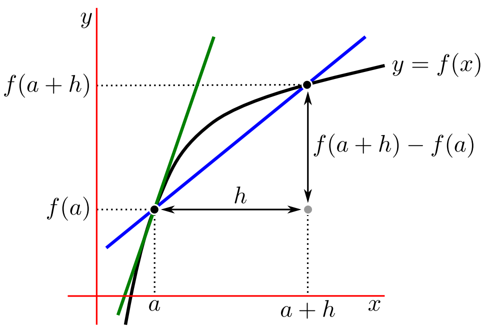
第二章 微分

2.1 導數與導函數



定義. 給定 $f(x)$, $a \in \operatorname{dom} f$. f 在 a 的導數 (derivative) $f'(a)$ 定義為

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

若 $f'(a)$ 存在, 則稱 f 在 a 可微 (分)(differentiable). f 的導函數 $f'(x)$ 定義為

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

註.

- 求導函數之過程稱為微分 (differentiation)：求 $f'(x) \iff f(x)$ (對 x) 微分
- 給定 $y=f(x)$, 其導函數可記為 $f'(x) = f' = y' = \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}f}{\mathrm{d}x} = \dfrac{\mathrm{d}}{\mathrm{d}x}f(x) = Df(x) = D_x f(x)$.
- f 在 a 的導數可記為 $f'(a) = \left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a}$.

例. 以極限定義求以下 $f(x)$ 之導函數 $f'(x)$, 當 $f(x)$ 為

1. x^2 2. x^4 3. $\frac{1}{x}$ 4. $\frac{1}{x^5}$ 5. $\frac{1}{x^2+3}$ 6. $\sqrt{x+1}$ 7. $\sqrt{x^2+1}$ 8. $\sqrt[3]{1-x^3}$

解.

1. $f'(x) = \lim\limits_{h\to 0} \dfrac{(x+h)^2-x^2}{h} = \lim\limits_{h\to 0} \dfrac{2xh+h^2}{h} = \lim\limits_{h\to 0} (2x+h) = 2x$.

2. $f'(x) = \lim\limits_{h\to 0} \dfrac{(x+h)^4-x^4}{h} = \lim\limits_{h\to 0} \dfrac{4x^3h+6x^2h^2+4xh^3+h^4}{h} = \lim\limits_{h\to 0} (4x^3+6x^2h+4xh^2+h^3) = 4x^3$.

3. $f'(x) = \lim\limits_{h\to 0} \dfrac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim\limits_{h\to 0} \dfrac{x-(x+h)}{h\,(x+h)\,x} = \lim\limits_{h\to 0} \dfrac{-h}{h\,(x+h)\,x} = \lim\limits_{h\to 0} \dfrac{-1}{(x+h)\,x} = \dfrac{-1}{x^2} = (-1)\,x^{-2}$.

4. $f'(x) = \lim\limits_{h\to 0} \dfrac{\frac{1}{(x+h)^5}-\frac{1}{x^5}}{h} = \lim\limits_{h\to 0} \dfrac{x^5-(x+h)^5}{h\,(x+h)^5\,x^5}$
$= \lim\limits_{h\to 0} \dfrac{(-h)(x^4+x^3(x+h)+x^2(x+h)^2+x^3(x+h)+(x+h)^4)}{h\,(x+h)^5\,x^5}$
$= \lim\limits_{h\to 0} \dfrac{-(x^4+x^3(x+h)+x^2(x+h)^2+x^3(x+h)+(x+h)^4)}{(x+h)^5\,x^5} = \dfrac{-5x^4}{x^{10}} = \dfrac{-5}{x^6} = (-5)\,x^{-6}$.

5. $f'(x) = \lim\limits_{h\to 0} \dfrac{\frac{1}{(x+h)^2+3}-\frac{1}{x^2+3}}{h} = \lim\limits_{h\to 0} \dfrac{(x^2+3)-((x+h)^2+3)}{h\,((x+h)^2+3)\,(x^2+3)} = \lim\limits_{h\to 0} \dfrac{(2x+h)\,(-h)}{h\,((x+h)^2+3)\,(x^2+3)}$
$= \lim\limits_{h\to 0} \dfrac{-(2x+h)}{((x+h)^2+3)(x^2+3)} = \dfrac{-2x}{(x^2+3)^2}$.

6. $f'(x) = \lim\limits_{h\to 0} \dfrac{\sqrt{x+h+1}-\sqrt{x+1}}{h} = \lim\limits_{h\to 0} \dfrac{h}{h\,(\sqrt{x+h+1}+\sqrt{x+1})} = \dfrac{1}{2\sqrt{x+1}}$.

7. $f'(x) = \lim\limits_{h\to 0} \dfrac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h} = \lim\limits_{h\to 0} \dfrac{h\,(2x+h)}{h\,(\sqrt{(x+h)^2+1}+\sqrt{x^2+1})} = \dfrac{x}{\sqrt{x^2+1}}$.

8. $f'(x)=\lim\limits_{h\to 0}\dfrac{\sqrt[3]{1-(x+h)^3}-\sqrt[3]{1-x^3}}{h}$ (使用 $a^3-b^3=(a-b)(a^2+ab+b^2)$)

$$=\lim_{h\to 0}\frac{1-(x+h)^3-(1-x^3)}{h\left(\left(\sqrt[3]{1-(x+h)^3}\right)^2+\sqrt[3]{1-(x+h)^3}\,\sqrt[3]{1-x^3}+\left(\sqrt[3]{1-x^3}\right)^2\right)}$$

$$=\lim_{h\to 0}\frac{(-h)(x^2+x(x+h)+(x+h)^2)}{h\left(\left(\sqrt[3]{1-(x+h)^3}\right)^2+\sqrt[3]{1-(x+h)^3}\,\sqrt[3]{1-x^3}+\left(\sqrt[3]{1-x^3}\right)^2\right)}=\frac{-3x^2}{3\left(\sqrt[3]{1-x^3}\right)^2}=\frac{-x^2}{\left(\sqrt[3]{1-x^3}\right)^2}.$$

結論. x^α ($\alpha\in\mathbb{R}$) 之導函數為 $\alpha\, x^{\alpha-1}$.

定義. 若 f 在 (a,b) 上每一點均有導數, 則稱 f 在 (a,b) 可微 (分) .

定理. 若 f 在 a 可微, 則 f 在 a 連續.

2.2 微分規則

定理 (四則運算)**.** 令 f, g 可微, $c\in\mathbb{R}$. 則

1. $(c)'=0$
2. $(c\,f)'=c\,f'$
3. $(f\pm g)'=f'\pm g'$
4. $(f\cdot g)'=f'\cdot g+f\cdot g'$
5. $\left(\dfrac{f}{g}\right)'=\dfrac{g\cdot f'-g'\cdot f}{g^2}$

例. 求導函數.

