A Collection of Integrals

Problem. $\int_{-2}^{2} |x^2-4x| \, \mathrm{d}x$

Solution. $\int_{-2}^{2} |x^2-4x| \, \mathrm{d}x = \int_{-2}^{2} |x(x-4)| \, \mathrm{d}x = \int_{-2}^{0} |x(x-4)| \, \mathrm{d}x + \int_{0}^{2} |x(x-4)| \, \mathrm{d}x = \int_{-2}^{0} x(x-4) \, \mathrm{d}x - \int_{0}^{2} x(x-4) \, \mathrm{d}x = \int_{-2}^{0} (x^2-4x) \, \mathrm{d}x + \int_{0}^{2} (4x-x^2) \, \mathrm{d}x = \int_{-2}^{0} (x^2-4x) \, \mathrm{d}x + \int_{0}^{2} (4x-x^2) \, \mathrm{d}x = \left(\frac{x^3}{3} - 2x^2\right)\Big|_{-2}^{0} + \left(2x^2 - \frac{x^3}{3}\right)\Big|_{0}^{2} = 0 - \left(\frac{(-2)^3}{3} - 2(-2)^2\right) + \left(2 \cdot 2^2 - \frac{2^3}{3}\right) = 16$

Problem. $\int_{0}^{5} \frac{3x-1}{x+2} \, \mathrm{d}x$

Solution. $\int_{0}^{5} \frac{3x-1}{x+2} \, \mathrm{d}x = \int_{0}^{5} \frac{3(x+2)-7}{x+2} \, \mathrm{d}x = 3 \cdot 5 - 7\int_{0}^{5} \frac{1}{x+2} \, \mathrm{d}x = 15 - 7\ln|x+2|\Big|_{0}^{5} = 15 - 7\ln\frac{7}{2}$

Problem. $\int_{0}^{4} \frac{x-1}{x^2-4x-5} \, \mathrm{d}x$

Solution. $\int_{0}^{4} \frac{x-1}{x^2-4x-5} \, \mathrm{d}x = \int_{0}^{4} \frac{(x+1)-2}{(x-5)(x+1)} \, \mathrm{d}x = \int_{0}^{4} \frac{1}{x-5} \, \mathrm{d}x - 2\int_{0}^{4} \frac{1}{(x-5)(x+1)} \, \mathrm{d}x = \int_{0}^{4} \frac{1}{x-5} \, \mathrm{d}x - \frac{1}{3}\int_{0}^{4} \left(\frac{1}{x-5} - \frac{1}{x+1}\right) \mathrm{d}x = \frac{2}{3}\int_{0}^{4} \frac{1}{x-5} \, \mathrm{d}x + \frac{1}{3}\int_{0}^{4} \frac{1}{x+1} \, \mathrm{d}x = \frac{2}{3}\ln|x-5|\Big|_{0}^{4} + \frac{1}{3}\ln|x+1|\Big|_{0}^{4} = -\frac{2}{3}\ln 5 + \frac{1}{3}\ln 5 = -\frac{1}{3}\ln 5.$

Problem. $\int \frac{x-1}{x^2-4x+5} \, \mathrm{d}x$

Solution. $\int \frac{x-1}{x^2-4x+5} \, \mathrm{d}x = \int \frac{(x-2)+1}{(x-2)^2+1} \, \mathrm{d}x = \underbrace{\int \frac{x-2}{(x-2)^2+1} \, \mathrm{d}x}_{\text{Let } u=(x-2)^2+1} + \int \frac{1}{(x-2)^2+1} \, \mathrm{d}x = \frac{1}{2}\ln((x-2)^2+1) + \tan^{-1}(x-2) = \frac{1}{2}\ln(x^2-4x+5) + \tan^{-1}(x-2)$

Problem. $\int \frac{x-1}{x^2-2x+5} \, \mathrm{d}x$

Solution. $\underbrace{\int \frac{x-1}{x^2-2x+5} \, \mathrm{d}x}_{\text{Let } u=x^2-2x+5} = \frac{1}{2}\ln(x^2-2x+5)$

Problem. $\int_{0}^{2} \frac{2x}{(x-3)^2} \, \mathrm{d}x$

Solution. $\int_{0}^{2} \frac{2x}{(x-3)^2} \, \mathrm{d}x = \int_{0}^{2} \left(\frac{2}{x-3} + \frac{6}{(x-3)^2}\right) \mathrm{d}x = 2\ln|x-3|\Big|_{0}^{2} - \frac{6}{x-3}\Big|_{0}^{2} = -2\ln 3 - 6\left(-1 + \frac{1}{3}\right) = 4 - 2\ln 3$

Problem. $\int \frac{3x^2-2}{x^3-2x-8} \, \mathrm{d}x$

Solution. $\underbrace{\int \frac{3x^2-2}{x^3-2x-8} \, \mathrm{d}x}_{\text{Let } u=x^3-2x-8} = \ln|x^3-2x-8|$

Problem. $\int_{2}^{3} \frac{x^3+1}{x^3-x^2} \, \mathrm{d}x$

Solution. $\int_{2}^{3} \frac{x^3+1}{x^3-x^2} \, \mathrm{d}x = \int_{2}^{3} \frac{(x^3-x^2)+x^2+1}{x^3-x^2} \, \mathrm{d}x = \int_{2}^{3} \left(1 + \frac{x^2}{x^3-x^2} + \frac{1}{x^3-x^2}\right) \mathrm{d}x = 1 + \int_{2}^{3} \frac{1}{x-1} \, \mathrm{d}x + \int_{2}^{3} \left(-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}\right) \mathrm{d}x = 1 + \ln|x-1|\Big|_{2}^{3} - \ln|x|\Big|_{2}^{3} + \frac{1}{x}\Big|_{2}^{3} + \ln|x-1|\Big|_{2}^{3} = 1 + \ln 2 - \frac{1}{6} - \ln\frac{3}{2} + \ln 2 = \frac{5}{6} + 3\ln 2 - \ln 3$

Problem. $\displaystyle\int \frac{1}{(x-2)(x^2+4)}\,\mathrm{d}x$

Solution. $\displaystyle\int \frac{1}{(x-2)(x^2+4)}\,\mathrm{d}x = \frac{1}{8}\int\left(\frac{1}{x-2}-\frac{x+2}{x^2+4}\right)\mathrm{d}x = \frac{1}{8}\ln|x-2| - \frac{1}{16}\ln(x^2+4) - \frac{1}{8}\tan^{-1}\frac{x}{2}$

Problem. $\displaystyle\int \frac{x}{x^4-a^4}\,\mathrm{d}x$

Solution. $\displaystyle\underbrace{\int \frac{x}{x^4-a^4}\,\mathrm{d}x}_{\text{Let } u=x^2} = \frac{1}{2}\int \frac{1}{u^2-a^4}\,\mathrm{d}u = \frac{1}{4a^2}\int\left(\frac{1}{u-a^2}-\frac{1}{u+a^2}\right)\mathrm{d}u = \frac{1}{4a^2}\ln\left|\frac{u-a^2}{u+a^2}\right| = \frac{1}{4a^2}\ln\left|\frac{x^2-a^2}{x^2+a^2}\right|$

Problem. $\displaystyle\int \frac{x}{x^4+x^2+1}\,\mathrm{d}x$

Solution. $\displaystyle\underbrace{\int \frac{x}{x^4+x^2+1}\,\mathrm{d}x}_{\text{Let } u=x^2} = \frac{1}{2}\int \frac{1}{u^2+u+1}\,\mathrm{d}u = \frac{1}{2}\int \frac{1}{(u+\frac{1}{2})^2+\frac{3}{4}}\,\mathrm{d}u = \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{2u+1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x^2+1}{\sqrt{3}}$

Problem. $\displaystyle\int \frac{1}{x(x^4+1)}\,\mathrm{d}x$

Solution. $\displaystyle\underbrace{\int \frac{1}{x(x^4+1)}\,\mathrm{d}x}_{\text{Let } u=x^2} = \frac{1}{2}\int \frac{1}{u(u^2+1)}\,\mathrm{d}u = \frac{1}{2}\int\left(\frac{1}{u}-\frac{u}{u^2+1}\right)\mathrm{d}u = \frac{1}{2}\ln|u| - \frac{1}{4}\ln(u^2+1) = \frac{1}{4}\ln\frac{x^4}{x^4+1}$

Problem. $\displaystyle\int_1^\infty \frac{x^2-3}{x^4+2x^2+9}\,\mathrm{d}x$

Solution. Note that $x^4+2x^2+9 = (x^2+3)^2-(2x)^2 = (x^2-2x+3)(x^2+2x+3)$, $\displaystyle\int_1^\infty \frac{x^2-3}{x^4+2x^2+9}\,\mathrm{d}x = \frac{1}{2}\int_1^\infty\left(\frac{x-1}{x^2-2x+3}-\frac{x+1}{x^2+2x+3}\right)\mathrm{d}x = \frac{1}{4}\ln\left|\frac{x^2-2x+3}{x^2+2x+3}\right|\Bigg|_1^\infty = \frac{\ln 3}{4}$

Problem. $\displaystyle\int \frac{1}{x^6-1}\,\mathrm{d}x$

Solution. $\displaystyle\frac{1}{x^6-1} = \frac{1}{2}\left(\frac{1}{x^3-1}-\frac{1}{x^3+1}\right)$. $\displaystyle\frac{1}{x^3-1} = \frac{a}{x-1}+\frac{bx+C}{x^2+x+1} \implies 1 = a(x^2+x+1)+(bx+C)(x-1) \implies a=\frac{1}{3},\ b=-\frac{1}{3},\ c=-\frac{2}{3}$; $\displaystyle\frac{1}{x^3+1} = \frac{a}{x+1}+\frac{bx+C}{x^2-x+1} \implies 1 = a(x^2-x+1)+(bx+C)(x+1) \implies a=\frac{1}{3},\ b=-\frac{1}{3},\ c=\frac{2}{3}$. So $\displaystyle\frac{1}{x^6-1} = \frac{1}{6}\left(\frac{1}{x-1}-\frac{x+2}{x^2+x+1}-\frac{1}{x+1}+\frac{x-2}{x^2-x+1}\right)$, $\displaystyle\int\frac{1}{x^6-1}\,\mathrm{d}x = \frac{1}{6}\int\left(\frac{1}{x-1}-\frac{x+2}{x^2+x+1}-\frac{1}{x+1}+\frac{x-2}{x^2-x+1}\right)\mathrm{d}x = \frac{1}{6}\int\left(\frac{1}{x-1}-\frac{\frac{1}{2}(2x+1)+\frac{3}{2}}{x^2+x+1}-\frac{1}{x+1}+\frac{\frac{1}{2}(2x-1)-\frac{3}{2}}{x^2-x+1}\right)\mathrm{d}x = \frac{1}{6}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{12}\ln\left|\frac{x^2-x+1}{x^2+x+1}\right| - \frac{1}{2\sqrt{3}}\left(\tan^{-1}\frac{2x+1}{\sqrt{3}}+\tan^{-1}\frac{2x-1}{\sqrt{3}}\right)$

