Appendix for:
Statistical performance of subgradient step-size update rules in Lagrangian relaxations of chance-constrained optimization models

Charlotte Ritter[a], Bismark Singh[b,*]

[a] *Department of Mathematics, Friedrich-Alexander-Universität Erlangen-Nürnberg, Erlangen, 91058, Germany*
[b] *School of Mathematical Sciences, University of Southampton, Southampton, S017 1BJ, UK*

There are two components to this appendix. In Appendix A, we provide a motivating example and algorithms we omitted in the main text to preserve space. In Appendix B, we provide tables reporting all computational results of the main text.

A. Algorithms

Here, we provide pseudocodes for the Lagrangian procedure of Section 2 and the three lower bounding procedures of Section 3. We first provide a motivating example that we refer to in Section 2.

Example S1 demonstrates that, unfortunately, in practice we cannot provide a strong optimality certificate as that given by the theoretical bound in the `BBP`.

Example S1. *Consider the following MIP:*

$$\begin{aligned} \mathcal{Z} = \max_{x,y} \quad & 2x + 3y \\ s.t. \quad & x + y \leq 1 \\ & 3x + y \leq 3 \\ & x, y \in \{0, 1\}. \end{aligned}$$

The optimal solution is $x = 0, y = 1$ with $\mathcal{Z} = 3$. Consider the following Lagrangian dual model:

$$\begin{aligned} \mathcal{Z}_D(\lambda) = \max_{x,y} \quad & 2x + 3y + \lambda(1 - x - y) \\ s.t. \quad & 3x + y \leq 3 \\ & x, y \in \{0, 1\}. \end{aligned}$$

Assume that the branch-and-bound solver terminates with the following feasible solution for the Lagrangian dual model: $x = 1, y = 0$ that provides $\underline{\mathcal{Z}_D}(\lambda) = 2, \forall \lambda \geq 0$. We note that this solution is feasible for the

*Corresponding author
Email address: `b.singh@southampton.ac.uk` (Bismark Singh)

true model. A corresponding upper bound is available from $x = 0, y = 1$ with $\overline{\mathcal{Z}_D}(\lambda) = 3, \forall \lambda \geq 0$. Since $\underline{\mathcal{Z}_D}(\lambda) < \mathcal{Z}$, the feasible solution reported by the solver cannot be used for the `BBP`. *Further, since the precise value of $\mathcal{Z}_D(\lambda) \in [\underline{\mathcal{Z}_D}(\lambda), \overline{\mathcal{Z}_D}(\lambda)]$ is unknown, the* `BBP` *is of no value.* □

Next, we provide the four algorithms. In our computational experiments we use the following setup; see, also, Section 5.1. We solve model (1) naively to an optimality gap of 0.1% using a time limit of 4200 seconds. We solve each iteration of the Lagrangian relaxation of model (3) to an optimality gap of 0.01% using a time limit of 2100 seconds per iteration. We allow at most ten iterations for the Lagrangian relaxation procedure with a total time limit of 4200 seconds. We stop the procedure if the total time after completion of any iteration exceeds 2250 seconds; else, the procedure might exceed the total time limit of 4200 seconds. In this way, our results for the Lagrangian relaxation are conservative.

In Algorithm S1, we use $iter = 10$, $time = 2250$, $\theta = 2$, $\psi = 0.001$ and $\iota = 0.0001$. For the `IR` and `AP` bounds in Algorithm S2 and Algorithm S3, respectively, we use $m = 20$, $\kappa = 0.1$, $time = 1800$, $iter = 30$, $\rho = 0.1$ and $p = 7200$. Here, we solve the SAAs of model (1) to an optimality gap of 4% with a time limit of 2100 seconds, the SAAs of model (6) to an optimality gap of 0.05% with a time limit of 2100 seconds, and the models with the x variables fixed to an optimality gap of 0% with a time limit of 1800 seconds. For the `QP` bound in Algorithm S4, we solve all the subproblems to an optimality gap of 0.01% using a time limit of 2100 seconds; although the iterations in Step 1 of Algorithm S4 are solved very quickly.

Algorithm S1 Lagrangian relaxation procedure of model (1)

Input: $iter$; $\underline{\mathcal{Z}}$; $time$; θ; ψ; ι; step-size rule r.

Output: $\mathcal{Z}_D$; δ.

