Classical Mechanics by John R. Taylor Notes

Chris Doble

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Contents

1 Newton's Laws of Motion

1.2 Space and Time

- In cartesian coordinates the basis vectors don't depend on time so their derivatives are $\mathbf{0}$. This means that

$$\begin{aligned}\frac{d}{dt}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) &= \frac{dx}{dt}\hat{\mathbf{x}} + x\frac{d\hat{\mathbf{x}}}{dt} + \frac{dy}{dt}\hat{\mathbf{y}} + y\frac{d\hat{\mathbf{y}}}{dt} + \frac{dz}{dt}\hat{\mathbf{z}} + z\frac{d\hat{\mathbf{z}}}{dt} \\ &= \frac{dx}{dt}\hat{\mathbf{x}} + \frac{dy}{dt}\hat{\mathbf{y}} + \frac{dz}{dt}\hat{\mathbf{z}}\end{aligned}$$

  as expected. However, in order coordinate systems (e.g. polar, spherical) the basis vectors may depend on time and their derivatives aren't $\mathbf{0}$.

1.4 Newton's First and Second Laws; Inertial Frames

- Newton's second law $\mathbf{F} = m\mathbf{a}$ can be restated as $\mathbf{F} = \dot{\mathbf{p}}$.
- An inertial frame is one where Newton's first law holds. Typically this means the frame isn't accelerating or rotating.

1.5 The Third Law and Conservation of Momentum

- Forces that act along the line joining two objects are called **central forces**.
- The **principle of conservation of momentum** states that if the net external force $\mathbf{F}_{\text{ext}}$ on an N-particle system is zero, the system's total momentum $\mathbf{P}$ is constant.

1.7 Two-Dimensional Polar Coordinates

- In two-dimensional polar coordinates, the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ depend on position and thus time. Their derivatives are

$$\begin{aligned}\frac{d\hat{\mathbf{r}}}{dt} &= \dot{\phi}\hat{\boldsymbol{\phi}} \\ \frac{d\hat{\boldsymbol{\phi}}}{dt} &= -\dot{\phi}\hat{\mathbf{r}}.\end{aligned}$$

Consequently, the derivatives of the position vector $\mathbf{r} = r\hat{\mathbf{r}}$ are

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= \frac{d}{dt}(r\hat{\mathbf{r}}) \\ &= \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \\ &= \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}\end{aligned}$$

and

$$\begin{aligned}\frac{d^2\mathbf{r}}{dt^2} &= \frac{d}{dt}(\dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}) \\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt} + \dot{r}\dot{\phi}\hat{\boldsymbol{\phi}} + r\ddot{\phi}\hat{\boldsymbol{\phi}} + r\dot{\phi}\frac{d\hat{\boldsymbol{\phi}}}{dt} \\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{r}\dot{\phi}\hat{\boldsymbol{\phi}} + r\ddot{\phi}\hat{\boldsymbol{\phi}} - r\dot{\phi}^2\hat{\mathbf{r}} \\ &= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\boldsymbol{\phi}}.\end{aligned}$$

- In light of the above, Newton's second law in polar coordinates can be written

$$\begin{aligned}F_r &= m(\ddot{r} - r\dot{\phi}^2) \\ F_\phi &= m(r\ddot{\phi} + 2\dot{r}\dot{\phi}).\end{aligned}$$

2 Projectiles and Charged Particles

2.1 Air Resistance

- Air resistance depends on the speed v of the moving object. For many objects the direction of the air resistance force $\mathbf{f}$ is opposite to $\mathbf{v}$, but not always. For example, the air resistance force on an airplane causes lift.

- An air resistance force can be described by the equation

$$\mathbf{f} = -f(v)\hat{\mathbf{v}}$$

  where $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$ gives the direction and $f(v)$ gives the magnitude.

- $f(v)$ can be approximated as

$$f(v) = f_{\text{lin}} + f_{\text{quad}} = bv + cv^2.$$

- The linear term f_{lin} arises from the viscous drag of the medium and is generally proportional to the projectile's linear size.

- The quadratic term f_{quad} arises from the fact that the projectile must accelerate the air with which it is continually colliding and it is proportional to the density of the medium and the cross-sectional area of the projectile.

- For a spherical projectile the coefficients b and c above have the form

$$b = \beta D \text{ and } c = \gamma D^2$$

where D is the diameter of the sphere and the coefficients β and γ depend on the nature of the medium. In air at STP they have approximate values

$$\beta = 1.6 \times 10^{-4}\,\mathrm{N\,s/m^2}$$

and

$$\gamma = 0.25\,\mathrm{N\,s^2/m^4}.$$

- Depending on the natures of the medium and projectile it's often possible to neglect one of the terms in $f(v)$. To determine if this is the case we can calculate their ratio. For example, for a spherical projectile at STP

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} = \frac{cv^2}{bv} = \frac{\gamma D}{\beta} v = (1.6 \times 10^3\,\mathrm{s/m^2}) Dv.$$

If the ratio is large f_{lin} can be ignored. If it's small f_{quad} can be ignored.

- The **Reynolds number** can be used to characterise the behaviour of an object in a fluid

$$R = \frac{\rho}{\mu} Dv$$

where ρ is the medium's density, μ is its viscosity, D is the linear dimension of the projectile (diameter for spherical projectiles), and v is the projectile's speed. The quadratic force f_{quad} is dominant when the Reynolds number R is large and the linear force f_{linear} is dominant when it is small.

2.2 Linear Air Resistance

- When the quadratic drag force is negligible the equation of motion becomes

$$\begin{aligned} \mathbf{F} &= \mathbf{W} - \mathbf{f} \\ m\mathbf{a} &= m\mathbf{g} - b\mathbf{v} \\ m\dot{\mathbf{v}} &= m\mathbf{g} - b\mathbf{v}. \end{aligned}$$

This is a first-order differential equation for $\mathbf{v}$ where the horizontal and vertical components can be separated to

$$\begin{aligned} m\dot{v}_x &= -bv_x \\ m\dot{v}_y &= mg - bv_y, \end{aligned}$$

each of which is easily solvable.

