問題解答

問題. 若 $\lim\limits_{x\to\frac{\pi}{6}} \dfrac{\sqrt{a+\cos^2 x}+a}{2\sin x-1}=b$, 求 $a+b=$**?**

解. 當 $x\to\dfrac{\pi}{6}$, $2\sin x-1=0$; 若此時 $\sqrt{a+\cos^2 x}+a$ 不為 0 則極限不存在, 故 $\sqrt{a+\dfrac{3}{4}}+a=0 \implies a+\dfrac{3}{4}=a^2 \implies a=\dfrac{3}{2}$ 或 $a=-\dfrac{1}{2} \implies a=-\dfrac{1}{2}$; $\lim\limits_{x\to\frac{\pi}{6}} \dfrac{\sqrt{\cos^2 x-\frac{1}{2}}-\frac{1}{2}}{2\sin x-1}=\lim\limits_{x\to\frac{\pi}{6}} \dfrac{\frac{-2\cos x\sin x}{2\sqrt{\cos^2 x-\frac{1}{2}}}}{2\cos x}=\lim\limits_{x\to\frac{\pi}{6}} \dfrac{-\sin x}{2\sqrt{\cos^2 x-\frac{1}{2}}}=-\dfrac{1}{2}=b$, 故 $a+b=-1$.

例題. 令 $T_1=\displaystyle\int \frac{\sin x}{a\cos x+b\sin x}\,\mathrm{d}x$, $T_2=\displaystyle\int \frac{\cos x}{a\cos x+b\sin x}\,\mathrm{d}x$, a, $b\neq 0$, 求 T_1, T_2.

解.

(a) $bT_1+aT_2=\displaystyle\int \frac{b\sin x}{a\cos x+b\sin x}\,\mathrm{d}x+\int \frac{a\cos x}{a\cos x+b\sin x}\,\mathrm{d}x=\int \frac{b\sin x+a\cos x}{a\cos x+b\sin x}\,\mathrm{d}x=\int 1\,\mathrm{d}x=x.$

(b) $-aT_1+bT_2=\displaystyle\int \frac{-a\sin x}{a\cos x+b\sin x}\,\mathrm{d}x+\int \frac{b\cos x}{a\cos x+b\sin x}\,\mathrm{d}x=\int \frac{-a\sin x+b\cos x}{a\cos x+b\sin x}\,\mathrm{d}x=\int \frac{\mathrm{d}u}{u}=\ln u=\ln|a\cos x+b\sin x|$ (令 $u=a\cos x+b\sin x$, 則 $\mathrm{d}u=(-a\sin x+b\cos x)\,\mathrm{d}x$) .

解 T_1, T_2 方程式 (a), (b) 得 $T_1=\dfrac{bx-a\ln|a\cos x+b\sin x|}{a^2+b^2}$, $T_2=\dfrac{ax+b\ln|a\cos x+b\sin x|}{a^2+b^2}$.

問題. 求 $\displaystyle\int \frac{1}{1+\tan\theta}\,\mathrm{d}\theta$.

解. $\displaystyle\int \frac{1}{1+\tan\theta}\,\mathrm{d}\theta=\int \frac{1}{1+\frac{\sin\theta}{\cos\theta}}\,\mathrm{d}\theta=\int \frac{\cos\theta}{\cos\theta+\sin\theta}\,\mathrm{d}\theta$, 為上題當 $a=b=1$ 之 T_1: 答案為 $\dfrac{\theta-\ln|\cos\theta+\sin\theta|}{2}$.

例題. 求 $\displaystyle\int \sec x\,\mathrm{d}x$.

