Classical Mechanics by John R. Taylor Problems

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August 2023

Contents

1 Newton's Laws of Motion

1.1

$$
\begin{aligned}
\mathbf{b} + \mathbf{c} &= 2\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}} \\
5\mathbf{b} + 2\mathbf{x} &= 7\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 2\hat{\mathbf{z}} \\
\mathbf{b} \cdot \mathbf{c} &= 1 \\
\mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \\
&= \hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}}
\end{aligned}
$$

1.5

$$
\begin{aligned}
\mathbf{v}_{\text{body}} &= \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}} \\
\mathbf{v}_{\text{face}} &= \hat{\mathbf{x}} + \hat{\mathbf{z}} \\
\mathbf{v}_{\text{body}} \cdot \mathbf{v}_{\text{face}} &= v_{\text{body}} v_{\text{face}} \cos\theta \\
2 &= \sqrt{6}\cos\theta \\
\cos\theta &= \frac{2}{\sqrt{6}} \\
\theta &= \arccos\frac{2}{\sqrt{6}} \\
&= 35.26^\circ
\end{aligned}
$$

1.11

The particle moves counterclockwise in an ellipse of width $2b$ and height $2c$. The angular speed is ω.

1.23

$$
\begin{aligned}
\mathbf{v} &= v\cos\theta\frac{\mathbf{b}}{b} - v\sin\theta\frac{\mathbf{b}\times\mathbf{c}}{bc} \\
&= \frac{\lambda}{b}\frac{\mathbf{b}}{b} - \frac{c}{b}\frac{\mathbf{b}\times\mathbf{c}}{bc} \\
&= \frac{\lambda\mathbf{b} - \mathbf{b}\times\mathbf{c}}{b^2}
\end{aligned}
$$

1.25

$$
\begin{aligned}
\frac{df}{dt} &= -3f \\
\frac{1}{f}\frac{df}{dt} &= -3 \\
\ln f &= -3t + c \\
f &= ce^{-3t}
\end{aligned}
$$

One constant.

1.35

$$
\begin{aligned}
F_x &= 0 \\
ma_x &= 0 \\
a_x &= 0 \\
v_x &= c_1 \\
&= v_o \cos\theta \\
r_x &= v_o \cos(\theta)t + c_2 \\
&= v_o \cos(\theta)t
\end{aligned}
$$

$$
\begin{aligned}
F_y &= 0 \\
ma_y &= 0 \\
a_y &= 0 \\
v_y &= c_3 \\
v_y &= 0 \\
r_y &= c_4 \\
r_y &= 0
\end{aligned}
$$

$$
\begin{aligned}
F_z &= -mg \\
ma_z &= -mg \\
a_z &= -g \\
v_z &= -gt + c_5 \\
&= v_o \sin\theta - gt \\
r_z &= v_o \sin(\theta)t - \frac{1}{2}gt^2 + c_6 \\
&= v_o \sin(\theta)t - \frac{1}{2}gt^2
\end{aligned}
$$

$$
\begin{aligned}
0 &= v_o \sin(\theta)t - \frac{1}{2}gt^2 \\
t &= \frac{2\sin(\theta)v_o}{g} \\
r_x &= v_o \cos(\theta)t \\
&= \frac{2\cos(\theta)\sin(\theta)v_o^2}{g} \\
&= \frac{\sin(2\theta)v_o^2}{g}
\end{aligned}
$$

1.37

(a)

$$
\begin{aligned}
F &= -mg\sin\theta \\
ma &= -mg\sin\theta \\
a &= -g\sin\theta \\
v &= c_1 - gt\sin\theta \\
&= v_o - gt\sin\theta \\
x &= v_o t - \frac{1}{2}gt^2\sin\theta
\end{aligned}
$$

(b)

$$t = \frac{2v_o}{g\sin\theta}$$

1.39

$$
\begin{aligned}
F_x &= -mg\sin\phi \\
ma_x &= -mg\sin\phi \\
a_x &= -g\sin\phi \\
v_x &= c_1 - gt\sin\phi \\
&= v_o\cos\theta - gt\sin\phi \\
r_x &= v_o t\cos\theta - \frac{1}{2}gt^2\sin\phi + c_2 \\
&= v_o t\cos\theta - \frac{1}{2}gt^2\sin\phi
\end{aligned}
$$

$$
\begin{aligned}
F_y &= -mg\cos\phi \\
ma_y &= -mg\cos\phi \\
a_y &= -g\cos\phi \\
v_y &= c_3 - gt\cos\phi \\
&= v_o\sin\theta - gt\cos\phi \\
r_y &= v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi + c_4 \\
&= v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi
\end{aligned}
$$

$$
\begin{aligned}
0 &= v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi \\
t &= \frac{2v_o\sin\theta}{g\cos\phi}
\end{aligned}
$$

$$
\begin{aligned}
r_x &= \frac{2v_o^2\cos\theta\sec\phi\sin\theta}{g} - \frac{2v_o^2\sec\phi\sin^2\theta\tan\phi}{g} \\
&= \frac{2v_o^2\sin\theta(\cos\theta\cos\phi - \sin\theta\sin\phi)}{g\cos^2\phi} \\
&= \frac{2v_o^2\sin\theta\cos(\theta+\phi)}{g\cos^2\phi}
\end{aligned}
$$

$$
\begin{aligned}
\frac{dr_x}{d\theta} &= \frac{2v_o^2}{g\cos^2\phi}[\cos\theta\cos(\theta+\phi) - \sin\theta\sin(\theta+\phi)] \\
&= \frac{2v_o^2\cos(2\theta+\phi)}{g\cos^2\phi} \\
0 &= \frac{2v_o^2\cos(2\theta+\phi)}{g\cos^2\phi} \\
&= \cos(2\theta+\phi) \\
2\theta+\phi &= \frac{\pi}{2} \\
\theta &= \frac{\pi}{4} - \frac{\phi}{2} \\
r_{x,\max} &= \frac{2v_o^2\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\phi}{2}+\phi\right)}{g\cos^2\phi} \\
&= \frac{v_o^2(1-\sin\phi)}{g\cos^2\phi} \\
&= \frac{v_o^2}{g(1+\sin\phi)}
\end{aligned}
$$

1.41

$$
\begin{aligned}
F &= ma \\
T &= m\frac{v^2}{R} \\
&= m\frac{(\omega R)^2}{R} \\
&= m\omega^2 R
\end{aligned}
$$

1.47

(a)

$$
\begin{aligned}
\rho &= \sqrt{x^2+y^2} \\
\phi &= \arctan\frac{y}{x} \\
z &= z
\end{aligned}
$$

ρ is the distance of P from the z-axis.

The use of r may be unfortunate because it suggests it's the distance of P from the origin.

(b) $\hat{\boldsymbol{\rho}}$ points away from the z-axis, $\hat{\boldsymbol{\phi}}$ points counter-clockwise around the z-axis, and $\hat{\mathbf{z}}$ points in the positive z direction.

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}} + \sqrt{x^2+y^2}\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$$

(c)

$$\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{r}}{dt} \\
&= \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\frac{d\hat{\boldsymbol{\rho}}}{dt} + \dot{z}\hat{\mathbf{z}} + z\frac{d\hat{\mathbf{z}}}{dt} \\
&= \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}} \\
\mathbf{a} &= \frac{d\mathbf{v}}{dt} \\
&= \ddot{\rho}\hat{\boldsymbol{\rho}} + \dot{\rho}\frac{d\hat{\boldsymbol{\rho}}}{dt} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}} + \rho\ddot{\phi}\hat{\boldsymbol{\phi}} + \rho\dot{\phi}\frac{d\hat{\boldsymbol{\phi}}}{dt} + \ddot{z}\hat{\mathbf{z}} \\
&= \ddot{\rho}\hat{\boldsymbol{\rho}} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}} + \rho\ddot{\phi}\hat{\boldsymbol{\phi}} - \rho\dot{\phi}^2\hat{\boldsymbol{\rho}} + \ddot{z}\hat{\mathbf{z}} \\
&= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\boldsymbol{\rho}} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\mathbf{z}}
\end{aligned}$$

2 Projectiles and Charged Particles

2.1

$$\begin{aligned}
1 &= (1.6\times10^3)Dv \\
v &= \frac{1}{(1.6\times10^3)D} \\
&= 8.9\,\mathrm{mm/s}
\end{aligned}$$

When $v \gg 1\,\mathrm{cm/s}$ the drag force can be treated as purely quadratic. For a beach ball this becomes $v \gg 1\,\mathrm{mm/s}$.

2.3

(a)

$$\begin{aligned}
\frac{f_{\text{quad}}}{f_{\text{lin}}} &= \frac{(1/4)\rho A v^2}{3\pi\eta D v} \\
&= \frac{\rho\pi\left(\frac{D}{2}\right)^2 v}{12\pi\eta D} \\
&= \frac{\rho D v}{48\eta} \\
&= \frac{R}{48}
\end{aligned}$$

(b)

$$R = \frac{Dv\rho}{\eta} \approx 0.01$$

2.5

$$\begin{aligned} v_y(t) &= v_{\text{ter}} + (v_{\text{yo}} - v_{\text{ter}})e^{-t/\tau} \\ &= v_{\text{ter}} + (2v_{\text{ter}} - v_{\text{ter}})e^{-t/\tau} \\ &= v_{\text{ter}}(1 + e^{-t/\tau}) \end{aligned}$$

The velocity starts at $2v_{\text{ter}}$ and asymptotically approaches v_{ter}.

2.7

$$\begin{aligned} F &= F(v) \\ m\dot{v} &= F(v) \\ m\frac{dv}{F(v)} &= dt \\ t &= \int_{v_{\text{o}}}^{v} m\frac{dv'}{F(v')} \end{aligned}$$

$$\begin{aligned} F &= F(v) \\ m\dot{v} &= F_{\text{o}} \\ v &= \frac{F_{\text{o}}}{m}t + c \end{aligned}$$

2.11

(a)

$$\begin{aligned}
m\dot{v} &= -mg - bv \\
\dot{v} &= -g - kv \\
\frac{1}{-g-kv}\dot{v} &= 1 \\
-\frac{1}{k}\ln(-g-kv) &= t + c \\
\ln(-g-kv) &= c - \frac{t}{\tau} \\
-g - kv &= Ae^{-t/\tau} \\
v &= \tau(-g - Ae^{-t/\tau}) \\
&= -v_{\text{ter}} - \tau A e^{-t/\tau}
\end{aligned}$$

$$\begin{aligned}
v_{\text{o}} &= -v_{\text{ter}} - \tau A \\
A &= -k(v_{\text{o}} + v_{\text{ter}})
\end{aligned}$$

$$v = -v_{\text{ter}} + (v_{\text{o}} + v_{\text{ter}})e^{-t/\tau}$$

$$y = -v_{\text{ter}}t - \tau(v_{\text{o}} + v_{\text{ter}})e^{-t/\tau} + c$$

$$\begin{aligned}
0 &= -\tau(v_{\text{o}} + v_{\text{ter}}) + c \\
c &= \tau(v_{\text{o}} + v_{\text{ter}})
\end{aligned}$$

$$y = -v_{\text{ter}}t + \tau(v_{\text{o}} + v_{\text{ter}})(1 - e^{-t/\tau})$$

(b)

$$
\begin{aligned}
0 &= -v_{\text{ter}} + (v_{\text{o}} + v_{\text{ter}})e^{-t/\tau} \\
e^{-t/\tau} &= \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} \\
-\frac{t}{\tau} &= \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} \\
t &= -\tau \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}}
\end{aligned}
$$

$$
\begin{aligned}
y_{\max} &= -v_{\text{ter}} \left(-\tau \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} \right) + \tau(v_{\text{o}} + v_{\text{ter}}) \left(1 - \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} \right) \\
&= \tau v_{\text{ter}} \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} + \tau(v_{\text{o}} + v_{\text{ter}} - v_{\text{ter}}) \\
&= \tau \left(v_{\text{o}} + v_{\text{ter}} \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} \right) \\
&= \tau \left[v_{\text{o}} - v_{\text{ter}} \ln \left(1 + \frac{v_{\text{o}}}{v_{\text{ter}}} \right) \right]
\end{aligned}
$$

(c)

$$
\begin{aligned}
y_{\max} &= \tau \left[v_{\text{o}} - v_{\text{ter}} \ln \left(1 + \frac{v_{\text{o}}}{v_{\text{ter}}} \right) \right] \\
&= \tau \left[v_{\text{o}} - g\tau \ln \left(1 + \frac{v_{\text{o}}}{g\tau} \right) \right] \\
&\approx \tau \left\{ v_{\text{o}} - g\tau \left[\frac{v_{\text{o}}}{g\tau} - \frac{1}{2} \left(\frac{v_{\text{o}}}{g\tau} \right)^2 \right] \right\} \\
&= \tau \left(v_{\text{o}} - v_{\text{o}} + \frac{1}{2} \frac{v_{\text{o}}^2}{g\tau} \right) \\
&= \frac{1}{2} \frac{v_{\text{o}}^2}{g}
\end{aligned}
$$

2.13

$$\begin{aligned}
v^2 &= \frac{2}{m}\int_{x_0}^{x} -kx'\,dx' \\
&= -\frac{2k}{m}\left(\frac{1}{2}x^2 - \frac{1}{2}x_0^2\right) \\
&= -\frac{k}{m}(x^2 - x_0^2) \\
v &= \sqrt{\frac{k}{m}(x_0^2 - x^2)} \\
&= \omega\sqrt{x_0^2 - x^2}
\end{aligned}$$

$$\begin{aligned}
\int_{x_0}^{x} \frac{1}{\sqrt{x_0^2 - x'^2}}\,dx' &= \int_0^t \omega\,dt \\
\arctan\frac{x}{\sqrt{x_0^2 - x^2}} - \arctan\frac{x_0}{\sqrt{x_0^2 - x_0^2}} &= \omega t \\
\arctan\frac{x}{\sqrt{x_0^2 - x^2}} &= \omega t + \frac{\pi}{2} \\
\frac{x}{\sqrt{x_0^2 - x^2}} &= \tan\left(\omega t + \frac{\pi}{2}\right) \\
&= -\cot\omega t \\
\frac{\sqrt{x_0^2 - x^2}}{x} &= -\tan\omega t \\
\sqrt{x_0^2 - x^2} &= -x\tan\omega t \\
x_0^2 - x^2 &= x^2\tan^2\omega t \\
x^2 &= \frac{x_0^2}{1 + \tan^2\omega t} \\
&= \frac{x_0^2\cos^2\omega t}{\cos^2\omega t + \sin^2\omega t} \\
&= x_0^2\cos^2\omega t \\
x &= x_0\cos\omega t
\end{aligned}$$

2.15

$$
\begin{aligned}
a_y &= -g \\
v_y &= v_{y0} - gt \\
y &= v_{y0}t - \frac{1}{2}gt^2 \\
0 &= v_{y0}t - \frac{1}{2}gt^2 \\
t &= \frac{2v_{y0}}{g} \\
x &= v_{x0}t \\
R &= \frac{2v_{x0}v_{y0}}{g}
\end{aligned}
$$

2.19

(a)

$$
\begin{aligned}
x &= v_{x0}t \\
y &= v_{y0}t - \frac{1}{2}gt^2 \\
&= \frac{v_{y0}}{v_{x0}}x - \frac{1}{2}g\left(\frac{x}{v_{x0}}\right)^2
\end{aligned}
$$

(b)

$$
\begin{aligned}
y &= \frac{v_{y0} + v_{\text{ter}}}{v_{x0}}x + v_{\text{ter}}\tau \ln\left(1 - \frac{x}{v_{x0}\tau}\right) \\
&\approx \frac{v_{y0}}{v_{x0}}x + \frac{g\tau}{v_{x0}}x - g\tau^2\left[\frac{x}{v_{x0}\tau} + \frac{1}{2}\left(\frac{x}{v_{x0}\tau}\right)^2\right] \\
&= \frac{v_{y0}}{v_{x0}}x - \frac{1}{2}g\left(\frac{x}{v_{x0}}\right)^2
\end{aligned}
$$

