Model order reduction via Proper orthogonal decomposition and Balanced truncation

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Singular value decomposition

- $\mathbb{C}^{m\times n}$: matrix with m rows, n columns,
- $\mathbb{O}_{m,n} \in \mathbb{C}^{m\times n}$: zero matrix.

Theorem

Let $M \in \mathbb{C}^{m\times n}$, $\operatorname{rank}(M) = r$. Then $\exists\ U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$ unitary, and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ s.t.

$$M = U \underbrace{\begin{pmatrix} \Sigma_r & \mathbb{O}_{r,n-r} \\ \mathbb{O}_{m-r,r} & \mathbb{O}_{m-r,n-r} \end{pmatrix}}_{\substack{\text{on the main diagonal:} \\ \text{the singular values of } M}} V^*,$$

where $\Sigma_r := \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$.

Remark: $\|M\| = \sigma_1$.

The POD problem

Let X be a separable Hilbert space.

POD idea: Reducing a set of data (or snapshots) by representing its essential information via a few "basis vectors".

What are the data? Example 1: $X = L^2(\Omega)$ or $H_0^1(\Omega)$

$$\begin{cases} \partial_t z - \Delta z = f & \text{in } \Omega \times (0, T) \\ z = 0 & \text{on } \partial\Omega \times (0, T) \\ z(0, \cdot) = z_0 & \text{in } \Omega \end{cases}$$

Snapshot set:

- $\{z(t) \colon t \in [0, T]\}$, or
- $\{z_k\}_{k=1}^N$, where $z_k := z(t_k)$, for $t_1, t_2, \ldots, t_N \in (0, T)$.

The POD problem

Let X be a separable Hilbert space.

POD idea: Reducing a set of data (or snapshots) by representing its essential information via a few "basis vectors".

What are the data? Example 2: $X = L^2(\Omega)$ or $H_0^1(\Omega)$

Let $\mathcal{I}$ be an interval of $\mathbb{R}_+$, and $\theta \in \mathcal{I}$.

$$\begin{cases} -\Delta z + \theta z = f & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega \end{cases}$$

Snapshot set:

- $\{z(\cdot\,;\theta) \colon \theta \in \mathcal{I}\}$, or
- $\{z_k\}_{k=1}^N$, where $z_k := z(\cdot\,;\theta_k)$, for $\theta_1, \theta_2, \ldots, \theta_N \in \mathcal{I}$.

The POD problem

Let X be a separable Hilbert space.

POD idea: Reducing a set of data (or snapshots) by representing its essential information via a few "basis vectors".

What are the data? Example 3: $X = \mathbb{C}^n$

$A \in \mathbb{C}^{n\times n}$ is the discretization of the Laplacian.

$$\begin{cases} \dot{z} = Az + f & \text{on } (0, T) \\ z(0) = z_0 \end{cases}$$

Snapshot set:

- $\{z(t) \colon t \in [0, T]\}$, or
- $\{z_k\}_{k=1}^N$, where $z_k := z(t_k)$ for $t_1, t_2, \dots, t_N \in (0, T)$.

The POD problem

Aim: For a fixed $\ell \in \mathbb{N}$, find an orthonormal (ON) set $\psi := \{\psi_k\}_{k=1}^{\ell}$ (called a "POD basis") s.t.

Discrete POD:

$$J(\psi) = \min_{\substack{\phi := \{\phi_k\}_{k=1}^{\ell} \\ \text{ON set of } \mathcal{V}_N}} J(\phi),$$

where $\mathcal{V}_N := \text{span}\{z_k\}_{k=1}^{N}$, and

$$J(\phi) := \sum_{j=1}^{N} \alpha_j \|z_j - P^{\phi} z_j\|_X^2,$$

where P^{ϕ} denotes the orthogonal projection onto $\text{span}\{\phi\}$.

The POD problem

Aim: For a fixed $\ell \in \mathbb{N}$, find an orthonormal (ON) set $\psi := \{\psi_k\}_{k=1}^{\ell}$ (called a "POD basis") s.t.

