

Introduction to Quantum Mechanics by David J. Griffiths Problems

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Part I

Theory

1 The Wave Function

1.1

(a)

$$\begin{aligned}
 \langle j^2 \rangle &= \sum j^2 P(j) \\
 &= 14^2 \frac{1}{14} + 15^2 \frac{1}{14} + 16^2 \frac{3}{14} + 22^2 \frac{2}{14} + 24^2 \frac{2}{14} + 25^2 \frac{5}{14} \\
 &= \frac{3217}{7} \\
 &\approx 459.571 \\
 \langle j \rangle^2 &= \left(\sum j P(j) \right)^2 \\
 &= 441
 \end{aligned}$$

(b)

$$\begin{aligned}
 \Delta j_{14} &= -7 \\
 \Delta j_{15} &= -6 \\
 \Delta j_{16} &= -5 \\
 \Delta j_{22} &= 1 \\
 \Delta j_{24} &= 3 \\
 \Delta j_{25} &= 4 \\
 \sigma^2 &= \sum (\Delta j)^2 P(j) \\
 &= \frac{130}{7} \\
 &\approx 18.571
 \end{aligned}$$

(c)

$$\sigma^2 = \sqrt{\langle j^2 \rangle - \langle j \rangle^2} = 18.571$$

1.2

(a)

$$\begin{aligned}
 \langle x^2 \rangle &= \int_0^h x^2 \rho(x) dx \\
 &= \int_0^h \frac{x^{3/2}}{2\sqrt{h}} dx \\
 &= \frac{1}{2\sqrt{h}} \left[\frac{2}{5} x^{5/2} \right]_0^h \\
 &= \frac{h^2}{5} \\
 \langle x \rangle^2 &= \frac{h^2}{9} \\
 \sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
 &= \sqrt{\frac{h^2}{5} - \frac{h^2}{9}} \\
 &= h \sqrt{\frac{4}{45}} \\
 &= \frac{2}{3\sqrt{5}} h
 \end{aligned}$$

(b)

$$\begin{aligned}
 1 - \int_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} \rho(x) dx &= 1 - \frac{1}{2\sqrt{h}} [2\sqrt{x}]_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} \\
 &= 1 - \frac{1}{\sqrt{h}} \left(\sqrt{\frac{1}{3}h + \frac{2}{3\sqrt{5}}h} - \sqrt{\frac{1}{3}h - \frac{2}{3\sqrt{5}}h} \right) \\
 &= 1 - \left(\sqrt{\frac{1}{3} + \frac{2}{3\sqrt{5}}} - \sqrt{\frac{1}{3} - \frac{2}{3\sqrt{5}}} \right) \\
 &\approx 0.393
 \end{aligned}$$

1.3

(a)

$$\begin{aligned}
 \rho(x) &= A e^{-\lambda(x-a)^2} \\
 1 &= \int_{-\infty}^{\infty} \rho(x) dx \\
 &= A \int_{-\infty}^{\infty} e^{-\lambda(x-a)^2} dx \\
 &= A \sqrt{\frac{\pi}{\lambda}} \\
 A &= \sqrt{\frac{\lambda}{\pi}}
 \end{aligned}$$

(b)

$$\begin{aligned}
 \langle x \rangle &= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x e^{-\lambda(x-a)^2} dx \\
 &= a \\
 \langle x^2 \rangle &= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2} dx \\
 &= a^2 + \frac{1}{2\lambda} \\
 \sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
 &= \sqrt{a^2 + \frac{1}{2\lambda} - a^2} \\
 &= \frac{1}{\sqrt{2\lambda}}
 \end{aligned}$$

1.4

(a)

$$\begin{aligned}
 1 &= \int_{-\infty}^{\infty} |\Psi(x, 0)|^2 dx \\
 &= \left(\frac{A}{a}\right)^2 \int_0^a x^2 dx + \left(\frac{A}{b-a}\right)^2 \int_a^b (b-x)^2 dx \\
 &= \frac{1}{3}A^2a + \left(\frac{A}{b-a}\right)^2 \left[-\frac{1}{3}(b-x)^3\right]_a^b \\
 &= \frac{1}{3}A^2a + \frac{1}{3}A^2(b-a) \\
 &= \frac{1}{3}A^2b \\
 A &= \sqrt{\frac{3}{b}}
 \end{aligned}$$

(c) $x = a$

(d)

$$\begin{aligned}
 \int_0^a |\Psi(x, 0)|^2 dx &= \frac{3}{a^2b} \left[\frac{1}{3}x^3\right]_0^a \\
 &= \frac{a}{b}
 \end{aligned}$$

(e)

$$\begin{aligned}
 \langle x \rangle &= \int_{-\infty}^{\infty} x |\Psi(x, 0)|^2 dx \\
 &= \frac{3}{a^2b} \left[\frac{1}{4}x^4\right]_0^a + \frac{3}{b(b-a)^2} \int_a^b x(b-x)^2 dx \\
 &= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \int_a^b (b^2x - 2bx^2 + x^3) dx \\
 &= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \left[\frac{1}{2}b^2x^2 - \frac{2}{3}bx^3 + \frac{1}{4}x^4\right]_a^b \\
 &= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \left(\frac{1}{2}b^4 - \frac{2}{3}b^4 + \frac{1}{4}b^4 - \frac{1}{2}a^2b^2 + \frac{2}{3}a^3b - \frac{1}{4}a^4\right) \\
 &= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \frac{1}{12}(b-a)^3(3a+b) \\
 &= \frac{3a^2}{4b} + \frac{1}{4b}(3ab + b^2 - 3a^2 - ab) \\
 &= \frac{1}{2}a + \frac{1}{4}b
 \end{aligned}$$

1.5

(a)

$$\Psi(x, t) = Ae^{-\lambda|x|}e^{-i\omega t}$$

$$\Psi(x, 0) = Ae^{-\lambda|x|}$$

$$\begin{aligned} 1 &= A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|} dx \\ &= 2A^2 \int_0^{\infty} e^{-2\lambda x} dx \\ &= 2A^2 \left[-\frac{1}{2\lambda} e^{-2\lambda x} \right]_0^{\infty} \\ &= \frac{A^2}{\lambda} \\ A &= \sqrt{\lambda} \end{aligned}$$

(b)

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{\infty} x \lambda e^{-2\lambda|x|} dx \\ &= \lambda \int_{-\infty}^{\infty} x e^{-2\lambda|x|} dx \\ &= 0 \\ \langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 \lambda e^{-2\lambda|x|} dx \\ &= 2\lambda \int_0^{\infty} x^2 e^{-2\lambda x} dx \\ &= \frac{1}{2\lambda^2} \end{aligned}$$

(c)

$$\begin{aligned} \sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\ &= \frac{1}{\sqrt{2}\lambda} \\ 1 - \int_{-\sigma}^{\sigma} \lambda e^{-2\lambda|x|} dx &= 1 - 2\lambda \int_0^{\sigma} e^{-2\lambda x} dx \\ &= 1 - 2\lambda \left[-\frac{1}{2\lambda} e^{-2\lambda x} \right]_0^{\sigma} \\ &= e^{-2\lambda\sigma} \\ &= e^{-\sqrt{2}} \\ &\approx 0.243 \end{aligned}$$

1.6

The chain rule requires that you apply it to both x and $|\Psi|^2$ which gives the same result

$$\begin{aligned}
 \frac{d\langle x \rangle}{dt} &= \frac{d}{dt} \int x |\Psi|^2 dx \\
 &= \int \frac{d}{dt} (x |\Psi|^2) dx \\
 &= \int \left(0 \cdot |\Psi|^2 + x \frac{\partial |\Psi|^2}{\partial t} \right) dx \\
 &= \int x \frac{\partial |\Psi|^2}{\partial t} dx
 \end{aligned}$$

