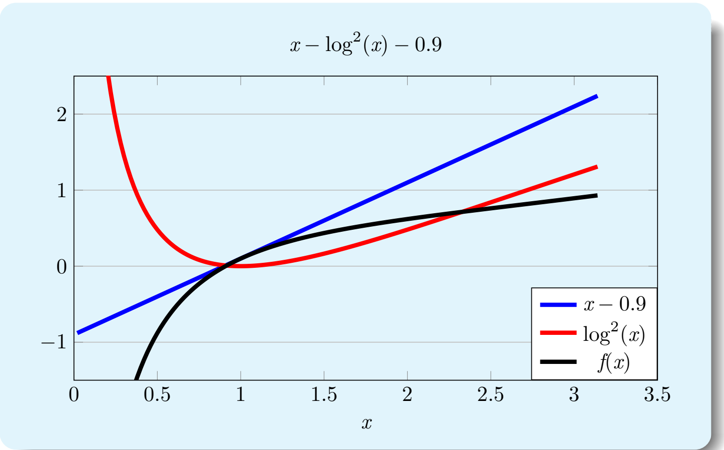
Unit 7 Nonlinear System Solutions

Numerical Analysis

May 12, 2015

Rootfinding of Nonlinear Equations

- Finding numerical solutions of nonlinear equations are needed in many applications. For example,
$$x - \log^2(x) = 0.9$$
For easy treatment, the equation is reformulated as
$$x - \log^2(x) - 0.9 = 0$$
Thus, we need to find the root of the nonlinear equation. In general, we write
$$f(x) = 0 \tag{7.1.1}$$
where $f(x)$ is a nonlinear equation. It is also assumed that $f(x)$ is continuous differentiable in our analysis.



Iterative Approaches

- A general approach to solving a nonlinear equation is the iterative approach.
- The aim is to generate a sequence of $x^{(k)}$ such that

$$\lim_{k\to\infty} x^{(k)} = x^*, \tag{7.1.2}$$

with $f(x^*) = 0$.

Definition. 7.1.1.

A sequence $\{x^{(k)}\}$ generated by a numerical method is said to converge to x^* with order $p \geq 1$ if there are constants $k_0, C > 0$ such that

$$\frac{\left|x^{(k+1)} - x^*\right|}{|x^{(k)} - x^*|^p} \leq C, \qquad k \geq k_0, \tag{7.1.3}$$

where k_0 is an integer. In this case, the method is said to be of order p. Note that if $p = 1$, then in order for $x^{(k)}$ to converge to x^* it is necessary $C < 1$, and C is called the convergence factor of the method.

- It is known that the convergence behavior of most iterative methods depend on the initial point x_0. Thus, they are local convergent in contrast to globally convergent methods, in which convergence holds for any choice of $x^{(0)}$.

Bisection Method

- A group of geometry based methods are based on the following theorem.

Theorem 7.1.2. Zeros for continuous functions.

Given a continuous function $f\colon [a, b] \to \mathbb{R}$ such that $f(a)f(b) < 0$, the there is a $x^* \in (a, b)$ such that $f(x^*) = 0$.

- The bisection method is then

Algorithm 7.1.3. Bisection Method.

Given a, b such that $f(a)f(b) < 0$, and a small $\epsilon > 0$, let

$a^{(0)} = a,\ b^{(0)} = b,\ x^{(0)} = (a^{(0)} + b^{(0)})/2,\ k = 0$,

while $(|x^{(k)} - a^{(k)}| > \epsilon)$ {

  if $(f(x^{(k)})f(a^{(k)}) \leq 0$ then {

    $a^{(k+1)} = a^{(k)},\ b^{(k+1)} = x^{(k)},$

  } else {

    $a^{(k+1)} = x^{(k)},\ b^{(k+1)} = b^{(k)},$

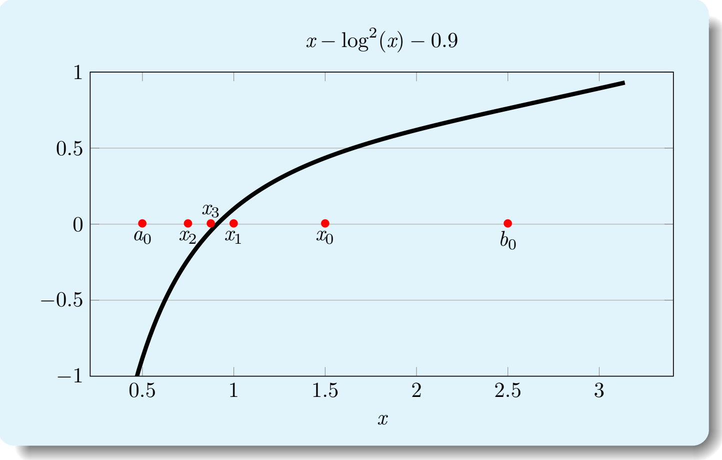
  }

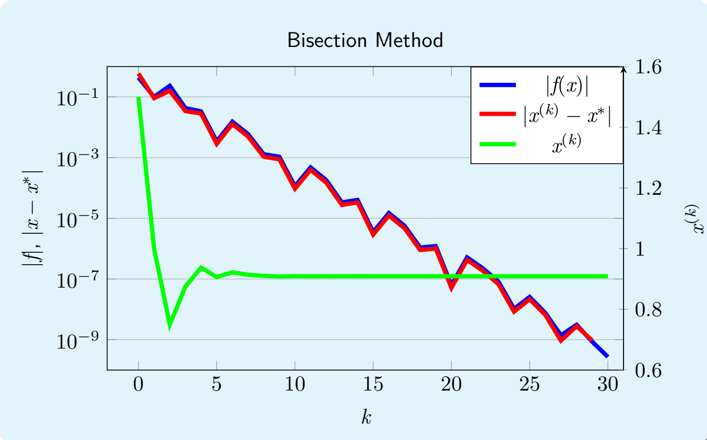
  $k = k + 1,$

} .

