Thesis Progress Report #5

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Agenda

Questions Answered

How many irreducible and primitive polynomials exist for extension fields $GF((2^n)^m)$?

- $(n,m) = (2,2) = 18$
- $(n,m) = (2,3) = 180$
- $(n,m) = (3,2) = 504$
- $(n,m) = (2,4) = 1800$
- $(n,m) = (4,2) = 10800$
- ...

Determining the algebraic complexity

- The AES S-box is a function $S(x) = L(x) \oplus b$, where $L(x)$ is a linear function over $GF(2)$.
- There are many ways to represent $S(x)$ as a polynomial equation:
  - Lagrangian interpolation
  - Polynomial linearization
  - q-ary polynomial deduction

Lagrangian Interpolation

For any function $F : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ with input $x_1, \ldots, x_n$ and output $y_1, \ldots, y_n$, we may find a polynomial representation $P(x)$ as follows:

$$P(x) = \sum_{i=0}^{k-1} P_i(x),$$

where

$$P_i(x) = y_i \prod_{j=1, j \neq i}^{k} \frac{x - x_j}{x_i - x_j}$$

A Simple Example

Let $F : GF(2^2) \rightarrow GF(2^2)$ be a function defined in $GF(2^2)/p(x) = x^2 + x + 1$ by the following map:

$$\begin{aligned} 0 &\rightarrow 1 \\ 1 &\rightarrow \alpha \\ \alpha &\rightarrow \alpha + 1 \\ \alpha + 1 &\rightarrow 0 \end{aligned}$$

For Lagrangian interpolation, we need polynomials $f_z(x)$ with the property $f_z(x) = 1$ and $f_z(y) = 0$ if $y \neq z$.

A Simple Example

Start by constructing the polynomial $g(x) = (x-1)(x-\alpha)(x-(\alpha+1))$. Note that if $x \in GF(2^2) \setminus \{0\}$, then $g(x) = 0$.

Therefore, we pick $f_0(x) = g(x)/g(0)$, where $g(0) = 1 \cdot \alpha \cdot (\alpha+1) = 1$

Thus, $f_0(x) = g(x)$, which makes this very easy. Expanding out $g(x)$ yields:

$$
\begin{aligned}
g(x) &= (x-1)(x-\alpha)(x-(\alpha+1)) \\
&= (x^2 - x - x\alpha + \alpha)(x-(\alpha+1)) \\
&= x^3 - x^2 - x^2\alpha + x\alpha - x^2\alpha - x\alpha - x\alpha^2 - \alpha^2 + x^2 - x - x\alpha + \alpha \quad =
\end{aligned}
$$

after reduction with $p(x) = x^2 + x + 1$, of course.

A Simple Example

We may find the other polynomials $f_1(x), f_\alpha(x), f_{\alpha+1}(x)$ by linear substitutions:

$$f_z(x) = f_0(x - z)$$

(A textbook informed me of this fact)

A Simple Example

Now we can do interpolation as follows:

$$\begin{aligned} q(x) &= F(0)f_0(x) + F(1)f_1(x) + F(\alpha)f_\alpha(x) + F(\alpha+1)f_{\alpha+1}(x) \\ &= x^2(\alpha+1)+1 \end{aligned}$$

A simple check...

$$\begin{aligned} q(\alpha) = (\alpha)^2(\alpha+1)+1 = \alpha^3+\alpha^2+1 &= \alpha+1 \\ q(1) = (1)^2(\alpha+1)+1 &= \alpha \\ q(0) = (0)^2(\alpha+1)+1 &= 1 \\ q(\alpha+1) = (\alpha+1)^2(\alpha+1)+1 = \alpha^3+\alpha+\alpha^2 &= 0 \end{aligned}$$

Lagrangian Lesson

The method is more symbolic than computational (at first glance), so perhaps there's a better way to estimate the complexity...

Polynomial Linearization

- Any linear function A over $GF(2^k)$ can be represented as a matrix multiplication
- Similarly, such functions can be represented by a linearized polynomial:

$$f(\alpha) = \sum_{i=0}^{k-1} \lambda_i \alpha^{2^i}$$

- Solve for λ_i by setting up and solving a system of linear equations
    - Select some α, compute $A(\alpha)$ and α^{2^i} for all $0 \leq i \leq k-1$
    - Solve for each λ_i using Gaussian elimination

Bounds on Algebraic Expression

The upper bound on the number of terms in an algebraic expression for affine-power functions

$$F(x) = A(P(x))$$

in $GF(2^n)$ is $n+1$

The forward AES S-box, $F(X) = L(x^{-1}) = L(x^{254})$, has 9 terms:

$$L(x) = \sum_{i=0}^{7} \lambda_i x^{2^i}$$

Increasing the Algebraic Complexity

- Affine-power-affine functions: $F(x) = A \circ P \circ A$
  - Increases algebraic complexity without affecting other cryptographic properties (strict avalanche, nonlinearity, differential uniformity, algebraic degree)
  - This increased the algebgraic complexity from 9 to 253
- Gray code augmentation: $F(x) = L \circ P \circ G$
  - A *gray code* is a binary numeral system where two successive values differ by a single bit
  - G is gray-code conversion from an element $x \in GF(2^n)$ to a corresponding gray-code
  - Conversion process: $y_i = x_{i+1} \oplus x_i$ and $y_n = x_n$
- Möbius transformation: $f(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in GF(2^k)$.