Solve LP relaxation of model (1); $\mathcal{Z}_D \leftarrow$ optimal objective function value; $\lambda \leftarrow$ optimal dual of constraint (2b).
$k \leftarrow 1$.
while $k \leq iter$ **do**
    Solve model (3); $\mathcal{Z}_D \leftarrow \min\{\mathcal{Z}_D, \overline{\mathcal{Z}_D}(\lambda)\}$; $\gamma \leftarrow \varepsilon - \sum_{\omega \in \Omega} p^\omega z^\omega$ with optimal z^ω.
    If no change in $\mathcal{Z}_D$ between current and previous two iterations, $\theta \leftarrow \frac{\theta}{2}$.
    Update Δ according to step-size rule r.
    $\lambda_{\text{new}} \leftarrow \max\{0, \lambda - \Delta\gamma\}$; $\zeta \leftarrow |\lambda_{\text{new}} - \lambda|$; $\lambda \leftarrow \lambda_{\text{new}}$.
    $\delta \leftarrow \frac{\mathcal{Z}_D - \underline{\mathcal{Z}}}{\mathcal{Z}_D}$.
    If $\delta \leq \psi$ or time $\geq time$ or $\zeta < \iota$, *STOP*.
    $k \leftarrow k + 1$; update time to the cumulative wall-clock time.

Algorithm S2 Iterative regularization bound of [3] for model (1)

Input: m; κ; *time*; *iter*; ρ; Oracle to generate independent scenarios for model (1); instance of model (1) with $|\Omega|$ scenarios.

Output: $\underline{\mathcal{Z}}$.

$\underline{\mathcal{Z}} \leftarrow 0$; $k \leftarrow 1$.
while time $\leq$ *time* and $k \leq$ *iter* **do**
  if $k = 1$ **then**
    Generate m independent scenarios from Oracle; solve SAA of model (1) with these m scenarios; $\hat{x} \leftarrow$ optimal x.
  else
    Generate m independent scenarios from Oracle; solve SAA of model (6) with these m scenarios; $\hat{x} \leftarrow$ optimal x.
  Solve input instance of model (1) with x fixed to $\hat{x}$; $z_m \leftarrow$ objective function value.
  $\underline{\mathcal{Z}} \leftarrow \max\{z_m, \underline{\mathcal{Z}}\}$.
  $m \leftarrow \lceil (1+\kappa)m \rceil$.
  $k \leftarrow k + 1$; update time to wall-clock time.

Algorithm S3 Aggregation bound for model (1)

Input: m; κ; *time*; *iter*; ρ; p; Oracle to generate independent scenarios for model (1); procedure `cluster`(a, b) that aggregates set of a scenarios into set of b scenarios and their respective weights, $a > b$; instance of model (1) with $|\Omega|$ scenarios.

Output: $\underline{\mathcal{Z}}$.

Generate $p >> |\Omega|$ independent scenarios from Oracle.
$\underline{\mathcal{Z}} \leftarrow 0$; $k \leftarrow 1$.
while time $\leq$ *time* and $k \leq$ *iter* **do**
  if k=1 **then**
    Use `cluster`(p, m); solve SAA of model (1) with these m scenarios and their respective weights; $\hat{x} \leftarrow$ optimal x.
  else
    Use `cluster`(p, m); solve SAA of model (6) with these m scenarios and their respective weights; $\hat{x} \leftarrow$ optimal x.
  Solve input instance of model (1) with x fixed to $\hat{x}$; let z_m denote the objective function value.
  $\underline{\mathcal{Z}} \leftarrow \max\{z_m, \underline{\mathcal{Z}}\}$.
  $m \leftarrow \lceil (1+\kappa)m \rceil$.
  $k \leftarrow k + 1$; update time to wall-clock time.

Algorithm S4 Quantile bound of [2, 1] for equally likely scenarios for model (1)

Input: instance of model (1) with $|\Omega|$ scenarios.

Output: $\underline{\mathcal{Z}}$.

1: Solve model (1) separately for each ω in the input instance with $z^\omega \leftarrow 0$.

2: Sort corresponding objective function values in ascending order; $z^\omega \leftarrow 1$ for first $\lfloor |\Omega|\varepsilon \rfloor$ scenarios, $z^\omega \leftarrow 0$ for rest.

3: Solve input instance of model (1) with z fixed from Step 2.

