

# Autoregressive Models

The *WaveNet*\* Architecture

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# WaveNet\*

Wavenet: A generative model for raw audio.

Aaron van den Oord, et al.

[@deepmind](#), 2016

## Contributions

- Models the wave-form directly (16kHz)
- Generates one time sample at a time
- Capable of capturing important audio structure at many time-scales
- Allows for conditional generation

Led to the **most natural-sounding** speech/audio at the time.

\*<https://arxiv.org/abs/1609.03499>

# Content

This talk covers

- an introduction to autoregressive models and their limitations,
- the architectural ideas to overcome those limitations, and
- few of existing improvements.

This talk is not

- about audio/speech (we use time series instead),
- a comprehensive state-of-the-art presentation on generative models.

Accompanying code: <https://github.com/cheind/autoregressive>

# Background

# Generative Models

Generative models build a distribution over the data itself. Consider a set of random variables

$$\mathbf{X} = \{X_1, X_2, X_3\},$$

then a generative model estimates

$$p(\mathbf{X}).$$

Given the joint distribution, we can generate *new* data via sampling

$$\mathbf{x} \sim p(\mathbf{X}).$$

In contrast, a discriminative models models conditional distributions, e.g.  $p(X_3 \mid X_2, X_1)$ .

## Chain Rule of Probability

Allows us to break down  $p(\mathbf{X})$  into a product of single-variable conditional distributions

$$\begin{aligned} p(\mathbf{X}) &= p(X_3 \mid X_2, X_1)p(X_2 \mid X_1)p(X_1) \\ &= p(X_1 \mid X_2, X_3)p(X_3 \mid X_2)p(X_2) \\ &\dots \end{aligned}$$

# Autoregressive Models

## Autoregressive Models

Given a set of (time-)ordered random variables  $\mathbf{X} = \{X_1, X_2, X_3 \dots, X_T\}$ , we represent their joint distribution as

$$\begin{aligned} p(\mathbf{X}) &= \prod_{i=1}^T p(X_i \mid \mathbf{X}_{j < i}) \\ &= p(X_1)p(X_2 \mid X_1)p(X_3 \mid X_2, X_1) \dots \end{aligned}$$

This induces a form of **causality**, as the distribution over a future variable depends on all previous observations. It also allows us to generate *new* data one sample point at a time (conditioned on all the previous ones).



## Lagged Autoregressive Models

For computational reasons, one usually limits the number of past observations influencing future predictions. An autoregressive model of order/lag  $R$  is defined as

$$X_t \mid \mathbf{X}_{j < t} = \theta_0 + \sum_{i=1}^R \theta_i X_{t-i} + \epsilon_t,$$

where  $\theta = \{\theta_0, \dots, \theta_R\}$  are the parameters of the model and  $\epsilon_t$  is (white) noise.

## Translation to Neural Networks

The definition of autoregressive models can be captured by a single fully connected neural layer

$$\begin{aligned} X_t \mid \mathbf{X}_{j < t} &= \theta_0 + \sum_{i=1}^R \theta_i X_{t-i} + \epsilon_t \\ &= \boldsymbol{\theta}^T \mathbf{h}_t + \epsilon_t, \end{aligned}$$

where  $\boldsymbol{\theta} = (\theta_0 \quad \theta_1 \quad \dots \quad \theta_R)$  are the weights including the bias, and  $\mathbf{h}_t = (1 \quad X_{t-1} \quad \dots \quad X_{t-R})$ .

## Deep models

For more model capacity, one might stack layers having multiple features, in which case we get something along the following line

$$\mathbf{H}_t^l = \sigma \left( \boldsymbol{\Theta}^l \mathbf{H}_t^{l-1} + \mathbf{E}_t^l \right),$$

where  $\sigma$  is a non-linearity and subscript  $l$  denotes the  $l$ -th layer.

## Limitations

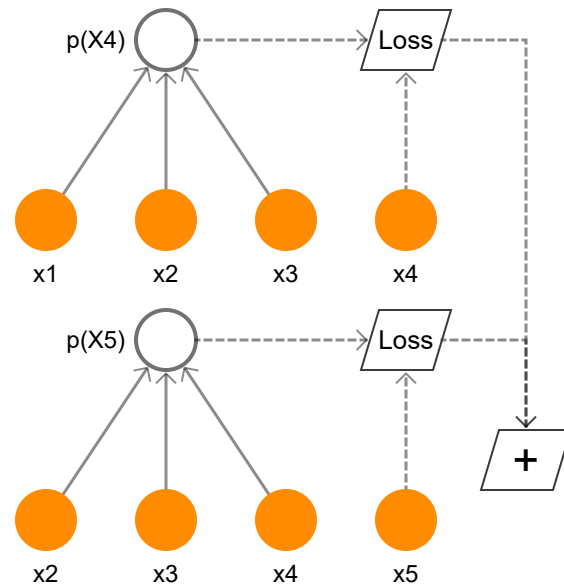
1. **Training** with linear layers is **inefficient** as autoregressive value needs to be computed for every possible window of size  $R$ .
2. The **number of weights** grows linearly with the receptive field of the model. For multi-time scale models (speech, audio) this becomes quickly an issue.

