Unofficial Solution to Practice Problem

1. Find the solution $u = u(x,t)$ to

$$\begin{cases} u_t + 3xu_x = 0 & \text{for} \quad -\infty < x < \infty, -\infty < t < \infty \\ u(x,0) = \sin x & \text{for} \quad -\infty < x < \infty \end{cases}$$

Solution. Observe that the solution of the PDE must be tangent to $(1, 3x)$ all the time. Therefore we have

$$\frac{dx}{dt} = \frac{3x}{1}$$

Solving for the ODE gives us

$$x = Ce^{3t}$$

Since $u_t + 3xu_x = 0$, the curve is independent of t. Solving for C gives us $C = e^{-3t}x$, therefore we can rewrite u as a function of one variable.

$$u(x,t) = u(Ce^{3t}, t) = u(C, 0) = f(e^{-3t}x)$$

Now we plug in the initial condition we have

$$u(x,0) = f(e^{-3\cdot 0}x) = \sin x \implies f(x) = \sin x,$$

therefore the solution to the PDE is

$$u(x,t) = \sin(e^{-3t}x)$$

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2. (a) Find the coefficients of the Fourier cosine Series of $f(x) = x$ on $[0, \pi]$.

Proof. Let $l = \pi$. We know the full Fourier Series of $f(x)$ is of the form

$$x = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}.$$

We shall compute for A_0 first

$$A_0 = \frac{2}{l}\int_0^l x \cdot 1 \,\mathrm{d}x = \pi.$$

Then for $n \neq 0$ we have

$$\begin{aligned}
A_n &= \frac{2}{l}\int_0^l x \cdot \cos\frac{n\pi x}{l}\,\mathrm{d}x \\
&= \frac{2}{l}\left[\frac{l}{n\pi}x \cdot \sin\frac{n\pi x}{l} + \frac{l^2}{(n\pi)^2}\cos\frac{n\pi x}{l}\right]_0^l \\
&= \frac{2}{\pi}\left[\frac{1}{n}x \cdot \sin nx + \frac{1}{n^2}\cos nx\right]_0^{\pi} \\
&= \frac{2}{\pi n^2}(\cos n\pi - \cos 0) \\
&= \frac{2}{\pi n^2}((-1)^n - 1)
\end{aligned}$$

Therefore the Fourier cosine series for x is

$$x = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{4}{2n\pi}\cos nx$$

Therefore we have

$$A_n = \begin{cases} -\frac{4}{n\pi} & n\,\text{odd} \\ 0 & n\,\text{even} \end{cases}$$

Therefore the Fourier cosine series for x is

$$x = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{4}{(2n+1)\pi}\cos(2n+1)x$$

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(b) Prove that the cosine series converges uniformly in $[0, \pi]$. (*Difficult!*)

(c) Does the Fourier sine series of f converges uniformly on $[0, \pi]$?

Proof. No,it does not converge. Suppose there exists a Fourier sine series that converge to x. Then the sine series must be of the form

$$f(x) = \sum_{n=1}^{\infty} A_n \sin nx$$

Then we compute

$$f(\pi) = \sum_{n=1}^{\infty} A_n \sin n\pi = 0$$

However, we have $f(\pi) = \pi$. Therefore the sine series does not converge to x on $[0, \pi]$. ■

3. Suppose $u = u(x, t)$ solves

$$\begin{cases} u_t - u_{xx} = f(u) & \text{for} \quad 0 < x < l, t \geq 0 \\ u(0, t) = u(l, t) = 0 & \text{for} \quad t \geq 0 \end{cases}$$

where $f(s) = -F'(s)$ is a given function. Show that

$$\frac{d}{dx} \int_0^l \frac{1}{2} u_x^2(x, t) + F(u(x, t))\, dx \leq 0.$$

4. Let $u = u(x, t)$ solves

$$\begin{cases} u_{xx} + u_{yy} = -\lambda u & \text{for} \quad x^2 + y^2 < 1 \\ u = 0 & \text{for} \quad x^2 + y^2 = 1 \end{cases}$$

where $\lambda > 0$.

   (a) State the polar form of the 2-dimensional Laplace operator.

   (b) If $u = R(r)\Theta(\theta)$, which ODEs do R and Θ solve?

   (c) Solve the ODE for Θ. (*You don't have to solve the more difficult ODE for R; its solutions are called Bessel functions.*)

5. Let $u = u(x, t)$ solve $u_t + \nabla \cdot \mathbf{F} = 0$ where $\mathbf{F} = \mathbf{F}(\mathbf{x}) = (F_1, F_2, \ldots, F_d)$ is a given vector field and let D be an open set. Find a formula for

$$\frac{d}{dt} \int_D u(\mathbf{x}, t)\, d\mathbf{x}$$

which only depends on the values of $\mathbf{F}$ on ∂D.

6. Consider the linear wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x, t) & \text{for} \quad -\infty < x < \infty, -\infty < t < \infty \\ u(x, 0) = \phi(x) & \text{for} \quad -\infty < x < \infty \\ u_t(x, 0) = \psi(x) & \text{for} -\infty < x < \infty \end{cases}$$

   (a) Sketch the domain of dependence of the point $(x, t) = (1, 1)$ in the (x, t)-plane in case $c = 1$. Do the same in the case $c = 2$.

   (b) State the general solution formula for this problem in terms of the data f, ϕ, ψ.

   (c) If $\phi = \psi = 0$ and $f \geq 0$, show that $u \geq 0$.

   (d) If $\psi = 0$ and $f = 0$, show that $\max_{x,t} |u| \leq \max_x |\phi|$.

$$\phi \quad \varphi \quad \psi \quad \epsilon \quad \varepsilon$$