第五章　偏微分

5.0 微積分函數分類

1. 單變數函數: $\mathbb{R} \to \mathbb{R}$
2. 向量值函數: $\mathbb{R} \to \mathbb{R}^n$, $n > 1$ (5.2)
3. 多變數函數: $\mathbb{R}^n \to \mathbb{R}$, $n > 1$ (5.3)
4. 多變數向量值函數: $\mathbb{R}^n \to \mathbb{R}^m$, $m, n > 1$

5.1 空間向量

定義 (符號).

- 向量: $\mathbf{a}$, $\mathbf{x}$
- 分量形: $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $a_1, a_2, a_3 \in \mathbb{R}$; $\mathbf{a} \in \mathbb{R}^3$.
- 長度: $|\mathbf{a}| = |\langle a_1, a_2, a_3 \rangle| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
- 三維直角座標系單位向量: $\widehat{\boldsymbol{\imath}} = \langle 1, 0, 0 \rangle$, $\widehat{\boldsymbol{\jmath}} = \langle 0, 1, 0 \rangle$, $\widehat{\mathbf{k}} = \langle 0, 0, 1 \rangle$
- $\mathbf{a} = \langle a_1, a_2, a_3 \rangle \equiv a_1 \widehat{\boldsymbol{\imath}} + a_2 \widehat{\boldsymbol{\jmath}} + a_3 \widehat{\mathbf{k}}$
- $\mathbf{0} = \langle 0, 0, \ldots, 0 \rangle$

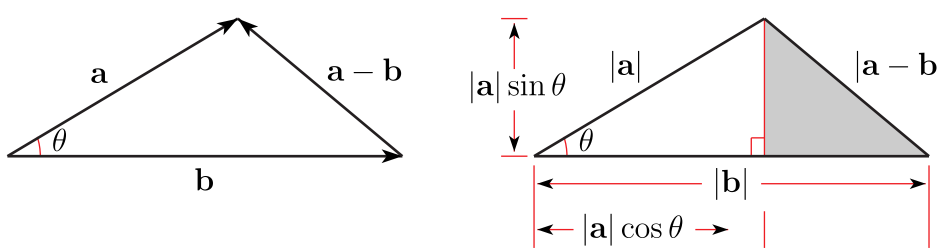
定義 (內積 / 點積). 給定 n ($n \geqslant 2$) 維向量 $\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$, $\mathbf{b} = \langle b_1, b_2, \ldots, b_n \rangle$, 則 $\mathbf{a}$ 與 $\mathbf{b}$ 的內積 / 點積 (inner product / dot product) 定義為 $\mathbf{a} \cdot \mathbf{b} = \displaystyle\sum_{i=1}^{n} a_i b_i$.

性質. 給定 $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, $n \geqslant 2$, $s \in \mathbb{R}$. 則

- $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$, $0 \leqslant \theta \leqslant \pi$ 為 $\mathbf{a}$ 與 $\mathbf{b}$ 的夾角.

且

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$
4. $\mathbf{0} \cdot \mathbf{a} = 0$
5. $\mathbf{a} \cdot \mathbf{b} = 0 \iff \mathbf{a} = \mathbf{0} \vee \mathbf{b} = \mathbf{0} \vee \mathbf{a} \perp \mathbf{b}$
6. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$, $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$



證. 由上圖, $|\mathbf{a} - \mathbf{b}|^2 = \big(|\mathbf{b}| - |\mathbf{a}| \cos\theta\big)^2 + \big(|\mathbf{a}| \sin\theta\big)^2 = |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}| \cos\theta + |\mathbf{a}|^2 \cos^2\theta + |\mathbf{a}|^2 \sin^2\theta = |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}| \cos\theta + |\mathbf{a}|^2$, 又 $|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,\mathbf{a} \cdot \mathbf{b} = |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}| \cos\theta + |\mathbf{a}|^2$, 故 $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$.

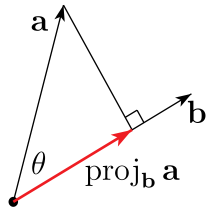
定義 (投影). 給定向量 $\mathbf{a}$, $\mathbf{b}$, 則 $\mathbf{a}$ 在 $\mathbf{b}$ 方向上的投影 (projection) $\operatorname{proj}_{\mathbf{b}} \mathbf{a}$ 定義為 $\operatorname{proj}_{\mathbf{b}} \mathbf{a} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} \mathbf{b}$.

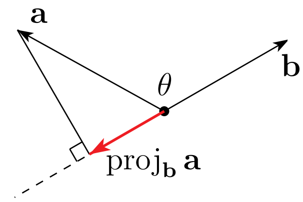
定義 (外積 / 叉積). 給定三維向量 $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, 則 $\mathbf{a}$ 與 $\mathbf{b}$ 的外積 / 叉積 (outer product / cross product) 定義為 $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \widehat{\boldsymbol{\imath}} & \widehat{\boldsymbol{\jmath}} & \widehat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$.

性質. 令 $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, $s \in \mathbb{R}$,

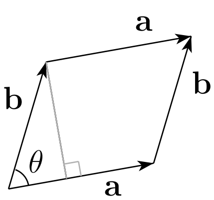
- $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$, $0 \leqslant \theta \leqslant \pi$ 為 $\mathbf{a}$ 與 $\mathbf{b}$ 的夾角; $|\mathbf{a} \times \mathbf{b}|$ 為 $\mathbf{a}$ 與 $\mathbf{b}$ 張成之平行四邊形面積.
- $\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta\, \widehat{\mathbf{n}}$, $0 \leqslant \theta \leqslant \pi$ 為 $\mathbf{a}$ 與 $\mathbf{b}$ 的夾角, $(\mathbf{a}, \mathbf{b}, \widehat{\mathbf{n}})$ 滿足右手定則且 $|\widehat{\mathbf{n}}| = 1$, $\widehat{\mathbf{n}} \perp \mathbf{a}$, $\widehat{\mathbf{n}} \perp \mathbf{b}$.
- $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ 為 $\mathbf{a}, \mathbf{b}, \mathbf{c}$ 張成之平行六面體體積.

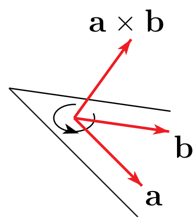
且

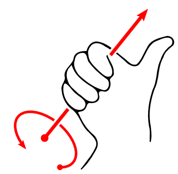


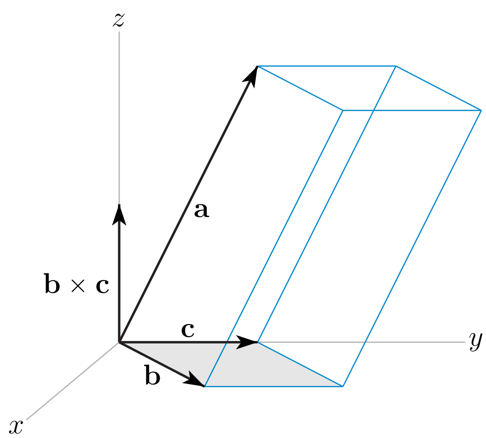


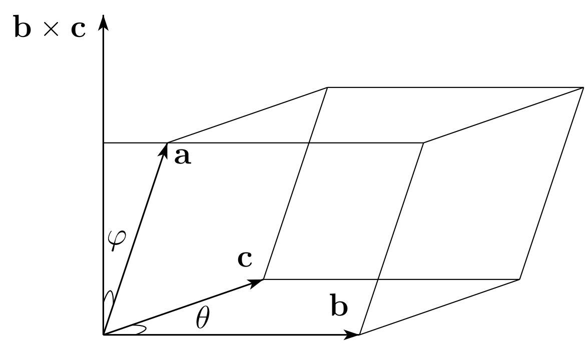
1. $\mathbf{a}\times\mathbf{b}\perp\mathbf{a}$, $\mathbf{a}\times\mathbf{b}\perp\mathbf{b}$
2. $\widehat{\imath}\times\widehat{\jmath}=\widehat{\mathbf{k}}$, $\widehat{\jmath}\times\widehat{\mathbf{k}}=\widehat{\imath}$, $\widehat{\mathbf{k}}\times\widehat{\imath}=\widehat{\jmath}$
3. $\mathbf{a}\times\mathbf{b}=\mathbf{0} \iff \mathbf{a}=\mathbf{0} \vee \mathbf{b}=\mathbf{0} \vee \mathbf{a}\parallel\mathbf{b}$
4. $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$
5. $(s\mathbf{a})\times\mathbf{b}=\mathbf{a}\times(s\mathbf{b})=s(\mathbf{a}\times\mathbf{b})$
6. $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$
7. $(\mathbf{a}+\mathbf{b})\times\mathbf{c}=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$
8. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$
9. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{b}\,(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}\,(\mathbf{a}\cdot\mathbf{b})$ (baccab 規則)











證.

- $|\mathbf{a}\times\mathbf{b}|^2=\big(\mathbf{a}\times\mathbf{b}\big)\cdot\big(\mathbf{a}\times\mathbf{b}\big)=(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2=a_2^2b_3^2-2a_2b_3a_3b_2+a_3^2b_2^2+a_3^2b_1^2-2a_3b_1a_1b_3+a_1^2b_3^2+a_1^2b_2^2-2a_1b_2a_2b_1+a_2^2b_1^2$, 而 $|\mathbf{a}|^2\,|\mathbf{b}|^2\sin^2\theta=|\mathbf{a}|^2\,|\mathbf{b}|^2(1-\cos^2\theta)=|\mathbf{a}|^2\,|\mathbf{b}|^2-(\mathbf{a}\cdot\mathbf{b})^2=\big(a_1^2+a_2^2+a_3^2\big)\big(b_1^2+b_2^2+b_3^2\big)-\big(a_1b_1+a_2b_2+a_3b_3\big)^2=a_1^2b_2^2+a_1^2b_3^2+a_2^2b_1^2+a_2^2b_3^2+a_3^2b_1^2+a_3^2b_2^2-\big(2a_1b_1a_2b_2+2a_1b_1a_3b_3+2a_2b_2a_3b_3\big)$, 故 $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}|\,|\mathbf{b}|\sin\theta$.
- $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})=a_1(a_2b_3-a_3b_2)+a_2(a_3b_1-a_1b_3)+a_3(a_1b_2-a_2b_1)=0$,
  $\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b})=b_1(a_2b_3-a_3b_2)+b_2(a_3b_1-a_1b_3)+b_3(a_1b_2-a_2b_1)=0$
- $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\langle a_1,a_2,a_3\rangle\cdot\langle b_2c_3-b_3c_2\,,\,b_3c_1-b_1c_3\,,\,b_1c_2-b_2c_1\rangle=a_1b_2c_3-a_1b_3c_2+a_2b_3c_1-a_2b_1c_3+a_3b_1c_2-a_3b_2c_1$, 又 $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\langle a_2b_3-a_3b_2\,,\,a_3b_1-a_1b_3\,,\,a_1b_2-a_2b_1\rangle\cdot\langle c_1,c_2,c_3\rangle=a_2b_3c_1-a_3b_2c_1+a_3b_1c_2-a_1b_3c_2+a_1b_2c_3-a_2b_1c_3$.

  另證: $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\langle a_1,a_2,a_3\rangle\cdot\det\begin{vmatrix}\widehat{\imath}&\widehat{\jmath}&\widehat{\mathbf{k}}\\ b_1&b_2&b_3\\ c_1&c_2&c_3\end{vmatrix}=a_1\det\begin{vmatrix}b_2&b_3\\c_2&c_3\end{vmatrix}-a_2\det\begin{vmatrix}b_1&b_3\\c_1&c_3\end{vmatrix}+a_3\det\begin{vmatrix}b_1&b_2\\c_1&c_2\end{vmatrix}$

  $=\det\begin{vmatrix}a_1&a_2&a_3\\ b_1&b_2&b_3\\ c_1&c_2&c_3\end{vmatrix}$, 而 $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\det\begin{vmatrix}\widehat{\imath}&\widehat{\jmath}&\widehat{\mathbf{k}}\\ a_1&a_2&a_3\\ b_1&b_2&b_3\end{vmatrix}\cdot\langle c_1,c_2,c_3\rangle=c_1\det\begin{vmatrix}a_2&a_3\\b_2&b_3\end{vmatrix}-c_2\det\begin{vmatrix}a_1&a_3\\b_1&b_3\end{vmatrix}+$

$c_3 \det\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \det\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$, 故 $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$.

- $\mathbf{b}\times\mathbf{c} = (b_2c_3-b_3c_2)\widehat{\boldsymbol{\imath}} - (b_1c_3-b_3c_1)\widehat{\boldsymbol{\jmath}} + (b_1c_2-b_2c_1)\widehat{\mathbf{k}}$, 故 $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ $= \det\begin{vmatrix} \widehat{\boldsymbol{\imath}} & \widehat{\boldsymbol{\jmath}} & \widehat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_2c_3-b_3c_2 & -b_1c_3+b_3c_1 & b_1c_2-b_2c_1 \end{vmatrix} = \widehat{\boldsymbol{\imath}}\left(a_2(b_1c_2-b_2c_1)-a_3(-b_1c_3+b_3c_1)\right) - \widehat{\boldsymbol{\jmath}}\left(a_1(b_1c_2-b_2c_1)-a_3(b_2c_3-b_3c_2)\right) + \widehat{\mathbf{k}}\left(a_1(-b_1c_3+b_3c_1)-a_2(b_2c_3-b_3c_2)\right)$. 而 $\mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b}) = (a_1c_1+a_2c_2+a_3c_3)(b_1\widehat{\boldsymbol{\imath}}+b_2\widehat{\boldsymbol{\jmath}}+b_3\widehat{\mathbf{k}}) - (a_1b_1+a_2b_2+a_3b_3)(c_1\widehat{\boldsymbol{\imath}}+c_2\widehat{\boldsymbol{\jmath}}+c_3\widehat{\mathbf{k}}) = \widehat{\boldsymbol{\imath}}\left(a_1b_1c_1+a_2b_1c_2+a_3b_1c_3-a_1b_1c_1-a_2b_2c_1-a_3b_3c_1\right) + \widehat{\boldsymbol{\jmath}}\left(a_1b_2c_1+a_2b_2c_2+a_3b_2c_3-a_1b_1c_2-a_2b_2c_2-a_3b_3c_2\right) + \widehat{\mathbf{k}}\left(a_1b_3c_1+a_2b_3c_2+a_3b_3c_3-a_1b_1c_3-a_2b_2c_3-a_3b_3c_3\right) = \widehat{\boldsymbol{\imath}}\left(a_2b_1c_2+a_3b_1c_3-a_2b_2c_1-a_3b_3c_1\right) + \widehat{\boldsymbol{\jmath}}\left(a_1b_2c_1+a_3b_2c_3-a_1b_1c_2-a_3b_3c_2\right) + \widehat{\mathbf{k}}\left(a_1b_3c_1+a_2b_3c_2-a_1b_1c_3-a_2b_2c_3\right)$, 故 $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\,(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}\,(\mathbf{a}\cdot\mathbf{b})$.

例. 令 $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$,

1. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = 0$
2. $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})\,(\mathbf{b}\cdot\mathbf{c})$
3. $(\mathbf{a}\times\mathbf{b})\cdot((\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})) = (\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}))^2$

解.

1. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\,(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}\,(\mathbf{a}\cdot\mathbf{b}) + \mathbf{c}\,(\mathbf{b}\cdot\mathbf{a}) - \mathbf{a}\,(\mathbf{b}\cdot\mathbf{c}) + \mathbf{a}\,(\mathbf{c}\cdot\mathbf{b}) - \mathbf{b}\,(\mathbf{c}\cdot\mathbf{a}) = 0$
2. $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \mathbf{c}\cdot(\mathbf{d}\times(\mathbf{a}\times\mathbf{b})) = \mathbf{c}\cdot(\mathbf{a}\,(\mathbf{d}\cdot\mathbf{b}) - \mathbf{b}\,(\mathbf{d}\cdot\mathbf{a})) = (\mathbf{c}\cdot\mathbf{a})\,(\mathbf{d}\cdot\mathbf{b}) - (\mathbf{c}\cdot\mathbf{b})\,(\mathbf{d}\cdot\mathbf{a}) = (\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})\,(\mathbf{b}\cdot\mathbf{c})$
3. $(\mathbf{a}\times\mathbf{b})\cdot((\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})) = (\mathbf{b}\times\mathbf{c})\cdot((\mathbf{c}\times\mathbf{a})\times(\mathbf{a}\times\mathbf{b})) = (\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\,((\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}) - \mathbf{b}\,((\mathbf{c}\times\mathbf{a})\cdot\mathbf{a})) = (\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\,((\mathbf{c}\times\mathbf{a})\cdot\mathbf{b})) = (\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}))^2$

性質 (常用公式)**.**

- 點 $p = (p_1, p_2, p_3)$ 與平面 $ax+by+cz+d=0$ 距離為 $\dfrac{|ap_1+bp_2+cp_3+d|}{\sqrt{a^2+b^2+c^2}}$.
- 若 $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, $\mathbf{c} = \langle c_1, c_2, c_3\rangle$, $\mathbf{d} = \langle d_1, d_2, d_3\rangle$, 三維空間中兩直線 $\langle a_1+b_1\,s, a_2+b_2\,s, a_3+b_3\,s\rangle$, $\langle c_1+d_1\,t, c_2+d_2\,t, c_3+d_3\,t\rangle$, $s, t\in\mathbb{R}$ 之距離為 $\dfrac{|(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d})|}{|\mathbf{b}\times\mathbf{d}|}$.

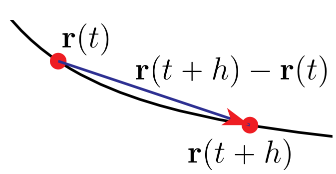
解.

- 平面 S : $ax+by+cz+d=0$ 法向量為 $\mathbf{n} = \langle a, b, c\rangle$; 點 $p = (p_1, p_2, p_3)$ 與投影至平面 S 之點 $o = (x, y, z)$ 所形成之向量平行於 $\mathbf{n}$, 故 $(x, y, z) = (p_1+a\,t, p_2+b\,t, p_3+c\,t)$, $t\in\mathbb{R}$ 為待定常數. 又 o 位於平面 S 上, 故 $a(p_1+a\,t)+b(p_2+b\,t)+c(p_3+c\,t)+d=0 \implies t = \dfrac{-(ap_1+bp_2+cp_3+d)}{a^2+b^2+c^2}$, 所求距離 $\overline{op} = |\langle a\,t, b\,t, c\,t\rangle| = \sqrt{a^2+b^2+c^2}\,|t| = \dfrac{|ap_1+bp_2+cp_3+d|}{\sqrt{a^2+b^2+c^2}}$.
- 兩直線點分別為 $\mathbf{a}$, $\mathbf{c}$, 方向向量分別為 $\mathbf{b}$, $\mathbf{d}$; $\mathbf{b}\times\mathbf{d}$ 同時垂直於兩直線, 所求距離即為 $|\mathrm{proj}_{\mathbf{b}\times\mathbf{d}}(\mathbf{a}-\mathbf{c})| = \dfrac{|(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d})|}{|\mathbf{b}\times\mathbf{d}|}$.

5.2 向量值函數

定義. 向量值函數 $\mathbf{r}(t):\mathbb{R}\to\mathbb{R}^n$, $n>1$, 其微分為

$$\mathbf{r}'(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) = \lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$$



若 $\mathbf{r}(t)=x(t)\,\widehat{\boldsymbol{\imath}}+y(t)\,\widehat{\boldsymbol{\jmath}}+z(t)\,\widehat{\mathbf{k}}$, 則 $\mathbf{r}'(t)=x'(t)\,\widehat{\boldsymbol{\imath}}+y'(t)\,\widehat{\boldsymbol{\jmath}}+z'(t)\,\widehat{\mathbf{k}}$.

註. 向量值函數多用在空間曲線表示式中: (曲線) $\equiv$ (位置向量).

定理 (微分規則)**.** 令 $\mathbf{a}(t)$, $\mathbf{b}(t)$ 為 $t\in\mathbb{R}$ 可微 $\mathbb{R}^n$ 向量值函數, α, $\beta\in\mathbb{R}$, $\gamma(t)$, $s(t)$ 為 $t\in\mathbb{R}$ 可微實函數, 則

1. (線性) $\dfrac{\mathrm{d}}{\mathrm{d}t}\big(\alpha\,\mathbf{a}(t)+\beta\,\mathbf{b}(t)\big)=\alpha\,\mathbf{a}'(t)+\beta\,\mathbf{b}'(t)$
2. (乘積) $\dfrac{\mathrm{d}}{\mathrm{d}t}\big(\gamma(t)\mathbf{b}(t)\big)=\gamma'(t)\mathbf{b}(t)+\gamma(t)\mathbf{b}'(t)$
3. (內積) $\dfrac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{a}(t)\cdot\mathbf{b}(t)\big)=\mathbf{a}'(t)\cdot\mathbf{b}(t)+\mathbf{a}(t)\cdot\mathbf{b}'(t)$
4. (外積) $\dfrac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{a}(t)\times\mathbf{b}(t)\big)=\mathbf{a}'(t)\times\mathbf{b}(t)+\mathbf{a}(t)\times\mathbf{b}'(t)$
5. (合成) $\dfrac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{a}\big(s(t)\big)\big)=\mathbf{a}'\big(s(t)\big)\,s'(t)$

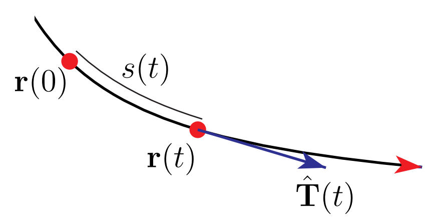
例. 若 $\mathbf{v}(t)=\mathbf{r}(t)\cdot(\mathbf{r}'(t)\times\mathbf{r}''(t))$, 證明 $\mathbf{v}'(t)=\mathbf{r}(t)\cdot(\mathbf{r}'(t)\times\mathbf{r}'''(t))$.

解. $\mathbf{v}'(t)=\big(\mathbf{r}(t)\cdot(\mathbf{r}'(t)\times\mathbf{r}''(t))\big)'=\mathbf{r}'(t)\cdot(\mathbf{r}'(t)\times\mathbf{r}''(t))+\mathbf{r}(t)\cdot(\mathbf{r}'(t)\times\mathbf{r}''(t))'=\mathbf{r}(t)\cdot(\mathbf{r}'(t)\times\mathbf{r}''(t))'=\mathbf{r}(t)\cdot(\mathbf{r}''(t)\times\mathbf{r}''(t)+\mathbf{r}'(t)\times\mathbf{r}'''(t))=\mathbf{r}(t)\cdot(\mathbf{r}'(t)\times\mathbf{r}'''(t))$.

性質. 給定曲線 $\mathbf{r}(t)$.

1. 令 $\widehat{\mathbf{T}}(t)$ 為曲線在點 $\mathbf{r}(t)$ 並指向 t 遞增方向之單位切線向量, 則 $\widehat{\mathbf{T}}(t)=\dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, $\mathbf{r}'(t)\neq\mathbf{0}$.
2. 令 $s(t)$ 為曲線介於點 $\mathbf{r}(0)$ 與 $\mathbf{r}(t)$ 間之弧長, 則

$$\frac{\mathrm{d}s}{\mathrm{d}t}(t)=\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right|$$
$$s(T)-s(T_0)=\int_{T_0}^{T}\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right|\,\mathrm{d}t$$



3. 若以弧長為參數, 亦即 $t=s$ 使得 $\dfrac{\mathrm{d}t}{\mathrm{d}s}=\dfrac{\mathrm{d}s}{\mathrm{d}s}=1$, 則 $\left|\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}(s)\right|=1$, $\widehat{\mathbf{T}}(s)=\mathbf{r}'(s)$.

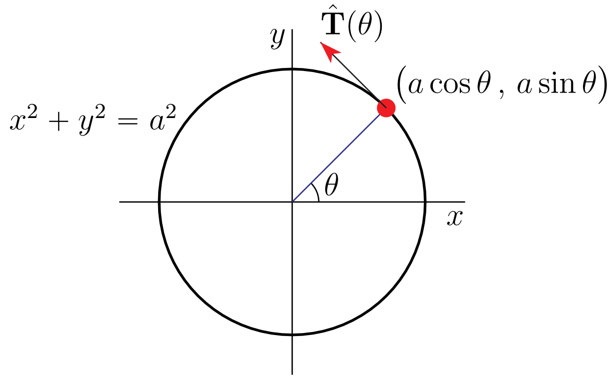
性質. 給定位置向量 $\mathbf{r}(t)=\langle x(t)\,,\,y(t)\,,\,z(t)\rangle$, 則時點 t 之

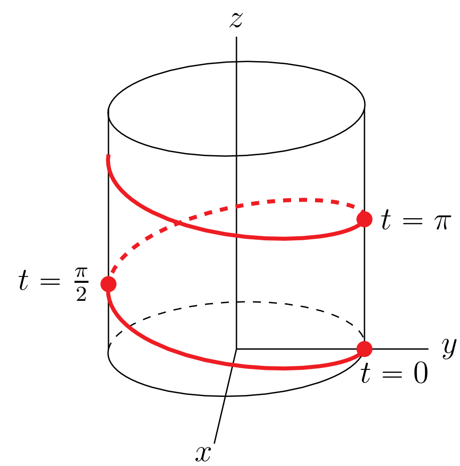
- 速度 $\mathbf{v}(t)=\mathbf{r}'(t)=x'(t)\,\widehat{\boldsymbol{\imath}}+y'(t)\,\widehat{\boldsymbol{\jmath}}+z'(t)\,\widehat{\mathbf{k}}$
- 速率 $\dfrac{\mathrm{d}s}{\mathrm{d}t}(t)=|\mathbf{v}(t)|=|\mathbf{r}'(t)|=\sqrt{(x'(t)^2+y'(t)^2+z'(t)^2}$
- 加速度 $\mathbf{a}(t)=\mathbf{r}''(t)=\mathbf{v}'(t)=x''(t)\,\widehat{\boldsymbol{\imath}}+y''(t)\,\widehat{\boldsymbol{\jmath}}+z''(t)\,\widehat{\mathbf{k}}$

時點 T_0 與 T 間經過距離為 $s(T)-s(T_0)=\displaystyle\int_{T_0}^{T}\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right|\mathrm{d}t=\int_{T_0}^{T}\sqrt{(x'(t)^2+y'(t)^2+z'(t)^2}\,\mathrm{d}t$

例. 圓 $x^2+y^2=a^2$ 之曲線表示式為 $\mathbf{r}(\theta)=\langle a\cos\theta,\,a\sin\theta\rangle$, $0\leqslant\theta\leqslant 2\pi$. $\mathbf{r}'(\theta)=\langle -a\sin\theta,\,a\cos\theta\rangle$, $\widehat{\mathbf{T}}(\theta)=\dfrac{\mathbf{r}'(\theta)}{|\mathbf{r}'(\theta)|}=\langle -\sin\theta,\,\cos\theta\rangle$, $\dfrac{\mathrm{d}s}{\mathrm{d}\theta}(\theta)=\big|\mathbf{r}'(\theta)\big|=a$, $s(\Theta)-s(0)=\displaystyle\int_0^{\Theta}\big|\mathbf{r}'(\theta)\big|\,\mathrm{d}\theta=a\Theta$.

例 (螺旋線弧長)**.** 求 $\mathbf{r}(t)=6\sin 2t\,\widehat{\boldsymbol{\imath}}+6\cos 2t\,\widehat{\boldsymbol{\jmath}}+5t\,\widehat{\mathbf{k}}$ 介於 $t=0$ 與 $t=\pi$ 間弧長.





解. $\mathbf{r}(t) = 6\sin 2t\,\widehat{\imath} + 6\cos 2t\,\widehat{\jmath} + 5t\,\widehat{\mathbf{k}} \implies \mathbf{r}'(t) = 12\cos 2t\,\widehat{\imath} - 12\sin 2t\,\widehat{\jmath} + 5\,\widehat{\mathbf{k}}$. 則 $\dfrac{\mathrm{d}s}{\mathrm{d}t}(t) = \big|\mathbf{r}'(t)\big| = \sqrt{12^2\cos^2 2t + 12^2\sin^2 2t + 5^2} = \sqrt{12^2+5^2} = 13$, $\widehat{\mathbf{T}}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{12}{13}\cos 2t\,\widehat{\imath} - \dfrac{12}{13}\sin 2t\,\widehat{\jmath} + \dfrac{5}{13}\,\widehat{\mathbf{k}}$,
$s(\pi) - s(0) = \displaystyle\int_0^{\pi} \big|\mathbf{r}'(t)\big|\,\mathrm{d}t = 13\pi$.

例. 求 $\mathbf{r}(t) = \left\langle e^{3t},\, e^{-3t},\, 3\sqrt{2}\,t \right\rangle$ 介於 $t=0$ 與 $t=\dfrac{1}{3}$ 間弧長.

解. $\mathbf{r}'(t) = \left\langle 3e^{3t},\, -3e^{-3t},\, 3\sqrt{2}\right\rangle$, $s\left(\dfrac{1}{3}\right) - s(0) = \displaystyle\int_0^{\frac{1}{3}} \big|\mathbf{r}'(t)\big|\,\mathrm{d}t = \int_0^{\frac{1}{3}} \sqrt{9e^{6t}+9e^{-6t}+18}\,\mathrm{d}t = 3\int_0^{\frac{1}{3}} \sqrt{e^{6t}+e^{-6t}+2}\,\mathrm{d}t$
$= 3\displaystyle\int_0^{\frac{1}{3}} \sqrt{(e^{3t}+e^{-3t})^2}\,\mathrm{d}t = 3\int_0^{\frac{1}{3}} (e^{3t}+e^{-3t})\,\mathrm{d}t = e^{3t} - e^{-3t}\Big|_0^{\frac{1}{3}} = e - \dfrac{1}{e}$.

例. 求 $\mathbf{r}(t) = \langle t,\, 2\,t,\, t^2\rangle$ 介於 $t=1$ 與 $t=3$ 間弧長.

解. $\mathbf{r}'(t) = \langle 1,\, 2,\, 2\,t\rangle$, $s(3)-s(1) = \displaystyle\int_1^3 \big|\mathbf{r}'(t)\big|\,\mathrm{d}t = \int_1^3 \sqrt{5+4t^2}\,\mathrm{d}t = \dfrac{6\sqrt{41} - 6 - 5\ln 5 + 5\ln(\sqrt{41}+6)}{4}$,
由 $\displaystyle\int \sqrt{x^2+a^2}\,\mathrm{d}x = \dfrac{x\sqrt{x^2+a^2} + a^2\ln\left|\sqrt{x^2+a^2}+x\right|}{2} + c$.

5.3 極限與微分

定義 (多變數函數)**.** 令 $U \subseteq \mathbb{R}^n$, $n > 1$, 從 $U \to \mathbb{R}$ 的映射 $f(x_1, x_2, \ldots, x_n) : U \to \mathbb{R}$ 稱為 U 上的 n 變數函數 (real-valued function of n variables), 其中 U 為定義域, $f(U)$ 為值域.

