

# Introduction to Quantum Mechanics by David J. Griffiths Problems

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## Part I

# Theory

## 1 The Wave Function

### 1.1

(a)

$$\begin{aligned}\langle j^2 \rangle &= \sum j^2 P(j) \\ &= 14^2 \frac{1}{14} + 15^2 \frac{1}{14} + 16^2 \frac{3}{14} + 22^2 \frac{2}{14} + 24^2 \frac{2}{14} + 25^2 \frac{5}{14} \\ &= \frac{3217}{7} \\ &\approx 459.571 \\ \langle j \rangle^2 &= \left( \sum j P(j) \right)^2 \\ &= 441\end{aligned}$$

(b)

$$\begin{aligned}\Delta j_{14} &= -7 \\ \Delta j_{15} &= -6 \\ \Delta j_{16} &= -5 \\ \Delta j_{22} &= 1 \\ \Delta j_{24} &= 3 \\ \Delta j_{25} &= 4 \\ \sigma^2 &= \sum (\Delta j)^2 P(j) \\ &= \frac{130}{7} \\ &\approx 18.571\end{aligned}$$

(c)

$$\sigma^2 = \sqrt{\langle j^2 \rangle - \langle j \rangle^2} = 18.571$$

## 1.2

(a)

$$\begin{aligned}
 \langle x^2 \rangle &= \int_0^h x^2 \rho(x) dx \\
 &= \int_0^h \frac{x^{3/2}}{2\sqrt{h}} dx \\
 &= \frac{1}{2\sqrt{h}} \left[ \frac{2}{5} x^{5/2} \right]_0^h \\
 &= \frac{h^2}{5} \\
 \langle x \rangle^2 &= \frac{h^2}{9} \\
 \sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
 &= \sqrt{\frac{h^2}{5} - \frac{h^2}{9}} \\
 &= h \sqrt{\frac{4}{45}} \\
 &= \frac{2}{3\sqrt{5}} h
 \end{aligned}$$

(b)

$$\begin{aligned}
 1 - \int_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} \rho(x) dx &= 1 - \frac{1}{2\sqrt{h}} [2\sqrt{x}]_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} \\
 &= 1 - \frac{1}{\sqrt{h}} \left( \sqrt{\frac{1}{3}h + \frac{2}{3\sqrt{5}}h} - \sqrt{\frac{1}{3}h - \frac{2}{3\sqrt{5}}h} \right) \\
 &= 1 - \left( \sqrt{\frac{1}{3} + \frac{2}{3\sqrt{5}}} - \sqrt{\frac{1}{3} - \frac{2}{3\sqrt{5}}} \right) \\
 &\approx 0.393
 \end{aligned}$$

### 1.3

(a)

$$\begin{aligned}\rho(x) &= A e^{-\lambda(x-a)^2} \\ 1 &= \int_{-\infty}^{\infty} \rho(x) dx \\ &= A \int_{-\infty}^{\infty} e^{-\lambda(x-a)^2} dx \\ &= A \sqrt{\frac{\pi}{\lambda}} \\ A &= \sqrt{\frac{\lambda}{\pi}}\end{aligned}$$

(b)

$$\begin{aligned}\langle x \rangle &= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x e^{-\lambda(x-a)^2} dx \\ &= a \\ \langle x^2 \rangle &= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2} dx \\ &= a^2 + \frac{1}{2\lambda} \\ \sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\ &= \sqrt{a^2 + \frac{1}{2\lambda} - a^2} \\ &= \frac{1}{\sqrt{2\lambda}}\end{aligned}$$

## 1.4

(a)

$$\begin{aligned}
 1 &= \int_{-\infty}^{\infty} |\Psi(x, 0)|^2 dx \\
 &= \left(\frac{A}{a}\right)^2 \int_0^a x^2 dx + \left(\frac{A}{b-a}\right)^2 \int_a^b (b-x)^2 dx \\
 &= \frac{1}{3}A^2a + \left(\frac{A}{b-a}\right)^2 \left[-\frac{1}{3}(b-x)^3\right]_a^b \\
 &= \frac{1}{3}A^2a + \frac{1}{3}A^2(b-a) \\
 &= \frac{1}{3}A^2b \\
 A &= \sqrt{\frac{3}{b}}
 \end{aligned}$$

