MATH 126 Homework 5, Due October 4th

Section 5.3

2. (a) On the interval $[-1, 1]$ show that the function x is orthogonal to the constant functions.

Solution. Take $f(x) = c$ to be a constant function for some $c \in \mathbb{R}$. Since $f(x)$ is even and x is odd, then $xf(x)$ must be odd, therefore we have

$$\int_{-1}^{1} xf(x)\,dx = 0.$$

Hence x is orthogonal to any constant function. □

(b) Find a quadratic that is orthogonal to both 1 and x.

Solution. For the sake of notation, we shall denote $\langle f, g\rangle$ as $\int_{-1}^{1} f(x)g(x)\,dx$. Apply Gram–Schmidt process (with help from Wolfram Alpha) we get

$$P_2(x) = \frac{x^2 - \frac{\langle x^2, x\rangle}{\langle x, x\rangle}\cdot x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle}\cdot 1}{\langle x^2, x^2\rangle} = \frac{1}{2}(3x^2 - 1).$$

as desired. □

(c) Find a cubic polynomial that is orthogonal to all quadratics. (*Theses are the first few Legendre polynomials.*)

Solution. For the sake of notation, let $P_2(x) = \frac{1}{2}(3x^2 - 1), P_1(x) = x, P_0(x) = 0$. Apply Gram–Schmidt process (with help from Wolfram Alpha) we get

$$\begin{aligned} P_3(x) &= \frac{x^3 - \frac{\langle x^3, P_2(x)\rangle}{\langle P_2(x), P_2(x)\rangle}\cdot P_2(x) - \frac{\langle x^3, P_1(x)\rangle}{\langle P_1(x), P_1(x)\rangle}\cdot P_1(x) - \frac{\langle x^3, P_0(x)\rangle}{\langle P_0(x), P_0(x)\rangle}\cdot P_0(x)}{\langle x^3, x^3\rangle} \\ &= \frac{1}{2}\left(5x^3 - 3x\right). \end{aligned}$$

□

3. Consider $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u(0,t) = 0, u_x(l,t) = 0$ and the initial conditions $u(x,0) = x, u_t(x,0) = 0$. Find the solution explicitly in series form.

Proof. Since $X(0) = 0$. We then can take $\tilde{X}$ to be the odd extension of X. Observe that $\tilde{X}'' + \lambda X = 0$, and $X'(-l) - X'(l) = 0$. Therefore the eigenvalues and the eigenfunctions are

$$\lambda_n = \frac{\left[\left(n + \frac{1}{2}\right)\pi\right]^2}{l^2}, X_n(x) = \sin\left[\frac{(n + \frac{1}{2})\pi x}{l}\right]$$

Therefore the general series solution to this PDE is

$$u(x,t) = \sum_{n=0}^{\infty}\left[A_n \cos\frac{\left(n + \frac{1}{2}\right)\pi ct}{l} + B_n \sin\frac{\left(n + \frac{1}{2}\right)\pi ct}{l}\right]\sin\frac{\left(n + \frac{1}{2}\right)\pi x}{l}$$

By the boundary condition we have $B_n \cos 0 = B_n \implies B_n = 0$ and

$$a_n = \frac{2}{l}\int_0^l \sin\left[\left(n + \frac{1}{2}\right)\pi x/l\right] \cdot x\,dx = \frac{(-1)^n 2^l}{\left[\left(n + \frac{1}{2}\right)\pi\right]^2}.$$

$\square$

5. (a) Show that the boundary conditions $u(0,t) = 0, u_x(l,t) = 0$ lead to the eigenfunctions $(sin(\pi x/2l), \sin(3\pi x/2l), \sin(5\pi x/2l), \ldots)$.

Proof. We shall proceed with separation of variables. Let $u(x,t) = X(x)T(t)$ we have

$$-X'' = \lambda X, X(0) = 0, X'(l) = 0.$$

By Theorem 3 we know that there is no negative eigenvalues. Suppose 0 is a eigenvalue, then suppose 0 is a an eigenvalue, then we have $u(0,t) = 0$, and it's not very useful. Now suppose $\lambda = \mu^2$ for some $\mu > 0$. then we have

$$X(x) = A\cos\mu X + B\sin\mu x.$$

using the boundary condition we have $A = 0$ and $B\mu\cos\mu l = 0$. Therefore we have