註. 若 $f(x_1, x_2, \ldots, x_n)$ 為 n 變數函數, 可將 f 視為

- n 個實變數 $x_1, x_2, \ldots, x_n$ 的函數
- 向量 $\mathbf{x} = \langle x_1, x_2, \ldots, x_n\rangle$ 的函數
- $\mathbb{R}^n$ 中之點 $(x_1, x_2, \ldots, x_n)$ 的函數

定義 (圖形, 等值曲線)**.** 令 $f(x,y)$ 為定義在 U 上的雙變數函數.

- 集合 $\left\{(x,y,z) \in \mathbb{R}^3 \,|\, z = f(x,y),\ (x,y) \in U\right\}$ 稱為 f 的圖形 (graph).
- 給定常數 $k \in \mathbb{R}$, 曲線 $f(x,y) = k$ 稱為 f 的等值曲線 (level / contour curve).

若 $w = f(x,y,z)$ 為三變數函數, $f(x,y,z) = k$ 稱為 f 的等值曲面 (level surface).

定義 (極限)**.** 令 f 為 n 變數函數. 若對任意 $\varepsilon > 0$ 存在 $\delta > 0$ 使得對所有 $\mathbf{x} \in \operatorname{dom} f$ 滿足

$$0 < |\mathbf{x} - \mathbf{a}| < \delta \implies |f(\mathbf{x}) - L| < \varepsilon$$

則稱 f 在 $\mathbf{a}$ 的極限為 L, 記為 $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$.

性質 (極限運算)**.** 令 $\mathbf{a} \in \mathbb{R}^n$, c, F, $G \in \mathbb{R}$, $D \subseteq \mathbb{R}^n$, f, $g : D \setminus \{\mathbf{a}\} \to \mathbb{R}$, $\gamma : \mathbb{R} \to \mathbb{R}$. 若 $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = F$, $\lim\limits_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = G$, $\lim\limits_{t\to F} \gamma(t) = \gamma(F)$, 則

1. $\lim\limits_{\mathbf{x}\to\mathbf{a}} \big[f(\mathbf{x}) \pm g(\mathbf{x})\big] = F \pm G$
2. $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})\, g(\mathbf{x}) = FG$
3. $\lim\limits_{\mathbf{x}\to\mathbf{a}} \dfrac{f(\mathbf{x})}{g(\mathbf{x})} = \dfrac{F}{G}$ 若 $G \neq \mathbf{0}$
4. $\lim\limits_{\mathbf{x}\to\mathbf{a}} \gamma\big(f(\mathbf{x})\big) = \gamma(F)$

例. 求 $\displaystyle\lim_{(x,y)\to(2,3)} \frac{x + \sin y}{x^2y^2 + 1}$.

解. $\displaystyle\lim_{(x,y)\to(2,3)} \big(x + \sin y\big) = \lim_{(x,y)\to(2,3)} x + \lim_{(x,y)\to(2,3)} \sin y = \lim_{(x,y)\to(2,3)} x + \sin\left(\lim_{(x,y)\to(2,3)} y\right) = 2 + \sin 3$, $\displaystyle\lim_{(x,y)\to(2,3)} \big(x^2y^2 + 1\big) = \lim_{(x,y)\to(2,3)} x^2y^2 + \lim_{(x,y)\to(2,3)} 1 = \Big(\lim_{(x,y)\to(2,3)} x\Big)\Big(\lim_{(x,y)\to(2,3)} x\Big)\Big(\lim_{(x,y)\to(2,3)} y\Big)\Big(\lim_{(x,y)\to(2,3)} y\Big) + 1 = 2^2 3^2 + 1 = 37$, $\displaystyle\lim_{(x,y)\to(2,3)} \frac{x + \sin y}{x^2y^2 + 1} = \frac{\lim\limits_{(x,y)\to(2,3)} (x + \sin y)}{\lim\limits_{(x,y)\to(2,3)} (x^2y^2 + 1)} = \frac{2 + \sin 3}{37}$

註.

- 單變數函數極限 $\lim\limits_{x\to a} f(x)$ 存在的充要條件為 $\lim\limits_{x\to a-} f(x)$ 與 $\lim\limits_{x\to a+} f(x)$ 均存在且相等.
- 多變數函數極限 $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ 存在的充要條件為任意趨近 $\mathbf{a}$ 之路徑的極限均存在且相等.
- 求 $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ 常可將 (x,y) 轉成極座標 $x = r\cos\theta$, $y = r\sin\theta$ 後令 $r \to 0$ 觀察.

例. 求 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2 + y^2}$.

解. $\displaystyle\frac{x^2y}{x^2+y^2} = \frac{(r\cos\theta)^2(r\sin\theta)}{r^2} = r\cos^2\theta\sin\theta$. 由 $\left|r\cos^2\theta\sin\theta\right| \leqslant r \to 0$ 當 $r \to 0$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0$.

例. 求 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$.

解. 由 $\displaystyle\frac{x^2-y^2}{x^2+y^2} = \frac{(r\cos\theta)^2 - (r\sin\theta)^2}{r^2} = \cos^2\theta - \sin^2\theta = \cos(2\theta)$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \text{DNE}$

例. 求 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2 + y^4}$.

解.

- 令 $y = mx$, $m \neq 0$, 則 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4} = \lim_{x\to0} \frac{2m^2x^3}{x^2 + m^4x^4} = \lim_{x\to0} \frac{2m^2x}{1 + m^4x^2} = 0$.
- 令 $x = y^2$, 則 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4} = \lim_{y\to0} \frac{2y^4}{2y^4} = 1$.
- 結論: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4} = \text{DNE}$

例. 求 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2 + y^2}$.

解. 由 $\displaystyle\frac{x^2y^2}{x^2+y^2} = \frac{(r\cos\theta)^2(r\sin\theta)^2}{r^2} = r^2\cos^2\theta\sin^2\theta \leqslant \frac{r^2}{4}$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0$.

例. 求 $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^3+y^3}$.

解.

- 由 $\dfrac{x^2y^2}{x^3+y^3}=\dfrac{(r\cos\theta)^2(r\sin\theta)^2}{r^3(\cos^3\theta+\sin^3\theta)}=r\dfrac{\cos^2\theta\sin^2\theta}{\cos^3\theta+\sin^3\theta}$, 但 $\dfrac{\cos^2\theta\sin^2\theta}{\cos^3\theta+\sin^3\theta}$ 非為有界 (取 $\theta=\dfrac{3\pi}{4}$ 時 $\cos^3\theta+\sin^3\theta=0$), $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^3+y^3}=\lim_{r\to0}r\frac{\cos^2\theta\sin^2\theta}{\cos^3\theta+\sin^3\theta}=\text{DNE}$.
- 另解:
  - 令 $y=mx$, $m\neq0$, 則 $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^3+y^3}=\lim_{x\to0}\frac{x^2m^2x^2}{x^3+m^3x^3}=\lim_{x\to0}\frac{m^2x}{1+m^3}=0$.
  - 令 $y=-xe^x$, 則 $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^3+y^3}=\lim_{x\to0}\frac{x^2x^2e^{2x}}{x^3-x^3e^{3x}}=\lim_{x\to0}\frac{xe^{2x}}{1-e^{3x}}=\lim_{x\to0}\frac{e^{2x}(1+2x)}{3e^{3x}}=\frac{1}{3}$.
  - 結論: $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^3+y^3}=\text{DNE}$

例. 求 $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^4+y^4}$.

解.

- 由 $\dfrac{x^2y^2}{x^4+y^4}=\dfrac{(r\cos\theta)^2(r\sin\theta)^2}{r^4(\cos^4\theta+\sin^4\theta)}=\dfrac{\cos^2\theta\sin^2\theta}{\cos^4\theta+\sin^4\theta}$, $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^4+y^4}=\lim_{r\to0}\frac{\cos^2\theta\sin^2\theta}{\cos^4\theta+\sin^4\theta}=\text{DNE}$.
- 另解: 令 $y=mx$, $m\neq0$, 則 $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^4+y^4}=\lim_{x\to0}\frac{x^2m^2x^2}{x^4+m^4x^4}=\lim_{x\to0}\frac{m^2}{1+m^4}=\text{DNE}$.

例. 若 $f(x,y)=\begin{cases}\frac{(2x-y)^2}{x-y}, & x\neq y\\ 0, & x=y\end{cases}$, 求 $\displaystyle\lim_{(x,y)\to(0,0)}f(x,y)$.

解.

- 令 $y=x-x^3$, $f(x,x-x^3)=\dfrac{(2x-x+x^3)^2}{x-x+x^3}=\dfrac{(x+x^3)^2}{x^3}=\dfrac{(1+x^2)^2}{x}\longrightarrow\begin{cases}+\infty, & x\to0+\\ -\infty, & x\to0-\end{cases}$
- 令 $y=x-ax^2$, $a\neq0$: $\displaystyle\lim_{x\to0}f(x,x-ax^2)=\lim_{x\to0}\frac{(2x-x+ax^2)^2}{x-x+ax^2}=\lim_{x\to0}\frac{(x+ax^2)^2}{ax^2}=\lim_{x\to0}\frac{(1+ax)^2}{a}=\frac{1}{a}$
- 結論: $\displaystyle\lim_{(x,y)\to(0,0)}f(x,y)=\text{DNE}$

定義 (偏導函數, 偏微分, 偏導數)**.**

- $f(x,y)$ 的 x-偏導函數定義為 $\dfrac{\partial f}{\partial x}(x,y)=\displaystyle\lim_{h\to0}\frac{f(x+h,y)-f(x,y)}{h}$; $f(x,y)$ 的 y-偏導函數定義為 $\dfrac{\partial f}{\partial y}(x,y)=\displaystyle\lim_{h\to0}\frac{f(x,y+h)-f(x,y)}{h}$.
- 求 $f(x,y)$ 之 x-偏導函數之過程稱作「$f(x,y)$ 對 x 偏微分」.
- $f(x,y)$ 在 (a,b) 的 y-偏導數記為 $\dfrac{\partial f}{\partial y}(a,b)\equiv\left.\dfrac{\partial f}{\partial y}\right|_{(a,b)}$.

註.

- $\dfrac{\partial f}{\partial y}(x,y)$ 亦可記為 $\dfrac{\partial f}{\partial y}$, $f_y(x,y)$, f_y, $D_yf(x,y)$, D_yf, $D_2f(x,y)$, D_2f.

- 求 $\dfrac{\partial f}{\partial y}(x,y)$: 將 $f(x,y)$ 中的 x 視作常數, 然後對 y 微分.
- 求 $\dfrac{\partial f}{\partial y}(a,b)$: 將 $f(x,y)$ 中的 x 視作常數, 然後對 y 微分並代入 $x=a$, $y=b$.
- 以上符號 / 運算均可直接推廣至維度 >2 狀況.

例. $f(x,y)=x^3+y^2+4xy^2$, 則 $\dfrac{\partial f}{\partial x}=\dfrac{\partial}{\partial x}(x^3)+\dfrac{\partial}{\partial x}(y^2)+\dfrac{\partial}{\partial x}(4xy^2)=3x^2+0+4y^2\dfrac{\partial}{\partial x}(x)=3x^2+4y^2$, $\dfrac{\partial f}{\partial y}=\dfrac{\partial}{\partial y}(x^3)+\dfrac{\partial}{\partial y}(y^2)+\dfrac{\partial}{\partial y}(4xy^2)=0+2y+4x\dfrac{\partial}{\partial y}(y^2)=2y+8xy$, $\dfrac{\partial f}{\partial x}(1,\,0)=3(1)^2+4(0)^2=3$, $\dfrac{\partial f}{\partial y}(1,\,0)=2(0)+8(1)(0)=0$.

例. $f(x,y)=y\cos x+xe^{xy}$, $\dfrac{\partial}{\partial x}e^{yx}=ye^{yx}$, $\dfrac{\partial f}{\partial x}(x,y)=y\dfrac{\partial}{\partial x}(\cos x)+e^{xy}\dfrac{\partial}{\partial x}(x)+x\dfrac{\partial}{\partial x}\big(e^{xy}\big)=-y\sin x+e^{xy}+xye^{xy}$, $\dfrac{\partial f}{\partial y}(x,y)=\cos x\dfrac{\partial}{\partial y}(y)+x\dfrac{\partial}{\partial y}\big(e^{xy}\big)=\cos x+x^2e^{xy}$

例. $f(x,y,z,t)=x\sin(y+2z)+t^2e^{3y}\ln z$, 則 $\dfrac{\partial f}{\partial x}(x,y,z,t)=\sin(y+2z)$, $\dfrac{\partial f}{\partial y}(x,y,z,t)=x\cos(y+2z)+3t^2e^{3y}\ln z$, $\dfrac{\partial f}{\partial z}(x,y,z,t)=2x\cos(y+2z)+\dfrac{t^2e^{3y}}{z}$, $\dfrac{\partial f}{\partial t}(x,y,z,t)=2te^{3y}\ln z$.

例. 若 $f(x,y)=\begin{cases}\frac{\cos x-\cos y}{x-y} & x\neq y\\ 0 & x=y\end{cases}$

- $\forall\, x\neq y$, $f_x=\dfrac{-\sin x(x-y)-(\cos x-\cos y)}{(x-y)^2}$; 無法由此求 $f_x(0,0)$.
- 由定義計算 $f_x(0,0)$: $f_x(0,0)=\lim\limits_{h\to0}\dfrac{f(0+h,0)-f(0,0)}{h}=\lim\limits_{h\to0}\dfrac{\frac{\cos h-1}{h-0}-0}{h}=\lim\limits_{h\to0}\dfrac{\cos h-1}{h^2}=\lim\limits_{h\to0}\dfrac{-\sin h}{2h}=\lim\limits_{h\to0}\dfrac{-\cos h}{2}=-\dfrac{1}{2}$.
- 由定義計算 $f_y(0,0)$: $f_y(0,0)=\lim\limits_{h\to0}\dfrac{f(0,0+h)-f(0,0)}{h}=\lim\limits_{h\to0}\dfrac{\frac{1-\cos h}{-h}-0}{h}=\lim\limits_{h\to0}\dfrac{\cos h-1}{h^2}=\lim\limits_{h\to0}\dfrac{-\sin h}{2h}=\lim\limits_{h\to0}\dfrac{-\cos h}{2}=-\dfrac{1}{2}$.
- $\lim\limits_{(x,\,y)\to(0,\,0)}\dfrac{\cos x-\cos y}{x-y}=\lim\limits_{(x,\,y)\to(0,\,0)}\dfrac{-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}}{x-y}=-\lim\limits_{(x,\,y)\to(0,\,0)}\sin\dfrac{x+y}{2}\lim\limits_{(x,\,y)\to(0,\,0)}\dfrac{\sin\frac{x-y}{2}}{\frac{x-y}{2}}=0$, 故 $f(x,y)$ 在 $(0,0)$ 連續.
- $f(x,y)$ 在 (a,a), $a\neq0$ 不連續: 由定義 $\lim\limits_{(x,\,y)\to(a,\,a)}f(x,y)=\sin a$, 但 $f(a,a)=0$.

例. 若 x, y, z 滿足方程式 $z^5+y^2e^z+e^{2x}=0$, 求 $\dfrac{\partial z}{\partial x}(0,0)$.

解. 局域下 z 為 x, y 之函數; 當 $x=y=0$, 原方程式為 $z(0,0)^5=-1\implies z(0,0)=-1$. 令 $z\equiv z(x,y)$ 代入原方程式並對 x 偏微分得 $5\,z(x,y)^4\,\dfrac{\partial z}{\partial x}(x,y)+y^2\,e^{z(x,y)}\,\dfrac{\partial z}{\partial x}(x,y)+2\,e^{2x}=0$; 代入 $(x,\,y)=(0,\,0)$ 得 $5\,z(0,0)^4\,\dfrac{\partial z}{\partial x}(0,0)+2=0$, 再由 $z(0,0)=-1$, $\dfrac{\partial z}{\partial x}(0,0)=-\dfrac{2}{5\,z(0,0)^4}=-\dfrac{2}{5}$.

