基礎微積分

基本概念

記號

$\forall$	對所有	for all
$\exists$	存在	there exists
$\exists!$	存在唯一	there exists uniquely
$\in$	屬於	belongs to
A $\implies$ B	若 A 則 B	if A then B
A $\iff$ B	A 等價於 B	A if and only if B
∞	無限大	infinity
$\vee$	或	or
$\wedge$	且	and
$\because$	因為	because
$\therefore$	所以	therefore

數

$\mathbb{N}$	自然數	natural number	$1, 2, 3, \ldots$
$\mathbb{Z}$	整數	integer	$\ldots, -2, -1, 0, 1, 2, \ldots$
$\mathbb{Q}$	有理數	rational number	$\dfrac{p}{q} : p, q \in \mathbb{Z}$
$\mathbb{R}$	實數	real number	

集合

$x \in S$	x 為集合 S 的元素	
$S_1 = \{x_1, x_2, \ldots\}$	列舉式	
$S_2 = \{x \mid x$ 滿足某性質$\}$	敘述式	
$S \cap T$	$\{x \mid x \in S \wedge x \in T\}$	交集 (intersection)
$S \cup T$	$\{x \mid x \in S \vee x \in T\}$	聯集 (union)
$S \setminus T$	$\{x \mid x \in S \wedge x \notin T\}$	差集 (difference)
$S \times T$	$\{(x, y) \mid x \in S \wedge y \in T\}$	積集 (Cartesian product)
$\varnothing$	空集合	
$S_1 \subset S_2,\ S_2 \supset S_1$	S_1 為 S_2 的真子集合	
$S_1 \subseteq S_2,\ S_2 \supseteq S_1$	S_1 為 S_2 的子集合	
$\cap_{i=1}^{n} S_i$	$S_1 \cap S_2 \cap \cdots \cap S_n$	
$\cup_{i=1}^{n} S_i$	$S_1 \cup S_2 \cup \cdots \cup S_n$	

不等式

性質. 令 $a,\ b,\ c \in \mathbb{R}$.

1. $a < b \implies a + c < b + c$
2. $a < b,\ c < d \implies a + c < b + d$
3. $a < b,\ c > 0 \implies ac < bc$
4. $a < b,\ c < 0 \implies ac > bc$
5. $0 < a < b \implies \dfrac{1}{a} > \dfrac{1}{b}$

例. 解下列不等式.

1. $2x-3<x+4<3x-2$
2. $x^3>x$
3. $(2-x)(1-x)^2x^3 \leqslant 0$
4. $-2<\dfrac{2x-3}{x+1}<1$

解.

1. $3 \leqslant x<7$
2. $x^3-x>0 \implies x(x^2-1)>0 \implies x(x+1)(x-1)>0 \implies x>1 \vee -1<x<0$
3. $(2-x)(1-x)^2x^3 \leqslant 0 \implies (x-2)(x-1)^2x^3 \geqslant 0 \implies x \geqslant 2 \vee x \leqslant 0 \vee x=1$
4. $-2<\dfrac{2x-3}{x+1}<1 \implies \Big(-2<\dfrac{2x-3}{x+1}\Big) \wedge \Big(\dfrac{2x-3}{x+1}<1\Big) \implies \Big(\dfrac{4x-1}{x+1}>0\Big) \wedge \Big(\dfrac{x-4}{x+1}<0\Big) \implies \Big(x<-1 \vee x>\dfrac{1}{4}\Big) \wedge (-1<x<4) \implies \dfrac{1}{4}<x<4$

絕對值

令 $a \in \mathbb{R}$; a 的絕對值 (absolute value) $|a|$ 定義為 $|a|=\begin{cases} a & 若\ a \geqslant 0 \\ -a & 若\ a<0 \end{cases}$

性質. 若 $a>0$, 則

1. $|x|=a \iff x=\pm a$
2. $|x|<a \iff -a<x<a$
3. $|x|>a \iff x<-a \vee x>a$

性質. 若 $a, b \in \mathbb{R}$, 則

1. $\sqrt{a^2}=|a|$
2. $|ab|=|a|\,|b|$
3. $\left|\dfrac{b}{a}\right|=\dfrac{|b|}{|a|}$
4. $|a+b| \leqslant |a|+|b|$
5. $\big|\,|a|-|b|\,\big| \leqslant |a-b|$

證.

- $(|a+b|)^2=(a+b)^2=a^2+2ab+b^2=|a|^2+2ab+|b|^2 \leqslant |a|^2+2|ab|+|b|^2=|a|^2+2|a|\,|b|+|b|^2=(|a|+|b|)^2$, 故 $|a+b| \leqslant |a|+|b|$.
- $|a|=|(a-b)+b| \leqslant |a-b|+|b| \implies |a|-|b| \leqslant |a-b|$; $|b|=|(b-a)+a| \leqslant |b-a|+|a|=|a-b|+|a| \implies |b|-|a| \leqslant |a-b|$. 故 $\big|\,|a|-|b|\,\big| \leqslant |a-b|$.

例. 解下列不等式與方程式.

1. $|5-2x|<3$
2. $\left|\dfrac{2x-1}{x+1}\right|=3$
3. $|x-1|-|x-10| \geqslant 5$

解.

1. $|5-2x|<3 \implies -3<5-2x<3 \implies -8<-2x<-2 \implies 1<x<4$
2. $\left|\dfrac{2x-1}{x+1}\right|=3 \implies \dfrac{2x-1}{x+1}=3 \vee \dfrac{2x-1}{x+1}=-3 \implies x=-4 \vee x=-\dfrac{2}{5}$
3. 當 $x<1$, $|x-1|-|x-10| \geqslant 5 \implies (1-x)-(10-x) \geqslant 5 \implies -9 \geqslant 5$, 不合. 當 $1 \leqslant x<10$, $|x-1|-|x-10| \geqslant 5 \implies (x-1)-(10-x) \geqslant 5 \implies 2x \geqslant 16 \implies x \geqslant 8$, 則 $8 \leqslant x<10$. 當 $x \geqslant 10$, $|x-1|-|x-10| \geqslant 5 \implies (x-1)-(x-10) \geqslant 5 \implies 9 \geqslant 5$ 恆成立. 綜上, $8 \leqslant x$.

函數

定義.

- 函數 (function) $f : A \to B$ 是一個對應關係: 對所有 $a \in A$, 存在唯一 $b \in B$, 使得 f 將 a 對應到 b. $\forall\, a \in A\; \exists!\, b \in B\; (f(a) = b)$.
- A: 定義域 (domain) ; $\operatorname{dom} f = A$
  B: 對應域 (codomain) ; $\operatorname{codom} f = B$
  $f(A) = \{f(a) \,|\, a \in A\} \subseteq B$: 值域 (range) ; $\operatorname{ran} f \equiv f(A)$

嵌射與蓋射

定義. 給定函數 $f : A \to B$.

- 若 $\forall\, x_1, x_2 \in A \wedge x_1 \neq x_2\; (f(x_1) \neq f(x_2))$, 則 f 為嵌射 (one-to-one, injective) .
- 若 $\forall\, b \in B\; \exists\, a \in A\; (f(a) = b)$, 則 f 為蓋射 (onto, surjective) .

函數圖形

定義. 若 $A,\ B \subseteq \mathbb{R}$, 則函數 $f : A \to B$ 稱為實數值函數 (real-valued function) , 集合 $\{(x, f(x)) \,|\, x \in A\}$ 稱為 f 的圖形 (graph) .

性質. 函數 / 圖形判斷法

- 垂直線判斷法: 函數圖形 $\iff$ 任一垂直線與其至多交於一點
- 水平線判斷法: 嵌射圖形 $\iff$ 任一水平線與其至多交於一點

函數特性

奇偶性

定義. 給定實數值函數 f:

- 若 $\forall x \in \operatorname{dom} f,\ f(-x) = f(x)$, 則 f 為偶函數 (even function) .
- 若 $\forall x \in \operatorname{dom} f,\ f(-x) = -f(x)$, 則 f 為奇函數 (odd function) .

反函數

定義. 若函數 f 為嵌射, 則其反函數 $f^{-1} : \operatorname{ran} f \to \operatorname{dom} f$ 定義為 $f^{-1}(b) = a \iff f(a) = b$, 其中 $a \in \operatorname{dom} f,\ b \in \operatorname{ran} f$.

性質. 反函數常用規則.

1. $f^{-1}(y) = x \iff f(x) = y$
2. $\operatorname{dom} f^{-1} = \operatorname{ran} f,\ \operatorname{ran} f^{-1} = \operatorname{dom} f$
3. $f^{-1}(x) \neq \dfrac{1}{f(x)} = (f(x))^{-1}$
4. $(f^{-1} \circ f)(x) = x,\ \forall\, x \in \operatorname{dom} f$
5. $(f \circ f^{-1})(y) = y,\ \forall\, y \in \operatorname{dom} f^{-1} = \operatorname{ran} f$
6. $y = f(x)$ 與 $y = f^{-1}(x)$ 之圖形對 $y = x$ 對稱.
7. 若 f 為嚴格遞增或嚴格遞減函數, 則 f 為嵌射 $\implies$ 存在 f^{-1}.

例. 求反函數.

1. 求 $f(x) = x^3 + 2$ 的反函數.
2. 求 $f(x) = x^2$, $x \geqslant 0$ 與 $x \leqslant 0$ 的反函數.
3. 求 $f(x) = \dfrac{1+9x}{4-x}$, $x < 4$ 的反函數.

解.

1. $y = x^3+2$; $x \longleftrightarrow y$: $x = y^3+2 \implies y^3 = x-2 \implies y = \sqrt[3]{x-2} \implies f^{-1}(x) = \sqrt[3]{x-2}$.
2. $y = x^2$, $x \geqslant 0$; $x \longleftrightarrow y$: $x = y^2 \implies y^2 = x \implies y = \sqrt{x} \implies f^{-1}(x) = \sqrt{x}$; $f^{-1} : [0, \infty) \to [0, \infty)$.
   $y = x^2$, $x \leqslant 0$; $x \longleftrightarrow y$: $x = y^2 \implies y^2 = x \implies y = -\sqrt{x} \implies f^{-1}(x) = -\sqrt{x}$; $f^{-1} : [0, \infty) \to (-\infty, 0]$.
3. $f(x) = \dfrac{1+9x}{4-x} \implies y = \dfrac{1+9x}{4-x}$; $x \longleftrightarrow y$: $x = \dfrac{1+9y}{4-y} \implies y = \dfrac{4x-1}{x+9} \implies f^{-1}(x) = \dfrac{4x-1}{x+9}$. 檢驗: $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = \dfrac{4 \cdot \frac{1+9x}{4-x} - 1}{\frac{1+9x}{4-x} + 9} = \dfrac{\frac{4+36x-4+x}{4-x}}{\frac{1+9x+36-9x}{4-x}} = \dfrac{37x}{37} = x$; $(f \circ f^{-1})(x) = f(f^{-1}(x)) = \dfrac{1 + 9 \cdot \frac{4x-1}{x+9}}{4 - \frac{4x-1}{x+9}} = \dfrac{\frac{x+9+36x-9}{x+9}}{\frac{4x+36-4x+1}{x+9}} = \dfrac{37x}{37} = x$.

指數函數

$y = f(x) = a^x$, $a > 0 \wedge a \neq 1$.

