

**Math 126, Fall 2019**  
**Introduction to Partial Differential Equation**  
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# Contents

<b>1</b>	<b>where PDEs Come From</b>	<b>1</b>
1.1	What is a partial differential equation? . . . . .	1
1.2	Review of Multivariable Calculus . . . . .	3
1.3	Differentiation . . . . .	5
1.4	Integration . . . . .	6
1.5	Derivatives of Integrals . . . . .	6
1.6	Integrals of derivatives . . . . .	7
1.7	Some First-order PDEs . . . . .	8
1.7.1	Constant Coefficient Linear Equation . . . . .	8
1.7.2	Variable Coefficient PDEs . . . . .	9
1.8	Motivation behind PDE . . . . .	10

# Logistics

**Question 0.1**

Why study PDE?

**Answer 0.2.** TL;DR. It's useful.

**Note 0.3.** Office hour : MWF 9-10AM 895 Evans, GSI Office hour : MW 1-3 PM 1049 Evans

## 1 where PDEs Come From

### 1.1 What is a partial differential equation?

**Example 1.1**

This is an example of ODE:

$$u = u(x), \quad \frac{d}{dx}u = u.$$

**Example 1.2**

A PDE consist of the form

$$u = u(x_1, x_2, \dots, x_d), \quad u_{x_k} = \frac{\partial u}{\partial x_k}$$

Where  $x_i$  are scalars.

**Example 1.3**

The most general form of a PDE of first order in two dimension, say  $u = u(x, y)$  and of the form

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0, \quad \text{or} \quad F(x, y, u, u_x, u_y) = 0$$

**Example 1.4**

The most general form of a PDE of second order in two dimension, say  $u = u(x, y)$  and of the form

$$G(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}) = 0$$

**Definition 1.5**

A vector  $x$  is defined as

$$x = \vec{x} = (x_1, x_2, \dots, x_n).$$

**Definition 1.6**

Let  $u$  be a function of vector  $x$  of  $n$ -dimension. The gradient of  $u$  is denoted as

$$\nabla u = (u_{x_1}, u_{x_2}, \dots, u_{x_n})$$

**Example 1.7** 1. Linear transport equation  $u_t + bu_t = 0, \quad b \in \mathbb{R}$

2. Burgher's Equation  $u_t + u \cdot u_x = 0$

3. Laplace's Equation  $u_{xx} + u_{yy} = 0$

4. Hermite Equation  $-(u_{xx} + u_{yy}) = \lambda u, \quad \lambda \in \mathbb{R}$

5. Wave with interaction  $u_{tt} - u_{xx} + u^3 = 0$

6. Linear diffusion with source  $u_t - u_{xx} - f(x, t) = 0$

7. Schroedinger's equation  $u_t - i \cdot u_{xx} = 0$

**Example 1.8 (Cauchy-Riemann Equation)**

$$\begin{cases} u_x &= u_y \\ u_y &= -u_x \end{cases}$$

**Definition 1.9 (Digression to Linear Algebra)**

Let  $\mathcal{L}$  be a operator in a function space  $V$ .  $\mathcal{L}$  is linear if

$$\mathcal{L}(u + v) = \mathcal{L}(u) + \mathcal{L}(v), \quad \mathcal{L}(cu) = c\mathcal{L}(u) \quad \forall v, u \in V, \quad \forall c \in \mathbb{F}.$$

**Definition 1.10**

A PDE is called homogeneous linear PDE if it's of the form  $\mathcal{L}(u) = 0$ . If it's the form  $\mathcal{L}(u) = f$ , then it's called inhomogeneous PDE.

**Remark 1.11**

Things that we are interested in

1. Find analytical formulas for some specific PDE's
2. Well-possessedness
  - Existence (Does there exists a solution?)
  - Uniqueness (Is this the only solution?)
  - Stability (If I change the data slightly, does the solution changes just by a little bit?)
3. Predicting qualitative (and sometimes quantitative) behavior of the solution without having a solution formula.
4. Devise an analyze numerical algorithms to approximate solutions.

**Example 1.12**

Consider the equation

$$\cos(xy)u_x + \sin(e^x)u_yy = e^{x^2 \sin(y)}$$

Let

$$\mathcal{L}(u(x, y)) = \cos(xy)u_x + \sin(e^x)u_{yy}$$

$\mathcal{L}$  is a linear operator, so the PDE is an inhomogeneous linear PDE.