Problem. $\displaystyle\int \frac{x^3}{(x+1)^{10}}\,\mathrm{d}x$

Solution. $\displaystyle\underbrace{\int \frac{x^3}{(x+1)^{10}}\,\mathrm{d}x}_{\text{Let } u=x+1} = \int\frac{(u-1)^3}{u^{10}}\,\mathrm{d}u = \int\frac{u^3-3u^2+3u-1}{u^{10}}\,\mathrm{d}u = \int\left(u^{-7}-3u^{-8}+3u^{-9}-u^{-10}\right)\mathrm{d}u = \frac{1}{-6}u^{-6}+3\,\frac{1}{7}u^{-7}-3\,\frac{1}{8}u^{-8}+\frac{1}{9}u^{-9} = -\frac{1}{6(x+1)^6}+\frac{3}{7(x+1)^7}-\frac{3}{8(x+1)^8}+\frac{1}{9(x+1)^9} = -\frac{84x^3+36x^2+9x+1}{504\,(x+1)^9}$

Problem. $\displaystyle\int \frac{x^4}{x^{10}+16}\,\mathrm{d}x$

Solution. $\displaystyle\underbrace{\int \frac{x^4}{x^{10}+16}\,\mathrm{d}x}_{\text{Let } u=x^5} = \frac{1}{5}\int\frac{1}{u^2+16}\,\mathrm{d}u = \frac{1}{20}\tan^{-1}\frac{u}{4} = \frac{1}{20}\tan^{-1}\frac{x^5}{4}$

Problem. $\displaystyle\int_{-1}^{\infty}\left(\frac{x^4}{1+x^6}\right)^2\mathrm{d}x$

Solution. $\displaystyle\int_{-1}^{\infty}\left(\frac{x^4}{1+x^6}\right)^2\mathrm{d}x=\underbrace{\int_{-1}^{\infty}\frac{x^8}{(1+x^6)^2}\mathrm{d}x}_{\text{Let } w=x^3}=\frac{1}{3}\int_{-1}^{\infty}\frac{w^2}{(1+w^2)^2}\mathrm{d}w$. Note that by evaluating $\displaystyle\int\frac{1}{1+w^2}\mathrm{d}w$ through integration by parts: let $u=\dfrac{1}{1+w^2}$, then $\mathrm{d}u=-\dfrac{2w}{(1+w^2)^2}\mathrm{d}w$; let $\mathrm{d}v=\mathrm{d}w$, then $v=w$. So $\displaystyle\int\frac{1}{1+w^2}\mathrm{d}w=\frac{1}{1+w^2}\cdot w+\int w\cdot\frac{2w}{(1+w^2)^2}\mathrm{d}w\implies\int\frac{w^2}{(1+w^2)^2}\mathrm{d}w=\frac{1}{2}\int\frac{1}{1+w^2}\mathrm{d}w-\frac{w}{2(1+w^2)}=\frac{1}{2}\Big(\tan^{-1}w-\frac{w}{1+w^2}\Big)$; Hence $\displaystyle\frac{1}{3}\int_{-1}^{\infty}\frac{w^2}{(1+w^2)^2}\mathrm{d}w=\frac{1}{6}\left(\tan^{-1}w-\frac{w}{1+w^2}\right)\Big|_{-1}^{\infty}=\frac{1}{6}\Big(\Big(\frac{\pi}{2}-0\Big)-\Big(-\frac{\pi}{4}+\frac{1}{2}\Big)\Big)=\frac{\pi}{8}-\frac{1}{12}$

Problem. $\displaystyle\int\frac{1}{x^n-x}\mathrm{d}x$

Solution. $\displaystyle\int\frac{1}{x^n-x}\mathrm{d}x=\int\frac{1}{x(x^{n-1}-1)}\mathrm{d}x=\int\left(\frac{x^{n-2}}{x^{n-1}-1}-\frac{1}{x}\right)\mathrm{d}x=\underbrace{\int\frac{x^{n-2}}{x^{n-1}-1}\mathrm{d}x}_{\text{Let } u=x^{n-1}-1}-\int\frac{1}{x}\mathrm{d}x=\frac{1}{n-1}\ln|x^{n-1}-1|-\ln|x|$

Problem. $\displaystyle\int_0^{\infty}\frac{x^n}{(1+x^2)^{1+\frac{n}{2}}}\mathrm{d}x,\ \forall\, n\in\mathbb{N}.$

Solution. $\displaystyle\underbrace{\int_0^{\infty}\frac{x^n}{(1+x^2)^{1+\frac{n}{2}}}\mathrm{d}x}_{\text{Let } x=\tan\theta}=\int_0^{\frac{\pi}{2}}\frac{\tan^n\theta}{\sec^2\theta\cdot\sec^n\theta}\cdot\sec^2\theta\,\mathrm{d}\theta=\int_0^{\frac{\pi}{2}}\sin^n\theta\,\mathrm{d}\theta=\begin{cases}\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3} & n \text{ odd}\\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2} & n\text{ even}\end{cases}$

Problem. $\displaystyle\int x\sqrt[3]{x+c}\,\mathrm{d}x$

Solution. $\displaystyle\underbrace{\int x\sqrt[3]{x+c}\,\mathrm{d}x}_{\text{Let } u=\sqrt[3]{x+c}}=\int(u^3-c)\cdot u\cdot 3u^2\,\mathrm{d}u=3\int u^6\,\mathrm{d}u-3c\int u^3\,\mathrm{d}u=\frac{3}{7}u^7-\frac{3c}{4}u^4=\frac{3}{7}(\sqrt[3]{x+c})^7-\frac{3c}{4}(\sqrt[3]{x+c})^4$

Problem. $\displaystyle\int_0^1(1+\sqrt{x})^8\,\mathrm{d}x$

Solution. $\displaystyle\underbrace{\int_0^1(1+\sqrt{x})^8\,\mathrm{d}x}_{\text{Let } u=1+\sqrt{x}}=\int_1^2u^8\cdot 2(u-1)\,\mathrm{d}u=2\int_1^2(u^9-u^8)\,\mathrm{d}u=2\left(\frac{u^{10}}{10}-\frac{u^9}{9}\right)\Big|_1^2=2\cdot\Big(\Big(\frac{2^{10}}{10}-\frac{2^9}{9}\Big)-\Big(\frac{1}{10}-\frac{1}{9}\Big)\Big)=\frac{4097}{45}$

Problem. $\displaystyle\int\frac{\sqrt{x}}{1+x^3}\mathrm{d}x$

Solution. $\displaystyle\underbrace{\int\frac{\sqrt{x}}{1+x^3}\mathrm{d}x}_{\text{Let } u=x^{\frac{3}{2}}}=\int\frac{\frac{2}{3}\mathrm{d}u}{1+u^2}=\frac{2}{3}\int\frac{\mathrm{d}u}{1+u^2}=\frac{2}{3}\tan^{-1}u=\frac{2}{3}\tan^{-1}x^{\frac{3}{2}}$

Problem. $\displaystyle\int\frac{\sqrt{x}}{1+\sqrt[3]{x}}\mathrm{d}x$

Solution. $\displaystyle\underbrace{\int\frac{\sqrt{x}}{1+\sqrt[3]{x}}\mathrm{d}x}_{\text{Let } u=x^{\frac{1}{6}}}=\int\frac{u^3}{1+u^2}\cdot 6u^5\,\mathrm{d}u=6\int\Big(u^6-u^4+u^2-1+\frac{1}{1+u^2}\Big)\mathrm{d}u=6\left(\frac{u^7}{7}-\frac{u^5}{5}+\frac{u^3}{3}-u+\tan^{-1}u\right)=6\left(\frac{x^{\frac{7}{6}}}{7}-\frac{x^{\frac{5}{6}}}{5}+\frac{x^{\frac{1}{2}}}{3}-x^{\frac{1}{6}}+\tan^{-1}x^{\frac{1}{6}}\right)$

Problem. $\displaystyle\int\frac{1}{\sqrt{x+1}+\sqrt{x}}\mathrm{d}x$

Solution. $\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,\mathrm{d}x = \int \frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}} \cdot \frac{1}{\sqrt{x+1}+\sqrt{x}}\,\mathrm{d}x = \int \frac{\sqrt{x+1}-\sqrt{x}}{(\sqrt{x+1})^2-(\sqrt{x})^2}\,\mathrm{d}x = \int (\sqrt{x+1}-\sqrt{x})\,\mathrm{d}x = \frac{2}{3}\big((x+1)^{\frac{3}{2}} - x^{\frac{3}{2}}\big)$

Problem. $\int \frac{1}{x\sqrt{4x+1}}\,\mathrm{d}x$

Solution. $\underbrace{\int \frac{1}{x\sqrt{4x+1}}\,\mathrm{d}x}_{\text{Let } u=\sqrt{4x+1}} = \int \frac{\frac{1}{2}u\,\mathrm{d}u}{\frac{u^2-1}{4}\cdot u} = \int \frac{2}{u^2-1}\,\mathrm{d}u = \int \Big(\frac{1}{u-1} - \frac{1}{u+1}\Big)\,\mathrm{d}u = \ln\Big|\frac{u-1}{u+1}\Big| = \ln\Big|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\Big|$

Problem. $\int \frac{1}{x^2\sqrt{4x+1}}\,\mathrm{d}x$

Solution. $\underbrace{\int \frac{1}{x^2\sqrt{4x+1}}\,\mathrm{d}x}_{\text{Let } u=\sqrt{4x+1}} = \int \frac{\frac{1}{2}u\,\mathrm{d}u}{\frac{(u^2-1)^2}{16}\cdot u} = \int \frac{8}{(u^2-1)^2}\,\mathrm{d}u = 2\int \Big(\frac{1}{u-1} - \frac{1}{u+1}\Big)^2\,\mathrm{d}u$
$= 2\int \Big(\frac{1}{(u-1)^2} + \frac{1}{(u+1)^2} - \frac{2}{(u-1)(u+1)}\Big)\,\mathrm{d}u = 2\int \Big(\frac{1}{(u-1)^2} + \frac{1}{(u+1)^2} + \frac{1}{u+1} - \frac{1}{u-1}\Big)\,\mathrm{d}u = 2\Big(-\frac{1}{u+1} - \frac{1}{u-1} + \ln\Big|\frac{u+1}{u-1}\Big|\Big) = -\frac{\sqrt{4x+1}}{x} + 2\ln\Big|\frac{\sqrt{4x+1}+1}{\sqrt{4x+1}-1}\Big|$