Continuous POD:

$$J(\psi) = \min_{\substack{\phi := \{\phi_k\}_{k=1}^{\ell} \\ \text{ON set of } \mathcal{V}}} J(\phi),$$

where $\mathcal{V} := \operatorname{span}\{z(t) \colon t \in [0, T]\}$, and

$$J(\phi) := \int_0^T \|z(t) - P^{\phi} z(t)\|_X^2 dt.$$

The POD problem

Aim: For a fixed $\ell \in \mathbb{N}$, find an orthonormal (ON) set $\psi := \{\psi_k\}_{k=1}^{\ell}$ (called a "POD basis") s.t.

Continuous POD:

$$J(\psi) = \min_{\substack{\phi := \{\phi_k\}_{k=1}^{\ell} \\ \text{ON set of } \mathcal{V}}} J(\phi),$$

where $\mathcal{V} := \operatorname{span}\{z(\theta)\colon \theta \in \mathcal{I}\}$, and

$$J(\phi) := \int_{\mathcal{I}} \|z(\theta) - P^{\phi} z(\theta)\|_X^2 d\theta.$$

The POD problem

Use of the POD basis: project the system onto $\mathrm{span}\{\psi_k\}_{k=1}^{\ell}$

$$z(t) \approx \sum_{k=1}^{\ell} \langle z(t)\,,\psi_k\rangle_X \psi_k$$

or

$$z(\cdot\,;\theta) \approx \sum_{k=1}^{\ell} \langle z(\cdot\,;\theta)\,,\psi_k\rangle_X \psi_k.$$

Existence of discrete POD

Proposition

The discrete POD problem has the same solutions as

$$\begin{cases} \underset{\phi := \{\phi_k\}_{k=1}^{\ell} \subset \mathcal{V}_N}{\arg\min} & -\sum_{j=1}^{N} \alpha_j \|P^{\phi} z_j\|_X^2 \\ \text{s.t.} \quad \mathbf{c}_k(\phi_1, \ldots, \phi_\ell) = 0, & \text{for } 1 \le k \le \ell, \end{cases}$$

where $$\mathbf{c}_k(\phi_1, \ldots, \phi_\ell) := \begin{pmatrix} \langle \phi_k \,, \phi_k \rangle_X - 1 \\ \langle \phi_{k+1} \,, \phi_k \rangle_X \\ \langle \phi_{k+2} \,, \phi_k \rangle_X \\ \vdots \\ \langle \phi_\ell \,, \phi_k \rangle_X \end{pmatrix}.$$

Existence of discrete POD

Necessary condition: If a POD basis $\psi := \{\psi_k\}_{k=1}^{\ell}$ exists, then $\exists\ \{\eta_k\}_{k=1}^{\ell}$ s.t.

$$\begin{cases} \dfrac{\partial L}{\partial \psi_m}(\psi_1, \dots, \psi_\ell, \eta_1, \dots, \eta_\ell) = 0 \\ \text{s.t. } \mathbf{c}_m(\psi_1, \dots, \psi_\ell) = 0 \end{cases} \qquad \text{for } 1 \le m \le \ell$$

where L is the Lagrangian:

$$L(\psi_1, \dots, \psi_\ell, \eta_1, \dots, \eta_l) := -\sum_{j=1}^{N} \alpha_j \|P^{\psi} z_j\|_X^2 + \sum_{k=1}^{\ell} \langle \mathbf{c}_k(\psi_1, \dots, \psi_\ell), \eta_k \rangle.$$

Consequently, $$\underbrace{\sum_{j=1}^{N} \alpha_j \langle \psi_m, z_j \rangle_X z_j}_{:= \mathcal{S}_N \psi_m} = \eta_m^1 \psi_m, \qquad \text{for } 1 \le m \le \ell.$$

Existence of discrete POD

Theorem

Let $\ell \leq d := \dim(\mathcal{V}_N)$. There exist eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ of $\mathcal{S}_N$ and associated eigenvectors $\{\psi_k\}_{k=1}^{\infty}$ s.t.