- The **terminal speed** of an object undergoing freefall and experiencing only linear drag is
$$v_{\text{ter}} = \frac{mg}{b}.$$
- The **characteristic time**
$$\tau = \frac{1}{k} = \frac{1}{b/m} = \frac{m}{b}$$
is a measure of the importance of air resistance.
  - For horizontal motion with drag it's a measure of the time it takes for the projectile to reach $1/e$ of its initial velocity.
  - For freefall with drag it's a measure of the time it would take the projectile to reach its terminal velocity if it didn't experience drag
$$v_{\text{ter}} = g\tau.$$
  - For freefall with drag it can also be used to gauge what percentage of its terminal velocity a projectile will reach after a certain time:

Time t	Percent of v_{ter}
0	0
τ	63%
2τ	86%
3τ	95%

    From this it can be seen that after $t = 3\tau$ the projectile has effectively reached its terminal velocity.

2.4 Quadratic Air Resistance

- Equations of motion for quadratic air resistance can be solved analytically when the projectile moves in one dimension, but can only be solved numerically when it moves in multiple dimensions.
- When a projectile moves in one dimension and only experiences the force of air resistance (i.e. there are no other forces), the equation of motion is
$$m\dot{v} = -cv^2.$$
Using separation of variables the solution can be found to be
$$v(t) = \frac{v_0}{1 + t/\tau}$$
where
$$\tau = \frac{m}{cv_0}.$$

- As in the linear case, τ is a measure of how long it takes for air resistance to slow down the projectile ($v = v_0/2$ at $t = \tau$).
- Integrating the equation for $v(t)$ gives

$$x(t) = v_0 \tau \ln\left(1 + \frac{t}{\tau}\right).$$

- When a projectile moves in one dimension and experiences the forces of air resistance and weight, the equation of motion (with y down) is

$$m\dot{v} = mg - cv^2.$$

Using separation of variables the solution can be found to be

$$v(t) = v_{\text{ter}} \tanh \frac{gt}{v_{\text{ter}}}$$

where

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}}.$$

- Integrating the equation for $v(t)$ gives

$$y = \frac{v_{\text{ter}}^2}{g} \ln\left(\cosh \frac{gt}{v_{\text{ter}}}\right).$$

2.5 Motion of a Charge in a Uniform Magnetic Field

- When a particle of charge q moves in a magnetic field $\mathbf{B} = (0, 0, B_z)$ with velocity $\mathbf{v} = (v_x, v_y, v_z)$ it experiences a force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = q(v_y B, -b_x B, 0).$$

This gives the coupled equations of motion

$$\begin{aligned} m\dot{v}_x &= qBv_y \\ m\dot{v}_y &= -qBv_x \\ m\dot{v}_z &= 0 \end{aligned}$$

or

$$\begin{aligned} \dot{v}_x &= \omega v_y \\ \dot{v}_y &= -\omega v_x \\ \dot{v}_z &= 0 \end{aligned}$$

where $\omega = qB/m$ is called the **cyclotron frequency**.

- If we define a complex value

$$\eta = v_x + iv_y,$$

its derivative is

$$\begin{aligned}\dot{\eta} &= \dot{v}_x + i\dot{v}_y \\ &= \omega v_y - i\omega v_x \\ &= -i\omega\eta\end{aligned}$$

which has the solution

$$\eta = Ae^{-i\omega t}.$$

3 Momentum and Angular Momentum

3.1 Conservation of Momentum

- The **principle of conservation of momentum** states that if the net external force $\mathbf{F}_{\text{ext}}$ on an N-particle system is zero, the system's total mechanical momentum $\mathbf{P} = \sum m_\alpha v_\alpha$ is constant.

3.2 Rockets

- Newton's second law for a rocket is

$$m\dot{v} = -\dot{m}v_{\text{ex}}$$

where $\dot{m}$ is the (negative) rate of change of the mass of the rocket and v_{ex} is the velocity of the exhaust. The quantity on the right hand side of the equation is called the **thrust**.

- The equation above can be solved by separation of variables giving

$$v - v_0 = v_{\text{ex}} \ln \frac{m_0}{m}$$

which is often called the **rocket equation**.

3.3 The Center of Mass

- The **centre of mass** of a system is defined to be

$$\mathbf{R} = \frac{1}{M} \sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha$$

where M is the total mass of all particles in the system, m_α is the mass of particle α, and $\mathbf{r}_\alpha$ is the vector from the origin to particle α.

- The total momentum of a system can be written in terms of its centre of mass
$$\mathbf{P} = \sum_\alpha \mathbf{p}_\alpha = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha = M\dot{\mathbf{R}}$$
i.e. the total momentum of N particles is equivalent to that of a single particle of mass M with velocity equal to that of the centre of mass.

- Differentiating the above we find
$$\begin{aligned}\frac{d}{dt}\mathbf{P} &= \frac{d}{dt}(M\dot{\mathbf{R}}) \\ \mathbf{F}_{\text{ext}} &= M\ddot{\mathbf{R}}\end{aligned}$$
i.e. the centre of mass moves as if it was a single particle of mass M subject to the net external force on the system.

- When a body is continuous the expression for its centre of mass becomes an integral
$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\, dm = \frac{1}{M}\int \rho\mathbf{r}\, dV.$$

3.4 Angular Momentum for a Single Particle

- The **angular momentum** of a particle relative to an origin O is
$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$
where $\mathbf{r}$ is measured relative to O.

- Taking the derivative of angular momentum gives
$$\begin{aligned}\frac{d}{dt}\mathbf{L} &= \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) \\ \dot{\mathbf{L}} &= \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} \\ &= \mathbf{r} \times \mathbf{F} \\ &= \boldsymbol{\tau}.\end{aligned}$$
In other words, the rate of change in angular momentum about an origin O is equal to the net torque about that origin.

- We can simplify some one-particle problems by choosing the origin such that the net torque is 0 and thus angular momentum is constant.

3.5 Angular Momentum for Several Particles

- The **total angular momentum** of a system is
$$\mathbf{L} = \sum_{\alpha=1}^{N} \mathbf{L}_\alpha = \sum_{\alpha=1}^{N} \mathbf{r}_\alpha \times \mathbf{p}_\alpha.$$

- Differentiating the above

$$\dot{\mathbf{L}} = \sum_\alpha \dot{\mathbf{L}}_\alpha = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_\alpha = \boldsymbol{\tau}_{\text{ext}}$$

  we find that the rate of change of the total angular momentum of the system is equal to the net torque on the system.

