Unit 7.2 Roots of Polynomials

Numerical Analysis

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Polynomials

- Polynomials of degree greater than one are nonlinear functions.
- The solution methods described in the previous section can be applied in finding roots of polynomials.
- A polynomial of degree n is usually written as

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \tag{7.2.1}$$

$$= \sum_{k=0}^{n} a_k x^k. \tag{7.2.2}$$

- In this course we assume all coefficients, a_k, are real.
  - But, the roots are not necessarily real.
  - Since all a_k are real, if a complex number z is a root to $P_n(x)$ then so is its complex conjugate $\bar{z}$.
- Evaluation of $P_n(x)$ can be done more efficiently by the following

$$P_n(x) = a_0 + x(a_1 + x(a_2 + \cdots + x(a_{n-1} + xa_n))). \tag{7.2.3}$$

- n multiplications and n additions are needed to evaluate $P_n(x)$.
- Derivative of $P_n(x)$ is

$$P'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1}. \tag{7.2.4}$$

Roots of Polynomials

- Some useful theorems for roots of polynomials.

Theorem 7.2.1. Descartes' rule of signs.

Given a polynomial $P_n(x)$ of degree n, let ν be the number of sign changes in the set of coefficients $\{a_j\}$ and k be the number of positive roots (counting with its multiplicity), then $k \le \nu$ and $\nu - k$ is an even number.

Theorem 7.2.2. Cauchy's Theorem.

All zeros of $P_n(x)$ are contained in the circle Γ in the complex plane

$$\Gamma = \{z \in \mathbb{C} : |z| \le 1 + \eta_k\}, \qquad \eta_k = \max_{0\le k\le n-1} \left|\frac{a_k}{a_n}\right|. \tag{7.2.5}$$

Theorem 7.2.3.

Let $P_n(x)$, $n \ge 2$, be a polynomial of degree n with real coefficients. If all roots, z_i, are real and

$$z_1 \ge z_2 \ge \cdots \ge z_n,$$

then Newton's method yields a strictly decreasing sequence $x^{(k)}$ converging to z_1 for any initial guess $x^{(0)} > z_1$.

Roots of Polynomials, II

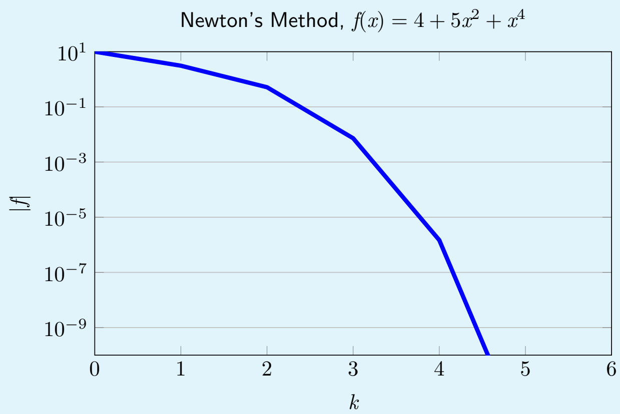
Theorem 7.2.4.

Let $P_n(x)$, $n \ge 2$, be a polynomial of degree n with real coefficients and all roots are real. Assuming $a_n > 0$ and z_1 is the largest root of $P_n'(x)$, then $P_n'''(x) \ge 0$ for $x \ge z_1$, i.e., P_n'(x) is a convex function for $x \ge z_1$.

- From the above, with a real $P_n(x)$, $n \ge 2$, assuming all roots are real, then one is able to use Newton's method to find the largest root z_1.
- If $P_n(x)$ has all real coefficients, then Newton's method is able to find a complex root if complex operations are employed in Newton's method.

Newton's method applied on $f(x) = 4 + 5x^2 + x^4$.

Iteration	x	$f(x)$
0	$1+i$	10
1	$0.4512+0.9390i$	3.1076
2	$0.0819+0.9763i$	0.5160
3	$0.0012+1.0004i$	0.0073
4	$-1.3969\text{e-}07+i$	1.478e-06
5	$9.4506\text{e-}15+i$	6.089e-14



Finding the First Root

- To find the first root of $P_n(x)$, let $x^{(0)}$ be an initial guess. Note that Eq. (7.2.3) can be rewritten as

$$P_n(x) = (((a_n x + a_{n-1})x + a_{n-2})x + \cdots)x + a_0 \tag{7.2.6}$$

And the first derivative, function Eq. (7.2.4), as

$$P_n'(x) = (((n\ a_n x + (n-1)a_{n-1})x + (n-2)a_{n-2})x + \cdots)x + a_1 \tag{7.2.7}$$

- Then Newton's iteration is

$$x^{(k+1)} = x^{(k)} - \frac{P_n(x^{(k)})}{P_n'(x^{(k)})} \tag{7.2.8}$$

- Note that for polynomial functions, the Newton's method is robust for most initial guesses with $P'(x^{(0)}) > 0$.
- Each iteration requires $2n$ multiplications, 1 division, $2n$ additions and 1 subtraction.
- If these operations are carried out using complex numbers then complex roots can also be found.
- Newton's iteration has the convergence rate of order 2.

