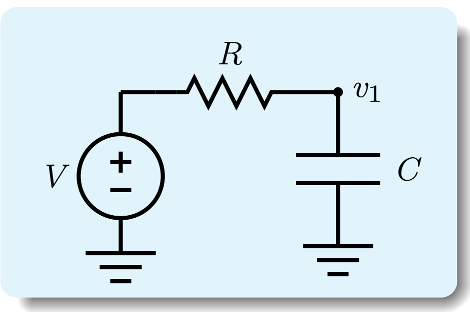
Unit 8.3. Solution Stability

Numerical Analysis

June 9, 2015

Exploring Large Step Sizes

- Higher order integration methods are more accurate
  - Smaller local truncation errors
  - Larger step sizes possible for similar accuracy
  - Step size should still be smaller than the RC time constant
- What would happen if larger step sizes, larger than RC time constant, are taken in solving the dynamic systems
- Example with the simple RC circuit



$$V(t) = 1, \quad t \geq 0,$$
$$v_1(0) = 0.$$

Analytical solution:

$$v_1(t) = 1 - \exp\left(\frac{-t}{RC}\right)$$

- Nodal equation:

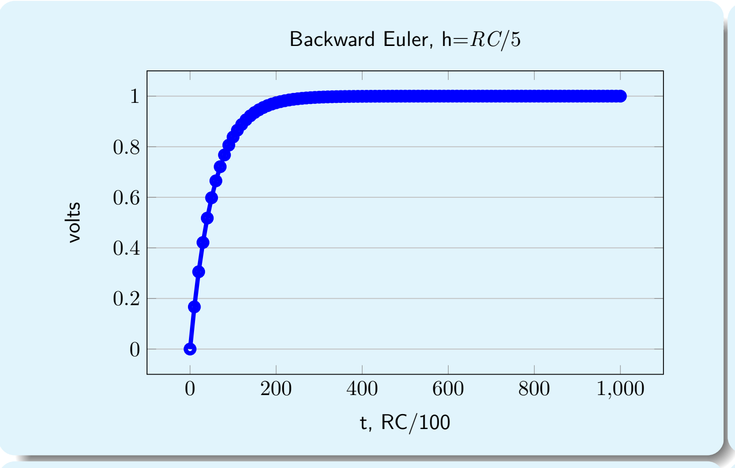
$$\frac{dv_1}{dt} = \frac{V - v_1}{RC}$$

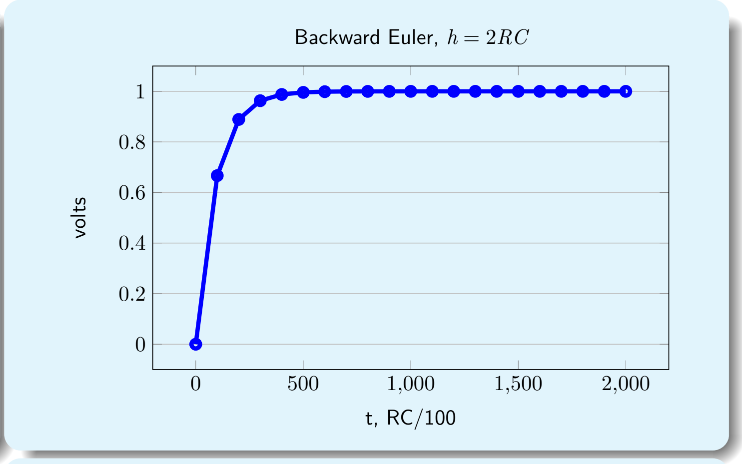
Let $x = v_1$, then

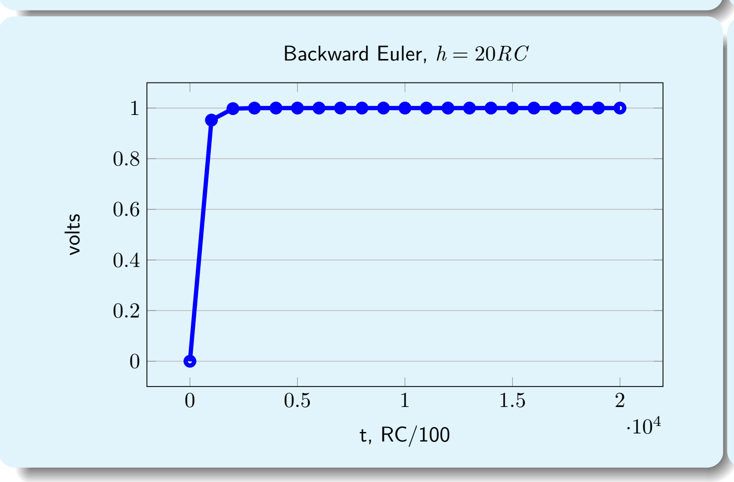
$$\frac{dx}{dt} = f(x, t)$$
$$f = \frac{V - x}{RC}$$

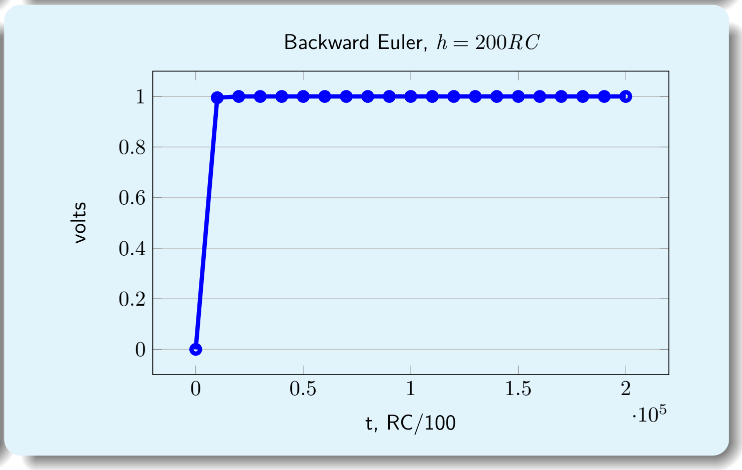
And x can be found using different integration methods.

Backward Euler vs. Step Size









Backward Integration with Large Steps

- Backward Euler method

$$x_{n+1} = x_n + hf_{n+1}$$
$$= x_n + \frac{h}{RC}(V - x_{n+1})$$

Let $y = \dfrac{h}{RC}$
$$x_{n+1} = \frac{x_n + yV}{1 + y}$$

As $y \to \infty$, $x_{n+1} \to V$.

  - Backward Euler, $x_n \to V$ for large h.

- Trapezoidal rule

$$x_{n+1} = x_n + h\frac{f_{n+1} + f_n}{2}$$
$$= x_n + \frac{h}{2RC}(V - x_{n+1} + V - x_n)$$

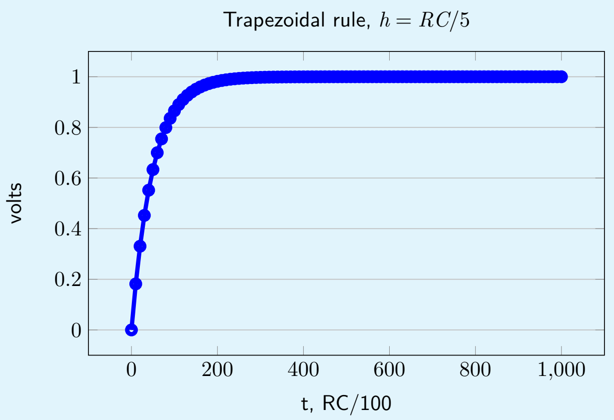
Let $y = \dfrac{h}{RC}$
$$x_{n+1} = \frac{(1 - y/2)x_n + yV}{1 + y/2}$$

