雙界限選擇權之評價：使用有限元素法

摘要

界限選擇權在保險商品創新與監理上具有廣泛應用。本文以有限元素法求解界限選擇權的評價問題，從推導並設定界限選擇權分別在 Black-Scholes 模型以及 Heston 隨機變動模型下的微分方程式、邊界條件與變分形式開始，針對雙界限選擇權寫出在此二模型下之理論解，並與使用 `FEniCS` 撰寫的有限元素法評價程式數值結果作比較。本文中 Heston 隨機波動模型下界限選擇權評價之相對誤差可達到 10^{-4} 數量級。

We use the finite element method to evaluate the barrier options which is essential for insurance product development and regulation. Variational forms and partial differential equations with corresponding boundary conditions arising from the double barrier options under the Black-Scholes and the Heston stochastic volatility models are derived; numerical results obtained via the subroutine `FEniCS` are compared against the published exact solutions. The relative error of the barrier option valuation under the Heston stochastic volatility model committed is about 10^{-4}.

關鍵字: 界限選擇權, Heston 隨機變動模型, 有限元素法, `FEniCS`

Keywords: barrier options, Heston stochastic volatility model, finite element methods, `FEniCS`

1. 研究動機與緒論

選擇權評價及其管理爲現代財務金融的基石之一，在風險管理與保險領域同樣扮演重要角色。傳統壽險精算著重在人身相關的機率分佈計算 — 生存率、患病率等等，保險公司的投資面管理少有著墨。但隨著金融環境演進，低利率、低投資報酬率與高變動率成爲常態，資產負債管理成爲保險公司能否永續經營的核心議題；傳統商品已不見容於目前市場，與選擇權概念相關的種種投資型商品漸成主流。另外，因應保險市場的變化，保險監理的政策工具必須多樣性，穩定市場的作爲勢必投入更多資金；應用選擇權或其他現代財務工程概念的財務安定機制成爲焦點。以上種種，了解並掌握選擇權評價技術一直是保險風控從業人員重要的學習目標。

依據市場標的物機率模型的差異，選擇權的評價方式主要有三種：Monte-Carlo / 機率法、離散格子點 (lattice) 法與微分方程式法。離散格子點法包括了常見的樹法，嚴格說來是微分方程式法的離散型表現，故選擇權評價方式概分爲機率法與微分方程式法。本文闡述選擇權評價微分方程式法的其中一種數值求解方式 — 有限元素法 (finite element method, FEM) — 並應用於界限選擇權 (barrier option)，特別是雙界限選擇權 (double barrier option) 在古典 Black-Scholes 模型與 Heston 隨機變動模型下的評價問題。

界限選擇權是一種路徑相關的選擇權，其給付不僅與到期日的償付函數相關，也與標的物在到期日前的指數變動相關。界限選擇權主要有兩種子型：knock-out, knock-in，分別規範的是在到

期日前指數碰觸到期初設定的界限後之後的行爲：結束、生效。界限選擇權的設計最主要是藉由選擇權發行前判斷標的物指數變動模式預先指定可能邊界與履約條件，以此設計降低該選擇權成本價格，吸引更多潛在的購買者。「碰觸界限則結束」的條款概念不只在保險商品設計上提供可能的想法，更可應用在保險公司的資產負債管理模式上 — 比方，如 Grosen and Jørgensen (2002) 文章所描述的：若將滿期保戶權益視爲依公司淨值狀況給付的選擇權，監理機構在期間針對保險公司淨值破底時所採取的強制接管與中止營運的措施恰恰符合 down-and-out 界限選擇權的機制，因此更符合現實狀況。

有限元素法是用於求解微分方程式邊界值問題的數值方法，自上世紀六零年代肇始，普遍用於一般工程界；與財務界更爲熟悉的有限差分法（finite difference method, FDM；財務應用可參考 Ikonen and Toivanen (2008); Tavella and Randall (2000)）相比，有限元素法的理論基礎更穩固、誤差推估更容易與精確，在局域數值變動劇烈的求解區域的可配適性更佳；缺點則是需要更多較迂迴的程式設計，不若有限差分法的直接。然而，高品質開源有限元素法函式庫如 `FEniCS`（Alnæs et al. (2015); Logg et al. (2012)），`FreeFEM`（Hecht (2012)）等等的出現徹底改變這個狀況：使用者只須輸入正確的問題敍述即可得到計算結果，一般問題敍述往往只須幾十行程式。在此我們使用 `FEniCS` 實作界限選擇權評價程式。

有限元素法應用在財務問題上的之主要參考書籍爲 Hilber, Reichmann, Schwab and Winter (2013); Achdou and Pironneau (2005); Topper (2005)。率先使用有限元素法求解 Heston 隨機變動模型標準歐式選擇權的文章爲 Winkler, Apel and Wystup (2002)；Xiong (2020) 進一步延伸至界限選擇權的評價。Zvan, Vetzal and Forsyth (2000) 回顧關於界限選擇權的微分方程式數值求解方法，但僅限於 Black-Scholes 模型。Lipton (2001) 的書籍介紹雙界限選擇權在特殊情形下的解；雙界限選擇權的一般解迄今仍未見文獻中。

本文從推導並設定界限選擇權分別在 Black-Scholes 模型以及 Heston 隨機變動模型下的微分方程式、邊界條件，與變分形式開始，針對雙界限選擇權寫出在此二模型下之理論解，並與我們使用 `FEniCS` 撰寫的有限元素法評價程式數值結果作比較。Heston 隨機波動模型下界限選擇權評價之相對誤差以目前粗略的離散情形下可達到 10^{-4} 數量級。所有的微分方程式法 — 有限元素法、有限差分法、樹法等等 — 同樣必須面對近似區域截切與邊界條件之引進問題，結果是等價的；剩下的選擇只有機率法，而欲達相當之精確度必須配合大量的模擬與變異數縮減（variance reduction），計算精度與效率或略遜於微分方程式法。因此，本文提出之使用 `FEniCS` 之有限元素法近似計算至少可作爲計算基準之用。

2. 有限元素法簡介

本節我們簡單介紹有限元素法的概念。考慮（橢圓形）PDE 問題的變分形式：求 $u \in V$ 滿足

$$a(u, v) = F(v) \quad \forall v \in V \tag{1}$$

求解的 Galerkin 方法如下。考慮 V 的子空間族 V_h（以 h 區隔）及其對應的離散解 u_h 使得

$$a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h \tag{2}$$

結合 (1)、(2) 我們得到 Galerkin 正交性

$$a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h \tag{3}$$

我們稱近似解 u_h 收斂到眞實解 u，如果以下成立:

$$\lim_{h\to 0}\|u-u_h\|=0$$

有限元素法的意旨即在於適當的建構 V 的有限維子空間 V_h 及其對應的離散解 u_h，使得 u_h 收斂到 u。建構過程的第一步在於將問題變數所在的空間 U 適當分割成有限元素 $\{K_i\}$ — 圖 1 爲典型的長方形區域有限元素分割範例 — 緊接著建構 V_h，使得對於 $v_h\in V_h$，$v_h|_{K_i}$ 爲定義在 K_i 上的「簡單」函數 — 比如多項式。經過此建構過程，近似解 u_h 可表示爲

$$u_h=\sum_{i=1}^{m}u_iw_i$$

其中 m 爲 V_h 的維度，$w_i, 1\leqslant i\leqslant m$ 爲 V_h 的基底，$u_i, 1\leqslant i\leqslant m$ 爲待求解常數。將此 u_h 表示式代回變分形式:

