

A Collection of Integrals

Problem 1. $\int_{-2}^2 |x^2 - 4x| \, dx$

Solution. $\int_{-2}^2 |x^2 - 4x| \, dx = \int_{-2}^2 |x(x - 4)| \, dx = \int_{-2}^0 |x(x - 4)| \, dx + \int_0^2 |x(x - 4)| \, dx = \int_{-2}^0 x(x - 4) \, dx - \int_0^2 x(x - 4) \, dx = \int_{-2}^0 (x^2 - 4x) \, dx + \int_0^2 (4x - x^2) \, dx = \int_{-2}^0 (x^2 - 4x) \, dx + \int_0^2 (4x - x^2) \, dx = \left(\frac{x^3}{3} - 2x^2 \right) \Big|_{-2}^0 + \left(2x^2 - \frac{x^3}{3} \right) \Big|_0^2 = 0 - \left(\frac{(-2)^3}{3} - 2(-2)^2 \right) + \left(2 \cdot 2^2 - \frac{2^3}{3} \right) = 16$

Problem 2. $\int_0^5 \frac{3x - 1}{x + 2} \, dx$

Solution. $\int_0^5 \frac{3x - 1}{x + 2} \, dx = \int_0^5 \frac{3(x + 2) - 7}{x + 2} \, dx = 3 \cdot 5 - 7 \int_0^5 \frac{1}{x + 2} \, dx = 15 - 7 \ln |x + 2| \Big|_0^5 = 15 - 7 \ln \frac{7}{2}$

Problem 3. $\int_0^4 \frac{x - 1}{x^2 - 4x - 5} \, dx$

Solution. $\int_0^4 \frac{x - 1}{x^2 - 4x - 5} \, dx = \int_0^4 \frac{(x + 1) - 2}{(x - 5)(x + 1)} \, dx = \int_0^4 \frac{1}{x - 5} \, dx - 2 \int_0^4 \frac{1}{(x - 5)(x + 1)} \, dx = \int_0^4 \frac{1}{x - 5} \, dx - \frac{1}{3} \int_0^4 \left(\frac{1}{x - 5} - \frac{1}{x + 1} \right) \, dx = \frac{2}{3} \int_0^4 \frac{1}{x - 5} \, dx + \frac{1}{3} \int_0^4 \frac{1}{x + 1} \, dx = \frac{2}{3} \ln |x - 5| \Big|_0^4 + \frac{1}{3} \ln |x + 1| \Big|_0^4 = -\frac{2}{3} \ln 5 + \frac{1}{3} \ln 5 = -\frac{1}{3} \ln 5.$

Problem 4. $\int \frac{x - 1}{x^2 - 4x + 5} \, dx$

Solution. $\int \frac{x - 1}{x^2 - 4x + 5} \, dx = \int \frac{(x - 2) + 1}{(x - 2)^2 + 1} \, dx = \underbrace{\int \frac{x - 2}{(x - 2)^2 + 1} \, dx}_{\text{Let } u = (x - 2)^2 + 1} + \int \frac{1}{(x - 2)^2 + 1} \, dx = \frac{1}{2} \ln((x - 2)^2 + 1) +$

$\tan^{-1}(x - 2) = \frac{1}{2} \ln(x^2 - 4x + 5) + \tan^{-1}(x - 2)$

Problem 5. $\int \frac{x - 1}{x^2 - 2x + 5} \, dx$

Solution. $\underbrace{\int \frac{x - 1}{x^2 - 2x + 5} \, dx}_{\text{Let } u = x^2 - 2x + 5} = \frac{1}{2} \ln(x^2 - 2x + 5)$

Problem 6. $\int_0^2 \frac{2x}{(x - 3)^2} \, dx$

Solution. $\int_0^2 \frac{2x}{(x - 3)^2} \, dx = \int_0^2 \left(\frac{2}{x - 3} + \frac{6}{(x - 3)^2} \right) \, dx = 2 \ln |x - 3| \Big|_0^2 - \frac{6}{x - 3} \Big|_0^2 = -2 \ln 3 - 6 \left(-1 + \frac{1}{3} \right) = 4 - 2 \ln 3$

Problem 7. $\int \frac{3x^2 - 2}{x^3 - 2x - 8} \, dx$

Solution. $\underbrace{\int \frac{3x^2 - 2}{x^3 - 2x - 8} \, dx}_{\text{Let } u = x^3 - 2x - 8} = \ln |x^3 - 2x - 8|$

Problem 8. $\int_2^3 \frac{x^3 + 1}{x^3 - x^2} \, dx$

Solution. $\int_2^3 \frac{x^3 + 1}{x^3 - x^2} \, dx = \int_2^3 \frac{(x^3 - x^2) + x^2 + 1}{x^3 - x^2} \, dx = \int_2^3 \left(1 + \frac{x^2}{x^3 - x^2} + \frac{1}{x^3 - x^2} \right) \, dx = 1 + \int_2^3 \frac{1}{x - 1} \, dx + \int_2^3 \left(-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x - 1} \right) \, dx = 1 + \ln |x - 1| \Big|_2^3 - \ln |x| \Big|_2^3 + \frac{1}{x} \Big|_2^3 + \ln |x - 1| \Big|_2^3 = 1 + \ln 2 - \frac{1}{6} - \ln \frac{3}{2} + \ln 2 = \frac{5}{6} + 3 \ln 2 - \ln 3$

Problem 9. $\int \frac{1}{(x-2)(x^2+4)} dx$

Solution. $\int \frac{1}{(x-2)(x^2+4)} dx = \frac{1}{8} \int \left(\frac{1}{x-2} - \frac{x+2}{x^2+4} \right) dx = \frac{1}{8} \ln|x-2| - \frac{1}{16} \ln(x^2+4) - \frac{1}{8} \tan^{-1} \frac{x}{2}$

Problem 10. $\int \frac{x}{x^4 - a^4} dx$

Solution. $\underbrace{\int \frac{x}{x^4 - a^4} dx}_{\text{Let } u=x^2} = \frac{1}{2} \int \frac{1}{u^2 - a^4} du = \frac{1}{4a^2} \int \left(\frac{1}{u - a^2} - \frac{1}{u + a^2} \right) du = \frac{1}{4a^2} \ln \left| \frac{u - a^2}{u + a^2} \right| = \frac{1}{4a^2} \ln \left| \frac{x^2 - a^2}{x^2 + a^2} \right|$

Problem 11. $\int \frac{x}{x^4 + x^2 + 1} dx$

Solution. $\underbrace{\int \frac{x}{x^4 + x^2 + 1} dx}_{\text{Let } u=x^2} = \frac{1}{2} \int \frac{1}{u^2 + u + 1} du = \frac{1}{2} \int \frac{1}{(u + \frac{1}{2})^2 + \frac{3}{4}} du = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1} \frac{2u+1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x^2+1}{\sqrt{3}}$

Problem 12. $\int \frac{1}{x(x^4+1)} dx$

Solution. $\underbrace{\int \frac{1}{x(x^4+1)} dx}_{\text{Let } u=x^2} = \frac{1}{2} \int \frac{1}{u(u^2+1)} du = \frac{1}{2} \int \left(\frac{1}{u} - \frac{u}{u^2+1} \right) du = \frac{1}{2} \ln|u| - \frac{1}{4} \ln(u^2+1) = \frac{1}{4} \ln \frac{x^4}{x^4+1}$

Problem 13. $\int_1^\infty \frac{x^2-3}{x^4+2x^2+9} dx$

Solution. Note that $x^4 + 2x^2 + 9 = (x^2 + 3)^2 - (2x)^2 = (x^2 - 2x + 3)(x^2 + 2x + 3)$, $\int_1^\infty \frac{x^2-3}{x^4+2x^2+9} dx = \frac{1}{2} \int_1^\infty \left(\frac{x-1}{x^2-2x+3} - \frac{x+1}{x^2+2x+3} \right) dx = \frac{1}{4} \ln \left| \frac{x^2-2x+3}{x^2+2x+3} \right| \Big|_1^\infty = \frac{\ln 3}{4}$

Problem 14. $\int \frac{1}{x^6-1} dx$

Solution. $\frac{1}{x^6-1} = \frac{1}{2} \left(\frac{1}{x^3-1} - \frac{1}{x^3+1} \right)$. $\frac{1}{x^3-1} = \frac{a}{x-1} + \frac{bx+C}{x^2+x+1} \implies 1 = a(x^2+x+1) + (bx+C)(x-1)$
 $\implies a = \frac{1}{3}, b = -\frac{1}{3}, c = -\frac{2}{3}; \frac{1}{x^3+1} = \frac{a}{x+1} + \frac{bx+C}{x^2-x+1} \implies 1 = a(x^2-x+1) + (bx+C)(x+1)$
 $\implies a = \frac{1}{3}, b = -\frac{1}{3}, c = \frac{2}{3}$. So $\frac{1}{x^6-1} = \frac{1}{6} \left(\frac{1}{x-1} - \frac{x+2}{x^2+x+1} - \frac{1}{x+1} + \frac{x-2}{x^2-x+1} \right)$, $\int \frac{1}{x^6-1} dx = \frac{1}{6} \int \left(\frac{1}{x-1} - \frac{x+2}{x^2+x+1} - \frac{1}{x+1} + \frac{x-2}{x^2-x+1} \right) dx = \frac{1}{6} \int \left(\frac{1}{x-1} - \frac{\frac{1}{2}(2x+1) + \frac{3}{2}}{x^2+x+1} - \frac{1}{x+1} + \frac{\frac{1}{2}(2x-1) - \frac{3}{2}}{x^2-x+1} \right) dx = \frac{1}{6} \ln \left| \frac{x-1}{x+1} \right| + \frac{1}{12} \ln \left| \frac{x^2-x+1}{x^2+x+1} \right| - \frac{1}{2\sqrt{3}} \left(\tan^{-1} \frac{2x+1}{\sqrt{3}} + \tan^{-1} \frac{2x-1}{\sqrt{3}} \right)$

