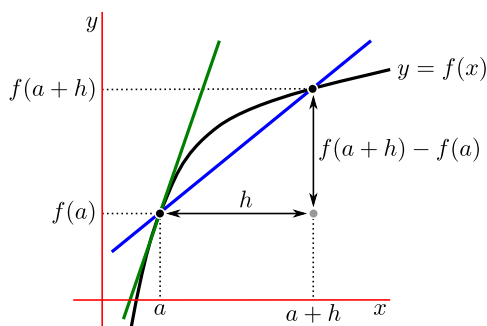# 第二章 微分

## 2.1 導數與導函數



**定義.** 給定 $f(x)$, $a \in \operatorname{dom} f$. $f$ 在 $a$ 的導數 (derivative) $f'(a)$ 定義為

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

若 $f'(a)$ 存在, 則稱 $f$ 在 $a$ 可微 (分)(differentiable). $f$ 的導函數 $f'(x)$ 定義為

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

**註.**

- 求導函數之過程稱為微分 (differentiation) : 求 $f'(x) \iff f(x)$ (對 $x$) 微分

- 給定 $y = f(x)$, 其導函數可記為 $f'(x) = f' = y' = \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}f}{\mathrm{d}x} = \dfrac{\mathrm{d}}{\mathrm{d}x} f(x) = Df(x) = D_x f(x)$.

- $f$ 在 $a$ 的導數可記為 $f'(a) = \dfrac{\mathrm{d}y}{\mathrm{d}x}\bigg|_{x=a}$.

**例.** 以極限定義求以下 $f(x)$ 之導函數 $f'(x)$, 當 $f(x)$ 為

1. $x^2$    2. $x^4$    3. $\frac{1}{x}$    4. $\frac{1}{x^5}$    5. $\frac{1}{x^2+3}$    6. $\sqrt{x+1}$    7. $\sqrt{x^2+1}$    8. $\sqrt[3]{1-x^3}$

**解.**

1. $f'(x) = \lim\limits_{h \to 0} \dfrac{(x+h)^2 - x^2}{h} = \lim\limits_{h \to 0} \dfrac{2xh + h^2}{h} = \lim\limits_{h \to 0} (2x + h) = 2x$.

2. $f'(x) = \lim\limits_{h \to 0} \dfrac{(x+h)^4 - x^4}{h} = \lim\limits_{h \to 0} \dfrac{4x^3 h + 6x^2 h^2 + 4xh^3 + h^4}{h} = \lim\limits_{h \to 0} (4x^3 + 6x^2 h + 4xh^2 + h^3) = 4x^3$.

3. $f'(x) = \lim\limits_{h \to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim\limits_{h \to 0} \dfrac{x - (x+h)}{h(x+h)x} = \lim\limits_{h \to 0} \dfrac{-h}{h(x+h)x} = \lim\limits_{h \to 0} \dfrac{-1}{(x+h)x} = \dfrac{-1}{x^2} = (-1)\, x^{-2}$.

4. $f'(x) = \lim\limits_{h \to 0} \dfrac{\frac{1}{(x+h)^5} - \frac{1}{x^5}}{h} = \lim\limits_{h \to 0} \dfrac{x^5 - (x+h)^5}{h(x+h)^5 x^5}$

   $= \lim\limits_{h \to 0} \dfrac{(-h)(x^4 + x^3(x+h) + x^2(x+h)^2 + x^3(x+h) + (x+h)^4)}{h(x+h)^5 x^5}$

   $= \lim\limits_{h \to 0} \dfrac{-(x^4 + x^3(x+h) + x^2(x+h)^2 + x^3(x+h) + (x+h)^4)}{(x+h)^5 x^5} = \dfrac{-5x^4}{x^{10}} = \dfrac{-5}{x^6} = (-5)\, x^{-6}$.

5. $f'(x) = \lim\limits_{h \to 0} \dfrac{\frac{1}{(x+h)^2+3} - \frac{1}{x^2+3}}{h} = \lim\limits_{h \to 0} \dfrac{(x^2+3) - ((x+h)^2 + 3)}{h((x+h)^2 + 3)(x^2+3)} = \lim\limits_{h \to 0} \dfrac{(2x+h)(-h)}{h((x+h)^2 + 3)(x^2+3)}$

   $= \lim\limits_{h \to 0} \dfrac{-(2x+h)}{((x+h)^2 + 3)(x^2+3)} = \dfrac{-2x}{(x^2+3)^2}$.

6. $f'(x) = \lim\limits_{h \to 0} \dfrac{\sqrt{x+h+1} - \sqrt{x+1}}{h} = \lim\limits_{h \to 0} \dfrac{h}{h(\sqrt{x+h+1} + \sqrt{x+1})} = \dfrac{1}{2\sqrt{x+1}}$.

7. $f'(x) = \lim\limits_{h \to 0} \dfrac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h} = \lim\limits_{h \to 0} \dfrac{h(2x+h)}{h(\sqrt{(x+h)^2+1} + \sqrt{x^2+1})} = \dfrac{x}{\sqrt{x^2+1}}$.

8. $f'(x) = \lim\limits_{h \to 0} \dfrac{\sqrt[3]{1-(x+h)^3} - \sqrt[3]{1-x^3}}{h}$ \qquad (使用 $a^3 - b^3 = (a-b)(a^2+ab+b^2)$)

$= \lim\limits_{h \to 0} \dfrac{1-(x+h)^3 - (1-x^3)}{h\left(\left(\sqrt[3]{1-(x+h)^3}\right)^2 + \sqrt[3]{1-(x+h)^3}\,\sqrt[3]{1-x^3} + \left(\sqrt[3]{1-x^3}\right)^2\right)}$

$= \lim\limits_{h \to 0} \dfrac{(-h)(x^2 + x(x+h) + (x+h)^2)}{h\left(\left(\sqrt[3]{1-(x+h)^3}\right)^2 + \sqrt[3]{1-(x+h)^3}\,\sqrt[3]{1-x^3} + \left(\sqrt[3]{1-x^3}\right)^2\right)} = \dfrac{-3x^2}{3\left(\sqrt[3]{1-x^3}\right)^2} = \dfrac{-x^2}{\left(\sqrt[3]{1-x^3}\right)^2}.$

**結論.** $x^\alpha$ $(\alpha \in \mathbb{R})$ 之導函數為 $\alpha\, x^{\alpha-1}$.

**定義.** 若 $f$ 在 $(a,b)$ 上每一點均有導數, 則稱 $f$ 在 $(a,b)$ 可微 (分) .

**定理.** 若 $f$ 在 $a$ 可微, 則 $f$ 在 $a$ 連續.

## 2.2 微分規則

**定理** (四則運算). 令 $f, g$ 可微, $c \in \mathbb{R}$. 則

1. $(c)' = 0$
2. $(c\,f)' = c\,f'$
3. $(f \pm g)' = f' \pm g'$
4. $(f \cdot g)' = f' \cdot g + f \cdot g'$
5. $\left(\dfrac{f}{g}\right)' = \dfrac{g \cdot f' - g' \cdot f}{g^2}$