1.8

$$\begin{aligned}
 i\hbar \frac{\partial}{\partial t} \left(e^{-iV_0 t/\hbar} \Psi \right) &= -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \left(e^{-iV_0 t/\hbar} \Psi \right) + (V + V_0) \left(e^{-iV_0 t/\hbar} \Psi \right) \\
 i\hbar \left(-\frac{iV_0}{\hbar} e^{-iV_0 t/\hbar} \Psi + e^{-iV_0 t/\hbar} \frac{\partial \Psi}{\partial t} \right) &= -\frac{\hbar^2}{2m} e^{-iV_0 t/\hbar} \frac{\partial^2 \Psi}{\partial x^2} + V e^{-iV_0 t/\hbar} \Psi + V_0 e^{-iV_0 t/\hbar} \Psi \\
 V_0 \Psi + i\hbar \frac{\partial \Psi}{\partial t} &= -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V \Psi + V_0 \Psi \\
 i\hbar \frac{\partial \Psi}{\partial t} &= -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V \Psi
 \end{aligned}$$

$$\begin{aligned}
 \langle Q(x, p) \rangle &= \int \left(e^{-iV_0 t/\hbar} \Psi \right)^* [Q(x, -i\hbar \partial/\partial x)] e^{-iV_0 t/\hbar} \Psi dx \\
 &= \int e^{iV_0 t/\hbar} \Psi^* [Q(x, -i\hbar \partial/\partial x)] e^{-iV_0 t/\hbar} \Psi dx \\
 &= \int \Psi^* [Q(x, -i\hbar \partial/\partial x)] \Psi dx
 \end{aligned}$$

No effect on the expectation value.

1.9

(a)

$$\begin{aligned}
 \Psi(x, t) &= Ae^{-a[(mx^2/\hbar)+it]} \\
 1 &= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)} dx \\
 &= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)} dx \\
 &= A^2 \sqrt{\frac{\pi\hbar}{2am}} \\
 A^2 &= \sqrt{\frac{2am}{\pi\hbar}} \\
 A &= \left(\frac{2am}{\pi\hbar}\right)^{1/4}
 \end{aligned}$$

(b)

$$\begin{aligned}
 \Psi &= Ae^{-a[(mx^2/\hbar)+it]} \\
 \frac{\partial \Psi}{\partial t} &= -ia\Psi \\
 \frac{\partial \Psi}{\partial x} &= -\frac{2amx}{\hbar}\Psi \\
 \frac{\partial^2 \Psi}{\partial x^2} &= -\frac{2am}{\hbar} \left(\Psi + x \frac{\partial \Psi}{\partial x} \right) \\
 &= -\frac{2am}{\hbar} \left(1 - \frac{2amx^2}{\hbar} \right) \Psi \\
 V\Psi &= i\hbar \frac{\partial \Psi}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} \\
 &= a\hbar\Psi - a\hbar \left(1 - \frac{2amx^2}{\hbar} \right) \Psi \\
 V &= a\hbar - a\hbar + 2a^2mx^2 \\
 &= 2a^2mx^2
 \end{aligned}$$

(c)

$$\begin{aligned}
\langle x \rangle &= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)} x \, dx \\
&= 0 \\
\langle x^2 \rangle &= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)} x^2 \, dx \\
&= 2A^2 \int_0^{\infty} e^{-2a(mx^2/\hbar)} x^2 \, dx \\
&= \frac{\hbar}{4am} \\
\langle p \rangle &= \int_{-\infty}^{\infty} \Psi^* \left[-i\hbar \frac{\partial}{\partial x} \right] \Psi \, dx \\
&= -i\hbar \int_{-\infty}^{\infty} A e^{-a[(mx^2/\hbar)-it]} \left(-\frac{2amx}{\hbar} A e^{-a[(mx^2/\hbar)+it]} \right) dx \\
&= 2iA^2 am \int_{-\infty}^{\infty} x e^{-2amx^2/\hbar} \, dx \\
&= 0 \\
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \Psi^* \left[-\hbar^2 \frac{\partial^2}{\partial x^2} \right] \Psi \, dx \\
&= -\hbar^2 \int_{-\infty}^{\infty} A e^{-a[(mx^2/\hbar)-it]} \left[-\frac{2am}{\hbar} \left(1 - \frac{2amx^2}{\hbar} \right) A e^{-a[(mx^2/\hbar)+it]} \right] dx \\
&= 2A^2 am\hbar \int_{-\infty}^{\infty} e^{-2amx^2/\hbar} \left(1 - \frac{2amx^2}{\hbar} \right) dx \\
&= am\hbar
\end{aligned}$$

(d)

$$\begin{aligned}
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{\hbar}{4am}} \\
\sigma_p &= \sqrt{am\hbar} \\
\sigma_x \sigma_p &= \sqrt{\frac{1}{4} \hbar^2} \\
&= \frac{1}{2} \hbar \\
&\geq \frac{1}{2} \hbar
\end{aligned}$$

Yes, this is consistent with Heisenberg's uncertainty principle.

1.10

(a)

$$P(0) = 0$$

$$\begin{aligned} P(1) &= \frac{2}{25} \\ &= 0.08 \end{aligned}$$

$$\begin{aligned} P(2) &= \frac{3}{25} \\ &= 0.12 \end{aligned}$$

$$\begin{aligned} P(3) &= \frac{1}{5} \\ &= 0.2 \end{aligned}$$

$$\begin{aligned} P(4) &= \frac{3}{25} \\ &= 0.12 \end{aligned}$$

$$\begin{aligned} P(5) &= \frac{3}{25} \\ &= 0.2 \end{aligned}$$

$$\begin{aligned} P(6) &= \frac{3}{25} \\ &= 0.2 \end{aligned}$$

$$\begin{aligned} P(7) &= \frac{1}{25} \\ &= 0.04 \end{aligned}$$

$$\begin{aligned} P(8) &= \frac{2}{25} \\ &= 0.08 \end{aligned}$$

$$\begin{aligned} P(9) &= \frac{3}{25} \\ &= 0.12 \end{aligned}$$

(b) The most probable digit is 3, the median digit is 4, and the average value is $\frac{118}{25} = 4.72$.

(c) $\sigma = 2.474$

1.14

(a)

$$\begin{aligned}
 P_{ab}(t) &= \int_a^b |\Psi(x, t)|^2 dx \\
 \frac{dP_{ab}}{dt} &= \frac{d}{dt} \int_a^b |\Psi(x, t)|^2 dx \\
 &= \int_a^b \frac{d}{dt} (|\Psi(x, t)|^2) dx \\
 &= \int_a^b \frac{\partial}{\partial x} \left[\frac{i\hbar}{2m} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) \right] dx \\
 &= J(a, t) - J(b, t)
 \end{aligned}$$

The units are s^{-1} .

(b)

$$\begin{aligned}
 \Psi(x, t) &= Ae^{-a[(mx^2/\hbar)+it]} \\
 \frac{\partial \Psi}{\partial x} &= -\frac{2amx}{\hbar} \Psi \\
 \Psi^*(x, t) &= Ae^{-a[(mx^2/\hbar)-it]} \\
 \frac{\partial \Psi^*}{\partial x} &= -\frac{2amx}{\hbar} \Psi^* \\
 J(x, t) &= \frac{i\hbar}{2m} \left(\Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi}{\partial x} \right) \\
 &= \frac{i\hbar}{2m} \left[\Psi \left(-\frac{2amx}{\hbar} \Psi^* \right) - \Psi^* \left(-\frac{2amx}{\hbar} \Psi \right) \right] \\
 &= 0
 \end{aligned}$$