Finding All roots

- Once a root, z_1, of $P_n(x)$ has been found, we can rewrite $P_n(x)$ as

$$P_n(x) = (x - z_1)P_{n-1}(x) \tag{7.2.9}$$

- Then to proceed to using Newton's method on $P_{n-1}(x)$ to find the rest of the roots.
- This process is called deflation.
- Note that

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

- Let

$$P_{n-1}(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0 \tag{7.2.10}$$

From Eq. (7.2.9), we have

$$\begin{aligned} P_n(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\ &= (x - z_1)(b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0) \end{aligned}$$

And

$$\begin{aligned} a_n &= b_{n-1}, \\ a_{n-1} &= b_{n-2} - b_{n-1}z_1, \\ a_{n-2} &= b_{n-3} - b_{n-2}z_1, \\ &\cdots \\ a_1 &= b_0 - b_1 z_1 \\ a_0 &= -b_0 z_1. \end{aligned}$$

$$\begin{aligned} b_{n-1} &= a_n, \\ b_{n-2} &= a_{n-1} + b_{n-1}z_1, \\ b_{n-3} &= a_{n-2} + b_{n-2}z_1, \\ &\cdots \\ b_0 &= a_1 + b_1 z_1. \end{aligned}$$

Finding All roots, II

- Thus, the coefficients of the deflated polynomial can be calculated using the recursive formulas

$$b_{n-1} = a_n, \tag{7.2.11}$$

$$b_j = a_{j+1} + z_1 b_{j+1}, \qquad j = n-2, \cdots, 0. \tag{7.2.12}$$

- Once the deflated polynomial $P_{n-1}(x)$ is found, then Newton's algorithm can be applied to find the next root.
- Note also that if we define

$$\begin{aligned} b_{-1} &= a_0 + z_1 b_0 \\ &= a_0 + z_1(a_1 + z_1 b_1) \\ &= a_0 + z_1(a_1 + z_1(a_2 + z_1 b_2)) \\ &= a_0 + z_1(a_1 + z_1(a_2 + z_1(a_3 + z_1(\cdots z_1(a_{n-1} + z_1 a_n))))) \\ &= P_n(z_1) \end{aligned} \tag{7.2.13}$$

  And when z_1 is a root of $P_n(x)$ then $b_{-1} = 0$.
- Thus, the value of the polynomial $P_n(x)$ can be calculated when deflation process in on going.

Finding All roots, III

- We need the derivative, $P'(x)$, when using Newton's method to find a root.
- Note that in Eq. (7.2.9) we define

$$P_n(x) = (x - z_1)P_{n-1}(x)$$

  Then

$$\begin{aligned} \frac{\mathrm{d}P_n(x)}{\mathrm{d}x} &= P_{n-1}(x) + (x - z_1)\frac{\mathrm{d}P_{n-1}(x)}{\mathrm{d}x} \\ P_n'(z_1) &= P_{n-1}(z_1) \\ &= b_{n-1}z_1^{n-1} + b_{n-2}z_1^{n-2} + \cdots + b_1 z_1 + b_0 \end{aligned} \tag{7.2.14}$$

- Therefore, the derivative of $P_n(z_1)$ can be obtained by evaluating the polynomial $P_{n-1}(z_1)$.
- Since $P_{n-1}(x)$ is also a polynomial (of degree $n-1$), the same deflation process to get the coefficients b_j's can be carried out to find its value.
  - Thus, another deflation process is carried out for $P_{n-2}(x)$ to get $c_{n-2}, c_{n-3}, \ldots, c_0, c_{-1}$, where c_{-1} is the value of the derivative.