Problem. $\int \frac{1}{x+4+4\sqrt{x+1}}\,\mathrm{d}x$

Solution. $\underbrace{\int \frac{1}{x+4+4\sqrt{x+1}}\,\mathrm{d}x}_{\text{Let } u=\sqrt{x+1}} = \int \frac{2u\,\mathrm{d}u}{u^2+3+4u} = \int \frac{2u}{(u+3)(u+1)}\,\mathrm{d}u = \int \Big(\frac{3}{u+3} - \frac{1}{u+1}\Big)^2\,\mathrm{d}u = \ln\Big|\frac{(u+3)^3}{u+1}\Big| = \ln\Big|\frac{(\sqrt{x+1}+3)^3}{\sqrt{x+1}+1}\Big|$

Problem. $\int \sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x$

Solution. $\int \sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x = \int \frac{\sqrt{1+x}}{\sqrt{1-x}}\,\mathrm{d}x = \int \frac{\sqrt{1+x}}{\sqrt{1-x}}\cdot\frac{\sqrt{1+x}}{\sqrt{1+x}}\,\mathrm{d}x = \int \frac{1+x}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x + \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x = \sin^{-1} x - \sqrt{1-x^2}$

Problem. $\int \frac{\sqrt{2-x}}{\sqrt{x}}\,\mathrm{d}x$

Solution. $\underbrace{\int \frac{\sqrt{2-x}}{\sqrt{x}}\,\mathrm{d}x}_{\text{Let } u=1-x} = -\int \frac{\sqrt{1+u}}{\sqrt{1-u}}\,\mathrm{d}u = \sqrt{1-u^2} - \sin^{-1} u = \sqrt{x(2-x)} - \sin^{-1}(1-x)$

Problem. $\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}+x^2}$

Solution. $\underbrace{\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}+x^2}}_{\text{Let } u=\sqrt{x}} = \int_0^\infty \frac{2u\,\mathrm{d}u}{u+u^4} = 2\int_0^\infty \frac{\mathrm{d}u}{1+u^3}$. From $\frac{1}{u^3+1} = \frac{\frac{1}{3}}{u+1} + \frac{-\frac{1}{3}u+\frac{2}{3}}{u^2-u+1} = \frac{\frac{1}{3}}{u+1} + \frac{-\frac{1}{3}(u-\frac{1}{2})+\frac{1}{2}}{u^2-u+1} = \frac{\frac{1}{3}}{u+1} + \frac{-\frac{1}{6}(2u-1)}{u^2-u+1} + \frac{\frac{1}{2}}{(u-\frac{1}{2})^2+\frac{3}{4}}$, $\int \frac{\mathrm{d}u}{u^3+1} = \frac{1}{3}\ln|u+1| - \frac{1}{6}\ln(u^2-u+1) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2u-1}{\sqrt{3}}$, so $\int_0^\infty \frac{\mathrm{d}x}{x^2+\sqrt{x}} = 2\int_0^\infty \frac{\mathrm{d}u}{u^3+1} = \frac{4\pi}{3\sqrt{3}}$ by $\lim\limits_{u\to\infty} \frac{(u+1)^2}{u^2-u+1} = 1$, $\lim\limits_{u\to 0} \frac{(u+1)^2}{u^2-u+1} = 1$, $\lim\limits_{u\to\infty} \tan^{-1}\frac{2u-1}{\sqrt{3}} = \frac{\pi}{2}$, and $\lim\limits_{u\to 0} \tan^{-1}\frac{2u-1}{\sqrt{3}} = -\frac{\pi}{6}$.

Problem. $\int \sqrt{3-2x-x^2}\,\mathrm{d}x$

Solution. $\int \sqrt{3-2x-x^2}\,\mathrm{d}x = \int \sqrt{4-1-2x-x^2}\,\mathrm{d}x = \underbrace{\int \sqrt{4-(x+1)^2}\,\mathrm{d}x}_{\text{Let } u=x+1} = \underbrace{\int \sqrt{4-u^2}\,\mathrm{d}u}_{\text{Let } u=2\sin\theta}$

$= \int 2\cos\theta\cdot 2\cos\theta\,\mathrm{d}\theta = 2\int(\cos 2\theta+1)\,\mathrm{d}\theta = \sin 2\theta + 2\theta = 2\sin\theta\cos\theta + 2\sin^{-1}\frac{u}{2} = 2\cdot\frac{u}{2}\cdot\frac{\sqrt{4-u^2}}{2} + 2\sin^{-1}\frac{u}{2} = \frac{(x+1)\sqrt{3-2x-x^2}}{2} + 2\sin^{-1}\frac{x+1}{2}$

Problem. $\int \frac{\mathrm{d}x}{(2x-1)\sqrt{x^2-x}}$

Solution. $\underbrace{\int \frac{\mathrm{d}x}{(2x-1)\sqrt{x^2-x}}}_{\text{Let } u=\sqrt{x^2-x}} = \int \frac{2u\,\mathrm{d}u}{(4u^2+1)u} = \tan^{-1}(2\sqrt{x^2-x})$

Problem. $\int \frac{2x^2+\sqrt{1-x^2}}{x\sqrt{1-x^2}}\,\mathrm{d}x$

Solution. $\int \frac{2x^2+\sqrt{1-x^2}}{x\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{2x}{\sqrt{1-x^2}}\,\mathrm{d}x + \int\frac{\mathrm{d}x}{x} = -2\sqrt{1-x^2} + \ln|x|$

Problem. $\int \frac{\mathrm{d}x}{(1-x^2)^{\frac{3}{2}}}$

Solution. $\underbrace{\int \frac{\mathrm{d}x}{(1-x^2)^{\frac{3}{2}}}}_{\text{Let } x=\sin\theta} = \int \frac{\cos\theta\,\mathrm{d}\theta}{\cos^3\theta} = \int \sec^2\theta\,\mathrm{d}\theta = \tan\theta = \frac{x}{\sqrt{1-x^2}}$

Problem. $\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$

Solution. $\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x = -\sqrt{1-x^2}\,\Big|_0^{\frac{1}{2}} = -\Big(\frac{\sqrt{3}}{2}-1\Big) = 1-\frac{\sqrt{3}}{2}$

Problem. $\int_0^{\frac{\sqrt{2}}{2}} \frac{x^2}{\sqrt{1-x^2}}\,\mathrm{d}x$

Solution. $\int_0^{\frac{\sqrt{2}}{2}} \frac{x^2}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_0^{\frac{\sqrt{2}}{2}} \frac{(x^2-1)+1}{\sqrt{1-x^2}}\,\mathrm{d}x = -\int_0^{\frac{\sqrt{2}}{2}} \sqrt{1-x^2}\,\mathrm{d}x + \sin^{-1}x\,\Big|_0^{\frac{\sqrt{2}}{2}} = -\Big(\frac{\pi}{8}+\frac{1}{4}\Big) + \frac{\pi}{4} = \frac{\pi}{8}-\frac{1}{4}$

Problem. $\int \frac{x}{1-x^2+\sqrt{1-x^2}}\,\mathrm{d}x$

Solution. $\underbrace{\int \frac{x}{1-x^2+\sqrt{1-x^2}}\,\mathrm{d}x}_{\text{Let } u=\sqrt{1-x^2}} = \int \frac{-u\,\mathrm{d}u}{u^2+u} = -\ln|u+1| = -\ln|\sqrt{1-x^2}+1|$

Problem. $\int \frac{1}{\sqrt{4x^2-4x-3}}\,\mathrm{d}x$

Solution. $\int \frac{1}{\sqrt{4x^2-4x-3}}\,\mathrm{d}x = \int \frac{1}{\sqrt{4x^2-4x+1-4}}\,\mathrm{d}x = \underbrace{\int \frac{1}{\sqrt{(2x-1)^2-2^2}}\,\mathrm{d}x}_{\text{Let } 2x-1=2\sec\theta} = \int \frac{\sec\theta\tan\theta\,\mathrm{d}\theta}{2\tan\theta}$

$= \frac{1}{2}\int\sec\theta\,\mathrm{d}\theta = \frac{1}{2}\ln|\sec\theta+\tan\theta| = \frac{1}{2}\ln\Big|x-\frac{1}{2}+\sqrt{x^2-x-\frac{3}{4}}\Big|$

Problem. $\int \frac{1}{x-\sqrt{1-x^2}}\,\mathrm{d}x$

Solution. $\underbrace{\int \frac{1}{x-\sqrt{1-x^2}}\,\mathrm{d}x}_{\text{Let } x=\sin\theta} = \int \frac{\cos\theta}{\sin\theta-\cos\theta}\,\mathrm{d}\theta = \frac{1}{2}\int \frac{2\cos\theta}{\sin\theta-\cos\theta}\,\mathrm{d}\theta = \frac{1}{2}\int \frac{(\cos\theta-\sin\theta)+(\cos\theta+\sin\theta)}{\sin\theta-\cos\theta}\,\mathrm{d}\theta$ $-\frac{1}{2}\theta+\frac{1}{2}\underbrace{\int\frac{\cos\theta+\sin\theta}{\sin\theta-\cos\theta}\,\mathrm{d}\theta}_{\text{Let } u=\sin\theta-\cos\theta} = \frac{1}{2}\left(-\theta+\ln|\sin\theta-\cos\theta|\right) = \frac{1}{2}\left(-\sin^{-1}x+\ln|x-\sqrt{1-x^2}|\right)$

Problem. $\int \frac{x}{\sqrt{3-x^4}}\,\mathrm{d}x$

Solution. $\underbrace{\int \frac{x}{\sqrt{3-x^4}}\,\mathrm{d}x}_{\text{Let } u=x^2} = \int \frac{\frac{\mathrm{d}u}{2}}{\sqrt{3-u^2}} = \frac{1}{2}\int\frac{\mathrm{d}u}{\sqrt{3-u^2}} = \frac{1}{2}\sin^{-1}\frac{u}{\sqrt{3}} = \frac{1}{2}\sin^{-1}\frac{x^2}{\sqrt{3}}$

Problem. $\int \frac{\mathrm{d}x}{\sqrt{1+\sqrt[3]{x}}}$

Solution. $\underbrace{\int \frac{\mathrm{d}x}{\sqrt{1+\sqrt[3]{x}}}}_{\text{Let } u=\sqrt{1+\sqrt[3]{x}}} = \int \frac{(u^2-1)^2\,6u\,\mathrm{d}u}{u} = 6\int (u^4-2u^2+1)\,\mathrm{d}u = \frac{6}{5}u^5-4u^3+6u = \frac{6}{5}\left(\sqrt{1+\sqrt[3]{x}}\right)^5 -$ $4\left(\sqrt{1+\sqrt[3]{x}}\right)^3+6\sqrt{1+\sqrt[3]{x}} = \frac{2}{5}\sqrt{1+\sqrt[3]{x}}\left(8-4\sqrt[3]{x}+6\sqrt[3]{x^2}\right)$

Problem. $\int_0^1 \left(\sqrt[3]{1-x^7}-\sqrt[7]{1-x^3}\right)\mathrm{d}x$

Solution. Let $u=\sqrt[3]{1-x^7} \implies x=\sqrt[7]{1-u^3}$. Integration by parts yield $\int_0^1 \sqrt[3]{1-x^7}\,\mathrm{d}x = \int_1^0 u\,\mathrm{d}(\sqrt[7]{1-u^3}) =$ $u\cdot\sqrt[7]{1-u^3}\Big|_1^0 - \int_1^0 \sqrt[7]{1-u^3}\,\mathrm{d}u = \int_0^1 \sqrt[7]{1-u^3}\,\mathrm{d}u$. So $\int_0^1\left(\sqrt[3]{1-x^7}-\sqrt[7]{1-x^3}\right)\mathrm{d}x = 0$.

