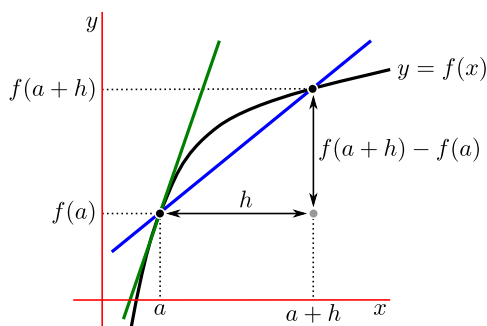


第二章 微分

2.1 導數與導函數



定義. 給定 $f(x)$, $a \in \text{dom } f$. f 在 a 的導數 (derivative) $f'(a)$ 定義為

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

若 $f'(a)$ 存在, 則稱 f 在 a 可微 (分) (differentiable). f 的導函數 $f'(x)$ 定義為

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

註.

- 求導函數之過程稱為微分 (differentiation): 求 $f'(x) \iff f(x)$ (對 x) 微分
- 給定 $y = f(x)$, 其導函數可記為 $f'(x) = f' = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$.
- f 在 a 的導數可記為 $f'(a) = \left. \frac{dy}{dx} \right|_{x=a}$.

例. 以極限定義求以下 $f(x)$ 之導函數 $f'(x)$, 當 $f(x)$ 為

1. x^2
2. x^4
3. $\frac{1}{x}$
4. $\frac{1}{x^5}$
5. $\frac{1}{x^2+3}$
6. $\sqrt{x+1}$
7. $\sqrt{x^2+1}$
8. $\sqrt[3]{1-x^3}$

解.

$$1. f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x.$$

$$2. f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^4 - x^4}{h} = \lim_{h \rightarrow 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} = \lim_{h \rightarrow 0} (4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3.$$

$$3. f'(x) = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \rightarrow 0} \frac{x - (x+h)}{h(x+h)x} = \lim_{h \rightarrow 0} \frac{-h}{h(x+h)x} = \lim_{h \rightarrow 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2} = (-1)x^{-2}.$$

$$\begin{aligned} 4. f'(x) &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^5} - \frac{1}{x^5}}{h} = \lim_{h \rightarrow 0} \frac{x^5 - (x+h)^5}{h(x+h)^5 x^5} \\ &= \lim_{h \rightarrow 0} \frac{(-h)(x^4 + x^3(x+h) + x^2(x+h)^2 + x^3(x+h) + (x+h)^4)}{h(x+h)^5 x^5} \\ &= \lim_{h \rightarrow 0} \frac{-(x^4 + x^3(x+h) + x^2(x+h)^2 + x^3(x+h) + (x+h)^4)}{(x+h)^5 x^5} = \frac{-5x^4}{x^{10}} = \frac{-5}{x^6} = (-5)x^{-6}. \end{aligned}$$

$$\begin{aligned} 5. f'(x) &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2+3} - \frac{1}{x^2+3}}{h} = \lim_{h \rightarrow 0} \frac{(x^2+3) - ((x+h)^2+3)}{h((x+h)^2+3)(x^2+3)} = \lim_{h \rightarrow 0} \frac{(2x+h)(-h)}{h((x+h)^2+3)(x^2+3)} \\ &= \lim_{h \rightarrow 0} \frac{-(2x+h)}{((x+h)^2+3)(x^2+3)} = \frac{-2x}{(x^2+3)^2}. \end{aligned}$$

$$6. f'(x) = \lim_{h \rightarrow 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h+1} + \sqrt{x+1})} = \frac{1}{2\sqrt{x+1}}.$$

$$7. f'(x) = \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h} = \lim_{h \rightarrow 0} \frac{h(2x+h)}{h(\sqrt{(x+h)^2+1} + \sqrt{x^2+1})} = \frac{x}{\sqrt{x^2+1}}.$$

$$\begin{aligned}
8. \quad f'(x) &= \lim_{h \rightarrow 0} \frac{\sqrt[3]{1-(x+h)^3} - \sqrt[3]{1-x^3}}{h} \quad (\text{使用 } a^3 - b^3 = (a-b)(a^2 + ab + b^2)) \\
&= \lim_{h \rightarrow 0} \frac{1 - (x+h)^3 - (1-x^3)}{h((\sqrt[3]{1-(x+h)^3})^2 + \sqrt[3]{1-(x+h)^3} \sqrt[3]{1-x^3} + (\sqrt[3]{1-x^3})^2)} \\
&= \lim_{h \rightarrow 0} \frac{(-h)(x^2 + x(x+h) + (x+h)^2)}{h((\sqrt[3]{1-(x+h)^3})^2 + \sqrt[3]{1-(x+h)^3} \sqrt[3]{1-x^3} + (\sqrt[3]{1-x^3})^2)} = \frac{-3x^2}{3(\sqrt[3]{1-x^3})^2} = \frac{-x^2}{(\sqrt[3]{1-x^3})^2}.
\end{aligned}$$

結論. x^α ($\alpha \in \mathbb{R}$) 之導函數為 $\alpha x^{\alpha-1}$.

定義. 若 f 在 (a, b) 上每一點均有導數, 則稱 f 在 (a, b) 可微 (分).

定理. 若 f 在 a 可微, 則 f 在 a 連續.

2.2 微分規則

定理 (四則運算). 令 f, g 可微, $c \in \mathbb{R}$. 則

$$\begin{aligned}
1. \quad (c)' &= 0 & 3. \quad (f \pm g)' &= f' \pm g' & 5. \quad \left(\frac{f}{g}\right)' &= \frac{g \cdot f' - g' \cdot f}{g^2} \\
2. \quad (cf)' &= c f' & 4. \quad (f \cdot g)' &= f' \cdot g + f \cdot g'
\end{aligned}$$

例. 求導函數.

$$1. \quad x^5 \qquad 2. \quad \frac{1}{x^2+3} \qquad 3. \quad \frac{x-1}{x+1} \qquad 4. \quad \sqrt{\frac{x-1}{x+1}}$$

解.