Example

- Example: $f(x) = x^3 - 6x^2 + 11x - 6$
- All roots are located in the circles: $|z| \le 1 + \eta_k$, $\eta_k = \max\{6, 11, 6\} = 11$.
- Thus to find the largest root, we can start from $z_1^{(0)} = 12$.
- The first 2 iterations to find the largest root

	$z_1^{(0)} = 12$		$z_1^{(1)} = 8.689$	
a_i	b_i	c_i	b_i	c_i
1				
-6	1		1	
11	6	1	2.689	1
-6	83	18	34.364	11.378
	990	299	292.590	133.227
		$z_1^{(1)} = 8.689$		$z_1^{(2)} = 6.493$

- Note that the updated solution is

$$z_1^{(k+1)} = z_1^{(k)} - b_{-1}/c_{-1}$$

Finding All roots, IV

Algorithm 7.2.5. Polynomial Roots.

Given an n-degree polynomial with coefficients $a_0, a_1, \cdots, a_n$, an initial guess $x^{(0)}$, a small number ϵ and an integer $maxiter$,

while $(n \ge 1)\{$

$\quad err = 1 + \epsilon; \quad k = 0;$

$\quad$ while $((err >= \epsilon)$ and $(k < maxiter))\{$

$\quad\quad b_{n-1} = a_n; \quad c_{n-2} = b_{n-1};$

$\quad\quad$ for $(j = n-2; j >= -1; j = j-1) \quad b_j = a_{j+1} + x^{(k)} b_{j+1};$

$\quad\quad$ for $(j = n-3; j >= -1; j = j-1) \quad c_j = b_{j+1} + x^{(k)} c_{j+1};$

