第七章 重積分

7.1 二重積分

概念與基本性質

二重積分 ≈ (帶符號) 體積: z 軸上方為正, 下方為負.

定義. 給定 $\Omega = [a,\, b] \times [c,\, d]$, $b \geqslant a$, $d \geqslant c$, $f : \Omega \to \mathbb{R}$.

- Ω 分割 $\mathbb{P} : a = x_0 < x_1 < x_2 < \cdots < x_n = b,\ c = y_0 < y_1 < y_2 < \cdots < y_m = d$
- $\Delta x_k = x_k - x_{k-1},\ \ \Delta y_l = y_l - y_{l-1},\ \ k = 1,\, 2,\, \ldots,\, n;\ \ l = 1,\, 2,\, \ldots,\, m$
- $\|\mathbb{P}\| = \max\{\Delta x_k \Delta y_l \mid 1 \leqslant k \leqslant n,\ 1 \leqslant l \leqslant m\}$
- 樣本點 (ξ_k, ζ_l): $x_{k-1} \leqslant \xi_k \leqslant x_k,\ \ y_{l-1} \leqslant \zeta_l \leqslant y_l,\ \ k = 1,\, 2,\, \ldots,\, n;\ \ l = 1,\, 2,\, \ldots,\, m.$
- $u_{k,l} = \max\{f(x,y) \mid x_{k-1} \leqslant x \leqslant x_k,\ y_{l-1} \leqslant y \leqslant y_l\},\ \ \ell_{k,l} = \min\{f(x,y) \mid x_{k-1} \leqslant x \leqslant x_k,\ y_{l-1} \leqslant y \leqslant y_l\}$, $k = 1,\, 2,\, \ldots,\, n;\ \ l = 1,\, 2,\, \ldots,\, m.$
- $R(f, \mathbb{P}) = \displaystyle\sum_{l=1}^{m}\sum_{k=1}^{n} f(\xi_k, \zeta_l)\, \Delta x_k \Delta y_l,\ \ U(f, \mathbb{P}) = \sum_{l=1}^{m}\sum_{k=1}^{n} u_{k,l}\, \Delta x_k \Delta y_l,\ \ L(f, \mathbb{P}) = \sum_{l=1}^{m}\sum_{k=1}^{n} \ell_{k,l}\, \Delta x_k \Delta y_l$;
  顯然 $L(f, \mathbb{P}) \leqslant R(f, \mathbb{P}) \leqslant U(f, \mathbb{P})$.
- 求 $\displaystyle\lim_{\|\mathbb{P}\| \to 0} R(f, \mathbb{P})$. 若對不同分割與樣本點選取此極限均存在且相等, 稱 f 在 Ω 可積 (分); $f(x,y)$ 在 Ω 的定積分為 $\displaystyle\int_\Omega f(x,y)\,\mathrm{d}A \equiv \lim_{\|\mathbb{P}\| \to 0} R(f, \mathbb{P})$.

性質.

- 若 f 在 Ω 有界, 且除了 Ω 中有限個平滑曲線外 f 在 Ω 連續, 則 f 在 Ω 可積分.
- 若 $\Omega = \Omega_1 \cap \Omega_2$ 且 Ω_1, Ω_2 均為矩形, f 在 Ω_1, Ω_2 均可積, 則 $\displaystyle\int_\Omega f\,\mathrm{d}A = \int_{\Omega_1} f\,\mathrm{d}A + \int_{\Omega_2} f\,\mathrm{d}A$.
- 若 f_1, f_2 均在 Ω 可積且 $f_1(x,y) \leqslant f_2(x,y)\ \forall\,(x,y) \in \Omega$, 則 $\displaystyle\int_\Omega f_1\,\mathrm{d}A \leqslant \int_\Omega f_2\,\mathrm{d}A$.
- 若 $\alpha,\, \beta \in \mathbb{R}$, f_1, f_2 均在 Ω 可積, 則 $\displaystyle\int_\Omega (\alpha f_1 + \beta f_2)\,\mathrm{d}A = \alpha\int_\Omega f_1\,\mathrm{d}A + \beta\int_\Omega f_2\,\mathrm{d}A$.

逐次積分

矩形積分區域

定理 (Fubini)**.** 若 $\Omega = [a,\, b] \times [c,\, d]$, $f : \Omega \to \mathbb{R}$ 為可積, 則

$$\int_\Omega f(x,y)\,\mathrm{d}A = \int_c^d \left\{\int_a^b f(x,y)\,\mathrm{d}x\right\}\mathrm{d}y = \int_a^b \left\{\int_c^d f(x,y)\,\mathrm{d}y\right\}\mathrm{d}x$$

例. 令 $\Omega = \{(x,\, y) \mid -1 \leqslant x \leqslant 1,\ -2 \leqslant y \leqslant 2\}$, 則 $\displaystyle\int_\Omega \sqrt{1-x^2}\,\mathrm{d}A = \int_{-2}^{2}\left\{\int_{-1}^{1}\sqrt{1-x^2}\,\mathrm{d}x\right\}\mathrm{d}y = \frac{\pi}{2} \times 4 = 2\pi$.

例. 若 $\Omega = [0,\, 2] \times [1,\, 3]$, 求 $\displaystyle\int_\Omega x^2 y\,\mathrm{d}A$.

解.

- 由 Fubini 定理 $\displaystyle\int_\Omega x^2 y\,\mathrm{d}A = \int_0^2\int_1^3 x^2 y\,\mathrm{d}y\,\mathrm{d}x = \int_1^3\int_0^2 x^2 y\,\mathrm{d}x\,\mathrm{d}y$.

- $\displaystyle\int_0^2\int_1^3 x^2y\,\mathrm{d}y\,\mathrm{d}x=\int_0^2 x^2\Big(\frac{y^2}{2}\Big|_1^3\Big)\,\mathrm{d}x=\int_0^2 x^2\Big(\frac{9}{2}-\frac{1}{2}\Big)\,\mathrm{d}x=\frac{4}{3}x^3\Big|_0^2=\frac{32}{3}.$
- $\displaystyle\int_1^3\int_0^2 x^2y\,\mathrm{d}x\,\mathrm{d}y=\int_1^3 y\Big(\frac{x^3}{3}\Big|_0^2\Big)\,\mathrm{d}y=\int_1^3 y\,\frac{8}{3}\,\mathrm{d}x=\frac{4}{3}y^2\Big|_1^3=\frac{36}{3}-\frac{4}{3}=\frac{32}{3}.$

例. 若 $\Omega=[0,\,1]\times[0,\,3]$, 求 $\displaystyle\int_\Omega e^{2x+y}\,\mathrm{d}A$.

解.

- 由 Fubini 定理 $\displaystyle\int_\Omega e^{2x+y}\,\mathrm{d}A=\int_0^3\int_0^1 e^{2x+y}\,\mathrm{d}x\,\mathrm{d}y=\int_0^1\int_0^3 e^{2x+y}\,\mathrm{d}y\,\mathrm{d}x.$
- $\displaystyle\int_0^3\int_0^1 e^{2x+y}\,\mathrm{d}x\,\mathrm{d}y=\int_0^3 e^y\left(\int_0^1 e^{2x}\,\mathrm{d}x\right)\mathrm{d}y=\int_0^3 e^y\Big(\frac{e^{2x}}{2}\Big|_0^1\Big)\,\mathrm{d}y=\frac{(e^2-1)(e^3-1)}{2}.$
- $\displaystyle\int_0^1\int_0^3 e^{2x+y}\,\mathrm{d}y\,\mathrm{d}x=\int_0^1 e^{2x}\left(\int_0^3 e^y\,\mathrm{d}y\right)\mathrm{d}x=\int_0^1 e^{2x}\Big(\frac{e^{2x}}{2}\Big|_0^3\Big)\,\mathrm{d}y=\frac{(e^2-1)(e^3-1)}{2}.$

例. 若 $\Omega=[0,\,\pi]\times[0,\,2\pi]$, 求 $\displaystyle\int_\Omega\sin(x+y)\,\mathrm{d}A$.

解.

- 由 Fubini 定理 $\displaystyle\int_\Omega\sin(x+y)\,\mathrm{d}A=\int_0^\pi\int_0^{2\pi}\sin(x+y)\,\mathrm{d}y\,\mathrm{d}x=\int_0^{2\pi}\int_0^\pi\sin(x+y)\,\mathrm{d}x\,\mathrm{d}y.$
- $\displaystyle\int_0^\pi\int_0^{2\pi}\sin(x+y)\,\mathrm{d}y\,\mathrm{d}x=\int_0^\pi\Big(-\cos(x+y)\Big|_0^{2\pi}\Big)\,\mathrm{d}x=-\int_0^\pi(\cos(x+2\pi)-\cos x)\,\mathrm{d}x=0.$
- $\displaystyle\int_0^{2\pi}\int_0^\pi\sin(x+y)\,\mathrm{d}x\,\mathrm{d}y=\int_0^{2\pi}\Big(-\cos(x+y)\Big|_0^\pi\Big)\,\mathrm{d}y=-\int_0^{2\pi}(\cos(\pi+y)-\cos y)\,\mathrm{d}y=-\int_0^{2\pi}(\cos\pi\cos y-\sin\pi\sin y-\cos y)\,\mathrm{d}y=2\int_0^{2\pi}\cos y\,\mathrm{d}y=0.$

例. 若 $\Omega=[1,\,2]\times[0,\,1]$, 求 $\displaystyle\int_\Omega ye^{xy}\,\mathrm{d}A$.

解.

- 由 Fubini 定理 $\displaystyle\int_\Omega ye^{xy}\,\mathrm{d}A=\int_0^1\int_1^2 ye^{xy}\,\mathrm{d}x\,\mathrm{d}y=\int_1^2\int_0^1 ye^{xy}\,\mathrm{d}y\,\mathrm{d}x.$
- $\displaystyle\int_0^1\int_1^2 ye^{xy}\,\mathrm{d}x\,\mathrm{d}y=\int_0^1 y\,\Big(\frac{e^{xy}}{y}\Big|_1^2\Big)\,\mathrm{d}y=\int_0^1(e^{2y}-e^y)\,\mathrm{d}y=\frac{e^2}{2}-e+\frac{1}{2}.$
- $\displaystyle\int_1^2\int_0^1 ye^{xy}\,\mathrm{d}y\,\mathrm{d}x=\int_1^2\Big(y\,\frac{e^{xy}}{x}-\frac{e^{xy}}{x^2}\Big|_0^1\Big)\,\mathrm{d}x=\int_1^2\Big(\underbrace{\frac{e^x}{x}-\frac{e^x}{x^2}}_{(\frac{e^x}{x})'=\frac{e^x}{x}-\frac{e^x}{x^2}}+\frac{1}{x^2}\Big)\,\mathrm{d}x=\Big(\frac{e^x}{x}-\frac{1}{x}\Big)\Big|_1^2=\frac{e^2}{2}-e+\frac{1}{2}.$

習題. 求下列重積分.

1. $\displaystyle\int_2^3\int_1^5(x+2y)\,\mathrm{d}x\,\mathrm{d}y$
2. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\cos y}e^x\sin y\,\mathrm{d}x\,\mathrm{d}y$
3. $\displaystyle\int_{-\pi}^\pi\int_0^2 r\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta$
4. $\displaystyle\int_0^{\ln 4}\int_0^{\ln 3}e^{x+y}\,\mathrm{d}x\,\mathrm{d}y$
5. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta}\rho^2\sin\theta\,\mathrm{d}\rho\,\mathrm{d}\theta$
6. $\displaystyle\int_0^1\int_{x^2}^{x^3}(x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x$
7. $\displaystyle\int_1^2\int_0^y x\sqrt{y^2-x^2}\,\mathrm{d}x\,\mathrm{d}y$
8. $\displaystyle\int_0^1\int_{y^4}^{y^2}\sqrt{\frac{y}{x}}\,\mathrm{d}x\,\mathrm{d}y$
9. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^a\frac{r}{\sqrt{a^2-r^2\cos^2\theta}}\,\mathrm{d}r\,\mathrm{d}\theta$

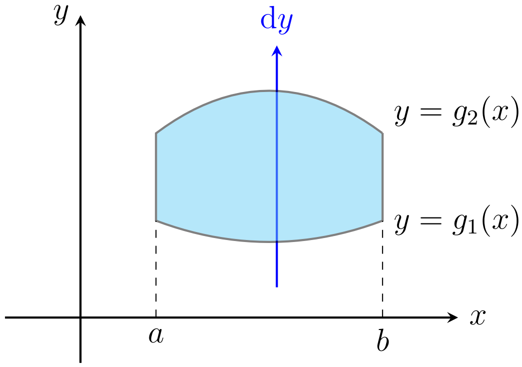
解.