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d > 0 = \lambda_{d+1} = \lambda_{d+2} = \ldots$$

The first ℓ eigenvectors $\psi := \{\psi_k\}_{k=1}^{\ell}$ form a POD basis, and

$$\sum_{j=1}^{N} \alpha_j \|z_j - P^{\psi} z_j\|_X^2 = \sum_{k=\ell+1}^{d} \lambda_k.$$

Proof.

Hilbert-Schmidt theorem for $\mathcal{S}_N$, decay of $(\lambda_k)_{k\geq 1}$, orthonormality of $\{\psi_k\}_{k=1}^{\infty}$, and the eigenvalue-eigenvector relation. $\square$

Discrete POD: the "snapshots method"

For $\mathcal{Y}_N \colon \mathbb{C}^N \to X$ defined by:

$$\mathcal{Y}_N v := \sum_{j=1}^{N} \sqrt{\alpha_j} v_j z_j,$$

- $S_N = \mathcal{Y}_N \mathcal{Y}_N^*$.
- $\mathcal{S}_N$ has the same nonzero eigenvalues as $\mathcal{K}_N := \mathcal{Y}_N^* \mathcal{Y}_N$,
- $\exists \; \{\lambda_k\}_{k=1}^N$ and $\{v_k\}_{k=1}^N$ eigenvalues and associated eigenvectors of $\mathcal{K}_N$ s.t.
$$\lambda_1 \geq \ldots \geq \lambda_d > 0 = \lambda_{d+1} = \ldots = \lambda_N.$$

Then, for $\psi_k := \dfrac{1}{\sqrt{\lambda_k}} \displaystyle\sum_{j=1}^{N} \sqrt{\alpha_j} v_k^j z_j, \quad \{\psi_k\}_{k=1}^{\ell}$ is a POD basis.

Link between POD and SVD

If X is finite dimensional: $X = \mathbb{C}^n$ (with $N << n$),

$$\mathcal{Y}_N \overset{\text{SVD}}{=} \begin{pmatrix} | & & | \\ \psi_1 & \dots & \psi_n \\ | & & | \end{pmatrix} \begin{pmatrix} \Sigma_d & \mathbb{O}_{d,N-d} \\ \mathbb{O}_{n-d,d} & \mathbb{O}_{n-d,N-d} \end{pmatrix} \begin{pmatrix} | & & | \\ v_1 & \dots & v_N \\ | & & | \end{pmatrix}^*$$

where $\Sigma_d := \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_d})$, for $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d > 0$.

Hence, for $\ell \leq d$, diagonalizing $\mathcal{K}_N$ allows to find the POD basis:

$\{\psi_k\}_{k=1}^{\ell}$, given by:

$$\psi_k := \frac{1}{\sqrt{\lambda_k}} \mathcal{Y}_N v_k.$$

Link between continuous and discrete POD

Recall that

$$\mathcal{S}_N x := \sum_{j=1}^{N} \alpha_j \langle x\,, z_j \rangle_X z_j.$$

Let $\mathcal{S} \colon X \to X$ be defined by

$$\mathcal{S}x := \int_0^T \langle x\,, z(t) \rangle_X z(t) dt.$$

$\mathcal{S}$ is diagonalizable, and the ℓ eigenvectors associated with the ℓ largest eigenvalues form a POD basis.

Link between continuous and discrete POD

Theorem

Let $z \in H^1(0, T; X)$ and for $h = \frac{T}{N-1}$, the time instances are

$$t_j := (j-1)h \quad \text{for } 1 \leq j \leq N,$$

and the weights are

$$\alpha_j = \begin{cases} \dfrac{h}{2} & \text{if } j = 1,\ j = N, \\ h & \text{if } j \in \{2, \dots, N-1\}. \end{cases}$$

Then $\mathcal{S}_N$ converges uniformly to $\mathcal{S}$: $\lim_{N \to \infty} \|\mathcal{S}_N - \mathcal{S}\| = 0.$