例. 若 x, y, z 滿足方程式 $x^2+y^2+z^2=1$, 證明 $x\,\dfrac{\partial z}{\partial x}+y\,\dfrac{\partial z}{\partial y}=z-\dfrac{1}{z}$.

解. 局域下 z 為 x, y 之函數; $x^2+y^2+z^2=1$ 對 x 偏微分得 $2x+2z\dfrac{\partial z}{\partial x}=0 \implies \dfrac{\partial z}{\partial x}=-\dfrac{x}{z}$; 對 y 偏微分得 $2y+2z\dfrac{\partial z}{\partial y}=0 \implies \dfrac{\partial z}{\partial x}=-\dfrac{y}{z}$. 故 $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=-\dfrac{x^2+y^2}{z}=\dfrac{z^2-1}{z}=z-\dfrac{1}{z}$.

例. 若 x, y, z 滿足方程式 $x\sin z-z^2y=1$, 求 $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$.

解. 局域下 z 為 x, y 之函數; $x\sin z-z^2y=1$ 對 x 偏微分得 $\sin z+x\cos z\dfrac{\partial z}{\partial x}-2yz\dfrac{\partial z}{\partial x}=0 \implies \dfrac{\partial z}{\partial x}=\dfrac{\sin z}{2yz-x\cos z}$; 對 y 偏微分得 $x\cos z\dfrac{\partial z}{\partial y}-2z\dfrac{\partial z}{\partial y}y-z^2=0 \implies \dfrac{\partial z}{\partial y}=\dfrac{z^2}{x\cos z-2yz}$.

定義 (高階偏導函數)**.** 給定可微雙變數函數 $f(x,y)$,

- $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)(x,y)=\dfrac{\partial^2 f}{\partial x^2}(x,y)=f_{xx}(x,y)$
- $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)(x,y)=\dfrac{\partial^2 f}{\partial x\partial y}(x,y)=f_{yx}(x,y)$
- $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)(x,y)=\dfrac{\partial^2 f}{\partial y\partial x}(x,y)=f_{xy}(x,y)$
- $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)(x,y)=\dfrac{\partial^2 f}{\partial y^2}(x,y)=f_{yy}(x,y)$

例. 令 $f(x,y)=e^{my}\cos(nx)$, 則

- $f_x=-ne^{my}\sin(nx)$
- $f_{xx}=-n^2e^{my}\cos(nx)$
- $f_{yx}=-mne^{my}\sin(nx)$
- $f_y=me^{my}\cos(nx)$
- $f_{yy}=m^2e^{my}\cos(nx)$
- $f_{xy}=-mne^{my}\sin(nx)$

例. 令 $f(x,y)=e^{\alpha x+\beta y}$, 則

- $f_x=\alpha e^{\alpha x+\beta y}$
- $f_{xx}=\alpha^2 e^{\alpha x+\beta y}$
- $f_{xy}=\alpha\beta e^{\alpha x+\beta y}$
- $f_y=\beta e^{\alpha x+\beta y}$
- $f_{yx}=\beta\alpha e^{\alpha x+\beta y}$
- $f_{yy}=\beta^2 e^{\alpha x+\beta y}$

對整數 $m,n\geqslant 0$, $\dfrac{\partial^{m+n}f}{\partial x^m\partial y^n}=\alpha^m\beta^n e^{\alpha x+\beta y}$.

例. 令 $f(x,y)=\ln(x^2+y^2)$, 則

- $f_x=\dfrac{2x}{x^2+y^2}$
- $f_{xx}=\dfrac{(x^2+y^2)\cdot 2-2x\cdot 2x}{(x^2+y^2)^2}=\dfrac{2\,(y^2-x^2)}{(x^2+y^2)^2}$
- $f_y=\dfrac{2y}{x^2+y^2}$
- $f_{yy}=\dfrac{(x^2+y^2)\cdot 2-2y\cdot 2y}{(x^2+y^2)^2}=\dfrac{2\,(x^2-y^2)}{(x^2+y^2)^2}$

$f(x,y)$ 滿足 Laplace 方程式 $f_{xx}+f_{yy}=0$.

例. 令 $f(x,y)=\tan^{-1}\dfrac{y}{x}$, 則

- $f_x=\dfrac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\dfrac{-y}{x^2}\right)=-\dfrac{y}{x^2+y^2}$
- $f_{xx}=\dfrac{y\cdot 2x}{(x^2+y^2)^2}=\dfrac{2xy}{(x^2+y^2)^2}$
- $f_y=\dfrac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\dfrac{1}{x}\right)=\dfrac{x}{x^2+y^2}$
- $f_{yy}=-\dfrac{x\cdot 2y}{(x^2+y^2)^2}=-\dfrac{2xy}{(x^2+y^2)^2}$

$f(x,y)$ 滿足 Laplace 方程式 $f_{xx}+f_{yy}=0$.

例. 若 $f(x_1,x_2,x_3,x_4)=x_1^4\,x_2^3\,x_3^2\,x_4$, 則

- $\dfrac{\partial^4 f}{\partial x_1\partial x_2\partial x_3\partial x_4}=\dfrac{\partial^3}{\partial x_1\partial x_2\partial x_3}\left(x_1^4x_2^3x_3^2\right)=\dfrac{\partial^2}{\partial x_1\partial x_2}\left(2x_1^4x_2^3x_3\right)=\dfrac{\partial}{\partial x_1}\left(6x_1^4x_2^2x_3\right)=24x_1^3x_2^2x_3$

- $\dfrac{\partial^4 f}{\partial x_4 \partial x_3 \partial x_2 \partial x_1} = \dfrac{\partial^3}{\partial x_4 \partial x_3 \partial x_2}\left(4x_1^3x_2^3x_3^2x_4\right) = \dfrac{\partial^2}{\partial x_4 \partial x_3}\left(12x_1^3x_2^2x_3^2x_4\right) = \dfrac{\partial}{\partial x_4}\left(24x_1^3x_2^2x_3x_4\right) = 24x_1^3x_2^2x_3$

定理 (Clairaut)**.** 若 $\dfrac{\partial^2 f}{\partial x \partial y}$ 與 $\dfrac{\partial^2 f}{\partial y \partial x}$ 均存在且在 $(x_0,\, y_0)$ 連續, 則 $\dfrac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \dfrac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$.

5.4 鏈鎖律

定理. 若 f 為 $x_1,\, x_2,\, \ldots,\, x_n$ 的可微函數, 而 x_j 是 $t_1,\, t_2,\, \ldots,\, t_m$ 的可微函數, $n,\, m \geqslant 1$, 則 f 為 $t_1,\, t_2,\, \ldots,\, t_m$ 的可微函數; 輔助函數 $F(t_1,\, t_2,\, \ldots,\, t_m) \equiv f(x_1(t_1,\, t_2,\, \ldots,\, t_m),\, x_2(t_1,\, t_2,\, \ldots,\, t_m),\, \ldots,\, x_n(t_1,\, t_2,\, \ldots,\, t_m))$, 則

$$\frac{\partial F}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

例 $(n = m = 2)$**.** 輔助函數 $F(s,t) \equiv f\big(x(s,t),\, y(s,t)\big)$, 則

$$\begin{aligned}
\frac{\partial F}{\partial s}(s,t) &= \frac{\partial f}{\partial x}\big(x(s,t),\, y(s,t)\big)\,\frac{\partial x}{\partial s}(s,t) + \frac{\partial f}{\partial y}\big(x(s,t),\, y(s,t)\big)\,\frac{\partial y}{\partial s}(s,t)\\
\frac{\partial F}{\partial t}(s,t) &= \frac{\partial f}{\partial x}\big(x(s,t),\, y(s,t)\big)\,\frac{\partial x}{\partial t}(s,t) + \frac{\partial f}{\partial y}\big(x(s,t),\, y(s,t)\big)\,\frac{\partial y}{\partial t}(s,t)
\end{aligned}$$

例. 若 $f(x,y) = e^{xy}$, $x(s,t) = s$, $y(s,t) = \cos t$; $F(s,t) \equiv f\big(x(s,t),\, y(s,t)\big)$, 求 $\dfrac{\partial F}{\partial s}$.

解.

- $\dfrac{\partial f}{\partial x} = y\, e^{xy} = y(s,t)\, e^{x(s,t)\, y(s,t)} = \cos t\, e^{s \cos t}$, $\dfrac{\partial f}{\partial y} = x\, e^{xy} = x(s,t)\, e^{x(s,t)\, y(s,t)} = s\, e^{s\cos t}$, $\dfrac{\partial x}{\partial s} = \dfrac{\partial s}{\partial s} = 1$, $\dfrac{\partial y}{\partial s} = \dfrac{\partial \cos t}{\partial s} = 0$, 故 $\dfrac{\partial F}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s} = \cos t\, e^{s\cos t}\cdot 1 + s\ e^{s\cos t}\cdot 0 = \cos t\ e^{s\cos t}$.
- 直接寫出 $F(s,t)$ 並對 s 偏微分: $F(s,t) = f\big(x(s,t),\, y(s,t)\big) = e^{s\cdot\cos t}$, $\dfrac{\partial F}{\partial s} = e^{s\cdot\cos t}\, \cos t$.

例. 若 $f(x,y) = x^2 - y^2$, $x(t) = \cos t$, $y(t) = \sin t$, 求 $\dfrac{\mathrm{d}f}{\mathrm{d}t}$.

解. 輔助函數 $F(t) \equiv f\big(x(t),\, y(t)\big)$, 則

- $\dfrac{\partial f}{\partial x} = 2x = 2\cos t$, $\dfrac{\partial f}{\partial y} = -2y = -2\sin t$, $\dfrac{\mathrm{d}x}{\mathrm{d}t} = -\sin t$, $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \cos t$, 故 $\dfrac{\mathrm{d}F}{\mathrm{d}t} = \dfrac{\partial f}{\partial x}\dfrac{\mathrm{d}x}{\mathrm{d}t} + \dfrac{\partial f}{\partial y}\dfrac{\mathrm{d}y}{\mathrm{d}t} = (2\cos t)(-\sin t) + (-2\sin t)(\cos t) = -4\ \sin t\cos t$.
- 直接寫出 $F(t)$ 並對 t 微分: $F(t) = f\big(x(t), y(t)\big) = x(t)^2 - y(t)^2 = \cos^2 t - \sin^2 t$, 故 $F'(t) = 2(\cos t)(-\sin t) - 2(\sin t)(\cos t) = -4\sin t\cos t$

例.

1. 令 $w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$, 求 $\dfrac{\mathrm{d}w}{\mathrm{d}t}$ 與 $\left.\dfrac{\mathrm{d}w}{\mathrm{d}t}\right|_{t=0}$.
2. 令 $w = x + 2y + z^2$, $x = \dfrac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$, 求 $\dfrac{\partial w}{\partial r}$ 與 $\dfrac{\partial w}{\partial s}$.
3. 令 $w = x^4 y + y^2 z^3$, $x = rse^t$, $y = rs^2e^{-t}$, $z = r^2 s\sin t$, 求 $\left.\dfrac{\partial w}{\partial s}\right|_{(r,\, s,\, t)=(2,\, 1,\, 0)}$.

解.

1. $\dfrac{\partial w}{\partial x}=y=\sin t,\ \dfrac{\partial w}{\partial y}=x=\cos t,\ \dfrac{\partial w}{\partial z}=1,\ \dfrac{\mathrm{d}x}{\mathrm{d}t}=-\sin t,\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=\cos t,\ \dfrac{\mathrm{d}z}{\mathrm{d}t}=1$, 故 $\dfrac{\mathrm{d}w}{\mathrm{d}t}=\dfrac{\partial w}{\partial x}\dfrac{\mathrm{d}x}{\mathrm{d}t}+\dfrac{\partial w}{\partial y}\dfrac{\mathrm{d}y}{\mathrm{d}t}+\dfrac{\partial w}{\partial z}\dfrac{\mathrm{d}z}{\mathrm{d}t}=(\sin t)(-\sin t)+(\cos t)(\cos t)+(1)(1)=\cos 2t+1,\ \dfrac{\mathrm{d}w}{\mathrm{d}t}\Big|_{t=0}=1+1=2.$

2. $\dfrac{\partial w}{\partial x}=1,\ \dfrac{\partial w}{\partial y}=2,\ \dfrac{\partial w}{\partial z}=2z=4r,\ \dfrac{\partial x}{\partial r}=\dfrac{1}{s},\ \dfrac{\partial y}{\partial r}=2r,\ \dfrac{\partial z}{\partial r}=2,\ \dfrac{\partial x}{\partial s}=-\dfrac{r}{s^2},\ \dfrac{\partial y}{\partial s}=\dfrac{1}{s},\ \dfrac{\partial z}{\partial s}=0$, 故

$$\begin{aligned}\frac{\partial w}{\partial r}&=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r}=(1)\Big(\frac{1}{s}\Big)+(2)(2r)+(4r)(2)=\frac{1}{s}+12r\\ \frac{\partial w}{\partial s}&=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}=(1)\Big(-\frac{r}{s^2}\Big)+(2)\Big(\frac{1}{s}\Big)+(4r)(0)=-\frac{r}{s^2}+\frac{2}{s}\end{aligned}$$

3. $\dfrac{\partial w}{\partial x}=4x^3y,\ \dfrac{\partial w}{\partial y}=x^4+2yz^3,\ \dfrac{\partial w}{\partial z}=3y^2z^2,\ \dfrac{\partial x}{\partial s}=re^t,\ \dfrac{\partial y}{\partial s}=2rse^{-t},\ \dfrac{\partial z}{\partial s}=r^2\sin t$. 當 $(r,\ s,\ t)=(2,\ 1,\ 0)$, $(x,\ y,\ z)=(2,\ 2,\ 0)$, 故

$$\begin{aligned}\frac{\partial w}{\partial s}\Big|_{(r,s,t)=(2,1,0)}&=\Big(\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}\Big)\Big|_{(r,s,t)=(2,1,0)}\\ &=(4\cdot 2^3\cdot 2)(2)+(2^4+0)(2\cdot 2\cdot 1)+(0)(0)=192\end{aligned}$$

例. 若 $z=f(x-y)$, 證明 $\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}=0$.

解. 令 $u=x-y$, 則 $\dfrac{\partial z}{\partial x}=\dfrac{\mathrm{d}z}{\mathrm{d}u}\dfrac{\partial u}{\partial x}=\dfrac{\mathrm{d}z}{\mathrm{d}u}(1)=\dfrac{\mathrm{d}z}{\mathrm{d}u},\ \dfrac{\partial z}{\partial y}=\dfrac{\mathrm{d}z}{\mathrm{d}u}\dfrac{\partial u}{\partial y}=\dfrac{\mathrm{d}z}{\mathrm{d}u}(-1)=-\dfrac{\mathrm{d}z}{\mathrm{d}u}$, 故 $\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}=0$.

例. 若 $z=f(x,y)$, $x=s+t$, $y=s-t$, 證明 $\left(\dfrac{\partial z}{\partial x}\right)^2-\left(\dfrac{\partial z}{\partial y}\right)^2=\dfrac{\partial z}{\partial s}\dfrac{\partial z}{\partial t}$.

解. $\dfrac{\partial z}{\partial s}=\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s}+\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}=\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y},\ \dfrac{\partial z}{\partial t}=\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t}+\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}=\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}$, 故 $\dfrac{\partial z}{\partial s}\dfrac{\partial z}{\partial t}=\left(\dfrac{\partial z}{\partial x}\right)^2-\left(\dfrac{\partial z}{\partial y}\right)^2$.

例. 若 $g(s,t)=f(s^2-t^2,t^2-s^2)$ 且 f 可微, 證明 $t\dfrac{\partial g}{\partial s}+s\dfrac{\partial g}{\partial t}=0$.

