第四章 積分

4.1 不定積分

定義 (反導函數). 給定 $F(x)$, 若 $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$, 則稱 $F(x)$ 為 $f(x)$ 的反導函數 (antiderivative) .

性質. 若 $F(x)$, $G(x)$ 分別為 $f(x)$, $g(x)$ 的反導函數, $c \in \mathbb{R}$. 則

- $F(x) + c$ 為 $f(x)$ 的反導函數.
- $c\,F(x)$ 為 $c\,f(x)$ 的反導函數.
- $F(x) + G(x)$ 為 $f(x) + g(x)$ 的反導函數.

結論.

- $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x) \implies \mathrm{d}F(x) = f(x)\cdot\mathrm{d}x \implies F(x) = \int f(x)\cdot\mathrm{d}x = \int f(x)\,\mathrm{d}x$
- $F(x)$ 為 $f(x)$ 的反導函數 $\iff$ $f(x)$ 的反導函數為 $F(x)$ $\iff$ $F(x)$ 的導函數為 $f(x)$ $\iff$ $F(x)$ (對 x) 的微分為 $f(x)$ $\iff$ $f(x)$ (對 x) 的 (不定) 積分為 $F(x)$
- $f(x)$ 的反導函數 $\equiv$ $f(x)$ (對 x) 的 (不定) 積分
- 基礎積分集: 以下 $\alpha \neq -1$, $a \neq 0$.

$f(x)$	x^α	$\frac{1}{x}$	e^{ax}	$\sin ax$	$\cos ax$	$\frac{1}{\sqrt{a^2-x^2}}$	$\frac{1}{a^2+x^2}$
$\int f(x)\,\mathrm{d}x$	$\frac{1}{\alpha+1}x^{\alpha+1}$	$\ln\lvert x\rvert$	$\frac{1}{a}e^{ax}$	$-\frac{1}{a}\cos ax$	$\frac{1}{a}\sin ax$	$\sin^{-1}\frac{x}{\lvert a\rvert}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$

- (Liouville) e^{-x^2}, $\frac{e^x}{x}$, $\frac{1}{\ln x}$, $\sin(x^2)$, $\cos(x^2)$, $\frac{\sin x}{x}$, $\frac{\cos x}{x}$, x^x 無 (初等函數形式之) 反導函數!

例.

1. $\int \sqrt{x}\,\mathrm{d}x = \frac{2}{3}x^{\frac{3}{2}} + c$
2. $\int x^\pi\,\mathrm{d}x = \frac{1}{\pi+1}x^{\pi+1} + c$
3. $\int \sin\pi x\,\mathrm{d}x = -\frac{1}{\pi}\cos\pi x + c$
4. $\int \cos xy\,\mathrm{d}x = \frac{1}{y}\sin xy + c$
5. $\int e^{-2x}\,\mathrm{d}x = -\frac{1}{2}e^{-2x} + c$
6. $\int \frac{1}{\sqrt{\pi - x^2}}\,\mathrm{d}x = \sin^{-1}\frac{x}{\sqrt{\pi}} + c$
7. $\int \frac{1}{e+u^2}\,\mathrm{d}u = \frac{1}{\sqrt{e}}\tan^{-1}\frac{u}{\sqrt{e}} + c$
8. $\int \left(\frac{\pi}{x} - e^{\pi x}\right)\mathrm{d}u = \pi\ln x - \frac{e^{\pi x}}{\pi} + c$
9. $\int \frac{3+x^2}{1+x^2}\,\mathrm{d}u = x + 2\tan^{-1}x + c$

習題. 求下列不定積分.

1. $\int \frac{x^3-1}{x^3}\,\mathrm{d}x = x + \frac{1}{2x^3} + c$
2. $\int 5 - \frac{1}{\sqrt{x}}\,\mathrm{d}x = 5x - 2\sqrt{x} + c$
3. $\int (t-1)(t+1)\,\mathrm{d}t = \frac{t^3}{3} - t + c$
4. $\int (\sqrt{x}+1)^2\,\mathrm{d}x = \frac{x^2}{2} + x + \frac{4x^{\frac{3}{2}}}{3} + c$
5. $\int x\sqrt{3x}\,\mathrm{d}x = \frac{2\sqrt{3}}{5}x^{\frac{5}{2}} + c$
6. $\int \frac{1}{x^3} - \frac{1}{x^5}\,\mathrm{d}x = \frac{-1}{2x^2} + \frac{1}{4x^4} + c$
7. $\int \sec^2 x - \sec x\tan x\,\mathrm{d}x = \tan x - \sec x + c$
8. $\int \frac{e^{3x}+1}{e^x+1}\,\mathrm{d}x = \frac{e^{2x}}{2} - e^x + x + c$

變數變換法

結論. $\dfrac{\mathrm{d}}{\mathrm{d}x}f(g(x)) = f'(g(x))\cdot g'(x) \implies \mathrm{d}f(g(x)) = f'(g(x))\cdot g'(x)\,\mathrm{d}x \implies f(g(x)) = \displaystyle\int f'(g(x))\cdot g'(x)\,\mathrm{d}x.$
令 $u = g(x)$, 則 $\dfrac{\mathrm{d}u}{\mathrm{d}x} = g'(x) \implies \mathrm{d}u = g'(x)\,\mathrm{d}x$; 故 $\displaystyle\int f'(g(x))\cdot g'(x)\,\mathrm{d}x = \int f'(u)\,\mathrm{d}u = f(u)+c = f(g(x))+c.$

例. 求 $\displaystyle\int \frac{x}{\sqrt{x+1}}\,\mathrm{d}x.$

解.

- (解一) 令 $u = x+1$, 則 $x = u-1$, $\mathrm{d}u = \mathrm{d}x$. 故 $\displaystyle\int \frac{x}{\sqrt{x+1}}\,\mathrm{d}x = \int \frac{u-1}{\sqrt{u}}\,\mathrm{d}u = \int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,\mathrm{d}u = \frac{2}{3}u^{\frac{3}{2}} - 2\sqrt{u} + c = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c.$
- (解二) $\displaystyle\int \frac{x}{\sqrt{x+1}}\,\mathrm{d}x = \int \frac{x+1-1}{\sqrt{x+1}}\,\mathrm{d}x = \int \sqrt{x+1} - \frac{1}{\sqrt{x+1}}\,\mathrm{d}x.$ 令 $u = x+1$, 則 $\mathrm{d}u = \mathrm{d}x$. 故 $\displaystyle\int \sqrt{x+1} - \frac{1}{\sqrt{x+1}}\,\mathrm{d}x = \int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,\mathrm{d}u = \frac{2}{3}u^{\frac{3}{2}} - 2\sqrt{u} + c = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c.$
- (解三) 令 $u = \sqrt{x+1}$, 則 $x = u^2 - 1$, $\mathrm{d}u = \dfrac{1}{2\sqrt{x+1}}\,\mathrm{d}x \implies \dfrac{1}{\sqrt{x+1}}\,\mathrm{d}x = 2\,\mathrm{d}u$. 故 $\displaystyle\int \frac{x}{\sqrt{x+1}}\,\mathrm{d}x = \int x\cdot\frac{1}{\sqrt{x+1}}\,\mathrm{d}x = \int (u^2-1)\cdot 2\,\mathrm{d}u = \frac{2}{3}u^3 - 2u + c = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c.$

例. 求 $\displaystyle\int \frac{x}{x^2+1}\,\mathrm{d}x.$

解. 令 $u = x^2+1$, 則 $\mathrm{d}u = 2\,x\,\mathrm{d}x \implies x\,\mathrm{d}x = \dfrac{1}{2}\,\mathrm{d}u$. 故 $\displaystyle\int \frac{x}{x^2+1}\,\mathrm{d}x = \int \frac{1}{x^2+1}\cdot x\,\mathrm{d}x = \int \frac{1}{u}\cdot\frac{1}{2}\,\mathrm{d}u = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln(x^2+1) + c.$

例. 求 $\displaystyle\int \frac{\sin(3\ln x)}{x}\,\mathrm{d}x.$

解. 令 $u = \ln x$, 則 $\mathrm{d}u = \dfrac{1}{x}\,\mathrm{d}x$. 故 $\displaystyle\int \frac{\sin(3\ln x)}{x}\,\mathrm{d}x = \int \sin(3\ln x)\cdot\frac{1}{x}\,\mathrm{d}x = \int \sin 3u\cdot\mathrm{d}u = -\frac{1}{3}\cos 3u + c = -\frac{1}{3}\cos(3\ln x) + c.$

例. 求 $\displaystyle\int e^x\sqrt{1+e^x}\,\mathrm{d}x.$

解. 令 $u = 1+e^x$, 則 $\mathrm{d}u = e^x\,\mathrm{d}x$. 故 $\displaystyle\int e^x\sqrt{1+e^x}\,\mathrm{d}x = \int \sqrt{1+e^x}\cdot e^x\,\mathrm{d}x = \int \sqrt{u}\cdot\mathrm{d}u = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}(1+e^x)^{\frac{3}{2}} + c$

例. 求 $\displaystyle\int \frac{e^x}{\sqrt{2-e^{2x}}}\,\mathrm{d}x.$

解. 令 $u = e^x$, 則 $\mathrm{d}u = e^x\,\mathrm{d}x$. 故 $\displaystyle\int \frac{e^x}{\sqrt{2-e^{2x}}}\,\mathrm{d}x = \int \frac{1}{\sqrt{2-u^2}}\,\mathrm{d}u = \sin^{-1}\frac{u}{\sqrt{2}} + c = \sin^{-1}\frac{e^x}{\sqrt{2}} + c.$

例. 求 $\displaystyle\int \frac{1}{\sqrt{e^{2x}-1}}\,\mathrm{d}x.$

解. $\displaystyle\int\frac{1}{\sqrt{e^{2x}-1}}\,\mathrm{d}x=\int\frac{1}{e^x\sqrt{1-e^{-2x}}}\,\mathrm{d}x=\int\frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,\mathrm{d}x$. 令 $u=e^{-x}$, 則 $\mathrm{d}u=-e^{-x}\,\mathrm{d}x \implies e^{-x}\,\mathrm{d}x=-\mathrm{d}u$; 故 $\displaystyle\int\frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,\mathrm{d}x=\int\frac{1}{\sqrt{1-e^{-2x}}}\cdot e^{-x}\,\mathrm{d}x=-\int\frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u=-\sin^{-1}u+c=-\sin^{-1}e^{-x}+c$.

例. 求 $\displaystyle\int\frac{1}{x^2+4x+5}\,\mathrm{d}x$.

解. $\displaystyle\int\frac{1}{x^2+4x+5}\,\mathrm{d}x=\int\frac{1}{(x+2)^2+1}\,\mathrm{d}x$. 令 $u=x+2$, 則 $\mathrm{d}u=\mathrm{d}x$; 故 $\displaystyle\int\frac{1}{(x+2)^2+1}\,\mathrm{d}x=\int\frac{1}{u^2+1}\,\mathrm{d}u=\tan^{-1}u+c=\tan^{-1}(x+2)+c$.

例. 求 $\displaystyle\int\frac{1}{\sqrt{4+2x-x^2}}\,\mathrm{d}x$

解. $\displaystyle\int\frac{1}{\sqrt{4+2x-x^2}}\,\mathrm{d}x=\int\frac{1}{\sqrt{5-(x-1)^2}}\,\mathrm{d}x$. 令 $u=x-1$, 則 $\mathrm{d}u=\mathrm{d}x$; 故 $\displaystyle\int\frac{1}{\sqrt{5-(x-1)^2}}\,\mathrm{d}x=\int\frac{1}{\sqrt{5-u^2}}\,\mathrm{d}u=\sin^{-1}\frac{u}{\sqrt5}+c=\sin^{-1}\frac{x-1}{\sqrt5}+c$.

例. 求 $\displaystyle\int\tan x\,\mathrm{d}x$.

解. $\displaystyle\int\tan x\,\mathrm{d}x=\int\frac{\sin x}{\cos x}\,\mathrm{d}x$. 令 $u=\cos x$, 則 $\mathrm{d}u=-\sin x\,\mathrm{d}x$; 故 $\displaystyle\int\frac{\sin x}{\cos x}\,\mathrm{d}x=\int\frac{-1}{u}\,\mathrm{d}u=-\ln|u|+c=-\ln|\cos x|+c=\ln|\sec x|+c$

例. 求 $\displaystyle\int\cos^5 ax\,\mathrm{d}x$, $a\neq 0$.

解. $\displaystyle\int\cos^5 ax\,\mathrm{d}x=\int(1-\sin^2 ax)^2\cos ax\,\mathrm{d}x$. 令 $u=\sin ax$, 則 $\mathrm{d}u=a\cos ax\,\mathrm{d}x$; 故 $\displaystyle\int(1-\sin^2 ax)^2\cos ax\,\mathrm{d}x=\frac1a\int(1-u^2)^2\,\mathrm{d}u=\frac1a\int(1-2u^2+u^4)\,\mathrm{d}u=\frac1a\left(u-\frac23u^3+\frac15u^5\right)+c=\frac1a\left(\sin ax-\frac23\sin^3 ax+\frac15\sin^5 ax\right)+c$.

例. 求 $\displaystyle\int\cos^4 ax\,\mathrm{d}x$, $a\neq 0$.

解. $\displaystyle\int\cos^4 ax\,\mathrm{d}x=\int\left(\frac{1+\cos 2ax}{2}\right)^2\mathrm{d}x=\frac14\int\left(1+2\cos 2ax+\cos^2 2ax\right)\mathrm{d}x=\frac14\int\left(1+2\cos 2ax+\frac{1+\cos 4ax}{2}\right)\mathrm{d}x$ $\displaystyle=\int\left(\frac38+\frac12\cos 2ax+\frac18\cos 4ax\right)\mathrm{d}x=\frac38x+\frac{1}{4a}\sin 2ax+\frac{1}{32a}\sin 4ax+c$

例. 求 $\displaystyle\int\sec x\,\mathrm{d}x$.

解. $\displaystyle\int\sec x\,\mathrm{d}x=\int\frac{\sec x\cdot(\sec x+\tan x)}{\sec x+\tan x}\,\mathrm{d}x$. 令 $u=\sec x+\tan x$, 則 $\mathrm{d}u=(\sec^2 x+\sec x\tan x)\,\mathrm{d}x$; 故 $\displaystyle\int\frac{\sec x\cdot(\sec x+\tan x)}{\sec x+\tan x}\,\mathrm{d}x=\int\frac1u\,\mathrm{d}u=\ln|u|+c=\ln|\sec x+\tan x|+c$

例. 令 $\displaystyle T_1=\int\frac{\sin x}{a\cos x+b\sin x}\,\mathrm{d}x$, $\displaystyle T_2=\int\frac{\cos x}{a\cos x+b\sin x}\,\mathrm{d}x$, a, $b\neq 0$, 求 T_1, T_2.

解.

(a) $\displaystyle bT_1+aT_2=\int\frac{b\sin x}{a\cos x+b\sin x}\,\mathrm{d}x+\int\frac{a\cos x}{a\cos x+b\sin x}\,\mathrm{d}x=\int\frac{b\sin x+a\cos x}{a\cos x+b\sin x}\,\mathrm{d}x=\int 1\,\mathrm{d}x=x$.

(b) $-aT_1+bT_2=\int\frac{-a\sin x}{a\cos x+b\sin x}\,\mathrm{d}x+\int\frac{b\cos x}{a\cos x+b\sin x}\,\mathrm{d}x=\int\frac{-a\sin x+b\cos x}{a\cos x+b\sin x}\,\mathrm{d}x=\int\frac{\mathrm{d}u}{u}=\ln u=\ln|a\cos x+b\sin x|$ (令 $u=a\cos x+b\sin x$, 則 $\mathrm{d}u=(-a\sin x+b\cos x)\,\mathrm{d}x)$.

解 T_1, T_2 方程式 (a), (b) 得 $T_1=\frac{bx-a\ln|a\cos x+b\sin x|}{a^2+b^2}$, $T_2=\frac{ax+b\ln|a\cos x+b\sin x|}{a^2+b^2}$.

習題. 以變數變換法求下列不定積分. 注意: 可能會因為常數項而跟此處答案不同.

1. $\int\frac{1}{\sqrt{2x-1}}\,\mathrm{d}x$
2. $\int\sqrt{7x+4}\,\mathrm{d}x$
3. $\int\sin(3x-1)\,\mathrm{d}x$
4. $\int e^{\pi x-1}\,\mathrm{d}x$
5. $\int(x^2-2x+1)^{\frac{1}{3}}\,\mathrm{d}x$
6. $\int\sin 3x\cos 3x\,\mathrm{d}x$
7. $\int\frac{\cos 3x}{\sin^2 3x}\,\mathrm{d}x$
8. $\int\frac{x}{\sqrt{1+2x^2}}\,\mathrm{d}x$
9. $\int x^2\sqrt{1-x}\,\mathrm{d}x$

解.

1. 令 $u=2x-1$, 則 $\mathrm{d}u=2\,\mathrm{d}x\implies\mathrm{d}x=\frac{1}{2}\mathrm{d}u$. 故 $\int\frac{1}{\sqrt{2x-1}}\,\mathrm{d}x=\int\frac{1}{\sqrt{u}}\cdot\frac{1}{2}\,\mathrm{d}u=\sqrt{u}+c=\sqrt{2x-1}+c$.
2. 令 $u=7x+4$, 則 $\mathrm{d}u=7\,\mathrm{d}x\implies\mathrm{d}x=\frac{1}{7}\mathrm{d}u$. 故 $\int\sqrt{7x+4}\,\mathrm{d}x=\int\sqrt{u}\cdot\frac{1}{7}\,\mathrm{d}u=\frac{1}{7}\cdot\frac{2}{3}u^{\frac{3}{2}}+c=\frac{2}{21}(7x+4)^{\frac{3}{2}}+c$.
3. 令 $u=3x-1$, 則 $\mathrm{d}u=3\,\mathrm{d}x\implies\mathrm{d}x=\frac{1}{3}\mathrm{d}u$. 故 $\int\sin(3x-1)\,\mathrm{d}x=\int\sin u\cdot\frac{1}{3}\,\mathrm{d}u=-\frac{1}{3}\cos u+c=\frac{-\cos(3x-1)}{3}+c$.
4. 令 $u=\pi x-1$, 則 $\mathrm{d}u=\pi\,\mathrm{d}x\implies\mathrm{d}x=\frac{1}{\pi}\mathrm{d}u$. 故 $\int e^{\pi x-1}\,\mathrm{d}x=\int e^u\cdot\frac{1}{\pi}\,\mathrm{d}u=\frac{1}{\pi}e^u+c=\frac{e^{\pi x-1}}{\pi}+c$.
5. 令 $u=x-1$, 則 $\mathrm{d}u=\mathrm{d}x$. 故 $\int(x^2-2x+1)^{\frac{1}{3}}\,\mathrm{d}x=\int(x-1)^{\frac{2}{3}}\,\mathrm{d}x=\int u^{\frac{2}{3}}\,\mathrm{d}u=\frac{3}{5}u^{\frac{5}{3}}+c=\frac{3}{5}(x-1)^{\frac{5}{3}}+c$.
6. 令 $u=\sin 3x$, 則 $\mathrm{d}u=3\cos 3x\,\mathrm{d}x\implies\cos 3x\,\mathrm{d}x=\frac{1}{3}\mathrm{d}u$. 故 $\int\sin 3x\cos 3x\,\mathrm{d}x=\int u\cdot\frac{1}{3}\,\mathrm{d}u=\frac{1}{3}\cdot\frac{1}{2}u^2+c=\frac{\sin^2 3x}{6}+c$.
7. 令 $u=\sin 3x$, 則 $\mathrm{d}u=3\cos 3x\,\mathrm{d}x\implies\cos 3x\,\mathrm{d}x=\frac{1}{3}\mathrm{d}u$. 故 $\int\frac{\cos 3x}{\sin^2 3x}\,\mathrm{d}x=\int\frac{1}{u^2}\cdot\frac{1}{3}\,\mathrm{d}u=-\frac{1}{3}\cdot\frac{1}{u}+c=-\frac{1}{3\sin 3x}+c$.
8. 令 $u=1+2x^2$, 則 $\mathrm{d}u=4x\,\mathrm{d}x\implies x\,\mathrm{d}x=\frac{1}{4}\mathrm{d}u$. 故 $\int\frac{x}{\sqrt{1+2x^2}}\,\mathrm{d}x=\int\frac{1}{\sqrt{1+2x^2}}\cdot x\,\mathrm{d}x=\int\frac{1}{\sqrt{u}}\cdot\frac{1}{4}\,\mathrm{d}u=\frac{1}{4}\cdot 2u^{\frac{1}{2}}+c=\frac{\sqrt{1+2x^2}}{2}+c$.
9. 令 $u=\sqrt{1-x}$, 則 $u^2=1-x\implies x=1-u^2$, $\mathrm{d}x=-2u\,\mathrm{d}u$. 故 $\int x^2\sqrt{1-x}\,\mathrm{d}x=\int(1-u^2)^2\cdot u\cdot(-2)u\,\mathrm{d}u=-2\int(1-u^2)^2\cdot u^2\,\mathrm{d}u=-2\int\left(u^2-2u^4+u^6\right)\mathrm{d}u=-\frac{2u^3}{3}+\frac{4u^5}{5}-\frac{2u^7}{7}+c=-\frac{2(1-x)^{\frac{3}{2}}}{3}+\frac{4(1-x)^{\frac{5}{2}}}{5}-\frac{2(1-x)^{\frac{7}{2}}}{7}$.

習題. 以變數變換法求下列不定積分, 其中 $a\neq b\neq 0$. 注意: 可能會因為常數項而跟此處答案不同.

1. $\int e^{2x}\sin e^{2x}\,\mathrm{d}x$
2. $\int xe^{-\frac{x^2}{2}}\,\mathrm{d}x$
3. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x$
4. $\int x^2 2^{x^3+1}\,\mathrm{d}x$
5. $\int \frac{\ln x}{x}\,\mathrm{d}x$
6. $\int \frac{x+1}{\sqrt{x^2+2x+3}}\,\mathrm{d}x$
7. $\int \frac{x^2}{2+x^6}\,\mathrm{d}x$
8. $\int \frac{1}{e^x+e^{-x}}\,\mathrm{d}x$
9. $\int \frac{x+1}{\sqrt{1-x^2}}\,\mathrm{d}x$
10. $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,\mathrm{d}x$
11. $\int \sin^4 x\cos^5 x\,\mathrm{d}x$
12. $\int \sin^2 x\cos^2 x\,\mathrm{d}x$
13. $\int \sin^{-\frac{2}{3}} x\cos^3 x\,\mathrm{d}x$
14. $\int \frac{\sin^3(\ln x)\cos^3(\ln x)}{x}\,\mathrm{d}x$
15. $\int \frac{\sin^3 x}{\cos^4 x}\,\mathrm{d}x$
16. $\int \cos ax\cos bx\,\mathrm{d}x$
17. $\int \sin ax\sin bx\,\mathrm{d}x$
18. $\int \sin ax\cos bx\,\mathrm{d}x$

解.