2.23

(a)

$$\begin{aligned}
v_{\text{ter}} &= \sqrt{\frac{mg}{c}} \\
&= \sqrt{\frac{mg}{\gamma D^2}} \\
&= \sqrt{\frac{mg}{0.25 D^2}} \\
&= \sqrt{\frac{\rho \frac{4}{3}\pi \left(\frac{D}{2}\right)^3 g}{0.25 D^2}} \\
&= \sqrt{\frac{4\pi\rho D g}{6}} \\
&= 22\,\text{m/s}
\end{aligned}$$

(b)

$$\begin{aligned}
m &= \rho V \\
&= \rho \frac{4}{3}\pi \left(\frac{D}{2}\right)^3 \\
&= \frac{\pi\rho D^3}{6} \\
D^2 &= \left(\frac{6m}{\pi\rho}\right)^{2/3} \\
v_{\text{ter}} &= \sqrt{\frac{mg}{0.25 D^2}} \\
&= \sqrt{\frac{mg}{0.25(6m/\pi\rho)^{2/3}}} \\
&= 140\,\text{m/s}
\end{aligned}$$

(c)

$$v_{\text{ter}} = 107\,\text{m/s}$$

2.27

$$
\begin{aligned}
m\dot{v} &= -mg\sin\theta - cv^2 \\
-\frac{\sqrt{m}\arctan\frac{\sqrt{c}v}{\sqrt{gm\sin\theta}}}{\sqrt{cg\sin\theta}} &= t + c_1 \\
\arctan\frac{\sqrt{c}v}{\sqrt{gm\sin\theta}} &= \sqrt{\frac{cg\sin\theta}{m}}(c_1 - t) \\
\frac{\sqrt{c}v}{\sqrt{gm\sin\theta}} &= \tan\left[\sqrt{\frac{cg\sin\theta}{m}}(c_1 - t)\right] \\
v &= \sqrt{\frac{gm\sin\theta}{c}}\tan\left[\sqrt{\frac{cg\sin\theta}{m}}(c_1 - t)\right]
\end{aligned}
$$

$$
\begin{aligned}
v_0 &= \sqrt{\frac{gm\sin\theta}{c}}\tan\left(\sqrt{\frac{cg\sin\theta}{m}}c_1\right) \\
c_1 &= \sqrt{\frac{m}{cg\sin\theta}}\arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right)
\end{aligned}
$$

$$
v = \sqrt{\frac{gm\sin\theta}{c}}\tan\left[\arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right) - \sqrt{\frac{cg\sin\theta}{m}}t\right]
$$

$$
\begin{aligned}
0 &= \sqrt{\frac{gm\sin\theta}{c}}\tan\left[\arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right) - \sqrt{\frac{cg\sin\theta}{m}}t\right] \\
\sqrt{\frac{cg\sin\theta}{m}}t &= \arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right) \\
t &= \sqrt{\frac{m}{cg\sin\theta}}\arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right)
\end{aligned}
$$

2.29

$$
\begin{aligned}
v(t) &= v_{\text{ter}}\tanh\frac{gt}{v_{\text{ter}}} \\
v(1) &= 9.6\,\text{m/s} \\
v(5) &= 38\,\text{m/s} \\
v(10) &= 48\,\text{m/s} \\
v(20) &= 50\,\text{m/s} \\
v(30) &= 50\,\text{m/s}
\end{aligned}
$$

2.31

(a)

$$\begin{aligned} v_{\text{ter}} &= \sqrt{\frac{mg}{c}} \\ &= \sqrt{\frac{mg}{0.25D^2}} \\ &= 20.2\,\text{m/s} \end{aligned}$$

(b)

$$\begin{aligned} y &= -30 + \frac{v_{\text{ter}}^2}{g} \ln\left(\cosh\frac{gt}{v_{\text{ter}}}\right) \\ 0 &= -30 + \frac{v_{\text{ter}}^2}{g} \ln\left(\cosh\frac{gt}{v_{\text{ter}}}\right) \\ t &= 2.78\,\text{s} \\ v(2.78) &= 17.6\,\text{m/s} \end{aligned}$$

2.33

(b)

$$\begin{aligned} \cosh z &= \frac{e^z + e^{-z}}{2} \\ &= \frac{1}{2}\left[\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots\right) + \left(1 - z + \frac{z^2}{2} - \frac{z^3}{6} + \cdots\right)\right] \\ &= 1 + \frac{z^2}{2} + \frac{z^4}{24} + \frac{z^6}{720} + \cdots \\ \cos iz &= 1 - \frac{(iz)^2}{2} + \frac{(iz)^4}{24} - \frac{(iz)^6}{720} + \cdots \\ &= 1 + \frac{z^2}{2} + \frac{z^4}{24} + \frac{z^6}{720} + \cdots \\ &= \cosh z \\ \sinh z &= -i \sin iz \end{aligned}$$

(c)

$$\begin{aligned}
\frac{d}{dz}\cosh z &= \frac{d}{dz}\left(\frac{e^z + e^{-z}}{2}\right)\\
&= \frac{e^z - e^{-z}}{2}\\
&= \sinh z\\
\frac{d}{dz}\sinh z &= \frac{d}{dz}\left(\frac{e^z - e^{-z}}{2}\right)\\
&= \frac{e^z + e^{-z}}{2}\\
&= \cosh z
\end{aligned}$$

(d)

$$\begin{aligned}
\cosh^2 z - \sinh^2 z &= \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2\\
&= \frac{1}{4}(e^{2z} + 2 + e^{-2z} - e^{2z} + 2 - e^{-2z})\\
&= 1
\end{aligned}$$

(e)

$$\begin{aligned}
\int \frac{1}{\sqrt{1+x^2}}\,dx &= \int \frac{\cosh z}{\sqrt{1+\sinh^2 z}}\,dz\\
&= \int 1\,dz\\
&= z\\
&= \operatorname{arcsinh} x
\end{aligned}$$

2.35

(a)

$$\begin{aligned}
m\dot{v} &= mg - cv^2 \\
\dot{v} &= g\left(1 - \frac{v^2}{v_{\text{ter}}^2}\right) \\
\int_0^v \frac{1}{1 - v'^2/v_{\text{ter}}^2}\, dv' &= \int_0^t g\, dt \\
v_{\text{ter}} \operatorname{arctanh} \frac{v}{v_{\text{ter}}} &= gt \\
v &= v_{\text{ter}} \tanh \frac{gt}{v_{\text{ter}}}
\end{aligned}$$

$$\begin{aligned}
y &= \int_0^t v_{\text{ter}} \tanh \frac{gt'}{v_{\text{ter}}} dt' \\
&= \frac{v_{\text{ter}}^2}{g} \ln\left[\cosh\left(\frac{gt}{v_{\text{ter}}}\right)\right]
\end{aligned}$$

(b)

$$\begin{aligned}
v &= g\tau \tanh \frac{t}{\tau} \\
y &= g\tau^2 \ln\left[\cosh\left(\frac{t}{\tau}\right)\right] \\
v(\tau) &= g\tau \tanh 1 \\
&= 0.76 v_{\text{ter}} \\
v(2\tau) &= 0.96 v_{\text{ter}} \\
v(3\tau) &= 0.99 v_{\text{ter}}
\end{aligned}$$

(c)

$$\begin{aligned}
y &= g\tau^2 \ln\left[\cosh\left(\frac{t}{\tau}\right)\right] \\
&= g\tau^2 \ln\left(\frac{e^{t/\tau} + e^{-t/\tau}}{2}\right) \\
&= g\tau^2 \ln\left(\frac{e^{t/\tau}}{2}\right) \\
&= g\tau^2 (\ln e^{t/\tau} - \ln 2) \\
&= g\tau t - g\tau^2 \ln 2 \\
&= v_{\text{ter}} t - g\tau^2 \ln 2
\end{aligned}$$

(d)

$$\begin{aligned} y &= \frac{(v_{\text{ter}})^2}{g} \ln\left[\cosh\left(\frac{gt}{v_{\text{ter}}}\right)\right] \\ &\approx \frac{(v_{\text{ter}})^2}{g} \ln\left[1 + \frac{1}{2}\left(\frac{gt}{v_{\text{ter}}}\right)^2\right] \\ &\approx \frac{(v_{\text{ter}})^2}{g} \frac{1}{2}\left(\frac{gt}{v_{\text{ter}}}\right)^2 \\ &= \frac{1}{2} g t^2 \end{aligned}$$

2.39

(a)

$$\begin{aligned} m\dot{v} &= -cv^2 - 3 \\ \int_{v_0}^{v} \frac{m}{-cv'^2 - 3}\, dv' &= \int_0^t dt' \\ \frac{m}{\sqrt{3c}}\left[\arctan\left(\sqrt{\frac{c}{3}} v_0\right) - \arctan\left(\sqrt{\frac{c}{3}} v\right)\right] &= t \end{aligned}$$

(b)

Speed	Time
15 m/s	6.34 s
10 m/s	18.4 s
5 m/s	48.3 s
0 m/s	142 s

2.41

$$
\begin{aligned}
m\dot{v} &= -mg - cv^2 \\
\dot{v} &= -g\left[1 + \left(\frac{v}{v_{\text{ter}}}\right)^2\right] \\
v\frac{dv}{dy} &= -g\left[1 + \left(\frac{v}{v_{\text{ter}}}\right)^2\right] \\
\int_{v_0}^{v} \frac{v'}{1 + (v'/v_{\text{ter}})^2}\,dv' &= \int_0^y -g\,dy' \\
\frac{1}{2}v_{\text{ter}}^2[\ln(v_{\text{ter}}^2 + v^2) - \ln(v_{\text{ter}}^2 + v_0^2)] &= -gy \\
\ln\frac{v_{\text{ter}}^2 + v^2}{v_{\text{ter}}^2 + v_0^2} &= -\frac{2gy}{v_{\text{ter}}^2} \\
\frac{v_{\text{ter}}^2 + v^2}{v_{\text{ter}}^2 + v_0^2} &= e^{-2gy/v_{\text{ter}}^2} \\
v &= \sqrt{(v_{\text{ter}}^2 + v_0^2)e^{-2gy/v_{\text{ter}}^2} - v_{\text{ter}}^2}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \sqrt{(v_{\text{ter}}^2 + v_0^2)e^{-2gy/v_{\text{ter}}^2} - v_{\text{ter}}^2} \\
v_{\text{ter}}^2 &= (v_{\text{ter}}^2 + v_0^2)e^{-2gy/v_{\text{ter}}^2} \\
\frac{v_{\text{ter}}^2}{v_{\text{ter}}^2 + v_0^2} &= e^{-2gy/v_{\text{ter}}^2} \\
-\frac{2gy}{v_{\text{ter}}^2} &= \ln\frac{v_{\text{ter}}^2}{v_{\text{ter}}^2 + v_0^2} \\
y &= -\frac{v_{\text{ter}}^2}{2g}\ln\frac{v_{\text{ter}}^2}{v_{\text{ter}}^2 + v_0^2} \\
&= \frac{v_{\text{ter}}^2}{2g}\ln\frac{v_{\text{ter}}^2 + v_0^2}{v_{\text{ter}}^2}
\end{aligned}
$$

$$
y_{\max} = 17.6\,\text{m}
$$

2.45

(a)

$$\begin{aligned}
z &= re^{i\theta} \\
&= r(\cos\theta + i\sin\theta) \\
&= \sqrt{x^2+y^2}\left[\cos\left(\arctan\frac{y}{x}\right) + i\sin\left(\arctan\frac{y}{x}\right)\right] \\
&= \sqrt{x^2+y^2}\left(\frac{1}{\sqrt{1+\frac{y^2}{x^2}}} + i\frac{y}{x\sqrt{1+\frac{y^2}{x^2}}}\right) \\
&= \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} + i\frac{y}{\sqrt{x^2+y^2}}\right) \\
&= x + iy
\end{aligned}$$

r is the distance between z and the origin, θ is the angle between the positive real axis and z.

(b)

$$z = \sqrt{3^2+4^2}e^{i\arctan 4/3} = 5e^{0.927i}$$

(c)

$$z = 2\cos-\frac{\pi}{3} + i2\sin-\frac{\pi}{3} = 1-\sqrt{3}i$$

2.47

(a)

$$\begin{aligned}
z+w &= 9+4i \\
z-w &= 3+12i \\
zw &= (6+8i)(3-4i) \\
&= 18-24i+24i+32 \\
&= 50 \\
\frac{z}{w} &= \frac{zw*}{ww*} \\
&= \frac{(6+8i)(3+4i)}{(3-4i)(3+4i)} \\
&= \frac{18+24i+24i-32}{9+12i-12i+16} \\
&= \frac{-14+48i}{25} \\
&= -\frac{14}{25}+\frac{48}{25}i
\end{aligned}$$

(b)

$$
\begin{aligned}
z + w &= \left(8\cos\frac{\pi}{3} + i8\sin\frac{\pi}{3}\right) + \left(4\cos\frac{\pi}{6} + i4\sin\frac{\pi}{6}\right) \\
&= (4 + 2\sqrt{3}) + i(4\sqrt{3} + 2) \\
z - w &= (4 - 2\sqrt{3}) + i(4\sqrt{3} - 2) \\
zw &= 32e^{i\pi/2} \\
&= 32i \\
\frac{z}{w} &= 2e^{i\pi/6} \\
&= \sqrt{3} + i
\end{aligned}
$$

2.49

(a)

$$
\begin{aligned}
z^2 &= (e^{i\theta})^2 \\
&= e^{i2\theta} \\
&= \cos 2\theta + i\sin 2\theta \\
z^2 &= (\cos\theta + i\sin\theta)^2 \\
&= \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta \\
&= \cos^2\theta + i\sin 2\theta - \sin^2\theta
\end{aligned}
$$

$$
\cos 2\theta = \cos^2\theta - \sin^2\theta
$$

$$
\begin{aligned}
\cos 2\theta + i\sin 2\theta &= \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta \\
i\sin 2\theta &= 2i\sin\theta\cos\theta \\
\sin 2\theta &= 2\sin\theta\cos\theta
\end{aligned}
$$

(b)

$$
\begin{aligned}
z^3 &= (e^{i\theta})^3 \\
&= e^{i3\theta} \\
&= \cos 3\theta + i \sin 3\theta \\
z^3 &= (\cos\theta + i\sin\theta)^3 \\
&= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta
\end{aligned}
$$

$$
\begin{aligned}
\cos 3\theta + i\sin 3\theta &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\
&= \cos\theta(\cos^2\theta - 3\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta) \\
&= \cos\theta(\cos^2\theta - 3\sin^2\theta) + i[3(1-\sin^2\theta)\sin\theta - \sin^3\theta] \\
&= \cos\theta(\cos^2\theta - 3\sin^2\theta) + i(3\sin\theta - 4\sin^3\theta) \\
\cos 3\theta &= \cos\theta(\cos^2\theta - 3\sin^2\theta)
\end{aligned}
$$

$$
\sin 3\theta = 3\sin\theta - 4\sin^3\theta
$$

2.53

$$
\begin{aligned}
\mathbf{B} &= B_z\hat{\mathbf{z}} \\
\mathbf{E} &= E_z\hat{\mathbf{z}} \\
\mathbf{v}\times\mathbf{B} &= B_z v_y\hat{\mathbf{x}} - B_z v_x\hat{\mathbf{y}} \\
\mathbf{F} &= q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \\
&= qB_z v_y\hat{\mathbf{x}} - qB_z v_x\hat{\mathbf{y}} + qE_z\hat{\mathbf{z}}
\end{aligned}
$$

$$
\begin{aligned}
m\dot{v}_x &= qB_z v_y \\
\dot{v}_x &= \omega v_y
\end{aligned}
$$

$$
\begin{aligned}
m\dot{v}_y &= -qB_z v_x \\
\dot{v}_y &= -\omega v_x
\end{aligned}
$$

$$
\begin{aligned}
m\dot{v}_z &= qE_z \\
\dot{v}_z &= \frac{q}{m}E_z
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\begin{pmatrix} v_x \\ v_y \end{pmatrix} &= \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} \\
\begin{pmatrix} v_x \\ v_y \end{pmatrix} &= c_1\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos\omega t - \begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin\omega t\right] + c_2\left[\begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos\omega t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin\omega t\right] \\
&= \begin{pmatrix} c_1\sin\omega t - c_2\cos\omega t \\ c_1\cos\omega t + c_2\sin\omega t \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
x &= -\frac{c_1}{\omega}\cos\omega t - \frac{c_2}{\omega}\sin\omega t + x_0 \\
y &= \frac{c_1}{\omega}\sin\omega t - \frac{c_2}{\omega}\cos\omega t + y_0
\end{aligned}
$$

$$
\begin{aligned}
v_z &= \frac{q}{m}E_z t + v_{z0} \\
z &= \frac{q}{2m}E_z t^2 + v_{z0}t + z_0
\end{aligned}
$$

The particle moves in a helix oriented along the z-axis.