Bisection Method, II

- Given the function
$$f(x) = x - \log^2(x) - 0.9$$
- The first few iterations of bisection method are shown below left.
- The bisection method terminates after m iterations for which
$$|x^{(m)} - x^*| \leq |b^{(m)} - a^{(m)}| \leq \epsilon. \tag{7.1.4}$$
- Let the absolute error at iteration k be
$$e^{(k)} = |x^{(k)} - x^*|. \tag{7.1.5}$$
The convergence behavior of the bisection method is also plotted below.





Bisection Method, III

- At iteration k, we have
$$|x^{(k)} - x^*| \leq b^{(k)} - a^{(k)} = \frac{b^{(k-1)} - a^{(k-1)}}{2} = 2^{-k} \times (b^{(0)} - a^{(0)}) \tag{7.1.6}$$
Thus, as $k \to \infty$, $x^{(k)} \to x^*$.
- Bisection method is convergent.
  - It is convergent if $f(a)f(b) \leq 0$, regardless of the value of a and b.
  - Bisection method converges globally.
- The bisection method terminates when
$$|x^{(m)} - x^*| \leq a^{(m)} - b^{(m)} \leq \epsilon.$$
From Eq (7.1.6), we have
$$\epsilon \leq 2^{-m} \times (b^{(0)} - a^{(0)}), \tag{7.1.7}$$
Or
$$m \geq \log_2\left(\frac{b-a}{\epsilon}\right). \tag{7.1.8}$$
Thus, it takes m iterations to reach the accuracy of ϵ regardless of what function we are solving.
  - Bisection method is convergent with a fixed rate.
- Also note from the figure the absolute error is not monotonically decreasing.

Taylor Series Expansion

- It is assume that $f(x^*) = 0$. If x is near x^* then we can expand $f(x)$ at x as

$$f(x^*) = 0 = f(x) + f'(\xi)(x^* - x), \tag{7.1.9}$$

with ξ between x and x^*. Or,

$$x^* = x - (f'(\xi))^{-1} f(x). \tag{7.1.10}$$

Thus, some iterative methods were developed based on the above equation

$$x^{(k+1)} = x^{(k)} - (f'(\xi))^{-1} f(x^{(k)}). \tag{7.1.11}$$

with proper approximation for $f'(\xi)$.

- A simple approximation of $f'(\xi)$ is simply

$$f'(\xi)) = \frac{f(b) - f(a)}{b - a}. \tag{7.1.12}$$

- This is the Chord method.

Chord Method

Algorithm 7.1.4. Chord Method.

Given a, b such that $f(a)f(b) < 0$, and a small $\epsilon > 0$, let

$g = \dfrac{f(b) - f(a)}{b - a}$, $x^{(0)} = b$, $k = 0$, $err^{(0)} = 1 + \epsilon$,