Problem. $\int_0^1 3(x-1)^2\left(\int_0^x \sqrt{1+(t-1)^4}\,\mathrm{d}t\right)\mathrm{d}x$

Solution. $\underbrace{\int_0^1 3(x-1)^2\left(\int_0^x \sqrt{1+(t-1)^4}\,\mathrm{d}t\right)\mathrm{d}x}_{\text{Let } u=\int_0^x\sqrt{1+(t-1)^4}\,\mathrm{d}t,\ \mathrm{d}v=3(x-1)^2\,\mathrm{d}x}$ $= \left(\int_0^x \sqrt{1+(t-1)^4}\,\mathrm{d}t\right)\cdot(x-1)^3\Big|_0^1 - \underbrace{\int_0^1 (x-1)^3\cdot\sqrt{1+(x-1)^4}\,\mathrm{d}x}_{\text{Let } u=1+(x-1)^4} = \frac{1}{4}\int_1^2\sqrt{u}\,\mathrm{d}u = \frac{2\sqrt{2}-1}{6}$

Problem. $\int \sin^3 x\cos^5 x\,\mathrm{d}x$

Solution. $\int \sin^3 x\cos^5 x\,\mathrm{d}x = \int \sin^3 x\cos^4 x\cos x\,\mathrm{d}x = \underbrace{\int \sin^3 x(1-\sin^2 x)^2\cos x\,\mathrm{d}x}_{\text{Let } u=\sin x} = \int u^3(1-u^2)^2\,\mathrm{d}u = \int (u^3-$ $2u^5+u^7)\,\mathrm{d}u = \frac{u^4}{4}-\frac{u^6}{3}+\frac{u^8}{8} = \frac{\sin^4 x}{4}-\frac{\sin^6 x}{3}+\frac{\sin^8 x}{8}$

Problem. $\int \sin x\cos(\cos x)\,\mathrm{d}x$

Solution. $\underbrace{\int \sin x\cos(\cos x)\,\mathrm{d}x}_{\text{Let } u=\cos x} = -\int \cos u\,\mathrm{d}u = -\sin u = -\sin(\cos x)$

Problem. $\int \tan^3 x\,\mathrm{d}x$

Solution. $\int \tan^3 x \, \mathrm{d}x = \int (\sec^2 x - 1) \tan x \, \mathrm{d}x = \int \sec^2 x \tan x \, \mathrm{d}x - \int \tan x \, \mathrm{d}x = \underbrace{\int \sec x \cdot \sec x \tan x \, \mathrm{d}x}_{\text{Let } u = \sec x} - \underbrace{\int \frac{\sin x}{\cos x} \, \mathrm{d}x}_{\text{Let } w = \cos x} = \frac{\sec^2 x}{2} + \ln|\cos x|$

Problem. $\int \tan^5 x \sec^4 x \, \mathrm{d}x$

Solution. $\int \tan^5 x \sec^4 x \, \mathrm{d}x = \int \tan^5 x \sec^2 x \sec^2 x \, \mathrm{d}x = \underbrace{\int \tan^5 x (\tan^2 x + 1) \sec^2 x \, \mathrm{d}x}_{\text{Let } u = \tan x} = \int (u^7 + u^5) \, \mathrm{d}u = \frac{u^8}{8} + \frac{u^6}{6} = \frac{\tan^8 x}{8} + \frac{\tan^6 x}{6}$

Problem. $\int \tan^5 x \sec^7 x \, \mathrm{d}x$

Solution. $\int \tan^5 x \sec^7 x \, \mathrm{d}x = \int \tan^4 x \sec^6 x \cdot \tan x \sec x \, \mathrm{d}x = \underbrace{\int (\sec^2 x - 1)^2 \sec^6 x \cdot \tan x \sec x \, \mathrm{d}x}_{\text{Let } u = \sec x} = \int (u^2 - 1)^2 u^6 \, \mathrm{d}u = \int (u^{10} - 2u^8 + u^6) \, \mathrm{d}u = \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} = \frac{\sec^{11} x}{11} - \frac{2\sec^9 x}{9} + \frac{\sec^7 x}{7}$

Problem. $\int_0^{\frac{\pi}{4}} \cos^2 x \tan^2 x \, \mathrm{d}x$

Solution. $\int_0^{\frac{\pi}{4}} \cos^2 x \tan^2 x \, \mathrm{d}x = \int_0^{\frac{\pi}{4}} \sin^2 x \, \mathrm{d}x = \int_0^{\frac{\pi}{4}} \frac{1 - \cos 2x}{2} \, \mathrm{d}x = \frac{\pi}{8} - \frac{1}{4} \sin 2x \Big|_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{1}{4}$

Problem. $\int (\sin x + \cos x)^2 \, \mathrm{d}x$

Solution. $\int (\sin x + \cos x)^2 \, \mathrm{d}x = \int (\sin^2 x + 2 \sin x \cos x + \cos^2 x) \, \mathrm{d}x = \int (1 + \sin 2x) \, \mathrm{d}x = x - \frac{\cos 2x}{2}$

Problem. $\int \sin 4x \cos 3x \, \mathrm{d}x$

Solution. $\int \sin 4x \cos 3x \, \mathrm{d}x = \frac{1}{2} \int (\sin(4x + 3x) + \sin(4x - 3x)) \, \mathrm{d}x = -\frac{1}{2} \left(\frac{\cos 7x}{7} + \cos x \right)$

Problem. $\int \sin x \sin 2x \sin 3x \, \mathrm{d}x$

Solution. $\int \sin x \sin 2x \sin 3x \, \mathrm{d}x = \frac{1}{2} \int (\cos(x - 2x) - \cos(x + 2x)) \sin 3x \, \mathrm{d}x = \frac{1}{2} \int (\cos x \sin 3x - \cos 3x \sin 3x) \, \mathrm{d}x = \frac{1}{4} \int (\sin(3x - x) + \sin(3x + x) - \sin 6x) \, \mathrm{d}x = \frac{1}{4} \int (\sin 2x + \sin 4x - \sin 6x) \, \mathrm{d}x = -\frac{\cos 2x}{8} - \frac{\cos 4x}{16} + \frac{\cos 6x}{24}$

Problem. $\int \sin \sqrt{ax} \, \mathrm{d}x$

Solution. $\underbrace{\int \sin \sqrt{ax} \, \mathrm{d}x}_{\text{Let } u = \sqrt{ax}} = \int \sin u \cdot \frac{2u}{a} \, \mathrm{d}u = \frac{2}{a} \int u \sin u \, \mathrm{d}u = \frac{2}{a} (\sin u - u \cos u) = \frac{2}{a} (\sin \sqrt{ax} - \sqrt{ax} \cos \sqrt{ax})$

Problem. $\int \frac{\cos^5 x}{\sqrt{\sin x}} \, \mathrm{d}x$

Solution. $\int \frac{\cos^5 x}{\sqrt{\sin x}} \, \mathrm{d}x = \int \frac{\cos^4 x \cos x}{\sqrt{\sin x}} \, \mathrm{d}x = \underbrace{\int \frac{(1 - \sin^2 x)^2 \cos x}{\sqrt{\sin x}} \, \mathrm{d}x}_{\text{Let } u = \sin x} = \int \frac{(1 - u^2)^2}{\sqrt{u}} \, \mathrm{d}u = \int (u^{-\frac{1}{2}} - 2u^{\frac{3}{2}} + u^{\frac{7}{2}}) \, \mathrm{d}u = 2u^{\frac{1}{2}} - \frac{4}{5} u^{\frac{5}{2}} + \frac{2}{9} u^{\frac{9}{2}} = 2 \sin^{\frac{1}{2}} x - \frac{4}{5} \sin^{\frac{5}{2}} x + \frac{2}{9} \sin^{\frac{9}{2}} x$

Problem. $\displaystyle\int \frac{\cos^6 x}{\sin^4 x}\,\mathrm{d}x$

Solution. $\displaystyle\int \frac{\cos^6 x}{\sin^4 x}\,\mathrm{d}x = \int \frac{(1-\sin^2 x)^3}{\sin^4 x}\,\mathrm{d}x = \int \frac{1-3\sin^2 x+3\sin^4 x-\sin^6 x}{\sin^4 x}\,\mathrm{d}x = \int (\csc^4 x - 3\csc^2 x + 3 - \sin^2 x)\,\mathrm{d}x = \Big(-\frac{1}{3}\cot^3 x - \cot x\Big) + 3\cot x + 3x - \Big(\frac{x}{2} - \frac{\sin 2x}{4}\Big) = -\frac{1}{3}\cot^3 x + 2\cot x + \frac{5x}{2} + \frac{\sin 2x}{4}$

Problem. $\displaystyle\int \frac{\tan x}{\sec^2 x}\,\mathrm{d}x$

Solution. $\displaystyle\int \frac{\tan x}{\sec^2 x}\,\mathrm{d}x = \int \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos^2 x}}\,\mathrm{d}x = \int \sin x\cos x\,\mathrm{d}x = \frac{1}{2}\int \sin 2x\,\mathrm{d}x = -\frac{1}{4}\cos 2x$

Problem. $\displaystyle\int \frac{\tan^3 x}{\cos^3 x}\,\mathrm{d}x$

Solution. $\displaystyle\int \frac{\tan^3 x}{\cos^3 x}\,\mathrm{d}x = \int \tan^2 x\sec^2 x\cdot\tan x\sec x\,\mathrm{d}x = \underbrace{\int (\sec^2 x - 1)\sec^2 x\cdot\tan x\sec x\,\mathrm{d}x}_{\text{Let } u=\sec x} = \int (u^2-1)\,u^2\,\mathrm{d}u = \int (u^4-u^2)\,\mathrm{d}u = \frac{u^5}{5} - \frac{u^3}{3} = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3}$

Problem. $\displaystyle\int \frac{1}{1-\cos x}\,\mathrm{d}x$

Solution. $\displaystyle\int \frac{1}{1-\cos x}\,\mathrm{d}x = \int \frac{1+\cos x}{1-\cos^2 x}\,\mathrm{d}x = \int \frac{1+\cos x}{\sin^2 x}\,\mathrm{d}x = \int (\csc^2 x + \csc x\cot x)\,\mathrm{d}x = -\cot x - \csc x$