Problem 15. $\int \frac{x^3}{(x+1)^{10}} dx$

Solution. $\underbrace{\int \frac{x^3}{(x+1)^{10}} dx}_{\text{Let } u=x+1} = \int \frac{(u-1)^3}{u^{10}} du = \int \frac{u^3 - 3u^2 + 3u - 1}{u^{10}} du = \int (u^{-7} - 3u^{-8} + 3u^{-9} - u^{-10}) du = \frac{1}{-6} u^{-6} + 3 \frac{1}{7} u^{-7} - 3 \frac{1}{8} u^{-8} + \frac{1}{9} u^{-9} = -\frac{1}{6(x+1)^6} + \frac{3}{7(x+1)^7} - \frac{3}{8(x+1)^8} + \frac{1}{9(x+1)^9} = -\frac{84x^3 + 36x^2 + 9x + 1}{504(x+1)^9}$

Problem 16. $\int \frac{x^4}{x^{10}+16} dx$

Solution. $\underbrace{\int \frac{x^4}{x^{10}+16} dx}_{\text{Let } u=x^5} = \frac{1}{5} \int \frac{1}{u^2+16} du = \frac{1}{20} \tan^{-1} \frac{u}{4} = \frac{1}{20} \tan^{-1} \frac{x^5}{4}$

Problem 17. $\int_{-1}^{\infty} \left(\frac{x^4}{1+x^6} \right)^2 dx$

Solution. $\int_{-1}^{\infty} \left(\frac{x^4}{1+x^6} \right)^2 dx = \underbrace{\int_{-1}^{\infty} \frac{x^8}{(1+x^6)^2} dx}_{\text{Let } w=x^3} = \frac{1}{3} \int_{-1}^{\infty} \frac{w^2}{(1+w^2)^2} dw.$ Note that by evaluating $\int \frac{1}{1+w^2} dw$

through integration by parts: let $u = \frac{1}{1+w^2}$, then $du = -\frac{2w}{(1+w^2)^2} dw$; let $dv = dw$, then $v = w$. So

$$\int \frac{1}{1+w^2} dw = \frac{1}{1+w^2} \cdot w + \int w \cdot \frac{2w}{(1+w^2)^2} dw \Rightarrow \int \frac{w^2}{(1+w^2)^2} dw = \frac{1}{2} \int \frac{1}{1+w^2} dw - \frac{w}{2(1+w^2)} = \frac{1}{2} \left(\tan^{-1} w - \frac{w}{1+w^2} \right);$$

Hence $\frac{1}{3} \int_{-1}^{\infty} \frac{w^2}{(1+w^2)^2} dw = \frac{1}{6} \left(\tan^{-1} w - \frac{w}{1+w^2} \right) \Big|_{-1}^{\infty} = \frac{1}{6} \left(\left(\frac{\pi}{2} - 0 \right) - \left(-\frac{\pi}{4} + \frac{1}{2} \right) \right) = \frac{\pi}{8} - \frac{1}{12}$

Problem 18. $\int \frac{1}{x^n - x} dx$

Solution. $\int \frac{1}{x^n - x} dx = \int \frac{1}{x(x^{n-1} - 1)} dx = \int \left(\frac{x^{n-2}}{x^{n-1} - 1} - \frac{1}{x} \right) dx = \underbrace{\int \frac{x^{n-2}}{x^{n-1} - 1} dx}_{\text{Let } u = x^{n-1} - 1} - \int \frac{1}{x} dx = \frac{1}{n-1} \ln |x^{n-1} -$

$1| - \ln |x|$

Problem 19. $\int_0^{\infty} \frac{x^n}{(1+x^2)^{1+\frac{n}{2}}} dx, \forall n \in \mathbb{N}.$

Solution. $\underbrace{\int_0^{\infty} \frac{x^n}{(1+x^2)^{1+\frac{n}{2}}} dx}_{\text{Let } x = \tan \theta} = \int_0^{\frac{\pi}{2}} \frac{\tan^n \theta}{\sec^2 \theta \cdot \sec^n \theta} \cdot \sec^2 \theta d\theta = \int_0^{\frac{\pi}{2}} \sin^n \theta d\theta = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} & n \text{ odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & n \text{ even} \end{cases}$

Problem 20. $\int x \sqrt[3]{x+c} dx$

Solution. $\underbrace{\int x \sqrt[3]{x+c} dx}_{\text{Let } u = \sqrt[3]{x+c}} = \int (u^3 - c) \cdot u \cdot 3u^2 du = 3 \int u^6 du - 3c \int u^3 du = \frac{3}{7} u^7 - \frac{3c}{4} u^4 = \frac{3}{7} (\sqrt[3]{x+c})^7 - \frac{3c}{4} (\sqrt[3]{x+c})^4$

Problem 21. $\int_0^1 (1 + \sqrt{x})^8 dx$

Solution. $\underbrace{\int_0^1 (1 + \sqrt{x})^8 dx}_{\text{Let } u = 1 + \sqrt{x}} = \int_1^2 u^8 \cdot 2(u-1) du = 2 \int_1^2 (u^9 - u^8) du = 2 \left(\frac{u^{10}}{10} - \frac{u^9}{9} \right) \Big|_1^2 = 2 \cdot \left(\left(\frac{2^{10}}{10} - \frac{2^9}{9} \right) - \left(\frac{1}{10} - \frac{1}{9} \right) \right) = \frac{4097}{45}$

Problem 22. $\int \frac{\sqrt{x}}{1+x^3} dx$

Solution. $\underbrace{\int \frac{\sqrt{x}}{1+x^3} dx}_{\text{Let } u = x^{\frac{3}{2}}} = \int \frac{\frac{2}{3} du}{1+u^2} = \frac{2}{3} \int \frac{du}{1+u^2} = \frac{2}{3} \tan^{-1} u = \frac{2}{3} \tan^{-1} x^{\frac{3}{2}}$

Problem 23. $\int \frac{\sqrt{x}}{1+\sqrt[3]{x}} dx$

Solution. $\underbrace{\int \frac{\sqrt{x}}{1+\sqrt[3]{x}} dx}_{\text{Let } u = x^{\frac{1}{6}}} = \int \frac{u^3}{1+u^2} \cdot 6u^5 du = 6 \int \left(u^6 - u^4 + u^2 - 1 + \frac{1}{1+u^2} \right) du = 6 \left(\frac{u^7}{7} - \frac{u^5}{5} + \frac{u^3}{3} - u + \tan^{-1} u \right) =$

$$6 \left(\frac{x^{\frac{7}{6}}}{7} - \frac{x^{\frac{5}{6}}}{5} + \frac{x^{\frac{1}{2}}}{3} - x^{\frac{1}{6}} + \tan^{-1} x^{\frac{1}{6}} \right)$$

Problem 24. $\int \frac{1}{\sqrt{x+1} + \sqrt{x}} dx$

Solution. $\int \frac{1}{\sqrt{x+1} + \sqrt{x}} dx = \int \frac{\sqrt{x+1} - \sqrt{x}}{\sqrt{x+1} - \sqrt{x}} \cdot \frac{1}{\sqrt{x+1} + \sqrt{x}} dx = \int \frac{\sqrt{x+1} - \sqrt{x}}{(\sqrt{x+1})^2 - (\sqrt{x})^2} dx = \int (\sqrt{x+1} - \sqrt{x}) dx = \frac{2}{3}((x+1)^{\frac{3}{2}} - x^{\frac{3}{2}})$

Problem 25. $\int \frac{1}{x\sqrt{4x+1}} dx$

Solution. $\underbrace{\int \frac{1}{x\sqrt{4x+1}} dx}_{\text{Let } u = \sqrt{4x+1}} = \int \frac{\frac{1}{2}u du}{\frac{u^2-1}{4} \cdot u} = \int \frac{2}{u^2-1} du = \int \left(\frac{1}{u-1} - \frac{1}{u+1} \right) du = \ln \left| \frac{u-1}{u+1} \right| = \ln \left| \frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1} \right|$

Problem 26. $\int \frac{1}{x^2\sqrt{4x+1}} dx$

Solution. $\underbrace{\int \frac{1}{x^2\sqrt{4x+1}} dx}_{\text{Let } u = \sqrt{4x+1}} = \int \frac{\frac{1}{2}u du}{\frac{(u^2-1)^2}{16} \cdot u} = \int \frac{8}{(u^2-1)^2} du = 2 \int \left(\frac{1}{u-1} - \frac{1}{u+1} \right)^2 du$
 $= 2 \int \left(\frac{1}{(u-1)^2} + \frac{1}{(u+1)^2} - \frac{2}{(u-1)(u+1)} \right) du = 2 \int \left(\frac{1}{(u-1)^2} + \frac{1}{(u+1)^2} + \frac{1}{u+1} - \frac{1}{u-1} \right) du = 2 \left(-\frac{1}{u+1} - \frac{1}{u-1} + \ln \left| \frac{u+1}{u-1} \right| \right) = -\frac{\sqrt{4x+1}}{x} + 2 \ln \left| \frac{\sqrt{4x+1}+1}{\sqrt{4x+1}-1} \right|$

