CLP 3

MULTIVARIABLE CALCULUS

EXERCISES

FELDMAN RECHNITZER YEAGER

CLP-3 Multivariable Calculus
Exercises

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▸▸ Legal stuff

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Contents

Chapter 1

Vectors and Geometry in Two and Three Dimensions

1.1 ▴ Points

▸▸ Stage 1

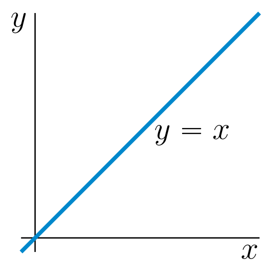
$\boxed{1.1.1}$ Describe the set of all points (x, y, z) in $\mathbb{R}^3$ that satisfy
(a) $x^2 + y^2 + z^2 = 2x - 4y + 4$
(b) $x^2 + y^2 + z^2 < 2x - 4y + 4$

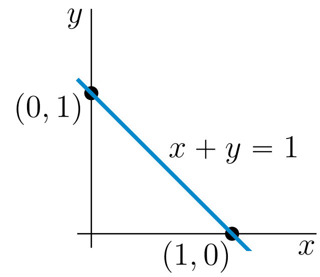
$\boxed{\text{Solution}}$ (a) The point (x, y, z) satisfies $x^2 + y^2 + z^2 = 2x - 4y + 4$ if and only if it satisfies $x^2 - 2x + y^2 + 4y + z^2 = 4$, or equivalently $(x-1)^2 + (y+2)^2 + z^2 = 9$. Since $\sqrt{(x-1)^2 + (y+2)^2 + z^2}$ is the distance from $(1, -2, 0)$ to (x, y, z), our point satisfies the given equation if and only if its distance from $(1, -2, 0)$ is three. So the set is the sphere of radius 3 centered on $(1, -2, 0)$.

(b) As in part (a), $x^2 + y^2 + z^2 < 2x - 4y + 4$ if and only if $(x-1)^2 + (y+2)^2 + z^2 < 9$. Hence our point satifies the given inequality if and only if its distance from $(1, -2, 0)$ is strictly smaller than three. The set is the interior of the sphere of radius 3 centered on $(1, -2, 0)$.

$\boxed{1.1.2}$ Describe and sketch the set of all points (x, y) in $\mathbb{R}^2$ that satisfy
(a) $x = y$
(b) $x + y = 1$
(c) $x^2 + y^2 = 4$
(d) $x^2 + y^2 = 2y$
(e) $x^2 + y^2 < 2y$

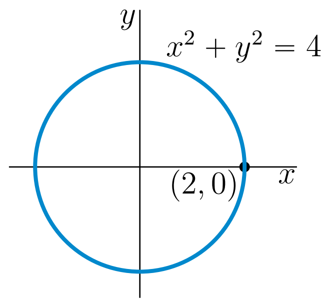
$\boxed{\text{Solution}}$ (a) $x = y$ is a straight line and passes through the points $(0, 0)$ and $(1, 1)$. So it is the straight line through the origin that makes an angle $45°$ with the x– and y–axes. It is sketched in the figure on the left below.

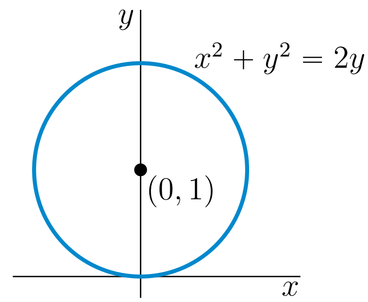




(b) $x + y = 1$ is the straight line through the points $(1, 0)$ and $(0, 1)$. It is sketched in the figure on the right above.

(c) $x^2 + y^2$ is the square of the distance from $(0, 0)$ to (x, y). So $x^2 + y^2 = 4$ is the circle with centre $(0, 0)$ and radius 2. It is sketched in the figure on the left below.



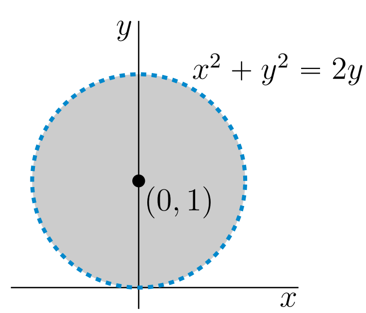


(d) The equation $x^2 + y^2 = 2y$ is equivalent to $x^2 + (y - 1)^2 = 1$. As $x^2 + (y - 1)^2$ is the square of the distance from $(0, 1)$ to (x, y), $x^2 + (y - 1)^2 = 1$ is the circle with centre $(0, 1)$ and radius 1. It is sketched in the figure on the right above.

(e) As in part (d),

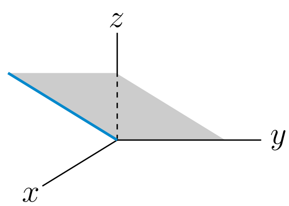
$$x^2 + y^2 < 2y \iff x^2 + y^2 - 2y < 0 \iff x^2 + y^2 - 2y + 1 < 1 \iff x^2 + (y - 1)^2 < 1$$

As $x^2 + (y - 1)^2$ is the square of the distance from $(0, 1)$ to (x, y), $x^2 + (y - 1)^2 < 1$ is the set of points whose distance from $(0, 1)$ is strictly less than 1. That is, it is the set of points strictly inside the circle with centre $(0, 1)$ and radius 1. That set is the shaded region (not including the dashed circle) in the sketch below.

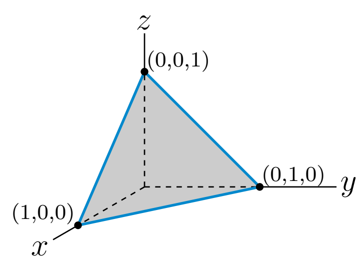


1.1.3 Describe the set of all points (x,y,z) in $\mathbb{R}^3$ that satisfy the following conditions. Sketch the part of the set that is in the first octant.
(a) $z = x$
(b) $x + y + z = 1$
(c) $x^2 + y^2 + z^2 = 4$
(d) $x^2 + y^2 + z^2 = 4$, $z = 1$
(e) $x^2 + y^2 = 4$
(f) $z = x^2 + y^2$

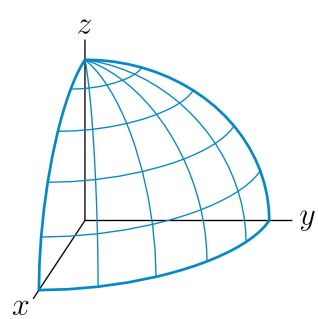
Solution (a) For each fixed y_0, $z = x,\ y = y_0$ is a straight line that lies in the plane, $y = y_0$ (which is parallel to the plane containing the x and z axes and is a distance y_0 from it). This line passes through $x = z = 0$ and makes an angle $45°$ with the xy–plane. Such a line (with $y_0 = 0$) is sketched in the figure below. The set $z = x$ is the union of all the lines $z = x,\ y = y_0$ with all values of y_0. As y_0 varies $z = x,\ y = y_0$ sweeps out the plane which contains the y–axis and which makes an angle $45°$ with the xy–plane. Here is a sketch of the part of the plane that is in the first octant.



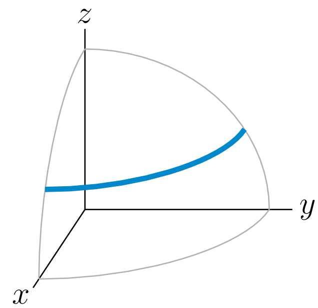
(b) $x + y + z = 1$ is the plane through the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Here is a sketch of the part of the plane that is in the first octant.



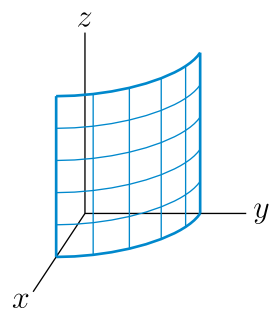
(c) $x^2 + y^2 + z^2$ is the square of the distance from $(0,0,0)$ to (x,y,z). So $x^2 + y^2 + z^2 = 4$ is the set of points whose distance from $(0,0,0)$ is 2. It is the sphere with centre $(0,0,0)$ and radius 2. Here is a sketch of the part of the sphere that is in the first octant.



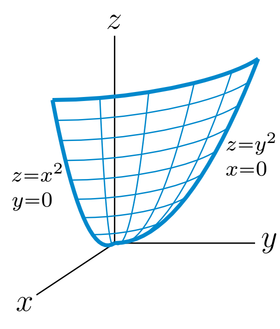
(d) $x^2 + y^2 + z^2 = 4$, $z = 1$ or equivalently $x^2 + y^2 = 3$, $z = 1$, is the intersection of the plane $z = 1$ with the sphere of centre $(0,0,0)$ and radius 2. It is a circle in the plane $z = 1$ that has centre $(0,0,1)$ and radius $\sqrt{3}$. The part of the circle in the first octant is the heavy quarter circle in the sketch



(e) For each fixed z_0, $x^2 + y^2 = 4$, $z = z_0$ is a circle in the plane $z = z_0$ with centre $(0,0,z_0)$ and radius 2. So $x^2 + y^2 = 4$ is the union of $x^2 + y^2 = 4,\ z = z_0$ for all possible values of z_0. It is a vertical stack of horizontal circles. It is the cylinder of radius 2 centered on the z–axis. Here is a sketch of the part of the cylinder that is in the first octant.



(f) For each fixed $z_0 \geqslant 0$, the curve $z = x^2 + y^2,\ z = z_0$ is the circle in the plane $z = z_0$ with centre $(0,0,z_0)$ and radius $\sqrt{z_0}$. As $z = x^2 + y^2$ is the union of $z = x^2 + y^2,\ z = z_0$ for all possible values of $z_0 \geqslant 0$, it is a vertical stack of horizontal circles. The intersection of the surface with the yz–plane is the parabola $z = y^2$. Here is a sketch of the part of the paraboloid that is in the first octant.



$\boxed{1.1.4}$ Let A be the point $(2,1,3)$.

(a) Find the distance from A to the xy-plane.
(b) Find the distance from A to the xz-plane.
(c) Find the distance from A to the point $(x,0,0)$ on the x-axis.
(d) Find the point on the x-axis that is closest to A.
(e) What is the distance from A to the x-axis?

Solution (a) The z coordinate of any point is the signed distance from the point to the xy-plane. So the distance from $(2,1,3)$ to the xy-plane is $|3| = 3$.

(b) The y coordinate of any point is the signed distance from the point to the xz-plane. So the distance from $(2,1,3)$ to the xz-plane is $|1| = 1$.

(c) The distance from $(2,1,3)$ to $(x,0,0)$ is

$$\sqrt{(2-x)^2+(1-0)^2+(3-0)^2} = \sqrt{(x-2)^2+10}$$

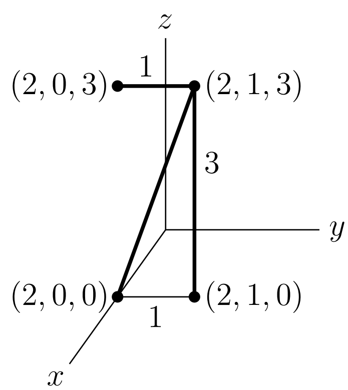
(d) Since $(x-2)^2 \geqslant 0$, the distance $\sqrt{(x-2)^2+10}$ is minimized when $x = 2$. Alternatively,

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{(x-2)^2+10} = \frac{x-2}{\sqrt{(x-2)^2+10}} = 0 \iff x = 2$$

So the point on the x-axis that is closest to A is $(2,0,0)$.

(e) As $(2,0,0)$ is the point on the x-axis that is nearest $(2,1,3)$, the distance from A to the x-axis is

$$\sqrt{(2-2)^2+(1-0)^2+(3-0)^2} = \sqrt{1^2+3^2} = \sqrt{10}$$



▸▸ Stage 2

1.1.5 Consider any triangle. Pick a coordinate system so that one vertex is at the origin and a second vertex is on the positive x–axis. Call the coordinates of the second vertex $(a,0)$ and those of the third vertex (b,c). Find the circumscribing circle (the circle that goes through all three vertices).

Solution Call the centre of the circumscribing circle $(\bar{x},\bar{y})$. This centre must be equidistant from the three vertices. So

$$\bar{x}^2+\bar{y}^2 = (\bar{x}-a)^2+\bar{y}^2 = (\bar{x}-b)^2+(\bar{y}-c)^2$$

or, subtracting $\bar{x}^2+\bar{y}^2$ from the three equal expressions,

$$0 = a^2-2a\bar{x} = b^2-2b\bar{x}+c^2-2c\bar{y}$$

which implies

$$\bar{x} = \frac{a}{2} \qquad \bar{y} = \frac{b^2 + c^2 - 2b\bar{x}}{2c} = \frac{b^2 + c^2 - ab}{2c}$$

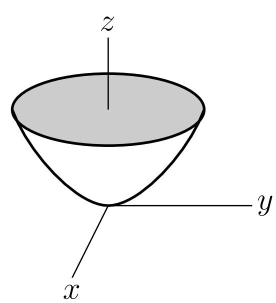
The radius is the distance from the vertex $(0,0)$ to the centre $(\bar{x}, \bar{y})$, which is $\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b^2+c^2-ab}{2c}\right)^2}$.

$\boxed{1.1.6}$ (*) A certain surface consists of all points $P = (x, y, z)$ such that the distance from P to the point $(0,0,1)$ is equal to the distance from P to the plane $z + 1 = 0$. Find an equation for the surface, sketch and describe it verbally.

$\boxed{\text{Solution}}$ The distance from P to the point $(0,0,1)$ is $\sqrt{x^2 + y^2 + (z-1)^2}$. The distance from P to the specified plane is $|z+1|$. Hence the equation of the surface is

$$x^2 + y^2 + (z-1)^2 = (z+1)^2 \text{ or } x^2 + y^2 = 4z$$

All points on this surface have $z \geqslant 0$. The set of points on the surface that have any fixed value, $z_0 \geqslant 0$, of z consists of a circle that is centred on the z–axis, is parallel to the xy-plane and has radius $2\sqrt{z_0}$. The surface consists of a stack of these circles, starting with a point at the origin and with radius increasing vertically. The surface is a paraboloid and is sketched below.



$\boxed{1.1.7}$ Show that the set of all points P that are twice as far from $(3, -2, 3)$ as from $(3/2, 1, 0)$ is a sphere. Find its centre and radius.

$\boxed{\text{Solution}}$ Let (x, y, z) be a point in P. The distances from (x, y, z) to $(3, -2, 3)$ and to $(3/2, 1, 0)$ are

$$\sqrt{(x-3)^2 + (y+2)^2 + (z-3)^2} \quad \text{and} \quad \sqrt{(x-3/2)^2 + (y-1)^2 + z^2}$$

respectively. To be in P, (x, y, z) must obey

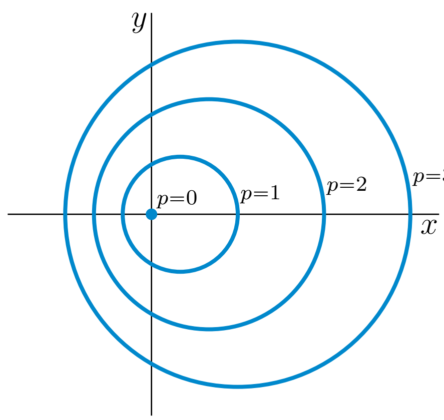
$$\begin{aligned}
\sqrt{(x-3)^2 + (y+2)^2 + (z-3)^2} &= 2\sqrt{(x-3/2)^2 + (y-1)^2 + z^2} \\
(x-3)^2 + (y+2)^2 + (z-3)^2 &= 4(x-3/2)^2 + 4(y-1)^2 + 4z^2 \\
x^2 - 6x + 9 + y^2 + 4y + 4 + z^2 - 6z + 9 &= 4x^2 - 12x + 9 + 4y^2 - 8y + 4 + 4z^2 \\
3x^2 - 6x + 3y^2 - 12y + 3z^2 + 6z - 9 &= 0 \\
x^2 - 2x + y^2 - 4y + z^2 + 2z - 3 &= 0 \\
(x-1)^2 + (y-2)^2 + (z+1)^2 &= 9
\end{aligned}$$

This is a sphere of radius 3 centered on $(1,2,-1)$.

▸▸ Stage 3

$\boxed{1.1.8}$ The pressure $p(x,y)$ at the point (x,y) is at least zero and is determined by the equation $x^2 - 2px + y^2 = 3p^2$. Sketch several isobars. An isobar is a curve with equation $p(x,y) = c$ for some constant $c \geqslant 0$.

$\boxed{\text{Solution}}$ For each fixed $c \geqslant 0$, the isobar $p(x,y) = c$ is the curve $x^2 - 2cx + y^2 = 3c^2$, or equivalently, $(x-c)^2 + y^2 = 4c^2$. This is a circle with centre $(c,0)$ and radius $2c$. Here is a sketch of the isobars $p(x,y) = c$ with $c = 0,1,2,3$.

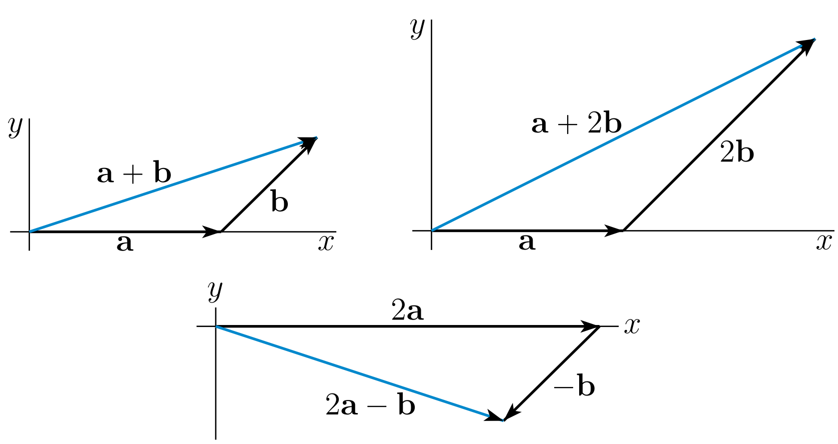


1.2▴ Vectors

▸▸ Stage 1

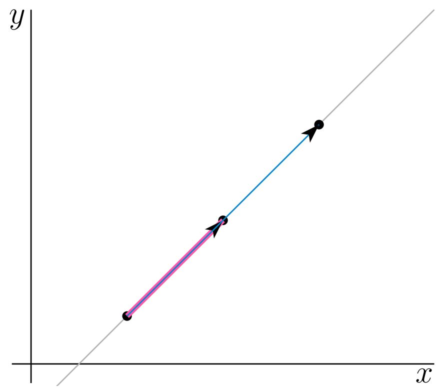
$\boxed{1.2.1}$ Let $\mathbf{a} = \langle 2,0\rangle$ and $\mathbf{b} = \langle 1,1\rangle$. Evaluate and sketch $\mathbf{a}+\mathbf{b}$, $\mathbf{a}+2\mathbf{b}$ and $2\mathbf{a}-\mathbf{b}$.

$\boxed{\text{Solution}}$ $\mathbf{a}+\mathbf{b} = \langle 3,1\rangle$, $\mathbf{a}+2\mathbf{b} = \langle 4,2\rangle$, $2\mathbf{a}-\mathbf{b} = \langle 3,-1\rangle$



$\boxed{1.2.2}$ Determine whether or not the given points are collinear (that is, lie on a common straight line)
(a) $(1,2,3)$, $(0,3,7)$, $(3,5,11)$
(b) $(0,3,-5)$, $(1,2,-2)$, $(3,0,4)$

Solution If three points are collinear, then the vector from the first point to the second point, and the vector from the first point to the third point must both be parallel to the line, and hence must be parallel to each other (i.e. must be multiples of each other).



(a) The vectors $\langle 0,3,7\rangle - \langle 1,2,3\rangle = \langle -1,1,4\rangle$ and $\langle 3,5,11\rangle - \langle 1,2,3\rangle = \langle 2,3,8\rangle$ are not parallel (i.e. are not multiples of each other), so the three points are not on the same line.

(b) The vectors $\langle 1,2,-2\rangle - \langle 0,3,-5\rangle = \langle 1,-1,3\rangle$ and $\langle 3,0,4\rangle - \langle 0,3,-5\rangle = \langle 3,-3,9\rangle$ are parallel (i.e. are multiples of each other), so the three points are on the same line.

$\boxed{1.2.3}$ Determine whether the given pair of vectors is perpendicular
(a) $\langle 1,3,2\rangle$, $\langle 2,-2,2\rangle$
(b) $\langle -3,1,7\rangle$, $\langle 2,-1,1\rangle$
(c) $\langle 2,1,1\rangle$, $\langle -1,4,2\rangle$

Solution By property 7 of Theorem 1.2.11 in the CLP-3 text,

$$\begin{array}{lrcl}
\text{(a)} & \langle 1,3,2\rangle\cdot\langle 2,-2,2\rangle = 1\times 2-3\times 2+2\times 2=0 & \implies & \text{perpendicular}\\
\text{(b)} & \langle -3,1,7\rangle\cdot\langle 2,-1,1\rangle = -3\times 2-1\times 1+7\times 1=0 & \implies & \text{perpendicular}\\
\text{(c)} & \langle 2,1,1\rangle\cdot\langle -1,4,2\rangle = -2\times 1+1\times 4+1\times 2=4\neq 0 & \implies & \text{not perpendicular}
\end{array}$$

$\boxed{1.2.4}$ Consider the vector $\mathbf{a} = \langle 3,4\rangle$.
(a) Find a unit vector in the same direction as $\mathbf{a}$.
(b) Find all unit vectors that are parallel to $\mathbf{a}$.
(c) Find all vectors that are parallel to $\mathbf{a}$ and have length 10.
(d) Find all unit vectors that are perpendicular to $\mathbf{a}$.

Solution (a) The vector $\mathbf{a}$ has length

$$|\langle 3,4\rangle| = \sqrt{3^2+4^2} = \sqrt{25} = 5$$

So the vector $\frac{1}{5}\langle 3,4\rangle$ has length 1 (i.e. is a unit vector) and is in the same direction as $\langle 3,4\rangle$.

(b) Recall, from Definition 1.2.5 in the CLP-3 text, that a vector is parallel to $\mathbf{a}$ if and only if it is of the form $s\mathbf{a}$ for some nonzero real number s. Such a vector is a unit vector if and only if

$$\begin{aligned} |s\mathbf{a}| = 1 \iff |s|\,|\langle 3,4\rangle| = 1 \iff |s| = \frac{1}{|\langle 3,4\rangle|} = \frac{1}{5} \\ \iff s = \pm\frac{1}{5} \end{aligned}$$

So there are two unit vectors that are parallel to $\mathbf{a}$, namely $\pm\frac{1}{5}\langle 3,4\rangle$.

(c) We have already found, in part (b), all vectors that are parallel to $\mathbf{a}$ and have length 1, namely $\pm\frac{1}{5}\langle 3,4\rangle$. To increase the lengths of those vectors to 10, we just need to multiply them by 10, giving $\pm\frac{10}{5}\langle 3,4\rangle = \pm 2\langle 3,4\rangle = \pm\langle 6,8\rangle$.

(d) A vector $\langle x,y\rangle$ is perpendicular to $\mathbf{a} = \langle 3,4\rangle$ if and only if

$$0 = \langle x,y\rangle \cdot \langle 3,4\rangle = 3x+4y \iff y = -\frac{3}{4}x \iff \langle x,y\rangle = \left\langle x, -\frac{3}{4}x\right\rangle = \frac{x}{4}\langle 4,-3\rangle$$

Such a vector is a unit vector if and only if

$$\begin{aligned} \frac{|x|}{4}\,|\langle 4,-3\rangle| = 1 \iff \frac{|x|}{4} = \frac{1}{|\langle 4,-3\rangle|} = \frac{1}{5} \\ \iff \frac{x}{4} = \pm\frac{1}{5} \end{aligned}$$

So there are two unit vectors that are perpendicular to $\mathbf{a}$, namely $\pm\frac{1}{5}\langle 4,-3\rangle$.

1.2.5 Consider the vector $\mathbf{b} = \langle 3,4,0\rangle$.

(a) Find a unit vector in the same direction as $\mathbf{b}$.
(b) Find all unit vectors that are parallel to $\mathbf{b}$.
(c) Find four different unit vectors that are perpendicular to $\mathbf{b}$.

Solution (a) The vector $\mathbf{b}$ has length

$$|\langle 3,4,0\rangle| = \sqrt{3^2+4^2+0^2} = \sqrt{25} = 5$$

So the vector $\frac{1}{5}\langle 3,4,0\rangle$ has length 1 (i.e. is a unit vector) and is in the same direction as $\langle 3,4,0\rangle$.

(b) Recall, from Definition 1.2.5 in the CLP-3 text, that a vector is parallel to $\mathbf{b}$ if and only if it is of the form $s\mathbf{b}$ for some nonzero real number s. Such a vector is a unit vector if and only if

$$\begin{aligned} |s\mathbf{b}| = 1 \iff |s|\,|\langle 3,4,0\rangle| = 1 \iff |s| = \frac{1}{|\langle 3,4,0\rangle|} = \frac{1}{5} \\ \iff s = \pm\frac{1}{5} \end{aligned}$$

So there are two unit vectors that are parallel to $\mathbf{b}$, namely $\pm\frac{1}{5}\langle 3,4,0\rangle$.

(c) A vector $\langle x,y,z\rangle$ is perpendicular to $\mathbf{a} = \langle 3,4,0\rangle$ if and only if

$$0 = \langle x,y,z\rangle \cdot \langle 3,4,0\rangle = 3x+4y \iff y = -\frac{3}{4}x \iff \langle x,y,z\rangle = \left\langle x, -\frac{3}{4}x, z\right\rangle$$

Such a vector is a unit vector if and only if

$$\left|\left\langle x, -\frac{3}{4}x, z\right\rangle\right| = 1 \iff \sqrt{x^2 + \frac{9}{16}x^2 + z^2} = 1 \iff \sqrt{\frac{25}{16}x^2 + z^2} = 1$$

There are infinitely many pairs x, z that obey $\sqrt{\frac{25}{16}x^2 + z^2} = 1$. We can easily get two of them by setting $x = 0$ and choosing z to obey $\sqrt{z^2} = 1$, i.e. choosing $z = \pm 1$. We can easily get two more of them by setting $z = 0$ and choosing x to obey $\sqrt{\frac{25}{16}x^2} = 1$, i.e. choosing $x = \pm\frac{4}{5}$. This gives us four vectors of length one that are perpendicular to $\mathbf{b}$, namely

$$\pm\langle 0,0,1\rangle \qquad \pm\left\langle \frac{4}{5}, -\frac{3}{4}\frac{4}{5}, 0\right\rangle = \pm\frac{1}{5}\langle 4,-3,0\rangle$$

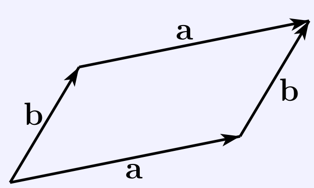
$\boxed{1.2.6}$ Let $\mathbf{a} = \langle a_1, a_2\rangle$. Compute the projection of $\mathbf{a}$ on $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$.

$\boxed{\text{Solution}}$ $\text{proj}_{\hat{\boldsymbol{\imath}}}\mathbf{a} = (\mathbf{a}\cdot\hat{\boldsymbol{\imath}})\hat{\boldsymbol{\imath}} = a_1\hat{\boldsymbol{\imath}}$ and $\text{proj}_{\hat{\boldsymbol{\jmath}}}\mathbf{a} = (\mathbf{a}\cdot\hat{\boldsymbol{\jmath}})\hat{\boldsymbol{\jmath}} = a_2\hat{\boldsymbol{\jmath}}$.

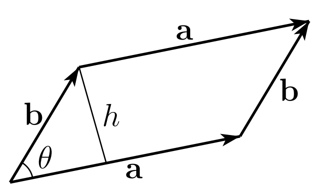
$\boxed{1.2.7}$ Does the triangle with vertices $(1,2,3)$, $(4,0,5)$ and $(3,6,4)$ have a right angle?

$\boxed{\text{Solution}}$ The vector from $(1,2,3)$ to $(4,0,5)$ is $\langle 3,-2,2\rangle$. The vector from $(1,2,3)$ to $(3,6,4)$ is $\langle 2,4,1\rangle$. The dot product between these two vectors is $\langle 3,-2,2\rangle \cdot \langle 2,4,1\rangle = 0$, so the vectors are perpendicular and the triangle does contain a right angle.

$\boxed{1.2.8}$ Show that the area of the parallelogram determined by the vectors $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a}\times\mathbf{b}|$.



$\boxed{\text{Solution}}$ The area of a parallelogram is the length of its base time its height. We can

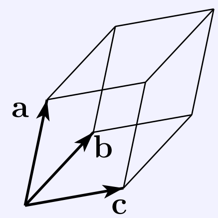


choose the base to be $\mathbf{a}$. Then, if θ is the angle between its sides $\mathbf{a}$ and $\mathbf{b}$, its height is $|\mathbf{b}|\sin\theta$. So

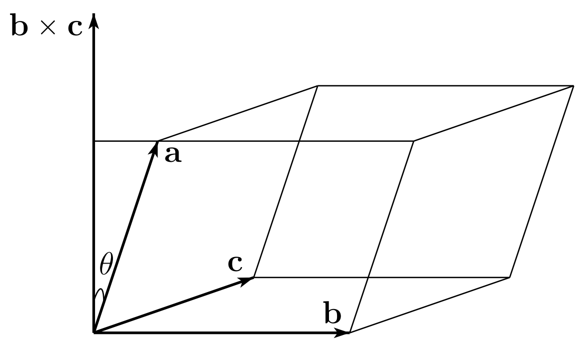
$$\text{area} = |\mathbf{a}||\mathbf{b}|\sin\theta = |\mathbf{a}\times\mathbf{b}|$$

$\boxed{1.2.9}$ Show that the volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is

$$|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$$



$\boxed{\text{Solution}}$ The volume of a parallelepiped is the area of its base time its height. We can choose the base to be the parallelogram determined by the vectors $\mathbf{b}$ and $\mathbf{c}$. It has area $|\mathbf{b}\times\mathbf{c}|$. The vector $\mathbf{b}\times\mathbf{c}$ is perpendicular to the base. Denote by θ the angle between $\mathbf{a}$



and the perpendicular $\mathbf{b}\times\mathbf{c}$. The height of the parallelepiped is $|\mathbf{a}||\cos\theta|$. So

$$\text{volume} = |\mathbf{a}|\,|\cos\theta|\,|\mathbf{b}\times\mathbf{c}| = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$$

$\boxed{1.2.10}$ Verify by direct computation that

(a) $\hat{\imath}\times\hat{\jmath}=\hat{\mathbf{k}}$, $\hat{\jmath}\times\hat{\mathbf{k}}=\hat{\imath}$, $\hat{\mathbf{k}}\times\hat{\imath}=\hat{\jmath}$

(b) $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{0}$

Solution (a)

$$\begin{aligned}\hat{\imath}\times\hat{\jmath} &= \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} = \hat{\imath}(0\times 0-0\times 1)-\hat{\jmath}(1\times 0-0\times 0)+\hat{\mathbf{k}}(1\times 1-0\times 0)\\ &= \hat{\mathbf{k}}\end{aligned}$$

$$\begin{aligned}\hat{\jmath}\times\hat{\mathbf{k}} &= \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \hat{\imath}(1\times 1-0\times 0)-\hat{\jmath}(0\times 1-0\times 0)+\hat{\mathbf{k}}(0\times 0-1\times 0)\\ &= \hat{\imath}\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{k}}\times\hat{\imath} &= \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix} = \hat{\imath}(0\times 0-1\times 0)-\hat{\jmath}(0\times 0-1\times 1)+\hat{\mathbf{k}}(0\times 0-0\times 1)\\ &= \hat{\jmath}\end{aligned}$$

(b)

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) &= a_1\big(a_2b_3-a_3b_2\big)-a_2\big(a_1b_3-a_3b_1\big)+a_3\big(a_1b_2-a_2b_1\big)=0\\ \mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}) &= b_1\big(a_2b_3-a_3b_2\big)-b_2\big(a_1b_3-a_3b_1\big)+b_3\big(a_1b_2-a_2b_1\big)=0\end{aligned}$$

1.2.11 Consider the following statement: "If $\mathbf{a}\neq\mathbf{0}$ and if $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}$ then $\mathbf{b}=\mathbf{c}$." If the statment is true, prove it. If the statement is false, give a counterexample.

Solution This statement is false. The two numbers $\mathbf{a}\cdot\mathbf{b}$, $\mathbf{a}\cdot\mathbf{c}$ are equal if and only if $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c})=0$. This in turn is the case if and only if $\mathbf{a}$ is perpendicular to $\mathbf{b}-\mathbf{c}$ (under the convention that $\mathbf{0}$ is perpendicular to all vectors). For example, if $\mathbf{a}=\langle 1,0,0\rangle$, $\mathbf{b}=\langle 0,1,0\rangle$, $\mathbf{c}=\langle 0,0,1\rangle$, then $\mathbf{b}-\mathbf{c}=\langle 0,1,-1\rangle$ is perpendicular to $\mathbf{a}$ so that $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}$.

1.2.12 Consider the following statement: "The vector $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is of the form $\alpha\mathbf{b}+\beta\mathbf{c}$ for some real numbers α and β." If the statement is true, prove it. If the statement is false, give a counterexample.

Solution This statement is true. In the event that $\mathbf{b}$ and $\mathbf{c}$ are parallel, $\mathbf{b}\times\mathbf{c}=\mathbf{0}$ so that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{0}=0\mathbf{b}+0\mathbf{c}$, so we may assume that $\mathbf{b}$ and $\mathbf{c}$ are not parallel. Then as α and β run over $\mathbb{R}$, the vector $\alpha\mathbf{b}+\beta\mathbf{c}$ runs over the plane that contains the origin and the vectors $\mathbf{b}$ and $\mathbf{c}$. Call this plane P. Because $\mathbf{d}=\mathbf{b}\times\mathbf{c}$ is nonzero and perpendicular to both $\mathbf{b}$ and $\mathbf{c}$, P is the plane that contains the origin and is perpendicular to $\mathbf{d}$. As $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{a}\times\mathbf{d}$ is always perpendicular to $\mathbf{d}$, it lies in P.

1.2.13 What geometric conclusions can you draw from $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\langle 1,2,3\rangle$?

Solution None. The given equation is nonsense. The left hand side is a number while the right hand side is a vector.

1.2.14 What geometric conclusions can you draw from $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$?

Solution If $\mathbf{b}$ and $\mathbf{c}$ are parallel, then $\mathbf{b} \times \mathbf{c} = \mathbf{0}$ and $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ for all $\mathbf{a}$. If $\mathbf{b}$ and $\mathbf{c}$ are not parallel, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ if and only if $\mathbf{a}$ is perpendicular to $\mathbf{d} = \mathbf{b} \times \mathbf{c}$. As we saw in question 12, the set of all vectors perpendicular to $\mathbf{d}$ is the plane consisting of all vectors of the form $\alpha\mathbf{b} + \beta\mathbf{c}$ with α and β real numbers. So $\mathbf{a}$ must be of this form.

1.2.15 Consider the three points $O = (0, 0)$, $A = (a, 0)$ and $B = (b, c)$.

(a) Sketch, in a single figure,
- the triangle with vertices O, A and B, and
- the circumscribing circle for the triangle (i.e. the circle that goes through all three vertices), and
- the vectors
  - $\overrightarrow{OA}$, from O to A,
  - $\overrightarrow{OB}$, from O to B,
  - $\overrightarrow{OC}$, from O to C, where C is the centre of the circumscribing circle.

Then add to the sketch and evaluate, from the sketch,
- the projection of the vector $\overrightarrow{OC}$ on the vector $\overrightarrow{OA}$, and
- the projection of the vector $\overrightarrow{OC}$ on the vector $\overrightarrow{OB}$.

(b) Determine C.

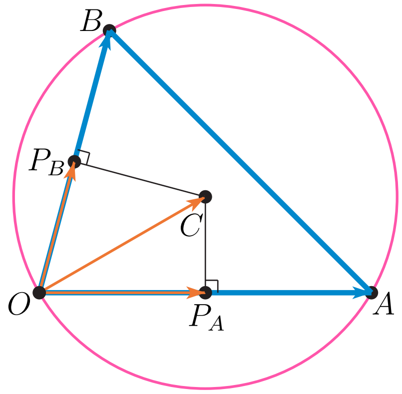
(c) Evaluate, using the formula (1.2.14) in the CLP-3 text,
- the projection of the vector $\overrightarrow{OC}$ on the vector $\overrightarrow{OA}$, and
- the projection of the vector $\overrightarrow{OC}$ on the vector $\overrightarrow{OB}$.

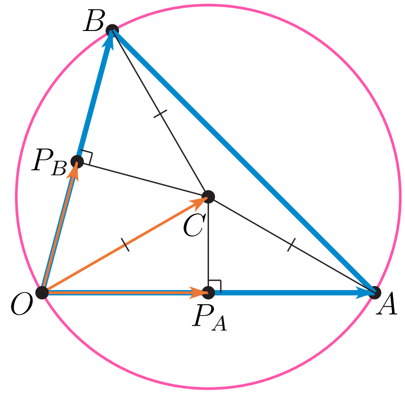
Solution (a) The sketch for part (a) is on the left below. To sketch the projections, we dropped perpendiculars

- from C to the line from O to A, and
- from C to the line from O to B.

By definition,

- $\text{proj}_{\overrightarrow{OA}}\overrightarrow{OC}$ is the vector $\overrightarrow{OP_A}$ from O to the point P_A, where the perpendicular from C to the line from O to A hits the line, and
- $\text{proj}_{\overrightarrow{OB}}\overrightarrow{OC}$ is the vector $\overrightarrow{OP_B}$ from O to the point P_B, where the perpendicular from C to the line from O to B hits the line.





To evaluate the projections we observe that the three lines from C to O, from C to A and from C to B all have exactly the same length (namely the radius of the circumscribing circle). Consequently (see the figure on the right above),

- the triangle OCA is an isoceles triangle, so that P_A is exactly the midpoint of the line segement from O to A. That is, P_A is $(a/2, 0)$ and

$$\text{proj}_{\overrightarrow{OA}}\,\overrightarrow{OC} = \overrightarrow{OP_A} = \langle a/2, 0\rangle$$

- Similarly, the triangle OCB is an isoceles triangle, so that P_B is exactly the midpoint of the line segement from O to B. That is P_A is $(b/2, c/2)$ and

$$\text{proj}_{\overrightarrow{OB}}\,\overrightarrow{OC} = \overrightarrow{OP_B} = \langle b/2, c/2\rangle$$

(b) Call the centre of the circumscribing circle $(\bar{x}, \bar{y})$. This centre must be equidistant from the three vertices. So

$$\bar{x}^2 + \bar{y}^2 = (\bar{x} - a)^2 + \bar{y}^2 = (\bar{x} - b)^2 + (\bar{y} - c)^2$$

or, subtracting $\bar{x}^2 + \bar{y}^2$ from all three expression,

$$0 = a^2 - 2a\bar{x} = b^2 - 2b\bar{x} + c^2 - 2c\bar{y}$$

which implies

$$\bar{x} = \frac{a}{2} \qquad\qquad \bar{y} = \frac{b^2 + c^2 - 2b\bar{x}}{2c} = \frac{b^2 + c^2 - ab}{2c}$$

(c) From part (b), we have

$$\overrightarrow{OA} \cdot \overrightarrow{OC} = \langle a, 0\rangle \cdot \left\langle \frac{a}{2}, \frac{b^2 + c^2 - ab}{2c}\right\rangle = \frac{a^2}{2} = \frac{1}{2}|\overrightarrow{OA}|^2$$
$$\overrightarrow{OB} \cdot \overrightarrow{OC} = \langle b, c\rangle \cdot \left\langle \frac{a}{2}, \frac{b^2 + c^2 - ab}{2c}\right\rangle = \frac{ab}{2} + \frac{b^2 + c^2 - ab}{2} = \frac{b^2 + c^2}{2} = \frac{1}{2}|\overrightarrow{OB}|^2$$

So, by Equation (1.2.14) in the CLP-3 text,

$$\text{proj}_{\overrightarrow{OA}}\,\overrightarrow{OC} = \frac{\overrightarrow{OA} \cdot \overrightarrow{OC}}{|\overrightarrow{OA}|^2}\overrightarrow{OA} = \frac{1}{2}\overrightarrow{OA} = \langle a/2, 0\rangle$$
$$\text{proj}_{\overrightarrow{OB}}\,\overrightarrow{OC} = \frac{\overrightarrow{OB} \cdot \overrightarrow{OC}}{|\overrightarrow{OB}|^2}\overrightarrow{OB} = \frac{1}{2}\overrightarrow{OB} = \langle b/2, c/2\rangle$$

▸▸ Stage 2

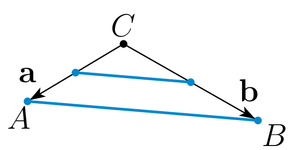
1.2.16 Find the equation of a sphere if one of its diameters has end points $(2,1,4)$ and $(4,3,10)$.

Solution The center of the sphere is $\frac{1}{2}\{(2,1,4)+(4,3,10)\} = (3,2,7)$. The diameter (i.e. twice the radius) is $|(2,1,4)-(4,3,10)| = |(-2,-2,-6)| = 2|(1,1,3)| = 2\sqrt{11}$. So the radius of the sphere is $\sqrt{11}$ and the equation of the sphere is

$$(x-3)^2+(y-2)^2+(z-7)^2 = 11$$

1.2.17 Use vectors to prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

Solution Call the vertices of the triangle A, B and C with C being the vertex that joins the two sides. We can always choose our coordinate system so that C is at the origin. Let $\mathbf{a}$ be the vector from C to A and $\mathbf{b}$ be the vector from C to B.



- Then the vector from C to the midpoint of the side from C to A is $\frac{1}{2}\mathbf{a}$ and
- the vector from C to the midpoint of the side from C to B is $\frac{1}{2}\mathbf{b}$ so that
- the vector joining the two midpoints is $\frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$.

As the vector from A to B is $\mathbf{b}-\mathbf{a} = 2\big[\frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}\big]$, the line joining the midpoints is indeed parallel to the third side and half its length.

1.2.18 Compute the areas of the parallelograms determined by the following vectors.
(a) $\langle -3,1\rangle$, $\langle 4,3\rangle$
(b) $\langle 4,2\rangle$, $\langle 6,8\rangle$

Solution (a) By (1.2.17) in the CLP-3 text, the area is

$$\left|\det\begin{bmatrix}-3 & 1\\ 4 & 3\end{bmatrix}\right| = \big|-3\times 3 - 1\times 4\big| = |-13| = 13$$

(b) By (1.2.17) in the CLP-3 text, the area is

$$\left|\det\begin{bmatrix}4 & 2\\ 6 & 8\end{bmatrix}\right| = \big|4\times 8 - 2\times 6\big| = 20$$

1.2.19 (*) Consider the plane W, defined by:

$$W \ : \ -x+3y+3z=6,$$

Find the area of the parallelogram on W defined by $0 \leqslant x \leqslant 3$, $0 \leqslant y \leqslant 2$.

Solution Note that

- the point on W with $x=0$, $y=0$ obeys $-0+3(0)+3z=6$ and so has $z=2$
- the point on W with $x=0$, $y=2$ obeys $-0+3(2)+3z=6$ and so has $z=0$
- the point on W with $x=3$, $y=0$ obeys $-3+3(0)+3z=6$ and so has $z=3$
- the point on W with $x=3$, $y=2$ obeys $-3+3(2)+3z=6$ and so has $z=1$

So the four corners of the parallelogram are $(0,0,2)$, $(0,2,0)$, $(3,0,3)$ and $(3,2,1)$. The vectors

$$\begin{aligned}
\mathbf{d}_1 &= \langle 0-0\,,\,2-0\,,\,0-2\rangle = \langle 0\,,\,2\,,\,-2\rangle \\
\mathbf{d}_2 &= \langle 3-0\,,\,0-0\,,\,3-2\rangle = \langle 3\,,\,0\,,\,1\rangle
\end{aligned}$$

form two sides of the paralleogram. So the area of the parallelogram is

$$\left|\mathbf{d}_1 \times \mathbf{d}_2\right| = \left|\det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 0 & 2 & -2 \\ 3 & 0 & 1 \end{bmatrix}\right| = \left|2\,\hat{\boldsymbol{\imath}} - 6\,\hat{\boldsymbol{\jmath}} - 6\hat{\mathbf{k}}\right| = \sqrt{76} = 2\sqrt{19}$$

1.2.20 Compute the volumes of the parallelepipeds determined by the following vectors.
(a) $\langle 4,1,-1\rangle$, $\langle -1,5,2\rangle$, $\langle 1,1,6\rangle$
(b) $\langle -2,1,2\rangle$, $\langle 3,1,2\rangle$, $\langle 0,2,5\rangle$

Solution (a) By (1.2.18) in the CLP-3 text, the volume is

$$\begin{aligned}
\left|\det\begin{bmatrix} 4 & 1 & -1 \\ -1 & 5 & 2 \\ 1 & 1 & 6 \end{bmatrix}\right| &= \left|4\det\begin{bmatrix} 5 & 2 \\ 1 & 6 \end{bmatrix} - 1\det\begin{bmatrix} -1 & 2 \\ 1 & 6 \end{bmatrix} + (-1)\det\begin{bmatrix} -1 & 5 \\ 1 & 1 \end{bmatrix}\right| \\
&= \left|4(30-2) - 1(-6-2) - 1(-1-5)\right| = 4\times 28 + 8 + 6 \\
&= 126
\end{aligned}$$

(b) By (1.2.18) in the CLP-3 text, the volume is

$$\begin{aligned}
\left|\det\begin{bmatrix} -2 & 1 & 2 \\ 3 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}\right| &= \left|-2\det\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} - 1\det\begin{bmatrix} 3 & 2 \\ 0 & 5 \end{bmatrix} + 2\det\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\right| \\
&= \left|-2(5-4) - 1(15-0) + 2(6-0)\right| = \left|-2-15+12\right| = \left|-5\right| \\
&= 5
\end{aligned}$$

1.2.21 Compute the dot product of the vectors **a** and **b**. Find the angle between them.
(a) $\mathbf{a} = \langle 1, 2 \rangle$, $\mathbf{b} = \langle -2, 3 \rangle$
(b) $\mathbf{a} = \langle -1, 1 \rangle$, $\mathbf{b} = \langle 1, 1 \rangle$
(c) $\mathbf{a} = \langle 1, 1 \rangle$, $\mathbf{b} = \langle 2, 2 \rangle$
(d) $\mathbf{a} = \langle 1, 2, 1 \rangle$, $\mathbf{b} = \langle -1, 1, 1 \rangle$
(e) $\mathbf{a} = \langle -1, 2, 3 \rangle$, $\mathbf{b} = \langle 3, 0, 1 \rangle$

Solution

(a) $\mathbf{a} \cdot \mathbf{b} = \langle 1, 2 \rangle \cdot \langle -2, 3 \rangle = 4$ $\quad \cos\theta = \dfrac{4}{\sqrt{5}\sqrt{13}} = .4961$ $\quad \theta = 60.25^\circ$

(b) $\mathbf{a} \cdot \mathbf{b} = \langle -1, 1 \rangle \cdot \langle 1, 1 \rangle = 0$ $\quad \cos\theta = \dfrac{0}{\sqrt{2}\sqrt{2}} = 0$ $\quad \theta = 90^\circ$

(c) $\mathbf{a} \cdot \mathbf{b} = \langle 1, 1 \rangle \cdot \langle 2, 2 \rangle = 4$ $\quad \cos\theta = \dfrac{4}{\sqrt{2}\sqrt{8}} = 1$ $\quad \theta = 0^\circ$

(d) $\mathbf{a} \cdot \mathbf{b} = \langle 1, 2, 1 \rangle \cdot \langle -1, 1, 1 \rangle = 2$ $\quad \cos\theta = \dfrac{2}{\sqrt{6}\sqrt{3}} = .4714$ $\quad \theta = 61.87^\circ$

(e) $\mathbf{a} \cdot \mathbf{b} = \langle -1, 2, 3 \rangle \cdot \langle 3, 0, 1 \rangle = 0$ $\quad \cos\theta = \dfrac{0}{\sqrt{14}\sqrt{10}} = 0$ $\quad \theta = 90^\circ$

1.2.22 Determine the angle between the vectors **a** and **b** if
(a) $\mathbf{a} = \langle 1, 2 \rangle$, $\mathbf{b} = \langle 3, 4 \rangle$
(b) $\mathbf{a} = \langle 2, 1, 4 \rangle$, $\mathbf{b} = \langle 4, -2, 1 \rangle$
(c) $\mathbf{a} = \langle 1, -2, 1 \rangle$, $\mathbf{b} = \langle 3, 1, 0 \rangle$

Solution By property 6 of Theorem 1.2.11 in the CLP-3 text,

(a) $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{1 \times 3 + 2 \times 4}{\sqrt{1+4}\sqrt{9+16}} = \dfrac{11}{5\sqrt{5}} = .9839 \quad \Longrightarrow \quad \theta = 10.3^\circ$

(b) $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{2 \times 4 - 1 \times 2 + 4 \times 1}{\sqrt{4+1+16}\sqrt{16+4+1}} = \dfrac{10}{21} = .4762 \quad \Longrightarrow \quad \theta = 61.6^\circ$

(c) $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{1 \times 3 - 2 \times 1 + 1 \times 0}{\sqrt{1+4+1}\sqrt{9+1}} = \dfrac{1}{\sqrt{60}} = .1291 \quad \Longrightarrow \quad \theta = 82.6^\circ$

1.2.23 Determine all values of y for which the given vectors are perpendicular.
(a) $\langle 2, 4 \rangle$, $\langle 2, y \rangle$
(b) $\langle 4, -1 \rangle$, $\langle y, y^2 \rangle$
(c) $\langle 3, 1, 1 \rangle$, $\langle 2, 5y, y^2 \rangle$

Solution

$$
\begin{array}{llll}
\text{(a)} & \langle 2,4\rangle \cdot \langle 2,y\rangle = 2\times 2+4\times y = 4+4y=0 & \iff & y=-1 \\
\text{(b)} & \langle 4,-1\rangle \cdot \left\langle y,y^2\right\rangle = 4\times y - 1\times y^2 = 4y-y^2=0 & \iff & y=0,4 \\
\text{(c)} & \langle 3,1,1\rangle \cdot \left\langle 2,5y,y^2\right\rangle = 6+5y+y^2=0 & \iff & y=-2,-3
\end{array}
$$

1.2.24 Let $\mathbf{u} = -2\hat{\imath} + 5\hat{\jmath}$ and $\mathbf{v} = \alpha\hat{\imath} - 2\hat{\jmath}$. Find α so that
(a) $\mathbf{u} \perp \mathbf{v}$
(b) $\mathbf{u} \| \mathbf{v}$
(c) The angle between $\mathbf{u}$ and $\mathbf{v}$ is $60°$.

Solution (a) We want $0 = \mathbf{u}\cdot\mathbf{v} = -2\alpha - 10$ or $\alpha = -5$.

(b) We want $-2/\alpha = 5/(-2)$ or $\alpha = 0.8$.

(c) We want $\mathbf{u}\cdot\mathbf{v} = -2\alpha - 10 = |\mathbf{u}|\,|\mathbf{v}|\,\cos 60° = \sqrt{29}\,\sqrt{\alpha^2+4}\,\frac{1}{2}$. Squaring both sides gives

$$
\begin{aligned}
4\alpha^2 + 40\alpha + 100 &= \frac{29}{4}(\alpha^2+4) \\
\implies \quad 13\alpha^2 - 160\alpha - 284 &= 0 \\
\implies \quad \alpha &= \frac{160 \pm \sqrt{160^2 + 4\times 13\times 284}}{26} \approx 13.88 \text{ or } -1.574
\end{aligned}
$$

Both of these α's give $\mathbf{u}\cdot\mathbf{v} < 0$ so no α works.

1.2.25 Define $\mathbf{a} = \langle 1,2,3\rangle$ and $\mathbf{b} = \langle 4,10,6\rangle$.
(a) Find the component of $\mathbf{b}$ in the direction $\mathbf{a}$.
(b) Find the projection of $\mathbf{b}$ on $\mathbf{a}$.
(c) Find the projection of $\mathbf{b}$ perpendicular to $\mathbf{a}$.

Solution (a) The component of $\mathbf{b}$ in the direction $\mathbf{a}$ is

$$
\mathbf{b}\cdot\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1\times 4 + 2\times 10 + 3\times 6}{\sqrt{1+4+9}} = \frac{42}{\sqrt{14}}
$$

(b) The projection of $\mathbf{b}$ on $\mathbf{a}$ is a vector of length $42/\sqrt{14}$ in direction $\mathbf{a}/|\mathbf{a}|$, namely $\frac{42}{14}\langle 1,2,3\rangle = \langle 3,6,9\rangle$.

(c) The projection of $\mathbf{b}$ perpendicular to $\mathbf{a}$ is $\mathbf{b}$ minus its projection on $\mathbf{a}$, namely $\langle 4,10,6\rangle - \langle 3,6,9\rangle = \langle 1,4,-3\rangle$.

1.2.26 Compute $\langle 1,2,3\rangle \times \langle 4,5,6\rangle$.

Solution

$$\langle 1,2,3\rangle \times \langle 4,5,6\rangle = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \hat{\imath}\,(2\times 6 - 3\times 5) - \hat{\jmath}\,(1\times 6 - 3\times 4) + \hat{\mathbf{k}}\,(1\times 5 - 2\times 4)$$
$$= -3\,\hat{\imath} + 6\,\hat{\jmath} - 3\,\hat{\mathbf{k}}$$

1.2.27 Calculate the following cross products.
(a) $\langle 1,-5,2\rangle \times \langle -2,1,5\rangle$
(b) $\langle 2,-3,-5\rangle \times \langle 4,-2,7\rangle$
(c) $\langle -1,0,1\rangle \times \langle 0,4,5\rangle$

Solution

(a) $$\det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & -5 & 2 \\ -2 & 1 & 5 \end{bmatrix} = \hat{\imath}\det\begin{bmatrix} -5 & 2 \\ 1 & 5 \end{bmatrix} - \hat{\jmath}\det\begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix} + \hat{\mathbf{k}}\det\begin{bmatrix} 1 & -5 \\ -2 & 1 \end{bmatrix}$$
$$= \hat{\imath}(-25-2) - \hat{\jmath}(5+4) + \hat{\mathbf{k}}(1-10) = \langle -27,-9,-9\rangle$$

(b) $$\det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 2 & -3 & -5 \\ 4 & -2 & 7 \end{bmatrix} = \hat{\imath}\det\begin{bmatrix} -3 & -5 \\ -2 & 7 \end{bmatrix} - \hat{\jmath}\det\begin{bmatrix} 2 & -5 \\ 4 & 7 \end{bmatrix} + \hat{\mathbf{k}}\det\begin{bmatrix} 2 & -3 \\ 4 & -2 \end{bmatrix}$$
$$= \hat{\imath}(-21-10) - \hat{\jmath}(14+20) + \hat{\mathbf{k}}(-4+12) = \langle -31,-34,8\rangle$$

(c) $$\det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ -1 & 0 & 1 \\ 0 & 4 & 5 \end{bmatrix} = \hat{\imath}\det\begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} - \hat{\jmath}\det\begin{bmatrix} -1 & 1 \\ 0 & 5 \end{bmatrix} + \hat{\mathbf{k}}\det\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$$
$$= \hat{\imath}(0-4) - \hat{\jmath}(-5-0) + \hat{\mathbf{k}}(-4-0) = \langle -4,5,-4\rangle$$

1.2.28 Let $\mathbf{p} = \langle -1,4,2\rangle$, $\mathbf{q} = \langle 3,1,-1\rangle$, $\mathbf{r} = \langle 2,-3,-1\rangle$. Check, by direct computation, that
(a) $\mathbf{p}\times\mathbf{p} = \mathbf{0}$
(b) $\mathbf{p}\times\mathbf{q} = -\mathbf{q}\times\mathbf{p}$
(c) $\mathbf{p}\times(3\mathbf{r}) = 3(\mathbf{p}\times\mathbf{r})$
(d) $\mathbf{p}\times(\mathbf{q}+\mathbf{r}) = \mathbf{p}\times\mathbf{q} + \mathbf{p}\times\mathbf{r}$
(e) $\mathbf{p}\times(\mathbf{q}\times\mathbf{r}) \neq (\mathbf{p}\times\mathbf{q})\times\mathbf{r}$

Solution

(a) $$\mathbf{p}\times\mathbf{p} = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ -1 & 4 & 2\\ -1 & 4 & 2\end{bmatrix} = \hat{\imath}(4\times 2-2\times 4)-\hat{\jmath}(2-(-2))+\hat{\mathbf{k}}(-4-(-4))$$
$$= \langle 0,0,0\rangle$$

(b) $$\mathbf{p}\times\mathbf{q} = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ -1 & 4 & 2\\ 3 & 1 & -1\end{bmatrix} = \hat{\imath}(-4-2)-\hat{\jmath}(1-6)+\hat{\mathbf{k}}(-1-12) = \langle -6,5,-13\rangle$$
$$\mathbf{q}\times\mathbf{p} = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ 3 & 1 & -1\\ -1 & 4 & 2\end{bmatrix} = \hat{\imath}(2+4)-\hat{\jmath}(6-1)+\hat{\mathbf{k}}(12+1) = \langle 6,-5,13\rangle$$

(c) $$\mathbf{p}\times(3\mathbf{r}) = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ -1 & 4 & 2\\ 6 & -9 & -3\end{bmatrix} = \hat{\imath}(-12+18)-\hat{\jmath}(3-12)+\hat{\mathbf{k}}(9-24) = \langle 6,9,-15\rangle$$
$$3(\mathbf{p}\times\mathbf{r}) = 3\det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ -1 & 4 & 2\\ 2 & -3 & -1\end{bmatrix} = 3\Big(\hat{\imath}(-4+6)-\hat{\jmath}(1-4)+\hat{\mathbf{k}}(3-8)\Big) = \langle 6,9,-15\rangle$$

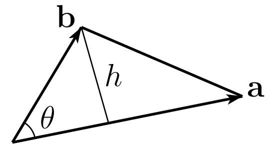
(d) As $\mathbf{q}+\mathbf{r} = \langle 5,-2,-2\rangle$
$$\mathbf{p}\times(\mathbf{q}+\mathbf{r}) = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ -1 & 4 & 2\\ 5 & -2 & -2\end{bmatrix} = \hat{\imath}(-8+4)-\hat{\jmath}(2-10)+\hat{\mathbf{k}}(2-20) = \langle -4,8,-18\rangle$$

Using the values of $\mathbf{p}\times\mathbf{q}$ and $3(\mathbf{p}\times\mathbf{r})$ computed in parts (b) and (c)
$$\mathbf{p}\times\mathbf{q}+\mathbf{p}\times\mathbf{r} = \langle -6,5,-13\rangle + \frac{1}{3}\langle 6,9,-15\rangle = \langle -4,8,-18\rangle$$

(e) $$\mathbf{q}\times\mathbf{r} = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ 3 & 1 & -1\\ 2 & -3 & -1\end{bmatrix} = \hat{\imath}(-1-3)-\hat{\jmath}(-3+2)+\hat{\mathbf{k}}(-9-2) = \langle -4,1,-11\rangle$$
$$\mathbf{p}\times(\mathbf{q}\times\mathbf{r}) = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ -1 & 4 & 2\\ -4 & 1 & -11\end{bmatrix} = \hat{\imath}(-44-2)-\hat{\jmath}(11+8)+\hat{\mathbf{k}}(-1+16) = \langle -46,-19,15\rangle$$
$$(\mathbf{p}\times\mathbf{q})\times\mathbf{r} = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ -6 & 5 & -13\\ 2 & -3 & -1\end{bmatrix} = \hat{\imath}(-5-39)-\hat{\jmath}(6+26)+\hat{\mathbf{k}}(18-10) = \langle -44,-32,8\rangle$$

1.2.29 Calculate the area of the triangle with vertices $(0,0,0)$, $(1,2,3)$ and $(3,2,1)$.

Solution Denote by θ the angle between the two vectors $\mathbf{a} = \langle 1,2,3\rangle$ and $\mathbf{b} = \langle 3,2,1\rangle$. The area of the triangle is one half times the length, $|\mathbf{a}|$, of its base times its height $h = |\mathbf{b}|\sin\theta$. Thus the area of the triangle is $\frac{1}{2}|\mathbf{a}|\,|\mathbf{b}|\,\sin\theta$. By property 2 of the cross



product in Theorem 1.2.23 of the CLP-3 text, $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta$. So

$$\begin{aligned} \text{area} &= \tfrac{1}{2}|\mathbf{a}\times\mathbf{b}| = \tfrac{1}{2}|\,\langle 1,2,3\rangle \times \langle 3,2,1\rangle\,| \\ &= \tfrac{1}{2}|\hat{\boldsymbol{\imath}}\,(2-6) - \hat{\boldsymbol{\jmath}}\,(1-9) + \hat{\mathbf{k}}\,(2-6)| \\ &= \tfrac{1}{2}\sqrt{16+64+16} \\ &= 2\sqrt{6} \end{aligned}$$

1.2.30 (*) A particle P of unit mass whose position in space at time t is $\mathbf{r}(t)$ has angular momentum $L(t) = \mathbf{r}(t)\times\mathbf{r}'(t)$. If $\mathbf{r}''(t) = \rho(t)\mathbf{r}(t)$ for a scalar function ρ, show that L is constant, i.e. does not change with time. Here $'$ denotes $\frac{\mathrm{d}}{\mathrm{d}t}$.

Solution The derivative of L is

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{r}(t)\times\mathbf{r}'(t)\big) = \mathbf{r}'(t)\times\mathbf{r}'(t) + \mathbf{r}(t)\times\mathbf{r}''(t) = \mathbf{r}'(t)\times\mathbf{r}'(t) + \mathbf{r}(t)\times\big(\rho(t)\mathbf{r}(t)\big)$$

Both terms vanish because the cross product of any two parallel vectors is zero. So $\frac{\mathrm{d}L}{\mathrm{d}t} = 0$ and $L(t)$ is independent of t.

▸▸ Stage 3

1.2.31 Show that the diagonals of a parallelogram bisect each other.

Solution The parallelogram determined by the vectors $\mathbf{a}$ and $\mathbf{b}$ has vertices $\mathbf{0}$, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}+\mathbf{b}$. As t varies from 0 to 1, $t(\mathbf{a}+\mathbf{b})$ traverses the diagonal from $\mathbf{0}$ to $\mathbf{a}+\mathbf{b}$. As s varies from 0 to 1, $\mathbf{a}+s(\mathbf{b}-\mathbf{a})$ traverses the diagonal from $\mathbf{a}$ to $\mathbf{b}$. These two straight lines meet when s and t are such that

$$t(\mathbf{a}+\mathbf{b}) = \mathbf{a}+s(\mathbf{b}-\mathbf{a})$$

or

$$(t+s-1)\mathbf{a} = (s-t)\mathbf{b}$$

Assuming that $\mathbf{a}$ and $\mathbf{b}$ are not parallel (i.e. the parallelogram has not degenerated to a line segment), this is the case only when $t+s-1=0$ and $s-t=0$. That is, $s=t=\frac{1}{2}$. So the two lines meet at their midpoints.

1.2.32 Consider a cube such that each side has length s. Name, in order, the four vertices on the bottom of the cube A, B, C, D and the corresponding four vertices on the top of the cube A', B', C', D'.

(a) Show that all edges of the tetrahedron $A'C'BD$ have the same length.

(b) Let E be the center of the cube. Find the angle between EA and EC.

Solution We may choose our coordinate axes so that $A = (0,0,0)$, $B = (s,0,0)$, $C = (s,s,0)$, $D = (0,s,0)$ and $A' = (0,0,s)$, $B' = (s,0,s)$, $C' = (s,s,s)$, $D' = (0,s,s)$.

(a) Then

$$\begin{aligned}
|A'C'| &= \big|\langle s,s,s\rangle - \langle 0,0,s\rangle\big| = \big|\langle s,s,0\rangle\big| &&= \sqrt{2}\,s\\
|A'B| &= \big|\langle s,0,0\rangle - \langle 0,0,s\rangle\big| = \big|\langle s,0,-s\rangle\big| &&= \sqrt{2}\,s\\
|A'D| &= \big|\langle 0,s,0\rangle - \langle 0,0,s\rangle\big| = \big|\langle 0,s,-s\rangle\big| &&= \sqrt{2}\,s\\
|C'B| &= \big|\langle s,0,0\rangle - \langle s,s,s\rangle\big| = \big|\langle 0,-s,-s\rangle\big| &&= \sqrt{2}\,s\\
|C'D| &= \big|\langle 0,s,0\rangle - \langle s,s,s\rangle\big| = \big|\langle -s,0,-s\rangle\big| &&= \sqrt{2}\,s\\
|BD| &= \big|\langle 0,s,0\rangle - \langle s,0,0\rangle\big| = \big|\langle -s,s,0\rangle\big| &&= \sqrt{2}\,s
\end{aligned}$$

(b) $E = \frac{1}{2}(s,s,s)$ so that $EA = \langle 0,0,0\rangle - \frac{1}{2}\langle s,s,s\rangle = -\frac{1}{2}\langle s,s,s\rangle$ and $EC = \langle s,s,0\rangle - \frac{1}{2}\langle s,s,s\rangle = \frac{1}{2}\langle s,s,-s\rangle$.

$$\cos\theta = \frac{-\langle s,s,s\rangle\cdot\langle s,s,-s\rangle}{|\langle s,s,s\rangle|\,|\langle s,s,-s\rangle|} = \frac{-s^2}{3s^2} = -\frac{1}{3} \qquad\Longrightarrow\qquad \theta = 109.5^\circ$$

1.2.33 Find the angle between the diagonal of a cube and the diagonal of one of its faces.

Solution Suppose that the cube has height, length and width s. We may choose our coordinate axes so that the vertices of the cube are at $(0,0,0)$, $(s,0,0)$, $(0,s,0)$, $(0,0,s)$, $(s,s,0)$, $(0,s,s)$, $(s,0,s)$ and (s,s,s).

We'll start with a couple of examples. The diagonal from $(0,0,0)$ to (s,s,s) is $\langle s,s,s\rangle$. One face of the cube has vertices $(0,0,0)$, $(s,0,0)$, $(0,s,0)$ and $(s,s,0)$. One diagonal of this face runs from $(0,0,0)$ to $(s,s,0)$ and hence is $\langle s,s,0\rangle$. The angle between $\langle s,s,s\rangle$ and $\langle s,s,0\rangle$ is

$$\cos^{-1}\left(\frac{\langle s,s,s\rangle\cdot\langle s,s,0\rangle}{|\langle s,s,s\rangle|\,|\langle s,s,0\rangle|}\right) = \cos^{-1}\left(\frac{2s^2}{\sqrt{3}s\,\sqrt{2}s}\right) = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) \approx 35.26^\circ$$

A second diagonal for the face with vertices $(0,0,0)$, $(s,0,0)$, $(0,s,0)$ and $(s,s,0)$ is that running from $(s,0,0)$ to $(0,s,0)$. This diagonal is $\langle -s,s,0\rangle$. The angle between $\langle s,s,s\rangle$ and $\langle -s,s,0\rangle$ is

$$\cos^{-1}\left(\frac{\langle s,s,s\rangle\cdot\langle -s,s,0\rangle}{|\langle s,s,s\rangle|\,|\langle -s,s,0\rangle|}\right) = \cos^{-1}\left(\frac{0}{\sqrt{3}s\,\sqrt{2}s}\right) = \cos^{-1}(0) = 90^\circ$$

Now we'll consider the general case. Note that every component of every vertex of the cube is either 0 or s. In general, two vertices of the cube are at opposite ends of a diagonal of the cube if all three components of the two vertices are different. For example, if one end of the diagonal is $(s,0,s)$, the other end is $(0,s,0)$. The diagonals of the cube are all of the form $\langle \pm s, \pm s, \pm s\rangle$. All of these diagonals are of length $\sqrt{3}s$. Two vertices are on the same face of the cube if one of their components agree. They are on opposite ends of a diagonal for the face if their other two components differ. For example $(0, s, s)$ and $(s, 0, s)$ are both on the face with $z = s$. Because the x components 0, s are different and the y components s, 0 are different, $(0, s, s)$ and $(s, 0, s)$ are the ends of a diagonal of the face with $z = s$. The diagonals of the faces with $z = 0$ or $z = s$ are $\langle \pm s, \pm s, 0\rangle$. The diagonals of the faces with $y = 0$ or $y = s$ are $\langle \pm s, 0, \pm s\rangle$. The diagonals of the faces with $x = 0$ or $x = s$ are $\langle 0, \pm s, \pm s\rangle$. All of these diagonals have length $\sqrt{2}s$. The dot product of one the cube diagonals $\langle \pm s, \pm s, \pm s\rangle$ with one of the face diagonals $\langle \pm s, \pm s, 0\rangle$, $\langle \pm s, 0, \pm s\rangle$, $\langle 0, \pm s, \pm s\rangle$ is of the form $\pm s^2 \pm s^2 + 0$ and hence must be either $2s^2$ or 0 or $-2s^2$. In general, the angle between a cube diagonal and a face diagonal is

$$\cos^{-1}\left(\frac{2s^2 \text{ or } 0 \text{ or } -2s^2}{\sqrt{3}s\,\sqrt{2}s}\right) = \cos^{-1}\left(\frac{2 \text{ or } 0 \text{ or } -2}{\sqrt{6}}\right) \approx 35.26^\circ \text{ or } 90^\circ \text{ or } 144.74^\circ.$$

$\boxed{1.2.34}$ Consider a skier who is sliding without friction on the hill $y = h(x)$ in a two dimensional world. The skier is subject to two forces. One is gravity. The other acts perpendicularly to the hill. The second force automatically adjusts its magnitude so as to prevent the skier from burrowing into the hill. Suppose that the skier became airborne at some (x_0, y_0) with $y_0 = h(x_0)$. How fast was the skier going?

$\boxed{\text{Solution}}$ Denote by $\big(x(t), y(t)\big)$ the position of the skier at time t. As long as the skier remains on the surface of the hill

$$\begin{aligned} y(t) &= h\big(x(t)\big) \\ \implies y'(t) &= h'\big(x(t)\big)\,x'(t) \\ \implies y''(t) &= h''\big(x(t)\big)\,x'(t)^2 + h'\big(x(t)\big)\,x''(t) \end{aligned}$$

So the velocity and acceleration vectors of the skier are

$$\begin{aligned} \mathbf{v}(t) &= \big\langle 1, h'\big(x(t)\big)\big\rangle\,x'(t) \\ \mathbf{a}(t) &= \big\langle 1, h'\big(x(t)\big)\big\rangle\,x''(t) + \big\langle 0, h''\big(x(t)\big)\big\rangle\,x'(t)^2 \end{aligned}$$

The skier is subject to two forces. One is gravity. The other acts perpendicularly to the hill and has a magnitude such that the skier remains on the surface of the hill. From the velocity vector of the skier (which remain tangential to the hill as long as the skier remains of the surface of the hill),we see that one vector normal to the hill at $\big(x(t), y(t)\big)$ is

$$\mathbf{n}(t) = \big\langle -h'\big(x(t)\big), 1\big\rangle$$

This vector is not a unit vector, but that's ok. By Newton's law of motion

$$m\mathbf{a} = -mg\,\hat{\boldsymbol{\jmath}} + p(t)\,\mathbf{n}(t)$$

for some function $p(t)$. Dot both sides of this equation with $\mathbf{n}(t)$.

$$m\mathbf{a}(t)\cdot\mathbf{n}(t) = -mg\hat{\boldsymbol{\jmath}}\cdot\mathbf{n}(t) + p(t)|\mathbf{n}(t)|^2$$

Substituting in

$$mh''\big(x(t)\big)\,x'(t)^2 = -mg + p(t)\left[1 + h'\big(x(t)\big)^2\right]$$
$$\implies p(t)\left[1 + h'\big(x(t)\big)^2\right] = m\Big(g + h''\big(x(t)\big)\,x'(t)^2\Big)$$

As long as $p(t) \geqslant 0$, the hill is pushing up in order to keep the skier on the surface. When $p(t)$ becomes negative, the hill has to pull on the skier in order to keep her on the surface. But the hill can't pull, so the skier becomes airborne instead. This happens when

$$g + h''\big(x(t)\big)x'(t)^2 = 0$$

That is when $x'(t) = \sqrt{-g/h''\big(x(t)\big)}$. At this time $x(t) = x_0$, $y(t) = y_0$ and the speed of the skier is

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{1 + h'\big(x_0\big)^2}\sqrt{-g/h''\big(x_0\big)}$$

$\boxed{1.2.35}$ A marble is placed on the plane $ax + by + cz = d$. The coordinate system has been chosen so that the positive z–axis points straight up. The coefficient c is nonzero and the coefficients a and b are not both zero. In which direction does the marble roll? Why were the conditions "$c \neq 0$" and "a, b not both zero" imposed?

$\boxed{\text{Solution}}$ The marble is subject to two forces. The first, gravity, is $-mg\,\hat{\mathbf{k}}$ with m being the mass of the marble. The second is the normal force imposed by the plane. This forces acts in a direction perpendicular to the plane. One vector normal to the plane is $a\,\hat{\boldsymbol{\imath}} + b\,\hat{\boldsymbol{\jmath}} + c\,\hat{\mathbf{k}}$. So the force due to the plane is $T\,\langle a, b, c\rangle$ with T determined by the property that the net force perpendicular to the plane must be exactly zero, so that the marble remains on the plane, neither digging into nor flying off of it. The projection of the gravitational force onto the normal vector $\langle a, b, c\rangle$ is

$$\frac{-mg\,\langle 0,0,1\rangle \cdot \langle a,b,c\rangle}{|\,\langle a,b,c\rangle\,|^2}\,\langle a,b,c\rangle = \frac{-mgc}{a^2+b^2+c^2}\,\langle a,b,c\rangle$$

The condition that determines T is thus

$$T\,\langle a,b,c\rangle + \frac{-mgc}{a^2+b^2+c^2}\,\langle a,b,c\rangle = 0 \implies T = \frac{mgc}{a^2+b^2+c^2}$$

The total force on the marble is then (ignoring friction – which will have no effect on the direction of motion)

$$\begin{aligned} T\,\langle a,b,c\rangle - mg\,\langle 0,0,1\rangle &= \frac{mgc}{a^2+b^2+c^2}\,\langle a,b,c\rangle - mg\,\langle 0,0,1\rangle \\ &= mg\frac{c\,\langle a,b,c\rangle - \langle 0,0,a^2+b^2+c^2\rangle}{a^2+b^2+c^2} \\ &= mg\frac{\langle ac, bc, -a^2-b^2\rangle}{a^2+b^2+c^2} \end{aligned}$$

The direction of motion $\langle ac, bc, -a^2 - b^2 \rangle$. If you want to turn this into a unit vector, just divide by $\sqrt{(a^2+b^2)(a^2+b^2+c^2)}$. Note that the direction vector in perpendicular $\langle a, b, c \rangle$ and hence is parallel to the plane. If $c = 0$, the plane is vertical. In this case, the marble doesn't roll – it falls straight down. If $a = b = 0$, the plane is horizontal. In this case, the marble doesn't roll — it remains stationary.

1.2.36 Show that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.

Solution By definition, the left and right hand sides are

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \langle a_1, a_2, a_3 \rangle \cdot \langle b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1 \rangle \\ &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \qquad \text{(lhs)} \end{aligned}$$

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle \cdot \langle c_1, c_2, c_3 \rangle \\ &= a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_1b_3c_2 + a_1b_2c_3 - a_2b_1c_3 \qquad \text{(rhs)} \end{aligned}$$

(lhs) and (rhs) are the same.

1.2.37 Show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

Solution By definition,

$$\mathbf{b} \times \mathbf{c} \ = (b_2c_3 - b_3c_2)\hat{\boldsymbol{\imath}} - (b_1c_3 - b_3c_1)\hat{\boldsymbol{\jmath}} + (b_1c_2 - b_2c_1)\hat{\mathbf{k}}$$

so that the left and right hand sides are

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \ &= \ \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & -b_1c_3 + b_3c_1 & b_1c_2 - b_2c_1 \end{bmatrix} \\ &= \hat{\boldsymbol{\imath}}\,[a_2(b_1c_2 - b_2c_1) - a_3(-b_1c_3 + b_3c_1)] \\ &\quad -\hat{\boldsymbol{\jmath}}\,[a_1(b_1c_2 - b_2c_1) - a_3(b_2c_3 - b_3c_2)] \\ &\quad +\hat{\mathbf{k}}\,[a_1(-b_1c_3 + b_3c_1) - a_2(b_2c_3 - b_3c_2)] \qquad \text{(lhs)} \\ (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \ &= (a_1c_1 + a_2c_2 + a_3c_3)(b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\mathbf{k}}) - (a_1b_1 + a_2b_2 + a_3b_3)(c_1\hat{\boldsymbol{\imath}} + c_2\hat{\boldsymbol{\jmath}} + c_3\hat{\mathbf{k}}) \\ &= \hat{\boldsymbol{\imath}}\,[a_1b_1c_1 + a_2b_1c_2 + a_3b_1c_3 - a_1b_1c_1 - a_2b_2c_1 - a_3b_3c_1] \\ &\quad +\hat{\boldsymbol{\jmath}}\,[a_1b_2c_1 + a_2b_2c_2 + a_3b_2c_3 - a_1b_1c_2 - a_2b_2c_2 - a_3b_3c_2] \\ &\quad +\hat{\mathbf{k}}\,[a_1b_3c_1 + a_2b_3c_2 + a_3b_3c_3 - a_1b_1c_3 - a_2b_2c_3 - a_3b_3c_3] \\ &= \hat{\boldsymbol{\imath}}\,[a_2b_1c_2 + a_3b_1c_3 - a_2b_2c_1 - a_3b_3c_1] \\ &\quad +\hat{\boldsymbol{\jmath}}\,[a_1b_2c_1 + a_3b_2c_3 - a_1b_1c_2 - a_3b_3c_2] \\ &\quad +\hat{\mathbf{k}}\,[a_1b_3c_1 + a_2b_3c_2 - a_1b_1c_3 - a_2b_2c_3] \qquad \text{(rhs)} \end{aligned}$$

(lhs) and (rhs) are the same.

1.2.38 Derive a formula for $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$ that involves dot but not cross products.

Solution By properties 9 and 10 of Theorem 1.2.23 in the CLP-3 text,

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= \mathbf{a}\cdot[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})] && \text{(by property 9 with } \mathbf{c}\to(\mathbf{c}\times\mathbf{d})\text{)}\\ &= \mathbf{a}\cdot[(\mathbf{b}\cdot\mathbf{d})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{d}] && \text{(by property 10)}\\ &= (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})\end{aligned}$$

So

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

1.2.39 A prism has the six vertices

$$\begin{aligned} A &= (1,0,0) & A' &= (5,0,1)\\ B &= (0,3,0) & B' &= (4,3,1)\\ C &= (0,0,4) & C' &= (4,0,5)\end{aligned}$$

(a) Verify that three of the faces are parallelograms. Are they rectangular?
(b) Find the length of AA'.
(c) Find the area of the triangle ABC.
(d) Find the volume of the prism.

Solution (a) $AA' = \langle 4,0,1\rangle$ and $BB' = \langle 4,0,1\rangle$ are opposite sides of the quadrilateral $AA'B'B$. They have the same length and direction. The same is true for $AB = \langle -1,3,0\rangle$ and $A'B' = \langle -1,3,0\rangle$. So $AA'B'B$ is a parallelogram. Because, $AA'\cdot AB = \langle 4,0,1\rangle\cdot\langle -1,3,0\rangle = -4 \neq 0$, the neighbouring edges of $AA'B'B$ are not perpendicular and so $AA'B'B$ is not a rectangle.

Similarly, the quadilateral $ACC'A'$ has opposing sides $AA' = \langle 4,0,1\rangle = CC' = \langle 4,0,1\rangle$ and $AC = \langle -1,0,4\rangle = A'C' = \langle -1,0,4\rangle$ and so is a parallelogram. Because $AA'\cdot AC = \langle 4,0,1\rangle\cdot\langle -1,0,4\rangle = 0$, the neighbouring edges of $ACC'A'$ are perpendicular, so $ACC'A'$ is a rectangle.

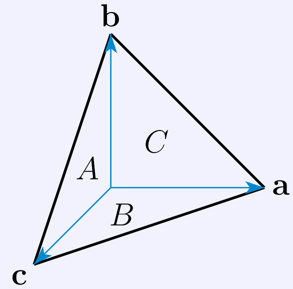
Finally, the quadilateral $BCC'B'$ has opposing sides $BB' = \langle 4,0,1\rangle = CC' = \langle 4,0,1\rangle$ and $BC = \langle 0,-3,4\rangle = B'C' = \langle 0,-3,4\rangle$ and so is a parallelogram. Because $BB'\cdot BC = \langle 4,0,1\rangle\cdot\langle 0,-3,4\rangle = 4 \neq 0$, the neighbouring edges of $BCC'B'$ are not perpendicular, so $BCC'B'$ is not a rectangle.

(b) The length of AA' is $|\,\langle 4,0,1\rangle\,| = \sqrt{16+1} = \sqrt{17}$.

(c) The area of a triangle is one half its base times its height. That is, one half times $|AB|$ times $|AC|\sin\theta$, where θ is the angle between AB and AC. This is precisely $\frac{1}{2}|AB\times AC| = \frac{1}{2}|\,\langle -1,3,0\rangle\times\langle -1,0,4\rangle\,| = \frac{1}{2}|\,\langle 12,4,3\rangle\,| = \frac{13}{2}$.

(d) The volume of the prism is the area of its base ABC, times its height, which is the length of AA' times the cosine of the angle between AA' and the normal to ABC. This coincides with $\frac{1}{2}\langle 12,4,3\rangle\cdot\langle 4,0,1\rangle = \frac{1}{2}(48+3) = \frac{51}{2}$, which is one half times the length of $\langle 12,4,3\rangle$ (the area of ABC) times the length of $\langle 4,0,1\rangle$ (the length of AA') times the cosine of the angle between $\langle 12,4,3\rangle$ and $\langle 4,0,1\rangle$ (the angle between the normal to ABC and AA').

1.2.40 (Three dimensional Pythagorean Theorem) A solid body in space with exactly four vertices is called a tetrahedron. Let A, B, C and D be the areas of the four faces of a tetrahedron. Suppose that the three edges meeting at the vertex opposite the face of area D are perpendicular to each other. Show that $D^2 = A^2 + B^2 + C^2$.



Solution Choose our coordinate axes so that the vertex opposite the face of area D is at the origin. Denote by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ the vertices opposite the sides of area A, B and C respectively. Then the face of area A has edges $\mathbf{b}$ and $\mathbf{c}$ so that $A = \frac{1}{2}|\mathbf{b}\times\mathbf{c}|$. Similarly $B = \frac{1}{2}|\mathbf{c}\times\mathbf{a}|$ and $C = \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$. The face of area D is the triangle spanned by $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$ so that

$$\begin{aligned} D &= \tfrac{1}{2}|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})| \\ &= \tfrac{1}{2}|\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}-\mathbf{b}\times\mathbf{a}| \\ &= \tfrac{1}{2}|\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}| \end{aligned}$$

By hypothesis, the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are all perpendicular to each other. Consequently the vectors $\mathbf{b}\times\mathbf{c}$ (which is a scalar times $\mathbf{a}$), $\mathbf{c}\times\mathbf{a}$ (which is a scalar times $\mathbf{b}$) and $\mathbf{a}\times\mathbf{b}$ (which is a scalar times $\mathbf{c}$) are also mutually perpendicular. So, when we multiply out

$$D^2 = \frac{1}{4}\big[\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}\big]\cdot\big[\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}\big]$$

all the cross terms vanish, leaving

$$D^2 = \frac{1}{4}\big[(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{b}\times\mathbf{c})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{c}\times\mathbf{a})+(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})\big] = A^2+B^2+C^2$$

1.2.41 (Three dimensional law of cosines) Let A, B, C and D be the areas of the four faces of a tetrahedron. Let α be the angle between the faces with areas B and C, β be the angle between the faces with areas A and C and γ be the angle between the faces with areas A and B. (By definition, the angle between two faces is the angle between the normal vectors to the faces.) Show that

$$D^2 = A^2+B^2+C^2-2BC\cos\alpha-2AC\cos\beta-2AB\cos\gamma$$

Solution As in problem 40,

$$D^2 = \frac{1}{4}\big[\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}\big]\cdot\big[\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}\big]$$

But now $(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{c})$, instead of vanishing, is $|\mathbf{b}\times\mathbf{c}| = 2A$ times $|\mathbf{a}\times\mathbf{c}| = 2B$ times the cosine of the angle between $\mathbf{b}\times\mathbf{c}$ (which is perpendicular to the face of area A) and $\mathbf{a}\times\mathbf{c}$ (which is perpendicular to the face of area B). That is

$$\begin{aligned}(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{c}) &= 4AB\cos\gamma\\ (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{b}) &= 4AC\cos\beta\\ (\mathbf{b}\times\mathbf{a})\cdot(\mathbf{c}\times\mathbf{a}) &= 4BC\cos\alpha\end{aligned}$$

(If you're worried about the signs, that is, if you are worried about why $(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{c}) = 4AB\cos\gamma$ rather than $(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{c}\times\mathbf{a}) = 4AB\cos\gamma$, note that when $\mathbf{a}\approx\mathbf{b}$, $(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{c})\approx|\mathbf{b}\times\mathbf{c}|^2$ is positive and $(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{c}\times\mathbf{a})\approx -|\mathbf{b}\times\mathbf{c}|^2$ is negative.) Now, expanding out

$$\begin{aligned}D^2 &= \frac{1}{4}\big[\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}\big]\cdot\big[\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}\big]\\ &= \frac{1}{4}\big[(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{b}\times\mathbf{c})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{c}\times\mathbf{a})+(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})\\ &\qquad +2(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{c}\times\mathbf{a})+2(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{b})+2(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{a}\times\mathbf{b})\big]\\ &= A^2+B^2+C^2-2AB\cos\gamma-2AC\cos\beta-2BC\cos\alpha\end{aligned}$$

1.3▲ Equations of Lines in 2d

▸▸ Stage 1

$\boxed{\text{1.3.1}}$ A line in $\mathbb{R}^2$ has direction $\mathbf{d}$ and passes through point $\mathbf{c}$. Which of the following gives its parametric equation: $\langle x,y\rangle = \mathbf{c}+t\mathbf{d}$, or $\langle x,y\rangle = \mathbf{c}-t\mathbf{d}$?

$\boxed{\text{Solution}}$ Since t can be any real number, these equation describe the same line. They're both valid. For example, the point given by the first parametric equation with $t=7$, namely $\mathbf{c}+7\mathbf{d}$, is exactly the same as the point given by the second parametric equation with $t=-7$, namely $\mathbf{c}-(-7)\mathbf{d}$.

$\boxed{\text{1.3.2}}$ A line in $\mathbb{R}^2$ has direction $\mathbf{d}$ and passes through point $\mathbf{c}$. Which of the following gives its parametric equation: $\langle x,y\rangle = \mathbf{c}+t\mathbf{d}$, or $\langle x,y\rangle = -\mathbf{c}+t\mathbf{d}$?

$\boxed{\text{Solution}}$ In contrast to Question 1, the sign on $\mathbf{c}$ does generally matter. $\mathbf{c}$ is required to be a point on the line, but except in particular circumstances, there's no reason to believe that $-\mathbf{c} = -\mathbf{c}+t\mathbf{d}\big|_{t=0}$ is a point on the line. Indeed $-\mathbf{c}$ is on the line if and only if there is a t with $\mathbf{c}+t\mathbf{d} = -\mathbf{c}$, i.e. $t\mathbf{d} = -2\mathbf{c}$. That is the case if and only if $\mathbf{d}$ is parallel to $\mathbf{c}$. So, only the first equation is correct in general.

1.3.3 Two points determine a line. Verify that the equations

$$\langle x-1, y-9\rangle = t\,\langle 8,4\rangle$$

and

$$\langle x-9, y-13\rangle = t\left\langle 1, \tfrac{1}{2}\right\rangle$$

describe the same line by finding two different points that lie on both lines.

Solution Here is one answer of many.

Setting $t=0$ in the first equation shows that $(1,9)$ is on the first line. To see that $(1,9)$ is also on the second line, we substitute $x=1$, $y=9$ into the second equation to give

$$\langle 1-9, 9-13\rangle = t\left\langle 1, \tfrac{1}{2}\right\rangle \qquad \text{or} \qquad \langle -8, -4\rangle = t\left\langle 1, \tfrac{1}{2}\right\rangle$$

This equation is satisfied when $t=-8$. So $(1,9)$ is on both lines.

Setting $t=0$ in the second equation shows that $(9,13)$ is on the second line. To see that $(9,13)$ is also on the first line, we substitute $x=9$, $y=13$ into the first equation to give

$$\langle 9-1, 13-9\rangle = t\,\langle 8,4\rangle \qquad \text{or} \qquad \langle 8,4\rangle = t\,\langle 8,4\rangle$$

This equation is satisfied when $t=1$. So $(9,13)$ is on both lines.

Since both lines pass through $(1,9)$ and $(9,13)$, the lines are identical.

1.3.4 A line in $\mathbb{R}^2$ has parametric equations

$$\begin{aligned} x-3 &= 9t \\ y-5 &= 7t \end{aligned}$$

There are many different ways to write the parametric equations of this line. If we rewrite the equations as

$$\begin{aligned} x-x_0 &= d_x t \\ y-y_0 &= d_y t \end{aligned}$$

what are all possible values of $\langle x_0, y_0\rangle$ and $\langle d_x, d_y\rangle$?

Solution $\langle d_x, d_y\rangle$ is the direction of the line, so it can be any non-zero scalar multiple of $\langle 9,7\rangle$.

$\langle x_0, y_0\rangle$ can be any point on the line. Describing these is the same as describing the line itself. We're trying to find all doubles $\langle x_0, y_0\rangle$ that obey

$$\begin{cases} x_0-3 &= 9t \\ y_0-5 &= 7t \end{cases}$$

for some real number t. That is,

$$\begin{aligned} t = \frac{x_0 - 3}{9} &= \frac{y_0 - 5}{7} \\ 7(x_0 - 3) &= 9(y_0 - 5) \\ 7x_0 + 24 &= 9y_0 \end{aligned}$$

Any of these steps could specify the possible values of $\langle x_0, y_0 \rangle$. Say, they can be any pair satisfying $7x_0 + 24 = 9y_0$.

▸▸ Stage 2

1.3.5 Find the vector parametric, scalar parametric and symmetric equations for the line containing the given point and with the given direction.
(a) point $(1,2)$, direction $\langle 3,2 \rangle$
(b) point $(5,4)$, direction $\langle 2,-1 \rangle$
(c) point $(-1,3)$, direction $\langle -1,2 \rangle$

Solution (a) The vector parametric equation is $\langle x,y \rangle = \langle 1,2 \rangle + t\,\langle 3,2 \rangle$. The scalar parametric equations are $x = 1+3t,\ y = 2+2t$. The symmetric equation is $\frac{x-1}{3} = \frac{y-2}{2}$.

(b) The vector parametric equation is $\langle x,y \rangle = \langle 5,4 \rangle + t\,\langle 2,-1 \rangle$. The scalar parametric equations are $x = 5+2t,\ y = 4-t$. The symmetric equation is $\frac{x-5}{2} = \frac{y-4}{-1}$.

(c) The vector parametric equation is $\langle x,y \rangle = \langle -1,3 \rangle + t\,\langle -1,2 \rangle$. The scalar parametric equations are $x = -1-t,\ y = 3+2t$. The symmetric equation is $\frac{x+1}{-1} = \frac{y-3}{2}$.

1.3.6 Find the vector parametric, scalar parametric and symmetric equations for the line containing the given point and with the given normal.
(a) point $(1,2)$, normal $\langle 3,2 \rangle$
(b) point $(5,4)$, normal $\langle 2,-1 \rangle$
(c) point $(-1,3)$, normal $\langle -1,2 \rangle$

Solution (a) The vector $\langle -2,3 \rangle$ is perpendicular to $\langle 3,2 \rangle$ (you can verify this by taking the dot product of the two vectors) and hence is a direction vector for the line. The vector parametric equation is $\langle x,y \rangle = \langle 1,2 \rangle + t\,\langle -2,3 \rangle$. The scalar parametric equations are $x = 1-2t,\ y = 2+3t$. The symmetric equation is $\frac{x-1}{-2} = \frac{y-2}{3}$.

(b) The vector $\langle 1,2 \rangle$ is perpendicular to $\langle 2,-1 \rangle$ and hence is a direction vector for the line. The vector parametric equation for the line is $\langle x,y \rangle = \langle 5,4 \rangle + t\,\langle 1,2 \rangle$. The scalar parametric equations are $x = 5+t,\ y = 4+2t$. The symmetric equation is $x-5 = \frac{y-4}{2}$.

(c) The vector $\langle 2,1 \rangle$ is perpendicular to $\langle -1,2 \rangle$ and hence is a direction vector for the line. The vector parametric equation is $\langle x,y \rangle = \langle -1,3 \rangle + t\,\langle 2,1 \rangle$. The scalar parametric equations are the two component equations $x = -1+2t,\ y = 3+t$. The symmetric equation is $\frac{x+1}{2} = y-3$.

1.3.7 Use a projection to find the distance from the point $(-2,3)$ to the line $3x-4y=-4$.

Solution $(0,1)$ is one point on the line $3x-4y=-4$. So $\langle -2-0, 3-1\rangle = \langle -2,2\rangle$ is a vector whose tail is on the line and whose head is at $(-2,3)$. $\langle 3,-4\rangle$ is a vector perpendicular to the line, so $\frac{1}{5}\langle 3,-4\rangle$ is a unit vector perpendicular to the line. The distance from $(-2,3)$ to the line is the length of the projection of $\langle -2,2\rangle$ on $\frac{1}{5}\langle 3,-4\rangle$, which is the magnitude of $\frac{1}{5}\langle 3,-4\rangle\cdot\langle -2,2\rangle$. So the distance is $14/5$.

1.3.8 Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the vertices of a triangle. By definition, a median of a triangle is a straight line that passes through a vertex of the triangle and through the midpoint of the opposite side.
(a) Find the parametric equations of the three medians.
(b) Do the three medians meet at a common point? If so, which point?

Solution (a) The midpoint of the side opposite $\mathbf{a}$ is $\frac{1}{2}(\mathbf{b}+\mathbf{c})$. The vector joining $\mathbf{a}$ to that midpoint is $\frac{1}{2}\mathbf{b}+\frac{1}{2}\mathbf{c}-\mathbf{a}$. The vector parametric equation of the line through $\mathbf{a}$ and $\frac{1}{2}(\mathbf{b}+\mathbf{c})$ is

$$\mathbf{x}(t)=\mathbf{a}+t\big(\tfrac{1}{2}\mathbf{b}+\tfrac{1}{2}\mathbf{c}-\mathbf{a}\big)$$

Similarly, for the other two medians (but using s and u as parameters, rather than t)

$$\mathbf{x}(s)=\mathbf{b}+s\big(\tfrac{1}{2}\mathbf{a}+\tfrac{1}{2}\mathbf{c}-\mathbf{b}\big)$$
$$\mathbf{x}(u)=\mathbf{c}+u\big(\tfrac{1}{2}\mathbf{a}+\tfrac{1}{2}\mathbf{b}-\mathbf{c}\big)$$

(b) The three medians meet at a common point if there are values of s,t and u such that

$$\begin{aligned}\mathbf{a}+t\big(\tfrac{1}{2}\mathbf{b}+\tfrac{1}{2}\mathbf{c}-\mathbf{a}\big) &= \mathbf{b}+s\big(\tfrac{1}{2}\mathbf{a}+\tfrac{1}{2}\mathbf{c}-\mathbf{b}\big) &= \mathbf{c}+u\big(\tfrac{1}{2}\mathbf{a}+\tfrac{1}{2}\mathbf{b}-\mathbf{c}\big)\\ (1-t)\mathbf{a}+\frac{t}{2}\mathbf{b}+\frac{t}{2}\mathbf{c} &= \frac{s}{2}\mathbf{a}+(1-s)\mathbf{b}+\frac{s}{2}\mathbf{c} &= \frac{u}{2}\mathbf{a}+\frac{u}{2}\mathbf{b}+(1-u)\mathbf{c}\end{aligned}$$

Assuming that the triangle has not degenerated to a line segment, this is the case if and only if the coefficients of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ match

$$\begin{aligned}1-t=\frac{s}{2}\quad &= \frac{u}{2}\\ \frac{t}{2}=1-s &= \frac{u}{2}\\ \frac{t}{2}=\frac{s}{2}\quad &= 1-u\end{aligned}$$

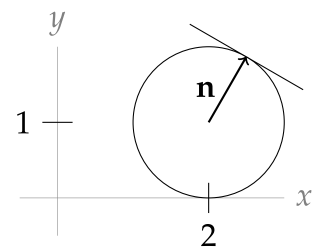
or

$$s=t=u,\ 1-t=\frac{t}{2}\implies s=t=u=\frac{2}{3}$$

The medians meet at $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$.

1.3.9 Let C be the circle of radius 1 centred at $(2,1)$. Find an equation for the line tangent to C at the point $\left(\frac{5}{2}, 1+\frac{\sqrt{3}}{2}\right)$.

Solution A normal vector to the line is the vector with its tail at the centre of C, $(2,1)$, and its head at $\left(\frac{5}{2}, 1+\frac{\sqrt{3}}{2}\right)$. So, we set $\mathbf{n} = \left\langle \frac{5}{2}, 1+\frac{\sqrt{3}}{2}\right\rangle - \langle 2,1\rangle = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle$.



We know one point on the line is $\left(\frac{5}{2}, 1+\frac{\sqrt{3}}{2}\right)$, so following Equation 1.3.3 in the CLP-3 text:

$$\begin{aligned}
n_x x + n_y y &= n_x x_0 + n_y y_0 \\
\frac{1}{2}x + \frac{\sqrt{3}}{2}y &= \frac{1}{2}\cdot\frac{5}{2} + \frac{\sqrt{3}}{2}\cdot\left(1+\frac{\sqrt{3}}{2}\right) \\
\frac{1}{2}x + \frac{\sqrt{3}}{2}y &= 2 + \frac{\sqrt{3}}{2} \\
x + \sqrt{3}y &= 4 + \sqrt{3}
\end{aligned}$$

1.4▲ Equations of Planes in 3d

▸▸ Stage 1

1.4.1 The vector $\hat{\mathbf{k}}$ is a normal vector (i.e. is perpendicular) to the plane $z = 0$. Find another nonzero vector that is normal to $z = 0$.

Solution We are looking for a vector that is perpendicular to $z = 0$ and hence is parallel to $\hat{\mathbf{k}}$. To be parallel of $\hat{\mathbf{k}}$, the vector has to be of the form $c\,\hat{\mathbf{k}}$ for some real number c. For the vector to be nonzero, we need $c \neq 0$ and for the vector to be different from $\hat{\mathbf{k}}$, we need $c \neq 1$. So three possible choices are $-\hat{\mathbf{k}}$, $2\,\hat{\mathbf{k}}$, $7.12345\,\hat{\mathbf{k}}$.

1.4.2 Consider the plane P with equation $3x + \frac{1}{2}y + z = 4$.

(a) Find the intersection of P with the y-axis.

(b) Find the intersection of P with the z-axis.

(c) Sketch the part of the intersection of P with the yz-plane that is in the first octant. (That is, with $x, y, z \geqslant 0$.)

Solution

(a) Each point on the y-axis is of the form $(0, y, 0)$. Such a point is on the plane P if

$$3(0) + \frac{1}{2}y + 0 = 4 \iff y = 8$$

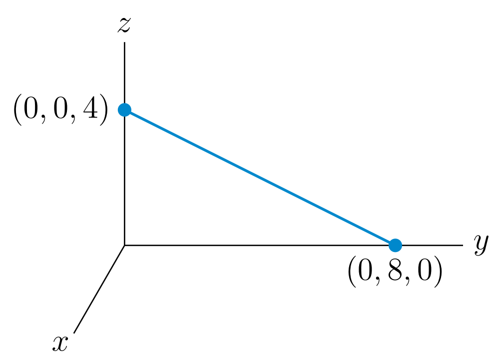
So the intersection of P with the y-axis is the single point $(0, 8, 0)$.

(b) Each point on the z-axis is of the form $(0, 0, z)$. Such a point is on the plane P if

$$3(0) + \frac{1}{2}(0) + z = 4 \iff z = 4$$

So the intersection of P with the z-axis is the single point $(0, 0, 4)$.

(c) The intersection of the plane P with the yz-plane is a line. We have shown in parts (a) and (b) that the points $(0, 8, 0)$ and $(0, 0, 4)$ are on that line. Here is a sketch of the part of that line that is in the first octant.



1.4.3

(a) Find the equation of the plane that passes through the origin and has normal vector $\langle 1, 2, 3\rangle$.
(b) Find the equation of the plane that passes through the point $(0, 0, 1)$ and has normal vector $\langle 1, 1, 3\rangle$.
(c) Find, if possible, the equation of a plane that passes through both $(1, 2, 3)$ and $(1, 0, 0)$ and has normal vector $\langle 4, 5, 6\rangle$.
(d) Find, if possible, the equation of a plane that passes through both $(1, 2, 3)$ and $(0, 3, 4)$ and has normal vector $\langle 2, 1, 1\rangle$.

Solution (a) If (x, y, z) is any point on the plane, then both the head and the tail of the vector from (x, y, z) to $(0, 0, 0)$, namely $\langle x, y, z\rangle$, lie in the plane. So the vector $\langle x, y, z\rangle$ must be perpendicular to $\langle 1, 2, 3\rangle$ and

$$0 = \langle x, y, z\rangle \cdot \langle 1, 2, 3\rangle = x + 2y + 3z$$

(b) If (x, y, z) is any point on the plane, then both the head and the tail of the vector from (x, y, z) to $(0, 0, 1)$, namely $\langle x, y, z - 1\rangle$, lie in the plane. So the vector $\langle x, y, z - 1\rangle$ must be perpendicular to $\langle 1, 1, 3\rangle$ and

$$0 = \langle x, y, z - 1\rangle \cdot \langle 1, 1, 3\rangle = x + y + 3(z - 1) \iff x + y + 3z = 3$$

(c) If both $(1,2,3)$ and $(1,0,0)$ are on the plane, then both the head and the tail of the vector from $(1,2,3)$ to $(1,0,0)$, namely $\langle 0,2,3\rangle$, lie in the plane. So the vector $\langle 0,2,3\rangle$ must be perpendicular to $\langle 4,5,6\rangle$. As

$$\langle 0,2,3\rangle \cdot \langle 4,5,6\rangle = 28 \neq 0$$

the vector $\langle 0,2,3\rangle$ is not perpendicular to $\langle 4,5,6\rangle$. So there is no plane that passes through both $(1,2,3)$ and $(1,0,0)$ and has normal vector $\langle 4,5,6\rangle$.

(d) If both $(1,2,3)$ and $(0,3,4)$ are on the plane, then both the head and the tail of the vector from $(1,2,3)$ to $(0,3,4)$, namely $\langle 1,-1,-1\rangle$, lie in the plane. So the vector $\langle 1,-1,-1\rangle$ must be perpendicular to $\langle 2,1,1\rangle$. As

$$\langle 1,-1,-1\rangle \cdot \langle 2,1,1\rangle = 0$$

the vector $\langle 1,-1,-1\rangle$ is indeed perpendicular to $\langle 2,1,1\rangle$. So there is a plane that passes through both $(1,2,3)$ and $(0,3,4)$ and has normal vector $\langle 2,1,1\rangle$. We now just have to build its equation.

If (x,y,z) is any point on the plane, then both the head and the tail of the vector from (x,y,z) to $(1,2,3)$, namely $\langle x-1,y-2,z-3\rangle$, lie in the plane. So the vector $\langle x-1,y-2,z-3\rangle$ must be perpendicular to $\langle 2,1,1\rangle$ and

$$0 = \langle x-1,y-2,z-3\rangle \cdot \langle 2,1,1\rangle = 2(x-1)+(y-2)+(z-3) \iff 2x+y+z=7$$

As a check, note that both $(x,y,z) = (1,2,3)$ and $(x,y,z) = (0,3,4)$ obey the equation $2x+y+z=7$.

$\boxed{1.4.4}$ $(*)$ Find the equation of the plane that contains $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

$\boxed{\text{Solution}}$ Solution 1: That's too easy. We just guess. The plane $x+y+z=1$ contains all three given points.

Solution 2: The plane does not pass through the origin. (You can see this by just making a quick sketch.) So the plane has an equation of the form $ax+by+cz=1$.

- For $(1,0,0)$ to be on the plane we need that

$$a(1)+b(0)+c(0)=1 \implies a=1$$

- For $(0,1,0)$ to be on the plane we need that

$$a(0)+b(1)+c(0)=1 \implies b=1$$

- For $(0,0,1)$ to be on the plane we need that

$$a(0)+b(0)+c(1)=1 \implies c=1$$

So the plane is $x+y+z=1$.

Solution 3: Both the head and the tail of the vector from $(1,0,0)$ to $(0,1,0)$, namely $\langle -1,1,0\rangle$, lie in the plane. Similarly, both the head and the tail of the vector from $(1,0,0)$ to $(0,0,1)$, namely $\langle -1,0,1\rangle$, lie in the plane. So the vector

$$\langle -1,1,0\rangle \times \langle -1,0,1\rangle = \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \langle 1,1,1\rangle$$

is a normal vector for the plane. As $(1,0,0)$ is a point in the plane,

$$\langle 1,1,1\rangle \cdot \langle x-1\,,\, y-0\,,\, z-0\rangle = 0 \qquad \text{or} \quad x+y+z=1$$

is an equation for the plane.

1.4.5

(a) Find the equation of the plane containing the points $(1,0,1)$, $(1,1,0)$ and $(0,1,1)$.
(b) Is the point $(1,1,1)$ on the plane?
(c) Is the origin on the plane?
(d) Is the point $(4,-1,-1)$ on the plane?

Solution (a) Solution 1: That's too easy. We just guess. The plane $x+y+z=2$ contains all three given points.

Solution 2: Both the head and the tail of the vector from $(1,0,1)$ to $(0,1,1)$, namely $\langle 1,-1,0\rangle$, lie in the plane. Similarly, both the head and the tail of the vector from $(1,1,0)$ to $(0,1,1)$, namely $\langle 1,0,-1\rangle$, lie in the plane. So the vector

$$\langle 1,-1,0\rangle \times \langle 1,0,-1\rangle = \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \langle 1,1,1\rangle$$

is a normal vector for the plane. As $(0,1,1)$ is a point in the plane,

$$\langle 1,1,1\rangle \cdot \langle x-0\,,\, y-1\,,\, z-1\rangle = 0 \qquad \text{or} \quad x+y+z=2$$

is an equation for the plane.

(b) Since

$$\Big[x+y+z\Big]_{(x,y,z)=(1,1,1)} = 3 \neq 2$$

the point $(1,1,1)$ is not on $x+y+z=2$.

(c) Since

$$\Big[x+y+z\Big]_{(x,y,z)=(0,0,0)} = 0 \neq 2$$

the origin is not on $x+y+z=2$.

(d) Since

$$\Big[x+y+z\Big]_{(x,y,z)=(4,-1,-1)} = 2$$

the point $(4,-1,-1)$ is on $x+y+z=2$.

$\boxed{1.4.6}$ What's wrong with the following exercise? "Find the equation of the plane containing $(1,2,3)$, $(2,3,4)$ and $(3,4,5)$."

$\boxed{\text{Solution}}$ The vector from $(1,2,3)$ to $(2,3,4)$, namely $\langle 1,1,1\rangle$ is parallel to the vector from $(1,2,3)$ to $(3,4,5)$, namely $\langle 2,2,2\rangle$. So the three given points are collinear. Precisely, all three points $(1,2,3)$, $(2,3,4)$ and $(3,4,5)$ are on the line $\langle x-1, y-2, z-3\rangle = t\,\langle 1,1,1\rangle$. There are many planes through that line.

▸▸ Stage 2

$\boxed{1.4.7}$ Find the plane containing the given three points.
(a) $(1,0,1),\ (2,4,6),\ (1,2,-1)$
(b) $(1,-2,-3),\ (4,-4,4),\ (3,2,-3)$
(c) $(1,-2,-3),\ (5,2,1),\ (-1,-4,-5)$

$\boxed{\text{Solution}}$ (a) The plane must be parallel to $\langle 2,4,6\rangle - \langle 1,0,1\rangle = \langle 1,4,5\rangle$ and to $\langle 1,2,-1\rangle - \langle 1,0,1\rangle = \langle 0,2,-2\rangle$. So its normal vector must be perpendicular to both $\langle 1,4,5\rangle$ and $\langle 0,2,-2\rangle$ and hence parallel to

$$\langle 1,4,5\rangle \times \langle 0,2,-2\rangle = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & 4 & 5 \\ 0 & 2 & -2 \end{bmatrix} = \langle -18,2,2\rangle$$

The plane is $9(x-1)-y-(z-1)=0$ or $9x-y-z=8$.

We can check this by observing that $(1,0,1),\ (2,4,6)$ and $(1,2,-1)$ all satisfy $9x-y-z=8$.

(b) The plane must be parallel to $\langle 4,-4,4\rangle - \langle 1,-2,-3\rangle = \langle 3,-2,7\rangle$ and to $\langle 3,2,-3\rangle - \langle 1,-2,-3\rangle = \langle 2,4,0\rangle$. So its normal vector must be perpendicular to both $\langle 3,-2,7\rangle$ and $\langle 2,4,0\rangle$ and hence parallel to

$$\langle 3,-2,7\rangle \times \langle 2,4,0\rangle = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 3 & -2 & 7 \\ 2 & 4 & 0 \end{bmatrix} = \langle -28,14,16\rangle$$

The plane is $14(x-1)-7(y+2)-8(z+3)=0$ or $14x-7y-8z=52$.

We can check this by observing that $(1,-2,-3),\ (4,-4,4)$ and $(3,2,-3)$ all satisfy $14x-7y-8z=52$.

(c) The plane must be parallel to $\langle 5,2,1\rangle - \langle 1,-2,-3\rangle = \langle 4,4,4\rangle$ and to $\langle -1,-4,-5\rangle - \langle 1,-2,-3\rangle = \langle -2,-2,-2\rangle$. My, my. These two vectors are parallel. So the three points are all on the same straight line. Any plane containing the line contains all three points. If $\langle a,b,c\rangle$ is any vector perpendicular to $\langle 1,1,1\rangle$ (i.e. which obeys $a+b+c=0$) then the plane $a(x-1)+b(y+2)+c(z+3)=0$ or

$a(x-1)+b(y+2)-(a+b)(z+3)=0$ or $ax+by-(a+b)z=4a+b$ contains the three given points.

We can check this by observing that $(1,-2,-3)$, $(5,2,1)$ and $(-1,-4,-5)$ all satisfy the equation $ax+by-(a+b)z=4a+b$ for all a and b.

1.4.8 Find the distance from the given point to the given plane.
(a) point $(-1,2,3)$, plane $x+y+z=7$
(b) point $(1,-4,3)$, plane $x-2y+z=5$

Solution (a) One point on the plane is $(0,0,7)$. The vector from $(-1,2,3)$ to $(0,0,7)$ is $\langle 0,0,7\rangle - \langle -1,2,3\rangle = \langle 1,-2,4\rangle$. A unit vector perpendicular to the plane is $\frac{1}{\sqrt{3}}\langle 1,1,1\rangle$. The distance from $(-1,2,3)$ to the plane is the length of the projection of $\langle 1,-2,4\rangle$ on $\frac{1}{\sqrt{3}}\langle 1,1,1\rangle$ which is

$$\frac{1}{\sqrt{3}}\langle 1,1,1\rangle \cdot \langle 1,-2,4\rangle = \frac{3}{\sqrt{3}} = \sqrt{3}$$

(b) One point on the plane is $(0,0,5)$. The vector from $(1,-4,3)$ to $(0,0,5)$ is $\langle 0,0,5\rangle - \langle 1,-4,3\rangle = \langle -1,4,2\rangle$. A unit vector perpendicular to the plane is $\frac{1}{\sqrt{6}}\langle 1,-2,1\rangle$. The distance from $(1,-4,3)$ to the plane is the length of the projection of $\langle -1,4,2\rangle$ on $\frac{1}{\sqrt{6}}\langle 1,-2,1\rangle$ which is the absolute value of

$$\frac{1}{\sqrt{6}}\langle 1,-2,1\rangle \cdot \langle -1,4,2\rangle = \frac{-7}{\sqrt{6}}$$

or $7/\sqrt{6}$.

1.4.9 (*) A plane Π passes through the points $A=(1,1,3)$, $B=(2,0,2)$ and $C=(2,1,0)$ in $\mathbb{R}^3$.
(a) Find an equation for the plane Π.
(b) Find the point E in the plane Π such that the line L through $D=(6,1,2)$ and E is perpendicular to Π.

Solution (a) The vector from C to A, namely $\langle 1-2\,,\,1-1\,,\,3-0\rangle = \langle -1\,,\,0\,,\,3\rangle$ lies entirely inside Π. The vector from C to B, namely $\langle 2-2\,,\,0-1\,,\,2-0\rangle = \langle 0\,,\,-1\,,\,2\rangle$ also lies entirely inside Π. Consequently, the vector

$$\langle -1\,,\,0\,,\,3\rangle \times \langle 0\,,\,-1\,,\,2\rangle = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ -1 & 0 & 3 \\ 0 & -1 & 2 \end{bmatrix} = \langle 3\,,\,2\,,\,1\rangle$$

is perpendicular to Π. The equation of Π is then

$$\langle 3\,,\,2\,,\,1\rangle \cdot \langle x-2\,,\,y-1\,,\,z\rangle = 0 \qquad \text{or} \qquad 3x+2y+z=8$$

(b) Let E be (x, y, z). Then the vector from D to E, namely $\langle x-6\,,\, y-1\,,\, z-2\rangle$ has to be parallel to the vector $\langle 3\,,\, 2\,,\, 1\rangle$, which is perpendicular to Π. That is, there must be a number t such that

$$\langle x-6\,,\, y-1\,,\, z-2\rangle = t\,\langle 3\,,\, 2\,,\, 1\rangle$$
$$\text{or } x = 6+3t,\ y = 1+2t,\ z = 2+t$$

As (x, y, z) must be in Π,

$$8 = 3x + 2y + z = 3(6+3t) + 2(1+2t) + (2+t) = 22 + 14t \implies t = -1$$

So $(x, y, z) = \big(6+3(-1)\,,\, 1+2(-1)\,,\, 2+(-1)\big) = \big(3\,,\, -1\,,\, 1\big)$.

$\boxed{1.4.10}$ (*) Let $A = (2, 3, 4)$ and let L be the line given by the equations $x + y = 1$ and $x + 2y + z = 3$.
(a) Write an equation for the plane containing A and perpendicular to L.
(b) Write an equation for the plane containing A and L.

$\boxed{\text{Solution}}$ We are going to need a direction vector for L in both parts (a) and (b). So we find one first.

- The vector $\langle 1, 1, 0\rangle$ is perpendicular to $x + y = 1$ and hence to L.
- The vector $\langle 1, 2, 1\rangle$ is perpendicular to $x + 2y + z = 3$ and hence to L.

So the vector

$$\langle 1, 1, 0\rangle \times \langle 1, 2, 1\rangle = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \langle 1\,,\, -1\,,\, 1\rangle$$

is a direction vector for L.

(a) The plane is to contain the point $(2, 3, 4)$ and is to have $\langle -1, 1, -1\rangle$ as a normal vector. So

$$\langle -1, 1, -1\rangle \cdot \langle x-2, y-3, z-4\rangle = 0 \qquad \text{or} \qquad x - y + z = 3$$

does the job.

(b) The plane is to contain the points $A = (2, 3, 4)$ and $(1, 0, 2)$ (which is on L) so that the vector $\langle 2-1, 3-0, 4-2\rangle = \langle 1, 3, 2\rangle$ is to be parallel to the plane. The direction vector of L, namely $\langle -1, 1, -1\rangle$, is also to be parallel to the plane. So the vector

$$\langle 1, 3, 2\rangle \times \langle -1, 1, -1\rangle = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix} = \langle -5\,,\, -1\,,\, 4\rangle$$

is to be normal to the plane. So

$$\langle -5, -1, 4\rangle \cdot \langle x-2, y-3, z-4\rangle = 0 \qquad \text{or} \qquad 5x + y - 4z = -3$$

does the job.

$\boxed{1.4.11}$ ($*$) Consider the plane $4x + 2y - 4z = 3$. Find all parallel planes that are distance 2 from the above plane. Your answers should be in the following form: $4x + 2y - 4z = C$.

$\boxed{\text{Solution}}$ All planes that are parallel to the plane $4x + 2y - 4z = 3$ must have $\langle 4, 2, -4\rangle$ as a normal vector and hence must have an equation of the form $4x + 2y - 4z = C$ for some constant C. We must find the C's for which the distance from $4x + 2y - 4z = 3$ to $4x + 2y - 4z = C$ is 2. One point on $4x + 2y - 4z = 3$ is $\big(0, \frac{3}{2}, 0\big)$. The two points (x', y', z') with

$$\overbrace{\left\langle x' - 0, y' - \frac{3}{2}, z' - 0\right\rangle}^{\text{vector from } (0,\frac{3}{2},0) \text{ to } (x',y',z')} = \pm 2 \overbrace{\frac{\langle 4, 2, -4\rangle}{\sqrt{16+4+16}}}^{\text{unit vector}} = \pm\frac{2}{6}\langle 4, 2, -4\rangle = \pm\left\langle \frac{4}{3}, \frac{2}{3}, -\frac{4}{3}\right\rangle$$

are the two points that are a distance 2 from $\big(0, \frac{3}{2}, 0\big)$ in the direction of the normal.The two points (x', y', z') are

$$\begin{aligned} &\left(0 + \frac{4}{3}, \frac{3}{2} + \frac{2}{3}, 0 - \frac{4}{3}\right) = \left(\frac{4}{3}, \frac{13}{6}, -\frac{4}{3}\right) \\ \text{and } &\left(0 - \frac{4}{3}, \frac{3}{2} - \frac{2}{3}, 0 + \frac{4}{3}\right) = \left(-\frac{4}{3}, \frac{5}{6}, \frac{4}{3}\right) \end{aligned}$$

These two points lie on the desired planes, so the two desired planes are

$$4x + 2y - 4z = \frac{4 \times 4}{3} + \frac{2 \times 13}{6} - \frac{-4 \times 4}{3} = \frac{32 + 26 + 32}{6} = 15$$

and

$$4x + 2y - 4z = \frac{4 \times (-4)}{3} + \frac{2 \times 5}{6} - \frac{4 \times 4}{3} = \frac{-32 + 10 - 32}{6} = -9$$

$\boxed{1.4.12}$ ($*$) Find the distance from the point $(1, 2, 3)$ to the plane that passes through the points $(0, 1, 1)$, $(1, -1, 3)$ and $(2, 0, -1)$.

$\boxed{\text{Solution}}$ The two vectors

$$\begin{aligned} \mathbf{a} &= \langle 1, -1, 3\rangle - \langle 0, 1, 1\rangle = \langle 1, -2, 2\rangle \\ \mathbf{b} &= \langle 2, 0, -1\rangle - \langle 0, 1, 1\rangle = \langle 2, -1, -2\rangle \end{aligned}$$

both lie entirely inside the plane. So the vector

$$\mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 1 & -2 & 2 \\ 2 & -1 & -2 \end{bmatrix} = \langle 6, 6, 3\rangle$$

is perpendicular to the plane. The vector $\mathbf{c} = \frac{1}{3}\langle 6, 6, 3\rangle = \langle 2, 2, 1\rangle$ is also perpendicular to the plane. The vector

$$\mathbf{d} = \langle 1, 2, 3\rangle - \langle 0, 1, 1\rangle = \langle 1, 1, 2\rangle$$

joins the point to the plane. So, if θ is the angle between $\mathbf{d}$ and $\mathbf{c}$, the distance is

$$|\mathbf{d}|\cos\theta = \frac{\mathbf{c}\cdot\mathbf{d}}{|\mathbf{c}|} = \frac{6}{\sqrt{9}} = 2$$

▸▸ Stage 3

$\boxed{1.4.13}$ (*) Consider two planes W_1, W_2, and a line M defined by:

$$W_1\ :\ -2x + y + z = 7, \qquad W_2\ :\ -x + 3y + 3z = 6, \qquad M\ :\ \frac{x}{2} = \frac{2y-4}{4} = z + 5.$$

(a) Find a parametric equation of the line of intersection L of W_1 and W_2.
(b) Find the distance from L to M .

Solution (a) Let's use z as the parameter and rename it to t. That is, $z = t$. Subtracting 2 times the W_2 equation from the W_1 equation gives

$$-5y - 5z = -5 \implies y = 1 - z = 1 - t \quad \text{or} \quad y - 1 = -t$$

Substituting the result into the equation for W_2 gives

$$-x + 3(1-t) + 3t = 6 \implies x = -3 \quad \text{or} \quad x + 3 = 0$$

So a parametric equation is

$$\langle x+3, y-1, z\rangle = t\,\langle 0, -1, 1\rangle$$

(b) Solution 1

We can also parametrize M using $z = t$:

$$x = 2z + 10 = 2t + 10,\ y = 2z + 12 = 2t + 12 \quad \implies \quad \langle x, y, z\rangle = \langle 10, 12, 0\rangle + t\,\langle 2, 2, 1\rangle$$

So one point on M is $(10, 12, 0)$ and one point on L is $(-3, 1, 0)$ and

$$\mathbf{v} = \langle (-3) - 10\,,\, 1 - 12\,,\, 0 - 0\rangle = \langle -13\,,\, -11\,,\, 0\rangle$$

is one vector from a point on M to a point on L.

The direction vectors of L and M are $\langle 0, -1, 1\rangle$ and $\langle 2, 2, 1\rangle$, respectively. The vector

$$\mathbf{n} = \langle 0, -1, 1\rangle \times \langle 2, 2, 1\rangle = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ 0 & -1 & 1\\ 2 & 2 & 1\end{bmatrix} = \langle -3\,,\, 2\,,\, 2\rangle$$

is then perpendicular to both L and M.

The distance from L to M is then the length of the projection of $\mathbf{v}$ on $\mathbf{n}$, which is

$$\frac{|\mathbf{v}\cdot\mathbf{n}|}{|\mathbf{n}|} = \frac{|39-22+0|}{\sqrt{9+4+4}} = \sqrt{17}$$

(b) Solution 2 We can also parametrize M using $z = s$:

$$x = 2z+10 = 2s+10,\ y = 2z+12 = 2s+12 \qquad \Longrightarrow \qquad \langle x,y,z\rangle = \langle 10,12,0\rangle + s\,\langle 2,2,1\rangle$$

The vector from the point $(-3\,,\,1-t\,,\,t)$ on L to the point $(10+2s\,,\,12+2s\,,\,s)$ on M is

$$\langle 13+2s\,,\,11+2s+t\,,\,s-t\rangle$$

So the distance from the point $(-3\,,\,1-t\,,\,t)$ on L to the point $(10+2s\,,\,12+2s\,,\,s)$ on M is the square root of

$$D(s,t) = (13+2s)^2 + (11+2s+t)^2 + (s-t)^2$$

That distance is minimized when

$$\begin{aligned} 0 &= \frac{\partial D}{\partial s} = 4(13+2s) + 4(11+2s+t) + 2(s-t) \\ 0 &= \frac{\partial D}{\partial t} = 2(11+2s+t) - 2(s-t) \end{aligned}$$

Cleaning up those equations gives

$$\begin{aligned} 18s+2t &= -96 \\ 2s+4t &= -22 \end{aligned}$$

or

$$9s+t = -48 \tag{E1}$$
$$s+2t = -11 \tag{E2}$$

Subtracting (E2) from twice (E1) gives

$$17s = -85 \implies s = -5$$

Substituting that into (E2) gives

$$2t = -11+5 \implies t = -3$$

Note that

$$\begin{aligned} 13+2s &= 3 \\ 11+2s+t &= -2 \\ s-t &= -2 \end{aligned}$$

So the distance is

$$\sqrt{D(-5,-3)} = \sqrt{3^2+(-2)^2+(-2)^2} = \sqrt{17}$$

$\boxed{1.4.14}$ Find the equation of the sphere which has the two planes $x+y+z=3,\ x+y+z=9$ as tangent planes if the center of the sphere is on the planes $2x-y=0,\ 3x-z=0$.

Solution The two planes $x+y+z=3$ and $x+y+z=9$ are parallel. The centre must be on the plane $x+y+z=6$ half way between them. So, the center is on $x+y+z=6$, $2x-y=0$ and $3x-z=0$. Solving these three equations, or equivalently,

$$y=2x,\ z=3x,\ x+y+z=6x=6$$

gives $(1,2,3)$ as the centre. $(1,1,1)$ is a point on $x+y+z=3$. $(3,3,3)$ is a point on $x+y+z=9$. So $\langle 2,2,2\rangle$ is a vector with tail on $x+y+x=3$ and head on $x+y+z=9$. Furthermore $\langle 2,2,2\rangle$ is perpendicular to the two planes. So the distance between the planes is $|\,\langle 2,2,2\rangle\,|=2\sqrt{3}$ and the radius of the sphere is $\sqrt{3}$. The sphere is

$$(x-1)^2+(y-2)^2+(z-3)^2=3$$

1.4.15 Find the equation of the plane that passes through the point $(-2,0,1)$ and through the line of intersection of $2x+3y-z=0,\ x-4y+2z=-5$.

Solution Set $y=0$ and then solve $2x+3y-z=0,\ x-4y+2z=-5$, i.e. $2x-z=0$, $x+2z=-5$, or equivalently

$$z=2x,\ x+2z=5x=-5$$

The result, $(-1,0,-2)$, is one point on the plane. Set $y=5$ and then solve $2x+3y-z=0,\ x-4y+2z=-5$, i.e. $2x+15-z=0$, $x-20+2z=-5$, or equivalently

$$z=2x+15,\ x-20+4x+30=-5$$

The result, $(-3,5,9)$, is another point on the plane. So three points on the plane are $(-2,0,1)$, $(-1,0,-2)$ and $(-3,5,9)$. $\langle -2+1\,,\,0-0\,,\,1+2\rangle=\langle -1,0,3\rangle$ and $\langle -2+3\,,\,0-5\,,\,1-9\rangle=\langle 1,-5,-8\rangle$ are two vectors having both head and tail in the plane.

$$\langle -1,0,3\rangle\times\langle 1,-5,-8\rangle=\det\begin{bmatrix}\hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\mathbf{k}}\\ -1 & 0 & 3\\ 1 & -5 & -8\end{bmatrix}=\langle 15,-5,5\rangle$$

is a vector perpendicular to the plane. $\frac{1}{5}\langle 15,-5,5\rangle=\langle 3,-1,1\rangle$ is also a vector perpendicular to the plane. The plane is

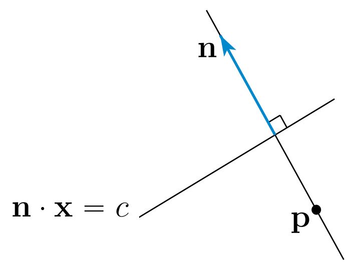
$$3(x+1)-(y-0)+(z+2)=0\quad\text{or}\quad 3x-y+z=-5$$

1.4.16 Find the distance from the point $\mathbf{p}$ to the plane $\mathbf{n}\cdot\mathbf{x}=c$.

Solution The vector $\mathbf{n}$ is perpendicular to the plane $\mathbf{n}\cdot\mathbf{x}=c$. So the line

$$\mathbf{x}(t)=\mathbf{p}+t\mathbf{n}$$

passes through $\mathbf{p}$ and is perpendicular to the plane. It crosses the plane at the value of t



which obeys

$$\mathbf{n}\cdot\mathbf{x}(t) = c \qquad \text{or} \qquad \mathbf{n}\cdot[\mathbf{p}+t\mathbf{n}] = c$$

namely

$$t = [c - \mathbf{n}\cdot\mathbf{p}]/|\mathbf{n}|^2$$

The vector

$$\mathbf{x}(t) - \mathbf{p} = t\,\mathbf{n} = \mathbf{n}\,[c - \mathbf{n}\cdot\mathbf{p}]/|\mathbf{n}|^2$$

has head on the plane $\mathbf{n}\cdot\mathbf{x} = c$, tail at $\mathbf{p}$, and is perpendicular to the plane. So the distance is the length of that vector, which is

$$|c - \mathbf{n}\cdot\mathbf{p}|/|\mathbf{n}|$$

1.4.17 Describe the set of points equidistant from $(1,2,3)$ and $(5,2,7)$.

Solution The distance from the point (x,y,z) to $(1,2,3)$ is $\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$ and the distance from $(x,y,z)$to $(5,2,7)$ is $\sqrt{(x-5)^2+(y-2)^2+(z-7)^2}$. Hence (x,y,z) is equidistant from $(1,2,3)$ and $(5,2,7)$ if and only if

$$\begin{aligned} & (x-1)^2+(y-2)^2+(z-3)^2 = (x-5)^2+(y-2)^2+(z-7)^2 \\ \iff\quad & x^2-2x+1+z^2-6z+9 = x^2-10x+25+z^2-14z+49 \\ \iff\quad & 8x+8z = 64 \\ \iff\quad & x+z = 8 \end{aligned}$$

This is the plane through $(3,2,5) = \frac{1}{2}(1,2,3) + \frac{1}{2}(5,2,7)$ with normal vector $\langle 1,0,1\rangle = \frac{1}{4}\big(\langle 5,2,7\rangle - \langle 1,2,3\rangle\big)$.

1.4.18 Describe the set of points equidistant from $\mathbf{a}$ and $\mathbf{b}$.

Solution The distance from the point $\mathbf{x}$ to $\mathbf{a}$ is $\sqrt{(\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{a})}$ and the distance from $\mathbf{x}$ to $\mathbf{b}$ is $\sqrt{(\mathbf{x}-\mathbf{b})\cdot(\mathbf{x}-\mathbf{b})}$. Hence $\mathbf{x}$ is equidistant from $\mathbf{a}$ and $\mathbf{b}$ if and only if

$$\begin{aligned} & (\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{a}) = (\mathbf{x}-\mathbf{b})\cdot(\mathbf{x}-\mathbf{b}) \\ \iff\quad & |\mathbf{x}|^2 - 2\mathbf{a}\cdot\mathbf{x} + |\mathbf{a}|^2 = |\mathbf{x}|^2 - 2\mathbf{b}\cdot\mathbf{x} + |\mathbf{b}|^2 \\ \iff\quad & 2(\mathbf{b}-\mathbf{a})\cdot\mathbf{x} = |\mathbf{b}|^2 - |\mathbf{a}|^2 \end{aligned}$$

This is the plane through $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$ with normal vector $\mathbf{b}-\mathbf{a}$.

1.4.19 (*) Consider a point $P(5,-10,2)$ and the triangle with vertices $A(0,1,1)$, $B(1,0,1)$ and $C(1,3,0)$.
(a) Compute the area of the triangle ABC.
(b) Find the distance from the point P to the plane containing the triangle.

Solution (a) One side of the triangle is $\overrightarrow{AB} = \langle 1,0,1\rangle - \langle 0,1,1\rangle = \langle 1,-1,0\rangle$. A second side of the triangle is $\overrightarrow{AC} = \langle 1,3,0\rangle - \langle 0,1,1\rangle = \langle 1,2,-1\rangle$. If the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is θ and if we take $\overrightarrow{AB}$ as the base of the triangle, then the triangle has base length $b = |\overrightarrow{AB}|$ and height $h = |\overrightarrow{AC}|\sin\theta$ and hence

$$\text{area} = \tfrac{1}{2}bh = \tfrac{1}{2}|\overrightarrow{AB}|\,|\overrightarrow{AC}|\sin\theta = \tfrac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}| = \tfrac{1}{2}|\,\langle 1,-1,0\rangle \times \langle 1,2,-1\rangle\,|$$

As

$$\langle 1,-1,0\rangle \times \langle 1,2,-1\rangle = \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 1 & -1 & 0 \\ 1 & 2 & -1 \end{bmatrix} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + 3\hat{\mathbf{k}}$$

we have

$$\text{area} = \tfrac{1}{2}|\,\langle 1,1,3\rangle\,| = \tfrac{1}{2}\sqrt{11} \approx 1.658$$

(b) A unit vector perpendicular to the plane containing the triangle is

$$\hat{\mathbf{n}} = \frac{\overrightarrow{AB}\times\overrightarrow{AC}}{|\overrightarrow{AB}\times\overrightarrow{AC}|} = \frac{\langle 1,1,3\rangle}{\sqrt{11}}$$

The distance from P to the plane containing the triangle is the length of the projection of $\overrightarrow{AP} = \langle 5,-10,2\rangle - \langle 0,1,1\rangle = \langle 5,-11,1\rangle$ on $\hat{\mathbf{n}}$. If θ the angle between $\overrightarrow{AP}$ and $\hat{\mathbf{n}}$, then this is

$$\text{distance} = |\overrightarrow{AP}|\,|\cos\theta| = \left|\overrightarrow{AP}\cdot\hat{\mathbf{n}}\right| = \left|\langle 5,-11,1\rangle \cdot \frac{\langle 1,1,3\rangle}{\sqrt{11}}\right| = \frac{3}{\sqrt{11}} \approx 0.9045$$

1.4.20 (*) Consider the sphere given by

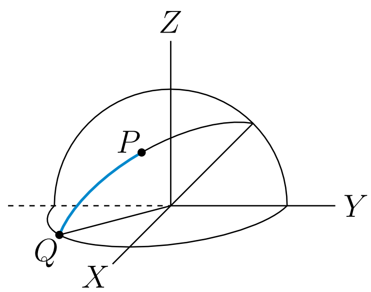
$$(x-1)^2 + (y-2)^2 + (z+1)^2 = 2$$

Suppose that you are at the point $(2,2,0)$ on S, and you plan to follow the shortest path on S to $(2,1,-1)$. Express your initial direction as a cross product.

Solution Switch to a new coordinate system with

$$X = x-1 \qquad Y = y-2 \qquad Z = z+1$$

In this new coordinate system, the sphere has equation $X^2+Y^2+Z^2 = 2$. So the sphere is centred at $(X,Y,Z) = (0,0,0)$ and has radius $\sqrt{2}$. In the new coordinate system, the initial point $(x,y,z) = (2,2,0)$ has $(X,Y,Z) = (1,0,1)$ and our final point $(x,y,z) = (2,1,-1)$ has $(X,Y,Z) = (1,-1,0)$. Call the initial point P and the final point Q. The shortest path will follow the great circle from P to Q. A great circle on a sphere is the intersection of the



sphere with a plane that contains the centre of the sphere. Our strategy for finding the initial direction will be based on two observations.

- The shortest path lies on the plane Π that contains the origin and the points P and Q. Since the shortest path lies on Π, our direction vector must also lie on Π and hence must be perpendicular to the normal vector to Π.
- The shortest path also remains on the sphere, so our initial direction must also be perpendicular to the normal vector to the sphere at our initial point P.

As our initial direction is perpendicular to the two normal vectors, it is parallel to their cross product.

So our main job is to find normal vectors to the plane Π and to the sphere at P.

- One way to find a normal vector to Π is to guess an equation for Π. As $(0,0,0)$ is on Π, $(0,0,0)$ must obey Π's equation. So Π's equation must be of the form $aX + bY + cZ = 0$. That $(X,Y,Z) = (1,0,1)$ is on Π forces $a + c = 0$. That $(X,Y,Z) = (1,-1,0)$ is on Π forces $a - b = 0$. So we may take $a = 1$, $b = 1$ and $c = -1$. That is, Π is $X + Y - Z = 0$. (Check that all three points $(0,0,0)$, $(1,0,1)$ and $(1,-1,0)$ do indeed obey $X + Y - Z = 0$.) A normal vector to Π is $\langle 1,1,-1\rangle$.
- A second way to find a normal vector to Π is to observe that both
  - the vector from $(0,0,0)$ to $(1,0,1)$, that is $\langle 1,0,1\rangle$, lies completely inside Π and
  - the vector from $(0,0,0)$ to $(1,-1,0)$, that is $\langle 1,-1,0\rangle$, lies completely inside Π.

  So the vector

$$\langle 1,0,1\rangle \times \langle 1,-1,0\rangle = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \hat{\imath} + \hat{\jmath} - \hat{\mathbf{k}}$$

  is perpendicular to Π.
- The vector from the centre of the sphere to the point P on the sphere is perpendicular to the sphere at P. So a normal vector to the sphere at our initial point $(X,Y,Z) = (1,0,1)$ is $\langle 1,0,1\rangle$.

Since our initial direction[1] must be perpendicular to both $\langle 1,1,-1\rangle$ and $\langle 1,0,1\rangle$, it must be one of $\pm \langle 1,1,-1\rangle \times \langle 1,0,1\rangle$. To get from $(1,0,1)$ to $(1,-1,0)$ by the shortest path, our Z

1 Note that the change of coordinates $X = x - 1$, $Y = y - 2$, $Z = z + 1$ has absolutely no effect on any velocity or direction vector. If our position at time t is $(x(t), y(t), z(t))$ in the original coordinate system, then it is $(X(t), Y(t), Z(t)) = (x(t) - 1, y(t) - 2, z(t) + 1)$ in the new coordinate system. The velocity vectors in the two coordinate systems $\langle x'(t), y'(t), z'(t)\rangle = \langle X'(t), Y'(t), Z'(t)\rangle$ are identical.

coordinate should decrease from 1 to 0. So the Z coordinate of our initial direction should be negative. This is the case for

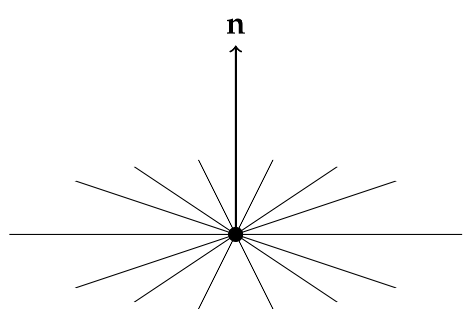
$$\langle 1,1,-1\rangle \times \langle 1,0,1\rangle = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} = \hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}}$$

1.5▲ Equations of Lines in 3d

▸▸ Stage 1

1.5.1 What is wrong with the following exercise?
"Give an equation for the line passing through the point $(3,1,3)$ that is normal to the vectors $\langle 4,-6,2\rangle$ and $\left\langle \frac{1}{3}, -\frac{1}{2}, \frac{1}{6}\right\rangle$."

Solution Note $12\left\langle \frac{1}{3}, -\frac{1}{2}, \frac{1}{6}\right\rangle = \langle 4,-6,2\rangle$. So, we actually only know one normal direction to the line we're supposed to be describing. That means there are actually infinitely many lines satisfying the given conditions.



1.5.2 Find, if possible, four lines in 3d with

- no two of the lines parallel to each other and
- no two of the lines intersecting.

Solution There are infinitely many correct answers. One is

$$L_1: \ \langle x\,,\, x\,,\, z-1\rangle = t\,\langle 1,0,0\rangle \qquad L_2: \ \langle x\,,\, x\,,\, z-2\rangle = t\,\langle 0,1,0\rangle$$
$$L_3: \ \langle x\,,\, x\,,\, z-3\rangle = t\,\langle 1,1,0\rangle \qquad L_4: \ \langle x\,,\, x\,,\, z-4\rangle = t\,\langle 1,-1,0\rangle$$

No two of the lines are parallel because

- L_1 and L_2 are not parallel because $\langle 1,0,0\rangle \times \langle 0,1,0\rangle = \langle 0,0,1\rangle \neq \mathbf{0}$.
- L_1 and L_3 are not parallel because $\langle 1,0,0\rangle \times \langle 1,1,0\rangle = \langle 0,0,1\rangle \neq \mathbf{0}$.
- L_1 and L_4 are not parallel because $\langle 1,0,0\rangle \times \langle 1,-1,0\rangle = \langle 0,0,-1\rangle \neq \mathbf{0}$.
- L_2 and L_3 are not parallel because $\langle 0,1,0\rangle \times \langle 1,1,0\rangle = \langle 0,0,-1\rangle \neq \mathbf{0}$.
- L_2 and L_4 are not parallel because $\langle 0,1,0\rangle \times \langle 1,-1,0\rangle = \langle 0,0,-1\rangle \neq \mathbf{0}$.

- L_3 and L_4 are not parallel because $\langle 1,1,0\rangle \times \langle 1,-1,0\rangle = \langle 0,0,-2\rangle \neq \mathbf{0}$.

No two of the lines intersect because

- every point on L_1 has $z = 1$ and
- every point on L_2 has $z = 2$ and
- every point on L_3 has $z = 3$ and
- every point on L_4 has $z = 4$.

▸▸ Stage 2

1.5.3 Find a vector parametric equation for the line of intersection of the given planes.
(a) $x - 2z = 3$ and $y + \frac{1}{2}z = 5$
(b) $2x - y - 2z = -3$ and $4x - 3y - 3z = -5$

Solution (a) The point (x,y,z) obeys both $x - 2z = 3$ and $y + \frac{1}{2}z = 5$ if and only if $\langle x,y,z\rangle = \left\langle 3+2z, 5-\frac{1}{2}z, z\right\rangle = \langle 3,5,0\rangle + \left\langle 2,-\frac{1}{2},1\right\rangle z$. So, introducing a new variable t obeying $t = z$, we get the vector parametric equation $\langle x,y,z\rangle = \langle 3,5,0\rangle + \left\langle 2,-\frac{1}{2},1\right\rangle t$.

(b) The point (x,y,z) obeys

$$\left\{\begin{matrix} 2x-y-2z=-3 \\ 4x-3y-3z=-5 \end{matrix}\right\} \iff \left\{\begin{matrix} 2x-y=2z-3 \\ 4x-3y=3z-5 \end{matrix}\right\} \iff \left\{\begin{matrix} 4x-2y=4z-6 \\ 4x-3y=3z-5 \end{matrix}\right\}$$

$$\iff \left\{\begin{matrix} 4x-2y=4z-6 \\ y=z-1 \end{matrix}\right\}$$

Hence the point (x,y,z) is on the line if and only if $\langle x,y,z\rangle = \left\langle \frac{1}{4}(2y+4z-6), z-1, z\right\rangle = \left\langle \frac{3}{2}z-2, z-1, z\right\rangle = \langle -2,-1,0\rangle + \left\langle \frac{3}{2},1,1\right\rangle z$. So, introducing a new variable t obeying $t = z$, we get the vector parametric equation $\langle x,y,z\rangle = \langle -2,-1,0\rangle + \left\langle \frac{3}{2},1,1\right\rangle t$.

1.5.4 Determine a vector equation for the line of intersection of the planes
(a) $x + y + z = 3$ and $x + 2y + 3z = 7$
(b) $x + y + z = 3$ and $2x + 2y + 2z = 7$

Solution (a) The normals to the two planes are $\langle 1,1,1\rangle$ and $\langle 1,2,3\rangle$ respectively. The line of intersection must have direction perpendicular to both of these normals. Its direction vector is

$$\langle 1,1,1\rangle \times \langle 1,2,3\rangle = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \langle 1,-2,1\rangle$$

Substituting $z = 0$ into the equations of the two planes and solving

$$\left\{ \begin{array}{c} x + y = 3 \\ x + 2y = 7 \end{array} \right\} \iff \left\{ \begin{array}{c} x = 3 - y \\ x + 2y = 7 \end{array} \right\} \iff \left\{ \begin{array}{c} x = 3 - y \\ 3 - y + 2y = 7 \end{array} \right\}$$

we see that $z = 0,\ y = 4, x = -1$ lies on both planes. The line of intersection is $\langle x, y, z\rangle = \langle -1, 4, 0\rangle + t\,\langle 1, -2, 1\rangle$. This can be checked by verifying that, for all values of t, $\langle x, y, z\rangle = \langle -1, 4, 0\rangle + t\,\langle 1, -2, 1\rangle$ satisfies both $x + y + z = 3$ and $x + 2y + 3z = 7$.

(b) The equation $x + y + z = 3$ is equivalent to $2x + 2y + 2z = 6$. So the two equations $x + y + z = 3$ and $2x + 2y + 2z = 7$ are mutually contradictory. They have no solution. The two planes are parallel and do not intersect.

1.5.5 In each case, determine whether or not the given pair of lines intersect. Also find all planes containing the pair of lines.
(a) $\langle x, y, z\rangle = \langle -3, 2, 4\rangle + t\,\langle -4, 2, 1\rangle$ and $\langle x, y, z\rangle = \langle 2, 1, 2\rangle + t\,\langle 1, 1, -1\rangle$
(b) $\langle x, y, z\rangle = \langle -3, 2, 4\rangle + t\,\langle -4, 2, 1\rangle$ and $\langle x, y, z\rangle = \langle 2, 1, -1\rangle + t\,\langle 1, 1, -1\rangle$
(c) $\langle x, y, z\rangle = \langle -3, 2, 4\rangle + t\,\langle -2, -2, 2\rangle$ and $\langle x, y, z\rangle = \langle 2, 1, -1\rangle + t\,\langle 1, 1, -1\rangle$
(d) $\langle x, y, z\rangle = \langle 3, 2, -2\rangle + t\,\langle -2, -2, 2\rangle$ and $\langle x, y, z\rangle = \langle 2, 1, -1\rangle + t\,\langle 1, 1, -1\rangle$

Solution (a) Note that the value of the parameter t in the equation $\langle x, y, z\rangle = \langle -3, 2, 4\rangle + t\,\langle -4, 2, 1\rangle$ need not have the same value as the parameter t in the equation $\langle x, y, z\rangle = \langle 2, 1, 2\rangle + t\,\langle 1, 1, -1\rangle$. So it is much safer to change the name of the parameter in the first equation from t to s. In order for a point (x, y, z) to lie on both lines we need

$$\langle -3, 2, 4\rangle + s\,\langle -4, 2, 1\rangle = \langle 2, 1, 2\rangle + t\,\langle 1, 1, -1\rangle$$

or equivalently, writing out the three component equations and moving all s's and t's to the left and constants to the right,

$$\begin{aligned} -4s - t &= 5 \\ 2s - t &= -1 \\ s + t &= -2 \end{aligned}$$

Adding the last two equations together gives $3s = -3$ or $s = -1$. Substituting this into the last equation gives $t = -1$. Note that $s = t = -1$ does indeed satisfy all three equations so that $\langle x, y, z\rangle = \langle -3, 2, 4\rangle - \langle -4, 2, 1\rangle = \langle 1, 0, 3\rangle$ lies on both lines. Any plane that contains the two lines must be parallel to both direction vectors $\langle -4, 2, 1\rangle$ and $\langle 1, 1, -1\rangle$. So its normal vector must be perpendicular to them, i.e. must be parallel to $\langle -4, 2, 1\rangle \times \langle 1, 1, -1\rangle = \langle -3, -3, -6\rangle = -3\,\langle 1, 1, 2\rangle$. The plane must contain $(1, 0, 3)$ and be perpendicular to $\langle 1, 1, 2\rangle$. Its equation is $\langle 1, 1, 2\rangle \cdot \langle x - 1, y, z - 3\rangle = 0$ or $x + y + 2z = 7$. This can be checked by verifying that $\langle -3, 2, 4\rangle + s\,\langle -4, 2, 1\rangle$ and $\langle 2, 1, 2\rangle + t\,\langle 1, 1, -1\rangle$ obey $x + y + 2z = 7$ for all s and t respectively.

(b) In order for a point (x, y, z) to lie on both lines we need

$$\langle -3, 2, 4\rangle + s\,\langle -4, 2, 1\rangle = \langle 2, 1, -1\rangle + t\,\langle 1, 1, -1\rangle$$

or equivalently, writing out the three component equations and moving all s's and t's to the left and constants to the right,

$$\begin{aligned} -4s - t &= 5 \\ 2s - t &= -1 \\ s + t &= -5 \end{aligned}$$

Adding the last two equations together gives $3s = -6$ or $s = -2$. Substituting this into the last equation gives $t = -3$. However, substituting $s = -2,\ t = -3$ into the first equation gives $11 = 5$, which is impossible. The two lines do not intersect. In order for two lines to lie in a common plane and not intersect, they must be parallel. So, in this case no plane contains the two lines.

(c) In order for a point (x, y, z) to lie on both lines we need

$$\langle -3, 2, 4\rangle + s\,\langle -2, -2, 2\rangle = \langle 2, 1, -1\rangle + t\,\langle 1, 1, -1\rangle$$

or equivalently, writing out the three component equations and moving all s's and t's to the left and constants to the right,

$$\begin{aligned} -2s - t &= 5 \\ -2s - t &= -1 \\ 2s + t &= -5 \end{aligned}$$

The first two equations are obviously contradictory. The two lines do not intersect. Any plane containing the two lines must be parallel to $\langle 1, 1, -1\rangle$ (and hence automatically parallel to $\langle -2, -2, 2\rangle = -2\,\langle 1, 1, -1\rangle$) and must also be parallel to the vector from the point $(-3, 2, 4)$, which lies on the first line, to the point $(2, 1, -1)$, which lies on the second. The vector is $\langle 5, -1, -5\rangle$. Hence the normal to the plane is $\langle 5, -1, -5\rangle \times \langle 1, 1, -1\rangle = \langle 6, 0, 6\rangle = 6\,\langle 1, 0, 1\rangle$. The plane perpendicular to $\langle 1, 0, 1\rangle$ containing $(2, 1, -1)$ is $\langle 1, 0, 1\rangle \cdot \langle x - 2, y - 1, z + 1\rangle = 0$ or $x + z = 1$.

(d) Again the two lines are parallel, since $\langle -2, -2, 2\rangle = -2\,\langle 1, 1, -1\rangle$. Furthermore the point $\langle 3, 2, -2\rangle = \langle 3, 2, -2\rangle + 0\,\langle -2, -2, 2\rangle = \langle 2, 1, -1\rangle + 1\,\langle 1, 1, -1\rangle$ lies on both lines. So the two lines not only intersect but are identical. Any plane that contains the point $(3, 2, -2)$ and is parallel to $\langle 1, 1, -1\rangle$ contains both lines. In general, the plane $ax + by + cz = d$ contains $(3, 2, -2)$ if and only if $d = 3a + 2b - 2c$ and is parallel to $\langle 1, 1, -1\rangle$ if and only if $\langle a, b, c\rangle \cdot \langle 1, 1, -1\rangle = a + b - c = 0$. So, for arbitrary a and b (not both zero) $ax + by + (a + b)z = a$ works.

1.5.6 Find the equation of the line through $(2, -1, -1)$ and parallel to each of the two planes $x + y = 0$ and $x - y + 2z = 0$. Express the equations of the line in vector and scalar parametric forms and in symmetric form.

Solution First observe that

- $\langle 1, 1, 0\rangle$ is perpendicular to $x + y = 0$ and hence to the line, and
- $\langle 1, -1, 2\rangle$ is perpendicular to $x - y + 2z = 0$ and hence to the line.

Consequently

$$\langle 1\,,1\,,0\rangle \times \langle 1\,,-1\,,2\rangle = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} = \langle 2\,,-2\,,-2\rangle$$

is perpendicular to both $\langle 1,1,0\rangle$ and $\langle 1,-1,2\rangle$. So $\frac{1}{2}\langle 2\,,-2\,,-2\rangle = \langle 1,-1,-1\rangle$ is also perpendicular to both $\langle 1,1,0\rangle$ and $\langle 1,-1,2\rangle$ and hence is parallel to the line. As the point $(2,-1,-1)$ is on the line, the vector equation of the line is

$$\langle x-2, y+1, z+1\rangle = t\,\langle 1,-1,-1\rangle$$

The scalar parametric equations for the line are

$$x-2=t,\ y+1=-t,\ z+1=-t \qquad \text{or} \qquad x=2+t,\ y=-1-t,\ z=-1-t$$

The symmetric equations for the line are

$$(t=)\frac{x-2}{1} = \frac{y+1}{-1} = \frac{z+1}{-1} \qquad \text{or} \qquad x-2=-y-1=-z-1$$

$\boxed{1.5.7}$ $(*)$ Let L be the line given by the equations $x+y=1$ and $x+2y+z=3$. Write a vector parametric equation for L.

$\boxed{\text{Solution}}$ Let's parametrize L using y, renamed to t, as the parameter. Then $y=t$, so that

$$x+y=1 \implies x+t=1 \implies x=1-t$$

and

$$x+2y+z=3 \implies 1-t+2t+z=3 \implies z=2-t$$

and

$$\langle x,y,z\rangle = \langle 1,0,2\rangle + t\,\langle -1,1,-1\rangle$$

is a vector parametric equation for L.

$\boxed{1.5.8}$

(a) Find a vector parametric equation for the line $x+2y+3z=11,\ x-2y+z=-1$.
(b) Find the distance from $(1,0,1)$ to the line $x+2y+3z=11,\ x-2y+z=-1$.

$\boxed{\text{Solution}}$ (a) The normal vectors to the two given planes are $\langle 1,2,3\rangle$ and $\langle 1,-2,1\rangle$ respectively. Since the line is to be contained in both planes, its direction vector must be perpendicular to both $\langle 1,2,3\rangle$ and $\langle 1,-2,1\rangle$, and hence must be parallel to

$$\langle 1,2,3\rangle \times \langle 1,-2,1\rangle = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & 2 & 3 \\ 1 & -2 & 1 \end{bmatrix} = \langle 8,2,-4\rangle$$

or to $\langle 4,1,-2\rangle$. Setting $z=0$ in $x+2y+3z=11,\ x-2y+z=-1$ and solving

$$\left\{\begin{matrix} x+2y=11 \\ x-2y=-1 \end{matrix}\right\} \iff \left\{\begin{matrix} 2y=11-x \\ x-2y=-1 \end{matrix}\right\} \iff \left\{\begin{matrix} 2y=11-x \\ x-(11-x)=-1 \end{matrix}\right\} \iff \left\{\begin{matrix} 2y=11-x \\ 2x=10 \end{matrix}\right\}$$

we see that $(5,3,0)$ is on the line. So the vector parametric equation of the line is $\langle x,y,z\rangle = \langle 5,3,0\rangle + t\,\langle 4,1,-2\rangle = \langle 5+4t, 3+t, -2t\rangle$.

(b) The vector from $(1,0,1)$ to the point $(5+4t,3+t,-2t)$ on the line is $\langle 4+4t, 3+t, -1-2t\rangle$. In order for $(5+4t,3+t,-2t)$ to be the point of the line closest to $(1,0,1)$, the vector $\langle 4+4t,3+t,-1-2t\rangle$ joining those two points must be perpendicular to the direction vector $\langle 4,1,-2\rangle$ of the line. (See Example 1.5.4 in the CLP-3 text.) This is the case when

$$\langle 4,1,-2\rangle\cdot\langle 4+4t,3+t,-1-2t\rangle = 0 \quad\text{or}\quad 16+16t+3+t+2+4t=0 \quad\text{or}\quad t=-1$$

The point on the line nearest $(1,0,1)$ is thus $(5+4t,3+t,-2t)\Big|_{t=-1} = (5-4,3-1,2) = (1,2,2)$. The distance from the point to the line is the length of the vector from $(1,0,1)$ to the point on the line nearest $(1,0,1)$. That vector is $\langle 1,2,2\rangle - \langle 1,0,1\rangle = \langle 0,2,1\rangle$. So the distance is $|\,\langle 0,2,1\rangle\,| = \sqrt{5}$.

1.5.9 Let L_1 be the line passing through $(1,-2,-5)$ in the direction of $\mathbf{d}_1 = \langle 2,3,2\rangle$. Let L_2 be the line passing through $(-3,4,-1)$ in the direction $\mathbf{d}_2 = \langle 5,2,4\rangle$.
(a) Find the equation of the plane P that contains L_1 and is parallel to L_2.
(b) Find the distance from L_2 to P.

Solution (a) The plane P must be parallel to both $\langle 2,3,2\rangle$ (since it contains L_1) and $\langle 5,2,4\rangle$ (since it is parallel to L_2). Hence

$$\langle 2,3,2\rangle\times\langle 5,2,4\rangle = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 2 & 3 & 2 \\ 5 & 2 & 4\end{bmatrix} = \langle 8,2,-11\rangle$$

is normal to P. As the point $(1,-2,-5)$ is on P, the equation of P is

$$\langle 8,2,-11\rangle\cdot\langle x-1,y+2,z+5\rangle = 0 \qquad\text{or}\qquad 8x+2y-11z=59$$

(b) As L_2 is parallel to P, the distance from L_2 to P is the same as the distance from any one point of L_2, for example $(-3,4,-1)$, to P. As $(1,-2,-5)$ is a point on P, the vector $\langle 1,-2,-5\rangle - \langle -3,4,-1\rangle = \langle 4,-6,-4\rangle$ has its head on P and tail at $(-3,4,-1)$ on L_2. The distance from L_2 to P is the length of the projection of the vector $\langle 4,-6,-4\rangle$ on the normal to P. (See Example 1.4.5 in the CLP-3 text.) This is

$$\left|\text{proj}_{\langle 8,2,-11\rangle}\,\langle 4,-6,-4\rangle\right| = \frac{|\,\langle 4,-6,-4\rangle\cdot\langle 8,2,-11\rangle\,|}{|\,\langle 8,2,-11\rangle\,|} = \frac{64}{\sqrt{189}} \approx 4.655$$

$\boxed{1.5.10}$ (∗) Let L be a line which is parallel to the plane $2x + y - z = 5$ and perpendicular to the line $x = 3 - t$, $y = 1 - 2t$ and $z = 3t$.
(a) Find a vector parallel to the line L.
(b) Find parametric equations for the line L if L passes through a point $Q(a, b, c)$ where $a < 0$, $b > 0$, $c > 0$, and the distances from Q to the xy–plane, the xz–plane and the yz–plane are 2, 3 and 4 respectively.

$\boxed{\text{Solution}}$ (a) The line L must be perpendicular both to $\langle 2, 1, -1\rangle$, which is a normal vector for the plane $2x + y - z = 5$, and to $\langle -1, -2, 3\rangle$, which is a direction vector for the line $x = 3 - t$, $y = 1 - 2t$ and $z = 3t$. Any such vector must be a nonzero constant times

$$\langle 2, 1, -1\rangle \times \langle -1, -2, 3\rangle = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 2 & 1 & -1 \\ -1 & -2 & 3 \end{bmatrix} = \langle 1, -5, -3\rangle$$

(b) For the point $Q(a, b, c)$

- to be a distance 2 from the xy–plane, it is necessary that $|c| = 2$, and
- to be a distance 3 from the xz–plane, it is necessary that $|b| = 3$, and
- to be a distance 4 from the yz–plane, it is necessary that $|a| = 4$.