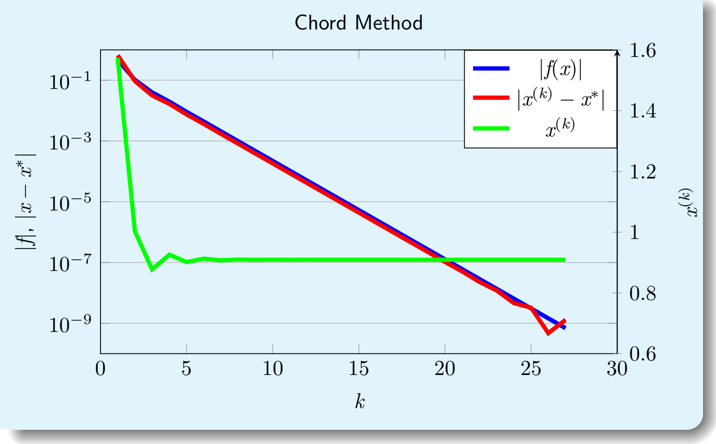
while $(err^{(k)} > \epsilon)$ {

$x^{(k+1)} = x^{(k)} - f(x^{(k)})/g$, $\quad k = k + 1$,

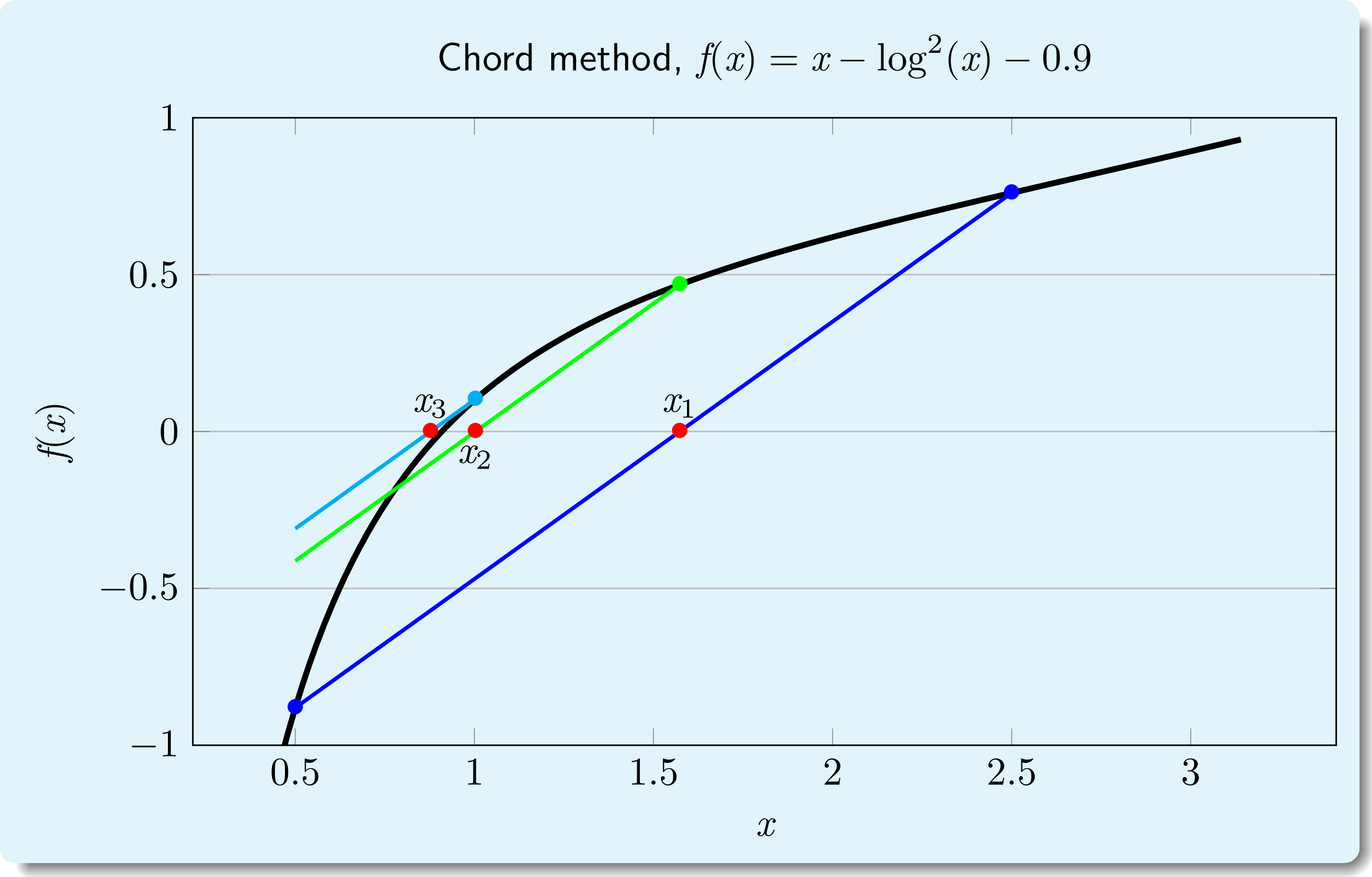
$err^{(k)} = |f(x^{(k)})|$,

} .

- $f'(\xi)$ is assumed to be constant for the chord method.
- Once $f'(\xi)$ is found, each iteration is rather quick
  - It is usually more efficient to use $1/f'(\xi)$ in the iterations.
- Overall convergence rate is slower, but the convergent behavior is smoother.

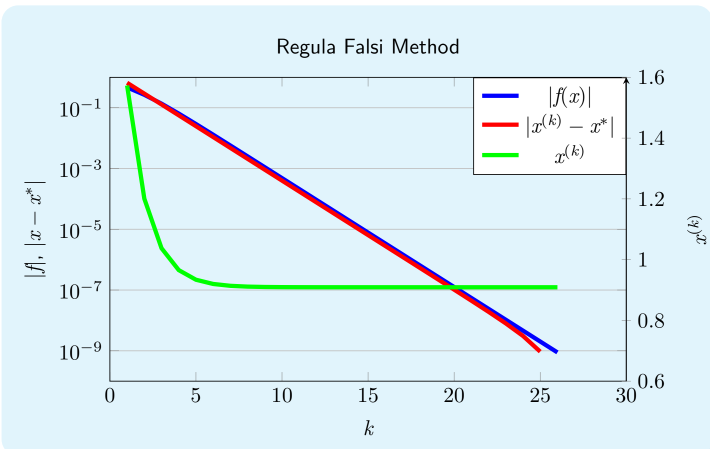


Chord Method, II



Regula Falsi Method

- The chord method was observed to have slow convergence rate with a constant approximation on $f'(\xi)$.
- The regula falsi, or false position, method recalculates $f'(\xi)$ every iteration.
- But it needs to enforce the condition $f(a)f(b) \leq 0$.
- Once the new point, x, is located the range, $[a, b]$, is updated and iteration carried out with new a and b.
- Smooth convergent with the regula falsi method.
- Note that for concave or convex functions $\{x_k\}$ approaches to x^* from one side.