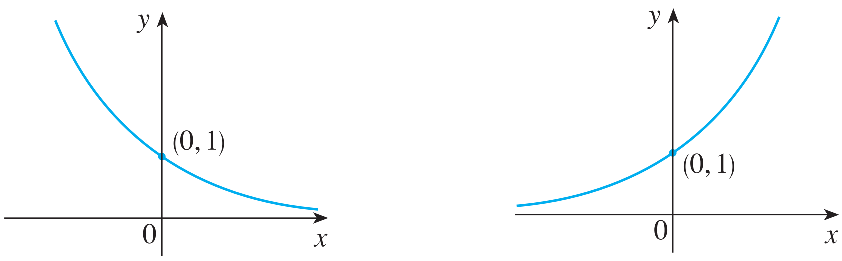


圖 1: $y = a^x$: 圖左 $0 < a < 1$, 圖右 $a > 1$

性質. 若 a, $b > 0$, x, $y \in \mathbb{R}$, 則

- $a^x \cdot a^y = a^{x+y}$
- $\dfrac{a^x}{a^y} = a^{x-y}$
- $a^x \cdot b^x = (ab)^x$
- $a^{-x} = \dfrac{1}{a^x}$
- $(a^x)^y = a^{xy} = (a^y)^x$
- $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$

對數函數

$y = f(x) = \log_a x$, $a > 0 \wedge a \neq 1$.

性質. 給定 $a > 0 \wedge a \neq 1$, $x > 0$, $y \in \mathbb{R}$.

- $\log_a x = y \iff a^y = x$
- $\log_a a^y = y$
- $a^{\log_a x} = x$

性質. 給定 $b > 0$, $x > 0$, $a > 0 \wedge a \neq 1$, $c > 0 \wedge c \neq 1$, $r \in \mathbb{R}$.

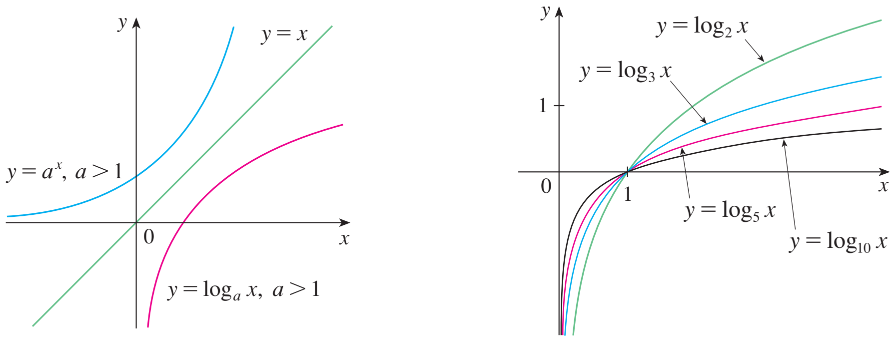


圖 2: $y = \log_a x$

- $\log_a bx = \log_a b + \log_a x$
- $\log_a x^r = r\log_a x$
- $\log_a x = \dfrac{\log_c x}{\log_c a}$

例. 解下列 x 的方程式與不等式.

1. $\log_{10} x + \log_{10}(x-21) = 2$ 2. $\log_2\left(x^2-2x-2\right) \leqslant 0$ 3. $3^{\log_3 7} - 4^{\log_4 2} = 5^{\log_5 x - \log_5 x^2}$

解.

1. $\log_{10}(x^2-21x) = \log_{10} 10^2 \implies x^2-21x-100=0 \implies (x-25)(x+4)=0 \implies x = 25 \vee x = -4$ (不合) .
2. $x^2-2x-2>0 \wedge x^2-2x-2 \leqslant 1 \implies x \in [-1, 1-\sqrt{3}) \cup (1+\sqrt{3}, 3]$.
3. $7-2 = \dfrac{1}{x} \implies x = \dfrac{1}{5}$.

例. 證明 $f(x) = \log_2\left(x+\sqrt{x^2+1}\right)$ 為奇函數, 並求其反函數.

解.

- $f(-x) = \log_2\left(-x+\sqrt{x^2+1}\right) = \log_2\left(\dfrac{\left(\sqrt{x^2+1}\right)^2-x^2}{\sqrt{x^2+1}+x}\right) = \log_2\left(\dfrac{1}{x+\sqrt{x^2+1}}\right) = -f(x)$.
- $y = \log_2\left(x+\sqrt{x^2+1}\right)$; $x \longleftrightarrow y$: $x = \log_2\left(y+\sqrt{y^2+1}\right) \implies 2^x - y = \sqrt{y^2+1} \implies 2^{2x} - 2\cdot 2^x y + y^2 = y^2+1 \implies y = \dfrac{2^x-2^{-x}}{2}$.

直觀極限

畫圖; 兩邊取接近值

$\lim\limits_{x\to 2} x = 2$

x	1.9	1.99	1.999	∘	2.001	2.01	2.1
$f(x)$	1.9	1.99	1.999	∘	2.001	2.01	2.1

$\lim\limits_{x\to 2} x^2 = 4$

x	1.9	1.99	1.999	∘	2.001	2.01	2.1
$f(x)$	3.61	3.9601	3.996	∘	4.004	4.040	4.41

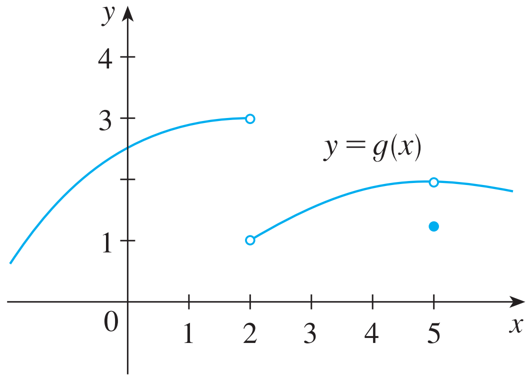
$\lim\limits_{x\to 2} \dfrac{x-2}{x^2+x-6} = 0.2$

x	1.9	1.99	1.999	∘	2.001	2.01	2.1
$f(x)$	0.20408	0.20040	0.20004	∘	0.19996	0.19960	0.19608

單側極限，在無限遠之極限，無窮極限

單側極限 (One-Sided Limits)

$$\lim_{x\to 2-} g(x) = 3, \qquad \lim_{x\to 5-} g(x) = 2$$
$$\lim_{x\to 2+} g(x) = 1, \qquad \lim_{x\to 5+} g(x) = 2$$
$$\lim_{x\to 2} g(x) = \text{DNE}, \qquad \lim_{x\to 5} g(x) = 2$$



例. $g(x) = \begin{cases} \sqrt{x-4} & 若 x > 4 \\ 8 - 2x & 若 x < 4 \end{cases}$, $\lim_{x\to 4} g(x) = 0$.

例. $f(x) = \sqrt{4 - x^2}$, $\lim_{x\to(-2)+} f(x) = 0$, $\lim_{x\to 2+} f(x) = \text{DNE}$, $\lim_{x\to 2-} f(x) = 0$.

定理. 若 $F \in \mathbb{R}$, $\lim_{x\to a} f(x) = F \iff \lim_{x\to a-} f(x) = \lim_{x\to a+} f(x) = F$.

在無限遠的極限 (Limits at Infinity)

例.
- $\lim_{x\to\infty} \frac{1}{x^\alpha} = 0$, $\alpha > 0$; $\lim_{x\to-\infty} \frac{1}{x^N} = 0$, $\forall N \in \mathbb{N}$
- $\lim_{x\to\infty} a^x = \infty$, $\lim_{x\to-\infty} a^x = 0$, $\lim_{x\to 0-} a^{\frac{1}{x}} = 0$, $\forall a > 1$.

無窮極限 (Infinite Limits)

例.
- $\lim_{x\to 0+} \frac{1}{x} = \infty$
- $\lim_{x\to 0-} \frac{1}{x} = -\infty$
- $\lim_{x\to 0} \frac{1}{x} = \text{DNE}$
- $\lim_{x\to 0} \frac{1}{x^2} = \infty$

極限運算

定理 (極限四則運算)**.** 若 $\lim_{x\to a} f(x) = F$, $\lim_{x\to a} g(x) = G$, 則

1. $\lim_{x\to a} c = c$, $\lim_{x\to a} x = a$
2. $\lim_{x\to a} k\,f(x) = k\,F$
3. $\lim_{x\to a} \big(f(x) \pm g(x)\big) = F \pm G$
4. $\lim_{x\to a} \big(f(x)\cdot g(x)\big) = F \cdot G$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{F}{G}$, 若 $G \neq 0$
6. $\lim_{x\to a} \big(f(x)\big)^\alpha = F^\alpha$, 若 $\alpha \in \mathbb{Q} \wedge F > 0$

當 F, G 存在 (非為無窮)，此定理敘述對「單側極限」及「無限遠之極限」均成立.

例.

- $\lim_{h\to 0} \frac{(2+h)^2 - 4}{2h} = \lim_{h\to 0} \frac{h^2 + 4h + 4 - 4}{2h} = \lim_{h\to 0} \frac{h^2 + 4h}{2h} = \lim_{h\to 0} \frac{h+4}{2} = 2$
- $\lim_{x\to 4} \frac{x^2 - 4x}{x^2 - 16} = \lim_{x\to 4} \frac{x(x-4)}{(x-4)(x+4)} = \lim_{x\to 4} \frac{x}{x+4} = \frac{1}{2}$
- $\lim_{x\to 2} \frac{x-2}{x^2 + x - 6} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+3)} = \lim_{x\to 2} \frac{1}{x+3} = \frac{1}{5}$

例. 求 $\displaystyle\lim_{x\to-1}\frac{\sqrt{x^2+8}-3}{x+1}$.

解. $\displaystyle\lim_{x\to-1}\frac{\sqrt{x^2+8}-3}{x+1}=\lim_{x\to-1}\frac{1}{x+1}\frac{(x^2+8)-3^2}{\sqrt{x^2+8}+3}=\lim_{x\to-1}\frac{1}{x+1}\frac{x^2-1}{\sqrt{x^2+8}+3}$
$\displaystyle=\lim_{x\to-1}\frac{1}{x+1}\frac{(x+1)(x-1)}{\sqrt{x^2+8}+3}=\lim_{x\to-1}\frac{x-1}{\sqrt{x^2+8}+3}=\frac{-2}{\sqrt{1+8}+3}=-\frac{1}{3}$.

例. 求 $\displaystyle\lim_{x\to3}\frac{\sqrt{x-2}-\sqrt{4-x}}{x-3}$.

解. $\displaystyle\lim_{x\to3}\frac{\sqrt{x-2}-\sqrt{4-x}}{x-3}=\lim_{x\to3}\frac{1}{x-3}\frac{(x-2)-(4-x)}{\sqrt{x-2}+\sqrt{4-x}}=\lim_{x\to3}\frac{1}{x-3}\frac{2x-6}{\sqrt{x-2}+\sqrt{4-x}}$
$\displaystyle=\lim_{x\to3}2\frac{1}{\sqrt{x-2}+\sqrt{4-x}}=\frac{2}{\sqrt{1}+\sqrt{1}}=1$.

例. $\displaystyle\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}=\lim_{x\to\infty}\frac{5+\frac{8}{x}-\frac{3}{x^2}}{3+\frac{2}{x^2}}=\frac{5+0+0}{3+0}=\frac{5}{3}=\lim_{x\to-\infty}\frac{5x^2+8x-3}{3x^2+2}$.

例. 求 $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+5x}-\sqrt{x^2-x}\right)$.