$$\mu_n = \frac{(n+\frac{1}{2})\pi x}{l}$$

Therefore the eigenfunction must be

$$X_n(x) = \sin\frac{\left(n+\frac{1}{2}\right)\pi x}{l}$$

$\square$

(b) If $\phi(x)$ is any function of $(0,l)$, derive the expansion

$$\phi(x) = \sum_{n=0}^{\infty}\left\{\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right\} \qquad (0 < x < l)$$

by the following method. Extend $\phi(x)$ to the function $\tilde{\phi}$ defined by $\tilde{\phi}(x) = \phi(x)$ for $0 \le x \le l$ and $\tilde{\phi}(x) = \phi(2l - x)$ for $l \le x \le 2l$. (*This means you that you extending it evenly across $x = l$.*) Writhe the Fourier sine series for $\tilde{\phi}(x)$ on the interval $(0, 2l)$ and write the formula for the coefficients.

Solution. Let $\tilde{\phi}$ to the extension of ϕ on $(0,2l)$, then the Fourier sine series is

$$\tilde{\phi}(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2l}$$

We then compute for B_k:

$$\begin{aligned}
\int_0^{2l}\tilde{\phi}(x)\sin\frac{k\pi x}{2l}\,dx &= \sum_{n=1}^{\infty}\int_0^{2l}\sin\frac{n\pi x}{2l}\sin\frac{k\pi x}{2l}\,dx \\
&= \frac{B_k}{2}\int_0^{2k} 1 + \cos\frac{k\pi x}{l}\,dx \\
&= B_k l
\end{aligned}$$

Therefore the coefficient for $\tilde{\phi}(x)$ is

$$B_n = \frac{1}{l}\int_0^{2l}\tilde{\phi}(x)\sin\frac{n\pi x}{2l}\,dx.$$

$\square$

(c) Show that every second coefficient vanishes.

Proof. Observe that by the symmetric properties of $\phi(x)$ we have

$$B_n = \frac{1}{l}\int_o^{2l} \tilde{\phi}(x)\sin\frac{n\pi x}{2l}\,dx = \frac{1}{l}\int_0^1 (1-(-1)^n)\phi(x)\sin\frac{n\pi x}{2l}\,dx$$

Then we n is even we have $B_n = 0$ as desired. □

(d) Rewrite the formula for C_n as an integral of the original function $\phi(x)$ on the interval $(0, l)$.

Proof. By part(c) we have

$$B_{2k} = 0, B_{2k+1} = \frac{2}{l}\int_0^l \phi(x)\sin\frac{(2k-1)\pi x}{2l}\,dx.$$

Therefore the Fourier series is

$$\tilde{\phi}(x) = \frac{2}{l}\sum_{k=1}^{\infty}\left[\int_0^l \phi(y)\sin\frac{(2k-1)\pi y}{2l} dy\right]\sin\frac{(2k-1)\pi x}{2l}$$

which gives the coefficient

$$C_n = \int_0^l \phi(x)\sin\frac{(2n-1)\pi x}{2l}\,dx.$$

□

8. Show directly that $(-X_1'X_2 + X_1X_2')|_a^b = 0$ if both X_1 and X_2 satisfies the same Robin boundary condition at $x = a$ and $x = b$.

Proof. Suppose both X_1 and X_2 satisfies the same Robin boundary condition at $x = b$, then we have

$$X_1'(a) - a_a X_1(a) = X_2'(a) - a_a X_2(a) = 0$$
$$X_1'(b) + a_b X_1(b) = X_2'(b) + a_b X_2(b) = 0$$

We then compute

$$\begin{aligned}(-X_1'X_2 + X_1X_2')|_a^b &= -X_1'(b)X_2(b) + X_1(b)X_2'(b) + X_1'(a)X_2(b) - X_1(a)'X_2(a)\\ &= (a_bX_1(b) - X_1(b))X_2(b) + (a_aX_2(a) - X_2(a))X_1(a)\\ &= 0 \cdot X_2(b) + 0 \cdot X_1(a)\\ &= 0\end{aligned}$$

As desired. □

9. Show that the boundary conditions

$$X(b) = \alpha X(a) + \beta X'(a) \quad \text{and} \quad X'(b) = \gamma X(a) + \delta X'(a)$$

on an interval $a \leq x \leq b$ are symmetric if and only if $\alpha\delta - \beta\gamma = 1$.

