infty \frac{e^{-ax^2 - \frac{t}{x^2}}}{x^2} dx = - \int_0^0 e^{-a \cdot \frac{t}{a} \cdot \frac{1}{u^2} - t \cdot \frac{a}{t} \cdot u^2} (-) \sqrt{\frac{a}{t}} du = -\sqrt{\frac{a}{t}} \int_0^\infty e^{-\frac{t}{u^2} - au^2} du = -\sqrt{\frac{a}{t}} F(t) \Rightarrow \int_0^b \frac{F'(t)}{F(t)} dt = -\sqrt{a} \int_0^b \frac{1}{\sqrt{t}} dt \Rightarrow \ln F(t) \Big|_0^b = -2\sqrt{ab} \Rightarrow F(b) = F(0) e^{-2\sqrt{ab}} = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}.$

定理 (Glasser's Master Theorem (GMT)). 若 f 在 $(-\infty, \infty)$ 可積, $\forall c > 0$, $\int_{-\infty}^\infty f(x - \frac{c}{x}) dx = \int_{-\infty}^\infty f(x) dx$.

證. 變數變換 $u = x - \frac{c}{x}$, 則 $x^2 - ux - c = 0 \Rightarrow x = \frac{u + \sqrt{u^2 + 4c}}{2} > 0$ 或 $x = \frac{u - \sqrt{u^2 + 4c}}{2} < 0$, 分別對應 $dx = \left(\frac{1}{2} + \frac{u}{\sqrt{u^2 + 4c}}\right) du$ 或 $dx = \left(\frac{1}{2} - \frac{u}{\sqrt{u^2 + 4c}}\right) du$; 積分範圍 x 分別由 $-\infty$ 至 $0-$, $0+$ 至 ∞ , u 則分別由 $-\infty$ 至 ∞ , $-\infty$ 至 ∞ . 故 $\int_{-\infty}^\infty f(x - \frac{c}{x}) dx = \int_{-\infty}^{0-} f(x - \frac{c}{x}) dx + \int_{0+}^\infty f(x - \frac{c}{x}) dx = \int_{-\infty}^\infty f(u) \left(\frac{1}{2} + \frac{u}{\sqrt{u^2 + 4c}}\right) du + \int_{-\infty}^\infty f(u) \left(\frac{1}{2} - \frac{u}{\sqrt{u^2 + 4c}}\right) du = \int_{-\infty}^\infty f(u) du = \int_{-\infty}^\infty f(x) dx$.

例 (GMT 範例). • $\int_0^\infty e^{-ax^2 - \frac{b}{x^2}} dx = \underbrace{\int_0^\infty e^{-(\sqrt{ax} - \frac{\sqrt{b}}{x})^2 - 2\sqrt{ab}} dx}_{\text{令 } u = \sqrt{ax}; \frac{du}{\sqrt{a}} = dx} = \frac{e^{-2\sqrt{ab}}}{\sqrt{a}} \int_0^\infty e^{-(u - \frac{\sqrt{ab}}{u})^2} du = \underbrace{\frac{e^{-2\sqrt{ab}}}{2\sqrt{a}} \int_{-\infty}^\infty e^{-u^2} du}_{\text{偶函數, } \sqrt{ab} > 0; \text{符合 GMT}} = \frac{e^{-2\sqrt{ab}}}{2\sqrt{a}} \int_{-\infty}^\infty e^{-u^2} du = \frac{e^{-2\sqrt{ab}}}{2\sqrt{a}} \cdot \sqrt{\pi} = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}.$

• $\int_{-\infty}^\infty \frac{x^2}{x^4 - 3x^2 + 4} dx = \int_{-\infty}^\infty \frac{1}{(x - \frac{2}{x})^2 + 1} dx = \int_{-\infty}^\infty \frac{1}{x^2 + 1} dx = \pi$

例. 若 $a > 0$, 求 $\int_0^\infty \frac{dx}{(x^2 + a^2)^2}, \int_0^\infty \frac{dx}{(x^2 + a^2)^3}$ 並推導 $\int_0^\infty \frac{dx}{(x^2 + a^2)^n}, n \in \mathbb{N}$.