1. x^5
2. $\frac{1}{x^2+3}$
3. $\frac{x-1}{x+1}$
4. $\sqrt{\frac{x-1}{x+1}}$

解.

1. $(x^5)'=(x\cdot x^2\cdot x^2)'=(x)'\cdot x^2\cdot x^2+x\cdot(x^2)'\cdot x^2+x\cdot x^2\cdot(x^2)'=x^4+x\cdot(2x)\cdot x^2+x^3\cdot(2x)=5x^4$
2. $\left(\dfrac{1}{x^2+3}\right)'=\dfrac{(x^2+3)\cdot(1)'-(x^2+3)'\cdot 1}{(x^2+3)^2}=\dfrac{-2x}{(x^2+3)^2}$
3. $\left(\dfrac{x-1}{x+1}\right)'=\dfrac{(x+1)\cdot(x-1)'-(x+1)'\cdot(x-1)}{(x+1)^2}=\dfrac{2}{(x+1)^2}$
4. $\left(\sqrt{\dfrac{x-1}{x+1}}\right)'=\left(\dfrac{\sqrt{x-1}}{\sqrt{x+1}}\right)'=\dfrac{\sqrt{x+1}\cdot(\sqrt{x-1})'-(\sqrt{x+1})'\cdot\sqrt{x-1}}{(\sqrt{x+1})^2}$
   $=\dfrac{\sqrt{x+1}\cdot\frac{1}{2\sqrt{x-1}}-\frac{1}{2\sqrt{x+1}}\cdot\sqrt{x-1}}{(\sqrt{x+1})^2}=\dfrac{\sqrt{x+1}\cdot\sqrt{x+1}-\sqrt{x-1}\cdot\sqrt{x-1}}{2\sqrt{x-1}(\sqrt{x+1})^3}=\dfrac{1}{\sqrt{(x-1)(x+1)^3}}$

定理 (鏈鎖律 (chain rule))**.** 若 $f(u)$ 在 $u=g(x)$ 可微, $g(x)$ 在 x 可微, 則 $f\circ g$ 在 x 可微:

$$(f\circ g)'(x)\;\equiv\;(f(g(x)))'=f'(g(x))\cdot g'(x)$$

例. 求導函數.

1. $(x^3-1)^{2025}$
2. $\sqrt{x^2+1}$
3. $\frac{1}{x^2+3}$
4. $\sqrt{\frac{x-1}{x+1}}$

解.

1. 令 $f(u)=u^{2025}$, $g(x)=x^3-1$, 則 $f'(u)=2025\,u^{2024}$, $(x^3-1)^{2025}=f(g(x))$.
   由鏈鎖律 $((x^3-1)^{2025})'=(f(g(x)))'=f'(g(x))\cdot g'(x)=2025\cdot(x^3-1)^{2024}\cdot(3x^2)$.
2. 令 $f(u)=\sqrt{u}=u^{\frac{1}{2}}$, $g(x)=x^2+1$, 則 $f'(u)=\dfrac{1}{2\sqrt{u}}$, $\sqrt{x^2+1}=f(g(x))$.
   由鏈鎖律 $(\sqrt{x^2+1})'=(f(g(x)))'=f'(g(x))\cdot g'(x)=\dfrac{1}{2\sqrt{x^2+1}}\cdot(2x)=\dfrac{x}{\sqrt{x^2+1}}$.

3. 令 $f(u)=\dfrac{1}{u}$, $g(x)=x^2+3$, 則 $f'(u)=\dfrac{-1}{u^2}$, $\dfrac{1}{x^2+3}=f(g(x))$.

由鏈鎖律 $\left(\dfrac{1}{x^2+3}\right)'=(f(g(x)))'=f'(g(x))\cdot g'(x)=\dfrac{-1}{(x^2+3)^2}\cdot(x^2+3)'=\dfrac{-2x}{(x^2+3)^2}$.

4. 令 $f(u)=\sqrt{u}$, $g(x)=\dfrac{x-1}{x+1}$, 則 $f'(u)=\dfrac{1}{2\sqrt{u}}$, $\sqrt{\dfrac{x-1}{x+1}}=f(g(x))$.

由鏈鎖律 $\left(\sqrt{\dfrac{x-1}{x+1}}\right)'=(f(g(x)))'=f'(g(x))\cdot g'(x)=\dfrac{1}{2\sqrt{\frac{x-1}{x+1}}}\cdot\left(\dfrac{x-1}{x+1}\right)'=\dfrac{1}{2\sqrt{\frac{x-1}{x+1}}}\dfrac{2}{(x+1)^2}=$ $\dfrac{1}{\sqrt{(x-1)(x+1)^3}}$.

結論. 若 $f(g(x))=x$, 等式兩邊對 x 微分 $\implies (f(g(x)))'=1 \implies f'(g(x))\cdot g'(x)=1 \implies g'(x)=\dfrac{1}{f'(g(x))}$.

例. f, g 為可微函數且 $f(g(x))=x$. 若 $f'(x)=1+(f(x))^2$, 求 $g'(x)$.

解. $f(g(x))=x$ 等式兩邊對 x 微分得 $f'(g(x))\cdot g'(x)=1 \implies g'(x)=\dfrac{1}{f'(g(x))}=\dfrac{1}{1+(f(g(x)))^2}=\dfrac{1}{1+x^2}$.

2.3 自然指數, 對數與微分

定義 (自然指數 e 與 e^x 微分)**.**

- 給定 $a>0$, 求 $f(x)=a^x$ 之導函數
- $\dfrac{\mathrm{d}f}{\mathrm{d}x}=\lim\limits_{h\to0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to0}\dfrac{a^{x+h}-a^x}{h}=\lim\limits_{h\to0}a^x\cdot\dfrac{a^h-1}{h}=a^x\cdot\lim\limits_{h\to0}\dfrac{a^h-1}{h}=a^x\cdot C(a)$
- 觀察: $C(a)$ 隨 a 遞增; 存在 $\frac{27}{10}<e<\frac{68}{25}$ 使 $C(e)=1$.

h	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
$\dfrac{2^h-1}{h}$	0.7177	0.6956	0.6934	0.6932	0.6931	0.6931	0.6931
$\dfrac{(\frac{5}{2})^h-1}{h}$	0.9596	0.9205	0.9167	0.9163	0.9163	0.9163	0.9163
$\dfrac{(\frac{27}{10})^h-1}{h}$	1.0442	0.9982	0.9937	0.9933	0.9933	0.9933	0.9933
$\dfrac{(\frac{68}{25})^h-1}{h}$	1.0524	1.0056	1.0011	1.0007	1.0006	1.0006	1.0006
$\dfrac{(\frac{28}{10})^h-1}{h}$	1.0845	1.0349	1.0301	1.0297	1.0296	1.0296	1.0296
$\dfrac{3^h-1}{h}$	1.1612	1.1047	1.0992	1.0987	1.0986	1.0986	1.0986

- $C(e)=1 \implies \lim\limits_{h\to0}\dfrac{e^h-1}{h}=1 \implies \dfrac{\mathrm{d}}{\mathrm{d}x}e^x=C(e)\cdot e^x \implies (e^x)'=e^x.\quad \ln x\equiv\log_e x$

性質. $(\ln|x|)'=\dfrac{1}{x}$.

