Assignment Presentation
Linear Algebra

Group 7

Submatrices and Partitioned Matrices

A submatrix of a matrix A is a matrix that can be obtained by striking out rows and/or columns of A.

- Any matrix is a submatrix of itself; it is the submatrix obtained by striking out zero rows and zero columns.

- Submatrices of a row or column vector, that is, of a matrix having one row or column, are themselves row or column vectors and are customarily referred to as subvectors.

Principal Submatrix

- A submatrix of an n*n matrix is called a principal submatrix if it can be obtained by striking out the same rows as columns (so that the ith row is struck out whenever the ith column is struck out, and vice versa).
- The r*r (principal) submatrix of an n*n matrix obtained by striking out its last n-r rows and columns is referred to as a leading principal submatrix.
- A principal submatrix of a symmetric matrix is symmetric, a principal submatrix of a diagonal matrix is diagonal, and a principal submatrix of an upper or lower triangular matrix is respectively upper or lower triangular, as is easily verified.

A matrix can be partitioned into submatrices by drawing horizontal or vertical lines between various of its rows or columns, in which case the matrix is called a partitioned matrix.

If a matrix, say is introduced as a partitioned matrix, there is an implicit assumption that each of the submatrices in the ith “row” of submatrices contains the same number of rows and similarly that each of the submatrices in the jth “column” of submatrices contains the same number of columns.

- Diagonal block
- Off-diagonal block
- Upper block-triangular matrix
- Lower block-triangular matrix
- Block-diagonal matrix
- Partitioned (row or column) vector

- Scalar multiples
- Transposes
- Sums
- Products
- Eigenvalues and Eigenvectors

Strassen Algorithm

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \times \underbrace{\begin{bmatrix} e & f \\ g & h \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}}_{C}$$

A, B and C are square matrices of size N x N

a, b, c and d are submatrices of A, of size N/2 x N/2

e, f, g and h are submatrices of B, of size N/2 x N/2

$$p1 = a(f - h)$$
$$p2 = (a + b)h$$
$$p3 = (c + d)e$$
$$p4 = d(g - e)$$
$$p5 = (a + d)(e + h)$$
$$p6 = (b - d)(g + h)$$
$$p7 = (a - c)(e + f)$$

The A x B can be calculated using above seven multiplications.
Following are values of four sub-matrices of result C

$$\underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \times \underset{B}{\begin{bmatrix} e & f \\ g & h \end{bmatrix}} = \underset{C}{\begin{bmatrix} p5 + p4 - p2 + p6 & p1 + p2 \\ p3 + p4 & p1 + p5 - p3 - p7 \end{bmatrix}}$$

A, B and C are square metrices of size N x N
a, b, c and d are submatrices of A, of size N/2 x N/2
e, f, g and h are submatrices of B, of size N/2 x N/2
p1, p2, p3, p4, p5, p6 and p7 are submatrices of size N/2 x N/2

A sparse matrix is a matrix in which most of the elements are zero. By contrast, if most of the elements are nonzero, then the matrix is considered dense.

