Chapter 5 Partial Differentiation

5.0 Classification of Calculus Functions

1. Single-variable: $\mathbb{R} \to \mathbb{R}$
2. Vector-valued: $\mathbb{R} \to \mathbb{R}^n$, $n > 1$ (5.2)
3. Multivariable: $\mathbb{R}^n \to \mathbb{R}$, $n > 1$ (5.3)
4. Multivariable vector-valued: $\mathbb{R}^n \to \mathbb{R}^m$, m, $n > 1$

5.1 Space Vectors

Definition (Notation)**.**

- Vector: $\mathbf{a}$, $\mathbf{x}$
- Component form: $\mathbf{a} = \langle a_1,\, a_2,\, a_3 \rangle$, $a_1,\, a_2,\, a_3 \in \mathbb{R}$; $\mathbf{a} \in \mathbb{R}^3$.
- Vector magnitude: $|\mathbf{a}| = |\, \langle a_1,\, a_2,\, a_3 \rangle \,| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
- Unit vectors in three-dimensional Cartesian coordinates: $\widehat{\boldsymbol{\imath}} = \langle 1,\, 0,\, 0 \rangle$, $\widehat{\boldsymbol{\jmath}} = \langle 0,\, 1,\, 0 \rangle$, $\widehat{\mathbf{k}} = \langle 0,\, 0,\, 1 \rangle$
- $\mathbf{a} = \langle a_1,\, a_2,\, a_3 \rangle \equiv a_1 \widehat{\boldsymbol{\imath}} + a_2 \widehat{\boldsymbol{\jmath}} + a_3 \widehat{\mathbf{k}}$

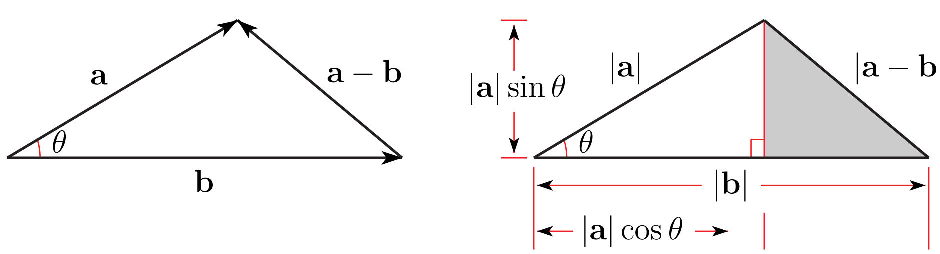
Definition (Inner/Dot Product)**.** Given n-dimensional vectors ($n \geqslant 2$) $\mathbf{a} = \langle a_1,\, a_2,\, \ldots,\, a_n \rangle$, $\mathbf{b} = \langle b_1,\, b_2,\, \ldots,\, b_n \rangle$, the inner/dot product of $\mathbf{a}$ and $\mathbf{b}$ is defined as $\mathbf{a} \cdot \mathbf{b} = \displaystyle\sum_{i=1}^{n} a_i\, b_i$.

Property. Given $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^n$, $n \geqslant 2$, $s \in \mathbb{R}$. Then

- $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\, |\mathbf{b}|\, \cos\theta$, where $0 \leqslant \theta \leqslant \pi$ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

And

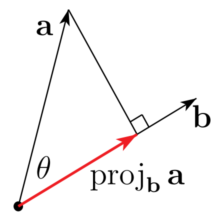
1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$
4. $\mathbf{0} \cdot a = 0$
5. $\mathbf{a} \cdot \mathbf{b} = 0 \iff \mathbf{a} = \mathbf{0} \,\vee\, \mathbf{b} = \mathbf{0} \,\vee\, \mathbf{a} \perp \mathbf{b}$
6. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$, $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$

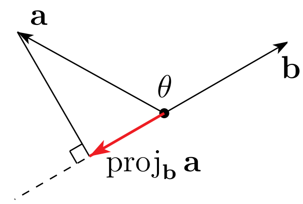


Proof. From the above figure, $|\mathbf{a} - \mathbf{b}|^2 = \big(|\mathbf{b}| - |\mathbf{a}|\, \cos\theta\big)^2 + \big(|\mathbf{a}| \sin\theta\big)^2 = |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}|\, \cos\theta + |\mathbf{a}|^2 \cos^2\theta + |\mathbf{a}|^2 \sin^2\theta = |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}|\, \cos\theta + |\mathbf{a}|^2$, and $|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,\mathbf{a} \cdot \mathbf{b} = |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}|\, \cos\theta + |\mathbf{a}|^2$, therefore $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\, \cos\theta$.

Definition (Projection)**.** Given vectors $\mathbf{a}$, $\mathbf{b}$, the projection of $\mathbf{a}$ onto $\mathbf{b}$ (denoted as $\operatorname{proj}_{\mathbf{b}} \mathbf{a}$) is defined as $\operatorname{proj}_{\mathbf{b}} \mathbf{a} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\, \mathbf{b}$.

Definition (Outer/Cross Product)**.** Given three-dimensional vectors $\mathbf{a} = \langle a_1,\, a_2,\, a_3 \rangle$, $\mathbf{b} = \langle b_1,\, b_2,\, b_3 \rangle$, the outer/cross product of $\mathbf{a}$ and $\mathbf{b}$ is defined as $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \widehat{\boldsymbol{\imath}} & \widehat{\boldsymbol{\jmath}} & \widehat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2 b_3 - a_3 b_2,\, a_3 b_1 - a_1 b_3,\, a_1 b_2 - a_2 b_1 \rangle$.



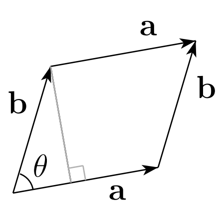


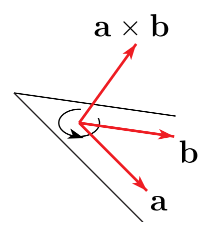
Property. For $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$, $s \in \mathbb{R}$,

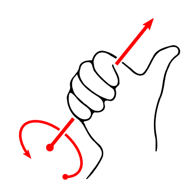
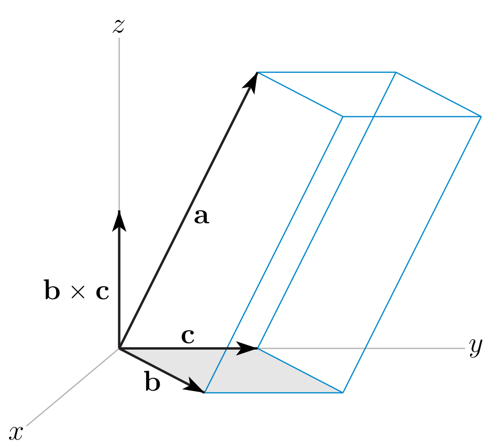
- $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta$, where $0 \leqslant \theta \leqslant \pi$ is the angle between $\mathbf{a}$ and $\mathbf{b}$; $|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$.
- $\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta\,\widehat{\mathbf{n}}$, where $0 \leqslant \theta \leqslant \pi$ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $(\mathbf{a},\, \mathbf{b},\, \widehat{\mathbf{n}})$ satisfies the right-hand rule and $|\widehat{\mathbf{n}}| = 1$, $\widehat{\mathbf{n}} \perp \mathbf{a}$, $\widehat{\mathbf{n}} \perp \mathbf{b}$.
- $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ is the volume of the parallelepiped formed by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

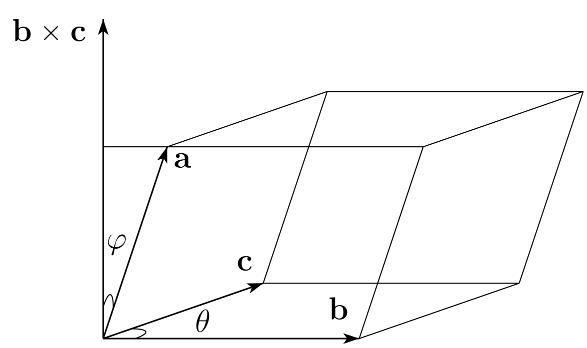
And

1. $\mathbf{a} \times \mathbf{b} \perp \mathbf{a}$, $\mathbf{a} \times \mathbf{b} \perp \mathbf{b}$
2. $\widehat{\boldsymbol{\imath}} \times \widehat{\boldsymbol{\jmath}} = \widehat{\mathbf{k}}$, $\widehat{\boldsymbol{\jmath}} \times \widehat{\mathbf{k}} = \widehat{\boldsymbol{\imath}}$, $\widehat{\mathbf{k}} \times \widehat{\boldsymbol{\imath}} = \widehat{\boldsymbol{\jmath}}$
3. $\mathbf{a} \times \mathbf{b} = 0 \iff \mathbf{a} = \mathbf{0} \;\vee\; \mathbf{b} = \mathbf{0} \;\vee\; \mathbf{a} \parallel \mathbf{b}$
4. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
5. $(s\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (s\mathbf{b}) = s(\mathbf{a} \times \mathbf{b})$
6. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
7. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
8. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$
9. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b})$ (baccab rule)







Proof.

- $|\mathbf{a} \times \mathbf{b}|^2 = \big(\mathbf{a} \times \mathbf{b}\big) \cdot \big(\mathbf{a} \times \mathbf{b}\big) = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 = a_2^2b_3^2 - 2a_2b_3a_3b_2 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_3b_1a_1b_3 + a_1^2b_3^2 + a_1^2b_2^2 - 2a_1b_2a_2b_1 + a_2^2b_1^2$, while $|\mathbf{a}|^2\,|\mathbf{b}|^2 \sin^2\theta = |\mathbf{a}|^2\,|\mathbf{b}|^2(1 - \cos^2\theta) = |\mathbf{a}|^2\,|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 = \big(a_1^2 + a_2^2 + a_3^2\big)\big(b_1^2 + b_2^2 + b_3^2\big) - \big(a_1b_1 + a_2b_2 + a_3b_3\big)^2 = a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 - \big(2a_1b_1a_2b_2 + 2a_1b_1a_3b_3 + 2a_2b_2a_3b_3\big)$, therefore $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$.
- $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) = 0$,
  $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = b_1(a_2b_3 - a_3b_2) + b_2(a_3b_1 - a_1b_3) + b_3(a_1b_2 - a_2b_1) = 0$

- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \langle a_1, a_2, a_3 \rangle \cdot \langle b_2c_3 - b_3c_2\,,\, b_3c_1 - b_1c_3\,,\, b_1c_2 - b_2c_1 \rangle = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$, and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \langle a_2b_3 - a_3b_2\,,\, a_3b_1 - a_1b_3\,,\, a_1b_2 - a_2b_1 \rangle \cdot \langle c_1, c_2, c_3 \rangle = a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_1b_3c_2 + a_1b_2c_3 - a_2b_1c_3$.

  Alternative proof: $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \langle a_1, a_2, a_3 \rangle \cdot \det \begin{vmatrix} \widehat{\imath} & \widehat{\jmath} & \widehat{\mathbf{k}} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \det \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \det \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \det \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$ $= \det \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, while $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \det \begin{vmatrix} \widehat{\imath} & \widehat{\jmath} & \widehat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot \langle c_1, c_2, c_3 \rangle = c_1 \det \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \det \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \det \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \det \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$, therefore $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.

- $\mathbf{b} \times \mathbf{c} = (b_2c_3 - b_3c_2)\widehat{\imath} - (b_1c_3 - b_3c_1)\widehat{\jmath} + (b_1c_2 - b_2c_1)\widehat{\mathbf{k}}$, thus $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ $= \det \begin{vmatrix} \widehat{\imath} & \widehat{\jmath} & \widehat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & -b_1c_3 + b_3c_1 & b_1c_2 - b_2c_1 \end{vmatrix} = \widehat{\imath}\left(a_2(b_1c_2 - b_2c_1) - a_3(-b_1c_3 + b_3c_1)\right) - \widehat{\jmath}\left(a_1(b_1c_2 - b_2c_1) - a_3(b_2c_3 - b_3c_2)\right) + \widehat{\mathbf{k}}\left(a_1(-b_1c_3 + b_3c_1) - a_2(b_2c_3 - b_3c_2)\right)$. While $\mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}) = (a_1c_1 + a_2c_2 + a_3c_3)(b_1\widehat{\imath} + b_2\widehat{\jmath} + b_3\widehat{\mathbf{k}}) - (a_1b_1 + a_2b_2 + a_3b_3)(c_1\widehat{\imath} + c_2\widehat{\jmath} + c_3\widehat{\mathbf{k}}) = \widehat{\imath}\left(a_1b_1c_1 + a_2b_1c_2 + a_3b_1c_3 - a_1b_1c_1 - a_2b_2c_1 - a_3b_3c_1\right) + \widehat{\jmath}\left(a_1b_2c_1 + a_2b_2c_2 + a_3b_2c_3 - a_1b_1c_2 - a_2b_2c_2 - a_3b_3c_2\right) + \widehat{\mathbf{k}}\left(a_1b_3c_1 + a_2b_3c_2 + a_3b_3c_3 - a_1b_1c_3 - a_2b_2c_3 - a_3b_3c_3\right) = \widehat{\imath}\,(a_2b_1c_2 + a_3b_1c_3 - a_2b_2c_1 - a_3b_3c_1) + \widehat{\jmath}\,(a_1b_2c_1 + a_3b_2c_3 - a_1b_1c_2 - a_3b_3c_2) + \widehat{\mathbf{k}}\,(a_1b_3c_1 + a_2b_3c_2 - a_1b_1c_3 - a_2b_2c_3)$, therefore $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b})$.

Ex. For $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in \mathbb{R}^3$,

1. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$
2. $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})\,(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})\,(\mathbf{b} \cdot \mathbf{c})$
3. $(\mathbf{a} \times \mathbf{b}) \cdot ((\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})) = (\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}))^2$

Sol.

1. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b}) + \mathbf{c}\,(\mathbf{b} \cdot \mathbf{a}) - \mathbf{a}\,(\mathbf{b} \cdot \mathbf{c}) + \mathbf{a}\,(\mathbf{c} \cdot \mathbf{b}) - \mathbf{b}\,(\mathbf{c} \cdot \mathbf{a}) = 0$
2. $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \mathbf{c} \cdot (\mathbf{d} \times (\mathbf{a} \times \mathbf{b})) = \mathbf{c} \cdot (\mathbf{a}\,(\mathbf{d} \cdot \mathbf{b}) - \mathbf{b}\,(\mathbf{d} \cdot \mathbf{a})) = (\mathbf{c} \cdot \mathbf{a})\,(\mathbf{d} \cdot \mathbf{b}) - (\mathbf{c} \cdot \mathbf{b})\,(\mathbf{d} \cdot \mathbf{a}) = (\mathbf{a} \cdot \mathbf{c})\,(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})\,(\mathbf{b} \cdot \mathbf{c})$
3. $(\mathbf{a} \times \mathbf{b}) \cdot ((\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})) = (\mathbf{b} \times \mathbf{c}) \cdot ((\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b})) = (\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a}\,((\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}) - \mathbf{b}\,((\mathbf{c} \times \mathbf{a}) \cdot \mathbf{a})) = (\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a}\,((\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b})) = (\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}))^2$

Property (Common Formulas)**.**

- The distance between point $p = (p_1,\, p_2,\, p_3)$ and plane $ax + by + cz + d = 0$ is $\dfrac{|ap_1 + bp_2 + cp_3 + d|}{\sqrt{a^2 + b^2 + c^2}}$.
- If $\mathbf{a} = \langle a_1,\, a_2,\, a_3 \rangle$, $\mathbf{b} = \langle b_1,\, b_2,\, b_3 \rangle$, $\mathbf{c} = \langle c_1,\, c_2,\, c_3 \rangle$, $\mathbf{d} = \langle d_1,\, d_2,\, d_3 \rangle$, the distance between two lines $\langle a_1 + b_1\, s,\, a_2 + b_2\, s,\, a_3 + b_3\, s \rangle$, $\langle c_1 + d_1\, t,\, c_2 + d_2\, t,\, c_3 + d_3\, t \rangle$, $s,\ t \in \mathbb{R}$ in three-dimensional space is $\dfrac{|(\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{d})|}{|\mathbf{b} \times \mathbf{d}|}$.

Sol.

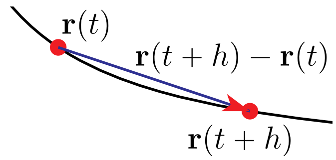
- The normal vector of plane $S:\ ax + by + cz + d = 0$ is $\mathbf{n} = \langle a,\, b,\, c \rangle$; the vector formed by point $p = (p_1,\, p_2,\, p_3)$ and its projection onto plane S, point $o = (x,\, y,\, z)$, is parallel to $\mathbf{n}$, so $(x,\, y,\, z) = (p_1 + a\, t,\, p_2 + b\, t,\, p_3 + c\, t)$, where $t \in \mathbb{R}$ is a constant to be determined. Since o is on plane S, $a(p_1 + a\, t) + b(p_2 + b\, t) + c(p_3 + c\, t) + d = 0 \implies t = \dfrac{-(ap_1 + bp_2 + cp_3 + d)}{a^2 + b^2 + c^2}$, the required distance $\overline{op} = |\,\langle a\, t,\, b\, t,\, c\, t \rangle\,| = \sqrt{a^2 + b^2 + c^2}\,|t| = \dfrac{|ap_1 + bp_2 + cp_3 + d|}{\sqrt{a^2 + b^2 + c^2}}$.

- Points on the two lines are $\mathbf{a}$ and $\mathbf{c}$ respectively, with direction vectors $\mathbf{b}$ and $\mathbf{d}$; $\mathbf{b}\times\mathbf{d}$ is perpendicular to both lines, so the required distance is $|\mathrm{proj}_{\mathbf{b}\times\mathbf{d}}(\mathbf{a}-\mathbf{c})| = \dfrac{|(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d})|}{|\mathbf{b}\times\mathbf{d}|}$.

5.2 Vector-Valued Functions

Definition. For a vector-valued function $\mathbf{r}(t):\mathbb{R}\to\mathbb{R}^n$, $n>1$, its derivative is

$$\mathbf{r}'(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) = \lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$$



If $\mathbf{r}(t) = x(t)\,\widehat{\boldsymbol{\imath}} + y(t)\,\widehat{\boldsymbol{\jmath}} + z(t)\,\widehat{\mathbf{k}}$, then $\mathbf{r}'(t) = x'(t)\,\widehat{\boldsymbol{\imath}} + y'(t)\,\widehat{\boldsymbol{\jmath}} + z'(t)\,\widehat{\mathbf{k}}$.

Remark. Vector-valued functions are often used in space curve expressions: (curve) $\equiv$ (position vector).

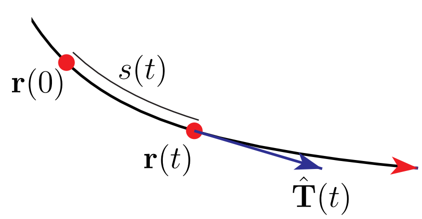
Theorem (Differentiation Rules)**.** Let $\mathbf{a}(t)$, $\mathbf{b}(t)$ be differentiable $\mathbb{R}^n$ vector-valued functions for $t\in\mathbb{R}$, α, $\beta\in\mathbb{R}$, $\gamma(t)$, $s(t)$ be differentiable real functions for $t\in\mathbb{R}$, then

1. (Linearity) $\dfrac{\mathrm{d}}{\mathrm{d}t}\big(\alpha\,\mathbf{a}(t)+\beta\,\mathbf{b}(t)\big) = \alpha\,\mathbf{a}'(t)+\beta\,\mathbf{b}'(t)$
2. (Product) $\dfrac{\mathrm{d}}{\mathrm{d}t}\big(\gamma(t)\mathbf{b}(t)\big) = \gamma'(t)\mathbf{b}(t)+\gamma(t)\mathbf{b}'(t)$
3. (Inner Product) $\dfrac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{a}(t)\cdot\mathbf{b}(t)\big) = \mathbf{a}'(t)\cdot\mathbf{b}(t)+\mathbf{a}(t)\cdot\mathbf{b}'(t)$
4. (Cross Product) $\dfrac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{a}(t)\times\mathbf{b}(t)\big) = \mathbf{a}'(t)\times\mathbf{b}(t)+\mathbf{a}(t)\times\mathbf{b}'(t)$
5. (Composition) $\dfrac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{a}\big(s(t)\big)\big) = \mathbf{a}'\big(s(t)\big)\,s'(t)$

Property. Given a curve $\mathbf{r}(t)$.

1. Let $\widehat{\mathbf{T}}(t)$ be the unit tangent vector of the curve at point $\mathbf{r}(t)$ pointing in the direction of increasing t, then $\widehat{\mathbf{T}}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, $\mathbf{r}'(t)\neq\mathbf{0}$.
2. Let $s(t)$ be the arc length of the curve between points $\mathbf{r}(0)$ and $\mathbf{r}(t)$, then

$$\frac{\mathrm{d}s}{\mathrm{d}t}(t) = \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right|$$
$$s(T)-s(T_0) = \int_{T_0}^{T}\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right|\,\mathrm{d}t$$



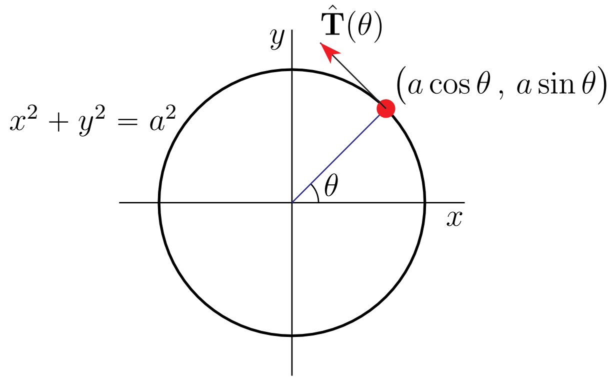
3. If arc length is used as the parameter, i.e., $t=s$ such that $\dfrac{\mathrm{d}t}{\mathrm{d}s}=\dfrac{\mathrm{d}s}{\mathrm{d}s}=1$, then $\left|\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}(s)\right|=1$, $\widehat{\mathbf{T}}(s)=\mathbf{r}'(s)$.

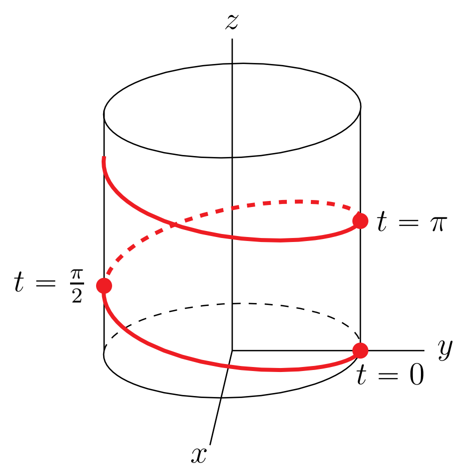
Property. Given a position vector $\mathbf{r}(t)=\langle x(t)\,,\,y(t)\,,\,z(t)\rangle$, at time t:

- Velocity $\mathbf{v}(t)=\mathbf{r}'(t)=x'(t)\,\widehat{\boldsymbol{\imath}}+y'(t)\,\widehat{\boldsymbol{\jmath}}+z'(t)\,\widehat{\mathbf{k}}$
- Speed $\dfrac{\mathrm{d}s}{\mathrm{d}t}(t)=|\mathbf{v}(t)|=|\mathbf{r}'(t)|=\sqrt{(x'(t)^2+y'(t)^2+z'(t)^2}$
- Acceleration $\mathbf{a}(t)=\mathbf{r}''(t)=\mathbf{v}'(t)=x''(t)\,\widehat{\boldsymbol{\imath}}+y''(t)\,\widehat{\boldsymbol{\jmath}}+z''(t)\,\widehat{\mathbf{k}}$

The distance traveled between times T_0 and T is $s(T)-s(T_0) = \displaystyle\int_{T_0}^{T} \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right| \mathrm{d}t = \int_{T_0}^{T} \sqrt{(x'(t)^2 + y'(t)^2 + z'(t)^2}\, \mathrm{d}t$

Ex. The curve expression for the circle $x^2 + y^2 = a^2$ is $\mathbf{r}(\theta) = \langle a\cos\theta,\, a\sin\theta\rangle$, $0 \leqslant \theta \leqslant 2\pi$. $\mathbf{r}'(\theta) = \langle -a\sin\theta,\, a\cos\theta\rangle$, $\widehat{\mathbf{T}}(\theta) = \dfrac{\mathbf{r}'(\theta)}{|\mathbf{r}'(\theta)|} = \langle -\sin\theta,\, \cos\theta\rangle$, $\dfrac{\mathrm{d}s}{\mathrm{d}\theta}(\theta) = \left|\mathbf{r}'(\theta)\right| = a$, $s(\Theta) - s(0) = \displaystyle\int_0^{\Theta} \left|\mathbf{r}'(\theta)\right| \mathrm{d}\theta = a\Theta.$





Ex (Helix Arc Length)**.** Find the arc length of $\mathbf{r}(t) = 6\sin 2t\,\widehat{\boldsymbol{\imath}} + 6\cos 2t\,\widehat{\boldsymbol{\jmath}} + 5t\,\widehat{\mathbf{k}}$ between $t = 0$ and $t = \pi$.

Sol.

- $\mathbf{r}(t) = 6\sin 2t\,\widehat{\boldsymbol{\imath}} + 6\cos 2t\,\widehat{\boldsymbol{\jmath}} + 5t\,\widehat{\mathbf{k}} \implies \mathbf{r}'(t) = 12\cos 2t\,\widehat{\boldsymbol{\imath}} - 12\sin 2t\,\widehat{\boldsymbol{\jmath}} + 5\,\widehat{\mathbf{k}}$. Then $\dfrac{\mathrm{d}s}{\mathrm{d}t}(t) = \left|\mathbf{r}'(t)\right| = \sqrt{12^2\cos^2 2t + 12^2\sin^2 2t + 5^2} = \sqrt{12^2 + 5^2} = 13$, $\widehat{\mathbf{T}}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{12}{13}\cos 2t\,\widehat{\boldsymbol{\imath}} - \dfrac{12}{13}\sin 2t\,\widehat{\boldsymbol{\jmath}} + \dfrac{5}{13}\,\widehat{\mathbf{k}}$,
  $s(\pi) - s(0) = \displaystyle\int_0^{\pi} \left|\mathbf{r}'(t)\right| \mathrm{d}t = 13\pi.$

Ex (Helix Arc Length)**.** Find the arc length of $\mathbf{r}(t) = 6\sin 2t\,\widehat{\boldsymbol{\imath}} + 6\cos 2t\,\widehat{\boldsymbol{\jmath}} + 5t\,\widehat{\mathbf{k}}$ between $t = 0$ and $t = \pi$.

Sol. $\mathbf{r}(t) = 6\sin 2t\,\widehat{\boldsymbol{\imath}} + 6\cos 2t\,\widehat{\boldsymbol{\jmath}} + 5t\,\widehat{\mathbf{k}} \implies \mathbf{r}'(t) = 12\cos 2t\,\widehat{\boldsymbol{\imath}} - 12\sin 2t\,\widehat{\boldsymbol{\jmath}} + 5\,\widehat{\mathbf{k}}$. Then $\dfrac{\mathrm{d}s}{\mathrm{d}t}(t) = \left|\mathbf{r}'(t)\right| = \sqrt{12^2\cos^2 2t + 12^2\sin^2 2t + 5^2} = \sqrt{12^2 + 5^2} = 13$, $\widehat{\mathbf{T}}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{12}{13}\cos 2t\,\widehat{\boldsymbol{\imath}} - \dfrac{12}{13}\sin 2t\,\widehat{\boldsymbol{\jmath}} + \dfrac{5}{13}\,\widehat{\mathbf{k}}$,
$s(\pi) - s(0) = \displaystyle\int_0^{\pi} \left|\mathbf{r}'(t)\right| \mathrm{d}t = 13\pi.$

Ex. Find the arc length of $\mathbf{r}(t) = \left\langle e^{3t},\, e^{-3t},\, 3\sqrt{2}\,t\right\rangle$ between $t = 0$ and $t = \dfrac{1}{3}$.