1.15

$$\begin{aligned}
\frac{d}{dt} \int_{-\infty}^{\infty} \Psi_1^* \Psi_2 \, dx &= \int_{-\infty}^{\infty} \left(\frac{\partial \Psi_1^*}{\partial t} \Psi_2 + \Psi_1^* \frac{\partial \Psi_2}{\partial t} \right) dx \\
&= \int_{-\infty}^{\infty} \left[\left(-i \frac{\hbar}{2m} \frac{\partial^2 \Psi_1^*}{\partial x^2} + i \frac{V}{\hbar} \Psi_1^* \right) \Psi_2 \right. \\
&\quad \left. + \Psi_1^* \left(i \frac{\hbar}{2m} \frac{\partial^2 \Psi_2}{\partial x^2} - i \frac{V}{\hbar} \Psi_2 \right) \right] dx \\
&= i \frac{\hbar}{2m} \int_{-\infty}^{\infty} \left(\Psi_1^* \frac{\partial^2 \Psi_2}{\partial x^2} - \frac{\partial^2 \Psi_1^*}{\partial x^2} \Psi_2 \right) dx \\
&= i \frac{\hbar}{2m} \left[\Psi_1^* \frac{\partial \Psi_2}{\partial x} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial}{\partial x} (\Psi_1^* \Psi_2) \, dx \right. \\
&\quad \left. - \frac{\partial \Psi_1^*}{\partial x} \Psi_2 \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{\partial}{\partial x} (\Psi_1^* \Psi_2) \, dx \right] \\
&= 0
\end{aligned}$$

1.16

(a)

$$\begin{aligned}
1 &= \int_{-a}^a A^2 (a^2 - x^2)^2 \, dx \\
&= A^2 \int_0^a (a^2 - x^2)^2 \, dx \\
&= \frac{16}{15} A^2 a^5 \\
A &= \sqrt{\frac{15}{16a^5}}
\end{aligned}$$

(b)

$$\begin{aligned}
\langle x \rangle &= \int_{-a}^a x A (a^2 - x^2) \, dx \\
&= 0
\end{aligned}$$

(c)

$$\begin{aligned}
\langle p \rangle &= \int_{-a}^a \Psi^* \left(-i \hbar \frac{\partial}{\partial x} \right) \Psi \, dx \\
&= 2i A^2 \hbar \int_{-a}^a x (a^2 - x^2) \, dx \\
&= 0
\end{aligned}$$

(d)

$$\begin{aligned}\langle x^2 \rangle &= \int_{-a}^a \Psi^* x^2 \Psi dx \\ &= A^2 \int_{-a}^a x^2 (a^2 - x^2)^2 dx \\ &= A^2 \frac{16}{105} a^7 \\ &= \frac{a^2}{7}\end{aligned}$$

(e)

$$\begin{aligned}\langle p^2 \rangle &= \int_{-a}^a \Psi^* \left(-\hbar^2 \frac{\partial^2}{\partial x^2} \right) \Psi dx \\ &= -\hbar^2 \int_{-a}^a A(a^2 - x^2)(-2A) dx \\ &= 4A^2 \hbar^2 \int_0^a (a^2 - x^2) dx \\ &= 4A^2 \hbar^2 \left[a^2 x - \frac{1}{3} x^3 \right]_0^a \\ &= 4A^2 \hbar^2 \left(a^3 - \frac{1}{3} a^3 \right) \\ &= \frac{8}{3} A^2 a^3 \hbar^2 \\ &= \frac{8}{3} \frac{15}{16a^5} a^3 \hbar^2 \\ &= \frac{5}{2} \frac{\hbar^2}{a^2}\end{aligned}$$

(f)

$$\begin{aligned}\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\ &= \sqrt{\frac{a^2}{7}} \\ &= \frac{a}{\sqrt{7}}\end{aligned}$$

(g)

$$\begin{aligned}\sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\ &= \sqrt{\frac{5}{2}} \frac{\hbar}{a}\end{aligned}$$

(h)

$$\begin{aligned}\sigma_x \sigma_p &= \sqrt{\frac{5}{14}} \hbar \\ &\geq \frac{1}{2} \hbar\end{aligned}$$

1.18

(a)

$$\begin{aligned}\frac{\hbar}{\sqrt{3mk_B T}} &> d \\ \frac{\sqrt{3mk_B T}}{\hbar} &< \frac{1}{d} \\ T_{\text{electron}} &< \frac{\hbar^2}{3d^2 m k_B} \\ &< 1.3 \times 10^5 \text{ K} \\ T_{\text{nuclei}} &< 2.5 \text{ K}\end{aligned}$$

(b)

$$\begin{aligned}PV &= Nk_B T \\ \frac{V}{N} &= \frac{k_B T}{P} \\ d &= \left(\frac{k_B T}{P} \right)^{1/3} \\ \frac{\hbar}{\sqrt{3mk_B T}} &> \left(\frac{k_B T}{P} \right)^{1/3} \\ T &< \frac{1}{k_B} \left(\frac{\hbar^2}{3m} \right)^{3/5} P^{2/5}\end{aligned}$$

2 Time-Independent Schrödinger Equation

2.1

(a)

$$\begin{aligned}\int_{-\infty}^{\infty} |\Psi|^2 dx &= \int_{-\infty}^{\infty} \Psi^* \Psi dx \\ &= \int_{-\infty}^{\infty} \psi^* e^{i(E_0 - i\Gamma)t/\hbar} \psi e^{-i(E_0 + i\Gamma)t/\hbar} dx \\ &= e^{2\Gamma t/\hbar} \int_{-\infty}^{\infty} |\psi|^2 dx\end{aligned}$$

In order for this to equal 1 for all t , Γ must be 0.

- (b) If $\psi(x)$ is a complex solution to the time-independent Schrödinger equation then so is $\psi^*(x)$ and $\psi(x) + \psi^*(x)$ which is real.

2.2

If ψ and its second derivative always have the same sign, ψ will increase or decrease without bound forever. This means there is no non-zero choice of constant A such that

$$\int_{-\infty}^{\infty} |A\psi|^2 dx = 1$$

and thus the equation can't be normalised.

The classical analog of this is statements is that the potential energy of a system can't exceed its total energy.

2.3

The time-independent Schrödinger equation in an infinite square well is

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi.$$

If $E = 0$ then $\psi = Ax + B$ which isn't normalisable.

If $E < 0$ then $\psi = Ae^{kt} + Be^{-kt}$ where $k \in \mathbb{R}$ which also isn't normalisable.

2.4

$$\begin{aligned}
\Psi_n(x, t) &= \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-i(n^2\pi^2\hbar/2ma^2)t} \\
\langle x \rangle &= \int_0^a \Psi_n^* x \Psi_n dx \\
&= \frac{2}{a} \int_0^a x \sin^2\left(\frac{n\pi}{a}x\right) dx \\
&= \frac{a}{2} \\
\langle x^2 \rangle &= \int_0^a \Psi_n^* x^2 \Psi_n dx \\
&= \frac{2}{a} \int_0^a x^2 \sin^2\left(\frac{n\pi}{a}x\right) dx \\
&= a^2 \left(\frac{1}{3} - \frac{1}{2n^2\pi^2} \right) \\
\langle p \rangle &= \int_0^a \Psi_n^* \left(-i\hbar \frac{\partial}{\partial x} \right) \Psi_n dx \\
&= -i \frac{2\hbar n\pi}{a^2} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \cos\left(\frac{n\pi}{a}x\right) dx \\
&= 0 \\
\langle p^2 \rangle &= \int_0^a \Psi_n^* \left(-\hbar^2 \frac{\partial^2}{\partial x^2} \right) \Psi_n dx \\
&= \frac{2\hbar^2 n^2 \pi^2}{a^3} \int_0^a \sin^2\left(\frac{n\pi}{a}x\right) dx \\
&= \left(\frac{n\pi\hbar}{a} \right)^2 \\
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \frac{a}{2} \sqrt{\frac{1}{3} - \frac{2}{n^2\pi^2}} \\
\sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\
&= \frac{n\pi\hbar}{a}
\end{aligned}$$

2.5

(a)

$$\begin{aligned}
 1 &= \int_0^a A^2 (\psi_1 + \psi_2)^2 dx \\
 &= A^2 \int_0^a (\psi_1^2 + 2\psi_1\psi_2 + \psi_2^2) dx \\
 &= \frac{2A^2}{a} \left[\int_0^a \sin^2\left(\frac{\pi}{a}x\right) dx + \int_0^a \sin^2\left(\frac{2\pi}{a}x\right) dx \right] \\
 &= 2A^2 \\
 A &= \frac{1}{\sqrt{2}}
 \end{aligned}$$

