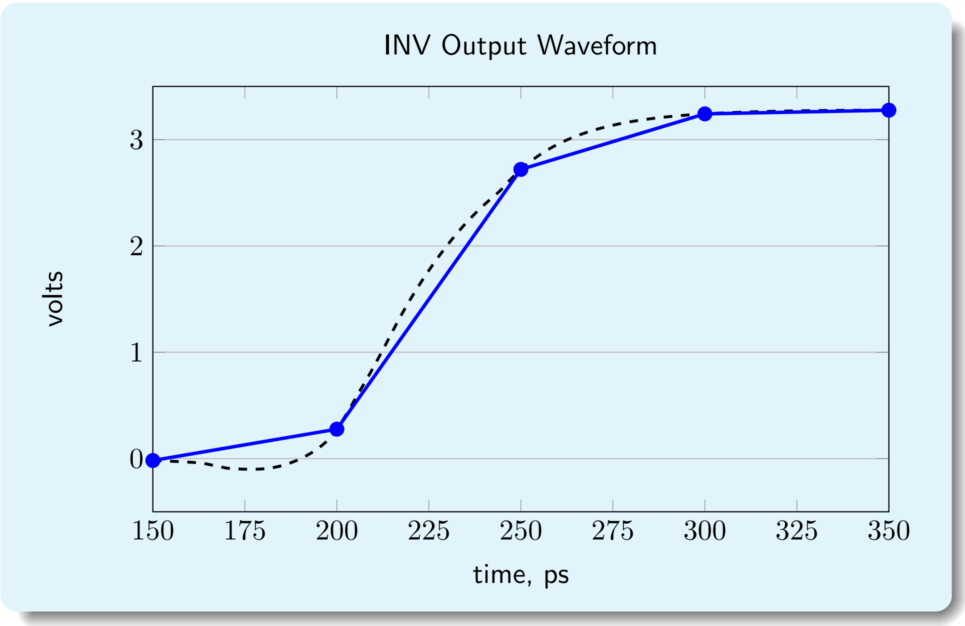
Unit 5.2 Spline Interpolations

Numerical Analysis

Apr. 30, 2015

Piecewise Linear Approximation



- Piecewise linear approximation of data points has been a popular approach
  - Exact solutions at support points
  - linear interpolation between support points
  - Simple and reasonable accurate

Spline Functions

Definition 5.2.1.

A partition of an interval $[a, b]$ is a set of points

$$\Delta : a = x_0 < x_1 < \cdots < x_n = b. \tag{5.2.1}$$

- A piecewise polynomial function $S : [a, b] \to \mathbb{R}$ is a set of polynomial functions, $\{S|I_i\}$, $I_i = [x_{i-1}, x_i], i = 1, 2, \dots, n$, where $S|I_i$ the restriction of S on I_i are polynomials.
- The piecewise linear approximation is an example.
  - $S|I_i$ are polynomial of degree 1.
- Piecewise approximations are continuous

$$S(x_i) = y_i$$

- But the derivatives are not continuous.

Cubic Spline Function

Definition 5.2.2. Cubic Spline

Given a partition Δ of $[a, b]$ and a set of support points $\{(x_i, y_i), i = 0, 1, \dots, n\}$, a cubic spline S_Δ on Δ is a real function $S_\Delta : [a, b] \to \mathbb{R}$ with the following properties:

1. $S_\Delta \in C^2[a, b]$, that is, S_Δ is twice continuously differentiable on $[a, b]$.
2. S_Δ coincides on every subinterval $[x_{i-1}, x_i], i = 1, 2, \dots, n$, with a polynomial of degree at most three.

- From the first property, the first and second derivatives of S_Δ are continuous. Thus, $S_\Delta|I_i$ and $S_\Delta|I_{i+1}$ have the same first and second derivatives at x_i.
- For each subinterval $I_i = [x_{i-1}, x_i]$, we have

$$S_\Delta(x)|I_i = a_0^{(i)} + a_1^{(i)}x + a_2^{(i)}x^2 + a_3^{(i)}x^3. \tag{5.2.2}$$

  There are n subintervals and hence $4n$ coefficients.
- At the $n-1$ support points, $x_i, i = 1, 2, \dots, n-1$ we have

$$S_\Delta|I_i(x_i) = S_\Delta|I_{i+1}(x_i), \qquad S'_\Delta|I_i(x_i) = S'_\Delta|I_{i+1}(x_i), \qquad S''_\Delta|I_i(x_i) = S''_\Delta|I_{i+1}(x_i).$$

  There are $3(n-1)$ constraints.
- How do we find all the coefficients?

Determining Cubic Spline Functions

- In determining the cubic spline function, we assume
  $\{(x_i, y_i), i = 0, 1, \ldots, n\}$ are the support points,
  $\Delta = \{x_i, i = 0, 1, \ldots, n\}$ is the partition,
  $I_i = [x_{i-1}, x_i], i = 1, \ldots, n$ are the subintervals,
  and $h_i = x_i - x_{i-1}$ is the length of the subinterval.
- We also denote the second derivatives at $x_i \in \Delta$ as

$$M_i = S''_\Delta(x_i). \tag{5.2.3}$$

  M_i is also referred to as the moment of $S_\Delta(x)$.
- Since S_Δ is twice continuously differentiable, the second derivative in the subinterval I_i can be expressed as

$$S''_\Delta(x) = M_{i-1}\frac{x_i - x}{h_i} + M_i\frac{x - x_{i-1}}{h_i}. \tag{5.2.4}$$

  Note that $S''_\Delta(x_i) = M_i$ and $S''_\Delta(x_{i-1}) = M_{i-1}$.

Moment and Cubic Spline Function

- Integrating Eq. (5.2.4), we have

$$S'_\Delta(x) = -M_{i-1}\frac{(x_i - x)^2}{2h_i} + M_i\frac{(x - x_{i-1})^2}{2h_i} + A_i. \tag{5.2.5}$$

$$S_\Delta(x) = M_{i-1}\frac{(x_i - x)^3}{6h_i} + M_i\frac{(x - x_{i-1})^3}{6h_i} + A_i(x - x_{i-1}) + B_i. \tag{5.2.6}$$

  for $x \in [x_{i-1}, x_i], i = 1, 2, \ldots, n$, where A_i, B_i are constants of integration.
- Since $S_\Delta(x_{i-1}) = y_{i-1}$ and $S_\Delta(x_i) = y_i$,

$$y_{i-1} = M_{i-1}\frac{h_i^2}{6} + B_i, \tag{5.2.7}$$

$$y_i = M_i\frac{h_i^2}{6} + A_i h_i + B_i. \tag{5.2.8}$$

  We have

$$B_i = y_{i-1} - M_{i-1}\frac{h_i^2}{6}, \tag{5.2.9}$$

$$\begin{aligned} A_i &= \frac{y_i - B_i}{h_i} - M_i\frac{h_i}{6} \\ &= \frac{y_i - y_{i-1}}{h_i} - \frac{h_i}{6}(M_i - M_{i-1}). \end{aligned} \tag{5.2.10}$$

