Advanced Engineering Mathematics Vectors, Matrices, and Vector Calculus by Dennis G. Zill Problems

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Contents

1 Vectors

1.1 Vectors in 2-Space

1.1.1

(a) $3\mathbf{a} = 6\mathbf{i} + 12\mathbf{j}$

(b) $\mathbf{a} + \mathbf{b} = \mathbf{i} + 8\mathbf{j}$

(c) $\mathbf{a} - \mathbf{b} = 3\mathbf{i}$

(d) $||\mathbf{a} + \mathbf{b}|| = \sqrt{1 + 8^2} = \sqrt{65}$

(e) $||\mathbf{a} - \mathbf{b}|| = 3$

1.1.9

(a) $4\mathbf{a} - 2\mathbf{b} = \langle 6, -14\rangle$

(b) $-3\mathbf{a} - 5\mathbf{b} = \langle 2, 4\rangle$

1.1.15

$\overrightarrow{P_1P_2} = \langle 2, 5\rangle$

1.1.19

$(1, 18)$

1.1.21

(a) Yes

(b) Yes

(c) Yes

(d) No

(e) Yes

(f) Yes

1.1.25

(a) $\frac{\mathbf{a}}{||\mathbf{a}||} = \frac{\langle 2,2\rangle}{\sqrt{2^2+2^2}} = \frac{1}{2\sqrt{2}}\langle 2, 2\rangle = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$

(b) $\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle$

1.1.31

$2\frac{\mathbf{a}}{||\mathbf{a}||} = 2\frac{\langle 3,7\rangle}{\sqrt{3^2+7^2}} = \frac{2}{\sqrt{58}}\langle 3, 7\rangle = \langle \frac{6}{\sqrt{58}}, \frac{14}{\sqrt{58}}\rangle$

1.1.37

$\mathbf{x} = -(\mathbf{a} + \mathbf{b})$

1.1.41

$$\begin{aligned}
\mathbf{a} &= 2\mathbf{i} + 3\mathbf{j} \\
\mathbf{b} &= \mathbf{i} + \mathbf{j} \\
\mathbf{c} &= \mathbf{i} - \mathbf{j}
\end{aligned}$$

$$\begin{aligned}
\mathbf{i} &= \frac{1}{2}(\mathbf{b} + \mathbf{c}) \\
\mathbf{j} &= \frac{1}{2}(\mathbf{b} - \mathbf{c}) \\
\mathbf{a} &= 2\left(\frac{1}{2}(\mathbf{b} + \mathbf{c})\right) + 3\left(\frac{1}{2}(\mathbf{b} - \mathbf{c})\right) \\
&= \mathbf{b} + \mathbf{c} + \frac{3}{2}\mathbf{b} - \frac{3}{2}\mathbf{c} \\
&= \frac{5}{2}\mathbf{b} - \frac{1}{2}\mathbf{c}
\end{aligned}$$

1.1.43

$$\begin{aligned}
y &= \frac{1}{4}x^2 + 1 \\
y(2) &= 2 \\
y' &= \frac{1}{2}x \\
y'(2) &= 1 \\
\mathbf{v} &= \pm\langle\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle
\end{aligned}$$

1.1.45

(a)

$$\begin{aligned}
\mathbf{F}_n &= \mathbf{F}\cos\theta \\
\mathbf{F}_g &= \mathbf{F}\sin\theta \\
||\mathbf{F}_f|| &= \mu||\mathbf{F}_n|| \\
||-\mathbf{F}_g|| &= \mu||\mathbf{F}_n|| \\
||-\mathbf{F}\sin\theta|| &= \mu||\mathbf{F}\cos\theta|| \\
||\mathbf{F}||\sin\theta &= \mu||\mathbf{F}||\cos\theta \\
\tan\theta &= \mu
\end{aligned}$$

(b) $\theta = \arctan \mu \approx 31^\circ$

1.1.47

$$\begin{aligned}
F_x &= \frac{qQ}{4\pi\epsilon_0} \int_{-a}^{a} \frac{L\,dy}{2a(L^2+y^2)^{3/2}} \\
&= \frac{LqQ}{8\pi\epsilon_0} \int_{-a}^{a} (L^2+y^2)^{-3/2}\,dy \\
&= \frac{LqQ}{8\pi\epsilon_0} \frac{2a}{L^2\sqrt{a^2+L^2}} \\
&= \frac{aqQ}{4\pi\epsilon_0 L\sqrt{a^2+L^2}} \\
F_y &= -\frac{qQ}{4\pi\epsilon_0} \int_{-a}^{a} \frac{y\,dy}{2a(L^2+y^2)^{3/2}} \\
&= 0 \\
\mathbf{F} &= \langle \frac{1}{4\pi\epsilon_0} \frac{qQ}{L\sqrt{a^2+L^2}}, 0\rangle
\end{aligned}$$

1.1.49

Let the three sides of the triangle be vectors **a**, **b**, and **c**. The triangle is closed so it must be the case that

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0.$$

This gives

$$\mathbf{c} = -(\mathbf{a} + \mathbf{b}).$$

The vector from the midpoint of side **a** to the midpoint of side **b** is

$$\left(\mathbf{a} + \frac{1}{2}\mathbf{b}\right) - \frac{1}{2}\mathbf{a} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$$

which is parallel with **c** and half its length.

1.2 Vectors in 3-Space

1.2.7

A plane at $z = 5$ parellel with the x-y plane.

1.2.9

A line parallel to the z axis at $x = 2$ and $y = 3$.

1.2.13

(a) $(0,5,4)$, $(-2,0,4)$, $(-2,5,0)$

(b) $(-2,5,-2)$

(c) $(3,5,4)$

1.2.15

The planes $x=0$, $y=0$, and $z=0$.

1.2.17

$(-1,2,-3)$

1.2.19

The planes $z=\pm 5$.

1.2.21

$\sqrt{(6-3)^2+(4+1)^2+(8-2)^2}=\sqrt{9+25+36}=\sqrt{70}$

1.2.31

$$\begin{aligned}\sqrt{(2-x)^2+(1-2)^2+(1-3)^2} &= \sqrt{21}\\ (2-x)^2+1+4 &= 21\\ (2-x)^2 &= 16\\ 2-x &= \pm 4\\ x &= 2\pm 4\\ &= -2 \text{ or } 6\end{aligned}$$

1.2.33

$\left(4,\frac{1}{2},\frac{3}{2}\right)$

1.2.37

$(-3,-6,1)$

1.3 Dot Product

1.3.1

$\mathbf{a}\cdot\mathbf{b}=12$

1.3.11

$\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} = \frac{12}{30}\mathbf{b} = \langle -\frac{2}{5}, \frac{4}{5}, 2\rangle$

1.3.13

$\mathbf{a}\cdot\mathbf{b} = ||\mathbf{a}||||\mathbf{b}||\cos\theta = 25\sqrt{2}$

1.3.17

$$\begin{aligned} \mathbf{a}\cdot\mathbf{v} &= 0 \\ 3x_1 + y_1 - 1 &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{b}\cdot\mathbf{v} &= 0 \\ -3x_1 + 2y_2 + 2 &= 0 \end{aligned}$$

$$\begin{aligned} 3y_2 + 1 &= 0 \\ y_2 &= -\frac{1}{3} \end{aligned}$$

$$\begin{aligned} 3x_1 - \frac{1}{3} - 1 &= 0 \\ x_1 &= \frac{4}{9} \end{aligned}$$

$$\mathbf{v} = \langle \frac{4}{9}, -\frac{1}{3}, 1\rangle$$

1.3.19

$$\begin{aligned} \mathbf{a}\cdot\mathbf{c} = \mathbf{a}\cdot\left(\mathbf{b} - \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||^2}\mathbf{a}\right) \\ = \mathbf{a}\cdot\mathbf{b} - \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||^2}\mathbf{a}\cdot\mathbf{a} \\ = 0 \end{aligned}$$

1.3.21

$$\begin{aligned}
||\mathbf{a}|| &= \sqrt{3^2 + 1^2} \\
&= \sqrt{10} \\
||\mathbf{b}|| &= \sqrt{2^2 + 2^2} \\
&= \sqrt{8} \\
&= 2\sqrt{2} \\
\mathbf{a} \cdot \mathbf{b} &= 4 \\
\theta &= \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} \\
&= \arccos \frac{4}{(\sqrt{10})(2\sqrt{2})} \\
&= \arccos \frac{1}{\sqrt{5}} \\
&\approx 63^\circ
\end{aligned}$$

1.3.25

$$\begin{aligned}
||\mathbf{a}|| &= \sqrt{1^2 + 2^2 + 3^3} \\
&= \sqrt{14} \\
\cos\alpha &= \frac{1}{\sqrt{14}} \\
\alpha &\approx 75^\circ \\
\cos\beta &= \frac{2}{\sqrt{14}} \\
\beta &\approx 58^\circ \\
\cos\gamma &= \frac{3}{\sqrt{14}} \\
\gamma &\approx 37^\circ
\end{aligned}$$

1.3.29

$$\begin{aligned}
\overrightarrow{AD} &= \langle s, -s, s\rangle \\
||\overrightarrow{AD}|| &= s\sqrt{3} \\
\overrightarrow{AB} &= \langle s, 0, 0\rangle \\
||\overrightarrow{AB}|| &= s \\
\theta &= \arccos \frac{\overrightarrow{AD}\cdot\overrightarrow{AB}}{||\overrightarrow{AD}||||\overrightarrow{AB}||} \\
&= \arccos \frac{s^2}{s^2\sqrt{3}} \\
&= \arccos \frac{1}{\sqrt{3}} \\
&\approx 55^\circ
\end{aligned}$$

1.3.33

$$\begin{aligned}
\mathrm{comp}_{\mathbf{b}}\mathbf{a} &= \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{b}||} \\
&= \frac{5}{7}
\end{aligned}$$

1.3.37

$$\begin{aligned}
\mathrm{comp}_{\overrightarrow{OP}}\mathbf{a} &= \frac{\mathbf{a}\cdot\overrightarrow{OP}}{||\overrightarrow{OP}||} \\
&= \frac{72}{\sqrt{109}}
\end{aligned}$$

1.3.39

$$\begin{aligned}
\mathrm{proj}_{\mathbf{b}}\mathbf{a} &= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} \\
&= \frac{35}{25}\mathbf{b} \\
&= \langle -\frac{21}{5}, \frac{28}{5}\rangle
\end{aligned}$$

1.3.43

$$\begin{aligned}
\mathbf{a}+\mathbf{b} &= \langle 3,4\rangle \\
\mathrm{proj}_{\mathbf{a}+\mathbf{b}}\mathbf{a} &= \left(\frac{\mathbf{a}\cdot(\mathbf{a}+\mathbf{b})}{(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})}\right)(\mathbf{a}+\mathbf{b}) \\
&= \frac{24}{25}(\mathbf{a}+\mathbf{b}) \\
&= \langle\frac{72}{25},\frac{96}{25}\rangle
\end{aligned}$$

1.3.45

$W=\mathbf{F}\cdot\mathbf{d}=Fd\cos\theta=1000$

1.3.47

(a) $W=0$

(b)

$$\begin{aligned}
||\mathbf{d}|| &= \sqrt{4^2+3^2} \\
&= 5 \\
\mathbf{F} &= F\hat{\mathbf{d}} \\
&= F\frac{\mathbf{d}}{||\mathbf{d}||} \\
&= F\langle\frac{4}{5},\frac{3}{5}\rangle \\
&= \langle 24,18\rangle \\
W &= \mathbf{F}\cdot\mathbf{d} \\
&= 150\,\mathrm{J}
\end{aligned}$$

1.4 Cross Product

1.4.1

$$\begin{aligned}
\mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 0 & 3 & 5 \end{vmatrix} \\
&= -5\mathbf{i}-5\mathbf{j}+3\mathbf{k}
\end{aligned}$$

1.4.11

$$\begin{aligned}\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & -4 \\ -3 & 1 & 1 \end{vmatrix} \\ &= 6\mathbf{i} + 14\mathbf{j} + 4\mathbf{k}\end{aligned}$$

1.4.17

(a)

$$\begin{aligned}\mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} \\ &= \mathbf{j} - \mathbf{k} \\ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 0 & 1 & -1 \end{vmatrix} \\ &= -\mathbf{i} + \mathbf{j} + \mathbf{k}\end{aligned}$$

1.4.19

$2\mathbf{k}$

1.4.21

$$\begin{aligned}\mathbf{k} \times (2\mathbf{i} - \mathbf{j}) &= (\mathbf{k} \times 2\mathbf{i}) - (\mathbf{k} \times \mathbf{j}) \\ &= \mathbf{i} + 2\mathbf{j}\end{aligned}$$

1.4.23

$$\begin{aligned}[(2\mathbf{k}) \times (3\mathbf{j})] \times (4\mathbf{j}) &= (-6\mathbf{i}) \times (4\mathbf{j}) \\ &= -24\mathbf{k}\end{aligned}$$

1.4.37

$12\mathbf{i} - 9\mathbf{j} + 18\mathbf{k}$

1.4.53

$$\begin{aligned}
\mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 6 & -6 \\ \frac{5}{2} & 3 & \frac{1}{2} \end{vmatrix} \\
&= 21\mathbf{i} - 14\mathbf{j} - 21\mathbf{k} \\
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= 4 \times 21 + 6 \times (-14) \\
&= 0
\end{aligned}$$

They are coplanar.