- The **principle of conservation of angular momentum** states that if the net external torque on a system is 0, the system's total angular momentum is constant. This assumes that all internal forces are central and obey Newton's third law.

- The principle of conservation of momentum and the result $\dot{\mathbf{L}} = \boldsymbol{\tau}_{\text{ext}}$ also hold if $\mathbf{L}$ and $\boldsymbol{\tau}_{\text{ext}}$ are measured about the centre of mass, even if the centre of mass is being accelerated and is thus not an interial frame.

4 Energy

4.1 Kinetic Energy and Work

- The **work-kinetic-energy theorem** states that the change in a particle's kinetic energy between two points is equal to the work done by the net force on the particle between those two points

$$\Delta K = \int_1^2 \mathbf{F} \cdot d\mathbf{r}.$$

4.2 Potential Energy and Conservative Forces

- A force $\mathbf{F}$ acting on a particle is considered **conservative** if:
  - $\mathbf{F}$ depends only on the particle's position $\mathbf{r}$ (and not on its velocity $\mathbf{v}$, time t, or any other variable), and
  - for any two points 1 and 2, the work done by $\mathbf{F}$ is the same for all paths between 1 and 2.

- Only conservative forces have associated **potential energy** functions.

- The potential energy function $U(\mathbf{r})$ of a conservative force $\mathbf{F}$ is defined as

$$U(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}')\, d\mathbf{r}'$$

  where $\mathbf{r}_0$ is an arbitrary point at which $U(\mathbf{r}_0)$ is defined to be 0.

- The **principle of conservation of energy** states that if all the forces acting on a particle are conservative, each with its corresponding potential energy function $U_i(\mathbf{r})$, the **total mechanical energy**

$$E = K + U = K + U_1(\mathbf{r}) + \cdots + U_n(\mathbf{r}),$$

is constant in time.

- If nonconservative forces do work then the total energy of the system changes by that amount

$$\Delta E = W_{\text{nc}}.$$

4.3 Force as the Gradient of Potential Energy

- A conservative force $\mathbf{F}$ can be expressed as the negative gradient of its potential energy function U

$$\mathbf{F} = -\nabla U.$$

4.4 The Second Condition that F be Conservative

- A force $\mathbf{F}$ is conservative if $\nabla \times \mathbf{F} = \mathbf{0}$.

4.5 Time-Dependent Potential Energy

- If a time-dependent force $\mathbf{F}(t)$ has the property $\nabla \times \mathbf{F}(t) = \mathbf{0}$ it's still possible to define an associated potential energy function $U(\mathbf{r}, t)$ where $\mathbf{F}(t) = -\nabla U(t)$ but it's no longer guaranteed that total mechanical energy is conserved over time.

4.8 Central Forces

- A central force is conservative if and only if it's spherically symmetric.

5 Oscillations

5.2 Simple Harmonic Motion

- The equation of motion for a harmonic oscillator

$$\ddot{x} = -\frac{k}{m}x = -\omega^2 x$$

can be solved in multiple ways:

  - the exponential solution

$$x = c_1 e^{i\omega t} + c_2 e^{-i\omega t},$$

– the sine and cosine solutions

$$x = c_1 \cos \omega t + c_2 \sin \omega t,$$

and

– the phase shifted cosine solution

$$x = A \cos(\omega t - \delta)$$

where

$$A = \sqrt{c_1^2 + c_2^2}$$

with c_1 and c_2 coming from the sine and cosine solutions above and

$$\delta = \arctan -\frac{c_1}{c_2}.$$

5.3 Two-Dimensional Oscillators

- An **isotropic harmonic oscillator** in $n > 1$ dimensional space experiences a restoring force directed towards the equilibrium position and with magnitude kr where r is the object's distance from equilibrium.
- In two dimensions an isotropic harmonic oscillator has general solutions

$$x(t) = A_x \cos \omega t$$
$$y(t) = A_y \cos(\omega t - \delta).$$

It was possible to eliminate the phase from $x(t)$ by redefining the origin of time but in general it isn't possible to also eliminate the phase from $y(t)$.

- An **anisotropic harmonic oscillator** is similar to an isotropic harmonic oscillator but the spring constants are different in different directions.

5.7 Fourier Series

- Any periodic function with period T can be represented as a Fourier series

$$f(t) = \sum_{n=0}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

where

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt$$
$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt$$
$$b_0 = 0$$
$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)\,dt$$
$$\omega = \frac{2\pi}{T}.$$

6 Calculus of Variations

- A **functional** is a mapping from a space X to the real or complex numbers. When X is the space of functions a functional is a "function of a function", i.e. it takes a function as an argument.
- The goal of the **calculus of variations** is to find maxima and minima of functionals, i.e. functions that maximise or minimise the value of the functional. This is analogous to finding real numbers that maximise or minimise a function in single-variable calculus.
- A functional of the form

  $$S = \int_{x_1}^{x_2} f[x, y(x), y'(x)]\,dx$$

  can be solved using the **Euler-Lagrange equation**

  $$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0.$$

- A solution to the Euler-Lagrange equation isn't guaranteed to be a minimum — it could be a maximum or an inflection point, as in single-variable calculus. In general it's difficult to determine the nature of a given solution so other methods (e.g. inspection) must be used.
- A functional with multiple functions as arguments, e.g.

  $$S = \int_{t_1}^{t_2} f[t, x(t), x'(t), y(t), y'(t)]\,dt,$$

  results in a Euler-Lagrange equation for each function, e.g.

  $$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial x'} = 0$$
  $$\frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial y'} = 0.$$

These can then be solved as above.

- Under Lagrangian mechanics, the independent variable is time t and the dependent variable(s) depend on the system under consideration. In general they're denoted $q_1, q_2, \ldots, q_n$ and are called **generalized coordinates**.

7 Lagrange's Equations

- Lagrangian mechanics has two advantages over Newtonian mechanics:
  - Lagrange's equations have the same form in all coordinate systems, and
  - Lagrange's equations omit the forces of constraint (e.g. the normal force that keeps a bead on a wire), simplifying calculations.