As $a < 0$, $b > 0$, $c > 0$, the point Q is $(-4, 3, 2)$ and the line L is

$$x = -4 + t \qquad y = 3 - 5t \qquad z = 2 - 3t$$

$\boxed{1.5.11}$ (∗) Let L be the line of intersection of the planes $x + y + z = 6$ and $x - y + 2z = 0$.
(a) Find the points in which the line L intersects the coordinate planes.
(b) Find parametric equations for the line through the point $(10, 11, 13)$ that is perpendicular to the line L and parallel to the plane $y = z$.

$\boxed{\text{Solution}}$ (a) The line L intersects the xy–plane when $x + y + z = 6$, $x - y + 2z = 0$, and $z = 0$. When $z = 0$ the equations of L reduce to $x + y = 6$, $x - y = 0$. So the intersection point is $(3, 3, 0)$.

The line L intersects the xz–plane when $x + y + z = 6$, $x - y + 2z = 0$, and $y = 0$. When $y = 0$ the equations of L reduce to $x + z = 6$, $x + 2z = 0$. Substituting $x = -2z$ into $x + z = 6$ gives $-z = 6$. So the intersection point is $(12, 0, -6)$.

The line L intersects the yz–plane when $x + y + z = 6$, $x - y + 2z = 0$, and $x = 0$. When $x = 0$ the equations of L reduce to $y + z = 6$, $-y + 2z = 0$. Substituting $y = 2z$ into $y + z = 6$ gives $3z = 6$. So the intersection point is $(0, 4, 2)$.

(b) Our main job is to find a direction vector $\mathbf{d}$ for the line.

- Since the line is to be parallel to $y = z$, $\mathbf{d}$ must be perpendicular to the normal vector for $y = z$, which is $\langle 0, 1, -1\rangle$.

- **d** must also be perpendicular to L. For a point (x, y, z) to be on L it must obey $x + y = 6 - z$ and $x - y = -2z$. Adding these two equations gives $2x = 6 - 3z$ and subtracting the second equation from the first gives $2y = 6 + z$. So for a point (x, y, z) to be on L it must obey $x = 3 - \frac{3z}{2}$, $y = 3 + \frac{z}{2}$. The point on L with $z = 0$ is $(3, 3, 0)$ and the point on L with $z = 2$ is $(0, 4, 2)$. So $\langle 0-3, 4-3, 2-0\rangle = \langle -3, 1, 2\rangle$ is a direction vector for L.

So **d** must be perpendicular to both $\langle 0, 1, -1\rangle$ and $\langle -3, 1, 2\rangle$ and so must be a nonzero constant times

$$\langle 0,1,-1\rangle \times \langle -3,1,2\rangle = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 0 & 1 & -1 \\ -3 & 1 & 2 \end{bmatrix} = \langle 3\,,\,3\,,\,3\rangle$$

We choose $\mathbf{d} = \frac{1}{3}\langle 3\,,\,3\,,\,3\rangle = \langle 1\,,\,1\,,\,1\rangle$. So

$$\langle x, y, z\rangle = \langle 10, 11, 13\rangle + t\,\langle 1\,,\,1\,,\,1\rangle$$

is a vector parametric equation for the line. We can also write this as $x = 10 + t$, $y = 11 + t$, $z = 13 + t$.

$\boxed{1.5.12}$ (∗) The line L has vector parametric equation $\mathbf{r}(t) = (2+3t)\hat{\imath} + 4t\hat{\jmath} - \hat{\mathbf{k}}$.
(a) Write the symmetric equations for L.
(b) Let α be the angle between the line L and the plane given by the equation $x - y + 2z = 0$. Find α.

$\boxed{\text{Solution}}$ (a) Since

$$\begin{aligned} x = 2 + 3t &\implies t = \frac{x-2}{3} \\ y = 4t &\implies t = \frac{y}{4} \end{aligned}$$

we have

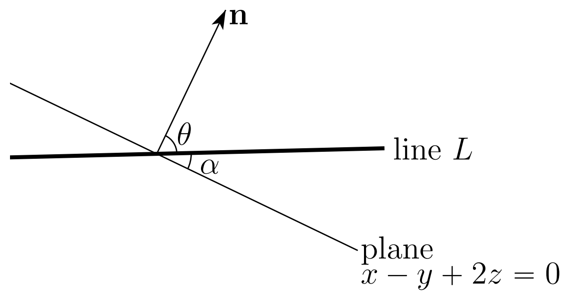
$$\frac{x-2}{3} = \frac{y}{4} \qquad z = -1$$

(b) The direction vector for the line $\mathbf{r}(t) = 2\hat{\imath} - \hat{\mathbf{k}} + t(3\hat{\imath} + 4\hat{\jmath})$ is $\mathbf{d} = 3\hat{\imath} + 4\hat{\jmath}$. A normal vector for the plane $x - y + 2z = 0$ is $\mathbf{n} = \pm(\hat{\imath} - \hat{\jmath} + 2\hat{\mathbf{k}})$. The angle θ between **d** and **n** obeys

$$\cos\theta = \frac{\mathbf{d}\cdot\mathbf{n}}{|\mathbf{d}|\,|\mathbf{n}|} = \frac{1}{5\sqrt{6}} \implies \theta = \arccos\frac{1}{5\sqrt{6}} \approx 1.49\,\text{radians}$$

(We picked $\mathbf{n} = -\hat{\imath} + \hat{\jmath} - 2\hat{\mathbf{k}}$ to make $0 \leqslant \theta \leqslant \frac{\pi}{2}$.) Then the angle between **d** and the plane is

$$\alpha = \frac{\pi}{2} - \arccos\frac{1}{5\sqrt{6}} \approx 0.08\,\text{radians}$$



1.5.13 (∗) Find the parametric equation for the line of intersection of the planes

$$x + y + z = 11 \qquad \text{and} \qquad x - y - z = 13.$$

Solution Let's use z as the parameter and call it t. Then $z = t$ and

$$\begin{aligned} x + y &= 11 - t \\ x - y &= 13 + t \end{aligned}$$

Adding the two equations gives $2x = 24$ and subtracting the second equation from the first gives $2y = -2 - 2t$. So

$$(x, y, z) = \big(12\,,\ -1 - t\,,\ t\big)$$

1.5.14 (∗)
(a) Find a point on the y-axis equidistant from $(2, 5, -3)$ and $(-3, 6, 1)$.
(b) Find the equation of the plane containing the point $(1, 3, 1)$ and the line $\mathbf{r}(t) = t\,\hat{\imath} + t\,\hat{\jmath} + (t + 2)\,\hat{\mathbf{k}}$.

Solution (a) The point $(0, y, 0)$, on the y–axis, is equidistant from $(2, 5, -3)$ and $(-3, 6, 1)$ if and only if

$$\begin{aligned} &\big|\,\langle 2, 5, -3\rangle - \langle 0, y, 0\rangle\,\big| = \big|\,\langle -3, 6, 1\rangle - \langle 0, y, 0\rangle\,\big| \\ &\iff 2^2 + (5 - y)^2 + (-3)^2 = (-3)^2 + (6 - y)^2 + 1^2 \\ &\iff 2y = 8 \\ &\iff y = 4 \end{aligned}$$

(b) The points $(1, 3, 1)$ and $\mathbf{r}(0) = (0, 0, 2)$ are both on the plane. Hence the vector $\langle 1, 3, 1\rangle - \langle 0, 0, 2\rangle = \langle 1, 3, -1\rangle$ joining them, and the direction vector of the line, namely $\langle 1, 1, 1\rangle$ are both parallel to the plane. So

$$\langle 1, 3, -1\rangle \times \langle 1, 1, 1\rangle = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \langle 4, -2, -2\rangle$$

is perpendicular to the plane. As the point $(0, 0, 2)$ is on the plane and the vector $\langle 4, -2, -2\rangle$ is perpendicular to the plane, the equation of the plane is

$$4(x - 0) - 2(y - 0) - 2(z - 2) = 0 \text{ or } 2x - y - z = -2$$

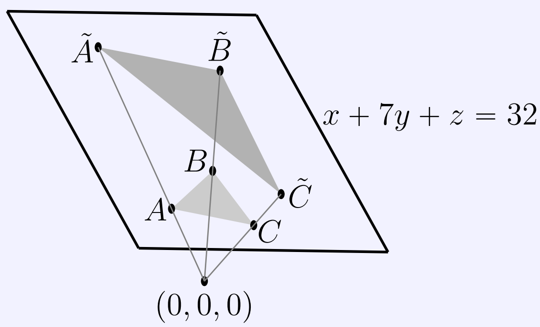
▸▸ Stage 3

1.5.15 (∗) Let $A = (0,2,2)$, $B = (2,2,2)$, $C = (5,2,1)$.

(a) Find the parametric equations for the line which contains A and is perpendicular to the triangle ABC.

(b) Find the equation of the set of all points P such that $\overrightarrow{PA}$ is perpendicular to $\overrightarrow{PB}$. This set forms a Plane/Line/Sphere/Cone/Paraboloid/Hyperboloid (circle one) in space.

(c) A light source at the origin shines on the triangle ABC making a shadow on the plane $x + 7y + z = 32$. (See the diagram.) Find $\tilde{A}$.



Solution (a) We are given one point on the line, so we just need a direction vector. That direction vector has to be perpendicular to the triangle ABC.

The fast way to get a direction vector is to observe that all three points A, B and C, and consequently the entire triangle ABC, are contained in the plane $y = 2$. A normal vector to that plane, and consequently a direction vector for the desired line, is $\hat{\boldsymbol{\jmath}}$.

Here is another, more mechanical, way to get a direction vector. The vector from A to B is $\langle 2-0\,,\,2-2\,,\,2-2\rangle = \langle 2,0,0\rangle$ and the vector from A to C is $\langle 5-0\,,\,2-2\,,\,1-2\rangle = \langle 5,0,-1\rangle$. So a vector perpendicular to the triangle ABC is

$$\langle 2,0,0\rangle \times \langle 5,0,-1\rangle = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 2 & 0 & 0 \\ 5 & 0 & -1 \end{bmatrix} = \langle 0\,,\,2\,,\,0\rangle$$

The vector $\frac{1}{2}\langle 0\,,\,2\,,\,0\rangle = \langle 0\,,\,1\,,\,0\rangle$ is also perpendicular to the triangle ABC.

So the specified line has to contain the point $(0,2,2)$ and have direction vector $\langle 0,1,0\rangle$. The parametric equations

$$\langle x,y,z\rangle = \langle 0,2,2\rangle + t\,\langle 0,1,0\rangle$$

or

$$x = 0,\ y = 2 + t,\ z = 2$$

do the job.

(b) Let P be the point (x,y,z). Then the vector from P to A is $\langle 0-x\,,\,2-y\,,\,2-z\rangle$ and the vector from P to B is $\langle 2-x\,,\,2-y\,,\,2-z\rangle$. These two vector are perpendicular if and only if

$$\begin{aligned}0 &= \langle -x\,,\,2-y\,,\,2-z\rangle\cdot\langle 2-x\,,\,2-y\,,\,2-z\rangle = x(x-2)+(y-2)^2+(z-2)^2\\ &= (x-1)^2-1+(y-2)^2+(z-2)^2\end{aligned}$$

This is a sphere.

(c) The light ray that forms $\tilde{A}$ starts at the origin, passes through A and then intersects the plane $x+7y+z=32$ at $\tilde{A}$. The line from the origin through A has vector parametric equation

$$\langle x,y,z\rangle = \langle 0,0,0\rangle + t\,\langle 0,2,2\rangle = \langle 0,2t,2t\rangle$$

This line intersects the plane $x+7y+z=32$ at the point whose value of t obeys

$$(0)+7\overbrace{(2t)}^{y}+\overbrace{(2t)}^{z}=32 \iff t=2$$

So $\tilde{A}$ is $(0,4,4)$.

$\boxed{1.5.16}$ Let P, Q, R and S be the vertices of a tetrahedron. Denote by $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$ the vectors from the origin to P, Q, R and S respectively. A line is drawn from each vertex to the centroid of the opposite face, where the centroid of a triangle with vertices $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$. Show that these four lines meet at $\frac{1}{4}(\mathbf{p}+\mathbf{q}+\mathbf{r}+\mathbf{s})$.

$\boxed{\text{Solution}}$ The face opposite $\mathbf{p}$ is the triangle with vertices $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$. The centroid of this triangle is $\frac{1}{3}(\mathbf{q}+\mathbf{r}+\mathbf{s})$. The direction vector of the line through $\mathbf{p}$ and the centroid $\frac{1}{3}(\mathbf{q}+\mathbf{r}+\mathbf{s})$ is $\frac{1}{3}(\mathbf{q}+\mathbf{r}+\mathbf{s})-\mathbf{p}$. The points on the line through $\mathbf{p}$ and the centroid $\frac{1}{3}(\mathbf{q}+\mathbf{r}+\mathbf{s})$ are those of the form

$$\mathbf{x}=\mathbf{p}+t\left[\frac{1}{3}(\mathbf{q}+\mathbf{r}+\mathbf{s})-\mathbf{p}\right]$$

for some real number t. Observe that when $t=\frac{3}{4}$

$$\mathbf{p}+t\left[\frac{1}{3}(\mathbf{q}+\mathbf{r}+\mathbf{s})-\mathbf{p}\right]=\frac{1}{4}(\mathbf{p}+\mathbf{q}+\mathbf{r}+\mathbf{s})$$

so that $\frac{1}{4}(\mathbf{p}+\mathbf{q}+\mathbf{r}+\mathbf{s})$ is on the line. The other three lines have vector parametric equations

$$\begin{aligned}\mathbf{x}&=\mathbf{q}+t\left[\frac{1}{3}(\mathbf{p}+\mathbf{r}+\mathbf{s})-\mathbf{q}\right]\\ \mathbf{x}&=\mathbf{r}+t\left[\frac{1}{3}(\mathbf{p}+\mathbf{q}+\mathbf{s})-\mathbf{r}\right]\\ \mathbf{x}&=\mathbf{s}+t\left[\frac{1}{3}(\mathbf{p}+\mathbf{q}+\mathbf{r})-\mathbf{s}\right]\end{aligned}$$

When $t = \frac{3}{4}$, each of the three right hand sides also reduces to $\frac{1}{4}(\mathbf{p}+\mathbf{q}+\mathbf{r}+\mathbf{s})$ so that $\frac{1}{4}(\mathbf{p}+\mathbf{q}+\mathbf{r}+\mathbf{s})$ is also on each of these three lines.

1.5.17 Calculate the distance between the lines $\frac{x+2}{3} = \frac{y-7}{-4} = \frac{z-2}{4}$ and $\frac{x-1}{-3} = \frac{y+2}{4} = \frac{z+1}{1}$.

Solution We'll use the procedure of Example 1.5.7 in the CLP-3 text. The vector

$$\langle 3,-4,4\rangle \times \langle -3,4,1\rangle = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 3 & -4 & 4 \\ -3 & 4 & 1 \end{bmatrix} = \langle -20,-15,0\rangle$$

is perpendicular to both lines. Hence so is $\mathbf{n} = -\frac{1}{5}\langle -20,-15,0\rangle = \langle 4,3,0\rangle$. The point $(-2,7,2)$ is on the first line and the point $(1,-2,-1)$ is on the second line. Hence $\mathbf{v} = \langle -2,7,2\rangle - \langle 1,-2,-1\rangle = \langle -3,9,3\rangle$ is a vector joining the two lines. The desired distance is the length of the projection of $\mathbf{v}$ on $\mathbf{n}$. This is

$$\left|\text{proj}_{\mathbf{n}}\mathbf{v}\right| = \frac{|\langle -3,9,3\rangle \cdot \langle 4,3,0\rangle|}{|\langle 4,3,0\rangle|} = \frac{15}{5} = 3$$

1.6▲ Curves and their Tangent Vectors

▸▸ Stage 1

Questions 1 through 5 provide practice with curve parametrization. Being comfortable with the algebra and interpretation of these descriptions are essential ingredients in working effectively with parametrizations.

1.6.1 Consider the following time-parametrized curve:

$$\mathbf{r}(t) = \left(\cos\left(\frac{\pi}{4}t\right)\ ,\ (t-5)^2\right)$$

List the three points $(-1/\sqrt{2},0)$, $(1,25)$, and $(0,25)$ in chronological order.

Solution We can find the time at which the curve hits a given point by considering the two equations that arise from the two coordinates. For the y-coordinate to be 0, we must have $(t-5)^2 = 0$, i.e. $t = 5$. So, the point $(-1/\sqrt{2},0)$ happens when $t = 5$.

Similarly, for the y-coordinate to be 25, we need $(t-5)^2 = 25$, so $(t-5) = \pm 5$. When $t = 0$, the curve hits $(1,25)$; when $t = 10$, the curve hits $(0,25)$.

So, in order, the curve passes through the points $(1,25)$, $(-1/\sqrt{2},0)$, and $(0,25)$.

1.6.2 At what points in the xy-plane does the curve $(\sin t, t^2)$ cross itself? What is the difference in t between the first time the curve crosses through a point, and the last?

Solution The curve "crosses itself" when the same coordinates occur for different values of t, say t_1 and t_2. So, we want to know when $\sin t_1 = \sin t_2$ and also $t_1^2 = t_2^2$. Since t_1 and t_2 should be different, the second equation tells us $t_2 = -t_1$. Then the first equation tells us $\sin t_1 = \sin t_2 = \sin(-t_1) = -\sin t_1$. That is, $\sin t_1 = -\sin t_1$, so $\sin t_1 = 0$. That happens whenever $t_1 = \pi n$ for an integer n.

So, the points at which the curve crosses itself are those points $(0, (\pi n)^2)$ where n is an integer. It passes such a point at times $t = \pi n$ and $t = -\pi n$. So, the curve hits this point $2\pi n$ time units apart.

1.6.3 Find the specified parametrization of the first quadrant part of the circle $x^2 + y^2 = a^2$.

(a) In terms of the y coordinate.
(b) In terms of the angle between the tangent line and the positive x-axis.
(c) In terms of the arc length from $(0, a)$.

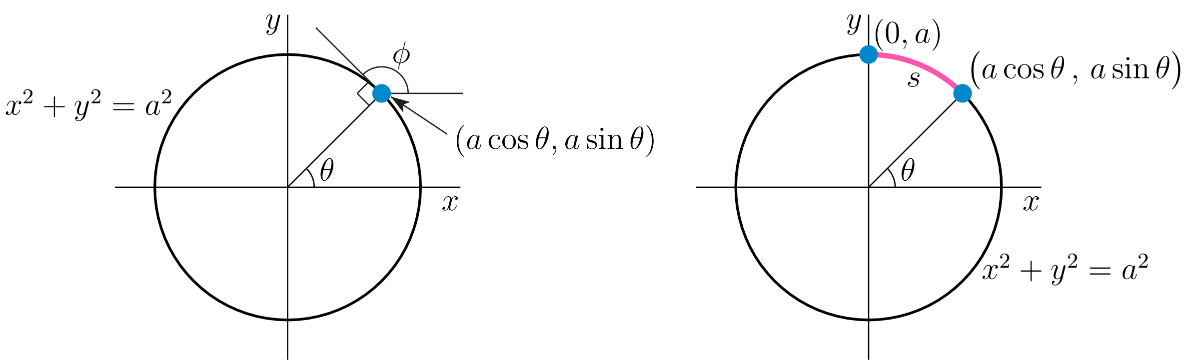
Solution (a) Since, on the specified part of the circle, $x = \sqrt{a^2 - y^2}$ and y runs from 0 to a, the parametrization is $\mathbf{r}(y) = \sqrt{a^2 - y^2}\,\hat{\imath} + y\,\hat{\jmath}$, $0 \leqslant y \leqslant a$.

(b) Let θ be the angle between

- the radius vector from the origin to the point $(a\cos\theta, a\sin\theta)$ on the circle and
- the positive x-axis.

The tangent line to the circle at $(a\cos\theta, a\sin\theta)$ is perpendicular to the radius vector and so makes angle $\phi = \frac{\pi}{2} + \theta$ with the positive x axis. (See the figure on the left below.) As $\theta = \phi - \frac{\pi}{2}$, the desired parametrization is

$$\big(x(\phi), y(\phi)\big) = \big(a\cos(\phi - \tfrac{\pi}{2}), a\sin(\phi - \tfrac{\pi}{2})\big) = \big(a\sin\phi, -a\cos\phi\big), \ \tfrac{\pi}{2} \leqslant \phi \leqslant \pi$$



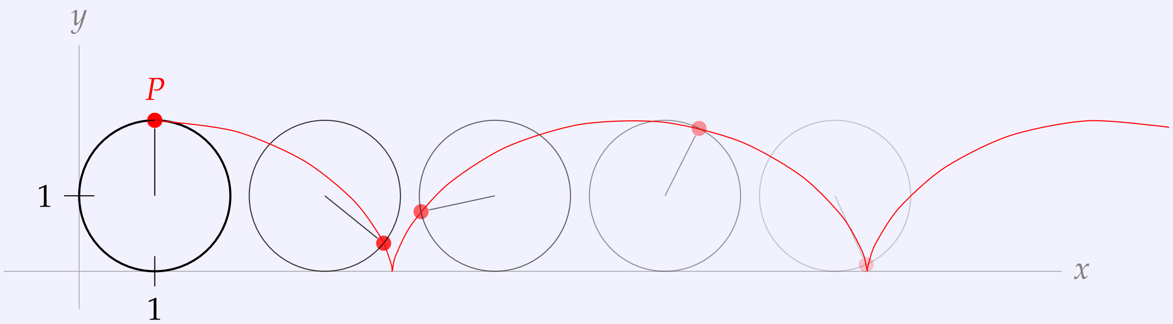
(c) Let θ be the angle between

- the radius vector from the origin to the point $(a\cos\theta, a\sin\theta)$ on the circle and
- the positive x-axis.

The arc from $(0, a)$ to $(a\cos\theta, a\sin\theta)$ subtends an angle $\frac{\pi}{2} - \theta$ and so has length $s = a\big(\frac{\pi}{2} - \theta\big)$. (See the figure on the right above.) Thus $\theta = \frac{\pi}{2} - \frac{s}{a}$ and the desired parametrization is

$$\big(x(s), y(s)\big) = \left(a\cos\left(\frac{\pi}{2} - \frac{s}{a}\right), a\sin\left(\frac{\pi}{2} - \frac{s}{a}\right)\right), \ 0 \leqslant s \leqslant \frac{\pi}{2}a$$

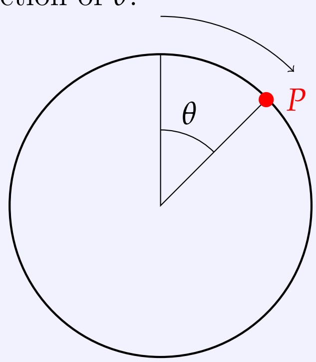
1.6.4



A circle of radius a rolls along the x-axis in the positive direction, starting with its centre at (a, a). In that position, we mark the topmost point on the circle P. As the circle moves, P moves with it. Let θ be the angle the circle has rolled–see the diagram below.

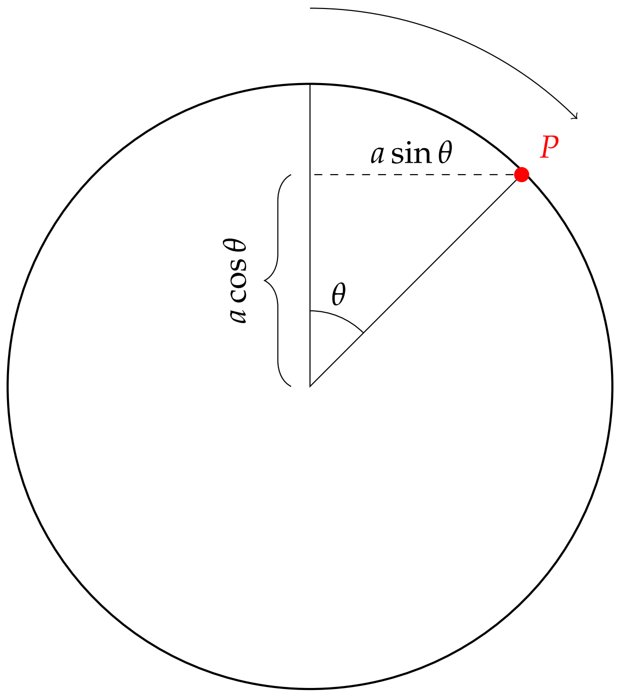
(a) Give the position of the centre of the circle as a function of θ.

(b) Give the position of P a function of θ.



Solution Pretend that the circle is a spool of thread. As the circle rolls it dispenses the thread along the ground. When the circle rolls θ radians it dispenses the arc length θa of thread and the circle advances a distance θa. So centre of the circle has moved θa units to the right from its starting point, $x = a$. The centre of the circle always has y-coordinate a. So, after rolling θ radians, the centre of the circle is at position $\mathbf{c}(\theta) = (a + a\theta, a)$.

Now, let's consider the position of P on the circle, after the circle has rolled θ radians.



From the diagram, we see that P is $a\cos\theta$ units above the centre of the circle, and $a\sin\theta$ units to the right of it. So, the position of P is $(a + a\theta + a\sin\theta, a + a\cos\theta)$.

Remark: this type of curve is known as a cycloid.

1.6.5 The curve C is defined to be the intersection of the ellipsoid

$$x^2 - \frac{1}{4}y^2 + 3z^2 = 1$$

and the plane

$$x + y + z = 0.$$

When y is very close to 0, and z is negative, find an expression giving z in terms of y.

Solution We aren't concerned with x, so we can eliminate it by solving for it in one equation, and plugging that into the other. Since C lies on the plane, $x = -y - z$, so:

$$\begin{aligned} 1 &= x^2 - \frac{1}{4}y^2 + 3z^2 = (-y-z)^2 - \frac{1}{4}y^2 + 3z^2 \\ &= \frac{3}{4}y^2 + 4z^2 + 2yz \end{aligned}$$

Completing the square,

$$\begin{aligned} 1 &= \frac{1}{2}y^2 + \left(2z + \frac{y}{2}\right)^2 \\ 1 - \frac{y^2}{2} &= \left(2z + \frac{y}{2}\right)^2 \end{aligned}$$

Since y is small, the left hand is close to 1 and the right hand side is close to $(2z)^2$. So $(2z^2) \approx 1$. Since z is negative, $z \approx -\frac{1}{2}$ and $2z + \frac{y}{2} < 0$. Also, $1 - \frac{y^2}{2}$ is positive, so it has a real square root.

$$\begin{aligned} -\sqrt{1-\frac{y^2}{2}} &= 2z + \frac{y}{2} \\ -\frac{1}{2}\sqrt{1-\frac{y^2}{2}} - \frac{y}{4} &= z \end{aligned}$$

1.6.6 A particle traces out a curve in space, so that its position at time t is

$$\mathbf{r}(t) = e^{-t}\,\hat{\boldsymbol{\imath}} + \frac{1}{t}\,\hat{\boldsymbol{\jmath}} + (t-1)^2(t-3)^2\,\hat{\mathbf{k}}$$

for $t > 0$.

Let the positive z axis point vertically upwards, as usual. When is the particle moving upwards, and when is it moving downwards? Is it moving faster at time $t = 1$ or at time $t = 3$?

Solution To determine whether the particle is rising or falling, we only need to consider its z-coordinate: $z(t) = (t-1)^2(t-3)^2$. Its derivative with respect to time is $z'(t) = 4(t-1)(t-2)(t-3)$. This is positive when $1 < t < 2$ and when $3 < t$, so the particle is increasing on $(1,2) \cup (3,\infty)$ and decreasing on $(0,1) \cup (2,3)$.

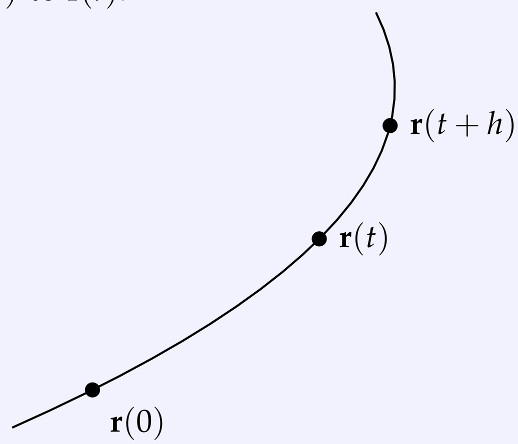
If $\mathbf{r}(t)$ is the position of the particle at time t, then its speed is $|\mathbf{r}'(t)|$. We differentiate:

$$\mathbf{r}'(t) = -e^{-t}\,\hat{\boldsymbol{\imath}} - \frac{1}{t^2}\,\hat{\boldsymbol{\jmath}} + 4(t-1)(t-2)(t-3)\hat{\mathbf{k}}$$

So, $\mathbf{r}(1) = -\frac{1}{e}\,\hat{\boldsymbol{\imath}} - 1\,\hat{\boldsymbol{\jmath}}$ and $\mathbf{r}(3) = -\frac{1}{e^3}\,\hat{\boldsymbol{\imath}} - \frac{1}{9}\,\hat{\boldsymbol{\jmath}}$. The absolute value of every component of $\mathbf{r}(1)$ is greater than or equal to that of the corresponding component of $\mathbf{r}(3)$, so $|\mathbf{r}(1)| > |\mathbf{r}(3)|$. That is, the particle is moving more swiftly at $t = 1$ than at $t = 3$.

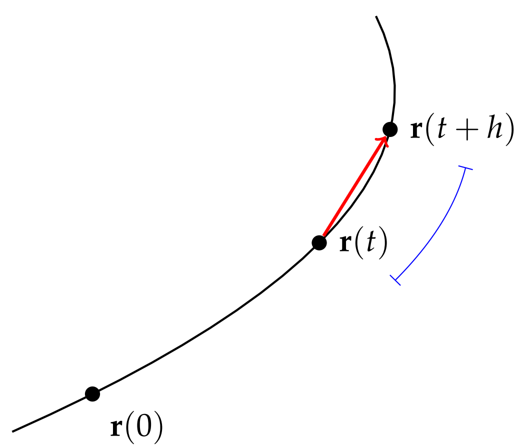
Note: We could also compute the sizes of both vectors directly: $|\mathbf{r}'(1)| = \sqrt{\left(\frac{1}{e}\right)^2 + (-1)^2}$, and $|\mathbf{r}'(3)| = \sqrt{\left(\frac{1}{e^3}\right)^2 + \left(-\frac{1}{9}\right)^2}$.

$\boxed{1.6.7}$ Below is the graph of the parametrized function $\mathbf{r}(t)$. Let $s(t)$ be the arclength along the curve from $\mathbf{r}(0)$ to $\mathbf{r}(t)$.



Indicate on the graph $s(t+h)-s(t)$ and $\mathbf{r}(t+h)-\mathbf{r}(t)$. Are the quantities scalars or vectors?

Solution



The red vector is $\mathbf{r}(t+h)-\mathbf{r}(t)$. The arclength of the segment indicated by the blue line is the (scalar) $s(t+h)-s(t)$.

Remark: as h approaches 0, the curve (if it's differentiable at t) starts to resemble a straight line, with the length of the vector $\mathbf{r}(t+h)-\mathbf{r}(t)$ approaching the scalar $s(t+h)-s(t)$. This step is crucial to understanding Lemma 1.6.12 in the CLP-3 text.

$\boxed{1.6.8}$ What is the relationship between velocity and speed in a vector-valued function of time?

Solution Velocity is a vector-valued quantity, so it has both a magnitude and a direction. Speed is a scalar–the magnitude of the velocity. It does not include a direction.

1.6.9 (*) Let $\mathbf{r}(t)$ be a vector valued function. Let $\mathbf{r}'$, $\mathbf{r}''$, and $\mathbf{r}'''$ denote $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$, $\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$ and $\frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}t^3}$, respectively. Express

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{r}''\big]$$

in terms of $\mathbf{r}$, $\mathbf{r}'$, $\mathbf{r}''$, and $\mathbf{r}'''$. Select the correct answer.

(a) $(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''$
(b) $(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}+(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{r}'''$
(c) $(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{r}'''$
(d) 0
(e) None of the above.

Solution By the product rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{r}''\big]=(\mathbf{r}'\times\mathbf{r}')\cdot\mathbf{r}''+(\mathbf{r}\times\mathbf{r}'')\cdot\mathbf{r}''+(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{r}'''$$

The first term vanishes because $\mathbf{r}'\times\mathbf{r}'=\mathbf{0}$. The second term vanishes because $\mathbf{r}\times\mathbf{r}''$ is perpendicular to $\mathbf{r}''$. So

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{r}''\big]=(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{r}'''$$

which is (c).

▸▸ Stage 2

1.6.10 (*) Find the speed of a particle with the given position function

$$\mathbf{r}(t)=5\sqrt{2}\,t\,\hat{\boldsymbol{\imath}}+e^{5t}\,\hat{\boldsymbol{\jmath}}-e^{-5t}\,\hat{\mathbf{k}}$$

Select the correct answer:

(a) $|\mathbf{v}(t)|=\big(e^{5t}+e^{-5t}\big)$
(b) $|\mathbf{v}(t)|=\sqrt{10+5e^{t}+5e^{-t}}$
(c) $|\mathbf{v}(t)|=\sqrt{10+e^{10t}+e^{-10t}}$
(d) $|\mathbf{v}(t)|=5\big(e^{5t}+e^{-5t}\big)$
(e) $|\mathbf{v}(t)|=5\big(e^{t}+e^{-t}\big)$

Solution We have

$$\mathbf{v}(t)=\mathbf{r}'(t)=5\sqrt{2}\,\hat{\boldsymbol{\imath}}+5e^{5t}\,\hat{\boldsymbol{\jmath}}+5e^{-5t}\,\hat{\mathbf{k}}$$

and hence

$$|\mathbf{v}(t)|=|\mathbf{r}'(t)|=5\big|\sqrt{2}\,\hat{\boldsymbol{\imath}}+e^{5t}\,\hat{\boldsymbol{\jmath}}+e^{-5t}\,\hat{\mathbf{k}}\big|=5\sqrt{2+e^{10t}+e^{-10t}}$$

Since $2+e^{10t}+e^{-10t}=\big(e^{5t}+e^{-5t}\big)^2$, that's (d).

$\boxed{1.6.11}$ Find the velocity, speed and acceleration at time t of the particle whose position is $\mathbf{r}(t)$. Describe the path of the particle.
(a) $\mathbf{r}(t) = a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath} + ct\,\hat{\mathbf{k}}$
(b) $\mathbf{r}(t) = a\cos t\sin t\,\hat{\imath} + a\sin^2 t\,\hat{\jmath} + a\cos t\,\hat{\mathbf{k}}$

$\boxed{\text{Solution}}$ (a) By definition,

$$\begin{aligned}
\mathbf{r}(t) &= a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath} + ct\,\hat{\mathbf{k}}\\
\mathbf{v}(t) = \mathbf{r}'(t) &= -a\sin t\,\hat{\imath} + a\cos t\,\hat{\jmath} + c\,\hat{\mathbf{k}}\\
\frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{v}(t)| &= \sqrt{a^2+c^2}\\
\mathbf{a}(t) = \mathbf{r}''(t) &= -a\cos t\,\hat{\imath} - a\sin t\,\hat{\jmath}
\end{aligned}$$

The $(x,y) = a(\cos t, \sin t)$ coordinates go around a circle of radius a and centre $(0,0)$ counterclockwise. One circle is completed for each increase of t by 2π. At the same time, the z coordinate increases at a constant rate. Each time the (x,y) coordinates complete one circle, the z coordinate increases by $2\pi c$. The path is a helix with radius a and with each turn having height $2\pi c$.

(b) By definition,

$$\begin{aligned}
\mathbf{r}(t) &= a\cos t\sin t\,\hat{\imath} + a\sin^2 t\,\hat{\jmath} + a\cos t\,\hat{\mathbf{k}}\\
&= \frac{a}{2}\sin 2t\,\hat{\imath} + a\frac{1-\cos 2t}{2}\,\hat{\jmath} + a\cos t\,\hat{\mathbf{k}}\\
\mathbf{v}(t) = \mathbf{r}'(t) &= a\cos 2t\,\hat{\imath} + a\sin 2t\,\hat{\jmath} - a\sin t\,\hat{\mathbf{k}}\\
\frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{v}(t)| &= a\sqrt{1+\sin^2 t}\\
\mathbf{a}(t) = \mathbf{r}''(t) &= -2a\sin 2t\,\hat{\imath} + 2a\cos 2t\,\hat{\jmath} - a\cos t\,\hat{\mathbf{k}}
\end{aligned}$$

The (x,y) coordinates go around a circle of radius $\frac{a}{2}$ and centre $\left(0,\frac{a}{2}\right)$ counterclockwise. At the same time the z coordinate oscillates over the interval between 1 and -1 half as fast.

$\boxed{1.6.12}$ (*)
(a) Let

$$\mathbf{r}(t) = \left(t^2, 3, \tfrac{1}{3}t^3\right)$$

Find the unit tangent vector to this parametrized curve at $t=1$, pointing in the direction of increasing t.
(b) Find the arc length of the curve from (a) between the points $(0,3,0)$ and $(1,3,-\frac{1}{3})$.

$\boxed{\text{Solution}}$ (a) Since $\mathbf{r}'(t) = (2t, 0, t^2)$, the specified unit tangent at $t=1$ is

$$\hat{\mathbf{T}}(1) = \frac{(2,0,1)}{\sqrt{5}}$$

(b) We are to find the arc length between $\mathbf{r}(0)$ and $\mathbf{r}(-1)$. As $\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{4t^2 + t^4}$, the

$$\text{arc length} = \int_{-1}^{0} \sqrt{4t^2 + t^4}\, \mathrm{d}t$$

The integrand is even, so

$$\text{arc length} = \int_0^1 \sqrt{4t^2 + t^4}\, \mathrm{d}t = \int_0^1 t\sqrt{4 + t^2}\, \mathrm{d}t = \left[\tfrac{1}{3}(4 + t^2)^{3/2}\right]_0^1 = \tfrac{1}{3}\left[5^{3/2} - 8\right]$$

$\boxed{1.6.13}$ Using Lemma 1.6.12 in the CLP-3 text, find the arclength of $\mathbf{r}(t) = \left(t, \sqrt{\frac{3}{2}}t^2, t^3\right)$ from $t = 0$ to $t = 1$.

$\boxed{\text{Solution}}$ By Lemma 1.6.12 in the CLP-3 text, the arclength of $\mathbf{r}(t)$ from $t = 0$ to $t = 1$ is $\int_0^1 \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right| \mathrm{d}t$. We'll calculate this in a few pieces to make the steps clearer.

$$\begin{aligned}
\mathbf{r}(t) &= \left(t, \sqrt{\frac{3}{2}}t^2, t^3\right) \\
\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) &= \left(1, \sqrt{6}t, 3t^2\right) \\
\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right| &= \sqrt{1^2 + (\sqrt{6}t)^2 + (3t^2)^2} = \sqrt{1 + 6t^2 + 9t^4} = \sqrt{(3t^2+1)^2} = 3t^2 + 1 \\
\int_0^1 \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right| \mathrm{d}t &= \int_0^1 \left(3t^2 + 1\right) \mathrm{d}t = 2
\end{aligned}$$

$\boxed{1.6.14}$ A particle's position at time t is given by $\mathbf{r}(t) = (t + \sin t, \cos t)$*. What is the magnitude of the acceleration of the particle at time t?

* The particle traces out a cycloid–see Question 4

$\boxed{\text{Solution}}$ Since $\mathbf{r}(t)$ is the position of the particle, its acceleration is $\mathbf{r}''(t)$.

$$\begin{aligned}
\mathbf{r}(t) &= (t + \sin t, \cos t) \\
\mathbf{r}'(t) &= (1 + \cos t, -\sin t) \\
\mathbf{r}''(t) &= (-\sin t, -\cos t) \\
|\mathbf{r}''(t)| &= \sqrt{\sin^2 t + \cos^2 t} = 1
\end{aligned}$$

The magnitude of acceleration is constant, but its direction is changing, since $\mathbf{r}''(t)$ is a vector with changing direction.

$\boxed{1.6.15}$ (∗) A curve in $\mathbb{R}^3$ is given by the vector equation $\mathbf{r}(t) = \left(2t\cos t, 2t\sin t, \frac{t^3}{3}\right)$
(a) Find the length of the curve between $t = 0$ and $t = 2$.
(b) Find the parametric equations of the tangent line to the curve at $t = \pi$.

Solution (a) The speed is

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{r}'(t)| &= \left|\left(2\cos t - 2t\sin t\,,\, 2\sin t + 2t\cos t\,,\, t^2\right)\right| \\ &= \sqrt{\left(2\cos t - 2t\sin t\right)^2 + \left(2\sin t + 2t\cos t\right)^2 + t^4} \\ &= \sqrt{4 + 4t^2 + t^4} \\ &= 2 + t^2\end{aligned}$$

so the length of the curve is

$$\text{length} = \int_0^2 \frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t = \int_0^2 (2 + t^2)\,\mathrm{d}t = \left[2t + \frac{t^3}{3}\right]_0^2 = \frac{20}{3}$$

(b) A tangent vector to the curve at $\mathbf{r}(\pi) = \big(-2\pi\,,\, 0\,,\, \pi^3/3\big)$ is

$$\mathbf{r}'(\pi) = \left(2\cos\pi - 2\pi\sin\pi\,,\, 2\sin\pi + 2\pi\cos\pi\,,\, \pi^2\right) = (-2\,,\, -2\pi\,,\, \pi^2)$$

So parametric equations for the tangent line at $\mathbf{r}(\pi)$ are

$$\begin{aligned}x(t) &= -2\pi - 2t \\ y(t) &= -2\pi t \\ z(t) &= \pi^3/3 + \pi^2 t\end{aligned}$$

$\boxed{1.6.16}$ (∗) Let $\mathbf{r}(t) = \big(3\cos t, 3\sin t, 4t\big)$ be the position vector of a particle as a function of time $t \geqslant 0$.
(a) Find the velocity of the particle as a function of time t.
(b) Find the arclength of its path between $t = 1$ and $t = 2$.

Solution (a) As $\mathbf{r}(t) = \big(3\cos t, 3\sin t, 4t\big)$, the velocity of the particle is

$$\mathbf{r}'(t) = \big(-3\sin t, 3\cos t, 4\big)$$

(b) As $\frac{\mathrm{d}s}{\mathrm{d}t}$, the rate of change of arc length per unit time, is

$$\frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{r}'(t)| = \big|\big(-3\sin t, 3\cos t, 4\big)\big| = 5$$

the arclength of its path between $t = 1$ and $t = 2$ is

$$\int_1^2 \mathrm{d}t\ \frac{\mathrm{d}s}{\mathrm{d}t}(t) = \int_1^2 \mathrm{d}t\ 5 = 5$$

$\boxed{1.6.17}$ (*) Consider the curve

$$\mathbf{r}(t) = \frac{1}{3}\cos^3 t\,\hat{\imath} + \frac{1}{3}\sin^3 t\,\hat{\jmath} + \sin^3 t\,\hat{\mathbf{k}}$$

(a) Compute the arc length of the curve from $t = 0$ to $t = \frac{\pi}{2}$.
(b) Compute the arc length of the curve from $t = 0$ to $t = \pi$.

Solution (a) As

$$\mathbf{r}'(t) = -\sin t\cos^2 t\,\hat{\imath} + \sin^2 t\cos t\,\hat{\imath} + 3\sin^2 t\cos t\,\hat{\mathbf{k}} = \sin t\cos t\big(-\cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + 3\sin t\,\hat{\mathbf{k}}\big)$$
$$\frac{ds}{dt}(t) = |\sin t\cos t|\sqrt{\cos^2 t + \sin^2 t + 9\sin^2 t} = |\sin t\cos t|\sqrt{1 + 9\sin^2 t}$$

the arclength from $t = 0$ to $t = \frac{\pi}{2}$ is

$$\begin{aligned}\int_0^{\pi/2}\frac{ds}{dt}(t)\,dt &= \int_0^{\pi/2}\sin t\cos t\sqrt{1+9\sin^2 t}\,dt\\ &= \frac{1}{18}\int_1^{10}\sqrt{u}\,du \qquad \text{with } u = 1+9\sin^2 t,\ du = 18\sin t\cos t\,dt\\ &= \frac{1}{18}\Big[\frac{2}{3}u^{3/2}\Big]_1^{10}\\ &= \frac{1}{27}\big(10\sqrt{10}-1\big)\end{aligned}$$

(b) The arclength from $t = 0$ to $t = \pi$ is

$$\begin{aligned}\int_0^{\pi}\frac{ds}{dt}(t)\,dt &= \int_0^{\pi}|\sin t\cos t|\sqrt{1+9\sin^2 t}\,dt \qquad \text{Don't forget the absolute value signs!}\\ &= 2\int_0^{\pi/2}|\sin t\cos t|\sqrt{1+9\sin^2 t}\,dt = 2\int_0^{\pi/2}\sin t\cos t\sqrt{1+9\sin^2 t}\,dt\end{aligned}$$

since the integrand is invariant under $t \to \pi - t$. So the arc length from $t = 0$ to $t = \pi$ is just twice the arc length from part (a), namely $\frac{2}{27}\big(10\sqrt{10}-1\big)$.

$\boxed{1.6.18}$ (*) Let $\mathbf{r}(t) = \big(\frac{1}{3}t^3, \frac{1}{2}t^2, \frac{1}{2}t\big)$, $t \geqslant 0$. Compute $s(t)$, the arclength of the curve at time t.

Solution Since

$$\begin{aligned}\mathbf{r}(t) &= \frac{t^3}{3}\,\hat{\imath} + \frac{t^2}{2}\,\hat{\jmath} + \frac{t}{2}\,\hat{\mathbf{k}}\\ \mathbf{r}'(t) &= t^2\,\hat{\imath} + t\,\hat{\jmath} + \frac{1}{2}\,\hat{\mathbf{k}}\\ \frac{ds}{dt}(t) = |\mathbf{r}'(t)| &= \sqrt{t^4 + t^2 + \frac{1}{4}} = \sqrt{\Big(t^2 + \frac{1}{2}\Big)^2} = t^2 + \frac{1}{2}\end{aligned}$$

the length of the curve is

$$s(t) = \int_0^t \frac{\mathrm{d}s}{\mathrm{d}t}(u)\,\mathrm{d}u = \int_0^t \left(u^2 + \frac{1}{2}\right)\mathrm{d}u = \frac{t^3}{3} + \frac{t}{2}$$

$\boxed{1.6.19}$ (*) Find the arc length of the curve $\mathbf{r}(t) = \left(t^m, t^m, t^{3m/2}\right)$ for $0 \leqslant a \leqslant t \leqslant b$, and where $m > 0$. Express your result in terms of m, a, and b.

$\boxed{\text{Solution}}$ Since

$$\begin{aligned}
\mathbf{r}(t) &= t^m\,\hat{\boldsymbol{\imath}} + t^m\,\hat{\boldsymbol{\jmath}} + t^{3m/2}\,\hat{\mathbf{k}} \\
\mathbf{r}'(t) &= mt^{m-1}\,\hat{\boldsymbol{\imath}} + mt^{m-1}\,\hat{\boldsymbol{\jmath}} + \frac{3m}{2}t^{3m/2-1}\,\hat{\mathbf{k}} \\
\frac{\mathrm{d}s}{\mathrm{d}t} = |\mathbf{r}'(t)| &= \sqrt{2m^2t^{2m-2} + \frac{9m^2}{4}t^{3m-2}} = mt^{m-1}\sqrt{2 + \frac{9}{4}t^m}
\end{aligned}$$

the arc length is

$$\begin{aligned}
\int_a^b \frac{\mathrm{d}s}{\mathrm{d}t}(t)\,\mathrm{d}t &= \int_a^b mt^{m-1}\sqrt{2 + \frac{9}{4}t^m}\,\mathrm{d}t \\
&= \frac{4}{9}\int_{2+\frac{9}{4}a^m}^{2+\frac{9}{4}b^m} \sqrt{u}\,\mathrm{d}u \qquad \text{with } u = 2 + \frac{9}{4}t^m,\ \mathrm{d}u = \frac{9m}{4}t^{m-1} \\
&= \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_{2+\frac{9}{4}a^m}^{2+\frac{9}{4}b^m} \\
&= \frac{8}{27}\left[\left(2 + \frac{9}{4}b^m\right)^{3/2} - \left(2 + \frac{9}{4}a^m\right)^{3/2}\right]
\end{aligned}$$

$\boxed{1.6.20}$ If a particle has constant mass m, position $\mathbf{r}$, and is moving with velocity $\mathbf{v}$, then its angular momentum is $\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$.
For a particle with mass $m = 1$ and position function $\mathbf{r} = (\sin t, \cos t, t)$, find $\left|\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|$.

$\boxed{\text{Solution}}$ Given the position of the particle, we can find its velocity:

$$\mathbf{v}(t) = \mathbf{r}'(t) = (\cos t, -\sin t, 1)$$

Applying the given formula,

$$\mathbf{L}(t) = \mathbf{r} \times \mathbf{v} = (\sin t, \cos t, t) \times (\cos t, -\sin t, 1).$$

Solution 1: We can first compute the cross product, then differentiate:

$$\begin{aligned}
\mathbf{L}(t) &= (\cos t + t\sin t)\hat{\boldsymbol{\imath}} + (t\cos t - \sin t)\hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}} \\
\mathbf{L}'(t) &= t\cos t\,\hat{\boldsymbol{\imath}} - t\sin t\,\hat{\boldsymbol{\jmath}} \\
|\mathbf{L}'(t)| &= \sqrt{t^2(\sin^2 t + \cos^2 t)} = \sqrt{t^2} = |t|
\end{aligned}$$

Solution 2: Using the product rule:

$$\begin{aligned}
\mathbf{L}'(t) &= \mathbf{r}'(t)\times\mathbf{v}(t)+\mathbf{r}(t)\times\mathbf{v}'(t)\\
&= \underbrace{\mathbf{r}'(t)\times\mathbf{r}'(t)}_{0}+\mathbf{r}(t)\times\mathbf{v}'(t)\\
&= (\sin t,\cos t,t)\times(-\sin t,-\cos t,0)\\
&= t\cos t\,\hat{\boldsymbol{\imath}}-t\sin t\,\hat{\boldsymbol{\jmath}}\\
|\mathbf{L}'(t)| &= \sqrt{t^2\cos^2 t+t^2\sin t^2}=|t|
\end{aligned}$$

$\boxed{1.6.21}$ (∗) Consider the space curve Γ whose vector equation is

$$\mathbf{r}(t)=t\sin(\pi t)\,\hat{\boldsymbol{\imath}}+t\cos(\pi t)\,\hat{\boldsymbol{\jmath}}+t^2\hat{\mathbf{k}}\qquad 0\leqslant t<\infty$$

This curve starts from the origin and eventually reaches the ellipsoid E whose equation is $2x^2+2y^2+z^2=24$.
(a) Determine the coordinates of the point P where Γ intersects E.
(b) Find the tangent vector of Γ at the point P.
(c) Does Γ intersect E at right angles? Why or why not?

$\boxed{\text{Solution}}$ (a) The curve intersects E when

$$2\big(t\sin(\pi t)\big)^2+2\big(t\cos(\pi t)\big)^2+\big(t^2\big)^2=24\iff 2t^2+t^4=24\iff (t^2-4)(t^2+6)=0$$

Since we need $t>0$, the desired time is $t=2$ and the corresponding point is $\mathbf{r}(2)=2\,\hat{\boldsymbol{\jmath}}+4\,\hat{\mathbf{k}}$.

(b) Since

$$\mathbf{r}'(t)=\big[\sin(\pi t)+\pi t\cos(\pi t)\big]\hat{\boldsymbol{\imath}}+\big[\cos(\pi t)-\pi t\sin(\pi t)\big]\hat{\boldsymbol{\jmath}}+2t\hat{\mathbf{k}}$$

a tangent vector to Γ at P is any nonzero multiple of

$$\mathbf{r}'(2)=2\pi\,\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}+4\,\hat{\mathbf{k}}$$

(c) A normal vector to E at P is

$$\boldsymbol{\nabla}(2x^2+2y^2+z^2)\big|_{(0,2,4)}=\langle 4x,4y,2z\rangle\,\big|_{(0,2,4)}=\langle 0,8,8\rangle$$

Since $\mathbf{r}'(2)$ and $\langle 0,8,8\rangle$ are not parallel, Γ and E do not intersect at right angles.

$\boxed{1.6.22}$ (∗) Suppose a particle in 3-dimensional space travels with position vector $\mathbf{r}(t)$, which satisfies $\mathbf{r}''(t)=-\mathbf{r}(t)$. Show that the "energy" $|\mathbf{r}(t)|^2+|\mathbf{r}'(t)|^2$ is constant (that is, independent of t).

Solution

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[|\mathbf{r}(t)|^2+|\mathbf{r}'(t)|^2\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)+\mathbf{r}'(t)\cdot\mathbf{r}'(t)\right]\\ &= 2\mathbf{r}(t)\cdot\mathbf{r}'(t)+2\mathbf{r}'(t)\cdot\mathbf{r}''(t)\\ &= 2\mathbf{r}'(t)\cdot\left[\mathbf{r}(t)+\mathbf{r}''(t)\right]\\ &= 0 \qquad \text{since } \mathbf{r}''(t)=-\mathbf{r}(t)\end{aligned}$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}\left[|\mathbf{r}(t)|^2+|\mathbf{r}'(t)|^2\right]=0$ for all t, $|\mathbf{r}(t)|^2+|\mathbf{r}'(t)|^2$ is independent of t.

▸▸ Stage 3

1.6.23 (∗) A particle moves along the curve $\mathcal{C}$ of intersection of the surfaces $z^2=12y$ and $18x=yz$ in the upward direction. When the particle is at $(1,3,6)$ its velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ are given by

$$\mathbf{v}=6\,\hat{\imath}+12\,\hat{\jmath}+12\,\hat{\mathbf{k}} \qquad \mathbf{a}=27\,\hat{\imath}+30\,\hat{\jmath}+6\,\hat{\mathbf{k}}$$

(a) Write a vector parametric equation for $\mathcal{C}$ using $u=\frac{z}{6}$ as a parameter.
(b) Find the length of $\mathcal{C}$ from $(0,0,0)$ to $(1,3,6)$.
(c) If $u=u(t)$ is the parameter value for the particle's position at time t, find $\frac{\mathrm{d}u}{\mathrm{d}t}$ when the particle is at $(1,3,6)$.
(d) Find $\frac{\mathrm{d}^2u}{\mathrm{d}t^2}$ when the particle is at $(1,3,6)$.

Solution (a) Since $z=6u$, $y=\frac{z^2}{12}=3u^2$ and $x=\frac{yz}{18}=u^3$,

$$\mathbf{r}(u)=u^3\,\hat{\imath}+3u^2\,\hat{\jmath}+6u\,\hat{\mathbf{k}}$$

(b)

$$\begin{aligned}\mathbf{r}'(u)&=3u^2\,\hat{\imath}+6u\,\hat{\jmath}+6\,\hat{\mathbf{k}}\\ \mathbf{r}''(u)&=6u\,\hat{\imath}+6\,\hat{\jmath}\\ \frac{\mathrm{d}s}{\mathrm{d}u}(u)&=|\mathbf{r}'(u)|=\sqrt{9u^4+36u^2+36}=3\big(u^2+2\big)\end{aligned}$$

$$\int_{\mathcal{C}} ds=\int_0^1\frac{\mathrm{d}s}{\mathrm{d}u}\,\mathrm{d}u=\int_0^1 3\big(u^2+2\big)\,\mathrm{d}u=\left[u^3+6u\right]_0^1=7$$

(c) Denote by $\mathbf{R}(t)$ the position of the particle at time t. Then

$$\mathbf{R}(t)=\mathbf{r}\big(u(t)\big) \implies \mathbf{R}'(t)=\mathbf{r}'\big(u(t)\big)\frac{\mathrm{d}u}{\mathrm{d}t}$$

In particular, if the particle is at $(1, 3, 6)$ at time t_1, then $u(t_1) = 1$ and

$$6\,\hat{\imath} + 12\,\hat{\jmath} + 12\,\hat{\mathbf{k}} = \mathbf{R}'(t_1) = \mathbf{r}'(1)\frac{\mathrm{d}u}{\mathrm{d}t}(t_1) = \big(3\,\hat{\imath} + 6\,\hat{\jmath} + 6\,\hat{\mathbf{k}}\big)\frac{\mathrm{d}u}{\mathrm{d}t}(t_1)$$

which implies that $\frac{\mathrm{d}u}{\mathrm{d}t}(t_1) = 2$.

(d) By the product and chain rules,

$$\mathbf{R}'(t) = \mathbf{r}'\big(u(t)\big)\frac{\mathrm{d}u}{\mathrm{d}t} \implies \mathbf{R}''(t) = \mathbf{r}''\big(u(t)\big)\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big)^2 + \mathbf{r}'\big(u(t)\big)\frac{\mathrm{d}^2u}{\mathrm{d}t^2}$$

In particular,

$$\begin{aligned}
27\,\hat{\imath} + 30\,\hat{\jmath} + 6\,\hat{\mathbf{k}} = \mathbf{R}''(t_1) &= \mathbf{r}''(1)\Big(\frac{\mathrm{d}u}{\mathrm{d}t}(t_1)\Big)^2 + \mathbf{r}'(1)\frac{\mathrm{d}^2u}{\mathrm{d}t^2}(t_1)\\
&= \big(6\,\hat{\imath} + 6\,\hat{\jmath}\big)2^2 + \big(3\,\hat{\imath} + 6\,\hat{\jmath} + 6\,\hat{\mathbf{k}}\big)\frac{\mathrm{d}^2u}{\mathrm{d}t^2}(t_1)
\end{aligned}$$

Simplifying

$$3\,\hat{\imath} + 6\,\hat{\jmath} + 6\,\hat{\mathbf{k}} = \big(3\,\hat{\imath} + 6\,\hat{\jmath} + 6\,\hat{\mathbf{k}}\big)\frac{\mathrm{d}^2u}{\mathrm{d}t^2}(t_1) \implies \frac{\mathrm{d}^2u}{\mathrm{d}t^2}(t_1) = 1$$

$\boxed{1.6.24}$ (∗) A particle of mass $m = 1$ has position $\mathbf{r}_0 = \frac{1}{2}\,\hat{\mathbf{k}}$ and velocity $\mathbf{v}_0 = \frac{\pi^2}{2}\,\hat{\imath}$ at time 0. It moves under a force

$$\mathbf{F}(t) = -3t\,\hat{\imath} + \sin t\,\hat{\jmath} + 2e^{2t}\,\hat{\mathbf{k}}.$$

(a) Determine the position $\mathbf{r}(t)$ of the particle depending on t.

(b) At what time after time $t = 0$ does the particle cross the plane $x = 0$ for the first time?

(c) What is the velocity of the particle when it crosses the plane $x = 0$ in part (b)?

$\boxed{\text{Solution}}$ (a) According to Newton,

$$m\mathbf{r}''(t) = \mathbf{F}(t) \qquad \text{so that} \qquad \mathbf{r}''(t) = -3t\,\hat{\imath} + \sin t\,\hat{\jmath} + 2e^{2t}\,\hat{\mathbf{k}}$$

Integrating once gives

$$\mathbf{r}'(t) = -3\frac{t^2}{2}\,\hat{\imath} - \cos t\,\hat{\jmath} + e^{2t}\,\hat{\mathbf{k}} + \mathbf{c}$$

for some constant vector $\mathbf{c}$. We are told that $\mathbf{r}'(0) = \mathbf{v}_0 = \frac{\pi^2}{2}\,\hat{\imath}$. This forces $\mathbf{c} = \frac{\pi^2}{2}\,\hat{\imath} + \hat{\jmath} - \hat{\mathbf{k}}$ so that

$$\mathbf{r}'(t) = \left(\frac{\pi^2}{2} - \frac{3t^2}{2}\right)\hat{\imath} + (1 - \cos t)\,\hat{\jmath} + \big(e^{2t} - 1\big)\,\hat{\mathbf{k}}$$

Integrating a second time gives

$$\mathbf{r}(t) = \left(\frac{\pi^2 t}{2} - \frac{t^3}{2}\right)\hat{\imath} + (t - \sin t)\,\hat{\jmath} + \left(\frac{1}{2}e^{2t} - t\right)\hat{\mathbf{k}} + \mathbf{c}$$

for some (other) constant vector $\mathbf{c}$. We are told that $\mathbf{r}(0) = \mathbf{r}_0 = \frac{1}{2}\hat{\mathbf{k}}$. This forces $\mathbf{c} = \mathbf{0}$ so that

$$\mathbf{r}(t) = \left(\frac{\pi^2 t}{2} - \frac{t^3}{2}\right)\hat{\boldsymbol{\imath}} + (t - \sin t)\,\hat{\boldsymbol{\jmath}} + \left(\frac{1}{2}e^{2t} - t\right)\hat{\mathbf{k}}$$

(b) The particle is in the plane $x = 0$ when

$$0 = \left(\frac{\pi^2 t}{2} - \frac{t^3}{2}\right) = \frac{t}{2}(\pi^2 - t^2) \iff t = 0, \pm\pi$$

So the desired time is $t = \pi$.

(c) At time $t = \pi$, the velocity is

$$\begin{aligned}\mathbf{r}'(\pi) &= \left(\frac{\pi^2}{2} - \frac{3\pi^2}{2}\right)\hat{\boldsymbol{\imath}} + (1 - \cos\pi)\,\hat{\boldsymbol{\jmath}} + \left(e^{2\pi} - 1\right)\hat{\mathbf{k}} \\ &= -\pi^2\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + \left(e^{2\pi} - 1\right)\hat{\mathbf{k}}\end{aligned}$$

$\boxed{1.6.25}$ (∗) Let C be the curve of intersection of the surfaces $y = x^2$ and $z = \frac{2}{3}x^3$. A particle moves along C with constant speed such that $\frac{\mathrm{d}x}{\mathrm{d}t} > 0$. The particle is at $(0, 0, 0)$ at time $t = 0$ and is at $(3, 9, 18)$ at time $t = \frac{7}{2}$.

(a) Find the length of the part of C between $(0, 0, 0)$ and $(3, 9, 18)$.
(b) Find the constant speed of the particle.
(c) Find the velocity of the particle when it is at $\left(1, 1, \frac{2}{3}\right)$.
(d) Find the acceleration of the particle when it is at $\left(1, 1, \frac{2}{3}\right)$.

Solution (a) Parametrize C by x. Since $y = x^2$ and $z = \frac{2}{3}x^3$,

$$\begin{aligned}\mathbf{r}(x) &= x\,\hat{\boldsymbol{\imath}} + x^2\,\hat{\boldsymbol{\jmath}} + \frac{2}{3}x^3\,\hat{\mathbf{k}} \\ \mathbf{r}'(x) &= \hat{\boldsymbol{\imath}} + 2x\,\hat{\boldsymbol{\jmath}} + 2x^2\,\hat{\mathbf{k}} \\ \mathbf{r}''(x) &= 2\,\hat{\boldsymbol{\jmath}} + 4x\,\hat{\mathbf{k}} \\ \frac{\mathrm{d}s}{\mathrm{d}x} &= |\mathbf{r}'(x)| = \sqrt{1 + 4x^2 + 4x^4} = 1 + 2x^2\end{aligned}$$

and

$$\int_C \mathrm{d}s = \int_0^3 \frac{\mathrm{d}s}{\mathrm{d}x}\,\mathrm{d}x = \int_0^3 \left(1 + 2x^2\right)\mathrm{d}x = \left[x + \frac{2}{3}x^3\right]_0^3 = 21$$

(b) The particle travelled a distance of 21 units in $\frac{7}{2}$ time units. This corresponds to a speed of $\frac{21}{7/2} = 6$.

(c) Denote by $\mathbf{R}(t)$ the position of the particle at time t. Then

$$\mathbf{R}(t) = \mathbf{r}\big(x(t)\big) \implies \mathbf{R}'(t) = \mathbf{r}'\big(x(t)\big)\frac{\mathrm{d}x}{\mathrm{d}t}$$

By parts (a) and (b) and the chain rule

$$6 = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}s}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = (1+2x^2)\frac{\mathrm{d}x}{\mathrm{d}t} \implies \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{6}{1+2x^2}$$

In particular, the particle is at $\left(1, 1, \frac{2}{3}\right)$ at $x = 1$. At this time $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{6}{1+2\times 1} = 2$ and

$$\mathbf{R}' = \mathbf{r}'(1)\frac{\mathrm{d}x}{\mathrm{d}t} = \left(\hat{\imath} + 2\,\hat{\jmath} + 2\,\hat{\mathbf{k}}\right)2 = 2\hat{\imath} + 4\,\hat{\jmath} + 4\,\hat{\mathbf{k}}$$

(d) By the product and chain rules,

$$\mathbf{R}'(t) = \mathbf{r}'\big(x(t)\big)\frac{\mathrm{d}x}{\mathrm{d}t} \implies \mathbf{R}''(t) = \mathbf{r}''\big(x(t)\big)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \mathbf{r}'\big(x(t)\big)\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

Applying $\frac{\mathrm{d}}{\mathrm{d}t}$ to $6 = \big(1 + 2x(t)^2\big)\frac{\mathrm{d}x}{\mathrm{d}t}(t)$ gives

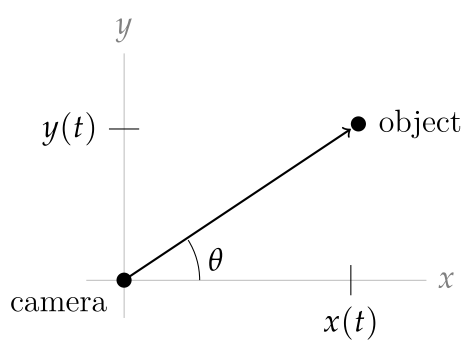
$$0 = 4x\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + (1+2x^2)\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

In particular, when $x = 1$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 2$, $0 = 4 \times 1\big(2\big)^2 + (3)\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ gives $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{16}{3}$ and

$$\mathbf{R}'' = \left(2\,\hat{\jmath} + 4\,\hat{\mathbf{k}}\right)(2)^2 - \left(\hat{\imath} + 2\,\hat{\jmath} + 2\,\hat{\mathbf{k}}\right)\frac{16}{3} = -\frac{8}{3}\left(2\hat{\imath} + \hat{\jmath} - 2\,\hat{\mathbf{k}}\right)$$

1.6.26 A camera mounted to a pole can swivel around in a full circle. It is tracking an object whose position at time t seconds is $x(t)$ metres east of the pole, and $y(t)$ metres north of the pole.
In order to always be pointing directly at the object, how fast should the camera be programmed to rotate at time t? (Give your answer in terms of $x(t)$ and $y(t)$ and their derivatives, in the units rad/sec.)

Solution The question is already set up as an xy–plane, with the camera at the origin, so the vector in the direction the camera is pointing is $(x(t), y(t))$. Let θ be the angle the camera makes with the positive x-axis (due east). The camera, the object, and the due-east direction (positive x-axis) make a right triangle.



$$\tan\theta = \frac{y}{x}$$

Differentiating implicitly with respect to t:

$$\begin{aligned} \sec^2\theta\,\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{xy' - yx'}{x^2} \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \cos^2\theta\left(\frac{xy'-yx'}{x^2}\right) = \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2\left(\frac{xy'-yx'}{x^2}\right) = \frac{xy'-yx'}{x^2+y^2} \end{aligned}$$

$\boxed{1.6.27}$ A projectile falling under the influence of gravity and slowed by air resistance proportional to its speed has position satisfying

$$\frac{d^2\mathbf{r}}{dt^2} = -g\hat{\mathbf{k}} - \alpha\frac{d\mathbf{r}}{dt}$$

where α is a positive constant. If $\mathbf{r} = \mathbf{r}_0$ and $\frac{d\mathbf{r}}{dt} = \mathbf{v}_0$ at time $t = 0$, find $\mathbf{r}(t)$. (Hint: Define $\mathbf{u}(t) = e^{\alpha t}\frac{d\mathbf{r}}{dt}(t)$ and substitute $\frac{d\mathbf{r}}{dt}(t) = e^{-\alpha t}\mathbf{u}(t)$ into the given differential equation to find a differential equation for $\mathbf{u}$.)

$\boxed{\text{Solution}}$ Define $\mathbf{u}(t) = e^{\alpha t}\frac{d\mathbf{r}}{dt}(t)$. Then

$$\begin{aligned} \frac{d\mathbf{u}}{dt}(t) &= \alpha e^{\alpha t}\frac{d\mathbf{r}}{dt}(t) + e^{\alpha t}\frac{d^2\mathbf{r}}{dt^2}(t) \\ &= \alpha e^{\alpha t}\frac{d\mathbf{r}}{dt}(t) - ge^{\alpha t}\hat{\mathbf{k}} - \alpha e^{\alpha t}\frac{d\mathbf{r}}{dt}(t) \\ &= -ge^{\alpha t}\hat{\mathbf{k}} \end{aligned}$$

Integrating both sides of this equation from $t = 0$ to $t = T$ gives

$$\begin{aligned} &\mathbf{u}(T) - \mathbf{u}(0) = -g\frac{e^{\alpha T}-1}{\alpha}\hat{\mathbf{k}} \\ &\implies \mathbf{u}(T) = \mathbf{u}(0) - g\frac{e^{\alpha T}-1}{\alpha}\hat{\mathbf{k}} = \frac{d\mathbf{r}}{dt}(0) - g\frac{e^{\alpha T}-1}{\alpha}\hat{\mathbf{k}} = \mathbf{v}_0 - g\frac{e^{\alpha T}-1}{\alpha}\hat{\mathbf{k}} \end{aligned}$$

Substituting in $\mathbf{u}(T) = e^{\alpha t}\frac{d\mathbf{r}}{dt}(T)$ and multiplying through by $e^{-\alpha T}$

$$\frac{d\mathbf{r}}{dt}(T) = e^{-\alpha T}\mathbf{v}_0 - g\frac{1-e^{-\alpha T}}{\alpha}\hat{\mathbf{k}}$$

Integrating both sides of this equation from $T = 0$ to $T = t$ gives

$$\begin{aligned} \mathbf{r}(t) - \mathbf{r}(0) &= \frac{e^{-\alpha t}-1}{-\alpha}\mathbf{v}_0 - g\frac{t}{\alpha}\hat{\mathbf{k}} + g\frac{e^{-\alpha t}-1}{-\alpha^2}\hat{\mathbf{k}} \\ \implies \mathbf{r}(t) &= \mathbf{r}_0 - \frac{e^{-\alpha t}-1}{\alpha}\mathbf{v}_0 + g\frac{1-\alpha t - e^{-\alpha t}}{\alpha^2}\hat{\mathbf{k}} \end{aligned}$$

$\boxed{1.6.28}$ (*) At time $t = 0$ a particle has position and velocity vectors $\mathbf{r}(0) = \langle -1, 0, 0 \rangle$ and $\mathbf{v}(0) = \langle 0, -1, 1 \rangle$. At time t, the particle has acceleration vector

$$\mathbf{a}(t) = \langle \cos t, \sin t, 0 \rangle$$

(a) Find the position of the particle after t seconds.
(b) Show that the velocity and acceleration of the particle are always perpendicular for every t.
(c) Find the equation of the tangent line to the particle's path at $t = -\pi/2$.
(d) True or False: None of the lines tangent to the path of the particle pass through $(0, 0, 0)$. Justify your answer.

$\boxed{\text{Solution}}$ (a) By definition,

$$\mathbf{v}'(t) = \mathbf{a}(t) = \langle \cos t, \sin t, 0 \rangle \implies \mathbf{v}(t) = \langle \sin t + c_1, -\cos t + c_2, c_3 \rangle$$

for some constants c_1, c_2, c_3. To satisfy $\mathbf{v}(0) = \langle 0, -1, 1 \rangle$, we need $c_1 = 0$, $c_2 = 0$ and $c_3 = 1$. So $\mathbf{v}(t) = \langle \sin t, -\cos t, 1 \rangle$. Similarly,

$$\mathbf{r}'(t) = \mathbf{v}(t) = \langle \sin t, -\cos t, 1 \rangle \implies \mathbf{r}(t) = \langle -\cos t + d_1, -\sin t + d_2, t + d_3 \rangle$$

for some constants d_1, d_2, d_3. To satisfy $\mathbf{r}(0) = \langle -1, 0, 0 \rangle$, we need $d_1 = 0$, $d_2 = 0$ and $d_3 = 0$. So $\mathbf{r}(t) = \langle -\cos t, -\sin t, t \rangle$.

(b) To test for orthogonality, we compute the dot product

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = \langle \sin t, -\cos t, 1 \rangle \cdot \langle \cos t, \sin t, 0 \rangle = \sin t \cos t - \cos t \sin t + 1 \times 0 = 0$$

so $\mathbf{v}(t) \perp \mathbf{a}(t)$ for all t.

(c) At $t = -\frac{\pi}{2}$ the particle is at $\mathbf{r}\big(-\frac{\pi}{2}\big) = \big\langle 0, 1, -\frac{\pi}{2} \big\rangle$ and has velocity $\mathbf{v}\big(-\frac{\pi}{2}\big) = \langle -1, 0, 1 \rangle$. So the tangent line must pass through $\big\langle 0, 1, -\frac{\pi}{2} \big\rangle$ and have direction vector $\langle -1, 0, 1 \rangle$. Here is a vector parametric equation for the tangent line.

$$\mathbf{r}(u) = \left\langle 0, 1, -\frac{\pi}{2} \right\rangle + u \langle -1, 0, 1 \rangle$$

(d) True. Look at the path followed by the particle from the top so that we only see x and y coordinates. The path we see (call this the projected path) is $x(t) = -\cos t$, $y(t) = -\sin t$, which is a circle of radius one centred on the origin. Any tangent line to any circle always remains outside the circle. So no tangent line to the projected path can pass through the $(0, 0)$. So no tangent line to the path followed by the particle can pass through the z–axis and, in particular, through $(0, 0, 0)$.

$\boxed{1.6.29}$ (*) The position of a particle at time t (measured in seconds s) is given by

$$\mathbf{r}(t) = t\cos\left(\frac{\pi t}{2}\right)\hat{\imath} + t\sin\left(\frac{\pi t}{2}\right)\hat{\jmath} + t\,\hat{\mathbf{k}}$$

(a) Show that the path of the particle lies on the cone $z^2 = x^2 + y^2$.
(b) Find the velocity vector and the speed at time t.
(c) Suppose that at time $t = 1$s the particle flies off the path on a line L in the direction tangent to the path. Find the equation of the line L.
(d) How long does it take for the particle to hit the plane $x = -1$ after it started moving along the straight line L?

$\boxed{\text{Solution}}$ (a) Since

$$x(t)^2 + y(t)^2 = t^2\cos^2\left(\tfrac{\pi t}{2}\right) + t^2\sin^2\left(\tfrac{\pi t}{2}\right) = t^2 \qquad \text{and} \qquad z(t)^2 = t^2$$

are the same, the path of the particle lies on the cone $z^2 = x^2 + y^2$.

(b) By definition,

$$\begin{aligned}
\text{velocity} = \mathbf{r}'(t) &= \left[\cos\left(\tfrac{\pi t}{2}\right) - \tfrac{\pi t}{2}\sin\left(\tfrac{\pi t}{2}\right)\right]\hat{\imath} + \left[\sin\left(\tfrac{\pi t}{2}\right) + \tfrac{\pi t}{2}\cos\left(\tfrac{\pi t}{2}\right)\right]\hat{\jmath} + \hat{\mathbf{k}} \\
\text{speed} = |\mathbf{r}'(t)| &= \sqrt{\left[\cos\left(\tfrac{\pi t}{2}\right) - \tfrac{\pi t}{2}\sin\left(\tfrac{\pi t}{2}\right)\right]^2 + \left[\sin\left(\tfrac{\pi t}{2}\right) + \tfrac{\pi t}{2}\cos\left(\tfrac{\pi t}{2}\right)\right]^2 + 1^2} \\
&= \Big[\cos^2\left(\tfrac{\pi t}{2}\right) - 2\tfrac{\pi t}{2}\cos\left(\tfrac{\pi t}{2}\right)\sin\left(\tfrac{\pi t}{2}\right) + \left(\tfrac{\pi t}{2}\right)^2\sin^2\left(\tfrac{\pi t}{2}\right) \\
&\qquad + \sin^2\left(\tfrac{\pi t}{2}\right) + 2\tfrac{\pi t}{2}\cos\left(\tfrac{\pi t}{2}\right)\sin\left(\tfrac{\pi t}{2}\right) + \left(\tfrac{\pi t}{2}\right)^2\cos^2\left(\tfrac{\pi t}{2}\right) + 1\Big]^{1/2} \\
&= \sqrt{2 + \frac{\pi^2 t^2}{4}}
\end{aligned}$$

(c) At $t = 1$, the particle is at $\mathbf{r}(1) = (0, 1, 1)$ and has velocity $\mathbf{r}'(1) = \left\langle -\frac{\pi}{2}, 1, 1\right\rangle$. So for $t \geqslant 1$, the particle is at

$$\langle x, y, z\rangle = \langle 0, 1, 1\rangle + (t-1)\left\langle -\frac{\pi}{2}, 1, 1\right\rangle$$

This is also a vector parametric equation for the line.

(d) Assume that the particle's speed remains constant as it flies along L. Then the x-coordinate of the particle at time t (for $t \geqslant 1$) is $-\frac{\pi}{2}(t-1)$. This takes the value -1 when $t - 1 = \frac{2}{\pi}$. So the particle hits $x = -1$, $\frac{2}{\pi}$ seconds after it flew off the cone.

$\boxed{1.6.30}$ (*)
(a) The curve $\mathbf{r}_1(t) = \left\langle 1+t, t^2, t^3\right\rangle$ and $\mathbf{r}_2(t) = \langle \cos t, \sin t, t\rangle$ intersect at the point $P(1, 0, 0)$. Find the angle of intersection between the curves at the point P.
(b) Find the distance between the line of intersection of the planes $x + y - z = 4$ and $2x - z = 4$ and the line $\mathbf{r}(t) = \langle t, -1+2t, 1+3t\rangle$.

$\boxed{\text{Solution}}$ (a) The tangent vectors to the two curves are

$$\mathbf{r}_1'(t) = \left\langle 1, 2t, 3t^2 \right\rangle \qquad \mathbf{r}_2'(t) = \langle -\sin t, \cos t, 1\rangle$$

Both curves pass through P at $t = 0$ and then the tangent vectors are

$$\mathbf{r}_1'(0) = \langle 1, 0, 0\rangle \qquad \mathbf{r}_2'(0) = \langle 0, 1, 1\rangle$$

So the angle of intersection, θ, is determined by

$$\begin{aligned}\mathbf{r}_1'(0)\cdot\mathbf{r}_2'(0) = |\mathbf{r}_1'(0)|\,|\mathbf{r}_2'(0)|\,\cos\theta &\implies \langle 1,0,0\rangle\cdot\langle 0,1,1\rangle = 1\cdot\sqrt{2}\cdot\cos\theta\\ &\implies \cos\theta = 0 \implies \theta = 90^\circ\end{aligned}$$

(b) Our strategy will be to

- find a vector $\mathbf{v}$ whose tail is on one line and whose head is on the other line and then
- find a vector $\mathbf{n}$ that is perpendicular to both lines.
- Then, if we denote by θ the angle between $\mathbf{v}$ and $\mathbf{n}$, the distance between the two lines is $|\mathbf{v}|\cos\theta = \frac{|\mathbf{v}\cdot\mathbf{n}|}{|\mathbf{n}|}$

Here we go

- So the first step is to find a $\mathbf{v}$.
  - One point on the line $\mathbf{r}(t) = \langle t, -1+2t, 1+3t\rangle$ is $\mathbf{r}(0) = \langle 0, -1, 1\rangle$.
  - (x, y, z) is on the other line if and only if $x + y - z = 4$ and $2x - z = 4$. In particular, if $z = 0$ then $x + y = 4$ and $2x = 4$ so that $x = 2$ and $y = 2$.
  - So the vector $\mathbf{v} = \langle 2-0\,,\, 2-(-1)\,,\, 0-1\rangle = \langle 2, 3, -1\rangle$ has its head on one line and its tail on the other line.
- Next we find a vector $\mathbf{n}$ that is perpendicular to both lines.
  - First we find a direction vector for the line $x + y - z = 4$, $2x - z = 4$. We already know that $x = y = 2$, $z = 0$ is on that line. We can find a second point on that line by choosing, for example, $z = 2$ and then solving $x + y = 6$, $2x = 6$ to get $x = 3$, $y = 3$. So one direction vector for the line $x + y - z = 4$, $2x - z = 4$ is $\mathbf{d}_1 = \langle 3-2\,,\, 3-2\,,\, 2-0\rangle = \langle 1, 1, 2\rangle$.
  - A second way to get a direction vector for the line $x + y - z = 4$, $2x - z = 4$ is to observe that $\langle 1, 1, -1\rangle$ is normal to $x + y - z = 4$ and so is perpendicular to the line and $\langle 2, 0, -1\rangle$ is normal to $2x - z = 4$ and so is also perpendicular to the line. So $\langle 1, 1, -1\rangle \times \langle 2, 0, -1\rangle$ is a direction vector for the line.
  - A direction vector for the line $\mathbf{r}(t) = \langle t, -1+2t, 1+3t\rangle$ is $\mathbf{d}_2 = \mathbf{r}'(t) = \langle 1, 2, 3\rangle$.
  - So
$$\mathbf{n} = \mathbf{d}_2 \times \mathbf{d}_1 = \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ 1 & 2 & 3\\ 1 & 1 & 2\end{bmatrix} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}}$$
    is perpendicular to both lines.

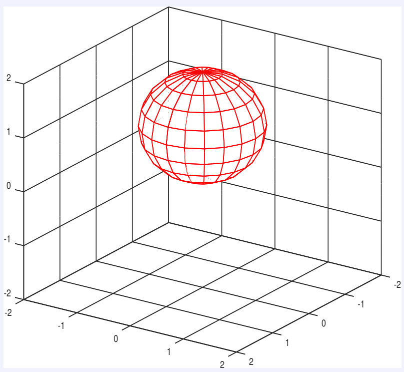
The distance between the two lines is then

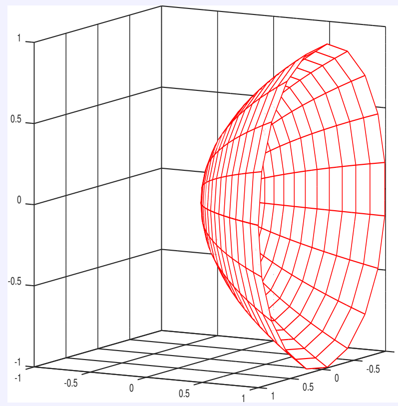
$$|\mathbf{v}|\cos\theta = \frac{|\mathbf{v}\cdot\mathbf{n}|}{|\mathbf{n}|} = \frac{\langle 2,3,-1\rangle\cdot\langle 1,1,-1\rangle}{|\langle 1,1,-1\rangle|} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

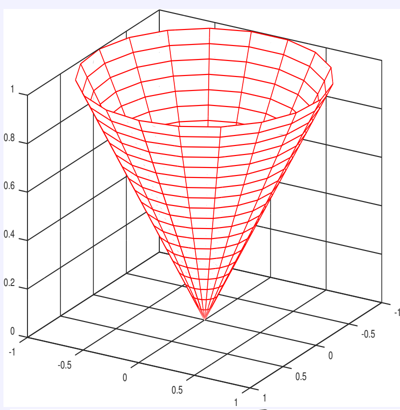
1.7▲ Sketching Surfaces in 3d

▸▸ Stage 1

$\boxed{1.7.1}$ (∗) Match the following equations and expressions with the corresponding pictures. Cartesian coordinates are (x, y, z), cylindrical coordinates are (r, θ, z), and spherical coordinates are (ρ, θ, φ).

(A)


(B)


(C)


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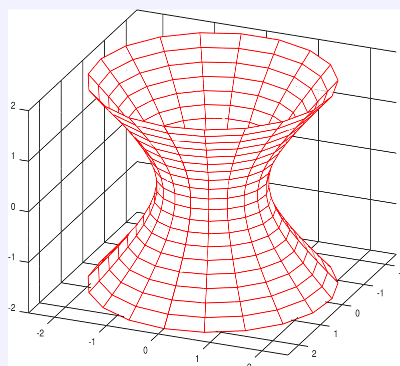

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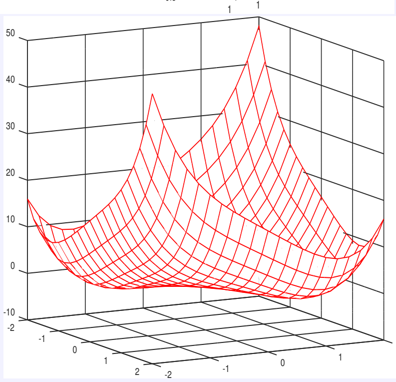
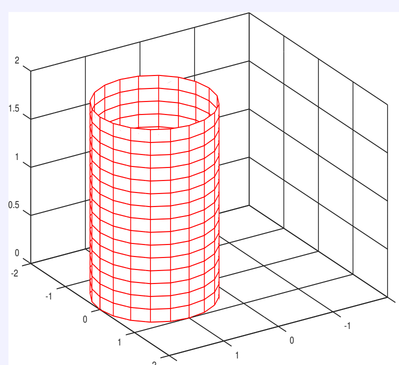


(F)


(a) $\varphi = \pi/3$ (b) $r = 2\cos\theta$ (c) $x^2 + y^2 = z^2 + 1$

(d) $y = x^2 + z^2$ (e) $\rho = 2\cos\varphi$ (f) $z = x^4 + y^4 - 4xy$

Solution (a) $\varphi = \frac{\pi}{3}$ is a surface of constant (spherical coordinate) φ. So it is a cone with vertex at the origin. We can express $\varphi = \frac{\pi}{3}$ in cartesian coordinates by observing that $0 \leqslant \varphi \leqslant \frac{\pi}{2}$ so that $z \geqslant 0$, and

$$\varphi = \frac{\pi}{3} \iff \tan\varphi = \frac{\sqrt{3}}{2} \iff \rho\sin\varphi = \frac{\sqrt{3}}{2}\rho\cos\varphi \iff \sqrt{x^2+y^2} = \frac{\sqrt{3}}{2}z$$

So the picture that corresponds to (a) is (C).

(b) As r and θ are cylindrical coordinates

$$r = 2\cos\theta \iff r^2 = 2r\cos\theta \iff x^2+y^2 = 2x \iff (x-1)^2+y^2 = 1$$

There is no z appearing in $(x-1)^2+y^2=1$. So every constant z cross–section of $(x-1)^2+y^2=1$ is a (horizontal) circle of radius 1 centred on the line $x=1$, $y=0$. It is a cylinder of radius 1 centred on the line $x=1$, $y=0$. So the picture that corresponds to (b) is (F).

(c) Each constant z cross–section of $x^2+y^2=z^2+1$ is a (horizontal) circle centred on the z–axis. The radius of the circle is 1 when $z=0$ and grows as z moves away from $z=0$. So $x^2+y^2=z^2+1$ consists of a bunch of (horizontal) circles stacked on top of each other, with the radius increasing with $|z|$. It is a hyperboloid of one sheet. The picture that corresponds to (c) is (D).

(d) Every point of $y=x^2+z^2$ has $y \geqslant 0$. Only (B) has that property. We can also observe that every constant y cross–section is a circle centred on $x=z=0$. The radius of the circle is zero when $y=0$ and increases as y increases. The surface $y=x^2+z^2$ is a paraboloid. The picture that corresponds to (d) is (B).

(e) As ρ and φ are spherical coordinates

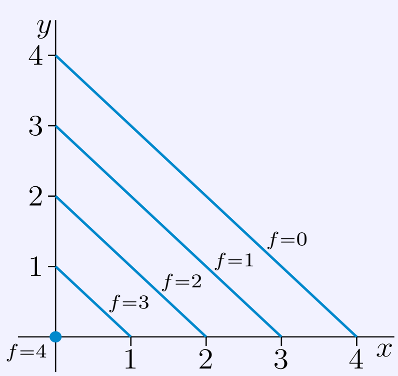
$$\rho = 2\cos\varphi \iff \rho^2 = 2\rho\cos\varphi \iff x^2+y^2+z^2 = 2z \iff x^2+y^2+(z-1)^2 = 1$$

This is the sphere of radius 1 centred on $(0,0,1)$. The picture that corresponds to (e) is (A).

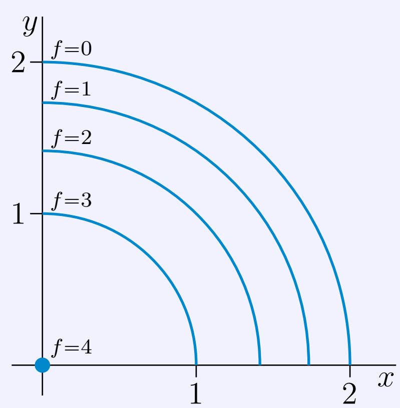
(f) The only possibility left is that the picture that corresponds to (f) is (E).

1.7.2 In each of (a) and (b) below, you are provided with a sketch of the first quadrant parts of a few level curves of some function $f(x,y)$. Sketch the first octant part of the corresponding graph $z=f(x,y)$.

(a)



(b)



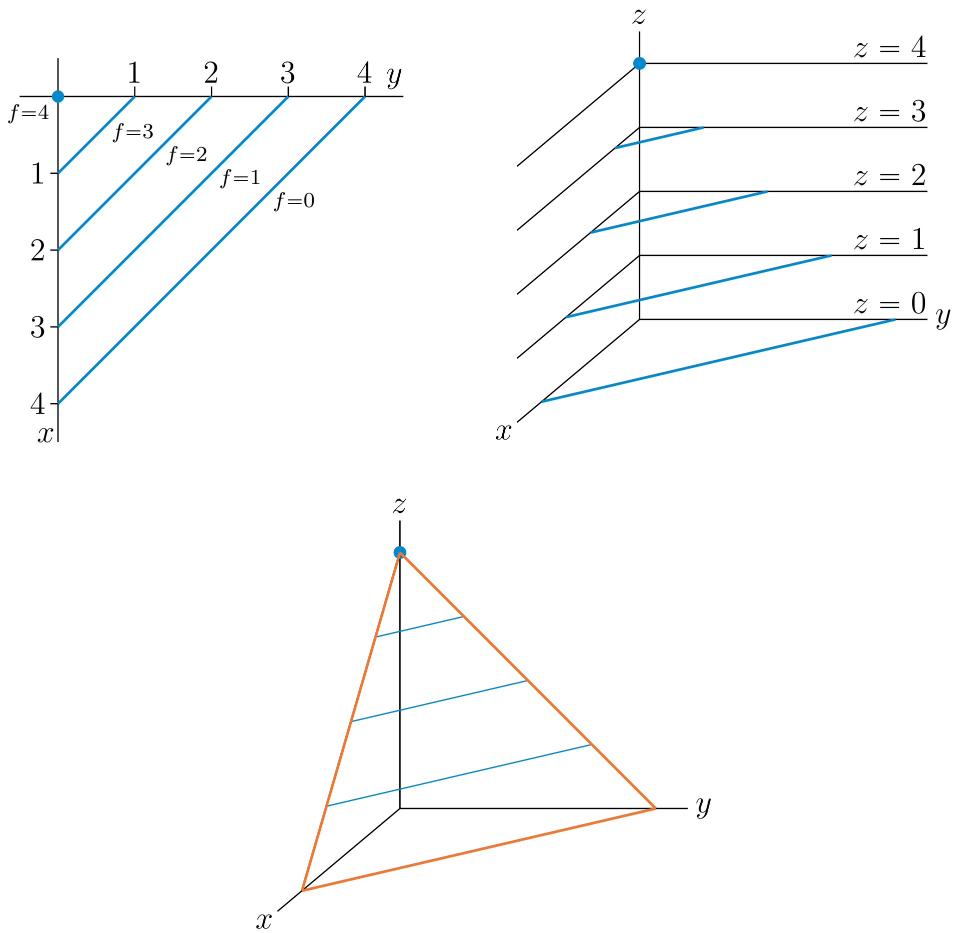
Solution Each solution below consists of three sketchs.

- In the first sketch, we just redraw the given level curves with the x- and y-axes reoriented so that the sketch looks like we are high on the z-axis looking down at the xy-plane.
- In the second sketch, we lift up each level curve $f(x,y) = C$ and draw it in the horizontal plane $z = C$. That is we draw

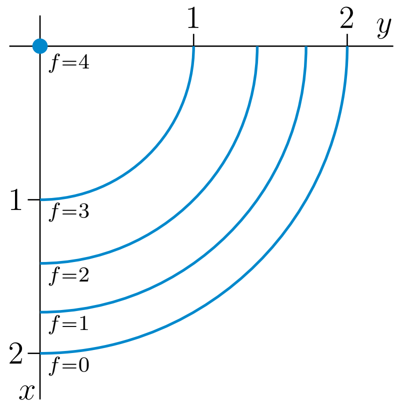
$$\begin{aligned}&\{\ (x,y,z) \mid f(x,y) = C,\ z = C,\ x \geqslant 0, y \geqslant 0\ \}\\ &\qquad = \{\ (x,y,z) \mid z = f(x,y),\ z = C,\ x \geqslant 0, y \geqslant 0\ \}\end{aligned}$$

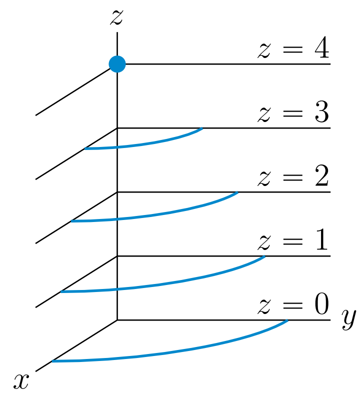
- Finally, in the third sketch, we draw the part of graph $z = f(x,y)$ in the first octant, just by "filling in the gaps in the second sketch".

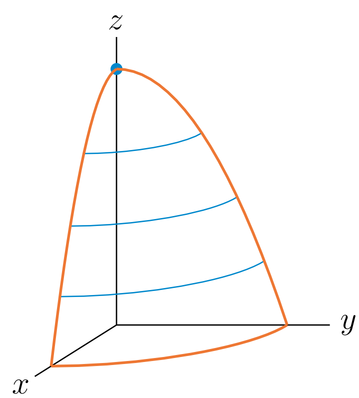
(a)



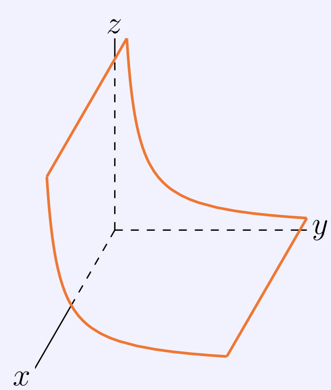
(b)



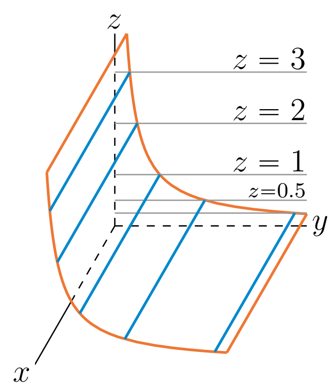




$\boxed{1.7.3}$ Sketch a few level curves for the function $f(x,y)$ whose graph $z = f(x,y)$ is sketched below.



$\boxed{\text{Solution}}$ We first add into the sketch of the graph the horizontal planes $z = C$, for $C = 3$, 2, 1, 0.5, 0.25.



To reduce clutter, for each C, we have drawn in only

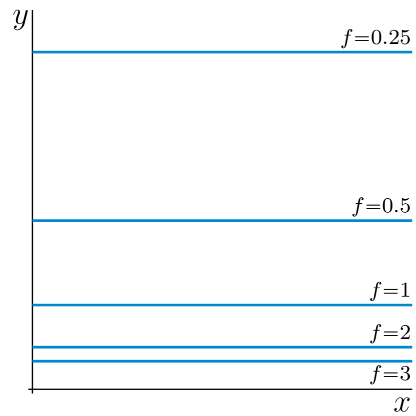
- the (gray) intersection of the horizontal plane $z = C$ with the yz–plane, i.e. with the vertical plane $x = 0$, and
- the (blue) intersection of the horizontal plane $z = C$ with the graph $z = f(x,y)$.

We have also omitted the label for the plane $z = 0.25$.

The intersection of the plane $z = C$ with the graph $z = f(x,y)$ is line

$$\left\{ \ (x,y,z) \ \middle| \ z = f(x,y), \ z = C \ \right\} = \left\{ \ (x,y,z) \ \middle| \ f(x,y) = C, \ z = C \ \right\}$$

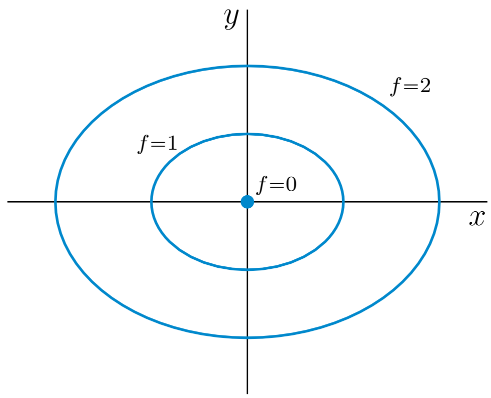
Drawing this line (which is parallel to the x-axis) in the xy-plane, rather than in the plane $z = C$, gives a level curve. Doing this for each of $C = 3$, 2, 1, 0.5, 0.25 gives five level curves.



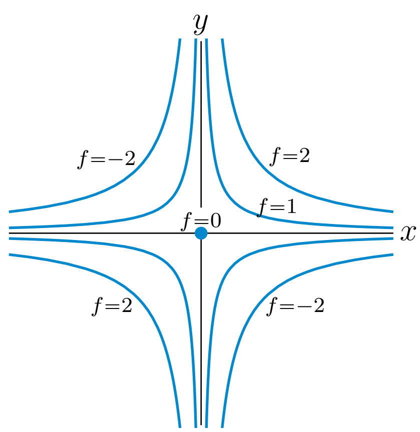
▸▸ Stage 2

$\boxed{1.7.4}$ Sketch some of the level curves of
(a) $f(x,y) = x^2 + 2y^2$
(b) $f(x,y) = xy$
(c) $f(x,y) = xe^{-y}$

Solution (a) For each fixed $c > 0$, the level curve $x^2 + 2y^2 = c$ is the ellipse centred on the origin with x semi axis $\sqrt{c}$ and y semi axis $\sqrt{c/2}$. If $c = 0$, the level curve $x^2 + 2y^2 = c = 0$ is the single point $(0,0)$.



(b) For each fixed $c \neq 0$, the level curve $xy = c$ is a hyperbola centred on the origin with asymptotes the x- and y-axes. If $c > 0$, any x and y obeying $xy = c > 0$ are of the same sign. So the hyperbola is contained in the first and third quadrants. If $c < 0$, any x and y obeying $xy = c > 0$ are of opposite sign. So the hyperbola is contained in the second and fourth quadrants. If $c = 0$, the level curve $xy = c = 0$ is the single point $(0, 0)$.



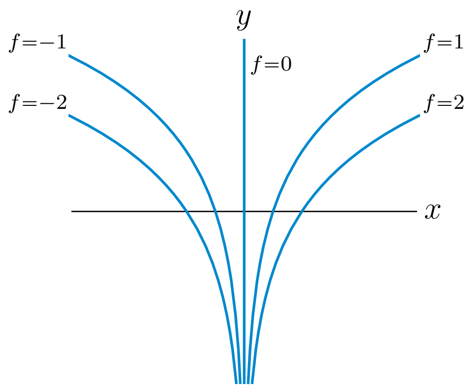
(c) For each fixed $c \neq 0$, the level curve $xe^{-y} = c$ is the logarithmic curve $y = -\ln \frac{c}{x}$. Note that, for $c > 0$, the curve

- is restricted to $x > 0$, so that $\frac{c}{x} > 0$ and $\ln \frac{c}{x}$ is defined, and that
- as $x \to 0^+$, y goes to $-\infty$, while
- as $x \to +\infty$, y goes to $+\infty$, and
- the curve crosses the x-axis (i.e. has $y = 0$) when $x = c$.

and for $c < 0$, the curve

- is restricted to $x < 0$, so that $\frac{c}{x} > 0$ and $\ln \frac{c}{x}$ is defined, and that
- as $x \to 0^-$, y goes to $-\infty$, while
- as $x \to -\infty$, y goes to $+\infty$, and
- the curve crosses the x-axis (i.e. has $y = 0$) when $x = c$.

If $c = 0$, the level curve $xe^{-y} = c = 0$ is the y-axis, $x = 0$.

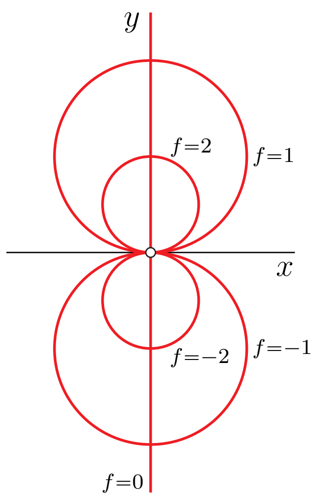


$\boxed{1.7.5}$ (*) Sketch the level curves of $f(x,y) = \frac{2y}{x^2+y^2}$.

$\boxed{\text{Solution}}$ If $C = 0$, the level curve $f = C = 0$ is just the line $y = 0$. If $C \neq 0$ (of either sign), we may rewrite the equation, $f(x,y) = \frac{2y}{x^2+y^2} = C$, of the level curve $f = C$ as

$$x^2 - \frac{2}{C}y + y^2 = 0 \iff x^2 + \left(y - \frac{1}{C}\right)^2 = \frac{1}{C^2}$$

which is the equation of the circle of radius $\frac{1}{|C|}$ centred on $\left(0\,,\,\frac{1}{C}\right)$.

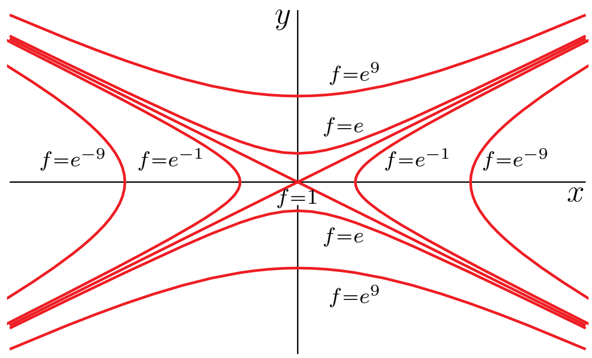


Remark. To be picky, the function $f(x,y) = \frac{2y}{x^2+y^2}$ is not defined at $(x,y) = (0,0)$. The question should have either specified that the domain of f excludes $(0,0)$ or have specified a value for $f(0,0)$. In fact, it is impossible to assign a value to $f(0,0)$ in such a way that $f(x,y)$ is continuous at $(0,0)$, because $\lim_{x\to 0} f(x,0) = 0$ while $\lim_{y\to 0} f(0,|y|) = \infty$. So it makes more sense to have the domain of f being $\mathbb{R}^2$ with the point $(0,0)$ removed. That's why there is a little hole at the origin in the above sketch.

$\boxed{1.7.6}$ (*) Draw a "contour map" of $f(x,y) = e^{-x^2+4y^2}$, showing all types of level curves that occur.

$\boxed{\text{Solution}}$ Observe that, for any constant C, the curve $-x^2 + 4y^2 = C$ is the level curve $f = e^C$.

- If $C = 0$, then $-x^2 + 4y^2 = C = 0$ is the pair of lines $y = \pm\frac{x}{2}$.
- If $C > 0$, then $-x^2 + 4y^2 = C > 0$ is the hyperbola $y = \pm\frac{1}{2}\sqrt{C + x^2}$.
- If $C < 0$, then $-x^2 + 4y^2 = C < 0$ is the hyperbola $x = \pm\sqrt{|C| + 4y^2}$.



$\boxed{1.7.7}$ (∗) A surface is given implicitly by

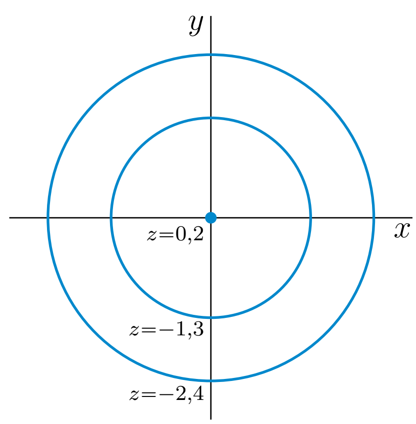
$$x^2 + y^2 - z^2 + 2z = 0$$

(a) Sketch several level curves $z =$constant.
(b) Draw a rough sketch of the surface.

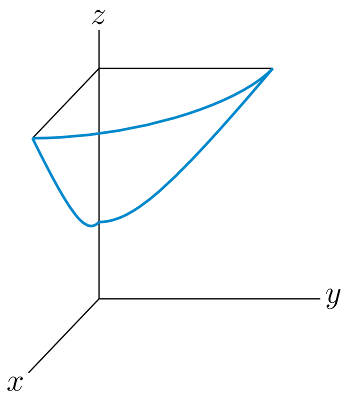
$\boxed{\text{Solution}}$ (a) We can rewrite the equation as

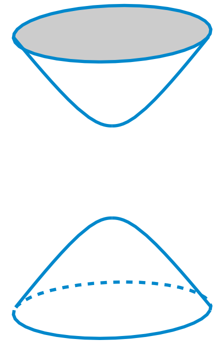
$$x^2 + y^2 = (z - 1)^2 - 1$$

The right hand side is negative for $|z - 1| < 1$, i.e. for $0 < z < 2$. So no point on the surface has $0 < z < 2$. For any fixed z, outside that range, the curve $x^2 + y^2 = (z - 1)^2 - 1$ is the circle of radius $\sqrt{(z - 1)^2 - 1}$ centred on the z–axis. That radius is 0 when $z = 0, 2$ and increases as z moves away from $z = 0, 2$. For very large $|z|$, the radius increases roughly linearly with $|z|$. Here is a sketch of some level curves.



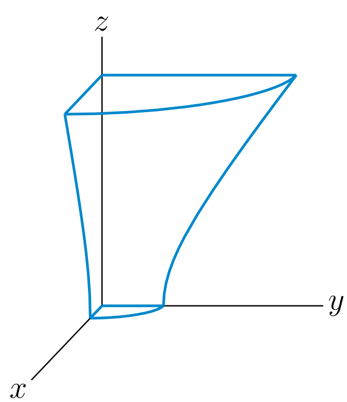
(b) The surface consists of two stacks of circles. One stack starts with radius 0 at $z = 2$. The radius increases as z increases. The other stack starts with radius 0 at $z = 0$. The radius increases as z decreases. This surface is a hyperboloid of two sheets. Here are two sketchs. The sketch on the left is of the part of the surface in the first octant. The sketch on the right of the full surface.

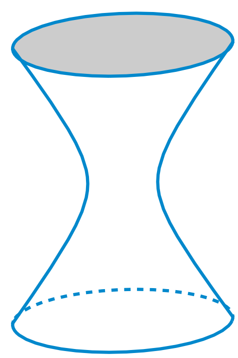




$\boxed{1.7.8}$ ($*$) Sketch the hyperboloid $z^2 = 4x^2 + y^2 - 1$.

$\boxed{\text{Solution}}$ For each fixed z, $4x^2 + y^2 = 1 + z^2$ is an ellipse. So the surface consists of a stack of ellipses one on top of the other. The semi axes are $\frac{1}{2}\sqrt{1+z^2}$ and $\sqrt{1+z^2}$. These are smallest when $z = 0$ (i.e. for the ellipse in the xy-plane) and increase as $|z|$ increases. The intersection of the surface with the xz-plane (i.e. with the plane $y = 0$) is the hyperbola $4x^2 - z^2 = 1$ and the intersection with the yz-pane (i.e. with the plane $x = 0$) is the hyperbola $y^2 - z^2 = 1$. Here are two sketches of the surface. The sketch on the left only shows the part of the surface in the first octant (with axes).





$\boxed{1.7.9}$ Describe the level surfaces of
(a) $f(x, y, z) = x^2 + y^2 + z^2$
(b) $f(x, y, z) = x + 2y + 3z$
(c) $f(x, y, z) = x^2 + y^2$

$\boxed{\text{Solution}}$ (a) If $c > 0$, $f(x, y, z) = c$, i.e. $x^2 + y^2 + z^2 = c$, is the sphere of radius $\sqrt{c}$ centered at the origin. If $c = 0$, $f(x, y, z) = c$ is just the origin. If $c < 0$, no (x, y, z) satisfies $f(x, y, z) = c$.

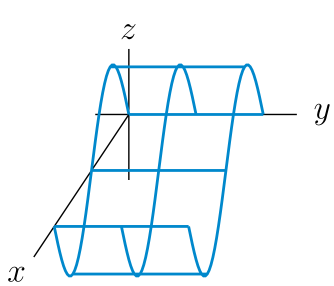
(b) $f(x, y, z) = c$, i.e. $x + 2y + 3z = c$, is the plane normal to $(1, 2, 3)$ passing through

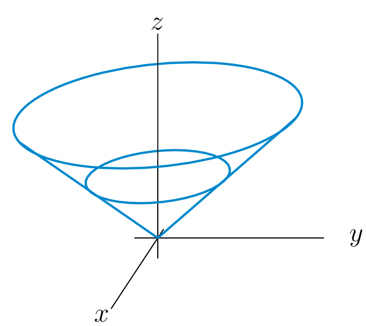
$(c, 0, 0)$.

(c) If $c > 0$, $f(x,y,z) = c$, i.e. $x^2 + y^2 = c$, is the cylinder parallel to the z-axis whose cross-section is a circle of radius $\sqrt{c}$ that is parallel to the xy-plane and is centered on the z-axis. If $c = 0$, $f(x,y,z) = c$ is the z-axis. If $c < 0$, no (x,y,z) satisfies $f(x,y,z) = c$.

1.7.10 Sketch the graphs of
(a) $f(x,y) = \sin x \qquad 0 \leqslant x \leqslant 2\pi,\ 0 \leqslant y \leqslant 1$
(b) $f(x,y) = \sqrt{x^2 + y^2}$
(c) $f(x,y) = |x| + |y|$

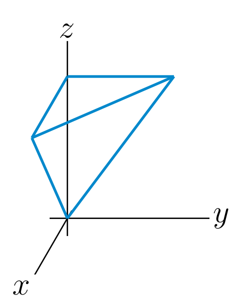
Solution (a) The graph is $z = \sin x$ with (x,y) running over $0 \leqslant x \leqslant 2\pi$, $0 \leqslant y \leqslant 1$. For each fixed y_0 between 0 and 1, the intersection of this graph with the vertical plane $y = y_0$ is the same sin graph $z = \sin x$ with x running from 0 to 2π. So the whole graph is just a bunch of 2-d sin graphs stacked side-by-side. This gives the graph on the left below.

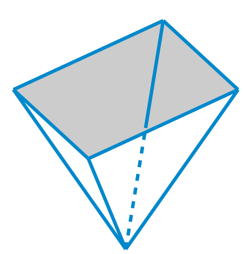




(b) The graph is $z = \sqrt{x^2 + y^2}$. For each fixed $z_0 \geqslant 0$, the intersection of this graph with the horizontal plane $z = z_0$ is the circle $\sqrt{x^2 + y^2} = z_0$. This circle is centred on the z-axis and has radius z_0. So the graph is the upper half of a cone. It is the sketch on the right above.

(c) The graph is $z = |x| + |y|$. For each fixed $z_0 \geqslant 0$, the intersection of this graph with the horizontal plane $z = z_0$ is the square $|x| + |y| = z_0$. The side of the square with $x, y \geqslant 0$ is the straight line $x + y = z_0$. The side of the square with $x \geqslant 0$ and $y \leqslant 0$ is the straight line $x - y = z_0$ and so on. The four corners of the square are $(\pm z_0, 0, z_0)$ and $(0, \pm z_0, z_0)$. So the graph is a stack of squares. It is an upside down four-sided pyramid. The part of the pyramid in the first octant (that is, $x, y, z \geqslant 0$) is the sketch below.





1.7.11 Sketch and describe the following surfaces.

(a) $4x^2 + y^2 = 16$

(b) $x + y + 2z = 4$

(c) $\frac{y^2}{9} + \frac{z^2}{4} = 1 + \frac{x^2}{16}$

(d) $y^2 = x^2 + z^2$

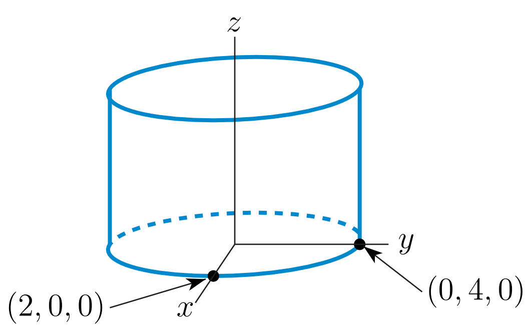
(e) $\frac{x^2}{9} + \frac{y^2}{12} + \frac{z^2}{9} = 1$

(f) $x^2 + y^2 + z^2 + 4x - by + 9z - b = 0$ where b is a constant.

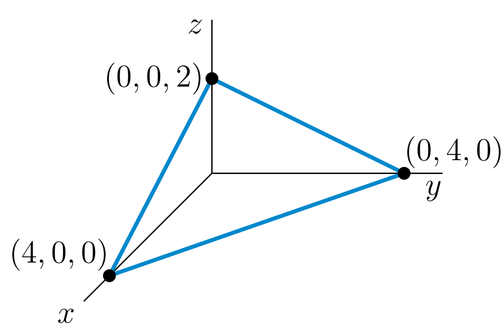
(g) $\frac{x}{4} = \frac{y^2}{4} + \frac{z^2}{9}$

(h) $z = x^2$

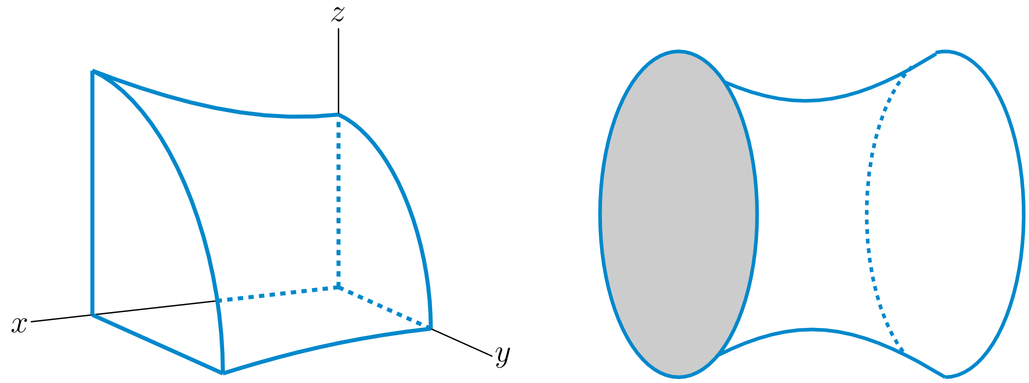
Solution (a) For each fixed z_0, the $z = z_0$ cross-section (parallel to the xy-plane) of this surface is an ellipse centered on the origin with one semiaxis of length 2 along the x-axis and one semiaxis of length 4 along the y-axis. So this is an elliptic cylinder parallel to the z-axis. Here is a sketch of the part of the surface above the xy–plane.



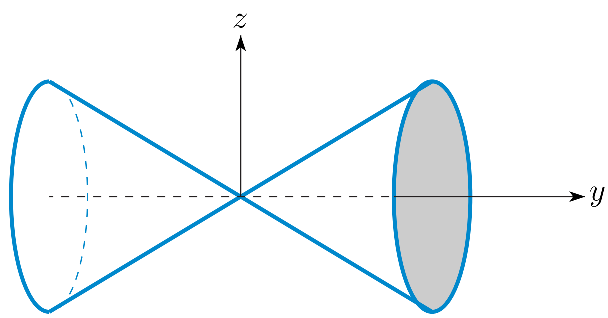
(b) This is a plane through $(4, 0, 0)$, $(0, 4, 0)$ and $(0, 0, 2)$. Here is a sketch of the part of the plane in the first octant.



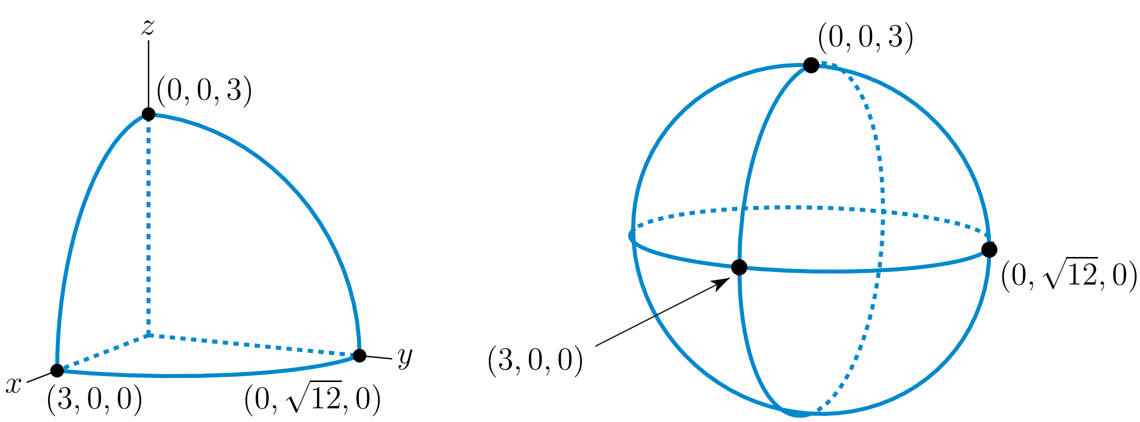
(c) For each fixed x_0, the $x = x_0$ cross-section parallel to the yz-plane is an ellipse with semiaxes $3\sqrt{1 + \frac{x_0^2}{16}}$ parallel to the y-axis and $2\sqrt{1 + \frac{x_0^2}{16}}$ parallel to the z-axis. As you move out along the x-axis, away from $x = 0$, the ellipses grow at a rate proportional to $\sqrt{1 + \frac{x^2}{16}}$, which for large x is approximately $\frac{|x|}{4}$. This is called a hyperboloid of one sheet. Its



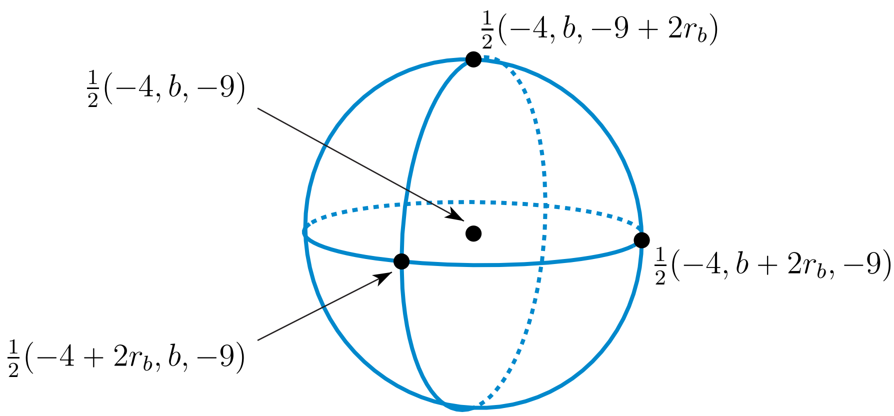
(d) For each fixed y_0, the $y = x_0$ cross-section (parallel to the xz-plane) is a circle of radius $|y|$ centred on the y-axis. When $y_0 = 0$ the radius is 0. As you move further from the xz-plane, in either direction, i.e. as $|y_0|$ increases, the radius grows linearly. The full surface consists of a bunch of these circles stacked sideways. This is a circular cone centred on the y-axis.



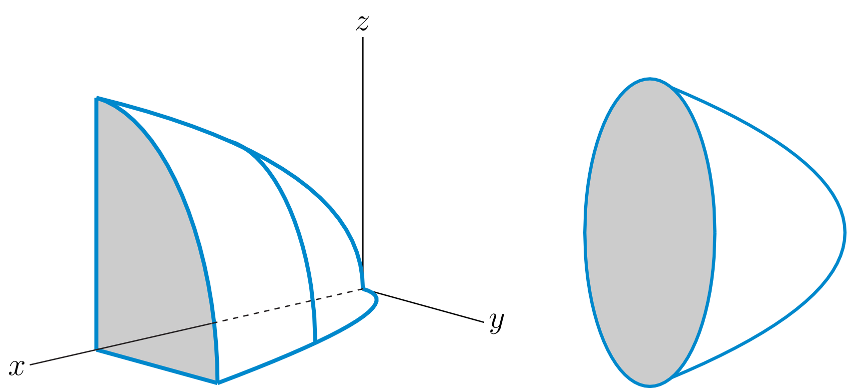
(e) This is an ellipsoid centered on the origin with semiaxes 3, $\sqrt{12} = 2\sqrt{3}$ and 3 along the x, y and z-axes, respectively.



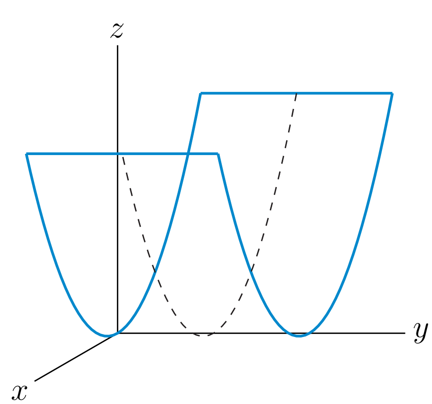
(f) Completing three squares, we have that $x^2 + y^2 + z^2 + 4x - by + 9z - b = 0$ if and only if $(x+2)^2 + \left(y - \frac{b}{2}\right)^2 + \left(z + \frac{9}{2}\right)^2 = b + 4 + \frac{b^2}{4} + \frac{81}{4}$. This is a sphere of radius $r_b = \frac{1}{2}\sqrt{b^2 + 4b + 97}$ centered on $\frac{1}{2}(-4, b, -9)$.



(g) There are no points on the surface with $x < 0$. For each fixed $x_0 > 0$ the cross-section $x = x_0$ parallel to the yz-plane is an ellipse centred on the x–axis with semiaxes $\sqrt{x_0}$ in the y-axis direction and $\frac{3}{2}\sqrt{x_0}$ in the z–axis direction. As you increase x_0, i.e. move out along the x-axis, the ellipses grow at a rate proportional to $\sqrt{x_0}$. This is an elliptic paraboloid with axis the x-axis.

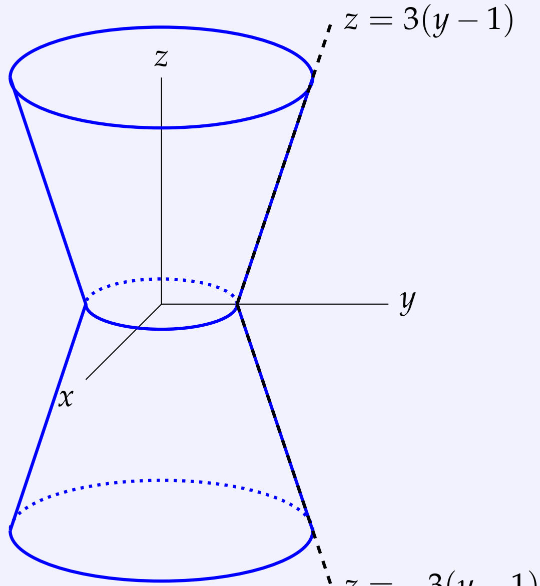


(h) This is called a parabolic cylinder. For any fixed y_0, the $y = y_0$ cross-section (parallel to the xz-plane) is the upward opening parabola $z = x^2$ which has vertex on the y-axis.



▸▸ Stage 3

1.7.12 The surface below has circular level curves, centred along the z-axis. The lines given are the intersection of the surface with the right half of the yz-plane. Give an equation for the surface.

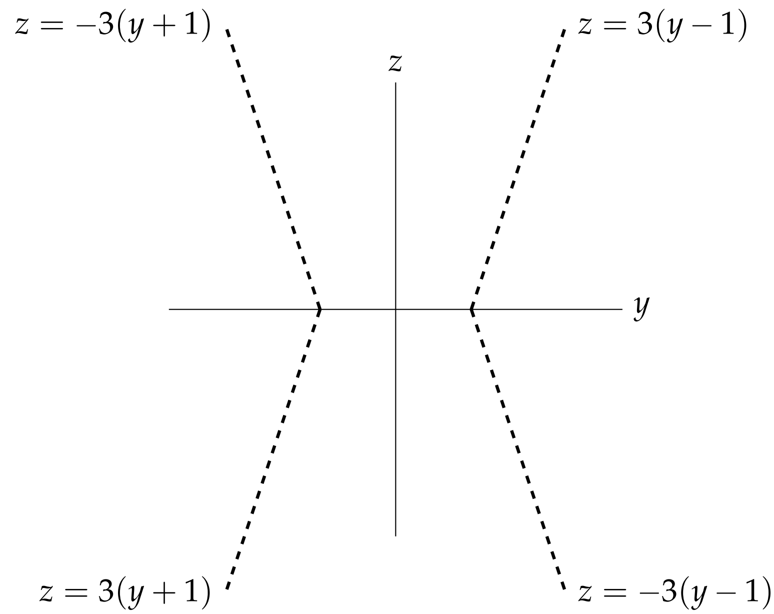


Solution Since the level curves are circles centred at the origin (in the xy-plane), when z is a constant, the equation will have the form $x^2 + y^2 = c$ for some constant. That is, our equation looks like

$$x^2 + y^2 = g(z),$$

where $g(z)$ is a function depending only on z.

Because our cross-sections are so nicely symmetric, we know the intersection of the figure with the left side of the yz-plane as well: $z = 3(-y-1) = -3(y+1)$ (when $z \geqslant 0$) and $z = -3(-y-1) = 3(y+1)$ (when $z < 0$). Below is the intersection of our surface with the yz plane.



Setting $x=0$, our equation becomes $y^2=g(z)$. Looking at the right side of the yz plane, this should lead to: $\left\{\begin{matrix} z=3(y-1) & \text{if } z\geqslant 0,\ y\geqslant 1 \\ z=-3(y-1) & \text{if } z<0,\ y\geqslant 1\end{matrix}\right\}$. That is:

$$\begin{aligned} |z| &= 3(y-1) \\ \frac{|z|}{3}+1 &= y \\ \left(\frac{|z|}{3}+1\right)^2 &= y^2 \qquad\qquad (*) \end{aligned}$$

A quick check: when we squared both sides of the equation in $(*)$, we added another solution, $\frac{|z|}{3}+1=-y$. Let's make sure we haven't diverged from our diagram.

$$\begin{aligned} & \left(\frac{|z|}{3}+1\right)^2 = y^2 \\ \Leftrightarrow\quad & \underbrace{\frac{|z|}{3}+1}_{\text{positive}} = \pm y \\ \Leftrightarrow\quad & \begin{cases} \frac{|z|}{3}+1=y & y>0 \\ \frac{|z|}{3}+1=-y & y<0 \end{cases} \\ \Leftrightarrow\quad & \begin{cases} \frac{|z|}{3}+1=y & y\geqslant 1 \\ \frac{|z|}{3}+1=-y & y\leqslant -1 \end{cases} \\ \Leftrightarrow\quad & \begin{cases} |z|=3(y-1) & y\geqslant 1 \\ |z|=-3(y+1) & y\leqslant -1 \end{cases} \end{aligned}$$

$$\iff \begin{cases} z = \pm \underbrace{3(y-1)}_{\text{positive}} & y \geqslant 1 \\ z = \pm \underbrace{3(y+1)}_{\text{negative}} & y \leqslant -1 \end{cases}$$

$$\iff \begin{cases} z = 3(y-1) & y \geqslant 1,\ z \geqslant 0 \\ z = -3(y-1) & y \geqslant 1,\ z \leqslant 0 \\ z = -3(y+1) & y \leqslant -1,\ z \geqslant 0 \\ z = 3(y+1) & y \leqslant -1,\ z \leqslant 0 \end{cases}$$

This matches our diagram eactly. So, all together, the equation of the surface is

$$x^2 + y^2 = \left(\frac{|z|}{3} + 1\right)^2$$

Chapter 2

Partial Derivatives

2.1 ▲ Limits

▸▸ Stage 1

2.1.1 Suppose $f(x,y)$ is a function such that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 10$.
True or false: $|f(0.1,0.1) - 10| < |f(0.2,0.2) - 10|$

Solution In general, this is false. Consider $f(x,y) = 12 - (1-10x)^2 - (1-10y)^2$.

- $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 12 - 1 - 1 = 10$ (the function is continuous)
- $f(0.1,0.1) = 12 - (1-1)^2 - (1-1)^2 = 12$
- $f(0.2,0.2) = 12 - (1-2)^2 - (1-2)^2 = 10$

We often (somewhat lazily) interpret the limit "$\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 10$" to mean that, as (x,y) gets closer and closer to the origin, $f(x,y)$ gets closer and closer to 10. This isn't exactly what the definition means, though. The definition tells us that, we can guarantee that $f(x,y)$ be very close to 10 by choosing (x,y) very close to $(0,0)$.

The function $f(x,y)$ can also be very close to 10 for some (x,y)'s that are not close to $(0,0)$. Moreover, we don't know how close to $(0,0)$ we have to be in order for $f(x,y)$ to be "very close" to 10.

2.1.2 A millstone pounds wheat into flour. The wheat sits in a basin, and the millstone pounds up and down.
Samples of wheat are taken from various places along the basin. Their diameters are measured and their position on the basin is recorded.
Consider this claim: "As the particles get very close to the millstone, the diameters of the particles approach 50 μm." In this context, describe the variables below from Definition 2.1.2 in the CLP-3 text.
(a) $\mathbf{x}$
(b) $\mathbf{a}$
(c) $\mathbf{L}$

Solution

(a) The function we're taking the limit of has its input as the position of the particle, and its output the size of the particle. So, $f(x,y)$ gives the size of particles found at position (x,y). In the definition, we write $\mathbf{x} = (x,y)$. So, $\mathbf{x}$ is the position in the basin the particle was taken from.

(b) Our claim deals with particles very close to where the millstone hits the basin, so $\mathbf{a}$ is the position in the basinfwhere the millstone hits.

(c) $\mathbf{L}$ is the limit of the function: in this case, 50 μm.

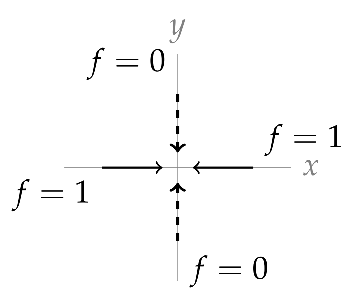
2.1.3 Let $f(x,y) = \dfrac{x^2}{x^2+y^2}$.
(a) Find a ray approaching the origin along which $f(x,y) = 1$.
(b) Find a ray approaching the origin along which $f(x,y) = 0$.
(c) What does the above work show about a limit of $f(x,y)$?

Solution

(a) By inspection, when $y = 0$, then $f(x,y) = 1$ as long as $x \neq 0$. So, if we follow the x-axis in towards the origin, $f(x,y) = 1$ along this route.

(b) Also by inspection, when $x = 0$, then $f(x,y) = 0$ as long as $y \neq 0$. So, if we follow the y-axis in towards the origin, $f(x,y) = 0$ along this route.

(c) Since two different directions give us different values as we approach the origin, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.



$\boxed{2.1.4}$ Let $f(x,y) = x^2 - y^2$

(a) Express the function in terms of the polar coordinates r and θ, and simplify.
(b) Suppose (x,y) is a distance of 1 from the origin. What are the largest and smallest values of $f(x,y)$?
(c) Let $r > 0$. Suppose (x,y) is a distance of r from the origin. What are the largest and smallest values of $f(x,y)$?
(d) Let $\epsilon > 0$. Find a positive value of r that guarantees $|f(x,y)| < \epsilon$ whenever (x,y) is at most r units from the origin.
(e) What did you just show?

Solution

(a) Since $x = r\cos\theta$ and $y = r\sin\theta$, we have that

$$f = x^2 - y^2 = r^2\cos^2\theta - r^2\sin^2\theta = r^2\cos(2\theta)$$

(b) When $r = 1$, $f = \cos(2\theta)$. So, $f(x,y)$ runs between -1 and 1. It smallest value is -1 and its largest value is $+1$.

(c) The distance from (x,y) to the origin is r (for $r \geqslant 0$). So, at a distance r, our function is $r^2\cos(2\theta)$. Then $f(x,y)$ runs over the interval $[-r^2, r^2]$. It smallest value is $-r^2$ and its largest value is $+r^2$.

(d) Using our answer to the last part, we have that $|f| \leqslant r^2$. So for $0 < r < \sqrt{\epsilon}$, we necessarily have that $|f(x,y)| < \epsilon$ whenever the distance from (x,y) to the origin is at most r.

(e) For every $\epsilon > 0$, if we choose (x,y) to be sufficiently close to $(0,0)$ (in particular, within a distance $r < \sqrt{\epsilon}$), then $f(x,y)$ is within distance ϵ of 0. By Definition 2.1.2 in the CLP-3 text, we have that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

$\boxed{2.1.5}$ Suppose $f(x,y)$ is a polynomial. Evaluate $\lim\limits_{(x,y)\to(a,b)} f(x,y)$, where $(a,b) \in \mathbb{R}^2$.

Solution By Theorem 2.1.6,$f(x,y)$ is continuous over its domain. The domain of a polynomial is everywhere; in this case, $\mathbb{R}^2$. So, $f(x,y)$ is continuous at (a,b). By the definition of continuity, $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

▸▸ Stage 2

2.1.6 Evaluate, if possible,

(a) $\displaystyle\lim_{(x,y)\to(2,-1)} \big(xy + x^2\big)$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$

(e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^4}$

(f) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(\sin x)\,(e^y-1)}{xy}$

Solution (a) $\displaystyle\lim_{(x,y)\to(2,-1)} \big(xy + x^2\big) = 2(-1) + 2^2 = 2$

(b) Switching to polar coordinates,

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2} = \lim_{\substack{r\to0^+\\0\le\theta<2\pi}} \frac{r\cos\theta}{r^2} = \lim_{\substack{r\to0^+\\0\le\theta<2\pi}} \frac{\cos\theta}{r} = \text{undefined}$$

since, for example,

- if $\theta = 0$, then
$$\lim_{\substack{r\to0^+\\ \theta=0}} \frac{\cos\theta}{r} = \lim_{r\to0^+} \frac{1}{r} = +\infty$$
- while if $\theta = \pi$, then
$$\lim_{\substack{r\to0^+\\ \theta=\pi}} \frac{\cos\theta}{r} = \lim_{r\to0^+} \frac{-1}{r} = -\infty$$

(c) Switching to polar coordinates,

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{\substack{r\to0^+\\0\le\theta<2\pi}} \frac{r^2\cos^2\theta}{r^2} = \lim_{\substack{r\to0^+\\0\le\theta<2\pi}} \cos^2\theta = \text{undefined}$$

since, for example,

- if $\theta = 0$, then
$$\lim_{\substack{r\to0^+\\ \theta=0}} \cos^2\theta = \lim_{r\to0^+} 1 = 1$$
- while if $\theta = \frac{\pi}{2}$, then
$$\lim_{\substack{r\to0^+\\ \theta=\pi/2}} \cos^2\theta = \lim_{r\to0^+} 0 = 0$$

(d) Switching to polar coordinates,

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{\substack{r\to 0+\\ 0\leqslant\theta<2\pi}} \frac{r^3\cos^3\theta}{r^2} = \lim_{\substack{r\to 0+\\ 0\leqslant\theta<2\pi}} r\cos^3\theta = 0$$

since $|\cos\theta| \leqslant 1$ for all θ.

(e) Switching to polar coordinates,

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^4} = \lim_{\substack{r\to 0+\\ 0\leqslant\theta<2\pi}} \frac{r^2\cos^2\theta\, r^2\sin^2\theta}{r^2\cos^2\theta + r^4\sin^4\theta} = \lim_{\substack{r\to 0+\\ 0\leqslant\theta<2\pi}} r^2\sin^2\theta \frac{\cos^2\theta}{\cos^2\theta + r^2\sin^4\theta}$$
$$= 0$$

Here, we used that

$$\left|\sin^2\theta \frac{\cos^2\theta}{\cos^2\theta + r^2\sin^4\theta}\right| \leqslant \left|\frac{\cos^2\theta}{\cos^2\theta + r^2\sin^4\theta}\right| \leqslant \left|\frac{\cos^2\theta}{\cos^2\theta}\right| \leqslant 1$$

for all $r > 0$.

(f) To start, observe that

$$\lim_{(x,y)\to(0,0)} \frac{(\sin x)\,(e^y-1)}{xy} = \left[\lim_{x\to 0} \frac{\sin x}{x}\right]\left[\lim_{y\to 0} \frac{e^y-1}{y}\right]$$

We may evaluate $\left[\lim\limits_{x\to 0} \frac{\sin x}{x}\right]$ by l'Hôpital's rule or by using the definition of the derivative to give

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x - \sin 0}{x - 0} = \frac{\mathrm{d}}{\mathrm{d}x}\sin x\Big|_{x=0} = \cos x\Big|_{x=0} = 1$$

Similarly, we may evaluate $\left[\lim\limits_{y\to 0} \frac{e^y-1}{y}\right]$ by l'Hôpital's rule or by using the definition of the derivative to give

$$\lim_{y\to 0} \frac{e^y-1}{y} = \lim_{y\to 0} \frac{e^y-e^0}{y-0} = \frac{\mathrm{d}}{\mathrm{d}y}e^y\Big|_{y=0} = e^y\Big|_{y=0} = 1$$

So all together

$$\lim_{(x,y)\to(0,0)} \frac{(\sin x)\,(e^y-1)}{xy} = \left[\lim_{x\to 0} \frac{\sin x}{x}\right]\left[\lim_{y\to 0} \frac{e^y-1}{y}\right] = [1]\ [1] = 1$$

$\boxed{2.1.7}$ (∗)

(a) Find the limit: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^8+y^8}{x^4+y^4}$.

(b) Prove that the following limit does not exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^8+y^{10}}$.

Solution (a) In polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, so that

$$\frac{x^8+y^8}{x^4+y^4} = \frac{r^8\cos^8\theta + r^8\sin^8\theta}{r^4\cos^4\theta + r^4\sin^4\theta} = r^4\frac{\cos^8\theta+\sin^8\theta}{\cos^4\theta+\sin^4\theta}$$

As

$$\begin{aligned}\frac{\cos^8\theta+\sin^8\theta}{\cos^4\theta+\sin^4\theta} \leqslant \frac{\cos^8\theta+2\cos^4\theta\sin^4\theta+\sin^8\theta}{\cos^4\theta+\sin^4\theta} &= \frac{\left(\cos^4\theta+\sin^4\theta\right)^2}{\cos^4\theta+\sin^4\theta} \\ = \cos^4\theta+\sin^4\theta \leqslant 2\end{aligned}$$

we have

$$0 \leqslant \frac{x^8+y^8}{x^4+y^4} \leqslant 2r^4$$

As $\lim\limits_{(x,y)\to(0,0)} 2r^4 = 0$, the squeeze theorem yields $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^8+y^8}{x^4+y^4} = 0$.

(b) In polar coordinates

$$\frac{xy^5}{x^8+y^{10}} = \frac{r^6\cos\theta\,\sin^5\theta}{r^8\cos^8\theta + r^{10}\sin^{10}\theta} = \frac{1}{r^2}\frac{\cos\theta\,\sin^5\theta}{\cos^8\theta + r^2\sin^{10}\theta}$$

As $(x,y)\to(0,0)$ the first fraction $\frac{1}{r^2}\to\infty$ but the second factor can take many different values. For example, if we send (x,y) towards the origin along the y–axis, i.e. with $\theta = \pm\frac{\pi}{2}$,

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{xy^5}{x^8+y^{10}} = \lim_{y\to 0}\frac{0}{y^{10}} = 0$$

but if we send (x,y) towards the origin along the line $y = x$, i.e. with $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$,

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{xy^5}{x^8+y^{10}} = \lim_{x\to 0}\frac{x^6}{x^8+x^{10}} = \lim_{x\to 0}\frac{1}{x^2}\frac{1}{1+x^2} = +\infty$$

and if we send (x,y) towards the origin along the line $y = -x$, i.e. with $\theta = -\frac{\pi}{4}, \frac{3\pi}{4}$,

$$\lim_{\substack{(x,y)\to(0,0)\\ y=-x}} \frac{xy^5}{x^8+y^{10}} = \lim_{x\to 0}\frac{-x^6}{x^8+x^{10}} = \lim_{x\to 0}-\frac{1}{x^2}\frac{1}{1+x^2} = -\infty$$

So $\frac{xy^5}{x^8+y^{10}}$ does not approach a single value as $(x,y)\to(0,0)$ and the limit does not exist.

2.1.8 (*) Evaluate each of the following limits or show that it does not exist.

(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x^2+y^2}$

(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^4}{x^2+y^4}$

$\boxed{\text{Solution}}$ (a) In polar coordinates

$$\frac{x^3 - y^3}{x^2 + y^2} = \frac{r^3 \cos^3\theta - r^3 \sin^3\theta}{r^2} = r\cos^3\theta - r\sin^3\theta$$

Since

$$\left| r\cos^3\theta - r\sin^3\theta \right| \leqslant 2r$$

and $2r \to 0$ as $r \to 0$, the limit exists and is 0.

(b) The limit as we approach $(0,0)$ along the x-axis is

$$\lim_{t\to 0} \frac{x^2 - y^4}{x^2 + y^4}\bigg|_{(x,y)=(t,0)} = \lim_{t\to 0} \frac{t^2 - 0^4}{t^2 + 0^4} = 1$$

On the other hand the limit as we approach $(0,0)$ along the y-axis is

$$\lim_{t\to 0} \frac{x^2 - y^4}{x^2 + y^4}\bigg|_{(x,y)=(0,t)} = \lim_{t\to 0} \frac{0^2 - t^4}{0^2 + t^4} = -1$$

These are different, so the limit as $(x,y) \to 0$ does not exist.

We can gain a more detailed understanding of the behaviour of $\frac{x^2-y^4}{x^2+y^4}$ near the origin by switching to polar coordinates.

$$\frac{x^2 - y^4}{x^2 + y^4} = \frac{r^2\cos^2\theta - r^4\sin^4\theta}{r^2\cos^2\theta + r^4\sin^4\theta} = \frac{\cos^2\theta - r^2\sin^4\theta}{\cos^2\theta + r^2\sin^4\theta}$$

Now fix any θ and let $r \to 0$ (so that we are approaching the origin along the ray that makes an angle θ with the positive x-axis). If $\cos\theta \neq 0$ (i.e. the ray is not part of the y-axis)

$$\lim_{r\to 0} \frac{\cos^2\theta - r^2\sin^4\theta}{\cos^2\theta + r^2\sin^4\theta} = \frac{\cos^2\theta}{\cos^2\theta} = 1$$

But if $\cos\theta = 0$ (i.e. the ray is part of the y-axis)

$$\lim_{r\to 0} \frac{\cos^2\theta - r^2\sin^4\theta}{\cos^2\theta + r^2\sin^4\theta} = \lim_{r\to 0} \frac{-r^2\sin^4\theta}{r^2\sin^4\theta} = \frac{-\sin^4\theta}{\sin^4\theta} = -1$$

▸▸ Stage 3

$\boxed{2.1.9}$ $(*)$ Evaluate each of the following limits or show that it does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2 + x^2y - y^2x + 2y^2}{x^2 + y^2}$

(b) $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x^2y^2 - 2x^2y + x^2}{(x^2 + y^2 - 2y + 1)^2}$

Solution (a) In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{2x^2 + x^2y - y^2x + 2y^2}{x^2 + y^2} = \frac{2r^2\cos^2\theta + r^3\cos^2\theta\sin\theta - r^3\cos\theta\sin^2\theta + 2r^2\sin^2\theta}{r^2}$$
$$= 2 + r\big[\cos^2\theta\sin\theta - \sin^2\theta\cos\theta\big]$$

As

$$r\big|\cos^2\theta\sin\theta - \sin^2\theta\cos\theta\big| \leqslant 2r \to 0 \text{ as } r \to 0$$

we have

$$\lim_{(x,y)\to(0,0)} \frac{2x^2 + x^2y - y^2x + 2y^2}{x^2 + y^2} = 2$$

(b) Since

$$\frac{x^2y^2 - 2x^2y + x^2}{(x^2 + y^2 - 2y + 1)^2} = \frac{x^2(y-1)^2}{\big[x^2 + (y-1)^2\big]^2}$$

and, in polar coordinates centred on $(0,1)$, $x = r\cos\theta$, $y = 1 + r\sin\theta$,

$$\frac{x^2(y-1)^2}{\big[x^2 + (y-1)^2\big]^2} = \frac{r^4\cos^2\theta\sin^2\theta}{r^4} = \cos^2\theta\sin^2\theta$$

we have that the limit does not exist. For example, if we send (x,y) to $(0,1)$ along the line $y = 1$, so that $\theta = 0$, we get the limit 0, while if we send (x,y) to $(0,1)$ along the line $y = x + 1$, so that $\theta = \frac{\pi}{4}$, we get the limit $\frac{1}{4}$.

2.1.10 Define, for all $(x,y) \neq (0,0)$, $f(x,y) = \frac{x^2y}{x^4+y^2}$.

(a) Let $0 \leqslant \theta < 2\pi$. Compute $\lim\limits_{r\to 0^+} f(r\cos\theta, r\sin\theta)$.

(b) Compute $\lim\limits_{x\to 0} f(x, x^2)$.

(c) Does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

Solution (a) We have

$$\begin{aligned}\lim_{r\to 0^+} f(r\cos\theta, r\sin\theta) &= \lim_{r\to 0^+} \frac{(r\cos\theta)^2(r\sin\theta)}{(r\cos\theta)^4 + (r\sin\theta)^2}\\ &= \lim_{r\to 0^+} r\ \frac{\cos^2\theta\sin\theta}{r^2\cos^4\theta + \sin^2\theta}\\ &= \lim_{r\to 0^+} r\ \lim_{r\to 0^+} \frac{\cos^2\theta\sin\theta}{r^2\cos^4\theta + \sin^2\theta}\end{aligned}$$

Observe that, if $\sin\theta = 0$, then

$$\frac{\cos^2\theta\sin\theta}{r^2\cos^4\theta + \sin^2\theta} = 0$$

for all $r \neq 0$. If $\sin\theta \neq 0$,

$$\lim_{r\to 0^+} \frac{\cos^2\theta \sin\theta}{r^2\cos^4\theta + \sin^2\theta} = \frac{\cos^2\theta\sin\theta}{\sin^2\theta} = \frac{\cos^2\theta}{\sin\theta}$$

So the limit $\lim_{r\to 0^+} \frac{\cos^2\theta\sin\theta}{r^2\cos^4\theta+\sin^2\theta}$ exists (and is finite) for all fixed θ and

$$\lim_{r\to 0^+} f(r\cos\theta, r\sin\theta) = 0$$

(b) We have

$$\lim_{x\to 0} f(x,x^2) = \lim_{x\to 0} \frac{x^2x^2}{x^4+(x^2)^2} = \lim_{x\to 0}\frac{x^4}{2x^4} = \frac{1}{2}$$

(c) Note that in part (a) we showed that as (x,y) approaches $(0,0)$ along any straight line, $f(x,y)$ approaches the limit zero. In part (b) we have just shown that as (x,y) approaches $(0,0)$ along the parabola $y=x^2$, $f(x,y)$ approaches the limit $\frac{1}{2}$, not zero. So $f(x,y)$ takes values very close to 0, for some (x,y)'s that are really near $(0,0)$ and also takes values very close to $\frac{1}{2}$, for other (x,y)'s that are really near $(0,0)$. There is no single number, L, with the property that $f(x,y)$ is really close to L for all (x,y) that are really close to $(0,0)$. So the limit does not exist.

2.1.11 (*) Compute the following limits or explain why they do not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$

(c) $\displaystyle\lim_{(x,y)\to(-1,1)} \frac{x^2+2xy^2+y^4}{1+y^4}$

(d) $\displaystyle\lim_{(x,y)\to(0,0)} |y|^x$

Solution (a) Since, in polar coordinates,

$$\frac{xy}{x^2+y^2} = \frac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\sin\theta$$

we have that the limit does not exist. For example,

- if we send (x,y) to $(0,0)$ along the positive x-axis, so that $\theta=0$, we get the limit $\sin\theta\cos\theta\big|_{\theta=0} = 0$,
- while if we send (x,y) to $(0,0)$ along the line $y=x$ in the first quadrant, so that $\theta=\frac{\pi}{4}$, we get the limit $\sin\theta\cos\theta\big|_{\theta=\pi/4} = \frac{1}{2}$.

(b) This limit does not exist, since if it were to exist the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{xy}{\sin(xy)}\,\frac{\sin(xy)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{xy}{\sin(xy)}\ \lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$$

would also exist. (Recall that $\lim_{t\to 0} \frac{\sin t}{t} = 1$.)

(c) Since

$$\lim_{(x,y)\to(-1,1)} \left[x^2 + 2xy^2 + y^4\right] = (-1)^2 + 2(-1)(1)^2 + (1)^4 = 0$$
$$\lim_{(x,y)\to(-1,1)} \left[1 + y^4\right] = 1 + (1)^4 = 2$$

and the second limit is nonzero,

$$\lim_{(x,y)\to(-1,1)} \frac{x^2 + 2xy^2 + y^4}{1 + y^4} = \frac{0}{2} = 0$$

(d) Since the limit along the positive x-axis

$$\lim_{\substack{t\to 0\\ t>0}} |y|^x \Big|_{(x,y)=(t,0)} = \lim_{\substack{t\to 0\\ t>0}} 0^t = \lim_{\substack{t\to 0\\ t>0}} 0 = 0$$

and the limit along the y-axis

$$\lim_{t\to 0} |y|^x \Big|_{(x,y)=(0,t)} = \lim_{t\to 0} |t|^0 = \lim_{t\to 0} 1 = 1$$

are different, the limit as $(x, y) \to 0$ does not exist.

2.2▲ Partial Derivatives

▸▸ Stage 1

2.2.1 Let $f(x, y) = e^x \cos y$. The following table gives some values of $f(x, y)$.

	$x = 0$	$x = 0.01$	$x = 0.1$
$y = -0.1$	0.99500	1.00500	1.09965
$y = -0.01$	0.99995	1.01000	1.10512
$y = 0$	1.0	1.01005	1.10517

(a) Find two different approximate values for $\frac{\partial f}{\partial x}(0, 0)$ using the data in the above table.
(b) Find two different approximate values for $\frac{\partial f}{\partial y}(0, 0)$ using the data in the above table.
(c) Evaluate $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$ exactly.

Solution

(a) By definition

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h}$$

One approximation to this is

$$\frac{\partial f}{\partial x}(0,0) \approx \frac{f(h,0)-f(0,0)}{h}\bigg|_{h=0.1} = \frac{1.10517-1}{0.1} = 1.0517$$

Another approximation to this is

$$\frac{\partial f}{\partial x}(0,0) \approx \frac{f(h,0)-f(0,0)}{h}\bigg|_{h=0.01} = \frac{1.01005-1}{0.01} = 1.005$$

(b) By definition

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}$$

One approximation to this is

$$\frac{\partial f}{\partial y}(0,0) \approx \frac{f(0,h)-f(0,0)}{h}\bigg|_{h=-0.1} = \frac{0.99500-1}{-0.1} = 0.0500$$

Another approximation to this is

$$\frac{\partial f}{\partial y}(0,0) \approx \frac{f(0,h)-f(0,0)}{h}\bigg|_{h=-0.01} = \frac{0.99995-1}{0.01} = .0050$$

(c) To take the partial derivative with respect to x at $(0,0)$, we set $y=0$, differentiate with respect to x and then set $x=0$. So

$$\frac{\partial f}{\partial x}(0,0) = \frac{\mathrm{d}}{\mathrm{d}x}e^x\cos 0\bigg|_{x=0} = e^x\big|_{x=0} = 1$$

To take the partial derivative with respect to y at $(0,0)$, we set $x=0$, differentiate with respect to y and then set $y=0$. So

$$\frac{\partial f}{\partial y}(0,0) = \frac{\mathrm{d}}{\mathrm{d}y}e^0\cos y\bigg|_{y=0} = \sin y\big|_{y=0} = 0$$

2.2.2 You are traversing an undulating landscape. Take the z-axis to be straight up towards the sky, the positive x-axis to be due south, and the positive y-axis to be due east. Then the landscape near you is described by the equation $z=f(x,y)$, with you at the point $(0,0,f(0,0))$. The function $f(x,y)$ is differentiable.
Suppose $f_y(0,0)<0$. Is it possible that you are at a summit? Explain.

Solution

If $f_y(0,0)<0$, then $f(0,y)$ decreases as y increases from 0. Thus moving in the positive y direction takes you downhill. This means you aren't at the lowest point in a valley, since you can still move downhill. On the other hand, as $f_y(0,0)<0$, $f(0,y)$ also decreases as y increases towards 0 from slightly negative values. Thus if you move in the negative

y-direction from $y = 0$, your height z will increase. So you are not at a locally highest point—you're not at a summit.

$\boxed{2.2.3}$ (*) Let

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x, y) \neq (0,0) \\ 0 & \text{if } (x, y) = (0,0) \end{cases}$$

Compute, directly from the definitions,

(a) $\frac{\partial f}{\partial x}(0,0)$

(b) $\frac{\partial f}{\partial y}(0,0)$

(c) $\frac{\mathrm{d}}{\mathrm{d}t} f(t,t)\Big|_{t=0}$

Solution (a) By definition

$$\begin{aligned}
\frac{\partial f}{\partial x}(0,0) &= \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{(\Delta x^2)(0)}{\Delta x^2 + 0^2} - 0}{\Delta x} \\
&= 0
\end{aligned}$$

(b) By definition

$$\begin{aligned}
\frac{\partial f}{\partial y}(0,0) &= \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{\frac{(0^2)(\Delta y)}{0^2 + \Delta y^2} - 0}{\Delta y} \\
&= 0
\end{aligned}$$

(c) By definition

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f(t,t)\Big|_{t=0} &= \lim_{t \to 0} \frac{f(t,t) - f(0,0)}{t} \\
&= \lim_{h \to 0} \frac{\frac{(t^2)(t)}{t^2+t^2} - 0}{t} \\
&= \lim_{t \to 0} \frac{t/2}{t} \\
&= \frac{1}{2}
\end{aligned}$$

▸▸ Stage 2

2.2.4 Find all first partial derivatives of the following functions and evaluate them at the given point.
(a) $f(x,y,z) = x^3y^4z^5 \qquad (0,-1,-1)$
(b) $w(x,y,z) = \ln\left(1+e^{xyz}\right) \qquad (2,0,-1)$
(c) $f(x,y) = \frac{1}{\sqrt{x^2+y^2}} \qquad (-3,4)$

Solution (a)

$$\begin{aligned} f_x(x,y,z) &= 3x^2y^4z^5 & f_x(0,-1,-1) &= 0\\ f_y(x,y,z) &= 4x^3y^3z^5 & f_y(0,-1,-1) &= 0\\ f_z(x,y,z) &= 5x^3y^4z^4 & f_z(0,-1,-1) &= 0 \end{aligned}$$

(b)

$$\begin{aligned} w_x(x,y,z) &= \frac{yze^{xyz}}{1+e^{xyz}} & w_x(2,0,-1) &= 0\\ w_y(x,y,z) &= \frac{xze^{xyz}}{1+e^{xyz}} & w_y(2,0,-1) &= -1\\ w_z(x,y,z) &= \frac{xye^{xyz}}{1+e^{xyz}} & w_z(2,0,-1) &= 0 \end{aligned}$$

(c)

$$\begin{aligned} f_x(x,y) &= -\frac{x}{(x^2+y^2)^{3/2}} & f_x(-3,4) &= \frac{3}{125}\\ f_y(x,y) &= -\frac{y}{(x^2+y^2)^{3/2}} & f_y(-3,4) &= -\frac{4}{125} \end{aligned}$$

2.2.5 Show that the function $z(x,y) = \frac{x+y}{x-y}$ obeys

$$x\frac{\partial z}{\partial x}(x,y) + y\frac{\partial z}{\partial y}(x,y) = 0$$

Solution By the quotient rule

$$\begin{aligned} \frac{\partial z}{\partial x}(x,y) &= \frac{(1)(x-y)-(x+y)(1)}{(x-y)^2} &= \frac{-2y}{(x-y)^2}\\ \frac{\partial z}{\partial y}(x,y) &= \frac{(1)(x-y)-(x+y)(-1)}{(x-y)^2} &= \frac{2x}{(x-y)^2} \end{aligned}$$

Hence

$$x\frac{\partial z}{\partial x}(x,y) + y\frac{\partial z}{\partial y}(x,y) = \frac{-2xy+2yx}{(x-y)^2} = 0$$

2.2.6 (*) A surface $z(x,y)$ is defined by $zy - y + x = \ln(xyz)$.
(a) Compute $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ in terms of x, y, z.
(b) Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(x,y,z) = (-1,-2,1/2)$.