4: $\underline{\mathcal{Z}} \leftarrow$ objective value.

B. Supplemental computational results

Table S1 presents results using a direct naive solution method of Model I and Model II. Table S2 presents the 95% confidence intervals for the objective function value, the time, and the optimality gap on solving model (1) naively. For details, see Section 5.2.

$\lvert\Omega\rvert$	ε	Objective	Time	Gap
100	0.01	278.54	53	-
	0.03	280.05	284	-
	0.05	280.98	401	-
	0.07	283.99	615	-
600	0.01	246.57	**T**	6.85%
	0.03	232.13	**T**	16.47%
	0.05	232.30	**T**	19.20%
	0.07	246.98	**T**	15.73%
900	0.01	224.91	**T**	16.14%
	0.03	224.83	**T**	19.83%
	0.05	225.12	**T**	21.15%
	0.07	224.83	**T**	23.29%
1500	0.01	238.57	**T**	10.37%
	0.03	230.36	**T**	17.37%
	0.05	233.60	**T**	18.70%
	0.07	230.33	**T**	21.53%

(a) Model I

$\lvert\Omega\rvert$	ε	Objective	Time	Gap
100	0.01	6,052.86	**T**	29.51%
	0.03	6,337.38	**T**	37.08%
	0.05	6,605.23	**T**	43.46%
	0.07	6,946.04	**T**	47.83%
600	0.01	4,441.43	**T**	53.30%
	0.03	4,150.03	**T**	62.99%
	0.05	4,234.92	**T**	66.95%
	0.07	4,326.37	**T**	70.06%
900	0.01	3,823.28	**T**	60.78%
	0.03	3,133.00	**T**	72.81%
	0.05	2,921.09	**T**	77.85%
	0.07	3,254.70	**T**	78.05%
1500	0.01	3,745.73	**T**	61.84%
	0.03	3,849.86	**T**	66.37%
	0.05	3,550.13	**T**	72.92%
	0.07	3,696.89	**T**	74.96%

(b) Model II

Table S1: Computational results for a naive solution method on instances of model (1). An entry of "**T**" denotes the instance could not be solved to optimality in the time limit, while an entry of "$-$" denotes the instance solved to optimality. For details, see Section 5.2.

Table S3 summarizes our comparison of the three lower bounding heuristics of Section 3 for Model I and Model II. For details, see Section 5.3. In the practical implementation, we deviate slightly from Algorithm S2 and S3 for the `IR` and `AP` bounds, respectively. We add the time required for Step 7 and 8 in Algorithm S2 and S3, respectively, after the completion of the algorithms. This affects only the last row of Table S3a where the total time slightly exceeds the limit of 2100 seconds; for further details on our implementation, see our code that is publicly available at our GitHub repository cited below. In Table S3, entries marked with a ✗ denote the instance is infeasible for the corresponding heuristic; i.e., the heuristic fails. The "Improvement" columns denote the relative improvement of the heuristic with respect to the best known feasible solution obtained by the naive solution method (Table S1); i.e, $100\frac{\text{heuristic - naive}}{\text{naive}}\%$, where heuristic and naive are the objective function values in Table S3 and Table S1, respectively.

Table S4 presents the 95% confidence intervals for the objective function value, the time, and the

$\lvert\Omega\rvert$	Objective CI	Time CI	Gap CI [%]
100	(280.94, 282.29)	(387, 453)	(0.01, 0.04)
600	(239.28, 244.64)	**T**	(14.77, 16.49)
900	(234.08, 240.19)	**T**	(49.19, 86.23)
1500	(233.42, 237.98)	**T**	(14.16, 37.89)

(a) Model I

$\lvert\Omega\rvert$	Objective CI	Time CI	Gap CI [%]
100	(6,385.63, 6,529.70)	**T**	(42.69, 43,53)
600	(4,588.53, 4,755.71)	**T**	(62.13, 63.38)
900	(4,296.65, 4,417.90)	**T**	(65.62, 66.55)
1500	(2,879.22, 3,308.35)	**T**	(74.92, 78.17)

(b) Model II

Table S2: Computational results for a naive solution method on 20 instances of model (1) for $\varepsilon = 0.05$. CI denotes the 95% confidence interval. For details, see Section 5.2.

improvement in relation to the naive solution of the three lower bounding techniques. Here, the trend we report in Table S3 is further validated. For Model I, the `QP` bound is computed very quickly; however, compared to the `IR` bound over the larger instances, the improvements for `QP` are much smaller. For example, consider the $|\Omega| = 1500$ regime for Model I. The `QP` bound has an improvement of at most 3.3%, while the `IR` bound has an improvement that is at least double (6.7%). We further observe that the `AP` bound is practically useless for Model I. For all instances, in each of the 20 batches with $|\Omega| \geq 600$, we fail to obtain a feasible solution with the `AP` bound.