$$\sum_{i=1}^{m}u_i\,a(w_i,w_j)=f(w_j),\quad 1\leqslant j\leqslant m.$$

解此一線性方程式組可得 $u_i, 1\leqslant i\leqslant m$，因此可得 u_h。

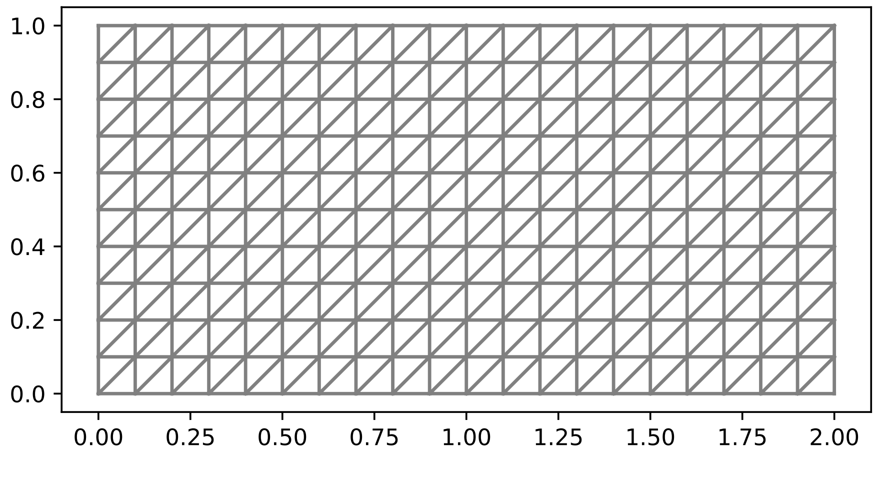

圖 1: 典型有限元素分割: 2×1 長方形區域分割爲 $2\times 20\times 10$ 個三角形元素。

應用有限元素法解微分方程式邊界值問題的最重要步驟即爲寫出微分方程式及其邊界條件的變分（弱問題）形式，再來選定求解區域適當的離散方式；使用 `FEniCS` 求解的流程非常貼近這樣的數學思維程序。在以下的章節，我們針對 Black-Scholes 模型以及 Heston 隨機變動模型分別發展其變分形式；這些資訊是構成 `FEniCS` 求解程式最核心的部份。

3. 邊界値問題: Black-Scholes 模型

假設市場由 n 個風險性資產 x_i, $i=1,2,\ldots,n$ 構成，其變動過程由以下的幾何布朗運動描述:

$$\mathrm{d}x_i=\mu_i x_i\,\mathrm{d}t+x_i\sigma_i\,\mathrm{d}W_i,\quad i=1,2,\ldots,n$$

其中 μ_i, σ_i 分別爲 x_i 的漂移係數與變異數，$\mathrm{d}W_j$ 爲 Wiener 過程，且 $\mathrm{d}\langle W_i, W_j\rangle_t = \rho_{ij}\,\mathrm{d}t$。考慮自洽（self-financing）的資產組合 Π:

$$\Pi = u(x_1, x_2, \dots, x_n, t) + \sum_{i=1}^{n} \Delta_i x_i.$$

則

$$\mathrm{d}\Pi = \mathrm{d}u + \sum_{i=1}^{n} \Delta_i\,\mathrm{d}x_i.$$

令矩陣 Γ 的 (i,j) 元素爲 $\Gamma_{ij} = \rho_{ij}\sigma_i\sigma_j$，使用 Itô 引理可得

$$\mathrm{d}u = \frac{\partial u}{\partial t}\,\mathrm{d}t + \sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\,\mathrm{d}x_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma_{ij}x_i x_j\frac{\partial^2 u}{\partial x_i \partial x_j}\,\mathrm{d}t$$

代入 $\mathrm{d}\Pi$，令 $\mathrm{d}x_i$ 中與 Wiener 過程有關的項係數爲零，我們得到

$$\Delta_i = -\frac{\partial u}{\partial x_i}.$$

此時只剩下 $\mathrm{d}t$ 項；利用 $\mathrm{d}\Pi = r\Pi\,\mathrm{d}t$，故

$$\begin{aligned}\mathrm{d}\Pi &= \left(\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma_{ij}x_i x_j\frac{\partial^2 u}{\partial x_i \partial x_j}\right)\mathrm{d}t \\ &= r\Pi\,\mathrm{d}t = r\left(u + \sum_{i=1}^{n}\Delta_i x_i\right) = r\left(u - \sum_{i=1}^{n}\frac{\partial u}{\partial x_i}x_i\right)\mathrm{d}t\end{aligned}$$

我們得到多維的 Black-Scholes 方程式

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma_{ij}x_i x_j\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} r x_i\frac{\partial u}{\partial x_i} - ru = 0. \tag{4}$$

透過變數變換可將 Black-Scholes 方程式變成熱方程式；推導過程如附錄 A。

令 $n = 1$，Black-Scholes 方程式爲

$$\frac{\partial u}{\partial t} - \frac{\sigma^2(x,t)\,x^2}{2}\frac{\partial^2 u}{\partial x^2} - r(t)\,x\frac{\partial u}{\partial x} + r(t)\,u = 0 \tag{5}$$

其中 $\sigma \equiv \sigma(x,t)$, $r \equiv r(t)$，此方程式的弱問題 / 變分形式推導如下。將 (5) 式乘上 $v(x)$ 並積分，使用部份積分公式 $\int U\mathrm{d}V = UV - \int V\mathrm{d}U$，

$$\begin{aligned}\int \frac{\sigma^2 x^2}{2}\frac{\partial^2 u}{\partial x^2}v\,\mathrm{d}x &= \int \underbrace{\frac{v\sigma^2 x^2}{2}}_{U}\underbrace{\frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x}_{\mathrm{d}V} = \underbrace{\frac{v\sigma^2 x^2}{2}}_{U}\underbrace{\frac{\partial u}{\partial x}}_{V} - \int \underbrace{\frac{\partial u}{\partial x}}_{V}\underbrace{\frac{\partial}{\partial x}\left(\frac{v\sigma^2 x^2}{2}\right)\mathrm{d}x}_{\mathrm{d}U} \\ &= \underbrace{\frac{v\sigma^2 x^2}{2}\frac{\partial u}{\partial x}}_{=0} - \int \frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial x}\frac{\sigma^2 x^2}{2} + v\sigma\frac{\partial \sigma}{\partial x}x^2 + v\sigma^2 x\right)\mathrm{d}x\end{aligned}$$

移項可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} uv\,\mathrm{d}x + \int_{\mathbb{R}_+} \frac{\sigma^2x^2}{2}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,\mathrm{d}x + \int_{\mathbb{R}_+}\left(\sigma^2 + x\sigma\frac{\partial\sigma}{\partial x} - r\right)x\frac{\partial u}{\partial x}v\,\mathrm{d}x + r\int_{\mathbb{R}_+} uv\,\mathrm{d}x = 0$$

令變分形式 $a_B(u,v)$ 爲

$$a_B(u,v) = \int_{\mathbb{R}_+} \frac{\sigma^2x^2}{2}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,\mathrm{d}x + \int_{\mathbb{R}_+}\left(\sigma^2 + x\sigma\frac{\partial\sigma}{\partial x} - r\right)x\frac{\partial u}{\partial x}v\,\mathrm{d}x + r\int_{\mathbb{R}_+} uv\,\mathrm{d}x$$

則一維 Black-Scholes 方程式的弱問題可寫作：求 $u \in V$ 使得

$$\left\langle \frac{\partial u}{\partial t}, v \right\rangle + a_B(u,v) = 0, \qquad \forall v \in V$$

其中 $\langle\cdot,\cdot\rangle$ 爲 V 的內積。

在一維 Black-Scholes 模型下的雙界限選擇權價格 $u(x,t)$ 除到期給付函數條件外另需滿足的邊界條件爲

$$u(a,t) = u(b,t) = 0 \qquad \forall t \tag{6}$$

其中 a, b 爲其上下界限值。

4. 邊界值問題：Heston 隨機變動模型

如附錄 B 之推導過程，Heston 隨機變動模型滿足之微分方程式爲

$$\frac{\partial u}{\partial t} - \nabla\cdot(A\,\nabla u) + b\cdot\nabla u + r\,u = 0 \tag{7}$$

其中

$$\nabla = \begin{pmatrix}\frac{\partial}{\partial s} & \frac{\partial}{\partial y}\end{pmatrix}^\top, \quad A = \frac{y}{2}\begin{pmatrix}1 & \rho\xi\\ \rho\xi & \xi^2\end{pmatrix}, \quad b = \begin{pmatrix}\frac{y+\rho\xi}{2} - r & \frac{\xi^2}{2} - \kappa(\vartheta - y)\end{pmatrix}^\top.$$

由散度定理我們有

$$\int_U \nabla\cdot(v\,A\nabla u)\ \mathrm{d}x = \int_{\partial U}(v\,A\nabla u)\cdot\mathsf{n}\,\mathrm{d}s$$