2.55

(a)

$$
\begin{aligned}
\mathbf{B} &= B\hat{\mathbf{z}} \\
\mathbf{E} &= E\hat{\mathbf{y}} \\
\mathbf{F} &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
&= Bqv_y\hat{\mathbf{x}} + q(E - Bv_x)\hat{\mathbf{y}} \\
\dot{v}_x &= \frac{Bq}{m}v_y \\
&= \omega v_y \\
\dot{v}_y &= \frac{Eq}{m} - \frac{Bq}{m}v_x \\
&= \frac{Eq}{m} - \omega v_x \\
\dot{v}_z &= 0
\end{aligned}
$$

The net force has no $\hat{\mathbf{z}}$ component, so the motion stays in the xy-plane.

(b)

$$
\begin{aligned}
0 &= \frac{Eq}{m} - \omega v_x \\
v_x &= \frac{Eq}{\omega m} \\
&= \frac{E}{B}
\end{aligned}
$$

(c)

$$\begin{aligned}
\begin{pmatrix} \dot{v}_x \\ \dot{v}_y \end{pmatrix} &= \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{Eq}{m} \end{pmatrix} \\
\mathbf{V}_c &= \begin{pmatrix} c_1 \sin \omega t - c_2 \cos \omega t \\ c_1 \cos \omega t + c_2 \sin \omega t \end{pmatrix} \\
\mathbf{V}_p &= \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{Eq}{m} \end{pmatrix} \\
&= \begin{pmatrix} c_4 \omega \\ -c_3 \omega + \frac{Eq}{m} \end{pmatrix} \\
c_3 &= \frac{Eq}{m\omega} \\
&= \frac{E}{B} \\
&= v_{\mathrm{dr}} \\
c_4 &= 0 \\
\mathbf{V}_p &= \begin{pmatrix} v_{\mathrm{dr}} \\ 0 \end{pmatrix} \\
\mathbf{V} &= \begin{pmatrix} c_1 \sin \omega t - c_2 \cos \omega t + v_{\mathrm{dr}} \\ c_1 \cos \omega t + c_2 \sin \omega t \end{pmatrix} \\
\begin{pmatrix} v_{x0} \\ 0 \end{pmatrix} &= \begin{pmatrix} -c_2 + v_{\mathrm{dr}} \\ c_1 \end{pmatrix} \\
c_1 &= 0 \\
c_2 &= v_{\mathrm{dr}} - v_{x0} \\
\mathbf{V} &= \begin{pmatrix} v_{\mathrm{dr}} + (v_{x0} - v_{\mathrm{dr}}) \cos \omega t \\ -(v_{x0} - v_{\mathrm{dr}}) \sin \omega t \end{pmatrix}
\end{aligned}$$

(d)

$$\begin{aligned}
x &= v_{\mathrm{dr}} t + \frac{v_{x0} - v_{\mathrm{dr}}}{\omega} \sin \omega t + x_0 \\
y &= \frac{v_{x0} - v_{\mathrm{dr}}}{\omega} \cos \omega t + y_0 \\
z &= z_0
\end{aligned}$$

3 Momentum and Angular Momentum

3.3

$$
\begin{aligned}
mv_0 &= \frac{m}{3}(v_1 + v_2\cos\theta + v_3\cos\theta) \\
3v_0 &= v_0 + \sqrt{2}v_2 \\
2v_0 &= \sqrt{2}v_2 \\
v_2 &= \sqrt{2}v_0 \\
\mathbf{v}_2 &= \sqrt{2}v_0\left(\cos\frac{\pi}{4}\hat{\mathbf{x}} + \sin\frac{\pi}{4}\hat{\mathbf{y}}\right) \\
&= v_0(\hat{\mathbf{x}} + \hat{\mathbf{y}}) \\
\mathbf{v}_3 &= v_0(\hat{\mathbf{x}} - \hat{\mathbf{y}})
\end{aligned}
$$

3.7

$$
\begin{aligned}
v &= v_{\text{ex}}\ln\frac{m_0}{m} \\
&= 2079\,\text{m/s} \\
F &= 25\,\text{MN} \\
W &= 19.6\,\text{MN}
\end{aligned}
$$

The thrust is 1.28 times the weight on Earth.

3.9

$$
\begin{aligned}
-\dot{m}v_{\text{ex}} &= m_0 g \\
v_{\text{ex}} &= -\frac{m_0 g}{\dot{m}} \\
&= 2352\,\text{m/s}
\end{aligned}
$$

3.11

(a)

$$
m\dot{v} = -\dot{m}v_{\text{ex}} + F_{\text{ext}}
$$

(b)

$$
\begin{aligned}
m\dot{v} &= -\dot{m}v_{\mathrm{ex}} - mg \\
\dot{v} &= -\frac{v_{\mathrm{ex}}}{m}\dot{m} - g \\
\int_0^t \dot{v}\,dt &= \int_0^t \left(-\frac{v_{\mathrm{ex}}}{m}\dot{m} - g\right)\,dt \\
\int_0^v dv' &= -v_{\mathrm{ex}}\int_{m_0}^{m} \frac{1}{m'}\,dm' - \int_0^t g\,dt' \\
v &= -v_{\mathrm{ex}}\ln\frac{m}{m_0} - gt \\
&= v_{\mathrm{ex}}\ln\frac{m_0}{m} - gt
\end{aligned}
$$

(c)

$$v = 903\,\mathrm{m/s}$$

It would be $2079\,\mathrm{m/s}$ without gravity (2.3 times larger).

(d) The rocket wouldn't take off until it was light enough (from burning fuel) that its thrust was greater than its weight.

3.13

$$
\begin{aligned}
v &= v_{\mathrm{ex}}\ln\frac{m_0}{m} - gt \\
&= v_{\mathrm{ex}}\ln\frac{m_0}{m_0 - kt} - gt \\
y &= v_{\mathrm{ex}}t - \frac{mv_{\mathrm{ex}}}{k}\ln\frac{m_0}{m} - \frac{1}{2}gt^2 \\
y(2\,\mathrm{min}) &= 40\,\mathrm{km}
\end{aligned}
$$

3.15

$$
\begin{aligned}
M &= m_1 + m_2 + m_3 \\
&= m_1 + m_1 + 10m_1 \\
&= 12m_1 \\
\mathbf{R} &= \frac{1}{12m_1}\left[m_1\begin{pmatrix}1\\1\\0\end{pmatrix} + m_2\begin{pmatrix}1\\-1\\0\end{pmatrix} + m_3\begin{pmatrix}0\\0\\0\end{pmatrix}\right] \\
&= \frac{1}{12m_1}\begin{pmatrix}2m_1\\0\\0\end{pmatrix} \\
&= \begin{pmatrix}\frac{1}{6}\\0\\0\end{pmatrix}
\end{aligned}
$$

3.17

If we let Earth be at the origin, then

$$
\begin{aligned}
R &= \frac{dM_m}{M_e + M_m} \\
&= 4630\,\mathrm{km}
\end{aligned}
$$

The centre of mass is inside Earth.

3.19

(a) No external forces apply during the explosion so the path of the centre of mass would be unchanged.

(b) 100 m before the target.

(c) No.

3.21

$$\begin{aligned}
\mathbf{R} &= \frac{1}{M}\int \mathbf{r}\,dm \\
&= \frac{2}{\sigma\pi R^2}\int \mathbf{r}\sigma\,dA \\
&= \frac{2}{\pi R^2}\int_0^R\int_0^\pi \mathbf{r}r\,d\phi\,dr \\
&= \frac{2}{\pi R^2}\int_0^R\int_0^\pi r(\cos\phi\hat{\mathbf{x}}+\sin\phi\hat{\mathbf{y}})r\,d\phi\,dr \\
&= \frac{2}{\pi R^2}\int_0^R r^2\int_0^\pi (\cos\phi\hat{\mathbf{x}}+\sin\phi\hat{\mathbf{y}})\,d\phi\,dr \\
&= \frac{2}{\pi R^2}\int_0^R r^2[\sin\phi\hat{\mathbf{x}}-\cos\phi\hat{\mathbf{y}}]_0^\pi\,dr \\
&= \frac{4}{\pi R^2}\int_0^R r^2\,dr\,\hat{\mathbf{y}} \\
&= \frac{4R}{3\pi}\hat{\mathbf{y}}
\end{aligned}$$

3.25

$$\begin{aligned}
L &= L_0 \\
I\omega &= I_0\omega_0 \\
mr^2\omega &= mr_0^2\omega_0 \\
\omega &= \left(\frac{r_0}{r}\right)^2\omega_0
\end{aligned}$$

3.29

$$\begin{aligned}
I\omega &= I_0\omega_0 \\
\frac{2}{5}\left(\frac{4}{3}\pi R^3\rho\right)R^2\omega &= \frac{2}{5}\left(\frac{4}{3}\pi R_0^3\rho\right)R_0^2\omega_0 \\
\omega &= \left(\frac{R_0}{R}\right)^5\omega_0
\end{aligned}$$

If the radius doubles the angular velocity is $\omega_0/32$.

3.31

$$\begin{aligned}
I &= \int r^2 \, dm \\
&= \int r^2 \sigma \, dA \\
&= \frac{M}{\pi R^2} \int_0^R \int_0^{2\pi} r^3 \, d\phi \, dr \\
&= \frac{1}{2} M R^2
\end{aligned}$$

3.33

$$\begin{aligned}
I &= \int r^2 \, dm \\
&= \int r^2 \sigma \, dA \\
&= \frac{M}{(2b)^2} \int_{-b}^{b} \int_{-b}^{b} (x^2 + y^2) \, dx \, dy \\
&= \frac{M}{4b^2} \int_{-b}^{b} \left[\frac{1}{3} x^3 + x y^2 \right]_{-b}^{b} \, dy \\
&= \frac{M}{4b} \int_{-b}^{b} \left(\frac{2}{3} b^2 + 2 y^2 \right) \, dy \\
&= \frac{M}{4b} \left[\frac{2}{3} b^2 y + \frac{2}{3} y^3 \right]_{-b}^{b} \\
&= \frac{2}{3} M b^2
\end{aligned}$$

3.35

(b)

$$\Gamma_{\text{ext}} = RMg\sin\gamma$$

$$\begin{aligned}
\dot{L} &= \Gamma_{\text{ext}} \\
I\dot{\omega} &= RMg\sin\gamma \\
\frac{3}{2}MR^2\dot{\omega} &= RMg\sin\gamma \\
\dot{\omega} &= \frac{2g\sin\gamma}{3R}
\end{aligned}$$

$$\begin{aligned}
\dot{v} &= R\dot{\omega} \\
&= \frac{2}{3}g\sin\gamma
\end{aligned}$$

(c)

$$\begin{aligned}
M\dot{v} &= Mg\sin\gamma - f \\
f &= M(g\sin\gamma - \dot{v})
\end{aligned}$$

$$\begin{aligned}
\Gamma_{\text{ext}} &= Rf \\
&= RM(g\sin\gamma - \dot{v})
\end{aligned}$$

$$\begin{aligned}
\dot{L} &= \Gamma_{\text{ext}} \\
I\dot{\omega} &= RM(g\sin\gamma - \dot{v}) \\
\frac{1}{2}MR^2\dot{\omega} &= RM(g\sin\gamma - \dot{v}) \\
\dot{\omega} &= \frac{2(g\sin\gamma - \dot{v})}{R}
\end{aligned}$$

$$\begin{aligned}
\dot{v} &= R\dot{\omega} \\
&= 2(g\sin\gamma - \dot{v}) \\
&= \frac{2}{3}g\sin\gamma
\end{aligned}$$

3.37

(b)

$$\begin{aligned}
\sum m_\alpha r'_\alpha &= \sum m_\alpha (r_\alpha - R) \\
&= \sum m_\alpha r_\alpha - \sum m_\alpha R \\
&= MR - MR \\
&= 0
\end{aligned}$$

4 Energy

4.3

(a)

$$\begin{aligned}
\int_P^Q \mathbf{F} \cdot d\mathbf{r} &= \int_P^O \mathbf{F} \cdot d\mathbf{r} + \int_O^Q \mathbf{F} \cdot d\mathbf{r} \\
&= 0
\end{aligned}$$

(b)

$$\begin{aligned}
x &= 1 - t \\
y &= t \\
\mathbf{r} &= (1-t)\hat{\mathbf{x}} + t\hat{\mathbf{y}} \\
d\mathbf{r} &= (-\hat{\mathbf{x}} + \hat{\mathbf{y}})\, dt \\
\mathbf{F} \cdot d\mathbf{r} &= (x + y)\, dt \\
&= [(1-t) + 1]\, dt \\
&= dt \\
\int_P^Q \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \, dt \\
&= 1
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{r} &= \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}} \\
d\mathbf{r} &= (-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}})\, d\phi \\
\mathbf{F} \cdot d\mathbf{r} &= (\sin^2\phi + \cos^2\phi)\, d\phi \\
&= d\phi \\
\int_P^Q \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} \, d\phi \\
&= \frac{\pi}{2}
\end{aligned}$$

4.7

(a)

$$\begin{aligned}
\mathbf{F} &= -m\gamma y^2 \hat{\mathbf{y}} \\
W &= \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} \\
&= -m\gamma \int_{y_1}^{y_2} y^2 \, dy \\
&= -\frac{1}{3} m\gamma (y_2^3 - y_1^3) \\
U(\mathbf{r}) &= \frac{1}{3} m\gamma y^3
\end{aligned}$$

(b) Assuming no friction

$$\begin{aligned}
\frac{1}{2} m v^2 &= \frac{1}{3} m\gamma h^3 \\
v &= \sqrt{\frac{2}{3}\gamma h^3}
\end{aligned}$$

4.9

(a)

$$\begin{aligned}
U(x) &= -\int_0^x -kx' \, dx' \\
&= \frac{1}{2} k x^2
\end{aligned}$$

4.11

(a)

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 0 \\
\frac{\partial f}{\partial y} &= 2ay + 2bz \\
\frac{\partial f}{\partial z} &= 2by + 2cz
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\partial g}{\partial x} &= -ay^2 z^3 \sin(axy^2 z^3) \\
\frac{\partial g}{\partial y} &= -2axyz^3 \sin(axy^2 z^3) \\
\frac{\partial g}{\partial z} &= -3axy^2 z^2 \sin(axy^2 z^3)
\end{aligned}$$

(c)

$$\frac{\partial h}{\partial x} = \frac{ax}{r}$$
$$\frac{\partial h}{\partial y} = \frac{ay}{r}$$
$$\frac{\partial h}{\partial z} = \frac{az}{r}$$

4.13

(a)

$$\nabla f = \frac{1}{r^2}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})$$

(b)

$$\nabla f = nr^{n-2}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})$$

(c)

$$\nabla f = \frac{g'(r)}{r}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})$$

4.15

$$\begin{aligned} df &= \nabla f \cdot d\mathbf{r} \\ &= (2,4,6) \cdot (0.01, 0.03, 0.05) \\ &= 0.44 \\ f(1.01, 1.03, 1.05) - f(1,1,1) &= 0.4494 \end{aligned}$$

4.19

(a) An ellipse that is two times wider than it is tall.