As in Table S3b, we reach the time limit in Table S4b for all methods for the computationally more challenging Model II. However, in contrast to Table S4a we observe that the `QP` bound lags behind not only the `IR` bound but also the `AP` bound. For larger instances of Model II, the upper limit of the CI for the `QP` bound is lower than the lower limit of the CI for both the `IR` and `AP` bounds. The `IR` bound performs the best here, consistently delivering the highest improvement CIs for all instances except the smallest instance with $|\Omega| = 100$. For the smallest instance, the `QP` performs better for both Model I and Model II, further validating our empirical claim that the `IR` and `AP` bounds are most suitable for the larger instances. Finally, we also observe that the improvement by the `IR` bound is significantly higher for Model II than for Model I. This is especially evident from the last rows of Table S4a and Table S4b where the improvements in Model II increase by an order of magnitude, e.g., from 6.7% to 69.5%.

Table S5 summarizes the results of the Welch-t test of Section 2.3 for Model I and Model II using the 20 batches of scenarios reported in Section 5.2 for each of the four scenario-size regimes. A ✓ denotes that we have statistically significant evidence to reject the null hypothesis of Section 2.3, while a ✗ denotes that we are unable to do so; i.e., a ✓ suggests Rule r performs better than Rule s.

		IR			AP			QP		
$\lvert\Omega\rvert$	ε	Objective	Time	Improvement	Objective	Time	Improvement	Objective	Time	Improvement
100	0.01	253.92	123	-8.84%	232.26	127	-16.62%	278.54	53	0.00%
	0.03	258.16	191	-7.82%	232.54	451	-16.96%	279.16	61	-0.32%
	0.05	265.16	245	-5.63%	232.79	429	-17.15%	280.43	57	-0.20%
	0.07	265.43	289	-6.54%	277.26	693	-2.37%	283.71	58	-0.10%
600	0.01	238.39	669	-3.32%	✗	244		248.50	13	0.78%
	0.03	240.03	768	3.40%	✗	552		251.93	16	8.53%
	0.05	254.61	829	9.60%	✗	583		252.48	18	8.69%
	0.07	258.43	873	4.64%	✗	846		253.08	21	2.47%
900	0.01	234.38	1024	4.21%	✗	267		244.89	30	8.88%
	0.03	239.71	1115	6.62%	✗	630		245.71	46	9.29%
	0.05	253.70	1194	12.70%	✗	573		245.97	254	9.26%
	0.07	255.05	1246	13.44%	✗	948		246.22	194	9.51%
1500	0.01	235.36	1815	-1.35%	✗	429		233.37	69	-2.18%
	0.03	239.46	1970	3.95%	✗	836		235.44	61	2.21%
	0.05	253.49	2083	8.51%	✗	739		235.63	59	0.87%
	0.07	254.93	2137	10.68%	✗	1260		239.96	56	4.18%

