FEniCS Course

Oct 25, 2018

Incompressible Navier-Stokes with Finite element

FEniCS PROJECT

FEniCS is an automated programming environment for differential equations

- C++/Python library
- Initiated 2003 in Chicago
- 1000–2000 monthly downloads
- Part of Debian and Ubuntu
- Licensed under the GNU LGPL

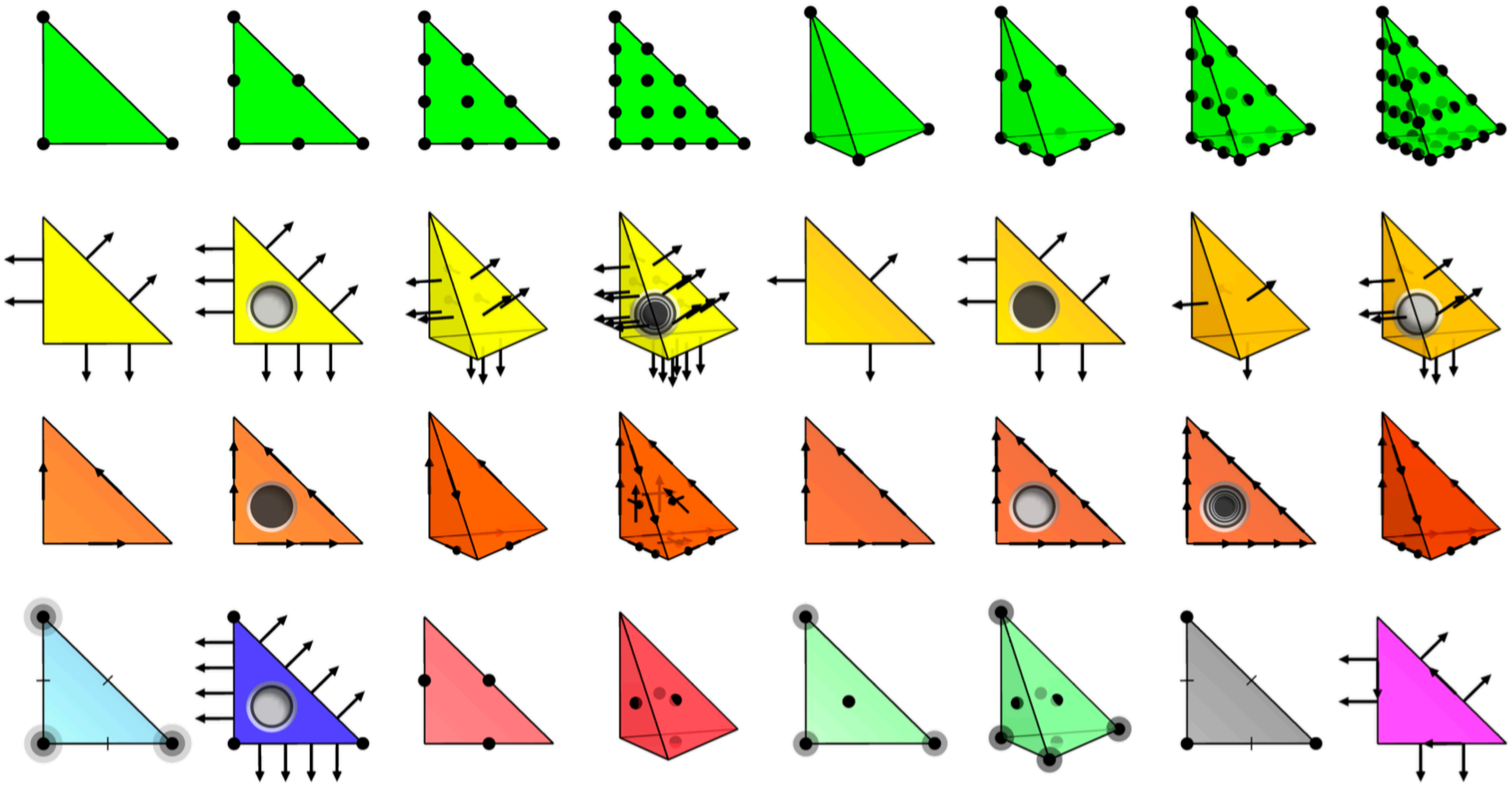
`http://fenicsproject.org/`

Collaborators

Simula Research Laboratory, University of Cambridge, University of Chicago, Texas Tech University, KTH Royal Institute of Technology, Chalmers University of Technology, Imperial College London, University of Oxford, Charles University in Prague, ...