1.5 Lines and Planes in 3-Space

1.5.1

$\mathbf{r} = \langle 1, 2, 1 \rangle + t\langle 2, 3, -3 \rangle$

1.5.7

$$\begin{aligned}
x &= 2 + 4t \\
y &= 3 - 4t \\
z &= 5 + 3t
\end{aligned}$$

1.5.13

$$\begin{aligned}
x &= 1 + 9t \\
y &= 4 + 10t \\
z &= -9 + 7t \\
\frac{x-1}{9} = \frac{y-4}{10} &= \frac{z+9}{7}
\end{aligned}$$

1.5.19

$$\begin{aligned}
x &= 4 + 3t \\
y &= 6 + \frac{1}{2}t \\
z &= -7 - \frac{3}{2}t \\
\frac{x-4}{3} = \frac{y-6}{1/2} &= -\frac{z+7}{3/2}
\end{aligned}$$

1.5.23

$$\begin{aligned} x &= 6 + 2t \\ y &= 4 - 3t \\ z &= -2 + 6t \end{aligned}$$

1.5.25

$$\begin{aligned} x &= 2 + t \\ y &= -2 \\ z &= 15 \end{aligned}$$

1.5.29

$(0, 5, 15)$, $(5, 0, \frac{15}{2})$, $(10, -5, 0)$

1.5.31

$$\begin{aligned} 4 + t_x &= 6 + 2t_x \\ t_x &= -2 \\ \\ 5 + t_y &= 11 + 4t_y \\ t_y &= -2 \\ \\ -1 + 2t_z &= -3 + t_z \\ t_z &= -2 \end{aligned}$$

$(2, 3, -5)$

1.5.35

$$\begin{aligned} \mathbf{a} &= \langle -1, 2, -2 \rangle \\ ||\mathbf{a}|| &= 3 \\ \mathbf{b} &= \langle 2, 3, -6 \rangle \\ ||\mathbf{b}|| &= 7 \\ \theta &= \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} \\ &\approx 40.37^\circ \end{aligned}$$

1.5.37

$$\begin{aligned}
\mathbf{a} &= \langle 1,1,1\rangle \\
\mathbf{b} &= \langle -2,1,-5\rangle \\
\mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ -2 & 1 & -5 \end{vmatrix} \\
&= \langle -6,3,3\rangle \\
x &= 4-6t \\
y &= 1+3t \\
z &= 6+3t
\end{aligned}$$

1.5.39

$$\begin{aligned}
\langle 2,-3,4\rangle\cdot(\mathbf{r}-\langle 5,1,3\rangle) &= 0 \\
2(x-5)-3(y-1)+4(z-3) &= 0 \\
2x-3y+4z-19 &= 0
\end{aligned}$$

1.5.45

$$\begin{aligned}
\mathbf{a} &= \langle 3,5,2\rangle \\
\mathbf{b} &= \langle 2,3,1\rangle \\
\mathbf{c} &= \langle -1,-1,4\rangle \\
\mathbf{a}-\mathbf{c} &= \langle 4,6,-2\rangle \\
\mathbf{b}-\mathbf{c} &= \langle 3,4,-3\rangle \\
(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 6 & -2 \\ 3 & 4 & -3 \end{vmatrix} \\
&= \langle -10,6,-2\rangle \\
\mathbf{n}\cdot(\mathbf{r}-\mathbf{c}) &= 0 \\
\langle -10,6,-2\rangle\cdot(\langle x,y,z\rangle-\langle -1,-1,4\rangle) &= 0 \\
-10(x+1)+6(y+1)-2(z-4) &= 0 \\
-10x+6y-2z+4 &= 0
\end{aligned}$$

1.5.51

$$\begin{aligned}
\langle 1,1,-4\rangle\cdot(\mathbf{r}-\langle 2,3,-5\rangle) &= 0 \\
(x-2)+(y-3)-4(z+5) &= 0 \\
x+y-4z &= 25
\end{aligned}$$

1.5.63

(a) Not perpendicular

(b) Not perpendicular

(c) Perpendicular

(d) Perpendicular

1.5.65

$$\begin{aligned} 5x - 4y - 9t &= 8 \\ x + 4y + 3t &= 4 \end{aligned}$$

$$\begin{aligned} 6x - 6t &= 12 \\ x &= 2 + t \end{aligned}$$

$$y = \frac{1}{2} - t$$

$$z = t$$

1.5.69

$$\begin{aligned} 2(1+2t) - 3(2-t) + 2(-3t) &= -7 \\ t &= -3 \\ x &= -5 \\ y &= 5 \\ z &= 9 \end{aligned}$$

1.5.73

$$
\begin{aligned}
x + y - 4t &= 2 \\
2x - y + t &= 10 \\
\\
3x - 3t &= 12 \\
x &= 4 + t \\
\\
2(4 + t) - y + t &= 10 \\
8 + 2t - y + t &= 10 \\
y &= -2 + 3t \\
\\
z &= t \\
\\
x &= 5 + t \\
y &= 6 + 3t \\
z &= -12 + t
\end{aligned}
$$

1.5.75

$$
\begin{aligned}
\mathbf{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 5 \\ 1 & 1 & 1 \end{vmatrix} \\
&= \langle -6, 2, 4 \rangle \\
\mathbf{n} \cdot (\mathbf{r} - \langle 4, 0, 1 \rangle) &= 0 \\
-6(x - 4) + 2y + 4(z - 1) &= 0 \\
-6x + 2y + 4z &= -20 \\
3x - y - 2z &= 10
\end{aligned}
$$

1.6 Vector Spaces

1.6.1

Violates axiom 6

1.6.3

Violates axiom 10

1.6.5

Vector space

1.6.7

Violates axiom 2

1.6.9

Vector space

1.6.11

Subspace

1.6.13

Not a subspace

1.6.15

Subspace

1.6.17

Subspace

1.6.19

Not a subspace

1.6.23

(a)

$$\begin{aligned} k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + k_3\mathbf{u}_3 &= \mathbf{0} \\ k_1\langle 1,0,0\rangle + k_2\langle 1,1,0\rangle + k_3\langle 1,1,1\rangle &= \mathbf{0} \\ \\ k_3 &= 0 \\ \\ k_2 + k_3 &= 0 \\ k_2 &= 0 \\ \\ k_1 + k_2 + k_3 &= 0 \\ k_1 &= 0 \end{aligned}$$

(b)

$$\mathbf{a} = 7\mathbf{u}_1 - 12\mathbf{u}_2 + 8\mathbf{u}_3$$

1.6.25

Dependent

1.6.27

Independent

1.6.29

$f(x)$ is undefined at $x = -3$ and $x = -1$.

1.6.31

$$
\begin{aligned}
||x|| &= \sqrt{(x,x)} \\
&= \sqrt{\int_0^{2\pi} x^2\,dx} \\
&= \sqrt{\left[\frac{1}{3}x^3\right]_0^{2\pi}} \\
&= \sqrt{\frac{8}{3}\pi^3} \\
||\sin x|| &= \sqrt{(\sin x, \sin x)} \\
&= \sqrt{\int_0^{2\pi} \sin^2 x\,dx} \\
&= \sqrt{\left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_0^{2\pi}} \\
&= \sqrt{\pi}
\end{aligned}
$$

1.7 Gram–Schmidt Orthogonalization Process

1.7.1

$$
\begin{aligned}
\langle \frac{12}{13}, \frac{5}{13} \rangle \cdot \langle \frac{5}{13}, -\frac{12}{13} \rangle &= 0 \\
\sqrt{\left(\frac{12}{13}\right)^2 + \left(\frac{5}{13}\right)^2} &= 1
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{u} &= \left(\langle 4, 2 \rangle \cdot \langle \frac{12}{13}, \frac{5}{13} \rangle \right) \langle \frac{12}{13}, \frac{5}{13} \rangle \\
&\quad + \left(\langle 4, 2 \rangle \cdot \langle \frac{5}{13}, -\frac{12}{13} \rangle \right) \langle \frac{5}{13}, -\frac{12}{13} \rangle \\
&= \left(\frac{58}{13} \right) \langle \frac{12}{13}, \frac{5}{13} \rangle - \left(\frac{4}{13} \right) \langle \frac{5}{13}, -\frac{12}{13} \rangle
\end{aligned}
$$

1.7.3

$$
\begin{aligned}
\langle 1, 0, 1 \rangle \cdot \langle 0, 1, 0 \rangle &= 0 \\
\langle 1, 0, 1 \rangle \cdot \langle -1, 0, 1 \rangle &= 0 \\
\langle 0, 1, 0 \rangle \cdot \langle -1, 0, 1 \rangle &= 0
\end{aligned}
$$

$$
\begin{aligned}
B' &= \{ \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle, \langle 0, 1, 0 \rangle, \langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle \} \\
\mathbf{u} &= -\frac{3}{\sqrt{2}} \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle + 7 \langle 0, 1, 0 \rangle - \frac{23}{\sqrt{2}} \langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle
\end{aligned}
$$

1.7.5

(a)

$$
\begin{aligned}
B &= \{\langle -3, 2\rangle, \langle -1, -1\rangle\} \\
\mathbf{v}_1 &= \mathbf{u}_1 \\
&= \langle -3, 2\rangle \\
\mathbf{v}_2 &= \mathbf{u}_2 - \operatorname{proj}_{\mathbf{v}_1} \mathbf{u}_2 \\
&= \langle -1, -1\rangle - \left(\frac{\langle -1, -1\rangle \cdot \langle -3, 2\rangle}{\langle -3, 2\rangle \cdot \langle -3, 2\rangle} \right) \langle -3, 2\rangle \\
&= \langle -1, -1\rangle - \frac{1}{13} \langle -3, 2\rangle \\
&= \langle -\frac{10}{13}, -\frac{15}{13}\rangle \\
\mathbf{w}_1 &= \langle -\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\rangle \\
\mathbf{w}_2 &= \sqrt{\frac{169}{325}} \langle -\frac{10}{13}, -\frac{15}{13}\rangle \\
&= \frac{\sqrt{13}}{5} \langle -\frac{10}{13}, -\frac{15}{13}\rangle \\
&= \langle -\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}\rangle
\end{aligned}
$$

1.7.9

$$
\begin{aligned}
B &= \{\langle 1,1,0\rangle, \langle 1,2,2\rangle, \langle 2,2,1\rangle\}\\
\mathbf{v}_1 &= \langle 1,1,0\rangle\\
\mathbf{v}_2 &= \mathbf{u}_2 - \text{proj}_{\mathbf{v}_1}\mathbf{u}_2\\
&= \langle 1,2,2\rangle - \left(\frac{\langle 1,2,2\rangle \cdot \langle 1,1,0\rangle}{\langle 1,1,0\rangle \cdot \langle 1,1,0\rangle}\right)\langle 1,1,0\rangle\\
&= \langle 1,2,2\rangle - \frac{3}{2}\langle 1,1,0\rangle\\
&= \langle -\frac{1}{2}, \frac{1}{2}, 2\rangle\\
\mathbf{v}_3 &= \mathbf{u}_3 - \text{proj}_{\mathbf{v}_1}\mathbf{u}_3 - \text{proj}_{\mathbf{v}_2}\mathbf{u}_3\\
&= \langle 2,2,1\rangle - \left(\frac{\langle 2,2,1\rangle \cdot \langle 1,1,0\rangle}{\langle 1,1,0\rangle \cdot \langle 1,1,0\rangle}\right)\langle 1,1,0\rangle\\
&\qquad - \left(\frac{\langle 2,2,1\rangle \cdot \langle -\frac{1}{2}, \frac{1}{2}, 2\rangle}{\langle -\frac{1}{2}, \frac{1}{2}, 2\rangle \cdot \langle -\frac{1}{2}, \frac{1}{2}, 2\rangle}\right)\langle -\frac{1}{2}, \frac{1}{2}, 2\rangle\\
&= \langle 2,2,1\rangle - 2\langle 1,1,0\rangle - \frac{4}{9}\langle -\frac{1}{2}, \frac{1}{2}, 2\rangle\\
&= \langle \frac{2}{9}, -\frac{2}{9}, \frac{1}{9}\rangle\\
\mathbf{w}_1 &= \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\rangle\\
\mathbf{w}_2 &= \langle -\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}\rangle\\
\mathbf{w}_3 &= 3\langle \frac{2}{9}, -\frac{2}{9}, \frac{1}{9}\rangle\\
&= \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\rangle
\end{aligned}
$$