(a) Model I

		IR			AP			QP		
$\lvert\Omega\rvert$	ε	Objective	Time	Improvement	Objective	Time	Improvement	Objective	Time	Improvement
100	0.01	4,951.30	**T**	-18.20%	4,578.96	**T**	-24.35%	6,051.36	**T**	-0.02%
	0.03	5,304.93	**T**	-16.29%	4,694.28	**T**	-25.93%	6,648.47	**T**	4.91%
	0.05	5,624.33	**T**	-14.85%	4,850.88	**T**	-26.56%	7,122.64	**T**	7.83%
	0.07	6,113.24	**T**	-11.99%	✗	**T**		7,620.29	**T**	9.71%
600	0.01	4,602.68	**T**	3.63%	4,431.45	**T**	-0.22%	4,371.01	**T**	-1.59%
	0.03	4,980.82	**T**	20.02%	4,725.91	**T**	13.88%	4,777.36	**T**	15.12%
	0.05	5,633.56	**T**	33.03%	5,138.94	**T**	21.35%	4,648.14	**T**	9.76%
	0.07	6,144.55	**T**	42.03%	✗	**T**		4,949.73	**T**	14.41%
900	0.01	4,547.32	**T**	18.94%	4,376.98	**T**	14.48%	4,267.61	**T**	11.62%
	0.03	5,083.08	**T**	62.24%	4,553.03	**T**	45.32%	4,534.42	**T**	44.73%
	0.05	5,601.94	**T**	91.78%	4,941.90	**T**	69.18%	4,477.66	**T**	53.29%
	0.07	6,113.63	**T**	87.84%	✗	**T**		4,660.55	**T**	43.19%
1500	0.01	4,603.75	**T**	22.91%	4,470.61	**T**	19.35%	4,069.86	**T**	8.65%
	0.03	5,081.49	**T**	31.99%	4,683.39	**T**	21.65%	4,104.78	**T**	6.62%
	0.05	5,543.66	**T**	56.15%	5,104.65	**T**	43.79%	4,058.15	**T**	14.31%
	0.07	6,056.66	**T**	63.83%	✗	**T**		4,589.12	**T**	24.13%

(b) Model II

Table S3: Computational results for three lower bounding heuristics on instances of model (1). A ✗ indicates the instance is infeasible for the heuristic. For details, see Section 3.

	IR			AP			QP		
$\lvert\Omega\rvert$	Objective CI	Time CI	Improvement CI [%]	Objective CI	Time CI	Improvement CI [%]	Objective CI	Time CI	Improvement CI [%]
100	(261.70, 263.43)	(244, 245)	(-7.08, -6.35)	(232.76, 232.97)	(430, 431)	(-17.45, -17.08)	(279.93, 281.63)	(4, 4)	(-0.44, -0.05)
600	(253.65, 254.17)	(822, 827)	(3.80, 6.20)	✗	✗	✗	(246.37, 249.76)	(44, 48)	(1.43, 3.72)
900	(253.61, 254.10)	(1194, 1198)	(5.39, 8.37)	✗	✗	✗	(244.88, 247.96)	(73, 81)	(2.13, 5.37)
1500	(253.72, 254.29)	(2087, 2096)	(6.73, 8.87)	✗	✗	✗	(239.06, 243.83)	(165, 184)	(1.60, 3.28)

(a) Model I

	IR			AP			QP		
$\lvert\Omega\rvert$	Objective CI	Time CI	Improvement CI [%]	Objective CI	Time CI	Improvement CI [%]	Objective CI	Time CI	Improvement CI [%]
100	(5,650.76, 5,871.67)	**T**	(-12.45, -9.07)	(5,110.87, 5,342.39)	**T**	(-20.98, -17.07)	(6,729.79, 6,852.93)	**T**	(4.60, 5.77)
600	(5,494.22, 5,612.70)	**T**	(17.37, 20.57)	(5,061.44, 5,184.64)	**T**	(7.66, 11.92)	(4,700.14, 4,896.32)	**T**	(-0.21, 5.96)
900	(5,545.61, 5,621.69)	**T**	(26.29, 30.22)	(5,144.17, 5,241.68)	**T**	(17.57, 20.95)	(4,344.65, 4,563.58)	**T**	(-0.60, 5.23)
1500	(5,574.85, 5,638.68)	**T**	(69.50, 103.09)	(5,148.49, 5,215.68)	**T**	(56.35, 88.31)	(3,184.35, 3,477.76)	**T**	(-0.92, 23.07)

(b) Model II

Table S4: Computational results for three lower bounding heuristics on 20 instances of model (1) for $\varepsilon = 0.05$. CI denotes the 95% confidence interval. For details, see Section 5.3.

Next, in Table S6, we repeat the above results but by running Algorithm S1 with a time limit of $time = 41000$ seconds. Compared to Table S5, we now have seven additional instances in Table S6b for Model I where we can reject the null hypothesis. For Model II, there are 16 such additional instances in Table S6a. This supports our premise that the previous time limit was not sufficiently high enough to allow for rejections. For Model I in Table S5a, Rule VI is statistically the best performer, however only for the smallest scenario regime of $|\Omega| = 100$. Now—given additional time as in Table S6b—Rule VI has rejections in its favor compared to all other rules for two scenario regimes $|\Omega| = \{100, 1500\}$. For Model II, given this markedly larger amount of additional time, Rule II has statistically significant evidence to perform better than all other rules for all regimes except the smallest one of $|\Omega| = 100$. We examine this particular scenario regime in greater detail. Here, Algorithm S1 manages to complete between five and eight iterations for the different instances.