1. $\displaystyle\int_2^3\int_1^5 (x+2y)\,\mathrm{d}x\,\mathrm{d}y = \int_2^3 \left(\frac{x^2}{2}+2yx\right)\Big|_1^5 \mathrm{d}y = \int_2^3 (12+8y)\,\mathrm{d}y = (12y+4y^2)\Big|_2^3 = 32$
2. $\displaystyle\int_{-\pi}^{\pi}\int_0^2 r\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta = \left(\int_{-\pi}^{\pi}\sin\theta\,\mathrm{d}\theta\right)\cdot\left(\int_0^2 r\,\mathrm{d}r\right) = 0$
3. $\displaystyle\int_0^{\ln 4}\int_0^{\ln 3} e^{x+y}\,\mathrm{d}x\,\mathrm{d}y = \left(\int_0^{\ln 4} e^y\,\mathrm{d}y\right)\cdot\left(\int_0^{\ln 3} e^x\,\mathrm{d}x\right) = (4-1)(3-1) = 6$
4. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\cos y} e^x\sin y\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{2}}\sin y\left(\int_0^{\cos y} e^x\,\mathrm{d}x\right)\mathrm{d}y = \int_0^{\frac{\pi}{2}}\sin y\,(e^{\cos y}-1)\,\mathrm{d}y = \big(-e^{\cos y}+\cos y\big)\Big|_0^{\frac{\pi}{2}} = e-2$
5. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta}\rho^2\sin\theta\,\mathrm{d}\rho\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}}\sin\theta\left(\int_0^{\cos\theta}\rho^2\,\mathrm{d}\rho\right)\mathrm{d}\theta = \int_0^{\frac{\pi}{2}}\sin\theta\,\frac{\cos^3\theta}{3}\,\mathrm{d}\theta = -\frac{\cos^4\theta}{12}\Big|_0^{\frac{\pi}{2}} = \frac{1}{12}$
6. $\displaystyle\int_0^1\int_{x^2}^{x^3}(x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\left(x^2y+\frac{y^3}{3}\right)\Big|_{y=x^2}^{y=x^3}\mathrm{d}x = \int_0^1\left(x^2\cdot x^3+\frac{x^9}{3}-x^2\cdot x^2-\frac{x^6}{3}\right)\mathrm{d}x = -\frac{1}{21}$
7. $\displaystyle\int_0^y x\sqrt{y^2-x^2}\,\mathrm{d}x$ 中, 令 $u=y^2-x^2$, 則 $\mathrm{d}u=-2x\,\mathrm{d}x \implies x\,\mathrm{d}x=\frac{-1}{2}\mathrm{d}u$. 積分範圍 x 由 0 至 y, 則變數變換 $u=y^2-x^2$ 由 $y^2-0^2=y^2$ 至 $y^2-y^2=0$. 則 $\displaystyle\int_0^y x\sqrt{y^2-x^2}\,\mathrm{d}x = \int_{y^2}^0\sqrt{u}\,\frac{-1}{2}\,\mathrm{d}u = \frac{1}{2}\int_0^{y^2}\sqrt{u}\,\mathrm{d}u = \frac{1}{2}\frac{2}{3}u^{\frac{3}{2}}\Big|_0^{y^2} = \frac{y^3}{3}$. 故 $\displaystyle\int_1^2\int_0^y x\sqrt{y^2-x^2}\,\mathrm{d}x\,\mathrm{d}y = \int_1^2\frac{y^3}{3}\,\mathrm{d}y = \frac{y^4}{12}\Big|_1^2 = \frac{5}{4}$.
8. $\displaystyle\int_0^1\int_{y^4}^{y^2}\sqrt{\frac{y}{x}}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\sqrt{y}\left(\int_{y^4}^{y^2}\frac{1}{\sqrt{x}}\,\mathrm{d}x\right)\mathrm{d}y = \int_0^1\sqrt{y}\,(2\sqrt{x})\Big|_{y^4}^{y^2}\mathrm{d}y = 2\int_0^1\left(y^{\frac{3}{2}}-y^{\frac{5}{2}}\right)\mathrm{d}y = \frac{8}{35}$
9. $\displaystyle\int_0^a\frac{r}{\sqrt{a^2-r^2\cos^2\theta}}\,\mathrm{d}r$ 中, 令 $u=a^2-r^2\cos^2\theta$, 則 $\mathrm{d}u=-2\cos^2\theta\,r\,\mathrm{d}r \implies r\,\mathrm{d}r=\frac{-1}{2\cos^2\theta}\mathrm{d}u$. 積分範圍 r 由 0 至 a, 則變數變換 $u=a^2-r^2\cos^2\theta$ 由 $a^2-0^2\cos^2\theta=a^2$ 至 $a^2-a^2\cos^2\theta=a^2\sin^2\theta$. 則 $\displaystyle\int_0^a\frac{r}{\sqrt{a^2-r^2\cos^2\theta}}\,\mathrm{d}r = \int_{a^2}^{a^2\sin^2\theta}\frac{1}{\sqrt{u}}\,\frac{-1}{2\cos^2\theta}\,\mathrm{d}u = \int_{a^2\sin^2\theta}^{a^2}\frac{1}{2\cos^2\theta\sqrt{u}}\,\mathrm{d}u = \frac{\sqrt{u}}{\cos^2\theta}\Big|_{u=a^2\sin^2\theta}^{u=a^2} = a\,\frac{1-\sin\theta}{\cos^2\theta}$.
   故 $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^a\frac{r}{\sqrt{a^2-r^2\cos^2\theta}}\,\mathrm{d}r\,\mathrm{d}\theta = a\int_0^{\frac{\pi}{2}}\frac{1-\sin\theta}{\cos^2\theta}\,\mathrm{d}\theta = a\,\frac{\sin\theta-1}{\cos\theta}\Big|_0^{\frac{\pi}{2}} = a\left(\lim_{\theta\to\frac{\pi}{2}}\frac{\sin\theta-1}{\cos\theta}+1\right) = a$.

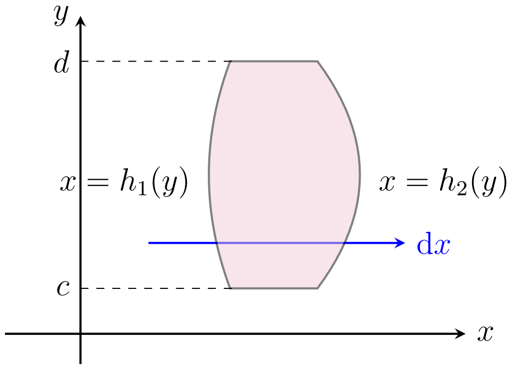
一般積分區域

結論 (基本區域積分)**.**

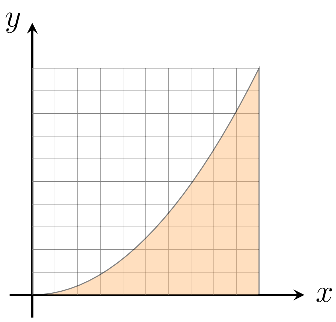
1. 選擇初始積分方向 v: 延 x 方向為 $\mathrm{d}x$, 延 y 方向為 $\mathrm{d}y$.
2. 想像積分方向 v 為一射線, 進入區域之邊界函數為積分下界, 離開區域之邊界函數為積分上界; 邊界函數本身不含變數 v.
3. 次第區域為原始區域延 v 之投影 (令 $v=$ (常數) 之方程式, 或邊界的交集); 若此投影區域維度 >1, 選擇積分方向.



$$\int_\Omega f(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

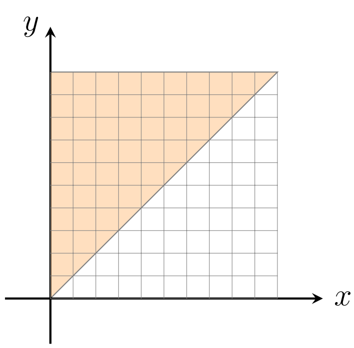


$$\int_\Omega f(x,y)\,\mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$



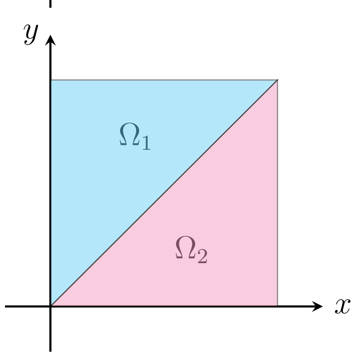
例. 求 $\displaystyle\int_\Omega x\cos y\,\mathrm{d}A$, Ω 為 $y=0$, $x=1$, 與 $y=x^2$ 圍成之區域.

解. $\displaystyle\int_\Omega x\cos y\,\mathrm{d}A = \int_0^1\int_0^{x^2} x\cos y\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 x\left(\sin y\,\Big|_0^{x^2}\right)\mathrm{d}x$
$\displaystyle= \int_0^1 x\sin x^2\,\mathrm{d}x = \frac{1-\cos 1}{2}$



例. 求 $\displaystyle\int_\Omega e^{-y^2}\,\mathrm{d}A$, $\Omega = \{(x,\ y)\,|\,0\leqslant y\leqslant 1,\ 0\leqslant x\leqslant y\}$.

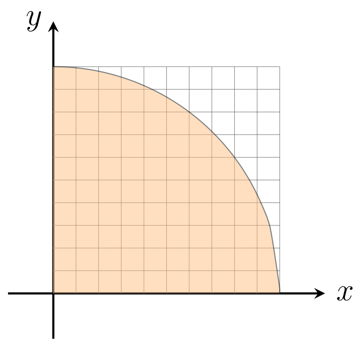
解. $\displaystyle\int_\Omega e^{-y^2}\,\mathrm{d}A = \int_0^1\int_0^y e^{-y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 ye^{-y^2}\,\mathrm{d}y = \frac{1-e^{-1}}{2}$



例. 求 $\displaystyle\int_0^1\int_0^1 e^{\max\{x^2,\,y^2\}}\,\mathrm{d}x\,\mathrm{d}y$.

解. $\displaystyle\int_0^1\int_0^1 e^{\max\{x^2,\,y^2\}}\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega_1} e^{y^2}\,\mathrm{d}A + \int_{\Omega_2} e^{x^2}\,\mathrm{d}A$
$\displaystyle= \int_0^1\int_0^y e^{y^2}\,\mathrm{d}x\,\mathrm{d}y + \int_0^1\int_0^x e^{x^2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 ye^{y^2}\,\mathrm{d}y + \int_0^1 xe^{x^2}\,\mathrm{d}x = e-1.$

積分順序交換



例. 求 $\int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{1-y^2}\,\mathrm{d}y\,\mathrm{d}x.$

解. $\int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{1-y^2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\int_0^{\sqrt{1-y^2}}\sqrt{1-y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1(1-y^2)\,\mathrm{d}y = \frac{2}{3}.$

$\int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{1-y^2}\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{2}\int_0^1(\sin^{-1}\sqrt{1-x^2}+x\sqrt{1-x^2})\,\mathrm{d}x = \frac{1}{2}\left(1+\frac{1}{3}\right) = \frac{2}{3}.$

例. 將下列積分以不同積分順序寫出.

1. $\int_0^2\int_{x^2}^{2x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$
2. $\int_0^1\int_{\sin^{-1}y}^{\frac{\pi}{2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$
3. $\int_{-1}^2\int_{x^2}^{x+2} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$
4. $\int_0^1\int_0^{2y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_1^3\int_0^{3-y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

解.