(b)

$$\begin{aligned}
 \Psi(x, t) &= \frac{1}{\sqrt{2}} \left[\sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right) e^{-i\omega t} + \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi}{a}x\right) e^{-4i\omega t} \right] \\
 |\Psi(x, t)|^2 &= \Psi^* \Psi \\
 &= \frac{1}{a} \left[\sin\left(\frac{\pi}{a}x\right) e^{i\omega t} + \sin\left(\frac{2\pi}{a}x\right) e^{4i\omega t} \right] \\
 &\quad \left[\sin\left(\frac{\pi}{a}x\right) e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right) e^{-4i\omega t} \right] \\
 &= \frac{1}{a} \left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) e^{-3i\omega t} \right. \\
 &\quad \left. + \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) e^{3i\omega t} + \sin^2\left(\frac{2\pi}{a}x\right) \right] \\
 &= \frac{1}{a} \left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right) \right. \\
 &\quad \left. + 2 \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) \cos(3\omega t) \right]
 \end{aligned}$$

(c)

$$\begin{aligned}
 \langle x \rangle &= \int_0^a \Psi^* x \Psi dx \\
 &= \int_0^a x |\Psi|^2 dx \\
 &= \frac{a}{2} \left[1 - \frac{32}{9\pi^2} \cos(3\omega t) \right]
 \end{aligned}$$

(d)

$$\begin{aligned}
 \langle p \rangle &= m \frac{d \langle x \rangle}{dt} \\
 &= \frac{16am\omega}{3\pi^2} \sin(3\omega t) \\
 &= \frac{8\hbar}{3a} \sin(3\omega t)
 \end{aligned}$$

(e) You can get E_1 or E_2 and the probability of getting each is $1/2$.

$H = \frac{1}{2}(E_1 + E_2)$ is the mean of the two possible energy values.

2.6

$$\begin{aligned}
 \Psi(x, 0) &= A[\psi_1 + e^{i\phi}\psi_2] \\
 1 &= \int_0^a |\Psi|^2 dx \\
 &= \int_0^a \Psi^* \Psi dx \\
 &= A^2 \int_0^a (\psi_1 + e^{-i\phi}\psi_2)(\psi_1 + e^{i\phi}\psi_2) dx \\
 &= A^2 \int_0^a (\psi_1^2 + e^{i\phi}\psi_1\psi_2 + e^{-i\phi}\psi_1\psi_2 + \psi_2^2) dx \\
 &= \frac{2A^2}{a} \int_0^a \left[\sin^2\left(\frac{\pi}{a}x\right) + e^{i\phi} \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) \right. \\
 &\quad \left. e^{-i\phi} \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right) \right] dx \\
 &= \frac{2A^2}{a} \int_0^a \left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) \cos\phi \right. \\
 &\quad \left. + \sin^2\left(\frac{2\pi}{a}x\right) \right] dx \\
 &= 2A^2 \\
 A &= \frac{1}{\sqrt{2}} \\
 \Psi(x, t) &= \frac{1}{\sqrt{a}} \left[\sin\left(\frac{\pi}{a}x\right) e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right) e^{i(\phi-4\omega t)} \right]
 \end{aligned}$$

$$\begin{aligned}
|\Psi|^2 &= \Psi^* \Psi \\
&= \frac{1}{a} \left[\sin\left(\frac{\pi}{a}x\right) e^{i\omega t} + \sin\left(\frac{2\pi}{a}x\right) e^{-i(\phi-4\omega t)} \right] \\
&\quad \left[\sin\left(\frac{\pi}{a}x\right) e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right) e^{i(\phi-4\omega t)} \right] \\
&= \frac{1}{a} \left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) e^{i(\phi-3\omega t)} \right. \\
&\quad \left. \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) e^{-i(\phi-3\omega t)} + \sin^2\left(\frac{2\pi}{a}x\right) \right] \\
&= \frac{1}{a} \left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right) \right. \\
&\quad \left. + 2 \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) \cos(\phi - 3\omega t) \right] \\
\langle x \rangle &= \int_0^a \Psi^* x \Psi dx \\
&= \int_0^a x |\Psi|^2 dx \\
&= \frac{a}{2} \left[1 - \frac{32}{9\pi^2} \cos(3\omega t - \phi) \right]
\end{aligned}$$

2.7

(a)

$$\begin{aligned}
1 &= \int_0^a |\Psi|^2 dx \\
&= A^2 \left[\int_0^{a/2} x^2 dx + \int_{a/2}^a (a-x)^2 dx \right] \\
&= A^2 \left\{ \frac{1}{3} \left[\frac{a}{2} \right]^3 + \left[-\frac{1}{3}(a-x)^3 \right]_{a/2}^a \right\} \\
&= A^2 \left(\frac{a^3}{24} + \frac{a^3}{24} \right) \\
&= \frac{A^2 a^3}{12} \\
A &= \frac{2\sqrt{3}}{\sqrt{a^3}}
\end{aligned}$$

(b)

$$\begin{aligned}
c_n &= \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \Psi(x, 0) dx \\
&= \sqrt{\frac{2}{a}} \left[\int_0^{a/2} \sin\left(\frac{n\pi}{a}x\right) Ax dx + \int_{a/2}^a \sin\left(\frac{n\pi}{a}x\right) A(a-x) dx \right] \\
&= \frac{2\sqrt{6}}{a^2} \left[\int_0^{a/2} x \sin\left(\frac{n\pi}{a}x\right) dx + \int_{a/2}^a (a-x) \sin\left(\frac{n\pi}{a}x\right) dx \right] \\
&= \frac{8\sqrt{6}}{n^2\pi^2} \sin^2\left(\frac{n\pi}{4}\right) \sin\left(\frac{n\pi}{2}\right) \\
&= \begin{cases} 0 & n \text{ even} \\ (-1)^{(n-1)/2} \frac{4\sqrt{6}}{n^2\pi^2} & n \text{ odd} \end{cases} \\
\Psi(x, t) &= \frac{4\sqrt{6}}{\pi^2} \sqrt{\frac{2}{a}} \sum_{n=1,3,5,\dots}^{\infty} (-1)^{(n-1)/2} \frac{1}{n^2} \sin\left(\frac{n\pi}{a}x\right) e^{-i(n^2\pi^2\hbar/2ma^2)t}
\end{aligned}$$

(c)

$$\begin{aligned}
|c_1|^2 &= \left(\frac{4\sqrt{6}}{\pi^2}\right)^2 \\
&\approx 0.985
\end{aligned}$$

(d)

$$\begin{aligned}
E_n &= \frac{n^2\pi^2\hbar^2}{2ma^2} \\
\langle H \rangle &= \sum_{n=0}^{\infty} |c_{2n+1}|^2 E_{2n+1} \\
&= \sum_{n=0}^{\infty} \left(\frac{4\sqrt{6}}{(2n+1)^2\pi^2}\right)^2 \frac{(2n+1)^2\pi^2\hbar^2}{2ma^2} \\
&= \sum_{n=0}^{\infty} \frac{48\hbar^2}{(2n+1)^2ma^2\pi^2} \\
&= \frac{48\hbar^2}{ma^2\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\
&= \frac{6\hbar^2}{ma^2}
\end{aligned}$$

2.8

$$\begin{aligned}
 1 &= \int_0^{a/2} |\Psi|^2 dx \\
 &= A^2 \int_0^{a/2} dx \\
 &= \frac{aA^2}{2} \\
 A &= \sqrt{\frac{2}{a}} \\
 c_n &= \frac{2}{a} \int_0^{a/2} \sin\left(\frac{n\pi}{a}x\right) dx \\
 |c_1|^2 &= \left(\frac{2}{\pi}\right)^2 \\
 &\approx 0.405
 \end{aligned}$$

