

Unit 1.4 Applications

Numerical Analysis

Mar. 19, 2015

Simple Resistor Network

- Linear system solution methods have many applications.
- Two examples are given in this unit.
 - Resistor network,
 - Equivalent resistance of a 2-D conductor.

- A simple resistor network is shown.

- $g_i = \frac{1}{r_i}$ is the conductance.

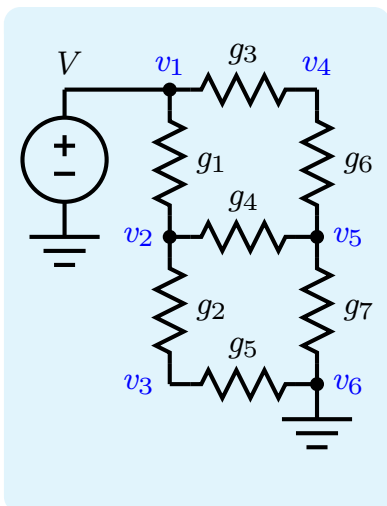
- Need to find the equivalent resistance of the network.

- In order to do that we need find the voltage on every node.
- This can be done by applying the Kirchhoff current law.

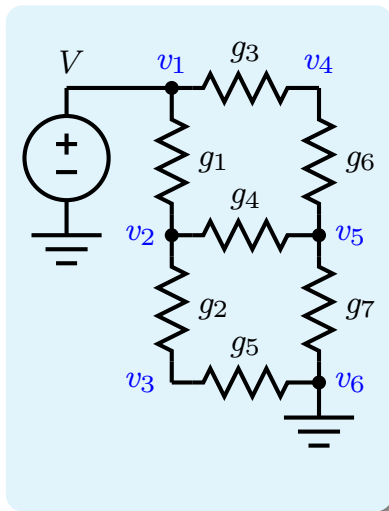
$$\sum_k j_{ik} = 0, \quad \text{for every node } i, \quad (1.4.1)$$

where j_{ik} is the current from node i to node k .

- There are 6 node voltages to be solved for and 6 equations.
- Two boundary conditions: $v_1 = V$ and $v_6 = 0$.



Simple Resistor Network – Equation Formulation



- Apply Kirchhoff Current Law and boundary conditions:

$$n_1 : v_1 = V,$$

$$n_2 : g_1(v_2 - v_1) + g_2(v_2 - v_3) + g_4(v_2 - v_5) = 0,$$

$$n_3 : g_2(v_3 - v_2) + g_5(v_3 - v_6) = 0,$$

$$n_4 : g_3(v_4 - v_1) + g_6(v_4 - v_5) = 0,$$

$$n_5 : g_6(v_5 - v_4) + g_4(v_5 - v_2) + g_7(v_5 - v_6) = 0,$$

$$n_6 : v_6 = 0.$$

- The linear system is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -g_1 & g_1 + g_2 + g_4 & -g_2 & 0 & -g_4 & 0 \\ 0 & -g_2 & g_2 + g_5 & 0 & 0 & -g_5 \\ -g_3 & 0 & 0 & g_3 + g_6 & -g_6 & 0 \\ 0 & -g_4 & 0 & -g_6 & g_4 + g_6 + g_7 & -g_7 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} V \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Simple Resistor Network – Equation Formulation, II

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -g_1 & g_1 + g_2 + g_4 & -g_2 & 0 & -g_4 & 0 \\ 0 & -g_2 & g_2 + g_5 & 0 & 0 & -g_5 \\ -g_3 & 0 & 0 & g_3 + g_6 & -g_6 & 0 \\ 0 & -g_4 & 0 & -g_6 & g_4 + g_6 + g_7 & -g_7 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} V \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- The system matrix can be solved using LU decomposition and forward and backward substitutions.
 - Conductance and applied voltage need to be given.
- The matrix is diagonally dominant.
 - The matrix is nonsingular.
 - Unique solution exists.
- The matrix is, however, not symmetric.

Simple Resistor Network – Forming Symmetric Matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_1 + g_2 + g_4 & -g_2 & 0 & -g_4 & 0 \\ 0 & -g_2 & g_2 + g_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_3 + g_6 & -g_6 & 0 \\ 0 & -g_4 & 0 & -g_6 & g_4 + g_6 + g_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} V \\ g_1 V \\ 0 \\ g_3 V \\ 0 \\ 0 \end{bmatrix}$$

- It is possible to transform the matrix into a symmetric form.
 - Move the nondiagonal entries of the first and the last columns to the right hand side.
- The matrix is now symmetric.
 - Cholesky factorization can be used for better solution efficiency.