解.

- 若 $x>0$, $\ln|x|=\ln x$ 且 $e^{\ln x}=x$. 令 $f(u)=e^u$, $g(x)=\ln|x|=\ln x$, 則 $f'(u)=e^u$, $f(g(x))=x$; 故 $g'(x)=\dfrac{1}{f'(g(x))} \implies (\ln|x|)'=(\ln x)'=\dfrac{1}{e^{\ln x}}=\dfrac{1}{x}$.
- 若 $x<0$, $\ln|x|=\ln(-x)$ 且 $e^{\ln(-x)}=-x$. 令 $f(u)=e^u$, $g(x)=\ln|x|=\ln(-x)$, 則 $f'(u)=e^u$, $f(g(x))=-x$; 故 $g'(x)=\dfrac{-1}{f'(g(x))} \implies (\ln|x|)'=(\ln(-x))'=\dfrac{-1}{e^{\ln(-x)}}=\dfrac{1}{x}$.

性質. $(a^x)'=a^x\cdot\ln a$, $\forall\, a>0$.

證. $a=e^{\log_e a}=e^{\ln a} \implies a^x=e^{x\ln a}$. 令 $f(u)=e^u$, $g(x)=x\ln a$, 則 $f'(u)=e^u$, $f(g(x))=e^{x\ln a}=a^x$; 故 $(f(g(x)))'=f'(g(x))\cdot g'(x)=e^{x\ln a}\cdot(x\ln a)'=a^x\cdot\ln a$.

結論. $C(a)=\ln a$.

性質. $(x^\alpha)'=\alpha\, x^{\alpha-1}$ $(\alpha\in\mathbb{R})$

解. $x^\alpha=e^{\ln x^\alpha}=e^{\alpha\ln x}$. 故 $(x^\alpha)'=(e^{\alpha\ln x})'=e^{\alpha\ln x}\cdot\left(\alpha\cdot\dfrac{1}{x}\right)=x^\alpha\cdot\alpha\cdot\dfrac{1}{x}=\alpha\, x^{\alpha-1}$.

例. 證明 $\displaystyle\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$.

解. 令 $f(x)=\ln(1+x)$, 則 $f(0)=0$, $f'(x)=\dfrac{1}{1+x}$, $f'(0)=1$. 故 $\displaystyle\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\frac{f(x)}{x}=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=f'(0)=1$.

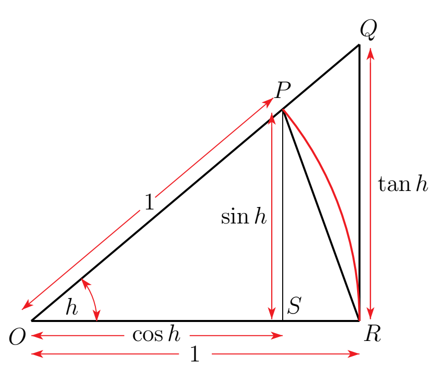
例. 令 $g(x)=e^x+x$, 求 $(g^{-1})'(e+1)$.

解. 令 $f(x)=g^{-1}(x)$, 則 $f(g(x))=g^{-1}(g(x))=x$. 等式兩邊對 x 微分得 $f'(g(x))\cdot g'(x)=1 \implies f'(g(x))=\dfrac{1}{g'(x)}$. 由 $g(1)=e+1$, $g'(x)=e^x+1$, $f'(e+1)=f'(g(1))=\dfrac{1}{g'(1)}=\dfrac{1}{e+1}$.

2.4 三角函數微分

性質. $\displaystyle\lim_{h\to 0}\frac{\sin h}{h}=1$

證.



取 $0<h\ll\frac{\pi}{2}$ 作圖如左. 比較面積 $\Delta OPR\leqslant\triangledown OPR\leqslant\Delta OQR$ $\implies \sin h\leqslant h\leqslant\dfrac{\sin h}{\cos h}$. 因 h, $\sin h$, $\cos h$ 均為正, 不等式同除 $\sin h$ 並取倒數及變向後得 $\cos h\leqslant\dfrac{\sin h}{h}\leqslant 1$. 由 $\displaystyle\lim_{h\to 0+}\cos h=1$ 與夾擠定理得 $\displaystyle\lim_{h\to 0+}\frac{\sin h}{h}=1$. 又 $\displaystyle\lim_{h\to 0-}\frac{\sin h}{h}=\lim_{(-h)\to 0+}\frac{\sin h}{h}=\underbrace{\lim_{(-h)\to 0+}\frac{\sin(-h)}{(-h)}}_{\sin\text{ 為奇函數}}=\underbrace{\lim_{H\to 0+}\frac{\sin H}{H}}_{\text{令 } H\equiv -h}=1$, 故 $\displaystyle\lim_{h\to 0}\frac{\sin h}{h}=1$.

性質. $\displaystyle\lim_{h\to 0}\frac{\cos h-1}{h}=0$

解. 由 $\cos h=\cos\left(\dfrac{h}{2}+\dfrac{h}{2}\right)=\cos^2\dfrac{h}{2}-\sin^2\dfrac{h}{2}=1-2\sin^2\dfrac{h}{2} \implies \cos h-1=-2\sin^2\dfrac{h}{2}$, 則 $\displaystyle\lim_{h\to 0}\frac{\cos h-1}{h}=\lim_{h\to 0}-\frac{2\sin^2\frac{h}{2}}{h}=-\lim_{h\to 0}\frac{\sin^2\frac{h}{2}}{\frac{h}{2}}=\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\cdot\sin\frac{h}{2}=\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\cdot\lim_{h\to 0}\sin\frac{h}{2}=\underbrace{\lim_{H\to 0}\frac{\sin H}{H}}_{\text{令 } H\equiv\frac{h}{2}}\cdot\lim_{h\to 0}\sin\frac{h}{2}=1\cdot 0=0$