# WaveNet

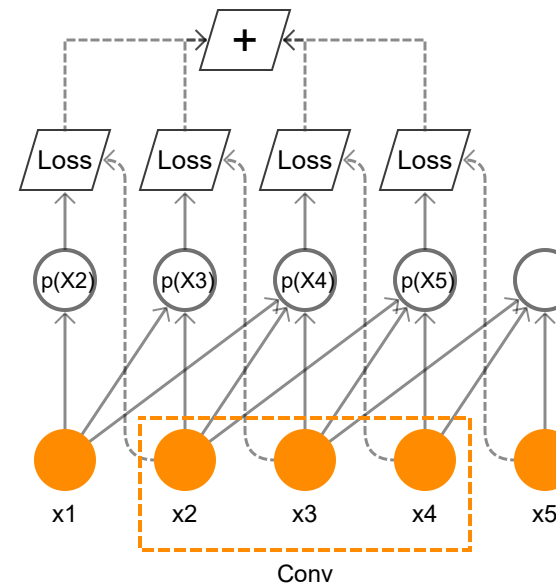
# Convolutions: Improving Training Efficiency

Interpret  $X_t \mid \mathbf{X}_{j < t}$  in terms of convolution. Allows for a fully-convolutional computation of all  $X_t$  in one sweep. Below illustration is for a model of  $R = 3$ .

*Fully Connected Approach*



*Convolutional Approach*

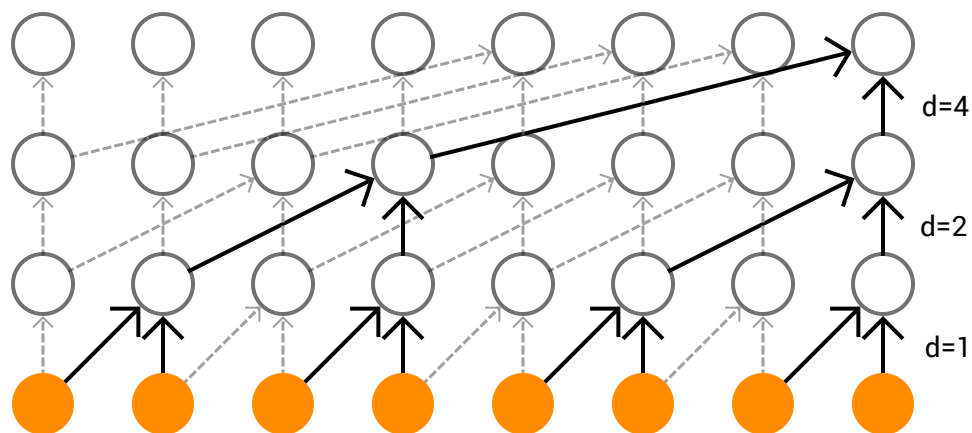


Need to be careful about (see Causal Padding slides)

- Ensure no data leakage happens (i.e input restricted to  $\mathbf{x}_{j < t}$ )
- How to handle variables  $X_t$ , where  $t < R$

# Dilated Convolutions: Exponential Receptive Fields

Receptive field of dilated convolutions grows exponentially while parameters increase only linearly. Figure below uses kernel size  $K_i = 2$ .



In general, each layer with dilation factor  $D_i$  and kernel size  $K_i$  adds

$$r_i = (K_i - 1)D_i$$

to the receptive field  $R = \sum_i r_i + 1$ .

Note, how each input (orange) within the receptive field is used exactly once.

## Dilated Convolutions: Number of parameters

Assume kernel size  $K_i = 2$  and a receptive field of  $R = 512$ . Then a vanilla convolution requires

$$R_{\text{vanilla}} = 512 \text{ parameters}$$

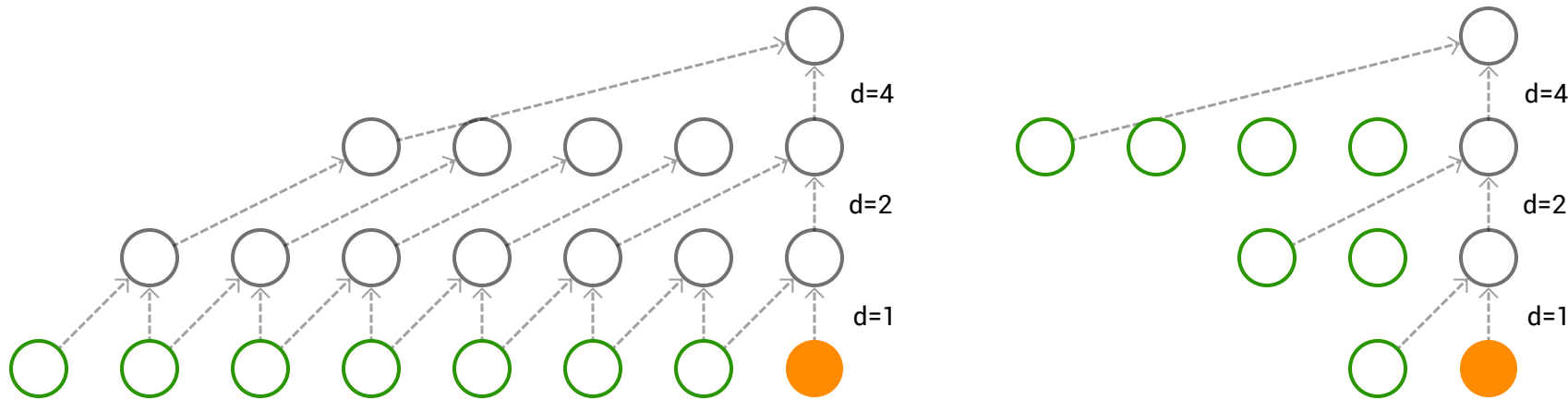
(without a bias), while a stacked dilated convolution requires

$$R_{\text{dilated}} = 2 * 9 = 18 \text{ parameters.}$$

**Note:** stacked dilated convolutions make use of all 512 inputs.

## Causal Padding

Causal padding (left-padding) ensures that convoluted features do not depend on future values and allows us to compute predictions for  $X_t$ , where  $t < R$ . Two possibilities: input-padding (left), layer-padding (right)