Table S7 summarizes our results of Algorithm S1 with $\underline{\mathcal{Z}}$ set to the largest of three lower bounds of Table S3. Here, the columns "Objective" and "Gap" denote the two outputs of Algorithm S1, $\mathcal{Z}_D$ and δ, respectively. The "Improvement" column denotes the relative reduction in the optimality gap compared to the optimality gap obtained naively; i.e., $100\,{}^{\mathrm{gap}_{\mathrm{naive}}-\mathrm{gap}_{\mathrm{LR}}}/_{\mathrm{gap}_{\mathrm{naive}}}\%$, where $\mathrm{gap}_{\mathrm{naive}}$ and $\mathrm{gap}_{\mathrm{LR}}$ are the optimality gaps in Table S1 and Table S7, respectively. Thus, a positive value of "Improvement" demonstrates value.

In Table S8, we present the results of the Welch t-test of Section 5.5. A ✓ denotes that we have statistically significant evidence to reject the null hypothesis of Section 2.3, while a ✗ denotes that we are unable to do so; i.e., a ✓ suggests Rule r performs better than Rule s. Thus, every entry in Table S5 results from the solution of 20 sets of optimization models.Similar to Table S5, every entry results from the solution of 20 sets of optimization models.

(r,s)	$\lvert\Omega\rvert$			
	100	600	900	1500
(I,II)	✓ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(I,III)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(I,IV)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(I,V)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(I,VI)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(II,I)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(II,III)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(II,IV)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(II,V)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(II,VI)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗

(r,s)	$\lvert\Omega\rvert$			
	100	600	900	1500
(III,I)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(III,II)	✓ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(III,IV)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(III,V)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(III,VI)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(IV,I)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(IV,II)	✓ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(IV,III)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(IV,V)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(IV,VI)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗

(r,s)	$\lvert\Omega\rvert$			
	100	600	900	1500
(V,I)	✗ | ✓	✗ | ✗	✗ | ✗	✗ | ✗
(V,II)	✗ | ✓	✗ | ✗	✗ | ✗	✗ | ✗
(V,III)	✗ | ✓	✗ | ✗	✗ | ✗	✗ | ✗
(V,IV)	✗ | ✓	✗ | ✗	✗ | ✗	✗ | ✗
(V,VI)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(VI,I)	✗ | ✓	✗ | ✗	✗ | ✗	✗ | ✗
(VI,II)	✓ | ✓	✗ | ✗	✗ | ✗	✗ | ✗
(VI,III)	✗ | ✓	✗ | ✗	✗ | ✗	✗ | ✗
(VI,IV)	✗ | ✓	✗ | ✗	✗ | ✗	✗ | ✗
(VI,V)	✓ | ✓	✗ | ✗	✗ | ✗	✗ | ✗

(a) Model I

(r,s)	$\lvert\Omega\rvert$			
	100	600	900	1500
(I,II)	✗ | ✓	✗ | ✗	✗ | ✗	✗ | ✗
(I,III)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(I,IV)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(I,V)	✗ | ✓	✗ | ✗	✗ | ✗	✗ | ✗
(I,VI)	✗ | ✓	✗ | ✗	✗ | ✗	✗ | ✗
(II,I)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(II,III)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(II,IV)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(II,V)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(II,VI)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗

(r,s)	$\lvert\Omega\rvert$			
	100	600	900	1500
(III,I)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(III,II)	✗ | ✓	✗ | ✗	✗ | ✗	✗ | ✗
(III,IV)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(III,V)	✗ | ✓	✗ | ✗	✗ | ✗	✗ | ✗
(III,VI)	✗ | ✓	✗ | ✗	✗ | ✗	✗ | ✗
(IV,I)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(IV,II)	✗ | ✓	✗ | ✗	✗ | ✗	✗ | ✗
(IV,III)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(IV,V)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(IV,VI)	✗ | ✓	✗ | ✗	✗ | ✗	✗ | ✗