Simple Resistor Network – System Formulation

- Formulation of the linear system for a resistor network

Algorithm 1.4.1. System Equation for a Resistor Network

Let the unknown vector be all node voltages, v_i , $i = 1, \dots, n$,
create an $n \times n$ matrix \mathbf{A} and an n -vector \mathbf{b} and initialize both to $\mathbf{0}$.

for each node i not connecting to voltage sources,

for each resistor, with conductance $g_k = \frac{1}{r_k}$, connecting node i and j ,

$$\mathbf{A}_{ii} = \mathbf{A}_{ii} + g_k,$$

$$\mathbf{A}_{ij} = \mathbf{A}_{ij} - g_k.$$

for each node i connecting to a fixed voltage V_i ,

$$\mathbf{A}_{ii} = 1,$$

$$\mathbf{A}_{ij} = 0, \quad j \neq i,$$

$$\mathbf{b}_i = V_i.$$

Simple Resistor Network – Symmetric System Formulation

Algorithm 1.4.2. Symmetric System Equation for a Resistor Network

Let the unknown vector be all node voltages, v_i , $i = 1, \dots, n$,
create an $n \times n$ matrix \mathbf{A} and an n -vector \mathbf{b} and initialize both to 0.

for each node i not connecting to voltage sources,

for each resistor, with conductance $g_k = \frac{1}{r_k}$, connecting node i and j ,

$$\mathbf{A}_{ii} = \mathbf{A}_{ii} + g_k,$$

$$\mathbf{A}_{ij} = \mathbf{A}_{ij} - g_k.$$

for each node i connecting to a fixed voltage V_i ,

$$\mathbf{A}_{ii} = 1,$$

$$\mathbf{A}_{ij} = 0, \quad j \neq i,$$

$$\mathbf{b}_i = V_i,$$

for each resistor, with conductance $g_k = \frac{1}{r_k}$, connecting node i and j ,

$$\mathbf{A}_{ji} = 0,$$

$$\mathbf{b}_i = \mathbf{b}_i + g_i V_i.$$

Simple Resistor Network, System Formulation, II

- The preceding algorithms generate system equations for general resistor network.
- The resulting linear system can be solved to get a unique solution if
 - At least one of the nodes is grounded ($V_i = 0$),
 - At least one of the non-grounded nodes is connecting to a fixed voltage.
- Multiple grounded nodes or multiple voltage supply nodes can still be solved to get a unique solution.
- Computation complexity $\mathcal{O}(mn)$, n is the number of nodes and m is the maximum number resistors connecting to a single node.
 - Each node needs to be processed
 - And the each resistor connecting to the node processed
- Data structures
 - Each node has a linked list for the resistors connecting to the node,
 - A linked list for the nodes connecting to a fixed voltage, including grounded nodes.

Algorithm 1.4.3. System Equation Using Stamps

Let the unknown vector be all node voltages, v_i , $i = 1, \dots, n$, create an $n \times n$ matrix \mathbf{A} and an n -vector \mathbf{b} and initialize both to $\mathbf{0}$.

For each resistor, with conductance $g_k = \frac{1}{r_k}$, connecting node i and j ,
if both nodes i and j are connecting to fixed voltages, V_i and V_j , respectively, then

$$\mathbf{A}_{ii} = 1, \mathbf{b}_i = V_i, \mathbf{A}_{jj} = 1, \mathbf{b}_j = V_j,$$

else if node i is connecting to a fixed voltage V_i

$$\mathbf{A}_{ii} = 1, \mathbf{b}_i = \mathbf{b}_i + g_k V_i,$$

else

$$\mathbf{A}_{ii} = \mathbf{A}_{ii} + g_k, \mathbf{A}_{ij} = \mathbf{A}_{ij} - g_k,$$

if node j is connecting to a fixed voltage V_j

$$\mathbf{A}_{jj} = 1, \mathbf{b}_j = \mathbf{b}_j + g_k V_j,$$

else

$$\mathbf{A}_{jj} = \mathbf{A}_{jj} + g_k, \mathbf{A}_{ji} = \mathbf{A}_{ji} - g_k.$$

Stamping Approach, II

- In the stamping approach a resistor with conductance g_k connecting nodes i and j , with neither node connecting to a fixed voltage, modifies four matrix entries

$$\begin{aligned} \mathbf{A}_{ii} &= \mathbf{A}_{ii} + g_k, & \mathbf{A}_{ij} &= \mathbf{A}_{ij} - g_k, \\ \mathbf{A}_{ji} &= \mathbf{A}_{ji} - g_k, & \mathbf{A}_{jj} &= \mathbf{A}_{jj} + g_k. \end{aligned}$$

This is the **stamp** a the resistor.

