Math 126, Fall 2019

Introduction to Partial Differential Equation

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Contents

Logistics

Question 0.1
Why study PDE?

Answer 0.2. TL;DR. It's useful.

Note 0.3. Office hour : MWF 9-10AM 895 Evans, GSI Office hour : MW 1-3 PM 1049 Evans

1 where PDEs Come From

1.1 What is a partial differential equation?

Example 1.1
This is an example of ODE:

$$u = u(x), \qquad \frac{d}{dx}u = u.$$

Example 1.2
A PDE consist of the form

$$u = u(x_1, x_2, \ldots, x_d), \qquad u_{x_k} = \frac{\partial u}{\partial x_k}$$

Where x_i are scalars.

Example 1.3
The most general form of a PDE of first order in two dimension, say $u = u(x, y)$ and of the form

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0, \quad \text{or} \quad F(x, y, u, u_x, u_y) = 0$$

Example 1.4
The most general form of a PDE of second order in two dimension, say $u = u(x, y)$ and of the form

$$G(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}) = 0$$

Definition 1.5
A vector x is defined as
$$x = \vec{x} = (x_1, x_2, \ldots, x_n).$$

Definition 1.6
Let u be a function of vector x of n-dimension. The gradient of u is denoted as
$$\nabla u = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})$$

Example 1.7
1. Linear transport equation $\quad u_t + bu_t = 0, \quad b \in \mathbb{R}$
2. Burgher's Equation $\quad u_t + u \cdot u_x = 0$
3. Laplace's Equation $\quad u_{xx} + u_{yy} = 0$
4. Hermite Equation $\quad -(u_{xx} + u_{yy}) = \lambda u, \quad \lambda \in \mathbb{R}$
5. Wave with interaction $\quad u_{tt} - u_{xx} + u^3 = 0$
6. Linear diffusion with source $\quad u_t - u_{xx} - f(x,t) = 0$
7. Schroedinger's equation $\quad u_t - i \cdot u_{xx} = 0$

Example 1.8 (Cauchy-Riemann Equation)
$$\begin{cases} u_x &= u_y \\ u_y &= -u_x \end{cases}$$

Definition 1.9 (Digression to Linear Algebra)
Let $\mathscr{L}$ be a operator in a function space V. $\mathscr{L}$ is linear if
$$\mathscr{L}(u+v) = \mathscr{L}(u) + \mathscr{L}(v), \quad \mathscr{L}(cu) = c\mathscr{L}(u) \qquad \forall v, u \in V, \quad \forall c \in \mathbb{F}.$$

Definition 1.10
A PDE is called homogeneous linear PDE if it's of the form $\mathscr{L}(u) = 0$. If it's the form $\mathscr{L}(u) = f$, then it's called inhomogeneous PDE.

Remark 1.11

Things that we are interested in

1. Find analytical formulas for some specific PDE's
2. Well-possessedness
   - Existence (Does there exists a solution?)
   - Uniqueness (Is this the only solution?)
   - Stability (If I change the data slightly, does the solution changes just by a little bit?)
3. Predicting qualitative (and sometimes quantitative) behavior of the solution without having a solution formula.
4. Devise an analyze numerical algorithms to approximate solutions.

Example 1.12

Consider the equation

$$\cos(xy)u_x + \sin\left(e^x\right)u_y y = e^{x^2 \sin(y)}$$

Let

$$\mathscr{L}(u(x,y)) = \cos(xy)u_x + \sin\left(e^x\right)u_{yy}$$

$\mathscr{L}$ is a linear operator, so the PDE is an inhomogeneous linear PDE.

Theorem 1.13 (Principle of superposition)

Let $u_1, u_2, \ldots, u_n$ be solutions of $\mathscr{L}(u_k) = 0$, and let $c_1, c_2, \ldots, c_n$ be scalars. then

$$u(x) = \sum_{i=1}^{n} c_i u_i(x) \quad \text{solves} \quad \mathscr{L}(u) = 0$$

Example 1.14 (Cool problem)

$$\begin{cases} u_t + u \cdot \nabla u - \mu \triangle u = -\nabla p \\ \operatorname{div} u = 0 \end{cases}$$

where $u = u(x,y,z,t)$, $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ is a velocity field, where p is pressure and μ is the viscosity of the liquid.

