

Situation: We have n data points (x_i, y_i) , $i = 1, \dots, n$ with $x_i \in \mathbb{R}^d$ and $y \in \{-1, 1\}$.

Probit Model: y_i is a realization of the random variable Y_i . Y_1, \dots, Y_n are independent. The distribution of Y_i is as follows:

$$\begin{aligned} P(Y_i = 1|x_i; \beta) &= \Phi(x_i^T \beta) \\ P(Y_i = -1|x_i; \beta) &= 1 - \Phi(x_i^T \beta) = \Phi(-x_i^T \beta) \end{aligned}$$

where $\beta \in \mathbb{R}^d$. It follows that

$$P(Y_i = y_i|x_i; \beta) = \Phi(y_i x_i^T \beta)$$

Likelihood: The likelihood of a parameter vector β is given as follows:

$$L(\beta) = \prod_{i=1}^n P(Y_i = y_i|x_i; \beta) = \prod_{i=1}^n \Phi(y_i x_i^T \beta)$$

The negative log-likelihood that we wish to minimize is:

$$\mathcal{L}(\beta) = - \sum_{i=1}^n \log \Phi(y_i x_i^T \beta)$$

The weighted case: We introduce sample weights $w_i \in \mathbb{R}_{>0}$ comprising a weight vector $w \in \mathbb{R}_{>0}^n$. Further, let $g(z) = -\log \Phi(-z)$. The objective function now becomes:

$$f_w(\beta) = \sum_{i=1}^n w_i g(-y_i x_i^T \beta)$$

To make the notation easier, we define $z_i = -y_i x_i^T$ and introduce the matrix $Z \in \mathbb{R}^{n \times d}$ with row vectors $Z_i = z_i$. This gives us:

$$f_w(\beta) = \sum_{i=1}^n w_i g(z_i \beta)$$

Gradient: The gradient of the objective function is needed during optimization. To derive it, we first need the derivative of $g(z)$:

$$g'(z) = \frac{d}{dz} -\log \Phi(-z) = \frac{\phi(z)}{\Phi(-z)}$$

Now we can calculate the gradient of the objective function as follows:

$$\frac{\partial f_w(\beta)}{\partial \beta} = \sum_{i=1}^n w_i \frac{\partial g(z_i \beta)}{\partial \beta} = \sum_{i=1}^n w_i z_i g'(z_i \beta)$$

Lemma 1. *Let $g(z) = -\log \Phi(-z)$. Then it holds for all $z \geq 0$ that:*

$$\frac{1}{2}z^2 \leq g(z)$$

Proof. The following relationship holds for all $z \geq 1$:

$$\begin{aligned} \Phi(-z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z} \exp\left(-\frac{1}{2}x^2\right) dx \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z} \frac{-x}{z} \exp\left(-\frac{1}{2}x^2\right) dx \\ &= \frac{1}{\sqrt{2\pi}z} \exp\left(-\frac{1}{2}z^2\right) \\ &\leq \exp\left(-\frac{1}{2}z^2\right) \end{aligned}$$

We therefore have for $z \geq 1$:

$$e^{g(z)} = e^{-\log \Phi(-z)} = \frac{1}{\Phi(-z)} \geq e^{\frac{1}{2}z^2}$$

Since $\exp(\cdot)$ is a monotonically increasing function, it follows that $g(z) \geq \frac{1}{2}z^2$ for all $z \geq 1$.

Let us now turn to the case when $0 \leq z \leq 1$. For $z = 0$ we have $g(0) > \frac{1}{2} > \frac{1}{2}0^2 = 0$ and for $z = 1$ we have $g(1) > 1 > \frac{1}{2}1^2 = \frac{1}{2}$. Since both $g(z)$ and $\frac{1}{2}z^2$ are continuous and monotonically increasing functions for $0 \leq z \leq 1$, it follows that $g(z) \geq \frac{1}{2}z^2$ for all $0 \leq z \leq 1$. \square

Lemma 2. *Let $g(z) = -\log \Phi(-z)$. Then it holds for all $z \geq 2$ that:*

$$g(z) \leq z^2$$

Proof. We first show that $\Phi(-z) \geq \frac{1}{\sqrt{2\pi}} \frac{z}{z^2+1} e^{-\frac{1}{2}z^2}$ for all $z \geq 0$. In order to prove this lower bound, we define $h(z) = \Phi(-z) - \frac{1}{\sqrt{2\pi}} \frac{z}{z^2+1} e^{-\frac{1}{2}z^2}$ and show that $h(z)$ is positive for all $z \geq 0$. The derivative $h'(z) = -\sqrt{\frac{2}{\pi}} \frac{e^{-\frac{1}{2}z^2}}{(z^2+1)^2}$ is negative for all z , so $h(z)$ is a monotonically decreasing function. Also, it clearly holds that $h(0) > 0$ and $\lim_{z \rightarrow \infty} h(z) = 0$. It follows that $h(z) \geq 0$ for all $z > 0$ which proves the lower bound.