Proof. Let X_1 and X_2 be the eigenfunctions that satisfy the boundary condition

$$\begin{cases} X_j(b) = \alpha X_j(a) + \beta X_j'(a) \\ X_j'(b) = \gamma X_j(a) + \delta X_j'(a) \end{cases}$$

For $j = 1, 2$. For the symmetric property we have

$$\begin{aligned} (X_1'X_2 - X_1X_2')\big|_a^b &= X_1'(b)X_2(b) - X_1(b)X_2'(b) - X_1'(a)X_2(a) + X_1(a)X_2'(a) \\ &= (\alpha\delta - \beta\gamma - 1)(X_1X_2)'(a). \end{aligned}$$

Therefore the boundary condition are symmetric iff $\alpha\delta - \beta\gamma = 1$.

$\square$

10. (*The Gram-Schmidt orthogonalization procedure*) If X_1, X_2 is any sequence (finite or infinite) of linearly independent vectors in any vector space with an inner product, it can be replaced by a sequence of linear combination that are mutually orthogonal. The idea is that at each step on subtracts off the components parallel to the previous vectors. The procedure is as follows. First we let $Z_1 = X_1/||X_1||$. Second, we define

$$Y_2 = X_2 - (X_2, Z_1)Z_1 \quad \text{and} \quad Z_2 = \frac{Y_2}{||Y_2||}$$

Third, we define

$$Y_3 = X_3 - (X_3, Z_2)Z_2 - (X_3, Z_1)Z_1 \quad \text{and} \quad Z_3 = \frac{Y_3}{||Y_3||},$$

and so on.

(a) Show that all the vectors $Z_1, Z_2, Z_3, \dots$ are orthogonal to each other.

Proof. We shall proceed with induction. We compute

$$\langle Z_2, Z_1 \rangle = \langle Z_2, Z_1 \rangle - \langle X_2, Z_1 \rangle \cdot \langle Z_1, Z_1 \rangle = 0.$$

Therefore the base case holds. Now suppose that the list

$$Z_1, Z_2, \dots, Z_k$$

are orthogonal to each other for some $k > 1$. Then for each $j = 1, 2, \dots, n$ we have

$$\begin{aligned}(Z_j, Z_{k+1}) &= \frac{\Big((Z_j, X_{k+1}) - \sum_{k=1}^{k} (X_{k+1}, Z_k)\,(Z_j, Z_k)\Big)}{\|Y_{k+1}\|} \\ &= \frac{((Z_j, X_{k+1}) - (Z_j, X_{k+1}))}{\|Y_{k+1}\|} \\ &= 0\end{aligned}$$

Therefore the inductive Hypothesis holds and the Gram Schmidt process works for all $n \in \mathbb{N}$.

□

(b) Apply the procedure to the pair of functions $\cos x + \cos 2x$ and $3\cos x - 4\cos 2x$ in the interval $(0, \pi)$ to get and orthogonal pair.

Solution. We compute

$$Z_1 = \frac{\cos x + \cos 2x}{\sqrt{\int_0^\pi (\cos x + \cos 2x)^2 dx}} = (\cos x + \cos 2x)/\sqrt{\pi}$$

$$Y_2 = 3\cos x - 4\cos 2x - Z_1 \int_0^\pi (3\cos x - 4\cos 2x) Z_1 dx = 7(\cos x - \cos 2x)/2$$

$$Z_2 = \frac{Y_2}{\sqrt{\int_0^\pi Y_2^2 dx}} = (\cos x - \cos 2x)/\sqrt{\pi}$$

Therefore we have $Z_1 = (\cos x + \cos 2x)/\sqrt{\pi}$, $Z_2 = (\cos x - \cos 2x)/\sqrt{\pi}$. □

Section 5.4

2. Consider any series of functions on any finite interval. Show that if it converges uniformly, then it also converges in the L^2 sense and in the point-wise sense.

Proof. Let $f_n(x)$ be a series that converges uniformly to $f(x)$ in some interval $[a, b]$. Then fix x_0 in $[a, b]$, for any $\varepsilon > 0$ there exists a positive N_0 such that $|f(x_0) - f_n(x_0)| < \varepsilon$ for all $n > N_0$. Therefore $\lim_{n\to\infty} |f(x_0) - f_n(x_0)| = 0$, or $f_n(x)$ converges point-wise.