Solution (a) We are told that $z(x,y)$ obeys

$$z(x,y)\,y - y + x = \ln\big(xy\,z(x,y)\big) \tag{$*$}$$

for all (x,y) (near $(-1,-2)$). Differentiating $(*)$ with respect to x gives

$$y\,\frac{\partial z}{\partial x}(x,y) + 1 = \frac{1}{x} + \frac{\frac{\partial z}{\partial x}(x,y)}{z(x,y)} \implies \frac{\partial z}{\partial x}(x,y) = \frac{\frac{1}{x} - 1}{y - \frac{1}{z(x,y)}}$$

or, dropping the arguments (x,y) and multiplying both the numerator and denominator by xz,

$$\frac{\partial z}{\partial x} = \frac{z - xz}{xyz - x} = \frac{z(1-x)}{x(yz-1)}$$

Differentiating $(*)$ with respect to y gives

$$z(x,y) + y\,\frac{\partial z}{\partial y}(x,y) - 1 = \frac{1}{y} + \frac{\frac{\partial z}{\partial y}(x,y)}{z(x,y)} \implies \frac{\partial z}{\partial y}(x,y) = \frac{\frac{1}{y} + 1 - z(x,y)}{y - \frac{1}{z(x,y)}}$$

or, dropping the arguments (x,y) and multiplying both the numerator and denominator by yz,

$$\frac{\partial z}{\partial y} = \frac{z + yz - yz^2}{y^2 z - y} = \frac{z(1+y-yz)}{y(yz-1)}$$

(b) When $(x,y,z) = (-1,-2,1/2)$,

$$\frac{\partial z}{\partial x}(-1,-2) = \left.\frac{\frac{1}{x} - 1}{y - \frac{1}{z}}\right|_{(x,y,z)=(-1,-2,1/2)} = \frac{\frac{1}{-1} - 1}{-2-2} = \frac{1}{2}$$

$$\frac{\partial z}{\partial y}(-1,-2) = \left.\frac{\frac{1}{y} + 1 - z}{y - \frac{1}{z}}\right|_{(x,y,z)=(-1,-2,1/2)} = \frac{\frac{1}{-2} + 1 - \frac{1}{2}}{-2-2} = 0$$

2.2.7 (*) Find $\frac{\partial U}{\partial T}$ and $\frac{\partial T}{\partial V}$ at $(1,1,2,4)$ if (T,U,V,W) are related by

$$(TU - V)^2 \ln(W - UV) = \ln 2$$

$\boxed{\text{Solution}}$ We are told that the four variables T, U, V, W obey the the single equation $(TU - V)^2 \ln(W - UV) = \ln 2$. So they are not all independent variables. Roughly speaking, we can treat any three of them as independent variables and solve the given equation for the fourth as a function of the three chosen independent variables.

We are first asked to find $\frac{\partial U}{\partial T}$. This implicitly tells to treat T, V and W as independent variables and to view U as a function $U(T, V, W)$ that obeys

$$\big(T\,U(T,V,W) - V\big)^2 \ln\big(W - U(T,V,W)\,V\big) = \ln 2 \tag{E1}$$

for all (T, U, V, W) sufficiently near $(1, 1, 2, 4)$. Differentiating (E1) with respect to T gives

$$\begin{aligned} 2\big(T\,U(T,V,W) - V\big)\left[U(T,V,W) + T\,\frac{\partial U}{\partial T}(T,V,W)\right] \ln\big(W - U(T,V,W)\,V\big) \\ - \big(T\,U(T,V,W) - V\big)^2 \frac{1}{W - U(T,V,W)\,V}\frac{\partial U}{\partial T}(T,V,W)\,V = 0 \end{aligned}$$

In particular, for $(T, U, V, W) = (1, 1, 2, 4)$,

$$\begin{aligned} 2\big((1)(1) - 2\big)\left[1 + (1)\frac{\partial U}{\partial T}(1,2,4)\right] \ln\big(4 - (1)(2)\big) \\ - \big((1)(1) - 2\big)^2 \frac{1}{4 - (1)(2)}\frac{\partial U}{\partial T}(1,2,4)\,(2) = 0 \end{aligned}$$

This simplifies to

$$-2\left[1 + \frac{\partial U}{\partial T}(1,2,4)\right]\ln(2) - \frac{\partial U}{\partial T}(1,2,4) = 0 \implies \frac{\partial U}{\partial T}(1,2,4) = -\frac{2\ln(2)}{1 + 2\ln(2)}$$

We are then asked to find $\frac{\partial T}{\partial V}$. This implicitly tells to treat U, V and W as independent variables and to view T as a function $T(U, V, W)$ that obeys

$$\big(T(U,V,W)\,U - V\big)^2 \ln\big(W - U\,V\big) = \ln 2 \tag{E2}$$

for all (T, U, V, W) sufficiently near $(1, 1, 2, 4)$. Differentiating (E2) with respect to V gives

$$\begin{aligned} 2\big(T(U,V,W)\,U - V\big)\left[\frac{\partial T}{\partial V}(U,V,W)\;U - 1\right] \ln\big(W - U\,V\big) \\ - \big(T(U,V,W)\,U - V\big)^2 \frac{U}{W - U\,V} = 0 \end{aligned}$$

In particular, for $(T, U, V, W) = (1, 1, 2, 4)$,

$$\begin{aligned} 2\big((1)(1) - 2\big)\left[(1)\frac{\partial T}{\partial V}(1,2,4) - 1\right] \ln\big(4 - (1)(2)\big) \\ - \big((1)(1) - 2\big)^2 \frac{1}{4 - (1)(2)} = 0 \end{aligned}$$

This simplifies to

$$-2\left[\frac{\partial T}{\partial V}(1,2,4)-1\right]\ln(2)-\frac{1}{2}=0 \implies \frac{\partial T}{\partial V}(1,2,4)=1-\frac{1}{4\ln(2)}$$

$\boxed{\text{2.2.8}}$ (∗) Suppose that $u = x^2 + yz$, $x = \rho r\cos(\theta)$, $y = \rho r\sin(\theta)$ and $z = \rho r$. Find $\frac{\partial u}{\partial r}$ at the point $(\rho_0, r_0, \theta_0) = (2, 3, \pi/2)$.

$\boxed{\text{Solution}}$ The function

$$\begin{aligned} u(\rho,r,\theta) &= \big[\rho r\cos\theta\big]^2 + \big[\rho r\sin\theta\big]\rho r \\ &= \rho^2 r^2\cos^2\theta + \rho^2 r^2\sin\theta \end{aligned}$$

So

$$\frac{\partial u}{\partial r}(\rho,r,\theta) = 2\rho^2 r\cos^2\theta + 2\rho^2 r\sin\theta$$

and

$$\frac{\partial u}{\partial r}(2,3,\pi/2) = 2(2^2)(3)(0)^2 + 2(2^2)(3)(1) = 24$$

$\boxed{\text{2.2.9}}$ Use the definition of the derivative to evaluate $f_x(0,0)$ and $f_y(0,0)$ for

$$f(x,y) = \begin{cases} \frac{x^2-2y^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

$\boxed{\text{Solution}}$ By definition

$$f_x(x_0,y_0) = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x} \qquad f_y(x_0,y_0) = \lim_{\Delta y\to 0}\frac{f(x_0, y_0+\Delta y) - f(x_0,y_0)}{\Delta y}$$

Setting $x_0 = y_0 = 0$,

$$\begin{aligned} f_x(0,0) &= \lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(\Delta x,0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{((\Delta x)^2 - 2\times 0^2)/(\Delta x - 0)}{\Delta x} \\ &= \lim_{\Delta x\to 0} 1 = 1 \\ f_y(0,0) &= \lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y} = \lim_{\Delta y\to 0}\frac{f(0,\Delta y)}{\Delta y} = \lim_{\Delta y\to 0}\frac{(0^2 - 2(\Delta y)^2)/(0-\Delta y)}{\Delta y} \\ &= \lim_{\Delta y\to 0} 2 = 2 \end{aligned}$$

▸▸ Stage 3

2.2.10 Let f be any differentiable function of one variable. Define $z(x,y) = f(x^2+y^2)$. Is the equation

$$y\frac{\partial z}{\partial x}(x,y) - x\frac{\partial z}{\partial y}(x,y) = 0$$

necessarily satisfied?

Solution As $z(x,y) = f(x^2+y^2)$

$$\frac{\partial z}{\partial x}(x,y) = 2xf'(x^2+y^2)$$
$$\frac{\partial z}{\partial y}(x,y) = 2yf'(x^2+y^2)$$

by the (ordinary single variable) chain rule. So

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = y(2x)f'(x^2+y^2) - x(2y)f'(x^2+y^2) = 0$$

and the differential equation is always satisfied, assuming that f is differentiable, so that the chain rule applies.

2.2.11 Define the function

$$f(x,y) = \begin{cases} \frac{(x+2y)^2}{x+y} & \text{if } x+y \neq 0 \\ 0 & \text{if } x+y = 0 \end{cases}$$

(a) Evaluate, if possible, $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.
(b) Is $f(x,y)$ continuous at $(0,0)$?

Solution By definition

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{\frac{(\Delta x+2\times 0)^2}{\Delta x+0}-0}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x} \\ &= 1\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{\frac{(0+2\Delta y)^2}{0+\Delta y} - 0}{\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{4\Delta y}{\Delta y} \\ &= 4\end{aligned}$$

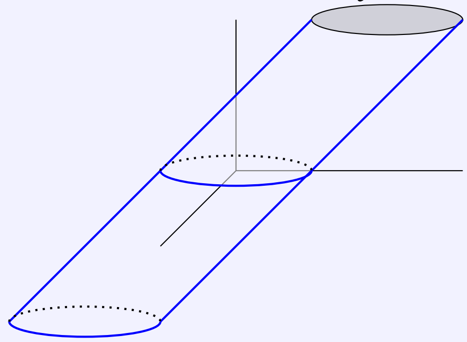
(b) $f(x,y)$ is not continuous at $(0,0)$, even though both partial derivatives exist there. To see this, make a change of coordinates from (x,y) to (X,y) with $X = x + y$ (the denominator). Of course, $(x,y) \to (0,0)$ if and only if $(X,y) \to (0,0)$. Now watch what happens when $(X,y) \to (0,0)$ with X a lot smaller than y. For example, $X = ay^2$. Then

$$\frac{(x+2y)^2}{x+y} = \frac{(X+y)^2}{X} = \frac{(ay^2+y)^2}{ay^2} = \frac{(1+ay)^2}{a} \to \frac{1}{a}$$

This depends on a. So approaching $(0,0)$ along different paths gives different limits. (You can see the same effect without changing coordinates by sending $(x,y) \to (0,0)$ with $x = -y + ay^2$.) Even more dramatically, watch what happens when $(X,y) \to (0,0)$ with $X = y^3$. Then

$$\frac{(x+2y)^2}{x+y} = \frac{(X+y)^2}{X} = \frac{(y^3+y)^2}{y^3} = \frac{\left(1+y^2\right)^2}{y} \to \pm\infty$$

2.2.12 Consider the cylinder whose base is the radius-1 circle in the xy-plane centred at $(0,0)$, and which slopes parallel to the line in the yz-plane given by $z = y$.



When you stand at the point $(0,-1,0)$, what is the slope of the surface if you look in the positive y direction? The positive x direction?

Solution Solution 1

Let's start by finding an equation for this surface. Every level curve is a horizontal circle of radius one, so the equation should be of the form

$$(x - f_1)^2 + (y - f_2)^2 = 1$$

where f_1 and f_2 are functions depending only on z. Since the centre of the circle at height z is at position $x = 0$, $y = z$, we see that the equation of our surface is

$$x^2 + (y - z)^2 = 1$$

The height of the surface at the point (x, y) is the $z(x, y)$ found by solving that equation. That is,

$$x^2 + \big(y - z(x, y)\big)^2 = 1 \tag{$*$}$$

We differentiate this equation implicitly to find $z_x(x, y)$ and $z_y(x, y)$ at the desired point $(x, y) = (0, -1)$. First, differentiating $(*)$ with respect to y gives

$$\begin{aligned} 0 + 2\big(y - z(x, y)\big)\big(1 - z_y(x, y)\big) &= 0 \\ 2(-1 - 0)\big(1 - z_y(0, -1)\big) &= 0 \qquad \text{at } (0, -1, 0) \end{aligned}$$

so that the slope looking in the positive y direction is $z_y(0, -1) = 1$. Similarly, differentiating $(*)$ with respect to x gives

$$\begin{aligned} 2x + 2\big(y - z(x, y)\big) \cdot \big(0 - z_x(x, y)\big) &= 0 \\ 2x &= 2\big(y - z(x, y)\big) \cdot z_x(x, y) \\ z_x(x, y) &= \frac{x}{y - z(x, y)} \\ z_x(0, -1) &= 0 \qquad \text{at } (0, -1, 0) \end{aligned}$$

The slope looking in the positive x direction is $z_x(0, -1) = 0$.

Solution 2

Standing at $(0, -1, 0)$ and looking in the positive y direction, the surface follows the straight line that

- passes through the point $(0, -1, 0)$, and
- is parallel to the central line $z = y, x = 0$ of the cylinder.

Shifting the central line one unit in the y-direction, we get the line $z = y + 1$, $x = 0$. (As a check, notice that $(0, -1, 0)$ is indeed on $z = y + 1$, $x = 0$.) The slope of this line is 1.

Standing at $(0, -1, 0)$ and looking in the positive x direction, the surface follows the circle $x^2 + y^2 = 1$, $z = 0$, which is the intersection of the cylinder with the xy-plane. As we move along that circle our z coordinate stays fixed at 0. So the slope in that direction is 0.

2.3▲ Higher Order Derivatives

▸▸ Stage 1

$\boxed{2.3.1}$ Let all of the third order partial derivatives of the function $f(x, y, z)$ exist and be continuous. Show that

$$f_{xyz}(x, y, z) = f_{xzy}(x, y, z) = f_{yxz}(x, y, z) = f_{yzx}(x, y, z) = f_{zxy}(x, y, z) = f_{zyx}(x, y, z)$$

Solution We have to derive a bunch of equalities.

- Fix any real number x and set $g(y,z) = f_x(x,y,z)$. By (Clairaut's) Theorem 2.3.4 in the CLP-3 text $g_{yz}(y,z) = g_{zy}(y,z)$, so
$$f_{xyz}(x,y,z) = g_{yz}(y,z) = g_{zy}(y,z) = f_{xzy}(x,y,z)$$
- For every fixed real number z, (Clairaut's) Theorem 2.3.4 in the CLP-3 text gives $f_{xy}(x,y,z) = f_{yx}(x,y,z)$. So
$$f_{xyz}(x,y,z) = \frac{\partial}{\partial z}f_{xy}(x,y,z) = \frac{\partial}{\partial z}f_{yx}(x,y,z) = f_{yxz}(x,y,z)$$
So far, we have
$$f_{xyz}(x,y,z) = f_{xzy}(x,y,z) = f_{yxz}(x,y,z)$$
- Fix any real number y and set $g(x,z) = f_y(x,y,z)$. By (Clairaut's) Theorem 2.3.4 in the CLP-3 text $g_{xz}(x,z) = g_{zx}(x,z)$. So
$$f_{yxz}(x,y,z) = g_{xz}(x,z) = g_{zx}(x,z) = f_{yzx}(x,y,z)$$
So far, we have
$$f_{xyz}(x,y,z) = f_{xzy}(x,y,z) = f_{yxz}(x,y,z) = f_{yzx}(x,y,z)$$
- For every fixed real number y, (Clairaut's) Theorem 2.3.4 in the CLP-3 text gives $f_{xz}(x,y,z) = f_{zx}(x,y,z)$. So
$$f_{xzy}(x,y,z) = \frac{\partial}{\partial y}f_{xz}(x,y,z) = \frac{\partial}{\partial y}f_{zx}(x,y,z) = f_{zxy}(x,y,z)$$
So far, we have
$$f_{xyz}(x,y,z) = f_{xzy}(x,y,z) = f_{yxz}(x,y,z) = f_{yzx}(x,y,z) = f_{zxy}(x,y,z)$$
- Fix any real number z and set $g(x,y) = f_z(x,y,z)$. By (Clairaut's) Theorem 2.3.4 in the CLP-3 text $g_{xy}(x,y) = g_{yx}(x,y)$. So
$$f_{zxy}(x,y,z) = g_{xy}(x,y) = g_{yx}(x,y) = f_{zxy}(x,y,z)$$
We now have all of
$$f_{xyz}(x,y,z) = f_{xzy}(x,y,z) = f_{yxz}(x,y,z) = f_{yzx}(x,y,z) = f_{zxy}(x,y,z) = f_{zxy}(x,y,z)$$

2.3.2 Find, if possible, a function $f(x,y)$ for which $f_x(x,y) = e^y$ and $f_y(x,y) = e^x$.

Solution No such $f(x,y)$ exists, because if it were to exist, then we would have that $f_{xy}(x,y) = f_{yx}(x,y)$. But
$$f_{xy}(x,y) = \frac{\partial}{\partial y}f_x(x,y) = \frac{\partial}{\partial y}e^y = e^y$$
$$f_{yx}(x,y) = \frac{\partial}{\partial x}f_y(x,y) = \frac{\partial}{\partial x}e^x = e^x$$
are not equal.

▸▸ Stage 2

2.3.3 Find the specified partial derivatives.
(a) $f(x,y) = x^2y^3$; $f_{xx}(x,y)$, $f_{xyy}(x,y)$, $f_{yxy}(x,y)$
(b) $f(x,y) = e^{xy^2}$; $f_{xx}(x,y)$, $f_{xy}(x,y)$, $f_{xxy}(x,y)$, $f_{xyy}(x,y)$
(c) $f(u,v,w) = \dfrac{1}{u+2v+3w}$, $\dfrac{\partial^3 f}{\partial u \partial v \partial w}(u,v,w)$, $\dfrac{\partial^3 f}{\partial u \partial v \partial w}(3,2,1)$

Solution (a) We have

$$f_x(x,y) = 2xy^3 \qquad f_{xx}(x,y) = 2y^3$$
$$f_{xy}(x,y) = 6xy^2 \qquad f_{yxy}(x,y) = f_{xyy}(x,y) = 12xy$$

(b) We have

$$\begin{aligned} f_x(x,y) = y^2e^{xy^2} \quad f_{xx}(x,y) &= y^4e^{xy^2} & f_{xxy}(x,y) &= 4y^3e^{xy^2} + 2xy^5e^{xy^2} \\ f_{xy}(x,y) &= 2ye^{xy^2} + 2xy^3e^{xy^2} & f_{xyy}(x,y) &= \big(2 + 4xy^2 + 6xy^2 + 4x^2y^4\big)e^{xy^2} \\ & & &= \big(2 + 10xy^2 + 4x^2y^4\big)e^{xy^2} \end{aligned}$$

(c) We have

$$\begin{aligned} \frac{\partial f}{\partial u}(u,v,w) &= -\frac{1}{(u+2v+3w)^2} \\ \frac{\partial^2 f}{\partial u\,\partial v}(u,v,w) &= \frac{4}{(u+2v+3w)^3} \\ \frac{\partial^3 f}{\partial u\,\partial v\,\partial w}(u,v,w) &= -\frac{36}{(u+2v+3w)^4} \end{aligned}$$

In particular

$$\frac{\partial^3 f}{\partial u\,\partial v\,\partial w}(3,2,1) = -\frac{36}{(3+2\times 2+3\times 1)^4} = -\frac{36}{10^4} = -\frac{9}{2500}$$

2.3.4 Find all second partial derivatives of $f(x,y) = \sqrt{x^2+5y^2}$.

Solution Let $f(x,y) = \sqrt{x^2+5y^2}$. Then

$$\begin{aligned} f_x &= \frac{x}{\sqrt{x^2+5y^2}} & f_{xx} &= \frac{1}{\sqrt{x^2+5y^2}} - \frac{1}{2}\frac{(x)(2x)}{(x^2+5y^2)^{3/2}} & f_{xy} &= -\frac{1}{2}\frac{(x)(10y)}{(x^2+5y^2)^{3/2}} \\ f_y &= \frac{5y}{\sqrt{x^2+5y^2}} & f_{yy} &= \frac{5}{\sqrt{x^2+5y^2}} - \frac{1}{2}\frac{(5y)(10y)}{(x^2+5y^2)^{3/2}} & f_{yx} &= -\frac{1}{2}\frac{(5y)(2x)}{(x^2+5y^2)^{3/2}} \end{aligned}$$

Simplifying, and in particular using that $\frac{1}{\sqrt{x^2+5y^2}} = \frac{x^2+5y^2}{(x^2+5y^2)^{3/2}}$,

$$f_{xx} = \frac{5y^2}{(x^2+5y^2)^{3/2}} \qquad f_{xy} = f_{yx} = -\frac{5xy}{(x^2+5y^2)^{3/2}} \qquad f_{yy} = \frac{5x^2}{(x^2+5y^2)^{3/2}}$$

2.3.5 Find the specified partial derivatives.
(a) $f(x,y,z) = \arctan\left(e^{\sqrt{xy}}\right)$; $f_{xyz}(x,y,z)$
(b) $f(x,y,z) = \arctan\left(e^{\sqrt{xy}}\right) + \arctan\left(e^{\sqrt{xz}}\right) + \arctan\left(e^{\sqrt{yz}}\right)$; $f_{xyz}(x,y,z)$
(c) $f(x,y,z) = \arctan\left(e^{\sqrt{xyz}}\right)$; $f_{xx}(1,0,0)$

Solution (a) As $f(x,y,z) = \arctan\left(e^{\sqrt{xy}}\right)$ is independent of z, we have $f_z(x,y,z) = 0$ and hence

$$f_{xyz}(x,y,z) = f_{zxy}(x,y,z) = 0$$

(b) Write $u(x,y,z) = \arctan\left(e^{\sqrt{xy}}\right)$, $v(x,y,z) = \arctan\left(e^{\sqrt{xz}}\right)$ and $w(x,y,z) = \arctan\left(e^{\sqrt{yz}}\right)$. Then

- As $u(x,y,z) = \arctan\left(e^{\sqrt{xy}}\right)$ is independent of z, we have $u_z(x,y,z) = 0$ and hence $u_{xyz}(x,y,z) = u_{zxy}(x,y,z) = 0$
- As $v(x,y,z) = \arctan\left(e^{\sqrt{xz}}\right)$ is independent of y, we have $v_y(x,y,z) = 0$ and hence $v_{xyz}(x,y,z) = v_{yxz}(x,y,z) = 0$
- As $w(x,y,z) = \arctan\left(e^{\sqrt{yz}}\right)$ is independent of x, we have $w_x(x,y,z) = 0$ and hence $w_{xyz}(x,y,z) = 0$

As $f(x,y,z) = u(x,y,z) + v(x,y,z) + w(x,y,z)$, we have

$$f_{xyz}(x,y,z) = u_{xyz}(x,y,z) + v_{xyz}(x,y,z) + w_{xyz}(x,y,z) = 0$$

(c) In the course of evaluating $f_{xx}(x,0,0)$, both y and z are held fixed at 0. Thus, if we set $g(x) = f(x,0,0)$, then $f_{xx}(x,0,0) = g''(x)$. Now

$$g(x) = f(x,0,0) = \arctan\left(e^{\sqrt{xyz}}\right)\Big|_{y=z=0} = \arctan(1) = \frac{\pi}{4}$$

for all x. So $g'(x) = 0$ and $g''(x) = 0$ for all x. In particular,

$$f_{xx}(1,0,0) = g''(1) = 0$$

2.3.6 (*) Let $f(r,\theta) = r^m \cos m\theta$ be a function of r and θ, where m is a positive integer.
(a) Find the second order partial derivatives f_{rr}, $f_{r\theta}$, $f_{\theta\theta}$ and evaluate their respective values at $(r,\theta) = (1,0)$.
(b) Determine the value of the real number λ so that $f(r,\theta)$ satisfies the differential equation

$$f_{rr} + \frac{\lambda}{r} f_r + \frac{1}{r^2} f_{\theta\theta} = 0$$

Solution (a) The first order derivatives are

$$f_r(r,\theta) = mr^{m-1}\cos m\theta \qquad f_\theta(r,\theta) = -mr^m \sin m\theta$$

The second order derivatives are

$$f_{rr}(r,\theta) = m(m-1)r^{m-2}\cos m\theta \quad f_{r\theta}(r,\theta) = -m^2r^{m-1}\sin m\theta \quad f_{\theta\theta}(r,\theta) = -m^2r^m\cos m\theta$$

so that

$$f_{rr}(1,0) = m(m-1),\ f_{r\theta}(1,0) = 0,\ f_{\theta\theta}(1,0) = -m^2$$

(b) By part (a), the expression

$$f_{rr} + \frac{\lambda}{r}f_r + \frac{1}{r^2}f_{\theta\theta} = m(m-1)r^{m-2}\cos m\theta + \lambda mr^{m-2}\cos m\theta - m^2r^{m-2}\cos m\theta$$

vanishes for all r and θ if and only if

$$m(m-1) + \lambda m - m^2 = 0 \iff m(\lambda-1) = 0 \iff \lambda = 1$$

▸▸ Stage 3

2.3.7 Let $\alpha > 0$ be a constant. Show that $u(x,y,z,t) = \frac{1}{t^{3/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)}$ satisfies the heat equation

$$u_t = \alpha\big(u_{xx} + u_{yy} + u_{zz}\big)$$

for all $t > 0$

Solution As

$$\begin{aligned}
u_t(x,y,z,t) &= -\frac{3}{2}\frac{1}{t^{5/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)} + \frac{1}{4\alpha\, t^{7/2}}(x^2+y^2+z^2)e^{-(x^2+y^2+z^2)/(4\alpha t)} \\
u_x(x,y,z,t) &= -\frac{x}{2\alpha\, t^{5/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)} \\
u_{xx}(x,y,z,t) &= -\frac{1}{2\alpha\, t^{5/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)} + \frac{x^2}{4\alpha^2\, t^{7/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)} \\
u_{yy}(x,y,z,t) &= -\frac{1}{2\alpha\, t^{5/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)} + \frac{y^2}{4\alpha^2\, t^{7/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)} \\
u_{zz}(x,y,z,t) &= -\frac{1}{2\alpha\, t^{5/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)} + \frac{z^2}{4\alpha^2\, t^{7/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)}
\end{aligned}$$

we have

$$\alpha\big(u_{xx} + u_{yy} + u_{zz}\big) = -\frac{3}{2\, t^{5/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)} + \frac{x^2+y^2+z^2}{4\alpha\, t^{7/2}}e^{-(x^2+y^2+z^2)/(4\alpha t)} = u_t$$

2.4▲ The Chain Rule

▸▸ Stage 1

2.4.1 Write out the chain rule for each of the following functions.

(a) $\frac{\partial h}{\partial x}$ for $h(x,y) = f\big(x, u(x,y)\big)$

(b) $\frac{\mathrm{d}h}{\mathrm{d}x}$ for $h(x) = f\big(x, u(x), v(x)\big)$

(c) $\frac{\partial h}{\partial x}$ for $h(x,y,z) = f\big(u(x,y,z), v(x,y), w(x)\big)$

Solution

(c) We'll start with part (c) and follow the procedure given in §2.4.1 in the CLP-3 text. We are to compute the derivative of $h(x,y,z) = f\big(u(x,y,z), v(x,y), w(x)\big)$ with respect to x. For this function, the template of Step 2 in §2.4.1 is

$$\frac{\partial h}{\partial x} = \frac{\partial f}{}\frac{}{\partial x}$$

Note that

- The function h appears once in the numerator on the left. The function f, from which h is constructed by a change of variables, appears once in the numerator on the right.
- The variable, x, in the denominator on the left appears once in the denominator on the right.

Now we fill in the blanks with every variable that makes sense. In particular, since f is a function of u, v and w, it may only be differentiated with respect to u, v and w. So we add together three copies of our template — one for each of u, v and w:

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}x}$$

Since w is a function of only one variable, we use the ordinary derivative symbol $\frac{\mathrm{d}w}{\mathrm{d}x}$, rather than the partial derivative symbol $\frac{\partial w}{\partial x}$ in the third copy. Finally we put in the only functional dependence that makes sense. The left hand side is a function of x, y and z, because h is a function of x, y and z. Hence the right hand side must also be a function of x, y and z. As f is a function of u, v and w, this is achieved by evaluating f at $u = u(x,y,z)$, $v = v(x,y)$ and $w = w(x)$.

$$\begin{aligned}\frac{\partial h}{\partial x}(x,y,z) = \frac{\partial f}{\partial u}\big(u(x,y,z), v(x,y), w(x)\big)\frac{\partial u}{\partial x}(x,y,z) + \frac{\partial f}{\partial v}\big(u(x,y,z), v(x,y), w(x)\big)\frac{\partial v}{\partial x}(x,y)\\ + \frac{\partial f}{\partial w}\big(u(x,y,z), v(x,y), w(x)\big)\frac{\mathrm{d}w}{\mathrm{d}x}(x)\end{aligned}$$

(a) We again follow the procedure given in §2.4.1 in the CLP-3 text. We are to compute the derivative of $h(x,y) = f\big(x, u(x,y)\big)$ with respect to x. For this function, the template of Step 2 in §2.4.1 is

$$\frac{\partial h}{\partial x} = \frac{\partial f}{}\frac{}{\partial x}$$

Now we fill in the blanks with every variable that makes sense. In particular, since f is a function of x and u, it may only be differentiated with respect to x, and u. So we add together two copies of our template — one for x and one for u:

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}$$

In $\frac{\mathrm{d}x}{\mathrm{d}x}$ we are to differentiate the (explicit) function x (i.e. the function $F(x) = x$) with respect to x. The answer is of course 1. So

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

Finally we put in the only functional depedence that makes sense. The left hand side is a function of x, and y, because h is a function of x and y. Hence the right hand side must also be a function of x and y. As f is a function of x, u, this is achieved by evaluating f at $u = u(x, y)$.

$$\frac{\partial h}{\partial x}(x, y) = \frac{\partial f}{\partial x}\big(x, u(x, y)\big) + \frac{\partial f}{\partial u}\big(x, u(x, y)\big)\frac{\partial u}{\partial x}(x, y)$$

(b) Yet again we follow the procedure given in §2.4.1 in the CLP-3 text. We are to compute the derivative of $h(x) = f\big(x, u(x), v(x)\big)$ with respect to x. For this function, the template of Step 2 in §2.4.1 is

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{\partial f}{}\frac{}{\partial x}$$

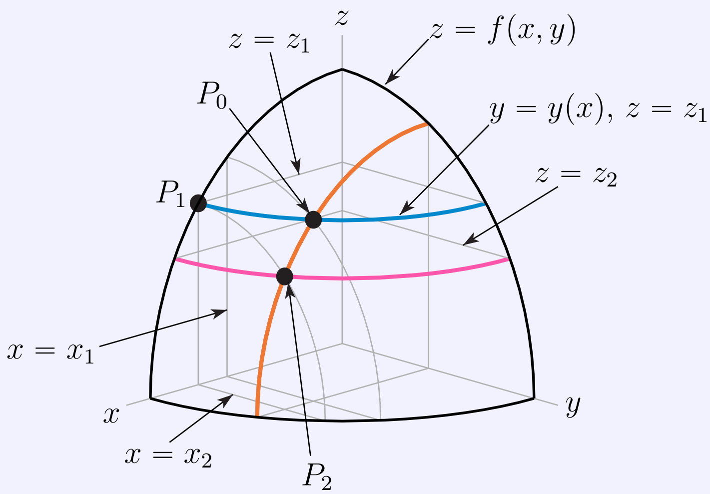
(As h is function of only one variable, we use the ordinary derivative symbol $\frac{\mathrm{d}h}{\mathrm{d}x}$ on the left hand side.) Now we fill in the blanks with every variable that makes sense. In particular, since f is a function of x, u and v, it may only be differentiated with respect to x, u and v. So we add together three copies of our template — one for each of x, u and v:

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}x} &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial f}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial f}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}x}\\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial f}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}x}\end{aligned}$$

Finally we put in the only functional depedence that makes sense.

$$\frac{\mathrm{d}h}{\mathrm{d}x}(x) = \frac{\partial f}{\partial x}\big(x, u(x), v(x)\big) + \frac{\partial f}{\partial u}\big(x, u(x), v(x)\big)\frac{\mathrm{d}u}{\mathrm{d}x}(x) + \frac{\partial f}{\partial v}\big(x, u(x), v(x)\big)\frac{\mathrm{d}v}{\mathrm{d}x}(x)$$

2.4.2 A piece of the surface $z = f(x, y)$ is shown below for some continuously differentiable function $f(x, y)$. The level curve $f(x, y) = z_1$ is marked with a blue line. The three points P_0, P_1, and P_2 lie on the surface.



On the level curve $z = z_1$, we can think of y as a function of x. Let $w(x) = f(x, y(x)) = z_1$. We approximate, at P_0, $f_x(x, y) \approx \frac{\Delta f}{\Delta x}$ and $\frac{dw}{dx}(x) \approx \frac{\Delta w}{\Delta x}$. Identify the quantities Δf, Δw, and Δx from the diagram.

Solution

To visualize, in a simplified setting, the situation from Example 2.4.10 in CLP3, note that $w'(x)$ is the rate of change of z as we slide along the blue line, while $f_x(x, y)$ is the change of z as we slide along the orange line.

In the partial derivative $f_x(x, y) \approx \frac{\Delta f}{\Delta x}$, we let x change, while y stays the same. Necessarily, that forces f to change as well. Starting at point P_0, if we move x but keep y fixed, we end up at P_2. According to the labels on the diagram, Δx is $x_2 - x_1$, and Δf is $z_2 - z_1$.

The function $w(x)$ is a constant function, so we expect $w'(x) = 0$. In the approximation $\frac{\mathrm{d}w}{\mathrm{d}x} \approx \frac{\Delta w}{\Delta x}$, we let x change, but w stays the same. Necessarily, to stay on the surface, this forces y to change. Starting at point P_0, if we move x but keep $z = f(x, y)$ fixed, we end up at P_1. According to the labels on the diagram, Δx is $x_2 - x_1$ again, and $\Delta w = z_1 - z_1 = 0$.

To compare the two situations, note the first case has $\Delta y = 0$ while the second case has $\Delta f = 0$.

2.4.3 (∗) Let $w = f(x, y, t)$ with x and y depending on t. Suppose that at some point (x, y) and at some time t, the partial derivatives f_x, f_y and f_t are equal to 2, -3 and 5 respectively, while $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 2$. Find and explain the difference between $\frac{\mathrm{d}w}{\mathrm{d}t}$ and f_t.

Solution We are told in the statement of the question that $w(t) = f\big(x(t), y(t), t\big)$. Applying the chain rule to $w(t) = f\big(x(t), y(t), t\big)$, by following the procedure given in

§2.4.1 in the CLP-3 tex, gives

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}t}(t) &= \frac{\partial f}{\partial x}\big(x(t),y(t),t\big)\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \frac{\partial f}{\partial y}\big(x(t),y(t),t\big)\frac{\mathrm{d}y}{\mathrm{d}t}(t) + \frac{\partial f}{\partial t}\big(x(t),y(t),t\big)\frac{\mathrm{d}t}{\mathrm{d}t}\\ &= \frac{\partial f}{\partial x}\big(x(t),y(t),t\big)\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \frac{\partial f}{\partial y}\big(x(t),y(t),t\big)\frac{\mathrm{d}y}{\mathrm{d}t}(t) + \frac{\partial f}{\partial t}\big(x(t),y(t),t\big)\end{aligned}$$

Substituting in the values given in the question

$$\frac{\mathrm{d}w}{\mathrm{d}t} = 2\times 1 - 3\times 2 + 5 = 1$$

On the other hand, we are told explicitly in the question that f_t is 5. The reason that f_t and $\frac{\mathrm{d}w}{\mathrm{d}t}$ are different is that

- f_t gives the rate of change of $f(x,y,t)$ as t varies while x and y are held fixed, but
- $\frac{\mathrm{d}w}{\mathrm{d}t}$ gives the rate of change of $f\big(x(t),y(t),t\big)$. For the latter all of $x = x(t)$, $y = y(t)$ and t are changing at once.

2.4.4 Thermodynamics texts use the relationship

$$\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial x}{\partial z}\right) = -1$$

Explain the meaning of this equation and prove that it is true.

Solution The basic assumption is that the three quantites x, y and z are not independent. Given any two of them, the third is uniquely determined. They are assumed to satisfy a relationship $F(x,y,z) = 0$, which can be solved to

- determine x as a function of y and z (say $x = f(y,z)$) and can alternatively be solved to
- determine y as a function of x and z (say $y = g(x,z)$) and can alternatively be solved to
- determine z as a function of x and y (say $z = h(x,y)$).

As an example, if $F(x,y,z) = xyz - 1$, then

- $F(x,y,z) = xyz - 1 = 0$ implies that $x = \frac{1}{yz} = f(y,z)$ and
- $F(x,y,z) = xyz - 1 = 0$ implies that $y = \frac{1}{xz} = g(x,z)$ and
- $F(x,y,z) = xyz - 1 = 0$ implies that $z = \frac{1}{xy} = h(x,y)$

In general, saying that $F(x,y,z) = 0$ determines $x = f(y,z)$ means that

$$F\big(f(y,z),y,z\big) = 0 \tag{$*$}$$

for all y and z. Set $\mathcal{F}(y,z) = F\big(f(y,z),y,z\big)$. Applying the chain rule to $\mathcal{F}(y,z) = F\big(f(y,z),y,z\big)$ (with y and z independent variables) gives

$$\frac{\partial \mathcal{F}}{\partial z}(y,z) = \frac{\partial F}{\partial x}\big(f(y,z),y,z\big)\frac{\partial f}{\partial z}(y,z) + \frac{\partial F}{\partial z}\big(f(y,z),y,z\big)$$

The equation $(*)$ says that $\mathcal{F}(y,z) = F\big(f(y,z),y,z\big) = 0$ for all y and z. So differentiating the equation $(*)$ with respect to z gives

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial z}(y,z) &= \frac{\partial F}{\partial x}\big(f(y,z),y,z\big)\frac{\partial f}{\partial z}(y,z) + \frac{\partial F}{\partial z}\big(f(y,z),y,z\big) = 0 \\ \implies \quad \frac{\partial f}{\partial z}(y,z) &= -\frac{\frac{\partial F}{\partial z}\big(f(y,z),y,z\big)}{\frac{\partial F}{\partial x}\big(f(y,z),y,z\big)} \end{aligned}$$

for all y and z. Similarly, differentiating $F\big(x,g(x,z),z\big) = 0$ with respect to x and $F\big(x,y,h(x,y)\big) = 0$ with respect to y gives

$$\frac{\partial g}{\partial x}(x,z) = -\frac{\frac{\partial F}{\partial x}\big(x,g(x,z),z\big)}{\frac{\partial F}{\partial y}\big(x,g(x,z),z\big)} \qquad \frac{\partial h}{\partial y}(x,y) = -\frac{\frac{\partial F}{\partial y}\big(x,y,h(x,y)\big)}{\frac{\partial F}{\partial z}\big(x,y,h(x,y)\big)}$$

If (x,y,z) is any point satisfying $F(x,y,z) = 0$ (so that $x = f(y,z)$ and $y = g(x,z)$ and $z = h(x,y)$), then

$$\frac{\partial f}{\partial z}(y,z) = -\frac{\frac{\partial F}{\partial z}(x,y,z)}{\frac{\partial F}{\partial x}(x,y,z)} \qquad \frac{\partial g}{\partial x}(x,z) = -\frac{\frac{\partial F}{\partial x}(x,y,z)}{\frac{\partial F}{\partial y}(x,y,z)} \qquad \frac{\partial h}{\partial y}(x,y) = -\frac{\frac{\partial F}{\partial y}(x,y,z)}{\frac{\partial F}{\partial z}(x,y,z)}$$

and

$$\begin{aligned} \frac{\partial f}{\partial z}(y,z)\ \frac{\partial g}{\partial x}(x,z)\ \frac{\partial h}{\partial y}(x,y) &= -\frac{\frac{\partial F}{\partial z}(x,y,z)}{\frac{\partial F}{\partial x}(x,y,z)}\ \frac{\frac{\partial F}{\partial x}(x,y,z)}{\frac{\partial F}{\partial y}(x,y,z)}\ \frac{\frac{\partial F}{\partial y}(x,y,z)}{\frac{\partial F}{\partial z}(x,y,z)} \\ &= -1 \end{aligned}$$

$\boxed{2.4.5}$ What is wrong with the following argument? Suppose that $w = f(x,y,z)$ and $z = g(x,y)$. By the chain rule,

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}$$

Hence $0 = \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}$ and so $\frac{\partial w}{\partial z} = 0$ or $\frac{\partial z}{\partial x} = 0$.

$\boxed{\text{Solution}}$ The problem is that $\frac{\partial w}{\partial x}$ is used to represent two completely different functions in the same equation. The careful way to write the equation is the following. Let $f(x,y,z)$ and $g(x,y)$ be continuously differentiable functions and define $w(x,y) = f\big(x,y,g(x,y)\big)$. By the chain rule,

$$\begin{aligned} \frac{\partial w}{\partial x}(x,y) &= \frac{\partial f}{\partial x}\big(x,y,g(x,y)\big)\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\big(x,y,g(x,y)\big)\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\big(x,y,g(x,y)\big)\frac{\partial g}{\partial x}(x,y) \\ &= \frac{\partial f}{\partial x}\big(x,y,g(x,y)\big) + \frac{\partial f}{\partial z}\big(x,y,g(x,y)\big)\frac{\partial g}{\partial x}(x,y) \end{aligned}$$

While $w(x,y) = f\big(x,y,g(x,y)\big)$, it is not true that $\frac{\partial w}{\partial x}(x,y) = \frac{\partial f}{\partial x}\big(x,y,g(x,y)\big)$. For example, take $f(x,y,z) = x - z$ and $g(x,y) = x$. Then $w(x,y) = f\big(x,y,g(x,y)\big) = x - g(x,y) = 0$ for all (x,y), so that $\frac{\partial w}{\partial x}(x,y) = 0$ while $\frac{\partial f}{\partial x}(x,y,z) = 1$ for all (x,y,z).

▸▸ Stage 2

2.4.6 Use two methods (one using the chain rule) to evaluate $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ given that the function $w = x^2 + y^2 + z^2$, with $x = st,\ y = s\cos t$ and $z = s\sin t$.

Solution Method 1: Since $w(s,t) = x(s,t)^2 + y(s,t)^2 + z(s,t)^2$ with $x(s,t) = st$, $y(s,t) = s\cos t$ and $z(s,t) = s\sin t$ we can write out $w(s,t)$ explicitly:

$$\begin{aligned} w(s,t) &= (st)^2 + (s\cos t)^2 + (s\sin t)^2 = s^2(t^2+1) \\ \implies w_s(s,t) &= 2s(t^2+1) \qquad \text{and} \qquad w_t(s,t) = s^2(2t) \end{aligned}$$

Method 2: The question specifies that $w(s,t) = x(s,t)^2 + y(s,t)^2 + z(s,t)^2$ with $x(s,t) = st$, $y(s,t) = s\cos t$ and $z(s,t) = s\sin t$. That is, $w(s,t) = W\big(x(s,t), y(s,t), z(s,t)\big)$ with $W(x,y,z) = x^2 + y^2 + z^2$. Applying the chain rule to $w(s,t) = W\big(x(s,t), y(s,t), z(s,t)\big)$ and noting that $\frac{\partial W}{\partial x} = 2x$, $\frac{\partial W}{\partial y} = 2y$, $\frac{\partial W}{\partial z} = 2z$, gives

$$\begin{aligned} \frac{\partial w}{\partial s}(s,t) &= \frac{\partial W}{\partial x}\big(x(s,t),y(s,t),z(s,t)\big)\frac{\partial x}{\partial s}(s,t) + \frac{\partial W}{\partial y}\big(x(s,t),y(s,t),z(s,t)\big)\frac{\partial y}{\partial s}(s,t) \\ &\qquad + \frac{\partial W}{\partial z}\big(x(s,t),y(s,t),z(s,t)\big)\frac{\partial z}{\partial s}(s,t) \\ &= 2x(s,t)\ x_s(s,t) + 2y(s,t)\ y_s(s,t) + 2z(s,t)\ z_s(s,t) \\ &= 2(st)\ t + 2(s\cos t)\ \cos t + 2(s\sin t)\ \sin t \\ &= 2st^2 + 2s \\ \frac{\partial w}{\partial t}(s,t) &= \frac{\partial W}{\partial x}\big(x(s,t),y(s,t),z(s,t)\big)\frac{\partial x}{\partial t}(s,t) + \frac{\partial W}{\partial y}\big(x(s,t),y(s,t),z(s,t)\big)\frac{\partial y}{\partial t}(s,t) \\ &\qquad + \frac{\partial W}{\partial z}\big(x(s,t),y(s,t),z(s,t)\big)\frac{\partial z}{\partial t}(s,t) \\ &= 2x(s,t)\ x_t(s,t) + 2y(s,t)\ y_t(s,t) + 2z(s,t)\ z_t(s,t) \\ &= 2(st)\ s + 2(s\cos t)\ (-s\sin t) + 2(s\sin t)\ (s\cos t) \\ &= 2s^2t \end{aligned}$$

2.4.7 Evaluate $\frac{\partial^3}{\partial x\partial y^2} f(2x+3y, xy)$ in terms of partial derivatives of f. You may assume that f is a smooth function so that the Chain Rule and Clairaut's Theorem on the equality of the mixed partial derivatives apply.

Solution By definition,

$$\frac{\partial^3}{\partial x\partial y^2} f(2x+3y,xy) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} f(2x+3y,xy)\right)\right]$$

We'll compute the derivatives from the inside out. Let's call $F(x,y) = f(2x+3y,xy)$ so that the innermost derivative is $G(x,y) = \frac{\partial}{\partial y} f(2x+3y, xy) = \frac{\partial}{\partial y} F(x,y)$. By the chain rule

$$\begin{aligned} G(x,y) = \frac{\partial}{\partial y} F(x,y) &= f_1(2x+3y, xy) \frac{\partial}{\partial y}(2x+3y) + f_2(2x+3y, xy) \frac{\partial}{\partial y}(xy) \\ &= 3f_1(2x+3y, xy) + x f_2(2x+3y, xy) \end{aligned}$$

Here the subscript 1 means take the partial derivative of f with respect to the first argument while holding the second argument fixed, and the subscript 2 means take the partial derivative of f with respect to the second argument while holding the first argument fixed. Next call the middle derivative $H(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} f(2x+3y, xy)\right)$ so that

$$\begin{aligned} H(x,y) &= \frac{\partial}{\partial y} G(x,y) \\ &= \frac{\partial}{\partial y}\Big(3f_1(2x+3y, xy) + x f_2(2x+3y, xy)\Big) \\ &= 3\frac{\partial}{\partial y}\Big(f_1(2x+3y, xy)\Big) + x\frac{\partial}{\partial y}\Big(f_2(2x+3y, xy)\Big) \end{aligned}$$

By the chain rule (twice),

$$\begin{aligned} \frac{\partial}{\partial y}\Big(f_1(2x+3y, xy)\Big) &= f_{11}(2x+3y, xy)\frac{\partial}{\partial y}(2x+3y) + f_{12}(2x+3y, xy)\frac{\partial}{\partial y}(xy) \\ &= 3f_{11}(2x+3y, xy) + x f_{12}(2x+3y, xy) \\ \frac{\partial}{\partial y}\Big(f_2(2x+3y, xy)\Big) &= f_{21}(2x+3y, xy)\frac{\partial}{\partial y}(2x+3y) + f_{22}(2x+3y, xy)\frac{\partial}{\partial y}(xy) \\ &= 3f_{21}(2x+3y, xy) + x f_{22}(2x+3y, xy) \end{aligned}$$

so that

$$\begin{aligned} H(x,y) &= 3\Big(3f_{11}(2x+3y, xy) + x f_{12}(2x+3y, xy)\Big) \\ &\qquad\qquad + x\Big(3f_{21}(2x+3y, xy) + x f_{22}(2x+3y, xy)\Big) \\ &= 9f_{11}(2x+3y, xy) + 6x f_{12}(2x+3y, xy) + x^2 f_{22}(2x+3y, xy) \end{aligned}$$

In the last equality we used that $f_{21}(2x+3y, xy) = f_{12}(2x+3y, xy)$. The notation f_{21} means first differentiate with respect to the second argument and then differentiate with respect to the first argument. For example, if $f(x,y) = e^{2y}\sin x$, then

$$f_{21}(x,y) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(e^{2y}\sin x\right)\right] = \frac{\partial}{\partial x}\left[2e^{2y}\sin x\right] = 2e^{y}\cos x$$

Finally, we get to

$$\begin{aligned}
\frac{\partial^3}{\partial x\partial y^2} f(2x+3y,xy) &= \frac{\partial}{\partial x} H(x,y) \\
&= \frac{\partial}{\partial x}\Big(9f_{11}(2x+3y,xy) + 6xf_{12}(2x+3y,xy) + x^2 f_{22}(2x+3y,xy)\Big) \\
&= 9\frac{\partial}{\partial x}\Big(f_{11}(2x+3y,xy)\Big) \\
&\quad + 6f_{12}(2x+3y,xy) + 6x\frac{\partial}{\partial x}\Big(f_{12}(2x+3y,xy)\Big) \\
&\quad + 2xf_{22}(2x+3y,xy) + x^2\frac{\partial}{\partial x}\Big(f_{22}(2x+3y,xy)\Big)
\end{aligned}$$

By three applications of the chain rule

$$\begin{aligned}
\frac{\partial^3}{\partial x\partial y^2} f(2x+3y,xy) &= 9\Big(2f_{111} + yf_{112}\Big) \\
&\quad + 6f_{12} + 6x\Big(2f_{121} + yf_{122}\Big) \\
&\quad + 2xf_{22} + x^2\Big(2f_{221} + yf_{222}\Big) \\
&= 6\, f_{12} + 2x\, f_{22} + 18\, f_{111} + (9y+12x)\, f_{112} + (6xy+2x^2)\, f_{122} + x^2 y\, f_{222}
\end{aligned}$$

All functions on the right hand side have arguments $(2x+3y,xy)$.

> 2.4.8 Find all second order derivatives of $g(s,t) = f(2s+3t, 3s-2t)$. You may assume that $f(x,y)$ is a smooth function so that the Chain Rule and Clairaut's Theorem on the equality of the mixed partial derivatives apply.

Solution The given function is

$$g(s,t) = f(2s+3t, 3s-2t)$$

The first order derivatives are

$$\begin{aligned}
g_s(s,t) &= 2f_1(2s+3t,3s-2t) + 3f_2(2s+3t,3s-2t) \\
g_t(s,t) &= 3f_1(2s+3t,3s-2t) - 2f_2(2s+3t,3s-2t)
\end{aligned}$$

The second order derivatives are

$$\begin{aligned}
g_{ss}(s,t) &= \frac{\partial}{\partial s}\Big(2f_1(2s+3t,3s-2t)+3f_2(2s+3t,3s-2t)\Big)\\
&= 2\Big(2f_{11}+3f_{12}\Big)+3\Big(2f_{21}+3f_{22}\Big)\\
&= 4f_{11}+6f_{12}+6f_{21}+9f_{22}\\
&= 4f_{11}(2s+3t,3s-2t)+12f_{12}(2s+3t,3s-2t)+9f_{22}(2s+3t,3s-2t)\\
g_{st}(s,t) &= \frac{\partial}{\partial t}\Big(2f_1(2s+3t,3s-2t)+3f_2(2s+3t,3s-2t)\Big)\\
&= 2\Big(3f_{11}-2f_{12}\Big)+3\Big(3f_{21}-2f_{22}\Big)\\
&= 6f_{11}(2s+3t,3s-2t)+5f_{12}(2s+3t,3s-2t)-6f_{22}(2s+3t,3s-2t)\\
g_{tt}(s,t) &= \frac{\partial}{\partial t}\Big(3f_1(2s+3t,3s-2t)-2f_2(2s+3t,3s-2t)\Big)\\
&= 3\Big(3f_{11}-2f_{12}\Big)-2\Big(3f_{21}-2f_{22}\Big)\\
&= 9f_{11}(2s+3t,3s-2t)-12f_{12}(2s+3t,3s-2t)+4f_{22}(2s+3t,3s-2t)
\end{aligned}$$

Here f_1 denotes the partial derivative of f with respect to its first argument, f_{12} is the result of first taking one partial derivative of f with respect to its first argument and then taking a partial derivative with respect to its second argument, and so on.

$\boxed{2.4.9}$ $(*)$ Assume that $f(x,y)$ satisfies Laplace's equation $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$. Show that this is also the case for the composite function $g(s,t)=f(s-t,s+t)$. That is, show that $\frac{\partial^2 g}{\partial s^2}+\frac{\partial^2 g}{\partial t^2}=0$. You may assume that $f(x,y)$ is a smooth function so that the Chain Rule and Clairaut's Theorem on the equality of the mixed partial derivatives apply.

$\boxed{\text{Solution}}$ By the chain rule,

$$\begin{aligned}
\frac{\partial g}{\partial s}(s,t) &= \frac{\partial}{\partial s}f(s-t,s+t)\\
&= \frac{\partial f}{\partial x}(s-t\,,\,s+t)\frac{\partial}{\partial s}(s-t)+\frac{\partial f}{\partial y}(s-t\,,\,s+t)\frac{\partial}{\partial s}(s+t)\\
&= \frac{\partial f}{\partial x}(s-t\,,\,s+t)+\frac{\partial f}{\partial y}(s-t\,,\,s+t)\\
\frac{\partial^2 g}{\partial s^2}(s,t) &= \frac{\partial}{\partial s}\left[\frac{\partial f}{\partial x}(s-t\,,\,s+t)\right]+\frac{\partial}{\partial s}\left[\frac{\partial f}{\partial y}(s-t\,,\,s+t)\right]\\
&= \frac{\partial^2 f}{\partial x^2}(s-t\,,\,s+t)+\frac{\partial^2 f}{\partial y\partial x}(s-t\,,\,s+t)\\
&\quad+\frac{\partial^2 f}{\partial x\partial y}(s-t\,,\,s+t)+\frac{\partial^2 f}{\partial y^2}(s-t\,,\,s+t)\\
&= \left\{\frac{\partial^2 f}{\partial x^2}(s-t\,,\,s+t)+2\frac{\partial^2 f}{\partial x\partial y}(s-t\,,\,s+t)+\frac{\partial^2 f}{\partial y^2}(s-t\,,\,s+t)\right\}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial g}{\partial t}(s,t) &= \frac{\partial}{\partial t} f(s-t, s+t) \\
&= \frac{\partial f}{\partial x}(s-t\,,\, s+t)\frac{\partial}{\partial t}(s-t) + \frac{\partial f}{\partial y}(s-t\,,\, s+t)\frac{\partial}{\partial t}(s+t) \\
&= -\frac{\partial f}{\partial x}(s-t\,,\, s+t) + \frac{\partial f}{\partial y}(s-t\,,\, s+t) \\
\frac{\partial^2 g}{\partial t^2}(s,t) &= -\frac{\partial}{\partial t}\left[\frac{\partial f}{\partial x}(s-t\,,\, s+t)\right] + \frac{\partial}{\partial t}\left[\frac{\partial f}{\partial y}(s-t\,,\, s+t)\right] \\
&= -\Big[-\frac{\partial^2 f}{\partial x^2}(s-t\,,\, s+t) + \frac{\partial^2 f}{\partial y\partial x}(s-t\,,\, s+t)\Big] \\
&\quad + \Big[-\frac{\partial^2 f}{\partial x\partial y}(s-t\,,\, s+t) + \frac{\partial^2 f}{\partial y^2}(s-t\,,\, s+t)\Big] \\
&= \left\{\frac{\partial^2 f}{\partial x^2}(s-t\,,\, s+t) - 2\frac{\partial^2 f}{\partial x\partial y}(s-t\,,\, s+t) + \frac{\partial^2 f}{\partial y^2}(s-t\,,\, s+t)\right\}
\end{aligned}$$

Suppressing the arguments

$$\begin{aligned}
\frac{\partial^2 g}{\partial s^2} + \frac{\partial^2 g}{\partial t^2} &= \left\{\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x\partial y} + \frac{\partial^2 f}{\partial y^2}\right\} + \left\{\frac{\partial^2 f}{\partial x^2} - 2\frac{\partial^2 f}{\partial x\partial y} + \frac{\partial^2 f}{\partial y^2}\right\} \\
&= 2\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right] \\
&= 0
\end{aligned}$$

as desired.

2.4.10 ($*$) Let $z = f(x,y)$ where $x = 2s+t$ and $y = s-t$. Find the values of the constants a, b and c such that

$$a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x\,\partial y} + c\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial s^2} + \frac{\partial^2 z}{\partial t^2}$$

You may assume that $z = f(x,y)$ is a smooth function so that the Chain Rule and Clairaut's Theorem on the equality of the mixed partial derivatives apply.

Solution The notation in the statement of this question is horrendous — the symbol z is used with two different meanings in one equation. On the left hand side, it is a function of x and y, and on the right hand side, it is a function of s and t. Unfortunately that abuse of notation is also very common. Let us undo the notation conflict by renaming the function of s and t to $F(s,t)$. That is,

$$F(s,t) = f\big(2s+t\,,\, s-t\big)$$

In this new notation, we are being asked to find a, b and c so that

$$a\frac{\partial^2 f}{\partial x^2} + b\frac{\partial^2 f}{\partial x\partial y} + c\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 F}{\partial s^2} + \frac{\partial^2 F}{\partial t^2}$$

with the arguments on the right hand side being (s,t) and the arguments on the left hand side being $\big(2s+t\,,\,s-t\big)$.

By the chain rule,

$$\begin{aligned}
\frac{\partial F}{\partial s}(s,t) &= \frac{\partial f}{\partial x}\big(2s+t\,,\,s-t\big)\frac{\partial}{\partial s}(2s+t) + \frac{\partial f}{\partial y}\big(2s+t\,,\,s-t\big)\frac{\partial}{\partial s}(s-t)\\
&= 2\frac{\partial f}{\partial x}\big(2s+t\,,\,s-t\big) + \frac{\partial f}{\partial y}\big(2s+t\,,\,s-t\big)\\
\frac{\partial^2 F}{\partial s^2}(s,t) &= 2\frac{\partial}{\partial s}\left[\frac{\partial f}{\partial x}\big(2s+t\,,\,s-t\big)\right] + \frac{\partial}{\partial s}\left[\frac{\partial f}{\partial y}\big(2s+t\,,\,s-t\big)\right]\\
&= 4\frac{\partial^2 f}{\partial x^2}\big(2s+t\,,\,s-t\big) + 2\frac{\partial^2 f}{\partial y\partial x}\big(2s+t\,,\,s-t\big)\\
&\quad + 2\frac{\partial^2 f}{\partial x\partial y}\big(2s+t\,,\,s-t\big) + \frac{\partial^2 f}{\partial y^2}\big(2s+t\,,\,s-t\big)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial F}{\partial t}(s,t) &= \frac{\partial f}{\partial x}\big(2s+t\,,\,s-t\big)\frac{\partial}{\partial t}(2s+t) + \frac{\partial f}{\partial y}\big(2s+t\,,\,s-t\big)\frac{\partial}{\partial t}(s-t)\\
&= \frac{\partial f}{\partial x}\big(2s+t\,,\,s-t\big) - \frac{\partial f}{\partial y}\big(2s+t\,,\,s-t\big)\\
\frac{\partial^2 F}{\partial t^2}(s,t) &= \frac{\partial}{\partial t}\left[\frac{\partial f}{\partial x}\big(2s+t\,,\,s-t\big)\right] - \frac{\partial}{\partial t}\left[\frac{\partial f}{\partial y}\big(2s+t\,,\,s-t\big)\right]\\
&= \frac{\partial^2 f}{\partial x^2}\big(2s+t\,,\,s-t\big) - \frac{\partial^2 f}{\partial y\partial x}\big(2s+t\,,\,s-t\big)\\
&\quad - \frac{\partial^2 f}{\partial x\partial y}\big(2s+t\,,\,s-t\big) + \frac{\partial^2 f}{\partial y^2}\big(2s+t\,,\,s-t\big)
\end{aligned}$$

Suppressing the arguments

$$\frac{\partial^2 F}{\partial s^2} + \frac{\partial^2 F}{\partial t^2} = 5\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x\partial y} + 2\frac{\partial^2 f}{\partial y^2}$$

Finally, translating back into the (horrendous) notation of the question

$$\frac{\partial^2 z}{\partial s^2} + \frac{\partial^2 z}{\partial t^2} = 5\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x\partial y} + 2\frac{\partial^2 z}{\partial y^2}$$

so that $a=5$ and $b=c=2$.

$\boxed{2.4.11}$ $(*)$ Let F be a function on $\mathbb{R}^2$. Denote points in $\mathbb{R}^2$ by (u,v) and the corresponding partial derivatives of F by $F_u(u,v)$, $F_v(u,v)$, $F_{uu}(u,v)$, $F_{uv}(u,v)$, etc.. Assume those derivatives are all continuous. Express

$$\frac{\partial^2}{\partial x\,\partial y}F(x^2-y^2,2xy)$$

in terms of partial derivatives of the function F.

Solution Let $u(x,y) = x^2 - y^2$, and $v(x,y) = 2xy$. Then $F(x^2-y^2, 2xy) = F\big(u(x,y), v(x,y)\big)$. By the chain rule

$$\begin{aligned}
\frac{\partial}{\partial y} F(x^2-y^2,2xy) &= \frac{\partial}{\partial y} F(u(x,y),v(xy)) \\
&= F_u(u(x,y),v(xy))\frac{\partial u}{\partial y}(x,y) + F_v(u(x,y),v(xy))\frac{\partial v}{\partial y}(x,y) \\
&= F_u(x^2-y^2,2xy)\ (-2y) + F_v(x^2-y^2,2xy)\,(2x) \\
\frac{\partial^2}{\partial x\,\partial y} F(x^2-y^2,2xy) &= \frac{\partial}{\partial x}\Big\{-2yF_u(x^2-y^2,2xy) + 2xF_v(x^2-y^2,2xy)\Big\} \\
&= -2y\frac{\partial}{\partial x}\Big[F_u(x^2-y^2,2xy)\Big] + 2F_v(x^2-y^2,2xy) \\
&\qquad + 2x\frac{\partial}{\partial x}\Big[F_v(x^2-y^2,2xy)\Big] \\
&= -4xy\,F_{uu}(x^2-y^2,2xy) - 4y^2F_{uv}(x^2-y^2,2xy) + 2F_v(x^2-y^2,2xy) \\
&\qquad + 4x^2F_{vu}(x^2-y^2,2xy) + 4xy\,F_{vv}(x^2-y^2,2xy) \\
&= 2\,F_v(x^2-y^2,2xy) - 4xy\,F_{uu}(x^2-y^2,2xy) \\
&\qquad + 4(x^2-y^2)\,F_{uv}(x^2-y^2,2xy) \\
&\qquad + 4xy\,F_{vv}(x^2-y^2,2xy)
\end{aligned}$$

2.4.12 (∗) $u(x,y)$ is defined as

$$u(x,y) = e^y\,F\big(xe^{-y^2}\big)$$

for an arbitrary function $F(z)$.
(a) If $F(z) = \ln(z)$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.
(b) For an arbitrary $F(z)$ show that $u(x,y)$ satisfies

$$2xy\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u$$

Solution For any (differentiable) function F, we have, by the chain and product rules,

$$\begin{aligned}
\frac{\partial u}{\partial x}(x,y) &= \frac{\partial}{\partial x}\Big[e^y\,F\big(xe^{-y^2}\big)\Big] = e^y\,\frac{\partial}{\partial x}\Big[F\big(xe^{-y^2}\big)\Big] \\
&= e^y\,F'\big(xe^{-y^2}\big)\,\frac{\partial}{\partial x}\Big(xe^{-y^2}\Big) \\
&= e^y\,F'\big(xe^{-y^2}\big)\ e^{-y^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial u}{\partial y}(x,y) &= \frac{\partial}{\partial y}\Big[e^y\, F\big(xe^{-y^2}\big)\Big] \\
&= e^y\, F\big(xe^{-y^2}\big) + e^y\, \frac{\partial}{\partial y}\Big[F\big(xe^{-y^2}\big)\Big] \\
&= e^y\, F\big(xe^{-y^2}\big) + e^y\, F'\big(xe^{-y^2}\big)\ \frac{\partial}{\partial y}\Big(xe^{-y^2}\Big) \\
&= e^y\, F\big(xe^{-y^2}\big) + e^y\, F'\big(xe^{-y^2}\big)\ (-2xy)e^{-y^2}
\end{aligned}$$

(a) In particular, when $F(z) = \ln(z)$, $F'(z) = \frac{1}{z}$ and

$$\begin{aligned}
\frac{\partial u}{\partial x}(x,y) &= e^y\, \frac{1}{xe^{-y^2}}\ e^{-y^2} = \frac{e^y}{x} \\
\frac{\partial u}{\partial y}(x,y) &= e^y\, \ln\big(xe^{-y^2}\big) + e^y\, \frac{1}{xe^{-y^2}}\ (-2xy)e^{-y^2} = e^y\, \ln\big(xe^{-y^2}\big) - 2ye^y \\
&= e^y\, \ln(x) - y^2\, e^y - 2ye^y
\end{aligned}$$

(b) In general

$$\begin{aligned}
2xy\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} &= 2xy\ e^y\, F'\big(xe^{-y^2}\big)\ e^{-y^2} + e^y\, F\big(xe^{-y^2}\big) + e^y\, F'\big(xe^{-y^2}\big)\ (-2xy)e^{-y^2} \\
&= e^y\, F\big(xe^{-y^2}\big) \\
&= u
\end{aligned}$$

$\boxed{\text{2.4.13}}$ (∗) Let $f(x)$ and $g(x)$ be two functions of x satisfying $f''(7) = -2$ and $g''(-4) = -1$. If $z = h(s,t) = f(2s+3t) + g(s-6t)$ is a function of s and t, find the value of $\frac{\partial^2 z}{\partial t^2}$ when $s = 2$ and $t = 1$.

$\boxed{\text{Solution}}$ By the chain rule,

$$\begin{aligned}
\frac{\partial h}{\partial t}(s,t) &= \frac{\partial}{\partial t}\big[f(2s+3t)\big] + \frac{\partial}{\partial t}\big[g(s-6t)\big] \\
&= f'(2s+3t)\frac{\partial}{\partial t}(2s+3t) + g(s-6t)\frac{\partial}{\partial t}(s-6t) \\
&= 3f'(2s+3t) - 6g'(s-6t) \\
\frac{\partial^2 h}{\partial t^2}(s,t) &= 3\frac{\partial}{\partial t}\big[f'(2s+3t)\big] - 6\frac{\partial}{\partial t}\big[g'(s-6t)\big] \\
&= 3\, f''(2s+3t)\frac{\partial}{\partial t}(2s+3t) - 6\, g''(s-6t)\frac{\partial}{\partial t}(s-6t) \\
&= 9f''(2s+3t) + 36g''(s-6t)
\end{aligned}$$

In particular

$$\frac{\partial^2 h}{\partial t^2}(2,1) = 9f''(7) + 36g''(-4) = 9(-2) + 36(-1) = -54$$

$\boxed{2.4.14}$ (*) Suppose that $w = f(xz, yz)$, where f is a differentiable function. Show that

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = z\frac{\partial w}{\partial z}$$

$\boxed{\text{Solution}}$ We'll first compute the first order partial derivatives of $w(x, y, z)$. Write $u(x, y, z) = xz$ and $v(x, y, z) = yz$ so that $w(x, y, z) = f\big(u(x, y, z), v(x, y, z)\big)$. By the chain rule,

$$\begin{aligned}
\frac{\partial w}{\partial x}(x,y,z) &= \frac{\partial}{\partial x}\big[f\big(u(x,y,z), v(x,y,z)\big)\big] \\
&= \frac{\partial f}{\partial u}\big(u(x,y,z), v(x,y,z)\big)\frac{\partial u}{\partial x}(x,y,z) + \frac{\partial f}{\partial v}\big(u(x,y,z), v(x,y,z)\big)\frac{\partial v}{\partial x}(x,y,z) \\
&= z\frac{\partial f}{\partial u}(xz, yz) \\
\frac{\partial w}{\partial y}(x,y,z) &= \frac{\partial}{\partial y}\big[f\big(u(x,y,z), v(x,y,z)\big)\big] \\
&= \frac{\partial f}{\partial u}\big(u(x,y,z), v(x,y,z)\big)\frac{\partial u}{\partial y}(x,y,z) + \frac{\partial f}{\partial v}\big(u(x,y,z), v(x,y,z)\big)\frac{\partial v}{\partial y}(x,y,z) \\
&= z\frac{\partial f}{\partial v}(xz, yz) \\
\frac{\partial w}{\partial z}(x,y,z) &= \frac{\partial}{\partial z}\big[f\big(u(x,y,z), v(x,y,z)\big)\big] \\
&= \frac{\partial f}{\partial u}\big(u(x,y,z), v(x,y,z)\big)\frac{\partial u}{\partial z}(x,y,z) + \frac{\partial f}{\partial v}\big(u(x,y,z), v(x,y,z)\big)\frac{\partial v}{\partial z}(x,y,z) \\
&= x\frac{\partial f}{\partial u}(xz, yz) + y\frac{\partial f}{\partial v}(xz, yz)
\end{aligned}$$

So

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = xz\frac{\partial f}{\partial u}(xz, yz) + yz\frac{\partial f}{\partial v}(xz, yz) = z\left[x\frac{\partial f}{\partial u}(xz, yz) + y\frac{\partial f}{\partial v}(xz, yz)\right] = z\frac{\partial w}{\partial z}$$

as desired.

$\boxed{2.4.15}$ (*) Suppose $z = f(x, y)$ has continuous second order partial derivatives, and $x = r\cos t$, $y = r\sin t$. Express the following partial derivatives in terms r, t, and partial derivatives of f.

(a) $\frac{\partial z}{\partial t}$

(b) $\frac{\partial^2 z}{\partial t^2}$

$\boxed{\text{Solution}}$ By definition $z(r, t) = f(r\cos t\,, r\sin t)$.

(a) By the chain rule

$$\begin{aligned}
\frac{\partial z}{\partial t}(r,t) &= \frac{\partial}{\partial t}\Big[f(r\cos t\,,\, r\sin t)\Big] \\
&= \frac{\partial f}{\partial x}(r\cos t\,,\, r\sin t)\frac{\partial}{\partial t}(r\cos t) + \frac{\partial f}{\partial y}(r\cos t\,,\, r\sin t)\frac{\partial}{\partial t}(r\sin t) \\
&= -r\sin t\,\frac{\partial f}{\partial x}(r\cos t\,,\, r\sin t) + r\cos t\,\frac{\partial f}{\partial y}(r\cos t\,,\, r\sin t)
\end{aligned}$$

(b) By linearity, the product rule and the chain rule

$$\begin{aligned}
\frac{\partial^2 z}{\partial t^2}(r,t) &= -\frac{\partial}{\partial t}\left[r\sin t\,\frac{\partial f}{\partial x}(r\cos t\,,\, r\sin t)\right] + \frac{\partial}{\partial t}\left[r\cos t\,\frac{\partial f}{\partial y}(r\cos t\,,\, r\sin t)\right] \\
&= -r\cos t\,\frac{\partial f}{\partial x}(r\cos t\,,\, r\sin t) - r\sin t\,\frac{\partial}{\partial t}\left[\frac{\partial f}{\partial x}(r\cos t\,,\, r\sin t)\right] \\
&\qquad - r\sin t\,\frac{\partial f}{\partial y}(r\cos t\,,\, r\sin t) + r\cos t\,\frac{\partial}{\partial t}\left[\frac{\partial f}{\partial y}(r\cos t\,,\, r\sin t)\right] \\
&= -r\cos t\,\frac{\partial f}{\partial x}(r\cos t\,,\, r\sin t) \\
&\qquad + r^2\sin^2 t\,\frac{\partial^2 f}{\partial x^2}(r\cos t\,,\, r\sin t) - r^2\sin t\cos t\,\frac{\partial^2 f}{\partial y\partial x}(r\cos t\,,\, r\sin t) \\
&\quad - r\sin t\,\frac{\partial f}{\partial y}(r\cos t\,,\, r\sin t) \\
&\qquad - r^2\sin t\cos t\,\frac{\partial^2 f}{\partial x\partial y}(r\cos t\,,\, r\sin t) + r^2\cos^2 t\,\frac{\partial^2 f}{\partial y^2}(r\cos t\,,\, r\sin t) \\
&= -r\cos t\,\frac{\partial f}{\partial x} - r\sin t\,\frac{\partial f}{\partial y} \\
&\qquad + r^2\sin^2 t\,\frac{\partial^2 f}{\partial x^2} - 2r^2\sin t\cos t\,\frac{\partial^2 f}{\partial x\partial y} + r^2\cos^2 t\,\frac{\partial^2 f}{\partial y^2}
\end{aligned}$$

with all of the partial derivatives of f evaluated at $(r\cos t\,,\, r\sin t)$.

2.4.16 (∗) Let $z = f(x,y)$, where $f(x,y)$ has continuous second-order partial derivatives, and

$$f_x(2,1) = 5,\qquad f_y(2,1) = -2,\qquad f_{xx}(2,1) = 2,\qquad f_{xy}(2,1) = 1,\qquad f_{yy}(2,1) = -4$$

Find $\frac{\mathrm{d}^2}{\mathrm{d}t^2}z\big(x(t),y(t)\big)$ when $x(t) = 2t^2$, $y(t) = t^3$ and $t = 1$.

Solution Write $w(t) = z\big(x(t),y(t)\big) = f\big(x(t),y(t)\big)$ with $x(t) = 2t^2$, $y(t) = t^3$. We are to compute $\frac{d^2w}{dt^2}(1)$. By the chain rule

$$\begin{aligned}\frac{dw}{dt}(t) &= \frac{d}{dt}f\big(x(t),y(t)\big)\\ &= f_x(x(t)\,,\,y(t))\,\frac{dx}{dt}(t) + f_y(x(t)\,,\,y(t))\,\frac{dy}{dt}(t)\\ &= 4t\,f_x(x(t)\,,\,y(t)) + 3t^2\,f_y(x(t)\,,\,y(t))\end{aligned}$$

By linearity, the product rule, and the chain rule,

$$\begin{aligned}\frac{d^2}{dt^2}f\big(x(t),y(t)\big) &= \frac{d}{dt}\left[4t\,f_x\big(x(t),y(t)\big)\right] + \frac{d}{dt}\left[3t^2\,f_y\big(x(t),y(t)\big)\right]\\ &= 4\,f_x\big(x(t),y(t)\big) + 4t\frac{d}{dt}\left[f_x\big(x(t),y(t)\big)\right]\\ &\quad + 6t\,f_y\big(x(t),y(t)\big) + 3t^2\frac{d}{dt}\left[f_y\big(x(t),y(t)\big)\right]\\ &= 4\,f_x(2t^2\,,\,t^3) + 4t\Big[f_{xx}\big(x(t),y(t)\big)\,\frac{dx}{dt}(t) + f_{xy}\big(x(t),y(t)\big)\,\frac{dy}{dt}(t)\Big]\\ &\quad + 6t\,f_y(2t^2\,,\,t^3) + 3t^2\Big[f_{yx}\big(x(t),y(t)\big)\,\frac{dx}{dt}(t) + f_{yy}\big(x(t),y(t)\big)\,\frac{dy}{dt}(t)\Big]\\ &= 4\,f_x(2t^2\,,\,t^3) + 16t^2\,f_{xx}(2t^2\,,\,t^3) + 12t^3\,f_{xy}(2t^2\,,\,t^3)\\ &\quad + 6t\,f_y(2t^2\,,\,t^3) + 12t^3\,f_{yx}(2t^2\,,\,t^3) + 9t^4\,f_{yy}(2t^2\,,\,t^3)\end{aligned}$$

In particular, when $t = 1$, and since $f_{xy}(2,1) = f_{yx}(2,1)$,

$$\begin{aligned}\frac{d^2}{dt^2}f\big(x(t),y(t)\big)\Big|_{t=1} &= 4\,(5) + 16\,(2) + 12\,(1)\\ &\quad + 6\,(-2) + 12\,(1) + 9\,(-4)\\ &= 28\end{aligned}$$

$\boxed{2.4.17}$ $(*)$ Assume that the function $F(x,y,z)$ satisfies the equation $\frac{\partial F}{\partial z} = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}$ and the mixed partial derivatives $\frac{\partial^2 F}{\partial x \partial y}$ and $\frac{\partial^2 F}{\partial y \partial x}$ are equal. Let A be some constant and let $G(\gamma,s,t) = F(\gamma+s,\gamma-s,At)$. Find the value of A such that $\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial \gamma^2} + \frac{\partial^2 G}{\partial s^2}$.

$\boxed{\text{Solution}}$ By the chain rule

$$\begin{aligned}\frac{\partial G}{\partial t}(\gamma,s,t) &= \frac{\partial}{\partial t}\Big[F(\gamma+s,\gamma-s,At)\Big]\\ &= \frac{\partial F}{\partial x}(\gamma+s,\gamma-s,At)\,\frac{\partial}{\partial t}(\gamma+s) + \frac{\partial F}{\partial y}(\gamma+s,\gamma-s,At)\,\frac{\partial}{\partial t}(\gamma-s)\\ &\qquad + \frac{\partial F}{\partial z}(\gamma+s,\gamma-s,At)\,\frac{\partial}{\partial t}(At)\\ &= A\,\frac{\partial F}{\partial z}(\gamma+s,\gamma-s,At)\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial G}{\partial \gamma}(\gamma,s,t) &= \frac{\partial}{\partial \gamma}\Big[F(\gamma+s,\gamma-s,At)\Big] \\
&= \frac{\partial F}{\partial x}(\gamma+s,\gamma-s,At)\,\frac{\partial}{\partial \gamma}(\gamma+s) + \frac{\partial F}{\partial y}(\gamma+s,\gamma-s,At)\,\frac{\partial}{\partial \gamma}(\gamma-s) \\
&\qquad + \frac{\partial F}{\partial z}(\gamma+s,\gamma-s,At)\,\frac{\partial}{\partial \gamma}(At) \\
&= \frac{\partial F}{\partial x}(\gamma+s,\gamma-s,At) + \frac{\partial F}{\partial y}(\gamma+s,\gamma-s,At) \qquad \text{(E1)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial G}{\partial s}(\gamma,s,t) &= \frac{\partial}{\partial s}\Big[F(\gamma+s,\gamma-s,At)\Big] \\
&= \frac{\partial F}{\partial x}(\gamma+s,\gamma-s,At)\,\frac{\partial}{\partial s}(\gamma+s) + \frac{\partial F}{\partial y}(\gamma+s,\gamma-s,At)\,\frac{\partial}{\partial s}(\gamma-s) \\
&\qquad + \frac{\partial F}{\partial z}(\gamma+s,\gamma-s,At)\,\frac{\partial}{\partial s}(At) \\
&= \frac{\partial F}{\partial x}(\gamma+s,\gamma-s,At) - \frac{\partial F}{\partial y}(\gamma+s,\gamma-s,At) \qquad \text{(E2)}
\end{aligned}$$

We can evaluate the second derivatives by applying the chain rule to the four terms on the right hand sides of

$$\begin{aligned}
\frac{\partial^2 G}{\partial \gamma^2}(\gamma,s,t) &= \frac{\partial}{\partial \gamma}\Big[\frac{\partial G}{\partial \gamma}(\gamma,s,t)\Big] = \frac{\partial}{\partial \gamma}\Big[\frac{\partial F}{\partial x}(\gamma+s,\gamma-s,At)\Big] + \frac{\partial}{\partial \gamma}\Big[\frac{\partial F}{\partial y}(\gamma+s,\gamma-s,At)\Big] \\
\frac{\partial^2 G}{\partial s^2}(\gamma,s,t) &= \frac{\partial}{\partial s}\Big[\frac{\partial G}{\partial s}(\gamma,s,t)\Big] = \frac{\partial}{\partial s}\Big[\frac{\partial F}{\partial x}(\gamma+s,\gamma-s,At)\Big] - \frac{\partial}{\partial s}\Big[\frac{\partial F}{\partial y}(\gamma+s,\gamma-s,At)\Big]
\end{aligned}$$

Alternatively, we can observe that replacing F by $\frac{\partial F}{\partial x}$ in (E1) and (E2) gives

$$\begin{aligned}
\frac{\partial}{\partial \gamma}\Big[\frac{\partial F}{\partial x}(\gamma+s,\gamma-s,At)\Big] &= \frac{\partial^2 F}{\partial x^2}(\gamma+s,\gamma-s,At) + \frac{\partial^2 F}{\partial y \partial x}(\gamma+s,\gamma-s,At) \\
\frac{\partial}{\partial s}\Big[\frac{\partial F}{\partial x}(\gamma+s,\gamma-s,At)\Big] &= \frac{\partial^2 F}{\partial x^2}(\gamma+s,\gamma-s,At) - \frac{\partial^2 F}{\partial y \partial x}(\gamma+s,\gamma-s,At)
\end{aligned}$$

replacing F by $\frac{\partial F}{\partial y}$ in (E1) and (E2) gives

$$\begin{aligned}
\frac{\partial}{\partial \gamma}\Big[\frac{\partial F}{\partial y}(\gamma+s,\gamma-s,At)\Big] &= \frac{\partial^2 F}{\partial x \partial y}(\gamma+s,\gamma-s,At) + \frac{\partial^2 F}{\partial y^2}(\gamma+s,\gamma-s,At) \\
\frac{\partial}{\partial s}\Big[\frac{\partial F}{\partial y}(\gamma+s,\gamma-s,At)\Big] &= \frac{\partial^2 F}{\partial x \partial y}(\gamma+s,\gamma-s,At) - \frac{\partial^2 F}{\partial y^2}(\gamma+s,\gamma-s,At)
\end{aligned}$$

Consequently

$$\begin{aligned}\frac{\partial^2 G}{\partial \gamma^2}(\gamma,s,t) &= \frac{\partial^2 F}{\partial x^2}(\gamma+s,\gamma-s,At) + \frac{\partial^2 F}{\partial y\partial x}(\gamma+s,\gamma-s,At)\\ &\quad + \frac{\partial^2 F}{\partial x\partial y}(\gamma+s,\gamma-s,At) + \frac{\partial^2 F}{\partial y^2}(\gamma+s,\gamma-s,At)\\ &= \frac{\partial^2 F}{\partial x^2}(\gamma+s,\gamma-s,At) + 2\frac{\partial^2 F}{\partial y\partial x}(\gamma+s,\gamma-s,At) + \frac{\partial^2 F}{\partial y^2}(\gamma+s,\gamma-s,At)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 G}{\partial s^2}(\gamma,s,t) &= \frac{\partial^2 F}{\partial x^2}(\gamma+s,\gamma-s,At) - \frac{\partial^2 F}{\partial y\partial x}(\gamma+s,\gamma-s,At)\\ &\quad - \Big[\frac{\partial^2 F}{\partial x\partial y}(\gamma+s,\gamma-s,At) - \frac{\partial^2 F}{\partial y^2}(\gamma+s,\gamma-s,At)\Big]\\ &= \frac{\partial^2 F}{\partial x^2}(\gamma+s,\gamma-s,At) - 2\frac{\partial^2 F}{\partial y\partial x}(\gamma+s,\gamma-s,At) + \frac{\partial^2 F}{\partial y^2}(\gamma+s,\gamma-s,At)\end{aligned}$$

So, suppressing the arguments,

$$\frac{\partial^2 G}{\partial \gamma^2} + \frac{\partial^2 G}{\partial s^2} - \frac{\partial G}{\partial t} = 2\frac{\partial^2 F}{\partial x^2} + 2\frac{\partial^2 F}{\partial y^2} - A\frac{\partial F}{\partial z} = 2\frac{\partial F}{\partial z} - A\frac{\partial F}{\partial z} = 0$$

if $A = 2$.

$\boxed{2.4.18}$ ($*$) Let $f(x)$ be a differentiable function, and suppose it is given that $f'(0) = 10$. Let $g(s,t) = f(as - bt)$, where a and b are constants. Evaluate $\frac{\partial g}{\partial s}$ at the point $(s,t) = (b,a)$, that is, find $\frac{\partial g}{\partial s}\big|_{(b,a)}$.

$\boxed{\text{Solution}}$ By the chain rule

$$\frac{\partial g}{\partial s}(s,t) = \frac{\partial}{\partial s}\big[f(as-bt)\big] = f'(as-bt)\frac{\partial}{\partial s}(as-bt) = af'(as-bt)$$

In particular

$$\frac{\partial g}{\partial s}(b,a) = af'(ab-ba) = af'(0) = 10a$$

$\boxed{2.4.19}$ ($*$) Let $f(u,v)$ be a differentiable function of two variables, and let z be a differentiable function of x and y defined implicitly by $f(xz,yz) = 0$. Show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -z$$

Solution We are told that the function $z(x,y)$ obeys

$$f\big(x\,z(x,y)\,,\,y\,z(x,y)\big) = 0 \tag{$*$}$$

for all x and y. By the chain rule,

$$\begin{aligned}
&\frac{\partial}{\partial x}\big[f\big(x\,z(x,y)\,,\,y\,z(x,y)\big)\big]\\
&\qquad = f_u\big(x\,z(x,y)\,,\,y\,z(x,y)\big)\frac{\partial}{\partial x}\big[x\,z(x,y)\big] + f_v\big(x\,z(x,y)\,,\,y\,z(x,y)\big)\frac{\partial}{\partial x}\big[y\,z(x,y)\big]\\
&\qquad = f_u\big(x\,z(x,y)\,,\,y\,z(x,y)\big)\,\big[z(x,y) + x\,z_x(x,y)\big] + f_v\big(x\,z(x,y)\,,\,y\,z(x,y)\big)\,y\,z_x(x,y)\\
&\frac{\partial}{\partial y}\big[f\big(x\,z(x,y)\,,\,y\,z(x,y)\big)\big]\\
&\qquad = f_u\big(x\,z(x,y)\,,\,y\,z(x,y)\big)\frac{\partial}{\partial y}\big[x\,z(x,y)\big] + f_v\big(x\,z(x,y)\,,\,y\,z(x,y)\big)\frac{\partial}{\partial y}\big[y\,z(x,y)\big]\\
&\qquad = f_u\big(x\,z(x,y)\,,\,y\,z(x,y)\big)\,x\,z_y(x,y) + f_v\big(x\,z(x,y)\,,\,y\,z(x,y)\big)\,\big[z(x,y) + y\,z_y(x,y)\big]
\end{aligned}$$

so differentiating $(*)$ with respect to x and with respect to y gives

$$\begin{aligned}
f_u\big(x\,z(x,y)\,,\,y\,z(x,y)\big)\,\big[z(x,y) + x\,z_x(x,y)\big] + f_v\big(x\,z(x,y)\,,\,y\,z(x,y)\big)\,yz_x(x,y) &= 0\\
f_u\big(x\,z(x,y)\,,\,y\,z(x,y)\big)\,x\,z_y(x,y) + f_v\big(x\,z(x,y)\,,\,y\,z(x,y)\big)\,\big[z(x,y) + y\,z_y(x,y)\big] &= 0
\end{aligned}$$

or, leaving out the arguments,

$$\begin{aligned}
f_u\,\big[z + x\,z_x\big] + f_v\,yz_x &= 0\\
f_u\,x\,z_y + f_v\,\big[z + y\,z_y\big] &= 0
\end{aligned}$$

Solving the first equation for z_x and the second for z_y gives

$$\begin{aligned}
z_x &= -\frac{z\,f_u}{x\,f_u + y\,f_v}\\
z_y &= -\frac{z\,f_v}{x\,f_u + y\,f_v}
\end{aligned}$$

so that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -\frac{xz\,f_u}{x\,f_u + y\,f_v} - \frac{yz\,f_v}{x\,f_u + y\,f_v} = -\frac{z\,(x\,f_u + y\,f_v)}{x\,f_u + y\,f_v} = -z$$

as desired.

Remark: This is of course under the assumption that $x\,f_u + y\,f_v$ is nonzero. That is equivalent, by the chain rule, to the assumption that $\frac{\partial}{\partial z}\big[f(xz,yz)\big]$ is non zero. That, in turn, is almost, but not quite, equivalent to the statement that $f(xz,yz) = 0$ is can be solved for z as a function of x and y.

2.4.20 $(*)$ Let $w(s,t) = u(2s+3t, 3s-2t)$ for some twice differentiable function $u = u(x,y)$.
(a) Find w_{ss} in terms of u_{xx}, u_{xy} , and u_{yy} (you can assume that $u_{xy} = u_{yx}$).
(b) Suppose $u_{xx} + u_{yy} = 0$. For what constant A will $w_{ss} = Aw_{tt}$?

Solution (a) By the chain rule

$$\begin{aligned} w_s(s,t) &= \frac{\partial}{\partial s}\big[u(2s+3t, 3s-2t)\big] \\ &= u_x(2s+3t, 3s-2t)\frac{\partial}{\partial s}\big(2s+3t\big) + u_y(2s+3t, 3s-2t)\frac{\partial}{\partial s}\big(3s-2t\big) \\ &= 2\,u_x(2s+3t, 3s-2t) + 3\,u_y(2s+3t, 3s-2t) \end{aligned}$$

and

$$\begin{aligned} w_{ss}(s,t) &= 2\frac{\partial}{\partial s}\big[u_x(2s+3t, 3s-2t)\big] + 3\frac{\partial}{\partial s}\big[u_y(2s+3t, 3s-2t)\big] \\ &= \big[4u_{xx}(2s+3t, 3s-2t) + 6u_{xy}(2s+3t, 3s-2t)\big] \\ &\qquad + \big[6u_{yx}(2s+3t, 3s-2t) + 9u_{yy}(2s+3t, 3s-2t)\big] \\ &= 4\,u_{xx}(2s+3t, 3s-2t) + 12\,u_{xy}(2s+3t, 3s-2t) + 9\,u_{yy}(2s+3t, 3s-2t) \end{aligned}$$

(b) Again by the chain rule

$$\begin{aligned} w_t(s,t) &= \frac{\partial}{\partial t}\big[u(2s+3t, 3s-2t)\big] \\ &= u_x(2s+3t, 3s-2t)\frac{\partial}{\partial t}\big(2s+3t\big) + u_y(2s+3t, 3s-2t)\frac{\partial}{\partial t}\big(3s-2t\big) \\ &= 3\,u_x(2s+3t, 3s-2t) - 2\,u_y(2s+3t, 3s-2t) \end{aligned}$$

and

$$\begin{aligned} w_{tt}(s,t) &= 3\frac{\partial}{\partial t}\big[u_x(2s+3t, 3s-2t)\big] - 2\frac{\partial}{\partial t}\big[u_y(2s+3t, 3s-2t)\big] \\ &= \big[9u_{xx}(2s+3t, 3s-2t) - 6u_{xy}(2s+3t, 3s-2t)\big] \\ &\qquad - \big[6u_{yx}(2s+3t, 3s-2t) - 4u_{yy}(2s+3t, 3s-2t)\big] \\ &= 9\,u_{xx}(2s+3t, 3s-2t) - 12\,u_{xy}(2s+3t, 3s-2t) + 4\,u_{yy}(2s+3t, 3s-2t) \end{aligned}$$

Consquently, for any constant A,

$$w_{ss} - Aw_{tt} = (4-9A)u_{xx} + (12+12A)u_{xy} + (9-4A)u_{yy}$$

Given that $u_{xx} + u_{yy} = 0$, this will be zero, as desired, if $A = -1$. (Then $(4-9A) = (9-4A) = 13$.)

2.4.21 $(*)$ Suppose that $f(x,y)$ is twice differentiable (with $f_{xy} = f_{yx}$), and $x = r\cos\theta$ and $y = r\sin\theta$.

(a) Evaluate f_θ, f_r and $f_{r\theta}$ in terms of r, θ and partial derivatives of f with respect to x and y.

(b) Let $g(x,y)$ be another function satisfying $g_x = f_y$ and $g_y = -f_x$. Express f_r and f_θ in terms of r, θ and g_r, g_θ.

$\boxed{\text{Solution}}$ This question uses bad (but standard) notation, in that the one symbol f is used for two different functions, namely $f(x,y)$ and $f(r,\theta) = f(x,y)\big|_{x=r\cos\theta,\, y=r\sin\theta}$. Let us undo this notation conflict by renaming the function of r and θ to $F(r,\theta)$. That is,

$$F(r,\theta) = f\big(r\cos\theta\,,\, r\sin\theta\big)$$

Similarly, rename g, viewed as a function of r and θ, to $G(r,\theta)$. That is,

$$G(r,\theta) = g\big(r\cos\theta\,,\, r\sin\theta\big)$$

In this new notation, we are being asked

- in part (a) to find F_θ, F_r and $F_{r\theta}$ in terms of r, θ, f_x and f_y, and
- in part (b) to express F_r and F_θ in terms of r, θ and G_r, G_θ.

(a) By the chain rule

$$\begin{aligned}
F_\theta(r,\theta) &= \frac{\partial}{\partial\theta}\Big[f\big(r\cos\theta\,,\, r\sin\theta\big)\Big] \\
&= f_x\big(r\cos\theta\,,\, r\sin\theta\big)\ \frac{\partial}{\partial\theta}\big(r\cos\theta\big) + f_y\big(r\cos\theta\,,\, r\sin\theta\big)\ \frac{\partial}{\partial\theta}\big(r\sin\theta\big) \\
&= -r\sin\theta\, f_x\big(r\cos\theta\,,\, r\sin\theta\big) + r\cos\theta\, f_y\big(r\cos\theta\,,\, r\sin\theta\big) \qquad \text{(E1)} \\
F_r(r,\theta) &= \frac{\partial}{\partial r}\Big[f\big(r\cos\theta\,,\, r\sin\theta\big)\Big] \\
&= f_x\big(r\cos\theta\,,\, r\sin\theta\big)\ \frac{\partial}{\partial r}\big(r\cos\theta\big) + f_y\big(r\cos\theta\,,\, r\sin\theta\big)\ \frac{\partial}{\partial r}\big(r\sin\theta\big) \\
&= \cos\theta\, f_x\big(r\cos\theta\,,\, r\sin\theta\big) + \sin\theta\, f_y\big(r\cos\theta\,,\, r\sin\theta\big) \qquad \text{(E2)} \\
F_{r\theta}(r,\theta) &= \frac{\partial}{\partial\theta}\big[F_r(r,\theta)\big] \\
&= \frac{\partial}{\partial\theta}\Big[\cos\theta\, f_x\big(r\cos\theta\,,\, r\sin\theta\big) + \sin\theta\, f_y\big(r\cos\theta\,,\, r\sin\theta\big)\Big] \\
&= -\sin\theta\ f_x\big(r\cos\theta\,,\, r\sin\theta\big) + \cos\theta\, \frac{\partial}{\partial\theta}\Big[f_x\big(r\cos\theta\,,\, r\sin\theta\big)\Big] \\
&\qquad + \cos\theta\ f_y\big(r\cos\theta\,,\, r\sin\theta\big) + \sin\theta\, \frac{\partial}{\partial\theta}\Big[f_y\big(r\cos\theta\,,\, r\sin\theta\big)\Big] \\
&= -\sin\theta\ f_x\big(r\cos\theta\,,\, r\sin\theta\big) \\
&\qquad\qquad + \cos\theta\Big[f_{xx}\big(r\cos\theta\,,\, r\sin\theta\big)\ \big(-r\sin\theta\big) + f_{xy}\big(r\cos\theta\,,\, r\sin\theta\big)\ \big(r\cos\theta\big)\Big] \\
&\quad + \cos\theta\ f_y\big(r\cos\theta\,,\, r\sin\theta\big) \\
&\qquad\qquad + \sin\theta\Big[f_{yx}\big(r\cos\theta\,,\, r\sin\theta\big)\ \big(-r\sin\theta\big) + f_{yy}\big(r\cos\theta\,,\, r\sin\theta\big)\ \big(r\cos\theta\big)\Big] \\
&= -\sin\theta\, f_x + \cos\theta\, f_y \\
&\qquad\qquad - r\sin\theta\cos\theta\, f_{xx} + r[\cos^2\theta - \sin^2\theta]\, f_{xy} + r\sin\theta\cos\theta\, f_{yy}
\end{aligned}$$

with the arguments of f_x, f_y, f_{xx}, f_{xy} and f_{yy} all being $\big(r\cos\theta\,,\, r\sin\theta\big)$.

(b) Replacing f by g in (E1) gives

$$\begin{aligned} G_\theta(r,\theta) &= \frac{\partial}{\partial\theta}\big[g\big(r\cos\theta\,,\,r\sin\theta\big)\big] \\ &= -r\sin\theta\, g_x\big(r\cos\theta\,,\,r\sin\theta\big) + r\cos\theta\, g_y\big(r\cos\theta\,,\,r\sin\theta\big) \\ &= -r\sin\theta\, f_y\big(r\cos\theta\,,\,r\sin\theta\big) - r\cos\theta\, f_x\big(r\cos\theta\,,\,r\sin\theta\big) \\ &= -r\frac{\partial}{\partial r}\big[f\big(r\cos\theta\,,\,r\sin\theta\big)\big] \qquad \text{by (E2)} \end{aligned}$$

Replacing f by g in (E2) gives

$$\begin{aligned} G_r(r,\theta) &= \frac{\partial}{\partial r}\big[g\big(r\cos\theta\,,\,r\sin\theta\big)\big] \\ &= \cos\theta\, g_x\big(r\cos\theta\,,\,r\sin\theta\big) + \sin\theta\, g_y\big(r\cos\theta\,,\,r\sin\theta\big) \\ &= \cos\theta\, f_y\big(r\cos\theta\,,\,r\sin\theta\big) - \sin\theta\, f_x\big(r\cos\theta\,,\,r\sin\theta\big) \\ &= \frac{1}{r}\frac{\partial}{\partial\theta}\big[f\big(r\cos\theta\,,\,r\sin\theta\big)\big] \qquad \text{by (E1)} \end{aligned}$$

or

$$\begin{aligned} \frac{\partial}{\partial r}\big[f\big(r\cos\theta\,,\,r\sin\theta\big)\big] &= -\frac{1}{r}\frac{\partial}{\partial\theta}\big[g\big(r\cos\theta\,,\,r\sin\theta\big)\big] \\ \frac{\partial}{\partial\theta}\big[f\big(r\cos\theta\,,\,r\sin\theta\big)\big] &= r\frac{\partial}{\partial r}\big[g\big(r\cos\theta\,,\,r\sin\theta\big)\big] \end{aligned}$$

2.4.22 (*) Suppose $f(x,y)$ is a differentiable function and we know

$$\nabla f(3,6) = \langle 7,8\rangle$$

Suppose also that

$$\nabla g(1,2) = \langle -1,4\rangle\,,$$

and

$$\nabla h(1,2) = \langle -5,10\rangle\,.$$

Assuming $g(1,2)=3$, $h(1,2)=6$, and $z(s,t) = f\big(g(s,t),h(s,t)\big)$, find

$$\nabla z(1,2)$$

Solution By the chain rule

$$\begin{aligned} \frac{\partial z}{\partial s}(s,t) &= \frac{\partial}{\partial s}f\big(g(s,t),h(s,t)\big) = \frac{\partial f}{\partial x}\big(g(s,t),h(s,t)\big)\frac{\partial g}{\partial s}(s,t) + \frac{\partial f}{\partial y}\big(g(s,t),h(s,t)\big)\frac{\partial h}{\partial s}(s,t) \\ \frac{\partial z}{\partial t}(s,t) &= \frac{\partial}{\partial t}f\big(g(s,t),h(s,t)\big) = \frac{\partial f}{\partial x}\big(g(s,t),h(s,t)\big)\frac{\partial g}{\partial t}(s,t) + \frac{\partial f}{\partial y}\big(g(s,t),h(s,t)\big)\frac{\partial h}{\partial t}(s,t) \end{aligned}$$

In particular

$$\begin{aligned}\frac{\partial z}{\partial s}(1,2) &= \frac{\partial f}{\partial x}\big(g(1,2),h(1,2)\big)\frac{\partial g}{\partial s}(1,2) + \frac{\partial f}{\partial y}\big(g(1,2),h(1,2)\big)\frac{\partial h}{\partial s}(1,2)\\ &= \frac{\partial f}{\partial x}(3,6)\frac{\partial g}{\partial s}(1,2) + \frac{\partial f}{\partial y}(3,6)\big)\frac{\partial h}{\partial s}(1,2)\\ &= 7\times(-1) + 8\times(-5) = -47\\ \frac{\partial z}{\partial t}(1,2) &= \frac{\partial f}{\partial x}\big(g(1,2),h(1,2)\big)\frac{\partial g}{\partial t}(1,2) + \frac{\partial f}{\partial y}\big(g(1,2),h(1,2)\big)\frac{\partial h}{\partial t}(1,2)\\ &= 7\times 4 + 8\times 10 = 108\end{aligned}$$

Hence $\boldsymbol{\nabla} z(1,2) = \langle -47, 108\rangle$.

$\boxed{2.4.23}$ (∗)

(a) Let f be an arbitrary differentiable function defined on the entire real line. Show that the function w defined on the entire plane as

$$w(x,y) = e^{-y} f(x-y)$$

satisfies the partial differential equation:

$$w + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 0$$

(b) The equations $x = u^3 - 3uv^2$, $y = 3u^2v - v^3$ and $z = u^2 - v^2$ define z as a function of x and y. Determine $\frac{\partial z}{\partial x}$ at the point $(u,v) = (2,1)$ which corresponds to the point $(x,y) = (2,11)$.

$\boxed{\text{Solution}}$ (a) By the product and chain rules

$$\begin{aligned}w_x(x,y) &= \frac{\partial}{\partial x}\big[e^{-y}f(x-y)\big] = e^{-y}\frac{\partial}{\partial x}\big[f(x-y)\big] = e^{-y}f'(x-y)\frac{\partial}{\partial x}(x-y)\\ &= e^{-y}f'(x-y)\\ w_y(x,y) &= \frac{\partial}{\partial y}\big[e^{-y}f(x-y)\big] = -e^{-y}f(x-y) + e^{-y}\frac{\partial}{\partial y}\big[f(x-y)\big]\\ &= -e^{-y}f(x-y) + e^{-y}f'(x-y)\frac{\partial}{\partial y}(x-y)\\ &= -e^{-y}f(x-y) - e^{-y}f'(x-y)\end{aligned}$$

Hence

$$w + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = e^{-y}f(x-y) + e^{-y}f'(x-y) - e^{-y}f(x-y) - e^{-y}f'(x-y) = 0$$

as desired.

(b) Think of $x = u^3 - 3uv^2$, $y = 3u^2v - v^3$ as two equations in the two unknowns u, v with x, y just being given parameters. The question implicitly tells us that those two equations can be solved for u, v in terms of x, y, at least near $(u,v) = (2,1)$, $(x,y) = (2,11)$. That is, the question implicitly tells us that the functions $u(x,y)$ and $v(x,y)$ are determined by

$$x = u(x,y)^3 - 3u(x,y)\,v(x,y)^2 \qquad y = 3u(x,y)^2 v(x,y) - v(x,y)^3$$

Applying $\frac{\partial}{\partial x}$ to both sides of the equation $x = u(x,y)^3 - 3u(x,y)v(x,y)^2$ gives

$$1 = 3\,u(x,y)^2\,\frac{\partial u}{\partial x}(x,y) - 3\frac{\partial u}{\partial x}(x,y)\,v(x,y)^2 - 6\,u(x,y)\,v(x,y)\,\frac{\partial v}{\partial x}(x,y)$$

Then applying $\frac{\partial}{\partial x}$ to both sides of $y = 3u(x,y)^2 v(x,y) - v(x,y)^3$ gives

$$0 = 6\,u(x,y)\,\frac{\partial u}{\partial x}(x,y)\,v(x,y) + 3\,u(x,y)^2\,\frac{\partial v}{\partial x}(x,y) - 3\,v(x,y)^2\,\frac{\partial v}{\partial x}(x,y)$$

Substituting in $x = 2$, $y = 11$, $u = 2$, $v = 1$ gives

$$1 = 12\frac{\partial u}{\partial x}(2,11) - 3\frac{\partial u}{\partial x}(2,11) - 12\frac{\partial v}{\partial x}(2,11) = 9\frac{\partial u}{\partial x}(2,11) - 12\frac{\partial v}{\partial x}(2,11)$$
$$0 = 12\frac{\partial u}{\partial x}(2,11) + 12\frac{\partial v}{\partial x}(2,11) - 3\frac{\partial v}{\partial x}(2,11) = 12\frac{\partial u}{\partial x}(2,11) + 9\frac{\partial v}{\partial x}(2,11)$$

From the second equation $\frac{\partial v}{\partial x}(2,11) = -\frac{4}{3}\frac{\partial u}{\partial x}(2,11)$. Substituting into the first equation gives

$$1 = 9\frac{\partial u}{\partial x}(2,11) - 12\left[-\frac{4}{3}\frac{\partial u}{\partial x}(2,11)\right] = 25\frac{\partial u}{\partial x}(2,11)$$

so that $\frac{\partial u}{\partial x}(2,11) = \frac{1}{25}$ and $\frac{\partial v}{\partial x}(2,11) = -\frac{4}{75}$. The question also tells us that $z(x,y) = u(x,y)^2 - v(x,y)^2$. Hence

$$\frac{\partial z}{\partial x}(x,y) = 2u(x,y)\frac{\partial u}{\partial x}(x,y) - 2v(x,y)\frac{\partial v}{\partial x}(x,y)$$
$$\implies \frac{\partial z}{\partial x}(2,11) = 4\frac{\partial u}{\partial x}(2,11) - 2\frac{\partial v}{\partial x}(2,11) = 4\frac{1}{25} + 2\frac{4}{75} = \frac{20}{75} = \frac{4}{15}$$

2.4.24 (∗) The equations

$$x^2 - y\cos(uv) = v$$
$$x^2 + y^2 - \sin(uv) = \frac{4}{\pi}u$$

define x and y implicitly as functions of u and v (i.e. $x = x(u,v)$, and $y = y(u,v)$) near the point $(x,y) = (1,1)$ at which $(u,v) = \left(\frac{\pi}{2},0\right)$.

(a) Find

$$\frac{\partial x}{\partial u} \text{ and } \frac{\partial y}{\partial u}$$

at $(u,v) = \left(\frac{\pi}{2},0\right)$.

(b) If $z = x^4 + y^4$, determine $\frac{\partial z}{\partial u}$ at the point $(u,v) = \left(\frac{\pi}{2},0\right)$.

$\boxed{\text{Solution}}$ (a) We are told that

$$x(u,v)^2 - y(u,v)\cos(uv) = v \qquad x(u,v)^2 + y(u,v)^2 - \sin(uv) = \frac{4}{\pi}u$$

Applying $\frac{\partial}{\partial u}$ to both equations gives

$$\begin{aligned} 2x(u,v)\frac{\partial x}{\partial u}(u,v) - \frac{\partial y}{\partial u}(u,v)\cos(uv) + v\,y(u,v)\sin(uv) &= 0 \\ 2x(u,v)\frac{\partial x}{\partial u}(u,v) + 2y(u,v)\frac{\partial y}{\partial u}(u,v) - v\cos(uv) &= \frac{4}{\pi} \end{aligned}$$

Setting $u = \frac{\pi}{2}$, $v = 0$, $x\big(\frac{\pi}{2},0\big) = 1$, $y\big(\frac{\pi}{2},0\big) = 1$ gives

$$2\frac{\partial x}{\partial u}\Big(\frac{\pi}{2},0\Big) - \frac{\partial y}{\partial u}\Big(\frac{\pi}{2},0\Big) = 0$$

$$2\frac{\partial x}{\partial u}\Big(\frac{\pi}{2},0\Big) + 2\frac{\partial y}{\partial u}\Big(\frac{\pi}{2},0\Big) = \frac{4}{\pi}$$

Substituting $\frac{\partial y}{\partial u}\big(\frac{\pi}{2},0\big) = 2\frac{\partial x}{\partial u}\big(\frac{\pi}{2},0\big)$, from the first equation, into the second equation gives $6\frac{\partial x}{\partial u}\big(\frac{\pi}{2},0\big) = \frac{4}{\pi}$ so that $\frac{\partial x}{\partial u}\big(\frac{\pi}{2},0\big) = \frac{2}{3\pi}$ and $\frac{\partial y}{\partial u}\big(\frac{\pi}{2},0\big) = \frac{4}{3\pi}$.

(b) We are told that $z(u,v) = x(u,v)^4 + y(u,v)^4$. So

$$\frac{\partial z}{\partial u}(u,v) = 4x(u,v)^3\ \frac{\partial x}{\partial u}(u,v) + 4y(u,v)^3\ \frac{\partial y}{\partial u}(u,v)$$

Substituting in $u = \frac{\pi}{2}$, $v = 0$, $x\big(\frac{\pi}{2},0\big) = 1$, $y\big(\frac{\pi}{2},0\big) = 1$ and using the results of part (a),

$$\begin{aligned} \frac{\partial z}{\partial u}\Big(\frac{\pi}{2},0\Big) &= 4\,x\Big(\frac{\pi}{2},0\Big)^3\ \frac{\partial x}{\partial u}\Big(\frac{\pi}{2},0\Big) + 4\,y\Big(\frac{\pi}{2},0\Big)^3\ \frac{\partial y}{\partial u}\Big(\frac{\pi}{2},0\Big) \\ &= 4\left(\frac{2}{3\pi}\right) + 4\left(\frac{4}{3\pi}\right) \\ &= \frac{8}{\pi} \end{aligned}$$

$\boxed{2.4.25}$ $(*)$ Let $f(u,v)$ be a differentiable function, and let $u = x+y$ and $v = x-y$. Find a constant, α, such that

$$(f_x)^2 + (f_y)^2 = \alpha\big((f_u)^2 + (f_v)^2\big)$$

$\boxed{\text{Solution}}$ This question uses bad (but standard) notation, in that the one symbol f is used for two different functions, namely $f(u,v)$ and $f(x,y) = f(u,v)\big|_{u=x+y,v=x-y}$. A better wording is

> Let $f(u,v)$ and $F(x,y)$ be differentiable functions such that $F(x,y) = f(x+y, x-y)$. Find a constant, α, such that
>
> $$F_x(x,y)^2 + F_y(x,y)^2 = \alpha\big\{f_u(x+y,x-y)^2 + f_v(x+y,x-y)^2\big\}$$

By the chain rule

$$\begin{aligned}
\frac{\partial F}{\partial x}(x,y) &= f_u(x+y,x-y)\frac{\partial}{\partial x}(x+y) + f_v(x+y,x-y)\frac{\partial}{\partial x}(x-y) \\
&= f_u(x+y,x-y) + f_v(x+y,x-y) \\
\frac{\partial F}{\partial y}(x,y) &= f_u(x+y,x-y)\frac{\partial}{\partial y}(x+y) + f_v(x+y,x-y)\frac{\partial}{\partial y}(x-y) \\
&= f_u(x+y,x-y) - f_v(x+y,x-y)
\end{aligned}$$

Hence

$$\begin{aligned}
F_x(x,y)^2 + F_y(x,y)^2 = \Big[f_u(x+y,x-y) + f_v(x+y,x-y)\Big]^2 \\
+ \Big[f_u(x+y,x-y) - f_v(x+y,x-y)\Big]^2 \\
= 2f_u(x+y,x-y)^2 + 2f_v(x+y,x-y)^2
\end{aligned}$$

So $\alpha = 2$ does the job.

▸▸ Stage 3

2.4.26 The wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

arises in many models involving wave-like phenomena. Let $u(x,t)$ and $v(\xi,\eta)$ be related by the change of variables

$$\begin{aligned}
u(x,t) &= v\big(\xi(x,t),\eta(x,t)\big) \\
\xi(x,t) &= x - ct \\
\eta(x,t) &= x + ct
\end{aligned}$$

(a) Show that $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$ if and only if $\frac{\partial^2 v}{\partial\xi\partial\eta} = 0$.
(b) Show that $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$ if and only if $u(x,t) = F(x-ct) + G(x+ct)$ for some functions F and G.
(c) Interpret $F(x-ct) + G(x+ct)$ in terms of travelling waves. Think of $u(x,t)$ as the height, at position x and time t, of a wave that is travelling along the x-axis.

Remark: Don't be thrown by the strange symbols ξ and η. They are just two harmless letters from the Greek alphabet, called "xi" and "eta" respectively.

$\boxed{\text{Solution}}$ Recall that $u(x,t) = v\big(\xi(x,t), \eta(x,t)\big)$. By the chain rule

$$\begin{aligned}
\frac{\partial u}{\partial x}(x,t) &= \frac{\partial v}{\partial \xi}\big(\xi(x,t),\eta(x,t)\big)\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\big(\xi(x,t),\eta(x,t)\big)\frac{\partial \eta}{\partial x} \\
&= \frac{\partial v}{\partial \xi}\big(\xi(x,t),\eta(x,t)\big) + \frac{\partial v}{\partial \eta}\big(\xi(x,t),\eta(x,t)\big) \\
\frac{\partial u}{\partial t}(x,t) &= \frac{\partial v}{\partial \xi}\big(\xi(x,t),\eta(x,t)\big)\frac{\partial \xi}{\partial t} + \frac{\partial v}{\partial \eta}\big(\xi(x,t),\eta(x,t)\big)\frac{\partial \eta}{\partial t} \\
&= -c\frac{\partial v}{\partial \xi}\big(\xi(x,t),\eta(x,t)\big) + c\frac{\partial v}{\partial \eta}\big(\xi(x,t),\eta(x,t)\big)
\end{aligned}$$

Again by the chain rule

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x,t) &= \frac{\partial}{\partial x}\Big[\frac{\partial v}{\partial \xi}\big(\xi(x,t),\eta(x,t)\big)\Big] + \frac{\partial}{\partial x}\Big[\frac{\partial v}{\partial \eta}\big(\xi(x,t),\eta(x,t)\big)\Big] \\
&= \frac{\partial^2 v}{\partial \xi^2}\big(\xi(x,t),\eta(x,t)\big)\frac{\partial \xi}{\partial x} + \frac{\partial^2 v}{\partial \eta \partial \xi}\big(\xi(x,t),\eta(x,t)\big)\frac{\partial \eta}{\partial x} \\
&\quad + \frac{\partial^2 v}{\partial \xi \partial \eta}\big(\xi(x,t),\eta(x,t)\big)\frac{\partial \xi}{\partial x} + \frac{\partial^2 v}{\partial \eta^2}\big(\xi(x,t),\eta(x,t)\big)\frac{\partial \eta}{\partial x} \\
&= \frac{\partial^2 v}{\partial \xi^2}\big(\xi(x,t),\eta(x,t)\big) + 2\frac{\partial^2 v}{\partial \xi \partial \eta}\big(\xi(x,t),\eta(x,t)\big) + \frac{\partial^2 v}{\partial \eta^2}\big(\xi(x,t),\eta(x,t)\big)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t) &= -c\frac{\partial}{\partial t}\Big[\frac{\partial v}{\partial \xi}\big(\xi(x,t),\eta(x,t)\big)\Big] + c\frac{\partial}{\partial t}\Big[\frac{\partial v}{\partial \eta}\big(\xi(x,t),\eta(x,t)\big)\Big] \\
&= -c\Big[\frac{\partial^2 v}{\partial \xi^2}\big(\xi(x,t),\eta(x,t)\big)\frac{\partial \xi}{\partial t} + \frac{\partial^2 v}{\partial \eta \partial \xi}\big(\xi(x,t),\eta(x,t)\big)\frac{\partial \eta}{\partial t}\Big] \\
&\quad + c\Big[\frac{\partial^2 v}{\partial \xi \partial \eta}\big(\xi(x,t),\eta(x,t)\big)\frac{\partial \xi}{\partial t} + \frac{\partial^2 v}{\partial \eta^2}\big(\xi(x,t),\eta(x,t)\big)\frac{\partial \eta}{\partial t}\Big] \\
&= c^2\frac{\partial^2 v}{\partial \xi^2}\big(\xi(x,t),\eta(x,t)\big) - 2c^2\frac{\partial^2 v}{\partial \xi \partial \eta}\big(\xi(x,t),\eta(x,t)\big) + c^2\frac{\partial^2 v}{\partial \eta^2}\big(\xi(x,t),\eta(x,t)\big)
\end{aligned}$$

so that

$$\frac{\partial^2 u}{\partial x^2}(x,t) - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}(x,t) = 4\frac{\partial^2 v}{\partial \xi \partial \eta}\big(\xi(x,t),\eta(x,t)\big)$$

Hence

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x,t) - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}(x,t) = 0 \text{ for all } (x,t) &\iff 4\frac{\partial^2 v}{\partial \xi \partial \eta}\big(\xi(x,t),\eta(x,t)\big) = 0 \text{ for all } (x,t) \\
&\iff \frac{\partial^2 v}{\partial \xi \partial \eta}\big(\xi,\eta\big) = 0 \text{ for all } (\xi,\eta)
\end{aligned}$$

(b) Now $\frac{\partial^2 v}{\partial \xi \partial \eta}\big(\xi,\eta\big) = \frac{\partial}{\partial \xi}\Big[\frac{\partial v}{\partial \eta}\Big] = 0$. Temporarily rename $\frac{\partial v}{\partial \eta} = w$. The equation $\frac{\partial w}{\partial \xi}(\xi,\eta) = 0$ says that, for each fixed η, $w(\xi,\eta)$ is a constant. The value of the constant may depend on

η. That is, $\frac{\partial v}{\partial \eta}(\xi,\eta) = w(\xi,\eta) = H(\eta)$, for some function H. (As a check, observe that $\frac{\partial}{\partial \xi}H(\eta) = 0$.) So the derivative of v with respect to η, (viewing ξ as a constant) is $H(\eta)$.

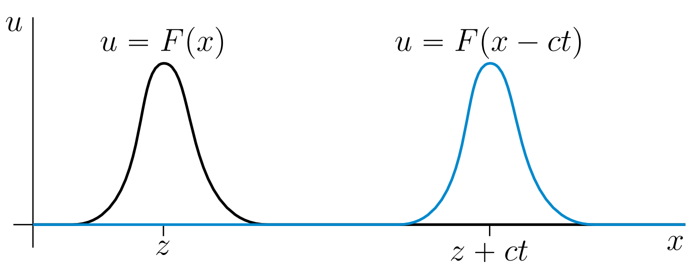
Let $G(\eta)$ be any function whose derivative is $H(\eta)$ (i.e. an indefinite integral of $H(\eta)$). Then $\frac{\partial}{\partial \eta}\big[v(\xi,\eta) - G(\eta)\big] = H(\eta) - H(\eta) = 0$. This is the case if and only if, for each fixed ξ, $v(\xi,\eta) - G(\xi,\eta)$ is a constant, independent of η. That is, if and only if

$$v(\xi,\eta) - G(\eta) = F(\xi)$$

for some function F. Hence

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x,t) - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}(x,t) = 0 \iff\ & \frac{\partial^2 v}{\partial \xi \partial \eta}(\xi,\eta) = 0 \text{ for all } (\xi,\eta)\\
\iff\ & v(\xi,\eta) = F(\xi) + G(\eta) \text{ for some functions } F \text{ and } G\\
\iff\ & u(x,t) = v\big(\xi(x,t),\eta(x,t)\big) = F\big(\xi(x,t)\big) + G\big(\eta(x,t)\big)\\
& \qquad = F(x-ct) + G(x+ct)
\end{aligned}$$

(c) We'll give the interpretation of $F(x-ct)$. The case $G(x+ct)$ is similar. Suppose that $u(x,t) = F(x-ct)$. Think of $u(x,t)$ as the height of water at position x and time t. Pick any number z. All points (x,t) in space time for which $x - ct = z$ have the same value of u, namely $F(z)$. So if you move so that your position is $x = z + ct$ (i.e. you move the right with speed c) you always see the same wave height. Thus $F(x-ct)$ represents a wave moving to the right with speed c.



Similarly, $G(x+ct)$ represents a wave moving to the left with speed c.

$\boxed{2.4.27}$ Evaluate

(a) $\frac{\partial y}{\partial z}$ if $e^{yz} - x^2 z \ln y = \pi$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $F(x,y,x^2-y^2) = 0$

(c) $\left(\frac{\partial y}{\partial x}\right)_u$ if $xyuv = 1$ and $x + y + u + v = 0$

$\boxed{\text{Solution}}$ (a) We are told to evaluate $\frac{\partial y}{\partial z}$. So y has to be a function of z and possibly some other variables. We are also told that x, y, and z are related by the single equation $e^{yz} - x^2 z \ln y = \pi$. So we are to think of x and z as being independent variables and think of $y(x,z)$ as being determined by solving $e^{yz} - x^2 z \ln y = \pi$ for y as a function of x and z. That is, the function $y(x,z)$ obeys

$$e^{y(x,z)\,z} - x^2 z \ln y(x,z) = \pi$$

for all x and z. Applying $\frac{\partial}{\partial z}$ to both sides of this equation gives

$$\begin{aligned}
&\Big[y(x,z) + z\frac{\partial y}{\partial z}(x,z)\Big]\, e^{y(x,z)\,z} - x^2 \ln y(x,z) - x^2 z\frac{1}{y(x,z)}\frac{\partial y}{\partial z}(x,z) = 0\\
\implies &\frac{\partial y}{\partial z}(x,z) = \frac{x^2 \ln y(x,z) - y(x,z)e^{y(x,z)\,z}}{ze^{y(x,z)\,z} - \frac{x^2 z}{y(x,z)}}
\end{aligned}$$

(b) We are told to evaluate $\frac{\mathrm{d}y}{\mathrm{d}x}$. So y has to be a function of the single variable x. We are also told that x and y are related by $F(x,y,x^2-y^2) = 0$. So the function $y(x)$ has to obey

$$F\big(x, y(x), x^2 - y(x)^2\big) = 0$$

for all x. Applying $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides of that equation and using the chain rule gives

$$\begin{aligned}
&F_1\big(x,y(x),x^2-y(x)^2\big)\ \frac{\mathrm{d}x}{\mathrm{d}x} + F_2\big(x,y(x),x^2-y(x)^2\big)\ \frac{\mathrm{d}y}{\mathrm{d}x}(x)\\
&\qquad\qquad + F_3\big(x,y(x),x^2-y(x)^2\big)\,\frac{\mathrm{d}}{\mathrm{d}x}\Big[x^2 - y(x)^2\Big] = 0\\
\implies &F_1\big(x,y(x),x^2-y(x)^2\big) + F_2\big(x,y(x),x^2-y(x)^2\big)\ \frac{\mathrm{d}y}{\mathrm{d}x}(x)\\
&\qquad\qquad + F_3\big(x,y(x),x^2-y(x)^2\big)\ \Big[2x - 2y(x)\frac{\mathrm{d}y}{\mathrm{d}x}(x)\Big] = 0\\
\implies &\frac{\mathrm{d}y}{\mathrm{d}x}(x) = -\frac{F_1\big(x,y(x),x^2-y(x)^2\big) + 2x\,F_3\big(x,y(x),x^2-y(x)^2\big)}{F_2\big(x,y(x),x^2-y(x)^2\big) - 2y(x)\,F_3\big(x,y(x),x^2-y(x)^2\big)}
\end{aligned}$$

(c) We are told to evaluate $\left(\frac{\partial y}{\partial x}\right)_u$, which is the partial derivative of y with respect to x with u being held fixed. So x and u have to be independent variables and y has to be a function of x and u.

Now the four variables x, y, u and v are related by the two equations $xyuv = 1$ and $x + y + u + v = 0$. As x and u are to be independent variables, $y = y(x,u)$, $v = v(x,u)$ are to be determined by solving $xyuv = 1$, $x + y + u + v = 0$ for y and v as functions of x and u. That is

$$\begin{aligned}
x\,y(x,u)\,u\,v(x,u) &= 1\\
x + y(x,u) + u + v(x,u) &= 0
\end{aligned}$$

for all x and u. Applying $\frac{\partial}{\partial x}$ to both sides of both of these equations gives

$$\begin{aligned}
y\,u\,v\ +\ x\,\frac{\partial y}{\partial x}\,u\,v\ +\ x\,y\,u\,\frac{\partial v}{\partial x} &= 0\\
1 + \frac{\partial y}{\partial x} + 0 + \frac{\partial v}{\partial x} &= 0
\end{aligned}$$

Substituting, $\frac{\partial v}{\partial x} = -1 - \frac{\partial y}{\partial x}$, from the second equation, into the first equation gives

$$y\,u\,v\ +\ x\,\frac{\partial y}{\partial x}\,u\,v - x\,y\,u\left(1 + \frac{\partial y}{\partial x}\right) = 0$$

Now u cannot be 0 because $x\,y(x,u)\,u\,v(x,u) = 1$. So

$$y\,v \;+\; x\,\frac{\partial y}{\partial x}\,v - x\,y\left(1+\frac{\partial y}{\partial x}\right) = 0 \implies \left(\frac{\partial y}{\partial x}\right)_u(x,u) = \frac{y(x,u)\,v(x,u) - x\,y(x,u)}{x\,y(x,u) - x\,v(x,u)}$$

2.5▲ Tangent Planes and Normal Lines

▸▸ Stage 1

$\boxed{2.5.1}$ Is it reasonable to say that the surfaces $x^2+y^2+(z-1)^2 = 1$ and $x^2+y^2+(z+1)^2 = 1$ are tangent to each other at $(0,0,0)$?

$\boxed{\text{Solution}}$ Write $F(x,y,z) = x^2+y^2+(z-1)^2-1$ and $G(x,y,z) = x^2+y^2+(z+1)^2-1$. Let S_1 denote the surface $F(x,y,z) = 0$ and S_2 denote the surface $G(x,y,z) = 0$. First note that $F(0,0,0) = G(0,0,0) = 0$ so that the point $(0,0,0)$ lies on both S_1 and S_2. The gradients of F and G are

$$\begin{aligned}
\boldsymbol{\nabla} F(x,y,z) &= \left\langle \frac{\partial F}{\partial x}(x,y,z)\,,\,\frac{\partial F}{\partial y}(x,y,z)\,,\,\frac{\partial F}{\partial z}(x,y,z)\right\rangle = \langle 2x\,,\,2y\,,\,2(z-1)\rangle\\
\boldsymbol{\nabla} G(x,y,z) &= \left\langle \frac{\partial G}{\partial x}(x,y,z)\,,\,\frac{\partial G}{\partial y}(x,y,z)\,,\,\frac{\partial G}{\partial z}(x,y,z)\right\rangle = \langle 2x\,,\,2y\,,\,2(z+1)\rangle
\end{aligned}$$

In particular,

$$\boldsymbol{\nabla} F(0,0,0) = \langle 0,0,-2\rangle \qquad \boldsymbol{\nabla} G(0,0,0) = \langle 0,0,2\rangle$$

so that the vector $\hat{\mathbf{k}} = -\frac{1}{2}\boldsymbol{\nabla} F(0,0,0) = \frac{1}{2}\boldsymbol{\nabla} G(0,0,0)$ is normal to both surfaces at $(0,0,0)$. So the tangent plane to both S_1 and S_2 at $(0,0,0)$ is

$$\hat{\mathbf{k}}\cdot\langle x-0, y-0, z-0\rangle = 0 \qquad \text{or} \qquad z = 0$$

Denote by P the plane $z = 0$. Thus S_1 is tangent to P at $(0,0,0)$ and P is tangent to S_2 at $(0,0,0)$. So it is reasonable to say that S_1 and S_2 are tangent at $(0,0,0)$.

$\boxed{2.5.2}$ Let the point $\mathbf{r}_0 = (x_0, y_0, z_0)$ lie on the surface $G(x,y,z) = 0$. Assume that $\boldsymbol{\nabla} G(x_0,y_0,z_0) \neq \mathbf{0}$. Suppose that the parametrized curve $\mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$ is contained in the surface and that $\mathbf{r}(t_0) = \mathbf{r}_0$. Show that the tangent line to the curve at $\mathbf{r}_0$ lies in the tangent plane to $G = 0$ at $\mathbf{r}_0$.

$\boxed{\text{Solution}}$ Denote by S the surface $G(x,y,z) = 0$ and by C the parametrized curve $\mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$. To start, we'll find the tangent plane to S at $\mathbf{r}_0$ and the tangent line to C at $\mathbf{r}_0$.

- The tangent vector to C at $\mathbf{r}_0$ is $\langle x'(t_0)\,,\,y'(t_0)\,,\,z'(t_0)\rangle$, so the parametric equations for the tangent line to C at $\mathbf{r}_0$ are

$$x - x_0 = tx'(t_0) \qquad y - y_0 = ty'(t_0) \qquad z - z_0 = tz'(t_0) \tag{E_1}$$

- The gradient $\left\langle \frac{\partial G}{\partial x}(x_0\,,\,y_0\,,\,z_0)\,,\,\frac{\partial G}{\partial y}(x_0\,,\,y_0\,,\,z_0)\,,\,\frac{\partial G}{\partial z}(x_0\,,\,y_0\,,\,z_0)\right\rangle$ is a normal vector to the surface S at (x_0, y_0, z_0). So the tangent plane to the surface S at (x_0, y_0, z_0) is
$$\left\langle \frac{\partial G}{\partial x}(x_0\,,\,y_0\,,\,z_0)\,,\,\frac{\partial G}{\partial y}(x_0\,,\,y_0\,,\,z_0)\,,\,\frac{\partial G}{\partial z}(x_0\,,\,y_0\,,\,z_0)\right\rangle\cdot\langle x-x_0\,,\,y-y_0\,,\,z-z_0\rangle = 0$$
or
$$\frac{\partial G}{\partial x}(x_0\,,\,y_0\,,\,z_0)\ (x-x_0)+\frac{\partial G}{\partial y}(x_0\,,\,y_0\,,\,z_0)\ (y-y_0)+\frac{\partial G}{\partial z}(x_0\,,\,y_0\,,\,z_0)\ (z-z_0)=0 \tag{E_2}$$

Next, we'll show that the tangent vector $\langle x'(t_0)\,,\,y'(t_0)\,,\,z'(t_0)\rangle$ to C at $\mathbf{r}_0$ and the normal vector $\left\langle \frac{\partial G}{\partial x}(x_0\,,\,y_0\,,\,z_0)\,,\,\frac{\partial G}{\partial y}(x_0\,,\,y_0\,,\,z_0)\,,\,\frac{\partial G}{\partial z}(x_0\,,\,y_0\,,\,z_0)\right\rangle$ to S at $\mathbf{r}_0$ are perpendicular to each other. To do so, we observe that, for every t, the point $\big(x(t), y(t), z(t)\big)$ lies on the surface $G(x, y, z) = 0$ and so obeys
$$G\big(x(t), y(t), z(t)\big) = 0$$
Differentiating this equation with respect to t gives, by the chain rule,
$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}t}G\big(x(t), y(t), z(t)\big)\\ &= \frac{\partial G}{\partial x}\big(x(t)\,,\,y(t)\,,\,z(t)\big)\ x'(t)+\frac{\partial G}{\partial y}\big(x(t)\,,\,y(t)\,,\,z(t)\big)\ y'(t)+\frac{\partial G}{\partial z}\big(x(t)\,,\,y(t)\,,\,z(t)\big)\ z'(t)\end{aligned}$$
Then setting $t = t_0$ gives
$$\frac{\partial G}{\partial x}(x_0\,,\,y_0\,,\,z_0)\ x'(t_0)+\frac{\partial G}{\partial y}(x_0\,,\,y_0\,,\,z_0)\ y'(t_0)+\frac{\partial G}{\partial z}(x_0\,,\,y_0\,,\,z_0)\ z'(t_0)=0 \tag{E_3}$$
Finally, we are in a position to show that if (x, y, z) is any point on the tangent line to C at $\mathbf{r}_0$, then (x, y, z) is also on the tangent plane to S at $\mathbf{r}_0$. As (x, y, z) is on the tangent line to C at $\mathbf{r}_0$ then there is a t such that, by (E_1),
$$\begin{aligned}&\frac{\partial G}{\partial x}(x_0\,,\,y_0\,,\,z_0)\ \{x-x_0\}+\frac{\partial G}{\partial y}(x_0\,,\,y_0\,,\,z_0)\ \{y-y_0\}+\frac{\partial G}{\partial z}(x_0\,,\,y_0\,,\,z_0)\ \{z-z_0\}\\ &= \frac{\partial G}{\partial x}(x_0\,,\,y_0\,,\,z_0)\ \{t\,x'(t_0)\}+\frac{\partial G}{\partial y}(x_0\,,\,y_0\,,\,z_0)\ \{t\,y'(t_0)\}+\frac{\partial G}{\partial z}(x_0\,,\,y_0\,,\,z_0)\ \{t\,z'(t_0)\}\\ &= t\left[\frac{\partial G}{\partial x}(x_0\,,\,y_0\,,\,z_0)\ x'(t_0)+\frac{\partial G}{\partial y}(x_0\,,\,y_0\,,\,z_0)\ y'(t_0)+\frac{\partial G}{\partial z}(x_0\,,\,y_0\,,\,z_0)\ z'(t_0)\right]=0\end{aligned}$$
by (E_3). That is, (x, y, z) obeys the equation, (E_2), of the tangent plane to S at $\mathbf{r}_0$ and so is on that tangent plane. So the tangent line to C at $\mathbf{r}_0$ is contained in the tangent plane to S at $\mathbf{r}_0$.

> $\boxed{2.5.3}$ Let $F(x_0, y_0, z_0) = G(x_0, y_0, z_0) = 0$ and let the vectors $\nabla F(x_0, y_0, z_0)$ and $\nabla G(x_0, y_0, z_0)$ be nonzero and not be parallel to each other. Find the equation of the normal plane to the curve of intersection of the surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$ at (x_0, y_0, z_0). By definition, that normal plane is the plane through (x_0, y_0, z_0) whose normal vector is the tangent vector to the curve of intersection at (x_0, y_0, z_0).

Solution Use S_1 to denote the surface $F(x,y,z)=0$, S_2 to denote the surface $G(x,y,z)=0$ and C to denote the curve of intersection of S_1 and S_2.

- Since C is contained in S_1, the tangent line to C at (x_0,y_0,z_0) is contained in the tangent plane to S_1 at (x_0,y_0,z_0), by Q[2]. In particular, any tangent vector, $\mathbf{t}$, to C at (x_0,y_0,z_0) must be perpendicular to $\nabla F(x_0,y_0,z_0)$, the normal vector to S_1 at (x_0,y_0,z_0).
- Since C is contained in S_2, the tangent line to C at (x_0,y_0,z_0) is contained in the tangent plane to S_2 at (x_0,y_0,z_0), by Q[2]. In particular, any tangent vector, $\mathbf{t}$, to C at (x_0,y_0,z_0) must be perpendicular to $\nabla G(x_0,y_0,z_0)$, the normal vector to S_2 at (x_0,y_0,z_0).

So any tangent vector to C at (x_0,y_0,z_0) must be perpendiular to both $\nabla F(x_0,y_0,z_0)$ and $\nabla G(x_0,y_0,z_0)$. One such tangent vector is

$$\mathbf{t}=\nabla F(x_0,y_0,z_0)\times\nabla G(x_0,y_0,z_0)$$

(Because the vectors $\nabla F(x_0,y_0,z_0)$ and $\nabla G(x_0,y_0,z_0)$ are nonzero and not parallel, $\mathbf{t}$ is nonzero.) So the normal plane in question passes through (x_0,y_0,z_0) and has normal vector $\mathbf{n}=\mathbf{t}$. Consquently, the normal plane is

$$\mathbf{n}\cdot\langle x-x_0\,,\,y-y_0\,,\,z-z_0\rangle=0 \qquad \text{where } \mathbf{n}=\mathbf{t}=\nabla F(x_0,y_0,z_0)\times\nabla G(x_0,y_0,z_0)$$

2.5.4 Let $f(x_0,y_0)=g(x_0,y_0)$ and let $\big\langle f_x(x_0,y_0),f_y(x_0,y_0)\big\rangle\ne\big\langle g_x(x_0,y_0),g_y(x_0,y_0)\big\rangle$. Find the equation of the tangent line to the curve of intersection of the surfaces $z=f(x,y)$ and $z=g(x,y)$ at $(x_0\,,\,y_0\,,\,z_0=f(x_0,y_0))$.

Solution Use S_1 to denote the surface $z=f(x,y)$, S_2 to denote the surface $z=g(x,y)$ and C to denote the curve of intersection of S_1 and S_2.

- Since C is contained in S_1, the tangent line to C at (x_0,y_0,z_0) is contained in the tangent plane to S_1 at (x_0,y_0,z_0), by Q[2]. In particular, any tangent vector, $\mathbf{t}$, to C at (x_0,y_0,z_0) must be perpendicular to $-f_x(x_0,y_0)\,\hat{\imath}-f_y(x_0,y_0)\,\hat{\jmath}+\hat{\mathbf{k}}$, the normal vector to S_1 at (x_0,y_0,z_0). (See Theorem 2.5.1 in the CLP-3 text.)
- Since C is contained in S_2, the tangent line to C at (x_0,y_0,z_0) is contained in the tangent plane to S_2 at (x_0,y_0,z_0), by Q[2]. In particular, any tangent vector, $\mathbf{t}$, to C at (x_0,y_0,z_0) must be perpendicular to $-g_x(x_0,y_0)\,\hat{\imath}-g_y(x_0,y_0)\,\hat{\jmath}+\hat{\mathbf{k}}$, the normal vector to S_2 at (x_0,y_0,z_0).

So any tangent vector to C at (x_0,y_0,z_0) must be perpendicular to both of the vectors $-f_x(x_0,y_0)\,\hat{\imath}-f_y(x_0,y_0)\,\hat{\jmath}+\hat{\mathbf{k}}$ and $-g_x(x_0,y_0)\,\hat{\imath}-g_y(x_0,y_0)\,\hat{\jmath}+\hat{\mathbf{k}}$. One such tangent vector is

$$\begin{aligned}
\mathbf{t}&=\big[-f_x(x_0,y_0)\,\hat{\imath}-f_y(x_0,y_0)\,\hat{\jmath}+\hat{\mathbf{k}}\big]\times\big[-g_x(x_0,y_0)\,\hat{\imath}-g_y(x_0,y_0)\,\hat{\jmath}+\hat{\mathbf{k}}\big]\\
&=\det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ -f_x(x_0,y_0) & -f_y(x_0,y_0) & 1\\ -g_x(x_0,y_0) & -g_y(x_0,y_0) & 1\end{bmatrix}\\
&=\big\langle g_y(x_0,y_0)-f_y(x_0,y_0)\,,\,f_x(x_0,y_0)-g_x(x_0,y_0)\,,\,f_x(x_0,y_0)g_y(x_0,y_0)-f_y(x_0,y_0)g_x(x_0,y_0)\big\rangle
\end{aligned}$$

So the tangent line in question passes through (x_0, y_0, z_0) and has direction vector $\mathbf{d} = \mathbf{t}$. Consquently, the tangent line is

$$\langle x - x_0 \,,\, y - y_0 \,,\, z - z_0 \rangle = t\,\mathbf{d}$$

or

$$\begin{aligned}
x &= x_0 + t\big[g_y(x_0,y_0) - f_y(x_0,y_0)\big] \\
y &= y_0 + t\big[f_x(x_0,y_0) - g_x(x_0,y_0)\big] \\
z &= z_0 + t\big[f_x(x_0,y_0)g_y(x_0,y_0) - f_y(x_0,y_0)g_x(x_0,y_0)\big]
\end{aligned}$$

▸▸ Stage 2

2.5.5 (*) Let $f(x,y) = \dfrac{x^2y}{x^4+2y^2}$. Find the tangent plane to the surface $z = f(x,y)$ at the point $\left(-1\,,\,1\,,\,\frac{1}{3}\right)$.

Solution We are going to use Theorem 2.5.1 in the CLP-3 text. To do so, we need the first order derivatives of $f(x,y)$ at $(x,y) = (-1,1)$. So we find them first.

$$\begin{aligned}
f_x(x,y) &= \frac{2xy}{x^4+2y^2} - \frac{x^2y(4x^3)}{(x^4+2y^2)^2} & f_x(-1,1) &= -\frac{2}{3} + \frac{4}{3^2} = -\frac{2}{9} \\
f_y(x,y) &= \frac{x^2}{x^4+2y^2} - \frac{x^2y(4y)}{(x^4+2y^2)^2} & f_y(-1,1) &= \frac{1}{3} - \frac{4}{3^2} = -\frac{1}{9}
\end{aligned}$$

The tangent plane is

$$\begin{aligned}
z &= f(-1,1) + f_x(-1,1)\,(x+1) + f_y(-1,1)\,(y-1) = \frac{1}{3} - \frac{2}{9}\,(x+1) - \frac{1}{9}\,(y-1) \\
&= \frac{2}{9} - \frac{2}{9}x - \frac{1}{9}y
\end{aligned}$$

or $2x + y + 9z = 2$.

2.5.6 (*) Find the tangent plane to

$$\frac{27}{\sqrt{x^2+y^2+z^2+3}} = 9$$

at the point $(2,1,1)$.

Solution The equation of the given surface is of the form $G(x,y,z) = 9$ with $G(x,y,z) = \frac{27}{\sqrt{x^2+y^2+z^2+3}}$. So, by Theorem 2.5.5 in the CLP-3 text, a normal vector to the surface at $(2,1,1)$ is

$$\begin{aligned}\nabla G(2,1,1) &= -\frac{1}{2}\,\frac{27}{(x^2+y^2+z^2+3)^{3/2}}\left(2x\,,\,2y\,,\,2z\right)\Big|_{(x,y,z)=(2,1,1)}\\ &= -\left\langle 2\,,\,1\,,\,1\right\rangle\end{aligned}$$

and the equation of the tangent plane is

$$-\left\langle 2\,,\,1\,,\,1\right\rangle\cdot\left\langle x-2\,,\,y-1\,,\,z-1\right\rangle = 0 \qquad \text{or} \qquad 2x+y+z=6$$

$\boxed{2.5.7}$ Find the equations of the tangent plane and the normal line to the graph of the specified function at the specified point.
(a) $f(x,y) = x^2 - y^2$ at $(-2,1)$
(b) $f(x,y) = e^{xy}$ at $(2,0)$

$\boxed{\text{Solution}}$ (a) The specified graph is $z = f(x,y) = x^2 - y^2$ or $F(x,y,z) = x^2 - y^2 - z = 0$. Observe that $f(-2,1) = 3$. The vector

$$\begin{aligned}\nabla F(-2,1,3) &= \left\langle F_x(x,y,z), F_y(x,y,z), F_z(x,y,z)\right\rangle\Big|_{(x,y,z)=(-2,1,3)}\\ &= \left\langle 2x, -2y, -1\right\rangle\Big|_{(x,y,z)=(-2,1,3)}\\ &= \left\langle -4, -2, -1\right\rangle\end{aligned}$$

is a normal vector to the graph at $(-2,1,3)$. So the tangent plane is

$$-4(x+2) - 2(y-1) - (z-3) = 0 \text{ or } 4x + 2y + z = -3$$

and the normal line is

$$\left\langle x,y,z\right\rangle = \left\langle -2,1,3\right\rangle + t\left\langle 4,2,1\right\rangle$$

(b) The specified graph is $z = f(x,y) = e^{xy}$ or $F(x,y,z) = e^{xy} - z = 0$. Observe that $f(2,0) = 1$. The vector

$$\begin{aligned}\nabla F(2,0,1) &= \left\langle F_x(x,y,z), F_y(x,y,z), F_z(x,y,z)\right\rangle\Big|_{(x,y,z)=(2,0,1)}\\ &= \left\langle ye^{xy}, xe^{xy}, -1\right\rangle\Big|_{(x,y,z)=(2,0,1)}\\ &= \left\langle 0, 2, -1\right\rangle\end{aligned}$$

is a normal vector to the graph at $(2,0,1)$. So the tangent plane is

$$0(x-2) + 2(y-0) - (z-1) = 0 \text{ or } 2y - z = -1$$

and the normal line is

$$\left\langle x,y,z\right\rangle = \left\langle 2,0,1\right\rangle + t\left\langle 0,2,-1\right\rangle$$

$\boxed{2.5.8}$ (∗) Consider the surface $z = f(x, y)$ defined implicitly by the equation $xyz^2 + y^2z^3 = 3 + x^2$. Use a 3–dimensional gradient vector to find the equation of the tangent plane to this surface at the point $(-1, 1, 2)$. Write your answer in the form $z = ax + by + c$, where a, b and c are constants.

$\boxed{\text{Solution}}$ We may use $G(x, y, z) = xyz^2 + y^2z^3 - 3 - x^2 = 0$ as an equation for the surface. Note that $(-1, 1, 2)$ really is on the surface since

$$G(-1, 1, 2) = (-1)(1)(2)^2 + (1)^2(2)^3 - 3 - (-1)^2 = -4 + 8 - 3 - 1 = 0$$

By Theorem 2.5.5 in the CLP-3 text, since

$$\begin{aligned} G_x(x, y, z) &= yz^2 - 2x & \qquad G_x(-1, 1, 2) &= 6 \\ G_y(x, y, z) &= xz^2 + 2yz^3 & \qquad G_y(-1, 1, 2) &= 12 \\ G_z(x, y, z) &= 2xyz + 3y^2z^2 & \qquad G_z(-1, 1, 2) &= 8 \end{aligned}$$

one normal vector to the surface at $(-1, 1, 2)$ is $\nabla G(-1, 1, 2) = \langle 6\,,\,12\,,\,8\rangle$ and an equation of the tangent plane to the surface at $(-1, 1, 2)$ is

$$\langle 6\,,\,12\,,\,8\rangle \cdot \langle x+1\,,\,y-1\,,\,z-2\rangle = 0 \qquad \text{or} \qquad 6x + 12y + 8z = 22$$

or

$$z = -\frac{3}{4}x - \frac{3}{2}y + \frac{11}{4}$$

$\boxed{2.5.9}$ (∗) A surface is given by

$$z = x^2 - 2xy + y^2.$$

(a) Find the equation of the tangent plane to the surface at $x = a$, $y = 2a$.
(b) For what value of a is the tangent plane parallel to the plane $x - y + z = 1$?

$\boxed{\text{Solution}}$ (a) The surface is $G(x, y, z) = z - x^2 + 2xy - y^2 = 0$. When $x = a$ and $y = 2a$ and (x, y, z) is on the surface, we have $z = a^2 - 2(a)(2a) + (2a)^2 = a^2$. So, by Theorem 2.5.5 in the CLP-3 text, a normal vector to this surface at $(a, 2a, a^2)$ is

$$\nabla G(a, 2a, a^2) = \langle -2x + 2y\,,\,2x - 2y\,,\,1\rangle\Big|_{(x,y,z)=(a,2a,a^2)} = \langle 2a\,,\,-2a\,,\,1\rangle$$

and the equation of the tangent plane is

$$\langle 2a\,,\,-2a\,,\,1\rangle \cdot \left\langle x - a\,,\,y - 2a\,,\,z - a^2\right\rangle = 0 \qquad \text{or} \qquad 2ax - 2ay + z = -a^2$$

(b) The two planes are parallel when their two normal vectors, namely $\langle 2a\,,\,-2a\,,\,1\rangle$ and $\langle 1\,,\,-1\,,\,1\rangle$, are parallel. This is the case if and only if $a = \frac{1}{2}$.

$\boxed{2.5.10}$ (∗) Find the tangent plane and normal line to the surface $z = f(x, y) = \frac{2y}{x^2+y^2}$ at $(x, y) = (-1, 2)$.

Solution The first order partial derivatives of f are

$$f_x(x,y) = -\frac{4xy}{(x^2+y^2)^2} \qquad f_x(-1,2) = \frac{8}{25}$$

$$f_y(x,y) = \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} \quad f_y(-1,2) = \frac{2}{5} - \frac{16}{25} = -\frac{6}{25}$$

So, by Theorem 2.5.1 in the CLP-3 text, a normal vector to the surface at $(x,y) = (-1,2)$ is $\left\langle \frac{8}{25}\,,\, -\frac{6}{25}\,,\, -1\right\rangle$. As $f(-1,2) = \frac{4}{5}$, the tangent plane is

$$\left\langle \frac{8}{25}\,,\, -\frac{6}{25}\,,\, -1\right\rangle \cdot \left\langle x+1\,,\, y-2\,,\, z-\frac{4}{5}\right\rangle = 0 \quad \text{or} \quad \frac{8}{25}x - \frac{6}{25}y - z = -\frac{8}{5}$$

and the normal line is

$$\langle x,y,z\rangle = \left\langle -1,2,\frac{4}{5}\right\rangle + t\left\langle \frac{8}{25}\,,\, -\frac{6}{25}\,,\, -1\right\rangle$$

2.5.11 (*) Find all the points on the surface $x^2 + 9y^2 + 4z^2 = 17$ where the tangent plane is parallel to the plane $x - 8z = 0$.

Solution A normal vector to the surface $x^2 + 9y^2 + 4z^2 = 17$ at the point (x,y,z) is $\langle 2x\,,\, 18y\,,\, 8z\rangle$. A normal vector to the plane $x - 8z = 0$ is $\langle 1\,,\, 0\,,\, -8\rangle$. So we want $\langle 2x\,,\, 18y\,,\, 8z\rangle$ to be parallel to $\langle 1\,,\, 0\,,\, -8\rangle$, i.e. to be a nonzero constant times $\langle 1\,,\, 0\,,\, -8\rangle$. This is the case whenever $y = 0$ and $z = -2x$ with $x \neq 0$. In addition, we want (x,y,z) to lie on the surface $x^2 + 9y^2 + 4z^2 = 17$. So we want $y = 0$, $z = -2x$ and

$$17 = x^2 + 9y^2 + 4z^2 = x^2 + 4(-2x)^2 = 17x^2 \implies x = \pm 1$$

So the allowed points are $\pm(1,0,-2)$.

2.5.12 (*) Let S be the surface $z = x^2 + 2y^2 + 2y - 1$. Find all points $P(x_0, y_0, z_0)$ on S with $x_0 \neq 0$ such that the normal line at P contains the origin $(0,0,0)$.

Solution The equation of S is of the form $G(x,y,z) = x^2 + 2y^2 + 2y - z = 1$. So one normal vector to S at the point (x_0, y_0, z_0) is

$$\nabla G(x_0, y_0, z_0) = 2x_0\,\hat{\imath} + (4y_0 + 2)\,\hat{\jmath} - \hat{\mathbf{k}}$$

and the normal line to S at (x_0, y_0, z_0) is

$$(x,y,z) = (x_0, y_0, z_0) + t\,\langle 2x_0\,,\, 4y_0 + 2\,,\, -1\rangle$$

For this normal line to pass through the origin, there must be a t with

$$(0,0,0) = (x_0, y_0, z_0) + t\,\langle 2x_0\,,\, 4y_0 + 2\,,\, -1\rangle$$

or

$$x_0 + 2x_0\,t = 0 \tag{E1}$$
$$y_0 + (4y_0 + 2)t = 0 \tag{E2}$$
$$z_0 - t = 0 \tag{E3}$$

Equation (E3) forces $t = z_0$. Substituting this into equations (E1) and (E2) gives

$$x_0(1 + 2z_0) = 0 \tag{E1}$$
$$y_0 + (4y_0 + 2)z_0 = 0 \tag{E2}$$

The question specifies that $x_0 \neq 0$, so (E1) forces $z_0 = -\frac{1}{2}$. Substituting $z_0 = -\frac{1}{2}$ into (E2) gives

$$-y_0 - 1 = 0 \implies y_0 = -1$$

Finally x_0 is determined by the requirement that (x_0, y_0, z_0) must lie on S and so must obey

$$z_0 = x_0^2 + 2y_0^2 + 2y_0 - 1 \implies -\frac{1}{2} = x_0^2 + 2(-1)^2 + 2(-1) - 1 \implies x_0^2 = \frac{1}{2}$$

So the allowed points P are $\big(\frac{1}{\sqrt{2}}, -1, -\frac{1}{2}\big)$ and $\big(-\frac{1}{\sqrt{2}}, -1, -\frac{1}{2}\big)$.

2.5.13 (∗) Find all points on the hyperboloid $z^2 = 4x^2 + y^2 - 1$ where the tangent plane is parallel to the plane $2x - y + z = 0$.

Solution Let (x_0, y_0, z_0) be a point on the hyperboloid $z^2 = 4x^2 + y^2 - 1$ where the tangent plane is parallel to the plane $2x - y + z = 0$. A normal vector to the plane $2x - y + z = 0$ is $\langle 2, -1, 1\rangle$. Because the hyperboloid is $G(x, y, z) = 4x^2 + y^2 - z^2 - 1$ and $\nabla G(x, y, z) = \langle 8x, 2y, -2z\rangle$, a normal vector to the hyperboloid at (x_0, y_0, z_0) is $\nabla G(x_0, y_0, z_0) = \langle 8x_0, 2y_0, -2z_0\rangle$. So (x_0, y_0, z_0) satisfies the required conditions if and only if there is a nonzero t obeying

$$\begin{aligned}
&\langle 8x_0, 2y_0, -2z_0\rangle = t\,\langle 2, -1, 1\rangle \text{ and } z_0^2 = 4x_0^2 + y_0^2 - 1\\
\iff\ & x_0 = \frac{t}{4},\ y_0 = z_0 = -\frac{t}{2} \text{ and } z_0^2 = 4x_0^2 + y_0^2 - 1\\
\iff\ & \frac{t^2}{4} = \frac{t^2}{4} + \frac{t^2}{4} - 1 \text{ and } x_0 = \frac{t}{4},\ y_0 = z_0 = -\frac{t}{2}\\
\iff\ & t = \pm 2 \qquad (x_0, y_0, z_0) = \pm\big(\tfrac{1}{2}, -1, -1\big)
\end{aligned}$$

2.5.14 Find a vector of length $\sqrt{3}$ which is tangent to the curve of intersection of the surfaces $z^2 = 4x^2 + 9y^2$ and $6x + 3y + 2z = 5$ at $(2, 1, -5)$.

Solution One vector normal to the surface $F(x, y, z) = 4x^2 + 9y^2 - z^2 = 0$ at $(2, 1, -5)$ is

$$\nabla F(2, 1, -5) = \langle 8x, 18y, -2z\rangle\Big|_{(2,1,-5)} = \langle 16, 18, 10\rangle$$

One vector normal to the surface $G(x,y,z) = 6x + 3y + 2z = 5$ at $(2,1,-5)$ is

$$\nabla G(2,1,-5) = \langle 6,3,2 \rangle$$

Now

- The curve lies in the surface $z^2 = 4x^2 + 9y^2$. So the tangent vector to the curve at $(2,1,-5)$ is perpendicular to the normal vector $\frac{1}{2}\langle 16,18,10 \rangle = \langle 8,9,5 \rangle$.
- The curve also lies in the surface $6x + 3y + 2z = 5$. So the tangent vector to the curve at $(2,1,-5)$ is also perpendicular to the normal vector $\langle 6,3,2 \rangle$.
- So the tangent vector to the curve at $(2,1,-5)$ is parallel to

$$\langle 8,9,5 \rangle \times \langle 6,3,2 \rangle = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 8 & 9 & 5 \\ 6 & 3 & 2 \end{bmatrix} = \langle 3,14,-30 \rangle$$

The desired vectors are

$$\pm\sqrt{3}\frac{\langle 3,14,-30 \rangle}{|\langle 3,14,-30 \rangle|} = \pm\sqrt{\frac{3}{1105}}\langle 3,14,-30 \rangle$$

▸▸ Stage 3

2.5.15 Find all horizontal planes that are tangent to the surface with equation

$$z = xye^{-(x^2+y^2)/2}$$

What are the largest and smallest values of z on this surface?

Solution Let (x_0,y_0,z_0) be any point on the surface. A vector normal to the surface at (x_0,y_0,z_0) is

$$\begin{aligned} &\nabla\Big(xye^{-(x^2+y^2)/2} - z\Big)\Big|_{(x_0,y_0,z_0)} \\ &\qquad = \Big\langle y_0e^{-(x_0^2+y_0^2)/2} - x_0^2y_0e^{-(x_0^2+y_0^2)/2}, x_0e^{-(x_0^2+y_0^2)/2} - x_0y_0^2e^{-(x_0^2+y_0^2)/2}, -1\Big\rangle \end{aligned}$$

The tangent plane to the surface at (x_0,y_0,z_0) is horizontal if and only if this vector is vertical, which is the case if and only if its x- and y-components are zero, which in turn is the case if and only if

$$\begin{aligned} &y_0(1-x_0^2) = 0 \text{ and } x_0(1-y_0^2) = 0 \\ &\iff \big\{y_0 = 0 \text{ or } x_0 = 1 \text{ or } x_0 = -1\big\} \text{ and } \big\{x_0 = 0 \text{ or } y_0 = 1 \text{ or } y_0 = -1\big\} \\ &\iff (x_0,y_0) = (0,0) \text{ or } (1,1) \text{ or } (1,-1) \text{ or } (-1,1) \text{ or } (-1,-1) \end{aligned}$$

The values of z_0 at these points are 0, e^{-1}, $-e^{-1}$, $-e^{-1}$ and e^{-1}, respectively. So the horizontal tangent planes are $z = 0$, $z = e^{-1}$ and $z = -e^{-1}$. At the highest and lowest points of the surface, the tangent plane is horizontal. So the largest and smallest values of z are e^{-1} and $-e^{-1}$, respectively.

2.5.16 (∗) Let S be the surface

$$xy - 2x + yz + x^2 + y^2 + z^3 = 7$$

(a) Find the tangent plane and normal line to the surface S at the point $(0,2,1)$.
(b) The equation defining S implicitly defines z as a function of x and y for (x,y,z) near $(0,2,1)$. Find expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Evaluate $\frac{\partial z}{\partial y}$ at $(x,y,z) = (0,2,1)$.
(c) Find an expression for $\frac{\partial^2 z}{\partial x \partial y}$.

Solution (a) A normal vector to the surface at $(0,2,1)$ is

$$\begin{aligned}\boldsymbol{\nabla}\big(xy - 2x + yz + x^2 + y^2 + z^3 - 7\big)\big|_{(0,2,1)} &= \left\langle y - 2 + 2x\,,\ x + z + 2y\,,\ y + 3z^2 \right\rangle\big|_{(0,2,1)} \\ &= \langle 0,5,5\rangle\end{aligned}$$

So the tangent plane is

$$0(x-0) + 5(y-2) + 5(z-1) = 0 \text{ or } y + z = 3$$

The vector parametric equations for the normal line are

$$\mathbf{r}(t) = \langle 0,2,1\rangle + t\,\langle 0,5,5\rangle$$

(b) Differentiating

$$xy - 2x + y\,z(x,y) + x^2 + y^2 + z(x,y)^3 = 7$$

gives

$$\begin{aligned} y - 2 + y\,z_x(x,y) + 2x + 3z(x,y)^2 z_x(x,y) = 0 &\implies z_x(x,y) = \frac{2-2x-y}{y+3z(x,y)^2} \\ x + z(x,y) + y\,z_y(x,y) + 2y + 3z(x,y)^2 z_y(x,y) = 0 &\implies z_y(x,y) = -\frac{x+2y+z(x,y)}{y+3z(x,y)^2}\end{aligned}$$

In particular, at $(0,2,1)$, $z_y(0,2) = -\frac{4+1}{2+3} = -1$.