1. 令 $u=e^{2x}$, 則 $\mathrm{d}u=2e^{2x}\,\mathrm{d}x \implies e^{2x}\,\mathrm{d}x=\frac{1}{2}\,\mathrm{d}u$. 故 $\int e^{2x}\sin e^{2x}\,\mathrm{d}x=\int \sin e^{2x}\cdot e^{2x}\,\mathrm{d}x=\int \sin u\cdot\frac{1}{2}\,\mathrm{d}u=-\frac{1}{2}\cos u+c=-\frac{1}{2}\cos e^{2x}+c$.
2. 令 $u=\frac{x^2}{2}$, 則 $\mathrm{d}u=x\,\mathrm{d}x$, 故 $\int xe^{-\frac{x^2}{2}}\,\mathrm{d}x=\int e^{-\frac{x^2}{2}}\cdot x\,\mathrm{d}x=\int e^{-u}\,\mathrm{d}u=-e^{-u}+c=-e^{-\frac{x^2}{2}}+c$
3. 令 $u=\sqrt{x}$, 則 $\mathrm{d}u=\frac{1}{2\sqrt{x}}\,\mathrm{d}x \implies \frac{1}{\sqrt{x}}\,\mathrm{d}x=2\,\mathrm{d}u$, 故 $\int\frac{\sin\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x=\int \sin\sqrt{x}\cdot\frac{1}{\sqrt{x}}\,\mathrm{d}x=\int \sin u\cdot 2\,\mathrm{d}u=-2\cos u+c=-2\cos\sqrt{x}+c$
4. 令 $u=x^3+1$, 則 $\mathrm{d}u=3x^2\,\mathrm{d}x \implies x^2\,\mathrm{d}x=\frac{1}{3}\,\mathrm{d}u$, 故 $\int x^2 2^{x^3+1}\,\mathrm{d}x=\int 2^{x^3+1}\cdot x^2\,\mathrm{d}x=\int 2^u\cdot\frac{1}{3}\,\mathrm{d}u=\frac{1}{3}\int e^{u\ln 2}\,\mathrm{d}u=\frac{1}{3\ln 2}e^{u\ln 2}+c=\frac{2^{x^3+1}}{3\ln 2}+c$
5. 令 $u=\ln x$, 則 $\mathrm{d}u=\frac{1}{x}\,\mathrm{d}x$, 故 $\int\frac{\ln x}{x}\,\mathrm{d}x=\int \ln x\cdot\frac{1}{x}\,\mathrm{d}x=\int u\,\mathrm{d}u=\frac{1}{2}u^2+c=\frac{1}{2}(\ln x)^2+c$
6. 令 $u=x^2+2x+3$, 則 $\mathrm{d}u=(2x+2)\,\mathrm{d}x \implies (x+1)\,\mathrm{d}x=\frac{1}{2}\,\mathrm{d}u$, 故 $\int\frac{x+1}{\sqrt{x^2+2x+3}}\,\mathrm{d}x=\int\frac{1}{\sqrt{x^2+2x+3}}\cdot(x+1)\,\mathrm{d}x=\int\frac{1}{\sqrt{u}}\cdot\frac{1}{2}\,\mathrm{d}u=\sqrt{u}+c=\sqrt{x^2+2x+3}+c$
7. 令 $u=x^3$, 則 $\mathrm{d}u=3x^2\,\mathrm{d}x \implies x^2\,\mathrm{d}x=\frac{1}{3}\,\mathrm{d}u$, 故 $\int\frac{x^2}{2+x^6}\,\mathrm{d}x=\int\frac{1}{2+x^6}\cdot x^2\,\mathrm{d}x=\int\frac{1}{2+u^2}\cdot\frac{1}{3}\,\mathrm{d}u=\frac{1}{3\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}}+c=\frac{1}{3\sqrt{2}}\tan^{-1}\frac{x^3}{\sqrt{2}}+c$
8. $\int\frac{1}{e^x+e^{-x}}\,\mathrm{d}x=\int\frac{e^x}{e^{2x}+1}\,\mathrm{d}x$. 令 $u=e^x$, 則 $\mathrm{d}u=e^x\,\mathrm{d}x$, 故 $\int\frac{e^x}{e^{2x}+1}\,\mathrm{d}x=\int\frac{1}{e^{2x}+1}\cdot e^x\,\mathrm{d}x=\int\frac{1}{u^2+1}\,\mathrm{d}u=\tan^{-1}u+c=\tan^{-1}e^x+c$.
9. $\int\frac{x+1}{\sqrt{1-x^2}}\,\mathrm{d}x=\int\frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x+\int\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$. 令 $u=\sqrt{1-x^2}$, 則 $\mathrm{d}u=\frac{-x}{\sqrt{1-x^2}}\,\mathrm{d}x \implies \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x=-\mathrm{d}u$, 故 $\int\frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x=\int -\mathrm{d}u=-u+c=-\sqrt{1-x^2}$, $\int\frac{x+1}{\sqrt{1-x^2}}\,\mathrm{d}x=\int\frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x+\int\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x=-\sqrt{1-x^2}+\sin^{-1}x+c$.

10. 令 $u=\tan x$, 則 $\mathrm{d}u=\sec^2 x\,\mathrm{d}x$, 故 $\displaystyle\int\frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,\mathrm{d}x=\int\frac{1}{\sqrt{1-\tan^2 x}}\cdot\sec^2 x\,\mathrm{d}x=\int\frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u=\sin^{-1}u+c=\sin^{-1}(\tan x)+c$

11. $\displaystyle\int\sin^4 x\cos^5 x\,\mathrm{d}x=\int\sin^4 x\cdot\cos^4 x\cdot\cos x\,\mathrm{d}x=\int\sin^4 x\cdot(1-\sin^2 x)^2\cdot\cos x\,\mathrm{d}x=\int\big(\sin^4 x-2\sin^6 x+\sin^8 x\big)\cdot\cos x\,\mathrm{d}x$. 令 $u=\sin x$, 則 $\mathrm{d}u=\cos x\,\mathrm{d}x$, 故 $\displaystyle\int\big(\sin^4 x-2\sin^6 x+\sin^8 x\big)\cdot\cos x\,\mathrm{d}x=\int\big(u^4-2u^6+u^8\big)\,\mathrm{d}u=\frac{u^5}{5}-\frac{2u^7}{7}+\frac{u^9}{9}+c=\frac{\sin^5 x}{5}-\frac{2\sin^7 x}{7}+\frac{\sin^9 x}{9}+c$

12. $\displaystyle\int\sin^2 x\cos^2 x\,\mathrm{d}x=\int\frac{1-\cos 2x}{2}\cdot\frac{1+\cos 2x}{2}\,\mathrm{d}x=\frac{1}{4}\int\big(1-\cos^2 2x\big)\,\mathrm{d}x=\frac{1}{4}\int\Big(1-\frac{1+\cos 4x}{2}\Big)\,\mathrm{d}x=\frac{1}{4}\int\Big(\frac{1}{2}+\frac{\cos 4x}{2}\Big)\,\mathrm{d}x=\frac{x}{8}-\frac{\sin 4x}{32}+c$

13. $\displaystyle\int\sin^{-\frac{2}{3}}x\cos^3 x\,\mathrm{d}x=\int\sin^{-\frac{2}{3}}x\cdot\cos^2 x\cdot\cos x\,\mathrm{d}x=\int\sin^{-\frac{2}{3}}x\cdot(1-\sin^2 x)\cdot\cos x\,\mathrm{d}x=\int\big(\sin^{-\frac{2}{3}}x-\sin^{\frac{4}{3}}x\big)\cdot\cos x\,\mathrm{d}x$. 令 $u=\sin x$, 則 $\mathrm{d}u=\cos x\,\mathrm{d}x$, 故 $\displaystyle\int\big(\sin^{-\frac{2}{3}}x-\sin^{\frac{4}{3}}x\big)\cdot\cos x\,\mathrm{d}x=\int\big(u^{-\frac{2}{3}}-u^{\frac{4}{3}}\big)\,\mathrm{d}u=3u^{\frac{1}{3}}-\frac{3}{7}u^{\frac{7}{3}}+c=3\sin^{\frac{1}{3}}x-\frac{3}{7}\sin^{\frac{7}{3}}x+c$

14. 令 $u=\ln x$, 則 $\mathrm{d}u=\dfrac{1}{x}\,\mathrm{d}x$, 故 $\displaystyle\int\frac{\sin^3(\ln x)\cos^3(\ln x)}{x}\,\mathrm{d}x=\int\sin^3(\ln x)\cos^3(\ln x)\cdot\frac{1}{x}\,\mathrm{d}x=\int\sin^3 u\cos^3 u\,\mathrm{d}u=\int\sin^3 u\cdot\cos^2 u\cdot\cos u\,\mathrm{d}u=\int\sin^3 u\big(1-\sin^2 u\big)\cdot\cos u\,\mathrm{d}u=\int\big(\sin^3 u-\sin^5 u\big)\cdot\cos u\,\mathrm{d}u$. 令 $w=\sin u$, 則 $\mathrm{d}w=\cos u\,\mathrm{d}u$, 故 $\displaystyle\int\big(\sin^3 u-\sin^5 u\big)\cdot\cos u\,\mathrm{d}u=\int\big(w^3-w^5\big)\,\mathrm{d}w=\frac{w^4}{4}-\frac{w^6}{6}+c=\frac{\sin^4(\ln x)}{4}-\frac{\sin^6(\ln x)}{6}+c$

15. $\displaystyle\int\frac{\sin^3 x}{\cos^4 x}\,\mathrm{d}x=\int\frac{\sin^2 x\cdot\sin x}{\cos^4 x}\,\mathrm{d}x=\int\frac{\sin^2 x}{\cos^4 x}\cdot\sin x\,\mathrm{d}x=\int\frac{1-\cos^2 x}{\cos^4 x}\cdot\sin x\,\mathrm{d}x=\int\Big(\frac{1}{\cos^4 x}-\frac{1}{\cos^2 x}\Big)\cdot\sin x\,\mathrm{d}x$. 令 $u=\cos x$, 則 $\mathrm{d}u=-\sin x\,\mathrm{d}x\implies\sin x\,\mathrm{d}x=-\mathrm{d}u$, 故 $\displaystyle\int\Big(\frac{1}{\cos^4 x}-\frac{1}{\cos^2 x}\Big)\cdot\sin x\,\mathrm{d}x=\int\Big(\frac{1}{u^4}-\frac{1}{u^2}\Big)(-)\,\mathrm{d}u=\int\Big(\frac{1}{u^2}-\frac{1}{u^4}\Big)\,\mathrm{d}u=-\frac{1}{u}+\frac{1}{3u^3}+c=-\frac{1}{\cos x}+\frac{1}{3\cos^3 x}+c$

16. 由積化和差公式 $\cos\alpha\cos\beta=\dfrac{1}{2}\big(\cos(\alpha-\beta)+\cos(\alpha+\beta)\big)$, $\displaystyle\int\cos ax\cos bx\,\mathrm{d}x=\int\frac{1}{2}\big(\cos(ax-bx)+\cos(ax+bx)\big)\,\mathrm{d}x=\int\frac{1}{2}\big(\cos(a-b)x+\cos(a+b)x\big)\,\mathrm{d}x=\frac{\sin(a-b)x}{2(a-b)}+\frac{\sin(a+b)x}{2(a+b)}+c.$

17. 由積化和差公式 $\sin\alpha\sin\beta=\dfrac{1}{2}\big(\cos(\alpha-\beta)-\cos(\alpha+\beta)\big)$, $\displaystyle\int\sin ax\sin bx\,\mathrm{d}x=\int\frac{1}{2}\big(\cos(ax-bx)-\cos(ax+bx)\big)\,\mathrm{d}x=\int\frac{1}{2}\big(\cos(a-b)x-\cos(a+b)x\big)\,\mathrm{d}x=\frac{\sin(a-b)x}{2(a-b)}-\frac{\sin(a+b)x}{2(a+b)}+c.$

18. 由積化和差公式 $\sin\alpha\cos\beta=\dfrac{1}{2}\big(\sin(\alpha-\beta)+\sin(\alpha+\beta)\big)$, $\displaystyle\int\sin ax\cos bx\,\mathrm{d}x=\int\frac{1}{2}\big(\sin(ax-bx)+\sin(ax+bx)\big)\,\mathrm{d}x=\int\frac{1}{2}\big(\sin(a-b)x+\sin(a+b)x\big)\,\mathrm{d}x=-\frac{\cos(a-b)x}{2(a-b)}-\frac{\cos(a+b)x}{2(a+b)}+c.$

部份積分法

結論. $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\,(u(x)\,v(x))=u(x)\,\frac{\mathrm{d}v(x)}{\mathrm{d}x}+v(x)\,\frac{\mathrm{d}u(x)}{\mathrm{d}x}\implies u(x)\,v(x)=\int u(x)\,\mathrm{d}v(x)+\int v(x)\,\mathrm{d}u(x)$
$\displaystyle\implies\int u\,\mathrm{d}v=u\,v-\int v\,\mathrm{d}u.$

例. 求 $\int xe^x \,\mathrm{d}x$.

解. 令 $u = x$, 則 $\mathrm{d}u = \mathrm{d}x$. 令 $\mathrm{d}v = e^x \,\mathrm{d}x$, 則 $v = e^x$. 故 $\int xe^x \,\mathrm{d}x = x \cdot e^x - \int e^x \,\mathrm{d}x = xe^x - e^x + c$.

例. 求 $\int \ln x \,\mathrm{d}x$.

解. 令 $u = \ln x$, 則 $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$. 令 $\mathrm{d}v = \mathrm{d}x$, 則 $v = x$. 故 $\int \ln x \,\mathrm{d}x = \ln x \cdot x - \int x \cdot \frac{1}{x} \,\mathrm{d}x = x \ln x - x + c$.

例. 求 $\int x \tan^{-1} x \,\mathrm{d}x$.

解. 令 $u = \tan^{-1} x$, 則 $\mathrm{d}u = \frac{1}{1+x^2}\mathrm{d}x$. 令 $\mathrm{d}v = x\,\mathrm{d}x$, 則 $v = \frac{x^2}{2}$. 故 $\int x \tan^{-1} x \,\mathrm{d}x = \tan^{-1} x \cdot \frac{x^2}{2} - \frac{1}{2}\int \frac{x^2}{1+x^2} \,\mathrm{d}x = \tan^{-1} x \cdot \frac{x^2}{2} - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) \mathrm{d}x = \tan^{-1} x \cdot \frac{x^2}{2} - \frac{1}{2}\big(x - \tan^{-1} x\big) + c$.

例. 求 $\int x^2 \sin x \,\mathrm{d}x$.

解. 令 $u = x^2$, 則 $\mathrm{d}u = 2x\,\mathrm{d}x$; 令 $\mathrm{d}v = \sin x \,\mathrm{d}x$, 則 $v = -\cos x$. 故 $\int x^2 \sin x \,\mathrm{d}x = x^2 \cdot (-\cos x) + 2\int x \cos x \,\mathrm{d}x$. 令 $u = x$, 則 $\mathrm{d}u = \mathrm{d}x$; 令 $\mathrm{d}v = \cos x \,\mathrm{d}x$, 則 $v = \sin x$. 故 $\int x \cos x \,\mathrm{d}x = x \cdot \sin x - \int \sin x \,\mathrm{d}x = x \cdot \sin x + \cos x$. 由上, $\int x^2 \sin x \,\mathrm{d}x = -x^2 \cos x + 2(x \sin x + \cos x) + c$.

例. 求 $\int \sin^{-1}\sqrt{1-x^2} \,\mathrm{d}x$, $x > 0$.

解. 令 $u = \sin^{-1}\sqrt{1-x^2}$, 則 $\mathrm{d}u = \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \cdot \frac{-x}{\sqrt{1-x^2}} \mathrm{d}x = \frac{1}{\sqrt{x^2}} \cdot \frac{-x}{\sqrt{1-x^2}} \mathrm{d}x = \frac{-1}{\sqrt{1-x^2}} \mathrm{d}x$; 令 $\mathrm{d}v = \mathrm{d}x$, 則 $v = x$. 故 $\int \sin^{-1}\sqrt{1-x^2} \,\mathrm{d}x = x \sin^{-1}\sqrt{1-x^2} + \int \frac{x}{\sqrt{1-x^2}} \,\mathrm{d}x = x \sin^{-1}\sqrt{1-x^2} - \sqrt{1-x^2}$.

例. 求 $\int e^{ax} \cos bx \,\mathrm{d}x$ 與 $\int e^{ax} \sin bx \,\mathrm{d}x$, a, $b \neq 0$.

解.

(a) 考慮 $\int e^{ax} \cos bx \,\mathrm{d}x$: 令 $u = \cos bx$, 則 $\mathrm{d}u = -b \sin bx \,\mathrm{d}x$; 令 $\mathrm{d}v = e^{ax} \,\mathrm{d}x$, 則 $v = \frac{1}{a} e^{ax}$, 故 $\int e^{ax} \cos bx \,\mathrm{d}x = \cos bx \cdot \frac{1}{a} e^{ax} + \frac{b}{a} \int e^{ax} \cdot \sin bx \,\mathrm{d}x$.

(b) 考慮 $\int e^{ax} \sin bx \,\mathrm{d}x$: 令 $u = \sin bx$, 則 $\mathrm{d}u = b \cos bx \,\mathrm{d}x$; 令 $\mathrm{d}v = e^{ax} \,\mathrm{d}x$, 則 $v = \frac{1}{a} e^{ax}$, 故 $\int e^{ax} \sin bx \,\mathrm{d}x = \sin bx \cdot \frac{1}{a} e^{ax} - \frac{b}{a} \int e^{ax} \cdot \cos bx \,\mathrm{d}x$.

令 $X = \int e^{ax} \cos bx \,\mathrm{d}x$, $Y = \int e^{ax} \sin bx \,\mathrm{d}x$, 由 (a)(b) $X = \cos bx \cdot \frac{1}{a} e^{ax} + \frac{b}{a} Y$, $Y = \sin bx \cdot \frac{1}{a} e^{ax} - \frac{b}{a} X$. 解 X, Y 得 $X = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$, $Y = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$.

部份積分法：列表

結論.

1. 將要微分函數寫左邊, 積分函數寫右邊; 左邊連續微分, 右邊連續積分
2. 依序左上連右下斜線函數相乘, 最底部水平兩邊函數相乘並積分, 符號正負相間
3. 將上式所得項全部加總即為所求積分

例. 求 $\displaystyle\int (x+3)\,e^{2x}\,\mathrm{d}x.$

解.

Diff		Int
$x+3$	$+$	e^{2x}
1	$-$	$\frac{1}{2}e^{2x}$
0		$\frac{1}{4}e^{2x}$

$$\int (x+3)\,e^{2x}\,\mathrm{d}x = (x+3)\,\frac{1}{2}e^{2x} - \frac{1}{4}e^{2x}$$

例. 求 $\displaystyle\int (x^2-2x)\,e^{kx}\,\mathrm{d}x.$

解.

Diff		Int
x^2-2x	$+$	e^{kx}
$2(x-1)$	$-$	$\frac{1}{k}e^{kx}$
2	$+$	$\frac{1}{k^2}e^{kx}$
0		$\frac{1}{k^3}e^{kx}$

$$\int (x^2-2x)\,e^{kx}\,\mathrm{d}x = (x^2-2x)\,\frac{1}{k}e^{kx} - 2(x-1)\frac{1}{k^2}e^{kx} + 2\frac{1}{k^3}e^{kx}$$

例. 求 $\displaystyle\int x^4\,\sin 2x\,\mathrm{d}x.$

解.

Diff		Int
x^4	$+$	$\sin 2x$
$4x^3$	$-$	$-\frac{1}{2}\cos 2x$
$12x^2$	$+$	$-\frac{1}{4}\sin 2x$
$24x$	$-$	$\frac{1}{8}\cos 2x$
24	$+$	$\frac{1}{16}\sin 2x$
0		$-\frac{1}{32}\cos 2x$

$$\begin{aligned}
&\int x^4\,\sin 2x\,\mathrm{d}x \\
&= -x^4\,\frac{1}{2}\cos 2x + 4x^3\,\frac{1}{4}\sin 2x + 12x^2\,\frac{1}{8}\cos 2x - 24x\,\frac{1}{16}\sin 2x - 24\,\frac{1}{32}\cos 2x \\
&= \left(-\frac{x^4}{2} + \frac{3x^2}{2} - \frac{3}{4}\right)\cos 2x + \left(x^3 - \frac{3x}{2}\right)\sin 2x
\end{aligned}$$

例. 求 $\displaystyle\int x^5\,e^{ax}\,\mathrm{d}x.$

解.

Diff		Int
x^5	$+$	e^{ax}
$5x^4$	$-$	$\frac{1}{a}e^{ax}$
$20x^3$	$+$	$\frac{1}{a^2}e^{ax}$
$60x^2$	$-$	$\frac{1}{a^3}e^{ax}$
$120x$	$+$	$\frac{1}{a^4}e^{ax}$
120	$-$	$\frac{1}{a^5}e^{ax}$
0		$\frac{1}{a^6}e^{ax}$

$$\begin{aligned}
&\int x^5\,e^{ax}\,\mathrm{d}x \\
&= x^5\,\frac{1}{a}e^{ax} - 5x^4\,\frac{1}{a^2}e^{ax} + 20x^3\,\frac{1}{a^3}e^{ax} - 60x^2\,\frac{1}{a^4}e^{ax} + 120x\,\frac{1}{a^5}e^{ax} - 120\,\frac{1}{a^6}e^{ax} \\
&= \left(\frac{x^5}{a} - \frac{5x^4}{a^2} + \frac{20x^3}{a^3} - \frac{60x^2}{a^4} + \frac{120x}{a^5} - \frac{120}{a^6}\right)e^{ax}
\end{aligned}$$

例. 求 $\displaystyle\int (\sin^{-1} x)^2\,\mathrm{d}x$.

解.

Diff		Int
$(\sin^{-1} x)^2$	$+$	1
$\frac{2\sin^{-1} x}{\sqrt{1-x^2}}$	$-$	x

Diff		Int
$2\sin^{-1} x$	$+$	$\frac{x}{\sqrt{1-x^2}}$
$\frac{2}{\sqrt{1-x^2}}$	$-$	$-\sqrt{1-x^2}$

$$\begin{aligned}
&\int (\sin^{-1} x)^2\,\mathrm{d}x = (\sin^{-1} x)^2 \cdot x - \int \frac{2\sin^{-1} x}{\sqrt{1-x^2}} \cdot x\,\mathrm{d}x \\
&= (\sin^{-1} x)^2 \cdot x - \int 2\sin^{-1} x \cdot \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x \\
&= (\sin^{-1} x)^2 \cdot x - \left(-2\sin^{-1} x \cdot \sqrt{1-x^2} + \int \frac{2}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2}\,\mathrm{d}x \right) \\
&= (\sin^{-1} x)^2 \cdot x + 2\sin^{-1} x \cdot \sqrt{1-x^2} - 2x
\end{aligned}$$

例. 求 $\displaystyle\int e^{ax}\cos bx\,\mathrm{d}x$, a, $b \neq 0$.