1. $\int_0^2\int_{x^2}^{2x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^4\int_{\sqrt{y}}^{\frac{y}{2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$
2. $\int_0^1\int_{\sin^{-1}y}^{\frac{\pi}{2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{2}}\int_0^{\sin x} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$
3. $\int_{-1}^2\int_{x^2}^{x+2} f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_1^4\int_{y-2}^{\sqrt{y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$
4. $\int_0^1\int_0^{2y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_1^3\int_0^{3-y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^2\int_{\frac{x}{2}}^{3-x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$

例. 證明 $\int_0^x\int_0^t F(u)\,\mathrm{d}u\,\mathrm{d}t = \int_0^x (x-u)F(u)\,\mathrm{d}u.$

解. $\int_0^x\int_0^t F(u)\,\mathrm{d}u\,\mathrm{d}t = \int_0^x\int_u^x F(u)\,\mathrm{d}t\,\mathrm{d}u = \int_0^x (x-u)F(u)\,\mathrm{d}u.$

例. 求 $\int_0^1\int_x^1 e^{-y^2}\,\mathrm{d}y\,\mathrm{d}x.$

解. $\int_0^1\int_x^1 e^{-y^2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\int_0^y e^{-y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 ye^{-y^2}\,\mathrm{d}y = \frac{1-e^{-1}}{2}$

例. 求 $\int_0^1\int_y^1 \frac{\sin x}{x}\,\mathrm{d}x\,\mathrm{d}y.$

解. $\int_0^1\int_y^1 \frac{\sin x}{x}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\int_0^x \frac{\sin x}{x}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \sin x\,\mathrm{d}x = 1-\cos 1.$

例. 求 $\int_0^1\int_{\sin^{-1}y}^{\frac{\pi}{2}} \cos x\sqrt{1+\cos^2 x}\,\mathrm{d}x\,\mathrm{d}y.$

解.

- $\int_0^1 \int_{\sin^{-1}y}^{\frac{\pi}{2}} \cos x\,\sqrt{1+\cos^2 x}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{2}} \int_0^{\sin x} \cos x\,\sqrt{1+\cos^2 x}\,\mathrm{d}y\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin x \cos x \sqrt{1+\cos^2 x}\,\mathrm{d}x$
  $= \int_0^1 u\sqrt{1+u^2}\,\mathrm{d}u = \int_1^2 \sqrt{v}\,\frac{1}{2}\,\mathrm{d}v = \frac{2\sqrt{2}-1}{3}$, 其中變數變換 $u = \cos x$ 與 $v = 1+u^2$.
- 令 $u = \sin x$, $\int_{\sin^{-1}y}^{\frac{\pi}{2}} \cos x\,\sqrt{1+\cos^2 x}\,\mathrm{d}x = \int_y^1 \sqrt{2-u^2}\,\mathrm{d}u$. 令 $u = \sqrt{2}\sin\theta$, 則 $\int_y^1 \sqrt{2-u^2}\,\mathrm{d}u = \int_{\sin^{-1}\frac{y}{\sqrt{2}}}^{\frac{\pi}{4}} 2\cos^2\theta\,\mathrm{d}\theta = \int_{\sin^{-1}\frac{y}{\sqrt{2}}}^{\frac{\pi}{4}} (1+\cos 2\theta)\,\mathrm{d}\theta = (\theta+\sin\theta\cos\theta)\Big|_{\sin^{-1}\frac{y}{\sqrt{2}}}^{\frac{\pi}{4}} = \frac{\pi}{4}+\frac{1}{2}-\sin^{-1}\frac{y}{\sqrt{2}}-\frac{y}{\sqrt{2}}\cdot\frac{\sqrt{2-y^2}}{\sqrt{2}} = \frac{\pi}{4}+\frac{1}{2}-\sin^{-1}\frac{y}{\sqrt{2}}-\frac{y\sqrt{2-y^2}}{2}$. 故 $\int_0^1 \int_{\sin^{-1}y}^{\frac{\pi}{2}} \cos x\,\sqrt{1+\cos^2 x}\,\mathrm{d}x\,\mathrm{d}y$
  $= \int_0^1 \left(\frac{\pi}{4}+\frac{1}{2}-\sin^{-1}\frac{y}{\sqrt{2}}-\frac{y\sqrt{2-y^2}}{2}\right)\mathrm{d}y = \frac{\pi}{4}+\frac{1}{2}-\left(\frac{\pi}{4}+1-\sqrt{2}\right)-\left(\frac{\sqrt{2}}{3}-\frac{1}{6}\right) = \frac{2\sqrt{2}-1}{3}$.

例. 求 $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,\mathrm{d}x$, $b > a > 0$.

解. 由 $\int_a^b e^{-xy}\,\mathrm{d}y = \frac{1}{-x}e^{-xy}\Big|_{y=a}^{y=b} = \frac{1}{-x}(e^{-xb}-e^{-xa}) = \frac{e^{-ax}-e^{-bx}}{x}$, $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,\mathrm{d}x = \int_0^\infty \left(\int_a^b e^{-xy}\,\mathrm{d}y\right)\mathrm{d}x = \int_a^b \left(\int_0^\infty e^{-xy}\,\mathrm{d}x\right)\mathrm{d}y = \int_a^b \frac{1}{y}\,\mathrm{d}y = \ln\frac{b}{a}$.

例. 求 $\int_0^\infty \frac{\tan^{-1}\pi x - \tan^{-1}x}{x}\,\mathrm{d}x$.

解. 由 $\int_x^{\pi x} \frac{1}{1+y^2}\,\mathrm{d}y = \tan^{-1}\pi x - \tan^{-1}x$, $\int_0^\infty \frac{\tan^{-1}\pi x - \tan^{-1}x}{x}\,\mathrm{d}x = \int_0^\infty \frac{1}{x}\left(\int_x^{\pi x}\frac{1}{1+y^2}\,\mathrm{d}y\right)\mathrm{d}x$
$= \int_0^\infty \int_x^{\pi x} \frac{1}{x(1+y^2)}\,\mathrm{d}y\,\mathrm{d}x = \int_0^\infty \int_{\frac{y}{\pi}}^{y} \frac{1}{x(1+y^2)}\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty \frac{1}{1+y^2}\left(\int_{\frac{y}{\pi}}^{y}\frac{1}{x}\,\mathrm{d}x\right)\mathrm{d}y = \int_0^\infty \frac{1}{1+y^2}\left(\ln y - \ln\frac{y}{\pi}\right)\mathrm{d}y = \int_0^\infty \frac{\ln\pi}{1+y^2}\,\mathrm{d}y = \frac{\pi}{2}\ln\pi$.

例. 求 $\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x$.

解. 由 $\int_0^\infty e^{-xy}\,\mathrm{d}y = \frac{1}{x}$, $x > 0$; 又 $\int e^{ax}\sin bx\,\mathrm{d}x = \frac{e^{ax}(a\cdot\sin bx - b\cdot\cos bx)}{a^2+b^2}$.
故 $\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \int_0^\infty \sin x\left(\int_0^\infty e^{-xy}\,\mathrm{d}y\right)\mathrm{d}x = \int_0^\infty\left(\int_0^\infty e^{-xy}\sin x\,\mathrm{d}y\right)\mathrm{d}x = \int_0^\infty\left(\int_0^\infty e^{-xy}\sin x\,\mathrm{d}x\right)\mathrm{d}y = \int_0^\infty \frac{e^{-xy}((-y)\cdot\sin x - 1\cdot\cos x)}{(-y)^2+1^2}\Big|_{x=0}^{x=\infty}\mathrm{d}y = \int_0^\infty \frac{0-(-1)}{1+y^2}\,\mathrm{d}y = \int_0^\infty \frac{1}{1+y^2}\,\mathrm{d}y = \tan^{-1}y\Big|_0^\infty = \frac{\pi}{2}$.

變數變換

單變數積分 $\int_{[a,b]} f(x)\,\mathrm{d}x$ 中, 令 $x = x(u)$, 則 $\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u$; $\int_{[a,b]} f(x)\,\mathrm{d}x = \underbrace{\int_{x^{-1}[a,b]}}_{\text{由小到大}} f(x(u))\left|\frac{\mathrm{d}x}{\mathrm{d}u}\right|\mathrm{d}u$.

例.

- 求 $\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}}$: 令 $x = \sin u$, 則 $\mathrm{d}x = \cos u\cdot\mathrm{d}u$; x 由 0 至 1, 則 u 由 $\sin^{-1}0 = 0$ 至 $\sin^{-1}1 = \frac{\pi}{2}$. 故 $\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int_0^{\frac{\pi}{2}} \frac{\cos u\cdot\mathrm{d}u}{\sqrt{1-\sin^2 u}} = \int_0^{\frac{\pi}{2}} \frac{\cos u\cdot\mathrm{d}u}{\cos u} = \int_0^{\frac{\pi}{2}} 1\cdot\mathrm{d}u = \frac{\pi}{2}$.

- 求 $\int_{\frac{1}{2}}^{1}(1-2x)^2\,\mathrm{d}x$: 令 $u=1-2x$, 亦即 $x=\frac{1}{2}-\frac{u}{2}$, 則 $\mathrm{d}x=\frac{-1}{2}\,\mathrm{d}u$; x 由 $\frac{1}{2}$ 至 1, 則 u 由 $1-2\cdot\frac{1}{2}=0$ 至 $1-2\cdot 1=-1$. 故 $\int_{\frac{1}{2}}^{1}(1-2x)^2\,\mathrm{d}x=\int_0^{-1}u^2\left(\frac{-1}{2}\right)\mathrm{d}u=\int_{-1}^{0}u^2\,\frac{1}{2}\,\mathrm{d}u=\frac{1}{6}$.

定理 (重積分變數變換)**.** 給定 $\Omega_{\mathbf{x}},\,\Omega_{\mathbf{u}}\subseteq\mathbb{R}^n$, $\mathbf{x}=\mathbf{x}(\mathbf{u}):\Omega_{\mathbf{u}}\to\Omega_{\mathbf{x}}$ 且

- $\mathbf{x}$ 為嵌射 (亦即 $\mathbf{x}(\Omega_{\mathbf{u}})=\Omega_{\mathbf{x}},\ \mathbf{x}^{-1}(\Omega_{\mathbf{x}})=\Omega_{\mathbf{u}}$)
- $\mathbf{x}$ 之各分量函數 x_j, $j=1,\,2\,\ldots,\,n$ 為連續可偏微分
- $\forall\,\mathbf{u}\in\Omega_{\mathbf{u}}$, $\mathbf{x}$ 之 Jacobian $J_{\mathbf{x}}(\mathbf{u})=\dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}(\mathbf{u})=\begin{vmatrix}\frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_n} & \cdots & \frac{\partial x_1}{\partial u_n}\\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_n} & \cdots & \frac{\partial x_2}{\partial u_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_n} & \cdots & \frac{\partial x_n}{\partial u_n}\end{vmatrix}(\mathbf{u})\neq 0$

則對於可積函數 $f:\Omega_{\mathbf{x}}\to\mathbb{R}$, $\displaystyle\int_{\Omega_{\mathbf{x}}}f(\mathbf{x})\,\mathrm{d}\mathbf{x}=\int_{\mathbf{x}^{-1}(\Omega_{\mathbf{x}})}f(\mathbf{x}(\mathbf{u}))\,\big|J_{\mathbf{x}}(\mathbf{u})\big|\,\mathrm{d}\mathbf{u}=\int_{\mathbf{x}^{-1}(\Omega_{\mathbf{x}})}f(\mathbf{x}(\mathbf{u}))\left|\frac{\partial\mathbf{x}}{\partial\mathbf{u}}\right|\mathrm{d}\mathbf{u}$.

例. 利用變數變換 $x=u^2-v^2$, $y=2uv$ 求 $\int_\Omega y\,\mathrm{d}A$, Ω 為 $y\geqslant 0$, $y^2=4-4x$, $y^2=4+4x$ 所圍成之區域.

解. 由 $\begin{cases}x=u^2-v^2\\ y=2uv\end{cases}$, Jacobian $J_{\mathbf{x}}(\mathbf{u})=\dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}(\mathbf{u})=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}2u & -2v\\ 2v & 2u\end{vmatrix}=4\,(u^2+v^2)$. 變數變換 $(x,\,y)\to(u,\,v)$ 後 Ω 由 $uv=0$, $u^2v^2+u^2-v^2-1=(v^2+1)(u^2-1)=0$, $u^2v^2-u^2+v^2-1=(u^2+1)(v^2-1)=0$ 圍成, 亦即 $\Omega=\{0\leqslant u\leqslant 1,\ 0\leqslant v\leqslant 1\}$, 故 $\int_\Omega y\,\mathrm{d}A=\int_0^1\int_0^1 2uv\,\Big|4\,(u^2+v^2)\Big|\,\mathrm{d}u\,\mathrm{d}v=2$.