Problem. $\displaystyle\int \frac{\mathrm{d}x}{3\sin x - 4\cos x}$

Solution. $3\sin x - 4\cos x = -5\left(\frac{4}{5}\cos x - \frac{3}{5}\sin x\right) = -5\cos(x+\alpha)$, where $\cos\alpha = \frac{4}{5}$. So $\displaystyle\int \frac{\mathrm{d}x}{3\sin x - 4\cos x} = -\frac{1}{5}\int \frac{\mathrm{d}x}{\cos(x+\alpha)} = -\frac{1}{5}\int \sec(x+\alpha)\,\mathrm{d}x = -\frac{1}{5}\ln|\sec(x+\alpha)+\tan(x+\alpha)|$

Problem. $\displaystyle\int \frac{\mathrm{d}x}{1+\sin x-\cos x}$

Solution. $\displaystyle\int \frac{\mathrm{d}x}{1+\sin x-\cos x} = \int \frac{1+\sin x+\cos x}{(1+\sin x)^2-\cos^2 x}\,\mathrm{d}x = \int \frac{(1+\sin x)+\cos x}{2\sin x(1+\sin x)}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{\sin x}\,\mathrm{d}x + \frac{1}{2}\underbrace{\int \frac{\cos x}{\sin x(1+\sin x)}\,\mathrm{d}x}_{\text{Let } u=\sin x} = -\frac{1}{2}\ln|\csc x+\cot x| + \frac{1}{2}\int \frac{\mathrm{d}u}{u(1+u)} = -\frac{1}{2}\ln|\csc x+\cot x| + \frac{1}{2}\int \Big(\frac{1}{u} - \frac{1}{u+1}\Big)\,\mathrm{d}u = -\frac{1}{2}\ln|\csc x+\cot x| + \frac{1}{2}\ln\left|\frac{\sin x}{\sin x+1}\right|$

Problem. $\displaystyle\int \frac{\cos x+\sin x}{\sin 2x}\,\mathrm{d}x$

Solution. $\displaystyle\int \frac{\cos x+\sin x}{\sin 2x}\,\mathrm{d}x = \int \frac{\cos x+\sin x}{2\sin x\cos x}\,\mathrm{d}x = \frac{1}{2}\int \Big(\frac{1}{\sin x}+\frac{1}{\cos x}\Big)\,\mathrm{d}x = \frac{1}{2}\int (\csc x+\sec x)\,\mathrm{d}x = \frac{1}{2}(-\ln|\csc x+\cot x| + \ln|\sec x+\tan x|) = \frac{1}{2}\ln\left|\frac{\sec x+\tan x}{\csc x+\cot x}\right| = \frac{1}{2}\ln\left|\frac{(1+\sin x)\sin x}{(1+\cos x)\cos x}\right|$

Problem. $\displaystyle\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,\mathrm{d}x$

Solution. $\displaystyle\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,\mathrm{d}x = \int \frac{\frac{1}{2}\sin 2x}{(\frac{1-\cos 2x}{2})^2 + (\frac{1+\cos 2x}{2})^2}\,\mathrm{d}x = \underbrace{\int \frac{\sin 2x}{1+\cos^2 2x}\,\mathrm{d}x}_{\text{Let } u=\cos 2x} = -\frac{1}{2}\int \frac{\mathrm{d}u}{1+u^2} = -\frac{1}{2}\tan^{-1} u = -\frac{1}{2}\tan^{-1}(\cos 2x)$

Problem. $\displaystyle\int \frac{\cos x}{\sqrt{\cos^2 x+2}}\,\mathrm{d}x$

Solution. $\displaystyle\int \frac{\cos x}{\sqrt{\cos^2 x+2}}\,\mathrm{d}x = \underbrace{\int \frac{\cos x}{\sqrt{3-\sin^2 x}}\,\mathrm{d}x}_{\text{Let } u=\sin x} = \int \frac{\mathrm{d}u}{\sqrt{3-u^2}} = \sin^{-1}\frac{u}{\sqrt{3}} = \sin^{-1}\Big(\frac{\sin x}{\sqrt{3}}\Big)$

Problem. $\displaystyle\int \frac{\sec x\cos 2x}{\sin x+\sec x}\,\mathrm{d}x$

Solution. $\displaystyle\int \frac{\sec x\cos 2x}{\sin x+\sec x}\,\mathrm{d}x = \int \frac{\cos 2x}{\sin x\cos x+1}\,\mathrm{d}x = \underbrace{\int \frac{\cos 2x}{\frac{1}{2}\sin 2x+1}\,\mathrm{d}x}_{\text{Let } u=\frac{1}{2}\sin 2x+1} = \int \frac{\mathrm{d}u}{u} = \ln\Big|\frac{1}{2}\sin 2x+1\Big|$

Problem. $\displaystyle\int \frac{\tan x}{\tan x+\sec x}\,\mathrm{d}x$

Solution. $\displaystyle\int \frac{\tan x}{\tan x+\sec x}\,\mathrm{d}x = \int \frac{\frac{\sin x}{\cos x}}{\frac{\sin x}{\cos x}+\frac{1}{\cos x}}\,\mathrm{d}x = \int \frac{\sin x}{1+\sin x}\,\mathrm{d}x = \int \frac{(1-\sin x)\sin x}{(1-\sin x)(1+\sin x)}\,\mathrm{d}x = \int \frac{\sin x-\sin^2 x}{\cos^2 x}\,\mathrm{d}x$
$\displaystyle= \int \frac{\sin x}{\cos^2 x}\,\mathrm{d}x - \int \frac{\sin^2 x}{\cos^2 x}\,\mathrm{d}x = \sec x - \int \tan^2 x\,\mathrm{d}x = \sec x - \int (\sec^2 x-1)\,\mathrm{d}x = \sec x - \tan x + x$

Problem. $\displaystyle\int \frac{1}{1-\tan^2 x}\,\mathrm{d}x$

Solution. $\displaystyle\int \frac{1}{1-\tan^2 x}\,\mathrm{d}x = \int \frac{1+\tan^2 x}{1-\tan^4 x}\,\mathrm{d}x = \underbrace{\int \frac{\sec^2 x}{1-\tan^4 x}\,\mathrm{d}x}_{\text{Let } u=\tan x} = \int \frac{\mathrm{d}u}{1-u^4} = \frac{1}{2}\int\Big(\frac{1}{1+u^2}+\frac{1}{1-u^2}\Big)\,\mathrm{d}u =$
$\displaystyle\frac{1}{2}\int\Big(\frac{1}{1+u^2}+\frac{1}{2}\Big(\frac{1}{1-u}+\frac{1}{1+u}\Big)\Big)\,\mathrm{d}u = \frac{1}{2}\tan^{-1}u + \frac{1}{4}\ln\Big|\frac{1+u}{1-u}\Big| = \frac{x}{2}+\frac{1}{4}\ln\Big|\frac{1+\tan x}{1-\tan x}\Big|$

Problem. $\displaystyle\int \frac{3+\sec^2 x+\sin x}{\tan x}\,\mathrm{d}x$

Solution. $\displaystyle\int \frac{3+\sec^2 x+\sin x}{\tan x}\,\mathrm{d}x = \int (3\cot x+2\csc 2x+\cos x)\,\mathrm{d}x = 3\ln|\sin x| - \ln|\csc 2x+\cot 2x| + \sin x =$
$4\ln|\sin x| - \ln|\cos x| + \sin x$

Problem. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1+4\cot x}{4-\cot x}\,\mathrm{d}x$

Solution. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1+4\cot x}{4-\cot x}\,\mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1+4\frac{\cos x}{\sin x}}{4-\frac{\cos x}{\sin x}}\,\mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\frac{\sin x+4\cos x}{\sin x}}{\frac{4\sin x-\cos x}{\sin x}}\,\mathrm{d}x = \underbrace{\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x+4\cos x}{4\sin x-\cos x}\,\mathrm{d}x}_{\text{Let } u=4\sin x-\cos x} = \int_{\frac{3}{\sqrt{2}}}^{4} \frac{\mathrm{d}u}{u} =$
$\displaystyle\frac{5}{2}\ln 2-\ln 3$

Problem. $\displaystyle\int \sqrt{\tan x}\,\mathrm{d}x$

Solution. $\displaystyle\underbrace{\int \sqrt{\tan x}\,\mathrm{d}x}_{\text{Let } u=\sqrt{\tan x}} = \int \frac{2u^2}{1+u^4}\,\mathrm{d}u.$ By $1+u^4 = 1+2u^2+u^4-2u^2 = (1+u^2)^2-(\sqrt{2}u)^2 = (1+\sqrt{2}u+u^2)(1-\sqrt{2}u+u^2)$, $\displaystyle\frac{2u^2}{1+u^4} = \frac{au+b}{1+\sqrt{2}u+u^2}+\frac{cu+d}{1-\sqrt{2}u+u^2} \implies (au+b)(1-\sqrt{2}u+u^2)+(cu+d)(1+\sqrt{2}u+u^2) = 2u^2$ $\implies (a+c)u^3+(-\sqrt{2}a+b+\sqrt{2}c+d)u^2+(a-\sqrt{2}b+c+\sqrt{2}d)u+(b+d) = 2u^2 \implies a+c=0,$ $-\sqrt{2}a+b+\sqrt{2}c+d=2,\ a-\sqrt{2}b+c+\sqrt{2}d=0,\ b+d=0 \implies a=-\frac{\sqrt{2}}{2},\ c=\frac{\sqrt{2}}{2},\ b=d=0$ $\displaystyle\implies \frac{2u^2}{1+u^4} = \frac{-\frac{1}{\sqrt{2}}u}{1+\sqrt{2}u+u^2}+\frac{\frac{1}{\sqrt{2}}u}{1-\sqrt{2}u+u^2} = \frac{-\frac{1}{2\sqrt{2}}(2u+\sqrt{2})+\frac{1}{2}}{1+\sqrt{2}u+u^2}+\frac{\frac{1}{2\sqrt{2}}(2u-\sqrt{2})+\frac{1}{2}}{1-\sqrt{2}u+u^2}.$ So $\displaystyle\int \frac{2u^2}{1+u^4}\,\mathrm{d}u =$
$\displaystyle\frac{1}{2\sqrt{2}}\ln\Big|\frac{1-\sqrt{2}u+u^2}{1+\sqrt{2}u+u^2}\Big| + \frac{1}{2}\int\Big(\frac{1}{(u+\frac{1}{\sqrt{2}})^2+(\frac{1}{\sqrt{2}})^2}+\frac{1}{(u-\frac{1}{\sqrt{2}})^2+(\frac{1}{\sqrt{2}})^2}\Big)\,\mathrm{d}u = \frac{1}{2\sqrt{2}}\ln\Big|\frac{1-\sqrt{2}u+u^2}{1+\sqrt{2}u+u^2}\Big|$