In the next step, we use this result to show that $e^{z^2} \cdot \Phi(-z) \geq 1$ for all $z \geq 2$:

$$\begin{aligned}
e^{z^2} \cdot \Phi(-z) &\geq e^{z^2} \frac{1}{\sqrt{2\pi}} \frac{z}{z^2 + 1} e^{-\frac{1}{2}z^2} \\
&= e^{\frac{1}{2}z^2} \frac{1}{\sqrt{2\pi}} \frac{z}{z^2 + 1} \\
&= e^{\frac{1}{2}z^2} \frac{1}{\frac{4}{3}(z^2 + 1)} \frac{\frac{4}{3}z}{\sqrt{2\pi}} \\
&\geq \frac{e^{\frac{1}{2}z^2}}{\frac{4}{3}(z^2 + 1)} \\
&\geq \frac{e^{\frac{1}{2}z^2}}{e^{\frac{1}{2}z^2}} \\
&= 1
\end{aligned}$$

From this it follows directly that $\frac{1}{\Phi(-z)} \leq e^{z^2}$ and thus we have for all $z \geq 2$:

$$e^{g(z)} = e^{-\log \Phi(-z)} = \frac{1}{\Phi(-z)} \leq e^{z^2}$$

Since $\exp(\cdot)$ is monotonically increasing, the claim that $g(z) \leq z^2$ for all $z \geq 2$ follows as a direct consequence. ¹ \square

Definition 1. Let $Z \in \mathbb{R}^{n \times d}$. Then we define

$$\mu_w(Z) = \sup_{\beta \in \mathbb{R}^d \setminus \{0\}} \frac{\|(\sqrt{D_w}Z\beta)^+\|_2^2}{\|(\sqrt{D_w}Z\beta)^-\|_2^2} = \sup_{\beta \in \mathbb{R}^d \setminus \{0\}} \frac{\|(\sqrt{D_w}Z\beta)^-\|_2^2}{\|(\sqrt{D_w}Z\beta)^+\|_2^2}$$

Z weighted by w is called μ -complex if $\mu_w(Z) \leq \mu$.

Lemma 3. Let $Z \in \mathbb{R}^{n \times d}$ weighted by $w \in \mathbb{R}_{>0}^n$ be μ -complex. Let U be an orthonormal basis for the columnspace of $\sqrt{D_w}Z$. If for index i , the supreme β in (TODO) satisfies $2 \leq z_i\beta$, then $w_i g(z_i\beta) \leq 2\|U_i\|_2^2(1 + \mu)f_w(\beta)$.

Proof. Let $\sqrt{D_w}Z = UR$, where U is an orthonormal basis for the columnspace of

¹The proofs of both lemmas were heavily inspired by <https://www.johndcook.com/blog/norm-dist-bounds/>.

$\sqrt{D_w}Z$. It follows from $2 \leq z_i\beta$ and from the monotonicity of g that

$$\begin{aligned}
w_i g(z_i\beta) &= w_i g\left(\frac{\sqrt{w_i}z_i\beta}{\sqrt{w_i}}\right) = w_i g\left(\frac{U_i R\beta}{\sqrt{w_i}}\right) \leq w_i g\left(\frac{\|U_i\|_2 \|R\beta\|_2}{\sqrt{w_i}}\right) \\
&= w_i g\left(\frac{\|U_i\|_2 \|UR\beta\|_2}{\sqrt{w_i}}\right) = w_i g\left(\frac{\|U_i\|_2 \|\sqrt{D_w}Z\beta\|_2}{\sqrt{w_i}}\right) \\
&\leq \|U_i\|_2^2 \|\sqrt{D_w}Z\beta\|_2^2 \leq \|U_i\|_2^2 (1 + \mu) \|(\sqrt{D_w}Z\beta)^+\|_2^2 \\
&= \|U_i\|_2^2 (1 + \mu) \sum_{j: \sqrt{w_j}z_j\beta \geq 0} w_j (z_j\beta)^2 \\
&\leq 2\|U_i\|_2^2 (1 + \mu) \sum_{j: \sqrt{w_j}z_j\beta \geq 0} w_j g(z_j\beta) \\
&\leq 2\|U_i\|_2^2 (1 + \mu) \sum_{j=1}^n w_j g(z_j\beta) \\
&= 2\|U_i\|_2^2 (1 + \mu) f_w(\beta)
\end{aligned}$$

□

Lemma 4. Let $Z \in \mathbb{R}^{n \times d}$ weighted by $w \in \mathbb{R}_{>0}^n$ be μ -complex. If for index i , the supreme β in (TODO) satisfies $z_i\beta \leq 2$, then $w_i g(z_i\beta) \leq \frac{w_i}{\mathcal{W}}(80 + 16\mu)f_w(\beta)$.