例. 求 $\int_\Omega(x+y)^2\,\mathrm{d}A$, Ω 為 $x+y=0$, $x+y=1$, $2x-y=0$, $2x-y=3$ 所圍成之平行四邊形.

解. 令 $\begin{cases}u=x+y\\ v=2x-y\end{cases}$, 則 $\begin{cases}x=\frac{1}{3}u+\frac{1}{3}v\\ y=\frac{2}{3}u-\frac{1}{3}v\end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u})=\dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}(\mathbf{u})=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{3} & \frac{1}{3}\\ \frac{2}{3} & \frac{-1}{3}\end{vmatrix}=-\frac{1}{3}$. 變數變換 $(x,\,y)\to(u,\,v)$ 後 $\Omega=\{0\leqslant u\leqslant 1,\ 0\leqslant v\leqslant 3\}$, 故 $\int_\Omega(x+y)^2\,\mathrm{d}A=\int_0^3\int_0^1 u^2\,\Big|-\frac{1}{3}\Big|\,\mathrm{d}u\,\mathrm{d}v=\frac{1}{3}$.

例. 求 $\int_\Omega\sqrt{x+y}\,\mathrm{d}A$, Ω 為頂點 $(0,\,0)$, $(a,\,0)$, $(0,\,a)$, $a>0$ 之三角形.

解. 令 $\begin{cases}u=x+y\\ v=x-y\end{cases}$, 則 $\begin{cases}x=\frac{1}{2}u+\frac{1}{2}v\\ y=\frac{1}{2}u-\frac{1}{2}v\end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u})=\dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}(\mathbf{u})=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{-1}{2}\end{vmatrix}=-\frac{1}{2}$. 變數變換前 Ω 由 $x+y=0$, $x+y=a$, $x=0$, $y=0$ 所圍成, 變數變換 $(x,\,y)\to(u,\,v)$ 後 Ω 由 $u=0$, $u=a$, $u+v=0$, $u-v=0$ 圍成, 故 $\int_\Omega\sqrt{x+y}\,\mathrm{d}A=\int_0^a\int_{-u}^{u}\sqrt{u}\,\Big|-\frac{1}{2}\Big|\,\mathrm{d}v\,\mathrm{d}u=\int_0^a u\sqrt{u}\,\mathrm{d}u=\frac{2\,a^{\frac{5}{2}}}{5}$.

例. 求 $\int_\Omega(x+y)\,e^{x-y}\,\mathrm{d}A$, Ω 為頂點 $(4,\,0)$, $(6,\,2)$, $(4,\,4)$, $(2,\,2)$ 之四邊形.

解. 令 $\begin{cases}u=x+y\\ v=x-y\end{cases}$, 則 $\begin{cases}x=\frac{1}{2}u+\frac{1}{2}v\\ y=\frac{1}{2}u-\frac{1}{2}v\end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u})=\dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}(\mathbf{u})=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{-1}{2}\end{vmatrix}=-\frac{1}{2}$. 變數變換前 Ω 由 $x-y=0$, $x-y=4$, $x+y=4$, $x+y=8$ 所圍成, 變數變換 $(x,\,y)\to(u,\,v)$ 後 Ω 由 $v=0$, $v=4$, $u=4$, $u=8$ 圍成, 故 $\int_\Omega(x+y)\,e^{x-y}\,\mathrm{d}A=\int_0^4\int_4^8 ue^v\Big|-\frac{1}{2}\Big|\,\mathrm{d}u\,\mathrm{d}v=\frac{1}{2}\int_0^4 e^v\,\mathrm{d}v\int_4^8 u\,\mathrm{d}u=12\,(e^4-1)$.

例. 求 $\displaystyle\int_\Omega e^{\frac{x+y}{x-y}}\,\mathrm{d}A$, Ω 為頂點 $(1,0)$, $(2,0)$, $(0,-2)$, $(0,-1)$ 之梯形.

解. 令 $\begin{cases}u=x+y\\ v=x-y\end{cases}$, 則 $\begin{cases}x=\frac{1}{2}u+\frac{1}{2}v\\ y=\frac{1}{2}u-\frac{1}{2}v\end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u})=\dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}(\mathbf{u})=\begin{vmatrix}\frac{\partial x}{\partial u}&\frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u}&\frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{-1}{2}\end{vmatrix}=-\dfrac{1}{2}$. 變數變換前 Ω 由 $x-y=1$, $x-y=2$, $x=0$, $y=0$ 所圍成, 變數變換 $(x,y)\to(u,v)$ 後 Ω 由 $v=1$, $v=2$, $u+v=0$, $u-v=0$ 圍成, 故 $\displaystyle\int_\Omega e^{\frac{x+y}{x-y}}\,\mathrm{d}A=\int_1^2\int_{-v}^{v}e^{\frac{u}{v}}\left|-\frac{1}{2}\right|\mathrm{d}u\,\mathrm{d}v=\frac{1}{2}\int_1^2\left(v\,e^{\frac{u}{v}}\Big|_{u=-v}^{u=v}\right)\mathrm{d}v=\frac{3\,(e-e^{-1})}{4}$.

例. 求 $y=x^2$, $y=2x^2$, $x=y^2$, $x=3y^2$ 所圍成之區域面積.

解. 令 $\begin{cases}u=\frac{y}{x^2}\\ v=\frac{x}{y^2}\end{cases}$, 則 $\begin{cases}x=(u^2v)^{-\frac{1}{3}}\\ y=(uv^2)^{-\frac{1}{3}}\end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u})=\dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}(\mathbf{u})=\begin{vmatrix}\frac{\partial x}{\partial u}&\frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u}&\frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}-\frac{1}{3}(u^2v)^{-\frac{4}{3}}\cdot 2uv&-\frac{1}{3}(u^2v)^{-\frac{4}{3}}\cdot u^2\\ -\frac{1}{3}(uv^2)^{-\frac{4}{3}}\cdot v^2&-\frac{1}{3}(uv^2)^{-\frac{4}{3}}\cdot 2uv\end{vmatrix}$ $=\dfrac{1}{3u^2v^2}$. 變數變換 $(x,y)\to(u,v)$ 後 Ω 由 $u=1$, $u=2$, $v=1$, $v=3$ 圍成, 亦即 $\Omega=\{1\leqslant u\leqslant 2,\ 1\leqslant v\leqslant 3\}$, 面積為 $\displaystyle\int_\Omega\mathrm{d}A=\int_1^3\int_1^2\left|\frac{1}{3u^2v^2}\right|\mathrm{d}u\,\mathrm{d}v=\frac{1}{9}$.

例. 求 $\displaystyle\int_\Omega\frac{xy}{1+x^2y^2}\,\mathrm{d}A$, Ω 為 $xy=1$, $xy=5$, $x=1$, $x=6$ 所圍成之區域.

解. 令 $\begin{cases}u=xy\\ v=x\end{cases}$, 則 $\begin{cases}x=v\\ y=\frac{u}{v}\end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u})=\dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}(\mathbf{u})=\begin{vmatrix}\frac{\partial x}{\partial u}&\frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u}&\frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}0&1\\ \frac{1}{v}&-\frac{u}{v^2}\end{vmatrix}=-\dfrac{1}{v}$. 變數變換 $(x,y)\to(u,v)$ 後 Ω 由 $u=1$, $u=5$, $v=1$, $v=6$ 圍成, 亦即 $\Omega=\{1\leqslant u\leqslant 5,\ 1\leqslant v\leqslant 6\}$, 故 $\displaystyle\int_\Omega\frac{xy}{1+x^2y^2}\,\mathrm{d}A=\int_1^6\int_1^5\frac{u}{1+u^2}\left|-\frac{1}{v}\right|\mathrm{d}u\,\mathrm{d}v=\int_1^6\frac{1}{v}\,\mathrm{d}v\int_1^5\frac{u}{1+u^2}\,\mathrm{d}u=\frac{\ln 6\cdot\ln 13}{2}$.

例. 求 $\displaystyle\int_\Omega\frac{y}{x}\,\mathrm{d}A$, Ω 為 $y=x$, $x^2+4y^2=4$, $y\geqslant 0$ 所圍成之區域.

解. 令 $\begin{cases}x=2r\cos\theta\\ y=r\sin\theta\end{cases}$, Jacobian $J_{\mathbf{x}}(\mathbf{u})=\dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}(\mathbf{u})=\begin{vmatrix}\frac{\partial x}{\partial r}&\frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r}&\frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}2\cos\theta&-2r\sin\theta\\ \sin\theta&r\cos\theta\end{vmatrix}=2r$. 變數變換 $(x,y)\to(r,\theta)$ 後 Ω 由 $r=0$, $r=1$, $\theta=0$, $\theta=\tan^{-1}2$ 圍成, 亦即 $\Omega=\{0\leqslant r\leqslant 1,\ 0\leqslant\theta\leqslant\tan^{-1}2\}$, 故 $\displaystyle\int_\Omega\frac{y}{x}\,\mathrm{d}A=\int_0^{\tan^{-1}2}\int_0^1\frac{1}{2}\tan\theta\,\big|2r\big|\,\mathrm{d}r\,\mathrm{d}\theta=\int_0^{\tan^{-1}2}\tan\theta\,\mathrm{d}\theta\int_0^1 r\,\mathrm{d}r=-\frac{1}{2}\ln|\cos\theta|\Big|_0^{\tan^{-1}2}=\frac{\ln 5}{4}$.

例. 求 $\displaystyle\int_\Omega\sin(9x^2+4y^2)\,\mathrm{d}A$, Ω 為 $9x^2+4y^2=1$, $y\geqslant 0$, $x\geqslant 0$ 所圍成之區域.

解. 令 $\begin{cases}x=\frac{1}{3}r\cos\theta\\ y=\frac{1}{2}r\sin\theta\end{cases}$, Jacobian $J_{\mathbf{x}}(\mathbf{u})=\dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}(\mathbf{u})=\begin{vmatrix}\frac{\partial x}{\partial r}&\frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r}&\frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\frac{1}{3}\cos\theta&-\frac{1}{3}r\sin\theta\\ \frac{1}{2}\sin\theta&\frac{1}{2}r\cos\theta\end{vmatrix}=\dfrac{1}{6}r$. 變數變換 $(x,y)\to(r,\theta)$ 後 Ω 由 $r=0$, $r=1$, $\theta=0$, $\theta=\dfrac{\pi}{2}$ 圍成, 亦即 $\Omega=\{0\leqslant r\leqslant 1,\ 0\leqslant\theta\leqslant\dfrac{\pi}{2}\}$, 故 $\displaystyle\int_\Omega\sin(9x^2+4y^2)\,\mathrm{d}A=\int_0^{\frac{\pi}{2}}\int_0^1\sin r^2\left|\frac{1}{6}r\right|\mathrm{d}r\,\mathrm{d}\theta=\frac{1}{6}\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^1 r\sin r^2\,\mathrm{d}r=\frac{\pi}{24}(1-\cos 1)$.

例. 求 $\displaystyle\int_0^1\int_0^{1-x}\sqrt{x+y}\,(y-2x)^2\,\mathrm{d}y\,\mathrm{d}x$.

解. 令 $\begin{cases} u = x+y \\ v = y-2x \end{cases}$, 則 $\begin{cases} x = \frac{1}{3}u - \frac{1}{3}v \\ y = \frac{2}{3}u + \frac{1}{3}v \end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{-1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \dfrac{1}{3}$. 變數變換前積分區域由 $x+y=0$, $x+y=1$, $x=0$, $y=0$ 所圍成, 變數變換 $(x, y) \to (u, v)$ 後積分區域由 $u=0$, $u=1$, $u-v=0$, $2u+v=0$ 圍成, 故 $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2 \,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_{-2u}^{u} \sqrt{u}\,v^2 \left|\frac{1}{3}\right| \mathrm{d}v\,\mathrm{d}u = \frac{1}{3}\int_0^1 \sqrt{u}\left(\frac{v^3}{3}\bigg|_{v=-2u}^{v=u}\right)\mathrm{d}u = \frac{2}{9}$.