Moment and Cubic Spline Function, II

- Substitute Eqs. (5.2.9) and (5.2.10) into (5.2.6) and rearrange $S_\Delta(x)$ into a polynomial form of

$$S_\Delta(x) = \alpha_i + \beta_i(x - x_{i-1}) + \gamma(x - x_{i-1})^2 + \delta(x - x_{i-1})^3, \qquad \text{for } x \in [x_{i-1}, x_i]. \quad (5.2.11)$$

It can be shown that

$$\alpha_i = y_{i-1}, \quad (5.2.12)$$

$$\beta_i = \frac{y_i - y_{i-1}}{h_i} - \frac{h_i}{6}(M_i + 2M_{i-1}), \quad (5.2.13)$$

$$\gamma = \frac{M_{i-1}}{2}, \quad (5.2.14)$$

$$\delta = \frac{M_i - M_{i-1}}{6h_i}. \quad (5.2.15)$$

- Thus, if the moments on each x_i, $i = 0, 1, \dots, n$, are known then the cubic spline function can be determined.

Moment and Cubic Spline Function, III

- To determine the moments, we will use Eq. (5.2.5) and the A_i found from Eq. (5.2.10).
- Consider two subintervals: $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$. The first derivative $S'_\Delta(x_i)$ should be equal for both subintervals.
- For $[x_{i-1}, x_i]$,

$$\begin{aligned} S'_\Delta(x_i) &= M_i \frac{h_i}{2} + A_i \\ &= M_i \frac{h_i}{2} + \frac{y_i - y_{i-1}}{h_i} - \frac{h_i}{6}(M_i - M_{i-1}) \\ &= \frac{h_i}{3} M_i + \frac{h_i}{6} M_{i-1} + \frac{y_i - y_{i-1}}{h_i} \end{aligned}$$

- For $[x_i, x_{i+1}]$,

$$\begin{aligned} S'_\Delta(x_i) &= -M_i \frac{h_{i+1}}{2} + A_{i+1} \\ &= -M_i \frac{h_{i+1}}{2} + \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{h_{i+1}}{6}(M_{i+1} - M_i) \\ &= -\frac{h_{i+1}}{3} M_i - \frac{h_{i+1}}{6} M_{i+1} + \frac{y_{i+1} - y_i}{h_{i+1}} \end{aligned}$$

- Thus, we have

$$\frac{h_i}{6} M_{i-1} + \frac{h_i + h_{i+1}}{3} M_i + \frac{h_{i+1}}{6} M_{i+1} = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}. \quad (5.2.16)$$

Moment and Cubic Spline Function, IV

- Equation (5.2.16) can be rewritten as

$$\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = d_i, \qquad i = 1, 2, \ldots, n-1. \tag{5.2.17}$$

where

$$\mu_i = \frac{h_i}{h_i + h_{i+1}}, \tag{5.2.18}$$

$$\lambda_i = \frac{h_{i+1}}{h_i + h_{i+1}}, \tag{5.2.19}$$

$$d_i = \frac{6}{h_i + h_{i+1}} \left(\frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i} \right). \tag{5.2.20}$$

- Equation (5.2.16) must be satisfied for $x = x_1, x_2, \ldots, x_{n-1}$. Thus, there are $n-1$ constraints.
- But, we have $n+1$ unknowns: $M_0, M_1, \ldots, M_n$.
- Two more constraints are needed to solve for all moments uniquely.

Boundary Conditions

- Popular additional constraints:

(A) Zero boundary moments:

$$\begin{aligned} M_0 &= 0, \\ M_n &= 0, \end{aligned} \tag{5.2.21}$$

(B) First derivative boundary conditions:

$$\begin{aligned} S'_\Delta(x_0) &= y'_0, \\ S'_\Delta(x_n) &= y'_n, \end{aligned} \tag{5.2.22}$$

(C) Periodical boundary condition:

$$\begin{aligned} M_0 &= M_n, \\ S'_\Delta(x_0) &= S'_\Delta(x_n), \\ y_0 &= y_n. \end{aligned} \tag{5.2.23}$$

Cubic Spline with Zero Boundary Moments

- With zero boundary moments, we get the following system of equations to solve for all moments, M_i, $i = 0, 1, \ldots, n$,

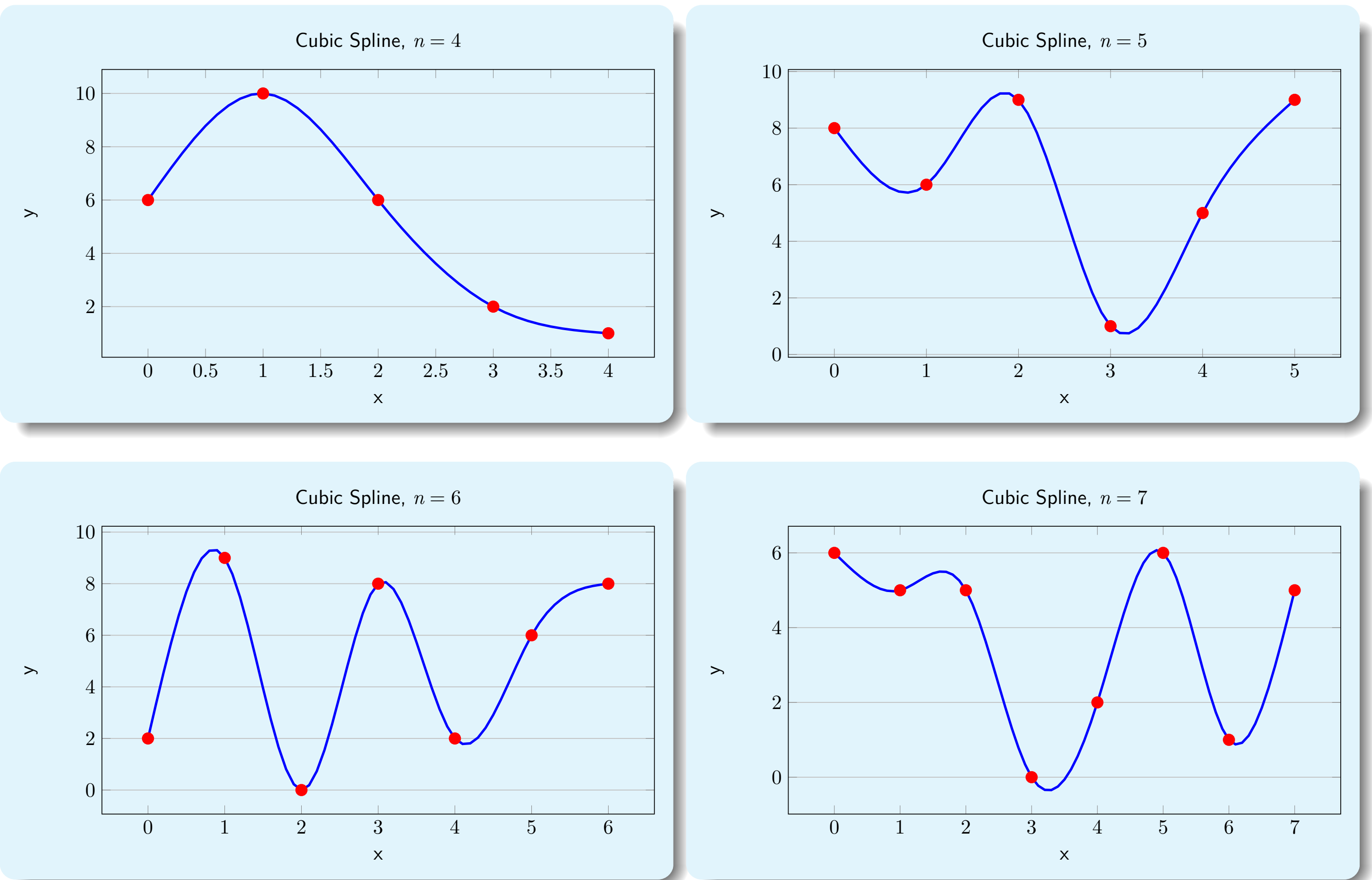
$$\begin{bmatrix} 2 & \lambda_0 & 0 & 0 & \cdots & & 0 \\ \mu_1 & 2 & \lambda_1 & 0 & \cdots & & 0 \\ 0 & \mu_2 & 2 & \lambda_2 & \cdots & & 0 \\ & \cdots & & & \ddots & & \vdots \\ & \cdots & & & \cdots & 2 & \lambda_{n-1} \\ 0 & 0 & \cdots & & \cdots & \mu_n & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix} \tag{5.2.24}$$