故

$$\begin{aligned}\int_U v\,\nabla\cdot(A\nabla u)\,\mathrm{d}x &= \int_U \nabla\cdot(v\,A\nabla u)\ \mathrm{d}x - \int_U \nabla v\cdot A\nabla u\,\mathrm{d}x\\ &= \int_{\partial U}(v\,A\nabla u)\cdot\mathsf{n}\,\mathrm{d}s - \int_U \nabla v\cdot A\nabla u\,\mathrm{d}x\end{aligned}$$

令變分形式 $a_H(u,v)$ 爲

$$a_H(u,v) = -\int_{\partial U}(v\,A\nabla u)\cdot\mathsf{n}\,\mathrm{d}s + \int_U \nabla v\cdot A\nabla u\,\mathrm{d}x + \int_U b\cdot\nabla u\,v\,\mathrm{d}x + r\int_U uv\,\mathrm{d}x \tag{8}$$

則 Heston 隨機變動模型的弱問題 / 變分形式可寫爲：求 $u \in V$ 使得

$$\left\langle \frac{\partial u}{\partial t}, v \right\rangle + a_H(u,v) = 0 \qquad \forall v \in V.$$

其中 $\langle \cdot, \cdot \rangle$ 爲 V 的內積。

在 Heston 隨機變動模型下的雙界限選擇權價格 $u(t,s,y)$ 除到期給付函數條件外另需滿足的邊界條件爲

$$u(t,a,y) = u(t,b,y) = 0 \qquad \forall t, y \tag{9}$$

其中 a, b 爲其上下界限值。

5．理論解

Black-Scholes 模型

文獻中已知最早的理論解公佈於 Kunitomo and Ikeda (1992)；以下等價的表示式摘自 Haug (2004, p.157–158)。給定期間 T （單位：年)、利率 r、變異數 σ、資產價格 S、持有成本/利差 (cost-of-carry) b (無配息時 $b = r$，配息時 $b = r - q$，q 爲配息率)、履約價格 X、上界限值 U、下界限值 L、描述上下界限的曲率值 δ_1, δ_2 在此均爲 0 （直線）的情況下，買權價值之公式爲

$$\begin{aligned} C = Se^{(b-r)T} \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{U^n}{L^n}\right)^{\mu_1} \left(\frac{L}{S}\right)^{\mu_2} (N(d_1) - N(d_2)) - \left(\frac{L^{n+1}}{U^n S}\right)^{\mu_3} (N(d_3) - N(d_4)) \right\} \\ - Xe^{-rT} \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{U^n}{L^n}\right)^{\mu_1 - 2} \left(\frac{L}{S}\right)^{\mu_2} \left(N\left(d_1 - \sigma\sqrt{T}\right) - N\left(d_2 - \sigma\sqrt{T}\right)\right) \right. \\ \left. - \left(\frac{L^{n+1}}{U^n S}\right)^{\mu_3 - 2} \left(N\left(d_3 - \sigma\sqrt{T}\right) - N\left(d_4 - \sigma\sqrt{T}\right)\right) \right\} \end{aligned} \tag{10}$$

而賣權價值之公式爲

$$\begin{aligned} P = Xe^{-rT} \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{U^n}{L^n}\right)^{\mu_1 - 2} \left(\frac{L}{S}\right)^{\mu_2} \left(N\left(y_1 - \sigma\sqrt{T}\right) - N\left(y_2 - \sigma\sqrt{T}\right)\right) \right. \\ \left. - \left(\frac{L^{n+1}}{U^n S}\right)^{\mu_3 - 2} \left(N\left(y_3 - \sigma\sqrt{T}\right) - N\left(y_4 - \sigma\sqrt{T}\right)\right) \right\} \\ - Se^{(b-r)T} \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{U^n}{L^n}\right)^{\mu_1} \left(\frac{L}{S}\right)^{\mu_2} (N(y_1) - N(y_2)) - \left(\frac{L^{n+1}}{U^n S}\right)^{\mu_3} (N(y_3) - N(y_4)) \right\} \end{aligned} \tag{11}$$

其中

$$
\begin{aligned}
d_1 &= \frac{\log\left(\frac{SU^{2n}}{XL^{2n}}\right)+\left(b+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} & d_2 &= \frac{\log\left(\frac{SU^{2n}}{FL^{2n}}\right)+\left(b+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\
d_3 &= \frac{\log\left(\frac{L^{2n+2}}{XSU^{2n}}\right)+\left(b+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} & d_4 &= \frac{\log\left(\frac{L^{2n+2}}{FSU^{2n}}\right)+\left(b+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\
y_1 &= \frac{\log\left(\frac{SU^{2n}}{EL^{2n}}\right)+\left(b+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} & y_2 &= \frac{\log\left(\frac{SU^{2n}}{XL^{2n}}\right)+\left(b+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\
y_3 &= \frac{\log\left(\frac{L^{2n+2}}{ESU^{2n}}\right)+\left(b+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} & y_4 &= \frac{\log\left(\frac{L^{2n+2}}{XSU^{2n}}\right)+\left(b+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\
\mu_1 &= \frac{2\left(b-\delta_2-n\left(\delta_1-\delta_2\right)\right)}{\sigma^2}+1 & \mu_3 &= \frac{2\left(b-\delta_2+n\left(\delta_1-\delta_2\right)\right)}{\sigma^2}+1 \\
\mu_2 &= \frac{2n\left(\delta_1-\delta_2\right)}{\sigma^2} & F &= Ue^{\delta_1 T}, \qquad E = Le^{\delta_2 T}
\end{aligned}
$$

Heston 隨機變動模型

Lipton (2001, Section 12.5.4) 推導使用 Green 函數與正交展開技術的解如下。給定期間 T (單位: 年)、資產價格 s、波動率 y、履約價格 K、上界限值 U、下界限值 L 與 Heston 模型之 r, q, κ, ϑ, ξ 且滿足 $\rho = 0$ 與 $r = q$ 之買權價值公式爲

$$
c(T,s,y) = e^{-rT}\sqrt{sK}\sum_{n=1}^{\infty} e^{2\frac{A(T,k_n)-B(T,k_n)y}{\xi^2}}\phi_n \sin\left(k_n \log\frac{s}{L}\right) \tag{12}
$$

其中

$$
\begin{aligned}
k_n = \frac{n\pi}{\log\frac{U}{L}}, \quad \mu(k) = -\frac{1}{2}\kappa, \quad \zeta(k) = \frac{1}{2}\sqrt{k^2\left(\xi^2+\frac{1}{4}\right)+\kappa^2} \\
A(\tau,k) = -\kappa\vartheta\left(\left(\mu(k)+\zeta(k)\right)\tau + \log\frac{-\mu(k)+\zeta(k)+\left(\mu(k)+\zeta(k)\right)e^{-2\zeta(k)\tau}}{2\zeta(k)}\right) \\
B(\tau,k) = \frac{\xi^2\left(k^2+\frac{1}{4}\right)\left(1-e^{-2\zeta(k)\tau}\right)}{4\left(-\mu(k)+\zeta(k)+\left(\mu(k)+\zeta(k)\right)e^{-2\zeta(k)\tau}\right)}
\end{aligned}
$$

且

$$
\phi_n = \frac{2\left(\left(-1\right)^{n+1}k_n\left(\sqrt{\frac{U}{K}}-\sqrt{\frac{K}{U}}\right)+\sin\left(k_n\log\frac{L}{K}\right)\right)}{\left(k_n^2+\frac{1}{4}\right)\log\frac{U}{L}}
$$

而同樣條件的賣權公式爲

$$
p(T,s,y) = e^{-rT}\sqrt{sK}\sum_{n=1}^{\infty} e^{2\frac{A(T,k_n)-B(T,k_n)y}{\xi^2}}\widehat{\phi_n} \sin\left(k_n \log\frac{s}{L}\right) \tag{13}
$$

其中

$$\widehat{\phi_n}=\frac{2\left(k_n\left(\sqrt{\frac{K}{L}}-\sqrt{\frac{L}{K}}\right)+\sin\left(k_n\log\frac{L}{K}\right)\right)}{\left(k_n^2+\frac{1}{4}\right)\log\frac{U}{L}}$$

以上的公式只在 $\rho=0$ 與 $r=q$ 情形下成立；就筆者所知，目前文獻中尙無完整 $\rho\neq 0$ 或 $r\neq q$ 的一般情形理論解。