(r,s)	$\lvert\Omega\rvert$			
	100	600	900	1500
(V,I)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(V,II)	✗ | ✓	✗ | ✗	✗ | ✗	✗ | ✗
(V,III)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(V,IV)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(V,VI)	✗ | ✓	✗ | ✗	✗ | ✗	✗ | ✗
(VI,I)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(VI,II)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(VI,III)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(VI,IV)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗
(VI,V)	✗ | ✗	✗ | ✗	✗ | ✗	✗ | ✗

(b) Model II

Table S5: Statistical performance of step-size rules for Algorithm S1 for $\varepsilon = 0.01 \mid \varepsilon = 0.05$. An entry of (r,s) in the first column denotes that the null hypothesis is $H_0 : \mu_r \geq \mu_s$. Rejection of the null hypothesis is denoted by a ✓, while a failure to reject is denoted by a ✗. For details, see Section 2 and Section 5.4.

(r,s)	$\|\Omega\|$			
	100	600	900	1500
(I,II)	✗	✗	✗	✗
(I,III)	✗	✗	✗	✗
(I,IV)	✗	✗	✗	✗
(I,V)	✗	✗	✗	✗
(I,VI)	✗	✗	✗	✗
(II,I)	✗	✗	✗	✗
(II,III)	✗	✗	✗	✗
(II,IV)	✗	✗	✗	✗
(II,V)	✗	✗	✗	✗
(II,VI)	✗	✗	✗	✗

(r,s)	$\|\Omega\|$			
	100	600	900	1500
(III,I)	✗	✗	✗	✗
(III,II)	✗	✗	✗	✗
(III,IV)	✗	✗	✗	✗
(III,V)	✗	✗	✗	✗
(III,VI)	✗	✗	✗	✗
(IV,I)	✗	✗	✗	✗
(IV,II)	✗	✗	✗	✗
(IV,III)	✗	✗	✗	✗
(IV,V)	✗	✗	✗	✗
(IV,VI)	✗	✗	✗	✗

(r,s)	$\|\Omega\|$			
	100	600	900	1500
(V,I)	✓	✗	✗	✗
(V,II)	✓	✗	✗	✗
(V,III)	✓	✗	✗	✗
(V,IV)	✓	✗	✗	✗
(V,VI)	✗	✗	✗	✗
(VI,I)	✓	✗	✗	✓
(VI,II)	✓	✓	✗	✓
(VI,III)	✓	✗	✗	✓
(VI,IV)	✓	✗	✗	✓
(VI,V)	✓	✓	✗	✓

(a) Model II

(r,s)	$\|\Omega\|$			
	100	600	900	1500
(I,II)	✓	✗	✗	✗
(I,III)	✗	✗	✗	✗
(I,IV)	✗	✗	✗	✗
(I,V)	✗	✗	✗	✗
(I,VI)	✗	✗	✗	✗
(II,I)	✗	✓	✓	✓
(II,III)	✗	✓	✓	✓
(II,IV)	✗	✓	✓	✓
(II,V)	✗	✓	✓	✓
(II,VI)	✗	✓	✓	✓

(r,s)	$\|\Omega\|$			
	100	600	900	1500
(III,I)	✓	✗	✗	✗
(III,II)	✓	✗	✗	✗
(III,IV)	✗	✗	✗	✗
(III,V)	✓	✗	✗	✗
(III,VI)	✗	✗	✗	✗
(IV,I)	✓	✗	✗	✗
(IV,II)	✓	✗	✗	✗
(IV,III)	✗	✗	✗	✗
(IV,V)	✓	✗	✗	✗
(IV,VI)	✗	✗	✗	✗

(r,s)	$\|\Omega\|$			
	100	600	900	1500
(V,I)	✗	✗	✗	✗
(V,II)	✓	✗	✗	✗
(V,III)	✗	✗	✗	✗
(V,IV)	✗	✗	✗	✗
(V,VI)	✗	✗	✗	✗
(VI,I)	✓	✗	✗	✗
(VI,II)	✓	✗	✗	✗
(VI,III)	✗	✗	✗	✗
(VI,IV)	✗	✗	✗	✗
(VI,V)	✓	✗	✗	✗

(b) Model I

Table S6: Analogous results to Table S5 for $\varepsilon = 0.05$ but with a time limit of $time = 41000$ in Algorithm S1.