Sol. $\mathbf{r}'(t) = \left\langle 3e^{3t},\, -3e^{-3t},\, 3\sqrt{2}\right\rangle$, $s\left(\dfrac{1}{3}\right) - s(0) = \displaystyle\int_0^{\frac{1}{3}} \left|\mathbf{r}'(t)\right| \mathrm{d}t = \int_0^{\frac{1}{3}} \sqrt{9e^{6t} + 9e^{-6t} + 18}\, \mathrm{d}t = 3\int_0^{\frac{1}{3}} \sqrt{e^{6t} + e^{-6t} + 2}\, \mathrm{d}t$
$= 3\displaystyle\int_0^{\frac{1}{3}} \sqrt{(e^{3t} + e^{-3t})^2}\, \mathrm{d}t = 3\int_0^{\frac{1}{3}} (e^{3t} + e^{-3t})\, \mathrm{d}t = e^{3t} - e^{-3t}\Big|_0^{\frac{1}{3}} = e - \frac{1}{e}.$

Ex. Find the arc length of $\mathbf{r}(t) = \left\langle t,\, 2t,\, t^2\right\rangle$ between $t = 1$ and $t = 3$.

Sol. $\mathbf{r}'(t) = \langle 1,\, 2,\, 2t\rangle$, $s(3) - s(1) = \displaystyle\int_1^3 \left|\mathbf{r}'(t)\right| \mathrm{d}t = \int_1^3 \sqrt{5 + 4t^2}\, \mathrm{d}t = \frac{6\sqrt{41} - 6 - 5\ln 5 + 5\ln(\sqrt{41} + 6)}{4}.$

Use $\displaystyle\int \sqrt{x^2 + a^2}\, \mathrm{d}x = \frac{x\sqrt{x^2 + a^2} + a^2\ln\left|\sqrt{x^2 + a^2} + x\right|}{2} + c.$

5.3 Limits and Differentiation

Definition (Multivariable Function)**.** Let $U \subseteq \mathbb{R}^n$, $n > 1$, a mapping $f(x_1, x_2, \dots, x_n) : U \to \mathbb{R}$ from $U \to \mathbb{R}$ is called an n-variable function (real-valued function of n variables) on U, where U is the domain and $f(U)$ is the range.

Remark. If $f(x_1, x_2, \dots, x_n)$ is an n-variable function, f can be viewed as

- A function of n real variables $x_1, x_2, \dots, x_n$
- A function of the vector $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$
- A function of the point $(x_1, x_2, \dots, x_n)$ in $\mathbb{R}^n$

Definition (Graph, Level Curve)**.** Let $f(x, y)$ be a two-variable function defined on U.

- The set $\big\{(x, y, z) \in \mathbb{R}^3 \,|\, z = f(x, y),\ (x, y) \in U\big\}$ is called the graph of f.
- Given a constant $k \in \mathbb{R}$, the curve $f(x, y) = k$ is called a level curve (or contour curve) of f.

If $w = f(x, y, z)$ is a three-variable function, $f(x, y, z) = k$ is called a level surface of f.

Definition (Limit)**.** Let f be an n-variable function. If for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $\mathbf{x} \in \operatorname{dom} f$ satisfying

$$0 < |\mathbf{x} - \mathbf{a}| < \delta \implies |f(\mathbf{x}) - L| < \varepsilon$$

then L is called the limit of f at $\mathbf{a}$, denoted as $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$.

Property (Limit Operations)**.** Let $\mathbf{a} \in \mathbb{R}^n$, $c,\ F,\ G \in \mathbb{R}$, $D \subseteq \mathbb{R}^n$, $f,\ g : D \setminus \{\mathbf{a}\} \to \mathbb{R}$, $\gamma : \mathbb{R} \to \mathbb{R}$. If $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = F$, $\lim\limits_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = G$, $\lim\limits_{t\to F} \gamma(t) = \gamma(F)$, then

1. $\lim\limits_{\mathbf{x}\to\mathbf{a}} \big[f(\mathbf{x}) \pm g(\mathbf{x})\big] = F \pm G$
2. $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})\, g(\mathbf{x}) = FG$
3. $\lim\limits_{\mathbf{x}\to\mathbf{a}} \dfrac{f(\mathbf{x})}{g(\mathbf{x})} = \dfrac{F}{G}$ if $G \neq \mathbf{0}$
4. $\lim\limits_{\mathbf{x}\to\mathbf{a}} \gamma\big(f(\mathbf{x})\big) = \gamma(F)$

Ex. Find $\lim\limits_{(x,\,y)\to(2,\,3)} \dfrac{x + \sin y}{x^2y^2 + 1}$.

Sol. $\lim\limits_{(x,\,y)\to(2,\,3)} \big(x + \sin y\big) = \lim\limits_{(x,\,y)\to(2,\,3)} x + \lim\limits_{(x,\,y)\to(2,\,3)} \sin y = \lim\limits_{(x,\,y)\to(2,\,3)} x + \sin\left(\lim\limits_{(x,\,y)\to(2,\,3)} y\right) = 2 + \sin 3$, $\lim\limits_{(x,\,y)\to(2,\,3)} \big(x^2y^2 + 1\big) = \lim\limits_{(x,\,y)\to(2,\,3)} x^2y^2 + \lim\limits_{(x,\,y)\to(2,\,3)} 1 = \left(\lim\limits_{(x,\,y)\to(2,\,3)} x\right)\left(\lim\limits_{(x,\,y)\to(2,\,3)} x\right)\left(\lim\limits_{(x,\,y)\to(2,\,3)} y\right)\left(\lim\limits_{(x,\,y)\to(2,\,3)} y\right) + 1 = 2^2 3^2 + 1 = 37$, $\lim\limits_{(x,\,y)\to(2,\,3)} \dfrac{x + \sin y}{x^2y^2 + 1} = \dfrac{\lim\limits_{(x,\,y)\to(2,\,3)} (x + \sin y)}{\lim\limits_{(x,\,y)\to(2,\,3)} (x^2y^2 + 1)} = \dfrac{2 + \sin 3}{37}$

Remark.

- The necessary and sufficient condition for the existence of the limit $\lim\limits_{x\to a} f(x)$ of a single-variable function is that $\lim\limits_{x\to a-} f(x)$ and $\lim\limits_{x\to a+} f(x)$ both exist and are equal.
- The necessary and sufficient condition for the existence of the limit $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ of a multivariable function is that the limits along any path approaching $\mathbf{a}$ all exist and are equal.
- To find $\lim\limits_{(x,\,y)\to(0,\,0)} f(x, y)$, it's often useful to convert (x, y) to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and then let $r \to 0$.

Ex. Find $\lim\limits_{(x,\,y)\to(0,\,0)} \dfrac{x^2y}{x^2 + y^2}$.

Sol. $\dfrac{x^2y}{x^2+y^2} = \dfrac{(r\cos\theta)^2(r\sin\theta)}{r^2} = r\cos^2\theta\sin\theta$. Since $\left|r\cos^2\theta\sin\theta\right| \leqslant r \to 0$ as $r \to 0$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0$.

Ex. Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$.

Sol. From $\dfrac{x^2-y^2}{x^2+y^2} = \dfrac{(r\cos\theta)^2-(r\sin\theta)^2}{r^2} = \cos^2\theta - \sin^2\theta = \cos(2\theta)$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \text{DNE}$

Ex. Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$.

Sol.

- Let $y = mx$, $m \neq 0$, then $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4} = \lim_{x\to0} \frac{2m^2x^3}{x^2+m^4x^4} = \lim_{x\to0} \frac{2m^2x}{1+m^4x^2} = 0$.
- Let $x = y^2$, then $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4} = \lim_{y\to0} \frac{2y^4}{2y^4} = 1$.
- Conclusion: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4} = \text{DNE}$

Ex. Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$.

Sol. From $\dfrac{x^2y^2}{x^2+y^2} = \dfrac{(r\cos\theta)^2(r\sin\theta)^2}{r^2} = r^2\cos^2\theta\sin^2\theta \leqslant \dfrac{r^2}{4}$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0$.

Ex. Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3}$.

Sol.

- From $\dfrac{x^2y^2}{x^3+y^3} = \dfrac{(r\cos\theta)^2(r\sin\theta)^2}{r^3(\cos^3\theta+\sin^3\theta)} = r\dfrac{\cos^2\theta\sin^2\theta}{\cos^3\theta+\sin^3\theta}$, but $\dfrac{\cos^2\theta\sin^2\theta}{\cos^3\theta+\sin^3\theta}$ is not bounded (take $\theta = \dfrac{3\pi}{4}$ when $\cos^3\theta+\sin^3\theta = 0$), $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3} = \lim_{r\to0} r\frac{\cos^2\theta\sin^2\theta}{\cos^3\theta+\sin^3\theta} = \text{DNE}$.
- Alternative solution:
  - Let $y = mx$, $m \neq 0$, then $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3} = \lim_{x\to0} \frac{x^2m^2x^2}{x^3+m^3x^3} = \lim_{x\to0} \frac{m^2x}{1+m^3} = 0$.
  - Let $y = -xe^x$, then $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3} = \lim_{x\to0} \frac{x^2x^2e^{2x}}{x^3-x^3e^{3x}} = \lim_{x\to0} \frac{xe^{2x}}{1-e^{3x}} = \lim_{x\to0} \frac{e^{2x}(1+2x)}{3e^{3x}} = \frac{1}{3}$.
  - Conclusion: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3} = \text{DNE}$

Ex. Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4}$.

Sol.

- From $\dfrac{x^2y^2}{x^4+y^4} = \dfrac{(r\cos\theta)^2(r\sin\theta)^2}{r^4(\cos^4\theta+\sin^4\theta)} = \dfrac{\cos^2\theta\sin^2\theta}{\cos^4\theta+\sin^4\theta}$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4} = \lim_{r\to0} \frac{\cos^2\theta\sin^2\theta}{\cos^4\theta+\sin^4\theta} = \text{DNE}$.
- Alternative solution: Let $y = mx$, $m \neq 0$, then $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4} = \lim_{x\to0} \frac{x^2m^2x^2}{x^4+m^4x^4} = \lim_{x\to0} \frac{m^2}{1+m^4} = \text{DNE}$.

Ex. If $f(x,y)=\begin{cases}\frac{(2x-y)^2}{x-y}, & x\neq y\\ 0, & x=y\end{cases}$, find $\lim\limits_{(x,\,y)\to(0,\,0)} f(x,y)$.

Sol.

- Let $y=x-x^3$, $f(x,x-x^3)=\dfrac{\left(2x-x+x^3\right)^2}{x-x+x^3}=\dfrac{\left(x+x^3\right)^2}{x^3}=\dfrac{\left(1+x^2\right)^2}{x}\longrightarrow\begin{cases}+\infty, & x\to 0+\\ -\infty, & x\to 0-\end{cases}$
- Let $y=x-ax^2$, $a\neq 0$: $\lim\limits_{x\to 0} f(x,x-ax^2)=\lim\limits_{x\to 0}\dfrac{\left(2x-x+ax^2\right)^2}{x-x+ax^2}=\lim\limits_{x\to 0}\dfrac{\left(x+ax^2\right)^2}{ax^2}=\lim\limits_{x\to 0}\dfrac{\left(1+ax\right)^2}{a}=\dfrac{1}{a}$
- Conclusion: $\lim\limits_{(x,\,y)\to(0,\,0)} f(x,y)=\text{DNE}$

Definition (Partial Derivative Function, Partial Differentiation, Partial Derivative)**.**

- The x-partial derivative function of $f(x,y)$ is defined as $\dfrac{\partial f}{\partial x}(x,y)=\lim\limits_{h\to 0}\dfrac{f(x+h,y)-f(x,y)}{h}$; the y-partial derivative function of $f(x,y)$ is defined as $\dfrac{\partial f}{\partial y}(x,y)=\lim\limits_{h\to 0}\dfrac{f(x,y+h)-f(x,y)}{h}$.
- The process of finding the x-partial derivative function of $f(x,y)$ is called ”partial differentiation of $f(x,y)$ with respect to x”.
- The y-partial derivative of $f(x,y)$ at (a,b) is denoted as $\dfrac{\partial f}{\partial y}(a,b)\equiv\left.\dfrac{\partial f}{\partial y}\right|_{(a,b)}$.

Remark.

- $\dfrac{\partial f}{\partial y}(x,y)$ can also be denoted as $\dfrac{\partial f}{\partial y}$, $f_y(x,y)$, f_y, $D_y f(x,y)$, $D_y f$, $D_2 f(x,y)$, $D_2 f$.
- To find $\dfrac{\partial f}{\partial y}(x,y)$: Treat x in $f(x,y)$ as a constant, then differentiate with respect to y.
- To find $\dfrac{\partial f}{\partial y}(a,b)$: Treat x in $f(x,y)$ as a constant, differentiate with respect to y, then substitute $x=a$, $y=b$.
- The above notation / operations can be directly extended to cases where the dimension is >2.

Ex. $f(x,y)=x^3+y^2+4xy^2$, then $\dfrac{\partial f}{\partial x}=\dfrac{\partial}{\partial x}(x^3)+\dfrac{\partial}{\partial x}(y^2)+\dfrac{\partial}{\partial x}(4xy^2)=3x^2+0+4y^2\dfrac{\partial}{\partial x}(x)=3x^2+4y^2$, $\dfrac{\partial f}{\partial y}=\dfrac{\partial}{\partial y}(x^3)+\dfrac{\partial}{\partial y}(y^2)+\dfrac{\partial}{\partial y}(4xy^2)=0+2y+4x\dfrac{\partial}{\partial y}(y^2)=2y+8xy$, $\dfrac{\partial f}{\partial x}(1,\,0)=3(1)^2+4(0)^2=3$, $\dfrac{\partial f}{\partial y}(1,\,0)=2(0)+8(1)(0)=0$.

Ex. $f(x,y)=y\cos x+xe^{xy}$, $\dfrac{\partial}{\partial x}e^{yx}=ye^{yx}$, $\dfrac{\partial f}{\partial x}(x,y)=y\dfrac{\partial}{\partial x}(\cos x)+e^{xy}\dfrac{\partial}{\partial x}(x)+x\dfrac{\partial}{\partial x}\left(e^{xy}\right)=-y\sin x+e^{xy}+xye^{xy}$, $\dfrac{\partial f}{\partial y}(x,y)=\cos x\dfrac{\partial}{\partial y}(y)+x\dfrac{\partial}{\partial y}\left(e^{xy}\right)=\cos x+x^2e^{xy}$

Ex. $f(x,y,z,t)=x\sin(y+2z)+t^2e^{3y}\ln z$, then $\dfrac{\partial f}{\partial x}(x,y,z,t)=\sin(y+2z)$, $\dfrac{\partial f}{\partial y}(x,y,z,t)=x\cos(y+2z)+3t^2e^{3y}\ln z$, $\dfrac{\partial f}{\partial z}(x,y,z,t)=2x\cos(y+2z)+\dfrac{t^2e^{3y}}{z}$, $\dfrac{\partial f}{\partial t}(x,y,z,t)=2te^{3y}\ln z$.