FEniCS is automated FEM

- Automated generation of basis functions
- Automated evaluation of variational forms
- Automated finite element assembly
- Automated adaptive error control



What is FEM?

The finite element method is a framework and a recipe for discretization of differential equations

- Ordinary differential equations
- Partial differential equations
- Integral equations

- A recipe for discretization of PDE
- PDE $\rightarrow Ax = b$
- Different bases, stabilization, error control, adaptivity

The FEM cookbook

(i)

$\boxed{\mathcal{A}u = f}$ *Partial differential equation*

Multiply by v

(ii)

$a(u, v) = L(v)$ *Continuous variational problem*

Take $V_h \subset V$

(iii)

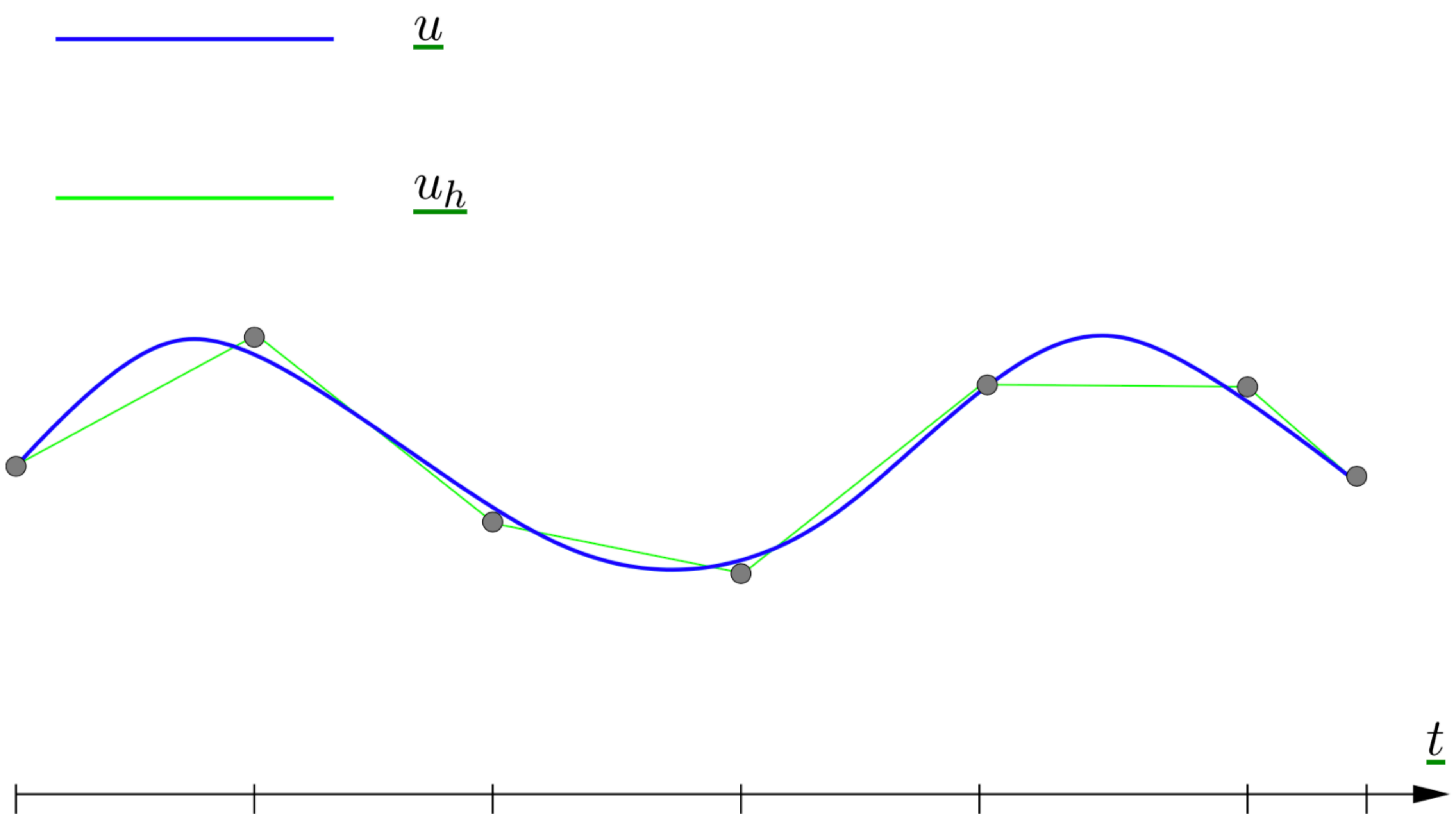
$\boxed{a(u_h, v) = L(v)}$ *Discrete variational problem*