7.1 Lagrange's Equations for Unconstrained Motion

- The **Lagrangian function** or **Lagrangian** is defined as

$$\mathcal{L} = K - U,$$

  i.e. the kinetic energy minus the potential energy.

- **Hamilton's principle** states that the actual path taken by a particle between points 1 and 2 in a given time interval t_1 to t_2 is such that the action integral

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

  is stationary when taken along the actual path, i.e. the actual path is the solution of the Euler-Lagrange equation when applied to the Lagrangian.

- A Lagrangian can be written in terms of any **generalized coordinates** q_1, q_2, q_3 providing each position $\mathbf{r}$ corresponds to a unique value (q_1, q_2, q_3) and vice versa.

- The derivative of the Lagrangian with respect to x

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial}{\partial x}(K - U) = \frac{\partial}{\partial x}\left(\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(x, y)\right) = -\frac{\partial U(x, y)}{\partial x} = F_x$$

  is the x component of the force while the derivative with respect to $\dot{x}$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}(K - U) = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(x, y)\right) = m\dot{x} = p_x$$

  is the x component of the momentum. The same applies to the y and z dimensions. When generalized coordinates q_1, q_2, q_3 are used the corresponding values behave like forces and momenta and are called **generalized forces** and **generalized momenta**, respectively.

- Another way of stating the above is

$$\frac{\partial \mathcal{L}}{\partial q_i} = (i\text{th component of generalized force})$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = (i\text{th component of generalized momentum}).$$

Using this terminology, the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

takes the form

$$(\text{generalized force}) = (\text{rate of change of generalized momentum}).$$

- For example, in 2D polar coordinates (r, ϕ) the generalized force for the ϕ coordinate is the torque on the particle and the generalized momentum is the angular momentum.

- Conservation laws can be derived from the Euler-Lagrange equations in generalized coordinates. For example, if the ith component of the generalized force is zero

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0$$

then the rate of change of the ith component of the generalized momentum is also zero and thus it doesn't change.

- If the relationship between $\mathbf{r}$ and the generalized coordinates $q_1, q_2, \ldots, q_n$ doesn't involve t the generalized coordinates are called **natural** and have some additional properties.

- The number of **degrees of freedom** of a system is the number of coordinates that can be independently varied in a small displacement, i.e. the number of independent "directions" in which the system can move from any given initial configuration.

- When the number of degrees of freedom of an N particle system is less than $3N$ (or $2N$ in two dimensions), the system is said to be **constrained**.

- When the number of degrees of freedom of a system matches the number of generalized coordinates required to model the system, it is said to be **holonomic**.

- In order to apply Lagrange's equations to a system its constraints must be holonomic, i.e. they must be expressible in the form

$$f(q_1, q_2, \ldots, q_n, t) = 0.$$

- The generalized coordinates can be measured relative to a non-inertial reference frame providing the Lagrangian $\mathcal{L} = K - U$ is originally written as inertial.

7.6 Generalized Momenta and Ignorable Coordinates

- When the Lagrangian is independent of a coordinate q_i, that coordinate is said to be **ignorable** or **cyclic**. When choosing coordinates, it is desirable to make as many ignorable as possible.

7.8 More about Conservation Laws

- If the Lagrangian is unchanged by spacial translation, the total momentum of the system is conserved.
- If the Lagrangian is unchanged by time translation, the total energy of the system is conserved.

7.9 Lagrange's Equations for Magnetic Forces

- For a given mechanical system with generalized coordinates $q = (q_1, \ldots, q_n)$, a **Lagrangian** $\mathcal{L}$ is a function $\mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$ of the coordinates and velocities, such that the correct equations of motion for the system are the Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \text{ for } i = 1, \ldots, n.$$

- It's important to note that the above does not define a unique Lagrangian function — any function $\mathcal{L}$ that gives the correct equations of motion is valid and has all the correct properties.
- The Lagrangian for a particle of charge q and mass m moving in electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ is

$$\mathcal{L} = \frac{1}{2}m\dot{\mathbf{r}}^2 - q(V - \dot{\mathbf{r}} \cdot \mathbf{A}).$$

8 Two-Body Central Force Problems

8.1 The Problem

- If two objects that experience a conservative central force, their potential energy depends only on the distance between them

$$U(\mathbf{r}_1, \mathbf{r}_2) = U(|\mathbf{r}_1 - \mathbf{r}_2|) = U(r)$$

and thus the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2 - U(r).$$

8.2 CM and Relative Coordinates; Reduced Mass

- It is simplest if the generalized coordinates are chosen to be the position of the centre of mass of the system

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}$$

and the relative position of the two bodies

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2.$$

- This results in a kinetic energy

$$K = \frac{1}{2}(M\dot{\mathbf{R}}^2 + \mu\dot{\mathbf{r}}^2)$$

where

$$\mu = \frac{m_1 m_2}{M} = \frac{m_1 m_2}{m_1 + m_2}$$

is the **reduced mass** of the system.

- The Lagrangian is then

$$\mathcal{L} = K - U = \frac{1}{2}M\dot{\mathbf{R}}^2 + \left[\frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)\right] = \mathcal{L}_{\text{cm}} + \mathcal{L}_{\text{rel}}$$

where each generalized coordinate only appears in one "sub-Lagrangian" and can be solved separately.

8.3 The Equations of Motion

- Because $\mathcal{L}_{\text{cm}}$ doesn't include $\mathbf{R}$ the equation of motion for the centre of mass is

$$M\ddot{\mathbf{R}} = \mathbf{0},$$

i.e. the centre of mass moves with constant velocity.

- The equation of relative motion is

$$\mu\ddot{\mathbf{r}} = -\nabla U(r),$$

i.e. the two bodies move as if they were a single particle of mass μ with potential energy $U(r)$.

- If we choose to use the inertial centre-of-mass reference frame, $\mathcal{L}_{\text{cm}} = 0$ and $\mathcal{L} = \mathcal{L}_{\text{rel}}$ becomes a one-body problem.

- The total angular momentum in the centre-of-mass frame is

$$\mathbf{L} = \mathbf{r} \times \mu \dot{\mathbf{r}}.$$

Because the total angular momentum is conserved — including its direction — this means that $\mathbf{r}$ and $\dot{\mathbf{r}}$ are confined to a plane that we can choose to be the xy plane. The three-dimensional two-body problem has been turned into a two-dimensional one-body problem.