**Theorem 1.13 (Principle of superposition)**

Let  $u_1, u_2, \dots, u_n$  be solutions of  $\mathcal{L}(u_k) = 0$ , and let  $c_1, c_2, \dots, c_n$  be scalars. then

$$u(x) = \sum_{i=1}^n c_i u_i(x) \quad \text{solves} \quad \mathcal{L}(u) = 0$$

**Example 1.14 (Cool problem)**

$$\begin{cases} u_t + u \cdot \nabla u - \mu \Delta u = -\nabla p \\ \operatorname{div} u = 0 \end{cases}$$

where  $u = u(x, y, z, t)$ ,  $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$  is a velocity field, where  $p$  is pressure and  $\mu$  is the viscosity of the liquid.

**1.2 Review of Multivariable Calculus**

**Definition 1.15**

Let  $\vec{x} \in \mathbb{R}^d$ , the **Euclidean length** of  $\vec{x}$  is defined as

$$|\vec{x}| = \sqrt{\sum_{k=1}^d x_k^2}$$

**Definition 1.16** (scalar product)

The dot product of two vectors  $x$  and  $\tilde{x}$  is defined as

$$x \cdot \tilde{x} = \sum_{k=1}^d x_k \cdot \tilde{x}_k$$

**Remark 1.17**

It's clear that  $|x|^2 = x \cdot x$ .

**Definition 1.18**

For  $r > 0$  and  $x_0 \in \mathbb{R}^d$  let

$$B_r = \{x \in \mathbb{R}^d : |x - x_0| < r\}$$

This is a open ball of radius  $r$  centered at  $x_0$ .

**Definition 1.19**

A set  $A \subset \mathbb{R}^d$  is called **open** if for each  $x \in A$ , there exists  $r > 0$  such that  $B_r(x) \subset A$ .

**Definition 1.20**

A set  $V \subset \mathbb{R}^d$  is called closed if  $\mathbb{R}^d \setminus V$  is open.

**Definition 1.21**

The **interior** of  $A$ , denoted as  $\text{int}A$  is are the points in  $A$  such that there exists  $r > 0$  with  $B_r(x) \subset A$ .

**Theorem 1.22**

A set is open if and only if  $\text{int}A = A$ .

**Definition 1.23**

The **closure** of  $A \subset \mathbb{R}^d$  is

$$\overline{A} := A \cup \{\text{limit points of } A\}$$

**Definition 1.24**

The **boundary** of the set  $A \subset \mathbb{R}^d$  is

$$\partial A := A \setminus \text{int} A$$

**Theorem 1.25** (Heine-Borel Theorem)

A set is closed in  $\mathbb{R}^d$  if and only if it's closed and bounded.

### 1.3 Differentiation

**Definition 1.26**

For  $u = u(\vec{x})$  we define the **gradient** of  $u$  as

$$\nabla u = \nabla u(x) := \left( \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d} \right)$$

**Definition 1.27** (Laplace's operator)

$$\Delta u = \Delta u(\vec{x}) := \sum_{k=1}^d \frac{\partial^2 u}{\partial x_k^2} = \sum_{k=1}^d u x_k \cdot x_k.$$

**Definition 1.28**

For a vector field  $F = F(\vec{x})$  we denote its divergence as

$$\text{div} F(x) = \sum_{k=1}^d \frac{\partial F_k}{\partial x_k}$$

**Remark 1.29**

$$\Delta u = \text{div} \nabla u.$$

**Exercise 1.30**

Compute the divergence of  $F(\vec{x}) = \frac{1}{2}\vec{x} = \frac{1}{2}(x_1, x_2)$ ,  $G(\vec{x}) = (-x_2, x_1)$ .

*Solution.* We have  $\text{div} F = 0$  and  $\text{div} G = 0$ . ■

## 1.4 Integration

### Definition 1.31

For  $\omega \subset \mathbb{R}^2, \Omega \subset \mathbb{R}^3$  we define the integral with respect to function  $f$  as

$$\int_{\omega} f(x, y) dx dy, \int_{\Omega} f(x, y, z) dx dy dz$$

### Definition 1.32

We define the volume of  $\Omega \subset \mathbb{R}^d$  as

$$\text{vol}_d(\Omega) = \int_{\Omega} 1 d\vec{x}$$

### Remark 1.33

If  $|f(x)| \leq M$  for all  $x \in \Omega$ , then

$$\left| \int_{\Omega} f(\vec{x}) d\vec{x} \right| \leq \int_{\Omega} |f(\vec{x})| d\vec{x} \leq M \text{vol}_d(\Omega)$$

## 1.5 Derivatives of Integrals

### Question 1.34

Let  $I(t)$  defined as

$$\int_{a(t)}^{b(t)} f(x, t) dx$$

What is  $\frac{dI}{dt}$

### Theorem 1.35

Suppose that  $a, b$  are independent of  $t$ . If both  $f$  and  $\frac{\partial f}{\partial t}$  are continuous on the rectangle  $[a, b] \times [c, d]$  then

$$\frac{d}{dt} I(t) = \int_a^b \frac{\partial f}{\partial t}(x, t) dx \quad \text{for } t \in [c, d].$$