$$\begin{aligned}
1. \quad (x^5)' &= (x \cdot x^2 \cdot x^2)' = (x)' \cdot x^2 \cdot x^2 + x \cdot (x^2)' \cdot x^2 + x \cdot x^2 \cdot (x^2)' = x^4 + x \cdot (2x) \cdot x^2 + x^3 \cdot (2x) = 5x^4 \\
2. \quad \left(\frac{1}{x^2+3}\right)' &= \frac{(x^2+3) \cdot (1)' - (x^2+3)' \cdot 1}{(x^2+3)^2} = \frac{-2x}{(x^2+3)^2} \\
3. \quad \left(\frac{x-1}{x+1}\right)' &= \frac{(x+1) \cdot (x-1)' - (x+1)' \cdot (x-1)}{(x+1)^2} = \frac{2}{(x+1)^2} \\
4. \quad \left(\sqrt{\frac{x-1}{x+1}}\right)' &= \left(\frac{\sqrt{x-1}}{\sqrt{x+1}}\right)' = \frac{\sqrt{x+1} \cdot (\sqrt{x-1})' - (\sqrt{x+1})' \cdot \sqrt{x-1}}{(\sqrt{x+1})^2} \\
&= \frac{\sqrt{x+1} \cdot \frac{1}{2\sqrt{x-1}} - \frac{1}{2\sqrt{x+1}} \cdot \sqrt{x-1}}{(\sqrt{x+1})^2} = \frac{\sqrt{x+1} \cdot \sqrt{x+1} - \sqrt{x-1} \cdot \sqrt{x-1}}{2\sqrt{x-1}(\sqrt{x+1})^3} = \frac{1}{\sqrt{(x-1)(x+1)^3}}
\end{aligned}$$

定理 (連鎖律 (chain rule)). 若 $f(u)$ 在 $u = g(x)$ 可微, $g(x)$ 在 x 可微, 則 $f \circ g$ 在 x 可微:

$$(f \circ g)'(x) \equiv (f(g(x)))' = f'(g(x)) \cdot g'(x)$$

例. 求導函數.

$$1. \quad (x^3 - 1)^{2025} \qquad 2. \quad \sqrt{x^2 + 1} \qquad 3. \quad \frac{1}{x^2+3} \qquad 4. \quad \sqrt{\frac{x-1}{x+1}}$$

解.

$$\begin{aligned}
1. \quad \text{令 } f(u) &= u^{2025}, g(x) = x^3 - 1, \text{ 則 } f'(u) = 2025 u^{2024}, (x^3 - 1)^{2025} = f(g(x)). \\
&\text{由連鎖律 } ((x^3 - 1)^{2025})' = (f(g(x)))' = f'(g(x)) \cdot g'(x) = 2025 \cdot (x^3 - 1)^{2024} \cdot (3x^2). \\
2. \quad \text{令 } f(u) &= \sqrt{u} = u^{\frac{1}{2}}, g(x) = x^2 + 1, \text{ 則 } f'(u) = \frac{1}{2\sqrt{u}}, \sqrt{x^2 + 1} = f(g(x)). \\
&\text{由連鎖律 } (\sqrt{x^2 + 1})' = (f(g(x)))' = f'(g(x)) \cdot g'(x) = \frac{1}{2\sqrt{x^2 + 1}} \cdot (2x) = \frac{x}{\sqrt{x^2 + 1}}.
\end{aligned}$$

3. 令 $f(u) = \frac{1}{u}$, $g(x) = x^2 + 3$, 則 $f'(u) = \frac{-1}{u^2}$, $\frac{1}{x^2 + 3} = f(g(x))$.

由鏈鎖律 $\left(\frac{1}{x^2 + 3}\right)' = (f(g(x)))' = f'(g(x)) \cdot g'(x) = \frac{-1}{(x^2 + 3)^2} \cdot (x^2 + 3)' = \frac{-2x}{(x^2 + 3)^2}$.

4. 令 $f(u) = \sqrt{u}$, $g(x) = \frac{x-1}{x+1}$, 則 $f'(u) = \frac{1}{2\sqrt{u}}$, $\sqrt{\frac{x-1}{x+1}} = f(g(x))$.

由鏈鎖律 $\left(\sqrt{\frac{x-1}{x+1}}\right)' = (f(g(x)))' = f'(g(x)) \cdot g'(x) = \frac{1}{2\sqrt{\frac{x-1}{x+1}}} \cdot \left(\frac{x-1}{x+1}\right)' = \frac{1}{2\sqrt{\frac{x-1}{x+1}}} \cdot \frac{2}{(x+1)^2} = \frac{1}{\sqrt{(x-1)(x+1)^3}}$.

結論. 若 $f(g(x)) = x$, 等式兩邊對 x 微分 $\Rightarrow (f(g(x)))' = 1 \Rightarrow f'(g(x)) \cdot g'(x) = 1 \Rightarrow g'(x) = \frac{1}{f'(g(x))}$.

例. f, g 為可微函數且 $f(g(x)) = x$. 若 $f'(x) = 1 + (f(x))^2$, 求 $g'(x)$.

解. $f(g(x)) = x$ 等式兩邊對 x 微分得 $f'(g(x)) \cdot g'(x) = 1 \Rightarrow g'(x) = \frac{1}{f'(g(x))} = \frac{1}{1 + (f(g(x)))^2} = \frac{1}{1 + x^2}$.

2.3 自然指數, 對數與微分

定義 (自然指數 e 與 e^x 微分).

- 給定 $a > 0$, 求 $f(x) = a^x$ 之導函數
- $\frac{df}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \rightarrow 0} a^x \cdot \frac{a^h - 1}{h} = a^x \cdot \lim_{h \rightarrow 0} \frac{a^h - 1}{h} = a^x \cdot C(a)$
- 觀察: $C(a)$ 隨 a 遞增; 存在 $\frac{27}{10} < e < \frac{68}{25}$ 使 $C(e) = 1$.

h	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
$\frac{2^h - 1}{h}$	0.7177	0.6956	0.6934	0.6932	0.6931	0.6931	0.6931
$\frac{(\frac{5}{2})^h - 1}{h}$	0.9596	0.9205	0.9167	0.9163	0.9163	0.9163	0.9163
$\frac{(\frac{27}{10})^h - 1}{h}$	1.0442	0.9982	0.9937	0.9933	0.9933	0.9933	0.9933
$\frac{(\frac{68}{25})^h - 1}{h}$	1.0524	1.0056	1.0011	1.0007	1.0006	1.0006	1.0006
$\frac{(\frac{28}{10})^h - 1}{h}$	1.0845	1.0349	1.0301	1.0297	1.0296	1.0296	1.0296
$\frac{3^h - 1}{h}$	1.1612	1.1047	1.0992	1.0987	1.0986	1.0986	1.0986

- $C(e) = 1 \Rightarrow \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1 \Rightarrow \frac{d}{dx} e^x = C(e) \cdot e^x \Rightarrow (e^x)' = e^x$. $\ln x \equiv \log_e x$

性質. $(\ln |x|)' = \frac{1}{x}$.