- The Lagrangian for this two-dimensional problem in polar coordinates is

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r).$$

Because this doesn't involve ϕ the Lagrange equation corresponding to ϕ is

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} = \text{const} = \ell$$

which is simply a statement of the conservation of angular momentum. The Lagrange equation corresponding to r is

$$\mu r \dot{\phi}^2 - \frac{dU}{dr} = \mu \ddot{r}.$$

8.4 The Equivalent One-Dimensional Problem

- Rearranging the ϕ equation we find

$$\dot{\phi} = \frac{\ell}{\mu r^2}$$

where ℓ is determined by initial conditions.

- The radial equation can be rewritten as

$$\begin{aligned} \mu \ddot{r} &= -\frac{dU}{dr} + \mu r \dot{\phi}^2 \\ &= -\frac{dU}{dr} + F_{\text{cf}} \end{aligned}$$

where F_{cf} is the fictitious centifugal force

$$F_{\text{cf}} = \mu r \dot{\phi}^2 = \frac{\ell^2}{\mu r^3} = -\frac{d}{dr}\left(\frac{\ell^2}{2\mu r^2}\right) = -\frac{dU_{\text{cf}}}{dr}.$$

- The radial equation can now be written in terms of the **effective potential energy**

$$\mu \ddot{r} = -\frac{d}{dr}[U(r) + U_{\text{cf}}(r)] = -\frac{d}{dr}U_{\text{eff}}(r).$$

- The total energy of the one-body system is

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\phi}^2 + U(r)$$

and this value is conserved.

9 Mechanics in Noninertial Frames

9.1 Acceleration without Rotation

- A noninertial frame of reference $\mathcal{S}$ has acceleration $\mathbf{A}$ relative to an inertial frame of reference $\mathcal{S}_0$. Newton's second law can be used in $\mathcal{S}$ providing we add an extra force-like term called the **inertial force**

$$m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{A}.$$

9.3 The Angular Velocity Vector

- The angular velocity vector $\boldsymbol{\omega}$ has direction equal to that of the axis of rotation (using the right-hand rule to disambiguate direction) and magnitude equal to the rate of rotation.

- If a body is rotating with angular velocity $\boldsymbol{\omega}$ about an axis through O, the velocity of a point P (position $\mathbf{r}$) fixed on the body is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}.$$

- Suppose there are two frames of reference 2 and 1. Frame 2 is rotating with angular velocity $\boldsymbol{\omega}_{21}$ relative to frame 1. A body 3 is rotating with angular velocities $\boldsymbol{\omega}_{31}$ and $\boldsymbol{\omega}_{32}$ relative to frames 1 and 2, respectively. These angular velocity vectors add such that

$$\boldsymbol{\omega}_{31} = \boldsymbol{\omega}_{32} + \boldsymbol{\omega}_{21}.$$

9.4 Time Derivatives in a Rotating Frame

- Given an inertial reference frame $\mathcal{S}_0$ and a noninertial reference frame $\mathcal{S}$ that is rotating with angular velocity $\boldsymbol{\Omega}$ relative to $\mathcal{S}_0$, the time derivative of a vector $\mathbf{Q}$ differs between the two with the relation

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}_0} = \left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}} + \boldsymbol{\Omega} \times \mathbf{Q}.$$

9.5 Newton's Second Law in a Rotating Frame

- Newton's second law in a noninertial frame rotating with angular velocity $\boldsymbol{\Omega}$ is

$$\begin{aligned} m\ddot{\mathbf{r}} &= \mathbf{F} + \mathbf{F}_{\text{cor}} + \mathbf{F}_{cf} \\ &= \mathbf{F} + 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega} \end{aligned}$$

  where $\mathbf{F}_{\text{cor}}$ is the **Coriolis force** and $\mathbf{F}_{\text{cf}}$ is the **centrifugal force**.

10 Rotational Motion of Rigid Bodies

10.1 Properties of the Centre of Mass

- The total angular momentum of a system of N particles is

$$\begin{aligned}\mathbf{L} &= \mathbf{L}_{\mathrm{O}} + \mathbf{L}_{\mathrm{CM}} \\ &= \mathbf{R} \times \mathbf{P} + \sum \mathbf{r}'_\alpha \times m_\alpha \dot{\mathbf{r}}'_\alpha\end{aligned}$$

  where $\mathbf{L}_{\mathrm{O}}$ is the angular momentum of the centre of mass of the system relative to the origin, $\mathbf{L}_{\mathrm{CM}}$ is the angular momentum of the particles of the system relative to its centre of mass, $\mathbf{R}$ is the centre of mass of the system, $\mathbf{P}$ is the linear momentum of the system, $\mathbf{r}'_\alpha$ is the position of partical α relative to the centre of mass of the system, m_α is the mass of particle α, and $\dot{\mathbf{r}}'_\alpha$ is the velocity of particle α relative to the centre of mass of the system.

- The values $\mathbf{L}_{\mathrm{O}}$ and $\mathbf{L}_{\mathrm{CM}}$ are independently conserved, with

$$\dot{\mathbf{L}}_{\mathrm{O}} = \mathbf{R} \times \mathbf{F}^{\mathrm{ext}} = \boldsymbol{\tau}^{\mathrm{ext}} \text{ (about origin)}$$

  and

$$\dot{\mathbf{L}}_{\mathrm{CM}} = \sum \mathbf{r}'_\alpha \times \mathbf{F}^{\mathrm{ext}}_\alpha = \boldsymbol{\tau}^{\mathrm{ext}} \text{ (about centre of mass).}$$

- The total kinetic energy of a system of N particles is

$$\begin{aligned}T &= T_{\mathrm{CM}} + T_{\text{relative to CM}} \\ &= \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \sum m_\alpha \dot{\mathbf{r}}^2_\alpha\end{aligned}$$

- The total potential energy of a system of N particles is

$$U = U^{\mathrm{ext}} + U^{\mathrm{int}}$$

  where U^{ext} is the total potential energy due to external forces and U^{int} is the total potential energies for all pairs of particles

$$U^{\mathrm{int}} = \sum_{\alpha<\beta} U_{\alpha\beta}(r_{\alpha\beta}).$$

  However, in a rigid body the distances between all particles are fixed so U^{int} is constant and can be ignored.