where μ_i, λ_i and d_i, are given by equations (5.2.18), (5.2.19) and (5.2.20) for $i = 1, 2, \ldots, n-1$ and

$$\lambda_0 = 0,$$
$$d_0 = 0,$$
$$\mu_n = 0,$$
$$d_n = 0.$$

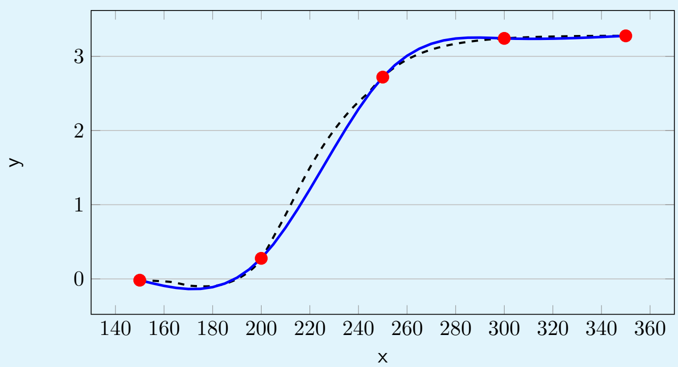
- Note that the matrix is tridiagonal and can be solved efficiently.
- Once all moments are found, then the spline function of Eq. (5.2.6) is obtained with A_i given by Eq. (5.2.10).