(c) Differentiating z_x with respect to y gives

$$\begin{aligned} z_{xy}(x,y) &= -\frac{1}{y+3z(x,y)^2} - \frac{2-2x-y}{[y+3z(x,y)^2]^2}\big(1 + 6z(x,y)z_y(x,y)\big) \\ &= -\frac{1}{y+3z(x,y)^2} - \frac{2-2x-y}{[y+3z(x,y)^2]^2}\left(1 - 6z(x,y)\frac{x+2y+z(x,y)}{y+3z(x,y)^2}\right)\end{aligned}$$

As an alternate solution, we could also differentiate z_y with respect to x. This gives

$$\begin{aligned} z_{yx}(x,y) &= -\frac{1+z_x(x,y)}{y+3z(x,y)^2} + \frac{x+2y+z(x,y)}{[y+3z(x,y)^2]^2}6z(x,y)z_x(x,y) \\ &= -\frac{1}{y+3z(x,y)^2}\left(1 + \frac{2-2x-y}{y+3z(x,y)^2}\right) + \frac{x+2y+z(x,y)}{[y+3z(x,y)^2]^2}6z(x,y)\frac{2-2x-y}{y+3z(x,y)^2}\end{aligned}$$

2.5.17 (*)
(a) Find a vector perpendicular at the point $(1,1,3)$ to the surface with equation $x^2+z^2=10$.
(b) Find a vector tangent at the same point to the curve of intersection of the surface in part (a) with surface $y^2+z^2=10$.
(c) Find parametric equations for the line tangent to that curve at that point.

Solution (a) A vector perpendicular to $x^2+z^2=10$ at $(1,1,3)$ is

$$\nabla(x^2+z^2)\big|_{(1,1,3)} = (2x\hat{\imath}+2z\hat{\mathbf{k}})\big|_{(1,1,3)} = 2\hat{\imath}+6\hat{\mathbf{k}} \text{ or } \frac{1}{2}\langle 2,0,6\rangle = \langle 1,0,3\rangle$$

(b) A vector perpendicular to $y^2+z^2=10$ at $(1,1,3)$ is

$$\nabla(y^2+z^2)\big|_{(1,1,3)} = (2y\hat{\jmath}+2z\hat{\mathbf{k}})\big|_{(1,1,3)} = 2\hat{\jmath}+6\hat{\mathbf{k}} \text{ or } \frac{1}{2}\langle 0,2,6\rangle = \langle 0,1,3\rangle$$

A vector is tangent to the specified curve at the specified point if and only if it perpendicular to both $(1,0,3)$ and $(0,1,3)$. One such vector is

$$\langle 0,1,3\rangle\times\langle 1,0,3\rangle = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ 0 & 1 & 3\\ 1 & 0 & 3\end{bmatrix} = \langle 3,3,-1\rangle$$

(c) The specified tangent line passes through $(1,1,3)$ and has direction vector $\langle 3,3,-1\rangle$ and so has vector parametric equation

$$\mathbf{r}(t) = \langle 1,1,3\rangle + t\,\langle 3,3,-1\rangle$$

2.5.18 (*) Let P be the point where the curve

$$\mathbf{r}(t) = t^3\,\hat{\imath} + t\,\hat{\jmath} + t^2\,\hat{\mathbf{k}}, \qquad (0 \leqslant t < \infty)$$

intersects the surface

$$z^3 + xyz - 2 = 0$$

Find the (acute) angle between the curve and the surface at P.

Solution $\mathbf{r}(t) = \langle x(t)\,,\, y(t)\,,\, z(t)\rangle$ intersects $z^3+xyz-2=0$ when

$$z(t)^3 + x(t)\,y(t)\,z(t) - 2 = 0 \iff \left(t^2\right)^3 + \left(t^3\right)(t)\left(t^2\right) - 2 = 0 \iff 2t^6 = 2 \iff t = 1$$

since t is required to be positive. The direction vector for the curve at $t=1$ is

$$\mathbf{r}'(1) = 3\,\hat{\imath} + \hat{\jmath} + 2\,\hat{\mathbf{k}}$$

A normal vector for the surface at $\mathbf{r}(1) = \langle 1,1,1\rangle$ is

$$\nabla(z^3+xyz)\big|_{(1,1,1)} = [yz\hat{\imath} + xz\hat{\jmath} + (3z^2+xy)\hat{\mathbf{k}}]_{(1,1,1)} = \hat{\imath}+\hat{\jmath}+4\hat{\mathbf{k}}$$

The angle θ between the curve and the normal vector to the surface is determined by

$$\begin{aligned} |\langle 3,1,2\rangle||\langle 1,1,4\rangle|\cos\theta = \langle 3,1,2\rangle\cdot\langle 1,1,4\rangle &\iff \sqrt{14}\sqrt{18}\cos\theta = 12 \\ &\iff \sqrt{7\times 36}\cos\theta = 12 \\ &\iff \cos\theta = \frac{2}{\sqrt{7}} \\ &\iff \theta = 40.89^\circ \end{aligned}$$

The angle between the curve and the surface is $90 - 40.89 = 49.11^\circ$ (to two decimal places).

$\boxed{2.5.19}$ Find the distance from the point $(1,1,0)$ to the circular paraboloid with equation $z = x^2 + y^2$.

$\boxed{\text{Solution}}$ Let (x,y,z) be any point on the paraboloid $z = x^2 + y^2$. The square of the distance from $(1,1,0)$ to this point is

$$\begin{aligned} D(x,y) &= (x-1)^2 + (y-1)^2 + z^2 \\ &= (x-1)^2 + (y-1)^2 + (x^2+y^2)^2 \end{aligned}$$

We wish to minimize $D(x,y)$. That is, to find the lowest point on the graph $z = D(x,y)$. At this lowest point, the tangent plane to $z = D(x,y)$ is horizontal. So at the minimum, the normal vector to $z = D(x,y)$ has x and y components zero. So

$$\begin{aligned} 0 &= \frac{\partial D}{\partial x}(x,y) = 2(x-1) + 2(x^2+y^2)(2x) \\ 0 &= \frac{\partial D}{\partial y}(x,y) = 2(y-1) + 2(x^2+y^2)(2y) \end{aligned}$$

By symmetry (or multiplying the first equation by y, multiplying the second equation by x and subtracting) the solution will have $x = y$ with

$$0 = 2(x-1) + 2(x^2+x^2)(2x) = 8x^3 + 2x - 2$$

Observe that the value of $8x^3 + 2x - 2 = 2(4x^3 + x - 1)$ at $x = \frac{1}{2}$ is 0. (See Appendix A.16 of the CLP-2 text for some useful tricks that can help you guess roots of polynomials with integer coefficients.) So $\left(x - \frac{1}{2}\right)$ is a factor of

$$4x^3 + x - 1 = 4\left(x^3 + \tfrac{x}{4} - \tfrac{1}{4}\right) = 4\left(x - \tfrac{1}{2}\right)\left(x^2 + \tfrac{1}{2}x + \tfrac{1}{2}\right)$$

and the minimizing (x,y) obeys $x = y$ and

$$0 = 8x^3 + 2x - 2 = 8\left(x - \tfrac{1}{2}\right)\left(x^2 + \tfrac{1}{2}x + \tfrac{1}{2}\right) = 0$$

By the quadratic root formula, $x^2 + \frac{1}{2}x + \frac{1}{2}$ has no real roots, so the only solution is $x = y = \frac{1}{2}$, $z = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$ and the distance is $\sqrt{\left(\frac{1}{2}-1\right)^2 + \left(\frac{1}{2}-1\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$.

2.6 ▲ Linear Approximations and Error

▸▸ Stage 1

2.6.1 Let x_0 and y_0 be constants and let m and n be integers. If $m < 0$ assume that $x_0 \neq 0$, and if $n < 0$ assume that $y_0 \neq 0$. Define $P(x,y) = x^m y^n$.

(a) Find the linear approximation to $P(x_0 + \Delta x, y_0 + \Delta y)$.

(b) Denote by

$$P_\% = 100\left|\frac{P(x_0+\Delta x, y_0+\Delta y) - P(x_0,y_0)}{P(x_0,y_0)}\right| \qquad x_\% = 100\left|\frac{\Delta x}{x_0}\right| \qquad y_\% = 100\left|\frac{\Delta y}{y_0}\right|$$

the percentage errors in P, x and y respectively. Use the linear approximation to find an (approximate) upper bound on $P_\%$ in terms of m, n, $x_\%$ and $y_\%$.

Solution (a) The first order partial derivatives of $P(x,y)$ at $x = x_0$ and $y = y_0$ are

$$P_x(x_0,y_0) = m x_0^{m-1} y_0^n \qquad P_y(x_0,y_0) = n x_0^m y_0^{n-1}$$

So, by (2.6.1) in the CLP-3 text, the linear approximation is

$$\begin{aligned} P(x_0+\Delta x, y_0+\Delta y) &\approx P(x_0,y_0) + P_x(x_0,y_0)\,\Delta x + P_y(x_0,y_0)\,\Delta y \\ &\approx P(x_0,y_0) + m x_0^{m-1} y_0^n\,\Delta x + n x_0^m y_0^{n-1}\,\Delta y \end{aligned}$$

(b) By part (a)

$$\frac{P(x_0+\Delta x, y_0+\Delta y) - P(x_0,y_0)}{P(x_0,y_0)} \approx \frac{m x_0^{m-1} y_0^n\,\Delta x + n x_0^m y_0^{n-1}\,\Delta y}{x_0^m y_0^n} = m\frac{\Delta x}{x_0} + n\frac{\Delta y}{y_0}$$

Hence

$$\begin{aligned} P_\% &\approx 100\left|m\frac{\Delta x}{x_0} + n\frac{\Delta y}{y_0}\right| \\ &\leqslant |m| 100\left|\frac{\Delta x}{x_0}\right| + |n| 100\left|\frac{\Delta y}{y_0}\right| \\ &\leqslant |m| x_\% + |n| y_\% \end{aligned}$$

Warning. The answer $m\,x_\% + n\,y_\%$, without absolute values on m and n, can be seriously wrong. As an example, suppose that $m = 1$, $n = -1$, $x_0 = y_0 = 1$, $\Delta x = 0.05$ and $\Delta y = -0.05$. Then

$$\begin{aligned} P_\% &\approx 100\left|m\frac{\Delta x}{x_0} + n\frac{\Delta y}{y_0}\right| \\ &= 100\left|(1)\frac{0.05}{1} + (-1)\frac{-0.05}{1}\right| \\ &= 10\% \end{aligned}$$

while

$$\begin{aligned} m\,x_{\%} + n\,y_{\%} &= m\,100\left|\frac{\Delta x}{x_0}\right| + n\,100\left|\frac{\Delta y}{y_0}\right| \\ &= (1)100\left|\frac{0.05}{1}\right| + (-1)100\left|\frac{-0.05}{1}\right| \\ &= 0 \end{aligned}$$

The point is that m and n being of opposite sign does not guarantee that there is a cancelation between the two terms of $m\frac{\Delta x}{x_0} + n\frac{\Delta y}{y_0}$, because $\frac{\Delta x}{x_0}$ and $\frac{\Delta y}{y_0}$ can also be of opposite sign.

2.6.2 Consider the following work.
We compute, approximately, the y-coordinate of the point whose polar coordinates are $r = 0.9$ and $\theta = 2^\circ$. In general, the y-coordinate of the point whose polar coordinates are r and θ is $Y(r, \theta) = r\sin\theta$. The partial derivatives

$$Y_r(r,\theta) = \sin\theta \qquad Y_\theta(r,\theta) = r\cos\theta$$

So the linear approximation to $Y(r_0 + \Delta r, \theta_0 + \Delta\theta)$ with $r_0 = 1$ and $\theta_0 = 0$ is

$$\begin{aligned} Y(1+\Delta r, 0+\Delta\theta) &\approx Y(1,0) + Y_r(1,0)\,\Delta r + Y_\theta(1,0)\,\Delta\theta \\ &= 0 \ + \ (0)\,\Delta r \ + \ (1)\Delta\theta \end{aligned}$$

Applying this with $\Delta r = -0.1$ and $\Delta\theta = 2$ gives the (approximate) y-coordinate

$$Y(0.9, 2) = Y(1-0.1\,,\,0+2) \approx 0 \ + \ (0)\,(-0.1) \ + \ (1)(2) = 2$$

This conclusion is ridiculous. We're saying that the y-coordinate is more than twice the distance from the point to the origin. What was the mistake?

Solution We used that $\frac{d}{d\theta}\sin\theta = \cos\theta$. That is true only if θ is given in radians, not degrees. (See Lemma 2.8.3 and Warning 3.4.23 in the CLP-1 text.) So we have to convert 2° to radians, which is $2 \times \frac{\pi}{180} = \frac{\pi}{90}$. The correct computation is

$$Y\big(0.9, \tfrac{\pi}{90}\big) = Y\big(1-0.1\,,\,0+\tfrac{\pi}{90}\big) \approx 0 \ + \ (0)\,(-0.1) \ + \ (1)\big(\tfrac{\pi}{90}\big) = \tfrac{\pi}{90} \approx 0.035$$

Just out of general interest, $0.9\sin\frac{\pi}{90} = 0.0314$ to four decimal places.

▸▸ Stage 2

2.6.3 Find an approximate value for $f(x, y) = \sin(\pi xy + \ln y)$ at $(0.01, 1.05)$ without using a calculator or computer.

Solution Apply the linear approximation

$f(0.01, 1.05) \approx f(0,1) + f_x(0,1)(0.01) + f_y(0,1)(0.05)$, with

$$\begin{aligned} f(x,y) &= \sin(\pi xy + \ln y) & f(0,1) &= \sin 0 = 0 \\ f_x(x,y) &= \pi y\cos(\pi xy + \ln y) & f_x(0,1) &= \pi\cos 0 = \pi \\ f_y(x,y) &= \left(\pi x + \frac{1}{y}\right)\cos(\pi xy + \ln y) & f_y(0,1) &= \cos 0 = 1 \end{aligned}$$

This gives

$$\begin{aligned} f(0.01, 1.05) &\approx f(0,1) + f_x(0,1)(0.01) + f_y(0,1)(0.05) = 0 + \pi(0.01) + 1(0.05) \\ &= 0.01\,\pi + 0.05 \approx 0.0814 \end{aligned}$$

2.6.4 (*) Let $f(x,y) = \dfrac{x^2y}{x^4+2y^2}$. Find an approximate value for $f(-0.9\,,\,1.1)$ without using a calculator or computer.

Solution We are going to need the first order derivatives of $f(x,y)$ at $(x,y) = (-1,1)$. So we find them first.

$$\begin{aligned} f_x(x,y) &= \frac{2xy}{x^4+2y^2} - \frac{x^2y(4x^3)}{(x^4+2y^2)^2} & f_x(-1,1) &= -\frac{2}{3} + \frac{4}{3^2} = -\frac{2}{9} \\ f_y(x,y) &= \frac{x^2}{x^4+2y^2} - \frac{x^2y(4y)}{(x^4+2y^2)^2} & f_y(-1,1) &= \frac{1}{3} - \frac{4}{3^2} = -\frac{1}{9} \end{aligned}$$

The linear approximation to $f(x,y)$ about $(-1,1)$ is

$$f(x,y) \approx= f(-1,1) + f_x(-1,1)\,(x+1) + f_y(-1,1)\,(y-1) = \frac{1}{3} - \frac{2}{9}\,(x+1) - \frac{1}{9}\,(y-1)$$

In particular

$$f(-0.9, 1.1) \approx \frac{1}{3} - \frac{2}{9}\,(0.1) - \frac{1}{9}\,(0.1) = \frac{27}{90} = 0.3$$

2.6.5 Four numbers, each at least zero and each at most 50, are rounded to the first decimal place and then multiplied together. Estimate the maximum possible error in the computed product.

Solution Let the four numbers be x_1, x_2, x_3 and x_4. Let the four rounded numbers be $x_1+\varepsilon_1$, $x_2+\varepsilon_2$, $x_3+\varepsilon_3$ and $x_4+\varepsilon_4$. Then $0 \leqslant x_1, x_2, x_3, x_4 \leqslant 50$ and $|\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3|, |\varepsilon_4| \leqslant 0.05$. If $P(x_1,x_2,x_3,x_4) = x_1x_2x_3x_4$, then the error in the product introduced by rounding is, using the four variable variant of the linear approximation (2.6.2) of the CLP-3 text,

$$\begin{aligned} &\big|P(x_1+\varepsilon_1, x_2+\varepsilon_2, x_3+\varepsilon_3, x_4+\varepsilon_4) - P(x_1,x_2,x_3,x_4)\big| \\ &\approx \Big|\frac{\partial P}{\partial x_1}(x_1,x_2,x_3,x_4)\varepsilon_1 + \frac{\partial P}{\partial x_2}(x_1,x_2,x_3,x_4)\varepsilon_2 + \frac{\partial P}{\partial x_3}(x_1,x_2,x_3,x_4)\varepsilon_3 + \frac{\partial P}{\partial x_4}(x_1,x_2,x_3,x_4)\varepsilon_4\Big| \\ &= \big|x_2x_3x_4\varepsilon_1 + x_1x_3x_4\varepsilon_2 + x_1x_2x_4\varepsilon_3 + x_1x_2x_3\varepsilon_4\big| \\ &\leqslant 4\times 50\times 50\times 50\times 0.05 = 25000 \end{aligned}$$

2.6.6 (*) One side of a right triangle is measured to be 3 with a maximum possible error of ± 0.1, and the other side is measured to be 4 with a maximum possible error of ± 0.2. Use the linear approximation to estimate the maximum possible error in calculating the length of the hypotenuse of the right triangle.

Solution Denote by x and y the lengths of sides with $x = 3 \pm 0.1$ and $y = 4 \pm 0.2$. Then the length of the hypotenuse is $f(x, y) = \sqrt{x^2 + y^2}$. Note that

$$\begin{aligned} f(x,y) &= \sqrt{x^2+y^2} & f(3,4) &= 5 \\ f_x(x,y) &= \frac{x}{\sqrt{x^2+y^2}} & f_x(3,4) &= \frac{3}{5} \\ f_y(x,y) &= \frac{y}{\sqrt{x^2+y^2}} & f_y(3,4) &= \frac{4}{5} \end{aligned}$$

By the linear approximation

$$f(x,y) \approx f(3,4) + f_x(3,4)\,(x-3) + f_y(3,4)\,(y-4) = 5 + \frac{3}{5}\,(x-3) + \frac{4}{5}\,(y-4)$$

So the approximate maximum error in calculating the length of the hypotenuse is

$$\frac{3}{5}\,(0.1) + \frac{4}{5}\,(0.2) = \frac{1.1}{5} = 0.22$$

2.6.7 (*) If two resistors of resistance R_1 and R_2 are wired in parallel, then the resulting resistance R satisfies the equation $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. Use the linear approximation to estimate the change in R if R_1 decreases from 2 to 1.9 ohms and R_2 increases from 8 to 8.1 ohms.

Solution The function $R(R_1, R_2)$ is defined implictly by

$$\frac{1}{R(R_1,R_2)} = \frac{1}{R_1} + \frac{1}{R_2} \tag{$*$}$$

In particular

$$\frac{1}{R(2,8)} = \frac{1}{2} + \frac{1}{8} = \frac{5}{8} \implies R(2,8) = \frac{8}{5}$$

We wish to use the linear approximation

$$R(R_1,R_2) \approx R(2,8) + \frac{\partial R}{\partial R_1}(2,8)\,(R_1-2) + \frac{\partial R}{\partial R_2}(2,8)\,(R_2-8)$$

To do so, we need the partial derivatives $\frac{\partial R}{\partial R_1}(2,8)$ and $\frac{\partial R}{\partial R_2}(2,8)$. To find them, we differentiate $(*)$ with respect to R_1 and R_2:

$$
\begin{aligned}
-\frac{1}{R(R_1,R_2)^2}\frac{\partial R}{\partial R_1}(R_1,R_2) &= -\frac{1}{R_1^2}\\
-\frac{1}{R(R_1,R_2)^2}\frac{\partial R}{\partial R_2}(R_1,R_2) &= -\frac{1}{R_2^2}
\end{aligned}
$$

Setting $R_1 = 2$ and $R_2 = 8$ gives

$$
\begin{aligned}
-\frac{1}{(8/5)^2}\frac{\partial R}{\partial R_1}(2,8) &= -\frac{1}{4} &&\implies &\frac{\partial R}{\partial R_1}(2,8) &= \frac{16}{25}\\
-\frac{1}{(8/5)^2}\frac{\partial R}{\partial R_2}(2,8) &= -\frac{1}{64} &&\implies &\frac{\partial R}{\partial R_2}(2,8) &= \frac{1}{25}
\end{aligned}
$$

So the specified change in R is

$$R(1.9,8.1) - R(2,8) \approx \frac{16}{25}(-0.1) + \frac{1}{25}(0.1) = -\frac{15}{250} = -0.06$$

$\boxed{2.6.8}$ The total resistance R of three resistors, R_1, R_2, R_3, connected in parallel is determined by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

If the resistances, measured in Ohms, are $R_1 = 25\Omega$, $R_2 = 40\Omega$ and $R_3 = 50\Omega$, with a possible error of 0.5% in each case, estimate the maximum error in the calculated value of R.

$\boxed{\text{Solution}}$ First, we compute the values of the partial derivatives of $R(R_1, R_2, R_3)$ at the measured values of R_1, R_2, R_3. Applying $\frac{\partial}{\partial R_i}$, with $i = 1, 2, 3$ to both sides of the defining equation

$$\frac{1}{R(R_1,R_2,R_3)} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

for $R(R_1, R_2, R_3)$ gives

$$
\begin{aligned}
-\frac{1}{R(R_1,R_2,R_3)^2}\ \frac{\partial R}{\partial R_1}(R_1,R_2,R_3) &= -\frac{1}{R_1^2}\\
-\frac{1}{R(R_1,R_2,R_3)^2}\ \frac{\partial R}{\partial R_2}(R_1,R_2,R_3) &= -\frac{1}{R_2^2}\\
-\frac{1}{R(R_1,R_2,R_3)^2}\ \frac{\partial R}{\partial R_3}(R_1,R_2,R_3) &= -\frac{1}{R_3^2}
\end{aligned}
$$

When $R_1 = 25\Omega$, $R_2 = 40\Omega$ and $R_3 = 50\Omega$

$$\frac{1}{R(25,40,50)} = \frac{1}{25} + \frac{1}{40} + \frac{1}{50} = \frac{8+5+4}{200} \implies R(25,40,50) = \frac{200}{17} = 11.765$$

Substituting in these values of R_1, R_2, R_3 and R,

$$\begin{aligned}
\frac{\partial R}{\partial R_1}(25,40,50) &= \frac{R(25,40,50)^2}{25^2} = \frac{64}{17^2} = 0.221\\
\frac{\partial R}{\partial R_2}(25,40,50) &= \frac{R(25,40,50)^2}{40^2} = \frac{25}{17^2} = 0.0865\\
\frac{\partial R}{\partial R_3}(25,40,50) &= \frac{R(25,40,50)^2}{50^2} = \frac{16}{17^2} = 0.0554
\end{aligned}$$

If the absolute errors in measuring R_1, R_2 and R_3 are denoted ε_1, ε_2 and ε_3, respectively, then, , using the linear approximation (2.6.2) of the CLP-3 text, the corresponding error E in R is

$$\begin{aligned}
E &= R(25+\varepsilon_1, 40+\varepsilon_2, 50+\varepsilon_3) - R(25,40,50)\\
&\approx \frac{\partial R}{\partial R_1}(25,40,50)\varepsilon_1 + \frac{\partial R}{\partial R_2}(25,40,50)\varepsilon_2 + \frac{\partial R}{\partial R_3}(25,40,50)\varepsilon_3
\end{aligned}$$

and obeys

$$\begin{aligned}
&|E| \leqslant \frac{64}{17^2}|\varepsilon_1| + \frac{25}{17^2}|\varepsilon_2| + \frac{16}{17^2}|\varepsilon_3|\\
\text{or}\quad &|E| \leqslant 0.221|\varepsilon_1| + 0.0865|\varepsilon_2| + 0.0554|\varepsilon_3|
\end{aligned}$$

We are told that the percentage error in each measurement is no more that 0.5%. So

$$|\varepsilon_1| \leqslant \frac{0.5}{100}25 = \frac{1}{8} = 0.125 \qquad |\varepsilon_2| \leqslant \frac{0.5}{100}40 = \frac{1}{5} = 0.2 \qquad |\varepsilon_3| \leqslant \frac{0.5}{100}50 = \frac{1}{4} = 0.25$$

so that

$$\begin{aligned}
&|E| \leqslant \frac{8}{17^2} + \frac{5}{17^2} + \frac{4}{17^2} = \frac{1}{17}\\
\text{or}\quad &|E| \leqslant 0.221\times 0.125 + 0.0865\times 0.2 + 0.0554\times 0.25 = 0.059
\end{aligned}$$

2.6.9 The specific gravity S of an object is given by $S = \frac{A}{A-W}$ where A is the weight of the object in air and W is the weight of the object in water. If $A = 20 \pm .01$ and $W = 12 \pm .02$ find the approximate percentage error in calculating S from the given measurements.

Solution By the linear approximation

$$\Delta S \approx \frac{\partial S}{\partial A}(20,12)\,\Delta A + \frac{\partial S}{\partial W}(20,12)\,\Delta W$$

with $S(A,W) = \frac{A}{A-W} = 1 + \frac{W}{A-W}$. So

$$\begin{aligned}
S(A,W) &= \frac{A}{A-W} & S(20,12) &= \frac{20}{8} = \frac{5}{2}\\
S_A(A,W) &= -\frac{W}{(A-W)^2} & S_A(20,12) &= -\frac{12}{8^2} = -\frac{3}{16}\\
S_W(A,W) &= \frac{A}{(A-W)^2} & S_W(20,12) &= \frac{20}{8^2} = \frac{5}{16}
\end{aligned}$$

For any given ΔA and ΔW, the percentage error is

$$\left|100\frac{\Delta S}{S}\right| = \left|100\frac{2}{5}\Big(-\frac{3}{16}\Delta A + \frac{5}{16}\Delta W\Big)\right|$$

We are told that $|\Delta A| \leqslant 0.01$ and $|\Delta W| \leqslant 0.02$. To maximize $\left|100\frac{2}{5}\big(-\frac{3}{16}\Delta A + \frac{5}{16}\Delta W\big)\right|$ take $\Delta A = -0.01$ and $\Delta W = +0.02$. So the maximum percentage error is

$$100\frac{2}{5}\left[-\frac{3}{16}(-0.01) + \frac{5}{16}(0.02)\right] = \frac{2}{5} \times \frac{13}{16} = \frac{13}{40} = 0.325\%$$

$\boxed{2.6.10}$ (*) The pressure in a solid is given by

$$P(s,r) = sr(4s^2 - r^2 - 2)$$

where s is the specific heat and r is the density. We expect to measure (s,r) to be approximately $(2,2)$ and would like to have the most accurate value for P. There are two different ways to measure s and r. Method 1 has an error in s of ± 0.01 and an error in r of ± 0.1, while method 2 has an error of ± 0.02 for both s and r.
Should we use method 1 or method 2? Explain your reasoning carefully.

$\boxed{\text{Solution}}$ The linear approximation to $P(s,r)$ at $(2,2)$ is

$$P(s,r) \approx P(2,2) + P_s(2,2)\,(s-2) + P_r(2,2)\,(r-2)$$

As

$$\begin{aligned}
P(2,2) &= (2)(2)\big[4(2)^2 - (2)^2 - 2\big] = 40 \qquad \text{(which we don't actually need)}\\
P_s(2,2) &= \Big[12s^2r - r^3 - 2r\Big]_{s=r=2} = 84\\
P_r(2,2) &= \Big[4s^3 - 3sr^2 - 2s\Big]_{s=r=2} = 4
\end{aligned}$$

the linear approximation is

$$P(s,r) \approx 40 + 84\,(s-2) + 4\,(r-2)$$

Under method 1, the maximum error in P will have magnitude at most (approximately)

$$84(0.01) + 4(0.1) = 1.24$$

Under method 2, the maximum error in P will have magnitude at most (approximately)

$$84(0.02) + 4(0.02) = 1.76$$

Method 1 is better.

$\boxed{2.6.11}$ A rectangular beam that is supported at its two ends and is subjected to a uniform load sags by an amount

$$S = C\frac{p\ell^4}{wh^3}$$

where $p = \text{load}$, $\ell = \text{length}$, $h = \text{height}$, $w = \text{width}$ and C is a constant. Suppose $p \approx 100$, $\ell \approx 4$, $w \approx .1$ and $h \approx .2$. Will the sag of the beam be more sensitive to changes in the height of the beam or to changes in the width of the beam.

Solution Using the four variable variant of the linear approximation (2.6.2) of the CLP-3 text,

$$\Delta S \approx \frac{\partial S}{\partial p}\Delta p + \frac{\partial S}{\partial \ell}\Delta \ell + \frac{\partial S}{\partial w}\Delta w + \frac{\partial S}{\partial h}\Delta h = C\frac{p\ell^4}{wh^3}\left[\frac{\Delta p}{p} + 4\frac{\Delta \ell}{\ell} - \frac{\Delta w}{w} - 3\frac{\Delta h}{h}\right]$$

When $w \approx 0.1$ and $h \approx 0.2$,

$$\frac{\Delta w}{w} \approx 10\Delta w \qquad 3\frac{\Delta h}{h} \approx 15\Delta h$$

So a change in height by $\Delta h = \varepsilon$ produces a change in sag of about $\Delta S = 15\varepsilon$ times $-C\frac{p\ell^4}{wh^3}$, while a change Δw in width by the same ε produces a change in sag of about $\Delta S = 10\varepsilon$ times the same $-C\frac{p\ell^4}{wh^3}$. The sag is more sensitive to Δh.

2.6.12 (*) Let $z = f(x,y) = \frac{2y}{x^2+y^2}$. Find an approximate value for $f(-0.8, 2.1)$.

Solution The first order partial derivatives of f are

$$f_x(x,y) = -\frac{4xy}{(x^2+y^2)^2} \qquad f_x(-1,2) = \frac{8}{25}$$

$$f_y(x,y) = \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} \qquad f_y(-1,2) = \frac{2}{5} - \frac{16}{25} = -\frac{6}{25}$$

The linear approximation of $f(x,y)$ about $(-1,2)$ is

$$\begin{aligned} f(x,y) &\approx f(-1,2) + f_x(-1,2)\,(x+1) + f_y(-1,2)\,(y-2) \\ &= \frac{4}{5} + \frac{8}{25}\,(x+1) - \frac{6}{25}\,(y-2) \end{aligned}$$

In particular, for $x = -0.8$ and $y = 2.1$,

$$\begin{aligned} f(-0.8, 2.1) &\approx \frac{4}{5} + \frac{8}{25}\,(0.2) - \frac{6}{25}\,(0.1) \\ &= 0.84 \end{aligned}$$

2.6.13 (*) Suppose that a function $z = f(x,y)$ is implicitly defined by an equation:

$$xyz + x + y^2 + z^3 = 0$$

(a) Find $\frac{\partial z}{\partial x}$.
(b) If $f(-1,1) < 0$, find the linear approximation of the function $z = f(x,y)$ at $(-1,1)$.
(c) If $f(-1,1) < 0$, use the linear approximation in (b) to approximate $f(-1.02, 0.97)$.

$\boxed{\text{Solution}}$ (a) The function $f(x,y)$ obeys

$$xy\,f(x,y) + x + y^2 + f(x,y)^3 = 0 \tag{$*$}$$

for all x and y (sufficiently close to $(-1,1)$). Differentiating $(*)$ with respect to x gives

$$y\,f(x,y) + xy\,f_x(x,y) + 1 + 3f(x,y)^2 f_x(x,y) = 0 \implies f_x(x,y) = -\frac{y\,f(x,y)+1}{3f(x,y)^2+xy}$$

Without knowing $f(x,y)$ explicitly, there's not much that we can do with this.

(b) $f(-1,1)$ obeys

$$(-1)(1)\,f(-1,1) + (-1) + (1)^2 + f(-1,1)^3 = 0 \iff f(-1,1)^3 - f(-1,1) = 0$$

Since $f(-1,1) < 0$ we may divide this equation by $f(-1,1) < 0$, giving $f(-1,1)^2 - 1 = 0$. Since $f(-1,1) < 0$, we must have $f(-1,1) = -1$. By part (a)

$$f_x(-1,1) = -\frac{(1)\,f(-1,1)+1}{3f(-1,1)^2 + (-1)(1)} = 0$$

To get the linear approximation, we still need $f_y(-1,1)$. Differentiating $(*)$ with respect to y gives

$$x\,f(x,y) + xy\,f_y(x,y) + 2y + 3f(x,y)^2 f_y(x,y) = 0$$

Then setting $x = -1$, $y = 1$ and $f(-1,1) = -1$ gives

$$(-1)\,(-1) + (-1)(1)\,f_y(-1,1) + 2(1) + 3(-1)^2 f_y(-1,1) = 0 \implies f_y(-1,1) = -\frac{3}{2}$$

So the linear approximation is

$$f(x,y) \approx f(-1,1) + f_x(-1,1)\,(x+1) + f_y(-1,1)\,(y-1) = -1 - \frac{3}{2}(y-1)$$

(c) By part (b),

$$f(-1.02, 0.97) \approx -1 - \frac{3}{2}(0.97-1) = -0.955$$

$\boxed{2.6.14}$ $(*)$ Let $z = f(x,y)$ be given implicitly by

$$e^z + yz = x + y.$$

(a) Find the differential $\mathrm{d}z$.
(b) Use linear approximation at the point $(1,0)$ to approximate $f(0.99, 0.01)$.

$\boxed{\text{Solution}}$ By definition, the differential at $x = a$, $y = b$ is

$$f_x(a,b)\,\mathrm{d}x + f_y(a,b)\,\mathrm{d}y$$

so we have to determine the partial derivatives $f_x(a,b)$ and $f_y(a,b)$. We are told that

$$e^{f(x,y)} + y\, f(x,y) = x + y$$

for all x and y. Differentiating this equation with respect to x and with respect to y gives, by the chain rule,

$$\begin{aligned} e^{f(x,y)} f_x(x,y) + y\, f_x(x,y) &= 1 \\ e^{f(x,y)} f_y(x,y) + f(x,y) + y\, f_y(x,y) &= 1 \end{aligned}$$

Solving the first equation for f_x and the second for f_y gives

$$\begin{aligned} f_x(x,y) &= \frac{1}{e^{f(x,y)} + y} \\ f_y(x,y) &= \frac{1 - f(x,y)}{e^{f(x,y)} + y} \end{aligned}$$

So the differential at $x = a$, $y = b$ is

$$\frac{\mathrm{d}x}{e^{f(a,b)} + b} + \frac{1 - f(a,b)}{e^{f(a,b)} + b}\,\mathrm{d}y$$

Since we can't solve explicitly for $f(a,b)$ for general a and b. There's not much more that we can do with this.

(b) In particular, when $a = 1$ and $b = 0$, we have

$$e^{f(1,0)} + 0\, f(1,0) = 1 + 0 \implies e^{f(1,0)} = 1 \implies f(1,0) = 0$$

and the linear approximation simpifies to

$$f\big(1 + \mathrm{d}x\,,\, \mathrm{d}y\big) \approx f(1,0) + \frac{\mathrm{d}x}{e^{f(1,0)} + 0} + \frac{1 - f(1,0)}{e^{f(1,0)} + 0}\,\mathrm{d}y = \mathrm{d}x + \mathrm{d}y$$

Choosing $\mathrm{d}x = -0.01$ and $\mathrm{d}y = 0.01$, we have

$$f\big(0.99\,,\, 0.01\big) \approx -0.01 + 0.01 = 0$$

$\boxed{2.6.15}$ $(*)$ Two sides and the enclosed angle of a triangle are measured to be $3 \pm .1$m, $4 \pm .1$m and $90 \pm 1^\circ$ respectively. The length of the third side is then computed using the cosine law $C^2 = A^2 + B^2 - 2AB\cos\theta$. What is the approximate maximum error in the computed value of C?

$\boxed{\text{Solution}}$ Let $C(A,B,\theta) = \sqrt{A^2 + B^2 - 2AB\cos\theta}$. Then $C\big(3, 4, \frac{\pi}{2}\big) = 5$. Differentiating $C^2 = A^2 + B^2 - 2AB\cos\theta$ gives

$$\begin{aligned} 2C\frac{\partial C}{\partial A}(A,B,\theta) &= 2A - 2B\cos\theta &\implies\quad& 10\frac{\partial C}{\partial A}\big(3,4,\tfrac{\pi}{2}\big) = 6 \\ 2C\frac{\partial C}{\partial B}(A,B,\theta) &= 2B - 2A\cos\theta &\implies\quad& 10\frac{\partial C}{\partial B}\big(3,4,\tfrac{\pi}{2}\big) = 8 \\ 2C\frac{\partial C}{\partial \theta}(A,B,\theta) &= 2AB\sin\theta &\implies\quad& 10\frac{\partial C}{\partial \theta}\big(3,4,\tfrac{\pi}{2}\big) = 24 \end{aligned}$$

Hence the approximate maximum error in the computed value of C is

$$\begin{aligned}|\Delta C| &\approx \left|\frac{\partial C}{\partial A}\big(3,4,\tfrac{\pi}{2}\big)\Delta A + \frac{\partial C}{\partial B}\big(3,4,\tfrac{\pi}{2}\big)\Delta B + \frac{\partial C}{\partial \theta}\big(3,4,\tfrac{\pi}{2}\big)\Delta\theta\right| \\ &\leqslant (0.6)(0.1) + (0.8)(0.1) + (2.4)\frac{\pi}{180} \\ &= \frac{\pi}{75} + 0.14 \leqslant 0.182\end{aligned}$$

$\boxed{2.6.16}$ (*) Use differentials to find a reasonable approximation to the value of $f(x,y) = xy\sqrt{x^2+y^2}$ at $x = 3.02$, $y = 3.96$. Note that $3.02 \approx 3$ and $3.96 \approx 4$.

$\boxed{\text{Solution}}$ Substituting $(x_0, y_0) = (3,4)$ and $(x,y) = (3.02, 3.96)$ into

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

gives

$$\begin{aligned}f(3.02, 3.96) &\approx f(3,4) + 0.02 f_x(3,4) - 0.04 f_y(3,4) \\ &= 60 + 0.02\left(20 + \frac{36}{5}\right) - 0.04\left(15 + \frac{48}{5}\right) \\ &= 59.560\end{aligned}$$

since

$$f_x(x,y) = y\sqrt{x^2+y^2} + \frac{x^2 y}{\sqrt{x^2+y^2}} \qquad f_y(x,y) = x\sqrt{x^2+y^2} + \frac{xy^2}{\sqrt{x^2+y^2}}$$

$\boxed{2.6.17}$ (*) Use differentials to estimate the volume of metal in a closed metal can with diameter 8cm and height 12cm if the metal is 0.04cm thick.

$\boxed{\text{Solution}}$ The volume of a cylinder of diameter d and height h is $V(d,h) = \pi\big(\frac{d}{2}\big)^2 h$. The wording of the question is a bit ambiguous in that it does not specify if the given dimensions are inside dimensions or outside dimensions. Assume that they are outside dimensions. Then the volume of the can, including the metal, is $V(8,12)$ and the volume of the interior, excluding the metal, is

$$\begin{aligned}V(8 - 2\times 0.04\,,\, 12 - 2\times 0.04) &\approx V(8,12) + V_d(8,12)(-2\times 0.04) + V_h(8,12)(-2\times 0.04) \\ &= V(8,12) + \frac{1}{2}\pi \times 8 \times 12 \times (-2\times 0.04) + \pi\left(\frac{8}{2}\right)^2(-2\times 0.04) \\ &= V(8,12) - \pi \times 128 \times 0.04\end{aligned}$$

So the volume of metal is approximately $\pi \times 128 \times 0.04 = 5.12\pi \approx 16.1$cc. (To this level of approximation, it doesn't matter whether the dimensions are inside or outside dimensions.)

2.6.18 (*) Let z be a function of x, y such that

$$z^3 - z + 2xy - y^2 = 0, \qquad z(2,4) = 1.$$

(a) Find the linear approximation to z at the point $(2,4)$.
(b) Use your answer in (a) to estimate the value of z at $(2.02, 3.96)$.

Solution (a) The function $z(x,y)$ obeys

$$z(x,y)^3 - z(x,y) + 2xy - y^2 = 0$$

for all (x,y) near $(2,4)$. Differentiating with respect to x and y

$$\begin{aligned} 3z(x,y)^2\frac{\partial z}{\partial x}(x,y) - \frac{\partial z}{\partial x}(x,y) + 2y &= 0 \\ 3z(x,y)^2\frac{\partial z}{\partial y}(x,y) - \frac{\partial z}{\partial y}(x,y) + 2x - 2y &= 0 \end{aligned}$$

Substituting in $x = 2$, $y = 4$ and $z(2,4) = 1$ gives

$$\begin{aligned} 3\frac{\partial z}{\partial x}(2,4) - \frac{\partial z}{\partial x}(2,4) + 8 = 0 &\iff \frac{\partial z}{\partial x}(2,4) = -4 \\ 3\frac{\partial z}{\partial y}(2,4) - \frac{\partial z}{\partial y}(2,4) + 4 - 8 = 0 &\iff \frac{\partial z}{\partial y}(2,4) = 2 \end{aligned}$$

The linear approximation is

$$\begin{aligned} z(x,y) &\approx z(2,4) + z_x(2,4)(x-2) + z_y(2,4)(y-4) = 1 - 4(x-2) + 2(y-4) \\ &= 1 - 4x + 2y \end{aligned}$$

(b) Substituting in $x = 2.02$ and $y = 3.96$ gives

$$z(2.02, 3.96) \approx 1 - 4 \times 0.02 + 2 \times (-0.04) = 0.84$$

▸▸ Stage 3

$\boxed{2.6.19}$ (*) Consider the surface given by:

$$z^3 - xyz^2 - 4x = 0.$$

(a) Find expressions for $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ as functions of x, y, z.
(b) Evaluate $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ at $(1, 1, 2)$.
(c) Measurements are made with errors, so that $x = 1 \pm 0.03$ and $y = 1 \pm 0.02$. Find the corresponding maximum error in measuring z.
(d) A particle moves over the surface along the path whose projection in the xy–plane is given in terms of the angle θ as

$$x(\theta) = 1 + \cos\theta,\ y(\theta) = \sin\theta$$

from the point $A : x = 2,\ y = 0$ to the point $B : x = 1,\ y = 1$. Find $\frac{\mathrm{d}z}{\mathrm{d}\theta}$ at points A and B.

Solution (a) We are told that

$$z(x,y)^3 - xy\,z(x,y)^2 - 4x = 0$$

for all (x, y) (sufficiently near $(1, 1)$). Differentiating this equation with respect to x gives

$$\begin{aligned}
3z(x,y)^2\,\frac{\partial z}{\partial x}(x,y) - y\,z(x,y)^2 - 2xy\,z(x,y)\frac{\partial z}{\partial x}(x,y) - 4 = 0\\
\implies \frac{\partial z}{\partial x} = \frac{4 + yz^2}{3z^2 - 2xyz}
\end{aligned}$$

and differentiating with respect to y gives

$$\begin{aligned}
3z(x,y)^2\,\frac{\partial z}{\partial y}(x,y) - x\,z(x,y)^2 - 2xy\,z(x,y)\frac{\partial z}{\partial y}(x,y) = 0\\
\implies \frac{\partial z}{\partial y} = \frac{xz^2}{3z^2 - 2xyz}
\end{aligned}$$

(b) When $(x, y, z) = (1, 1, 2)$,

$$\frac{\partial z}{\partial x}(1,1) = \frac{4 + (1)(2)^2}{3(2)^2 - 2(1)(1)(2)} = 1 \qquad \frac{\partial z}{\partial y}(1,1) = \frac{(1)(2)^2}{3(2)^2 - 2(1)(1)(2)} = \frac{1}{2}$$

(c) Under the linear approximation at $(1, 1)$

$$z(x,y) \approx z(1,1) + z_x(1,1)\,(x-1) + z_y\,(1,1)\,(y-1) = 2 + (x-1) + \frac{1}{2}(y-1)$$

So errors of ± 0.03 in x and ± 0.02 in y leads of errors of about

$$\pm\left[0.03 + \frac{1}{2}(0.02)\right] = \pm 0.04$$

in z.

(d) By the chain rule

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}z\big(x(\theta),y(\theta)\big) &= z_x\big(x(\theta),y(\theta)\big)\,x'(\theta) + z_y\big(x(\theta),y(\theta)\big)\,y'(\theta)\\ &= -z_x\big(1+\cos\theta,\sin\theta\big)\,\sin\theta + z_y\big(1+\cos\theta,\sin\theta\big)\,\cos\theta\end{aligned}$$

At A, $x=2$, $y=0$, $z=2$ (since $z^3-(2)(0)z^2-4(2)=0$) and $\theta=0$, so that

$$\frac{\partial z}{\partial x}(2,0) = \frac{4+(0)(2)^2}{3(2)^2-2(2)(0)(2)} = \frac{1}{3} \qquad \frac{\partial z}{\partial y}(2,0) = \frac{(2)(2)^2}{3(2)^2-2(2)(0)(2)} = \frac{2}{3}$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = -\frac{1}{3}\sin(0) + \frac{2}{3}\cos(0) = \frac{2}{3}$$

At B, $x=1$, $y=1$, $z=2$ and $\theta=\frac{\pi}{2}$, so that, by part (b),

$$\frac{\partial z}{\partial x}(1,1) = 1 \qquad \frac{\partial z}{\partial y}(1,1) = \frac{1}{2}$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = -\sin\frac{\pi}{2} + \frac{1}{2}\cos\frac{\pi}{2} = -1$$

2.6.20 (∗) Consider the function f that maps each point (x,y) in $\mathbb{R}^2$ to ye^{-x}.

(a) Suppose that $x=1$ and $y=e$, but errors of size 0.1 are made in measuring each of x and y. Estimate the maximum error that this could cause in $f(x,y)$.

(b) The graph of the function f sits in $\mathbb{R}^3$, and the point $(1,e,1)$ lies on that graph. Find a nonzero vector that is perpendicular to that graph at that point.

Solution We are going to need the first order partial derivatives of $f(x,y)=ye^{-x}$ at $(x,y)=(1,e)$. Here they are.

$$\begin{aligned} f_x(x,y) &= -ye^{-x} & \qquad f_x(1,e) &= -e\,e^{-1} = -1\\ f_y(x,y) &= e^{-x} & f_y(1,e) &= e^{-1}\end{aligned}$$

(a) The linear approximation to $f(x,y)$ at $(x,y)=(1,e)$ is

$$f(x,y) \approx f(1,e) + f_x(1,e)\,(x-1) + f_y(1,e)\,(y-e) = 1-(x-1)+e^{-1}(y-e)$$

The maximum error is then approximately

$$-1(-0.1)+e^{-1}(0.1) = \frac{1+e^{-1}}{10}$$

(b) The equation of the graph is $g(x, y, z) = f(x, y) - z = 0$. Any vector that is a nonzero constant times

$$\nabla g(1, e, 1) = \langle f_x(1, e)\,,\, f_y(1, e)\,,\, -1 \rangle = \left\langle -1\,,\, e^{-1}\,,\, -1 \right\rangle$$

is perpendicular to $g = 0$ at $(1, e, 1)$.

$\boxed{2.6.21}$ (∗) A surface is defined implicitly by $z^4 - xy^2z^2 + y = 0$.

(a) Compute $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ in terms of x, y, z.

(b) Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(x, y, z) = (2, -1/2, 1)$.

(c) If x decreases from 2 to 1.94, and y increases from -0.5 to -0.4, find the approximate change in z from 1.

(d) Find the equation of the tangent plane to the surface at the point $(2, -1/2, 1)$.

$\boxed{\text{Solution}}$ (a) We are told that for all x, y (with (x, y, z) near $(2, -1/2, 1)$), the function $z(x, y)$ obeys

$$z(x, y)^4 - xy^2 z(x, y)^2 + y = 0 \tag{$*$}$$

Differentiating (∗) with respect to x gives

$$\begin{aligned} 4z(x, y)^3 \frac{\partial z}{\partial x}(x, y) - y^2 z(x, y)^2 - 2xy^2 z(x, y) \frac{\partial z}{\partial x}(x, y) &= 0 \\ \implies \frac{\partial z}{\partial x}(x, y) &= \frac{y^2 z(x, y)^2}{4z(x, y)^3 - 2xy^2 z(x, y)} \end{aligned}$$

Similarly, differentiating this equation with respect to x gives

$$\begin{aligned} 4z(x, y)^3 \frac{\partial z}{\partial y}(x, y) - 2xy\, z(x, y)^2 - 2xy^2 z(x, y) \frac{\partial z}{\partial y}(x, y) + 1 &= 0 \\ \implies \frac{\partial z}{\partial y}(x, y) &= \frac{2xy\, z(x, y)^2 - 1}{4z(x, y)^3 - 2xy^2 z(x, y)} \end{aligned}$$

(b) Substituting $(x, y, z) = (2, -1/2, 1)$ into the results of part (a) gives

$$\begin{aligned} \frac{\partial z}{\partial x}(2, -1/2) &= \frac{1/4}{4 - 1} = \frac{1}{12} \\ \frac{\partial z}{\partial y}(2, -1/2) &= \frac{-2 - 1}{4 - 1} = -1 \end{aligned}$$

(c) Under the linear approximation about $(2, -1/2)$,

$$\begin{aligned} f(x, y) &\approx f(2, -1/2) + f_x(2, -1/2)\,(x - 2) + f_y(2, -1/2)\,(y + 1/2) \\ &= 1 + \frac{1}{12}(x - 2) - (y + 0.5) \end{aligned}$$

In particular

$$f(1.94, -0.4) \approx 1 - \frac{0.06}{12} - 0.1$$

so that

$$f(1.94, -0.4) - 1 \approx -0.105$$

(d) The tangent plane is

$$\begin{aligned} z &= f(2, -1/2) + f_x(2, -1/2)\,(x-2) + f_y(2, -1/2)\,(y+1/2) \\ &= 1 + \frac{1}{12}(x-2) - (y+0.5) \end{aligned}$$

or

$$\frac{x}{12} - y - z = -\frac{1}{3}$$

$\boxed{2.6.22}$ (∗) A surface $z = f(x, y)$ has derivatives $\frac{\partial f}{\partial x} = 3$ and $\frac{\partial f}{\partial y} = -2$ at $(x, y, z) = (1, 3, 1)$.

(a) If x increases from 1 to 1.2, and y decreases from 3 to 2.6, find the change in z using a linear approximation.

(b) Find the equation of the tangent plane to the surface at the point $(1, 3, 1)$.

$\boxed{\text{Solution}}$ (a) The linear approximation to $f(x, y)$ at $(1, 3)$ is

$$f(x, y) \approx f(1, 3) + f_x(1, 3)\,(x-1) + f_y(1, 3)\,(y-3) = 1 + 3(x-1) - 2(y-3)$$

So the change is z is approximately

$$3(1.2-1) - 2(2.6-3) = 1.4$$

(b) The equation of the tangent plane is

$$z = f(1, 3) + f_x(1, 3)\,(x-1) + f_y(1, 3)\,(y-3) = 1 + 3(x-1) - 2(y-3)$$

or

$$3x - 2y - z = -4$$

$\boxed{2.6.23}$ (∗) According to van der Waal's equation, a gas satisfies the equation

$$(pV^2 + 16)(V - 1) = TV^2,$$

where p, V and T denote pressure, volume and temperature respectively. Suppose the gas is now at pressure 1, volume 2 and temperature 5. Find the approximate change in its volume if p is increased by 0.2 and T is increased by 0.3.

Solution Think of the volume as being the function $V(p,T)$ of pressure and temperature that is determined implicitly (at least for $p \approx 1$, $T \approx 5$ and $V \approx 2$) by the equation

$$\big(pV(p,T)^2 + 16\big)\big(V(p,T) - 1\big) = TV(p,T)^2 \tag{$*$}$$

To determine the approximate change in V, we will use the linear approximation to $V(p,T)$ at $p = 1$, $T = 5$. So we will need the partial derivatives $V_p(1,5)$ and $V_T(1,5)$. As the equation $(*)$ is valid for all p near 1 and T near 5, we may differentiate $(*)$ with respect to p, giving

$$\big(V^2 + 2pVV_p\big)\big(V-1\big) + \big(pV^2 + 16\big)V_p = 2TVV_p$$

and we may also differentiate $(*)$ with respect to T, giving

$$\big(2pVV_T\big)\big(V-1\big) + \big(pV^2 + 16\big)V_T = V^2 + 2TVV_T$$

In particular, when $p = 1$, $V = 2$, $T = 5$,

$$\begin{aligned} \big(4 + 4V_p(1,5)\big)\big(2-1\big) + \big(4+16\big)V_p(1,5) &= 20V_p(1,5) &&\implies V_p(1,5) = -1 \\ 4V_T(1,5)\big(2-1\big) + \big(4+16\big)V_T(1,5) &= 4 + 20V_T(1,5) &&\implies V_T(1,5) = 1 \end{aligned}$$

so that the change in V is

$$V(1.2\,,\,5.3) - V(1,5) \approx V_p(1,5)\,(0.2) + V_T(1,5)\,(0.3) = -0.2 + 0.3 = 0.1$$

2.6.24 $(*)$ Consider the function $f(x,y) = e^{-x^2+4y^2}$.

(a) Find the equation of the tangent plane to the graph $z = f(x,y)$ at the point where $(x,y) = (2,1)$.

(b) Find the tangent plane approximation to the value of $f(1.99, 1.01)$ using the tangent plane from part (a).

Solution Since

$$\begin{aligned} f_x(2,1) &= -2xe^{-x^2+4y^2}\Big|_{(x,y)=(2,1)} = -4 \\ f_y(2,1) &= \quad 8ye^{-x^2+4y^2}\Big|_{(x,y)=(2,1)} = 8 \end{aligned}$$

The tangent plane to $z = f(x,y)$ at $(2,1)$ is

$$\begin{aligned} z &= f(2,1) + f_x(2,1)\,(x-2) + f_y(2,1)\,(y-1) = 1 - 4(x-2) + 8(y-1) \\ &= 1 - 4x + 8y \end{aligned}$$

and the tangent plane approximation to the value of $f(1.99, 1.01)$ is

$$f(1.99, 1.01) \approx 1 - 4(1.99 - 2) + 8(1.01 - 1) = 1.12$$

$\boxed{2.6.25}$ ($*$) Let $z = f(x, y) = \ln(4x^2 + y^2)$.

(a) Use a linear approximation of the function $z = f(x, y)$ at $(0, 1)$ to estimate $f(0.1, 1.2)$.

(b) Find a point $P(a, b, c)$ on the graph of $z = f(x, y)$ such that the tangent plane to the graph of $z = f(x, y)$ at the point P is parallel to the plane $2x + 2y - z = 3$.

$\boxed{\text{Solution}}$ (a) The linear approximation to $f(x, y)$ at (a, b) is

$$\begin{aligned} f(x,y) &\approx f(a,b) + f_x(a,b)\,(x-a) + f_y(a,b)\,(y-b) \\ &= \ln(4a^2+b^2) + \frac{8a}{4a^2+b^2}\,(x-a) + \frac{2b}{4a^2+b^2}\,(y-b) \end{aligned}$$

In particular, for $a = 0$ and $b = 1$,

$$f(x,y) \approx 2\,(y-1)$$

and, for $x = 0.1$ and $y = 1.2$,

$$f(0.1, 1.2) \approx 0.4$$

(b) The point (a, b, c) is on the surface $z = f(x, y)$ if and only if

$$c = f(a,b) = \ln(4a^2+b^2)$$

Note that this forces $4a^2 + b^2$ to be nonzero. The tangent plane to the surface $z = f(x, y)$ at the point (a, b, c) is parallel to the plane $2x + 2y - z = 3$ if and only if $\langle 2, 2, -1\rangle$ is a normal vector for the tangent plane. That is, there is a nonzero number t such that

$$\langle 2\,,\,2\,,\,-1\rangle = t\,\big\langle f_x(a,b)\,,\,f_y(a,b)\,,\,-1\big\rangle = t\left\langle \frac{8a}{4a^2+b^2}\,,\,\frac{2b}{4a^2+b^2}\,,\,-1\right\rangle$$

For the z–coordinates to be equal, t must be 1. Then, for the x– and y–coordinates to be equal, we need

$$\begin{aligned} \frac{8a}{4a^2+b^2} &= 2 \\ \frac{2b}{4a^2+b^2} &= 2 \end{aligned}$$

Note that these equations force both a and b to be nonzero. Dividing these equations gives $\frac{8a}{2b} = 1$ and hence $b = 4a$. Substituting $b = 4a$ into either of the two equations gives

$$\frac{8a}{20a^2} = 2 \implies a = \frac{1}{5}$$

So $a = \frac{1}{5}$, $b = \frac{4}{5}$ and

$$c = \ln\left(\frac{4}{5^2} + \frac{4^2}{5^2}\right) = \ln\frac{4}{5}$$

2.6.26 (*)
(a) Find the equation of the tangent plane to the surface $x^2z^3 + y\sin(\pi x) = -y^2$ at the point $P = (1, 1, -1)$.
(b) Let z be defined implicitly by $x^2z^3 + y\sin(\pi x) = -y^2$. Find $\frac{\partial z}{\partial x}$ at the point $P = (1, 1, -1)$.
(c) Let z be the same implicit function as in part (ii), defined by the equation $x^2z^3 + y\sin(\pi x) = -y^2$. Let $x = 0.97$, and $y = 1$. Find the approximate value of z.

Solution (a) The surface has equation $G(x, y, z) = x^2z^3 + y\sin(\pi x) + y^2 = 0$. So a normal vector to the surface at $(1, 1-1)$ is

$$\begin{aligned}\nabla G(1, 1, -1) &= \big[\big(2xz^3 + \pi y\cos(\pi x)\big)\hat{\imath} + \big(\sin(\pi x) + 2y\big)\hat{\jmath} + 3z^2x^2\,\hat{\mathbf{k}}\big]_{(x,y,z)=(1,1,-1)} \\ &= \big(-2-\pi\big)\hat{\imath} + 2\,\hat{\jmath} + 3\,\hat{\mathbf{k}}\end{aligned}$$

So the equation of the tangent plane is

$$\big(-2-\pi\big)(x-1) + 2(y-1) + 3(z+1) = 0 \quad \text{or} \quad -(2+\pi)x + 2y + 3z = -\pi - 3$$

(b) The functions $z(x, y)$ obeys

$$x^2z(x, y)^3 + y\sin(\pi x) + y^2 = 0$$

for all x and y. Differentiating this equation with respect to x gives

$$2xz(x, y)^3 + 3x^2z(x, y)^2\frac{\partial z}{\partial x}(x, y) + \pi y\cos(\pi x) = 0$$

Evaluating at $(1, 1, -1)$ gives

$$-2 + 3\frac{\partial z}{\partial x}(1, 1) - \pi = 0 \implies \frac{\partial z}{\partial x}(1, 1) = \frac{\pi + 2}{3}$$

(c) Using the linear approximation about $(x, y) = (1, 1)$,

$$z(x, 1) \approx z(1, 1) + \frac{\partial z}{\partial x}(1, 1)\ (x-1)$$

gives

$$z(0.97, 1) \approx -1 + \frac{\pi + 2}{3}\ (-0.03) = -1 - \frac{\pi + 2}{100} = -\frac{\pi + 102}{100}$$

2.6.27 (*) The surface $x^4 + y^4 + z^4 + xyz = 17$ passes through $(0, 1, 2)$, and near this point the surface determines x as a function, $x = F(y, z)$, of y and z.
(a) Find F_y and F_z at $(x, y, z) = (0, 1, 2)$.
(b) Use the tangent plane approximation (also known as linear, first order or differential approximation) to find the approximate value of x (near 0) such that $(x, 1.01, 1.98)$ lies on the surface.

Solution (a) The function $F(y,z)$ obeys $F(y,z)^4 + y^4 + z^4 + F(y,z)yz = 17$ for all y and z near $y = 1$, $z = 2$. Applying the derivatives $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ to this equation gives

$$\begin{aligned} 4F(y,z)^3 F_y(y,z) + 4y^3 + F_y(y,z)yz + F(y,z)z &= 0 \\ 4F(y,z)^3 F_z(y,z) + 4z^3 + F_z(y,z)yz + F(y,z)y &= 0 \end{aligned}$$

Substiututing $F(1,2) = 0$, $y = 1$ and $z = 2$ gives

$$\begin{aligned} 4 + 2F_y(1,2) = 0 &\implies F_y(1,2) = -2 \\ 32 + 2F_z(1,2) = 0 &\implies F_z(1,2) = -16 \end{aligned}$$

(b) Using the tangent plane to $x = F(y,z)$ at $y = 1$ and $z = 2$, which is

$$x \approx F(1,2) + F_y(1,2)\,(y-1) + F_z(1,2)\,(z-2)$$

with $y = 1.01$ and $z = 1.98$ gives

$$\begin{aligned} x = F(1.01, 1.98) &\approx F(1,2) + F_y(1,2)(1.01-1) + F_z(1,2)(1.98-2) \\ &= 0 - 2(.01) - 16(-0.02) = 0.3 \end{aligned}$$

2.7▲ Directional Derivatives and the Gradient

▸▸ Stage 1

2.7.1 (∗) Find the directional derivative of $f(x,y,z) = e^{xyz}$ in the $\langle 0,1,1\rangle$ direction at the point $(0,1,1)$.

Solution The partial derivatives, at a general point (x,y,z) and also at the point of interest $(0,1,1)$, are

$$\begin{aligned} f_x(x,y,z) &= yze^{xyz} & f_x(0,1,1) &= 1 \\ f_y(x,y,z) &= xze^{xyz} & f_x(0,1,1) &= 0 \\ f_z(x,y,z) &= xye^{xyz} & f_z(0,1,1) &= 0 \end{aligned}$$

So $\nabla f(0,1,1) = \langle 1,0,0\rangle$ and the specified directional derivative is

$$D_{\frac{\langle 0,1,1\rangle}{\sqrt{2}}} = \langle 1,0,0\rangle \cdot \frac{\langle 0,1,1\rangle}{\sqrt{2}} = 0$$

2.7.2 (∗) Find $\nabla\big(y^2 + \sin(xy)\big)$.

Solution In two dimensions, write $g(x,y) = y^2 + \sin(xy)$. Then

$$\nabla g = \langle g_x\,,\, g_y\rangle = \langle y\cos(xy)\,,\, 2y + x\cos(xy)\rangle$$

In three dimensions, write $g(x,y,z) = y^2 + \sin(xy)$. Then

$$\nabla g = \langle g_x\,,\, g_y\,,\, g_z\rangle = \langle y\cos(xy)\,,\, 2y + x\cos(xy)\,,\, 0\rangle$$

▸▸ Stage 2

2.7.3 Find the rate of change of the given function at the given point in the given direction.
(a) $f(x,y) = 3x - 4y$ at the point $(0,2)$ in the direction $-2\hat{\imath}$.
(b) $f(x,y,z) = x^{-1} + y^{-1} + z^{-1}$ at $(2,-3,4)$ in the direction $\hat{\imath} + \hat{\jmath} + \hat{\mathbf{k}}$.

Solution (a) The gradient of f is $\nabla f(x,y) = \langle 3,-4\rangle$. So the specified rate of change is

$$\langle 3,-4\rangle \cdot \frac{\langle -2,0\rangle}{|\langle -2,0\rangle|} = -3$$

(b) The gradient of f is $\nabla f(x,y,z) = \left\langle -x^{-2}, -y^{-2}, -z^{-2}\right\rangle$. In particular, the gradient of f at the point $(2,-3,4)$ is $\nabla f(2,-3,4) = \left\langle -\frac{1}{4}, -\frac{1}{9}, -\frac{1}{16}\right\rangle$. So the specified rate of change is

$$\left\langle -\frac{1}{4}, -\frac{1}{9}, -\frac{1}{16}\right\rangle \cdot \frac{\langle 1,1,1\rangle}{\sqrt{3}} = -\frac{61}{144\sqrt{3}} \approx -0.2446$$

2.7.4 In what directions at the point $(2,0)$ does the function $f(x,y) = xy$ have the specified rates of change?
(a) -1
(b) -2
(c) -3

Solution The gradient of $f(x,y)$ is $\nabla f(x,y) = \langle y,x\rangle$. In particular, the gradient of f at the point $(2,0)$ is $\nabla f(2,0) = \langle 0,2\rangle$. So the rate of change in the direction that makes angle θ with respect to the x-axis, that is, in the direction $\langle \cos\theta, \sin\theta\rangle$ is

$$\langle \cos\theta, \sin\theta\rangle \cdot \nabla f(2,0) = \langle \cos\theta, \sin\theta\rangle \cdot \langle 0,2\rangle = 2\sin\theta$$

(a) To get a rate -1, we need

$$\sin\theta = -\frac{1}{2} \implies \theta = -30^\circ,\ -150^\circ$$

So the desired directions are

$$\langle \cos\theta, \sin\theta\rangle = \left\langle \pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle$$

(b) To get a rate -2, we need

$$\sin\theta = -1 \implies \theta = -90^\circ$$

So the desired direction is

$$\langle \cos\theta, \sin\theta\rangle = \langle 0,-1\rangle$$

(c) To get a rate -3, we need

$$\sin\theta = -\frac{3}{2}$$

No θ obeys this, since $-1 \leqslant \sin\theta \leqslant 1$ for all θ. So no direction works!

2.7.5 Find $\nabla f(a,b)$ given the directional derivatives

$$D_{(\hat{\imath}+\hat{\jmath})/\sqrt{2}}f(a,b) = 3\sqrt{2} \qquad D_{(3\hat{\imath}-4\hat{\jmath})/5}f(a,b) = 5$$

Solution Denote $\nabla f(a,b) = \langle \alpha, \beta\rangle$. We are told that

$$\langle\alpha,\beta\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle = 3\sqrt{2} \quad \text{or} \qquad \alpha+\beta = 6$$
$$\langle\alpha,\beta\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle = 5 \qquad \text{or} \quad 3\alpha - 4\beta = 25$$

Adding 4 times the first equation to the second equation gives $7\alpha = 49$. Substituting $\alpha = 7$ into the first equation gives $\beta = -1$. So $\nabla f(a,b) = \langle 7, -1\rangle$.

2.7.6 (*) You are standing at a location where the surface of the earth is smooth. The slope in the southern direction is 4 and the slope in the south–eastern direction is $\sqrt{2}$. Find the slope in the eastern direction.

Solution Use a coordinate system with the positive y–axis pointing north, with the positive x–axis pointing east and with our current location being $x = y = 0$. Denote by $z(x,y)$ the elevation of the earth's surface at (x,y). We are told that

$$\nabla z(0,0)\cdot(-\hat{\jmath}) = 4$$
$$\nabla z(0,0)\cdot\left(\frac{\hat{\imath}-\hat{\jmath}}{\sqrt{2}}\right) = \sqrt{2}$$

The first equation implies that $z_y(0,0) = -4$ and the second equation implies that

$$\frac{z_x(0,0) - z_y(0,0)}{\sqrt{2}} = \sqrt{2} \implies z_x(0,0) = z_y(0,0) + 2 = -2$$

So the slope in the eastern direction is

$$\nabla z(0,0)\cdot\hat{\imath} = z_x(0,0) = -2$$

2.7.7 (*) Assume that the directional derivative of $w = f(x,y,z)$ at a point P is a maximum in the direction of the vector $2\hat{\imath} - \hat{\jmath} + \hat{\mathbf{k}}$, and the value of the directional derivative in that direction is $3\sqrt{6}$.

(a) Find the gradient vector of $w = f(x,y,z)$ at P.

(b) Find the directional derivative of $w = f(x,y,z)$ at P in the direction of the vector $\hat{\imath} + \hat{\jmath}$

Solution (a) Use $\nabla f(P)$ to denote the gradient vector of f at P. We are told that

- directional derivative of f at P is a maximum in the direction $2\hat{\imath} - \hat{\jmath} + \hat{\mathbf{k}}$, which implies that $\nabla f(P)$ is parallel to $2\hat{\imath} - \hat{\jmath} + \hat{\mathbf{k}}$, and
- the magnitude of the directional derivative in that direction is $3\sqrt{6}$, which implies that $|\nabla f(P)| = 3\sqrt{6}$.

So

$$\nabla f(P) = 3\sqrt{6}\frac{2\hat{\imath} - \hat{\jmath} + \hat{\mathbf{k}}}{|2\hat{\imath} - \hat{\jmath} + \hat{\mathbf{k}}|} = 6\hat{\imath} - 3\hat{\jmath} + 3\hat{\mathbf{k}}$$

(b) The directional derivative of f at P in the direction $\hat{\imath} + \hat{\jmath}$ is

$$\nabla f(P) \cdot \frac{\hat{\imath} + \hat{\jmath}}{|\hat{\imath} + \hat{\jmath}|} = \frac{1}{\sqrt{2}}\big(6\hat{\imath} - 3\hat{\jmath} + 3\hat{\mathbf{k}}\big) \cdot \big(\hat{\imath} + \hat{\jmath}\big) = \frac{3}{\sqrt{2}}$$

2.7.8 (*) A hiker is walking on a mountain with height above the $z = 0$ plane given by

$$z = f(x, y) = 6 - xy^2$$

The positive x–axis points east and the positive y–axis points north, and the hiker starts from the point $P(2, 1, 4)$.

(a) In what direction should the hiker proceed from P to ascend along the steepest path? What is the slope of the path?
(b) Walking north from P, will the hiker start to ascend or descend? What is the slope?
(c) In what direction should the hiker walk from P to remain at the same height?

Solution (a) The gradient of f at $(x, y) = (2, 1)$ is

$$\nabla f(2, 1) = \left\langle -y^2\,,\, -2xy \right\rangle\Big|_{(x,y)=(2,1)} = \langle -1\,,\, -4\rangle$$

So the path of steepest ascent is in the direction $-\frac{1}{\sqrt{17}}\langle 1\,,\, 4\rangle$, which is a little west of south. The slope is

$$|\nabla f(2, 1)| = |\,\langle -1\,,\, -4\rangle\,| = \sqrt{17}$$

(b) The directional derivative in the north direction is

$$D_{\langle 0,1\rangle} f(2, 1) = \nabla f(2, 1) \cdot \langle 0, 1\rangle = \langle -1\,,\, -4\rangle \cdot \langle 0, 1\rangle = -4$$

So the hiker descends with slope $|-4| = 4$.

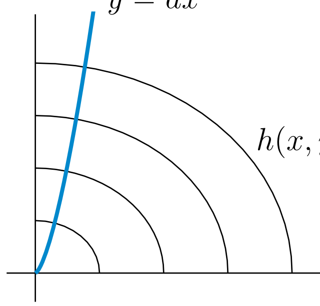
(c) To contour, i.e. remain at the same height, the hiker should walk in a direction perpendicular to $\nabla f(2, 1) = \langle -1\,,\, -4\rangle$. Two unit vectors perpendicular to $\langle -1\,,\, -4\rangle$ are $\pm\frac{1}{\sqrt{17}}\langle 4, -1\rangle$.

2.7.9 Two hikers are climbing a (small) mountain whose height is $z = 1000 - 2x^2 - 3y^2$. They start at $(1, 1, 995)$ and follow the path of steepest ascent. Their (x, y) coordinates obey $y = ax^b$ for some constants a, b. Determine a and b.

Solution The gradient of $h(x,y) = 1000 - 2x^2 - 3y^2$ is $\nabla h(x,y) = (-4x, -6y)$. This gradient (which points in the direction of steepest ascent) must be parallel to the tangent to $y = ax^b$ at all points on $y = ax^b$. A tangent to $y = ax^b$ is $\left\langle 1, \frac{\mathrm{d}y}{\mathrm{d}x} \right\rangle = \langle 1, abx^{b-1} \rangle$.

$$\langle -4x, -6y \rangle \parallel \left\langle 1, abx^{b-1} \right\rangle \implies \frac{abx^{b-1}}{1} = \frac{-6y}{-4x} \implies \frac{3}{2}y = abx^b$$

This is true at all points on $y = ax^b$ if and only if $b = \frac{3}{2}$. As $(1,1)$ must also be on $y = ax^b$, we need $1 = a1^b$, which forces $a = 1$, $b = \frac{3}{2}$. Here is a contour map showing the hiking trail.



2.7.10 (*) A mosquito is at the location $(3,2,1)$ in $\mathbb{R}^3$. She knows that the temperature T near there is given by $T = 2x^2 + y^2 - z^2$.

(a) She wishes to stay at the same temperature, but must fly in some initial direction. Find a direction in which the initial rate of change of the temperature is 0.

(b) If you and another student both get correct answers in part (a), must the directions you give be the same? Why or why not?

(c) What initial direction or directions would suit the mosquito if she wanted to cool down as fast as possible?

Solution (a) The temperature gradient at $(3,2,1)$ is

$$\nabla T(3,2,1) = \langle 4x\,,\, 2y\,,\, -2z \rangle \Big|_{(x,y,z)=(3,2,1)} = \langle 12\,,\, 4\,,\, -2 \rangle$$

She wishes to fly in a direction that is perpendicular to $\nabla T(3,2,1)$. That is, she wishes to fly in a direction $\langle a\,,\, b\,,\, c \rangle$ that obeys

$$0 = \langle 12\,,\, 4\,,\, -2 \rangle \cdot \langle a\,,\, b\,,\, c \rangle = 12a + 4b - 2c$$

Any nonzero $\langle a\,,\, b\,,\, c \rangle$ that obeys $12a + 4b - 2c = 0$ is an allowed direction. Four allowed unit vectors are $\pm\frac{\langle 0\,,1\,,2 \rangle}{\sqrt{5}}$ and $\pm\frac{\langle 1\,,-3\,,0 \rangle}{\sqrt{10}}$.

(b) No they need not be the same. Four different explicit directions were given in part (a).

(c) To cool down as quickly as possible, she should move in the direction opposite to the temperature gradient. A unit vector in that direction is $-\frac{\langle 6\,,2\,,-1 \rangle}{\sqrt{41}}$.

2.7.11 (*)

The air temperature $T(x,y,z)$ at a location (x,y,z) is given by:

$$T(x,y,z) = 1 + x^2 + yz.$$

(a) A bird passes through $(2,1,3)$ travelling towards $(4,3,4)$ with speed 2. At what rate does the air temperature it experiences change at this instant?

(b) If instead the bird maintains constant altitude ($z = 3$) as it passes through $(2,1,3)$ while also keeping at a fixed air temperature, $T = 8$, what are its two possible directions of travel?

Solution The temperature gradient at $(2,1,3)$ is

$$\nabla T(2,1,3) = \langle 2x\,,\, z\,,\, y\rangle\Big|_{(x,y,z)=(2,1,3)} = \langle 4\,,\, 3\,,\, 1\rangle$$

(a) The bird is flying in the direction $\langle 4-2\,,\, 3-1\,,\, 4-3\rangle = \langle 2\,,\, 2\,,\, 1\rangle$ at speed 2 and so has velocity $\mathbf{v} = 2\frac{\langle 2,2,1\rangle}{|\langle 2,2,1\rangle|} = \frac{2}{3}\langle 2\,,\, 2\,,\, 1\rangle$. The rate of change of air temperature experienced by the bird at that instant is

$$\nabla T(2,1,3)\cdot\mathbf{v} = \frac{2}{3}\langle 4\,,\, 3\,,\, 1\rangle\cdot\langle 2\,,\, 2\,,\, 1\rangle = 10$$

(b) To maintain constant altitude (while not being stationary), the bird's direction of travel has to be of the form $\langle a\,,\, b\,,\, 0\rangle$, for some constants a and b, not both zero. To keep the air temperature fixed, its direction of travel has to be perpendicular to $\nabla T(2,1,3) = \langle 4\,,\, 3\,,\, 1\rangle$. So a and b have to obey

$$0 = \langle a\,,\, b\,,\, 0\rangle\cdot\langle 4\,,\, 3\,,\, 1\rangle = 4a + 3b \iff b = -\frac{4}{3}a$$

and the direction of travel has to be a nonzero constant times $\langle 3\,,-4\,,0\rangle$. The two such unit vectors are $\pm\frac{1}{5}\langle 3\,,-4\,,0\rangle$.

2.7.12 (*) Let $f(x,y) = 2x^2 + 3xy + y^2$ be a function of x and y.

(a) Find the maximum rate of change of $f(x,y)$ at the point $P\left(1,-\frac{4}{3}\right)$.

(b) Find the directions in which the directional derivative of $f(x,y)$ at the point $P\left(1,-\frac{4}{3}\right)$ has the value $\frac{1}{5}$.

Solution We are going to need, in both parts of this question, the gradient of $f(x,y)$ at $(x,y) = \left(1,-\frac{4}{3}\right)$. So we find it first.

$$f_x(x,y) = 4x + 3y \qquad f_x(1,-4/3) = 0$$
$$f_y(x,y) = 3x + 2y \qquad f_y(1,-4/3) = \frac{1}{3}$$

so $\nabla f\left(1,-\frac{4}{3}\right) = \left\langle 0,\frac{1}{3}\right\rangle$.

(a) The maximum rate of change of f at P is

$$\left|\boldsymbol{\nabla} f\left(1,-\tfrac{4}{3}\right)\right| = \left|\left\langle 0, \tfrac{1}{3}\right\rangle\right| = \tfrac{1}{3}$$

(b) If $\langle a,b\rangle$ is a unit vector, the directional derivative of f at P in the direction $\langle a,b\rangle$ is

$$D_{\langle a,b\rangle} f\left(1,-\tfrac{4}{3}\right) = \boldsymbol{\nabla} f\left(1,-\tfrac{4}{3}\right)\cdot\langle a,b\rangle = \left\langle 0,\tfrac{1}{3}\right\rangle\cdot\langle a,b\rangle = \tfrac{b}{3}$$

So we need $\frac{b}{3} = \frac{1}{5}$ and hence $b = \frac{3}{5}$. For $\langle a,b\rangle$ to be a unit vector, we also need

$$a^2 + b^2 = 1 \iff a^2 = 1 - b^2 = 1 - \frac{3^2}{5^2} = \frac{16}{25} \iff a = \pm\frac{4}{5}$$

So the allowed directions are $\left\langle \pm\frac{4}{5}, \frac{3}{5}\right\rangle$.

$\boxed{2.7.13}$ (*) The temperature $T(x,y)$ at a point of the xy–plane is given by

$$T(x,y) = ye^{x^2}$$

A bug travels from left to right along the curve $y = x^2$ at a speed of 0.01m/sec. The bug monitors $T(x,y)$ continuously. What is the rate of change of T as the bug passes through the point $(1,1)$?

$\boxed{\text{Solution}}$ The slope of $y = x^2$ at $(1,1)$ is $\frac{\mathrm{d}}{\mathrm{d}x}x^2\Big|_{x=1} = 2$. So a unit vector in the bug's direction of motion is $\frac{\langle 1,2\rangle}{\sqrt{5}}$ and the bug's velocity vector is $\mathbf{v} = 0.01\frac{\langle 1,2\rangle}{\sqrt{5}}$.

The temperature gradient at $(1,1)$ is

$$\boldsymbol{\nabla} T(1,1) = \left\langle 2xye^{x^2}, e^{x^2}\right\rangle\Big|_{(x.y)=(1,1)} = \langle 2e, e\rangle$$

and the rate of change of T (per unit time) that the bug feels as it passes through the point $(1,1)$ is

$$\boldsymbol{\nabla} T(1,1)\cdot\mathbf{v} = \frac{0.01}{\sqrt{5}}\langle 2e, e\rangle\cdot\langle 1,2\rangle = \frac{0.04e}{\sqrt{5}}$$

$\boxed{2.7.14}$ (*) Suppose the function $T = F(x,y,z) = 3 + xy - y^2 + z^2 - x$ describes the temperature at a point (x,y,z) in space, with $F(3,2,1) = 3$.

(a) Find the directional derivative of T at $(3,2,1)$, in the direction of the point $(0,1,2)$.

(b) At the point $(3,2,1)$, in what direction does the temperature decrease most rapidly?

(c) Moving along the curve given by $x = 3e^t$, $y = 2\cos t$, $z = \sqrt{1+t}$, find $\frac{\mathrm{d}T}{\mathrm{d}t}$, the rate of change of temperature with respect to t, at $t = 0$.

(d) Suppose $\hat{\imath} + 5\hat{\jmath} + a\hat{\mathbf{k}}$ is a vector that is tangent to the temperature level surface $T(x,y,z) = 3$ at $(3,2,1)$. What is a?

Solution (a) We are to find the directional derivative in the direction

$$\langle 0-3\,,\,1-2\,,\,2-1\rangle = \langle -3\,,\,-1\,,\,1\rangle$$

As the gradient of F is

$$\nabla F(x,y,z) = \langle y-1\,,\,x-2y\,,\,2z\rangle$$

the directional derivative is

$$\begin{aligned} D_{\frac{\langle -3,-1,1\rangle}{|\langle -3,-1,1\rangle|}} F(3,2,1) &= \nabla F(3,2,1)\cdot \frac{\langle -3\,,\,-1\,,\,1\rangle}{|\,\langle -3\,,\,-1\,,\,1\rangle\,|} \\ &= \langle 2-1\,,\,3-2(2)\,,\,2(1)\rangle \cdot \frac{\langle -3\,,\,-1\,,\,1\rangle}{|\,\langle -3\,,\,-1\,,\,1\rangle\,|} = \langle 1\,,\,-1\,,\,2\rangle\cdot\frac{\langle -3\,,\,-1\,,\,1\rangle}{\sqrt{11}} \\ &= 0 \end{aligned}$$

(b) The temperature decreases most rapidly in the direction opposite the gradient. A unit vector in that direction is

$$-\frac{\nabla F(3,2,1)}{|\nabla F(3,2,1)|} = -\frac{\langle 1\,,\,-1\,,\,2\rangle}{|\,\langle 1\,,\,-1\,,\,2\rangle\,|} = \frac{1}{\sqrt{6}}\langle -1\,,\,1\,,\,-2\rangle$$

(c) The velocity vector at time 0 is

$$\mathbf{v} = \big\langle x'(0)\,,\,y'(0)\,,\,z'(0)\big\rangle = \left\langle 3e^t\,,\,-2\sin t\,,\,\frac{1}{2\sqrt{1+t}}\right\rangle\Big|_{t=0} = \left\langle 3\,,\,0\,,\,\frac{1}{2}\right\rangle$$

So the rate of change of temperature with respect to t at $t=0$ is

$$\nabla F(3,2,1)\cdot\mathbf{v} = \langle 1\,,\,-1\,,\,2\rangle\cdot\left\langle 3\,,\,0\,,\,\frac{1}{2}\right\rangle = 4$$

(d) For $\hat{\imath}+5\hat{\jmath}+a\hat{\mathbf{k}}$ to be tangent to the level surface $F(x,y,z)=3$ at $(3,2,1)$, $\hat{\imath}+5\hat{\jmath}+a\hat{\mathbf{k}}$ must be perpendicular to $\nabla F(3,2,1)$. So

$$0 = \langle 1\,,\,5\,,\,a\rangle\cdot\langle 1\,,\,-1\,,\,2\rangle = -4+2a$$

So $a=2$.

2.7.15 (∗) Let

$$\begin{aligned} f(x,y,z) &= (2x+y)e^{-(x^2+y^2+z^2)} \\ g(x,y,z) &= xz+y^2+yz+z^2 \end{aligned}$$

(a) Find the gradients of f and g at $(0,1,-1)$.
(b) A bird at $(0,1,-1)$ flies at speed 6 in the direction in which $f(x,y,z)$ increases most rapidly. As it passes through $(0,1,-1)$, how quickly does $g(x,y,z)$ appear (to the bird) to be changing?
(c) A bat at $(0,1,-1)$ flies in the direction in which $f(x,y,z)$ and $g(x,y,z)$ do not change, but z increases. Find a vector in this direction.

Solution (a) The first order partial derivatives of f and g are

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y,z) &= 2e^{-(x^2+y^2+z^2)} - 2x(2x+y)e^{-(x^2+y^2+z^2)} &&\implies \frac{\partial f}{\partial x}(0,1,-1) = 2e^{-2}\\
\frac{\partial f}{\partial y}(x,y,z) &= e^{-(x^2+y^2+z^2)} - 2y(2x+y)e^{-(x^2+y^2+z^2)} &&\implies \frac{\partial f}{\partial y}(0,1,-1) = -e^{-2}\\
\frac{\partial f}{\partial z}(x,y,z) &= -2z(2x+y)e^{-(x^2+y^2+z^2)} &&\implies \frac{\partial f}{\partial z}(0,1,-1) = 2e^{-2}\\
\frac{\partial g}{\partial x}(x,y,z) &= z &&\implies \frac{\partial g}{\partial x}(0,1,-1) = -1\\
\frac{\partial g}{\partial y}(x,y,z) &= 2y+z &&\implies \frac{\partial g}{\partial y}(0,1,-1) = 1\\
\frac{\partial g}{\partial z}(x,y,z) &= x+y+2z &&\implies \frac{\partial g}{\partial z}(0,1,-1) = -1
\end{aligned}$$

so that gradients are

$$\nabla f(0,1,-1) = e^{-2}\langle 2,-1,2\rangle \qquad \nabla g(0,1,-1) = \langle -1,1,-1\rangle$$

(b) The bird's velocity is the vector of length 6 in the direction of $\nabla f(0,1,-1)$, which is

$$\mathbf{v} = 6\frac{\langle 2,-1,2\rangle}{|\langle 2,-1,2\rangle|} = \langle 4,-2,4\rangle$$

The rate of change of g (per unit time) seen by the bird is

$$\nabla g(0,1,-1)\cdot\mathbf{v} = \langle -1,1,-1\rangle\cdot\langle 4,-2,4\rangle = -10$$

(c) The direction of flight for the bat has to be perpendicular to both $\nabla f(0,1,-1) = e^{-2}\langle 2,-1,2\rangle$ and $\nabla g(0,1,-1) = \langle -1,1,-1\rangle$. Any vector which is a non zero constant times

$$\langle 2,-1,2\rangle\times\langle -1,1,-1\rangle = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ 2 & -1 & 2\\ -1 & 1 & -1\end{bmatrix} = \langle -1\,,\,0\,,\,1\rangle$$

is perpendicular to both $\nabla f(0,1,-1)$ and $\nabla g(0,1,-1)$. In addition, the direction of flight for the bat must have a positive z–component. So any vector which is a (strictly) positive constant times $\langle -1\,,\,0\,,\,1\rangle$ is fine.

2.7.16 ($*$) A bee is flying along the curve of intersection of the surfaces $3z+x^2+y^2=2$ and $z=x^2-y^2$ in the direction for which z is increasing. At time $t=2$, the bee passes through the point $(1,1,0)$ at speed 6.
(a) Find the velocity (vector) of the bee at time $t=2$.
(b) The temperature T at position (x,y,z) at time t is given by $T=xy-3x+2yt+z$. Find the rate of change of temperature experienced by the bee at time $t=2$.

Solution (a) Let's use $\mathbf{v}$ to denote the bee's velocity vector at time $t=2$.

- The bee's direction of motion is tangent to the curve. That tangent is perpendicular to both the normal vector to $3z + x^2 + y^2 = 2$ at $(1,1,0)$, which is

$$\langle 2x\,,\, 2y\,,\, 3\rangle \Big|_{(x,y,z)=(1,1,0)} = \langle 2\,,\, 2\,,\, 3\rangle$$

  and the normal vector to $z = x^2 - y^2$ at $(1,1,0)$, which is

$$\langle 2x\,,\, -2y\,,\, -1\rangle \Big|_{(x,y,z)=(1,1,0)} = \langle 2\,,\, -2\,,\, -1\rangle$$

  So $\mathbf{v}$ has to be some constant times

$$\langle 2\,,\, 2\,,\, 3\rangle \times \langle 2\,,\, -2\,,\, -1\rangle = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 2 & 2 & 3 \\ 2 & -2 & -1 \end{bmatrix} = \langle 4\,,\, 8\,,\, -8\rangle$$

  or, equivalently, some constant times $\langle 1\,,\, 2\,,\, -2\rangle$.

- Since the z–component of $\mathbf{v}$ has to be positive, $\mathbf{v}$ has to be a positive constant times $\langle -1\,,\, -2\,,\, 2\rangle$.

- Since the speed has to be 6, $\mathbf{v}$ has to have length 6.

As $|\,\langle -1\,,\, -2\,,\, 2\rangle\,| = 3$

$$\mathbf{v} = 2\,\langle -1\,,\, -2\,,\, 2\rangle = \langle -2\,,\, -4\,,\, 4\rangle$$

(b) Solution 1: Suppose that the bee is at $\big(x(t), y(t), z(t)\big)$ at time t. Then the temperature that the bee feels at time t is

$$T\big(x(t), y(t), z(t), t\big) = x(t)y(t) - 3x(t) + 2y(t)t + z(t)$$

Then the rate of change of temperature (per unit time) felt by the bee at time $t = 2$ is

$$\frac{\mathrm{d}}{\mathrm{d}t} T\big(x(t), y(t), z(t), t\big)\Big|_{t=2} = x'(2)y(2) + x(2)y'(2) - 3x'(2) + 2y'(2)2 + 2y(2) + z'(2)$$

Recalling that, at time $t = 2$, the bee is at $(1,1,0)$ and has velocity $\langle -2\,,\, -4\,,\, 4\rangle$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} T\big(x(t), y(t), z(t), t\big)\Big|_{t=2} &= (-2)(1) + (1)(-4) - 3(-2) + 2(-4)2 + 2(1) + 4 \\ &= -10 \end{aligned}$$

(b) Solution 2: Suppose that the bee is at $\big(x(t), y(t), z(t)\big)$ at time t. Then the temperature that the bee feels at time t is

$$T\big(x(t), y(t), z(t), t\big)$$

By the chain rule, the rate of change of temperature (per unit time) felt by the bee at time $t = 2$ is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} T\big(x(t), y(t), z(t), t\big)\Big|_{t=2} = \Big[&\frac{\partial T}{\partial x}\big(x(t), y(t), z(t), t\big)\, x'(t) + \frac{\partial T}{\partial y}\big(x(t), y(t), z(t), t\big)\, y'(t)\\ &+ \frac{\partial T}{\partial z}\big(x(t), y(t), z(t), t\big)\, z'(t) + \frac{\partial T}{\partial t}\big(x(t), y(t), z(t), t\big)\Big]_{t=2}\end{aligned}$$

Recalling that $T = xy - 3x + 2yt + z$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} T\big(x(t), y(t), z(t), t\big)\Big|_{t=2} = [y(2) - 3]x'(2) + [x(2) + 2 \times 2]y'(2) + z'(2) + 2y(2)$$

Also recalling that, at time $t = 2$, the bee is at $(1, 1, 0)$ and has velocity $\langle -2\,, -4\,, 4\rangle$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} T\big(x(t), y(t), z(t), t\big)\Big|_{t=2} &= [-2](-2) + [5](-4) + 4 + 2\\ &= -10\end{aligned}$$

2.7.17 (*) The temperature at a point (x, y, z) is given by $T(x, y, z) = 5e^{-2x^2-y^2-3z^2}$, where T is measured in centigrade and x, y, z in meters.

(a) Find the rate of change of temperature at the point $P(1, 2, -1)$ in the direction toward the point $(1, 1, 0)$.

(b) In which direction does the temperature decrease most rapidly?

(c) Find the maximum rate of decrease at P.

Solution (a) We are to find the rate of change of $T(x, y, z)$ at $(1, 2, -1)$ in the direction $\langle 1, 1, 0\rangle - \langle 1, 2, -1\rangle = \langle 0, -1, 1\rangle$. That rate of change (per unit distance) is the directional derivative

$$D_{\frac{\langle 0,-1,1\rangle}{\sqrt{2}}} T(1, 2, -1) = \boldsymbol{\nabla} T(1, 2, -1) \cdot \frac{\langle 0, -1, 1\rangle}{\sqrt{2}}$$

As

$$\begin{aligned}\frac{\partial T}{\partial x}(x, y, z) &= -20x\, e^{-2x^2-y^2-3z^2} & \frac{\partial T}{\partial x}(1, 2, -1) &= -20\, e^{-9}\\ \frac{\partial T}{\partial y}(x, y, z) &= -10y\, e^{-2x^2-y^2-3z^2} & \frac{\partial T}{\partial y}(1, 2, -1) &= -20\, e^{-9}\\ \frac{\partial T}{\partial z}(x, y, z) &= -30z\, e^{-2x^2-y^2-3z^2} & \frac{\partial T}{\partial z}(1, 2, -1) &= 30\, e^{-9}\end{aligned}$$

the directional derivative

$$D_{\frac{\langle 0,-1,1\rangle}{\sqrt{2}}} T(1, 2, -1) = e^{-9}\, \langle -20, -20, 30\rangle \cdot \frac{\langle 0, -1, 1\rangle}{\sqrt{2}} = \frac{50}{\sqrt{2}} e^{-9} = 25\sqrt{2}\, e^{-9}$$

(b) The direction of maximum rate of decrease is $-\boldsymbol{\nabla} T(1, 2, -1)$. A unit vector in that direction is $\frac{\langle 2,2,-3\rangle}{\sqrt{17}}$.

(c) The maximum rate of decrease at P is $-|\boldsymbol{\nabla} T(1,2,-1)| = -10e^{-9}|\langle -2,-2,3\rangle| = -10\sqrt{17}e^{-9}$.

$\boxed{2.7.18}$ (∗) The directional derivative of a function $w = f(x,y,z)$ at a point P in the direction of the vector $\hat{\boldsymbol{\imath}}$ is 2, in the direction of the vector $\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}$ is $-\sqrt{2}$, and in the direction of the vector $\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}}$ is $-\frac{5}{\sqrt{3}}$. Find the direction in which the function $w = f(x,y,z)$ has the maximum rate of change at the point P . What is this maximum rate of change?

$\boxed{\text{Solution}}$ Denote by $\langle a\,,\,b\,,\,c\rangle$ the gradient of the function f at P. We are told

$$\begin{aligned}\langle a\,,\,b\,,\,c\rangle\cdot\langle 1\,,\,0\,,\,0\rangle &= 2\\ \langle a\,,\,b\,,\,c\rangle\cdot\frac{1}{\sqrt{2}}\langle 1\,,\,1\,,\,0\rangle &= -\sqrt{2}\\ \langle a\,,\,b\,,\,c\rangle\cdot\frac{1}{\sqrt{3}}\langle 1\,,\,1\,,\,1\rangle &= -\frac{5}{\sqrt{3}}\end{aligned}$$

Simplifying

$$\begin{aligned}a &= 2\\ a+b &= -2\\ a+b+c &= -5\end{aligned}$$

From these equations we read off, in order, $a = 2$, $b = -4$ and $c = -3$. The function f has maximum rate of change at P in the direction if the gradient of f. The unit vector in that direction is

$$\frac{\langle 2\,,\,-4\,,\,-3\rangle}{|\langle 2\,,\,-4\,,\,-3\rangle|} = \frac{\langle 2\,,\,-4\,,\,-3\rangle}{\sqrt{29}}$$

The maximum rate of change is the magnitude of the gradient, which is $\sqrt{29}$.

$\boxed{2.7.19}$ (∗) Suppose it is known that the direction of the fastest increase of the function $f(x,y)$ at the origin is given by the vector $\langle 1,2\rangle$. Find a unit vector u that is tangent to the level curve of $f(x,y)$ that passes through the origin.

$\boxed{\text{Solution}}$ We are told that the direction of fastest increase for the function $f(x,y)$ at the origin is given by the vector $\langle 1,2\rangle$. This implies that $\boldsymbol{\nabla} f(0,0)$ is parallel to $\langle 1,2\rangle$. This in turn implies that $\langle 1,2\rangle$ is normal to the level curve of $f(x,y)$ that passes through the origin. So $\langle 2,-1\rangle$, being perpendicular to $\langle 1,2\rangle$, is tangent to the level curve of $f(x,y)$ that passes through the origin. The unit vectors that are parallel to $\langle 2,-1\rangle$ are $\pm\frac{1}{\sqrt{5}}\langle 2,-1\rangle$.

$\boxed{2.7.20}$ (∗) The shape of a hill is given by $z = 1000 - 0.02x^2 - 0.01y^2$. Assume that the x–axis is pointing East, and the y–axis is pointing North, and all distances are in metres.

(a) What is the direction of the steepest ascent at the point $(0, 100, 900)$? (The answer should be in terms of directions of the compass).

(b) What is the slope of the hill at the point $(0, 100, 900)$ in the direction from (a)?

(c) If you ride a bicycle on this hill in the direction of the steepest descent at 5 m/s, what is the rate of change of your altitude (with respect to time) as you pass through the point (0, 100, 900)?

$\boxed{\text{Solution}}$ Write $h(x, y) = 1000 - 0.02\,x^2 - 0.01y^2$ so that the hill is $z = h(x, y)$.

(a) The direction of steepest ascent at $(0, 100, 900)$ is the direction of maximum rate of increase of $h(x, y)$ at $(0, 100)$ which is $\nabla h(0, 100) = \langle 0\,,\, -0.01(2)(100)\rangle = \langle 0\,,\, -2\rangle$. In compass directions that is South.

(b) The slope of the hill there is

$$\nabla h(0, 100) \cdot \langle 0, -1\rangle = -\frac{\partial h}{\partial y}(0, 100) = 2$$

(c) Denote by $\big(x(t), y(t), z(t)\big)$ your position at time t and suppose that you are at $(0, 100, 900)$ at time $t = 0$. Then we know

- $z(t) = 1000 - 0.02\,x(t)^2 - 0.01y(t)^2$, so that $z'(t) = -0.04\,x(t)x'(t) - 0.02y(t)y'(t)$, since you are on the hill and
- $x'(0) = 0$ and $y'(0) > 0$ since you are going in the direction of steepest descent and
- $x'(0)^2 + y'(0)^2 + z'(0)^2 = 25$ since you are moving at speed 5.

Since $x(0)$ and $y(0) = 100$, we have $z'(0) = -0.02(100)y'(0) = -2y'(0)$. So

$$\begin{aligned} 25 = x'(0)^2 + y'(0)^2 + z'(0)^2 = 5\,y'(0)^2 &\implies y'(0) = \sqrt{5} \\ &\implies \big\langle x'(0)\,,\, y'(0)\,,\, z'(0)\big\rangle = \Big\langle 0\,,\, \sqrt{5}\,,\, -2\sqrt{5}\Big\rangle \end{aligned}$$

and your rate of change of altitude is

$$\frac{\mathrm{d}}{\mathrm{d}t} h\big(x(t)\,,\, y(t)\big)\Big|_{t=0} = \nabla h(0, 100) \cdot \big\langle x'(0)\,,\, y'(0)\big\rangle = \langle 0\,,\, -2\rangle \cdot \Big\langle 0\,,\, \sqrt{5}\Big\rangle = -2\sqrt{5}$$

$\boxed{2.7.21}$ (∗) Let the pressure P and temperature T at a point (x, y, z) be

$$P(x, y, z) = \frac{x^2 + 2y^2}{1 + z^2}, \qquad T(x, y, z) = 5 + xy - z^2$$

(a) If the position of an airplane at time t is

$$(x(t), y(t), z(t)) = (2t, t^2 - 1, \cos t)$$

find $\frac{\mathrm{d}}{\mathrm{d}t}(PT)^2$ at time $t = 0$ as observed from the airplane.

(b) In which direction should a bird at the point $(0, -1, 1)$ fly if it wants to keep both P and T constant. (Give one possible direction vector. It does not need to be a unit vector.)

(c) An ant crawls on the surface $z^3 + zx + y^2 = 2$. When the ant is at the point $(0, -1, 1)$, in which direction should it go for maximum increase of the temperature $T = 5 + xy - z^2$? Your answer should be a vector $\langle a, b, c\rangle$, not necessarily of unit length. (Note that the ant cannot crawl in the direction of the gradient because that leads off the surface. The direction vector $\langle a, b, c\rangle$ has to be on the tangent plane to the surface.)

Solution Reading through the question as a whole we see that we will need

- for part (a), the gradient of PT at $(2t, t^2 - 1, \cos t)\Big|_{t=0} = (0, -1, 1)$
- for part (b), the gradients of both P and T at $(0, -1, 1)$ and
- for part (c), the gradient of T at $(0, -1, 1)$ and the gradient of $S = z^3 + xz + y^2$ at $(0, -1, 1)$ (to get the normal vector to the surface at that point).

So, by way of preparation, let's compute all of these gradients.

$$\begin{aligned}
\nabla P(x, y, z) &= \frac{2x}{1 + z^2}\hat{\imath} + \frac{4y}{1 + z^2}\hat{\jmath} - \frac{(x^2 + 2y^2)2z}{(1 + z^2)^2}\hat{\mathbf{k}} & \nabla P(0, -1, 1) &= -2\hat{\jmath} - \hat{\mathbf{k}} \\
\nabla T(x, y, z) &= y\,\hat{\imath} + x\,\hat{\jmath} - 2z\,\hat{\mathbf{k}} & \nabla T(0, -1, 1) &= -\hat{\imath} - 2\hat{\mathbf{k}} \\
\nabla S(x, y, z) &= z\,\hat{\imath} + 2y\,\hat{\jmath} + (x + 3z^2)\,\hat{\mathbf{k}} & \nabla S(0, -1, 1) &= \hat{\imath} - 2\hat{\jmath} + 3\,\hat{\mathbf{k}}
\end{aligned}$$

To get the gradient of PT we use the product rule

$$\nabla(PT)(x, y, z) = T(x, y, z)\,\nabla P(x, y, z) + P(x, y, z)\,\nabla T(x, y, z)$$

so that

$$\begin{aligned}
\nabla(PT)(0, -1, 1) &= T(0, -1, 1)\,\nabla P(0, -1, 1) + P(0, -1, 1)\,\nabla T(0, -1, 1) \\
&= (5 + 0 - 1)\big(-2\hat{\jmath} - \hat{\mathbf{k}}\big) + \frac{0 + 2}{1 + 1}\big(-\hat{\imath} - 2\hat{\mathbf{k}}\big) \\
&= -\hat{\imath} - 8\,\hat{\jmath} - 6\,\hat{\mathbf{k}}
\end{aligned}$$

(a) Since $\frac{\mathrm{d}}{\mathrm{d}t}(PT)^2 = 2(PT)\frac{\mathrm{d}}{\mathrm{d}t}(PT)$, and the velocity vector of the plane at time 0 is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle 2t, t^2-1, \cos t\right\rangle\Big|_{t=0} = \langle 2, 2t, -\sin t\rangle\Big|_{t=0} = \langle 2,0,0\rangle$$

we have

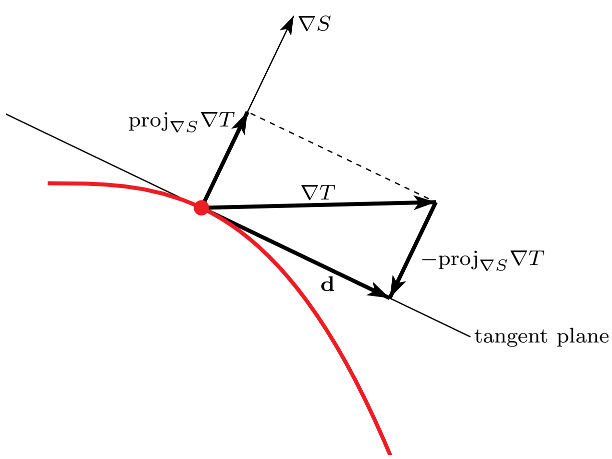
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(PT)^2\Big|_{t=0} &= 2\,P(0,-1,1)\,T(0,-1,1)\ \nabla(PT)(0,-1,1)\cdot\langle 2,0,0\rangle \\ &= 2\ \frac{0+2}{1+1}\ (5+0-1)\ \langle -1,-8,-6\rangle\cdot\langle 2,0,0\rangle \\ &= -16\end{aligned}$$

(b) The direction should be perpendicular to $\nabla P(0,-1,1)$ (to keep P constant) and should also be perpendicular to $\nabla T(0,-1,1)$ (to keep T constant). So any nonzero constant times

$$\begin{aligned}\pm\nabla P(0,-1,1)\times\nabla T(0,-1,1) &= \pm\langle 0\,,\,-2\,,\,-1\rangle\times\langle -1\,,\,0\,,\,-2\rangle = \pm\det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ 0 & 2 & 1\\ 1 & 0 & 2\end{bmatrix} \\ &= \pm\langle 4\,,\,1\,,\,-2\rangle\end{aligned}$$

are allowed directions.

(c) We want the direction to be as close as possible to $\nabla T(0,-1,1) = \langle -1\,,\,0\,,\,-2\rangle$ while still being tangent to the surface, i.e. being perpendicular to the normal vector $\nabla S(0,-1,1) = \langle 1\,,\,-2\,,\,3\rangle$. We can get that optimal direction by subtracting from $\nabla T(0,-1,1)$ the projection of $\nabla T(0,-1,1)$ onto the normal vector.



The projection of $\nabla T(0,-1,1)$ onto the normal vector $\nabla S(0,-1,1)$ is

$$\begin{aligned}\mathrm{proj}_{\nabla S(0,-1,1)}\nabla T(0,-1,1) &= \frac{\nabla T(0,-1,1)\cdot\nabla S(0,-1,1)}{|\nabla S(0,-1,1)|^2}\nabla S(0,-1,1) \\ &= \frac{\langle -1\,,\,0\,,\,-2\rangle\cdot\langle 1\,,\,-2\,,\,3\rangle}{|\,\langle 1\,,\,-2\,,\,3\rangle\,|^2}\,\langle 1\,,\,-2\,,\,3\rangle \\ &= \frac{-7}{14}\,\langle 1\,,\,-2\,,\,3\rangle\end{aligned}$$

So the optimal direction is

$$\begin{aligned}\mathbf{d} &= \boldsymbol{\nabla} T(0,-1,1) - \text{proj}_{\boldsymbol{\nabla} S(0,-1,1)} \boldsymbol{\nabla} T(0,-1,1) \\ &= \langle -1\,,\,0\,,\,-2\rangle - \frac{-7}{14}\,\langle 1\,,\,-2\,,\,3\rangle \\ &= \left\langle -\frac{1}{2}\,,\,-1\,,\,-\frac{1}{2}\right\rangle\end{aligned}$$

So any positive non zero multiple of $-\langle 1\,,\,2\,,\,1\rangle$ will do. Note, as a check, that $-\langle 1\,,\,2\,,\,1\rangle$ has dot product zero, i.e. is perpendicular to, $\boldsymbol{\nabla} S(0,-1,1) = \langle 1\,,\,-2\,,\,3\rangle$.

$\boxed{2.7.22}$ ($*$) Suppose that $f(x,y,z)$ is a function of three variables and let $\mathbf{u} = \frac{1}{\sqrt{6}}\langle 1,1,2\rangle$ and $\mathbf{v} = \frac{1}{\sqrt{3}}\langle 1,-1,-1\rangle$ and $\mathbf{w} = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle$. Suppose that at a point (a,b,c),

$$\begin{aligned}D_{\mathbf{u}} f &= 0 \\ D_{\mathbf{v}} f &= 0 \\ D_{\mathbf{w}} f &= 4\end{aligned}$$

Find $\boldsymbol{\nabla} f$ at (a,b,c).

$\boxed{\text{Solution}}$ Write $\boldsymbol{\nabla} f(a,b,c) = \langle F,G,H\rangle$. We are told that

$$\begin{aligned}D_{\mathbf{u}} f &= \frac{1}{\sqrt{6}}\langle 1,1,2\rangle\cdot\langle F,G,H\rangle = 0 \\ D_{\mathbf{v}} f &= \frac{1}{\sqrt{3}}\langle 1,-1,-1\rangle\cdot\langle F,G,H\rangle = 0 \\ D_{\mathbf{w}} f &= \frac{1}{\sqrt{3}}\langle 1,1,1\rangle\cdot\langle F,G,H\rangle = 4\end{aligned}$$

so that

$$F + G + 2H = 0 \tag{E1}$$
$$F - G - H = 0 \tag{E2}$$
$$F + G + H = 4\sqrt{3} \tag{E3}$$

Adding (E2) and (E3) gives $2F = 4\sqrt{3}$ or $F = 2\sqrt{3}$. Substituting $F = 2\sqrt{3}$ into (E1) and (E2) gives

$$G + 2H = -2\sqrt{3} \tag{E1}$$
$$-G - H = -2\sqrt{3} \tag{E2}$$

Adding (E1) and (E2) gives $H = -4\sqrt{3}$ and substituting $H = -4\sqrt{3}$ back into (E2) gives $G = 6\sqrt{3}$. All together

$$\boldsymbol{\nabla} f(a,b,c) = \sqrt{3}\,\langle 2,6,-4\rangle$$

$\boxed{2.7.23}$ (∗) The elevation of a hill is given by the equation $f(x,y) = x^2y^2e^{-x-y}$. An ant sits at the point $(1,1,e^{-2})$.

(a) Find the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ that maximizes

$$\lim_{t\to 0} \frac{f\big((1,1)+t\mathbf{u}\big) - f(1,1)}{t}$$

(b) Find a vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ pointing in the direction of the path that the ant could take in order to stay on the same elevation level e^{-2}.

(c) Find a vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ pointing in the direction of the path that the ant should take in order to maximize its instantaneous rate of level increase.

$\boxed{\text{Solution}}$ (a) The expression $\lim_{t\to 0} \frac{f((1,1)+t\mathbf{u})-f(1,1)}{t}$ is the directional derivative of f at $(1,1)$ in the direction $\mathbf{u}$, which is $D_{\mathbf{u}}f(1,1) = \nabla f(1,1)\cdot\mathbf{u}$. This is mazimized when $\mathbf{u}$ is parallel to $\nabla f(1,1)$. Since

$$f_x(x,y) = 2xy^2e^{-x-y} - x^2y^2e^{-x-y} \qquad f_y(x,y) = 2x^2ye^{-x-y} - x^2y^2e^{-x-y}$$

we have

$$\nabla f(1,1) = e^{-2}\langle 1,1\rangle$$

so that the desired unit vector $\mathbf{u}$ is $\frac{1}{\sqrt{2}}\langle 1,1\rangle$.

(b) In order to remain at elevation e^{-2}, the ant must move so that $D_{\mathbf{u}}f(1,1) = 0$. This is the case if $\mathbf{u} \perp \nabla f(1,1)$. For example, we can take $\mathbf{u} = \langle 1,-1\rangle$. When the ant moves in this direction, while remaining on the surface of the hill, its vertical component of velocity is zero. So $\mathbf{v} = c\langle 1,-1,0\rangle$ for any nonzero constant c.

(c) In order to maximize its instantaneous rate of level increase, the ant must choose the x and y coordinates of its velocity vector in the same direction as $\nabla f(1,1)$. Namely $\mathbf{u} = c\langle 1,1\rangle$ for any $c > 0$. To make $\mathbf{u}$ a unit vector, we choose $c = \frac{1}{\sqrt{2}}$. The corresponding value of the z coordinate of its velocity vector is the rate of change of f per unit horizontal distance travelled, which is the directional derivative

$$D_{\mathbf{u}}f(1,1) = \nabla f(1,1)\cdot\mathbf{u} = e^{-2}\langle 1,1\rangle\cdot\langle c,c\rangle = 2ce^{-2}$$

So $\mathbf{v} = \frac{1}{\sqrt{2}}\langle 1,1,2e^{-2}\rangle$. Any positive multiple of this vector is also a correct answer.

2.7.24 (*) Let the temperature in a region of space be given by $T(x,y,z) = 3x^2 + \frac{1}{2}y^2 + 2z^2$ degrees.

(a) A sparrow is flying along the curve $\mathbf{r}(s) = \left(\frac{1}{3}s^3, 2s, s^2\right)$ at a constant speed of 3ms^{-1}. What is the velocity of the sparrow when $s = 1$?

(b) At what rate does the sparrow feel the temperature is changing at the point $A\left(\frac{1}{3}, 2, 1\right)$ for which $s = 1$.

(c) At the point $A\left(\frac{1}{3}, 2, 1\right)$ in what direction will the temperature be decreasing at maximum rate?

(d) An eagle crosses the path of the sparrow at $A\left(\frac{1}{3}, 2, 1\right)$, is moving at right angles to the path of the sparrow, and is also moving in a direction in which the temperature remains constant. In what directions could the eagle be flying as it passes through the point A?

Solution (a) The direction of motion at $s = 1$ is given by the tangent vector

$$\mathbf{r}'(s) = \left\langle s^2, 2, 2s \right\rangle \big|_{s=1} = \langle 1, 2, 2 \rangle$$

Since the length of the velocity vector must be 3,

$$\text{velocity} = \mathbf{v} = 3\frac{\langle 1,2,2 \rangle}{|\langle 1,2,2 \rangle|} = \langle 1,2,2 \rangle$$

(b) The rate of change of temperature per unit distance felt by the sparrow at $s = 1$ is $\nabla T\left(\frac{1}{3}, 2, 1\right) \cdot \frac{\mathbf{v}}{|\mathbf{v}|}$. The rate of change of temperature per unit time felt by the sparrow at $s = 1$ is

$$\begin{aligned}\nabla T\left(\frac{1}{3}, 2, 1\right) \cdot \frac{\mathbf{v}}{|\mathbf{v}|}\,|\mathbf{v}| = \nabla T\left(\frac{1}{3}, 2, 1\right) \cdot \mathbf{v} &= \mathbf{v} \cdot \langle 6x, y, 4z \rangle \Big|_{\left(\frac{1}{3},2,1\right)} \\ &= \langle 1,2,2 \rangle \cdot \langle 2,2,4 \rangle = 14^\circ/\text{s}\end{aligned}$$

(c) The temperature decreases at maximum rate in the direction opposite the temperature gradient, which is (any positive constant times) $-\langle 2,2,4 \rangle$.

(d) The eagle is moving at right angles to the direction of motion of the sparrow, which is $\langle 1,2,2 \rangle$. As the eagle is also moving in a direction for which the temperature remains constant, it must be moving perpendicularly to the temperature gradient, $\langle 2,2,4 \rangle$. So the direction of the eagle must be (a posiitve constant times) one of

$$\pm \langle 1,2,2 \rangle \times \langle 2,2,4 \rangle = \pm \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & 2 & 2 \\ 2 & 2 & 4 \end{bmatrix} = \pm \langle 4, 0, -2 \rangle$$

or equivalently, any positive constant times $\pm \langle 2, 0, -1 \rangle$.

$\boxed{2.7.25}$ (∗) Assume that the temperature T at a point (x, y, z) near a flame at the origin is given by

$$T(x, y, z) = \frac{200}{1 + x^2 + y^2 + z^2}$$

where the coordinates are given in meters and the temperature is in degrees Celsius. Suppose that at some moment in time, a moth is at the point $(3, 4, 0)$ and is flying at a constant speed of $1\mathrm{m/s}$ in the direction of maximum increase of temperature.

(a) Find the velocity vector $\mathbf{v}$ of the moth at this moment.

(b) What rate of change of temperature does the moth feel at that moment?

$\boxed{\text{Solution}}$ (a) The moth is moving the direction of the temperature gradient at $(3, 4, 0)$, which is

$$\boldsymbol{\nabla} T(3, 4, 0) = -200 \frac{2x\hat{\boldsymbol{\imath}} + 2y\hat{\boldsymbol{\jmath}} + 2z\hat{\mathbf{k}}}{(1 + x^2 + y^2 + z^2)^2}\bigg|_{(3,4,0)} = -400 \frac{3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}}{26^2}$$

Since the speed of the moth is $1\mathrm{m/s}$ its velocity vector is a vector of length one in direction $-\frac{400}{26^2}\langle 3, 4, 0\rangle$ and hence is $\mathbf{v} = -\frac{\langle 3,4,0\rangle}{|\langle 3,4,0\rangle|} = -\left\langle \frac{3}{5}, \frac{4}{5}, 0\right\rangle$.

(b) The rate of change of temperature (per unit time) the moth feels at that time is

$$\boldsymbol{\nabla} T(3, 4, 0) \cdot \mathbf{v} = \frac{400}{26^2}\langle 3, 4, 0\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5}, 0\right\rangle = \frac{400 \times 25}{26^2 \times 5} = \frac{500}{169} \approx 2.96^\circ/\mathrm{s}$$

$\boxed{2.7.26}$ (∗) We say that u is inversely proportional to v if there is a constant k so that $u = k/v$. Suppose that the temperature T in a metal ball is inversely proportional to the distance from the centre of the ball, which we take to be the origin. The temperature at the point $(1, 2, 2)$ is 120°.

(a) Find the constant of proportionality.

(b) Find the rate of change of T at $(1, 2, 2)$ in the direction towards the point $(2, 1, 3)$.

(c) Show that at most points in the ball, the direction of greatest increase is towards the origin.

$\boxed{\text{Solution}}$ (a) We are told that $T(x, y, z) = \frac{k}{|\langle x,y,z\rangle|} = \frac{k}{\sqrt{x^2+y^2+z^2}}$ for some constant k and that

$$120 = T(1, 2, 2) = \frac{k}{|\langle 1, 2, 2\rangle|} \implies k = 120 \times \sqrt{1 + 2^2 + 2^2} = 360$$

(b) The (unit) direction from $(1, 2, 2)$ to $(2, 1, 3)$ is $\mathbf{d} = \frac{\langle 2,1,3\rangle - \langle 1,2,2\rangle}{|\langle 2,1,3\rangle - \langle 1,2,2\rangle|} = \frac{\langle 1,-1,1\rangle}{|\langle 1,-1,1\rangle|} = \frac{1}{\sqrt{3}}\langle 1, -1, 1\rangle$. The desired rate of change of temperature is

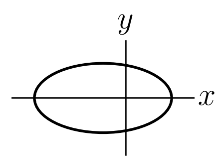
$$\begin{aligned}
D_{\mathbf{d}} T(1, 2, 2) = \boldsymbol{\nabla} T(1, 2, 2) \cdot \mathbf{d} &= -360 \frac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\mathbf{k}}}{(x^2 + y^2 + z^2)^{3/2}}\bigg|_{\langle 1,2,2\rangle} \cdot \mathbf{d} \\
&= -360 \frac{\langle 1, 2, 2\rangle}{27} \cdot \frac{\langle 1, -1, 1\rangle}{\sqrt{3}} = -\frac{40}{3\sqrt{3}} \approx -7.70
\end{aligned}$$

degrees per unit distance.

(c) At (x, y, z), the direction of greatest increase is in the direction of the temperature gradient at (x, y, z), which is $\nabla T(x, y, z) = -360\frac{x\hat{\imath}+y\hat{\jmath}+z\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}}$ and which points opposite to the radius vector. That is, it points towards the origin. This argument only fails at $(x, y, z) = (0, 0, 0)$, where the gradient, and indeed $T(x, y, z)$, is not defined.

2.7.27 (*) The depth of a lake in the xy-plane is equal to $f(x, y) = 32 - x^2 - 4x - 4y^2$ meters.
(a) Sketch the shoreline of the lake in the xy-plane.
Your calculus instructor is in the water at the point $(-1, 1)$. Find a unit vector which indicates in which direction he should swim in order to:
(b) stay at a constant depth?
(c) increase his depth as rapidly as possible (i.e. be most likely to drown)?

Solution (a) The shoreline is $f(x, y) = 0$ or $x^2 + 4x + 4y^2 = 32$ or $(x + 2)^2 + 4y^2 = 36$, which is an ellipse centred on $(-2, 0)$ with semiaxes 6 in the x-direction and 3 in the y-direction.



(b,c) The gradient of f at $(-1, 1)$ is

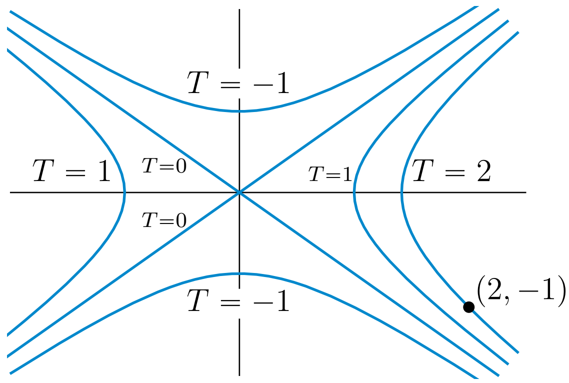
$$\nabla f(-1, 1) = \big[(-2x - 4)\,\hat{\imath} - 8y\,\hat{\jmath}\big]_{(-1,1)} = -2\,\hat{\imath} - 8\,\hat{\jmath}$$

To remain at constant depth, he should swim perpendicular to the depth gradient. So he should swim in direction $\pm\frac{1}{\sqrt{17}}\langle 4, -1\rangle$. To increase his depth as rapidly as possible, he should swim in the direction of the depth gradient, which is $-\frac{1}{\sqrt{17}}\langle 1, 4\rangle$.

▸▸ Stage 3

2.7.28 The temperature $T(x, y)$ at points of the xy-plane is given by $T(x, y) = x^2 - 2y^2$.
(a) Draw a contour diagram for T showing some isotherms (curves of constant temperature).
(b) In what direction should an ant at position $(2, -1)$ move if it wishes to cool off as quickly as possible?
(c) If the ant moves in that direction at speed v at what rate does its temperature decrease?
(d) What would the rate of decrease of temperature of the ant be if it moved from $(2, -1)$ at speed v in direction $\langle -1, -2\rangle$?
(e) Along what curve through $(2, -1)$ should the ant move to continue experiencing maximum rate of cooling?

$\boxed{\text{Solution}}$ (a) The curve on which the temperature is T_0 is $x^2 - 2y^2 = T_0$. If $T_0 = 0$, this is the pair of straight lines $y = \pm\frac{x}{\sqrt{2}}$. If $T_0 > 0$, it is a hyperbola on which $x^2 = 2y^2 + T_0 \geqslant T_0$. If $T_0 < 0$, it is a hyperbola on which $2y^2 = x^2 - T_0 \geqslant |T_0|$. Here is a sketch which show the isotherms $T = 0,\ 1,\ -1$ as well as the branch of the $T = 2$ isotherm that contains the ant's location $(2, -1)$.



Note that the temperature gradient is $\boldsymbol{\nabla} T(x, y) = \langle 2x, -4y\rangle$. In particular, the temperature gradient at $(2, -1)$ is $\boldsymbol{\nabla} T(2, -1) = \langle 4, 4\rangle$.

(b) To achieve maximum rate of cooling, the ant should move in the direction opposite the temperature gradient at $(2, -1)$. So the direction of maximum rate of cooling is

$$-\frac{\langle 4, 4\rangle}{4\sqrt{2}} = \frac{\langle -1, -1\rangle}{\sqrt{2}}$$

(c) If the ant moves in the direction of part (b), its rate of cooling per unit distance is $|\boldsymbol{\nabla} T(2, -1)| = |\langle 4, 4\rangle| = 4\sqrt{2}$. It the ant is moving at speed v, its rate of cooling per unit time is $4\sqrt{2}\,v$.

(d) If the ant moves from $(2, -1)$ in direction $\langle -1, -2\rangle$ its temperature increases at the rate

$$D_{\frac{\langle -1,-2\rangle}{\sqrt{5}}} T(2, -1) = \langle 4, 4\rangle \cdot \frac{\langle -1, -2\rangle}{\sqrt{5}} = -\frac{12}{\sqrt{5}}$$

per unit distance. So, if the ant is moving at speed v, its rate of decrease of temperature per unit time is $\frac{12}{\sqrt{5}}\,v$

(e) Suppose that the ant moves along the curve $y = y(x)$. For the ant to always experience maximum rate of cooling (or maximum rate of heating), the tangent to this curve must be parallel to $\boldsymbol{\nabla} T(x, y)$ at every point of the curve. A tangent to the curve at (x, y) is $\left\langle 1, \frac{\mathrm{d}y}{\mathrm{d}x}(x)\right\rangle$. This is parallel to $\boldsymbol{\nabla} T(x, y) = \langle 2x, -4y\rangle$ when

$$\frac{\frac{\mathrm{d}y}{\mathrm{d}x}}{1} = \frac{-4y}{2x} \implies \frac{dy}{y} = -2\frac{dx}{x} \implies \ln y = -2\ln x + C \implies y = C'x^{-2}$$

To pass through $(2, -1)$, we need $C' = -4$, so $y = -\frac{4}{x^2}$.

$\boxed{2.7.29}$ (∗) Consider the function $f(x,y,z) = x^2 + \cos(yz)$.

(a) Give the direction in which f is increasing the fastest at the point $(1,0,\pi/2)$.

(b) Give an equation for the plane T tangent to the surface

$$S = \{ \ (x,y,z) \mid f(x,y,z) = 1 \ \}$$

at the point $(1,0,\pi/2)$.

(c) Find the distance between T and the point $(0,1,0)$.

(d) Find the angle between the plane T and the plane

$$P = \{ \ (x,y,z) \mid x + z = 0 \ \}.$$

$\boxed{\text{Solution}}$ The first order partial derivatives of f, both at a general point (x,y,z) and at the point $(1,0,\pi/2)$, are

$$\begin{aligned} f_x(x,y,z) &= 2x & f_x(1,0,\pi/2) &= 2 \\ f_y(x,y,z) &= -z\sin(yz) & f_y(1,0,\pi/2) &= 0 \\ f_z(x,y,z) &= -y\sin(yz) & f_z(1,0,\pi/2) &= 0 \end{aligned}$$

(a) The rate of increase of f is largest in the direction of $\nabla f(1,0,\pi/2) = \langle 2,0,0\rangle$. A unit vector in that direction is $\hat{\imath}$.

(b) The gradient vector $\nabla f(1,0,\pi/2) = \langle 2,0,0\rangle$ is a normal vector to the surface $f = 1$ at $(1,0,\pi/2)$. So the specified tangent plane is

$$\langle 2,0,0\rangle \cdot \langle x-1\,,\,y-0\,,\,z-\pi/2\rangle = 0 \qquad \text{or} \qquad x = 1$$

(c) The vector from the point $(0,1,0)$ to the point $(1,1,0)$, on T, is $\langle 1,0,0\rangle$, which is perpendicular to T. So $(1,1,0)$ is the point on T nearest $(0,1,0)$ and the distance from $(0,1,0)$ to T is $|\langle 1,0,0\rangle| = 1$.

(d) The vector $\langle 1,0,1\rangle$ is perpendicular to the plane $x+z=0$. So the angle between the planes T and $x+z=0$ is the same as the angle θ between the vectors $\langle 1,0,0\rangle$ and $\langle 1,0,1\rangle$, which obeys

$$\begin{aligned} &|\langle 1,0,0\rangle|\,|\langle 1,0,1\rangle| \ \cos\theta = |\langle 1,0,0\rangle \cdot \langle 1,0,1\rangle| = 1 \\ &\implies \cos\theta = \frac{1}{\sqrt{2}} \implies \theta = \frac{\pi}{4} \end{aligned}$$

$\boxed{2.7.30}$ (∗) A function $T(x,y,z)$ at $P = (2,1,1)$ is known to have $T(P) = 5$, $T_x(P) = 1$, $T_y(P) = 2$, and $T_z(P) = 3$.

(a) A bee starts flying at P and flies along the unit vector pointing towards the point $Q = (3,2,2)$. What is the rate of change of $T(x,y,z)$ in this direction?

(b) Use the linear approximation of T at the point P to approximate $T(1.9,1,1.2)$.

(c) Let $S(x,y,z) = x + z$. A bee starts flying at P; along which unit vector direction should the bee fly so that the rate of change of $T(x,y,z)$ and of $S(x,y,z)$ are both zero in this direction?

Solution (a) We are being asked for the directional derivative of T in the direction of the unit vector from $P = (2,1,1)$ to $Q = (3,2,2)$, which is $\frac{\langle 1,1,1\rangle}{\sqrt{3}}$. That directional derivative is

$$\nabla T(P) \cdot \frac{\langle 1,1,1\rangle}{\sqrt{3}} = \langle 1,2,3\rangle \cdot \frac{\langle 1,1,1\rangle}{\sqrt{3}} = 2\sqrt{3}$$

(b) The linear approximation to T at P is

$$\begin{aligned} T(2+\Delta x\,,\,1+\Delta y\,,\,1+\Delta z) &\approx T(P) + T_x(P)\,\Delta x + T_y(P)\,\Delta y + T_z(P)\,\Delta z \\ &= 5 + \Delta x + 2\,\Delta y + 3\,\Delta z \end{aligned}$$

Applying this with $\Delta x = -0.1$, $\Delta y = 0$, $\Delta z = 0.2$ gives

$$T(1.9\,,\,1\,,\,1.2) \approx 5 + (-0.1) + 2\,(0) + 3\,(0.2) = 5.5$$

(c) For the rate of change of T to be zero, the direction of motion must be perpendicular to $\nabla T(P) = \langle 1,2,3\rangle$. For the rate of change of S to also be zero, the direction of motion must also be perpendicular to $\nabla S(P) = \langle 1,0,1\rangle$. The vector

$$\langle 1,2,3\rangle \times \langle 1,0,1\rangle = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix} = \langle 2,2,-2\rangle$$

is perpendicular to both $\nabla T(P)$ and $\nabla S(P)$. So the desired unit vectors are $\pm\frac{\langle 1,1,-1\rangle}{\sqrt{3}}$.

2.7.31 (*) Consider the functions $F(x,y,z) = z^3 + xy^2 + xz$ and $G(x,y,z) = 3x - y + 4z$. You are standing at the point $P(0,1,2)$.

(a) You jump from P to $Q(0.1\,,\,0.9\,,\,1.8)$. Use the linear approximation to determine approximately the amount by which F changes.

(b) You jump from P in the direction along which G increases most rapidly. Will F increase or decrease?

(c) You jump from P in a direction $\langle a\,,\,b\,,\,c\rangle$ along which the rates of change of F and G are both zero. Give an example of such a direction (need not be a unit vector).

Solution We are going to need the gradients of both F and G at $(0,1,2)$. So we compute

$$\begin{aligned} &\frac{\partial F}{\partial x}(x,y,z) = y^2 + z &\quad &\frac{\partial F}{\partial y}(x,y,z) = 2xy &\quad &\frac{\partial F}{\partial z}(x,y,z) = 3z^2 + x \\ &\frac{\partial G}{\partial x}(x,y,z) = 3 & &\frac{\partial G}{\partial y}(x,y,z) = -1 & &\frac{\partial G}{\partial z}(x,y,z) = 4 \end{aligned}$$

and then

$$\nabla F(0,1,2) = \langle 3,0,12\rangle \qquad \nabla G(0,1,2) = \langle 3,-1,4\rangle$$

(a) The linear approximation to F at $(0,1,2)$ is

$$\begin{aligned} F(x,y,z) &\approx F(0,1,2) + F_x(0,1,2)\,x + F_y(0,1,2)\,(y-1) + F_z(0,1,2)\,(z-2) \\ &= 8 + 3x + 12(z-2) \end{aligned}$$

In particular

$$F(0.1\,,\,0.9\,,\,1.8) - F(0,1,2) \approx 3(0.1) + 12(-0.2) = -2.1$$

(b) The direction along which G increases most rapidly at P is $\nabla G(0,1,2) = \langle 3,-1,4\rangle$. The directional derivative of F in that direction is

$$D_{\frac{\langle 3,-1,4\rangle}{\sqrt{26}}} F(0,1,2) = \nabla F(0,1,2)\cdot\frac{\langle 3,-1,4\rangle}{\sqrt{26}} = \langle 3,0,12\rangle\cdot\frac{\langle 3,-1,4\rangle}{\sqrt{26}} > 0$$

So F increases.

(c) For the rate of change of F to be zero, $\langle a\,,\,b\,,\,c\rangle$ must be perpendicular to $\nabla F(0,1,2) = \langle 3,0,12\rangle$.

For the rate of change of G to be zero, $\langle a\,,\,b\,,\,c\rangle$ must be perpendicular to $\nabla G(0,1,2) = \langle 3,-1,4\rangle$.

So any nonzero constant times

$$\det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ 3 & 0 & 12\\ 3 & -1 & 4\end{bmatrix} = \langle 12\,,\,24\,,\,-3\rangle = 3\,\langle 4\,,\,8\,,\,-1\rangle$$

is an allowed direction.

$\boxed{2.7.32}$ (∗) A meteor strikes the ground in the heartland of Canada. Using satellite photographs, a model

$$z = f(x,y) = -\frac{100}{x^2+2x+4y^2+11}$$

of the resulting crater is made and a plan is drawn up to convert the site into a tourist attraction. A car park is to be built at $(4,5)$ and a hiking trail is to be made. The trail is to start at the car park and take the steepest route to the bottom of the crater.

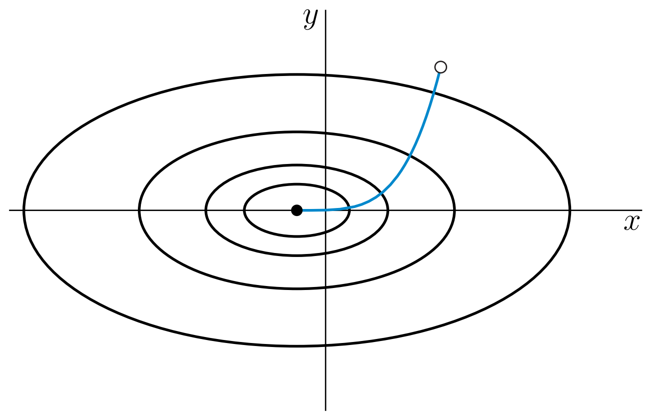
(a) Sketch a map of the proposed site clearly marking the car park, a few level curves for the function f and the trail.

(b) In which direction does the trail leave the car park?

$\boxed{\text{Solution}}$ (a) Since

$$z = -\frac{100}{x^2+2x+4y^2+11} = -\frac{100}{(x+1)^2+4y^2+10}$$

the bottom of the crater is at $x=-1$, $y=0$ (where the denominator is a minimum) and the contours (level curves) are ellipses having equations $(x+1)^2+4y^2 = C$. In the sketch below, the filled dot represents the bottom of the crater and the open dot represents the car park. The contours sketched are (from inside out) $z = -7.5, -5, -2.5, -1$. Note that the trail crosses the contour lines at right angles.



(b) The trail is to be parallel to

$$\nabla z = \frac{100}{(x^2 + 2x + 4y^2 + 11)^2}(2x + 2, 8y)$$

At the car park $\nabla z(4, 5) \parallel \langle 10, 40\rangle \parallel \langle 1, 4\rangle$. To move towards the bottom of the crater, we should leave in the direction $-\langle 1, 4\rangle$.

$\boxed{2.7.33}$ (*) You are standing at a lone palm tree in the middle of the Exponential Desert. The height of the sand dunes around you is given in meters by

$$h(x, y) = 100e^{-(x^2+2y^2)}$$

where x represents the number of meters east of the palm tree (west if x is negative) and y represents the number of meters north of the palm tree (south if y is negative).

(a) Suppose that you walk 3 meters east and 2 meters north. At your new location, $(3, 2)$, in what direction is the sand dune sloping most steeply downward?

(b) If you walk north from the location described in part (a), what is the instantaneous rate of change of height of the sand dune?

(c) If you are standing at $(3, 2)$ in what direction should you walk to ensure that you remain at the same height?

(d) Find the equation of the curve through $(3, 2)$ that you should move along in order that you are always pointing in a steepest descent direction at each point of this curve.

$\boxed{\text{Solution}}$ We have

$$\nabla h(x, y) = -200e^{-(x^2+2y^2)}\langle x, 2y\rangle \text{ and, in particular, } \nabla h(3, 2) = -200e^{-17}\langle 3, 4\rangle$$

(a) At $(3, 2)$ the dune slopes downward the most steeply in the direction opposite $\nabla h(3, 2)$, which is (any positive multiple of) $\langle 3, 4\rangle$.

(b) The rate is $D_{\hat{\jmath}}h(3, 2) = \nabla h(3, 2) \cdot \hat{\jmath} = -800e^{-17}$.

(c) To remain at the same height, you should walk perpendicular to $\nabla h(3, 2)$. So you should walk in one of the directions $\pm\left(\frac{4}{5}, -\frac{3}{5}\right)$.

(d) Suppose that you are walking along a steepest descent curve. Then the direction from (x, y) to $(x + \mathrm{d}x, y + \mathrm{d}y)$, with $(\mathrm{d}x, \mathrm{d}y)$ infinitesmal, must be opposite to $\nabla h(x, y) = -200e^{-(x^2+2y^2)}(x, 2y)$. Thus $(\mathrm{d}x, \mathrm{d}y)$ must be parallel to $(x, 2y)$ so that the slope

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y}{x} \implies \frac{\mathrm{d}y}{y} = 2\frac{\mathrm{d}x}{x} \implies \ln y = 2\ln x + C$$

We must choose C to obey $\ln 2 = 2\ln 3 + C$ in order to pass through the point $(3, 2)$. Thus $C = \ln\frac{2}{9}$ and the curve is $\ln y = 2\ln x + \ln\frac{2}{9}$ or $y = \frac{2}{9}x^2$.

2.7.34 (*) Let $f(x, y)$ be a differentiable function with $f(1, 2) = 7$. Let

$$\mathbf{u} = \frac{3}{5}\hat{\boldsymbol{\imath}} + \frac{4}{5}\hat{\boldsymbol{\jmath}}, \qquad \mathbf{v} = \frac{3}{5}\hat{\boldsymbol{\imath}} - \frac{4}{5}\hat{\boldsymbol{\jmath}}$$

be unit vectors. Suppose it is known that the directional derivatives $D_{\mathbf{u}}f(1, 2)$ and $D_{\mathbf{v}}f(1, 2)$ are equal to 10 and 2 respectively.

(a) Show that the gradient vector ∇f at $(1, 2)$ is $10\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}}$.

(b) Determine the rate of change of f at $(1, 2)$ in the direction of the vector $\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$.

(c) Using the tangent plane approximation, estimate the value of $f(1.01, 2.05)$.

Solution (a) Denote $\nabla f(1, 2) = \langle a, b\rangle$. We are told that

$$D_{\mathbf{u}}f(1, 2) = \mathbf{u}\cdot(a, b) = \frac{3}{5}a + \frac{4}{5}b = 10$$
$$D_{\mathbf{v}}f(1, 2) = \mathbf{v}\cdot(a, b) = \frac{3}{5}a - \frac{4}{5}b = 2$$

Adding these two equations gives $\frac{6}{5}a = 12$, which forces $a = 10$, and subtracting the two equations gives $\frac{8}{5}b = 8$, which forces $b = 5$, as desired.

(b) The rate of change of f at $(1, 2)$ in the direction of the vector $\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$ is

$$\frac{\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}}{|\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}|}\cdot\nabla f(1, 2) = \frac{1}{\sqrt{5}}\langle 1, 2\rangle\cdot\langle 10, 5\rangle = 4\sqrt{5} \approx 8.944$$

(c) Applying (2.6.1 in the CLP3 text, which is

$$f\big(x_0 + \Delta x\,,\, y_0 + \Delta y\big) \approx f\big(x_0\,,\, y_0\big) + \frac{\partial f}{\partial x}\big(x_0\,,\, y_0\big)\,\Delta x + \frac{\partial f}{\partial y}\big(x_0\,,\, y_0\big)\,\Delta y$$

with $x_0 = 1$, $\Delta x = 0.01$, $y_0 = 2$, and $\Delta y = 0.05$, gives

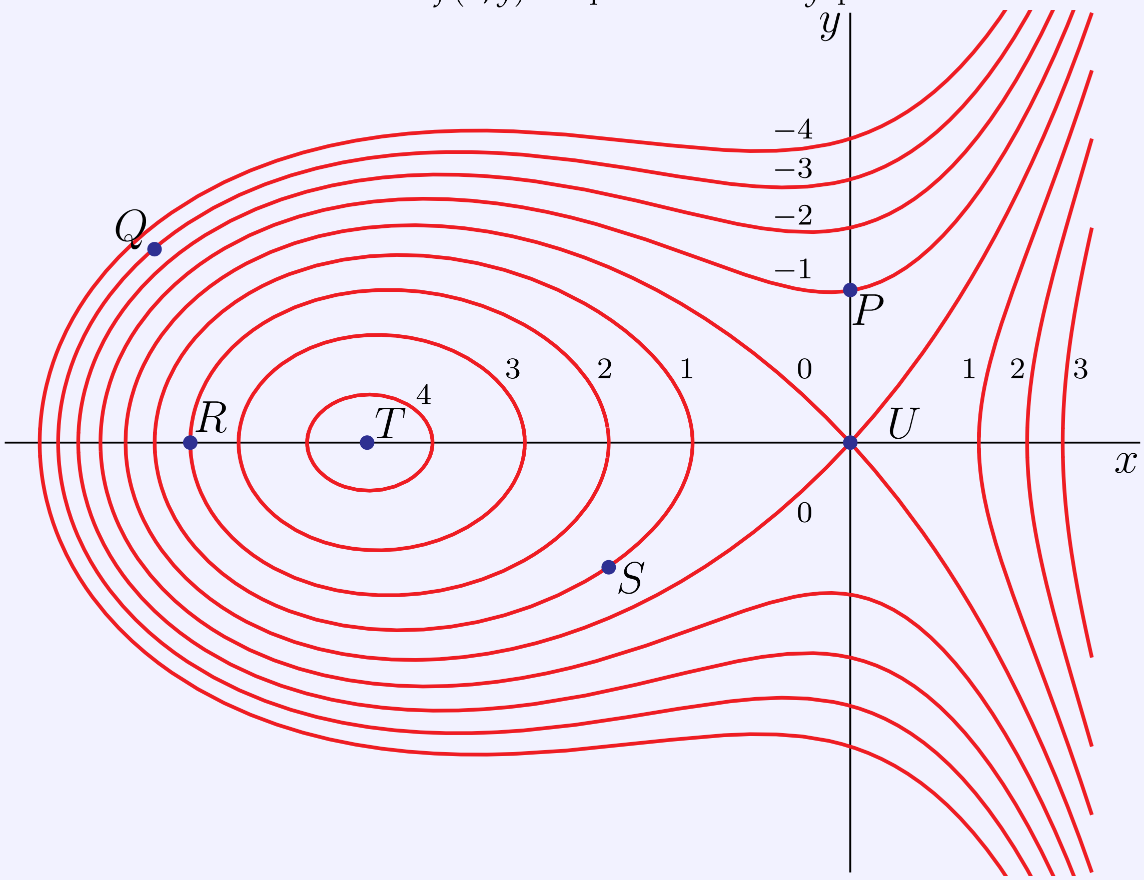
$$\begin{aligned} f(1.01, 2.05) &\approx f(1, 2) + f_x(1, 2)\times(1.01 - 1) + f_y(1, 2)\times(2.05 - 2)\\ &= 7 + 10\times 0.01 + 5\times 0.05\\ &= 7.35 \end{aligned}$$

2.9▲ Maximum and Minimum Values

▸▸ Stage 1

$\boxed{2.9.1}$ (*)

(a) Some level curves of a function $f(x, y)$ are plotted in the xy–plane below.



For each of the four statmemements below, circle the letters of all points in the diagram where the situation applies. For example, if the statement were "These points are on the y–axis", you would circle both P and U, but none of the other letters. You may assume that a local maximum occurs at point T.

(i) ∇f is zero	P R S T U
(ii) f has a saddle point	P R S T U
(iii) the partial derivative f_y is positive	P R S T U
(iv) the directional derivative of f in the direction $\langle 0, -1 \rangle$ is negative	P R S T U

(b) The diagram below shows three "y traces" of a graph $z = F(x, y)$ plotted on xz–axes. (Namely the intersections of the surface $z = F(x, y)$ with the three planes ($y = 1.9$, $y = 2$, $y = 2.1$). For each statement below, circle the correct word.

(i) the first order partial derivative $F_x(1, 2)$ is	positive/negative/zero (circle one)
(ii) F has a critical point at $(2, 2)$	true/false (circle one)
(iii) the second order partial derivative $F_{xy}(1, 2)$ is	positive/negative/zero (circle one)

z

Solution a) (i) ∇f is zero at critical points. The point T is a local maximum and the point U is a saddle point. The remaining points P, R, S, are not critical points.

(a) (ii) Only U is a saddle point.

(a) (iii) We have $f_y(x, y) > 0$ if f increases as you move vertically upward through (x, y). Looking at the diagram, we see

$$f_y(P) < 0 \qquad f_y(Q) < 0 \qquad f_y(R) = 0 \qquad f_y(S) > 0 \qquad f_y(T) = 0 \qquad f_y(U) = 0$$

So only S works.

(a) (iv) The directional derivative of f in the direction $\langle 0, -1\rangle$ is $\nabla f \cdot \langle 0, -1\rangle = -f_y$. It is negative if and only if $f_y > 0$. So, again, only S works.

(b) (i) The function $z = F(x, 2)$ is increasing at $x = 1$, because the $y = 2.0$ graph in the diagram has positive slope at $x = 1$. So $F_x(1, 2) > 0$.

(b) (ii) The function $z = F(x, 2)$ is also increasing (though slowly) at $x = 2$, because the $y = 2.0$ graph in the diagram has positive slope at $x = 2$. So $F_x(2, 2) > 0$. So F does not have a critical point at $(2, 2)$.

(b) (iii) From the diagram the looks like $F_x(1, 1.9) > F_x(1, 2.0) > F_x(1, 2.1)$. That is, it looks like the slope of the $y = 1.9$ graph at $x = 1$ is larger than the slope of the $y = 2.0$ graph at $x = 1$, which in turn is larger than the slope of the $y = 2.1$ graph at $x = 1$. So it looks like $F_x(1, y)$ decreases as y increases through $y = 2$, and consequently $F_{xy}(1, 2) < 0$.

2.9.2 Find the high and low points of the surface $z = \sqrt{x^2 + y^2}$ with (x, y) varying over the square $|x| \leqslant 1$, $|y| \leqslant 1$. Discuss the values of z_x, z_y there. Do not evaluate any derivatives in answering this question.

Solution The height $\sqrt{x^2 + y^2}$ at (x, y) is the distance from (x, y) to $(0, 0)$. So the minimum height is zero at $(0, 0, 0)$. The surface is a cone. The cone has a point at $(0, 0, 0)$ and the derivatives z_x and z_y do not exist there. The maximum height is achieved when (x, y) is as far as possible from $(0, 0)$. The highest points are at $(\pm 1, \pm 1, \sqrt{2})$. There z_x and z_y exist but are not zero. These points would not be the highest points if it were not for the restriction $|x|, |y| \leqslant 1$.

2.9.3 If t_0 is a local minimum or maximum of the smooth function $f(t)$ of one variable (t runs over all real numbers) then $f'(t_0) = 0$. Derive an analogous necessary condition for $\mathbf{x}_0$ to be a local minimum or maximium of the smooth function $g(\mathbf{x})$ restricted to points on the line $\mathbf{x} = \mathbf{a} + t\mathbf{d}$. The test should involve the gradient of $g(\mathbf{x})$.

Solution Define $f(t) = g(\mathbf{a} + t\mathbf{d})$ and determine t_0 by $\mathbf{x}_0 = \mathbf{a} + t_0\mathbf{d}$. Then $f'(t) = \nabla g(\mathbf{a} + t\mathbf{d}) \cdot \mathbf{d}$. To see this, write $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{d} = \langle d_1, d_2, d_3\rangle$. Then

$$f(t) = g(a_1 + td_1, a_2 + td_2, a_3 + td_3)$$

So, by the chain rule,

$$\begin{aligned} f'(t) &= \frac{\partial g}{\partial x}(a_1+td_1, a_2+td_2, a_3+td_3)\, d_1 + \frac{\partial g}{\partial y}(a_1+td_1, a_2+td_2, a_3+td_3)\, d_2 \\ &\qquad\qquad\qquad\qquad + \frac{\partial g}{\partial z}(a_1+td_1, a_2+td_2, a_3+td_3)\, d_3 \\ &= \nabla g(\mathbf{a}+t\mathbf{d})\cdot\mathbf{d} \end{aligned}$$

Then $\mathbf{x}_0$ is a local max or min of the restriction of g to the specified line if and only if t_0 is a local max or min of $f(t)$. If so, $f'(t_0)$ necessarily vanishes. So if $\mathbf{x}_0$ is a local max or min of the restriction of g to the specified line, then $\nabla g(\mathbf{x}_0)\cdot\mathbf{d} = 0$, i.e. $\nabla g(\mathbf{x}_0)\perp\mathbf{d}$, and $\mathbf{x}_0 = \mathbf{a}+t_0\mathbf{d}$ for some t_0. The second condition is to ensure that x_0 lies on the line.

▸▸ Stage 2

2.9.4 (∗) Let $z = f(x,y) = \left(y^2-x^2\right)^2$.

(a) Make a reasonably accurate sketch of the level curves in the xy–plane of $z = f(x,y)$ for $z = 0$, 1 and 16. Be sure to show the units on the coordinate axes.

(b) Verify that $(0,0)$ is a critical point for $z = f(x,y)$, and determine from part (a) or directly from the formula for $f(x,y)$ whether $(0,0)$ is a local minimum, a local maximum or a saddle point.

(c) Can you use the Second Derivative Test to determine whether the critical point $(0,0)$ is a local minimum, a local maximum or a saddle point? Give reasons for your answer.

Solution (a)

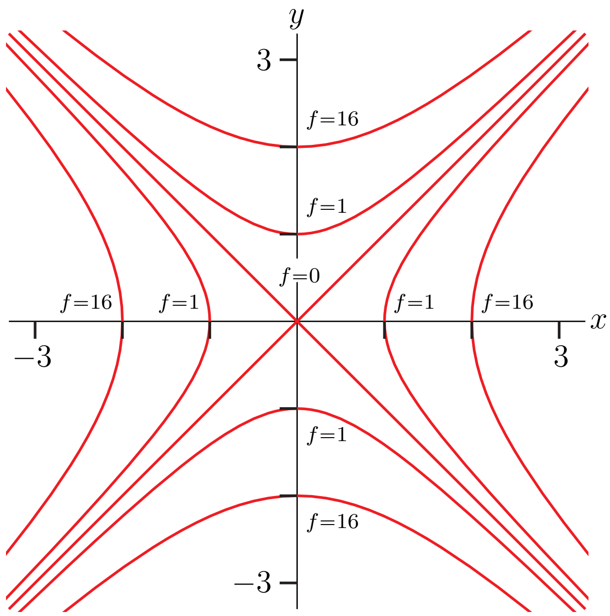
- The level curve $z = 0$ is $y^2 - x^2 = 0$, which is the pair of 45° lines $y = \pm x$.

- When $C > 0$, the level curve $z = C^4$ is $\left(y^2-x^2\right)^2 = C^4$, which is the pair of hyperbolae $y^2-x^2 = C^2$, $y^2-x^2 = -C^2$ or

$$y = \pm\sqrt{x^2+C^2} \qquad x = \pm\sqrt{y^2+C^2}$$

  The hyperbola $y^2-x^2 = C^2$ crosses the y–axis (i.e. the line $x = 0$) at $(0, \pm C)$. The hyperbola $y^2-x^2 = -C^2$ crosses the x–axis (i.e. the line $y = 0$) at $(\pm C, 0)$.

Here is a sketch showing the level curves $z = 0$, $z = 1$ (i.e. $C = 1$), and $z = 16$ (i.e. $C = 2$).



(b) As $f_x(x,y) = -4x(y^2 - x^2)$ and $f_y(x,y) = 4y(y^2 - x^2)$, we have $f_x(0,0) = f_y(0,0) = 0$ so that $(0,0)$ is a critical point. Note that

- $f(0,0) = 0$,
- $f(x,y) \geqslant 0$ for all x and y.

So $(0,0)$ is a local (and also absolute) minimum.

(c) Note that

$$\begin{aligned} f_{xx}(x,y) &= -4y^2 + 12x^2 & f_{xx}(x,y) &= 0 \\ f_{yy}(x,y) &= 12y^2 - 4x^2 & f_{yy}(x,y) &= 0 \\ f_{xy}(x,y) &= -8xy & f_{xx}(x,y) &= 0 \end{aligned}$$

As $f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 0$, the Second Derivative Test (Theorem 2.9.16 in the CLP-3 text) tells us absolutely nothing.

$\boxed{2.9.5}$ (∗) Use the Second Derivative Test to find all values of the constant c for which the function $z = x^2 + cxy + y^2$ has a saddle point at $(0,0)$.

$\boxed{\text{Solution}}$ Write $f(x,y) = x^2 + cxy + y^2$. Then

$$\begin{aligned} f_x(x,y) &= 2x + cy & f_x(0,0) &= 0 \\ f_y(x,y) &= cx + 2y & f_y(0,0) &= 0 \\ f_{xx}(x,y) &= 2 \\ f_{xy}(x,y) &= c \\ f_{yy}(x,y) &= 2 \end{aligned}$$

As $f_x(0,0) = f_y(0,0) = 0$, we have that $(0,0)$ is always a critical point for f. According to the Second Derivative Test, $(0,0)$ is also a saddle point for f if

$$f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 < 0 \iff 4 - c^2 < 0 \iff |c| > 2$$

As a remark, the Second Derivative Test provides no information when the expression $f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 0$, i.e. when $c = \pm 2$. But when $c = \pm 2$,

$$f(x,y) = x^2 \pm 2xy + y^2 = (x \pm y)^2$$

and f has a local minimum, not a saddle point, at $(0,0)$.

2.9.6 (∗) Find and classify all critical points of the function

$$f(x,y) = x^3 - y^3 - 2xy + 6.$$

Solution To find the critical points we will need the gradient of f, and to apply the second derivative test of Theorem 2.9.16 in the CLP-3 text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} &f = x^3 - y^3 - 2xy + 6 \\ &f_x = 3x^2 - 2y \qquad f_{xx} = 6x \qquad f_{xy} = -2 \\ &f_y = -3y^2 - 2x \qquad f_{yy} = -6y \qquad f_{yx} = -2 \end{aligned}$$

The critical points are the solutions of

$$f_x = 3x^2 - 2y = 0 \qquad f_y = -3y^2 - 2x = 0$$

Substituting $y = \frac{3}{2}x^2$, from the first equation, into the second equation gives

$$\begin{aligned} -3\left(\frac{3}{2}x^2\right)^2 - 2x = 0 &\iff -2x\left(\frac{3^3}{2^3}x^3 + 1\right) = 0 \\ &\iff x = 0,\ -\frac{2}{3} \end{aligned}$$

So there are two critical points: $(0,0)$, $\left(-\frac{2}{3}, \frac{2}{3}\right)$.

The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$0 \times 0 - (-2)^2 < 0$		saddle point
$\left(-\frac{2}{3}, \frac{2}{3}\right)$	$(-4) \times (-4) - (-2)^2 > 0$	-4	local max

2.9.7 (∗) Find all critical points for $f(x,y) = x(x^2 + xy + y^2 - 9)$. Also find out which of these points give local maximum values for $f(x,y)$, which give local minimum values, and which give saddle points.

Solution To find the critical points we will need the gradient of f, and to apply the second derivative test of Theorem 2.9.16 in the CLP-3 text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} f &= x^3 + x^2y + xy^2 - 9x \\ f_x &= 3x^2 + 2xy + y^2 - 9 \qquad & f_{xx} &= 6x + 2y \qquad & f_{xy} &= 2x + 2y \\ f_y &= x^2 + 2xy & f_{yy} &= 2x & f_{yx} &= 2x + 2y \end{aligned}$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The critical points are the solutions of

$$\begin{aligned} f_x &= 3x^2 + 2xy + y^2 - 9 = 0 && \text{(E1)} \\ f_y &= x(x+2y) \qquad\qquad = 0 && \text{(E2)} \end{aligned}$$

Equation (E2) is satisfied if at least one of $x = 0$, $x = -2y$.

- If $x = 0$, equation (E1) reduces to $y^2 - 9 = 0$, which is satisfied if $y = \pm 3$.
- If $x = -2y$, equation (E1) reduces to

  $$0 = 3(-2y)^2 + 2(-2y)y + y^2 - 9 = 9y^2 - 9$$

  which is satisfied if $y = \pm 1$.

So there are four critical points: $(0,3)$, $(0,-3)$, $(-2,1)$ and $(2,-1)$. The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,3)$	$(6) \times (0) - (6)^2 < 0$		saddle point
$(0,-3)$	$(-6) \times (0) - (-6)^2 < 0$		saddle point
$(-2,1)$	$(-10) \times (-4) - (-2)^2 > 0$	-10	local max
$(2,-1)$	$(10) \times (4) - (2)^2 > 0$	10	local min

2.9.8 $(*)$ Find the largest and smallest values of x^2y^2z in the part of the plane $2x + y + z = 5$ where $x \geqslant 0$, $y \geqslant 0$ and $z \geqslant 0$. Also find all points where those extreme values occur.

Solution The region of interest is

$$D = \big\{ \ (x,y,z) \ \big| \ x \geqslant 0, \ y \geqslant 0, \ z \geqslant 0, \ 2x + y + z = 5 \ \big\}$$

First observe that, on the boundary of this region, at least one of x, y and z is zero. So $f(x,y,z) = x^2y^2z$ is zero on the boundary. As f takes values which are strictly bigger than zero at all points of D that are not on the boundary, the minimum value of f is 0 on

$$\partial D = \big\{ \ (x,y,z) \ \big| \ x \geqslant 0, \ y \geqslant 0, \ z \geqslant 0, \ 2x + y + z = 5, \ \text{at least one of } x,y,z \text{ zero} \ \big\}$$

The maximum value of f will be taken at a critical point. On D

$$f = x^2y^2(5-2x-y) = 5x^2y^2 - 2x^3y^2 - x^2y^3$$

So the critical points are the solutions of

$$\begin{aligned} 0 &= f_x(x,y) = 10xy^2 - 6x^2y^2 - 2xy^3 \\ 0 &= f_y(x,y) = 10x^2y - 4x^3y - 3x^2y^2 \end{aligned}$$

or, dividing by the first equation by xy^2 and the second equation by x^2y, (recall that $x, y \neq 0$)

$$\begin{aligned} 10 - 6x - 2y &= 0 \qquad \text{or} \qquad 3x + y = 5 \\ 10 - 4x - 3y &= 0 \qquad \text{or} \qquad 4x + 3y = 10 \end{aligned}$$

Substituting $y = 5 - 3x$, from the first equation, into the second equation gives

$$4x + 3(5-3x) = 10 \implies -5x + 15 = 10 \implies x = 1,\ y = 5 - 3(1) = 2$$

So the maximum value of f is $(1)^2(2)^2(5-2-2) = 4$ at $(1, 2, 1)$.

2.9.9 Find and classify all the critical points of $f(x,y) = x^2 + y^2 + x^2y + 4$.

Solution To find the critical points we will need the gradient of f, and to apply the second derivative test of Theorem 2.9.16 in the CLP-3 text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} & f = x^2 + y^2 + x^2y + 4 \\ & f_x = 2x + 2xy \qquad f_{xx} = 2 + 2y \qquad f_{xy} = 2x \\ & f_y = 2y + x^2 \qquad f_{yy} = 2 \end{aligned}$$

The critical points are the solutions of

$$\begin{aligned} & f_x = 0 && f_y = 0 \\ \iff\ & 2x(1+y) = 0 && 2y + x^2 = 0 \\ \iff\ & x = 0 \text{ or } y = -1 && 2y + x^2 = 0 \end{aligned}$$

When $x = 0$, y must be 0. When $y = -1$, x^2 must be 2. So, there are three critical points: $(0,0)$, $\big(\pm\sqrt{2}, -1\big)$.