性質. $(\sin x)' = \cos x$

證. $(\sin x)' = \lim\limits_{h\to 0}\dfrac{\sin(x+h)-\sin x}{h} = \lim\limits_{h\to 0}\dfrac{\sin x\cos h + \cos x \sin h - \sin x}{h} = \lim\limits_{h\to 0} \sin x\,\dfrac{\cos h - 1}{h} + \cos x\,\dfrac{\sin h}{h} = \sin x \lim\limits_{h\to 0}\dfrac{\cos h - 1}{h} + \cos x \lim\limits_{h\to 0}\dfrac{\sin h}{h} = \cos x$

性質. $(\cos x)' = -\sin x$

證. $(\cos x)' = \left(\sin\left(\dfrac{\pi}{2}-x\right)\right)' = \cos\left(\dfrac{\pi}{2}-x\right)\cdot\left(\dfrac{\pi}{2}-x\right)' = -\sin x$

性質. $(\tan x)' = \sec^2 x$

證. $(\tan x)' = \left(\dfrac{\sin x}{\cos x}\right)' = \dfrac{\cos x\cdot\cos x - (-\sin x)\cdot \sin x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$

性質. $(\sec x)' = \sec x \tan x$

證. $(\sec x)' = \left(\dfrac{1}{\cos x}\right)' = \dfrac{\cos x\cdot 0 - (-\sin x)\cdot 1}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x}\,\dfrac{\sin x}{\cos x} = \sec x\tan x$

性質. $(\cot x)' = -\csc^2 x$

證. $(\cot x)' = \left(\dfrac{\cos x}{\sin x}\right)' = \dfrac{\sin x\cdot(-\sin x) - \cos x\cdot\cos x}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$

性質. $(\csc x)' = -\csc x \cot x$

證. $(\csc x)' = \left(\dfrac{1}{\sin x}\right)' = \dfrac{\sin x\cdot 0 - (\cos x)\cdot 1}{\sin^2 x} = -\dfrac{\cos x}{\sin^2 x} = -\dfrac{1}{\sin x}\,\dfrac{\cos x}{\sin x} = -\csc x\cot x$

2.5 反三角函數微分

性質. $(\sin^{-1} x)' = \dfrac{1}{\sqrt{1-x^2}}$

證. $\sin(\sin^{-1} x) = x$, $x\in[-1,1]$. 令 $f(u) = \sin u$, $g(x) = \sin^{-1} x$, 則 $f'(u) = \cos u$, $f(g(x)) = x$; 故 $g'(x) = \dfrac{1}{f'(g(x))} \implies (\sin^{-1} x)' = \dfrac{1}{\cos(\sin^{-1} x)} = \dfrac{1}{\sqrt{1-x^2}}$.

性質. $(\cos^{-1} x)' = -\dfrac{1}{\sqrt{1-x^2}}$

證. $\cos(\cos^{-1} x) = x$, $x\in[-1,1]$. 令 $f(u) = \cos u$, $g(x) = \cos^{-1} x$, 則 $f'(u) = -\sin u$, $f(g(x)) = x$; 故 $g'(x) = \dfrac{1}{f'(g(x))} \implies (\cos^{-1} x)' = \dfrac{1}{-\sin(\cos^{-1} x)} = -\dfrac{1}{\sqrt{1-x^2}}$.

性質. $(\tan^{-1} x)' = \dfrac{1}{1+x^2}$

證. $\tan(\tan^{-1} x) = x$, $x\in(-\infty,\infty)$. 令 $f(u) = \tan u$, $g(x) = \tan^{-1} x$, 則 $f'(u) = \sec^2 u$, $f(g(x)) = x$; 故 $g'(x) = \dfrac{1}{f'(g(x))} \implies (\tan^{-1} x)' = \dfrac{1}{\sec^2(\tan^{-1} x)} = \dfrac{1}{1+x^2}$.

性質. $(\cot^{-1} x)' = -\dfrac{1}{1+x^2}$

證. $\cot(\cot^{-1}x)=x,\ x\in(-\infty,\infty)$. 令 $f(u)=\cot u,\ g(x)=\cot^{-1}x$, 則 $f'(u)=-\csc^2 u,\ f(g(x))=x$; 故 $g'(x)=\dfrac{1}{f'(g(x))} \implies (\cot^{-1}x)'=\dfrac{1}{-\csc^2(\cot^{-1}x)}=-\dfrac{1}{1+x^2}$.

性質. $(\sec^{-1}x)'=\dfrac{1}{x\sqrt{x^2-1}}$

證. $\sec(\sec^{-1}x)=x,\ x\in(1,\infty)\ \vee\ (-\infty,-1)$. 令 $f(u)=\sec u,\ g(x)=\sec^{-1}x$, 則 $f'(u)=\sec u\tan u$, $f(g(x))=x$; 故 $g'(x)=\dfrac{1}{f'(g(x))} \implies (\sec^{-1}x)'=\dfrac{1}{\sec(\sec^{-1}x)\tan(\sec^{-1}x)}=\dfrac{1}{x\sqrt{x^2-1}}$.

性質. $(\csc^{-1}x)'=-\dfrac{1}{x\sqrt{x^2-1}}$

證. $\csc(\csc^{-1}x)=x,\ x\in(1,\infty)\ \vee\ (-\infty,-1)$. 令 $f(u)=\csc u,\ g(x)=\csc^{-1}x$, 則 $f'(u)=-\csc u\cot u$, $f(g(x))=x$; 故 $g'(x)=\dfrac{1}{f'(g(x))} \implies (\csc^{-1}x)'=-\dfrac{1}{\csc(\csc^{-1}x)\cot(\csc^{-1}x)}=-\dfrac{1}{x\sqrt{x^2-1}}$.

註. 由定義 $\sec:[0,\frac{\pi}{2})\cup[\pi,\frac{3\pi}{2})\to(-\infty,-1]\cup[1,\infty)$, $\csc:(0,\frac{\pi}{2}]\cup(\pi,\frac{3\pi}{2}]\to(-\infty,-1]\cup[1,\infty)$, 其反三角函數為 $\sec^{-1}:(-\infty,-1]\cup[1,\infty)\to[0,\frac{\pi}{2})\cup[\pi,\frac{3\pi}{2})$, $\csc^{-1}:(-\infty,-1]\cup[1,\infty)\to(0,\frac{\pi}{2}]\cup(\pi,\frac{3\pi}{2}]$, 故 $\tan(\sec^{-1}x)$ 與 $\cot(\csc^{-1}x)$ 恆為正值. 依此, 若 $u=\sec^{-1}x$, 則 $\tan^2 u=\sec^2 u-1 \implies \tan u=\sqrt{\sec^2 u-1}=\sqrt{x^2-1} \implies \tan(\sec^{-1}x)=\sqrt{x^2-1}$ (開平方僅需取正值). 同理, 若 $u=\csc^{-1}x$, 則 $\cot^2 u=\csc^2 u-1 \implies \cot u=\sqrt{\csc^2 u-1}=\sqrt{x^2-1} \implies \cot(\csc^{-1}x)=\sqrt{x^2-1}$. 若初始定義 $\sec:[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]\to(-\infty,-1]\cup[1,\infty)$, $\csc:(0,\frac{\pi}{2}]\cup[\frac{\pi}{2},\pi)\to(-\infty,-1]\cup[1,\infty)$ 則 $\tan(\sec^{-1}x)$ 與 $\cot(\csc^{-1}x)$ 之正負將依 x 之正負而定: 此時 $(\sec^{-1}x)'=\dfrac{1}{|x|\sqrt{x^2-1}}$, $(\csc^{-1}x)'=-\dfrac{1}{|x|\sqrt{x^2-1}}$.