Problem 27. $\int \frac{1}{x+4+4\sqrt{x+1}} dx$

Solution. $\underbrace{\int \frac{1}{x+4+4\sqrt{x+1}} dx}_{\text{Let } u = \sqrt{x+1}} = \int \frac{2u du}{u^2+3+4u} = \int \frac{2u}{(u+3)(u+1)} du = \int \left(\frac{3}{u+3} - \frac{1}{u+1} \right)^2 du = \ln \left| \frac{(u+3)^3}{u+1} \right| = \ln \left| \frac{(\sqrt{x+1}+3)^3}{\sqrt{x+1}+1} \right|$

Problem 28. $\int \sqrt{\frac{1+x}{1-x}} dx$

Solution. $\int \sqrt{\frac{1+x}{1-x}} dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}} dx = \int \frac{1+x}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-x^2}} dx + \int \frac{x}{\sqrt{1-x^2}} dx = \sin^{-1} x - \sqrt{1-x^2}$

Problem 29. $\int \frac{\sqrt{2-x}}{\sqrt{x}} dx$

Solution. $\underbrace{\int \frac{\sqrt{2-x}}{\sqrt{x}} dx}_{\text{Let } u = 1-x} = - \int \frac{\sqrt{1+u}}{\sqrt{1-u}} du = \sqrt{1-u^2} - \sin^{-1} u = \sqrt{x(2-x)} - \sin^{-1}(1-x)$

Problem 30. $\int_0^\infty \frac{dx}{\sqrt{x}+x^2}$

Solution. $\underbrace{\int_0^\infty \frac{dx}{\sqrt{x}+x^2}}_{\text{Let } u = \sqrt{x}} = \int_0^\infty \frac{2u du}{u+u^4} = 2 \int_0^\infty \frac{du}{1+u^3}$. From $\frac{1}{u^3+1} = \frac{\frac{1}{3}}{u+1} + \frac{-\frac{1}{3}u + \frac{2}{3}}{u^2-u+1} = \frac{\frac{1}{3}}{u+1} + \frac{-\frac{1}{3}(u-\frac{1}{2}) + \frac{1}{2}}{u^2-u+1} = \frac{\frac{1}{3}}{u+1} + \frac{-\frac{1}{6}(2u-1)}{u^2-u+1} + \frac{\frac{1}{2}}{(u-\frac{1}{2})^2 + \frac{3}{4}}$, $\int \frac{du}{u^3+1} = \frac{1}{3} \ln|u+1| - \frac{1}{6} \ln(u^2-u+1) + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2u-1}{\sqrt{3}}$, so $\int_0^\infty \frac{dx}{x^2+\sqrt{x}} = 2 \int_0^\infty \frac{du}{u^3+1} = \frac{4\pi}{3\sqrt{3}}$ by $\lim_{u \rightarrow \infty} \frac{(u+1)^2}{u^2-u+1} = 1$, $\lim_{u \rightarrow 0} \frac{(u+1)^2}{u^2-u+1} = 1$, $\lim_{u \rightarrow \infty} \tan^{-1} \frac{2u-1}{\sqrt{3}} = \frac{\pi}{2}$, and $\lim_{u \rightarrow 0} \tan^{-1} \frac{2u-1}{\sqrt{3}} = -\frac{\pi}{6}$.

Problem 31. $\int \sqrt{3-2x-x^2} dx$

Solution.
$$\int \sqrt{3-2x-x^2} \, dx = \int \sqrt{4-1-2x-x^2} \, dx = \underbrace{\int \sqrt{4-(x+1)^2} \, dx}_{\text{Let } u = x+1} = \underbrace{\int \sqrt{4-u^2} \, du}_{\text{Let } u = 2 \sin \theta}$$

$$= \int 2 \cos \theta \cdot 2 \cos \theta \, d\theta = 2 \int (\cos 2\theta + 1) \, d\theta = \sin 2\theta + 2\theta = 2 \sin \theta \cos \theta + 2 \sin^{-1} \frac{u}{2} = 2 \cdot \frac{u}{2} \cdot \frac{\sqrt{4-u^2}}{2} + 2 \sin^{-1} \frac{u}{2} =$$

$$\frac{(x+1)\sqrt{3-2x-x^2}}{2} + 2 \sin^{-1} \frac{x+1}{2}$$

Problem 32.
$$\int \frac{dx}{(2x-1)\sqrt{x^2-x}}$$

Solution.
$$\underbrace{\int \frac{dx}{(2x-1)\sqrt{x^2-x}}}_{\text{Let } u = \sqrt{x^2-x}} = \int \frac{2u \, du}{(4u^2+1)u} = \tan^{-1}(2\sqrt{x^2-x})$$

Problem 33.
$$\int \frac{2x^2 + \sqrt{1-x^2}}{x\sqrt{1-x^2}} \, dx$$

Solution.
$$\int \frac{2x^2 + \sqrt{1-x^2}}{x\sqrt{1-x^2}} \, dx = \int \frac{2x}{\sqrt{1-x^2}} \, dx + \int \frac{dx}{x} = -2\sqrt{1-x^2} + \ln|x|$$

Problem 34.
$$\int \frac{dx}{(1-x^2)^{\frac{3}{2}}}$$

Solution.
$$\underbrace{\int \frac{dx}{(1-x^2)^{\frac{3}{2}}}}_{\text{Let } x = \sin \theta} = \int \frac{\cos \theta \, d\theta}{\cos^3 \theta} = \int \sec^2 \theta \, d\theta = \tan \theta = \frac{x}{\sqrt{1-x^2}}$$

Problem 35.
$$\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} \, dx$$

Solution.
$$\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} \, dx = -\sqrt{1-x^2} \Big|_0^{\frac{1}{2}} = -\left(\frac{\sqrt{3}}{2} - 1\right) = 1 - \frac{\sqrt{3}}{2}$$

Problem 36.
$$\int_0^{\frac{\sqrt{2}}{2}} \frac{x^2}{\sqrt{1-x^2}} \, dx$$

Solution.
$$\int_0^{\frac{\sqrt{2}}{2}} \frac{x^2}{\sqrt{1-x^2}} \, dx = \int_0^{\frac{\sqrt{2}}{2}} \frac{(x^2-1)+1}{\sqrt{1-x^2}} \, dx = -\int_0^{\frac{\sqrt{2}}{2}} \sqrt{1-x^2} \, dx + \sin^{-1} x \Big|_0^{\frac{\sqrt{2}}{2}} = -\left(\frac{\pi}{8} + \frac{1}{4}\right) + \frac{\pi}{4} = \frac{\pi}{8} - \frac{1}{4}$$

Problem 37.
$$\int \frac{x}{1-x^2+\sqrt{1-x^2}} \, dx$$

Solution.
$$\underbrace{\int \frac{x}{1-x^2+\sqrt{1-x^2}} \, dx}_{\text{Let } u = \sqrt{1-x^2}} = \int \frac{-u \, du}{u^2+u} = -\ln|u+1| = -\ln|\sqrt{1-x^2}+1|$$

Problem 38.
$$\int \frac{1}{\sqrt{4x^2-4x-3}} \, dx$$

Solution.
$$\int \frac{1}{\sqrt{4x^2-4x-3}} \, dx = \int \frac{1}{\sqrt{4x^2-4x+1-4}} \, dx = \underbrace{\int \frac{1}{\sqrt{(2x-1)^2-2^2}} \, dx}_{\text{Let } 2x-1 = 2 \sec \theta} = \int \frac{\sec \theta \tan \theta \, d\theta}{2 \tan \theta}$$

$$= \frac{1}{2} \int \sec \theta \, d\theta = \frac{1}{2} \ln|\sec \theta + \tan \theta| = \frac{1}{2} \ln \left| x - \frac{1}{2} + \sqrt{x^2 - x - \frac{3}{4}} \right|$$

Problem 39.
$$\int \frac{1}{x - \sqrt{1-x^2}} \, dx$$

Solution. $\underbrace{\int \frac{1}{x - \sqrt{1-x^2}} dx}_{\text{Let } x = \sin \theta} = \int \frac{\cos \theta}{\sin \theta - \cos \theta} d\theta = \frac{1}{2} \int \frac{2 \cos \theta}{\sin \theta - \cos \theta} d\theta = \frac{1}{2} \int \frac{(\cos \theta - \sin \theta) + (\cos \theta + \sin \theta)}{\sin \theta - \cos \theta} d\theta$
 $= -\frac{1}{2}\theta + \frac{1}{2} \underbrace{\int \frac{\cos \theta + \sin \theta}{\sin \theta - \cos \theta} d\theta}_{\text{Let } u = \sin \theta - \cos \theta} = \frac{1}{2}(-\theta + \ln |\sin \theta - \cos \theta|) = \frac{1}{2}(-\sin^{-1} x + \ln |x - \sqrt{1-x^2}|)$

Problem 40. $\int \frac{x}{\sqrt{3-x^4}} dx$

Solution. $\underbrace{\int \frac{x}{\sqrt{3-x^4}} dx}_{\text{Let } u = x^2} = \int \frac{\frac{du}{2}}{\sqrt{3-u^2}} = \frac{1}{2} \int \frac{du}{\sqrt{3-u^2}} = \frac{1}{2} \sin^{-1} \frac{u}{\sqrt{3}} = \frac{1}{2} \sin^{-1} \frac{x^2}{\sqrt{3}}$

Problem 41. $\int \frac{dx}{\sqrt{1+\sqrt[3]{x}}}$

Solution. $\underbrace{\int \frac{dx}{\sqrt{1+\sqrt[3]{x}}}}_{\text{Let } u = \sqrt{1+\sqrt[3]{x}}} = \int \frac{(u^2-1)^2 6u du}{u} = 6 \int (u^4 - 2u^2 + 1) du = \frac{6}{5}u^5 - 4u^3 + 6u = \frac{6}{5} \left(\sqrt{1+\sqrt[3]{x}} \right)^5 - 4 \left(\sqrt{1+\sqrt[3]{x}} \right)^3 + 6\sqrt{1+\sqrt[3]{x}} = \frac{2}{5} \sqrt{1+\sqrt[3]{x}} (8 - 4\sqrt[3]{x} + 6\sqrt[3]{x^2})$

Problem 42. $\int_0^1 (\sqrt[3]{1-x^7} - \sqrt[7]{1-x^3}) dx$

Solution. Let $u = \sqrt[3]{1-x^7} \implies x = \sqrt[7]{1-u^3}$. Integration by parts yield $\int_0^1 \sqrt[3]{1-x^7} dx = \int_1^0 u d(\sqrt[7]{1-u^3}) = u \cdot \sqrt[7]{1-u^3} \Big|_1^0 - \int_1^0 \sqrt[7]{1-u^3} du = \int_0^1 \sqrt[7]{1-u^3} du$. So $\int_0^1 (\sqrt[3]{1-x^7} - \sqrt[7]{1-x^3}) dx = 0$.