General Maiorana-McFarland Construction

- Concatenate small affine functions to form higher-order functions
- (Hopefully) the result is an equally strong Boolean function
- All MM Boolean functions have an annihilator of degree $(n - r + 1)$, where r is the number of variables of affine functions which are used (concatenated) to construct the function
- As r decreases the annihilator degree increases, making algebraic attacks easier (it simplifies the equations)

Linear Codes

- A $[n, k, d]$-code (binary code) is a subspace of $\mathbb{F}_2^n = GF(2)^n$
  - n is the length, k is the rank, d is the minimum (Hamming) distance between each codeword in the subspace
- The vectors of a binary linear code are called the *codewords*
- As a subspace, there exists a basis **B** for the code, which is often represented as a *generator matrix* **G**
- Many codes of cryptographic interest: Hamming, Walsh-Hadamard, . . .

Candidate Codes

- Hamming Code: a special type of binary $[n, k, 3]$ code
  - Mainly used for error detection/correction, but we can use it for resilient BF constructions
- Hadamard Code: a special type of binary $[2^k, k, 2^{k-1}]$ code

Construction Idea for t-resilient

- Let $f_1, \ldots, f_{2^{n-r}}$ be 2^{n-r} affine Boolean functions of length 2^r (i.e. the truth table has 2^r entries)
- Concatenating $f_1, \ldots, f_{2^{n-r}}$ yeilds a string of length 2^n
- Let $g(x_n, \ldots, x_{r+1})$ be a nonlinear function and let $h(x_r, \ldots, x_1)$ be a linear (affine) function, and let $f(x_n, \ldots, x_1) = g(x_n, \ldots, x_{r+1}) \oplus h(x_r, \ldots, x_1)$

*Note: all Boolean functions are $(t+1)$ degenerate, for reasons that are discussed in the paper :-)

Construction Idea for t-resilient

- Select a $[n = u, k = m, d = t + 1]$ code and construct a $(2^m - 1) \times m$ matrix with codewords from C s.t. $\{c_1 D_{i,1} \oplus \cdots \oplus c_m D_{i,m} : i \leq 1 \leq 2^m - 1\} = C \setminus \{\bar{0}\}$. Let $L(C)$ be a $(2^m - 1) \times m$ matrix whose entries are u-variable functions defined by $L_{i,j}(x_1, \ldots, x_u))$
- Define an (p, m) S-box with component functions $G_1, \ldots, G_m$, and let $L(C, k, l)$ be an $(l - k + 1) \times m$ matrix whose i, jth entry is

$$G_j(y_1, \ldots, y_p) \oplus L_{k+i-1,j}(x_1, \ldots, x_u).$$

Construction Continued

- If $l - k + 1 = 2^r$ then $G \oplus L(C, k, l)$ is an $(r + p + u, m)$ S-box:

$$F_j(z_1, \ldots, z_r, y_1, \ldots, y_p, x_1, \ldots, x_u) = G_j(y_1, \ldots, y_p) \oplus L_{k+i-1,j}(x_1, \ldots, x_u)$$

- Goal: Let $m = 16$, find other parameters that make the construction "work"
- Need to select good $(p, 16)$ S-boxes $G_1, \ldots, G_m$ and *find* a good $[n, 16, t+1]$ code word

Software Optimizations for S-Box

- Extended Euclidean Algorithm - Straightforward
- Binary Extended Euclidean Algorithm - Optimized version of EEA for fields of characteristic 2
- Normal basis conversion with Fermat's Theorem - Two matrix multiplications with some shifting and multiplying
- Almost Inverse Algorithm - Compute $A^{-1}x^k \mod f(x)$ and then reduce by x^k
- Bitsliced implementation - Carnright investigates this technique with his normal basis optimizations
- LUTs - Not a goal, but always an option...

Software Optimizations for S-Box - Metrics

These can be captured with gprof for different platforms...