(c)  $x = a$

(d)

$$\begin{aligned}
 \int_0^a |\Psi(x, 0)|^2 dx &= \frac{3}{a^2b} \left[\frac{1}{3}x^3\right]_0^a \\
 &= \frac{a}{b}
 \end{aligned}$$

(e)

$$\begin{aligned}
 \langle x \rangle &= \int_{-\infty}^{\infty} x |\Psi(x, 0)|^2 dx \\
 &= \frac{3}{a^2b} \left[\frac{1}{4}x^4\right]_0^a + \frac{3}{b(b-a)^2} \int_a^b x(b-x)^2 dx \\
 &= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \int_a^b (b^2x - 2bx^2 + x^3) dx \\
 &= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \left[\frac{1}{2}b^2x^2 - \frac{2}{3}bx^3 + \frac{1}{4}x^4\right]_a^b \\
 &= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \left(\frac{1}{2}b^4 - \frac{2}{3}b^4 + \frac{1}{4}b^4 - \frac{1}{2}a^2b^2 + \frac{2}{3}a^3b - \frac{1}{4}a^4\right) \\
 &= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \frac{1}{12}(b-a)^3(3a+b) \\
 &= \frac{3a^2}{4b} + \frac{1}{4b}(3ab + b^2 - 3a^2 - ab) \\
 &= \frac{1}{2}a + \frac{1}{4}b
 \end{aligned}$$

## 1.5

(a)

$$\Psi(x, t) = Ae^{-\lambda|x|}e^{-i\omega t}$$

$$\Psi(x, 0) = Ae^{-\lambda|x|}$$

$$\begin{aligned} 1 &= A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|} dx \\ &= 2A^2 \int_0^{\infty} e^{-2\lambda x} dx \\ &= 2A^2 \left[ -\frac{1}{2\lambda} e^{-2\lambda x} \right]_0^{\infty} \\ &= \frac{A^2}{\lambda} \\ A &= \sqrt{\lambda} \end{aligned}$$

(b)

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{\infty} x \lambda e^{-2\lambda|x|} dx \\ &= \lambda \int_{-\infty}^{\infty} x e^{-2\lambda|x|} dx \\ &= 0 \\ \langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 \lambda e^{-2\lambda|x|} dx \\ &= 2\lambda \int_0^{\infty} x^2 e^{-2\lambda x} dx \\ &= \frac{1}{2\lambda^2} \end{aligned}$$

(c)

$$\begin{aligned} \sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\ &= \frac{1}{\sqrt{2}\lambda} \\ 1 - \int_{-\sigma}^{\sigma} \lambda e^{-2\lambda|x|} dx &= 1 - 2\lambda \int_0^{\sigma} e^{-2\lambda x} dx \\ &= 1 - 2\lambda \left[ -\frac{1}{2\lambda} e^{-2\lambda x} \right]_0^{\sigma} \\ &= e^{-2\lambda\sigma} \\ &= e^{-\sqrt{2}} \\ &\approx 0.243 \end{aligned}$$

## 1.6

The chain rule requires that you apply it to both  $x$  and  $|\Psi|^2$  which gives the same result

$$\begin{aligned}
 \frac{d\langle x \rangle}{dt} &= \frac{d}{dt} \int x |\Psi|^2 dx \\
 &= \int \frac{d}{dt} (x |\Psi|^2) dx \\
 &= \int \left( 0 \cdot |\Psi|^2 + x \frac{\partial |\Psi|^2}{\partial t} \right) dx \\
 &= \int x \frac{\partial |\Psi|^2}{\partial t} dx
 \end{aligned}$$