**例.** 求導函數.

1. $x^5$
2. $\frac{1}{x^2+3}$
3. $\frac{x-1}{x+1}$
4. $\sqrt{\frac{x-1}{x+1}}$

**解.**

1. $(x^5)' = (x \cdot x^2 \cdot x^2)' = (x)' \cdot x^2 \cdot x^2 + x \cdot (x^2)' \cdot x^2 + x \cdot x^2 \cdot (x^2)' = x^4 + x \cdot (2x) \cdot x^2 + x^3 \cdot (2x) = 5x^4$

2. $\left(\dfrac{1}{x^2+3}\right)' = \dfrac{(x^2+3) \cdot (1)' - (x^2+3)' \cdot 1}{(x^2+3)^2} = \dfrac{-2x}{(x^2+3)^2}$

3. $\left(\dfrac{x-1}{x+1}\right)' = \dfrac{(x+1) \cdot (x-1)' - (x+1)' \cdot (x-1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$

4. $\left(\sqrt{\dfrac{x-1}{x+1}}\right)' = \left(\dfrac{\sqrt{x-1}}{\sqrt{x+1}}\right)' = \dfrac{\sqrt{x+1} \cdot (\sqrt{x-1})' - (\sqrt{x+1})' \cdot \sqrt{x-1}}{(\sqrt{x+1})^2}$

$= \dfrac{\sqrt{x+1} \cdot \frac{1}{2\sqrt{x-1}} - \frac{1}{2\sqrt{x+1}} \cdot \sqrt{x-1}}{(\sqrt{x+1})^2} = \dfrac{\sqrt{x+1} \cdot \sqrt{x+1} - \sqrt{x-1} \cdot \sqrt{x-1}}{2\sqrt{x-1}(\sqrt{x+1})^3} = \dfrac{1}{\sqrt{(x-1)(x+1)^3}}$

**定理** (鏈鎖律 (chain rule) ). 若 $f(u)$ 在 $u = g(x)$ 可微, $g(x)$ 在 $x$ 可微, 則 $f \circ g$ 在 $x$ 可微:

$$(f \circ g)'(x) \equiv (f(g(x)))' = f'(g(x)) \cdot g'(x)$$

**例.** 求導函數.

1. $(x^3-1)^{2025}$
2. $\sqrt{x^2+1}$
3. $\frac{1}{x^2+3}$
4. $\sqrt{\frac{x-1}{x+1}}$

**解.**

1. 令 $f(u) = u^{2025}$, $g(x) = x^3 - 1$, 則 $f'(u) = 2025\,u^{2024}$, $(x^3-1)^{2025} = f(g(x))$.
   由鏈鎖律 $((x^3-1)^{2025})' = (f(g(x)))' = f'(g(x)) \cdot g'(x) = 2025 \cdot (x^3-1)^{2024} \cdot (3x^2)$.

2. 令 $f(u) = \sqrt{u} = u^{\frac{1}{2}}$, $g(x) = x^2 + 1$, 則 $f'(u) = \dfrac{1}{2\sqrt{u}}$, $\sqrt{x^2+1} = f(g(x))$.

   由鏈鎖律 $(\sqrt{x^2+1})' = (f(g(x)))' = f'(g(x)) \cdot g'(x) = \dfrac{1}{2\sqrt{x^2+1}} \cdot (2x) = \dfrac{x}{\sqrt{x^2+1}}$.

3. 令 $f(u) = \dfrac{1}{u}$, $g(x) = x^2 + 3$, 則 $f'(u) = \dfrac{-1}{u^2}$, $\dfrac{1}{x^2 + 3} = f(g(x))$.

   由鏈鎖律 $\left(\dfrac{1}{x^2 + 3}\right)' = (f(g(x)))' = f'(g(x)) \cdot g'(x) = \dfrac{-1}{(x^2 + 3)^2} \cdot (x^2 + 3)' = \dfrac{-2x}{(x^2 + 3)^2}$.

4. 令 $f(u) = \sqrt{u}$, $g(x) = \dfrac{x-1}{x+1}$, 則 $f'(u) = \dfrac{1}{2\sqrt{u}}$, $\sqrt{\dfrac{x-1}{x+1}} = f(g(x))$.

   由鏈鎖律 $\left(\sqrt{\dfrac{x-1}{x+1}}\right)' = (f(g(x)))' = f'(g(x)) \cdot g'(x) = \dfrac{1}{2\sqrt{\frac{x-1}{x+1}}} \cdot \left(\dfrac{x-1}{x+1}\right)' = \dfrac{1}{2\sqrt{\frac{x-1}{x+1}}} \dfrac{2}{(x+1)^2} = \dfrac{1}{\sqrt{(x-1)(x+1)^3}}$.

**結論.** 若 $f(g(x)) = x$, 等式兩邊對 $x$ 微分 $\implies (f(g(x)))' = 1 \implies f'(g(x)) \cdot g'(x) = 1 \implies g'(x) = \dfrac{1}{f'(g(x))}$.

**例.** $f, g$ 為可微函數且 $f(g(x)) = x$. 若 $f'(x) = 1 + (f(x))^2$, 求 $g'(x)$.

**解.** $f(g(x)) = x$ 等式兩邊對 $x$ 微分得 $f'(g(x)) \cdot g'(x) = 1 \implies g'(x) = \dfrac{1}{f'(g(x))} = \dfrac{1}{1 + (f(g(x)))^2} = \dfrac{1}{1 + x^2}$.

**例.** 令 $f(x) = e^x + x$, 求 $(f^{-1})'(e + 1)$.

**解.** 令 $g(x) = f^{-1}(x)$, 則 $g(f(x)) = f^{-1}(f(x)) = x$. 等式兩邊對 $x$ 微分得 $g'(f(x)) \cdot f'(x) = 1 \implies g'(f(x)) = \dfrac{1}{f'(x)}$. 由 $f(1) = e + 1$, $g'(e + 1) = g'(f(1)) = \dfrac{1}{f'(1)} = \dfrac{1}{e + 1}$.

**例.** 若 $F(x) = f(x \cdot f(x \cdot f(x)))$, 且 $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$, $f'(2) = 5$, $f'(3) = 6$, 求 $F'(1)$.

**解.** 由鏈鎖律 $\left(f(x \cdot f(x \cdot f(x)))\right)' = f'(x \cdot f(x \cdot f(x))) \cdot (x \cdot f(x \cdot f(x)))'$. 又 $(x \cdot f(x \cdot f(x)))' = f(x \cdot f(x)) + x \cdot f'(x \cdot f(x)) \cdot (x \cdot f(x))' = f(x \cdot f(x)) + x \cdot f'(x \cdot f(x)) \cdot (x \cdot f'(x) + f(x))$. 故 $F'(1) = \left(f(x \cdot f(x \cdot f(x)))\right)'\big|_{x=1} = f'(x \cdot f(x \cdot f(x))) \cdot (f(x \cdot f(x)) + x \cdot f'(x \cdot f(x)) \cdot (x \cdot f'(x) + f(x)))\big|_{x=1} = f'(f(f(1))) \cdot (f(f(1)) + f'(f(1)) \cdot (f'(1) + f(1))) = 6 \cdot (3 + 5 \cdot (4 + 2)) = 198$.

