Math 126, Fall 2019

Introduction to Partial Differential Equation

Tim Laux, 3106 Etcheverry, 9-10AM

Contents

Logistics

Question 0.1

Why study PDE?

Answer 0.2. TL;DR. It's useful.

Note 0.3. Office hour : MWF 9-10AM 895 Evans, GSI Office hour : MW 1-3 PM 1049 Evans

1 where PDEs Come From

1.1 What is a partial differential equation?

Example 1.1

This is an example of ODE:

$$u = u(x), \qquad \frac{d}{dx}u = u.$$

Example 1.2

A PDE consist of the form

$$u = u(x_1, x_2, \ldots, x_d), \qquad u_{x_k} = \frac{\partial u}{\partial x_k}$$

Where x_i are scalars.

Example 1.3

The most general form of a PDE of first order in two dimension, say $u = u(x, y)$ and of the form

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0, \quad \text{or} \quad F(x, y, u, u_x, u_y) = 0$$

Example 1.4

The most general form of a PDE of second order in two dimension, say $u = u(x, y)$ and of the form

$$G(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}) = 0$$

Definition 1.5
A vector x is defined as

$$x = \vec{x} = (x_1, x_2, \ldots, x_n).$$

Definition 1.6
Let u be a function of vector x of n-dimension. The gradient of u is denoted as

$$\nabla u = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})$$

Example 1.7

1. Linear transport equation $\quad u_t + bu_t = 0, \quad b \in \mathbb{R}$
2. Burgher's Equation $\quad u_t + u \cdot u_x = 0$
3. Laplace's Equation $\quad u_{xx} + u_{yy} = 0$
4. Hermite Equation $\quad -(u_{xx} + u_{yy}) = \lambda u, \quad \lambda \in \mathbb{R}$
5. Wave with interaction $\quad u_{tt} - u_{xx} + u^3 = 0$
6. Linear diffusion with source $\quad u_t - u_{xx} - f(x,t) = 0$
7. Schroedinger's equation $\quad u_t - i \cdot u_{xx} = 0$

Example 1.8 (Cauchy-Riemann Equation)

$$\begin{cases} u_x & = u_y \\ u_y & = -u_x \end{cases}$$

Definition 1.9 (Digression to Linear Algebra)
Let $\mathscr{L}$ be a operator in a function space V. $\mathscr{L}$ is linear if

$$\mathscr{L}(u+v) = \mathscr{L}(u) + \mathscr{L}(v), \quad \mathscr{L}(cu) = c\mathscr{L}(u) \qquad \forall v, u \in V, \quad \forall c \in \mathbb{F}.$$

Definition 1.10
A PDE is called homogeneous linear PDE if it's of the form $\mathscr{L}(u) = 0$. If it's the form $\mathscr{L}(u) = f$, then it's called inhomogeneous PDE.

Remark 1.11

Things that we are interested in

1. Find analytical formulas for some specific PDE's
2. Well-possessedness
   - Existence (Does there exists a solution?)
   - Uniqueness (Is this the only solution?)
   - Stability (If I change the data slightly, does the solution changes just by a little bit?)
3. Predicting qualitative (and sometimes quantitative) behavior of the solution without having a solution formula.
4. Devise an analyze numerical algorithms to approximate solutions.

Example 1.12

Consider the equation

$$\cos(xy)u_x + \sin\left(e^x\right)u_y y = e^{x^2 \sin(y)}$$

Let

$$\mathscr{L}(u(x,y)) = \cos(xy)u_x + \sin\left(e^x\right)u_{yy}$$

$\mathscr{L}$ is a linear operator, so the PDE is an inhomogeneous linear PDE.

Theorem 1.13 (Principle of superposition)

Let $u_1, u_2, \ldots, u_n$ be solutions of $\mathscr{L}(u_k) = 0$, and let $c_1, c_2, \ldots, c_n$ be scalars. then

$$u(x) = \sum_{i=1}^{n} c_i u_i(x) \quad \text{solves} \quad \mathscr{L}(u) = 0$$

Example 1.14 (Cool problem)

$$\begin{cases} u_t + u \cdot \nabla u - \mu \triangle u &= -\nabla p \\ \operatorname{div} u &= 0 \end{cases}$$

where $u = u(x,y,z,t)$, $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ velocity field, where p is pressure and μ is the viscosity of the liquid.