解. 由 $F(a) = \int_0^\infty \frac{dx}{x^2 + a^2} = \frac{\pi}{2a} = \frac{\pi}{2} a^{-1}$, $F'(a) = \frac{d}{da} \left(\frac{\pi}{2} a^{-1} \right) = \frac{\pi}{2} (-1) a^{-2} = (-1) 2a \int_0^\infty \frac{dx}{(x^2 + a^2)^2} \Rightarrow \int_0^\infty \frac{dx}{(x^2 + a^2)^2} = \frac{\pi}{4a^3}$; 等式兩邊再對 a 微分 $\Rightarrow (-2)(2a) \int_0^\infty \frac{dx}{(x^2 + a^2)^3} = (-3) \frac{\pi}{4a^4} \Rightarrow \int_0^\infty \frac{dx}{(x^2 + a^2)^3} = \frac{3\pi}{16a^5}$. 由數學歸納法可得 $\int_0^\infty \frac{dx}{(x^2 + a^2)^n} = \frac{\pi}{(2a)^{2n-1}} \binom{2(n-1)}{n-1}, n \in \mathbb{N}$.

例. $\int_0^\infty e^{-tx} \frac{\sin x}{x} dx = \frac{\pi}{2} - \tan^{-1} t, \forall t > 0$

證. 令 $F(t) = \int_0^\infty e^{-tx} \frac{\sin x}{x} dx$, 則 $F'(t) = - \int_0^\infty e^{-tx} \sin x dx = - \frac{e^{-tx}(-t \sin x - \cos x)}{1+t^2} \Big|_{x=0}^{x=\infty} = - \frac{1}{1+t^2} \Rightarrow F(t) = -\tan^{-1} t + c$. 令 $t \rightarrow \infty$, $F(\infty) = 0 = -\tan^{-1} \infty + c = -\frac{\pi}{2} + c \Rightarrow c = \frac{\pi}{2}$.

註. 上式令 $t \rightarrow 0+$ 得 $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$; 由變數變換 $u = ax$ 得 $\int_0^\infty \frac{\sin ax}{x} dx = \frac{\pi}{2} \operatorname{sgn}(a), \forall a \neq 0$.

例. $\int_0^\infty e^{-ax} \frac{\sin bx}{x} dx = \tan^{-1} \frac{b}{a}, \forall a, b > 0$.

證. 令 $F(t) = \int_0^\infty e^{-ax} \frac{\sin tx}{x} dx$, 則 $F(0) = 0$; $F'(t) = \int_0^\infty e^{-ax} \cos tx dx = \frac{a}{t^2 + a^2} \Rightarrow \int_0^b F'(t) dt = a \int_0^b \frac{dt}{t^2 + a^2} \Rightarrow F(b) - F(0) = \tan^{-1} \frac{t}{a} \Big|_{t=0}^{t=b} \Rightarrow F(b) = \tan^{-1} \frac{b}{a}$.

例. $\int_0^\infty \frac{\sin ax \sin bx}{x^2} dx = \frac{\pi}{2} \min(a, b), \forall a, b > 0$.

證. 由 $\sin ax \sin bx = \frac{\cos(a-b)x - \cos(a+b)x}{2}$, $\int_0^\infty \frac{\sin ax \sin bx}{x^2} dx = \frac{1}{2} \int_0^\infty \frac{\cos(a-b)x - \cos(a+b)x}{x^2} dx$.

設 $F(t) = \int_0^\infty \frac{1 - \cos tx}{x^2} dx$, 則 $F(0) = 0$. $F'(t) = \int_0^\infty \frac{\sin tx}{x} dx = \frac{\pi}{2} \operatorname{sgn}(t) \Rightarrow F(t) = \frac{\pi|t|}{2}$.