$\quad\quad f = b_{-1}; \quad f' = c_{-1};$

$\quad\quad x^{(k+1)} = x^{(k)} - \dfrac{f}{f'};$

$\quad\quad err = |f|; \quad k = k + 1;$

$\quad \}$

$\quad z_n = x^{(k)};$

$\quad$ for $(j = 0; j < n; j = j+1) \quad a_j = b_j;$

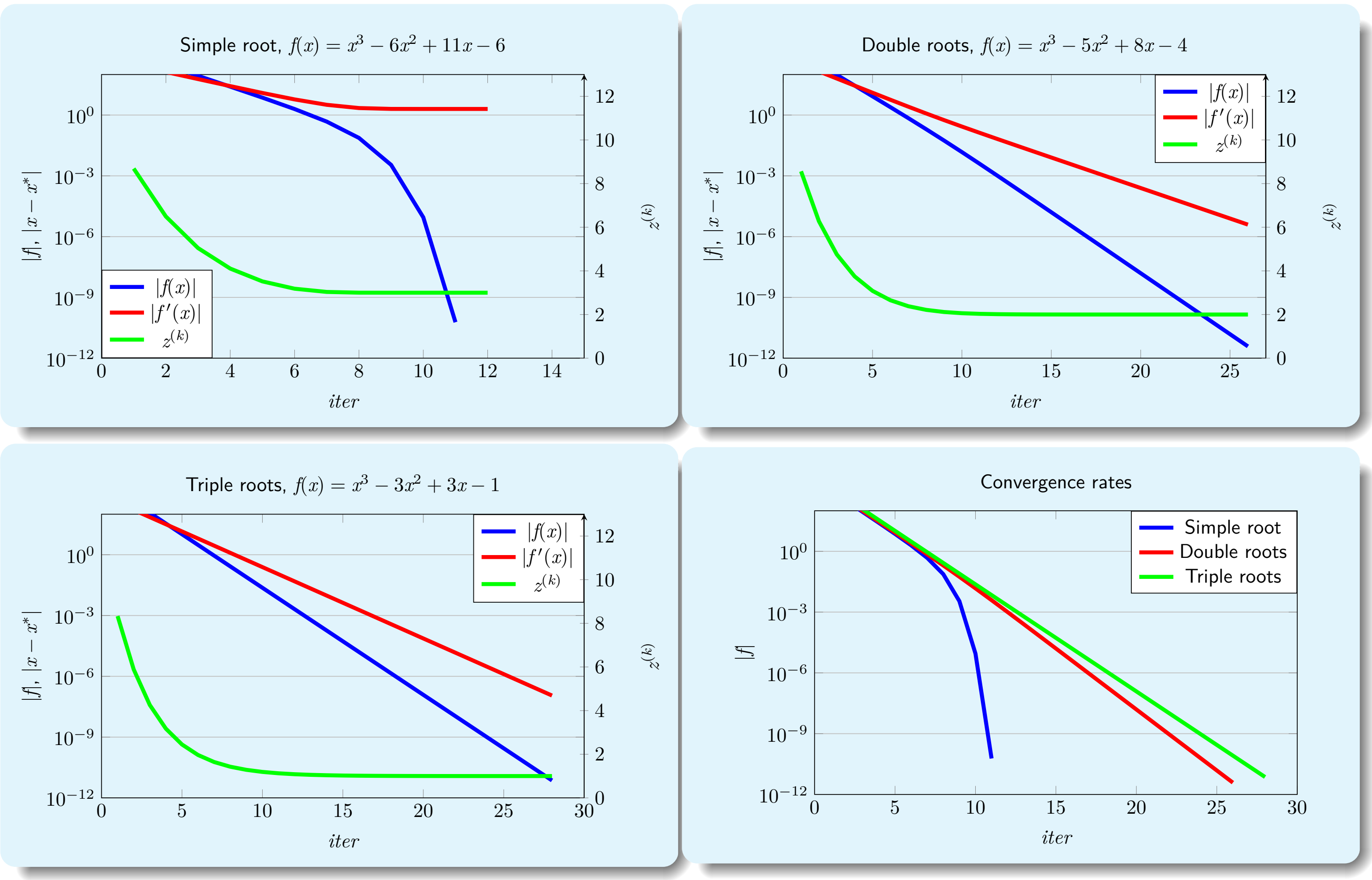
$\quad x^{(0)} = z_n; \quad n = n - 1;$

$\}$

Finding All roots, V

- The preceding algorithm finds all roots of a polynomial of degree n, assuming
  - The coefficients of the polynomial are $a_0, a_1, \cdots, a_n$,
  - Initial guess is given to be $x^{(0)}$,
  - All roots are real
- The roots found are $z_1, z_2, \cdots, z_n$.
- In the algorithm, the polynomial deflation process are repeated twice
  - From a_j to b_j to find $P_n(x^{(k)})$,
  - From b_j to c_j to find $P_n'(x^{(k)})$.
- The inner while loop is simply the Newton's method
- Once a root is found, deflated coefficients, b_j, are copied to a_j and the degree is reduced by 1, then the Newton's method is repeated to find the next root with the initial guess of z_n.
- This algorithm works well if all roots are simple (multiplicity of 1)
  - For any root, z_j, with multiplicity greater than one then $f'(z_j) \to 0$ and Newton's method can be slow or the solution accuracy is low
- This algorithm can find complex roots if it is implemented with complex arithmetic operations

Multiple Roots Example



Quadratic Method

- The Newton-Horner deflation process is convergent and can find all roots of a polynomial.
- If the polynomial has double roots, then the deflation process can be slow – linear convergence.
- To find complex roots, the algorithm needs to be implemented using complex arithmetic.
- A different approach, Lin's quadratic method, can be applied to get complex conjugate solutions without using complex arithmetic.
- Given the polynomial as before

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0. \tag{7.2.15}$$

- We assume $P_n(x)$ can be factorized as

$$P_n(x) = (x^2 + px + q)(b_{n-2} x^{n-2} + b_{n-3} x^{n-3} + \cdots + b_1 x + b_0) + Rx + S. \tag{7.2.16}$$

In case that $x^2 + px + q$ is a factor of $P(x)$, then

$$R = 0, \qquad S = 0. \tag{7.2.17}$$

Quadratic Method, II

- By equating the coefficients of the same power of x in Eqs. (7.2.15) and (7.2.16), we have

$$\begin{aligned} a_n &= b_{n-2} & b_{n-2} &= a_n \\ a_{n-1} &= b_{n-3} + p b_{n-2} & b_{n-3} &= a_{n-1} - p b_{n-2} \\ a_{n-2} &= b_{n-4} + p b_{n-3} + q b_{n-2} & b_{n-4} &= a_{n-2} - p b_{n-3} - q b_{n-2} \\ &\cdots & &\cdots \\ a_2 &= b_0 + p b_1 + q b_2 & b_0 &= a_2 - p b_1 - q b_2 \\ a_1 &= p b_0 + q b_1 + R & R &= a_1 - p b_0 - q b_1 \\ a_0 &= q b_0 + S & S &= a_0 - q b_0 \end{aligned}$$

Or in recursive form b_i can be found by

$$\begin{aligned} b_n &= 0, \\ b_{n-1} &= 0, \\ b_j &= a_{j+2} - p b_{j+1} - q b_{j+2}, \qquad j = n-2, \ldots, 0. \end{aligned} \tag{7.2.18}$$