## 2.3 自然指數, 對數與微分

**定義** (自然指數 $e$ 與 $e^x$ 微分).

- 給定 $a > 0$, 求 $f(x) = a^x$ 之導函數

- $\dfrac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{a^{x+h} - a^x}{h} = \lim_{h \to 0} a^x \cdot \dfrac{a^h - 1}{h} = a^x \cdot \lim_{h \to 0} \dfrac{a^h - 1}{h} = a^x \cdot C(a)$

- 觀察: $C(a)$ 隨 $a$ 遞增; 存在 $\dfrac{27}{10} < e < \dfrac{68}{25}$ 使 $C(e) = 1$.

- $C(e) = 1 \implies \lim_{h \to 0} \dfrac{e^h - 1}{h} = 1 \implies \dfrac{\mathrm{d}}{\mathrm{d}x} e^x = C(e) \cdot e^x \implies (e^x)' = e^x$. $\ln x \equiv \log_e x$

**性質.** $(\ln|x|)' = \dfrac{1}{x}$.

**解.**

- 若 $x > 0$, $\ln|x| = \ln x$ 且 $e^{\ln x} = x$. 令 $f(u) = e^u$, $g(x) = \ln|x| = \ln x$, 則 $f'(u) = e^u$, $f(g(x)) = x$; 故 $g'(x) = \dfrac{1}{f'(g(x))} \implies (\ln|x|)' = (\ln x)' = \dfrac{1}{e^{\ln x}} = \dfrac{1}{x}$.

| $h$ | $10^{-1}$ | $10^{-2}$ | $10^{-3}$ | $10^{-4}$ | $10^{-5}$ | $10^{-6}$ | $10^{-7}$ |
|---|---|---|---|---|---|---|---|
| $\dfrac{2^h-1}{h}$ | 0.7177 | 0.6956 | 0.6934 | 0.6932 | 0.6931 | 0.6931 | 0.6931 |
| $\dfrac{(\frac{5}{2})^h-1}{h}$ | 0.9596 | 0.9205 | 0.9167 | 0.9163 | 0.9163 | 0.9163 | 0.9163 |
| $\dfrac{(\frac{27}{10})^h-1}{h}$ | 1.0442 | 0.9982 | 0.9937 | 0.9933 | 0.9933 | 0.9933 | 0.9933 |
| $\dfrac{(\frac{68}{25})^h-1}{h}$ | 1.0524 | 1.0056 | 1.0011 | 1.0007 | 1.0006 | 1.0006 | 1.0006 |
| $\dfrac{(\frac{28}{10})^h-1}{h}$ | 1.0845 | 1.0349 | 1.0301 | 1.0297 | 1.0296 | 1.0296 | 1.0296 |
| $\dfrac{3^h-1}{h}$ | 1.1612 | 1.1047 | 1.0992 | 1.0987 | 1.0986 | 1.0986 | 1.0986 |

- 若 $x < 0$, $\ln|x| = \ln(-x)$ 且 $e^{\ln(-x)} = -x$. 令 $f(u) = e^u$, $g(x) = \ln|x| = \ln(-x)$, 則 $f'(u) = e^u$, $f(g(x)) = -x$; 故 $g'(x) = \dfrac{-1}{f'(g(x))} \implies (\ln|x|)' = (\ln(-x))' = \dfrac{-1}{e^{\ln(-x)}} = \dfrac{1}{x}$.

**性質.** $(a^x)' = a^x \cdot \ln a$, $\forall\, a > 0$.

**證.** $a = e^{\log_e a} = e^{\ln a} \implies a^x = e^{x\ln a}$. 令 $f(u) = e^u$, $g(x) = x\ln a$, 則 $f'(u) = e^u$, $f(g(x)) = e^{x\ln a} = a^x$; 故 $(f(g(x)))' = f'(g(x)) \cdot g'(x) = e^{x\ln a} \cdot (x\ln a)' = a^x \cdot \ln a$.

**結論.** $C(a) = \ln a$.

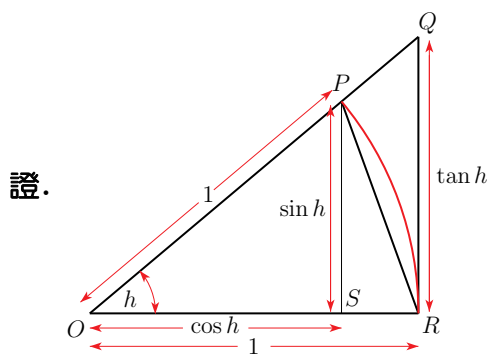**性質.** $(x^\alpha)' = \alpha\, x^{\alpha-1}$ $(\alpha \in \mathbb{R})$

**解.** $x^\alpha = e^{\ln x^\alpha} = e^{\alpha \ln x}$. 故 $(x^\alpha)' = (e^{\alpha\ln x})' = e^{\alpha\ln x} \cdot \left(\alpha \cdot \dfrac{1}{x}\right) = x^\alpha \cdot \alpha \cdot \dfrac{1}{x} = \alpha\, x^{\alpha-1}$.

**例.** 證明 $\displaystyle\lim_{x\to 0} \dfrac{\ln(1+x)}{x} = 1$.

**解.** 令 $f(x) = \ln(1+x)$, 則 $f(0) = 0$, $f'(x) = \dfrac{1}{1+x}$, $f'(0) = 1$. 故 $\displaystyle\lim_{x\to 0} \dfrac{\ln(1+x)}{x} = \lim_{x\to 0} \dfrac{f(x)}{x} = \lim_{x\to 0} \dfrac{f(x)-f(0)}{x-0} = f'(0) = 1$.

## 2.4 三角函數微分

**性質.** $\displaystyle\lim_{h\to 0} \dfrac{\sin h}{h} = 1$

**證.**



取 $0 < h \ll \frac{\pi}{2}$ 作圖如左. 比較面積 $\Delta OPR \leqslant \triangledown OPR \leqslant \Delta OQR$ $\implies \sin h \leqslant h \leqslant \dfrac{\sin h}{\cos h}$. 因 $h$, $\sin h$, $\cos h$ 均為正, 不等式同除 $\sin h$ 並取倒數及變向後得 $\cos h \leqslant \dfrac{\sin h}{h} \leqslant 1$. 由 $\displaystyle\lim_{h\to 0+} \cos h = 1$ 與夾擠定理得 $\displaystyle\lim_{h\to 0+} \dfrac{\sin h}{h} = 1$. 又 $\displaystyle\lim_{h\to 0-} \dfrac{\sin h}{h} = \lim_{(-h)\to 0+} \dfrac{\sin h}{h} = \underbrace{\lim_{(-h)\to 0+} \dfrac{\sin(-h)}{(-h)}}_{\sin\ 為奇函數} = \underbrace{\lim_{H\to 0+} \dfrac{\sin H}{H}}_{令\ H \equiv -h} = 1$, 故 $\displaystyle\lim_{h\to 0} \dfrac{\sin h}{h} = 1$.