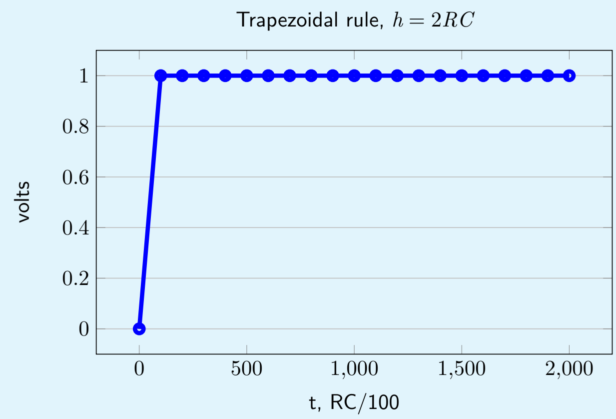
As $y \to \infty$, $x_{n+1} \to 2V - x_n$, and

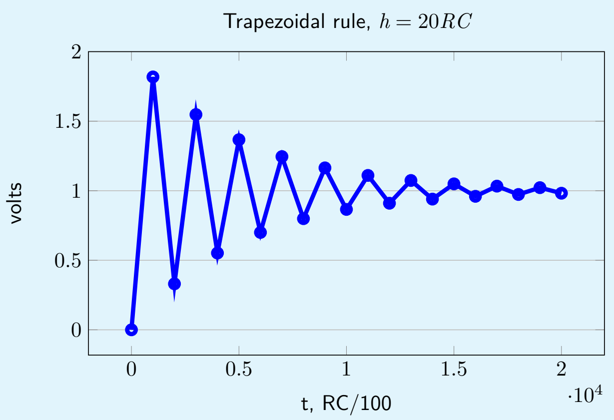
$$x_{n+1} \to 2V - x_n = 2V - (2V - x_{n-1}) = x_{n-1}$$

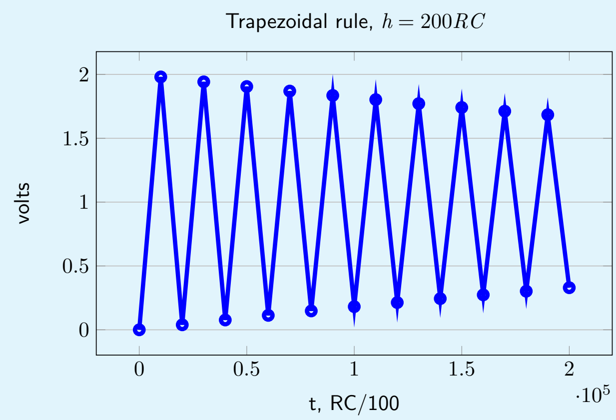
  - Trapezoidal rule, x_n between 0 and $2V$, and $x_n = x_{n-2}$ for large h.

Trapezoidal Rule vs. Step Size









3rd Order Backward Integration with Large Steps

- Using 3rd order backward integration method

$$(1 + \frac{5y}{12})x_{n+1} = (1 - \frac{8y}{12})x_n + \frac{y}{12}x_{n-1} + yV$$

For large h and hence y

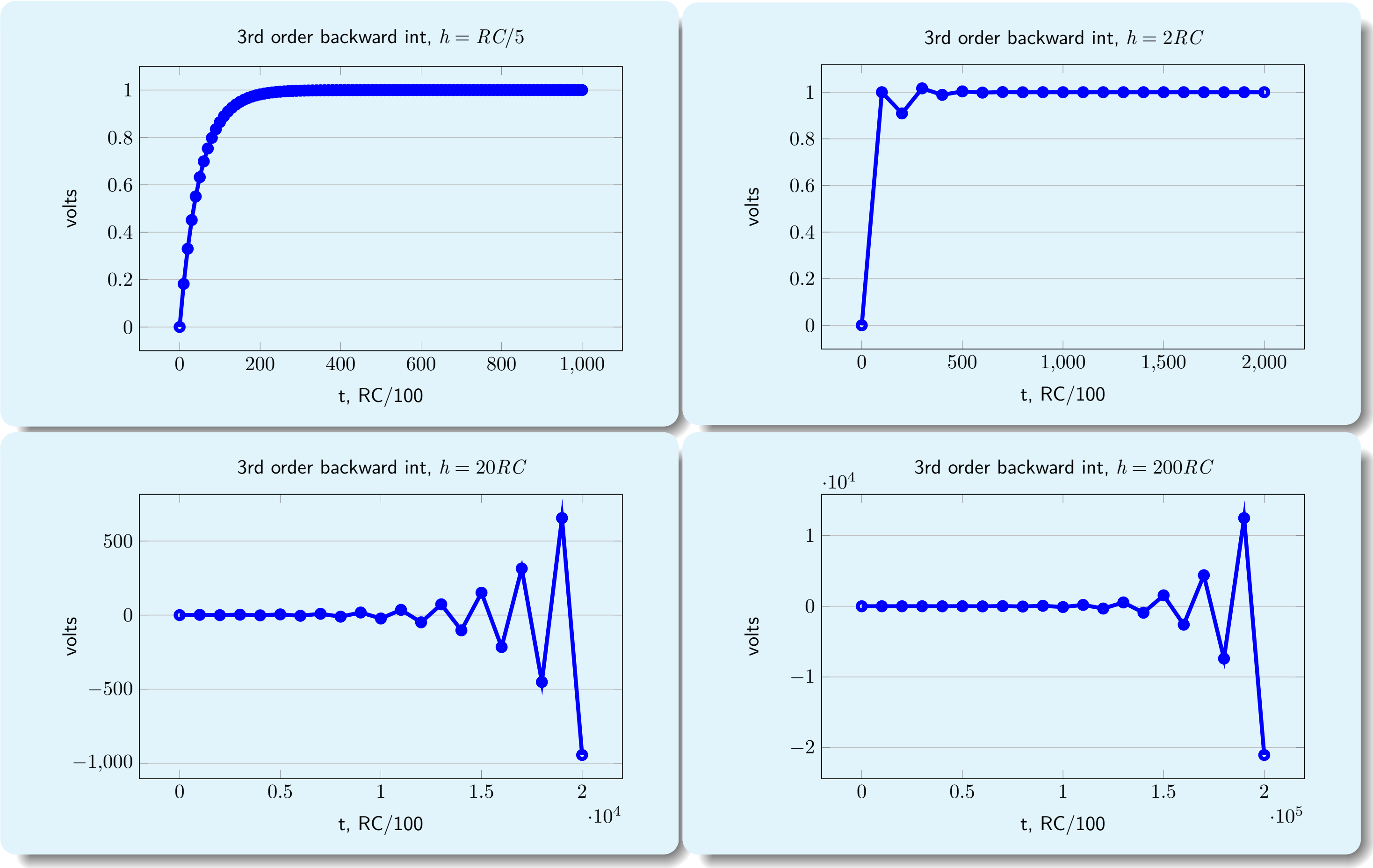
$$\begin{aligned} x_{n+1} &= \frac{(1 - 8y/12)x_n + (y/12)x_{n-1} + yV}{1 + 5y/12} \\ &= \frac{-8x_n + x_{n-1} + 12V}{5} \end{aligned}$$

And,

$$\begin{aligned} x_{n+1} &= -\frac{8}{5}x_n + \frac{1}{5}x_{n-1} + \frac{12}{5}V \\ &= \frac{64}{25}x_{n-1} - \frac{8}{25}x_{n-2} - \frac{96}{25}V + \frac{1}{5}x_{n-1} + \frac{12}{5}V \\ &= \frac{69}{25}x_{n-1} - \frac{8}{25}x_{n-2} - \frac{36}{25}V \end{aligned}$$

- Even when x_{n-1}, x_{n-2} are small, x_{n+1} can be larger than V in magnitude.
- When x_{n-1} becomes significant, it can be amplified further.