解. $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+5x}-\sqrt{x^2-x}\right)=\lim_{x\to\infty}\frac{(x^2+5x)-(x^2-x)}{\sqrt{x^2+5x}+\sqrt{x^2-x}}=\lim_{x\to\infty}\frac{6x}{\sqrt{x^2\left(1+\frac{5}{x}\right)}+\sqrt{x^2\left(1-\frac{1}{x}\right)}}$
$\displaystyle=\lim_{x\to\infty}\frac{6x}{|x|\sqrt{1+\frac{5}{x}}+|x|\sqrt{1-\frac{1}{x}}}=\lim_{x\to\infty}\frac{6x}{x\sqrt{1+\frac{5}{x}}+x\sqrt{1-\frac{1}{x}}}=\lim_{x\to\infty}\frac{6}{\sqrt{1+\frac{5}{x}}+\sqrt{1-\frac{1}{x}}}$
$\displaystyle=\frac{6}{\sqrt{1+0}+\sqrt{1-0}}=3$

例. 求 $\displaystyle\lim_{x\to-\infty}\frac{3x}{\sqrt{4x^2+x}-2x}$.

解. 變數變換 $y=-x \implies x=-y$: $\displaystyle\lim_{x\to-\infty}\frac{3x}{\sqrt{4x^2+x}-2x}=\lim_{y\to\infty}\frac{-3y}{\sqrt{4y^2-y}+2y}=\lim_{y\to\infty}\frac{-3y}{\sqrt{y^2\left(4-\frac{1}{y}\right)}+2y}=\lim_{y\to\infty}\frac{-3y}{|y|\sqrt{4-\frac{1}{y}}+2y}=\lim_{y\to\infty}\frac{-3y}{y\sqrt{4-\frac{1}{y}}+2y}=\lim_{y\to\infty}\frac{-3}{\sqrt{4-\frac{1}{y}}+2}=-\frac{3}{4}$.

連續性

定義. 給定 f, $a\in\operatorname{dom}f$.

- 若 $\displaystyle\lim_{x\to a}f(x)$ 存在且 $\displaystyle f(a)=\lim_{x\to a}f(x)$, 則稱 f 在 a *連續*.
- 若 $\displaystyle\lim_{x\to a+}f(x)$ 存在且 $\displaystyle f(a)=\lim_{x\to a+}f(x)$, 則稱 f 在 a *左連續*.
- 若 $\displaystyle\lim_{x\to a-}f(x)$ 存在且 $\displaystyle f(a)=\lim_{x\to a-}f(x)$, 則稱 f 在 a *右連續*.

$f(x)$ 在 a 連續 $\displaystyle\iff\lim_{x\to a}f(x)=f\left(\lim_{x\to a}x\right)$

定義 (端點連續性)**.** 給定 f, $\operatorname{dom}f=[a,b]$.

- 若 f 在 b 左連續, 則稱 f 在 b 連續.
- 若 f 在 a 右連續, 則稱 f 在 a 連續.

定義 (連續函數)**.**

- 若 f 在區間 I 之每一點均連續, 則稱 f 在 I 連續.
- 若 f 在 $\operatorname{dom} f$ 之每一點均連續, 則稱 f 為*連續函數*.

定理 (五則運算仍連續)**.**

- 若 f, g 在 a 連續, 則 $f \pm g,\ f \cdot g,\ kf,\ f^\alpha,\ \dfrac{f}{g}$ (若 $g(a) \neq 0$) 均在 a 連續.
- 若 f 在 a 連續, 且 g 在 $f(a)$ 連續, 則 $g \circ f$ 在 a 連續.

註. 多項式, 指數, 對數函數及其五則運算之組合均為連續函數.

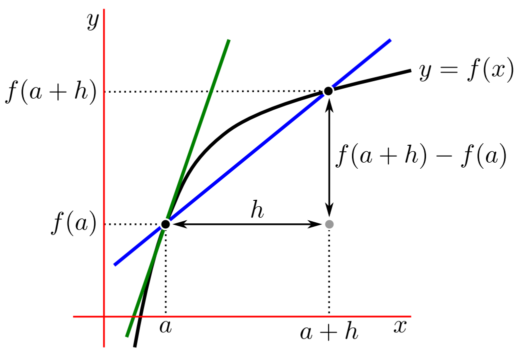
例. 若 $f(x) = \begin{cases} x+2 & \text{當 } x < a \\ x^2 & \text{當 } x \geqslant a \end{cases}$ 為連續函數, 求 a.

解. 若 f 為連續函數, 則 f 在 a 連續 $\implies \lim\limits_{x \to a-} f(x) = \lim\limits_{x \to a+} f(x) \implies a + 2 = a^2 \implies a = -1 \vee a = 2$.

例. 若 $f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{當 } x < 2 \\ ax^2 - bx + 3 & \text{當 } 2 \leqslant x < 3 \\ 2x - a + b & \text{當 } x \geqslant 3 \end{cases}$ 為連續函數, 求 a, b.

解. 若 f 為連續函數, 則 f 在 $2, 3$ 均連續 $\implies \big(\lim\limits_{x \to 2-} f(x) = \lim\limits_{x \to 2+} f(x)\big) \wedge \big(\lim\limits_{x \to 3-} f(x) = \lim\limits_{x \to 3+} f(x)\big) \implies (4 = 4a - 2b + 3) \wedge (9a - 3b + 3 = 6 - a + b) \implies a = \dfrac{1}{2},\ b = \dfrac{1}{2}$.

導數與導函數



定義. 給定 $f(x)$, $a \in \operatorname{dom} f$. f 在 a 的導數 (derivative) $f'(a)$ 定義為

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

若 $f'(a)$ 存在, 則稱 f 在 a 可微 (分)(differentiable). f 的導函數 $f'(x)$ 定義為

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

註.

- 求導函數之過程稱為微分 (differentiation) : 求 $f'(x) \iff f(x)$ (對 x) 微分
- 給定 $y = f(x)$, 其導函數可記為 $f'(x) = f' = y' = \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}f}{\mathrm{d}x} = \dfrac{\mathrm{d}}{\mathrm{d}x} f(x) = Df(x) = D_x f(x)$.
- f 在 a 的導數可記為 $f'(a) = \left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a}$.

例. 求以下 $f(x)$ 之導函數 $f'(x)$。

1. $f(x) = x$ 2. $f(x) = x^2$ 3. $f(x) = x^4$ 4. $f(x) = \frac{1}{x}$ 5. $f(x) = \frac{1}{x^5}$ 6. $f(x) = \frac{1}{x^2+3}$

解.

1. $f'(x) = \lim_{h\to 0} \dfrac{(x+h)-x}{h} = \lim_{h\to 0} \dfrac{h}{h} = \lim_{h\to 0} 1 = 1.$
2. $f'(x) = \lim_{h\to 0} \dfrac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \dfrac{2xh+h^2}{h} = \lim_{h\to 0} (2x+h) = 2x.$
3. $f'(x) = \lim_{h\to 0} \dfrac{(x+h)^4-x^4}{h} = \lim_{h\to 0} \dfrac{4x^3h+6x^2h^2+4xh^3+h^4}{h} = \lim_{h\to 0} (4x^3+6x^2h+4xh^2+h^3) = 4x^3.$
4. $f'(x) = \lim_{h\to 0} \dfrac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0} \dfrac{x-(x+h)}{h\,(x+h)\,x} = \lim_{h\to 0} \dfrac{-h}{h\,(x+h)\,x} = \lim_{h\to 0} \dfrac{-1}{(x+h)\,x} = \dfrac{-1}{x^2} = (-1)\,x^{-2}.$
5. $f'(x) = \lim_{h\to 0} \dfrac{\frac{1}{(x+h)^5}-\frac{1}{x^5}}{h} = \lim_{h\to 0} \dfrac{x^5-(x+h)^5}{h\,(x+h)^5\,x^5}$
$= \lim_{h\to 0} \dfrac{(-h)(x^4+x^3(x+h)+x^2(x+h)^2+x^3(x+h)+(x+h)^4)}{h\,(x+h)^5\,x^5}$
$= \lim_{h\to 0} \dfrac{-(x^4+x^3(x+h)+x^2(x+h)^2+x^3(x+h)+(x+h)^4)}{(x+h)^5\,x^5} = \dfrac{-5x^4}{x^{10}} = \dfrac{-5}{x^6} = (-5)\,x^{-6}.$
6. $f'(x) = \lim_{h\to 0} \dfrac{\frac{1}{(x+h)^2+3}-\frac{1}{x^2+3}}{h} = \lim_{h\to 0} \dfrac{(x^2+3)-((x+h)^2+3)}{h\,((x+h)^2+3)\,(x^2+3)} = \lim_{h\to 0} \dfrac{(2x+h)\,(-h)}{h\,((x+h)^2+3)\,(x^2+3)}$
$= \lim_{h\to 0} \dfrac{-(2x+h)}{((x+h)^2+3)(x^2+3)} = \dfrac{-2x}{(x^2+3)^2}.$

結論. x^α $(\alpha \in \mathbb{R})$ 之導函數為 $\alpha\, x^{\alpha-1}$.

定義. 若 f 在 (a,b) 上每一點均有導數, 則稱 f 在 (a,b) 可微 (分) .

定理. 若 f 在 a 可微, 則 f 在 a 連續.

微分規則

定理 (四則運算)**.** 令 f, g 可微, $c \in \mathbb{R}$. 則

1. $(c)' = 0$
2. $(c\,f)' = c\,f'$
3. $(f \pm g)' = f' \pm g'$
4. $(f \cdot g)' = f' \cdot g + f \cdot g'$
5. $\left(\dfrac{f}{g}\right)' = \dfrac{f' \cdot g - f \cdot g'}{g^2}$

例. 求導函數。

1. x^5 2. $\frac{x-1}{x+1}$ 3. $\frac{1}{x^2+3}$

解.

1. $(x^5)' = (x \cdot x^2 \cdot x^2)' = (x)' \cdot x^2 \cdot x^2 + x \cdot (x^2)' \cdot x^2 + x \cdot x^2 \cdot (x^2)' = x^4 + x \cdot (2x) \cdot x^2 + x^3 \cdot (2x) = 5x^4$
2. $\left(\dfrac{x-1}{x+1}\right)' = \dfrac{(x+1)\cdot(x-1)' - (x-1)\cdot(x+1)'}{(x+1)^2} = \dfrac{2}{(x+1)^2}$
3. $\left(\dfrac{1}{x^2+3}\right)' = \dfrac{(x^2+3)\cdot(1)' - 1\cdot(x^2+3)'}{(x^2+3)^2} = \dfrac{-2x}{(x^2+3)^2}$

定理 (鏈鎖律 (chain rule))**.** 若 $f(u)$ 在 $u = g(x)$ 可微, $g(x)$ 在 x 可微, 則 $f \circ g$ 在 x 可微:

$$(f \circ g)'(x) \;\equiv\; (f(g(x)))' = f'(g(x)) \cdot g'(x)$$

例. 求導函數.

1. $(x^3-1)^{2023}$ 2. $\sqrt{x^2+1}$ 3. $\frac{1}{x^2+3}$ 4. $\sqrt{\frac{x-1}{x+1}}$

解.