Examples

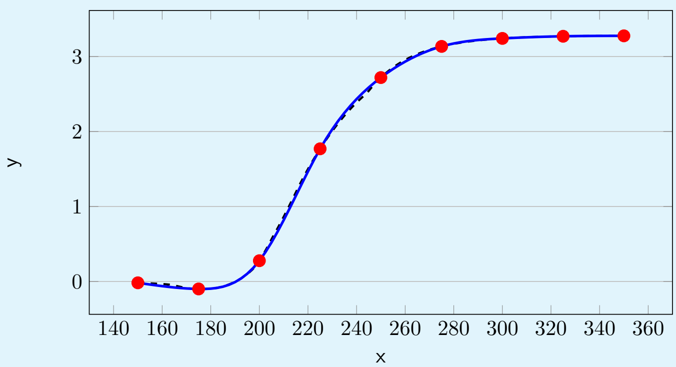


Examples, II

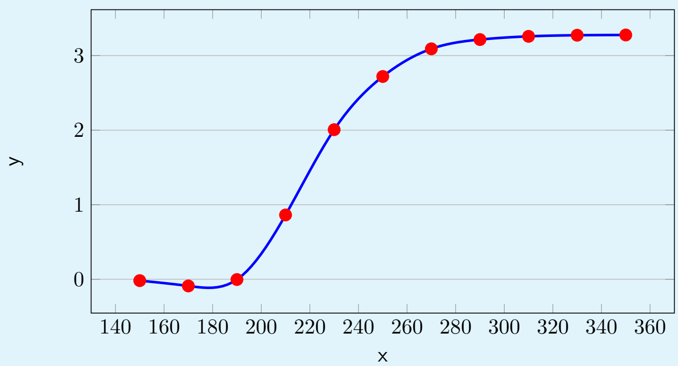
Cubic Spline, $n = 5$



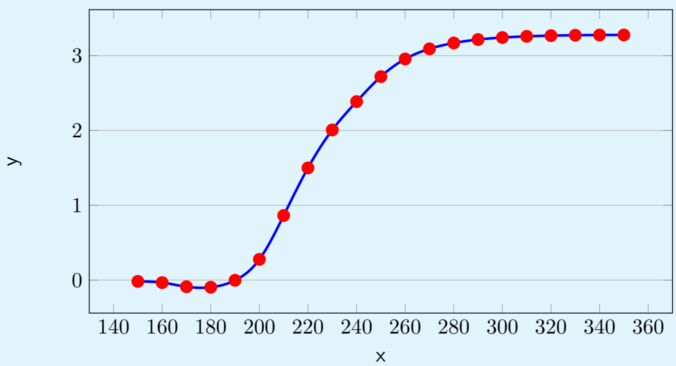
Cubic Spline, $n = 9$



Cubic Spline, $n = 11$

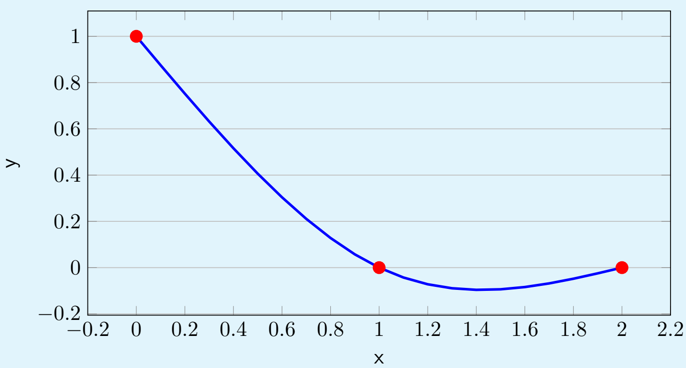


Cubic Spline, $n = 21$

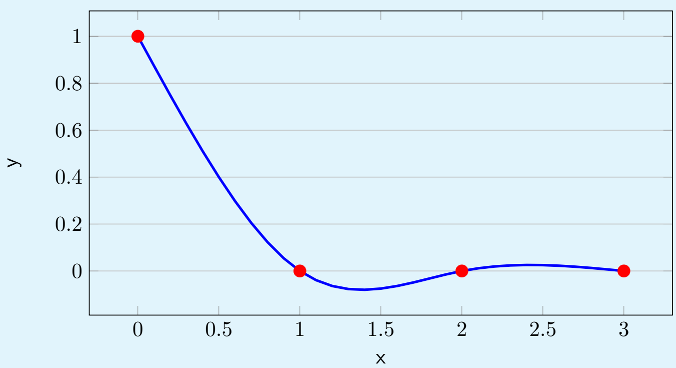


Examples, III

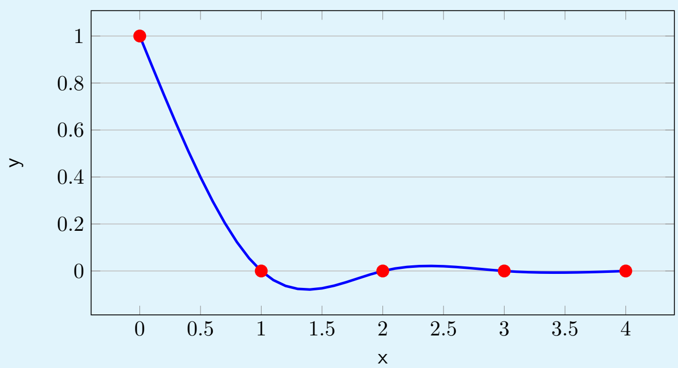
Cubic Spline



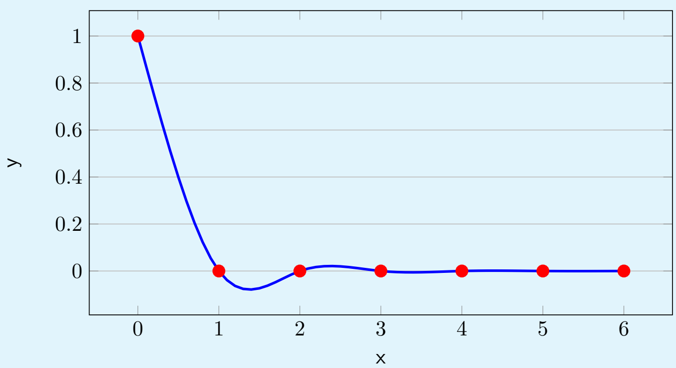
Cubic Spline



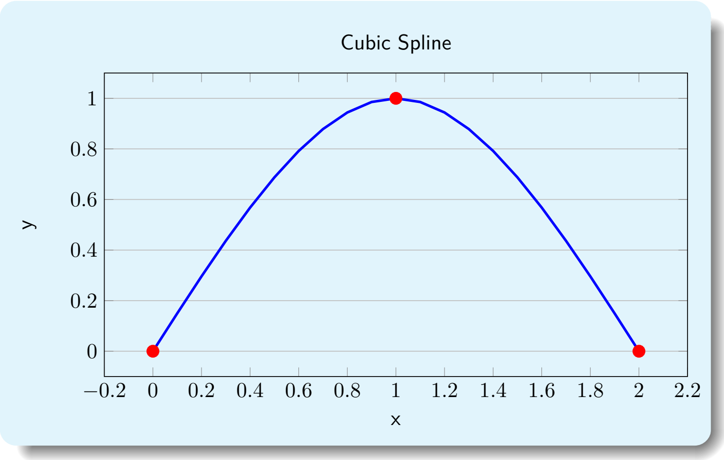
Cubic Spline

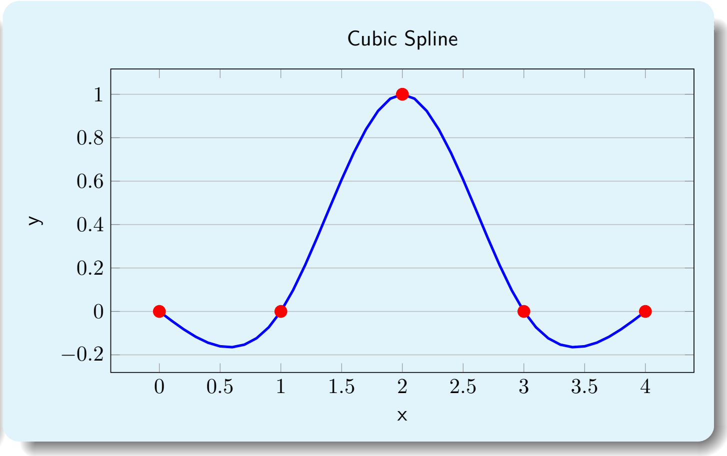


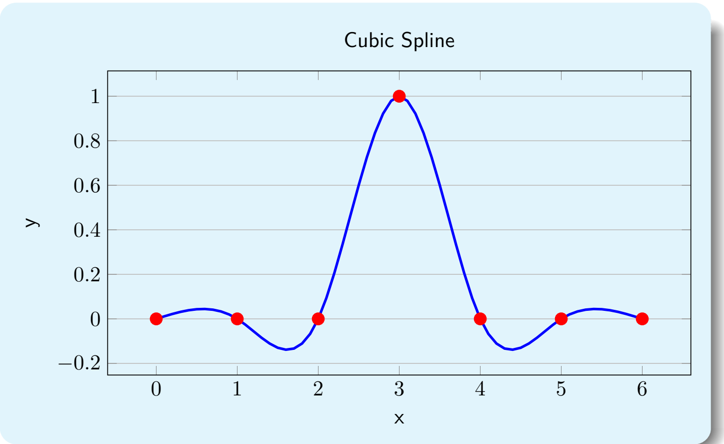
Cubic Spline

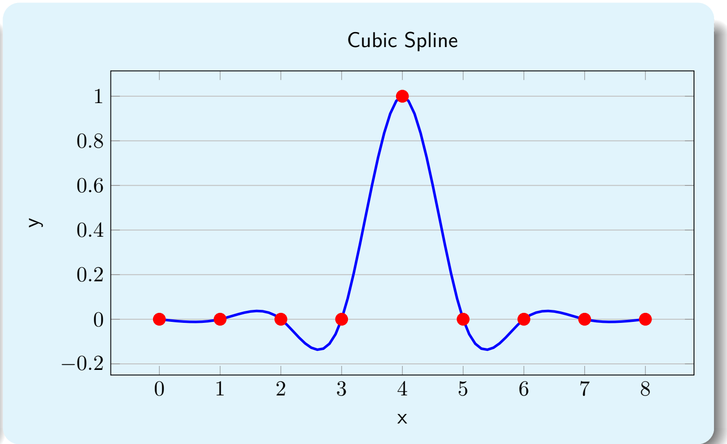


Examples, IV









First Derivative Boundary Conditions

- In case of boundary conditions are specified by first derivatives, we will use Eq. (5.2.5).
  At $x = x_0$, the derivatives is

$$y_0' = -M_0\frac{h_1}{2} + \frac{y_1 - y_0}{h_1} - \frac{h_1}{6}(M_1 - M_0)$$
$$= -\frac{h_1}{3}M_0 - \frac{h_1}{6}M_1 + \frac{y_1 - y_0}{h_1}.$$