例. 若 Ω 為頂點 $(0, 0)$, $(1, 0)$, $(0, 1)$ 之三角形, f 為可積, 證明 $\displaystyle\int_\Omega f(x+y)\,\mathrm{d}A = \int_0^1 u\,f(u)\,\mathrm{d}u$.

解. 令 $\begin{cases} u = x+y \\ v = x \end{cases}$, 則 $\begin{cases} x = v \\ y = u-v \end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$. 變數變換前 Ω 由 $x+y=0$, $x+y=1$, $x=0$, $y=0$ 所圍成, 變數變換 $(x, y) \to (u, v)$ 後 Ω 由 $u=0$, $u=1$, $v=0$, $u-v=0$ 圍成, 故 $\displaystyle\int_\Omega f(x+y)\,\mathrm{d}A = \int_0^1 \int_0^u f(u)\,\big|-1\big|\,\mathrm{d}v\,\mathrm{d}u = \int_0^1 u\,f(u)\,\mathrm{d}u$.

例. 1. 證明 $\displaystyle\int_0^1 \int_0^1 \frac{1}{1-xy}\,\mathrm{d}x\,\mathrm{d}y = \sum_{n=1}^\infty \frac{1}{n^2}$. 2. 以變數變換證明 $\displaystyle\int_0^1 \int_0^1 \frac{1}{1-xy}\,\mathrm{d}x\,\mathrm{d}y = \frac{\pi^2}{6}$.

解.

1. $|xy| < 1$ 時 $\dfrac{1}{1-xy} = \displaystyle\sum_{n=0}^\infty (xy)^n$ 且可逐項積分; 代入 $\displaystyle\int_0^1\int_0^1 \frac{1}{1-xy}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\int_0^1 \sum_{n=0}^\infty (xy)^n\,\mathrm{d}x\,\mathrm{d}y = \sum_{n=0}^\infty \int_0^1 x^n\,\mathrm{d}x \int_0^1 y^n\,\mathrm{d}y = \sum_{n=0}^\infty \frac{1}{(n+1)^2} = \sum_{n=1}^\infty \frac{1}{n^2}$.
2. 令 $\begin{cases} x = \frac{u-v}{\sqrt{2}} \\ y = \frac{u+v}{\sqrt{2}} \end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = 1$. 變數變換前積分區域由 $x=0$, $x=1$, $y=0$, $y=1$ 所圍成, 變數變換 $(x, y) \to (u, v)$ 後積分區域由 $u-v=0$, $u-v=\sqrt{2}$, $u+v=0$, $u+v=\sqrt{2}$ 圍成, 故 $\displaystyle\int_0^1\int_0^1 \frac{1}{1-xy}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\sqrt{2}}{2}}\int_{-u}^{u} \frac{\mathrm{d}v\,\mathrm{d}u}{1-\frac{u-v}{\sqrt{2}}\frac{u+v}{\sqrt{2}}} + \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}}\int_{-\sqrt{2}+u}^{\sqrt{2}-u} \frac{\mathrm{d}v\,\mathrm{d}u}{1-\frac{u-v}{\sqrt{2}}\frac{u+v}{\sqrt{2}}} = 2\left(\int_0^{\frac{\sqrt{2}}{2}}\int_{-u}^{u} \frac{\mathrm{d}v\,\mathrm{d}u}{2-u^2+v^2} + \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}}\int_{-\sqrt{2}+u}^{\sqrt{2}-u} \frac{\mathrm{d}v\,\mathrm{d}u}{2-u^2+v^2}\right) = 2\left(\int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{2-u^2}}\left(\tan^{-1}\frac{v}{\sqrt{2-u^2}}\bigg|_{v=-u}^{v=u}\right)\mathrm{d}u\right.$ $\displaystyle\left. + \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}}\left(\tan^{-1}\frac{v}{\sqrt{2-u^2}}\bigg|_{v=-\sqrt{2}+u}^{v=\sqrt{2}-u}\right)\mathrm{d}u\right)$ $\displaystyle= 4\left(\int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{2-u^2}}\tan^{-1}\frac{u}{\sqrt{2-u^2}}\,\mathrm{d}u + \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}}\tan^{-1}\frac{\sqrt{2}-u}{\sqrt{2-u^2}}\,\mathrm{d}u\right)$. 令 $u = \sqrt{2}\sin\theta$, 則 $\mathrm{d}u = \sqrt{2}\cos\theta\,\mathrm{d}\theta$; 化簡 $\displaystyle 4\left(\int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{2-u^2}}\tan^{-1}\frac{u}{\sqrt{2-u^2}}\,\mathrm{d}u + \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}}\tan^{-1}\frac{\sqrt{2}-u}{\sqrt{2-u^2}}\,\mathrm{d}u\right)$ $\displaystyle= 4\left(\int_0^{\frac{\pi}{6}} \tan^{-1}(\tan\theta)\,\mathrm{d}\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \tan^{-1}\left(\frac{1-\sin\theta}{\cos\theta}\right)\mathrm{d}\theta\right) = 4\left(\int_0^{\frac{\pi}{6}} \theta\,\mathrm{d}\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\mathrm{d}\theta\right) = \frac{\pi^2}{6}$; 其中 $\dfrac{1-\sin\theta}{\cos\theta} = \dfrac{1-\cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)} = \dfrac{2\sin^2\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)}{2\sin\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\cos\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)} = \tan\dfrac{1}{2}\left(\dfrac{\pi}{2}-\theta\right)$.

極座標二重積分

令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$.

例. 求 $\int_\Omega (3x+4y^2)\,\mathrm{d}A$, Ω 為 $x^2+y^2=1$, $x^2+y^2=4$ 與 $y\geqslant 0$ 所圍成之區域.

解. 變數變換 $(x,\,y)\to(r,\,\theta)$ 後 Ω 由 $r=1$, $r=2$, $\theta=0$, $\theta=\pi$ 圍成, 故 $\int_\Omega (3x+4y^2)\,\mathrm{d}A=\int_0^\pi\int_1^2(3r\cos\theta+4r^2\sin^2\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta=\int_0^\pi(7\,\cos\theta+15\,\sin^2\theta)\,\mathrm{d}\theta=\frac{15\,\pi}{2}$.

例. 求 $z=1-x^2-y^2$ 與 $z\geqslant 0$ 所圍成之體積.

解. 體積為 $\int_\Omega(1-x^2-y^2)\,\mathrm{d}A$, Ω 為單位圓 $x^2+y^2=1$. 變數變換 $(x,\,y)\to(r,\,\theta)$ 後 Ω 由 $r=0$, $r=1$, $\theta=0$, $\theta=2\pi$ 圍成, 故 $\int_\Omega(1-x^2-y^2)\,\mathrm{d}A=\int_0^{2\pi}\int_0^1(1-r^2)\,r\,\mathrm{d}r\,\mathrm{d}\theta=\frac{\pi}{2}$.

例. 求 $\int_\Omega \frac{y^2}{x^2}\,\mathrm{d}A$, Ω 為 $0<a^2\leqslant x^2+y^2\leqslant b^2$ 與 $x\geqslant y\geqslant 0$ 所圍成之區域.

解. 變數變換 $(x,\,y)\to(r,\,\theta)$ 後 Ω 由 $r=a$, $r=b$, $\theta=0$, $\theta=\frac{\pi}{4}$ 圍成, 故 $\int_\Omega\frac{y^2}{x^2}\,\mathrm{d}A=\int_0^{\frac{\pi}{4}}\int_a^b\tan^2\theta\cdot r\,\mathrm{d}r\,\mathrm{d}\theta=\frac{b^2-a^2}{2}\int_0^{\frac{\pi}{4}}\tan^2\theta\,\mathrm{d}\theta=\frac{b^2-a^2}{2}\int_0^{\frac{\pi}{4}}(\sec^2\theta-1)\,\mathrm{d}\theta=\frac{b^2-a^2}{2}(\tan\theta-\theta)\Big|_0^{\frac{\pi}{4}}=\frac{b^2-a^2}{2}\Big(1-\frac{\pi}{4}\Big)$.

例. 反變數變換: $\int_0^{\frac{\pi}{2}}\int_0^a\frac{r}{\sqrt{a^2-r^2\cos^2\theta}}\,\mathrm{d}r\,\mathrm{d}\theta=\int_0^a\int_0^{\sqrt{a^2-x^2}}\frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}y\,\mathrm{d}x=\int_0^a\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\,\mathrm{d}x=a$.

例. 若 $a>0$, 求

1. $\int_0^\infty e^{-ax^2}\,\mathrm{d}x$.
2. $\int_0^\infty x\,e^{-ax^2}\,\mathrm{d}x$.
3. $\int_0^\infty x^2\,e^{-ax^2}\,\mathrm{d}x$.

解.

1. $\left(\int_{-\infty}^\infty e^{-x^2}\,\mathrm{d}x\right)^2=\left(\int_{-\infty}^\infty e^{-x^2}\,\mathrm{d}x\right)\left(\int_{-\infty}^\infty e^{-y^2}\,\mathrm{d}y\right)=\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y$. 變數變換 $(x,\,y)\to(r,\,\theta)$ 後得 $\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y=\int_0^{2\pi}\int_0^\infty e^{-r^2}r\,\mathrm{d}r\,\mathrm{d}\theta=2\pi\left(\frac{-e^{-r^2}}{2}\Big|_0^\infty\right)=\pi$, 故 $\int_{-\infty}^\infty e^{-x^2}\,\mathrm{d}x=\sqrt{\pi}$; 由 e^{-x^2} 之對稱性, $\int_0^\infty e^{-x^2}\,\mathrm{d}x=\frac{\sqrt{\pi}}{2}$. 在 $\int_0^\infty e^{-ax^2}\,\mathrm{d}x$ 中, 令 $w=\sqrt{a}\,x\implies x=\frac{w}{\sqrt{a}}\implies \mathrm{d}x=\frac{1}{\sqrt{a}}\,\mathrm{d}w$; 積分範圍 x 由 0 至 ∞, 變數變換後 w 由 $\sqrt{a}{\cdot}0=0$ 至 $\sqrt{a}{\cdot}\infty=\infty$, 則 $\int_0^\infty e^{-ax^2}\,\mathrm{d}x=\frac{1}{\sqrt{a}}\int_0^\infty e^{-w^2}\,\mathrm{d}w=\frac{\sqrt{\pi}}{2\sqrt{a}}$.
2. 令 $w=a\,x^2$, 則 $\mathrm{d}w=2\,a\,x\,\mathrm{d}x\implies x\,\mathrm{d}x=\frac{\mathrm{d}w}{2a}$. 積分範圍 x 由 0 至 ∞, 則變數變換後 w 由 $a\cdot 0^2=0$ 至 $a\cdot\infty^2=\infty$, 故 $\int_0^\infty x\,e^{-\alpha\,x^2}\,\mathrm{d}x=\int_0^\infty e^{-w}\,\frac{\mathrm{d}w}{2a}=\frac{1}{2a}\int_0^\infty e^{-w}\,\mathrm{d}w=\frac{1}{2a}\left(-\,e^{-w}\Big|_0^\infty\right)=\frac{1}{2a}$.
3. 令 $w=\sqrt{a}\,x\implies x=\frac{w}{\sqrt{a}}\implies \mathrm{d}x=\frac{1}{\sqrt{a}}\,\mathrm{d}w$; 積分範圍 x 由 0 至 ∞, 變數變換後 w 由 $\sqrt{a}\cdot 0=0$ 至 $\sqrt{a}\cdot\infty=\infty$, 故 $\int_0^\infty x^2\,e^{-ax^2}\,\mathrm{d}x=\int_0^\infty\frac{w^2}{a}\,e^{-w^2}\,\frac{1}{\sqrt{a}}\,\mathrm{d}w=\frac{1}{a^{\frac{3}{2}}}\int_0^\infty w^2\,e^{-w^2}\,\mathrm{d}w$. 令 $u=-\frac{1}{2}\,w$, 則 $\mathrm{d}u=-\frac{1}{2}\,\mathrm{d}w$. 令 $\mathrm{d}v=-2\,w\,e^{-w^2}\,\mathrm{d}w$, 則 $v=e^{-w^2}$. 故 $\frac{1}{a^{\frac{3}{2}}}\int_0^\infty w^2\,e^{-w^2}\,\mathrm{d}w=\frac{1}{a^{\frac{3}{2}}}\Big(-\frac{1}{2}w\cdot e^{-w^2}\Big|_0^\infty+\frac{1}{2}\int_0^\infty e^{-w^2}\,\mathrm{d}w\Big)=\frac{1}{a^{\frac{3}{2}}}\cdot\frac{1}{2}\cdot\frac{\sqrt{\pi}}{2}=\frac{\sqrt{\pi}}{4a^{\frac{3}{2}}}$.