- Extended Euclidean Algorithm - TODO
- Binary Extended Euclidean Algorithm - TODO
- Normal basis conversion with Fermat's Theorem - TODO
- Almost Inverse Algorithm - TODO
- Bitsliced implementation - TODO
- LUTs - ;-)

Complexity of Finite Field Multipliers

- Claim: for small fields (e.g. $GF(2^k), k \leq 32$) the *arithmetic* procedures for software implementations **are not** affected by the field polynomial.
  - Advanced algorithms such as the "comb" multiplier target fields where single elements cannot fit within a single word
- This is not true for hardware...
  - If we're going for area optimized designs, we want serial modules, otherwise we want parallel modules
  - Some bases yield more efficient arithmetic operations than others
  - This leads us to Optimal Normal Bases

Inverse by Fermat's Theorem

By Fermat's Theorem, $\alpha^{-1} \equiv \alpha^{2^k-2}$

$$2^{m-2} = 2 + 2^2 + 2^3 + \cdots + 2^{m-1}$$

This leads us to a simple square and multiply algorithm...

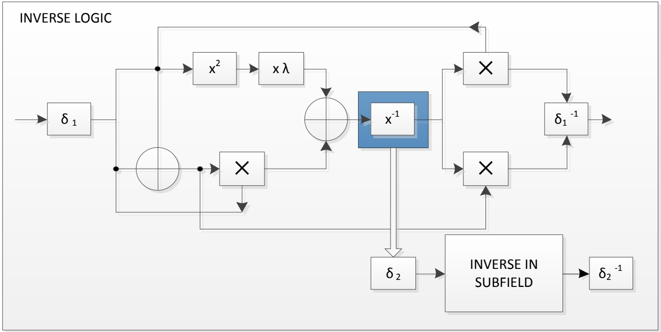
$$\alpha^{-1} = \alpha^2 \cdot \alpha^{2^2} \cdot \alpha^{2^3} \cdots \cdot \alpha^{2^{m-1}}$$

In a normal basis the cycle complexity is $\mathcal{O}(k)$ for computing the successive powers of α, but the area complexity depends on the type of multiplier used (e.g. using a ONB Type II basis one can implement a parallel multiplier with $1.5(k^2 - k)$ XOR gates [1])

Inverse by Composite Field Computation

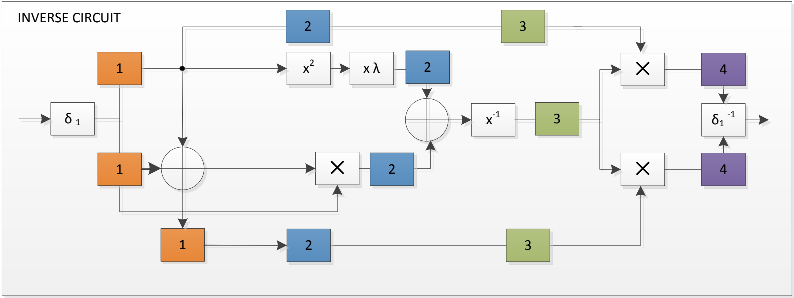
$$(bx+c)^{-1} = b(b^2B+bcA+c^2)^{-1}x+(c+bA)(b^2B+bcA+c^2)^{-1}$$

with $A = 1$ and $B = \lambda$



Inverse by Composite Field Computation (continued)

5-stage pipeline design



Optimal Pipeline Selection Strategy (for FPGAs)

Algorithm 1 Pipeline Optimization Strategy

```
E_c = Throughput(Mbits/s)/Area
Opt ← False
while Opt = False do
    Remove the pipeline state that contributes the lowest frequency reduction
    Reimplement and resynthesize the design
    E_n = Throughput(Mbits/s)/Area
    if E_c > E_n then
        Opt = True
    end if
end while
```

Inverse by Composite Field Computation (continued)

The next step is to synthesize the design and gather hardware metrics.

- LUT count (FPGA - captured with Xilinx tools)
- Register count (FPGA - captured with Xilinx tools)
- Slice count (FPGA - captured with Xilinx tools)
- Throughput in cycles/byte (FPGA - captured with Xilinx tools)
- Power consumption (ASIC - captured with Synopsys) :-)

References

1 Sunar, Berk, and Cetin Kaya Koc. "An efficient optimal normal basis type II multiplier." Computers, IEEE Transactions on 50.1 (2001): 83-87.

Action Items (perhaps overly ambitious...)

- Optimize Galois field software for more efficient calculation of polynomials and transformation matrices
- Finish composite field decomposition chapter
- Polynomial and normal basis conversion code and preparation for OSG execution
- Literature survey of S-box constructions and code for estimating algebraic complexity
- Complete the exhaustive list of all polynomials $P(x)$, $Q(y)$, and $R(z)$ and the corresponding list of all transformation matrices (using OSG!)
- Hardware metrics of regular and non-pipelined 16-bit inverse of composite field inverse
- Implement Carnright's normal basis S-box
- $(16, 16)$-Boolean function code using the prescribed approach