It can be rewritten as

$$\frac{h_1}{3}M_0 + \frac{h_1}{6}M_1 = \frac{y_1 - y_0}{h_1} - y_0'.$$

Or

$$2M_0 + \lambda_0 M_1 = d_0, \qquad (5.2.25)$$

with

$$\lambda_0 = 1, \qquad d_0 = \frac{6}{h_1}\left(\frac{y_1 - y_0}{h_1} - y_0'\right). \qquad (5.2.26)$$

Similarly, we find at $x = x_n$ the boundary condition becomes

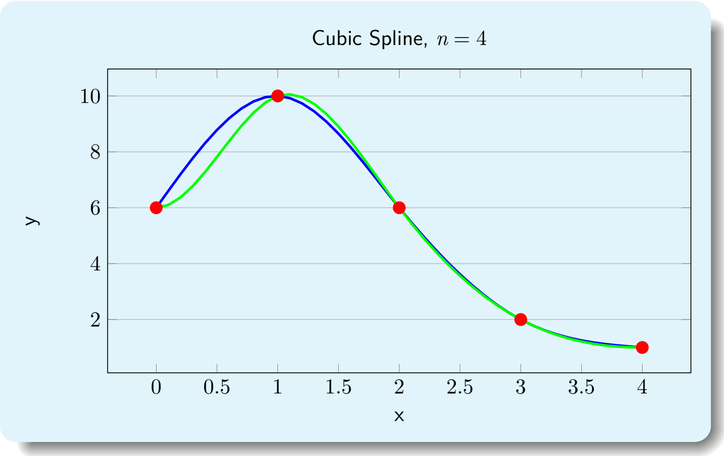
$$\mu_n M_{n-1} + 2M_n = d_n, \qquad (5.2.27)$$

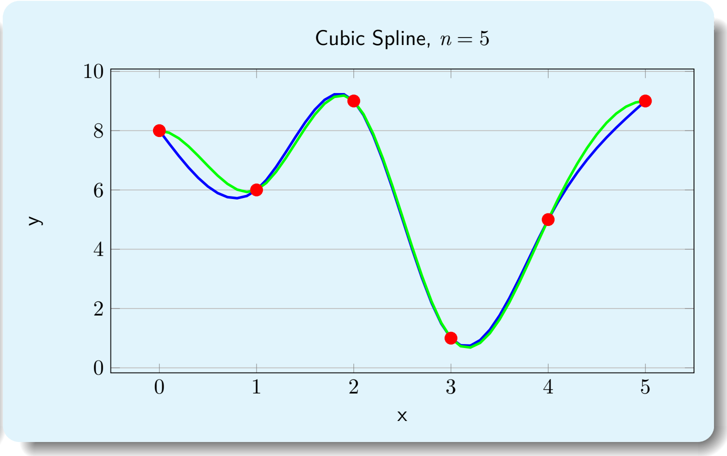
with

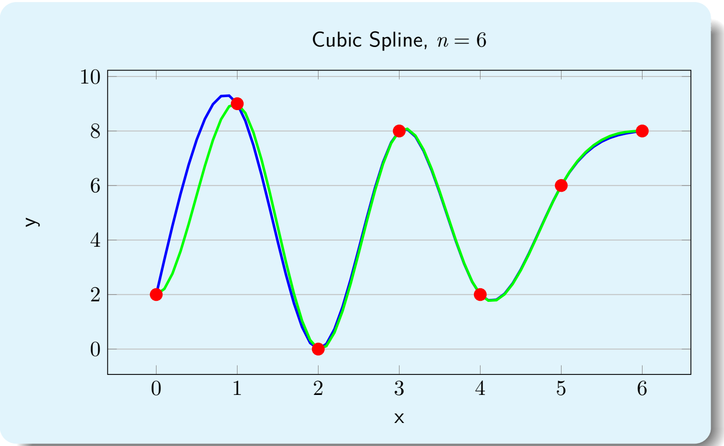
$$\mu_n = 1, \qquad d_n = \frac{6}{h_n}\left(y_n' - \frac{y_n - y_{n-1}}{h_n}\right). \qquad (5.2.28)$$

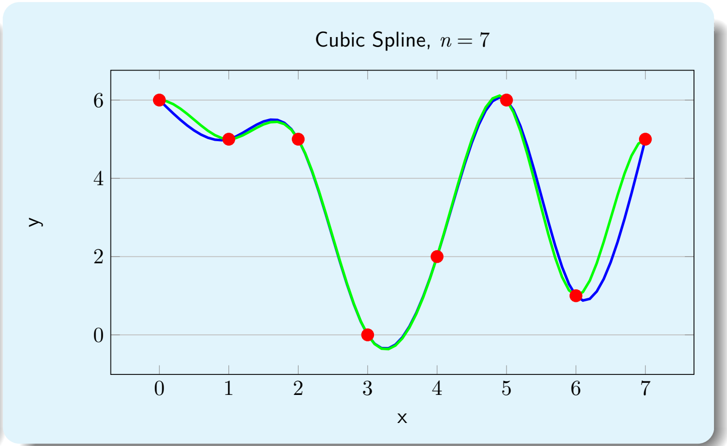
Now the form of Eq. (5.2.35) is obtained to solved for M_i, $i = 0, \ldots, n$.