Let $u_h = \sum_j U_j \phi_j$

(iv)

$\boxed{AU = b}$ *System of discrete equations*

Finite element function spaces



The incompressible Navier–Stokes equations

Constitutive equations

$$
\begin{aligned}
\rho(\dot{u} + u \cdot \nabla u) - \nabla \cdot \sigma(u, p) &= f \qquad \text{in } \Omega \times (0, T] \\
\nabla \cdot u &= 0 \qquad \text{in } \Omega \times (0, T]
\end{aligned}
$$

Boundary conditions

$$
\begin{aligned}
u &= g_{\mathrm{D}} \qquad \text{on } \Gamma_{\mathrm{D}} \times (0, T] \\
\sigma \cdot n &= t_{\mathrm{N}} \qquad \text{on } \Gamma_{\mathrm{N}} \times (0, T]
\end{aligned}
$$

Initial condition

$$
u(\cdot, 0) = u_0 \quad \text{in } \Omega
$$

Mixed variational formulation of Navier–Stokes

Multiply the momentum equation by a test function v and integrate by parts:

$$\int_\Omega \rho(\dot{u}+u\cdot\nabla u)\cdot v \,\mathrm{d}x + \int_\Omega \sigma(u,p) : \varepsilon(v)\,\mathrm{d}x = \int_\Omega f\cdot v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} t_{\mathrm{N}}\cdot v\,\mathrm{d}s$$

Short-hand notation: $\langle \cdot, \cdot \rangle$ is L^2-inner product

$$\langle \rho\dot{u}, v\rangle + \langle \rho u \cdot \nabla u, v\rangle + \langle \sigma(u,p), \varepsilon(v)\rangle = \langle f, v\rangle + \langle t_{\mathrm{N}}, v\rangle_{\Gamma_{\mathrm{N}}}$$

Multiply the continuity equation by a test function q and sum up: find $(u,p) \in V$ such that

$$\langle \rho\dot{u}, v\rangle + \langle \rho u \cdot \nabla u, v\rangle + \langle \sigma(u,p), \varepsilon(v)\rangle + \langle \nabla \cdot u, q\rangle = \langle f, v\rangle + \langle t_{\mathrm{N}}, v\rangle_{\Gamma_{\mathrm{N}}}$$

for all $(v,q) \in \hat{V}$

Discrete mixed variational form of Navier–Stokes

Time-discretization leads to a *saddle-point* problem on each time step:

$$\begin{bmatrix} M + \Delta t A + \Delta t N(U) & \Delta t B \\ \Delta t B^\top & 0 \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

- Efficient solution of the saddle-point problem relies on the efficiency of special-purpose preconditioners (Uzawa iteration, Schur complement preconditioners, ...)
- We will use another approach (simpler and often more efficient)

Splitting scheme for Navier-Stokes: Core idea

- Solving the full coupled system for the velocity and the pressure simultaneously is computationally expensive.
- To reduce computational cost, iterative *splitting schemes* are an attractive alternative.
- Splitting schemes are typically based on solving for the velocity and the pressure separately.
- We will consider a splitting scheme solving three different (smaller!) systems at each time step n:
  1. Compute the *tentative velocity*
  2. Compute the *pressure*
  3. Compute the *corrected velocity*
- Next slides show how the scheme is derived – time to pay close attention!

A splitting scheme for Navier-Stokes (Summary)

For each n, given u^{n-1} and $p^{n-3/2}$,

- **Step 1**: Compute the tentative velocity $u^\star$ from

$$\rho D_t u^\star + \rho u^{n-1} \cdot \nabla u^{n-1} - \nabla \cdot \sigma(u^{n-1/2}, p^{n-3/2}) = f^{n-1/2}$$

- **Step 2**: Compute the pressure $p^{n-1/2}$ from

$$-k_n \Delta p^{n-1/2} = -k_n \Delta p^{n-3/2} - \rho \nabla \cdot u^\star$$

- **Step 3**: Compute the corrected velocity u^n from

$$\rho u^n = \rho u^\star - k_n \nabla (p^{n-1/2} - p^{n-3/2})$$

Weak formulation of N-S splitting scheme

For $n = 1, 2, \ldots, N$, given u^0 and $p^{-1/2}$:

1. Compute $u^\star$ with $u^\star|_{\Gamma_D} = g_D$ solving

$$\langle \rho D_t^n u^\star, v\rangle + \langle \rho u^{n-1} \cdot \nabla u^{n-1}, v\rangle + \langle \sigma(u^{n-\frac{1}{2}}, p^{n-3/2}), \varepsilon(v)\rangle$$
$$= \langle f^{n-1/2}, v\rangle - \langle p^{n-3/2} n, v\rangle_{\partial\Omega}$$

for all v such that $v|_{\Gamma_D} = 0$.