解.

- 若 $x > 0$, $\ln|x| = \ln x$ 且 $e^{\ln x} = x$. 令 $f(u) = e^u$, $g(x) = \ln|x| = \ln x$, 則 $f'(u) = e^u$, $f(g(x)) = x$; 故 $g'(x) = \frac{1}{f'(g(x))} \Rightarrow (\ln|x|)' = (\ln x)' = \frac{1}{e^{\ln x}} = \frac{1}{x}$.
- 若 $x < 0$, $\ln|x| = \ln(-x)$ 且 $e^{\ln(-x)} = -x$. 令 $f(u) = e^u$, $g(x) = \ln|x| = \ln(-x)$, 則 $f'(u) = e^u$, $f(g(x)) = -x$; 故 $g'(x) = \frac{-1}{f'(g(x))} \Rightarrow (\ln|x|)' = (\ln(-x))' = \frac{-1}{e^{\ln(-x)}} = \frac{1}{x}$.

性質. $(a^x)' = a^x \cdot \ln a$, $\forall a > 0$.

證. $a = e^{\log_e a} = e^{\ln a} \Rightarrow a^x = e^{x \ln a}$. 令 $f(u) = e^u$, $g(x) = x \ln a$, 則 $f'(u) = e^u$, $f(g(x)) = e^{x \ln a} = a^x$; 故 $(f(g(x)))' = f'(g(x)) \cdot g'(x) = e^{x \ln a} \cdot (x \ln a)' = a^x \cdot \ln a$.

結論. $C(a) = \ln a$.

性質. $(x^\alpha)' = \alpha x^{\alpha-1}$ ($\alpha \in \mathbb{R}$)

解. $x^\alpha = e^{\ln x^\alpha} = e^{\alpha \ln x}$. 故 $(x^\alpha)' = (e^{\alpha \ln x})' = e^{\alpha \ln x} \cdot \left(\alpha \cdot \frac{1}{x}\right) = x^\alpha \cdot \alpha \cdot \frac{1}{x} = \alpha x^{\alpha-1}$.

例. 證明 $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$.

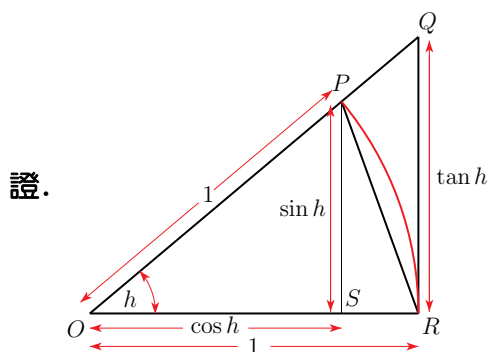
解. 令 $f(x) = \ln(1+x)$, 則 $f(0) = 0$, $f'(x) = \frac{1}{1+x}$, $f'(0) = 1$. 故 $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{f(x)}{x} = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = 1$.

例. 令 $g(x) = e^x + x$, 求 $(g^{-1})'(e+1)$.

解. 令 $f(x) = g^{-1}(x)$, 則 $f(g(x)) = g^{-1}(g(x)) = x$. 等式兩邊對 x 微分得 $f'(g(x)) \cdot g'(x) = 1 \Rightarrow f'(g(x)) = \frac{1}{g'(x)}$. 由 $g(1) = e+1$, $g'(x) = e^x + 1$, $f'(e+1) = f'(g(1)) = \frac{1}{g'(1)} = \frac{1}{e+1}$.

2.4 三角函數微分

性質. $\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$



取 $0 < h \ll \frac{\pi}{2}$ 作圖如左. 比較面積 $\triangle OPR \leq \nabla OPR \leq \triangle OQR$
 $\Rightarrow \sin h \leq h \leq \frac{\sin h}{\cos h}$. 因 $h, \sin h, \cos h$ 均為正, 不等式同除 $\sin h$ 並取倒數及變向後得 $\cos h \leq \frac{\sin h}{h} \leq 1$. 由 $\lim_{h \rightarrow 0+} \cos h = 1$ 與夾擠定理得 $\lim_{h \rightarrow 0+} \frac{\sin h}{h} = 1$. 又 $\lim_{h \rightarrow 0-} \frac{\sin h}{h} = \lim_{(-h) \rightarrow 0+} \frac{\sin h}{h} = \lim_{(-h) \rightarrow 0+} \frac{\sin(-h)}{(-h)} = \lim_{H \rightarrow 0+} \frac{\sin H}{H} = 1$, 故 $\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$.
sin 為奇函數 令 $H \equiv -h$

性質. $\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0$

解. 由 $\cos h = \cos\left(\frac{h}{2} + \frac{h}{2}\right) = \cos^2 \frac{h}{2} - \sin^2 \frac{h}{2} = 1 - 2\sin^2 \frac{h}{2} \Rightarrow \cos h - 1 = -2\sin^2 \frac{h}{2}$, 則 $\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = \lim_{h \rightarrow 0} -\frac{2\sin^2 \frac{h}{2}}{h} = -\lim_{h \rightarrow 0} \frac{\sin^2 \frac{h}{2}}{\frac{h}{2}} = \lim_{h \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \cdot \sin \frac{h}{2} = \lim_{h \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \cdot \lim_{h \rightarrow 0} \sin \frac{h}{2} = \lim_{H \rightarrow 0} \frac{\sin H}{H} \cdot \lim_{h \rightarrow 0} \sin \frac{h}{2} = 1 \cdot 0 = 0$
令 $H \equiv \frac{h}{2}$

性質. $(\sin x)' = \cos x$

證. $(\sin x)' = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} = \lim_{h \rightarrow 0} \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} =$
 $\sin x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h} = \cos x$