- Computational complexity for stamping all the resistor is $\mathcal{O}(N_R)$, where N_R is the number of resistors.
- Stamps can also be added for a voltage source.
- For voltage source with a fixed value V_k connecting nodes i and j , an extra unknown for the current of the voltage source, i_k , is added, and the boundary condition is treated as an extra equation of the system.

$$\begin{aligned} v_i - v_j &= V_k, \\ i_k + \sum_{r_{ij}} g_{ij}(v_i - v_j) &= 0, \\ -i_k + \sum_{r_{ji}} g_{ji}(v_j - v_i) &= 0. \end{aligned}$$

where the second equation is the Kirchhoff current law at node i , and the third equation is the Kirchhoff current law at node j .

Stamping Approach, III

- Thus, the voltage stamps for a voltage source connecting nodes i and j , and the current equation k are

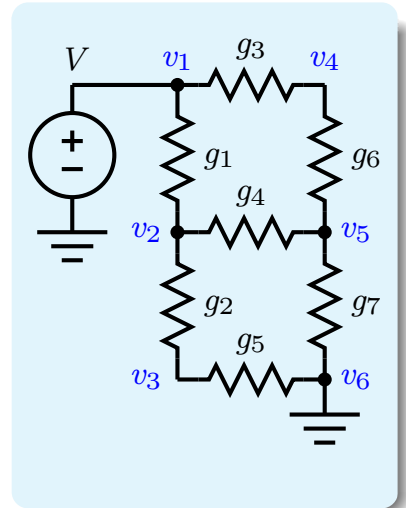
$$\begin{aligned} \mathbf{A}_{ki} &= 1, & \mathbf{A}_{kj} &= -1, & \mathbf{b}_k &= V_k, \\ \mathbf{A}_{ik} &= 1, \\ \mathbf{A}_{jk} &= -1. \end{aligned}$$

- Note that voltage stamps involve right hand side vector.

Example

- Unknown vector = $[v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad i_V]$
- System equation

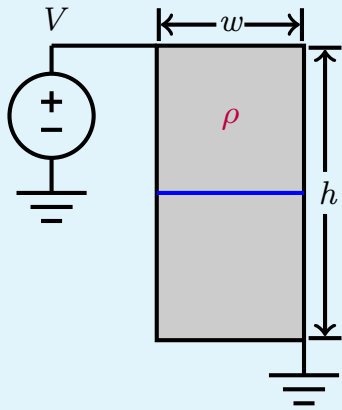
$$\begin{bmatrix} g_1 + g_3 & -g_1 & 0 & -g_3 & 0 & 1 \\ -g_1 & g_1 + g_2 + g_4 & -g_2 & 0 & -g_4 & 0 \\ 0 & -g_2 & g_2 + g_5 & 0 & 0 & 0 \\ -g_3 & 0 & 0 & g_3 + g_6 & -g_6 & 0 \\ 0 & -g_4 & 0 & -g_6 & g_4 + g_6 + g_7 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ i_V \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ V \end{bmatrix}$$



Stamping Approach, IV

- This approach of creating a linear system of equations for the resistor network is known as **modified nodal approach**.
- This approach has been adopted in some SPICE programs.
- In this approach, the ground node voltage is not an unknown, but the current of each voltage source is.
- The resulting system matrix is still symmetric.
- But it is not diagonal dominant.
- LU decomposition is still an effective approach to solving the system.
 - Many fill-ins created for in the last column and last row (borders).
- But, Cholesky decomposition may not be applicable.

Effective Resistance of Conductor



- Given a piece of conductor with resistivity ρ then it can be approximated by the resistor network.
- If uniform current is flown from left to right in conductor, then the resistance is

$$r = \frac{\rho w}{h} \quad (1.4.2)$$

- Since the current flow is not uniform, we need to discretize the conductor into small pieces to get approximated solution.
- The example shown bisects the conductor into two pieces.
- It can be shown that