**性質.** $\lim\limits_{h \to 0} \dfrac{\cos h - 1}{h} = 0$

**解.** 由 $\cos h = \cos\left(\dfrac{h}{2} + \dfrac{h}{2}\right) = \cos^2\dfrac{h}{2} - \sin^2\dfrac{h}{2} = 1 - 2\sin^2\dfrac{h}{2} \implies \cos h - 1 = -2\sin^2\dfrac{h}{2}$, 則 $\lim\limits_{h \to 0} \dfrac{\cos h - 1}{h} =$ $\lim\limits_{h \to 0} -\dfrac{2\sin^2\frac{h}{2}}{h} = -\lim\limits_{h \to 0} \dfrac{\sin^2\frac{h}{2}}{\frac{h}{2}} = \lim\limits_{h \to 0} \dfrac{\sin\frac{h}{2}}{\frac{h}{2}} \cdot \sin\dfrac{h}{2} = \lim\limits_{h \to 0} \dfrac{\sin\frac{h}{2}}{\frac{h}{2}} \cdot \lim\limits_{h \to 0} \sin\dfrac{h}{2} = \underbrace{\lim\limits_{H \to 0} \dfrac{\sin H}{H}}_{\text{令 } H \equiv \frac{h}{2}} \cdot \lim\limits_{h \to 0} \sin\dfrac{h}{2} = 1 \cdot 0 = 0$

**性質.** $(\sin x)' = \cos x$

**證.** $(\sin x)' = \lim\limits_{h \to 0} \dfrac{\sin(x+h) - \sin x}{h} = \lim\limits_{h \to 0} \dfrac{\sin x \cos h + \cos x \sin h - \sin x}{h} = \lim\limits_{h \to 0} \sin x \dfrac{\cos h - 1}{h} + \cos x \dfrac{\sin h}{h} =$ $\sin x \lim\limits_{h \to 0} \dfrac{\cos h - 1}{h} + \cos x \lim\limits_{h \to 0} \dfrac{\sin h}{h} = \cos x$

**性質.** $(\cos x)' = -\sin x$

**證.** $(\cos x)' = \left(\sin\left(\dfrac{\pi}{2} - x\right)\right)' = \cos\left(\dfrac{\pi}{2} - x\right) \cdot \left(\dfrac{\pi}{2} - x\right)' = -\sin x$

**性質.** $(\tan x)' = \sec^2 x$

**證.** $(\tan x)' = \left(\dfrac{\sin x}{\cos x}\right)' = \dfrac{\cos x \cdot \cos x - (-\sin x) \cdot \sin x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$

**性質.** $(\sec x)' = \sec x \tan x$

**證.** $(\sec x)' = \left(\dfrac{1}{\cos x}\right)' = \dfrac{\cos x \cdot 0 - (-\sin x) \cdot 1}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x}\dfrac{\sin x}{\cos x} = \sec x \tan x$

**性質.** $(\cot x)' = -\csc^2 x$

**證.** $(\cot x)' = \left(\dfrac{\cos x}{\sin x}\right)' = \dfrac{\sin x \cdot (-\sin x) - \cos x \cdot \cos x}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$

**性質.** $(\csc x)' = -\csc x \cot x$

**證.** $(\csc x)' = \left(\dfrac{1}{\sin x}\right)' = \dfrac{\sin x \cdot 0 - (\cos x) \cdot 1}{\sin^2 x} = -\dfrac{\cos x}{\sin^2 x} = -\dfrac{1}{\sin x}\dfrac{\cos x}{\sin x} = -\csc x \cot x$

## 2.5 反三角函數微分

**性質.** $(\sin^{-1} x)' = \dfrac{1}{\sqrt{1 - x^2}}$

**證.** $\sin(\sin^{-1} x) = x$, $x \in [-1, 1]$. 令 $f(u) = \sin u$, $g(x) = \sin^{-1} x$, 則 $f'(u) = \cos u$, $f(g(x)) = x$; 故 $g'(x) = \dfrac{1}{f'(g(x))} \implies (\sin^{-1} x)' = \dfrac{1}{\cos(\sin^{-1} x)} = \dfrac{1}{\sqrt{1 - x^2}}$.

**性質.** $(\cos^{-1} x)' = -\dfrac{1}{\sqrt{1 - x^2}}$

**證.** $\cos(\cos^{-1} x) = x$, $x \in [-1, 1]$. 令 $f(u) = \cos u$, $g(x) = \cos^{-1} x$, 則 $f'(u) = -\sin u$, $f(g(x)) = x$; 故 $g'(x) = \dfrac{1}{f'(g(x))} \implies (\cos^{-1} x)' = \dfrac{1}{-\sin(\cos^{-1} x)} = -\dfrac{1}{\sqrt{1 - x^2}}$.

**性質.** $(\tan^{-1} x)' = \dfrac{1}{1 + x^2}$

**證.** $\tan(\tan^{-1} x) = x$, $x \in (-\infty, \infty)$. 令 $f(u) = \tan u$, $g(x) = \tan^{-1} x$, 則 $f'(u) = \sec^2 u$, $f(g(x)) = x$; 故 $g'(x) = \dfrac{1}{f'(g(x))} \implies (\tan^{-1} x)' = \dfrac{1}{\sec^2(\tan^{-1} x)} = \dfrac{1}{1+x^2}$.

**性質.** $(\cot^{-1} x)' = -\dfrac{1}{1+x^2}$

**證.** $\cot(\cot^{-1} x) = x$, $x \in (-\infty, \infty)$. 令 $f(u) = \cot u$, $g(x) = \cot^{-1} x$, 則 $f'(u) = -\csc^2 u$, $f(g(x)) = x$; 故 $g'(x) = \dfrac{1}{f'(g(x))} \implies (\cot^{-1} x)' = \dfrac{1}{-\csc^2(\cot^{-1} x)} = -\dfrac{1}{1+x^2}$.

**性質.** $(\sec^{-1} x)' = \dfrac{1}{x\sqrt{x^2-1}}$

**證.** $\sec(\sec^{-1} x) = x$, $x \in (1, \infty) \vee (-\infty, -1)$. 令 $f(u) = \sec u$, $g(x) = \sec^{-1} x$, 則 $f'(u) = \sec u \tan u$, $f(g(x)) = x$; 故 $g'(x) = \dfrac{1}{f'(g(x))} \implies (\sec^{-1} x)' = \dfrac{1}{\sec(\sec^{-1} x)\tan(\sec^{-1} x)} = \dfrac{1}{x\sqrt{x^2-1}}$.

**性質.** $(\csc^{-1} x)' = -\dfrac{1}{x\sqrt{x^2-1}}$

**證.** $\csc(\csc^{-1} x) = x$, $x \in (1, \infty) \vee (-\infty, -1)$. 令 $f(u) = \csc u$, $g(x) = \csc^{-1} x$, 則 $f'(u) = -\csc u \cot u$, $f(g(x)) = x$; 故 $g'(x) = \dfrac{1}{f'(g(x))} \implies (\csc^{-1} x)' = -\dfrac{1}{\csc(\csc^{-1} x)\cot(\csc^{-1} x)} = -\dfrac{1}{x\sqrt{x^2-1}}$.

