# 第七章 重積分

## 7.1 二重積分

### 概念與基本性質

二重積分 $\approx$ (帶符號) 體積: $z$ 軸上方為正, 下方為負.

**定義.** 給定 $\Omega = [a, b] \times [c, d]$, $b \geqslant a$, $d \geqslant c$, $f : \Omega \to \mathbb{R}$.

- $\Omega$ 分割 $\mathbb{P} : a = x_0 < x_1 < x_2 < \cdots < x_n = b$, $c = y_0 < y_1 < y_2 < \cdots < y_m = d$

- $\Delta x_k = x_k - x_{k-1}$, $\Delta y_l = y_l - y_{l-1}$, $k = 1, 2, \ldots, n$; $l = 1, 2, \ldots, m$

- $\|\mathbb{P}\| = \max \{\Delta x_k \Delta y_l \mid 1 \leqslant k \leqslant n, 1 \leqslant l \leqslant m\}$

- 樣本點 $(\xi_k, \zeta_l)$: $x_{k-1} \leqslant \xi_k \leqslant x_k$, $y_{l-1} \leqslant \zeta_l \leqslant y_l$, $k = 1, 2, \ldots, n$; $l = 1, 2, \ldots, m$.

- $u_{k,l} = \max \{f(x, y) \mid x_{k-1} \leqslant x \leqslant x_k, y_{l-1} \leqslant y \leqslant y_l\}$, $\ell_{k,l} = \min \{f(x, y) \mid x_{k-1} \leqslant x \leqslant x_k, y_{l-1} \leqslant y \leqslant y_l\}$, $k = 1, 2, \ldots, n$; $l = 1, 2, \ldots, m$.

- $R(f, \mathbb{P}) = \sum_{l=1}^{m} \sum_{k=1}^{n} f(\xi_k, \zeta_l) \, \Delta x_k \Delta y_l$, $U(f, \mathbb{P}) = \sum_{l=1}^{m} \sum_{k=1}^{n} u_{k,l} \, \Delta x_k \Delta y_l$, $L(f, \mathbb{P}) = \sum_{l=1}^{m} \sum_{k=1}^{n} \ell_{k,l} \, \Delta x_k \Delta y_l$; 顯然 $L(f, \mathbb{P}) \leqslant R(f, \mathbb{P}) \leqslant U(f, \mathbb{P})$.

- 求 $\lim_{\|\mathbb{P}\| \to 0} R(f, \mathbb{P})$. 若對不同分割與樣本點選取此極限均存在且相等, 稱 $f$ 在 $\Omega$ 可積 (分); $f(x, y)$ 在 $\Omega$ 的定積分為 $\displaystyle\int_{\Omega} f(x, y) \, \mathrm{d}A \equiv \lim_{\|\mathbb{P}\| \to 0} R(f, \mathbb{P})$.

**性質.**

- 若 $f$ 在 $\Omega$ 有界, 且除了 $\Omega$ 中有限個平滑曲線外 $f$ 在 $\Omega$ 連續, 則 $f$ 在 $\Omega$ 可積分.

- 若 $\Omega = \Omega_1 \cap \Omega_2$ 且 $\Omega_1, \Omega_2$ 均為矩形, $f$ 在 $\Omega_1, \Omega_2$ 均可積, 則 $\displaystyle\int_{\Omega} f \, \mathrm{d}A = \int_{\Omega_1} f \, \mathrm{d}A + \int_{\Omega_2} f \, \mathrm{d}A$.

- 若 $f_1, f_2$ 均在 $\Omega$ 可積且 $f_1(x, y) \leqslant f_2(x, y) \, \forall (x, y) \in \Omega$, 則 $\displaystyle\int_{\Omega} f_1 \, \mathrm{d}A \leqslant \int_{\Omega} f_2 \, \mathrm{d}A$.

- 若 $\alpha, \beta \in \mathbb{R}$, $f_1, f_2$ 均在 $\Omega$ 可積, 則 $\displaystyle\int_{\Omega} (\alpha f_1 + \beta f_2) \, \mathrm{d}A = \alpha \int_{\Omega} f_1 \, \mathrm{d}A + \beta \int_{\Omega} f_2 \, \mathrm{d}A$.

### 逐次積分

#### 矩形積分區域

**定理** (Fubini). 若 $\Omega = [a, b] \times [c, d]$, $f : \Omega \to \mathbb{R}$ 為可積, 則

$$\int_{\Omega} f(x, y) \, \mathrm{d}A = \int_c^d \left\{ \int_a^b f(x, y) \, \mathrm{d}x \right\} \mathrm{d}y = \int_a^b \left\{ \int_c^d f(x, y) \, \mathrm{d}y \right\} \mathrm{d}x$$

**例.** 令 $\Omega = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$, 則 $\displaystyle\int_{\Omega} \sqrt{1 - x^2} \, \mathrm{d}A = \int_{-2}^{2} \left\{ \int_{-1}^{1} \sqrt{1 - x^2} \, \mathrm{d}x \right\} \mathrm{d}y = \frac{\pi}{2} \times 4 = 2\pi$.

**例.** 若 $\Omega = [0, 2] \times [1, 3]$, 求 $\displaystyle\int_{\Omega} x^2 y \, \mathrm{d}A$.

**解.**

- 由 Fubini 定理 $\displaystyle\int_{\Omega} x^2 y \, \mathrm{d}A = \int_0^2 \int_1^3 x^2 y \, \mathrm{d}y \, \mathrm{d}x = \int_1^3 \int_0^2 x^2 y \, \mathrm{d}x \, \mathrm{d}y$.

- $\int_0^2 \int_1^3 x^2 y \, dy \, dx = \int_0^2 x^2 \Big(\frac{y^2}{2} \Big|_1^3\Big) \, dx = \int_0^2 x^2 \Big(\frac{9}{2} - \frac{1}{2}\Big) \, dx = \frac{4}{3} x^3 \Big|_0^2 = \frac{32}{3}.$

- $\int_1^3 \int_0^2 x^2 y \, dx \, dy = \int_1^3 y \Big(\frac{x^3}{3} \Big|_0^2\Big) \, dy = \int_1^3 y \frac{8}{3} \, dx = \frac{4}{3} y^2 \Big|_1^3 = \frac{36}{3} - \frac{4}{3} = \frac{32}{3}.$

**例.** 若 $\Omega = [0,\,1] \times [0,\,3]$, 求 $\int_\Omega e^{2x+y} \, dA$.

**解.**

- 由 Fubini 定理 $\int_\Omega e^{2x+y} \, dA = \int_0^3 \int_0^1 e^{2x+y} \, dx \, dy = \int_0^1 \int_0^3 e^{2x+y} \, dy \, dx.$

- $\int_0^3 \int_0^1 e^{2x+y} \, dx \, dy = \int_0^3 e^y \Big(\int_0^1 e^{2x} \, dx\Big) \, dy = \int_0^3 e^y \Big(\frac{e^{2x}}{2} \Big|_0^1\Big) \, dy = \frac{(e^2-1)(e^3-1)}{2}.$

- $\int_0^1 \int_0^3 e^{2x+y} \, dy \, dx = \int_0^1 e^{2x} \Big(\int_0^3 e^y \, dy\Big) \, dx = \int_0^1 e^{2x} \Big(\frac{e^{2x}}{2} \Big|_0^3\Big) \, dy = \frac{(e^2-1)(e^3-1)}{2}.$

**例.** 若 $\Omega = [0,\,\pi] \times [0,\,2\pi]$, 求 $\int_\Omega \sin(x+y) \, dA$.

**解.**

- 由 Fubini 定理 $\int_\Omega \sin(x+y) \, dA = \int_0^\pi \int_0^{2\pi} \sin(x+y) \, dy \, dx = \int_0^{2\pi} \int_0^\pi \sin(x+y) \, dx \, dy.$

- $\int_0^\pi \int_0^{2\pi} \sin(x+y) \, dy \, dx = \int_0^\pi \Big(-\cos(x+y)\Big|_0^{2\pi}\Big) \, dx = -\int_0^\pi (\cos(x+2\pi) - \cos x) \, dx = 0.$

- $\int_0^{2\pi} \int_0^\pi \sin(x+y) \, dx \, dy = \int_0^{2\pi} \Big(-\cos(x+y)\Big|_0^\pi\Big) \, dy = -\int_0^{2\pi} (\cos(\pi+y) - \cos y) \, dy = -\int_0^{2\pi} (\cos\pi\cos y -$ $\sin\pi\sin y - \cos y) \, dy = 2\int_0^{2\pi} \cos y \, dy = 0.$

**例.** 若 $\Omega = [1,\,2] \times [0,\,1]$, 求 $\int_\Omega y e^{xy} \, dA$.

**解.**

- 由 Fubini 定理 $\int_\Omega y e^{xy} \, dA = \int_0^1 \int_1^2 y e^{xy} \, dx \, dy = \int_1^2 \int_0^1 y e^{xy} \, dy \, dx.$

- $\int_0^1 \int_1^2 y e^{xy} \, dx \, dy = \int_0^1 y \Big(\frac{e^{xy}}{y}\Big|_1^2\Big) \, dy = \int_0^1 (e^{2y} - e^y) \, dy = \frac{e^2}{2} - e + \frac{1}{2}.$

- $\int_1^2 \int_0^1 y e^{xy} \, dy \, dx = \int_1^2 \Big(y\frac{e^{xy}}{x} - \frac{e^{xy}}{x^2}\Big|_0^1\Big) \, dx = \int_1^2 \Big(\underbrace{\frac{e^x}{x} - \frac{e^x}{x^2}}_{(\frac{e^x}{x})' = \frac{e^x}{x} - \frac{e^x}{x^2}} + \frac{1}{x^2}\Big) \, dx = \Big(\frac{e^x}{x} - \frac{1}{x}\Big)\Big|_1^2 = \frac{e^2}{2} - e + \frac{1}{2}.$

**習題.** 求下列重積分.

1. $\int_2^3 \int_1^5 (x+2y) \, dx \, dy$

2. $\int_0^{\ln 4} \int_0^{\ln 3} e^{x+y} \, dx \, dy$

3. $\int_0^1 \int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}} \, dy \, dx$

4. $\int_0^{\frac{\pi}{2}} \int_0^{\cos y} e^x \sin y \, dx \, dy$

5. $\int_0^{\frac{\pi}{2}} \int_0^{\cos\theta} \rho^2 \sin\theta \, d\rho \, d\theta$

6. $\int_0^1 \int_{x^2}^{x^3} (x^2+y^2) \, dy \, dx$

7. $\int_1^2 \int_0^y x\sqrt{y^2-x^2} \, dx \, dy$

8. $\int_0^1 \int_{y^4}^{y^2} \sqrt{\frac{y}{x}} \, dx \, dy$

9. $\int_0^{\frac{\pi}{2}} \int_0^a \frac{r}{\sqrt{a^2-r^2\cos^2\theta}} \, dr \, d\theta$
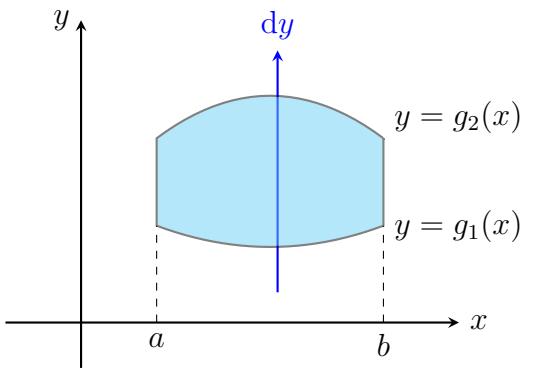
**解.**

1. $\displaystyle\int_2^3\int_1^5 (x+2y)\,\mathrm{d}x\,\mathrm{d}y = \int_2^3\Big(\frac{x^2}{2}+2yx\Big)\Big|_1^5\,\mathrm{d}y = \int_2^3(12+8y)\,\mathrm{d}y = (12y+4y^2)\Big|_2^3 = 32$