(b)

$$\begin{aligned} \nabla f &= (2x, 8y, 0) \\ \nabla f|_{(1,1,1)} &= (2, 8, 0) \\ \mathbf{n} &= (1,4,0)/\sqrt{17} \end{aligned}$$

4.21

$$
\begin{aligned}
\mathbf{F} &= -\frac{GMm}{r^2}\hat{\mathbf{r}} \\
\nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{GMm}{r^3}x & -\frac{GMm}{r^3}y & -\frac{GMm}{r^3}z \end{vmatrix} \\
&= \left[\frac{\partial}{\partial y}\left(-\frac{GMm}{r^3}z\right) - \frac{\partial}{\partial z}\left(-\frac{GMm}{r^3}y\right)\right]\hat{\mathbf{x}} \\
&\quad - \left[\frac{\partial}{\partial x}\left(-\frac{GMm}{r^3}z\right) - \frac{\partial}{\partial z}\left(-\frac{GMm}{r^3}x\right)\right]\hat{\mathbf{y}} \\
&\quad + \left[\frac{\partial}{\partial x}\left(-\frac{GMm}{r^3}y\right) - \frac{\partial}{\partial y}\left(-\frac{GMm}{r^3}x\right)\right]\hat{\mathbf{z}} \\
&= -GMm\left[\left(-\frac{3yz}{r^5} + \frac{3yz}{r^5}\right)\hat{\mathbf{x}} + \left(-\frac{3xz}{r^5} + \frac{3xz}{r^5}\right)\hat{\mathbf{y}}\right. \\
&\quad \left.\left(-\frac{3xy}{r^5} + \frac{3xy}{r^4}\right)\hat{\mathbf{z}}\right] \\
&= \mathbf{0} \\
U(\mathbf{r}) &= -\int_{\boldsymbol{\infty}}^{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r}' \\
&= -\int_{\infty}^{r} -\frac{GMm}{r'^2}\,dr' \\
&= GMm\left[-\frac{1}{r'}\right]_{\infty}^{r} \\
&= GMm\left(-\frac{1}{r} + \frac{1}{\infty}\right) \\
&= -\frac{GMm}{r}
\end{aligned}
$$

4.23

(a)

$$
\begin{aligned}
\nabla \times \mathbf{F} &= \mathbf{0} \\
U &= -\int_{\mathbf{0}}^{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r} \\
&= -\left(\int_0^x kx\,dx + \int_0^y 2ky\,dy + \int_0^z 3kz\,dz\right) \\
&= -\frac{1}{2}k(x^2 + 2y^2 + 3z^2)
\end{aligned}
$$

(b)

$$\begin{aligned}
\nabla \times \mathbf{F} &= \mathbf{0} \\
U &= -\int_{\mathbf{0}}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} \\
&= -\left(\int_0^x ky\,dx + \int_0^y kx\,dy\right) \\
&= -kxy
\end{aligned}$$

(c)

$$\nabla \times \mathbf{F} = 2k\hat{\mathbf{z}}$$

Not conservative

4.29

(a) The mass will oscillate around $x = 0$.

(b)

$$\begin{aligned}
t &= \sqrt{\frac{m}{2}} \int_0^A \frac{dx}{\sqrt{kA^4 - kx^4}} \\
&= \sqrt{\frac{m}{2k}} \int_0^A \frac{dx}{\sqrt{A^4 - x^4}} \\
\tau &= 4t
\end{aligned}$$

(d) $\tau \approx 3.71\,\mathrm{s}$

4.31

(a)

$$E = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_2 - m_1)gx$$

(b)

$$(m_1 + m_2)\ddot{x} = (m_1 - m_2)g$$

$$\begin{aligned}
c &= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_2 - m_1)gx \\
0 &= (m_1 + m_2)\dot{x}\ddot{x} + (m_2 - m_1)g\dot{x} \\
(m_1 + m_2)\ddot{x} &= (m_1 - m_2)g
\end{aligned}$$

4.35

(a)

$$E = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}^2 + (m_2 - m_1)gx$$

(b)

$$0 = \left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}\ddot{x} + (m_2 - m_1)g\dot{x}$$

$$\left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} = (m_1 - m_2)g$$

$$\begin{aligned} m_1\ddot{x} &= m_1 g - T_2 \\ T_2 &= m_1 g - m_1\ddot{x} \end{aligned}$$

$$\begin{aligned} m_2\ddot{x} &= T_1 - m_2 g \\ T_1 &= m_2\ddot{x} + m_2 g \end{aligned}$$

$$\begin{aligned} \omega &= -\frac{\dot{x}}{R} \\ \dot{\omega} &= -\frac{\ddot{x}}{R} \end{aligned}$$

$$\begin{aligned} I\dot{\omega} &= (T_1 - T_2)R \\ -I\frac{\ddot{x}}{R} &= \left(m_2\ddot{x} + m_2 g - m_1 g - m_1\ddot{x}\right)R \\ \left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} &= (m_1 - m_2)g \end{aligned}$$

4.37

(a)

$$U(\phi) = MgR(1 - \cos\phi) - mgR\phi$$

(b)

$$\begin{aligned}\frac{dU(\phi)}{d\phi} &= MgR\sin\phi - mgR \\ &= gR(M\sin\phi - m) \\ 0 &= gR(M\sin\phi - m) \\ m &= M\sin\phi\end{aligned}$$

$$\frac{d^2U(\phi)}{d\phi^2} = MgR\cos\phi$$

There is a position of equilibrium at $\phi = \arcsin\frac{m}{M}$. It is stable if $\phi < \frac{\pi}{2}$, i.e. $m < M$.

4.51

$$\begin{aligned}U(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) &= U_{\text{int}} + U_{\text{ext}} \\ &= [U_{12}(\mathbf{r}_1-\mathbf{r}_2) + U_{13}(\mathbf{r}_1-\mathbf{r}_3) + U_{14}(\mathbf{r}_1-\mathbf{r}_4) + U_{23}(\mathbf{r}_2-\mathbf{r}_3) \\ &\quad + U_{24}(\mathbf{r}_2-\mathbf{r}_4) + U_{34}(\mathbf{r}_3-\mathbf{r}_4)] + [U_1(\mathbf{r}_1) + U_2(\mathbf{r}_2) + U_3(\mathbf{r}_3) \\ &\quad + U_4(\mathbf{r}_4)] \\ \mathbf{F}_3 &= \mathbf{F}_{3,\text{int}} + \mathbf{F}_{3,\text{ext}} \\ &= [\mathbf{F}_{13} + \mathbf{F}_{23} + \mathbf{F}_{34}] + \mathbf{F}_{3,\text{ext}} \\ &= -\nabla_3 U_{13} - \nabla_3 U_{23} - \nabla_3 U_{34} - \nabla_3 U_{3,\text{ext}} \\ &= -\nabla_3 U\end{aligned}$$

4.53

(a)

$$\begin{aligned}F &= ma \\ \frac{ke^2}{r^2} &= m\frac{v^2}{r} \\ v^2 &= \frac{ke^2}{mr} \\ K &= \frac{1}{2}mv^2 \\ &= \frac{ke^2}{2r} \\ U &= -\frac{ke^2}{r} \\ K &= -\frac{1}{2}U\end{aligned}$$

(b)

$$
\begin{aligned}
E &= K_1 + K_2 + U_{12} + U_{1p} + U_{2p} \\
&= \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 - ke^2\left(\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_{12}}\right)
\end{aligned}
$$

(c)

$$
\begin{aligned}
E_{\text{before}} &= K_1 + K_2 + U_1 + U_2 + U_{12} \\
&= T_2 + \frac{ke^2}{2r} - \frac{ke^2}{r} \\
&= T_2 - \frac{ke^2}{2r}
\end{aligned}
$$

$$
\begin{aligned}
E_{\text{after}} &= K_1' + K_2' + U_1 + U_2 + U_{12} \\
&= T_1' + \frac{ke^2}{2r'} - \frac{ke^2}{r'} \\
&= T_1' - \frac{ke^2}{2r'} \\
T_2 - \frac{ke^2}{2r} &= T_1' - \frac{ke^2}{2r'} \\
T_1' &= T_2 + \frac{ke^2}{2}\left(\frac{1}{r'} - \frac{1}{r}\right)
\end{aligned}
$$

5 Oscillations

5.3

$$
\begin{aligned}
U(\phi) &= mgl(1 - \cos\phi) \\
&\approx mgl\left(1 - 1 + \frac{1}{2}\phi^2\right) \\
&= \frac{1}{2}mgl\phi^2 \\
k &= mgl
\end{aligned}
$$

5.5

$$
\begin{aligned}
x(t) &= C_1 e^{i\omega t} + C_2 e^{-i\omega t} \\
&= C_1(\cos\omega t + i\sin\omega t) + C_2(\cos\omega t - i\sin\omega t) \\
&= (C_1 + C_2)\cos\omega t + i(C_1 - C_2)\sin\omega t \\
&= B_1\cos\omega t + B_2\sin\omega t \\
B_1 &= C_1 + C_2 \\
B_2 &= i(C_1 - C_2)
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= A\cos(\omega t - \delta) \\
A &= \sqrt{B_1^2 + B_2^2} \\
\delta &= \arctan\frac{B_2}{B_1}
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= C\,\mathrm{Re}\,e^{\omega t} \\
C &= Ae^{-i\delta}
\end{aligned}
$$

5.7

(a) $B_1 = x_0$, $B_2 = \frac{v_0}{\omega}$

(b)

$$
\begin{aligned}
\omega &= \sqrt{\frac{k}{m}} \\
&= 10\,\mathrm{rad/s} \\
B_1 &= 3.0\,\mathrm{m} \\
B_2 &= 5.0\,\mathrm{m}
\end{aligned}
$$

(c) $x = 0\,\mathrm{m}$ at $t = 0.26\,\mathrm{s}$, $\dot{x} = 0\,\mathrm{m/s}$ at $t = 0.10\,\mathrm{s}$.

5.9

$$\begin{aligned}
\frac{1}{2}kA^2 &= \frac{1}{2}mv^2 \\
\frac{k}{m} &= \left(\frac{v}{A}\right)^2 \\
\tau &= \frac{2\pi}{\omega} \\
&= \frac{2\pi}{\sqrt{k/m}} \\
&= \frac{2\pi}{v/A} \\
&= 1.05\,\mathrm{s}
\end{aligned}$$

5.11

$$\begin{aligned}
\frac{1}{2}kx_1^2 + \frac{1}{2}mv_1^2 &= \frac{1}{2}kx_2^2 + \frac{1}{2}mv_2^2 \\
kx_1^2 + mv_1^2 &= kx_2^2 + mv_2^2 \\
\frac{k}{m}x_1^2 + v_1^2 &= \frac{k}{m}x_2^2 + v_2^2 \\
\omega^2(x_1^2 - x_2^2) &= v_2^2 - v_1^2 \\
\omega &= \sqrt{\frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}kA^2 &= \frac{1}{2}kx_1^2 + \frac{1}{2}mv_1^2 \\
A^2 &= x_1^2 + \frac{m}{k}v_1^2 \\
A &= \sqrt{x_1^2 + \frac{v_1^2}{\omega^2}} \\
&= \sqrt{x_1^2 + v_1^2\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}} \\
&= \sqrt{\frac{x_1^2(v_2^2 - v_1^2) + v_1^2(x_1^2 - x_2^2)}{v_2^2 - v_1^2}} \\
&= \sqrt{\frac{x_2^2v_1^2 - x_1^2v_2^2}{v_1^2 - v_2^2}}
\end{aligned}$$

5.13

$$
\begin{aligned}
U(r) &= U_0\left(\frac{r}{R} + \lambda^2\frac{R}{r}\right)\\
\frac{dU(r)}{dr} &= U_0\left(\frac{1}{R} - \lambda^2\frac{R}{r^2}\right)\\
0 &= \frac{dU(r_0)}{dr}\\
&= U_0\left(\frac{1}{R} - \lambda^2\frac{R}{r_0^2}\right)\\
\frac{1}{R} &= \lambda^2\frac{R}{r_0^2}\\
r_0 &= \lambda R
\end{aligned}
$$

$$
\begin{aligned}
U(r_0+x) - U(r_0) &= U_0\left(\frac{r_0+x}{R} + \lambda^2\frac{R}{r_0+x}\right) - U_0\left(\frac{r_0}{R} + \lambda^2\frac{R}{r_0}\right)\\
&= U_0\left[\frac{1}{R}x + \lambda^2 R\left(\frac{1}{r_0+x} - \frac{1}{r_0}\right)\right]\\
&\approx U_0\left[\frac{1}{R}x + \lambda^2 R\left(\frac{1}{r_0} - \frac{x}{r_0^2} + \frac{x^2}{r_0^3} - \frac{1}{r_0}\right)\right]\\
&= U_0\left[\frac{1}{R}x + \lambda^2 R\left(\frac{x^2}{r_0^3} - \frac{x}{r_0^2}\right)\right]\\
&= U_0\left[\frac{1}{R}x + \lambda^2 R\left(\frac{x^2}{(\lambda R)^3} - \frac{x}{(\lambda R)^2}\right)\right]\\
&= \frac{U_0 x^2}{\lambda R^2}\\
&= \frac{1}{2}\left(\frac{2U_0}{\lambda R^2}\right)x^2
\end{aligned}
$$

$$
\begin{aligned}
\omega &= \sqrt{\frac{k}{m}}\\
&= \sqrt{\frac{2U_0}{\lambda m R^2}}
\end{aligned}
$$

5.17

(a)

$$
\begin{aligned}
x(t) &= A_x \cos\omega_x t \\
y(t) &= A_y \cos(\omega_y t - \delta) \\
\frac{\omega_x}{\omega_y} &= \frac{p}{q} \\
\omega_x \tau &= 2\pi p \\
\omega_y \tau &= 2\pi q \\
(\omega_x + \omega_y)\tau &= 2\pi(p+q) \\
\tau &= \frac{2\pi(p+q)}{\omega_x + \omega_y} \\
x(\tau) &= A_x \cos\left(\omega_x \frac{2\pi(p+q)}{\omega_x + \omega_y}\right) \\
&= A_x \cos\left(\frac{2\pi(p+q)}{1 + \omega_y/\omega_x}\right) \\
&= A_x \cos\left(\frac{2\pi(p+q)}{1 + q/p}\right) \\
&= A_x \cos\left(2\pi \frac{p(p+q)}{p+q}\right) \\
y(\tau) &= A_y \cos\left(\omega_y \frac{2\pi(p+q)}{\omega_x + \omega_y} - \delta\right) \\
&= A_y \cos\left(2\pi \frac{p+q}{1 + \omega_x/\omega_y} - \delta\right) \\
&= A_y \cos\left(2\pi \frac{q(p+q)}{p+q} - \delta\right)
\end{aligned}
$$

5.23

$$
\begin{aligned}
\frac{dE}{dt} &= m\dot{x}\ddot{x} + kx\dot{x} \\
&= \dot{x}(m\ddot{x} + kx) \\
&= -b\dot{x}^2
\end{aligned}
$$

5.25

(a)

$$
\tau = \frac{2\pi}{\omega_1}
$$

(b)

$$\begin{aligned} 0 &= Ae^{-\beta t}\cos\omega_1 t \\ &= \cos\omega_1 t \\ t &= \frac{\frac{\pi}{2} + n\pi}{\omega_1}, n \in \mathbb{Z} \end{aligned}$$

The time between successive zeroes is π/ω_1. The period τ is twice this.