## 1.8

$$\begin{aligned}
 i\hbar \frac{\partial}{\partial t} \left( e^{-iV_0 t/\hbar} \Psi \right) &= -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \left( e^{-iV_0 t/\hbar} \Psi \right) + (V + V_0) \left( e^{-iV_0 t/\hbar} \Psi \right) \\
 i\hbar \left( -\frac{iV_0}{\hbar} e^{-iV_0 t/\hbar} \Psi + e^{-iV_0 t/\hbar} \frac{\partial \Psi}{\partial t} \right) &= -\frac{\hbar^2}{2m} e^{-iV_0 t/\hbar} \frac{\partial^2 \Psi}{\partial x^2} + V e^{-iV_0 t/\hbar} \Psi + V_0 e^{-iV_0 t/\hbar} \Psi \\
 V_0 \Psi + i\hbar \frac{\partial \Psi}{\partial t} &= -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V \Psi + V_0 \Psi \\
 i\hbar \frac{\partial \Psi}{\partial t} &= -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V \Psi
 \end{aligned}$$

$$\begin{aligned}
 \langle Q(x, p) \rangle &= \int \left( e^{-iV_0 t/\hbar} \Psi \right)^* [Q(x, -i\hbar \partial/\partial x)] e^{-iV_0 t/\hbar} \Psi dx \\
 &= \int e^{iV_0 t/\hbar} \Psi^* [Q(x, -i\hbar \partial/\partial x)] e^{-iV_0 t/\hbar} \Psi dx \\
 &= \int \Psi^* [Q(x, -i\hbar \partial/\partial x)] \Psi dx
 \end{aligned}$$

No effect on the expectation value.

## 1.9

(a)

$$\begin{aligned}
 \Psi(x, t) &= A e^{-a[(mx^2/\hbar) + it]} \\
 1 &= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)} dx \\
 &= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)} dx \\
 &= A^2 \sqrt{\frac{\pi \hbar}{2am}} \\
 A^2 &= \sqrt{\frac{2am}{\pi \hbar}} \\
 A &= \left( \frac{2am}{\pi \hbar} \right)^{1/4}
 \end{aligned}$$

(b)

$$\begin{aligned}
 \Psi &= A e^{-a[(mx^2/\hbar) + it]} \\
 \frac{\partial \Psi}{\partial t} &= -ia\Psi \\
 \frac{\partial \Psi}{\partial x} &= -\frac{2amx}{\hbar} \Psi \\
 \frac{\partial^2 \Psi}{\partial x^2} &= -\frac{2am}{\hbar} \left( \Psi + x \frac{\partial \Psi}{\partial x} \right) \\
 &= -\frac{2am}{\hbar} \left( 1 - \frac{2amx^2}{\hbar} \right) \Psi \\
 V\Psi &= i\hbar \frac{\partial \Psi}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} \\
 &= a\hbar \Psi - a\hbar \left( 1 - \frac{2amx^2}{\hbar} \right) \Psi \\
 V &= a\hbar - a\hbar + 2a^2mx^2 \\
 &= 2a^2mx^2
 \end{aligned}$$



(c)

$$\begin{aligned}
\langle x \rangle &= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)} x \, dx \\
&= 0 \\
\langle x^2 \rangle &= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)} x^2 \, dx \\
&= 2A^2 \int_0^{\infty} e^{-2a(mx^2/\hbar)} x^2 \, dx \\
&= \frac{\hbar}{4am} \\
\langle p \rangle &= \int_{-\infty}^{\infty} \Psi^* \left[ -i\hbar \frac{\partial}{\partial x} \right] \Psi \, dx \\
&= -i\hbar \int_{-\infty}^{\infty} A e^{-a[(mx^2/\hbar)-it]} \left( -\frac{2amx}{\hbar} A e^{-a[(mx^2/\hbar)+it]} \right) dx \\
&= 2iA^2 am \int_{-\infty}^{\infty} x e^{-2amx^2/\hbar} \, dx \\
&= 0 \\
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \Psi^* \left[ -\hbar^2 \frac{\partial^2}{\partial x^2} \right] \Psi \, dx \\
&= -\hbar^2 \int_{-\infty}^{\infty} A e^{-a[(mx^2/\hbar)-it]} \left[ -\frac{2am}{\hbar} \left( 1 - \frac{2amx^2}{\hbar} \right) A e^{-a[(mx^2/\hbar)+it]} \right] dx \\
&= 2A^2 am\hbar \int_{-\infty}^{\infty} e^{-2amx^2/\hbar} \left( 1 - \frac{2amx^2}{\hbar} \right) dx \\
&= am\hbar
\end{aligned}$$

(d)

$$\begin{aligned}
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{\hbar}{4am}} \\
\sigma_p &= \sqrt{am\hbar} \\
\sigma_x \sigma_p &= \sqrt{\frac{1}{4} \hbar^2} \\
&= \frac{1}{2} \hbar \\
&\geq \frac{1}{2} \hbar
\end{aligned}$$

Yes, this is consistent with Heisenberg's uncertainty principle.