3rd Order Backward Int. vs. Step Size



Higher Order Integration methods

- Given k'th order backward integration method, the solution is

$$x_{n+1} = x_n + h(b_0 f_{n+1} + b_1 f_n + b_2 f_{n-1} + \cdots + b_{k-1} f_{n-k+2})$$

- For the simple RC circuit, we have $f_n = \dfrac{V - x_n}{RC}$

$$x_{n+1} = x_n + y\Big(b_0(V - x_{n+1}) + b_1(V - x_n) + b_2(V - x_{n-1}) + \cdots + b_{k-1}(V - x_{n-k+2})\Big)$$

And, $\quad (1 + b_0 y)x_{n+1} = (1 - b_1 y)x_n - b_2 y x_{n-1} - \cdots - b_{k-1} y x_{n-k+2} + yV$

For large h and hence y

$$\begin{aligned}
x_{n+1} &= -\frac{b_1}{b_0}x_n - \frac{b_2}{b_0}x_{n-1} - \cdots - \frac{b_{k-1}}{b_0}x_{n-k+2} + \frac{V}{b_0} \\
x_{n+2} &= -\frac{b_1}{b_0}x_{n+1} - \frac{b_2}{b_0}x_n - \cdots - \frac{b_{k-1}}{b_0}x_{n-k+3} + \frac{V}{b_0} \\
&= -\frac{b_1}{b_0}\left(-\frac{b_1}{b_0}x_n - \frac{b_2}{b_0}x_{n-1} - \cdots - \frac{b_{k-1}}{b_0}x_{n-k+2} + \frac{V}{b_0}\right) \\
&\quad - \frac{b_2}{b_0}x_n - \cdots - \frac{b_{k-1}}{b_0}x_{n-k+3} + \frac{V}{b_0}
\end{aligned}$$

Higher Order Integration methods, II

- Thus,

$$x_{n+2} = \frac{b_1^2 - b_0 b_2}{b_0^2} x_n - \cdots + \frac{b_0 - b_1}{b_0^2} V$$

- To keep x_{n+2} bounded

$$|b_1^2 - b_0 b_2| \le b_0^2 \tag{8.3.1}$$
$$|b_0 - b_1| \le b_0^2 \tag{8.3.2}$$

- For trapezoidal rule

$$b_0 = \frac{1}{2}, \quad b_1 = \frac{1}{2}$$

Both Eqs. (8.3.1) and (8.3.2) are satisfied.

- Trapezoidal rule is stable for large h.

Higher Order Integration methods, II

- For 3rd order backward integration method

$$b_0 = \frac{5}{12}, \quad b_1 = \frac{8}{12}, \quad b_2 = \frac{-1}{12}$$

$$b_1^2 - b_0 b_2 = \frac{64+5}{144} > b_0^2 = \frac{25}{144}$$
$$b_0 - b_1 = \frac{-3}{12} < b_0^2 = \frac{25}{64}$$

- Eq. (8.3.1) is not satisfied.
- 3rd order backward integration is unstable for large h.

Higher Order Integration methods, III

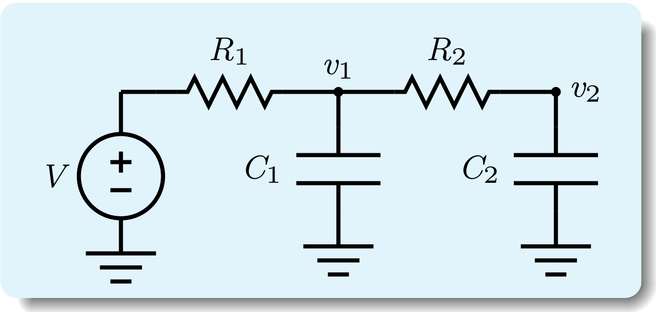
- For 4th order backward integration method

$$b_0 = \frac{9}{24}, \quad b_1 = \frac{19}{24}, \quad b_2 = \frac{-5}{24}$$

$$b_1^2 - b_0 b_2 = \frac{361 + 45}{576} > b_0^2 = \frac{81}{576}$$
$$b_0 - b_1 = \frac{-10}{24} > b_0^2 = \frac{81}{576}$$

- Both Eqs. (8.3.1) and (8.3.2) are not satisfied.
- 4th order backward integration is unstable for large h.
- For higher order backward integration methods, they are not stable for large h.

Stiff Differential Equations



$$V(t) = 1, \qquad t \geq 0,$$
$$v_1(0) = 0, \qquad v_2(0) = 0,$$
$$C_1 = 1\text{pF}, \qquad R_1 = 50\Omega,$$
$$C_2 = 1\text{pF}, \qquad R_2 = 50\text{K}\Omega.$$

- This circuit has two time constants with three orders of magnitudes difference.
- R_1 and C_1 are parasitics in the circuit and the voltage v_1 is usually not critical for the circuit.
- System equation of the circuit

$$C_1 \frac{dv_1}{dt} + \frac{v_1 - V}{R_1} + \frac{v_1 - v_2}{R_2} = 0$$
$$C_2 \frac{dv_2}{dt} + \frac{v_2 - v_1}{R_2} = 0$$

Stiff Differential Equations, II

- It can be written as

$$\frac{dv_1}{dt} = -(\frac{1}{R_1C_1} + \frac{1}{R_2C_1})v_1 + \frac{v_2}{R_2C_1} + \frac{V}{R_1C_1}$$
$$\frac{dv_2}{dt} = \frac{v_1}{R_2C_2} - \frac{v_2}{R_2C_2}$$

- Using trapezoidal rule

$$\begin{aligned} v_1(t+h) =& v_1(t) + h\Big(-(\frac{1}{R_1C_1} + \frac{1}{R_2C_1})v_1(t+h) + \frac{v_2(t+h)}{R_2C_1} + \frac{V(t+h)}{R_1C_1} \\ & -(\frac{1}{R_1C_1} + \frac{1}{R_2C_1})v_1(t) + \frac{v_2(t)}{R_2C_1} + \frac{V(t)}{R_1C_1}\Big)/2 \\ v_2(t+h) =& v_2(t) + h\left(\frac{v_1(t+h)}{R_2C_2} - \frac{v_2(t+h)}{R_2C_2} + \frac{v_1(t)}{R_2C_2} - \frac{v_2(t)}{R_2C_2}\right)/2 \end{aligned}$$

- Let $y_1 = \dfrac{h}{2R_1C_1}$, $y_2 = \dfrac{h}{2R_2C_2}$, $y_3 = \dfrac{h}{2R_2C_1}$

$$(1+y_1+y_3)v_1(t+h) - y_3v_2(t+h) = (1-y_1-y_3)v_1(t) + y_3v_2(t) + 2y_1V$$
$$-y_2v_1(t+h) + (1+y_2)v_2(t+h) = y_2v_1(t) + (1-y_2)v_2(t)$$