Link between continuous and discrete POD

Theorem

Let $\{\psi_k^N\}_{k=1}^{\infty}$ be the eigenvectors of $\mathcal{S}_N$ associated with $\{\lambda_k^N\}_{k=1}^{\infty}$ (in decreasing order) and $\{\psi_k^{\infty}\}_{k=1}^{\infty}$ the eigenvectors of $\mathcal{S}$ associated with $\{\lambda_k^{\infty}\}_{k=1}^{\infty}$ (in decreasing order). If ℓ is s.t. $\lambda_\ell^{\infty} \neq \lambda_{\ell+1}^{\infty}$, then

$$\lim_{N\to\infty} \lambda_k^N = \lambda_k^{\infty}, \qquad \lim_{N\to\infty} \psi_k^N = \psi_k^{\infty}$$

for any $k \in \{1, \dots, \ell\}$. Moreover,

$$\lim_{N\to\infty} \Big(\sum_{k=\ell+1}^{d(N)} \lambda_k^N \Big) = \sum_{m=\ell+1}^{d} \lambda_m^{\infty}.$$

Balanced truncation: context

Finite dimensional linear system:

$$\begin{cases} \dot{z} = Az + Bu, \quad z(0) = z_0 \\ y = Cz + Du \end{cases}$$

State: $z(t) \in \mathbb{C}^n$

Input: $u(t) \in \mathbb{C}^m$

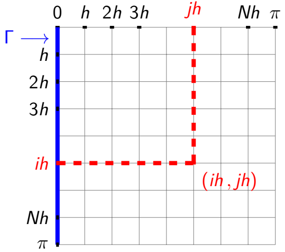
Output: $y(t) \in \mathbb{C}^p$

Assumptions

- A is Hurwitz.
- The system is minimal: (A, B) is controllable and (A, C) is observable.

Example: space discretization HE

$$HE : \begin{cases} \partial_t z = \Delta z & \text{in } \Omega \times (0,T) \\ z = 0 & \text{on } [\partial\Omega \setminus \Gamma] \times (0,T) \\ z = u & \text{on } \Gamma \times (0,T) \\ y = \partial_n z & \text{on } \Gamma \times (0,T) \\ z(0,\cdot) = z^0 & \text{in } \Omega. \end{cases}$$



where $\Omega = (0,\pi) \times (0,\pi)$.

Observation and control on $\Gamma = \{(x_1, 0) \colon 0 < x_1 < \pi\}$.

Example: space semi-discretization of (P)

$$\Longrightarrow \begin{cases} \dot{z} = Az + Bu \\ y = Cz + Dz \\ z(0) = z^0 \end{cases} \qquad \text{where} \qquad \begin{array}{r} z(t) \in \mathbb{C}^{N^2} \\ u(t) \in \mathbb{C}^{N} \\ y(t) \in \mathbb{C}^{N} \end{array}$$

$$A := \frac{1}{h^2} \begin{pmatrix} M & I_N & & 0 \\ I_N & \ddots & \ddots & \\ & \ddots & \ddots & I_N \\ 0 & & I_N & M \end{pmatrix}, \quad M := \begin{pmatrix} -4 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -4 \end{pmatrix},$$

$$B := \frac{1}{h^2} \begin{pmatrix} I_N \\ \mathbb{O}_{N^2-N,N} \end{pmatrix}, \quad C := -\frac{1}{h} \begin{pmatrix} I_N & \mathbb{O}_{N,N^2-N} \end{pmatrix}, \quad D = \frac{1}{h} I_N.$$

Order reduction

Let $\ell \in \mathbb{N}$. From the system

$$\begin{cases} \dot{z} = Az + Bu \\ y = Cz + Du \end{cases}$$

we want to obtain an approximate system

$$\begin{cases} \dot{z}_\ell = A_\ell z_\ell + B_\ell u \\ y = C_\ell z_\ell + Du \end{cases}$$