**註.** 由定義 $\sec : [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2}) \to (-\infty, -1] \cup [1, \infty)$, $\csc : (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}] \to (-\infty, -1] \cup [1, \infty)$, 其反三角函數為 $\sec^{-1} : (-\infty, -1] \cup [1, \infty) \to [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$, $\csc^{-1} : (-\infty, -1] \cup [1, \infty) \to (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$, 故 $\tan(\sec^{-1} x)$ 與 $\cot(\csc^{-1} x)$ 恆為正值. 依此, 若 $u = \sec^{-1} x$, 則 $\tan^2 u = \sec^2 u - 1 \implies \tan u = \sqrt{\sec^2 u - 1} = \sqrt{x^2 - 1} \implies \tan(\sec^{-1} x) = \sqrt{x^2 - 1}$ (開平方僅需取正值). 同理, 若 $u = \csc^{-1} x$, 則 $\cot^2 u = \csc^2 u - 1 \implies \cot u = \sqrt{\csc^2 u - 1} = \sqrt{x^2 - 1} \implies \cot(\csc^{-1} x) = \sqrt{x^2 - 1}$. 若初始定義 $\sec : [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi] \to (-\infty, -1] \cup [1, \infty)$, $\csc : (0, \frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi) \to (-\infty, -1] \cup [1, \infty)$ 則 $\tan(\sec^{-1} x)$ 與 $\cot(\csc^{-1} x)$ 之正負將依 $x$ 之正負而定: 此時 $(\sec^{-1} x)' = \dfrac{1}{|x|\sqrt{x^2-1}}$, $(\csc^{-1} x)' = -\dfrac{1}{|x|\sqrt{x^2-1}}$.

### 常用初等函數微分公式

| $f(x)$ | $e^x$ | $\ln|x|$ | $x^\alpha$ | $\sin x$ | $\cos x$ | $\tan x$ | $\sec x$ | $\sin^{-1}x$ | $\cos^{-1}x$ | $\tan^{-1}x$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $f'(x)$ | $e^x$ | $\dfrac{1}{x}$ | $\alpha\, x^{\alpha-1}$ | $\cos x$ | $-\sin x$ | $\sec^2 x$ | $\sec x \tan x$ | $\dfrac{1}{\sqrt{1-x^2}}$ | $-\dfrac{1}{\sqrt{1-x^2}}$ | $\dfrac{1}{1+x^2}$ |

## 2.6 對數微分法

**性質.** $(\ln g(x))' = \dfrac{g'(x)}{g(x)}$.

**證.** 令 $f(u) = \ln u$, 則 $f'(u) = \dfrac{1}{u}$, $\ln g(x) = f(g(x))$. 由鏈鎖律 $(f(g(x)))' = f'(g(x)) \cdot g'(x) \implies (\ln g(x))' = \dfrac{1}{g(x)} \cdot g'(x) = \dfrac{g'(x)}{g(x)}$.

**例.** $f(x) = \sqrt[4]{\dfrac{(x^4+12)(x^5-x^2+2)}{x^3+1}}$, 求 $f'(x)$.

**解.** $\ln f(x) = \dfrac{1}{4}\ln\dfrac{(x^4+12)(x^5-x^2+2)}{x^3+1} = \dfrac{1}{4}\big(\ln(x^4+12)+\ln(x^5-x^2+2)-\ln(x^3+1)\big)$; 等式兩邊對 $x$ 微分得 $\dfrac{f'(x)}{f(x)} = \dfrac{1}{4}\left(\dfrac{4x^3}{x^4+12}+\dfrac{5x^4-2x}{x^5-x^2+2}-\dfrac{3x^2}{x^3+1}\right) \implies f'(x) = \sqrt[4]{\dfrac{(x^4+12)(x^5-x^2+2)}{x^3+1}}\cdot\dfrac{1}{4}\left(\dfrac{4x^3}{x^4+12}+\dfrac{5x^4-2x}{x^5-x^2+2}-\dfrac{3x^2}{x^3+1}\right)$.

**例.** $f(x) = \dfrac{e^{-x}\cos^2 x}{x^2+x+1}$, 求 $f'(x)$.

**解.** $\ln f(x) = \ln(e^{-x}\cos^2 x) - \ln(x^2+x+1) = \ln(e^{-x}) + \ln(\cos^2 x) - \ln(x^2+x+1) = -x + 2\ln(\cos x) - \ln(x^2+x+1)$; 等式兩邊對 $x$ 微分得 $\dfrac{f'(x)}{f(x)} = -1 - 2\tan x - \dfrac{2x+1}{x^2+x+1} \implies f'(x) = -\dfrac{e^{-x}\cos^2 x}{x^2+x+1}\cdot\left(1 + 2\tan x + \dfrac{2x+1}{x^2+x+1}\right)$.

**例.** 給定 $f(x)$, 求 $f'(x)$.

- $f(x) = (\sin x)^{\ln x}$
- $f(x) = (\tan x)^{\frac{1}{x}}$
- $f(x) = (\cos x)^{\sin x}$

**解.**

- $\log f(x) = \ln x \cdot \ln(\sin x) \implies f'(x) = (\sin x)^{\ln x}\cdot\left(\dfrac{1}{x}\cdot\ln(\sin x) + \ln x \cdot \dfrac{\cos x}{\sin x}\right) = (\sin x)^{\ln x}\cdot\left(\dfrac{\ln(\sin x)}{x} + \ln x \cdot \cot x\right)$

- $\log f(x) = \dfrac{1}{x}\cdot\ln(\tan x) \implies f'(x) = (\tan x)^{\frac{1}{x}}\cdot\left(-\dfrac{1}{x^2}\cdot\ln(\tan x)+\dfrac{1}{x}\cdot\dfrac{\sec^2 x}{\tan x}\right) = (\tan x)^{\frac{1}{x}}\cdot\left(-\dfrac{\ln(\tan x)}{x^2}+\dfrac{\sec x \csc x}{x}\right)$

- $\log f(x) = \sin x \cdot \ln(\cos x) \implies f'(x) = (\cos x)^{\sin x}\cdot\left(\cos x \cdot \ln(\cos x) + \sin x \cdot \dfrac{-\sin x}{\cos x}\right)$

## 2.7 隱函數微分

**例.** 求圓 $x^2 + y^2 = 25$ 上點 $(3,-4)$ 之切線方程式.

**解.**

- (顯函數微分) 在點 $(3,-4)$ 附近 $y = -\sqrt{25-x^2} \implies y' = \dfrac{x}{\sqrt{25-x^2}}$, 故點 $(3,-4)$ 之切線斜率為 $\dfrac{3}{\sqrt{25-3^2}} = \dfrac{3}{4}$, 切線方程式為 $(y+4) = \dfrac{3}{4}(x-3)$.