$\int_0^\infty \frac{\cos(a-b)x - \cos(a+b)x}{x^2} dx = F(a+b) - F(a-b) = \frac{\pi(a+b)}{2} - \frac{\pi|a-b|}{2} = \frac{\pi}{2} \cdot 2 \min(a, b) = \pi \min(a, b)$,

故 $\int_0^\infty \frac{\sin ax \sin bx}{x^2} dx = \frac{1}{2} \cdot \pi \min(a, b) = \frac{\pi}{2} \min(a, b)$.

例. $\int_0^\infty \frac{\sin^2 ax}{x^2} dx = \frac{\pi a}{2}, \forall a > 0$.

證. 令 $F(t) = \int_0^\infty \frac{\sin^2 tx}{x^2} dx, t > 0$, 則 $F(0) = 0$; $F'(t) = \int_0^\infty \frac{\partial}{\partial t} \left(\frac{\sin^2 tx}{x^2} \right) dx = \int_0^\infty \frac{2 \sin tx \cos tx \cdot x}{x^2} dx = \int_0^\infty \frac{\sin 2tx}{x} dx = \frac{\pi}{2} \Rightarrow \int_0^a F'(t) dt = \frac{\pi}{2} \int_0^a dt \Rightarrow F(a) - F(0) = \frac{\pi a}{2} \Rightarrow F(a) = \frac{\pi a}{2}$.

例. $\int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx = \frac{\pi(b-a)}{2} \quad \forall a, b > 0$.

證. 設 $F(t) = \int_0^\infty \frac{1 - \cos tx}{x^2} dx, t > 0$, 則 $F(0) = 0$; $F'(t) = \int_0^\infty \frac{\sin tx}{x} dx = \frac{\pi}{2} \Rightarrow \int_0^t F'(\tau) d\tau = \frac{\pi}{2} \int_0^t d\tau \Rightarrow F(t) - F(0) = \frac{\pi t}{2} \Rightarrow F(t) = \frac{\pi t}{2}$; $\int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx = F(b) - F(a) = \frac{\pi(b-a)}{2}$.

定理 (Frullani 公式). 若 f 在 $(0, \infty)$ 連續, 且 $f(0+)$ 和 $f(\infty) = \lim_{x \rightarrow \infty} f(x)$ 都存在且有限, 則 $\forall a, b > 0$

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = (f(\infty) - f(0+)) \ln \frac{a}{b}.$$

證. 令 $F(t) = \int_0^\infty \frac{f(tx) - f(bx)}{x} dx$, 則 $F(b) = 0$; 變數變換 $u = tx, F'(t) = \int_0^\infty f'(tx) dx = \frac{1}{t} \int_0^\infty f'(u) du = \frac{1}{t} (f(\infty) - f(0+))$. 則 $\int_b^a F'(t) dt = (f(\infty) - f(0+)) \int_b^a \frac{1}{t} dt \Rightarrow F(a) = (f(\infty) - f(0+)) \ln \frac{a}{b}$.

註. $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \ln \frac{b}{a}$, $\int_0^\infty \frac{\tan^{-1} ax - \tan^{-1} bx}{x} dx = \frac{\pi}{2} \ln \frac{a}{b}$ 可分別由 Frullani 公式求得: 分別取 $f(x) = e^{-x}$; $f(0+) = 1$, $f(\infty) = 0$ 及取 $f(x) = \tan^{-1} x$; $f(0+) = 0$, $f(\infty) = \frac{\pi}{2}$.

例. $\int_0^{\frac{\pi}{2}} \ln(a^2 \cos^2 \theta + b^2 \sin^2 \theta) d\theta = \pi \ln \frac{a+b}{2}$, $\forall a, b > 0$.