1.2 Review of Multivariable Calculus

Definition 1.15

Let $\vec{x} \in \mathbb{R}^d$, the **Euclidean length** of $\vec{x}$ is defined as

$$|\vec{x}| = \sqrt{\sum_{k=1}^{d} x_k^2}$$

Definition 1.16 (scalar product)

The dot product of two vectors x and $\tilde{x}$ is defined as

$$x \cdot \tilde{x} = \sum_{k=1}^{d} x_k \cdot \tilde{x}_k$$

Remark 1.17

It's clear that $|x|^2 = x \cdot x$.

Definition 1.18

For $r > 0$ and $x_0 \in \mathbb{R}^d$ let

$$B_r = \left\{x \in \mathbb{R}^d : |x - x_0| < r\right\}$$

This is a open ball of radius r centered at x_0.

Definition 1.19

A set $A \subset \mathbb{R}^d$ is called **open** if for each $x \in A$, there exists $r > 0$ such that $B_r(x) \subset A$.

Definition 1.20

A set $V \subset \mathbb{R}^d$ is called closed if $\mathbb{R}^d \setminus V$ is open.

Definition 1.21

The **interior** of A, denoted as $\text{int}A$ is are the points in A such that there exists $r > 0$ with $B_r(x) \subset A$.

Theorem 1.22

A set is open if and only if $\text{int}A = A$.

Definition 1.23
The **closure** of $A \subset \mathbb{R}^d$ is

$$A^- := A \cup \{ \text{ limit points of } A\}$$

Definition 1.24
The boundary of the set $A \subset \mathbb{R}^d$ is

$$\partial A := A^- \setminus \text{int}A$$

Theorem 1.25 (Heine-Borel Theorem)
A set is closed in $\mathbb{R}^d$ if and only if it's closed and bounded.

1.3 Differentiation

Definition 1.26
For $u = u(\vec{x})$ we define the **gradient** of u as

$$\nabla u = \nabla u(x) := \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_d}\right)$$

Definition 1.27 (Laplace's operator)
$\triangle u = \triangle u(\vec{x}) := \sum_{k=1}^{d} \frac{\partial^2 u}{\partial x_k^2} = \sum_{k=1}^{d} u x_k \cdot x_k$.

Definition 1.28
For a vector field $F = F(\vec{x})$ we denote its divergence as

$$\text{div}F(x) = \sum_{k=1}^{d} \frac{\partial F_k}{\partial x_k}$$

Remark 1.29
$\triangle u = \text{div}\nabla u$.

Exercise 1.30
Compute the divergence of $F(\vec{x}) = \frac{1}{2}\vec{x} = \frac{1}{2}(x_1, x_2), G(\vec{x}) = (-x_2, x_1)$.

Solution. We have $\text{div}F = 0$ and $\text{div}G = 0$. ∎

1.4 Integration

Definition 1.31

For $\omega \subset \mathbb{R}^2, \Omega \subset \mathbb{R}^3$ we definite the integral with respect to function f as

$$\int_\omega f(x,y)\,dx\,dy, \int_\Omega f(x,y,z)\,dx\,dy\,dz$$

Definition 1.32

We define the volume of $\Omega \subset \mathbb{R}^d$ as

$$\mathrm{vol}_d(\Omega) = \int_\Omega 1 d\vec{x}$$

Remark 1.33

If $|f(x)| \leq M$ for all $x \in \Omega$, then

$$\left|\int_\Omega f(\vec{x})\,d\vec{x}\right| \leq \int_\Omega |f(\vec{x})|\,d\vec{x} \leq M \mathrm{vol}_d(\Omega)$$