10.2 Rotation about a Fixed Axis

- For a system of N particles rotating around the z axis, the angular momentum of the system is is

$$\begin{aligned}\mathbf{L} &= \sum_\alpha \boldsymbol{\ell}_\alpha \\ &= \sum_\alpha m_\alpha \mathbf{r}_\alpha \times \mathbf{v}_\alpha \\ &= \sum_\alpha m_\alpha \omega(-z_\alpha x_\alpha, -z_\alpha y_\alpha, x_\alpha^2 + y_\alpha^2).\end{aligned}$$

  Thus, $\mathbf{L}$ may have $\mathbf{x}$ and/or $\mathbf{y}$ components and $\mathbf{L} = I\boldsymbol{\omega}$ doesn't hold.

- The angular momentum can be expressed as

$$\mathbf{L} = (I_{xz}\omega, I_{yz}\omega, I_{zz}\omega)$$

  where

$$\begin{aligned}I_{xz} &= -\sum_\alpha m_\alpha x_\alpha z_\alpha, \\ I_{yz} &= -\sum_\alpha m_\alpha y_\alpha z_\alpha, \text{ and} \\ I_{zz} &= \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2)\end{aligned}$$

  are called the **products of inertia** of the body. The notation means that e.g. I_{xz} is the x component of the angular momentum for a body rotating around the z axis.

10.3 Rotation about Any Axis; the Inertia Tensor

- Continuing from the above, if a body has angular velocity

$$\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$$

  its angular momentum is

$$\begin{aligned}L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z\end{aligned}$$

  where

$$\begin{aligned}I_{xx} &= \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2), \\ I_{xy} &= -\sum_\alpha m_\alpha x_\alpha y_\alpha,\end{aligned}$$

  and the other values follow the same pattern.

- The above can also be written

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$$

where $\mathbf{I}$ is the **moment of inertia tensor**

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}.$$

10.4 Principle Axes of Inertia

- In general, the angular momentum $\mathbf{L}$ of a spinning object is not parallel to $\boldsymbol{\omega}$. However for any rigid body and any point O there are three perpendicular axes through O such that $\mathbf{I}$ is diagonal when $\boldsymbol{\omega}$ is parallel to one of these axes, so is $\mathbf{L}$. These are called the **principle axes**.

- The three moments of inertia about the three principle axes are called the **principle moments**.

- If we choose the principle axes of a rigid body as our frame of reference rotation calculations are simplified but we're using a noninertial frame of reference.

10.5 Finding the Principle Axes; Eigenvalue Equations

- If $\boldsymbol{\omega}$ is parallel to a principle axis, then $\mathbf{L} = \mathbf{I}\boldsymbol{\omega} = \lambda\boldsymbol{\omega}$. This can be converted to an eigenvalue equation

$$(\mathbf{I} - \lambda\mathbf{1})\boldsymbol{\omega} = 0$$

where the eigenvectors are the principle axes. Thus to find the principle axes of a rigid body about a point we must calculate an initial intertia tensor then calculate its eigenvectors. The updated inertia tensor will be

$$\mathbf{I} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

10.7 Euler's Equations

- The axes x, y, and z are chosen to be in an inertial frame called the **space frame** while the axes $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are chosen to align with the body's principle axes and this is is a non-inertial frame known as the **body frame**. If the body is rotating around a fixed point that is taken to be the origin of the body frame, otherwise the body's centre of mass is used.

- Suppose a body is rotating with angular velocity $\boldsymbol{\omega}$ and has principle moments of inertia λ_1, λ_2, and λ_3, then its angular momentum in the body frame is

$$\mathbf{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3).$$

If a torque $\boldsymbol{\Gamma}$ acts on the body, then in the space frame

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{space}} = \boldsymbol{\Gamma}.$$

In the body frame this becomes

$$\begin{aligned}\left(\frac{d\mathbf{L}}{dt}\right)_{\text{space}} &= \left(\frac{d\mathbf{L}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \mathbf{L} \\ &= \dot{\mathbf{L}} + \boldsymbol{\omega} \times \mathbf{L}.\end{aligned}$$

Equating the previous two equations gives

$$\dot{\mathbf{L}} + \boldsymbol{\omega} + \mathbf{L} = \boldsymbol{\Gamma}$$

or

$$\begin{aligned}\lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3 &= \Gamma_1 \\ \lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_3\omega_1 &= \Gamma_2 \\ \lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1\omega_2 &= \Gamma_3.\end{aligned}$$

where all quantities are in the body frame. This is known as **Euler's equation** and it is the equation of motion for the rotation of the body.

10.8 Euler's Equations with Zero Torque

- If a body has three different principle moments of inertia and it is rotating about one of its principle axes, its angular velocity will not change. Conversely, if it isn't rotating about one of its principle axes, its angular velocity will change.

11 Coupled Oscillators and Normal Modes

11.1 Two Masses and Three Springs

- Two carts of masses m_1 and m_2 are connected to walls via springs of spring constant k_1 and k_3, respectively. The carts are joined together by another spring of spring constant k_2. The equations of motion for the carts are

$$\begin{aligned}m_1\ddot{x}_1 &= -(k_1 + k_2)x_1 + k_2x_2 \\ m_2\ddot{x}_2 &= k_2x_1 - (k_2 + k_3)x_2\end{aligned}$$

or

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}.$$

Assuming a solution of the form

$$\begin{aligned}\mathbf{z}(t) &= \begin{pmatrix}\alpha_1 e^{i(\omega t-\delta_1)}\\ \alpha_2 e^{i(\omega t-\delta_2)}\end{pmatrix}\\ &= \begin{pmatrix}\alpha_1 e^{-i\delta_1} e^{i\omega t}\\ \alpha_2 e^{-i\delta_2} e^{i\omega t}\end{pmatrix}\\ &= \begin{pmatrix}\alpha_1 e^{-i\delta_1}\\ \alpha_2 e^{-i\delta_2}\end{pmatrix} e^{i\omega t}\\ &= \mathbf{a}e^{i\omega t}\end{aligned}$$

and substituting it into the equations of motion gives

$$\begin{aligned}-\omega^2\mathbf{M}\mathbf{a}e^{i\omega t} &= -\mathbf{K}\mathbf{a}e^{i\omega t}\\ (\mathbf{K}-\omega^2\mathbf{M})\mathbf{a} &= 0\end{aligned}$$

where the solutions ω give the **normal frequencies** of the system.

