# Math 126, Fall 2019

## Introduction to Partial Differential Equation

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## Contents

# Logistics

**Question 0.1**
Why study PDE?

**Answer 0.2.** TL;DR. It's useful.

**Note 0.3.** Office hour : MWF 9-10AM 895 Evans, GSI Office hour : MW 1-3 PM 1049 Evans

# 1 where PDEs Come From

## 1.1 What is a partial differential equation?

**Example 1.1**
This is an example of ODE:

$$u = u(x), \qquad \frac{d}{dx}u = u.$$

**Example 1.2**
A PDE consist of the form

$$u = u(x_1, x_2, \ldots, x_d), \qquad u_{x_k} = \frac{\partial u}{\partial x_k}$$

Where $x_i$ are scalars.

**Example 1.3**
The most general form of a PDE of first order in two dimension, say $u = u(x, y)$ and of the form

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0, \quad \text{or} \quad F(x, y, u, u_x, u_y) = 0$$

**Example 1.4**
The most general form of a PDE of second order in two dimension, say $u = u(x, y)$ and of the form

$$G(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}) = 0$$

**Definition 1.5**

A vector $x$ is defined as

$$x = \vec{x} = (x_1, x_2, \ldots, x_n).$$

**Definition 1.6**

Let $u$ be a function of vector $x$ of $n$-dimension. The gradient of $u$ is denoted as

$$\nabla u = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})$$

**Example 1.7**

1. Linear transport equation $\quad u_t + bu_t = 0, \quad b \in \mathbb{R}$
2. Burgher's Equation $\quad u_t + u \cdot u_x = 0$
3. Laplace's Equation $\quad u_{xx} + u_{yy} = 0$
4. Hermite Equation $\quad -(u_{xx} + u_{yy}) = \lambda u, \quad \lambda \in \mathbb{R}$
5. Wave with interaction $\quad u_{tt} - u_{xx} + u^3 = 0$
6. Linear diffusion with source $\quad u_t - u_{xx} - f(x,t) = 0$
7. Schroedinger's equation $\quad u_t - i \cdot u_{xx} = 0$

**Example 1.8** (Cauchy-Riemann Equation)

$$\begin{cases} u_x & = u_y \\ u_y & = -u_x \end{cases}$$

**Definition 1.9** (Digression to Linear Algebra)

Let $\mathscr{L}$ be a operator in a function space $V$. $\mathscr{L}$ is linear if

$$\mathscr{L}(u+v) = \mathscr{L}(u) + \mathscr{L}(v), \quad \mathscr{L}(cu) = c\mathscr{L}(u) \qquad \forall v, u \in V, \quad \forall c \in \mathbb{F}.$$

**Definition 1.10**

A PDE is called homogeneous linear PDE if it's of the form $\mathscr{L}(u) = 0$. If it's the form $\mathscr{L}(u) = f$, then it's called inhomogeneous PDE.

**Remark 1.11**

Things that we are interested in

1. Find analytical formulas for some specific PDE's
2. Well-possessedness
   - Existence (Does there exists a solution?)
   - Uniqueness (Is this the only solution?)
   - Stability (If I change the data slightly, does the solution changes just by a little bit?)
3. Predicting qualitative (and sometimes quantitative) behavior of the solution without having a solution formula.
4. Devise an analyze numerical algorithms to approximate solutions.

**Example 1.12**

Consider the equation

$$\cos(xy)u_x + \sin\left(e^x\right)u_y y = e^{x^2\sin(y)}$$

Let

$$\mathscr{L}(u(x,y)) = \cos(xy)u_x + \sin\left(e^x\right)u_{yy}$$

$\mathscr{L}$ is a linear operator, so the PDE is an inhomogeneous linear PDE.

**Theorem 1.13** (Principle of superposition)

Let $u_1, u_2, \ldots, u_n$ be solutions of $\mathscr{L}(u_k) = 0$, and let $c_1, c_2, \ldots, c_n$ be scalars. then

$$u(x) = \sum_{i=1}^{n} c_i u_i(x) \quad \text{solves} \quad \mathscr{L}(u) = 0$$

**Example 1.14** (Cool problem)

$$\begin{cases} u_t + u \cdot \nabla u - \mu \triangle u = -\nabla p \\ \operatorname{div} u = 0 \end{cases}$$

where $u = u(x,y,z,t)$, $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ is a velocity field, where $p$ is pressure and $\mu$ is the viscosity of the liquid.

# Review of Multivariable Calculus

**Definition 1.15**
Let $\vec{x} \in \mathbb{R}^d$, the **Euclidean length** of $\vec{x}$ is defined as

$$|\vec{x}| = \sqrt{\sum_{k=1}^{d} x_k^2}$$

**Definition 1.16** (scalar product)
The dot product of two vectors $x$ and $\tilde{x}$ is defined as

$$x \cdot \tilde{x} = \sum_{k=1}^{d} x_k \cdot \tilde{x}_k$$

**Remark 1.17**
It's clear that $|x|^2 = x \cdot x$.

**Definition 1.18**
For $r > 0$ and $x_0 \in \mathbb{R}^d$ let

$$B_r = \left\{ x \in \mathbb{R}^d : |x - x_0| < r \right\}$$

This is a open ball of radius $r$ centered at $x_0$.

**Definition 1.19**
A set $A \subset \mathbb{R}^d$ is called **open** if for each $x \in A$, there exists $r > 0$ such that $B_r(x) \subset A$.