1.7.17

$$\begin{aligned}
B &= \{1, x, x^2\} \\
\mathbf{v}_1 &= \mathbf{u}_1 \\
&= 1 \\
\mathbf{v}_2 &= \mathbf{u}_2 - \operatorname{proj}_{\mathbf{v}_1}\mathbf{u}_2 \\
&= \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\
&= x - \frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1} dx} \\
&= x - \frac{\left[\frac{1}{2}x^2\right]_{-1}^{1}}{2} \\
&= x \\
\mathbf{v}_3 &= \mathbf{u}_3 - \operatorname{proj}_{\mathbf{v}_1}\mathbf{u}_3 - \operatorname{proj}_{\mathbf{v}_2}\mathbf{u}_3 \\
&= \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\
&= x^2 - \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} dx} - \frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx}x \\
&= x^2 - \frac{\left[\frac{1}{3}x^3\right]_{-1}^{1}}{2} - \frac{\left[\frac{1}{4}x^4\right]_{-1}^{1}}{\left[\frac{1}{3}x^3\right]_{-1}^{1}}x \\
&= x^2 - \frac{1}{3}
\end{aligned}$$

1.7.19

$$
\begin{aligned}
||\mathbf{v}_1||^2 &= \int_{-1}^{1} \, dx \\
&= 2 \\
\mathbf{w}_1 &= \frac{1}{\sqrt{2}} \\
||\mathbf{v}_2||^2 &= \int_{-1}^{1} x^2 \, dx \\
&= \left[\frac{1}{3}x^3\right]_{-1}^{1} \\
&= \frac{2}{3} \\
\mathbf{w}_2 &= \frac{3}{\sqrt{6}}x \\
||\mathbf{v}_3||^2 &= \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 \, dx \\
&= \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) \, dx \\
&= \left[\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x\right]_{-1}^{1} \\
&= \frac{1}{5} - \frac{2}{9} + \frac{1}{9} + \frac{1}{5} - \frac{2}{9} + \frac{1}{9} \\
&= \frac{2}{5} - \frac{2}{9} \\
&= \frac{8}{45} \\
\mathbf{w}_3 &= \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3}\right) \\
&= \frac{5}{2\sqrt{10}} \left(3x^2 - 1\right)
\end{aligned}
$$

1.7.21

$$\begin{aligned}
(\mathbf{p}, \mathbf{w}_1) &= \int_{-1}^{1} \frac{1}{\sqrt{2}}(9x^2 - 6x + 5)\,dx \\
&= \frac{1}{\sqrt{2}}\left[3x^3 - 3x^2 + 5x\right]_{-1}^{1} \\
&= \frac{1}{\sqrt{2}}(3 - 3 + 5 + 3 + 3 + 5) \\
&= \frac{16}{\sqrt{2}} \\
(\mathbf{p}, \mathbf{w}_2) &= \int_{-1}^{1} \frac{3}{\sqrt{6}}x(9x^2 - 6x + 5)\,dx \\
&= \frac{3}{\sqrt{6}}\left[\frac{9}{4}x^4 - 2x^3 + \frac{5}{2}x^2\right]_{-1}^{1} \\
&= \frac{3}{\sqrt{6}}\left(\frac{9}{4} - 2 + \frac{5}{2} - \frac{9}{4} - 2 - \frac{5}{2}\right) \\
&= \frac{3}{\sqrt{6}}\left(\frac{9}{4} - \frac{8}{4} + \frac{10}{4} - \frac{9}{4} - \frac{8}{4} - \frac{10}{4}\right) \\
&= -\frac{12}{\sqrt{6}} \\
(\mathbf{p}, \mathbf{w}_3) &= \int_{-1}^{1} \frac{5}{2\sqrt{10}}(3x^2 - 1)(9x^2 - 6x + 5)\,dx \\
&= \frac{5}{2\sqrt{10}}\int_{-1}^{1}(27x^4 - 18x^3 + 6x^2 + 6x - 5)\,dx \\
&= \frac{5}{2\sqrt{10}}\left[\frac{27}{5}x^5 - \frac{9}{2}x^4 + 2x^3 + 3x^2 - 5x\right]_{-1}^{1} \\
&= \frac{5}{2\sqrt{10}}\left(\frac{27}{5} - \frac{9}{2} + 2 + 3 - 5 + \frac{27}{5} + \frac{9}{2} + 2 - 3 - 5\right) \\
&= \frac{5}{2\sqrt{10}}\left(\frac{54}{10} - \frac{45}{10} + \frac{20}{10} + \frac{30}{10} - \frac{50}{10} + \frac{54}{10} + \frac{45}{10} + \frac{20}{10} - \frac{30}{10} - \frac{50}{10}\right) \\
&= \frac{5}{2\sqrt{10}}\frac{48}{10} \\
&= \frac{12}{\sqrt{10}} \\
\mathbf{p} &= \frac{16}{\sqrt{2}}\mathbf{w}_1 - \frac{12}{\sqrt{6}}\mathbf{w}_2 + \frac{12}{\sqrt{10}}\mathbf{w}_3
\end{aligned}$$

1.8 Chapter in Review

1.8.1

True

1.8.3

$$\mathbf{u} = \langle 5, -2, 1 \rangle$$
$$\mathbf{v} = \langle 2, 3, -4 \rangle$$

False

1.8.5

True

1.8.7

True

1.8.9

True

1.8.11

$9\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

1.8.13

$$\begin{aligned}(-\mathbf{k}) \times (5\mathbf{j}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -1 \\ 0 & 5 & 0 \end{vmatrix} \\ &= 5\mathbf{i}\end{aligned}$$

1.8.15

$$||-12\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}|| = \sqrt{12^2 + 4^2 + 6^2} = 14$$

1.8.17

$\langle -6, 1, -7 \rangle$

1.8.19

$$\begin{aligned} x &= 1+t \\ y &= -2+3t \\ z &= -1+2t \end{aligned}$$

$$\begin{aligned} x+2y-z &= 13 \\ (1+t)+2(-2+3t)-(-1+2t) &= 13 \\ 1+t-4+6t+1-2t &= 13 \\ -2+5t &= 13 \\ t &= 3 \end{aligned}$$

$$\begin{aligned} x &= 4 \\ y &= 7 \\ z &= 5 \end{aligned}$$

1.8.21

$$\begin{aligned} \overrightarrow{P_1P_2} &= \overrightarrow{P_2} - \overrightarrow{P_1} \\ \overrightarrow{P_2} &= \overrightarrow{P_1P_2} + \overrightarrow{P_1} \\ &= \langle 3,5,-4\rangle + \langle 2,1,7\rangle \\ &= \langle 5,6,3\rangle \end{aligned}$$

1.8.23

$\mathbf{a}\cdot\mathbf{b} = -36\sqrt{2}$

1.8.25

$x = 12$, $y = -8$, $z = 6$

1.8.27

$$\begin{aligned} \frac{1}{2}(\mathbf{a}\times\mathbf{b}) &= \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & -1 & 2 \end{vmatrix} \\ &= \frac{1}{2}\langle 5,-4,-7\rangle \\ &= \langle \frac{5}{2},-2,-\frac{7}{2}\rangle \end{aligned}$$

The area is $\sqrt{\left(\frac{5}{2}\right)^2 + (-2)^2 + \left(-\frac{7}{2}\right)^2} = \frac{3}{2}\sqrt{10}$

1.8.29

2

1.8.31

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -2 & 1 \end{vmatrix} \\
&= \langle 1, -1, -3 \rangle \\
||\mathbf{a} \times \mathbf{b}|| &= \sqrt{11} \\
\mathrm{norm}(\mathbf{a} \times \mathbf{b}) &= \langle \frac{1}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, -\frac{3}{\sqrt{11}} \rangle
\end{aligned}$$

1.8.33

$$\mathrm{comp}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||} = \frac{10}{5} = 2$$

1.8.35

$$\begin{aligned}
\mathbf{a} &= \langle 1, 2, -2 \rangle \\
\mathbf{b} &= \langle 4, 3, 0 \rangle \\
\mathbf{a} + \mathbf{b} &= \langle 5, 5, -2 \rangle \\
\mathrm{proj}_{\mathbf{a}}(\mathbf{a} + \mathbf{b}) &= \left(\frac{(\mathbf{a} + \mathbf{b}) \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}} \right) \mathbf{a} \\
&= \frac{19}{9} \langle 1, 2, -2 \rangle \\
&= \langle \frac{19}{9}, \frac{38}{9}, -\frac{38}{9} \rangle
\end{aligned}$$

1.8.37

(a)

(b) A plane with normal $\mathbf{a}$

1.8.39

$$\frac{x-7}{4} = \frac{y-3}{-2} = \frac{z+5}{6}$$

1.8.41

$$\begin{aligned}\langle -2, 3, 1\rangle \cdot \langle 2, 1, 1\rangle &= -4 + 3 + 1\\ &= 0\end{aligned}$$

$$\begin{aligned}1 - 2t &= 1 + 2s\\ t &= -s\end{aligned}$$

$$\begin{aligned}3t &= -4 + s\\ t &= -\frac{4}{3} + \frac{s}{3}\\ &= -\frac{4}{3} - \frac{t}{3}\\ \frac{4}{3}t &= -\frac{4}{3}\\ t &= -1\end{aligned}$$

$$s = 1$$

$$\langle 3, -3, 0\rangle$$

1.8.43

$$\begin{aligned}\mathbf{u} &= \langle 1, 4, -2\rangle\\ \mathbf{v} &= \langle 1, 1, 3\rangle\\ \mathbf{n} &= \mathbf{u} \times \mathbf{v}\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 4 & -2\\ 1 & 1 & 3\end{vmatrix}\\ &= \langle 14, -5, -3\rangle\end{aligned}$$

$$\begin{aligned}\mathbf{n} \cdot (\mathbf{r} - \mathbf{v}) &= 0\\ \langle 14, -5, -3\rangle \cdot (\langle x, y, z\rangle - \langle 1, 1, 3\rangle) &= 0\\ 14(x - 1) - 5(y - 1) - 3(z - 3) &= 0\\ 14x - 5y - 3z &= 0\end{aligned}$$

1.8.45

$$\begin{aligned}\mathbf{F} &= \langle \frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}}, 0\rangle \\ \mathbf{d} &= \langle 3, 3, 0\rangle \\ \mathbf{F}\cdot\mathbf{d} &= 30\sqrt{2}\,\mathrm{J}\end{aligned}$$

1.8.47

$$\begin{aligned}\mathbf{F}_1 &= \langle 200, 0, 0\rangle \\ \mathbf{F}_2 &= \langle \frac{200}{\sqrt{2}}, \frac{200}{\sqrt{2}}, 0\rangle \\ \mathbf{F}_2 &= \mathbf{F}_1 + \mathbf{F}_3 \\ \mathbf{F}_3 &= \mathbf{F}_2 - \mathbf{F}_1 \\ &= \langle \frac{200}{\sqrt{2}}, \frac{200}{\sqrt{2}}, 0\rangle - \langle 200, 0, 0\rangle \\ &= \langle \frac{200}{\sqrt{2}} - 200, \frac{200}{\sqrt{2}}, 0\rangle \\ ||\mathbf{F}_3|| &= \sqrt{\left(\frac{200}{\sqrt{2}} - 200\right)^2 + \left(\frac{200}{\sqrt{2}}\right)^2} \\ &= \sqrt{\frac{40000}{2} - \frac{80000}{\sqrt{2}} + 40000 + \frac{40000}{2}} \\ &= 200\sqrt{2\left(1 - \frac{1}{\sqrt{2}}\right)} \\ &\approx 153\,\mathrm{lb}\end{aligned}$$

2 Matrices

2.1 Matrix Algebra

2.1.1

2×4

2.1.3

3×3

2.1.5

3×4

2.1.7

No

2.1.9

No

2.1.11

$$\begin{aligned} x &= y - 2 \\ 3x - 2 &= y \end{aligned}$$

$$\begin{aligned} 2x - 2 &= 2 \\ 2x &= 4 \\ x &= 2 \end{aligned}$$

$$\begin{aligned} 2 &= y - 2 \\ y &= 4 \end{aligned}$$

2.1.13

$$\begin{aligned} c_{23} &= 9 \\ c_{12} &= 12 \end{aligned}$$

2.1.15

(a) $\begin{pmatrix} 2 & 11 \\ 2 & -1 \end{pmatrix}$

(b) $\begin{pmatrix} -6 & 1 \\ 14 & -19 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 28 \\ 12 & -12 \end{pmatrix}$

2.1.17

(a) $\begin{pmatrix} -11 & 6 \\ 17 & -22 \end{pmatrix}$

(b) $\begin{pmatrix} -32 & 27 \\ -4 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 19 & -18 \\ -30 & 31 \end{pmatrix}$

(d) $\begin{pmatrix} 19 & 6 \\ 3 & 22 \end{pmatrix}$

2.1.21

(a) 180

(b) $\begin{pmatrix} 4 & 8 & 10 \\ 8 & 16 & 20 \\ 10 & 20 & 25 \end{pmatrix}$

(c) $\begin{pmatrix} 6 \\ 12 \\ -5 \end{pmatrix}$

2.1.23

(a) $\begin{pmatrix} 7 & 38 \\ 10 & 75 \end{pmatrix}$

(b) $\begin{pmatrix} 7 & 38 \\ 10 & 75 \end{pmatrix}$

2.1.25

$\begin{pmatrix} -14 \\ 1 \end{pmatrix}$

2.1.27

$\begin{pmatrix} -38 \\ -2 \end{pmatrix}$

2.1.29

4×5

2.1.41

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + x_{22}x_2 &= b_2 \end{aligned}$$