解.

Diff		Int
$\cos bx$	$+$	e^{ax}
$-b\sin bx$	$-$	$\frac{1}{a}e^{ax}$
$-b^2\cos bx$	$+$	$\frac{1}{a^2}e^{ax}$

$$\begin{aligned}
&\int e^{ax}\cos bx\,\mathrm{d}x = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}e^{ax}\sin bx - \frac{b^2}{a^2}\int e^{ax}\cos bx\,\mathrm{d}x \\
&\Longrightarrow \int e^{ax}\cos bx\,\mathrm{d}x = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}
\end{aligned}$$

例. 求 $\displaystyle\int e^{ax}\sin bx\,\mathrm{d}x$, a, $b \neq 0$.

解.

Diff		Int
$\sin bx$	$+$	e^{ax}
$b\cos bx$	$-$	$\frac{1}{a}e^{ax}$
$-b^2\sin bx$	$+$	$\frac{1}{a^2}e^{ax}$

$$\begin{aligned}
&\int e^{ax}\sin bx\,\mathrm{d}x = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx\,\mathrm{d}x \\
&\Longrightarrow \int e^{ax}\sin bx\,\mathrm{d}x = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}
\end{aligned}$$

遞迴式

例. 令 $I_n = \displaystyle\int \frac{1}{(x^2+a^2)^n}\,\mathrm{d}x$, $n \in \mathbb{N}$, 則 $I_{n+1} = \dfrac{x}{2na^2(x^2+a^2)^n} + \dfrac{2n-1}{2na^2}I_n$.

解. 令 $u = \dfrac{1}{(x^2+a^2)^n}$, 則 $\mathrm{d}u = -2n\dfrac{x}{(x^2+a^2)^{n+1}}\,\mathrm{d}x$; $\mathrm{d}v = \mathrm{d}x$, 則 $v = x$. 故 $I_n = \displaystyle\int \frac{1}{(x^2+a^2)^n}\,\mathrm{d}x = \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2}{(x^2+a^2)^{n+1}}\,\mathrm{d}x = \frac{x}{(x^2+a^2)^n} + 2n\int \frac{(x^2+a^2-a^2)}{(x^2+a^2)^{n+1}}\,\mathrm{d}x = \frac{x}{(x^2+a^2)^n} + 2nI_n - 2na^2I_{n+1} \Longrightarrow 2na^2I_{n+1} = \frac{x}{(x^2+a^2)^n} + (2n-1)I_n \Longrightarrow I_{n+1} = \frac{x}{2na^2(x^2+a^2)^n} + \frac{2n-1}{2na^2}I_n$.

註 (使用例). $I_1 = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$, $I_2 = \dfrac{1}{2a^2}I_1 + \dfrac{x}{2a^2(x^2+a^2)} = \dfrac{1}{2a^3}\tan^{-1}\dfrac{x}{a} + \dfrac{x}{2a^2(x^2+a^2)}$, $I_3 = \dfrac{3}{4a^2}I_2 + \dfrac{x}{4a^2(x^2+a^2)^2} = \dfrac{3}{4a^2}\left(\dfrac{1}{2a^3}\tan^{-1}\dfrac{x}{a} + \dfrac{x}{2a^2(x^2+a^2)}\right) + \dfrac{x}{4a^2(x^2+a^2)^2} = \dfrac{3}{8a^5}\tan^{-1}\dfrac{x}{a} + \dfrac{3x}{8a^4(x^2+a^2)} + \dfrac{x}{4a^2(x^2+a^2)^2}$.

例. 令 $J_n = \displaystyle\int \sin^n x\,\mathrm{d}x$, $n \in \mathbb{N}$, $n \geqslant 2$, 則 $J_n = \dfrac{-\sin^{n-1}x\cos x}{n} + \dfrac{n-1}{n}J_{n-2}$.

解. 令 $u=\sin^{n-1}x$, 則 $\mathrm{d}u=(n-1)\sin^{n-2}x\,\cos x\,\mathrm{d}x$; $\mathrm{d}v=\sin x\,\mathrm{d}x$, 則 $v=-\cos x$. 故 $J_n=\int\sin^n x\,\mathrm{d}x=-\sin^{n-1}x\cdot\cos x+(n-1)\int\cos x\cdot\sin^{n-2}x\cos x\,\mathrm{d}x=-\sin^{n-1}x\cos x+(n-1)\int\sin^{n-2}x\cos^2 x\,\mathrm{d}x=-\sin^{n-1}x\cos x+(n-1)\int\sin^{n-2}x\cdot(1-\sin^2 x)\,\mathrm{d}x=-\sin^{n-1}x\cos x+(n-1)\int\sin^{n-2}x\,\mathrm{d}x-(n-1)\int\sin^n x\,\mathrm{d}x=-\sin^{n-1}x\cos x+(n-1)J_{n-2}+(1-n)J_n \implies nJ_n=-\sin^{n-1}x\cos x+(n-1)J_{n-2} \implies J_n=\dfrac{-\sin^{n-1}x\cos x}{n}+\dfrac{n-1}{n}\,J_{n-2}$.

註 (使用例)**.** $J_2=\dfrac{-\sin x\cos x}{2}+\dfrac{1}{2}\,J_0=\dfrac{-\sin x\cos x}{2}+\dfrac{x}{2}$, $J_3=\dfrac{-\sin^2 x\cos x}{3}+\dfrac{2}{3}\,J_1=\dfrac{-\sin^2 x\cos x}{3}-\dfrac{2\cos x}{3}$, $J_4=\dfrac{-\sin^3 x\cos x}{4}+\dfrac{3}{4}\,J_2=\dfrac{-\sin^3 x\cos x}{4}+\dfrac{3\sin x\cos x}{8}-\dfrac{3x}{8}$.

例. 令 $K_n=\int\sec^{2n+1}\theta\,\mathrm{d}\theta$, $n\in\mathbb{N}$, $n\geqslant 1$, 則 $K_n=\dfrac{\sec^{2n-1}\theta\tan\theta}{2n}+\dfrac{2n-1}{2n}\,K_{n-1}$.

解. 令 $u=\sec^{2n-1}\theta$, 則 $\mathrm{d}u=(2n-1)\sec^{2n-2}\theta\cdot\sec\theta\tan\theta\,\mathrm{d}\theta=(2n-1)\sec^{2n-1}\theta\tan\theta\,\mathrm{d}\theta$; 令 $\mathrm{d}v=\sec^2\theta\,\mathrm{d}\theta$, 則 $v=\tan\theta$. 故 $K_n=\int\sec^{2n+1}\theta\,\mathrm{d}\theta=\sec^{2n-1}\theta\cdot\tan\theta-\int\tan\theta\cdot(2n-1)\sec^{2n-1}\theta\tan\theta\,\mathrm{d}\theta=\sec^{2n-1}\theta\tan\theta-(2n-1)\int\tan^2\theta\cdot\sec^{2n-1}\theta\,\mathrm{d}\theta=\sec^{2n-1}\theta\tan\theta-(2n-1)\int(\sec^2\theta-1)\cdot\sec^{2n-1}\theta\,\mathrm{d}\theta=\sec^{2n-1}\theta\tan\theta-(2n-1)\int\sec^{2n+1}\theta\,\mathrm{d}\theta+(2n-1)\int\sec^{2n-1}\theta\,\mathrm{d}\theta \implies K_n=\sec^{2n-1}\theta\tan\theta-(2n-1)K_n+(2n-1)K_{n-1} \implies K_n=\dfrac{\sec^{2n-1}\theta\tan\theta}{2n}+\dfrac{2n-1}{2n}K_{n-1}$.

註 (使用例)**.** $K_0=\int\sec\theta\,\mathrm{d}\theta=\ln|\sec\theta+\tan\theta|$, $\int\sec^3\theta\,\mathrm{d}\theta=K_1=\dfrac{\sec\theta\tan\theta}{2}+\dfrac{1}{2}\,K_0=\dfrac{1}{2}\big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\big)$, $\int\sec^5\theta\,\mathrm{d}\theta=K_2=\dfrac{\sec^3\theta\tan\theta}{4}+\dfrac{3}{4}\,K_1=\dfrac{\sec^3\theta\tan\theta}{4}+\dfrac{3}{8}\big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\big)$.

習題. 以部份積分法求下列不定積分. 注意: 可能會因為常數項而跟此處答案不同.

1. $\displaystyle\int\frac{\sin^{-1}x}{x^2}\,\mathrm{d}x$
2. $\displaystyle\int\ln(x+\sqrt{1+x^2})\,\mathrm{d}x$
3. $\displaystyle\int x^3\ln x\,\mathrm{d}x$
4. $\displaystyle\int x(\ln x)^3\,\mathrm{d}x$
5. $\displaystyle\int x^2\tan^{-1}x\,\mathrm{d}x$
6. $\displaystyle\int\frac{xe^x}{(x+1)^2}\,\mathrm{d}x$
7. $\displaystyle\int x^5e^{-x^2}\,\mathrm{d}x$
8. $\displaystyle\int xe^{\sqrt{x}}\,\mathrm{d}x$
9. $\displaystyle\int(2x^2+1)e^{x^2}\,\mathrm{d}x$
10. $\displaystyle\int\sin(\ln x)\,\mathrm{d}x$
11. $\displaystyle\int x^2\ln\frac{1-x}{1+x}\,\mathrm{d}x$
12. $\displaystyle\int\frac{\ln x}{\sqrt{1+x}}\,\mathrm{d}x$

解.

1. 令 $u=\sin^{-1}x$, 則 $\mathrm{d}u=\dfrac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$; $\mathrm{d}v=\dfrac{1}{x^2}\,\mathrm{d}x$, 則 $v=\dfrac{-1}{x}$. 故 $\displaystyle\int\frac{\sin^{-1}x}{x^2}\,\mathrm{d}x=\sin^{-1}x\cdot\frac{-1}{x}-\int\frac{-1}{x}\cdot\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x=-\frac{\sin^{-1}x}{x}+\int\frac{1}{x\sqrt{1-x^2}}\,\mathrm{d}x$. 令 $w=\sqrt{1-x^2}$, 則 $-x^2=w^2-1$, $\mathrm{d}w=\dfrac{-x}{\sqrt{1-x^2}}\,\mathrm{d}x$. 故 $\displaystyle\int\frac{1}{x\sqrt{1-x^2}}\,\mathrm{d}x=\int\frac{1}{-x^2}\cdot\frac{-x}{\sqrt{1-x^2}}\,\mathrm{d}x=\int\frac{1}{w^2-1}\,\mathrm{d}w=\frac{1}{2}\int\Big(\frac{1}{w-1}-\frac{1}{w+1}\Big)\,\mathrm{d}w=\frac{1}{2}\big(\ln|w-1|-\ln|w+1|\big)=\frac{1}{2}\ln\left|\frac{w-1}{w+1}\right|=\frac{1}{2}\ln\left|\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right|=\frac{1}{2}\ln\left|\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right|=\frac{1}{2}\ln\left|\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right|$.

$\left.\dfrac{1-\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right| = \dfrac{1}{2}\ln\left|\dfrac{(1-\sqrt{1-x^2})^2}{x^2}\right| = \ln\left|\dfrac{1-\sqrt{1-x^2}}{x}\right|.$
以上, $\displaystyle\int \frac{\sin^{-1}x}{x^2}\,\mathrm{d}x = -\frac{\sin^{-1}x}{x} + \int \frac{1}{x\sqrt{1-x^2}}\,\mathrm{d}x = -\frac{\sin^{-1}x}{x} + \ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + c.$

2. 令 $u = \ln(x+\sqrt{1+x^2})$, 則 $\mathrm{d}u = \dfrac{1}{x+\sqrt{1+x^2}}\cdot\big(x+\sqrt{1+x^2}\big)'\,\mathrm{d}x = \dfrac{1}{x+\sqrt{1+x^2}}\cdot\Big(1+\dfrac{x}{\sqrt{1+x^2}}\Big)\,\mathrm{d}x = \dfrac{1}{\sqrt{1+x^2}}\,\mathrm{d}x$; $\mathrm{d}v = \mathrm{d}x$, 則 $v = x$. 故 $\displaystyle\int \ln(x+\sqrt{1+x^2})\,\mathrm{d}x = \ln(x+\sqrt{1+x^2})\cdot x - \int x\cdot\frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = \ln(x+\sqrt{1+x^2})\,x - \sqrt{1+x^2} + c.$

3. 令 $u = \ln x$, 則 $\mathrm{d}u = \dfrac{1}{x}\,\mathrm{d}x$; $\mathrm{d}v = x^3\,\mathrm{d}x$, 則 $v = \dfrac{x^4}{4}$. 故 $\displaystyle\int x^3\ln x\,\mathrm{d}x = \ln x\cdot\frac{x^4}{4} - \int\frac{x^4}{4}\cdot\frac{1}{x}\,\mathrm{d}x = \ln x\cdot\frac{x^4}{4} - \int\frac{x^3}{4}\,\mathrm{d}x = \ln x\cdot\frac{x^4}{4} - \frac{x^4}{16} + c.$

4. 令 $u = (\ln x)^3$, 則 $\mathrm{d}u = 3(\ln x)^2\cdot\dfrac{1}{x}\,\mathrm{d}x$; $\mathrm{d}v = x\,\mathrm{d}x$, 則 $v = \dfrac{x^2}{2}$. 故 $\displaystyle\int x(\ln x)^3\,\mathrm{d}x = (\ln x)^3\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot 3(\ln x)^2\cdot\frac{1}{x}\,\mathrm{d}x = (\ln x)^3\cdot\frac{x^2}{2} - \frac{3}{2}\int x(\ln x)^2\,\mathrm{d}x$. 令 $u = (\ln x)^2$, 則 $\mathrm{d}u = 2\ \ln x\cdot\dfrac{1}{x}\,\mathrm{d}x$; $\mathrm{d}v = x\,\mathrm{d}x$, 則 $v = \dfrac{x^2}{2}$. 故 $\displaystyle\int x(\ln x)^2\,\mathrm{d}x = (\ln x)^2\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot 2\ln x\cdot\frac{1}{x}\,\mathrm{d}x = (\ln x)^2\cdot\frac{x^2}{2} - \int x\ln x\,\mathrm{d}x$. 令 $u = \ln x$, 則 $\mathrm{d}u = \dfrac{1}{x}\,\mathrm{d}x$; $\mathrm{d}v = x\,\mathrm{d}x$, 則 $v = \dfrac{x^2}{2}$. 故 $\displaystyle\int x\ln x\,\mathrm{d}x = \ln x\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{1}{x}\,\mathrm{d}x = \ln x\cdot\frac{x^2}{2} - \frac{1}{2}\int x\,\mathrm{d}x = \ln x\cdot\frac{x^2}{2} - \frac{x^2}{4}$. 以上, $\displaystyle\int x(\ln x)^3\,\mathrm{d}x = (\ln x)^3\cdot\frac{x^2}{2} - \frac{3}{2}\Big((\ln x)^2\cdot\frac{x^2}{2} - \big(\ln x\cdot\frac{x^2}{2} - \frac{x^2}{4}\big)\Big) = \frac{x^2}{2}\big((\ln x)^3 - \frac{3(\ln x)^2}{2} + \frac{3\ln x}{2} - \frac{3}{4}\big) + c$

5. 令 $u = \tan^{-1}x$, 則 $\mathrm{d}u = \dfrac{1}{1+x^2}\,\mathrm{d}x$; $\mathrm{d}v = x^2\,\mathrm{d}x$, 則 $v = \dfrac{x^3}{3}$. 故 $\displaystyle\int x^2\tan^{-1}x\,\mathrm{d}x = \tan^{-1}x\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{1}{x^2+1}\,\mathrm{d}x$. 又 $\displaystyle\int\frac{x^3}{3}\cdot\frac{1}{x^2+1}\,\mathrm{d}x = \frac{1}{3}\int\frac{(x^3+x)-x}{x^2+1}\,\mathrm{d}x = \frac{1}{3}\int\frac{x(x^2+1)-x}{x^2+1}\,\mathrm{d}x = \frac{1}{3}\int\big(x - \frac{x}{x^2+1}\big)\,\mathrm{d}x$. 令 $w = x^2+1$, 則 $\mathrm{d}w = 2x\,\mathrm{d}x \implies x\,\mathrm{d}x = \dfrac{1}{2}\,\mathrm{d}w$, 故 $\displaystyle\int\frac{x}{x^2+1}\,\mathrm{d}x = \int\frac{1}{x^2+1}\cdot x\,\mathrm{d}x = \int\frac{1}{w}\cdot\frac{1}{2}\,\mathrm{d}w = \frac{1}{2}\ln w + c = \frac{1}{2}\ln(x^2+1) + c$. 以上, $\displaystyle\int x^2\tan^{-1}x\,\mathrm{d}x = \tan^{-1}x\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{1}{x^2+1}\,\mathrm{d}x = \tan^{-1}x\cdot\frac{x^3}{3} - \frac{x^2}{6} + \frac{\ln(x^2+1)}{6} + c.$

6. $\displaystyle\int\frac{xe^x}{(x+1)^2}\,\mathrm{d}x = \int\frac{(x+1-1)e^x}{(x+1)^2}\,\mathrm{d}x = \int\frac{e^x}{x+1}\,\mathrm{d}x - \int\frac{e^x}{(x+1)^2}\,\mathrm{d}x$. 令 $u = e^x$, 則 $\mathrm{d}u = e^x\,\mathrm{d}x$; $\mathrm{d}v = \dfrac{1}{(x+1)^2}\,\mathrm{d}x$, 則 $v = \dfrac{-1}{x+1}$. 故 $\displaystyle\int\frac{e^x}{(x+1)^2}\,\mathrm{d}x = e^x\cdot\frac{-1}{x+1} + \int\frac{1}{x+1}\cdot e^x\,\mathrm{d}x = \frac{-e^x}{x+1} + \int\frac{e^x}{x+1}\,\mathrm{d}x$; 原式 $\displaystyle\int\frac{xe^x}{(x+1)^2}\,\mathrm{d}x = \int\frac{e^x}{x+1}\,\mathrm{d}x - \int\frac{e^x}{(x+1)^2}\,\mathrm{d}x = \int\frac{e^x}{x+1}\,\mathrm{d}x - \Big(\frac{-e^x}{x+1} + \int\frac{e^x}{x+1}\,\mathrm{d}x\Big) = \frac{e^x}{x+1}.$

7. 令 $w = x^2$, 則 $\mathrm{d}w = 2x\,\mathrm{d}x \implies x\,\mathrm{d}x = \dfrac{1}{2}\,\mathrm{d}w$, 故 $\displaystyle\int x^5e^{-x^2}\,\mathrm{d}x = \int e^{-x^2}\cdot(x^2)^2\cdot x\,\mathrm{d}x = \int e^{-w}\cdot w^2\cdot\frac{1}{2}\,\mathrm{d}w = \frac{1}{2}\int w^2\,e^{-w}\,\mathrm{d}w = -\frac{1}{2}\,e^{-w}(w^2+2w+2) = -\frac{1}{2}\,e^{-x^2}(x^4+2x^2+2) + c.$

Diff		Int
w^2	$+$	e^{-w}
$2w$	$-$	$-e^{-w}$
2	$+$	e^{-w}
0		$-e^{-w}$

$$\int w^2 e^{-w}\,\mathrm{d}w = -w^2\,e^{-w} - 2w\,e^{-w} - 2\,e^{-w} = -e^{-w}(w^2+2w+2)$$

8. 令 $w=\sqrt{x}$, 則 $w^2=x$, $\mathrm{d}x = 2w\,\mathrm{d}w$, 故 $\displaystyle\int xe^{\sqrt{x}}\,\mathrm{d}x = \int w^2 e^w\cdot 2w\,\mathrm{d}w = 2\int w^3 e^w\,\mathrm{d}w = 2\,e^w(w^3-3w^2+6w-6) = 2\,e^{\sqrt{x}}\,(x\sqrt{x}-3x+6\sqrt{x}-6)+c$

Diff		Int
w^3	$+$	e^w
$3w^2$	$-$	e^w
$6w$	$+$	e^w
6	$-$	e^w
0		e^w

$$\int w^3\,e^w\,\mathrm{d}w = w^3\,e^w - 3w^2\,e^w + 6w\,e^w - 6\,e^w = e^w(w^3-3w^2+6w-6)$$

9. $\displaystyle\int(2x^2+1)\,e^{x^2}\,\mathrm{d}x = \int 2x^2\,e^{x^2}\,\mathrm{d}x + \int e^{x^2}\,\mathrm{d}x$. 令 $u=x$, 則 $\mathrm{d}u=\mathrm{d}x$; $\mathrm{d}v = 2x\,e^{x^2}\,\mathrm{d}x$, 則 $v=e^{x^2}$. 故 $\displaystyle\int 2x^2\,e^{x^2}\,\mathrm{d}x = x\cdot e^{x^2} - \int e^{x^2}\mathrm{d}x$; 原式 $\displaystyle\int 2x^2\,e^{x^2}\,\mathrm{d}x + \int e^{x^2}\,\mathrm{d}x = x\cdot e^{x^2} - \int e^{x^2}\mathrm{d}x + \int e^{x^2}\,\mathrm{d}x = xe^{x^2}$

10. 令 $w=\ln x$, 則 $e^w = x$, $\mathrm{d}x = e^w\,\mathrm{d}w$, 故 $\displaystyle\int\sin(\ln x)\,\mathrm{d}x = \int\sin w\cdot e^w\,\mathrm{d}w = \int e^w\,\sin w\,\mathrm{d}w = \frac{e^w(\sin w-\cos w)}{2} = \frac{x\,(\sin(\ln x)-\cos(\ln x))}{2}+c$

Diff		Int
$\sin w$	$+$	e^w
$\cos w$	$-$	e^w
$-\sin w$	$+$	e^w

$$\int e^w\sin w\,\mathrm{d}w = e^w\sin w - e^w\cos w - \int e^w\sin w\,\mathrm{d}w$$
$$\implies \int e^w\sin w\,\mathrm{d}w = \frac{e^w(\sin w-\cos w)}{2}$$

11. 令 $u=\ln\dfrac{1-x}{1+x}$, 則 $\mathrm{d}u = \dfrac{1+x}{1-x}\cdot\left(\dfrac{1-x}{1+x}\right)'\mathrm{d}x = \dfrac{1+x}{1-x}\cdot\dfrac{(1+x)\cdot(-1)-(1-x)}{(1+x)^2}\,\mathrm{d}x = \dfrac{2}{x^2-1}\,\mathrm{d}x$; $\mathrm{d}v = x^2\,\mathrm{d}x$, 則 $v=\dfrac{x^3}{3}$. 故 $\displaystyle\int x^2\ln\frac{1-x}{1+x}\,\mathrm{d}x = \ln\frac{1-x}{1+x}\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{2}{x^2-1}\,\mathrm{d}x$. 又 $\displaystyle\int\frac{x^3}{3}\cdot\frac{2}{x^2-1}\,\mathrm{d}x = \frac{2}{3}\int\frac{x^3}{x^2-1}\,\mathrm{d}x = \frac{2}{3}\int\left(x+\frac{x}{x^2-1}\right)\mathrm{d}x = \frac{x^2}{3}+\frac{2}{3}\int\frac{x}{x^2-1}\,\mathrm{d}x$. 令 $w=x^2-1$, 則 $\mathrm{d}w = 2x\,\mathrm{d}x \implies x\,\mathrm{d}x = \dfrac{1}{2}\,\mathrm{d}w$, 故 $\displaystyle\int\frac{x}{x^2-1}\,\mathrm{d}x = \int\frac{1}{x^2-1}\cdot x\,\mathrm{d}x = \int\frac{1}{w}\cdot\frac{1}{2}\,\mathrm{d}w = \frac{1}{2}\ln w + c = \frac{1}{2}\ln(x^2-1)+c$. 以上, $\displaystyle\int x^2\ln\frac{1-x}{1+x}\,\mathrm{d}x = \frac{1+x}{1-x}\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{1}{x^2-1}\,\mathrm{d}x = \ln\frac{1-x}{1+x}\cdot\frac{x^3}{3} - \frac{x^2+\ln(x^2-1)}{3}+c$.