Now for all $n \in \mathbb{N}$ we have

$$0 \leq \int_a^b |f(x) - f_n(x)|^2 \, dx \leq \int_a^b \max_{a \leq x \leq b} |f(x) - f_n(x)|^2 \, dx$$

Since $f_n(x)$ converges point-wise to $f(x)$, taking the limit gives us

$$\lim_{n\to\infty} 0 \leq \int_a^b |f(x) - f_n(x)|^2 \, dx \leq \int_a^b \max_{a \leq x \leq b} |f(x) - f_n(x)|^2 \, dx = (b - a) \cdot 0^2 = 0$$

Therefore $\lim_{n\to\infty} 0 \leq \int_a^b |f(x) - f_n(x)|^2 \, dx = 0$ by the Squeeze Theorem, or $f(x)$ converge in the L^2 sense. $\square$

3. Let γ_n be a sequence of constants tending to ∞. Let $f_n(x)$ be the sequence of functions defined as follows: $f_n(\frac{1}{2}) = 0, f_n(x) = \gamma_n$ in the interval $\left[\frac{1}{2} - \frac{1}{n}, \frac{1}{2}\right)$, let $f_n(x) = -\gamma_n$ in the interval $\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{n}\right]$ and let $f_n(x) = 0$ elsewhere. Show that:

   (a) $f_n(x) \to 0$ point-wise.

   Proof. Take $x_0 \in \mathbb{R}$. If $x_0 = \frac{1}{2}$ Then we can see that $f_n(x_0) = 0$ for all $n \in \mathbb{N}$. Now suppose $x_0 \neq \frac{1}{2}$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, then there exists N_0 such that $|x_0 - \frac{1}{2}| < \frac{1}{n}$ for all $n > N_0$. Therefore we have $f(x) = 0$ for all $n > N_0$. Since the choice of x_0 is arbitrary, we have $f_n \to 0$ point-wise. ☐

   (b) The convergence is not uniform.

   Proof. We pick x_n to be $\frac{1}{2} + \frac{1}{n}$. Then we can see that $\lim_{n\to\infty} = \frac{1}{2}$. However, we have $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \gamma_n = \infty$. Therefore the convergence is not uniform. ☐

   (c) $f_n(x) \to 0$ in the L^2 sense if $\gamma_n = n^{1/3}$.

   Proof. We compute

$$\int |f_n(x)|^2 \, dx = \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} \gamma_n^2 \, dx + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} \gamma_n^2 \, dx = \frac{2\gamma_n^2}{n}.$$

   Now let $\gamma_n = n^{1/3}$ we hvae

$$\lim_{n\to\infty} \frac{2\left(n^{1/3}\right)^2}{n} = 0.$$

   Therefore $f_n \to 0$ in the L^2 sense if $\gamma_n = n^{1/3}$. ☐

   (d) $f_n(x)$ does not converge in the L^2 sense if $\gamma_n = n$.

   Proof. Following part(c), let $\gamma_n = n$, we have

$$\lim_{n\to\infty} \frac{2n^2}{n} = \lim_{n\to\infty} 2n = \infty.$$

   Therefore $f_n(x) \not\to 0$ in the L^2 sense if $\gamma_n = n$. ☐

4. Let

$$g_n(x) = \begin{cases} 1 \text{ in the interval } \left[\dfrac{1}{4} - \dfrac{1}{n^2}, \dfrac{1}{4} + \dfrac{1}{n^2}\right) & \text{for odd } n; \\ 1 \text{ in the interval } \left[\dfrac{3}{4} - \dfrac{1}{n^2}, \dfrac{3}{4} + \dfrac{1}{n^2}\right) & \text{for even } n; \\ 0 & \text{for all other } x. \end{cases}$$

Show that $g_n(x) \to 0$ in the L^2 sense but that $g_n(x)$ does not tend to zero in the point-wise sense.

Proof. Suppose n is odd, then we have

$$\lim_{n\to\infty} \int |f_{2n+1}^2(x)|^2 \, dx = \int_{\frac{1}{4}-\frac{1}{(2n+1)^2}}^{\frac{1}{4}+\frac{1}{(2n+1)^2}} 1^2 \, dx = \lim_{n\to\infty} \frac{2}{(2n+1)^2} = 0.$$

Now suppose n is even, then we have

$$\lim_{n\to\infty} \int |f_{2n}(x)|^2 \, dx = \int_{\frac{3}{4}-\frac{1}{(2n)^2}}^{\frac{3}{4}+\frac{1}{(2n)^2}} 1^2 \, dx = \lim_{n\to\infty} \frac{2}{(2n)^2} = 0.$$