**Theorem 1.36**

If  $f$  and  $\frac{\partial f}{\partial t}$  are continuous on the rectangle  $[A, B] \times [c, d]$ , where  $[a(t), b(t)] \subset [A, B]$  for all  $t \in [c, d]$ , and if  $a(t), b(t)$  are differentiable functions on the interval  $[c, d]$  then

$$\frac{d}{dt}I(t) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x, t) dx + b'(t)f(b(t), t) - a'(t)f(a(t), t) \quad \text{for } t \in [c, d]$$

**Remark 1.37**

Theorem 2.2 works for any dimensions. What about theorem 2.3?

**1.6 Integrals of derivatives****Theorem 1.38 (Divergence theorem)**

Let  $\Omega$  be a “nice” open set in  $\mathbb{R}^d$ , let  $n(\vec{x})$  denote the upward pointing normal of  $\partial\Omega$ . If  $F$  is continuously differentiable in  $\Omega$ , and continuous in  $\bar{\Omega}$ , then

$$\int_{\Omega} \operatorname{div} F dx = \int_{\partial\Omega} F(\vec{x}) \cdot n(\vec{x}) dS(x).$$

**Remark 1.39**

When  $d = 1$ , this is really the fundamental theorem of calculus.

$$\int_a^b F'(x) dx = F(b) - F(a)$$

**Theorem 1.40 (The first vanishing Theorem)**

If  $f \geq 0$  in  $\Omega$  and continuous and  $\int_{\Omega} f(\vec{x}) d\vec{x} = 0$ , then  $f = 0$  in  $\Omega$

**Theorem 1.41 (The second vanishing Theorem)**

If  $f$  is continuous in  $\Omega$  and for all  $D \subset \Omega$  that

$$\int_D f(\vec{x}) d\vec{x} = 0$$

then  $f = 0$  in  $\Omega$ .

*Proof.* Suppose there exists a point  $x$  such that  $f(x) \neq 0$ . Pick a region  $D$  around  $x$  and we can see that  $\int_D f(x) dx \neq 0$ , a contradiction. ■

## 1.7 Some First-order PDEs

### 1.7.1 Constant Coefficient Linear Equation

#### Example 1.42

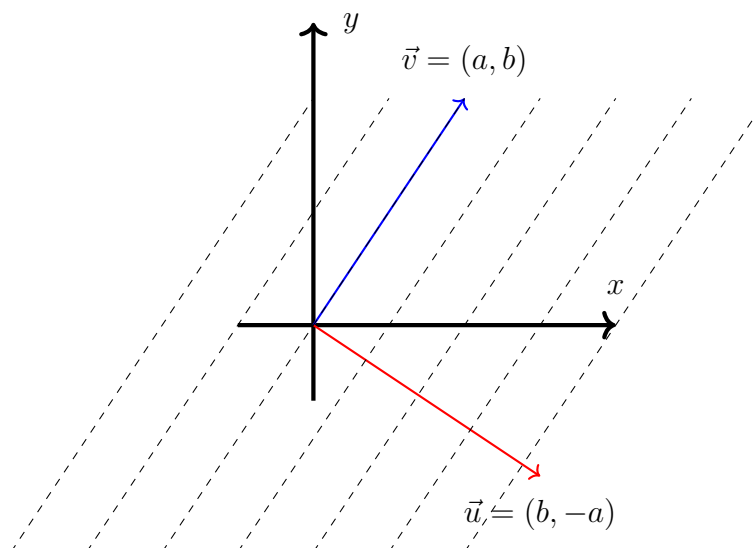
Find  $u = u(x, y)$  such that

$$au_x + bu_y = 0 \quad \text{for } a, b \in \mathbb{R}$$

*Geometric Method.* Notice that we can rewrite the equation as

$$(a, b) \cdot \nabla u = 0$$

Then  $u$  is constant in the direction  $\vec{v} = (a, b)$ . Let  $\vec{w} = (b, -a)$  so that we can form a basis in  $\mathbb{R}^2$ .



since  $u$  is constant on every one of the dashed lines. Then there exists  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$u(x, y) = f(\vec{w} \cdot (x, y)) = f(bx - ay)$$