常用初等函數微分公式

$f(x)$	e^x	$\ln\lvert x\rvert$	x^α	$\sin x$	$\cos x$	$\tan x$	$\sec x$	$\sin^{-1}x$	$\cos^{-1}x$	$\tan^{-1}x$
$f'(x)$	e^x	$\dfrac{1}{x}$	$\alpha\,x^{\alpha-1}$	$\cos x$	$-\sin x$	$\sec^2 x$	$\sec x\tan x$	$\dfrac{1}{\sqrt{1-x^2}}$	$-\dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{1}{1+x^2}$

2.6 對數微分法

性質. $(\ln g(x))'=\dfrac{g'(x)}{g(x)}$.

證. 令 $f(u)=\ln u$, 則 $f'(u)=\dfrac{1}{u}$, $\ln g(x)=f(g(x))$. 由鏈鎖律 $(f(g(x)))'=f'(g(x))\cdot g'(x) \implies (\ln g(x))'=\dfrac{1}{g(x)}\cdot g'(x)=\dfrac{g'(x)}{g(x)}$.

例. $f(x)=\sqrt[4]{\dfrac{(x^4+12)(x^5-x^2+2)}{x^3+1}}$, 求 $f'(x)$.

解. $\ln f(x)=\dfrac{1}{4}\ln\dfrac{(x^4+12)(x^5-x^2+2)}{x^3+1}=\dfrac{1}{4}\big(\ln(x^4+12)+\ln(x^5-x^2+2)-\ln(x^3+1)\big)$; 等式兩邊對 x 微分得 $\dfrac{f'(x)}{f(x)}=\dfrac{1}{4}\left(\dfrac{4x^3}{x^4+12}+\dfrac{5x^4-2x}{x^5-x^2+2}-\dfrac{3x^2}{x^3+1}\right) \implies f'(x)=\sqrt[4]{\dfrac{(x^4+12)(x^5-x^2+2)}{x^3+1}}\cdot\dfrac{1}{4}\left(\dfrac{4x^3}{x^4+12}+\dfrac{5x^4-2x}{x^5-x^2+2}-\dfrac{3x^2}{x^3+1}\right)$.

例. $f(x)=\dfrac{e^{-x}\cos^2 x}{x^2+x+1}$, 求 $f'(x)$.

解. $\ln f(x)=\ln(e^{-x}\cos^2 x)-\ln(x^2+x+1)=\ln(e^{-x})+\ln(\cos^2 x)-\ln(x^2+x+1)=-x+2\ln(\cos x)-\ln(x^2+x+1)$; 等式兩邊對 x 微分得 $\dfrac{f'(x)}{f(x)}=-1-2\tan x-\dfrac{2x+1}{x^2+x+1} \implies f'(x)=-\dfrac{e^{-x}\cos^2 x}{x^2+x+1}\cdot\Big(1+2\tan x+\dfrac{2x+1}{x^2+x+1}\Big)$.

例. 給定 $f(x)$, 求 $f'(x)$.

- $f(x)=(\sin x)^{\ln x}$
- $f(x)=(\tan x)^{\frac{1}{x}}$
- $f(x)=(\cos x)^{\sin x}$

解.

- $\log f(x)=\ln x\cdot\ln(\sin x) \implies f'(x)=(\sin x)^{\ln x}\cdot\Big(\dfrac{1}{x}\cdot\ln(\sin x)+\ln x\cdot\dfrac{\cos x}{\sin x}\Big)=(\sin x)^{\ln x}\cdot\Big(\dfrac{\ln(\sin x)}{x}+\ln x\cdot\cot x\Big)$
- $\log f(x)=\dfrac{1}{x}\cdot\ln(\tan x) \implies f'(x)=(\tan x)^{\frac{1}{x}}\cdot\Big(-\dfrac{1}{x^2}\cdot\ln(\tan x)+\dfrac{1}{x}\cdot\dfrac{\sec^2 x}{\tan x}\Big)=(\tan x)^{\frac{1}{x}}\cdot\Big(-\dfrac{\ln(\tan x)}{x^2}+\dfrac{\sec x\csc x}{x}\Big)$
- $\log f(x)=\sin x\cdot\ln(\cos x) \implies f'(x)=(\cos x)^{\sin x}\cdot\Big(\cos x\cdot\ln(\cos x)+\sin x\cdot\dfrac{-\sin x}{\cos x}\Big)$

2.7 隱函數微分

例. 求圓 $x^2+y^2=25$ 上點 $(3,-4)$ 之切線方程式.

解.

- (顯函數微分) 在點 $(3,-4)$ 附近 $y=-\sqrt{25-x^2} \implies y'=\dfrac{x}{\sqrt{25-x^2}}$, 故點 $(3,-4)$ 之切線斜率為 $\dfrac{3}{\sqrt{25-3^2}}=\dfrac{3}{4}$, 切線方程式為 $(y+4)=\dfrac{3}{4}(x-3)$.
- (隱函數微分) 令點 $(3,-4)$ 附近 y 為 x 之函數 $(y=y(x))$, 圓方程式寫作 $x^2+y(x)^2=25$; 兩邊同對 x 微分: $2x+2\,y(x)\cdot y'(x)=0 \implies y'(x)=-\dfrac{x}{y(x)}$. 點 $(3,-4)$ 之切線斜率為 $y'(3)=-\dfrac{3}{y(3)}=\dfrac{3}{4}$, 切線方程式為 $(y+4)=\dfrac{3}{4}(x-3)$.

例. 若 $xy+e^x+e^y=1$, 求 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

解. 令 y 為 x 之函數 $(y=y(x))$, 等式寫作 $x\cdot y(x)+e^x+e^{y(x)}=1$; 兩邊對 x 微分: $x\cdot y'(x)+y(x)+e^x+e^{y(x)}\cdot y'(x)=0 \implies (x+e^{y(x)})\cdot y'(x)=-(y(x)+e^x) \implies \dfrac{\mathrm{d}y}{\mathrm{d}x}\equiv y'(x)=-\dfrac{y(x)+e^x}{x+e^{y(x)}}$.