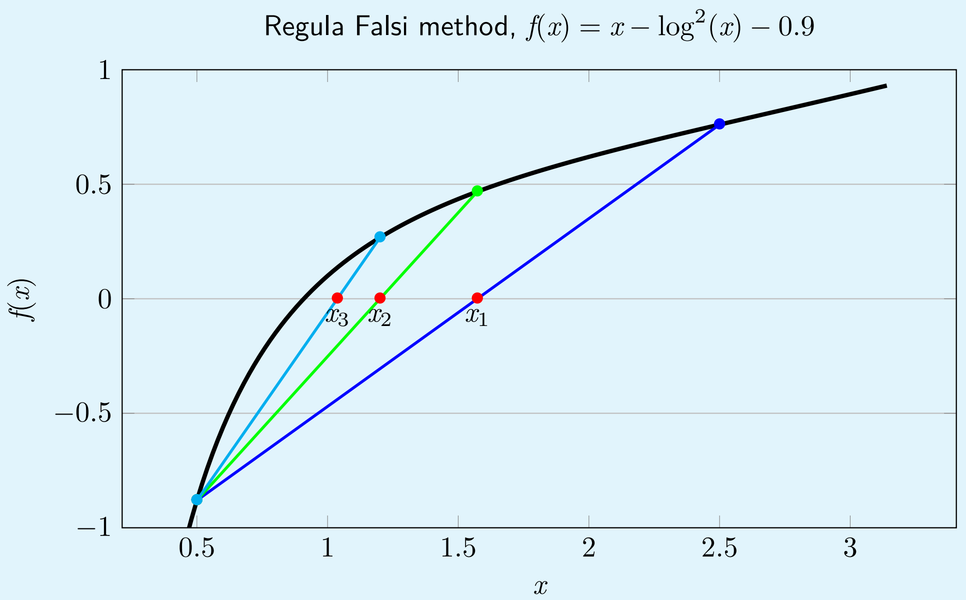
Regula Falsi Method, II

Algorithm 7.1.5. Regula Falsi Method.

Given a, b such that $f(a)f(b) < 0$, and a small $\epsilon > 0$, let

$$
\begin{aligned}
& a^{(0)} = a,\ b^{(0)} = b,\ k = 0\ ,\ err^{(0)} = 1 + \epsilon, \\
& \text{while } (err^{(k)} > \epsilon)\ \{ \\
& \quad x^{(k+1)} = a^{(k)} - f(a^{(k)}) \frac{b^{(k)} - a^{(k)}}{f(b^{(k)}) - f(a^{(k)})}\ , \\
& \quad \text{if } (f(x^{(k+1)}) f(a^{(k)}) \le 0 \text{ then } \{ \\
& \quad\quad a^{(k+1)} = a^{(k)},\ b^{(k+1)} = x^{(k+1)}, \\
& \quad \}\ \text{else}\ \{ \\
& \quad\quad a^{(k+1)} = x^{(k+1)},\ b^{(k+1)} = b^{(k)}, \\
& \quad \} \\
& \quad k = k + 1, \\
& \quad err^{(k)} = |f(x^{(k)})|, \\
& \}\ .
\end{aligned}
$$

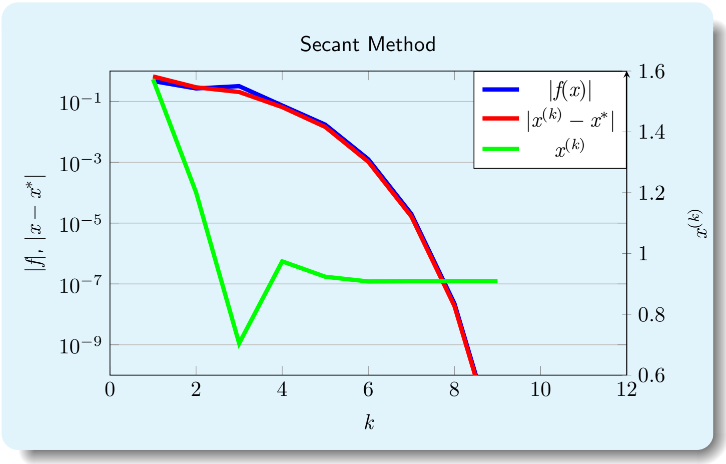
Regula Falsi Method, III



- The sequence generated by the regula falsi method falls in the interval $[a, b]$, thus, the regula falsi method is globally convergent if $f(a) \cdot f(b) < 0$.
- The regula falsi method is convergent with order 1 (linear convergent).

Secant Method

- The regula falsi method was observed to converge from one side.
  - $f'(\xi)$ is not approaching $f'(x^*)$.
- The secant method calculates $f'(\xi)$ using the last two points, $x^{(k-1)}$ and $x^{(k-2)}$.
  - It dose not maintain the region $[a, b]$;
  - $f(x^{(k-1)})f(x^{(k-2)}) \leq 0$ is not required
- Faster convergence if initial guess is close to x^*.



Secant Method, II

Algorithm 7.1.6. Secant Method.