Ex. If $f(x,y)=\begin{cases}\frac{\cos x-\cos y}{x-y} & x\neq y\\ 0 & x=y\end{cases}$

- $\forall\, x \neq y,\ f_x = \dfrac{-\sin x(x-y) - (\cos x - \cos y)}{(x-y)^2}$; we cannot use this to find $f_x(0,0)$.
- Calculate $f_x(0,0)$ by definition: $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{\cos h - 1}{h-0} - 0}{h} = \lim\limits_{h\to 0} \dfrac{\cos h - 1}{h^2} = \lim\limits_{h\to 0} \dfrac{-\sin h}{2h} = \lim\limits_{h\to 0} \dfrac{-\cos h}{2} = -\dfrac{1}{2}$.
- Calculate $f_y(0,0)$ by definition: $f_y(0,0) = \lim\limits_{h\to 0} \dfrac{f(0,0+h) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{1-\cos h}{-h} - 0}{h} = \lim\limits_{h\to 0} \dfrac{\cos h - 1}{h^2} = \lim\limits_{h\to 0} \dfrac{-\sin h}{2h} = \lim\limits_{h\to 0} \dfrac{-\cos h}{2} = -\dfrac{1}{2}$.
- $\lim\limits_{(x,\,y)\to(0,\,0)} \dfrac{\cos x - \cos y}{x-y} = \lim\limits_{(x,\,y)\to(0,\,0)} \dfrac{-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}}{x-y} = -\lim\limits_{(x,\,y)\to(0,\,0)} \sin\dfrac{x+y}{2} \lim\limits_{(x,\,y)\to(0,\,0)} \dfrac{\sin\frac{x-y}{2}}{\frac{x-y}{2}} = 0$, so $f(x,y)$ is continuous at $(0,0)$.
- $f(x,y)$ is not continuous at (a,a), $a \neq 0$: By definition $\lim\limits_{(x,\,y)\to(a,\,a)} f(x,y) = \sin a$, but $f(a,a) = 0$.

Ex. If x, y, z satisfy the equation $z^5 + y^2 e^z + e^{2x} = 0$, find $\dfrac{\partial z}{\partial x}(0,0)$.

Sol. Locally, z is a function of x and y; when $x = y = 0$, the original equation becomes $z(0,0)^5 = -1 \implies z(0,0) = -1$. Let $z \equiv z(x,y)$ substitute into the original equation and partially differentiate with respect to x to get $5\,z(x,y)^4 \dfrac{\partial z}{\partial x}(x,y) + y^2\, e^{z(x,y)} \dfrac{\partial z}{\partial x}(x,y) + 2\,e^{2x} = 0$; substitute $(x,\ y) = (0,\ 0)$ to get $5\,z(0,0)^4 \dfrac{\partial z}{\partial x}(0,0) + 2 = 0$, then from $z(0,0) = -1$, $\dfrac{\partial z}{\partial x}(0,0) = -\dfrac{2}{5\,z(0,0)^4} = -\dfrac{2}{5}$.

Ex. If x, y, z satisfy the equation $x^2 + y^2 + z^2 = 1$, prove that $x\,\dfrac{\partial z}{\partial x} + y\,\dfrac{\partial z}{\partial y} = z - \dfrac{1}{z}$.

Sol. Locally, z is a function of x and y; $x^2 + y^2 + z^2 = 1$ partially differentiated with respect to x gives $2x + 2z\,\dfrac{\partial z}{\partial x} = 0 \implies \dfrac{\partial z}{\partial x} = -\dfrac{x}{z}$; partially differentiated with respect to y gives $2y + 2z\,\dfrac{\partial z}{\partial y} = 0 \implies \dfrac{\partial z}{\partial x} = -\dfrac{y}{z}$. Therefore $x\,\dfrac{\partial z}{\partial x} + y\,\dfrac{\partial z}{\partial y} = -\dfrac{x^2+y^2}{z} = \dfrac{z^2-1}{z} = z - \dfrac{1}{z}$.

Ex. If x, y, z satisfy the equation $x\sin z - z^2 y = 1$, find $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$.

Sol. Locally, z is a function of x and y; $x\sin z - z^2 y = 1$ partially differentiated with respect to x gives $\sin z + x\cos z\,\dfrac{\partial z}{\partial x} - 2yz\,\dfrac{\partial z}{\partial x} = 0 \implies \dfrac{\partial z}{\partial x} = \dfrac{\sin z}{2yz - x\cos z}$; partially differentiated with respect to y gives $x\cos z\,\dfrac{\partial z}{\partial y} - 2z\,\dfrac{\partial z}{\partial y}\,y - z^2 = 0 \implies \dfrac{\partial z}{\partial y} = \dfrac{z^2}{x\cos z - 2yz}$.

Definition (Higher-Order Partial Derivatives)**.** Given a differentiable two-variable function $f(x,y)$,

- $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)(x,y) = \dfrac{\partial^2 f}{\partial x^2}(x,y) = f_{xx}(x,y)$
- $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)(x,y) = \dfrac{\partial^2 f}{\partial x \partial y}(x,y) = f_{yx}(x,y)$
- $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)(x,y) = \dfrac{\partial^2 f}{\partial y \partial x}(x,y) = f_{xy}(x,y)$
- $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)(x,y) = \dfrac{\partial^2 f}{\partial y^2}(x,y) = f_{yy}(x,y)$

Ex. Let $f(x,y) = e^{my}\cos(nx)$, then

- $f_x = -ne^{my}\sin(nx)$
- $f_y = me^{my}\cos(nx)$
- $f_{xx} = -n^2e^{my}\cos(nx)$
- $f_{yy} = m^2e^{my}\cos(nx)$
- $f_{yx} = -mne^{my}\sin(nx)$
- $f_{xy} = -mne^{my}\sin(nx)$

Ex. Let $f(x,y) = e^{\alpha x+\beta y}$, then

- $f_x = \alpha e^{\alpha x+\beta y}$
- $f_y = \beta e^{\alpha x+\beta y}$
- $f_{xx} = \alpha^2 e^{\alpha x+\beta y}$
- $f_{yx} = \beta\alpha e^{\alpha x+\beta y}$
- $f_{xy} = \alpha\beta e^{\alpha x+\beta y}$
- $f_{yy} = \beta^2 e^{\alpha x+\beta y}$

For integers $m, n \geqslant 0$, $\dfrac{\partial^{m+n} f}{\partial x^m \partial y^n} = \alpha^m \beta^n e^{\alpha x+\beta y}$.

Ex. Let $f(x,y) = \ln(x^2+y^2)$, then

- $f_x = \dfrac{2x}{x^2+y^2}$
- $f_y = \dfrac{2y}{x^2+y^2}$
- $f_{xx} = \dfrac{(x^2+y^2)\cdot 2 - 2x\cdot 2x}{(x^2+y^2)^2} = \dfrac{2\left(y^2-x^2\right)}{(x^2+y^2)^2}$
- $f_{yy} = \dfrac{(x^2+y^2)\cdot 2 - 2y\cdot 2y}{(x^2+y^2)^2} = \dfrac{2\left(x^2-y^2\right)}{(x^2+y^2)^2}$

$f(x,y)$ satisfies the Laplace equation $f_{xx} + f_{yy} = 0$.

Ex. Let $f(x,y) = \tan^{-1}\dfrac{y}{x}$, then

- $f_x = \dfrac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\dfrac{-y}{x^2}\right) = -\dfrac{y}{x^2+y^2}$
- $f_y = \dfrac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\dfrac{1}{x}\right) = \dfrac{x}{x^2+y^2}$
- $f_{xx} = \dfrac{y\cdot 2x}{(x^2+y^2)^2} = \dfrac{2xy}{(x^2+y^2)^2}$
- $f_{yy} = -\dfrac{x\cdot 2y}{(x^2+y^2)^2} = -\dfrac{2xy}{(x^2+y^2)^2}$

$f(x,y)$ satisfies the Laplace equation $f_{xx} + f_{yy} = 0$.

Ex. If $f(x_1,x_2,x_3,x_4) = x_1^4\,x_2^3\,x_3^2\,x_4$, then

- $\dfrac{\partial^4 f}{\partial x_1\partial x_2\partial x_3\partial x_4} = \dfrac{\partial^3}{\partial x_1\partial x_2\partial x_3}\left(x_1^4x_2^3x_3^2\right) = \dfrac{\partial^2}{\partial x_1\partial x_2}\left(2x_1^4x_2^3x_3\right) = \dfrac{\partial}{\partial x_1}\left(6x_1^4x_2^2x_3\right) = 24x_1^3x_2^2x_3$
- $\dfrac{\partial^4 f}{\partial x_4\partial x_3\partial x_2\partial x_1} = \dfrac{\partial^3}{\partial x_4\partial x_3\partial x_2}\left(4x_1^3x_2^3x_3^2x_4\right) = \dfrac{\partial^2}{\partial x_4\partial x_3}\left(12x_1^3x_2^2x_3^2x_4\right) = \dfrac{\partial}{\partial x_4}\left(24x_1^3x_2^2x_3x_4\right) = 24x_1^3x_2^2x_3$

Theorem (Clairaut's Theorem)**.** If $\dfrac{\partial^2 f}{\partial x\partial y}$ and $\dfrac{\partial^2 f}{\partial y\partial x}$ both exist and are continuous at $(x_0,\, y_0)$, then $\dfrac{\partial^2 f}{\partial x\partial y}(x_0,y_0) = \dfrac{\partial^2 f}{\partial y\partial x}(x_0,y_0)$.

5.4 Chain Rule

Theorem. If f is a differentiable function of $x_1,\, x_2,\, \ldots,\, x_n$, and each x_j is a differentiable function of $t_1,\, t_2,\, \ldots,\, t_m$, $n,\, m \geqslant 1$, then f is a differentiable function of $t_1,\, t_2,\, \ldots,\, t_m$; with auxiliary function $F(t_1,\, t_2,\, \ldots,\, t_m) \equiv f(x_1(t_1,\, t_2,\, \ldots,\, t_m),\, x_2(t_1,\, t_2,\, \ldots,\, t_m),\, \ldots,\, x_n(t_1,\, t_2,\, \ldots,\, t_m))$, we have

$$\frac{\partial F}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Ex $(n = m = 2)$**.** Auxiliary function $F(s,t) \equiv f\big(x(s,t),\, y(s,t)\big)$, then

$$\frac{\partial F}{\partial s}(s,t) = \frac{\partial f}{\partial x}\big(x(s,t),\, y(s,t)\big)\,\frac{\partial x}{\partial s}(s,t) + \frac{\partial f}{\partial y}\big(x(s,t),\, y(s,t)\big)\,\frac{\partial y}{\partial s}(s,t)$$
$$\frac{\partial F}{\partial t}(s,t) = \frac{\partial f}{\partial x}\big(x(s,t),\, y(s,t)\big)\,\frac{\partial x}{\partial t}(s,t) + \frac{\partial f}{\partial y}\big(x(s,t),\, y(s,t)\big)\,\frac{\partial y}{\partial t}(s,t)$$

Ex. If $f(x,y)=e^{xy}$, $x(s,t)=s$, $y(s,t)=\cos t$; $F(s,t)\equiv f\big(x(s,t),\,y(s,t)\big)$, find $\dfrac{\partial F}{\partial s}$.

Sol.

- $\dfrac{\partial f}{\partial x}=y\,e^{xy}=y(s,t)\,e^{x(s,t)\,y(s,t)}=\cos t\,e^{s\cos t}$, $\dfrac{\partial f}{\partial y}=x\,e^{xy}=x(s,t)\,e^{x(s,t)\,y(s,t)}=s\,e^{s\cos t}$, $\dfrac{\partial x}{\partial s}=\dfrac{\partial s}{\partial s}=1$, $\dfrac{\partial y}{\partial s}=\dfrac{\partial \cos t}{\partial s}=0$, therefore $\dfrac{\partial F}{\partial s}=\dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s}+\dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s}=\cos t\,e^{s\cos t}\cdot 1+s\;e^{s\cos t}\cdot 0=\cos t\;e^{s\cos t}$.
- Alternatively, write out $F(s,t)$ directly and partially differentiate with respect to s: $F(s,t)=f\big(x(s,t),\,y(s,t)\big)$ $e^{s\cdot\cos t}$, $\dfrac{\partial F}{\partial s}=e^{s\cdot\cos t}\;\cos t$.

Ex. If $f(x,y)=x^2-y^2$, $x(t)=\cos t$, $y(t)=\sin t$, find $\dfrac{\mathrm{d}f}{\mathrm{d}t}$.

Sol. Auxiliary function $F(t)\equiv f\big(x(t),\,y(t)\big)$, then

- $\dfrac{\partial f}{\partial x}=2x=2\cos t$, $\dfrac{\partial f}{\partial y}=-2y=-2\sin t$, $\dfrac{\mathrm{d}x}{\mathrm{d}t}=-\sin t$, $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\cos t$, therefore $\dfrac{\mathrm{d}F}{\mathrm{d}t}=\dfrac{\partial f}{\partial x}\dfrac{\mathrm{d}x}{\mathrm{d}t}+\dfrac{\partial f}{\partial y}\dfrac{\mathrm{d}y}{\mathrm{d}t}=(2\cos t)(-\sin t)+(-2\sin t)(\cos t)=-4\,\sin t\cos t$.
- Alternatively, write out $F(t)$ directly and differentiate with respect to t: $F(t)=f\big(x(t),y(t)\big)=x(t)^2-y(t)^2=\cos^2 t-\sin^2 t$, therefore $F'(t)=2(\cos t)(-\sin t)-2(\sin t)(\cos t)=-4\sin t\cos t$

Ex.

1. Let $w=xy+z$, $x=\cos t$, $y=\sin t$, $z=t$, find $\dfrac{\mathrm{d}w}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}w}{\mathrm{d}t}\Big|_{t=0}$.
2. Let $w=x+2y+z^2$, $x=\dfrac{r}{s}$, $y=r^2+\ln s$, $z=2r$, find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$.
3. Let $w=x^4y+y^2z^3$, $x=rse^t$, $y=rs^2e^{-t}$, $z=r^2s\sin t$, find $\dfrac{\partial w}{\partial s}\Big|_{(r,\,s,\,t)=(2,\,1,\,0)}$.