2.1.43

$$\begin{aligned} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} ax + \frac{1}{2}by \\ \frac{1}{2}bx + cy \end{pmatrix} \\ &= ax^2 + \frac{1}{2}bxy + \frac{1}{2}bxy + cy^2 \\ &= ax^2 + bxy + cy^2 \end{aligned}$$

2.1.45

$\langle -1, 1\rangle$

2.1.47

$\langle -2, 0\rangle$

2.1.49

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

2.1.51

(b)

$$\begin{pmatrix} x_S \\ y_S \\ z_S \end{pmatrix} = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

2.2 Systems of Linear Algebraic Equations

2.2.1

$$\left(\begin{array}{cc|c} 1 & -1 & 11 \\ 4 & 3 & -5 \end{array}\right)$$
$$\left(\begin{array}{cc|c} 1 & -1 & 11 \\ 0 & 7 & -49 \end{array}\right)$$
$$\left(\begin{array}{cc|c} 1 & -1 & 11 \\ 0 & 1 & -7 \end{array}\right)$$
$$\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -7 \end{array}\right)$$

2.2.5

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 2 & 3 & 5 & 7 \\ 1 & -2 & 3 & -11 \end{array}\right)$$
$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 0 & 5 & 7 & 13 \\ 0 & -1 & 4 & -8 \end{array}\right)$$
$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 0 & 1 & \frac{7}{5} & \frac{13}{5} \\ 0 & -1 & 4 & -8 \end{array}\right)$$
$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 0 & 1 & \frac{7}{5} & \frac{13}{5} \\ 0 & 0 & \frac{27}{5} & -\frac{27}{5} \end{array}\right)$$
$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 0 & 1 & \frac{7}{5} & \frac{13}{5} \\ 0 & 0 & 1 & -1 \end{array}\right)$$
$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right)$$
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

2.3 Rank of a Matrix

2.3.1

$$\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -\frac{1}{3} \\ 1 & 3 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & \frac{10}{3} \end{pmatrix}$$

Rank 2

2.3.3

$$\begin{pmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ -1 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}$$
$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 6 & 3 & 9 \\ -1 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}$$
$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Rank 1

2.3.5

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 4 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 3 \\ 0 & 3 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 3 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 9 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

Rank 3

2.3.7

$$\begin{pmatrix} 1 & -2 \\ 3 & -6 \\ 7 & -1 \\ 4 & 5 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -2 \\ 0 & 13 \\ 0 & 13 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -2 \\ 0 & 13 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Rank 2

2.3.11

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 5 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & -3 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -3 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

Linearly independent

2.3.15

5

2.3.17

$\text{rank}(\mathbf{A}) = 2$

2.4 Determinants

2.4.1

9

2.4.3

1

2.4.5

$$\begin{aligned} M_{33} &= \begin{vmatrix} 0 & 2 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} \\ &= -2 \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} \\ &= 2 \end{aligned}$$

2.4.7

$$\begin{aligned} C_{34} &= (-1)^{3+4} \begin{vmatrix} 0 & 2 & 4 \\ 1 & 2 & -2 \\ 1 & 1 & 1 \end{vmatrix} \\ &= -\left(-2 \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} + 4 \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \right) \\ &= 10 \end{aligned}$$

2.4.9

-7

2.4.11

17

2.4.13

$$\begin{aligned} (1-\lambda)(2-\lambda) - 6 &= 2 - \lambda - 2\lambda + \lambda^2 - 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda + 1)(\lambda - 4) \end{aligned}$$

2.4.15

-48

2.4.23

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} y & z \\ 3 & 4 \end{vmatrix} - \begin{vmatrix} x & z \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} x & y \\ 2 & 3 \end{vmatrix}$$
$$= 4y - 3z - 4x + 2z + 3x - 2y$$
$$= -x + 2y - z$$

2.4.29

$$\begin{aligned} \begin{vmatrix} (-3-\lambda) & 10 \\ 2 & (5-\lambda) \end{vmatrix} &= 0 \\ (-3-\lambda)(5-\lambda) - 20 &= 0 \\ -15 + 3\lambda - 5\lambda + \lambda^2 - 20 &= 0 \\ \lambda^2 - 2\lambda - 35 &= 0 \\ (\lambda - 7)(\lambda + 5) &= 0 \\ \lambda &= -5 \text{ or } 7 \end{aligned}$$

2.5 Properties of Determinants

2.5.1

8.5.4

2.5.3

8.5.7

2.5.5

8.5.5

2.5.7

8.5.3

2.5.9

8.5.1

2.5.11

-5

2.5.13

-5

2.5.15

5

2.5.17

80

2.5.19

-105

2.5.25

$$\begin{aligned}
\mathbf{AA} &= \mathbf{I} \\
\det \mathbf{A} \cdot \det \mathbf{A} &= \det \mathbf{I} \\
(\det \mathbf{A})^2 &= 1 \\
\det \mathbf{A} &= \pm 1
\end{aligned}$$

2.5.27

$$\begin{aligned}
\begin{vmatrix} a & a+1 & a+2 \\ b & b+1 & b+2 \\ c & c+1 & c+2 \end{vmatrix} &= \begin{vmatrix} a & 1 & 2 \\ b & 1 & 2 \\ c & 1 & 2 \end{vmatrix} \\
&= \begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} \\
&= 0
\end{aligned}$$

2.5.29

$$\begin{aligned}
\begin{vmatrix} 1 & 1 & 5 \\ 4 & 3 & 6 \\ 0 & -1 & 1 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 5 \\ 0 & -1 & -14 \\ 0 & -1 & 1 \end{vmatrix} \\
&= -\begin{vmatrix} 1 & 1 & 5 \\ 0 & 1 & 14 \\ 0 & 0 & 15 \end{vmatrix} \\
&= -15
\end{aligned}$$

2.5.37

$$
\begin{aligned}
\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & c^2-a^2-(b+a)(c-a) \end{vmatrix} \\
&= (b-a)(c^2-a^2-(b+a)(c-a)) \\
&= (b-a)(c^2-a^2-bc+ab-ac+a^2) \\
&= (b-a)(c^2+ab-ac-bc) \\
&= (b-a)(c-a)(c-b)
\end{aligned}
$$

2.5.39

$$
\begin{aligned}
a_{21}C_{11}+a_{22}C_{12}+a_{23}C_{13} &= (-1)(4)+(2)(5)+(1)(-6) \\
&= 0 \\
a_{13}C_{12}+a_{23}C_{22}+a_{33}C_{32} &= (2)(5)+(1)(-7)+(1)(-3) \\
&= 0
\end{aligned}
$$

2.5.41

$$
\begin{aligned}
\mathbf{A}+\mathbf{B} &= \begin{pmatrix} 10 & 0 \\ 0 & -3 \end{pmatrix} \\
\det(\mathbf{A}+\mathbf{B}) &= -30 \\
\det\mathbf{A} &= 10 \\
\det\mathbf{B} &= -31 \\
-30 &\neq 10-31
\end{aligned}
$$

2.6 Inverse of a Matrix

2.6.3

$$
\begin{aligned}
\det\mathbf{A} &= 9 \\
\mathbf{A}^{-1} &= \frac{1}{9}\begin{pmatrix} 1 & 1 \\ -4 & 5 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{9} & \frac{1}{9} \\ -\frac{4}{9} & \frac{5}{9} \end{pmatrix}
\end{aligned}
$$

2.6.5

$$\begin{aligned} \det \mathbf{A} &= 12 \\ \mathbf{A}^{-1} &= \frac{1}{12}\begin{pmatrix} 2 & 0 \\ 3 & 6 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{6} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \end{aligned}$$

2.6.7

$$\begin{aligned} \det \mathbf{A} &= (1)\begin{vmatrix} 4 & 4 \\ -1 & 1 \end{vmatrix} - (3)\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} + (5)\begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} \\ &= 8 + 6 - 30 \\ &= -16 \\ \mathbf{A}^{-1} &= -\frac{1}{16}\begin{pmatrix} 8 & 2 & -6 \\ -8 & -4 & 4 \\ -8 & 6 & -2 \end{pmatrix}^T \\ &= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{8} & \frac{1}{4} & -\frac{3}{8} \\ \frac{3}{8} & -\frac{1}{4} & \frac{1}{8} \end{pmatrix} \end{aligned}$$

2.6.15

$$\left(\begin{array}{cc|cc} 6 & -2 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & -\frac{1}{3} & \frac{1}{6} & 0 \\ 0 & 1 & 0 & \frac{1}{4} \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{6} & \frac{1}{12} \\ 0 & 1 & 0 & \frac{1}{4} \end{array}\right)$$

2.6.17

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right)$$
$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -12 & -5 & 1 \end{array}\right)$$
$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{5}{12} & -\frac{1}{12} \end{array}\right)$$
$$\left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & \frac{5}{12} & -\frac{1}{12} \end{array}\right)$$

2.6.27

$$\begin{aligned} (\mathbf{AB})^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} \\ &= \begin{pmatrix} \frac{2}{3} & \frac{4}{3} \\ -\frac{1}{3} & \frac{5}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{5}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ -\frac{17}{12} & \frac{55}{12} \end{pmatrix} \end{aligned}$$

2.6.29

$$\begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$$

2.6.31

$$\begin{aligned} \begin{pmatrix} 4 & -3 \\ x & -4 \end{pmatrix} &= \frac{1}{3x-16} \begin{pmatrix} -4 & -x \\ 3 & 4 \end{pmatrix}^T \\ &= \frac{1}{3x-16} \begin{pmatrix} -4 & 3 \\ -x & 4 \end{pmatrix} \\ -1 &= \frac{1}{3x-16} \\ 16-3x &= 1 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

2.6.45

$$\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}4\\ 14\end{pmatrix}$$
$$\begin{aligned}\begin{pmatrix}x_1\\ x_2\end{pmatrix} &= -\frac{1}{3}\begin{pmatrix}-1 & -1\\ -2 & 1\end{pmatrix}\begin{pmatrix}4\\ 14\end{pmatrix}\\ &= -\frac{1}{3}\begin{pmatrix}-18\\ 6\end{pmatrix}\\ &= \begin{pmatrix}6\\ -2\end{pmatrix}\end{aligned}$$

2.6.49

$$\begin{pmatrix}1 & 0 & 1\\ 1 & 1 & 1\\ 5 & -1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}-4\\ 0\\ 6\end{pmatrix}$$
$$\begin{aligned}\det \mathbf{A} &= \begin{vmatrix}1 & 1\\ -1 & 0\end{vmatrix} + \begin{vmatrix}1 & 1\\ 5 & -1\end{vmatrix}\\ &= 1 - 6\\ &= -5\end{aligned}$$
$$\begin{aligned}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} &= -\frac{1}{5}\begin{pmatrix}1 & 5 & -6\\ -1 & -5 & 1\\ -1 & 0 & 1\end{pmatrix}^T\begin{pmatrix}-4\\ 0\\ 6\end{pmatrix}\\ &= -\frac{1}{5}\begin{pmatrix}1 & -1 & -1\\ 5 & -5 & 0\\ -6 & 1 & 1\end{pmatrix}\begin{pmatrix}-4\\ 0\\ 6\end{pmatrix}\\ &= -\frac{1}{5}\begin{pmatrix}-10\\ -20\\ 30\end{pmatrix}\\ &= \begin{pmatrix}2\\ 4\\ -6\end{pmatrix}\end{aligned}$$

2.6.55

$$\begin{aligned}\det\begin{pmatrix}1 & 2 & -1\\ 4 & -1 & 1\\ 5 & 1 & -2\end{pmatrix} &= (1)\begin{vmatrix}-1 & 1\\ 1 & -2\end{vmatrix} - (2)\begin{vmatrix}4 & 1\\ 5 & -2\end{vmatrix} + (-1)\begin{vmatrix}4 & -1\\ 5 & 1\end{vmatrix}\\ &= 1 + 26 - 9\\ &= 18\end{aligned}$$

Only trivial solution

2.7 Cramer's Rule

2.7.1

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix} \\
\mathbf{B} &= \begin{pmatrix} 3 \\ -6 \end{pmatrix} \\
\det \mathbf{A} &= 10 \\
\det \mathbf{A}_1 &= -6 \\
\det \mathbf{A}_2 &= 12 \\
x_1 &= -\frac{3}{5} \\
x_2 &= \frac{6}{5}
\end{aligned}
$$

2.7.11

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 2-k & k \\ k & 3-k \end{pmatrix} \\
\mathbf{B} &= \begin{pmatrix} 4 \\ 3 \end{pmatrix} \\
\det \mathbf{A} &= (2-k)(3-k) - k^2 \\
&= 6 - 5k \\
\det \mathbf{A}_1 &= 4(3-k) - 3k \\
&= 12 - 7k \\
\det \mathbf{A}_2 &= 3(2-k) - 4k \\
&= 6 - 7k \\
x_1 &= \frac{12-7k}{6-5k} \\
x_2 &= \frac{6-7k}{6-5k}
\end{aligned}
$$