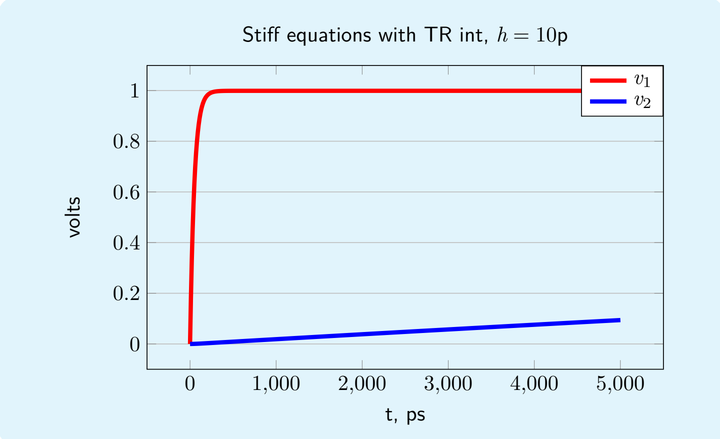
Stiff Differential Equations, III

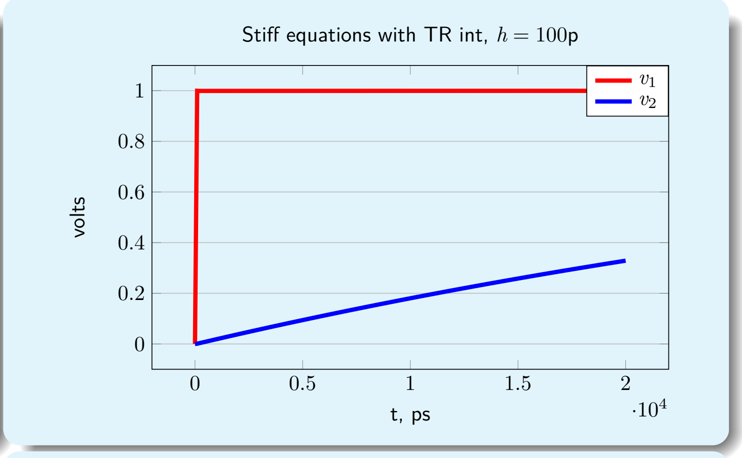
- Or in matrix form

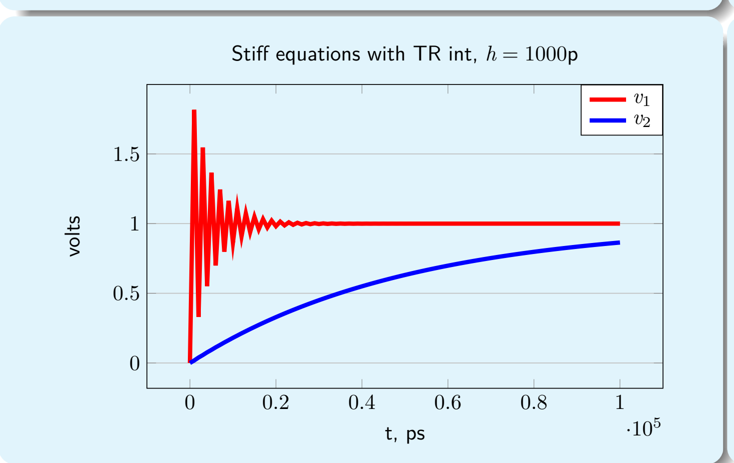
$$\begin{bmatrix} 1+y_1+y_3 & -y_3 \\ -y_2 & 1+y_2 \end{bmatrix} \begin{bmatrix} v_1(t+h) \\ v_2(t+h) \end{bmatrix} = \begin{bmatrix} (1-y_1-y_3)v_1(t) + y_3v_2(t) + 2y_1V \\ y_2v_1(t) + (1-y_2)v_2(t) \end{bmatrix}$$

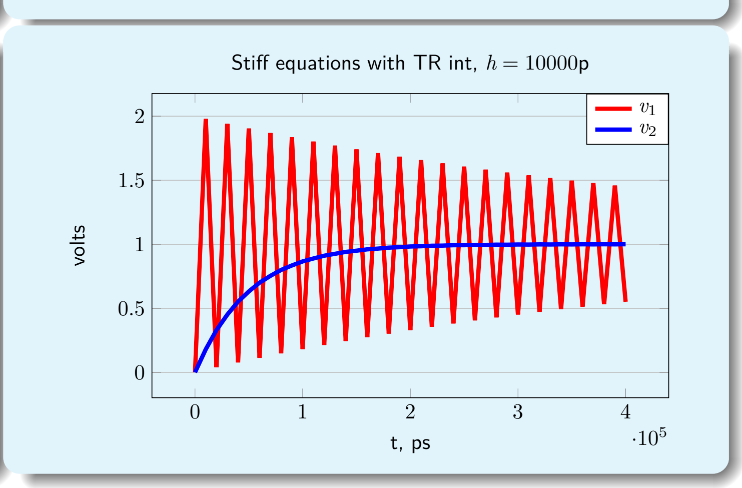
- Given initial conditions $v_1(0)$, $v_2(0)$, one can solve for $v_1(h)$, $v_2(h)$ and then $v_1(2h)$, $v_2(2h)$ and so on.
- It can be seen from the following plots that
  - Both $v_1(t)$ and $v_2(t)$ can be solved accurately with relative small h.
  - For larger h, $v_2(t)$ solution is still stable but not $v_1(t)$.
  - For large h, $v_1(t)$ can have unphysical solutions.
- To get physical $v_1(t)$, the time step h is dominated by the smallest time constant.

Stiff Differential Equations, III









Stiff Differential Equations, IV

- In today's circuit simulations, it is not uncommon to see stiff state equations.
  - Large and small time constants coexist in a single circuit.
  - Back annotated circuits could be easily be stiff.
- Trapezoidal rule is accurate if the time stamp taken is small
  - Relative to the node time constants,
  - Long simulation time maybe required.
- Trapezoidal rule with large time steps could result in false oscillation.
  - For large time constant part of the circuit, the solution might still be accurate.
- Higher order integration methods might not be stable if the time step is too large compared to the smallest time constants.

Gear's Integration Methods

- Gear sought for stable integration method for large time steps
- General form of k'th order Gear method is

$$x(t+h) = \alpha_1 x(t) + \alpha_2 x(t-h) + \cdots + \alpha_k x(t-kh+h) + h\alpha_{k+1} f(t+h). \quad (8.3.3)$$

- Since $b_1 = b_2 = \cdots = 0$, Gear's methods are stable.
- First order Gear's method

$$\begin{aligned} x &= a_0 + a_1 t \\ f(t) &= a_1 \\ x(t+h) &= a_0 + a_1(t+h) && (8.3.4) \\ &= \alpha_1 x(t) + h\alpha_2 f(t+h) \\ &= \alpha_1(a_0 + a_1 t) + h\alpha_2 a_1 && (8.3.5) \end{aligned}$$

- Equating coefficients for a_0 among Eqs. (8.3.4) and (8.3.5):

$$1 = \alpha_1.$$

- Equating coefficients for a_1: $\quad t + h = t + h\alpha_2,$

$$\alpha_2 = 1.$$

- Thus, the first order Gear's method is

$$x(t+h) = x(t) + hf(t+h). \quad (8.3.6)$$

Gear's Integration Methods, 2nd Order

- The first order Gear's integration method is the same as the backward Euler method.
- 2nd order Gear's method

$$\begin{aligned} x(t) =& a_1 + a_1 t + a_2 t^2 \\ f(t) =& a_1 + 2a_2 t \\ x(t+h) =& a_0 + a_1(t+h) + a_2(t+h)^2 && (8.3.7) \\ =& \alpha_1 x(t) + \alpha_2 x(t-h) + h\alpha_3 f(t+h) \\ =& \alpha_1(a_0 + a_1 t + a_2 t^2) + \alpha_2(a_0 + a_1(t-h) + a_2(t-h)^2) \\ & + h\alpha_3(a_1 + 2a_2(t+h)) \\ =& a_0(\alpha_1 + \alpha_2) + a_1(\alpha_1 t + \alpha_2(t-h) + h\alpha_3) \\ & + a_2(\alpha_1 t^2 + \alpha_2(t-h)^2 + 2h\alpha_3(t+h)) \\ =& a_0(\alpha_1 + \alpha_2) + a_1((\alpha_1 + \alpha_2)t + h(-\alpha_2 + \alpha_3)) \\ & + a_2((\alpha_1 + \alpha_2)t^2 + th(-2\alpha_2 + 2\alpha_3) + h^2(\alpha_2 + 2\alpha_3)) && (8.3.8) \end{aligned}$$

Gear's Integration Methods, 2nd Order, II

- Equating coefficients of Eqs. (8.3.7) and (8.3.8)

$$\begin{aligned}\alpha_1 + \alpha_2 &= 1\\ -\alpha_2 + \alpha_3 &= 1\\ \alpha_2 + 2\alpha_3 &= 1\end{aligned}$$

- We have $\alpha_1 = \frac{4}{3}$, $\alpha_2 = \frac{-1}{3}$, $\alpha_3 = \frac{2}{3}$.
- Thus, the 2nd order Gear's integration method is

$$x(t+h) = \frac{4}{3}x(t) - \frac{1}{3}x(t-h) + \frac{2}{3}hf(t+h). \tag{8.3.9}$$

- Note that it can also be formulated as

$$x(t+h) = x(t) + \frac{1}{3}(x(t) - x(t-h)) + \frac{2}{3}hf(t+h).$$

Gear's Integration Methods, 2nd Order LTE

- To find the LTE for the 2nd order integration method, consider $x(t)$ with t^3 term

$$\begin{aligned} x(t) =& a_0 + a_1 t + a_2 t^2 + a_3 t^3 \\ x(t+h) =& a_0 + a_1(t+h) + a_2(t+h)^2 + a_3(t+h)^3 \qquad (8.3.10)\\ =&(4x(t) - x(t-h) + 2hf(t+h))/3 \\ =&\Big(4a_0 + 4a_1 t + 4a_2 t^2 + 4a_3 t^3 - a_0 - a_1(t-h) - a_2(t-h)^2 - a_3(t-h)^3 \\ &+ h\Big(2a_2 + 4a_2(t+h) + 6a_3(t+h)^2\Big)\Big)/3 \qquad (8.3.11)\end{aligned}$$

- Consider the coefficients for a_3
  In (8.3.10): $(t+h)^3 = t^3 + 3t^2h + 3th^2 + h^3$
  In (8.3.11): $(4t^3 - (t-h)^3 + 6h(t+h)^2)/3 = (3t^3 + 9t^2h + 9th^2 + 7h^3)/3$
- LTE: $\frac{4h^3}{3}a_3 = \frac{4h^3}{3}\frac{x'''}{6}$
- Compare to the LTE of trapezoidal rule: $\frac{h^3}{2}a_3$
  - 2nd order Gear's method is not as accurate.