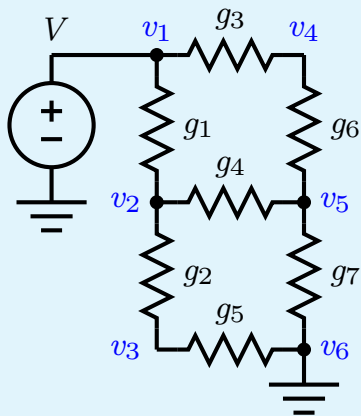
$$r_1 = r_2 = r_6 = r_7 = \rho \frac{h}{w}$$

$$r_3 = r_5 = \rho \frac{4w}{h}$$

$$r_4 = \rho \frac{2w}{h}$$

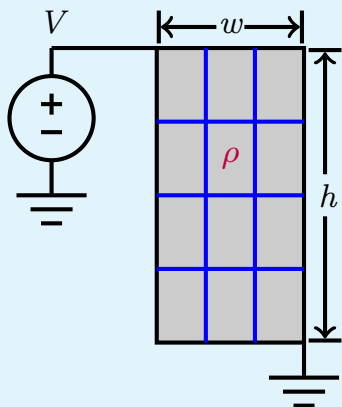
- And, $g_i = \frac{1}{r_i}$, as usual.
- Then the linear system can be solved to find the solution.
- The equivalent resistance of the configuration given is

$$R_{eq} = \frac{V}{i_V}. \quad (1.4.3)$$



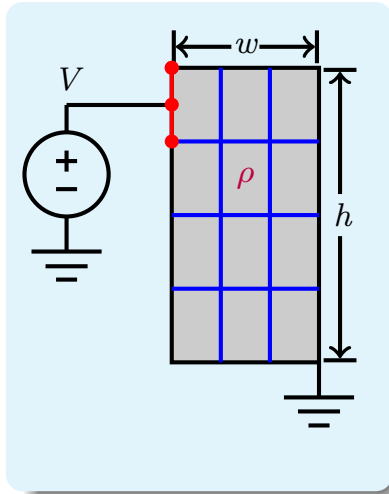
Effective Resistance of Conductor, II

- The preceding example bisects the conductor into two pieces, with the total number of grid points equals to 6.
 - The number of grid points in x -direction, N_x , is 2.
 - And the number of grid points in y -direction, N_y , is 3.



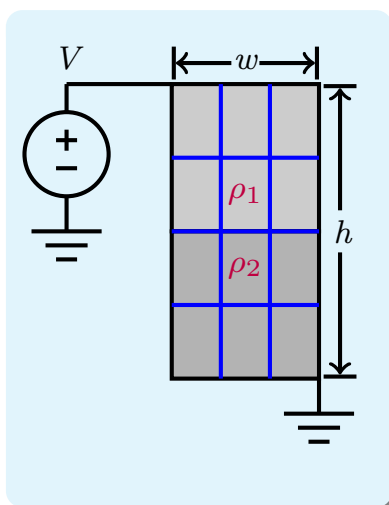
- The equivalent resistance calculated in this case is not very accurate.
- It is straightforward to increase the number of grid points to increase the accuracy.
 - The larger number of grid points, the solution accuracy is usually better.
 - But the solution time and storage space are getting larger.
 - The number of variables increases with $N_x \times N_y$.
 - The time to solve the system is then $\mathcal{O}(N_x^3 \times N_y^3)$, if full matrix LU decomposition method is employed.
 - Sparse matrix techniques can very helpful in reducing the linear system solution time.

Effective Resistance of Conductor, III



- For general problems, the boundary condition can be different.
- As shown, the voltage is applied to two grid points.
- One approach to solve this problem is to have two voltage sources connecting to each grid point individually.
 - Extra current variables are needed.
 - Total current supplied are the sum of these currents.
 - Formulation of the linear system using resistor stamp method is straightforward.
- The other approach is to treat these two grid points as a single variable.
 - Only one voltage source is needed.
 - Smaller matrix can be obtained.
 - Formulation of the matrix is a little more complicated.
- Either method can deliver good solution.
- Multiple voltage source or multiple ground nodes can also be formulated and solved.

Effective Resistance of Conductor, IV



- The conductor may consist of different materials with different resistivity, as shown.
- In this case, the resistor stamping function needs to be able to calculate the resistance (and conductance) based on the material of the region.
- Once that is done, the solution method is identical to the preceding discussions.
- It is also possible that a void ($R = \infty$) is placed in the conductor.
 - In this case, the conductance is 0.
 - If a grid point is completely inside the void region, this grid point should be eliminated.

- Resistor network
 - General resistor network can be solved
- Resistor stamping
 - Systematic formulation of the linear system
- Equivalent resistance of a conductor
 - Can solve general interconnect resistance calculation.
- Linear system solution has many applications
 - Try to apply this solution method in your own research.