1.5 Derivatives of Integrals

Question 1.34

Let $I(t)$ defined as

$$\int_{a(t)}^{b(t)} f(x,t)\,dx$$

What is $\frac{dI}{dt}$

Theorem 1.35

Suppose that a, b are independent of t. If both f and $\frac{\partial f}{\partial t}$ are continuous on the rectangle $[a,b] \times [c,d]$ then

$$\frac{d}{dt} I(t) = \int_a^b \frac{\partial t}{\partial t}(x,t)\,dx \qquad \text{for} \quad t \in [c,d].$$

Theorem 1.36

If f and $\frac{\partial f}{\partial t}$ are continuous on the rectangle $[A, B] \times [c, d]$, where $[a(t), b(t)] \subset [A, B]$ for all $t \in [c, d]$, and if $a(t), b(t)$ are differentiable functions on the interval $[c, d]$ then

$$\frac{d}{dt} I(t) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x, t)\, dx + b'(t) f(b(t), t) - a'(t) f(a(t), t) \quad \text{for} \quad t \in [c, d]$$

Remark 1.37

Theorem 2.2 works for any dimensions. What about theorem 2.3?

1.6 Integrals of derivatives

Theorem 1.38 (Divergence theorem)

Let Ω be a "nice" open set in $\mathbb{R}^d$, let $n(\vec{x})$ denote the upward pointing normal of $\partial\Omega$. If F is continuously differentiable in Ω, and continuous in $\overline{\Omega}$, then

$$\int_\Omega \operatorname{div} F \, dx = \int_{\partial\Omega} F(\vec{x}) \cdot n(\vec{x}) dS(x).$$

Remark 1.39

When $d = 1$, this is really the fundamental theorem of calculus.

$$\int_a^n F'(x)\, dx = F(b) - F(a)$$

Theorem 1.40 (The first vanishing Theorem)

If $f \geq 0$ in Ω and continuous and $\int_\Omega f(\vec{x})\, d\vec{x} = 0$, then $f = 0$ in Ω

Theorem 1.41 (The second vanishing Theorem)

If f is continuous in Ω and for all $D \subset \Omega$ that

$$\int_D f(\vec{x})\, d\vec{x} = 0$$

then $f = 0$ in Ω.

Proof. Suppose there exists a point x such that $f(x) \neq 0$. Pick a region D around x and we can see that $\int_D f(x)\, dx \neq 0$, a contradiction. ∎

1.7 Some First-order PDEs

1.7.1 Constant Coefficient Linear Equation

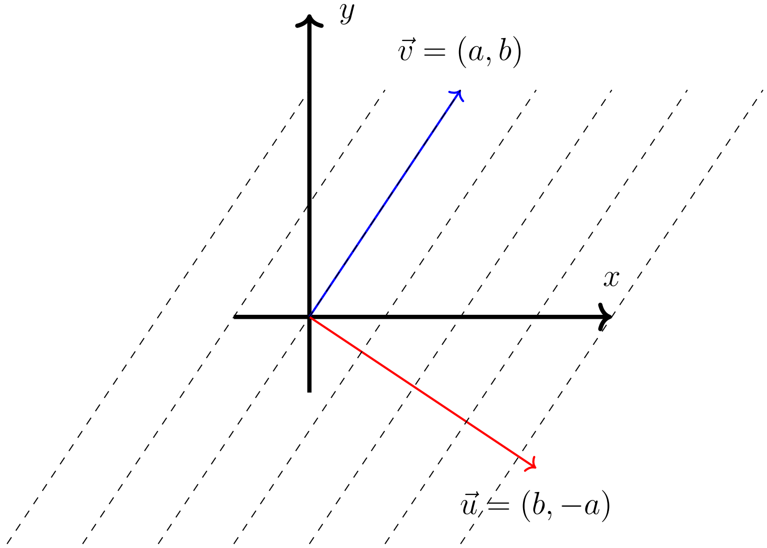
Example 1.42
Find $u = u(x, y)$ such that

$$au_x + bu_y = 0 \qquad \text{for } a, b \in \mathbb{R}$$

Geometric Method. Notice that we can rewrite the equation as

$$(a, b) \cdot \nabla u = 0$$

Then u is constant in the direction $\vec{v} = (a, b)$. Let $\vec{w} = (b, -a)$ so that we can form a basis in $\mathbb{R}^2$.