6. 數值結果

我們使用 `FEniCS` 撰寫有限元素法程式求取數值解，同時與雙界限選擇權理論價格公式 (10), (11), (12), (13) 做比較；完整 `FEniCS` 程式如附錄 C，所得數值結果如表 1, 2, 3, 4。`FEniCS` 程式是由求解區域與函數空間、分割有限元素、弱問題 / 變分形式及邊界條件之指定所組成。由於選擇權評價問題爲起始值問題，邊界爲無限大，欲使用有限元素法必須先裁取適當的求解區域。一維 Black-Scholes 模型的雙界限選擇權問題選擇上下界即可，Heston 隨機變動模型的雙界限選擇權問題在 s 方向可選擇限制在上下界，而 y 方向必須做適當選取，同時必須指定在 y 上下界處之邊界條件。在我們的計算中 $y\in[y_{\min},y_{\max}]$，$y_{\min}=0$, $y_{\max}=3$, 而邊界條件爲

$$\left.\frac{\partial u(t,s,y)}{\partial \mathsf{n}}\right|_{y=y_{\max},\,y_{\min}}=0$$

表 1, 2 爲 Black-Scholes 模型下在不同期限 T 年、變異數 σ 與價格上下界之買賣權價格結果。FEM 方法使用 1000 個元素，$\Delta t=10^{-4}$, $r=0.1$, $s=100$, $K=100$，與理論解間的相對誤差小於 10^{-5}。表 3, 4 爲 Heston 隨機變動模型下在不同期限 T 年與價格上下界之買賣權價格結果。FEM 方法使用 100×100 個元素，$\Delta t=10^{-3}$, $r=q=0.03$, $\kappa=1.5$, $\vartheta=0.1$, $\xi=0.5$, $s=100$, $y=0.12$, $K=100$，與理論解間的相對誤差小於 10^{-2}。與 Black-Scholes 結果相比顯著較差的原因之一是元素分割數與時間步距 Δt 的選取，更重要的原因在於無可避免的近似區域截切與邊界條件之引進。

7. 結論與展望

本文介紹有限元素法在雙界限選擇權評價問題上的應用並給出可用的 `FEniCS` 程式。本文計算程式並沒有特別考慮執行效率問題，純以直覺明瞭與正確爲目的。誠如文中所述，Heston 隨機變動模型下雙界限選擇權的一般問題迄今仍無公開的解析解，有限元素法可提供相對正確的數值解。高度精確的有限元素法數值解需要更精確的近似邊界截切與邊界條件指定，這方面仍是後續研究的重心。針對風險管理與保險學門的研究者而言，掌握有限元素法並配上合適的軟體可大大加速正確評價結果的產出。後續文章我們將探討有限元素法應用在其他的奇異（exotic）選擇權與美式選擇權的評價與敏感性 / 避險問題。

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T	lower	upper	σ	FEM	analytic	error
0.25	50	150	0.15	4.351471	4.351472	0.000000
0.25	60	140	0.15	4.350455	4.350456	0.000000
0.25	70	130	0.15	4.313875	4.313879	0.000001
0.25	80	120	0.15	3.751593	3.751600	0.000002
0.25	90	110	0.15	1.205466	1.205465	0.000001
0.25	50	150	0.25	6.164448	6.164454	0.000001
0.25	60	140	0.25	5.850012	5.850021	0.000002
0.25	70	130	0.25	4.829310	4.829317	0.000002
0.25	80	120	0.25	2.638713	2.638713	0.000000
0.25	90	110	0.25	0.309826	0.309824	0.000008
0.25	50	150	0.35	7.037269	7.037281	0.000002
0.25	60	140	0.35	5.772596	5.772603	0.000001
0.25	70	130	0.35	3.776464	3.776464	0.000000
0.25	80	120	0.35	1.490282	1.490279	0.000002
0.25	90	110	0.35	0.047743	0.047742	0.000029
0.50	50	150	0.15	6.985280	6.985283	0.000000
0.50	60	140	0.15	6.808261	6.808265	0.000001
0.50	70	130	0.15	5.969749	5.969756	0.000001
0.50	80	120	0.15	3.580449	3.580450	0.000000
0.50	90	110	0.15	0.553662	0.553661	0.000002
0.50	50	150	0.25	7.933573	7.933580	0.000001
0.50	60	140	0.25	6.338327	6.338331	0.000001
0.50	70	130	0.25	4.000403	4.000403	0.000000
0.50	80	120	0.25	1.509811	1.509809	0.000001
0.50	90	110	0.25	0.044082	0.044081	0.000015
0.50	50	150	0.35	6.508810	6.508812	0.000000
0.50	60	140	0.35	4.384068	4.384066	0.000000
0.50	70	130	0.35	2.256340	2.256337	0.000001
0.50	80	120	0.35	0.563534	0.563532	0.000004
0.50	90	110	0.35	0.001090	0.001090	0.000057

表 1: Black-Scholes 模型下雙界限買權價格: $s = 100$, $r = 0.1$, $K = 100$, $\Delta t = 10^{-4}$, 1000 個元素。

T	lower	upper	σ	FEM	analytic	error
0.25	50	150	0.15	1.882477	1.882479	0.000001
0.25	60	140	0.15	1.882477	1.882479	0.000001
0.25	70	130	0.15	1.882464	1.882465	0.000001
0.25	80	120	0.15	1.860006	1.860008	0.000001
0.25	90	110	0.15	0.947268	0.947268	0.000000
0.25	50	150	0.25	3.785484	3.785486	0.000001
0.25	60	140	0.25	3.784522	3.784525	0.000001
0.25	70	130	0.25	3.701441	3.701445	0.000001
0.25	80	120	0.25	2.686629	2.686632	0.000001
0.25	90	110	0.25	0.344898	0.344895	0.000008
0.25	50	150	0.35	5.719055	5.719058	0.000001
0.25	60	140	0.35	5.606031	5.606038	0.000001
0.25	70	130	0.35	4.647194	4.647200	0.000001
0.25	80	120	0.35	2.071860	2.071857	0.000001
0.25	90	110	0.35	0.057764	0.057762	0.000029
0.50	50	150	0.15	2.137408	2.137408	0.000000
0.50	60	140	0.15	2.137401	2.137402	0.000000
0.50	70	130	0.15	2.132486	2.132486	0.000000
0.50	80	120	0.15	1.888268	1.888269	0.000001
0.50	90	110	0.15	0.455530	0.455529	0.000002
0.50	50	150	0.25	4.703256	4.703257	0.000000
0.50	60	140	0.25	4.623585	4.623587	0.000001
0.50	70	130	0.25	3.894419	3.894421	0.000001
0.50	80	120	0.25	1.785111	1.785110	0.000001
0.50	90	110	0.25	0.049140	0.049139	0.000015
0.50	50	150	0.35	7.168276	7.168280	0.000001
0.50	60	140	0.35	6.106217	6.106221	0.000001
0.50	70	130	0.35	3.586824	3.586824	0.000000
0.50	80	120	0.35	0.824370	0.824367	0.000003
0.50	90	110	0.35	0.001319	0.001319	0.000057

表 2: Black-Scholes 模型下雙界限賣權價格: $s = 100$, $r = 0.1$, $K = 100$, $\Delta t = 10^{-4}$, 1000 個元素。

T	lower	upper	FEM	analytic	error
0.25	50	150	5.743766	5.750610	0.0012
0.25	60	140	4.891759	4.896052	0.0009
0.25	70	130	3.462751	3.462611	0.0000
0.25	80	120	1.567859	1.564969	0.0018
0.25	90	110	0.109954	0.108847	0.0102
0.50	50	150	5.520852	5.521398	0.0001
0.50	60	140	4.092226	4.091332	0.0002
0.50	70	130	2.414123	2.412510	0.0007
0.50	80	120	0.809275	0.807802	0.0018
0.50	90	110	0.025838	0.025693	0.0057
1.00	50	150	4.247493	4.246693	0.0002
1.00	60	140	2.770113	2.769151	0.0003
1.00	70	130	1.351024	1.350143	0.0007
1.00	80	120	0.309542	0.309018	0.0017
1.00	90	110	0.003382	0.003351	0.0092

表 3: Heston 隨機變動模型下雙界限買權價格: $s = 100$, $y = 0.12$, $K = 100$, $r = q = 0.03$, $\kappa = 1.5$, $\vartheta = 0.1$, $\xi = 0.5$, $\Delta t = 10^{-3}$, 100×100 元素。