1. 令 $f(u)=u^{2023}$, $g(x)=x^3-1$, 則 $f'(u)=2023\,u^{2022}$, $(x^3-1)^{2023}=f(g(x))$.
由鏈鎖律 $((x^3-1)^{2023})'=(f(g(x)))'=f'(g(x))\cdot g'(x)=2023\cdot(x^3-1)^{2022}\cdot(3x^2)$.
2. 令 $f(u)=\sqrt{u}=u^{\frac{1}{2}}$, $g(x)=x^2+1$, 則 $f'(u)=\dfrac{1}{2\sqrt{u}}$, $\sqrt{x^2+1}=f(g(x))$.
由鏈鎖律 $(\sqrt{x^2+1})'=(f(g(x)))'=f'(g(x))\cdot g'(x)=\dfrac{1}{2\sqrt{x^2+1}}\cdot(2x)=\dfrac{x}{\sqrt{x^2+1}}$.
3. 令 $f(u)=\dfrac{1}{u}$, $g(x)=x^2+3$, 則 $f'(u)=\dfrac{-1}{u^2}$, $\dfrac{1}{x^2+3}=f(g(x))$.
由鏈鎖律 $\left(\dfrac{1}{x^2+3}\right)'=(f(g(x)))'=f'(g(x))\cdot g'(x)=\dfrac{-1}{(x^2+3)^2}\cdot(x^2+3)'=\dfrac{-2x}{(x^2+3)^2}$.
4. 令 $f(u)=\sqrt{u}$, $g(x)=\dfrac{x-1}{x+1}$, 則 $f'(u)=\dfrac{1}{2\sqrt{u}}$, $\sqrt{\dfrac{x-1}{x+1}}=f(g(x))$.
由鏈鎖律 $\left(\sqrt{\dfrac{x-1}{x+1}}\right)'=(f(g(x)))'=f'(g(x))\cdot g'(x)=\dfrac{1}{2\sqrt{\frac{x-1}{x+1}}}\cdot\left(\dfrac{x-1}{x+1}\right)'=\dfrac{1}{2\sqrt{\frac{x-1}{x+1}}}\dfrac{2}{(x+1)^2}=$
$\dfrac{1}{\sqrt{(x-1)(x+1)^3}}$.

結論. 若 $f(g(x))=x$, 則等式兩邊對 x 微分 $\implies (f(g(x)))'=1 \implies f'(g(x))\cdot g'(x)=1 \implies g'(x)=\dfrac{1}{f'(g(x))}$.

自然指數, 對數與微分

定義 (自然指數 e 與 e^x 微分).

- 給定 $a>0$, 求 $f(x)=a^x$ 之導函數
- $\dfrac{\mathrm{d}f}{\mathrm{d}x}=\lim\limits_{h\to0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to0}\dfrac{a^{x+h}-a^x}{h}=\lim\limits_{h\to0}a^x\cdot\dfrac{a^h-1}{h}=a^x\cdot\lim\limits_{h\to0}\dfrac{a^h-1}{h}$
- 令 $C(a)=\lim\limits_{h\to0}\dfrac{a^h-1}{h}$, 則 $\dfrac{\mathrm{d}}{\mathrm{d}x}a^x=C(a)\cdot a^x$
- 觀察: $C(a)$ 隨 a 遞增; 對於某個介於 2, 3 間的 a, $C(a)=1$.

h	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
$\dfrac{2^h-1}{h}$	0.7177	0.6956	0.6934	0.6932	0.6931	0.6931	0.6931
$\dfrac{3^h-1}{h}$	1.1612	1.1047	1.0992	1.0987	1.0986	1.0986	1.0986
$\dfrac{5^h-1}{h}$	1.7462	1.6225	1.6107	1.6096	1.6095	1.6094	1.6094
$\dfrac{10^h-1}{h}$	2.5893	2.3293	2.3052	2.3028	2.3026	2.3026	2.3026

- 定義 $C(e)=1 \implies \lim\limits_{h\to0}\dfrac{e^h-1}{h}=1$, 則 $\dfrac{\mathrm{d}}{\mathrm{d}x}e^x=C(e)\cdot e^x \implies (e^x)'=e^x$. $\ln x\equiv\log_e x$

性質. $(\ln|x|)' = \dfrac{1}{x}$.

解.

- 若 $x > 0$, $\ln|x| = \ln x$ 且 $e^{\ln x} = x$. 令 $f(u) = e^u$, $g(x) = \ln|x| = \ln x$, 則 $f'(u) = e^u$, $f(g(x)) = x$; 故 $g'(x) = \dfrac{1}{f'(g(x))} \implies (\ln|x|)' = (\ln x)' = \dfrac{1}{e^{\ln x}} = \dfrac{1}{x}$.
- 若 $x < 0$, $\ln|x| = \ln(-x)$ 且 $e^{\ln(-x)} = -x$. 令 $f(u) = e^u$, $g(x) = \ln|x| = \ln(-x)$, 則 $f'(u) = e^u$, $f(g(x)) = -x$; 故 $g'(x) = \dfrac{-1}{f'(g(x))} \implies (\ln|x|)' = (\ln(-x))' = \dfrac{-1}{e^{\ln(-x)}} = \dfrac{1}{x}$.

性質. $(a^x)' = a^x \cdot \ln a$, $\forall\, a > 0$.

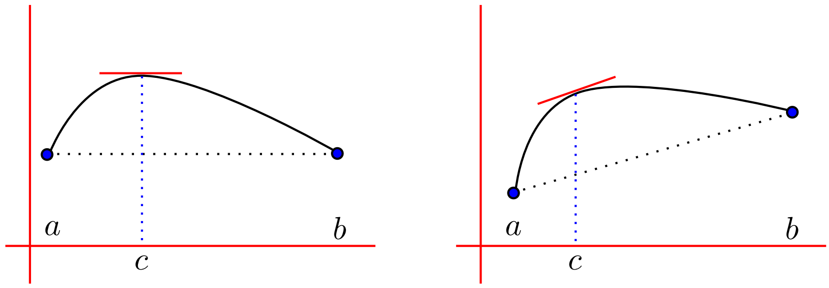
證. $a = e^{\log_e a} \equiv e^{\ln a} \implies a^x = e^{x\ln a}$. 令 $f(u) = e^u$, $g(x) = x\ln a$, 則 $f'(u) = e^u$, $f(g(x)) = e^{x\ln a} = a^x$; 故 $(f(g(x)))' = f'(g(x)) \cdot g'(x) = e^{x\ln a} \cdot (x\ln a)' = a^x \cdot \ln a$.

結論. $C(a) = \ln a$.

性質. $(x^\alpha)' = \alpha\, x^{\alpha-1}$ $(\alpha \in \mathbb{R})$

解. $x^\alpha = e^{\ln x^\alpha} = e^{\alpha\ln x}$. 故 $(x^\alpha)' = (e^{\alpha\ln x})' = e^{\alpha\ln x} \cdot \big(\alpha \cdot \dfrac{1}{x}\big) = x^\alpha \cdot \alpha \cdot \dfrac{1}{x} = \alpha\, x^{\alpha-1}$.

定理 (均值定理 (mean-value theorem, MVT))**.** 若 $f(x)$ 在 $[a,b]$ 連續, 在 (a,b) 可微, 則存在 $c \in (a,b)$ 使 $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.



性質. $f(x)$ 在 $[a,b]$ 連續, 在 (a,b) 可微. 若

1. $\forall\, x \in (a,b)$ $f'(x) > 0$, 則 f 在 $[a,b]$ 嚴格遞增.
2. $\forall\, x \in (a,b)$ $f'(x) < 0$, 則 f 在 $[a,b]$ 嚴格遞減.
3. $\forall\, x \in (a,b)$ $f'(x) \geqslant 0$, 則 f 在 $[a,b]$ 遞增.
4. $\forall\, x \in (a,b)$ $f'(x) \leqslant 0$, 則 f 在 $[a,b]$ 遞減.

證. 令 $x,\ y \in [a,b]$, $x < y$. 由 MVT $\exists\, c \in (x,y) \subseteq (a,b)$ 使 $f(y) - f(x) = f'(c)\,(y-x)$. 又 $y - x > 0$, $f(y) - f(x)$ 與 $f'(c)$ 同號.

L'Hôpital 法則

定理 (L'Hôpital 法則 (LHR))**.** 若 f 與 g 為實可微函數, 且在 (a,b) 上 $g'(x) \neq 0$ $(a,\ b \in \overline{\mathbb{R}})$. 假設

$$\lim_{x\to a+} f(x) = \lim_{x\to a+} g(x) = 0 \quad \Big(\frac{0}{0}\ \text{型}\Big) \qquad \text{或} \qquad \lim_{x\to a+} g(x) = \infty \quad \Big(\frac{\infty}{\infty}\ \text{型}\Big)$$

若 $\displaystyle\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L \in \overline{\mathbb{R}}$, 則 $\displaystyle\lim_{x\to a+} \frac{f(x)}{g(x)} = L$.

不定型	$\frac{0}{0}$	$\frac{\infty}{\infty}$	$0\cdot\infty$	$\infty-\infty$	0^0	∞^0	1^∞
範例	$\lim\limits_{x\to1}\frac{5x^4-4x^2-1}{10-x-9x^3}$	$\lim\limits_{x\to\infty}\frac{x^2}{e^x}$	$\lim\limits_{x\to0+}x\ln\frac{1}{x}$	$\lim\limits_{x\to1+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$	$\lim\limits_{x\to0+}x^x$	$\lim\limits_{x\to\infty}x^{\frac{1}{x}}$	$\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$

例 $(\frac{0}{0})$**.** 求 $\lim\limits_{x\to1}\dfrac{5x^4-4x^2-1}{10-x-9x^3}$.

解. $\lim\limits_{x\to1}\dfrac{5x^4-4x^2-1}{10-x-9x^3}=\lim\limits_{x\to1}\dfrac{20x^3-8x}{-1-27x^2}=-\dfrac{3}{7}$.

例 $(\frac{\infty}{\infty})$**.** 求 $\lim\limits_{x\to\infty}\dfrac{x^2}{e^x}$.

解. $\lim\limits_{x\to\infty}\dfrac{x^2}{e^x}=\lim\limits_{x\to\infty}\dfrac{2x}{e^x}=\lim\limits_{x\to\infty}\dfrac{2}{e^x}=0$.

例 $(0\cdot\infty)$**.** 求 $\lim\limits_{x\to0+}x\ln\dfrac{1}{x}$.

解. $\lim\limits_{x\to0+}x\ln\dfrac{1}{x}=\lim\limits_{x\to0+}\dfrac{\ln\frac{1}{x}}{\frac{1}{x}}=\lim\limits_{x\to0+}\dfrac{-\frac{1}{x}}{-\frac{1}{x^2}}=\lim\limits_{x\to0+}x=0$.

例 $(\infty-\infty)$**.** 求 $\lim\limits_{x\to1+}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right)$.

解. $\lim\limits_{x\to1+}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right)=\lim\limits_{x\to1+}\dfrac{x\ln x-(x-1)}{(x-1)\ln x}=\lim\limits_{x\to1+}\dfrac{1+\ln x-1}{(x-1)\frac{1}{x}+\ln x}=\lim\limits_{x\to1+}\dfrac{\frac{1}{x}}{\frac{1}{x^2}+\frac{1}{x}}=\dfrac{1}{2}$.

例 (0^0)**.** 求 $\lim\limits_{x\to0+}x^x$.

解. 求 $\lim\limits_{x\to0+}x^x=\lim\limits_{x\to0+}\exp\{x\ln x\}=\exp\left\{\lim\limits_{x\to0+}x\ln x\right\}=\exp\left\{-\lim\limits_{x\to0+}x\ln\dfrac{1}{x}\right\}=e^0=1$.

例 (∞^0)**.** 求 $\lim\limits_{x\to\infty}x^{\frac{1}{x}}$.