since u is constant on every one of the dashed lines. Then there exists $f : \mathbb{R} \to \mathbb{R}$ such that

$$u(x, y) = f(\vec{w} \cdot (x, y)) = f(bx - ay)$$

■

Brute Force Method. We can simply compute

$$\begin{aligned} x' &= ax + by \\ y' &= bx - ay \end{aligned}$$

then we have

$$\frac{\partial x'}{\partial x} = a \qquad \frac{\partial x'}{\partial y} = b$$

$$\frac{\partial y'}{\partial x} = b \qquad \frac{\partial y'}{\partial y} = -b$$

By the chain rule we have

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = au_{x'} + bu_{y'}$$
$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = bu_{x'} - au_{y'}$$

Hence
$$0 = au_x + bu_y = a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) - (a^2 \cdot b^2)$$
Therefore $u_{x'} = 0$, so $u(x', y') = f(y') = f(bx - ay)$, which gives us the same answer as the geometric method.

■

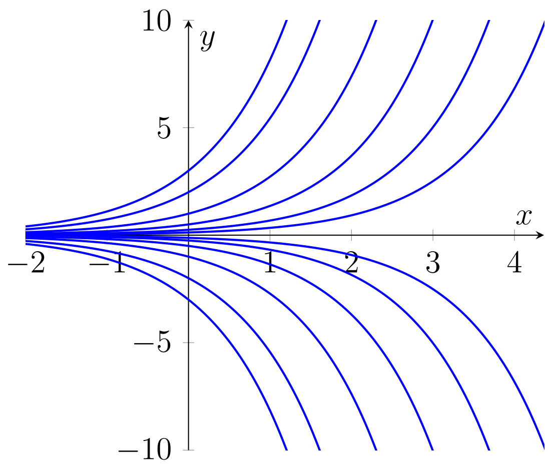
1.7.2 Variable Coefficient PDEs

Example 1.43

Find $u = u(x, y)$ such that
$$u_x + yu_y = 0$$

Solution. Similar to the first PDE, we can rewrite the equation as

$$(x, y) \cdot \nabla u = 0$$



The blue lines are called characteristic curves.

Hence consider curves with slope y
$$\frac{dy}{dx} = y.$$

Solving for the ODE gives us
$$y = Ce^x \qquad C \in \mathbb{R}.$$

Notice that u is constant on the curves.
$$\frac{d}{dx}\left(u\left(x, Ce^x\right)\right) = u_x + u_y \cdot \frac{dy}{dx} = u_x + u_y y = 0.$$

Hence

$$u(x, Ce^x) = u(0, Ce^0) = u(0, C)$$

is independent of x. Now given (x, y) in the place, we compute

$$u(x, y) = u(x, Ce^x) = u(0, C) = u(0, e^{-x}y)$$

choose $C \in \mathbb{R}$, we have $C = e^{-x}y \iff y = Ce^x$. Hence

$$u(x, y) = f(e^{-x}y)$$

where f is a elementary function of one variable. ■

Exercise 1.44

Find the solution of the example above such that

$$u(0, y) = y^5$$

Solution.

$$u(0, y) = y^5 \implies u(x, y) = (e^{-x}y)^5 = y^5 e^{-5x}$$

■

1.8 Motivation behind PDE

Example 1.45 (Transport)

Suppose a pipe with a pollutant suspend in the water and the water is moving along side the pipe to the right at a rate c, then the concentration of the pollution at time t and point x can be modeled as

$$u_t + cu_x = 0.$$

Example 1.46 (Vibrating String)

Check the derivation in textbook which gives the wave equation. Suppose a string is plucked, then the displacement of the string can be modeled as

$$u_{tt} = c^2 u_{xx}.$$

The three dimensional version, a vibrating drumhead can be expressed as

$$u_{tt} = c^2 \left(u_{xx} + u_{yy}\right)$$

Example 1.47 (Diffusion)