The system is inconsistent when $k = \frac{6}{5}$

2.7.13

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} \cos 25^\circ & -\cos 15^\circ \\ \sin 25^\circ & \sin 15^\circ \end{pmatrix} \\
\mathbf{B} &= \begin{pmatrix} 0 \\ 300 \end{pmatrix} \\
\det \mathbf{A} &= \cos 25^\circ \sin 15^\circ + \cos 15^\circ \sin 25^\circ \\
&= \sin 40^\circ \\
\det \mathbf{A}_1 &= 300 \cos 15^\circ \\
\det \mathbf{A}_2 &= 300 \cos 25^\circ \\
T_1 &= \frac{300 \cos 15^\circ}{\sin 40^\circ} \\
&\approx 451\,\mathrm{lb} \\
T_2 &= \frac{300 \cos 25^\circ}{\sin 40^\circ} \\
&\approx 423\,\mathrm{lb}
\end{aligned}$$

2.8 The Eigenvalue Problem

2.8.1

$\mathbf{K}_3$ with $\lambda = -1$

2.8.3

$\mathbf{K}_3$ with $\lambda = 0$

2.8.5

$\mathbf{K}_2$ with $\lambda = 3$
$\mathbf{K}_3$ with $\lambda = 1$

2.8.7

$$\begin{aligned}
\det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} -1-\lambda & 2 \\ -7 & 8-\lambda \end{vmatrix} \\
&= (-1-\lambda)(8-\lambda) + 14 \\
&= -8 + \lambda - 8\lambda + \lambda^2 + 14 \\
&= \lambda^2 - 7\lambda + 6 \\
&= (\lambda - 1)(\lambda - 6) \\
\lambda_1 &= 1 \\
\lambda_2 &= 6
\end{aligned}$$

$$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ -7 & 7 & 0 \end{array}\right)$$

$x_1 = x_2$

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -7 & 2 & 0 \\ -7 & 2 & 0 \end{array}\right)$$

$x_1 = \frac{2}{7}x_2$

$$\mathbf{X}_2 = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

Nonsingular

2.8.9

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} -8-\lambda & -1 \\ 16 & -\lambda \end{vmatrix} \\ &= -\lambda(-8-\lambda) + 16 \\ &= 8\lambda + \lambda^2 + 16 \\ &= (\lambda+4)^2 \\ \lambda_1 = \lambda_2 &= -4 \end{aligned}$$

$$\left(\begin{array}{cc|c} -4 & -1 & 0 \\ 16 & 4 & 0 \end{array}\right)$$

$x_1 = -\frac{1}{4}x_2$

$$\mathbf{X}_1 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

Nonsingular

2.8.11

$$\begin{aligned}
\det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} -1-\lambda & 2 \\ -5 & 1-\lambda \end{vmatrix} \\
&= (-1-\lambda)(1-\lambda) + 10 \\
&= -1 + \lambda - \lambda + \lambda^2 + 10 \\
&= \lambda^2 + 9 \\
&= (\lambda - 3i)(\lambda + 3i) \\
\lambda_1 &= 3i \\
\lambda_2 &= -3i
\end{aligned}$$

$$\left(\begin{array}{cc|c} -1-3i & 2 & 0 \\ -5 & 1-3i & 0 \end{array}\right)$$

$$\mathbf{X}_1 = \begin{pmatrix} 1-3i \\ 5 \end{pmatrix}$$

$$\mathbf{X}_2 = \begin{pmatrix} 1+3i \\ 5 \end{pmatrix}$$

Nonsingular

2.8.23

$$\begin{aligned}
\det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{vmatrix} \\
&= (5-\lambda)^2 - 1 \\
&= 25 - 10\lambda + \lambda^2 - 1 \\
&= \lambda^2 - 10\lambda + 24 \\
&= (\lambda - 4)(\lambda - 6) \\
\lambda_1 &= 4 \\
\lambda_2 &= 6
\end{aligned}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\mathbf{X_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{aligned} \lambda_1' &= \frac{1}{4} \\ \lambda_2' &= \frac{1}{6} \\ \mathbf{X}_1' &= \mathbf{X}_1 \\ \mathbf{X}_2' &= \mathbf{X}_2 \end{aligned}$$

2.9 Powers of Matrices

2.9.1

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= (1-\lambda)(5-\lambda) + 8 \\ &= 5 - \lambda - 5\lambda + \lambda^2 + 8 \\ &= \lambda^2 - 6\lambda + 13 \\ \mathbf{A}^2 &= 6\mathbf{A} - 13\mathbf{I} \\ \begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix} &= 6 \begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix} - 13 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} -7 & -12 \\ 24 & 17 \end{pmatrix} &= \begin{pmatrix} -7 & -12 \\ 24 & 17 \end{pmatrix} \end{aligned}$$

2.9.3

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix} \\
\det(\mathbf{A} - \lambda\mathbf{I}) &= (-1-\lambda)(4-\lambda) - 6 \\
&= -4 + \lambda - 4\lambda + \lambda^2 - 6 \\
&= \lambda^2 - 3\lambda - 10 \\
&= (\lambda - 5)(\lambda + 2)
\end{aligned}
$$

$$
\begin{aligned}
\lambda^m &= c_0 + c_1\lambda \\
(-2)^m &= c_0 - 2c_1 \\
(5)^m &= c_0 + 5c_1
\end{aligned}
$$

$$
\begin{aligned}
(-2)^m + \frac{2}{5}(5)^m &= \frac{7}{5}c_0 \\
\frac{5}{7}(-2)^m + \frac{2}{7}(5)^m &= c_0
\end{aligned}
$$

$$
\begin{aligned}
(5)^m - (-2)^m &= 7c_1 \\
\frac{1}{7}(5)^m - \frac{1}{7}(-2)^m &= c_1
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A}^m &= \frac{1}{7}\begin{pmatrix} 5^m + 6(-2)^m & 3(5)^m - 3(-2)^m \\ 2(5)^m - 2(-2)^m & 6(5)^m + (-2)^m \end{pmatrix} \\
\mathbf{A}^3 &= \frac{1}{7}\begin{pmatrix} 5^3 + 6(-2)^3 & 3(5)^3 - 3(-2)^3 \\ 2(5)^3 - 2(-2)^3 & 6(5)^3 + (-2)^3 \end{pmatrix} \\
&= \begin{pmatrix} 11 & 57 \\ 38 & 106 \end{pmatrix}
\end{aligned}
$$

2.9.5

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 8 & 5 \\ 4 & 0 \end{pmatrix} \\
\det(\mathbf{A} - \lambda\mathbf{I}) &= -\lambda(8-\lambda) - 20 \\
&= -8\lambda + \lambda^2 - 20 \\
&= \lambda^2 - 8\lambda - 20 \\
&= (\lambda - 10)(\lambda + 2)
\end{aligned}
$$

$$
\begin{aligned}
\lambda^m &= c_0 + c_1\lambda \\
(-2)^m &= c_0 - 2c_1 \\
10^m &= c_0 + 10c_1
\end{aligned}
$$

$$
\begin{aligned}
10^m + 5(-2)^m &= 6c_0 \\
\frac{1}{6}(10^m + 5(-2)^m) &= c_0 \\
\frac{1}{12}(2 \cdot 10^m + 10(-2)^m) &= c_0
\end{aligned}
$$

$$
\begin{aligned}
10^m - (-2)^m &= 12c_1 \\
\frac{1}{12}(10^m - (-2)^m) &= c_1
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A}^m &= \frac{1}{12}\begin{pmatrix} 10 \cdot 10^m + 2(-2)^m & 5 \cdot 10^m - 5(-2)^m \\ 4 \cdot 10^m - 4(-2)^m & 2 \cdot 10^m + 10(-2)^m \end{pmatrix} \\
\mathbf{A}^5 &= \begin{pmatrix} 83328 & 41680 \\ 33344 & 16640 \end{pmatrix}
\end{aligned}
$$

2.9.11

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 7 & 3 \\ -3 & 1 \end{pmatrix} \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= (7-\lambda)(1-\lambda) + 9 \\
&= 7 - 7\lambda - \lambda + \lambda^2 + 9 \\
&= \lambda^2 - 8\lambda + 16 \\
&= (\lambda - 4)^2 \\
\\
\lambda^m &= c_0 + c_1 \lambda \\
m\lambda^{m-1} &= c_1 \\
\\
4^{m-1} m &= c_1 \\
\\
4^m &= c_0 + 4^m m \\
4^m (1-m) &= c_0 \\
\\
\mathbf{A}^m &= \begin{pmatrix} 4^m(1-m) + 7 \cdot 4^{m-1} m & 3 \cdot 4^{m-1} m \\ -3 \cdot 4^{m-1} m & 4^m(1-m) + 4^{m-1} m \end{pmatrix} \\
&= 4^m \begin{pmatrix} 1 - m + \frac{7}{4} m & \frac{3}{4} m \\ -\frac{3}{4} m & 1 - m + \frac{1}{4} m \end{pmatrix} \\
&= 4^m \begin{pmatrix} 1 + \frac{3}{4} m & \frac{3}{4} m \\ -\frac{3}{4} m & 1 - \frac{3}{4} m \end{pmatrix} \\
\mathbf{A}^6 &= 4^6 \begin{pmatrix} 1 + \frac{3}{4} 6 & \frac{3}{4} 6 \\ -\frac{3}{4} 6 & 1 - \frac{3}{4} 6 \end{pmatrix} \\
&= 4^5 \begin{pmatrix} 4 + 18 & 18 \\ -18 & 4 - 18 \end{pmatrix} \\
&= 4^5 \begin{pmatrix} 22 & 18 \\ -18 & -14 \end{pmatrix} \\
&= \begin{pmatrix} 22528 & 18432 \\ -18432 & -14336 \end{pmatrix}
\end{aligned}
$$

2.9.13

(a)

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \\
\det(\mathbf{A} - \lambda\mathbf{I}) &= (1-\lambda)(3-\lambda) - 3 \\
&= 3 - \lambda - 3\lambda + \lambda^2 - 3 \\
&= \lambda^2 - 4\lambda \\
&= \lambda(\lambda - 4)
\end{aligned}
$$

$$
\begin{aligned}
\lambda^m &= c_1\lambda \\
4^m &= 4c_1 \\
c_1 &= 4^{m-1}
\end{aligned}
$$

$$
\mathbf{A}^m = 4^{m-1}\mathbf{A}
$$

2.9.15

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix} \\
\det(\mathbf{A} - \lambda\mathbf{I}) &= (2-\lambda)(3-\lambda) + 4 \\
&= 6 - 2\lambda - 3\lambda + \lambda^2 + 4 \\
&= \lambda^2 - 5\lambda + 10 \\
10\mathbf{I} &= 5\mathbf{A} - \mathbf{A}^2 \\
\mathbf{I} &= \frac{1}{2}\mathbf{A} - \frac{1}{10}\mathbf{A}^2 \\
\mathbf{A}^{-1} &= \frac{1}{2}\mathbf{I} - \frac{1}{10}\mathbf{A} \\
&= \begin{pmatrix} \frac{3}{10} & \frac{2}{5} \\ -\frac{1}{10} & \frac{1}{5} \end{pmatrix}
\end{aligned}
$$

2.10 Orthogonal Matrices

2.10.5

Orthogonal

2.10.7

Orthogonal

2.10.9

Not orthogonal

2.10.11

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 9 \\ 9 & 1 \end{pmatrix} \\
\det(\mathbf{A} - \lambda\mathbf{I}) &= (1-\lambda)^2 - 81 \\
&= \lambda^2 - 2\lambda + 1 - 81 \\
&= \lambda^2 - 2\lambda - 80 \\
&= (\lambda - 10)(\lambda + 8)
\end{aligned}
$$

$$\begin{pmatrix} 9 & 9 \\ 9 & 9 \end{pmatrix}$$

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -9 & 9 \\ 9 & -9 \end{pmatrix}$$

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

2.10.13

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \\
\det(\mathbf{A} - \lambda\mathbf{I}) &= (1-\lambda)(9-\lambda) - 9 \\
&= 9 - \lambda - 9\lambda + \lambda^2 - 9 \\
&= \lambda^2 - 10\lambda \\
&= \lambda(\lambda - 10)
\end{aligned}
$$

$$\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$$

$$\mathbf{X}_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -9 & 3 \\ 3 & -1 \end{pmatrix}$$

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}$$

2.10.19

$$\begin{aligned} \frac{3}{5}a + \frac{4}{5}b &= 0 \\ 3a &= -4b \\ a &= -\frac{4}{3}b \\ a &= -\frac{4}{5} \\ b &= \frac{3}{5} \end{aligned}$$