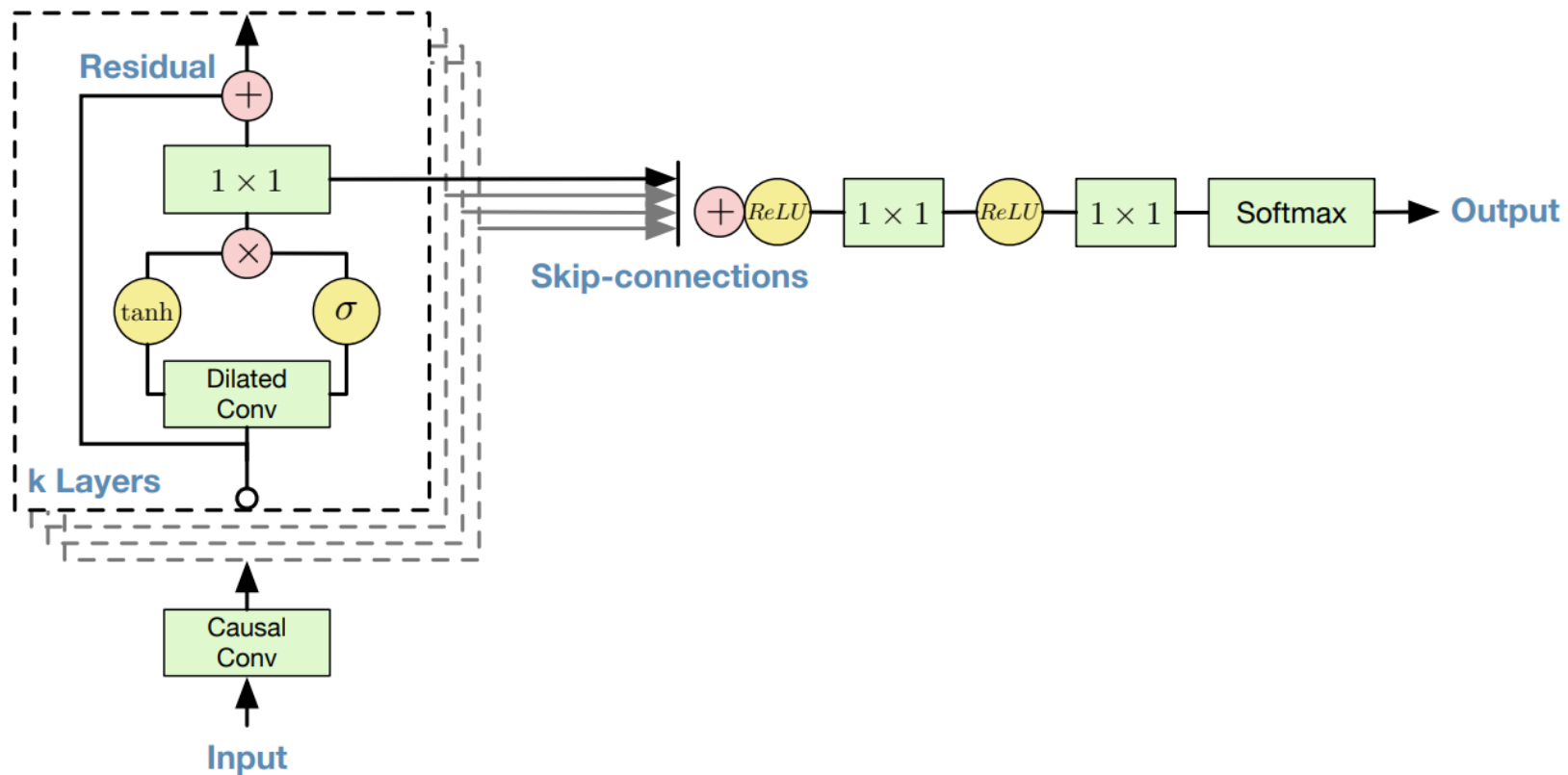


In general, a total of  $P = R - 1$  padding values are required.