Problem 43. $\int_0^1 3(x-1)^2 \left(\int_0^x \sqrt{1+(t-1)^4} dt \right) dx$

Solution. $\underbrace{\int_0^1 3(x-1)^2 \left(\int_0^x \sqrt{1+(t-1)^4} dt \right) dx}_{\text{Let } u = \int_0^x \sqrt{1+(t-1)^4} dt, dv = 3(x-1)^2 dx} = \left(\int_0^x \sqrt{1+(t-1)^4} dt \right) \cdot (x-1)^3 \Big|_0^1 - \underbrace{\int_0^1 (x-1)^3 \cdot \sqrt{1+(x-1)^4} dx}_{\text{Let } u = 1+(x-1)^4} = \frac{1}{4} \int_1^2 \sqrt{u} du = \frac{2\sqrt{2}-1}{6}$

Problem 44. $\int \sin^3 x \cos^5 x dx$

Solution. $\int \sin^3 x \cos^5 x dx = \int \sin^2 x \cos^4 x \cos x dx = \underbrace{\int \sin^2 x (1 - \sin^2 x)^2 \cos x dx}_{\text{Let } u = \sin x} = \int u^2 (1 - u^2)^2 du = \int (u^3 - 2u^5 + u^7) du = \frac{u^4}{4} - \frac{u^6}{3} + \frac{u^8}{8} = \frac{\sin^4 x}{4} - \frac{\sin^6 x}{3} + \frac{\sin^8 x}{8}$

Problem 45. $\int \sin x \cos(\cos x) dx$

Solution. $\underbrace{\int \sin x \cos(\cos x) dx}_{\text{Let } u = \cos x} = - \int \cos u du = -\sin u = -\sin(\cos x)$

Problem 46. $\int \tan^3 x dx$

Solution. $\int \tan^3 x \, dx = \int (\sec^2 x - 1) \tan x \, dx = \int \sec^2 x \tan x \, dx - \int \tan x \, dx = \underbrace{\int \sec x \cdot \sec x \tan x \, dx}_{\text{Let } u = \sec x} - \underbrace{\int \frac{\sin x}{\cos x} \, dx}_{\text{Let } w = \cos x} = \frac{\sec^2 x}{2} + \ln |\cos x|$

Problem 47. $\int \tan^5 x \sec^4 x \, dx$

Solution. $\int \tan^5 x \sec^4 x \, dx = \int \tan^5 x \sec^2 x \sec^2 x \, dx = \underbrace{\int \tan^5 x (\tan^2 x + 1) \sec^2 x \, dx}_{\text{Let } u = \tan x} = \int (u^7 + u^5) \, du = \frac{u^8}{8} + \frac{u^6}{6} = \frac{\tan^8 x}{8} + \frac{\tan^6 x}{6}$

Problem 48. $\int \tan^5 x \sec^7 x \, dx$

Solution. $\int \tan^5 x \sec^7 x \, dx = \int \tan^4 x \sec^6 x \cdot \tan x \sec x \, dx = \underbrace{\int (\sec^2 x - 1)^2 \sec^6 x \cdot \tan x \sec x \, dx}_{\text{Let } u = \sec x} = \int (u^2 - 1)^2 u^6 \, du = \int (u^{10} - 2u^8 + u^6) \, du = \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} = \frac{\sec^{11} x}{11} - \frac{2\sec^9 x}{9} + \frac{\sec^7 x}{7}$

Problem 49. $\int_0^{\frac{\pi}{4}} \cos^2 x \tan^2 x \, dx$

Solution. $\int_0^{\frac{\pi}{4}} \cos^2 x \tan^2 x \, dx = \int_0^{\frac{\pi}{4}} \sin^2 x \, dx = \int_0^{\frac{\pi}{4}} \frac{1 - \cos 2x}{2} \, dx = \frac{\pi}{8} - \frac{1}{4} \sin 2x \Big|_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{1}{4}$

Problem 50. $\int (\sin x + \cos x)^2 \, dx$

Solution. $\int (\sin x + \cos x)^2 \, dx = \int (\sin^2 x + 2 \sin x \cos x + \cos^2 x) \, dx = \int (1 + \sin 2x) \, dx = x - \frac{\cos 2x}{2}$

Problem 51. $\int \sin 4x \cos 3x \, dx$

Solution. $\int \sin 4x \cos 3x \, dx = \frac{1}{2} \int (\sin(4x + 3x) + \sin(4x - 3x)) \, dx = -\frac{1}{2} \left(\frac{\cos 7x}{7} + \cos x \right)$

Problem 52. $\int \sin x \sin 2x \sin 3x \, dx$

Solution. $\int \sin x \sin 2x \sin 3x \, dx = \frac{1}{2} \int (\cos(x - 2x) - \cos(x + 2x)) \sin 3x \, dx = \frac{1}{2} \int (\cos x \sin 3x - \cos 3x \sin x) \, dx = \frac{1}{4} \int (\sin(3x - x) + \sin(3x + x) - \sin 6x) \, dx = \frac{1}{4} \int (\sin 2x + \sin 4x - \sin 6x) \, dx = -\frac{\cos 2x}{8} - \frac{\cos 4x}{16} + \frac{\cos 6x}{24}$

Problem 53. $\int \sin \sqrt{ax} \, dx$

Solution. $\underbrace{\int \sin \sqrt{ax} \, dx}_{\text{Let } u = \sqrt{ax}} = \int \sin u \cdot \frac{2u}{a} \, du = \frac{2}{a} \int u \sin u \, du = \frac{2}{a} (\sin u - u \cos u) = \frac{2}{a} (\sin \sqrt{ax} - \sqrt{ax} \cos \sqrt{ax})$

Problem 54. $\int \frac{\cos^5 x}{\sqrt{\sin x}} \, dx$

Solution. $\int \frac{\cos^5 x}{\sqrt{\sin x}} \, dx = \int \frac{\cos^4 x \cos x}{\sqrt{\sin x}} \, dx = \underbrace{\int \frac{(1 - \sin^2 x)^2 \cos x}{\sqrt{\sin x}} \, dx}_{\text{Let } u = \sin x} = \int \frac{(1 - u^2)^2}{\sqrt{u}} \, du = \int (u^{-\frac{1}{2}} - 2u^{\frac{3}{2}} + u^{\frac{7}{2}}) \, du = 2u^{\frac{1}{2}} - \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{9}u^{\frac{9}{2}} = 2\sin^{\frac{1}{2}} x - \frac{4}{5}\sin^{\frac{5}{2}} x + \frac{2}{9}\sin^{\frac{9}{2}} x$

Problem 55. $\int \frac{\cos^6 x}{\sin^4 x} dx$

Solution. $\int \frac{\cos^6 x}{\sin^4 x} dx = \int \frac{(1 - \sin^2 x)^3}{\sin^4 x} dx = \int \frac{1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x}{\sin^4 x} dx = \int (\csc^4 x - 3\csc^2 x + 3 - \sin^2 x) dx = \left(-\frac{1}{3}\cot^3 x - \cot x\right) + 3\cot x + 3x - \left(\frac{x}{2} - \frac{\sin 2x}{4}\right) = -\frac{1}{3}\cot^3 x + 2\cot x + \frac{5x}{2} + \frac{\sin 2x}{4}$

Problem 56. $\int \frac{\tan x}{\sec^2 x} dx$

Solution. $\int \frac{\tan x}{\sec^2 x} dx = \int \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos^2 x}} dx = \int \sin x \cos x dx = \frac{1}{2} \int \sin 2x dx = -\frac{1}{4} \cos 2x$

Problem 57. $\int \frac{\tan^3 x}{\cos^3 x} dx$

Solution. $\int \frac{\tan^3 x}{\cos^3 x} dx = \int \tan^2 x \sec^2 x \cdot \tan x \sec x dx = \underbrace{\int (\sec^2 x - 1) \sec^2 x \cdot \tan x \sec x dx}_{\text{Let } u = \sec x} = \int (u^2 - 1) u^2 du = \int (u^4 - u^2) du = \frac{u^5}{5} - \frac{u^3}{3} = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3}$