2.10.21

(b) $\lambda_1 = -2, \lambda_2 = -2, \lambda_3 = 4$

(c)

$$\begin{aligned}
\mathbf{W}_1 &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \\
\mathbf{V}_2 &= \mathbf{K}_2 - (\mathbf{K}_2 \cdot \mathbf{W}_1)\mathbf{W}_1 \\
&= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix} \\
|\mathbf{V}_2| &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (-1)^2} \\
&= \sqrt{\frac{3}{2}} \\
\mathbf{W}_2 &= \sqrt{\frac{2}{3}} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix}
\end{aligned}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

2.11 Approximation of Eigenvalues

2.11.1

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \\
\mathbf{X}_0 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathbf{A}^5\mathbf{X}_0 &= 32 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathbf{K}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\lambda_1 &= \frac{\mathbf{A}\mathbf{X}_5 \cdot \mathbf{X}_5}{\mathbf{X}_5 \cdot \mathbf{X}_5} \\
&= 2
\end{aligned}$$

2.11.3

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 2 & 4 \\ 3 & 13 \end{pmatrix} \\
\mathbf{X}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathbf{X}_2 &= \begin{pmatrix} \frac{3}{8} \\ 1 \end{pmatrix} \\
\mathbf{X}_3 &= \begin{pmatrix} 0.3363 \\ 1 \end{pmatrix} \\
\mathbf{X}_4 &= \begin{pmatrix} 0.3335 \\ 1 \end{pmatrix} \\
\mathbf{X}_5 &= \begin{pmatrix} 0.3333 \\ 1 \end{pmatrix} \\
\mathbf{K}_1 &= \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \\
\lambda_1 &= \frac{\mathbf{A}\mathbf{K}_1 \cdot \mathbf{K}_1}{\mathbf{K}_1 \cdot \mathbf{K}_1} \\
&= 14
\end{aligned}$$

2.11.7

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} \\
\mathbf{X}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathbf{A}^5\mathbf{X}_1 &= \begin{pmatrix} 0.5008 \\ 1 \end{pmatrix} \\
\lambda_1 &= 7 \\
\mathbf{K}_1 &= \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \\
\mathbf{B} &= \mathbf{A} - \lambda_1 \mathbf{K}_1 \mathbf{K}_1^T \\
&= \begin{pmatrix} \frac{8}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{2}{5} \end{pmatrix} \\
\mathbf{B}^5\mathbf{X}_1 &= \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \\
\lambda_2 &= 2
\end{aligned}$$

2.11.11

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} \\
\det \mathbf{A} &= 1 \\
\mathbf{A}^{-1} &= \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix} \\
\mathbf{X}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
(\mathbf{A}^{-1})^5\mathbf{X}_1 &= \begin{pmatrix} 1 \\ -0.7913 \end{pmatrix} \\
\lambda_1' &\approx 4.78 \\
\lambda_1 &\approx 0.21
\end{aligned}$$

2.11.13

(a)

$$\begin{aligned}
EI\frac{d^2y}{dx^2} + Py &= 0 \\
EI\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + Py_i &= 0 \\
EI(y_{i+1} - 2y_i + y_{i-1}) + Ph^2 y_i &= 0
\end{aligned}$$

(b)

$$\begin{aligned} EI(y_2 - 2y_1) + Ph^2 y_1 &= 0 \\ EI(2y_1 - y_2) &= Ph^2 y_1 \end{aligned}$$

$$\begin{aligned} EI(y_3 - 2y_2 + y_1) + Ph^2 y_2 &= 0 \\ EI(-y_1 + 2y_2 - y_3) &= Ph^2 y_2 \end{aligned}$$

$$\begin{aligned} EI(-2y_3 + y_2) + Ph^2 y_3 &= 0 \\ EI(-y_2 + 2y_3) &= Ph^2 y_3 \end{aligned}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \frac{PL^2}{16EI} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

(c)

$$\begin{aligned} \det \mathbf{A} &= 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} \\ &= 4 \\ \mathbf{A}^{-1} &= \frac{1}{\det \mathbf{A}} \operatorname{adj} \mathbf{A} \\ &= \frac{1}{4} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}^T \\ &= \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix} \end{aligned}$$

(d)

$$\begin{aligned} \mathbf{X}_1 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ (\mathbf{A}^{-1})^6 \mathbf{X}_1 &= \begin{pmatrix} 0.7071 \\ 1 \\ 0.7071 \end{pmatrix} \\ \lambda_1' &\approx 1.7071 \\ \lambda_1 &\approx 0.59 \end{aligned}$$

(e)

$$\lambda_1 = \frac{PL^2}{16EI}$$
$$P = \frac{16EI\lambda_1}{L^2}$$
$$\approx \frac{9.44EI}{L^2}$$

2.12 Diagonalization

2.12.1

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \\
\det(\mathbf{A} - \lambda\mathbf{I}) &= (2-\lambda)(4-\lambda) - 3 \\
&= 8 - 2\lambda - 4\lambda + \lambda^2 - 3 \\
&= \lambda^2 - 6\lambda + 5 \\
&= (\lambda - 5)(\lambda - 1) \\
\lambda_1 &= 1 \\
\lambda_2 &= 5
\end{aligned}$$

$$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$$

$$\mathbf{X}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}$$

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{aligned}
\mathbf{P} &= \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \\
\mathbf{P}^{-1} &= \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \\
\mathbf{D} &= \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}
\end{aligned}$$

2.12.3

Not diagonalisable

2.12.5

$$\mathbf{P} = \begin{pmatrix} 13 & 1 \\ 2 & 1 \end{pmatrix}$$
$$\mathbf{D} = \begin{pmatrix} -7 & 0 \\ 0 & 4 \end{pmatrix}$$

2.12.7

$$\mathbf{P} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$\mathbf{D} = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

2.12.21

$$\mathbf{P} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

2.12.23

$$\mathbf{P} = \begin{pmatrix} \sqrt{\frac{2}{5}} & -\sqrt{\frac{5}{2}} \\ 1 & 1 \end{pmatrix}$$
$$\mathbf{D} = \begin{pmatrix} 10 & 0 \\ 0 & 3 \end{pmatrix}$$

2.12.35

$$\begin{aligned} \mathbf{A} &= \mathbf{PDP}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \end{aligned}$$

2.12.39

$$
\begin{aligned}
\mathbf{D} &= \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \\
\mathbf{P} &= \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \\
\mathbf{A}^5 &= \mathbf{P}\mathbf{D}^5\mathbf{P}^{-1} \\
&= \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 32 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \\
&= \begin{pmatrix} 21 & 11 \\ 22 & 10 \end{pmatrix}
\end{aligned}
$$

2.13 LU-Factorisation

2.13.1

$$
\begin{aligned}
\mathbf{A} &= \mathbf{L}\mathbf{U} \\
\begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} \\
&= \begin{pmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \end{pmatrix} \\
u_{11} &= 2 \\
u_{12} &= -2 \\
l_{21} &= \frac{1}{2} \\
u_{22} &= 3 \\
\mathbf{L} &= \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \\
\mathbf{U} &= \begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix}
\end{aligned}
$$

2.13.3

$$
\begin{aligned}
\begin{pmatrix} -1 & 4 \\ 2 & 2 \end{pmatrix} &= \begin{pmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \end{pmatrix} \\
u_{11} &= -1 \\
u_{12} &= 4 \\
l_{21} &= -2 \\
u_{22} &= 10 \\
\mathbf{L} &= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \\
\mathbf{U} &= \begin{pmatrix} -1 & 4 \\ 0 & 10 \end{pmatrix}
\end{aligned}
$$

2.13.11

$$
\begin{aligned}
\begin{pmatrix} 3 & 9 \\ 1 & 11 \end{pmatrix} &\Rightarrow \begin{pmatrix} 3 & 9 \\ 0 & 8 \end{pmatrix} \\
\mathbf{L} &= \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{pmatrix} \\
\mathbf{U} &= \begin{pmatrix} 3 & 9 \\ 0 & 8 \end{pmatrix}
\end{aligned}
$$

2.13.13

$$
\begin{aligned}
\begin{pmatrix} -4 & -2 \\ -1 & -3 \end{pmatrix} &\Rightarrow \begin{pmatrix} -4 & -2 \\ 0 & -\frac{5}{2} \end{pmatrix} \\
\mathbf{L} &= \begin{pmatrix} 1 & 0 \\ \frac{1}{4} & 1 \end{pmatrix} \\
\mathbf{U} &= \begin{pmatrix} -4 & -2 \\ 0 & -\frac{5}{2} \end{pmatrix}
\end{aligned}
$$

2.13.21

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix}$$

$$\mathbf{LY} = \mathbf{B}$$

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$y_1 = 1$$

$$y_2 = -\frac{5}{2}$$

$$\mathbf{UX} = \mathbf{Y}$$

$$\begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{5}{2} \end{pmatrix}$$

$$x_2 = -\frac{5}{6}$$

$$2x_1 - 2\left(-\frac{5}{6}\right) = 1$$

$$2x_1 = -\frac{2}{3}$$

$$x_1 = -\frac{1}{3}$$

$$\mathbf{X} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{5}{6} \end{pmatrix}$$

2.13.23

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} -1 & 4 \\ 0 & 10 \end{pmatrix}$$

$$\mathbf{LY} = \mathbf{B}$$

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix}$$

$$y_1 = 15$$

$$y_2 = 35$$

$$\mathbf{UX} = \mathbf{Y}$$

$$\begin{pmatrix} -1 & 4 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 35 \end{pmatrix}$$

$$x_2 = \frac{7}{2}$$

$$-x_1 + 4\frac{7}{2} = 15$$

$$x_1 = -1$$

$$\mathbf{X} = \begin{pmatrix} -1 \\ \frac{7}{2} \end{pmatrix}$$

2.13.31

$$\begin{aligned}
\mathbf{LY} &= \mathbf{B}\\
\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix} &= \begin{pmatrix}2\\4\\1\end{pmatrix}\\
y_1 &= 2\\
y_2 &= 2\\
y_3 &= -3\\
\mathbf{UX} &= \mathbf{Y}\\
\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} &= \begin{pmatrix}2\\2\\-3\end{pmatrix}\\
x_3 &= -3\\
x_2 &= 5\\
x_1 &= 0\\
\mathbf{X} &= \begin{pmatrix}0\\5\\-3\end{pmatrix}
\end{aligned}$$

2.14 Cryptography

2.14.1

$$\begin{aligned}
\mathbf{B} &= \mathbf{AM}\\
&= \begin{pmatrix}1&2\\1&1\end{pmatrix}\begin{pmatrix}19&5&14&4&0\\8&5&12&16&0\end{pmatrix}\\
&= \begin{pmatrix}35&15&38&36&0\\27&10&26&20&0\end{pmatrix}
\end{aligned}$$

2.14.7

$$\begin{aligned}
\mathbf{A}^{-1} &= \begin{pmatrix}2&-3\\-5&8\end{pmatrix}\\
\mathbf{M} &= \mathbf{A^{-1}B}\\
&= \begin{pmatrix}19&20&21&4&25\\0&8&1&18&4\end{pmatrix}\\
&= \text{STUDY HARD}
\end{aligned}$$

2.14.11

$$\mathbf{A^{-1}B} = \mathbf{M}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 17 & 16 \\ -30 & -31 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ m_{21} & m_{22} \end{pmatrix}$$

$$17a_{11} - 30a_{12} = 4$$

$$16a_{11} - 31a_{12} = 1$$

$$17a_{11} - \frac{30}{31}16a_{11} = 4 - \frac{30}{31}$$

$$527a_{11} - 480a_{11} = 124 - 30$$

$$47a_{11} = 94$$

$$a_{11} = 2$$

$$-30a_{12} + \frac{17}{16}31a_{12} = 4 - \frac{17}{16}$$

$$-480a_{12} + 527a_{12} = 64 - 17$$

$$47a_{12} = 47$$

$$a_{12} = 1$$

$$\begin{pmatrix} 2 & 1 \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 5 & 25 \\ -6 & -50 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ 1 & 25 \end{pmatrix}$$

$$5a_{21} - 6a_{22} = 1$$

$$25a_{21} - 50a_{22} = 25$$

$$5a_{21} - \frac{6}{50}25a_{21} = 1 - \frac{6}{50}25$$

$$250a_{21} - 150a_{21} = 50 - 150$$

$$100a_{21} = -100$$

$$a_{21} = -1$$

$$4a_{22} = -4$$

$$a_{22} = -1$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\mathbf{A^{-1}B} = \text{DAD I NEED MONEY TODAY}$$

2.15 Chapter in Review

2.15.1

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{pmatrix}$$

2.15.3

$$\mathbf{AB} = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$$
$$\mathbf{BA} = \begin{pmatrix} 11 \end{pmatrix}$$

2.15.5

False

2.15.7

$$\det\left(\frac{1}{2}\mathbf{A}\right) = \frac{5}{8}$$
$$\det -\mathbf{A}^T = -5$$

2.15.9

0

2.15.11

False

2.15.13

True

2.15.15

False

2.15.17

True

2.15.19

False

2.15.23

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}^2$$

2.15.25

$$\left(\begin{array}{ccc|c} 5 & -1 & 1 & -9 \\ 2 & 4 & 0 & 27 \\ 1 & 1 & 5 & 9 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{5} & \frac{1}{5} & -\frac{9}{5} \\ 1 & 2 & 0 & \frac{27}{2} \\ 1 & 1 & 5 & 9 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{5} & \frac{1}{5} & -\frac{9}{5} \\ 0 & \frac{11}{5} & -\frac{1}{5} & \frac{153}{10} \\ 0 & \frac{6}{5} & \frac{24}{5} & \frac{54}{5} \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{5} & \frac{1}{5} & -\frac{9}{5} \\ 0 & 1 & -\frac{1}{11} & \frac{153}{22} \\ 0 & 1 & 4 & 9 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{5} & \frac{1}{5} & -\frac{9}{5} \\ 0 & 1 & -\frac{1}{11} & \frac{153}{22} \\ 0 & 0 & \frac{45}{11} & \frac{45}{22} \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{5} & \frac{1}{5} & -\frac{9}{5} \\ 0 & 1 & -\frac{1}{11} & \frac{153}{22} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{5} & 0 & -\frac{19}{10} \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$$

$$\mathbf{X} = \begin{pmatrix} -\frac{1}{2} \\ 7 \\ \frac{1}{2} \end{pmatrix}$$

2.15.29

240

2.15.31

$$\begin{vmatrix} 1 & -1 & 1 \\ 5 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = (1)\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + (1)\begin{vmatrix} 5 & -1 \\ 1 & 1 \end{vmatrix} + (1)\begin{vmatrix} 5 & 1 \\ 1 & 2 \end{vmatrix}$$
$$= 18$$

The matrix is nonsingular so the system only has the trivial solution.