解. $\lim\limits_{x\to\infty}x^{\frac{1}{x}}=\lim\limits_{x\to\infty}\exp\left\{\ln x^{\frac{1}{x}}\right\}=\lim\limits_{x\to\infty}\exp\left\{\dfrac{1}{x}\ln x\right\}=\exp\left\{\lim\limits_{x\to\infty}\dfrac{\ln x}{x}\right\}=\exp\left\{\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}}{1}\right\}=e^0=1$.

例 (1^∞)**.** 求 $\lim\limits_{x\to\infty}\left(1+\dfrac{a}{x}\right)^{bx}$, $a,\ b\in\mathbb{R}$.

解. $\lim\limits_{x\to\infty}\left(1+\dfrac{a}{x}\right)^{bx}=\lim\limits_{x\to\infty}\exp\left\{\ln\left(1+\dfrac{a}{x}\right)^{bx}\right\}=\lim\limits_{x\to\infty}\exp\left\{bx\,\ln\left(1+\dfrac{a}{x}\right)\right\}=\exp\left\{\lim\limits_{x\to\infty}bx\,\ln\left(1+\dfrac{a}{x}\right)\right\}=\exp\left\{b\lim\limits_{x\to\infty}\dfrac{\ln(1+\frac{a}{x})}{\frac{1}{x}}\right\}=\exp\left\{b\lim\limits_{x\to\infty}\dfrac{\frac{1}{1+\frac{a}{x}}\cdot\frac{-a}{x^2}}{\frac{-1}{x^2}}\right\}=\exp\left\{b\lim\limits_{x\to\infty}\dfrac{a}{1+\frac{a}{x}}\right\}=e^{ba}$.

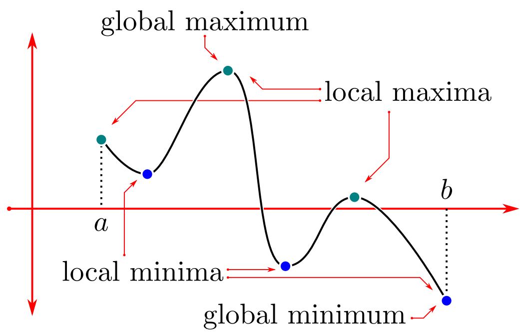
例 (循環形)**.** 求 $\lim\limits_{x\to\infty}\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$.

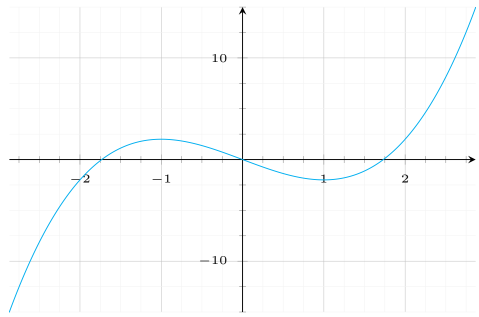
解. $\lim\limits_{x\to\infty}\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$ 為 $(\frac{\infty}{\infty})$ 型, 理應可使用 LHR, 但 $\lim\limits_{x\to\infty}\dfrac{e^x+e^{-x}}{e^x-e^{-x}}=\lim\limits_{x\to\infty}\dfrac{e^x-e^{-x}}{e^x+e^{-x}}=\lim\limits_{x\to\infty}\dfrac{e^x+e^{-x}}{e^x-e^{-x}}=\ldots$, 無限循環; $\lim\limits_{x\to\infty}\dfrac{e^x+e^{-x}}{e^x-e^{-x}}=\lim\limits_{x\to\infty}\dfrac{e^x\left(1+e^{-2x}\right)}{e^x\left(1-e^{-2x}\right)}=\lim\limits_{x\to\infty}\dfrac{1+e^{-2x}}{1-e^{-2x}}=1$.

極值問題

定義. 給定 $f : I \to \mathbb{R}$, $B(x,h) \equiv \{y \mid |y-x| < h\}$.

- f 在 $x_{\mathrm{M}} \in I$ 有最大值 (global maximum) $f(x_{\mathrm{M}})$: $f(x_{\mathrm{M}}) \geqslant f(x)$, $\forall x \in I$.
- f 在 $x_{\mathrm{m}} \in I$ 有最小值 (global minimum) $f(x_{\mathrm{m}})$: $f(x_{\mathrm{m}}) \leqslant f(x)$, $\forall x \in I$.
- f 在 $x_0 \in I$ 有極大值 (local maximum) $f(x_0)$: $\exists h_0 > 0$ 使 $f(x_0) \geqslant f(x)$, $\forall x \in B(x_0,h_0) \cap I$.
- f 在 $x_1 \in I$ 有極小值 (local minimum) $f(x_1)$: $\exists h_1 > 0$ 使 $f(x_1) \leqslant f(x)$, $\forall x \in B(x_1,h_1) \cap I$.





例.

- $f(x) = \dfrac{1}{x}$ 在 $I = \mathbb{R}$ 沒有最大值, 最小值.
- $f(x) = x$ 在 $I = (0,1)$ 沒有最大值, 最小值.
- $f(x) = x$ 在 $I = [0,1]$ 有最大值 1, 最小值 0.
- 若 $I = [-3, 3]$, $f(x) = x^3 - 3x$ 在 $x = 3$ 有最大值 18, 在 $x = -3$ 有最小值 -18, 在 $x = -1$ 有極大值 2, 在 $x = 1$ 有極小值 -2.

定理. 若 f 在 $c \in \operatorname{dom} f$ 有極值, 且 $f'(c)$ 存在, 則 $f'(c) = 0$.

結論. 設 $f : I \to \mathbb{R}$ 在 $x_0 \in I$ 有極值, 則 x_0 為以下三情形之一:

- 臨界點 (critical point) : $f'(x_0) = 0$.
- 奇異點 (singular point) : f 在 x_0 不可微.
- I 的邊界點 (boundary).

例. 求 $f(x) = x^3 - 3x^2 - 9x + 2$ 在 $[-2, 2]$ 的最大值與最小值.

解.

- f 在有限閉區間 $[-2, 2]$ 連續, 故在 $[-2, 2]$ 有最大值, 最小值.
-
  - $f'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3)$, 在 $[-2, 2]$ 之臨界點為 -1: $f(-1) = 7$.
  - f 在 $[-2, 2]$ 可微, 故無奇異點.
  - $[-2, 2]$ 的邊界點為 -2 與 2; $f(-2) = 0$, $f(2) = -20$.

  故最大值: $f(-1) = 7$, 最小值: $f(2) = -20$.

Taylor 展開式

定義.

- 給定 $f \in C^\infty(a,b)$，$x_0 \in (a,b)$，$\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ 稱為 $f(x)$ 在 $x=x_0$ 之 Tayler 級數 / 展開式；若 $x_0 = 0$ 稱為 $f(x)$ 之 MacLaurin 級數 / 展開式。
- 給定 $f \in C^N(a,b)$，$x_0 \in (a,b)$, $0 \leqslant n \leqslant N$，$\displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^n$ 稱為 $f(x)$ 在 $x=x_0$ 之 n 階 Tayler 多項式；若 $x_0 = 0$ 稱為 $f(x)$ 之 n 階 MacLaurin 多項式。

例. 常用 Maclaurin 級數。

1. $\displaystyle\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \cdots = \sum_{n=0}^{\infty} x^n, \forall\, |x| < 1.$
2. $\displaystyle e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \forall\, x \in \mathbb{R}.$

不定積分

定義 (反導函數)**.** 給定 $F(x)$, 若 $\dfrac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$, 則稱 $F(x)$ 為 $f(x)$ 的反導函數 (antiderivative)
.

性質. 若 $F(x)$, $G(x)$ 分別為 $f(x)$, $g(x)$ 的反導函數, $c \in \mathbb{R}$. 則

- $F(x) + c$ 為 $f(x)$ 的反導函數.
- $c\,F(x)$ 為 $c\,f(x)$ 的反導函數.
- $F(x) + G(x)$ 為 $f(x) + g(x)$ 的反導函數.

結論.

- $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x) \implies \mathrm{d}F(x) = f(x)\cdot\mathrm{d}x \implies F(x) = \int f(x)\cdot\mathrm{d}x = \int f(x)\,\mathrm{d}x$
- $F(x)$ 為 $f(x)$ 的反導函數 $\iff$ $f(x)$ 的反導函數為 $F(x)$ $\iff$ $F(x)$ 的導函數為 $f(x)$ $\iff$ $F(x)$ (對 x) 的微分為 $f(x)$ $\iff$ $f(x)$ (對 x) 的 (不定) 積分為 $F(x)$
- $f(x)$ 的反導函數 $\equiv$ $f(x)$ (對 x) 的 (不定) 積分
- 基礎積分集: 以下 $\alpha \neq -1$, $a \neq 0$.

$f(x)$	x^α	$\frac{1}{x}$	e^{ax}
$\int f(x)\,\mathrm{d}x$	$\frac{1}{\alpha+1}x^{\alpha+1}$	$\ln\|x\|$	$\frac{1}{a}e^{ax}$

- (Liouville) e^{-x^2}, $\dfrac{e^x}{x}$, $\dfrac{1}{\ln x}$, x^x 無 (初等函數形式之) 反導函數!

例.

1. $\int \sqrt{x}\,\mathrm{d}x = \frac{2}{3}\,x^{\frac{3}{2}} + c$
2. $\int x^{\pi}\,\mathrm{d}x = \frac{1}{\pi+1}\,x^{\pi+1} + c$
3. $\int e^{-2x}\,\mathrm{d}x = -\frac{1}{2}\,e^{-2x} + c$
4. $\int \left(\frac{\pi}{x} - e^{\pi x}\right)\mathrm{d}u = \pi \ln x - \frac{e^{\pi x}}{\pi} + c$

習題. 求下列不定積分.

1. $\int \frac{x^3-1}{x^3}\,\mathrm{d}x = x + \frac{1}{2x^3} + c$
2. $\int 5 - \frac{1}{\sqrt{x}}\,\mathrm{d}x = 5x - 2\sqrt{x} + c$
3. $\int (t-1)(t+1)\,\mathrm{d}t = \frac{t^3}{3} - t + c$
4. $\int (\sqrt{x}+1)^2\,\mathrm{d}x = \frac{x^2}{2} + x + \frac{4x^{\frac{3}{2}}}{3} + c$
5. $\int x\sqrt{3x}\,\mathrm{d}x = \frac{2\sqrt{3}}{5}\,x^{\frac{5}{2}} + c$
6. $\int \frac{1}{x^3} - \frac{1}{x^5}\,\mathrm{d}x = \frac{-1}{2x^2} + \frac{1}{4x^4} + c$

變數變換法

結論. $\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x)) \cdot g'(x) \implies \mathrm{d}f(g(x)) = f'(g(x)) \cdot g'(x)\,\mathrm{d}x \implies f(g(x)) = \int f'(g(x)) \cdot g'(x)\,\mathrm{d}x$. 令 $u = g(x)$, 則 $\frac{\mathrm{d}}{\mathrm{d}x} u = g'(x) \implies \mathrm{d}u = g'(x)\,\mathrm{d}x$; 故 $\int f'(g(x)) \cdot g'(x)\,\mathrm{d}x = \int f'(u)\,\mathrm{d}u = f(u) + c = f(g(x)) + c$.

例. 求 $\int \frac{x}{\sqrt{x+1}}\,\mathrm{d}x$.

解.