Problem 58. $\int \frac{1}{1 - \cos x} dx$

Solution. $\int \frac{1}{1 - \cos x} dx = \int \frac{1 + \cos x}{1 - \cos^2 x} dx = \int \frac{1 + \cos x}{\sin^2 x} dx = \int (\csc^2 x + \csc x \cot x) dx = -\cot x - \csc x$

Problem 59. $\int \frac{dx}{3 \sin x - 4 \cos x}$

Solution. $3 \sin x - 4 \cos x = -5 \left(\frac{4}{5} \cos x - \frac{3}{5} \sin x \right) = -5 \cos(x + \alpha)$, where $\cos \alpha = \frac{4}{5}$. So $\int \frac{dx}{3 \sin x - 4 \cos x} = -\frac{1}{5} \int \frac{dx}{\cos(x + \alpha)} = -\frac{1}{5} \int \sec(x + \alpha) dx = -\frac{1}{5} \ln |\sec(x + \alpha) + \tan(x + \alpha)|$

Problem 60. $\int \frac{dx}{1 + \sin x - \cos x}$

Solution. $\int \frac{dx}{1 + \sin x - \cos x} = \int \frac{1 + \sin x + \cos x}{(1 + \sin x)^2 - \cos^2 x} dx = \int \frac{(1 + \sin x) + \cos x}{2 \sin x (1 + \sin x)} dx = \frac{1}{2} \int \frac{1}{\sin x} dx + \frac{1}{2} \int \frac{\cos x}{\sin x (1 + \sin x)} dx = -\frac{1}{2} \ln |\csc x + \cot x| + \frac{1}{2} \int \frac{du}{u(1+u)} = -\frac{1}{2} \ln |\csc x + \cot x| + \frac{1}{2} \int \left(\frac{1}{u} - \frac{1}{u+1} \right) du = -\frac{1}{2} \ln |\csc x + \cot x| + \frac{1}{2} \ln \left| \frac{\sin x}{\sin x + 1} \right|$

Problem 61. $\int \frac{\cos x + \sin x}{\sin 2x} dx$

Solution. $\int \frac{\cos x + \sin x}{\sin 2x} dx = \int \frac{\cos x + \sin x}{2 \sin x \cos x} dx = \frac{1}{2} \int \left(\frac{1}{\sin x} + \frac{1}{\cos x} \right) dx = \frac{1}{2} \int (\csc x + \sec x) dx = \frac{1}{2} (-\ln |\csc x + \cot x| + \ln |\sec x + \tan x|) = \frac{1}{2} \ln \left| \frac{\sec x + \tan x}{\csc x + \cot x} \right| = \frac{1}{2} \ln \left| \frac{(1 + \sin x) \sin x}{(1 + \cos x) \cos x} \right|$

Problem 62. $\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$

Solution. $\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx = \int \frac{\frac{1}{2} \sin 2x}{\left(\frac{1 - \cos 2x}{2}\right)^2 + \left(\frac{1 + \cos 2x}{2}\right)^2} dx = \underbrace{\int \frac{\sin 2x}{1 + \cos^2 2x} dx}_{\text{Let } u = \cos 2x} = -\frac{1}{2} \int \frac{du}{1 + u^2} = -\frac{1}{2} \tan^{-1} u = -\frac{1}{2} \tan^{-1}(\cos 2x)$

Problem 63. $\int \frac{\cos x}{\sqrt{\cos^2 x + 2}} dx$

Solution. $\int \frac{\cos x}{\sqrt{\cos^2 x + 2}} dx = \int \underbrace{\frac{\cos x}{\sqrt{3 - \sin^2 x}}}_{\text{Let } u = \sin x} dx = \int \frac{du}{\sqrt{3 - u^2}} = \sin^{-1} \frac{u}{\sqrt{3}} = \sin^{-1} \left(\frac{\sin x}{\sqrt{3}} \right)$

Problem 64. $\int \frac{\sec x \cos 2x}{\sin x + \sec x} dx$

Solution. $\int \frac{\sec x \cos 2x}{\sin x + \sec x} dx = \int \frac{\cos 2x}{\sin x \cos x + 1} dx = \int \underbrace{\frac{\cos 2x}{\frac{1}{2} \sin 2x + 1}}_{\text{Let } u = \frac{1}{2} \sin 2x + 1} dx = \int \frac{du}{u} = \ln \left| \frac{1}{2} \sin 2x + 1 \right|$

Problem 65. $\int \frac{\tan x}{\tan x + \sec x} dx$

Solution. $\int \frac{\tan x}{\tan x + \sec x} dx = \int \frac{\frac{\sin x}{\cos x}}{\frac{\sin x}{\cos x} + \frac{1}{\cos x}} dx = \int \frac{\sin x}{1 + \sin x} dx = \int \frac{(1 - \sin x) \sin x}{(1 - \sin x)(1 + \sin x)} dx = \int \frac{\sin x - \sin^2 x}{\cos^2 x} dx$
 $= \int \frac{\sin x}{\cos^2 x} dx - \int \frac{\sin^2 x}{\cos^2 x} dx = \sec x - \int \tan^2 x dx = \sec x - \int (\sec^2 x - 1) dx = \sec x - \tan x + x$

Problem 66. $\int \frac{1}{1 - \tan^2 x} dx$

Solution. $\int \frac{1}{1 - \tan^2 x} dx = \int \frac{1 + \tan^2 x}{1 - \tan^4 x} dx = \int \underbrace{\frac{\sec^2 x}{1 - \tan^4 x}}_{\text{Let } u = \tan x} dx = \int \frac{du}{1 - u^4} = \frac{1}{2} \int \left(\frac{1}{1 + u^2} + \frac{1}{1 - u^2} \right) du =$
 $\frac{1}{2} \int \left(\frac{1}{1 + u^2} + \frac{1}{2} \left(\frac{1}{1 - u} + \frac{1}{1 + u} \right) \right) du = \frac{1}{2} \tan^{-1} u + \frac{1}{4} \ln \left| \frac{1 + u}{1 - u} \right| = \frac{x}{2} + \frac{1}{4} \ln \left| \frac{1 + \tan x}{1 - \tan x} \right|$

Problem 67. $\int \frac{3 + \sec^2 x + \sin x}{\tan x} dx$

Solution. $\int \frac{3 + \sec^2 x + \sin x}{\tan x} dx = \int (3 \cot x + 2 \csc 2x + \cos x) dx = 3 \ln |\sin x| - \ln |\csc 2x + \cot 2x| + \sin x =$
 $4 \ln |\sin x| - \ln |\cos x| + \sin x$

Problem 68. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1 + 4 \cot x}{4 - \cot x} dx$

Solution. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1 + 4 \cot x}{4 - \cot x} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1 + 4 \frac{\cos x}{\sin x}}{4 - \frac{\cos x}{\sin x}} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\frac{\sin x + 4 \cos x}{\sin x}}{\frac{4 \sin x - \cos x}{\sin x}} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \underbrace{\frac{\sin x + 4 \cos x}{4 \sin x - \cos x}}_{\text{Let } u = 4 \sin x - \cos x} dx = \int_{\frac{3}{\sqrt{2}}}^4 \frac{du}{u} =$

$\frac{5}{2} \ln 2 - \ln 3$

Problem 69. $\int \sqrt{\tan x} dx$

Solution. $\underbrace{\int \sqrt{\tan x} dx}_{\text{Let } u = \sqrt{\tan x}} = \int \frac{2u^2}{1 + u^4} du. \text{ By } 1 + u^4 = 1 + 2u^2 + u^4 - 2u^2 = (1 + u^2)^2 - (\sqrt{2}u)^2 = (1 + \sqrt{2}u + u^2)(1 -$

$\sqrt{2}u + u^2), \frac{2u^2}{1 + u^4} = \frac{au + b}{1 + \sqrt{2}u + u^2} + \frac{cu + d}{1 - \sqrt{2}u + u^2} \implies (au + b)(1 - \sqrt{2}u + u^2) + (cu + d)(1 + \sqrt{2}u + u^2) = 2u^2$
 $\implies (a + c)u^3 + (-\sqrt{2}a + b + \sqrt{2}c + d)u^2 + (a - \sqrt{2}b + c + \sqrt{2}d)u + (b + d) = 2u^2 \implies a + c = 0,$
 $-\sqrt{2}a + b + \sqrt{2}c + d = 2, a - \sqrt{2}b + c + \sqrt{2}d = 0, b + d = 0 \implies a = -\frac{\sqrt{2}}{2}, c = \frac{\sqrt{2}}{2}, b = d = 0$
 $\implies \frac{2u^2}{1 + u^4} = \frac{-\frac{1}{\sqrt{2}}u}{1 + \sqrt{2}u + u^2} + \frac{\frac{1}{\sqrt{2}}u}{1 - \sqrt{2}u + u^2} = \frac{-\frac{1}{2\sqrt{2}}(2u + \sqrt{2}) + \frac{1}{2}}{1 + \sqrt{2}u + u^2} + \frac{\frac{1}{2\sqrt{2}}(2u - \sqrt{2}) + \frac{1}{2}}{1 - \sqrt{2}u + u^2}. \text{ So } \int \frac{2u^2}{1 + u^4} du =$
 $\frac{1}{2\sqrt{2}} \ln \left| \frac{1 - \sqrt{2}u + u^2}{1 + \sqrt{2}u + u^2} \right| + \frac{1}{2} \int \left(\frac{1}{(u + \frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2} + \frac{1}{(u - \frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2} \right) du = \frac{1}{2\sqrt{2}} \ln \left| \frac{1 - \sqrt{2}u + u^2}{1 + \sqrt{2}u + u^2} \right|$

$$+\frac{1}{\sqrt{2}}\left(\tan^{-1}(\sqrt{2}u+1)+\tan^{-1}(\sqrt{2}u-1)\right)=\frac{1}{2\sqrt{2}}\ln\left|\frac{1-\sqrt{2\tan x}+\tan x}{1+\sqrt{2\tan x}+\tan x}\right|+\frac{1}{\sqrt{2}}\left(\tan^{-1}(\sqrt{2\tan x}+1)+\tan^{-1}(\sqrt{2\tan x}-1)\right)$$