Proof. Let $K^- = \{j \in [n] \mid z_j\beta \leq -1\}$ and $K^+ = \{j \in [n] \mid z_j\beta > -1\}$. Note that $g(-1) > \frac{1}{10}$ and $g(z_i\beta) \leq g(2) < 4$. Also, $\sum_{j \in K^+} w_j + \sum_{j \in K^-} w_j = \mathcal{W}$. Thus, if $\sum_{j \in K^+} w_j \geq \frac{1}{2}\mathcal{W}$ then

$$f_w(\beta) = \sum_{j=1}^n w_j g(z_j\beta) \geq \sum_{j \in K^+} w_j g(z_j\beta) \geq \frac{\sum_{j \in K^+} w_j}{10} \geq \frac{\mathcal{W}}{20} = \frac{\mathcal{W}}{20w_i} w_i \geq \frac{\mathcal{W}}{80w_i} w_i g(z_i\beta)$$

If on the other hand $\sum_{j \in K^+} w_j < \frac{1}{2}\mathcal{W}$, then $\sum_{j \in K^-} w_j \geq \frac{1}{2}\mathcal{W}$. Thus

$$\begin{aligned}
f_w(\beta) &= \sum_{j=1}^n w_j g(z_j \beta) \geq \sum_{j: z_j \beta > 0} w_j g(z_j \beta) \geq \frac{1}{2} \sum_{j: z_j \beta > 0} w_j (z_j \beta)^2 \\
&= \frac{1}{2} \|(\sqrt{D_w} Z \beta)^+\|_2^2 \geq \frac{1}{2\mu} \|(\sqrt{D_w} Z \beta)^-\|_2^2 \\
&= \frac{1}{2\mu} \sum_{j: z_j \beta < 0} w_j (z_j \beta)^2 \\
&\geq \frac{1}{2\mu} \sum_{j \in K^-} w_j (z_j \beta)^2 \\
&\geq \frac{1}{2\mu} \sum_{j \in K^-} w_j \\
&\geq \frac{\mathcal{W}}{4\mu} \\
&\geq \frac{\mathcal{W}}{16\mu w_i} w_i g(z_i \beta)
\end{aligned}$$

Adding both bounds, we get that for $z_i \beta \leq 2$:

$$w_i g(z_i \beta) \leq f_w(\beta) \frac{80w_i}{\mathcal{W}} + f_w(\beta) \frac{16\mu w_i}{\mathcal{W}} = \frac{w_i}{\mathcal{W}} (80 + 16\mu) f_w(\beta)$$

□

Lemma 5. *Let $Z \in \mathbb{R}^{n \times d}$ weighted by $w \in \mathbb{R}_{>0}^n$ be μ -complex. Let U be an orthonormal basis for the columnspace of $\sqrt{D_w} Z$. For each $i \in [n]$, the sensitivity of $g_i(\beta) = g(z_i \beta)$ is bounded by $\varsigma_i \leq s_i = (80 + 16\mu)(\|U_i\|_2^2 + \frac{w_i}{\mathcal{W}})$. The total sensitivity is bounded by $\mathfrak{S} \leq 192\mu d$.*

Proof.

$$\begin{aligned}
\varsigma_i &= \sup_{\beta} \frac{w_i g(z_i \beta)}{f_w(\beta)} \leq \sup_{\beta} \frac{2\|U_i\|_2^2(1 + \mu)f_w(\beta) + \frac{w_i}{\mathcal{W}}(80 + 16\mu)f_w(\beta)}{f_w(\beta)} \\
&= 2\|U_i\|_2^2(1 + \mu) + \frac{w_i}{\mathcal{W}}(80 + 16\mu) \\
&\leq \|U_i\|_2^2(80 + 16\mu) + \frac{w_i}{\mathcal{W}}(80 + 16\mu) \\
&= (80 + 16\mu)(\|U_i\|_2^2 + \frac{w_i}{\mathcal{W}})
\end{aligned}$$

$$\begin{aligned}
\mathfrak{S} &= \sum_{i=1}^n \varsigma_i \leq (80 + 16\mu) \sum_{i=1}^n \|U_i\|_2^2 + \frac{w_i}{\mathcal{W}} \\
&= (80 + 16\mu)(\|U\|_F^2 + 1) \\
&= (80 + 16\mu)(d + 1) \\
&\leq 96\mu(d + 1) \\
&\leq 192\mu d
\end{aligned}$$

□

References