11.2 Identical Springs and Equal Masses

- Because the equations of motion are linear, a linear combination of solutions is also a solution. The general solution is thus a linear combination of a fundamental set of solutions.
- **Normal coordinates** are alternate coordinates for the system that result in separate differential equations.

11.6 Three Coupled Pendulums

- Choosing **natural units** is the process of choosing a system of units such that uninteresting parameters, e.g. mass m and length L, are equal to one unit of mass/length/etc. This means they have the value 1 and can be dropped from equations.

12 Nonlinear Mechanics and Chaos

12.1 Linearity and Nonlinearity

- For a system to exhibit chaos its equations of motion must be nonlinar.
- Nonlinearity is essential but not sufficient for chaos, e.g. the equation of motion of a simple pendulum without the small angle approximation.
- Nonlinear equations are complicated, in part, because the superposition principle doesn't apply — you can't just take the linear combination of n independent solutions as the general solution.

12.2 The Driven Damped Pendulum DDP

- The equation of motion for the DDP is

$$\begin{aligned} I\ddot{\phi} &= \Gamma \\ mL^2\ddot{\phi} &= -Lbv - mgL\sin\phi + LF(t) \\ &= -bL^2\dot{\phi} - mgL\sin\phi + LF(t). \end{aligned}$$

- If we assume $F(t) = F_0\cos\omega t$ the above becomes

$$\begin{aligned} \ddot{\phi} + \frac{b}{m}\dot{\phi} + \frac{g}{L}\sin\phi &= \frac{F_0}{mL}\cos\omega t \\ \ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2\sin\phi &= \gamma\omega_0^2\cos\omega t \end{aligned}$$

  where

$$\beta = \frac{b}{2m}$$

  is the **damping constant** of the pendulum,

$$\omega_0 = \sqrt{\frac{g}{L}}$$

  is the **natural frequency** of the pendulum, and

$$\gamma = \frac{F_0}{mL\omega_0^2} = \frac{F_0}{mg}$$

  is the **drive strength** of the driving force.

- The drive strength is the ratio of the amplitude of the driving force to the weight of the pendulum. When $\gamma < 1$ we would expect the driving force to produce small movement and when $\gamma \geq 1$ large movement.

12.3 Some Expected Features of the DDP

- When the initial angle, angular velocity, and driving force are small, the small angle approximation is valid and the system reaches a steady state after the transient response dies out.

- When the drive strength is increased but is still less than 1 the system still exhibits periodic motion, but it is composed of harmonics of the driving force rather than a single cosine.

12.4 The DDP: Approach to Chaos

- As the drive strength is increased over 1 the transient behaviour becomes wild and less predictable.

- Continuing to increase the drive strength results in a steady state period two times the driving period — the system is said to have **period two**. This isn't possible with harmonics alone so **subharmonics** (sinusoidal terms with frequency ω/n rather than $n\omega$) must be present.
- Continuing to increase the drive strength results in a steady state with period three.
- Changing the initial conditions of the DDP and keeping all other values the same can result in different steady state solutions. This differs from the linear case where there is only one possible steady state solution.
- If initial conditions are kept constant and the drive strength is increased we observe a **period-doubling cascade**, i.e. the period continually doubles as the drive strength increases.
- Period-doubling cascades occur across many nonlinear systems and the parameter that determines when the cascades occur (the drive strength above) is called the **control parameter**.
- In many systems, as the control parameter changes there are certain **threshold values** or **bifurcation points** at which the period doubles. As the threshold value continues to change the rate at which the period doubles increases. If γ_n represents bifurcation point n then the intervals between successive bifurcation points are related by

  $$\gamma_{n+1} - \gamma_n \approx \frac{1}{\delta}(\gamma_n - \gamma_{n-1})$$

  where $\delta = 4.669\,201\,6$ is called the **Feigenbaum number**.
- This implies that the intervals between successive bifucation points shrink as the control parameter increases, approaching a finite limit γ_c as $n \to \infty$.
- If $\gamma > \gamma_c$ chaos sets in. This is why the period-doubling cascade is sometimes called a **route to chaos**.

12.5 Chaos and Sensitivity to Initial Conditions

- Linear systems aren't sensitive to initial conditions in that two systems with different initial conditions will exponentially approach the same steady state.
- Two nonlinear systems may exponentially approach the same steady state if the control parameter is in the correct range. If it has exceeded γ_c and is in chaos mode, the difference between the two systems will increase exponentially. Nonlinear systems are extremely sensitive to initial conditions.

- The difference between the two systems can be described by

$$\Delta\phi(t) \sim Ke^{\lambda t}$$

where the $\sim$ symbol indicates that the difference remains within the envelope $Ke^{\lambda t}$, K is a positive constant, and λ is called the **Liapunov exponent**. If the long-term behaviour is stable λ is negative. If it's chaotic λ is positive.

- As we increase γ the system moves through regions of chaos and stability.

12.6 Bifurcation Diagrams

- A bifurcation diagram is a plot of the dependent variable $\phi(t)$ vs. the control parameter γ with the goal of visualising regions of periodicity and chaos as γ changes. It is constructed by following these steps:

  1. Solve the nonlinear system for each value of γ to be shown on the diagram.
  2. Simulate each solution from $t = 0$ to $t = t_{\max}$ where $t_{\max}$ is large enough that all transient behaviour has died out. The simulations should all have the same initial conditions.
  3. Sample each solution a large number of times at one-cycle intervals between $t_{\min}$ and $t_{\max}$ where $t_{\min}$ is large enough that all transient behaviour has died out. If the solution is periodic a small number of values will appear multiple times, if it is chaotic each value will be different.
  4. Plot those values as points on the diagram at the appropriate $(\gamma, \phi(t))$ coordinates.