- (隱函數微分) 令點 $(3,-4)$ 附近 $y$ 為 $x$ 之函數 $(y=y(x))$, 圓方程式寫作 $x^2 + y(x)^2 = 25$; 兩邊同對 $x$ 微分: $2x + 2\,y(x)\cdot y'(x) = 0 \implies y'(x) = -\dfrac{x}{y(x)}$. 點 $(3,-4)$ 之切線斜率為 $y'(3) = -\dfrac{3}{y(3)} = \dfrac{3}{4}$, 切線方程式為 $(y+4) = \dfrac{3}{4}(x-3)$.

**例.** 若 $xy + e^x + e^y = 1$, 求 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

**解.** 令 $y$ 為 $x$ 之函數 $(y=y(x))$, 等式寫作 $x\cdot y(x) + e^x + e^{y(x)} = 1$; 兩邊對 $x$ 微分: $x\cdot y'(x) + y(x) + e^x + e^{y(x)}\cdot y'(x) = 0 \implies (x+e^{y(x)})\cdot y'(x) = -(y(x)+e^x) \implies \dfrac{\mathrm{d}y}{\mathrm{d}x} \equiv y'(x) = -\dfrac{y(x)+e^x}{x+e^{y(x)}}$.

**例.** 若 $x^y = y^x$, 求 $y'$.

**解.** 令 $y$ 為 $x$ 之函數 $(y = y(x))$ , 等式寫作 $x^{y(x)} = y(x)^x$ $\implies$ $e^{y(x)\ln x} = e^{x\ln y(x)}$; 兩邊對 $x$ 微分:
$e^{y(x)\ln x} \cdot \left(y'(x)\ln x + \dfrac{y(x)}{x}\right) = e^{x\ln y(x)} \cdot \left(\ln y(x) + x \cdot \dfrac{y'(x)}{y(x)}\right)$ $\implies$ $y'(x)\ln x + \dfrac{y(x)}{x} = \ln y(x) + x \cdot \dfrac{y'(x)}{y(x)}$
$\implies$ $y'\ln x + \dfrac{y}{x} = \ln y + x\dfrac{y'}{y}$ $\implies$ $y'\left(\ln x - \dfrac{x}{y}\right) = \ln y - \dfrac{y}{x}$ $\implies$ $y' = \dfrac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}$.

**例.** 若 $y = \ln(x^2 + y^2)$, 求 $y'$.

**解.** 令 $y$ 為 $x$ 之函數 $(y = y(x))$ , 等式寫作 $y(x) = \ln\left(x^2 + y(x)^2\right)$; 兩邊對 $x$ 微分: $y'(x) = \dfrac{2x + 2\,y(x)\,y'(x)}{x^2 + y(x)^2}$
$\implies$ $y' = \dfrac{2x + 2yy'}{x^2 + y^2}$ $\implies$ $(x^2 + y^2)y' = 2x + 2yy'$ $\implies$ $(x^2 + y^2 - 2y)y' = 2x$ $\implies$ $y' = \dfrac{2x}{x^2 + y^2 - 2y}$.

**例.** 若 $x^2 e^y + 4x\cos y = 5$, 求在 $y = 0$ 時之 $y'$.

**解.** 令 $y$ 為 $x$ 之函數 $(y = y(x))$ , 等式寫作 $x^2 e^{y(x)} + 4x\cos y(x) = 5$; 兩邊對 $x$ 微分: $2x \cdot e^{y(x)} + x^2 \cdot e^{y(x)} \cdot y'(x) + 4\cos y(x) - 4x \cdot \sin y(x) \cdot y'(x) = 0$. 當 $y(x) = 0$, 上式為 $2x \cdot e^0 + x^2 \cdot e^0 \cdot y'(x) + 4\cos 0 - 4x \cdot \sin 0 \cdot y'(x) = 0$ $\implies$ $2x + x^2 \cdot y'(x) + 4 = 0$ $\implies$ $y'(x) = -\dfrac{4 + 2x}{x^2}$. 由 $x^2 e^y + 4x\cos y = 5$ 知 $y = 0$ 時 $x^2 + 4x = 5$ $\implies$ $x = -5 \vee x = 1$, 則 $y'(-5) = -\dfrac{4 - 10}{(-5)^2} = \dfrac{6}{25}$, $y'(1) = -\dfrac{4 + 2}{1^2} = -6$.

**例.** 若 $x^4 + y^4 = 16$, 求 $y''$.

**解.** 等式兩邊對 $x$ 微分得 $4x^3 + 4y^3 y' = 0$ $\implies$ $y' = -\dfrac{x^3}{y^3}$; 兩邊再對 $x$ 微分得 $y'' = -\dfrac{y^3 \cdot 3x^2 - 3y^2 \cdot y' \cdot x^3}{y^6}$
$\implies$ $y'' = -\dfrac{y^3 \cdot 3x^2 - 3y^2 \cdot (-\frac{x^3}{y^3}) \cdot x^3}{y^6} = -\dfrac{3(x^2 y^4 + x^6)}{y^7} = -\dfrac{3x^2(y^4 + x^4)}{y^7} = -\dfrac{3x^2 \cdot 16}{y^7} = -\dfrac{48x^2}{y^7}$.

**例** (0 階 Bessel 函數). 若 $J(x)$ 滿足 $J(0) = 1$ 與 $xJ''(x) + J'(x) + xJ(x) = 0$, 求 $J'(0)$ 與 $J''(0)$.

**解.** 等式 $xJ''(x) + J'(x) + xJ(x) = 0$ 代入 $x = 0$ 得 $0 \cdot J''(0) + J'(0) + 0 \cdot J(0) = 0$ $\implies$ $J'(0) = 0$; 等式兩邊對 $x$ 微分可得 $xJ'''(x) + J''(x) + J''(x) + J(x) + xJ'(x) = 0$, 代入 $x = 0$ 得 $J''(0) + 0 \cdot J'''(0) + J''(0) + J(0) + 0 \cdot J'(0) = 0$ $\implies$ $2J''(0) + 1 = 0$ $\implies$ $J''(0) = -\frac{1}{2}$.

**習題** (隱函數微分). 求 $y'$.

1. $x^3 + y^3 = 1 \implies y' = -\dfrac{x^2}{y^2}$

2. $2\sqrt{x} + \sqrt{y} = 3 \implies y' = -\dfrac{2\sqrt{y}}{\sqrt{x}}$

3. $x^2 + xy - y^2 = 4 \implies y' = \dfrac{2x + y}{2y - x}$

4. $xe^y = x - y \implies y' = \dfrac{1 - e^y}{xe^y + 1}$

5. $e^{\frac{x}{y}} = x - y \implies y' = \dfrac{y(y - e^{\frac{x}{y}})}{y^2 - xe^{\frac{x}{y}}}$

6. $y\cos x = x^2 + y^2 \implies y' = \dfrac{2x + y\sin x}{\cos x - 2y}$

7. $4\cos x \sin y = 1 \implies y' = \tan x \tan y$

8. $e^y \sin x = x + xy \implies y' = \dfrac{1 + y - e^y \cos x}{e^y \sin x - x}$