證. 令 $F(a, b) = \int_0^{\frac{\pi}{2}} \ln(a^2 \cos^2 \theta + b^2 \sin^2 \theta) d\theta$, $\frac{\partial F}{\partial a} = \int_0^{\frac{\pi}{2}} \frac{2a \cos^2 \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta$, $\frac{\partial F}{\partial b} = \int_0^{\frac{\pi}{2}} \frac{2b \sin^2 \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta$.
 令 $K = \int_0^{\frac{\pi}{2}} \frac{\cos^2 \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta$, $L = \int_0^{\frac{\pi}{2}} \frac{\sin^2 \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta$. 變數變換 $t = \tan \theta$, 則 $\cos^2 \theta = \frac{1}{1+t^2}$, $\sin^2 \theta = \frac{t^2}{1+t^2}$, $d\theta = \frac{dt}{1+t^2}$, t 範圍從 0 至 ∞ ; $K+L = \int_0^{\frac{\pi}{2}} \frac{1}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta = \int_0^\infty \frac{1}{a^2 + b^2 t^2} dt = \frac{\pi}{2ab}$. 又 $a^2 K + b^2 L = \int_0^{\frac{\pi}{2}} \frac{a^2 \cos^2 \theta + b^2 \sin^2 \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta = \frac{\pi}{2}$, 解得 $K = \frac{\pi}{2a(a+b)}$, $L = \frac{\pi}{2b(a+b)}$; 代入得 $\frac{\partial F}{\partial a} = \frac{\partial F}{\partial b} = \frac{\pi}{a+b}$, $F(a, b) = \pi \ln(a+b) + C$. 令 $a = b$, $F(a, a) = \pi \ln a = \pi \ln(2a) + C \Rightarrow C = -\pi \ln 2$, 故 $F(a, b) = \pi \ln(a+b) - \pi \ln 2 = \pi \ln \frac{a+b}{2}$

例. $\int_0^{\frac{\pi}{2}} \ln(\sin \theta) d\theta = \int_0^{\frac{\pi}{2}} \ln(\cos \theta) d\theta = -\frac{\pi}{2} \ln 2$, $\int_0^\pi \ln(2 \pm 2 \cos \theta) d\theta = 0$.

證. 變數變換 $u = \frac{\pi}{2} - \theta$, $\int_0^{\frac{\pi}{2}} \ln(\sin \theta) d\theta = \int_{\frac{\pi}{2}}^0 \ln(\sin(\frac{\pi}{2} - u))(-1) du = \int_0^{\frac{\pi}{2}} \ln(\cos u) du = \int_0^{\frac{\pi}{2}} \ln(\cos \theta) d\theta$;
 接連變數變換 $x = 2\theta$, $u = \frac{\pi}{2} - x$, $\int_0^{\frac{\pi}{2}} \ln(\sin 2\theta) d\theta = \frac{1}{2} \int_0^\pi \ln(\sin x) dx = \frac{1}{2} \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \ln(\cos u)(-1) du = \int_0^{\frac{\pi}{2}} \ln(\cos u) du = \int_0^{\frac{\pi}{2}} \ln(\cos \theta) d\theta$; $\int_0^{\frac{\pi}{2}} \ln(\sin \theta) d\theta = \int_0^{\frac{\pi}{2}} \ln(\sin 2\theta) d\theta = \int_0^{\frac{\pi}{2}} \ln(\sin \theta) d\theta + \int_0^{\frac{\pi}{2}} \ln(\cos \theta) d\theta + \frac{\pi}{2} \ln 2 = \int_0^{\frac{\pi}{2}} \ln(\cos \theta) d\theta \Rightarrow \int_0^{\frac{\pi}{2}} \ln(\sin \theta) d\theta = -\frac{\pi}{2} \ln 2$. $\int_0^\pi \ln(2 + 2 \cos \theta) d\theta = \int_0^\pi \ln\left(4 \cos^2 \frac{\theta}{2}\right) d\theta = 2\pi \ln 2 + 2 \int_0^\pi \ln\left(\cos \frac{\theta}{2}\right) d\theta = 2\pi \ln 2 + 4 \int_0^{\frac{\pi}{2}} \ln(\cos \theta) d\theta = 0$; $\int_0^\pi \ln(2 - 2 \cos \theta) d\theta = \int_0^\pi \ln\left(4 \sin^2 \frac{\theta}{2}\right) d\theta = 2\pi \ln 2 + 2 \int_0^\pi \ln\left(\sin \frac{\theta}{2}\right) d\theta = 2\pi \ln 2 + 4 \int_0^{\frac{\pi}{2}} \ln(\sin \theta) d\theta = 0$;