曲面表面積

結論. 若曲面 S 由 $z=f(x,y)$, $(x,y)\in\Omega$ 所定義, 則 S 的表面積為 $\displaystyle\int_\Omega\sqrt{1+f_x^2+f_y^2}\,\mathrm{d}A$.

證 (sketch)**.** 以切平面近似曲面 S; S 參數式為 $\mathbf{r}=(x,y,f(x,y))$, x, y 方向切向量分別為 $\mathbf{r}_x=(1,0,f_x)$, $\mathbf{r}_y=(0,1,f_y)$; $|\mathbf{r}_x\times\mathbf{r}_y|=|(-f_x,-f_y,1)|=\sqrt{1+f_x^2+f_y^2}$.

例. 求半球面 $z=\sqrt{a^2-x^2-y^2}$ 與圓柱體 $(x-\frac{a}{2})^2+y^2=(\frac{a}{2})^2$ 相交區域之表面積.

解. 由 $z=f(x,y)=\sqrt{a^2-x^2-y^2}$, $f_x=\dfrac{-x}{\sqrt{a^2-x^2-y^2}}$, $f_y=\dfrac{-y}{\sqrt{a^2-x^2-y^2}}$, $\mathrm{d}S=\sqrt{1+f_x^2+f_y^2}\,\mathrm{d}x\,\mathrm{d}y=\sqrt{1+\dfrac{x^2}{a^2-x^2-y^2}+\dfrac{y^2}{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y=\sqrt{\dfrac{a^2}{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y$. 所求表面積為 $\displaystyle\int_\Omega\mathrm{d}S$, $\Omega=\big\{(x,y)\,\big|\,(x-\frac{a}{2})^2+y^2\leqslant(\frac{a}{2})^2\big\}$; $\displaystyle\int_\Omega\mathrm{d}S=\int_\Omega\sqrt{\frac{a^2}{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y=2\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}\sqrt{\frac{a^2}{a^2-r^2}}\,r\,\mathrm{d}r\,\mathrm{d}\theta$

$\displaystyle=2a\int_0^{\frac{\pi}{2}}\Big(-\sqrt{a^2-r^2}\,\Big|_{r=0}^{r=a\cos\theta}\Big)\,\mathrm{d}\theta=2a^2\int_0^{\frac{\pi}{2}}(a-\sin\theta)\,\mathrm{d}\theta=a^2(\pi-2).$

7.2 三重積分

一般區域積分

例. 若 E 為中心 0, 半徑 a 之球體, 寫出 $\displaystyle\int_E f(x,y,z)\,\mathrm{d}V$.

解. $\displaystyle\int_E f(x,y,z)\,\mathrm{d}V$

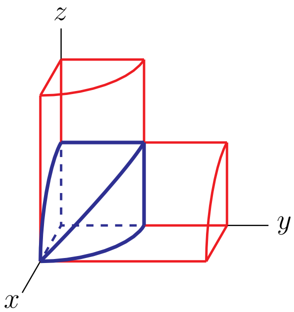
$$=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}}f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x=\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}}f(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$
$$=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-z^2}}^{\sqrt{a^2-x^2-z^2}}f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x=\int_{-a}^{a}\int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}}\int_{-\sqrt{a^2-x^2-z^2}}^{\sqrt{a^2-x^2-z^2}}f(x,y,z)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$$
$$=\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-y^2-z^2}}^{\sqrt{a^2-y^2-z^2}}f(x,y,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y=\int_{-a}^{a}\int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}}\int_{-\sqrt{a^2-y^2-z^2}}^{\sqrt{a^2-y^2-z^2}}f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

例. 若 E 為第一卦限中 $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}\leqslant 1$ 圍成之區域, $a,\ b,\ c>0$, 寫出 $\displaystyle\int_E f(x,y,z)\,\mathrm{d}V$.

解. $\displaystyle\int_E f(x,y,z)\,\mathrm{d}V$

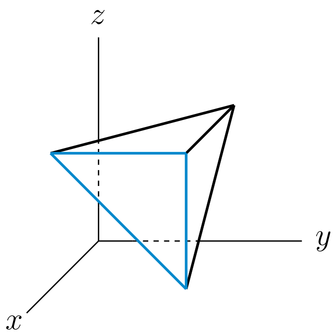
$$=\int_0^a\int_0^{b-\frac{bx}{a}}\int_0^{c-\frac{cx}{a}-\frac{cy}{b}}f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x=\int_0^b\int_0^{a-\frac{ay}{b}}\int_0^{c-\frac{cx}{a}-\frac{cy}{b}}f(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$
$$=\int_0^a\int_0^{c-\frac{cx}{a}}\int_0^{b-\frac{bx}{a}-\frac{bz}{c}}f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x=\int_0^c\int_0^{a-\frac{az}{c}}\int_0^{b-\frac{bx}{a}-\frac{bz}{c}}f(x,y,z)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$$
$$=\int_0^b\int_0^{c-\frac{cy}{b}}\int_0^{a-\frac{ay}{b}-\frac{az}{c}}f(x,y,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y=\int_0^c\int_0^{b-\frac{bz}{c}}\int_0^{a-\frac{ay}{b}-\frac{az}{c}}f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

例. 求 $x^2+y^2=a^2$ 與 $x^2+z^2=a^2$ 交集區域之表面積與體積.

解. 

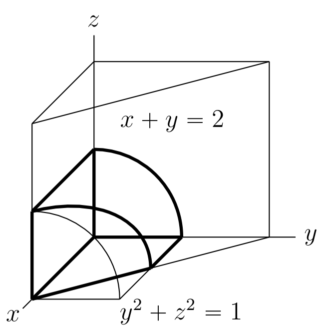
$$(\text{表面積}) = 16\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{1+\left(\frac{-x}{\sqrt{a^2-x^2}}\right)^2}\,\mathrm{d}y\,\mathrm{d}x = 16\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{\frac{a^2}{a^2-x^2}}\,\mathrm{d}y\,\mathrm{d}x$$
$$= 16a\int_0^a\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\,\mathrm{d}x = 16a^2.$$
$$(\text{體積}) = 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2}}\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = 8\int_0^a(a^2-x^2)\,\mathrm{d}x = \frac{16a^3}{3}.$$

例. 若 E 為 $x=1$, $y=1$, $z=1$, $x+y+z=2$ 圍成之四面體, 求 $\displaystyle\int_E x\,\mathrm{d}V$.

解. 

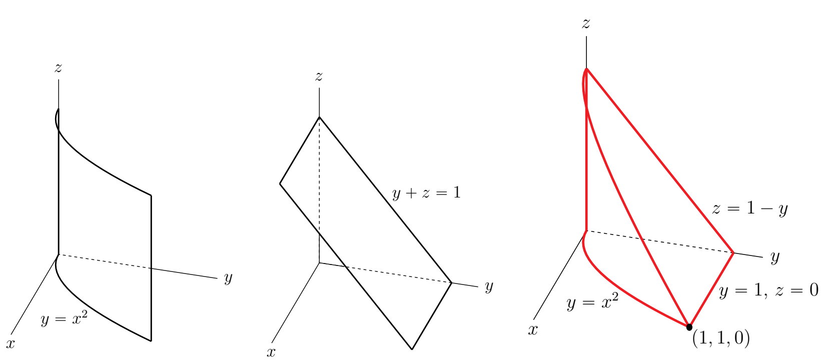
$$\int_E x\,\mathrm{d}V = \int_0^1\int_{1-x}^1\int_{2-x-y}^1 x\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\int_{1-x}^1 x\,(x+y-1)\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1\int_{1-x}^1(x^2-x+xy)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\left((x^2-x)\,x + x\,\frac{1-(1-x)^2}{2}\right)\mathrm{d}x$$
$$= \int_0^1\left(x^3-x^2+x\,\frac{2x-x^2}{2}\right)\mathrm{d}x = \int_0^1\frac{x^3}{2}\,\mathrm{d}x = \frac{1}{8}$$

例. 若 E 為第一卦限中 $y^2+z^2=1$, $x+y=2$, $y=0$, $z=0$ 圍成之區域, 求 $\displaystyle\int_E z\,\mathrm{d}V$.

解. 

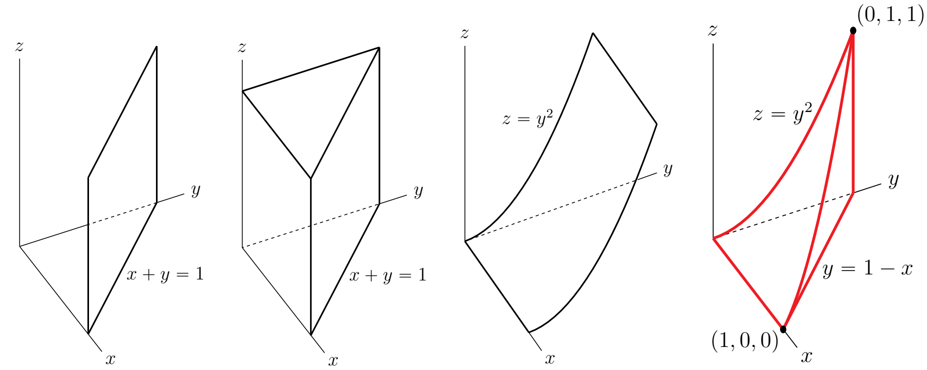
$$\int_E z\,\mathrm{d}V = \int_0^1\int_0^{2-y}\int_0^{\sqrt{1-y^2}} z\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\frac{1}{2}(1-y^2)(2-y)\,\mathrm{d}y$$
$$= \frac{1}{2}\int_0^1(2-y-2y^2+y^3)\,\mathrm{d}y = \frac{1}{2}\left(2-\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right) = \frac{1}{2}\cdot\frac{13}{12} = \frac{13}{24}$$

例. 若 E 為第一卦限中 $y=x^2$, $y+z=1$ 圍成之區域, 求 $\displaystyle\int_E x\,\mathrm{d}V$.

解. 

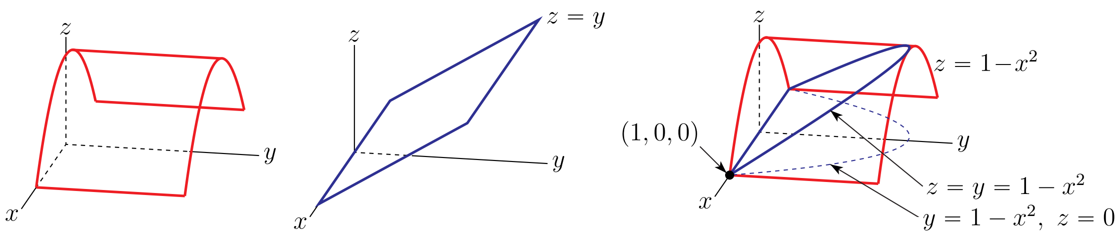
$$\int_E x\,\mathrm{d}V = \int_0^1\int_{x^2}^1\int_0^{1-y} x\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1\int_{x^2}^1 x\,(1-y)\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1\left(x(1-x^2) - x\left(\frac{y^2}{2}\Bigg|_{y=x^2}^{y=1}\right)\right)\mathrm{d}x$$
$$= \int_0^1\left(x-x^3+\frac{x^5}{2}-\frac{x}{2}\right)\mathrm{d}x = \frac{1}{12}$$

例. 若 E 為第一卦限中 $z=y^2$, $x+y=1$ 圍成之區域, 求 $\displaystyle\int_E z\,\mathrm{d}V$.

解. 