T	lower	upper	FEM	analytic	error
0.25	50	150	6.609884	6.614373	0.0007
0.25	60	140	6.360951	6.368411	0.0012
0.25	70	130	5.173619	5.179524	0.0011
0.25	80	120	2.448555	2.446362	0.0009
0.25	90	110	0.144522	0.143062	0.0102
0.50	50	150	8.546129	8.550847	0.0006
0.50	60	140	7.115637	7.119144	0.0005
0.50	70	130	4.390767	4.390461	0.0001
0.50	80	120	1.342222	1.340050	0.0016
0.50	90	110	0.034015	0.033822	0.0057
1.00	50	150	9.248777	9.250651	0.0002
1.00	60	140	6.176992	6.177137	0.0000
1.00	70	130	2.793040	2.791924	0.0004
1.00	80	120	0.530510	0.529676	0.0016
1.00	90	110	0.004456	0.004416	0.0093

表 4: Heston 隨機變動模型下雙界限賣權價格: $s = 100$, $y = 0.12$, $K = 100$, $r = q = 0.03$, $\kappa = 1.5$, $\vartheta = 0.1$, $\xi = 0.5$, $\Delta t = 10^{-3}$, 100×100 元素。

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附錄 A. Black-Scholes 方程式與熱方程之等價性

令 $s_i = \log x_i, i = 1, 2, \ldots, n$, 亦即 $\frac{\partial}{\partial x_i} = \frac{\partial s_i}{\partial x_i}\frac{\partial}{\partial s_i} = \frac{1}{x_i}\frac{\partial}{\partial s_i}$, 可得

$$\frac{\partial u}{\partial x_i} = \frac{1}{x_i}\frac{\partial u}{\partial s_i}, \quad \frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_j}\right) = \frac{1}{x_i}\frac{\partial}{\partial s_i}\left(\frac{1}{x_j}\frac{\partial u}{\partial s_j}\right) = \frac{1}{x_i}\frac{1}{x_j}\frac{\partial^2 u}{\partial s_i \partial s_j}$$

且

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{1}{x_i}\frac{\partial u}{\partial s_i}\right) = \frac{-1}{x_i^2}\frac{\partial u}{\partial s_i} + \frac{1}{x_i}\underbrace{\frac{\partial}{\partial x_i}}_{=\frac{1}{x_i}\frac{\partial}{\partial s_i}}\left(\frac{\partial u}{\partial s_i}\right) = \frac{-1}{x_i^2}\frac{\partial u}{\partial s_i} + \frac{1}{x_i^2}\frac{\partial^2 u}{\partial s_i^2}$$

亦即

$$x_i\frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial s_i}, \quad x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial s_i \partial s_j}, \quad x_i^2\frac{\partial^2 u}{\partial x_i^2} = -\frac{\partial u}{\partial s_i} + \frac{\partial^2 u}{\partial s_i^2}.$$

(4) 變成

$$\frac{\partial u}{\partial t}+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma_{ij}\frac{\partial^2 u}{\partial s_i\partial s_j}+\sum_{i=1}^{n}\left(r-\frac{1}{2}\Gamma_{ii}\right)\frac{\partial u}{\partial s_i}-ru=0.$$

可寫作

$$\frac{\partial u}{\partial t}+\frac{1}{2}\,\nabla_s\cdot(\Gamma\,\nabla_s u)+b\cdot\nabla_s u-r\,u=0 \tag{A.1}$$

其中

$$\nabla_s=\begin{pmatrix}\frac{\partial}{\partial s_1} & \frac{\partial}{\partial s_2} & \dots & \frac{\partial}{\partial s_n}\end{pmatrix}^\top,\quad b=\begin{pmatrix}r-\frac{1}{2}\Gamma_{11} & r-\frac{1}{2}\Gamma_{22} & \dots & r-\frac{1}{2}\Gamma_{nn}\end{pmatrix}^\top.$$

Γ 爲 $n\times n$ 對稱矩陣，標準的可對角線化結果(如 Abadir and Magnus (2005, Exercise 7.46))顯示存在 $n\times n$ 正交矩陣 M 使 $M^\top M=MM^\top=I$ 且 $M^\top\Gamma M=\Lambda,\Lambda=\mathrm{diag}(\lambda_1,\lambda_2,\dots,\lambda_n)$；其中 $\lambda_i,i=1,2,\dots,n$ 爲 Γ 的（正）特徵值. 再令 $z=Ms$；$z=(z_1,z_2,\dots,z_n)^\top$，$s=(s_1,s_2,\dots,s_n)^\top$。定義 $\nabla_z=\begin{pmatrix}\frac{\partial}{\partial z_1} & \frac{\partial}{\partial z_2} & \dots & \frac{\partial}{\partial z_n}\end{pmatrix}^\top$，由基本的多變數微分鏈鎖律 $\nabla_z=M^\top\nabla_s$，(A.1) 變成

$$\frac{\partial u}{\partial t}+\frac{1}{2}\,\nabla_z\cdot(\Lambda\,\nabla_z u)+M^\top b\cdot\nabla_z u-r\,u=0 \tag{A.2}$$

令 $\varphi(z,t)=e^{\omega\cdot z+\beta t}$ and $u(z,t)=\varphi(z,t)\,v(z,t)$；$\beta$ 與行向量 ω 待定。留意到 $\frac{\partial\varphi}{\partial t}=\beta\,\varphi$，

$$\frac{\partial u}{\partial t}=\varphi\,\frac{\partial v}{\partial t}+\frac{\partial\varphi}{\partial t}v=\varphi\,\frac{\partial v}{\partial t}+\beta\,\varphi\,v$$

使用恆等式

$$\begin{aligned}\nabla_z(\psi_1\,\psi_2)&=\psi_1\nabla_z\psi_2+\psi_2\nabla_z\psi_1 && \text{對向量函數 } \psi_1(z),\,\psi_2(z)\\ \nabla_z\cdot(\psi\,a)&=\nabla_z\psi\cdot a+\psi\,\nabla_z\cdot a && \text{對純量函數 } \psi(z) \text{ 及向量函數 } a(z)\end{aligned}$$

其中 $\nabla_z\varphi=\varphi\,\omega$，

$$\nabla_z u=\varphi\,\nabla_z v+v\,\nabla_z\varphi=\varphi\,\nabla_z v+v\,\varphi\,\omega$$

且

$$\begin{aligned}\nabla_z\cdot(\Lambda\,\nabla_z u)&=\nabla_z\cdot(\varphi\,\Lambda\,\nabla_z v+v\,\varphi\,\Lambda\,\omega)\\&=\nabla_z\cdot(\varphi\,\Lambda\,\nabla_z v)+\nabla_z\cdot(v\,\varphi\,\Lambda\,\omega)\\&=\nabla_z\varphi\cdot\Lambda\,\nabla_z v+\varphi\,\nabla_z\cdot(\Lambda\,\nabla_z v)+\nabla_z(v\,\varphi)\cdot\Lambda\,\omega\\&=\varphi\,\omega\cdot\Lambda\,\nabla_z v+\varphi\,\nabla_z\cdot(\Lambda\,\nabla_z v)+v\,\nabla_z\varphi\cdot\Lambda\,\omega+\varphi\,\nabla_z v\cdot\Lambda\,\omega\\&=\varphi\,\omega\cdot\Lambda\,\nabla_z v+\varphi\,\nabla_z\cdot(\Lambda\,\nabla_z v)+v\,\varphi\,\omega\cdot\Lambda\,\omega+\varphi\,\nabla_z v\cdot\Lambda\,\omega\end{aligned}$$