The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$2 \times 2 - 0^2 > 0$	$2 > 0$	local min
$(\sqrt{2}, -1)$	$0 \times 2 - (2\sqrt{2})^2 < 0$		saddle point
$(-\sqrt{2}, -1)$	$0 \times 2 - (-2\sqrt{2})^2 < 0$		saddle point

2.9.10 (*) Find all saddle points, local minima and local maxima of the function

$$f(x,y) = x^3 + x^2 - 2xy + y^2 - x.$$

Solution To find the critical points we will need the gradient of f, and to apply the second derivative test of Theorem 2.9.16 in the CLP-3 text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned}
f &= x^3 + x^2 - 2xy + y^2 - x \\
f_x &= 3x^2 + 2x - 2y - 1 \qquad & f_{xx} &= 6x + 2 \qquad & f_{xy} &= -2 \\
f_y &= -2x + 2y & f_{yy} &= 2 & f_{yx} &= -2
\end{aligned}$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The critical points are the solutions of

$$\begin{aligned}
f_x &= 3x^2 + 2x - 2y - 1 = 0 \qquad \text{(E1)} \\
f_y &= -2x + 2y \qquad\qquad\;\; = 0 \qquad \text{(E2)}
\end{aligned}$$

Substituting $y = x$, from (E2), into (E1) gives

$$3x^2 - 1 = 0 \iff x = \pm\frac{1}{\sqrt{3}} = 0$$

So there are two critical points: $\pm\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$(2\sqrt{3}+2) \times (2) - (-2)^2 > 0$	$2\sqrt{3}+2 > 0$	local min
$-\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$(-2\sqrt{3}+2) \times (2) - (-2)^2 < 0$		saddle point

2.9.11 (*) For the surface

$$z = f(x,y) = x^3 + xy^2 - 3x^2 - 4y^2 + 4$$

Find and classify [as local maxima, local minima, or saddle points] all critical points of $f(x,y)$.

Solution To find the critical points we will need the gradient of f and to apply the second derivative test of Theorem 2.9.16 in the CLP-3 text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned}
f &= x^3 + xy^2 - 3x^2 - 4y^2 + 4 \\
f_x &= 3x^2 + y^2 - 6x \qquad & f_{xx} &= 6x - 6 \qquad & f_{xy} &= 2y \\
f_y &= 2xy - 8y & f_{yy} &= 2x - 8 & f_{yx} &= 2y
\end{aligned}$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The critical points are the solutions of

$$f_x = 3x^2 + y^2 - 6x = 0 \qquad f_y = 2(x-4)y = 0$$

The second equation is satisfied if at least one of $x = 4$, $y = 0$ are satisfied.

- If $x = 4$, the first equation reduces to $y^2 = -24$, which has no real solutions.
- If $y = 0$, the first equation reduces to $3x(x-2) = 0$, which is satisfied if either $x = 0$ or $x = 2$.

So there are two critical points: $(0,0)$, $(2,0)$.

The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$(-6) \times (-8) - (0)^2 > 0$	-6	local max
$(2,0)$	$6 \times (-4) - (0)^2 < 0$		saddle point

2.9.12 Find the maximum and minimum values of $f(x,y) = xy - x^3y^2$ when (x,y) runs over the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

Solution The specified function and its first order derivatives are

$$f(x,y) = xy - x^3y^2 \qquad f_x(x,y) = y - 3x^2y^2 \qquad f_y(x,y) = x - 2x^3y$$

- First, we find the critical points.

$$\begin{aligned} f_x = 0 &\iff y(1-3x^2y) = 0 &&\iff y = 0 \text{ or } 3x^2y = 1 \\ f_y = 0 &\iff x(1-2x^2y) = 0 &&\iff x = 0 \text{ or } 2x^2y = 1 \end{aligned}$$

  - If $y = 0$, we cannot have $2x^2y = 1$, so we must have $x = 0$.
  - If $3x^2y = 1$, we cannot have $x = 0$, so we must have $2x^2y = 1$. Dividing gives $1 = \frac{3x^2y}{2x^2y} = \frac{3}{2}$ which is impossible.

  So the only critical point in the square is $(0,0)$. There $f = 0$.

- Next, we look at the part of the boundary with $x = 0$. There $f = 0$.
- Next, we look at the part of the boundary with $y = 0$. There $f = 0$.
- Next, we look at the part of the boundary with $x = 1$. There $f = y - y^2$. As $\frac{\mathrm{d}}{\mathrm{d}y}(y - y^2) = 1 - 2y$, the max and min of $y - y^2$ for $0 \leqslant y \leqslant 1$ must occur either at $y = 0$, where $f = 0$, or at $y = \frac{1}{2}$, where $f = \frac{1}{4}$, or at $y = 1$, where $f = 0$.
- Next, we look at the part of the boundary with $y = 1$. There $f = x - x^3$. As $\frac{\mathrm{d}}{\mathrm{d}x}(x - x^3) = 1 - 3x^2$, the max and min of $x - x^3$ for $0 \leqslant x \leqslant 1$ must occur either at $x = 0$, where $f = 0$, or at $x = \frac{1}{\sqrt{3}}$, where $f = \frac{2}{3\sqrt{3}}$, or at $x = 1$, where $f = 0$.

All together, we have the following candidates for max and min.

point	$(0,0)$	$x=0$	$y=0$	$(1,0)$	$(1,\frac{1}{2})$	$(1,1)$	$(0,1)$	$(\frac{1}{\sqrt{3}},1)$	$(1,1)$
value of f	0	0	0	0	$\frac{1}{4}$	0	0	$\frac{2}{3\sqrt{3}}$	0
	min	min	min	min		min	min	max	min

The largest and smallest values of f in this table are

$$\min = 0 \qquad \max = \frac{2}{3\sqrt{3}} \approx 0.385$$

2.9.13 The temperature at all points in the disc $x^2+y^2 \leqslant 1$ is given by $T(x,y) = (x+y)e^{-x^2-y^2}$. Find the maximum and minimum temperatures at points of the disc.

Solution The specified temperature and its first order derivatives are

$$\begin{aligned} T(x,y) &= (x+y)e^{-x^2-y^2} \\ T_x(x,y) &= (1-2x^2-2xy)e^{-x^2-y^2} \\ T_y(x,y) &= (1-2xy-2y^2)e^{-x^2-y^2} \end{aligned}$$

- First, we find the critical points.

$$\begin{aligned} T_x = 0 &\iff 2x(x+y) = 1 \\ T_y = 0 &\iff 2y(x+y) = 1 \end{aligned}$$

  As $x+y$ may not vanish, this forces $x=y$ and then $x=y=\pm\frac{1}{2}$. So the only critical points are $(\frac{1}{2},\frac{1}{2})$ and $(-\frac{1}{2},-\frac{1}{2})$.

- On the boundary $x=\cos\theta$ and $y=\sin\theta$, so $T = (\cos\theta+\sin\theta)e^{-1}$. This is a periodic function and so takes its max and min at zeroes of $\frac{dT}{d\theta} = \big(-\sin\theta+\cos\theta\big)e^{-1}$. That is, when $\sin\theta = \cos\theta$, which forces $\sin\theta = \cos\theta = \pm\frac{1}{\sqrt{2}}$.

All together, we have the following candidates for max and min.

point	$(\frac{1}{2},\frac{1}{2})$	$(-\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$
value of T	$\frac{1}{\sqrt{e}} \approx 0.61$	$-\frac{1}{\sqrt{e}}$	$\frac{\sqrt{2}}{e} \approx 0.52$	$-\frac{\sqrt{2}}{e}$
	max	min		

The largest and smallest values of T in this table are

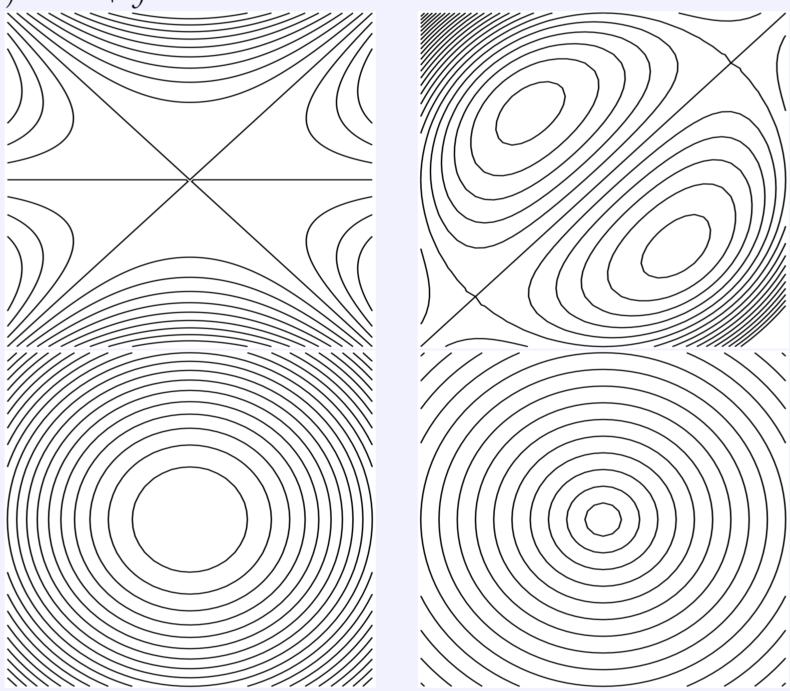
$$\min = -\frac{1}{\sqrt{e}} \qquad \max = \frac{1}{\sqrt{e}}$$

$\boxed{2.9.14}$ (*)

(a) For the function $z = f(x,y) = x^3 + 3xy + 3y^2 - 6x - 3y - 6$. Find and classify as [local maxima, local minima, or saddle points] all critical points of $f(x,y)$.

(b) The images below depict level sets $f(x,y) = c$ of the functions in the list at heights $c = 0, 0.1, 0.2, \ldots, 1.9, 2$. Label the pictures with the corresponding function and mark the critical points in each picture. (Note that in some cases, the critical points might not be drawn on the images already. In those cases you should add them to the picture.)

(i) $f(x,y) = (x^2 + y^2 - 1)(x - y) + 1$
(ii) $f(x,y) = \sqrt{x^2 + y^2}$
(iii) $f(x,y) = y(x + y)(x - y) + 1$
(iv) $f(x,y) = x^2 + y^2$



$\boxed{\text{Solution}}$ (a) To find the critical points we will need the gradient of f and to apply the second derivative test of Theorem 2.9.16 in the CLP-3 text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} f &= x^3 + 3xy + 3y^2 - 6x - 3y - 6 \\ f_x &= 3x^2 + 3y - 6 \qquad f_{xx} = 6x \qquad f_{xy} = 3 \\ f_y &= 3x + 6y - 3 \qquad f_{yy} = 6 \qquad f_{yx} = 3 \end{aligned}$$

The critical points are the solutions of

$$f_x = 3x^2 + 3y - 6 = 0 \qquad f_y = 3x + 6y - 3 = 0$$

Subtracting the second equation from 2 times the first equation gives

$$6x^2 - 3x - 9 = 0 \iff 3(2x - 3)(x + 1) = 0 \iff x = \frac{3}{2},\ -1$$

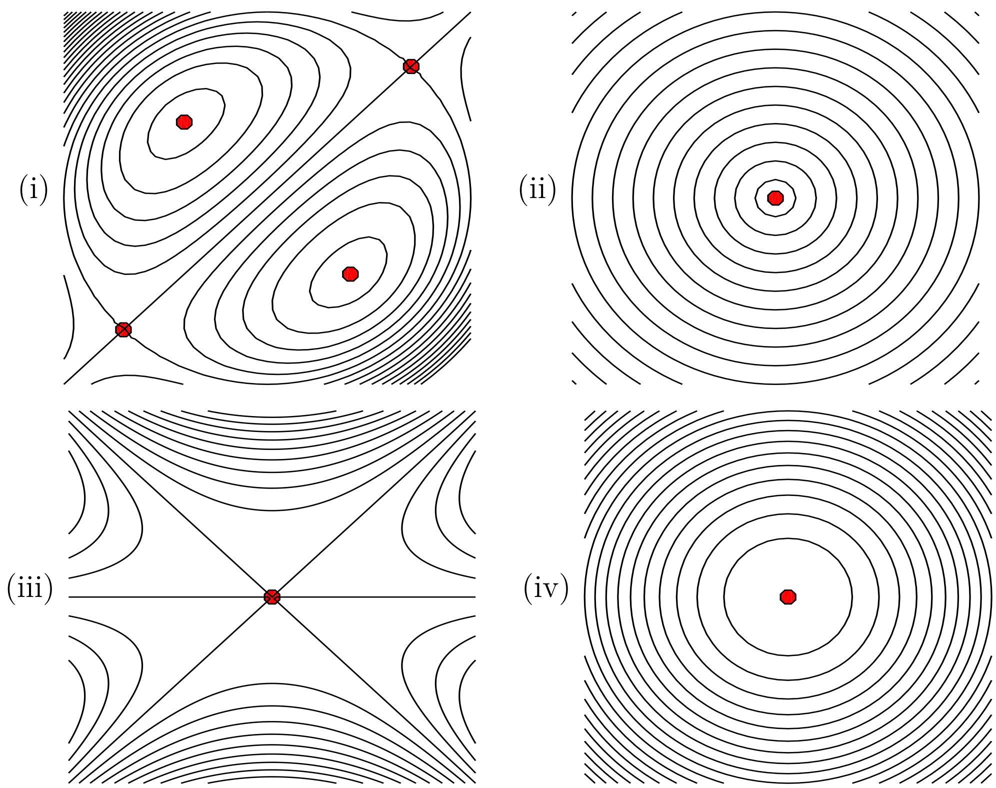
Since $y = \frac{1-x}{2}$ (from the second equation), the critical points are $(\frac{3}{2}, -\frac{1}{4})$, $(-1, 1)$ and the classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(\frac{3}{2}, -\frac{1}{4})$	$(9) \times (6) - (3)^2 > 0$	9	local min
$(-1, 1)$	$(-6) \times (6) - (3)^2 < 0$		saddle point

(b) Both of the functions $f(x, y) = \sqrt{x^2 + y^2}$ (i.e. (ii)) and $f(x, y) = x^2 + y^2$ (i.e. (iv)) are invariant under rotations around the $(0, 0)$. So their level curves are circles centred on the origin. In polar coordinates $\sqrt{x^2 + y^2}$ is r. So the sketched level curves of the function in (ii) are $r = 0, 0.1, 0.2, \ldots, 1.9, 2$. They are equally spaced. So at this point, we know that the third picture goes with (iv) and the fourth picture goes with (ii).

Notice that the lines $x = y$, $x = -y$ and $y = 0$ are all level curves of the function $f(x, y) = y(x + y)(x - y) + 1$ (i.e. of (iii)) with $f = 1$. So the first picture goes with (iii). And the second picture goes with (i).

Here are the pictures with critical points marked on them. There are saddle points where level curves cross and there are local max's or min's at "bull's eyes".



$\boxed{2.9.15}$ (∗) Let the function

$$f(x, y) = x^3 + 3xy + 3y^2 - 6x - 3y - 6$$

Classify as [local maxima, minima or saddle points] all critical points of $f(x, y)$.

$\boxed{\text{Solution}}$ To find the critical points we will need the gradient of f, and to apply the second derivative test of Theorem 2.9.16 in the CLP-3 text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} f &= x^3 + 3xy + 3y^2 - 6x - 3y - 6 \\ f_x &= 3x^2 + 3y - 6 \qquad f_{xx} = 6x \qquad f_{xy} = 3 \\ f_y &= 3x + 6y - 3 \qquad f_{yy} = 6 \qquad f_{yx} = 3 \end{aligned}$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The critical points are the solutions of

$$f_x = 3x^2 + 3y - 6 = 0 \tag{E1}$$
$$f_y = 3x \ + 6y - 3 = 0 \tag{E2}$$

Subtracting equation (E2) from twice equation (E1) gives

$$6x^2 - 3x - 9 = 0 \iff (2x-3)(3x+3) = 0$$

So we must have either $x = \frac{3}{2}$ or $x = -1$.

- If $x = \frac{3}{2}$, (E2) reduces to $\frac{9}{2} + 6y - 3 = 0$ so $y = -\frac{1}{4}$.
- If $x = -1$, (E2) reduces to $-3 + 6y - 3 = 0$ so $y = 1$.

So there are two critical points: $\left(\frac{3}{2}, -\frac{1}{4}\right)$ and $(-1, 1)$.

The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$\left(\frac{3}{2}, -\frac{1}{4}\right)$	$(9) \times (6) - (3)^2 > 0$	9	local min
$(-1, 1)$	$(-6) \times (6) - (3)^2 < 0$		saddle point

$\boxed{2.9.16}$ (*) Let $h(x, y) = y(4 - x^2 - y^2)$.

(a) Find and classify the critical points of $h(x, y)$ as local maxima, local minima or saddle points.

(b) Find the maximum and minimum values of $h(x, y)$ on the disk $x^2 + y^2 \leqslant 1$.

Solution (a) To find the critical points we will need the gradient of h and to apply the second derivative test of Theorem 2.9.16 in the CLP-3 text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} h &= y(4 - x^2 - y^2) \\ h_x &= -2xy \qquad h_{xx} = -2y \qquad h_{xy} = -2x \\ h_y &= 4 - x^2 - 3y^2 \qquad h_{yy} = -6y \qquad h_{yx} = -2x \end{aligned}$$

(Of course, h_{xy} and h_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The critical points are the solutions of

$$h_x = -2xy = 0 \qquad h_y = 4 - x^2 - 3y^2 = 0$$

The first equation is satisfied if at least one of $x = 0$, $y = 0$ are satisfied.

- If $x = 0$, the second equation reduces to $4 - 3y^2 = 0$, which is satisfied if $y = \pm\frac{2}{\sqrt{3}}$.
- If $y = 0$, the second equation reduces to $4 - x^2 = 0$ which is satisfied if $x = \pm 2$.

So there are four critical points: $\left(0, \frac{2}{\sqrt{3}}\right)$, $\left(0, -\frac{2}{\sqrt{3}}\right)$, $(2, 0)$, $(-2, 0)$.

The classification is

critical point	$h_{xx}h_{yy} - h_{xy}^2$	h_{xx}	type
$\left(0, \frac{2}{\sqrt{3}}\right)$	$\left(\frac{-4}{\sqrt{3}}\right) \times \left(-\frac{12}{\sqrt{3}}\right) - (0)^2 > 0$	$\frac{-4}{\sqrt{3}}$	local max
$\left(0, -\frac{2}{\sqrt{3}}\right)$	$\left(\frac{4}{\sqrt{3}}\right) \times \left(\frac{12}{\sqrt{3}}\right) - (0)^2 > 0$	$\frac{4}{\sqrt{3}}$	local min
$(2, 0)$	$0 \times 0 - (-4)^2 < 0$		saddle point
$(-2, 0)$	$0 \times 0 - (4)^2 < 0$		saddle point

(b) The absolute max and min can occur either in the interior of the disk or on the boundary of the disk. The boundary of the disk is the circle $x^2 + y^2 = 1$.

- Any absolute max or min in the interior of the disk must also be a local max or min and, since there are no singular points, must also be a critical point of h. We found all of the critical points of h in part (a). Since $2 > 1$ and $\frac{2}{\sqrt{3}} > 1$ none of the critical points are in the disk.
- At each point of $x^2 + y^2 = 1$ we have $h(x, y) = 3y$ with $-1 \leqslant y \leqslant 1$. Clearly the maximum value is 3 (at $(0, 1)$) and the minimum value is -3 (at $(0, -1)$).

So all together, the maximum and minimum values of $h(x, y)$ in $x^2 + y^2 \leqslant 1$ are 3 (at $(0, 1)$) and -3 (at $(0, -1)$), respectively.

2.9.17 $(*)$ Find the absolute maximum and minimum values of the function $f(x, y) = 5 + 2x - x^2 - 4y^2$ on the rectangular region

$$R = \left\{ (x, y) \;\middle|\; -1 \leqslant x \leqslant 3,\ -1 \leqslant y \leqslant 1 \right\}$$

Solution The maximum and minimum must either occur at a critical point or on the boundary of R.

- The critical points are the solutions of

$$0 = f_x(x, y) = 2 - 2x$$
$$0 = f_y(x, y) = -8y$$

So the only critical point is $(1, 0)$.

- On the side $x = -1$, $-1 \leqslant y \leqslant 1$ of the boundary of R

  $$f(-1, y) = 2 - 4y^2$$

  This function decreases as $|y|$ increases. So its maximum value on $-1 \leqslant y \leqslant 1$ is achieved at $y = 0$ and its minimum value is achieved at $y = \pm 1$.

- On the side $x = 3$, $-1 \leqslant y \leqslant 1$ of the boundary of R

  $$f(3, y) = 2 - 4y^2$$

  This function decreases as $|y|$ increases. So its maximum value on $-1 \leqslant y \leqslant 1$ is achieved at $y = 0$ and its minimum value is achieved at $y = \pm 1$.

- On both sides $y = \pm 1$, $-1 \leqslant x \leqslant 3$ of the boundary of R

  $$f(x, \pm 1) = 1 + 2x - x^2 = 2 - (x - 1)^2$$

  This function decreases as $|x - 1|$ increases. So its maximum value on $-1 \leqslant x \leqslant 3$ is achieved at $x = 1$ and its minimum value is achieved at $x = 3$ and $x = -1$ (both of whom are a distance 2 from $x = 1$).

So we have the following candidates for the locations of the min and max

point	$(1, 0)$	$(-1, 0)$	$(1, \pm 1)$	$(-1, \pm 1)$	$(3, 0)$	$(3, \pm 1)$
value of f	6	2	2	-2	2	-2
	max			min		min

So the minimum is -2 and the maximum is 6.

$\boxed{2.9.18}$ $(*)$ Find the minimum of the function $h(x, y) = -4x - 2y + 6$ on the closed bounded domain defined by $x^2 + y^2 \leqslant 1$.

$\boxed{\text{Solution}}$ Since $\nabla h = \langle -4\,, -2 \rangle$ is never zero, h has no critical points and the minimum of h on the disk $x^2 + y^2 \leqslant 1$ must be taken on the boundary, $x^2 + y^2 = 1$, of the disk. To find the minimum on the boundary, we parametrize $x^2 + y^2 \leqslant 1$ by $x = \cos\theta$, $y = \sin\theta$ and find the minimum of

$$H(\theta) = -4\cos\theta - 2\sin\theta + 6$$

Since

$$0 = H'(\theta) = 4\sin\theta - 2\cos\theta \implies x = \cos\theta = 2\sin\theta = 2y$$

So

$$1 = x^2 + y^2 = 4y^2 + y^2 = 5y^2 \implies y = \pm\frac{1}{\sqrt{5}},\ x = \pm\frac{2}{\sqrt{5}}$$

At these two points

$$h = 6 - 4x - 2y = 6 - 10y = 6 \mp \frac{10}{\sqrt{5}} = 6 \mp 2\sqrt{5}$$

The minimum is $6 - 2\sqrt{5}$.

$\boxed{2.9.19}$ (*) Let $f(x,y) = xy(x+y-3)$.

(a) Find all critical points of f, and classify each one as a local maximum, a local minimum, or saddle point.

(b) Find the location and value of the absolute maximum and minimum of f on the triangular region $x \geqslant 0$, $y \geqslant 0$, $x+y \leqslant 8$.

$\boxed{\text{Solution}}$ (a) Thinking a little way ahead, to find the critical points we will need the gradient of f and to apply the second derivative test of Theorem 2.9.16 in the CLP-3 text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} f &= xy(x+y-3) \\ f_x &= 2xy + y^2 - 3y \qquad & f_{xx} &= 2y \qquad & f_{xy} &= 2x+2y-3 \\ f_y &= x^2 + 2xy - 3x \qquad & f_{yy} &= 2x \qquad & f_{yx} &= 2x+2y-3 \end{aligned}$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The critical points are the solutions of

$$f_x = y(2x+y-3) = 0 \qquad f_y = x(x+2y-3) = 0$$

The first equation is satisfied if at least one of $y=0$, $y=3-2x$ are satisfied.

- If $y=0$, the second equation reduces to $x(x-3)=0$, which is satisfied if either $x=0$ or $x=3$.
- If $y=3-2x$, the second equation reduces to $x(x+6-4x-3) = x(3-3x) = 0$ which is satisfied if $x=0$ or $x=1$.

So there are four critical points: $(0,0)$, $(3,0)$, $(0,3)$, $(1,1)$.

The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$0 \times 0 - (-3)^2 < 0$		saddle point
$(3,0)$	$0 \times 6 - (3)^2 < 0$		saddle point
$(0,3)$	$6 \times 0 - (3)^2 < 0$		saddle point
$(1,1)$	$2 \times 2 - (1)^2 > 0$	2	local min

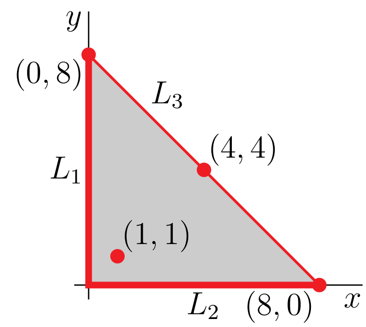
(b) The absolute max and min can occur either in the interior of the triangle or on the boundary of the triangle. The boundary of the triangle consists of the three line segments.

$$\begin{aligned} L_1 &= \{\ (x,y)\ \big|\ x=0,\ 0 \leqslant y \leqslant 8\ \} \\ L_2 &= \{\ (x,y)\ \big|\ y=0,\ 0 \leqslant x \leqslant 8\ \} \\ L_3 &= \{\ (x,y)\ \big|\ x+y=8,\ 0 \leqslant x \leqslant 8\ \} \end{aligned}$$

- Any absolute max or min in the interior of the triangle must also be a local max or min and, since there are no singular points, must also be a critical point of f. We found all of the critical points of f in part (a). Only one of them, namely $(1,1)$ is in the interior of the triangle. (The other three critical points are all on the boundary of the triangle.) We have $f(1,1) = -1$.
- At each point of L_1 we have $x = 0$ and so $f(x,y) = 0$.
- At each point of L_2 we have $y = 0$ and so $f(x,y) = 0$.
- At each point of L_3 we have $f(x,y) = x(8-x)(5) = 40x - 5x^2 = 5[8x - x^2]$ with $0 \leqslant x \leqslant 8$. As $\frac{\mathrm{d}}{\mathrm{d}x}\big(40x - 5x^2\big) = 40 - 10x$, the max and min of $40x - 5x^2$ on $0 \leqslant x \leqslant 8$ must be one of $5\big[8x - x^2\big]_{x=0} = 0$ or $5\big[8x - x^2\big]_{x=8} = 0$ or $5\big[8x - x^2\big]_{x=4} = 80$.

So all together, we have the following candidates for max and min, with the max and min indicated.

point(s)	$(1,1)$	L_1	L_2	$(0,8)$	$(8,0)$	$(4,4)$
value of f	-1	0	0	0	0	80
	min					max



2.9.20 (*) Find and classify the critical points of $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 4$.

Solution Thinking a little way ahead, to find the critical points we will need the gradient of f, and to apply the second derivative test of Theorem 2.9.16 in the CLP-3 text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned}
f &= 3x^2y + y^3 - 3x^2 - 3y^2 + 4 \\
f_x &= 6xy - 6x \qquad & f_{xx} &= 6y - 6 \qquad & f_{xy} &= 6x \\
f_y &= 3x^2 + 3y^2 - 6y \qquad & f_{yy} &= 6y - 6 \qquad & f_{yx} &= 6x
\end{aligned}$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The critical points are the solutions of

$$f_x = 6x(y-1) = 0 \qquad f_y = 3x^2 + 3y^2 - 6y = 0$$

The first equation is satisfied if at least one of $x = 0$, $y = 1$ are satisfied.

- If $x = 0$, the second equation reduces to $3y^2 - 6y = 0$, which is satisfied if either $y = 0$ or $y = 2$.
- If $y = 1$, the second equation reduces to $3x^2 - 3 = 0$ which is satisfied if $x = \pm 1$.

So there are four critical points: $(0,0)$, $(0,2)$, $(1,1)$, $(-1,1)$.

The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$(-6) \times (-6) - (0)^2 > 0$	-6	local max
$(0,2)$	$6 \times 6 - (0)^2 > 0$	6	local min
$(1,1)$	$0 \times 0 - (6)^2 < 0$		saddle point
$(-1,1)$	$0 \times 0 - (-6)^2 < 0$		saddle point

$\boxed{\text{2.9.21}}$ (*) Consider the function

$$f(x,y) = 2x^3 - 6xy + y^2 + 4y$$

(a) Find and classify all of the critical points of $f(x,y)$.
(b) Find the maximum and minimum values of $f(x,y)$ in the triangle with vertices $(1,0)$, $(0,1)$ and $(1,1)$.

$\boxed{\text{Solution}}$ (a) Since

$$\begin{aligned} f &= 2x^3 - 6xy + y^2 + 4y \\ f_x &= 6x^2 - 6y \qquad & f_{xx} &= 12x \qquad & f_{xy} &= -6 \\ f_y &= -6x + 2y + 4 \qquad & f_{yy} &= 2 \end{aligned}$$

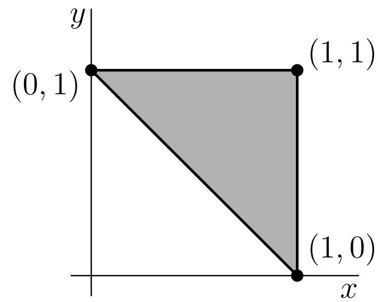
the critical points are the solutions of

$$\begin{aligned} & f_x = 0 && f_y = 0 \\ \iff\quad & y = x^2 && y - 3x + 2 = 0 \\ \iff\quad & y = x^2 && x^2 - 3x + 2 = 0 \\ \iff\quad & y = x^2 && x = 1 \text{ or } 2 \end{aligned}$$

So, there are two critical points: $(1,1)$, $(2,4)$.

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(1,1)$	$12 \times 2 - (-6)^2 < 0$		saddle point
$(2,4)$	$24 \times 2 - (-6)^2 > 0$	24	local min

(b) There are no critical points in the interior of the allowed region, so both the maximum and the minimum occur only on the boundary. The boundary consists of the line segments (i) $x = 1$, $0 \leqslant y \leqslant 1$, (ii) $y = 1$, $0 \leqslant x \leqslant 1$ and (iii) $y = 1 - x$, $0 \leqslant x \leqslant 1$.



- First, we look at the part of the boundary with $x = 1$. There $f = y^2 - 2y + 2$. As $\frac{d}{dy}(y^2 - 2y + 2) = 2y - 2$ vanishes only at $y = 1$, the max and min of $y^2 - 2y + 2$ for $0 \leqslant y \leqslant 1$ must occur either at $y = 0$, where $f = 2$, or at $y = 1$, where $f = 1$.
- Next, we look at the part of the boundary with $y = 1$. There $f = 2x^3 - 6x + 5$. As $\frac{d}{dx}(2x^3 - 6x + 5) = 6x^2 - 6$, the max and min of $2x^3 - 6x + 5$ for $0 \leqslant x \leqslant 1$ must occur either at $x = 0$, where $f = 5$, or at $x = 1$, where $f = 1$.
- Next, we look at the part of the boundary with $y = 1 - x$. There $f = 2x^3 - 6x(1-x) + (1-x)^2 + 4(1-x) = 2x^3 + 7x^2 - 12x + 5$. As $\frac{d}{dx}(2x^3 + 7x^2 - 12x + 5) = 6x^2 + 14x - 12 = 2\big(3x^2 + 7x - 6\big) = 2(3x-2)(x+3)$, the max and min of $2x^3 + 7x^2 - 12x + 5$ for $0 \leqslant x \leqslant 1$ must occur either at $x = 0$, where $f = 5$, or at $x = 1$, where $f = 2$, or at $x = \frac{2}{3}$, where $f = 2(\frac{8}{27}) - 6(\frac{2}{3})(\frac{1}{3}) + \frac{1}{9} + \frac{4}{3} = \frac{16-36+3+36}{27} = \frac{19}{27}$.

So all together, we have the following candidates for max and min, with the max and min indicated.

point	$(1,0)$	$(1,1)$	$(0,1)$	$(\frac{2}{3},\frac{1}{3})$
value of f	2	1	5	$\frac{19}{27}$
			max	min

$\boxed{2.9.22}$ (*) Find all critical points of the function $f(x,y) = x^4 + y^4 - 4xy + 2$, and for each determine whether it is a local minimum, maximum or saddle point.

$\boxed{\text{Solution}}$ We have

$$\begin{aligned} f(x,y) &= x^4 + y^4 - 4xy + 2 & f_x(x,y) &= 4x^3 - 4y & f_{xx}(x,y) &= 12x^2 \\ & & f_y(x,y) &= 4y^3 - 4x & f_{yy}(x,y) &= 12y^2 \\ & & & & f_{xy}(x,y) &= -4 \end{aligned}$$

At a critical point

$$\begin{aligned} f_x(x,y) = f_y(x,y) = 0 &\iff y = x^3 \text{ and } x = y^3 \\ &\iff x = x^9 \text{ and } y = x^3 \\ &\iff x(x^8 - 1) = 0,\ y = x^3 \\ &\iff (x,y) = (0,0) \text{ or } (1,1) \text{ or } (-1,-1) \end{aligned}$$

Here is a table giving the classification of each of the three critical points.

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$0 \times 0 - (-4)^2 < 0$		saddle point
$(1,1)$	$12 \times 12 - (-4)^2 > 0$	12	local min
$(-1,-1)$	$12 \times 12 - (-4)^2 > 0$	12	local min

2.9.23 (*) Let

$$f(x,y) = xy(x+2y-6)$$

(a) Find every critical point of $f(x,y)$ and classify each one.
(b) Let D be the region in the plane between the hyperbola $xy = 4$ and the line $x + 2y - 6 = 0$. Find the maximum and minimum values of $f(x,y)$ on D.

Solution (a) We have

$$\begin{aligned}
f(x,y) &= xy(x+2y-6) & f_x(x,y) &= 2xy + 2y^2 - 6y & f_{xx}(x,y) &= 2y \\
& & f_y(x,y) &= x^2 + 4xy - 6x & f_{yy}(x,y) &= 4x \\
& & & & f_{xy}(x,y) &= 2x + 4y - 6
\end{aligned}$$

At a critical point

$$\begin{aligned}
f_x(x,y) = f_y(x,y) = 0 &\iff 2y(x+y-3) = 0 \text{ and } x(x+4y-6) = 0 \\
&\iff \{y = 0 \text{ or } x+y = 3\} \text{ and } \{x = 0 \text{ or } x + 4y = 6\} \\
&\iff \{x = y = 0\} \text{ or } \{y = 0,\ x + 4y = 6\} \\
&\qquad \text{or } \{x+y = 3,\ x = 0\} \text{ or } \{x + y = 3,\ x + 4y = 6\} \\
&\iff (x,y) = (0,0) \text{ or } (6,0) \text{ or } (0,3) \text{ or } (2,1)
\end{aligned}$$

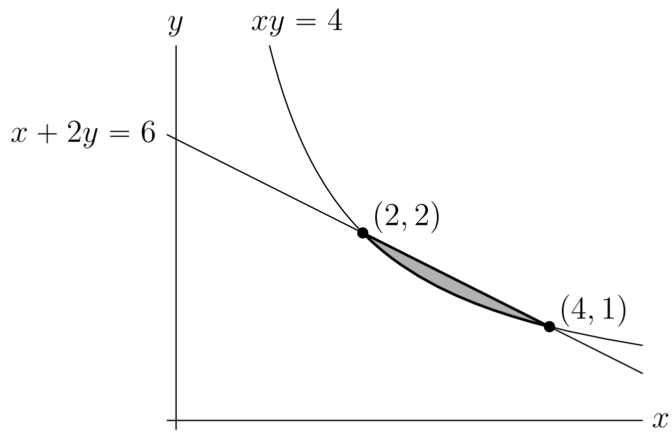
Here is a table giving the classification of each of the four critical points.

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$0 \times 0 - (-6)^2 < 0$		saddle point
$(6,0)$	$0 \times 24 - 6^2 < 0$		saddle point
$(0,3)$	$6 \times 0 - 6^2 < 0$		saddle point
$(2,1)$	$2 \times 8 - 2^2 > 0$	2	local min

(b) Observe that $xy = 4$ and $x + 2y = 6$ intersect when $x = 6 - 2y$ and

$$\begin{aligned}
(6-2y)y = 4 &\iff 2y^2 - 6y + 4 = 0 \iff 2(y-1)(y-2) = 0 \\
&\iff (x,y) = (4,1) \text{ or } (2,2)
\end{aligned}$$

The shaded region in the sketch below is D.



None of the critical points are in D. So the max and min must occur at either $(2, 2)$ or $(4, 1)$ or on $xy = 4$, $2 < x < 4$ (in which case $F(x) = f\big(x, \frac{4}{x}\big) = 4\big(x + \frac{8}{x} - 6\big)$ obeys $F'(x) = 4 - \frac{32}{x^2} = 0 \iff x = \pm 2\sqrt{2}$) or on $x + 2y = 6$, $2 < x < 4$ (in which case $f(x, y)$ is identically zero). So the min and max must occur at one of

(x, y)	$f(x, y)$
$(2, 2)$	$2 \times 2(2 + 2 \times 2 - 6) = 0$
$(4, 1)$	$4 \times 1(4 + 2 \times 1 - 6) = 0$
$(2\sqrt{2}, \sqrt{2})$	$4(2\sqrt{2} + 2\sqrt{2} - 6) < 0$

The maximum value is 0 and the minimum value is $4(4\sqrt{2} - 6) \approx -1.37$.

$\boxed{2.9.24}$ $(*)$ Find all the critical points of the function

$$f(x, y) = x^4 + y^4 - 4xy$$

defined in the xy-plane. Classify each critical point as a local minimum, maximum or saddle point.

$\boxed{\text{Solution}}$ We have

$$\begin{aligned} f(x,y) &= x^4 + y^4 - 4xy & f_x(x,y) &= 4x^3 - 4y & f_{xx}(x,y) &= 12x^2 \\ && f_y(x,y) &= 4y^3 - 4x & f_{yy}(x,y) &= 12y^2 \\ &&&& f_{xy}(x,y) &= -4 \end{aligned}$$

At a critical point

$$\begin{aligned} f_x(x,y) = f_y(x,y) = 0 &\iff y = x^3 \text{ and } x = y^3 \iff x = x^9 \text{ and } y = x^3 \\ &\iff x(x^8 - 1) = 0,\ y = x^3 \\ &\iff (x,y) = (0,0) \text{ or } (1,1) \text{ or } (-1,-1) \end{aligned}$$

Here is a table giving the classification of each of the three critical points.

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$0 \times 0 - (-4)^2 < 0$		saddle point
$(1,1)$	$12 \times 12 - (-4)^2 > 0$	12	local min
$(-1,-1)$	$12 \times 12 - (-4)^2 > 0$	12	local min

$\boxed{2.9.25}$ ($*$) A metal plate is in the form of a semi-circular disc bounded by the x-axis and the upper half of $x^2 + y^2 = 4$. The temperature at the point (x, y) is given by $T(x, y) = \ln\left(1 + x^2 + y^2\right) - y$. Find the coldest point on the plate, explaining your steps carefully. (Note: $\ln 2 \approx 0.693$, $\ln 5 \approx 1.609$)

$\boxed{\text{Solution}}$ The coldest point must be either on the boundary of the plate or in the interior of the plate.

- On the semi–circular part of the boundary $0 \leqslant y \leqslant 2$ and $x^2 + y^2 = 4$ so that $T = \ln\left(1 + x^2 + y^2\right) - y = \ln 5 - y$. The smallest value of $\ln 5 - y$ is taken when y is as large as possible, i.e. when $y = 2$, and is $\ln 5 - 2 \approx -0.391$.
- On the flat part of the boundary, $y = 0$ and $-2 \leqslant x \leqslant 2$ so that $T = \ln\left(1 + x^2 + y^2\right) - y = \ln\left(1 + x^2\right)$. The smallest value of $\ln\left(1 + x^2\right)$ is taken when x is as small as possible, i.e. when $x = 0$, and is 0.
- If the coldest point is in the interior of the plate, it must be at a critical point of $T(x, y)$. Since

$$T_x(x, y) = \frac{2x}{1 + x^2 + y^2} \qquad T_y(x, y) = \frac{2y}{1 + x^2 + y^2} - 1$$

  a critical point must have $x = 0$ and $\frac{2y}{1+x^2+y^2} - 1 = 0$, which is the case if and only if $x = 0$ and $2y - 1 - y^2 = 0$. So the only critical point is $x = 0,\ y = 1$, where $T = \ln 2 - 1 \approx -0.307$.

Since $-0.391 < -0.307 < 0$, the coldest temperture is -0.391 and the coldest point is $(0, 2)$.

$\boxed{2.9.26}$ ($*$) Find all the critical points of the function

$$f(x, y) = x^3 + xy^2 - x$$

defined in the xy-plane. Classify each critical point as a local minimum, maximum or saddle point. Explain your reasoning.

$\boxed{\text{Solution}}$ We have

$$\begin{aligned} f(x,y) &= x^3 + xy^2 - x \qquad & f_x(x,y) &= 3x^2 + y^2 - 1 \qquad & f_{xx}(x,y) &= 6x \\ & & f_y(x,y) &= 2xy & f_{yy}(x,y) &= 2x \\ & & & & f_{xy}(x,y) &= 2y \end{aligned}$$

At a critical point

$$\begin{aligned} f_x(x,y) = f_y(x,y) = 0 &\iff xy = 0 \text{ and } 3x^2 + y^2 = 1 \\ &\iff \{x = 0 \text{ or } y = 0\} \text{ and } 3x^2 + y^2 = 1 \\ &\iff (x,y) = (0,1) \text{ or } (0,-1) \text{ or } \left(\frac{1}{\sqrt{3}}, 0\right) \text{ or } \left(-\frac{1}{\sqrt{3}}, 0\right) \end{aligned}$$

Here is a table giving the classification of each of the four critical points.

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,1)$	$0 \times 0 - 2^2 < 0$		saddle point
$(0,-1)$	$0 \times 0 - (-2)^2 < 0$		saddle point
$(\frac{1}{\sqrt{3}}, 0)$	$2\sqrt{3} \times \frac{2}{\sqrt{3}} - 0^2 > 0$	$2\sqrt{3}$	local min
$(-\frac{1}{\sqrt{3}}, 0)$	$-2\sqrt{3} \times (-\frac{2}{\sqrt{3}}) - 0^2 > 0$	$-2\sqrt{3}$	local max

2.9.27 (*) Consider the function $g(x,y) = x^2 - 10y - y^2$.
(a) Find and classify all critical points of g.
(b) Find the absolute extrema of g on the bounded region given by

$$x^2 + 4y^2 \leqslant 16, \; y \leqslant 0$$

Solution (a) We have

$$\begin{aligned} g(x,y) &= x^2 - 10y - y^2 & g_x(x,y) &= 2x & g_{xx}(x,y) &= 2 \\ & & g_y(x,y) &= -10 - 2y & g_{yy}(x,y) &= -2 \\ & & & & g_{xy}(x,y) &= 0 \end{aligned}$$

At a critical point

$$g_x(x,y) = g_y(x,y) = 0 \iff 2x = 0 \text{ and } -10 - 2y = 0 \iff (x,y) = (0,-5)$$

Since $g_{xx}(0,-5)g_{yy}(0,-5) - g_{xy}(0,-5)^2 = 2 \times (-2) - 0^2 < 0$, the critical point is a saddle point.

(b) The extrema must be either on the boundary of the region or in the interior of the region.

- On the semi-elliptical part of the boundary $-2 \leqslant y \leqslant 0$ and $x^2 + 4y^2 = 16$ so that $g = x^2 - 10y - y^2 = 16 - 10y - 5y^2 = 21 - 5(y+1)^2$. This has a minimum value of 16 (at $y = 0, -2$) and a maximum value of 21 (at $y = -1$). You could also come to this conclusion by checking the critical point of $16 - 10y - 5y^2$ (i.e. solving $\frac{d}{dy}(16 - 10y - 5y^2) = 0$) and checking the end points of the allowed interval (namely $y = 0$ and $y = -2$).
- On the flat part of the boundary $y = 0$ and $-4 \leqslant x \leqslant 4$ so that $g = x^2$. The smallest value is taken when $x = 0$ and is 0 and the largest value is taken when $x = \pm 4$ and is 16.

- If an extremum is in the interior of the plate, it must be at a critical point of $g(x,y)$. The only critical point is not in the prescribed region.

Here is a table giving all candidates for extrema:

(x,y)	$g(x,y)$
$(0,-2)$	16
$(\pm 4,0)$	16
$(\pm\sqrt{12},-1)$	21
$(0,0)$	0

From the table the smallest value of g is 0 at $(0,0)$ and the largest value is 21 at $(\pm 2\sqrt{3},-1)$.

2.9.28 (∗) Find and classify all critical points of

$$f(x,y) = x^3 - 3xy^2 - 3x^2 - 3y^2$$

Solution We have

$$f(x,y) = x^3 - 3xy^2 - 3x^2 - 3y^2 \qquad \begin{aligned} f_x(x,y) &= 3x^2 - 3y^2 - 6x \\ f_y(x,y) &= -6xy - 6y \end{aligned} \qquad \begin{aligned} f_{xx}(x,y) &= 6x - 6 \\ f_{yy}(x,y) &= -6x - 6 \\ f_{xy}(x,y) &= -6y \end{aligned}$$

At a critical point

$$\begin{aligned} f_x(x,y) = f_y(x,y) = 0 &\iff 3(x^2 - y^2 - 2x) = 0 \text{ and } -6y(x+1) = 0 \\ &\iff \{x = -1 \text{ or } y = 0\} \text{ and } x^2 - y^2 - 2x = 0 \\ &\iff (x,y) = (-1,\sqrt{3}) \text{ or } (-1,-\sqrt{3}) \text{ or } (0,0) \text{ or } (2,0) \end{aligned}$$

Here is a table giving the classification of each of the four critical points.

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$(-6)\times(-6) - 0^2 > 0$	-6	local max
$(2,0)$	$6\times(-18) - 0^2 < 0$		saddle point
$(-1,\sqrt{3})$	$(-12)\times 0 - (-6\sqrt{3})^2 < 0$		saddle point
$(-1,-\sqrt{3})$	$(-12)\times 0 - (6\sqrt{3})^2 < 0$		saddle point

2.9.29 (∗) Find the maximum value of

$$f(x,y) = xye^{-(x^2+y^2)/2}$$

on the quarter-circle $D = \{\ (x,y)\ \big|\ x^2 + y^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0\ \}$.

Solution The maximum must be either on the boundary of D or in the interior of D.

- On the circular part of the boundary $r = 2$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$ (in polar coordinates) so that $f = r^2 \cos\theta \sin\theta e^{-r^2/2} = 2\sin(2\theta)e^{-2}$. This has a maximum value of $2e^{-2}$ at $\theta = \frac{\pi}{4}$ or $x = y = \sqrt{2}$.
- On the two flat parts of the boundary $x = 0$ or $y = 0$ so that $f = 0$.
- If the maximum is in the interior of D, it must be at a critical point of $f(x, y)$. Since

$$f_x(x,y) = e^{-(x^2+y^2)/2}\big[y - x^2y\big] \qquad f_y(x,y) = e^{-(x^2+y^2)/2}\big[x - xy^2\big]$$

  (x, y) is a critical point if and only if

$$\begin{aligned} &y(1-x^2) = 0 \text{ and } x(1-y^2) = 0 \\ &\iff \{y = 0 \text{ or } x = 1 \text{ or } x = -1\} \text{ and } \{x = 0 \text{ or } y = 1 \text{ or } y = -1\} \end{aligned}$$

  There are two critical points with $x, y \geqslant 0$, namely $(0, 0)$ and $(1, 1)$. The first of these is on the boundary of D and the second is in the interior of D.

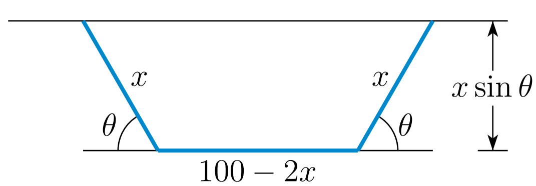
Here is a table giving all candidates for the maximum:

(x,y)	$g(x,y)$
$(\sqrt{2},\sqrt{2})$	$2e^{-2} \approx 0.271$
$(x,0)$	0
$(0,y)$	0
$(1,1)$	$e^{-1} \approx 0.368$

Since $e > 2$, we have that $2e^{-2} = e^{-1}\frac{2}{e} < e^{-1}$ and the largest value is e^{-1}.

2.9.30 Equal angle bends are made at equal distances from the two ends of a 100 metre long fence, so that the resulting three segment fence can be placed along an existing wall to make an enclosure of trapezoidal shape. What is the largest possible area for such an enclosure?

Solution Suppose that the bends are made a distance x from the ends of the fence and that the bends are through an angle θ. Here is a sketch of the enclosure.



It consists of a rectangle, with side lengths $100 - 2x$ and $x \sin\theta$, together with two triangles, each of height $x \sin\theta$ and base length $x \cos\theta$. So the enclosure has area

$$\begin{aligned} A(x,\theta) &= (100-2x)x\sin\theta + 2\cdot\tfrac{1}{2}\cdot x\sin\theta\cdot x\cos\theta \\ &= (100x - 2x^2)\sin\theta + \tfrac{1}{2}x^2\sin(2\theta) \end{aligned}$$

The maximize the area, we need to solve

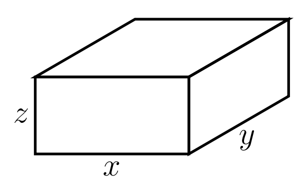
$$\begin{aligned} 0 = A_x &= (100-4x)\sin\theta + x\sin(2\theta) &&\implies & (100-4x)+2x\cos\theta &= 0\\ 0 = A_\theta &= (100x-2x^2)\cos\theta + x^2\cos(2\theta) &&\implies & (100-2x)\cos\theta + x\cos(2\theta) &= 0 \end{aligned}$$

Here we have used that the fence of maximum area cannot have $\sin\theta = 0$ or $x = 0$, because in either of these two cases, the area enclosed will be zero. The first equation forces $\cos\theta = -\frac{100-4x}{2x}$ and hence $\cos(2\theta) = 2\cos^2\theta - 1 = \frac{(100-4x)^2}{2x^2} - 1$. Substituting these into the second equation gives

$$\begin{aligned} &-(100-2x)\frac{100-4x}{2x} + x\Big[\frac{(100-4x)^2}{2x^2} - 1\Big] = 0\\ \implies &-(100-2x)(100-4x) + (100-4x)^2 - 2x^2 = 0\\ \implies &\qquad 6x^2 - 200x = 0\\ \implies &\qquad x = \frac{100}{3} \quad \cos\theta = -\frac{-100/3}{200/3} = \frac{1}{2} \quad \theta = 60^\circ\\ &A = \Big(100\frac{100}{3} - 2\frac{100^2}{3^2}\Big)\frac{\sqrt{3}}{2} + \frac{1}{2}\frac{100^2}{3^2}\frac{\sqrt{3}}{2} = \frac{2500}{\sqrt{3}} \end{aligned}$$

2.9.31 Find the most economical shape of a rectangular box that has a fixed volume V and that has no top.

Solution Suppose that the box has side lengths x, y and z. Here is a sketch.



Because the box has to have volume V we need that $V = xyz$. We wish to minimize the area $A = xy + 2yz + 2xz$ of the four sides and bottom. Substituting in $z = \frac{V}{xy}$,

$$\begin{aligned} A &= xy + 2\frac{V}{x} + 2\frac{V}{y}\\ A_x &= y - 2\frac{V}{x^2}\\ A_y &= x - 2\frac{V}{y^2} \end{aligned}$$

To minimize, we want $A_x = A_y = 0$, which is the case when $yx^2 = 2V$, $xy^2 = 2V$. This forces $yx^2 = xy^2$. Since $V = xyz$ is nonzero, neither x nor y may be zero. So $x = y = (2V)^{1/3}$, $z = 2^{-2/3}V^{1/3}$.

▸▸ Stage 3

2.9.32 (*) The temperature $T(x,y)$ at a point of the xy–plane is given by

$$T(x,y) = 20 - 4x^2 - y^2$$

(a) Find the maximum and minimum values of $T(x,y)$ on the disk D defined by $x^2 + y^2 \leqslant 4$.
(b) Suppose an ant lives on the disk D. If the ant is initially at point $(1,1)$, in which direction should it move so as to increase its temperature as quickly as possible?
(c) Suppose that the ant moves at a velocity $\mathbf{v} = \langle -2, -1 \rangle$. What is its rate of increase of temperature as it passes through $(1,1)$?
(d) Suppose the ant is constrained to stay on the curve $y = 2 - x^2$. Where should the ant go if it wants to be as warm as possible?

Solution (a) The maximum and minimum can occur either in the interior of the disk or on the boundary of the disk. The boundary of the disk is the circle $x^2 + y^2 = 4$.

- Any absolute max or min in the interior of the disk must also be a local max or min and, since there are no singular points, must also be a critical point of h. Since $T_x = -8x$ and $T_y = -2y$, the only critical point is $(x,y) = (0,0)$, where $T = 20$. Since $4x^2 + y^2 \geqslant 0$, we have $T(x,y) = 20 - 4x^2 - y^2 \leqslant 20$. So the maximum value of T (even in $\mathbb{R}^2$) is 20.

- At each point of $x^2 + y^2 = 4$ we have $T(x,y) = 20 - 4x^2 - y^2 = 20 - 4x^2 - (4 - x^2) = 16 - 3x^2$ with $-2 \leqslant x \leqslant 2$. So T is a minimum when x^2 is a maximum. Thus the minimum value of T on the disk is $16 - 3(\pm 2)^2 = 4$.

So all together, the maximum and minimum values of $T(x,y)$ in $x^2 + y^2 \leqslant 4$ are 20 (at $(0,0)$) and 4 (at $(\pm 2, 0)$), respectively.

(b) To increase its temperature as quickly as possible, the ant should move in the direction of the temperature gradient $\boldsymbol{\nabla} T(1,1) = \langle -8x, -2y \rangle \Big|_{(x,y)=(1,1)} = \langle -8, -2 \rangle$. A unit vector in that direction is $\frac{1}{\sqrt{17}} \langle -4, -1 \rangle$.

(c) The ant's rate f increase of temperature (per unit time) is

$$\boldsymbol{\nabla} T(1,1) \cdot \langle -2, -1 \rangle = \langle -8, -2 \rangle \cdot \langle -2, -1 \rangle = 18$$

(d) We are being asked to find the $(x,y) = (x, 2 - x^2)$ which maximizes

$$T\big(x, 2 - x^2\big) = 20 - 4x^2 - \big(2 - x^2\big)^2 = 16 - x^4$$

The maximum of $16 - x^4$ is obviously 16 at $x = 0$. So the ant should go to $\big(0, 2 - 0^2\big) = (0,2)$.

2.9.33 (*) Consider the function

$$f(x, y) = 3kx^2y + y^3 - 3x^2 - 3y^2 + 4$$

where $k > 0$ is a constant. Find and classify all critical points of $f(x, y)$ as local minima, local maxima, saddle points or points of indeterminate type. Carefully distinguish the cases $k < \frac{1}{2}$, $k = \frac{1}{2}$ and $k > \frac{1}{2}$.

Solution To find the critical points we will need the gradient of f and to apply the second derivative test of Theorem 2.9.16 in the CLP-3 text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned}
f &= 3kx^2y + y^3 - 3x^2 - 3y^2 + 4 \\
f_x &= 6kxy - 6x \qquad & f_{xx} &= 6ky - 6 \qquad & f_{xy} &= 6kx \\
f_y &= 3kx^2 + 3y^2 - 6y \qquad & f_{yy} &= 6y - 6 \qquad & f_{yx} &= 6kx
\end{aligned}$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The critical points are the solutions of

$$f_x = 6x(ky - 1) = 0 \qquad f_y = 3kx^2 + 3y^2 - 6y = 0$$

The first equation is satisfied if at least one of $x = 0$, $y = 1/k$ are satisfied. (Recall that $k > 0$.)

- If $x = 0$, the second equation reduces to $3y(y - 2) = 0$, which is satisfied if either $y = 0$ or $y = 2$.
- If $y = 1/k$, the second equation reduces to $3kx^2 + \frac{3}{k^2} - \frac{6}{k} = 3kx^2 + \frac{3}{k^2}(1 - 2k) = 0$.

Case $k < \frac{1}{2}$: If $k < \frac{1}{2}$, then $\frac{3}{k^2}(1 - 2k) > 0$ and the equation $3kx^2 + \frac{3}{k^2}(1 - 2k) = 0$ has no real solutions. In this case there are two critical points: $(0, 0)$, $(0, 2)$ and the classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0, 0)$	$(-6) \times (-6) - (0)^2 > 0$	-6	local max
$(0, 2)$	$(12k - 6) \times 6 - (0)^2 < 0$		saddle point

Case $k = \frac{1}{2}$: If $k = \frac{1}{2}$, then $\frac{3}{k^2}(1 - 2k) = 0$ and the equation $3kx^2 + \frac{3}{k^2}(1 - 2k) = 0$ reduces to $3kx^2 = 0$ which has as its only solution $x = 0$. We have already seen this third critical point, $x = 0$, $y = 1/k = 2$. So there are again two critical points: $(0, 0)$, $(0, 2)$ and the classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0, 0)$	$(-6) \times (-6) - (0)^2 > 0$	-6	local max
$(0, 2)$	$(12k - 6) \times 6 - (0)^2 = 0$		unknown

Case $k > \frac{1}{2}$: If $k > \frac{1}{2}$, then $\frac{3}{k^2}(1-2k) < 0$ and the equation $3kx^2 + \frac{3}{k^2}(1-2k) = 0$ reduces to $3kx^2 = \frac{3}{k^2}(2k-1)$ which has two solutions, namely $x = \pm\sqrt{\frac{1}{k^3}(2k-1)}$. So there are four critical points: $(0,0)$, $(0,2)$, $\left(\sqrt{\frac{1}{k^3}(2k-1)}\,,\,\frac{1}{k}\right)$ and $\left(-\sqrt{\frac{1}{k^3}(2k-1)}\,,\,\frac{1}{k}\right)$ and the classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$(-6)\times(-6) - (0)^2 > 0$	-6	local max
$(0,2)$	$(12k-6)\times 6 - (0)^2 > 0$	$12k-6 > 0$	local min
$\left(\sqrt{\frac{1}{k^3}(2k-1)}\,,\,\frac{1}{k}\right)$	$(6-6)\times(\frac{6}{k}-6) - (>0)^2 < 0$		saddle point
$\left(-\sqrt{\frac{1}{k^3}(2k-1)}\,,\,\frac{1}{k}\right)$	$(6-6)\times(\frac{6}{k}-6) - (<0)^2 < 0$		saddle point

2.9.34 (*)

(a) Show that the function $f(x,y) = 2x + 4y + \frac{1}{xy}$ has exactly one critical point in the first quadrant $x > 0$, $y > 0$, and find its value at that point.

(b) Use the second derivative test to classify the critical point in part (a).

(c) Hence explain why the inequality $2x + 4y + \frac{1}{xy} \geqslant 6$ is valid for all positive real numbers x and y.

Solution (a) For $x, y > 0$,

$$f_x = 2 - \frac{1}{x^2 y} = 0 \iff y = \frac{1}{2x^2}$$
$$f_y = 4 - \frac{1}{xy^2} = 0$$

Substituting $y = \frac{1}{2x^2}$, from the first equation, into the second gives $4 - 4x^3 = 0$ which forces $x = 1$, $y = \frac{1}{2}$. At $x = 1$, $y = \frac{1}{2}$,

$$f\left(1, \tfrac{1}{2}\right) = 2 + 2 + 2 = 6$$

(b) The second derivatives are

$$f_{xx}(x,y) = \frac{2}{x^3 y} \qquad f_{xy}(x,y) = \frac{1}{x^2 y^2} \qquad f_{yy}(x,y) = \frac{2}{xy^3}$$

In particular

$$f_{xx}\left(1, \tfrac{1}{2}\right) = 4 \qquad f_{xy}\left(1, \tfrac{1}{2}\right) = 4 \qquad f_{yy}\left(1, \tfrac{1}{2}\right) = 16$$

Since $f_{xx}\left(1, \frac{1}{2}\right) f_{yy}\left(1, \frac{1}{2}\right) - f_{xy}\left(1, \frac{1}{2}\right)^2 = 4 \times 16 - 4^2 = 48 > 0$ and $f_{xx}\left(1, \frac{1}{2}\right) = 4 > 0$, the point $\left(1, \frac{1}{2}\right)$ is a local minimum.

(c) As x or y tends to infinity (with the other at least zero), $2x + 4y$ tends to $+\infty$. As (x, y) tends to any point on the first quadrant part of the x- and y–axes, $\frac{1}{xy}$ tends to $+\infty$. Hence as x or y tends to the boundary of the first quadrant (counting infinity as part of the boundary),

$f(x,y)$ tends to $+\infty$. As a result $\left(1,\frac{1}{2}\right)$ is a global (and not just local) minimum for f in the first quadrant. Hence $f(x,y) \geqslant f\left(1,\frac{1}{2}\right) = 6$ for all $x, y > 0$.

2.9.35 An experiment yields data points (x_i, y_i), $i = 1, 2, \cdots, n$. We wish to find the straight line $y = mx + b$ which "best" fits the data. The definition of "best" is "minimizes the root mean square error", i.e. minimizes $\sum_{i=1}^{n}(mx_i + b - y_i)^2$. Find m and b.

Solution We wish to choose m and b so as to minimize the (square of the) rms error $E(m,b) = \sum_{i=1}^{n}(mx_i + b - y_i)^2$.

$$\begin{aligned} 0 = \frac{\partial E}{\partial m} &= \sum_{i=1}^{n} 2(mx_i + b - y_i)x_i = m\Big[\sum_{i=1}^{n} 2x_i^2\Big] + b\Big[\sum_{i=1}^{n} 2x_i\Big] - \Big[\sum_{i=1}^{n} 2x_iy_i\Big] \\ 0 = \frac{\partial E}{\partial b} &= \sum_{i=1}^{n} 2(mx_i + b - y_i) \quad = m\Big[\sum_{i=1}^{n} 2x_i\Big] + b\Big[\sum_{i=1}^{n} 2\Big] - \Big[\sum_{i=1}^{n} 2y_i\Big] \end{aligned}$$

There are a lot of symbols in those two equations. But remember that only two of them, namely m and b, are unknowns. All of the x_i's and y_i's are given data. We can make the equations look a lot less imposing if we define $S_x = \sum_{i=1}^{n} x_i$, $S_y = \sum_{i=1}^{n} y_i$, $S_{x^2} = \sum_{i=1}^{n} x_i^2$ and $S_{xy} = \sum_{i=1}^{n} x_iy_i$. In terms of this notation, the two equations are (after dividing by two)

$$S_{x^2}\, m + S_x\, b = S_{xy} \tag{1}$$
$$S_x\, m + n\, b = S_y \tag{2}$$

This is a system of two linear equations in two unknowns. One way[1] to solve them, is to use one of the two equations to solve for one of the two unknowns in terms of the other unknown. For example, equation (2) gives that

$$b = \frac{1}{n}\big(S_y - S_x\, m\big)$$

If we now substitute this into equation (1) we get

$$S_{x^2}\, m + \frac{S_x}{n}\big(S_y - S_x\, m\big) = S_{xy} \implies \left(S_{x^2} - \frac{S_x^2}{n}\right) m = S_{xy} - \frac{S_xS_y}{n}$$

which is a single equation in the single unkown m. We can easily solve it for m. It tells us that

$$m = \frac{nS_{xy} - S_xS_y}{nS_{x^2} - S_x^2}$$

Then substituting this back into $b = \frac{1}{n}\big(S_y - S_x\, m\big)$ gives us

$$b = \frac{S_y}{n} - \frac{S_x}{n}\left(\frac{nS_{xy} - S_xS_y}{nS_{x^2} - S_x^2}\right) = \frac{S_yS_{x^2} - S_xS_{xy}}{nS_{x^2} - S_x^2}$$

1 This procedure is probably not the most efficient one. But it has the advantage that it always works, it does not require any ingenuity on the part of the solver, and it generalizes easily to larger linear systems of equations.

2.10▲ Lagrange Multipliers

▸▸ Stage 1

2.10.1 (∗)
(a) Does the function $f(x,y) = x^2 + y^2$ have a maximum or a minimum on the curve $xy = 1$? Explain.
(b) Find all maxima and minima of $f(x,y)$ on the curve $xy = 1$.

Solution (a) $f(x,y) = x^2 + y^2$ is the square of the distance from the point (x,y) to the origin. There are points on the curve $xy = 1$ that have either x or y arbitrarily large and so whose distance from the origin is arbitrarily large. So f has no maximum on the curve. On the other hand f will have a minimum, achieved at the points of $xy = 1$ that are closest to the origin.

(b) On the curve $xy = 1$ we have $y = \frac{1}{x}$ and hence $f = x^2 + \frac{1}{x^2}$. As

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + \frac{1}{x^2}\right) = 2x - \frac{2}{x^3} = \frac{2}{x^3}(x^4 - 1)$$

and as no point of the curve has $x = 0$, the minimum is achieved when $x = \pm 1$. So the minima are at $\pm(1,1)$, where f takes the value 2.

2.10.2 The surface S is given by the equation $g(x,y,z) = 0$. You are walking on S measuring the function $f(x,y,z)$ as you go. You are currently at the point (x_0,y_0,z_0) where f takes its largest value on S, and are walking in the direction $\mathbf{d} \neq \mathbf{0}$. Because you are walking on S, the vector $\mathbf{d}$ is tangent to S at (x_0,y_0,z_0).
(a) What is the directional derivative of f at (x_0,y_0,z_0) in the direction $\mathbf{d}$? Do not use the method of Lagrange multipliers.
(b) What is the directional derivative of f at (x_0,y_0,z_0) in the direction $\mathbf{d}$? This time use the method of Lagrange multipliers.

Solution (a) As you leave (x_0,y_0,z_0) walking in the direction $\mathbf{d} \neq \mathbf{0}$, f has to be decreasing, or at least not increasing, because f takes its largest value on S at (x_0,y_0,z_0). So the directional derivative

$$D_{\mathbf{d}/|\mathbf{d}|} f(x_0,y_0,z_0) = \boldsymbol{\nabla} f(x_0,y_0,z_0) \cdot \frac{\mathbf{d}}{|\mathbf{d}|} \leqslant 0 \tag{E1}$$

As you leave (x_0,y_0,z_0) walking in the direction $-\mathbf{d} \neq \mathbf{0}$, f also has to be decreasing, or at least not increasing, because f still takes its largest value on S at (x_0,y_0,z_0). So the directional derivative

$$D_{-\mathbf{d}/|\mathbf{d}|} f(x_0,y_0,z_0) = -\boldsymbol{\nabla} f(x_0,y_0,z_0) \cdot \frac{\mathbf{d}}{|\mathbf{d}|} \leqslant 0 \tag{E2}$$

(E1) and (E2) can both be true only if the directional derivative

$$D_{\mathbf{d}/|\mathbf{d}|} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \frac{\mathbf{d}}{|\mathbf{d}|} = 0$$

(b) By Definition 2.7.5 in the CLP-3 text, the directional derivative is

$$D_{\mathbf{d}/|\mathbf{d}|} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \frac{\mathbf{d}}{|\mathbf{d}|}$$

- As (x_0, y_0, z_0) is a local maximum for f on S, the method of Lagrange multipliers, Theorem 2.10.2 in the CLP-3 text, gives that $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$ for some λ.
- By Theorem 2.5.5, the vector $\nabla g(x_0, y_0, z_0)$ is perpendicular to the surface S at (x_0, y_0, z_0), and, in particular, is perpendicular to the vector $\mathbf{d}$, which after all is tangent to the surface S at (x_0, y_0, z_0).

So $\nabla g(x_0, y_0, z_0) \cdot \mathbf{d} = 0$ and the directional derivative

$$D_{\mathbf{d}/|\mathbf{d}|} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \frac{\mathbf{d}}{|\mathbf{d}|} = 0$$

▸▸ Stage 2

2.10.3 Find the maximum and minimum values of the function $f(x, y, z) = x + y - z$ on the sphere $x^2 + y^2 + z^2 = 1$.

Solution We are to find the maximum and minimum of $f(x, y, z) = x + y - z$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. According to the method of Lagrange multipliers, we need to find all solutions to

$$f_x = 1 = 2\lambda x = \lambda g_x \implies x = \frac{1}{2\lambda} \tag{E1}$$

$$f_y = 1 = 2\lambda y = \lambda g_y \implies y = \frac{1}{2\lambda} \tag{E2}$$

$$f_z = -1 = 2\lambda z = \lambda g_z \implies z = -\frac{1}{2\lambda} \tag{E3}$$

$$x^2 + y^2 + z^2 = 1 \implies 3\left(\frac{1}{2\lambda}\right)^2 = 1 \implies \lambda = \pm\frac{\sqrt{3}}{2} \tag{E4}$$

Thus the critical points are $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, where $f = -\sqrt{3}$ and $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, where $f = \sqrt{3}$. So, the max is $f = \sqrt{3}$ and the min is $f = -\sqrt{3}$.

2.10.4 Find a, b and c so that the volume $\frac{4\pi}{3}abc$ of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ passing through the point $(1, 2, 1)$ is as small as possible.

Solution The ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ passes through the point $(1,2,1)$ if and only if $\frac{1}{a^2}+\frac{4}{b^2}+\frac{1}{c^2}=1$. We are to minimize $f(a,b,c)=\frac{4}{3}\pi abc$ subject to the constraint that $g(a,b,c)=\frac{1}{a^2}+\frac{4}{b^2}+\frac{1}{c^2}-1=0$. According to the method of Lagrange multipliers, we need to find all solutions to

$$f_a=\frac{4}{3}\pi bc=-\frac{2\lambda}{a^3}=\lambda g_a \implies \frac{3}{2\pi}\lambda=-a^3bc \tag{E1}$$
$$f_b=\frac{4}{3}\pi ac=-\frac{8\lambda}{b^3}=\lambda g_b \implies \frac{3}{2\pi}\lambda=-\frac{1}{4}ab^3c \tag{E2}$$
$$f_c=\frac{4}{3}\pi ab=-\frac{2\lambda}{c^3}=\lambda g_c \implies \frac{3}{2\pi}\lambda=-abc^3 \tag{E3}$$
$$\frac{1}{a^2}+\frac{4}{b^2}+\frac{1}{c^2}=1 \tag{E4}$$

The equations $-\frac{3}{2\pi}\lambda=a^3bc=\frac{1}{4}ab^3c$ force $b=2a$ (since we want $a,b,c>0$). The equations $-\frac{3}{2\pi}\lambda=a^3bc=abc^3$ force $a=c$. Hence, by (E4),

$$1=\frac{1}{a^2}+\frac{4}{b^2}+\frac{1}{c^2}=\frac{3}{a^2} \implies a=c=\sqrt{3},\ b=2\sqrt{3}$$

2.10.5 (*) Use the Method of Lagrange Multipliers to find the minimum value of $z=x^2+y^2$ subject to $x^2y=1$. At which point or points does the minimum occur?

Solution So we are to minimize $f(x,y)=x^2+y^2$ subject to the constraint $g(x,y)=x^2y-1=0$. According to the method of Lagrange multipliers, we need to find all solutions to

$$f_x=2x=2\lambda xy=\lambda g_x \tag{E1}$$
$$f_y=2y=\lambda x^2=\lambda g_y \tag{E2}$$
$$x^2y=1 \tag{E3}$$

- Equation (E1), $2x(1-\lambda y)=0$, gives that either $x=0$ or $\lambda y=1$.
- But substituting $x=0$ in (E3) gives $0=1$, which is impossible.
- Also note that $\lambda=0$ is impossible, since substituting $\lambda=0$ in (E1) and (E2) gives $x=y=0$, which violates (E3).
- So $y=\frac{1}{\lambda}$.
- Substituting $y=\frac{1}{\lambda}$ into (E2) gives $\frac{2}{\lambda}=\lambda x^2$ or $x^2=\frac{2}{\lambda^2}$. So $x=\pm\frac{\sqrt{2}}{\lambda}$.
- Substituting $y=\frac{1}{\lambda}$, $x=\pm\frac{\sqrt{2}}{\lambda}$ into (E3) gives $\frac{2}{\lambda^3}=1$ or $\lambda^3=2$ or $\lambda=\sqrt[3]{2}$.
- $\lambda=2^{1/3}$ gives $x=\pm 2^{\frac{1}{2}-\frac{1}{3}}=\pm 2^{\frac{1}{6}}$ and $y=2^{-\frac{1}{3}}$.

So the two critical points are $\big(2^{\frac{1}{6}}, 2^{-\frac{1}{3}}\big)$ and $\big(-2^{\frac{1}{6}}, 2^{-\frac{1}{3}}\big)$. For both of these critical points,

$$x^2+y^2=2^{\frac{1}{3}}+2^{-\frac{2}{3}}=2^{\frac{1}{3}}+\frac{1}{2}2^{\frac{1}{3}}=\frac{3}{2}\sqrt[3]{2}=\frac{3}{\sqrt[3]{4}}$$

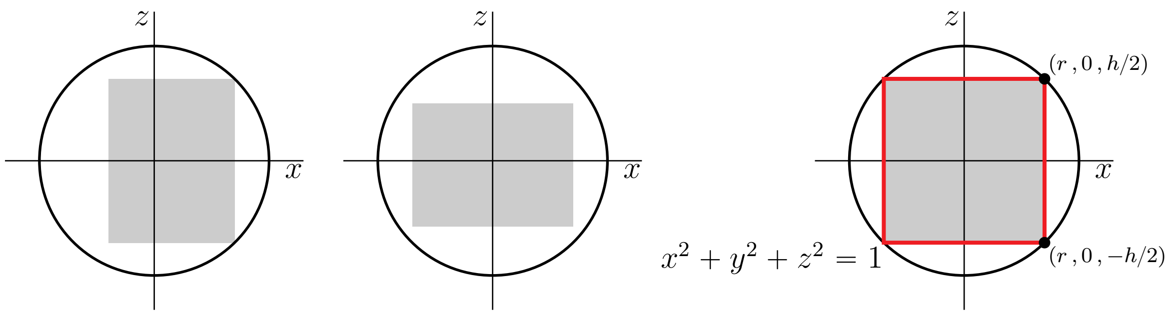
2.10.6 (*) Use the Method of Lagrange Multipliers to find the radius of the base and the height of a right circular cylinder of maximum volume which can be fit inside the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution Let r and h denote the radius and height, respectively, of the cylinder. We can always choose our coordinate system so that the axis of the cylinder is parallel to the z–axis.

- If the axis of the cylinder does not lie exactly on the z–axis, we can enlarge the cylinder sideways. (See the figure on the left below. It shows the $y = 0$ cross–section of the cylinder.) So we can assume that the axis of the cylinder lies on the z–axis
- If the top and/or the bottom of the cylinder does not touch the sphere $x^2 + y^2 + z^2 = 1$, we can enlarge the cylinder vertically. (See the central figure below.)
- So we may assume that the cylinder is

$$\left\{\ (x, y, z)\ \middle|\ x^2 + y^2 \leqslant r^2,\ -h/2 \leqslant z \leqslant h/2\ \right\}$$

with $r^2 + (h/2)^2 = 1$. See the figure on the right below.



So we are to maximize the volume, $f(r, h) = \pi r^2 h$, of the cylinder subject to the constraint $g(r, h) = r^2 + \frac{h^2}{4} - 1 = 0$. According to the method of Lagrange multipliers, we need to find all solutions to

$$f_r = 2\pi rh = 2\lambda r = \lambda g_r \tag{E1}$$

$$f_h = \pi r^2 = \lambda \frac{h}{2} = \lambda g_h \tag{E2}$$

$$r^2 + \frac{h^2}{4} = 1 \tag{E3}$$

Equation (E1), $2r(\pi h - \lambda) = 0$, gives that either $r = 0$ or $\lambda = \pi h$. Clearly $r = 0$ cannot give the maximum volume, so $\lambda = \pi h$. Substituting $\lambda = \pi h$ into (E2) gives

$$\pi r^2 = \frac{1}{2}\pi h^2 \implies r^2 = \frac{h^2}{2}$$

Substituting $r^2 = \frac{h^2}{2}$ into (E3) gives

$$\frac{h^2}{2} + \frac{h^2}{4} = 1 \implies h^2 = \frac{4}{3}$$

Clearly both r and h have to be positive, so $h = \frac{2}{\sqrt{3}}$ and $r = \sqrt{\frac{2}{3}}$.

$\boxed{2.10.7}$ (*) Use the method of Lagrange Multipliers to find the maximum and minimum values of

$$f(x,y) = xy$$

subject to the constraint

$$x^2 + 2y^2 = 1.$$

$\boxed{\text{Solution}}$ For this problem the objective function is $f(x,y) = xy$ and the constraint function is $g(x,y) = x^2 + 2y^2 - 1$. To apply the method of Lagrange multipliers we need ∇f and ∇g. So we start by computing the first order derivatives of these functions.

$$f_x = y \qquad f_y = x \qquad g_x = 2x \qquad g_y = 4y$$

So, according to the method of Lagrange multipliers, we need to find all solutions to

$$y = \lambda(2x) \tag{E1}$$

$$x = \lambda(4y) \tag{E2}$$

$$x^2 + 2y^2 - 1 = 0 \tag{E3}$$

First observe that none of x, y, λ can be zero, because if at least one of them is zero, then (E1) and (E2) force $x = y = 0$, which violates (E3). Dividing (E1) by (E2) gives $\frac{y}{x} = \frac{x}{2y}$ so that $x^2 = 2y^2$ or $x = \pm\sqrt{2}\,y$. Then (E3) gives

$$2y^2 + 2y^2 = 1 \iff y = \pm\frac{1}{2}$$

The method of Lagrange multipliers, Theorem 2.10.2 in the CLP-3 text, gives that the only possible locations of the maximum and minimum of the function f are $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2}\right)$. So the maximum and minimum values of f are $\frac{1}{2\sqrt{2}}$ and $-\frac{1}{2\sqrt{2}}$, respectively.

$\boxed{2.10.8}$ (*) Find the maximum and minimum values of $f(x,y) = x^2 + y^2$ subject to the constraint $x^4 + y^4 = 1$.

$\boxed{\text{Solution}}$ This is a constrained optimization problem with the objective function being $f(x,y) = x^2 + y^2$ and the constraint function being $g(x,y) = x^4 + y^4 - 1$. By Theorem 2.10.2 in the CLP-3 text, any minimum or maximum (x,y) must obey the Lagrange multiplier equations

$$f_x = 2x = 4\lambda x^3 = \lambda g_x \tag{E1}$$

$$f_y = 2y = 4\lambda y^3 = \lambda g_y \tag{E2}$$

$$x^4 + y^4 = 1 \tag{E3}$$

for some real number λ. By equation (E1), $2x(1 - 2\lambda x^2) = 0$, which is obeyed if and only if at least one of $x = 0$, $2\lambda x^2 = 1$ is obeyed. Similarly, by equation (E2), $2y(1 - 2\lambda y^2) = 0$, which is obeyed if and only if at least one of $y = 0$, $2\lambda y^2 = 1$ is obeyed.

- If $x=0$, (E3) reduces to $y^4=1$ or $y=\pm 1$. At both $\big(0,\pm 1\big)$ we have $f\big(0,\pm 1\big)=1$.
- If $y=0$, (E3) reduces to $x^4=1$ or $x=\pm 1$. At both $\big(\pm 1,0\big)$ we have $f\big(\pm 1,0\big)=1$.
- If both x and y are nonzero, we have $x^2=\frac{1}{2\lambda}=y^2$. Then (E3) reduces to
$$2x^4=1$$
so that $x^2=y^2=\frac{1}{\sqrt{2}}$ and $x=\pm 2^{-1/4}$, $y=\pm 2^{-1/4}$. At all four of these points, we have $f=\sqrt{2}$.

So the minimum value of f on $x^4+y^4=1$ is 1 and the maximum value of f on $x^4+y^4=1$ is $\sqrt{2}$.

$\boxed{2.10.9}$ ($*$) Use Lagrange multipliers to find the points on the sphere $z^2+x^2+y^2-2y-10=0$ closest to and farthest from the point $(1,-2,1)$.

$\boxed{\text{Solution}}$ The function $f(x,y,z)=(x-1)^2+(y+2)^2+(z-1)^2$ gives the square of the distance from the point (x,y,z) to the point $(1,-2,1)$. So it suffices to find the (x,y,z) which minimizes $f(x,y,z)=(x-1)^2+(y+2)^2+(z-1)^2$ subject to the constraint $g(x,y,z)=z^2+x^2+y^2-2y-10=0$. By Theorem 2.10.2 in the CLP-3 text, any local minimum or maximum (x,y,z) must obey the Lagrange multiplier equations

$$f_x=2(x-1)=2\lambda x=\lambda g_x \tag{E1}$$
$$f_y=2(y+2)=2\lambda(y-1)=\lambda g_y \tag{E2}$$
$$f_z=2(z-1)=2\lambda z=\lambda g_z \tag{E3}$$
$$z^2+x^2+y^2-2y=10 \tag{E4}$$

for some real number λ. Now

$$\begin{aligned}
\text{(E1)} &\implies x=\frac{1}{1-\lambda}\\
\text{(E2)} &\implies y=-\frac{2+\lambda}{1-\lambda}\\
\text{(E3)} &\implies z=\frac{1}{1-\lambda}
\end{aligned}$$

(Note that λ cannot be 1, because if it were (E1) would reduce to $-2=0$.) Substituting these into (E4), and using that

$$y-2=-\frac{2+\lambda}{1-\lambda}-\frac{2-2\lambda}{1-\lambda}=-\frac{4-\lambda}{1-\lambda}$$

gives

$$\begin{aligned}
&\frac{1}{(1-\lambda)^2}+\frac{1}{(1-\lambda)^2}+\frac{2+\lambda}{1-\lambda}\,\frac{4-\lambda}{1-\lambda}=10\\
\iff& 2+(2+\lambda)(4-\lambda)=10(1-\lambda)^2\\
\iff& 11\lambda^2-22\lambda=0\\
\iff& \lambda=0 \text{ or } \lambda=2
\end{aligned}$$

When $\lambda = 0$, we have $(x, y, z) = (1, -2, 1)$ (nasty!), which gives distance zero and so is certainly the closest point. When $\lambda = 2$, we have $(x, y, z) = (-1, 4, -1)$, which does not give distance zero and so is certainly the farthest point.

2.10.10 (*) Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = x^2 + y^2 - \frac{1}{20}z^2$ on the curve of intersection of the plane $x + 2y + z = 10$ and the paraboloid $x^2 + y^2 - z = 0$.

Solution We are to maximize and minimize $f(x, y, z) = x^2 + y^2 - \frac{1}{20}z^2$ subject to the constraints $g(x, y, z) = x + 2y + z - 10 = 0$ and $h(x, y, z) = x^2 + y^2 - z = 0$. By Theorem 2.10.8 in the CLP-3 text, any local minimum or maximum (x, y, z) must obey the double Lagrange multiplier equations

$$f_x = 2x = \lambda + 2\mu x = \lambda g_x + \mu h_x \tag{E1}$$
$$f_y = 2y = 2\lambda + 2\mu y = \lambda g_y + \mu h_y \tag{E2}$$
$$f_z = -\frac{z}{10} = \lambda - \mu = \lambda g_z + \mu h_z \tag{E3}$$
$$x + 2y + z = 10 \tag{E4}$$
$$x^2 + y^2 - z = 0 \tag{E5}$$

for some real numbers λ and μ.

Equation (E1) gives $2(1-\mu)x = \lambda$ and equation (E2) gives $(1-\mu)y = \lambda$. So

$$2(1-\mu)x = (1-\mu)y \implies (1-\mu)(2x - y) = 0$$

So at least one of $\mu = 1$ and $y = 2x$ must be true.

- If $\mu = 1$, equations (E1) and (E2) both reduce to $\lambda = 0$ and then the remaining equations reduce to

$$-\frac{z}{10} = -1 \tag{E3}$$
$$x + 2y + z = 10 \tag{E4}$$
$$x^2 + y^2 - z = 0 \tag{E5}$$

  Then (E3) implies $z = 10$, and (E4) in turn implies $x + 2y + 10 = 10$ so that $x = -2y$. Finally, substituting $z = 10$ and $x = -2y$ into (E5) gives

$$4y^2 + y^2 - 10 = 0 \iff 5y^2 = 10 \iff y = \pm\sqrt{2}$$

- If $y = 2x$, equations (E4) and (E5) reduce to

$$5x + z = 10 \tag{E4}$$
$$5x^2 - z = 0 \tag{E5}$$

  Substituting $z = 5x^2$, from (E5), into (E4) gives

$$5x^2 + 5x - 10 = 0 \iff x^2 + x - 2 = 0 \iff (x+2)(x-1) = 0$$

So we have either $x = -2$, $y = 2x = -4$, $z = 5x^2 = 20$ or $x = 1$, $y = 2x = 2$, $z = 5x^2 = 5$. (In both cases, we could now solve (E1) and (E3) for λ and μ, but we don't care what the values of λ and μ are.)

So we have the following candidates for the locations of the min and max

point	$(-2\sqrt{2}, \sqrt{2}, 10)$	$(2\sqrt{2}, -\sqrt{2}, 10)$	$(-2, -4, 20)$	$(1, 2, 5)$
value of f	$8+2-5$	$8+2-5$	$4+16-20$	$1+4-\frac{25}{20}$
	max	max	min	

So the maximum is 5 and the minimum is 0.

$\boxed{2.10.11}$ (*) Find the point $P = (x, y, z)$ (with x, y and $z > 0$) on the surface $x^3y^2z = 6\sqrt{3}$ that is closest to the origin.

$\boxed{\text{Solution}}$ The function $f(x, y, z) = x^2 + y^2 + z^2$ gives the square of the distance from the point (x, y, z) to the origin. So it suffices to find the (x, y, z) (in the first octant) which minimizes $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = x^3y^2z - 6\sqrt{3} = 0$. To start, we'll find the minimizers in all of $\mathbb{R}^3$. By Theorem 2.10.2 in the CLP-3 text, any local minimum or maximum (x, y, z) must obey the Lagrange multiplier equations

$$f_x = 2x = 3\lambda x^2y^2z = \lambda g_x \tag{E1}$$
$$f_y = 2y = 2\lambda x^3yz = \lambda g_y \tag{E2}$$
$$f_z = 2z = \lambda x^3y^2 = \lambda g_z \tag{E3}$$
$$x^3y^2z = 6\sqrt{3} \tag{E4}$$

for some real number λ.

Multiplying (E1) by $2x$, (E2) by $3y$, and (E3) by $6z$ gives

$$4x^2 = 6\lambda x^3y^2z \tag{E1'}$$
$$6y^2 = 6\lambda x^3y^2z \tag{E2'}$$
$$12z^2 = 6\lambda x^3y^2z \tag{E3'}$$

The three right hand sides are all identical. So the three left hand sides must all be equal.

$$4x^2 = 6y^2 = 12z^2 \iff x = \pm\sqrt{3}\,z,\ y = \pm\sqrt{2}\,z$$

Equation (E4) forces x and z to have the same sign. So we must have $x = \sqrt{3}\,z$ and $y = \pm\sqrt{2}\,z$. Substituting this into (E4) gives

$$\big(\sqrt{3}\,z\big)^3\big(\pm\sqrt{2}\,z\big)^2 z = 6\sqrt{3} \iff z^6 = 1 \iff z = \pm 1$$

So our minimizer (in all of $\mathbb{R}^3$) must be one of $\big(\sqrt{3}\,,\ \pm\sqrt{2}\,,\ 1\big)$ or $\big(-\sqrt{3}\,,\ \pm\sqrt{2}\,,\ -1\big)$. All of these points give exactly the same value of f (namely $3+2+1 = 6$). That is all four points are a distance $\sqrt{6}$ from the origin and all other points on $x^3y^2z = 6\sqrt{3}$ have distance from the origin strictly greater than $\sqrt{6}$. So the first octant point on $x^3y^2z = 6\sqrt{3}$ that is closest to the origin is $\big(\sqrt{3}\,,\ \sqrt{2}\,,\ 1\big)$.

2.10.12 (∗) Find the maximum value of $f(x,y,z) = xyz$ on the ellipsoid

$$g(x,y,z) = x^2 + xy + y^2 + 3z^2 = 9$$

Specify all points at which this maximum value occurs.

Solution This is a constrained optimization problem with the objective function being

$$f(x,y,z) = xyz$$

and the constraint function being

$$G(x,y,z) = x^2 + xy + y^2 + 3z^2 - 9$$

By Theorem 2.10.2 in the CLP-3 text, any local minimum or maximum (x,y,z) must obey the Lagrange multiplier equations

$$\begin{aligned} f_x = yz &= \lambda(2x+y) = \lambda G_x && \text{(E1)} \\ f_y = xz &= \lambda(2y+x) = \lambda G_y && \text{(E2)} \\ f_z = xy &= 6\lambda z = \lambda G_z && \text{(E3)} \\ x^2 + xy + y^2 + 3z^2 &= 9 && \text{(E4)} \end{aligned}$$

for some real number λ.

- If $\lambda = 0$, then, by (E1), $yz = 0$ so that $f(x,y,z) = xyz = 0$. This cannot possibly be the maximum value of f because there are points (x,y,z) on $g(x,y,z) = 9$ (for example $x = y = 1$, $z = \sqrt{2}$) with $f(x,y,z) > 0$.
- If $\lambda \neq 0$, then multiplying (E1) by x, (E2) by y, and (E3) by z gives

$$\begin{aligned} xyz = \lambda(2x^2 + xy) = \lambda(2y^2 + xy) = 6\lambda z^2 &\implies 2x^2 + xy = 2y^2 + xy = 6z^2 \\ &\implies x = \pm y,\ z^2 = \frac{1}{6}(2x^2 + xy) \end{aligned}$$

  - If $x = y$, then $z^2 = \frac{x^2}{2}$ and, by (E4)

$$x^2 + x^2 + x^2 + \frac{3}{2}x^2 = 9 \implies x^2 = 2 \implies x = y = \pm\sqrt{2},\ z = \pm 1$$

    For these points

$$f(x,y,z) = 2z = \begin{cases} 2 & \text{if } z = 1 \\ -2 & \text{if } z = -1 \end{cases}$$

  - If $x = -y$, then $z^2 = \frac{x^2}{6}$ and, by (E4)

$$x^2 - x^2 + x^2 + \frac{x^2}{2} = 9 \implies x^2 = 6 \implies x = -y = \pm\sqrt{6},\ z = \pm 1$$

    For these points

$$f(x,y,z) = -6z = \begin{cases} -6 & \text{if } z = 1 \\ 6 & \text{if } z = -1 \end{cases}$$

So the maximum is 6 and is achieved at $\big(\sqrt{6}\,,\,-\sqrt{6}\,,\,-1\big)$ and $\big(-\sqrt{6}\,,\,\sqrt{6}\,,\,-1\big)$.

$\boxed{2.10.13}$ (*) Find the radius of the largest sphere centred at the origin that can be inscribed inside (that is, enclosed inside) the ellipsoid

$$2(x+1)^2 + y^2 + 2(z-1)^2 = 8$$

$\boxed{\text{Solution}}$ In order for a sphere of radius r centred on the origin to be enclosed in the ellipsoid, every point of the ellipsoid must be at least a distance r from the origin. So the largest allowed r is the distance from the origin to the nearest point on the ellipsoid.

We have to minimize $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $g(x,y,z) = 2(x+1)^2 + y^2 + 2(z-1)^2 - 8$. By Theorem 2.10.2 in the CLP-3 text, any local minimum or maximum (x,y,z) must obey the Lagrange multiplier equations

$$\begin{aligned} f_x = 2x &= 4\lambda(x+1) = \lambda g_x && \text{(E1)}\\ f_y = 2y &= 2\lambda y = \lambda g_y && \text{(E2)}\\ f_z = 2z &= 4\lambda(z-1) = \lambda g_z && \text{(E3)}\\ 2(x+1)^2 + y^2 + 2(z-1)^2 &= 8 && \text{(E4)} \end{aligned}$$

for some real number λ.

By equation (E2), $2y(1-\lambda) = 0$, which is obeyed if and only if at least one of $y = 0$, $\lambda = 1$ is obeyed.

- If $y = 0$, the remaining equations reduce to

$$\begin{aligned} x &= 2\lambda(x+1) && \text{(E1)}\\ z &= 2\lambda(z-1) && \text{(E3)}\\ (x+1)^2 + (z-1)^2 &= 4 && \text{(E4)} \end{aligned}$$

  Note that 2λ cannot be 1 — if it were, (E1) would reduce to $0 = 1$. So equation (E1) gives

$$x = \frac{2\lambda}{1-2\lambda} \qquad \text{or} \qquad x+1 = \frac{1}{1-2\lambda}$$

  Equation (E3) gives

$$z = -\frac{2\lambda}{1-2\lambda} \qquad \text{or} \qquad z-1 = -\frac{1}{1-2\lambda}$$

  Substituting $x+1 = \frac{1}{1-2\lambda}$ and $z-1 = -\frac{1}{1-2\lambda}$ into (E4) gives

$$\begin{aligned} \frac{1}{(1-2\lambda)^2} + \frac{1}{(1-2\lambda)^2} = 4 &\iff \frac{1}{(1-2\lambda)^2} = 2\\ &\iff \frac{1}{1-2\lambda} = \pm\sqrt{2} \end{aligned}$$

  So we now have two candidates for the location of the max and min, namely $(x,y,z) = \big(-1+\sqrt{2}, 0, 1-\sqrt{2}\big)$ and $(x,y,z) = \big(-1-\sqrt{2}, 0, 1+\sqrt{2}\big)$.

- If $\lambda = 1$, the remaining equations reduce to

$$x = 2(x+1) \tag{E1}$$
$$z = 2(z-1) \tag{E3}$$
$$2(x+1)^2 + y^2 + 2(z-1)^2 = 8 \tag{E4}$$

  Equation (E1) gives $x = -2$ and equation (E3) gives $z = 2$. Substituting these into (E4) gives

$$2 + y^2 + 2 = 8 \iff y^2 = 4 \iff y = \pm 2$$

So we have the following candidates for the locations of the min and max

point	$(-1+\sqrt{2}, 0, 1-\sqrt{2})$	$(-1-\sqrt{2}, 0, 1+\sqrt{2})$	$(-2, 2, 2)$	$(-2, -2, 2)$
value of f	$2(3-2\sqrt{2})$	$2(3+2\sqrt{2})$	12	12
	min		max	max

Recalling that $f(x, y, z)$ is the square of the distance from (x, y, z) to the origin, the maximum allowed radius for the enclosed sphere is $\sqrt{6-4\sqrt{2}} \approx 0.59$.

$\boxed{2.10.14}$ (*) Let C be the intersection of the plane $x + y + z = 2$ and the sphere $x^2 + y^2 + z^2 = 2$.

(a) Use Lagrange multipliers to find the maximum value of $f(x, y, z) = z$ on C.

(b) What are the coordinates of the lowest point on C?

$\boxed{\text{Solution}}$ (a) We are to maximize $f(x, y, z) = z$ subject to the constraints $g(x, y, z) = x + y + z - 2 = 0$ and $h(x, y, z) = x^2 + y^2 + z^2 - 2 = 0$. By Theorem 2.10.8 in the CLP-3 text, any local minimum or maximum (x, y, z) must obey the double Lagrange multiplier equations

$$f_x = 0 = \lambda + 2\mu x = \lambda g_x + \mu h_x \tag{E1}$$
$$f_y = 0 = \lambda + 2\mu y = \lambda g_y + \mu h_y \tag{E2}$$
$$f_z = 1 = \lambda + 2\mu z = \lambda g_z + \mu h_z \tag{E3}$$
$$x + y + z = 2 \tag{E4}$$
$$x^2 + y^2 + z^2 = 2 \tag{E5}$$

for some real numbers λ and μ. Subtracting (E2) from (E1) gives $2\mu(x - y) = 0$. So at least one of $\mu = 0$ and $y = x$ must be true.

- If $\mu = 0$, equations (E1) and (E3) reduce to $\lambda = 0$ and $\lambda = 1$, which is impossible. So $\mu \neq 0$.

- If $y = x$, equations (E2) through (E5) reduce to

$$\lambda + 2\mu x = 0 \tag{E2}$$
$$\lambda + 2\mu z = 1 \tag{E3}$$
$$2x + z = 2 \tag{E4}$$
$$2x^2 + z^2 = 2 \tag{E5}$$

By (E4), $x = \frac{2-z}{2}$. Substituting this into (E5) gives

$$2\frac{(2-z)^2}{4} + z^2 = 2 \iff (2-z)^2 + 2z^2 = 4 \iff 3z^2 - 4z = 0$$
$$\iff z = 0,\ \frac{4}{3}$$

The maximum z is thus $\frac{4}{3}$.

(b) Presumably the "lowest point" is the point with the minimal z–coordinate. By our work in part (a), we have that the minimal value of z on C is 0. We have also already seen in part (a) that $y = x$. When $z = 0$, (E4) reduces to $2x = 2$. So the desired point is $(1, 1)$.

2.10.15 (*)

(a) Use Lagrange multipliers to find the extreme values of

$$f(x, y, z) = (x-2)^2 + (y+2)^2 + (z-4)^2$$

on the sphere $x^2 + y^2 + z^2 = 6$.

(b) Find the point on the sphere $x^2 + y^2 + z^2 = 6$ that is farthest from the point $(2, -2, 4)$.

Solution (a) This is a constrained optimization problem with the objective function being $f(x, y, z) = (x-2)^2 + (y+2)^2 + (z-4)^2$ and the constraint function being $g(x, y, z) = x^2 + y^2 + z^2 - 6$. By Theorem 2.10.2 in the CLP-3 text, any local minimum or maximum (x, y, z) must obey the Lagrange multiplier equations

$$f_x = 2(x-2) = 2\lambda x = \lambda g_x \tag{E1}$$
$$f_y = 2(y+2) = 2\lambda y = \lambda g_y \tag{E2}$$
$$f_z = 2(z-4) = 2\lambda z = \lambda g_z \tag{E3}$$
$$x^2 + y^2 + z^2 = 6 \tag{E4}$$

for some real number λ. Simplifying

$$x - 2 = \lambda x \tag{E1}$$
$$y + 2 = \lambda y \tag{E2}$$
$$z - 4 = \lambda z \tag{E2}$$
$$x^2 + y^2 + z^2 = 6 \tag{E4}$$

Note that we cannot have $\lambda = 1$, because then (E1) would reduce to $-2 = 0$. Substituting $x = \frac{2}{1-\lambda}$, from (E1), and $y = \frac{-2}{1-\lambda}$, from (E2), and $z = \frac{4}{1-\lambda}$, from (E3), into (E4) gives

$$\frac{4}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} + \frac{16}{(1-\lambda)^2} = 6 \iff (1-\lambda)^2 = 4 \iff 1 - \lambda = \pm 2$$

and hence

$$(x, y, z) = \pm\frac{(2, -2, 4)}{2} = \pm(1, -1, 2)$$

So we have the following candidates for the locations of the min and max

point	$(1,-1,2)$	$-(1,-1,2)$
value of f	6	54
	min	max

So the minimum is 6 and the maximum is 54.

(b) $f(x,y,z)$ is the square of the distance from (x,y,z) to $(2,-2,4)$. So the point on the sphere $x^2+y^2+z^2=6$ that is farthest from the point $(2,-2,4)$ is the point from part (a) that maximizes f, which is $(-1,1,-2)$.

2.10.16 (*)
(a) Find the minimum of the function

$$f(x,y,z) = (x-2)^2 + (y-1)^2 + z^2$$

subject to the constraint $x^2+y^2+z^2=1$, using the method of Lagrange multipliers.
(b) Give a geometric interpretation of this problem.

Solution (a) This is a constrained optimization problem with the objective function being $f(x,y,z) = (x-2)^2+(y-1)^2+z^2$ and the constraint function being $g(x,y,z) = x^2+y^2+z^2-1$. By Theorem 2.10.2 in the CLP-3 text, any local minimum or maximum (x,y,z) must obey the Lagrange multiplier equations

$$f_x = 2(x-2) = 2\lambda x = \lambda g_x \qquad \text{(E1)}$$
$$f_y = 2(y-1) = 2\lambda y = \lambda g_y \qquad \text{(E2)}$$
$$f_z = 2z = 2\lambda z = \lambda g_z \qquad \text{(E3)}$$
$$x^2+y^2+z^2 = 1 \qquad \text{(E4)}$$

for some real number λ. By equation (E3), $2z(1-\lambda)=0$, which is obeyed if and only if at least one of $z=0$, $\lambda=1$ is obeyed.

- If $z=0$ and $\lambda \neq 1$, the remaining equations reduce to

$$x-2 = \lambda x \qquad \text{(E1)}$$
$$y-1 = \lambda y \qquad \text{(E2)}$$
$$x^2+y^2 = 1 \qquad \text{(E4)}$$

Substituting $x = \frac{2}{1-\lambda}$, from (E1), and $y = \frac{1}{1-\lambda}$, from (E2), into (E3) gives

$$\frac{4}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} = 1 \iff (1-\lambda)^2 = 5 \iff 1-\lambda = \pm\sqrt{5}$$

and hence

$$(x,y,z) = \pm\frac{1}{\sqrt{5}}(2,1,0)$$

To aid in the evaluation of $f(x,y,z)$ at these points note that, at these points,

$$x-2=\lambda x=\frac{2\lambda}{1-\lambda}, \qquad y-1=\lambda y=\frac{\lambda}{1-\lambda}$$
$$\implies f(x,y,z)=\frac{4\lambda^2}{(1-\lambda)^2}+\frac{\lambda^2}{(1-\lambda)^2}=\frac{5\lambda^2}{(1-\lambda)^2}=\lambda^2=\big(1\mp\sqrt{5}\big)^2$$

- If $\lambda=1$, the remaining equations reduce to

$$x-2=x \tag{E1}$$
$$y-1=y \tag{E2}$$
$$x^2+y^2+z^2=1 \tag{E4}$$

Since $-2\neq 0$ and $-1\neq 0$, neither (E1) nor (E2) has any solution.

So we have the following candidates for the locations of the min and max

point	$\frac{1}{\sqrt{5}}(2,1,0)$	$-\frac{1}{\sqrt{5}}(2,1,0)$
value of f	$\big(1-\sqrt{5}\big)^2$	$\big(1+\sqrt{5}\big)^2$
	min	max

So the minimum is $\big(\sqrt{5}-1\big)^2=6-2\sqrt{5}$.

(b) The function $f(x,y,z)=(x-2)^2+(y-1)^2+z^2$ is the square of the distance from the point (x,y,z) to the point $(2,1,0)$. So the minimum of f subject to the constraint $x^2+y^2+z^2=1$ is the square of the distance from $(2,1,0)$ to the point on the sphere $x^2+y^2+z^2=1$ that is nearest $(2,1,0)$.

$\boxed{2.10.17}$ (∗) Use Lagrange multipliers to find the minimum and maximum values of $(x+z)e^y$ subject to $x^2+y^2+z^2=6$.

$\boxed{\text{Solution}}$ For this problem the objective function is $f(x,y,z)=(x+z)e^y$ and the constraint function is $g(x,y,z)=x^2+y^2+z^2-6$. To apply the method of Lagrange multipliers we need ∇f and ∇g. So we start by computing the first order derivatives of these functions.

$$f_x=e^y \qquad f_y=(x+z)e^y \qquad f_z=e^y \qquad g_x=2x \qquad g_y=2y \qquad g_z=2z$$

So, according to the method of Lagrange multipliers, we need to find all solutions to

$$e^y=\lambda(2x) \tag{E1}$$
$$(x+z)e^y=\lambda(2y) \tag{E2}$$
$$e^y=\lambda(2z) \tag{E3}$$
$$x^2+y^2+z^2-6=0 \tag{E4}$$

First notice that, since $e^y\neq 0$, equation (E1) guarantees that $\lambda\neq 0$ and $x\neq 0$ and equation (E3) guarantees that $z\neq 0$ too.

- So dividing (E1) by (E3) gives $\frac{x}{z} = 1$ and hence $x = z$.
- Then subbing $x = z$ into (E2) gives $2ze^y = \lambda(2y)$. Dividing this equation by (E3) gives $2z = \frac{y}{z}$ or $y = 2z^2$.
- Then subbing $x = z$ and $y = 2z^2$ into (E4) gives

$$z^2 + 4z^4 + z^2 - 6 = 0 \iff 4z^4 + 2z^2 - 6 = 0 \iff (2z^2 + 3)(2z^2 - 2) = 0$$

- As $2z^2 + 3 > 0$, we must have $2z^2 - 2 = 0$ or $z = \pm 1$.

Recalling that $x = z$ and $y = 2z^2$, the method of Lagrange multipliers, Theorem 2.10.2 in the CLP-3 text, gives that the only possible locations of the maximum and minimum of the function f are $(1, 2, 1)$ and $(-1, 2, -1)$. To complete the problem, we only have to compute f at those points.

point	$(1,2,1)$	$(-1,2,-1)$
value of f	$2e^2$	$-2e^2$
	max	min

Hence the maximum value of $(x + z)e^y$ on $x^2 + y^2 + z^2 = 6$ is $2e^2$ and the minimum value is $-2e^2$.

$\boxed{2.10.18}$ (*) Find the points on the ellipse $2x^2 + 4xy + 5y^2 = 30$ which are closest to and farthest from the origin.

$\boxed{\text{Solution}}$ Let (x, y) be a point on $2x^2 + 4xy + 5y^2 = 30$. We wish to maximize and minimize $x^2 + y^2$ subject to $2x^2 + 4xy + 5y^2 = 30$. Define
$L(x, y, \lambda) = x^2 + y^2 - \lambda(2x^2 + 4xy + 5y^2 - 30)$. Then

$$\begin{aligned}
0 = L_x &= 2x - \lambda(4x + 4y) &&\implies & (1 - 2\lambda)x - 2\lambda y &= 0 && (1)\\
0 = L_y &= 2y - \lambda(4x + 10y) &&\implies & -2\lambda x + (1 - 5\lambda)y &= 0 && (2)\\
0 = L_\lambda &= 2x^2 + 4xy + 5y^2 - 30
\end{aligned}$$

Note that λ cannot be zero because if it is, (1) forces $x = 0$ and (2) forces $y = 0$, but $(0, 0)$ is not on the ellipse. So equation (1) gives $y = \frac{1-2\lambda}{2\lambda}x$. Substituting this into equation (2) gives $-2\lambda x + \frac{(1-5\lambda)(1-2\lambda)}{2\lambda}x = 0$. To get a nonzero (x, y) we need

$$-2\lambda + \frac{(1 - 5\lambda)(1 - 2\lambda)}{2\lambda} = 0 \iff 0 = -4\lambda^2 + (1 - 5\lambda)(1 - 2\lambda) = 6\lambda^2 - 7\lambda + 1 = (6\lambda - 1)(\lambda - 1)$$

So λ must be either 1 or $\frac{1}{6}$. Substituting these into either (1) or (2) gives

$$\begin{aligned}
\lambda = 1 &\implies -x - 2y = 0 \implies x = -2y \implies 8y^2 - 8y^2 + 5y^2 = 30 \implies y = \pm\sqrt{6}\\
\lambda = \frac{1}{6} &\implies \frac{2}{3}x - \frac{1}{3}y = 0 \implies y = 2x \implies 2x^2 + 8x^2 + 20x^2 = 30 \implies x = \pm 1
\end{aligned}$$

The farthest points are $\pm\sqrt{6}(-2, 1)$. The nearest points are $\pm(1, 2)$.

$\boxed{2.10.19}$ Find the ends of the major and minor axes of the ellipse $3x^2 - 2xy + 3y^2 = 4$.

$\boxed{\text{Solution}}$ Let (x, y) be a point on $3x^2 - 2xy + 3y^2 = 4$. This point is at the end of a major axis when it maximizes its distance from the centre, $(0, 0)$, of the ellipse. It is at the end of a minor axis when it minimizes its distance from $(0, 0)$. So we wish to maximize and minimize $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = 3x^2 - 2xy + 3y^2 - 4 = 0$. According to the method of Lagrange multipliers, we need to find all solutions to

$$f_x = 2x = \lambda(6x - 2y) \quad = \lambda g_x \quad \implies \quad (1 - 3\lambda)x + \lambda y = 0 \tag{E1}$$

$$f_y = 2y = \lambda(-2x + 6y) = \lambda g_y \quad \implies \quad \lambda x + (1 - 3\lambda)y = 0 \tag{E2}$$

$$3x^2 - 2xy + 3y^2 = 4 \tag{E3}$$

To start, let's concentrate on the first two equations. Pretend for a couple of minutes, that we already know the value of λ and are trying to find x and y. The system of equations $(1 - 3\lambda)x + \lambda y = 0$, $\lambda x + (1 - 3\lambda)y = 0$ has one obvious solution. Namely $x = y = 0$. But this solution is not acceptable because it does not satisfy the equation of the ellipse. If you have already taken a linear algebra course, you know that a system of two linear homogeneous equations in two unknowns has a nonzero solution if and only if the determinant of the matrix of coefficients is zero. (You use this when you find eigenvalues and eigenvectors.) For the equations of interest, this is

$$\det \begin{bmatrix} 1 - 3\lambda & \lambda \\ \lambda & 1 - 3\lambda \end{bmatrix} = (1 - 3\lambda)^2 - \lambda^2 = (1 - 2\lambda)(1 - 4\lambda) = 0 \implies \lambda = \frac{1}{2}, \frac{1}{4}$$

Even if you have not already taken a linear algebra course, you also come to this conclusion directly when you try to solve the equations. Note that λ cannot be zero because if it is, (E1) forces $x = 0$ and (E2) forces $y = 0$. So equation (E1) gives $y = -\frac{1-3\lambda}{\lambda}x$. Substituting this into equation (E2) gives $\lambda x - \frac{(1-3\lambda)^2}{\lambda}x = 0$. To get a nonzero (x, y) we need

$$\lambda - \frac{(1 - 3\lambda)^2}{\lambda} = 0 \iff \lambda^2 - (1 - 3\lambda)^2 = 0$$

By either of these two methods, we now know that λ must be either $\frac{1}{2}$ or $\frac{1}{4}$. Substituting these into either (E1) or (E2) and then using (E3) gives

$$\lambda = \frac{1}{2} \implies -\frac{1}{2}x + \frac{1}{2}y = 0 \implies x = y \implies 3x^2 - 2x^2 + 3x^2 = 4 \implies x = \pm 1$$

$$\lambda = \frac{1}{4} \implies \frac{1}{4}x + \frac{1}{4}y = 0 \implies x = -y \implies 3x^2 + 2x^2 + 3x^2 = 4 \implies x = \pm\frac{1}{\sqrt{2}}$$

The ends of the minor axes are $\pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. The ends of the major axes are $\pm(1, 1)$.

$\boxed{2.10.20}$ $(*)$ A closed rectangular box with a volume of 96 cubic meters is to be constructed of two materials. The material for the top costs twice as much per square meter as that for the sides and bottom. Use the method of Lagrange multipliers to find the dimensions of the least expensive box.

Solution Let the box have dimensions $x \times y \times z$. Use units of money so that the sides and bottom cost one unit per square meter and the top costs two units per square meter. Then the top costs $2xy$, the bottom costs xy and the four sides cost $2xz + 2yz$. We are to find the x, y and z that minimize the cost $f(x,y,z) = 2xy + xy + 2xz + 2yz$ subject to the constraint that $g(x,y,z) = xyz - 96 = 0$. By the method of Lagrange multipliers (Theorem 2.10.2 in the CLP-3 text), the minimizing x, y, z must obey

$$\begin{aligned} f_x &= 3y + 2z = \lambda yz = \lambda g_x \\ f_y &= 3x + 2z = \lambda xz = \lambda g_y \\ f_z &= 2x + 2y = \lambda xy = \lambda g_z \\ xyz &- 96 = 0 \end{aligned}$$

Multiplying the first equation by x, the second equation by y and the third equation by z and then substituting in $xyz = 96$ gives

$$\begin{aligned} 3xy + 2xz &= 96\lambda \\ 3xy + 2yz &= 96\lambda \\ 2xz + 2yz &= 96\lambda \end{aligned}$$

Subtracting the second equation from the first gives $2z(x - y) = 0$. Since $z = 0$ is impossible, we must have $x = y$. Substituting this in,

$$3x^2 + 2xz = 96\lambda \qquad 4xz = 96\lambda$$

Subtracting,

$$\begin{aligned} 3x^2 - 2xz = 0 \implies z = \frac{3}{2}x \implies 96 = xyz = \frac{3}{2}x^3 \implies x^3 = 64 \\ \implies x = y = 4,\ z = 6 \text{ meters} \end{aligned}$$

2.10.21 (∗) Consider the unit sphere

$$S = \left\{ (x,y,z) \mid x^2 + y^2 + z^2 = 1 \right\}$$

in $\mathbb{R}^3$. Assume that the temperature at a point (x,y,z) of S is

$$T(x,y,z) = 40xy^2z$$

Find the hottest and coldest temperatures on S.

Solution We are to find the x, y and z that minimize the temperature $T(x,y,z) = 40xy^2z$ subject to the constraint that $g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$. By the method of Lagrange multipliers (Theorem 2.10.2 in the CLP-3 text), the minimizing x, y, z must obey

$$\begin{aligned} T_x &= 40y^2z = \lambda(2x) = \lambda g_x \\ T_y &= 80xyz = \lambda(2y) = \lambda g_y \\ T_z &= 40xy^2 = \lambda(2z) = \lambda g_z \\ x^2 + y^2 &+ z^2 - 1 = 0 \end{aligned}$$

Multiplying the first equation by x, the second equation by $y/2$ and the third equation by z gives

$$\begin{aligned} 40xy^2z &= 2x^2\lambda \\ 40xy^2z &= y^2\lambda \\ 40xy^2z &= 2z^2\lambda \end{aligned}$$

Hence we must have

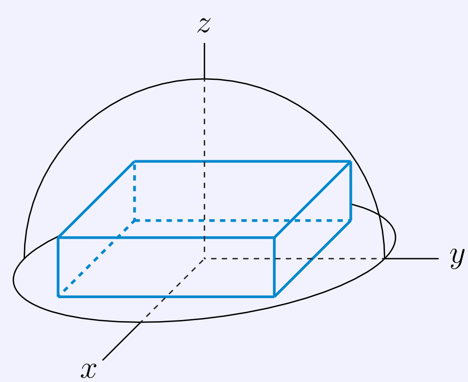
$$2x^2\lambda = y^2\lambda = 2z^2\lambda$$

- If $\lambda = 0$, then $40y^2z = 0,\ 80xyz = 0,\ 40xy^2 = 0$ which is possible only if at least one of x, y, z is zero so that $T(x, y, z) = 0$.
- If $\lambda \neq 0$, then

$$\begin{aligned} 2x^2 = y^2 = 2z^2 &\implies 1 = x^2 + y^2 + z^2 = x^2 + 2x^2 + x^2 = 4x^2 \\ &\implies x = \pm\tfrac{1}{2},\ y^2 = \tfrac{1}{2},\ z = \pm\tfrac{1}{2} \\ &\implies T = 40\big(\pm\tfrac{1}{2}\big)\tfrac{1}{2}\big(\pm\tfrac{1}{2}\big) = \pm 5 \end{aligned}$$

(The sign of x and z need not be the same.) So the hottest temperature is $+5$ and the coldest temperature is -5.

$\boxed{2.10.22}$ $(*)$ Find the dimensions of the box of maximum volume which has its faces parallel to the coordinate planes and which is contained inside the region $0 \leqslant z \leqslant 48 - 4x^2 - 3y^2$.



$\boxed{\text{Solution}}$ The optimal box will have vertices $(\pm x, \pm y, 0)$, $(\pm x, \pm y,\ z)$ with $x, y, z > 0$ and $z = 48 - 4x^2 - 3y^2$. (If the lower vertices are not in the xy–plane, the volume of the box can be increased by lowering the bottom of the box to the xy–plane. If any of the four upper vertices are not on the hemisphere, the volume of the box can be increased by moving the upper vertices outwards to the hemisphere.) The volume of this box will be $(2x)(2y)z$. So we are to find the x, y and z that maximize the volume $f(x, y, z) = 4xyz$ subject to the constraint that $g(x, y, z) = 48 - 4x^2 - 3y^2 - z = 0$. By the method of

Lagrange multipliers (Theorem 2.10.2 in the CLP-3 text), the minimizing x, y, z must obey

$$\begin{aligned} f_x &= 4yz = -8\lambda x = \lambda g_x \\ f_y &= 4xz = -6\lambda y = \lambda g_y \\ f_z &= 4xy = -\lambda \quad\;\; = \lambda g_z \\ 48 &- 4x^2 - 3y^2 - z = 0 \end{aligned}$$

Multiplying the first equation by x, the second equation by y and the third equation by z gives

$$\begin{aligned} 4xyz &= -8\lambda x^2 \\ 4xyz &= -6\lambda y^2 \\ 4xyz &= -\lambda z \end{aligned}$$

This forces $8\lambda x^2 = 6\lambda y^2 = \lambda z$. Since λ cannot be zero (because that would force $4xyz = 0$), this in turn gives $8x^2 = 6y^2 = z$. Substituting in to the fourth equation gives

$$48 - \frac{z}{2} - \frac{z}{2} - z = 0 \implies 2z = 48 \implies z = 24,\ 8x^2 = 24,\ 6y^2 = 24$$

The dimensions of the box of biggest volume are $2x = 2\sqrt{3}$ by $2y = 4$ by $z = 24$.

$\boxed{2.10.23}$ (∗) A rectangular bin is to be made of a wooden base and heavy cardboard with no top. If wood is three times more expensive than cardboard, find the dimensions of the cheapest bin which has a volume of 12m^3.

$\boxed{\text{Solution}}$ Use units of money for which cardboard costs one unit per square meter. Then, if the bin has dimensions $x \times y \times z$, it costs $3xy + 2xz + 2yz$. We are to find the x, y and z that minimize the cost $f(x,y,z) = 3xy + 2xz + 2yz$ subject to the constraint that $g(x,y,z) = xyz - 12 = 0$. By the method of Lagrange multipliers (Theorem 2.10.2 in the CLP-3 text), the minimizing x, y, z must obey

$$\begin{aligned} f_x &= 3y + 2z = \lambda yz = \lambda g_x \\ f_y &= 3x + 2z = \lambda xz = \lambda g_y \\ f_z &= 2x + 2y = \lambda xy = \lambda g_z \\ & xyz - 12 = 0 \end{aligned}$$

Multiplying the first equation by x, the second equation by y and the third equation by z and then substituting in $xyz = 12$ gives

$$\begin{aligned} 3xy + 2xz &= 12\lambda \\ 3xy + 2yz &= 12\lambda \\ 2xz + 2yz &= 12\lambda \end{aligned}$$

Subtracting the second equation from the first gives $2z(x - y) = 0$. Since $z = 0$ is impossible, we must have $x = y$. Substituting this in

$$3x^2 + 2xz = 12\lambda \qquad 4xz = 12\lambda$$

Subtracting

$$3x^2 - 2xz = 0 \implies z = \frac{3}{2}x \implies 12 = xyz = \frac{3}{2}x^3 \implies x^3 = 8$$
$$\implies x = y = 2,\ z = 3 \text{ meters}$$

$\boxed{2.10.24}$ (∗) A closed rectangular box having a volume of 4 cubic metres is to be built with material that costs \$8 per square metre for the sides but \$12 per square metre for the top and bottom. Find the least expensive dimensions for the box.

$\boxed{\text{Solution}}$ If the box has dimensions $x \times y \times z$, it costs $24xy + 16xz + 16yz$. We are to find the x, y and z that minimize the cost $f(x,y,z) = 24xy + 16xz + 16yz$ subject to the constraint that $g(x,y,z) = xyz - 4 = 0$. By the method of Lagrange multipliers (Theorem 2.10.2 in the CLP-3 text), the minimizing x, y, z must obey

$$\begin{aligned} f_x = 24y + 16z &= \lambda yz = \lambda g_x \\ f_y = 24x + 16z &= \lambda xz = \lambda g_y \\ f_z = 16x + 16y &= \lambda xy = \lambda g_z \\ xyz - 4 &= 0 \end{aligned}$$

Multiplying the first equation by x, the second equation by y and the third equation by z and then substituting in $xyz = 4$ gives

$$\begin{aligned} 24xy + 16xz &= 4\lambda \\ 24xy + 16yz &= 4\lambda \\ 16xz + 16yz &= 4\lambda \end{aligned}$$

Subtracting the second equation from the first gives $16z(x - y) = 0$. Since $z = 0$ is impossible, we must have $x = y$. Subbing this in

$$24x^2 + 16xz = 4\lambda \qquad 32xz = 4\lambda$$

Subtracting

$$24x^2 - 16xz = 0 \implies z = \frac{3}{2}x \implies 4 = xyz = \frac{3}{2}x^3 \implies x^3 = \frac{8}{3}$$
$$\implies x = y = \frac{2}{\sqrt[3]{3}},\ z = 3^{2/3}\text{metres}$$

$\boxed{2.10.25}$ (∗) Suppose that a, b, c are all greater than zero and let D be the pyramid bounded by the plane $ax + by + cz = 1$ and the 3 coordinate planes. Use the method of Lagrange multipliers to find the largest possible volume of D if the plane $ax + by + cz = 1$ is required to pass through the point $(1, 2, 3)$. (The volume of a pyramid is equal to one-third of the area of its base times the height.)

Solution The vertices of the pyramid are $(0,0,0)$, $\big(\frac{1}{a},0,0\big)$, $\big(0,\frac{1}{b},0\big)$ and $\big(0,0,\frac{1}{c}\big)$. So the base of the pyramid is a triangle of area $\frac{1}{2}\frac{1}{a}\frac{1}{b}$ and the height of the pyramid is $\frac{1}{c}$. So the volume of the pyramid is $\frac{1}{6abc}$. The plane passes through $(1,2,3)$ if and only if $a+2b+3c=1$. Thus we are to find the a, b and c that maximize the volume $f(a,b,c)=\frac{1}{6abc}$ subject to the constraint that $g(a,b,c)=a+2b+3c-1=0$. By the method of Lagrange multipliers (Theorem 2.10.2 in the CLP-3 text), the maximizing a, b, c must obey

$$\begin{aligned}
f_a &= -\frac{1}{6a^2bc} = \lambda\ = \lambda g_a &&\iff && 6\lambda a^2bc = -1\\
f_b &= -\frac{1}{6ab^2c} = 2\lambda = \lambda g_b &&\iff && 6\lambda ab^2c = -\tfrac{1}{2}\\
f_c &= -\frac{1}{6abc^2} = 3\lambda = \lambda g_c &&\iff && 6\lambda abc^2 = -\frac{1}{3}\\
&a+2b+3c=1
\end{aligned}$$

Dividing the first two equations gives $\frac{a}{b}=2$ and dividing the first equation by the third gives $\frac{a}{c}=3$. Substituting $b=\frac{1}{2}a$ and $c=\frac{1}{3}a$ in to the final equation gives

$$a+2b+3c=3a=1 \implies a=\frac{1}{3},\ b=\frac{1}{6},\ c=\frac{1}{9}$$

and the maximum volume is $\frac{3\times6\times9}{6}=27$.

▸▸ Stage 3

2.10.26 (*) Use Lagrange multipliers to find the minimum distance from the origin to all points on the intersection of the curves

$$\begin{aligned}
g(x,y,z) &= x-z-4=0\\
\text{and } h(x,y,z) &= x+y+z-3=0
\end{aligned}$$

Solution We'll find the minimum distance^2 and then take the square root. That is, we'll find the minimum of $f(x,y,z)=x^2+y^2+z^2$ subject to the constraints $g(x,y,z)=x-z-4=0$ and $h(x,y,z)=x+y+z-3=0$. By Theorem 2.10.8 in the CLP-3 text, any local minimum or maximum (x,y,z) must obey the double Lagrange multiplier equations

$$f_x = 2x = \lambda+\mu = \lambda g_x + \mu h_x \tag{E1}$$
$$f_y = 2y = \mu = \lambda g_y + \mu h_y \tag{E2}$$
$$f_z = 2z = -\lambda+\mu = \lambda g_z + \mu h_z \tag{E3}$$
$$x-z=4 \tag{E4}$$
$$x+y+z=3 \tag{E5}$$

for some real numbers λ and μ. Adding (E1)and (E3) and then subtracting 2 times (E2) gives

$$2x-4y+2z=0 \qquad \text{or} \qquad x-2y+z=0 \tag{E6}$$

Substituting $x = 4 + z$ (from (E4)) into (E5) and (E6) gives

$$y + 2z = -1 \tag{E5'}$$
$$-2y + 2z = -4 \tag{E6'}$$

Substituting $y = -1 - 2z$ (from (E5')) into (E6') gives

$$6z = -6 \implies z = -1 \implies y = -1 - 2(-1) = 1 \implies x = 4 + (-1) = 3$$

So the closest point is $(3, 1, -1)$ and the minimum distance is $\sqrt{3^2 + 1^2 + (-1)^2} = \sqrt{11}$.

$\boxed{2.10.27}$ (∗) Find the largest and smallest values of

$$f(x, y, z) = 6x + y^2 + xz$$

on the sphere $x^2 + y^2 + z^2 = 36$. Determine all points at which these values occur.

$\boxed{\text{Solution}}$ Solution 1: This is a constrained optimization problem with objective function $f(x, y, z) = 6x + y^2 + xz$ and constraint function $g(x, y, z) = x^2 + y^2 + z^2 - 36$. By Theorem 2.10.2 in the CLP-3 text, any local minimum or maximum (x, y, z) must obey the Lagrange multiplier equations

$$f_x = 6 + z = 2\lambda x = \lambda g_x \tag{E1}$$
$$f_y = 2y = 2\lambda y = \lambda g_y \tag{E2}$$
$$f_z = x = 2\lambda z = \lambda g_z \tag{E3}$$
$$x^2 + y^2 + z^2 = 36 \tag{E4}$$

for some real number λ. By equation (E2), $y(1 - \lambda) = 0$, which is obeyed if and only if at least one of $y = 0$, $\lambda = 1$ is obeyed.

- If $y = 0$, the remaining equations reduce to

$$6 + z = 2\lambda x \tag{E1}$$
$$x = 2\lambda z \tag{E3}$$
$$x^2 + z^2 = 36 \tag{E4}$$

Substituting (E3) into (E1) gives $6 + z = 4\lambda^2 z$, which forces $4\lambda^2 \neq 1$ (since $6 \neq 0$) and gives $z = \frac{6}{4\lambda^2-1}$ and then $x = \frac{12\lambda}{4\lambda^2-1}$. Substituting this into (E4) gives

$$\begin{aligned}
\frac{144\lambda^2}{(4\lambda^2-1)^2} + \frac{36}{(4\lambda^2-1)^2} &= 36 \\
\frac{4\lambda^2}{(4\lambda^2-1)^2} + \frac{1}{(4\lambda^2-1)^2} &= 1 \\
4\lambda^2 + 1 &= (4\lambda^2-1)^2
\end{aligned}$$

Write $\mu = 4\lambda^2$. Then this last equation is

$$\begin{aligned} \mu + 1 = \mu^2 - 2\mu + 1 &\iff \mu^2 - 3\mu = 0 \\ &\iff \mu = 0, 3 \end{aligned}$$

When $\mu = 0$, we have $z = \frac{6}{\mu-1} = -6$ and $x = 0$ (by (E4)). When $\mu = 3$, we have $z = \frac{6}{\mu-1} = 3$ and then $x = \pm\sqrt{27} = \pm 3\sqrt{3}$ (by (E4)).

- If $\lambda = 1$, the remaining equations reduce to

$$6 + z = 2x \tag{E1}$$
$$x = 2z \tag{E3}$$
$$x^2 + y^2 + z^2 = 36 \tag{E4}$$

Substituting (E3) into (E1) gives $6 + z = 4z$ and hence $z = 2$. Then (E3) gives $x = 4$ and (E4) gives $4^2 + y^2 + 2^2 = 36$ or $y^2 = 16$ or $y = \pm 4$.

So we have the following candidates for the locations of the min and max

point	$(0,0,-6)$	$(3\sqrt{3},0,3)$	$(-3\sqrt{3},0,3)$	$(4,4,2)$	$(4,-4,2)$
value of f	0	$27\sqrt{3}$	$-27\sqrt{3}$	48	48
			min	max	max

Solution 2: On the sphere we have $y^2 = 36 - x^2 - z^2$ and hence $f = 36 + 6x + xz - x^2 - z^2$ and $x^2 + z^2 \leqslant 36$. So it suffices to find the max and min of $h(x,z) = 36 + 6x + xz - x^2 - z^2$ on the disk $D = \{ (x,z) \mid x^2 + z^2 \leqslant 36 \}$.

- If a max or min occurs at an interior point (x,z) of D, then (x,z) must be a critical point of h and hence must obey

$$\begin{aligned} h_x &= 6 + z - 2x = 0 \\ h_z &= x - 2z = 0 \end{aligned}$$

Substituting $x = 2z$ into the first equation gives $6 - 3z = 0$ and hence $z = 2$ and $x = 4$.

- If a max or min occurs a point (x,z) on the boundary of D, we have $x^2 + z^2 = 36$ and hence $x = \pm\sqrt{36 - z^2}$ and $h = 6x + zx = \pm(6+z)\sqrt{36 - z^2}$ with $-6 \leqslant z \leqslant 6$. So the max or min can occur either when $z = -6$ or $z = +6$ or at a z obeying

$$0 = \frac{\mathrm{d}}{\mathrm{d}z}\big[(6+z)\sqrt{36 - z^2}\big] = \sqrt{36 - z^2} - \frac{z(6+z)}{\sqrt{36 - z^2}}$$

or equivalently

$$\begin{aligned} 36 - z^2 - z(6+z) &= 0 \\ 2z^2 + 6z - 36 &= 0 \\ z^2 + 3z - 18 &= 0 \\ (z+6)(z-3) &= 0 \end{aligned}$$

So the max or min can occur either when $z = \pm 6$ or $z = 3$.

So we have the following candidates for the locations of the min and max

point	$(0,0,\pm 6)$	$(3\sqrt{3},0,3)$	$(-3\sqrt{3},0,3)$	$(4,4,2)$	$(4,-4,2)$
value of f	0	$27\sqrt{3}$	$-27\sqrt{3}$	48	48
			min	max	max

$\boxed{2.10.28}$ $(*)$

The temperature in the plane is given by $T(x,y) = e^y\big(x^2+y^2\big)$.

(a) (i) Give the system of equations that must be solved in order to find the warmest and coolest point on the circle $x^2+y^2 = 100$ by the method of Lagrange multipliers.

(ii) Find the warmest and coolest points on the circle by solving that system.

(b) (i) Give the system of equations that must be solved in order to find the critical points of $T(x,y)$.

(ii) Find the critical points by solving that system.

(c) Find the coolest point on the solid disc $x^2+y^2 \leqslant 100$.

$\boxed{\text{Solution}}$ By way of preparation, we have

$$\frac{\partial T}{\partial x}(x,y) = 2x\,e^y \qquad \frac{\partial T}{\partial y}(x,y) = e^y\big(x^2+y^2+2y\big)$$

(a) (i) For this problem the objective function is $T(x,y) = e^y\big(x^2+y^2\big)$ and the constraint function is $g(x,y) = x^2+y^2-100$. According to the method of Lagrange multipliers, Theorem 2.10.2 in the CLP-3 text, we need to find all solutions to

$$T_x = 2x\,e^y = \lambda(2x) = \lambda g_x \qquad \text{(E1)}$$
$$T_y = e^y\big(x^2+y^2+2y\big) = \lambda(2y) = \lambda g_y \qquad \text{(E2)}$$
$$x^2+y^2 = 100 \qquad \text{(E3)}$$

(a) (ii) According to equation (E1), $2x(e^y-\lambda) = 0$. This condition is satisfied if and only if at least one of $x = 0$, $\lambda = e^y$ is obeyed.

- If $x = 0$, then equation (E3) reduces to $y^2 = 100$, which is obeyed if $y = \pm 10$. Equation (E2) then gives the corresponding values for λ, which we don't need.
- If $\lambda = e^y$, then equation (E2) reduces to

  $$e^y\big(x^2+y^2+2y\big) = (2y)e^y \iff e^y\big(x^2+y^2\big) = 0$$

  which conflicts with (E3). So we can't have $\lambda = e^y$.

So the only possible locations of the maximum and minimum of the function T are $(0,10)$ and $(0,-10)$. To complete the problem, we only have to compute T at those points.

point	$(0,10)$	$(0,-10)$
value of T	$100e^{10}$	$100e^{-10}$
	max	min

Hence the maximum value of $T(x,y) = e^y\big(x^2+y^2\big)$ on $x^2+y^2=100$ is $100e^{10}$ at $(0,10)$ and the minimum value is $100e^{-10}$ at $(0,-10)$.

We remark that, on $x^2+y^2=100$, the objective function $T(x,y) = e^y\big(x^2+y^2\big) = 100e^y$. So of course the maximum value of T is achieved when y is a maximum, i.e. when $y=10$, and the minimum value of T is achieved when y is a minimum, i.e. when $y=-10$.

(b) (i) By definition, the point (x,y) is a critical point of $T(x,y)$ if ane only if

$$T_x = 2x\,e^y = 0 \tag{E1}$$
$$T_y = e^y\big(x^2+y^2+2y\big) = 0 \tag{E2}$$

(b) (ii) Equation (E1) forces $x=0$. When $x=0$, equation (E2) reduces to

$$e^y\big(y^2+2y\big) = 0 \iff y(y+2)=0 \iff y=0 \text{ or } y=-2$$

So there are two critical points, namely $(0,0)$ and $(0,-2)$.

(c) Note that $T(x,y) = e^y\big(x^2+y^2\big) \geqslant 0$ on all of $\mathbb{R}^2$. As $T(x,y)=0$ only at $(0,0)$, it is obvious that $(0,0)$ is the coolest point.

In case you didn't notice that, here is a more conventional solution.

The coolest point on the solid disc $x^2+y^2 \leqslant 100$ must either be on the boundary, $x^2+y^2=100$, of the disc or be in the interior, $x^2+y^2<100$, of the disc.

In part (a) (ii) we found that the coolest point on the boundary is $(0,-10)$, where $T=100e^{-10}$.

If the coolest point is in the interior, it must be a critical point and so must be either $(0,0)$, where $T=0$, or $(0,-2)$, where $T=4e^{-2}$.

So the coolest point is $(0,0)$.

$\boxed{2.10.29}$ (*)

(a) By finding the points of tangency, determine the values of c for which $x+y+z=c$ is a tangent plane to the surface $4x^2+4y^2+z^2=96$.

(b) Use the method of Lagrange Multipliers to determine the absolute maximum and minimum values of the function $f(x,y,z)=x+y+z$ along the surface $g(x,y,z)=4x^2+4y^2+z^2=96$.

(c) Why do you get the same answers in (a) and (b)?

$\boxed{\text{Solution}}$ (a) A normal vector to $F(x,y,z)=4x^2+4y^2+z^2=96$ at (x_0,y_0,z_0) is $\nabla F(x_0,y_0,z_0) = \langle 8x_0, 8y_0, 2z_0\rangle$. (Note that this normal vector is never the zero vector because $(0,0,0)$ is not on the surface.) So the tangent plane to $4x^2+4y^2+z^2=96$ at (x_0,y_0,z_0) is

$$8x_0(x-x_0)+8y_0(y-y_0)+2z_0(z-z_0)=0 \quad\text{or}\quad 8x_0x+8y_0y+2z_0z = 8x_0^2+8y_0^2+2z_0^2$$

This plane is of the form $x+y+z=c$ if and only if $8x_0=8y_0=2z_0$. A point (x_0,y_0,z_0) with $8x_0=8y_0=2z_0$ is on the surface $4x^2+4y^2+z^2=96$ if and only if

$$4x_0^2+4y_0^2+z_0^2 = 4x_0^2+4x_0^2+(4x_0)^2 = 96 \iff 24x_0^2=96 \iff x_0^2=4 \iff x_0=\pm 2$$

When $x_0 = \pm 2$, we have $y_0 = \pm 2$ and $z_0 = \pm 8$ (upper signs go together and lower signs go together) so that the tangent plane $8x_0x + 8y_0y + 2z_0z = 8x_0^2 + 8y_0^2 + 2z_0^2$ is

$$\begin{aligned} 8(\pm 2)x + 8(\pm 2)y + 2(\pm 8)z = 8(\pm 2)^2 + 8(\pm 2)^2 + 2(\pm 8)^2 \quad &\text{or} \quad \pm x \pm y \pm z = 2 + 2 + 8 \\ &\text{or} \quad x + y + z = \mp 12 \\ &\implies c = \pm 12 \end{aligned}$$

(b) We are to find the x, y and z that minimize or maximize $f(x,y,z) = x + y + z$ subject to the constraint that $g(x,y,z) = 4x^2 + 4y^2 + z^2 - 96 = 0$. By the method of Lagrange multipliers (Theorem 2.10.2 in the CLP-3 text), the minimizing/maximizing x, y, z must obey

$$\begin{aligned} f_x = 1 = \lambda(8x) &= \lambda g_x \\ f_y = 1 = \lambda(8y) &= \lambda g_y \\ f_z = 1 = \lambda(2z) &= \lambda g_z \\ 4x^2 + 4y^2 + z^2 - 96 \quad &= 0 \end{aligned}$$

The first three equations give

$$x = \frac{1}{8\lambda} \qquad y = \frac{1}{8\lambda} \qquad z = \frac{1}{2\lambda} \qquad \text{with } \lambda \neq 0$$

Substituting this into the fourth equation gives

$$\begin{aligned} 4\left(\frac{1}{8\lambda}\right)^2 + 4\left(\frac{1}{8\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 96 &\iff \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{4}\right)\frac{1}{\lambda^2} = 96 \\ &\iff \lambda^2 = \frac{3}{8}\frac{1}{96} = \frac{1}{8 \times 32} \\ &\iff \lambda = \pm\frac{1}{16} \end{aligned}$$

Hence $x = \pm 2$, $y = \pm 2$ and $z = \pm 8$ so that the largest and smallest values of $x + y + z$ on $4x^2 + 4y^2 + z^2 - 96$ are $\pm 2 \pm 2 \pm 8$ or ± 12.

(c) The level surfaces of $x + y + z$ are planes with equation of the form $x + y + z = c$. To find the largest (smallest) value of $x + y + z$ on $4x^2 + 4y^2 + z^2 = 96$ we keep increasing (decreasing) c until we get to the largest (smallest) value of c for which the plane $x + y + z = c$ intersects $4x^2 + 4y^2 + z^2 = 96$. For this value of c, $x + y + z = c$ is tangent to $4x^2 + 4y^2 + z^2 = 96$.

$\boxed{2.10.30}$ Let $f(x,y)$ have continuous partial derivatives. Consider the problem of finding local minima and maxima of $f(x,y)$ on the curve $xy = 1$.

- Define $g(x,y) = xy - 1$. According to the method of Lagrange multipliers, if (x,y) is a local minimum or maximum of $f(x,y)$ on the curve $xy = 1$, then there is a real number λ such that

$$\nabla f(x,y) = \lambda \nabla g(x,y), \quad g(x,y) = 0 \tag{E1}$$

- On the curve $xy = 1$, we have $y = \frac{1}{x}$ and $f(x,y) = f\big(x, \frac{1}{x}\big)$. Define $F(x) = f\big(x, \frac{1}{x}\big)$. If $x \neq 0$ is a local minimum or maximum of $F(x)$, we have that

$$F'(x) = 0 \tag{E2}$$

Show that (E1) is equivalent to (E2), in the sense that

$$\begin{aligned} &\text{there is a } \lambda \text{ such that } (x,y,\lambda) \text{ obeys (E1)} \\ \text{if and only if}& \\ &x \neq 0 \text{ obeys (E2) and } y = 1/x. \end{aligned}$$

$\boxed{\text{Solution}}$ Note that if (x,y) obeys $g(x,y) = xy - 1 = 0$, then x is necessarily nonzero. So we may assume that $x \neq 0$. Then

$$\begin{aligned}
&\text{There is a } \lambda \text{ such that } (x,y,\lambda) \text{ obeys (E1)} \\
&\quad\iff \text{there is a } \lambda \text{ such that } f_x(x,y) = \lambda g_x(x,y), \quad f_y(x,y) = \lambda g_y(x,y), \quad g(x,y) = 0 \\
&\quad\iff \text{there is a } \lambda \text{ such that } f_x(x,y) = \lambda y, \quad f_y(x,y) = \lambda x, \quad xy = 1 \\
&\quad\iff \text{there is a } \lambda \text{ such that } \frac{1}{y} f_x(x,y) = \frac{1}{x} f_y(x,y) = \lambda, \quad xy = 1 \\
&\quad\iff \frac{1}{y} f_x(x,y) = \frac{1}{x} f_y(x,y), \quad xy = 1 \\
&\quad\iff x f_x\Big(x, \frac{1}{x}\Big) = \frac{1}{x} f_y\Big(x, \frac{1}{x}\Big), \quad y = \frac{1}{x} \\
&\quad\iff F'(x) = \frac{\mathrm{d}}{\mathrm{d}x} f\Big(x, \frac{1}{x}\Big) = f_x\Big(x, \frac{1}{x}\Big) - \frac{1}{x^2} f_y\Big(x, \frac{1}{x}\Big) = 0, \quad y = \frac{1}{x}
\end{aligned}$$

Chapter 3

Multiple Integrals

3.1 ▲ Double Integrals

▸▸ Stage 1

$\boxed{3.1.1}$ For each of the following, evaluate the given double integral without using iteration. Instead, interpret the integral as, for example, an area or a volume.

(a) $\displaystyle\int_{-1}^{3}\int_{-4}^{1} \mathrm{d}y\,\mathrm{d}x$

(b) $\displaystyle\int_{0}^{2}\int_{0}^{\sqrt{4-y^2}} \mathrm{d}x\,\mathrm{d}y$

(c) $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}} \sqrt{9-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y$

$\boxed{\text{Solution}}$ (a) The given double integral $\int_{-1}^{3}\int_{-4}^{1}\mathrm{d}y\,\mathrm{d}x = \iint_R \mathrm{d}x\,\mathrm{d}y$ where

$$R = \left\{\ (x,y)\ \middle|\ -1 \leqslant x \leqslant 3,\ -4 \leqslant y \leqslant 1\ \right\}$$

and so the integral is the area of a rectangle with sides of lengths 4 and 5. Thus $\int_{-1}^{3}\int_{-4}^{1}\mathrm{d}y\,\mathrm{d}x = 4\times 5 = 20$.

(b) The given double integral $\displaystyle\int_{0}^{2}\int_{0}^{\sqrt{4-y^2}} \mathrm{d}x\,\mathrm{d}y = \iint_R \mathrm{d}x\,\mathrm{d}y$ where

$$\begin{aligned} R &= \left\{\ (x,y)\ \middle|\ 0 \leqslant y \leqslant 2,\ 0 \leqslant x \leqslant \sqrt{4-y^2}\ \right\} \\ &= \left\{\ (x,y)\ \middle|\ x \geqslant 0,\ y \geqslant 0,\ y \leqslant 2,\ x^2+y^2 \leqslant 4\ \right\} \end{aligned}$$

So R is the first quadrant part of the circular disk of radius 2 centred on $(0,0)$. The area of the full disk is $\pi\, 2^2 = 4\pi$. The given integral is one quarter of that, which is π.

(c) The given double integral $\displaystyle\int_{-3}^{3}\int_0^{\sqrt{9-y^2}} \sqrt{9-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathcal{R}} z(x,y)\,\mathrm{d}x\,\mathrm{d}y$ where $z(x,y) = \sqrt{9-x^2-y^2}$ and

$$\begin{aligned}\mathcal{R} &= \left\{\ (x,y)\ \middle|\ -3 \leqslant y \leqslant 3,\ 0 \leqslant x \leqslant \sqrt{9-y^2}\ \right\}\\ &= \left\{\ (x,y)\ \middle|\ x \geqslant 0,\ -3 \leqslant y \leqslant 3,\ \ x^2+y^2 \leqslant 9\ \right\}\end{aligned}$$

So $\mathcal{R}$ is the right half of the circular disk of radius 3 centred on $(0,0)$. By Equation (3.1.9) in the CLP-3 text, the given integral is the volume of the solid

$$\begin{aligned}\mathcal{V} &= \left\{\ (x,y,z)\ \middle|\ (x,y) \in \mathcal{R},\ 0 \leqslant z \leqslant \sqrt{9-x^2-y^2}\ \right\}\\ &= \left\{\ (x,y,z)\ \middle|\ (x,y) \in \mathcal{R},\ z \geqslant 0,\ x^2+y^2+z^2 \leqslant 9\right\}\end{aligned}$$

Thus $\mathcal{V}$ is the one quarter of the spherical ball of radius 3 and centre $(0,0,0)$ with $x \geqslant 0$ and $z \geqslant 0$. So

$$\int_{-3}^{3}\int_0^{\sqrt{9-y^2}} \sqrt{9-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{4}\left(\frac{4}{3}\pi 3^3\right) = 9\pi$$

$\boxed{3.1.2}$ Let $f(x,y) = 12x^2y^3$. Evaluate

(a) $\displaystyle\int_0^3 f(x,y)\,\mathrm{d}x$

(b) $\displaystyle\int_0^2 f(x,y)\,\mathrm{d}y$

(c) $\displaystyle\int_0^2\int_0^3 f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

(d) $\displaystyle\int_0^3\int_0^2 f(x,y)\,\mathrm{d}y\,\mathrm{d}x$

(e) $\displaystyle\int_0^3\int_0^2 f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

Solution (a) The integral with respect to x treats y as a constant. So

$$\int_0^3 f(x,y)\,\mathrm{d}x = \int_0^3 12x^2y^3\,\mathrm{d}x = \Big[4x^3y^3\Big]_{x=0}^{x=3} = 108y^3$$

(b) The integral with respect to y treats x as a constant. So

$$\int_0^2 f(x,y)\,\mathrm{d}y = \int_0^2 12x^2y^3\,\mathrm{d}y = \Big[3x^2y^4\Big]_{y=0}^{y=2} = 48x^2$$

(c) By part (a)

$$\begin{aligned}\int_0^2\int_0^3 f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^2\left[\int_0^3 f(x,y)\,\mathrm{d}x\right]\mathrm{d}y = \int_0^2 108y^3\,\mathrm{d}y = \Big[27y^4\Big]_{y=0}^{y=2}\\ &= 27 \times 16 = 432\end{aligned}$$

(d) By part (b)

$$\int_0^3 \int_0^2 f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^3 \left[\int_0^2 f(x,y)\,\mathrm{d}y\right]\mathrm{d}x = \int_0^3 48x^2\,\mathrm{d}y = \Big[16x^3\Big]_{x=0}^{x=3}$$
$$= 16 \times 27 = 432$$

(e) This time

$$\int_0^3 \int_0^2 f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^3 \left[\int_0^2 12x^2y^3\,\mathrm{d}x\right]\mathrm{d}y = \int_0^3 \Big[4x^3y^3\Big]_0^2\mathrm{d}y = \int_0^3 32y^3\mathrm{d}y$$
$$= \Big[8y^4\Big]_0^3 = 8 \times 81 = 648$$

▸▸ Stage 2

Questions 3 through 8 provide practice with limits of integration for double integrals in Cartesian coordinates.

3.1.3 For each of the following, evaluate the given double integral using iteration.

(a) $\displaystyle\iint_R (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$ where R is the rectangle $0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b$ where $a > 0$ and $b > 0$.

(b) $\displaystyle\iint_T (x-3y)\,\mathrm{d}x\,\mathrm{d}y$ where T is the triangle with vertices $(0,0),\ (a,0),\ (0,b)$.

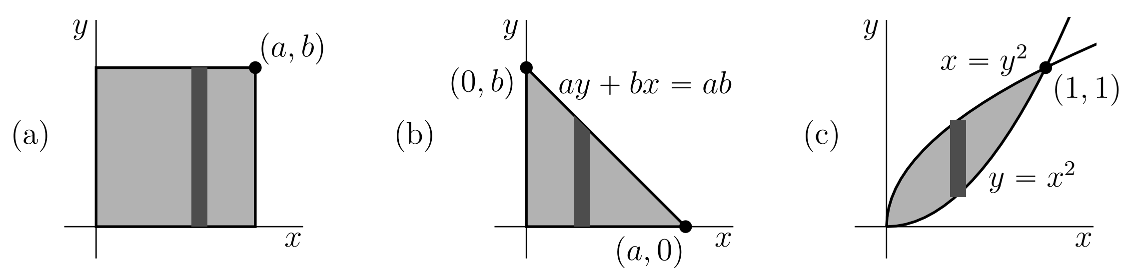
(c) $\displaystyle\iint_R xy^2\,\mathrm{d}x\,\mathrm{d}y$ where R is the finite region in the first quadrant bounded by the curves $y=x^2$ and $x=y^2$.

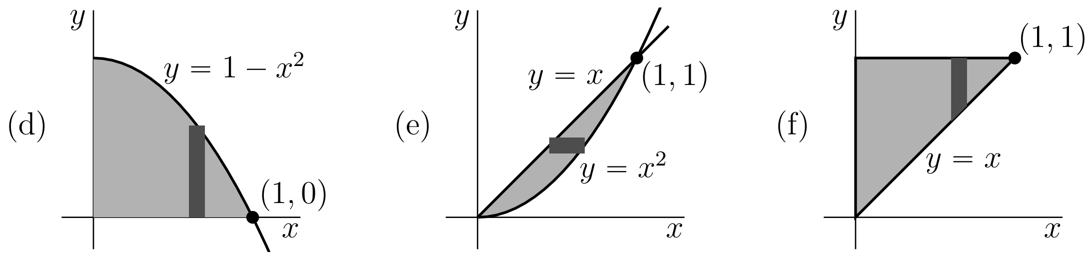
(d) $\displaystyle\iint_D x\cos y\,\mathrm{d}x\,\mathrm{d}y$ where D is the finite region in the first quadrant bounded by the coordinate axes and the curve $y = 1-x^2$.

(e) $\displaystyle\iint_R \frac{x}{y}e^y\,\mathrm{d}x\,\mathrm{d}y$ where R is the region $0 \leqslant x \leqslant 1,\ x^2 \leqslant y \leqslant x$.

(f) $\displaystyle\iint_T \frac{xy}{1+x^4}\,\mathrm{d}x\,\mathrm{d}y$ where T is the triangle with vertices $(0,0),\ (0,1),\ (1,1)$.

Solution The following figures show the domains of integration for the integrals in this problem.





(a) $$\iint_R (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y = \int_0^a \mathrm{d}x \int_0^b \mathrm{d}y\ (x^2 + y^2) = \int_0^a \mathrm{d}x\ \left(x^2 b + \frac{1}{3}b^3\right)$$
$$= \frac{1}{3}\big(a^3 b + ab^3\big)$$

(b) $$\iint_T (x - 3y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^a \mathrm{d}x \int_0^{b(1-\frac{x}{a})} \mathrm{d}y\,(x - 3y)$$
$$= \int_0^a \mathrm{d}x\ \left[bx\left(1 - \frac{x}{a}\right) - \frac{3}{2}b^2\left(1 - \frac{x}{a}\right)^2\right]$$
$$= \left[\frac{b}{2}x^2 - \frac{b}{3a}x^3 + \frac{a}{2}b^2\left(1 - \frac{x}{a}\right)^3\right]_0^a$$
$$= \frac{a^2 b}{2} - \frac{a^2 b}{3} - \frac{ab^2}{2} = \frac{a^2 b}{6} - \frac{ab^2}{2}$$

(c) $$\iint_R xy^2\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \mathrm{d}x \int_{x^2}^{\sqrt{x}} \mathrm{d}y\ xy^2 = \frac{1}{3}\int_0^1 \mathrm{d}x\ x\big(x^{3/2} - x^6\big) = \frac{1}{3}\left(\frac{2}{7} - \frac{1}{8}\right)$$
$$= \frac{3}{56}$$

(d) $$\iint_D x\cos y\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^{1-x^2} \mathrm{d}y\ x\cos y = \int_0^1 \mathrm{d}x\ x\sin(1 - x^2)$$
$$= \frac{1}{2}\Big[\cos(1 - x^2)\Big]_0^1 = \frac{1}{2}(1 - \cos 1)$$

(e) $$\iint_R \frac{x}{y}e^y\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \mathrm{d}y \int_y^{\sqrt{y}} \mathrm{d}x\ \frac{x}{y}e^y = \int_0^1 \mathrm{d}y\ \frac{y - y^2}{2y}e^y = \frac{1}{2}\int_0^1 \mathrm{d}y\ (1 - y)e^y$$
$$= \frac{1}{2}\Big[-ye^y + 2e^y\Big]_0^1 = \frac{1}{2}(e - 2)$$

(f) $$\iint_T \frac{xy}{1 + x^4}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \mathrm{d}x \int_x^1 \mathrm{d}y\ \frac{xy}{1 + x^4} = \frac{1}{2}\int_0^1 \mathrm{d}x\ \frac{x(1 - x^2)}{1 + x^4}$$
$$= \frac{1}{4}\int_0^1 \mathrm{d}t\ \frac{1 - t}{1 + t^2} \text{ where } t = x^2$$
$$= \frac{1}{4}\left[\arctan t - \frac{1}{2}\ln(1 + t^2)\right]_0^1 = \frac{1}{4}\left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right)$$

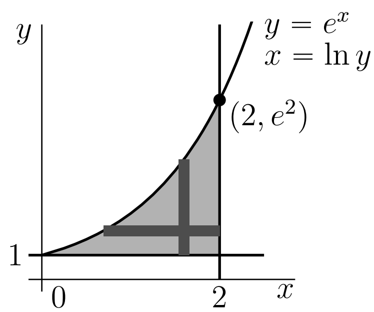
$\boxed{3.1.4}$ For each of the following integrals (i) sketch the region of integration, (ii) write an equivalent double integral with the order of integration reversed and (iii) evaluate both double integrals.

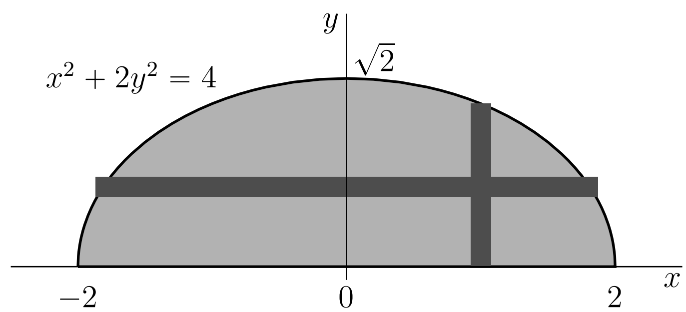
(a) $\displaystyle\int_0^2 \mathrm{d}x \int_1^{e^x} \mathrm{d}y$

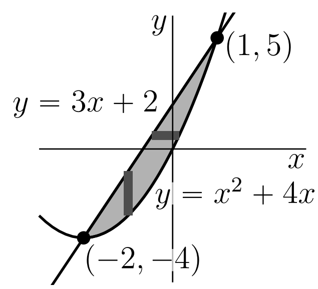
(b) $\displaystyle\int_0^{\sqrt{2}} \mathrm{d}y \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} \mathrm{d}x\ y$

(c) $\displaystyle\int_{-2}^{1} \mathrm{d}x \int_{x^2+4x}^{3x+2} \mathrm{d}y$

$\boxed{\text{Solution}}$ The following figures show the domains of integration for the integrals in this problem.

(a)


(b)


(c)


(a)
$$\int_0^2 \mathrm{d}x \int_1^{e^x} \mathrm{d}y = \int_0^2 \mathrm{d}x\ \left[e^x - 1\right] = \left[e^x - x\right]_0^2 = e^2 - 3$$
$$\int_1^{e^2} \mathrm{d}y \int_{\ln y}^{2} \mathrm{d}x = \int_1^{e^2} \mathrm{d}y\ \left[2 - \ln y\right] = \left[2y - y\ln y + y\right]_1^{e^2} = e^2 - 3$$

(b)
$$\int_0^{\sqrt{2}} \mathrm{d}y \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} \mathrm{d}x\ y = \int_0^{\sqrt{2}} \mathrm{d}y\ 2y\sqrt{4-2y^2} = -\frac{1}{3}\left[\left(4-2y^2\right)^{3/2}\right]_0^{\sqrt{2}} = \frac{8}{3}$$
$$\int_{-2}^{2} \mathrm{d}x \int_0^{\sqrt{2-\frac{x^2}{2}}} \mathrm{d}y\ y = \int_{-2}^{2} \mathrm{d}x\ \left[1 - \frac{x^2}{4}\right] = 2\int_0^2 \mathrm{d}x\ \left[1 - \frac{x^2}{4}\right] = 2\left[x - \frac{x^3}{12}\right]_0^2 = \frac{8}{3}$$

(c)
$$\int_{-2}^{1} \mathrm{d}x \int_{x^2+4x}^{3x+2} \mathrm{d}y = \int_{-2}^{1} \mathrm{d}x\ \big[-x^2 - x + 2\big] = \Big[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\Big]_{-2}^{1} = \frac{9}{2}$$
$$\int_{-4}^{5} \mathrm{d}y \int_{\frac{y-2}{3}}^{-2+\sqrt{4+y}} \mathrm{d}x = \int_{-4}^{5} \mathrm{d}y\ \Big[-\frac{4}{3} - \frac{y}{3} + \sqrt{4+y}\Big] = \Big[-\frac{4y}{3} - \frac{y^2}{6} + \frac{2}{3}(4+y)^{\frac{3}{2}}\Big]_{-4}^{5}$$
$$= \frac{9}{2}$$

In part (c), we used that the equation $y = x^2 + 4x$ is equivalent to $y + 4 = (x+2)^2$ and hence to $x = -2 \pm \sqrt{y+4}$.