$+\frac{1}{\sqrt{2}}\left(\tan^{-1}(\sqrt{2}u+1)+\tan^{-1}(\sqrt{2}u-1)\right)=\frac{1}{2\sqrt{2}}\ln\left|\frac{1-\sqrt{2\tan x}+\tan x}{1+\sqrt{2\tan x}+\tan x}\right|+\frac{1}{\sqrt{2}}\left(\tan^{-1}(\sqrt{2\tan x}+1)+\tan^{-1}(\sqrt{2\tan x}-1)\right)$

Problem. $\int \frac{\sin 2x}{1+\tan x}\,\mathrm{d}x$

Solution. $\int \frac{\sin 2x}{1+\tan x}\,\mathrm{d}x=\int \frac{2\sin x\cos x}{1+\frac{\sin x}{\cos x}}\,\mathrm{d}x=\int \frac{2\sin x\cos^2 x}{\cos x+\sin x}\,\mathrm{d}x=\int \frac{2\sin x\cos^2 x}{\cos x+\sin x}\cdot\frac{\cos x-\sin x}{\cos x-\sin x}\,\mathrm{d}x$
$=\int \frac{\cos^2 x\sin 2x-\frac{1}{2}\sin^2 2x}{\cos 2x}\,\mathrm{d}x=\int \frac{\frac{1+\cos 2x}{2}\cdot\sin 2x-\frac{1}{2}(1-\cos^2 2x)}{\cos 2x}\,\mathrm{d}x=\frac{1}{2}\int \frac{(1+\cos 2x)\cdot\sin 2x-(1-\cos^2 2x)}{\cos 2x}\,\mathrm{d}x=$
$\frac{1}{2}\int\left(\frac{\sin 2x}{\cos 2x}+\sin 2x-\sec 2x+\cos 2x\right)\mathrm{d}x=\frac{1}{4}\left(\ln|\sec 2x|-\cos 2x-\ln|\sec 2x+\tan 2x|+\sin 2x\right)$

Problem. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\sqrt{\frac{\tan x}{\sin 2x}}\,\mathrm{d}x$

Solution. $\int\sqrt{\frac{\tan x}{\sin 2x}}\,\mathrm{d}x=\int\sqrt{\frac{\frac{\sin x}{\cos x}}{2\sin x\cos x}}\,\mathrm{d}x=\frac{1}{\sqrt{2}}\int\sec x\,\mathrm{d}x=\frac{1}{\sqrt{2}}\ln|\sec x+\tan x|$, so $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\sqrt{\frac{\tan x}{\sin 2x}}\,\mathrm{d}x=$ $\frac{1}{\sqrt{2}}\ln|\sec x+\tan x|\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}=\frac{1}{\sqrt{2}}\ln\frac{2+\sqrt{3}}{\sqrt{2}+1}$

Problem. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sqrt{\tan x}}{\sin 2x}\,\mathrm{d}x$

Solution. By $(\sqrt{\tan x})'=\frac{1}{2\sqrt{\tan x}}\cdot\sec^2 x=\frac{\sec^2 x\sqrt{\tan x}}{2\tan x}=\frac{\sqrt{\tan x}}{\sin 2x}$, $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sqrt{\tan x}}{\sin 2x}\,\mathrm{d}x=\sqrt{\tan x}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}=\sqrt[4]{3}-1$

Problem. $\int\tan^5 x\,\sqrt[3]{\cos x}\,\mathrm{d}x$

Solution. $\int\tan^5 x\,\sqrt[3]{\cos x}\,\mathrm{d}x=\int\frac{\sin^5 x}{\cos^5 x}\sqrt[3]{\cos x}\,\mathrm{d}x=\underbrace{\int(1-\cos^2 x)^2\cos^{\frac{1}{3}-5}x\cdot\sin x\,\mathrm{d}x}_{\text{Let } u=\cos x}=-\int\left(u^{-\frac{14}{3}}-2u^{-\frac{8}{3}}+u^{-\frac{2}{3}}\right)\mathrm{d}u=\frac{3}{11}u^{-\frac{11}{3}}-\frac{6}{5}u^{-\frac{5}{3}}-3u^{\frac{1}{3}}=\frac{3}{11}\cos^{-\frac{11}{3}}x-\frac{6}{5}\cos^{-\frac{5}{3}}x-3\cos^{\frac{1}{3}}x$

Problem. $\int x\sin^2 x\,\mathrm{d}x$

Solution. $\int x\sin^2 x\,\mathrm{d}x=\frac{1}{2}\int(x-x\cos 2x)\,\mathrm{d}x=\frac{1}{2}\left(x\cdot\frac{1}{2}\sin 2x+\frac{1}{4}\cos 2x\right)=\frac{x\sin 2x}{4}+\frac{\cos 2x}{8}$

Problem. $\int(x+\sin x)^2\,\mathrm{d}x$

Solution. $\int(x+\sin x)^2\,\mathrm{d}x=\int\left(x^2+2x\sin x+\sin^2 x\right)\mathrm{d}x=\frac{x^3}{6}+2\int x\sin x\,\mathrm{d}x+\frac{1}{2}\int\left(1-\cos 2x\right)\mathrm{d}x=\frac{x^3}{6}+2\left(\sin x-x\cos x\right)+\frac{x}{2}-\frac{1}{4}\sin 2x$

Problem. $\int x\sin^2 x\cos x\,\mathrm{d}x$

Solution. $\int x\sin^2 x\cos x\,\mathrm{d}x=\int x\cdot\frac{1-\cos 2x}{2}\cdot\cos x\,\mathrm{d}x=\frac{1}{2}\int\left(x\cos x-x\cos 2x\cos x\right)\mathrm{d}x=\frac{1}{2}\int\Big(x\cos x-x\cdot\frac{1}{2}\big(\cos(2x-x)+\cos(2x+x)\big)\Big)\mathrm{d}x=\frac{1}{4}\int\left(2x\cos x-x\cos x-x\cos 3x\right)\mathrm{d}x=\frac{1}{4}\int x\left(\cos x-\cos 3x\right)\mathrm{d}x=\frac{1}{4}\left(x\Big(\sin x-\frac{1}{3}\sin 3x\Big)+\cos x-\frac{1}{9}\cos 3x\right)$

Problem. $\int x\tan^2 x\,\mathrm{d}x$

Solution. $\int x\tan^2 x\,\mathrm{d}x = \int x(\sec^2 x-1)\,\mathrm{d}x = \underbrace{\int x\sec^2 x\,\mathrm{d}x}_{\text{Let } u=x,\,\mathrm{d}v=\sec^2 x\,\mathrm{d}x} - \frac{x^2}{2} = x\tan x + \ln|\cos x| - \frac{x^2}{2}$

Problem. $\int_{-1}^{1} x^8\sin x\,\mathrm{d}x$

Solution. $\int_{-1}^{1} x^8\sin x\,\mathrm{d}x = 0$ for $x^8\sin x$ is odd.

Problem. $\int x\sin^{-1}x\,\mathrm{d}x$

Solution. $\underbrace{\int x\sin^{-1}x\,\mathrm{d}x}_{\text{Let } u=\sin^{-1}x,\,\mathrm{d}v=\mathrm{d}x} = \sin^{-1}x\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{x^2\sin^{-1}x}{2} - \frac{1}{2}\int\frac{(x^2-1)+1}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{x^2\sin^{-1}x}{2} + \frac{1}{2}\underbrace{\int\sqrt{1-x^2}\,\mathrm{d}x}_{\text{Let } x=\sin\theta} - \frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{x^2\sin^{-1}x}{2} + \frac{1}{2}\left(\frac{\theta}{2}+\frac{1}{4}\sin 2\theta\right) - \frac{1}{2}\sin^{-1}x = \frac{x^2\sin^{-1}x}{2} + \frac{1}{4}\left(\sin^{-1}x + x\sqrt{1-x^2}\right) - \frac{1}{2}\sin^{-1}x = \frac{x^2\sin^{-1}x}{2} + \frac{x\sqrt{1-x^2}}{4} - \frac{\sin^{-1}x}{4}$

Problem. $\int x^{-2}\tan^{-1}x\,\mathrm{d}x$

Solution. $\underbrace{\int x^{-2}\tan^{-1}x\,\mathrm{d}x}_{\text{Let } u=\tan^{-1}x,\,\mathrm{d}v=x^{-2}\mathrm{d}x} = \tan^{-1}x\cdot\frac{-1}{x} + \int\frac{1}{x}\cdot\frac{1}{1+x^2}\,\mathrm{d}x = -\frac{\tan^{-1}x}{x} + \int\left(\frac{1}{x}-\frac{x}{1+x^2}\right)\mathrm{d}x = -\frac{\tan^{-1}x}{x} + \ln|x| - \underbrace{\int\frac{x}{1+x^2}\,\mathrm{d}x}_{\text{Let } u=1+x^2} = -\frac{\tan^{-1}x}{x} + \ln|x| - \int\frac{\frac{\mathrm{d}u}{2}}{u} = -\frac{\tan^{-1}x}{x} + \ln|x| - \frac{1}{2}\ln(1+x^2)$

Problem. $\int_1^3 \frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x$

Solution. $\underbrace{\int_1^3 \frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x}_{\text{Let } u=\sqrt{x}} = \int_1^{\sqrt{3}}\frac{\tan^{-1}u}{u}\cdot 2u\,\mathrm{d}u = 2\int_1^{\sqrt{3}}\tan^{-1}u\,\mathrm{d}u$. From $\underbrace{\int\tan^{-1}w\,\mathrm{d}w}_{\text{Let } u=\tan^{-1}w,\,\mathrm{d}v=\mathrm{d}w} = \tan^{-1}w\cdot w - \int w\cdot\frac{1}{1+w^2}\,\mathrm{d}w = w\tan^{-1}w - \frac{1}{2}\ln(1+w^2)$, $2\int_1^{\sqrt{3}}\tan^{-1}u\,\mathrm{d}u = \left(2u\tan^{-1}u - \ln(1+u^2)\right)\Big|_1^{\sqrt{3}} = \frac{(4\sqrt{3}-3)\pi}{6} - \ln 2$