Order reduction

$$\begin{cases} \dot{z} = Az + Bu \\ y = Cz + Du \end{cases}$$

Relations between inputs and outputs:

$$y(t) = \int_0^t Ce^{(t-\tau)A}Bu(\tau)d\tau + Du(t) \quad \text{and} \quad (\mathcal{L}y)(s) = G(s)(\mathcal{L}u)(s)$$

Transfer function

$$G(s) := C(sI_n - A)^{-1}B + D := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

$G \in \mathrm{Hol}(\mathbb{C}_+)$, continuous bounded in $\mathrm{clos}(\mathbb{C}_+)$, thus

$$\|G\|_{H^\infty_{p\times m}} = \sup_{w\in\mathbb{R}} \|G(iw)\|.$$

Order reduction

$$\begin{cases} \dot{z} = Az + Bu \\ y = Cz + Du \end{cases} \quad \rightarrow \quad \begin{cases} \dot{z}_\ell = A_\ell z_\ell + B_\ell u \\ y = C_\ell z_\ell + Du \end{cases}$$

$$G(s) = C(sI_n - A)^{-1}B + D \qquad G_\ell(s) = C_\ell(sI_\ell - A_\ell)^{-1}B_\ell + D$$

Definition (H^∞-error of approximation)

$$\mathrm{err}(G, G_\ell) := \|G - G_\ell\|_{H^\infty_{p\times m}}$$

Order reduction

$$\begin{cases} \dot{z} = Az + Bu \\ y = Cz + \cancel{Du} \end{cases} \quad \to \quad \begin{cases} \dot{z}_\ell = A_\ell z_\ell + B_\ell u \\ y = C_\ell z_\ell + \cancel{Du} \end{cases}$$

$$G(s) = C(sI_n - A)^{-1}B + \cancel{D} \qquad G_\ell(s) = C_\ell(sI_\ell - A_\ell)^{-1}B_\ell + \cancel{D}$$

Definition (H^∞-error of approximation)

$$\mathrm{err}(G, G_\ell) := \|G - G_\ell\|_{H^\infty_{p\times m}}$$

In this new context, we also define:

$$g(t) := Ce^{tA}B \qquad g_\ell(t) := C_\ell e^{tA_\ell}B_\ell$$

Operators

- Controllability Gramian $\mathcal{R}$, Observability Gramian $\mathcal{Q}$

$$\mathcal{R}x := \int_0^\infty e^{tA}BB^*e^{tA^*}xdt,$$

and

$$\mathcal{Q}x := \int_0^\infty e^{tA^*}C^*Ce^{tA}xdt.$$

- Hankel operator $\Gamma_g \colon L^2(\mathbb{R}_+;\mathbb{C}^m) \to L^2(\mathbb{R}_+;\mathbb{C}^p)$

$$(\Gamma_g u)(t) := \chi_{[0,+\infty)}(t)\int_0^{+\infty} g(t+\tau)u(\tau)d\tau$$

Property: $\|\Gamma_g\| \leq \|G\|_{H^\infty_{p\times m}}$

Input map and output map

Usefulness:

1. quantifying controllability and observability,
2. rewritting the operators.

Input map and output map

1. Quantifying controllability and observability:

Output map: $\Psi : \mathbb{C}^n \to L^2(\mathbb{R}_+, \mathbb{C}^p), \quad (\Psi x)(t) := Ce^{tA}x.$

Interpretation: $y = \Psi x$ is the output obtained, when the initial state is $x \in \mathbb{C}^n$, in the system

$$\begin{cases} \dot{z} = Az, & z(0) = x \\ y = Cz \end{cases}$$

Definition (more observable)

Let x_1, $x_2 \in \mathbb{C}^n$ of norm 1. We say that x_1 is more observable than x_2 if

$$\|\Psi x_1\|_{L^2(\mathbb{R}_+,\mathbb{C}^p)} > \|\Psi x_2\|_{L^2(\mathbb{R}_+,\mathbb{C}^p)}.$$