性質. $(\cos x)' = -\sin x$

證. $(\cos x)' = \left(\sin \left(\frac{\pi}{2} - x \right) \right)' = \cos \left(\frac{\pi}{2} - x \right) \cdot \left(\frac{\pi}{2} - x \right)' = -\sin x$

性質. $(\tan x)' = \sec^2 x$

證. $(\tan x)' = \left(\frac{\sin x}{\cos x} \right)' = \frac{\cos x \cdot \cos x - (-\sin x) \cdot \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

性質. $(\sec x)' = \sec x \tan x$

證. $(\sec x)' = \left(\frac{1}{\cos x} \right)' = \frac{\cos x \cdot 0 - (-\sin x) \cdot 1}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \frac{\sin x}{\cos x} = \sec x \tan x$

性質. $(\cot x)' = -\csc^2 x$

證. $(\cot x)' = \left(\frac{\cos x}{\sin x} \right)' = \frac{\sin x \cdot (-\sin x) - \cos x \cdot \cos x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$

性質. $(\csc x)' = -\csc x \cot x$

證. $(\csc x)' = \left(\frac{1}{\sin x} \right)' = \frac{\sin x \cdot 0 - (\cos x) \cdot 1}{\sin^2 x} = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x} \frac{\cos x}{\sin x} = -\csc x \cot x$

2.5 反三角函數微分

性質. $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$

證. $\sin(\sin^{-1} x) = x, x \in [-1, 1]$. 令 $f(u) = \sin u, g(x) = \sin^{-1} x$, 則 $f'(u) = \cos u, f(g(x)) = x$; 故
 $g'(x) = \frac{1}{f'(g(x))} \Rightarrow (\sin^{-1} x)' = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1-x^2}}.$

性質. $(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$

證. $\cos(\cos^{-1} x) = x, x \in [-1, 1]$. 令 $f(u) = \cos u, g(x) = \cos^{-1} x$, 則 $f'(u) = -\sin u, f(g(x)) = x$; 故
 $g'(x) = \frac{1}{f'(g(x))} \Rightarrow (\cos^{-1} x)' = \frac{1}{-\sin(\cos^{-1} x)} = -\frac{1}{\sqrt{1-x^2}}.$

性質. $(\tan^{-1} x)' = \frac{1}{1+x^2}$

證. $\tan(\tan^{-1} x) = x, x \in (-\infty, \infty)$. 令 $f(u) = \tan u, g(x) = \tan^{-1} x$, 則 $f'(u) = \sec^2 u, f(g(x)) = x$; 故
 $g'(x) = \frac{1}{f'(g(x))} \Rightarrow (\tan^{-1} x)' = \frac{1}{\sec^2(\tan^{-1} x)} = \frac{1}{1+x^2}.$

性質. $(\cot^{-1} x)' = -\frac{1}{1+x^2}$

證. $\cot(\cot^{-1} x) = x, x \in (-\infty, \infty)$. 令 $f(u) = \cot u, g(x) = \cot^{-1} x$, 則 $f'(u) = -\csc^2 u, f(g(x)) = x$; 故 $g'(x) = \frac{1}{f'(g(x))} \Rightarrow (\cot^{-1} x)' = \frac{1}{-\csc^2(\cot^{-1} x)} = -\frac{1}{1+x^2}$.

性質. $(\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}}$

證. $\sec(\sec^{-1} x) = x, x \in (1, \infty) \cup (-\infty, -1)$. 令 $f(u) = \sec u, g(x) = \sec^{-1} x$, 則 $f'(u) = \sec u \tan u, f(g(x)) = x$; 故 $g'(x) = \frac{1}{f'(g(x))} \Rightarrow (\sec^{-1} x)' = \frac{1}{\sec(\sec^{-1} x) \tan(\sec^{-1} x)} = \frac{1}{x\sqrt{x^2-1}}$.

性質. $(\csc^{-1} x)' = -\frac{1}{x\sqrt{x^2-1}}$

證. $\csc(\csc^{-1} x) = x, x \in (1, \infty) \cup (-\infty, -1)$. 令 $f(u) = \csc u, g(x) = \csc^{-1} x$, 則 $f'(u) = -\csc u \cot u, f(g(x)) = x$; 故 $g'(x) = \frac{1}{f'(g(x))} \Rightarrow (\csc^{-1} x)' = -\frac{1}{\csc(\csc^{-1} x) \cot(\csc^{-1} x)} = -\frac{1}{x\sqrt{x^2-1}}$.

註. 由定義 $\sec : [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2}) \rightarrow (-\infty, -1] \cup [1, \infty)$, $\csc : (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}] \rightarrow (-\infty, -1] \cup [1, \infty)$, 其反三角函數為 $\sec^{-1} : (-\infty, -1] \cup [1, \infty) \rightarrow [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$, $\csc^{-1} : (-\infty, -1] \cup [1, \infty) \rightarrow (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$, 故 $\tan(\sec^{-1} x)$ 與 $\cot(\csc^{-1} x)$ 恆為正值. 依此, 若 $u = \sec^{-1} x$, 則 $\tan^2 u = \sec^2 u - 1 \Rightarrow \tan u = \sqrt{\sec^2 u - 1} = \sqrt{x^2 - 1} \Rightarrow \tan(\sec^{-1} x) = \sqrt{x^2 - 1}$ (開平方僅需取正值). 同理, 若 $u = \csc^{-1} x$, 則 $\cot^2 u = \csc^2 u - 1 \Rightarrow \cot u = \sqrt{\csc^2 u - 1} = \sqrt{x^2 - 1} \Rightarrow \cot(\csc^{-1} x) = \sqrt{x^2 - 1}$. 若初始定義 $\sec : [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi] \rightarrow (-\infty, -1] \cup [1, \infty)$, $\csc : (0, \frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi) \rightarrow (-\infty, -1] \cup [1, \infty)$ 則 $\tan(\sec^{-1} x)$ 與 $\cot(\csc^{-1} x)$ 之正負將依 x 之正負而定: 此時 $(\sec^{-1} x)' = \frac{1}{|x|\sqrt{x^2-1}}, (\csc^{-1} x)' = -\frac{1}{|x|\sqrt{x^2-1}}$.