3rd Order Gear's Method

- 3rd order Gear's method

$$
\begin{aligned}
x =& a_0 + a_1 t + a_2 t^2 + a_3 t^3 \\
f(t) =& a_1 + 2a_2 t + 3a_3 t^2 \\
x(t+h) =& a_0 + a_1(t+h) + a_2(t+h)^2 + a_3(t+h)^3 \qquad (8.3.12)\\
=& \alpha_1 x(t) + \alpha_2 x(t-h) + \alpha_3 x(t-2h) + h\alpha_4 f(t+h) \\
=& \alpha_1(a_0 + a_1 t + a_2 t^2 + a_3 t^3) + \alpha_2(a_0 + a_1(t-h) + a_2(t-h)^2 + a_3(t-h)^3) \\
& + \alpha_3(a_0 + a_1(t-2h) + a_2(t-2h)^2 + a_3(t-2h)^3) \\
& + h\alpha_4(a_1 + 2a_2(t+h) + 3a_3(t+h)^2) \qquad (8.3.13)
\end{aligned}
$$

- To equate Eqs. (8.3.12) and (8.3.13)

$$
\begin{aligned}
a_0 :\quad & 1 = \alpha_1 + \alpha_2 + \alpha_3 \\
a_1 :\quad & t + h = (\alpha_1 + \alpha_2 + \alpha_3)t + (-\alpha_2 - 2\alpha_3 + \alpha_4)h \\
a_2 :\quad & t^2 + 2th + h^2 = (\alpha_1 + \alpha_2 + \alpha_3)t^2 - (-2\alpha_2 - 4\alpha_3 + 2\alpha_4)th \\
& \quad + (\alpha_2 + 4\alpha_3 + 2\alpha_4)h^2 \\
a_3 :\quad & t^3 + 3t^2 h + 3th^2 + h^3 = (\alpha_1 + \alpha_2 + \alpha_3)t^3 + (-3\alpha_2 - 6\alpha_3 + 3\alpha_4)t^2 h \\
& \quad + (3\alpha_2 + 12\alpha_3 + 6\alpha_4)th^2 + (-\alpha_2 - 8\alpha_3 + 3\alpha_4)h^3
\end{aligned}
$$

3rd Order Gear's Method, II

- We have

$$
\begin{aligned}
\alpha_1 + \alpha_2 + \alpha_3 &= 1, \\
-\alpha_2 - 2\alpha_3 + \alpha_4 &= 1, \\
\alpha_2 + 4\alpha_3 + 2\alpha_4 &= 1, \\
-\alpha_2 - 8\alpha_3 + 3\alpha_4 &= 1.
\end{aligned}
$$

- Or in matrix form

$$
\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 4 & 2 \\ 0 & -1 & -8 & 3 \end{bmatrix}
\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

$$
\alpha_1 = \frac{18}{11}, \quad \alpha_2 = \frac{-9}{11}, \quad \alpha_3 = \frac{2}{11}, \quad \alpha_4 = \frac{6}{11}.
$$

- Thus, the 3rd order Gear's integration method is

$$
x(t+h) = \frac{18}{11}x(t) - \frac{9}{11}x(t-h) + \frac{2}{11}x(t-2h) + \frac{6}{11}h \cdot f(t+h). \qquad (8.3.14)
$$

k'th Order Gear's Method

- The k'th order Gear's method is

$$x(t+h) = \alpha_1 x(t) + \alpha_2 x(t-h) + \alpha_3 x(t-2h) + \cdots + \alpha_k x(t-kh+h) + \alpha_{k+1} f(t+h) \tag{8.3.15}$$

- The coefficients can be found by solving the following equation.

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & -1 & -2 & \cdots & -k+1 & 1 \\ 0 & 1 & 4 & \cdots & (k-1)^2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (-1)^k & (-2)^k & \cdots & (-k+1)^k & k \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{k+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{8.3.16}$$

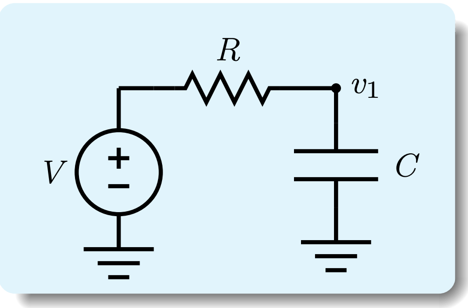
- And the local truncation errors for k'th order Gear's method is

$$\begin{aligned} LTE =& \Big((-1)^{k+1}\alpha_2 + (-2)^{k+1}\alpha_3 + \cdots + (-k+1)^{k+1}\alpha_k \\ &+ (k+1)\alpha_{k+1} - 1 \Big) a_{k+1} h^{k+1} \\ =& \left(\sum_{i=2}^{k} (-i+1)^{k+1}\alpha_i + (k+1)\alpha_{k+1} - 1 \right) a_{k+1} h^{k+1} \end{aligned} \tag{8.3.17}$$

  - Note. This equation can be regarded as an extra row of Eq. (8.3.16).

Gear's Method Example

- Application of 2nd order Gear's method.



$V(t) = 1, \quad t \geq 0,$

$v_1(0) = 0.$

Analytical solution:

$v_1(t) = 1 - \exp(\frac{-t}{RC})$

- Let $x = v_1$, then

$$\frac{dx}{dt} = f(x, t)$$

$$f = \frac{V - x}{RC}$$

- Applying 2nd order Gear's method

$$\begin{aligned} x(t+h) =& \frac{4}{3}x(t) - \frac{1}{3}x(t-h) + \frac{2h}{3}f(t+h) \\ =& \frac{4}{3}x(t) - \frac{1}{3}x(t-h) - \frac{2h}{3RC}(V - x(t+h)) \end{aligned}$$

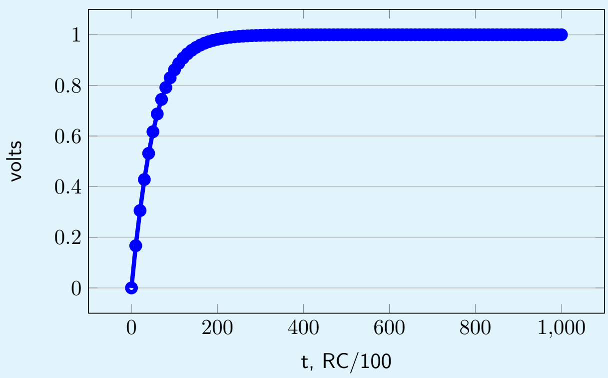
Let $y = \frac{h}{RC}$

$$(1 + \frac{2y}{3})x(t+h) = \frac{4}{3}x(t) - \frac{1}{3}x(t-h) + \frac{2y}{3}V$$

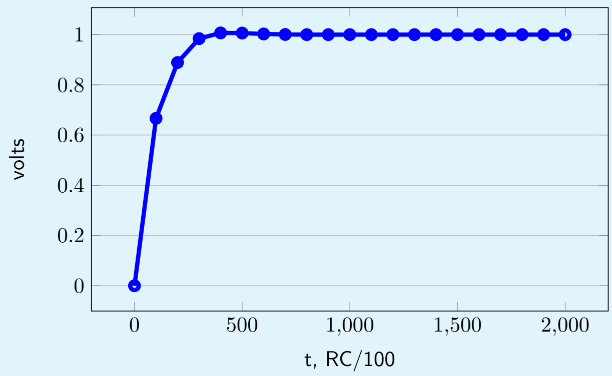
As $y \to \infty$, $x(t+h) \to V$, and the 2nd order Gear's method is stable.