12. 令 $u=\ln x$, 則 $\mathrm{d}u = \dfrac{1}{x}\,\mathrm{d}x$; $\mathrm{d}v = \dfrac{1}{\sqrt{1+x}}\,\mathrm{d}x$, 則 $v = 2\sqrt{1+x}$. 故 $\displaystyle\int\frac{\ln x}{\sqrt{1+x}}\,\mathrm{d}x = \ln x\cdot 2\sqrt{1+x} - 2\int\frac{\sqrt{1+x}}{x}\,\mathrm{d}x$. 令 $w=\sqrt{1+x}$, 則 $w^2 = 1+x \implies x = w^2-1$, $2w\,\mathrm{d}w = \mathrm{d}x$, 故 $\displaystyle\int\frac{\sqrt{1+x}}{x}\,\mathrm{d}x = \int\frac{w}{w^2-1}\cdot 2w\,\mathrm{d}w = 2\int\frac{w^2}{w^2-1}\,\mathrm{d}w = 2\int\frac{(w^2-1)+1}{w^2-1}\,\mathrm{d}w = 2w+2\int\frac{1}{w^2-1}\,\mathrm{d}w = 2\sqrt{1+x}+$

$2\int \frac{1}{w^2-1}\,\mathrm{d}w$. 由 $\frac{1}{w^2-1} = \frac{1}{2}\Big(\frac{1}{w-1} - \frac{1}{w+1}\Big)$, $\int \frac{1}{w^2-1}\,\mathrm{d}w = \frac{1}{2}\int \Big(\frac{1}{w-1} - \frac{1}{w+1}\Big)\,\mathrm{d}w = \frac{1}{2}\big(\ln|w-1| - \ln|w+1|\big) = \frac{1}{2}\big(\ln|\sqrt{1+x}-1| - \ln|\sqrt{1+x}+1|\big)$. 以上, $\int \frac{\ln x}{\sqrt{1+x}}\,\mathrm{d}x = \ln x\cdot 2\sqrt{1+x} - 2\big(2\sqrt{1+x} + 2\big(\frac{1}{2}\big(\ln|\sqrt{1+x}-1| - \ln|\sqrt{1+x}+1|\big)\big)\big) = \ln x\cdot 2\sqrt{1+x} - 4\sqrt{1+x} - 2\ln|\sqrt{1+x}-1| + 2\ln|\sqrt{1+x}+1| + c$.

4.2 定積分

定積分 ≈ (帶符號) 面積: x 軸上方為正, 下方為負.

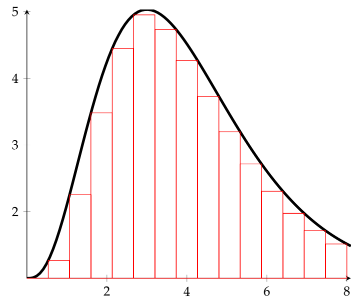
定義. 給定 $f:[a,\,b]\to\mathbb{R}$.

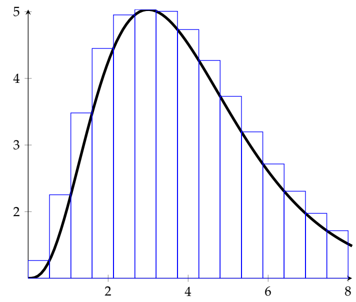
- $[a,\,b]$ 分割 $\mathbb{P}: a = x_0 < x_1 < x_2 < \cdots < x_n = b$
- $\Delta x_k = x_k - x_{k-1}$, $k = 1,\,2,\,\ldots,\,n$; $\|\mathbb{P}\| = \max\{\,|\Delta x_k| \mid 1\leqslant k\leqslant n\}$
- 樣本點 ξ_k: $x_{k-1}\leqslant \xi_k\leqslant x_k$, $k = 1,\,2,\,\ldots,\,n$
- $u_k = \sup\{\,f(x) \mid x_{k-1}\leqslant x\leqslant x_k\}$, $l_k = \inf\{\,f(x) \mid x_{k-1}\leqslant x\leqslant x_k\}$, $k = 1,\,2,\,\ldots,\,n$
- $R(f,\mathbb{P}) = \sum_{k=1}^{n} f(\xi_k)\,\Delta x_k$, $U(f,\mathbb{P}) = \sum_{k=1}^{n} u_k\,\Delta x_k$, $L(f,\mathbb{P}) = \sum_{k=1}^{n} l_k\,\Delta x_k$;
  顯然 $L(f,\mathbb{P})\leqslant R(f,\mathbb{P})\leqslant U(f,\mathbb{P})$.
- 求 $\lim_{\|\mathbb{P}\|\to 0} R(f,\mathbb{P})$. 若對不同分割與樣本點選取此極限均存在且相等, 稱 f 在 $[a,b]$ 可積 (分) ; $f(x)$ 在 $[a,\,b]$ 的定積分 $\int_a^b f(x)\,\mathrm{d}x \equiv \lim_{\|\mathbb{P}\|\to 0} R(f,\mathbb{P})$

註.

- 在 $\int_a^b f(x)\,\mathrm{d}x$ 中, a 為*積分下限* (lower limit of integration) , b 為*積分上限* (upper limit of integration) , $f(x)$ 為*被積分式* (integrand) , x 為*積分變數* (variable of integration) .
- $\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(t)\,\mathrm{d}t = \int_a^b f(u)\,\mathrm{d}u$ (定積分數值與積分變數無關)

結論. 若 f 在 $[a,\,b]$ 連續, 則 f 在 $[a,\,b]$ 可積.





性質. 令 f, g 在包含 a, b, c 之區間為可積, α, $\beta\in\mathbb{R}$. 則

1. $\int_a^a f(x)\,\mathrm{d}x = 0$

2. $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$

3. $\displaystyle\int_a^b \big(\alpha\, f(x) + \beta\, g(x)\big)\,\mathrm{d}x = \alpha\int_a^b f(x)\,\mathrm{d}x + \beta\int_a^b g(x)\,\mathrm{d}x$

4. $\displaystyle\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x$

5. $\displaystyle\int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b g(x)\,\mathrm{d}x$, 若 $f(x) \leqslant g(x)\ \forall\, x \in [a,b],\ a \leqslant b$

6. $\displaystyle\left|\int_a^b f(x)\,\mathrm{d}x\right| \leqslant \int_a^b |\,f(x)\,|\ \mathrm{d}x, \quad a \leqslant b$

7. $f(x)$ 為奇函數: $\displaystyle\int_{-a}^a f(x)\,\mathrm{d}x = 0$

8. $f(x)$ 為偶函數: $\displaystyle\int_{-a}^a f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x$

例.

1. $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-x^2}\,\mathrm{d}x = \pi$ (半徑 $\sqrt{2}$ 之半圓面積)

2. 定義 $\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$, 則 $\displaystyle\int_{-1}^2 \operatorname{sgn}(x)\,\mathrm{d}x = 2\cdot 1 - 1\cdot 1 = 1$

3. $\displaystyle\int_{-\pi}^{\pi} \sin(x^7 - 5x^3)\,\mathrm{d}x = 0$ ($\sin(x^7-5x^3)$ 為奇函數)

4. $\displaystyle\int_{-6}^{6} e^{-x^4}\sin(\sin x)\,\mathrm{d}x = 0$ ($e^{-x^4}\sin(\sin x)$ 為奇函數)

5. $\displaystyle\int_{-4}^{4} \big(e^x - e^{-x}\big)\,\mathrm{d}x = 0$ ($e^x - e^{-x}$ 為奇函數)

6. $\displaystyle\int_{-2025}^{2025} \big(e^{9x^5-2x^7} - e^{-9x^5+2x^7}\big)\,\mathrm{d}x = 0$ ($e^{9x^5-2x^7} - e^{-9x^5+2x^7}$ 為奇函數)

7. $\displaystyle\int_{-a}^{a} |x|\,\mathrm{d}x = 2\int_0^a |x|\,\mathrm{d}x = a^2$ ($|x|$ 為偶函數; 兩 $a\times a$ 等腰直角三角形面積)

8. 定義 $\displaystyle\int_1^x \frac{1}{\tau}\,\mathrm{d}\tau = \ln x$, 則 $\displaystyle\int_{\frac14}^3 \frac{1}{x}\,\mathrm{d}x = \int_{\frac14}^1 \frac{1}{x}\,\mathrm{d}x + \int_1^3 \frac{1}{x}\,\mathrm{d}x = -\int_1^{\frac14} \frac{1}{x}\,\mathrm{d}x + \int_1^3 \frac{1}{x}\,\mathrm{d}x = \ln 12$

9. $\displaystyle\int_0^2 \sqrt{4-x^2}\cdot\operatorname{sgn}(1-x)\,\mathrm{d}x = \int_0^1 \sqrt{4-x^2}\,\mathrm{d}x - \int_1^2 \sqrt{4-x^2}\,\mathrm{d}x = \left(\frac{4\pi}{12} + \frac{\sqrt{3}}{2}\right) - \left(\frac{4\pi}{6} - \frac{\sqrt{3}}{2}\right) = \sqrt{3} - \frac{\pi}{3}$

10. $\displaystyle\forall\, n \in \mathbb{N},\ \int_n^{n+1} \lfloor x \rfloor\,\mathrm{d}x = n$

以極限定義求定積分

結論. $\displaystyle\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$

證. 由 $k(k+1) = \dfrac{k(k+1)(k+2)-(k-1)k(k+1)}{3}$, $\displaystyle\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$. 又 $k^2 = k(k+1)-k$, $\displaystyle\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n}\big(k(k+1)-k\big) = \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2} = \frac{n(n+1)(2n+1)}{6}$.

例. 求 $\displaystyle\int_0^1 x^2\,\mathrm{d}x$.

解. 建立 $[0,\,1]$ 分割 $\mathbb{P}:\Big\{0, \dfrac{1}{n}, \dfrac{2}{n}, \ldots, \dfrac{n}{n}\Big\}$, 則 $\Delta x_k = \dfrac{1}{n}\ \forall\, k=1,2,\ldots,n$ 且 $\|P\| \to 0$ 當 $n\to\infty$. $f(x)=x^2$ 且 f 在 $[0,\,1]$ 嚴格遞增, $\displaystyle U(f,\mathbb{P}) = \sum_{k=1}^{n} u_k\,\Delta x_k = \sum_{k=1}^{n}\frac{k^2}{n^2}\cdot\frac{1}{n} = \frac{1}{n^3}\sum_{k=1}^{n}k^2$, $\displaystyle L(f,\mathbb{P}) = \sum_{k=1}^{n} l_k\,\Delta x_k = \sum_{k=1}^{n}\frac{(k-1)^2}{n^2}\cdot\frac{1}{n} = \frac{1}{n^3}\sum_{k=1}^{n-1}k^2$, $L(f,\mathbb{P}) \leqslant R(f,\mathbb{P}) \leqslant U(f,\mathbb{P})$. 由 $\displaystyle\sum_{k=1}^{n}k^2 = \frac{n(n+1)(2n+1)}{6}$, $\displaystyle\lim_{n\to\infty} U(f,\mathbb{P}) = \lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}k^2 = \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}$, $\displaystyle\lim_{n\to\infty} L(f,\mathbb{P}) = \lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n-1}k^2 = \lim_{n\to\infty}\frac{(n-1)n(2n-1)}{6n^3} = \frac{1}{3}$, 故由夾擊定理 $\displaystyle\lim_{n\to\infty} R(f,\mathbb{P}) = \frac{1}{3}$.

結論. 若 $r\in\mathbb{R}$, $\displaystyle\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$.

證. 令 $\displaystyle s = \sum_{k=0}^{n} r^k$, 則 $\displaystyle r\,s = \sum_{k=1}^{n+1} r^k$; $r\,s - s = r^{n+1}-1 \implies s = \dfrac{r^{n+1}-1}{r-1}$.

例. 求 $\displaystyle\int_0^1 e^x\,\mathrm{d}x$.

解. 建立 $[0,\,1]$ 分割 $\mathbb{P}:\Big\{0, \dfrac{1}{n}, \dfrac{2}{n}, \ldots, \dfrac{n}{n}\Big\}$, 則 $\Delta x_k = \dfrac{1}{n}\ \forall\, k=1,2,\ldots,n$ 且 $\|P\|\to 0$ 當 $n\to\infty$. $f(x)=e^x$ 且 f 在 $[0,\,1]$ 嚴格遞增, $\displaystyle U(f,\mathbb{P}) = \sum_{k=1}^{n} u_k\,\Delta x_k = \sum_{k=1}^{n} e^{\frac{k}{n}}\cdot\frac{1}{n} = \frac{1}{n}\frac{e^{\frac{1}{n}}\big(e^{\frac{n}{n}}-1\big)}{e^{\frac{1}{n}}-1} = \frac{e-1}{n}\frac{e^{\frac{1}{n}}}{e^{\frac{1}{n}}-1}$, $\displaystyle L(f,\mathbb{P}) = \sum_{k=1}^{n} l_k\,\Delta x_k = \sum_{k=1}^{n} e^{\frac{k-1}{n}}\cdot\frac{1}{n} = \frac{1}{n}\frac{e^0\big(e^{\frac{n}{n}}-1\big)}{e^{\frac{1}{n}}-1} = \frac{e-1}{n}\frac{1}{e^{\frac{1}{n}}-1}$, $L(f,\mathbb{P}) \leqslant R(f,\mathbb{P}) \leqslant U(f,\mathbb{P})$. $\displaystyle\lim_{n\to\infty} U(f,\mathbb{P}) = \lim_{n\to\infty}\frac{e-1}{n}\frac{e^{\frac{1}{n}}}{e^{\frac{1}{n}}-1} = (e-1)\lim_{x\to0+}\frac{xe^x}{e^x-1} = (e-1)\lim_{x\to0+}\frac{e^x+xe^x}{e^x} = e-1$, $\displaystyle\lim_{n\to\infty} L(f,\mathbb{P}) = \lim_{n\to\infty}\frac{e-1}{n}\frac{1}{e^{\frac{1}{n}}-1} = (e-1)\lim_{x\to0+}\frac{x}{e^x-1} = (e-1)\lim_{x\to0+}\frac{1}{e^x} = e-1$, 故由夾擊定理 $\displaystyle\lim_{n\to\infty} R(f,\mathbb{P}) = e-1$.

結論. $\displaystyle\sum_{k=1}^{n}\cos kx = \frac{1}{2}\left(\frac{\sin\big(n+\frac{1}{2}\big)x}{\sin\frac{x}{2}} - 1\right)$

證. 考慮 $\displaystyle\sum_{k=1}^{n} e^{ikx} = \frac{e^{ix}(1-e^{inx})}{1-e^{ix}}$, 則 $\displaystyle\sum_{k=1}^{n}\cos kx = \Re\Big\{\sum_{k=1}^{n}e^{ikx}\Big\} = \Re\Big\{\frac{e^{ix}(1-e^{inx})}{1-e^{ix}}\Big\} = \Re\Big\{\frac{e^{ix}\,e^{\frac{inx}{2}}\big(e^{-\frac{inx}{2}}-e^{\frac{inx}{2}}\big)}{e^{\frac{ix}{2}}\big(e^{-\frac{ix}{2}}-e^{\frac{ix}{2}}\big)}\Big\} = \frac{\cos\frac{(n+1)x}{2}\,\sin\frac{nx}{2}}{\sin\frac{x}{2}} = \frac{\sin\big(\frac{nx}{2}+\frac{(n+1)x}{2}\big)+\sin\big(\frac{nx}{2}+\frac{(n+1)x}{2}\big)}{2\,\sin\frac{x}{2}} = \frac{\sin\big(n+\frac{1}{2}\big)x - \sin\frac{x}{2}}{2\,\sin\frac{x}{2}} = \frac{1}{2}\left(\frac{\sin\big(n+\frac{1}{2}\big)x}{\sin\frac{x}{2}} - 1\right)$.

例. 求 $\displaystyle\int_0^{\frac{\pi}{2}}\cos x\,\mathrm{d}x$.

解. 建立 $\left[0, \frac{\pi}{2}\right]$ 分割 $\mathbb{P}: \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \ldots, \frac{n\pi}{2n}\right\}$, 則 $\Delta x_k = \frac{\pi}{2n}\ \forall k = 1, 2, \ldots, n$ 且 $\|P\| \to 0$ 當 $n \to \infty$, 則積分為 $\lim\limits_{n\to\infty} R(f, \mathbb{P}) = \sum\limits_{k=1}^{n} f(\xi_k)\,\Delta x_k = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \cos\frac{k\pi}{2n} \cdot \frac{\pi}{2n} = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \cos\left(k\left(\frac{\pi}{2n}\right)\right) \cdot \frac{\pi}{2n} =$ $\lim\limits_{n\to\infty} \frac{1}{2}\left(\frac{\sin\left(\left(n+\frac{1}{2}\right)\cdot\frac{\pi}{2n}\right)}{\sin\left(\frac{1}{2}\cdot\frac{\pi}{2n}\right)} - 1\right) \cdot \frac{\pi}{2n} = \lim\limits_{n\to\infty} \frac{\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi}{2n}\right)\frac{1}{2}\cdot\frac{\pi}{2n}}{\sin\left(\frac{1}{2}\cdot\frac{\pi}{2n}\right)} = \lim\limits_{n\to\infty} \sin\left(\frac{1}{2} + \frac{1}{4n}\right)\pi \cdot \lim\limits_{n\to\infty} \frac{\frac{1}{2}\cdot\frac{\pi}{2n}}{\sin\left(\frac{1}{2}\cdot\frac{\pi}{2n}\right)} =$ $\sin\frac{\pi}{2} \cdot 1 = 1.$

微積分基本定理

複習: 連續函數性質

定理 (中間值定理)**.** 若 f 在 $[a, b]$ 連續, 則對任意介於 $f(a)$ 與 $f(b)$ 之間的數 d, 存在 $c \in [a, b]$ 使得 $f(c) = d$.

定理 (最大最小值定理)**.** 若函數在定義域為有限閉區間, 或有限閉區間的有限聯集連續, 則函數在定義域上有最大值及最小值.

積分均值定理

定理 (積分均值定理)**.** 設 f 在 $[a, b]$ 連續, 則存在 $c \in (a, b)$ 使得 $\int_a^b f(x)\,\mathrm{d}x = f(c) \cdot (b - a)$.

證. **第一步: 建立上下界.** 因 f 在 $[a, b]$ 連續, 根據最大最小值定理, f 在 $[a, b]$ 有最大值 M 及最小值 m, 亦即 $\exists x_{\mathrm{m}}, x_{\mathrm{M}}$ 使 $m = f(x_{\mathrm{m}}) \leqslant f(x) \leqslant f(x_{\mathrm{M}}) = M,\ \forall x \in [a, b]$.

第二步: 積分的夾擠. 對不等式 $m \leqslant f(x) \leqslant M$ 在 $[a, b]$ 上積分, 得 $m(b - a) \leqslant \int_a^b f(x)\,\mathrm{d}x \leqslant M(b - a)$; 除以 $(b - a) > 0$ 得 $m \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant M$.

第三步: 應用中間值定理. 令 $\mu = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$: 我們已證明 $m \leqslant \mu \leqslant M$. 因為 f 連續且 $f(x_{\mathrm{m}}) = m$, $f(x_{\mathrm{M}}) = M$, 由中間值定理, 存在 c 介於 x_{m} 與 x_{M} 間 (故 $c \in [a, b]$), 使 $f(c) = \mu = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \implies$ $\int_a^b f(x)\,\mathrm{d}x = f(c) \cdot (b - a)$.

定理 (微積分基本定理 I (積分的微分))**.** 設 f 在 $[a, b]$ 上連續. 定義 $F(x) = \int_a^x f(t)\,\mathrm{d}t,\ x \in [a, b]$, 則 F 在 (a, b) 可微, 且 $F'(x) = f(x)$.

證. 對於 $x \in (a, b)$ 及充分小的 $h \neq 0$, 考慮 $\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t$. 因 f 在 $[a, b]$ 連續, 由積分均值定理, 存在 c_h 介於 x 與 $x + h$ 間, 使 $\int_x^{x+h} f(t)\,\mathrm{d}t = f(c_h) \cdot h$, 故 $\frac{F(x+h) - F(x)}{h} = f(c_h)$. 當 $h \to 0$, $c_h \to x$. 由 f 的連續性, $f(c_h) \to f(x)$. 故 $F'(x) = \lim\limits_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x)$.

定理 (微積分基本定理 II (牛頓–萊布尼茲公式))**.** 設 f 在 $[a, b]$ 連續, 且 G 是 f 的任一反導函數 (即 $G'(x) = f(x)$), 則 $\int_a^b f(x)\,\mathrm{d}x = G(b) - G(a)$.

證. 由微積分基本定理 I, $F(x) = \int_a^x f(t)\,\mathrm{d}t$ 滿足 $F'(x) = f(x)$. 因為 G 也是 f 的反導函數, F 與 G 相差一個常數: $F(x) = G(x) + C$. 由 $F(a) = \int_a^a f(t)\,\mathrm{d}t = 0$ 得 $0 = G(a) + C \implies C = -G(a)$. 故 $\int_a^b f(x)\,\mathrm{d}x = F(b) = G(b) + C = G(b) - G(a)$.