$\|\Omega\|$	ε	Objective	Time	Gap	Improvement
100	0.01	278.54	101	-	-
	0.03	284.30	69	1.81%	-
	0.05	290.47	135	3.45%	-
	0.07	296.99	63	4.47%	-
600	0.01	252.55	**T**	1.60%	76.64%
	0.03	273.42	**T**	7.86%	52.28%
	0.05	284.88	**T**	10.63%	44.64%
	0.07	291.25	**T**	11.27%	28.35%
900	0.01	264.84	**T**	7.53%	52.47%
	0.03	277.81	**T**	11.56%	41.70%
	0.05	286.26	**T**	11.37%	46.24%
	0.07	292.49	**T**	12.80%	45.04%
1500	0.01	266.12	**T**	11.56%	-11.48%
	0.03	279.52	**T**	14.33%	17.50%
	0.05	287.34	**T**	11.78%	37.01%
	0.07	293.32	**T**	13.09%	39.21%

(a) Model I

$\|\Omega\|$	ε	Objective	Time	Gap	Improvement
100	0.01	8,898.50	**T**	32.00%	-8.44%
	0.03	10,617.61	**T**	37.38%	-0.83%
	0.05	12,207.23	**T**	41.65%	4.16%
	0.07	14,632.89	**T**	47.92%	-0.19%
600	0.01	9,646.32	**T**	52.29%	1.91%
	0.03	11,334.21	**T**	56.05%	11.01%
	0.05	12,939.53	**T**	56.46%	15.67%
	0.07	14,556.63	**T**	57.79%	17.52%
900	0.01	9,819.28	**T**	53.69%	11.67%
	0.03	11,506.56	**T**	55.82%	23.33%
	0.05	13,147.37	**T**	57.39%	26.27%
	0.07	14,764.62	**T**	58.59%	24.93%
1500	0.01	9,904.72	**T**	53.52%	13.45%
	0.03	11,594.00	**T**	56.17%	15.36%
	0.05	13,228.31	**T**	58.09%	20.34%
	0.07	14,845.89	**T**	59.20%	21.02%

(b) Model II

Table S7: Computational results of Algorithm S1 on instances of model (1) using r = Rule VI and $\underline{\mathcal{Z}}$ = the maximum of the three lower bounds in Table S3. The "Gap" column denotes the relative optimality gap between $\underline{\mathcal{Z}}$ and the upper bound obtained using the Lagrangian procedure of Algorithm S1. The "Improvement" column denotes the relative reduction in the optimality gap as compared to Table S1. A "-" indicates that an improvement cannot be computed as the instance solved to optimality naively. For details, see Section 5.5.

r	$\|\Omega\|$			
	100	600	900	1500
I	✗	✓	✓	✓
II	✗	✓	✓	✓
III	✗	✓	✓	✓
IV	✗	✓	✓	✓
V	✗	✓	✓	✓
VI	✗	✓	✓	✓

(a) Model I

r	$\|\Omega\|$			
	100	600	900	1500
I	✓	✓	✓	✓
II	✗	✓	✓	✓
III	✗	✓	✓	✓
IV	✗	✓	✓	✓
V	✗	✓	✓	✓
VI	✗	✓	✓	✓

(b) Model II

Table S8: Statistical performance of step-size rules for Algorithm S1 for $\varepsilon = 0.05$. An entry of Rule r in the first column denotes that the null hypothesis is $H_0 : \mu_r \geq \mu_{\text{naive}}$. Rejection of the null hypothesis is denoted by a ✓, while a failure to reject is denoted by a ✗. For details, see Section 5.5.

References

[1] Ahmed, S., Luedtke, J., Song, Y., Xie, W., 2017. Nonanticipative duality, relaxations, and formulations for chance-constrained stochastic programs. Mathematical Programming 162, 51–81. doi:10.1007/s10107-016-1029-z.

[2] Singh, B., Knueven, B., 2021. Lagrangian relaxation based heuristics for a chance-constrained optimization model of a hybrid solar-battery storage system. Journal of Global Optimization 80, 965–989. doi:10.1007/s10898-021-01041-y.

[3] Singh, B., Morton, D.P., Santoso, S., 2018. An adaptive model with joint chance constraints for a hybrid wind-conventional generator system. Computational Management Science 15, 563—-582. doi:10.1007/s10287-018-0309-x.