Quadratic Method, III

- Again, $x^2 + px + q$ is a factor of $P_n(x)$ if and only if

$$R = a_1 - pb_0 - qb_1 = 0 \tag{7.2.19}$$
$$S = a_0 - qb_0 = 0 \tag{7.2.20}$$

- Lin's quadratic method sets

$$q^{(k+1)} = \frac{a_0}{b_0} \tag{7.2.21}$$
$$p^{(k+1)} = \frac{a_1 - qb_1}{b_0} \tag{7.2.22}$$

Or

$$q^{(k+1)} = \frac{a_0 - b_0 q^{(k)}}{b_0} + q^{(k)} \tag{7.2.23}$$
$$p^{(k+1)} = \frac{a_1 - b_0 p^{(k)} + qb_1}{b_0} + p^{(k)} \tag{7.2.24}$$

- This leads to the iterations

$$p^{(k+1)} = p^{(k)} + \frac{R}{b_0} \tag{7.2.25}$$
$$q^{(k+1)} = q^{(k)} + \frac{S}{b_0} \tag{7.2.26}$$

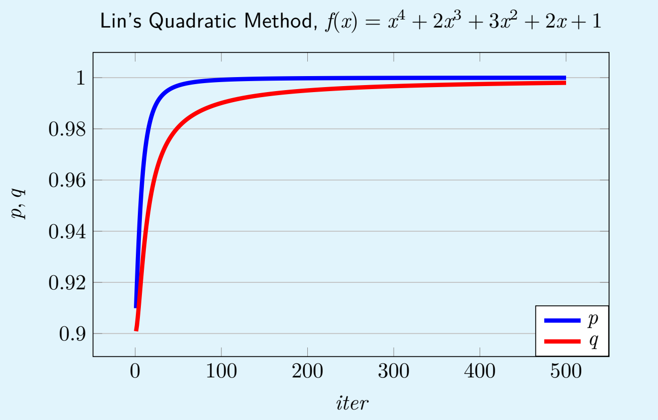
Quadratic Method, IV

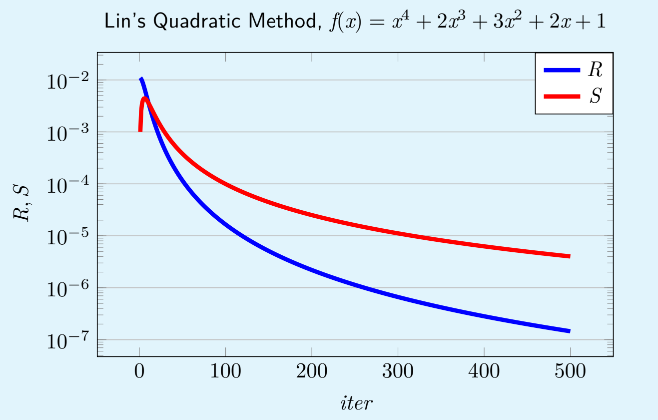
- If Eqs. (7.2.25) and (7.2.26) converge, then $x^2 + px + q$ is a quadratic factor of $P_n(x)$.
- If $p^{(0)} = q^{(0)} = 0$ then

$$p^{(1)} = \frac{a_1}{a_2}, \tag{7.2.27}$$
$$q^{(1)} = \frac{a_0}{a_2}. \tag{7.2.28}$$

- The iterative process using these initial guesses tends to produce the smallest roots of $P_n(x)$ for the quadratic $x^2 + px + q$.
- If one selects the initial guesses: $p^{(0)} = a_{n-1}/a_n$, $q^{(0)} = a_{n-2}/a_n$, then the iterative process tends to find the largest roots of $P_n(x)$ for the quadratic $x^2 + px + q$.
    - The iterative process using these initial guesses is less robust, and more divergence could be observed.
- Once the quadratic factor, $x^2 + px + q$, is found then the real roots or the complex conjugates can be calculated quickly.
- $P_n(x)$ can be deflated again to the $n-2$ polynomial $P_{n-2}(x) = b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + \cdots + b_1 x + b_0$.
- The same process can be carried out on $P_{n-2}(x)$ for the next factor or factors.
- Thus, all the roots for $P_n(x)$ can be found.

Quadratic Method, Example





- Example: factorize

$$P_4(x) = x^4 + 2x^3 + 3x^2 + 2x + 1 = (x^2 + px + q)P_2(x).$$

- Convergence of Lin's quadratic method is observed.
- But, convergence is very slow.
  - This method has the convergence order of 1.

Summary

- Polynomial and evaluating $P_n(x)$
- Location of roots of polynomials
- Finding the first root
- Horner deflation process
- Newton-Horner algorithm
- Lin's quadratic method for double roots or roots of complex conjugates