(c)

$$\begin{aligned} e^{-\beta\tau} &= e^{-(\omega_0/2)(2\pi/\omega_1)} \\ &= e^{-\pi\omega_0/\omega_1} \\ &= e^{-\pi\omega_0/\sqrt{\omega_0^2 - (\omega_0/2)^2}} \\ &= e^{-\pi\omega_0/\sqrt{3\omega_0^2/4}} \\ &= e^{-\pi\sqrt{4/3}} \\ &\approx 0.027 \end{aligned}$$

5.29

$$
\begin{aligned}
\tau_0 &= 1\,\mathrm{s} \\
\omega_0 &= 2\pi f \\
&= \frac{2\pi}{\tau} \\
&= 2\pi\,\mathrm{rad/s} \\
\frac{1}{2} &= e^{-\beta\tau_1} \\
&= e^{-2\pi\beta/\omega_1} \\
&= e^{-2\pi\beta/\sqrt{\omega_0^2-\beta^2}} \\
\ln\frac{1}{2} &= -\frac{2\pi\beta}{\sqrt{\omega_0^2-\beta^2}} \\
\sqrt{\omega_0^2-\beta^2}\ln\frac{1}{2} &= -2\pi\beta \\
(\omega_0^2-\beta^2)\ln^2\frac{1}{2} &= 4\pi^2\beta^2 \\
\omega_0^2\ln^2\frac{1}{2} &= \left(4\pi^2+\ln^2\frac{1}{2}\right)\beta^2 \\
\beta &= \pm\frac{\ln\frac{1}{2}}{\sqrt{4\pi^2+\ln^2\frac{1}{2}}}\omega_0 \\
&\approx 0.11\omega_0 \\
\tau_1 &= \frac{2\pi}{\omega_1} \\
&= \frac{2\pi}{\sqrt{\omega_0^2-\beta^2}} \\
&= \frac{2\pi}{\sqrt{\omega_0^2-0.0121\omega_0^2}} \\
&\approx 1.006\,\mathrm{s}
\end{aligned}
$$

5.43

(a)

$$
\begin{aligned}
4mg &= 4kx \\
k &= \frac{mg}{x} \\
&\approx 4\times10^4\,\mathrm{N/m}
\end{aligned}
$$

(b)

$$
\omega_0 = \sqrt{\frac{2k}{m}} = 40\,\mathrm{rad/s} \approx 6\,\mathrm{Hz}
$$

(c)

$$5\,\mathrm{m/s} \approx 18\,\mathrm{km/h}$$

6 Calculus of Variations

6.5

$$
\begin{aligned}
|AP| &= \frac{|AB|}{2} \operatorname{crd}\left(\frac{\pi}{2} - \theta\right) \\
&= \frac{|AB|}{2} 2 \sin\left(\frac{\frac{\pi}{2} - \theta}{2}\right) \\
&= |AB| \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
|PB| &= \frac{|AB|}{2} \operatorname{crd}\left(\frac{\pi}{2} + \theta\right) \\
&= \frac{|AB|}{2} 2 \sin\left(\frac{\frac{\pi}{2} + \theta}{2}\right) \\
&= |AB| \sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
S &= |AP| + |PB| \\
&= |AB| \left[\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + \sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{dS}{d\theta} &= |AB| \left[-\frac{1}{2} \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + \frac{1}{2} \cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right] \\
&= \frac{1}{2}|AB| \left[\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right) - \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{dS}{d\theta} &= 0 \\
\frac{\pi}{4} + \frac{\theta}{2} &= \frac{\pi}{4} - \frac{\theta}{2} \\
\theta &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^2S}{d\theta^2} &= -\frac{1}{4}|AB| \left[\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + \sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right] \\
\left.\frac{d^2S}{d\theta^2}\right|_{\theta=0} &= -\frac{\sqrt{2}}{4}|AB|
\end{aligned}
$$

$\frac{dS}{d\theta} = 0$ and $\frac{d^2S}{d\theta^2} < 0$ at $\theta = 0$ so it is a maximum.

6.7

$$
\begin{aligned}
S &= \int_P^Q dS \\
&= \int_P^Q \sqrt{(R\,d\phi)^2 + (dz)^2} \\
&= \int_{z_1}^{z_2} \sqrt{1 + \left(R\frac{d\phi}{dz}\right)^2}\, dz \\
f\left(z, \phi, \frac{d\phi}{dz}\right) &= \sqrt{1 + \left(R\frac{d\phi}{dz}\right)^2} \\
\frac{\partial f}{\partial \phi} &= 0 \\
\frac{\partial f}{\partial \phi'} &= \frac{R^2 \frac{d\phi}{dz}}{\sqrt{1 + \left(R\frac{d\phi}{dz}\right)^2}} \\
\frac{\partial f}{\partial \phi} - \frac{d}{dz}\frac{\partial f}{\partial \phi'} &= 0 \\
\frac{R^2 \frac{d\phi}{dz}}{\sqrt{1 + \left(R\frac{d\phi}{dz}\right)^2}} &= c_1 \\
R^4 \left(\frac{d\phi}{dz}\right)^2 &= c_1 \left[1 + \left(R\frac{d\phi}{dz}\right)^2\right] \\
(R^4 - c_1 R^2)\left(\frac{d\phi}{dz}\right)^2 &= c_1 \\
\frac{d\phi}{dz} &= \sqrt{\frac{c_1}{R^4 - c_1 R^2}} \\
&= c_2 \\
\phi &= c_2 z + c_3
\end{aligned}
$$

6.9

$$
\begin{aligned}
f(x,y,y') &= y'^2 + yy' + y^2 \\
\frac{\partial f}{\partial y} &= y' + 2y \\
\frac{\partial f}{\partial y'} &= 2y' + y \\
\frac{d}{dx}\frac{\partial f}{\partial y'} &= 2y'' + y' \\
\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} &= 0 \\
y' + 2y - 2y'' - y' &= 0 \\
y'' &= y \\
y &= c_1 e^x + c_2 e^{-x} \\
y(0) &= 0 \\
&= c_1 + c_2 \\
y(1) &= 1 \\
&= c_1 e + c_2 e^{-1} \\
1 &= -c_2 e + c_2 e^{-1} \\
&= c_2(e^{-1} - e) \\
c_2 &= \frac{1}{e^{-1} - e} \\
c_1 &= \frac{1}{e - e^{-1}} \\
y &= \frac{e^x}{e - e^{-1}} + \frac{e^{-x}}{e^{-1} - e} \\
&= \frac{e^x - e^{-x}}{e - e^{-1}} \\
&= \frac{\sinh x}{\sinh 1}
\end{aligned}
$$

6.11

$$
\begin{aligned}
f(x, y, y') &= \sqrt{x(1 + y'^2)} \\
\frac{\partial f}{\partial y} &= 0 \\
\frac{\partial f}{\partial y'} &= \frac{\sqrt{x}y'}{\sqrt{1 + y'^2}} \\
\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} &= 0 \\
\frac{\sqrt{x}y'}{\sqrt{1 + y'^2}} &= c_1 \\
xy'^2 &= c_1(1 + y'^2) \\
(x - c_1)y'^2 &= c_1 \\
y' &= \sqrt{\frac{c_1}{x - c_1}} \\
y &= 2c_1\sqrt{\frac{x - c_1}{c_1}} + c_2
\end{aligned}
$$

7 Lagrange's Equations

7.1

$$
\begin{aligned}
K &= \frac{1}{2}m\mathbf{v}^2 \\
&= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\
U &= mgz \\
\mathcal{L} &= K - U \\
&= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \\
F_x &= \frac{\partial \mathcal{L}}{\partial x} \\
&= 0 \\
F_y &= \frac{\partial \mathcal{L}}{\partial y} \\
&= 0 \\
F_z &= \frac{\partial \mathcal{L}}{\partial z} \\
&= -mg
\end{aligned}
$$

7.3

$$
\begin{aligned}
\mathcal{L} &= K - U \\
&= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \\
m\ddot{x} &= -kx
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial y} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \\
m\ddot{y} &= -ky
\end{aligned}
$$

The mass is a harmonic oscillator in each dimension.

7.5

$$
\begin{aligned}
df &= \frac{\partial f}{\partial r}\,dr + \frac{\partial f}{\partial \theta}\,d\theta \\
d\mathbf{r} &= dr + r\,d\theta \\
df &= \nabla f \cdot d\mathbf{r} \\
&= (\nabla f)_r\,dr + (\nabla f)_\theta r\,d\theta \\
(\nabla f)_r &= \frac{\partial f}{\partial r} \\
(\nabla f)_\theta &= \frac{1}{r}\frac{\partial f}{\partial \theta} \\
\nabla f &= \frac{\partial f}{\partial r}\,dr + \frac{1}{r}\frac{\partial f}{\partial \theta}\,d\theta
\end{aligned}
$$

7.7

(a)

$$m_i\ddot{\mathbf{r}}_i = -\nabla U_i$$

(b)

$$\mathcal{L}(\mathbf{r}_1, \ldots, \mathbf{r}_n, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_n) = \frac{1}{2}m_1\dot{r}_1^2 + \ldots + \frac{1}{2}m_n\dot{r}_n^2 - U(\mathbf{r}_1, \ldots, \mathbf{r}_n)$$

7.9

$$
\begin{aligned}
x &= R\cos\phi \\
y &= R\sin\phi \\
\phi &= \arctan\frac{y}{x}
\end{aligned}
$$

7.11

$$\begin{aligned}
x &= A\cos\omega t + l\sin\phi \\
y &= l\cos\phi \\
\phi &= \arctan\frac{x - A\cos\omega t}{y}
\end{aligned}$$

7.15

$$\begin{aligned}
\mathcal{L} &= K - U \\
&= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_2 g x \\
\frac{\partial\mathcal{L}}{\partial x} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} \\
(m_1 + m_2)\ddot{x} &= m_2 g \\
\ddot{x} &= \frac{m_2}{m_1 + m_2} g
\end{aligned}$$

7.17

$$\begin{aligned}
\mathcal{L} &= K - U \\
&= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}m_2\dot{x}^2 - (m_2 g x - m_1 g x) \\
&= \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}^2 + (m_1 - m_2) g x \\
\frac{\partial\mathcal{L}}{\partial x} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} \\
(m_1 - m_2) g &= \left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} \\
\ddot{x} &= \frac{(m_1 - m_2) g}{m_1 + m_2 + \frac{I}{R^2}}
\end{aligned}$$

7.21

$$\begin{aligned}
\mathcal{L} &= K - U \\
&= \frac{1}{2}mv^2 \\
&= \frac{1}{2}m[\dot{r}^2 + (r\dot{\phi})^2] \\
&= \frac{1}{2}m(\dot{r}^2 + r^2\omega^2) \\
\frac{\partial \mathcal{L}}{\partial r} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \\
m\ddot{r} &= mr\omega^2 \\
\ddot{r} &= r\omega^2 \\
r &= c_1 e^{\omega t} + c_2 e^{-\omega t}
\end{aligned}$$

If the bead is initially at rest at the origin

$$\begin{aligned}
0 &= c_1 + c_2 \\
0 &= c_1\omega - c_2\omega \\
&= c_1 - c_2 \\
c_1 &= 0 \\
c_2 &= 0
\end{aligned}$$

thus the bead stays at the origin.

If it is released from a point $r_0 > 0$

$$\begin{aligned}
r_0 &= c_1 + c_2 \\
0 &= c_1 - c_2 \\
c_1 &= c_2 \\
&= \frac{r_0}{2} \\
r &= \frac{r_0}{2}(e^{\omega t} + e^{-\omega t})
\end{aligned}$$

r eventually grows as $\frac{r_0}{2}e^{\omega t}$.

7.23

$$\begin{aligned}
K &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}m(\dot{x}+\dot{X})^2 \\
&= \frac{1}{2}m(\dot{x}-A\omega\sin\omega t)^2 \\
&= \frac{1}{2}m(\dot{x}^2-2\dot{x}A\omega\sin\omega t+A^2\omega^2\sin^2\omega t) \\
U &= \frac{1}{2}kx^2 \\
\mathcal{L} &= K-U \\
&= \frac{1}{2}m(\dot{x}^2-2\dot{x}A\omega\sin\omega t+A^2\omega^2\sin^2\omega t)-\frac{1}{2}kx^2 \\
\frac{\partial\mathcal{L}}{\partial x} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} \\
-kx &= \frac{d}{dt}\left[\frac{1}{2}m\left(2\dot{x}-2A\omega\sin\omega t\right)\right] \\
&= m\ddot{x}-Am\omega^2\cos\omega t \\
\ddot{x}+\frac{k}{m}x &= A\omega^2\cos\omega t \\
\ddot{x}+\omega_0^2x &= B\cos\omega t
\end{aligned}$$

7.27

$$
\begin{aligned}
K &= \frac{1}{2}(4m)\dot{x}_1^2 + \frac{1}{2}(3m)(\dot{x}_1 - \dot{x}_2)^2 + \frac{1}{2}m(\dot{x}_1 + \dot{x}_2)^2 \\
&= 2m\dot{x}_1^2 + \frac{3}{2}m(\dot{x}_1^2 - 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2) + \frac{1}{2}m(\dot{x}_1^2 + 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2) \\
&= \frac{1}{2}m(4\dot{x}_1^2 + 3\dot{x}_1^2 - 6\dot{x}_1\dot{x}_2 + 3\dot{x}_2^2 + \dot{x}_1^2 + 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2) \\
&= 2m(2\dot{x}_1^2 - \dot{x}_1\dot{x}_2 + \dot{x}_2^2) \\
U &= 4mgx_1 + 3mg(x_2 - x_1) - mg(x_1 + x_2) \\
&= mg(4x_1 + 3x_2 - 3x_1 - x_1 - x_2) \\
&= 2mgx_2 \\
\mathcal{L} &= K - U \\
&= 2m(2\dot{x}_1^2 - \dot{x}_1\dot{x}_2 + \dot{x}_2^2) - 2mgx_2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \\
0 &= \frac{d}{dt}(8m\dot{x}_1 - 2m\dot{x}_2) \\
&= 8m\ddot{x}_1 - 2m\ddot{x}_2 \\
\ddot{x}_1 &= \frac{1}{4}\ddot{x}_2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_2} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_2} \\
-2mg &= \frac{d}{dt}(-2m\dot{x}_1 + 4m\dot{x}_2) \\
&= -2m\ddot{x}_1 + 4m\ddot{x}_2 \\
\ddot{x}_2 &= \frac{1}{2}\ddot{x}_1 - \frac{1}{2}g
\end{aligned}
$$

$$
\begin{aligned}
\ddot{x}_1 &= \frac{1}{4}\left(\frac{1}{2}\ddot{x}_1 - \frac{1}{2}g\right) \\
&= \frac{1}{8}\ddot{x}_1 - \frac{1}{8}g \\
\frac{7}{8}\ddot{x}_1 &= -\frac{1}{8}g \\
\ddot{x}_1 &= -\frac{1}{7}g
\end{aligned}
$$

The acceleration of the mass $4m$ is $\frac{1}{7}g$ downwards.