Examples, with Zero First Derivatives









Periodic Boundary Conditions

- In the case of periodic boundary condition, the point (x_n, y_n) should be identical to (x_0, y_0).
  - $y_n = y_0,\ y_n' = y_0',\ M_n = M_0.$
- Furthermore, the points repeat themselves after n points

$$y_{n+k} = y_k, \qquad y'_{n+k} = y'_k, \qquad M_{n+k} = M_k. \tag{5.2.29}$$

- Thus, Eq. (5.2.17) can be extended to $i = n$ as following

$$\frac{h_n}{6} M_{n-1} + \frac{h_n + h_1}{3} M_n + \frac{h_1}{6} M_1 = \frac{y_1 - y_n}{h_1} - \frac{y_n - y_{n-1}}{h_n}. \tag{5.2.30}$$

- Or

$$\mu_n M_{n-1} + 2M_n + \lambda_n M_1 = d_n, \tag{5.2.31}$$

with

$$\mu_n = \frac{h_n}{h_n + h_1}, \tag{5.2.32}$$

$$\lambda_n = \frac{h_1}{h_n + h_1}, \tag{5.2.33}$$

$$d_n = \frac{6}{h_n + h_1}\left(\frac{y_1 - y_n}{h_1} - \frac{y_n - y_{n-1}}{h_n}\right). \tag{5.2.34}$$

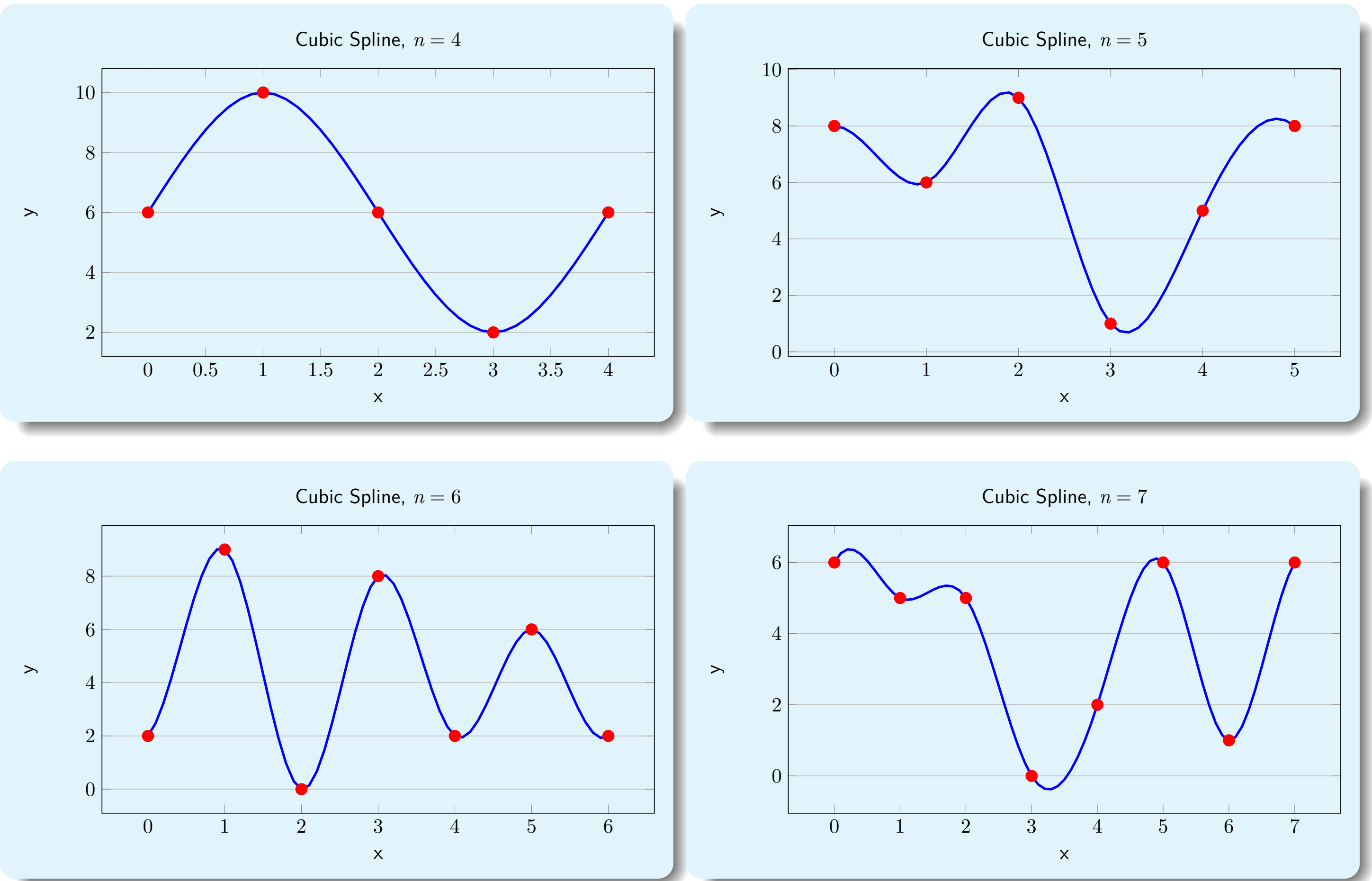
Periodic Boundary Conditions, II

- Thus, the system of equations to solve for $M_i,\ i = 1, 2, \dots, n$ is

$$\begin{bmatrix} 2 & \lambda_1 & 0 & \cdots & 0 & \mu_1 \\ \mu_2 & 2 & \lambda_2 & \cdots & 0 & 0 \\ \cdots & & & \ddots & & \vdots \\ \cdots & & & \cdots & 2 & \lambda_{n-1} \\ \lambda_n & 0 & \cdots & \cdots & \mu_n & 2 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix} \tag{5.2.35}$$

  with μ_i, λ_i and d_i defined in Eqs. (5.2.18 – 5.2.20), and (5.2.32 – 5.2.34).
- Note that M_0 needs not be solved for, and thus the number of variables is one smaller than other boundary conditions.
- But, the matrix is no longer tridiagonal.

Examples, Periodic Boundary Conditions



Cubic Spline Functions

- Cubic spline functions provides a smooth interpolation to the support points.
- The matrix of the linear system to solve for the moments is mostly tridiagonal
  - Formulating the matrix is straightforward
  - The system can be solved efficiently
- Three types of boundary condition provide unique solution of the spline functions.
  - Boundary conditions of moment or first derivative can be mixed
- The support points need not be equally spaced.
  - More support points in rapidly changing regions can improve accuracy.

Cubic Spline Properties

Theorem. 5.2.3. Minimum norm property.

If $f \in C^2([a,b])$ and S is the cubic spline interpolating function on f with the zero moment boundary condition, then

$$\int_a^b [S''(x)]^2\,dx \le \int_a^b [f''(x)]^2\,dx, \tag{5.2.36}$$

and the equality holds if and only if $f = S$.

Theorem. 5.2.4.

if $f \in C^2([a,b])$ and S_f is a cubic spline interpolation of f in $[a,b]$ with $S_f'(a) = f'(a)$ and $S_f'(b) = f'(b)$ then

$$\int_a^b [f''(x) - S_f''(x)]^2\,dx \le \int_a^b [f''(x) - S''(x)]^2\,dx, \tag{5.2.37}$$

where S is any cubic spline interpolating f.