Gear 2 Method vs. Step Size

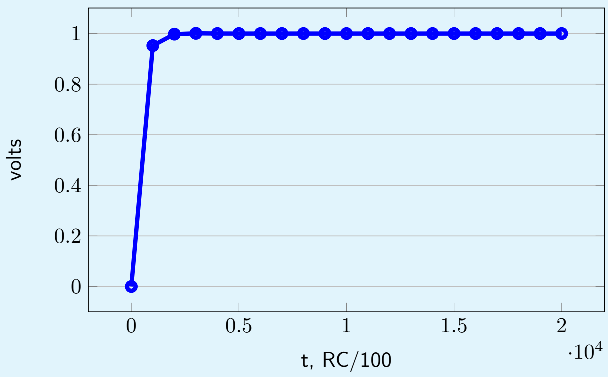
2nd order Gear's method, $h = RC/5$



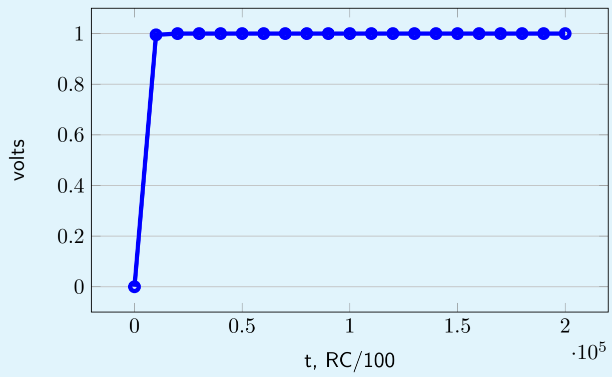
2nd order Gear's method, $h = 2RC$



2nd order Gear's method, $h = 20RC$

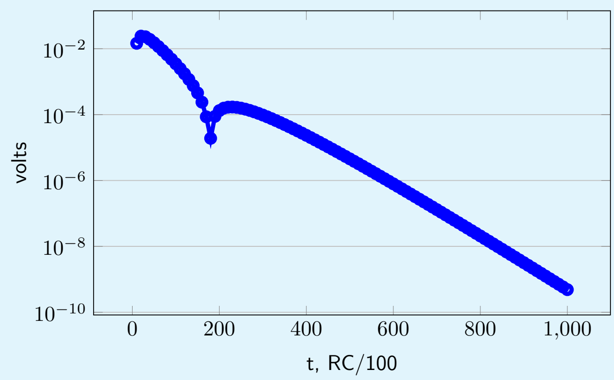


2nd order Gear's method, $h = 200RC$

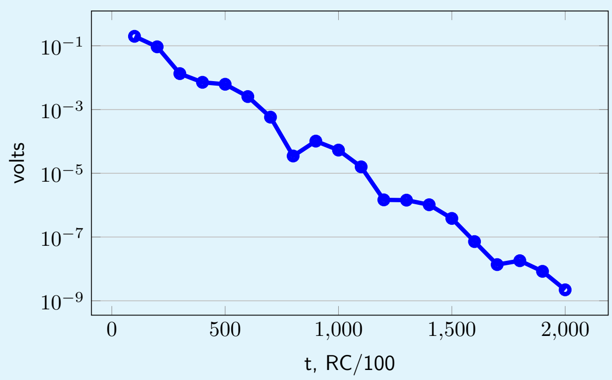


Error of Gear 2 Method

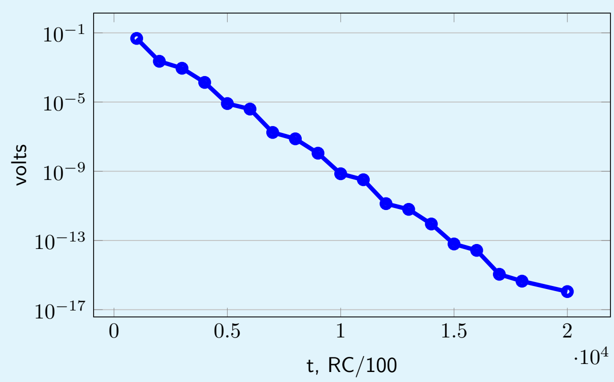
2nd order Gear's method Error, $h = RC/5$



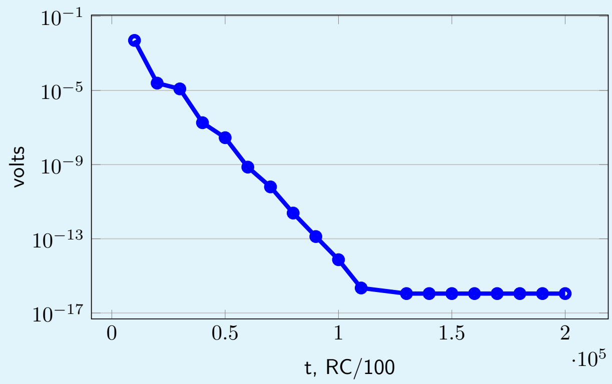
2nd order Gear's method Error, $h = 2RC$



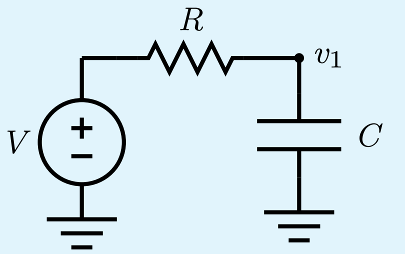
2nd order Gear's method error, $h = 20RC$



2nd order Gear's method error, $h = 200RC$



Gear 3 Method Example



$$V(t) = 1, \quad t \geq 0,$$
$$v_1(0) = 0.$$

Analytical solution:
$$v_1(t) = 1 - \exp(\frac{-t}{RC})$$

- Let $x = v_1$, then

$$\frac{dx}{dt} = f(x, t)$$
$$f = \frac{V - x}{RC}$$

- Applying 3rd order Gear's method

$$x(t+h) = \frac{18}{11}x(t) - \frac{9}{11}x(t-h) + \frac{2}{11}x(t-2h) + \frac{6h}{11}f(t+h)$$

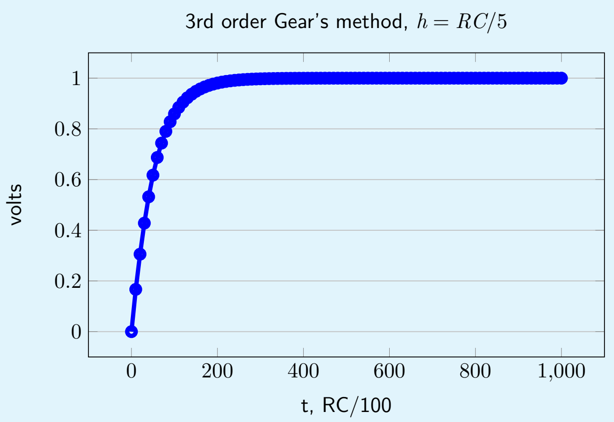
Let $y = \frac{h}{RC}$

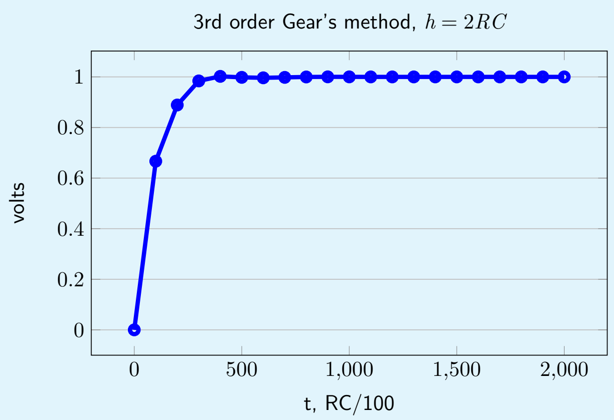
$$(1 + \frac{6y}{11})x(t+h) = \frac{18}{11}x(t) - \frac{9}{11}x(t-h) + \frac{2}{11}x(t-2h) + \frac{6y}{11}V$$