2.9

$$\begin{aligned}
 \Psi(x, 0) &= Ax(a - x) \\
 \langle H \rangle &= \int_0^a \Psi(x, 0)^* \hat{H} \Psi(x, 0) dx \\
 &= \int_0^a \Psi(x, 0)^* \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \Psi(x, 0) dx \\
 &= \frac{A^2 \hbar^2}{m} \int_0^a x(a - x) dx \\
 &= \frac{30 \hbar^2}{ma^5} \frac{a^3}{6} \\
 &= \frac{5 \hbar^2}{ma^2}
 \end{aligned}$$

2.10

(a)

$$\begin{aligned}
 \psi_2(x) &= \frac{1}{\sqrt{2!}}(\hat{a}_+)\psi_1 \\
 &= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\hbar m\omega}} \left(-\hbar \frac{d}{dx} + m\omega x\right) \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}} x e^{-\frac{m\omega}{2\hbar}x^2} \\
 &= \frac{1}{\sqrt{2\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(-\hbar \frac{d}{dx} + m\omega x\right) x e^{-\frac{m\omega}{2\hbar}x^2} \\
 &= \frac{1}{\sqrt{2\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left[-\hbar \left(e^{-\frac{m\omega}{2\hbar}x^2} - \frac{m\omega}{\hbar} x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right) + m\omega x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right] \\
 &= \frac{1}{\sqrt{2}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(\frac{2m\omega}{\hbar} x^2 - 1\right) e^{-\frac{m\omega}{2\hbar}x^2}
 \end{aligned}$$

2.11

(a)

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} \psi_0^* x \psi_0 dx \\
&= \alpha^2 \int_{-\infty}^{\infty} x e^{-\frac{m\omega}{\hbar} x^2} dx \\
&= 0 \\
\langle p \rangle &= m \frac{d\langle x \rangle}{dt} \\
&= 0 \\
\langle x^2 \rangle &= \int_{-\infty}^{\infty} \psi_0^* x^2 \psi_0 dx \\
&= \alpha^2 \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar} x^2} dx \\
&= \frac{\hbar}{2m\omega} \\
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \psi_0^* \left(-\hbar^2 \frac{d^2}{dx^2} \right) \psi_0 dx \\
&= -\hbar^2 \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \int_{-\infty}^{\infty} e^{-\frac{m\omega}{2\hbar} x^2} \frac{d}{dx} \left(-\frac{m\omega}{\hbar} x e^{-\frac{m\omega}{2\hbar} x^2} \right) dx \\
&= \hbar^2 \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \frac{m\omega}{\hbar} \int_{-\infty}^{\infty} e^{-\frac{m\omega}{2\hbar} x^2} \left(e^{-\frac{m\omega}{2\hbar} x^2} - \frac{m\omega}{\hbar} x^2 e^{-\frac{m\omega}{2\hbar} x^2} \right) dx \\
&= \hbar^2 \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \frac{m\omega}{\hbar} \int_{-\infty}^{\infty} \left(1 - \frac{m\omega}{\hbar} x^2 \right) e^{-\frac{m\omega}{2\hbar} x^2} dx \\
&= \hbar^2 \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \frac{m\omega}{\hbar} \frac{\hbar\sqrt{\pi}}{2\sqrt{\hbar m\omega}} \\
&= \frac{1}{2} m\hbar\omega
\end{aligned}$$

$$\begin{aligned}
\psi_1(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}} x e^{-\frac{m\omega}{2\hbar} x^2} \\
\langle x \rangle &= 0 \\
\langle p \rangle &= m \frac{d\langle x \rangle}{dt} \\
&= 0 \\
\langle x^2 \rangle &= \int_{-\infty}^{\infty} \psi_1^* x^2 \psi_1 dx \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{2m\omega}{\hbar} \int_{-\infty}^{\infty} x^4 e^{-\frac{m\omega}{\hbar} x^2} dx \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{2m\omega}{\hbar} \frac{3}{4} \sqrt{\pi} \left(\frac{\hbar}{m\omega}\right)^{5/2} \\
&= \frac{3}{2} \frac{\hbar}{m\omega} \\
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \psi_1^* \left(-\hbar^2 \frac{d^2}{dx^2}\right) \psi_1 dx \\
&= -\hbar^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{2m\omega}{\hbar} \int_{-\infty}^{\infty} x e^{-\frac{m\omega}{2\hbar} x^2} \frac{d}{dx} \left(e^{-\frac{m\omega}{2\hbar} x^2} - \frac{m\omega}{\hbar} x^2 e^{-\frac{m\omega}{2\hbar} x^2}\right) dx \\
&= -\hbar^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{2m\omega}{\hbar} \int_{-\infty}^{\infty} x e^{-\frac{m\omega}{2\hbar} x^2} \left[-\frac{m\omega}{\hbar} x e^{-\frac{m\omega}{2\hbar} x^2} - \frac{2m\omega}{\hbar} x e^{-\frac{m\omega}{2\hbar} x^2} + \left(\frac{m\omega}{\hbar}\right)^2 x^3 e^{-\frac{m\omega}{2\hbar} x^2}\right] dx \\
&= 2\hbar^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right)^2 \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar} x^2} \left(3 - \frac{m\omega}{\hbar} x^2\right) dx \\
&= 2\hbar^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right)^2 \frac{3}{4} \sqrt{\pi} \left(\frac{\hbar}{m\omega}\right)^{3/2} \\
&= \frac{3}{2} m\hbar\omega
\end{aligned}$$

(b)

$$\begin{aligned}\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\ &= \sqrt{\frac{\hbar}{2m\omega}} \\ \sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\ &= \sqrt{\frac{m\hbar\omega}{2}} \\ \sigma_x \sigma_p &= \frac{\hbar}{2} \\ \sigma_x &= \sqrt{\frac{3\hbar}{2m\omega}} \\ \sigma_p &= \sqrt{\frac{3m\hbar\omega}{2}} \\ \sigma_x \sigma_p &= \frac{3}{2}\hbar\end{aligned}$$

(c)

$$\begin{aligned}\langle T \rangle &= \frac{\langle p^2 \rangle}{2m} \\ &= \frac{\hbar\omega}{4} \\ \langle V \rangle &= \frac{1}{2}m\omega^2 \langle x^2 \rangle \\ &= \frac{1}{4}\hbar\omega \\ \langle T \rangle &= \frac{\langle p^2 \rangle}{2m} \\ &= \frac{3}{4}\hbar\omega \\ \langle V \rangle &= \frac{1}{2}m\omega^2 \langle x^2 \rangle \\ &= \frac{3}{4}\hbar\omega\end{aligned}$$

2.12

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} \psi_n^* x \psi_n dx \\
&= \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}_+ + \hat{a}_-) \psi_n dx \\
&= \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{\infty} \psi_n^* (\sqrt{n+1} \psi_{n+1} + \sqrt{n} \psi_{n-1}) dx \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\langle p \rangle &= \int_{-\infty}^{\infty} \psi_n^* p \psi_n dx \\
&= i \sqrt{\frac{\hbar m \omega}{2}} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}_+ - \hat{a}_-) \psi_n dx \\
&= i \sqrt{\frac{\hbar m \omega}{2}} \int_{-\infty}^{\infty} \psi_n^* (\sqrt{n+1} \psi_{n+1} - \sqrt{n} \psi_{n-1}) dx \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} \psi_n^* x^2 \psi_n dx \\
&= \frac{\hbar}{2m\omega} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}_+^2 + \hat{a}_+ \hat{a}_- + \hat{a}_- \hat{a}_+ + \hat{a}_-^2) \psi_n dx \\
&= \frac{\hbar}{2m\omega} (2n+1) \int_{-\infty}^{\infty} |\psi_n|^2 dx \\
&= \frac{\hbar}{m\omega} \left(n + \frac{1}{2} \right)
\end{aligned}$$

$$\begin{aligned}
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \psi_n^* p^2 \psi_n dx \\
&= -\frac{\hbar m \omega}{2} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}_+^2 - \hat{a}_+ \hat{a}_- - \hat{a}_- \hat{a}_+ + \hat{a}_-^2) \psi_n dx \\
&= \frac{\hbar m \omega}{2} (2n+1) \int_{-\infty}^{\infty} |\psi_n|^2 dx \\
&= \hbar m \omega \left(n + \frac{1}{2} \right)
\end{aligned}$$

$$\begin{aligned}
\langle T \rangle &= \left\langle \frac{p^2}{2m} \right\rangle \\
&= \frac{1}{2} \hbar \omega \left(n + \frac{1}{2} \right)
\end{aligned}$$

$$\begin{aligned}
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{\hbar}{m\omega} \left(n + \frac{1}{2} \right)} \\
\sigma_p &= \sqrt{\hbar m\omega \left(n + \frac{1}{2} \right)} \\
\sigma_x \sigma_p &= (2n + 1) \frac{\hbar}{2} \\
&\geq \frac{\hbar}{2}
\end{aligned}$$