例 (以 FTC 求定積分)**.**

1. x^2 之反導函數為 $\frac{x^3}{3}$, 故 $\int_0^1 x^2\,\mathrm{d}x = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$.
2. e^x 之反導函數為 e^x, 故 $\int_0^1 e^x\,\mathrm{d}x = e^1 - e^0 = e - 1$.
3. $\cos x$ 之反導函數為 $\sin x$, 故 $\int_0^{\frac{\pi}{2}} \cos x\,\mathrm{d}x = \sin\frac{\pi}{2} - \sin 0 = 1$.

結論 (定積分變數變換)**.**

- 求反導函數後代入: $\int_a^b f'(g(x))\,g'(x)\,\mathrm{d}x = f(g(x))\Big|_{x=a}^{x=b} = f(g(b)) - f(g(a))$
- 變數變換並改變積分範圍: $\int_a^b f'(g(x))\,g'(x)\,\mathrm{d}x = \int_a^b f'(g(x))\,\mathrm{d}g(x) = \int_{g(a)}^{g(b)} f'(u)\,\mathrm{d}u = f(u)\Big|_{u=g(a)}^{u=g(b)} = f(g(b)) - f(g(a))$

例. 求 $\int_0^1 x^3(1+x^4)^3\,\mathrm{d}x$.

解.

- 求反導函數後代入: 令 $u = 1 + x^4$, 則 $\mathrm{d}u = 4x^3\,\mathrm{d}x \implies x^3\,\mathrm{d}x = \frac{\mathrm{d}u}{4}$, 故 $\int x^3(1+x^4)^3\,\mathrm{d}x = \int (1+x^4)^3\,x^3\,\mathrm{d}x = \int u^3\,\frac{\mathrm{d}u}{4} = \frac{u^4}{16} + c = \frac{(1+x^4)^4}{16} + c$. 故 $\int_0^1 x^3(1+x^4)^3\,\mathrm{d}x = \frac{(1+x^4)^4}{16}\Big|_{x=0}^{x=1} = \frac{(1+1^4)^4 - (1+0^4)^4}{16} = \frac{15}{16}$.
- 變數變換並改變積分範圍: 令 $u = 1 + x^4$, 則 $\mathrm{d}u = 4x^3\,\mathrm{d}x \implies x^3\,\mathrm{d}x = \frac{\mathrm{d}u}{4}$. 積分範圍 x 由 0 至 1, 則變數變換後 u 由 $1 + 0^4 = 1$ 至 $1 + 1^4 = 2$, 故 $\int_0^1 x^3(1+x^4)^3\,\mathrm{d}x = \int_0^1 (1+x^4)^3\,x^3\,\mathrm{d}x = \int_1^2 u^3\,\frac{\mathrm{d}u}{4} = \frac{1}{4}\int_1^2 u^3\,\mathrm{d}u = \frac{1}{16}u^4\Big|_{u=1}^{u=2} = \frac{2^4 - 1^4}{16} = \frac{15}{16}$.

例. 求 $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{1+x^2}}\,\mathrm{d}x$.

解.

- 求反導函數後代入: 令 $u = 1 + x^2$, 則 $\mathrm{d}u = 2x\,\mathrm{d}x \implies 4x\,\mathrm{d}x = 2\,\mathrm{d}u$, 故 $\int \frac{4x}{\sqrt{1+x^2}}\,\mathrm{d}x = \int \frac{2}{\sqrt{u}}\,\mathrm{d}u = 4\sqrt{u} + c = 4\sqrt{1+x^2} + c$. 故 $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{1+x^2}}\,\mathrm{d}x = 4\sqrt{1+x^2}\Big|_{x=0}^{x=\sqrt{3}} = 4\sqrt{1+3} - 4\sqrt{1} = 4$.
- 變數變換並改變積分範圍: 令 $u = 1 + x^2$, 則 $\mathrm{d}u = 2x\,\mathrm{d}x \implies 4x\,\mathrm{d}x = 2\,\mathrm{d}u$. 積分範圍 x 由 0 至 $\sqrt{3}$, 則變數變換後 u 由 $1 + 0^2 = 1$ 至 $1 + (\sqrt{3})^2 = 4$, 故 $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{1+x^2}}\,\mathrm{d}x = \int_1^4 \frac{2}{\sqrt{u}}\,\mathrm{d}u = 4\sqrt{u}\Big|_{u=1}^{u=4} = 4\,(\sqrt{4} - \sqrt{1}) = 4$.

例. 求 $\displaystyle\int_0^{\pi} 3\cos^2 x \sin x \,\mathrm{d}x$.

解.

- 求反導函數後代入: 令 $u=\cos x$, 則 $\mathrm{d}u=-\sin x\,\mathrm{d}x \implies \sin x\,\mathrm{d}x=-\mathrm{d}u$, 故 $\displaystyle\int 3\cos^2 x\sin x\,\mathrm{d}x = -3\int u^2\,\mathrm{d}u = -u^3+c = -\cos^3 x + c$. 故 $\displaystyle\int_0^{\pi} 3\cos^2 x\sin x\,\mathrm{d}x = -\cos^3 x\,\Big|_{x=0}^{x=\pi} = -(\cos^3\pi - \cos^3 0) = -((-1)^3-1^3)=2$.
- 變數變換並改變積分範圍: 令 $u=\cos x$, 則 $\mathrm{d}u=-\sin x\,\mathrm{d}x \implies \sin x\,\mathrm{d}x=-\mathrm{d}u$. 積分範圍 x 由 0 至 π, 則變數變換後 u 由 $\cos 0=1$ 至 $\cos\pi=-1$, 故 $\displaystyle\int_0^{\pi} 3\cos^2 x\sin x\,\mathrm{d}x = -\int_1^{-1} 3u^2\,\mathrm{d}u = -u^3\,\Big|_{u=1}^{u=-1} = -((-1)^3-1^3)=2$.

例. 若 f 在 $[a,b]$ 二次可微且 $f(a)=f(b)=0$, 證明 $\displaystyle\int_a^b (x-a)(b-x)\,f''(x)\,\mathrm{d}x = -2\int_a^b f(x)\,\mathrm{d}x$.

解. $\displaystyle\int_a^b (x-a)(b-x)\,f''(x)\,\mathrm{d}x = \big((x-a)(b-x)\,f'(x)-(a+b-2x)\,f(x)\big)\Big|_a^b - 2\int_a^b f(x)\,\mathrm{d}x = \big((b-a)(b-b)\,f'(b)-(a+b-2b)\,f(b)\big) - \big((a-a)(b-a)\,f'(a)-(a+b-2a)\,f(a)\big) - 2\int_a^b f(x)\,\mathrm{d}x = -2\int_a^b f(x)\,\mathrm{d}x$.

Diff		Int
$(x-a)(b-x)$		$f''(x)$
$a+b-2x$	$+$	$f'(x)$
-2	$-$	$f(x)$
0	$+$	$\int f(x)\,\mathrm{d}x$

$$\begin{aligned}&\int (x-a)(b-x)\,f''(x)\,\mathrm{d}x\\ &= \big((x-a)(b-x)\,f'(x)-(a+b-2x)\,f(x)\big) - 2\int f(x)\,\mathrm{d}x\end{aligned}$$

性質. 令 $\displaystyle F(x)=\int_{v(x)}^{u(x)} f(\tau)\,\mathrm{d}\tau$, 則 $F'(x)=f(u(x))\cdot u'(x)-f(v(x))\cdot v'(x)$.

證. 令 $a\in\mathbb{R}$, $\displaystyle F(x)=\int_{v(x)}^{u(x)} f(\tau)\,\mathrm{d}\tau = \int_a^{u(x)} f(\tau)\,\mathrm{d}\tau - \int_a^{v(x)} f(\tau)\,\mathrm{d}\tau$. 令 $\displaystyle G(x)\equiv\int_a^x f(\tau)\,\mathrm{d}\tau$, 則 $G'(x)=f(x)$, $\displaystyle F(x)=\int_a^{u(x)} f(\tau)\,\mathrm{d}\tau - \int_a^{v(x)} f(\tau)\,\mathrm{d}\tau = G(u(x))-G(v(x))$; 故 $F'(x)=(G(u(x))-G(v(x)))' = G'(u(x))\cdot u'(x)-G'(v(x))\cdot v'(x) = f(u(x))\cdot u'(x)-f(v(x))\cdot v'(x)$.

例.

1. $\displaystyle F(x)=\int_1^x \frac{1}{1+\tau^4}\,\mathrm{d}\tau \implies F'(x)=\frac{1}{1+x^4}$
2. $\displaystyle F(x)=\int_2^{\sqrt{x}} \sin\tau\,\mathrm{d}\tau \implies F'(x)=\sin\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$
3. $\displaystyle F(x)=\int_x^{2x} \tau^3\,\mathrm{d}\tau \implies F'(x)=(2x)^3\cdot 2 - x^3\cdot 1 = 15x^3$
4. $\displaystyle F(x)=\int_{\sin x}^{\tan^{-1}x} e^{\tau^2}\,\mathrm{d}\tau \implies F'(x)=e^{(\tan^{-1}x)^2}\cdot\frac{1}{1+x^2} - e^{\sin^2 x}\cdot\cos x$

例. 若 $\displaystyle g(x)=\int_0^{\cos x}(1+\sin(t^2))\,\mathrm{d}t$, $\displaystyle f(x)=\int_0^{g(x)}\frac{1}{\sqrt{1+t^3}}\,\mathrm{d}t$, 求 $f'\!\left(\frac{\pi}{2}\right)$.

解. $f'(x) = \dfrac{1}{\sqrt{1+g(x)^3}} \cdot g'(x)$, $g'(x) = (1+\sin(\cos^2 x)) \cdot (-\sin x)$. 代入 $x = \dfrac{\pi}{2}$, $g\left(\dfrac{\pi}{2}\right) = 0$, $g'\left(\dfrac{\pi}{2}\right) = -1$, 故 $f'\left(\dfrac{\pi}{2}\right) = -1$.

例. 若 $\displaystyle\int_0^{x^2} f(t)\,\mathrm{d}t = x\sin\pi x$, 求 $f'(9)$.

解. $\displaystyle\int_0^{x^2} f(t)\,\mathrm{d}t = x\sin\pi x$ 兩邊對 x 微分得 $f(x^2)\cdot 2x = \sin\pi x + x\cdot\pi\cos\pi x$. 兩邊再對 x 微分得 $(f'(x^2)\cdot 2x)\cdot 2x + f(x^2)\cdot 2 = \pi\cos\pi x + \pi\cos\pi x - x\cdot\pi^2\sin\pi x$. 代入 $x=3$, 則 $(f'(9)\cdot 2\cdot 3)\cdot 2\cdot 3 + f(9)\cdot 2 = \pi\cos 3\pi + \pi\cos 3\pi - 3\cdot\pi^2\sin 3\pi \implies f'(9)\cdot 36 + f(9)\cdot 2 = -2\pi$. 又 $f(3^2)\cdot(2\cdot 3) = \sin 3\pi + 3\cdot\pi\cos 3\pi \implies f(9) = -\dfrac{\pi}{2}$, 故 $f'(9) = -\dfrac{\pi}{36}$.

例. 求函數 f 與 $a\in\mathbb{R}$ 使 $6 + \displaystyle\int_a^x \frac{f(t)}{t^2}\,\mathrm{d}t = 2\sqrt{x}$, $\forall x > 0$.

解. $6 + \displaystyle\int_a^x \frac{f(t)}{t^2}\,\mathrm{d}t = 2\sqrt{x}$ 兩邊對 x 微分得 $\dfrac{f(x)}{x^2} = \dfrac{1}{\sqrt{x}} \implies f(x) = x^{\frac{3}{2}}$. 代入原式得 $6 + \displaystyle\int_a^x \frac{t^{\frac{3}{2}}}{t^2}\,\mathrm{d}t = 6 + \int_a^x \frac{1}{\sqrt{t}}\,\mathrm{d}t = 6 + 2\sqrt{t}\Big|_a^x = 6 + 2\sqrt{x} - 2\sqrt{a} = 2\sqrt{x} \implies a = 9$.

例. 求下列極限.

1. $\displaystyle\lim_{x\to 0}\frac{\int_0^x(\sec t - 1)\,\mathrm{d}t}{x^3}$
2. $\displaystyle\lim_{x\to\infty}\frac{\int_{x^2}^0 e^{t-x^2}(2t^2+1)\,\mathrm{d}t}{x^4}$
3. $\displaystyle\lim_{x\to 0}\frac{1}{x}\int_0^{\tan x} f(u)(\sin x - \cos u)\,\mathrm{d}u$, f 為連續函數

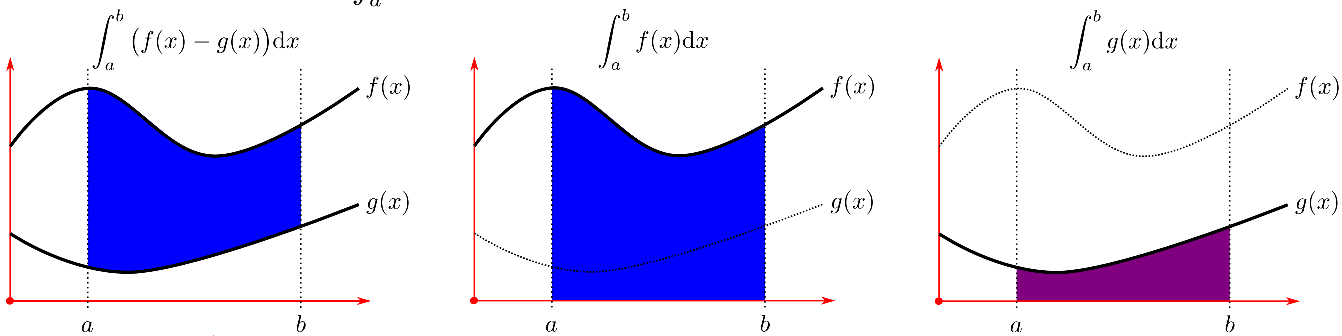
解.

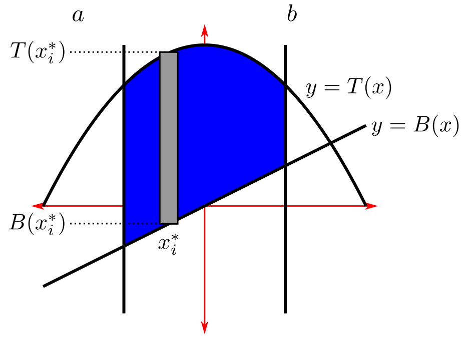
1. $\displaystyle\lim_{x\to 0}\frac{\int_0^x(\sec t - 1)\,\mathrm{d}t}{x^3} = \lim_{x\to 0}\frac{\sec x - 1}{3x^2} = \lim_{x\to 0}\frac{\sec x\tan x}{6x} = \lim_{x\to 0}\frac{\sec x\tan x\cdot\tan x + \sec x\cdot\sec^2 x}{6} = \frac{1}{6}$.
2. $\displaystyle\lim_{x\to\infty}\frac{\int_{x^2}^0 e^{t-x^2}(2t^2+1)\,\mathrm{d}t}{x^4} = \lim_{x\to\infty}\frac{\int_{x^2}^0 e^{t}(2t^2+1)\,\mathrm{d}t}{e^{x^2}x^4} = \lim_{x\to\infty}\frac{-e^{x^2}(2(x^2)^2+1)\cdot 2x}{e^{x^2}2x\cdot x^4 + e^{x^2}\cdot 4x^3} = \lim_{x\to\infty}\frac{-(4x^5+2x)}{2x^5+4x^3} = -2$.
3. $\displaystyle\lim_{x\to 0}\frac{1}{x}\int_0^{\tan x} f(u)(\sin x - \cos u)\,\mathrm{d}u = \lim_{x\to 0}\frac{\int_0^{\tan x} f(u)(\sin x - \cos u)\,\mathrm{d}u}{x}$
$\displaystyle= \lim_{x\to 0}\frac{\sin x\int_0^{\tan x} f(u)\,\mathrm{d}u - \int_0^{\tan x} f(u)\cos u\,\mathrm{d}u}{x}$
$\displaystyle= \lim_{x\to 0}\frac{\cos x\int_0^{\tan x} f(u)\,\mathrm{d}u + \sin x\cdot f(\tan x)\cdot\sec^2 x - f(\tan x)\cos(\tan x)\cdot\sec^2 x}{1} = -f(0)$

4.3 面積與體積

面積

結論. 當 $f(x) \geqslant 0\ \forall\, x \in [a,\, b]$, $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ 為 $y = f(x)$, x 軸, $x = a$, 與 $x = b$ 所圍成之面積.



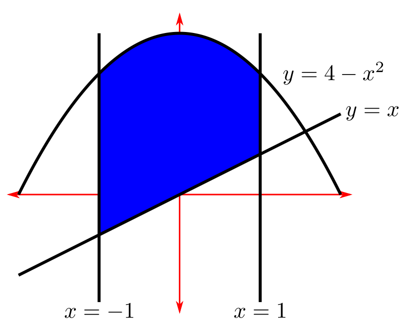


$$(\text{面積}) \approx \sum_{i=1}^{n} \big(T(x_i^*) - B(x_i^*)\big)\,\Delta x,\ \ \Delta x = \frac{b-a}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \big(T(x_i^*) - B(x_i^*)\big)\,\Delta x = \int_a^b \big(T(x) - B(x)\big)\,\mathrm{d}x$$

例. 求以 $y = 4 - x^2$, $y = x$, $x = -1$, 與 $x = 1$ 圍成之區域面積.

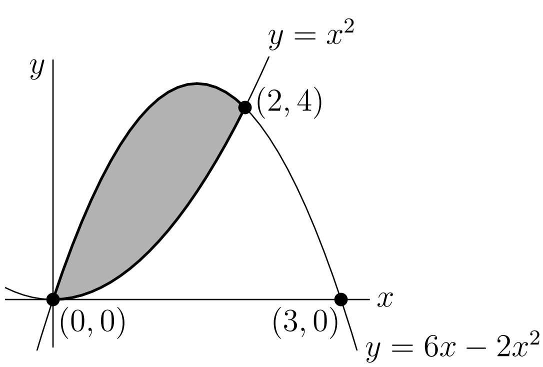
解.



$$(\text{面積}) = \int_{-1}^{1} \big((4 - x^2) - x\big)\,\mathrm{d}x = \frac{22}{3}$$

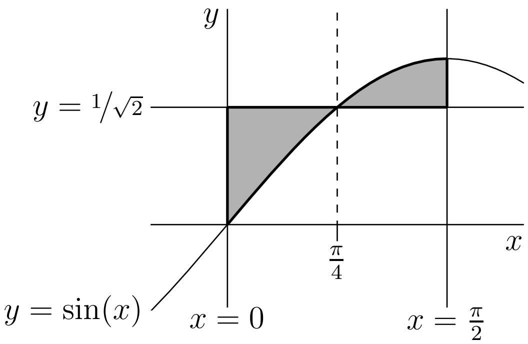
例. 求 $y = x^2$ 與 $y = 6x - 2x^2$ 圍成之區域面積.

解.

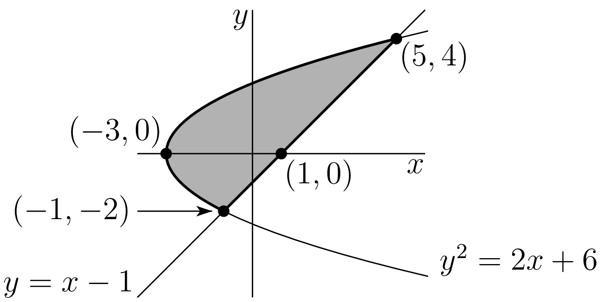


$$(\text{面積}) = \int_0^2 \big((6x - 2x^2) - x^2\big)\,\mathrm{d}x = 4$$

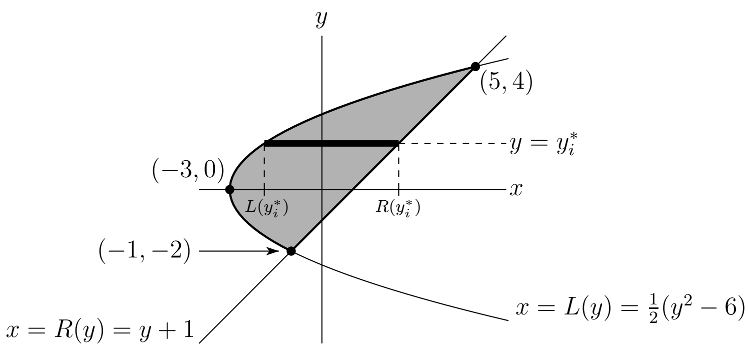
例. 求 $y = \dfrac{1}{\sqrt{2}}$ 與 $y = \sin x$ 在 x 從 0 至 $\dfrac{\pi}{2}$ 範圍內圍成之區域面積.

解. 

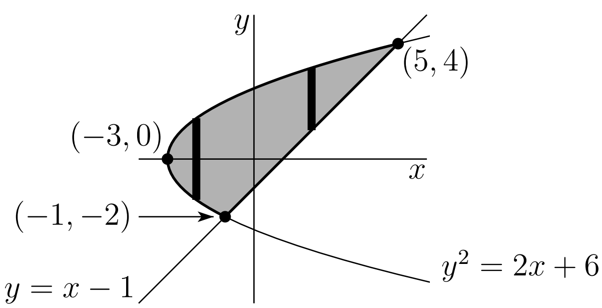
$$(\text{面積}) = \int_0^{\frac{\pi}{4}} \left(\frac{1}{\sqrt{2}} - \sin x\right) \mathrm{d}x + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\sin x - \frac{1}{\sqrt{2}}\right) \mathrm{d}x = \sqrt{2} - 1$$

例. 

求 $y^2 = 2x + 6$ 與 $y = x - 1$ 圍成之區域面積.

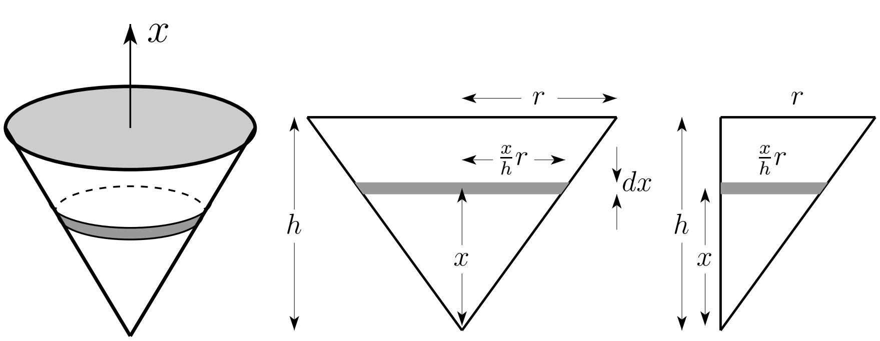
解. 

$$(\text{面積}) = \int_{-2}^{4} \Big((y+1) - \frac{1}{2}\left(y^2 - 6\right)\Big) \mathrm{d}y = 18$$



$$(\text{面積}) = \int_{-3}^{-1} 2\sqrt{2x+6}\, \mathrm{d}x + \int_{-1}^{5} \left(\sqrt{2x+6} - x + 1\right) \mathrm{d}x = 18$$

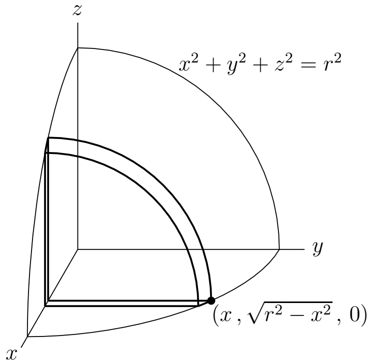
例. 求高 h 與底半徑 r 之圓錐體體積.