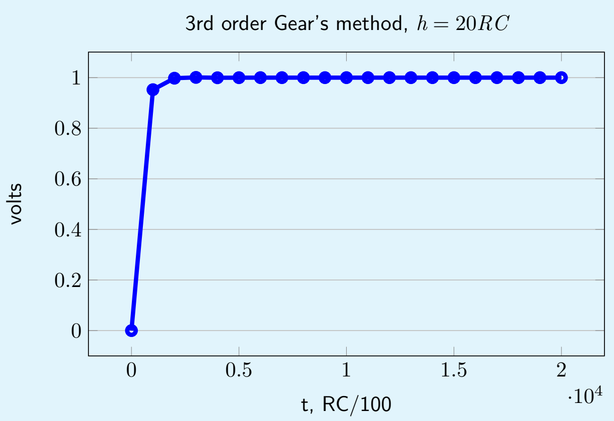
As $y \to \infty$, $x(t+h) \to V$, and the 3rd order Gear's method is stable.

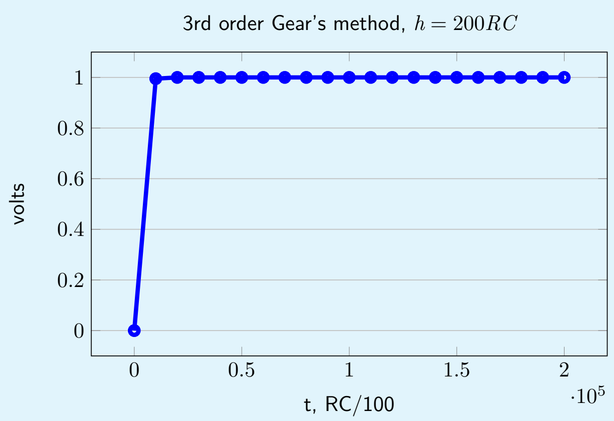
- All Gear's methods are stable.

Gear 3 Method vs. Step Size

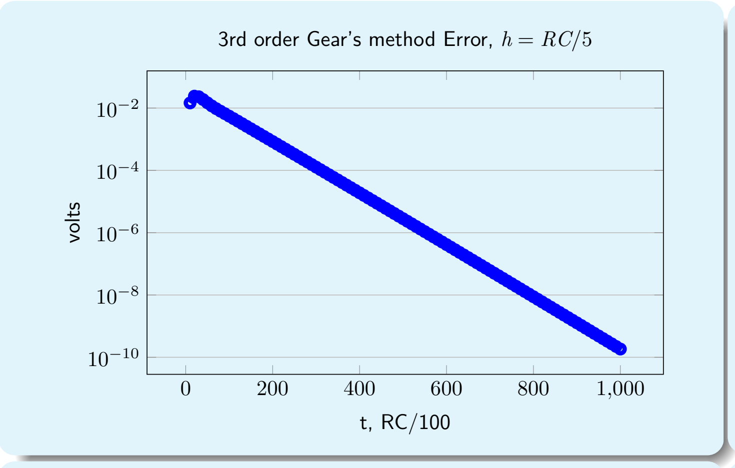


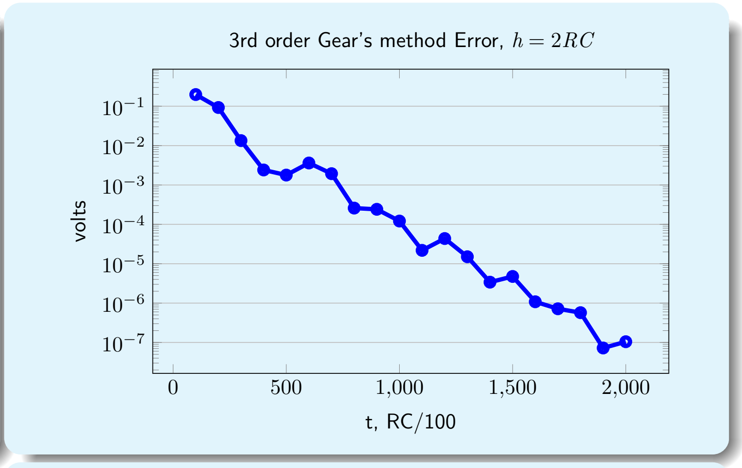


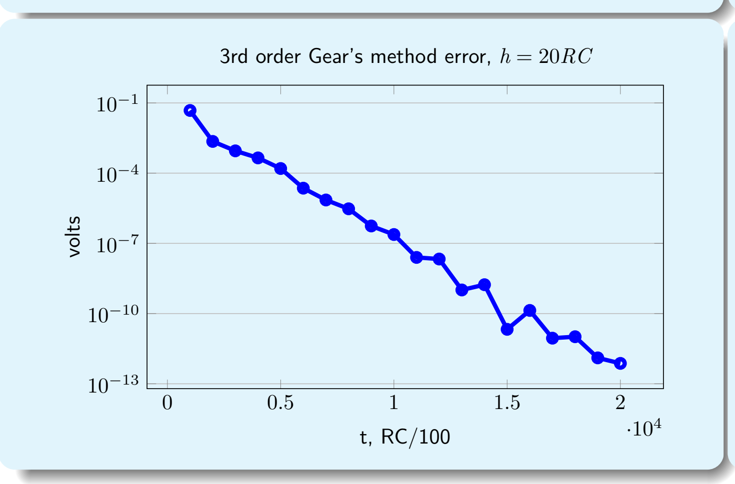


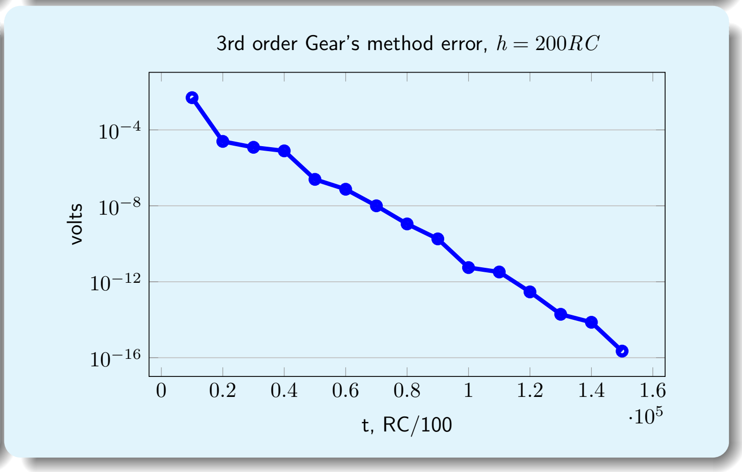


Error of Gear 3 Method

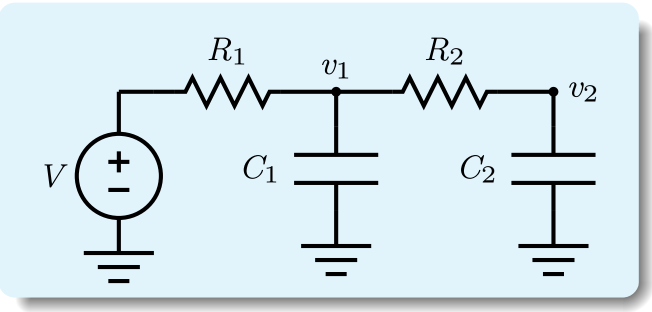








Solving Stiff Differential Equations Using Gear 2



$$
\begin{aligned}
V(t) &= 1, & t &\geq 0, \\
v_1(0) &= 0, & v_2(0) &= 0, \\
C_1 &= 1\text{pF}, & R_1 &= 50\Omega, \\
C_2 &= 1\text{pF}, & R_2 &= 50\text{K}\Omega.
\end{aligned}
$$