解. $\displaystyle\int \sec x\,\mathrm{d}x=\int \frac{\sec x\cdot(\sec x+\tan x)}{\sec x+\tan x}\,\mathrm{d}x$. 令 $u=\sec x+\tan x$, 則 $\mathrm{d}u=(\sec^2 x+\sec x\tan x)\,\mathrm{d}x$; 故 $\displaystyle\int \frac{\sec x\cdot(\sec x+\tan x)}{\sec x+\tan x}\,\mathrm{d}x=\int \frac{1}{u}\,\mathrm{d}u=\ln|u|+c=\ln|\sec x+\tan x|+c$

例題. 令 $K_n=\displaystyle\int \sec^{2n+1}\theta\,\mathrm{d}\theta$, $n\in\mathbb{N}$, $n\geqslant 1$, 則 $K_n=\dfrac{\sec^{2n-1}\theta\tan\theta}{2n}+\dfrac{2n-1}{2n}K_{n-1}$.

解. 令 $u=\sec^{2n-1}\theta$, 則 $\mathrm{d}u=(2n-1)\sec^{2n-2}\theta\cdot\sec\theta\tan\theta\,\mathrm{d}\theta=(2n-1)\sec^{2n-1}\theta\tan\theta\,\mathrm{d}\theta$; 令 $\mathrm{d}v=\sec^2\theta\,\mathrm{d}\theta$, 則 $v=\tan\theta$. 故 $K_n=\displaystyle\int \sec^{2n+1}\theta\,\mathrm{d}\theta=\sec^{2n-1}\theta\cdot\tan\theta-\int \tan\theta\cdot(2n-1)\sec^{2n-1}\theta\tan\theta\,\mathrm{d}\theta=\sec^{2n-1}\theta\tan\theta-(2n-1)\int \tan^2\theta\cdot\sec^{2n-1}\theta\,\mathrm{d}\theta=\sec^{2n-1}\theta\tan\theta-(2n-1)\int (\sec^2\theta-1)\cdot\sec^{2n-1}\theta\,\mathrm{d}\theta=\sec^{2n-1}\theta\tan\theta-(2n-1)\int \sec^{2n+1}\theta\,\mathrm{d}\theta+(2n-1)\int \sec^{2n-1}\theta\,\mathrm{d}\theta \implies K_n=\sec^{2n-1}\theta\tan\theta-(2n-1)K_n+(2n-1)K_{n-1} \implies K_n=\dfrac{\sec^{2n-1}\theta\tan\theta}{2n}+\dfrac{2n-1}{2n}K_{n-1}$.

註 (使用例)**.** $K_0=\displaystyle\int \sec\theta\,\mathrm{d}\theta=\ln|\sec\theta+\tan\theta|$, $\displaystyle\int \sec^3\theta\,\mathrm{d}\theta=K_1=\frac{\sec\theta\tan\theta}{2}+\frac{1}{2}K_0=\frac{1}{2}\big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\big)$, $\displaystyle\int \sec^5\theta\,\mathrm{d}\theta=K_2=\frac{\sec^3\theta\tan\theta}{4}+\frac{3}{4}K_1=\frac{\sec^3\theta\tan\theta}{4}+\frac{3}{8}\big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\big)$.

註 (三角函數代換).

- 遇 $\sqrt{a^2-x^2}$, 考慮 $x=a\sin\theta \implies \theta=\sin^{-1}\dfrac{x}{a}$, $\mathrm{d}x=a\cos\theta\,\mathrm{d}\theta$
- 遇 $\sqrt{a^2+x^2}$, 考慮 $x=a\tan\theta \implies \theta=\tan^{-1}\dfrac{x}{a}$, $\mathrm{d}x=a\sec^2\theta\,\mathrm{d}\theta$
- 遇 $\sqrt{x^2-a^2}$, 考慮 $x=a\sec\theta \implies \theta=\sec^{-1}\dfrac{x}{a}$, $\mathrm{d}x=a\sec\theta\,\tan\theta\,\mathrm{d}\theta$
- 遇 $\sin x, \cos x$ 之有理式, 考慮 $u=\tan\dfrac{x}{2}$, 由以下化為 u 之有理式:
  - $\sin x=2\sin\dfrac{x}{2}\cos\dfrac{x}{2}=2\dfrac{u}{\sqrt{1+u^2}}\dfrac{1}{\sqrt{1+u^2}}=\dfrac{2u}{1+u^2}$
  - $\cos x=2\cos^2\dfrac{x}{2}-1=2\cdot\dfrac{1}{1+u^2}-1=\dfrac{1-u^2}{1+u^2}$
  - $\mathrm{d}u=\dfrac{1}{2}\sec^2\dfrac{x}{2}\,\mathrm{d}x \implies \mathrm{d}x=\dfrac{2}{1+u^2}\,\mathrm{d}u$

例題. 若 $a\neq 0$, 求下列不定積分 (注意積分常數).