例. 若 $x^y=y^x$, 求 y'.

解. 令 y 為 x 之函數 $(y=y(x))$, 等式寫作 $x^{y(x)}=y(x)^x \implies e^{y(x)\ln x}=e^{x\ln y(x)}$; 兩邊對 x 微分: $e^{y(x)\ln x}\cdot\Big(y'(x)\ln x+\dfrac{y(x)}{x}\Big)=e^{x\ln y(x)}\cdot\Big(\ln y(x)+x\cdot\dfrac{y'(x)}{y(x)}\Big) \implies y'(x)\ln x+\dfrac{y(x)}{x}=\ln y(x)+x\cdot\dfrac{y'(x)}{y(x)} \implies y'\ln x+\dfrac{y}{x}=\ln y+x\dfrac{y'}{y} \implies y'\Big(\ln x-\dfrac{x}{y}\Big)=\ln y-\dfrac{y}{x} \implies y'=\dfrac{\ln y-\frac{y}{x}}{\ln x-\frac{x}{y}}$.

例. 若 $y=\ln(x^2+y^2)$, 求 y'.

解. 令 y 為 x 之函數 $(y=y(x))$, 等式寫作 $y(x)=\ln\left(x^2+y(x)^2\right)$; 兩邊對 x 微分: $y'(x)=\dfrac{2x+2\,y(x)\,y'(x)}{x^2+y(x)^2}$
$\Longrightarrow\ y'=\dfrac{2x+2yy'}{x^2+y^2}\ \Longrightarrow\ (x^2+y^2)y'=2x+2yy'\ \Longrightarrow\ (x^2+y^2-2y)y'=2x\ \Longrightarrow\ y'=\dfrac{2x}{x^2+y^2-2y}$.

例. 若 $x^2e^y+4x\cos y=5$, 求在 $y=0$ 時之 y'.

解. 令 y 為 x 之函數 $(y=y(x))$, 等式寫作 $x^2e^{y(x)}+4x\cos y(x)=5$; 兩邊對 x 微分: $2x\cdot e^{y(x)}+x^2\cdot e^{y(x)}\cdot y'(x)+4\,\cos y(x)-4x\cdot\sin y(x)\cdot y'(x)=0$. 當 $y(x)=0$, 上式為 $2x\cdot e^0+x^2\cdot e^0\cdot y'(x)+4\,\cos 0-4x\cdot\sin 0\cdot y'(x)=0\ \Longrightarrow\ 2x+x^2\cdot y'(x)+4=0\ \Longrightarrow\ y'(x)=-\dfrac{4+2x}{x^2}$. 由 $x^2e^y+4x\cos y=5$ 知 $y=0$ 時 $x^2+4x=5\ \Longrightarrow\ x=-5\vee x=1$, 則 $y'(-5)=-\dfrac{4-10}{(-5)^2}=\dfrac{6}{25}$, $y'(1)=-\dfrac{4+2}{1^2}=-6$.

例. 若 $x^4+y^4=16$, 求 y''.

解. 等式兩邊對 x 微分得 $4x^3+4y^3y'=0\ \Longrightarrow\ y'=-\dfrac{x^3}{y^3}$; 兩邊再對 x 微分得 $y''=-\dfrac{y^3\cdot 3x^2-3y^2\cdot y'\cdot x^3}{y^6}$
$\Longrightarrow\ y''=-\dfrac{y^3\cdot 3x^2-3y^2\cdot(-\frac{x^3}{y^3})\cdot x^3}{y^6}=-\dfrac{3(x^2y^4+x^6)}{y^7}=-\dfrac{3x^2(y^4+x^4)}{y^7}=-\dfrac{3x^2\cdot 16}{y^7}=-\dfrac{48x^2}{y^7}$.

例 (0 階 Bessel 函數)**.** 若 $J(x)$ 滿足 $J(0)=1$ 與 $xJ''(x)+J'(x)+xJ(x)=0$, 求 $J'(0)$ 與 $J''(0)$.

解. 等式 $xJ''(x)+J'(x)+xJ(x)=0$ 代入 $x=0$ 得 $0\cdot J''(0)+J'(0)+0\cdot J(0)=0\ \Longrightarrow\ J'(0)=0$; 等式兩邊對 x 微分可得 $xJ'''(x)+J''(x)+J''(x)+J(x)+xJ'(x)=0$, 代入 $x=0$ 得 $J''(0)+0\cdot J'''(0)+J''(0)+J(0)+0\cdot J'(0)=0\ \Longrightarrow\ 2J''(0)+1=0\ \Longrightarrow\ J''(0)=-\frac{1}{2}$.

習題 (隱函數微分)**.** 求 y'.

1. $x^3+y^3=1\ \Longrightarrow\ y'=-\dfrac{x^2}{y^2}$
2. $2\sqrt{x}+\sqrt{y}=3\ \Longrightarrow\ y'=-\dfrac{2\sqrt{y}}{\sqrt{x}}$
3. $x^2+xy-y^2=4\ \Longrightarrow\ y'=\dfrac{2x+y}{2y-x}$
4. $xe^y=x-y\ \Longrightarrow\ y'=\dfrac{1-e^y}{xe^y+1}$
5. $e^{\frac{x}{y}}=x-y\ \Longrightarrow\ y'=\dfrac{y(y-e^{\frac{x}{y}})}{y^2-xe^{\frac{x}{y}}}$
6. $y\cos x=x^2+y^2\ \Longrightarrow\ y'=\dfrac{2x+y\sin x}{\cos x-2y}$
7. $4\cos x\sin y=1\ \Longrightarrow\ y'=\tan x\tan y$
8. $e^y\sin x=x+xy\ \Longrightarrow\ y'=\dfrac{1+y-e^y\cos x}{e^y\sin x-x}$
9. $\cos(xy)=1+\sin y\ \Longrightarrow\ y'=-\dfrac{y\sin(xy)}{x\sin(xy)+\cos y}$
10. $\sqrt{x+y}=1+x^2y^2\ \Longrightarrow\ y'=\dfrac{4xy^2\sqrt{x+y}-1}{1-4x^2y\sqrt{x+y}}$
11. $2x^3+x^2y-xy^3=2\ \Longrightarrow\ y'=\dfrac{y^3-6x^2-2xy}{x^2-3xy^2}$
12. $x^4(x+y)=y^2(3x-y)\ \Longrightarrow\ y'=\dfrac{3y^2-5x^4-4x^3y}{x^4+3y^2-6xy}$
13. $x\sin y+y\sin x=1\ \Longrightarrow\ y'=\dfrac{-\sin y-y\cos x}{x\cos y+\sin x}$
14. $e^y\cos x=1+\sin(xy)\ \Longrightarrow\ y'=\dfrac{e^y\sin x+y\cos(xy)}{e^y\cos x-x\cos(xy)}$

習題 (隱函數微分)**.** 求 y''.