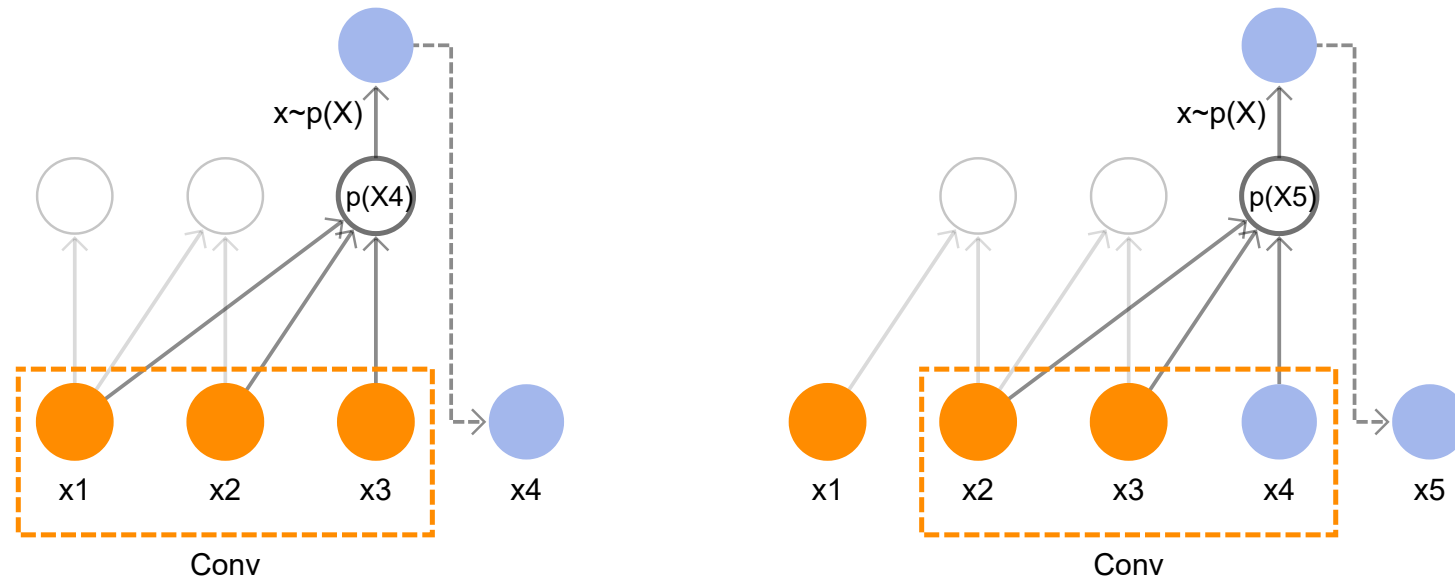
# Full Architecture

WaveNet combines stacked dilated convolutions, causal padding and gated activation functions to predict a categorical distribution for  $X_t | \mathbf{X}_{j < t}$  in parallel.



# Data Generation

New data is generated one sample at a time. The figure below shows two steps for a model with  $R = 3$



Remarks:

- Generation is inefficient - requires  $R$  inputs but uses only the last output.
- Generation involves sampling from the distribution.

## Extensions

## Conditional WaveNets\*

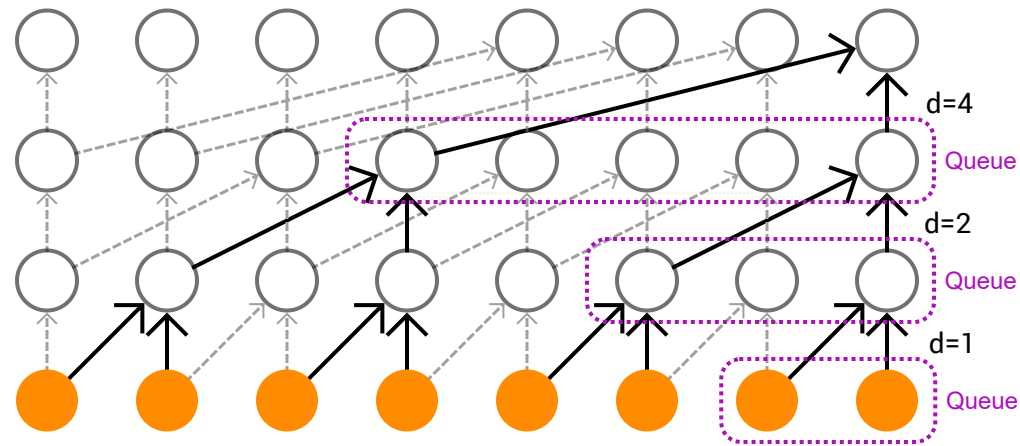
Condition the model on additional external input

$$\begin{aligned} p(\mathbf{X} \mid \mathbf{y}) &= \prod_{i=1}^T p(X_i \mid \mathbf{X}_{j < i}, \mathbf{y}) \\ &= p(X_1 \mid \mathbf{y}) p(X_2 \mid X_1, \mathbf{y}) p(X_3 \mid X_2, X_1, \mathbf{y}) \dots \end{aligned}$$

to change generative behavior. For example  $\mathbf{y}$  might represent speaker identity in which case the model would generate data wrt. the given speaker.

## Faster Generation\*

Relies on sparsity of access during computation. Introduce *queues* (i.e rolling buffers of size  $r_i + 1$ ) to store intermediate outputs. During generation only use oldest in queue and update queue.



Similar to updates in recurrent neural nets.

## Train Unrolling\*

In training, WaveNet uses a one-step rolling-origin loss which can causes substantial drift.

### Idea

A n-step loss would allow the model to correct its own drift.

I.e we want to apply n-step generation and backprop through all samples.

### Issue

How to backprop through a random sample from a categorical distribution?

# Train Unrolling

## Reparametrization Idea

Note if  $x_t \sim \mathcal{N}(\mu, \sigma)$ , which we can express as  $x_t \sim \mathcal{N}(0, 1)\sigma + \mu$ .

Now  $\frac{\partial}{\partial \mu}, \frac{\partial}{\partial \sigma}$  exist and randomness becomes an input (for which we do not require gradients).