- (解一) 令 $u = x+1$, 則 $x = u-1$, $\mathrm{d}u = \mathrm{d}x$. 故 $\int \frac{x}{\sqrt{x+1}}\,\mathrm{d}x = \int \frac{u-1}{\sqrt{u}}\,\mathrm{d}u = \int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,\mathrm{d}u = \frac{2}{3}\,u^{\frac{3}{2}} - 2\sqrt{u} + c = \frac{2}{3}\,(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c$.
- (解二) $\int \frac{x}{\sqrt{x+1}}\,\mathrm{d}x = \int \frac{x+1-1}{\sqrt{x+1}}\,\mathrm{d}x = \int \sqrt{x+1} - \frac{1}{\sqrt{x+1}}\,\mathrm{d}x$. 令 $u = x+1$, 則 $\mathrm{d}u = \mathrm{d}x$. 故 $\int \sqrt{x+1} - \frac{1}{\sqrt{x+1}}\,\mathrm{d}x = \int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,\mathrm{d}u = \frac{2}{3}\,u^{\frac{3}{2}} - 2\sqrt{u} + c = \frac{2}{3}\,(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c$.
- (解三) 令 $u = \sqrt{x+1}$, 則 $x = u^2 - 1$, $\mathrm{d}u = \frac{1}{2\sqrt{x+1}}\,\mathrm{d}x \implies \frac{1}{\sqrt{x+1}}\,\mathrm{d}x = 2\,\mathrm{d}u$. 故 $\int \frac{x}{\sqrt{x+1}}\,\mathrm{d}x = \int x \cdot \frac{1}{\sqrt{x+1}}\,\mathrm{d}x = \int (u^2-1) \cdot 2\,\mathrm{d}u = \frac{2}{3}\,u^3 - 2u + c = \frac{2}{3}\,(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c$.

例. 求 $\int \frac{x}{x^2+1}\,\mathrm{d}x$.

解. 令 $u = x^2+1$, 則 $\mathrm{d}u = 2\,x\,\mathrm{d}x \implies x\,\mathrm{d}x = \frac{1}{2}\mathrm{d}u$. 故 $\int \frac{x}{x^2+1}\,\mathrm{d}x = \int \frac{1}{x^2+1} \cdot x\,\mathrm{d}x = \int \frac{1}{u} \cdot \frac{1}{2}\,\mathrm{d}u = \frac{1}{2} \ln|u| + c = \frac{1}{2} \ln(x^2+1) + c$.

例. 求 $\int e^x \sqrt{1+e^x}\,\mathrm{d}x$.

解. 令 $u=1+e^x$, 則 $\mathrm{d}u=e^x\,\mathrm{d}x$. 故 $\int e^x\sqrt{1+e^x}\,\mathrm{d}x=\int\sqrt{1+e^x}\cdot e^x\,\mathrm{d}x=\int\sqrt{u}\cdot\mathrm{d}u=\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+c=\frac{2}{3}(1+e^x)^{\frac{3}{2}}+c$

習題. 以變數變換法求下列不定積分.

1. $\int\frac{1}{\sqrt{2x-1}}\,\mathrm{d}x=\sqrt{2x-1}+c$
2. $\int\sqrt{7x+4}\,\mathrm{d}x=\frac{2}{21}\,(7x+4)^{\frac{3}{2}}+c$
3. $\int e^{\pi x-1}\,\mathrm{d}x=\frac{e^{\pi x-1}}{\pi}+c$
4. $\int(x^2-2x+1)^{\frac{1}{3}}\,\mathrm{d}x=\frac{3}{5}\,(x-1)^{\frac{5}{3}}+c$
5. $\int\frac{x}{\sqrt{1+2x^2}}\,\mathrm{d}x=\frac{\sqrt{1+2x^2}}{2}+c$
6. $\int x^2\sqrt{1-x}\,\mathrm{d}x=-\frac{2\,(1-x)^{\frac{3}{2}}}{3}+\frac{4\,(1-x)^{\frac{5}{2}}}{5}-\frac{2\,(1-x)^{\frac{7}{2}}}{7}$

解.

1. 令 $u=2x-1$, 則 $\mathrm{d}u=2\,\mathrm{d}x\implies\mathrm{d}x=\frac{1}{2}\mathrm{d}u$. 故 $\int\frac{1}{\sqrt{2x-1}}\,\mathrm{d}x=\int\frac{1}{\sqrt{u}}\cdot\frac{1}{2}\mathrm{d}u=\sqrt{u}+c=\sqrt{2x-1}+c$.
2. 令 $u=7x+4$, 則 $\mathrm{d}u=7\,\mathrm{d}x\implies\mathrm{d}x=\frac{1}{7}\mathrm{d}u$. 故 $\int\sqrt{7x+4}\,\mathrm{d}x=\int\sqrt{u}\cdot\frac{1}{7}\mathrm{d}u=\frac{1}{7}\cdot\frac{2}{3}\,u^{\frac{3}{2}}+c=\frac{2}{21}\,(7x+4)^{\frac{3}{2}}+c$.
3. 令 $u=\pi x-1$, 則 $\mathrm{d}u=\pi\,\mathrm{d}x\implies\mathrm{d}x=\frac{1}{\pi}\mathrm{d}u$. 故 $\int e^{\pi x-1}\,\mathrm{d}x=\int e^u\cdot\frac{1}{\pi}\mathrm{d}u=\frac{1}{\pi}\,e^u+c=\frac{e^{\pi x-1}}{\pi}+c$.
4. 令 $u=x-1$, 則 $\mathrm{d}u=\mathrm{d}x$. 故 $\int(x^2-2x+1)^{\frac{1}{3}}\,\mathrm{d}x=\int(x-1)^{\frac{2}{3}}\,\mathrm{d}x=\int u^{\frac{2}{3}}\,\mathrm{d}u=\frac{3}{5}\,u^{\frac{5}{3}}+c=\frac{3}{5}\,(x-1)^{\frac{5}{3}}+c$.
5. 令 $u=1+2x^2$, 則 $\mathrm{d}u=4x\,\mathrm{d}x\implies x\,\mathrm{d}x=\frac{1}{4}\mathrm{d}u$. 故 $\int\frac{x}{\sqrt{1+2x^2}}\,\mathrm{d}x=\int\frac{1}{\sqrt{1+2x^2}}\cdot x\,\mathrm{d}x=\int\frac{1}{\sqrt{u}}\cdot\frac{1}{4}\mathrm{d}u=\frac{1}{4}\cdot 2\,u^{\frac{1}{2}}+c=\frac{\sqrt{1+2x^2}}{2}+c$.
6. 令 $u=\sqrt{1-x}$, 則 $u^2=1-x\implies x=1-u^2$, $\mathrm{d}x=-2\,u\,\mathrm{d}u$. 故 $\int x^2\sqrt{1-x}\,\mathrm{d}x=\int(1-u^2)^2\cdot u\cdot(-2)\,u\,\mathrm{d}u=-2\int(1-u^2)^2\cdot u^2\,\mathrm{d}u=-2\int\left(u^2-2u^4+u^6\right)\mathrm{d}u=-\frac{2\,u^3}{3}+\frac{4\,u^5}{5}-\frac{2\,u^7}{7}+c=-\frac{2\,(1-x)^{\frac{3}{2}}}{3}+\frac{4\,(1-x)^{\frac{5}{2}}}{5}-\frac{2\,(1-x)^{\frac{7}{2}}}{7}$.

習題. 以變數變換法求下列不定積分.

1. $\int xe^{-\frac{x^2}{2}}\,\mathrm{d}x=-e^{-\frac{x^2}{2}}+c$
2. $\int x^2 2^{x^3+1}\,\mathrm{d}x=\frac{2^{x^3+1}}{3\ln 2}+c$
3. $\int\frac{\ln x}{x}\,\mathrm{d}x=\frac{1}{2}\,(\ln x)^2+c$
4. $\int\frac{x+1}{\sqrt{x^2+2x+3}}\,\mathrm{d}x=\sqrt{x^2+2x+3}+c$

解.

1. 令 $u = \frac{x^2}{2}$, 則 $\mathrm{d}u = x\,\mathrm{d}x$, 故 $\int x e^{-\frac{x^2}{2}}\,\mathrm{d}x = \int e^{-\frac{x^2}{2}} \cdot x\,\mathrm{d}x = \int e^{-u}\,\mathrm{d}u = -e^{-u} + c = -e^{-\frac{x^2}{2}} + c$
2. 令 $u = x^3 + 1$, 則 $\mathrm{d}u = 3x^2\,\mathrm{d}x \implies x^2\,\mathrm{d}x = \frac{1}{3}\,\mathrm{d}u$, 故 $\int x^2 2^{x^3+1}\,\mathrm{d}x = \int 2^{x^3+1} \cdot x^2\,\mathrm{d}x = \int 2^u \cdot \frac{1}{3}\,\mathrm{d}u = \frac{1}{3}\int e^{u\ln 2}\,\mathrm{d}u = \frac{1}{3\ln 2} e^{u\ln 2} + c = \frac{2^{x^3+1}}{3\ln 2} + c$
3. 令 $u = \ln x$, 則 $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$, 故 $\int \frac{\ln x}{x}\,\mathrm{d}x = \int \ln x \cdot \frac{1}{x}\,\mathrm{d}x = \int u\,\mathrm{d}u = \frac{1}{2}u^2 + c = \frac{1}{2}(\ln x)^2 + c$
4. 令 $u = x^2 + 2x + 3$, 則 $\mathrm{d}u = (2x+2)\,\mathrm{d}x \implies (x+1)\,\mathrm{d}x = \frac{1}{2}\,\mathrm{d}u$, 故 $\int \frac{x+1}{\sqrt{x^2+2x+3}}\,\mathrm{d}x = \int \frac{1}{\sqrt{x^2+2x+3}} \cdot (x+1)\,\mathrm{d}x = \int \frac{1}{\sqrt{u}} \cdot \frac{1}{2}\,\mathrm{d}u = \sqrt{u} + c = \sqrt{x^2+2x+3} + c$

部份積分法

結論. $\frac{\mathrm{d}}{\mathrm{d}x}\left(u(x)\,v(x)\right) = u(x)\,\frac{\mathrm{d}v(x)}{\mathrm{d}x} + v(x)\,\frac{\mathrm{d}u(x)}{\mathrm{d}x} \implies u(x)\,v(x) = \int u(x)\,\mathrm{d}v(x) + \int v(x)\,\mathrm{d}u(x)$ $\implies \int u\,\mathrm{d}v = u\,v - \int v\,\mathrm{d}u.$

例. 求 $\int x\,e^x\,\mathrm{d}x.$

解. 令 $u = x$, 則 $\mathrm{d}u = \mathrm{d}x$. 令 $\mathrm{d}v = e^x\,\mathrm{d}x$, 則 $v = e^x$. 故 $\int x\,e^x\,\mathrm{d}x = x \cdot e^x - \int e^x\,\mathrm{d}x = xe^x - e^x + c.$

例. 求 $\int \ln x\,\mathrm{d}x.$

解. 令 $u = \ln x$, 則 $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$. 令 $\mathrm{d}v = \mathrm{d}x$, 則 $v = x$. 故 $\int \ln x\,\mathrm{d}x = \ln x \cdot x - \int x \cdot \frac{1}{x}\,\mathrm{d}x = x\ln x - x + c.$

列表法

結論.

1. 將要微分函數寫左邊, 積分函數寫右邊; 左邊連續微分, 右邊連續積分
2. 依序左上連右下斜線函數相乘, 最底部水平兩邊函數相乘並積分, 符號正負相間
3. 將上式所得項全部加總即為所求積分

例. 求 $\int (x+3)\,e^{2x}\,\mathrm{d}x.$

解.