解. 

$$(\text{體積}) = \int_0^h \pi \left(\frac{x}{h}\, r\right)^2 \mathrm{d}x = \frac{1}{3}\pi r^2 h$$

例. 求半徑 r 之三維球體體積.

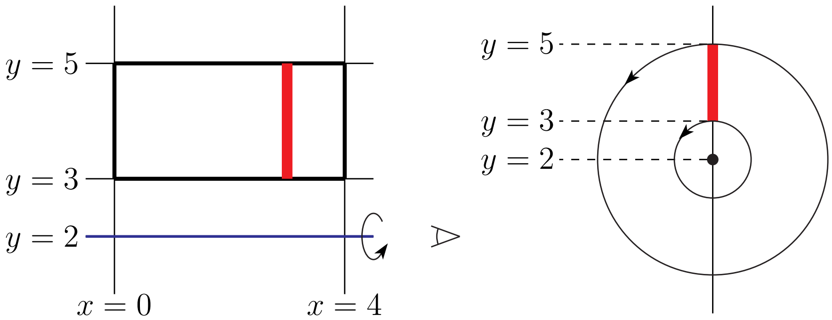
解.



$$(\text{體積}) = 8\cdot\int_0^r \frac{\pi}{4}\left(\sqrt{r^2-x^2}\right)^2 \mathrm{d}x = \frac{4}{3}\pi r^3$$

例. 求以 $y=3$, $y=5$, $x=0$ 與 $x=4$ 圍成之區域繞 $y=2$ 旋轉而成之旋轉體體積.

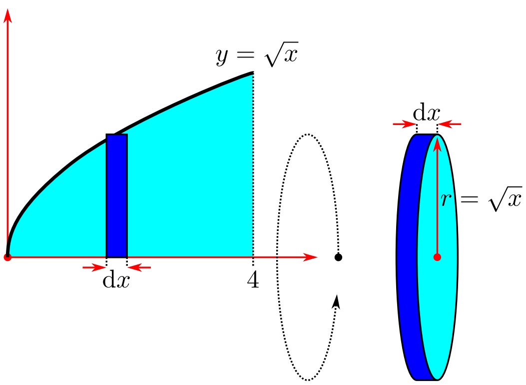
解.

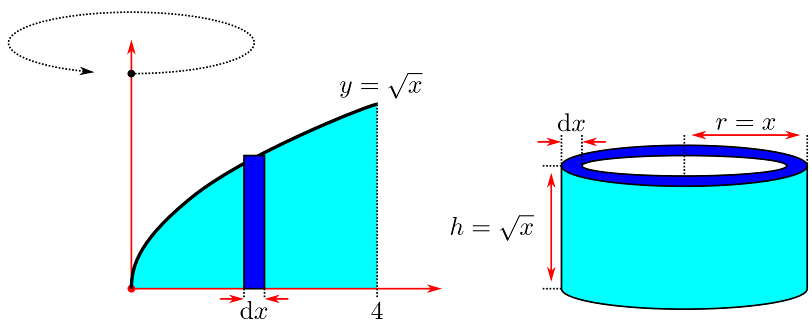


$$(\text{體積}) = \int_0^4 \pi(3^2-1^2)\,\mathrm{d}x = 32\pi$$

例. 求以 $y=\sqrt{x}$, $y=0$, $x=0$ 與 $x=4$ 圍成之區域繞 (i) $y=0$ (ii) $x=0$ 旋轉而成之旋轉體體積.

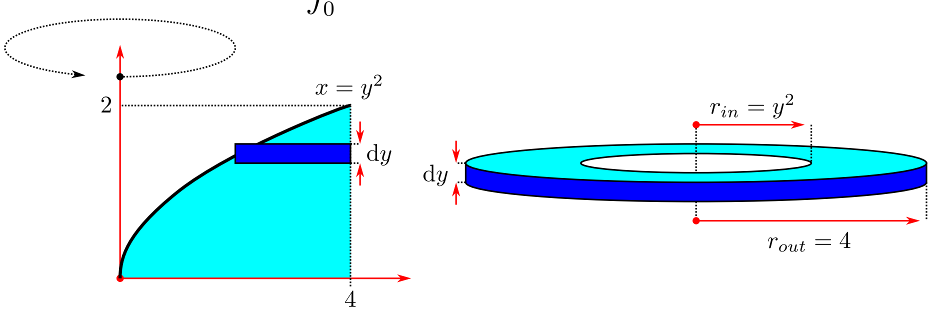
解.





$$(\text{體積}) = \int_0^4 \pi\left(\sqrt{x}\right)^2 \mathrm{d}x = 8\pi$$

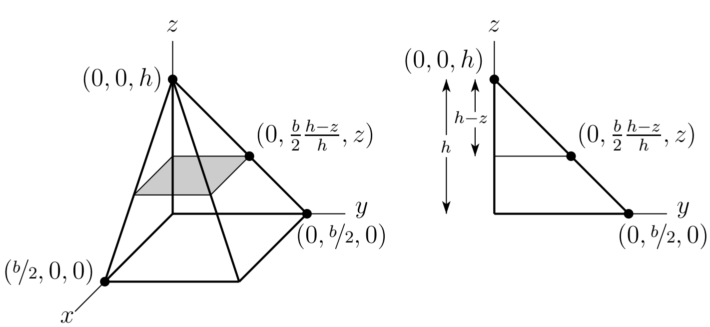
$$(\text{體積}) = \int_0^4 2\pi\cdot x\cdot\sqrt{x}\,\mathrm{d}x = \frac{128\pi}{5}$$



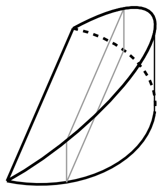
$$(\text{體積}) = \int_0^2 \pi(4^2-(y^2)^2)\,\mathrm{d}y = \frac{128\pi}{5}$$

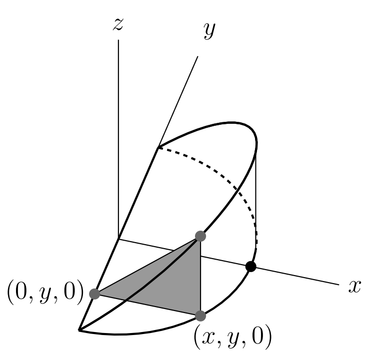
例. 求高為 h, 底面為邊長 b 正方形之錐體體積.

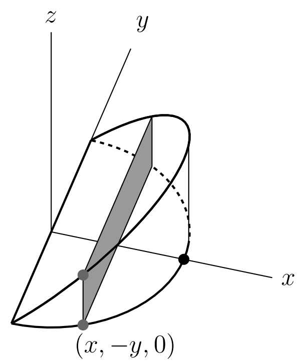
解.



$$(\text{體積}) = \int_0^h \left(b-\frac{z}{h}\,b\right)^2 \mathrm{d}z = \frac{1}{3}\,b^2h$$

例.  將一半徑為 a 之圓柱體水平橫切, 再對其底面圓心 45° 角斜切, 求如圖所示結果體積.

解. 



$$(\text{體積}) = \int_{-a}^{a} \frac{1}{2}\left(\sqrt{a^2-y^2}\right)^2 \mathrm{d}y = \frac{2}{3}a^3$$

$$(\text{體積}) = \int_{0}^{a} x \cdot 2\sqrt{a^2-x^2}\,\mathrm{d}x = \frac{2}{3}a^3$$

4.4 積分技巧

部份分式

例 (動機)**.** 若 $a \neq 0$, 求 $\displaystyle\int \frac{1}{x^2-a^2}\,\mathrm{d}x$.

解. 由 $\dfrac{1}{x^2-a^2} = \dfrac{1}{2a}\left(\dfrac{1}{x-a} - \dfrac{1}{x+a}\right)$, $\displaystyle\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \int \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)\mathrm{d}x = \frac{1}{2a}\big(\ln|x-a| - \ln|x+a|\big) = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

結論.

- 實係數多項式可分解成不可約的一次及二次因式的乘積.
- 有理式可寫成多項式與真分式之和.
- 若 $\dfrac{p(x)}{q(x)}$ 為一真分式, $q(x) = (x+a_1)^{m_1}(x+a_2)^{m_2}\cdots(x+a_k)^{m_k}\cdot(x^2+b_1x+c_1)^{n_1}(x^2+b_2x+c_2)^{n_2}\cdots(x^2+b_lx+c_l)^{n_l}$, 其中 $(x+a_i)$, $(x^2+b_ix+c_i)$ 均相異, $(x^2+b_ix+c_i)$ 為不可分解之二次式 $(b_i^2-4c_i<0)$, 則

$$\begin{aligned}
\frac{p(x)}{q(x)} = \; & \frac{\alpha_{11}}{x+a_1} + \frac{\alpha_{12}}{(x+a_1)^2} + \cdots + \frac{\alpha_{1m_1}}{(x+a_1)^{m_1}} + \\
& \frac{\alpha_{21}}{x+a_2} + \frac{\alpha_{22}}{(x+a_2)^2} + \cdots + \frac{\alpha_{2m_2}}{(x+a_2)^{m_2}} + \cdots + \\
& \frac{\alpha_{k1}}{x+a_k} + \frac{\alpha_{k2}}{(x+a_k)^2} + \cdots + \frac{\alpha_{km_k}}{(x+a_k)^{m_k}} + \\
& \frac{\beta_{11}x+\gamma_{11}}{x^2+b_1x+c_1} + \frac{\beta_{12}x+\gamma_{12}}{(x^2+b_1x+c_1)^2} + \cdots + \frac{\beta_{1n_1}x+\gamma_{1n_1}}{(x^2+b_1x+c_1)^{n_1}} + \\
& \frac{\beta_{21}x+\gamma_{21}}{x^2+b_2x+c_2} + \frac{\beta_{22}x+\gamma_{22}}{(x^2+b_2x+c_2)^2} + \cdots + \frac{\beta_{2n_2}x+\gamma_{2n_2}}{(x^2+b_1x+c_1)^{n_2}} + \cdots + \\
& \frac{\beta_{l1}x+\gamma_{l1}}{x^2+b_lx+c_l} + \frac{\beta_{l2}x+\gamma_{l2}}{(x^2+b_lx+c_l)^2} + \cdots + \frac{\beta_{ln_l}x+\gamma_{ln_l}}{(x^2+b_lx+c_l)^{n_l}}
\end{aligned}$$

  其中 α_{ij}, β_{ij}, $\gamma_{ij} \in \mathbb{R}$.

註. 任一有理函數之積分可分解為多項式積分與以下兩型積分:

- $\displaystyle\int \frac{1}{(x+a)^n}\,\mathrm{d}x,\ n\in\mathbb{N}$
- $\displaystyle\int \frac{f(x)}{(x^2+bx+c)^n}\,\mathrm{d}x,\ f(x)=1$ 或 $f(x)=x;\ b^2-4c<0,\ n\in\mathbb{N}$

例. 求 $\displaystyle\int \frac{x}{x^2-5x+6}\,\mathrm{d}x.$

解. $\displaystyle\frac{x}{x^2-5x+6}=\frac{x}{(x-3)(x-2)}=\frac{A}{x-3}+\frac{B}{x-2}\implies x=A\,(x-2)+B\,(x-3).$ 代入 $x=3\implies 3=A$; 代入 $x=2\implies 2=B(2-3)\implies B=-2$, 故 $\displaystyle\frac{x}{x^2-5x+6}=\frac{3}{x-3}-\frac{2}{x-2}$, $\displaystyle\int\frac{x}{x^2-5x+6}\,\mathrm{d}x=\int\left(\frac{3}{x-3}-\frac{2}{x-2}\right)\mathrm{d}x=3\ln|x-3|-2\ln|x-2|$

例. 求 $\displaystyle\int \frac{1}{x^3+1}\,\mathrm{d}x.$

解. $\displaystyle\frac{1}{x^3+1}=\frac{1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}\implies 1=A\,(x^2-x+1)+(Bx+C)\,(x+1).$ 代入 $x=-1\implies 1=A((-1)^2-(-1)+1)=3A\implies A=\dfrac{1}{3}$; 代入 $x=0\implies 1=A+C\implies C=\dfrac{2}{3}$; 代入 $x=1\implies 1=A+(B+C)\cdot 2\implies 1=\dfrac{1}{3}+\left(B+\dfrac{2}{3}\right)\cdot 2\implies B=-\dfrac{1}{3}$; 故 $\displaystyle\frac{1}{x^3+1}=\frac{1}{3}\left(\frac{1}{x+1}-\frac{x-2}{x^2-x+1}\right)$, $\displaystyle\int\frac{1}{x^3+1}\,\mathrm{d}x=\int\frac{1}{3}\left(\frac{1}{x+1}-\frac{x-2}{x^2-x+1}\right)\mathrm{d}x=\int\frac{1}{3}\left(\frac{1}{x+1}-\frac{\frac{1}{2}\,(2x-1)-\frac{3}{2}}{x^2-x+1}\right)\mathrm{d}x=\frac{1}{3}\int\frac{1}{x+1}\,\mathrm{d}x-\frac{1}{6}\int\frac{2x-1}{x^2-x+1}\,\mathrm{d}x-\frac{1}{2}\int\frac{1}{x^2-x+1}\,\mathrm{d}x=\frac{1}{3}\ln|x+1|-\frac{1}{6}\ln(x^2-x+1)-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}$

例. 求 $\displaystyle\int \frac{1}{x^4+4}\,\mathrm{d}x.$

解. 由 $x^4+4=(x^2+2)^2-(2x)^2=(x^2-2x+2)(x^2+2x+2)$, $\displaystyle\frac{1}{x^4+4}=\frac{Ax+B}{x^2-2x+2}+\frac{Cx+D}{x^2+2x+2}\implies (Ax+B)(x^2+2x+2)+(Cx+D)(x^2-2x+2)=1\implies (A+C)x^3+(2A+B-2C+D)x^2+(2A+2B+2C-2D)x+(2B+2D)=1\implies A+C=0,\ 2A+B-2C+D=0,\ 2A+2B+2C-2D=0,\ 2B+2D=1\implies A=-\frac{1}{8}$, $B=\dfrac{1}{4}$, $C=\dfrac{1}{8}$, $D=\dfrac{1}{4}$. 故 $\displaystyle\frac{1}{x^4+4}=\frac{1}{8}\left(\frac{x+2}{x^2+2x+2}-\frac{x-2}{x^2-2x+2}\right)$, $\displaystyle\int\frac{1}{x^4+4}\,\mathrm{d}x=\frac{1}{8}\int\left(\frac{x+2}{x^2+2x+2}-\frac{x-2}{x^2-2x+2}\right)\mathrm{d}x=\frac{1}{8}\int\left(\frac{(x+1)+1}{(x+1)^2+1}-\frac{(x-1)-1}{(x-1)^2+1}\right)\mathrm{d}x=\frac{1}{16}\ln\frac{x^2+2x+2}{x^2-2x+2}+\frac{1}{8}\big(\tan^{-1}(x+1)-\tan^{-1}(x-1)\big)$

例. 求 $\displaystyle\int \frac{1}{\cos^3 x}\,\mathrm{d}x.$

解. $\displaystyle\int\frac{1}{\cos^3 x}\,\mathrm{d}x=\int\frac{\cos x}{\cos^4 x}\,\mathrm{d}x=\int\frac{\cos x}{(1-\sin^2 x)^2}\,\mathrm{d}x.$ 令 $u=\sin x$, 則 $\mathrm{d}u=\cos x\,\mathrm{d}x$, 則 $\displaystyle\int\frac{\cos x}{(1-\sin^2 x)^2}\,\mathrm{d}x=\int\frac{1}{(1-u^2)^2}\,\mathrm{d}u.$ $\displaystyle\frac{1}{(1-u^2)^2}=\frac{1}{(u-1)^2(u+1)^2}=\frac{A}{u-1}+\frac{B}{(u-1)^2}+\frac{C}{u+1}+\frac{D}{(u+1)^2}\implies 1=A\,(u-1)(u+1)^2+B\,(u+1)^2+C\,(u-1)^2(u+1)+D\,(u-1)^2.$ 代入 $u=1\implies 1=4B\implies B=\dfrac{1}{4}$; 代入 $u=-1\implies 1=4D\implies D=\dfrac{1}{4}$; 代入 $u=0\implies 1=-A+B+C+D\implies 1=-A+\dfrac{1}{4}+C+\dfrac{1}{4}\implies\dfrac{1}{2}=-A+C$; 代入 $u=2\implies 1=9A+9B+3C+D\implies 1=9A+\dfrac{9}{4}+3C+\dfrac{1}{4}\implies A=-\dfrac{1}{4},\ C=\dfrac{1}{4}$; 則 $\displaystyle\frac{1}{(1-u^2)^2}=\frac{1}{4}\left(\frac{-1}{u-1}+\frac{1}{(u-1)^2}+\frac{1}{u+1}+\frac{1}{(u+1)^2}\right)$, $\displaystyle\int\frac{1}{(1-u^2)^2}\,\mathrm{d}u=\int\frac{1}{4}\left(\frac{-1}{u-1}+\frac{1}{(u-1)^2}+\frac{1}{u+1}+\frac{1}{(u+1)^2}\right)\mathrm{d}x=\frac{1}{4}\left(-\ln|u-1|-\frac{1}{u-1}+\ln|u+1|-\frac{1}{u+1}\right)=\frac{1}{4}\left(-\ln|\sin x-1|-\frac{1}{\sin x-1}+\ln|\sin x+1|-\frac{1}{\sin x+1}\right).$

部份分式另解: $\frac{1}{(1-u^2)^2}=\left(\frac{1}{u^2-1}\right)^2=\left(\frac{1}{(u-1)(u+1)}\right)^2=\frac{1}{4}\left(\frac{1}{u-1}-\frac{1}{u+1}\right)^2=\frac{1}{4}\left(\frac{1}{(u-1)^2}-\frac{2}{(u-1)(u+1)}+\frac{1}{(u+1)^2}\right)=\frac{1}{4}\left(\frac{1}{(u-1)^2}-\left(\frac{1}{u-1}-\frac{1}{u+1}\right)+\frac{1}{(u+1)^2}\right)=\frac{1}{4}\left(\frac{-1}{u-1}+\frac{1}{(u-1)^2}+\frac{1}{u+1}+\frac{1}{(u+1)^2}\right)$.

三角函數代換

結論.

- 遇 $\sqrt{a^2-x^2}$, 考慮 $x=a\sin\theta \implies \theta=\sin^{-1}\frac{x}{a}$, $\mathrm{d}x=a\cos\theta\,\mathrm{d}\theta$
- 遇 $\sqrt{a^2+x^2}$, 考慮 $x=a\tan\theta \implies \theta=\tan^{-1}\frac{x}{a}$, $\mathrm{d}x=a\sec^2\theta\,\mathrm{d}\theta$
- 遇 $\sqrt{x^2-a^2}$, 考慮 $x=a\sec\theta \implies \theta=\sec^{-1}\frac{x}{a}$, $\mathrm{d}x=a\sec\theta\,\tan\theta\,\mathrm{d}\theta$
- 遇 $\sin x$, $\cos x$ 之有理式, 考慮 $u=\tan\frac{x}{2}$, 由以下化為 u 之有理式:
  - $\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}=2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}}=\frac{2u}{1+u^2}$
  - $\cos x=2\cos^2\frac{x}{2}-1=2\cdot\frac{1}{1+u^2}-1=\frac{1-u^2}{1+u^2}$
  - $\mathrm{d}u=\frac{1}{2}\sec^2\frac{x}{2}\,\mathrm{d}x \implies \mathrm{d}x=\frac{2}{1+u^2}\,\mathrm{d}u$

例. 若 $a\neq 0$, 求下列不定積分.

1. $\int\sqrt{a^2-x^2}\,\mathrm{d}x$
2. $\int\sqrt{x^2+a^2}\,\mathrm{d}x$
3. $\int x^2\sqrt{x^2+a^2}\,\mathrm{d}x$
4. $\int\frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x$
5. $\int\frac{1}{x^2\sqrt{x^2+a^2}}\,\mathrm{d}x$
6. $\int\sqrt{x^2-a^2}\,\mathrm{d}x$
7. $\int\frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x$
8. $\int\frac{1}{\tan x+\sin x}\,\mathrm{d}x$
9. $\int\frac{1}{a+\sin x}\,\mathrm{d}x$, $a>1$

解.

1. 令 $x=a\sin\theta$, 則 $\int\sqrt{a^2-x^2}\,\mathrm{d}x=\int\sqrt{a^2-a^2\sin^2\theta}\,a\cos\theta\,\mathrm{d}\theta=a^2\int\cos^2\theta\,\mathrm{d}\theta=a^2\int\frac{1+\cos2\theta}{2}\,\mathrm{d}\theta$ $=\frac{a^2}{2}\theta+\frac{a^2}{4}\sin2\theta=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{a^2}{2}\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}$.
2. 令 $x=a\tan\theta$, 則 $\int\sqrt{x^2+a^2}\,\mathrm{d}x=\int a\sec\theta\cdot a\sec^2\theta\,\mathrm{d}\theta=a^2\int\sec^3\theta\,\mathrm{d}\theta=\frac{a^2}{2}\left(\sec\theta\cdot\tan\theta+\ln|\sec\theta+\tan\theta|\right)=\frac{a^2}{2}\left(\frac{x}{a}\cdot\frac{\sqrt{x^2+a^2}}{a}+\ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right|\right)=\frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\ln\left|\sqrt{x^2+a^2}+x\right|-\frac{a^2}{2}\ln|a|$
3. 令 $x=a\tan\theta$, 則 $\int x^2\sqrt{x^2+a^2}\,\mathrm{d}x=\int a\sec^2\theta\cdot a^2\tan^2\theta\cdot a\sec\theta\,\mathrm{d}\theta=a^4\int\sec^3\theta\tan^2\theta\,\mathrm{d}\theta$ $=a^4\int\sec^3\theta\,(\sec^2\theta-1)\,\mathrm{d}\theta=a^4\int(\sec^5\theta-\sec^3\theta)\,\mathrm{d}\theta=\frac{a^4}{4}\left(\sec^3\theta\tan\theta-\frac{1}{2}\left(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right)\right)$ $=\frac{a^4}{4}\left(\sec^3\theta\tan\theta-\frac{1}{2}\left(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right)\right)=\frac{x(x^2+a^2)^{\frac{3}{2}}}{4}-\frac{a^2x\sqrt{x^2+a^2}}{8}-\frac{a^4\ln|\sqrt{x^2+a^2}+x|}{8}+\frac{a^4\ln|a|}{8}$.