例 (Poisson 積分). $\int_0^\pi \ln(1 - 2r \cos \theta + r^2) d\theta = \begin{cases} 0 & |r| \leq 1 \\ 2\pi \ln|r| & |r| > 1 \end{cases}$

證. 設 $F(r) = \int_0^\pi \ln(1 - 2r \cos \theta + r^2) d\theta$, $\forall |r| \neq 1$, $F(0) = \int_0^\pi \ln 1 d\theta = 0$. 令 $u = \tan \frac{\theta}{2}$, 則 $\cos \theta = \frac{1-u^2}{1+u^2}$, $d\theta = \frac{2}{1+u^2} du$, $u : 0 \rightarrow \infty$. 由 $1 - 2r \cos \theta + r^2 = 1 - 2r \cdot \frac{1-u^2}{1+u^2} + r^2 = \frac{(1+u^2) - 2r(1-u^2) + r^2(1+u^2)}{1+u^2} = \frac{(1-r)^2 + (1+r)^2 u^2}{1+u^2}$, $-2 \cos \theta + 2r = -2 \cdot \frac{1-u^2}{1+u^2} + 2r = \frac{2(r-1) + 2(1+r)u^2}{1+u^2}$, $F'(r) = \int_0^\pi \frac{-2 \cos \theta + 2r}{1 - 2r \cos \theta + r^2} d\theta = \int_0^\infty \frac{2(r-1) + 2(1+r)u^2}{(1-r)^2 + (1+r)^2 u^2} \cdot \frac{2}{1+u^2} du = 4 \int_0^\infty \frac{(r-1) + (1+r)u^2}{((1-r)^2 + (1+r)^2 u^2)(1+u^2)} du$. 求部份分式:
 $\frac{(r-1) + (1+r)u^2}{((1-r)^2 + (1+r)^2 u^2)(1+u^2)} = \frac{A}{(1-r)^2 + (1+r)^2 u^2} + \frac{B}{1+u^2}$, 則 $A = \frac{r^2-1}{2r}$, $B = \frac{1}{2r}$.
 $\int_0^\infty \frac{du}{(1-r)^2 + (1+r)^2 u^2} = \frac{1}{(1+r)^2} \int_0^\infty \frac{du}{\frac{(1-r)^2}{(1+r)^2} + u^2} = \frac{1}{(1+r)^2} \frac{|1+r|}{|1-r|} \cdot \frac{\pi}{2} = \frac{1}{|1+r| \cdot |1-r|} \cdot \frac{\pi}{2} = \frac{\pi}{2} \cdot \frac{1}{|1-r^2|}$,

故 $F'(r) = 4 \int_0^\infty \frac{(r-1)+(1+r)u^2}{((1-r)^2+(1+r)^2u^2)(1+u^2)} du = 4 \cdot \frac{\pi}{2} \left(\frac{A}{|1-r^2|} + B \right) = \frac{\pi}{r} \left(\frac{r^2-1}{|r^2-1|} + 1 \right)$. 故 $F'(r) = \begin{cases} 0 & |r| < 1 \\ \frac{2\pi}{r} & |r| > 1 \end{cases}$, $F(r) = \begin{cases} 0 & |r| < 1 \\ 2\pi \ln|r| & |r| > 1 \end{cases}$; $|r|=1$ 情形由 $\int_0^\pi \ln(2 \pm 2 \cos \theta) d\theta = 0$ 含括.