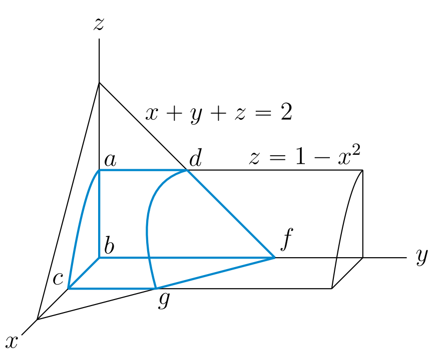
$$\int_E z\,\mathrm{d}V = \int_0^1\int_0^{1-x}\int_0^{y^2} z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\int_0^{1-x}\frac{y^4}{2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\frac{(1-x)^5}{10}\,\mathrm{d}x = \frac{1}{60}$$

例. 若 E 為 $z=1-x^2$, $y=z$, $y=0$, $z=0$ 圍成之區域, 求 $\int_E \mathrm{d}V$.

解. 

$$\int_E \mathrm{d}V = \int_{-1}^1\int_0^{1-x^2}\int_y^{1-x^2}\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_{-1}^1\frac{(1-x^2)^2}{2}\,\mathrm{d}x = \frac{8}{15}$$

例. 若 E 為 $x=0$, $y=0$, $z=0$, $x+y+z=2$, $x^2+z=1$ 圍成之區域, 求 $\int_E x\,\mathrm{d}V$.

解. 

$$\begin{aligned}\int_E x\,\mathrm{d}V &= \int_0^1\int_0^{1-x^2}\int_0^{2-x-z} x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x = \int_0^1\int_0^{1-x^2}(2-x-z)\,x\,\mathrm{d}z\,\mathrm{d}x\\ &= \int_0^1\left((2-x)\,x\,(1-x^2) - x\,\frac{(1-x^2)^2}{2}\right)\mathrm{d}x\\ &= \frac12\int_0^1(-x^5+2x^4-2x^3-2x^2+3x)\,\mathrm{d}x\\ &= \frac12\left(-\frac16+\frac25-\frac12-\frac23+\frac32\right) = \frac{17}{60}\end{aligned}$$

例. 若 E 為 $x+y+z=1$, $x=0$, $y=0$, $z=0$ 圍成之區域, 求 $\int_E z\,\mathrm{d}V$.

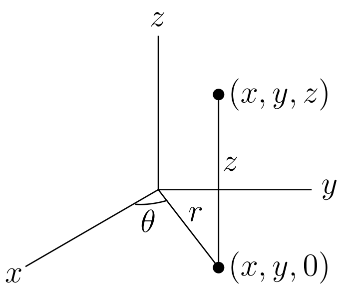
解. $\int_E z\,\mathrm{d}V = \int_0^1\int_0^{1-x}\int_0^{1-x-y} z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\int_0^{1-x}\frac{(1-x-y)^2}{2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 -\frac{(1-x-y)^3}{6}\Big|_{y=0}^{y=1-x}\,\mathrm{d}x$
$= \int_0^1\frac16(1-x)^3\,\mathrm{d}x = -\frac{1}{24}(1-x)^4\Big|_0^1 = \frac{1}{24}.$

例. 若 E 為 $x+2y+z=2$, $x=2y$, $x=0$, $z=0$ 圍成之區域, 求 $\int_E y\,\mathrm{d}V$.

解. $\int_E y\,\mathrm{d}V = \int_0^1\int_{\frac{x}{2}}^{\frac{2-x}{2}}\int_0^{2-x-2y} y\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\int_{\frac{x}{2}}^{\frac{2-x}{2}}(2-x-2y)\,y\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\left(\frac{(2-x)y^2}{2}-\frac{2y^3}{3}\right)\Big|_{\frac{x}{2}}^{\frac{2-x}{2}}\,\mathrm{d}x =$
$\int_0^1\left(\frac{2-x}{2}\left(\left(\frac{2-x}{2}\right)^2-\left(\frac{x}{2}\right)^2\right)-\frac23\left(\left(\frac{2-x}{2}\right)^3-\left(\frac{x}{2}\right)^3\right)\right)\mathrm{d}x = \int_0^1\left(\frac{2-x}{2}(1-x)-\frac23(1-x)\left(\left(\frac{2-x}{2}\right)^2+\right.\right.$
$\left.\left.\frac{2-x}{2}\cdot\frac{x}{2}+\left(\frac{x}{2}\right)^2\right)\right)\mathrm{d}x = \frac16\int_0^1(x^3-3x+2)\,\mathrm{d}x = \frac16\left(\frac14-\frac32+2\right) = \frac18.$

變數變換

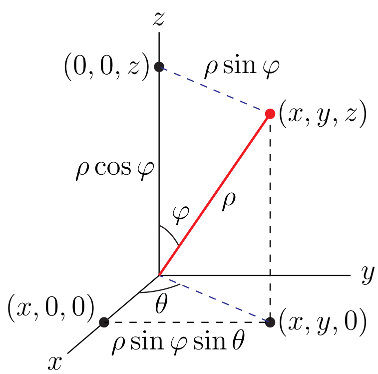
柱面座標



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \iff \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\frac{y}{x} \\ z = z \end{cases}$$

$$\text{Jacobian } J_{\mathbf{x}}(\mathbf{u}) = \frac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

球面座標



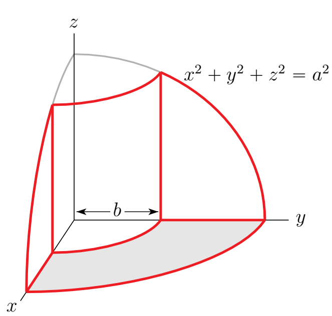
$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases} \iff \begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\frac{y}{x} \\ \varphi = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} \end{cases}$$

$$\text{Jacobian } J_{\mathbf{x}}(\mathbf{u}) = \frac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix} = -\rho^2\sin\varphi$$

例. 半徑為 a 之球中心對稱鑽半徑為 b 之圓孔, $a>b>0$, 求球剩下的體積.

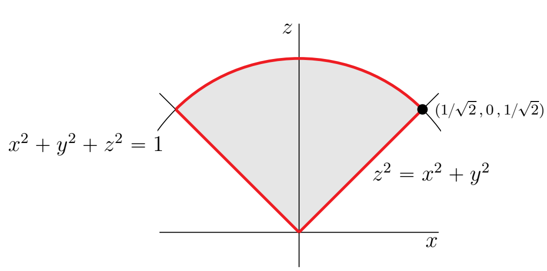
解.



- 柱面座標: $\displaystyle\int_E \mathrm{d}V = 8\int_0^{\frac{\pi}{2}}\int_b^a\int_0^{\sqrt{a^2-r^2}} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta = \frac{4\pi}{3}(a^2-b^2)^{\frac{3}{2}}$
- 球面座標: $\displaystyle\int_E \mathrm{d}V = 2\int_{\sin^{-1}\frac{b}{a}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_{\frac{b}{\sin\varphi}}^a \rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi = \frac{4\pi}{3}(a^2-b^2)^{\frac{3}{2}}$

例. 若 E 為 $x^2+y^2\leqslant z^2$, $x^2+y^2+z^2\leqslant 1$ 與 $z\geqslant 0$ 圍成之區域, 求 $\displaystyle\int_E \sqrt{x^2+y^2+z^2}\,\mathrm{d}V$.

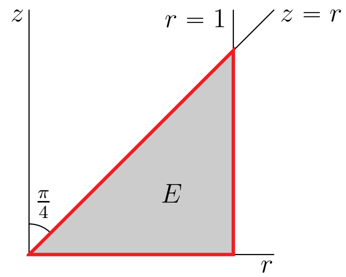
解.



- 柱面座標: $\displaystyle\int_E \sqrt{x^2+y^2+z^2}\,\mathrm{d}V$
  $$= \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\int_r^{\sqrt{1-r^2}} \sqrt{r^2+z^2}\cdot r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$
- 球面座標: $\displaystyle\int_E \sqrt{x^2+y^2+z^2}\,\mathrm{d}V$
  $$= \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^1 \rho\cdot\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi = \frac{\pi}{2}\left(1-\frac{1}{\sqrt{2}}\right)$$

例. 若 E 為 $0 \leqslant z \leqslant \sqrt{x^2+y^2}$ 與 $x^2+y^2 \leqslant 1$ 圍成之區域, 求 $\displaystyle\int_E z\sqrt{x^2+y^2+z^2}\,\mathrm{d}V$.

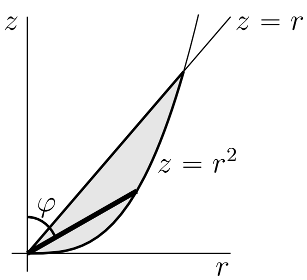
解.



- 柱面座標: $\displaystyle\int_E z\sqrt{x^2+y^2+z^2}\,\mathrm{d}V = \int_0^{2\pi}\int_0^1\int_0^r z\sqrt{r^2+z^2}\cdot r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$
$$= 2\pi\int_0^1 \frac{r}{3}(r^2+z^2)^{\frac{3}{2}}\Big|_{z=0}^{z=r}\mathrm{d}r = \frac{2\pi}{3}\int_0^1 r\cdot(2^{\frac{3}{2}}-1)\,r^3\,\mathrm{d}r = \frac{2\pi\,(2^{\frac{3}{2}}-1)}{15}$$
- 球面座標: $\displaystyle\int_E z\sqrt{x^2+y^2+z^2}\,\mathrm{d}V = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{\frac{1}{\sin\varphi}}\rho\cos\varphi\cdot\rho\cdot\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$

例. 若 E 為 $z=x^2+y^2$ 與 $z \leqslant \sqrt{x^2+y^2}$ 圍成之區域, 求 $\displaystyle\int_E z\,(x^2+y^2+z^2)\,\mathrm{d}V$.

解.



- 柱面座標: $\displaystyle\int_E z\,(x^2+y^2+z^2)\,\mathrm{d}V = \int_0^{2\pi}\int_0^1\int_{r^2}^r z\,(r^2+z^2)\cdot r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$
$$= 2\pi\int_0^1\int_{r^2}^r (r^3z+rz^3)\,\mathrm{d}z\,\mathrm{d}r = 2\pi\int_0^1\left(\frac{r^3}{2}(r^2-r^4)+\frac{r}{4}(r^4-r^8)\right)\mathrm{d}r = \frac{3\pi}{40}$$
- 球面座標: $z=x^2+y^2 \implies \rho\cos\varphi=\rho^2\sin^2\varphi \implies \rho=\dfrac{\cos\varphi}{\sin^2\varphi}$;
$z=\sqrt{x^2+y^2} \implies \rho\cos\varphi=\rho\sin\varphi \implies \tan\varphi=1$;
$$\int_E z\,(x^2+y^2+z^2)\,\mathrm{d}V = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{\frac{\cos\varphi}{\sin^2\varphi}}\rho\cos\varphi\cdot\rho^2\cdot\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$$

例. 若 E 為 $y=x^2+z^2$ 與 $y=4$ 圍成之區域, 求 $\displaystyle\int_E \sqrt{x^2+z^2}\,\mathrm{d}V$.

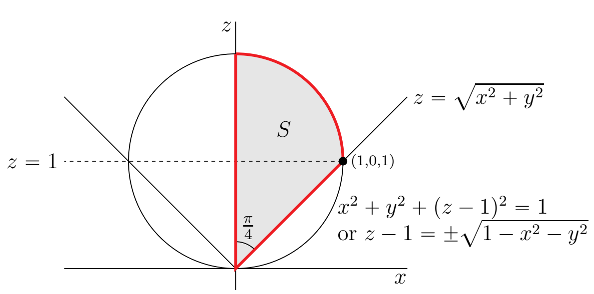
解. 令 $\begin{cases} x=r\cos\theta \\ z=r\sin\theta \end{cases}$, 投影之 Ω 為 $x^2+z^2 \leqslant 4$, 則 $\displaystyle\int_E \sqrt{x^2+z^2}\,\mathrm{d}V = \int_\Omega\int_{x^2+z^2}^4 \sqrt{x^2+z^2}\,\mathrm{d}y\,\mathrm{d}A$
$$= \int_\Omega (4-(x^2+z^2))\sqrt{x^2+z^2}\,\mathrm{d}A = \int_0^{2\pi}\int_0^2(4-r^2)\,r\cdot r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi\int_0^2(4r^2-r^4)\,\mathrm{d}r = 2\pi\cdot\left(\frac{4\cdot 2^3}{3}-\frac{2^5}{5}\right) = \frac{128\pi}{15}.$$

例. 若 $E=\{(x,y,z)\,|\,x^2+y^2+(z-1)^2 \leqslant 1\}$, 求 $\displaystyle\int_E (x^2+y^2+z^2)^{\frac{5}{2}}\,\mathrm{d}V$.