Input map and output map

1. Quantifying controllability and observability:

Input map: $\Phi : L^2(\mathbb{R}_+, \mathbb{C}^m) \to \mathbb{C}^n, \quad \Phi u := \int_0^\infty e^{\tau A} B u(\tau) d\tau$

Interpretation: $\Phi u = z^\infty(0)$ where

$$\begin{cases} \dot{z}^\infty(t) = A z^\infty(t) + B u(-t) & t \in \mathbb{R}_- \\ \text{s.t.} \quad \lim_{\tau \to -\infty} z^\infty(\tau) = 0. \end{cases}$$

Proposition

For any $x \in \mathbb{C}^n$, the control $u_{opt} := \Phi^* \mathcal{R}^{-1} x$ satisfies

$$\|u_{opt}\|_{L^2(\mathbb{R}_+, \mathbb{C}^m)} = \min_{\substack{u \in L^2(\mathbb{R}_+, \mathbb{C}^m) \\ \Phi u = x}} \|u\|_{L^2(\mathbb{R}_+, \mathbb{C}^m)}$$

Input map and output map

1. Quantifying controllability and observability:

Input map: $\Phi : L^2(\mathbb{R}_+, \mathbb{C}^m) \to \mathbb{C}^n, \quad \Phi u := \int_0^\infty e^{\tau A} B u(\tau) d\tau$

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Definition (more controllable)

Let x_1, $x_2 \in \mathbb{C}^n$ of norm 1. We say that x_1 is more controllable than x_2 if

$$\|\Phi^* \mathcal{R}^{-1} x_1\|_{L^2(\mathbb{R}_+,\mathbb{C}^m)} < \|\Phi^* \mathcal{R}^{-1} x_2\|_{L^2(\mathbb{R}_+,\mathbb{C}^m)}.$$

Input map and output map

2. Rewritting the operators:

- Gramians:

$$\mathcal{R} = \Phi\Phi^*, \quad \mathcal{Q} = \Psi^*\Psi.$$

- Hankel operator:

$$\Gamma_g = \Psi\Phi.$$

$\Rightarrow$ $\Gamma_g^*\Gamma_g$ and $\mathcal{R}\mathcal{Q}$ have the same nonzero eigenvalues.

Balanced system

Specific change of variable in the state space:

Example 1: for $\mathcal{R}^{\frac{1}{2}}\mathcal{Q}\mathcal{R}^{\frac{1}{2}} \overset{\text{diag.}}{=} U\Sigma^2 U^*$, set

$$T := \Sigma^{\frac{1}{2}} U^* \mathcal{R}^{-\frac{1}{2}}, \quad S := T^{-1}$$

Example 2: for $\mathcal{R} = XX^*$, $\mathcal{Q} = YY^*$, $Y^*X \overset{\text{SVD}}{=} U\Sigma V^*$, set

$$T := \Sigma^{-\frac{1}{2}} U^* Y^*, \quad S := T^{-1}$$

where $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$, for $\sigma_1 \geq \dots \geq \sigma_n > 0$.

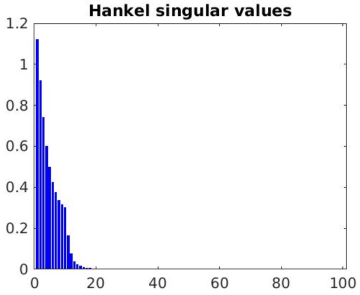
Then,
$$\underbrace{T\mathcal{R}T^*}_{:=\widetilde{\mathcal{R}}} = \underbrace{S^*\mathcal{Q}S}_{:=\widetilde{\mathcal{Q}}} = \Sigma.$$

Hankel singular values

- G and g are not influenced by a change of variable.
- $\mathcal{R}\mathcal{Q}$, $\widetilde{\mathcal{R}}\widetilde{\mathcal{Q}}$ and $\Gamma_g^*\Gamma_g$ have the same nonzero eigenvalues:

$$\sigma_1{}^2 \geq \ldots \geq \sigma_n{}^2 > 0.$$

- $\sigma_1 = \|\Gamma_g\| \leq \|G\|_{H^\infty_{p\times m}}$.