2. $\displaystyle\int_0^{\ln 4}\int_0^{\ln 3} e^{x+y}\,\mathrm{d}x\,\mathrm{d}y = \Big(\int_0^{\ln 4} e^y\,\mathrm{d}y\Big)\cdot\Big(\int_0^{\ln 3} e^x\,\mathrm{d}x\Big) = (4-1)(3-1) = 6$

3. $\displaystyle\int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 x\Big(\int_0^1 \frac{y}{\underbrace{\sqrt{x^2+y^2+1}}_{\text{令 }u=y^2+x^2+1,\ \mathrm{d}u=2y\mathrm{d}y}}\,\mathrm{d}y\Big)\mathrm{d}x = \int_0^1 x\Big(\frac{1}{2}\int_{x^2+1}^{x^2+2}\frac{1}{\sqrt{u}}\,\mathrm{d}u\Big)\mathrm{d}x$

   $\displaystyle = \int_0^1 x(\sqrt{x^2+2}-\sqrt{x^2+1})\,\mathrm{d}x = \int_0^1 x\underbrace{\sqrt{x^2+2}}_{\text{令 }u=x^2+2}\,\mathrm{d}x - \int_0^1 x\underbrace{\sqrt{x^2+1}}_{\text{令 }u=x^2+1}\,\mathrm{d}x = \sqrt{3}-\frac{4}{3}\sqrt{2}+\frac{1}{3}$

4. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\cos y} e^x\sin y\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{2}}\sin y\Big(\int_0^{\cos y} e^x\,\mathrm{d}x\Big)\mathrm{d}y = \int_0^{\frac{\pi}{2}}\sin y\,(e^{\cos y}-1)\,\mathrm{d}y = \big(-e^{\cos y}+\cos y\big)\Big|_0^{\frac{\pi}{2}} = e-2$

5. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta}\rho^2\sin\theta\,\mathrm{d}\rho\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}}\sin\theta\Big(\int_0^{\cos\theta}\rho^2\,\mathrm{d}\rho\Big)\mathrm{d}\theta = \int_0^{\frac{\pi}{2}}\sin\theta\,\frac{\cos^3\theta}{3}\,\mathrm{d}\theta = -\frac{\cos^4\theta}{12}\Big|_0^{\frac{\pi}{2}} = \frac{1}{12}$

6. $\displaystyle\int_0^1\int_{x^2}^{x^3}(x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\Big(x^2 y+\frac{y^3}{3}\Big)\Big|_{y=x^2}^{y=x^3}\,\mathrm{d}x = \int_0^1\Big(x^2\cdot x^3 + \frac{x^9}{3} - x^2\cdot x^2 - \frac{x^6}{3}\Big)\mathrm{d}x = -\frac{1}{21}$

7. $\displaystyle\int_0^y x\sqrt{y^2-x^2}\,\mathrm{d}x$ 中, 令 $u=y^2-x^2$, 則 $\mathrm{d}u = -2x\,\mathrm{d}x \implies x\,\mathrm{d}x = \frac{-1}{2}\,\mathrm{d}u$. 積分範圍 $x$ 由 $0$ 至 $y$, 則變

   數變換 $u=y^2-x^2$ 由 $y^2-0^2=y^2$ 至 $y^2-y^2=0$. 則 $\displaystyle\int_0^y x\sqrt{y^2-x^2}\,\mathrm{d}x = \int_{y^2}^0 \sqrt{u}\,\frac{-1}{2}\,\mathrm{d}u = \frac{1}{2}\int_0^{y^2}\sqrt{u}\,\mathrm{d}u =$

   $\displaystyle\frac{1}{2}\frac{2}{3}u^{\frac{3}{2}}\Big|_0^{y^2} = \frac{y^3}{3}$. 故 $\displaystyle\int_1^2\int_0^y x\sqrt{y^2-x^2}\,\mathrm{d}x\,\mathrm{d}y = \int_1^2\frac{y^3}{3}\,\mathrm{d}y = \frac{y^4}{12}\Big|_1^2 = \frac{5}{4}$.

8. $\displaystyle\int_0^1\int_{y^4}^{y^2}\sqrt{\frac{y}{x}}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\sqrt{y}\Big(\int_{y^4}^{y^2}\frac{1}{\sqrt{x}}\,\mathrm{d}x\Big)\mathrm{d}y = \int_0^1\sqrt{y}\,(2\sqrt{x})\Big|_{y^4}^{y^2}\,\mathrm{d}y = 2\int_0^1\big(y^{\frac{3}{2}}-y^{\frac{5}{2}}\big)\,\mathrm{d}y = \frac{8}{35}$

9. $\displaystyle\int_0^a \frac{r}{\sqrt{a^2-r^2\cos^2\theta}}\,\mathrm{d}r$ 中, 令 $u=a^2-r^2\cos^2\theta$, 則 $\mathrm{d}u=-2\cos^2\theta\,r\,\mathrm{d}r \implies r\,\mathrm{d}r = \frac{-1}{2\cos^2\theta}\,\mathrm{d}u$. 積分
   範圍 $r$ 由 $0$ 至 $a$, 則變數變換 $u=a^2-r^2\cos^2\theta$ 由 $a^2-0^2\cos^2\theta=a^2$ 至 $a^2-a^2\cos^2\theta=a^2\sin^2\theta$. 則
   $\displaystyle\int_0^a \frac{r}{\sqrt{a^2-r^2\cos^2\theta}}\,\mathrm{d}r = \int_{a^2}^{a^2\sin^2\theta}\frac{1}{\sqrt{u}}\frac{-1}{2\cos^2\theta}\,\mathrm{d}u = \int_{a^2\sin^2\theta}^{a^2}\frac{1}{2\cos^2\theta\sqrt{u}}\,\mathrm{d}u = \frac{\sqrt{u}}{\cos^2\theta}\Big|_{u=a^2\sin^2\theta}^{u=a^2} = a\,\frac{1-\sin\theta}{\cos^2\theta}$.
   故 $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^a \frac{r}{\sqrt{a^2-r^2\cos^2\theta}}\,\mathrm{d}r\,\mathrm{d}\theta = a\int_0^{\frac{\pi}{2}}\frac{1-\sin\theta}{\cos^2\theta}\,\mathrm{d}\theta = a\,\frac{\sin\theta-1}{\cos\theta}\Big|_0^{\frac{\pi}{2}} = a\Big(\lim_{\theta\to\frac{\pi}{2}}\frac{\sin\theta-1}{\cos\theta}+1\Big) = a$.
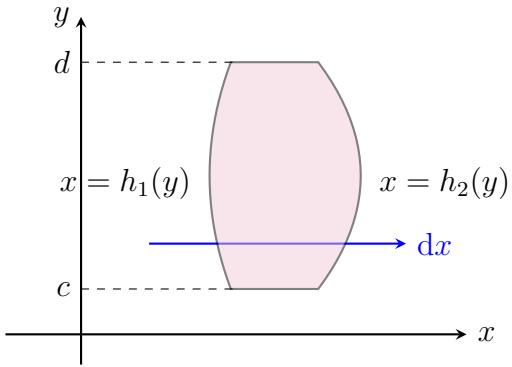
## 一般積分區域

**結論** (基本區域積分).
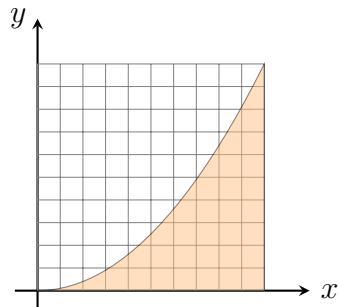
1. 選擇初始積分方向 $v$: 延 $x$ 方向為 $\mathrm{d}x$, 延 $y$ 方向為 $\mathrm{d}y$.

2. 想像積分方向 $v$ 為一射線, 進入區域之邊界函數為積分下界, 離開區域之邊界函數為積分上界; 邊界函數本身不含變數 $v$.

3. 次第區域為原始區域延 $v$ 之投影 (令 $v=$ (常數) 之方程式, 或邊界的交集); 若此投影區域維度 $>1$, 選擇積分方向.

$$\int_{\Omega} f(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$
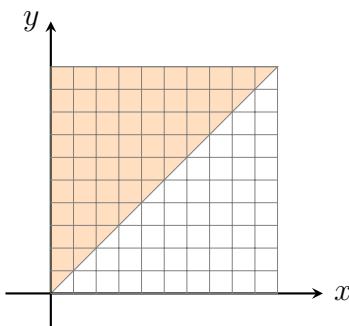


$$\int_{\Omega} f(x,y)\,\mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
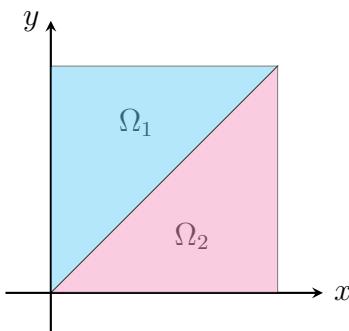


**例.** 求 $\displaystyle\int_{\Omega} x\cos y\,\mathrm{d}A$, $\Omega$ 為 $y=0$, $x=1$, 與 $y=x^2$ 圍成之區域.

**解.** $\displaystyle\int_{\Omega} x\cos y\,\mathrm{d}A = \int_0^1 \int_0^{x^2} x\cos y\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 x\left(\left.\sin y\,\right|_0^{x^2}\right)\mathrm{d}x$
$\displaystyle = \int_0^1 x\sin x^2\,\mathrm{d}x = \frac{1-\cos 1}{2}$



**例.** 求 $\displaystyle\int_{\Omega} e^{-y^2}\,\mathrm{d}A$, $\Omega = \{(x,y)\,|\,0\leqslant y\leqslant 1,\ 0\leqslant x\leqslant y\}$.
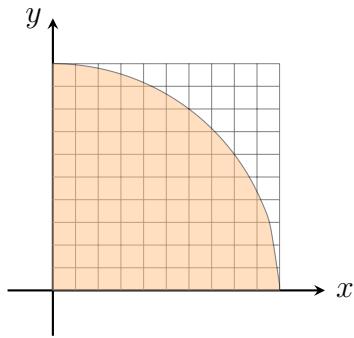
**解.** $\displaystyle\int_{\Omega} e^{-y^2}\,\mathrm{d}A = \int_0^1 \int_0^y e^{-y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 y e^{-y^2}\,\mathrm{d}y = \frac{1-e^{-1}}{2}$



**例.** 求 $\displaystyle\int_0^1 \int_0^1 e^{\max\{x^2,y^2\}}\,\mathrm{d}x\,\mathrm{d}y$.

**解.** $\displaystyle\int_0^1 \int_0^1 e^{\max\{x^2,y^2\}}\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega_1} e^{y^2}\,\mathrm{d}A + \int_{\Omega_2} e^{x^2}\,\mathrm{d}A$
$\displaystyle = \int_0^1 \int_0^y e^{y^2}\,\mathrm{d}x\,\mathrm{d}y + \int_0^1 \int_0^x e^{x^2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 y e^{y^2}\,\mathrm{d}y + \int_0^1 x e^{x^2}\,\mathrm{d}x = e-1.$

## 積分順序交換



**例.** 求 $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\,\mathrm{d}y\,\mathrm{d}x$.

**解.** $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1-y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 (1-y^2)\,\mathrm{d}y = \frac{2}{3}$.

$\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{2}\int_0^1 (\sin^{-1}\sqrt{1-x^2} + x\sqrt{1-x^2})\,\mathrm{d}x = \frac{1}{2}\left(1+\frac{1}{3}\right) = \frac{2}{3}$.