7.29

$$
\begin{aligned}
x &= R\cos\omega t + l\sin\phi \\
\dot{x} &= -R\omega\sin\omega t + l\dot{\phi}\cos\phi \\
y &= R\sin\omega t - l\cos\phi \\
\dot{y} &= R\omega\cos\omega t + l\dot{\phi}\sin\phi \\
K &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\
&= \frac{1}{2}m[R^2\omega^2 + l^2\dot{\phi}^2 + 2R\omega l\dot{\phi}\sin(\phi - \omega t)] \\
U &= mgy \\
&= mg(R\sin\omega t - l\cos\phi) \\
\mathcal{L} &= K - U \\
&= \frac{1}{2}m[R^2\omega^2 + l^2\dot{\phi}^2 + 2R\omega l\dot{\phi}\sin(\phi - \omega t)] - mg(R\sin\omega t - l\cos\phi)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\phi} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} \\
mR\omega l\dot{\phi}\cos(\phi - \omega t) - mgl\sin\phi &= \frac{d}{dt}[ml^2\dot{\phi} + mR\omega l\sin(\phi - \omega t)] \\
&= ml^2\ddot{\phi} + mR\omega l\cos(\phi - \omega t)(\dot{\phi} - \omega) \\
l\ddot{\phi} &= R\omega^2\cos(\phi - \omega t) - g\sin\phi
\end{aligned}
$$

7.31

(a)

$$\begin{aligned}
X &= x + L\sin\phi \\
\dot{X} &= \dot{x} + L\dot{\phi}\cos\phi \\
y &= -L\cos\phi \\
\dot{y} &= L\dot{\phi}\sin\phi
\end{aligned}$$

$$\begin{aligned}
K &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Mv^2 \\
&= \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}LM\dot{\phi}(2\dot{x}\cos\phi + L\dot{\phi})
\end{aligned}$$

$$U = \frac{1}{2}kx^2 - MgL\cos\phi$$

$$\mathcal{L} = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}LM\dot{\phi}(2\dot{x}\cos\phi + L\dot{\phi}) - \frac{1}{2}kx^2 + MgL\cos\phi$$

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial x} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} \\
-kx &= (m+M)\ddot{x} + LM(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\phi} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} \\
M(L\ddot{\phi} + \ddot{x}\cos\phi) &= -Mg\sin\phi
\end{aligned}$$

(b)

$$\begin{aligned}
-kx &= (m+M)\ddot{x} + LM(\ddot{\phi} - \phi\dot{\phi}^2) \\
&\approx (m+M)\ddot{x} + LM\ddot{\phi}
\end{aligned}$$

$$M(L\ddot{\phi} + \ddot{x}) = -Mg\phi$$

7.33

$$
\begin{aligned}
X &= -x\cos\omega t \\
\dot{X} &= -\dot{x}\cos\omega t + \omega x\sin\omega t \\
y &= x\sin\omega t \\
\dot{y} &= \dot{x}\sin\omega t + \omega x\cos\omega t \\
v^2 &= \dot{X}^2 + \dot{y}^2 \\
&= (-\dot{x}\cos\omega t + \omega x\sin\omega t)^2 + (\dot{x}\sin\omega t + \omega x\cos\omega t)^2 \\
&= \dot{x}^2 + \omega^2 x^2 \\
K &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2) \\
U &= mgy \\
&= mgx\sin\omega t \\
\mathcal{L} &= K - U \\
&= \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2) - mgx\sin\omega t \\
\frac{\partial\mathcal{L}}{\partial x} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} \\
\ddot{x} - \omega^2 x &= -g\sin\omega t \\
x &= x_0\cosh\omega t + \frac{g}{2\omega^2}(\sin\omega - \sinh\omega t)
\end{aligned}
$$

7.35

$$
\begin{aligned}
x &= R\cos\omega t + R\cos(\omega t + \phi) \\
\dot{x} &= -R[\omega\sin\omega t + (\omega + \dot{\phi})\sin(\omega t + \phi)] \\
y &= R[\sin\omega t + \sin(\omega t + \phi)] \\
\dot{y} &= R[\omega\cos\omega t + (\omega + \dot{\phi})\cos(\omega t + \phi)] \\
v^2 &= R^2[\omega^2 + (\omega + \dot{\phi})^2 + 2\omega(\omega + \dot{\phi})\cos\phi] \\
\mathcal{L} &= K - U \\
&= \frac{1}{2}mv^2 \\
&= \frac{1}{2}mR^2[\omega^2 + (\omega + \dot{\phi})^2 + 2\omega(\omega + \dot{\phi})\cos\phi]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\
-m\omega(\omega + \dot{\phi})R^2 \sin\phi &= mR^2(\ddot{\phi} - \omega\dot{\phi}\sin\phi) \\
-\omega^2 \sin\phi &= \ddot{\phi}
\end{aligned}
$$

This is the equation of motion for a simple pendulum. For small ϕ the angular frequency is ω.

7.37

(a)

$$
\begin{aligned}
K &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}m\dot{r}^2 \\
&= m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 \\
U &= mgr \\
\mathcal{L} &= K - U \\
&= m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - mgr
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial r} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \\
mr\dot{\phi}^2 - mg &= \frac{d}{dt}(2m\dot{r}) \\
&= 2m\ddot{r}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\
0 &= \frac{d}{dt}(mr^2\dot{\phi}) \\
\ell &= mr^2\dot{\phi}
\end{aligned}
$$

(c)

$$\dot{\phi} = \frac{\ell}{mr^2}$$

$$\begin{aligned}
mr\left(\frac{\ell}{mr^2}\right)^2 - mg &= 2m\ddot{r} \\
\frac{\ell^2}{mr^3} - mg &= 2m\ddot{r} \\
m\ddot{r} &= \frac{\ell^2}{2mr^3} - \frac{mg}{2}
\end{aligned}$$

$$\begin{aligned}
\frac{\ell^2}{2mr_0^3} - \frac{mg}{2} &= 0 \\
\frac{2mr_0^3}{\ell^2} &= \frac{2}{mg} \\
r_0 &= \sqrt[3]{\frac{\ell^2}{m^2 g}}
\end{aligned}$$

7.39

(a)

$$\begin{aligned}
K &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}I_\theta\dot{\theta}^2 + \frac{1}{2}I_\phi\dot{\phi}^2 \\
&= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}m(r\sin\theta)^2\dot{\phi}^2 \\
&= \frac{1}{2}m[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2] \\
\mathcal{L} &= K - U \\
&= \frac{1}{2}m[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2] - U(r)
\end{aligned}$$

(b)

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial r} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \\
mr\dot{\theta}^2 + mr\sin^2\theta\dot{\phi}^2 - \frac{dU(r)}{dr} &= m\ddot{r}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \\
mr^2\sin\theta\cos\theta\dot{\phi}^2 &= \frac{d}{dt}(mr^2\dot{\theta})
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\
0 &= \frac{d}{dt}(mr^2\sin^2\theta\dot{\phi})
\end{aligned}
$$

(c) The motion remains in the equatorial plane.

(d) The motion remains on that line of longitude.

7.41

$$
\begin{aligned}
z &= k\rho^2 \\
\dot{z} &= 2k\rho\dot{\rho} \\
\phi &= \omega t \\
\dot{\phi} &= \omega \\
K &= \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}I_\phi\omega^2 + \frac{1}{2}m\dot{z}^2 \\
&= \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\omega^2 + \frac{1}{2}m(2k\rho\dot{\rho})^2 \\
&= \frac{1}{2}m[\dot{\rho}^2 + (\rho\omega)^2 + (2k\rho\dot{\rho})^2] \\
U &= mgz \\
&= mgk\rho^2 \\
\mathcal{L} &= K - U \\
&= \frac{1}{2}m[\dot{\rho}^2 + (\rho\omega)^2 + (2k\rho\dot{\rho})^2] - mgk\rho^2 \\
\frac{\partial\mathcal{L}}{\partial\rho} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\rho}} \\
m\rho\omega^2 + 4mk^2\rho\dot{\rho}^2 - 2mgk\rho &= \frac{d}{dt}(m\dot{\rho} + 4mk^2\rho^2\dot{\rho}) \\
&= m\ddot{\rho} + 8mk^2\rho\dot{\rho}^2 + 4mk^2\rho^2\ddot{\rho} \\
(1 + 4k^2\rho^2)\ddot{\rho} + 4k^2\rho\dot{\rho}^2 &= (\omega^2 - 2gk)\rho
\end{aligned}
$$

8 Two-Body Central Force Problems

8.1

$$
\begin{aligned}
\mathbf{r}_1 &= \mathbf{R} + \frac{m_2}{M}\mathbf{r} \\
&= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M} + \frac{m_2}{M}(\mathbf{r}_1 - \mathbf{r}_2) \\
&= \frac{(m_1 + m_2)\mathbf{r}_1}{M} \\
&= \mathbf{r}_1 \\
\mathbf{r}_2 &= \mathbf{R} - \frac{m_1}{M}\mathbf{r} \\
&= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M} - \frac{m_1}{M}(\mathbf{r}_1 - \mathbf{r}_2) \\
&= \frac{(m_1 + m_2)\mathbf{r}_2}{M} \\
&= \mathbf{r}_2 \\
K &= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 \\
&= \frac{1}{2}M\left(\frac{m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2}{M}\right)^2 + \frac{1}{2}\frac{m_1m_2}{M}(\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2)^2 \\
&= \frac{1}{2}\frac{m_1^2\dot{\mathbf{r}}_1^2 + m_1m_2\dot{\mathbf{r}}_1\dot{\mathbf{r}}_2 + m_2^2\dot{\mathbf{r}}_2^2}{M} + \frac{1}{2}\frac{m_1m_2}{M}(\dot{\mathbf{r}}_1^2 - 2\dot{\mathbf{r}}_1\dot{\mathbf{r}}_2 + \dot{\mathbf{r}}_2^2) \\
&= \frac{1}{2}\left(\frac{m_1^2 + m_1m_2}{M}\dot{\mathbf{r}}_1^2 + \frac{m_2^2 + m_1m_2}{M}\dot{\mathbf{r}}_2^2\right) \\
&= \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2
\end{aligned}
$$

8.2

(a)

$$
\begin{aligned}
\mathcal{L} &= K - U \\
&= \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - m_1 g z_1 - m_2 g z_2 - U(r) \\
&= \frac{1}{2}m_1\left(\dot{\mathbf{R}} + \frac{m_2}{M}\dot{\mathbf{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\mathbf{R}} - \frac{m_1}{M}\dot{\mathbf{r}}\right)^2 - m_1 g\left(Z + \frac{m_2}{M}z\right) \\
&\quad - m_2 g\left(Z - \frac{m_1}{M}z\right) - U(r) \\
&= \frac{1}{2}m_1\left(\dot{\mathbf{R}}^2 + 2\frac{m_2}{M}\dot{\mathbf{r}}\dot{\mathbf{R}} + \frac{m_2^2}{M^2}\dot{\mathbf{r}}^2\right) + \frac{1}{2}m_2\left(\dot{\mathbf{R}}^2 - 2\frac{m_1}{M}\dot{\mathbf{r}}\dot{\mathbf{R}} + \frac{m_1^2}{M^2}\dot{\mathbf{r}}^2\right) \\
&\quad - m_1 g\left(Z + \frac{m_2}{M}z\right) - m_2 g\left(Z - \frac{m_1}{M}z\right) - U(r) \\
&= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - MgZ - U(r) \\
&= \left(\frac{1}{2}M\dot{\mathbf{R}}^2 - MgZ\right) + \left(\frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)\right) \\
&= \mathcal{L}_{\text{cm}} + \mathcal{L}_{\text{rel}}
\end{aligned}
$$

8.3

$$
\begin{aligned}
\dot{\mathbf{R}} &= \frac{m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2}{M} \\
\mathcal{L}_{\mathrm{cm}} &= \frac{1}{2}M\dot{\mathbf{R}}^2 - MgZ \\
&= \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - MgZ \\
\frac{\partial\mathcal{L}_{\mathrm{cm}}}{\partial Z} &= \frac{d}{dt}\frac{\partial\mathcal{L}_{\mathrm{cm}}}{\partial\dot{Z}} \\
-Mg &= M\ddot{Z} \\
Z &= \dot{R}_0 t - \frac{1}{2}gt^2 \\
&= \frac{m_1}{M}v_0 t - \frac{1}{2}gt^2
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{\mathrm{rel}} &= \frac{1}{2}\mu\dot{\mathbf{r}}^2 - \frac{1}{2}k(r-L)^2 \\
\frac{\partial\mathcal{L}_{\mathrm{rel}}}{\partial r} &= \frac{d}{dt}\frac{\partial\mathcal{L}_{\mathrm{rel}}}{\partial\dot{r}} \\
-k(r-L) &= \mu\ddot{r} \\
\omega &= \sqrt{\frac{k}{\mu}} \\
r &= \frac{v_0}{\omega}\sin\omega t + L
\end{aligned}
$$

$$
\begin{aligned}
z_1 &= Z + \frac{m_2}{M}r \\
&= \frac{m_1}{M}v_0 t - \frac{1}{2}gt^2 + \frac{m_2}{M}\left(\frac{v_0}{\omega}\sin\omega t + L\right)
\end{aligned}
$$

$$
\begin{aligned}
z_2 &= Z - \frac{m_1}{M}r \\
&= \frac{m_1}{M}v_0 t - \frac{1}{2}gt^2 - \frac{m_1}{M}\left(\frac{v_0}{\omega}\sin\omega t + L\right)
\end{aligned}
$$

8.7

(a)

$$\begin{aligned} m_1 \frac{v^2}{r} &= \frac{Gm_1 m_2}{r^2} \\ v &= \sqrt{\frac{Gm_2}{r}} \\ T &= \frac{2\pi r}{v} \\ &= 2\pi r \sqrt{\frac{r}{Gm_2}} \end{aligned}$$

(b)

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{GM\mu}{r} \\ \frac{\partial \mathcal{L}}{\partial r} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \\ \mu r \dot{\phi}^2 - \frac{GM\mu}{r^2} &= \mu \ddot{r} \\ r\left(\frac{v}{r}\right)^2 &= \frac{GM}{r} \\ v &= \sqrt{\frac{GM}{r}} \\ T &= \frac{2\pi r}{v} \\ &= \frac{2\pi r^{3/2}}{\sqrt{GM}} \end{aligned}$$

This is equal to the period found in part (a) when $m_2 \to \infty$.

(c) $T = 0.703\,\text{years}$

8.9

(a)

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - \frac{1}{2}k(|\mathbf{r}_1 - \mathbf{r}_2| - L)^2 \\
&= \frac{1}{2}m_1\left(\dot{\mathbf{R}} + \frac{m_2}{M}\dot{\mathbf{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\mathbf{R}} - \frac{m_1}{M}\dot{\mathbf{r}}\right)^2 - \frac{1}{2}k(r-L)^2 \\
&= \frac{1}{2}m_1\left(\dot{\mathbf{R}}^2 + 2\frac{m_2}{M}\dot{\mathbf{r}}\dot{\mathbf{R}} + \frac{m_2^2}{M^2}\dot{\mathbf{r}}^2\right) + \frac{1}{2}m_2\left(\dot{\mathbf{R}}^2 - 2\frac{m_1}{M}\dot{\mathbf{r}}\dot{\mathbf{R}} + \frac{m_1^2}{M^2}\dot{\mathbf{r}}^2\right) \\
&\quad - \frac{1}{2}k(r-L)^2 \\
&= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - \frac{1}{2}k(r-L)^2 \\
&= \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{2}k(r-L)^2
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial X} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{X}} \\
0 &= M\ddot{X} \\
X &= \dot{X}_0 t + X_0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial Y} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{Y}} \\
0 &= M\ddot{Y} \\
Y &= \dot{Y}_0 t + Y_0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial Z} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{Z}} \\
0 &= M\ddot{Z} \\
Z &= \dot{Z}_0 t + Z_0
\end{aligned}
$$

(c)

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial r} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \\
\mu r\dot{\phi}^2 - k(r-L) &= \mu\ddot{r}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\
0 &= \frac{d}{dt}(\mu r^2\dot{\phi}) \\
\mu r^2\dot{\phi} &= \ell
\end{aligned}
$$

r remains constant:

$$
\begin{aligned}
\mu r\dot{\phi}^2 - k(r-L) &= 0 \\
\dot{\phi} &= \sqrt{\frac{k(r-L)}{\mu r}} \\
r &= \frac{\mu r\dot{\phi}^2}{k} + L
\end{aligned}
$$

ϕ remains constant:

$$
\begin{aligned}
\mu\ddot{r} &= -k(r-L) \\
\ddot{r} &= -\frac{k}{\mu}(r-L) \\
&= -\frac{2k}{m_1}(r-L) \\
&= -\omega^2(r-L) \\
\omega &= \sqrt{\frac{2k}{m_1}} \\
r &= L + A\cos(\omega t - \delta)
\end{aligned}
$$

(b)

$$\begin{aligned}
U_{\text{eff}}(r) &= U(r) + U_{\text{cf}}(r) \\
&= \frac{1}{2}kr^2 + \frac{\ell^2}{2\mu r^2} \\
\frac{dU_{\text{eff}}(r)}{dr} &= kr - \frac{\ell^2}{\mu r^3} \\
0 &= kr - \frac{\ell^2}{\mu r^3} \\
r &= \sqrt[4]{\frac{\ell^2}{\mu k}}
\end{aligned}$$

(c) $\omega = \sqrt{\frac{4k}{\mu}}$

8.15

$$\begin{aligned}
\tau^2 &= \frac{4\pi^2\mu}{\gamma}a^3 \\
&= \frac{4\pi^2\mu}{G\mu M}a^3 \\
&= \frac{4\pi^2}{G(M_s + m)}a^3
\end{aligned}$$

The constant would vary around 0.101%.