Problem. $\int\sqrt{1-x^2}\sin^{-1}x\,\mathrm{d}x$

Solution. First note that $\underbrace{\int\sqrt{1-x^2}\,\mathrm{d}x}_{\text{Let } u=\sin\theta} = \int\cos\theta\cdot\cos\theta\,\mathrm{d}\theta = \frac{1}{2}\int(1+\cos 2\theta)\,\mathrm{d}\theta = \frac{\theta}{2}+\frac{\sin 2\theta}{4} = \frac{1}{2}\left(\theta+\sin\theta\cos\theta\right) = \frac{1}{2}\left(\sin^{-1}x + x\sqrt{1-x^2}\right)$, then $\underbrace{\int\sqrt{1-x^2}\sin^{-1}x\,\mathrm{d}x}_{\text{Let } u=\sin^{-1}x,\,\mathrm{d}v=\sqrt{1-x^2}\,\mathrm{d}x} = \sin^{-1}x\cdot\frac{1}{2}\left(\sin^{-1}x+x\sqrt{1-x^2}\right) - \frac{1}{2}\int\left(\sin^{-1}x+x\sqrt{1-x^2}\right)\cdot\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{1}{2}\sin^{-1}x\left(\sin^{-1}x+x\sqrt{1-x^2}\right) - \frac{1}{2}\int\left(\frac{\sin^{-1}x}{\sqrt{1-x^2}}+x\right)\mathrm{d}x = \frac{1}{2}\sin^{-1}x\left(\sin^{-1}x+x\sqrt{1-x^2}\right) - \frac{x^2}{4} - \frac{1}{2}\underbrace{\int\frac{\sin^{-1}x}{\sqrt{1-x^2}}\,\mathrm{d}x}_{\text{Let } u=\sin^{-1}x} = \frac{1}{2}\sin^{-1}x\left(\sin^{-1}x+x\sqrt{1-x^2}\right) - \frac{x^2}{4} - \frac{1}{4}(\sin^{-1}x)^2 = \frac{1}{4}(\sin^{-1}x)^2 + \frac{1}{2}x\sqrt{1-x^2}\,\sin^{-1}x - \frac{x^2}{4}$

Problem. $\int_0^1 \frac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}}\,\mathrm{d}x$

Solution. By $(\sin^{-1}\sqrt{x})' = \dfrac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x(1-x)}}$, $\displaystyle\int_0^1 \frac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}}\,\mathrm{d}x = 2\int_0^1 \sin^{-1}\sqrt{x}\,\mathrm{d}(\sin^{-1}\sqrt{x})$
$= (\sin^{-1}\sqrt{x})^2\Big|_0^1 = \dfrac{\pi^2}{4}$

Problem. $\displaystyle\int e^{x+e^x}\,\mathrm{d}x$

Solution. $\displaystyle\int e^{x+e^x}\,\mathrm{d}x = \underbrace{\int e^{e^x}\cdot e^x\,\mathrm{d}x}_{\text{Let } u=e^x} = \int e^u\,\mathrm{d}u = e^u = e^{e^x}$

Problem. $\displaystyle\int \frac{1}{e^{3x}-e^x}\,\mathrm{d}x$

Solution. $\displaystyle\int \frac{1}{e^{3x}-e^x}\,\mathrm{d}x = \underbrace{\int \frac{e^{-3x}}{1-e^{-2x}}\,\mathrm{d}x}_{\text{Let } u=e^{-x}} = \int \frac{u^2}{u^2-1}\,\mathrm{d}u = \int \left(1+\frac{1}{2}\left(\frac{1}{u-1}-\frac{1}{u+1}\right)\right)\mathrm{d}u = u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| =$
$\displaystyle e^{-x} + \frac{1}{2}\ln\left|\frac{e^{-x}-1}{e^{-x}+1}\right|$

Problem. $\displaystyle\int \frac{e^{2x}}{1+e^x}\,\mathrm{d}x$

Solution. $\displaystyle\underbrace{\int \frac{e^{2x}}{1+e^x}\,\mathrm{d}x}_{\text{Let } u=e^x} = \int \frac{u}{1+u}\,\mathrm{d}u = \int \left(1-\frac{1}{1+u}\right)\mathrm{d}u = u - \ln|1+u| = e^x - \ln(1+e^x)$

Problem. $\displaystyle\int \frac{1}{1+2e^x-e^{-x}}\,\mathrm{d}x$

Solution. $\displaystyle\int \frac{1}{1+2e^x-e^{-x}}\,\mathrm{d}x = \underbrace{\int \frac{e^x}{e^x+2e^{2x}-1}\,\mathrm{d}x}_{\text{Let } u=e^x} = \int \frac{1}{u+2u^2-1}\,\mathrm{d}u = \int \left(\frac{-\frac{1}{3}}{u+1}+\frac{\frac{2}{3}}{2u-1}\right)\mathrm{d}u = \frac{1}{3}\ln\left|\frac{2u-1}{u+1}\right| =$
$\displaystyle \frac{1}{3}\ln\left|\frac{2e^x-1}{e^x+1}\right|$

Problem. $\displaystyle\int \frac{e^{2x}}{1+e^{4x}}\,\mathrm{d}x$

Solution. $\displaystyle\underbrace{\int \frac{e^{2x}}{1+e^{4x}}\,\mathrm{d}x}_{\text{Let } u=e^{2x}} = \frac{1}{2}\int \frac{\mathrm{d}u}{1+u^2} = \frac{1}{2}\tan^{-1}u = \frac{1}{2}\tan^{-1}e^{2x}$

Problem. $\displaystyle\int e^x\sqrt{1+e^x}\,\mathrm{d}x$

Solution. $\displaystyle\underbrace{\int e^x\sqrt{1+e^x}\,\mathrm{d}x}_{\text{Let } u=1+e^x} = \int \sqrt{u}\,\mathrm{d}u = \frac{2}{3}u^{\frac{3}{2}} = \frac{2}{3}(1+e^x)^{\frac{3}{2}}$

Problem. $\displaystyle\int \sqrt{1+e^x}\,\mathrm{d}x$

Solution. $\displaystyle\underbrace{\int \sqrt{1+e^x}\,\mathrm{d}x}_{\text{Let } u=\sqrt{1+e^x}} = \int u\cdot\frac{2u}{u^2-1}\,\mathrm{d}u = \int \frac{2(u^2-1)+2}{u^2-1}\,\mathrm{d}u = 2u+\int \left(\frac{1}{u-1}-\frac{1}{u+1}\right)\mathrm{d}u = 2u+\ln\left|\frac{u-1}{u+1}\right| =$
$\displaystyle 2\sqrt{1+e^x} + \ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right|$

Problem. $\displaystyle\int \frac{2e^{2x}-e^x}{\sqrt{3e^{2x}-6e^x-1}}\,\mathrm{d}x$

Solution. $\int \frac{2e^{2x}-e^x}{\sqrt{3e^{2x}-6e^x-1}}\,\mathrm{d}x = \int \frac{(2e^{2x}-2e^x)+e^x}{\sqrt{3e^{2x}-6e^x-1}}\,\mathrm{d}x = \frac{1}{3}\underbrace{\int \frac{6e^{2x}-6e^x}{\sqrt{3e^{2x}-6e^x-1}}\,\mathrm{d}x}_{\text{Let } u=3e^{2x}-6e^x-1} + \underbrace{\int \frac{e^x}{\sqrt{3e^{2x}-6e^x-1}}\,\mathrm{d}x}_{\text{Let } w=e^x} =$ $\frac{1}{3}\int \frac{\mathrm{d}u}{\sqrt{u}} + \int \frac{\mathrm{d}w}{\sqrt{3w^2-6w-1}} = \frac{2}{3}\sqrt{3e^{2x}-6e^x-1} + \int \frac{\mathrm{d}w}{\sqrt{3(w-1)^2-4}} = \frac{2}{3}\sqrt{3e^{2x}-6e^x-1}$ $+\frac{1}{\sqrt{3}}\int \frac{\mathrm{d}w}{\sqrt{(w-1)^2-(\frac{2}{\sqrt{3}})^2}} = \frac{2}{3}\sqrt{3e^{2x}-6e^x-1} + \frac{1}{\sqrt{3}}\ln\left|\frac{e^x-1}{\frac{2}{\sqrt{3}}} + \frac{\sqrt{(e^x-1)^2-(\frac{2}{\sqrt{3}})^2}}{\frac{2}{\sqrt{3}}}\right| = \frac{2}{3}\sqrt{3e^{2x}-6e^x-1} +$ $\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}(e^x-1)}{2} + \frac{\sqrt{3(e^x-1)^2-4}}{2}\right|$ by $\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \int \frac{a\sec\theta\tan\theta}{a\tan\theta}\,\mathrm{d}\theta = \int \sec\theta\,\mathrm{d}\theta = \ln|\sec\theta+\tan\theta| =$ $\ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right|$

Problem. $\int e^{\sqrt[3]{x}}\,\mathrm{d}x$

Solution. $\underbrace{\int e^{\sqrt[3]{x}}\,\mathrm{d}x}_{\text{Let } u=\sqrt[3]{x}} = \int e^u \cdot 3u^2\,\mathrm{d}u = (3u^2-6u+6)\,e^u = (3(\sqrt[3]{x})^2 - 6\sqrt[3]{x}+6)\,e^{\sqrt[3]{x}}$

Problem. $\int \sqrt{x}e^{\sqrt{x}}\,\mathrm{d}x$

Solution. $\underbrace{\int \sqrt{x}e^{\sqrt{x}}\,\mathrm{d}x}_{\text{Let } u=\sqrt{x}} = \int u\cdot e^u\cdot 2u\,\mathrm{d}u = \int 2u^2e^u\,\mathrm{d}u = (2u^2-4u+4)\,e^u = (2x-4\sqrt{x}+4)\,e^{\sqrt{x}}$

Problem. $\int x^5e^{-x^3}\,\mathrm{d}x$

Solution. $\underbrace{\int x^5e^{-x^3}\,\mathrm{d}x}_{\text{Let } u=x^3} = \frac{1}{3}\int u\cdot e^{-u}\,\mathrm{d}u = -\frac{u+1}{3}\,e^u = -\frac{x^3+1}{3}\,e^{-x^3}$

Problem. $\int x^3e^{-2x}\,\mathrm{d}x$

Solution. $\int x^3e^{-2x}\,\mathrm{d}x = -\Big(\frac{x^3}{2}+\frac{3x^2}{4}+\frac{3x}{4}+\frac{3}{8}\Big)\,e^{-2x}$

Problem. $\int 27^{3x+1}\,\mathrm{d}x$

Solution. $\int 27^{3x+1}\,\mathrm{d}x = \int e^{(3x+1)\ln 27}\,\mathrm{d}x = 27\int e^{3\ln 27\cdot x}\,\mathrm{d}x = \frac{27}{3\ln 27}\,e^{3\ln 27\cdot x} = \frac{27^{3x+1}}{3\ln 27}$

Problem. $\int_{-1}^{1} \frac{e^{\tan^{-1}x}}{1+x^2}\,\mathrm{d}x$

Solution. $\underbrace{\int_{-1}^{1} \frac{e^{\tan^{-1}x}}{1+x^2}\,\mathrm{d}x}_{\text{Let } u=\tan^{-1}x} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^u\,\mathrm{d}u = e^{\frac{\pi}{4}} - e^{-\frac{\pi}{4}}$

Problem. $\int e^x\sin(ax-3)\,\mathrm{d}x$

Solution. $\int e^x\sin(ax-3)\,\mathrm{d}x = -\frac{1}{a}\,e^x\cos(ax-3) + \frac{1}{a^2}\,e^x\sin(ax-3) - \frac{1}{a^2}\int e^x\sin(ax-3)\,\mathrm{d}x$
$\implies \int e^x\sin(ax-3)\,\mathrm{d}x = \frac{e^x\,(\sin(ax-3)-a\cos(ax-3))}{a^2+1}$