2. Compute $p^{n-1/2}$ with $p^{n-1/2}|_{\Gamma_N} = \bar{p}$

$$k_n \langle \nabla p^{n-1/2}, \nabla q\rangle = k_n \langle \nabla p^{n-3/2}, \nabla q\rangle - \langle \rho \nabla \cdot u^\star, q\rangle$$

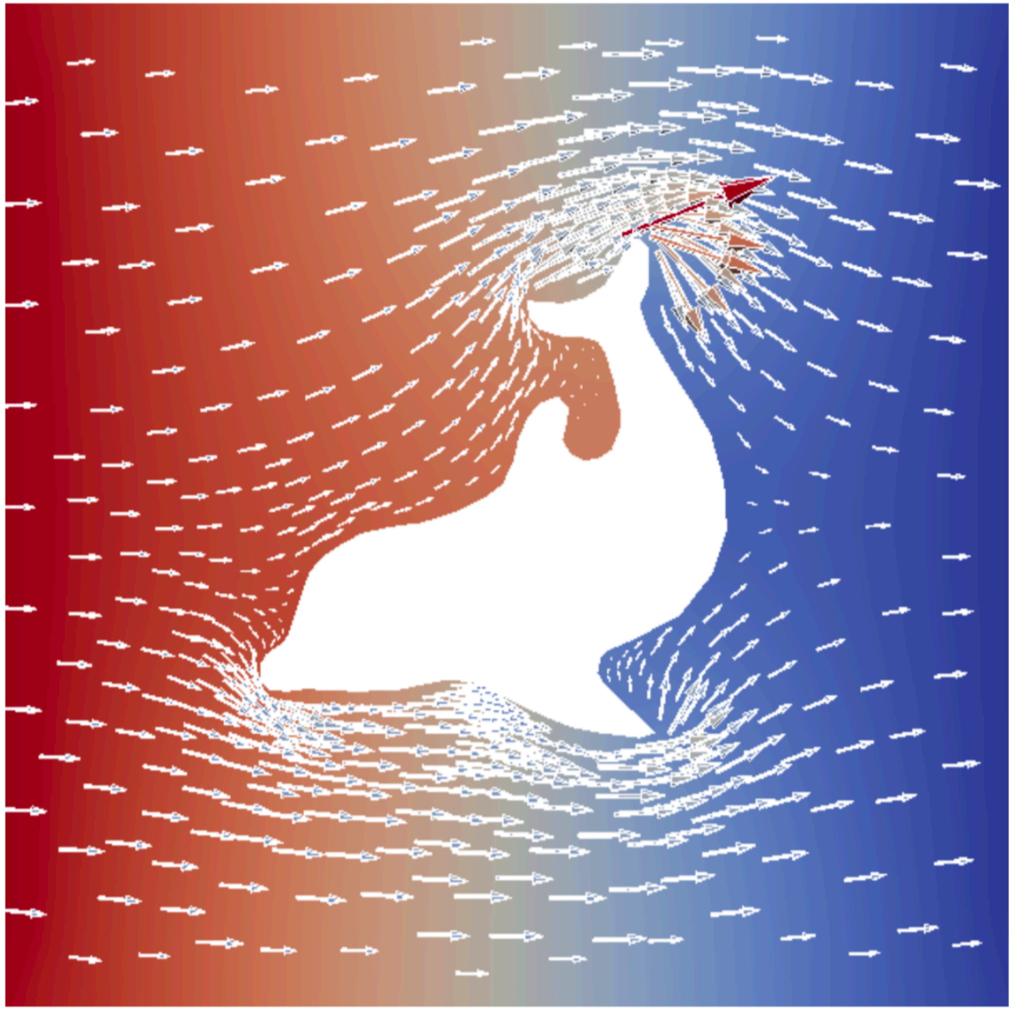
for all q such that $q|_{\Gamma_N} = 0$

3. Compute u^n solving

$$\langle \rho u^n, v\rangle = \langle \rho u^\star, v\rangle - k_n \langle \nabla (p^{n-1/2} - p^{n-3/2}), v\rangle$$

for all v.

Solve the incompressible Navier–Stokes equations for the flow of water around a dolphin. The water is initially at rest and the flow is driven by a pressure gradient.



Let’s start