  If a particular value of γ results in a periodic solution there will be a small number of points on that vertical line. If it results in a chaotic solution there will be many. This results in a diagram that shows how the periodicity of the solutions change with γ.

- Sometimes solutions can exhibit "rolling" behaviour where $\phi(t)$ increases by a multiple of 2π each period. The behaviour may still be periodic but the differing values of $\phi(t)$ render the bifurcation diagram useless. This can be avoided by instead plotting γ vs $\dot{\phi}(t)$.

12.7 State-Space Orbits

- A **state-space orbit** is the path traced out on the $\phi(t)$, $\phi'(t)$ plane as t evolves. Sometimes this visualisation makes it easier to understand the behaviour of the system.

- The **configuration space** of an n-dimensional mechanical system is the n-dimensional space of its n position coordinates $q_1, q_2, \ldots, q_n$.

- The **state space** of an n-dimensional mechanical system is the $2n$-dimensional space of the coordinates $q_1, q_2, \ldots, q_n$ and the velocities $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$.
- The **state** of a mechanical system is a specification of motion that is complete enough to uniquely determine the motion at all later times. In the case of the DDP this is ϕ and $\dot{\phi}$.

13 Hamiltonian Mechanics

13.1 The Basic Variables

- In the Lagrangian, the generalized momentum of coordinate q_i is

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}.$$

- The **Hamiltonian function** is defined as

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}.$$

- If the generalized coordinates are natural, i.e. the relationship between the generalized coordinates and cartesian coordinates doesn't depend on time, the Hamiltonian is equal to the total energy of the system and can also be calculated as

$$\mathcal{H} = T + U.$$

- Under the Lagrangian formalism the state of the system is described by the variables

$$(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n).$$

  Under the Hamiltonian formalism it is described by the variables

$$(q_1, \ldots, q_n, p_1, \ldots, p_n).$$

13.2 Hamilton's Equations for One-Dimensional Systems

- Hamilton's equations for a one-dimensional system are

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} \text{ and } \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}.$$

- For a one-dimensional system, the Lagrangian formalism gives a single second-order differential equation while the Hamiltonian formalism gives two first-order differential equations.

13.3 Hamilton's Equations in Several Dimensions

- Hamilton's equations for a multi-dimensional system are

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \text{ and } \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \text{ for } i = 1, \ldots, n.$$

- The process of applying the Hamiltonian formalism to a multi-dimensional system is similar to that of a one-dimensional system:

  1. Determine the Lagrangian $\mathcal{L} = T - U$.
  2. Determine the generalized momenta

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \text{ for } i = 1, \ldots, n.$$

  3. Treat the generalized momenta equations as a system of n equations for the n generalized velocities $\dot{q}_i$ and solve.
  4. Define the Hamiltonian as

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}.$$

  5. Apply Hamilton's equations as above.

14 Collision Theory

14.1 The Scattering Angle and Impact Parameter

- In a collision experiment, a projectile approaches a target from so far away that its energy is entirely kinetic. As the projectile approaches the target a force (e.g. the Coulomb force) may be exerted, deflecting the projectile's path, or the projectile may hit the target and be deflected that way.

- The **scattering angle** θ is the angle between the incoming and outgoing velocities of the projectile. $\theta = 0$ corresponds to no deflection and $\theta = \pi$ corresponds to a head-on collision.

- The **impact parameter** b is the perpendicular distance between the projectile's incoming velocity and a parallel line through the centre of the target. In other words, it's the closest the projectile would come to the target if it weren't deflected.

- For contact-based collisions, the impact parameter b must be less than the radius of the target in order for any deflection to occur (otherwise there will be no collision).

- For a given value of b there will be a unique corresponding value of θ. The main theoretical task of collision theory is to find the relation $\theta = \theta(b)$ between these two variables.
- It's possible to experimentally measure θ, e.g. with a cloud chamber, but not b.

14.2 The Collision Cross Section

- When a projectile is approaching a collection of targets it "sees" the cross-sectional area of those targets. If the target density per unit area is n_{tar} then in an area A there will be $n_{\text{tar}}A$ targets. If the cross sectional area of a target is then $\sigma = \pi R^2$ the total area of those targets will be $n_{\text{tar}}A\sigma$ and therefore the probability that the projectile will hit a target is given by the percentage of the area taken up by the targets which is

$$\frac{n_{\text{tar}}A\sigma}{A} = n_{\text{tar}}\sigma.$$

- If we fire a large number N_{inc} of projectiles and measure how many are scattered N_{sc} we can use the equation

$$N_{\text{sc}} = N_{\text{inc}}n_{\text{tar}}\sigma$$

to determine the cross sectional area of the targets σ.

- If you divide both sides of the equation by Δt you get

$$R_{\text{sc}} = R_{\text{inc}}n_{\text{tar}}\sigma$$

which relates the rate of scattered projectiles to the rate of incoming projectiles rather than absolute numbers.

- Nuclear cross sections are often measured in units of **barns** where

$$1\,\text{barn} = 10 \times 10^{-28}\,\text{m}^2.$$

14.3 Generalizations of the Cross Section

- If you consider a projectile of radius R_1 and a target of radius R_2, scattering will only occur if $b \leq R_1 + R_2$. It is equivalent to say that the center of the projectile must lie inside a circle centred on the target with radius $R_1 + R_2$ and area $\sigma = \pi(R_1 + R_2)^2$. Thus the relation above can still be used, but σ is now a property of both the target and the projectile and can be thought of as the effective area of the target for scattering the projectile. This value may change for different projectiles.
- Several things may happen other than scattering when a projectile is fired at a target, each of which has its own associated cross section that measures the probability of that thing occurring:

  - the projectile may be absorbed e.g. a target atom may gain an electron (capture cross section),
  - the projectile may ionise the target atom (ionisation cross section),
  - the projectile may cause the target nucleus to fission (fission cross section),
  - etc.

- If the projectile doesn't affect the target's internal motions the collision is said to be **elastic**. If it does the collision is said to be **inelastic**, e.g. if a target atom absorbs some of a projectile electron's kinetic energy to raise one of its own electrons to a higher energy level.

- Some cross sections vary with projectile energy, e.g. if the projectile doesn't have some minimum amount if kinetic energy it won't be able to ionise the target atom and the ionisation cross section is thus 0.