Special structure

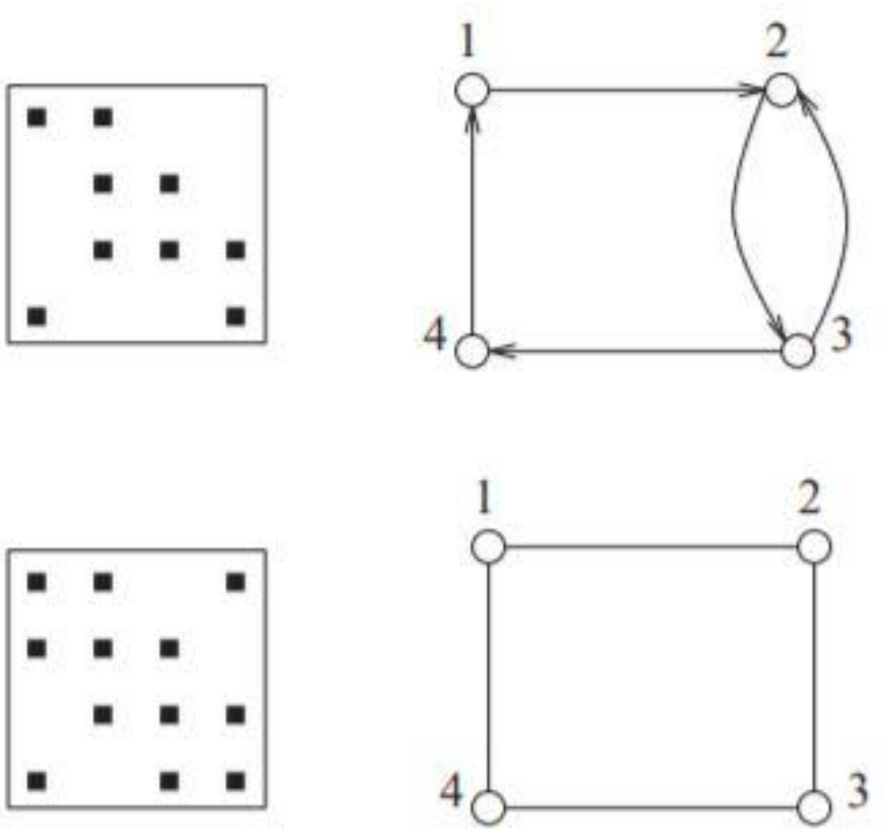
- Diagonal
- Symmetric
- Banded

Banded matrices

- The lower bandwidth of a matrix A is the smallest number p such that the entry a[i,j] = 0 whenever $i > j+p$.
- The upper bandwidth is the smallest number p such that a[i,j] = 0 whenever $i < j-p$.
- For example, a tridiagonal matrix has lower bandwidth 1 and upper bandwidth 1.

Graph representations

- The adjacency graph of a sparse matrix is a graph $G = (V, E)$, whose n vertices in V represent the n unknowns and the edges represent the binary relation a_{ij} is non-zero / ‘**equation i involves unknown j**’.



Graphs of two 4x4 sparse matrices

U-Bahn-Netz
Wien
Stand: 2013
U1
Leopoldau
Großfeldsiedlung
Aderklaaer Straße
Rennbahnweg
Kagraner Platz
Kagran
Alte Donau
Kaisermühlen
Donauinsel
Vorgartenstraße
Praterstern
Nestroyplatz
Schwedenplatz
Stephans-platz
Karlsplatz
Taubstummengasse
Hauptbahnhof
Keplerplatz
U1
Reumannplatz
U4
Heiligenstadt
Spittelau
Friedensbrücke
Roßauer Lände
Schottenring
Landstraße
Stadtpark
Kettenbrücken-gasse
Pilgramgasse
Margaretengürtel
Längenfeldgasse
Meidling Hauptstraße
Schönbrunn
Hietzing
Braunschweiggasse
Unter St. Veit
Ober St. Veit
Hütteldorf
U4
U6
Floridsdorf
Neue Donau
Handelskai
Dresdner Straße
Jägerstraße
Nußdorfer Straße
Währinger Straße
Michelbeuern
Alser Straße
Josefstädter Straße
Thaliastraße
Burggasse
Westbahnhof
Gumpendorfer Straße
Niederhofstraße
Bahnhof Meidling
Tscherttegasse
Am Schöpfwerk
Alterlaa
Erlaaer Straße
Perfektastraße
U6
Siebenhirten
U3
Ottakring
Kendlerstraße
Hütteldorfer Straße
Johnstraße
Schweglerstraße
Zieglergasse
Neubaugasse
Volkstheater
Herrengasse
Stubentor
Rochusgasse
Kardinal-Nagl-Platz
Schlachthausgasse
Erdberg
Gasometer
Zippererstraße
Enkplatz
Simmering
U3
U2
Taborstraße
Messe-Prater
Krieau
Stadion
Donaumarina
Donaustadtbrücke
Stadlau
Hardeggasse
Donauspital
Aspernstraße
Hausfeldstraße
Aspern Nord
Seestadt
U2
Schottentor
Rathaus
Museums-quartier
U2

Permutations and Reorderings

- Interchange matrix
- Permutation matrix
- Reordering

Column permuted linear system

$$A = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & a_{42} & 0 & a_{44} \end{pmatrix}$$

$\pi = \{1,3,2,4\}$

$$\begin{pmatrix} a_{11} & a_{13} & 0 & 0 \\ 0 & a_{23} & a_{22} & a_{24} \\ a_{31} & a_{33} & a_{32} & 0 \\ 0 & 0 & a_{42} & a_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

Symmetric permutations

The resulting matrix corresponds to renaming, or relabeling, or reordering the unknowns and then reordering the equations in the same manner.

$$A = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & a_{42} & 0 & a_{44} \end{pmatrix}$$

$\pi = \{1,3,2,4\}$

$$\begin{pmatrix} a_{11} & a_{13} & 0 & 0 \\ a_{31} & a_{33} & a_{32} & 0 \\ 0 & a_{23} & a_{22} & a_{24} \\ 0 & 0 & a_{42} & a_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3 \\ b_2 \\ b_4 \end{pmatrix}.$$

A permutation is equivalent to relabeling the vertices of the graph without altering the edges.