## Reparametrization of Categorical Distributions

Similar reparametrization exists for  $x_t \sim \text{Cat}(\pi_1, \dots, \pi_C)$

using Gumbel distribution\*, which allows us to write

$$x_t \sim g(\text{Gumbel}(0, 1), \pi_1, \dots, \pi_C, \tau),$$

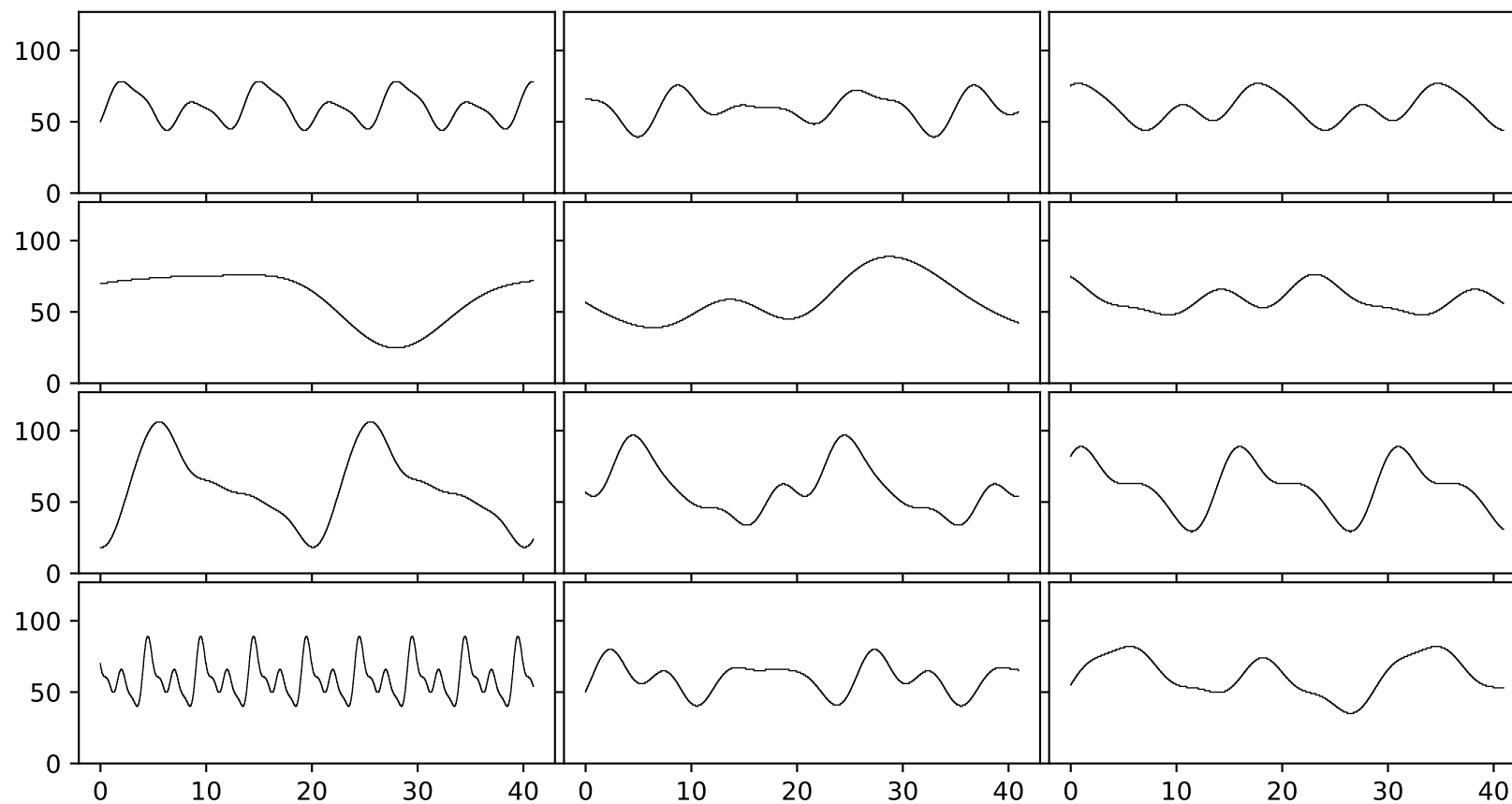
such that  $\frac{\partial g}{\partial \pi_i}$  exists.

# Experiments



# Setup

Instead of audio waveforms as input, using a Fourier dataset with randomized coefficients, number of terms and periodicity  
(sampling: 50Hz, quantization: 127 bins, encoding one-hot)