Diff		Int
$x+3$		e^{2x}
	$+$	
1		$\frac{1}{2}e^{2x}$
	$-$	
0		$\frac{1}{4}e^{2x}$

$$\int (x+3)\,e^{2x}\,\mathrm{d}x = (x+3)\,\frac{1}{2}e^{2x} - \frac{1}{4}e^{2x}$$

例. 求 $\int (x^2 - 2x)\,e^{kx}\,\mathrm{d}x.$

解.

Diff		Int
x^2-2x	$+$	e^{kx}
$2(x-1)$	$-$	$\frac{1}{k}e^{kx}$
2	$+$	$\frac{1}{k^2}e^{kx}$
0		$\frac{1}{k^3}e^{kx}$

$$\int (x^2-2x)\, e^{kx}\, \mathrm{d}x = (x^2-2x)\,\frac{1}{k}e^{kx} - 2(x-1)\frac{1}{k^2}e^{kx} + 2\frac{1}{k^3}e^{kx}$$

例. 求 $\displaystyle\int x^5\, e^{ax}\, \mathrm{d}x$.

解.

Diff		Int
x^5	$+$	e^{ax}
$5x^4$	$-$	$\frac{1}{a}e^{ax}$
$20x^3$	$+$	$\frac{1}{a^2}e^{ax}$
$60x^2$	$-$	$\frac{1}{a^3}e^{ax}$
$120x$	$+$	$\frac{1}{a^4}e^{ax}$
120	$-$	$\frac{1}{a^5}e^{ax}$
0		$\frac{1}{a^6}e^{ax}$

$$\begin{aligned}&\int x^5\, e^{ax}\, \mathrm{d}x\\ &= x^5\,\frac{1}{a}e^{ax} - 5x^4\,\frac{1}{a^2}e^{ax} + 20x^3\,\frac{1}{a^3}e^{ax} - 60x^2\,\frac{1}{a^4}e^{ax} + 120x\,\frac{1}{a^5}e^{ax} - 120\,\frac{1}{a^6}e^{ax}\\ &= \left(\frac{x^5}{a} - \frac{5x^4}{a^2} + \frac{20x^3}{a^3} - \frac{60x^2}{a^4} + \frac{120x}{a^5} - \frac{120}{a^6}\right) e^{ax}\end{aligned}$$

習題. 以部份積分法求下列不定積分. 注意: 可能會因為常數項而跟此處答案不同.

1. $\displaystyle\int x^3 \ln x\, \mathrm{d}x = \frac{x^4}{4}\ln x - \frac{x^4}{16} + c$
2. $\displaystyle\int x(\ln x)^3\, \mathrm{d}x = \frac{x^2}{2}\left((\ln x)^3 - \frac{3(\ln x)^2}{2} + \frac{3\ln x}{2} - \frac{3}{4}\right) + c$
3. $\displaystyle\int x^5 e^{-x^2}\, \mathrm{d}x = -\frac{1}{2}e^{-x^2}(x^4+2x^2+2) + c$
4. $\displaystyle\int x e^{\sqrt{x}}\, \mathrm{d}x = 2\, e^{\sqrt{x}}\left(x\sqrt{x} - 3x + 6\sqrt{x} - 6\right) + c$
5. $\displaystyle\int \frac{x e^x}{(x+1)^2}\, \mathrm{d}x = \frac{e^x}{x+1} + c$

解.

1. 令 $u = \ln x$, 則 $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$; $\mathrm{d}v = x^3\,\mathrm{d}x$, 則 $v = \frac{x^4}{4}$. 故 $\displaystyle\int x^3 \ln x\, \mathrm{d}x = \ln x \cdot \frac{x^4}{4} - \int \frac{x^4}{4}\cdot\frac{1}{x}\,\mathrm{d}x = \ln x\cdot\frac{x^4}{4} - \int\frac{x^3}{4}\,\mathrm{d}x = \ln x\cdot\frac{x^4}{4} - \frac{x^4}{16} + c$.
2. 令 $u = (\ln x)^3$, 則 $\mathrm{d}u = 3(\ln x)^2\cdot\frac{1}{x}\,\mathrm{d}x$; $\mathrm{d}v = x\,\mathrm{d}x$, 則 $v = \frac{x^2}{2}$. 故 $\displaystyle\int x(\ln x)^3\,\mathrm{d}x = (\ln x)^3\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot 3(\ln x)^2\cdot\frac{1}{x}\,\mathrm{d}x = (\ln x)^3\cdot\frac{x^2}{2} - \frac{3}{2}\int x(\ln x)^2\,\mathrm{d}x$. 令 $u = (\ln x)^2$, 則 $\mathrm{d}u = 2\,\ln x\cdot\frac{1}{x}\,\mathrm{d}x$; $\mathrm{d}v = x\,\mathrm{d}x$, 則 $v = \frac{x^2}{2}$. 故 $\displaystyle\int x(\ln x)^2\,\mathrm{d}x = (\ln x)^2\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot 2\ln x\cdot\frac{1}{x}\,\mathrm{d}x = (\ln x)^2\cdot\frac{x^2}{2} - \int x\ln x\,\mathrm{d}x$. 令 $u = \ln x$, 則 $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$; $\mathrm{d}v = x\,\mathrm{d}x$, 則 $v = \frac{x^2}{2}$. 故 $\displaystyle\int x\ln x\,\mathrm{d}x = \ln x\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{1}{x}\,\mathrm{d}x = \ln x\cdot\frac{x^2}{2} - \frac{1}{2}\int x\,\mathrm{d}x = \ln x\cdot\frac{x^2}{2} - \frac{x^2}{4}$. 以上,

$$\int x(\ln x)^3\,\mathrm{d}x = (\ln x)^3\cdot\frac{x^2}{2}-\frac{3}{2}\Big((\ln x)^2\cdot\frac{x^2}{2}-\big(\ln x\cdot\frac{x^2}{2}-\frac{x^2}{4}\big)\Big)=\frac{x^2}{2}\Big((\ln x)^3-\frac{3(\ln x)^2}{2}+\frac{3\ln x}{2}-\frac{3}{4}\Big)+c$$

3. 令 $w=x^2$, 則 $\mathrm{d}w=2x\,\mathrm{d}x \implies x\,\mathrm{d}x=\frac{1}{2}\mathrm{d}w$, 故 $\int x^5e^{-x^2}\,\mathrm{d}x=\int e^{-x^2}\cdot(x^2)^2\cdot x\,\mathrm{d}x=\int e^{-w}\cdot w^2\cdot\frac{1}{2}\mathrm{d}w=\frac{1}{2}\int w^2\,e^{-w}\,\mathrm{d}w=-\frac{1}{2}e^{-w}(w^2+2w+2)=-\frac{1}{2}e^{-x^2}(x^4+2x^2+2)+c.$

Diff		Int
w^2	$+$	e^{-w}
$2w$	$-$	$-e^{-w}$
2	$+$	e^{-w}
0		$-e^{-w}$

$$\int w^2e^{-w}\,\mathrm{d}w=-w^2\,e^{-w}-2w\,e^{-w}-2\,e^{-w}=-e^{-w}(w^2+2w+2)$$

4. 令 $w=\sqrt{x}$, 則 $w^2=x$, $\mathrm{d}x=2w\,\mathrm{d}w$, 故 $\int xe^{\sqrt{x}}\,\mathrm{d}x=\int w^2e^w\cdot 2w\,\mathrm{d}w=2\int w^3e^w\,\mathrm{d}w=2\,e^w(w^3-3w^2+6w-6)=2\,e^{\sqrt{x}}\,(x\sqrt{x}-3x+6\sqrt{x}-6)+c$

Diff		Int
w^3	$+$	e^w
$3w^2$	$-$	e^w
$6w$	$+$	e^w
6	$-$	e^w
0		e^w

$$\int w^3\,e^w\,\mathrm{d}w=w^3\,e^w-3w^2\,e^w+6w\,e^w-6\,e^w=e^w(w^3-3w^2+6w-6)$$

5. $\int\frac{xe^x}{(x+1)^2}\,\mathrm{d}x=\int\frac{(x+1-1)e^x}{(x+1)^2}\,\mathrm{d}x=\int\frac{e^x}{x+1}\,\mathrm{d}x-\int\frac{e^x}{(x+1)^2}\,\mathrm{d}x$. 令 $u=e^x$, 則 $\mathrm{d}u=e^x\,\mathrm{d}x$; $\mathrm{d}v=\frac{1}{(x+1)^2}\,\mathrm{d}x$, 則 $v=\frac{-1}{x+1}$. 故 $\int\frac{e^x}{(x+1)^2}\,\mathrm{d}x=e^x\cdot\frac{-1}{x+1}+\int\frac{1}{x+1}\cdot e^x\,\mathrm{d}x=\frac{-e^x}{x+1}+\int\frac{e^x}{x+1}\,\mathrm{d}x$; 原式 $\int\frac{xe^x}{(x+1)^2}\,\mathrm{d}x=\int\frac{e^x}{x+1}\,\mathrm{d}x-\int\frac{e^x}{(x+1)^2}\,\mathrm{d}x=\int\frac{e^x}{x+1}\,\mathrm{d}x-\Big(\frac{-e^x}{x+1}+\int\frac{e^x}{x+1}\,\mathrm{d}x\Big)=\frac{e^x}{x+1}.$

定積分

定積分 $\approx$ (帶符號) 面積: x 軸上方為正, 下方為負.

定義. 給定 $f:[a,\,b]\to\mathbb{R}$.

- $[a,\,b]$ 分割 $\mathbb{P}: a=x_0<x_1<x_2<\cdots<x_n=b$
- $\Delta x_k=x_k-x_{k-1}$, $k=1,\,2,\,\ldots,\,n$; $\|\mathbb{P}\|=\max\{\,|\Delta x_k|\mid 1\leqslant k\leqslant n\}$
- 樣本點 ξ_k: $x_{k-1}\leqslant\xi_k\leqslant x_k$, $k=1,\,2,\,\ldots,\,n$
- $u_k=\sup\{\,f(x)\mid x_{k-1}\leqslant x\leqslant x_k\}$, $l_k=\inf\{\,f(x)\mid x_{k-1}\leqslant x\leqslant x_k\}$, $k=1,\,2,\,\ldots,\,n$
- $R(f,\mathbb{P})=\sum_{k=1}^n f(\xi_k)\,\Delta x_k$, $U(f,\mathbb{P})=\sum_{k=1}^n u_k\,\Delta x_k$, $L(f,\mathbb{P})=\sum_{k=1}^n l_k\,\Delta x_k$;
  顯然 $L(f,\mathbb{P})\leqslant R(f,\mathbb{P})\leqslant U(f,\mathbb{P})$.

- 求 $\lim_{\|\mathbb{P}\|\to 0} R(f,\mathbb{P})$. 若對不同分割與樣本點選取此極限均存在且相等, 稱 f 在 $[a,b]$ 可積(分) ; $f(x)$ 在 $[a,b]$ 的定積分 $\int_a^b f(x)\,\mathrm{d}x \equiv \lim_{\|\mathbb{P}\|\to 0} R(f,\mathbb{P})$

註.

- 在 $\int_a^b f(x)\,\mathrm{d}x$ 中, a 為*積分下限* (lower limit of integration) , b 為*積分上限* (upper limit of integration) , $f(x)$ 為*被積分式* (integrand) , x 為*積分變數* (variable of integration) .
- $\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(t)\,\mathrm{d}t = \int_a^b f(u)\,\mathrm{d}u$ (定積分數值與積分變數無關)

結論. 若 f 在 $[a,b]$ 連續, 則 f 在 $[a,b]$ 可積.

