Probit Regression for Large Data Sets via Coresets

Alexander Munteanu Simon Omlor Christian Peters

TU Dortmund University, Germany

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- with n labels: $y_1, \ldots, y_n \in \{-1, 1\}$

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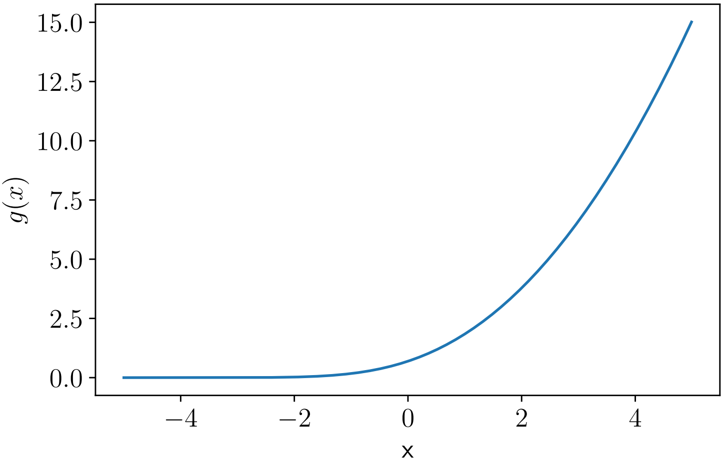
Loss function (negative log-likelihood):

$$f(\beta) = \sum_{i=1}^{n} \ln \left(\frac{1}{\Phi(y_i x_i^T \beta)} \right)$$

The Probit Loss



$$g(x) = \ln\left(\frac{1}{\Phi(-x)}\right)$$

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Conventional optimization algorithms become inefficient!

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- Results on coreset must be close to results on original data
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  - $\Rightarrow$ Otherwise useless!

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Main Goal: Develop efficient coreset construction algorithms!

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(1) ...approximate the data well:

- Let $f(\beta)$ be the original loss and $\tilde{f}(\beta)$ be the loss on the coreset
- Then for $\epsilon > 0$ and for all $\beta \in \mathbb{R}^d$ we want:

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(2) ...are significantly smaller than our data:

- Coreset sizes logarithmic in n would be a great success
- Given a dataset with $n = 1,000,000,000$ observations, $\log(n) \leq 21$
- Even better: Independent of n!

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$\Rightarrow$ **Need to restrict the class of data sets under study!**

- Slightly adapt the concept of μ-complexity from [Munteanu et al., 2018] to probit regression

μ-Complexity

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μ is a useful parameter introduced by [Munteanu et al., 2018]:

$$\mu = \sup_{\beta \in \mathbb{R}^d \setminus \{0\}} \frac{\sum_{y_i x_i^T \beta > 0} (x_i^T \beta)^2}{\sum_{y_i x_i^T \beta < 0} (x_i^T \beta)^2}$$

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We show how μ can be used to find small coresets!

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- It is known to fail when the data has outliers

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Our approach: Use importance sampling based on the sensitivity[2] of each observation!

[2] see [Langberg and Schulman, 2010]

The Sensitivity Framework

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- Well known algorithmic framework for coreset construction via importance sampling introduced by [Feldman and Langberg, 2011].
- The sensitivity is the worst-case importance of an observation
- Sampling proportionally to upper bounds on the sensitivities can lead to small coresets

$\Rightarrow$ Need to derive tight upper bounds!

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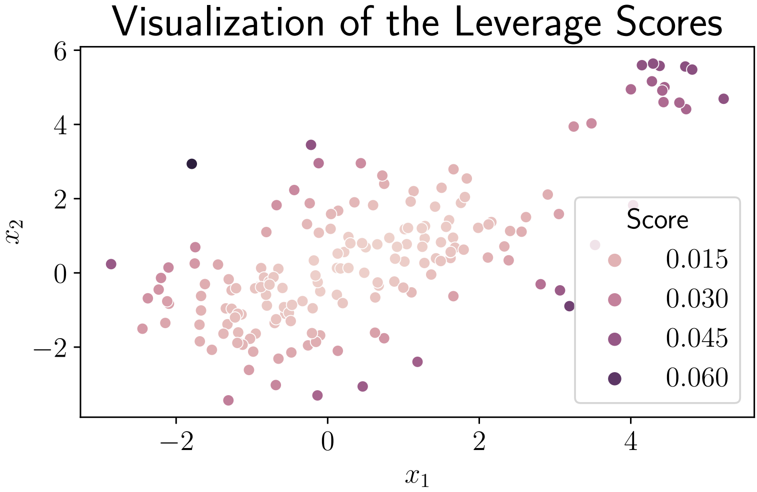
We show that the statistical leverage scores can be used to bound the sensitivities of high-loss points!

Statistical Leverage Scores



First Algorithm: Fast Leverage Score Sampling

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$$\boxed{\textbf{Coreset size: } O\left(\frac{\mu d^2}{\epsilon^2}\right)\textbf{, independent of } n\textbf{!}}$$

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- Use approximation techniques by [Chhaya et al., 2020]
- Increases coreset size by a factor of $\log(\sigma_{max})$
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Requires only one pass over the data set!