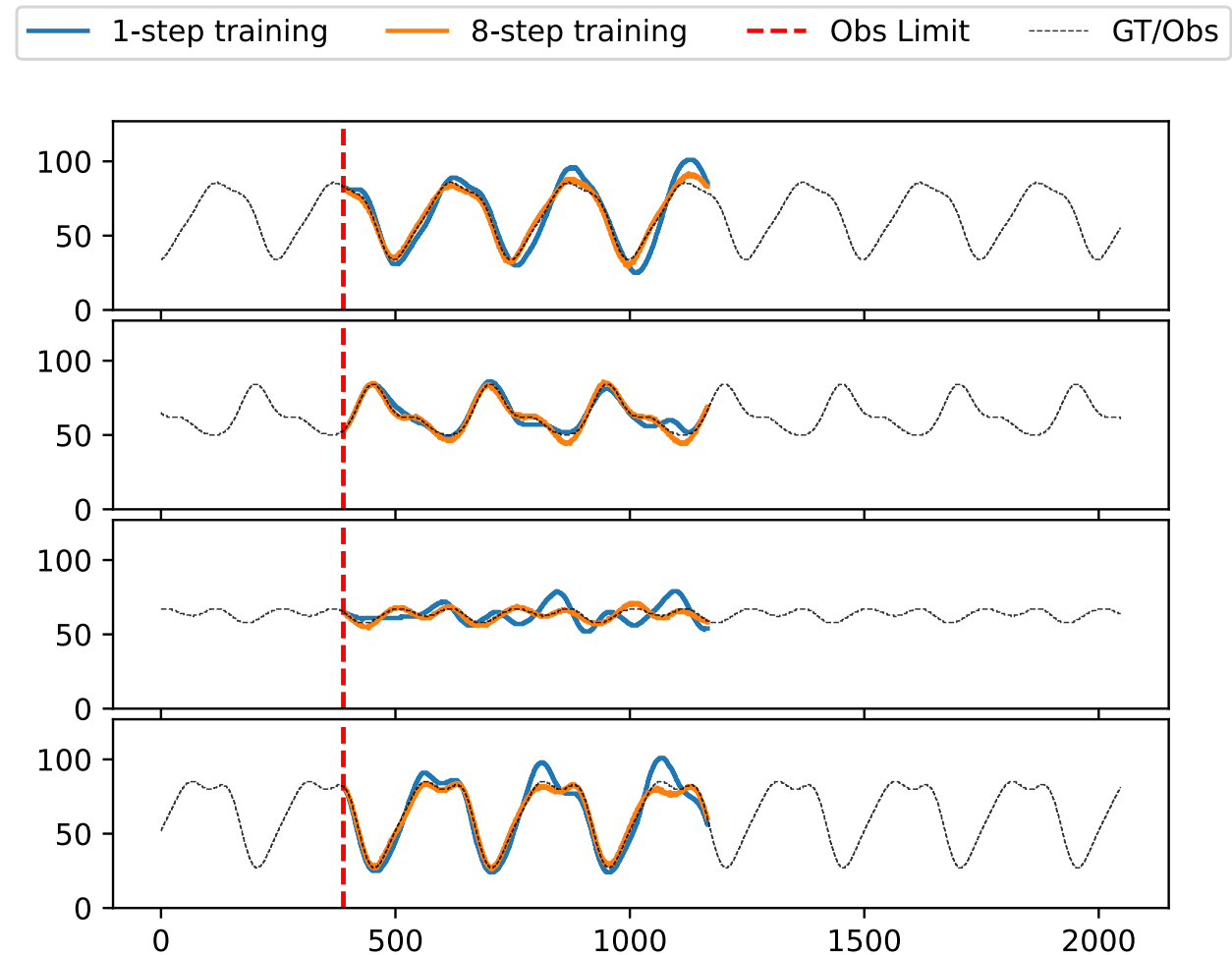


## Train-Unrolling Results

N-step forecast comparison between two models trained with and without unrolling on Fourier-series dataset with up to 4 terms.

### Conclusion

- (+) Decreases generative drift
- (+) Improves recreation of higher frequency patterns
- (-) Increases training time (rolling origin)
- (-) Sparser losses

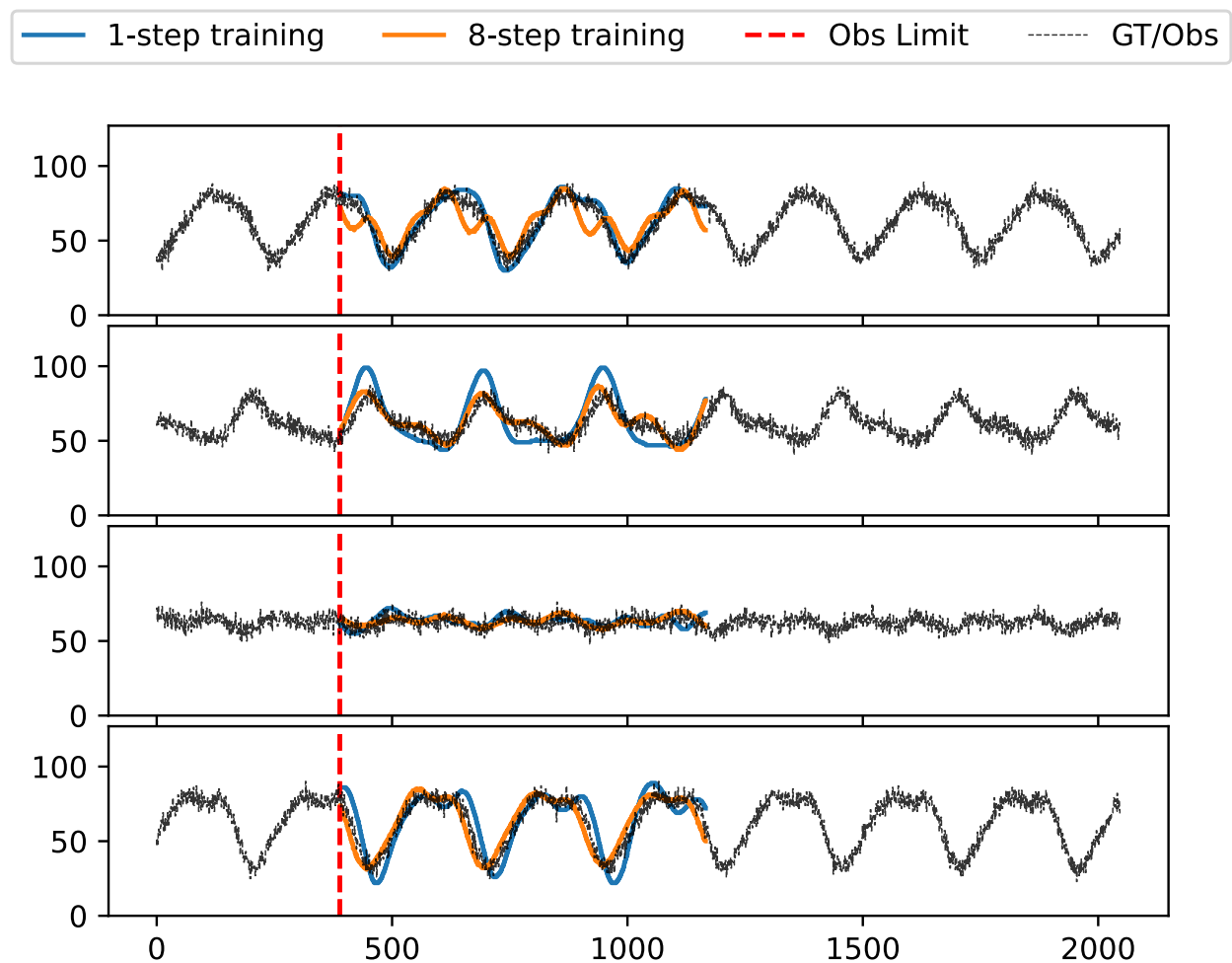


## Noisy Train-Unrolling Results

N-step prediction based on noisy observations - comparison between two models trained with and without unrolling on a clean Fourier series dataset with up to 4 terms.