解. 代入球面座標於 $x^2+y^2+(z-1)^2 \leqslant 1 \implies (\rho\sin\varphi\cos\theta)^2+(\rho\sin\varphi\sin\theta)^2+(\rho\cos\varphi-1)^2 \leqslant 1 \implies \rho^2\sin^2\varphi+\rho^2\cos^2\varphi-2\rho\cos\varphi+1 \leqslant 1 \implies \rho^2 \leqslant 2\rho\cos\varphi \implies \rho \leqslant 2\cos\varphi$, 故 $\displaystyle\int_E (x^2+y^2+z^2)^{\frac{5}{2}}\,\mathrm{d}V =$
$$\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{2\cos\varphi}\rho^5\cdot\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi = 2\pi\int_0^{\frac{\pi}{2}}\frac{(2\cos\varphi)^8}{8}\sin\varphi\,\mathrm{d}\varphi = \frac{64\pi}{9}(-\cos^9\varphi)\Big|_0^{\frac{\pi}{2}} = \frac{64\pi}{9}.$$

例. 若 E 為 $z=\sqrt{x^2+y^2}$ 與 $x^2+y^2+(z-1)^2=1$ 於第一卦限圍成之區域, 求其體積.

解.



- 柱面座標: $\displaystyle\int_E \mathrm{d}V = \int_0^{\frac{\pi}{2}}\int_0^1\int_r^{1+\sqrt{1-r^2}} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta = \frac{\pi}{4}$
- 球面座標: $\displaystyle\int_E \mathrm{d}V = \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}}\int_0^{2\cos\varphi}\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi = \frac{\pi}{4}$

7.3 綜合問題

例. 已知 $\forall\, 0<\alpha<1$, $\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x=\frac{\pi}{\sin\alpha\pi}$, 證明 $\Gamma(\alpha)\,\Gamma(1-\alpha)=\dfrac{\pi}{\sin\alpha\pi}$.

解. $\Gamma(\alpha)\,\Gamma(1-\alpha)=\displaystyle\int_0^\infty e^{-t}t^{\alpha-1}\,\mathrm{d}t\int_0^\infty e^{-s}s^{-\alpha}\,\mathrm{d}s=\int_0^\infty\!\!\int_0^\infty e^{-(t+s)}t^{\alpha-1}s^{-\alpha}\,\mathrm{d}s\,\mathrm{d}t$. 令 $\begin{cases}u=s+t\\ v=\frac{t}{s}\end{cases}$, 則 $\begin{cases}s=\frac{u}{1+v}\\ t=\frac{uv}{1+v}\end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u})=\dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}(\mathbf{u})=\begin{vmatrix}\frac{\partial s}{\partial u}&\frac{\partial s}{\partial v}\\ \frac{\partial t}{\partial u}&\frac{\partial t}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{1+v}&\frac{-u}{(1+v)^2}\\ \frac{v}{1+v}&\frac{u}{(1+v)^2}\end{vmatrix}=\dfrac{u}{(1+v)^2}$. 變數變換 $(s,t)\to(u,v)$ 後積分範圍仍為 $0<u<\infty$, $0<v<\infty$, 故 $\displaystyle\int_0^\infty\!\!\int_0^\infty e^{-(t+s)}t^{\alpha-1}s^{-\alpha}\,\mathrm{d}s\,\mathrm{d}t=\int_0^\infty\!\!\int_0^\infty e^{-(t+s)}\Big(\frac{t}{s}\Big)^\alpha t^{-1}\,\mathrm{d}s\,\mathrm{d}t=\int_0^\infty\!\!\int_0^\infty e^{-u}v^\alpha\frac{1+v}{uv}\frac{u}{(1+v)^2}\,\mathrm{d}u\,\mathrm{d}v=\int_0^\infty\!\!\int_0^\infty e^{-u}\frac{v^{\alpha-1}}{1+v}\,\mathrm{d}u\,\mathrm{d}v=\int_0^\infty e^{-u}\,\mathrm{d}u\int_0^\infty\frac{v^{\alpha-1}}{1+v}\,\mathrm{d}v=\int_0^\infty\frac{v^{\alpha-1}}{1+v}\,\mathrm{d}v=\frac{\pi}{\sin\alpha\pi}$.

例. Beta 函數定義為 $B(m,n)=\displaystyle\int_0^1 t^{m-1}(1-t)^{n-1}\,\mathrm{d}t\;\forall\, m,n>0$. 證明 $B(m,n)=B(n,m)=\dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

證. $\Gamma(x)=\displaystyle\int_0^\infty e^{-t}t^{x-1}\,\mathrm{d}t$; 令 $t=u^2$, 則 $\Gamma(x)=2\displaystyle\int_0^\infty e^{-u^2}u^{2x-1}\,\mathrm{d}u$.

$$\Gamma(m)\,\Gamma(n)=\Big(2\int_0^\infty e^{-u^2}u^{2m-1}\,\mathrm{d}u\Big)\Big(2\int_0^\infty e^{-v^2}v^{2n-1}\,\mathrm{d}v\Big)=4\int_0^\infty\!\!\int_0^\infty e^{-(u^2+v^2)}u^{2m-1}v^{2n-1}\,\mathrm{d}u\,\mathrm{d}v$$
$$=4\int_0^{\frac{\pi}{2}}\!\!\int_0^\infty e^{-r^2}(r\cos\theta)^{2m-1}(r\sin\theta)^{2n-1}\,r\,\mathrm{d}r\mathrm{d}\theta=\Big(2\int_0^\infty e^{-r^2}r^{2(m+n)-1}\,\mathrm{d}r\Big)\Big(2\int_0^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta\,\mathrm{d}\theta\Big)$$
$=\Gamma(m+n)\cdot 2\displaystyle\int_0^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta\,\mathrm{d}\theta$. 令 $t=\cos^2\theta$, 則 $\mathrm{d}t=-2\cos\theta\sin\theta\,\mathrm{d}\theta$; 故 $2\displaystyle\int_0^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta\,\mathrm{d}\theta$
$$=\int_0^{\frac{\pi}{2}}\cos^{2m-2}\theta\sin^{2n-2}\theta\big(-2\cos\theta\sin\theta\,\mathrm{d}\theta\big)=-\int_1^0 t^{m-1}(1-t)^{n-1}\,\mathrm{d}t=\int_0^1 t^{m-1}(1-t)^{n-1}\,\mathrm{d}t=B(m,n).$$

例. 設 $V_n(a)$ 為半徑 a 之 n 維球體積, $n\geqslant 1$, $a>0$; 證明 $V_n(1)=\dfrac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

證. $V_n(a)=\displaystyle\int_{x_1^2+x_2^2+\cdots+x_n^2\leqslant a^2}\mathrm{d}x_1\,\mathrm{d}x_2\,\cdots\,\mathrm{d}x_n$. 變數變換 $x_i=au_i\;\forall\, i=1,2,\ldots,n$, 則 Jacobian 為 a^n, 積分範圍變為 $u_1^2+u_2^2+\cdots+u_n^2\leqslant 1$, 故 $V_n(a)=a^n\,V_n(1)$. $V_n(1)=\displaystyle\int_{u_1^2+u_2^2+\cdots+u_n^2\leqslant 1}\mathrm{d}u_1\,\mathrm{d}u_2\,\cdots\,\mathrm{d}u_n$
$$=\int_{u_n^2\leqslant 1}\Big(\int_{u_1^2+u_2^2+\cdots+u_{n-1}^2\leqslant 1-u_n^2}\mathrm{d}u_1\,\mathrm{d}u_2\,\cdots\,\mathrm{d}u_{n-1}\Big)\mathrm{d}u_n=\int_{-1}^1 V_{n-1}\big(\sqrt{1-u_n^2}\big)\,\mathrm{d}u_n=V_{n-1}(1)\cdot\int_{-1}^1(1-u_n^2)^{\frac{n-1}{2}}\,\mathrm{d}u_n$$
$=V_{n-1}(1)\cdot 2\displaystyle\int_0^1(1-u_n^2)^{\frac{n-1}{2}}\,\mathrm{d}u_n$. 令 $t=u_n^2\implies u_n=\sqrt{t}\implies \mathrm{d}u_n=\dfrac{\mathrm{d}t}{2\sqrt{t}}$, 則 $2\displaystyle\int_0^1(1-u_n^2)^{\frac{n-1}{2}}\,\mathrm{d}u_n=\int_0^1(1-t)^{\frac{n-1}{2}}t^{-\frac{1}{2}}\,\mathrm{d}t=B\Big(\frac{n+1}{2},\frac{1}{2}\Big)=\frac{\Gamma(\frac{n+1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n+2}{2})}$. 故 $V_n(1)=\dfrac{\Gamma(\frac{n+1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n+2}{2})}V_{n-1}(1)=\dfrac{\Gamma(\frac{n+1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n+2}{2})}\cdot\dfrac{\Gamma(\frac{n}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})}V_{n-2}(1)=\dfrac{\Gamma(\frac{n+1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n+2}{2})}\cdot\dfrac{\Gamma(\frac{n}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})}\cdot\dfrac{\Gamma(\frac{n-1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})}V_{n-3}(1)=\cdots\cdots\cdots=\dfrac{\Gamma(\frac{3}{2})\cdot\big(\Gamma(\frac{1}{2})\big)^{n-1}}{\Gamma(\frac{n+2}{2})}V_1(1)=\dfrac{\frac{1}{2}\sqrt{\pi}\cdot(\sqrt{\pi})^{n-1}}{\Gamma(\frac{n}{2}+1)}\cdot 2=\dfrac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

例. 設 $V_n(a)$ 為 n 維區域 $|x_1|+|x_2|+\cdots+|x_n|\leqslant a$ 之體積, $n\geqslant 1$, $a>0$; 證明 $V_n(a)=a^n\dfrac{2^n}{n!}$.

證. $V_n(a) = \displaystyle\int_{|x_1|+|x_2|+\cdots+|x_n|\leqslant a} \mathrm{d}x_1\,\mathrm{d}x_2\,\cdots\,\mathrm{d}x_n$. 變數變換 $x_i = au_i\ \forall\, i = 1, 2, \ldots, n$, 則 Jacobian 為 a^n, 積分範圍變為 $|u_1| + |u_2| + \cdots + |u_n| \leqslant 1$, 故 $V_n(a) = a^n\, V_n(1)$. $V_n(1) = \displaystyle\int_{|u_1|+|u_2|+\cdots+|u_n|\leqslant 1} \mathrm{d}u_1\,\mathrm{d}u_2\,\cdots\,\mathrm{d}u_n$ $= \displaystyle\int_{|u_n|\leqslant 1}\left(\int_{|u_1|+|u_2|+\cdots+|u_{n-1}|\leqslant 1-|u_n|} \mathrm{d}u_1\,\mathrm{d}u_2\,\cdots\,\mathrm{d}u_{n-1}\right)\mathrm{d}u_n = \int_{-1}^{1} V_{n-1}(1-|u_n|)\,\mathrm{d}u_n = V_{n-1}(1)\int_{-1}^{1}(1-|u_n|)^{n-1}\,\mathrm{d}u_n$ $= V_{n-1}(1)\left(\displaystyle\int_{-1}^{0}(1+u_n)^{n-1}\,\mathrm{d}u_n + \int_{0}^{1}(1-u_n)^{n-1}\,\mathrm{d}u_n\right) = \frac{2}{n}\,V_{n-1}(1)$. 故 $V_n(1) = \dfrac{2^{n-1}}{n\cdot(n-1)\cdots 2}\,V_1(1) = \dfrac{2^{n-1}}{n!}\cdot 2 = \dfrac{2^n}{n!}$, $V_n(a) = a^n\dfrac{2^n}{n!}$.