4. 令 $x = a\tan\theta$, 則 $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \int \frac{1}{a\sec\theta}\cdot a\sec^2\theta\,\mathrm{d}\theta = \int \sec\theta\,\mathrm{d}\theta = \ln|\sec\theta+\tan\theta|$ $\displaystyle= \ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right| = \ln|\sqrt{x^2+a^2}+x| - \ln|a|.$

5. 令 $x = a\tan\theta$, 則 $\displaystyle\int \frac{1}{x^2\sqrt{x^2+a^2}}\,\mathrm{d}x = \int \frac{a\sec^2\theta}{a^2\tan^2\theta\cdot a\sec\theta}\,\mathrm{d}\theta = \int \frac{\sec\theta}{a^2\tan^2\theta}\,\mathrm{d}\theta = \int \frac{\cos\theta}{a^2\sin^2\theta}\,\mathrm{d}\theta$ $\displaystyle= -\frac{1}{a^2\sin\theta} = -\frac{\sqrt{x^2+a^2}}{a^2x}.$

6. 令 $x = a\sec\theta$, 則 $\displaystyle\int \sqrt{x^2-a^2}\,\mathrm{d}x = \int a\tan\theta\cdot a\sec\theta\tan\theta\,\mathrm{d}\theta = a^2\int \sec\theta\tan^2\theta\,\mathrm{d}\theta = a^2\int \sec\theta\,(\sec^2\theta - 1)\,\mathrm{d}\theta = a^2\left(\int \sec^3\theta\,\mathrm{d}\theta - \int \sec\theta\,\mathrm{d}\theta\right) = \frac{a^2}{2}\left(\sec\theta\cdot\tan\theta + \int \sec\theta\,\mathrm{d}\theta - 2\int \sec\theta\,\mathrm{d}\theta\right) = \frac{a^2}{2}\left(\sec\theta\cdot\tan\theta - \int \sec\theta\,\mathrm{d}\theta\right) = \frac{a^2}{2}\big(\sec\theta\cdot\tan\theta - \ln|\sec\theta+\tan\theta|\big) = \frac{a^2}{2}\left(\frac{x}{a}\cdot\frac{\sqrt{x^2-a^2}}{a} - \ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|\right) = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right| = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\big|\sqrt{x^2-a^2}+x\big| + \frac{a^2}{2}\ln|a|.$

7. 令 $x = a\sec\theta$, 則 $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \int \frac{a\sec\theta\tan\theta}{a\tan\theta}\,\mathrm{d}\theta = \int \sec\theta\,\mathrm{d}\theta = \ln|\sec\theta+\tan\theta|$ $\displaystyle= \ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right| = \ln|\sqrt{x^2-a^2}+x| - \ln|a|.$

8. 令 $u = \tan\dfrac{x}{2}$, 則 $\displaystyle\int \frac{1}{\tan x+\sin x}\,\mathrm{d}x = \int \frac{1}{\frac{2u}{1-u^2}+\frac{2u}{1+u^2}}\frac{2}{1+u^2}\,\mathrm{d}u = \int \frac{1-u^2}{2u}\,\mathrm{d}u = \frac{\ln|u|}{2} - \frac{u^2}{4} = \frac{\ln|\tan\frac{x}{2}|}{2} - \frac{\tan^2\frac{x}{2}}{4}$

9. 令 $u = \tan\dfrac{x}{2}$, 則 $\displaystyle\int \frac{1}{a+\sin x}\,\mathrm{d}x = \int \frac{1}{a+\frac{2u}{1+u^2}}\,\frac{2}{1+u^2}\,\mathrm{d}u = 2\int \frac{1}{au^2+2u+a}\,\mathrm{d}u = \frac{2}{a\sqrt{\frac{a^2-1}{a^2}}}\tan^{-1}\frac{u+\frac{1}{a}}{\sqrt{\frac{a^2-1}{a^2}}}$ $\displaystyle= \frac{2}{\sqrt{a^2-1}}\tan^{-1}\frac{au+1}{\sqrt{a^2-1}} = \frac{2}{\sqrt{a^2-1}}\tan^{-1}\frac{a\tan\frac{x}{2}+1}{\sqrt{a^2-1}}.$

4.5 瑕積分

定義 (瑕積分 (improper integral))**.**

- 無限區間 (第一型) 瑕積分
  - 若 $f(x)$ 在 $[a,\infty)$ 連續, 則 $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x.$
  - 若 $f(x)$ 在 $(-\infty,b]$ 連續, 則 $\displaystyle\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{a\to-\infty}\int_a^b f(x)\,\mathrm{d}x.$
  - 若 $f(x)$ 在 $(-\infty,\infty)$ 連續, 則任取 $c\in\mathbb{R}$, $\displaystyle\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^c f(x)\,\mathrm{d}x + \int_c^\infty f(x)\,\mathrm{d}x.$
- 不連續點 (第二型) 瑕積分
  - 若 $f(x)$ 在 $(a,b]$ 連續, 則 $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \lim_{c\to a+}\int_c^b f(x)\,\mathrm{d}x.$
  - 若 $f(x)$ 在 $[a,b)$ 連續, 則 $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \lim_{c\to b-}\int_a^c f(x)\,\mathrm{d}x.$

- 令 $c \in (a, b)$. 若 $f(x)$ 在 $[a, c) \cup (c, b]$ 連續且在 $x = c$ 不連續, 則 $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$.

例. 1. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x$. 2. $\int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x$. 3. $\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$.

解.

1. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x = 2\int_0^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x = 2\lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,\mathrm{d}x = 2\lim_{b\to\infty}\tan^{-1} b = 2\cdot\frac{\pi}{2} = \pi$.
2. $\int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,\mathrm{d}x = \lim_{b\to\infty}\Big(-\frac{1}{b}+1\Big) = 1$
3. $\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{c\to 0+}\int_c^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{c\to 0+} 2\sqrt{x}\Big|_c^1 = \lim_{c\to 0+}(2-2\sqrt{c}) = 2$

例. 證明 $\forall\, n \in \mathbb{N}$, $\int_0^1 (\ln x)^n\,\mathrm{d}x = (-1)^n n!$.

解. 使用數學歸納法: $n = 1$ 時 $\int_0^1 \ln x\,\mathrm{d}x = (x\ln x - x)\Big|_0^1 = -1 - \lim_{x\to 0+} x\ln x = -1 + \lim_{x\to 0+}\frac{\ln\frac{1}{x}}{\frac{1}{x}} = -1 + \lim_{y\to\infty}\frac{\ln y}{y} = -1 = (-1)^1\,1!$. 令等式在 $n-1$ 成立: $\int_0^1 (\ln x)^{n-1}\,\mathrm{d}x = (-1)^{n-1}(n-1)!$, 則 $\int_0^1 (\ln x)^n\,\mathrm{d}x = x\,(\ln x)^n\Big|_0^1 - n\int_0^1 x\cdot(\ln x)^{n-1}\cdot\frac{1}{x}\,\mathrm{d}x = 0 - \lim_{x\to 0+} x\,(\ln x)^n + (-1)\cdot n\cdot(-1)^{n-1}(n-1)! = -\lim_{x\to 0+} x\,(\ln x)^n + (-1)^n n!$. 反覆使用 L'Hôpital 法則得 $\lim_{x\to 0+} x\,(\ln x)^n = (-1)^n\cdot\lim_{x\to 0+}\frac{(\ln\frac{1}{x})^n}{\frac{1}{x}} = (-1)^n\lim_{y\to\infty}\frac{(\ln y)^n}{y} = (-1)^n\cdot n\lim_{y\to\infty}\frac{(\ln y)^{n-1}}{y} = (-1)^n\cdot n(n-1)\lim_{y\to\infty}\frac{(\ln y)^{n-2}}{y} = \cdots = (-1)^n\,n!\lim_{y\to\infty}\frac{1}{y} = 0$, 故 $\int_0^1 (\ln x)^n\,\mathrm{d}x = (-1)^n n!$ 成立.

定理. 給定 $0 < a < \infty$.

- $\int_a^{\infty} \frac{1}{x^p}\,\mathrm{d}x$ 當 $p > 1$ 收斂至 $\frac{a^{1-p}}{p-1}$, 當 $p \leqslant 1$ 發散至 ∞.
- $\int_0^a \frac{1}{x^p}\,\mathrm{d}x$ 當 $p < 1$ 收斂至 $\frac{a^{1-p}}{1-p}$, 當 $p \geqslant 1$ 發散至 ∞.

定理. 令 $-\infty \leqslant a < b \leqslant \infty$, f, g 在 (a, b) 連續, 且 $0 \leqslant f(x) \leqslant g(x)\ \forall\, x$.

- 若 $\int_a^b g(x)\,\mathrm{d}x$ 收斂, $\int_a^b f(x)\,\mathrm{d}x$ 收斂.
- 若 $\int_a^b f(x)\,\mathrm{d}x$ 發散, $\int_a^b g(x)\,\mathrm{d}x$ 發散.

定理. 令 f, g 在 $[a, \infty)$, $a \in \mathbb{R}$ 連續, 均為正值, 且 $\lim_{x\to\infty}\frac{f(x)}{g(x)}$ 存在, 則 $\int_a^{\infty} f(x)\,\mathrm{d}x$ 與 $\int_a^{\infty} g(x)\,\mathrm{d}x$ 同斂散.

例. 證明 $\int_0^{\infty} e^{-x^2}\,\mathrm{d}x$ 收斂.

解. 由 $e^{-x^2} \leqslant 1\ \forall\, 0 \leqslant x < 1$ 及 $e^{-x^2} \leqslant e^{-x}\ \forall\, x \geqslant 1$, $\int_0^{\infty} e^{-x^2}\,\mathrm{d}x = \int_0^1 e^{-x^2}\,\mathrm{d}x + \int_1^{\infty} e^{-x^2}\,\mathrm{d}x \leqslant \int_0^1 1\,\mathrm{d}x + \int_1^{\infty} e^{-x}\,\mathrm{d}x = 1 + \frac{1}{e}$, 故 $\int_0^{\infty} e^{-x^2}\,\mathrm{d}x$ 收斂.

例. 定義 Γ 函數 $\Gamma(x) \equiv \int_0^{\infty} t^{x-1}e^{-t}\,\mathrm{d}t$; 已知 $\int_0^{\infty} e^{-t^2}\,\mathrm{d}t = \frac{\sqrt{\pi}}{2}$.

1. 證明 $\Gamma(x)$ 收斂, $\forall x>0$.
2. 證明 $\Gamma(x+1)=x\,\Gamma(x)$, $\forall x>0$.
3. 證明 $\Gamma(n+1)=n!$, $\forall n\in\mathbb{N}$.
4. 求 $\Gamma\Big(\dfrac{1}{2}\Big)$ 與 $\Gamma\Big(\dfrac{3}{2}\Big)$.

解.

1. $\displaystyle\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t=\int_0^1 t^{x-1}e^{-t}\,\mathrm{d}t+\int_1^\infty t^{x-1}e^{-t}\,\mathrm{d}t$. $\displaystyle\int_1^\infty t^{x-1}e^{-t}\,\mathrm{d}t$ 收斂, 因為
   - $\displaystyle\int_0^1 t^{x-1}e^{-t}\,\mathrm{d}t\leqslant\int_0^1 t^{x-1}\,\mathrm{d}t=\frac{t^x}{x}\bigg|_{t=0}^{t=1}=\frac{1}{x}$, $\displaystyle\int_0^1 t^{x-1}e^{-t}\,\mathrm{d}t$ 收斂.
   - $\displaystyle\int_1^\infty\frac{1}{t^2}\,\mathrm{d}t$ 收斂, $\displaystyle\lim_{t\to\infty}\frac{t^{x-1}e^{-t}}{\frac{1}{t^2}}=\lim_{t\to\infty}\frac{t^{x+1}}{e^t}=0$, $\displaystyle\int_1^\infty t^{x-1}e^{-t}\,\mathrm{d}t$ 收斂.
2. $\displaystyle\Gamma(x+1)=\int_0^\infty t^xe^{-t}\,\mathrm{d}t=\lim_{\substack{a\to0+\\b\to\infty}}\int_a^b t^xe^{-t}\,\mathrm{d}t$. 令 $u=t^x$, 則 $\mathrm{d}u=xt^{x-1}\,\mathrm{d}t$. 令 $\mathrm{d}v=e^{-t}\,\mathrm{d}t$, 則 $v=-e^{-t}$.
   故 $\displaystyle\lim_{\substack{a\to0+\\b\to\infty}}\int_a^b t^xe^{-t}\,\mathrm{d}t=\lim_{\substack{a\to0+\\b\to\infty}}\Big(-t^xe^{-t}\Big|_a^b+\int_a^b e^{-t}\cdot xt^{x-1}\,\mathrm{d}t\Big)=0+x\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t=x\,\Gamma(x)$.
3. 由上 $\Gamma(n+1)=n\,\Gamma(n)=n(n-1)\,\Gamma(n-2)=\cdots=n(n-1)\cdots2\,\Gamma(1)$, 又 $\displaystyle\Gamma(1)=\lim_{b\to\infty}\int_0^b e^{-t}\,\mathrm{d}t=\lim_{b\to\infty}-e^{-t}\Big|_0^b=\lim_{b\to\infty}1-e^{-b}=1$, 得證.
4. (a) $\displaystyle\Gamma\Big(\frac{1}{2}\Big)=\int_0^\infty t^{-\frac{1}{2}}e^{-t}\,\mathrm{d}t=\int_0^\infty\frac{e^{-t}}{\sqrt{t}}\,\mathrm{d}t$. 令 $u=\sqrt{t}$, 則 $t=u^2$, $\displaystyle\mathrm{d}u=\frac{1}{2\sqrt{t}}\,\mathrm{d}t\implies\frac{1}{\sqrt{t}}\,\mathrm{d}t=2\,\mathrm{d}u$. 積分範圍 t 由 0 至 ∞, 則變數變換後 u 由 $\sqrt{0}=0$ 至 $\sqrt{\infty}=\infty$, 故 $\displaystyle\Gamma\Big(\frac{1}{2}\Big)=\int_0^\infty\frac{e^{-t}}{\sqrt{t}}\,\mathrm{d}t=2\int_0^\infty e^{-u^2}\,\mathrm{d}u=2\cdot\frac{\sqrt{\pi}}{2}=\sqrt{\pi}$.
   (b) 由 $\Gamma(x+1)=x\,\Gamma(x)$, $\displaystyle\Gamma\Big(\frac{3}{2}\Big)=\Gamma\Big(\frac{1}{2}+1\Big)=\frac{1}{2}\,\Gamma\Big(\frac{1}{2}\Big)=\frac{1}{2}\sqrt{\pi}$.

4.6 Leibniz 積分法則: 積分號下取微分

定理 (Leibniz 積分法則)**.** 設 $(X,\mathcal{A},\mu)$ 為測度空間, $U\subseteq\mathbb{R}$ 為開區間, 函數 $f:X\times U\to\mathbb{R}$ (或 $\mathbb{C}$) 滿足

(i) **可測性**: 對每個 $t\in U$, 函數 $x\mapsto f(x,t)$ 是 $\mathcal{A}$-可測的.
(ii) **可微性**: 對 μ-幾乎所有 $x\in X$, 函數 $t\mapsto f(x,t)$ 在 U 上可微.
(iii) **控制條件**: 存在函數 $g\in L^1(\mu)$ 使得對所有 $t\in U$ 及 μ-幾乎所有 $x\in X$, $\displaystyle\Big|\frac{\partial f}{\partial t}(x,t)\Big|\leqslant g(x)$.

則

(a) 對每個 $t\in U$, 函數 $x\mapsto f(x,t)$ 屬於 $L^1(\mu)$.
(b) 函數 $\displaystyle F(t):=\int_X f(x,t)\,\mathrm{d}\mu(x)$ 在 U 上可微.
(c) 微分與積分可交換次序: $\displaystyle\frac{\mathrm{d}F}{\mathrm{d}t}(t)=\int_X\frac{\partial f}{\partial t}(x,t)\,\mathrm{d}\mu(x)$.

重要特例

系理 (Lebesgue 測度下的經典形式)**.** 設 $U, V \subseteq \mathbb{R}$ 為開區間, $f : U \times V \to \mathbb{R}$ 滿足

(i) f 與 $\dfrac{\partial f}{\partial t}$ 在 $U \times V$ 上連續;

(ii) 存在可積函數 $g : U \to [0, \infty)$ 使得 $\left|\dfrac{\partial f}{\partial t}(x,t)\right| \leqslant g(x)$ 對所有 $(x,t) \in U \times V$ 成立.

則 $F(t) = \displaystyle\int_U f(x,t)\,\mathrm{d}x$ 在 V 上可微, 且 $F'(t) = \displaystyle\int_U \frac{\partial f}{\partial t}(x,t)\,\mathrm{d}x$.

系理 (緊支撐或有界區間情形)**.** 設 $[a,b] \subset \mathbb{R}$ 為有界閉區間, $V \subseteq \mathbb{R}$ 為開區間, $f : [a,b] \times V \to \mathbb{R}$ 滿足

(i) f 對 x 可積, 對 t 可微;

(ii) $\dfrac{\partial f}{\partial t}$ 在 $[a,b] \times V$ 上連續.

則 $\displaystyle\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b f(x,t)\,\mathrm{d}x = \int_a^b \frac{\partial f}{\partial t}(x,t)\,\mathrm{d}x$.

證. 連續函數在緊集 $[a,b] \times K$ ($K \subset V$ 為任意緊子區間) 上有界, 故控制條件自動滿足.

應用範例

例. $\displaystyle\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}$.

證. 令 $F(t) = \displaystyle\int_0^\infty \frac{e^{-t^2(1+x^2)}}{1+x^2}\,\mathrm{d}x$, $t \geqslant 0$; $F(\infty) = 0$, $F(0) = \displaystyle\int_0^\infty \frac{\mathrm{d}x}{1+x^2} = \tan^{-1} x\Big|_0^\infty = \tan^{-1}\infty - \tan^{-1} 0 = \frac{\pi}{2}$.
則 $F'(t) = \displaystyle\int_0^\infty \frac{-2\,t\,(1+x^2)\,e^{-t^2(1+x^2)}}{1+x^2}\,\mathrm{d}x = \underbrace{-2\,t\,e^{-t^2}\int_0^\infty e^{-t^2x^2}\,\mathrm{d}x}_{\text{令 } u=t\,x,\text{ 則 } \mathrm{d}u=t\,\mathrm{d}x} = -2\,e^{-t^2}\underbrace{\int_0^\infty e^{-u^2}\,\mathrm{d}u}_{=I} = -2\,e^{-t^2} I \implies$
$\displaystyle\int_0^b F'(t)\,\mathrm{d}t = -2I\int_0^b e^{-t^2}\,\mathrm{d}t \implies F(b) - F(0) = -2I\int_0^b e^{-t^2}\,\mathrm{d}t$. 令 $b \to \infty$, 則 $F(\infty) - F(0) = -2I^2 \implies$
$0 - \dfrac{\pi}{2} = -2I^2 \implies I = \dfrac{\sqrt{\pi}}{2}$.

例. $\displaystyle\int_0^\infty x^{2n}e^{-ax^2}\,\mathrm{d}x = \frac{(2n-1)!!}{2^{n+1}}\sqrt{\frac{\pi}{a^{2n+1}}} = \frac{(2n)!}{n!\,4^n}\sqrt{\frac{\pi}{a^{2n+1}}}$, $\forall\, a > 0$, $n \in \mathbb{N}$, 其中 $(2n-1)!! = 1\cdot 3\cdot 5\cdots(2n-1)$.

證. 令 $F(a) = \displaystyle\int_0^\infty e^{-ax^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}$, 則 $F'(a) = -\displaystyle\int_0^\infty x^2 e^{-ax^2}\,\mathrm{d}x = -\frac{1}{2}\sqrt{\pi}\cdot\big(-\frac{1}{2}\big)a^{-\frac{3}{2}} = \frac{\sqrt{\pi}}{4a^{\frac{3}{2}}} \implies$
$\displaystyle\int_0^\infty x^2 e^{-ax^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{4a^{\frac{3}{2}}}$, $F''(a) = \displaystyle\int_0^\infty x^4 e^{-ax^2}\,\mathrm{d}x = \frac{3\sqrt{\pi}}{8a^{\frac{5}{2}}}$; 可歸納出通式為 $(-1)^n F^{(n)}(a) = \displaystyle\int_0^\infty x^{2n}e^{-ax^2}\,\mathrm{d}x$. 由 $\dfrac{\mathrm{d}^n}{\mathrm{d}a^n}(a^{-\frac{1}{2}}) = (-1)^n\dfrac{(2n-1)!!}{2^n}a^{-\frac{2n+1}{2}}$, 得證.

例. $\displaystyle\int_0^\infty e^{-ax^2}\cos bx\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-\frac{b^2}{4a}}$, $\forall\, a > 0$.

證. 令 $F(b) = \displaystyle\int_0^\infty e^{-ax^2}\cos bx\,\mathrm{d}x$, 則 $F(0) = \dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}$; $F'(b) = -\displaystyle\int_0^\infty x\,e^{-ax^2}\sin bx\,\mathrm{d}x$. 令 $u = \sin bx$, 則 $\mathrm{d}u = b\cos bx$; $\mathrm{d}v = -xe^{-ax^2}\,\mathrm{d}x$, 則 $v = \dfrac{1}{2a}e^{-ax^2}$; 故 $F'(b) = \sin bx\cdot\dfrac{1}{2a}e^{-ax^2}\Big|_{x=0}^{x=\infty} - \dfrac{b}{2a}\displaystyle\int_0^\infty e^{-ax^2}\cos bx\,\mathrm{d}x = -\frac{b}{2a}F(b)$
$\implies F(b) = F(0)\,e^{-\frac{b^2}{4a}}$.