解. 令 $u(s,t)=s^2-t^2$, $v(s,t)=t^2-s^2$, 則 $g(s,t)=f(u(s,t),v(s,t))$. 由鏈鎖律

$$\begin{aligned}\frac{\partial g}{\partial s}&=\frac{\partial f}{\partial u}\frac{\partial u}{\partial s}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial s}=\frac{\partial f}{\partial u}\cdot(2s)+\frac{\partial f}{\partial v}\cdot(-2s)\\ \frac{\partial g}{\partial t}&=\frac{\partial f}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial t}=\frac{\partial f}{\partial u}\cdot(-2t)+\frac{\partial f}{\partial v}\cdot(2t)\end{aligned}$$

故 $t\dfrac{\partial g}{\partial s}+s\dfrac{\partial g}{\partial t}=t\left(\dfrac{\partial f}{\partial u}\cdot(2s)+\dfrac{\partial f}{\partial v}\cdot(-2s)\right)+s\left(\dfrac{\partial f}{\partial u}\cdot(-2t)+\dfrac{\partial f}{\partial v}\cdot(2t)\right)=0.$

例. 若函數 $f(x,y)$ 滿足 $f(tx,ty)=t^nf(x,y)$, $t\neq 0$, $n\in\mathbb{N}$, 稱 $f(x,y)$ 為 n 次齊次函數; 證明

1. $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}=nf$.
2. $x^2\dfrac{\partial^2 f}{\partial x^2}+2xy\dfrac{\partial^2 f}{\partial x\partial y}+y^2\dfrac{\partial^2 f}{\partial y^2}=n(n-1)f$.

解. 令 $u(x,y,t)=tx$, $v(x,y,t)=ty$, 則 $f(u(x,y,t),v(x,y,t))=t^nf(x,y)$.

1. 等式 $f(u(x,y,t),v(x,y,t))=t^nf(x,y)$ 兩邊對 t 微分 $\implies \dfrac{\partial f}{\partial u}(tx,ty)\dfrac{\partial u}{\partial t}+\dfrac{\partial f}{\partial v}(tx,ty)\dfrac{\partial v}{\partial t}=nt^{n-1}f(x,y)$ $\implies x\dfrac{\partial f}{\partial u}(tx,ty)+y\dfrac{\partial f}{\partial v}(tx,ty)=nt^{n-1}f(x,y)$. 令 $t=1$ 則 $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}=nf$.

2. 等式 $x\dfrac{\partial f}{\partial u}(tx,ty)+y\dfrac{\partial f}{\partial v}(tx,ty)=nt^{n-1}f(x,y)$ 兩邊對 t 微分 $\Longrightarrow$ $\dfrac{\partial}{\partial t}\left(x\dfrac{\partial f}{\partial u}(tx,ty)+y\dfrac{\partial f}{\partial v}(tx,ty)\right)=n(n-1)t^{n-2}f(x,y)\Longrightarrow x\dfrac{\partial}{\partial t}\left(\dfrac{\partial f}{\partial u}(tx,ty)\right)+y\dfrac{\partial}{\partial t}\left(\dfrac{\partial f}{\partial v}(tx,ty)\right)=n(n-1)t^{n-2}f(x,y)$. 又 $\dfrac{\partial}{\partial t}\left(\dfrac{\partial f}{\partial u}(tx,ty)\right)=\dfrac{\partial^2 f}{\partial u^2}(tx,ty)\dfrac{\partial u}{\partial t}+\dfrac{\partial^2 f}{\partial v\partial u}(tx,ty)\dfrac{\partial v}{\partial t}=x\dfrac{\partial^2 f}{\partial u^2}(tx,ty)+y\dfrac{\partial^2 f}{\partial v\partial u}(tx,ty)$, $\dfrac{\partial}{\partial t}\left(\dfrac{\partial f}{\partial v}(tx,ty)\right)=\dfrac{\partial^2 f}{\partial u\partial v}(tx,ty)\dfrac{\partial u}{\partial t}+\dfrac{\partial^2 f}{\partial v^2}(tx,ty)\dfrac{\partial v}{\partial t}=x\dfrac{\partial^2 f}{\partial u\partial v}(tx,ty)+y\dfrac{\partial^2 f}{\partial v^2}(tx,ty)$, 故 $x\dfrac{\partial}{\partial t}\left(\dfrac{\partial f}{\partial u}(tx,ty)\right)+y\dfrac{\partial}{\partial t}\left(\dfrac{\partial f}{\partial v}(tx,ty)\right)=x^2\dfrac{\partial^2 f}{\partial u^2}(tx,ty)+xy\dfrac{\partial^2 f}{\partial v\partial u}(tx,ty)+yx\dfrac{\partial^2 f}{\partial u\partial v}(tx,ty)+y^2\dfrac{\partial^2 f}{\partial v^2}(tx,ty)=n(n-1)t^{n-2}f(x,y)$; 令 $t=1$ 則 $x^2\dfrac{\partial^2 f}{\partial x^2}+2xy\dfrac{\partial^2 f}{\partial x\partial y}+y^2\dfrac{\partial^2 f}{\partial y^2}=n(n-1)f$.

例. 若 $u=f(x,y)$, $x=e^s\cos t$, $y=e^s\sin t$, 證明 $\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial u}{\partial y}\right)^2=e^{-2s}\left(\left(\dfrac{\partial u}{\partial s}\right)^2+\left(\dfrac{\partial u}{\partial t}\right)^2\right)$.

解. 由鏈鎖律

$$\begin{aligned}\frac{\partial u}{\partial s}&=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}=\frac{\partial u}{\partial x}\cdot(e^s\cos t)+\frac{\partial u}{\partial y}\cdot(e^s\sin t)\\ \frac{\partial u}{\partial t}&=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}=\frac{\partial u}{\partial x}\cdot(-e^s\sin t)+\frac{\partial u}{\partial y}\cdot(e^s\cos t)\end{aligned}$$

故

$$\begin{aligned}\left(\frac{\partial u}{\partial s}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2&=\left(\frac{\partial u}{\partial x}\cdot(e^s\cos t)+\frac{\partial u}{\partial y}\cdot(e^s\sin t)\right)^2+\left(\frac{\partial u}{\partial x}\cdot(-e^s\sin t)+\frac{\partial u}{\partial y}\cdot(e^s\cos t)\right)^2\\ &=\left(\frac{\partial u}{\partial x}\right)^2e^{2s}\cos^2 t+2\,\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}e^{2s}\cos t\sin t+\left(\frac{\partial u}{\partial y}\right)^2e^{2s}\sin^2 t\\ &\quad+\left(\frac{\partial u}{\partial x}\right)^2e^{2s}\sin^2 t-2\,\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}e^{2s}\sin t\cos t+\left(\frac{\partial u}{\partial y}\right)^2e^{2s}cos^2t=e^{2s}\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right)\end{aligned}$$

例. 若 $z=f(x,y)$, $x=r\cos\theta$, $y=r\sin\theta$, 證明 $\left(\dfrac{\partial z}{\partial x}\right)^2+\left(\dfrac{\partial z}{\partial y}\right)^2=\left(\dfrac{\partial z}{\partial r}\right)^2+\dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2$.

解. 由鏈鎖律

$$\begin{aligned}\frac{\partial z}{\partial r}&=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta\\ \frac{\partial z}{\partial \theta}&=\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial z}{\partial x}\cdot(-r\sin\theta)+\frac{\partial z}{\partial y}\cdot(r\cos\theta)\end{aligned}$$

故

$$\begin{aligned}\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2&=\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial x}\cdot(-r\sin\theta)+\frac{\partial z}{\partial y}\cdot(r\cos\theta)\right)^2\\ &=\left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta+2\,\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta+\left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta\\ &\quad+\left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta-2\,\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta+\left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\end{aligned}$$

例. 若 $z=u(x,y)$, $x=r^2+s^2$, $y=2rs$, 求 $\dfrac{\partial z}{\partial r}$, $\dfrac{\partial^2 z}{\partial r^2}$, $\dfrac{\partial^2 z}{\partial s\partial r}$.

解. 由鏈鎖律

$$\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = 2r\,\frac{\partial u}{\partial x} + 2s\,\frac{\partial u}{\partial y} \\
\frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r}\left(2r\,\frac{\partial u}{\partial x} + 2s\,\frac{\partial u}{\partial y}\right) = 2\,\frac{\partial u}{\partial x} + 2r\,\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) + 2s\,\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right) \\
\frac{\partial^2 z}{\partial s\partial r} &= \frac{\partial}{\partial s}\left(2r\,\frac{\partial u}{\partial x} + 2s\,\frac{\partial u}{\partial y}\right) = 2\,\frac{\partial u}{\partial y} + 2s\,\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right) + 2r\,\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)
\end{aligned}$$

又

$$\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 u}{\partial x^2}\cdot(2r) + \frac{\partial^2 u}{\partial y\partial x}\cdot(2s) \\
\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 u}{\partial x\partial y}\cdot(2r) + \frac{\partial^2 u}{\partial y^2}\cdot(2s) \\
\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial s} = \frac{\partial^2 u}{\partial x^2}\cdot(2s) + \frac{\partial^2 u}{\partial y\partial x}\cdot(2r) \\
\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial s} = \frac{\partial^2 u}{\partial x\partial y}\cdot(2s) + \frac{\partial^2 u}{\partial y^2}\cdot(2r)
\end{aligned}$$

故

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2} &= 2\,\frac{\partial u}{\partial x} + 2r\,\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) + 2s\,\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right) \\
&= 2\,\frac{\partial u}{\partial x} + 2r\left(\frac{\partial^2 u}{\partial x^2}\cdot(2r) + \frac{\partial^2 u}{\partial y\partial x}\cdot(2s)\right) + 2s\left(\frac{\partial^2 u}{\partial x\partial y}\cdot(2r) + \frac{\partial^2 u}{\partial y^2}\cdot(2s)\right) \\
&= 2\,\frac{\partial u}{\partial x} + 4r^2\,\frac{\partial^2 u}{\partial x^2} + 8rs\,\frac{\partial^2 u}{\partial x\partial y} + 4s^2\,\frac{\partial^2 u}{\partial y^2} \\
\frac{\partial^2 z}{\partial s\partial r} &= 2\,\frac{\partial u}{\partial y} + 2s\,\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right) + 2r\,\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right) \\
&= 2\,\frac{\partial u}{\partial y} + 2s\left(\frac{\partial^2 u}{\partial x\partial y}\cdot(2s) + \frac{\partial^2 u}{\partial y^2}\cdot(2r)\right) + 2r\left(\frac{\partial^2 u}{\partial x^2}\cdot(2s) + \frac{\partial^2 u}{\partial y\partial x}\cdot(2r)\right) \\
&= 2\,\frac{\partial u}{\partial y} + 4rs\,\frac{\partial^2 u}{\partial x^2} + 4(r^2+s^2)\,\frac{\partial^2 u}{\partial x\partial y} + 4rs\,\frac{\partial^2 u}{\partial y^2}
\end{aligned}$$

例. 若 $z = u(x,y)$, $x = g(s,t)$, $y = h(s,t)$, 證明

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\,\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2$$

解. 令 $z = U(s,t) = u(x(s,t),\, y(s,t))$, 由鏈鎖律

$$\begin{aligned}
\frac{\partial U}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} \\
\frac{\partial^2 U}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\right) = \frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial x}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial u}{\partial y}\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)
\end{aligned}$$

又

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial t} = \frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial t} + \frac{\partial^2 u}{\partial y\partial x}\frac{\partial y}{\partial t} \\
\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial t} = \frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial t} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial t}
\end{aligned}$$

故

$$\begin{aligned}\frac{\partial^2 U}{\partial t^2} &= \frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial x}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 u}{\partial y \partial x}\frac{\partial y}{\partial t}\right) + \frac{\partial y}{\partial t}\left(\frac{\partial^2 u}{\partial x \partial y}\frac{\partial x}{\partial t} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial t}\right)\\ &= \frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial u}{\partial y}\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 u}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\,\frac{\partial^2 u}{\partial x \partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 u}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2\end{aligned}$$

例. 若 $f(x,t) = g(x+at) + h(x-at)$, 其中 g, h 可二次微分, 證明 f 滿足波動方程式 $\dfrac{\partial^2 f}{\partial t^2} = a^2\dfrac{\partial^2 f}{\partial x^2}$.

解. 令 $u(x,t) = x+at$, $v(x,t) = x-at$, $f\big(u(x,t), v(x,t)\big) = g(u(x,t)) + h(v(x,t))$. 由鏈鎖律

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t}\\ &= g'(u(x,t))\cdot a + h'(v(x,t))\cdot(-a) = a\,g'(x+at) - a\,h'(x-at) = a\,g'(u(x,t)) - a\,h'(v(x,t))\\ \frac{\partial^2 f}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right)\\ &= \frac{\partial}{\partial u}\big(a\,g'(u) - a\,h'(v)\big)\big(u(x,t), v(x,t)\big)\,\frac{\partial u}{\partial t} + \frac{\partial}{\partial v}\big(a\,g'(u) - a\,h'(v)\big)\big(u(x,t), v(x,t)\big)\frac{\partial v}{\partial t}\\ &= a\,g''(u(x,t))\cdot a - a\,h''(v(x,t))\cdot(-a)\\ &= a^2\left(g''(x+at) + h''(x-at)\right)\\ \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\\ &= g'(u(x,t))\cdot 1 + h'(v(x,t))\cdot(1) = g'(x+at) + h'(x-at) = g'(u(x,t)) + h'(v(x,t))\\ \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\\ &= \frac{\partial}{\partial x}\big(g'(u) + h'(v)\big)\big(u(x,t), v(x,t)\big)\,\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\big(g'(u) + h'(v)\big)\big(u(x,t), v(x,t)\big)\frac{\partial v}{\partial x}\\ &= g''(u(x,t))\cdot 1 + h''(v(x,t))\cdot 1\\ &= g''(x+at) + h''(x-at)\end{aligned}$$

故 $\dfrac{\partial^2 f}{\partial t^2} = a^2\dfrac{\partial^2 f}{\partial x^2}$.

例. 若 $u = f(x,y)$, $x = e^s\cos t$, $y = e^s\sin t$, 證明 $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = e^{-2s}\left(\dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 u}{\partial t^2}\right)$.