代入 (A.2)，我們可得

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \frac{1}{2}\,\nabla_z \cdot (\Lambda\,\nabla_z u) + M^\top b \cdot \nabla_z u - r\,u \\
&\quad = \varphi\bigg(\frac{\partial v}{\partial t} + \beta\,v + \frac{1}{2}\big(\omega \cdot \Lambda\,\nabla_z v + \nabla_z \cdot (\Lambda\,\nabla_z v) + \nabla_z v \cdot \Lambda\,\omega + v\,\omega \cdot \Lambda\,\omega\big) \\
&\qquad\qquad + M^\top b \cdot (\nabla_z v + v\,\omega) - r\,v\bigg) \\
&\quad = \varphi\bigg(\frac{\partial v}{\partial t} + \frac{1}{2}\,\nabla_z \cdot (\Lambda\,\nabla_z v) + \underbrace{\frac{1}{2}\,\omega \cdot \Lambda\,\nabla_z v}_{=\frac{1}{2}\,\nabla_z v \cdot \Lambda\,\omega} + \frac{1}{2}\,\nabla_z v \cdot \Lambda\,\omega + \frac{1}{2}\,v\,\omega \cdot \Lambda\,\omega \\
&\qquad\qquad + M^\top b \cdot (\nabla_z v + v\,\omega) - r\,v + \beta\,v\bigg) \\
&\quad = \varphi\bigg(\frac{\partial v}{\partial t} + \frac{1}{2}\,\nabla_z \cdot (\Lambda\,\nabla_z v) + \nabla_z v \cdot \big(\Lambda\,\omega + M^\top b\big) \\
&\qquad\qquad + v\left(\frac{1}{2}\,\omega \cdot \Lambda\,\omega + M^\top b \cdot \omega - r + \beta\right)\bigg) = 0
\end{aligned}
$$

爲建立 $\frac{1}{2}\,\omega \cdot \Lambda\,\nabla_z v = \frac{1}{2}\,\nabla_z v \cdot \Lambda\,\omega$,

$$
\begin{aligned}
\frac{1}{2}\,\omega \cdot \Lambda\,\nabla_z v &= \frac{1}{2}\,\Lambda\,\nabla_z v \cdot \omega \quad \text{（內積交換性：} a \cdot b = b \cdot a\text{）} \\
&= \frac{1}{2}\,(\Lambda\,\nabla_z v)^\top\,\omega \quad \text{（內積對應矩陣運算：} a \cdot b = a^\top b\text{）} \\
&= \frac{1}{2}\,(\nabla_z v)^\top\,\Lambda^\top \omega \quad \text{（矩陣轉置規則：} (ab)^\top = b^\top a^\top\text{）} \\
&= \frac{1}{2}\,(\nabla_z v)^\top\,\Lambda\,\omega \quad \text{（} \Lambda \text{ 爲對角線矩陣，故 } \Lambda^\top = \Lambda\text{）} \\
&= \frac{1}{2}\,\nabla_z v \cdot \Lambda\,\omega \quad \text{（矩陣運算對應內積：} a^\top b = a \cdot b\text{）}
\end{aligned}
$$

爲消去 $\nabla_z v$ 與 v 相關項，令

$$
\begin{aligned}
\Lambda\,\omega + M^\top b &= 0 \\
\frac{1}{2}\,\omega \cdot \Lambda\,\omega + M^\top b \cdot \omega - r + \beta &= 0
\end{aligned}
$$

則

$$
\omega = -\Lambda^{-1} M^\top b
$$

與

$$
\begin{aligned}
\beta &= r - M^\top b \cdot \omega - \frac{1}{2}\,\omega \cdot \Lambda\,\omega \\
&= r - M^\top b \cdot \left(-\Lambda^{-1} M^\top b\right) - \frac{1}{2}\left(-\Lambda^{-1} M^\top b\right) \cdot \Lambda \left(-\Lambda^{-1} M^\top b\right) \\
&= r + \left(M^\top b\right)^\top \Lambda^{-1} M^\top b - \frac{1}{2}\left(\Lambda^{-1} M^\top b\right)^\top \Lambda \Lambda^{-1} M^\top b \\
&= r + b^\top M \Lambda^{-1} M^\top b - \frac{1}{2} b^\top M \Lambda^{-1} \Lambda \Lambda^{-1} M^\top b \\
&= r + \frac{1}{2} b^\top M \Lambda^{-1} M^\top b = r + \frac{1}{2} b^\top \Gamma^{-1} b
\end{aligned}
$$

方程式變爲

$$
\frac{\partial v}{\partial t} + \frac{1}{2}\,\nabla_z \cdot \left(\Lambda\,\nabla_z v\right) = 0
$$

最後令 $y = \sqrt{2}\,\Lambda^{-\frac{1}{2}} z$, $y = \begin{pmatrix} y_1 & y_2 & \dots & y_n \end{pmatrix}^\top$ and $\nabla_y = \begin{pmatrix} \frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} & \dots & \frac{\partial}{\partial y_n} \end{pmatrix}^\top$, 我們得到 $\nabla_y = \frac{1}{\sqrt{2}} \Lambda^{\frac{1}{2}} \nabla_z$ 且

$$
\frac{\partial v}{\partial t} - \Delta_y v = 0
$$

其中 $\Delta_y = \sum_{i=1}^n \frac{\partial^2}{\partial y_i^2}$。多維熱方程初始值問題

$$
\begin{cases}
\frac{\partial v}{\partial t} - \Delta_y v = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
v = g & \text{on } \mathbb{R}^n \times \{t = 0\}
\end{cases}
$$

的解可由 Fourier 變換得到（可參照 Evans (2010, p.192)），最後結果爲

$$
v(y,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|y-\zeta|^2}{4t}} g(\zeta)\,\mathrm{d}\zeta, \quad \forall y \in \mathbb{R}^n,\ t > 0.
$$

由此可得出 Black-Scholes 起始值問題之解。出現在 Guillaume (2019), Jiang (2005, (7.3.22), p.209) 等等的多維 Black-Scholes 公式基本都由以上步驟產生。

附錄 B. Heston 隨機變動模型方程式

Heston 隨機變動模型首先發表於 Heston (1993)；Rouah (2013) 爲介紹此模型的專書。以下推導主要參考 Fouque, Papanicolaou, Sircar and Sølna (2011)。

假設單因子隨機變動模型如下:

$$
\begin{aligned}
\mathrm{d}x &= x\mu(y)\,\mathrm{d}t + x\sigma(y)\,\mathrm{d}W_x \\
\mathrm{d}y &= a(y)\,\mathrm{d}t + b(y)\,\mathrm{d}W_y
\end{aligned}
$$

其中 $\mathrm{d}\langle W_x, W_y\rangle_t = \rho\,\mathrm{d}t$, $\rho \in [-1,1]$。我們可分解 W_y

$$W_y = \rho W_x + \sqrt{1-\rho^2}W_0$$

使 W_0 獨立於 W_x，故

$$\mathrm{d}y = a(y)\,\mathrm{d}t + b(y)\rho\,\mathrm{d}W_x + b(y)\sqrt{1-\rho^2}\,\mathrm{d}W_0$$

與 Black-Schole 模型不同的是，我們現在需要額外的避險項。令 $u_1(x,y,t)$、$u_2(x,y,t)$ 分別爲到期日 T_1, T_2 的歐式選擇權價格；自洽的資產組合 Π 爲

$$\Pi = \alpha x + \beta u_1 + \gamma u_2.$$

則

$$\mathrm{d}\Pi = \alpha\,\mathrm{d}x + \beta\,\mathrm{d}u_1 + \gamma\,\mathrm{d}u_2 \tag{B.1}$$

給定函數 $u(x,y,t)$，使用 Itô 引理的結果爲

$$\begin{aligned}
\mathrm{d}u &= \frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y + \frac{\partial u}{\partial t}\,\mathrm{d}t + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x\,\mathrm{d}x + \frac{1}{2}\frac{\partial^2 u}{\partial y^2}\,\mathrm{d}y\,\mathrm{d}y + \frac{\partial^2 u}{\partial x\partial y}\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y + \left(\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma(y)^2x^2 + \frac{1}{2}\frac{\partial^2 u}{\partial y^2}b(y)^2 + \frac{\partial^2 u}{\partial x\partial y}\sigma(y)b(y)\rho x\right)\mathrm{d}t \\
&= \frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y + \mathcal{L}u\,\mathrm{d}t
\end{aligned}$$

運算子 $\mathcal{L}$ 定義爲

$$\mathcal{L} \equiv \frac{\partial}{\partial t} + \frac{1}{2}\sigma(y)^2x^2\frac{\partial^2}{\partial x^2} + \frac{1}{2}b(y)^2\frac{\partial^2}{\partial y^2} + \sigma(y)b(y)\rho x\frac{\partial^2}{\partial x\partial y} \tag{B.2}$$

代入 (B.1)

$$\mathrm{d}\Pi = \alpha\,\mathrm{d}x + \beta\left(\mathcal{L}u_1\,\mathrm{d}t + \frac{\partial u_1}{\partial x}\,\mathrm{d}x + \frac{\partial u_1}{\partial y}\,\mathrm{d}y\right) + \gamma\left(\mathcal{L}u_2\,\mathrm{d}t + \frac{\partial u_2}{\partial x}\,\mathrm{d}x + \frac{\partial u_2}{\partial y}\,\mathrm{d}y\right)$$