Problem. $\int \frac{xe^x}{\sqrt{1+e^x}}\,\mathrm{d}x$

Solution. $\underbrace{\int \frac{xe^x}{\sqrt{1+e^x}}\,\mathrm{d}x}_{\text{Let } u=x,\, \mathrm{d}v=\frac{e^x}{\sqrt{1+e^x}}\,\mathrm{d}x} = x\cdot 2\sqrt{1+e^x} - 2\int \sqrt{1+e^x}\,\mathrm{d}x = 2x\sqrt{1+e^x} - 2\int \sqrt{1+e^x}\,\mathrm{d}x = 2x\sqrt{1+e^x} - 4\sqrt{1+e^x} - 2\ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right|$

Problem. $\int_0^\infty xe^{-x}\sin x\,\mathrm{d}x$

Solution. From $\int e^{ax}\sin bx\,\mathrm{d}x = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$, $\int e^{ax}\cos bx\,\mathrm{d}x = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$, $\int e^{-x}\sin x\,\mathrm{d}x = -\frac{e^{-x}(\sin x+\cos x)}{2}$, $\int e^{-x}\cos x\,\mathrm{d}x = \frac{e^{-x}(\sin x - \cos x)}{2}$, by tabulation $\int xe^{-x}\sin x\,\mathrm{d}x = x\cdot -\frac{e^{-x}(\sin x+\cos x)}{2} + \frac{1}{2}\Big(-\frac{e^{-x}(\sin x+\cos x)}{2} + \frac{e^{-x}(\sin x-\cos x)}{2}\Big) = -\frac{e^{-x}}{2}(x\sin x + (x+1)\,\cos x)$, so $\int_0^\infty xe^{-x}\sin x\,\mathrm{d}x = \frac{1}{2}$.

Problem. $\int x^2\ln(1+x)\,\mathrm{d}x$

Solution. $\underbrace{\int x^2\ln(1+x)\,\mathrm{d}x}_{\text{Let } u=\ln(x+1),\, \mathrm{d}v=x^2\,\mathrm{d}x} = \ln(x+1)\cdot\frac{x^3}{3} - \frac{1}{3}\int x^3\cdot\frac{1}{1+x}\,\mathrm{d}x = \ln(x+1)\cdot\frac{x^3}{3} - \frac{1}{3}\int\Big(x^2 - x + 1 - \frac{1}{x+1}\Big)\,\mathrm{d}x = \ln(x+1)\cdot\frac{x^3}{3} - \frac{x^3}{9} + \frac{x^2}{6} - \frac{x}{3} + \frac{\ln(x+1)}{3}$

Problem. $\int \ln\sqrt{x-1}\,\mathrm{d}x$

Solution. $\underbrace{\int \ln\sqrt{x-1}\,\mathrm{d}x}_{\text{Let } u=\ln\sqrt{x-1}} = \int u\cdot 2e^{2u}\,\mathrm{d}u = \Big(u-\frac{1}{2}\Big)\,e^{2u} = \Big(\ln\sqrt{x-1} - \frac{1}{2}\Big)(x-1)$

Problem. $\int \ln(\sqrt{x}+\sqrt{1+x})\,\mathrm{d}x$

Solution. $\underbrace{\int \ln(\sqrt{x}+\sqrt{1+x})\,\mathrm{d}x}_{\text{Let } u=\ln(\sqrt{x}+\sqrt{1+x}),\, \mathrm{d}v=\mathrm{d}x} = \ln(\sqrt{x}+\sqrt{1+x})\cdot x - \int x\cdot\frac{1}{2\sqrt{x(1+x)}}\,\mathrm{d}x = x\ln(\sqrt{x}+\sqrt{1+x}) - \underbrace{\frac{1}{2}\int\sqrt{\frac{x}{1+x}}\,\mathrm{d}x}_{\text{Let } u=\sqrt{1+x}} = x\ln(\sqrt{x}+\sqrt{1+x}) - \frac{1}{2}\int\frac{\sqrt{u^2-1}}{u}\cdot 2u\,\mathrm{d}u = x\ln(\sqrt{x}+\sqrt{1+x}) - \underbrace{\int\sqrt{u^2-1}\,\mathrm{d}u}_{\text{Let } u=\sec\theta} = x\ln(\sqrt{x}+\sqrt{1+x}) - \frac{1}{2}\,\big(u\sqrt{u^2-1} - \ln|u+\sqrt{u^2-1}|\big) = \Big(x+\frac{1}{2}\Big)\,\ln(\sqrt{x}+\sqrt{1+x}) - \frac{1}{2}\sqrt{x(1+x)}$

Problem. $\int \frac{\sqrt{1+\ln x}}{x\ln x}\,\mathrm{d}x$

Solution. $\underbrace{\int \frac{\sqrt{1+\ln x}}{x\ln x}\,\mathrm{d}x}_{\text{Let } u=\ln x} = \underbrace{\int\frac{\sqrt{1+u}}{u}\,\mathrm{d}u}_{\text{Let } v=\sqrt{1+u}} = \int\frac{v}{v^2-1}\,2v\,\mathrm{d}v = \int\frac{2v^2}{v^2-1}\,\mathrm{d}v = \int\Big(\frac{2v^2-2}{v^2-1} + \frac{2}{v^2-1}\Big)\,\mathrm{d}v = 2v + \int\Big(\frac{1}{v-1} - \frac{1}{v+1}\Big)\,\mathrm{d}v = 2v + \ln\left|\frac{v-1}{v+1}\right| = 2\sqrt{1+\ln x} + \ln\left|\frac{\sqrt{1+\ln x}-1}{\sqrt{1+\ln x}+1}\right|$

Problem. $\int \frac{\mathrm{d}x}{x\sqrt{\ln x}}$

Solution. $\underbrace{\int \frac{\mathrm{d}x}{x\sqrt{\ln x}}}_{\text{Let } u=\ln x} = \int\frac{\mathrm{d}u}{\sqrt{u}} = 2\sqrt{u} = 2\sqrt{\ln x}$

Problem. $\int \frac{\ln(x+1)}{x^2}\,\mathrm{d}x$

Solution. $\underbrace{\int \frac{\ln(x+1)}{x^2}\,\mathrm{d}x}_{\text{Let } u=\ln(x+1),\, \mathrm{d}v=\frac{1}{x^2}\,\mathrm{d}x} = \ln(x+1)\cdot\frac{-1}{x} + \int \frac{1}{x}\cdot\frac{1}{x+1}\,\mathrm{d}x = \frac{-\ln(x+1)}{x} + \int \left(\frac{1}{x} - \frac{1}{x+1}\right)\mathrm{d}x$
$= \frac{-\ln(x+1)}{x} + \ln\frac{x}{x+1}$

Problem. $\int \frac{x\ln x}{\sqrt{x^2-1}}\,\mathrm{d}x$

Solution. $\underbrace{\int \frac{x\ln x}{\sqrt{x^2-1}}\,\mathrm{d}x}_{\text{Let } u=\ln x,\, \mathrm{d}v=\frac{x}{\sqrt{x^2-1}}\,\mathrm{d}x} = \ln x\cdot\sqrt{x^2-1} - \int \sqrt{x^2-1}\cdot\frac{1}{x}\,\mathrm{d}x = \sqrt{x^2-1}\ln x - \underbrace{\int \frac{\sqrt{x^2-1}}{x}\,\mathrm{d}x}_{\text{Let } u=\sqrt{x^2-1}} = \sqrt{x^2-1}\ln x - \int \frac{u^2}{u^2+1}\,\mathrm{d}u = \sqrt{x^2-1}\ln x - \int \left(1 - \frac{1}{u^2+1}\right)\mathrm{d}u = \sqrt{x^2-1}\ln x - u + \tan^{-1} u = \sqrt{x^2-1}\ln x - \sqrt{x^2-1} + \tan^{-1}\sqrt{x^2-1}$

Problem. $\int \frac{\mathrm{d}x}{x(1+\ln x)\sqrt{(\ln x)(2+\ln x)}}$

Solution. $\underbrace{\int \frac{\mathrm{d}x}{x(1+\ln x)\sqrt{(\ln x)(2+\ln x)}}}_{\text{Let } u=1+\ln x} = \underbrace{\int \frac{\mathrm{d}u}{u\sqrt{u^2-1}}}_{\text{Let } w=\ln u} = \int w\,\mathrm{d}w = \frac{w^2}{2} = \frac{(\ln\sin x)^2}{2}$

Problem. $\int \cot x \ln(\sin x)\,\mathrm{d}x$

Solution. $\int \cot x\ln\sin x\,\mathrm{d}x = \underbrace{\int \frac{\cos x}{\sin x}\ln\sin x\,\mathrm{d}x}_{\text{Let } u=\sin x} = \underbrace{\int \frac{\ln u}{u}\,\mathrm{d}u}_{\text{Let } w=\ln u} = \int w\,\mathrm{d}w = \frac{w^2}{2} = \frac{(\ln\sin x)^2}{2}$

Problem. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\sin x\cos x}\,\mathrm{d}x$

Solution. Note that $(\ln(\tan x))' = \frac{1}{\sin x\cos x}$, so $\int \frac{\ln(\tan x)}{\sin x\cos x}\,\mathrm{d}x = \int \ln(\tan x)\,\mathrm{d}(\ln(\tan x)) = \frac{1}{2}(\ln(\tan x))^2$; $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\sin x\cos x}\,\mathrm{d}x = \frac{1}{2}(\ln(\tan x))^2\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{(\ln 3)^2}{8}$.

Problem. $\int \frac{\cot x}{\ln\sin x}\,\mathrm{d}x$

Solution. $\int \frac{\cot x}{\ln\sin x}\,\mathrm{d}x = \underbrace{\int \frac{\cos x}{\sin x\ln\sin x}\,\mathrm{d}x}_{\text{Let } u=\sin x} = \underbrace{\int \frac{\mathrm{d}u}{u\ln u}}_{\text{Let } w=\ln u} = \int \frac{\mathrm{d}w}{w} = \ln|w| = \ln|\ln(\sin x)|$

Problem. $\int (1+\ln x)\sqrt{1+(x\ln x)^2}\,\mathrm{d}x$

Solution. $\underbrace{\int (1+\ln x)\sqrt{1+(x\ln x)^2}\,\mathrm{d}x}_{\text{Let } u=x\ln x} = \int \sqrt{1+u^2}\,\mathrm{d}u = \frac{u\sqrt{u^2+1} + \ln|\sqrt{u^2+1}+u|}{2}$
$= \frac{x\ln x\sqrt{(x\ln x)^2+1} + \ln|\sqrt{(x\ln x)^2+1} + x\ln x|}{2}$