2.13

(a)

$$\begin{aligned}
\Psi(x, 0) &= A[3\psi_0(x) + 4\psi_1(x)] \\
1 &= \int_{-\infty}^{\infty} |\Psi(x, 0)|^2 dx \\
&= A^2 \int_{-\infty}^{\infty} [9\psi_0(x)^2 + 24\psi_0(x)\psi_1(x) + 16\psi_1(x)^2] dx \\
&= 25A^2 \\
A &= \frac{1}{5}
\end{aligned}$$

(b)

$$\begin{aligned}
\Psi(x, t) &= \frac{1}{5}[3\psi_0(x)e^{-i\omega t/2} + 4\psi_1(x)e^{-3i\omega t/2}] \\
|\Psi(x, t)|^2 &= \Psi(x, t)^* \Psi(x, t) \\
&= \frac{1}{25}[3\psi_0(x)e^{i\omega t/2} + 4\psi_1(x)e^{3i\omega t/2}][3\psi_0(x)e^{-i\omega t/2} + 4\psi_1(x)e^{-3i\omega t/2}] \\
&= \frac{1}{25}[9\psi_0(x)^2 + 12\psi_0(x)\psi_1(x)e^{-i\omega t} + 12\psi_0(x)\psi_1(x)e^{i\omega t} + 16\psi_1(x)^2] \\
&= \frac{1}{25}[9\psi_0(x)^2 + 16\psi_1(x)^2 + 24\psi_0(x)\psi_1(x)\cos\omega t]
\end{aligned}$$

(c)

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} \Psi^* x \Psi dx \\
&= \frac{1}{25} \int_{-\infty}^{\infty} x(9\psi_0^2 + 16\psi_1^2 + 24\psi_0\psi_1 \cos \omega t) dx \\
&= \frac{24}{25} \int_{-\infty}^{\infty} x\psi_0\psi_1 \cos(\omega t) dx \\
&= \frac{24}{25} \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \sqrt{\frac{2m\omega}{\hbar}} \cos(\omega t) \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar} x^2} dx \\
&= \frac{24}{25} \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \sqrt{\frac{2m\omega}{\hbar}} \cos(\omega t) \frac{1}{2} \sqrt{\pi} \left(\frac{\hbar}{m\omega} \right)^{3/2} \\
&= \frac{24}{25} \sqrt{\frac{\hbar}{2m\omega}} \cos(\omega t) \\
\langle p \rangle &= m \frac{d\langle x \rangle}{dt} \\
&= -\frac{24}{25} \sqrt{\frac{\hbar m\omega}{2}} \sin(\omega t) \\
\frac{d\langle p \rangle}{dt} &= -\frac{24}{25} \sqrt{\frac{\hbar m\omega}{2}} \omega \cos(\omega t) \\
V &= \frac{1}{2} m\omega^2 x^2 \\
\frac{\partial V}{\partial \theta} &= m\omega^2 x \\
\left\langle -\frac{\partial V}{\partial x} \right\rangle &= -m\omega^2 \langle x \rangle \\
&= -\frac{24}{25} \sqrt{\frac{\hbar m\omega}{2}} \omega \cos(\omega t) \\
&= \frac{d\langle p \rangle}{dt}
\end{aligned}$$

(d)

$$\begin{aligned}
E_0 &= \frac{\hbar\omega}{2} \\
P(E_0) &= \frac{9}{25} \\
E_1 &= \frac{3\hbar\omega}{2} \\
P(E_1) &= \frac{16}{25}
\end{aligned}$$

2.14

$$1 - \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\sqrt{\hbar/m\omega}}^{\sqrt{\hbar/m\omega}} e^{-m\omega x^2/\hbar} dx = 1 - \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \sqrt{\frac{\pi\hbar}{m\omega}} \operatorname{erf} 1$$

$$= 0.157$$

2.15

$$a_{j+2} = \frac{-2(n-j)}{(j+1)(j+2)} a_j$$

$$a_3 = -\frac{4}{3} a_1$$

$$a_5 = \frac{4}{15} a_1$$

$$H_5(\xi) = a_1 \left(\xi - \frac{4}{3} \xi^3 + \frac{4}{15} \xi^5 \right)$$

$$= \frac{1}{120} a_1 (120\xi - 160\xi^3 + 32\xi^5)$$

$$= 32\xi^5 - 160\xi^3 + 120\xi$$

$$a_2 = -6a_0$$

$$a_4 = \frac{-8}{12} a_2$$

$$= 4a_0$$

$$a_6 = \frac{-4}{30} a_4$$

$$= -\frac{8}{15} a_0$$

$$H_6(\xi) = a_0 \left(1 - 6\xi^2 + 4\xi^4 - \frac{8}{15} \xi^6 \right)$$

$$= \frac{1}{120} a_0 (120 - 720\xi^2 + 480\xi^4 - 64\xi^6)$$

$$= 64\xi^6 - 480\xi^4 + 720\xi^2 - 120$$

2.17

$$\begin{aligned}
Ae^{ikx} + Be^{-ikx} &= A[\cos(kx) + i\sin(kx)] + B[\cos(kx) - i\sin(kx)] \\
&= (A + B)\cos(kx) + i(A - B)\sin(kx) \\
C &= A + B \\
D &= i(A - B) \\
-iD &= A - B \\
A &= \frac{C - iD}{2} \\
B &= \frac{C + iD}{2}
\end{aligned}$$

2.18

$$\begin{aligned}
\Psi_k(x, t) &= Ae^{i\left(kx - \frac{\hbar k^2}{2m}t\right)} \\
J(x, t) &= \frac{i\hbar}{2m} \left(\Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi}{\partial x} \right) \\
&= \frac{\hbar k |A|^2}{m}
\end{aligned}$$

The probability travels in the same direction as the wave.

2.20

(a)

$$\begin{aligned}
\Psi(x, 0) &= Ae^{-a|x|} \\
1 &= \int_{-\infty}^{\infty} \Psi^* \Psi \, dx \\
&= |A|^2 \int_{-\infty}^{\infty} e^{-2a|x|} \, dx \\
&= 2|A|^2 \int_0^{\infty} e^{-2ax} \, dx \\
&= \frac{|A|^2}{a} \\
A &= \sqrt{a}
\end{aligned}$$

(b)

$$\begin{aligned}
\phi(k) &= \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|-ikx} dx \\
&= \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} [\cos(kx) - i \sin(kx)] dx \\
&= \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} \cos(kx) dx \\
&= \sqrt{\frac{a}{2\pi}} 2 \int_0^{\infty} e^{-ax} \cos(kx) dx \\
&= \sqrt{\frac{a}{2\pi}} \frac{2a}{a^2 + k^2}
\end{aligned}$$

(c)

$$\Psi(x, t) = \frac{a^{3/2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{a^2 + k^2} e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)} dk$$

2.21

(a)

$$\begin{aligned}
\Psi(x, 0) &= A e^{-ax^2} \\
1 &= A^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx \\
&= \sqrt{\frac{\pi}{2a}} A^2 \\
A &= \left(\frac{2a}{\pi}\right)^{1/4}
\end{aligned}$$