■

*Brute Force Method.* We can simply compute

$$x' = ax + by$$

$$y' = bx - ay$$

then we have

$$\frac{\partial x'}{\partial x} = a$$

$$\frac{\partial y'}{\partial x} = b$$

$$\frac{\partial x'}{\partial y} = b$$

$$\frac{\partial y'}{\partial y} = -a$$

By the chain rule we have

$$\begin{aligned} u_x &= \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'} \frac{\partial y'}{\partial x} = au_{x'} + bu_{y'} \\ u_y &= \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x'} \frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'} \frac{\partial y'}{\partial y} = bu_{x'} - au_{y'} \end{aligned}$$

Hence

$$0 = au_x + bu_y = a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) - (a^2 \cdot b^2)$$

Therefore  $u_{x'} = 0$ , so  $u(x', y') = f(y') = f(bx - ay)$ , which gives us the same answer as the geometric method. ■

### 1.7.2 Variable Coefficient PDEs

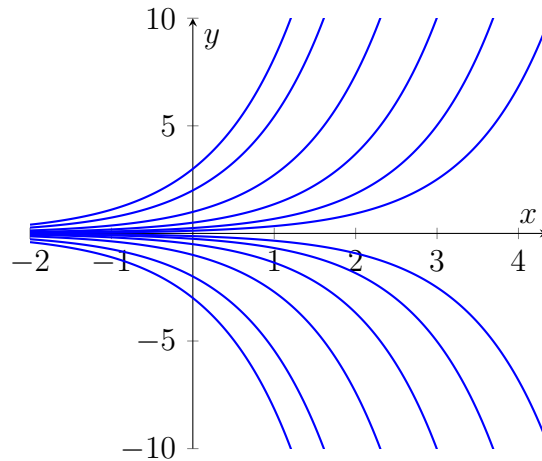
#### Example 1.43

Find  $u = u(x, y)$  such that

$$u_x + yu_y = 0$$

*Solution.* Similar to the first PDE, we can rewrite the equation as

$$(x, y) \cdot \nabla u = 0$$



The blue lines are called characteristic curves.

Hence consider curves with slope  $y$

$$\frac{dy}{dx} = y.$$

Solving for the ODE gives us

$$y = Ce^x \quad C \in \mathbb{R}.$$

Notice that  $u$  is constant on the curves.

$$\frac{d}{dx} (u(x, Ce^x)) = u_x + u_y \cdot \frac{dy}{dx} = u_x + u_y y = 0.$$

Hence

$$u(x, Ce^x) = u(0, Ce^0) = u(0, C)$$

is independent of  $x$ . Now given  $(x, y)$  in the place, we compute

$$u(x, y) = u(x, Ce^x) = u(0, C) = u(0, e^{-x}y)$$

choose  $C \in \mathbb{R}$ , we have  $C = e^{-x}y \iff y = Ce^x$ . Hence

$$u(x, y) = f(e^{-x}y)$$

where  $f$  is a elementary function of one variable. ■

### Exercise 1.44

Find the solution of the example above such that

$$u(0, y) = y^5$$

*Solution.*

$$u(0, y) = y^5 \implies u(x, y) = (e^{-x}y)^5 = y^5 e^{-5x}$$
■

## 1.8 Motivation behind PDE

### Example 1.45 (Transport)

Suppose a pipe with a pollutant suspend in the water and the water is moving along side the pipe to the right at a rate  $c$ , then the concentration of the pollution at time  $t$  and point  $x$  can be modeled as

$$u_t + cu_x = 0.$$

### Example 1.46 (Vibrating String)

Check the derivation in textbook which gives the wave equation. Suppose a string is plucked, then the displacement of the string can be modeled as

$$u_{tt} = c^2 u_{xx}.$$

The three dimensional version, a vibrating drumhead can be expressed as

$$u_{tt} = c^2 (u_{xx} + u_{yy})$$

**Example 1.47 (Diffusion)**

Suppose a chemical substance is diffusing through the liquid. The mass of the substance at time  $t$  in any given cross section  $[x_0, x_1]$  is given by

$$M(t) = \int_{x_0}^{x_1} u(x, t) dx$$

We then have

$$\frac{dM}{dt} = \text{flow in} - \text{flow out} = ku_x(x_1, t) - ku_x(x_0, t) = \int_{x_0}^{x_1} u_t(x, t) dx.$$

Differentiating with respect to  $x$  gives

$$u_t = ku_{xx}.$$

In dim  $d$  we have

$$\forall D \subset D_0 \quad \int_D u_t(x, t) dx \stackrel{\text{Fluid's Law}}{=} \int_{\partial D} k(n \cdot \nabla u) dX \stackrel{\text{div thm}}{=} \int_D \operatorname{div}(k \nabla u) dx = \int_D k \Delta u dx$$

By the second vanishing theorem we have

$$u_t - k \Delta u = 0 \quad \text{in } D_0$$