Given $x^{(-1)}$, $x^{(0)}$ and a small $\epsilon > 0$, let

$k = 0$, $err^{(0)} = 1 + \epsilon$,

while $(err^{(k)} > \epsilon)$ {

$$x^{(k+1)} = x^{(k)} - f(x^{(k)}) \frac{x^{(k)} - x^{(k-1)}}{f(x^{(k)}) - f(x^{(k-1)})} ,$$

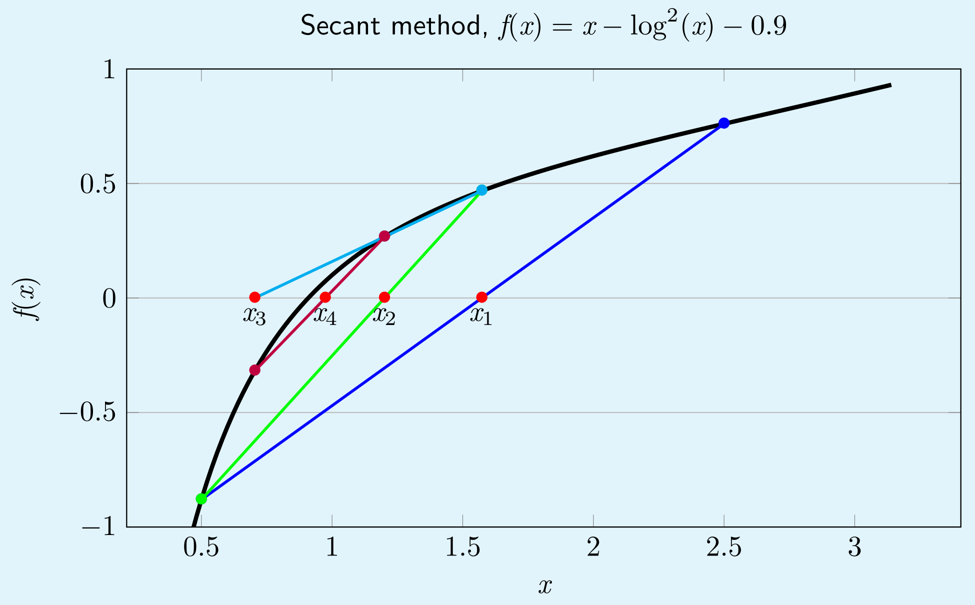
$k = k + 1$,

$err^{(k)} = |f(x^{(k)})|$,

} .

- It is not required $f(x^{(k-1)})f(x^{(k)}) < 0$, it is possible that $|x^{(k)}| \gg 1$ and the iteration diverges
- Secant method is not global convergent
  - Local convergent only
  - Initial guesses, $x^{(-1)}$ and $x^{(0)}$, need to be close to x^* to ensure a converged solution
- Note also that the rate of convergence improves as $x^{(k)}$ is getting closer to x^*

Secant Method, III

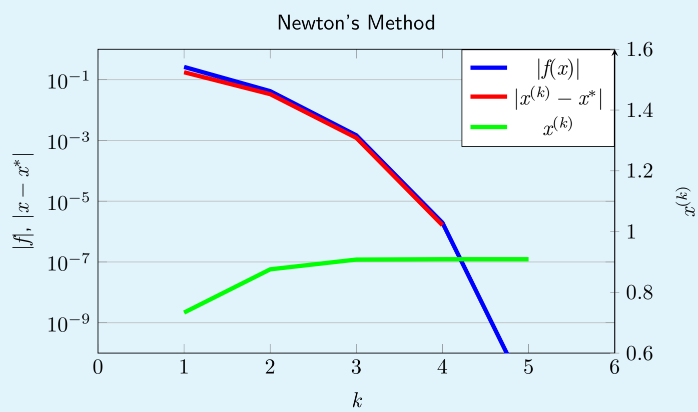


Theorem 7.1.7.

If $f(x) \in C^2$ for $x \in [a, b]$ and $f(x^*) = 0$ with $f'(x^*) \neq 0$, then if $x^{(-1)}$ and $x^{(0)}$ are sufficiently close to x^*, the sequence generated by secant method converges to x^* with the order $p = (1 + \sqrt{5})/2 \approx 1.63$.

Newton's Method

- The chord, regula falsi and secant methods approximate $f'(\xi)$ with different formulas to get converged solution
- As $x^{(k)} \to x^*$ and $f'(\xi) \to f'(x^*)$ the convergence rate improves in secant method
- Newton's method calculates $f'(x^{(k)})$ in the place of $f'(\xi)$
- Faster convergence rate is thus obtained



Newton's Method, II

Algorithm 7.1.8. Newton's Method.