(b)

$$\begin{aligned}
\phi(k) &= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-(ax^2 + ikx)} dx \\
&= \frac{1}{(2\pi a)^{1/4}} e^{-k^2/4a} \\
\Psi(x, t) &= \frac{1}{\sqrt{2\pi}} \frac{1}{(2\pi a)^{1/4}} \int_{-\infty}^{\infty} e^{-\frac{k^2}{4a} + i\left(kx - \frac{\hbar k^2}{2m} t\right)} dk \\
\Psi(x, t) &= \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\gamma} e^{-ax^2/\gamma^2}
\end{aligned}$$

(c)

$$\begin{aligned}
|\Psi(x, t)|^2 &= \Psi^* \Psi \\
&= \left(\frac{2a}{\pi}\right)^{1/2} \frac{1}{\gamma^*} e^{-ax^2/(\gamma^*)^2} \frac{1}{\gamma} e^{-ax^2/\gamma^2} \\
&= \left(\frac{2a}{\pi}\right)^{1/2} \frac{1}{\sqrt{1-2i\hbar a t/m}} e^{-ax^2/(1-2i\hbar a t/m)} \\
&\quad \frac{1}{\sqrt{1+2i\hbar a t/m}} e^{-ax^2/(1+2i\hbar a t/m)} \\
&= \left(\frac{2a}{\pi}\right)^{1/2} \frac{1}{\sqrt{1+(2\hbar a t/m)^2}} e^{-2ax^2/[1+(2\hbar a t/m)^2]} \\
&= \sqrt{\frac{2}{\pi}} w e^{-2w^2 x^2}
\end{aligned}$$

As t increases $|\Psi|^2$ flattens out and broadens.

(d)

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} \Psi^* x \Psi dx \\
&= 0 \\
\langle p \rangle &= m \frac{d\langle x \rangle}{dt} \\
&= 0 \\
\langle x^2 \rangle &= \int_{-\infty}^{\infty} \Psi^* x^2 \Psi dx \\
&= \sqrt{\frac{2}{\pi}} w \int_{-\infty}^{\infty} x^2 e^{-2w^2 x^2} dx \\
&= \frac{1}{4w^2}
\end{aligned}$$

2.22

(a) -25

(b) 1

(c) 0

2.26

$$\begin{aligned}
 F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-ikx} dx \\
 &= \frac{1}{\sqrt{2\pi}} \\
 f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk
 \end{aligned}$$

2.29

$$\begin{aligned}
 \psi(x) &= \begin{cases} Fe^{-\kappa x} & x > a \\ C \sin(lx) & 0 < x < a \\ -\psi(-x) & x < 0 \end{cases} \\
 Fe^{-\kappa a} &= C \sin(la) \\
 -\kappa Fe^{-\kappa a} &= lC \cos(la) \\
 -\frac{1}{\kappa} &= \frac{1}{l} \tan(la) \\
 \tan z &= -\frac{l}{\kappa} \\
 &= -\frac{la}{\kappa a} \\
 &= -\frac{z}{\sqrt{z_0^2 - z^2}}
 \end{aligned}$$

For large z_0 the intersections occur just below $z_n = n\pi$ so

$$\begin{aligned}
 z &= la \\
 n\pi &\approx \frac{\sqrt{2m(E + V_0)}}{\hbar} a \\
 E + V_0 &\approx \frac{n^2 \pi^2 \hbar^2}{2ma^2}.
 \end{aligned}$$

As z_0 decreases there are fewer and fewer bound states. When $z_0 < \pi/2$ there are no odd bound states.

2.30

$$\begin{aligned}
\psi(x) &= \begin{cases} Fe^{-\kappa x} & x > a \\ D \cos(lx) & 0 < x < a \\ \psi(-x) & x < 0 \end{cases} \\
1 &= \int_{-\infty}^{\infty} |\psi|^2 dx \\
&= 2 \left(|D|^2 \int_0^a \cos^2(lx) dx + |F|^2 \int_a^{\infty} e^{-2\kappa x} dx \right) \\
&= 2 \left[|D|^2 \frac{2al + \sin(2al)}{4l} + \frac{|F|^2}{2\kappa} e^{-2\kappa a} \right] \\
&= |D|^2 \left[a + \frac{\sin(2al)}{2l} + \frac{\cos^2(al)}{\kappa} \right] \\
&= |D|^2 \left[a + \frac{2 \sin(al) \cos(al)}{2l} + \frac{\cos^3(al)}{l \sin(al)} \right] \\
&= |D|^2 \left\{ a + \frac{\cos(al)}{l \sin(al)} [\sin^2(al) + \cos^2(al)] \right\} \\
&= |D|^2 \left[a + \frac{\cos(al)}{l \sin(al)} \right] \\
&= |D|^2 \left[a + \frac{1}{l \tan(al)} \right] \\
&= |D|^2 \left[a + \frac{1}{\kappa} \right] \\
D &= \frac{1}{\sqrt{a + 1/\kappa}}
\end{aligned}$$

$$\begin{aligned}
1 &= \left\{ \frac{1}{a + 1/\kappa} \left[a + \frac{\sin(2al)}{2l} \right] + |F|^2 \frac{e^{-2\kappa a}}{\kappa} \right\} \\
\frac{(Fe^{-\kappa a})^2}{\kappa} &= 1 - \frac{1}{a + 1/\kappa} \left[a + \frac{\sin(2al)}{2l} \right] \\
(Fe^{-\kappa a})^2 &= \kappa - \frac{\kappa}{a + 1/\kappa} \left[a + \frac{\sin(2al)}{2l} \right] \\
&= \frac{\kappa a + 1 - \kappa a - \kappa \sin(al) \cos(al)/l}{a + 1/\kappa} \\
&= \frac{1 - \sin^2(al)}{a + 1/\kappa} \\
F &= \frac{e^{\kappa a} \cos(al)}{\sqrt{a + 1/\kappa}}
\end{aligned}$$

2.31

$$\begin{aligned}1 &= 2aV_0 \\ V_0 &= \frac{1}{2a} \\ z_0 &= \frac{a}{\hbar} \sqrt{2mV_0} \\ &= \frac{a}{\hbar} \sqrt{\frac{m}{a}} \\ &= \frac{\sqrt{am}}{\hbar} \\ \lim_{a \rightarrow 0} z_0 &= 0\end{aligned}$$

2.34

(a)

$$\begin{aligned}
 V(x) &= \begin{cases} 0 & x \leq 0 \\ V_0 & x > 0 \end{cases} \\
 -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} &= E\psi \\
 \frac{d^2\psi}{dx^2} &= -\frac{2mE}{\hbar^2}\psi \\
 &= -k^2\psi \\
 k &= \frac{\sqrt{2mE}}{\hbar} \\
 \psi &= Ae^{ikx} + Be^{-ikx} \\
 -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V_0\psi &= E\psi \\
 \frac{d^2\psi}{dx^2} &= -\frac{2m(E - V_0)}{\hbar^2}\psi \\
 &= \kappa^2\psi \\
 \kappa &= \frac{\sqrt{2m(V_0 - E)}}{\hbar} \\
 \psi &= Fe^{-\kappa x} \\
 A + B &= F \\
 ik(A - B) &= -\kappa F \\
 F &= -i\frac{k}{\kappa}(A - B) \\
 A + B &= -i\frac{k}{\kappa}(A - B) \\
 \left(1 - i\frac{k}{\kappa}\right)B &= -\left(1 + i\frac{k}{\kappa}\right)A \\
 B &= -\frac{1 + ik/\kappa}{1 - ik/\kappa}A \\
 R &= \frac{|B|^2}{|A|^2} \\
 &= \left(-\frac{1 + ik/\kappa}{1 - ik/\kappa}\right)\left(-\frac{1 - ik/\kappa}{1 + ik/\kappa}\right) \\
 &= 1
 \end{aligned}$$

(b)