### 1.10

(a)

$$P(0) = 0$$

$$\begin{aligned} P(1) &= \frac{2}{25} \\ &= 0.08 \end{aligned}$$

$$\begin{aligned} P(2) &= \frac{3}{25} \\ &= 0.12 \end{aligned}$$

$$\begin{aligned} P(3) &= \frac{1}{5} \\ &= 0.2 \end{aligned}$$

$$\begin{aligned} P(4) &= \frac{3}{25} \\ &= 0.12 \end{aligned}$$

$$\begin{aligned} P(5) &= \frac{3}{25} \\ &= 0.2 \end{aligned}$$

$$\begin{aligned} P(6) &= \frac{3}{25} \\ &= 0.2 \end{aligned}$$

$$\begin{aligned} P(7) &= \frac{1}{25} \\ &= 0.04 \end{aligned}$$

$$\begin{aligned} P(8) &= \frac{2}{25} \\ &= 0.08 \end{aligned}$$

$$\begin{aligned} P(9) &= \frac{3}{25} \\ &= 0.12 \end{aligned}$$

(b) The most probable digit is 3, the median digit is 4, and the average value is  $\frac{118}{25} = 4.72$ .

(c)  $\sigma = 2.474$

### 1.14

(a)

$$\begin{aligned}
 P_{ab}(t) &= \int_a^b |\Psi(x, t)|^2 dx \\
 \frac{dP_{ab}}{dt} &= \frac{d}{dt} \int_a^b |\Psi(x, t)|^2 dx \\
 &= \int_a^b \frac{d}{dt} (|\Psi(x, t)|^2) dx \\
 &= \int_a^b \frac{\partial}{\partial x} \left[ \frac{i\hbar}{2m} \left( \Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) \right] dx \\
 &= J(a, t) - J(b, t)
 \end{aligned}$$

The units are  $s^{-1}$ .

(b)

$$\begin{aligned}
 \Psi(x, t) &= Ae^{-a[(mx^2/\hbar)+it]} \\
 \frac{\partial \Psi}{\partial x} &= -\frac{2amx}{\hbar} \Psi \\
 \Psi^*(x, t) &= Ae^{-a[(mx^2/\hbar)-it]} \\
 \frac{\partial \Psi^*}{\partial x} &= -\frac{2amx}{\hbar} \Psi^* \\
 J(x, t) &= \frac{i\hbar}{2m} \left( \Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi}{\partial x} \right) \\
 &= \frac{i\hbar}{2m} \left[ \Psi \left( -\frac{2amx}{\hbar} \Psi^* \right) - \Psi^* \left( -\frac{2amx}{\hbar} \Psi \right) \right] \\
 &= 0
 \end{aligned}$$

1.15

$$\begin{aligned}
\frac{d}{dt} \int_{-\infty}^{\infty} \Psi_1^* \Psi_2 dx &= \int_{-\infty}^{\infty} \left( \frac{\partial \Psi_1^*}{\partial t} \Psi_2 + \Psi_1^* \frac{\partial \Psi_2}{\partial t} \right) dx \\
&= \int_{-\infty}^{\infty} \left[ \left( -i \frac{\hbar}{2m} \frac{\partial^2 \Psi_1^*}{\partial x^2} + i \frac{V}{\hbar} \Psi_1^* \right) \Psi_2 \right. \\
&\quad \left. + \Psi_1^* \left( i \frac{\hbar}{2m} \frac{\partial^2 \Psi_2}{\partial x^2} - i \frac{V}{\hbar} \Psi_2 \right) \right] dx \\
&= i \frac{\hbar}{2m} \int_{-\infty}^{\infty} \left( \Psi_1^* \frac{\partial^2 \Psi_2}{\partial x^2} - \frac{\partial^2 \Psi_1^*}{\partial x^2} \Psi_2 \right) dx \\
&= i \frac{\hbar}{2m} \left[ \Psi_1^* \frac{\partial \Psi_2}{\partial x} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial}{\partial x} (\Psi_1^* \Psi_2) dx \right. \\
&\quad \left. - \frac{\partial \Psi_1^*}{\partial x} \Psi_2 \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{\partial}{\partial x} (\Psi_1^* \Psi_2) dx \right] \\
&= 0
\end{aligned}$$