Problem 70. $\int \frac{\sin 2x}{1+\tan x} dx$

Solution. $\int \frac{\sin 2x}{1+\tan x} dx = \int \frac{2\sin x \cos x}{1+\frac{\sin x}{\cos x}} dx = \int \frac{2\sin x \cos^2 x}{\cos x + \sin x} dx = \int \frac{2\sin x \cos^2 x}{\cos x + \sin x} \cdot \frac{\cos x - \sin x}{\cos x - \sin x} dx$
 $= \int \frac{\cos^2 x \sin 2x - \frac{1}{2} \sin^2 2x}{\cos 2x} dx = \int \frac{\frac{1+\cos 2x}{2} \cdot \sin 2x - \frac{1}{2}(1-\cos^2 2x)}{\cos 2x} dx = \frac{1}{2} \int \frac{(1+\cos 2x) \cdot \sin 2x - (1-\cos^2 2x)}{\cos 2x} dx =$
 $\frac{1}{2} \int \left(\frac{\sin 2x}{\cos 2x} + \sin 2x - \sec 2x + \cos 2x \right) dx = \frac{1}{4} \left(\ln |\sec 2x| - \cos 2x - \ln |\sec 2x + \tan 2x| + \sin 2x \right)$

Problem 71. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{\frac{\tan x}{\sin 2x}} dx$

Solution. $\int \sqrt{\frac{\tan x}{\sin 2x}} dx = \int \sqrt{\frac{\frac{\sin x}{\cos x}}{2\sin x \cos x}} dx = \frac{1}{\sqrt{2}} \int \sec x dx = \frac{1}{\sqrt{2}} \ln |\sec x + \tan x|$, so $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{\frac{\tan x}{\sin 2x}} dx =$
 $\frac{1}{\sqrt{2}} \ln |\sec x + \tan x| \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{1}{\sqrt{2}} \ln \frac{2+\sqrt{3}}{\sqrt{2}+1}$

Problem 72. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\sin 2x} dx$

Solution. By $(\sqrt{\tan x})' = \frac{1}{2\sqrt{\tan x}} \cdot \sec^2 x = \frac{\sec^2 x \sqrt{\tan x}}{2 \tan x} = \frac{\sqrt{\tan x}}{\sin 2x}$, $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\sin 2x} dx = \sqrt{\tan x} \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \sqrt[4]{3} - 1$

Problem 73. $\int \tan^5 x \sqrt[3]{\cos x} dx$

Solution. $\int \tan^5 x \sqrt[3]{\cos x} dx = \int \frac{\sin^5 x}{\cos^5 x} \sqrt[3]{\cos x} dx = \underbrace{\int (1-\cos^2 x)^2 \cos^{\frac{1}{3}-5} x \cdot \sin x dx}_{\text{Let } u = \cos x} = - \int (u^{-\frac{14}{3}} - 2u^{-\frac{8}{3}} +$
 $u^{-\frac{2}{3}}) du = \frac{3}{11} u^{-\frac{11}{3}} - \frac{6}{5} u^{-\frac{5}{3}} - 3u^{\frac{1}{3}} = \frac{3}{11} \cos^{-\frac{11}{3}} x - \frac{6}{5} \cos^{-\frac{5}{3}} x - 3 \cos^{\frac{1}{3}} x$

Problem 74. $\int x \sin^2 x dx$

Solution. $\int x \sin^2 x dx = \frac{1}{2} \int (x - x \cos 2x) dx = \frac{1}{2} \left(x \cdot \frac{1}{2} \sin 2x + \frac{1}{4} \cos 2x \right) = \frac{x \sin 2x}{4} + \frac{\cos 2x}{8}$

Problem 75. $\int (x + \sin x)^2 dx$

Solution. $\int (x + \sin x)^2 dx = \int (x^2 + 2x \sin x + \sin^2 x) dx = \frac{x^3}{6} + 2 \int x \sin x dx + \frac{1}{2} \int (1 - \cos 2x) dx = \frac{x^3}{6} +$
 $2(\sin x - x \cos x) + \frac{x}{2} - \frac{1}{4} \sin 2x$

Problem 76. $\int x \sin^2 x \cos x dx$

Solution. $\int x \sin^2 x \cos x dx = \int x \cdot \frac{1-\cos 2x}{2} \cdot \cos x dx = \frac{1}{2} \int (x \cos x - x \cos 2x \cos x) dx = \frac{1}{2} \int (x \cos x -$
 $x \cdot \frac{1}{2} (\cos(2x-x) + \cos(2x+x))) dx = \frac{1}{4} \int (2x \cos x - x \cos x - x \cos 3x) dx = \frac{1}{4} \int x (\cos x - \cos 3x) dx =$
 $\frac{1}{4} \left(x \left(\sin x - \frac{1}{3} \sin 3x \right) + \cos x - \frac{1}{9} \cos 3x \right)$

Problem 77. $\int x \tan^2 x dx$

Solution. $\int x \tan^2 x \, dx = \int x(\sec^2 x - 1) \, dx = \underbrace{\int x \sec^2 x \, dx}_{\text{Let } u = x, dv = \sec^2 x \, dx} - \frac{x^2}{2} = x \tan x + \ln |\cos x| - \frac{x^2}{2}$

Problem 78. $\int_{-1}^1 x^8 \sin x \, dx$

Solution. $\int_{-1}^1 x^8 \sin x \, dx = 0$ for $x^8 \sin x$ is odd.

Problem 79. $\int x \sin^{-1} x \, dx$

Solution. $\underbrace{\int x \sin^{-1} x \, dx}_{\text{Let } u = \sin^{-1} x, dv = dx} = \sin^{-1} x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{\sqrt{1-x^2}} \, dx = \frac{x^2 \sin^{-1} x}{2} - \frac{1}{2} \int \frac{(x^2 - 1) + 1}{\sqrt{1-x^2}} \, dx = \frac{x^2 \sin^{-1} x}{2} +$
 $\frac{1}{2} \underbrace{\int \sqrt{1-x^2} \, dx}_{\text{Let } x = \sin \theta} - \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} \, dx = \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left(\frac{\theta}{2} + \frac{1}{4} \sin 2\theta \right) - \frac{1}{2} \sin^{-1} x = \frac{x^2 \sin^{-1} x}{2} + \frac{1}{4} \left(\sin^{-1} x + x \sqrt{1-x^2} \right) -$
 $\frac{1}{2} \sin^{-1} x = \frac{x^2 \sin^{-1} x}{2} + \frac{x \sqrt{1-x^2}}{4} - \frac{\sin^{-1} x}{4}$

Problem 80. $\int x^{-2} \tan^{-1} x \, dx$

Solution. $\underbrace{\int x^{-2} \tan^{-1} x \, dx}_{\text{Let } u = \tan^{-1} x, dv = x^{-2} \, dx} = \tan^{-1} x \cdot \frac{-1}{x} + \int \frac{1}{x} \cdot \frac{1}{1+x^2} \, dx = -\frac{\tan^{-1} x}{x} + \int \left(\frac{1}{x} - \frac{x}{1+x^2} \right) \, dx = -\frac{\tan^{-1} x}{x} +$
 $\ln |x| - \underbrace{\int \frac{x}{1+x^2} \, dx}_{\text{Let } u = 1+x^2} = -\frac{\tan^{-1} x}{x} + \ln |x| - \int \frac{\frac{du}{2}}{u} = -\frac{\tan^{-1} x}{x} + \ln |x| - \frac{1}{2} \ln(1+x^2)$

Problem 81. $\int_1^3 \frac{\tan^{-1} \sqrt{x}}{\sqrt{x}} \, dx$

Solution. $\underbrace{\int_1^3 \frac{\tan^{-1} \sqrt{x}}{\sqrt{x}} \, dx}_{\text{Let } u = \sqrt{x}} = \int_1^{\sqrt{3}} \frac{\tan^{-1} u}{u} \cdot 2u \, du = 2 \int_1^{\sqrt{3}} \tan^{-1} u \, du.$ From $\underbrace{\int \tan^{-1} w \, dw}_{\text{Let } u = \tan^{-1} w, dv = dw} = \tan^{-1} w \cdot w -$
 $\int w \cdot \frac{1}{1+w^2} \, dw = w \tan^{-1} w - \frac{1}{2} \ln(1+w^2), 2 \int_1^{\sqrt{3}} \tan^{-1} u \, du = (2u \tan^{-1} u - \ln(1+u^2)) \Big|_1^{\sqrt{3}} = \frac{(4\sqrt{3}-3)\pi}{6} - \ln 2$