**例.** 將下列積分以不同積分順序寫出.

1. $\displaystyle\int_0^2 \int_{x^2}^{2x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$

2. $\displaystyle\int_0^1 \int_{\sin^{-1}y}^{\frac{\pi}{2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

3. $\displaystyle\int_{-1}^2 \int_{x^2}^{x+2} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$

4. $\displaystyle\int_0^1 \int_0^{2y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_1^3 \int_0^{3-y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

**解.**

1. $\displaystyle\int_0^2 \int_{x^2}^{2x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^4 \int_{\sqrt{y}}^{\frac{y}{2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

2. $\displaystyle\int_0^1 \int_{\sin^{-1}y}^{\frac{\pi}{2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{2}} \int_0^{\sin x} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

3. $\displaystyle\int_{-1}^2 \int_{x^2}^{x+2} f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_1^4 \int_{y-2}^{\sqrt{y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

4. $\displaystyle\int_0^1 \int_0^{2y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_1^3 \int_0^{3-y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^2 \int_{\frac{x}{2}}^{3-x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$

**例.** 證明 $\displaystyle\int_0^x \int_0^t F(u)\,\mathrm{d}u\,\mathrm{d}t = \int_0^x (x-u)F(u)\,\mathrm{d}u$.

**解.** $\displaystyle\int_0^x \int_0^t F(u)\,\mathrm{d}u\,\mathrm{d}t = \int_0^x \int_u^x F(u)\,\mathrm{d}t\,\mathrm{d}u = \int_0^x (x-u)F(u)\,\mathrm{d}u$.

**例.** 求 $\displaystyle\int_0^1 \int_x^1 e^{-y^2}\,\mathrm{d}y\,\mathrm{d}x$.

**解.** $\displaystyle\int_0^1 \int_x^1 e^{-y^2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_0^y e^{-y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 y e^{-y^2}\,\mathrm{d}y = \frac{1-e^{-1}}{2}$

**例.** 求 $\displaystyle\int_0^1 \int_y^1 \frac{\sin x}{x}\,\mathrm{d}x\,\mathrm{d}y$.

**解.** $\displaystyle\int_0^1 \int_y^1 \frac{\sin x}{x}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^x \frac{\sin x}{x}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \sin x\,\mathrm{d}x = 1 - \cos 1$.

**例.** 求 $\displaystyle\int_0^1 \int_{\sin^{-1}y}^{\frac{\pi}{2}} \cos x\sqrt{1+\cos^2 x}\,\mathrm{d}x\,\mathrm{d}y$.

**解.**

- $\int_0^1 \int_{\sin^{-1} y}^{\frac{\pi}{2}} \cos x \sqrt{1 + \cos^2 x}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{2}} \int_0^{\sin x} \cos x \sqrt{1 + \cos^2 x}\,\mathrm{d}y\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin x \cos x \sqrt{1 + \cos^2 x}\,\mathrm{d}x$

  $= \int_0^1 u\sqrt{1 + u^2}\,\mathrm{d}u = \int_1^2 \sqrt{v}\,\frac{1}{2}\,\mathrm{d}v = \dfrac{2\sqrt{2} - 1}{3}$, 其中變數變換 $u = \cos x$ 與 $v = 1 + u^2$.

- 令 $u = \sin x$, $\int_{\sin^{-1} y}^{\frac{\pi}{2}} \cos x \sqrt{1 + \cos^2 x}\,\mathrm{d}x = \int_y^1 \sqrt{2 - u^2}\,\mathrm{d}u$. 令 $u = \sqrt{2}\sin\theta$, 則 $\int_y^1 \sqrt{2 - u^2}\,\mathrm{d}u =$

  $\int_{\sin^{-1}\frac{y}{\sqrt{2}}}^{\frac{\pi}{4}} 2\cos^2\theta\,\mathrm{d}\theta = \int_{\sin^{-1}\frac{y}{\sqrt{2}}}^{\frac{\pi}{4}} (1 + \cos 2\theta)\,\mathrm{d}\theta = (\theta + \sin\theta\cos\theta)\Big|_{\sin^{-1}\frac{y}{\sqrt{2}}}^{\frac{\pi}{4}} = \dfrac{\pi}{4} + \dfrac{1}{2} - \sin^{-1}\dfrac{y}{\sqrt{2}} - \dfrac{y}{\sqrt{2}}\cdot\dfrac{\sqrt{2 - y^2}}{\sqrt{2}} =$

  $\dfrac{\pi}{4} + \dfrac{1}{2} - \sin^{-1}\dfrac{y}{\sqrt{2}} - \dfrac{y\sqrt{2 - y^2}}{2}$. 故 $\int_0^1 \int_{\sin^{-1} y}^{\frac{\pi}{2}} \cos x \sqrt{1 + \cos^2 x}\,\mathrm{d}x\,\mathrm{d}y$

  $= \int_0^1 \left( \dfrac{\pi}{4} + \dfrac{1}{2} - \sin^{-1}\dfrac{y}{\sqrt{2}} - \dfrac{y\sqrt{2 - y^2}}{2} \right)\mathrm{d}y = \dfrac{\pi}{4} + \dfrac{1}{2} - \left( \dfrac{\pi}{4} + 1 - \sqrt{2} \right) - \left( \dfrac{\sqrt{2}}{3} - \dfrac{1}{6} \right) = \dfrac{2\sqrt{2} - 1}{3}$.

**例.** 求 $\int_0^\infty \dfrac{e^{-ax} - e^{-bx}}{x}\,\mathrm{d}x$, $b > a > 0$.

**解.** 由 $\int_a^b e^{-xy}\,\mathrm{d}y = \dfrac{1}{-x}e^{-xy}\Big|_{y=a}^{y=b} = \dfrac{1}{-x}(e^{-xb} - e^{-xa}) = \dfrac{e^{-ax} - e^{-bx}}{x}$, $\int_0^\infty \dfrac{e^{-ax} - e^{-bx}}{x}\,\mathrm{d}x = \int_0^\infty \left( \int_a^b e^{-xy}\,\mathrm{d}y \right)\mathrm{d}x =$

$\int_a^b \left( \int_0^\infty e^{-xy}\,\mathrm{d}x \right)\mathrm{d}y = \int_a^b \dfrac{1}{y}\,\mathrm{d}y = \ln\dfrac{b}{a}$.

**例.** 求 $\int_0^\infty \dfrac{\tan^{-1}\pi x - \tan^{-1} x}{x}\,\mathrm{d}x$.

**解.** 由 $\int_x^{\pi x} \dfrac{1}{1 + y^2}\,\mathrm{d}y = \tan^{-1}\pi x - \tan^{-1} x$, $\int_0^\infty \dfrac{\tan^{-1}\pi x - \tan^{-1} x}{x}\,\mathrm{d}x = \int_0^\infty \dfrac{1}{x}\left( \int_x^{\pi x} \dfrac{1}{1 + y^2}\,\mathrm{d}y \right)\mathrm{d}x$

$= \int_0^\infty \int_x^{\pi x} \dfrac{1}{x(1 + y^2)}\,\mathrm{d}y\,\mathrm{d}x = \int_0^\infty \int_{\frac{y}{\pi}}^y \dfrac{1}{x(1 + y^2)}\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty \dfrac{1}{1 + y^2}\left( \int_{\frac{y}{\pi}}^y \dfrac{1}{x}\,\mathrm{d}x \right)\mathrm{d}y = \int_0^\infty \dfrac{1}{1 + y^2}\left( \ln y - \ln\dfrac{y}{\pi} \right)\mathrm{d}y =$

$\int_0^\infty \dfrac{\ln\pi}{1 + y^2}\,\mathrm{d}y = \dfrac{\pi}{2}\ln\pi$.

**例.** 求 $\int_0^\infty \dfrac{\sin x}{x}\,\mathrm{d}x$.

**解.** 由 $\int_0^\infty e^{-xy}\,\mathrm{d}y = \dfrac{1}{x}$, $x > 0$; 又 $\int e^{ax}\sin bx\,\mathrm{d}x = \dfrac{e^{ax}(a\cdot\sin bx - b\cdot\cos bx)}{a^2 + b^2}$.

故 $\int_0^\infty \dfrac{\sin x}{x}\,\mathrm{d}x = \int_0^\infty \sin x \left( \int_0^\infty e^{-xy}\,\mathrm{d}y \right)\mathrm{d}x = \int_0^\infty \left( \int_0^\infty e^{-xy}\sin x\,\mathrm{d}y \right)\mathrm{d}x = \int_0^\infty \left( \int_0^\infty e^{-xy}\sin x\,\mathrm{d}x \right)\mathrm{d}y =$

$\int_0^\infty \dfrac{e^{-xy}((-y)\cdot\sin x - 1\cdot\cos x)}{(-y)^2 + 1^2}\Big|_{x=0}^{x=\infty}\,\mathrm{d}y = \int_0^\infty \dfrac{0 - (-1)}{1 + y^2}\,\mathrm{d}y = \int_0^\infty \dfrac{1}{1 + y^2}\,\mathrm{d}y = \tan^{-1} y\Big|_0^\infty = \dfrac{\pi}{2}$.

### 變數變換

單變數積分 $\int_{[a,b]} f(x)\,\mathrm{d}x$ 中, 令 $x = x(u)$, 則 $\mathrm{d}x = \dfrac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u$; $\int_{[a,b]} f(x)\,\mathrm{d}x = \underbrace{\int_{x^{-1}[a,b]}}_{\text{由小到大}} f(x(u))\left| \dfrac{\mathrm{d}x}{\mathrm{d}u} \right|\,\mathrm{d}u$.

**例.**

- 求 $\int_0^1 \dfrac{\mathrm{d}x}{\sqrt{1 - x^2}}$: 令 $x = \sin u$, 則 $\mathrm{d}x = \cos u\cdot\mathrm{d}u$; $x$ 由 $0$ 至 $1$, 則 $u$ 由 $\sin^{-1} 0 = 0$ 至 $\sin^{-1} 1 = \dfrac{\pi}{2}$. 故

  $\int_0^1 \dfrac{\mathrm{d}x}{\sqrt{1 - x^2}} = \int_0^{\frac{\pi}{2}} \dfrac{\cos u\cdot\mathrm{d}u}{\sqrt{1 - \sin^2 u}} = \int_0^{\frac{\pi}{2}} \dfrac{\cos u\cdot\mathrm{d}u}{\cos u} = \int_0^{\frac{\pi}{2}} 1\cdot\mathrm{d}u = \dfrac{\pi}{2}$.

- 求 $\int_{\frac{1}{2}}^{1}(1-2x)^2\,\mathrm{d}x$: 令 $u=1-2x$, 亦即 $x=\frac{1}{2}-\frac{u}{2}$, 則 $\mathrm{d}x=\frac{-1}{2}\,\mathrm{d}u$; $x$ 由 $\frac{1}{2}$ 至 $1$, 則 $u$ 由 $1-2\cdot\frac{1}{2}=0$ 至 $1-2\cdot 1=-1$. 故 $\int_{\frac{1}{2}}^{1}(1-2x)^2\,\mathrm{d}x=\int_{0}^{-1}u^2\left(\frac{-1}{2}\right)\mathrm{d}u=\int_{-1}^{0}u^2\frac{1}{2}\,\mathrm{d}u=\frac{1}{6}$.

**定理** (重積分變數變換). 給定 $\Omega_{\mathbf{x}}$, $\Omega_{\mathbf{u}}\subseteq\mathbb{R}^n$, $\mathbf{x}=\mathbf{x}(\mathbf{u}):\Omega_{\mathbf{u}}\to\Omega_{\mathbf{x}}$ 且

- $\mathbf{x}$ 為嵌射 (亦即 $\mathbf{x}(\Omega_{\mathbf{u}})=\Omega_{\mathbf{x}}$, $\mathbf{x}^{-1}(\Omega_{\mathbf{x}})=\Omega_{\mathbf{u}}$)

- $\mathbf{x}$ 之各分量函數 $x_j$, $j=1,2\ldots,n$ 為連續可偏微分

- $\forall\,\mathbf{u}\in\Omega_{\mathbf{u}}$, $\mathbf{x}$ 之 Jacobian $J_{\mathbf{x}}(\mathbf{u})=\dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}(\mathbf{u})=\begin{vmatrix}\frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_n} & \cdots & \frac{\partial x_1}{\partial u_n}\\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_n} & \cdots & \frac{\partial x_2}{\partial u_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_n} & \cdots & \frac{\partial x_n}{\partial u_n}\end{vmatrix}(\mathbf{u})\neq 0$

則對於可積函數 $f:\Omega_{\mathbf{x}}\to\mathbb{R}$, $\displaystyle\int_{\Omega_{\mathbf{x}}}f(\mathbf{x})\,\mathrm{d}\mathbf{x}=\int_{\mathbf{x}^{-1}(\Omega_{\mathbf{x}})}f(\mathbf{x}(\mathbf{u}))\left|J_{\mathbf{x}}(\mathbf{u})\right|\mathrm{d}\mathbf{u}=\int_{\mathbf{x}^{-1}(\Omega_{\mathbf{x}})}f(\mathbf{x}(\mathbf{u}))\left|\dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}\right|\mathrm{d}\mathbf{u}$.