9. $\cos(xy) = 1 + \sin y \implies y' = -\dfrac{y\sin(xy)}{x\sin(xy) + \cos y}$

10. $\sqrt{x + y} = 1 + x^2 y^2 \implies y' = \dfrac{4xy^2\sqrt{x + y} - 1}{1 - 4x^2 y\sqrt{x + y}}$

11. $2x^3 + x^2 y - xy^3 = 2 \implies y' = \dfrac{y^3 - 6x^2 - 2xy}{x^2 - 3xy^2}$

12. $x^4(x + y) = y^2(3x - y) \implies y' = \dfrac{3y^2 - 5x^4 - 4x^3 y}{x^4 + 3y^2 - 6xy}$

13. $x\sin y + y\sin x = 1 \implies y' = \dfrac{-\sin y - y\cos x}{x\cos y + \sin x}$

14. $e^y \cos x = 1 + \sin(xy) \implies y' = \dfrac{e^y \sin x + y\cos(xy)}{e^y \cos x - x\cos(xy)}$

**習題** (隱函數微分). 求 $y''$.

1. $9x^2 + y^2 = 9 \implies y'' = -\dfrac{81}{y^3}$

2. $\sqrt{x} + \sqrt{y} = 1 \implies y'' = \dfrac{1}{2x\sqrt{x}}$

3. $x^3 + y^3 = 1 \implies y'' = -\dfrac{2x}{y^5}$

4. $x^4 + y^4 = a^4 \implies y'' = -\dfrac{3a^4 x^2}{y^7}$

**習題** (基礎微分運算). 求下列導函數.

1. $\left((3x^2 + 5)^{-3}\right)' = \dfrac{-18x}{(3x^2 + 5)^4}$

2. $\left(\dfrac{1}{(2x-3)^2}\right)' = \dfrac{-4}{(2x-3)^3}$

3. $\left(\dfrac{1}{x^2 - 4}\right)' = \dfrac{-2x}{(x^2 - 4)^2}$

4. $\left(\dfrac{4}{3x^2 - x + 5}\right)' = \dfrac{4(1 - 6x)}{(3x^2 - x + 5)^2}$

5. $\left(\dfrac{x}{\sqrt{x^2 + 1}}\right)' = \dfrac{1}{(x^2 + 1)^{\frac{3}{2}}}$

6. $\left(\dfrac{\sqrt{x+2}}{\sqrt{x+1}}\right)' = \dfrac{-1}{2\sqrt{x+2}\,(\sqrt{x+1})^3}$

7. $\left(\dfrac{x}{x^2 - 1}\right)' = \dfrac{-(x^2 + 1)}{(x^2 - 1)^2}$

8. $\left(\dfrac{\sin x}{x}\right)' = \dfrac{x\cos x - \sin x}{x^2}$

9. $\left(\sin^3(5x + 4)\right)' = 15\sin^2(5x + 4)\cos(5x + 4)$

10. $(x\sin x)' = x\cos x + \sin x$

11. $(x^2 \cos 2x)' = -2x^2 \sin 2x + 2x\cos 2x$

12. $(x\sin x^2)' = \sin x^2 + 2x^2 \cos x^2$

13. $\left(\sin^3 x^2\right)' = 6x \cdot \sin^2 x^2 \cdot \cos x^2$

14. $\left(\sqrt{1 - \sin x^2}\right)' = \dfrac{-x\cos x^2}{\sqrt{1 - \sin x^2}}$

15. $\left(\tan^{-1}\dfrac{x}{2}\right)' = \dfrac{2}{x^2 + 4}$

16. $\left(\tan^{-1}\dfrac{2}{x}\right)' = \dfrac{-2}{x^2 + 4}$

17. $\left(\tan^{-1} e^{2x}\right)' = \dfrac{2e^{2x}}{e^{4x} + 1}$

18. $\left(\ln(\tan^{-1} x)\right)' = \dfrac{1}{(x^2 + 1)\tan^{-1} x}$

19. $\left(\tan^{-1}\dfrac{a + x}{1 - ax}\right)' = \dfrac{1}{x^2 + 1}$

20. $\left((\ln(2x - 1))^3\right)' = \dfrac{6\left(\ln(2x - 1)\right)^2}{2x - 1}$

21. $(\ln\cos x)' = -\tan x$

22. $\left(\ln\dfrac{x - 1}{x + 1}\right)' = \dfrac{2}{x^2 - 1}$

23. $\left(e^{\frac{1}{x}}\right)' = -\dfrac{1}{x^2} e^{\frac{1}{x}}$

24. $\left(e^{\sin 2x}\right)' = 2e^{\sin 2x}\cos 2x$

25. $\left(3^{\sin \pi x}\right)' = \pi\ln 3 \cdot \cos\pi x \cdot 3^{\sin\pi x}$

26. $(\ln(\ln x))' = \dfrac{1}{x\ln x}$

27. $(x^x)' = x^x(1 + \ln x)$

28. $\left(x^{\ln x}\right)' = 2x^{\ln x - 1} \cdot \ln x$

29. $\left(\dfrac{2x^2 + 3x - 1}{x - 2}\right)' = \dfrac{2x^2 - 8x - 5}{(x - 2)^2}$

30. $\left(\tan^2 x\right)' = 2\tan x\sec^2 x$

31. $\left(\dfrac{1}{1 + e^{-x}}\right)' = \dfrac{e^{-x}}{(1 + e^{-x})^2}$

32. $(x^3 \ln x)' = 3x^2 \ln x + x^2$

33. $\left(\sqrt{1 - x^2}\right)' = -\dfrac{x}{\sqrt{1 - x^2}}$

34. $(e^x \sin x)' = e^x \sin x + e^x \cos x$

35. $(\sin 2x\cos 3x)' = 2\cos 2x\cos 3x - 3\sin 2x\sin 3x$

36. $(\ln\tan x)' = \dfrac{1}{\sin x\cos x}$

37. $(x^{\sin x})' = x^{\sin x}\left(\sin x \cdot \dfrac{1}{x} + \ln x \cdot \cos x\right)$

38. $\left(\dfrac{x}{1 + x^2}\right)' = \dfrac{1 - x^2}{(1 + x^2)^2}$

39. $\left(\dfrac{1}{\ln x}\right)' = -\dfrac{1}{x(\ln x)^2}$

40. $(x^4 e^{-x})' = e^{-x}(4x^3 - x^4)$

41. $\left(\dfrac{\sin x}{\cos x + 2}\right)' = \dfrac{1 + 2\cos x}{(\cos x + 2)^2}$

42. $(\ln(\sec x + \tan x))' = \sec x$

43. $\left((1 - x^2)^3\right)' = -6x(1 - x^2)^2$

44. $\left(e^{2x}\cos\pi x\right)' = e^{2x}(2\cos\pi x - \pi\sin\pi x)$

45. $\left(\dfrac{x^2-1}{x^2+1}\right)' = \dfrac{4x}{(x^2+1)^2}$

46. $\left(\dfrac{1}{1-\sin 2x}\right)' = \dfrac{2\cos 2x}{(1-\sin 2x)^2}$

47. $\left(x^3\cos\pi x\right)' = 3x^2\cos\pi x - \pi x^3\sin\pi x$

48. $(\cos(x\ln x))' = -(1+\ln x)\sin(x\ln x)$

49. $\left(\dfrac{e^{\pi x}-e^{-\pi x}}{2}\right)' = \dfrac{\pi(e^{\pi x}+e^{-\pi x})}{2}$