Problem 82. $\int \sqrt{1-x^2} \sin^{-1} x \, dx$

Solution. First note that $\underbrace{\int \sqrt{1-x^2} \, dx}_{\text{Let } u = \sin \theta} = \int \cos \theta \cdot \cos \theta \, d\theta = \frac{1}{2} \int (1 + \cos 2\theta) \, d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} = \frac{1}{2} (\theta + \sin \theta \cos \theta) =$
 $\frac{1}{2} (\sin^{-1} x + x \sqrt{1-x^2}),$ then $\underbrace{\int \sqrt{1-x^2} \sin^{-1} x \, dx}_{\text{Let } u = \sin^{-1} x, dv = \sqrt{1-x^2} \, dx} = \sin^{-1} x \cdot \frac{1}{2} (\sin^{-1} x + x \sqrt{1-x^2}) - \frac{1}{2} \int (\sin^{-1} x + x \sqrt{1-x^2}) \cdot$
 $\frac{1}{\sqrt{1-x^2}} \, dx = \frac{1}{2} \sin^{-1} x (\sin^{-1} x + x \sqrt{1-x^2}) - \frac{1}{2} \int \left(\frac{\sin^{-1} x}{\sqrt{1-x^2}} + x \right) \, dx = \frac{1}{2} \sin^{-1} x (\sin^{-1} x + x \sqrt{1-x^2}) - \frac{x^2}{4} -$
 $\frac{1}{2} \underbrace{\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} \, dx}_{\text{Let } u = \sin^{-1} x} = \frac{1}{2} \sin^{-1} x (\sin^{-1} x + x \sqrt{1-x^2}) - \frac{x^2}{4} - \frac{1}{4} (\sin^{-1} x)^2 = \frac{1}{4} (\sin^{-1} x)^2 + \frac{1}{2} x \sqrt{1-x^2} \sin^{-1} x - \frac{x^2}{4}$

Problem 83. $\int_0^1 \frac{\sin^{-1} \sqrt{x}}{\sqrt{x(1-x)}} \, dx$

Solution. By $(\sin^{-1}\sqrt{x})' = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(1-x)}}$, $\int_0^1 \frac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}} dx = 2 \int_0^1 \sin^{-1}\sqrt{x} d(\sin^{-1}\sqrt{x})$
 $= (\sin^{-1}\sqrt{x})^2 \Big|_0^1 = \frac{\pi^2}{4}$

Problem 84. $\int e^{x+e^x} dx$

Solution. $\int e^{x+e^x} dx = \underbrace{\int e^{e^x} \cdot e^x dx}_{\text{Let } u = e^x} = \int e^u du = e^u = e^{e^x}$

Problem 85. $\int \frac{1}{e^{3x} - e^x} dx$

Solution. $\int \frac{1}{e^{3x} - e^x} dx = \underbrace{\int \frac{e^{-3x}}{1 - e^{-2x}} dx}_{\text{Let } u = e^{-x}} = \int \frac{u^2}{u^2 - 1} du = \int \left(1 + \frac{1}{2} \left(\frac{1}{u-1} - \frac{1}{u+1}\right)\right) du = u + \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| =$
 $e^{-x} + \frac{1}{2} \ln \left| \frac{e^{-x} - 1}{e^{-x} + 1} \right|$

Problem 86. $\int \frac{e^{2x}}{1 + e^x} dx$

Solution. $\underbrace{\int \frac{e^{2x}}{1 + e^x} dx}_{\text{Let } u = e^x} = \int \frac{u}{1 + u} du = \int \left(1 - \frac{1}{1 + u}\right) du = u - \ln |1 + u| = e^x - \ln(1 + e^x)$

Problem 87. $\int \frac{1}{1 + 2e^x - e^{-x}} dx$

Solution. $\int \frac{1}{1 + 2e^x - e^{-x}} dx = \underbrace{\int \frac{e^x}{e^x + 2e^{2x} - 1} dx}_{\text{Let } u = e^x} = \int \frac{1}{u + 2u^2 - 1} du = \int \left(\frac{-\frac{1}{3}}{u+1} + \frac{\frac{2}{3}}{2u-1}\right) du = \frac{1}{3} \ln \left| \frac{2u-1}{u+1} \right| =$
 $\frac{1}{3} \ln \left| \frac{2e^x - 1}{e^x + 1} \right|$

Problem 88. $\int \frac{e^{2x}}{1 + e^{4x}} dx$

Solution. $\underbrace{\int \frac{e^{2x}}{1 + e^{4x}} dx}_{\text{Let } u = e^{2x}} = \frac{1}{2} \int \frac{du}{1 + u^2} = \frac{1}{2} \tan^{-1} u = \frac{1}{2} \tan^{-1} e^{2x}$

Problem 89. $\int e^x \sqrt{1 + e^x} dx$

Solution. $\underbrace{\int e^x \sqrt{1 + e^x} dx}_{\text{Let } u = 1 + e^x} = \int \sqrt{u} du = \frac{2}{3} u^{\frac{3}{2}} = \frac{2}{3} (1 + e^x)^{\frac{3}{2}}$

Problem 90. $\int \sqrt{1 + e^x} dx$

Solution. $\underbrace{\int \sqrt{1 + e^x} dx}_{\text{Let } u = \sqrt{1 + e^x}} = \int u \cdot \frac{2u}{u^2 - 1} du = \int \frac{2(u^2 - 1) + 2}{u^2 - 1} du = 2u + \int \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du = 2u + \ln \left| \frac{u-1}{u+1} \right| =$
 $2\sqrt{1 + e^x} + \ln \left| \frac{\sqrt{1 + e^x} - 1}{\sqrt{1 + e^x} + 1} \right|$

Problem 91. $\int \frac{2e^{2x} - e^x}{\sqrt{3e^{2x} - 6e^x - 1}} dx$

Solution.
$$\int \frac{2e^{2x} - e^x}{\sqrt{3e^{2x} - 6e^x - 1}} dx = \int \frac{(2e^{2x} - 2e^x) + e^x}{\sqrt{3e^{2x} - 6e^x - 1}} dx = \frac{1}{3} \underbrace{\int \frac{6e^{2x} - 6e^x}{\sqrt{3e^{2x} - 6e^x - 1}} dx}_{\text{Let } u = 3e^{2x} - 6e^x - 1} + \underbrace{\int \frac{e^x}{\sqrt{3e^{2x} - 6e^x - 1}} dx}_{\text{Let } w = e^x}$$

$$\frac{1}{3} \int \frac{du}{\sqrt{u}} + \int \frac{dw}{\sqrt{3w^2 - 6w - 1}} = \frac{2}{3} \sqrt{3e^{2x} - 6e^x - 1} + \int \frac{dw}{\sqrt{3(w-1)^2 - 4}} = \frac{2}{3} \sqrt{3e^{2x} - 6e^x - 1}$$

$$+ \frac{1}{\sqrt{3}} \int \frac{dw}{\sqrt{(w-1)^2 - (\frac{2}{\sqrt{3}})^2}} = \frac{2}{3} \sqrt{3e^{2x} - 6e^x - 1} + \frac{1}{\sqrt{3}} \ln \left| \frac{e^x - 1}{\frac{2}{\sqrt{3}}} + \frac{\sqrt{(e^x - 1)^2 - (\frac{2}{\sqrt{3}})^2}}{\frac{2}{\sqrt{3}}} \right| = \frac{2}{3} \sqrt{3e^{2x} - 6e^x - 1} +$$

$$\frac{1}{\sqrt{3}} \ln \left| \frac{\sqrt{3}(e^x - 1)}{2} + \frac{\sqrt{3(e^x - 1)^2 - 4}}{2} \right| \text{ by } \int \frac{1}{\sqrt{x^2 - a^2}} dx = \int \frac{a \sec \theta \tan \theta}{a \tan \theta} d\theta = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| =$$

$$\ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right|$$

Problem 92. $\int e^{\sqrt[3]{x}} dx$

Solution.
$$\underbrace{\int e^{\sqrt[3]{x}} dx}_{\text{Let } u = \sqrt[3]{x}} = \int e^u \cdot 3u^2 du = (3u^2 - 6u + 6) e^u = (3(\sqrt[3]{x})^2 - 6\sqrt[3]{x} + 6) e^{\sqrt[3]{x}}$$

Problem 93. $\int \sqrt{x} e^{\sqrt{x}} dx$

Solution.
$$\underbrace{\int \sqrt{x} e^{\sqrt{x}} dx}_{\text{Let } u = \sqrt{x}} = \int u \cdot e^u \cdot 2u du = \int 2u^2 e^u du = (2u^2 - 4u + 4) e^u = (2x - 4\sqrt{x} + 4) e^{\sqrt{x}}$$

Problem 94. $\int x^5 e^{-x^3} dx$

Solution.
$$\underbrace{\int x^5 e^{-x^3} dx}_{\text{Let } u = x^3} = \frac{1}{3} \int u \cdot e^{-u} du = -\frac{u+1}{3} e^u = -\frac{x^3+1}{3} e^{-x^3}$$

Problem 95. $\int x^3 e^{-2x} dx$

Solution.
$$\int x^3 e^{-2x} dx = -\left(\frac{x^3}{2} + \frac{3x^2}{4} + \frac{3x}{4} + \frac{3}{8}\right) e^{-2x}$$

Problem 96. $\int 27^{3x+1} dx$

Solution.
$$\int 27^{3x+1} dx = \int e^{(3x+1) \ln 27} dx = 27 \int e^{3 \ln 27 \cdot x} dx = \frac{27}{3 \ln 27} e^{3 \ln 27 \cdot x} = \frac{27^{3x+1}}{3 \ln 27}$$

Problem 97. $\int_{-1}^1 \frac{e^{\tan^{-1} x}}{1+x^2} dx$

Solution.
$$\underbrace{\int_{-1}^1 \frac{e^{\tan^{-1} x}}{1+x^2} dx}_{\text{Let } u = \tan^{-1} x} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^u du = e^{\frac{\pi}{4}} - e^{-\frac{\pi}{4}}$$