**例.** 利用變數變換 $x=u^2-v^2$, $y=2uv$ 求 $\displaystyle\int_{\Omega}y\,\mathrm{d}A$, $\Omega$ 為 $y\geqslant 0$, $y^2=4-4x$, $y^2=4+4x$ 所圍成之區域.

**解.** 由 $\begin{cases}x=u^2-v^2\\ y=2uv\end{cases}$, Jacobian $J_{\mathbf{x}}(\mathbf{u})=\dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}(\mathbf{u})=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}2u & -2v\\ 2v & 2u\end{vmatrix}=4\left(u^2+v^2\right)$. 變數變換 $(x,y)\to$ $(u,v)$ 後 $\Omega$ 由 $uv=0$, $u^2v^2+u^2-v^2-1=(v^2+1)(u^2-1)=0$, $u^2v^2-u^2+v^2-1=(u^2+1)(v^2-1)=0$ 圍成, 亦即 $\Omega=\{0\leqslant u\leqslant 1,\ 0\leqslant v\leqslant 1\}$, 故 $\displaystyle\int_{\Omega}y\,\mathrm{d}A=\int_{0}^{1}\int_{0}^{1}2uv\left|4\left(u^2+v^2\right)\right|\mathrm{d}u\,\mathrm{d}v=2$.

**例.** 求 $\displaystyle\int_{\Omega}(x+y)^2\,\mathrm{d}A$, $\Omega$ 為 $x+y=0$, $x+y=1$, $2x-y=0$, $2x-y=3$ 所圍成之平行四邊形.

**解.** 令 $\begin{cases}u=x+y\\ v=2x-y\end{cases}$, 則 $\begin{cases}x=\frac{1}{3}u+\frac{1}{3}v\\ y=\frac{2}{3}u-\frac{1}{3}v\end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u})=\dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}(\mathbf{u})=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{3} & \frac{1}{3}\\ \frac{2}{3} & \frac{-1}{3}\end{vmatrix}=-\frac{1}{3}$. 變數變換 $(x,y)\to(u,v)$ 後 $\Omega=\{0\leqslant u\leqslant 1,\ 0\leqslant v\leqslant 3\}$, 故 $\displaystyle\int_{\Omega}(x+y)^2\,\mathrm{d}A=\int_{0}^{3}\int_{0}^{1}u^2\left|-\frac{1}{3}\right|\mathrm{d}u\,\mathrm{d}v=\frac{1}{3}$.

**例.** 求 $\displaystyle\int_{\Omega}\sqrt{x+y}\,\mathrm{d}A$, $\Omega$ 為頂點 $(0,0)$, $(a,0)$, $(0,a)$, $a>0$ 之三角形.

**解.** 令 $\begin{cases}u=x+y\\ v=x-y\end{cases}$, 則 $\begin{cases}x=\frac{1}{2}u+\frac{1}{2}v\\ y=\frac{1}{2}u-\frac{1}{2}v\end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u})=\dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}(\mathbf{u})=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{-1}{2}\end{vmatrix}=-\frac{1}{2}$. 變數變換前 $\Omega$ 由 $x+y=0$, $x+y=a$, $x=0$, $y=0$ 所圍成, 變數變換 $(x,y)\to(u,v)$ 後 $\Omega$ 由 $u=0$, $u=a$, $u+v=0$, $u-v=0$ 圍成, 故 $\displaystyle\int_{\Omega}\sqrt{x+y}\,\mathrm{d}A=\int_{0}^{a}\int_{-u}^{u}\sqrt{u}\left|-\frac{1}{2}\right|\mathrm{d}v\,\mathrm{d}u=\int_{0}^{a}u\sqrt{u}\,\mathrm{d}u=\frac{2\,a^{\frac{5}{2}}}{5}$.

**例.** 求 $\displaystyle\int_{\Omega}(x+y)\,e^{x-y}\,\mathrm{d}A$, $\Omega$ 為頂點 $(4,0)$, $(6,2)$, $(4,4)$, $(2,2)$ 之四邊形.

**解.** 令 $\begin{cases}u=x+y\\ v=x-y\end{cases}$, 則 $\begin{cases}x=\frac{1}{2}u+\frac{1}{2}v\\ y=\frac{1}{2}u-\frac{1}{2}v\end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u})=\dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}(\mathbf{u})=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{-1}{2}\end{vmatrix}=-\frac{1}{2}$. 變數變換前 $\Omega$ 由 $x-y=0$, $x-y=4$, $x+y=4$, $x+y=8$ 所圍成, 變數變換 $(x,y)\to(u,v)$ 後 $\Omega$ 由 $v=0$, $v=4$, $u=4$, $u=8$ 圍成, 故 $\displaystyle\int_{\Omega}(x+y)\,e^{x-y}\,\mathrm{d}A=\int_{0}^{4}\int_{4}^{8}ue^v\left|-\frac{1}{2}\right|\mathrm{d}u\,\mathrm{d}v=\frac{1}{2}\int_{0}^{4}e^v\,\mathrm{d}v\int_{4}^{8}u\,\mathrm{d}u=12\left(e^4-1\right)$.

**例.** 求 $\displaystyle\int_\Omega e^{\frac{x+y}{x-y}}\,\mathrm{d}A$, $\Omega$ 為頂點 $(1,0)$, $(2,0)$, $(0,-2)$, $(0,-1)$ 之梯形.

**解.** 令 $\begin{cases} u = x+y \\ v = x-y \end{cases}$, 則 $\begin{cases} x = \frac{1}{2}u + \frac{1}{2}v \\ y = \frac{1}{2}u - \frac{1}{2}v \end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{vmatrix} = -\dfrac{1}{2}$. 變數變換前 $\Omega$ 由 $x - y = 1$, $x - y = 2$, $x = 0$, $y = 0$ 所圍成, 變數變換 $(x,y) \to (u,v)$ 後 $\Omega$ 由 $v = 1$, $v = 2$, $u + v = 0$, $u - v = 0$ 圍成, 故 $\displaystyle\int_\Omega e^{\frac{x+y}{x-y}}\,\mathrm{d}A = \int_1^2 \int_{-v}^{v} e^{\frac{u}{v}}\left|-\frac{1}{2}\right|\,\mathrm{d}u\,\mathrm{d}v = \frac{1}{2}\int_1^2 \left(v\,e^{\frac{u}{v}}\Big|_{u=-v}^{u=v}\right)\mathrm{d}v = \frac{3\left(e - e^{-1}\right)}{4}$.

**例.** 求 $y = x^2$, $y = 2x^2$, $x = y^2$, $x = 3y^2$ 所圍成之區域面積.

**解.** 令 $\begin{cases} u = \frac{y}{x^2} \\ v = \frac{x}{y^2} \end{cases}$, 則 $\begin{cases} x = (u^2 v)^{-\frac{1}{3}} \\ y = (u v^2)^{-\frac{1}{3}} \end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{3}(u^2 v)^{-\frac{4}{3}}\cdot 2uv & -\frac{1}{3}(u^2 v)^{-\frac{4}{3}}\cdot u^2 \\ -\frac{1}{3}(u v^2)^{-\frac{4}{3}}\cdot v^2 & -\frac{1}{3}(u v^2)^{-\frac{4}{3}}\cdot 2uv \end{vmatrix}$ $= \dfrac{1}{3u^2 v^2}$. 變數變換 $(x,y) \to (u,v)$ 後 $\Omega$ 由 $u = 1$, $u = 2$, $v = 1$, $v = 3$ 圍成, 亦即 $\Omega = \{1 \leqslant u \leqslant 2,\ 1 \leqslant v \leqslant 3\}$, 面積為 $\displaystyle\int_\Omega \mathrm{d}A = \int_1^3 \int_1^2 \left|\frac{1}{3u^2 v^2}\right|\,\mathrm{d}u\,\mathrm{d}v = \frac{1}{9}$.

**例.** 求 $\displaystyle\int_\Omega \frac{xy}{1 + x^2 y^2}\,\mathrm{d}A$, $\Omega$ 為 $xy = 1$, $xy = 5$, $x = 1$, $x = 6$ 所圍成之區域.

**解.** 令 $\begin{cases} u = xy \\ v = x \end{cases}$, 則 $\begin{cases} x = v \\ y = \frac{u}{v} \end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\dfrac{1}{v}$. 變數變換 $(x,y) \to (u,v)$ 後 $\Omega$ 由 $u = 1$, $u = 5$, $v = 1$, $v = 6$ 圍成, 亦即 $\Omega = \{1 \leqslant u \leqslant 5,\ 1 \leqslant v \leqslant 6\}$, 故 $\displaystyle\int_\Omega \frac{xy}{1 + x^2 y^2}\,\mathrm{d}A =$ $\displaystyle\int_1^6 \int_1^5 \frac{u}{1 + u^2}\left|-\frac{1}{v}\right|\,\mathrm{d}u\,\mathrm{d}v = \int_1^6 \frac{1}{v}\,\mathrm{d}v \int_1^5 \frac{u}{1 + u^2}\,\mathrm{d}u = \frac{\ln 6 \cdot \ln 13}{2}$.

**例.** 求 $\displaystyle\int_\Omega \frac{y}{x}\,\mathrm{d}A$, $\Omega$ 為 $y = x$, $x^2 + 4y^2 = 4$, $y \geqslant 0$ 所圍成之區域.

**解.** 令 $\begin{cases} x = 2r\cos\theta \\ y = r\sin\theta \end{cases}$, Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2\cos\theta & -2r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = 2r$. 變數變換 $(x,y) \to (r,\theta)$ 後 $\Omega$ 由 $r = 0$, $r = 1$, $\theta = 0$, $\theta = \tan^{-1} 2$ 圍成, 亦即 $\Omega = \{0 \leqslant r \leqslant 1,\ 0 \leqslant \theta \leqslant \tan^{-1} 2\}$, 故 $\displaystyle\int_\Omega \frac{y}{x}\,\mathrm{d}A = \int_0^{\tan^{-1} 2}\int_0^1 \frac{1}{2}\tan\theta\,|2r|\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\tan^{-1} 2}\tan\theta\,\mathrm{d}\theta \int_0^1 r\,\mathrm{d}r = -\frac{1}{2}\ln|\cos\theta|\Big|_0^{\tan^{-1} 2} = \frac{\ln 5}{4}$.

**例.** 求 $\displaystyle\int_\Omega \sin(9x^2 + 4y^2)\,\mathrm{d}A$, $\Omega$ 為 $9x^2 + 4y^2 = 1$, $y \geqslant 0$, $x \geqslant 0$ 所圍成之區域.