Therefore for all n we have $\lim_{n\to\infty} g_n^2(x) \, dx = 0$. Hence $g_n(x)$ converge in the L^2 sense. Now pick $x_0 \in \mathbb{R}$. Then since $\lim_{n\to\infty} \frac{1}{4} - \frac{1}{n^2} = 0$, there exists N_0 such that for all $n > N_0$ we have $|x_0 - \frac{1}{4}| < \frac{1}{n^2}$. Then we have $\lim_{n\to\infty} f_n(x_0) = 1$ for all odd $n > 0$. Therefore $g_n(x) \not\to 0$ in the point-wise sense. $\square$

8. Consider the Fourier sine series of each of the following functions. In this exercise do not compute the coefficients but use the general convergence theorems (Theorems 2, 3, and 4) to discuss the convergence of each of the series in the point-wise, uniform, and L 2 senses.

   (a) $f(x) = x^3$ on $(0, l)$.

   Proof. **Uniform:** We can see that $f(x) = x^3, f'(x) = 3x^2, f''(x) = 6x$ exists are all continuous and we can see that $f(l) = l^3 \neq 0$, therefore the boundary conditions are satisfied here. By Theorem 2 the Fourier Sine series converges uniformly to $f(x)$ on $(0, l)$.
   Point-wise: Since $f(x)$ converges uniformly, by Problem 2 in Section 5.4 it must also converge point-wise.
   L^2 **sense:** Since $f(x)$ converges uniformly, by Problem 2 in Section 5.4 it must also converge in the L^2 sense.

   □

   (b) $f(x) = lx - x^2$ on $(0, l)$.

   Proof. **Uniform:** We can see that $f(x) = lx - x^2, f'(x) =; = 2x, f'''(x) = -2$ exist and are all continuous and we can see that $f(0) = f(l) = 0$. Therefore $f(x)$ satisfies the boundary condition. By Theorem 2 the Fourier Sine series converge uniformly to $f(x)$ on $(0, l)$.
   Point-wise: Since $f(x)$ converges uniformly, by Problem 2 in Section 5.4 it must also converge point-wise.
   L^2 **sense:** Since $f(x)$ converges uniformly, by Problem 2 in Section 5.4 it must also converge in the L^2 sense.

   □

   (c) $f(x) = x^{-2}$ on $(0, l)$.

   Proof. **Uniform:** Since $f(x)$ does not converge in the L^2 sense, it cannot converge uniformly by Problem 2 in Section 5.4.
   Point-wise: Observe that $f(x)$ is piece wise continuous on $(0, l)$ and $f(0) = 0$. $f'(x) = -2x^{-3}$ implies the derivative exists and is continuous on $[0, l]$. By Theorem 4 the Fourier Sine series converges point-wise on $(0, l)$.
   L^2 **sense:** We compute

$$\int_0^l \left(x^{-2}\right)^2 dx = -\frac{l^{-3}}{3} + \lim_{x \to 0} \frac{x^{-3}}{3} = \infty.$$

   Since the integral is not finite, then by Theorem 3 $f(x)$ does not converge in the L^2 sense. □

11. (*Term by term integration*)

(a) If $f(x)$ is a piecewise continuous function in $[-l,l]$,show that its indefinite integral $F(x)=\int_{-l}^{x} f(s)\,ds$ has a full Fourier series that converges point-wise.

Proof. Let the full Fourier series of $f(x)$ be

$$f(x)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{l}+B_n\sin\frac{n\pi x}{l}\right)$$

Integrating term by term gives us

$$F(x)=\frac{1}{2}A_0x+\frac{1}{2}A_0l+\sum_{n=1}^{\infty}\frac{l(-1)^n}{n\pi}B_n+\sum_{n=1}^{\infty}\frac{l}{n\pi}\left(A_n\sin\frac{n\pi x}{l}-B_n\cos\frac{n\pi x}{l}\right)$$

Observe that $F(x)-\frac{1}{2}A_0x$ is a full Fourier Series and $\frac{1}{2}A_0x$ has a full Fourier series on $(-l,l)$ by previous homework. Therefore $F(x)$ has a full Fourier series. Now we want to show that the series converges point-wise. We can see that the full Fourier Series Converges point-wise by Theorem 4. By part(b) we can see that the Fourier series must converge point-wise since the full Fourier series of $f(x)$ converges point-wise. □

(b) Write this convergent series for $f(x)$ explicitly in terms of the Fourier coefficients a_0,a_n,b_n of $f(x)$.
(*Hint:* Apply a convergence Theorem. Write the formulas for the coefficients and integrate by parts.)