令 $\mathrm{d}W_0$ 項係數爲零可得

$$\beta\frac{\partial u_1}{\partial y} + \gamma\frac{\partial u_2}{\partial y} = 0,$$

故

$$\gamma = -\beta\frac{\frac{\partial u_1}{\partial y}}{\frac{\partial u_2}{\partial y}}. \tag{B.3}$$

令 $\mathrm{d}W_x$ 項係數爲零可得

$$\alpha + \beta\frac{\partial u_1}{\partial x} + \gamma\frac{\partial u_2}{\partial x} = 0$$

故

$$\alpha = -\beta \frac{\partial u_1}{\partial x} - \gamma \frac{\partial u_2}{\partial x}. \tag{B.4}$$

由 $\mathrm{d}\Pi = r\Pi\,\mathrm{d}t$,

$$\mathrm{d}\Pi = (\beta \mathcal{L} u_1 + \gamma \mathcal{L} u_2)\,\mathrm{d}t = r\Pi\,\mathrm{d}t = r\left(\alpha x + \beta u_1 + \gamma u_2\right)\mathrm{d}t$$

故

$$\beta \mathcal{L} u_1 + \gamma \mathcal{L} u_2 = r\left(\alpha x + \beta u_1 + \gamma u_2\right).$$

由 (B.3)、(B.4) 可進一步化簡爲

$$\frac{1}{\frac{\partial u_1}{\partial y}} \widehat{\mathcal{L}} u_1 = \frac{1}{\frac{\partial u_2}{\partial y}} \widehat{\mathcal{L}} u_2, \quad \text{其中} \quad \widehat{\mathcal{L}} \equiv \mathcal{L} + rx\frac{\partial}{\partial x} - r. \tag{B.5}$$

(B.5) 式左方只與 T_1 有關，而 (B.5) 式右方只與 T_2 有關，故 (B.5) 要成立，左右兩方必須等於一個與 T_1, T_2 無關的函數才行。令這個函數爲 $-a(y) + \Lambda(x, y, t)$，其中 $\Lambda(x, y, t)$ 代表變異風險價值。然後 $u_i, i = 1, 2$ 滿足

$$\widehat{\mathcal{L}} u + \left(a(y) - \Lambda(x, y, t)\right)\frac{\partial u}{\partial y} = 0.$$

由 (B.2) 與 (B.5),

$$\begin{aligned}\frac{\partial u}{\partial t} + \frac{1}{2}\sigma(y)^2 x^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} b(y)^2 \frac{\partial^2 u}{\partial y^2} + \sigma(y) b(y) \rho x \frac{\partial^2 u}{\partial x \partial y} \\ + rx\frac{\partial u}{\partial x} + \left(a(y) - \Lambda(x, y, t)\right)\frac{\partial u}{\partial y} - ru = 0\end{aligned} \tag{B.6}$$

由變數變換 $s = \log x$，亦即 $\frac{\partial}{\partial x} = \frac{\partial s}{\partial x}\frac{\partial}{\partial s} = \frac{1}{x}\frac{\partial}{\partial s}$，我們有 $\frac{\partial u}{\partial x} = \frac{1}{x}\frac{\partial u}{\partial s}$, $\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{x}\frac{\partial^2 u}{\partial s \partial y}$ 且

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial u}{\partial s}\right) = \frac{-1}{x^2}\frac{\partial u}{\partial s} + \frac{1}{x}\underbrace{\frac{\partial}{\partial x}}_{=\frac{1}{x}\frac{\partial}{\partial s}}\left(\frac{\partial u}{\partial s}\right) = \frac{-1}{x^2}\frac{\partial u}{\partial s} + \frac{1}{x^2}\frac{\partial^2 u}{\partial s^2}$$

故

$$x\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}, \quad x\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial s \partial y}, \quad x^2\frac{\partial^2 u}{\partial x^2} = -\frac{\partial u}{\partial s} + \frac{\partial^2 u}{\partial s^2}.$$

而 (B.6) 變成

$$\begin{aligned}\frac{\partial u}{\partial t} + \frac{1}{2}\sigma(y)^2 \frac{\partial^2 u}{\partial s^2} + \frac{1}{2} b(y)^2 \frac{\partial^2 u}{\partial y^2} + \sigma(y) b(y) \rho \frac{\partial^2 u}{\partial s \partial y} \\ + \left(r - \frac{1}{2}\sigma(y)^2\right)\frac{\partial u}{\partial s} + \left(a(y) - \Lambda(s, y, t)\right)\frac{\partial u}{\partial y} - ru = 0\end{aligned} \tag{B.7}$$

在 Heston 隨機變動模型中 $\mu(y)=r$，$\sigma(y)=\sqrt{y}$，$a(y)=\kappa(\vartheta-y)$，$b(y)=\xi\sqrt{y}$，且 $\Lambda(s,y,t)\equiv 0$；(B.6)，(B.7) 爲

$$\frac{\partial u}{\partial t}+\frac{1}{2}yx^2\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}\xi^2 y\frac{\partial^2 u}{\partial y^2}+\xi y\rho x\frac{\partial^2 u}{\partial x\partial y}+rx\frac{\partial u}{\partial x}+\kappa(\vartheta-y)\frac{\partial u}{\partial y}-ru=0 \tag{B.8}$$

與

$$\frac{\partial u}{\partial t}+\frac{1}{2}y\frac{\partial^2 u}{\partial s^2}+\frac{1}{2}\xi^2 y\frac{\partial^2 u}{\partial y^2}+\rho\xi y\frac{\partial^2 u}{\partial s\partial y}+\left(r-\frac{1}{2}y\right)\frac{\partial u}{\partial s}+\kappa(\vartheta-y)\frac{\partial u}{\partial y}-ru=0 \tag{B.9}$$