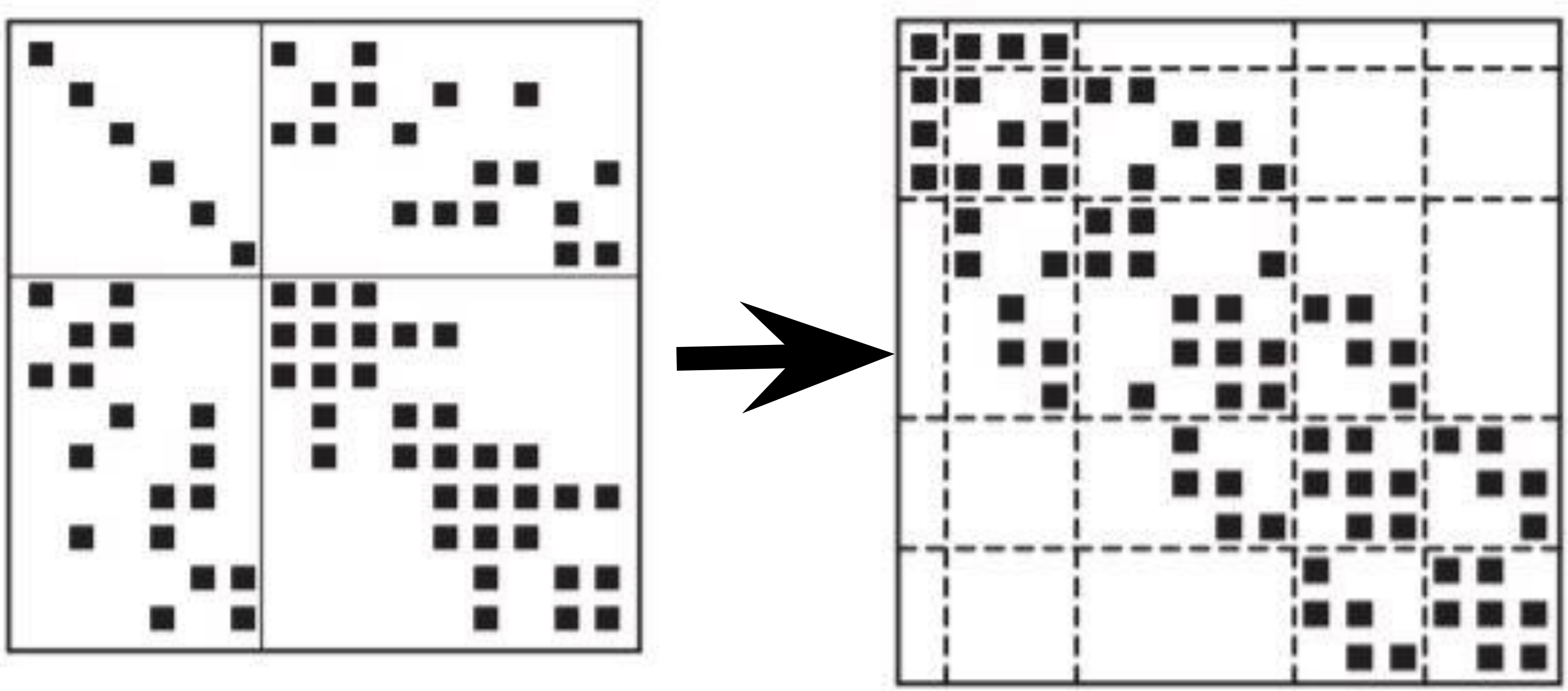
BFS

```
Initialize S={v}, seen=1, π(seen)=v, Mark v;
While seen < n Do
        S_new = NULL
        For each node v in S do
                For each unmarked w in adj(v) do
                        Add w to S_new ;
                        Mark w;
                        π(++ seen)=w;
                Enddo
                S= S_new
        EndDo
EndWhile
```

Cuthill - McKee (G) - Queue implementation

1. Find an initial node v for the traversal
2. Initialize Q= {v} , Mark v;
3. While |Q| < n Do
4. head ++;
5. For each unmarked w in adj(h) , going from lowest to highest degree
6. Append w to Q;
7. Mark w;
8. EndDo
9. EndWhile

Effect of Cuthill-Mckee



Independent set ordering

1. Set S=NULL
2. For j=1,2,.....,n Do:
3. If node j is not marked then
4. S = S U { j }
5. Mark j and all its nearest neighbours
6. EndIf
7. EndDo

1. Set S = NULL . Find an ordering $i_1,i_2,....,i_n$ of the nodes by increasing order.
2. For j=1,2,...,n Do :
3. If node i_j is not marked then
4. $S = S \cup \{ i_j \}$
5. Mark i_j and all its nearest neighbors
6. EndIf
7. EndDo

References

- Matrix Algebra from a Statistician's Perspective - David A. Harville
- Iterative Methods for Sparse Linear Systems - Yousef Saad