**解.** 令 $\begin{cases} x = \frac{1}{3}r\cos\theta \\ y = \frac{1}{2}r\sin\theta \end{cases}$, Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{1}{3}\cos\theta & -\frac{1}{3}r\sin\theta \\ \frac{1}{2}\sin\theta & \frac{1}{2}r\cos\theta \end{vmatrix} = \frac{1}{6}r$. 變數變換 $(x,y) \to (r,\theta)$ 後 $\Omega$ 由 $r = 0$, $r = 1$, $\theta = 0$, $\theta = \frac{\pi}{2}$ 圍成, 亦即 $\Omega = \{0 \leqslant r \leqslant 1,\ 0 \leqslant \theta \leqslant \frac{\pi}{2}\}$, 故 $\displaystyle\int_\Omega \sin(9x^2 + 4y^2)\,\mathrm{d}A =$ $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^1 \sin r^2\left|\frac{1}{6}r\right|\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{6}\int_0^{\frac{\pi}{2}}\mathrm{d}\theta \int_0^1 r\sin r^2\,\mathrm{d}r = \frac{\pi}{24}(1 - \cos 1)$.

**例.** 求 $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x + y}\,(y - 2x)^2\,\mathrm{d}y\,\mathrm{d}x$.

**解.** 令 $\begin{cases} u = x + y \\ v = y - 2x \end{cases}$, 則 $\begin{cases} x = \frac{1}{3}u - \frac{1}{3}v \\ y = \frac{2}{3}u + \frac{1}{3}v \end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{-1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \dfrac{1}{3}$. 變數變換

前積分區域由 $x + y = 0$, $x + y = 1$, $x = 0$, $y = 0$ 所圍成, 變數變換 $(x, y) \to (u, v)$ 後積分區域由 $u = 0$, $u = 1$, $u - v = 0$, $2u + v = 0$ 圍成, 故 $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2 \,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_{-2u}^u \sqrt{u}\,v^2 \left|\frac{1}{3}\right| \mathrm{d}v\,\mathrm{d}u =$

$\dfrac{1}{3} \displaystyle\int_0^1 \sqrt{u} \left( \dfrac{v^3}{3}\, \Big|_{v=-2u}^{v=u} \right) \mathrm{d}u = \dfrac{2}{9}$.

**例.** 若 $\Omega$ 為頂點 $(0, 0)$, $(1, 0)$, $(0, 1)$ 之三角形, $f$ 為可積, 證明 $\displaystyle\int_\Omega f(x+y) \,\mathrm{d}A = \int_0^1 u\,f(u) \,\mathrm{d}u$.

**解.** 令 $\begin{cases} u = x + y \\ v = x \end{cases}$, 則 $\begin{cases} x = v \\ y = u - v \end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$. 變數變換前 $\Omega$ 由

$x + y = 0$, $x + y = 1$, $x = 0$, $y = 0$ 所圍成, 變數變換 $(x, y) \to (u, v)$ 後 $\Omega$ 由 $u = 0$, $u = 1$, $v = 0$, $u - v = 0$

圍成, 故 $\displaystyle\int_\Omega f(x+y) \,\mathrm{d}A = \int_0^1 \int_0^u f(u) \,\big| -1 \big| \,\mathrm{d}v\,\mathrm{d}u = \int_0^1 u\,f(u) \,\mathrm{d}u$.

**例.**　1. 證明 $\displaystyle\int_0^1 \int_0^1 \dfrac{1}{1-xy} \,\mathrm{d}x\,\mathrm{d}y = \sum_{n=1}^\infty \dfrac{1}{n^2}$.　　　　2. 以變數變換證明 $\displaystyle\int_0^1 \int_0^1 \dfrac{1}{1-xy} \,\mathrm{d}x\,\mathrm{d}y = \dfrac{\pi^2}{6}$.

**解.**

1. $|xy| < 1$ 時 $\dfrac{1}{1-xy} = \displaystyle\sum_{n=0}^\infty (xy)^n$ 且可逐項積分; 代入 $\displaystyle\int_0^1 \int_0^1 \dfrac{1}{1-xy} \,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^1 \sum_{n=0}^\infty (xy)^n \,\mathrm{d}x\,\mathrm{d}y =$

    $\displaystyle\sum_{n=0}^\infty \int_0^1 x^n \,\mathrm{d}x \int_0^1 y^n \,\mathrm{d}y = \sum_{n=0}^\infty \dfrac{1}{(n+1)^2} = \sum_{n=1}^\infty \dfrac{1}{n^2}$.

2. 令 $\begin{cases} x = \frac{u-v}{\sqrt{2}} \\ y = \frac{u+v}{\sqrt{2}} \end{cases}$; Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = 1$. 變數變換前積分區域由 $x = 0$,

    $x = 1$, $y = 0$, $y = 1$ 所圍成, 變數變換 $(x, y) \to (u, v)$ 後積分區域由 $u - v = 0$, $u - v = \sqrt{2}$,

    $u + v = 0$, $u + v = \sqrt{2}$ 圍成, 故 $\displaystyle\int_0^1 \int_0^1 \dfrac{1}{1-xy} \,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\sqrt{2}}{2}} \int_{-u}^u \dfrac{\mathrm{d}v\,\mathrm{d}u}{1 - \frac{u-v}{\sqrt{2}} \frac{u+v}{\sqrt{2}}} + \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \int_{-\sqrt{2}+u}^{\sqrt{2}-u} \dfrac{\mathrm{d}v\,\mathrm{d}u}{1 - \frac{u-v}{\sqrt{2}} \frac{u+v}{\sqrt{2}}} =$

    $2 \left( \displaystyle\int_0^{\frac{\sqrt{2}}{2}} \int_{-u}^u \dfrac{\mathrm{d}v\,\mathrm{d}u}{2 - u^2 + v^2} + \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \int_{-\sqrt{2}+u}^{\sqrt{2}-u} \dfrac{\mathrm{d}v\,\mathrm{d}u}{2 - u^2 + v^2} \right) = 2 \left( \displaystyle\int_0^{\frac{\sqrt{2}}{2}} \dfrac{1}{\sqrt{2-u^2}} \left( \tan^{-1} \dfrac{v}{\sqrt{2-u^2}}\, \Big|_{v=-u}^{v=u} \right) \mathrm{d}u \right.$

    $\left. + \displaystyle\int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \dfrac{1}{\sqrt{2-u^2}} \left( \tan^{-1} \dfrac{v}{\sqrt{2-u^2}}\, \Big|_{v=-\sqrt{2}+u}^{v=\sqrt{2}-u} \right) \mathrm{d}u \right)$

    $= 4 \left( \displaystyle\int_0^{\frac{\sqrt{2}}{2}} \dfrac{1}{\sqrt{2-u^2}} \tan^{-1} \dfrac{u}{\sqrt{2-u^2}} \,\mathrm{d}u + \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \dfrac{1}{\sqrt{2-u^2}} \tan^{-1} \dfrac{\sqrt{2}-u}{\sqrt{2-u^2}} \,\mathrm{d}u \right)$. 令 $u = \sqrt{2} \sin\theta$, 則 $\mathrm{d}u =$

    $\sqrt{2} \cos\theta \,\mathrm{d}\theta$; 化簡 $4 \left( \displaystyle\int_0^{\frac{\sqrt{2}}{2}} \dfrac{1}{\sqrt{2-u^2}} \tan^{-1} \dfrac{u}{\sqrt{2-u^2}} \,\mathrm{d}u + \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \dfrac{1}{\sqrt{2-u^2}} \tan^{-1} \dfrac{\sqrt{2}-u}{\sqrt{2-u^2}} \,\mathrm{d}u \right)$

    $= 4 \left( \displaystyle\int_0^{\frac{\pi}{6}} \tan^{-1}(\tan\theta) \,\mathrm{d}\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \tan^{-1} \left( \dfrac{1 - \sin\theta}{\cos\theta} \right) \mathrm{d}\theta \right) = 4 \left( \displaystyle\int_0^{\frac{\pi}{6}} \theta \,\mathrm{d}\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \dfrac{1}{2} \left( \dfrac{\pi}{2} - \theta \right) \mathrm{d}\theta \right) = \dfrac{\pi^2}{6}$; 其中

    $\dfrac{1 - \sin\theta}{\cos\theta} = \dfrac{1 - \cos\left(\frac{\pi}{2} - \theta\right)}{\sin\left(\frac{\pi}{2} - \theta\right)} = \dfrac{2 \sin^2 \frac{1}{2}\left(\frac{\pi}{2} - \theta\right)}{2 \sin \frac{1}{2}\left(\frac{\pi}{2} - \theta\right) \cos \frac{1}{2}\left(\frac{\pi}{2} - \theta\right)} = \tan \dfrac{1}{2}\left(\dfrac{\pi}{2} - \theta\right)$.

## 極座標二重積分

令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, Jacobian $J_{\mathbf{x}}(\mathbf{u}) = \dfrac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$.

**例.** 求 $\displaystyle\int_\Omega (3x+4y^2)\,\mathrm{d}A$, $\Omega$ 為 $x^2+y^2=1$, $x^2+y^2=4$ 與 $y\geqslant 0$ 所圍成之區域.

**解.** 變數變換 $(x,y)\to(r,\theta)$ 後 $\Omega$ 由 $r=1$, $r=2$, $\theta=0$, $\theta=\pi$ 圍成, 故 $\displaystyle\int_\Omega(3x+4y^2)\,\mathrm{d}A=\int_0^\pi\int_1^2(3r\cos\theta+$
$4r^2\sin^2\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta=\displaystyle\int_0^\pi(7\cos\theta+15\sin^2\theta)\,\mathrm{d}\theta=\frac{15\,\pi}{2}$.

**例.** 求 $z=1-x^2-y^2$ 與 $z\geqslant 0$ 所圍成之體積.

**解.** 體積為 $\displaystyle\int_\Omega(1-x^2-y^2)\,\mathrm{d}A$, $\Omega$ 為單位圓 $x^2+y^2=1$. 變數變換 $(x,y)\to(r,\theta)$ 後 $\Omega$ 由 $r=0$, $r=1$, $\theta=0$,
$\theta=2\pi$ 圍成, 故 $\displaystyle\int_\Omega(1-x^2-y^2)\,\mathrm{d}A=\int_0^{2\pi}\int_0^1(1-r^2)\,r\,\mathrm{d}r\,\mathrm{d}\theta=\frac{\pi}{2}$.

**例.** 求 $\displaystyle\int_\Omega\frac{y^2}{x^2}\,\mathrm{d}A$, $\Omega$ 為 $0<a^2\leqslant x^2+y^2\leqslant b^2$ 與 $x\geqslant y\geqslant 0$ 所圍成之區域.

**解.** 變數變換 $(x,y)\to(r,\theta)$ 後 $\Omega$ 由 $r=a$, $r=b$, $\theta=0$, $\theta=\dfrac{\pi}{4}$ 圍成, 故 $\displaystyle\int_\Omega\frac{y^2}{x^2}\,\mathrm{d}A=\int_0^{\frac{\pi}{4}}\int_a^b\tan^2\theta\cdot r\,\mathrm{d}r\,\mathrm{d}\theta=$
$\dfrac{b^2-a^2}{2}\displaystyle\int_0^{\frac{\pi}{4}}\tan^2\theta\,\mathrm{d}\theta=\frac{b^2-a^2}{2}\int_0^{\frac{\pi}{4}}(\sec^2\theta-1)\,\mathrm{d}\theta=\frac{b^2-a^2}{2}(\tan\theta-\theta)\Big|_0^{\frac{\pi}{4}}=\frac{b^2-a^2}{2}\Big(1-\frac{\pi}{4}\Big)$.

**例.** 反變數變換: $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^a\frac{r}{\sqrt{a^2-r^2\cos^2\theta}}\,\mathrm{d}r\,\mathrm{d}\theta=\int_0^a\int_0^{\sqrt{a^2-x^2}}\frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}y\,\mathrm{d}x=\int_0^a\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\,\mathrm{d}x=a$.

## 曲面表面積

**結論.** 若曲面 $S$ 由 $z=f(x,y)$, $(x,y)\in\Omega$ 所定義, 則 $S$ 的表面積為 $\displaystyle\int_\Omega\sqrt{1+f_x^2+f_y^2}\,\mathrm{d}A$.