1.16

(a)

$$\begin{aligned}
1 &= \int_{-a}^a A^2 (a^2 - x^2)^2 dx \\
&= A^2 \int_0^a (a^2 - x^2)^2 dx \\
&= \frac{16}{15} A^2 a^5 \\
A &= \sqrt{\frac{15}{16a^5}}
\end{aligned}$$

(b)

$$\begin{aligned}
\langle x \rangle &= \int_{-a}^a x A (a^2 - x^2) dx \\
&= 0
\end{aligned}$$

(c)

$$\begin{aligned}
\langle p \rangle &= \int_{-a}^a \Psi^* \left( -i \hbar \frac{\partial}{\partial x} \right) \Psi dx \\
&= 2i A^2 \hbar \int_{-a}^a x (a^2 - x^2) dx \\
&= 0
\end{aligned}$$

(d)

$$\begin{aligned}\langle x^2 \rangle &= \int_{-a}^a \Psi^* x^2 \Psi \, dx \\ &= A^2 \int_{-a}^a x^2 (a^2 - x^2)^2 \, dx \\ &= A^2 \frac{16}{105} a^7 \\ &= \frac{a^2}{7}\end{aligned}$$

(e)

$$\begin{aligned}\langle p^2 \rangle &= \int_{-a}^a \Psi^* \left( -\hbar^2 \frac{\partial^2}{\partial x^2} \right) \Psi \, dx \\ &= -\hbar^2 \int_{-a}^a A(a^2 - x^2)(-2A) \, dx \\ &= 4A^2 \hbar^2 \int_0^a (a^2 - x^2) \, dx \\ &= 4A^2 \hbar^2 \left[ a^2 x - \frac{1}{3} x^3 \right]_0^a \\ &= 4A^2 \hbar^2 \left( a^3 - \frac{1}{3} a^3 \right) \\ &= \frac{8}{3} A^2 a^3 \hbar^2 \\ &= \frac{8}{3} \frac{15}{16a^5} a^3 \hbar^2 \\ &= \frac{5}{2} \frac{\hbar^2}{a^2}\end{aligned}$$

(f)

$$\begin{aligned}\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\ &= \sqrt{\frac{a^2}{7}} \\ &= \frac{a}{\sqrt{7}}\end{aligned}$$

(g)

$$\begin{aligned}\sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\ &= \sqrt{\frac{5}{2}} \frac{\hbar}{a}\end{aligned}$$

(h)

$$\begin{aligned}\sigma_x \sigma_p &= \sqrt{\frac{5}{14}} \hbar \\ &\geq \frac{1}{2} \hbar\end{aligned}$$

**1.18**

(a)

$$\begin{aligned}\frac{h}{\sqrt{3mk_B T}} &> d \\ \frac{\sqrt{3mk_B T}}{h} &< \frac{1}{d} \\ T_{\text{electron}} &< \frac{h^2}{3d^2 m k_B} \\ &< 1.3 \times 10^5 \text{ K} \\ T_{\text{nuclei}} &< 2.5 \text{ K}\end{aligned}$$

(b)

$$\begin{aligned}PV &= Nk_B T \\ \frac{V}{N} &= \frac{k_B T}{P} \\ d &= \left( \frac{k_B T}{P} \right)^{1/3} \\ \frac{h}{\sqrt{3mk_B T}} &> \left( \frac{k_B T}{P} \right)^{1/3} \\ T &< \frac{1}{k_B} \left( \frac{h^2}{3m} \right)^{3/5} P^{2/5}\end{aligned}$$