1. $\displaystyle\int\sqrt{a^2-x^2}\,\mathrm{d}x$ 2. $\displaystyle\int\sqrt{x^2+a^2}\,\mathrm{d}x$ 3. $\displaystyle\int\frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x$ 4. $\displaystyle\int\sqrt{x^2-a^2}\,\mathrm{d}x$ 5. $\displaystyle\int\frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x$

解.

1. 令 $x=a\sin\theta$, 則 $\displaystyle\int\sqrt{a^2-x^2}\,\mathrm{d}x=\int\sqrt{a^2-a^2\sin^2\theta}\,a\cos\theta\,\mathrm{d}\theta=a^2\int\cos^2\theta\,\mathrm{d}\theta=a^2\int\frac{1+\cos 2\theta}{2}\,\mathrm{d}\theta$ $\displaystyle=\frac{a^2}{2}\theta+\frac{a^2}{4}\sin 2\theta=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{a^2}{2}\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}.$
2. 令 $x=a\tan\theta$, 則 $\displaystyle\int\sqrt{x^2+a^2}\,\mathrm{d}x=\int a\sec\theta\cdot a\sec^2\theta\,\mathrm{d}\theta=a^2\int\sec^3\theta\,\mathrm{d}\theta=\frac{a^2}{2}\big(\sec\theta\cdot\tan\theta+\ln|\sec\theta+\tan\theta|\big)=\frac{a^2}{2}\left(\frac{x}{a}\cdot\frac{\sqrt{x^2+a^2}}{a}+\ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right|\right)=\frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\ln\left|\sqrt{x^2+a^2}+x\right|-\frac{a^2}{2}\ln|a|$
3. 令 $x=a\tan\theta$, 則 $\displaystyle\int\frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x=\int\frac{1}{a\sec\theta}\cdot a\sec^2\theta\,\mathrm{d}\theta=\int\sec\theta\,\mathrm{d}\theta=\ln|\sec\theta+\tan\theta|$ $\displaystyle=\ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right|=\ln|\sqrt{x^2+a^2}+x|-\ln|a|.$
4. 令 $x=a\sec\theta$, 則 $\displaystyle\int\sqrt{x^2-a^2}\,\mathrm{d}x=\int a\tan\theta\cdot a\sec\theta\tan\theta\,\mathrm{d}\theta=a^2\int\sec\theta\tan^2\theta\,\mathrm{d}\theta=a^2\int\sec\theta\,(\sec^2\theta-1)\,\mathrm{d}\theta=a^2\left(\int\sec^3\theta\,\mathrm{d}\theta-\int\sec\theta\,\mathrm{d}\theta\right)=\frac{a^2}{2}\left(\sec\theta\cdot\tan\theta+\int\sec\theta\,\mathrm{d}\theta-2\int\sec\theta\,\mathrm{d}\theta\right)=\frac{a^2}{2}\left(\sec\theta\cdot\tan\theta-\int\sec\theta\,\mathrm{d}\theta\right)=\frac{a^2}{2}\big(\sec\theta\cdot\tan\theta-\ln|\sec\theta+\tan\theta|\big)=\frac{a^2}{2}\left(\frac{x}{a}\cdot\frac{\sqrt{x^2-a^2}}{a}-\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|\right)=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\ln\left|\sqrt{x^2-a^2}+x\right|+\frac{a^2}{2}\ln|a|.$
5. 令 $x=a\sec\theta$, 則 $\displaystyle\int\frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x=\int\frac{a\sec\theta\tan\theta}{a\tan\theta}\,\mathrm{d}\theta=\int\sec\theta\,\mathrm{d}\theta=\ln|\sec\theta+\tan\theta|$ $\displaystyle=\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|=\ln|\sqrt{x^2-a^2}+x|-\ln|a|.$

問題. 求 $\displaystyle\int\sqrt{x^2+x+1}\,\mathrm{d}x.$

解. $\int\sqrt{x^2+x+1}\,\mathrm{d}x=\int\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,\mathrm{d}x$, 亦即上例題 2. 將 x 代為 $x+\frac{1}{2}$ 與 $a=\frac{\sqrt{3}}{2}$ 之結果.