**證** (sketch). 以切平面近似曲面 $S$; $S$ 參數式為 $\mathbf{r}=(x,y,f(x,y))$, $x$, $y$ 方向切向量分別為 $\mathbf{r}_x=(1,0,f_x)$,
$\mathbf{r}_y=(0,1,f_y)$; $|\mathbf{r}_x\times\mathbf{r}_y|=|(-f_x,-f_y,1)|=\sqrt{1+f_x^2+f_y^2}$.

**例.** 求半球面 $z=\sqrt{a^2-x^2-y^2}$ 與圓柱體 $(x-\frac{a}{2})^2+y^2=(\frac{a}{2})^2$ 相交區域之表面積.

**解.** 由 $z=f(x,y)=\sqrt{a^2-x^2-y^2}$, $f_x=\dfrac{-x}{\sqrt{a^2-x^2-y^2}}$, $f_y=\dfrac{-y}{\sqrt{a^2-x^2-y^2}}$, $\mathrm{d}S=\sqrt{1+f_x^2+f_y^2}\,\mathrm{d}x\,\mathrm{d}y=$
$\sqrt{1+\dfrac{x^2}{a^2-x^2-y^2}+\dfrac{y^2}{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y=\sqrt{\dfrac{a^2}{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y$. 所求表面積為 $\displaystyle\int_\Omega\mathrm{d}S$, $\Omega=\big\{(x,y)\,\big|\,(x-$
$\frac{a}{2})^2+y^2\leqslant(\frac{a}{2})^2\big\}$; $\displaystyle\int_\Omega\mathrm{d}S=\int_\Omega\sqrt{\frac{a^2}{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y=2\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}\sqrt{\frac{a^2}{a^2-r^2}}\,r\,\mathrm{d}r\,\mathrm{d}\theta$
$=2a\displaystyle\int_0^{\frac{\pi}{2}}\Big(-\sqrt{a^2-r^2}\,\Big|_{r=0}^{r=a\cos\theta}\Big)\,\mathrm{d}\theta=2a^2\int_0^{\frac{\pi}{2}}(a-\sin\theta)\,\mathrm{d}\theta=a^2(\pi-2)$.

# 7.2 三重積分

## 一般區域積分

**例.** 若 $E$ 為中心 $0$, 半徑 $a$ 之球體, 寫出 $\displaystyle\int_E f(x,y,z)\,\mathrm{d}V$.

**解.** $\displaystyle\int_E f(x,y,z)\,\mathrm{d}V$

$\displaystyle = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$

$\displaystyle = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-z^2}}^{\sqrt{a^2-x^2-z^2}} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x = \int_{-a}^{a}\int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}}\int_{-\sqrt{a^2-x^2-z^2}}^{\sqrt{a^2-x^2-z^2}} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$

$\displaystyle = \int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-y^2-z^2}}^{\sqrt{a^2-y^2-z^2}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y = \int_{-a}^{a}\int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}}\int_{-\sqrt{a^2-y^2-z^2}}^{\sqrt{a^2-y^2-z^2}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

**例.** 若 $E$ 為第一卦限中 $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}\leqslant 1$ 圍成之區域, $a,\ b,\ c>0$, 寫出 $\displaystyle\int_E f(x,y,z)\,\mathrm{d}V$.
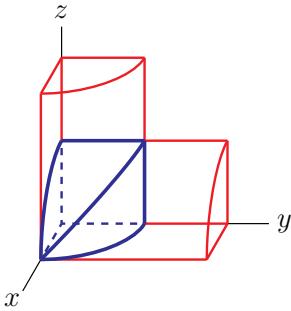
**解.** $\displaystyle\int_E f(x,y,z)\,\mathrm{d}V$

$\displaystyle = \int_0^a\int_0^{b-\frac{bx}{a}}\int_0^{c-\frac{cx}{a}-\frac{cy}{b}} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^b\int_0^{a-\frac{ay}{b}}\int_0^{c-\frac{cx}{a}-\frac{cy}{b}} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$

$\displaystyle = \int_0^a\int_0^{c-\frac{cx}{a}}\int_0^{b-\frac{bx}{a}-\frac{bz}{c}} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x = \int_0^c\int_0^{a-\frac{az}{c}}\int_0^{b-\frac{bx}{a}-\frac{bz}{c}} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$

$\displaystyle = \int_0^b\int_0^{c-\frac{cy}{b}}\int_0^{a-\frac{ay}{b}-\frac{az}{c}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y = \int_0^c\int_0^{b-\frac{bz}{c}}\int_0^{a-\frac{ay}{b}-\frac{az}{c}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

**例.** 求 $x^2+y^2=a^2$ 與 $x^2+z^2=a^2$ 交集區域之表面積與體積.

**解.**
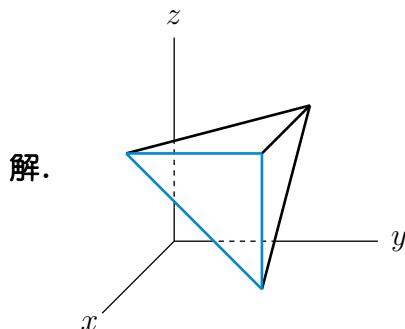


$\displaystyle(\text{表面積}) = 16\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{1+\left(\frac{-x}{\sqrt{a^2-x^2}}\right)^2}\,\mathrm{d}y\,\mathrm{d}x = 16\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{\frac{a^2}{a^2-x^2}}\,\mathrm{d}y\,\mathrm{d}x$

$\displaystyle\qquad = 16a\int_0^a\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\,\mathrm{d}x = 16a^2.$

$\displaystyle(\text{體積}) = 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2}}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = 8\int_0^a(a^2-x^2)\,\mathrm{d}x = \frac{16a^3}{3}.$

**例.** 若 $E$ 為 $x=1,\ y=1,\ z=1,\ x+y+z=2$ 圍成之四面體, 求 $\displaystyle\int_E x\,\mathrm{d}V$.

**解.**
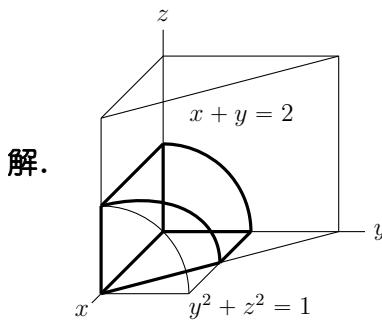


$\displaystyle\int_E x\,\mathrm{d}V = \int_0^1\int_{1-x}^1\int_{2-x-y}^1 x\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\int_{1-x}^1 x\,(x+y-1)\,\mathrm{d}y\,\mathrm{d}x$

$\displaystyle\qquad = \int_0^1\int_{1-x}^1 (x^2-x+xy)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\left((x^2-x)\,x + x\,\frac{1-(1-x)^2}{2}\right)\mathrm{d}x$

$\displaystyle\qquad = \int_0^1\left(x^3-x^2+x\,\frac{2x-x^2}{2}\right)\mathrm{d}x = \int_0^1\frac{x^3}{2}\,\mathrm{d}x = \frac{1}{8}$
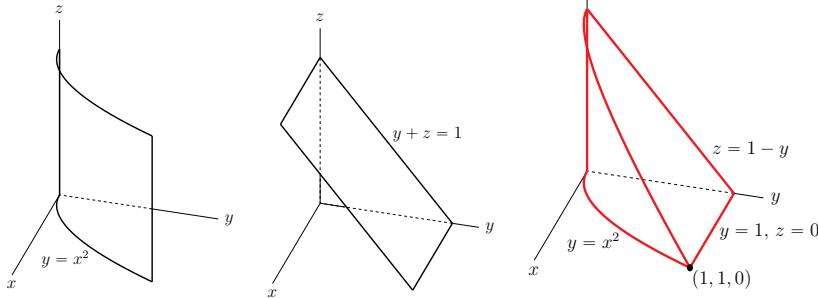
**例.** 若 $E$ 為第一卦限中 $y^2+z^2=1,\ x+y=2,\ y=0,\ z=0$ 圍成之區域, 求 $\displaystyle\int_E z\,\mathrm{d}V$.

**解.**



$$\int_E z\,\mathrm{d}V = \int_0^1 \int_0^{2-y} \int_0^{\sqrt{1-y^2}} z\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \frac{1}{2}(1-y^2)(2-y)\,\mathrm{d}y$$

$$= \frac{1}{2}\int_0^1 (2-y-2y^2+y^3)\,\mathrm{d}y = \frac{1}{2}\left(2-\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right) = \frac{1}{2}\cdot\frac{13}{12} = \frac{13}{24}$$

**例.** 若 $E$ 為第一卦限中 $y=x^2$, $y+z=1$ 圍成之區域, 求 $\displaystyle\int_E x\,\mathrm{d}V$.

**解.**
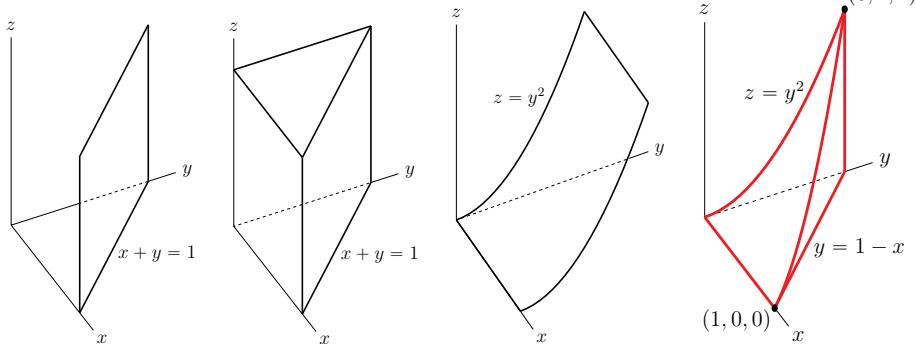


$$\int_E x\,\mathrm{d}V = \int_0^1 \int_{x^2}^1 \int_0^{1-y} x\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^1 \int_{x^2}^1 x\,(1-y)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^1 \left( x(1-x^2) - x\left(\frac{y^2}{2}\Big|_{y=x^2}^{y=1}\right)\right)\mathrm{d}x$$

$$= \int_0^1 \left(x - x^3 + \frac{x^5}{2} - \frac{x}{2}\right)\mathrm{d}x = \frac{1}{12}$$

**例.** 若 $E$ 為第一卦限中 $z=y^2$, $x+y=1$ 圍成之區域, 求 $\displaystyle\int_E z\,\mathrm{d}V$.

**解.**
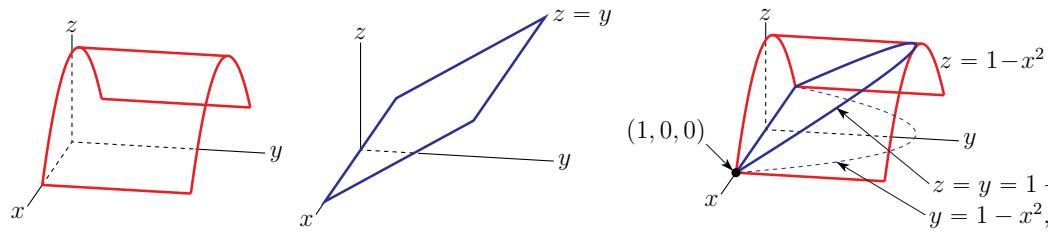


$$\int_E z\,\mathrm{d}V = \int_0^1 \int_0^{1-x} \int_0^{y^2} z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^1 \int_0^{1-x} \frac{y^4}{2}\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^1 \frac{(1-x)^5}{10}\,\mathrm{d}x = \frac{1}{60}$$
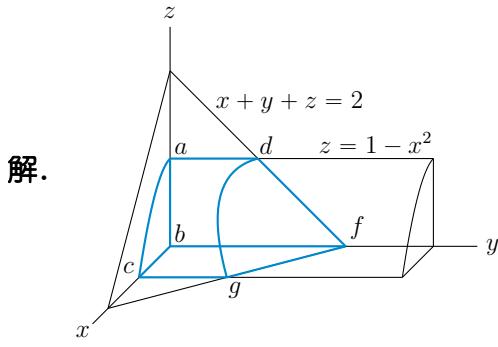
**例.** 若 $E$ 為 $z=1-x^2$, $y=z$, $y=0$, $z=0$ 圍成之區域, 求 $\displaystyle\int_E \mathrm{d}V$.