$\boxed{3.1.5}$ (*) Combine the sum of the two iterated double integrals

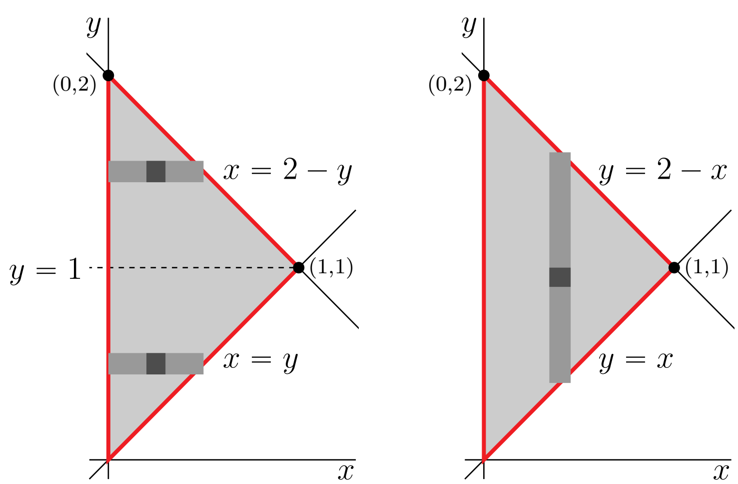
$$\int_{y=0}^{y=1} \int_{x=0}^{x=y} f(x,y)\ \mathrm{d}x\,\mathrm{d}y + \int_{y=1}^{y=2} \int_{x=0}^{x=2-y} f(x,y)\ \mathrm{d}x\,\mathrm{d}y$$

into a single iterated double integral with the order of integration reversed.

Solution In the given integrals

- y runs for 0 to 2, and
- for each fixed y between 0 and 1, x runs from 0 to y and
- for each fixed y between 1 and 2, x runs from 0 to $2-y$

The figure on the left below contains a sketch of that region together with the generic horizontal slices that were used to set up the given integrals.



To reverse the order of integration, we switch to vertical, rather than horizontal, slices, as in the figure on the right above. Looking at that figure, we see that

- x runs for 0 to 1, and
- for each fixed x in that range, y runs from x to $2-x$.

So the desired integral is

$$\int_{x=0}^{x=1} \int_{y=x}^{y=2-x} f(x,y)\ \mathrm{d}y\, \mathrm{d}x$$

$\boxed{3.1.6}$ (*) Consider the integral

$$\int_0^1 \int_x^1 e^{x/y}\ \mathrm{d}y\ \mathrm{d}x$$

(a) Sketch the domain of integration.
(b) Evaluate the integral by reversing the order of integration.

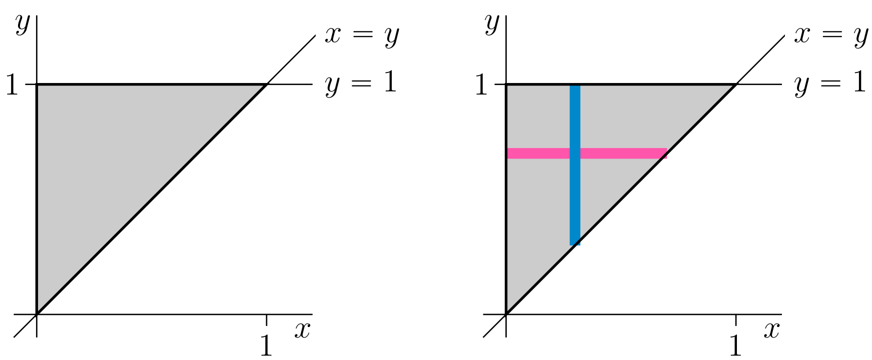
$\boxed{\text{Solution}}$ (a) In the given integral

- x runs from 0 to 1 and
- for each fixed x between 0 and 1, y runs from x to 1

So the domain of integration is

$$D = \left\{\ (x,y)\ \middle|\ 0 \leqslant x \leqslant 1,\ x \leqslant y \leqslant 1\ \right\}$$

It is sketched in the figure on the left below.



(b) The given integral decomposed the domain of integration into vertical strips like the blue strip in the figure on the right above. To reverse the order of integration, we instead use horizontal strips. Looking at the pink strip in the figure on the right above, we see that this entails

- having y run from 0 to 1 and
- for each fixed y between 0 and 1, having x run from 0 to y

This gives

$$\int_0^1 \mathrm{d}y \int_0^y \mathrm{d}x\ e^{x/y} = \int_0^1 \mathrm{d}y\ \left[ye^{x/y}\right]_0^y = \int_0^1 \mathrm{d}y\ y(e-1) = \frac{1}{2}(e-1)$$

$\boxed{3.1.7}$ (*) The integral I is defined as

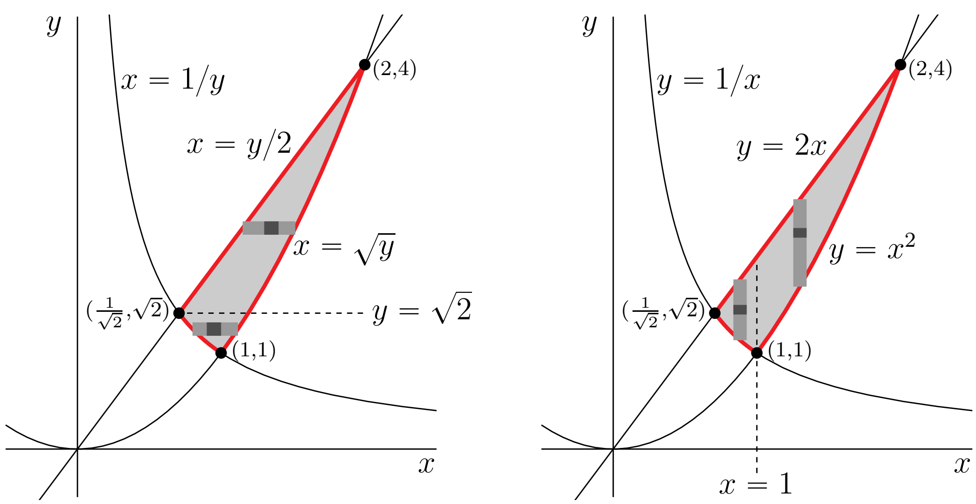
$$I = \iint\limits_R f(x,y)\, \mathrm{d}A = \int_1^{\sqrt{2}} \int_{1/y}^{\sqrt{y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{\sqrt{2}}^4 \int_{y/2}^{\sqrt{y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

(a) Sketch the region R.
(b) Re–write the integral I by reversing the order of integration.
(c) Compute the integral I when $f(x,y) = x/y$.

Solution (a) On R

- y runs from 1 to 4 (from 1 to $\sqrt{2}$ in the first integral and from $\sqrt{2}$ to 4 in the second).
- For each fixed y between 1 and $\sqrt{2}$, x runs from $\frac{1}{y}$ to $\sqrt{y}$ and
- for each fixed y between $\sqrt{2}$ and 4, x runs from $\frac{y}{2}$ to $\sqrt{y}$.

The figure on the left below is a sketch of R, together with generic horizontal strips as were used in setting up the integral.



(b) To reverse the order of integration we use vertical strips as in the figure on the right above. Looking at that figure, we see that, on R,

- x runs from $1/\sqrt{2}$ to 2.
- For each fixed x between $1/\sqrt{2}$ and 1, y runs from $\frac{1}{x}$ to $2x$ and
- for each fixed x between 1 and 2, y runs from x^2 to $2x$.

So

$$I = \int_{1/\sqrt{2}}^1 \int_{1/x}^{2x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x + \int_1^2 \int_{x^2}^{2x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

(c) When $f(x,y) = \frac{x}{y}$,

$$\begin{aligned}
I &= \int_1^{\sqrt{2}} \int_{1/y}^{\sqrt{y}} \frac{x}{y}\,\mathrm{d}x\,\mathrm{d}y + \int_{\sqrt{2}}^4 \int_{y/2}^{\sqrt{y}} \frac{x}{y}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_1^{\sqrt{2}} \frac{1}{y}\left[\frac{y}{2} - \frac{1}{2y^2}\right]\,\mathrm{d}y + \int_{\sqrt{2}}^4 \frac{1}{y}\left[\frac{y}{2} - \frac{y^2}{8}\right]\,\mathrm{d}y \\
&= \left[\frac{y}{2} + \frac{1}{4y^2}\right]_1^{\sqrt{2}} + \left[\frac{y}{2} - \frac{y^2}{16}\right]_{\sqrt{2}}^4 = \frac{1}{\sqrt{2}} + \frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 2 - 1 - \frac{1}{\sqrt{2}} + \frac{1}{8} \\
&= \frac{1}{2}
\end{aligned}$$

$\boxed{3.1.8}$ (∗) A region E in the xy–plane has the property that for all continuous functions f

$$\iint\limits_E f(x,y)\,\mathrm{d}A = \int_{x=-1}^{x=3}\left[\int_{y=x^2}^{y=2x+3} f(x,y)\mathrm{d}y\right]\mathrm{d}x$$

(a) Compute $\iint_E x\,\mathrm{d}A$.
(b) Sketch the region E.
(c) Set up $\iint_E x\,\mathrm{d}A$ as an integral or sum of integrals in the opposite order.

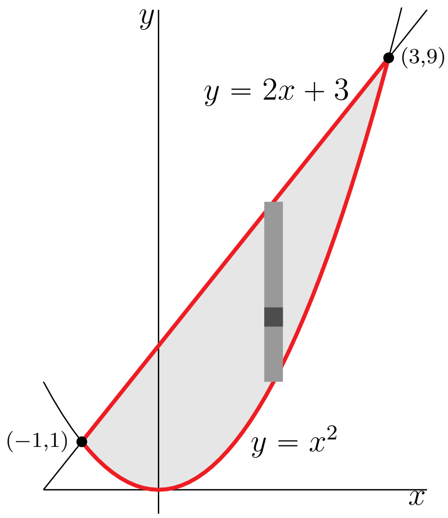
Solution (a) When $f(x,y) = x$,

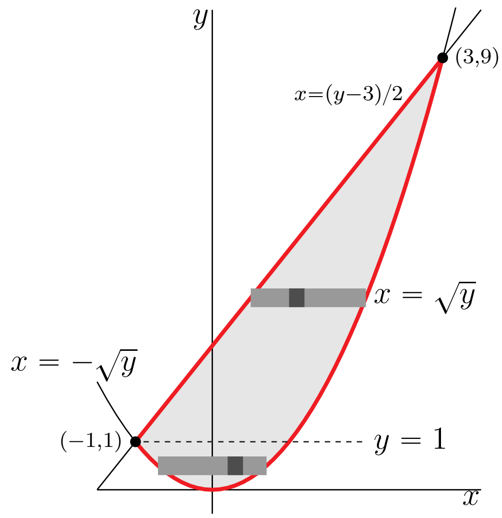
$$\begin{aligned}
\int_{x=-1}^{x=3}\left[\int_{y=x^2}^{y=2x+3} x\mathrm{d}y\right]\mathrm{d}x &= \int_{x=-1}^{x=3}\left[x(2x+3-x^2)\right]\mathrm{d}x \\
&= \left[\frac{2x^3}{3} + \frac{3x^2}{2} - \frac{x^4}{4}\right]_{-1}^3 = 18 + \frac{27}{2} - \frac{81}{4} + \frac{2}{3} - \frac{3}{2} + \frac{1}{4} \\
&= 18 + 12 - 20 + \frac{2}{3} = \frac{32}{3}
\end{aligned}$$

(b) On the region E

- x runs from -1 to 3 and
- for each x in that range, y runs from x^2 to $2x+3$

Here are two sketches of E, with the left one including a generic vertical strip as was used in setting up the given integral.





(c) To reverse the order of integration we use horizontal strips as in the figure on the right above. Looking at that figure, we see that, on the region E,

- y runs from 0 to 9 and
- for each y between 0 and 1, x runs from $-\sqrt{y}$ to $\sqrt{y}$
- for each y between 1 and 9, x runs from $(y-3)/2$ to $\sqrt{y}$

So

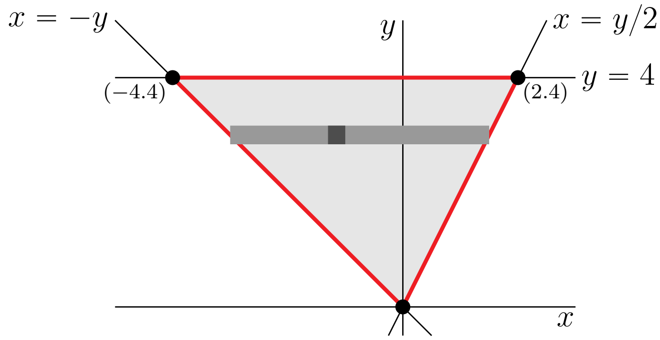
$$I = \int_0^1 \mathrm{d}y \int_{-\sqrt{y}}^{\sqrt{y}} \mathrm{d}x\ x + \int_1^9 \mathrm{d}y \int_{(y-3)/2}^{\sqrt{y}} \mathrm{d}x\ x$$

$\boxed{3.1.9}$ $(*)$ Calculate the integral:

$$\iint_D \sin(y^2)\ \mathrm{d}A$$

where D is the region bounded by $x + y = 0$, $2x - y = 0$, and $y = 4$.

$\boxed{\text{Solution}}$ The antiderivative of the function $\sin(y^2)$ cannot be expressed in terms of familiar functions. So we do not want the inside integral to be over y. So we'll use horizontal slices as in the figure



On the domain of integration

- y runs from 0 to 4, and
- for each fixed y in that range, x runs from $-y$ to $y/2$

The given integral

$$\begin{aligned}\iint\limits_D \sin(y^2)\ \mathrm{d}A &= \int_0^4 \mathrm{d}y \int_{-y}^{y/2} \mathrm{d}x\ \sin(y^2)\\ &= \int_0^4 \mathrm{d}y\ \frac{3}{2}y\ \sin(y^2)\\ &= \left[-\frac{3}{4}\cos(y^2)\right]_0^4\\ &= \frac{3}{4}\big[1-\cos(16)\big]\end{aligned}$$

3.1.10 (*) Consider the integral

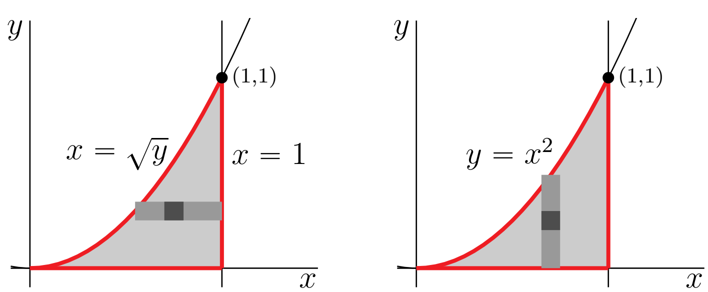
$$I = \int_0^1 \int_{\sqrt{y}}^1 \frac{\sin(\pi x^2)}{x}\ \mathrm{d}x\,\mathrm{d}y$$

(a) Sketch the region of integration.
(b) Evaluate I.

Solution (a) On the domain of integration

- y runs from 0 to 1 and
- for each fixed y in that range, x runs from $\sqrt{y}$ to 1.

The figure on the left below is a sketch of that domain, together with a generic horizontal strip as was used in setting up the integral.



(b) The inside integral, $\int_{\sqrt{y}}^1 \frac{\sin(\pi x^2)}{x}\ \mathrm{d}x$, in the given form of I looks really nasty. So let's try exchanging the order of integration. Looking at the figure on the right above, we see that, on the domain of integration,

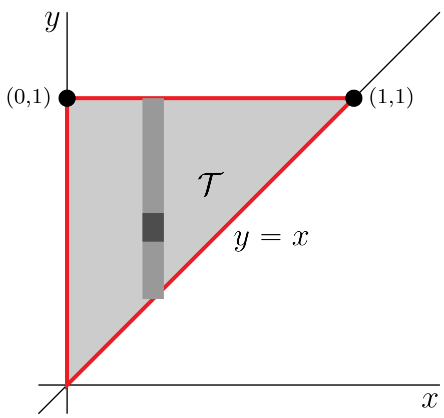
- x runs from 0 to 1 and
- for each fixed x in that range, y runs from 0 to x^2.

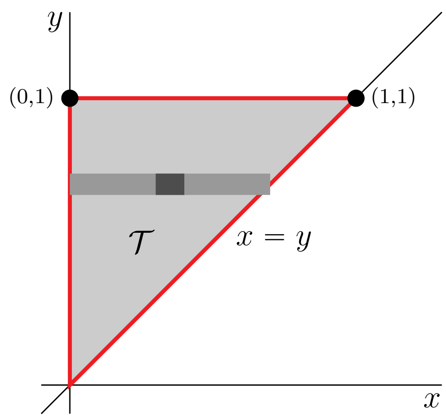
So

$$\begin{aligned}
I &= \int_0^1 \mathrm{d}x \int_0^{x^2} \mathrm{d}y\ \frac{\sin(\pi x^2)}{x} \\
&= \int_0^1 \mathrm{d}x\ x\sin(\pi x^2) \\
&= \Big[-\frac{\cos(\pi x^2)}{2\pi}\Big]_0^1 \qquad \text{(Looks pretty rigged!)} \\
&= \frac{1}{\pi}
\end{aligned}$$

3.1.11 (*) Let I be the double integral of the function $f(x,y) = y^2 \sin xy$ over the triangle with vertices $(0,0)$, $(0,1)$ and $(1,1)$ in the xy–plane.
(a) Write I as an iterated integral in two different ways.
(b) Evaluate I.

Solution (a) Let's call the triangle $\mathcal{T}$. Here are two sketches of $\mathcal{T}$, one including a generic vertical strip and one including a generic horizontal strip. Notice that the equation of the line through $(0,0)$ and $(1,1)$ is $y = x$.





First, we'll set up the integral using vertical strips. Looking at the figure on the left above, we see that, on $\mathcal{T}$,

- x runs from 0 to 1 and
- for each x in that range, y runs from x to 1.

So the integral

$$I = \int_0^1 \mathrm{d}x \int_x^1 \mathrm{d}y\ y^2 \sin xy$$

Next, we'll set up the integral using horizontal strips. Looking at the figure on the right above, we see that, on $\mathcal{T}$,

- y runs from 0 to 1 and

- for each y in that range, x runs from 0 to y.

So the integral

$$I = \int_0^1 \mathrm{d}y \int_0^y \mathrm{d}x\ y^2 \sin xy$$

(b) To evaluate the inside integral, $\int_x^1 \mathrm{d}y\ y^2 \sin xy$, of the vertical strip version, will require two integration by parts to get rid of the y^2. So we'll use the horizontal strip version.

$$\begin{aligned}
I &= \int_0^1 \mathrm{d}y \int_0^y \mathrm{d}x\ y^2 \sin xy \\
&= \int_0^1 \mathrm{d}y\ \Big[-y\cos xy\Big]_0^y \\
&= \int_0^1 \mathrm{d}y\ \big[y - y\cos y^2\big] \\
&= \left[\frac{y^2}{2} - \frac{\sin y^2}{2}\right]_0^1 \qquad \text{(Look's pretty rigged!)} \\
&= \frac{1-\sin 1}{2}
\end{aligned}$$

$\boxed{3.1.12}$ (*) Find the volume (V) of the solid bounded above by the surface

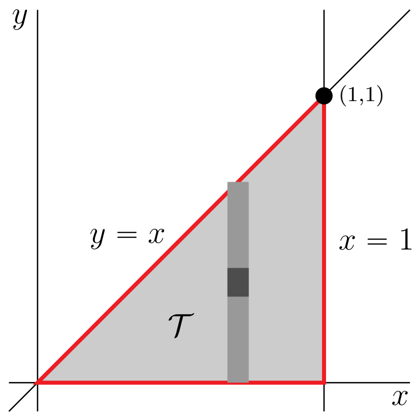
$$z = f(x, y) = e^{-x^2},$$

below by the plane $z = 0$ and over the triangle in the xy–plane formed by the lines $x = 1$, $y = 0$ and $y = x$.

$\boxed{\text{Solution}}$ If we call the triangular base region $\mathcal{T}$, then the volume is

$$V = \iint_{\mathcal{T}} f(x, y)\ \mathrm{d}A = \iint_{\mathcal{T}} e^{-x^2}\ \mathrm{d}x\,\mathrm{d}y$$

If we set up the integral using horizontal slices, so that the inside integral is the x–integral, there will be a big problem — the integrand e^{-x^2} does not have an obvious anti–derivative. (In fact its antiderivative cannot be expressed in terms of familiar functions.) So let's try vertical slices as in the sketch



Looking at that sketch we see that

- x runs from 0 to 1, and
- for each x in that range, y runs from 0 to x.

So the integral is

$$\begin{aligned} V &= \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y\; e^{-x^2} \\ &= \int_0^1 \mathrm{d}x\; xe^{-x^2} \\ &= \left[-\frac{1}{2}e^{-x^2}\right]_0^1 \\ &= \frac{1-e^{-1}}{2} \end{aligned}$$

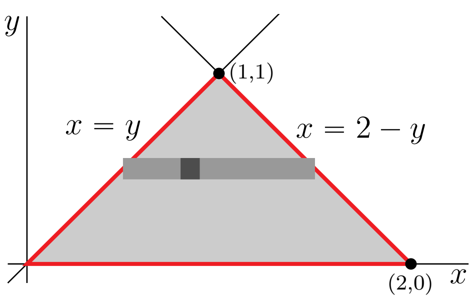
$\boxed{3.1.13}$ (*) Consider the integral $I = \int_0^1 \int_y^{2-y} \frac{y}{x}\,\mathrm{d}x\,\mathrm{d}y$.

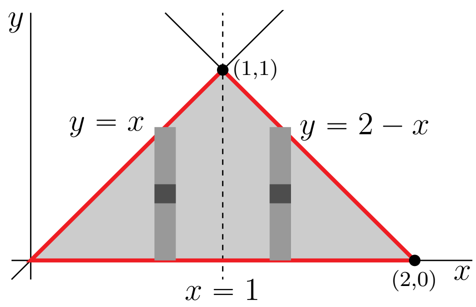
(a) Sketch the region of integration.
(b) Interchange the order of integration.
(c) Evaluate I.

$\boxed{\text{Solution}}$ (a) On the domain of integration

- y runs from 0 to 1 and
- for each y in that range x runs from y to $2-y$. So the left hand side of the domain is the line $x = y$ and the right hand side of the domain is $x = 2-y$.

The figure on the left below is a sketch of that domain, together with a generic horizontal strip as was used in setting up the integral.





(b) To reverse the order of integration we use vertical, rather than horizontal, strips. Looking at the figure on the right above, we see that, in the domain of integration

- x runs from 0 to 2 and
- for each x between 0 and 1, y runs from 0 to x, while
- for each x between 1 and 2, y runs from 0 to $2-x$.

So the integral

$$I = \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y\ \frac{y}{x} + \int_1^2 \mathrm{d}x \int_0^{2-x} \mathrm{d}y\ \frac{y}{x}$$

(c) Using the answer to part (b)

$$\begin{aligned}
I &= \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y\ \frac{y}{x} + \int_1^2 \mathrm{d}x \int_0^{2-x} \mathrm{d}y\ \frac{y}{x} \\
&= \frac{1}{2}\int_0^1 \mathrm{d}x\ x + \frac{1}{2}\int_1^2 \mathrm{d}x\ \frac{(2-x)^2}{x} \\
&= \frac{1}{4} + \frac{1}{2}\int_1^2 \mathrm{d}x\ \left(\frac{4}{x} - 4 + x\right) \\
&= \frac{1}{4} + \frac{1}{2}\left[4\ln 2 - 4 + \frac{4-1}{2}\right] \\
&= 2\ln 2 - 1
\end{aligned}$$

$\boxed{3.1.14}$ (∗) For the integral

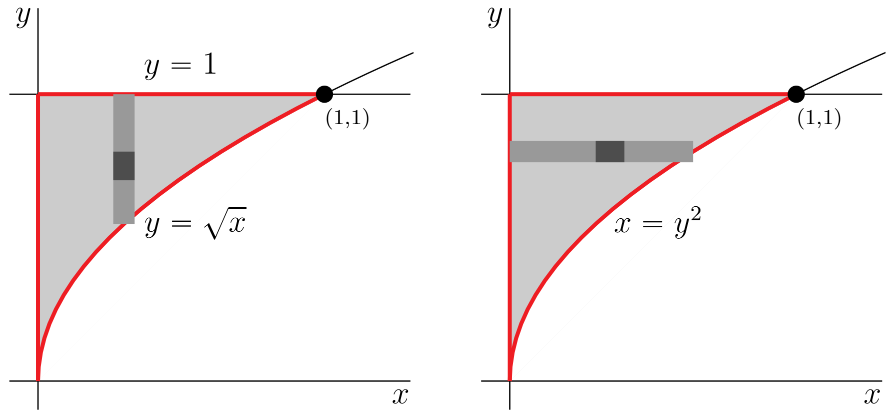
$$I = \int_0^1 \int_{\sqrt{x}}^1 \sqrt{1+y^3}\ \mathrm{d}y\,\mathrm{d}x$$

(a) Sketch the region of integration.
(b) Evaluate I.

$\boxed{\text{Solution}}$ (a) On the domain of integration,

- x runs from 0 to 1, and
- for each fixed x in that range, y runs from $\sqrt{x}$ to 1. We may rewrite $y = \sqrt{x}$ as $x = y^2$, which is a rightward opening parabola.

Here are two sketches of the domain of integration, which we call D. The left hand sketch also shows a vertical slice, as was used in setting up the integral.



(b) The inside integral, $\int_{\sqrt{x}}^{1}\sqrt{1+y^3}\,\mathrm{d}y$, of the given integral looks pretty nasty. So let's reverse the order of integration, by using horizontal, rather than vertical, slices. Looking at the figure on the right above, we see that

- y runs from 0 to 1, and
- for each fixed y in that range x runs from 0 to y^2.

So

$$\begin{aligned}
I &= \int_0^1 \mathrm{d}y \int_0^{y^2} \mathrm{d}x\ \sqrt{1+y^3} \\
&= \int_0^1 \mathrm{d}y\ y^2\sqrt{1+y^3} \\
&= \int_1^2 \frac{\mathrm{d}u}{3}\ \sqrt{u} \qquad \text{with } u = 1+y^3,\ \mathrm{d}u = 3y^2\,\mathrm{d}y. \text{ Looks pretty rigged!} \\
&= \frac{1}{3}\left[\frac{u^{3/2}}{3/2}\right]_1^2 \\
&= \frac{2\big(2\sqrt{2}-1\big)}{9}
\end{aligned}$$

$\boxed{3.1.15}$ $(*)$

(a) D is the region bounded by the parabola $y^2 = x$ and the line $y = x-2$. Sketch D and evaluate J where

$$J = \iint_D 3y\ \mathrm{d}A$$

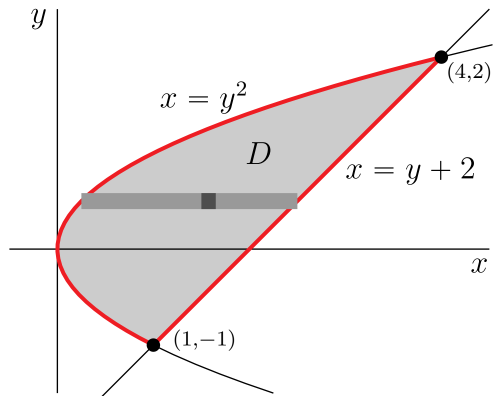
(b) Sketch the region of integration and then evaluate the integral I :

$$I = \int_0^4 \int_{\frac{1}{2}\sqrt{x}}^{1} e^{y^3}\ \mathrm{d}y\,\mathrm{d}x$$

$\boxed{\text{Solution}}$ (a) Observe that the parabola $y^2 = x$ and the line $y = x - 2$ meet when $x = y + 2$ and

$$y^2 = y + 2 \iff y^2 - y - 2 = 0 \iff (y-2)(y+1) = 0$$

So the points of intersection of $x = y^2$ and $y = x - 2$ are $(1, -1)$ and $(4, 2)$. Here is a sketch of D.



To evaluate J, we'll use horizontal slices as in the figure above. (If we were to use vertical slices we would have to split the integral in two, with $0 \leqslant x \leqslant 1$ in one part and $1 \leqslant x \leqslant 4$ in the other.) From the figure, we see that, on D,

- y runs from -1 to 2 and
- for each fixed y in that range, x runs from y^2 to $y + 2$.

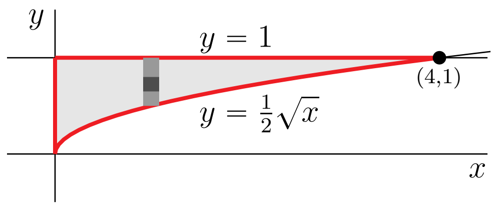
Hence

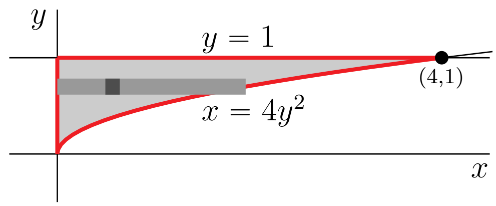
$$\begin{aligned}
J = \iint_D 3y \, \mathrm{d}A &= \int_{-1}^{2} \mathrm{d}y \int_{y^2}^{y+2} \mathrm{d}x \; 3y \\
&= 3\int_{-1}^{2} \mathrm{d}y \; y(y + 2 - y^2) \\
&= 3\left[\frac{y^3}{3} + y^2 - \frac{y^4}{4}\right]_{-1}^{2} \\
&= 3\left[\frac{8}{3} + 4 - 4 + \frac{1}{3} - 1 + \frac{1}{4}\right] \\
&= \frac{27}{4}
\end{aligned}$$

(b) On the domain of integration,

- x runs from 0 to 4 and
- for each fixed x in that range, y runs from $\frac{1}{2}\sqrt{x}$ to 1.

The figure on the left below is a sketch of that domain, together with a generic vertical strip as was used in setting up the integral.





The inside integral, over y, looks pretty nasty because e^{y^3} does not have an obvious antiderivative. So let's reverse the order of integration. That is, let's use horizontal, rather than vertical, strips. From the figure on the right above, we see that, on the domain of integration

- y runs from 0 to 1 and
- for each fixed y in that range, x runs from 0 to $4y^2$.

So

$$\begin{aligned} I &= \int_0^1 \mathrm{d}y \int_0^{4y^2} \mathrm{d}x\ e^{y^3} \\ &= \int_0^1 \mathrm{d}y\ 4y^2 e^{y^3} \\ &= \frac{4}{3}\int_0^1 \mathrm{d}u\ e^u \qquad \text{with } u = y^3,\ \mathrm{d}u = 3y^2\,\mathrm{d}y \qquad \text{(Looks rigged!)} \\ &= \frac{4}{3}\big[e-1\big] \end{aligned}$$

$\boxed{\text{3.1.16}}$ (*) Consider the iterated integral

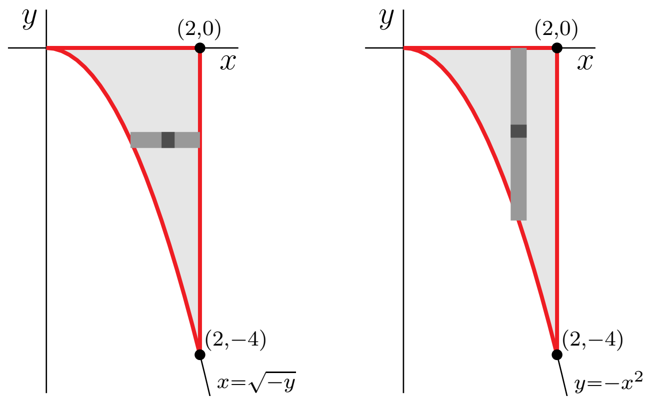
$$\int_{-4}^{0}\int_{\sqrt{-y}}^{2} \cos(x^3)\,\mathrm{d}x\,\mathrm{d}y$$

(a) Draw the region of integration.
(b) Evaluate the integral.

$\boxed{\text{Solution}}$ (a) On the domain of integration

- y runs from -4 to 0 and
- for each y in that range, x runs from $\sqrt{-y}$ (when $y = -x^2$) to 2.

The figure on the left below provides a sketch of the domain of integration. It also shows the generic horizontal slice that was used to set up the given iterated integral.



(b) The inside integral, $\int_{\sqrt{-y}}^{2} \cos(x^3)\,\mathrm{d}x$ looks nasty. So let's reverse the order of integration and use vertical, rather than horizontal, slices. From the figure on the right above, on the domain of integration,

- x runs from 0 to 2 and
- for each x in that range, y runs from $-x^2$ to 0.

So the integral

$$\begin{aligned}\int_{-4}^{0}\int_{\sqrt{-y}}^{2} \cos(x^3)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^2 \mathrm{d}x \int_{-x^2}^{0} \mathrm{d}y\ \cos(x^3)\\ &= \int_0^2 \mathrm{d}x\ x^2\ \cos(x^3) = \left[\frac{\sin(x^3)}{3}\right]_0^2\\ &= \frac{\sin(8)}{3}\end{aligned}$$

$\boxed{3.1.17}$ (∗)

(a) Combine the sum of the iterated integrals

$$I = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\ \mathrm{d}x\,\mathrm{d}y + \int_1^4 \int_{y-2}^{\sqrt{y}} f(x,y)\ \mathrm{d}x\,\mathrm{d}y$$

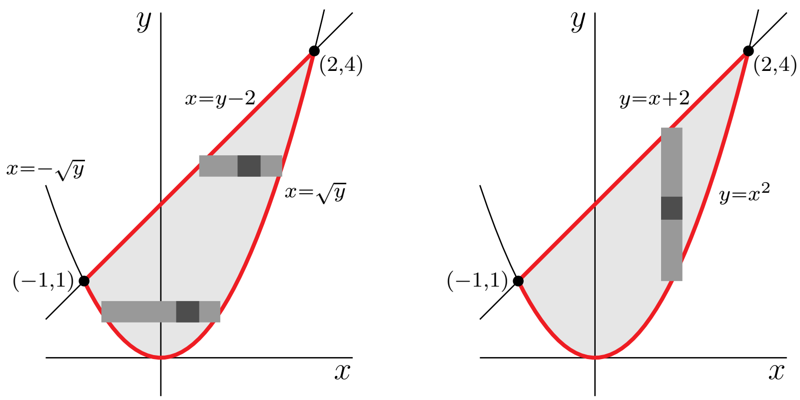
into a single iterated integral with the order of integration reversed.

(b) Evaluate I if $f(x,y) = \frac{e^x}{2-x}$.

$\boxed{\text{Solution}}$ (a) On the domain of integration

- y runs from 0 to 4 and
- for each y in the range $0 \leqslant y \leqslant 1$, x runs from $-\sqrt{y}$ to $\sqrt{y}$ and
- for each y in the range $1 \leqslant y \leqslant 4$, x runs from $y-2$ to $\sqrt{y}$.

Both figures below provide sketches of the domain of integration.



To reverse the order of integration observe, from the figure on the right above that, on the domain of integration,

- x runs from -1 to 2 and
- for each x in that range, y runs from x^2 to $x+2$.

So the integral

$$I = \int_{-1}^{2}\int_{x^2}^{x+2} f(x,y)\ \mathrm{d}y\,\mathrm{d}x$$

(b) We'll use the integral with the order of integration reversed that we found in part (a). When $f(x,y) = \frac{e^x}{2-x}$

$$\begin{aligned}
I &= \int_{-1}^{2}\int_{x^2}^{x+2} \frac{e^x}{2-x}\ \mathrm{d}y\,\mathrm{d}x \\
&= \int_{-1}^{2} (x+2-x^2)\frac{e^x}{2-x}\ \mathrm{d}x = -\int_{-1}^{2}(x-2)(x+1)\frac{e^x}{2-x}\ \mathrm{d}x \\
&= \int_{-1}^{2}(x+1)e^x\ \mathrm{d}x \\
&= \Big[xe^x\Big]_{-1}^{2} \\
&= 2e^2 + \frac{1}{e}
\end{aligned}$$

$\boxed{3.1.18}$ $(*)$ Let

$$I = \int_0^4 \int_{\sqrt{y}}^{\sqrt{8-y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

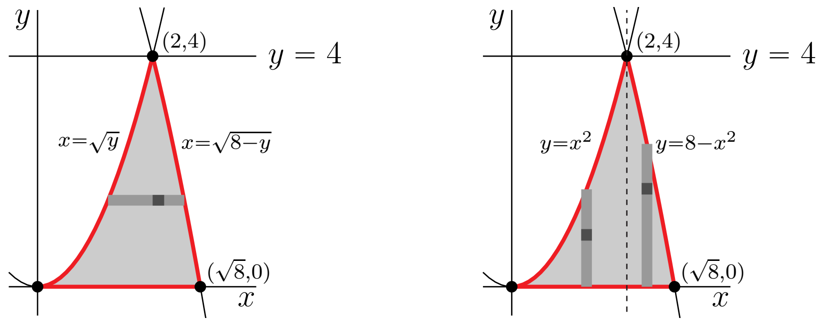
(a) Sketch the domain of integration.
(b) Reverse the order of integration.
(c) Evaluate the integral for $f(x,y) = \frac{1}{(1+y)^2}$.

Solution On the domain of integration

- y runs from 0 to 4. In inequalities, $0 \leqslant y \leqslant 4$.

- For each fixed y in that range, x runs from $\sqrt{y}$ to $\sqrt{8-y}$. In inequalities, that is $\sqrt{y} \leqslant x \leqslant \sqrt{8-y}$, or $y \leqslant x^2 \leqslant 8-y$.

Here are two sketchs of the domain of integration.



(b) To reverse the order we observe, from the figure on the right above, that, on the domain of integration,

- x runs from 0 to $\sqrt{8}$. In inequalities, $0 \leqslant x \leqslant \sqrt{8}$.
- For each fixed x between 0 and 2, y runs from 0 to x^2. In inequalities, that is $0 \leqslant y \leqslant x^2$.
- For each fixed x between 2 and $\sqrt{8}$, y runs from 0 to $8-x^2$. In inequalities, that is $0 \leqslant y \leqslant 8-x^2$.

So the integral is

$$\int_0^2 \int_0^{x^2} f(x,y)\,\mathrm{d}y\,\mathrm{d}x + \int_2^{\sqrt{8}} \int_0^{8-x^2} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

(c) We'll use the form of part (b).

$$\begin{aligned}
&\int_0^2 \int_0^{x^2} \frac{1}{(1+y)^2}\,\mathrm{d}y\,\mathrm{d}x + \int_2^{\sqrt{8}} \int_0^{8-x^2} \frac{1}{(1+y)^2}\,\mathrm{d}y\,\mathrm{d}x \\
&= -\int_0^2 \left[\frac{1}{1+y}\right]_0^{x^2} \mathrm{d}x - \int_2^{\sqrt{8}} \left[\frac{1}{1+y}\right]_0^{8-x^2} \mathrm{d}x \\
&= \int_0^2 \left[1 - \frac{1}{1+x^2}\right] \mathrm{d}x + \int_2^{\sqrt{8}} \left[1 - \frac{1}{9-x^2}\right] \mathrm{d}x \\
&= \sqrt{8} - \arctan x\Big|_0^2 - \frac{1}{6}\int_2^{\sqrt{8}} \left[\frac{1}{3+x} + \frac{1}{3-x}\right] \mathrm{d}x \\
&= \sqrt{8} - \arctan 2 - \frac{1}{6}\Big[\ln(3+x) - \ln(3-x)\Big]_2^{\sqrt{8}} \\
&= \sqrt{8} - \arctan 2 - \frac{1}{6}\left[\ln\frac{3+\sqrt{8}}{3-\sqrt{8}} - \ln 5\right]
\end{aligned}$$

$\boxed{3.1.19}$ (*) Evaluate

$$\int_{-1}^{0}\int_{-2}^{2x} e^{y^2}\,\mathrm{d}y\,\mathrm{d}x$$

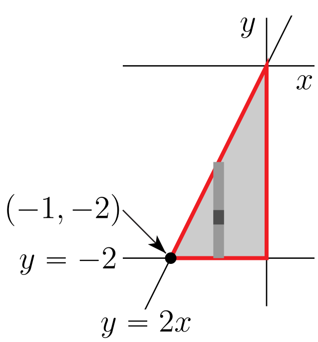
$\boxed{\text{Solution}}$ The antiderivative of the function e^{-y^2} cannot be expressed in terms of elementary functions. So the inside integral $\int_{-2}^{2x} e^{y^2}\,\mathrm{d}y$ cannot be evaluated using standard calculus 2 techniques. The trick for dealing with this integral is to reverse the order of integration. On the domain of integration

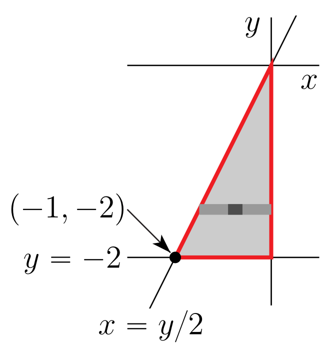
- x runs from -1 to 0. In inequalities, $-1 \leqslant x \leqslant 0$.
- For each fixed x in that range, y runs from -2 to $2x$. In inequalities, $-2 \leqslant y \leqslant 2x$.

The domain of integration, namely

$$\left\{\ (x,y)\ \middle|\ -1 \leqslant x \leqslant 0,\ -2 \leqslant y \leqslant 2x\ \right\}$$

is sketched in the figure on the left below.





Looking at the figure on the right above, we see that we can also express the domain of integration as

$$\left\{\ (x,y)\ \middle|\ -2 \leqslant y \leqslant 0,\ y/2 \leqslant x \leqslant 0\ \right\}$$

So the integral

$$\begin{aligned}\int_{-1}^{0}\int_{-2}^{2x} e^{y^2}\,\mathrm{d}y\,\mathrm{d}x &= \int_{-2}^{0}\int_{y/2}^{0} e^{y^2}\,\mathrm{d}x\,\mathrm{d}y \\ &= -\frac{1}{2}\int_{-2}^{0} ye^{y^2}\,\mathrm{d}y \\ &= -\frac{1}{2}\left[\frac{1}{2}e^{y^2}\right]_{-2}^{0} \\ &= \frac{1}{4}\left[e^4 - 1\right]\end{aligned}$$

$\boxed{3.1.20}$ (*) Let

$$I = \int_0^2 \int_0^x f(x,y)\ \mathrm{d}y\,\mathrm{d}x + \int_2^6 \int_0^{\sqrt{6-x}} f(x,y)\ \mathrm{d}y\,\mathrm{d}x$$

Express I as an integral where we integrate first with respect to x.

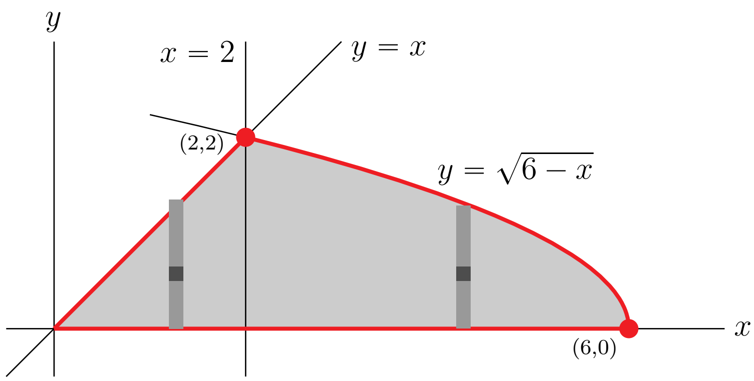
$\boxed{\text{Solution}}$ We first have to get a picture of the domain of integration. The first integral has domain of integration

$$\big\{\ (x,y)\ \big|\ 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant x\ \big\}$$

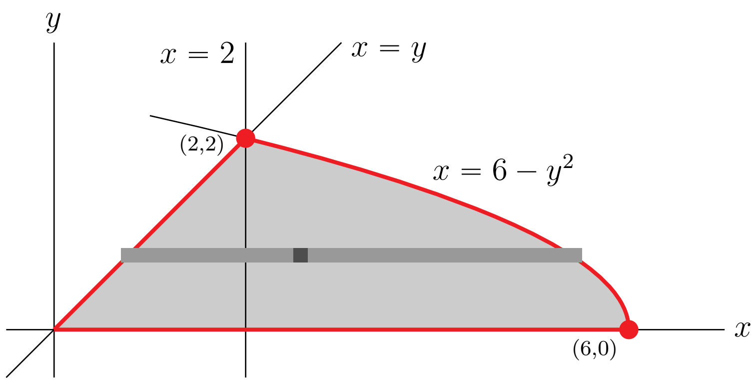
and the second integral has domain of integration

$$\big\{\ (x,y)\ \big|\ 2 \leqslant x \leqslant 6,\ 0 \leqslant y \leqslant \sqrt{6-x}\ \big\}$$

Here is a sketch. The domain of integration for the first integral is the shaded triangular region to the left of $x = 2$ and the domain of integration for the second integral is the shaded region to the right of $x = 2$.



To exchange the order of integration, we use horizontal slices as in the figure below.



The bottom slice has $y = 0$ and the top slice has $y = 2$. On the slice at height y, x runs from y to $6 - y^2$. So

$$I = \int_0^2 \int_y^{6-y^2} f(x,y)\ \mathrm{d}x\,\mathrm{d}y$$

$\boxed{3.1.21}$ (∗) Consider the domain D above the x–axis and below parabola $y = 1 - x^2$ in the xy–plane.

(a) Sketch D.

(b) Express

$$\iint_D f(x,y)\,\mathrm{d}A$$

as an iterated integral corresponding to the order $\mathrm{d}x\,\mathrm{d}y$. Then express this integral as an iterated integral corresponding to the order $\mathrm{d}y\,\mathrm{d}x$.

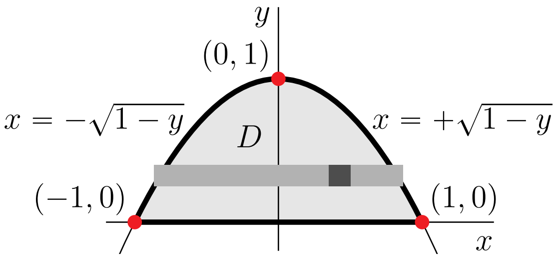
(c) Compute the integral in the case $f(x,y) = e^{x-(x^3/3)}$.

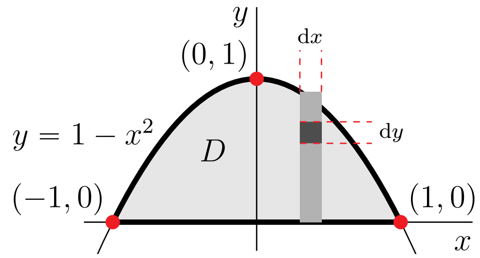
$\boxed{\text{Solution}}$ (a), (b) Looking at the figure on the left below, we see that we can write the domain

$$D = \left\{ (x,y) \;\middle|\; 0 \leqslant y \leqslant 1,\ -\sqrt{1-y} \leqslant x \leqslant \sqrt{1-y} \right\}$$

So

$$\iint_D f(x,y)\,\mathrm{d}A = \int_0^1 \mathrm{d}y \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \mathrm{d}x\ f(x,y) = \int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$





Looking at the figure on the right above, we see that we can write the domain

$$D = \left\{ (x,y) \;\middle|\; -1 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x^2 \right\}$$

So

$$\iint_D f(x,y)\,\mathrm{d}A = \int_{-1}^1 \mathrm{d}x \int_0^{1-x^2} \mathrm{d}y\ f(x,y) = \int_{-1}^1 \int_0^{1-x^2} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

(c) Using the second form from part (b),

$$\begin{aligned}
\iint_D e^{x-(x^3/3)}\,\mathrm{d}A &= \int_{-1}^1 \mathrm{d}x \int_0^{1-x^2} \mathrm{d}y\ e^{x-(x^3/3)} \\
&= \int_{-1}^1 (1-x^2)e^{x-(x^3/3)}\,\mathrm{d}x \\
&= \int_{-2/3}^{2/3} e^u\,\mathrm{d}u \qquad \text{with } u = x - \frac{x^3}{3},\ \mathrm{d}u = \left(1-x^2\right)\mathrm{d}x \\
&= e^{2/3} - e^{-2/3}
\end{aligned}$$

$\boxed{3.1.22}$ (∗) Let $I = \int_0^1 \int_{x^2}^1 x^3 \ \sin(y^3)\ \mathrm{d}y\ \mathrm{d}x$.

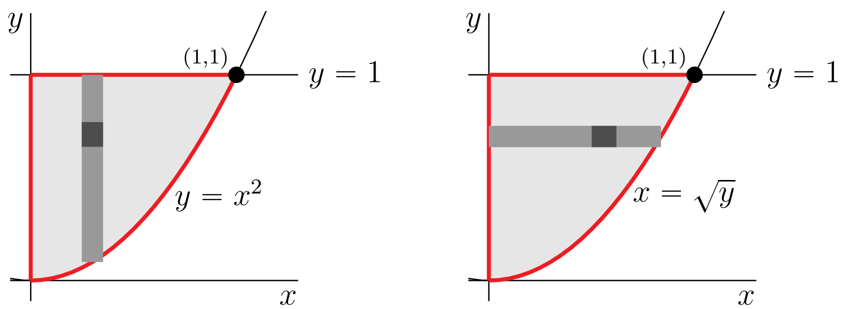
(a) Sketch the region of integration in the xy–plane. Label your sketch sufficiently well that one could use it to determine the limits of double integration.

(b) Evaluate I.

Solution (a) On the domain of integration,

- x runs from 0 to 1 and
- for each fixed x in that range, y runs from x^2 to 1.

The figure on the left below is a sketch of that domain, together with a generic vertical strip as was used in setting up the integral.



(b) As it stands, the inside integral, over y, looks pretty nasty because $\sin(y^3)$ does not have an obvious antiderivative. So let's reverse the order of integration. The given integral was set up using vertical strips. So, to reverse the order of integration, we use horizontal strips as in the figure on the right above. Looking at that figure we see that, on the domain of integration,

- y runs from 0 to 1 and
- for each fixed y in that range, x runs from 0 to $\sqrt{y}$.

So

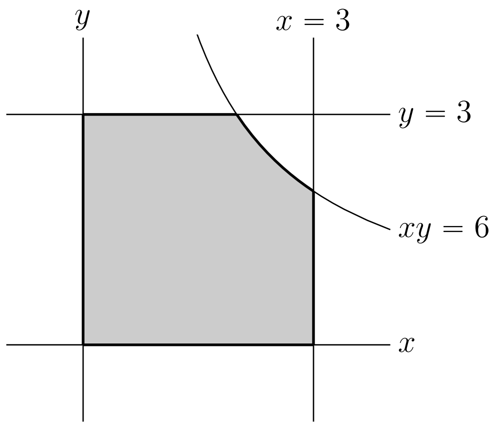
$$\begin{aligned}
I &= \int_0^1 \mathrm{d}y \int_0^{\sqrt{y}} \mathrm{d}x\ x^3 \ \sin(y^3) \\
&= \int_0^1 \mathrm{d}y\ \sin(y^3) \left[\frac{x^4}{4}\right]_0^{\sqrt{y}} \\
&= \frac{1}{4}\int_0^1 \mathrm{d}y\ y^2 \sin(y^3) \\
&= \frac{1}{4}\left[-\frac{\cos(y^3)}{3}\right]_0^1 \\
&= \frac{1-\cos(1)}{12}
\end{aligned}$$

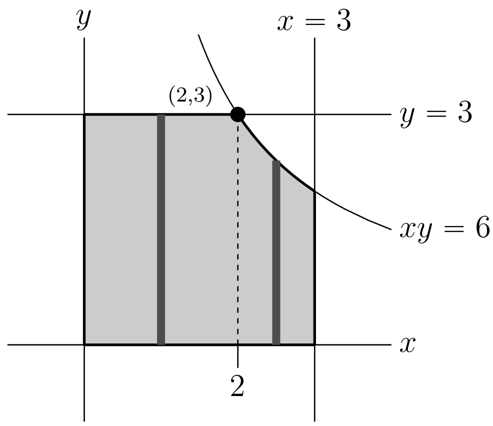
$\boxed{3.1.23}$ (∗) Consider the solid under the surface $z = 6 - xy$, bounded by the five planes $x = 0$, $x = 3$, $y = 0$, $y = 3$, $z = 0$. Note that no part of the solid lies below the x–y plane.

(a) Sketch the base of the solid in the xy–plane. Note that it is not a square!

(b) Compute the volume of the solid.

$\boxed{\text{Solution}}$ (a) The solid is the set of all (x,y,z) obeying $0 \leqslant x \leqslant 3$, $0 \leqslant y \leqslant 3$ and $0 \leqslant z \leqslant 6 - xy$. The base of this region is the set of all (x,y) for which there is a z such that (x,y,z) is in the solid. So the base is the set of all (x,y) obeying $0 \leqslant x \leqslant 3$, $0 \leqslant y \leqslant 3$ and $6 - xy \geqslant 0$, i.e. $xy \leqslant 6$. This region is sketched in the figure on the left below.





(b) We'll deompose the base region into vertical strips as in the figure on the right above. Observe that the line $y = 3$ intersects the curve $xy = 6$ at the point $(2,3)$ and that on the base

- x runs from 0 to 3 and that
- for each fixed x between 0 and 2, y runs from 0 to 3, while
- for each fixed x between 2 and 3, y runs from 0 to $6/x$

and that, for each (x,y) in the base, z runs from 0 to $6 - xy$. So the

$$\begin{aligned}
\text{Volume} &= \int_0^2 \mathrm{d}x \int_0^3 \mathrm{d}y\ (6 - xy) + \int_2^3 \mathrm{d}x \int_0^{6/x} \mathrm{d}y\ (6 - xy) \\
&= \int_0^2 \mathrm{d}x\ \left[6y - \frac{1}{2}xy^2\right]_0^3 + \int_2^3 \mathrm{d}x\ \left[6y - \frac{1}{2}xy^2\right]_0^{6/x} \\
&= \int_0^2 \mathrm{d}x\ \left[18 - \frac{9}{2}x\right] + \int_2^3 \mathrm{d}x\ \left[\frac{36}{x} - \frac{18}{x}\right] \\
&= \left[18x - \frac{9}{4}x^2\right]_0^2 + \left[18 \ln x\right]_2^3 = 27 + 18 \ln \frac{3}{2} \approx 34.30
\end{aligned}$$

3.1.24 (*) Evaluate the following integral:

$$\int_{-2}^{2}\int_{x^2}^{4}\cos\left(y^{3/2}\right)\,\mathrm{d}y\,\mathrm{d}x$$

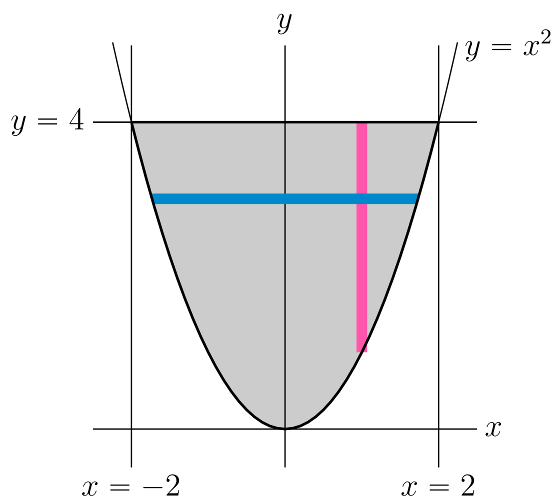
Solution In the given integral

- x runs from -2 to 2 and
- for each fixed x between -2 and 2, y runs from x^2 to 4

So the domain of integration is

$$D = \left\{\ (x,y)\ \middle|\ -2 \leqslant x \leqslant 2,\ x^2 \leqslant y \leqslant 4\ \right\}$$

This is sketched below.



The inside integral, $\int_{x^2}^{4}\cos\left(y^{3/2}\right)\,\mathrm{d}y$, in the given integral looks really nasty. So let's try exchanging the order of integration. The given integral was formed by decomposing the domain of integration D into horizontal strips, like the blue strip in the figure above. To exchange the order of integration we instead decompose the domain of integration D into vertical strips, like the pink strip in the figure above. To do so, we observe that, on D,

- y runs from 0 to 4 and
- for each fixed y between 0 and 4, x runs from $-\sqrt{y}$ to $\sqrt{y}$.

That is, we reexpress the domain of integration as

$$D = \left\{\ (x,y)\ \middle|\ 0 \leqslant y \leqslant 4,\ -\sqrt{y} \leqslant x \leqslant \sqrt{y}\ \right\}$$

and the given integral as

$$\begin{aligned}\int_{-2}^{2}\int_{x^2}^{4}\cos\left(y^{3/2}\right)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^4 \mathrm{d}y\int_{-\sqrt{y}}^{\sqrt{y}}\mathrm{d}x\ \cos\left(y^{3/2}\right)\\ &= \int_0^4 \mathrm{d}y\ 2\sqrt{y}\cos\left(y^{3/2}\right)\\ &= \frac{4}{3}\int_0^8 \mathrm{d}t\ \cos t \quad \text{where } t = y^{3/2},\ \mathrm{d}t = \frac{3}{2}\sqrt{y}\ \mathrm{d}y\\ &= \frac{4}{3}\sin t\Big|_0^8 = \frac{4}{3}\sin 8 \approx 1.319\end{aligned}$$

$\boxed{3.1.25}$ (*) Consider the volume above the xy-plane that is inside the circular cylinder $x^2+y^2=2y$ and underneath the surface $z=8+2xy$.

(a) Express this volume as a double integral I, stating clearly the domain over which I is to be taken.

(b) Express in Cartesian coordinates, the double integral I as an iterated intergal in two different ways, indicating clearly the limits of integration in each case.

(c) How much is this volume?

$\boxed{\text{Solution}}$ (a) We may rewrite the equation $x^2+y^2=2y$ of the cylinder as $x^2+(y-1)^2=1$. We are (in part (c)) to find the volume of the set

$$V = \left\{\ (x,y,z)\ \middle|\ x^2+(y-1)^2 \leqslant 1,\ 0 \leqslant z \leqslant 8+2xy\ \right\}$$

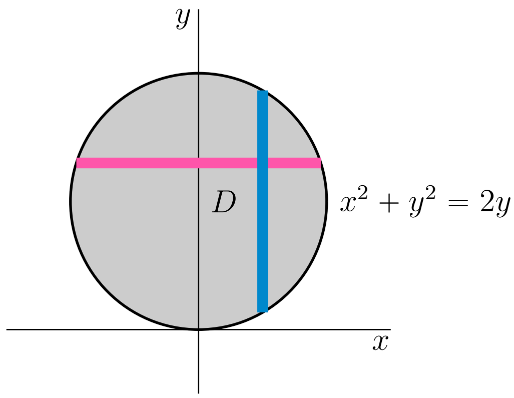
When we look at this solid from far above (so that we can't see z) we see the set of points (x,y) that obey $x^2+(y-1)^2 \leqslant 1$ and $8+2xy \geqslant 0$ (so that there is at least one allowed z for that (x,y)). All points in $x^2+(y-1)^2 \leqslant 1$ have $-1 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 2$ and hence $-2 \leqslant xy \leqslant 2$ and $8+2xy \geqslant 0$. So the domain of integration consists of the full disk

$$D = \left\{\ (x,y)\ \middle|\ x^2+(y-1)^2 \leqslant 1\ \right\}$$

The volume is

$$I = \iint_D (8+2xy)\ \mathrm{d}x\mathrm{d}y$$

(b) We can express the double integral over D as iterated integrals by decomposing D into horizontal strips, like the pink strip in the figure below, and also by decomposing D into blue strips, like the blue strip in the figure below.



For horizontal strips, we use that, on D

- y runs from 0 to 2 and,
- for each fixed y between 0 and 2, x runs from $-\sqrt{2y-y^2}$ to $\sqrt{2y-y^2}$

so that

$$D = \left\{ (x,y) \mid 0 \leqslant y \leqslant 2,\ -\sqrt{2y-y^2} \leqslant x \leqslant \sqrt{2y-y^2} \right\}$$

For vertical strips, we use that, on D

- x runs from -1 to 1 and,
- for each fixed x between -1 and 1, y runs from $1-\sqrt{1-x^2}$ to $1+\sqrt{1-x^2}$

so that

$$D = \left\{ (x,y) \mid -1 \leqslant x \leqslant 1,\ 1-\sqrt{1-x^2} \leqslant x \leqslant 1+\sqrt{1-x^2} \right\}$$

Thus

$$\begin{aligned} I &= \int_0^2 \mathrm{d}y \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \mathrm{d}x\ (8+2xy) \\ &= \int_{-1}^1 \mathrm{d}x \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} \mathrm{d}y\ (8+2xy) \end{aligned}$$

(c) Since $\iint_D 8\ \mathrm{d}x\mathrm{d}y$ is just 8 times the area of D, which is π,

$$\begin{aligned} \text{Volume} &= 8\pi + \int_0^2 \mathrm{d}y \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \mathrm{d}x\ 2xy = 8\pi + 2\int_0^2 \mathrm{d}y\ y \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \mathrm{d}x\ x \\ &= 8\pi \end{aligned}$$

because $\int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \mathrm{d}x\ x = 0$ for all y, because the integrand is odd and the domain of integration is even.

3.1.26 (∗) Evaluate the following integral:

$$\int_0^9 \int_{\sqrt{y}}^3 \sin(\pi x^3)\ \mathrm{d}x\mathrm{d}y$$

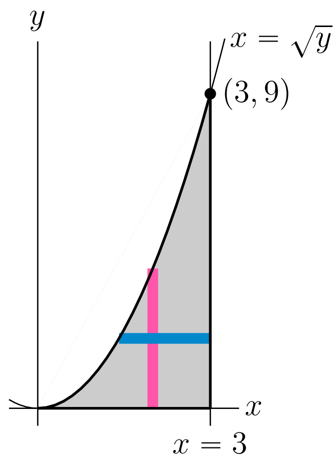
$\boxed{\text{Solution}}$ In the given integral

- y runs from 0 to 9 and
- for each fixed y between 0 and 9, x runs from $\sqrt{y}$ to 3

So the domain of integration is

$$D = \left\{ (x,y) \;\middle|\; 0 \leqslant y \leqslant 9,\ \sqrt{y} \leqslant x \leqslant 3 \right\}$$

This is sketched below.



The inside integral, $\int_{\sqrt{y}}^{3} \sin\left(\pi x^3\right)\,\mathrm{d}x$, in the given integral looks really nasty. So let's try exchanging the order of integration. The given integral was formed by decomposing the domain of integration D into horizontal strips, like the blue strip in the figure above. To exchange the order of integration we instead decompose the domain of integration D into vertical strips, like the pink strip in the figure above. To do so, we observe that, on D,

- x runs from 0 to 3 and
- for each fixed x between 0 and 3, y runs from 0 to x^2.

That is, we reexpress the domain of integration as

$$D = \left\{ (x,y) \;\middle|\; 0 \leqslant x \leqslant 3,\ 0 \leqslant y \leqslant x^2 \right\}$$

and the given integral as

$$\begin{aligned}
\int_0^9 \int_{\sqrt{y}}^3 \sin(\pi x^3)\,\mathrm{d}x\mathrm{d}y &= \int_0^3 \mathrm{d}x \int_0^{x^2} \mathrm{d}y\ \sin(\pi x^3) \\
&= \int_0^3 \mathrm{d}x\ x^2 \sin(\pi x^3) \\
&= \frac{1}{3\pi}\int_0^{27\pi} \mathrm{d}t\ \sin t \quad \text{where } t = \pi x^3,\ \mathrm{d}t = 3\pi x^2\,\mathrm{d}x \\
&= -\frac{1}{3\pi}\cos t\Big|_0^{27\pi} = -\frac{1}{3\pi}\cos t\Big|_0^{\pi} = \frac{2}{3\pi} \approx 0.212
\end{aligned}$$

$\boxed{3.1.27}$ (*) The iterated integral

$$I = \int_0^1 \left[\int_{-\sqrt{x}}^{\sqrt{x}} \sin\left(y^3 - 3y\right) \mathrm{d}y \right] \mathrm{d}x$$

is equal to $\iint_R \sin\left(y^3 - 3y\right) dA$ for a suitable region R in the xy-plane.

(a) Sketch the region R.

(b) Write the integral I with the orders of integration reversed, and with suitable limits of integration.

(c) Find I.

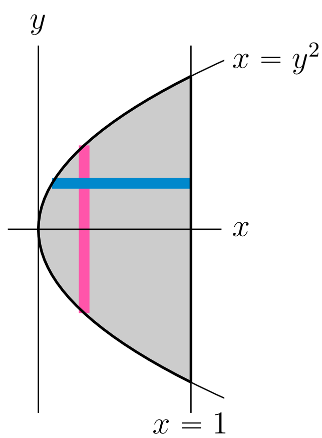
$\boxed{\text{Solution}}$ (a) In the given integral

- x runs from 0 to 1, and
- for each fixed x between 0 and 1, y runs from $-\sqrt{x}$ to $\sqrt{x}$.

So the region

$$R = \left\{ (x, y) \mid 0 \leqslant x \leqslant 1,\ -\sqrt{x} \leqslant y \leqslant \sqrt{x} \right\}$$

It is sketched below.



(b) The given integral was formed by decomposing the domain of integration R into vertical strips, like the pink strip in the figure above. To exchange the order of integration we instead decompose the domain of integration R into horizontal strips, like the blue strip in the figure above. To do so, we observe that, on R,

- y runs from -1 to 1, and
- for each fixed y between -1 and 1, x runs from y^2 to 1.

So

$$I = \int_{-1}^1 \left[\int_{y^2}^1 \sin\left(y^3 - 3y\right) \mathrm{d}x \right] \mathrm{d}y$$

(c) The easy way to evaluate I is to observe that, since $\sin\left(y^3 - 3y\right)$ is odd under $y \to -y$, the integral

$$\int_{-\sqrt{x}}^{\sqrt{x}} \sin\left(y^3 - 3y\right) \mathrm{d}y = 0$$

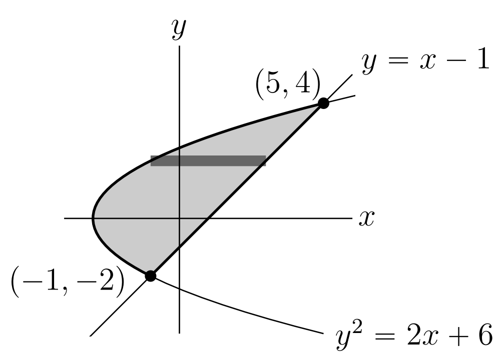
for all x. Hence $I = 0$. The hard way is

$$\begin{aligned}
I &= \int_{-1}^{1} \left[\int_{y^2}^{1} \sin\left(y^3 - 3y\right) \mathrm{d}x\right] \mathrm{d}y \\
&= \int_{-1}^{1} (1 - y^2) \sin\left(y^3 - 3y\right) \mathrm{d}y \\
&= \int_{2}^{-2} \sin t \, \frac{\mathrm{d}t}{-3} \qquad \text{where } t = y^3 - 3y,\ \mathrm{d}t = 3(y^2 - 1)\,\mathrm{d}y \\
&= \frac{1}{3} \cos t \,\Big|_{2}^{-2} = 0
\end{aligned}$$

again, since $\cos$ is even.

3.1.28 (*) Find the double integral of the function $f(x, y) = xy$ over the region bounded by $y = x - 1$ and $y^2 = 2x + 6$.

Solution The parabola $y^2 = 2x + 6$ and the line $y = x - 1$ meet when $x = y + 1$ with $y^2 = 2(y + 1) + 6$ or $y^2 - 2y - 8 = (y - 4)(y + 2) = 0$. So they meet at $(-1, -2)$ and $(5, 4)$. The domain of integration is sketched below.



On this domain

- y runs from -2 to 4, and
- for each fixed y between -2 and 4, x runs from $\frac{y^2}{2} - 3$ to $y + 1$.

So the integral is

$$\begin{aligned}
\int_{-2}^{4} \mathrm{d}y \int_{y^2/2-3}^{y+1} \mathrm{d}x\ xy &= \int_{-2}^{4} \mathrm{d}y\ \frac{1}{2}x^2 y\Big|_{y^2/2-3}^{y+1} \\
&= \frac{1}{2}\int_{-2}^{4} \mathrm{d}y\ \left[y^3+2y^2+y-\frac{1}{4}y^5+3y^3-9y\right] \\
&= \frac{1}{2}\int_{-2}^{4} \mathrm{d}y\ \left[-8y+2y^2+4y^3-\frac{1}{4}y^5\right] \\
&= \left[-2y^2+\frac{1}{3}y^3+\frac{1}{2}y^4-\frac{1}{48}y^6\right]_{-2}^{4} \\
&= -2(16-4)+\frac{1}{3}(64+8)+\frac{1}{2}(256-16)-\frac{1}{48}(4096-64) \\
&= -24+24+120-84 = 36
\end{aligned}$$

▸▸ Stage 3

$\boxed{\text{3.1.29}}$ Find the volume of the solid inside the cylinder $x^2+2y^2=8$, above the plane $z=y-4$ and below the plane $z=8-x$.

$\boxed{\text{Solution}}$ Looking down from the top, we see the cylinder $x^2+2y^2 \leqslant 8$. That gives the base region. The top of the solid, above any fixed (x,y) in the base region, is at $z=8-x$ (this is always positive because x never gets bigger than $\sqrt{8}$) . The bottom of the solid, below any fixed (x,y) in the base region, is at $z=y-4$ (this is always negative because y is always smaller than 2). So the height of the solid at any (x,y) is

$$z_{\mathrm{top}} - z_{\mathrm{bottom}} = (8-x)-(y-4) = 12-x-y$$

The volume is

$$\int_{-2}^{2} \mathrm{d}y \int_{-\sqrt{8-2y^2}}^{\sqrt{8-2y^2}} \mathrm{d}x\ (12-x-y)$$

Recall, from Theorem 1.2.11 in the CLP-2 text, that if $f(x)$ is an odd function (meaning that $f(-x)=-f(x)$ for all x), then $\int_{-a}^{a} f(x)\ \mathrm{d}x = 0$ (because the two integrals $\int_0^a f(x)\ \mathrm{d}x$ and $\int_{-a}^{0} f(x)\ \mathrm{d}x$ have the same magnitude but opposite signs). Applying this twice gives

$$\int_{-\sqrt{8-2y^2}}^{\sqrt{8-2y^2}} \mathrm{d}x\ x = 0 \text{ and } \int_{-2}^{2} \mathrm{d}y \int_{-\sqrt{8-2y^2}}^{\sqrt{8-2y^2}} \mathrm{d}x\ y = \int_{-2}^{2} \mathrm{d}y\ 2y\sqrt{8-2y^2} = 0$$

since x and $y\sqrt{8-2y^2}$ are both odd. Thus

$$\int_{-2}^{2} \mathrm{d}y \int_{-\sqrt{8-2y^2}}^{\sqrt{8-2y^2}} \mathrm{d}x\ (-x-y) = 0 \implies \text{Volume} = \int_{-2}^{2} \mathrm{d}y \int_{-\sqrt{8-2y^2}}^{\sqrt{8-2y^2}} \mathrm{d}x\ 12$$

so that the volume is just 12 times the area of the ellipse $x^2+2y^2=8$, which is

$$12\big(\pi\sqrt{8}\,2\big) = 48\sqrt{2}\,\pi$$

3.2▲ Double Integrals in Polar Coordinates

▸▸ Stage 1

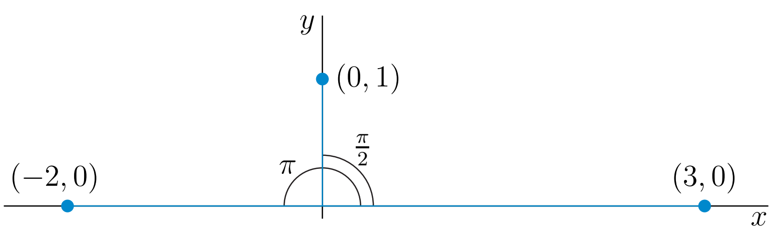
3.2.1 Consider the points

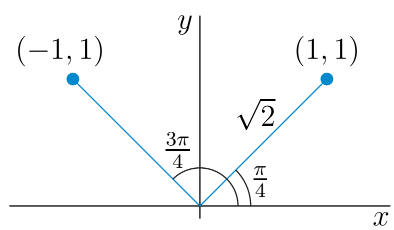
$$(x_1, y_1) = (3, 0) \qquad (x_2, y_2) = (1, 1) \qquad (x_3, y_3) = (0, 1)$$
$$(x_4, y_4) = (-1, 1) \qquad (x_5, y_5) = (-2, 0)$$

For each $1 \leqslant i \leqslant 5$,

- sketch, in the xy-plane, the point (x_i, y_i) and
- find the polar coordinates r_i and θ_i, with $0 \leqslant \theta_i < 2\pi$, for the point (x_i, y_i).

Solution The left hand sketch below contains the points, (x_1, y_1), (x_3, y_3), (x_5, y_5), that are on the axes. The right hand sketch below contains the points, (x_2, y_2), (x_4, y_4), that are not on the axes.





Recall that the polar coordinates r, θ are related to the cartesian coordinates x, y, by $x = r\cos\theta$, $y = r\sin\theta$. So $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$ (assuming that $x \neq 0$) and

$$\begin{aligned}
(x_1, y_1) = (3, 0) &\implies r_1 = 3,\ \tan\theta_1 = 0 \implies \theta_1 = 0 \text{ as } (x_1, y_1) \text{ is on the positive } x\text{-axis}\\
(x_2, y_2) = (1, 1) &\implies r_2 = \sqrt{2},\ \tan\theta_2 = 1 \implies \theta_2 = \frac{\pi}{4} \text{ as } (x_2, y_2) \text{ is in the first octant}\\
(x_3, y_3) = (0, 1) &\implies r_3 = 1,\ \cos\theta_3 = 0 \implies \theta_3 = \frac{\pi}{2} \text{ as } (x_3, y_3) \text{ is on the positive } y\text{-axis}\\
(x_4, y_4) = (-1, 1) &\implies r_4 = \sqrt{2},\ \tan\theta_4 = -1 \implies \theta_4 = \frac{3\pi}{4} \text{ as } (x_4, y_4) \text{ is in the third octant}\\
(x_5, y_5) = (-2, 0) &\implies r_5 = 2,\ \tan\theta_5 = 0 \implies \theta_5 = \pi \text{ as } (x_5, y_5) \text{ is on the negative } x\text{-axis}
\end{aligned}$$

3.2.2

(a) Find all pairs (r, θ) such that

$$(-2, 0) = \big(r\cos\theta\,,\, r\sin\theta\big)$$

(b) Find all pairs (r, θ) such that

$$(1, 1) = \big(r\cos\theta\,,\, r\sin\theta\big)$$

(c) Find all pairs (r, θ) such that

$$(-1, -1) = \big(r\cos\theta\,,\, r\sin\theta\big)$$

Solution Note that the distance from the point $\big(r\cos\theta\,,\, r\sin\theta\big)$ to the origin is

$$\sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = \sqrt{r^2} = |r|$$

Thus r can be either the distance to the origin or minus the distance to the origin.

(a) The distance from $(-2, 0)$ to the origin is 2. So either $r = 2$ or $r = -2$.

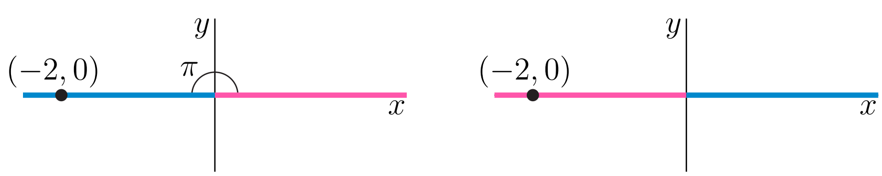
- If $r = 2$, then θ must obey

$$\begin{aligned}(-2, 0) = \big(2\cos\theta\,,\, 2\sin\theta\big) &\iff \sin\theta = 0,\ \cos\theta = -1\\ &\iff \theta = n\pi,\ n \text{ integer },\ \cos\theta = -1\\ &\iff \theta = n\pi,\ n \text{ odd integer}\end{aligned}$$

- If $r = -2$, then θ must obey

$$\begin{aligned}(-2, 0) = \big(-2\cos\theta\,,\, -2\sin\theta\big) &\iff \sin\theta = 0,\ \cos\theta = 1\\ &\iff \theta = n\pi,\ n \text{ integer },\ \cos\theta = 1\\ &\iff \theta = n\pi,\ n \text{ even integer}\end{aligned}$$

In the figure on the left below, the blue half-line is the set of all points with polar coordinates $\theta = \pi$, $r > 0$ and the pink half-line is the set of all points with polar coordinates $\theta = \pi$, $r < 0$. In the figure on the right below, the blue half-line is the set of all points with polar coordinates $\theta = 0$, $r > 0$ and the pink half-line is the set of all points with polar coordinates $\theta = 0$, $r < 0$.



(b) The distance from $(1, 1)$ to the origin is $\sqrt{2}$. So either $r = \sqrt{2}$ or $r = -\sqrt{2}$.

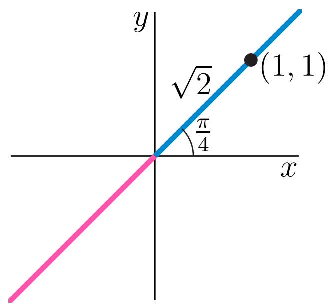
- If $r = \sqrt{2}$, then θ must obey

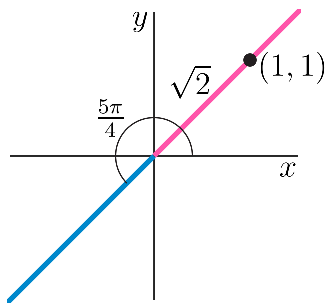
$$\begin{aligned}(1, 1) = \big(\sqrt{2}\,\cos\theta\,,\, \sqrt{2}\,\sin\theta\big) &\iff \sin\theta = \cos\theta = {}^1\!/\!\sqrt{2}\\ &\iff \theta = {}^\pi\!/\!_4 + 2n\pi,\ n \text{ integer}\end{aligned}$$

- If $r = -\sqrt{2}$, then θ must obey

$$(1,1) = \big(-\sqrt{2}\cos\theta\,,\ -\sqrt{2}\sin\theta\big) \iff \sin\theta = \cos\theta = -1/\sqrt{2}$$
$$\iff \theta = 5\pi/4 + 2n\pi,\ n \text{ integer}$$

In the figure on the left below, the blue half-line is the set of all points with polar coordinates $\theta = \frac{\pi}{4}$, $r > 0$ and the pink half-line is the set of all points with polar coordinates $\theta = \frac{\pi}{4}$, $r < 0$. In the figure on the right below, the blue half-line is the set of all points with polar coordinates $\theta = \frac{5\pi}{4}$, $r > 0$ and the pink half-line is the set of all points with polar coordinates $\theta = \frac{5\pi}{4}$, $r < 0$.





(c) The distance from $(-1,-1)$ to the origin is $\sqrt{2}$. So either $r = \sqrt{2}$ or $r = -\sqrt{2}$.

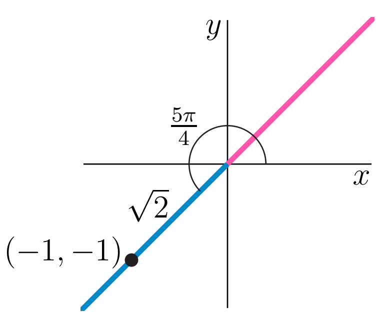
- If $r = \sqrt{2}$, then θ must obey

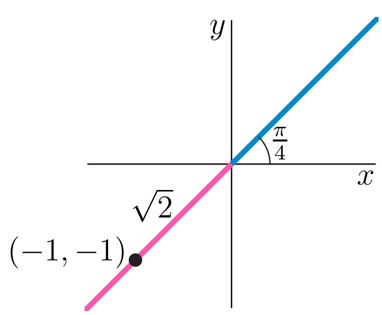
$$(-1,-1) = \big(\sqrt{2}\cos\theta\,,\ \sqrt{2}\sin\theta\big) \iff \sin\theta = \cos\theta = -1/\sqrt{2}$$
$$\iff \theta = 5\pi/4 + 2n\pi,\ n \text{ integer}$$

- If $r = -\sqrt{2}$, then θ must obey

$$(-1,-1) = \big(-\sqrt{2}\cos\theta\,,\ -\sqrt{2}\sin\theta\big) \iff \sin\theta = \cos\theta = 1/\sqrt{2}$$
$$\iff \theta = \pi/4 + 2n\pi,\ n \text{ integer}$$

In the figure on the left below, the blue half-line is the set of all points with polar coordinates $\theta = \frac{5\pi}{4}$, $r > 0$ and the pink half-line is the set of all points with polar coordinates $\theta = \frac{5\pi}{4}$, $r < 0$. In the figure on the right below, the blue half-line is the set of all points with polar coordinates $\theta = \frac{\pi}{4}$, $r > 0$ and the pink half-line is the set of all points with polar coordinates $\theta = \frac{\pi}{4}$, $r < 0$.





3.2.3 Consider the points

$$(x_1, y_1) = (3, 0) \qquad (x_2, y_2) = (1, 1) \qquad (x_3, y_3) = (0, 1)$$
$$(x_4, y_4) = (-1, 1) \qquad (x_5, y_5) = (-2, 0)$$

Also define, for each angle θ, the vectors

$$\hat{\mathbf{e}}_r(\theta) = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \qquad \hat{\mathbf{e}}_\theta(\theta) = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$$

(a) Determine, for each angle θ, the lengths of the vectors $\hat{\mathbf{e}}_r(\theta)$ and $\hat{\mathbf{e}}_\theta(\theta)$ and the angle between the vectors $\hat{\mathbf{e}}_r(\theta)$ and $\hat{\mathbf{e}}_\theta(\theta)$. Compute $\hat{\mathbf{e}}_r(\theta) \times \hat{\mathbf{e}}_\theta(\theta)$ (viewing $\hat{\mathbf{e}}_r(\theta)$ and $\hat{\mathbf{e}}_\theta(\theta)$ as vectors in three dimensions with zero $\hat{\mathbf{k}}$ components).

(b) For each $1 \leqslant i \leqslant 5$, sketch, in the xy-plane, the point (x_i, y_i) and the vectors $\hat{\mathbf{e}}_r(\theta_i)$ and $\hat{\mathbf{e}}_\theta(\theta_i)$. In your sketch of the vectors, place the tails of the vectors $\hat{\mathbf{e}}_r(\theta_i)$ and $\hat{\mathbf{e}}_\theta(\theta_i)$ at (x_i, y_i).

Solution (a) The lengths are

$$|\hat{\mathbf{e}}_r(\theta)| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$
$$|\hat{\mathbf{e}}_\theta(\theta)| = \sqrt{(-\sin\theta)^2 + \cos^2\theta = 1}$$

As

$$\hat{\mathbf{e}}_r(\theta) \cdot \hat{\mathbf{e}}_\theta(\theta) = (\cos\theta)(-\sin\theta) + (\sin\theta)(\cos\theta) = 0$$

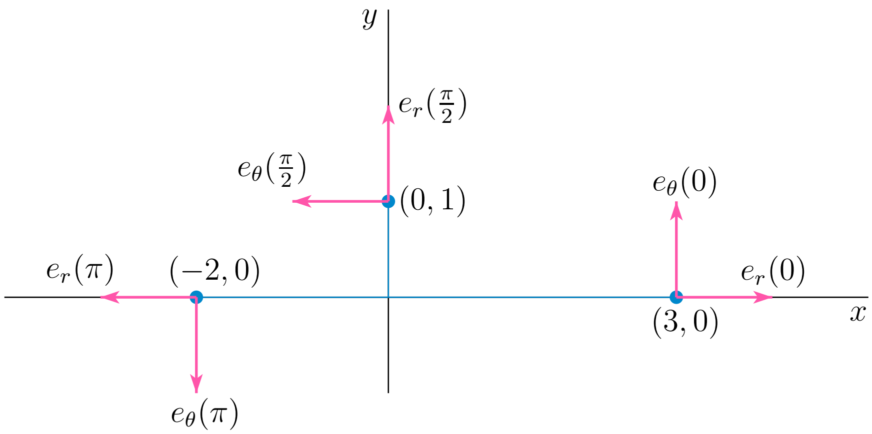
the two vectors are perpendicular and the angle between them is $\frac{\pi}{2}$. The cross product is

$$\hat{\mathbf{e}}_r(\theta) \times \hat{\mathbf{e}}_\theta(\theta) = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} = \hat{\mathbf{k}}$$

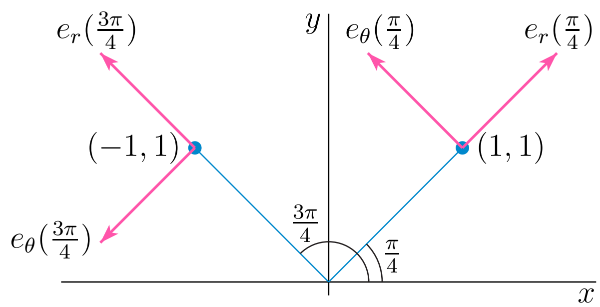
(b) Note that for θ determined by $x = r\cos\theta$, $y = r\sin\theta$,

- the vector $\hat{\mathbf{e}}_r(\theta)$ is a unit vector in the same direction as the vector from $(0, 0)$ to (x, y) and
- the vector $\hat{\mathbf{e}}_\theta(\theta)$ is a unit vector that is perpendicular to $\hat{\mathbf{e}}_r(\theta)$.
- The y-component of $\hat{\mathbf{e}}_\theta(\theta)$ has the same sign as the x-component of $\hat{\mathbf{e}}_r(\theta)$. The x-component of $\hat{\mathbf{e}}_\theta(\theta)$ has opposite sign to that of the y-component of $\hat{\mathbf{e}}_r(\theta)$.

Here is a sketch of (x_i, y_i), $\hat{\mathbf{e}}_r(\theta_i)$, $\hat{\mathbf{e}}_\theta(\theta_i)$ for $i = 1, 3, 5$ (the points on the axes)



and here is a sketch (to a different scale) of (x_i, y_i), $\hat{\mathbf{e}}_r(\theta_i)$, $\hat{\mathbf{e}}_\theta(\theta_i)$ for $i = 2, 4$ (the points off the axes).

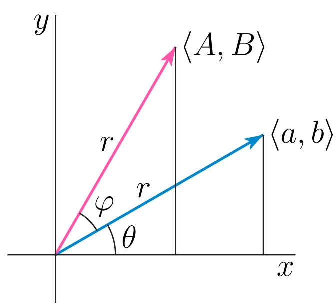


$\boxed{3.2.4}$ Let $\langle a, b\rangle$ be a vector. Let r be the length of $\langle a, b\rangle$ and θ be the angle between $\langle a, b\rangle$ and the x-axis.

(a) Express a and b in terms of r and θ.

(b) Let $\langle A, B\rangle$ be the vector gotten by rotating $\langle a, b\rangle$ by an angle φ about its tail. Express A and B in terms of a, b and φ.

$\boxed{\text{Solution}}$ Here is a sketch of $\langle a, b\rangle$ and $\langle A, B\rangle$.



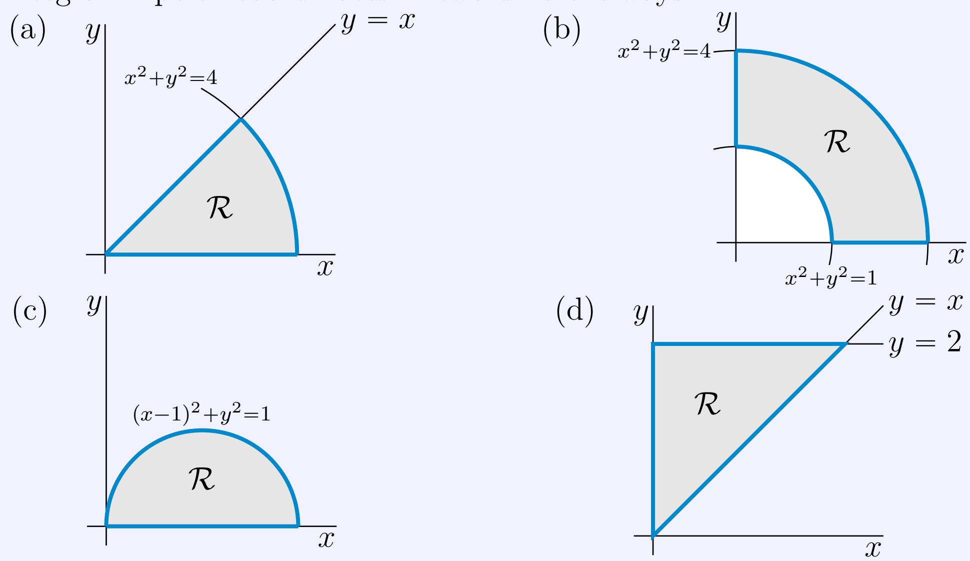
(a) From the sketch,

$$a = r\cos\theta$$
$$b = r\sin\theta$$

(b) The length of the vector $\langle A, B\rangle$ is again r and the angle between $\langle A, B\rangle$ and the x-axis is $\theta + \varphi$. So

$$\begin{aligned} A &= r\cos(\theta+\varphi) = r\cos\theta\cos\varphi - r\sin\theta\sin\varphi = a\cos\varphi - b\sin\varphi \\ B &= r\sin(\theta+\varphi) = r\sin\theta\cos\varphi + r\cos\theta\sin\varphi = b\cos\varphi + a\sin\varphi \end{aligned}$$

$\boxed{3.2.5}$ For each of the regions $\mathcal{R}$ sketched below, express $\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ as an iterated integral in polar coordinates in two different ways.



$\boxed{\text{Solution}}$ (a) The region

$$\mathcal{R} = \left\{ (x,y) \;\middle|\; 0 \leqslant x^2+y^2 \leqslant 4,\ 0 \leqslant y \leqslant x \right\}$$

In polar coordinates,

- the circle $x^2+y^2=4$ becomes $r^2=4$ or $r=2$ and
- the line $y=x$ becomes $r\sin\theta = r\cos\theta$ or $\tan\theta=1$ or $\theta=\frac{\pi}{4}$.

Thus the domain of integration is

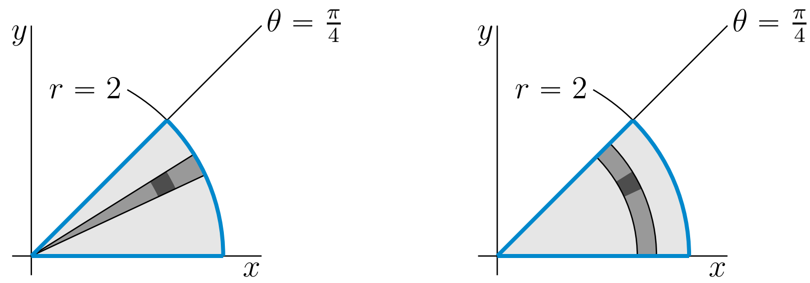
$$\mathcal{R} = \left\{ (r\cos\theta, r\sin\theta) \;\middle|\; 0 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant \tfrac{\pi}{4} \right\}$$

On this domain,

- θ runs from 0 to $\frac{\pi}{4}$.
- For each fixed θ in that range, r runs from 0 to 2, as in the figure on the left below.

In polar coordinates $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$, so that

$$\iint\limits_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\pi/4} \mathrm{d}\theta \int_0^2 \mathrm{d}r\ r\ f(r\cos\theta, r\sin\theta)$$



Alternatively, on $\mathcal{R}$,

- r runs from 0 to 2.
- For each fixed r in that range, θ runs from 0 to $\frac{\pi}{4}$, as in the figure on the right above.

So

$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^2 \mathrm{d}r \int_0^{\pi/4} \mathrm{d}\theta\ r\ f(r\cos\theta, r\sin\theta)$$

(b) The region

$$\mathcal{R} = \left\{\ (x,y)\ \middle|\ 1 \leqslant x^2+y^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0\ \right\}$$

In polar coordinates,

- the circle $x^2+y^2=1$ becomes $r^2=1$ or $r=1$ and
- the circle $x^2+y^2=4$ becomes $r^2=4$ or $r=2$ and
- the positive x-axis, $x \geqslant 0$, $y=0$, becomes $\theta=0$ and
- the positive y-axis, $x=0$, $y \geqslant 0$, becomes $\theta = \frac{\pi}{2}$.

Thus the domain of integration is

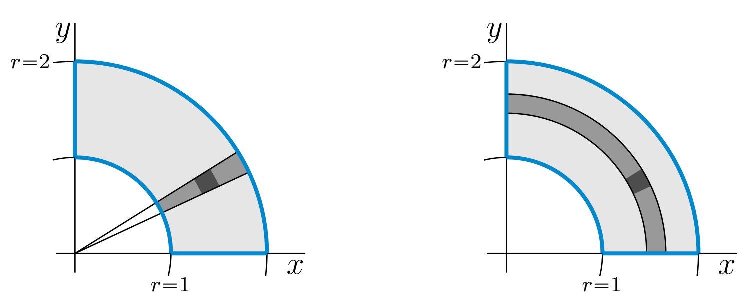
$$\mathcal{R} = \left\{\ (r\cos\theta, r\sin\theta)\ \middle|\ 1 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant \tfrac{\pi}{2}\ \right\}$$

On this domain,

- θ runs from 0 to $\frac{\pi}{2}$.
- For each fixed θ in that range, r runs from 1 to 2, as in the figure on the left below.

In polar coordinates $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$, so that

$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\pi/2} \mathrm{d}\theta \int_1^2 \mathrm{d}r\ r\ f(r\cos\theta, r\sin\theta)$$



Alternatively, on $\mathcal{R}$,

- r runs from 1 to 2.
- For each fixed r in that range, θ runs from 0 to $\frac{\pi}{2}$, as in the figure on the right above.

So

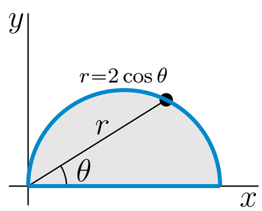
$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_1^2 \mathrm{d}r \int_0^{\pi/2} \mathrm{d}\theta\ r\ f(r\cos\theta, r\sin\theta)$$

(c) The region

$$\mathcal{R} = \left\{\ (x,y)\ \middle|\ (x-1)^2 + y^2 \leqslant 1,\ y \geqslant 0\ \right\}$$

In polar coordinates, the circle $(x-1)^2 + y^2 = 1$, or $x^2 - 2x + y^2 = 0$, is $r^2 - 2r\cos\theta = 0$ or $r = 2\cos\theta$. Note that, on $r = 2\cos\theta$,

- when $\theta = 0$, $r = 2$ and
- as θ increases from 0 towards $\pi/2$, r decreases but remains strictly bigger than 0 (look at the figure below), until
- when $\theta = \frac{\pi}{2}$, $r = 0$.



Thus the domain of integration is

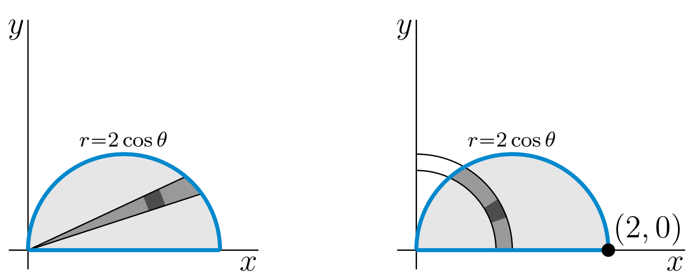
$$\mathcal{R} = \left\{\ (r\cos\theta, r\sin\theta)\ \middle|\ 0 \leqslant \theta \leqslant \tfrac{\pi}{2},\ 0 \leqslant r \leqslant 2\cos\theta\ \right\}$$

On this domain,

- θ runs from 0 to $\frac{\pi}{2}$.
- For each fixed θ in that range, r runs from 0 to $2\cos\theta$, as in the figure on the left below.

In polar coordinates $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$, so that

$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\pi/2} \mathrm{d}\theta \int_0^{2\cos\theta} \mathrm{d}r\ r\ f(r\cos\theta, r\sin\theta)$$



Alternatively, on $\mathcal{R}$,

- r runs from 0 (at the point $(0,0)$) to 2 (at the point $(2,0)$).
- For each fixed r in that range, θ runs from 0 to $\arccos\frac{r}{2}$ (which was gotten by solving $r = 2\cos\theta$ for θ as a function of r), as in the figure on the right above.

So

$$\iint\limits_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^2 \mathrm{d}r \int_0^{\arccos\frac{r}{2}} \mathrm{d}\theta\ r\ f(r\cos\theta, r\sin\theta)$$

(d) The region

$$\mathcal{R} = \left\{\ (x,y)\ \middle|\ 0 \leqslant y \leqslant 2,\ 0 \leqslant x \leqslant y\ \right\}$$

In polar coordinates,

- the line $y = 2$ becomes $r\sin\theta = 2$ and
- the positive y-axis, $x = 0$, $y \geqslant 0$, becomes $\theta = \frac{\pi}{2}$ and
- the line $y = x$ becomes $r\sin\theta = r\cos\theta$ or $\tan\theta = 1$ or $\theta = \frac{\pi}{4}$.

Thus the domain of integration is

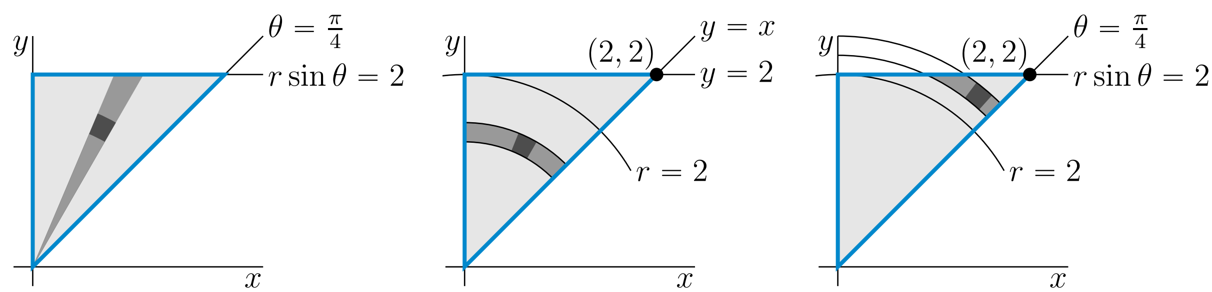
$$\mathcal{R} = \left\{\ (r\cos\theta, r\sin\theta)\ \middle|\ \tfrac{\pi}{4} \leqslant \theta \leqslant \tfrac{\pi}{2},\ 0 \leqslant r\sin\theta \leqslant 2\ \right\}$$

On this domain,

- θ runs from $\frac{\pi}{4}$ to $\frac{\pi}{2}$.
- For each fixed θ in that range, r runs from 0 to $\frac{2}{\sin\theta}$, as in the figure on the left below.

In polar coordinates $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$, so that

$$\iint\limits_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\pi/4}^{\pi/2} \mathrm{d}\theta \int_0^{2/\sin\theta} \mathrm{d}r\ r\ f(r\cos\theta, r\sin\theta)$$



Alternatively, on $\mathcal{R}$,

- r runs from 0 (at the point $(0,0)$) to $2\sqrt{2}$ (at the point $(2,2)$).
- For each fixed r between 0 and 2, θ runs from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, as in the central figure above.
- For each fixed r between 2 and $2\sqrt{2}$, θ runs from $\frac{\pi}{4}$ to $\arcsin\frac{2}{r}$ (which was gotten by solving $r\sin\theta = 2$ for θ as a function of r), as in the figure on the right above.

So

$$\iint\limits_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^2 \mathrm{d}r \int_{\pi/4}^{\pi/2} \mathrm{d}\theta\; r\, f(r\cos\theta, r\sin\theta) + \int_2^{2\sqrt{2}} \mathrm{d}r \int_{\pi/4}^{\arcsin\frac{2}{r}} \mathrm{d}\theta\; r\, f(r\cos\theta, r\sin\theta)$$

3.2.6 Sketch the domain of integration in the xy-plane for each of the following polar coordinate integrals.

(a) $\displaystyle\int_1^2 \mathrm{d}r \int_{-\pi/4}^{\pi/4} \mathrm{d}\theta\; r\; f(r\cos\theta, r\sin\theta)$

(b) $\displaystyle\int_0^{\pi/4} \mathrm{d}\theta \int_0^{\frac{2}{\sin\theta+\cos\theta}} \mathrm{d}r\; r\; f(r\cos\theta, r\sin\theta)$

(c) $\displaystyle\int_0^{2\pi} \mathrm{d}\theta \int_0^{\frac{3}{\sqrt{\cos^2\theta+9\sin^2\theta}}} \mathrm{d}r\; r\; f(r\cos\theta, r\sin\theta)$

Solution (a) Let D denote the domain of integration. The symbols $\displaystyle\int_1^2 \mathrm{d}r \int_{-\pi/4}^{\pi/4} \mathrm{d}\theta$ say that, on D,

- r runs from 1 to 2 and
- for each r in that range, θ runs from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$.

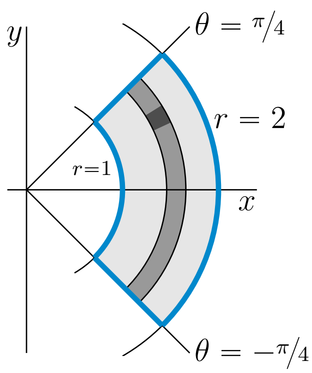
In Cartesian coordinates

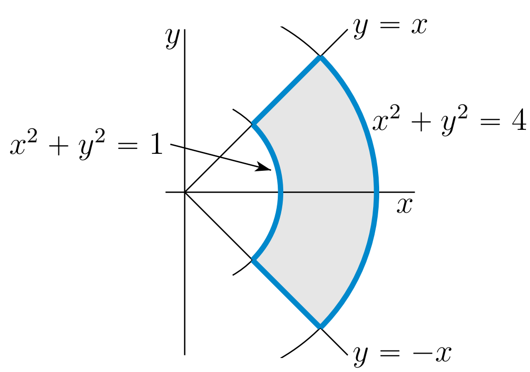
- $r = 1$ is the circle $x^2 + y^2 = 1$ and
- $r = 2$ is the circle $x^2 + y^2 = 4$ and
- $\theta = \frac{\pi}{4}$ is the ray $y = x$, $x \geqslant 0$ and
- $\theta = -\frac{\pi}{4}$ is the ray $y = -x$, $x \geqslant 0$.

So

$$D = \left\{ (x,y) \;\middle|\; 1 \leqslant x^2 + y^2 \leqslant 4,\ -x \leqslant y \leqslant x,\ x \geqslant 0 \right\}$$

Here are two sketches. D is the shaded region in the sketch on the right.





(b) Let D denote the domain of integration. The symbols $\int_0^{\pi/4} \mathrm{d}\theta \int_0^{\frac{2}{\sin\theta+\cos\theta}} \mathrm{d}r$ say that, on D,

- θ runs from 0 to $\pi/4$ and
- for each θ in that range, r runs from 0 to $\frac{2}{\sin\theta+\cos\theta}$.

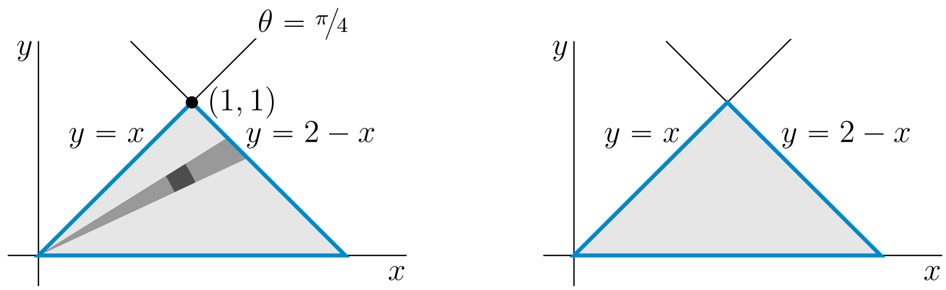
In Cartesian coordinates

- $\theta = 0$ is the positive x-axis and
- $\theta = \pi/4$ is the ray $y = x$, $x \geqslant 0$ and
- $r = \frac{2}{\sin\theta+\cos\theta}$, or equivalently $r\cos\theta + r\sin\theta = 2$, is the line $x + y = 2$.

Looking at the sketch on the left below, we see that, since the lines $y = x$ and $x + y = 2$ cross at $(1, 1)$,

$$D = \left\{ (x, y) \;\middle|\; 0 \leqslant y \leqslant 1,\ y \leqslant x \leqslant 2 - y \right\}$$

D is the shaded region in the sketch on the right.



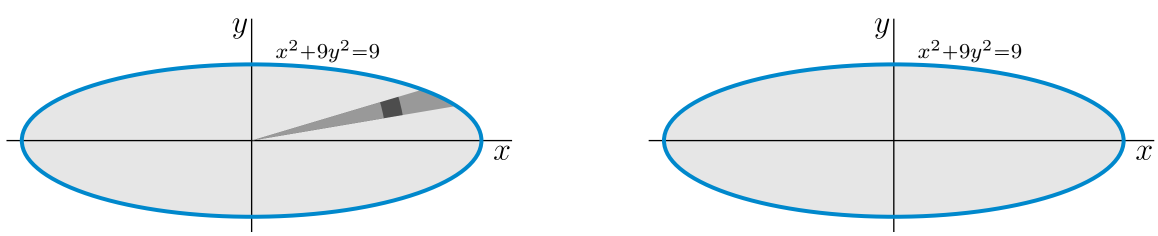
(c) Let D denote the domain of integration. The symbols $\int_0^{2\pi} d\theta \int_0^{\frac{3}{\sqrt{\cos^2\theta+9\sin^2\theta}}} dr$ say that, on D,

- θ runs all the way from 0 to 2π and
- for each θ, r runs from 0 to $\frac{3}{\sqrt{\cos^2\theta+9\sin^2\theta}}$.

In Cartesian coordinates

- $r = \frac{3}{\sqrt{\cos^2\theta+9\sin^2\theta}}$, or equivalently $r^2\cos^2\theta + 9r^2\sin^2\theta = 9$, is the ellipse $x^2 + 9y^2 = 9$.

So D is the interior of the ellipse $x^2 + 9y^2 = 9$ and D is the shaded region in the sketch on the right.



▸▸ Stage 2

3.2.7 Use polar coordinates to evaluate each of the following integrals.

(a) $\iint_S (x+y)\mathrm{d}x\,\mathrm{d}y$ where S is the region in the first quadrant lying inside the disc $x^2+y^2 \leqslant a^2$ and under the line $y=\sqrt{3}x$.

(b) $\iint_S x\ \mathrm{d}x\,\mathrm{d}y$, where S is the disc segment $x^2+y^2 \leqslant 2,\ x \geqslant 1$.

(c) $\iint_T (x^2+y^2)\mathrm{d}x\,\mathrm{d}y$ where T is the triangle with vertices $(0,0),(1,0)$ and $(1,1)$.

(d) $\displaystyle\iint_{x^2+y^2\leqslant 1} \ln(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$

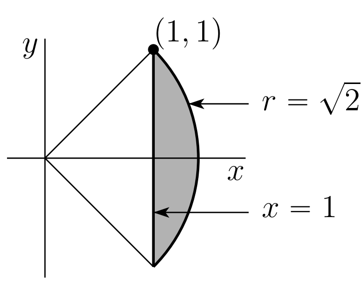
Solution (a) In polar coordinates, the domain of integration, $x^2+y^2 \leqslant a^2$, $0 \leqslant y \leqslant \sqrt{3}x$, becomes

$$r \leqslant a,\ 0 \leqslant r\sin\theta \leqslant \sqrt{3}r\cos\theta \qquad \text{or} \qquad r \leqslant a,\ 0 \leqslant \theta \leqslant \arctan\sqrt{3} = \frac{\pi}{3}$$

The integral is

$$\begin{aligned}\iint_S (x+y)\mathrm{d}x\,\mathrm{d}y &= \int_0^a \mathrm{d}r \int_0^{\frac{\pi}{3}} \mathrm{d}\theta\ r\,(r\cos\theta + r\sin\theta)\\ &= \int_0^a \mathrm{d}r\ r^2\Big[\sin\theta - \cos\theta\Big]_0^{\frac{\pi}{3}} = \frac{a^3}{3}\left[\frac{\sqrt{3}}{2} - \frac{1}{2} + 1\right] = \frac{a^3}{6}\big[\sqrt{3}+1\big]\end{aligned}$$

(b) In polar coordinates, the domain of integration, $x^2+y^2 \leqslant 2$, $x \geqslant 1$,



becomes

$$r \leqslant \sqrt{2},\ r\cos\theta \geqslant 1 \qquad \text{or} \qquad \frac{1}{\cos\theta} \leqslant r \leqslant \sqrt{2}$$

For $\frac{1}{\cos\theta} \leqslant r \leqslant \sqrt{2}$ to be nonempty, we need $\cos\theta \leqslant \frac{1}{\sqrt{2}}$ or $|\theta| \leqslant \frac{\pi}{4}$. By symmetry under

$y \to -y$, the integral is

$$\begin{aligned}\iint_S x\ \mathrm{d}x\,\mathrm{d}y &= 2\int_0^{\frac{\pi}{4}} \mathrm{d}\theta \int_{\frac{1}{\cos\theta}}^{\sqrt{2}} \mathrm{d}r\ r\,(r\cos\theta) \\ &= 2\int_0^{\frac{\pi}{4}} \mathrm{d}\theta\ \cos\theta\ \frac{r^3}{3}\bigg|_{\frac{1}{\cos\theta}}^{\sqrt{2}} = \frac{2}{3}\int_0^{\frac{\pi}{4}} \mathrm{d}\theta\ \Big[2^{3/2}\cos\theta - \frac{1}{\cos^2\theta}\Big] \\ &= \frac{2}{3}\Big[2^{3/2}\sin\theta - \tan\theta\Big]_0^{\frac{\pi}{4}} = \frac{2}{3}\big[2^{3/2}\frac{1}{\sqrt{2}} - 1\big] = \frac{2}{3}\end{aligned}$$

(c) In polar coordinates, the triangle with vertices $(0,0), (1,0)$ and $(1,1)$ has sides $\theta = 0$, $\theta = \frac{\pi}{4}$ and $r = \frac{1}{\cos\theta}$ (which is the polar coordinates version of $x = 1$). The integral is

$$\begin{aligned}\iint_T (x^2+y^2)\ \mathrm{d}x\,\mathrm{d}y &= \int_0^{\frac{\pi}{4}} \mathrm{d}\theta \int_0^{\frac{1}{\cos\theta}} \mathrm{d}r\ r(r^2) \\ &= \int_0^{\frac{\pi}{4}} \mathrm{d}\theta\ \frac{r^4}{4}\bigg|_0^{\frac{1}{\cos\theta}} = \frac{1}{4}\int_0^{\frac{\pi}{4}} \mathrm{d}\theta\ \frac{1}{\cos^4\theta} = \frac{1}{4}\int_0^{\frac{\pi}{4}} \mathrm{d}\theta\ \sec^4\theta \\ &= \frac{1}{4}\int_0^{\frac{\pi}{4}} \mathrm{d}\theta\ \sec^2\theta\big(1+\tan^2\theta\big) = \frac{1}{4}\int_0^1 dt\ \big(1+t^2\big) \text{ where } t = \tan\theta \\ &= \frac{1}{4}\left[t + \frac{t^3}{3}\right]_0^1 = \frac{1}{4}\,\frac{4}{3} = \frac{1}{3}\end{aligned}$$

(d) In polar coordinates, the domain of integration, $x^2+y^2 \leqslant 1$, becomes $r \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$. So

$$\begin{aligned}\iint_{x^2+y^2\leqslant 1} \ln(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \mathrm{d}r\ r\ln r^2 = 2\pi\int_0^1 \mathrm{d}r\ r\ln r^2 = \pi\int_0^1 \mathrm{d}s\ \ln s \text{ where } s = r^2 \\ &= \pi\Big[s\ln s - s\Big]_0^1 = -\pi\end{aligned}$$

To be picky, $\ln s$ tends to $-\infty$ as s tends to 0. So $\int_0^1 \mathrm{d}s\ \ln s$ is an improper integral. The careful way to evaluate it is

$$\int_0^1 \mathrm{d}s\ \ln s = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \mathrm{d}s\ \ln s = \lim_{\varepsilon\to 0^+}\Big[s\ln s - s\Big]_\varepsilon^1 = \lim_{\varepsilon\to 0^+}\Big[-1 - \varepsilon\ln\varepsilon + \varepsilon\Big] = -1$$

That $\lim\limits_{\varepsilon\to 0^+} \varepsilon\ln\varepsilon = 0$ was shown in Example 3.7.15 of the CLP-1 text.

$\boxed{3.2.8}$ Find the volume lying inside the sphere $x^2+y^2+z^2 = 2$ and above the paraboloid $z = x^2+y^2$.

Solution The top surface $x^2+y^2+z^2=2$ meets the bottom surface $z=x^2+y^2$ when z obeys $x^2+y^2=z=2-z^2$. That is, when $0=z^2+z-2=(z-1)(z+2)$. The root $z=-2$ is inconsistent with $z=x^2+y^2\geqslant 0$. So the top and bottom surfaces meet at the circle $z=1$, $x^2+y^2=1$.

In polar coordinates, the top surface is $z^2=2-r^2$, or equivalently $z=\sqrt{2-r^2}$, and the bottom surface is $z=r^2$. So the height of the volume above the point with polar coordinates (r,θ) is $\sqrt{2-r^2}-r^2$ and

$$\begin{aligned}\text{Volume} &= \int_0^1 \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\ r\left[\sqrt{2-r^2}-r^2\right] = 2\pi\int_0^1 \mathrm{d}r\ r\left[\sqrt{2-r^2}-r^2\right]\\ &= 2\pi\left[-\frac{1}{3}(2-r^2)^{3/2}-\frac{r^4}{4}\right]_0^1 = 2\pi\left[-\frac{1}{3}-\frac{1}{4}+\frac{1}{3}2^{3/2}\right]\\ &= \pi\left[\frac{4}{3}\sqrt{2}-\frac{7}{6}\right]\approx 2.26\end{aligned}$$

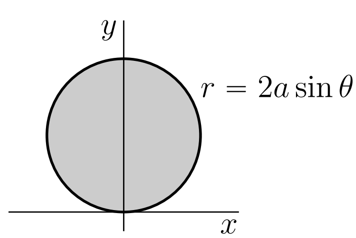
In Cartesian coordinates

$$\text{Volume} = 4\int_0^1 \mathrm{d}x\int_0^{\sqrt{1-x^2}} \mathrm{d}y\ \left[\sqrt{2-x^2-y^2}-x^2-y^2\right]$$

The y integral can be done using the substitution $y=\sqrt{2-x^2}\cos t$, but it is easier to use polar coordinates.

3.2.9 Let $a>0$. Find the volume lying inside the cylinder $x^2+(y-a)^2=a^2$ and between the upper and lower halves of the cone $z^2=x^2+y^2$.

Solution For this region x and y run over the interior of the cylinder $x^2+(y-a)^2=a^2$. For each (x,y) inside the cylinder, z runs from $-\sqrt{x^2+y^2}$ to $\sqrt{x^2+y^2}$. As $x^2+(y-a)^2=a^2$ if and only if $x^2+y^2-2ay=0$, the cylinder has equation $r^2=2ar\sin\theta$, or equivalently, $r=2a\sin\theta$, in polar coordinates. Thus (r,θ) runs over

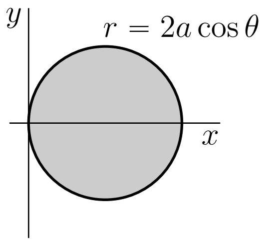


$0\leqslant\theta\leqslant\pi$, $0\leqslant r\leqslant 2a\sin\theta$ and for each (r,θ) in this region z runs from $-r$ to r. By symmetry under $x\to -x$, the volume is

$$\begin{aligned}\text{Volume} &= 2\int_0^{\frac{\pi}{2}} \mathrm{d}\theta\int_0^{2a\sin\theta} \mathrm{d}r\ r\big[r-(-r)\big] = 4\int_0^{\frac{\pi}{2}} \mathrm{d}\theta\int_0^{2a\sin\theta} \mathrm{d}r\ r^2 = \frac{4}{3}\int_0^{\frac{\pi}{2}} \mathrm{d}\theta\ (2a\sin\theta)^3\\ &= \frac{32}{3}a^3\int_0^{\frac{\pi}{2}} \mathrm{d}\theta\ \sin\theta(1-\cos^2\theta) = -\frac{32}{3}a^3\int_1^0 dt\ (1-t^2)\ \text{where } t=\cos\theta\\ &= -\frac{32}{3}a^3\left[t-\frac{t^3}{3}\right]_1^0 = \frac{64}{9}a^3\end{aligned}$$

3.2.10 Let $a > 0$. Find the volume common to the cylinders $x^2 + y^2 \leqslant 2ax$ and $z^2 \leqslant 2ax$.

Solution The figure below shows the top view of the specified solid. (x, y) runs over the interior of the circle $x^2 + y^2 = 2ax$. For each fixed (x, y) in this disk, z runs from $-\sqrt{2ax}$ to $+\sqrt{2ax}$. In polar coordinates, the circle is $r^2 = 2ar\cos\theta$ or $r = 2a\cos\theta$. The solid is



symmetric under $y \to -y$ and $z \to -z$, so we can restrict to $y \geqslant 0$, $z \geqslant 0$ and multiply by 4. The volume is

$$\begin{aligned}
\text{Volume} &= 4\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{2a\cos\theta} \mathrm{d}r\ r\sqrt{2ar\cos\theta} \\
&= 4\int_0^{\frac{\pi}{2}} \mathrm{d}\theta\ \sqrt{2a\cos\theta}\ \frac{2}{5}\ r^{5/2}\Big|_0^{2a\cos\theta} \\
&= \frac{8}{5}\int_0^{\frac{\pi}{2}} \mathrm{d}\theta\ \big(2a\cos\theta\big)^3 = \frac{64}{5}a^3\int_0^{\frac{\pi}{2}} \mathrm{d}\theta\ \cos\theta\big(1-\sin^2\theta\big) \\
&= \frac{64}{5}a^3\int_0^1 dt\ \big(1-t^2\big) \qquad \text{where } t = \sin\theta \\
&= \frac{64}{5}a^3\left[t - \frac{t^3}{3}\right]_0^1 = \frac{128}{15}a^3
\end{aligned}$$

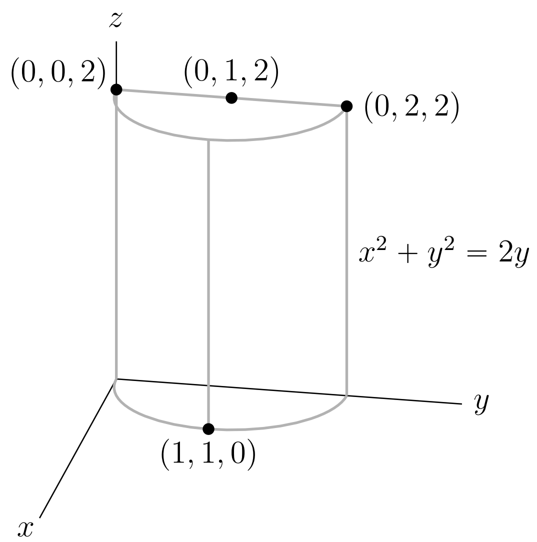
3.2.11 (∗) Consider the region E in 3–dimensions specified by the inequalities $x^2 + y^2 \leqslant 2y$ and $0 \leqslant z \leqslant \sqrt{x^2 + y^2}$.

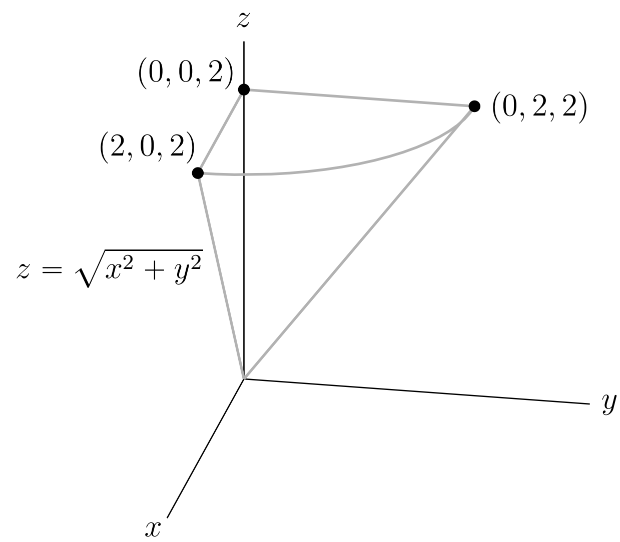
(a) Draw a reasonably accurate picture of E in 3–dimensions. Be sure to show the units on the coordinate axes.

(b) Use polar coordinates to find the volume of E. Note that you will be "using polar coordinates" if you solve this problem by means of cylindrical coordinates.
Hint: $\int \sin^n u\ \mathrm{d}u = -\frac{1}{n}\sin^{n-1} u\ \cos u + \frac{n-1}{n}\int \sin^{n-2} u\ \mathrm{d}u$

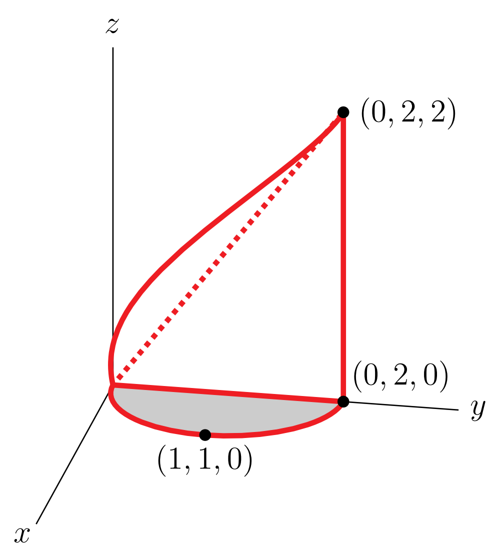
Solution (a)

- The equation $x^2 + y^2 \leqslant 2y$ is equivalent to the equation $x^2 + (y-1)^2 = 1$, which is the equation of the cylinder whose $z = z_0$ cross–section is the horizontal circle of radius 1, centred on $x = 0$, $y = 1$, $z = z_0$. The part of this cylinder in the first octant is sketched in the figure on the left below.
- $z \leqslant \sqrt{x^2 + y^2}$ is the equation of the cone with vertex $(0, 0, 0)$, and axis the positive z–axis, whose radius at height $z = 2$ is 2. The part of this cone in the first octant is sketched in the figure on the right below.





The region E is the part of the cylinder that is above the xy–plane (since $z \geqslant 0$) outside the cone (since $z \leqslant \sqrt{x^2+y^2}$). The part of E that is in the first octant is outlined in red in the figure below. Both $x^2+y^2 \leqslant 2y$ and $0 \leqslant z \leqslant \sqrt{x^2+y^2}$ are invariant under $x \to -x$. So E is also invariant under $x \to -x$. That is, E is symmetric about the yz–plane and contains, in the octant $x \leqslant 0$, $y \geqslant 0$, $z \geqslant 0$, a mirror image of the first octant part of E.



(b) In polar coordinates, $x^2+y^2 \leqslant 2y$ becomes

$$r^2 \leqslant 2r\sin\theta \iff r \leqslant 2\sin\theta$$

Let us denote by D the base region of the part of E in the first octant (i.e. the shaded region in the figure above). Think of D as being part of the xy–plane. In polar coordinates, on D

- θ runs from 0 to $\frac{\pi}{2}$. (Recall that D is contained in the first quadrant.)
- For each θ in that range, r runs from 0 to $2\sin\theta$.

Because

- in polar coordinates $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$, and

- the height of E above each point (x,y) in D is $\sqrt{x^2+y^2}$, or, in polar coordinates, r, and
- the volume of E is twice the volume of the part of E in the first octant,

we have

$$\begin{aligned}
\text{Volume}(E) &= 2\int_0^{\frac{\pi}{2}} d\theta \int_0^{2\sin\theta} dr\ r^2 \\
&= \frac{16}{3}\int_0^{\frac{\pi}{2}} d\theta\ \sin^3\theta = \frac{16}{3}\int_0^{\frac{\pi}{2}} d\theta\ \sin\theta\ \big(1-\cos^2\theta\big) \\
&= -\frac{16}{3}\int_1^0 du\ \big(1-u^2\big) \qquad \text{with } u=\cos\theta,\ du=-\sin\theta\ d\theta \\
&= \frac{16}{3}\left[1-\frac{1}{3}\right] \\
&= \frac{32}{9}
\end{aligned}$$

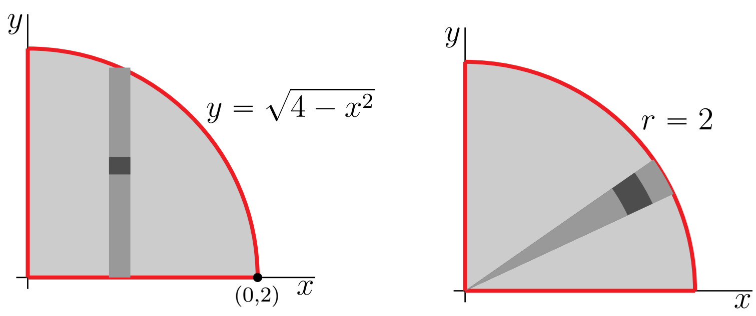
3.2.12 (*) Evaluate the iterated double integral

$$\int_{x=0}^{x=2}\int_{y=0}^{y=\sqrt{4-x^2}} (x^2+y^2)^{\frac{3}{2}}\ dy\,dx$$

Solution On the domain of integration

- x runs for 0 to 2, and
- for each fixed x in that range, y runs from 0 to $\sqrt{4-x^2}$. The equation $y=\sqrt{4-x^2}$ is equivalent to $x^2+y^2=4$, $y\geqslant 0$.

This domain is sketched in the figure on the left below.



Considering that

- the integrand, $(x^2+y^2)^{\frac{3}{2}}$, is invariant under rotations about the origin and
- the outer curve, $x^2+y^2=4$, is invariant under rotations about the origin

we'll use polar coordinates. In polar coordinates,

- the outer curve, $x^2+y^2=4$, is $r=2$, and

- the integrand, $(x^2+y^2)^{\frac{3}{2}}$ is r^3, and
- $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$

Looking at the figure on the right above, we see that the given integral is, in polar coordinates,

$$\int_0^{\pi/2} \mathrm{d}\theta \int_0^2 \mathrm{d}r\ r(r^3) = \frac{\pi}{2}\,\frac{2^5}{5} = \frac{16\pi}{5}$$

$\boxed{3.2.13}$ (∗)

(a) Sketch the region $\mathcal{L}$ (in the first quadrant of the xy–plane) with boundary curves

$$x^2+y^2=2,\ x^2+y^2=4,\ y=x,\ y=0.$$

The mass of a thin lamina with a density function $\rho(x,y)$ over the region $\mathcal{L}$ is given by

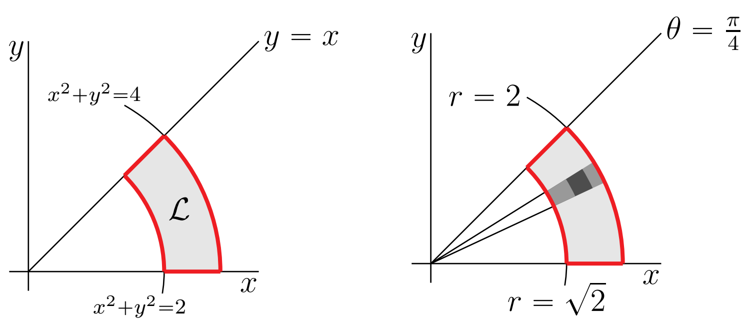
$$M = \iint_{\mathcal{L}} \rho(x,y)\,\mathrm{d}A$$

(b) Find an expression for M as an integral in polar coordinates.

(c) Find M when

$$\rho(x,y) = \frac{2xy}{x^2+y^2}$$

$\boxed{\text{Solution}}$ (a) The region $\mathcal{L}$ is sketched in the figure on the leflt below.



(b) In polar coordinates

- the circle $x^2+y^2=2$ is $r^2=2$ or $r=\sqrt{2}$, and
- the circle $x^2+y^2=4$ is $r^2=4$ or $r=2$, and
- the line $y=x$ is $r\sin\theta = r\cos\theta$, or $\tan\theta=1$, or (for the part in the first quadrant) $\theta=\frac{\pi}{4}$, and
- the positive x–axis ($y=0$, $x\geqslant 0$) is $\theta=0$

Looking at the figure on the right above, we see that, in $\mathcal{L}$,

- θ runs from 0 to $\frac{\pi}{4}$, and

- for each fixed θ in that range, r runs from $\sqrt{2}$ to 2.
- $\mathrm{d}A$ is $r\,\mathrm{d}r\,\mathrm{d}\theta$

So

$$M = \int_0^{\pi/4} \mathrm{d}\theta \int_{\sqrt{2}}^2 \mathrm{d}r\ r\,\rho(r\cos\theta\,,\ r\sin\theta)$$

(c) When

$$\rho = \frac{2xy}{x^2+y^2} = \frac{2r^2\cos\theta\ \sin\theta}{r^2} = \sin(2\theta)$$

we have

$$\begin{aligned} M &= \int_0^{\pi/4} \mathrm{d}\theta \int_{\sqrt{2}}^2 \mathrm{d}r\ r\ \sin(2\theta) \\ &= \left[\int_0^{\pi/4} \sin(2\theta)\ \mathrm{d}\theta\right]\left[\int_{\sqrt{2}}^2 r\ \mathrm{d}r\right] \\ &= \left[-\frac{1}{2}\cos(2\theta)\right]_0^{\pi/4}\left[\frac{r^2}{2}\right]_{\sqrt{2}}^2 = \frac{1}{2}\ \frac{4-2}{2} \\ &= \frac{1}{2} \end{aligned}$$

$\boxed{3.2.14}$ (∗) Evaluate $\displaystyle\iint_{\mathbb{R}^2} \frac{1}{(1+x^2+y^2)^2}\ \mathrm{d}A$.

$\boxed{\text{Solution}}$ We'll use polar coordinates. The domain of integration is

$$\mathbb{R}^2 = \left\{\ (r\cos\theta\,,\ r\sin\theta)\ \middle|\ 0 \leqslant r < \infty,\ 0 \leqslant \theta \leqslant 2\pi\ \right\}$$

The given integral is improper, so we'll start by integrating r from 0 to an arbitrary $R > 0$, and then we'll take the limit $R \to \infty$. In polar coordinates, the integrand $\frac{1}{(1+x^2+y^2)^2} = \frac{1}{(1+r^2)^2}$, and $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$, so

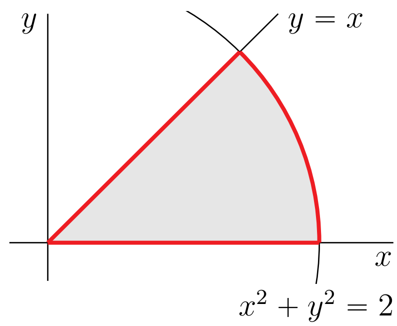
$$\begin{aligned} \iint_{\mathbb{R}^2} \frac{1}{(1+x^2+y^2)^2}\ \mathrm{d}A &= \lim_{R\to\infty} \int_0^{2\pi} \mathrm{d}\theta \int_0^R \mathrm{d}r \frac{r}{(1+r^2)^2} \\ &= \lim_{R\to\infty} \int_0^{2\pi} \mathrm{d}\theta \left[-\frac{1}{2(1+r^2)}\right]_0^R \\ &= \lim_{R\to\infty} 2\pi\left[\frac{1}{2} - \frac{1}{2(1+R^2)}\right] \\ &= \pi \end{aligned}$$

3.2.15 (∗) Evaluate the double integral

$$\iint_D y\sqrt{x^2+y^2}\,\mathrm{d}A$$

over the region $D = \{\ (x,y)\ \big|\ x^2+y^2 \leqslant 2,\ 0 \leqslant y \leqslant x\ \}$.

Solution Let's switch to polar coordinates. In polar coordinates, the circle $x^2+y^2=2$ is $r=\sqrt{2}$ and the line $y=x$ is $\theta=\frac{\pi}{4}$.



In polar coordinates $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$, so the integral

$$\begin{aligned}
\iint_D y\sqrt{x^2+y^2}\,\mathrm{d}A &= \int_0^{\pi/4}\mathrm{d}\theta\int_0^{\sqrt{2}}\mathrm{d}r\ r\ \overbrace{r\sin\theta}^{y}\ \overbrace{r}^{\sqrt{x^2+y^2}}\\
&= \int_0^{\pi/4}\mathrm{d}\theta\ \sin\theta\ \left[\frac{r^4}{4}\right]_0^{\sqrt{2}}\\
&= \Big[-\cos\theta\Big]_0^{\pi/4}\\
&= 1-\frac{1}{\sqrt{2}}
\end{aligned}$$

3.2.16 (∗) This question is about the integral

$$\int_0^1\int_{\sqrt{3}y}^{\sqrt{4-y^2}} \ln\left(1+x^2+y^2\right)\,\mathrm{d}x\,\mathrm{d}y$$

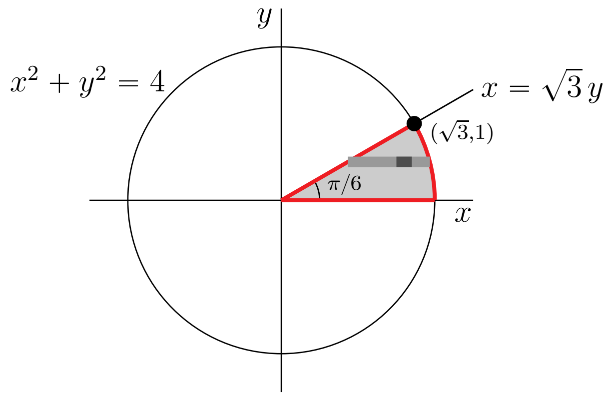
(a) Sketch the domain of integration.
(b) Evaluate the integral by transforming to polar coordinates.

Solution (a) On the domain of integration

- y runs from 0 to 1. In inequalities, $0 \leqslant y \leqslant 1$.
- For each fixed y in that range, x runs from $\sqrt{3}\,y$ to $\sqrt{4-y^2}$. In inequalities, that is

$\sqrt{3}\,y \leqslant x \leqslant \sqrt{4-y^2}$. Note that the inequalities $x \leqslant \sqrt{4-y^2}$, $x \geqslant 0$ are equivalent to $x^2+y^2 \leqslant 4$, $x \geqslant 0$.

Note that the line $x = \sqrt{3}\,y$ and the circle $x^2+y^2 \leqslant 4$ intersect when $3y^2+y^2=4$, i.e. $y=\pm 1$. Here is a sketch.



(b) In polar coordinates, the circle $x^2+y^2=4$ is $r=2$ and the line $x=\sqrt{3}\,y$, i.e. $\frac{y}{x}=\frac{1}{\sqrt{3}}$, is $\tan\theta = \frac{1}{\sqrt{3}}$ or $\theta = \frac{\pi}{6}$. As $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$, the domain of integration is

$$\left\{ \ (r\cos\theta, r\sin\theta) \ \middle|\ 0 \leqslant \theta \leqslant \pi/6,\ 0 \leqslant r \leqslant 2 \ \right\}$$

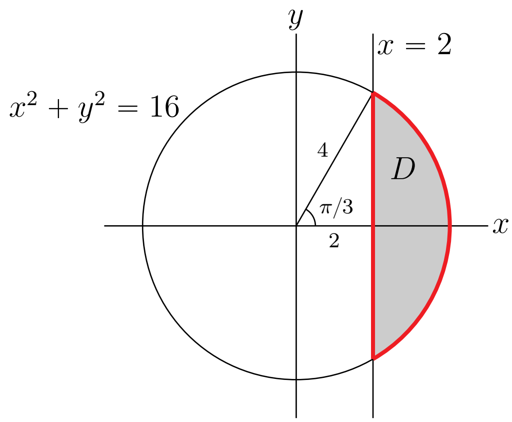
and

$$\begin{aligned}
\int_0^1 \int_{\sqrt{3}y}^{\sqrt{4-y^2}} \ln\left(1+x^2+y^2\right)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^2 \mathrm{d}r \int_0^{\pi/6} \mathrm{d}\theta\ r\ \ln(1+r^2) = \frac{\pi}{6}\int_0^2 \mathrm{d}r\ r\ \ln(1+r^2) \\
&= \frac{\pi}{12}\int_1^5 \mathrm{d}u\,\ln(u) \qquad \text{with } u=1+r^2,\ \mathrm{d}u = 2r\,\mathrm{d}r \\
&= \frac{\pi}{12}\Big[u\ln(u)-u\Big]_1^5 \\
&= \frac{\pi}{12}\big[5\ln(5)-4\big]
\end{aligned}$$

$\boxed{3.2.17}$ (∗) Let D be the region in the xy–plane bounded on the left by the line $x=2$ and on the right by the circle $x^2+y^2=16$. Evaluate

$$\iint_D \left(x^2+y^2\right)^{-3/2}\,\mathrm{d}A$$

$\boxed{\text{Solution}}$ Here is a sketch of D.



We'll use polar coordinates. In polar coordinates the circle $x^2 + y^2 = 16$ is $r = 4$ and the line $x = 2$ is $r\cos\theta = 2$. So

$$D = \left\{ (r\cos\theta\,,\, r\sin\theta) \;\middle|\; -\frac{\pi}{3} \leqslant \theta \leqslant \frac{\pi}{3},\ \frac{2}{\cos\theta} \leqslant r \leqslant 4 \right\}$$

and, as $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$, the specified integral is

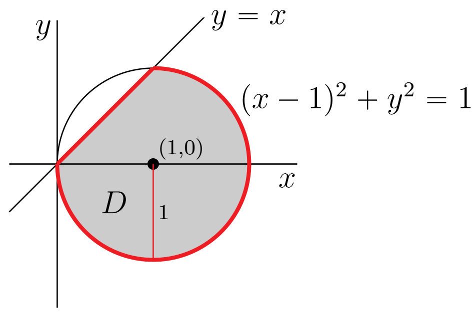
$$\begin{aligned}
\iint_D \left(x^2+y^2\right)^{-3/2} \mathrm{d}A &= \int_{-\pi/3}^{\pi/3} \mathrm{d}\theta \int_{2/\cos\theta}^{4} \mathrm{d}r\; r\frac{1}{r^3} \\
&= \int_{-\pi/3}^{\pi/3} \mathrm{d}\theta\ \left[-\frac{1}{r}\right]_{2/\cos\theta}^{4} \\
&= \int_{-\pi/3}^{\pi/3} \mathrm{d}\theta\ \left[\frac{\cos\theta}{2} - \frac{1}{4}\right] \\
&= \left[\frac{\sin\theta}{2} - \frac{\theta}{4}\right]_{-\pi/3}^{\pi/3} \\
&= \frac{\sqrt{3}}{2} - \frac{\pi}{6}
\end{aligned}$$

3.2.18 $(*)$ In the xy–plane, the disk $x^2 + y^2 \leqslant 2x$ is cut into 2 pieces by the line $y = x$. Let D be the larger piece.

(a) Sketch D including an accurate description of the center and radius of the given disk. Then describe D in polar coordinates (r, θ).

(b) Find the volume of the solid below $z = \sqrt{x^2 + y^2}$ and above D.

Solution (a) The inequality $x^2 + y^2 \leqslant 2x$ is equivalent to $(x-1)^2 + y^2 \leqslant 1$ and says that (x, y) is to be inside the disk of radius 1 centred on $(1, 0)$. Here is a sketch.



In polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$ so that the line $y = x$ is $\theta = \frac{\pi}{4}$ and the circle $x^2 + y^2 = 2x$ is

$$r^2 = 2r\cos\theta \qquad \text{or} \qquad r = 2\cos\theta$$

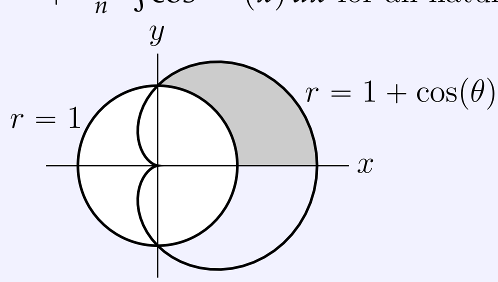
Consequently

$$D = \left\{ \ (r\cos\theta\,,\, r\sin\theta) \ \middle| \ -\pi/2 \leqslant \theta \leqslant \pi/4,\ 0 \leqslant r \leqslant 2\cos\theta \ \right\}$$

(b) The solid has height $z = r$ above the point in D with polar coordinates r, θ. So the

$$\begin{aligned}
\text{Volume} &= \iint\limits_D r \ \mathrm{d}A = \iint\limits_D r^2 \ \mathrm{d}r\,\mathrm{d}\theta = \int_{-\pi/2}^{\pi/4} \mathrm{d}\theta \int_0^{2\cos\theta} \mathrm{d}r\ r^2 \\
&= \frac{8}{3}\int_{-\pi/2}^{\pi/4} \mathrm{d}\theta\ \cos^3\theta = \frac{8}{3}\int_{-\pi/2}^{\pi/4} \mathrm{d}\theta\ \cos\theta\big[1 - \sin^2\theta\big] \\
&= \frac{8}{3}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_{-\pi/2}^{\pi/4} \\
&= \frac{8}{3}\left[\left(\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right) - \left(-1 + \frac{1}{3}\right)\right] \\
&= \frac{40}{18\sqrt{2}} + \frac{16}{9}
\end{aligned}$$

$\boxed{3.2.19}$ $(*)$ Let D be the shaded region in the diagram. Find the average distance of points in D from the origin. You may use that
$\int \cos^n(x)\,dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$ for all natural numbers $n \geqslant 2$.



$\boxed{\text{Solution}}$ We'll use polar coordinates. In D

- θ runs from 0 to $\frac{\pi}{2}$ and

- for each fixed θ between 0 and $\frac{\pi}{2}$, r runs from 1 to $1+\cos(\theta)$.

So the area of D is

$$\text{area} = A = \int_0^{\pi/2} \mathrm{d}\theta \int_1^{1+\cos\theta} \mathrm{d}r\ r = \int_0^{\pi/2} \mathrm{d}\theta\ \frac{1}{2}r^2\bigg|_1^{1+\cos\theta} = \int_0^{\pi/2} \mathrm{d}\theta\ \left[\frac{1}{2}\cos^2\theta + \cos\theta\right]$$

We are interested in the average value of r on D, which is

$$\begin{aligned}\text{ave dist} &= \frac{1}{A}\int_0^{\pi/2} \mathrm{d}\theta \int_1^{1+\cos\theta} \mathrm{d}r\ r^2 = \frac{1}{A}\int_0^{\pi/2} \mathrm{d}\theta\ \frac{1}{3}r^3\bigg|_1^{1+\cos\theta} \\ &= \frac{1}{A}\int_0^{\pi/2} \mathrm{d}\theta\ \left[\frac{1}{3}\cos^3\theta + \cos^2\theta + \cos\theta\right]\end{aligned}$$

Now we evaluate the integrals of the various powers of cosine.

$$\begin{aligned}\int_0^{\pi/2} \cos\theta\ \mathrm{d}\theta &= \sin\theta\bigg|_0^{\pi/2} = 1 \\ \int_0^{\pi/2} \cos^2\theta\ \mathrm{d}\theta &= \frac{\cos\theta\sin\theta}{2}\bigg|_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2} \mathrm{d}\theta = \frac{\pi}{4} \\ \int_0^{\pi/2} \cos^3\theta\ \mathrm{d}\theta &= \frac{\cos^2\theta\sin\theta}{3}\bigg|_0^{\pi/2} + \frac{2}{3}\int_0^{\pi/2} \cos\theta\ \mathrm{d}\theta = \frac{2}{3}\end{aligned}$$

So $A = \frac{\pi}{8} + 1$ and

$$\text{ave dist} = \frac{8}{\pi+8}\left[\frac{2}{9} + \frac{\pi}{4} + 1\right] = 2\frac{\pi + 44/9}{\pi + 8} \approx 1.442$$

▸▸ Stage 3

$\boxed{3.2.20}$ (*) Let G be the region in $\mathbb{R}^2$ given by

$$\begin{gathered} x^2 + y^2 \leqslant 1 \\ 0 \leqslant x \leqslant 2y \\ y \leqslant 2x \end{gathered}$$

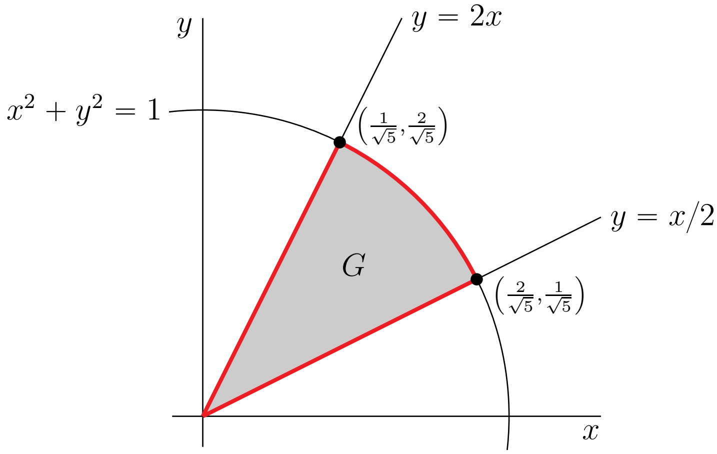
(a) Sketch the region G.
(b) Express the integral $\iint_G f(x,y)\ \mathrm{d}A$ a sum of iterated integrals $\iint f(x,y)\ \mathrm{d}x\mathrm{d}y$.
(c) Express the integral $\iint_G f(x,y)\ \mathrm{d}A$ as an iterated integral in polar coordinates (r,θ) where $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

$\boxed{\text{Solution}}$ (a) Observe that

- the condition $x^2 + y^2 \leqslant 1$ restricts G to the interior of the circle of radius 1 centred on the origin, and

- the conditions $0 \leqslant x \leqslant 2y$ restricts G to $x \geqslant 0$, $y \geqslant 0$, i.e. to the first quadrant, and
- the conditions $x \leqslant 2y$ and $y \leqslant 2x$ restrict $\frac{x}{2} \leqslant y \leqslant 2x$. So G lies below the (steep) line $y = 2x$ and lies above the (not steep) line $y = \frac{x}{2}$.

Here is a sketch of G



(b) Observe that the line $y = 2x$ crosses the circle $x^2 + y^2 = 1$ at a point (x, y) obeying

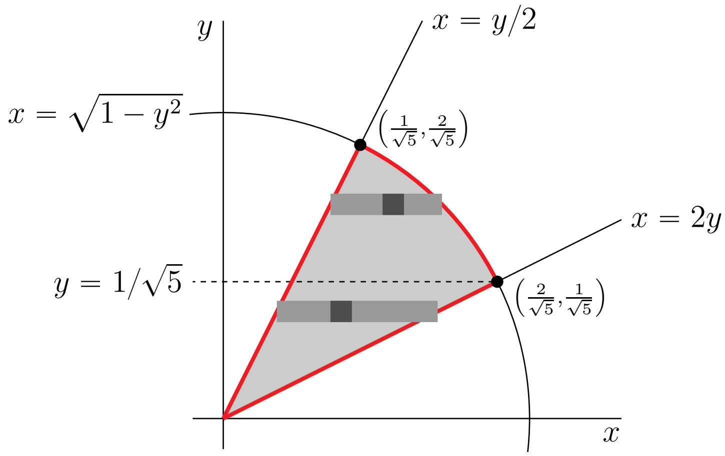
$$x^2 + (2x)^2 = x^2 + y^2 = 1 \implies 5x^2 = 1$$

and that the line $x = 2y$ crosses the circle $x^2 + y^2 = 1$ at a point (x, y) obeying

$$(2y)^2 + y^2 = x^2 + y^2 = 1 \implies 5y^2 = 1$$

So the intersection point of $y = 2x$ and $x^2 + y^2 = 1$ in the first octant is $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and the intersection point of $x = 2y$ and $x^2 + y^2 = 1$ in the first octant is $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$.

We'll set up the iterated integral using horizontal strips as in the sketch



Looking at that sketch, we see that, on G,

- y runs from 0 to $\frac{2}{\sqrt{5}}$, and
- for each fixed y between 0 and $\frac{1}{\sqrt{5}}$, x runs from $\frac{y}{2}$ to $2y$, and

- for each fixed y between $\frac{1}{\sqrt{5}}$ and $\frac{2}{\sqrt{5}}$ x runs from $\frac{y}{2}$ to $\sqrt{1-y^2}$.

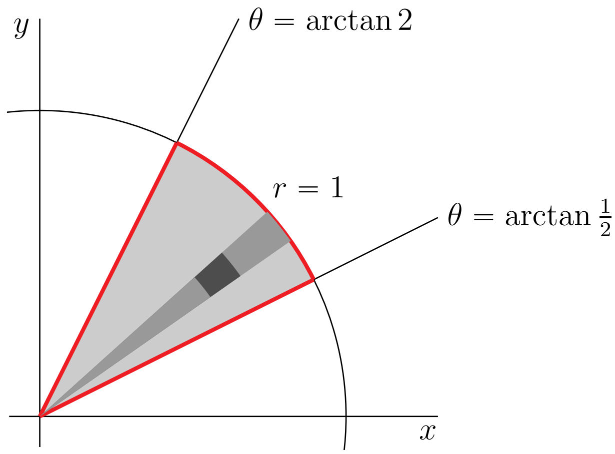
So

$$\iint_G f(x,y)\ \mathrm{d}A = \int_0^{\frac{1}{\sqrt{5}}} \mathrm{d}y \int_{y/2}^{2y} \mathrm{d}x\ f(x,y) + \int_{\frac{1}{\sqrt{5}}}^{\frac{2}{\sqrt{5}}} \mathrm{d}y \int_{y/2}^{\sqrt{1-y^2}} \mathrm{d}x\ f(x,y)$$

(b) In polar coordinates

- the equation $x^2+y^2=1$ becomes $r=1$, and
- the equation $y=x/2$ becomes $r\sin\theta = \frac{r}{2}\cos\theta$ or $\tan\theta = \frac{1}{2}$, and
- the equation $y=2x$ becomes $r\sin\theta = 2r\cos\theta$ or $\tan\theta = 2$.

Looking at the sketch



we see that, on G,

- θ runs from $\arctan\frac{1}{2}$ to $\arctan 2$, and
- for each fixed θ in that range, r runs from 0 to 1.

As $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$, and $x = r\cos\theta$, $y = r\sin\theta$,

$$\iint_G f(x,y)\ \mathrm{d}A = \int_{\arctan\frac{1}{2}}^{\arctan 2} \mathrm{d}\theta \int_0^1 \mathrm{d}r\ r\,f(r\cos\theta, r\sin\theta)$$

$\boxed{3.2.21}$ (∗) Consider

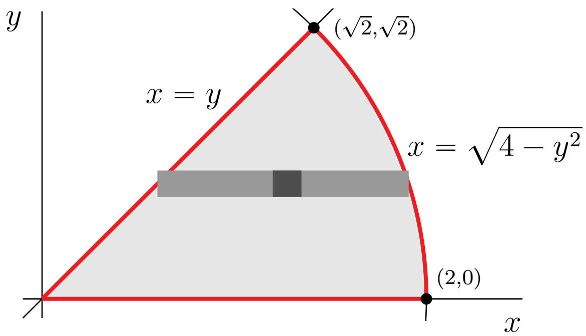
$$J = \int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{y}{x} e^{x^2+y^2}\ \mathrm{d}x\,\mathrm{d}y$$

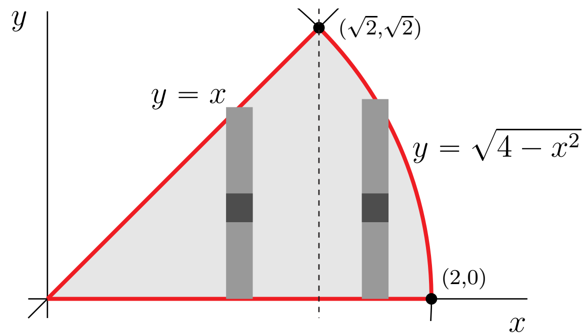
(a) Sketch the region of integration.
(b) Reverse the order of integration.
(c) Evaluate J by using polar coordinates.

$\boxed{\text{Solution}}$ (a) On the domain of integration

- y runs from 0 to $\sqrt{2}$ and
- for each y in that range, x runs from y to $\sqrt{4-y^2}$. We can rewrite $x = \sqrt{4-y^2}$ in the more familiar form $x^2 + y^2 = 4$, $x \geqslant 0$.

The figure on the left below provides a sketch of the domain of integration. It also shows the generic horizontal slice that was used to set up the given iterated integral.





(b) To reverse the order of integration observe, we use vertical, rather than horizontal slices. From the figure on the right above that, on the domain of integration,

- x runs from 0 to 2 and
- for each x in the range $0 \leqslant x \leqslant \sqrt{2}$, y runs from 0 to x.
- for each x in the range $\sqrt{2} \leqslant x \leqslant 2$, y runs from 0 to $\sqrt{4-x^2}$.

So the integral

$$J = \int_0^{\sqrt{2}} \int_0^x \frac{y}{x} e^{x^2+y^2}\, \mathrm{d}y\, \mathrm{d}x + \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-x^2}} \frac{y}{x} e^{x^2+y^2}\, \mathrm{d}y\, \mathrm{d}x$$

(c) In polar coordinates, the line $y = x$ is $\theta = \frac{\pi}{4}$, the circle $x^2 + y^2 = 4$ is $r = 2$, and $\mathrm{d}x\, \mathrm{d}y = r\, \mathrm{d}r\, \mathrm{d}\theta$. So

$$\begin{aligned}
J &= \int_0^{\pi/4} \mathrm{d}\theta \int_0^2 \mathrm{d}r\ r\ \overbrace{\frac{r\sin\theta}{r\cos\theta}}^{\frac{y}{x}} e^{r^2} \\
&= \int_0^{\pi/4} \mathrm{d}\theta\ \frac{\sin\theta}{\cos\theta} \left[\frac{1}{2} e^{r^2}\right]_0^2 \\
&= -\frac{1}{2}\left[e^4 - 1\right] \int_1^{1/\sqrt{2}} \mathrm{d}u\ \frac{1}{u} \qquad \text{with } u = \cos\theta,\ \mathrm{d}u = -\sin\theta\, \mathrm{d}\theta \\
&= -\frac{1}{2}\left[e^4 - 1\right]\ \Big[\ln|u|\Big]_1^{1/\sqrt{2}} \\
&= \frac{1}{4}\left[e^4 - 1\right] \ln 2
\end{aligned}$$

3.2.22 Find the volume of the region in the first octant below the paraboloid

$$z = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Solution The paraboloid hits the xy–plane at $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\begin{aligned} \text{Volume} &= \int_0^a \mathrm{d}x \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \mathrm{d}y \ \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \\ &= b\int_0^a \mathrm{d}x \int_0^{\sqrt{1-\frac{x^2}{a^2}}} \mathrm{d}v \ \left(1 - \frac{x^2}{a^2} - v^2\right) \qquad \text{where } y = bv \end{aligned}$$

Think of this integral as being of the form

$$b\int_0^a \mathrm{d}x \ g(x) \quad \text{with} \quad g(x) = \int_0^{\sqrt{1-\frac{x^2}{a^2}}} \mathrm{d}v \ \left(1 - \frac{x^2}{a^2} - v^2\right)$$

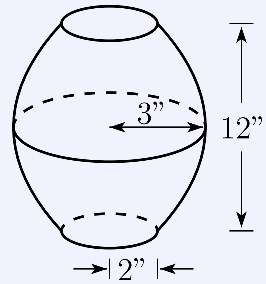
Then, substituting $x = au$,

$$\begin{aligned} \text{Volume} &= ab\int_0^1 \mathrm{d}u \int_0^{\sqrt{1-u^2}} \mathrm{d}v \ \left(1 - u^2 - v^2\right) \\ &= ab \iint\limits_{\substack{u^2+v^2\leqslant 1 \\ u,v\geqslant 0}} \mathrm{d}u\mathrm{d}v \ \left(1 - u^2 - v^2\right) \end{aligned}$$

Now switch to polar coordinates using $u = r\cos\theta$, $v = r\sin\theta$.

$$\text{Volume} = ab\int_0^1 \mathrm{d}r \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \ r\left(1 - r^2\right) = ab \ \frac{\pi}{2} \ \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = \frac{\pi}{8}ab$$

3.2.23 A symmetrical coffee percolator holds 24 cups when full. The interior has a circular cross-section which tapers from a radius of 3” at the centre to 2” at the base and top, which are 12” apart. The bounding surface is parabolic. Where should the mark indicating the 6 cup level be placed?



Solution Let $r(z)$ be the radius of the urn at height z above its middle. Because the bounding surface of the urn is parabolic, $r(z)$ must be a quadratic function of z that varies between 3 at $z = 0$ and 2 at $z = \pm 6$. That is, $r(z)$ must be of the form $r(z) = az^2 + bz + c$. The condition that $r(0) = 3$ tells us that $c = 3$. The conditions that $r(\pm 6) = 2$ tells us that

$$6^2 a + 6b + 3 = 2$$
$$6^2 a - 6b + 3 = 2$$

So $b = 0$ and $6^2 a = -1$ so that $a = -\frac{1}{6^2}$. All together $r(z) = 3 - \left(\frac{z}{6}\right)^2$.

Slice the urn into horzontal slices, with the slice at height z a disk of radius $r(z)$ and thickness $\mathrm{d}z$ and hence of volume $\pi r(z)^2 \mathrm{d}z$. The volume to height z_0 is

$$V(z) = \int_{-6}^{z_0} \mathrm{d}z\ \pi r(z)^2 = \int_{-6}^{z_0} \mathrm{d}z\ \pi \left[3 - \frac{z^2}{36}\right]^2 = \pi \left[9z - \frac{z^3}{18} + \frac{z^5}{5 \times 36^2}\right]_{-6}^{z_0}$$

We are told that the mark is to be at the 6 cup level and that the urn holds 24 cups. So the mark is to be at the height z_0 for which the volume, $V(z_0)$, is one quarter of the total volume, $V(6)$. That is, we are to choose z_0 so that $V(z_0) = \frac{1}{4} V(6)$ or

$$\pi \left[9z - \frac{z^3}{18} + \frac{z^5}{5 \times 36^2}\right]_{-6}^{z_0} = \frac{\pi}{4} \left[9z - \frac{z^3}{18} + \frac{z^5}{5 \times 36^2}\right]_{-6}^{6} = \frac{\pi}{2} \left[9 \times 6 - \frac{6^3}{18} + \frac{6^5}{5 \times 36^2}\right]$$

or

$$9z_0 - \frac{z_0^3}{18} + \frac{z_0^5}{5 \times 36^2} = -\frac{1}{2} \left[9 \times 6 - \frac{6^3}{18} + \frac{6^5}{5 \times 36^2}\right] = -21.60$$

Since $\left[9z_0 - \frac{z_0^3}{18} + \frac{z_0^5}{6480}\right]_{z_0=-2.495} = -21.61$ and $\left[9z_0 - \frac{z_0^3}{18} + \frac{z_0^5}{6480}\right]_{z_0=-2.490} = -21.57$, there is a solution $z_0 = -2.49$ (to two decimal places). The mark should be about 3.5” above the bottom.