性質. 令 f, g 在包含 a, b, c 之區間為可積, $\alpha, \beta \in \mathbb{R}$. 則

1. $\int_a^a f(x)\,\mathrm{d}x = 0$
2. $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$
3. $\int_a^b \big(\alpha\, f(x)+\beta\, g(x)\big)\,\mathrm{d}x = \alpha\int_a^b f(x)\,\mathrm{d}x + \beta\int_a^b g(x)\,\mathrm{d}x$
4. $\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x$
5. $\int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b g(x)\,\mathrm{d}x$, 若 $f(x) \leqslant g(x)\ \forall\, x\in[a,b]$, $a\leqslant b$
6. $\left|\int_a^b f(x)\,\mathrm{d}x\right| \leqslant \int_a^b |\,f(x)\,|\,\mathrm{d}x, \quad a\leqslant b$
7. $f(x)$ 為奇函數: $\int_{-a}^a f(x)\,\mathrm{d}x = 0$
8. $f(x)$ 為偶函數: $\int_{-a}^a f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x$

例.

1. $\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-x^2}\,\mathrm{d}x = \pi$ (半徑 $\sqrt{2}$ 之半圓面積)
2. $\int_{-4}^{4} \big(e^x - e^{-x}\big)\,\mathrm{d}x = 0$ ($e^x - e^{-x}$ 為奇函數)
3. $\int_{-2024}^{2024} \big(e^{9x^5-2x^7} - e^{-9x^5+2x^7}\big)\,\mathrm{d}x = 0$ ($e^{9x^5-2x^7} - e^{-9x^5+2x^7}$ 為奇函數)
4. $\int_{-a}^{a} |x|\,\mathrm{d}x = 2\int_0^a |x|\,\mathrm{d}x = a^2$ ($|x|$ 為偶函數; 兩 $a\times a$ 等腰直角三角形面積)
5. 定義 $\int_1^x \frac{1}{\tau}\,\mathrm{d}\tau \equiv \ln x$, 則 $\int_{\frac14}^3 \frac{1}{x}\,\mathrm{d}x = \int_{\frac14}^1 \frac{1}{x}\,\mathrm{d}x + \int_1^3 \frac{1}{x}\,\mathrm{d}x = -\int_1^{\frac14} \frac{1}{x}\,\mathrm{d}x + \int_1^3 \frac{1}{x}\,\mathrm{d}x = \ln 12$

微積分基本定理

定理 (微積分基本定理 (Fundamental Theorem of Calculus, FTC))**.**

1. 若 f 在 $[a,b]$ 連續, 令 $F(x) = \int_a^x f(\tau)\,\mathrm{d}\tau$ 且 $a\leqslant x\leqslant b$, 則 $F'(x) = f(x)\ \forall\, x\in[a,b]$.
2. 若 $G'(x) = f(x)\ \forall\, x\in[a,b]$, 則 $\int_a^b f(x)\,\mathrm{d}x = G(b)-G(a) \equiv G(x)\Big|_a^b$.

註. 由 FTC, $\int_a^b f(x)\,\mathrm{d}x$ 可由 f 的反導函數 (不定積分) 得出, 不需繁複極限計算!

例 (以 FTC 求定積分)**.**

1. x^2 之反導函數為 $\frac{x^3}{3}$, 故 $\int_0^1 x^2\,\mathrm{d}x = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$.
2. e^x 之反導函數為 e^x, 故 $\int_0^1 e^x\,\mathrm{d}x = e^1 - e^0 = e - 1$.

結論 (定積分變數變換)**.**

- 求反導函數後代入: $\int_a^b f'(g(x))\,g'(x)\,\mathrm{d}x = f(g(x))\Big|_{x=a}^{x=b} = f(g(b)) - f(g(a))$
- 變數變換並改變積分範圍: $\int_a^b f'(g(x))\,g'(x)\,\mathrm{d}x = \int_a^b f'(g(x))\,\mathrm{d}g(x) = \int_{g(a)}^{g(b)} f'(u)\,\mathrm{d}u = f(u)\Big|_{u=g(a)}^{u=g(b)} = f(g(b)) - f(g(a))$

例. 求 $\int_0^1 x^3(1+x^4)^3\,\mathrm{d}x$.

解.

- 求反導函數後代入: 令 $u = 1 + x^4$, 則 $\mathrm{d}u = 4x^3\,\mathrm{d}x \implies x^3\,\mathrm{d}x = \frac{\mathrm{d}u}{4}$, 故 $\int x^3(1+x^4)^3\,\mathrm{d}x = \int (1+x^4)^3\,x^3\,\mathrm{d}x = \int u^3\,\frac{\mathrm{d}u}{4} = \frac{u^4}{16} + c = \frac{(1+x^4)^4}{16} + c$. 故 $\int_0^1 x^3(1+x^4)^3\,\mathrm{d}x = \frac{(1+x^4)^4}{16}\Big|_{x=0}^{x=1} = \frac{(1+1^4)^4 - (1+0^4)^4}{16} = \frac{15}{16}$.
- 變數變換並改變積分範圍: 令 $u = 1 + x^4$, 則 $\mathrm{d}u = 4x^3\,\mathrm{d}x \implies x^3\,\mathrm{d}x = \frac{\mathrm{d}u}{4}$. 積分範圍 x 由 0 至 1, 則變數變換後 u 由 $1 + 0^4 = 1$ 至 $1 + 1^4 = 2$, 故 $\int_0^1 x^3(1+x^4)^3\,\mathrm{d}x = \int_0^1 (1+x^4)^3\,x^3\,\mathrm{d}x = \int_1^2 u^3\,\frac{\mathrm{d}u}{4} = \frac{1}{4}\int_1^2 u^3\,\mathrm{d}u = \frac{1}{16}u^4\Big|_{u=1}^{u=2} = \frac{2^4 - 1^4}{16} = \frac{15}{16}$.

例. 求 $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{1+x^2}}\,\mathrm{d}x$.

解.

- 求反導函數後代入: 令 $u = 1 + x^2$, 則 $\mathrm{d}u = 2x\,\mathrm{d}x \implies 4x\,\mathrm{d}x = 2\,\mathrm{d}u$, 故 $\int \frac{4x}{\sqrt{1+x^2}}\,\mathrm{d}x = \int \frac{2}{\sqrt{u}}\,\mathrm{d}u = 4\sqrt{u} + c = 4\sqrt{1+x^2} + c$. 故 $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{1+x^2}}\,\mathrm{d}x = 4\sqrt{1+x^2}\Big|_{x=0}^{x=\sqrt{3}} = 4\sqrt{1+3} - 4\sqrt{1} = 4$.
- 變數變換並改變積分範圍: 令 $u = 1 + x^2$, 則 $\mathrm{d}u = 2x\,\mathrm{d}x \implies 4x\,\mathrm{d}x = 2\,\mathrm{d}u$. 積分範圍 x 由 0 至 $\sqrt{3}$, 則變數變換後 u 由 $1 + 0^2 = 1$ 至 $1 + (\sqrt{3})^2 = 4$, 故 $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{1+x^2}}\,\mathrm{d}x = \int_1^4 \frac{2}{\sqrt{u}}\,\mathrm{d}u = 4\sqrt{u}\Big|_{u=1}^{u=4} = 4\,(\sqrt{4} - \sqrt{1}) = 4$.

例. 若 f 在 $[a,b]$ 二次可微且 $f(a) = f(b) = 0$, 證明 $\int_a^b (x-a)(b-x)\,f''(x)\,\mathrm{d}x = -2\int_a^b f(x)\,\mathrm{d}x$.

解. $\displaystyle\int_a^b (x-a)(b-x)\,f''(x)\,\mathrm{d}x = \big((x-a)(b-x)\,f'(x) - (a+b-2x)\,f(x)\big)\Big|_a^b - 2\int_a^b f(x)\,\mathrm{d}x =$ $\displaystyle\big((b-a)(b-b)\,f'(b)-(a+b-2b)\,f(b)\big)-\big((a-a)(b-a)\,f'(a)-(a+b-2a)\,f(a)\big)-2\int_a^b f(x)\,\mathrm{d}x =$ $\displaystyle -2\int_a^b f(x)\,\mathrm{d}x.$

Diff		Int
$(x-a)(b-x)$	$+$	$f''(x)$
$a+b-2x$	$-$	$f'(x)$
-2	$+$	$f(x)$
0		$\displaystyle\int f(x)\,\mathrm{d}x$

$$\begin{aligned}&\int (x-a)(b-x)\,f''(x)\,\mathrm{d}x\\ &= \big((x-a)(b-x)\,f'(x) - (a+b-2x)\,f(x)\big) - 2\int f(x)\,\mathrm{d}x\end{aligned}$$

性質. 令 $\displaystyle F(x) = \int_{v(x)}^{u(x)} f(\tau)\,\mathrm{d}\tau$, 則 $F'(x) = f(u(x))\cdot u'(x) - f(v(x))\cdot v'(x)$.

證. 令 $a\in\mathbb{R}$, $\displaystyle F(x) = \int_{v(x)}^{u(x)} f(\tau)\,\mathrm{d}\tau = \int_a^{u(x)} f(\tau)\,\mathrm{d}\tau - \int_a^{v(x)} f(\tau)\,\mathrm{d}\tau$. 令 $\displaystyle G(x) \equiv \int_a^x f(\tau)\,\mathrm{d}\tau$, 則 $G'(x) = f(x)$, $\displaystyle F(x) = \int_a^{u(x)} f(\tau)\,\mathrm{d}\tau - \int_a^{v(x)} f(\tau)\,\mathrm{d}\tau = G(u(x)) - G(v(x))$; 故 $F'(x) = (G(u(x)) - G(v(x)))' = G'(u(x))\cdot u'(x) - G'(v(x))\cdot v'(x) = f(u(x))\cdot u'(x) - f(v(x))\cdot v'(x)$.

例.

1. $\displaystyle F(x) = \int_1^x \frac{1}{1+\tau^4}\,\mathrm{d}\tau \implies F'(x) = \frac{1}{1+x^4}$
2. $\displaystyle F(x) = \int_x^{2x} \tau^3\,\mathrm{d}\tau \implies F'(x) = (2x)^3\cdot 2 - x^3\cdot 1 = 15x^3$

積分技巧: 部份分式

例. 若 $a\neq 0$, 求 $\displaystyle\int \frac{1}{x^2-a^2}\,\mathrm{d}x$.

解. 由 $\displaystyle\frac{1}{x^2-a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$, $\displaystyle\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \int \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)\mathrm{d}x = \frac{1}{2a}\big(\ln|x-a| - \ln|x+a|\big)$

例. 求 $\displaystyle\int \frac{x}{x^2-5x+6}\,\mathrm{d}x$.

解. $\displaystyle\frac{x}{x^2-5x+6} = \frac{x}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} \implies x = A\,(x-2) + B\,(x-3)$. 代入 $x = 3 \implies 3 = A$; 代入 $x = 2 \implies 2 = B(2-3) \implies B = -2$, 故 $\displaystyle\frac{x}{x^2-5x+6} = \frac{3}{x-3} - \frac{2}{x-2}$, $\displaystyle\int \frac{x}{x^2-5x+6}\,\mathrm{d}x = \int \left(\frac{3}{x-3} - \frac{2}{x-2}\right)\mathrm{d}x = 3\ln|x-3| - 2\ln|x-2|$