1. $9x^2+y^2=9\ \Longrightarrow\ y''=-\dfrac{81}{y^3}$
2. $\sqrt{x}+\sqrt{y}=1\ \Longrightarrow\ y''=\dfrac{1}{2x\sqrt{x}}$
3. $x^3+y^3=1\ \Longrightarrow\ y''=-\dfrac{2x}{y^5}$
4. $x^4+y^4=a^4\ \Longrightarrow\ y''=-\dfrac{3a^4x^2}{y^7}$

習題 (基礎微分運算). 求下列導函數.

1. $\left((3x^2+5)^{-3}\right)' = \dfrac{-18x}{(3x^2+5)^4}$
2. $\left(\dfrac{1}{(2x-3)^2}\right)' = \dfrac{-4}{(2x-3)^3}$
3. $\left(\dfrac{1}{x^2-4}\right)' = \dfrac{-2x}{(x^2-4)^2}$
4. $\left(\dfrac{4}{3x^2-x+5}\right)' = \dfrac{4\,(1-6x)}{(3x^2-x+5)^2}$
5. $\left(\dfrac{x}{\sqrt{x^2+1}}\right)' = \dfrac{1}{(x^2+1)^{\frac{3}{2}}}$
6. $\left(\dfrac{\sqrt{x+2}}{\sqrt{x+1}}\right)' = \dfrac{-1}{2\sqrt{x+2}\,(\sqrt{x+1})^3}$
7. $\left(\dfrac{x}{x^2-1}\right)' = \dfrac{-(x^2+1)}{(x^2-1)^2}$
8. $\left(\dfrac{\sin x}{x}\right)' = \dfrac{x\cos x-\sin x}{x^2}$
9. $\left(\sin^3(5x+4)\right)' = 15\sin^2(5x+4)\cos(5x+4)$
10. $(x\sin x)' = x\cos x+\sin x$
11. $(x^2\cos 2x)' = -2x^2\sin 2x+2x\cos 2x$
12. $(x\sin x^2)' = \sin x^2+2x^2\cos x^2$
13. $\left(\sin^3 x^2\right)' = 6x\cdot\sin^2 x^2\cdot\cos x^2$
14. $\left(\sqrt{1-\sin x^2}\right)' = \dfrac{-x\cos x^2}{\sqrt{1-\sin x^2}}$
15. $\left(\tan^{-1}\dfrac{x}{2}\right)' = \dfrac{2}{x^2+4}$
16. $\left(\tan^{-1}\dfrac{2}{x}\right)' = \dfrac{-2}{x^2+4}$
17. $\left(\tan^{-1}e^{2x}\right)' = \dfrac{2e^{2x}}{e^{4x}+1}$
18. $\left(\ln(\tan^{-1}x)\right)' = \dfrac{1}{(x^2+1)\tan^{-1}x}$
19. $\left(\tan^{-1}\dfrac{a+x}{1-ax}\right)' = \dfrac{1}{x^2+1}$
20. $\left((\ln(2x-1))^3\right)' = \dfrac{6\,(\ln(2x-1))^2}{2x-1}$
21. $(\ln\cos x)' = -\tan x$
22. $\left(\ln\dfrac{x-1}{x+1}\right)' = \dfrac{2}{x^2-1}$
23. $\left(e^{\frac{1}{x}}\right)' = -\dfrac{1}{x^2}e^{\frac{1}{x}}$
24. $\left(e^{\sin 2x}\right)' = 2\,e^{\sin 2x}\cos 2x$
25. $\left(3^{\sin\pi x}\right)' = \pi\ln 3\cdot\cos\pi x\cdot 3^{\sin\pi x}$
26. $(\ln(\ln x))' = \dfrac{1}{x\ln x}$
27. $(x^x)' = x^x(1+\ln x)$
28. $\left(x^{\ln x}\right)' = 2x^{\ln x-1}\cdot\ln x$
29. $\left(\dfrac{2x^2+3x-1}{x-2}\right)' = \dfrac{2x^2-8x-5}{(x-2)^2}$
30. $\left(\tan^2 x\right)' = 2\tan x\sec^2 x$
31. $\left(\dfrac{1}{1+e^{-x}}\right)' = \dfrac{e^{-x}}{(1+e^{-x})^2}$
32. $(x^3\ln x)' = 3x^2\ln x+x^2$
33. $\left(\sqrt{1-x^2}\right)' = -\dfrac{x}{\sqrt{1-x^2}}$
34. $(e^x\sin x)' = e^x\sin x+e^x\cos x$
35. $(\sin 2x\cos 3x)' = 2\cos 2x\cos 3x-3\sin 2x\sin 3x$
36. $(\ln\tan x)' = \dfrac{1}{\sin x\cos x}$
37. $\left(x^{\sin x}\right)' = x^{\sin x}\left(\sin x\cdot\dfrac{1}{x}+\ln x\cdot\cos x\right)$
38. $\left(\dfrac{x}{1+x^2}\right)' = \dfrac{1-x^2}{(1+x^2)^2}$
39. $\left(\dfrac{1}{\ln x}\right)' = -\dfrac{1}{x(\ln x)^2}$
40. $\left(x^4e^{-x}\right)' = e^{-x}(4x^3-x^4)$
41. $\left(\dfrac{\sin x}{\cos x+2}\right)' = \dfrac{1+2\cos x}{(\cos x+2)^2}$
42. $(\ln(\sec x+\tan x))' = \sec x$
43. $\left((1-x^2)^3\right)' = -6x(1-x^2)^2$
44. $\left(e^{2x}\cos\pi x\right)' = e^{2x}(2\cos\pi x-\pi\sin\pi x)$
45. $\left(\dfrac{x^2-1}{x^2+1}\right)' = \dfrac{4x}{(x^2+1)^2}$
46. $\left(\dfrac{1}{1-\sin 2x}\right)' = \dfrac{2\cos 2x}{(1-\sin 2x)^2}$
47. $(x^3\cos\pi x)' = 3x^2\cos\pi x-\pi x^3\sin\pi x$
48. $(\cos(x\ln x))' = -(1+\ln x)\sin(x\ln x)$
49. $\left(\dfrac{e^{\pi x}-e^{-\pi x}}{2}\right)' = \dfrac{\pi(e^{\pi x}+e^{-\pi x})}{2}$
50. $\left(\dfrac{x^3+1}{x^2+1}\right)' = \dfrac{x^4+3x^2-2x}{(x^2+1)^2}$