常用初等函數微分公式

$f(x)$	e^x	$\ln x $	x^α	$\sin x$	$\cos x$	$\tan x$	$\sec x$	$\sin^{-1} x$	$\cos^{-1} x$	$\tan^{-1} x$
$f'(x)$	e^x	$\frac{1}{x}$	$\alpha x^{\alpha-1}$	$\cos x$	$-\sin x$	$\sec^2 x$	$\sec x \tan x$	$\frac{1}{\sqrt{1-x^2}}$	$-\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$

2.6 對數微分法

性質. $(\ln g(x))' = \frac{g'(x)}{g(x)}$.

證. 令 $f(u) = \ln u$, 則 $f'(u) = \frac{1}{u}, \ln g(x) = f(g(x))$. 由鏈鎖律 $(f(g(x)))' = f'(g(x)) \cdot g'(x) \Rightarrow (\ln g(x))' = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}$.

例. $f(x) = \sqrt[4]{\frac{(x^4+12)(x^5-x^2+2)}{x^3+1}}$, 求 $f'(x)$.

解. $\ln f(x) = \frac{1}{4} \ln \frac{(x^4+12)(x^5-x^2+2)}{x^3+1} = \frac{1}{4} (\ln(x^4+12) + \ln(x^5-x^2+2) - \ln(x^3+1))$; 等式兩邊對 x 微分得 $\frac{f'(x)}{f(x)} = \frac{1}{4} \left(\frac{4x^3}{x^4+12} + \frac{5x^4-2x}{x^5-x^2+2} - \frac{3x^2}{x^3+1} \right) \Rightarrow f'(x) = \sqrt[4]{\frac{(x^4+12)(x^5-x^2+2)}{x^3+1}} \cdot \frac{1}{4} \left(\frac{4x^3}{x^4+12} + \frac{5x^4-2x}{x^5-x^2+2} - \frac{3x^2}{x^3+1} \right)$.

例. $f(x) = \frac{e^{-x} \cos^2 x}{x^2 + x + 1}$, 求 $f'(x)$.

解. $\ln f(x) = \ln(e^{-x} \cos^2 x) - \ln(x^2 + x + 1) = \ln(e^{-x}) + \ln(\cos^2 x) - \ln(x^2 + x + 1) = -x + 2 \ln(\cos x) - \ln(x^2 + x + 1)$; 等式兩邊對 x 微分得 $\frac{f'(x)}{f(x)} = -1 - 2 \tan x - \frac{2x + 1}{x^2 + x + 1} \Rightarrow f'(x) = -\frac{e^{-x} \cos^2 x}{x^2 + x + 1} \cdot \left(1 + 2 \tan x + \frac{2x + 1}{x^2 + x + 1}\right)$.

例. 給定 $f(x)$, 求 $f'(x)$.

- $f(x) = (\sin x)^{\ln x}$
- $f(x) = (\tan x)^{\frac{1}{x}}$
- $f(x) = (\cos x)^{\sin x}$

解.

- $\log f(x) = \ln x \cdot \ln(\sin x) \Rightarrow f'(x) = (\sin x)^{\ln x} \cdot \left(\frac{1}{x} \cdot \ln(\sin x) + \ln x \cdot \frac{\cos x}{\sin x}\right) = (\sin x)^{\ln x} \cdot \left(\frac{\ln(\sin x)}{x} + \ln x \cdot \cot x\right)$
- $\log f(x) = \frac{1}{x} \cdot \ln(\tan x) \Rightarrow f'(x) = (\tan x)^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2} \cdot \ln(\tan x) + \frac{1}{x} \cdot \frac{\sec^2 x}{\tan x}\right) = (\tan x)^{\frac{1}{x}} \cdot \left(-\frac{\ln(\tan x)}{x^2} + \frac{\sec x \csc x}{x}\right)$
- $\log f(x) = \sin x \cdot \ln(\cos x) \Rightarrow f'(x) = (\cos x)^{\sin x} \cdot \left(\cos x \cdot \ln(\cos x) + \sin x \cdot \frac{-\sin x}{\cos x}\right)$

2.7 隱函數微分

例. 求圓 $x^2 + y^2 = 25$ 上點 $(3, -4)$ 之切線方程式.

解.

- (顯函數微分) 在點 $(3, -4)$ 附近 $y = -\sqrt{25 - x^2} \Rightarrow y' = \frac{x}{\sqrt{25 - x^2}}$, 故點 $(3, -4)$ 之切線斜率為 $\frac{3}{\sqrt{25 - 3^2}} = \frac{3}{4}$, 切線方程式為 $(y + 4) = \frac{3}{4}(x - 3)$.
- (隱函數微分) 令點 $(3, -4)$ 附近 y 為 x 之函數 ($y = y(x)$), 圓方程式寫作 $x^2 + y(x)^2 = 25$; 兩邊同對 x 微分: $2x + 2y(x) \cdot y'(x) = 0 \Rightarrow y'(x) = -\frac{x}{y(x)}$. 點 $(3, -4)$ 之切線斜率為 $y'(3) = -\frac{3}{y(3)} = \frac{3}{4}$, 切線方程式為 $(y + 4) = \frac{3}{4}(x - 3)$.

例. 若 $xy + e^x + e^y = 1$, 求 $\frac{dy}{dx}$.

解. 令 y 為 x 之函數 ($y = y(x)$), 等式寫作 $x \cdot y(x) + e^x + e^{y(x)} = 1$; 兩邊對 x 微分: $x \cdot y'(x) + y(x) + e^x + e^{y(x)} \cdot y'(x) = 0 \Rightarrow (x + e^{y(x)}) \cdot y'(x) = -(y(x) + e^x) \Rightarrow \frac{dy}{dx} \equiv y'(x) = -\frac{y(x) + e^x}{x + e^{y(x)}}$.

例. 若 $x^y = y^x$, 求 y' .