Given $x^{(0)}$ and a small $\epsilon > 0$, let

$k = 0$, $err^{(0)} = 1 + \epsilon$,

while $(err^{(k)} > \epsilon)$ {

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$

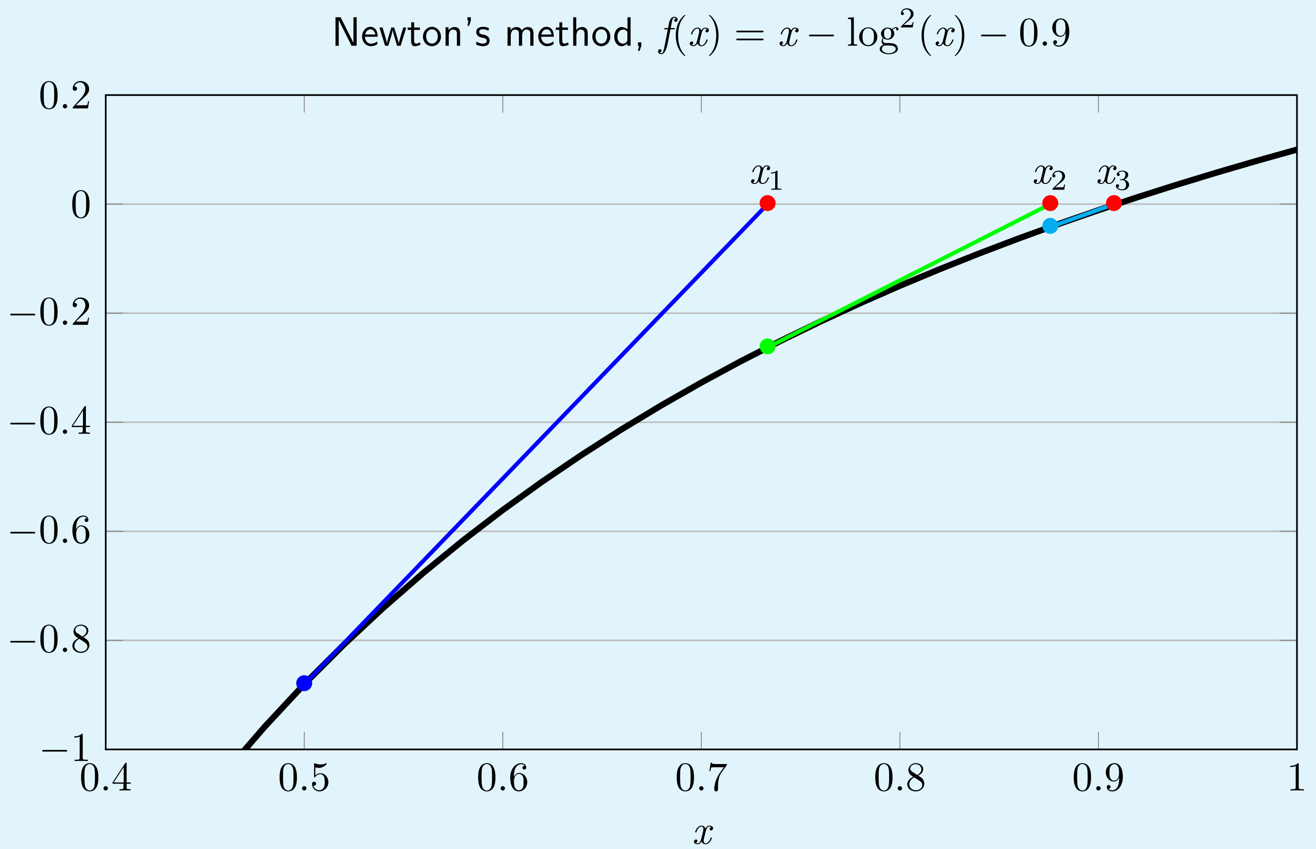
$k = k + 1$,

$err^{(k)} = |f(x^{(k)})|$,

} .

- In Newton's method, the derivative need to be evaluated at each iteration
- $f'(x^{(k)})$ may be expensive to evaluate
- But with explicit $f'(x^{(k)})$ the convergence rate improves
- Only one initial guess is needed, $x^{(0)}$.
- The initial guess needs to be close to x^*, otherwise Newton's iteration may diverge
  - Newton's method is local convergent only
  - with initial guess $x^{(0)} = 2.5$ Newton's method may not converge at all

Newton's Method, III



Newton's Method, IV

- To find the convergence order of Newton's method, we need to compare $|x^{(k+1)} - x^*|$ and $|x^{(k)} - x^*|$.

$$x^{(k+1)} - x^* = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})} - x^* \qquad (7.1.13)$$

Note that by Taylor series expansion

$$f(x^*) = f(x^{(k)}) + (x^* - x^{(k)})f'(x^{(k)}) + \frac{(x^* - x^{(k)})^2}{2} f''(\xi_k) = 0 \qquad (7.1.14)$$

Thus

$$\frac{f(x^{(k)})}{f'(x^{(k)})} = x^{(k)} - x^* - \frac{(x^* - x^{(k)})^2}{2} \cdot \frac{f''(\xi_k)}{f'(x^{(k)})} \qquad (7.1.15)$$