51. $\left(e^{1-2x^2}\right)' = -4x\,e^{1-2x^2}$

52. $\left(\ln(x^2+x+1)\right)' = \dfrac{2x+1}{x^2+x+1}$

53. $\left(\dfrac{1}{x\ln x}\right)' = -\dfrac{1+\ln x}{(x\ln x)^2}$

54. $\left(\dfrac{x}{e^x+1}\right)' = \dfrac{e^x+1-xe^x}{(e^x+1)^2}$

55. $\left(e^{\tan\pi x}\right)' = \pi e^{\tan\pi x}\sec^2\pi x$

56. $\left(\dfrac{1}{\sqrt{1-2x^2}}\right)' = \dfrac{2x}{(1-2x^2)^{\frac{3}{2}}}$

57. $\left(\sin x\ln(\cos x)\right)' = \cos x\ln(\cos x) - \sin x\tan x$

58. $\left(\dfrac{x^2}{\sqrt{1+2x^2}}\right)' = \dfrac{2x(x^2+1)}{(1+2x^2)^{\frac{3}{2}}}$

59. $\left(e^{\pi x}\ln 2x\right)' = e^{\pi x}\left(\pi\ln 2x + \dfrac{1}{x}\right)$

60. $\left(\dfrac{\tan 2x}{x}\right)' = \dfrac{2x\sec^2 2x - \tan 2x}{x^2}$

61. $\left(\dfrac{1}{1+\cos 2x}\right)' = \dfrac{2\sin 2x}{(1+\cos 2x)^2}$

62. $\left(x^2e^{-x^2}\right)' = 2xe^{-x^2} - 2x^3e^{-x^2}$

63. $\left(\ln\left(e^{-x}+xe^{-x}\right)\right)' = -\dfrac{x}{1+x}$

64. $\left(\dfrac{e^{-\pi x}-1}{e^{-\pi x}+1}\right)' = \dfrac{-2\pi e^{\pi x}}{(e^{\pi x}+1)^2} = \dfrac{-2\pi e^{-\pi x}}{(e^{-\pi x}+1)^2}$

65. $\left(\tan(\pi\ln x)\right)' = \dfrac{\pi\sec^2(\pi\ln x)}{x}$

66. $\left(e^{\sin x}\cos x\right)' = e^{\sin x}(\cos^2 x - \sin x)$

67. $\left(\dfrac{1}{\ln(1+x)}\right)' = -\dfrac{1}{(1+x)(\ln(1+x))^2}$

68. $\left((\sin x+\cos x)^2\right)' = 2(\sin x+\cos x)(\cos x-\sin x)$

69. $\left(x^3\tan x\right)' = 3x^2\tan x + x^3\sec^2 x$

70. $\left(\dfrac{\ln x}{x}\right)' = \dfrac{1-\ln x}{x^2}$

71. $\left(e^{x^2}\sin x\right)' = e^{x^2}(2x\sin x+\cos x)$

72. $\left(\dfrac{1}{1+e^x}\right)' = -\dfrac{e^x}{(1+e^x)^2}$

73. $\left(\ln\sin x\right)' = \cot x$

74. $\left(\sin^2 x\cos^2 x\right)' = 2\sin x\cos^3 x - 2\sin^3 x\cos x$

75. $\left(\dfrac{x^2}{1+x^4}\right)' = \dfrac{2x(1-x^4)}{(1+x^4)^2}$

76. $\left(\dfrac{1}{\sqrt{x^2-1}}\right)' = -\dfrac{x}{(x^2-1)^{\frac{3}{2}}}$

77. $\left(x\sin\dfrac{1}{x}\right)' = \sin\dfrac{1}{x} - \dfrac{1}{x}\cos\dfrac{1}{x}$

78. $\left(\dfrac{e^x}{1+e^x}\right)' = \dfrac{e^x}{(1+e^x)^2}$

79. $\left(\ln\left(x+\sqrt{x^2+a^2}\right)\right)' = \dfrac{1}{\sqrt{x^2+a^2}}$

80. $\left(\cos(x^2+1)\right)' = -2x\sin(x^2+1)$

81. $\left(\dfrac{x^3-3x+1}{x^2-1}\right)' = \dfrac{x^4-2x+3}{(x^2-1)^2}$

82. $\left(\dfrac{x}{\sqrt{1-x^2}}\right)' = \dfrac{1}{(1-x^2)^{\frac{3}{2}}}$

83. $\left(e^{\pi x}\sin^2 x\right)' = e^{\pi x}(\pi\sin^2 x + 2\sin x\cos x)$

84. $\left(\dfrac{1}{x^3+1}\right)' = -\dfrac{3x^2}{(x^3+1)^2}$

85. $\left(\sin^{-1}\dfrac{x}{\sqrt{1+x^2}}\right)' = \dfrac{1}{1+x^2}$

86. $\left(\dfrac{\ln x}{1+\ln x}\right)' = \dfrac{1}{x(1+\ln x)^2}$

87. $\left(x^2\cos\dfrac{1}{x}\right)' = 2x\cos\dfrac{1}{x} + \sin\dfrac{1}{x}$

88. $\left(e^{x^2}\cos x^2\right)' = 2xe^{x^2}(\cos x^2 - \sin x^2)$

89. $\left(\dfrac{\sin x+\cos x}{\sin x-\cos x}\right)' = -\dfrac{2}{(\sin x-\cos x)^2}$

90. $\left(\sin e^x\right)' = e^x\cos e^x$

91. $\left(e^{\sin^2 x}\right)' = 2e^{\sin^2 x}\sin x\cos x$

92. $\left(\sin x\cos 2x\right)' = \cos x\cos 2x - 2\sin x\sin 2x$

93. $\left(\ln(\sec x+\tan x)\right)' = \sec x$

94. $\left(\dfrac{x}{\sin x}\right)' = \dfrac{\sin x - x\cos x}{\sin^2 x}$

95. $\left(x\ln x - x\right)' = \ln x$

96. $\left(\ln(\cos\ln x)\right)' = -\dfrac{\tan\ln x}{x}$

97. $\left(\dfrac{e^x-e^{-x}}{e^x+e^{-x}}\right)' = \dfrac{4}{(e^x+e^{-x})^2}$

98. $\left(\sin^{-1}\dfrac{2x}{1+x^2}\right)' = \dfrac{2}{1+x^2}\,\dfrac{|1-x^2|}{1-x^2}$

99. $\left(\dfrac{\ln(1+x^2)}{x}\right)' = \dfrac{2}{1+x^2} - \dfrac{\ln(1+x^2)}{x^2}$

100. $\left(e^{\pi x}\tan^{-1}\pi x\right)' = \pi e^{\pi x}\left(\tan^{-1}\pi x + \dfrac{1}{1+\pi^2x^2}\right)$