解. 令 y 為 x 之函數 ($y = y(x)$), 等式寫作 $x^{y(x)} = y(x)^x \Rightarrow e^{y(x) \ln x} = e^{x \ln y(x)}$; 兩邊對 x 微分: $e^{y(x) \ln x} \cdot \left(y'(x) \ln x + \frac{y(x)}{x}\right) = e^{x \ln y(x)} \cdot \left(\ln y(x) + x \cdot \frac{y'(x)}{y(x)}\right) \Rightarrow y'(x) \ln x + \frac{y(x)}{x} = \ln y(x) + x \cdot \frac{y'(x)}{y(x)} \Rightarrow y' \ln x + \frac{y}{x} = \ln y + x \frac{y'}{y} \Rightarrow y' \left(\ln x - \frac{x}{y}\right) = \ln y - \frac{y}{x} \Rightarrow y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}$.

例. 若 $y = \ln(x^2 + y^2)$, 求 y' .

解. 令 y 為 x 之函數 ($y = y(x)$), 等式寫作 $y(x) = \ln(x^2 + y(x)^2)$; 兩邊對 x 微分: $y'(x) = \frac{2x + 2y(x)y'(x)}{x^2 + y(x)^2}$
 $\Rightarrow y' = \frac{2x + 2yy'}{x^2 + y^2} \Rightarrow (x^2 + y^2)y' = 2x + 2yy' \Rightarrow (x^2 + y^2 - 2y)y' = 2x \Rightarrow y' = \frac{2x}{x^2 + y^2 - 2y}$.

例. 若 $x^2e^y + 4x \cos y = 5$, 求在 $y = 0$ 時之 y' .

解. 令 y 為 x 之函數 ($y = y(x)$), 等式寫作 $x^2e^{y(x)} + 4x \cos y(x) = 5$; 兩邊對 x 微分: $2x \cdot e^{y(x)} + x^2 \cdot e^{y(x)} \cdot y'(x) + 4 \cos y(x) - 4x \cdot \sin y(x) \cdot y'(x) = 0$. 當 $y(x) = 0$, 上式為 $2x \cdot e^0 + x^2 \cdot e^0 \cdot y'(x) + 4 \cos 0 - 4x \cdot \sin 0 \cdot y'(x) = 0 \Rightarrow 2x + x^2 \cdot y'(x) + 4 = 0 \Rightarrow y'(x) = -\frac{4 + 2x}{x^2}$. 由 $x^2e^y + 4x \cos y = 5$ 知 $y = 0$ 時 $x^2 + 4x = 5 \Rightarrow x = -5 \vee x = 1$, 則 $y'(-5) = -\frac{4 - 10}{(-5)^2} = \frac{6}{25}$, $y'(1) = -\frac{4 + 2}{1^2} = -6$.

例. 若 $x^4 + y^4 = 16$, 求 y'' .

解. 等式兩邊對 x 微分得 $4x^3 + 4y^3y' = 0 \Rightarrow y' = -\frac{x^3}{y^3}$; 兩邊再對 x 微分得 $y'' = -\frac{y^3 \cdot 3x^2 - 3y^2 \cdot y' \cdot x^3}{y^6}$
 $\Rightarrow y'' = -\frac{y^3 \cdot 3x^2 - 3y^2 \cdot (-\frac{x^3}{y^3}) \cdot x^3}{y^6} = -\frac{3(x^2y^4 + x^6)}{y^7} = -\frac{3x^2(y^4 + x^4)}{y^7} = -\frac{3x^2 \cdot 16}{y^7} = -\frac{48x^2}{y^7}$.

例 (0 階 Bessel 函數). 若 $J(x)$ 滿足 $J(0) = 1$ 與 $xJ''(x) + J'(x) + xJ(x) = 0$, 求 $J'(0)$ 與 $J''(0)$.

解. 等式 $xJ''(x) + J'(x) + xJ(x) = 0$ 代入 $x = 0$ 得 $0 \cdot J''(0) + J'(0) + 0 \cdot J(0) = 0 \Rightarrow J'(0) = 0$; 等式兩邊對 x 微分可得 $xJ'''(x) + J''(x) + J''(x) + J(x) + xJ'(x) = 0$, 代入 $x = 0$ 得 $J''(0) + 0 \cdot J'''(0) + J''(0) + J(0) + 0 \cdot J'(0) = 0 \Rightarrow 2J''(0) + 1 = 0 \Rightarrow J''(0) = -\frac{1}{2}$.

習題 (隱函數微分). 求 y' .

$$1. x^3 + y^3 = 1 \Rightarrow y' = -\frac{x^2}{y^2}$$

$$8. e^y \sin x = x + xy \Rightarrow y' = \frac{1 + y - e^y \cos x}{e^y \sin x - x}$$

$$2. 2\sqrt{x} + \sqrt{y} = 3 \Rightarrow y' = -\frac{2\sqrt{y}}{\sqrt{x}}$$

$$9. \cos(xy) = 1 + \sin y \Rightarrow y' = -\frac{y \sin(xy)}{x \sin(xy) + \cos y}$$

$$3. x^2 + xy - y^2 = 4 \Rightarrow y' = \frac{2x + y}{2y - x}$$

$$10. \sqrt{x+y} = 1 + x^2y^2 \Rightarrow y' = \frac{4xy^2\sqrt{x+y} - 1}{1 - 4x^2y\sqrt{x+y}}$$

$$4. xe^y = x - y \Rightarrow y' = \frac{1 - e^y}{xe^y + 1}$$

$$11. 2x^3 + x^2y - xy^3 = 2 \Rightarrow y' = \frac{y^3 - 6x^2 - 2xy}{x^2 - 3xy^2}$$

$$5. e^{\frac{x}{y}} = x - y \Rightarrow y' = \frac{y(y - e^{\frac{x}{y}})}{y^2 - xe^{\frac{x}{y}}}$$

$$12. x^4(x+y) = y^2(3x-y) \Rightarrow y' = \frac{3y^2 - 5x^4 - 4x^3y}{x^4 + 3y^2 - 6xy}$$

$$6. y \cos x = x^2 + y^2 \Rightarrow y' = \frac{2x + y \sin x}{\cos x - 2y}$$

$$13. x \sin y + y \sin x = 1 \Rightarrow y' = \frac{-\sin y - y \cos x}{x \cos y + \sin x}$$

$$7. 4 \cos x \sin y = 1 \Rightarrow y' = \tan x \tan y$$

$$14. e^y \cos x = 1 + \sin(xy) \Rightarrow y' = \frac{e^y \sin x + y \cos(xy)}{e^y \cos x - x \cos(xy)}$$

習題 (隱函數微分). 求 y'' .

$$1. 9x^2 + y^2 = 9 \Rightarrow y'' = -\frac{81}{y^3}$$

$$3. x^3 + y^3 = 1 \Rightarrow y'' = -\frac{2x}{y^5}$$

$$2. \sqrt{x} + \sqrt{y} = 1 \Rightarrow y'' = \frac{1}{2x\sqrt{x}}$$

$$4. x^4 + y^4 = a^4 \Rightarrow y'' = -\frac{3a^4x^2}{y^7}$$

習題 (基礎微分運算). 求下列導函數.