Proof. Let A_n,B_n denote the coefficient of the full Fourier series, then we compute

$$\begin{aligned}A_n&=\frac{1}{L}\int_{-L}^{L}F(x)\cos\left(\frac{n\pi}{L}x\right)dx\\&=\frac{1}{n\pi}\int_{-L}^{L}\left(\int_{-L}^{x}f(s)ds\right)d\left(\sin\left(\frac{n\pi}{L}x\right)\right)\\&=-\frac{1}{n\pi}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi}{L}x\right)dx\\&=-\frac{L}{n\pi}b_n\end{aligned}$$

Using a similar logic we can show that

$$B_n=\frac{L}{n\pi}a_n.$$

therefore the full Fourier series of $F(x)$ can be expressed as

$$F(x)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}-\frac{L}{n\pi}b_n\cos\left(\frac{n\pi}{L}x\right)+\frac{L}{n\pi}a_n\sin\left(\frac{n\pi}{L}x\right).$$

□

18. Consider a solution of the wave equation with $c = 1$ on $[0, l]$ with homogeneous Dirichlet or Neumann boundary conditions.

(a) Show that is energy $E = \frac{1}{2}\int_0^l (u_t^2 + u_x^2)$ is a constant.

Proof. It is suffice to show that $\frac{dE}{dt} = 0$ to show that E is constant. We compute

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2}\frac{d}{dt}\int_0^l \left(u_t^2 + u_x^2\right)\, dx \\ &= \int_0^1 u_t \cdot u_{tt} + u_x \cdot u_{xt}\, dx \\ &= \int_0^1 (u_t u_{xx})\, dx + \int_0^l u_x u_{xt}\, dx \\ &= \int_0^l (u_t u_{xx})\, dx + [u_x u_t]|_0^l - \int_0^l (u_t u_{xx})\, dx \\ &= u_x(0)u_t(l) - u_x(0)u_t(0) \end{aligned}$$

by the Dirichlet and Neumann condition we have

$$\mathbf{D} : u(0,t) = u(l,t) = 0 \Longrightarrow u_t(0,t) = u_t(l,t) = 0$$
$$\mathbf{N} : u_x(0,t) = u_x(l,t) = 0$$

Therefore $\frac{dE}{dt} = 0$ and E is constant. □

(b) Let $E_n(t)$ be the energy of its nth harmonic (the nth term in the expansion). Show that $E = \sum E_n$. (*Hint:* Use the orthogonality. Assume that you can integrate term by term.)

Proof. Let h_n be the nth harmonic, then we have the following expansion

$$u(x,t) = \sum_{n=1}^{\infty} h_n.$$

We also have the energy of the nth harmonic

$$E_n = \frac{1}{2}\int_0^l (h_n)_t^2 + (h_n)_x^2\, dx.$$

Then we compute

$$\begin{aligned} E &= \frac{1}{2}\int_0^l u_t^2 + u_x^2\, dx \\ &= \frac{1}{2}\int_0^l \sum_{n=1}^{\infty}(h_n)_t^2 + \sum_{n=1}^{\infty}(h_n)_x^2\, dx \\ &= \sum_{n=1}^{\infty}\frac{1}{2}\int_0^l (h_n)_t^2 + (h_n)_x^2\, dx \\ &= \sum_{n=1}^{\infty} E_n \end{aligned}$$

□

Section 5.5

3. Prove the inequality $l\int_0^l (f'(x))^2 \, dx \geq [f(l) - f(0)]^2$ for any real function $f(x)$ whose derivative $f'(x)$ is continuous. [*Hint:* Use Schwarz's inequality with the pair $f'(x)$ and 1.]

   Proof. Following the hint we have

$$l\int_0^l (f'(x))^2 \, dx = \int_0^1 1^2 \, dx \int_0^l (f'(x))^2 \, dx = ||1||^2 \cdot ||' f(x)||^2 \overset{(1)}{\geq} |\langle f', 1\rangle|^2 = [f(l) - f(0)]^2.$$

   Where (1) is by Schwartz inequality. □

5. Prove the *Schwarz inequality* for infinite series:

$$\sum a_n b_n \leq \left(\sum a_n^2\right)^{1/2} \left(\sum b_n^2\right)^{1/2}.$$

(*Hint:* See the hint in Exercise 2. Prove it first for finite series (ordinary sums) and then pass to the limit.