Sol.

1. $\dfrac{\partial w}{\partial x}=y=\sin t$, $\dfrac{\partial w}{\partial y}=x=\cos t$, $\dfrac{\partial w}{\partial z}=1$, $\dfrac{\mathrm{d}x}{\mathrm{d}t}=-\sin t$, $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\cos t$, $\dfrac{\mathrm{d}z}{\mathrm{d}t}=1$, therefore $\dfrac{\mathrm{d}w}{\mathrm{d}t}=\dfrac{\partial w}{\partial x}\dfrac{\mathrm{d}x}{\mathrm{d}t}+\dfrac{\partial w}{\partial y}\dfrac{\mathrm{d}y}{\mathrm{d}t}+\dfrac{\partial w}{\partial z}\dfrac{\mathrm{d}z}{\mathrm{d}t}=(\sin t)(-\sin t)+(\cos t)(\cos t)+(1)(1)=\cos 2t+1$, $\dfrac{\mathrm{d}w}{\mathrm{d}t}\Big|_{t=0}=1+1=2$.
2. $\dfrac{\partial w}{\partial x}=1$, $\dfrac{\partial w}{\partial y}=2$, $\dfrac{\partial w}{\partial z}=2z=4r$, $\dfrac{\partial x}{\partial r}=\dfrac{1}{s}$, $\dfrac{\partial y}{\partial r}=2r$, $\dfrac{\partial z}{\partial r}=2$, $\dfrac{\partial x}{\partial s}=-\dfrac{r}{s^2}$, $\dfrac{\partial y}{\partial s}=\dfrac{1}{s}$, $\dfrac{\partial z}{\partial s}=0$, therefore

$$\begin{aligned}\frac{\partial w}{\partial r}&=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r}=(1)\Big(\frac{1}{s}\Big)+(2)(2r)+(4r)(2)=\frac{1}{s}+12r\\ \frac{\partial w}{\partial s}&=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}=(1)\Big(-\frac{r}{s^2}\Big)+(2)\Big(\frac{1}{s}\Big)+(4r)(0)=-\frac{r}{s^2}+\frac{2}{s}\end{aligned}$$

3. $\dfrac{\partial w}{\partial x}=4x^3y$, $\dfrac{\partial w}{\partial y}=x^4+2yz^3$, $\dfrac{\partial w}{\partial z}=3y^2z^2$, $\dfrac{\partial x}{\partial s}=re^t$, $\dfrac{\partial y}{\partial s}=2rse^{-t}$, $\dfrac{\partial z}{\partial s}=r^2\sin t$. When $(r,\,s,\,t)=(2,\,1,\,0)$, $(x,\,y,\,z)=(2,\,2,\,0)$, therefore

$$\begin{aligned}\frac{\partial w}{\partial s}\Big|_{(r,\,s,\,t)=(2,\,1,\,0)}&=\Big(\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}\Big)\Big|_{(r,\,s,\,t)=(2,\,1,\,0)}\\&=(4\cdot 2^3\cdot 2)(2)+(2^4+0)(2\cdot 2\cdot 1)+(0)(0)=192\end{aligned}$$

Ex. If $z=f(x-y)$, prove that $\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}=0$.

Sol. Let $u = x-y$, then $\dfrac{\partial z}{\partial x} = \dfrac{\mathrm{d}z}{\mathrm{d}u}\dfrac{\partial u}{\partial x} = \dfrac{\mathrm{d}z}{\mathrm{d}u}(1) = \dfrac{\mathrm{d}z}{\mathrm{d}u}$, $\dfrac{\partial z}{\partial y} = \dfrac{\mathrm{d}z}{\mathrm{d}u}\dfrac{\partial u}{\partial y} = \dfrac{\mathrm{d}z}{\mathrm{d}u}(-1) = -\dfrac{\mathrm{d}z}{\mathrm{d}u}$, therefore $\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y} = 0$.

Ex. If $z = f(x, y)$, $x = s + t$, $y = s - t$, prove that $\left(\dfrac{\partial z}{\partial x}\right)^2 - \left(\dfrac{\partial z}{\partial y}\right)^2 = \dfrac{\partial z}{\partial s}\dfrac{\partial z}{\partial t}$.

Sol. $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s}+\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s} = \dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}$, $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t}+\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t} = \dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}$, therefore $\dfrac{\partial z}{\partial s}\dfrac{\partial z}{\partial t} = \left(\dfrac{\partial z}{\partial x}\right)^2-\left(\dfrac{\partial z}{\partial y}\right)^2$.

Ex. If $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, prove that $t\dfrac{\partial g}{\partial s} + s\dfrac{\partial g}{\partial t} = 0$.

Sol. Let $u(s, t) = s^2 - t^2$, $v(s, t) = t^2 - s^2$, then $g(s, t) = f(u(s, t), v(s, t))$. By the chain rule

$$\begin{aligned}
\frac{\partial g}{\partial s} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial s} = \frac{\partial f}{\partial u}\cdot(2s) + \frac{\partial f}{\partial v}\cdot(-2s) \\
\frac{\partial g}{\partial t} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} = \frac{\partial f}{\partial u}\cdot(-2t) + \frac{\partial f}{\partial v}\cdot(2t)
\end{aligned}$$

Therefore $t\dfrac{\partial g}{\partial s} + s\dfrac{\partial g}{\partial t} = t\left(\dfrac{\partial f}{\partial u}\cdot(2s) + \dfrac{\partial f}{\partial v}\cdot(-2s)\right) + s\left(\dfrac{\partial f}{\partial u}\cdot(-2t) + \dfrac{\partial f}{\partial v}\cdot(2t)\right) = 0$.

Ex. If $u = f(x, y)$, $x = e^s\cos t$, $y = e^s\sin t$, prove that $\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 = e^{-2s}\left(\left(\dfrac{\partial u}{\partial s}\right)^2 + \left(\dfrac{\partial u}{\partial t}\right)^2\right)$.

Sol. By the chain rule

$$\begin{aligned}
\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial u}{\partial x}\cdot(e^s\cos t) + \frac{\partial u}{\partial y}\cdot(e^s\sin t) \\
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial u}{\partial x}\cdot(-e^s\sin t) + \frac{\partial u}{\partial y}\cdot(e^s\cos t)
\end{aligned}$$

Therefore

$$\begin{aligned}
\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 &= \left(\frac{\partial u}{\partial x}\cdot(e^s\cos t) + \frac{\partial u}{\partial y}\cdot(e^s\sin t)\right)^2 + \left(\frac{\partial u}{\partial x}\cdot(-e^s\sin t) + \frac{\partial u}{\partial y}\cdot(e^s\cos t)\right)^2 \\
&= \left(\frac{\partial u}{\partial x}\right)^2 e^{2s}\cos^2 t + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}e^{2s}\cos t\sin t + \left(\frac{\partial u}{\partial y}\right)^2 e^{2s}\sin^2 t \\
&\quad + \left(\frac{\partial u}{\partial x}\right)^2 e^{2s}\sin^2 t - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}e^{2s}\sin t\cos t + \left(\frac{\partial u}{\partial y}\right)^2 e^{2s}cos^2t = e^{2s}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right)
\end{aligned}$$

Ex. If $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, prove that $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 = \left(\dfrac{\partial z}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2$.

Sol. By the chain rule

$$\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \\
\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}\cdot(-r\sin\theta) + \frac{\partial z}{\partial y}\cdot(r\cos\theta)
\end{aligned}$$

Therefore

$$\begin{aligned}
\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 &= \left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial x}\cdot(-r\sin\theta) + \frac{\partial z}{\partial y}\cdot(r\cos\theta)\right)^2 \\
&= \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta \\
&\quad + \left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2
\end{aligned}$$

Ex. If $z = u(x,y)$, $x = r^2 + s^2$, $y = 2rs$, find $\dfrac{\partial z}{\partial r}$, $\dfrac{\partial^2 z}{\partial r^2}$, $\dfrac{\partial^2 z}{\partial s \partial r}$.

Sol. By the chain rule

$$\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = 2r\,\frac{\partial u}{\partial x} + 2s\,\frac{\partial u}{\partial y} \\
\frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r}\left(2r\,\frac{\partial u}{\partial x} + 2s\,\frac{\partial u}{\partial y}\right) = 2\,\frac{\partial u}{\partial x} + 2r\,\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) + 2s\,\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right) \\
\frac{\partial^2 z}{\partial s \partial r} &= \frac{\partial}{\partial s}\left(2r\,\frac{\partial u}{\partial x} + 2s\,\frac{\partial u}{\partial y}\right) = 2\,\frac{\partial u}{\partial y} + 2s\,\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right) + 2r\,\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)
\end{aligned}$$

Also

$$\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 u}{\partial x^2}\cdot(2r) + \frac{\partial^2 u}{\partial y \partial x}\cdot(2s) \\
\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 u}{\partial x \partial y}\cdot(2r) + \frac{\partial^2 u}{\partial y^2}\cdot(2s) \\
\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial s} = \frac{\partial^2 u}{\partial x^2}\cdot(2s) + \frac{\partial^2 u}{\partial y \partial x}\cdot(2r) \\
\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial s} = \frac{\partial^2 u}{\partial x \partial y}\cdot(2s) + \frac{\partial^2 u}{\partial y^2}\cdot(2r)
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2} &= 2\,\frac{\partial u}{\partial x} + 2r\,\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) + 2s\,\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right) \\
&= 2\,\frac{\partial u}{\partial x} + 2r\left(\frac{\partial^2 u}{\partial x^2}\cdot(2r) + \frac{\partial^2 u}{\partial y \partial x}\cdot(2s)\right) + 2s\left(\frac{\partial^2 u}{\partial x \partial y}\cdot(2r) + \frac{\partial^2 u}{\partial y^2}\cdot(2s)\right) \\
&= 2\,\frac{\partial u}{\partial x} + 4r^2\,\frac{\partial^2 u}{\partial x^2} + 8rs\,\frac{\partial^2 u}{\partial x \partial y} + 4s^2\,\frac{\partial^2 u}{\partial y^2} \\
\frac{\partial^2 z}{\partial s \partial r} &= 2\,\frac{\partial u}{\partial y} + 2s\,\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right) + 2r\,\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right) \\
&= 2\,\frac{\partial u}{\partial y} + 2s\left(\frac{\partial^2 u}{\partial x \partial y}\cdot(2s) + \frac{\partial^2 u}{\partial y^2}\cdot(2r)\right) + 2r\left(\frac{\partial^2 u}{\partial x^2}\cdot(2s) + \frac{\partial^2 u}{\partial y \partial x}\cdot(2r)\right) \\
&= 2\,\frac{\partial u}{\partial y} + 4rs\,\frac{\partial^2 u}{\partial x^2} + 4(r^2+s^2)\,\frac{\partial^2 u}{\partial x \partial y} + 4rs\,\frac{\partial^2 u}{\partial y^2}
\end{aligned}$$

Ex. If $z = u(x,y)$, $x = g(s,t)$, $y = h(s,t)$, prove that

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\,\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2$$

Sol. Let $z = U(s,t) = u(x(s,t),\, y(s,t))$, then by the chain rule

$$\begin{aligned}
\frac{\partial U}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} \\
\frac{\partial^2 U}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\right) = \frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial x}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial u}{\partial y}\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)
\end{aligned}$$

Also

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial t}+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial t}=\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial t}+\frac{\partial^2 u}{\partial y\partial x}\frac{\partial y}{\partial t}$$
$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial t}+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial t}=\frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial t}+\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial t}$$

Therefore

$$\begin{aligned}\frac{\partial^2 U}{\partial t^2}&=\frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial t^2}+\frac{\partial x}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial^2 y}{\partial t^2}+\frac{\partial^2 u}{\partial y\partial x}\frac{\partial y}{\partial t}\right)+\frac{\partial y}{\partial t}\left(\frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial t}+\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial t}\right)\\&=\frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial t^2}+\frac{\partial u}{\partial y}\frac{\partial^2 y}{\partial t^2}+\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2+2\,\frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t}+\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2\end{aligned}$$

Ex. If $f(x,t)=g(x+at)+h(x-at)$, where g, h are twice differentiable, prove that f satisfies the wave equation $\dfrac{\partial^2 f}{\partial t^2}=a^2\dfrac{\partial^2 f}{\partial x^2}$.

Sol. Let $u(x,t)=x+at$, $v(x,t)=x-at$, $f\big(u(x,t),v(x,t)\big)=g(u(x,t))+h(v(x,t))$. By the chain rule

$$\begin{aligned}\frac{\partial f}{\partial t}&=\frac{\partial f}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial t}\\&=g'(u(x,t))\cdot a+h'(v(x,t))\cdot(-a)=a\,g'(x+at)-a\,h'(x-at)=a\,g'(u(x,t))-a\,h'(v(x,t))\\\frac{\partial^2 f}{\partial t^2}&=\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right)\\&=\frac{\partial}{\partial u}\big(a\,g'(u)-a\,h'(v)\big)\big(u(x,t),v(x,t)\big)\frac{\partial u}{\partial t}+\frac{\partial}{\partial v}\big(a\,g'(u)-a\,h'(v)\big)\big(u(x,t),v(x,t)\big)\frac{\partial v}{\partial t}\\&=a\,g''(u(x,t))\cdot a-a\,h''(v(x,t))\cdot(-a)\\&=a^2\left(g''(x+at)+h''(x-at)\right)\\\frac{\partial f}{\partial x}&=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\\&=g'(u(x,t))\cdot 1+h'(v(x,t))\cdot(1)=g'(x+at)+h'(x-at)=g'(u(x,t))+h'(v(x,t))\\\frac{\partial^2 f}{\partial x^2}&=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\\&=\frac{\partial}{\partial x}\big(g'(u)+h'(v)\big)\big(u(x,t),v(x,t)\big)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\big(g'(u)+h'(v)\big)\big(u(x,t),v(x,t)\big)\frac{\partial v}{\partial x}\\&=g''(u(x,t))\cdot 1+h''(v(x,t))\cdot 1\\&=g''(x+at)+h''(x-at)\end{aligned}$$

Therefore $\dfrac{\partial^2 f}{\partial t^2}=a^2\dfrac{\partial^2 f}{\partial x^2}$.