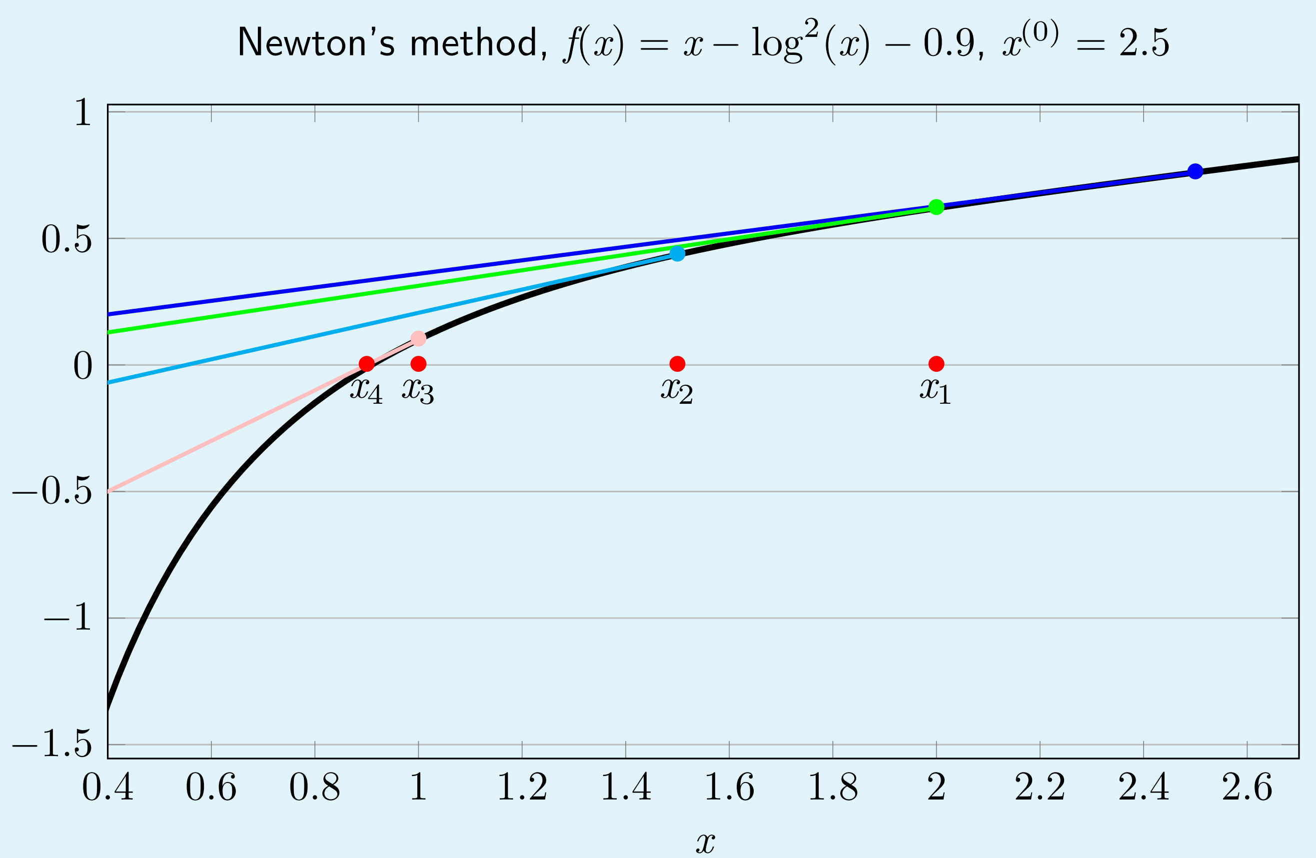
And

$$x^{(k+1)} - x^* = \frac{(x^* - x^{(k)})^2}{2} \cdot \frac{f''(\xi_k)}{f'(x^{(k)})} \qquad (7.1.16)$$

$$\frac{x^{(k+1)} - x^*}{(x^{(k)} - x^*)^2} = \frac{f''(\xi_k)}{2f'(x^{(k)})} \qquad (7.1.17)$$

If $f'(x^*)$ and $f''(x^*)$ both are finite and nonzero, then Newton's method has the order 2 convergence.

Newton's Method, V



Newton's Method, VI

- If initial guess is far away from x^* then Newton's method may not convergent
- Step limiting can help in some cases

Algorithm 7.1.9. Newton's Method with Step Limiting.

Given $x^{(0)}$, S_{limit} and a small $\epsilon > 0$, let

$k = 0$, $err^{(0)} = 1 + \epsilon$,

while $(err^{(k)} > \epsilon)$ {

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})} ,$$

if $(x^{(k+1)} > x^{(k)} + S_{limit})$ then $x^{(k+1)} = x^{(k)} + S_{limit}$,

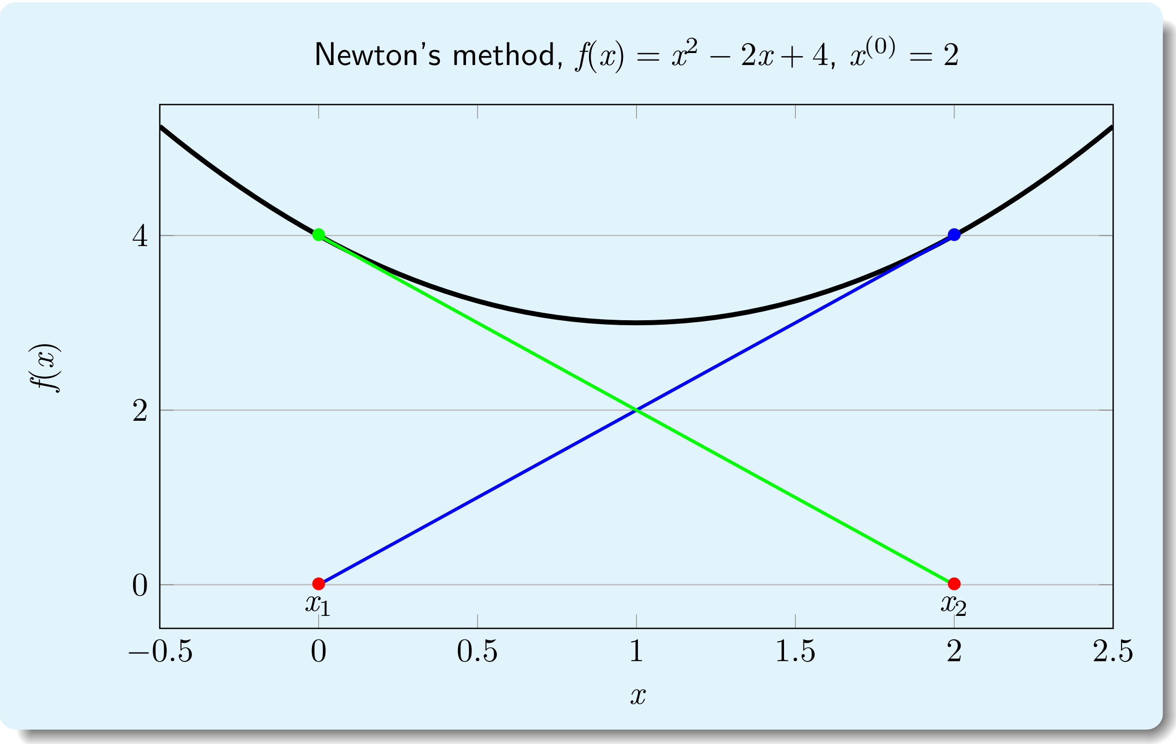
else if $(x^{(k+1)} < x^{(k)} - S_{limit})$ then $x^{(k+1)} = x^{(k)} - S_{limit}$,