Proof. Let's prove the finite case first. Let

$$f(x) = \sum_{i=1}^{n} (a_i x + b_i)^2.$$

Observe that $(a_k x + b_k)^2 \geq 0$. Therefore $f(x) \geq 0$ for all k in $1, 2, \ldots, n$. Also we have

$$f(x) = \sum_{i=1}^{n} a_i^2 + 2\left(\sum_{i=1}^{n} a_i b_i\right) + \sum_{i=1}^{n} b_i^2.$$

Since $f(x) \geq 0$ for all x and $f(x)$ is a quadratic then we have $\Delta \leq 0$, therefore we have.

$$\begin{aligned}
\left(2\sum_{i=1}^{n} a_i b_i\right)^2 - 4\left(\sum_{i=1}^{n} a_i^2\right) \cdot \left(\sum_{i=1}^{n} b_i^2\right) &\leq 0 \\
4\left(\sum_{i=1}^{n} a_i b_i\right)^2 &\leq 4\left(\sum_{i=1}^{n} a_i^2\right) \cdot \left(\sum_{i=1}^{n} b_i^2\right) \\
\left(\sum_{i=1}^{n} a_i b_i\right)^2 &\leq \left(\sum_{i=1}^{n} a_i^2\right) \cdot \left(\sum_{i=1}^{n} b_i^2\right) \\
\sum_{i=1}^{n} a_i b_i &\leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \cdot \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}
\end{aligned}$$

Now following the hint and let n tends to infinity and we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i b_i \leq \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}$$

to obtain

$$\sum a_n b_n \leq \left(\sum a_n^2\right)^{1/2} \left(\sum b_n^2\right)^{1/2}$$

as desired. ☐

12. Show that if $f(x)$ is a C^1 function in $[-\pi, \pi]$ that satisfies the periodic BC and if $\int_{-\pi}^{\pi} |f|^2\, dx \leq \int_{-\pi}^{\pi} |f'|^2\, dx$. (*Hint:* Use Parseval's equality.)

Proof. By Parseval's equality we have

$$\begin{aligned}
\int_{-\pi}^{\pi} |f(x)^2|\, dx &= \sum_{n=1}^{\infty} \left[|A_n|^2 \int_{-\pi}^{\pi} |\cos nx|^2\, dx + |B_n|^2 \int_{-\pi}^{\pi} |\sin nx|^2\, dx \right] \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left[A_n^2 \int_{-\pi}^{\pi} \cos 2nx\, dx + \int_{-\pi}^{\pi} 1 - \sin 2nx\, dx \right] \\
&= \pi \sum_{n=1}^{\infty} \left[A_n^2 + B_n^2 \right]
\end{aligned}$$

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Using a similar logic we can show that

$$\int_{-\pi}^{\pi} |f'(x)|^2\, dx = \pi \sum_{n=1}^{\infty} \left[(A_n')^2 + (B_n')^2 \right]$$

Now we want show that $B_n = \frac{1}{n} A_n'$. We compute

$$\begin{aligned}
B_n &= \frac{1}{\pi} \int_{\pi}^{\pi} f(x) \sin nx\, dx \\
&= -\frac{1}{n\pi} f(x) \cos nx \Big|_{-\pi}^{\pi} + \frac{1}{n\pi} \int_{-\pi}^{\pi} f'(x) \cos nx\, dx \\
&= \frac{1}{n\pi} \int_{-\pi}^{\pi} f'(x) \cos nx\, dx = A_n'
\end{aligned}$$

Similarly, we can show that $A_n = -\frac{1}{n} B_n'$. Therefore we have

$$\begin{aligned}
\int_{-\pi}^{\pi} |f(x)|^2\, dx &= \pi \sum_{n=1}^{\infty} \left[A_n^2 + B_n^2 \right] \\
&= \pi \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{n=1}^{\infty} \left[(A_n')^2 + (B_n')^2 \right] \\
&\leq \pi \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \left[(A_n')^2 + (B_n')^2 \right] \\
&= \int_{-\pi}^{\pi} |f'(x)|^2\, dx
\end{aligned}$$

As desired.

$\square$