解. 由鏈鎖律

$$\begin{aligned}\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial u}{\partial x}\cdot(e^s\cos t) + \frac{\partial u}{\partial y}\cdot(e^s\sin t)\\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial u}{\partial x}\cdot(-e^s\sin t) + \frac{\partial u}{\partial y}\cdot(e^s\cos t)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial s^2} = \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial s}\right) &= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\cdot(e^s\cos t) + \frac{\partial u}{\partial y}\cdot(e^s\sin t)\right)\\ &= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)\cdot(e^s\cos t) + \frac{\partial u}{\partial x}\cdot(e^s\cos t) + \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right)\cdot(e^s\sin t) + \frac{\partial u}{\partial y}\cdot(e^s\sin t)\\ \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) &= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\cdot(-e^s\sin t) + \frac{\partial u}{\partial y}\cdot(e^s\cos t)\right)\\ &= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)\cdot(-e^s\sin t) + \frac{\partial u}{\partial x}\cdot(-e^s\cos t) + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)\cdot(e^s\cos t) + \frac{\partial u}{\partial y}\cdot(-e^s\sin t)\end{aligned}$$

又

$$\begin{aligned}
\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial s} = \frac{\partial^2 u}{\partial x^2}\cdot(e^s\cos t) + \frac{\partial^2 u}{\partial y\partial x}\cdot(e^s\sin t)\\
\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial s} = \frac{\partial^2 u}{\partial x\partial y}\cdot(e^s\cos t) + \frac{\partial^2 u}{\partial y^2}\cdot(e^s\sin t)\\
\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial t} = \frac{\partial^2 u}{\partial x^2}\cdot(-e^s\sin t) + \frac{\partial^2 u}{\partial y\partial x}\cdot(e^s\cos t)\\
\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial t} = \frac{\partial^2 u}{\partial x\partial y}\cdot(-e^s\sin t) + \frac{\partial^2 u}{\partial y^2}\cdot(e^s\cos t)
\end{aligned}$$

故

$$\begin{aligned}
\frac{\partial^2 u}{\partial s^2} &= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)\cdot(e^s\cos t) + \frac{\partial u}{\partial x}\cdot(e^s\cos t) + \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right)\cdot(e^s\sin t) + \frac{\partial u}{\partial y}\cdot(e^s\sin t)\\
&= \left(\frac{\partial^2 u}{\partial x^2}\cdot(e^s\cos t) + \frac{\partial^2 u}{\partial y\partial x}\cdot(e^s\sin t)\right)\cdot(e^s\cos t) + \frac{\partial u}{\partial x}\cdot(e^s\cos t)\\
&\quad + \left(\frac{\partial^2 u}{\partial x\partial y}\cdot(e^s\cos t) + \frac{\partial^2 u}{\partial y^2}\cdot(e^s\sin t)\right)\cdot(e^s\sin t) + \frac{\partial u}{\partial y}\cdot(e^s\sin t)\\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)\cdot(-e^s\sin t) + \frac{\partial u}{\partial x}\cdot(-e^s\cos t) + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)\cdot(e^s\cos t) + \frac{\partial u}{\partial y}\cdot(-e^s\sin t)\\
&= \left(\frac{\partial^2 u}{\partial x^2}\cdot(-e^s\sin t) + \frac{\partial^2 u}{\partial y\partial x}\cdot(e^s\cos t)\right)\cdot(-e^s\sin t) + \frac{\partial u}{\partial x}\cdot(-e^s\cos t)\\
&\quad + \left(\frac{\partial^2 u}{\partial x\partial y}\cdot(-e^s\sin t) + \frac{\partial^2 u}{\partial y^2}\cdot(e^s\cos t)\right)\cdot(e^s\cos t) + \frac{\partial u}{\partial y}\cdot(-e^s\sin t)
\end{aligned}$$

可得 $\dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 u}{\partial t^2} = e^{2s}\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$.

例. 若 $z = u(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$, 求 $\dfrac{\partial^2 z}{\partial\theta\partial r}$, $\dfrac{\partial^2 z}{\partial r\partial\theta}$.

解.

- 求 $\dfrac{\partial^2 z}{\partial\theta\partial r}$: 由鏈鎖律

$$\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\\
\frac{\partial^2 z}{\partial\theta\partial r} &= \frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)\\
&= \frac{\partial u}{\partial x}(-\sin\theta) + \frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\right)\cos\theta + \frac{\partial u}{\partial y}\,\cos\theta + \frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial y}\right)\sin\theta
\end{aligned}$$

又

$$\begin{aligned}
\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial\theta} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial\theta} = \frac{\partial^2 u}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 u}{\partial y\partial x}(r\cos\theta)\\
\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial y}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial\theta} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial\theta} = \frac{\partial^2 u}{\partial x\partial y}(-r\sin\theta) + \frac{\partial^2 u}{\partial y^2}(r\cos\theta)
\end{aligned}$$

故

$$
\begin{aligned}
\frac{\partial^2 z}{\partial\theta\partial r} &= \frac{\partial u}{\partial x}(-\sin\theta) + \frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\right)\cos\theta + \frac{\partial u}{\partial y}\cos\theta + \frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial y}\right)\sin\theta \\
&= \frac{\partial u}{\partial x}(-\sin\theta) + \left(\frac{\partial^2 u}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 u}{\partial y\partial x}(r\cos\theta)\right)\cos\theta \\
&\quad + \frac{\partial u}{\partial y}\cos\theta + \left(\frac{\partial^2 u}{\partial x\partial y}(-r\sin\theta) + \frac{\partial^2 u}{\partial y^2}(r\cos\theta)\right)\sin\theta \\
&= \frac{\partial u}{\partial y}\cos\theta - \frac{\partial u}{\partial x}\sin\theta + r\sin\theta\cos\theta\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2}\right) + r(\cos^2\theta - \sin^2\theta)\frac{\partial^2 u}{\partial x\partial y}
\end{aligned}
$$

- 求 $\dfrac{\partial^2 z}{\partial r\partial\theta}$: 由鏈鎖律

$$
\begin{aligned}
\frac{\partial z}{\partial\theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial\theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) \\
\frac{\partial^2 z}{\partial r\partial\theta} &= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)\right) \\
&= \frac{\partial u}{\partial x}(-\sin\theta) + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)(-r\sin\theta) + \frac{\partial u}{\partial y}\cos\theta + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right)(r\cos\theta)
\end{aligned}
$$

又

$$
\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 u}{\partial x^2}\cos\theta + \frac{\partial^2 u}{\partial y\partial x}\sin\theta \\
\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 u}{\partial x\partial y}\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin\theta
\end{aligned}
$$

故

$$
\begin{aligned}
\frac{\partial^2 z}{\partial r\partial\theta} &= \frac{\partial u}{\partial x}(-\sin\theta) + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)(-r\sin\theta) + \frac{\partial u}{\partial y}\cos\theta + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right)(r\cos\theta) \\
&= \frac{\partial u}{\partial x}(-\sin\theta) + \left(\frac{\partial^2 u}{\partial x^2}\cos\theta + \frac{\partial^2 u}{\partial y\partial x}\sin\theta\right)(-r\sin\theta) \\
&\quad + \frac{\partial u}{\partial y}\cos\theta + \left(\frac{\partial^2 u}{\partial x\partial y}\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin\theta\right)(r\cos\theta) \\
&= \frac{\partial u}{\partial y}\cos\theta - \frac{\partial u}{\partial x}\sin\theta + r\sin\theta\cos\theta\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2}\right) + r(\cos^2\theta - \sin^2\theta)\frac{\partial^2 u}{\partial x\partial y}
\end{aligned}
$$

例. 若 $z = u(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, 證明 $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial z}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 z}{\partial\theta^2}$.

解. 令 $z = U(r,\theta) = u(x(r,\theta),\, y(r,\theta))$, 由鏈鎖律

$$\begin{aligned}
\frac{\partial U}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\
\frac{\partial^2 U}{\partial r^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)\frac{\partial y}{\partial r} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)\cos\theta + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)\sin\theta \\
&= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 u}{\partial x\partial y}\sin\theta\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta \\
\frac{\partial U}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) \\
\frac{\partial^2 U}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)\right) \\
&= \frac{\partial u}{\partial x}(-r\cos\theta) - r\sin\theta\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial u}{\partial y}(-r\sin\theta) + r\cos\theta\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial u}{\partial y}(-r\sin\theta) - r\sin\theta\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right) + r\cos\theta\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial u}{\partial y}(-r\sin\theta) - r\sin\theta\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y\partial x}\frac{\partial y}{\partial \theta}\right) + r\cos\theta\left(\frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \theta}\right) \\
&= \frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial u}{\partial y}(-r\sin\theta) - r\sin\theta\left(\frac{\partial^2 u}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 u}{\partial y\partial x}(r\cos\theta)\right) \\
&\qquad + r\cos\theta\left(\frac{\partial^2 u}{\partial x\partial y}(-r\sin\theta) + \frac{\partial^2 u}{\partial y^2}(r\cos\theta)\right) \\
&= \frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial u}{\partial y}(-r\sin\theta) + r^2\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2r^2\sin\theta\cos\theta\frac{\partial^2 u}{\partial y\partial x} + r^2\cos^2\theta\frac{\partial^2 u}{\partial y^2}
\end{aligned}$$

由上,

$$\begin{aligned}
\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} &= \left(\frac{\partial^2 u}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 u}{\partial x\partial y}\sin\theta\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta\right) + \frac{1}{r}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) \\
&\quad + \frac{1}{r^2}\left(\frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial u}{\partial y}(-r\sin\theta) + r^2\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2r^2\sin\theta\cos\theta\frac{\partial^2 u}{\partial y\partial x} + r^2\cos^2\theta\frac{\partial^2 u}{\partial y^2}\right) \\
&= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}
\end{aligned}$$

5.5 方向導數與梯度

以下 $S \subseteq \mathbb{R}^n$, 函數 $f : S \to \mathbb{R}$ 可微, $\mathbf{c} \in S$. $\mathbb{R}^n$ 中直角座標系單位向量記為 $\mathbf{e}_1,\, \mathbf{e}_2,\, \ldots,\, \mathbf{e}_n$.

定義 (方向導數, 梯度)**.**

- 給定 $\mathbf{u} \in \mathbb{R}^n$, f 在 $\mathbf{c}$ 之 $\mathbf{u}$ 方向導數 (directional derivative) 為 $D_{\mathbf{u}}f(\mathbf{c}) = \lim_{h\to 0}\dfrac{f(\mathbf{c}+h\mathbf{u}) - f(\mathbf{c})}{h}$.
- f 之偏導數 $f_i(\mathbf{c}) = D_{\mathbf{e}_i}f(\mathbf{c})$, $i = 1,\, 2,\, \ldots,\, n$.
- f 在 $\mathbf{c}$ 之梯度 (gradient) 為 $\nabla f(\mathbf{c}) = \langle f_1(\mathbf{c}),\, f_2(\mathbf{c}),\, \ldots,\, f_n(\mathbf{c})\rangle$.

性質. $D_{\mathbf{u}}f(\mathbf{c}) = \nabla f(\mathbf{c}) \cdot \mathbf{u}$.

證. 令 $g(x) = f(\mathbf{c}+x\mathbf{u}) = f(v_1+xu_1,\, v_2+xu_2,\, \ldots,\, v_n+xu_n)$, $D_{\mathbf{u}}f(\mathbf{c}) = \lim\limits_{h\to 0}\dfrac{f(\mathbf{c}+h\mathbf{u})-f(\mathbf{c})}{h} = \lim\limits_{h\to 0}\dfrac{g(h)-g(0)}{h} = g'(x)\big|_{x=0}$. 由鏈鎖律, $g'(x)\big|_{x=0} = \displaystyle\sum_{i=1}^{n} f_i(\mathbf{c}+x\mathbf{u})\,\frac{\mathrm{d}(v_i+xu_i)}{\mathrm{d}x}\bigg|_{x=0} = \sum_{i=1}^{n} f_i(\mathbf{c}+x\mathbf{u})\,u_i\bigg|_{x=0} = \nabla f(\mathbf{c})\cdot\mathbf{u}$.

性質. 給定 $\mathbb{R}^n$ 曲面 $G(\mathbf{x})=0$. 令 $\mathbf{x}_0\in\mathbb{R}^n$ 使 $G(\mathbf{x}_0)=0$ ($\mathbf{x}_0$ 位於 $G(\mathbf{x})$ 上), 則 $\nabla G(\mathbf{x}_0)$ 在 $\mathbf{x}_0$ 與 $G(\mathbf{x})$ 垂直.

證. 令 $\mathbb{R}^n$ 曲線 $\mathbf{r}(t) = \langle x_1(t),\, x_2(t),\, \ldots,\, x_n(t)\rangle$ 位於 $G(\mathbf{x})$ 上且通過 $\mathbf{x}_0$, 則 $G(\mathbf{r}(t))=0$, 且存在 $t_0\in\mathbb{R}$ 使 $\mathbf{r}(t_0)=\mathbf{x}_0$. 由鏈鎖律, $G(\mathbf{r}(t))=0$ 等式兩邊對 t 微分並代入 $t=t_0$ 得

$$\frac{\mathrm{d}G(\mathbf{r}(t))}{\mathrm{d}t}\bigg|_{t=t_0} = 0 \implies \sum_{i=1}^{n} G_i(\mathbf{r}(t))\,x_i'(t)\bigg|_{t=t_0} = 0 \implies \nabla G(\mathbf{r}(t_0))\cdot\mathbf{r}'(t_0) = 0$$

例. 求 $z = x^2+5xy-2y^2$ 在 $(1,2,3)$ 之切平面方程式.

解. $f(x,y,z) = x^2+5xy-2y^2-z=0$, 故 $\nabla f = (2x+5y)\widehat{\boldsymbol{\imath}} + (-4y+5x)\widehat{\boldsymbol{\jmath}} - \widehat{\mathbf{k}}$, $\nabla f(1,2,3) = \langle 12,\, -3,\, -1\rangle$, 切平面方程式為 $12(x-1)-3(y-2)-(z-3)=0 \implies 12x-3y-z=3$.

例. 求 $z^3+xyz-2=0$ 在 $(1,1,1)$ 之切平面方程式.

解. $f(x,y,z) = z^3+xyz-2=0$, 故 $\nabla f = yz\widehat{\boldsymbol{\imath}} + xz\widehat{\boldsymbol{\jmath}} + (3z^2+xy)\widehat{\mathbf{k}}$, $\nabla f(1,1,1) = \langle 1,\, 1,\, 4\rangle$, 切平面方程式為 $(x-1)+(y-1)+4(z-1)=0 \implies x+y+4z=6$.

5.6 極值問題

定義 (極值定義)**.** 定義 $B(\mathbf{x},h) \equiv \{\mathbf{y}\in\mathbb{R}^n \mid |\mathbf{y}-\mathbf{x}|<h\}$,

- f 在 $\mathbf{x}_\mathrm{M}\in S$ 有最大值 (global maximum) $f(\mathbf{x}_\mathrm{M})$: $f(\mathbf{x}_\mathrm{M}) \geqslant f(\mathbf{x})$, $\forall\,\mathbf{x}\in S$.
- f 在 $\mathbf{x}_\mathrm{m}\in S$ 有最小值 (global minimum) $f(\mathbf{x}_\mathrm{m})$: $f(\mathbf{x}_\mathrm{m}) \leqslant f(\mathbf{x})$, $\forall\,\mathbf{x}\in S$.
- f 在 $\mathbf{x}_0\in S$ 有極大值 (local maximum) $f(\mathbf{x}_0)$: $\exists\,h_0>0$ 使 $f(\mathbf{x}_0) \geqslant f(\mathbf{x})$, $\forall\,\mathbf{x}\in B(\mathbf{x}_0,h_0)\cap S$.
- f 在 $\mathbf{x}_1\in S$ 有極小值 (local minimum) $f(\mathbf{x}_1)$: $\exists\,h_1>0$ 使 $f(\mathbf{x}_1) \leqslant f(\mathbf{x})$, $\forall\,\mathbf{x}\in B(\mathbf{x}_1,h_1)\cap S$.

定理 (極值必要條件)**.** 若 f 在 S 之內點 $\mathbf{c}$ 有極值, 則 $\nabla f(\mathbf{c}) = \mathbf{0}$.

證. 若 $\mathbf{c} = (c_1,\, c_2,\, \ldots,\, c_n)$, 令 $g_j(t) \equiv f(c_1,\, c_2,\, \ldots,\, c_{j-1},\, t,\, c_{j+1},\, \ldots,\, c_n)$, $j=1,\,2,\,\ldots,\,n$. 因 f 在 $\mathbf{c}$ 有極值 $f(\mathbf{c}) = g_j(c_j)$, g_j 在 c_j 有極值 $\implies g_j'(t)\big|_{t=c_j} = 0 \implies D_jf(\mathbf{c}) = 0\;\forall\,j$, 故 $\nabla f(\mathbf{c}) = \mathbf{0}$.

結論. 若 f 在 $\mathbf{c}\in S$ 有極值, 則 $\mathbf{c}$ 為以下兩情形之一:

- $\mathbf{c}$ 為 f 之臨界點 (critical point) : $\mathbf{c}$ 為 S 之內點且 $\nabla f(\mathbf{c}) = \mathbf{0}$.
- $\mathbf{c}$ 為 S 之邊界點 (boundary).