1. $((3x^2 + 5)^{-3})' = \frac{-18x}{(3x^2 + 5)^4}$
2. $\left(\frac{1}{(2x - 3)^2}\right)' = \frac{-4}{(2x - 3)^3}$
3. $\left(\frac{1}{x^2 - 4}\right)' = \frac{-2x}{(x^2 - 4)^2}$
4. $\left(\frac{4}{3x^2 - x + 5}\right)' = \frac{4(1 - 6x)}{(3x^2 - x + 5)^2}$
5. $\left(\frac{x}{\sqrt{x^2 + 1}}\right)' = \frac{1}{(x^2 + 1)^{\frac{3}{2}}}$
6. $\left(\frac{\sqrt{x + 2}}{\sqrt{x + 1}}\right)' = \frac{-1}{2\sqrt{x + 2}(\sqrt{x + 1})^3}$
7. $\left(\frac{x}{x^2 - 1}\right)' = \frac{-(x^2 + 1)}{(x^2 - 1)^2}$
8. $\left(\frac{\sin x}{x}\right)' = \frac{x \cos x - \sin x}{x^2}$
9. $(\sin^3(5x + 4))' = 15 \sin^2(5x + 4) \cos(5x + 4)$
10. $(x \sin x)' = x \cos x + \sin x$
11. $(x^2 \cos 2x)' = -2x^2 \sin 2x + 2x \cos 2x$
12. $(x \sin x^2)' = \sin x^2 + 2x^2 \cos x^2$
13. $(\sin^3 x^2)' = 6x \cdot \sin^2 x^2 \cdot \cos x^2$
14. $(\sqrt{1 - \sin x^2})' = \frac{-x \cos x^2}{\sqrt{1 - \sin x^2}}$
15. $\left(\tan^{-1} \frac{x}{2}\right)' = \frac{2}{x^2 + 4}$
16. $\left(\tan^{-1} \frac{2}{x}\right)' = \frac{-2}{x^2 + 4}$
17. $(\tan^{-1} e^{2x})' = \frac{2e^{2x}}{e^{4x} + 1}$
18. $(\ln(\tan^{-1} x))' = \frac{1}{(x^2 + 1) \tan^{-1} x}$
19. $\left(\tan^{-1} \frac{a + x}{1 - ax}\right)' = \frac{1}{x^2 + 1}$
20. $((\ln(2x - 1))^3)' = \frac{6(\ln(2x - 1))^2}{2x - 1}$
21. $(\ln \cos x)' = -\tan x$
22. $\left(\ln \frac{x - 1}{x + 1}\right)' = \frac{2}{x^2 - 1}$
23. $(e^{\frac{1}{x}})' = -\frac{1}{x^2} e^{\frac{1}{x}}$

24. $(e^{\sin 2x})' = 2e^{\sin 2x} \cos 2x$
25. $(3^{\sin \pi x})' = \pi \ln 3 \cdot \cos \pi x \cdot 3^{\sin \pi x}$
26. $(\ln(\ln x))' = \frac{1}{x \ln x}$
27. $(x^x)' = x^x(1 + \ln x)$
28. $(x^{\ln x})' = 2x^{\ln x - 1} \cdot \ln x$
29. $\left(\frac{2x^2 + 3x - 1}{x - 2}\right)' = \frac{2x^2 - 8x - 5}{(x - 2)^2}$
30. $(\tan^2 x)' = 2 \tan x \sec^2 x$
31. $\left(\frac{1}{1 + e^{-x}}\right)' = \frac{e^{-x}}{(1 + e^{-x})^2}$
32. $(x^3 \ln x)' = 3x^2 \ln x + x^2$
33. $(\sqrt{1 - x^2})' = -\frac{x}{\sqrt{1 - x^2}}$
34. $(e^x \sin x)' = e^x \sin x + e^x \cos x$
35. $(\sin 2x \cos 3x)' = 2 \cos 2x \cos 3x - 3 \sin 2x \sin 3x$
36. $(\ln \tan x)' = \frac{1}{\sin x \cos x}$
37. $(x^{\sin x})' = x^{\sin x} \left(\sin x \cdot \frac{1}{x} + \ln x \cdot \cos x \right)$
38. $\left(\frac{x}{1 + x^2}\right)' = \frac{1 - x^2}{(1 + x^2)^2}$
39. $\left(\frac{1}{\ln x}\right)' = -\frac{1}{x(\ln x)^2}$
40. $(x^4 e^{-x})' = e^{-x}(4x^3 - x^4)$
41. $\left(\frac{\sin x}{\cos x + 2}\right)' = \frac{1 + 2 \cos x}{(\cos x + 2)^2}$
42. $(\ln(\sec x + \tan x))' = \sec x$
43. $((1 - x^2)^3)' = -6x(1 - x^2)^2$
44. $(e^{2x} \cos \pi x)' = e^{2x}(2 \cos \pi x - \pi \sin \pi x)$
45. $\left(\frac{x^2 - 1}{x^2 + 1}\right)' = \frac{4x}{(x^2 + 1)^2}$
46. $\left(\frac{1}{1 - \sin 2x}\right)' = \frac{2 \cos 2x}{(1 - \sin 2x)^2}$
47. $(x^3 \cos \pi x)' = 3x^2 \cos \pi x - \pi x^3 \sin \pi x$
48. $(\cos(x \ln x))' = -(1 + \ln x) \sin(x \ln x)$
49. $\left(\frac{e^{\pi x} - e^{-\pi x}}{2}\right)' = \frac{\pi(e^{\pi x} + e^{-\pi x})}{2}$
50. $\left(\frac{x^3 + 1}{x^2 + 1}\right)' = \frac{x^4 + 3x^2 - 2x}{(x^2 + 1)^2}$