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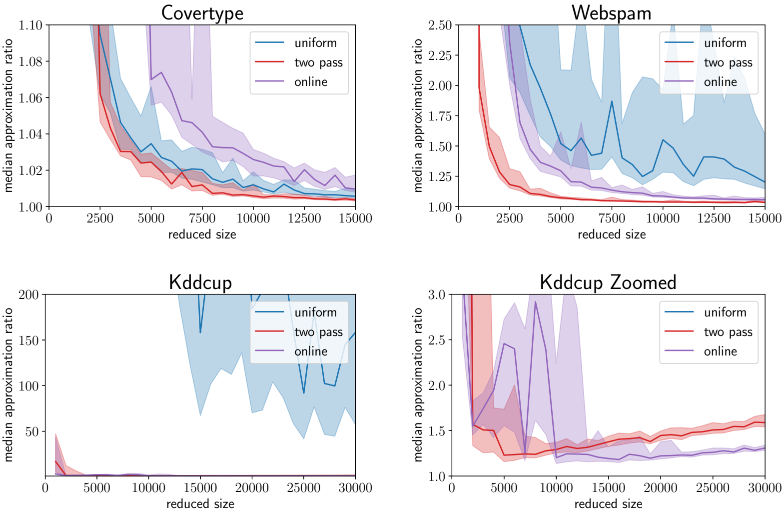
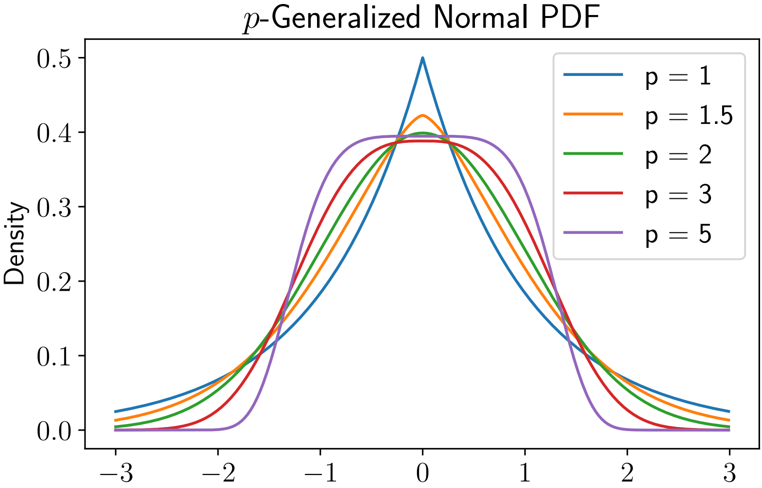
Use approximation ratio for evaluation:

- Let $\tilde{f}(\beta)$ be the loss function on the coreset and let $f(\beta)$ be the original loss
- Let β^{opt} be the solution of the original problem
- Optimize $\tilde{f}(\beta)$ to find solution $\tilde{\beta}$ of the reduced problem
- Compute ratio $\frac{f(\tilde{\beta})}{f(\beta^{opt})}$

Experiments



Extensions – p-Generalized Probit Model[3]



[3]See [Kalke and Richter, 2013] for a definition of the p-generalized normal distribution

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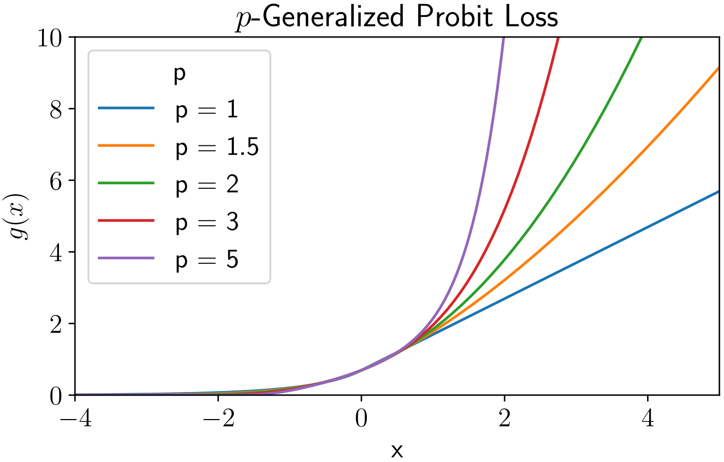

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Extensions – p-Generalized Probit Model

Three adaptations are required:

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Coreset size: $\begin{cases} O\left(\frac{\mu d^p}{\epsilon^2}\right), & for\ p \in [1,2) \\ O\left(\frac{\mu d^{2p}}{\epsilon^2}\right), & for\ p \in (2,\infty) \end{cases}$

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- Based on fast coreset construction algorithms and modern sketching techniques

(2) Demonstrated that our methods outperform standard uniform sampling

- By conducting experiments on well known real-world datasets

(3) Introduced the p-generalized probit model as a flexible framework for modeling binary data

- Enables you to control the tail behavior of the distribution

Literature I

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