問題. 求 $\int\sqrt{x^2-6x+5}\,\mathrm{d}x$.

解. $\int\sqrt{x^2-6x+5}\,\mathrm{d}x=\int\sqrt{(x-3)^2-4}\,\mathrm{d}x$, 亦即上例題 4. 將 x 代為 $x-3$ 與 $a=2$ 之結果.

問題. 求 $\int\frac{x}{\sqrt{x^2+2x+5}}\,\mathrm{d}x$

解. $\int\frac{x}{\sqrt{x^2+2x+5}}\,\mathrm{d}x=\int\frac{(x+1)-1}{\sqrt{(x+1)^2+4}}\,\mathrm{d}x=\int\frac{x+1}{\sqrt{(x+1)^2+4}}\,\mathrm{d}x-\int\frac{1}{\sqrt{(x+1)^2+4}}\,\mathrm{d}x$. 令 $u=x+1$, 則 $\mathrm{d}u=\mathrm{d}x$; 故 $\int\frac{x+1}{\sqrt{(x+1)^2+4}}\,\mathrm{d}x-\int\frac{1}{\sqrt{(x+1)^2+4}}\,\mathrm{d}x=\int\frac{u}{\sqrt{u^2+4}}\,\mathrm{d}u-\int\frac{1}{\sqrt{u^2+4}}\,\mathrm{d}u$; 第一項積分做變數變換令 $t=u^2+4$, 第二項積分為上例題 3. $a=2$ 之結果.

問題. 求 $\int\frac{x}{\sqrt{1-2x-x^2}}\,\mathrm{d}x$

解. $\int\frac{x}{\sqrt{1-2x-x^2}}\,\mathrm{d}x=\int\frac{(x+1)-1}{\sqrt{2-(x+1)^2}}\,\mathrm{d}x=\int\frac{x+1}{\sqrt{2-(x+1)^2}}\,\mathrm{d}x-\int\frac{1}{\sqrt{2-(x+1)^2}}\,\mathrm{d}x$. 令 $u=x+1$, 則 $\mathrm{d}u=\mathrm{d}x$; 故 $\int\frac{x+1}{\sqrt{2-(x+1)^2}}\,\mathrm{d}x-\int\frac{1}{\sqrt{2-(x+1)^2}}\,\mathrm{d}x=\int\frac{u}{\sqrt{2-u^2}}\,\mathrm{d}u-\int\frac{1}{\sqrt{2-u^2}}\,\mathrm{d}u$; 第一項積分做變數變換令 $t=2-u^2$, 第二項積分為標準積分 $\int\frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x=\sin^{-1}\frac{x}{a}$ 當 $a=\sqrt{2}$ 之結果.

問題. 求 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{2+2\sin\theta}\,\mathrm{d}\theta$.