Ex. If $u=f(x,y)$, $x=e^s\cos t$, $y=e^s\sin t$, prove that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=e^{-2s}\left(\dfrac{\partial^2 u}{\partial s^2}+\dfrac{\partial^2 u}{\partial t^2}\right)$.

Sol. By the chain rule

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}=\frac{\partial u}{\partial x}\cdot(e^s\cos t)+\frac{\partial u}{\partial y}\cdot(e^s\sin t)$$
$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}=\frac{\partial u}{\partial x}\cdot(-e^s\sin t)+\frac{\partial u}{\partial y}\cdot(e^s\cos t)$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial s^2} = \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial s}\right) &= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\cdot(e^s\cos t) + \frac{\partial u}{\partial y}\cdot(e^s\sin t)\right) \\
&= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)\cdot(e^s\cos t) + \frac{\partial u}{\partial x}\cdot(e^s\cos t) + \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right)\cdot(e^s\sin t) + \frac{\partial u}{\partial y}\cdot(e^s\sin t) \\
\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) &= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\cdot(-e^s\sin t) + \frac{\partial u}{\partial y}\cdot(e^s\cos t)\right) \\
&= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)\cdot(-e^s\sin t) + \frac{\partial u}{\partial x}\cdot(-e^s\cos t) + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)\cdot(e^s\cos t) + \frac{\partial u}{\partial y}\cdot(-e^s\sin t)
\end{aligned}$$

Also

$$\begin{aligned}
\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial s} = \frac{\partial^2 u}{\partial x^2}\cdot(e^s\cos t) + \frac{\partial^2 u}{\partial y\partial x}\cdot(e^s\sin t) \\
\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial s} = \frac{\partial^2 u}{\partial x\partial y}\cdot(e^s\cos t) + \frac{\partial^2 u}{\partial y^2}\cdot(e^s\sin t) \\
\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial t} = \frac{\partial^2 u}{\partial x^2}\cdot(-e^s\sin t) + \frac{\partial^2 u}{\partial y\partial x}\cdot(e^s\cos t) \\
\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial t} = \frac{\partial^2 u}{\partial x\partial y}\cdot(-e^s\sin t) + \frac{\partial^2 u}{\partial y^2}\cdot(e^s\cos t)
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial^2 u}{\partial s^2} &= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)\cdot(e^s\cos t) + \frac{\partial u}{\partial x}\cdot(e^s\cos t) + \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right)\cdot(e^s\sin t) + \frac{\partial u}{\partial y}\cdot(e^s\sin t) \\
&= \left(\frac{\partial^2 u}{\partial x^2}\cdot(e^s\cos t) + \frac{\partial^2 u}{\partial y\partial x}\cdot(e^s\sin t)\right)\cdot(e^s\cos t) + \frac{\partial u}{\partial x}\cdot(e^s\cos t) \\
&\quad + \left(\frac{\partial^2 u}{\partial x\partial y}\cdot(e^s\cos t) + \frac{\partial^2 u}{\partial y^2}\cdot(e^s\sin t)\right)\cdot(e^s\sin t) + \frac{\partial u}{\partial y}\cdot(e^s\sin t) \\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)\cdot(-e^s\sin t) + \frac{\partial u}{\partial x}\cdot(-e^s\cos t) + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)\cdot(e^s\cos t) + \frac{\partial u}{\partial y}\cdot(-e^s\sin t) \\
&= \left(\frac{\partial^2 u}{\partial x^2}\cdot(-e^s\sin t) + \frac{\partial^2 u}{\partial y\partial x}\cdot(e^s\cos t)\right)\cdot(-e^s\sin t) + \frac{\partial u}{\partial x}\cdot(-e^s\cos t) \\
&\quad + \left(\frac{\partial^2 u}{\partial x\partial y}\cdot(-e^s\sin t) + \frac{\partial^2 u}{\partial y^2}\cdot(e^s\cos t)\right)\cdot(e^s\cos t) + \frac{\partial u}{\partial y}\cdot(-e^s\sin t)
\end{aligned}$$

We can obtain $\dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 u}{\partial t^2} = e^{2s}\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$.

Ex. If $z = u(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$, find $\dfrac{\partial^2 z}{\partial\theta\partial r}$, $\dfrac{\partial^2 z}{\partial r\partial\theta}$.

Sol.

- To find $\dfrac{\partial^2 z}{\partial\theta\partial r}$: By the chain rule

$$\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\
\frac{\partial^2 z}{\partial\theta\partial r} &= \frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) \\
&= \frac{\partial u}{\partial x}(-\sin\theta) + \frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\right)\cos\theta + \frac{\partial u}{\partial y}\ \cos\theta + \frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial y}\right)\sin\theta
\end{aligned}$$

Also

$$\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial\theta} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial\theta} = \frac{\partial^2 u}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 u}{\partial y\partial x}(r\cos\theta)$$

$$\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial\theta} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial\theta} = \frac{\partial^2 u}{\partial x\partial y}(-r\sin\theta) + \frac{\partial^2 u}{\partial y^2}(r\cos\theta)$$

Therefore

$$\begin{aligned}
\frac{\partial^2 z}{\partial\theta\partial r} &= \frac{\partial u}{\partial x}(-\sin\theta) + \frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\right)\cos\theta + \frac{\partial u}{\partial y}\cos\theta + \frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial y}\right)\sin\theta \\
&= \frac{\partial u}{\partial x}(-\sin\theta) + \left(\frac{\partial^2 u}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 u}{\partial y\partial x}(r\cos\theta)\right)\cos\theta \\
&+ \frac{\partial u}{\partial y}\cos\theta + \left(\frac{\partial^2 u}{\partial x\partial y}(-r\sin\theta) + \frac{\partial^2 u}{\partial y^2}(r\cos\theta)\right)\sin\theta \\
&= \frac{\partial u}{\partial y}\cos\theta - \frac{\partial u}{\partial x}\sin\theta + r\sin\theta\cos\theta\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2}\right) + r(\cos^2\theta - \sin^2\theta)\frac{\partial^2 u}{\partial x\partial y}
\end{aligned}$$

- To find $\dfrac{\partial^2 z}{\partial r\partial\theta}$: By the chain rule

$$\begin{aligned}
\frac{\partial z}{\partial\theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial\theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) \\
\frac{\partial^2 z}{\partial r\partial\theta} &= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)\right) \\
&= \frac{\partial u}{\partial x}(-\sin\theta) + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)(-r\sin\theta) + \frac{\partial u}{\partial y}\cos\theta + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right)(r\cos\theta)
\end{aligned}$$

Also

$$\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 u}{\partial x^2}\cos\theta + \frac{\partial^2 u}{\partial y\partial x}\sin\theta$$

$$\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 u}{\partial x\partial y}\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin\theta$$

Therefore

$$\begin{aligned}
\frac{\partial^2 z}{\partial r\partial\theta} &= \frac{\partial u}{\partial x}(-\sin\theta) + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)(-r\sin\theta) + \frac{\partial u}{\partial y}\cos\theta + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right)(r\cos\theta) \\
&= \frac{\partial u}{\partial x}(-\sin\theta) + \left(\frac{\partial^2 u}{\partial x^2}\cos\theta + \frac{\partial^2 u}{\partial y\partial x}\sin\theta\right)(-r\sin\theta) \\
&+ \frac{\partial u}{\partial y}\cos\theta + \left(\frac{\partial^2 u}{\partial x\partial y}\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin\theta\right)(r\cos\theta) \\
&= \frac{\partial u}{\partial y}\cos\theta - \frac{\partial u}{\partial x}\sin\theta + r\sin\theta\cos\theta\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2}\right) + r(\cos^2\theta - \sin^2\theta)\frac{\partial^2 u}{\partial x\partial y}
\end{aligned}$$

Ex. If $z = u(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, prove that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial z}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 z}{\partial\theta^2}$.

Sol. Let $z = U(r,\theta) = u(x(r,\theta),\, y(r,\theta))$, then by the chain rule

$$
\begin{aligned}
\frac{\partial U}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\
\frac{\partial^2 U}{\partial r^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)\frac{\partial y}{\partial r} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)\cos\theta + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)\sin\theta \\
&= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 u}{\partial x\partial y}\sin\theta\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta \\
\frac{\partial U}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) \\
\frac{\partial^2 U}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)\right) \\
&= \frac{\partial u}{\partial x}(-r\cos\theta) - r\sin\theta\,\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial u}{\partial y}(-r\sin\theta) + r\cos\theta\,\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial u}{\partial y}(-r\sin\theta) - r\sin\theta\,\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right) + r\cos\theta\,\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial u}{\partial y}(-r\sin\theta) - r\sin\theta\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y\partial x}\frac{\partial y}{\partial \theta}\right) + r\cos\theta\left(\frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \theta}\right) \\
&= \frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial u}{\partial y}(-r\sin\theta) - r\sin\theta\left(\frac{\partial^2 u}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 u}{\partial y\partial x}(r\cos\theta)\right) \\
&\qquad + r\cos\theta\left(\frac{\partial^2 u}{\partial x\partial y}(-r\sin\theta) + \frac{\partial^2 u}{\partial y^2}(r\cos\theta)\right) \\
&= \frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial u}{\partial y}(-r\sin\theta) + r^2\sin^2\theta\,\frac{\partial^2 u}{\partial x^2} - 2r^2\sin\theta\cos\theta\,\frac{\partial^2 u}{\partial y\partial x} + r^2\cos^2\theta\,\frac{\partial^2 u}{\partial y^2}
\end{aligned}
$$

From the above,

$$
\begin{aligned}
\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} &= \left(\frac{\partial^2 u}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 u}{\partial x\partial y}\sin\theta\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta\right) + \frac{1}{r}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) \\
&\quad + \frac{1}{r^2}\left(\frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial u}{\partial y}(-r\sin\theta) + r^2\sin^2\theta\,\frac{\partial^2 u}{\partial x^2} - 2r^2\sin\theta\cos\theta\,\frac{\partial^2 u}{\partial y\partial x} + r^2\cos^2\theta\,\frac{\partial^2 u}{\partial y^2}\right) \\
&= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}
\end{aligned}
$$

5.5 Directional Derivatives and Gradients

Definition. Let $S \subseteq \mathbb{R}^n$, $\mathbf{c} \in S$, $\mathbf{u} \in \mathbb{R}^n$, and $f : S \to \mathbb{R}$ be a differentiable function; $\mathbf{e}_1$, $\mathbf{e}_2$, $\ldots$, $\mathbf{e}_n$ are the unit vectors in the $\mathbb{R}^n$ Cartesian coordinate system.

- The directional derivative of f at $\mathbf{c}$ in the direction of $\mathbf{u}$ is $D_{\mathbf{u}}f(\mathbf{c}) = \lim_{h\to 0}\dfrac{f(\mathbf{c}+h\mathbf{u}) - f(\mathbf{c})}{h}$.
- The partial derivatives of f are $f_i(\mathbf{c}) = D_{\mathbf{e}_i}f(\mathbf{c})$, $i = 1, 2, \ldots, n$.
- The gradient of f at $\mathbf{c}$ is $\nabla f(\mathbf{c}) = \langle f_1(\mathbf{c}), f_2(\mathbf{c}), \ldots, f_n(\mathbf{c})\rangle$.

Property. $D_{\mathbf{u}}f(\mathbf{c}) = \nabla f(\mathbf{c}) \cdot \mathbf{u}$.

Proof. Let $g(x) = f(\mathbf{c} + x\mathbf{u}) = f(v_1 + xu_1,\, v_2 + xu_2,\, \ldots,\, v_n + xu_n)$, $D_{\mathbf{u}}f(\mathbf{c}) = \lim\limits_{h\to 0} \dfrac{f(\mathbf{c}+h\mathbf{u}) - f(\mathbf{c})}{h} = \lim\limits_{h\to 0} \dfrac{g(h) - g(0)}{h} = g'(x)\,\Big|_{x=0}$. By the chain rule, $g'(x)\,\Big|_{x=0} = \displaystyle\sum_{i=1}^{n} f_i(\mathbf{c} + x\mathbf{u})\, \frac{\mathrm{d}(v_i + xu_i)}{\mathrm{d}x}\Bigg|_{x=0} = \sum_{i=1}^{n} f_i(\mathbf{c} + x\mathbf{u})\, u_i\Bigg|_{x=0} = \nabla f(\mathbf{c}) \cdot \mathbf{u}$.

Property. Given a surface $G(\mathbf{x}) = 0$ in $\mathbb{R}^n$. Let $\mathbf{x}_0 \in \mathbb{R}^n$ such that $G(\mathbf{x}_0) = 0$ (i.e., $\mathbf{x}_0$ is on $G(\mathbf{x})$), then $\nabla G(\mathbf{x}_0)$ is perpendicular to $G(\mathbf{x})$ at $\mathbf{x}_0$.

Proof. Let $\mathbf{r}(t) = \langle x_1(t),\, x_2(t),\, \ldots,\, x_n(t)\rangle$ be a curve in $\mathbb{R}^n$ on $G(\mathbf{x})$ and passing through $\mathbf{x}_0$, then $G(\mathbf{r}(t)) = 0$, and there exists $t_0 \in \mathbb{R}$ such that $\mathbf{r}(t_0) = \mathbf{x}_0$. By the chain rule, differentiating both sides of $G(\mathbf{r}(t)) = 0$ with respect to t and substituting $t = t_0$ gives

$$\frac{\mathrm{d}G(\mathbf{r}(t))}{\mathrm{d}t}\Bigg|_{t=t_0} = 0 \implies \sum_{i=1}^{n} G_i(\mathbf{r}(t))\, x_i'(t)\Bigg|_{t=t_0} = 0 \implies \nabla G(\mathbf{r}(t_0)) \cdot \mathbf{r}'(t_0) = 0$$

Ex. Find the equation of the tangent plane to $z = x^2 + 5xy - 2y^2$ at $(1, 2, 3)$.

Sol. $f(x, y, z) = x^2 + 5xy - 2y^2 - z = 0$, so $\nabla f = (2x + 5y)\,\widehat{\boldsymbol{\imath}} + (-4y + 5x)\,\widehat{\boldsymbol{\jmath}} - \widehat{\mathbf{k}}$, $\nabla f(1, 2, 3) = \langle 12,\, -3,\, -1\rangle$, the equation of the tangent plane is $12(x - 1) - 3(y - 2) - (z - 3) = 0 \implies 12x - 3y - z = 3$.