Suppose a chemical substance is diffusing through the liquid. The mass of the substance at time t in any given cross section $[x_0, x_1]$ is given by

$$M(t) = \int_{x_0}^{x_1} u(x,t)dx$$

We then have

$$\frac{dM}{dt} = \text{flow in} - \text{flow out} = ku_x(x_1,t) - ku_x(x_0,t) = \int_{x_0}^{x_1} u_t(x,t)dx.$$

Differentiating with respect to x gives

$$u_t = ku_{xx}.$$

In $\dim d$ we have

$$\forall D \subset D_0 \qquad \int_D u_t(x,t)dx \overset{\text{Fick's Law}}{=} \int_{\partial D} k(n\cdot\nabla u)\,dX \overset{\text{div thm}}{=} \int_D \operatorname{div}(k\nabla u)\,dx = \int_D k\triangle u\,dx$$

By the second vanishing theorem we have

$$u_t - k\triangle u = 0 \qquad \text{in} \quad D_0$$

1.9 Initial and boundary conditions, well-posed problems

We have seen a couple of PDE's

- Convection $u_x + cu_y = 0$.
- Vibrating string $u_{tt} = c^2 u_{xx}$
- Vibrating drumhead $utt = c^2\left(u_{xx} + u_{yy}\right)$
- Diffusion $u_t = k\triangle u$
- Heat $\rho c u_t = \kappa \triangle u$
- Stationary heat flow or vibration $\nabla u = 0$

Initial Condition

Specify the physical data at a particular time t_0.

$$u(x, t_0) = \phi(x)$$

where $\phi : \mathbb{R}^d \to \mathbb{R}$ is a given function.

Example 1.48

Let's consider the heat equation. We can set $\phi(x)$ as the initial condition.

Let's consider the wave equation. Since it's a second order DPE, we need a set of initial conditions

$$u(x, t_0) = \phi(x) \quad \text{initial position,} \qquad u_t(x, t_0) = \psi(x) \quad \text{initial velocity}$$

Boundary Conditions

We have three important kind of boundary conditions

- u is specified ("*Dirichlet* condition")
- $\frac{\partial u}{\partial n}$ is specified ("*Neumann* Condition")
- $\frac{\partial u}{\partial n} + au$ is specified ("*Robin* condition")

By specified we mean the condition is equal to some f, where f is a function of time. In the special case of $f = 0$, this is the **homogeneous** condition.

Example 1.49

Take a vibrating string in a region $D = \{x \in \mathbb{R} : 0 \leq x \leq l\}$. We have the following

(D) $u(0, t) = g(t), u(l, t) = l(t)$.

Keeping the end of the vibrating string fixed.

(N) $u_x(0, t) = g(t), u_x(l, t) = l(t)$. *The left end can move up and down.*

(R) $u_x(0, t) + a(t)u(0, t) = g(t), u_x(l, t) + a(t)u(l, t) = l(t)$.

The left end is attached to a spring.

Example 1.50

Take the diffusion equation and suppose D is a container. Take $S = \partial D$. If no substance can exit or enter the container though S, then by Fick's Law we have

$$\vec{n} \cdot \nabla u = 0 \, \text{on} \, S = \partial D = \frac{\partial u}{\partial n}$$

If the container is permeable and the substance leaving the container through ∂D is immediately washed away then $u = 0$ on $S = \partial D$, a homogeneous Dirichlet condition.

Example 1.51

Take the heat equation and let D to be the classroom. Suppose that it's perfectly insulated, then there is no heat flow through $\partial D = S \implies \frac{\partial u}{\partial n} = 0$ on S, a Neumann condition.

Example 1.52

Suppose D is immersed in a large heat bath of temperature $g(t)$ and assume perfect conduction across $S = \partial D \implies u = g(t)$ on S, this is a Dirichlet condition.

Example 1.53

Take

$$u_t = u_{xx} + u_{yy} \text{ in } Dx(0, \infty)$$

Suppose the initial condition $\frac{\partial u}{\partial n} = 0$ on $\partial Dx(0, \infty)$and $u = \phi$ on $Dx(t = 0)$.