定義 (Hessian 矩陣)**.** 給定 S 之內點 $\mathbf{c}$,

$$\mathbf{H}(f,\mathbf{c}) = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix}, \qquad f_{ij} = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{c}), \qquad i,\, j = 1,\, 2,\, \ldots,\, n.$$

定義 (矩陣正 / 負定性)**.** 給定 $n\times n$ 實對稱矩陣 $\mathbf{A}$. 對任意 $\mathbf{v}\in\mathbb{R}^n \neq \mathbf{0}$, 若

- $\mathbf{v}\mathbf{A}\mathbf{v}^\top > 0$: $\mathbf{A}$ 正定 (positive-definite)
- $\mathbf{v}\mathbf{A}\mathbf{v}^\top < 0$: $\mathbf{A}$ 負定 (negative-definite)
- $\mathbf{v}\mathbf{A}\mathbf{v}^\top \geqslant 0$: $\mathbf{A}$ 半正定 (positive-semidefinite)
- $\mathbf{v}\mathbf{A}\mathbf{v}^\top \leqslant 0$: $\mathbf{A}$ 半負定 (negative-semidefinite)

定理 (二階導數判定法)**.** 給定 f 之臨界點 $\mathbf{c}$, 若

- $\mathbf{H}(f,\mathbf{c})$ 為正定: f 在 $\mathbf{c}$ 有極小值.
- $\mathbf{H}(f,\mathbf{c})$ 為負定: f 在 $\mathbf{c}$ 有極大值.

結論. 考慮 2×2 對稱矩陣 $\mathbf{A}=\begin{pmatrix}\alpha & \gamma\\ \gamma & \beta\end{pmatrix}$: 令 $\mathbf{v}=\langle x,\, y\rangle$, $\mathbf{v}\mathbf{A}\mathbf{v}^\top=\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}\alpha & \gamma\\ \gamma & \beta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=x\,(\alpha x+\gamma y)+y\,(\gamma x+\beta y)=\alpha x^2+2\gamma xy+\beta y^2=\alpha\left(x+\dfrac{\gamma}{\alpha}\,y\right)^2+\dfrac{\alpha\beta-\gamma^2}{\alpha}\,y^2$. 定義 $D=\det\mathbf{A}=\alpha\beta-\gamma^2$, 若

- $D>0$ 且 $\alpha>0$: $\mathbf{A}$ 為正定.
- $D>0$ 且 $\alpha<0$: $\mathbf{A}$ 為負定.

結論. 令 $S\subseteq\mathbb{R}^2$, $\mathbf{c}$ 為 $f(x,y)$ 之臨界點, $D=f_{xx}(\mathbf{c})\cdot f_{yy}(\mathbf{c})-(f_{xy}(\mathbf{c}))^2$. 若

- $D>0$ 且 $f_{xx}(\mathbf{c})>0$: f 在 $\mathbf{c}$ 有極小值.
- $D>0$ 且 $f_{xx}(\mathbf{c})<0$: f 在 $\mathbf{c}$ 有極大值.
- $D<0$: $\mathbf{c}$ 為鞍點 (saddle point).

例. 求 $f(x,y)=x^3+xy^2-3x^2-4y^2+4$ 之臨界點並予以分類.

解. 由 $f_x(x,y)=3x^2+y^2-6x$, $f_y(x,y)=2xy-8y$, 臨界點為同時滿足以上二式為零之 (x,y). 故 $\left\{3x^2+y^2-6x=0\right\}\ \wedge\ \left\{y\,(x-4)=0\right\}\implies\left\{3x^2+y^2-6x=0\ \wedge\ y=0\right\}\ \vee\ \left\{3x^2+y^2-6x=0\ \wedge\ x=4\right\}\implies\left\{3x^2-6x=0\ \wedge\ y=0\right\}\ \vee\ \left\{3\cdot 4^2+y^2-6\cdot 4=0\ \wedge\ x=4\right\}$, 臨界點為 $(0,0)$, $(2,0)$. 又 $f_{xx}=6x-6$, $f_{yy}=2x-8$, $f_{xy}=f_{yx}=2y$, 分類如下表:

臨界點	$f_{xx}f_{yy}-f_{xy}^2$	f_{xx}	分類
$(0,0)$	$(-6)\times(-8)-(0)^2>0$	-6	極大值
$(2,0)$	$6\times(-4)-0^2<0$		鞍點

例. 求 $f(x,y)=xy\,(5x+y-15)$ 之臨界點並予以分類.

解.

$$f_x(x,y)=y\,(5x+y-15)+xy\,(5)=y\,(5x+y-15)+y\,(5x)=y\,(10x+y-15)$$
$$f_y(x,y)=x\,(5x+y-15)+xy\,(1)=x\,(5x+y-15)+x\,(y)=x\,(5x+2y-15)$$

臨界點為同時滿足以上二式為零之 (x,y). 故 $\left\{y=0\ \vee\ 10x+y=15\right\}\ \wedge\ \left\{x=0\ \vee\ 5x+2y=15\right\}\implies\left\{y=0\ \wedge\ x=0\right\}\ \vee\ \left\{y=0\ \wedge\ 5x+2y=15\right\}\ \vee\ \left\{10x+y=15\ \wedge\ x=0\right\}\ \vee\ \left\{10x+y=15\ \wedge\ 5x+2y=15\right\}$, 臨界點為 $(0,0)$, $(3,0)$, $(0,15)$, $(1,5)$. 又 $f_{xx}=10y$, $f_{yy}=2x$, $f_{xy}=f_{yx}=10x+2y-15$, 分類如下表:

臨界點	$f_{xx}f_{yy}-f_{xy}^2$	f_{xx}	分類
$(0,0)$	$0\times 0-(-15)^2<0$		鞍點
$(3,0)$	$0\times 6-15^2<0$		鞍點
$(0,15)$	$150\times 0-15^2<0$		鞍點
$(1,5)$	$50\times 2-5^2>0$	50	極小值

例. 求 $f(x,y)=(x+y)\,e^{-x^2-y^2}$ 在 $S:\ x^2+y^2\leqslant 1$ 上的最大值, 最小值.

解. 由於 f 可微, 無奇異點; f 之極值發生在臨界點 ($\mathbf{c}\in S$, $\nabla f(\mathbf{c})=0$) 與 S 之邊界點 ($x^2+y^2=1$) 上.

- 由 $f_x(x,y) = e^{-x^2-y^2} + (x+y)\,e^{-x^2-y^2}\,(-2x) = (-2x^2-2xy+1)\,e^{-x^2-y^2}$, $f_y(x,y) = e^{-x^2-y^2} + (x+y)\,e^{-x^2-y^2}\,(-2y) = (-2y^2-2xy+1)\,e^{-x^2-y^2}$, 臨界點 (x,y) 滿足 $2x^2+2xy=1$ 與 $2y^2+2xy=1$, 解得 $(x,y) = \left(\frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{2}, -\frac{1}{2}\right)$.
- 邊界點 $x^2+y^2=1$: 令 $x=\cos t$, $y=\sin t$, $0 \leqslant t \leqslant 2\pi$, 則 $f(x,y)$ 變為 $g(t) \equiv (\cos t+\sin t)\,e^{-1}$; $g'(t) = (-\sin t+\cos t)\,e^{-1} = 0$ 解得 $t = \frac{\pi}{4}, \frac{5\pi}{4}$; 又邊界 $t=0,\, 2\pi$, 亦即 $(x,y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), (1,0)$.

候選點	$f(x,y)$	分類
$\left(\frac{1}{2}, \frac{1}{2}\right)$	$e^{-\frac{1}{2}}$	最大値
$\left(-\frac{1}{2}, -\frac{1}{2}\right)$	$-e^{-\frac{1}{2}}$	最小値
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\sqrt{2}\,e^{-1}$	
$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$-\sqrt{2}\,e^{-1}$	
$(1,0)$	e^{-1}	

例. 求 $f(x,y) = x^3+xy^2-3x^2-4y^2+4$ 在 $S:\ x^2+y^2 \leqslant 1$ 上的最大値, 最小値.

解. 由於 f 可微, 無奇異點; f 之極値發生在臨界點 ($\mathbf{c} \in S$, $\nabla f(\mathbf{c}) = 0$) 與 S 之邊界點 ($x^2+y^2=1$) 上.

- 由 $f_x(x,y) = 3x^2+y^2-6x$, $f_y(x,y) = 2xy-8y$, 臨界點 (x,y) 滿足 $3x^2+y^2-6x=0$ 與 $2xy-8y=0$, 解得 $(x,y) = (0,0),\ (2,0)$; $(2,0)$ 在 S 外不合.
- 邊界點 $x^2+y^2=1$: $y^2 = 1-x^2$ 代入則 $f(x,y)$ 變為 $g(x) = x^3+x\,(1-x^2)-3x^2-4\,(1-x^2)+4 = x+x^2$, $-1 \leqslant x \leqslant 1$; $g'(x) = 1+2x = 0$ 解得 $x = -\frac{1}{2}$, 亦即 $g(x)$ 之極値發生在 $x = \pm 1$ 與 $-\frac{1}{2} \implies (x,y) = \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right),\ (1,0),\ (-1,0)$.

候選點	$f(x,y)$	分類
$(0,0)$	4	最大値
$\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$	$-\frac{1}{4}$	最小値
$(1,0)$	2	
$(-1,0)$	0	

例. 求 $f(x,y) = xy-x^3y^2$ 在 $S:\ 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1$ 上的最大値, 最小値.

解. 由於 f 可微, 無奇異點; f 之極値發生在臨界點 ($\mathbf{c} \in S$, $\nabla f(\mathbf{c}) = 0$) 與 S 之邊界點上.

- 由 $f_x(x,y) = y-3x^2y^2$, $f_y(x,y) = x-2x^3y$, 臨界點 (x,y) 滿足 $y-3x^2y^2 = y\,(1-3x^2y) = 0$ 與 $x-2x^3y = x\,(1-2x^2y) = 0$, 故 $y=0 \,\vee\, 1-3x^2y=0$ 及 $x=0 \,\vee\, 1-2x^2y=0$; 解得 $(x,y) = (0,0)$.
- 邊界點由 $L_1:\ x=0 \,\wedge\, 0 \leqslant y \leqslant 1$, $L_2:\ y=0 \,\wedge\, 0 \leqslant x \leqslant 1$, $L_3:\ x=1 \,\wedge\, 0 \leqslant y \leqslant 1$, $L_4:\ y=1 \,\wedge\, 0 \leqslant x \leqslant 1$ 組成.
  - L_1: $f(x,y) = 0$.
  - L_2: $f(x,y) = 0$.
  - L_3: $x=1$, $0 \leqslant y \leqslant 1$, $f(x,y)$ 變為 $g(y) = y-y^2$, $g'(y) = 1-2y = 0$ 解得 $y = \frac{1}{2}$, 亦即 $g(y)$ 之極値發生在 $y = 0,\ 1,\ \frac{1}{2} \implies (x,y) = (1,0),\ (1,1),\ \left(1, \frac{1}{2}\right)$
  - L_4: $y=1$, $0 \leqslant x \leqslant 1$, $f(x,y)$ 變為 $h(x) = x-x^3$, $h'(x) = 1-3x^2 = 0$ 解得 $x = \pm\frac{1}{\sqrt{3}}$ (負不合), 亦即 $h(x)$ 之極値發生在 $x = 0,\ 1,\ \frac{1}{\sqrt{3}} \implies (x,y) = (0,1),\ (1,1),\ \left(\frac{1}{\sqrt{3}}, 1\right)$.

候選點	$f(x,y)$	分類
$(0, 0\leqslant y\leqslant 1)$	0	最小值
$(0\leqslant x\leqslant 1, 0)$	0	最小值
$(0,0)$	0	最小值
$(1,0)$	0	最小值
$(1,1)$	0	最小值
$\left(1,\frac{1}{2}\right)$	$\frac{1}{4}$	
$(0,1)$	0	最小值
$\left(\frac{1}{\sqrt{3}},1\right)$	$\frac{2}{3\sqrt{3}}$	最大值

例. 求 $f(x,y)=xy+2x+y$ 在由 $(0,0)$, $(1,0)$, $(0,2)$ 形成之三角形區域 S 上的最大值, 最小值.

解. 由於 f 可微, 無奇異點; f 之極值發生在臨界點 ($\mathbf{c}\in S$, $\nabla f(\mathbf{c})=0$) 與 S 之邊界點上.

- 由 $f_x(x,y)=y+2$, $f_y(x,y)=x+1$, 臨界點 (x,y) 滿足 $y+2=0$ 與 $x+1=0$, 故 $(x,y)=(-1,-2)$.
- 邊界點由 $L_1:\ x=0\ \wedge\ 0\leqslant y\leqslant 2$, $L_2:\ y=0\ \wedge\ 0\leqslant x\leqslant 1$, $L_3:\ (1,0)$ — $(0,2)$ 組成.
  - L_1: $(x,y)=(0,0)$, $(0,2)$.
  - L_2: $(x,y)=(0,0)$, $(1,0)$.
  - L_3: $y=-2x+2$, $0\leqslant x\leqslant 1$, $f(x,y)$ 變為 $g(x)=x\,(-2x+2)+2x+(-2x+2)=-2x^2+2x+2$, $g'(x)=-4x+2=0$ 解得 $x=\frac{1}{2}$, 亦即 $g(x)$ 之極值發生在 $x=0,\ 1,\ \frac{1}{2}\implies(x,y)=(0,2),\ (1,0),\ \left(\frac{1}{2},1\right)$

候選點	$f(x,y)$	分類
$(0,0)$	0	最小值
$(0,2)$	2	
$(1,0)$	2	
$\left(\frac{1}{2},1\right)$	$\frac{5}{2}$	最大值

例. 求 $f(x,y)=xy\,e^{-\frac{x^2+y^2}{2}}$ 在 $S:\ \left\{(x,y)\,|\,x^2+y^2\leqslant 4,\ x\geqslant 0,\ y\geqslant 0\right\}$ 上的最大值, 最小值.

解. 由於 f 可微, 無奇異點; f 之極值發生在臨界點 ($\mathbf{c}\in S$, $\nabla f(\mathbf{c})=0$) 與 S 之邊界點上.

- 由 $f_x(x,y)=y\,e^{-\frac{x^2+y^2}{2}}+xy\,e^{-\frac{x^2+y^2}{2}}(-x)=y(1-x^2)\,e^{-\frac{x^2+y^2}{2}}$, $f_y(x,y)=x\,e^{-\frac{x^2+y^2}{2}}+xy\,e^{-\frac{x^2+y^2}{2}}(-y)=x(1-y^2)\,e^{-\frac{x^2+y^2}{2}}$, 臨界點 (x,y) 滿足 $y\,(1-x^2)=0$ 與 $x\,(1-y^2)=0$, 解得 $(x,y)=(0,0),\ (1,1),\ (1,-1),\ (-1,1)$, 僅 $(0,0)$, $(1,1)$ 於 S 內.
- 邊界點由 $L_1:\ x=0\ \wedge\ 0\leqslant y\leqslant 2$, $L_2:\ y=0\ \wedge\ 0\leqslant x\leqslant 2$, $L_3:\ x^2+y^2=4$ 於第一象限所組成.
  - L_1: $f(x,y)=0$.
  - L_2: $f(x,y)=0$.
  - L_3: 令 $x=2\cos t$, $y=2\sin t$, $0\leqslant t\leqslant\frac{\pi}{2}$, 則 $f(x,y)$ 變為 $g(t)\equiv 4\cos t\sin t\,e^{-2}$; $g'(t)=\cos 2t\,4e^{-2}=0$ 解得 $t=\frac{\pi}{4}$; 又邊界 $t=0,\ \frac{\pi}{2}$, 亦即 $(x,y)=(\sqrt{2},\sqrt{2}),\ (2,0),\ (0,2)$.