51. $(e^{1-2x^2})' = -4x e^{1-2x^2}$
52. $(\ln(x^2 + x + 1))' = \frac{2x + 1}{x^2 + x + 1}$
53. $\left(\frac{1}{x \ln x}\right)' = -\frac{1 + \ln x}{(x \ln x)^2}$
54. $\left(\frac{x}{e^x + 1}\right)' = \frac{e^x + 1 - x e^x}{(e^x + 1)^2}$
55. $(e^{\tan \pi x})' = \pi e^{\tan \pi x} \sec^2 \pi x$
56. $\left(\frac{1}{\sqrt{1-2x^2}}\right)' = \frac{2x}{(1-2x^2)^{\frac{3}{2}}}$
57. $(\sin x \ln(\cos x))' = \cos x \ln(\cos x) - \sin x \tan x$
58. $\left(\frac{x^2}{\sqrt{1+2x^2}}\right)' = \frac{2x(x^2 + 1)}{(1 + 2x^2)^{\frac{3}{2}}}$
59. $(e^{\pi x} \ln 2x)' = e^{\pi x} \left(\pi \ln 2x + \frac{1}{x}\right)$
60. $\left(\frac{\tan 2x}{x}\right)' = \frac{2x \sec^2 2x - \tan 2x}{x^2}$
61. $\left(\frac{1}{1 + \cos 2x}\right)' = \frac{2 \sin 2x}{(1 + \cos 2x)^2}$
62. $(x^2 e^{-x^2})' = 2x e^{-x^2} - 2x^3 e^{-x^2}$
63. $(\ln(e^{-x} + x e^{-x}))' = -\frac{x}{1 + x}$
64. $\left(\frac{e^{-\pi x} - 1}{e^{-\pi x} + 1}\right)' = \frac{-2\pi e^{\pi x}}{(e^{\pi x} + 1)^2} = \frac{-2\pi e^{-\pi x}}{(e^{-\pi x} + 1)^2}$
65. $(\tan(\pi \ln x))' = \frac{\pi \sec^2(\pi \ln x)}{x}$
66. $(e^{\sin x} \cos x)' = e^{\sin x}(\cos^2 x - \sin x)$
67. $\left(\frac{1}{\ln(1+x)}\right)' = -\frac{1}{(1+x)(\ln(1+x))^2}$
68. $((\sin x + \cos x)^2)' = 2(\sin x + \cos x)(\cos x - \sin x)$
69. $(x^3 \tan x)' = 3x^2 \tan x + x^3 \sec^2 x$
70. $\left(\frac{\ln x}{x}\right)' = \frac{1 - \ln x}{x^2}$
71. $(e^{x^2} \sin x)' = e^{x^2}(2x \sin x + \cos x)$
72. $\left(\frac{1}{1 + e^x}\right)' = -\frac{e^x}{(1 + e^x)^2}$
73. $(\ln \sin x)' = \cot x$
74. $(\sin^2 x \cos^2 x)' = 2 \sin x \cos^3 x - 2 \sin^3 x \cos x$
75. $\left(\frac{x^2}{1 + x^4}\right)' = \frac{2x(1 - x^4)}{(1 + x^4)^2}$
76. $\left(\frac{1}{\sqrt{x^2 - 1}}\right)' = -\frac{x}{(x^2 - 1)^{\frac{3}{2}}}$
77. $\left(x \sin \frac{1}{x}\right)' = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$
78. $\left(\frac{e^x}{1 + e^x}\right)' = \frac{e^x}{(1 + e^x)^2}$
79. $(\ln(x + \sqrt{x^2 + a^2}))' = \frac{1}{\sqrt{x^2 + a^2}}$
80. $(\cos(x^2 + 1))' = -2x \sin(x^2 + 1)$
81. $\left(\frac{x^3 - 3x + 1}{x^2 - 1}\right)' = \frac{x^4 - 2x + 3}{(x^2 - 1)^2}$
82. $\left(\frac{x}{\sqrt{1-x^2}}\right)' = \frac{1}{(1-x^2)^{\frac{3}{2}}}$
83. $(e^{\pi x} \sin^2 x)' = e^{\pi x}(\pi \sin^2 x + 2 \sin x \cos x)$
84. $\left(\frac{1}{x^3 + 1}\right)' = -\frac{3x^2}{(x^3 + 1)^2}$
85. $\left(\sin^{-1} \frac{x}{\sqrt{1+x^2}}\right)' = \frac{1}{1+x^2}$
86. $\left(\frac{\ln x}{1 + \ln x}\right)' = \frac{1}{x(1 + \ln x)^2}$
87. $\left(x^2 \cos \frac{1}{x}\right)' = 2x \cos \frac{1}{x} + \sin \frac{1}{x}$
88. $(e^{x^2} \cos x^2)' = 2x e^{x^2}(\cos x^2 - \sin x^2)$
89. $\left(\frac{\sin x + \cos x}{\sin x - \cos x}\right)' = -\frac{2}{(\sin x - \cos x)^2}$
90. $(\sin e^x)' = e^x \cos e^x$
91. $(e^{\sin^2 x})' = 2e^{\sin^2 x} \sin x \cos x$
92. $(\sin x \cos 2x)' = \cos x \cos 2x - 2 \sin x \sin 2x$
93. $(\ln(\csc x + \cot x))' = -\csc x$
94. $\left(\frac{x}{\sin x}\right)' = \frac{\sin x - x \cos x}{\sin^2 x}$
95. $(x \ln x - x)' = \ln x$
96. $(\ln(\cos \ln x))' = -\frac{\tan \ln x}{x}$
97. $\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)' = \frac{4}{(e^x + e^{-x})^2}$
98. $\left(\sin^{-1} \frac{2x}{1+x^2}\right)' = \frac{2}{1+x^2} \frac{|1-x^2|}{1-x^2}$
99. $\left(\frac{\ln(1+x^2)}{x}\right)' = \frac{2}{1+x^2} - \frac{\ln(1+x^2)}{x^2}$
100. $(e^{\pi x} \tan^{-1} \pi x)' = \pi e^{\pi x} \left(\tan^{-1} \pi x + \frac{1}{1 + \pi^2 x^2}\right)$