2.15.35

$$\begin{aligned}
\det \mathbf{A} &= \begin{vmatrix} 1 & 2 & -3 \\ 2 & -4 & 3 \\ 0 & 4 & 6 \end{vmatrix} \\
&= (1)\begin{vmatrix} -4 & 3 \\ 4 & 6 \end{vmatrix} - (2)\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} + (-3)\begin{vmatrix} 2 & -4 \\ 0 & 4 \end{vmatrix} \\
&= -36 - 24 - 24 \\
&= -84 \\
x_1 &= \frac{\det \mathbf{A}_1}{\det \mathbf{A}} \\
&= -\frac{1}{2} \\
x_2 &= \frac{\det \mathbf{A}_2}{\det \mathbf{A}} \\
&= \frac{1}{4} \\
x_3 &= \frac{\det \mathbf{A}_3}{\det \mathbf{A}} \\
&= \frac{2}{3}
\end{aligned}$$

2.15.37

$$
\begin{aligned}
\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} X \\ Y \end{pmatrix} \\
x &= \frac{\begin{vmatrix} X & \sin\theta \\ Y & \cos\theta \end{vmatrix}}{\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}} \\
&= X\cos\theta - Y\sin\theta \\
y &= \begin{vmatrix} \cos\theta & X \\ -\sin\theta & Y \end{vmatrix} \\
&= X\sin\theta + Y\cos\theta
\end{aligned}
$$

2.15.39

$$
\begin{aligned}
\begin{pmatrix} 2 & 3 & -1 \\ 1 & -2 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 6 \\ -3 \\ 9 \end{pmatrix} \\
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \frac{1}{3} \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 1 \\ 4 & 6 & 7 \end{pmatrix} \begin{pmatrix} 6 \\ -3 \\ 9 \end{pmatrix} \\
&= \begin{pmatrix} 7 \\ 5 \\ 23 \end{pmatrix}
\end{aligned}
$$

2.15.41

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= (1-\lambda)(3-\lambda) - 8 \\
&= 3 - \lambda - 3\lambda + \lambda^2 - 8 \\
&= \lambda^2 - 4\lambda - 5 \\
&= (\lambda - 5)(\lambda + 1) \\
\lambda_1 &= -1 \\
\lambda_2 &= 5
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \\
\mathbf{K}_1 &= \begin{pmatrix} 1 & -1 \end{pmatrix} \\
\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \\
\mathbf{K}_2 &= \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}
\end{aligned}$$

2.15.43

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \\
\det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} \\
&= (3-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & 3-\lambda \end{vmatrix} + 4\begin{vmatrix} 2 & -\lambda \\ 4 & 2 \end{vmatrix} \\
&= (3-\lambda)(-\lambda(3-\lambda)-4) - 2(2(3-\lambda)-8) + 4(4+4\lambda) \\
&= (3-\lambda)(\lambda^2 - 3\lambda - 4) - 2(-2-2\lambda) + 16 + 16\lambda \\
&= 3\lambda^2 - 9\lambda - 12 - \lambda^3 + 3\lambda^2 + 4\lambda + 4 + 4\lambda + 16 + 16\lambda \\
&= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 \\
&= -(\lambda - 8)(\lambda + 1)^2
\end{aligned}$$

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{2}{5} & -\frac{4}{5} \\ 1 & -4 & 1 \\ 1 & \frac{1}{2} & -\frac{5}{4} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & -\frac{18}{5} & \frac{9}{5} \\ 0 & \frac{9}{10} & -\frac{9}{20} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{X}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$
$$\mathbf{X}_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

2.15.47

$$a^2 + b^2 + c^2 = 1$$

$$-a\frac{1}{\sqrt{2}} + c\frac{1}{\sqrt{2}} = 0$$
$$a = c$$

$$a\frac{1}{\sqrt{3}} + b\frac{1}{\sqrt{3}} + c\frac{1}{\sqrt{3}} = 0$$
$$a + b + c = 0$$

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

2.15.57

$$\begin{pmatrix}1 & 1 & 1\\ 1 & -2 & 3\\ 2 & 0 & -3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}6\\ 2\\ 3\end{pmatrix}$$

$$\begin{aligned}
\mathbf{A} &= \mathbf{LU}\\
\begin{pmatrix}1 & 1 & 1\\ 1 & -2 & 3\\ 2 & 0 & -3\end{pmatrix} &= \begin{pmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{pmatrix}\begin{pmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{pmatrix}\\
&= \begin{pmatrix}u_{11} & u_{12} & u_{13}\\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23}\\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{pmatrix}\\
u_{11} &= 1\\
u_{12} &= 1\\
u_{13} &= 1\\
l_{21} &= 1\\
u_{22} &= -3\\
u_{23} &= 2\\
l_{31} &= 2\\
l_{32} &= \frac{2}{3}\\
u_{33} &= -\frac{19}{3}\\
\mathbf{L} &= \begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 2 & \frac{2}{3} & 1\end{pmatrix}\\
\mathbf{U} &= \begin{pmatrix}1 & 1 & 1\\ 0 & -3 & 2\\ 0 & 0 & -\frac{19}{3}\end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
\mathbf{AX} &= \mathbf{B} \\
\mathbf{LUX} &= \mathbf{B} \\
\mathbf{LY} &= \mathbf{B} \\
\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & \frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &= \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} \\
\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &= \begin{pmatrix} 6 \\ -4 \\ -\frac{19}{3} \end{pmatrix} \\
\mathbf{UX} &= \mathbf{Y} \\
\begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & -\frac{19}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 6 \\ -4 \\ -\frac{19}{3} \end{pmatrix} \\
\mathbf{X} &= \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}
\end{aligned}
$$

3 Vector Calculus

3.1 Vector Functions

3.1.11

$$
\begin{aligned}
x &= t \\
y &= t \\
z &= 2t^2 \\
\mathbf{r} &= \langle t, t, 2t^2 \rangle
\end{aligned}
$$

3.1.13

$$
\begin{aligned}
x &= 3\cos t \\
z &= 9 - 9\cos^2 t \\
&= 9(1 - \cos^2 t) \\
&= 9\sin^2 t \\
9\cos^2 t + y^2 &= 9 \\
y^2 &= 9\sin^2 t \\
y &= 3\sin t \\
\mathbf{r} &= \langle 3\cos t, 3\sin t, 9\sin^2 t \rangle
\end{aligned}
$$

3.1.15

$$\begin{aligned}\mathbf{r}(t) &= \langle \frac{\sin 2t}{t}, (t-2)^5, t\ln t\rangle \\ \lim_{t\to 0^+}\mathbf{r}(t) &= \langle 2, -32, 0\rangle\end{aligned}$$

3.1.17

$$\begin{aligned}\mathbf{r}(t) &= \langle \ln t, 1, 0\rangle \\ \mathbf{r}'(t) &= \langle \frac{1}{t}, 0, 0\rangle \\ \mathbf{r}''(t) &= \langle -\frac{1}{t^2}, 0, 0\rangle\end{aligned}$$

3.1.19

$$\begin{aligned}\mathbf{r}(t) &= \langle te^{2t}, t^3, 4t^2-t\rangle \\ \mathbf{r}'(t) &= \langle e^{2t}+2te^{2t}, 3t^2, 8t-1\rangle \\ \mathbf{r}''(t) &= \langle 4e^{2t}+4te^{2t}, 6t, 8\rangle\end{aligned}$$

3.1.25

$$\begin{aligned}x &= 2+t \\ y &= 2+2t \\ z &= \frac{8}{3}+4t\end{aligned}$$

3.1.27

$$\frac{d}{dt}[\mathbf{r}(t)\times\mathbf{r}'(t)] = \mathbf{r}(t)\times\mathbf{r}''(t)$$

3.1.29

$$\begin{aligned}\frac{d}{dt}[\mathbf{r}(t)\cdot(\mathbf{r}'(t)\times\mathbf{r}''(t))] &= \mathbf{r}(t)\cdot\frac{d}{dt}(\mathbf{r}'(t)\times\mathbf{r}''(t)) \\ &= \mathbf{r}(t)\cdot(\mathbf{r}'(t)\times\mathbf{r}'''(t))\end{aligned}$$

3.1.31

$$\frac{d}{dt}\left[\mathbf{r}_1(2t)+\mathbf{r}_2\left(\frac{1}{t}\right)\right] = 2\mathbf{r}_1'(2t) - \frac{1}{t^2}\mathbf{r}_2'\left(\frac{1}{t}\right)$$

3.1.33

$$\int_{-1}^{2} \langle t, 3t^2, 4t^3 \rangle \, dt = \langle \frac{3}{2}, 9, 15 \rangle$$

3.1.35

$$\int \langle te^t, -e^{-2t}, te^{t^2} \rangle \, dt = \langle e^t(t-1), \frac{1}{2}e^{-2t}, \frac{1}{2}e^{t^2} \rangle + \mathbf{c}$$

3.1.37

$$\begin{aligned}
\mathbf{r}(t) &= \langle 6t, 3t^2, t^3 \rangle + \mathbf{c} \\
\mathbf{r}(0) &= \mathbf{r}_0 \\
\mathbf{c} &= \mathbf{r}_0 \\
\mathbf{r}(t) &= \langle 6t+1, 3t^2-2, t^3+1 \rangle
\end{aligned}$$

3.1.39

$$\begin{aligned}
\mathbf{r}'(t) &= \langle 6(t^2-1), 7-6\sqrt{t}, 2(t-1) \rangle \\
\mathbf{r}(t) &= \langle 6\left(\frac{1}{3}t^3 - t + 1\right), 7t - 4t^{3/2} - 3, 2\left(\frac{1}{2}t^2 - t\right) \rangle
\end{aligned}$$

3.1.41

$$\begin{aligned}
\mathbf{r}'(t) &= \langle -a\sin t, a\cos t, c \rangle \\
s &= \int_0^{2\pi} ||\mathbf{r}'(t)|| \, dt \\
&= \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t + c^2} \, dt \\
&= \int_0^{2\pi} \sqrt{a^2 + c^2} \, dt \\
&= 2\pi\sqrt{a^2+c^2}
\end{aligned}$$

3.1.43

$$\begin{aligned}
\mathbf{r}'(t) &= \langle e^t(\cos 2t - 2\sin 2t), e^t(\sin 2t + 2\cos 2t), e^t \rangle \\
s &= \int_0^{3\pi} ||\mathbf{r}'(t)||\, dt \\
&= \int_0^{3\pi} \sqrt{(e^t(\cos 2t - 2\sin 2t))^2 + (e^t(\sin 2t + 2\cos 2t))^2 + (e^t)^2}\, dt \\
&= \int_0^{3\pi} e^t \sqrt{(\cos 2t - 2\sin 2t)^2 + (\sin 2t + 2\cos 2t)^2 + 1}\, dt \\
&= \int_0^{3\pi} e^t \sqrt{5\cos^2 2t + 5\sin^2 2t + 1}\, dt \\
&= \sqrt{6} \int_0^{3\pi} e^t\, dt \\
&= \sqrt{6}(e^{3\pi} - 1)
\end{aligned}$$

3.1.45

$$\begin{aligned}
\mathbf{r}(t) &= \langle a\cos t, a\sin t \rangle \\
\mathbf{r}'(t) &= \langle -a\sin t, a\cos t \rangle \\
||\mathbf{r}'(t)|| &= \sqrt{a^2\sin^2 t + a^2\cos^2 t} \\
&= a \\
s &= \int_0^t a\, du \\
&= at \\
t &= \frac{s}{a} \\
\mathbf{r}(s) &= \langle a\cos\frac{s}{a}, a\sin\frac{s}{a} \rangle \\
\mathbf{r}'(s) &= \langle -\sin\frac{s}{a}, \cos\frac{s}{a} \rangle \\
||\mathbf{r}'(s)|| &= \sqrt{\sin^2\frac{s}{a} + \cos^2\frac{s}{a}} \\
&= 1
\end{aligned}$$

3.1.47

$$
\begin{aligned}
||\mathbf{r}(t)|| &= c \\
\sqrt{f(t)^2+g(t)^2+h(t)^2} &= c \\
\frac{1}{\sqrt{f(t)^2+g(t)^2+h(t)^2}}(2f(t)f'(t)+2g(t)g'(t)+2h(t)h'(t)) &= 0 \\
\frac{2(f(t)f'(t)+g(t)g'(t)+h(t)h'(t))}{||\mathbf{r}(t)||} &= 0 \\
\mathbf{r}(t)\cdot\mathbf{r}'(t) &= 0
\end{aligned}
$$

Either $||\mathbf{r}'(t)|| = 0$ or they're perpendicular.