- System equation of the circuit

$$
\begin{aligned}
\frac{dv_1}{dt} &= (-\frac{1}{R_1C_1} - \frac{1}{R_2C_1})v_1 + \frac{v_2}{R_2C_1} + \frac{V}{R_1C_1} \\
\frac{dv_2}{dt} &= \frac{v_1}{R_2C_2} - \frac{v_2}{R_2C_2}
\end{aligned}
$$

- With Gear-2 method

$$
\begin{aligned}
v_1(t+h) =& \frac{4}{3}v_1(t) - \frac{1}{3}v_1(t-h) + \frac{2h}{3}\Bigg((-\frac{1}{R_1C_1} - \frac{1}{R_2C_1})v_1(t+h) + \frac{v_2(t+h)}{R_2C_1} \\
&+ \frac{V}{R_1C_1}\Bigg) \\
v_2(t+h) =& \frac{4}{3}v_2(t) - \frac{1}{3}v_2(t-h) + \frac{2h}{3}\left(\frac{v_1(t+h)}{R_2C_2} - \frac{v_2(t+h)}{R_2C_2}\right)
\end{aligned}
$$

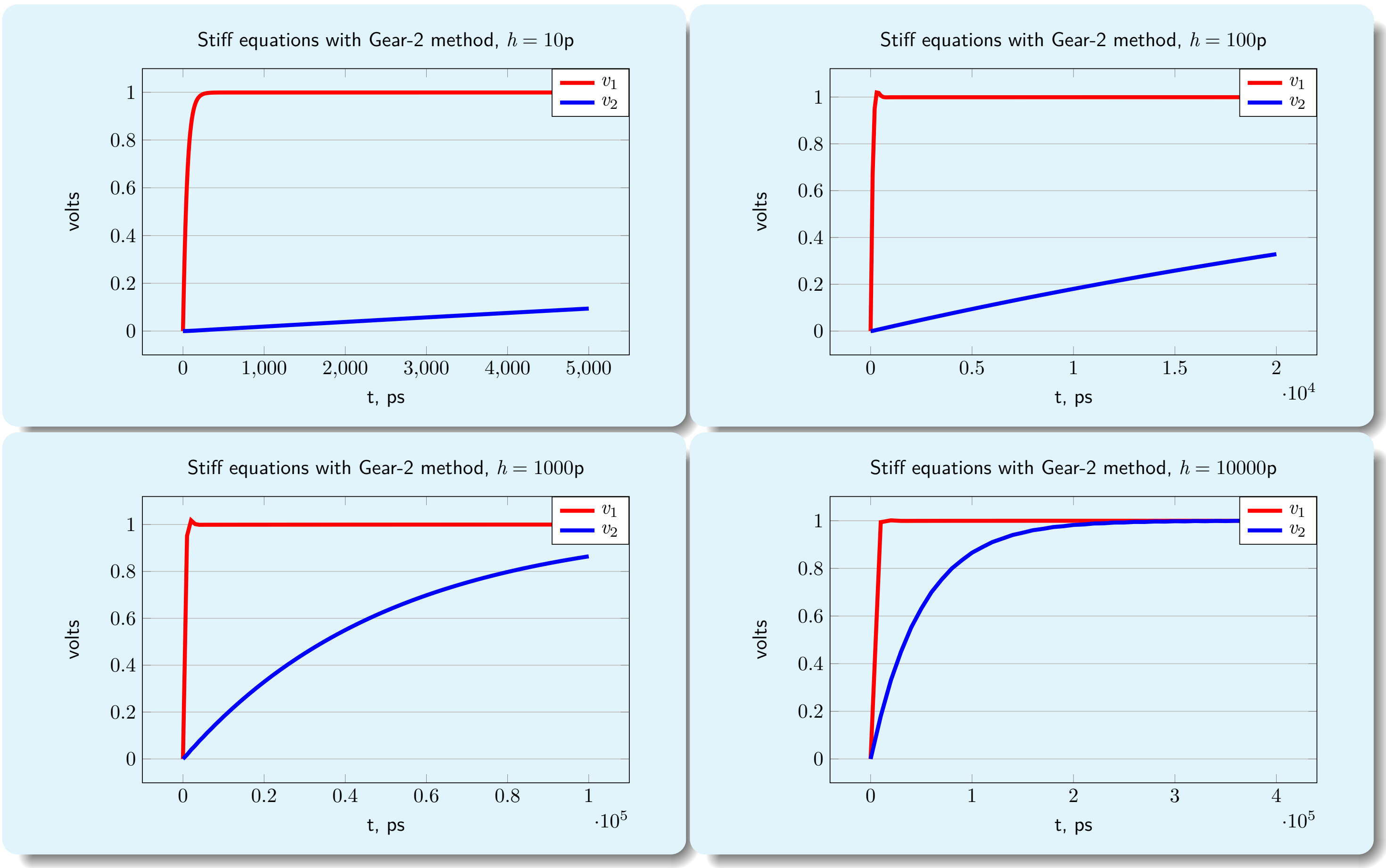
Solving Stiff Differential Equations Using Gear 2, II

- Let $y_1 = \dfrac{h}{R_1 C_1}$, $y_2 = \dfrac{h}{R_2 C_2}$, $y_3 = \dfrac{h}{R_2} C_1$, then in matrix from

$$\begin{bmatrix} 1 + 2y_1/3 + 2y_3/3 & -2y_3/3 \\ -2y_2/3 & 1 + 2y_2/3 \end{bmatrix} \begin{bmatrix} v_1(t+h) \\ v_2(t+h) \end{bmatrix} = \begin{bmatrix} 4v_1(t)/3 - v_1(t-h)/3 + 2y_1 V \\ 4v_2(t)/3 - v_2(t-h)/3 \end{bmatrix}$$

- This linear system can be solved for $v_1(t+h)$ and $v_2(t+h)$ Given $v_1(t)$, $v_2(t)$, $v_1(t-h)$ and $v_1(t-h)$.
- The first time cannot be solved using Gear-2 method
  - Usually, backward Euler is applied instead
- As seen from the following figures, the 2nd order Gear's method can be applied to get accurate solutions even with large time steps.

Solving Stiff Differential Equations Using Gear 2, III



Capacitance Stamps with Gear-2 Method

- 2nd order Gear's method is

$$x(t+h) = \frac{4}{3}x(t) - \frac{1}{3}x(t-h) + \frac{2h}{3}f(t+h)$$

- In capacitor case

$$\begin{aligned}
C\frac{dV_C}{dt} &= I_C \\
x(t) &= V_C(t) \\
f(t) &= I_C(t) \\
V_C(t+h) &= \frac{4}{3}V_C(t) - \frac{1}{3}V_C(t) + \frac{2h}{3}I_C(t+h) \\
I_C(t+h) &= \frac{3C}{2h}V_C(t+h) - \frac{C}{2h}(4V_C(t) - V_C(t-h))
\end{aligned}$$

- Capacitance stamps with Gear-2 method

$$\begin{bmatrix} 3C/2h & -3C/2h \\ -3C/2h & 3C/2h \end{bmatrix} \begin{bmatrix} V_P(t+h) \\ V_N(t+h) \end{bmatrix} = \begin{bmatrix} 2C/h(4V_C(t) - V_C(t-h)) \\ -2C/h(4V_C(t) - V_C(t-h)) \end{bmatrix}$$

where the capacitor is assumed to connect nodes V_P and V_N, and $V_C(t) = V_P(t) - V_N(t)$.

Summary

- ODE solution methods with large step sizes are explored
- Trapezoidal method is accurate with small time steps but unstable for large step sizes.
- Higher order backward integration methods are unstable with large step sizes.
- Still systems can be common in real life applications.
- Gear's integration methods are developed for stable solution with large step sizes.
  - Can be applied to stiff systems.
  - Large step sizes can be explored for better accuracy and efficiency trade off.