3.2.24 (*) Consider the surface S given by $z = e^{x^2+y^2}$.

(a) Compute the volume under S and above the disk $x^2 + y^2 \leqslant 9$ in the xy-plane.

(b) The volume under S and above a certain region R in the xy-plane is

$$\int_0^1 \left(\int_0^y e^{x^2+y^2}\,\mathrm{d}x\right)\mathrm{d}y + \int_1^2 \left(\int_0^{2-y} e^{x^2+y^2}\,\mathrm{d}x\right)\mathrm{d}y$$

Sketch R and express the volume as a single iterated integral with the order of integration reversed. Do not compute either integral in part (b).

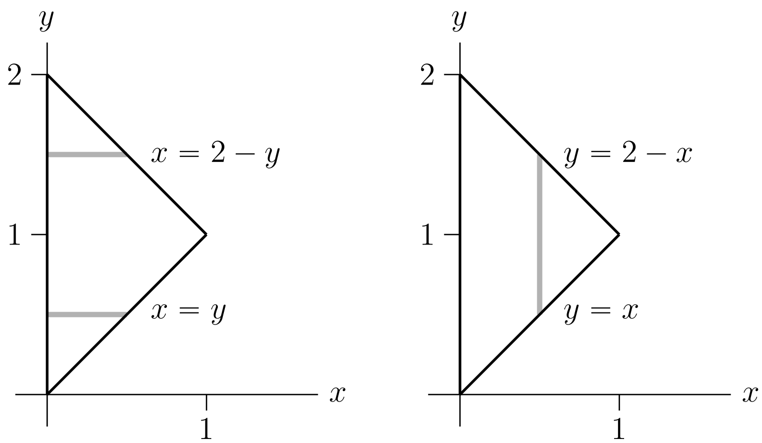
Solution (a) In polar coordinates, the base region $x^2 + y^2 \leqslant 9$ is $r \leqslant 3$, $0 \leqslant \theta \leqslant 2\pi$. So the

$$\begin{aligned} \text{Volume} &= \iint_{x^2+y^2 \leqslant 9} e^{x^2+y^2}\,\mathrm{d}x\mathrm{d}y = \int_0^3 \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\ r e^{r^2} = 2\pi \int_0^3 \mathrm{d}r\ r e^{r^2} = \pi e^{r^2}\Big|_0^3 \\ &= \pi\left(e^9 - 1\right) \approx 25{,}453 \end{aligned}$$

(b) The two integrals have domains

$$\left\{ (x, y) \;\middle|\; 0 \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant y \right\} \qquad \left\{ (x, y) \;\middle|\; 1 \leqslant y \leqslant 2,\ 0 \leqslant x \leqslant 2 - y \right\}$$

The union of those two domains (as well as horizontal strips that were used in setting up the two given integrals) is sketched in the figure on the left below.



To reverse the order of integration, we decompose the domain using vertical strips as in the figure on the right above. As

- x runs from 0 to 1 and
- for each fixed x between 0 and 1, y runs from x to $2 - x$.

we have that the

$$\text{Volume} = \int_0^1 \mathrm{d}x \int_x^{2-x} \mathrm{d}y\ e^{x^2+y^2}$$

3.3▲ Applications of Double Integrals

▸▸ Stage 1

$\boxed{3.3.1}$ For each of the following, evaluate the given double integral without using iteration. Instead, interpret the integral in terms of, for example, areas or average values.
(a) $\iint_D (x + 3)\ \mathrm{d}x\,\mathrm{d}y$, where D is the half disc $0 \leqslant y \leqslant \sqrt{4 - x^2}$
(b) $\iint_R (x + y)\ \mathrm{d}x\,\mathrm{d}y$ where R is the rectangle $0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b$

$\boxed{\text{Solution}}$ (a) $\iint_D x\ \mathrm{d}x\,\mathrm{d}y = 0$ because x is odd under $x \to -x$, i.e. under reflection about the y–axis, while the domain of integration is symmetric about the y–axis. $\iint_D 3\ \mathrm{d}x\,\mathrm{d}y$ is the three times the area of a half disc of radius 2. So, $\iint_D (x + 3)\mathrm{d}x\,\mathrm{d}y = 3 \times \frac{1}{2} \times \pi 2^2 = 6\pi$.

(s) $\iint_R x\ \mathrm{d}x\,\mathrm{d}y / \iint_R \mathrm{d}x\,\mathrm{d}y$ is the average value of x in the rectangle R, namely $\frac{a}{2}$. Similarly, $\iint_R y\ \mathrm{d}x\,\mathrm{d}y / \iint_R \mathrm{d}x\,\mathrm{d}y$ is the average value of y in the rectangle R, namely $\frac{b}{2}$. $\iint_R \mathrm{d}x\,\mathrm{d}y$ is area of the rectangle R, namely ab. So,

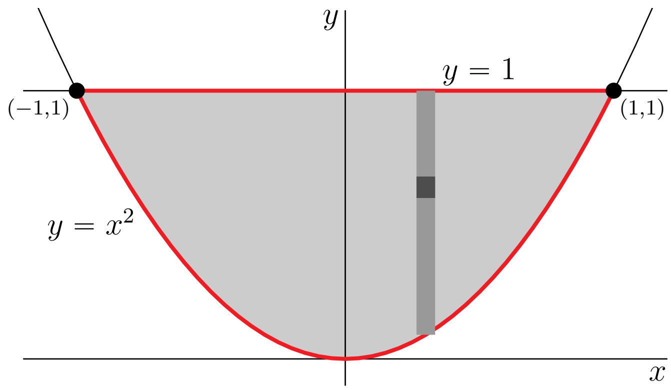
- $\iint_R x \, \mathrm{d}x \, \mathrm{d}y = \frac{a}{2} \iint_R \mathrm{d}x \, \mathrm{d}y = \frac{a}{2} ab$ and
- $\iint_R y \, \mathrm{d}x \, \mathrm{d}y = \frac{b}{2} \iint_R \mathrm{d}x \, \mathrm{d}y = \frac{b}{2} ab$

and $\iint_R (x+y) \mathrm{d}x \, \mathrm{d}y = \frac{1}{2} ab(a+b)$.

▸▸ Stage 2

3.3.2 (*) Find the centre of mass of the region D in the xy–plane defined by the inequalities $x^2 \leqslant y \leqslant 1$, assuming that the mass density function is given by $\rho(x,y) = y$.

Solution Here is a sketch of D.



By definition, the centre of mass is $(\bar{x}, \bar{y})$, with $\bar{x}$ and $\bar{y}$ being the weighted averages of the x and y–coordinates, respectively, over D. That is,

$$\bar{x} = \frac{\iint_D x \, \rho(x,y) \, \mathrm{d}A}{\iint_D \rho(x,y) \, \mathrm{d}A} \qquad \bar{y} = \frac{\iint_D y \, \rho(x,y) \, \mathrm{d}A}{\iint_D \rho(x,y) \, \mathrm{d}A}$$

By symmetry under reflection in the y–axis, we have $\bar{x} = 0$. So we just have to determine $\bar{y}$. We'll evaluate the integrals using vertical strips as in the figure above. Looking at that figure, we see that

- x runs from -1 to 1, and
- for each fixed x in that range, y runs from x^2 to 1.

So the denominator is

$$\begin{aligned} \iint_D \rho(x,y) \, \mathrm{d}A &= \int_{-1}^{1} \mathrm{d}x \int_{x^2}^{1} \mathrm{d}y \, \overbrace{y}^{\rho(x,y)} \\ &= \frac{1}{2} \int_{-1}^{1} \mathrm{d}x \, (1 - x^4) = \int_0^1 \mathrm{d}x \, (1 - x^4) \\ &= \frac{4}{5} \end{aligned}$$

and the numerator of $\bar{y}$ is

$$\begin{aligned}\iint_D y\,\rho(x,y)\ \mathrm{d}A &= \int_{-1}^{1} \mathrm{d}x \int_{x^2}^{1} \mathrm{d}y\ y\,\overbrace{y}^{\rho(x,y)} \\ &= \frac{1}{3}\int_{-1}^{1} \mathrm{d}x\ (1-x^6) = \frac{2}{3}\int_0^1 \mathrm{d}x\ (1-x^6) \\ &= \frac{2}{3}\,\frac{6}{7} = \frac{4}{7}\end{aligned}$$

All together, $\bar{x} = 0$ and

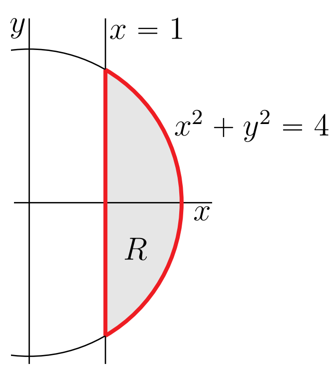
$$\bar{y} = \frac{\frac{4}{7}}{\frac{4}{5}} = \frac{5}{7}$$

$\boxed{3.3.3}$ (∗) Let R be the region bounded on the left by $x = 1$ and on the right by $x^2 + y^2 = 4$. The density in R is

$$\rho(x,y) = \frac{1}{\sqrt{x^2+y^2}}$$

(a) Sketch the region R.
(b) Find the mass of R.
(c) Find the centre-of-mass of R.
Note: You may use the result $\int \sec(\theta)\ \mathrm{d}\theta = \ln|\sec\theta + \tan\theta| + C$.

$\boxed{\text{Solution}}$ (a) Here is a sketch of R.



(b) Considering that

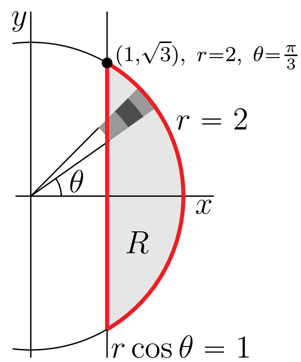
- $\rho(x,y)$ is invariant under rotations about the origin and
- the outer curve $x^2 + y^2 = 4$ is invariant under rotations about the origin and
- the given hint involves a θ integral

we'll use polar coordinates.

Observe that the line $x = 1$ and the circle $x^2 + y^2 = 4$ intersect when

$$1 + y^2 = 4 \iff y = \pm\sqrt{3}$$

and that the polar coordinates of the point $(x, y) = \big(1, \sqrt{3}\big)$ are $r = \sqrt{x^2 + y^2} = 2$ and $\theta = \arctan \frac{y}{x} = \arctan \sqrt{3} = \frac{\pi}{3}$. Looking at the sketch



we see that, on R,

- θ runs from $-\frac{\pi}{3}$ to $\frac{\pi}{3}$ and
- for each fixed θ in that range, r runs from $\frac{1}{\cos\theta} = \sec\theta$ to 2.
- In polar coordinates, $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$, and
- the density $\rho = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$

So the mass is

$$\begin{aligned} M = \iint_R \rho(x, y)\ \mathrm{d}A &= \int_{-\pi/3}^{\pi/3} \mathrm{d}\theta \int_{\sec\theta}^{2} \mathrm{d}r\ \frac{r}{r} = \int_{-\pi/3}^{\pi/3} \mathrm{d}\theta\ \big[2 - \sec\theta\big] \\ &= 2\int_0^{\pi/3} \mathrm{d}\theta\ \big[2 - \sec\theta\big] \\ &= 2\Big[2\theta - \ln\big(\sec\theta + \tan\theta\big)\Big]_0^{\pi/3} \\ &= 2\left[\frac{2\pi}{3} - \ln\big(2 + \sqrt{3}\big) + \ln\big(1 + 0\big)\right] \\ &= \frac{4\pi}{3} - 2\ln\big(2 + \sqrt{3}\big) \end{aligned}$$

(c) By definition, the centre of mass is $(\bar{x}, \bar{y})$, with $\bar{x}$ and $\bar{y}$ being the weighted averages of the x and y–coordinates, respectively, over R. That is,

$$\bar{x} = \frac{\iint_R x\ \rho(x, y)\ \mathrm{d}A}{\iint_R \rho(x, y)\ \mathrm{d}A} \qquad \bar{y} = \frac{\iint_R y\ \rho(x, y)\ \mathrm{d}A}{\iint_R \rho(x, y)\ \mathrm{d}A}$$

By symmetry under reflection in the x–axis, we have $\bar{y} = 0$. So we just have to determine

$\bar{x}$. The numerator is

$$\begin{aligned}
\iint_R x\ \rho(x,y)\ \mathrm{d}A &= \int_{-\pi/3}^{\pi/3} \mathrm{d}\theta \int_{\sec\theta}^{2} \mathrm{d}r\ \frac{r}{r}\ \overbrace{r\cos\theta}^{x} \\
&= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \mathrm{d}\theta\ \big[4-\sec^2\theta\big]\cos\theta = \int_0^{\pi/3} \mathrm{d}\theta\ \big[4\cos\theta-\sec\theta\big] \\
&= \Big[4\sin\theta - \ln\big(\sec\theta+\tan\theta\big)\Big]_0^{\pi/3} \\
&= \Big[4\frac{\sqrt{3}}{2} - \ln\big(2+\sqrt{3}\big) + \ln\big(1+0\big)\Big] \\
&= 2\sqrt{3} - \ln\big(2+\sqrt{3}\big)
\end{aligned}$$

All together, $\bar{y} = 0$ and

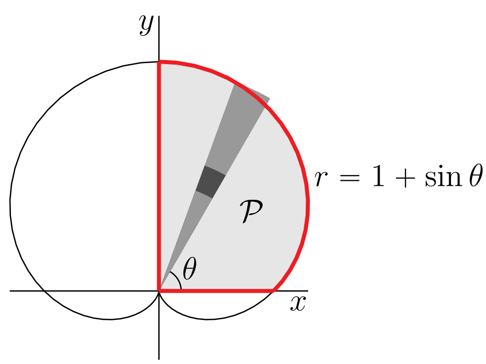
$$\bar{x} = \frac{2\sqrt{3}-\ln\big(2+\sqrt{3}\big)}{\frac{4\pi}{3} - 2\ln\big(2+\sqrt{3}\big)} \approx 1.38$$

$\boxed{3.3.4}$ (∗) A thin plate of uniform density 1 is bounded by the positive x and y axes and the cardioid $\sqrt{x^2+y^2} = r = 1+\sin\theta$, which is given in polar coordinates. Find the x–coordinate of its centre of mass.

$\boxed{\text{Solution}}$ Let's call the plate $\mathcal{P}$. By definition, the x–coordinate of its centre of mass is

$$\bar{x} = \frac{\iint_{\mathcal{P}} x\ \mathrm{d}A}{\iint_{\mathcal{P}} \mathrm{d}A}$$

Here is a sketch of the plate.



The cardiod is given to us in polar coordinates, so let's evaluate the integrals in polar coordinates. Looking at the sketch above, we see that, on $\mathcal{P}$,

- θ runs from 0 to $\pi/2$ and
- for each fixed θ in that range, r runs from 0 to $1+\sin\theta$.
- In polar coordinates $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$

So the two integrals of interest are

$$\begin{aligned}
\iint_{\mathcal{P}} \mathrm{d}A &= \int_0^{\pi/2} \mathrm{d}\theta \int_0^{1+\sin\theta} \mathrm{d}r\ r \\
&= \frac{1}{2}\int_0^{\pi/2} \mathrm{d}\theta\ \big(1 + 2\sin\theta + \sin^2\theta\big) \\
&= \frac{1}{2}\frac{\pi}{2} + \Big[-\cos\theta\Big]_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2} \mathrm{d}\theta\ \frac{1-\cos(2\theta)}{2} \\
&= \frac{\pi}{4} + 1 + \frac{1}{4}\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\pi/2} \\
&= \frac{3\pi}{8} + 1
\end{aligned}$$

and

$$\begin{aligned}
\iint_{\mathcal{P}} x\ \mathrm{d}A &= \int_0^{\pi/2} \mathrm{d}\theta \int_0^{1+\sin\theta} \mathrm{d}r\ r\ \overbrace{(r\cos\theta)}^{x} \\
&= \frac{1}{3}\int_0^{\pi/2} \mathrm{d}\theta\ \big(1+\sin\theta\big)^3 \cos\theta \\
&= \frac{1}{3}\int_1^2 \mathrm{d}u\ u^3 \qquad \text{with } u = 1+\sin\theta,\ \mathrm{d}u = \cos\theta\,\mathrm{d}\theta \\
&= \frac{1}{12}\big[2^4 - 1^4\big] \\
&= \frac{5}{4}
\end{aligned}$$

All together

$$\bar{x} = \frac{\frac{5}{4}}{\frac{3\pi}{8}+1} = \frac{10}{3\pi+8} \approx 0.57$$

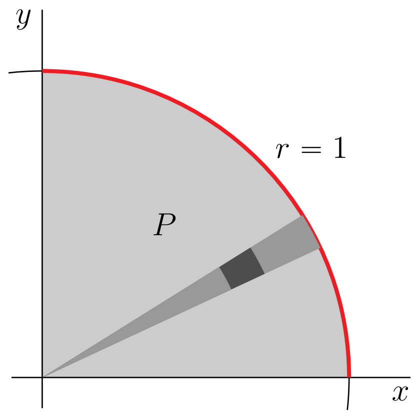
For an efficient, sneaky, way to evaluate $\int_0^{\pi/2} \sin^2\theta\ \mathrm{d}\theta$, see Remark 3.3.5 in the CLP-3 text.

$\boxed{3.3.5}$ (*) A thin plate of uniform density k is bounded by the positive x and y axes and the circle $x^2 + y^2 = 1$. Find its centre of mass.

$\boxed{\text{Solution}}$ Call the plate P. By definition, the centre of mass is $(\bar{x}, \bar{y})$, with $\bar{x}$ and $\bar{y}$ being the weighted averages of the x and y–coordinates, respectively, over P. That is,

$$\bar{x} = \frac{\iint_P x\ \rho(x,y)\ \mathrm{d}A}{\iint_P \rho(x,y)\ \mathrm{d}A} \qquad \bar{y} = \frac{\iint_P y\ \rho(x,y)\ \mathrm{d}A}{\iint_P \rho(x,y)\ \mathrm{d}A}$$

with $\rho(x,y) = k$. Here is a sketch of P.



By symmetry under reflection in the line $y = x$, we have $\bar{y} = \bar{x}$. So we just have to determine

$$\bar{x} = \frac{\iint_P x \, \mathrm{d}A}{\iint_P \mathrm{d}A}$$

The denominator is just one quarter of the area of circular disk of radius 1. That is, $\iint_P \mathrm{d}A = \frac{\pi}{4}$. We'll evaluate the numerator using polar coordinates as in the figure above. Looking at that figure, we see that

- θ runs from 0 to $\frac{\pi}{2}$, and
- for each fixed θ in that range, r runs from 0 to 1.

As $\mathrm{d}A = r \, \mathrm{d}r \, \mathrm{d}\theta$, and $x = r\cos\theta$, the numerator

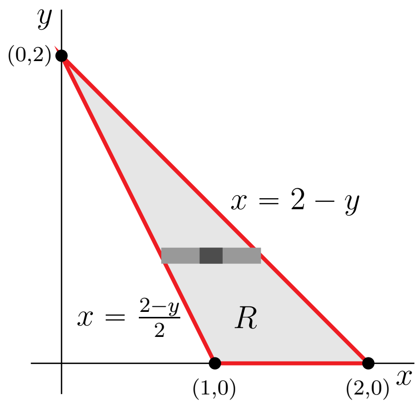
$$\begin{aligned}\iint_P x \, \mathrm{d}A &= \int_0^{\pi/2} \mathrm{d}\theta \int_0^1 \mathrm{d}r \; r \overbrace{r\cos\theta}^{x} = \left[\int_0^{\pi/2} \mathrm{d}\theta \; \cos\theta\right] \left[\int_0^1 \mathrm{d}r \; r^2\right] \\ &= \Big[\sin\theta\Big]_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^1 \\ &= \frac{1}{3}\end{aligned}$$

All together

$$\bar{x} = \bar{y} = \frac{1/3}{\pi/4} = \frac{4}{3\pi}$$

$\boxed{3.3.6}$ (*) Let R be the triangle with vertices $(0,2)$, $(1,0)$, and $(2,0)$. Let R have density $\rho(x,y) = y^2$. Find $\bar{y}$, the y–coordinate of the center of mass of R. You do not need to find $\bar{x}$.

$\boxed{\text{Solution}}$ Here is a sketch of R.



Note that

- the equation of the straight line through $(2,0)$ and $(0,2)$ is $y = 2 - x$, or $x = 2 - y$. (As a check note that both points $(2,0)$ and $(0,2)$ are on $x = 2 - y$.
- The equation of the straight line through $(1,0)$ and $(0,2)$ is $y = 2 - 2x$, or $x = \frac{2-y}{2}$. (As a check note that both points $(0,2)$ and $(1,0)$ are on $x = \frac{2-y}{2}$.

By definition, the y–coordinate of the center of mass of R is the weighted average of y over R, which is

$$\bar{y} = \frac{\iint_R y\,\rho(x,y)\,\mathrm{d}A}{\iint_R \rho(x,y)\,\mathrm{d}A} = \frac{\iint_R y^3\,\mathrm{d}A}{\iint_R y^2\,\mathrm{d}A}$$

On R,

- y runs from 0 to 2. That is, $0 \leqslant y \leqslant 2$.
- For each fixed y in that range, x runs from $\frac{2-y}{2}$ to $2 - y$. In inequalities, that is $\frac{2-y}{2} \leqslant x \leqslant 2 - y$.

Thus

$$R = \left\{ (x,y) \;\middle|\; 0 \leqslant y \leqslant 2,\ \frac{2-y}{2} \leqslant x \leqslant 2 - y \right\}$$

For both $n = 2$ and $n = 3$, we have

$$\begin{aligned}
\iint_R y^n\,\mathrm{d}A &= \int_0^2 \mathrm{d}y \int_{\frac{2-y}{2}}^{2-y} \mathrm{d}x\ y^n \\
&= \int_0^2 \mathrm{d}y\ y^n \frac{2-y}{2} \\
&= \frac{1}{2}\left[\frac{2y^{n+1}}{n+1} - \frac{y^{n+2}}{n+2}\right]_0^2 \\
&= \frac{1}{2}\left[\frac{2^{n+2}}{n+1} - \frac{2^{n+2}}{n+2}\right] \\
&= \frac{2^{n+1}}{(n+1)(n+2)}
\end{aligned}$$

So

$$\bar{y} = \frac{\iint_R y^3 \,\mathrm{d}A}{\iint_R y^2 \,\mathrm{d}A} = \frac{\frac{2^4}{(4)(5)}}{\frac{2^3}{(3)(4)}} = \frac{6}{5}$$

$\boxed{3.3.7}$ (*) The average distance of a point in a plane region D to a point (a,b) is defined by

$$\frac{1}{A(D)} \iint_D \sqrt{(x-a)^2 + (y-b)^2} \,\mathrm{d}x\,\mathrm{d}y$$

where $A(D)$ is the area of the plane region D. Let D be the unit disk $1 \geqslant x^2 + y^2$. Find the average distance of a point in D to the center of D.

Solution By the definition given in the statement with $(a,b) = (0,0)$, the average is

$$\frac{1}{A(D)} \iint_D \sqrt{x^2 + y^2} \,\mathrm{d}x\,\mathrm{d}y$$

The denominator $A(D) = \pi$. We'll use polar coordinates to evaluate the numerator.

$$\begin{aligned}
\iint_D \sqrt{x^2 + y^2} \,\mathrm{d}x\,\mathrm{d}y &= \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \mathrm{d}r \; r\sqrt{r^2\cos^2\theta + r^2\sin^2\theta} \\
&= \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \mathrm{d}r \; r^2 = \int_0^{2\pi} \mathrm{d}\theta \; \frac{1}{3} \\
&= \frac{2\pi}{3}
\end{aligned}$$

So the average is

$$\frac{\frac{2\pi}{3}}{\pi} = \frac{2}{3}$$

$\boxed{3.3.8}$ (*) A metal crescent is obtained by removing the interior of the circle defined by the equation $x^2 + y^2 = x$ from the metal plate of constant density 1 occupying the unit disc $x^2 + y^2 \leqslant 1$.
(a) Find the total mass of the crescent.
(b) Find the x-coordinate of its center of mass.
You may use the fact that $\int_{-\pi/2}^{\pi/2} \cos^4(\theta) \,\mathrm{d}\theta = \frac{3\pi}{8}$.

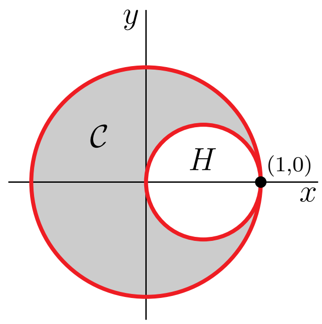
Solution Note that $x^2 + y^2 = x$ is equivalent to $\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$, which is the circle of radius $\frac{1}{2}$ centred on $\left(\frac{1}{2}, 0\right)$. Let's call the crescent $\mathcal{C}$ and write

$$\begin{aligned}
D &= \left\{ \ (x,y) \ \middle| \ x^2 + y^2 \leqslant 1 \ \right\} \\
H &= \left\{ \ (x,y) \ \middle| \ \left(x - \tfrac{1}{2}\right)^2 + y^2 \leqslant \tfrac{1}{4} \ \right\}
\end{aligned}$$

so that

$$\mathcal{C} = D \backslash H$$

meaning that $\mathcal{C}$ is the disk D with the "hole" H removed. Here is a sketch.



(a) As D is a disk of radius 1, it has area π. As H is a disk of radius $1/2$, it has area $\pi/4$. As $\mathcal{C}$ has density 1,

$$\begin{aligned}\mathrm{Mass}(\mathcal{C}) &= \iint_{\mathcal{C}} \mathrm{d}A = \iint_{D} \mathrm{d}A - \iint_{H} \mathrm{d}A \\ &= \pi - \frac{\pi}{4} \\ &= \frac{3\pi}{4}\end{aligned}$$

(b) Recall that, by definition, the x–coordinate of the centre of mass of $\mathcal{C}$ is the average value of x over $\mathcal{C}$, which is

$$\bar{x} = \frac{\iint_{\mathcal{C}} x \,\mathrm{d}A}{\iint_{\mathcal{C}} \mathrm{d}A}$$

We have already found that $\iint_{\mathcal{C}} \mathrm{d}A = \frac{3\pi}{4}$. So we have to determine the numerator

$$\iint_{\mathcal{C}} x \,\mathrm{d}A = \iint_{D} x \,\mathrm{d}A - \iint_{H} x \,\mathrm{d}A$$

As x is an odd function and D is invariant under $x \to -x$, $\iint_D x\,\mathrm{d}A = 0$. So we just have to determine $\iint_H x\,\mathrm{d}A$. To do so we'll work in polar coordinates, so that $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$. In polar coordinates $x^2 + y^2 = x$ is $r^2 = r\cos\theta$ or $r = \cos\theta$. So, looking at the figure above (just before the solution to part (a)), on the domain of integration,

- θ runs from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.
- For each fixed θ in that range, r runs from 0 to $\cos\theta$.

So the integral is

$$\begin{aligned}\iint\limits_H x\,\mathrm{d}A &= \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int_0^{\cos\theta} \mathrm{d}r\; r\,\overbrace{(r\cos\theta)}^{x} \\ &= \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta\; \frac{\cos^4\theta}{3} \\ &= \frac{\pi}{8}\end{aligned}$$

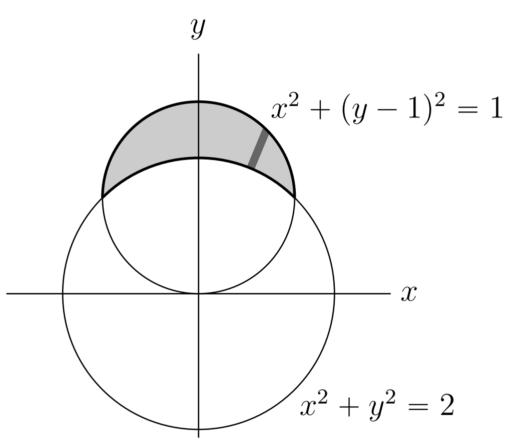
So all together

$$\bar{x} = \frac{\iint_{\mathcal{C}} x\,\mathrm{d}A}{\iint_{\mathcal{C}} \mathrm{d}A} = \frac{\iint_D x\,\mathrm{d}A - \iint_H x\,\mathrm{d}A}{\iint_{\mathcal{C}} \mathrm{d}A} = \frac{0 - \frac{\pi}{8}}{\frac{3\pi}{4}} = -\frac{1}{6}$$

$\boxed{3.3.9}$ (*) Let D be the region in the xy–plane which is inside the circle $x^2 + (y-1)^2 = 1$ but outside the circle $x^2 + y^2 = 2$. Determine the mass of this region if the density is given by

$$\rho(x,y) = \frac{2}{\sqrt{x^2+y^2}}$$

$\boxed{\text{Solution}}$ The domain is pictured below.



The two circles intersect when $x^2 + y^2 = 2$ and

$$x^2 + (y-1)^2 = 2 - y^2 + (y-1)^2 = 1 \iff -2y + 3 = 1 \iff y = 1 \text{ and } x = \pm 1$$

In polar coordinates $x^2 + y^2 = 2$ is $r = \sqrt{2}$ and $x^2 + (y-1)^2 = x^2 + y^2 - 2y + 1 = 1$ is $r^2 - 2r\sin\theta = 0$ or $r = 2\sin\theta$. The two curves intersect when $r = \sqrt{2}$ and $\sqrt{2} = 2\sin\theta$ so that $\theta = \frac{\pi}{4}$ or $\frac{3}{4}\pi$. So

$$D = \left\{\ (r\cos\theta, r\sin\theta)\ \middle|\ \tfrac{1}{4}\pi \leqslant \theta \leqslant \tfrac{3}{4}\pi,\ \sqrt{2} \leqslant r \leqslant 2\sin\theta\ \right\}$$

and, as the density is $\frac{2}{r}$,

$$\begin{aligned}\text{mass} &= \int_{\pi/4}^{3\pi/4} \mathrm{d}\theta \int_{\sqrt{2}}^{2\sin\theta} \mathrm{d}r\ r\frac{2}{r} = 2\int_{\pi/4}^{3\pi/4} \mathrm{d}\theta\ \big[2\sin\theta - \sqrt{2}\,\big] = 4\int_{\pi/4}^{\pi/2} \mathrm{d}\theta\ \big[2\sin\theta - \sqrt{2}\,\big] \\ &= 4\Big[-2\cos\theta - \sqrt{2}\theta\Big]_{\pi/4}^{\pi/2} = 4\sqrt{2} - \sqrt{2}\pi \approx 1.214\end{aligned}$$

▸▸ Stage 3

3.3.10 (*) Let a, b and c be positive numbers, and let T be the triangle whose vertices are $(-a, 0)$, $(b, 0)$ and $(0, c)$.

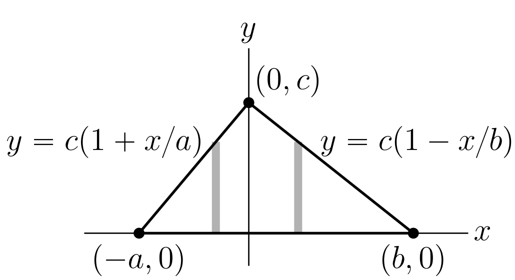
(a) Assuming that the density is constant on T, find the center of mass of T.

(b) The medians of T are the line segments which join a vertex of T to the midpoint of the opposite side. It is a well known fact that the three medians of any triangle meet at a point, which is known as the centroid of T. Show that the centroid of T is its centre of mass.

Solution (a) The side of the triangle from $(-a, 0)$ to $(0, c)$ is straight line that passes through those two points. As $y = 0$ when $x = -a$, the line must have an equation of the form $y = K(x + a)$ for some constant K. Since $y = c$ when $x = 0$, the constant $K = \frac{c}{a}$. So that the equation is $y = \frac{c}{a}(x + a)$. has equation $cx - ay = -ac$. Similarly the side of the triangle from $(b, 0)$ to $(0, c)$ has equation $y = \frac{c}{b}(b - x)$. The triangle has area $A = \frac{1}{2}(a + b)c$. It has centre of mass $(\bar{x}, \bar{y})$ with

$$\bar{x} = \frac{1}{A}\iint_T x\ \mathrm{d}x\mathrm{d}y \qquad \bar{y} = \frac{1}{A}\iint_T y\ \mathrm{d}x\mathrm{d}y$$

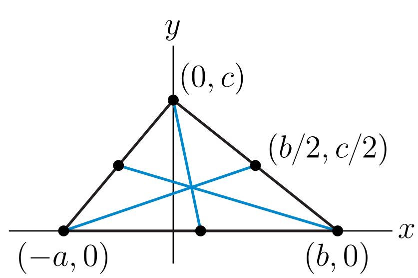
To evaluate the integrals we'll decompose the triangle into vertical strips as in the figure



$$\begin{aligned}
\bar{x} &= \frac{1}{A}\iint_T x\ \mathrm{d}x\mathrm{d}y \\
&= \frac{1}{A}\left(\int_{-a}^{0}\mathrm{d}x\int_0^{c+\frac{c}{a}x}\mathrm{d}y\ x + \int_0^b \mathrm{d}x\int_0^{c-\frac{c}{b}x}\mathrm{d}y\ x\right) \\
&= \frac{1}{A}\left(\int_{-a}^{0}\mathrm{d}x\ x\left(c+\frac{c}{a}x\right) + \int_0^b \mathrm{d}x\ x\left(c-\frac{c}{b}x\right)\right) \\
&= \frac{1}{A}\left(\left[\frac{1}{2}cx^2+\frac{c}{3a}x^3\right]_{-a}^{0} + \left[\frac{1}{2}cx^2-\frac{c}{3b}x^3\right]_0^b\right) \\
&= 2\frac{\frac{1}{2}c(b^2-a^2)+\frac{c}{3}(a^2-b^2)}{(a+b)c} = \frac{1}{3}(b-a) \\
\bar{y} &= \frac{1}{A}\iint_T y\ \mathrm{d}x\mathrm{d}y \\
&= \frac{1}{A}\left(\int_{-a}^{0}\mathrm{d}x\int_0^{c+\frac{c}{a}x}\mathrm{d}y\ y + \int_0^b \mathrm{d}x\int_0^{c-\frac{c}{b}x}\mathrm{d}y\ y\right) \\
&= \frac{1}{A}\left(\int_{-a}^{0}\mathrm{d}x\ \frac{1}{2}\left(c+\frac{c}{a}x\right)^2 + \int_0^b \mathrm{d}x\ \frac{1}{2}\left(c-\frac{c}{b}x\right)^2\right) \\
&= \frac{1}{A}\left(\frac{a}{6c}\left[c+\frac{c}{a}x\right]^3\Big|_{-a}^{0} - \frac{b}{6c}\left(c-\frac{c}{b}x\right)^3\Big|_0^b\right) \\
&= 2\frac{\frac{ac^2}{6}+\frac{bc^2}{6}}{(a+b)c} = \frac{c}{3}
\end{aligned}$$

(b) The midpoint of the side opposite $(-a,0)$ is $\frac{1}{2}\left[(b,0)+(0,c)\right] = \frac{1}{2}(b,c)$. The vector from $(-a,0)$ to $\frac{1}{2}(b,c)$ is $\frac{1}{2}\langle b,c\rangle - \langle -a,0\rangle = \left\langle a+\frac{b}{2},\frac{c}{2}\right\rangle$. So the line joining these two points has vector parametric equation

$$\mathbf{r}(t) = \langle -a,0\rangle + t\left\langle a+\frac{1}{2}b\,,\ \frac{1}{2}c\right\rangle$$



The point $(\bar{x},\bar{y})$ lies on this line since

$$\mathbf{r}\left(\frac{2}{3}\right) = \left(\frac{1}{3}(b-a)\,,\ \frac{c}{3}\right) = (\bar{x},\bar{y})$$

Similarly, the midpoint of the side opposite $(b,0)$ is $\frac{1}{2}(-a,c)$. The line joining these two points has vector parametric equation

$$\mathbf{r}(t) = \langle b, 0\rangle + t\left\langle -b - \frac{1}{2}a\,,\, \frac{1}{2}c\right\rangle$$

The point $(\bar{x}, \bar{y})$ lies on this line too, since

$$\mathbf{r}\left(\frac{2}{3}\right) = \left(\frac{1}{3}(b-a), \frac{c}{3}\right) = (\bar{x}, \bar{y})$$

It is not really necessary to check that $(\bar{x}, \bar{y})$ lies on the third median, but let's do it anyway. The midpoint of the side opposite $(0, c)$ is $\frac{1}{2}(b-a, 0)$. The line joining these two points has vector parametric equation

$$\mathbf{r}(t) = \langle 0, c\rangle + t\left\langle \frac{b}{2} - \frac{a}{2}, -c\right\rangle$$

The point $(\bar{x}, \bar{y})$ lies on this median too, since

$$\mathbf{r}\left(\frac{2}{3}\right) = \left(\frac{1}{3}(b-a), \frac{c}{3}\right) = (\bar{x}, \bar{y})$$

3.4▲ Surface Area

▸▸ Stage 1

$\boxed{3.4.1}$ Let $0 < \theta < \frac{\pi}{2}$, and $a, b > 0$. Denote by S the part of the surface $z = y\,\tan\theta$ with $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$.
(a) Find the surface area of S without using any calculus.
(b) Find the surface area of S by using Theorem 3.4.2 in the CLP-3 text.

Solution (a) S is the part of the plane $z = y\,\tan\theta$ that lies above the rectangle in the xy-plane with vertices $(0,0)$, $(a,0)$, $(0,b)$, (a,b). So S is the rectangle with vertices $(0,0,0)$, $(a,0,0)$, $(0,b,b\tan\theta)$, $(a,b,b\tan\theta)$. So it has side lengths

$$\begin{aligned} |\,\langle a,0,0\rangle - \langle 0,0,0\rangle\,| &= a \\ |\,\langle 0,b,b\tan\theta\rangle - \langle 0,0,0\rangle\,| &= \sqrt{b^2 + b^2\tan^2\theta} \end{aligned}$$

and hence area $ab\sqrt{1+\tan^2\theta} = ab\sec\theta$.

(b) S is the part of the surface $z = f(x,y)$ with $f(x,y) = y\,\tan\theta$ and with (x,y) running over

$$\mathcal{D} = \left\{\ (x,y)\ \middle|\ 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b\ \right\}$$

Hence by Theorem 3.4.2 in the CLP-3 text

$$\begin{aligned} \text{Area}(S) &= \iint_{\mathcal{D}} \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\ \mathrm{d}x\,\mathrm{d}y \\ &= \int_0^a \mathrm{d}x \int_0^b \mathrm{d}y\ \sqrt{1 + 0^2 + \tan^2\theta} \\ &= ab\sqrt{1+\tan^2\theta} = ab\sec\theta \end{aligned}$$

$\boxed{3.4.2}$ Let $c > 0$. Denote by S the part of the surface $ax + by + cz = d$ with (x, y) running over the region D in the xy-plane. Find the surface area of S, in terms of a, b, c, d and $A(D)$, the area of the region D.

$\boxed{\text{Solution}}$ S is the part of the surface $z = f(x, y)$ with $f(x, y) = \frac{d-ax-by}{c}$ and with (x, y) running over D. Hence by Theorem 3.4.2 in the CLP-3 text

$$\begin{aligned}\text{Area}(S) &= \iint_D \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, \mathrm{d}x\, \mathrm{d}y \\ &= \iint_D \sqrt{1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}} \\ &= \frac{\sqrt{a^2 + b^2 + c^2}}{c} A(D)\end{aligned}$$

$\boxed{3.4.3}$ Let $a, b, c > 0$. Denote by S the triangle with vertices $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$.

(a) Find the surface area of S in three different ways, each using Theorem 3.4.2 in the CLP-3 text.

(b) Denote by T_{xy} the projection of S onto the xy-plane. (It is the triangle with vertices $(0, 0, 0)$ $(a, 0, 0)$ and $(0, b, 0)$.) Similarly use T_{xz} to denote the projection of S onto the xz-plane and T_{yz} to denote the projection of S onto the yz-plane. Show that

$$\text{Area}(S) = \sqrt{\text{Area}(T_{xy})^2 + \text{Area}(T_{xz})^2 + \text{Area}(T_{yz})^2}$$

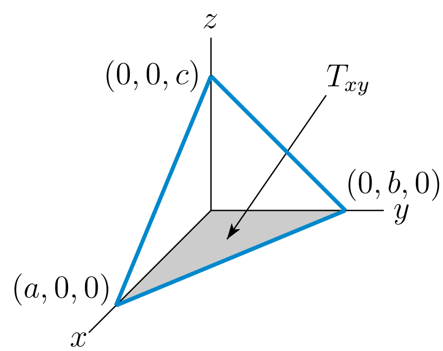
$\boxed{\text{Solution}}$ Note that all three vertices $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ lie on the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. So the triangle is part of that plane.

Method 1. S is the part of the surface $z = f(x, y)$ with $f(x, y) = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$ and with (x, y) running over the triangle T_{xy} in the xy-plane with vertices $(0, 0, 0)$ $(a, 0, 0)$ and $(0, b, 0)$. Hence by part a of Theorem 3.4.2 in the CLP-3 text

$$\begin{aligned}\text{Area}(S) &= \iint_{T_{xy}} \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, \mathrm{d}x\, \mathrm{d}y \\ &= \iint_{T_{xy}} \sqrt{1 + \frac{c^2}{a^2} + \frac{c^2}{b^2}}\, \mathrm{d}x\, \mathrm{d}y \\ &= \sqrt{1 + \frac{c^2}{a^2} + \frac{c^2}{b^2}}\, A(T_{xy})\end{aligned}$$

where $A(T_{xy})$ is the area of T_{xy}. Since the triangle T_{xy} has base a and height b (see the figure below), it has area $\frac{1}{2}ab$. So

$$\text{Area}(S) = \frac{1}{2}\sqrt{1 + \frac{c^2}{a^2} + \frac{c^2}{b^2}}\, ab = \frac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}$$



Method 2. S is the part of the surface $x = g(y,z)$ with $g(y,z) = a\left(1 - \frac{y}{b} - \frac{z}{c}\right)$ and with (y,z) running over the triangle T_{yz} in the yz-plane with vertices $(0,0,0)$ $(0,b,0)$ and $(0,0,c)$. Hence by part b of Theorem 3.4.2 in the CLP-3 text

$$\begin{aligned}\text{Area}(S) &= \iint_{T_{yz}} \sqrt{1 + g_y(y,z)^2 + g_z(y,z)^2}\ \mathrm{d}y\,\mathrm{d}z \\ &= \iint_{T_{yz}} \sqrt{1 + \frac{a^2}{b^2} + \frac{a^2}{c^2}}\ \mathrm{d}y\,\mathrm{d}z \\ &= \sqrt{1 + \frac{a^2}{b^2} + \frac{a^2}{c^2}}\ A(T_{yz})\end{aligned}$$

where $A(T_{yz})$ is the area of T_{yz}. Since T_{yz} has base b and height c, it has area $\frac{1}{2}bc$. So

$$\text{Area}(S) = \frac{1}{2}\sqrt{1 + \frac{a^2}{b^2} + \frac{a^2}{c^2}}\ bc = \frac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}$$

Method 3. S is the part of the surface $y = h(x,z)$ with $h(x,z) = b\left(1 - \frac{x}{a} - \frac{z}{c}\right)$ and with (x,z) running over the triangle T_{xz} in the xz-plane with vertices $(0,0,0)$ $(a,0,0)$ and $(0,0,c)$. Hence by part c of Theorem 3.4.2 in the CLP-3 text

$$\begin{aligned}\text{Area}(S) &= \iint_{T_{xz}} \sqrt{1 + h_x(x,z)^2 + h_z(x,z)^2}\ \mathrm{d}x\,\mathrm{d}z \\ &= \iint_{T_{xz}} \sqrt{1 + \frac{b^2}{a^2} + \frac{b^2}{c^2}}\ \mathrm{d}x\,\mathrm{d}z \\ &= \sqrt{1 + \frac{b^2}{a^2} + \frac{b^2}{c^2}}\ A(T_{xz})\end{aligned}$$

where $A(T_{xz})$ is the area of T_{xz}. Since T_{xz} has base a and height c, it has area $\frac{1}{2}ac$. So

$$\text{Area}(S) = \frac{1}{2}\sqrt{1 + \frac{b^2}{a^2} + \frac{b^2}{c^2}}\ bc = \frac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}$$

(b) We have already seen in the solution to part (a) that

$$\text{Area}(T_{xy}) = \frac{ab}{2} \qquad \text{Area}(T_{xz}) = \frac{ac}{2} \qquad \text{Area}(T_{yz}) = \frac{bc}{2}$$

Hence

$$\begin{aligned}\text{Area}(S) &= \sqrt{\frac{a^2b^2}{4} + \frac{a^2c^2}{4} + \frac{b^2c^2}{4}} \\ &= \sqrt{\text{Area}(T_{xy})^2 + \text{Area}(T_{xz})^2 + \text{Area}(T_{yz})^2}\end{aligned}$$

▸▸ Stage 2

$\boxed{3.4.4}$ (*) Find the area of the part of the surface $z = y^{3/2}$ that lies above $0 \leqslant x, y \leqslant 1$.

$\boxed{\text{Solution}}$ For the surface $z = f(x,y) = y^{3/2}$,

$$\mathrm{d}S = \sqrt{1 + f_x^2 + f_y^2}\ \mathrm{d}x\mathrm{d}y = \sqrt{1 + \Big(\frac{3}{2}\sqrt{y}\Big)^2}\ \mathrm{d}x\mathrm{d}y = \sqrt{1 + \frac{9}{4}y}\ \mathrm{d}x\mathrm{d}y$$

by Theorem 3.4.2.a in the CLP-3 text, So the area is

$$\begin{aligned}\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\ \sqrt{1 + \frac{9}{4}y} &= \int_0^1 \mathrm{d}x\ \frac{8}{27}\Big[\Big(1 + \frac{9}{4}y\Big)^{3/2}\Big]_0^1 = \int_0^1 \mathrm{d}x\ \frac{8}{27}\Big[\Big(\frac{13}{4}\Big)^{3/2} - 1\Big] \\ &= \frac{8}{27}\left[\left(\frac{13}{4}\right)^{3/2} - 1\right]\end{aligned}$$

$\boxed{3.4.5}$ (*) Find the surface area of the part of the paraboloid $z = a^2 - x^2 - y^2$ which lies above the xy–plane.

$\boxed{\text{Solution}}$ First observe that any point (x,y,z) on the paraboliod lies above the xy-plane if and only if

$$0 \leqslant z = a^2 - x^2 - y^2 \iff x^2 + y^2 \leqslant a^2$$

That is, if and only if (x,y) lies in the circular disk of radius a centred on the origin. The equation of the paraboloid is of the form $z = f(x,y)$ with $f(x,y) = a^2 - x^2 - y^2$. So, by Theorem 3.4.2.a in the CLP-3 text,

$$\begin{aligned}\text{Surface area} &= \iint\limits_{x^2+y^2 \leqslant a^2} \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\ \mathrm{d}x\,\mathrm{d}y \\ &= \iint\limits_{x^2+y^2 \leqslant a^2} \sqrt{1 + 4x^2 + 4y^2}\ \mathrm{d}x\,\mathrm{d}y\end{aligned}$$

Switching to polar coordinates,

$$\begin{aligned}
\text{Surface area} &= \int_0^a \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \; r\sqrt{1+4r^2} \\
&= 2\pi \int_0^a \mathrm{d}r \; r\sqrt{1+4r^2} \\
&= 2\pi \int_1^{1+4a^2} \frac{\mathrm{d}s}{8} \sqrt{s} \qquad \text{with } s = 1+4r^2,\ \mathrm{d}s = 8r\,\mathrm{d}r \\
&= \frac{\pi}{4}\,\frac{2}{3} s^{3/2}\Big|_{s=1}^{s=1+4a^2} \\
&= \frac{\pi}{6}\Big[\big(1+4a^2\big)^{3/2} - 1\Big]
\end{aligned}$$

$\boxed{3.4.6}$ (*) Find the area of the portion of the cone $z^2 = x^2 + y^2$ lying between the planes $z = 2$ and $z = 3$.

$\boxed{\text{Solution}}$ First observe that any point (x, y, z) on the cone lies between the planes $z = 2$ and $z = 3$ if and only if $4 \leqslant x^2 + y^2 \leqslant 9$. The equation of the cone can be rewritten in the form $z = f(x, y)$ with $f(x, y) = \sqrt{x^2 + y^2}$. Note that

$$f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \qquad f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

So, by Theorem 3.4.2.a in the CLP-3 text,

$$\begin{aligned}
\text{Surface area} &= \iint_{4\leqslant x^2+y^2\leqslant 9} \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\; \mathrm{d}x\,\mathrm{d}y \\
&= \iint_{4\leqslant x^2+y^2\leqslant 9} \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\; \mathrm{d}x\,\mathrm{d}y \\
&= \sqrt{2} \iint_{4\leqslant x^2+y^2\leqslant 9} \mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

Now the domain of integration is a circular washer with outside radius 3 and inside radius 2 and hence of area $\pi(3^2 - 2^2) = 5\pi$. So the surface area is $5\sqrt{2}\pi$.

$\boxed{3.4.7}$ (*) Determine the surface area of the surface given by $z = \frac{2}{3}\big(x^{3/2} + y^{3/2}\big)$, over the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

$\boxed{\text{Solution}}$ The equation of the surface is of the form $z = f(x, y)$ with $f(x, y) = \frac{2}{3}\big(x^{3/2} + y^{3/2}\big)$. Note that

$$f_x(x,y) = \sqrt{x} \qquad f_y(x,y) = \sqrt{y}$$

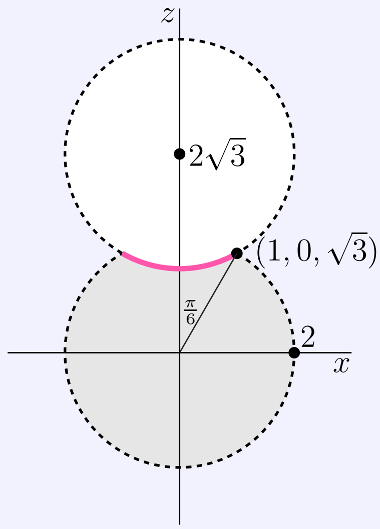
So, by Theorem 3.4.2.a in the CLP-3 text,

$$\begin{aligned}
\text{Surface area} &= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \ \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2} \\
&= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \ \sqrt{1+x+y} \\
&= \int_0^1 \mathrm{d}x \ \Big[\frac{2}{3}(1+x+y)^{3/2}\Big]_{y=0}^{y=1} \\
&= \frac{2}{3}\int_0^1 \mathrm{d}x \ \big[(2+x)^{3/2} - (1+x)^{3/2}\big] \\
&= \frac{2}{3}\ \frac{2}{5}\Big[(2+x)^{5/2} - (1+x)^{5/2}\Big]_{x=0}^{x=1} \\
&= \frac{4}{15}\big[3^{5/2} - 2^{5/2} - 2^{5/2} + 1^{5/2}\big] \\
&= \frac{4}{15}\big[9\sqrt{3} - 8\sqrt{2} + 1\big]
\end{aligned}$$

$\boxed{3.4.8}$ (*)

(a) To find the surface area of the surface $z = f(x, y)$ above the region D, we integrate $\iint_D F(x, y)\, \mathrm{d}A$. What is $F(x, y)$?

(b) Consider a "Death Star", a ball of radius 2 centred at the origin with another ball of radius 2 centred at $(0, 0, 2\sqrt{3})$ cut out of it. The diagram below shows the slice where $y = 0$.



(i) The Rebels want to paint part of the surface of Death Star hot pink; specifically, the concave part (indicated with a thick line in the diagram). To help them determine how much paint is needed, carefully fill in the missing parts of this integral:

$$\text{surface area} = \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \_\_\_\_\_\_\_ \, \mathrm{d}r \, \mathrm{d}\theta$$

(ii) What is the total surface area of the Death Star?

$\boxed{\text{Solution}}$ (a) By Theorem 3.4.2.a in the CLP-3 text, $F(x, y) = \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}$.

(b) (i) The "dimple" to be painted is part of the upper sphere $x^2 + y^2 + \left(z - 2\sqrt{3}\right)^2 = 4$. It is on the bottom half of the sphere and so has equation $z = f(x, y) = 2\sqrt{3} - \sqrt{4 - x^2 - y^2}$. Note that

$$f_x(x, y) = \frac{x}{\sqrt{4 - x^2 - y^2}} \qquad f_y(x, y) = \frac{y}{\sqrt{4 - x^2 - y^2}}$$

The point on the dimple with the largest value of x is $(1, 0, \sqrt{3})$. (It is marked by a dot in the figure above.) The dimple is invariant under rotations around the z–axis and so has

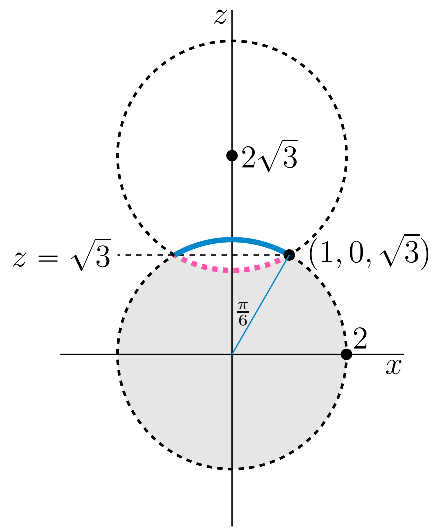
(x, y) running over $x^2 + y^2 \leqslant 1$. So, by Theorem 3.4.2.a in the CLP-3 text,

$$\begin{aligned}\text{Surface area} &= \iint_{x^2+y^2\leqslant 1} \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\, \mathrm{d}x\,\mathrm{d}y \\ &= \iint_{x^2+y^2\leqslant 1} \sqrt{1 + \frac{x^2}{4-x^2-y^2} + \frac{y^2}{4-x^2-y^2}}\, \mathrm{d}x\,\mathrm{d}y \\ &= \iint_{x^2+y^2\leqslant 1} \frac{2}{\sqrt{4-x^2-y^2}}\, \mathrm{d}x\,\mathrm{d}y\end{aligned}$$

Switching to polar coordinates,

$$\text{Surface area} = \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \mathrm{d}r\, \frac{2r}{\sqrt{4-r^2}}$$

(b) (ii) Observe that if we flip the dimple up by reflecting it in the plane $z = \sqrt{3}$, as in the figure below, the “Death Star” becomes a perfect ball of radius 2.



The area of the pink dimple in the figure above is identical to the area of the blue cap in that figure. So the total surface area of the Death Star is exactly the surface area of a sphere of radius $a = 2$ and so (see Example 3.4.5 in the CLP-3 text) is $4\pi a^2 = 4\pi 2^2 = 16\pi$.

$\boxed{3.4.9}$ (*) Find the area of the cone $z^2 = x^2 + y^2$ between $z = 1$ and $z = 16$.

$\boxed{\text{Solution}}$ On the upper half of the cone

$$z = f(x,y) = \sqrt{x^2+y^2} \qquad f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \qquad f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

so that

$$\mathrm{d}S = \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\, \mathrm{d}x\,\mathrm{d}y = \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\, \mathrm{d}x\,\mathrm{d}y = \sqrt{2}\, \mathrm{d}x\,\mathrm{d}y$$

and

$$\begin{aligned}\text{Area} &= \iint\limits_{1\leqslant x^2+y^2\leqslant 16^2} \sqrt{2}\,\mathrm{d}x\,\mathrm{d}y \\ &= \sqrt{2}\Big[\text{area of } \{\ (x,y)\ |\ x^2+y^2 \leqslant 16^2\ \} - \text{area of } \{\ (x,y)\ |\ x^2+y^2 \leqslant 1\ \}\Big] \\ &= \sqrt{2}\left[\pi 16^2 - \pi 1^2\right] = 255\sqrt{2}\pi \approx 1132.9\end{aligned}$$

$\boxed{3.4.10}$ (*) Find the surface area of that part of the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ which lies within the cylinder $\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$.

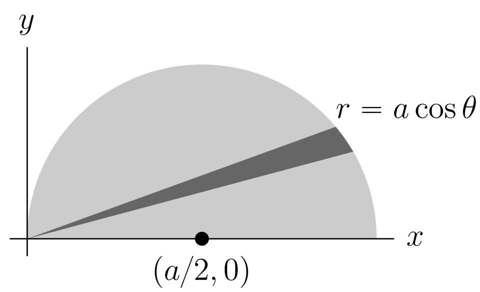
$\boxed{\text{Solution}}$ We are to find the surface area of part of a hemisphere. On the hemisphere

$$z = f(x,y) = \sqrt{a^2 - x^2 - y^2} \qquad f_x(x,y) = -\frac{x}{\sqrt{a^2 - x^2 - y^2}} \qquad f_y(x,y) = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}$$

so that

$$\begin{aligned}\mathrm{d}S &= \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\,\mathrm{d}x\,\mathrm{d}y = \sqrt{1 + \frac{x^2}{a^2-x^2-y^2} + \frac{y^2}{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y \\ &= \sqrt{\frac{a^2}{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

In polar coordinates, this is $\mathrm{d}S = \frac{a}{\sqrt{a^2-r^2}}\,r\,\mathrm{d}r\,\mathrm{d}\theta$. We are to find the surface area of the part of the hemisphere that is inside the cylinder, $x^2 - ax + y^2 = 0$, which is polar coordinates is becomes $r^2 - ar\cos\theta = 0$ or $r = a\cos\theta$. The top half of the domain of integration is sketched below.



So the

$$\begin{aligned}\text{Surface Area} &= 2\int_0^{\pi/2} \mathrm{d}\theta \int_0^{a\cos\theta} \mathrm{d}r\ r\frac{a}{\sqrt{a^2-r^2}} = 2a\int_0^{\pi/2} \mathrm{d}\theta\ \Big[-\sqrt{a^2-r^2}\Big]_0^{a\cos\theta} \\ &= 2a\int_0^{\pi/2} \mathrm{d}\theta\ \left[a - a\sin\theta\right] \\ &= 2a^2\Big[\theta + \cos\theta\Big]_0^{\pi/2} = a^2[\pi - 2]\end{aligned}$$

3.5▲ Triple Integrals

▸▸ Stage 1

3.5.1 Evaluate the integral

$$\iint\limits_R \sqrt{b^2 - y^2}\,\mathrm{d}x\,\mathrm{d}y \quad \text{where } R \text{ is the rectangle } 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b$$

without using iteration. Instead, interpret the integral geometrically.

Solution $\iint_R \sqrt{b^2 - y^2}\,\mathrm{d}x\,\mathrm{d}y = \iiint_V \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where

$$\begin{aligned} V &= \left\{ (x,y,z) \mid 0 \leqslant z \leqslant \sqrt{b^2 - y^2},\ 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b \right\} \\ &= \left\{ (x,y,z) \mid y^2 + z^2 \leqslant b^2,\ 0 \leqslant x \leqslant a,\ y \geqslant 0,\ z \geqslant 0 \right\} \end{aligned}$$

Now $y^2 + z^2 \leqslant b^2$ is a cylinder of radius b centered on the x–axis and the part of $y^2 + z^2 \leqslant b^2$, with $y \geqslant 0,\ z \geqslant 0$ is one quarter of this cylinder. It has cross–sectional area $\frac{1}{4}\pi b^2$. V is the part of this quarter–cylinder with $0 \leqslant x \leqslant a$. It has length a and cross–sectional area $\frac{1}{4}\pi b^2$. So, $\iint_R \sqrt{b^2 - y^2}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{4}\pi a b^2$.

3.5.2 (*) Find the total mass of the rectangular box $[0,1] \times [0,2] \times [0,3]$ (that is, the box defined by the inequalities $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$, $0 \leqslant z \leqslant 3$), with density function $h(x,y,z) = x$.

Solution The mass is

$$\int_0^1 \mathrm{d}x \int_0^2 \mathrm{d}y \int_0^3 \mathrm{d}z\ x = 6 \int_0^1 \mathrm{d}x\ x = 3$$

▸▸ Stage 2

3.5.3 Evaluate $\iiint\limits_R x\ \mathrm{d}V$ where R is the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Solution The domain of integration is

$$V = \left\{ (x,y,z) \mid x,y,z \geqslant 0,\ \tfrac{x}{a} + \tfrac{y}{b} + \tfrac{z}{c} \leqslant 1 \right\}$$

- In V, $\frac{z}{c} \leqslant 1 - \frac{x}{a} - \frac{y}{b}$ and $x,y \geqslant 0$, so the biggest value of z in V is achieved when $x = y = 0$ and is c. Thus, in V, z runs from 0 to c.

- For each fixed $0 \leqslant z \leqslant c$, (x, y) takes all values in

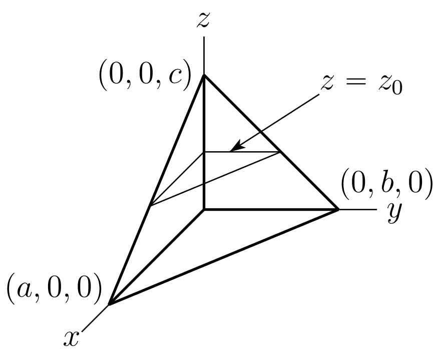
$$D_z = \left\{ \ (x,y) \ \middle| \ x,y \geqslant 0,\ \tfrac{x}{a} + \tfrac{y}{b} \leqslant 1 - \tfrac{z}{c} \ \right\}$$

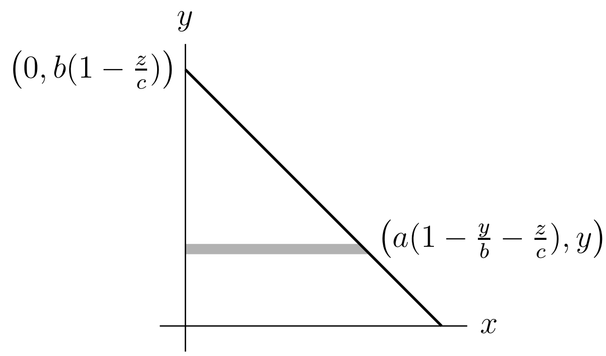
  The biggest value of y on D_z is achieved when $x = 0$ and is $b\left(1 - \frac{z}{c}\right)$. Thus, on D_z, y runs from 0 to $b\left(1 - \frac{z}{c}\right)$.

- For each fixed $0 \leqslant z \leqslant c$ and $0 \leqslant y \leqslant b\left(1 - \frac{z}{c}\right)$, x runs over

$$D_{y,z} = \left\{ \ x \ \middle| \ 0 \leqslant x \leqslant a\left(1 - \tfrac{y}{b} - \tfrac{z}{c}\right) \ \right\}$$

  This is pictured in the figure on the right below.





So the specified integral is

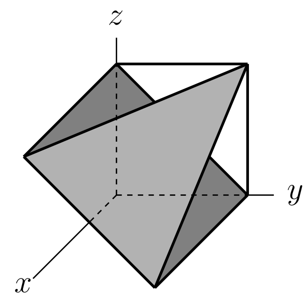
$$\begin{aligned}
\iiint_R x \ \mathrm{d}V &= \int_0^c \mathrm{d}z \iint_{D_z} \mathrm{d}x\,\mathrm{d}y\ x = \int_0^c \mathrm{d}z \int_0^{b(1-\frac{z}{c})} \mathrm{d}y \int_{D_{y,z}} \mathrm{d}x\ x \\
&= \int_0^c \mathrm{d}z \int_0^{b(1-\frac{z}{c})} \mathrm{d}y \int_0^{a(1-\frac{y}{b}-\frac{z}{c})} \mathrm{d}x\ x = \int_0^c \mathrm{d}z \int_0^{b(1-\frac{z}{c})} \mathrm{d}y\ \frac{a^2}{2}\left(1 - \frac{y}{b} - \frac{z}{c}\right)^2 \\
&= \int_0^c \mathrm{d}z\ \left[-\frac{a^2 b}{6}\left(1 - \frac{y}{b} - \frac{z}{c}\right)^3\right]_0^{b(1-\frac{z}{c})} = \int_0^c \mathrm{d}z\ \frac{a^2 b}{6}\left(1 - \frac{z}{c}\right)^3 \\
&= \left[-\frac{a^2 bc}{24}\left(1 - \frac{z}{c}\right)^4\right]_0^c = \frac{a^2 bc}{24}
\end{aligned}$$

3.5.4 Evaluate $\iiint_R y \ \mathrm{d}V$ where R is the portion of the cube $0 \leqslant x, y, z \leqslant 1$ lying above the plane $y + z = 1$ and below the plane $x + y + z = 2$.

Solution The domain of integration is

$$R = \left\{ \ (x,y,z) \ \middle| \ 0 \leqslant x,y,z \leqslant 1,\ z \geqslant 1 - y,\ z \leqslant 2 - x - y \ \right\}$$

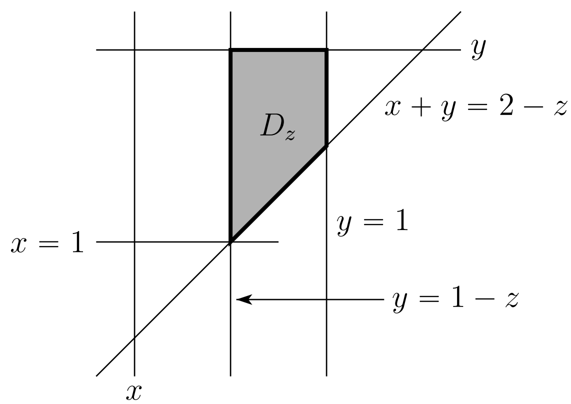
In the figure on the below, the more darkly shaded region is part of $z = 1 - y$ and the more lightly shaded region is part of $z = 2 - x - y$.



- In R, z runs from 0 (for example $(0,1,0)$ is in R) to 1 (for example $(0,0,1)$ is in R).
- For each fixed $0 \leqslant z \leqslant 1$, (x,y) runs over

$$D_z = \left\{ (x,y) \;\middle|\; 0 \leqslant x,y \leqslant 1,\ y \geqslant 1-z,\ x+y \leqslant 2-z \right\}$$

Here is a sketch of a top view of D_z.



On D_z, y runs from $1-z$ to 1.

- For each fixed $0 \leqslant z \leqslant 1$ and $1-z \leqslant y \leqslant 1$, x runs from 0 to $2-y-z$.

So the specified integral is

$$\begin{aligned}\iiint_R y\ \mathrm{d}V &= \int_0^1 \mathrm{d}z \iint_{D_z} \mathrm{d}x\mathrm{d}y\ y = \int_0^1 \mathrm{d}z \int_{1-z}^1 \mathrm{d}y \int_0^{2-y-z} \mathrm{d}x\ y = \int_0^1 \mathrm{d}z \int_{1-z}^1 \mathrm{d}y\ y(2-y-z) \\
&= -\int_0^1 \mathrm{d}z \int_z^0 du\ (1-u)(1+u-z) \qquad \text{where } u = 1-y \\
&= \int_0^1 \mathrm{d}z \int_0^z du\ (1-u^2-z+uz) = \int_0^1 \mathrm{d}z\ \left(z - \frac{z^3}{3} - z^2 + \frac{z^3}{2}\right) \\
&= \frac{1}{2} - \frac{1}{12} - \frac{1}{3} + \frac{1}{8} = \frac{5}{24}\end{aligned}$$

3.5.5 For each of the following, express the given iterated integral as an iterated integral in which the integrations are performed in the order: first z, then y, then x.

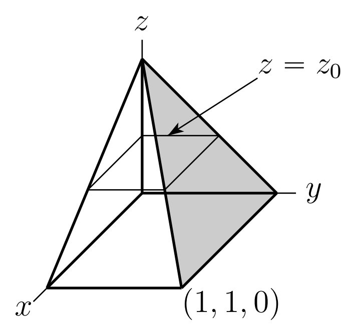
(a) $\int_0^1 \mathrm{d}z \int_0^{1-z} \mathrm{d}y \int_0^{1-z} \mathrm{d}x\ f(x,y,z)$

(b) $\int_0^1 \mathrm{d}z \int_{\sqrt{z}}^1 \mathrm{d}y \int_0^y \mathrm{d}x\ f(x,y,z)$

Solution (a) The domain of integration is

$$\begin{aligned} V &= \left\{ (x,y,z) \mid 0 \leqslant z \leqslant 1,\ 0 \leqslant y \leqslant 1-z,\ 0 \leqslant x \leqslant 1-z \right\} \\ &= \left\{ (x,y,z) \mid x,y,z \geqslant 0,\ x+z \leqslant 1,\ y+z \leqslant 1 \right\} \end{aligned}$$

This is sketched in the figure below. The front face is $x+z=1$ and the lightly shaded right face is $y+z=1$.

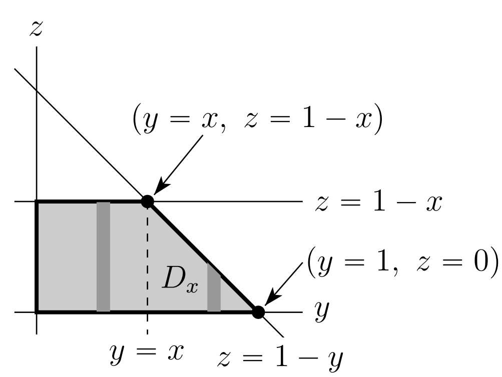


In V,

- x takes all values between 0 and 1.
- For each fixed $0 \leqslant x \leqslant 1$, (y,z) takes all values in

$$D_x = \left\{ (y,z) \mid y,z \geqslant 0,\ z \leqslant 1-x,\ y+z \leqslant 1 \right\}$$

Here is a sketch of D_x.



- Looking at the sketch above, we see that, on D_x, y runs from 0 to 1 and
  - for each fixed y between 0 and x, z runs from 0 to $1-x$ and
  - for each fixed y between x and 1, z runs from 0 to $1-y$

So the integral is, in the new order,

$$\begin{aligned}\iiint_V f(x,y,z)\, \mathrm{d}V &= \int_0^1 \mathrm{d}x \iint_{D_x} \mathrm{d}y\, \mathrm{d}z\ f(x,y,z) \\ &= \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \int_0^{1-x} \mathrm{d}z\ f(x,y,z) + \int_0^1 \mathrm{d}x \int_x^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z\ f(x,y,z)\end{aligned}$$

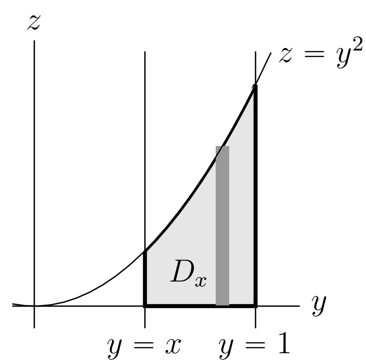
(b) The domain of integration is

$$\begin{aligned}V &= \{\ (x,y,z)\ \big|\ 0 \leqslant z \leqslant 1,\ \sqrt{z} \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant y\ \} \\ &= \{\ (x,y,z)\ \big|\ 0 \leqslant z \leqslant y^2,\ 0 \leqslant x \leqslant y \leqslant 1\ \}\end{aligned}$$

In this region, x takes all values between 0 and 1. For each fixed x between 0 and 1, (y,z) takes all values in

$$D_x = \{\ (y,z)\ \big|\ 0 \leqslant z \leqslant y^2,\ x \leqslant y \leqslant 1\ \}$$

Here is a sketch of D_x.



In the new order, the integral is

$$\int_0^1 \mathrm{d}x \iint_{D_x} \mathrm{d}y\, \mathrm{d}z\ f(x,y,z) = \int_0^1 \mathrm{d}x \int_x^1 \mathrm{d}y \int_0^{y^2} \mathrm{d}z\ f(x,y,z)$$

$\boxed{3.5.6}$ (*) A triple integral $\iiint_E f\ \mathrm{d}V$ is given in iterated form by

$$\int_{y=-1}^{y=1} \int_{z=0}^{z=1-y^2} \int_{x=0}^{2-y-z} f(x,y,z)\ \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}y$$

(a) Draw a reasonably accurate picture of E in 3–dimensions. Be sure to show the units on the coordinate axes.

(b) Rewrite the triple integral $\iiint_E f\ \mathrm{d}V$ as one or more iterated triple integrals in the order

$$\int_{y=}^{y=} \int_{x=}^{x=} \int_{z=}^{z=} f(x,y,z)\ \mathrm{d}z\, \mathrm{d}x\, \mathrm{d}y$$

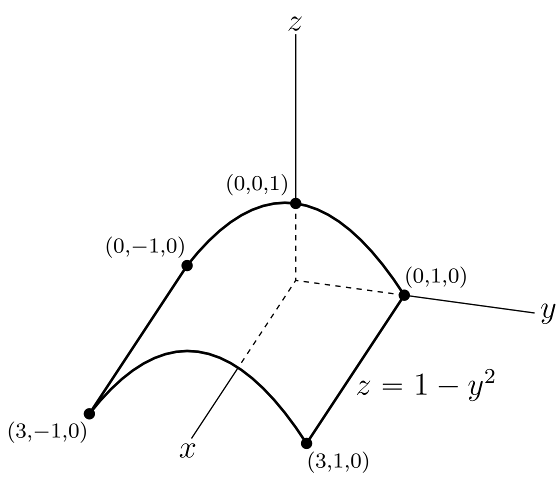
Solution (a) In the domain of integration for the given integral

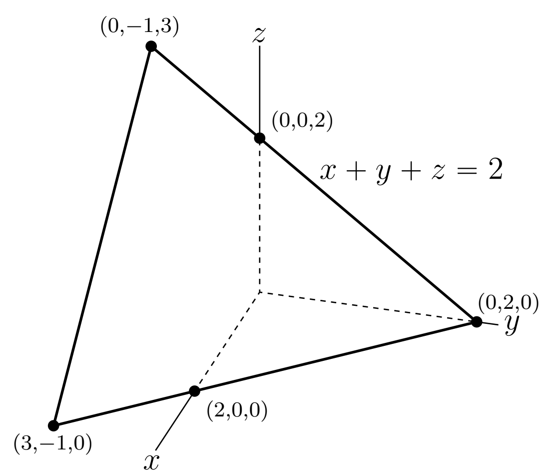
- y runs from -1 to 1, and
- for each fixed y in that range z runs from 0 to $1-y^2$, and
- for each fixed y and z as above, x runs from 0 to $2-y-z$.

That is,

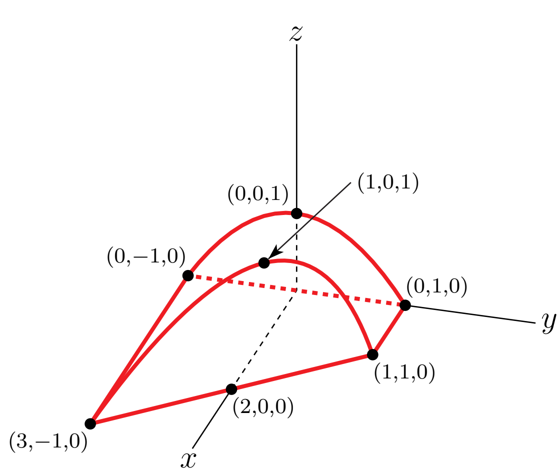
$$E = \left\{ \ (x,y,z) \ \middle| \ -1 \leqslant y \leqslant 1,\ 0 \leqslant z \leqslant 1-y^2,\ 0 \leqslant x \leqslant 2-y-z \ \right\}$$

- Each constant x cross–section of the surface $z = 1-y^2$ is an upside down parabola. So the surface $z = 1-y^2$ consists of a bunch of copies of the parabola $z = 1-y^2$ stacked front to back. The figure of the left below provides a sketch of $z = 1-y^2$.
- The surface $x = 2-y-z$, or equivalenty, $x+y+z = 2$ is a plane. It passes through the points $(2,0,0)$, $(0,2,0)$ and $(0,0,2)$. It is sketched in the figure on the right below. We know that our domain of integration extends to $y = -1$, so we have chosen to include in the sketch the part of the plane in $x \geqslant 0$, $y \geqslant -1$, $z \geqslant 0$.





The domain E is constructed by using the plane $x+y+z = 2$ to chop the front off of the "tunnel" $0 \leqslant z \leqslant 1-y^2$. It is outlined in red in the figure below.



(b) We are to change the order of integration so that the outside integral is over y (the same as the given integral), the middle integral is over x, and the inside integral is over over z.

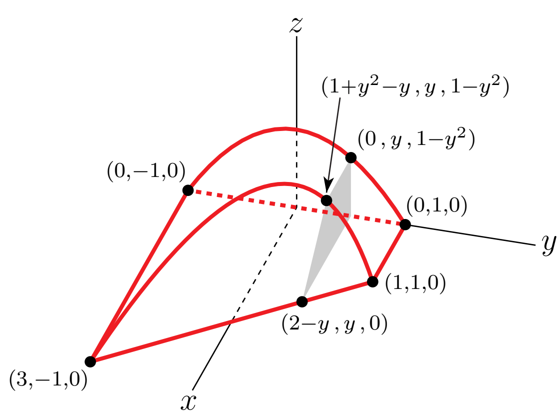
- We still have y running from -1 to 1.
- For each fixed y in that range, (x,z) runs over

$$E_y = \left\{ \ (x,z) \ \middle| \ 0 \leqslant z \leqslant 1-y^2, \ 0 \leqslant x+z \leqslant 2-y \ \right\}$$

- The biggest value of x in E_y is $2-y$. It is achieved when $z=0$. You can also see this in the figure below. The shaded region in that figure is E_y.
- For each fixed x and y as above, z runs over

$$E_{x,y} = \left\{ \ z \ \middle| \ 0 \leqslant z \leqslant 1-y^2, \ 0 \leqslant z \leqslant 2-x-y \ \right\}$$

  That is, z runs from 0 to the smaller of $1-y^2$ and $2-x-y$. Note that $1-y^2 \leqslant 2-x-y$ if and only if $x \leqslant 1+y^2-y$.
- So if $0 \leqslant x \leqslant 1+y^2-y$, z runs from 0 to $1-y^2$ and if $1+y^2-y \leqslant x \leqslant 2-y$, z runs from 0 to $2-x-y$.



So the integral is

$$\int_{y=-1}^{y=1} \int_{x=0}^{x=1+y^2-y} \int_{z=0}^{z=1-y^2} f(x,y,z) \ \mathrm{d}z \, \mathrm{d}x \, \mathrm{d}y$$
$$+ \int_{y=-1}^{y=1} \int_{x=1+y^2-y}^{x=2-y} \int_{z=0}^{z=2-x-y} f(x,y,z) \ \mathrm{d}z \, \mathrm{d}x \, \mathrm{d}y$$

$\boxed{3.5.7}$ (*) A triple integral $\iiint_E f(x,y,z)\,\mathrm{d}V$ is given in the iterated form

$$J = \int_0^1 \int_0^{1-\frac{x}{2}} \int_0^{4-2x-4z} f(x,y,z)\ \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$$

(a) Sketch the domain E in 3–dimensions. Be sure to show the units.
(b) Rewrite the integral as one or more iterated integrals in the form

$$J = \int_{y=}^{y=} \int_{x=}^{x=} \int_{z=}^{z=} f(x,y,z)\ \mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

Solution (a) In the given integral J,

- x runs from 0 to 1,
- for each fixed x in that range, z runs from 0 to $1 - \frac{x}{2}$, and
- for each fixed x and z as above, y runs from 0 to $4 - 2x - 4z$.

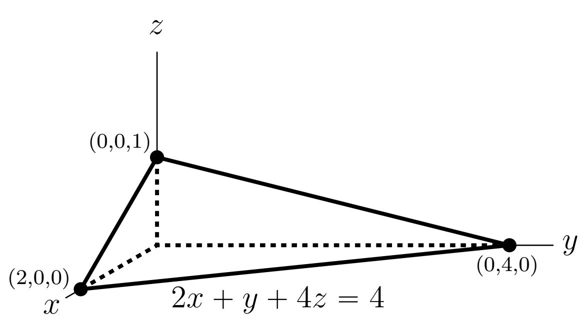
So

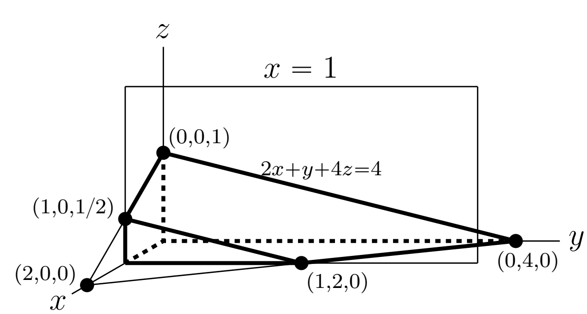
$$E = \left\{ (x,y,z) \;\middle|\; 0 \leqslant x \leqslant 1,\ 0 \leqslant z \leqslant 1 - \tfrac{x}{2},\ 0 \leqslant y \leqslant 4 - 2x - 4z \right\}$$

Notice that the condition $y \leqslant 4 - 2x - 4z$ can be rewritten as $z \leqslant 1 - \frac{x}{2} - \frac{y}{4}$. When $y \geqslant 0$, this implies that $z \leqslant 1 - \frac{x}{2}$, so that we can drop the condition $z \leqslant 1 - \frac{x}{2}$ from our description of E:

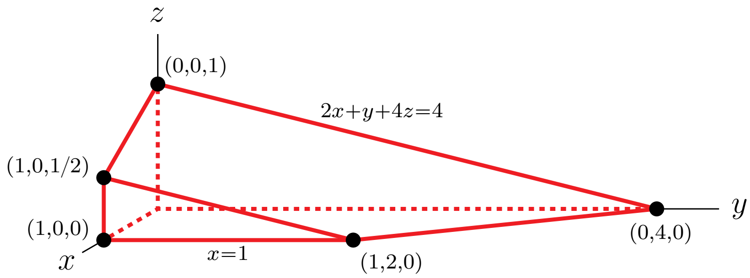
$$E = \left\{ (x,y,z) \;\middle|\; 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 4 - 2x - 4z,\ z \geqslant 0 \right\}$$

First, we figure out what E looks like. The plane $2x + y + 4z = 4$ intersects the x–, y– and z–axes at $(2,0,0)$, $(0,4,0)$ and $(0,0,1)$, respectively. That plane is shown in the sketch on the left below. The set of points $\left\{ (x,y,z) \;\middle|\; x,y,z \geqslant 0,\ y \leqslant 4 - 2x - 4z \right\}$ is outlined with heavy lines.

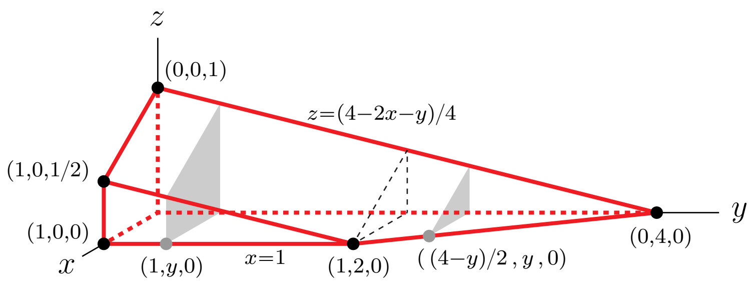




So it only remains to impose the condtion $x \leqslant 1$, which chops off the front bit of the tetrahedron. This is done in the sketch on the right above. Here is a cleaned up sketch of E.



(b) We are to reorder the integration so that the outside integral is over y, the middle integral is over x, and the inside integral is over z. Looking at the figure below,



we see that

- y runs from 0 to 4, and
- for each fixed y in that range, (x,z) runs over

$$\big\{\ (x,z)\ \big|\ 0\leqslant x\leqslant 1,\ 2x+4z\leqslant 4-y,\ z\geqslant 0\ \big\}$$

- for each fixed y between 0 and 2 (as in the left hand shaded bit in the figure above)
  - x runs from 0 to 1, and then
  - for each fixed x in that range, z runs from 0 to $\frac{4-2x-y}{4}$.
- for each fixed y between 2 and 4 (as in the right hand shaded bit in the figure above)
  - x runs from 0 to $\frac{4-y}{2}$ (the line of intersection of the plane $2x+y+4z=4$ and the xy–plane is $z=0$, $2x+y=4$), and then
  - for each fixed x in that range, z runs from 0 to $\frac{4-2x-y}{4}$.

So the integral

$$J=\int_{y=0}^{y=2}\int_{x=0}^{x=1}\int_{z=0}^{z=\frac{4-2x-y}{4}}f(x,y,z)\ \mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y+\int_{y=2}^{y=4}\int_{x=0}^{x=\frac{4-y}{2}}\int_{z=0}^{z=\frac{4-2x-y}{4}}f(x,y,z)\ \mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

$\boxed{3.5.8}$ (∗) Write the integral given below 5 other ways, each with a different order of integration.

$$I=\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y}f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

$\boxed{\text{Solution}}$ Let's use V to denote the domain of integration for the given integral. On V

- x runs from 0 to 1, and
- for each fixed x in that range, y runs from $\sqrt{x}$ to 1. In particular $0 \leqslant y \leqslant 1$. We can rewrite $y = \sqrt{x}$ as $x = y^2$ (with $y \geqslant 0$).
- For each fixed x and y as above, z runs from 0 to $1 - y$.

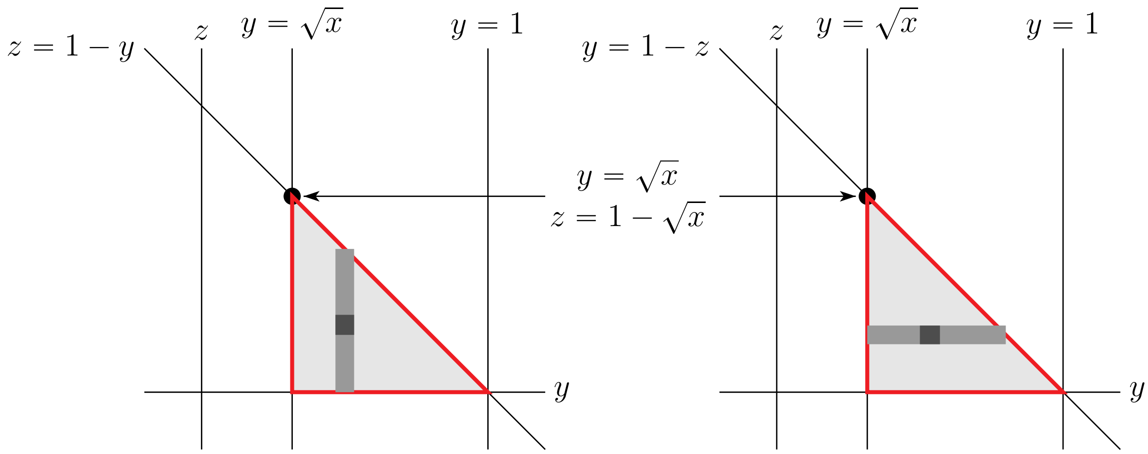
So

$$\begin{aligned} V &= \left\{ \ (x,y,z) \ \middle| \ 0 \leqslant x \leqslant 1,\ \sqrt{x} \leqslant y \leqslant 1,\ 0 \leqslant z \leqslant 1-y \ \right\} \\ &= \left\{ \ (x,y,z) \ \middle| \ x,z \geqslant 0,\ x \leqslant 1,\ y \geqslant \sqrt{x},\ y \leqslant 1,\ z \leqslant 1-y \ \right\} \end{aligned}$$

Outside integral is with respect to x: We have already seen that $0 \leqslant x \leqslant 1$ and that, for each fixed x in that range, (y,z) runs over

$$V_x = \left\{ \ (y,z) \ \middle| \ \sqrt{x} \leqslant y \leqslant 1,\ 0 \leqslant z \leqslant 1-y \ \right\}$$

Here are two sketches of V_x. The sketch on the left shows a vertical strip as was used in setting up the integral given in the statement of this problem. To reverse the order of the



y– and z–integrals we use horizontal strips as in the figure on the right above. Looking at that figure, we see that, on V_x,

- z runs from 0 to $1 - \sqrt{x}$, and
- for each fixed z in that range, y runs from $\sqrt{x}$ to $1 - z$.

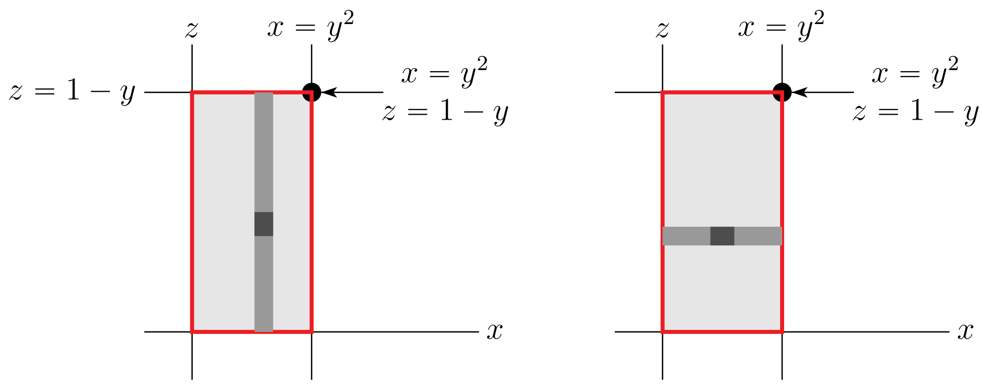
So

$$I = \int_0^1 \mathrm{d}x \int_0^{1-\sqrt{x}} \mathrm{d}z \int_{\sqrt{x}}^{1-z} \mathrm{d}y\ f(x,y,z) = \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\ \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$$

Outside integral is with respect to y: Looking at the figures above we see that, for each $0 \leqslant x \leqslant 1$, y runs from $\sqrt{x}$ to 1 on V_x. As x runs from 0 to 1 in V, we have that $\sqrt{x}$ also runs from 0 to 1 on V, so that y runs from 0 to 1 on V. Reviewing the definition of V, we see that, for each fixed $0 \leqslant y \leqslant 1$, (x,z) runs over

$$V_y = \left\{ \ (x,z) \ \middle| \ 0 \leqslant x \leqslant y^2,\ 0 \leqslant z \leqslant 1-y \ \right\}$$

Here are two sketches of V_y. Looking at the figure on the left (with the vertical strip), we



see that, on V_y,

- x runs from 0 to y^2, and
- for each fixed x in that range, z runs from 0 to $1-y$.

So

$$I = \int_0^1 \mathrm{d}y \int_0^{y^2} \mathrm{d}x \int_0^{1-y} \mathrm{d}z \; f(x,y,z) = \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x,y,z) \; \mathrm{d}z \, \mathrm{d}x \, \mathrm{d}y$$

Looking at the figure on the right above (with the horizontal strip), we see that, on V_y,

- z runs from 0 to $1-y$.
- for each fixed z in that range, x runs from 0 to y^2.

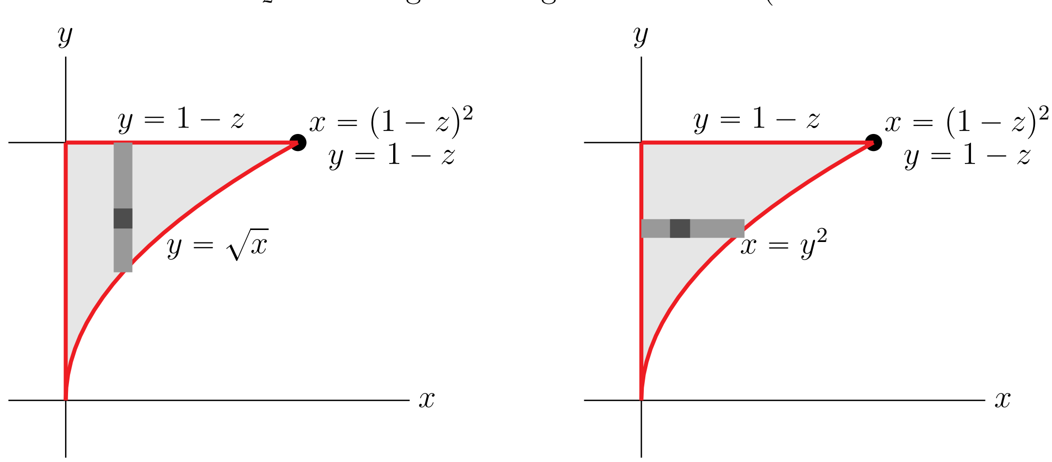
So

$$I = \int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \int_0^{y^2} \mathrm{d}x \; f(x,y,z) = \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z) \; \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}y$$

Outside integral is with respect to z: Looking at the sketches of V_x above we see that, for each $0 \leqslant x \leqslant 1$, z runs from 0 to $1-\sqrt{x}$ on V_x. As x runs from 0 to 1 in V, $1-\sqrt{x}$ also runs between 0 to 1 on V, so that z runs from 0 to 1 on V. Reviewing the definition of V, we see that, for each fixed $0 \leqslant z \leqslant 1$, (x, y) runs over

$$V_z = \left\{ \ (x,y) \ \middle| \ 0 \leqslant x \leqslant y^2, \ \sqrt{x} \leqslant y \leqslant 1-z \ \right\}$$

Here are two sketches of V_z. Looking at the figure on the left (with the vertical strip), we



see that, on V_z,

- x runs from 0 to $(1-z)^2$, and
- for each fixed x in that range, y runs from $\sqrt{x}$ to $1-z$.

So

$$I = \int_0^1 \mathrm{d}z \int_0^{(1-z)^2} \mathrm{d}x \int_{\sqrt{x}}^{1-z} \mathrm{d}y\ f(x,y,z) = \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\ \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$$

Looking at the figure on the right above (with the horizontal strip), we see that, on V_z,

- y runs from 0 to $1-z$.
- for each fixed y in that range, x runs from 0 to y^2.

So

$$I = \int_0^1 \mathrm{d}z \int_0^{1-z} \mathrm{d}y \int_0^{y^2} \mathrm{d}x\ f(x,y,z) = \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Summary: We have found that

$$\begin{aligned}
I &= \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x &&= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\ \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x \\
&= \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x,y,z)\ \mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y &&= \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)\ \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y \\
&= \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\ \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z &&= \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z
\end{aligned}$$

3.5.9 (*) Let $I = \iiint\limits_E f(x,y,z)\ \mathrm{d}V$ where E is the tetrahedron with vertices $(-1,0,0)$, $(0,0,0)$, $(0,0,3)$ and $(0,-2,0)$.

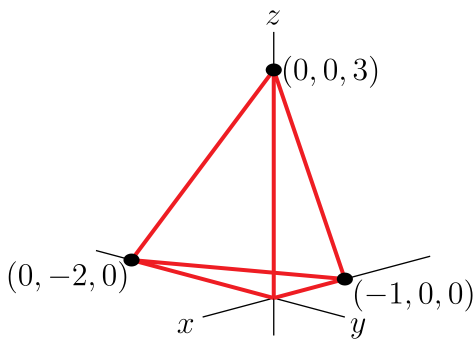
(a) Rewrite the integral I in the form

$$I = \int_{x=}^{x=} \int_{y=}^{y=} \int_{z=}^{z=} f(x,y,z)\ \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

(b) Rewrite the integral I in the form

$$I = \int_{z=}^{z=} \int_{x=}^{x=} \int_{y=}^{y=} f(x,y,z)\ \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$$

Solution First we have to get some idea as to what E looks like. Here is a sketch.



We are going to need the equation of the plane that contains the points $(-1,0,0)$, $(0,-2,0)$ and $(0,0,3)$. This plane does not contain the origin and so has an equation of the form $ax+by+cz=1$.

- $(-1,0,0)$ lies on the plane $ax+by+cz=1$ if and only if $a(-1)+b(0)+c(0)=1$. So $a=-1$.
- $(0,-2,0)$ lies on the plane $ax+by+cz=1$ if and only if $a(0)+b(-2)+c(0)=1$. So $b=-\frac{1}{2}$.
- $(0,0,3)$ lies on the plane $ax+by+cz=1$ if and only if $a(0)+b(0)+c(3)=1$. So $c=\frac{1}{3}$.

So the plane that contains the points $(-1,0,0)$, $(0,-2,0)$ and $(0,0,3)$ is $-x-\frac{y}{2}+\frac{z}{3}=1$.

We can now get a detailed mathematical description of E. A point (x,y,z) is in E if and only if

- (x,y,z) lies above the xy–plane, i.e. $z\geqslant 0$, and
- (x,y,z) lies to the left of the xz–plane, i.e. $y\leqslant 0$, and
- (x,y,z) lies behind the yz–plane, i.e. $x\leqslant 0$, and
- (x,y,z) lies on the same side of the plane $-x-\frac{y}{2}+\frac{z}{3}=1$ as the origin. That is $-x-\frac{y}{2}+\frac{z}{3}\leqslant 1$. (Go ahead and check that $(0,0,0)$ obeys this inequality.)

So

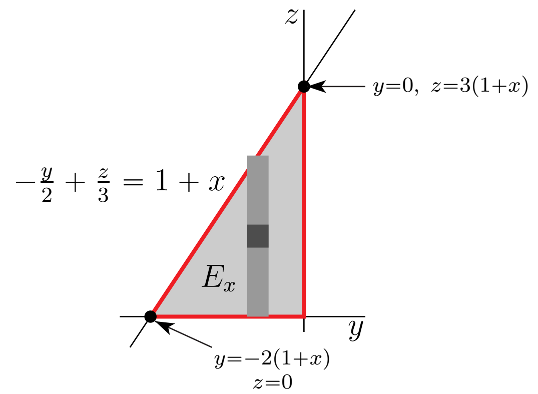
$$E=\left\{\ (x,y,z)\ \middle|\ x\leqslant 0,\ y\leqslant 0,\ z\geqslant 0,\ -x-\frac{y}{2}+\frac{z}{3}\leqslant 1\ \right\}$$

(a) Note that we want the outside integral to be the x–integral. On E

- x runs from -1 to 0 and
- for each fixed x in that range (y,z) runs over

$$E_x=\left\{\ (y,z)\ \middle|\ y\leqslant 0,\ z\geqslant 0,\ -\frac{y}{2}+\frac{z}{3}\leqslant 1+x\ \right\}$$

Here is a sketch of E_x.



- On E_x, y runs from $-2(1+x)$ to 0 and
- for each fixed such y, z runs from 0 to $3(1+x+y/2)$

So

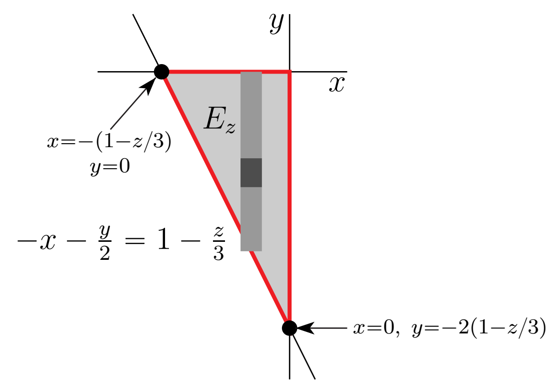
$$I=\int_{x=-1}^{x=0}\int_{y=-2(1+x)}^{y=0}\int_{z=0}^{z=3(1+x+y/2)} f(x,y,z)\ \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

(b) This time we want the outside integral to be the z–integral. Looking back at the sketch of E, we see that, on E,

- z runs from 0 to 3 and
- for each fixed z in that range (x,y) runs over

$$E_z=\left\{\ (x,y)\ \middle|\ x\leqslant 0,\ y\leqslant 0,\ -x-\tfrac{y}{2}\leqslant 1-\tfrac{z}{3}\ \right\}$$

Here is a sketch of E_z.



- On E_z, x runs from $-(1-z/3)$ to 0 and
- for each fixed such x, y runs from $-2(1+x-z/3)$ to 0

So

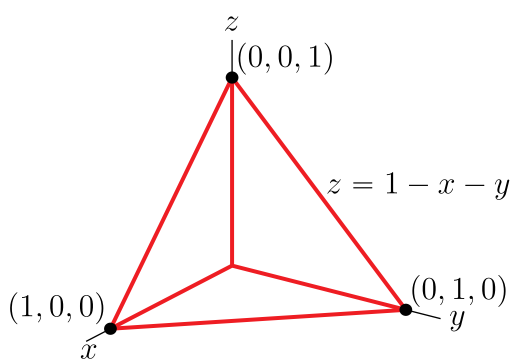
$$I=\int_{z=0}^{z=3}\int_{x=-(1-z/3)}^{x=0}\int_{y=-2(1+x-z/3)}^{y=0} f(x,y,z)\ \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$$

$\boxed{3.5.10}$ (∗) Let T denote the tetrahedron bounded by the coordinate planes $x = 0$, $y = 0$, $z = 0$ and the plane $x + y + z = 1$. Compute

$$K = \iiint_T \frac{1}{(1 + x + y + z)^4}\, \mathrm{d}V$$

$\boxed{\text{Solution}}$ The plane $x + y + z = 1$ intersects the coordinate plane $z = 0$ along the line $x + y = 1$, $z = 0$. So

$$\begin{aligned} T &= \left\{ (x, y, z) \mid x \geqslant 0,\ y \geqslant 0,\ x + y \leqslant 1,\ 0 \leqslant z \leqslant 1 - x - y \right\} \\ &= \left\{ (x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x,\ 0 \leqslant z \leqslant 1 - x - y \right\} \end{aligned}$$

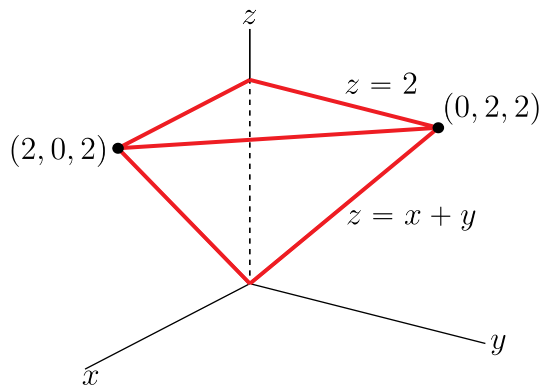


and

$$\begin{aligned} K &= \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int_0^{1-x-y} \mathrm{d}z\ \frac{1}{(1 + x + y + z)^4} \\ &= \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\ \left[-\frac{1}{3(1 + x + y + z)^3} \right]_{z=0}^{z=1-x-y} \\ &= \frac{1}{3} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\ \left[\frac{1}{(1 + x + y)^3} - \frac{1}{2^3} \right] \\ &= \frac{1}{3} \int_0^1 \mathrm{d}x\ \left[-\frac{1}{2(1 + x + y)^2} - \frac{y}{2(4)} \right]_{y=0}^{y=1-x} \\ &= \frac{1}{6} \int_0^1 \mathrm{d}x\ \left[\frac{1}{(1 + x)^2} - \frac{1}{2^2} - \frac{1 - x}{4} \right] = \frac{1}{6} \int_0^1 \mathrm{d}x\ \left[\frac{1}{(1 + x)^2} - \frac{1}{2} + \frac{x}{4} \right] \\ &= \frac{1}{6} \left[-\frac{1}{1 + x} - \frac{x}{2} + \frac{x^2}{8} \right]_{x=0}^{x=1} = \frac{1}{6} \left[1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{8} \right] \\ &= \frac{1}{48} \end{aligned}$$

$\boxed{3.5.11}$ (∗) Let E be the portion of the first octant which is above the plane $z = x + y$ and below the plane $z = 2$. The density in E is $\rho(x, y, z) = z$. Find the mass of E.

$\boxed{\text{Solution}}$ Note that the planes $z = x + y$ and $z = 2$ intersect along the line $x + y = 2$, $z = 2$.



So

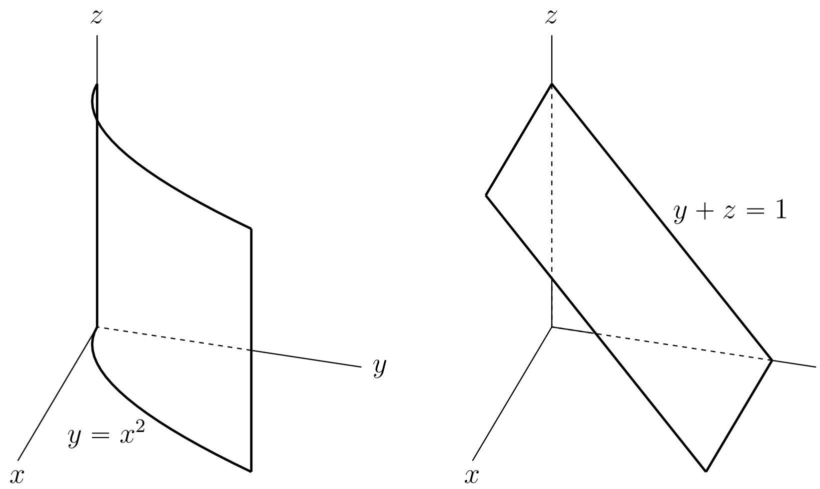
$$\begin{aligned} E &= \{ (x, y, z) \mid x \geqslant 0,\ y \geqslant 0,\ x + y \leqslant 2,\ x + y \leqslant z \leqslant 2 \} \\ &= \{ (x, y, z) \mid 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 2 - x,\ x + y \leqslant z \leqslant 2 \} \end{aligned}$$

and the mass of E is

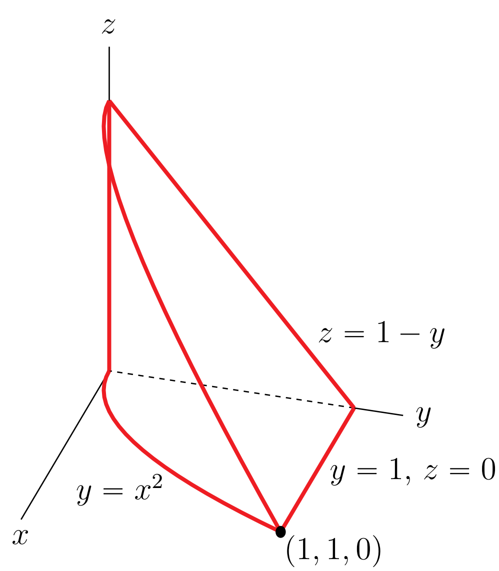
$$\begin{aligned} \iiint_E \rho(x, y, z)\, dV &= \int_0^2 dx \int_0^{2-x} dy \int_{x+y}^2 dz\ z \\ &= \frac{1}{2} \int_0^2 dx \int_0^{2-x} dy\ \left[4 - (x + y)^2\right] \\ &= \frac{1}{2} \int_0^2 dx\ \left[4(2 - x) - \frac{\big(x + (2 - x)\big)^3 - x^3}{3}\right] \\ &= \frac{1}{2} \left[4(2)(2) - 2(2)^2 - \frac{8}{3}(2) + \frac{2^4}{12}\right] = \frac{1}{2} \left[8 - \frac{16}{3} + \frac{4}{3}\right] \\ &= 2 \end{aligned}$$

$\boxed{3.5.12}$ (∗) Evaluate the triple integral $\iiint_E x\ dV$, where E is the region in the first octant bounded by the parabolic cylinder $y = x^2$ and the planes $y + z = 1$, $x = 0$, and $z = 0$.

$\boxed{\text{Solution}}$ First, we need to develop an understanding of what E looks like. Here are sketches of the parabolic cylinder $y = x^2$, on the left, and the plane $y + z = 1$, on the right.



E is constructed by using the plane $y + z = 1$ to chop the top off of the parabolic cylinder $y = x^2$. Here is a sketch.



So

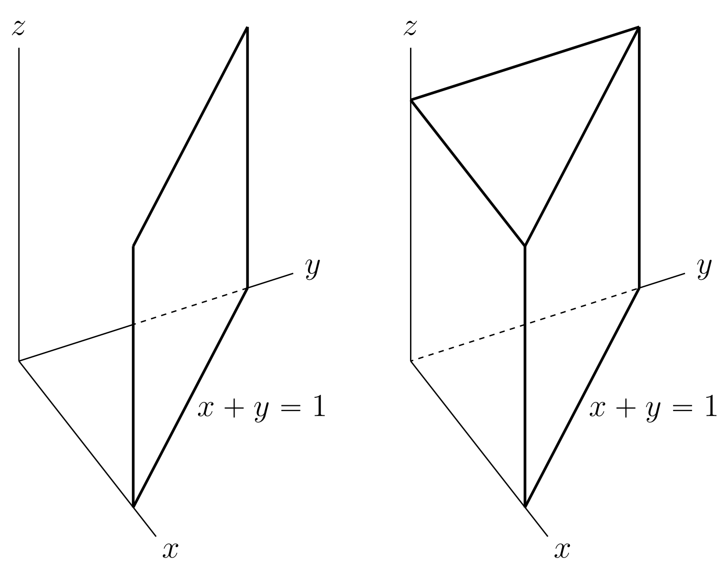
$$E = \left\{ \ (x, y, z) \ \middle| \ 0 \leqslant x \leqslant 1, \ x^2 \leqslant y \leqslant 1, \ 0 \leqslant z \leqslant 1 - y \ \right\}$$

and the integral

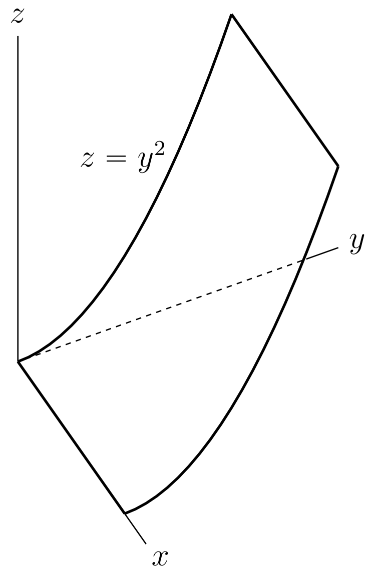
$$\begin{aligned}\iiint_E x \, \mathrm{d}V &= \int_0^1 \mathrm{d}x \int_{x^2}^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}z \; x \\ &= \int_0^1 \mathrm{d}x \int_{x^2}^1 \mathrm{d}y \; x(1-y) \\ &= \int_0^1 \mathrm{d}x \; x \left[y - \frac{y^2}{2} \right]_{x^2}^1 \\ &= \int_0^1 \mathrm{d}x \; \left[\frac{x}{2} - x^3 + \frac{x^5}{2} \right] \\ &= \frac{1}{4} - \frac{1}{4} + \frac{1}{12} \\ &= \frac{1}{12}\end{aligned}$$

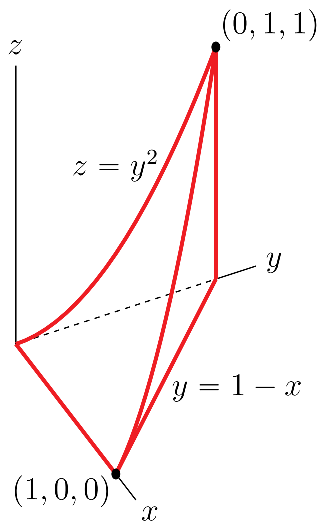
$\boxed{3.5.13}$ (∗) Let E be the region in the first octant bounded by the coordinate planes, the plane $x + y = 1$ and the surface $z = y^2$. Evaluate $\iiint_E z \, \mathrm{d}V$.

$\boxed{\text{Solution}}$ First, we need to develop an understanding of what E looks like. Here are sketches of the plane $x + y = 1$, on the left, and of the “tower” bounded by the coordinate planes $x = 0$, $y = 0$, $z = 0$ and the plane $x + y = 1$, on the right.



Now here is the parabolic cylinder $z = y^2$ on the left. E is constructed by using the parabolic cylinder $z = y^2$ to chop the top off of the tower $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$, $x + y \leqslant 1$. The figure on the right is a sketch.





So

$$E = \left\{ (x, y, z) \;\middle|\; 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x,\ 0 \leqslant z \leqslant y^2 \right\}$$

and the integral

$$\begin{aligned}
\iiint_E x \, \mathrm{d}V &= \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int_0^{y^2} \mathrm{d}z \; z \\
&= \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \; \frac{y^4}{2} \\
&= \int_0^1 \mathrm{d}x \; \frac{(1-x)^5}{10} \\
&= \left[-\frac{(1-x)^6}{60} \right]_0^1 \\
&= \frac{1}{60}
\end{aligned}$$

$\boxed{3.5.14}$ $(*)$ Evaluate $\iiint_R yz^2 e^{-xyz} \, \mathrm{d}V$ over the rectangular box

$$R = \left\{ (x, y, z) \;\middle|\; 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3 \right\}$$

Solution The integral

$$\begin{aligned}\iiint_R yz^2e^{-xyz}\,\mathrm{d}V &= \int_0^3 \mathrm{d}z\int_0^2 \mathrm{d}y\int_0^1 \mathrm{d}x\ yz^2e^{-xyz}\\ &= \int_0^3 \mathrm{d}z\int_0^2 \mathrm{d}y\ \Big[-ze^{-xyz}\Big]_{x=0}^{x=1} = \int_0^3 \mathrm{d}z\int_0^2 \mathrm{d}y\ \Big[z-ze^{-yz}\Big]\\ &= \int_0^3 \mathrm{d}z\ \Big[zy+e^{-yz}\Big]_{y=0}^{y=2} = \int_0^3 \mathrm{d}z\ \Big[2z+e^{-2z}-1\Big]\\ &= \Big[z^2-\frac{1}{2}e^{-2z}-z\Big]_0^3 = \frac{13}{2}-\frac{e^{-6}}{2}\end{aligned}$$

3.5.15 (*)

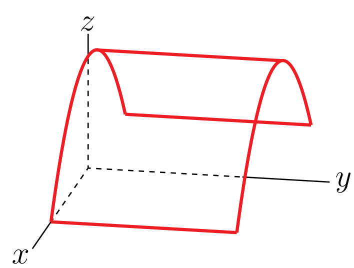
(a) Sketch the surface given by the equation $z = 1 - x^2$.

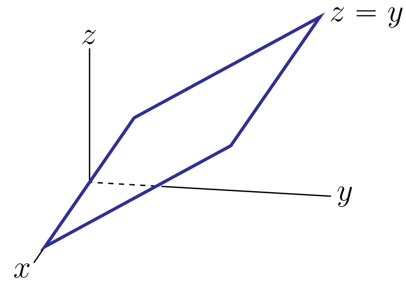
(b) Let E be the solid bounded by the plane $y = 0$, the cylinder $z = 1 - x^2$, and the plane $y = z$. Set up the integral

$$\iiint_E f(x,y,z)\,\mathrm{d}V$$

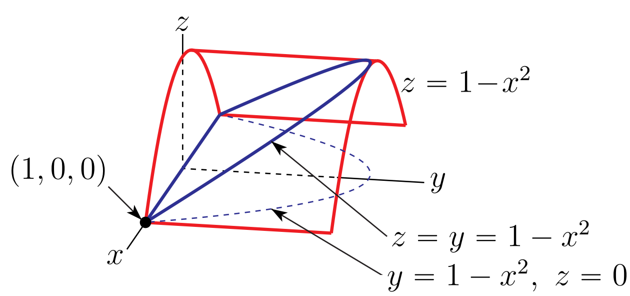
as an iterated integral.

Solution (a) Each constant y cross section of $z = 1 - x^2$ is an upside down parabola. So the surface is a bunch of upside down parabolas stacked side by side. The figure on the left below is a sketch of the part of the surface with $y \geqslant 0$ and $z \geqslant 0$ (both of which conditions will be required in part (b)).





(b) The figure on the right above is a sketch of the plane $y = z$. It intersects the surface $z = 1 - x^2$ in the solid blue sloped parabolic curve in the figure below.



Observe that, on the curve $z = 1 - x^2$, $z = y$, we have $y = 1 - x^2$. So that when one looks at the solid E from high on the z–axis, one sees

$$\left\{ (x, y) \mid 0 \leqslant y \leqslant 1 - x^2 \right\}$$

The $y = 1 - x^2$ boundary of that region is the dashed blue line in the xy–plane in the figure above. So

$$E = \left\{ (x, y, z) \;\middle|\; -1 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x^2,\ y \leqslant z \leqslant 1 - x^2 \right\}$$

and the integral

$$\iiint_E f(x, y, z)\, \mathrm{d}V = \int_{-1}^{1} \mathrm{d}x \int_0^{1-x^2} \mathrm{d}y \int_y^{1-x^2} \mathrm{d}z\ f(x, y, z)$$

$\boxed{3.5.16}$ (*) Let

$$J = \int_0^1 \int_0^x \int_0^y f(x, y, z)\ \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x$$

Express J as an integral where the integrations are to be performed in the order x first, then y, then z.

$\boxed{\text{Solution}}$ In the integral J,

- x runs from 0 to 1. In inequalities, $0 \leqslant x \leqslant 1$.
- Then, for each fixed x in that range, y runs from 0 to x. In inequalities, $0 \leqslant y \leqslant x$.
- Then, for each fixed x and y in those ranges, z runs from 0 to y. In inequalities, $0 \leqslant z \leqslant y$.

These inequalties can be combined into

$$0 \leqslant z \leqslant y \leqslant x \leqslant 1 \tag{*}$$

We wish to reverse the order of integration so that the z–integral is on the outside, the y–integral is in the middle and the x–integral is on the inside.

- The smallest z compatible with (*) is $z = 0$ and the largest z compatible with (*) is $z = 1$ (when $x = y = z = 1$). So $0 \leqslant z \leqslant 1$.
- Then, for each fixed z in that range, (x, y) run over $z \leqslant y \leqslant x \leqslant 1$. In particular, the smallest allowed y is $y = z$ and the largest allowed y is $y = 1$ (when $x = y = 1$). So $z \leqslant y \leqslant 1$.
- Then, for each fixed y and z in those ranges, x runs over $y \leqslant x \leqslant 1$.

So

$$J = \int_0^1 \int_z^1 \int_y^1 f(x, y, z)\ \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

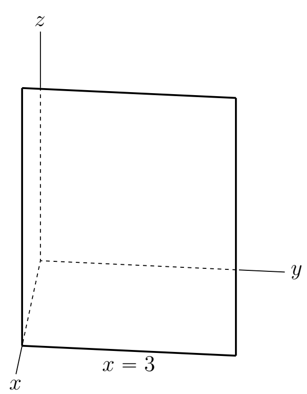
$\boxed{3.5.17}$ (∗) Let E be the region bounded by $z = 2x$, $z = y^2$, and $x = 3$. The triple integral $\iiint f(x, y, z)\, \mathrm{d}V$ can be expressed as an iterated integral in the following three orders of integration. Fill in the limits of integration in each case. No explanation required.

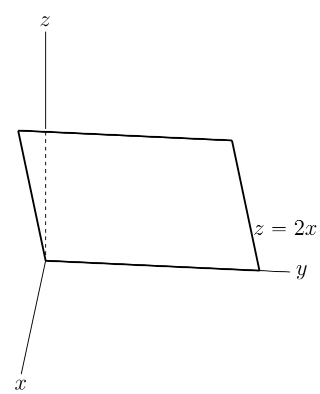
$$\int_{y=}^{y=} \int_{x=}^{x=} \int_{z=}^{z=} f(x, y, z)\, \mathrm{d}z\, \mathrm{d}x\, \mathrm{d}y$$

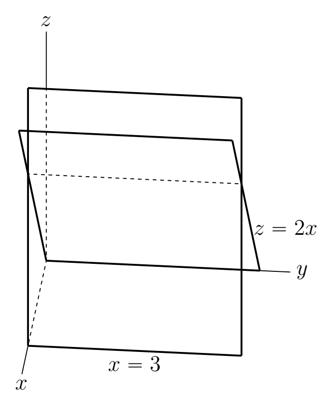
$$\int_{y=}^{y=} \int_{z=}^{z=} \int_{x=}^{x=} f(x, y, z)\, \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}y$$

$$\int_{z=}^{z=} \int_{x=}^{x=} \int_{y=}^{y=} f(x, y, z)\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}z$$

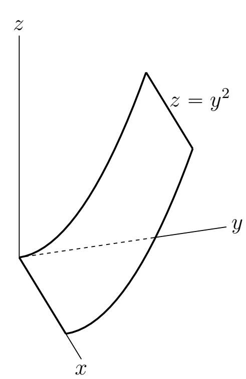
$\boxed{\text{Solution}}$ The hard part of this problem is figuring out what E looks like. First here are separate sketches of the plane $x = 3$ and the plane $z = 2x$ followed by a sketch of the two planes together.



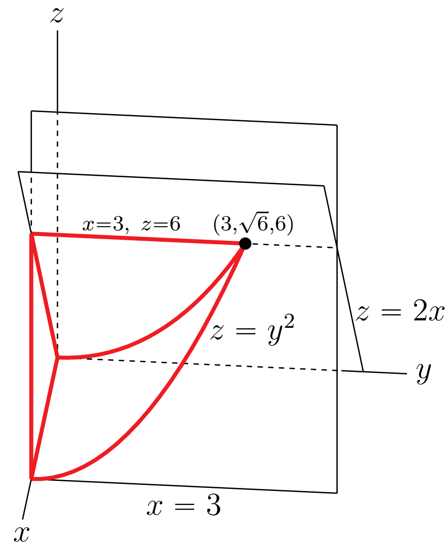




Next for the parabolic cylinder $z = y^2$. It is a bunch of parabolas $z = y^2$ stacked side by side along the x–axis. Here is a sketch of the part of $z = y^2$ in the first octant.



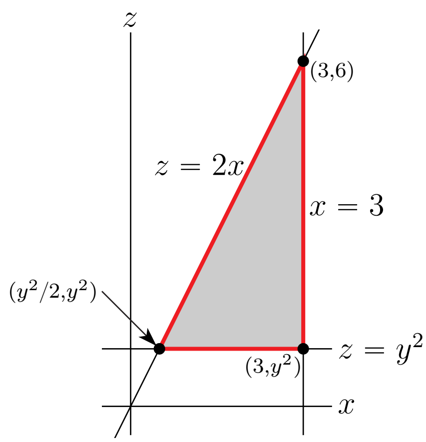
Finally, here is a sketch of the part of E in the first octant. E does have a second half gotten from the sketch by reflecting it in the xz–plane, i.e. by replacing $y \to -y$.



So[1]

$$E = \left\{ (x,y,z) \mid x \leqslant 3,\ -\sqrt{6} \leqslant y \leqslant \sqrt{6},\ y^2 \leqslant z \leqslant 2x \right\}$$

Order $\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$: On E, y runs from $-\sqrt{6}$ to $\sqrt{6}$. For each fixed y in this range (x,z) runs over $E_y = \left\{ (x,z) \mid x \leqslant 3,\ y^2 \leqslant z \leqslant 2x \right\}$. Here is a sketch of E_y.



From the sketch

$$E_y = \left\{ (x,z) \mid y^2/2 \leqslant x \leqslant 3,\ y^2 \leqslant z \leqslant 2x \right\}$$

and the integral is

$$\int_{y=-\sqrt{6}}^{y=\sqrt{6}} \int_{x=y^2/2}^{x=3} \int_{z=y^2}^{z=2x} f(x,y,z)\ \mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

Order $\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y$: Also from the sketch of E_y above

$$E_y = \left\{ (x,z) \mid y^2 \leqslant z \leqslant 6,\ z/2 \leqslant x \leqslant 3 \right\}$$

and the integral is

$$\int_{y=-\sqrt{6}}^{y=\sqrt{6}} \int_{z=y^2}^{z=6} \int_{x=z/2}^{x=3} f(x,y,z)\ \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y$$

1 The question doesn't specify on which side of the three surfaces E lies. When in doubt take the finite region bounded by the given surfaces. That's what we have done.

Order $dy\,dx\,dz$: From the sketch of the part of E in the first octant, we see that, on E, z runs from 0 to 6. For each fixed z in this range (x,y) runs over

$$\begin{aligned} E_z &= \{\ (x,y)\ \big|\ x \leqslant 3,\ -\sqrt{6} \leqslant y \leqslant \sqrt{6},\ y^2 \leqslant z \leqslant 2x\ \} \\ &= \{\ (x,y)\ \big|\ z/2 \leqslant x \leqslant 3,\ y^2 \leqslant z\ \} \\ &= \{\ (x,y)\ \big|\ z/2 \leqslant x \leqslant 3,\ -\sqrt{z} \leqslant y \leqslant \sqrt{z}\ \} \end{aligned}$$

So the integral is

$$\int_{z=0}^{z=6} \int_{x=z/2}^{x=3} \int_{y=-\sqrt{z}}^{y=\sqrt{z}} f(x,y,z)\ dy\,dx\,dz$$

$\boxed{3.5.18}$ (*) Let E be the region inside the cylinder $x^2+y^2=1$, below the plane $z=y$ and above the plane $z=-1$. Express the integral

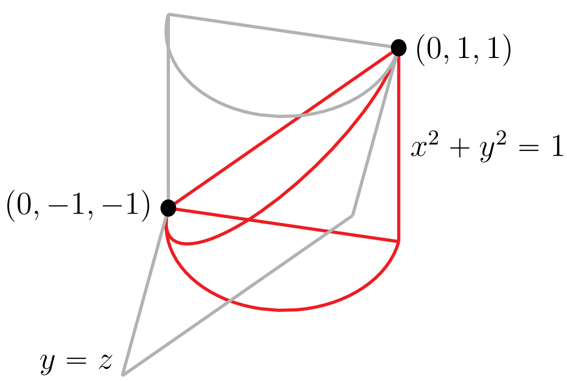
$$\iiint_E f(x,y,z)\ dV$$

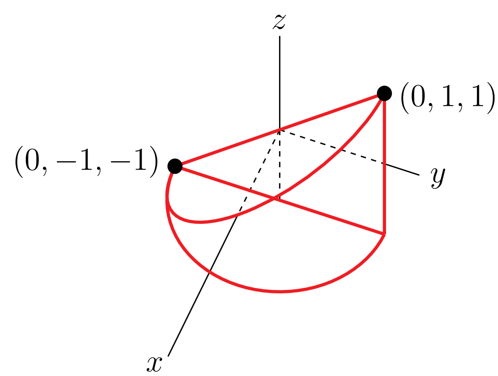
as three different iterated integrals corresponding to the orders of integration: (a) $dz\,dx\,dy$, (b) $dx\,dy\,dz$, and (c) $dy\,dz\,dx$.

Solution (a) The region E is

$$E = \{\ (x,y,z)\ \big|\ x^2+y^2 \leqslant 1,\ -1 \leqslant z \leqslant y\ \}$$

Here is are sketches, one without axes and one with axes, of the front half of E, outlined in red.

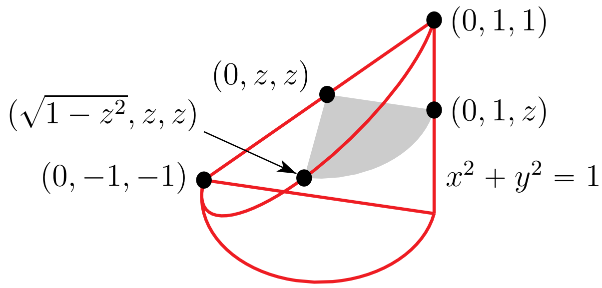




The integral

$$\begin{aligned} \iiint_E f(x,y,z)\ dV &= \int_{x^2+y^2\leqslant 1} dx\,dy \int_{-1}^{y} dz\ f(x,y,z) \\ &= \int_{-1}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx \int_{-1}^{y} dz\ f(x,y,z) \\ &= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{-1}^{y} f(x,y,z)\ dz\,dx\,dy \end{aligned}$$

(b) Here is a sketch of (the front half of) a constant z slice of E.



Note that

- in E, z runs from -1 to 1.
- Once z has been fixed, x and y must obey $x^2 + y^2 \leqslant 1$, $z \leqslant y \leqslant 1$

So

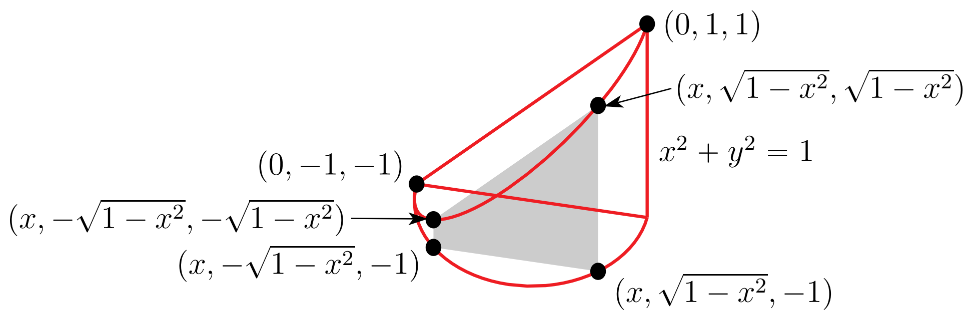
$$E = \left\{ (x, y, z) \;\middle|\; -1 \leqslant z \leqslant 1,\ z \leqslant y \leqslant 1,\ -\sqrt{1-y^2} \leqslant x \leqslant \sqrt{1-y^2} \right\}$$

and

$$\iiint_E f(x, y, z)\, \mathrm{d}V = \int_{-1}^{1} \mathrm{d}z \int_z^1 \mathrm{d}y \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \mathrm{d}x\ f(x, y, z)$$

$$= \int_{-1}^{1} \int_z^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

(c) Here is a sketch of a constant x slice of E.

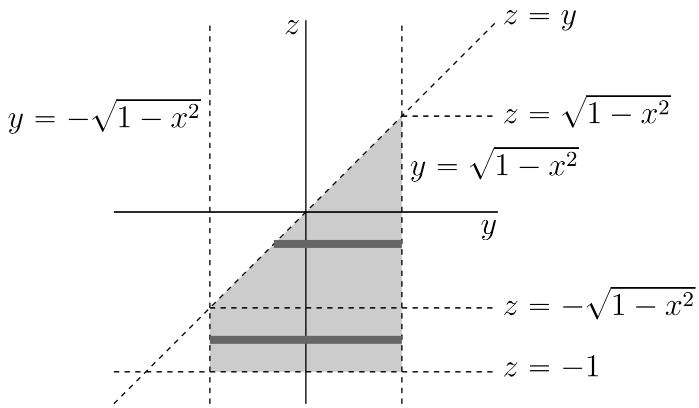


Note that

- in E, x runs from -1 to 1.
- Once x has been fixed, y and z must obey

$$-\sqrt{1-x^2} \leqslant y \leqslant \sqrt{1-x^2} \qquad -1 \leqslant z \leqslant y$$

Here is a sketch.



Note that

- z runs from -1 to $\sqrt{1-x^2}$.
- For each z between -1 and $-\sqrt{1-x^2}$, y runs from $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$, while
- for each z between $-\sqrt{1-x^2}$ and $\sqrt{1-x^2}$, y runs from z to $\sqrt{1-x^2}$.

So

$$\begin{aligned}\iiint_E f(x,y,z)\ \mathrm{d}V &= \int_{-1}^{1}\mathrm{d}x\int_{-1}^{-\sqrt{1-x^2}}\mathrm{d}z\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\mathrm{d}y\ f(x,y,z)\\ &\quad+\int_{-1}^{1}\mathrm{d}x\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\mathrm{d}z\int_{z}^{\sqrt{1-x^2}}\mathrm{d}y\ f(x,y,z)\end{aligned}$$

or

$$\begin{aligned}\iiint_E f(x,y,z)\ \mathrm{d}V &= \int_{-1}^{1}\int_{-1}^{-\sqrt{1-x^2}}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y,z)\ \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x\\ &\quad+\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{z}^{\sqrt{1-x^2}} f(x,y,z)\ \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x\end{aligned}$$

$\boxed{3.5.19}$ (∗) Let E be the region bounded by the planes $y = 0$, $y = 2$, $y + z = 3$ and the surface $z = x^2$. Consider the intergal

$$I = \iiint_E f(x, y, z)\ \mathrm{d}V$$

Fill in the blanks below. In each part below, you may need only one integral to express your answer. In that case, leave the other blank.

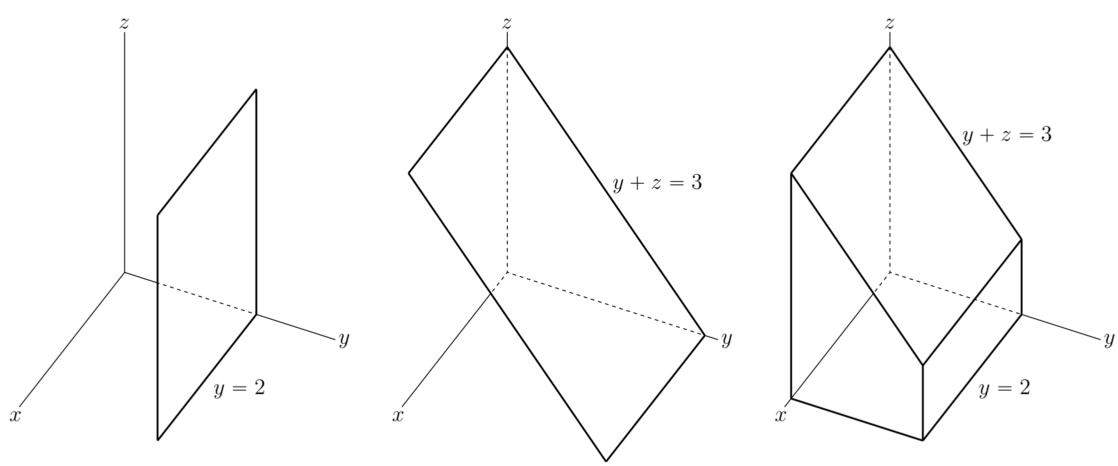
(a) $I = \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} f(x, y, z)\ \mathrm{d}z\, \mathrm{d}x\, \mathrm{d}y + \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} f(x, y, z)\ \mathrm{d}z\, \mathrm{d}x\, \mathrm{d}y$

(b) $I = \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} f(x, y, z)\ \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z + \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} f(x, y, z)\ \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$

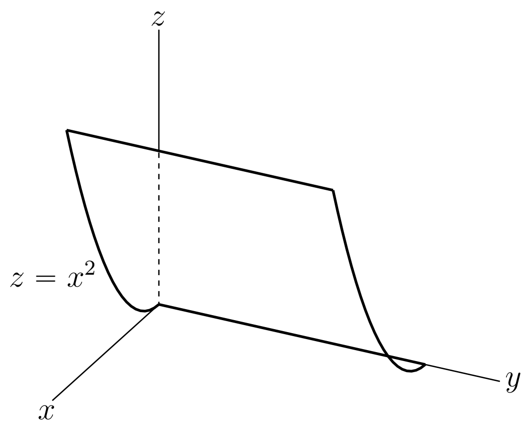
(c) $I = \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} f(x, y, z)\ \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}z + \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} f(x, y, z)\ \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}z$

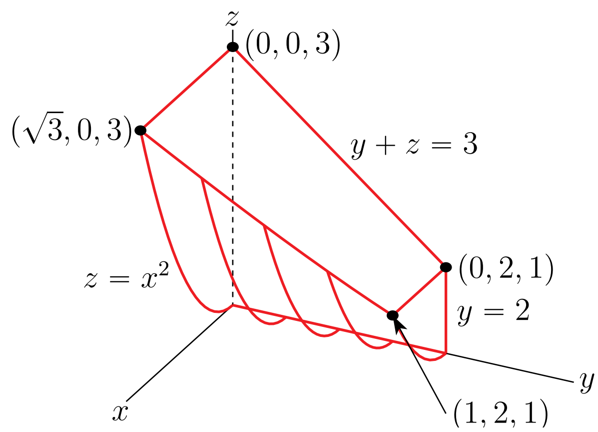
Solution First, we need to develop an understanding of what E looks like. Note that all of the equations $y = 0$, $y = 2$, $y + z = 3$ and $z = x^2$ are invariant under $x \to -x$. So E is invariant under $x \to -x$, i.e. is symmetric about the yz–plane. We'll sketch the first octant (i.e. $x, y, z \geqslant 0$) part of E. There is also a $x \leqslant 0$, $y \geqslant 0$, $z \geqslant 0$ part.

Here are sketches of the plane $y = 2$, on the left, the plane $y + z = 3$ in the centre and of the "tunnel" bounded by the coordinate planes $x = 0$, $y = 0$, $z = 0$ and the planes $y = 2$, $y + z = 3$, on the right.



Now here is the parabolic cylinder $z = x^2$ on the left. E is constructed by using the parabolic cylinder $z = x^2$ to chop the front off of the tunnel $x \geqslant 0$, $0 \leqslant y \leqslant 2$, $z \geqslant 0$, $y + z \leqslant 3$. The figure on the right is a sketch.





So

$$E = \{\ (x,y,z)\ \big|\ 0 \leqslant y \leqslant 2,\ x^2 \leqslant z \leqslant 3-y\ \}$$

(a) On E

- y runs from 0 to 2.
- For each fixed y in that range, (x,z) runs over $\{\ (x,z)\ \big|\ x^2 \leqslant z \leqslant 3-y\ \}$.
- In particular, the largest x^2 is $3-y$ (when $z = 3-y$). So x runs from $-\sqrt{3-y}$ to $\sqrt{3-y}$.
- For fixed y and x as above, z runs from x^2 to $3-y$.

so that

$$I = \iiint\limits_E f(x,y,z)\ \mathrm{d}V = \int_0^2 \int_{-\sqrt{3-y}}^{\sqrt{3-y}} \int_{x^2}^{3-y} f(x,y,z)\ \mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

(b) On E

- z runs from 0 to 3.
- For each fixed z in that range, (x,y) runs over

  $$\{\ (x,y)\ \big|\ 0 \leqslant y \leqslant 2,\ x^2 \leqslant z \leqslant 3-y\ \} = \{\ (x,y)\ \big|\ 0 \leqslant y \leqslant 2,\ y \leqslant 3-z,\ x^2 \leqslant z\ \}$$

  In particular, y runs from 0 to the minimum of 2 and $3-z$.
- So if $0 \leqslant z \leqslant 1$ (so that $3-z \geqslant 2$), (x,y) runs over $\{\ (x,y)\ \big|\ 0 \leqslant y \leqslant 2,\ x^2 \leqslant z\ \}$, while
- if $1 \leqslant z \leqslant 3$, (so that $3-z \leqslant 2$), (x,y) runs over $\{\ (x,y)\ \big|\ 0 \leqslant y \leqslant 3-z,\ x^2 \leqslant z\ \}$,

so that

$$I = \int_0^1 \int_0^2 \int_{-\sqrt{z}}^{\sqrt{z}} f(x,y,z)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_1^3 \int_0^{3-z} \int_{-\sqrt{z}}^{\sqrt{z}} f(x,y,z)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

(c) On E

- z runs from 0 to 3.
- For each fixed z in that range, (x,y) runs over

$$\{\ (x,y)\ \big|\ 0 \leqslant y \leqslant 2,\ x^2 \leqslant z \leqslant 3-y\ \}$$

  In particular, y runs from 0 to the minimum of 2 and $3-z$.

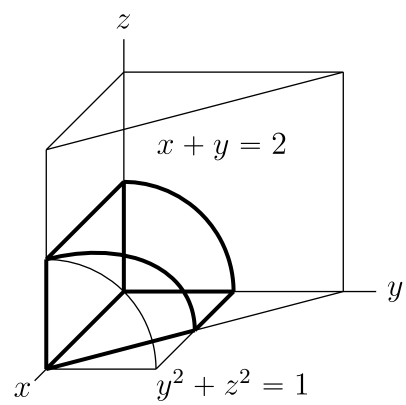
- So if $0 \leqslant z \leqslant 1$ (so that $3-z \geqslant 2$), (x,y) runs over $\{\ (x,y)\ \big|\ 0 \leqslant y \leqslant 2,\ x^2 \leqslant z\ \}$, while
- if $1 \leqslant z \leqslant 3$, (so that $3-z \leqslant 2$), (x,y) runs over $\{\ (x,y)\ \big|\ 0 \leqslant y \leqslant 3-z,\ x^2 \leqslant z\ \}$,

so that

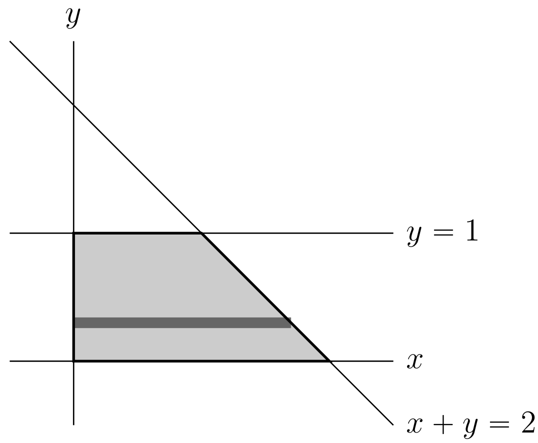
$$I = \int_0^1 \int_{-\sqrt{z}}^{\sqrt{z}} \int_0^2 f(x,y,z)\ \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z + \int_1^3 \int_{-\sqrt{z}}^{\sqrt{z}} \int_0^{3-z} f(x,y,z)\ \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$$

$\boxed{3.5.20}$ (*) Evaluate $\iiint_E z\,\mathrm{d}V$, where E is the region bounded by the planes $y=0$, $z=0$ $x+y=2$ and the cylinder $y^2+z^2=1$ in the first octant.

$\boxed{\text{Solution}}$ The cylinder $y^2+z^2=1$ is centred on the x axis. The part of the cylinder in the first octant intersects the plane $z=0$ in the line $y=1$, intersects to plane $y=0$ in the line $z=1$ and intersects the plane $x=0$ in the quarter circle $y^2+z^2=1$, $x=0$, $y,z \geqslant 0$. Here is a sketch of E.



Viewed from above, the region E is bounded by the lines $x=0$, $y=0$, $x+y=2$ and $y=1$. This base region is pictured below.



To set up the domain of integration, let's decompose the base region into horizontal strips as in the figure above. On the base region

- y runs from 0 to 1 and
- for each fixed y between 0 and 1, x runs from 0 to $2-y$.
- For each fixed (x,y) in the base region z runs from 0 to $\sqrt{1-y^2}$

So

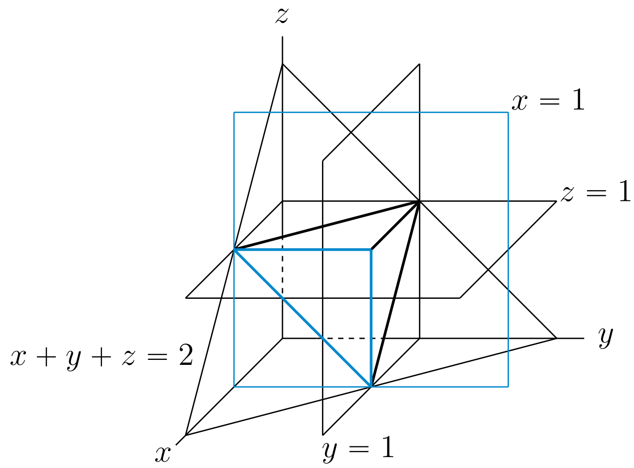
$$E = \left\{ (x,y,z) \,\middle|\, 0 \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant 2-y,\ 0 \leqslant z \leqslant \sqrt{1-y^2} \right\}$$

and

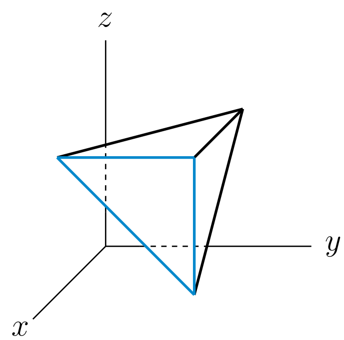
$$\begin{aligned}
\iiint_E z\,\mathrm{d}V &= \int_0^1 \mathrm{d}y \int_0^{2-y} \mathrm{d}x \int_0^{\sqrt{1-y^2}} \mathrm{d}z\ z \\
&= \int_0^1 \mathrm{d}y \int_0^{2-y} \mathrm{d}x\ \frac{1}{2}z^2\Big|_0^{\sqrt{1-y^2}} \\
&= \int_0^1 \mathrm{d}y \int_0^{2-y} \mathrm{d}x\ \frac{1}{2}(1-y^2) \\
&= \int_0^1 \mathrm{d}y\ \frac{1}{2}(1-y^2)(2-y) = \frac{1}{2}\int_0^1 \mathrm{d}y\ \left(2-y-2y^2+y^3\right) \\
&= \frac{1}{2}\left[2-\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right] = \frac{13}{24} \approx 0.5417
\end{aligned}$$

$\boxed{3.5.21}$ (∗) Find $\iiint_D x\,\mathrm{d}V$ where D is the tetrahedron bounded by the planes $x=1$, $y=1$, $z=1$, and $x+y+z=2$.

$\boxed{\text{Solution}}$ The planes $x=1$, $y=1$, $z=1$, and $x+y+z=2$ and the region D are sketched below.



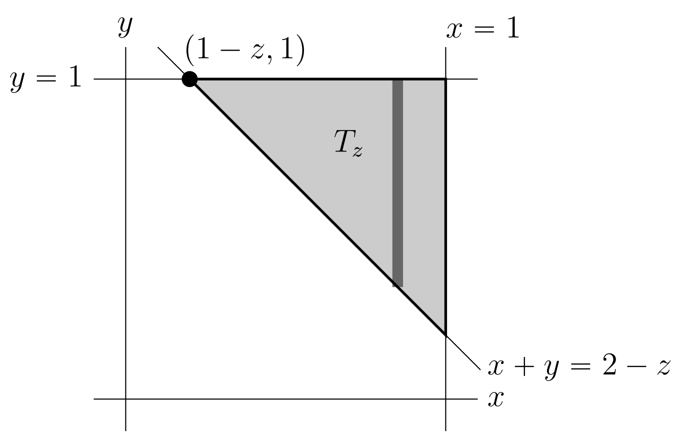
And here is a sketch of D without the planes cluttering up the figure.



On D

- z runs from 0 to 1 and
- for each fixed z, between 0 and 1, (x, y) runs over the triangle T_z bounded by $x = 1$, $y = 1$ and $x + y = 2 - z$. Observe that when $z = 0$, this triangle is just a point (the bottom vertex of the tetrahedron). As z increases, the triangle grows, reaching its maximum size when $z = 1$.

Here is a sketch of T_z.



In setting up the domain of integration, we'll decompose, for each $0 \leqslant z \leqslant 1$, T_z into vertical strips as in the figure above. On T_z

- x runs from $1-z$ to 1 and
- for each fixed x between $1-z$ and 1, y runs from $2-x-z$ to 1

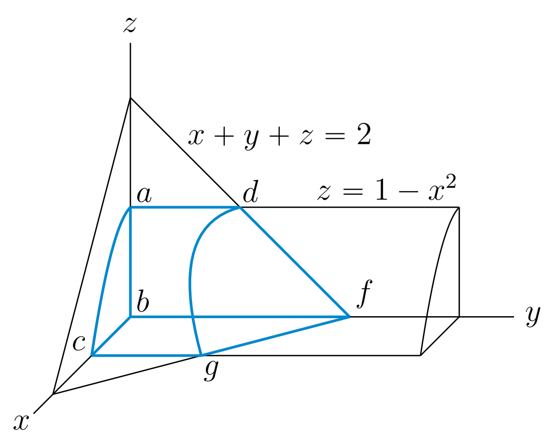
so that

$$\begin{aligned}
\iiint_D x\,\mathrm{d}V &= \int_0^1 \mathrm{d}z \iint_{T_z} \mathrm{d}x\,\mathrm{d}y\ x = \int_0^1 \mathrm{d}z \int_{1-z}^1 \mathrm{d}x \int_{2-x-z}^1 \mathrm{d}y\ x \\
&= \int_0^1 \mathrm{d}z \int_{1-z}^1 \mathrm{d}x\ x(x+z-1) \\
&= \int_0^1 \mathrm{d}z\ \left[\frac{1}{3}x^3 + \frac{1}{2}x^2(z-1)\right]_{1-z}^1 \\
&= \int_0^1 \mathrm{d}z\ \left[\frac{1}{3} + \frac{1}{2}(z-1) - \frac{1}{3}(1-z)^3 - \frac{1}{2}(1-z)^2(z-1)\right] \\
&= \int_0^1 \mathrm{d}z\ \left[\frac{1}{3} + \frac{1}{2}(z-1) - \frac{1}{6}(z-1)^3\right] \\
&= \left[\frac{1}{3}z + \frac{1}{4}(z-1)^2 - \frac{1}{24}(z-1)^4\right]_0^1 = \frac{1}{3} - \frac{1}{4} + \frac{1}{24} = \frac{3}{24} \\
&= \frac{1}{8} = 0.125
\end{aligned}$$

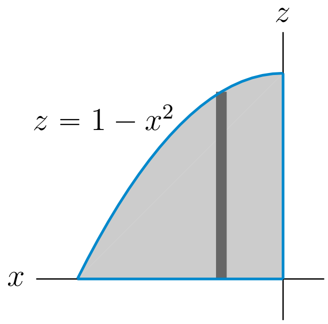
$\boxed{3.5.22}$ (*) The solid region T is bounded by the planes $x=0$, $y=0$, $z=0$, and $x+y+z=2$ and the surface $x^2+z=1$.
(a) Draw the region indicating coordinates of all corners.
(b) Calculate $\iiint_T x\,\mathrm{d}V$.

$\boxed{\text{Solution}}$ (a) Here is a 3d sketch of the region. The coordinates of the labelled corners are

$$a=(0,0,1) \quad b=(0,0,0) \quad c=(1,0,0) \quad d=(0,1,1) \quad f=(0,2,0) \quad g=(1,1,0)$$



(b) Here is a sketch of the side view of T, looking down the y axis.



We'll set up the limits of integration by using it as the base region. We decompose the base region into vertical strips as in the figure above. On the base region

- x runs from 0 to 1 and
- for each fixed x between 0 and 1, z runs from 0 to $1 - x^2$.
- In T, for each fixed (x, y) in the base region, y runs from 0 to $2 - x - z$.

So

$$\begin{aligned}
\iiint_T x\,\mathrm{d}V &= \int_0^1 \mathrm{d}x \int_0^{1-x^2} \mathrm{d}z \int_0^{2-x-z} \mathrm{d}y\ x = \int_0^1 \mathrm{d}x \int_0^{1-x^2} \mathrm{d}z\ (2-x-z)x \\
&= \int_0^1 \mathrm{d}x\ \left[x(2-x)(1-x^2) - \frac{1}{2}x(1-x^2)^2\right] \\
&= \int_0^1 \mathrm{d}x\ \left[2x - x^2 - 2x^3 + x^4 - \frac{1}{2}x + x^3 - \frac{1}{2}x^5\right] \\
&= \int_0^1 \mathrm{d}x\ \left[\frac{3}{2}x - x^2 - x^3 + x^4 - \frac{1}{2}x^5\right] \\
&= \left[\frac{3}{4}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{12}x^6\right]_0^1 \\
&= \frac{3}{4} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{12} = \frac{17}{60}
\end{aligned}$$

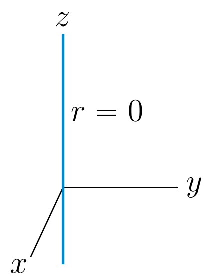
3.6▲ Triple Integrals in Cylindrical Coordinates

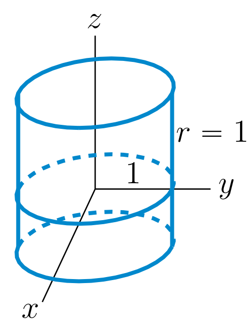
▸▸ Stage 1

3.6.1 Use (r, θ, z) to denote cylindrical coordinates.
(a) Draw $r = 0$.
(b) Draw $r = 1$.
(c) Draw $\theta = 0$.
(d) Draw $\theta = \pi/4$.

Solution (a), (b) Since the cylindrical coordinate $r(x, y, z)$ of a point (x, y, z) is the distance, $\sqrt{x^2+y^2}$, from (x,y,z) to the z-axis, the sets

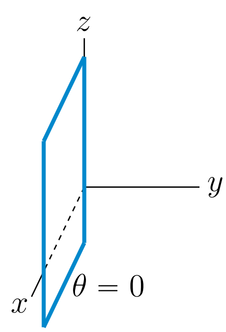
$$\begin{aligned}\{\ (x,y,z)\ \big|\ r(x,y,z)=0\ \} &= \{\ (x,y,z)\ \big|\ x^2+y^2=0\ \} = \{\ (x,y,z)\ \big|\ x=y=0\ \}\\ &= \text{the } z\text{-axis}\\ \{\ (x,y,z)\ \big|\ r(x,y,z)=1\ \} &= \{\ (x,y,z)\ \big|\ x^2+y^2=1\ \}\\ &= \text{the cylinder of radius 1 centred on the } z\text{-axis}\end{aligned}$$

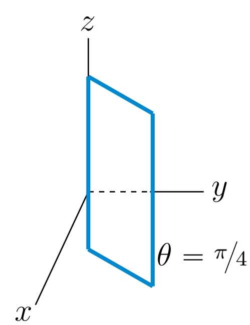




(c), (d) Since the cylindrical coordinate $\theta(x,y,z)$ of a point (x,y,z) is the angle between the positive x-axis and the line from $(0,0,0)$ to $(x,y,0)$, the sets

$$\begin{aligned}\{\ (x,y,z)\ \big|\ \theta(x,y,z)=0\ \} &= \text{the half of the } xz\text{-plane with } x>0\\ \{\ (x,y,z)\ \big|\ \theta(x,y,z)={}^{\pi}\!/\!_{4}\ \} &= \text{the half of the plane } y=x \text{ with } x>0\end{aligned}$$



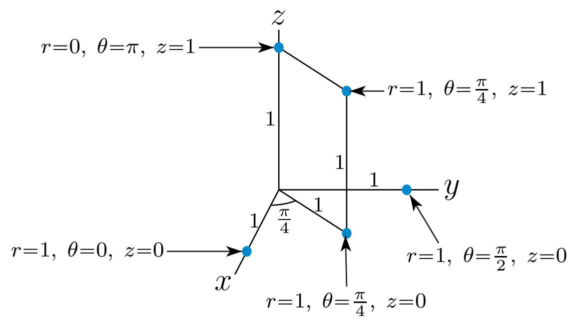


$\boxed{3.6.2}$ Sketch the points with the specified cylindrical coordinates.

(a) $r=1$, $\theta=0$, $z=0$
(b) $r=1$, $\theta=\frac{\pi}{4}$, $z=0$
(c) $r=1$, $\theta=\frac{\pi}{2}$, $z=0$
(d) $r=0$, $\theta=\pi$, $z=1$
(e) $r=1$, $\theta=\frac{\pi}{4}$, $z=1$

$\boxed{\text{Solution}}$ The sketch is below. To help build up this sketch, it is useful to recall the following facts.

- The cylindrical coordinate r is the distance of the point from the z-axis. In particular all points with $r=0$ lie on the z-axis (for all values of θ).
- The cylindrical coordinate z is the distance of the point from the xy-plane. In particular all points with $z=0$ lie on the xy-plane.



$\boxed{3.6.3}$ Convert from cylindrical to Cartesian coordinates.

(a) $r = 1$, $\theta = 0$, $z = 0$
(b) $r = 1$, $\theta = \frac{\pi}{4}$, $z = 0$
(c) $r = 1$, $\theta = \frac{\pi}{2}$, $z = 0$
(d) $r = 0$, $\theta = \pi$, $z = 1$
(e) $r = 1$, $\theta = \frac{\pi}{4}$, $z = 1$

$\boxed{\text{Solution}}$ (a) When $\theta = 0$, $\sin\theta = 0$ and $\cos\theta = 1$, so that the polar coordinates $r = 1$, $\theta = 0$, $z = 0$ correspond to the Cartesian coordinates

$$(x, y, z) = (r\cos\theta, r\sin\theta, z) = (1 \times \cos 0,\ 1 \times \sin 0,\ 0) = (1, 0, 0)$$

(b) When $\theta = \frac{\pi}{4}$, $\sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$, so that the polar coordinates $r = 1$, $\theta = \frac{\pi}{4}$, $z = 0$ correspond to the Cartesian coordinates

$$(x, y, z) = (r\cos\theta, r\sin\theta, z) = \left(1 \times \cos\frac{\pi}{4},\ 1 \times \sin\frac{\pi}{4},\ 0\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

(c) When $\theta = \frac{\pi}{2}$, $\sin\theta = 1$ and $\cos\theta = 0$, so that the polar coordinates $r = 1$, $\theta = \frac{\pi}{2}$, $z = 0$ correspond to the Cartesian coordinates

$$(x, y, z) = (r\cos\theta, r\sin\theta, z) = \left(1 \times \cos\frac{\pi}{2},\ 1 \times \sin\frac{\pi}{2},\ 0\right) = (0, 1, 0)$$

(d) When $\theta = \pi$, $\sin\theta = 0$ and $\cos\theta = -1$, so that the polar coordinates $r = 0$, $\theta = \pi$, $z = 1$ correspond to the Cartesian coordinates

$$(x, y, z) = (r\cos\theta, r\sin\theta, z) = (0 \times \cos\pi,\ 0 \times \sin\pi,\ 1) = (0, 0, 1)$$

(e) When $\theta = \frac{\pi}{4}$, $\sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$, so that the polar coordinates $r = 1$, $\theta = \frac{\pi}{4}$, $z = 1$ correspond to the Cartesian coordinates

$$(x, y, z) = (r\cos\theta, r\sin\theta, z) = \left(1 \times \cos\frac{\pi}{4},\ 1 \times \sin\frac{\pi}{4},\ 1\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right)$$

$\boxed{3.6.4}$ Convert from Cartesian to cylindrical coordinates.

(a) $(1, 1, 2)$
(b) $(-1, -1, 2)$
(c) $(-1, \sqrt{3}, 0)$
(d) $(0, 0, 1)$

Solution (a) The cylindrical coordinates must obey

$$1 = x = r\cos\theta \qquad 1 = y = r\sin\theta \qquad 2 = z$$

So $z = 2$, $r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\tan\theta = \frac{y}{x} = \frac{1}{1} = 1$. Recall that $\tan\left(\frac{\pi}{4} + k\pi\right) = 1$ for all integers k. As $(x, y) = (1, 1)$ lies in the first quadrant, $0 \leqslant \theta \leqslant \frac{\pi}{2}$. So $\theta = \frac{\pi}{4}$ (plus possibly any integer multiple of 2π).