式 (7) 即爲 (B.9)。

附錄 C. FEniCS 程式碼

Black-Scholes 模型

```python
from dolfin import *
from numpy import sqrt, exp, log
import scipy.stats as st

N = st.norm.cdf

def bs(s0, K, sigma, r, T, q=0, cp=1):
    # cp=1: call, cp=-1: put
    d1 = (log(s0 / K) + (r - q + 0.5 * sigma ** 2) * T) / (sigma *
    ↪ sqrt(T))
    d2 = d1 - sigma * sqrt(T)
    return cp * s0 * exp(-q * T) * N(cp * d1) - cp * K * exp(-r * T) *
    ↪ N(cp * d2)

# c.f. Haug E. G. The Complete Guide to Option Pricing Formulas 2ed
↪ pp.157--pp.158
def dbo_bs_anly(sigma, r, T, L, U, S, X, cp=1, q=0, delta1=0,
↪ delta2=0, n_terms=10):
    b = r if q == 0 else r - q

    def mu1(n):
        return 2 * (b - delta2 - n * (delta1 - delta2)) / (sigma**2) +
1
    def mu2(n):
        return 2 * n * (delta1 - delta2) / (sigma**2)
    def mu3(n):
        return 2 * (b - delta2 + n * (delta1 - delta2)) / (sigma**2) +
1

    F = U * exp(delta1 * T)
```

```python
    E = L * exp(delta2 * T)

    def d1(n):
        return (log((S * U**(2*n)) / (X * L**(2*n))) + (b + sigma**2 / 2) * T) / (sigma * sqrt(T))
    def d2(n):
        return (log((S * U**(2*n)) / (F * L**(2*n))) + (b + sigma**2 / 2) * T) / (sigma * sqrt(T))
    def d3(n):
        return (log((L**(2*n + 2)) / (X * S * U**(2*n))) + (b + sigma**2 / 2) * T) / (sigma * sqrt(T))
    def d4(n):
        return (log((L**(2*n + 2)) / (F * S * U**(2*n))) + (b + sigma**2 / 2) * T) / (sigma * sqrt(T))
    def y1(n):
        return (log((S * U**(2*n)) / (E * L**(2*n))) + (b + sigma**2 / 2) * T) / (sigma * sqrt(T))
    def y2(n):
        return (log((S * U**(2*n)) / (X * L**(2*n))) + (b + sigma**2 / 2) * T) / (sigma * sqrt(T))
    def y3(n):
        return (log((L**(2*n + 2)) / (E * S * U**(2*n))) + (b + sigma**2 / 2) * T) / (sigma * sqrt(T))
    def y4(n):
        return (log((L**(2*n + 2)) / (X * S * U**(2*n))) + (b + sigma**2 / 2) * T) / (sigma * sqrt(T))

    if cp == 1: # call
        return S * exp((b - r) * T) * sum([((U**n) / (L**n))**mu1(n) * (L / S)**mu2(n) * (N(d1(n)) - N(d2(n))) - ((L**(n+1)) / (U**n * S))**mu3(n) * (N(d3(n)) - N(d4(n))) for n in range(-n_terms, n_terms + 1)]) - X * exp(-r * T) * sum([((U**n)/(L**n))**(mu1(n) - 2) * (L/S)**mu2(n) * (N(d1(n) - sigma * sqrt(T)) - N(d2(n) - sigma * sqrt(T))) - ((L**(n+1)) / (U**n * S))**(mu3(n) - 2) * (N(d3(n) - sigma * sqrt(T)) - N(d4(n) - sigma * sqrt(T))) for n in range(-n_terms, n_terms + 1)])

    else: # put
        return X * exp(-r * T) * sum([((U**n)/(L**n))**(mu1(n) - 2) * (L/S)**mu2(n) * (N(y1(n) - sigma * sqrt(T)) - N(y2(n) - sigma * sqrt(T))) - ((L**(n+1)) / (U**n * S))**(mu3(n) - 2) * (N(y3(n) - sigma * sqrt(T)) - N(y4(n) - sigma * sqrt(T))) for n in range(-n_terms, n_terms + 1)]) - S * exp((b - r) * T) * sum([((U**n) / (L**n))**mu1(n) * (L / S)**mu2(n) * (N(y1(n)) - N(y2(n))) - ((L**(n+1)) / (U**n * S))**mu3(n) * (N(y3(n)) - N(y4(n))) for n in range(-n_terms, n_terms + 1)])
```

```python

def dbo_bs_fem(s0, K, sigma, r, T, dt, lb, ub, cp=1):
    n_el = 1000  # number of elements
    mesh = IntervalMesh(n_el, lb, ub)
    V = FunctionSpace(mesh, 'Lagrange', 2)

    # bc as s -> lb
    def boundary_lb(x, on_boundary):
        return on_boundary and near(x[0], lb, 1e-10)
    bc_lb = DirichletBC(V, Constant(0), boundary_lb)

    # bc as s -> ub
    def boundary_ub(x, on_boundary):
        return on_boundary and near(x[0], ub, 1e-10)
    bc_ub = DirichletBC(V, Constant(0), boundary_ub)

    bcs = [bc_lb, bc_ub]

    u0 = interpolate(Expression('fmax(cp * (x[0] - K), 0)', degree = 2, K = K, cp = cp), V)
    u = TrialFunction(V)
    v = TestFunction(V)
    el = V.ufl_element()
    exp1 = Expression('0.5 * pow(sigma, 2) * pow(x[0], 2) * dt', sigma=sigma, dt=dt, element=el)
    exp2 = Expression('(-r + pow(sigma, 2)) * x[0] * dt', r=r, sigma=sigma, dt=dt, element=el)
    exp3 = Expression('r * dt', r=r, dt=dt, element=el)
    a = u * v * dx + exp1 * u.dx(0) * v.dx(0) * dx + exp2 * u.dx(0) * v * dx + exp3 * u * v * dx
    L = u0 * v * dx

    A, b = None, None
    _ = Function(V)
    for i in range(1, int(T / dt) + 1):
        A, b = assemble_system(a, L, bcs)
        solve(A, _.vector(), b)
        u0.assign(_)
    return _(s0)
```

Heston 隨機變動模型

```python
from numpy import exp, log, sin, sqrt, pi
from dolfin import *
set_log_active(False)

# Lipton formula is valid only when rho = 0 !!!
def dbo_heston_anly(s0=100., y0=0.12, K=85., r=0.03, kappa=1.5, theta=0.1, xi=0.5, cp=1, T=1., L=65., U=135., n_terms=30000):
    ans = 0.
    for n in range(1, n_terms + 1):
        kn = pi * n / log(U / L)
        mu = - 1. / 2. * kappa
        zeta = 1. / 2. * sqrt(kn**2 * xi**2 + kappa**2 + xi**2 / 4.)
        AA = -1. * kappa * theta * (mu + zeta) * T - kappa * theta * log((-mu + zeta + (mu + zeta) * exp(-2. * zeta * T)) / (2. * zeta))
        BB = (xi ** 2 * (kn ** 2 + 1. / 4.) * (1 - exp(-2. * zeta * T))) / (4. * (-mu + zeta + (mu + zeta) * exp(-2. * zeta * T)))
        if cp == 1:
            phin = 2. * ((-1)**(n+1) * kn * (sqrt(U / K) - sqrt(K / U)) + sin(kn * log(L / K))) / ((kn ** 2 + 1. / 4.) * log(U / L))
        else:
            phin = 2. * (kn * (sqrt(K / L) - sqrt(L / K)) + sin(kn * log(L / K))) /  ((kn ** 2 + 1. / 4.) * log(U / L))
        ans += exp(2. * (AA - BB * y0) / (xi**2)) * phin * sin(kn * log(s0 / L))

    return exp(- r * T) * sqrt(s0 * K) * ans

def dbo_heston_fem(s0=100., y0=0.12, K=85., r=0.03, q=0.03, kappa=1.5, theta=0.1, xi=0.5, rho=0.0, cp=1, T=1., L=65., U=135., dt=1./100):
    s0 = log(s0 / K)
    mesh_size = (100, 100)
    domain = ((log(L / K), log(U / K)), (0., 3))
    s_min, s_max = domain[0]
    y_min, y_max = domain[1]

    mesh = RectangleMesh(Point(s_min, y_min), Point(s_max, y_max), mesh_size[0], mesh_size[1], 'right/left')
    V = FunctionSpace(mesh, 'Lagrange', 2)
    n = FacetNormal(mesh)

    ###################################################################
```

```
    ## boundary conditions
    ##################################################################

    # bc as s -> s_min
    def boundary_s_min(x, on_boundary):
        return on_boundary and near(x[0], s_min, 1e-14)

    class BoundaryValues_s_min(UserExpression):

        def set_t(self, t):
            self.t = t

        def value_shape(self):
            return ()

        def eval(self, values, x):
            values[0] = 0

    u_s_min = BoundaryValues_s_min(degree=2)
    bc_s_min = DirichletBC(V, u_s_min, boundary_s_min)

    # bc as s -> s_max
    def boundary_s_max(x, on_boundary):
        return on_boundary and near(x[0], s_max, 1e-14)

    class BoundaryValues_s_max(UserExpression):

        def set_t(self, t):
            self.t = t

        def value_shape(self):
            return ()

        def eval(self, values, x):
            values[0] = 0

    u_s_max = BoundaryValues_s_max(degree=2)
    bc_s_max = DirichletBC(V, u_s_max, boundary_s_max)

    bcs = [bc_s_min, bc_s_max,]

    ##################################################################
    ##
    ##################################################################

    u0 = interpolate(Expression('fmax(cp * (K * exp(x[0]) - K), 0)',
    ↪  degree=2, cp=cp, K=K), V)
```

```python

    s, y = SpatialCoordinate(mesh)
    u = TrialFunction(V)
    v = TestFunction(V)
    A = as_matrix(((y / 2, rho * xi * y / 2), (rho * xi * y / 2, xi ** 2 * y / 2)))
    b = as_vector(((y + rho * xi) / 2 - (r - q), xi ** 2 / 2 - kappa * (theta - y)))
    a = u * v * dx - dot(A * grad(u), n) * v * dt * ds + dot(A * grad(u), grad(v)) * dt * dx + dot(b, grad(u)) * v * dt * dx + r * u * v * dt * dx
    l = u0 * v * dx

    A_, b_ = None, None
    u = Function(V)

    for i in range(1, int(T / dt) + 1):
        t = i * dt
        u_s_min.set_t(t)
        u_s_max.set_t(t)

        A_, b_ = assemble_system(a, l, bcs)
        solve(A_, u.vector(), b_)

        u0.assign(u)

    return u(s0, y0)
```