### Conclusion

- (+) Both models capture global trends
- (-) Accuracy of both modes decreases

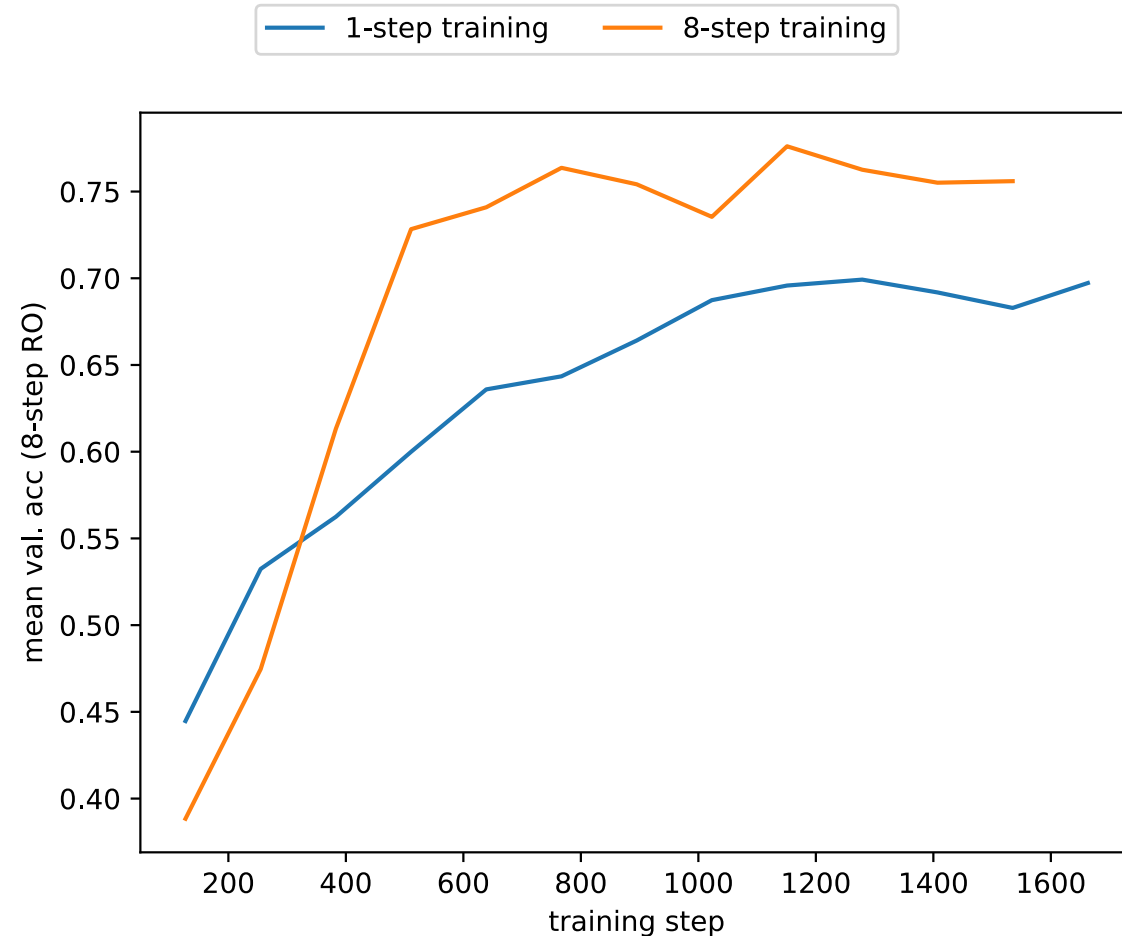


## Train-Unrolling Validation Acc. Results

8-step rolling origin validation comparison between models trained with and without unrolling on Fourier-series dataset with up to 4 terms.