8.19

$$\begin{aligned}
\frac{r_{\text{min}}}{r_{\text{max}}} &= \frac{1-\epsilon}{1+\epsilon} \\
0.712 &= \frac{1-\epsilon}{1+\epsilon} \\
0.712(1+\epsilon) &= 1-\epsilon \\
1.712\epsilon &= 0.288 \\
\epsilon &= 0.168
\end{aligned}$$

$$\begin{aligned}
c &= a(1-\epsilon^2) \\
&= \frac{r_{\text{min}} + r_{\text{max}}}{2}(1-\epsilon^2) \\
&= 7795\,\text{km} \\
&= 1424\,\text{km above the Earth's surface}
\end{aligned}$$

9 Mechanics in Noninertial Frames

9.1

$\theta = \arctan \frac{A}{g}$ from vertical.

9.3

(a)

$$\begin{aligned}
d &= d_0 - R_e \\
&= d_0(1 - R_e/d_0) \\
d^{-2} &\approx \frac{1 + 2\frac{R_e}{d_0}}{d_0^2} \\
\mathbf{F}_{\text{tid}} &= -GM_m m\left(\frac{\hat{\mathbf{d}}}{d^2} - \frac{\hat{\mathbf{d}}_0}{d_0^2}\right) \\
&\approx -GM_m m\left(\frac{1 + 2\frac{R_e}{d_0}}{d_0^2} - \frac{1}{d_0^2}\right)\hat{\mathbf{x}} \\
&= -\frac{2GM_m m R_e}{d_0^3}\hat{\mathbf{x}} \\
&\approx -(1.1 \times 10^{-6})m\hat{\mathbf{x}} \\
\frac{F_{\text{tid}}}{mg} &\approx 1.1 \times 10^{-7}
\end{aligned}$$

(b) Same magnitude, opposite direction.

9.9

$$\begin{aligned}
\dot{\mathbf{r}} &= v_0(-\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{z}}) \\
\boldsymbol{\Omega} &= \Omega\hat{\mathbf{z}} \\
\dot{\mathbf{r}} \times \boldsymbol{\Omega} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -v_0\cos\theta & 0 & v_0\sin\theta \\ 0 & 0 & \Omega \end{vmatrix} \\
&= v_0\Omega\cos\theta\hat{\mathbf{y}} \\
\mathbf{F}_{\text{cor}} &= 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega} \\
&= 2mv_0\Omega\cos\theta\hat{\mathbf{y}} \\
\frac{F_{\text{cor}}}{W} &= \frac{2mv_0\Omega\cos\theta}{mg} \\
&\approx 1.1\%
\end{aligned}$$

9.13

$$\begin{aligned}
\tan\alpha &= \frac{g_{\text{tan}}}{g_{\text{rad}}} \\
&= \frac{R_e\Omega^2 \sin\theta\cos\theta}{g} \\
&= \frac{R_e\Omega^2 \sin 2\theta}{2g} \\
\alpha_{\text{max}} &= \frac{R_e\Omega^2}{2g} \\
&= 1.7\times 10^{-3}\,\text{rad} \\
\alpha_{\text{min}} &= 0\,\text{rad}
\end{aligned}$$

9.19

(a) As seen from the ground, the puck moves in a straight line. As seen from the merry-go-round, the puck accelerates to the right and radially outwards — it accelerates radially outwards due to the centrifugal force and to the right due to the coriolis force.

(b) As seen from the ground, the puck is stationary. As seen from the merry-go-round, the puck moves in a clockwise circle.

9.25

$$\begin{aligned}
\alpha &= \arctan\frac{2mv\Omega}{mg} \\
&\approx 2.2\times 10^{-3}\,\text{rad} \\
&\approx 0.13^\circ
\end{aligned}$$

The plumb line is deflected to the left.

10 Rotational Motion of Rigid Bodies

10.3

In the equation for the centre of mass the four points at the base of the pyramid combine to a zero vector. The centre of mass is thus on the z axis $\frac{H}{5}$ units up from the origin.

10.5

By symmetry the centre of mass will be on the z axis. Its z coordinate is

$$
\begin{aligned}
z &= \frac{1}{M} \int z' \, dm \\
&= \frac{1}{M} \int z' M \frac{dV}{V} \\
&= \int_0^R z' \frac{\pi(R^2 - z'^2)}{\frac{2}{3}\pi R^3} \, dz' \\
&= \frac{3}{2R^3} \int_0^R (R^2 z' - z'^3) \, dz' \\
&= \frac{3}{2R^3} \left[\frac{1}{2} R^2 z'^2 - \frac{1}{4} z'^4 \right]_0^R \\
&= \frac{3}{2R^3} \left(\frac{1}{2} R^4 - \frac{1}{4} R^4 \right) \\
&= \frac{3}{8} R
\end{aligned}
$$

10.7

(a)

$$
\begin{aligned}
V &= \int dV \\
&= \int_0^R \int_0^{\theta_0} \int_0^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi \\
&= 2\pi \int_0^R r^2 [-\cos\theta]_0^{\theta_0} \, dr \\
&= 2\pi (1 - \cos\theta_0) \int_0^R r^2 \, dr \\
&= \frac{2}{3} \pi R^3 (1 - \cos\theta_0)
\end{aligned}
$$

(b) By symmetry the centre of mass will lie on the z axis, so all other dimensions will cancel and can be ignored.

$$
\begin{aligned}
z &= \frac{1}{M}\int z'\,dm \\
&= \frac{1}{M}\int r\cos\theta M\frac{dV}{V} \\
&= \frac{1}{V}\int_0^R\int_0^{\theta_0}\int_0^{2\pi} r^3\cos\theta\sin\theta\,dr\,d\theta\,d\phi \\
&= \frac{\pi}{V}\int_0^R r^3\left[-\frac{1}{2}\cos 2\theta\right]_0^{\theta_0}\,dr \\
&= \frac{\pi(1-\cos 2\theta_0)}{2V}\left[\frac{1}{4}r^4\right]_0^R \\
&= \frac{\pi R^4(1-\cos 2\theta_0)}{8V} \\
&= \frac{3R}{16}\cdot\frac{1-\cos 2\theta_0}{1-\cos\theta_0}
\end{aligned}
$$

10.9

$$
\begin{aligned}
I &= \int \rho^2\,dm \\
&= \int \rho^2\varrho\,dV \\
&= \frac{M}{\pi R^2 L}\int_0^R\int_0^{2\pi}\int_0^L \rho^3\,dz\,d\phi\,d\rho \\
&= \frac{2M}{R^2}\left[\frac{1}{4}\rho^4\right]_0^R \\
&= \frac{1}{2}MR^2
\end{aligned}
$$

10.13

(a)

$$
L_z = I\dot{\theta}
$$

$$
\begin{aligned}
\dot{L}_z &= \Gamma_z \\
I\ddot{\theta} &= -mga\sin\theta \\
\ddot{\theta} &\approx -\frac{mga}{I}\theta \\
\omega_0 &= \sqrt{\frac{mga}{I}}
\end{aligned}
$$

(b)

$$l = \frac{I}{ma}$$

10.15

(a)

$$\begin{aligned}
I &= \int \rho^2 \, dm \\
&= \int_0^a \int_0^a \int_0^a (x^2 + y^2) \frac{M}{a^3} \, dx \, dy \, dz \\
&= \frac{M}{a^2} \int_0^a \left[\frac{1}{3} x^3 + x y^2 \right]_0^a \, dy \\
&= \frac{M}{a^2} \left[\frac{1}{3} a^3 y + \frac{1}{3} a y^3 \right]_0^a \\
&= \frac{2}{3} M a^2
\end{aligned}$$

(b)

$$\begin{aligned}
K_i + U_i &= K_f + U_f \\
M g y_i &= \frac{1}{2} I \omega_f^2 + M g y_f \\
\frac{\sqrt{2} M g a}{2} &= \frac{1}{3} M a^2 \omega_f^2 + \frac{1}{2} M g a \\
\omega_f &= \sqrt{\frac{3g(\sqrt{2} - 1)}{2a}}
\end{aligned}$$

10.23

Any product of inertia involving z is immediately 0.

$$\begin{aligned}
I_{zz} &= I_{xx} + I_{yy} \\
\int_V (x^2 + y^2) \, dm &= \int_V (y^2 + z^2) \, dm + \int_V (x^2 + z^2) \, dm \\
&= \int_V (x^2 + y^2) \, dm
\end{aligned}$$

10.25

(a) Because the body is symmetric about the origin on all three principle axes, the products of inertia all cancel to 0. The moments of inertia are

$$\begin{aligned}
I_{xx} &= \int (y^2 + z^2)\, dm \\
&= \frac{M}{8abc} \int_{-a}^{a} \int_{-b}^{b} \int_{-c}^{c} (y^2 + z^2)\, dz\, dy\, dx \\
&= \frac{M}{4bc} \int_{-b}^{b} \left[y^2 z + \frac{1}{3} z^3 \right]_{-c}^{c} dy \\
&= \frac{M}{2b} \int_{-b}^{b} \left(y^2 + \frac{1}{3} c^2 \right) dy \\
&= \frac{M}{2b} \left[\frac{1}{3} y^3 + \frac{1}{3} c^2 y \right]_{-b}^{b} \\
&= \frac{M}{2b} \left(\frac{2}{3} b^3 + \frac{2}{3} c^2 b \right) \\
&= \frac{1}{3} M (b^2 + c^2) \\
I_{yy} &= \frac{1}{3} M (a^2 + c^2) \\
I_{zz} &= \frac{1}{3} M (a^2 + b^2) \\
\mathbf{I} &= \frac{1}{3} M \begin{pmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix}
\end{aligned}$$

(b)

$$\mathbf{I} = \frac{1}{3} M \begin{pmatrix} 4(b^2 + c^2) & -3ab & -3ac \\ -3ab & 4(a^2 + c^2) & -3bc \\ -3ac & -3bc & 4(a^2 + b^2) \end{pmatrix}$$

(c)

$$\begin{aligned}
\mathbf{L} &= \mathbf{I} \boldsymbol{\omega} \\
&= \frac{1}{3} M \begin{pmatrix} 4(b^2 + c^2) & -3ab & -3ac \\ -3ab & 4(a^2 + c^2) & -3bc \\ -3ac & -3bc & 4(a^2 + b^2) \end{pmatrix} \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} \\
&= \frac{1}{3} M \omega \begin{pmatrix} 4(b^2 + c^2) \\ -3ab \\ -3ac \end{pmatrix}
\end{aligned}$$

10.35

(a)

$$\mathbf{I} = a^2 m \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix}$$

(b)

$$\begin{aligned} \det(\mathbf{I} - \lambda \mathbf{1}) &= (5a^2 m - \lambda)(7a^2 m - \lambda)(10a^2 m - \lambda) \\ \lambda_1 &= 5a^2 m \\ \lambda_2 &= 7a^2 m \\ \lambda_3 &= 10a^2 m \\ \mathbf{K}_1 &= (0, -1, 1) \\ \mathbf{K}_2 &= (0, 1, 1) \\ \mathbf{K}_3 &= (1, 0, 0) \end{aligned}$$

10.37

(a)

$$
\begin{aligned}
I_{xx} &= 24\int_0^1 \int_0^{1-x} y^2\,dy\,dx \\
&= 24\int_0^1 \left[\frac{1}{3}y^3\right]_0^{1-x} dx \\
&= 8\int_0^1 (1-x)^3\,dx \\
&= 8\left[-\frac{1}{4}(1-x)^4\right]_0^1 \\
&= 2 \\
I_{xy} &= 24\int_0^1 \int_0^{1-x} xy\,dy\,dx \\
&= 24\int_0^1 \left[\frac{1}{2}xy^2\right]_0^{1-x} dx \\
&= 12\int_0^1 x(1-x)^2\,dx \\
&= 12\int_0^1 x(1-2x+x^2)\,dx \\
&= 12\left[\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4\right]_0^1 \\
&= 1 \\
I_{xz} &= 0 \\
I_{yy} &= 2 \\
I_{yz} &= 0 \\
I_{zz} &= 24\int_0^1 \int_0^{1-x} (x^2+y^2)\,dy\,dx \\
&= 24\int_0^1 \left[x^2 y + \frac{1}{3}y^3\right]_0^{1-x} dx \\
&= 24\int_0^1 \left[x^2(1-x) + \frac{1}{3}(1-x)^3\right] dy \\
&= 24\left[\frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{12}(1-x)^4\right]_0^1 \\
&= 4 \\
\mathbf{I} &= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}
\end{aligned}
$$

(b)

$$\begin{aligned}
\lambda_1 &= 4 \\
\lambda_2 &= 3 \\
\lambda_3 &= 1 \\
\mathbf{K}_1 &= (0,0,1) \\
\mathbf{K}_2 &= (1,1,0) \\
\mathbf{K}_3 &= (-1,1,0)
\end{aligned}$$

10.39

$$\begin{aligned}
\lambda_3 &= \frac{3Mr^2}{10} \\
\Omega &= \frac{MgR}{\lambda_3\omega} \\
&= \frac{10gh}{4r^2\omega} \\
&\approx 21\,\mathrm{rad/s}
\end{aligned}$$

11 Coupled Oscillators and Normal Modes

11.1

(a)

$$\begin{aligned}
k_1(l_1 - L_1) &= k_2(l_2 - L_2) \\
k_2(l_2 - L_2) &= k_3(l_3 - L_3)
\end{aligned}$$

(b)

$$\begin{aligned}
m_1\ddot{x}_1 &= -(k_1+k_2)x_1 + k_2x_2 \\
m_2\ddot{x}_2 &= k_2x_1 - (k_2+k_3)x_2
\end{aligned}$$

$$\begin{aligned}
m_1\ddot{x}_1 &= k_1(l_1 - L_1 - x_1) - k_2(l_2 - L_2 - x_2 + x_1) \\
&= k_1(l_1 - L_1) - k_1x_1 - k_2(l_2 - L_2) + k_2x_2 - k_2x_1 \\
&= -(k_1+k_2)x_1 + k_2x_2 + k_1(l_1 - L_1) - k_2(l_2 - L_2) \\
&= -(k_1+k_2)x_1 + k_2x_2 \\
m_2\ddot{x}_2 &= k_2(l_2 - L_2 - x_2 + x_1) - k_3(l_3 - L_3 + x_2) \\
&= k_2(l_2 - L_2) + k_2x_1 - k_2x_2 - k_3(l_3 - L_3) - k_3x_2 \\
&= k_2x_1 - (k_2+k_3)x_2 + k_2(l_2 - L_2) - k_3(l_3 - L_3) \\
&= k_2x_1 - (k_2+k_3)x_2
\end{aligned}$$

11.3

$$\begin{aligned}
\mathbf{M} &= \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \\
\mathbf{K} &= \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \\
\mathbf{K} - \omega^2 \mathbf{M} &= \begin{bmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 + k_3 - \omega^2 m_2 \end{bmatrix} \\
0 &= \det(\mathbf{K} - \omega^2 \mathbf{M}) \\
&= (k_1 + k_2 - \omega^2 m_1)(k_2 + k_3 - \omega^2 m_2) - k_2^2 \\
\omega^2 &= \frac{1}{2m_1 m_2} \Big(m_1(k_2 + k_3) + m_2(k_1 + k_2) \\
&\qquad \pm \sqrt{m_1^2(k_2+k_3)^2 + m_2^2(k_1+k_2)^2 - 2m_1m_2(k_2k_3 + k_1k_3 + k_1k_2 - k_2^2)} \Big)
\end{aligned}$$

11.5

(a)

$$\begin{aligned}
m\ddot{x}_1 &= -kx_1 + k(x_2 - x_1) \\
&= -2kx_1 + kx_2 \\
m\ddot{x}_2 &= -k(x_2 - x_1) \\
&= kx_1 - kx_2
\end{aligned}$$

$$\begin{aligned}
\mathbf{M}\ddot{\mathbf{x}} &= -\mathbf{K}\mathbf{x} \\
\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} &= - \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
\mathbf{K} - \omega^2 \mathbf{M} &= \begin{bmatrix} 2k - \omega^2 m & -k \\ -k & k - \omega^2 m \end{bmatrix} \\
0 &= \det(\mathbf{K} - \omega^2 \mathbf{M}) \\
&= (2k - \omega^2 m)(k - \omega^2 m) - k^2 \\
\omega &= \sqrt{\frac{(3 \pm \sqrt{5})k}{2m}} \\
&= \sqrt{\frac{3 \pm \sqrt{5}}{2}\omega_0} \\
\omega_1 &= 0.62\omega_0 \\
\omega_2 &= 1.62\omega_0
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{0} &= (\mathbf{K} - \omega_1^2 \mathbf{M})\mathbf{a} \\
&= \begin{bmatrix} 1.62 a_1 k - a_2 k \\ 0.62 a_2 k - a_1 k \end{bmatrix} \\
a_2 &= 1.62 a_1
\end{aligned}$$

In the first mode, the carts oscillate in phase with cart two having 1.62 times the amplitude of cart one.

$$\begin{aligned}
\mathbf{0} &= (\mathbf{K} - \omega_2^2 \mathbf{M})\mathbf{a} \\
&= \begin{bmatrix} -0.62 a_1 k - a_2 k \\ -1.62 a_2 k - a_1 k \end{bmatrix} \\
a_2 &= -0.62 a_1
\end{aligned}$$

In the second mode, the carts oscillate out of phase with cart one having 1.62 times the ampltiude of cart two.