解. 令 $u=\tan\frac{\theta}{2}$, 則 $\sin\theta=\frac{2u}{1+u^2}$; $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{2+2\sin\theta}\,\mathrm{d}\theta=\int_{-1}^{1}\sqrt{2+2\,\frac{2u}{1+u^2}}\,\frac{2}{1+u^2}\,\mathrm{d}u$ $=\int_{-1}^{1}\sqrt{\frac{2+4u+2u^2}{1+u^2}}\,\frac{2}{1+u^2}\,\mathrm{d}u=\int_{-1}^{1}\sqrt{\frac{2\,(1+u)^2}{1+u^2}}\,\frac{2}{1+u^2}\,\mathrm{d}u=2\sqrt{2}\int_{-1}^{1}\frac{1+u}{(1+u^2)^{\frac{3}{2}}}\,\mathrm{d}u=4\sqrt{2}\int_{0}^{1}\frac{1}{(1+u^2)^{\frac{3}{2}}}\,\mathrm{d}u$. 令 $u=\tan\theta$, 則 $\mathrm{d}u=\sec^2\theta\,\mathrm{d}\theta$, $(1+u^2)^{\frac{3}{2}}=\sec^3\theta$, 故 $4\sqrt{2}\int_{0}^{1}\frac{1}{(1+u^2)^{\frac{3}{2}}}\,\mathrm{d}u=4\sqrt{2}\int_{0}^{\frac{\pi}{4}}\cos\theta\,\mathrm{d}\theta=4$.

問題. $\int_{\sqrt{2}}^{\infty}\left(\frac{a}{\sqrt{x^2-1}}-\frac{x}{x^2+1}\right)\mathrm{d}x$ 在 a 為何值收斂? 又收斂值為何?

解. 由上例題 5. 知 $\int\frac{a}{\sqrt{x^2-1}}\,\mathrm{d}x=a\ln|\sqrt{x^2-1}+x|$; $\int\frac{x}{x^2+1}\,\mathrm{d}x$ 中, 令 $u=x^2+1$, 則 $\mathrm{d}u=2\,x\,\mathrm{d}x\implies x\,\mathrm{d}x=\frac{1}{2}\,\mathrm{d}u$, $\int\frac{x}{x^2+1}\,\mathrm{d}x=\int\frac{1}{x^2+1}\cdot x\,\mathrm{d}x=\int\frac{1}{u}\cdot\frac{1}{2}\,\mathrm{d}u=\frac{1}{2}\ln|u|+c=\frac{1}{2}\ln(x^2+1)$. 故 $\int_{\sqrt{2}}^{\infty}\left(\frac{a}{\sqrt{x^2-1}}-\frac{x}{x^2+1}\right)\mathrm{d}x=\left(a\ln(\sqrt{x^2-1}+x)-\ln(\sqrt{x^2+1})\right)\Big|_{\sqrt{2}}^{\infty}=\ln\frac{(\sqrt{x^2-1}+x)^a}{\sqrt{x^2+1}}\Big|_{\sqrt{2}}^{\infty}$ 收斂, 故 $a=1$, 收斂值為 $\ln 2+\ln(\sqrt{6}-\sqrt{3})$.

問題. 求 $\int\frac{\sin\theta\cos\theta}{\sin^4\theta+1}\,\mathrm{d}\theta$.

解. 令 $u=\sin\theta$, 則 $\mathrm{d}u=\cos\theta\,\mathrm{d}\theta$. 故 $\int\frac{\sin\theta\cos\theta}{\sin^4\theta+1}\,\mathrm{d}\theta=\int\frac{u}{u^4+1}\,\mathrm{d}u$. 令 $u^2=t$, 則 $u\,\mathrm{d}u=\frac{1}{2}\,\mathrm{d}t$, 原積分 $=\frac{1}{2}\int\frac{1}{t^2+1}\,\mathrm{d}t=\frac{1}{2}\tan^{-1}t=\frac{1}{2}\tan^{-1}(\sin^2\theta)+c$.

問題. $\int \frac{\sin^{-1}x}{x^2}\,\mathrm{d}x = -\frac{\sin^{-1}x}{x} - \ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + c$