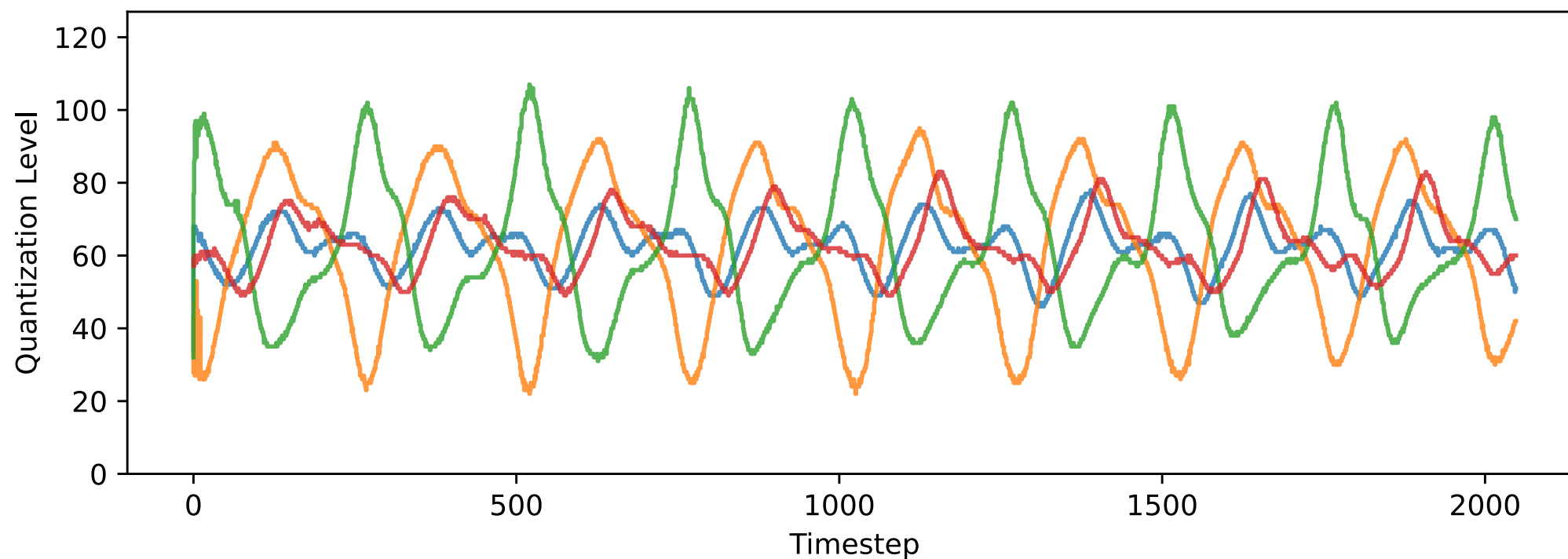
### Conclusion

- (+) Generally higher validation acc. at earlier training epochs.
- (+) Similar picture if validation unrolling > train unrolling steps.



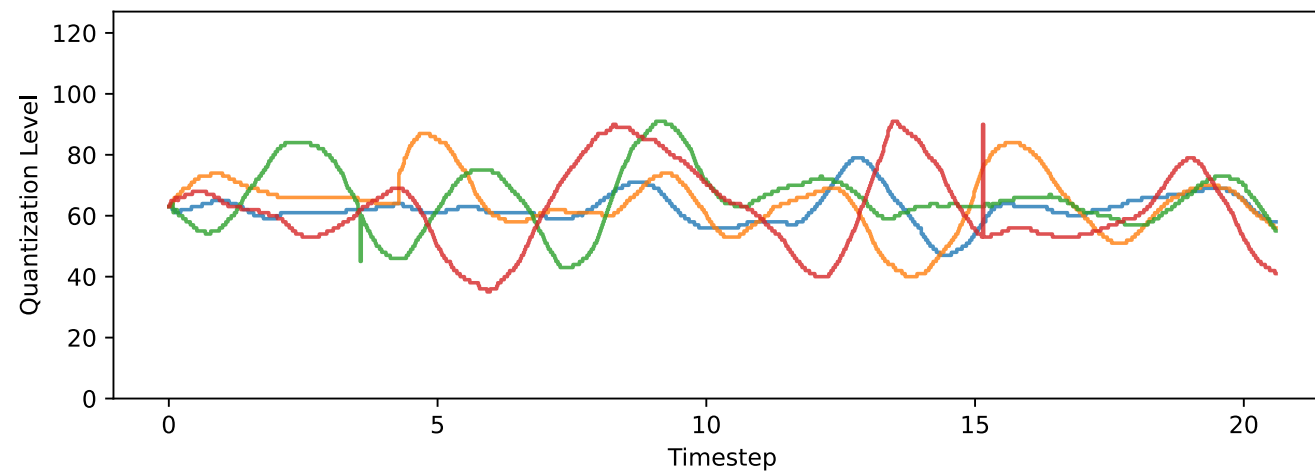
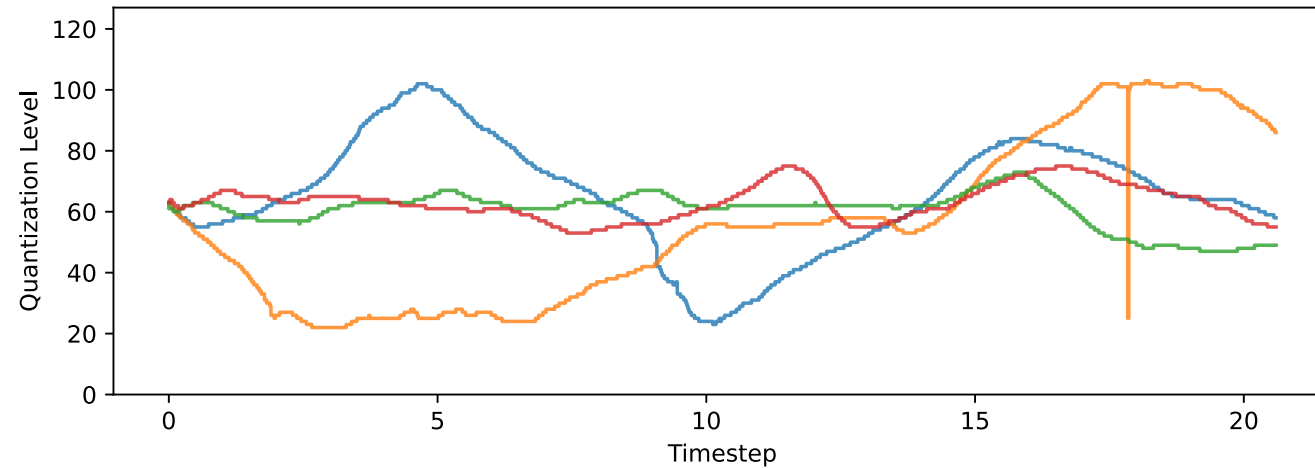
## Generative Results

The following graph shows four samples drawn from the models' prior distribution (periodicity fixed in training).



# Conditional Generative Results

The following graphs depict samples using different periodicity conditions: Large period (~20secs), short periods (~5secs). Model trained without unrolling.



## Runtime Performance Results

The plot to the left shows default (blue) and fast (orange) sample generation\* using 64 wave-channels, 8 quantization levels and 32 batch-size.

### Conclusion

- (+) Fast method avoids exponential inference time as layer depth increases.
- (-) Code overhead is considerable.