Ex. Find the equation of the tangent plane to $z^3 + xyz - 2 = 0$ at $(1, 1, 1)$.

Sol. $f(x, y, z) = z^3 + xyz - 2 = 0$, so $\nabla f = yz\,\widehat{\boldsymbol{\imath}} + xz\,\widehat{\boldsymbol{\jmath}} + (3z^2 + xy)\,\widehat{\mathbf{k}}$, $\nabla f(1, 1, 1) = \langle 1,\, 1,\, 4\rangle$, the equation of the tangent plane is $(x - 1) + (y - 1) + 4(z - 1) = 0 \implies x + y + 4z = 6$.

5.6 Extremum Problems

Definition. Given $S \subseteq \mathbb{R}^n$, $f : S \to \mathbb{R}$, $B(\mathbf{x}, h) \equiv \{\mathbf{y} \in \mathbb{R}^n \mid |\mathbf{y} - \mathbf{x}| < h\}$.

- f has a global maximum $f(\mathbf{x}_{\mathrm{M}})$ at $\mathbf{x}_{\mathrm{M}} \in S$ if: $f(\mathbf{x}_{\mathrm{M}}) \geqslant f(\mathbf{x})$, $\forall\, \mathbf{x} \in S$.
- f has a global minimum $f(\mathbf{x}_{\mathrm{m}})$ at $\mathbf{x}_{\mathrm{m}} \in S$ if: $f(\mathbf{x}_{\mathrm{m}}) \leqslant f(\mathbf{x})$, $\forall\, \mathbf{x} \in S$.
- f has a local maximum $f(\mathbf{x}_0)$ at $\mathbf{x}_0 \in S$ if: $\exists\, h_0 > 0$ such that $f(\mathbf{x}_0) \geqslant f(\mathbf{x})$, $\forall\, \mathbf{x} \in B(\mathbf{x}_0, h_0) \cap S$.
- f has a local minimum $f(\mathbf{x}_1)$ at $\mathbf{x}_1 \in S$ if: $\exists\, h_1 > 0$ such that $f(\mathbf{x}_1) \leqslant f(\mathbf{x})$, $\forall\, \mathbf{x} \in B(\mathbf{x}_1, h_1) \cap S$.

Theorem (Necessary Condition for Extremum)**.** Given $S \subseteq \mathbb{R}^n$ and a differentiable function $f : S \to \mathbb{R}$, if f has an extremum at an interior point $\mathbf{c}$ of S, then $\nabla f(\mathbf{c}) = \mathbf{0}$.

Proof. If $\mathbf{c} = (c_1,\, c_2,\, \ldots,\, c_n)$, let $g_j(t) \equiv f(c_1,\, c_2,\, \ldots,\, c_{j-1},\, t,\, c_{j+1},\, \ldots,\, c_n)$, $j = 1,\, 2,\, \ldots,\, n$. Since f has an extremum at $\mathbf{c}$, $f(\mathbf{c}) = g_j(c_j)$, g_j has an extremum at $c_j \implies g'(t)\,\Big|_{t=c_j} = 0 \implies f_j(\mathbf{c}) = 0\ \forall\, j$, therefore $\nabla f(\mathbf{c}) = \mathbf{0}$.

Conclusion. Given $S \subseteq \mathbb{R}^n$, if $f : S \to \mathbb{R}$ has an extremum at $\mathbf{c} \in S$, then $\mathbf{c}$ is one of the following three types:

- Critical point: $\nabla f(\mathbf{c}) = \mathbf{0}$.
- Singular point: f is not differentiable at $\mathbf{c}$.
- Boundary point of S.

Definition (Hessian Matrix)**.** Given $S \subseteq \mathbb{R}^n$, an interior point $\mathbf{c}$ of S, and a differentiable function $f : S \to \mathbb{R}$,

$$\mathbf{H}(f, \mathbf{c}) = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix}, \qquad f_{ij} = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{c}), \qquad i,\, j = 1,\, 2,\, \ldots,\, n.$$

Definition (Matrix Positive/Negative Definiteness)**.** Given an $n \times n$ real symmetric matrix $\mathbf{A}$. For any $\mathbf{v} \in \mathbb{R}^n \neq \mathbf{0}$, if

- $\mathbf{v}\mathbf{A}\mathbf{v}^\top > 0$: $\mathbf{A}$ is positive-definite
- $\mathbf{v}\mathbf{A}\mathbf{v}^\top < 0$: $\mathbf{A}$ is negative-definite
- $\mathbf{v}\mathbf{A}\mathbf{v}^\top \geqslant 0$: $\mathbf{A}$ is positive-semidefinite
- $\mathbf{v}\mathbf{A}\mathbf{v}^\top \leqslant 0$: $\mathbf{A}$ is negative-semidefinite

Definition (Minor)**.** Given an $n \times n$ matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ and minor $\mathbf{A}\begin{pmatrix} i_1, i_2, \cdots, i_k \\ j_1, j_2, \cdots, j_k \end{pmatrix} =$ $\begin{vmatrix} a_{i_1j_1} & a_{i_1j_2} & \cdots & a_{i_1j_k} \\ a_{i_2j_1} & a_{i_2j_2} & \cdots & a_{i_2j_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_kj_1} & a_{i_kj_2} & \cdots & a_{i_kj_k} \end{vmatrix}$, $1 \leqslant k \leqslant n$, $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$, $1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant n$.

- $\Delta_k \equiv \mathbf{A}\begin{pmatrix} i_1, i_2, \cdots, i_k \\ i_1, i_2, \cdots, i_k \end{pmatrix}$ is the k-th order principal minor of A.
- $M_k \equiv \mathbf{A}\begin{pmatrix} 1, 2, \cdots, k \\ 1, 2, \cdots, k \end{pmatrix}$ is the k-th order leading principal minor of A.

Theorem (Criteria for Matrix Positive/Negative Definiteness)**.** Given an $n \times n$ real symmetric matrix $\mathbf{A}$, then $\forall\, k \leqslant n$,

- $\mathbf{A}$ is positive-definite $\iff M_k > 0$
- $\mathbf{A}$ is negative-definite $\iff (-1)^k M_k > 0$
- $\mathbf{A}$ is positive-semidefinite $\iff \Delta_k \geqslant 0$
- $\mathbf{A}$ is negative-semidefinite $\iff (-1)^k \Delta_k \geqslant 0$

Ex. Consider the matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$: Let $\mathbf{v} = \langle a, b, c \rangle$, $\mathbf{v}\mathbf{A}\mathbf{v}^\top = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} =$ $\begin{pmatrix} 2a-b & -a+2b-c & -b+2c \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (2a-b)a + (-a+2b-c)b + (-b+2c)c = 2a^2 - 2ab + 2b^2 - 2bc + 2c^2 =$ $a^2 + (a-b)^2 + (b-c)^2 + c^2 > 0$, except when $a = b = c = 0$, so it is positive-definite. Also, $\mathbf{A}$'s M_1, M_2, M_3 are 2, $\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$, $\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 4$ respectively, by the above criteria $\mathbf{A}$ is positive-definite.

Theorem (Second Derivative Test)**.** Given $S \subseteq \mathbb{R}^n$ and a differentiable function $f : S \to \mathbb{R}$, and f at an interior point $\mathbf{c}$ of S has $\nabla f(\mathbf{c}) = 0$.

- If $\mathbf{H}(f, \mathbf{c})$ is positive-definite, then f has a local minimum at $\mathbf{c}$.
- If $\mathbf{H}(f, \mathbf{c})$ is negative-definite, then f has a local maximum at $\mathbf{c}$.

Conclusion. Given $S \subseteq \mathbb{R}^2$ and a differentiable function $f : S \to \mathbb{R}$, and f at an interior point (a, b) of S has $\nabla f(a, b) = 0$. Let

$$D = f_{xx}(a, b) \cdot f_{yy}(a, b) - (f_{xy}(a, b))^2$$

- If $D > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).
- If $D > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).
- If $D < 0$, then (a, b) is a saddle point.

Ex. Find the critical points of $f(x,y) = x^3 + xy^2 - 3x^2 - 4y^2 + 4$ and classify them.

Sol. From $f_x(x,y) = 3x^2 + y^2 - 6x$, $f_y(x,y) = 2xy - 8y$, the critical points are (x,y) that simultaneously satisfy these two equations being zero. Therefore

$$\left\{3x^2 + y^2 - 6x = 0\right\} \vee \left\{y\,(x-4) = 0\right\} \implies \left\{y = 0,\ 3x^2 - 6x = 0\right\} \vee \left\{x = 4,\ 3 \cdot 4^2 + y^2 + 6 \cdot 4 = 0\right\}$$

So the critical points are $(0,0)$, $(2,0)$. Also $f_{xx} = 6x - 6$, $f_{yy} = 2x - 8$, $f_{xy} = f_{yx} = 2y$, classified as follows:

Critical Point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	Classification
$(0,0)$	$(-6) \times (-8) - (0)^2 > 0$	-6	Local maximum
$(2,0)$	$6 \times (-4) - 0^2 < 0$		Saddle point

Ex. Find the critical points of $f(x,y) = xy\,(5x + y - 15)$ and classify them.

Sol.

$$\begin{aligned} f_x(x,y) &= y\,(5x + y - 15) + xy\,(5) = y\,(5x + y - 15) + y\,(5x) = y\,(10x + y - 15) \\ f_y(x,y) &= x\,(5x + y - 15) + xy\,(1) = x\,(5x + y - 15) + x\,(y) = x\,(5x + 2y - 15) \end{aligned}$$

The critical points are (x,y) that simultaneously satisfy these two equations being zero. Therefore

$$\left\{y = 0 \ \vee\ 10x + y = 15\right\} \vee \left\{x = 0 \ \vee\ 5x + 2y = 15\right\}$$

$$\implies \left\{y = 0,\ x = 0\right\} \vee \left\{y = 0,\ 5x + 2y = 15\right\} \vee \left\{10x + y = 15,\ x = 0\right\} \vee \left\{10x + y = 15,\ 5x + 2y = 15\right\}$$

So the critical points are $(0,0)$, $(3,0)$, $(0,15)$, $(1,5)$. Also $f_{xx} = 10y$, $f_{yy} = 2x$, $f_{xy} = f_{yx} = 10x + 2y - 15$, classified as follows:

Critical Point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	Classification
$(0,0)$	$0 \times 0 - (-15)^2 < 0$		Saddle point
$(3,0)$	$0 \times 6 - 15^2 < 0$		Saddle point
$(0,15)$	$150 \times 0 - 15^2 < 0$		Saddle point
$(1,5)$	$50 \times 2 - 5^2 > 0$	50	Local minimum

Ex. Find the maximum and minimum values of $f(x,y) = (x+y)\,e^{-x^2-y^2}$ on $S:\ x^2 + y^2 \leqslant 1$.

Sol. Since f is differentiable, it has no singular points; the extrema of f occur at critical points ($\mathbf{c} \in S$, $\nabla f(\mathbf{c}) = 0$) and boundary points of S ($x^2 + y^2 = 1$).

- From $f_x(x,y) = e^{-x^2-y^2} + (x+y)\,e^{-x^2-y^2}(-2x) = (-2x^2 - 2xy + 1)\,e^{-x^2-y^2}$, $f_y(x,y) = e^{-x^2-y^2} + (x+y)\,e^{-x^2-y^2}(-2y) = (-2y^2 - 2xy + 1)\,e^{-x^2-y^2}$, the critical points (x,y) satisfy $2x^2 + 2xy = 1$ and $2y^2 + 2xy = 1$, solving gives $(x,y) = \left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(-\frac{1}{2}, -\frac{1}{2}\right)$.
- Boundary points $x^2 + y^2 = 1$: Let $x = \cos t$, $y = \sin t$, $0 \leqslant t \leqslant 2\pi$, then $f(x,y)$ becomes $g(t) \equiv (\cos t + \sin t)\,e^{-1}$; $g'(t) = (-\sin t + \cos t)\,e^{-1} = 0$ solves to $t = \frac{\pi}{4}, \frac{5\pi}{4}$; also consider boundary $t = 0,\ 2\pi$, i.e., $(x,y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $(1,0)$.

Candidate Point	$f(x,y)$	Classification
$\left(\frac{1}{2}, \frac{1}{2}\right)$	$e^{-\frac{1}{2}}$	Maximum
$\left(-\frac{1}{2}, -\frac{1}{2}\right)$	$-e^{-\frac{1}{2}}$	Minimum
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\sqrt{2}\,e^{-1}$	
$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$-\sqrt{2}\,e^{-1}$	
$(1,0)$	e^{-1}	

Ex. Find the maximum and minimum values of $f(x,y) = x^3 + xy^2 - 3x^2 - 4y^2 + 4$ on $S: x^2 + y^2 \leqslant 1$.

Sol. Since f is differentiable, it has no singular points; the extrema of f occur at critical points ($\mathbf{c} \in S$, $\nabla f(\mathbf{c}) = 0$) and boundary points of S $(x^2 + y^2 = 1)$.

- From $f_x(x,y) = 3x^2 + y^2 - 6x$, $f_y(x,y) = 2xy - 8y$, the critical points (x,y) satisfy $3x^2 + y^2 - 6x = 0$ and $2xy - 8y = 0$, solving gives $(x,y) = (0,0),\ (2,0)$; $(2,0)$ is outside S and not applicable.
- Boundary points $x^2 + y^2 = 1$: Substitute $y^2 = 1 - x^2$ then $f(x,y)$ becomes $g(x) = x^3 + x\,(1 - x^2) - 3x^2 - 4\,(1 - x^2) + 4 = x + x^2$, $-1 \leqslant x \leqslant 1$; $g'(x) = 1 + 2x = 0$ solves to $x = -\frac{1}{2}$, i.e., the extrema of $g(x)$ occur at $x = \pm 1$ and $-\frac{1}{2} \implies (x,y) = \big(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\big),\ (1,0),\ (-1,0)$.

Candidate Point	$f(x,y)$	Classification
$(0,0)$	4	Maximum
$\big(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\big)$	$-\frac{1}{4}$	Minimum
$(1,0)$	2	
$(-1,0)$	0	

Ex. Find the maximum and minimum values of $f(x,y) = xy - x^3y^2$ on $S: 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1$.

Sol. Since f is differentiable, it has no singular points; the extrema of f occur at critical points ($\mathbf{c} \in S$, $\nabla f(\mathbf{c}) = 0$) and boundary points of S.