**解.**



$$\int_E \mathrm{d}V$$

$$= \int_{-1}^1 \int_0^{1-x^2} \int_y^{1-x^2} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_{-1}^1 \frac{(1-x^2)^2}{2}\,\mathrm{d}x = \frac{8}{15}$$

**例.** 若 $E$ 為 $x=0$, $y=0$, $z=0$, $x+y+z=2$, $x^2+z=1$ 圍成之區域, 求 $\displaystyle\int_E x\,\mathrm{d}V$.

解.

$$\int_E x\,\mathrm{d}V = \int_0^1 \int_0^{1-x^2} \int_0^{2-x-z} x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x = \int_0^1 \int_0^{1-x^2} (2-x-z)\,x\,\mathrm{d}z\,\mathrm{d}x$$

$$= \int_0^1 \left( (2-x)\,x\,(1-x^2) - x\,\frac{(1-x^2)^2}{2} \right)\mathrm{d}x$$

$$= \frac{1}{2}\int_0^1 (-x^5 + 2x^4 - 2x^3 - 2x^2 + 3x)\,\mathrm{d}x$$

$$= \frac{1}{2}\left( -\frac{1}{6} + \frac{2}{5} - \frac{1}{2} - \frac{2}{3} + \frac{3}{2} \right) = \frac{17}{60}$$

例. 若 $E$ 為 $x+y+z=1$, $x=0$, $y=0$, $z=0$ 圍成之區域, 求 $\int_E z\,\mathrm{d}V$.
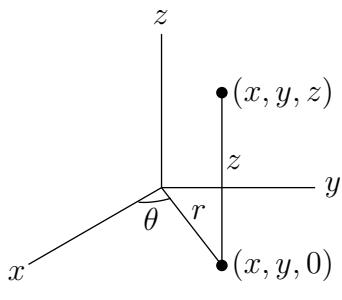
解. $\int_E z\,\mathrm{d}V = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_0^{1-x} \frac{(1-x-y)^2}{2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \left. -\frac{(1-x-y)^3}{6}\right|_{y=0}^{y=1-x}\mathrm{d}x$

$= \int_0^1 \frac{1}{6}(1-x)^3\,\mathrm{d}x = -\frac{1}{24}(1-x)^4\Big|_0^1 = \frac{1}{24}.$

例. 若 $E$ 為 $x+2y+z=2$, $x=2y$, $x=0$, $z=0$ 圍成之區域, 求 $\int_E y\,\mathrm{d}V$.

解. $\int_E y\,\mathrm{d}V = \int_0^1 \int_{\frac{x}{2}}^{\frac{2-x}{2}} \int_0^{2-x-2y} y\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_{\frac{x}{2}}^{\frac{2-x}{2}} (2-x-2y)\,y\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \left. \left( \frac{(2-x)y^2}{2} - \frac{2y^3}{3} \right)\right|_{\frac{x}{2}}^{\frac{2-x}{2}}\mathrm{d}x =$

$\int_0^1 \left( \frac{2-x}{2}\left( \left(\frac{2-x}{2}\right)^2 - \left(\frac{x}{2}\right)^2 \right) - \frac{2}{3}\left( \left(\frac{2-x}{2}\right)^3 - \left(\frac{x}{2}\right)^3 \right) \right)\mathrm{d}x = \int_0^1 \left( \frac{2-x}{2}(1-x) - \frac{2}{3}(1-x)\left( \left(\frac{2-x}{2}\right)^2 + \right.\right.$

$\left.\left. \frac{2-x}{2}\cdot\frac{x}{2} + \left(\frac{x}{2}\right)^2 \right) \right)\mathrm{d}x = \frac{1}{6}\int_0^1 (x^3 - 3x + 2)\,\mathrm{d}x = \frac{1}{6}\left( \frac{1}{4} - \frac{3}{2} + 2 \right) = \frac{1}{8}.$

## 變數變換
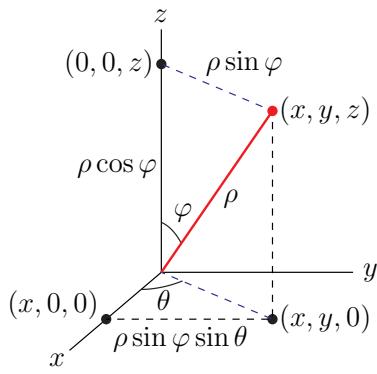
### 柱面座標



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad \Longleftrightarrow \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\frac{y}{x} \\ z = z \end{cases}$$

$$\text{Jacobian } J_{\mathbf{x}}(\mathbf{u}) = \frac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

# 球面座標
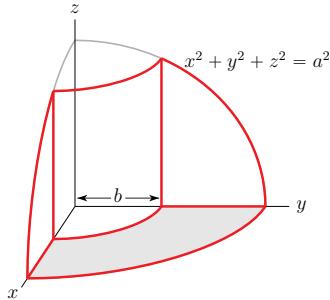


$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases} \iff \begin{cases} \rho = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1} \frac{y}{x} \\ \varphi = \tan^{-1} \frac{\sqrt{x^2+y^2}}{z} \end{cases}$$

$$\text{Jacobian } J_{\mathbf{x}}(\mathbf{u}) = \frac{\partial \mathbf{x}}{\partial \mathbf{u}}(\mathbf{u}) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\varphi \cos\theta & -\rho \sin\varphi \sin\theta & \rho \cos\varphi \cos\theta \\ \sin\varphi \sin\theta & \rho \sin\varphi \cos\theta & \rho \cos\varphi \sin\theta \\ \cos\varphi & 0 & -\rho \sin\varphi \end{vmatrix} = -\rho^2 \sin\varphi$$

**例.** 半徑為 $a$ 之球中心對稱鑽半徑為 $b$ 之圓孔, $a > b > 0$, 求球剩下的體積.

**解.**
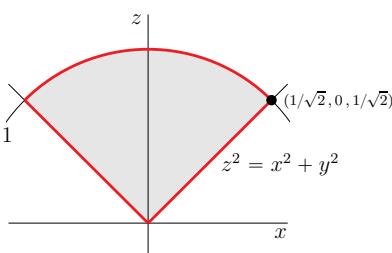


- 柱面座標: $\displaystyle\int_E \mathrm{d}V = 8\int_0^{\frac{\pi}{2}}\int_b^a \int_0^{\sqrt{a^2-r^2}} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta = \frac{4\pi}{3}(a^2-b^2)^{\frac{3}{2}}$

- 球面座標: $\displaystyle\int_E \mathrm{d}V = 2\int_{\sin^{-1}\frac{b}{a}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_{\frac{b}{\sin\varphi}}^a \rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi = \frac{4\pi}{3}(a^2-b^2)^{\frac{3}{2}}$

**例.** 若 $E$ 為 $x^2 + y^2 \leqslant z^2$, $x^2 + y^2 + z^2 \leqslant 1$ 與 $z \geqslant 0$ 圍成之區域, 求 $\displaystyle\int_E \sqrt{x^2 + y^2 + z^2}\,\mathrm{d}V$.
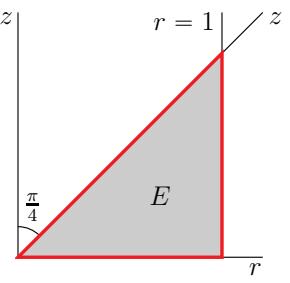
**解.**



- 柱面座標: $\displaystyle\int_E \sqrt{x^2 + y^2 + z^2}\,\mathrm{d}V$
$$= \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\int_r^{\sqrt{1-r^2}} \sqrt{r^2+z^2}\cdot r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

- 球面座標: $\displaystyle\int_E \sqrt{x^2 + y^2 + z^2}\,\mathrm{d}V$
$$= \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^1 \rho\cdot\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi = \frac{\pi}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$$
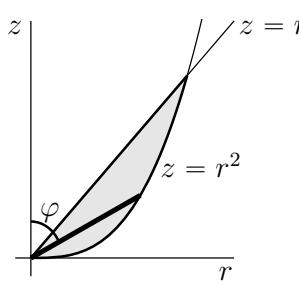
**例.** 若 $E$ 為 $0 \leqslant z \leqslant \sqrt{x^2 + y^2}$ 與 $x^2 + y^2 \leqslant 1$ 圍成之區域, 求 $\displaystyle\int_E z\sqrt{x^2 + y^2 + z^2}\,\mathrm{d}V$.

**解.**



- 柱面座標: $\displaystyle\int_E z\sqrt{x^2 + y^2 + z^2}\,\mathrm{d}V = \int_0^{2\pi}\int_0^1\int_0^r z\sqrt{r^2+z^2}\cdot r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$
$$= 2\pi\int_0^1 \frac{r}{3}(r^2+z^2)^{\frac{3}{2}}\Big|_{z=0}^{z=r}\mathrm{d}r = \frac{2\pi}{3}\int_0^1 r\cdot(2^{\frac{3}{2}}-1)r^3\,\mathrm{d}r = \frac{2\pi\,(2^{\frac{3}{2}}-1)}{15}$$

- 球面座標: $\displaystyle\int_E z\sqrt{x^2 + y^2 + z^2}\,\mathrm{d}V = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{\frac{1}{\sin\varphi}} \rho\cos\varphi\cdot\rho\cdot\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$

**例.** 若 $E$ 為 $z = x^2 + y^2$ 與 $z \leqslant \sqrt{x^2 + y^2}$ 圍成之區域, 求 $\displaystyle\int_E z\,(x^2 + y^2 + z^2)\,\mathrm{d}V$.

**解.**

- 柱面座標: $\displaystyle\int_E z\,(x^2+y^2+z^2)\,\mathrm{d}V=\int_0^{2\pi}\int_0^1\int_{r^2}^{r} z\,(r^2+z^2)\cdot r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$

$$=2\pi\int_0^1\int_{r^2}^{r}(r^3z+rz^3)\,\mathrm{d}z\,\mathrm{d}r=2\pi\int_0^1\left(\frac{r^3}{2}(r^2-r^4)+\frac{r}{4}(r^4-r^8)\right)\mathrm{d}r=\frac{3\pi}{40}$$

- 球面座標: $z=x^2+y^2 \implies \rho\cos\varphi=\rho^2\sin^2\varphi \implies \rho=\dfrac{\cos\varphi}{\sin^2\varphi};$

$z=\sqrt{x^2+y^2} \implies \rho\cos\varphi=\rho\sin\varphi \implies \tan\varphi=1;$

$$\int_E z\,(x^2+y^2+z^2)\,\mathrm{d}V=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{\frac{\cos\varphi}{\sin^2\varphi}}\rho\cos\varphi\cdot\rho^2\cdot\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$$
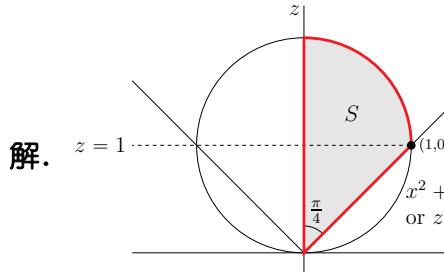
**例.** 若 $E$ 為 $y=x^2+z^2$ 與 $y=4$ 圍成之區域, 求 $\displaystyle\int_E\sqrt{x^2+z^2}\,\mathrm{d}V.$

**解.** 令 $\begin{cases}x=r\cos\theta\\z=r\sin\theta\end{cases}$, 投影之 $\Omega$ 為 $x^2+z^2\leqslant 4$, 則 $\displaystyle\int_E\sqrt{x^2+z^2}\,\mathrm{d}V=\int_\Omega\int_{x^2+z^2}^{4}\sqrt{x^2+z^2}\,\mathrm{d}y\,\mathrm{d}A$

$$=\int_\Omega(4-(x^2+z^2))\sqrt{x^2+z^2}\,\mathrm{d}A=\int_0^{2\pi}\int_0^2(4-r^2)\,r\cdot r\,\mathrm{d}r\,\mathrm{d}\theta=2\pi\int_0^2(4r^2-r^4)\,\mathrm{d}r=2\pi\cdot\left(\frac{4\cdot2^3}{3}-\frac{2^5}{5}\right)=\frac{128\pi}{15}.$$