Well-posed

Definition 1.54 (informal)

A problem (of Solving PDE) with certian boundary conditions is well-posed if

1. Existence
2. Uniqueness
3. Stability

Example 1.55

From last lecture we had

$$\begin{cases} au_x bu_y & = 0 \\ u(0, y) & = f(y) \end{cases}$$

has the unique solution $u(x, y) = f\left(e^{-x}y\right)$, so existence and uniqueness is check.

Let $f, \tilde{f}$ be two initial conditions, then

$$|u(x, y)| - \tilde{u}(x, y)| = |f\left(e^{-x}y\right) - \tilde{f}\left(e^{-x}y\right)|$$

so for example

$$\max |u(x, y)| - \tilde{u}(x, y)| \leq \max |f\left(e^{-x}y\right) - \tilde{f}\left(e^{-x}y\right)|$$

Then stability is satisfied as well.

Example 1.56 (A famous example)

Take $\triangle u = u_{xx} + u_{yy} = 0$. TODO 1.3

2 Waves and Diffusion

2.1 The wave equation

Recall that the wave equation in in dimension

$$u_{tt} = c^2 u_{xx} \quad \text{for} \quad -\infty < x, t < \infty$$

Observe that

$$0 = u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) v =: v$$

We can equivalently write our second order PDE as

$$\begin{cases} u_t + cu_x &= v \quad (2) \\ v_t - cv_x &= 0 \quad (1) \end{cases}$$

We know that the general solution to (1) is

$$v(x,t) = h(x+ct)$$

where h is an arbitrary function of one variable. Then substituting v into (2) gives us

$$u_t + cu_x = h(x+ct)$$

We know that one particular solution is given by $u(x,t) = f(x,t)$, where

$$f'(s) = \frac{h(s)}{2t}$$

To that, we can add any of the homogeneous solution

$$u_t + cu_x = 0 \implies u(x,t) = f(x+ct) + g(x-ct)$$

Hence we have shown that

$$u(x,t) = f(x+ct) - g(x-ct)$$

where f, g are arbitrary function.

2.1.1 Characteristic coordinates

Take

$$\xi = x + ct \qquad \eta = x - ct$$

By the chain rule we have

$$\partial_x = \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \partial_\xi + \partial_\eta$$

$$\partial_t = \frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} = c\partial_\xi - c\partial_\eta$$

Hence

$$\partial_t - c\partial_x = -2c\partial_\eta \qquad \partial_t + c\partial_x = 2c\partial_\xi$$

SO the wave equation is of the form

$$0 = (\partial_t - c\partial_x)(\partial_t + cd_x)u = (-4c\partial_\xi)(2c\partial_\eta)u = -4c^2 \partial u_{\xi\eta}$$

Since $-4c^2 \neq 0$, we have $u_{\xi\eta} = 0$. So $u(x,y) = f(\xi) + g(\eta)$.

2.1.2 Initial Value Problem

Take

$$\begin{cases} u_{tt} &= c^2 u_{xx} \\ u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \end{cases}$$

where $\phi(x) = \sin x$, and $\psi(x) = 0$. From the general solution we put $t = 0$ and obtain.

$$\phi(x) = f(x) + g(x)$$

differential by t we get

$$\psi(x) = cf'(x) - cg'(x)$$

differentiate ϕ and divide ψ by c we get

$$\phi' s = f' + g' \qquad \frac{1}{c}\psi = f' - g'$$

Solving for f' and g' gives us

$$f' = \frac{1}{2}\left(\phi' + \frac{\psi}{c}\right) \qquad g' = \frac{1}{2}\left(\phi' - \frac{\psi}{c}\right)$$

Integrate with respect to s gives us

$$f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \psi + A \qquad f(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi + B$$

where A, B are constants. Since $\phi(x) = f(x) + g(x)$, we have $A + B = 0$. Let $s = x + ct$ and $s = x - ct$ we get

$$u(x,t) = \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_0^{x+ct} \psi + \frac{1}{2}\phi(x-ct) - \frac{1}{2c}\int_0^{x-ct} \psi$$

which is reduced to

$$\boxed{u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)d(s)}$$

Example 2.1

Take $\phi = 0$ and $\psi = \cos x$. Solve for the wave equation.

2.2 Causality and Energy