Problem 98. $\int e^x \sin(ax-3) dx$

Solution.
$$\int e^x \sin(ax-3) dx = -\frac{1}{a} e^x \cos(ax-3) + \frac{1}{a^2} e^x \sin(ax-3) - \frac{1}{a^2} \int e^x \sin(ax-3) dx$$

$$\implies \int e^x \sin(ax-3) dx = \frac{e^x (\sin(ax-3) - a \cos(ax-3))}{a^2 + 1}$$

Problem 99. $\int \frac{xe^x}{\sqrt{1+e^x}} dx$

Solution. $\underbrace{\int \frac{xe^x}{\sqrt{1+e^x}} dx}_{\text{Let } u=x, dv=\frac{e^x}{\sqrt{1+e^x}} dx} = x \cdot 2\sqrt{1+e^x} - 2 \int \sqrt{1+e^x} dx = 2x\sqrt{1+e^x} - 2 \int \sqrt{1+e^x} dx = 2x\sqrt{1+e^x} - 4\sqrt{1+e^x} - 2 \ln \left| \frac{\sqrt{1+e^x} - 1}{\sqrt{1+e^x} + 1} \right|$

Problem 100. $\int_0^\infty xe^{-x} \sin x dx$

Solution. From $\int e^{ax} \sin bx dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$, $\int e^{ax} \cos bx dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$, $\int e^{-x} \sin x dx = -\frac{e^{-x}(\sin x + \cos x)}{2}$, $\int e^{-x} \cos x dx = \frac{e^{-x}(\sin x - \cos x)}{2}$, by tabulation $\int xe^{-x} \sin x dx = x \cdot -\frac{e^{-x}(\sin x + \cos x)}{2} + \frac{1}{2} \left(-\frac{e^{-x}(\sin x + \cos x)}{2} + \frac{e^{-x}(\sin x - \cos x)}{2} \right) = -\frac{e^{-x}}{2}(x \sin x + (x+1) \cos x)$, so $\int_0^\infty xe^{-x} \sin x dx = \frac{1}{2}$.

Problem 101. $\int x^2 \ln(1+x) dx$

Solution. $\underbrace{\int x^2 \ln(1+x) dx}_{\text{Let } u=\ln(1+x), dv=x^2 dx} = \ln(1+x) \cdot \frac{x^3}{3} - \frac{1}{3} \int x^3 \cdot \frac{1}{1+x} dx = \ln(1+x) \cdot \frac{x^3}{3} - \frac{1}{3} \int \left(x^2 - x + 1 - \frac{1}{x+1} \right) dx = \ln(1+x) \cdot \frac{x^3}{3} - \frac{x^3}{9} + \frac{x^2}{6} - \frac{x}{3} + \frac{\ln(x+1)}{3}$

Problem 102. $\int \ln \sqrt{x-1} dx$

Solution. $\underbrace{\int \ln \sqrt{x-1} dx}_{\text{Let } u=\ln \sqrt{x-1}} = \int u \cdot 2e^{2u} du = \left(u - \frac{1}{2}\right) e^{2u} = \left(\ln \sqrt{x-1} - \frac{1}{2}\right)(x-1)$

Problem 103. $\int \ln(\sqrt{x} + \sqrt{1+x}) dx$

Solution. $\underbrace{\int \ln(\sqrt{x} + \sqrt{1+x}) dx}_{\text{Let } u=\ln(\sqrt{x}+\sqrt{1+x}), dv=dx} = \ln(\sqrt{x} + \sqrt{1+x}) \cdot x - \int x \cdot \frac{1}{2\sqrt{x(1+x)}} dx = x \ln(\sqrt{x} + \sqrt{1+x}) - \underbrace{\frac{1}{2} \int \sqrt{\frac{x}{1+x}} dx}_{\text{Let } u=\sqrt{1+x}} = x \ln(\sqrt{x} + \sqrt{1+x}) - \frac{1}{2} \int \frac{\sqrt{u^2-1}}{u} \cdot 2u du = x \ln(\sqrt{x} + \sqrt{1+x}) - \underbrace{\int \sqrt{u^2-1} du}_{\text{Let } u=\sec \theta} = x \ln(\sqrt{x} + \sqrt{1+x}) - \frac{1}{2} (u\sqrt{u^2-1} - \ln|u + \sqrt{u^2-1}|) = \left(x + \frac{1}{2}\right) \ln(\sqrt{x} + \sqrt{1+x}) - \frac{1}{2} \sqrt{x(1+x)}$

Problem 104. $\int \frac{\sqrt{1+\ln x}}{x \ln x} dx$

Solution. $\underbrace{\int \frac{\sqrt{1+\ln x}}{x \ln x} dx}_{\text{Let } u=\ln x} = \underbrace{\int \frac{\sqrt{1+u}}{u} du}_{\text{Let } v=\sqrt{1+u}} = \int \frac{v}{v^2-1} 2v dv = \int \frac{2v^2}{v^2-1} dv = \int \left(\frac{2v^2-2}{v^2-1} + \frac{2}{v^2-1} \right) dv = 2v + \int \left(\frac{1}{v-1} - \frac{1}{v+1} \right) dv = 2v + \ln \left| \frac{v-1}{v+1} \right| = 2\sqrt{1+\ln x} + \ln \left| \frac{\sqrt{1+\ln x}-1}{\sqrt{1+\ln x}+1} \right|$

Problem 105. $\int \frac{dx}{x\sqrt{\ln x}}$

Solution. $\underbrace{\int \frac{dx}{x\sqrt{\ln x}}}_{\text{Let } u=\ln x} = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} = 2\sqrt{\ln x}$

Problem 106. $\int \frac{\ln(x+1)}{x^2} dx$

Solution.
$$\underbrace{\int \frac{\ln(x+1)}{x^2} dx}_{\text{Let } u = \ln(x+1), dv = \frac{1}{x^2} dx} = \ln(x+1) \cdot \frac{-1}{x} + \int \frac{1}{x} \cdot \frac{1}{x+1} dx = \frac{-\ln(x+1)}{x} + \int \left(\frac{1}{x} - \frac{1}{x+1} \right) dx$$

$$= \frac{-\ln(x+1)}{x} + \ln \frac{x}{x+1}$$

Problem 107.
$$\int \frac{x \ln x}{\sqrt{x^2-1}} dx$$

Solution.
$$\underbrace{\int \frac{x \ln x}{\sqrt{x^2-1}} dx}_{\text{Let } u = \ln x, dv = \frac{x}{\sqrt{x^2-1}} dx} = \ln x \cdot \sqrt{x^2-1} - \int \sqrt{x^2-1} \cdot \frac{1}{x} dx = \sqrt{x^2-1} \ln x - \underbrace{\int \frac{\sqrt{x^2-1}}{x} dx}_{\text{Let } u = \sqrt{x^2-1}}$$

$$\int \frac{u^2}{u^2+1} du = \sqrt{x^2-1} \ln x - \int \left(1 - \frac{1}{u^2+1} \right) du = \sqrt{x^2-1} \ln x - u + \tan^{-1} u = \sqrt{x^2-1} \ln x - \sqrt{x^2-1} + \tan^{-1} \sqrt{x^2-1}$$

Problem 108.
$$\int \frac{dx}{x(1+\ln x)\sqrt{(\ln x)(2+\ln x)}}$$

Solution.
$$\underbrace{\int \frac{dx}{x(1+\ln x)\sqrt{(\ln x)(2+\ln x)}}}_{\text{Let } u = 1+\ln x} = \int \frac{du}{u\sqrt{u^2-1}} = \int \frac{dw}{w} = \frac{w^2}{2} = \frac{(\ln \sin x)^2}{2}$$

Problem 109.
$$\int \cot x \ln(\sin x) dx$$

Solution.
$$\int \cot x \ln \sin x dx = \underbrace{\int \frac{\cos x}{\sin x} \ln \sin x dx}_{\text{Let } u = \sin x} = \underbrace{\int \frac{\ln u}{u} du}_{\text{Let } w = \ln u} = \int w dw = \frac{w^2}{2} = \frac{(\ln \sin x)^2}{2}$$

Problem 110.
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\sin x \cos x} dx$$

Solution. Note that $(\ln(\tan x))' = \frac{1}{\sin x \cos x}$, so $\int \frac{\ln(\tan x)}{\sin x \cos x} dx = \int \ln(\tan x) d(\ln(\tan x)) = \frac{1}{2}(\ln(\tan x))^2$;

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\sin x \cos x} dx = \frac{1}{2}(\ln(\tan x))^2 \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{(\ln 3)^2}{8}.$$

Problem 111.
$$\int \frac{\cot x}{\ln \sin x} dx$$

Solution.
$$\int \frac{\cot x}{\ln \sin x} dx = \underbrace{\int \frac{\cos x}{\sin x \ln \sin x} dx}_{\text{Let } u = \sin x} = \underbrace{\int \frac{du}{u \ln u}}_{\text{Let } w = \ln u} = \int \frac{dw}{w} = \ln |w| = \ln |\ln(\sin x)|$$

Problem 112.
$$\int (1+\ln x)\sqrt{1+(x \ln x)^2} dx$$

Solution.
$$\underbrace{\int (1+\ln x)\sqrt{1+(x \ln x)^2} dx}_{\text{Let } u = x \ln x} = \int \sqrt{1+u^2} du = \frac{u\sqrt{u^2+1} + \ln |\sqrt{u^2+1} + u|}{2}$$

$$= \frac{x \ln x \sqrt{(x \ln x)^2 + 1} + \ln |\sqrt{(x \ln x)^2 + 1} + x \ln x|}{2}$$