**Definition 1.20**
A set $V \subset \mathbb{R}^d$ is called closed if $\mathbb{R}^d \setminus V$ is open.

**Definition 1.21**
The **interior** of $A$, denoted as $\text{int}A$ is are the points in $A$ such that there exists $r > 0$ with $B_r(x) \subset A$.

**Theorem 1.22**
A set is open if and only if $\mathrm{int} A = A$.

**Definition 1.23**
The **closure** of $A \subset \mathbb{R}^d$ is

$$\overline{A} := A \cup \{ \text{ limit points of } A\}$$

**Definition 1.24**
The boundary of the set $A \subset \mathbb{R}^d$ is

$$\partial A := A \setminus \mathrm{int} A$$

**Theorem 1.25** (Heine-Borel Theorem)
A set is closed in $\mathbb{R}^d$ if and only if it's closed and bounded.

## 1.2 Differentiation

**Definition 1.26**
For $u = u(\vec{x})$ we define the **gradient** of $u$ as

$$\nabla u = \nabla u(x) := \left( \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_d} \right)$$

**Definition 1.27** (Laplace's operator)
$\triangle u = \triangle u(\vec{x}) := \sum_{k=1}^{d} \frac{\partial^2 u}{\partial x_k^2} = \sum_{k=1}^{d} u x_k \cdot x_k$.

**Definition 1.28**
For a vector field $F = F(\vec{x})$ we denote its divergence as

$$\mathrm{div} F(x) = \sum_{k=1}^{d} \frac{\partial F_k}{\partial x_k}$$

**Remark 1.29**
$\triangle u = \mathrm{div} \nabla u$.

**Exercise 1.30**

Compute the divergence of $F(\vec{x}) = \frac{1}{2}\vec{x} = \frac{1}{2}(x_1, x_2), G(\vec{x}) = (-x_2, x_1)$.

*Solution.* We have $\text{div}F = 0$ and $\text{div}G = 0$. ∎

## 1.3 Integration

**Definition 1.31**

For $\omega \subset \mathbb{R}^2, \Omega \subset \mathbb{R}^3$ we definite the integral with respect to function $f$ as

$$\int_\omega f(x,y)\,dx\,dy, \int_\Omega f(x,y,z)\,dx\,dy\,dz$$

**Definition 1.32**

We define the volume of $\Omega \subset\in \mathbb{R}^d$ as

$$\text{vol}_d(\Omega) = \int_\Omega 1 d\vec{x}$$

**Remark 1.33**

If $|f(x)| \leq M$ for all $x \in \Omega$, then

$$\left| \int_\Omega f(\vec{x})\,d\vec{x} \right| \leq \int_\Omega |f(\vec{x})|\,d\vec{x} \leq M\text{vol}_d(\Omega)$$

# 2 Derivatives of Integrals

**Question 2.1**

Let $I(t)$ defined as

$$\int_{a(t)}^{b(t)} f(x,t)\,dx$$

What is $\frac{dI}{dt}$

**Theorem 2.2**

Suppose that $a, b$ are independent of $t$. If both $f$ and $\frac{\partial f}{\partial t}$ are continuous on the rectangle $[a,b] \times [c,d]$ then

$$\frac{d}{dt}I(t) = \int_a^b \frac{\partial t}{\partial t}(x,t)\,dx \qquad \text{for} \quad t \in [c,d].$$

**Theorem 2.3**

If $f$ and $\frac{\partial f}{\partial t}$ are continuous on the rectangle $[A, B] \times [c, d]$, where $[a(t), b(t)] \subset [A, B]$ for all $t \in [c, d]$, and if $a(t), b(t)$ are differentiable functions on the interval $[c, d]$ then

$$\frac{d}{dt} I(t) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x, t)\, dx + b'(t) f(b(t), t) - a'(t) f(a(t), t) \quad \text{for} \quad t \in [c, d]$$

**Remark 2.4**

Theorem 2.2 works for any dimensions. What about theorem 2.3?

# 3 Integrals of derivatives

**Theorem 3.1** (Divergence theorem)

Let $\Omega$ be a "nice" open set in $\mathbb{R}^d$, let $n(\vec{x})$ denote the upward pointing normal of $\partial\Omega$. If $F$ is continuously differentiable in $\Omega$, and continuous in $\overline{\Omega}$, then

$$\int_{\Omega} \text{div} F\, dx = \int_{\partial\Omega} F(\vec{x}) \cdot n(\vec{x}) dS(x).$$

**Remark 3.2**

When $d = 1$, this is really the fundamental theorem of calculus.

$$\int_a^n F'(x)\, dx = F(b) - F(a)$$

**Theorem 3.3** (The first vanishing Theorem)

If $f \geq 0$ in $\Omega$ and continuous and $\int_{\Omega} f(\vec{x})\, d\vec{x} = 0$, then $f = 0$ in $\Omega$

**Theorem 3.4** (The second vanishing Theorem)

If $f$ is continuous in $\Omega$ and for all $D \subset \Omega$ that

$$\int_D f(\vec{x})\, d\vec{x} = 0$$

then $f = 0$ in $\Omega$.

*Proof.* Suppose there exists a point $x$ such that $f(x) \neq 0$. Pick a region $D$ around $x$ and we can see that $\int_D f(x)\, dx \neq 0$, a contradiction. ■