Cubic Spline Properties, II

Theorem. 5.2.5.

If $f \in C^4([a,b])$ and Δ is a partition of $[a,b]$ with $h_i = x_i - x_{i-1}$ and

$$h_{max} = \max_i h_i,$$
$$h_{min} = \min_i h_i,$$
$$\beta = \frac{h_{max}}{h_{min}},$$

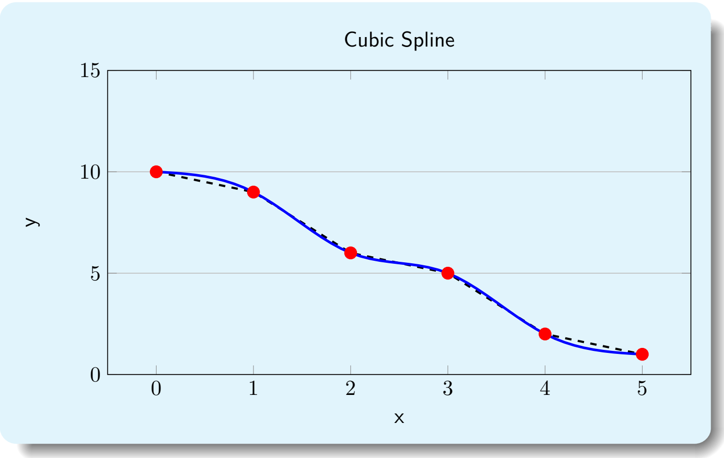
Let $S_\Delta(x)$ be the cubic spline interpolating f. Then

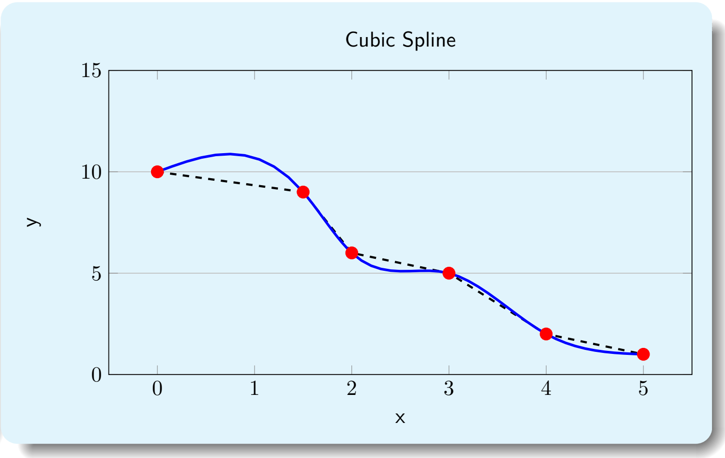
$$\|f^{(r)} - S_\Delta^{(r)}\|_\infty \le C_r h_{max}^{4-r} \|f^{(4)}\|_\infty, \qquad r = 0, 1, 2, 3, \qquad (5.2.38)$$

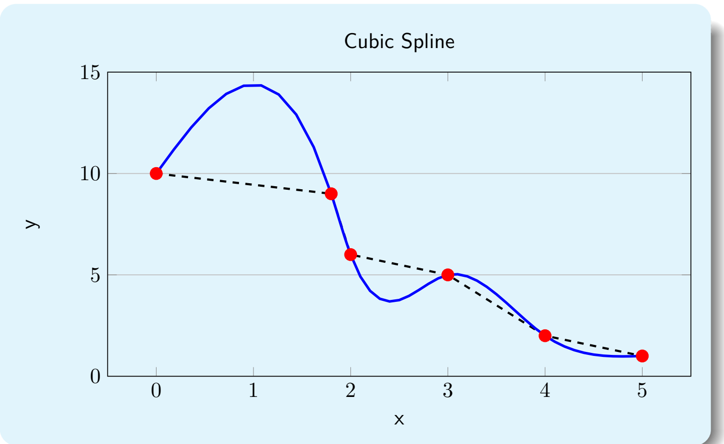
with $C_0 = 5/384$, $C_1 = 1/24$, $C_2 = 3/8$ and $C_3 = (\beta + 1/\beta)/2$.

- Thus, as $h_{max} \to 0$, $S_\Delta(x)$ converges to $f(x)$ and so do $S'_\Delta(x)$ and $S''_\Delta(x)$.
- And, $S'''_\Delta(x)$ converges to $f'''(x)$ if β is bounded.

Cubic Spline With Large β Ratio

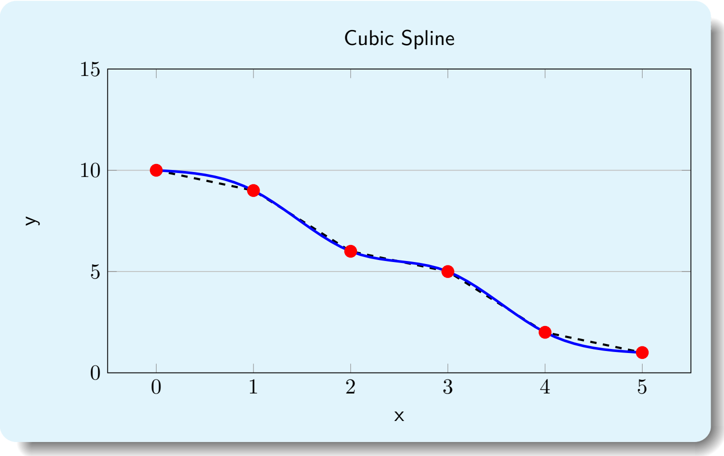


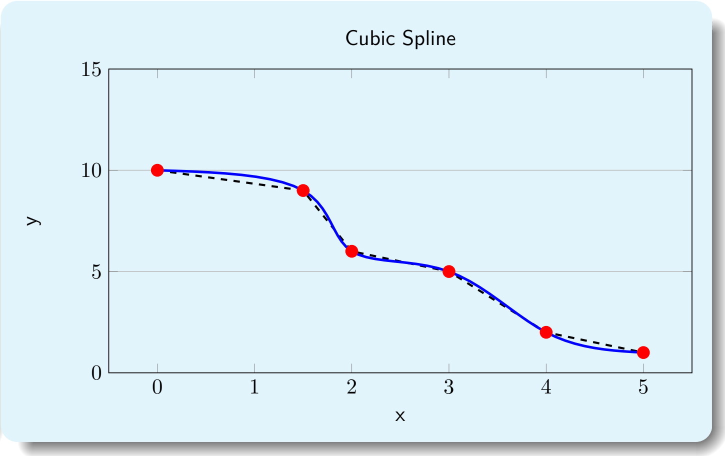


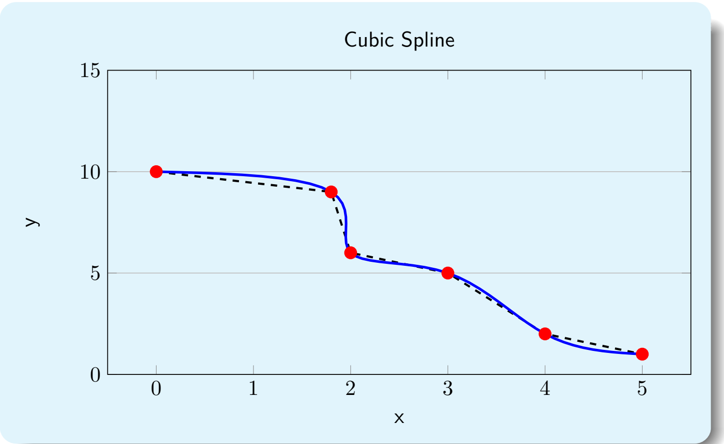


- When $\beta = h_{max}/h_{min}$ increases, cubic spline interpolation may result in local oscillation phenomenon.