解. 令 $u = \sin^{-1}x$, 則 $\mathrm{d}u = \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$; $\mathrm{d}v = \frac{1}{x^2}\,\mathrm{d}x$, 則 $v = \frac{-1}{x}$. 故 $\int \frac{\sin^{-1}x}{x^2}\,\mathrm{d}x = \sin^{-1}x \cdot \frac{-1}{x} - \int \frac{-1}{x} \cdot \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = -\frac{\sin^{-1}x}{x} + \int \frac{1}{x\sqrt{1-x^2}}\,\mathrm{d}x$. 令 $w = \sqrt{1-x^2}$, 則 $-x^2 = w^2 - 1$, $\mathrm{d}w = \frac{-x}{\sqrt{1-x^2}}\,\mathrm{d}x$. 故 $\int \frac{1}{x\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{1}{-x^2} \cdot \frac{-x}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{1}{w^2-1}\,\mathrm{d}w = \frac{1}{2}\int \left(\frac{1}{w-1} - \frac{1}{w+1}\right)\mathrm{d}w = \frac{1}{2}\big(\ln|w-1| - \ln|w+1|\big) = \frac{1}{2}\ln\left|\frac{w-1}{w+1}\right| = \frac{1}{2}\ln\left|\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right| = \frac{1}{2}\ln\left|\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right| = \frac{1}{2}\ln\left|\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} \cdot \frac{1-\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right| = \frac{1}{2}\ln\left|\frac{(1-\sqrt{1-x^2})^2}{x^2}\right| = \ln\left|\frac{1-\sqrt{1-x^2}}{x}\right|$. 以上, $\int \frac{\sin^{-1}x}{x^2}\,\mathrm{d}x = -\frac{\sin^{-1}x}{x} + \int \frac{1}{x\sqrt{1-x^2}}\,\mathrm{d}x = -\frac{\sin^{-1}x}{x} + \ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + c$.

問題. $\int (x+1)^2 e^{\frac{x^2}{2}}\,\mathrm{d}x = (x+2)\,e^{\frac{x^2}{2}}$

解. $\int (x+1)^2\,e^{\frac{x^2}{2}}\,\mathrm{d}x = \int x^2\,e^{\frac{x^2}{2}}\,\mathrm{d}x + \int 2x\,e^{\frac{x^2}{2}}\,\mathrm{d}x + \int e^{\frac{x^2}{2}}\,\mathrm{d}x$. 在 $\int x^2\,e^{\frac{x^2}{2}}\,\mathrm{d}x$ 中, 令 $u = x$, 則 $\mathrm{d}u = \mathrm{d}x$; $\mathrm{d}v = x\,e^{\frac{x^2}{2}}\,\mathrm{d}x$, 則 $v = e^{\frac{x^2}{2}}$. 故 $\int x^2\,e^{\frac{x^2}{2}}\,\mathrm{d}x = x \cdot e^{\frac{x^2}{2}} - \int e^{\frac{x^2}{2}}\mathrm{d}x$; 原式 $= \int x^2\,e^{\frac{x^2}{2}}\,\mathrm{d}x + \int 2x\,e^{\frac{x^2}{2}}\,\mathrm{d}x + \int e^{\frac{x^2}{2}}\,\mathrm{d}x = x \cdot e^{\frac{x^2}{2}} - \int e^{\frac{x^2}{2}}\,\mathrm{d}x + 2e^{\frac{x^2}{2}} + \int e^{\frac{x^2}{2}}\,\mathrm{d}x = (x+2)\,e^{\frac{x^2}{2}}$

問題. $\int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)\mathrm{d}x = \frac{e^x}{x}$

解. $\int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)\mathrm{d}x = \int \frac{e^x}{x}\,\mathrm{d}x + \int e^x\frac{-1}{x^2}\,\mathrm{d}x$. 在 $\int e^x\frac{-1}{x^2}\,\mathrm{d}x$ 中, 令 $u = e^x$, 則 $\mathrm{d}u = e^x\,\mathrm{d}x$; $\mathrm{d}v = \frac{-1}{x^2}\,\mathrm{d}x$, 則 $v = \frac{1}{x}$. 故 $\int e^x\frac{-1}{x^2}\,\mathrm{d}x = e^x \cdot \frac{1}{x} - \int \frac{e^x}{x}\,\mathrm{d}x$; 原式 $= \int \frac{e^x}{x}\,\mathrm{d}x + \int e^x\frac{-1}{x^2}\,\mathrm{d}x = \int \frac{e^x}{x}\,\mathrm{d}x + e^x \cdot \frac{1}{x} - \int \frac{e^x}{x}\,\mathrm{d}x = \frac{e^x}{x}$