**例.** 若 $E=\{(x,y,z)\,|\,x^2+y^2+(z-1)^2\leqslant 1\}$, 求 $\displaystyle\int_E(x^2+y^2+z^2)^{\frac{5}{2}}\,\mathrm{d}V.$

**解.** 代入球面座標於 $x^2+y^2+(z-1)^2\leqslant 1 \implies (\rho\sin\varphi\cos\theta)^2+(\rho\sin\varphi\sin\theta)^2+(\rho\cos\varphi-1)^2\leqslant 1 \implies$
$\rho^2\sin^2\varphi+\rho^2\cos^2\varphi-2\rho\cos\varphi+1\leqslant 1 \implies \rho^2\leqslant 2\rho\cos\varphi \implies \rho\leqslant 2\cos\varphi$, 故 $\displaystyle\int_E(x^2+y^2+z^2)^{\frac{5}{2}}\,\mathrm{d}V=$

$$\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{2\cos\varphi}\rho^5\cdot\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi=2\pi\int_0^{\frac{\pi}{2}}\frac{(2\cos\varphi)^8}{8}\sin\varphi\,\mathrm{d}\varphi=\frac{64\pi}{9}(-\cos^9\varphi)\Big|_0^{\frac{\pi}{2}}=\frac{64\pi}{9}.$$

**例.** 若 $E$ 為 $z=\sqrt{x^2+y^2}$ 與 $x^2+y^2+(z-1)^2=1$ 於第一卦限圍成之區域, 求其體積.

**解.**



- 柱面座標: $\displaystyle\int_E\mathrm{d}V=\int_0^{\frac{\pi}{2}}\int_0^1\int_r^{1+\sqrt{1-r^2}}r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta=\frac{\pi}{4}$

- 球面座標: $\displaystyle\int_E\mathrm{d}V=\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}}\int_0^{2\cos\varphi}\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi=\frac{\pi}{4}$

# 7.3 綜合問題

**例.** 已知 $\forall\,0<\alpha<1,\ \displaystyle\int_0^\infty\frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x=\frac{\pi}{\sin\alpha\pi}$, 證明 $\Gamma(\alpha)\,\Gamma(1-\alpha)=\dfrac{\pi}{\sin\alpha\pi}.$

**解.** $\displaystyle\Gamma(\alpha)\,\Gamma(1-\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}\,\mathrm{d}t\int_0^\infty e^{-s}s^{-\alpha}\,\mathrm{d}s=\int_0^\infty\int_0^\infty e^{-(t+s)}t^{\alpha-1}s^{-\alpha}\,\mathrm{d}s\,\mathrm{d}t.$ 令 $\begin{cases}u=s+t\\v=\frac{t}{s}\end{cases}$, 則 $\begin{cases}s=\frac{u}{1+v}\\t=\frac{uv}{1+v}\end{cases};$

Jacobian $J_{\mathbf{x}}(\mathbf{u})=\dfrac{\partial\mathbf{x}}{\partial\mathbf{u}}(\mathbf{u})=\begin{vmatrix}\frac{\partial s}{\partial u}&\frac{\partial s}{\partial v}\\\frac{\partial t}{\partial u}&\frac{\partial t}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{1+v}&\frac{-u}{(1+v)^2}\\\frac{v}{1+v}&\frac{u}{(1+v)^2}\end{vmatrix}=\dfrac{u}{(1+v)^2}.$ 變數變換 $(s,t)\to(u,v)$ 後積分

範圍仍為 $0<u<\infty,\ 0<v<\infty$, 故 $\displaystyle\int_0^\infty\int_0^\infty e^{-(t+s)}t^{\alpha-1}s^{-\alpha}\,\mathrm{d}s\,\mathrm{d}t=\int_0^\infty\int_0^\infty e^{-(t+s)}\left(\frac{t}{s}\right)^\alpha t^{-1}\,\mathrm{d}s\,\mathrm{d}t=$

$\displaystyle\int_0^\infty\int_0^\infty e^{-u}v^\alpha\frac{1+v}{uv}\frac{u}{(1+v)^2}\,\mathrm{d}u\,\mathrm{d}v=\int_0^\infty\int_0^\infty e^{-u}\frac{v^{\alpha-1}}{1+v}\,\mathrm{d}u\,\mathrm{d}v=\int_0^\infty e^{-u}\,\mathrm{d}u\int_0^\infty\frac{v^{\alpha-1}}{1+v}\,\mathrm{d}v=\int_0^\infty\frac{v^{\alpha-1}}{1+v}\,\mathrm{d}v=\frac{\pi}{\sin\alpha\pi}.$

**例.** Beta 函數定義為 $B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,\mathrm{d}t \ \forall\, m,n > 0$. 證明 $B(m,n) = B(n,m) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

**證.** $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,\mathrm{d}t$; 令 $t = u^2$, 則 $\Gamma(x) = 2\int_0^\infty e^{-u^2}u^{2x-1}\,\mathrm{d}u$.

$\Gamma(m)\,\Gamma(n) = \left(2\int_0^\infty e^{-u^2}u^{2m-1}\,\mathrm{d}u\right)\left(2\int_0^\infty e^{-v^2}v^{2n-1}\,\mathrm{d}v\right) = 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}u^{2m-1}v^{2n-1}\,\mathrm{d}u\,\mathrm{d}v$

$= 4\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}(r\cos\theta)^{2m-1}(r\sin\theta)^{2n-1}r\,\mathrm{d}r\mathrm{d}\theta = \left(2\int_0^\infty e^{-r^2}r^{2(m+n)-1}\,\mathrm{d}r\right)\left(2\int_0^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta\,\mathrm{d}\theta\right)$

$= \Gamma(m+n)\cdot 2\int_0^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta\,\mathrm{d}\theta$. 令 $t = \cos^2\theta$, 則 $\mathrm{d}t = -2\cos\theta\sin\theta\,\mathrm{d}\theta$; 故 $2\int_0^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta\,\mathrm{d}\theta$

$= \int_0^{\frac{\pi}{2}}\cos^{2m-2}\theta\sin^{2n-2}\theta\big(-2\cos\theta\sin\theta\,\mathrm{d}\theta\big) = -\int_1^0 t^{m-1}(1-t)^{n-1}\,\mathrm{d}t = \int_0^1 t^{m-1}(1-t)^{n-1}\,\mathrm{d}t = B(m,n)$.

**例.** 設 $V_n(a)$ 為半徑 $a$ 之 $n$ 維球體積, $n \geqslant 1$, $a > 0$; 證明 $V_n(1) = \dfrac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

**證.** $V_n(a) = \displaystyle\int_{x_1^2+x_2^2+\cdots+x_n^2\leqslant a^2}\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n$. 變數變換 $x_i = au_i\ \forall\, i = 1,2,\ldots,n$, 則 Jacobian 為 $a^n$, 積分範圍變為 $u_1^2+u_2^2+\cdots+u_n^2\leqslant 1$, 故 $V_n(a) = a^n\,V_n(1)$. $V_n(1) = \displaystyle\int_{u_1^2+u_2^2+\cdots+u_n^2\leqslant 1}\mathrm{d}u_1\,\mathrm{d}u_2\cdots\mathrm{d}u_n$

$= \displaystyle\int_{u_n^2\leqslant 1}\left(\int_{u_1^2+u_2^2+\cdots+u_{n-1}^2\leqslant 1-u_n^2}\mathrm{d}u_1\,\mathrm{d}u_2\cdots\mathrm{d}u_{n-1}\right)\mathrm{d}u_n = \int_{-1}^1 V_{n-1}\big(\sqrt{1-u_n^2}\big)\,\mathrm{d}u_n = V_{n-1}(1)\cdot\int_{-1}^1(1-u_n^2)^{\frac{n-1}{2}}\,\mathrm{d}u_n$

$= V_{n-1}(1)\cdot 2\displaystyle\int_0^1(1-u_n^2)^{\frac{n-1}{2}}\,\mathrm{d}u_n$. 令 $t = u_n^2 \implies u_n = \sqrt{t} \implies \mathrm{d}u_n = \dfrac{\mathrm{d}t}{2\sqrt{t}}$, 則 $2\displaystyle\int_0^1(1-u_n^2)^{\frac{n-1}{2}}\,\mathrm{d}u_n =$

$\displaystyle\int_0^1(1-t)^{\frac{n-1}{2}}t^{-\frac{1}{2}}\,\mathrm{d}t = B\Big(\dfrac{n+1}{2},\dfrac{1}{2}\Big) = \dfrac{\Gamma(\frac{n+1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n+2}{2})}$. 故 $V_n(1) = \dfrac{\Gamma(\frac{n+1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n+2}{2})}V_{n-1}(1) = \dfrac{\Gamma(\frac{n+1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n+2}{2})}\cdot$

$\dfrac{\Gamma(\frac{n}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})}V_{n-2}(1) = \dfrac{\Gamma(\frac{n+1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n+2}{2})}\cdot\dfrac{\Gamma(\frac{n}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})}\cdot\dfrac{\Gamma(\frac{n-1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})}V_{n-3}(1) = \cdots\cdots\cdots = \dfrac{\Gamma(\frac{3}{2})\cdot\big(\Gamma(\frac{1}{2})\big)^{n-1}}{\Gamma(\frac{n+2}{2})}V_1(1) =$

$\dfrac{\frac{1}{2}\sqrt{\pi}\cdot(\sqrt{\pi})^{n-1}}{\Gamma(\frac{n}{2}+1)}\cdot 2 = \dfrac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

**例.** 設 $V_n(a)$ 為 $n$ 維區域 $|x_1|+|x_2|+\cdots+|x_n|\leqslant a$ 之體積, $n \geqslant 1$, $a > 0$; 證明 $V_n(a) = a^n\dfrac{2^n}{n!}$.

**證.** $V_n(a) = \displaystyle\int_{|x_1|+|x_2|+\cdots+|x_n|\leqslant a}\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n$. 變數變換 $x_i = au_i\ \forall\, i = 1,2,\ldots,n$, 則 Jacobian 為 $a^n$, 積分範圍變為 $|u_1|+|u_2|+\cdots+|u_n|\leqslant 1$, 故 $V_n(a) = a^n\,V_n(1)$. $V_n(1) = \displaystyle\int_{|u_1|+|u_2|+\cdots+|u_n|\leqslant 1}\mathrm{d}u_1\,\mathrm{d}u_2\cdots\mathrm{d}u_n$

$= \displaystyle\int_{|u_n|\leqslant 1}\left(\int_{|u_1|+|u_2|+\cdots+|u_{n-1}|\leqslant 1-|u_n|}\mathrm{d}u_1\,\mathrm{d}u_2\cdots\mathrm{d}u_{n-1}\right)\mathrm{d}u_n = \int_{-1}^1 V_{n-1}(1-|u_n|)\,\mathrm{d}u_n = V_{n-1}(1)\int_{-1}^1(1-|u_n|)^{n-1}\,\mathrm{d}u_n$

$= V_{n-1}(1)\left(\displaystyle\int_{-1}^0(1+u_n)^{n-1}\,\mathrm{d}u_n + \int_0^1(1-u_n)^{n-1}\,\mathrm{d}u_n\right) = \dfrac{2}{n}V_{n-1}(1)$. 故 $V_n(1) = \dfrac{2^{n-1}}{n\cdot(n-1)\cdots 2}V_1(1) =$

$\dfrac{2^{n-1}}{n!}\cdot 2 = \dfrac{2^n}{n!}$, $V_n(a) = a^n\dfrac{2^n}{n!}$.