50. $\left(\dfrac{x^3+1}{x^2+1}\right)' = \dfrac{x^4+3x^2-2x}{(x^2+1)^2}$

51. $\left(e^{1-2x^2}\right)' = -4x\,e^{1-2x^2}$

52. $(\ln(x^2+x+1))' = \dfrac{2x+1}{x^2+x+1}$

53. $\left(\dfrac{1}{x\ln x}\right)' = -\dfrac{1+\ln x}{(x\ln x)^2}$

54. $\left(\dfrac{x}{e^x+1}\right)' = \dfrac{e^x+1-xe^x}{(e^x+1)^2}$

55. $(e^{\tan\pi x})' = \pi e^{\tan\pi x}\sec^2\pi x$

56. $\left(\dfrac{1}{\sqrt{1-2x^2}}\right)' = \dfrac{2x}{(1-2x^2)^{\frac{3}{2}}}$

57. $(\sin x\ln(\cos x))' = \cos x\ln(\cos x) - \sin x\tan x$

58. $\left(\dfrac{x^2}{\sqrt{1+2x^2}}\right)' = \dfrac{2x(x^2+1)}{(1+2x^2)^{\frac{3}{2}}}$

59. $(e^{\pi x}\ln 2x)' = e^{\pi x}\left(\pi\ln 2x + \dfrac{1}{x}\right)$

60. $\left(\dfrac{\tan 2x}{x}\right)' = \dfrac{2x\sec^2 2x - \tan 2x}{x^2}$

61. $\left(\dfrac{1}{1+\cos 2x}\right)' = \dfrac{2\sin 2x}{(1+\cos 2x)^2}$

62. $\left(x^2 e^{-x^2}\right)' = 2xe^{-x^2} - 2x^3 e^{-x^2}$

63. $(\ln(e^{-x}+xe^{-x}))' = -\dfrac{x}{1+x}$

64. $\left(\dfrac{e^{-\pi x}-1}{e^{-\pi x}+1}\right)' = \dfrac{-2\pi e^{\pi x}}{(e^{\pi x}+1)^2} = \dfrac{-2\pi e^{-\pi x}}{(e^{-\pi x}+1)^2}$

65. $(\tan(\pi\ln x))' = \dfrac{\pi\sec^2(\pi\ln x)}{x}$

66. $(e^{\sin x}\cos x)' = e^{\sin x}(\cos^2 x - \sin x)$

67. $\left(\dfrac{1}{\ln(1+x)}\right)' = -\dfrac{1}{(1+x)(\ln(1+x))^2}$

68. $((\sin x+\cos x)^2)' = 2(\sin x+\cos x)(\cos x-\sin x)$

69. $(x^3\tan x)' = 3x^2\tan x + x^3\sec^2 x$

70. $\left(\dfrac{\ln x}{x}\right)' = \dfrac{1-\ln x}{x^2}$

71. $\left(e^{x^2}\sin x\right)' = e^{x^2}(2x\sin x + \cos x)$

72. $\left(\dfrac{1}{1+e^x}\right)' = -\dfrac{e^x}{(1+e^x)^2}$

73. $(\ln\sin x)' = \cot x$

74. $\left(\sin^2 x\cos^2 x\right)' = 2\sin x\cos^3 x - 2\sin^3 x\cos x$

75. $\left(\dfrac{x^2}{1+x^4}\right)' = \dfrac{2x(1-x^4)}{(1+x^4)^2}$

76. $\left(\dfrac{1}{\sqrt{x^2-1}}\right)' = -\dfrac{x}{(x^2-1)^{\frac{3}{2}}}$

77. $\left(x\sin\dfrac{1}{x}\right)' = \sin\dfrac{1}{x} - \dfrac{1}{x}\cos\dfrac{1}{x}$

78. $\left(\dfrac{e^x}{1+e^x}\right)' = \dfrac{e^x}{(1+e^x)^2}$

79. $\left(\ln\left(x+\sqrt{x^2+a^2}\right)\right)' = \dfrac{1}{\sqrt{x^2+a^2}}$

80. $(\cos(x^2+1))' = -2x\sin(x^2+1)$

81. $\left(\dfrac{x^3-3x+1}{x^2-1}\right)' = \dfrac{x^4-2x+3}{(x^2-1)^2}$

82. $\left(\dfrac{x}{\sqrt{1-x^2}}\right)' = \dfrac{1}{(1-x^2)^{\frac{3}{2}}}$

83. $\left(e^{\pi x}\sin^2 x\right)' = e^{\pi x}(\pi\sin^2 x + 2\sin x\cos x)$

84. $\left(\dfrac{1}{x^3+1}\right)' = -\dfrac{3x^2}{(x^3+1)^2}$

85. $\left(\sin^{-1}\dfrac{x}{\sqrt{1+x^2}}\right)' = \dfrac{1}{1+x^2}$

86. $\left(\dfrac{\ln x}{1+\ln x}\right)' = \dfrac{1}{x(1+\ln x)^2}$

87. $\left(x^2\cos\dfrac{1}{x}\right)' = 2x\cos\dfrac{1}{x} + \sin\dfrac{1}{x}$

88. $\left(e^{x^2}\cos x^2\right)' = 2xe^{x^2}(\cos x^2 - \sin x^2)$

89. $\left(\dfrac{\sin x+\cos x}{\sin x-\cos x}\right)' = -\dfrac{2}{(\sin x-\cos x)^2}$

90. $(\sin e^x)' = e^x\cos e^x$

91. $\left(e^{\sin^2 x}\right)' = 2e^{\sin^2 x}\sin x\cos x$

92. $(\sin x\cos 2x)' = \cos x\cos 2x - 2\sin x\sin 2x$

93. $(\ln(\sec x+\tan x))' = \sec x$

94. $\left(\dfrac{x}{\sin x}\right)' = \dfrac{\sin x - x \cos x}{\sin^2 x}$

95. $\left(x \ln x - x\right)' = \ln x$

96. $\left(\ln(\cos \ln x)\right)' = -\dfrac{\tan \ln x}{x}$

97. $\left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right)' = \dfrac{4}{(e^x + e^{-x})^2}$

98. $\left(\sin^{-1}\dfrac{2x}{1+x^2}\right)' = \dfrac{2}{1+x^2}\dfrac{|1-x^2|}{1-x^2}$

99. $\left(\dfrac{\ln(1+x^2)}{x}\right)' = \dfrac{2}{1+x^2} - \dfrac{\ln(1+x^2)}{x^2}$

100. $\left(e^{\pi x} \tan^{-1}\pi x\right)' = \pi e^{\pi x}\left(\tan^{-1}\pi x + \dfrac{1}{1+\pi^2 x^2}\right)$