11.7

(a)

$$\begin{aligned}
\mathbf{x}(t) &= A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1) + A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2) \\
&= (B_1 \cos \omega_1 t + C_1 \sin \omega_1 t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (B_2 \cos \omega_2 t + C_2 \sin \omega_2 t) \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}$$

(b)

$$\begin{aligned}
\begin{bmatrix} A \\ A \end{bmatrix} &= \begin{bmatrix} B_1 + B_2 \\ B_1 - B_2 \end{bmatrix} \\
B_1 &= A \\
B_2 &= 0 \\
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} C_1 \omega_1 + C_2 \omega_2 \\ C_1 \omega_1 - C_2 \omega_2 \end{bmatrix} \\
C_1 &= 0 \\
C_2 &= 0
\end{aligned}$$

11.9

(a)

$$
\begin{aligned}
x_1 &= \xi_1 + \xi_2 \\
\ddot{x}_1 &= \ddot{\xi}_1 + \ddot{\xi}_2
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= \xi_1 - \xi_2 \\
\ddot{x}_2 &= \ddot{\xi}_1 - \ddot{\xi}_2
\end{aligned}
$$

$$
\begin{aligned}
m\ddot{x}_1 &= -2kx_1 + kx_2 \\
m(\ddot{\xi}_1 + \ddot{\xi}_2) &= -2k(\xi_1 + \xi_2) + k(\xi_1 - \xi_2) \\
&= -k\xi_1 - 3k\xi_2
\end{aligned}
$$

$$
\begin{aligned}
m\ddot{x}_2 &= kx_1 - 2kx_2 \\
m(\ddot{\xi}_1 - \ddot{\xi}_2) &= k(\xi_1 + \xi_2) - 2k(\xi_1 - \xi_2) \\
&= -k\xi_1 + 3k\xi_2
\end{aligned}
$$

$$
\begin{aligned}
2m\ddot{\xi}_1 &= -2k\xi_1 \\
m\ddot{\xi}_1 &= -k\xi_1
\end{aligned}
$$

$$
\begin{aligned}
2m\ddot{\xi}_2 &= -6k\xi_2 \\
m\ddot{\xi}_2 &= -3k\xi_2
\end{aligned}
$$

$$
\begin{aligned}
\xi_1 &= A_1 \cos\left(\sqrt{\frac{k}{m}}t - \delta_1\right) \\
\xi_2 &= A_2 \cos\left(\sqrt{\frac{3k}{m}}t - \delta_2\right)
\end{aligned}
$$

11.15

$$
\begin{aligned}
\mathcal{L} &= T - U \\
&= \frac{1}{2}(m_1+m_2)L_1^2\dot{\phi_1}^2 + m_2L_1L_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2) + \frac{1}{2}m_2L_2^2\dot{\phi}_2^2 \\
&\quad - (m_1+m_2)gL_1(1-\cos\phi_1) - m_2gL_2(1-\cos\phi_2) \\
\ddot{\phi}_1 &= \frac{-2gm_1\sin\phi_1 + gm_2\cos(\phi_1-\phi_2)\sin\phi_2 - m_2\sin(\phi_1-\phi_2)(L_1\cos(\phi_1-\phi_2)\dot{\phi}_1^2 + L_2\dot{\phi}_2^2)}{L_1(m_1+m_2-m_2\cos^2(\phi_1-\phi_2))} \\
\ddot{\phi}_2 &= \frac{gm_1\sin(2\phi_1-\phi_2) - gm_2\sin\phi_2 + L_1(m_1+m_2)\sin(\phi_1-\phi_2)\dot{\phi}_1^2 + \frac{1}{2}L_2m_2\sin[2(\phi_1-\phi_2)]\dot{\phi}_2^2}{L_2(m_1+m_2-m_2\cos^2(\phi_1-\phi_2))}
\end{aligned}
$$

11.17

(a)

$$
\mathbf{M}\ddot{\boldsymbol{\phi}} = -\mathbf{K}\boldsymbol{\phi}
$$

$$
\begin{bmatrix} (m_1+m_2)L_1^2 & m_2L_1L_2 \\ m_2L_1L_2 & m_2L_2^2 \end{bmatrix}\begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{bmatrix} = -\begin{bmatrix} (m_1+m_2)gL_1 & 0 \\ 0 & m_2gL_2 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}
$$

$$
0 = \det(\mathbf{K} - \omega^2\mathbf{M})
$$

$$
\begin{aligned}
\omega^2 &= \frac{9g \pm 3g}{8L} \\
&= \frac{3}{4}\omega_0^2 \text{ or } \frac{3}{2}\omega_0^2
\end{aligned}
$$

(b)

$$
\begin{aligned}
\omega_1 &= \sqrt{\frac{3}{4}}\omega_0 \\
\omega_2 &= \sqrt{\frac{3}{2}}\omega_0 \\
\mathbf{z}(t) &= A_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}\cos(\omega_1 t - \delta_1) + A_2\begin{bmatrix} 1 \\ -3 \end{bmatrix}\cos(\omega_2 t - \delta_2) \\
\begin{bmatrix} 0 \\ \alpha \end{bmatrix} &= A_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}\cos\delta_1 + A_2\begin{bmatrix} 1 \\ -3 \end{bmatrix}\cos\delta_2 \\
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= A_1\omega_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}\sin\delta_1 + A_2\omega_2\begin{bmatrix} 1 \\ -3 \end{bmatrix}\sin\delta_2 \\
\mathbf{z}(t) &= \frac{\alpha}{6}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\cos\omega_1 t - \begin{bmatrix} 1 \\ -3 \end{bmatrix}\cos\omega_2 t\right)
\end{aligned}
$$

12 Nonlinear Mechanics and Chaos

12.1

(a)

$$\begin{aligned}\dot{x} &= 2\sqrt{x-1}\\ \frac{1}{\sqrt{x-1}}\dot{x} &= 2\\ x &= 1+(t+c)^2\end{aligned}$$

(b) There is no value of c you can choose so $1+(t+c)^2 = 1$ for all t.

(c)

$$\begin{aligned}Ax_1 &= A + A(t+c)^2\\ \frac{d}{dt}(Ax_1) &= 2A(t+c)\\ 2A(t+c) &\neq 2\sqrt{A+A(t+c)^2-1}\end{aligned}$$

$$\begin{aligned}Ax_2 &= A\\ \frac{d}{dt}(Ax_2) &= 0\\ 0 &\neq 2\sqrt{A-1}\end{aligned}$$

12.13

$$\begin{aligned}K &= 10^{-4}\\ Ke^{14.5\lambda} &= 1\\ e^{14.5\lambda} &= 10^4\\ 14.5\lambda &= \ln 10^4\\ \lambda &= \frac{\ln 10^4}{14.5}\\ &\approx 0.64\end{aligned}$$

12.13

(a)

$$\begin{aligned}x &= A\cos(\omega_0 t - \delta)\\ \dot{x} &= -A\omega_0 \sin(\omega_0 t - \delta)\end{aligned}$$

The state-space orbit is an ellipse and it is traced clockwise.

13 Hamiltonian Mechanics

13.1

$$
\begin{aligned}
T &= \frac{1}{2}m\dot{x}^2 \\
U &= 0 \\
\mathcal{L} &= T - U \\
&= \frac{1}{2}m\dot{x}^2 \\
p &= \frac{\partial \mathcal{L}}{\partial \dot{x}} \\
&= m\dot{x} \\
\dot{x} &= \frac{p}{m} \\
\mathcal{H} &= p\dot{x} - \mathcal{L} \\
&= \frac{p^2}{m} - \frac{p^2}{2m} \\
&= \frac{p^2}{2m} \\
\dot{x} &= \frac{\partial \mathcal{H}}{\partial p} \\
&= \frac{p}{m} \\
x &= x_0 + \frac{p}{m}t \\
\dot{p} &= \frac{\partial \mathcal{H}}{\partial x} \\
&= 0 \\
p &= p_0
\end{aligned}
$$

13.3

$$
\begin{aligned}
T &= \frac{1}{2}\left(m_1 + m_2 + \frac{1}{2}M\right)\dot{x}^2 \\
U &= (m_2 - m_1)gx \\
\mathcal{L} &= T - U \\
&= \frac{1}{2}\left(m_1 + m_2 + \frac{1}{2}M\right)\dot{x}^2 - (m_2 - m_1)gx \\
p &= \frac{\partial \mathcal{L}}{\partial \dot{x}} \\
&= \left(m_1 + m_2 + \frac{1}{2}M\right)\dot{x} \\
\dot{x} &= \frac{p}{m_1 + m_2 + \frac{1}{2}M} \\
\mathcal{H} &= p\dot{x} - \mathcal{L} \\
&= \frac{p^2}{2\left(m_1 + m_2 + \frac{1}{2}M\right)} + (m_2 - m_1)gx \\
\dot{p} &= \frac{\partial \mathcal{H}}{\partial x} \\
&= (m_2 - m_1)g \\
\dot{x} &= \frac{\partial \mathcal{H}}{\partial p} \\
&= \frac{p}{m_1 + m_2 + \frac{1}{2}M} \\
\ddot{x} &= \frac{\dot{p}}{m_1 + m_2 + \frac{1}{2}M} \\
&= \frac{m_2 - m_1}{m_1 + m_2 + \frac{1}{2}M}g
\end{aligned}
$$

13.5

$$
\begin{aligned}
v^2 &= \dot{\rho}^2 + (\rho\dot{\phi})^2 + \dot{z}^2 \\
&= R^2\dot{\phi}^2 + c^2\dot{\phi}^2 \\
&= (c^2 + R^2)\dot{\phi}^2 \\
T &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}m(c^2 + R^2)\dot{\phi}^2 \\
U &= mgz \\
&= cmg\phi \\
\mathcal{L} &= T - U \\
&= \frac{1}{2}m(c^2 + R^2)\dot{\phi}^2 - cmg\phi \\
p &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\
&= m(c^2 + R^2)\dot{\phi} \\
\dot{\phi} &= \frac{p}{m(c^2 + R^2)} \\
\mathcal{H} &= p\dot{\phi} - \mathcal{L} \\
&= \frac{p^2}{2m(c^2 + R^2)} + cmg\phi \\
\dot{p} &= -\frac{\partial \mathcal{H}}{\partial \phi} \\
&= -cmg \\
\ddot{\phi} &= \frac{\dot{p}}{m(c^2 + R^2)} \\
&= -\frac{cg}{c^2 + R^2} \\
\ddot{z} &= c\ddot{\phi} \\
&= -\frac{c^2 g}{c^2 + R^2}
\end{aligned}
$$

13.7

(a)

$$
\begin{aligned}
v^2 &= \dot{x}^2 + \dot{y}^2 \\
&= [1 + h'(x)^2]\dot{x}^2 \\
T &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}m[1 + h'(x)^2]\dot{x}^2 \\
U &= mgh(x) \\
\mathcal{L} &= T - U \\
&= \frac{1}{2}m[1 + h'(x)^2]\dot{x}^2 - mgh(x) \\
p &= \frac{\partial \mathcal{L}}{\partial \dot{x}} \\
&= m[1 + h'(x)^2]\dot{x} \\
\dot{x} &= \frac{p}{m[1 + h'(x)^2]} \\
\mathcal{H} &= T + U \\
&= \frac{1}{2}m[1 + h'(x)^2]\dot{x}^2 + mgh(x) \\
&= \frac{p^2}{2m[1 + h'(x)^2]} + mgh(x)
\end{aligned}
$$

(b)

$$
\begin{aligned}
\dot{x} &= \frac{\partial \mathcal{H}}{\partial p} \\
&= \frac{p}{m[1 + h'(x)^2]} \\
\dot{p} &= -\frac{\partial \mathcal{H}}{\partial x} \\
&= \frac{p^2 h'(x) h''(x)}{m[1 + h'(x)^2]^2} - mgh'(x)
\end{aligned}
$$

13.9

$$
\begin{aligned}
T &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\
U &= mgy \\
\mathcal{L} &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy \\
p_x &= m\dot{x} \\
\dot{x} &= \frac{p_x}{m} \\
p_y &= m\dot{y} \\
\dot{y} &= \frac{p_y}{m} \\
\mathcal{H} &= \frac{p_x^2 + p_y^2}{2m} + mgy \\
\dot{x} &= \frac{p_x}{m} \\
\dot{p}_x &= 0 \\
\dot{y} &= \frac{p_y}{m} \\
\dot{p}_y &= -mg
\end{aligned}
$$

13.11

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}m[(V+\dot{x})^2 + \dot{y}^2 + \dot{z}^2] - mgz \\
p_x &= m(V+\dot{x}) \\
\dot{x} &= \frac{p_x}{m} - V \\
p_y &= m\dot{y} \\
\dot{y} &= \frac{p_y}{m} \\
p_z &= m\dot{z} \\
\dot{z} &= \frac{p_z}{m} \\
\mathcal{H} &= \sum_{i=1}^{3} p_i \dot{q}_i - \mathcal{L} \\
&= p_x\left(\frac{p_x}{m} - V\right) + \frac{p_y^2}{m} + \frac{p_z^2}{m} - \frac{1}{2}m[\left(\frac{p_x}{m}\right)^2 + \left(\frac{p_y}{m}\right)^2 + \left(\frac{p_z}{m}\right)^2] + mgz \\
&= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) - p_x V + mgz \\
T + U &= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + mgz \\
&\neq \mathcal{H}
\end{aligned}$$

13.13

$$
\begin{aligned}
v^2 &= \dot{\rho}^2 + (\rho\dot{\phi})^2 + \dot{z}^2 \\
&= R^2\dot{\phi}^2 + \dot{z}^2 \\
T &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{z}^2) \\
U &= -\int_0^r -kr'\,dr' \\
&= \frac{1}{2}kr^2 \\
&= \frac{1}{2}k(R^2 + z^2) \\
\mathcal{L} &= T - U \\
&= \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{z}^2) - \frac{1}{2}k(R^2 + z^2) \\
p_\phi &= mR^2\dot{\phi} \\
\dot{\phi} &= \frac{p_\phi}{mR^2} \\
p_z &= m\dot{z} \\
\dot{z} &= \frac{p_z}{m} \\
\mathcal{H} &= T + U \\
&= \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{z}^2) + \frac{1}{2}k(R^2 + z^2) \\
&= \frac{1}{2}m\left[R^2\left(\frac{p_\phi}{mR^2}\right)^2 + \left(\frac{p_z}{m}\right)^2\right] + \frac{1}{2}k(R^2 + z^2) \\
&= \frac{1}{2m}\left(\frac{p_\phi^2}{R^2} + p_z^2\right) + \frac{1}{2}k(R^2 + z^2) \\
\dot{\phi} &= \frac{p_\phi}{mR^2} \\
\dot{p}_\phi &= 0 \\
\dot{z} &= \frac{p_z}{m} \\
\dot{p}_z &= -kz \\
m\ddot{z} &= -kz \\
z &= A\cos(\omega t - \delta)
\end{aligned}
$$