例. $\displaystyle\int_0^\infty xe^{-ax^2}\sin bx\,\mathrm{d}x=\frac{b}{4a}\sqrt{\frac{\pi}{a}}e^{-\frac{b^2}{4a}},\ \forall\, a>0.$

證. 令 $\displaystyle F(b)=\int_0^\infty e^{-ax^2}\cos bx\,\mathrm{d}x=\frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-\frac{b^2}{4a}}$, 則 $\displaystyle F'(b)=-\int_0^\infty xe^{-ax^2}\sin bx\,\mathrm{d}x=\frac{1}{2}\sqrt{\frac{\pi}{a}}\cdot e^{-\frac{b^2}{4a}}\cdot\Big(-\frac{b}{2a}\Big)=-\frac{b}{4a}\sqrt{\frac{\pi}{a}}e^{-\frac{b^2}{4a}}.$

例. $\displaystyle\int_0^\infty e^{-ax^2-\frac{b}{x^2}}\,\mathrm{d}x=\frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-2\sqrt{ab}},\ \ \forall\, a,b>0.$

證. 令 $\displaystyle F(t)=\int_0^\infty e^{-ax^2-\frac{t}{x^2}}\,\mathrm{d}x,\ t>0$, 則 $\displaystyle F(0)=\frac{1}{2}\sqrt{\frac{\pi}{a}};\ F'(t)=-\int_0^\infty\frac{e^{-ax^2-\frac{t}{x^2}}}{x^2}\,\mathrm{d}x.$ 變數變換 $\displaystyle u=\sqrt{\frac{t}{a}}\,\frac{1}{x}$, 則 $\displaystyle x=\sqrt{\frac{t}{a}}\,\frac{1}{u},\ -\sqrt{\frac{a}{t}}\,\mathrm{d}u=\frac{\mathrm{d}x}{x^2}$, 積分範圍 x 從 $0+$ 到 ∞, 則 $\displaystyle u=\sqrt{\frac{t}{a}}\,\frac{1}{x}$ 從 ∞ 到 0, $\displaystyle F'(t)=\int_0^\infty\frac{e^{-ax^2-\frac{t}{x^2}}}{x^2}\,\mathrm{d}x=\int_\infty^0 e^{-a\cdot\frac{t}{a}\cdot\frac{1}{u^2}-t\cdot\frac{a}{t}\cdot u^2}(-)\sqrt{\frac{a}{t}}\,\mathrm{d}u=\sqrt{\frac{a}{t}}\int_0^\infty e^{-\frac{t}{u^2}-au^2}\,\mathrm{d}u=\sqrt{\frac{a}{t}}\,F(t)\implies\int_0^b\frac{F'(t)}{F(t)}\,\mathrm{d}t=\sqrt{a}\int_0^b\frac{1}{\sqrt{t}}\,\mathrm{d}t\implies\ln F(t)\Big|_0^b=-2\sqrt{ab}\implies F(b)=F(0)\,e^{-2\sqrt{ab}}=\frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-2\sqrt{ab}}.$

例 (Glasser's Master Theorem (GMT))**.** 若 f 在 $(-\infty,\infty)$ 可積, $\displaystyle\forall\, c>0,\ \int_{-\infty}^\infty f\Big(x-\frac{c}{x}\Big)\,\mathrm{d}x=\int_{-\infty}^\infty f(x)\,\mathrm{d}x.$

證. 變數變換 $\displaystyle u=x-\frac{c}{x}$, 則 $\displaystyle x^2-ux-c=0\implies x=\frac{u+\sqrt{u^2+4c}}{2}>0$ 或 $\displaystyle x=\frac{u-\sqrt{u^2+4c}}{2}<0$, 分別對應 $\displaystyle\mathrm{d}x=\Big(\frac{1}{2}+\frac{u}{\sqrt{u^2+4c}}\Big)\mathrm{d}u$ 或 $\displaystyle\mathrm{d}x=\Big(\frac{1}{2}-\frac{u}{\sqrt{u^2+4c}}\Big)\mathrm{d}u$; 積分範圍 x 分別由 $-\infty$ 至 $0-$, $0+$ 至 ∞, u 則分別由 $-\infty$ 至 ∞, $-\infty$ 至 ∞. 故 $\displaystyle\int_{-\infty}^\infty f\Big(x-\frac{c}{x}\Big)\,\mathrm{d}x=\int_{-\infty}^{0-}f\Big(x-\frac{c}{x}\Big)\,\mathrm{d}x+\int_{0+}^\infty f\Big(x-\frac{c}{x}\Big)\,\mathrm{d}x=\int_{-\infty}^\infty f(u)\Big(\frac{1}{2}+\frac{u}{\sqrt{u^2+4c}}\Big)\mathrm{d}u+\int_{-\infty}^\infty f(u)\Big(\frac{1}{2}+\frac{u}{\sqrt{u^2+4c}}\Big)\mathrm{d}u=\int_{-\infty}^\infty f(u)\,\mathrm{d}u=\int_{-\infty}^\infty f(x)\,\mathrm{d}x.$

註 (使用 GMT)**.** $\displaystyle\int_0^\infty e^{-ax^2-\frac{b}{x^2}}\,\mathrm{d}x=\underbrace{\int_0^\infty e^{-(\sqrt{a}x-\frac{\sqrt{b}}{x})^2-2\sqrt{ab}}\,\mathrm{d}x}_{\text{令 }u=\sqrt{a}x,\ ;\ \frac{\mathrm{d}u}{\sqrt{a}}=\mathrm{d}x}=\frac{e^{-2\sqrt{ab}}}{\sqrt{a}}\int_0^\infty e^{-(u-\frac{\sqrt{ab}}{u})^2}\,\mathrm{d}u=\underbrace{\frac{e^{-2\sqrt{ab}}}{2\sqrt{a}}\int_{-\infty}^\infty e^{-(u-\frac{\sqrt{ab}}{u})^2}\,\mathrm{d}u}_{\text{偶函數, }\sqrt{ab}>0\text{; 符合 GMT}}=\frac{e^{-2\sqrt{ab}}}{2\sqrt{a}}\int_{-\infty}^\infty e^{-u^2}\,\mathrm{d}u=\frac{e^{-2\sqrt{ab}}}{2\sqrt{a}}\cdot\sqrt{\pi}=\frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-2\sqrt{ab}}.$

例. 若 $a>0$, 求 $\displaystyle\int_0^\infty\frac{\mathrm{d}x}{(x^2+a^2)^2},\int_0^\infty\frac{\mathrm{d}x}{(x^2+a^2)^3}$ 並推導 $\displaystyle\int_0^\infty\frac{\mathrm{d}x}{(x^2+a^2)^n},\ n\in\mathbb{N}.$

解. 由 $\displaystyle F(a)=\int_0^\infty\frac{\mathrm{d}x}{x^2+a^2}=\frac{\pi}{2a}=\frac{\pi}{2}\,a^{-1},\ F'(a)=\frac{\mathrm{d}}{\mathrm{d}a}\Big(\frac{\pi}{2}\,a^{-1}\Big)=\frac{\pi}{2}(-1)\,a^{-2}=(-1)2a\int_0^\infty\frac{\mathrm{d}x}{(x^2+a^2)^2}\implies\int_0^\infty\frac{\mathrm{d}x}{(x^2+a^2)^2}=\frac{\pi}{4\,a^3}$; 等式兩邊再對 a 微分 $\displaystyle\implies(-2)(2a)\int_0^\infty\frac{\mathrm{d}x}{(x^2+a^2)^3}=(-3)\frac{\pi}{4\,a^4}\implies\int_0^\infty\frac{\mathrm{d}x}{(x^2+a^2)^3}=\frac{3\,\pi}{16\,a^5}.$ 由數學歸納法可得 $\displaystyle\int_0^\infty\frac{\mathrm{d}x}{(x^2+a^2)^n}=\frac{\pi}{(2a)^{2n-1}}\binom{2(n-1)}{n-1},\ n\in\mathbb{N}.$

例. $\displaystyle\int_0^\infty e^{-tx}\,\frac{\sin x}{x}\,\mathrm{d}x=\frac{\pi}{2}-\tan^{-1}t,\ \forall\, t>0$

證. 令 $\displaystyle F(t)=\int_0^\infty e^{-tx}\,\frac{\sin x}{x}\,\mathrm{d}x$, 則 $\displaystyle F'(t)=-\int_0^\infty e^{-tx}\sin x\,\mathrm{d}x=-\frac{e^{-tx}(-t\sin x-\cos x)}{1+t^2}\Big|_{x=0}^{x=\infty}=-\frac{1}{1+t^2}\implies F(t)=-\tan^{-1}t+c.$ 令 $t\to\infty$, $\displaystyle F(\infty)=0=-\tan^{-1}\infty+c=-\frac{\pi}{2}+c\implies c=\frac{\pi}{2}.$

註. 上式令 $t \to 0+$ 得 $\displaystyle\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}$; 由變數變換 $u = ax$ 得 $\displaystyle\int_0^\infty \frac{\sin ax}{x}\,\mathrm{d}x = \frac{\pi}{2}\operatorname{sgn}(a), \forall\, a \neq 0.$

例. $\displaystyle\int_0^\infty e^{-ax}\frac{\sin bx}{x}\,\mathrm{d}x = \tan^{-1}\frac{b}{a},\ \forall\, a, b > 0.$

證. 令 $\displaystyle F(b) = \int_0^\infty e^{-ax}\frac{\sin bx}{x}\,\mathrm{d}x$, 則 $F(0) = 0$; $\displaystyle F'(b) = \int_0^\infty e^{-ax}\cos bx\,\mathrm{d}x = \frac{a}{a^2+b^2} \implies F(b) = \tan^{-1}\frac{b}{a}.$

例. $\displaystyle\int_0^\infty \frac{\sin ax\sin bx}{x^2}\,\mathrm{d}x = \frac{\pi}{2}\min(a,b),\ \forall\, a, b > 0.$

證. 由 $\displaystyle\sin ax\sin bx = \frac{\cos(a-b)x - \cos(a+b)x}{2}$, $\displaystyle\int_0^\infty \frac{\sin ax\sin bx}{x^2}\,\mathrm{d}x = \frac{1}{2}\int_0^\infty \frac{\cos(a-b)x - \cos(a+b)x}{x^2}\,\mathrm{d}x.$
設 $\displaystyle F(c) = \int_0^\infty \frac{1-\cos cx}{x^2}\,\mathrm{d}x$, 則 $F(0) = 0$. $\displaystyle F'(c) = \int_0^\infty \frac{\sin cx}{x}\,\mathrm{d}x = \frac{\pi}{2}\operatorname{sgn}(c)$, $\displaystyle F(c) = \frac{\pi|c|}{2}.$
$\displaystyle\int_0^\infty \frac{\cos(a-b)x - \cos(a+b)x}{x^2}\,\mathrm{d}x = F(a+b) - F(|a-b|) = \frac{\pi(a+b)}{2} - \frac{\pi|a-b|}{2} = \frac{\pi}{2}\cdot 2\min(a,b) = \pi\min(a,b),$
故 $\displaystyle\int_0^\infty \frac{\sin ax\sin bx}{x^2}\,\mathrm{d}x = \frac{1}{2}\cdot\pi\min(a,b) = \frac{\pi}{2}\min(a,b).$

例. $\displaystyle\int_0^\infty \frac{\sin^2 ax}{x^2}\,\mathrm{d}x = \frac{\pi a}{2}, \forall\, a > 0.$

證. 令 $\displaystyle F(a) = \int_0^\infty \frac{\sin^2 ax}{x^2}\,\mathrm{d}x$, 則 $F(0) = 0$; $\displaystyle F'(a) = \int_0^\infty \frac{\partial}{\partial a}\Big(\frac{\sin^2 ax}{x^2}\Big)\,\mathrm{d}x = \int_0^\infty \frac{2\sin ax\cos ax\cdot x}{x^2}\,\mathrm{d}x = \int_0^\infty \frac{\sin 2ax}{x}\,\mathrm{d}x = \frac{\pi}{2}$, 故 $\displaystyle F(a) = \frac{\pi a}{2}.$

例. $\displaystyle\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,\mathrm{d}x = \frac{\pi(b-a)}{2}\quad \forall\, a, b > 0.$

證. 設 $\displaystyle F(a) = \int_0^\infty \frac{1-\cos ax}{x^2}\,\mathrm{d}x$, 則 $F(0) = 0$; $\displaystyle F'(a) = \int_0^\infty \frac{\sin ax}{x}\,\mathrm{d}x = \frac{\pi}{2}$, 故 $\displaystyle F(a) = \frac{\pi a}{2}$, $\displaystyle\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,\mathrm{d}x$
$\displaystyle = F(b) - F(a) = \frac{\pi(b-a)}{2}.$

例 (Frullani 公式)**.** 若 f 在 $(0,\infty)$ 連續, 且 $f(0+)$ 和 $\displaystyle f(\infty) = \lim_{x\to\infty} f(x)$ 都存在且有限, 則 $\forall\, a, b > 0$
$\displaystyle\int_0^\infty \frac{f(ax) - f(bx)}{x}\,\mathrm{d}x = \big(f(0+) - f(\infty)\big)\ln\frac{b}{a}.$

證. 令 $\displaystyle F(a) = \int_0^\infty \frac{f(ax) - f(bx)}{x}\,\mathrm{d}x$, 則 $F(b) = 0$; 變數變換 $u = ax$, $\displaystyle F'(a) = \int_0^\infty f'(ax)\,\mathrm{d}x = \frac{1}{a}\int_0^\infty f'(u)\,\mathrm{d}u = \frac{1}{a}\big(f(\infty) - f(0+)\big)$. 則 $\displaystyle\int_b^a F'(t)\,\mathrm{d}t = \big(f(\infty) - f(0+)\big)\int_b^a \frac{1}{t}\,\mathrm{d}t \implies F(a) = \big(f(\infty) - f(0+)\big)\ln\frac{a}{b}.$

註. $\displaystyle\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,\mathrm{d}x = \ln\frac{b}{a}$, $\displaystyle\int_0^\infty \frac{\tan^{-1}ax - \tan^{-1}bx}{x}\,\mathrm{d}x = \frac{\pi}{2}\ln\frac{a}{b}$ 可分別由 Frullani 公式求得: 分別取 $f(x) = e^{-x}$; $f(0+) = 1$, $f(\infty) = 0$ 及取 $f(x) = \tan^{-1}(x)$; $f(0+) = 0$, $\displaystyle f(\infty) = \frac{\pi}{2}.$

例. $\displaystyle\int_0^{\frac{\pi}{2}} \ln(a^2\cos^2\theta + b^2\sin^2\theta)\,\mathrm{d}\theta = \pi\ln\frac{a+b}{2},\ \forall\, a, b > 0.$

證. 令 $F(a,b)=\displaystyle\int_0^{\frac{\pi}{2}}\ln(a^2\cos^2\theta+b^2\sin^2\theta)\,\mathrm{d}\theta$, $\dfrac{\partial F}{\partial a}=\displaystyle\int_0^{\frac{\pi}{2}}\frac{2a\cos^2\theta}{a^2\cos^2\theta+b^2\sin^2\theta}\,\mathrm{d}\theta$, $\dfrac{\partial F}{\partial b}=\displaystyle\int_0^{\frac{\pi}{2}}\frac{2b\sin^2\theta}{a^2\cos^2\theta+b^2\sin^2\theta}\,\mathrm{d}\theta$. 令 $K=\displaystyle\int_0^{\frac{\pi}{2}}\frac{\cos^2\theta}{a^2\cos^2\theta+b^2\sin^2\theta}\,\mathrm{d}\theta$, $L=\displaystyle\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta}{a^2\cos^2\theta+b^2\sin^2\theta}\,\mathrm{d}\theta$. 變數變換 $t=\tan\theta$, 則 $\cos^2\theta=\dfrac{1}{1+t^2}$, $\sin^2\theta=\dfrac{t^2}{1+t^2}$, $\mathrm{d}\theta=\dfrac{\mathrm{d}t}{1+t^2}$, t 範圍從 0 至 ∞; $K+L=\displaystyle\int_0^{\frac{\pi}{2}}\frac{1}{a^2\cos^2\theta+b^2\sin^2\theta}\,\mathrm{d}\theta=\int_0^{\infty}\frac{1}{a^2+b^2t^2}\,\mathrm{d}t=\frac{\pi}{2ab}$. 又 $a^2K+b^2L=\displaystyle\int_0^{\frac{\pi}{2}}\frac{a^2\cos^2\theta+b^2\sin^2\theta}{a^2\cos^2\theta+b^2\sin^2\theta}\,\mathrm{d}\theta=\frac{\pi}{2}$, 解得 $K=\dfrac{\pi}{2a(a+b)}$, $L=\dfrac{\pi}{2b(a+b)}$; 代入得 $\dfrac{\partial F}{\partial a}=\dfrac{\partial F}{\partial b}=\dfrac{\pi}{a+b}$, $F(a,b)=\pi\ln(a+b)+C$. 令 $a=b$, $F(a,a)=\pi\ln a=\pi\ln(2a)+C\implies C=-\pi\ln 2$, 故 $F(a,b)=\pi\ln(a+b)-\pi\ln 2=\pi\ln\dfrac{a+b}{2}$

例. $\displaystyle\int_0^{\frac{\pi}{2}}\ln(\sin\theta)\,\mathrm{d}\theta=\int_0^{\frac{\pi}{2}}\ln(\cos\theta)\,\mathrm{d}\theta=-\frac{\pi}{2}\ln 2$, $\displaystyle\int_0^{\pi}\ln(2\pm 2\cos\theta)\,\mathrm{d}\theta=0$.

證. 變數變換 $u=\dfrac{\pi}{2}-\theta$, $\displaystyle\int_0^{\frac{\pi}{2}}\ln(\sin\theta)\,\mathrm{d}\theta=\int_{\frac{\pi}{2}}^{0}\ln\left(\sin\left(\frac{\pi}{2}-u\right)\right)(-1)\,\mathrm{d}u=\int_0^{\frac{\pi}{2}}\ln(\cos u)\,\mathrm{d}u=\int_0^{\frac{\pi}{2}}\ln(\cos\theta)\,\mathrm{d}\theta$; 接連變數變換 $x=2\theta$, $u=\dfrac{\pi}{2}-x$, $\displaystyle\int_0^{\frac{\pi}{2}}\ln(\sin 2\theta)\,\mathrm{d}\theta=\frac{1}{2}\int_0^{\pi}\ln(\sin x)\,\mathrm{d}x=\frac{1}{2}\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}}\ln(\cos u)(-1)\,\mathrm{d}u=\int_0^{\frac{\pi}{2}}\ln(\cos u)\,\mathrm{d}u=\int_0^{\frac{\pi}{2}}\ln(\cos\theta)\,\mathrm{d}\theta$; $\displaystyle\int_0^{\frac{\pi}{2}}\ln(\sin 2\theta)\,\mathrm{d}\theta=\int_0^{\frac{\pi}{2}}\ln(\sin\theta)\,\mathrm{d}\theta+\int_0^{\frac{\pi}{2}}\ln(\cos\theta)\,\mathrm{d}\theta+\frac{\pi}{2}\ln 2=\int_0^{\frac{\pi}{2}}\ln(\cos\theta)\,\mathrm{d}\theta$ $\displaystyle\implies\int_0^{\frac{\pi}{2}}\ln(\sin\theta)\,\mathrm{d}\theta=-\frac{\pi}{2}\ln 2$. $\displaystyle\int_0^{\pi}\ln(2+2\cos\theta)\,\mathrm{d}\theta=\int_0^{\pi}\ln\left(4\cos^2\frac{\theta}{2}\right)\mathrm{d}\theta=2\pi\ln 2+2\int_0^{\pi}\ln\left(\cos\frac{\theta}{2}\right)\mathrm{d}\theta=2\pi\ln 2+4\int_0^{\frac{\pi}{2}}\ln(\cos\theta)\,\mathrm{d}\theta=0$; $\displaystyle\int_0^{\pi}\ln(2-2\cos\theta)\,\mathrm{d}\theta=\int_0^{\pi}\ln\left(4\sin^2\frac{\theta}{2}\right)\mathrm{d}\theta=2\pi\ln 2+2\int_0^{\pi}\ln\left(\sin\frac{\theta}{2}\right)\mathrm{d}\theta=2\pi\ln 2+4\int_0^{\frac{\pi}{2}}\ln(\sin\theta)\,\mathrm{d}\theta=0$;

例 (Poisson 積分)**.** $\displaystyle\int_0^{\pi}\ln(1-2r\cos\theta+r^2)\,\mathrm{d}\theta=\begin{cases}0 & |r|\leqslant 1\\ 2\pi\ln|r| & |r|>1\end{cases}$

證. 設 $F(r)=\displaystyle\int_0^{\pi}\ln(1-2r\cos\theta+r^2)\,\mathrm{d}\theta$, $\forall\,|r|\neq 1$, $F(0)=\displaystyle\int_0^{\pi}\ln 1\,\mathrm{d}\theta=0$. 令 $u=\tan\dfrac{\theta}{2}$, 則 $\cos\theta=\dfrac{1-u^2}{1+u^2}$, $\mathrm{d}\theta=\dfrac{2}{1+u^2}\,\mathrm{d}u$, $u:0\to\infty$. 由 $1-2r\cos\theta+r^2=1-2r\cdot\dfrac{1-u^2}{1+u^2}+r^2=\dfrac{(1+u^2)-2r(1-u^2)+r^2(1+u^2)}{1+u^2}=\dfrac{(1-r)^2+(1+r)^2u^2}{1+u^2}$, $-2\cos\theta+2r=-2\cdot\dfrac{1-u^2}{1+u^2}+2r=\dfrac{2(r-1)+2(1+r)u^2}{1+u^2}$, $F'(r)=\displaystyle\int_0^{\pi}\frac{-2\cos\theta+2r}{1-2r\cos\theta+r^2}\,\mathrm{d}\theta=\int_0^{\infty}\frac{2(r-1)+2(1+r)u^2}{(1-r)^2+(1+r)^2u^2}\cdot\frac{2}{1+u^2}\,\mathrm{d}u=4\int_0^{\infty}\frac{(r-1)+(1+r)u^2}{((1-r)^2+(1+r)^2u^2)(1+u^2)}\,\mathrm{d}u$. 求部份分式: $\dfrac{(r-1)+(1+r)u^2}{((1-r)^2+(1+r)^2u^2)(1+u^2)}=\dfrac{A}{(1-r)^2+(1+r)^2u^2}+\dfrac{B}{1+u^2}$, 則 $A=\dfrac{r^2-1}{2r}$, $B=\dfrac{1}{2r}$. $\displaystyle\int_0^{\infty}\frac{\mathrm{d}u}{(1-r)^2+(1+r)^2u^2}=\frac{1}{(1+r)^2}\int_0^{\infty}\frac{\mathrm{d}u}{\frac{(1-r)^2}{(1+r)^2}+u^2}=\frac{1}{(1+r)^2}\frac{|1+r|}{|1-r|}\cdot\frac{\pi}{2}=\frac{1}{|1+r|\cdot|1-r|}\cdot\frac{\pi}{2}=\frac{\pi}{2}\cdot\frac{1}{|1-r^2|}$, 故 $F'(r)=4\displaystyle\int_0^{\infty}\frac{(r-1)+(1+r)u^2}{((1-r)^2+(1+r)^2u^2)(1+u^2)}\,\mathrm{d}u=4\cdot\frac{\pi}{2}\Big(\frac{A}{|1-r^2|}+B\Big)=\frac{\pi}{r}\Big(\frac{r^2-1}{|r^2-1|}+1\Big)$. 故 $F'(r)=\begin{cases}0 & |r|<1\\ \frac{2\pi}{r} & |r|>1\end{cases}$, $F(r)=\begin{cases}0 & |r|<1\\ 2\pi\ln|r| & |r|>1\end{cases}$; $|r|=1$ 情形由 $\displaystyle\int_0^{\pi}\ln(2\pm 2\cos\theta)\,\mathrm{d}\theta=0$ 含括.