$k = k + 1$,

$err^{(k)} = |f(x^{(k)})|$,

} .

Newton's Method, VII

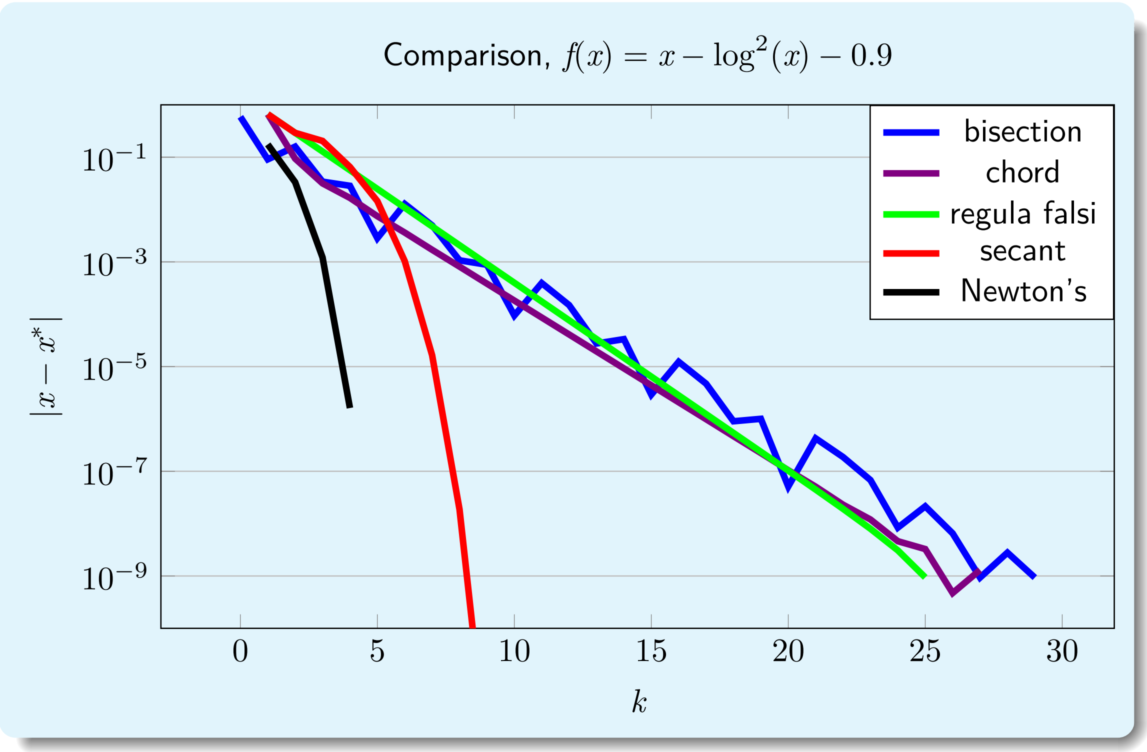


- Newton's method is not guaranteed to converge
  - Oscillation of solution point
- Newton's method is convergent only if $x^{(0)}$ is close to x^*.

Algorithms for Solving Nonlinear Equations

- Bisection method
- Chord method
- Regula falsi method
  - Global convergent
  - Need a and b with $f(a) \cdot f(b) < 0$
- Secant method
  - Need $x^{(-1)}$ and $x^{(0)}$
  - Local convergent
- Newton's method
  - Need $x^{(0)}$
  - Need $f'(x^{(k)})$
  - Local convergent

Comparisons



- Newton's method has the best convergence rate
  - May need more function evaluation due to $f'(x^{(k)})$
- Secant has also good convergence rate
- Chord method appears to have the slowest convergence rate

Summary

- Nonlinear equation solutions
- Iterative methods
- Bisection method
- Chord method
- Regula falsi method
- Secant method
- Newton's method
    - Newton's method with step limiting
    - Oscillation problem