(b) The cylindrical coordinates must obey

$$-1 = x = r\cos\theta \qquad -1 = y = r\sin\theta \qquad 2 = z$$

So $z = 2$, $r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$ and $\tan\theta = \frac{y}{x} = \frac{-1}{-1} = 1$. Recall that $\tan\left(\frac{\pi}{4} + k\pi\right) = 1$ for all integers k. As $(x, y) = (-1, -1)$ lies in the third quadrant, $\pi \leqslant \theta \leqslant \frac{3\pi}{2}$. So $\theta = \frac{5\pi}{4}$ (plus possibly any integer multiple of 2π).

(c) The cylindrical coordinates must obey

$$-1 = x = r\cos\theta \qquad \sqrt{3} = y = r\sin\theta \qquad 0 = z$$

So $z = 0$, $r = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = 2$ and $\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$. Recall that $\tan\left(\frac{2\pi}{3} + k\pi\right) = -\sqrt{3}$ for all integers k. As $(x, y) = (-1, \sqrt{3})$ lies in the second quadrant, $\frac{\pi}{2} \leqslant \theta \leqslant \pi$. So $\theta = \frac{2\pi}{3}$ (plus possibly any integer multiple of 2π).

(d) The cylindrical coordinates must obey

$$0 = x = r\cos\theta \qquad 0 = y = r\sin\theta \qquad 1 = z$$

So $z = 0$, $r = \sqrt{0^2 + 0^2} = 0$ and θ is completely arbitrary.

3.6.5 Rewrite the following equations in cylindrical coordinates.
(a) $z = 2xy$
(b) $x^2 + y^2 + z^2 = 1$
(c) $(x - 1)^2 + y^2 = 1$

Solution (a) As $x = r\cos\theta$ and $y = r\sin\theta$,

$$z = 2xy \iff z = 2r^2\cos\theta\sin\theta = r^2\sin(2\theta)$$

(b) As $x = r\cos\theta$ and $y = r\sin\theta$,

$$x^2 + y^2 + z^2 = 1 \iff r^2\cos^2\theta + r^2\sin^2\theta + z^2 = 1 \iff r^2 + z^2 = 1$$

(c) As $x = r\cos\theta$ and $y = r\sin\theta$,

$$\begin{aligned}(x-1)^2 + y^2 = 1 &\iff (r\cos\theta - 1)^2 + (r\sin\theta)^2 = 1 \\ &\iff r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta = 1 \\ &\iff r^2 = 2r\cos\theta \iff r = 2\cos\theta \text{ or } r = 0 \\ &\iff r = 2\cos\theta\end{aligned}$$

Note that the solution $r = 0$ is included in $r = 2\cos\theta$ — just choose $\theta = \frac{\pi}{2}$.

▸▸ Stage 2

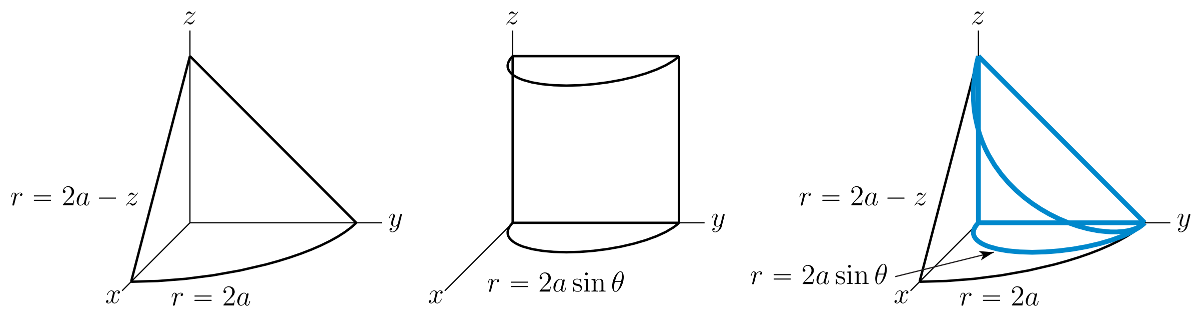
3.6.6 Use cylindrical coordinates to evaluate the volumes of each of the following regions.

(a) Above the xy–plane, inside the cone $z = 2a - \sqrt{x^2+y^2}$ and inside the cylinder $x^2 + y^2 = 2ay$.

(b) Above the xy–plane, under the paraboloid $z = 1 - x^2 - y^2$ and in the wedge $-x \leqslant y \leqslant \sqrt{3}x$.

(c) Above the paraboloid $z = x^2 + y^2$ and below the plane $z = 2y$.

Solution (a) In cylindrical coordinates, the cone $z = 2a - \sqrt{x^2+y^2}$ is $z = 2a - r$ and the cylinder $x^2 + y^2 = 2ay$ is $r^2 = 2ar\sin\theta$ or $r = 2a\sin\theta$. The figures below show the parts of the cone, the cylinder and the intersection, respectively, that are in the first octant.



The specified region is

$$V = \left\{ (r\cos\theta\,,\, r\sin\theta\,,\, z) \;\middle|\; 0 \leqslant \theta \leqslant \pi,\ r \leqslant 2a\sin\theta,\ 0 \leqslant z \leqslant 2a - r \right\}$$

By symmetry under $x \to -x$, the full volume is twice the volume in the first octant.

So the

$$\begin{aligned}
\text{Volume} &= 2\int_0^{\frac{\pi}{2}} d\theta \int_0^{2a\sin\theta} dr\ r \int_0^{2a-r} dz \\
&= 2\int_0^{\frac{\pi}{2}} d\theta \int_0^{2a\sin\theta} dr\ r(2a-r) \\
&= 2\int_0^{\frac{\pi}{2}} d\theta\ \left[4a^3\sin^2\theta - \frac{8a^3}{3}\sin^3\theta\right] \\
&= 8a^3\left[\int_0^{\frac{\pi}{2}} d\theta\ \frac{1-\cos(2\theta)}{2} + \frac{2}{3}\int_1^0 dt\ (1-t^2)\right] \qquad \text{where } t = \cos\theta \\
&= 8a^3\left[\frac{\pi}{4} - \frac{2}{3}\left(1 - \frac{1}{3}\right)\right] = a^3\left(2\pi - \frac{32}{9}\right)
\end{aligned}$$

For an efficient, sneaky, way to evaluate $\int_0^{\frac{\pi}{2}} d\theta\ \sin^2\theta$, see Remark 3.3.5 in the CLP-3 text.

(b) The domain of integration is

$$V = \left\{ (x,y,z) \;\middle|\; -x \leqslant y \leqslant \sqrt{3}x,\ 0 \leqslant z \leqslant 1 - x^2 - y^2 \right\}$$

Recall that in polar coordinates $\frac{y}{x} = \tan\theta$. So the boundaries of the wedge $-x \leqslant y \leqslant \sqrt{3}x$, or equivalently $-1 \leqslant \frac{y}{x} \leqslant \sqrt{3}$, correspond, in polar coordinates, to $\theta = \tan^{-1}(-1) = -\frac{\pi}{4}$ and $\theta = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$. In cylindrical coordinates, the paraboloid $z = 1 - x^2 - y^2$ becomes $z = 1 - r^2$. There are z's that obey $0 \leqslant z \leqslant 1 - r^2$ if and only if $r \leqslant 1$. So, in cylindrical coordinates,

$$V = \left\{ (r\cos\theta, r\sin\theta, z) \;\middle|\; -\tfrac{\pi}{4} \leqslant \theta \leqslant \tfrac{\pi}{3},\ 0 \leqslant r \leqslant 1,\ 0 \leqslant z \leqslant 1 - r^2 \right\}$$

and

$$\text{Volume} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} d\theta \int_0^1 dr\ r \int_0^{1-r^2} dz = \left(\frac{\pi}{3} + \frac{\pi}{4}\right) \int_0^1 dr\ r(1 - r^2) = \frac{7}{12}\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{7}{48}\pi$$

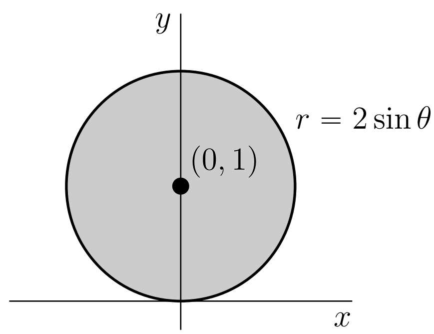
(c) The region is

$$V = \left\{ (x, y, z) \;\middle|\; x^2 + y^2 \leqslant z \leqslant 2y \right\}$$

There are z's that obey $x^2 + y^2 \leqslant z \leqslant 2y$ if and only if

$$x^2 + y^2 \leqslant 2y \iff x^2 + y^2 - 2y \leqslant 0 \iff x^2 + (y-1)^2 \leqslant 1$$

This disk is sketched in the figure



In cylindrical coordinates,

- the bottom, $z = x^2 + y^2$, is $z = r^2$,
- the top, $z = 2y$, is $z = 2r\sin\theta$, and
- the disk $x^2 + y^2 \leqslant 2y$ is $r^2 \leqslant 2r\sin\theta$, or equivalently $r \leqslant 2\sin\theta$,

so that, looking at the figure above,

$$V = \left\{ (r\cos\theta, r\sin\theta, z) \;\middle|\; 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant r \leqslant 2\sin\theta,\ r^2 \leqslant z \leqslant 2r\sin\theta \right\}$$

By symmetry under $x \to -x$, the full volume is twice the volume in the first octant so that

$$\begin{aligned}\text{Volume} &= 2\int_0^{\frac{\pi}{2}} d\theta \int_0^{2\sin\theta} dr\ r \int_{r^2}^{2r\sin\theta} dz = 2\int_0^{\frac{\pi}{2}} d\theta \int_0^{2\sin\theta} dr\ r(2r\sin\theta - r^2) \\ &= 2\int_0^{\frac{\pi}{2}} d\theta\ \left(\frac{2^4}{3} - \frac{2^4}{4}\right)\sin^4\theta\end{aligned}$$

To integrate[2] $\sin^4\theta$, we use the double angle formulae $\sin^2 x = \frac{1-\cos(2x)}{2}$ and $\cos^2 x = \frac{1+\cos(2x)}{2}$ to write

$$\begin{aligned}\sin^4\theta &= \left[\frac{1-\cos(2\theta)}{2}\right]^2 \\ &= \frac{1}{4} - \frac{1}{2}\cos(2\theta) + \frac{1}{4}\cos^2(2\theta) \\ &= \frac{1}{4} - \frac{1}{2}\cos(2\theta) + \frac{1}{8}\left(1+\cos(4\theta)\right) \\ &= \frac{3}{8} - \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)\end{aligned}$$

So

$$\text{Volume} = 2\,\frac{2^4}{12}\left[\frac{3}{8}\theta - \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta)\right]_0^{\frac{\pi}{2}} = 2\,\frac{2^4}{12}\,\frac{3}{16}\pi = \frac{\pi}{2}$$

$\boxed{3.6.7}$ (∗) Let E be the region bounded between the parabolic surfaces $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$ and within the cylinder $x^2 + y^2 \leqslant 1$. Calculate the integral of $f(x,y,z) = (x^2+y^2)^{3/2}$ over the region E.

$\boxed{\text{Solution}}$ Note that the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$ intersect when $z = x^2 + y^2 = 1$. We'll use cylindrical coordinates. Then $x^2 + y^2 = r^2$, $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$, and

$$E = \left\{\ (r\cos\theta\,,\, r\sin\theta\,,\, z))\ \middle|\ 0 \leqslant r \leqslant 1,\ r^2 \leqslant z \leqslant 2 - r^2,\ 0 \leqslant \theta \leqslant 2\pi\ \right\}$$

so that

$$\begin{aligned}\iiint_E f(x,y,z)\ \mathrm{d}V &= \int_0^1 \mathrm{d}r \int_{r^2}^{2-r^2} \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta\ r\ \overbrace{r^3}^{f} \\ &= 2\pi\int_0^1 \mathrm{d}r\ r^4\left(2 - r^2 - r^2\right) \\ &= 2\pi\left[2\frac{1^5}{5} - 2\frac{1^7}{7}\right] \\ &= \frac{8\pi}{35}\end{aligned}$$

$\boxed{3.6.8}$ (∗) Let E be the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$. Find the centroid of E.

2 For a general discussion of trigonometric integrals see §1.8 in the CLP-2 text. In particular the integral $\int \cos^4 x\ \mathrm{d}x$ is evaluated in Example 1.8.8 in the CLP-2 text.

$\boxed{\text{Solution}}$ Observe that both the sphere $x^2+y^2+z^2=2$ and the paraboloid $z=x^2+y^2$ are invariant under rotations around the z–axis. So E is invariant under rotations around the z–axis and the centroid (centre of mass) of E will lie on the z–axis. Thus $\bar{x}=\bar{y}=0$ and we just have to find

$$\bar{z}=\frac{\iiint_E z\ \mathrm{d}V}{\iiint_E \mathrm{d}V}$$

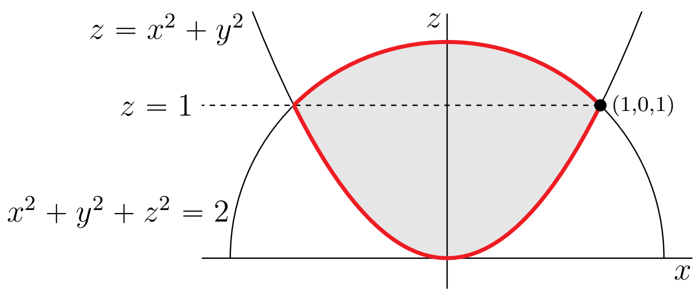
The surfaces $z=x^2+y^2$ and $x^2+y^2+z^2=2$ intersect when $z=x^2+y^2$ and

$$z+z^2=2 \iff z^2+z-2=0 \iff (z+2)(z-1)=0$$

Since $z=x^2+y^2\geqslant 0$, the surfaces intersect on the circle $z=1$, $x^2+y^2=2$. So

$$E=\Big\{\ (x,y,z)\ \Big|\ x^2+y^2\leqslant 1,\ x^2+y^2\leqslant z\leqslant\sqrt{2-x^2-y^2}\ \Big\}$$

Here is a sketch of the $y=0$ cross section of E.



Let's use cylindrical coordinates to do the two integrals. In cylindrical coordinates

- $E=\Big\{\ (r\cos\theta, r\sin\theta, z)\ \Big|\ 0\leqslant r\leqslant 1,\ 0\leqslant\theta\leqslant 2\pi,\ r^2\leqslant z\leqslant\sqrt{2-r^2}\ \Big\}$, and
- $\mathrm{d}V$ is $r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$

so, for $n=0,1$ (we'll try to do both integrals at the same time)

$$\begin{aligned}\iiint_E z^n\ \mathrm{d}V &= \int_0^1 \mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\int_{r^2}^{\sqrt{2-r^2}}\mathrm{d}z\ r\ z^n\\ &= 2\pi\int_0^1 \mathrm{d}r\ r\begin{cases}\sqrt{2-r^2}-r^2 & \text{if } n=0\\ \frac{1}{2}\big(2-r^2-r^4\big) & \text{if } n=1\end{cases}\end{aligned}$$

Since

$$\int_0^1 \mathrm{d}r\ r\sqrt{2-r^2}=\Big[-\frac{1}{3}(2-r^2)^{3/2}\Big]_0^1=\frac{1}{3}(2\sqrt{2}-1)$$

we have

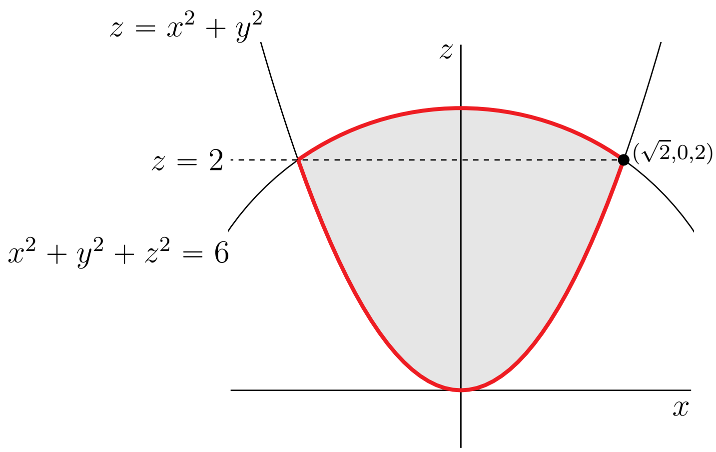
$$\iiint_E z^n\ \mathrm{d}V=2\pi\left\{\begin{matrix}\frac{1}{3}(2\sqrt{2}-1)-\frac{1}{4} & \text{if } n=0\\ \frac{1}{2}-\frac{1}{8}-\frac{1}{12} & \text{if } n=1\end{matrix}\right\}=2\pi\begin{cases}\frac{2}{3}\sqrt{2}-\frac{7}{12} & \text{if } n=0\\ \frac{7}{24} & \text{if } n=1\end{cases}$$

and $\bar{x} = \bar{y} = 0$ and

$$\bar{z} = \frac{\iiint_E z \, \mathrm{d}V}{\iiint_E \mathrm{d}V} = \frac{\frac{7}{24}}{\frac{2}{3}\sqrt{2} - \frac{7}{12}} = \frac{7}{16\sqrt{2} - 14} \approx 0.811$$

$\boxed{3.6.9}$ (∗) Let E be the smaller of the two solid regions bounded by the surfaces $z = x^2 + y^2$ and $x^2 + y^2 + z^2 = 6$. Evaluate $\iiint_E (x^2 + y^2) \, \mathrm{d}V$.

$\boxed{\text{Solution}}$ Note that both surfaces are invariant under rotations about the z–axis. Here is a sketch of the $y = 0$ cross section of E.



The surfaces $z = x^2 + y^2$ and $x^2 + y^2 + z^2 = 6$ intersect when $z = x^2 + y^2$ and

$$z + z^2 = 6 \iff z^2 + z - 6 = 0 \iff (z + 3)(z - 2) = 0$$

Since $z = x^2 + y^2 \geqslant 0$, the surfaces intersect on the circle $z = 2$, $x^2 + y^2 = 2$. So

$$E = \left\{ \ (x, y, z) \ \middle| \ x^2 + y^2 \leqslant 2, \ x^2 + y^2 \leqslant z \leqslant \sqrt{6 - x^2 - y^2} \ \right\}$$

Let's use cylindrical coordinates to do the integral. In cylindrical coordinates

- $E = \left\{ \ (r\cos\theta, r\sin\theta, z) \ \middle| \ r \leqslant \sqrt{2}, \ 0 \leqslant \theta \leqslant 2\pi, \ r^2 \leqslant z \leqslant \sqrt{6 - r^2} \ \right\}$, and
- $\mathrm{d}V$ is $r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z$

so

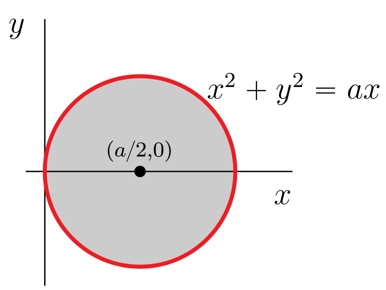
$$\begin{aligned}
\iiint_E (x^2+y^2)\,\mathrm{d}V &= \int_0^{\sqrt{2}} \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \int_{r^2}^{\sqrt{6-r^2}} \mathrm{d}z\ r\ r^2 \\
&= 2\pi \int_0^{\sqrt{2}} \mathrm{d}r\ r^3\big(\sqrt{6-r^2}-r^2\big) = 2\pi \int_0^{\sqrt{2}} \mathrm{d}r\, r\, r^2\sqrt{6-r^2} - 2\pi \int_0^{\sqrt{2}} \mathrm{d}r\ r^5 \\
&= 2\pi \int_6^4 \frac{\mathrm{d}u}{-2}\ (6-u)\sqrt{u} - 2\pi\frac{2^3}{6} \qquad \text{with } u = 6-r^2,\ \mathrm{d}u = -2r\,\mathrm{d}r \\
&= -\pi\left[6\frac{u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right]_6^4 - \frac{8\pi}{3} \\
&= -\pi\left[4\big(8-6\sqrt{6}\big) - \frac{2}{5}\big(32-36\sqrt{6}\big)\right] - \frac{8\pi}{3} \\
&= \pi\left[\frac{64}{5} - 32 - \frac{8}{3} + \left(24 - \frac{72}{5}\right)\sqrt{6}\right] \\
&= \pi\left[\frac{48}{5}\sqrt{6} - \frac{328}{15}\right] \approx 1.65\pi
\end{aligned}$$

$\boxed{3.6.10}$ $(*)$ Let $a > 0$ be a fixed positive real number. Consider the solid inside both the cylinder $x^2+y^2 = ax$ and the sphere $x^2+y^2+z^2 = a^2$. Compute its volume. You may use that $\int \sin^3(\theta) = \frac{1}{12}\cos(3\theta) - \frac{3}{4}\cos(\theta) + C$

$\boxed{\text{Solution}}$ We'll use cylindrical coordinates. In cylindrical coordinates

- the sphere $x^2+y^2+z^2 = a^2$ becomes $r^2+z^2 = a^2$ and
- the circular cylinder $x^2+y^2 = ax$ (or equivalently $(x-a/2)^2+y^2 = a^2/4$) becomes $r^2 = ar\cos\theta$ or $r = a\cos\theta$.

Here is a sketch of the top view of the solid.



The solid is

$$\left\{\ (r\cos\theta\,,\ r\sin\theta\,,\ z)\ \middle|\ -\pi/2 \leqslant \theta \leqslant \pi/2\,,\ 0 \leqslant r \leqslant a\cos\theta\,,\ -\sqrt{a^2-r^2} \leqslant z \leqslant \sqrt{a^2-r^2}\ \right\}$$

By symmetry, the volume of the specified solid is four times the volume of the solid

$$\left\{\ (r\cos\theta\,,\ r\sin\theta\,,\ z)\ \middle|\ 0 \leqslant \theta \leqslant \pi/2\,,\ 0 \leqslant r \leqslant a\cos\theta\,,\ 0 \leqslant z \leqslant \sqrt{a^2-r^2}\ \right\}$$

Since $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$, the volume of the solid is

$$
\begin{aligned}
4\int_0^{\pi/2}\mathrm{d}\theta\int_0^{a\cos\theta}\mathrm{d}r\int_0^{\sqrt{a^2-r^2}}\mathrm{d}z\ r &= 4\int_0^{\pi/2}\mathrm{d}\theta\int_0^{a\cos\theta}\mathrm{d}r\ r\sqrt{a^2-r^2}\\
&= -\frac{4}{3}\int_0^{\pi/2}\mathrm{d}\theta\ \left(a^2-r^2\right)^{3/2}\Big|_0^{a\cos\theta}\\
&= \frac{4}{3}\int_0^{\pi/2}\mathrm{d}\theta\ \left[a^3-\left(a^2-a^2\cos^2\theta\right)^{3/2}\right]\\
&= \frac{4a^3}{3}\int_0^{\pi/2}\mathrm{d}\theta\ \left[1-\sin^3\theta\right]\\
&= \frac{4a^3}{3}\left[\theta-\frac{1}{12}\cos(3\theta)+\frac{3}{4}\cos\theta\right]_0^{\pi/2}\\
&= \frac{4a^3}{3}\left[\frac{\pi}{2}+\frac{1}{12}-\frac{3}{4}\right]\\
&= \frac{4a^3}{3}\left[\frac{\pi}{2}-\frac{2}{3}\right]
\end{aligned}
$$

$\boxed{3.6.11}$ (∗) Let E be the solid lying above the surface $z = y^2$ and below the surface $z = 4 - x^2$. Evaluate

$$\iiint_E y^2\ \mathrm{d}V$$

You may use the half angle formulas:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}, \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$\boxed{\text{Solution}}$ Note that the surfaces meet when $z = y^2 = 4 - x^2$ and then (x, y) runs over the circle $x^2 + y^2 = 4$. So the domain of integration is

$$E = \left\{\ (x,y,z)\ \middle|\ x^2+y^2 \leqslant 4,\ y^2 \leqslant z \leqslant 4-x^2\ \right\}$$

Let's switch to cylindrical coordinates. Then

$$E = \left\{\ (r\cos\theta, r\sin\theta, z)\ \middle|\ 0 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant 2\pi,\ r^2\sin^2\theta \leqslant z \leqslant 4-r^2\cos^2\theta\ \right\}$$

and, since $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$,

$$
\begin{aligned}
\iiint_E y^2\ \mathrm{d}V &= \int_0^2\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\int_{r^2\sin^2\theta}^{4-r^2\cos^2\theta}\mathrm{d}z\ r\ \overbrace{r^2\sin^2\theta}^{y^2}\\
&= \int_0^2\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\ r^3\sin^2\theta\left[4-r^2\cos^2\theta-r^2\sin^2\theta\right]\\
&= \int_0^2\mathrm{d}r\ \left[4r^3-r^5\right]\int_0^{2\pi}\mathrm{d}\theta\ \frac{1-\cos(2\theta)}{2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^2 \mathrm{d}r\ \left[4r^3 - r^5\right]\ \left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{2\pi} \\
&= \pi\left[r^4 - \frac{r^6}{6}\right]_0^2 \\
&= \frac{16\pi}{3}
\end{aligned}
$$

For an efficient, sneaky, way to evaluate $\int_0^{2\pi}\sin^2\theta\ \mathrm{d}\theta$, see Remark 3.3.5 in the CLP-3 text.

3.6.12 The centre of mass $(\bar{x},\bar{y},\bar{z})$ of a body B having density $\rho(x,y,z)$ (units of mass per unit volume) at (x,y,z) is defined to be

$$\bar{x} = \frac{1}{M}\iiint_B x\rho(x,y,z)\ \mathrm{d}V \quad \bar{y} = \frac{1}{M}\iiint_B y\rho(x,y,z)\ \mathrm{d}V \quad \bar{z} = \frac{1}{M}\iiint_B z\rho(x,y,z)\ \mathrm{d}V$$

where

$$M = \iiint_B \rho(x,y,z)\ \mathrm{d}V$$

is the mass of the body. So, for example, $\bar{x}$ is the weighted average of x over the body. Find the centre of mass of the part of the solid ball $x^2+y^2+z^2 \leqslant a^2$ with $x \geqslant 0$, $y \geqslant 0$ and $z \geqslant 0$, assuming that the density ρ is constant.

Solution By symmetry, $\bar{x} = \bar{y} = \bar{z}$, so it suffices to compute, for example, $\bar{z}$. The mass of the body is the density, ρ, times its volume, which is one eighth of the volume of a sphere. So

$$M = \frac{\rho}{8}\,\frac{4}{3}\pi a^3$$

In cylindrical coordinates, the equation of the spherical surface of the body is $r^2+z^2 = a^2$. The part of the body at height z above the xy–plane is one quarter of a disk of radius $\sqrt{a^2-z^2}$. The numerator of $\bar{z}$ is

$$
\begin{aligned}
\iiint_B z\rho\ \mathrm{d}V &= \rho\int_0^a \mathrm{d}z\int_0^{\pi/2} d\theta\int_0^{\sqrt{a^2-z^2}} dr\ r\ z = \rho\int_0^a \mathrm{d}z\int_0^{\pi/2} d\theta\ z\ \frac{r^2}{2}\bigg|_0^{\sqrt{a^2-z^2}} \\
&= \frac{\rho}{2}\int_0^a \mathrm{d}z\int_0^{\pi/2} d\theta\ z(a^2-z^2) \qquad = \frac{\pi}{4}\rho\int_0^a \mathrm{d}z\ z(a^2-z^2) \\
&= \frac{\pi}{4}\rho\left[a^2\frac{z^2}{2} - \frac{z^4}{4}\right]_0^a = \frac{\pi}{16}\rho a^4
\end{aligned}
$$

Dividing by $M = \frac{\pi}{6}\rho a^3$ gives $\bar{x} = \bar{y} = \bar{z} = \frac{3}{8}a$.

$\boxed{3.6.13}$ (∗) A sphere of radius $2\mathrm{m}$ centred on the origin has variable density $\frac{5}{\sqrt{3}}(z^2+1)\mathrm{kg}/\mathrm{m}^3$. A hole of diameter 1m is drilled through the sphere along the z–axis.

(a) Set up a triple integral in cylindrical coordinates giving the mass of the sphere after the hole has been drilled.

(b) Evaluate this integral.

Solution (a) In cylindrical coordinates the equation of a sphere of radius 2 centred on the origin is $r^2+z^2=2^2$. Since $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$ and $\mathrm{d}m = \frac{5}{\sqrt{3}}(z^2+1)r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$ and the hole has radius $1/2$, the integral is

$$\text{mass} = \int_{1/2}^{2}\mathrm{d}r\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}}\mathrm{d}z\int_0^{2\pi}\mathrm{d}\theta\ \frac{5}{\sqrt{3}}(z^2+1)r$$

(b) By part (a)

$$\begin{aligned}\text{mass} = \int_{1/2}^{2}\mathrm{d}r\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}}\mathrm{d}z\int_0^{2\pi}\mathrm{d}\theta\ \frac{5}{\sqrt{3}}(z^2+1)r &= 4\pi\frac{5}{\sqrt{3}}\int_{1/2}^{2}\mathrm{d}r\ r\int_0^{\sqrt{4-r^2}}\mathrm{d}z\ (z^2+1)\\
&= 4\pi\frac{5}{\sqrt{3}}\int_{1/2}^{2}\mathrm{d}r\ r\left[\frac{z^3}{3}+z\right]_0^{\sqrt{4-r^2}}\\
&= 4\pi\frac{5}{\sqrt{3}}\int_{1/2}^{2}\mathrm{d}r\ r\left[\frac{1}{3}(4-r^2)^{3/2}+(4-r^2)^{1/2}\right]\end{aligned}$$

Make the change of variables $s=4-r^2$, $\mathrm{d}s=-2r\,\mathrm{d}r$. This gives

$$\begin{aligned}\text{mass} &= 4\pi\frac{5}{\sqrt{3}}\int_{15/4}^{0}\frac{\mathrm{d}s}{-2}\ \left[\frac{1}{3}s^{3/2}+s^{1/2}\right] = -2\pi\frac{5}{\sqrt{3}}\left[\frac{2}{15}s^{5/2}+\frac{2}{3}s^{3/2}\right]_{15/4}^{0}\\
&= 2\pi\frac{5}{\sqrt{3}}\left[\frac{2}{15}\frac{15^{5/2}}{32}+\frac{2}{3}\frac{15^{3/2}}{8}\right]\\
&= 2\pi\frac{5}{\sqrt{3}}\left[\frac{1}{16}+\frac{1}{12}\right]15^{3/2} = \frac{525}{24}\sqrt{5}\pi \approx 153.7\mathrm{kg}\end{aligned}$$

$\boxed{3.6.14}$ (∗) Consider the finite solid bounded by the three surfaces: $z=e^{-x^2-y^2}$, $z=0$ and $x^2+y^2=4$.

(a) Set up (but do not evaluate) a triple integral in rectangular coordinates that describes the volume of the solid.

(b) Calculate the volume of the solid using any method.

Solution (a) The solid consists of all (x,y,z) with

- (x,y) running over the disk $x^2+y^2\leqslant 4$ and
- for each fixed (x,y) obeying $x^2+y^2\leqslant 4$, z running from 0 to $e^{-x^2-y^2}$

On the disk $x^2 + y^2 \leqslant 4$,

- x runs from -2 to 2 and
- for each fixed x obeying $-2 \leqslant x \leqslant 2$, y runs from $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$

So

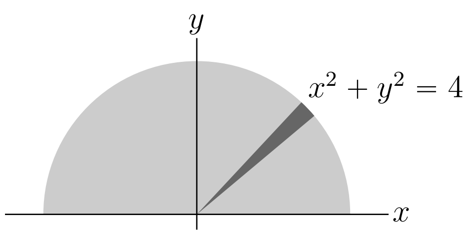
$$\text{Volume} = \int_{-2}^{2} \mathrm{d}x \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \mathrm{d}y \int_0^{e^{-x^2-y^2}} \mathrm{d}z$$

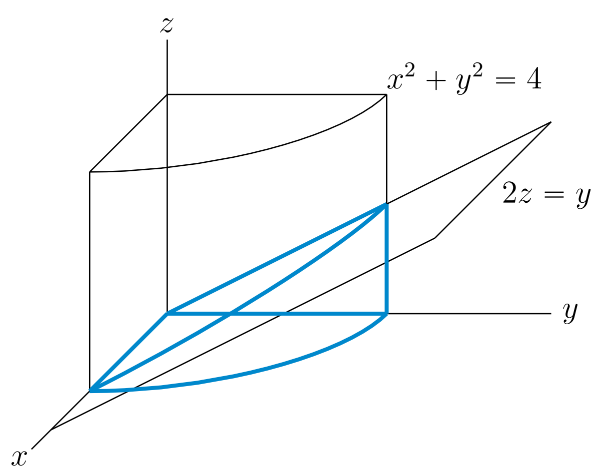
(b) Switching to cylindrical coordinates

$$\begin{aligned}\text{Volume} &= \int_0^2 \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \int_0^{e^{-r^2}} \mathrm{d}z\ r = \int_0^2 \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\ re^{-r^2} = \int_0^2 \mathrm{d}r\ 2\pi\ re^{-r^2} \\ &= -\pi e^{-r^2}\Big|_0^2 = \pi\left[1 - e^{-4}\right] \approx 3.084\end{aligned}$$

$\boxed{3.6.15}$ (*) Find the volume of the solid which is inside $x^2 + y^2 = 4$, above $z = 0$ and below $2z = y$.

$\boxed{\text{Solution}}$ The solid consists of the set of all points (x, y, z) such that $x^2 + y^2 \leqslant 4$ and $0 \leqslant z \leqslant \frac{y}{2}$. In particular $y \geqslant 0$. When we look at the solid from above, we see all (x, y) with $x^2 + y^2 \leqslant 4$ and $y \geqslant 0$. This is sketched in the figure on the left below.





We'll use cylindrical coordinates. In the base region (the shaded region in the figure on the left above)

- θ runs from 0 to π and
- for each fixed θ between 0 and π, r runs from 0 to 2.
- For each fixed point $(x, y) = (r\cos\theta, r\sin\theta)$ in the base region, z runs from 0) to $\frac{y}{2} = \frac{r\sin\theta}{2}$.

So the volume is

$$\int_0^\pi \mathrm{d}\theta \int_0^2 \mathrm{d}r \int_0^{r\sin\theta/2} dz\ r = \int_0^\pi \mathrm{d}\theta \int_0^2 \mathrm{d}r\ \frac{1}{2}r^2\sin\theta = \int_0^\pi \mathrm{d}\theta\ \frac{r^3}{6}\sin\theta\Big|_0^2 = \frac{4}{3}\int_0^\pi \mathrm{d}\theta\ \sin\theta$$
$$= -\frac{4}{3}\cos\theta\Big|_0^\pi = \frac{8}{3}$$

▸▸ Stage 3

$\boxed{3.6.16}$ (∗) The density of hydrogen gas in a region of space is given by the formula

$$\rho(x,y,z) = \frac{z+2x^2}{1+x^2+y^2}$$

(a) At $(1,0,-1)$, in which direction is the density of hydrogen increasing most rapidly?
(b) You are in a spacecraft at the origin. Suppose the spacecraft flies in the direction of $\langle 0,0,1\rangle$. It has a disc of radius 1, centred on the spacecraft and deployed perpendicular to the direction of travel, to catch hydrogen. How much hydrogen has been collected by the time that the spacecraft has traveled a distance 2?
You may use the fact that $\int_0^{2\pi}\cos^2\theta\ \mathrm{d}\theta = \pi$.

$\boxed{\text{Solution}}$ (a) The direction of maximum rate of increase is $\nabla\rho(1,0,-1)$. As

$$\begin{aligned}
\frac{\partial\rho}{\partial x}(x,y,z) &= \frac{4x}{1+x^2+y^2} - \frac{2x(z+2x^2)}{(1+x^2+y^2)^2} & \frac{\partial\rho}{\partial x}(1,0,-1) &= \frac{4}{2} - \frac{2(-1+2)}{(2)^2} = \frac{3}{2}\\
\frac{\partial\rho}{\partial y}(x,y,z) &= -\frac{2y(z+2x^2)}{(1+x^2+y^2)^2} & \frac{\partial\rho}{\partial y}(1,0,-1) &= 0\\
\frac{\partial\rho}{\partial z}(x,y,z) &= \frac{1}{1+x^2+y^2} & \frac{\partial\rho}{\partial z}(1,0,-1) &= \frac{1}{2}
\end{aligned}$$

So $\nabla\rho(1,0,-1) = \frac{1}{2}(3,0,1)$. The unit vector in this direction is $\frac{1}{\sqrt{10}}(3,0,1)$.

(b) The region swept by the space craft is, in cylindrical coordinates,

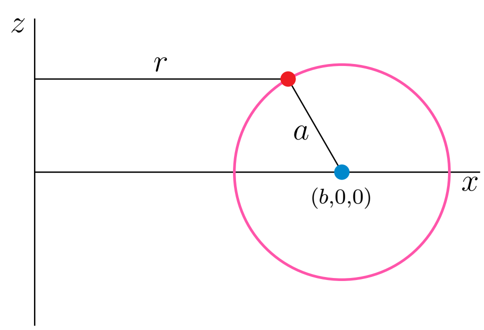
$$V = \big\{\ (r\cos\theta\,,\ r\sin\theta\,,\ z)\ \big|\ 0\leqslant\theta\leqslant 2\pi,\ 0\leqslant r\leqslant 1,\ 0\leqslant z\leqslant 2\ \big\}$$

and the amount of hydrogen collected is

$$
\begin{aligned}
\iiint_V \rho \, \mathrm{d}V &= \iiint_V \frac{z + 2r^2\cos^2\theta}{1+r^2} r\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\
&= \int_0^2 \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \mathrm{d}r \; \frac{zr + (2\cos^2\theta)r^3}{1+r^2} \\
&= \int_0^2 \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \mathrm{d}r \; \left[z\frac{r}{1+r^2} + 2r\cos^2\theta - \cos^2\theta \frac{2r}{1+r^2} \right] \\
&\qquad\qquad \text{since } \frac{r^3}{1+r^2} = \frac{r+r^3-r}{1+r^2} = r - \frac{r}{1+r^2} \\
&= \int_0^2 \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \; \left[\frac{z}{2}\ln(1+r^2) + r^2\cos^2\theta - \ln(1+r^2)\cos^2\theta \right]_0^1 \\
&= \int_0^2 \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \; \left[\frac{\ln 2}{2} z + \cos^2\theta - \ln(2)\cos^2\theta \right] \\
&= \int_0^2 \mathrm{d}z \; [(\pi \ln 2)z + \pi - \pi\ln 2] \\
&= 2\pi\ln 2 + 2\pi - 2\pi\ln 2 \\
&= 2\pi
\end{aligned}
$$

3.6.17 A torus of mass M is generated by rotating a circle of radius a about an axis in its plane at distance b from the centre $(b > a)$. The torus has constant density. Find the moment of inertia about the axis of rotation. By definition the moment of intertia is $\iiint r^2 \mathrm{d}m$ where $\mathrm{d}m$ is the mass of an infinitesimal piece of the solid and r is its distance from the axis.

Solution We may choose our coordinate axes so that the torus is constructed by rotating the circle $(x-b)^2 + z^2 = a^2$ (viewed as lying in the xz–plane) about the z–axis. On this circle, x runs from $b-a$ to $b+a$. In cylindrical coordinates, the torus has equation



$(r-b)^2 + z^2 = a^2$. (Recall that the cylindrical coordinate r of a point is its distance from the z–axis.) On this torus,

- r runs from $b-a$ to $b+a$.
- For each fixed r, z runs from $-\sqrt{a^2-(r-b)^2}$ to $\sqrt{a^2-(r-b)^2}$.

As the torus is symmetric about the xy–plane, its volume is twice that of the volume of the part with $z \geqslant 0$.

$$\begin{aligned}
\text{Volume} &= 2\int_0^{2\pi} d\theta \int_{b-a}^{b+a} \mathrm{d}r\ r \int_0^{\sqrt{a^2-(r-b)^2}} \mathrm{d}z \\
&= 2\int_0^{2\pi} d\theta \int_{b-a}^{b+a} \mathrm{d}r\ r\sqrt{a^2-(r-b)^2} \\
&= 4\pi\int_{-a}^{a} \mathrm{d}s\ (s+b)\sqrt{a^2-s^2} \qquad \text{where } s=r-b
\end{aligned}$$

As $s\sqrt{a^2-s^2}$ is odd under $s \to -s$, $\int_{-a}^{a} \mathrm{d}s\ s\sqrt{a^2-s^2} = 0$. Also, $\int_{-a}^{a} \mathrm{d}s\ \sqrt{a^2-s^2}$ is precisely the area of the top half of a circle of radius a. So

$$\text{Volume } = 4b\pi\int_{-a}^{a} \mathrm{d}s\ \sqrt{a^2-s^2} = 2\pi^2a^2b$$

So the mass density of the torus is $\frac{M}{2\pi^2a^2b}$ and $\mathrm{d}m = \frac{M}{2\pi^2a^2b}\,\mathrm{d}V = \frac{M}{2\pi^2a^2b}\,r\,\mathrm{d}r\,d\theta\,\mathrm{d}z$ and

$$\begin{aligned}
\text{moment of inertia} &= 2\int_0^{2\pi} d\theta \int_{b-a}^{b+a} \mathrm{d}r\ r \int_0^{\sqrt{a^2-(r-b)^2}} \mathrm{d}z\ \frac{M}{2\pi^2a^2b}r^2 \\
&= \frac{M}{\pi^2a^2b}\int_0^{2\pi} d\theta \int_{b-a}^{b+a} \mathrm{d}r\ r^3\sqrt{a^2-(r-b)^2} \\
&= \frac{2M}{\pi a^2b}\int_{-a}^{a} \mathrm{d}s\ (s+b)^3\sqrt{a^2-s^2} \qquad \text{where } s=r-b \\
&= \frac{2M}{\pi a^2b}\int_{-a}^{a} \mathrm{d}s\ (s^3+3s^2b+3sb+b^3)\sqrt{a^2-s^2}
\end{aligned}$$

Again, by oddness, the s^3 and $3sb$ integrals are zero. For the others, substitute in $s = a\sin t$, $\mathrm{d}s = a\cos t$.

$$\begin{aligned}
\text{moment} &= \frac{2M}{\pi a^2b}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (a\cos t\,\mathrm{d}t)\ (3a^2b\sin^2 t + b^3)a\cos t = \frac{2M}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}t\ (3a^2\sin^2 t + b^2)\cos^2 t \\
&= \frac{4M}{\pi}\int_0^{\frac{\pi}{2}} \mathrm{d}t\ (3a^2\cos^2 t - 3a^2\cos^4 t + b^2\cos^2 t) \qquad \text{since } \sin^2 t = 1-\cos^2 t
\end{aligned}$$

To integrate[3] $\cos^2 t$ and $\cos^4 t$, we use the double angle formulae $\sin^2 x = \frac{1-\cos(2x)}{2}$ and $\cos^2 x = \frac{1+\cos(2x)}{2}$ to write

$$\cos^2 t = \frac{1+\cos(2t)}{2}$$

3 For a general discussion of trigonometric integrals see §1.8 in the CLP-2 text. In particular the integral $\int\cos^4 x\ \mathrm{d}x$ is evaluated in Example 1.8.8 in the CLP-2 text. For an efficient, sneaky, way to evaluate $\int_0^{\frac{\pi}{2}} \cos^2 t\ \mathrm{d}t$ see Remark 3.3.5 in the CLP-3 text.

and

$$\begin{aligned}\cos^4 t &= \left[\frac{1+\cos(2t)}{2}\right]^2\\ &= \frac{1}{4}+\frac{1}{2}\cos(2t)+\frac{1}{4}\cos^2(2t)\\ &= \frac{1}{4}+\frac{1}{2}\cos(2t)+\frac{1}{8}\left(1+\cos(4t)\right)\\ &= \frac{3}{8}+\frac{1}{2}\cos(2t)+\frac{1}{8}\cos(4t)\end{aligned}$$

So

$$\begin{aligned}\text{moment} &= \frac{4M}{\pi}\left[3a^2\left(\frac{t}{2}+\frac{\sin(2t)}{4}\right)-3a^2\left(\frac{3t}{8}+\frac{1}{4}\sin(2t)+\frac{1}{32}\sin(4t)\right)+b^2\left(\frac{t}{2}+\frac{\sin(2t)}{4}\right)\right]_0^{\frac{\pi}{2}}\\ &= \frac{4M}{\pi}\left[3a^2\frac{\pi}{4}-3a^2\frac{3\pi}{16}+b^2\frac{\pi}{4}\right]=M\left(\frac{3}{4}a^2+b^2\right)\end{aligned}$$

3.7▲ Triple Integrals in Spherical Coordinates

▸▸ Stage 1

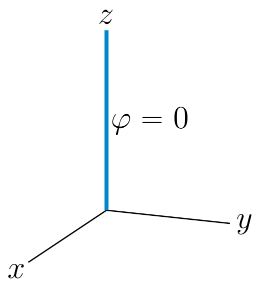
3.7.1 Use (ρ,θ,φ) to denote spherical coordinates.
(a) Draw $\varphi=0$.
(b) Draw $\varphi={}^{\pi}/_{4}$.
(c) Draw $\varphi={}^{\pi}/_{2}$.
(d) Draw $\varphi={}^{3\pi}/_{4}$.
(e) Draw $\varphi=\pi$.

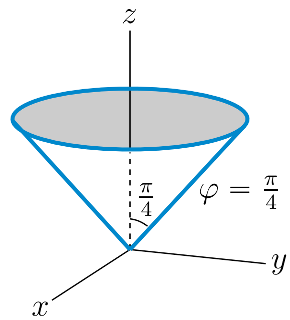
Solution Since the spherical coordinate $\varphi(x,y,z)$ of a point (x,y,z) is the angle between the positive z-axis and the radius vector from $(0,0,0)$ to (x,y,z), the sets

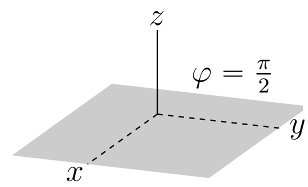
$$\begin{aligned}\left\{\ (x,y,z)\ \middle|\ \varphi(x,y,z)=0\ \right\} &= \text{the positive } z\text{-axis}\\ \left\{\ (x,y,z)\ \middle|\ \varphi(x,y,z)={}^{\pi}/_{2}\ \right\} &= \text{the } xy\text{-plane}\\ \left\{\ (x,y,z)\ \middle|\ \varphi(x,y,z)=\pi\ \right\} &= \text{the negative } z\text{-axis}\end{aligned}$$

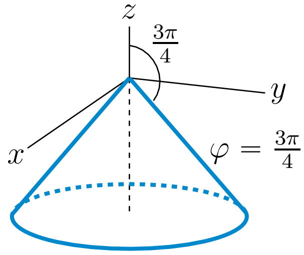
Alternatively, $\tan\varphi(x,y,z)=\frac{z}{\sqrt{x^2+y^2}}$, so that, for any $0<\Phi<\pi$,

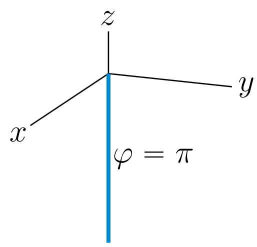
$$\begin{aligned}\left\{\ (x,y,z)\ \middle|\ \varphi(x,y,z)=\Phi\ \right\} &= \left\{\ (x,y,z)\ \middle|\ z=\tan\Phi\sqrt{x^2+y^2}\ \right\}\\ &= \text{the cone that makes the angle } \Phi \text{ with the positive } z\text{-axis}\end{aligned}$$









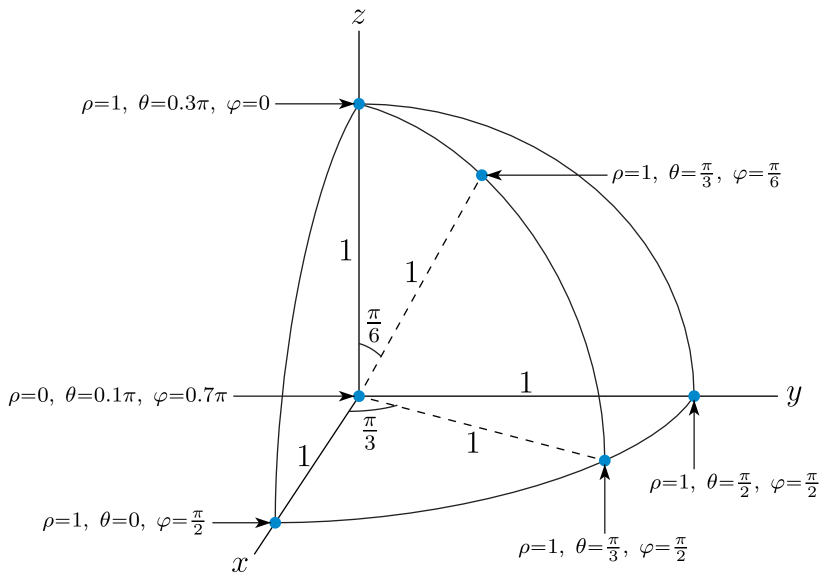


3.7.2 Sketch the point with the specified spherical coordinates.
(a) $\rho = 0$, $\theta = 0.1\pi$, $\varphi = 0.7\pi$
(b) $\rho = 1$, $\theta = 0.3\pi$, $\varphi = 0$
(c) $\rho = 1$, $\theta = 0$, $\varphi = \frac{\pi}{2}$
(d) $\rho = 1$, $\theta = \frac{\pi}{3}$, $\varphi = \frac{\pi}{2}$
(e) $\rho = 1$, $\theta = \frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$
(f) $\rho = 1$, $\theta = \frac{\pi}{3}$, $\varphi = \frac{\pi}{6}$

Solution The sketch is below. To help build up this sketch, it is useful to recall the following facts.

- The spherical coordinate ρ is the distance of the point from the origin $(0, 0, 0)$. In particular if $\rho = 0$, then the point is the origin (regardless of the values of θ and φ). If $\rho = 1$ then the point lies on the sphere of radius 1 centred on the origin.

- The spherical coordinate φ is the angle between the positive z-axis and the radial line segment from the origin to (x, y, z). In particular, all points with $\varphi = 0$ lie on the positive z–axis (regardless of the value of θ). All points with $\varphi = \frac{\pi}{2}$ lie in the xy-plane.



$\boxed{3.7.3}$ Convert from Cartesian to spherical coordinates.

(a) $(-2, 0, 0)$

(b) $(0, 3, 0)$

(c) $(0, 0, -4)$

(d) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{3}\right)$

$\boxed{\text{Solution}}$ (a) The point $(-2, 0, 0)$

- lies in the xy-plane (i.e. has $z = \rho\cos\varphi = 0$) and so has $\varphi = \frac{\pi}{2}$ and
- lies on the negative x-axis and so has $\theta = \pi$ and
- is a distance 2 from the origin and so has $\rho = 2$.

(b) The point $(0, 3, 0)$

- lies in the xy-plane (i.e. has $z = \rho\cos\varphi = 0$) and so has $\varphi = \frac{\pi}{2}$ and
- lies on the positive y-axis and so has $\theta = \frac{\pi}{2}$ and
- is a distance 3 from the origin and so has $\rho = 3$.

(c) The point $(0, 0, -4)$

- lies on the negative z-axis and so has $\varphi = \pi$ and θ arbitrary and
- is a distance 4 from the origin and so has $\rho = 4$.

(d) The point $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{3}\right)$

- has $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$ and
- has $\sqrt{3} = z = \rho\cos\varphi = 2\cos\varphi$ so that $\cos\varphi = \frac{\sqrt{3}}{2}$ and $\varphi = \frac{\pi}{6}$ and

- has $-\frac{1}{\sqrt{2}} = x = \rho \sin\varphi \cos\theta = 2\big(\frac{1}{2}\big)\cos\theta$ so that $\cos\theta = -\frac{1}{\sqrt{2}}$. As $\Big(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big)$ is in the second quadrant, we have $\frac{\pi}{2} \leqslant \theta \leqslant \pi$ and so $\theta = \frac{3\pi}{4}$.

$\boxed{3.7.4}$ Convert from spherical to Cartesian coordinates.
(a) $\rho = 1$, $\theta = \frac{\pi}{3}$, $\varphi = \frac{\pi}{6}$
(b) $\rho = 2$, $\theta = \frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$

$\boxed{\text{Solution}}$ (a) The Cartesian coordinates corresponding to $\rho = 1$, $\theta = \frac{\pi}{3}$, $\varphi = \frac{\pi}{6}$ are

$$x = \rho \sin\varphi \cos\theta = \sin\frac{\pi}{6}\cos\frac{\pi}{3} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$
$$y = \rho \sin\varphi \sin\theta = \sin\frac{\pi}{6}\sin\frac{\pi}{3} = \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$$
$$z = \rho\cos\varphi = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

(b) The Cartesian coordinates corresponding to $\rho = 2$, $\theta = \frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$ are

$$x = \rho \sin\varphi \cos\theta = 2\sin\frac{\pi}{2}\cos\frac{\pi}{2} = 0$$
$$y = \rho \sin\varphi \sin\theta = 2\sin\frac{\pi}{2}\sin\frac{\pi}{2} = 2$$
$$z = \rho\cos\varphi = 2\cos\frac{\pi}{2} = 0$$

Alternatively, we could just observe that

- as $\varphi = \frac{\pi}{2}$ the point lies in the xy-plane and so has $z = 0$ and
- as $\rho = 2$, $\theta = \frac{\pi}{2}$ the point lies on the positive y-axis and is a distance 2 from the origin and so is $(0, 2, 0)$.

$\boxed{3.7.5}$ Rewrite the following equations in spherical coordinates.
(a) $z^2 = 3x^2 + 3y^2$
(b) $x^2 + y^2 + (z-1)^2 = 1$
(c) $x^2 + y^2 = 4$

$\boxed{\text{Solution}}$ (a) In spherical coordinates

$$\begin{aligned} z^2 = 3x^2 + 3y^2 &\iff \rho^2\cos^2\varphi = 3\rho^2\sin^2\varphi\cos^2\theta + 3\rho^2\sin^2\varphi\sin^2\theta = 3\rho^2\sin^2\varphi \\ &\iff \tan^2\varphi = \frac{1}{3} \iff \tan\varphi = \pm\frac{1}{\sqrt{3}} \\ &\iff \varphi = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \end{aligned}$$

The surface $z^2 = 3x^2 + 3y^2$ is a cone. The upper half of the cone, i.e. the part with $z \geqslant 0$, is $\varphi = \frac{\pi}{6}$. The lower half of the cone, i.e. the part with $z \leqslant 0$, is $\varphi = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

(b) In spherical coordinates

$$\begin{aligned} x^2+y^2+(z-1)^2=1 &\iff \rho^2\sin^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\sin^2\theta+\big(\rho\cos\varphi-1\big)^2=1 \\ &\iff \rho^2\sin^2\varphi+\rho^2\cos^2\varphi-2\rho\cos\varphi=0 \\ &\iff \rho^2-2\rho\cos\varphi=0 \\ &\iff \rho=2\cos\varphi \end{aligned}$$

(c) In spherical coordinates

$$\begin{aligned} x^2+y^2=4 &\iff \rho^2\sin^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\sin^2\theta=4 \\ &\iff \rho^2\sin^2\varphi=4 \\ &\iff \rho\sin\varphi=2 \end{aligned}$$

since $\rho\geqslant 0$ and $0\leqslant\varphi\leqslant\pi$ so that $\sin\varphi\geqslant 0$.

$\boxed{3.7.6}$ (∗) Using spherical coordinates and integration, show that the volume of the sphere of radius 1 centred at the origin is $4\pi/3$.

$\boxed{\text{Solution}}$ In spherical coordinates, the sphere in question is

$$B=\left\{\ (\rho\sin\varphi\cos\theta\,,\ \rho\sin\varphi\sin\theta\,,\ \rho\cos\varphi)\ \middle|\ 0\leqslant\rho\leqslant 1\,,\ 0\leqslant\varphi\leqslant\pi\,,\ 0\leqslant\theta\leqslant 2\pi\ \right\}$$

As $\mathrm{d}V=\rho^2\sin\varphi\ \mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta$,

$$\begin{aligned} \text{Volume}(S)=\iiint_B \mathrm{d}V &= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\varphi\int_0^1\mathrm{d}\rho\ \rho^2\sin\varphi \\ &= \left[\int_0^{2\pi}\mathrm{d}\theta\right]\left[\int_0^{\pi}\mathrm{d}\varphi\ \sin\varphi\right]\left[\int_0^1\mathrm{d}\rho\ \rho^2\right] \\ &= 2\pi\Big[-\cos\varphi\Big]_0^{\pi}\left[\frac{\rho^3}{3}\right]_0^1=(2\pi)(2)\left(\frac{1}{3}\right) \\ &= \frac{4\pi}{3} \end{aligned}$$

▸▸ Stage 2

$\boxed{3.7.7}$ (∗) Consider the region E in 3–dimensions specified by the spherical inequalities

$$1\leqslant\rho\leqslant 1+\cos\varphi$$

(a) Draw a reasonably accurate picture of E in 3–dimensions. Be sure to show the units on the coordinates axes.
(b) Find the volume of E.

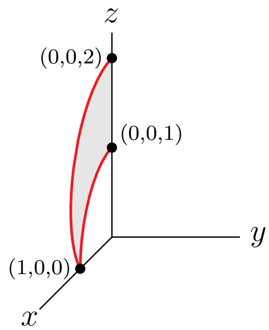
$\boxed{\text{Solution}}$ (a) First observe that both boundaries of E, namely $\rho = 1$ and $\rho = 1 + \cos\varphi$, are independent of the spherical coordinate θ. So E is invariant under rotations about the z–axis. To sketch E we

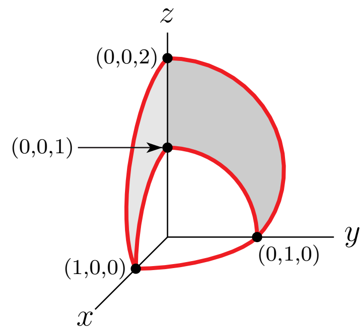
- first sketch the part of the boundary of E with $\theta = 0$ (i.e. in the half of the xz–plane with $x > 0$), and then
- rotate about the z–axis.

The part of the boundary of E with $\theta = 0$ (i.e. in the half–plane $y = 0$, $x \geqslant 0$), consists of two curves.

- $\rho = 1 + \cos\varphi$, $\theta = 0$:
  - When $\varphi = 0$ (i.e. on the positive z–axis), We have $\cos\varphi = 1$ and hence $\rho = 2$. So this curve starts at $(0, 0, 2)$.
  - As φ increases $\cos\varphi$, and hence ρ, decreases.
  - When φ is $\frac{\pi}{2}$ (i.e. in the xy–plane), we have $\cos\varphi = 0$ and hence $\rho = 1$.
  - When $\frac{\pi}{2} < \varphi \leqslant \pi$, we have $\cos\varphi < 0$ and hence $\rho < 1$. All points in E are required to obey $\rho \geqslant 1$. So this part of the boundary stops at the point $(1, 0, 0)$ in the xy–plane.
  - The curve $\rho = 1 + \cos\varphi$, $\theta = 0$, $0 \leqslant \varphi \leqslant \frac{\pi}{2}$ is sketched in the figure on the left below. It is the outer curve from $(0, 0, 2)$ to $(1, 0, 0)$.
- $\rho = 1$, $\theta = 0$:
  - The surface $\rho = 1$ is the sphere of radius 1 centred on the origin.
  - As we observed above, the conditions $1 \leqslant \rho \leqslant 1 + \cos\varphi$ force $0 \leqslant \varphi \leqslant \frac{\pi}{2}$, i.e. $z \geqslant 0$.
  - The sphere $\rho = 1$ intersects the quarter plane $y = 0$, $x \geqslant 0$, $z \geqslant 0$, in the quarter circle centred on the origin that starts at $(0, 0, 1)$ on the z–axis and ends at $(1, 0, 0)$ in the xy–plane.
  - The curve $\rho = 1$, $\theta = 0$, $0 \leqslant \varphi \leqslant \frac{\pi}{2}$ is sketched in the figure on the left below. It is the inner curve from $(0, 0, 1)$ to $(1, 0, 0)$.

To get E, rotate the shaded region in the figure on the left below about the z-axis. The part of E in the first octant is sketched in the figure on the right below. The part of E in the xz–plane (with $x \geqslant 0$) is lightly shaded and the part of E in the yz–plane (with $y \geqslant 0$) is shaded a little more darkly.





(b) In E

- φ runs from 0 (i.e. the positive z–axis) to $\frac{\pi}{2}$ (i.e. the xy–plane).
- For each φ in that range ρ runs from 1 to $1+\cos\varphi$ and θ runs from 0 to 2π.
- In spherical coordinates $\mathrm{d}V = \rho^2\, \sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$.

So

$$\begin{aligned}
\text{Volume}(E) &= \int_0^{\pi/2} \mathrm{d}\varphi \int_1^{1+\cos\varphi} \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta\ \rho^2 \sin\varphi \\
&= 2\pi \int_0^{\pi/2} \mathrm{d}\varphi\ \sin\varphi\ \frac{(1+\cos\varphi)^3 - 1^3}{3} \\
&= -\frac{2\pi}{3}\int_2^1 \left(u^3-1\right)\,\mathrm{d}u \qquad \text{with } u = 1+\cos\varphi,\ \mathrm{d}u = -\sin\varphi\,\mathrm{d}\varphi \\
&= -\frac{2\pi}{3}\left[\frac{u^4}{4} - u\right]_2^1 \\
&= -\frac{2\pi}{3}\left[\frac{1}{4} - 1 - 4 + 2\right] \\
&= \frac{11\pi}{6}
\end{aligned}$$

$\boxed{3.7.8}$ (∗) Use spherical coordinates to evaluate the integral

$$I = \iiint_D z\ \mathrm{d}V$$

where D is the solid enclosed by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 4$. That is, (x, y, z) is in D if and only if $\sqrt{x^2+y^2} \leqslant z$ and $x^2+y^2+z^2 \leqslant 4$.

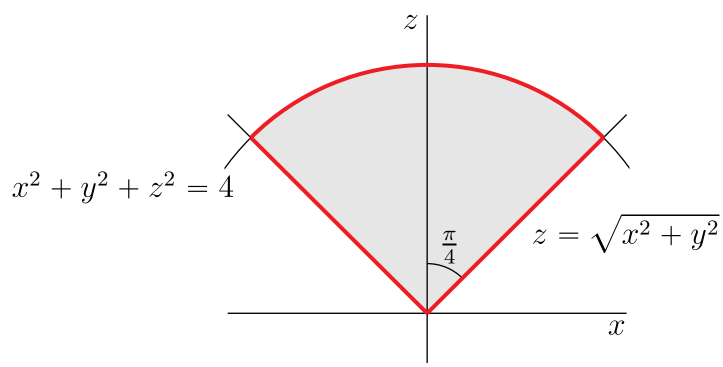
$\boxed{\text{Solution}}$ Recall that in spherical coordinates,

$$\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ y &= \rho \sin\varphi \sin\theta \\ z &= \rho \cos\varphi \\ x^2 + y^2 &= \rho^2 \sin^2\varphi \end{aligned}$$

so that $x^2 + y^2 + z^2 = 4$ becomes $\rho = 2$, and $\sqrt{x^2+y^2} = z$ becomes

$$\rho \sin\varphi = \rho\cos\varphi \iff \tan\varphi = 1 \iff \varphi = \frac{\pi}{4}$$

Here is a sketch of the $y = 0$ cross–section of D.



Looking at the figure above, we see that, on D

- φ runs from 0 (the positive z–axis) to $\frac{\pi}{4}$ (on the cone), and
- for each φ is that range, ρ runs from 0 to 2 and θ runs from 0 to 2π.

So

$$D = \left\{ \ (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) \ \middle| \ 0 \leqslant \varphi \leqslant \pi/4,\ 0 \leqslant \theta \leqslant 2\pi,\ \rho \leqslant 2 \ \right\}$$

and, as $\mathrm{d}V = \rho^2 \sin\varphi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\varphi$,

$$\begin{aligned} I &= \int_0^{\pi/4} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^2 \mathrm{d}\rho \ \rho^2 \sin\varphi \ \overbrace{\rho\cos\varphi}^{z} \\ &= \int_0^{\pi/4} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^2 \mathrm{d}\rho \ \rho^3 \sin\varphi \ \cos\varphi \\ &= 2\pi \ \frac{2^4}{4} \int_0^{\pi/4} \mathrm{d}\varphi \ \sin\varphi \ \cos\varphi \\ &= 2\pi \ \frac{2^4}{4} \left[\frac{\sin^2\varphi}{2} \right]_0^{\pi/4} \\ &= 2\pi \end{aligned}$$

3.7.9 Use spherical coordinates to find

(a) The volume inside the cone $z = \sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2=a^2$.

(b) $\iiint_R x\,\mathrm{d}V$ and $\iiint_R z\,\mathrm{d}V$ over the part of the sphere of radius a that lies in the first octant.

(c) The mass of a spherical planet of radius a whose density at distance ρ from the center is $\delta = A/(B+\rho^2)$.

(d) The volume enclosed by $\rho = a(1-\cos\varphi)$. Here ρ and φ refer to the usual spherical coordinates.

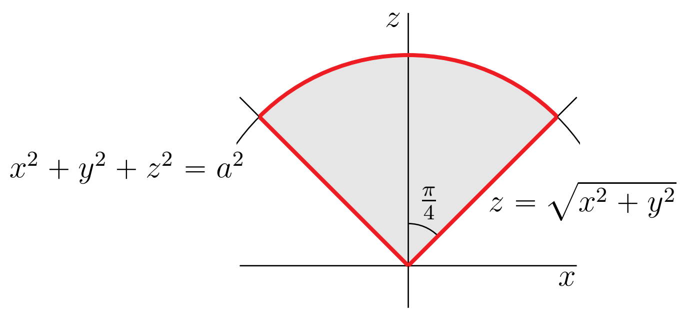
Solution (a) Recall that in spherical coordinates,

$$\begin{aligned} x &= \rho\sin\varphi\cos\theta \\ y &= \rho\sin\varphi\sin\theta \\ z &= \rho\cos\varphi \\ x^2+y^2 &= \rho^2\sin^2\varphi \end{aligned}$$

so that $x^2+y^2+z^2=a^2$ becomes $\rho=a$, and $\sqrt{x^2+y^2}=z$ becomes

$$\rho\sin\varphi = \rho\cos\varphi \iff \tan\varphi = 1 \iff \varphi = \frac{\pi}{4}$$

Here is a sketch of the $y=0$ cross–section of the specified region.



Looking at the figure above, we see that, on that region,

- φ runs from 0 (the positive z–axis) to $\frac{\pi}{4}$ (on the cone), and
- for each φ is that range, ρ runs from 0 to a and θ runs from 0 to 2π.

so that

$$\begin{aligned} \text{Volume} &= \int_0^a \mathrm{d}\rho \int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{4}}\mathrm{d}\varphi\ \rho^2\sin\varphi = \Big\{\int_0^a \mathrm{d}\rho\ \rho^2\Big\}\Big\{\int_0^{2\pi}\mathrm{d}\theta\Big\}\Big\{\int_0^{\frac{\pi}{4}}\mathrm{d}\varphi\ \sin\varphi\Big\} \\ &= \frac{a^3}{3}\,2\pi\,\Big[-\cos\varphi\Big]_0^{\frac{\pi}{4}} = 2\pi\frac{a^3}{3}\left(1-\frac{1}{\sqrt{2}}\right) \end{aligned}$$

(b) The part of the sphere in question is

$$\begin{aligned} R &= \big\{\ (x,y,z)\ \big|\ x^2+y^2+z^2\leqslant a^2,\ x\geqslant 0,\ y\geqslant 0,\ z\geqslant 0\ \big\} \\ &= \big\{\ (\rho\sin\varphi\cos\theta\,,\,\rho\sin\varphi\sin\theta\,,\,\rho\cos\varphi)\ \big|\ \rho\leqslant a,\ 0\leqslant\varphi\leqslant\tfrac{\pi}{2},\ 0\leqslant\theta\leqslant\tfrac{\pi}{2}\ \big\} \end{aligned}$$

By symmetry, the two specified integrals are equal, and are

$$\begin{aligned}\int_0^a \mathrm{d}\rho\ \rho^2 \int_0^{\frac{\pi}{2}} \mathrm{d}\varphi\ \sin\varphi \int_0^{\frac{\pi}{2}} \mathrm{d}\theta\ \overbrace{\rho\cos\varphi}^{z} &= \frac{a^4}{4}\frac{\pi}{2}\int_0^{\frac{\pi}{2}} \mathrm{d}\varphi\ \sin\varphi\cos\varphi \\ &= \frac{\pi a^4}{8}\int_0^1 \mathrm{d}t\ t \qquad \text{where } t=\sin\varphi,\ \mathrm{d}t=\cos\varphi\,\mathrm{d}\varphi \\ &= \frac{\pi a^4}{16}\end{aligned}$$

(c) The planet in question is

$$\begin{aligned}P &= \left\{\ (x,y,z)\ \middle|\ x^2+y^2+z^2 \leqslant a^2\ \right\} \\ &= \left\{\ (\rho\sin\varphi\cos\theta\,,\ \rho\sin\varphi\sin\theta\,,\ \rho\cos\varphi)\ \middle|\ \rho\leqslant a,\ 0\leqslant\varphi\leqslant\pi,\ 0\leqslant\theta\leqslant 2\pi\ \right\}\end{aligned}$$

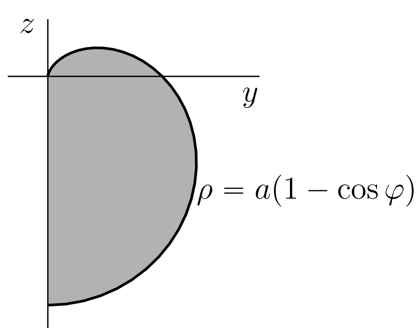
So the

$$\begin{aligned}\text{mass} &= \int_0^a \mathrm{d}\rho\ \rho^2\int_0^\pi \mathrm{d}\varphi\ \sin\varphi\int_0^{2\pi}\mathrm{d}\theta\ \overbrace{\frac{A}{B+\rho^2}}^{\text{density}} = 2\pi A\Big\{\int_0^\pi \mathrm{d}\varphi\ \sin\varphi\Big\}\Big\{\int_0^a \mathrm{d}\rho\ \frac{\rho^2}{B+\rho^2}\Big\} \\ &= 4\pi A\int_0^a \mathrm{d}\rho\ \left(1-\frac{B}{B+\rho^2}\right) \\ &= 4\pi Aa - 4\pi A\sqrt{B}\int_0^{a/\sqrt{B}} \mathrm{d}s\ \frac{1}{1+s^2} \qquad \text{where } \rho=\sqrt{B}\,s,\ \mathrm{d}\rho=\sqrt{B}\,\mathrm{d}s \\ &= 4\pi A\left(a-\sqrt{B}\tan^{-1}\frac{a}{\sqrt{B}}\right)\end{aligned}$$

(d) Observe that

- when $\varphi=0$ (i.e. on the positive z–axis), $\cos\varphi=1$ so that $\rho=a(1-\cos\varphi)=0$ and
- as φ increases from 0 to $\frac{\pi}{2}$, $\cos\varphi$ decreases so that $\rho=a(1-\cos\varphi)$ increases and
- when $\varphi=\frac{\pi}{2}$ (i.e. on the xy–plane), $\cos\varphi=0$ so that $\rho=a(1-\cos\varphi)=a$ and
- as φ increases from $\frac{\pi}{2}$ to π, $\cos\varphi$ continues to decrease so that $\rho=a(1-\cos\varphi)$ increases still more and
- when $\varphi=\pi$ (i.e. on the negative z–axis), $\cos\varphi=-1$ so that $\rho=a(1-\cos\varphi)=2a$

So we have the following sketch of the intersection of the specified volume with the right half of the yz–plane.



The volume in question is invariant under rotations about the z–axis so that

$$\begin{aligned}
\text{Volume} &= \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi\ \sin\varphi \int_0^{a(1-\cos\varphi)} d\rho\ \rho^2 \\
&= 2\pi\frac{a^3}{3}\int_0^{\pi} d\varphi\ \sin\varphi(1-\cos\varphi)^3 \\
&= 2\pi\frac{a^3}{3}\int_0^{2} dt\ t^3 \qquad \text{where } t = 1-\cos\varphi,\ dt = \sin\varphi\, d\varphi \\
&= 2\pi\ \frac{a^3}{3}\ \frac{2^4}{4} = \frac{8}{3}\pi a^3
\end{aligned}$$

$\boxed{3.7.10}$ (*) Consider the hemispherical shell bounded by the spherical surfaces

$$x^2+y^2+z^2 = 9 \qquad \text{and} \qquad x^2+y^2+z^2 = 4$$

and above the plane $z = 0$. Let the shell have constant density D.
(a) Find the mass of the shell.
(b) Find the location of the center of mass of the shell.

$\boxed{\text{Solution}}$ Let's use H to denote the hemispherical shell. On that shell, the spherical coordinate φ runs from 0 (on the z–axis) to $\pi/2$ (on the xy–plane, $z = 0$) and the spherical coordinate ρ runs from 2, on $x^2+y^2+z^2 = 4$, to 3, on $x^2+y^2+z^2 = 9$. So, in spherical coordinates,

$$H = \left\{\ (\rho\sin\varphi\cos\theta\,,\,\rho\sin\varphi\sin\theta\,,\,\rho\cos\varphi)\ \middle|\ 2\leqslant\rho\leqslant 3,\ 0\leqslant\varphi\leqslant\pi/2,\ 0\leqslant\theta\leqslant 2\pi\ \right\}$$

(a) In spherical coordinates $dV = \rho^2\sin\varphi\ d\rho\, d\varphi\, d\theta$, so that, as the density is the constant D,

$$\begin{aligned}
\text{Mass}(H) &= \int_2^3 d\rho\int_0^{2\pi} d\theta\int_0^{\pi/2} d\varphi\ D\ \rho^2\sin\varphi \\
&= D\left[\int_2^3 d\rho\ \rho^2\right]\left[\int_0^{2\pi} d\theta\right]\left[\int_0^{\pi/2} d\varphi\ \sin\varphi\right] \\
&= D\left[\frac{3^3}{3}-\frac{2^3}{3}\right]\ [2\pi]\ \big[\cos 0-\cos(\pi/2)\big] \\
&= \frac{38}{3}\pi D
\end{aligned}$$

We could have gotten the same result by expressing the mass as

- one half, times
- the density D, times
- the difference between the volume of a sphere of radius 3 and a sphere of radius 2.

That is

$$\text{Mass}(H) = \frac{1}{2}D\left[\frac{4}{3}\pi 3^3 - \frac{4}{3}\pi 2^3\right] = \frac{38}{3}\pi D$$

(b) By definition, the centre of mass is $(\bar{x}, \bar{y}, \bar{z})$ where $\bar{x}$, $\bar{y}$ and $\bar{z}$ are the weighted averages of x, y and z, respectively, over H. That is

$$\bar{x} = \frac{\iiint_H x\,D\,\mathrm{d}V}{\iiint_H D\,\mathrm{d}V} \qquad \bar{y} = \frac{\iiint_H y\,D\,\mathrm{d}V}{\iiint_H D\,\mathrm{d}V} \qquad \bar{z} = \frac{\iiint_H z\,D\,\mathrm{d}V}{\iiint_H D\,\mathrm{d}V}$$

As H is invariant under reflection in the yz–plane (i.e. under $x \to -x$) we have $\bar{x} = 0$. As H is also invariant under reflection in the xz–plane (i.e. under $y \to -y$) we have $\bar{y} = 0$. So we just have to find $\bar{z}$. We have already found the denominator in part (a), so we just have evaluate the numerator

$$\begin{aligned}
\iiint_H z\,D\,\mathrm{d}V &= \int_2^3 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/2} \mathrm{d}\varphi\ D\ \rho^2 \sin\varphi\ \overbrace{\rho\cos\varphi}^{z} \\
&= D\left[\int_2^3 \mathrm{d}\rho\ \rho^3\right]\left[\int_0^{2\pi} \mathrm{d}\theta\right]\left[\int_0^{\pi/2} \mathrm{d}\varphi\ \sin\varphi\ \cos\varphi\right] \\
&= D\left[\frac{3^4}{4} - \frac{2^4}{4}\right]\ [2\pi]\ \left[\frac{1}{2}\sin^2\frac{\pi}{2} - \frac{1}{2}\sin^2 0\right] \\
&= \frac{81-16}{4}\pi D = \frac{65}{4}\pi D
\end{aligned}$$

All together

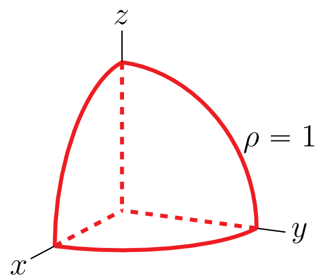
$$\bar{x} = \bar{y} = 0 \qquad \bar{z} = \frac{\frac{65}{4}\pi D}{\frac{38}{3}\pi D} = \frac{195}{152} \approx 1.28$$

$\boxed{3.7.11}$ $(*)$ Let

$$I = \iiint_T xz\ \mathrm{d}V$$

where T is the eighth of the sphere $x^2 + y^2 + z^2 \leqslant 1$ with $x, y, z \geqslant 0$.
(a) Sketch the volume T.
(b) Express I as a triple integral in spherical coordinates.
(c) Evaluate I by any method.

$\boxed{\text{Solution}}$ (a) Here is a sketch



(b) On T,

- the spherical coordinate φ runs from 0 (the positive z–xis) to $\pi/2$ (the xy–plane), and
- for each fixed φ in that range, θ runs from 0 to $\pi/2$, and
- for each fixed φ and θ, the spherical coordinate ρ runs from 0 to 1.
- In spherical coordinates $\mathrm{d}V = \rho^2\,\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$ and

$$xz = \big(\rho\sin\varphi\cos\theta\big)\big(\rho\cos\varphi\big) = \rho^2\sin\varphi\ \cos\varphi\ \cos\theta$$

So

$$I = \int_0^{\pi/2}\mathrm{d}\varphi\int_0^{\pi/2}\mathrm{d}\theta\int_0^1\mathrm{d}\rho\ \rho^4\sin^2\varphi\cos\varphi\ \cos\theta$$

(c) In spherical coordinates,

$$\begin{aligned} I &= \left[\int_0^{\pi/2}\mathrm{d}\varphi\ \sin^2\varphi\cos\varphi\right]\left[\int_0^{\pi/2}\mathrm{d}\theta\ \cos\theta\right]\left[\int_0^1\mathrm{d}\rho\ \rho^4\right] \\ &= \left[\frac{\sin^3\varphi}{3}\right]_0^{\pi/2}\ [\sin\theta]_0^{\pi/2}\left[\frac{\rho^5}{5}\right]_0^1 \\ &= \frac{1}{15} \end{aligned}$$

$\boxed{3.7.12}$ (∗) Evaluate $W = \iiint_Q xz\ \mathrm{d}V$, where Q is an eighth of the sphere $x^2+y^2+z^2 \leqslant 9$ with $x,\ y,\ z \geqslant 0$.

$\boxed{\text{Solution}}$ We'll use spherical coordinates. On Q,

- the spherical coordinate φ runs from 0 (the positive z–axis) to $\frac{\pi}{2}$ (the xy–plane),
- the spherical coordinate θ runs from 0 (the half of the xz–plane with $x \geqslant 0$) to $\frac{\pi}{2}$ (the half of the yz–plane with $y \geqslant 0$) and
- the spherical coordinate ρ runs from 0 to 3.

As $\mathrm{d}V = \rho^2 \sin\varphi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\varphi$,

$$\begin{aligned}
W = \iiint_Q xz \, \mathrm{d}V &= \int_0^3 \mathrm{d}\rho \int_0^{\pi/2} \mathrm{d}\theta \int_0^{\pi/2} \mathrm{d}\varphi \; \rho^2 \sin\varphi \; \overbrace{\rho\sin\varphi\cos\theta}^{x} \, \overbrace{\rho\cos\varphi}^{z} \\
&= \int_0^3 \mathrm{d}\rho \int_0^{\pi/2} \mathrm{d}\theta \; \rho^4 \cos\theta \left[\frac{\sin^3\varphi}{3}\right]_{\varphi=0}^{\varphi=\pi/2} \\
&= \frac{1}{3}\int_0^3 \mathrm{d}\rho \; \rho^4 \Big[\sin\theta\Big]_0^{\pi/2} \\
&= \frac{3^5}{15} = \frac{81}{5}
\end{aligned}$$

$\boxed{3.7.13}$ (*) Evaluate $\iiint_{\mathbb{R}^3} \left[1 + (x^2+y^2+z^2)^3\right]^{-1} \, \mathrm{d}V$.

$\boxed{\text{Solution}}$ Let's use spherical coordinates. This is an improper integral. So, to be picky, we'll take the limit as $R \to \infty$ of the integral over $0 \leqslant \rho \leqslant R$.

$$\begin{aligned}
\iiint_{\mathbb{R}^3} \left[1 + (x^2+y^2+z^2)^3\right]^{-1} \, \mathrm{d}V &= \lim_{R\to\infty} \int_0^R \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \mathrm{d}\varphi \; \rho^2 \sin\varphi \; \frac{1}{1+\rho^6} \\
&= \lim_{R\to\infty} \int_0^R \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \; \frac{\rho^2}{1+\rho^6} \Big[-\cos\varphi\Big]_0^{\pi} \\
&= 4\pi \lim_{R\to\infty} \int_0^R \mathrm{d}\rho \; \frac{\rho^2}{1+\rho^6} \\
&= \frac{4\pi}{3} \lim_{R\to\infty} \int_0^{R^3} \mathrm{d}u \; \frac{1}{1+u^2} \qquad \text{with } u = \rho^3, \; \mathrm{d}u = 3\rho^2 \, \mathrm{d}\rho \\
&= \frac{4\pi}{3} \lim_{R\to\infty} \Big[\arctan u\Big]_0^{R^3} \\
&= \frac{2\pi^2}{3} \qquad \text{since } \lim_{R\to\infty} \arctan R^3 = \frac{\pi}{2}
\end{aligned}$$

$\boxed{3.7.14}$ (*) Evaluate

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{1-\sqrt{1-x^2-y^2}}^{1+\sqrt{1-x^2-y^2}} (x^2+y^2+z^2)^{5/2} \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x$$

by changing to spherical coordinates.

$\boxed{\text{Solution}}$ On the domain of integration

- x runs from -1 to 1.

- For each fixed x in that range, y runs from $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$. In inequalities, that is $-\sqrt{1-x^2} \leqslant y \leqslant \sqrt{1-x^2}$, which is equivalent to $x^2+y^2 \leqslant 1$.
- For each fixed (x,y) obeying $x^2+y^2 \leqslant 1$, z runs from $1-\sqrt{1-x^2-y^2}$ to $1+\sqrt{1-x^2-y^2}$. In inequalities, that is $1-\sqrt{1-x^2-y^2} \leqslant z \leqslant 1+\sqrt{1-x^2-y^2}$, which is equivalent to $x^2+y^2+(z-1)^2 \leqslant 1$.

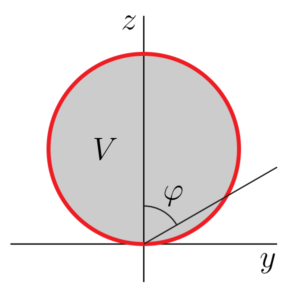
So the domain of integration is

$$V = \left\{ \ (x,y,z) \ \middle| \ x^2+y^2+(z-1)^2 \leqslant 1 \ \right\}$$

In spherical coordinates, the condition $x^2+y^2+(z-1)^2 \leqslant 1$ is

$$\begin{aligned} &(\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2 + (\rho\cos\varphi-1)^2 \leqslant 1 \\ \iff\ & \rho^2\sin^2\varphi + (\rho\cos\varphi-1)^2 \leqslant 1 \\ \iff\ & \rho^2\sin^2\varphi + \rho^2\cos^2\varphi - 2\rho\cos\varphi + 1 \leqslant 1 \\ \iff\ & \rho^2 \leqslant 2\rho\cos\varphi \\ \iff\ & \rho \leqslant 2\cos\varphi \end{aligned}$$

Note that V is contained in the upper half, $z \geqslant 0$, of $\mathbb{R}^3$ and that the xy–plane in tangent to V. So as (x,y,z) runs over V, the spherical coordinate φ runs from 0 (the positive z–axis) to $\pi/2$ (the xy–plane). Here is a sketch of the side view of V.

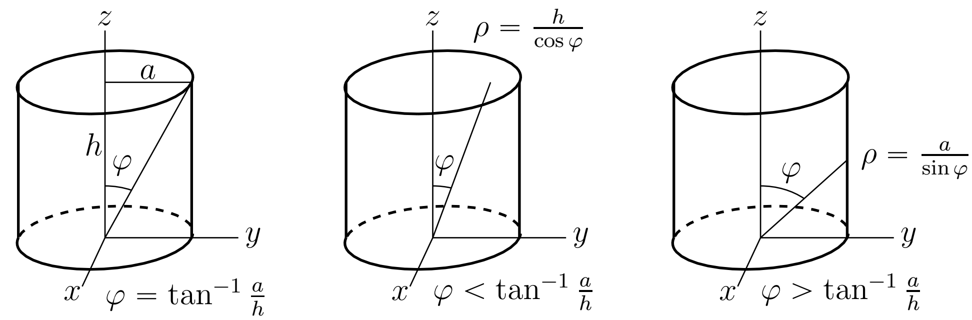


As $\mathrm{d}V = \rho^2\sin\varphi\ \mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta$ and $\big(x^2+y^2+z^2\big)^{5/2} = \rho^5$, the integral is

$$\begin{aligned} \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{1-\sqrt{1-x^2-y^2}}^{1+\sqrt{1-x^2-y^2}} (x^2+y^2+z^2)^{5/2}\ \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x &= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi/2}\mathrm{d}\varphi\int_0^{2\cos\varphi}\mathrm{d}\rho\ \rho^2\sin\varphi\ \rho^5 \\ &= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi/2}\mathrm{d}\varphi\ \frac{2^8\cos^8\varphi}{8}\sin\varphi \\ &= 32\int_0^{2\pi}\mathrm{d}\theta\ \left[-\frac{\cos^9\varphi}{9}\right]_0^{\pi/2} \\ &= \frac{32}{9}(2\pi) = \frac{64\pi}{9} \end{aligned}$$

$\boxed{3.7.15}$ Evaluate the volume of a circular cylinder of radius a and height h by means of an integral in spherical coordinates.

Solution The top of the cylinder has equation $z = h$, i.e. $\rho\cos\varphi = h$. The side of the cylinder has equation $x^2 + y^2 = a^2$, i.e. $\rho\sin\varphi = a$. The bottom of the cylinder has equation $z = 0$, i.e. $\varphi = \frac{\pi}{2}$.



For each fixed φ, θ runs from 0 to 2π and ρ runs from 0 to either $\frac{h}{\cos\varphi}$ (at the top of the can, if $\varphi < \tan^{-1}\frac{a}{h}$) or $\frac{a}{\sin\varphi}$ (at the side of the can, if $\varphi > \tan^{-1}\frac{a}{h}$). So the

$$\begin{aligned}
\text{Volume} &= \int_0^{\tan^{-1}\frac{a}{h}} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^{h/\cos\varphi} \mathrm{d}\rho\ \rho^2\sin\varphi + \int_{\tan^{-1}\frac{a}{h}}^{\frac{\pi}{2}} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^{a/\sin\varphi} \mathrm{d}\rho\ \rho^2\sin\varphi \\
&= 2\pi\int_0^{\tan^{-1}\frac{a}{h}} \mathrm{d}\varphi\ \frac{h^3\sin\varphi}{3\cos^3\varphi} + 2\pi\int_{\tan^{-1}\frac{a}{h}}^{\frac{\pi}{2}} \mathrm{d}\varphi\ \frac{a^3\sin\varphi}{3\sin^3\varphi} \\
&= 2\pi\Big\{\int_0^{\frac{a}{h}} \mathrm{d}t\ \frac{h^3}{3}t - \int_{\frac{h}{a}}^0 \mathrm{d}s\ \frac{a^3}{3}\Big\} \\
&\qquad\qquad \text{where } t = \tan\varphi,\ \mathrm{d}t = \sec^2\varphi\,\mathrm{d}\varphi,\ s = \cot\varphi,\ \mathrm{d}s = -\csc^2\varphi\,\mathrm{d}\varphi \\
&= 2\pi\left\{\frac{h^3}{3}\frac{1}{2}\left(\frac{a}{h}\right)^2 + \frac{a^3}{3}\frac{h}{a}\right\} = 2\pi\left\{\frac{ah^2}{6} + \frac{a^2h}{3}\right\} = \pi a^2 h
\end{aligned}$$

3.7.16 (∗) Let B denote the region inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $x^2 + y^2 = z^2$. Compute the moment of inertia

$$\iiint_B z^2\,\mathrm{d}V$$

Solution In spherical coordinates,

$$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$$

so that the sphere $x^2 + y^2 + z^2 = 4$ is $\rho^2 = 4$ or $\rho = 2$ and the cone $x^2 + y^2 = z^2$ is $\rho^2\sin^2\varphi = \rho^2\cos^2\varphi$ or $\tan\varphi = \pm 1$ or $\varphi = \frac{\pi}{4}, \frac{3\pi}{4}$. So

$$\begin{aligned}
\text{moment} &= \int_0^2 \mathrm{d}\rho \int_0^{\pi/4} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta\ \rho^2\sin\varphi\ (\rho\cos\varphi)^2 = 2\pi\int_0^2 \mathrm{d}\rho\ \rho^4 \int_0^{\pi/4} \mathrm{d}\varphi\ \sin\varphi\cos^2\varphi \\
&= 2\pi\left[\frac{\rho^5}{5}\right]_0^2 \left[-\frac{1}{3}\cos^3\varphi\right]_0^{\pi/4} = \frac{64}{15}\pi\left(1 - \frac{1}{2\sqrt{2}}\right) \approx 8.665
\end{aligned}$$

3.7.17 (*)
(a) Evaluate $\iiint\limits_{\Omega} z\,\mathrm{d}V$ where Ω is the three dimensional region in the first octant $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$, occupying the inside of the sphere $x^2+y^2+z^2=1$.
(b) Use the result in part (a) to quickly determine the centroid of a hemispherical ball given by $z \geqslant 0$, $x^2+y^2+z^2 \leqslant 1$.

Solution (a) In spherical coordinates,

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$$

so that

- the sphere $x^2+y^2+z^2=1$ is $\rho=1$,
- the xy-plane, $z=0$, is $\phi=\frac{\pi}{2}$,
- the positive half of the xz-plane, $y=0$, $x>0$, is $\theta=0$ and
- the positive half of the yz-plane, $x=0$, $y>0$, is $\theta=\frac{\pi}{2}$.

So

$$\begin{aligned}\iiint\limits_{\Omega} z\,\mathrm{d}V &= \int_0^1 \mathrm{d}\rho \int_0^{\pi/2} d\phi \int_0^{\pi/2} \mathrm{d}\theta\ \rho^2\sin\phi\,\overbrace{(\rho\cos\phi)}^{z}\\ &= \frac{\pi}{2}\int_0^1 \mathrm{d}\rho \int_0^{\pi/2} d\phi\ \rho^3\sin\phi\cos\phi\\ &= \frac{\pi}{2}\int_0^1 \mathrm{d}\rho\ \rho^3\ \frac{1}{2}\sin^2\phi\Big|_0^{\pi/2} = \frac{\pi}{4}\int_0^1 \mathrm{d}\rho\ \rho^3 = \frac{\pi}{16}\end{aligned}$$

(b) The hemispherical ball given by $z \geqslant 0$, $x^2+y^2+z^2 \leqslant 1$ (call it H) has centroid $(\bar{x},\bar{y},\bar{z})$ with $\bar{x}=\bar{y}=0$ (by symmetry) and

$$\bar{z} = \frac{\iiint_H z\,\mathrm{d}V}{\iiint_H \mathrm{d}V} = \frac{4\iiint_\Omega z\,\mathrm{d}V}{\frac{1}{2}\times\frac{4}{3}\pi} = \frac{\frac{\pi}{4}}{\frac{2\pi}{3}} = \frac{3}{8}$$

3.7.18 (*) Consider the top half of a ball of radius 2 centred at the origin. Suppose that the ball has variable density equal to $9z$ units of mass per unit volume.
(a) Set up a triple integral giving the mass of this half–ball.
(b) Find out what fraction of that mass lies inside the cone

$$z = \sqrt{x^2+y^2}$$

Solution (a) In spherical coordinates,

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$$

the sphere $x^2+y^2+z^2=4$ is $\rho^2=4$ or $\rho=2$ and the xy-plane is $\phi=\frac{\pi}{2}$. So

$$\text{mass} = \int_0^2 d\rho \int_0^{\pi/2} d\phi \int_0^{2\pi} d\theta\ \rho^2 \sin\phi\ \overbrace{(9\rho\cos\phi)}^{\text{density}}$$

(b) The mass of the half ball is

$$9\int_0^2 d\rho \int_0^{\pi/2} d\phi \int_0^{2\pi} d\theta\ \rho^3 \sin\phi\cos\phi = 9\left[\int_0^2 d\rho\ \rho^3\right]\left[\int_0^{\pi/2} d\phi\ \sin\phi\cos\phi\right]\left[\int_0^{2\pi} d\theta\right]$$

In spherical coordinates, the cone $x^2+y^2=z^2$ is $\rho^2\sin^2\phi=\rho^2\cos^2\phi$ or $\tan\phi=\pm1$ or $\phi=\frac{\pi}{4},\ \frac{3\pi}{4}$. So the mass of the part that is inside the cone is

$$9\int_0^2 d\rho \int_0^{\pi/4} d\phi \int_0^{2\pi} d\theta\ \rho^3 \sin\phi\cos\phi = 9\left[\int_0^2 d\rho\ \rho^3\right]\left[\int_0^{\pi/4} d\phi\ \sin\phi\cos\phi\right]\left[\int_0^{2\pi} d\theta\right]$$

The fraction inside the cone is

$$\frac{\int_0^{\pi/4} d\phi\ \sin\phi\cos\phi}{\int_0^{\pi/2} d\phi\ \sin\phi\cos\phi} = \frac{\frac{1}{2}\sin^2\phi\big|_0^{\pi/4}}{\frac{1}{2}\sin^2\phi\big|_0^{\pi/2}} = \frac{1}{2}$$

▸▸ Stage 3

3.7.19 (∗) Find the limit or show that it does not exist

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^4}$$

Solution In spherical coordinates, $x=\rho\sin\varphi\cos\theta$, $y=\rho\sin\varphi\sin\theta$, $z=\rho\cos\varphi$ so that

$$\begin{aligned}&\frac{\rho^2\sin^2\varphi\cos\theta\sin\theta+\rho^3\sin\varphi\sin\theta\ \cos^2\varphi+\rho^3\sin\varphi\ \cos^2\varphi}{\rho^2\sin^2\varphi+\rho^4\cos^4\varphi}\\ &\qquad\qquad = \frac{\sin^2\varphi\cos\theta\sin\theta+\rho\sin\varphi\sin\theta\ \cos^2\varphi+\rho\sin\varphi\cos\theta\ \cos^2\varphi}{\sin^2\varphi+\rho^2\cos^4\varphi}\end{aligned}$$

As $(x,y,z)\to(0,0,0)$, the radius $\rho\to0$ and the second and third terms in the numerator and the second term in the denominator converge to 0. But that leaves

$$\frac{\sin^2\varphi\cos\theta\sin\theta}{\sin^2\varphi} = \cos\theta\sin\theta$$

which takes many different values. In particular, if we send $(x,y,z)\to(0,0,0)$ along either the x– or y–axis, that is with $z=0$ and either $x=0$ or $y=0$, then

$$\left.\frac{xy+yz^2+xz^2}{x^2+y^2+z^4}\right|_{\substack{x=0\text{ or }y=0\\ z=0}} = 0$$

converges to 0. But, if we send $(x,y,z) \to (0,0,0)$ along the line $y = x$, $z = 0$

$$\frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}\bigg|_{\substack{y=x\\z=0}} = \frac{x^2}{2x^2} = \frac{1}{2}$$

converges to $1/2$. So $\frac{xy+yz^2+xz^2}{x^2+y^2+z^4}$ does not approach a single value as $(x,y,z) \to (0,0,0)$ and the limit does not exist.

$\boxed{3.7.20}$ (∗) A certain solid V is a right–circular cylinder. Its base is the disk of radius 2 centred at the origin in the xy–plane. It has height 2 and density $\sqrt{x^2+y^2}$.
A smaller solid U is obtained by removing the inverted cone, whose base is the top surface of V and whose vertex is the point $(0,0,0)$.
(a) Use cylindrical coordinates to set up an integral giving the mass of U.
(b) Use spherical coordinates to set up an integral giving the mass of U.
(c) Find that mass.

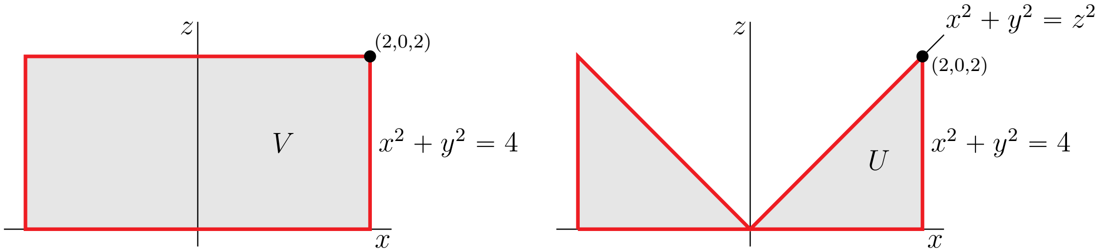
Solution The disk of radius 2 centred at the origin in the xy–plane is $x^2+y^2 \leqslant 4$. So

$$V = \big\{ (x,y,z) \bigm| x^2+y^2 \leqslant 4,\ 0 \leqslant z \leqslant 2 \big\}$$

The cone with vertex at the origin that contains the top edge, $x + y^2 = 4$, $z = 2$, of U is $x^2+y^2 = z^2$. So

$$U = \big\{ (x,y,z) \bigm| x^2+y^2 \leqslant 4,\ 0 \leqslant z \leqslant 2,\ x^2+y^2 \geqslant z^2 \big\}$$

Here are sketches of the $y = 0$ cross–section of V, on the left, and U, on the right.



(a) In cylindrical coordinates, $x^2+y^2 \leqslant 4$ becomes $r \leqslant 2$ and $x^2+y^2 \geqslant z^2$ is $r \geqslant |z|$, and the density is $\sqrt{x^2+y^2} = r$. So

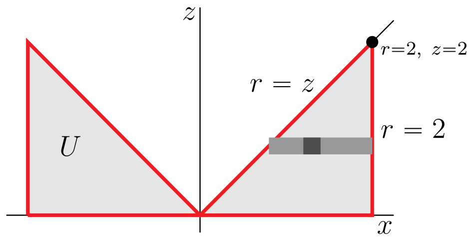
$$U = \big\{ (r\cos\theta, r\sin\theta, z) \bigm| r \leqslant 2,\ 0 \leqslant z \leqslant 2,\ r \geqslant z \big\}$$

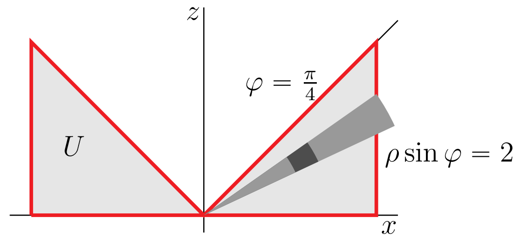
Looking at the figure on the left below, we see that, on U

- z runs from 0 to 2, and
- for each z is that range, r runs from z to 2 and θ runs from 0 to 2π.
- $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$

So

$$\text{Mass} = \int_0^2 \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_z^2 \mathrm{d}r\ r\ \overbrace{r}^{\text{density}} = \int_0^2 \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_z^2 \mathrm{d}r\ r^2$$





(b) Recall that in spherical coordinates,

$$\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ y &= \rho \sin\varphi \sin\theta \\ z &= \rho \cos\varphi \\ x^2 + y^2 &= \rho^2 \sin^2\varphi \end{aligned}$$

so that $x^2 + y^2 \leqslant 4$ becomes $\rho \sin\varphi \leqslant 2$, and $x^2 + y^2 \geqslant z^2$ becomes

$$\rho \sin\varphi \geqslant \rho \cos\varphi \iff \tan\varphi \geqslant 1 \iff \varphi \geqslant \frac{\pi}{4}$$

and the density $\sqrt{x^2 + y^2} = \rho \sin\varphi$. So

$$U = \left\{ (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi) \;\middle|\; \pi/4 \leqslant \varphi \leqslant \pi/2,\ 0 \leqslant \theta \leqslant 2\pi,\ \rho \sin\varphi \leqslant 2 \right\}$$

Looking at the figure on the right above, we see that, on U

- φ runs from $\frac{\pi}{4}$ (on the cone) to $\frac{\pi}{2}$ (on the xy–plane), and
- for each φ is that range, ρ runs from 0 to $\frac{2}{\sin\varphi}$ and θ runs from 0 to 2π.
- $\mathrm{d}V = \rho^2 \sin\varphi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\varphi$

So

$$\begin{aligned} \text{Mass} &= \int_{\pi/4}^{\pi/2} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^{2/\sin\varphi} \mathrm{d}\rho\ \rho^2 \sin\varphi\ \overbrace{\rho \sin\varphi}^{\text{density}} \\ &= \int_{\pi/4}^{\pi/2} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^{2/\sin\varphi} \mathrm{d}\rho\ \rho^3 \sin^2\varphi \end{aligned}$$

(c) We'll use the cylindrical form.

$$\begin{aligned} \text{Mass} &= \int_0^2 \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_z^2 \mathrm{d}r\ r^2 \\ &= 2\pi \int_0^2 \mathrm{d}z\ \frac{8 - z^3}{3} \\ &= \frac{2\pi}{3} \left[16 - \frac{2^4}{4} \right] \\ &= 8\pi \end{aligned}$$

$\boxed{3.7.21}$ (*) A solid is bounded below by the cone $z=\sqrt{x^2+y^2}$ and above by the sphere $x^2+y^2+z^2=2$. It has density $\delta(x,y,z)=x^2+y^2$.

(a) Express the mass M of the solid as a triple integral, with limits, in cylindrical coordinates.

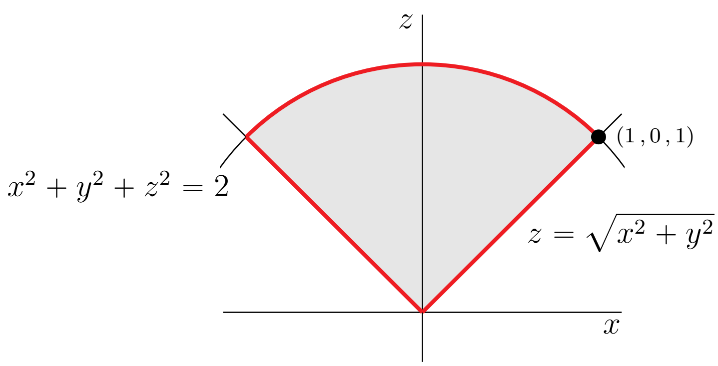
(b) Same as (a) but in spherical coordinates.

(c) Evaluate M.

$\boxed{\text{Solution}}$ (a) Call the solid V. In cylindrical coordinates

- $x^2+y^2+z^2 \leqslant 2$ is $r^2+z^2 \leqslant 2$ and
- $\sqrt{x^2+y^2} \leqslant z$ is $r \leqslant z$ and
- the density $\delta = r^2$, and
- $\mathrm{d}V$ is $r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$

Observe that $r^2+z^2=2$ and $r=z$ intersect when $2r^2=2$ so that $r=z=1$. Here is a sketch of the $y=0$ cross–section of E.



So

$$V = \left\{ \ (r\cos\theta\,,\, r\sin\theta\,,\, z) \ \middle|\ 0 \leqslant r \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi,\ r \leqslant z \leqslant \sqrt{2-r^2}\ \right\}$$

and

$$M = \iiint_V \rho(x,y,z)\ \mathrm{d}V = \int_0^1 \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \int_r^{\sqrt{2-r^2}} \mathrm{d}z\ r\ \overbrace{(r^2)}^{\delta} = \int_0^1 \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \int_r^{\sqrt{2-r^2}} \mathrm{d}z\ r^3$$

(b) In spherical coordinates

- $x^2+y^2+z^2 \leqslant 2$ is $\rho \leqslant \sqrt{2}$, and
- $\sqrt{x^2+y^2} \leqslant z$ is $\rho\sin\varphi \leqslant \rho\cos\varphi$, or $\tan\varphi \leqslant 1$ or $\varphi \leqslant \frac{\pi}{4}$, and
- the density $x^2+y^2 = \rho^2\sin^2\varphi$, and
- $\mathrm{d}V$ is $\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$

So

$$V = \left\{ \ (\rho\sin\varphi\cos\theta\,,\, \rho\sin\varphi\sin\theta\,,\, \rho\cos\varphi) \ \middle|\ 0 \leqslant \rho \leqslant \sqrt{2},\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \varphi \leqslant \pi/4\ \right\}$$

and, since the integrand $x^2 + y^2 = \rho^2 \sin^2 \varphi$,

$$M = \iiint_V (x^2 + y^2)\ \mathrm{d}V = \int_0^{\sqrt{2}} \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/4} \mathrm{d}\varphi\ \rho^2 \sin\varphi\ \rho^2 \sin^2\varphi$$
$$= \int_0^{\sqrt{2}} \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/4} \mathrm{d}\varphi\ \rho^4 \sin^3\varphi$$

(c) We'll use the spherical coordinate form.

$$\begin{aligned} M &= \int_0^{\sqrt{2}} \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/4} \mathrm{d}\varphi\ \rho^4 \sin^3\varphi \\ &= \int_0^{\sqrt{2}} \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/4} \mathrm{d}\varphi\ \rho^4 \sin\varphi \big[1 - \cos^2\varphi\big] \\ &= 2\pi \int_0^{\sqrt{2}} \mathrm{d}\rho\ \rho^4 \left[-\cos\varphi + \frac{\cos^3\varphi}{3}\right]_0^{\pi/4} = 2\pi \left[\frac{2}{3} - \frac{5}{6\sqrt{2}}\right] \int_0^{\sqrt{2}} \mathrm{d}\rho\ \rho^4 \\ &= 2\pi \frac{4\sqrt{2}}{5} \left[\frac{2}{3} - \frac{5}{6\sqrt{2}}\right] = \pi \left[\frac{16\sqrt{2}}{15} - \frac{4}{3}\right] \approx 0.5503 \end{aligned}$$

$\boxed{3.7.22}$ (∗) Let

$$I = \iiint_E xz\ \mathrm{d}V$$

where E is the eighth of the sphere $x^2 + y^2 + z^2 \leqslant 1$ with $x, y, z \geqslant 0$.
(a) Express I as a triple integral in spherical coordinates.
(b) Express I as a triple integral in cylindrical coordinates.
(c) Evaluate I by any method.

$\boxed{\text{Solution}}$ (a) On E,

- the spherical coordinate φ runs from 0 (the positive z–xis) to $\pi/2$ (the xy–plane), and
- for each fixed φ in that range, θ runs from 0 to $\pi/2$, and
- for each fixed φ and θ, the spherical coordinate ρ runs from 0 to 1.
- In spherical coordinates $\mathrm{d}V = \rho^2\ \sin\varphi\, \mathrm{d}\rho\, \mathrm{d}\theta\, \mathrm{d}\varphi$ and

$$xz = \big(\rho \sin\varphi \cos\theta\big)\big(\rho\cos\varphi\big) = \rho^2 \sin\varphi\ \ \cos\varphi\ \ \cos\theta$$

So

$$I = \int_0^{\pi/2} \mathrm{d}\varphi \int_0^{\pi/2} \mathrm{d}\theta \int_0^1 \mathrm{d}\rho\ \rho^4 \sin^2\varphi \cos\varphi\ \ \cos\theta$$

(b) In cylindrical coordinates, the condition $x^2 + y^2 + z^2 \leqslant 1$ becomes $r^2 + z^2 \leqslant 1$. So, on E

- the cylindrical coordinate z runs from 0 (in the xy–plane) to 1 (at $(0,0,1)$) and
- for each fixed z in that range, θ runs from 0 to $\pi/2$ and
- for each such fixed z and θ, the cylindrical coordinate r runs from 0 to $\sqrt{1-z^2}$ (recall that $r^2+z^2 \leqslant 1$).
- In cylindrical coordinates $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$ and

$$xz = \big(r\cos\theta\big)\big(z\big) = r\,z\,\cos\theta$$

So

$$I = \int_0^1 \mathrm{d}z \int_0^{\pi/2} \mathrm{d}\theta \int_0^{\sqrt{1-z^2}} \mathrm{d}r\ r^2\,z\,\cos\theta$$

(c) Both spherical and cylindrical integrals are straight forward to evaluate. Here are both. First, in spherical coordinates,

$$\begin{aligned}
I &= \left[\int_0^{\pi/2} \mathrm{d}\varphi\ \sin^2\varphi\cos\varphi\right]\left[\int_0^{\pi/2} \mathrm{d}\theta\ \cos\theta\right]\left[\int_0^1 \mathrm{d}\rho\ \rho^4\right]\\
&= \left[\frac{\sin^3\varphi}{3}\right]_0^{\pi/2} \big[\sin\theta\big]_0^{\pi/2} \left[\frac{\rho^5}{5}\right]_0^1\\
&= \frac{1}{15}
\end{aligned}$$

Now in cylindrical coordinates

$$\begin{aligned}
I &= \int_0^1 \mathrm{d}z \int_0^{\pi/2} \mathrm{d}\theta \int_0^{\sqrt{1-z^2}} \mathrm{d}r\ r^2\,z\,\cos\theta\\
&= \frac{1}{3}\int_0^1 \mathrm{d}z \int_0^{\pi/2} \mathrm{d}\theta\ z(1-z^2)^{3/2}\ \cos\theta\\
&= \frac{1}{3}\int_0^1 \mathrm{d}z\ z(1-z^2)^{3/2}\\
&= \frac{1}{3}\left[-\frac{1}{2}\ \frac{(1-z^2)^{5/2}}{5/2}\right]_0^1\\
&= \frac{1}{15}
\end{aligned}$$

$\boxed{3.7.23}$ (∗) Let

$$I = \iiint_T (x^2 + y^2)\ \mathrm{d}V$$

where T is the solid region bounded below by the cone $z = \sqrt{3x^2 + 3y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 9$.

(a) Express I as a triple integral in spherical coordinates.
(b) Express I as a triple integral in cylindrical coordinates.
(c) Evaluate I by any method.

Solution (a) Recall that in spherical coordinates

$$x = \rho\,\sin\varphi\,\cos\theta \qquad y = \rho\,\sin\varphi\,\sin\theta \qquad z = \rho\,\cos\varphi$$

so that

- $x^2 + y^2 + z^2 \leqslant 9$ is $\rho \leqslant 3$, and
- $\sqrt{3x^2 + 3y^2} \leqslant z$ is $\sqrt{3}\rho\sin\varphi \leqslant \rho\cos\varphi$, or $\tan\varphi \leqslant \frac{1}{\sqrt{3}}$ or $\varphi \leqslant \frac{\pi}{6}$, and
- the integrand $x^2 + y^2 = \rho^2\sin^2\varphi$, and
- $\mathrm{d}V$ is $\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$

So

$$T = \left\{\ (\rho\sin\varphi\cos\theta\,,\ \rho\sin\varphi\sin\theta\,,\ \rho\cos\varphi)\ \middle|\ 0 \leqslant \rho \leqslant 3,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \varphi \leqslant {}^{\pi}/_{6}\ \right\}$$

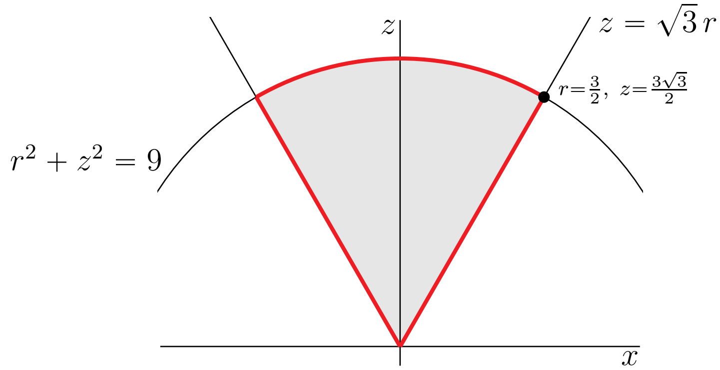
and,

$$\begin{aligned} I &= \iiint_T (x^2 + y^2)\ \mathrm{d}V = \int_0^3 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/6} \mathrm{d}\varphi\ \rho^2\sin\varphi\ \overbrace{\rho^2\sin^2\varphi}^{x^2+y^2} \\ &= \int_0^3 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/6} \mathrm{d}\varphi\ \rho^4\sin^3\varphi \end{aligned}$$

(b) In cylindrical coordinates

- $x^2 + y^2 + z^2 \leqslant 9$ is $r^2 + z^2 \leqslant 9$ and
- $\sqrt{3x^2 + 3y^2} \leqslant z$ is $\sqrt{3}\,r \leqslant z$ and
- the integand $x^2 + y^2 = r^2$, and
- $\mathrm{d}V$ is $r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$

Observe that $r^2 + z^2 = 9$ and $\sqrt{3}\,r = z$ intersect when $r^2 + 3r^2 = 9$ so that $r = \frac{3}{2}$ and $z = \frac{3\sqrt{3}}{2}$. Here is a sketch of the $y = 0$ cross–section of T.



So

$$T = \Big\{ \ (r\cos\theta\,,\, r\sin\theta\,,\, z) \ \Big|\ 0 \leqslant r \leqslant \tfrac{3}{2},\ 0 \leqslant \theta \leqslant 2\pi,\ \sqrt{3}\,r \leqslant z \leqslant \sqrt{9-r^2}\ \Big\}$$

and

$$I = \iiint_V (x^2+y^2)\ \mathrm{d}V = \int_0^{3/2} \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \int_{\sqrt{3}\,r}^{\sqrt{9-r^2}} \mathrm{d}z\ r\ \overbrace{(r^2)}^{x^2+y^2} = \int_0^{3/2} \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \int_{\sqrt{3}\,r}^{\sqrt{9-r^2}} \mathrm{d}z\ \ r^3$$

(c) We'll use the spherical coordinate form.

$$\begin{aligned}
I &= \int_0^3 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/6} \mathrm{d}\varphi\ \rho^4 \sin^3\varphi \\
&= \int_0^3 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/6} \mathrm{d}\varphi\ \rho^4 \sin\varphi\,[1-\cos^2\varphi] \\
&= 2\pi \int_0^3 \mathrm{d}\rho\ \rho^4 \left[-\cos\varphi + \frac{\cos^3\varphi}{3}\right]_0^{\pi/6} = 2\pi\left[-\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{8} + 1 - \frac{1}{3}\right] \int_0^3 \mathrm{d}\rho\ \rho^4 \\
&= 2\pi \frac{3^5}{5}\left[\frac{2}{3} - \frac{3\sqrt{3}}{8}\right] = \overbrace{81}^{3^4}\,\pi\left[\frac{4}{5} - \frac{9\sqrt{3}}{20}\right] \approx 5.24
\end{aligned}$$

$\boxed{3.7.24}$ ($*$) Let E be the "ice cream cone" $x^2+y^2+z^2 \leqslant 1$, $x^2+y^2 \leqslant z^2$, $z \geqslant 0$. Consider

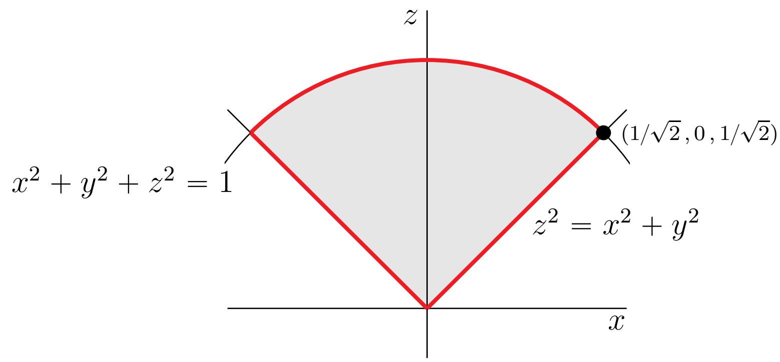
$$J = \iiint_E \sqrt{x^2+y^2+z^2}\ \mathrm{d}V$$

(a) Write J as an iterated integral, with limits, in cylindrical coordinates.
(b) Write J as an iterated integral, with limits, in spherical coordinates.
(c) Evaluate J.

$\boxed{\text{Solution}}$ (a) In cylindrical coordinates

- $x^2+y^2+z^2 \leqslant 1$ is $r^2+z^2 \leqslant 1$ and
- $x^2+y^2 \leqslant z^2$ is $r^2 \leqslant z^2$ and
- $\mathrm{d}V$ is $r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$

Observe that $r^2+z^2=1$ and $r^2=z^2$ intersect when $r^2=z^2=\frac{1}{2}$. Here is a sketch of the $y=0$ cross–section of E.



So

$$E = \Big\{ \ (r\cos\theta\,,\ r\sin\theta\,,\ z) \ \Big|\ 0 \leqslant r \leqslant 1/\sqrt{2},\ 0 \leqslant \theta \leqslant 2\pi,\ r \leqslant z \leqslant \sqrt{1-r^2} \ \Big\}$$

and

$$J = \iiint_E \sqrt{x^2+y^2+z^2}\ \mathrm{d}V = \int_0^{1/\sqrt{2}} \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \int_r^{\sqrt{1-r^2}} \mathrm{d}z\ r\sqrt{r^2+z^2}$$

(b) In spherical coordinates

- $x^2+y^2+z^2 \leqslant 1$ is $\rho \leqslant 1$ and
- $x^2+y^2 \leqslant z^2$ is $\rho^2\sin^2\varphi \leqslant \rho^2\cos^2\varphi$, or $\tan\varphi \leqslant 1$ or $\varphi \leqslant \frac{\pi}{4}$, and
- $\mathrm{d}V$ is $\rho^2 \sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$

So

$$E = \Big\{ \ (\rho\sin\varphi\cos\theta\,,\ \rho\sin\varphi\sin\theta\,,\ \rho\cos\varphi) \ \Big|\ 0 \leqslant \rho \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \varphi \leqslant \pi/4 \ \Big\}$$

and, since the integrand $\sqrt{x^2+y^2+z^2}=\rho$,

$$\begin{aligned} J &= \iiint_E \sqrt{x^2+y^2+z^2}\ \mathrm{d}V = \int_0^1 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/4} \mathrm{d}\varphi\ \rho^2\sin\varphi\ \rho \\ &= \int_0^1 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/4} \mathrm{d}\varphi\ \rho^3\sin\varphi \end{aligned}$$

(c) We'll use the spherical coordinate form to evaluate

$$\begin{aligned} J &= \int_0^1 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/4} \mathrm{d}\varphi\ \rho^3\sin\varphi \\ &= 2\pi \int_0^1 \mathrm{d}\rho\ \rho^3 \Big[-\cos\varphi\Big]_0^{\pi/4} = 2\pi\ \frac{1}{4}\ \left[1-\frac{1}{\sqrt{2}}\right] \\ &= \frac{\pi}{2}\ \left[1-\frac{1}{\sqrt{2}}\right] \end{aligned}$$

$\boxed{3.7.25}$ (∗) The body of a snowman is formed by the snowballs $x^2 + y^2 + z^2 = 12$ (this is its body) and $x^2 + y^2 + (z-4)^2 = 4$ (this is its head).

(a) Find the volume of the snowman by subtracting the intersection of the two snow balls from the sum of the volumes of the snow balls. [Recall that the volume of a sphere of radius r is $\frac{4\pi}{3}r^3$.]

(b) We can also calculate the volume of the snowman as a sum of the following triple integrals:

1.
$$\int_0^{\frac{2\pi}{3}} \int_0^{2\pi} \int_0^2 \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

2.
$$\int_0^{2\pi} \int_0^{\sqrt{3}} \int_{\sqrt{3}r}^{4-\frac{r}{\sqrt{3}}} r \, dz \, dr \, d\theta$$

3.
$$\int_{\frac{\pi}{6}}^{\pi} \int_0^{2\pi} \int_0^{2\sqrt{3}} \rho^2 \sin(\varphi) \, d\rho \, d\theta \, d\varphi$$

Circle the right answer from the underlined choices and fill in the blanks in the following descriptions of the region of integration for each integral. [Note: We have translated the axes in order to write down some of the integrals above. The equations you specify should be those before the translation is performed.]

i. The region of integration in (1) is a part of the snowman's

body/head/body and head.

It is the solid enclosed by the

sphere/cone defined by the equation ____________

and the

sphere/cone defined by the equation _______________.

ii. The region of integration in (2) is a part of the snowman's

body/head/body and head.

It is the solid enclosed by the

sphere/cone defined by the equation ____________

and the

sphere/cone defined by the equation _______________.

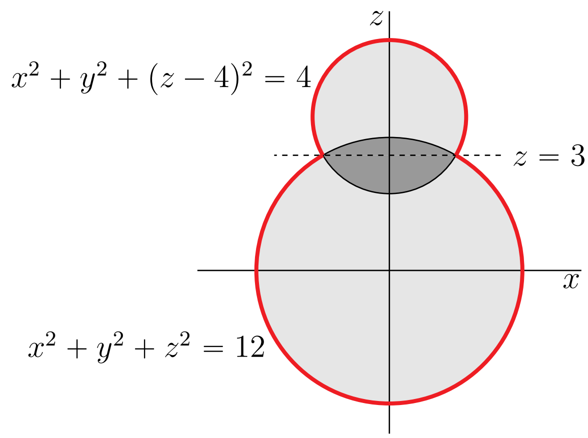
iii. The region of integration in (3) is a part of the snowman's

body/head/body and head.

It is the solid enclosed by the

sphere/cone defined by the equation ____________

$\boxed{\text{Solution}}$ (a) As a check, the body of the snow man has radius $\sqrt{12} = 2\sqrt{3} \approx 3.46$, which is between 2 (the low point of the head) and 4 (the center of the head). Here is a sketch of a side view of the snowman.



We want to determine the volume of the intersection of the body and the head, whose side view is the darker shaded region in the sketch.

- The outer boundary of the body and the outer boundary of the head intersect when both $x^2 + y^2 + z^2 = 12$ and $x^2 + y^2 + (z-4)^2 = 4$. Subtracting the second equation from the first gives

$$z^2 - (z-4)^2 = 12 - 4 \iff 8z - 16 = 8 \iff z = 3$$

  Then substituting $z = 3$ into either equation gives $x^2 + y^2 = 3$. So the intersection of the outer boundaries of the head and body (i.e. the neck) is the circle $x^2 + y^2 = 3$, $z = 3$.

- The top boundary of the intersection is part of the top half of the snowman's body and so has equation $z = +\sqrt{12 - x^2 - y^2}$.

- The bottom boundary of the intersection is part of the bottom half of the snowman's head, and so has equation $z = 4 - \sqrt{4 - x^2 - y^2}$

The intersection of the head and body is thus

$$\mathcal{V} = \left\{ \ (x, y, z \mid x^2 + y^2 \leqslant 3,\ 4 - \sqrt{4 - x^2 - y^2} \leqslant z \leqslant \sqrt{12 - x^2 - y^2} \ \right\}$$

We'll compute the volume of $\mathcal{V}$ using cylindrical coordinates

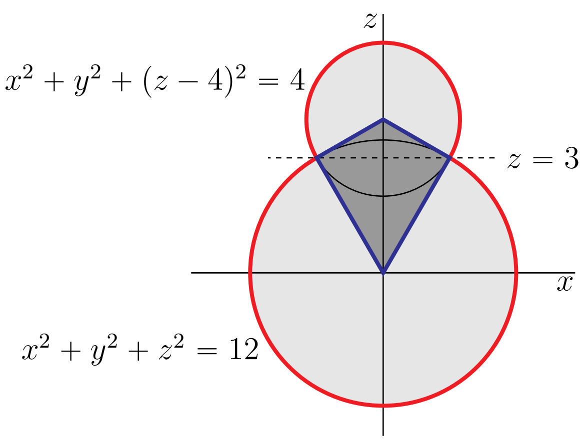
$$\begin{aligned}
\text{Volume}(\mathcal{V}) &= \int_0^{\sqrt{3}} \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \int_{4-\sqrt{4-r^2}}^{\sqrt{12-r^2}} \mathrm{d}z\ r \\
&= \int_0^{\sqrt{3}} \mathrm{d}r\ 2\pi\ r\big[\sqrt{12 - r^2} - 4 + \sqrt{4 - r^2}\big] \\
&= 2\pi \left[-\frac{1}{3}\left(12 - r^2\right)^{3/2} - 2r^2 - \frac{1}{3}\left(4 - r^2\right)^{3/2} \right]_0^{\sqrt{3}} \\
&= 2\pi \left[-\frac{1}{3}(9)^{3/2} - 2(3) - \frac{1}{3}(1)^{3/2} + \frac{1}{3}(12)^{3/2} + 2(0)^2 + \frac{1}{3}(4)^{3/2} \right]
\end{aligned}$$

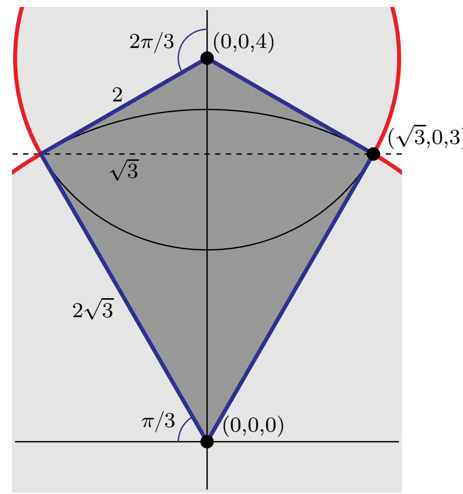
$$= 2\pi\left[-9-6-\frac{1}{3}+\frac{1}{3}(12)^{3/2}+\frac{8}{3}\right]$$
$$= 2\pi\left[\frac{1}{3}(12)^{3/2}-\frac{38}{3}\right]$$

So the volume of the snowman is

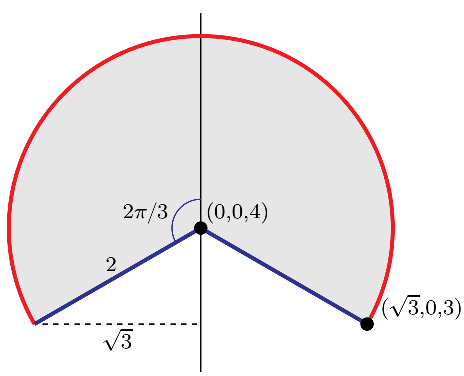
$$\frac{4\pi}{3}(12)^{3/2}+\frac{4\pi}{3}2^3-2\pi\left[\frac{1}{3}(12)^{3/2}-\frac{38}{3}\right]$$
$$=\frac{2\pi}{3}\left[(12)^{3/2}+54\right]$$

(b) The figure on the left below is another side view of the snowman. This time it is divided into a lighter gray top part, a darker gray middle part and a lighter gray bottom part. The figure on the right below is an enlarged view of the central part of the figure on the left.





i. The top part is the Pac–Man



part of the snowman's head. It is the part of the sphere

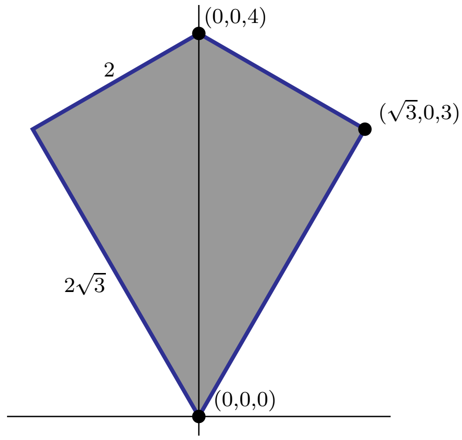
$$x^2+y^2+(z-4)^2\leqslant 4$$

that is above the cone

$$z-4=-\sqrt{\frac{x^2+y^2}{3}}$$

(which contains the points $(0,0,4)$ and $(\sqrt{3},0,3)$).

ii. The middle part is the diamond shaped



part of the snowman's head and body. It is bounded on the top by the cone

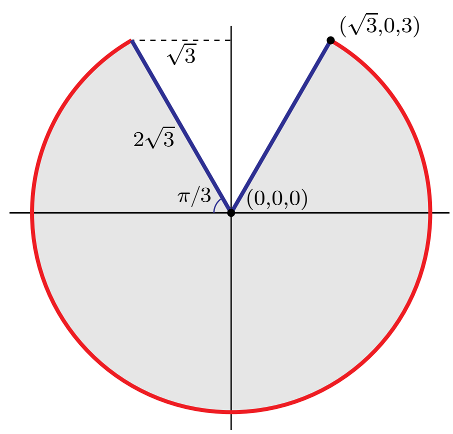
$$z - 4 = -\sqrt{\frac{x^2+y^2}{3}}$$

(which contains the points $(0,0,4)$ and $(\sqrt{3},0,3)$) and is bounded on the bottom by the cone

$$z = \sqrt{3(x^2+y^2)}$$

(which contains the points $(0,0,0)$ and $(\sqrt{3},0,3)$).

iii. The bottom part is the Pac–Man



part of the snowman's body. It is the part of the sphere

$$x^2 + y^2 + z^2 \leqslant 12$$

that is below the cone

$$z = \sqrt{3(x^2+y^2)}$$

(which contains the points $(0,0,0)$ and $(\sqrt{3},0,3)$).

$\boxed{3.7.26}$ (∗)

(a) Find the volume of the solid inside the surface defined by the equation $\rho = 8\sin(\varphi)$ in spherical coordinates.
You may use that

$$\int \sin^4(\varphi) = \frac{1}{32}\big(12\varphi - 8\sin(2\varphi) + \sin(4\varphi)\big) + C$$

(b) Sketch this solid or describe what it looks like.

$\boxed{\text{Solution}}$ (a) Recall that, in spherical coordinates, φ runs from 0 (that's the positive z–axis) to π (that's the negative z–axis), θ runs from 0 to 2π (θ is the regular polar or cylindrical coordinate) and $\mathrm{d}V = \rho^2\ \sin\varphi\ \mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$. So

$$\begin{aligned}
\text{Volume} &= \int_0^\pi \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^{8\sin\varphi} \mathrm{d}\rho\ \rho^2\ \sin\varphi \\
&= \int_0^\pi \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta\ \frac{(8\sin\varphi)^3}{3}\sin\varphi \\
&= \frac{2(8^3)\,\pi}{3} \int_0^\pi \mathrm{d}\varphi\ \sin^4\varphi \\
&= \frac{2(8^3)\,\pi}{3}\left[\frac{1}{32}\big(12\varphi - 8\sin(2\varphi) + \sin(4\varphi)\big)\right]_0^\pi \\
&= \frac{2(8^3)\,\pi}{3}\ \frac{12\,\pi}{32} = 128\pi^2
\end{aligned}$$

(b) Fix any φ between 0 and π. If $\rho = 8\sin\varphi$, then as θ runs from 0 to 2π,

$$\begin{aligned}
(x,y,z) &= \big(\rho\sin\varphi\cos\theta\,,\ \rho\sin\varphi\sin\theta\,,\ \rho\cos\varphi\big) \\
&= \big(8\sin^2\varphi\cos\theta\,,\ 8\sin^2\varphi\sin\theta\,,\ 8\sin\varphi\cos\varphi\big) \\
&= \big(R\cos\theta\,,\ R\sin\theta\,,\ Z\big) \qquad \text{with } R = 8\sin^2\varphi,\ Z = 8\sin\varphi\cos\varphi
\end{aligned}$$

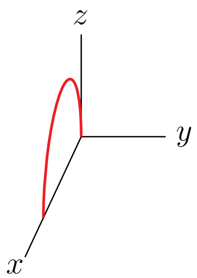
sweeps out a circle of radius $R = 8\sin^2\varphi$ contained in the plane $z = Z = 8\sin\varphi\cos\varphi$ and centred on $\big(0,0,Z = 8\sin\varphi\cos\varphi\big)$. So the surface is a bunch of circles stacked one on top of the other. It is a surface of revolution. We can sketch it by

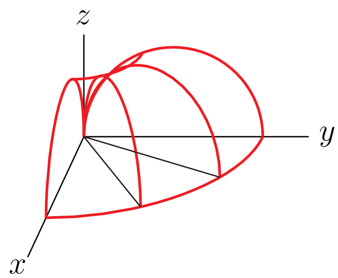
- first sketching the $\theta = 0$ section of the surface (that's the part of the surface in the right half of the xz–plane)
- and then rotate the result about the z–axis.

The $\theta = 0$ part of the surface is

$$\begin{aligned}
&\big\{\ (x,y,z)\ \big|\ x = 8\sin^2\varphi,\ y = 0,\ z = 8\sin\varphi\cos\varphi,\ 0 \leqslant \varphi \leqslant \pi\ \big\} \\
&= \big\{\ (x,y,z)\ \big|\ x = 4 - 4\cos(2\varphi),\ y = 0,\ z = 4\sin(2\varphi),\ 0 \leqslant \varphi \leqslant \pi\ \big\}
\end{aligned}$$

It's a circle of radius 4, contained in the xz–plane (i.e. $y = 0$) and centred on $(4,0,0)$! The figure on the left below is a sketch of the top half of the circle. When we rotate the circle about the z–axis we get a torus (a donut) but with the hole in the centre shrunk to a point. The figure on the right below is a sketch of the part of the torus in the first octant.





$\boxed{3.7.27}$ (∗) Let E be the solid

$$0 \leqslant z \leqslant \sqrt{x^2+y^2}, \qquad x^2+y^2 \leqslant 1,$$

and consider the integral

$$I = \iiint_E z\sqrt{x^2+y^2+z^2}\,\mathrm{d}V.$$

(a) Write the integral I in cylindrical coordinates.
(b) Write the integral I in spherical coordinates.
(c) Evaluate the integral I using either form.

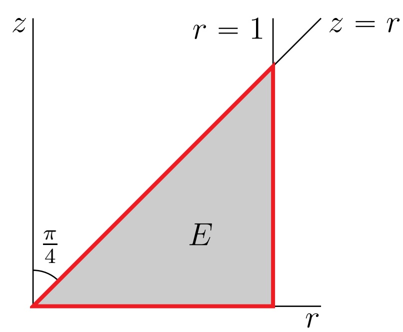
$\boxed{\text{Solution}}$ (a) In cylindrical coordinates $0 \leqslant z \leqslant \sqrt{x^2+y^2}$ becomes $0 \leqslant z \leqslant r$, and $x^2+y^2 \leqslant 1$ becomes $0 \leqslant r \leqslant 1$. So

$$E = \left\{ \ (r\cos\theta\,,\, r\sin\theta\,,\, z) \ \middle|\ 0 \leqslant r \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant z \leqslant r \ \right\}$$

and, since $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$,

$$I = \iiint_E z\sqrt{x^2+y^2+z^2}\,\mathrm{d}V = \int_0^1 \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \int_0^r \mathrm{d}z\ r\ z\ \sqrt{\underbrace{r^2+z^2}_{x^2+y^2+z^2}}$$

(b) Here is a sketch of a constant θ section of E.



Recall that the spherical coordinate φ is the angle between the z–axis and the radius vector. So, in spherical coordinates $z = r$ (which makes an angle $\frac{\pi}{4}$ with the z axis) becomes $\varphi = \frac{\pi}{4}$, and the plane $z = 0$, i.e. the xy–plane, becomes $\varphi = \frac{\pi}{2}$, and $r = 1$ becomes $\rho \sin\varphi = 1$. So

$$E = \left\{ (\rho\sin\varphi\cos\theta\,,\ \rho\sin\varphi\sin\theta\,,\ \rho\cos\varphi) \ \middle|\ \frac{\pi}{4} \leqslant \varphi \leqslant \frac{\pi}{2},\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \rho \leqslant \frac{1}{\sin\varphi} \right\}$$

and, since $\mathrm{d}V = \rho^2 \sin\varphi \,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$,

$$\begin{aligned} I &= \iiint_E z\sqrt{x^2+y^2+z^2}\ \mathrm{d}V = \int_{\pi/4}^{\pi/2} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^{1/\sin\varphi} \mathrm{d}\rho\ \rho^2\sin\varphi\ \overbrace{\rho\cos\varphi}^{z}\ \rho \\ &= \int_{\pi/4}^{\pi/2} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^{1/\sin\varphi} \mathrm{d}\rho\ \rho^4\sin\varphi\ \cos\varphi \end{aligned}$$

(c) We'll integrate using the spherical coordinate version.

$$\begin{aligned} I &= \int_{\pi/4}^{\pi/2} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^{1/\sin\varphi} \mathrm{d}\rho\ \rho^4\sin\varphi\ \cos\varphi \\ &= \int_{\pi/4}^{\pi/2} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta\ \frac{1}{5\sin^5\varphi}\sin\varphi\ \cos\varphi \\ &= \frac{2\pi}{5}\int_{\pi/4}^{\pi/2} \mathrm{d}\varphi\ \frac{\cos\varphi}{\sin^4\varphi} \\ &= \frac{2\pi}{5}\int_{1/\sqrt{2}}^{1} \frac{\mathrm{d}u}{u^4} \qquad \text{with } u = \sin\varphi,\ \mathrm{d}u = \cos\varphi\,\mathrm{d}\varphi \\ &= \frac{2\pi}{5}\left[\frac{u^{-3}}{-3}\right]_{1/\sqrt{2}}^{1} \\ &= \frac{2(2\sqrt{2}-1)\pi}{15} \end{aligned}$$

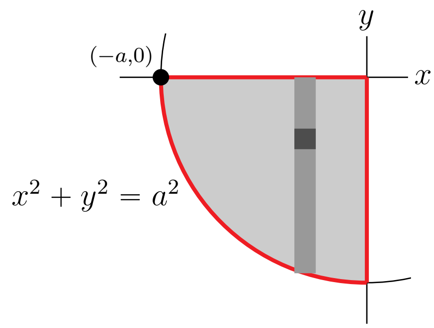
3.7.28 (∗) Consider the iterated integral

$$I = \int_{-a}^{0}\int_{-\sqrt{a^2-x^2}}^{0}\int_0^{\sqrt{a^2-x^2-y^2}} \left(x^2+y^2+z^2\right)^{2014}\ \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

where a is a positive constant.
(a) Write I as an iterated integral in cylindrical coordinates.
(b) Write I as an iterated integral in spherical coordinates.
(c) Evaluate I using whatever method you prefer.

Solution The main step is to figure out what the domain of integration looks like.

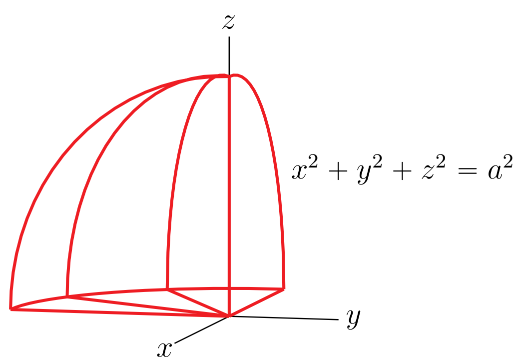
- The outside integral says that x runs from $-a$ to 0.
- The middle integrals says that, for each x in that range, y runs from $-\sqrt{a^2-x^2}$ to 0. We can rewrite $y = -\sqrt{a^2-x^2}$ in the more familiar form $x^2+y^2 = a^2$, $y \leqslant 0$. So (x, y) runs over the third quadrant part of the disk of radius a, centred on the origin.



- Finally, the inside integral says that, for each (x, y) in the quarter disk, z runs from 0 to $\sqrt{a^2 - x^2 - y^2}$. We can also rewrite $z = \sqrt{a^2 - x^2 - y^2}$ in the more familiar form $x^2 + y^2 + z^2 = a^2$, $z \geqslant 0$.

So the domain of integration is the part of the interior of the sphere of radius a, centred on the origin, that lies in the octant $x \leqslant 0$, $y \leqslant 0$, $z \geqslant 0$.

$$\begin{aligned} V &= \left\{ \ (x,y,z) \ \middle| \ -a \leqslant x \leqslant 0, \ -\sqrt{a^2-x^2} \leqslant y \leqslant 0, \ 0 \leqslant z \leqslant \sqrt{a^2-x^2-y^2} \ \right\} \\ &= \left\{ \ (x,y,z) \ \middle| \ x^2+y^2+z^2 \leqslant a^2, \ x \leqslant 0, \ y \leqslant 0, \ z \geqslant 0 \ \right\} \end{aligned}$$



(a) Note that, in V, (x, y) is restricted to the third quadrant, which in cylindrical coordinates is $\pi \leqslant \theta \leqslant \frac{3\pi}{2}$. So, in cylindrical coordinates,

$$\begin{aligned} V &= \left\{ (r\cos\theta, r\sin\theta, z) \ \middle| \ r^2+z^2 \leqslant a^2, \ \pi \leqslant \theta \leqslant \frac{3\pi}{2}, \ z \geqslant 0 \right\} \\ &= \left\{ (r\cos\theta, r\sin\theta, z) \ \middle| \ 0 \leqslant z \leqslant a, \ \pi \leqslant \theta \leqslant \frac{3\pi}{2}, \ 0 \leqslant r \leqslant \sqrt{a^2-z^2} \right\} \end{aligned}$$

and

$$\begin{aligned} I &= \iiint_V \left(x^2+y^2+z^2\right)^{2014} \mathrm{d}V = \iiint_V \left(r^2+z^2\right)^{2014} r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\ &= \int_0^a \mathrm{d}z \int_\pi^{3\pi/2} \mathrm{d}\theta \int_0^{\sqrt{a^2-z^2}} \mathrm{d}r \ r\left(r^2+z^2\right)^{2014} \end{aligned}$$

(b) The spherical coordinate φ runs from 0 (when the radius vector is along the positive

z–axis) to $\pi/2$ (when the radius vector lies in the xy–plane) so that

$$I = \iiint_V \left(x^2+y^2+z^2\right)^{2014}\,\mathrm{d}V = \iiint_V \rho^{2\times 2014}\,\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$$
$$= \int_0^{\pi/2}\mathrm{d}\varphi\int_\pi^{3\pi/2}\mathrm{d}\theta\int_0^a\mathrm{d}\rho\ \rho^{4030}\sin\varphi$$

(c) Using the spherical coordinate version

$$\begin{aligned} I &= \int_0^{\pi/2}\mathrm{d}\varphi\int_\pi^{3\pi/2}\mathrm{d}\theta\int_0^a\mathrm{d}\rho\ \rho^{4030}\sin\varphi \\ &= \frac{a^{4031}}{4031}\int_0^{\pi/2}\mathrm{d}\varphi\int_\pi^{3\pi/2}\mathrm{d}\theta\ \sin\varphi \\ &= \frac{a^{4031}\pi}{8062}\int_0^{\pi/2}\mathrm{d}\varphi\ \sin\varphi \\ &= \frac{a^{4031}\pi}{8062} \end{aligned}$$

3.7.29 (*) The solid E is bounded below by the paraboloid $z = x^2+y^2$ and above by the cone $z=\sqrt{x^2+y^2}$. Let

$$I = \iiint_E z\left(x^2+y^2+z^2\right)\,\mathrm{d}V$$

(a) Write I in terms of cylindrical coordinates. Do not evaluate.
(b) Write I in terms of spherical coordinates. Do not evaluate.
(c) Calculate I.

Solution (a) In cylindrical coordinates, the paraboloid is $z=r^2$ and the cone is $z=r$. The two meet when $r^2=r$. That is, when $r=0$ and when $r=1$. So, in cylindrical coordinates

$$I = \int_0^1\mathrm{d}r\ r\int_0^{2\pi}\mathrm{d}\theta\int_{r^2}^r\mathrm{d}z\ z(r^2+z^2)$$

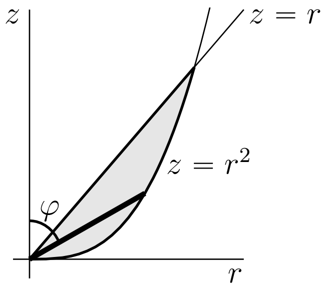
(b) In spherical coordinates, the paraboloid is

$$\rho\cos\varphi = \rho^2\sin^2\varphi \qquad \text{or} \qquad \rho = \frac{\cos\varphi}{\sin^2\varphi}$$

and the cone is

$$\rho\cos\varphi = \rho\sin\varphi \quad \text{or} \quad \tan\varphi = 1 \quad \text{or} \quad \varphi = \frac{\pi}{4}$$

The figure below shows a constant θ cross–section of E. Looking at that figure, we see that φ runs from $\frac{\pi}{4}$ (i.e. the cone) to $\frac{\pi}{2}$ (i.e. the xy–plane).



So, is spherical coordinates,

$$\begin{aligned} I &= \int_{\pi/4}^{\pi/2} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^{\cos\varphi/\sin^2\varphi} \mathrm{d}\rho\ \rho^2 \sin\varphi\ \overbrace{\rho\cos\varphi}^{z}\ \overbrace{\rho^2}^{x^2+y^2+z^2} \\ &= \int_{\pi/4}^{\pi/2} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^{\cos\varphi/\sin^2\varphi} \mathrm{d}\rho\ \rho^5 \sin\varphi\cos\varphi \end{aligned}$$

(c) The cylindrical coordinates integral looks easier.

$$\begin{aligned} I &= \int_0^1 \mathrm{d}r\ r \int_0^{2\pi} \mathrm{d}\theta \int_{r^2}^{r} \mathrm{d}z\ z(r^2+z^2) \\ &= \int_0^1 \mathrm{d}r\ r \int_0^{2\pi} \mathrm{d}\theta\ \left[r^2\frac{z^2}{2} + \frac{z^4}{4}\right]_{r^2}^{r} \\ &= 2\pi \int_0^1 \mathrm{d}r\ r \left[r^2\frac{r^2}{2} + \frac{r^4}{4} - r^2\frac{r^4}{2} - \frac{r^8}{4}\right] \\ &= 2\pi\left[\frac{1}{12} + \frac{1}{24} - \frac{1}{16} - \frac{1}{40}\right] = \frac{\pi}{2}\left[\frac{1}{3} + \frac{1}{6} - \frac{1}{4} - \frac{1}{10}\right] \\ &= \frac{3\pi}{40} \end{aligned}$$

$\boxed{3.7.30}$ (∗) Let S be the region on the first octant (so that $x, y, z \geqslant 0$) which lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $(z-1)^2 + x^2 + y^2 = 1$. Let V be its volume.

(a) Express V as a triple integral in cylindrical coordinates.
(b) Express V as an triple integral in spherical coordinates.
(c) Calculate V using either of the integrals above.

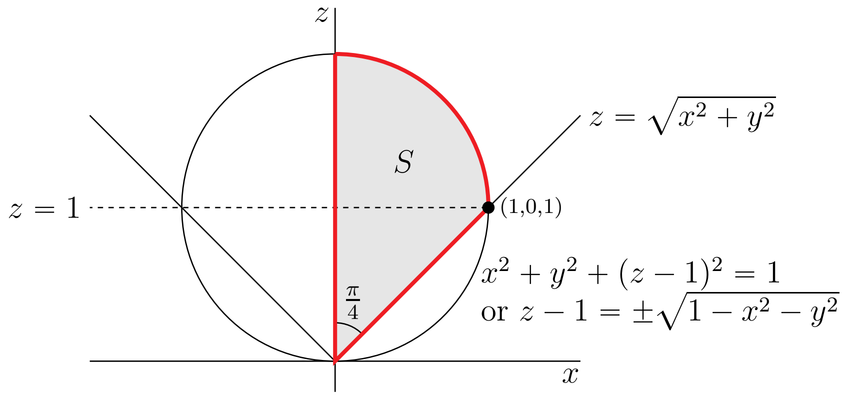
$\boxed{\text{Solution}}$ Note that both the sphere $x^2+y^2+(z-1)^2 = 1$ and the cone $z = \sqrt{x^2+y^2}$ are invariant under rotations around the z–axis. The sphere $x^2+y^2+(z-1)^2 = 1$ and the cone $z = \sqrt{x^2+y^2}$ intersect when $z = \sqrt{x^2+y^2}$, so that $x^2+y^2 = z^2$, and

$$\begin{aligned} x^2+y^2+(z-1)^2 = z^2+(z-1)^2 = 1 &\iff 2z^2-2z = 0 \iff 2z(z-1) = 0 \\ &\iff z = 0, 1 \end{aligned}$$

So the surfaces intersect on the circle $z = 1$, $x^2+y^2 = 1$ and

$$S = \left\{\ (x,y,z)\ \middle|\ x, y \geqslant 0,\ x^2+y^2 \leqslant 1,\ \sqrt{x^2+y^2} \leqslant z \leqslant 1 + \sqrt{1-x^2-y^2}\ \right\}$$

Here is a sketch of the $y = 0$ cross section of S.



(a) In cylindrical coordinates

- the condition $x, y \geqslant 0$ is $0 \leqslant \theta \leqslant \pi/2$,
- the condition $x^2 + y^2 \leqslant 1$ is $r \leqslant 1$, and
- the conditions $\sqrt{x^2 + y^2} \leqslant z \leqslant 1 + \sqrt{1 - x^2 - y^2}$ are $r \leqslant z \leqslant 1 + \sqrt{1 - r^2}$, and
- $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$.

So

$$V = \iiint_S \mathrm{d}V = \int_0^1 \mathrm{d}r \int_0^{\pi/2} \mathrm{d}\theta \int_r^{1+\sqrt{1-r^2}} \mathrm{d}z\ r$$

(b) In spherical coordinates,

- the cone $z = \sqrt{x^2 + y^2}$ becomes

$$\rho\,\cos\varphi = \sqrt{\rho^2 \sin^2\varphi\,\cos^2\theta + \rho^2 \sin^2\varphi\,\sin^2\theta} = \rho\sin\varphi \iff \tan\varphi = 1 \iff \varphi = \frac{\pi}{4}$$

- so that, on S, the spherical coordinate φ runs from $\varphi = 0$ (the positive z –axis) to $\varphi = \pi/4$ (the cone $z = \sqrt{x^2 + y^2}$), which keeps us above the cone,
- the condition $x, y \geqslant 0$ is $0 \leqslant \theta \leqslant \pi/2$,
- the condition $x^2 + y^2 + (z-1)^2 \leqslant 1$, (which keeps us inside the sphere), becomes

$$\begin{aligned}
&\rho^2 \sin^2\varphi\,\cos^2\theta + \rho^2 \sin^2\varphi\,\sin^2\theta + \big(\rho\cos\varphi - 1\big)^2 \leqslant 1 \\
&\iff \rho^2\,\sin^2\varphi + \rho^2\cos^2\varphi - 2\rho\cos\varphi + 1 \leqslant 1 \\
&\iff \rho^2 - 2\rho\cos\varphi \leqslant 0 \\
&\iff \rho \leqslant 2\cos\varphi
\end{aligned}$$

- and $\mathrm{d}V = \rho^2 \sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$.

So

$$V = \iiint_S \mathrm{d}V = \int_0^{\pi/4} \mathrm{d}\varphi \int_0^{\pi/2} \mathrm{d}\theta \int_0^{2\cos\varphi} \mathrm{d}\rho\ \rho^2 \sin\varphi$$

(c) We'll evaluate V using the spherical coordinate integral of part (b).

$$\begin{aligned} V &= \int_0^{\pi/4} \mathrm{d}\varphi \int_0^{\pi/2} \mathrm{d}\theta \int_0^{2\cos\varphi} \mathrm{d}\rho\ \rho^2 \sin\varphi \\ &= \frac{8}{3} \int_0^{\pi/4} \mathrm{d}\varphi \int_0^{\pi/2} \mathrm{d}\theta\ \cos^3\varphi\ \sin\varphi \\ &= \frac{8}{3}\ \frac{\pi}{2}\ \left[-\frac{\cos^4\varphi}{4}\right]_0^{\pi/4} \\ &= \frac{\pi}{3}\left[1 - \frac{1}{(\sqrt{2})^4}\right] \\ &= \frac{\pi}{4} \end{aligned}$$

$\boxed{3.7.31}$ (*) A solid is bounded below by the cone $z = \sqrt{3x^2 + 3y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 9$. It has density $\delta(x, y, z) = x^2 + y^2$.

(a) Express the mass m of the solid as a triple integral in cylindrical coordinates.

(b) Express the mass m of the solid as a triple integral in spherical coordinates.

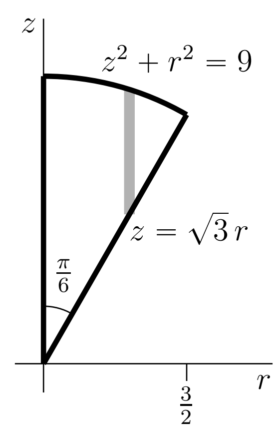
(c) Evaluate m.

$\boxed{\text{Solution}}$ (a) In cylindrical coordinates, the density of is $\delta = x^2 + y^2 = r^2$, the bottom of the solid is at $z = \sqrt{3x^2 + 3y^2} = \sqrt{3}\,r$ and the top of the solid is at $z = \sqrt{9 - x^2 - y^2} = \sqrt{9 - r^2}$. The top and bottom meet when

$$\sqrt{3}\,r = \sqrt{9 - r^2} \iff 3r^2 = 9 - r^2 \iff 4r^2 = 9 \iff r = \frac{3}{2}$$

The mass is

$$m = \int_0^{2\pi} \mathrm{d}\theta \int_0^{3/2} \mathrm{d}r\ r \int_{\sqrt{3}\,r}^{\sqrt{9-r^2}} dz\ \overbrace{r^2}^{\delta}$$



(b) In spherical coordinates, the density of is $\delta = x^2 + y^2 = \rho^2 \sin^2\varphi$, the bottom of the solid is at

$$z = \sqrt{3}\,r \iff \rho\cos\varphi = \sqrt{3}\,\rho\sin\varphi \iff \tan\varphi = \frac{1}{\sqrt{3}} \iff \varphi = \frac{\pi}{6}$$

and the top of the solid is at $x^2 + y^2 + z^2 = \rho^2 = 9$. The mass is

$$m = \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/6} \mathrm{d}\varphi \int_0^{3} \mathrm{d}\rho\ \left(\rho^2\sin\varphi\right) \overbrace{\left(\rho^2\sin^2\varphi\right)}^{\delta}$$

(c) Solution 1: Making the change of variables $s = \cos\varphi$, $\mathrm{d}s = -\sin\varphi\ \mathrm{d}\varphi$, in the integral of part (b),

$$\begin{aligned}
m &= \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/6} \mathrm{d}\varphi \int_0^{3} \mathrm{d}\rho\ \rho^4 \sin\varphi\big(1-\cos^2\varphi\big) \\
&= \frac{3^5}{5} \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/6} \mathrm{d}\varphi\ \sin\varphi\big(1-\cos^2\varphi\big) \\
&= -\frac{3^5}{5} \int_0^{2\pi} \mathrm{d}\theta \int_1^{\sqrt{3}/2} \mathrm{d}s\ \left(1-s^2\right) \\
&= -\frac{3^5}{5} \int_0^{2\pi} \mathrm{d}\theta\ \left[s - \frac{s^3}{3}\right]_1^{\sqrt{3}/2} \\
&= 2\pi\frac{3^5}{5}\left[1 - \frac{1}{3} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{8}\right] \\
&= 2\pi\frac{3^5}{5}\left[\frac{2}{3} - \frac{3\sqrt{3}}{8}\right]
\end{aligned}$$

(c) Solution 2: As an alternate solution, we can also evaluate the integral of part (a).

$$\begin{aligned}
m &= \int_0^{2\pi} \mathrm{d}\theta \int_0^{3/2} \mathrm{d}r\ r \int_{\sqrt{3}\,r}^{\sqrt{9-r^2}} dz\ r^2 \\
&= \int_0^{2\pi} \mathrm{d}\theta \int_0^{3/2} \mathrm{d}r\ r^3\big(\sqrt{9-r^2} - \sqrt{3}\,r\big) \\
&= 2\pi \int_0^{3/2} \mathrm{d}r\ r^3\big(\sqrt{9-r^2} - \sqrt{3}\,r\big)
\end{aligned}$$

The second term

$$-2\pi \int_0^{3/2} \mathrm{d}r\ \sqrt{3}\,r^4 = -2\pi\sqrt{3}\,\frac{r^5}{5}\bigg|_0^{3/2} = -2\pi\sqrt{3}\,\frac{3^5}{5\times 2^5}$$

For the first term, we substitute $s = 9 - r^2$, $\mathrm{d}s = -2r\,\mathrm{d}r$.

$$\begin{aligned}
2\pi \int_0^{3/2} \mathrm{d}r\ r^3\sqrt{9-r^2} &= 2\pi \int_9^{27/4} \overbrace{\frac{\mathrm{d}s}{-2}}^{r\,\mathrm{d}r}\ \overbrace{(9-s)}^{r^2}\sqrt{s} = -\pi\left[6s^{3/2} - \frac{2}{5}s^{5/2}\right]_9^{27/4} \\
&= -\pi\left[\frac{3^5\sqrt{3}}{4} - 2\times 3^4 - \frac{3^7}{2^4 5}\sqrt{3} + 2\frac{3^5}{5}\right]
\end{aligned}$$

Adding the two terms together,

$$\begin{aligned} m &= -2\pi\frac{3^5}{5}\frac{\sqrt{3}}{32} - 2\pi\frac{3^5}{5}\frac{5\sqrt{3}}{8} + 2\pi\frac{3^5}{5}\frac{5}{3} + 2\pi\frac{3^5}{5}\frac{9\sqrt{3}}{32} - 2\pi\frac{3^5}{5} \\ &= 2\pi\frac{3^5}{5}\left[\left(\frac{5}{3} - 1\right) - \sqrt{3}\left(\frac{1}{32} + \frac{5}{8} - \frac{9}{32}\right)\right] \\ &= 2\pi\frac{3^5}{5}\left[\frac{2}{3} - \frac{3\sqrt{3}}{8}\right] \end{aligned}$$