Parametric Cubic Spline Interpolations







- Parametric spline can eliminate the local oscillation.
- Smoothness of cubic spline function is still retained.

Parametric Cubic Spline Interpolations, II

- In the cubic spline interpolation, one assumes x is the independent variable and y is a function of x.
- In the parametric cubic spline interpolation, both x and y are assumed to be functions of a parameter t.
- Spline interpolations of t, x and t, y are carried out, thus $(x(t), y(t))$ can be obtained.
- For example, given the support points:

  (0,10), (1.8,9), (2,6), (3,5), (4,2), (5,1)

  cubic spline interpolation can be carried out.
  - Cubic spline interpolation.
- First, perform cubic spline interpolation on

  (0,0), (1,1.8), (2,2), (3,3), (4,4), (5,5)

  to get $x = x(t)$.
- Next perform cubic spline interpolation on

  (0,10), (1,9), (2,6), (3,5), (4,2), (5,1)

  to get $y = y(t)$.
- Combining both, we get $(x(t), y(t))$.

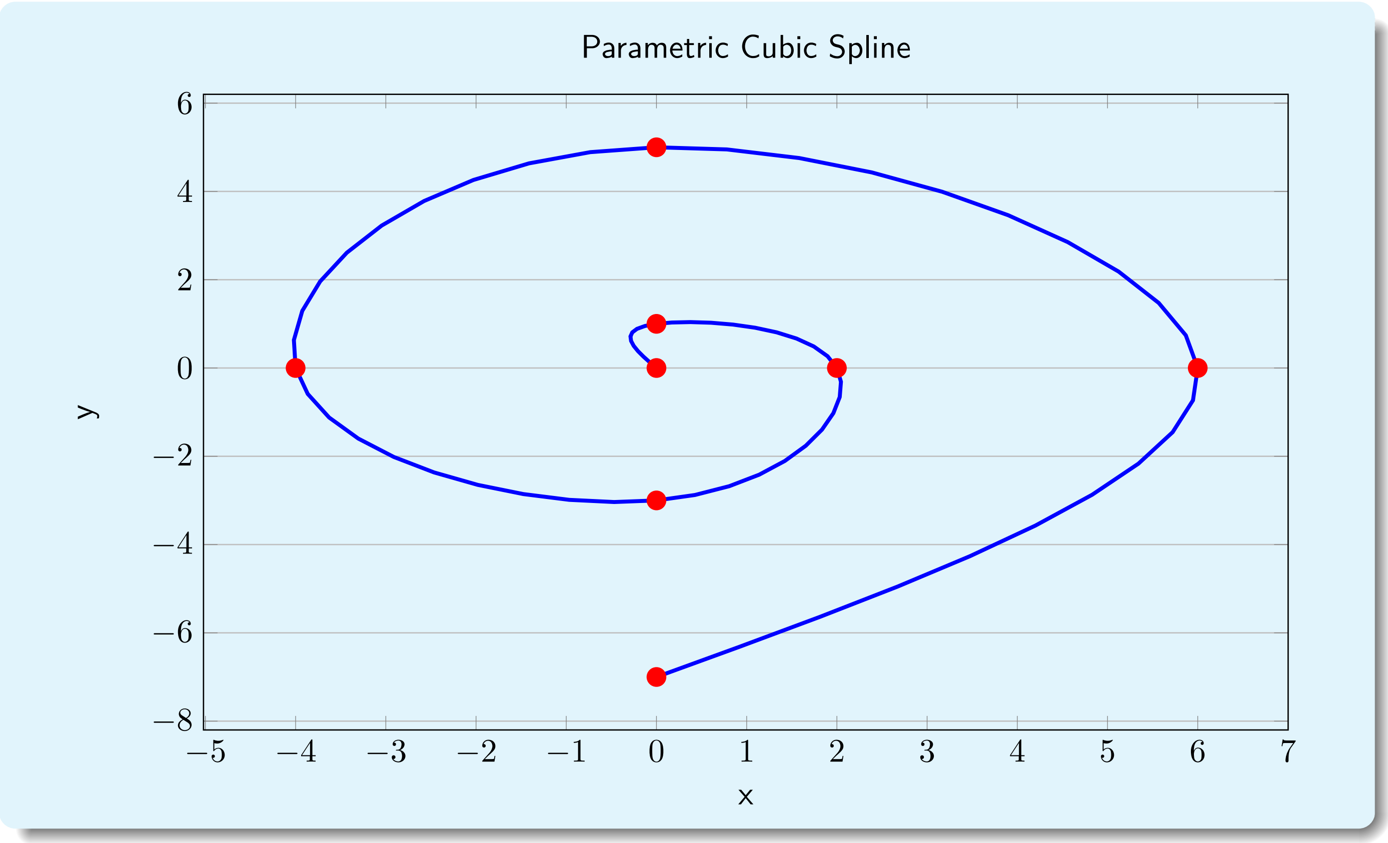
Parametric Cubic Spline Interpolations, III

- A common practice to construct the parameter t is to set t to be the path length, that is, let

$$\begin{aligned} t_0 &= 0, \\ t_i &= t_{i-1} + \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}, \qquad t = 1, \ldots, n. \end{aligned} \tag{5.2.39}$$

- Note that parametric spline interpolation is not guaranteed to have $1 \to 1$ mapping, that is, for a x, $f(x)$ may not be unique.
- Given an $\bar{x}$ to find $y(\bar{x})$ is more involved.
  - Need to find $\bar{t}$ such that $x(\bar{t}) = \bar{x}$, then find $y(\bar{t})$.
- But, parametric spline can be used to construct spiral paths.

Parametric Cubic Spline Interpolations, III



Summary

- Piecewise interpolations using lower order polynomials
  - Piecewise linear function
- Cubic spline functions
- Boundary conditions
  - Zero-moment boundary condition
  - First derivative boundary condition
  - Periodic boundary condition
- Cubic spline properties
- Parametric cubic spline function