3.2 Motion on a Curve

3.2.1

$$
\begin{aligned}
\mathbf{r}(t) &= \langle t^2, \frac{1}{4}t^4, 0\rangle \\
\mathbf{r}'(t) &= \langle 2t, t^3, 0\rangle \\
||\mathbf{r}'(1)|| &= ||\langle 2,1,0\rangle|| \\
&= \sqrt{2^2+1^2+0^2} \\
&= \sqrt{5}
\end{aligned}
$$

3.2.9

$$
\begin{aligned}
\mathbf{r}(t) &= \langle t^2, t^3-2t, t^2-5t\rangle \\
z &= 0 \\
t^2-5t &= 0 \\
t(t-5) &= 0 \\
t_1 &= 0 \\
t_2 &= 5 \\
\mathbf{r}(0) &= \langle 0,0,0\rangle \\
\mathbf{r}(5) &= \langle 25,115,0\rangle \\
\mathbf{r}'(t) &= \langle 2t, 3t^2-2, 2t-5\rangle \\
\mathbf{r}'(0) &= \langle 0,-2,-5\rangle \\
\mathbf{r}'(5) &= \langle 10,73,5\rangle \\
\mathbf{r}''(t) &= \langle 2,6t,2\rangle \\
\mathbf{r}''(0) &= \langle 2,0,2\rangle \\
\mathbf{r}''(5) &= \langle 2,30,2\rangle
\end{aligned}
$$

3.2.11

(a)

$$\begin{aligned}
\mathbf{r}''(t) &= \langle 0, -g \rangle \\
\mathbf{r}'(t) &= \langle 240\sqrt{3}, 240 - gt \rangle \\
\mathbf{r}(t) &= \langle 240\sqrt{3}t, 240t - \frac{1}{2}gt^2 \rangle \\
x(t) &= 240\sqrt{3}t \\
y(t) &= 240t - \frac{1}{2}gt^2 \\
&= 240t - 16t^2
\end{aligned}$$

(b)

$$\begin{aligned}
240 - 32t &= 0 \\
t &= \frac{15}{2} \\
y\left(\frac{15}{2}\right) &= 240\frac{15}{2} - 16\left(\frac{15}{2}\right)^2 \\
&= 1800 - 900 \\
&= 900\,\text{ft}
\end{aligned}$$

(c)

$$\begin{aligned}
y(t) &= 0 \\
240t - 16t^2 &= 0 \\
t(240 - 16t) &= 0 \\
t_1 &= 0 \\
t_2 &= 15 \\
x(15) &= 240\sqrt{3}(15) \\
&= 3600\sqrt{3} \\
&\approx 6235\,\text{ft}
\end{aligned}$$

(d)

$$\begin{aligned}
||\mathbf{v}(15)|| &= \sqrt{(240\sqrt{3})^2 + (240 - 32(15))^2} \\
&= 480\,\text{ft/s}
\end{aligned}$$

3.2.23

$$\begin{aligned}
\mathbf{r}(t) &= \langle r_0 \cos\omega t, r_0 \sin\omega t\rangle \\
\mathbf{v}(t) &= \langle -\omega r_0 \sin\omega t, \omega r_0 \cos\omega t\rangle \\
v &= ||\mathbf{v}(t)|| \\
&= \sqrt{(-\omega r_0 \sin\omega t)^2 + (\omega r_0 \cos\omega t)^2} \\
&= \omega r_0 \\
\mathbf{a}(t) &= \langle -\omega^2 r_0 \cos\omega t, -\omega^2 r_0 \sin\omega t\rangle \\
&= -\omega^2 \mathbf{r}(t) \\
a &= ||\mathbf{a}(t)|| \\
&= || - \omega^2 \mathbf{r}(t)|| \\
&= \omega^2 r_0 \\
&= \frac{v^2}{r_0}
\end{aligned}$$

3.2.25

$$\begin{aligned}
m'g &= mg - ma \\
m' &= m\left(1 - \frac{v^2}{gr}\right) \\
&\approx 191.3\,\mathrm{lb}
\end{aligned}$$

3.2.27

$$\mathbf{v}(t) = \langle 6t^2x, -4ty^2, 2t(z+1)\rangle$$

$$\begin{aligned}
\frac{dx}{dt} &= 6t^2x \\
\frac{1}{x}\frac{dx}{dt} &= 6t^2 \\
\ln x &= 2t^3 + c_1 \\
x &= c_1 e^{2t^3}
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dt} &= -4ty^2 \\
\frac{1}{y^2}\frac{dy}{dt} &= -4t \\
-\frac{1}{y} &= -2t^2 + c_2 \\
y &= \frac{1}{2t^2 + c_2}
\end{aligned}$$

$$\begin{aligned}
\frac{dz}{dt} &= 2t(z+1) \\
\frac{1}{z+1}\frac{dz}{dt} &= 2t \\
\ln(z+1) &= t^2 + c_3 \\
z+1 &= c_3 e^{t^2} \\
z &= c_3 e^{t^2} - 1
\end{aligned}$$

$$\mathbf{r}(t) = \langle c_1 e^{2t^3}, \frac{1}{2t^2 + c_2}, c_3 e^{t^2} - 1\rangle$$

3.2.29

(a)

$$\begin{aligned}
\boldsymbol{\tau} &= \mathbf{r} \times \mathbf{F} \\
&= \mathbf{r} \times \left(-k\frac{Mm}{r^2}\frac{\mathbf{r}}{r}\right) \\
&= -\frac{kMm}{r^3}(\mathbf{r} \times \mathbf{r}) \\
&= \mathbf{0}
\end{aligned}$$

(b) Torque is the derivative of angular momentum with respect to time. If there's no torque angular momentum doesn't change.

3.3 Curvature and Components of Acceleration

3.3.1

$$\begin{aligned}
\mathbf{r}(t) &= \langle t\cos t - \sin t, t\sin t + \cos t, t^2\rangle \\
\mathbf{r}'(t) &= \langle -t\sin t, t\cos t, 2t\rangle \\
||\mathbf{r}'(t)|| &= \sqrt{(-t\sin t)^2 + (t\cos t)^2 + (2t)^2} \\
&= t\sqrt{\sin^2 t + \cos^2 t + 4} \\
&= \sqrt{5}t \\
\mathbf{T} &= \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} \\
&= \frac{1}{\sqrt{5}}\langle -\sin t, \cos t, 2\rangle
\end{aligned}$$

3.3.3

$$
\begin{aligned}
\mathbf{r}(t) &= \langle a\cos t, a\sin t, ct\rangle \\
\mathbf{r}'(t) &= \langle -a\sin t, a\cos t, c\rangle \\
||\mathbf{r}'(t)|| &= \sqrt{(-a\sin t)^2 + (a\cos t)^2 + (c)^2} \\
&= \sqrt{a^2 + c^2} \\
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} \\
&= \frac{1}{\sqrt{a^2+c^2}}\langle -a\sin t, a\cos t, c\rangle \\
\frac{d\mathbf{T}}{dt} &= \frac{1}{\sqrt{a^2+c^2}}\langle -a\cos t, -a\sin t, 0\rangle \\
\left|\left|\frac{d\mathbf{T}}{dt}\right|\right| &= \frac{a}{\sqrt{a^2+c^2}} \\
\mathbf{N} &= \frac{d\mathbf{T}/dt}{||d\mathbf{T}/dt||} \\
&= \langle -\cos t, -\sin t, 0\rangle \\
\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\
&= -\frac{1}{\sqrt{a^2+c^2}}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -a\sin t & a\cos t & c \\ \cos t & \sin t & 0 \end{vmatrix} \\
&= \frac{1}{\sqrt{a^2+c^2}}\langle c\sin t, -c\cos t, a\rangle \\
\kappa &= \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||} \\
&= \frac{a}{a^2+c^2}
\end{aligned}
$$

3.3.5

$$\begin{aligned}
\mathbf{r}(t) &= \langle 2\cos t, 2\sin t, 3t\rangle \\
\mathbf{r}(\pi/4) &= \langle \sqrt{2}, \sqrt{2}, \frac{3\pi}{4}\rangle \\
\mathbf{B}(t) &= \frac{1}{\sqrt{13}}\langle 3\sin t, -3\cos t, 2\rangle \\
\mathbf{B}(\pi/4) &= \frac{1}{\sqrt{13}}\langle \frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 2\rangle
\end{aligned}$$

$$\begin{aligned}
0 &= \mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0) \\
&= \mathbf{B}(\pi/4)\cdot(\mathbf{r}-\mathbf{r}(\pi/4)) \\
&= \frac{1}{\sqrt{13}}\langle \frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 2\rangle \cdot \left(\langle x, y, z\rangle - \langle \sqrt{2}, \sqrt{2}, \frac{3\pi}{4}\rangle\right) \\
&= \frac{1}{\sqrt{13}}\langle \frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 2\rangle \cdot \langle x-\sqrt{2}, y-\sqrt{2}, z-\frac{3\pi}{4}\rangle \\
&= \frac{3}{\sqrt{26}}(x-\sqrt{2}) - \frac{3}{\sqrt{26}}(y-\sqrt{2}) + \frac{2}{\sqrt{13}}\left(z-\frac{3\pi}{4}\right) \\
&= \frac{3}{\sqrt{2}}x - 3 - \frac{3}{\sqrt{2}}y + 3 + 2z - \frac{3\pi}{2} \\
\frac{3\pi}{2} &= \frac{3}{\sqrt{2}}x - \frac{3}{\sqrt{2}}y + 2z \\
3\pi &= 3\sqrt{2}x - 3\sqrt{2}y + 4z
\end{aligned}$$

3.3.7

$$\begin{aligned}
\mathbf{r}(t) &= \langle 1, t, t^2 \rangle \\
\mathbf{r}'(t) &= \langle 0, 1, 2t \rangle \\
||\mathbf{r}'(t)|| &= \sqrt{1+4t^2} \\
\mathbf{r}''(t) &= \langle 0, 0, 2 \rangle \\
a_T &= \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{||\mathbf{r}'(t)||} \\
&= \frac{4t}{\sqrt{1+4t^2}} \\
\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} \\
&= \langle 2, 0, 0 \rangle \\
a_N &= \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||} \\
&= \frac{2}{\sqrt{1+4t^2}}
\end{aligned}$$

3.3.17

$$\begin{aligned}
\mathbf{r}(t) &= \langle a\cos t, b\sin t, ct \rangle \\
\mathbf{r}'(t) &= \langle -a\sin t, b\cos t, c \rangle \\
||\mathbf{r}'(t)|| &= \sqrt{(-a\sin t)^2 + (b\cos t)^2 + (c)^2} \\
&= \sqrt{a^2\sin^2 t + b^2\cos^2 t + c^2} \\
\mathbf{r}''(t) &= \langle -a\cos t, -b\sin t, 0 \rangle \\
\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -a\sin t & b\cos t & c \\ -a\cos t & -b\sin t & 0 \end{vmatrix} \\
&= \langle bc\sin t, -ac\cos t, ab \rangle \\
||\mathbf{r}'(t) \times \mathbf{r}''(t)|| &= \sqrt{(bc\sin t)^2 + (-ac\cos t)^2 + (ab)^2} \\
&= \sqrt{b^2c^2\sin^2 t + a^2c^2\cos^2 t + a^2b^2} \\
\kappa &= \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)}{||\mathbf{r}'(t)||^3} \\
&= \frac{\sqrt{b^2c^2\sin^2 t + a^2c^2\cos^2 t + a^2b^2}}{(a^2\sin^2 t + b^2\cos^2 t + c^2)^{3/2}}
\end{aligned}$$

3.3.23

$$\begin{aligned} y &= x^2 \\ \kappa &= \frac{2}{(1+4x^2)^{3/2}} \\ \rho &= \frac{1}{\kappa} \\ &= \frac{(1+4x^2)^{3/2}}{2} \\ \kappa(0) &= 2 \\ \kappa(1) &= \frac{2}{5\sqrt{5}} \end{aligned}$$

The curve is sharper at $(0, 0)$.