Definition (Hankel singular values)

They are the square roots of the eigenvalues of $\mathcal{R}\mathcal{Q}$.

Truncation

For $T = \begin{pmatrix} T1 \\ T2 \end{pmatrix}$ with $T_1 \in \mathbb{C}^{\ell \times n}$, and $S = \begin{pmatrix} S_1 & S_2 \end{pmatrix}$ with $S_1 \in \mathbb{C}^{n \times \ell}$,

$$\text{Balanced system:} \quad \begin{cases} \dot{z} = (TAS)z + (TB)u \\ y = (CS)z \end{cases}$$

Minimal norm control u_{opt}

$$\|\Phi^* \mathcal{R}^{-1} x\|^2_{L^2(\mathbb{R}_+, \mathbb{C}^m)} = \sum_{k=1}^n \frac{1}{\sigma_k} |\langle x \, , e_k \rangle_{\mathbb{C}^n}|^2$$

Observation y for initial state x

$$\|\Psi x\|^2_{L^2(\mathbb{R}_+, \mathbb{C}^p)} = \sum_{k=1}^n \sigma_k |\langle x \, , e_k \rangle_{\mathbb{C}^n}|^2$$

Truncation

For $T = \begin{pmatrix} T1 \\ T2 \end{pmatrix}$ with $T_1 \in \mathbb{C}^{\ell \times n}$, and $S = \begin{pmatrix} S_1 & S_2 \end{pmatrix}$ with $S_1 \in \mathbb{C}^{n \times \ell}$,

$$\text{Balanced system:} \quad \begin{cases} \dot{z} = (TAS)z + (TB)u \\ y = (CS)z \end{cases}$$

$$\text{Truncated system:} \quad \begin{cases} \dot{z}_\ell = (T_1 A S_1) z_\ell + T_1 B u \\ y = C S_1 z_\ell \end{cases}$$

Proposition

In the truncated system, $T_1 A S_1$ is **Hurwitz**, the Gramians are **balanced** and the Hankel singular values are $\sigma_1, \ldots, \sigma_\ell$.

Errors bounds

Theorem (Error bounds)

$$\sigma_{\ell+1} \leq \left\| G - G_\ell \right\|_{H^\infty_{p\times m}} \leq 2 \sum_{k=1}^{m} \sigma_{i_k}.$$

where $\{\sigma_{i_k}\}_{k=1}^m$ are the unrepeated elements of $\{\sigma_{\ell+1}, \ldots, \sigma_n\}$.

Proof.

- Show that: $\sigma_{\ell+1} \leq \|\Gamma_g - \Gamma_{g_\ell}\|$.
- **Truncate iteratively** directions corresponding to the same Hankel singular value σ_{i_k}
  $+$ estimate the error committed at each step by $2\sigma_{i_k}$:
  $$\left\| G^{i_{k-1}} - G^{i_k} \right\|_{H^\infty_{p\times m}} \leq \left\| E \right\|_{H^\infty_{(p+m)\times(p+m)}} = 2\sigma_{i_k}$$

$\square$

Errors bounds

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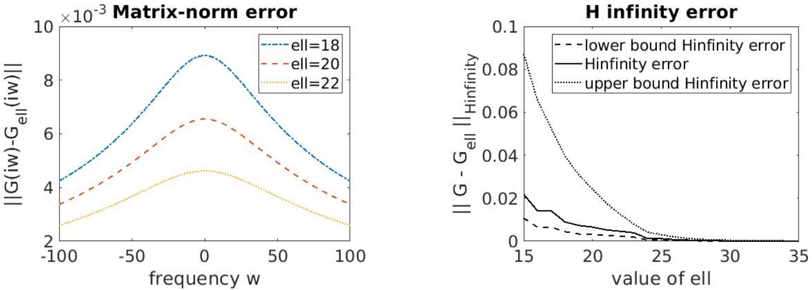

Figure: Error and error estimates

Thank you for your attention.