候選點	$f(x,y)$	分類
$(0,0)$	0	最小值
$(1,1)$	e^{-1}	最大值
$(0, 0 \leqslant y \leqslant 2)$	0	最小值
$(0 \leqslant x \leqslant 2, 0)$	0	最小值
$(\sqrt{2}, \sqrt{2})$	$2e^{-2}$	
$(2,0)$	0	最小值
$(0,2)$	0	最小值

5.7 Lagrange 乘數法

定理. 給定開集 $S \subseteq \mathbb{R}^n$, 可微函數 $f : S \to \mathbb{R}$ 與 $g_j : S \to \mathbb{R}$, $j = 1, 2, \ldots, m$, $m < n$, 及 $X_0 = \{\mathbf{x} \in S \mid g_j(\mathbf{x}) = 0,\ j = 1, 2, \ldots, m\}$. 若 f 在 $\mathbf{x}_0 \in S \cap X_0$ 有極值且 $\det\left(D_i g_j(\mathbf{x}_0)\right) \neq 0$, 則

$$\exists \lambda_1, \lambda_2, \ldots, \lambda_m \quad 使 \quad D_i f(\mathbf{x}_0) + \sum_{j=1}^m \lambda_j D_i g_j(\mathbf{x}_0) = 0, \quad i = 1, 2, \ldots, n$$

註. 令 $\mathcal{L} \equiv f + \sum_{j=1}^m \lambda_j g_j$, 上述充分條件可寫作

$$\begin{aligned} D_i \mathcal{L}(\mathbf{x}_0) &= 0, \quad i = 1, 2, \ldots, n \\ g_j(\mathbf{x}_0) &= 0, \quad j = 1, 2, \ldots, m \end{aligned}$$

例. 求 $x^2 - 10x - y^2$ 在 $x^2 + 4y^2 = 16$ 上的最大值, 最小值.

解. 令 $\mathcal{L} = x^2 - 10x - y^2 + \lambda(x^2 + 4y^2 - 16)$, 則

$$\frac{\partial \mathcal{L}}{\partial x} = 2x - 10 + 2\lambda x = 0 \implies x - 5 + \lambda x = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial y} = -2y + 8\lambda y = 0 \implies -y + 4\lambda y = 0 \tag{2}$$

$$x^2 + 4y^2 - 16 = 0 \tag{3}$$

由 (2) $(1 - 4\lambda)y = 0$, 則 $y = 0 \vee \lambda = \frac{1}{4}$. 若 $y = 0$, 由 (3) $x = \pm 4$; 若 $\lambda = \frac{1}{4}$, 由 (1) $(1+\lambda)x = 5 \implies x = 4$, 代入 (3) 得 $y = 0$. 故極值點為 $(x,y) = (4,0), (-4,0)$; $x^2 - 10x - y^2$ 最大值為 56 ($(x,y) = (-4,0)$), 最小值為 -24 ($(x,y) = (4,0)$).

例. 求 $x^2 = y^2 + z^2$ 上距離 $(0,1,3)$ 最近的點.

解. 距離平方函數為 $x^2 + (y-1)^2 + (z-3)^2$, 限制式為 $x^2 - y^2 - z^2 = 0$. 令 $\mathcal{L} = x^2 + (y-1)^2 + (z-3)^2 + \lambda(x^2 - y^2 - z^2)$, 則

$$\frac{\partial \mathcal{L}}{\partial x} = 2x + 2\lambda x = 0 \implies (1+\lambda)x = 0 \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2(y-1) - 2\lambda y = 0 \implies (1-\lambda)y = 1 \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial z} = 2(z-3) - 2\lambda z = 0 \implies (1-\lambda)z = 3 \tag{6}$$

$$x^2 - y^2 - z^2 = 0 \tag{7}$$

由 (4) $(1+\lambda)x=0$, 則 $x=0 \ \vee\ \lambda=-1$. 若 $x=0$, 由 (7) $y=z=0$; 若 $\lambda=-1$, 由 (5) $y=\frac{1}{2}$, 由 (6) $z=\frac{3}{2}$, 代入 (7) 得 $x=\pm\sqrt{\frac{5}{2}}$. 故極値點為 $(x,y,z)=(0,0,0)$, $\Big(\pm\sqrt{\frac{5}{2}},\frac{1}{2},\frac{3}{2}\Big)$; 距離平方 $x^2+(y-1)^2+(z-3)^2$ 最小値為 5, 當 $(x,y,z)=\Big(\pm\sqrt{\frac{5}{2}},\frac{1}{2},\frac{3}{2}\Big)$.

例. 求 $f(x,y,z)=(x+z)\,e^y$ 在 $x^2+y^2+z^2=6$ 上的最大値, 最小値.

解. 令 $\mathcal{L}=(x+z)\,e^y+\lambda\,(x^2+y^2+z^2-6)$, 則

$$\frac{\partial\mathcal{L}}{\partial x}=e^y+2\lambda x=0 \tag{8}$$

$$\frac{\partial\mathcal{L}}{\partial y}=(x+z)\,e^y+2\lambda y=0 \tag{9}$$

$$\frac{\partial\mathcal{L}}{\partial z}=e^y+2\lambda z=0 \tag{10}$$

$$x^2+y^2+z^2-6=0 \tag{11}$$

由 (8), (10) $2\,\lambda(x-z)=0$, 則 $\lambda=0 \ \vee\ x=z$. 若 $\lambda=0$, 則由 (8) $e^y=0$ 不合, 故 $x=z$. 由 (16) $e^y=-2\lambda x$, 代入 (9) $2x\,(-2\lambda x)+2\lambda y=0 \implies y=2x^2$, 代入 (11) 得 $x^2+4x^4+x^2=6 \implies (4x^2+6)(x^2-1)=0 \implies x=\pm1$. 故極値點為 $(x,y,z)=(1,2,1)$, $(-1,2,-1)$; $(x+z)\,e^y$ 最大値為 $2e^2$ $((x,y,z)=(1,2,1))$, 最小値為 $-2e^2$ $((x,y,z)=(-1,2,-1))$.

例. 若 L 為 $z^2=x^2+y^2$ 與 $x-2z=3$ 相交的曲線, 求 L 上與原點距離最短之點與最短距離.

解. 距離平方函數為 $x^2+y^2+z^2$, 限制式為 $x^2+y^2-z^2=0$ 與 $x-2z-3=0$. 令 $\mathcal{L}=x^2+y^2+z^2+\lambda_1\,(x^2+y^2-z^2)+\lambda_2\,(x-2z-3)$, 則

$$\frac{\partial\mathcal{L}}{\partial x}=2x+2\lambda_1 x+\lambda_2=0 \implies 2\,(1+\lambda_1)x+\lambda_2=0 \tag{12}$$

$$\frac{\partial\mathcal{L}}{\partial y}=2y+2\lambda_1 y=0 \implies (1+\lambda_1)y=0 \tag{13}$$

$$\frac{\partial\mathcal{L}}{\partial z}=2z-2\lambda_1 z-2\lambda_2=0 \implies (1-\lambda_1)z-\lambda_2=0 \tag{14}$$

$$x^2+y^2-z^2=0 \tag{15}$$

$$x-2z-3=0 \tag{16}$$

由 (13) $(1+\lambda_1)y=0$, 則 $y=0 \ \vee\ \lambda_1=-1$. 若 $y=0$, 由 (15) $x^2=z^2 \implies x=\pm z$. 若 $x=z$, 由 (16) $x=z=-3$. 若 $x=-z$, 由 (16) $x=1$, $z=-1$; 若 $\lambda_1=-1$, 由 (12) $\lambda_2=0$, 由 (14) $z=0$, 代入 (15) 得 $x=y=0$, 與 (16) 不合. 故極値點為 $(x,y,z)=(-3,0,-3)$, $(1,0,-1)$; 距離平方 $x^2+y^2+z^2$ 最小値為 2 (最短距離為 $\sqrt{2}$) , 當 $(x,y,z)=(1,0,-1)$.

例. 求 $\displaystyle\sum_{k=1}^{n} x_k\,y_k$ 在 $\displaystyle\sum_{k=1}^{n} x_k^2=1$ 與 $\displaystyle\sum_{k=1}^{n} y_k^2=1$ 下之最大值。

解. 目標函數為 $\displaystyle\sum_{k=1}^{n} x_k\,y_k$, 限制式為 $\displaystyle\sum_{k=1}^{n} x_k^2=1$, $\displaystyle\sum_{k=1}^{n} y_k^2=1$; 令 $\mathcal{L}=\displaystyle\sum_{k=1}^{n} x_k\,y_k+\lambda_1\Big(\sum_{k=1}^{n} x_k^2-1\Big)+\lambda_2\Big(\sum_{k=1}^{n} y_k^2-1\Big)$, 則 $\forall\, i=1,\,2,\,\ldots,\,n$, $\dfrac{\partial\mathcal{L}}{\partial x_i}=y_i+2\lambda_1 x_i=0$, $\dfrac{\partial\mathcal{L}}{\partial y_i}=x_i+2\lambda_2 y_i=0$. 代入限制式得 $1=\displaystyle\sum_{k=1}^{n} x_k^2=\sum_{k=1}^{n}(-2\lambda_2 y_k)^2=4\lambda_2^2\sum_{k=1}^{n} y_k^2=4\lambda_2^2 \implies \lambda_2=\pm\frac{1}{2}$ 與 $1=\displaystyle\sum_{k=1}^{n} y_k^2=\sum_{k=1}^{n}(-2\lambda_1 x_k)^2=4\lambda_1^2\sum_{k=1}^{n} x_k^2=4\lambda_1^2 \implies \lambda_1=\pm\frac{1}{2}$. 故 $x_i=y_i$ 或 $x_i=-y_i$, $i=1,\,2,\,\ldots,\,n \implies \displaystyle\sum_{k=1}^{n} x_k\,y_k$ 之最大値為 1.

例.

- 求 $\sum_{k=1}^{n} x_k\, y_k$ 在 $\sum_{k=1}^{n} x_k^2 = 1$ 與 $\sum_{k=1}^{n} y_k^2 = 1$ 下之最大值.
- 證明 Cauchy 不等式: $\sum_{k=1}^{n} a_k b_k \leqslant \Big(\sum_{k=1}^{n} a_k^2\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{n} b_k^2\Big)^{\frac{1}{2}}$.

解.

- 目標函數為 $\sum_{k=1}^{n} x_k\, y_k$, 限制式為 $\sum_{k=1}^{n} x_k^2 = 1$, $\sum_{k=1}^{n} y_k^2 = 1$; 令 $\mathcal{L} = \sum_{k=1}^{n} x_k\, y_k + \lambda_1\Big(\sum_{k=1}^{n} x_k^2 - 1\Big) + \lambda_2\Big(\sum_{k=1}^{n} y_k^2 - 1\Big)$, 則 $\forall\, i = 1, 2, \ldots, n$, $\frac{\partial\mathcal{L}}{\partial x_i} = y_i + 2\lambda_1 x_i = 0$, $\frac{\partial\mathcal{L}}{\partial y_i} = x_i + 2\lambda_2 y_i = 0$. 代入限制式得 $1 = \sum_{k=1}^{n} x_k^2 = \sum_{k=1}^{n}(-2\lambda_2 y_k)^2 = 4\lambda_2^2\sum_{k=1}^{n} y_k^2 = 4\lambda_2^2 \implies \lambda_2 = \pm\frac{1}{2}$ 與 $1 = \sum_{k=1}^{n} y_k^2 = \sum_{k=1}^{n}(-2\lambda_1 x_k)^2 = 4\lambda_1^2\sum_{k=1}^{n} x_k^2 = 4\lambda_1^2 \implies \lambda_1 = \pm\frac{1}{2}$. 故 $x_i = y_i$ 或 $x_i = -y_i$, $\forall\, i = 1, 2, \ldots, n \implies \sum_{k=1}^{n} x_k\, y_k$ 之最大值為 $\sum_{k=1}^{n} x_k^2 = 1$.
- WLOG 令 $x_i = \frac{a_i}{\big(\sum_{k=1}^n a_k^2\big)^{\frac{1}{2}}}$, $y_i = \frac{b_i}{\big(\sum_{k=1}^n b_k^2\big)^{\frac{1}{2}}}$, $i = 1, 2, \ldots, n$, 則 $\sum_{k=1}^{n} x_k^2 = 1$, $\sum_{k=1}^{n} y_k^2 = 1$, 故由上結果 $\sum_{k=1}^{n} x_k\, y_k = \sum_{i=1}^{n} \frac{a_i}{\big(\sum_{k=1}^n a_k^2\big)^{\frac{1}{2}}}\frac{b_i}{\big(\sum_{k=1}^n b_k^2\big)^{\frac{1}{2}}} \leqslant 1 \implies \sum_{k=1}^{n} a_k b_k = \sum_{i=1}^{n} a_i b_i \leqslant \Big(\sum_{k=1}^{n} a_k^2\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{n} b_k^2\Big)^{\frac{1}{2}}$.

例. 求 $\Big(\prod_{k=1}^{n} x_k\Big)^{\frac{1}{n}}$ 在 $\sum_{k=1}^{n} x_k = c$ 且 $c > 0$, $x_i > 0$, $i = 1, 2, \ldots, n$ 下之最大值。

解. 目標函數為 $\Big(\prod_{k=1}^{n} x_k\Big)^{\frac{1}{n}}$, 限制式為 $\sum_{k=1}^{n} x_k - c = 0$; 令 $\mathcal{L} = \Big(\prod_{k=1}^{n} x_k\Big)^{\frac{1}{n}} + \lambda\Big(\sum_{k=1}^{n} x_k - c\Big)$, 則 $\forall\, i = 1, 2, \ldots, n$, $\frac{\partial\mathcal{L}}{\partial x_i} = \frac{1}{n}\Big(\prod_{k=1}^{n} x_k\Big)^{\frac{1}{n}-1}\cdot\frac{\prod_{k=1}^n x_k}{x_i} + \lambda = 0 \implies \Big(\prod_{k=1}^{n} x_k\Big)^{\frac{1}{n}} + n\,\lambda\, x_i = 0 \implies x_1 = x_2 = \cdots = x_n$. 代入限制式, 則 $x_1 = x_2 = \cdots = x_n = \frac{c}{n}$, $\Big(\prod_{k=1}^{n} x_k\Big)^{\frac{1}{n}}$ 最大值為 $\frac{c}{n}$.

例. 一等腰三角形與長方形合併成之五邊形邊長為 L, 求面積最大之各邊長度.

解. 令 $\mathcal{L} = 2xy + y^2\tan\theta + \lambda\,(L - 2x - 2y - 2y\sec\theta)$, 則

$$\frac{\partial\mathcal{L}}{\partial x} = 2y - 2\lambda = 0 \implies y = \lambda \tag{17}$$

$$\frac{\partial\mathcal{L}}{\partial y} = 2x + 2y\tan\theta - 2\lambda - 2\lambda\sec\theta = 0 \tag{18}$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = y^2\sec^2\theta - 2y\lambda\sec\theta\tan\theta = 0 \tag{19}$$

$$2x + 2y + 2y\sec\theta = L \tag{20}$$

(17) 代入 (19) 得 $\lambda^2\sec\theta(\sec\theta - 2\tan\theta) = 0$; $\lambda \neq 0$ 且 $\sec\theta \neq 0 \implies 1 - 2\sin\theta = 0 \implies \theta = \frac{\pi}{6}$. 代入 (18), (20) 分別得 $2x + 2y\frac{1}{\sqrt{3}} - 2y - 2y\frac{2}{\sqrt{3}} = 0$, $2x + 2y + 2y\frac{2}{\sqrt{3}} = L$, 解得 $x = \frac{3-\sqrt{3}}{6}L$, $y = \frac{2-\sqrt{3}}{2}L$.