- From $f_x(x,y) = y - 3x^2y^2$, $f_y(x,y) = x - 2x^3y$, the critical points (x,y) satisfy $y - 3x^2y^2 = y\,(1 - 3x^2y) = 0$ and $x - 2x^3y = x\,(1 - 2x^2y) = 0$, so $y = 0 \ \vee\ 1 - 3x^2y = 0$ and $x = 0 \ \vee\ 1 - 2x^2y = 0$; solving gives $(x,y) = (0,0)$.
- The boundary points consist of $L_1: \ x = 0 \ \wedge\ 0 \leqslant y \leqslant 1$, $L_2: \ y = 0 \ \wedge\ 0 \leqslant x \leqslant 1$, $L_3: \ x = 1 \ \wedge\ 0 \leqslant y \leqslant 1$, $L_4: \ y = 1 \ \wedge\ 0 \leqslant x \leqslant 1$.
  - L_1: $f(x,y) = 0$.
  - L_2: $f(x,y) = 0$.
  - L_3: $x = 1$, $0 \leqslant y \leqslant 1$, $f(x,y)$ becomes $g(y) = y - y^2$, $g'(y) = 1 - 2y = 0$ solves to $y = \frac{1}{2}$, i.e., the extrema of $g(y)$ occur at $y = 0,\ 1,\ \frac{1}{2} \implies (x,y) = (1,0),\ (1,1),\ \big(1, \frac{1}{2}\big)$
  - L_4: $y = 1$, $0 \leqslant x \leqslant 1$, $f(x,y)$ becomes $h(x) = x - x^3$, $h'(x) = 1 - 3x^2 = 0$ solves to $x = \pm\frac{1}{\sqrt{3}}$ (negative not applicable), i.e., the extrema of $h(x)$ occur at $x = 0,\ 1,\ \frac{1}{\sqrt{3}} \implies (x,y) = (0,1),\ (1,1),\ \big(\frac{1}{\sqrt{3}}, 1\big)$.

Candidate Point	$f(x,y)$	Classification
$(0, 0 \leqslant y \leqslant 1)$	0	Minimum
$(0 \leqslant x \leqslant 1, 0)$	0	Minimum
$(0,0)$	0	Minimum
$(1,0)$	0	Minimum
$(1,1)$	0	Minimum
$\big(1, \frac{1}{2}\big)$	$\frac{1}{4}$	
$(0,1)$	0	Minimum
$\big(\frac{1}{\sqrt{3}}, 1\big)$	$\frac{2}{3\sqrt{3}}$	Maximum

Ex. Find the maximum and minimum values of $f(x,y)=xy+2x+y$ in the triangular region S formed by $(0,0)$, $(1,0)$, $(0,2)$.

Sol. Since f is differentiable, it has no singular points; the extrema of f occur at critical points ($\mathbf{c}\in S$, $\nabla f(\mathbf{c})=0$) and boundary points of S.

- From $f_x(x,y)=y+2$, $f_y(x,y)=x+1$, the critical points (x,y) satisfy $y+2=0$ and $x+1=0$, so $(x,y)=(-1,-2)$.
- The boundary points consist of $L_1:\ x=0\ \wedge\ 0\leqslant y\leqslant 2$, $L_2:\ y=0\ \wedge\ 0\leqslant x\leqslant 1$, $L_3:\ (1,0)$ — $(0,2)$.
  - L_1: $(x,y)=(0,0),\ (0,2)$.
  - L_2: $(x,y)=(0,0),\ (1,0)$.
  - L_3: $y=-2x+2$, $0\leqslant x\leqslant 1$, $f(x,y)$ becomes $g(x)=x\,(-2x+2)+2x+(-2x+2)=-2x^2+2x+2$, $g'(x)=-4x+2=0$ solves to $x=\frac{1}{2}$, i.e., the extrema of $g(x)$ occur at $x=0,\ 1,\ \frac{1}{2}\implies(x,y)=(0,2),\ (1,0),\ \left(\frac{1}{2},1\right)$

Candidate Point	$f(x,y)$	Classification
$(0,0)$	0	Minimum
$(0,2)$	2	
$(1,0)$	2	
$\left(\frac{1}{2},1\right)$	$\frac{5}{2}$	Maximum

Ex. Find the maximum and minimum values of $f(x,y)=xy\,e^{-\frac{x^2+y^2}{2}}$ on $S:\ \left\{(x,y)\,|\,x^2+y^2\leqslant 4,\ x\geqslant 0,\ y\geqslant 0\right\}$.

Sol. Since f is differentiable, it has no singular points; the extrema of f occur at critical points ($\mathbf{c}\in S$, $\nabla f(\mathbf{c})=0$) and boundary points of S.

- From $f_x(x,y)=y\,e^{-\frac{x^2+y^2}{2}}+xy\,e^{-\frac{x^2+y^2}{2}}(-x)=y(1-x^2)\,e^{-\frac{x^2+y^2}{2}}$, $f_y(x,y)=x\,e^{-\frac{x^2+y^2}{2}}+xy\,e^{-\frac{x^2+y^2}{2}}(-y)=x(1-y^2)\,e^{-\frac{x^2+y^2}{2}}$, the critical points (x,y) satisfy $y\,(1-x^2)=0$ and $x\,(1-y^2)=0$, solving gives $(x,y)=(0,0),\ (1,1),\ (1,-1),\ (-1,1),\ (-1,-1)$; only $(0,0),\ (1,1)$ are inside S.
- The boundary points consist of $L_1:\ x=0\ \wedge\ 0\leqslant y\leqslant 2$, $L_2:\ y=0\ \wedge\ 0\leqslant x\leqslant 2$, $L_3:\ x^2+y^2=4$ in the first quadrant.
  - L_1: $f(x,y)=0$.
  - L_2: $f(x,y)=0$.
  - L_3: Let $x=2\cos t$, $y=2\sin t$, $0\leqslant t\leqslant\frac{\pi}{2}$, then $f(x,y)$ becomes $g(t)\equiv 4\cos t\sin t\,e^{-2}$; $g'(t)=\cos 2t\,4e^{-2}=0$ solves to $t=\frac{\pi}{4}$; also consider boundary $t=0,\ \frac{\pi}{2}$, i.e., $(x,y)=(\sqrt{2},\sqrt{2}),\ (2,0),\ (0,2)$.

Candidate Point	$f(x,y)$	Classification
$(0,0)$	0	Minimum
$(1,1)$	e^{-1}	Maximum
$(0,0\leqslant y\leqslant 2)$	0	Minimum
$(0\leqslant x\leqslant 2,0)$	0	Minimum
$(\sqrt{2},\sqrt{2})$	$2e^{-2}$	
$(2,0)$	0	Minimum
$(0,2)$	0	Minimum

5.7 Lagrange Multiplier Method

Theorem. Given an open set $S \subseteq \mathbb{R}^n$, differentiable functions $f : S \to \mathbb{R}$ and $g_j : S \to \mathbb{R}$, $j = 1, 2, \ldots, m$, $m < n$, and $X_0 = \{\mathbf{x} \in S \mid g_j(\mathbf{x}) = 0,\ j = 1, 2, \ldots, m\}$. If f has an extremum at $\mathbf{x}_0 \in S \cap X_0$ and $\det\big(D_i g_j(\mathbf{x}_0)\big) \neq 0$, then

$$\exists\, \lambda_1, \lambda_2, \ldots, \lambda_m \quad \text{such that} \quad D_i f(\mathbf{x}_0) + \sum_{j=1}^{m} \lambda_j D_i g_j(\mathbf{x}_0) = 0, \quad i = 1, 2, \ldots, n$$

Remark. Let $\mathcal{L} \equiv f + \sum_{j=1}^{m} \lambda_j\, g_j$, the above sufficient condition can be written as

$$\begin{aligned} D_i\, \mathcal{L}(\mathbf{x}_0) &= 0, \quad i = 1, 2, \ldots, n \\ g_j(\mathbf{x}_0) &= 0, \quad j = 1, 2, \ldots, m \end{aligned}$$

Ex. Find the maximum and minimum values of $x^2 - 10x - y^2$ on $x^2 + 4y^2 = 16$.

Sol. Let $\mathcal{L} = x^2 - 10x - y^2 + \lambda\,(x^2 + 4y^2 - 16)$, then

$$\frac{\partial \mathcal{L}}{\partial x} = 2x - 10 + 2\lambda x = 0 \implies x - 5 + \lambda x = 0 \tag{1}$$
$$\frac{\partial \mathcal{L}}{\partial y} = -2y + 8\lambda y = 0 \implies -y + 4\lambda y = 0 \tag{2}$$
$$x^2 + 4y^2 - 16 = 0 \tag{3}$$

From (2) $(1-4\lambda)y = 0$, so $y = 0 \vee \lambda = \frac{1}{4}$. If $y = 0$, from (3) $x = \pm 4$; if $\lambda = \frac{1}{4}$, from (1) $(1+\lambda)\,x = 5 \implies x = 4$, substituting into (3) gives $y = 0$. Therefore, the extremum points are $(x, y) = (4, 0)$, $(-4, 0)$; $x^2 - 10x - y^2$ has a maximum value of 56 (at $(x, y) = (-4, 0)$), and a minimum value of -24 (at $(x, y) = (4, 0)$).

Ex. Find the point on $x^2 = y^2 + z^2$ that is closest to $(0, 1, 3)$.

Sol. The square of the distance function is $x^2 + (y-1)^2 + (z-3)^2$, with the constraint $x^2 - y^2 - z^2 = 0$. Let $\mathcal{L} = x^2 + (y-1)^2 + (z-3)^2 + \lambda(x^2 - y^2 - z^2)$, then

$$\frac{\partial \mathcal{L}}{\partial x} = 2x + 2\lambda x = 0 \implies (1+\lambda)x = 0 \tag{4}$$
$$\frac{\partial \mathcal{L}}{\partial y} = 2\,(y-1) - 2\lambda y = 0 \implies (1-\lambda)y = 1 \tag{5}$$
$$\frac{\partial \mathcal{L}}{\partial z} = 2\,(z-3) - 2\lambda z = 0 \implies (1-\lambda)z = 3 \tag{6}$$
$$x^2 - y^2 - z^2 = 0 \tag{7}$$

From (4) $(1+\lambda)x = 0$, so $x = 0 \ \vee\ \lambda = -1$. If $x = 0$, from (7) $y = z = 0$; if $\lambda = -1$, from (5) $y = \frac{1}{2}$, from (6) $z = \frac{3}{2}$, substituting into (7) gives $x = \pm\sqrt{\frac{5}{2}}$. Therefore, the extremum points are $(x, y, z) = (0, 0, 0)$, $\left(\pm\sqrt{\frac{5}{2}}, \frac{1}{2}, \frac{3}{2}\right)$; the minimum value of the square of the distance $x^2 + (y-1)^2 + (z-3)^2$ is 5, occurring at $(x, y, z) = \left(\pm\sqrt{\frac{5}{2}}, \frac{1}{2}, \frac{3}{2}\right)$.

Ex. Find the maximum and minimum values of $f(x, y, z) = (x + z)\, e^{y}$ on $x^2 + y^2 + z^2 = 6$.

Sol. Let $\mathcal{L} = (x+z)\,e^y + \lambda\,(x^2+y^2+z^2-6)$, then

$$\frac{\partial \mathcal{L}}{\partial x} = e^y + 2\lambda x = 0 \tag{8}$$
$$\frac{\partial \mathcal{L}}{\partial y} = (x+z)\,e^y + 2\lambda y = 0 \tag{9}$$
$$\frac{\partial \mathcal{L}}{\partial z} = e^y + 2\lambda z = 0 \tag{10}$$
$$x^2+y^2+z^2-6 = 0 \tag{11}$$

From (8), (10) $2\,\lambda(x-z)=0$, so $\lambda = 0 \;\vee\; x = z$. If $\lambda = 0$, then from (8) $e^y = 0$ which is impossible, so $x = z$. From (8) $e^y = -2\lambda x$, substituting into (9) $2x\,(-2\lambda x) + 2\lambda y = 0 \implies y = 2x^2$, substituting into (11) gives $x^2 + 4x^4 + x^2 = 6 \implies (4x^2+6)(x^2-1) = 0 \implies x = \pm 1$. Therefore, the extremum points are $(x,y,z) = (1,2,1),\ (-1,2,-1)$; $(x+z)\,e^y$ has a maximum value of $2e^2$ (at $(x,y,z) = (1,2,1)$), and a minimum value of $-2e^2$ (at $(x,y,z) = (-1,2,-1)$).

Ex. If L is the curve of intersection of $z^2 = x^2+y^2$ and $x - 2z = 3$, find the point on L that is closest to the origin and the shortest distance.

Sol. The square of the distance function is $x^2+y^2+z^2$, with constraints $x^2+y^2-z^2 = 0$ and $x-2z-3 = 0$. Let $\mathcal{L} = x^2+y^2+z^2+\lambda_1\,(x^2+y^2-z^2)+\lambda_2\,(x-2z-3)$, then

$$\frac{\partial \mathcal{L}}{\partial x} = 2x + 2\lambda_1 x + \lambda_2 = 0 \implies 2\,(1+\lambda_1)x + \lambda_2 = 0 \tag{12}$$
$$\frac{\partial \mathcal{L}}{\partial y} = 2y + 2\lambda_1 y = 0 \implies (1+\lambda_1)y = 0 \tag{13}$$
$$\frac{\partial \mathcal{L}}{\partial z} = 2z - 2\lambda_1 z - 2\lambda_2 = 0 \implies (1-\lambda_1)z - \lambda_2 = 0 \tag{14}$$
$$x^2+y^2-z^2 = 0 \tag{15}$$
$$x - 2z - 3 = 0 \tag{16}$$

From (13) $(1+\lambda_1)y = 0$, so $y = 0 \;\vee\; \lambda_1 = -1$. If $y = 0$, from (15) $x^2 = z^2 \implies x = \pm z$. If $x = z$, from (16) $x = z = -3$. If $x = -z$, from (16) $x = 1$, $z = -1$; if $\lambda_1 = -1$, from (12) $\lambda_2 = 0$, from (14) $z = 0$, substituting into (15) gives $x = y = 0$, which contradicts (16). Therefore, the extremum points are $(x,y,z) = (-3,0,-3)$, $(1,0,-1)$; the minimum value of the square of the distance $x^2+y^2+z^2$ is 2 (shortest distance is $\sqrt{2}$), occurring at $(x,y,z) = (1,0,-1)$.