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \\ Fe^{ilx} & x > 0 \end{cases}$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$$l = \frac{\sqrt{2m(E - V_0)}}{\hbar}$$

$$A + B = F$$

$$ik(A - B) = ilF$$

$$F = \frac{k}{l}(A - B)$$

$$A + B = \frac{k}{l}(A - B)$$

$$\left(\frac{k}{l} + 1\right)B = \left(\frac{k}{l} - 1\right)A$$

$$B = \frac{k/l - 1}{k/l + 1}A$$

$$R = \frac{|B|^2}{|A|^2}$$

$$= \left(\frac{k/l - 1}{k/l + 1}\right)^2$$

$$= \left(\frac{k - l}{k + l}\right)^2$$

$$= \frac{(k - l)^4}{(k^2 - l^2)^2}$$

$$k^2 - l^2 = \frac{2mE}{\hbar^2} - \frac{2m(E - V_0)}{\hbar^2}$$

$$= \frac{2m}{\hbar^2}V_0$$

$$k - l = \frac{\sqrt{2m}}{\hbar}(\sqrt{E} - \sqrt{E - V_0})$$

$$R = \frac{(\sqrt{E} - \sqrt{E - V_0})^4}{V_0^2}$$

(d)

$$\begin{aligned} B &= F - A \\ F &= \frac{k}{l}(A - F + A) \\ \left(1 + \frac{k}{l}\right) F &= \frac{2k}{l} A \\ F &= \frac{2k}{k+l} A \\ \frac{l}{k} &= \sqrt{\frac{E - V_0}{E}} \\ T &= \left| \frac{F}{A} \right|^2 \frac{l}{k} \\ &= \left(\frac{2k}{k+l} \right)^2 \frac{l}{k} \\ &= \frac{4kl}{(k+l)^2} \\ &= \frac{4kl(k-l)^2}{(k^2 - l^2)^2} \\ &= \frac{4\sqrt{E}\sqrt{E - V_0}(\sqrt{E} - \sqrt{E - V_0})^2}{V_0^2} \\ T + R &= \frac{4kl}{(k+l)^2} + \frac{(k-l)^2}{(k+l)^2} \\ &= \frac{4kl + k^2 - 2kl + l^2}{(k+l)^2} \\ &= \frac{k^2 + 2kl + l^2}{(k+l)^2} \\ &= \frac{(k+l)^2}{(k+l)^2} \\ &= 1 \end{aligned}$$

2.35

(a)

$$\begin{aligned}
 V(x) &= \begin{cases} 0 & x < 0 \\ -V_0 & x > 0 \end{cases} \\
 -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} &= E\psi \\
 \frac{d^2\psi}{dx^2} &= -\frac{2mE}{\hbar^2}\psi \\
 &= -k^2\psi \\
 k &= \frac{\sqrt{2mE}}{\hbar} \\
 \psi &= Ae^{ikx} + Be^{-ikx} \\
 -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} - V_0\psi &= E\psi \\
 \frac{d^2\psi}{dx^2} &= -\frac{2m}{\hbar^2}(E + V_0)\psi \\
 &= -l^2\psi \\
 l &= \frac{\sqrt{2m(E + V_0)}}{\hbar} \\
 \psi &= Fe^{ilx} \\
 A + B &= F \\
 ik(A - B) &= ilF \\
 k(A - B) &= l(A + B) \\
 (k + l)B &= (k - l)A \\
 B &= \frac{k - l}{k + l}A \\
 R &= \left| \frac{B}{A} \right|^2 \\
 &= \left(\frac{k - l}{k + l} \right)^2 \\
 &= \left(\frac{\sqrt{E} - \sqrt{E + V_0}}{\sqrt{E} + \sqrt{E + V_0}} \right)^2 \\
 &= \frac{1}{9}
 \end{aligned}$$

(c)

$$T = 1 - R = \frac{8}{9}$$

2.36

$$\begin{aligned}
 V(x) &= \begin{cases} 0 & |x| < a \\ \infty & |x| > a \end{cases} \\
 -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} &= E\psi \\
 \frac{d^2\psi}{dx^2} &= -\frac{2mE}{\hbar^2}\psi \\
 &= -k^2\psi \\
 k &= \frac{\sqrt{2mE}}{\hbar} \\
 \psi &= A \sin kx + B \cos kx \\
 0 &= -A \sin ka + B \cos ka \\
 0 &= A \sin ka + B \cos ka \\
 B \cos ka &= 0 \\
 k &= \frac{n\pi}{2a}, \quad n = 1, 3, 5, \dots \\
 E &= \frac{n^2\pi^2\hbar^2}{2m(2a)^2} \\
 \psi &= B \cos\left(\frac{n\pi}{2a}x\right), \quad n = 1, 3, 5, \dots \\
 1 &= |B|^2 \int_{-a}^a \cos^2\left(\frac{n\pi}{2a}x\right) dx \\
 B &= \frac{1}{\sqrt{a}} \\
 \psi &= \frac{1}{\sqrt{a}} \cos\left(\frac{n\pi}{2a}x\right), \quad n = 1, 3, 5, \dots \\
 A \sin ka &= 0 \\
 k &= \frac{n\pi}{2a}, \quad n = 2, 4, 6, \dots \\
 E &= \frac{n^2\pi^2\hbar^2}{2m(2a)^2} \\
 \psi &= A \sin\left(\frac{n\pi}{2a}x\right), \quad n = 2, 4, 6, \dots \\
 1 &= |A|^2 \int_{-a}^a \sin^2\left(\frac{n\pi}{2a}x\right) dx \\
 A &= \frac{1}{\sqrt{a}} \\
 \psi &= \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi}{2a}x\right), \quad n = 2, 4, 6, \dots
 \end{aligned}$$

2.37

$$\begin{aligned}
\Psi(x, 0) &= A \sin^3 \left(\frac{\pi}{a} x \right) \\
&= A \left[\frac{3}{4} \sin \left(\frac{\pi}{a} x \right) - \frac{1}{4} \sin \left(\frac{3\pi}{a} x \right) \right] \\
&= A \sqrt{\frac{a}{2}} \left[\frac{3}{4} \psi_1(x) - \frac{1}{4} \psi_3(x) \right] \\
1 &= |A|^2 \frac{a}{2} \int_0^a \left[\frac{3}{4} \psi_1(x) - \frac{1}{4} \psi_3(x) \right]^2 dx \\
&= |A|^2 \frac{a}{2} \int_0^a \left[\frac{9}{16} \psi_1(x)^2 - \frac{3}{8} \psi_1(x) \psi_3(x) + \frac{1}{16} \psi_3(x)^2 \right] dx \\
&= \frac{5}{16} a |A|^2 \\
A &= \frac{4}{\sqrt{5a}} \\
\Psi(x, 0) &= \frac{1}{\sqrt{10}} [3\psi_1(x) - \psi_3(x)] \\
\Psi(x, t) &= \frac{1}{\sqrt{10}} [3\psi_1(x) e^{-iE_1 t/\hbar} - \psi_3(x) e^{-iE_3 t/\hbar}] \\
\langle x \rangle &= \int_0^a \Psi^* x \Psi dx \\
&= \frac{1}{10} \int_0^a x \left(9\psi_1^2 + \psi_3^2 - 3\psi_1\psi_3 e^{-i(E_3-E_1)t/\hbar} - 3\psi_1\psi_3 e^{-i(E_1-E_3)t/\hbar} \right) dx \\
&= \frac{1}{10} \int_0^a x \left[9\psi_1^2 + \psi_3^2 - 6\psi_1\psi_3 \cos \left(\frac{E_3 - E_1}{\hbar} t \right) \right] dx \\
&= \frac{1}{10} \left[9 \langle x \rangle_1 + \langle x \rangle_3 - 6 \cos \left(\frac{E_3 - E_1}{\hbar} t \right) \int_0^a x \psi_1 \psi_3 dx \right] \\
&= \frac{1}{10} \left[\frac{9}{2} a + \frac{1}{2} a \right] \\
&= \frac{a}{2} \\
P(E_1) &= \frac{9}{10} \\
P(E_3) &= \frac{1}{10} \\
\langle E \rangle &= E_1 P(E_1) + E_3 P(E_3) \\
&= \frac{9\pi^2 \hbar^2}{20ma^2} + \frac{9\pi^2 \hbar^2}{20ma^2} \\
&= \frac{9\pi^2 \hbar^2}{10ma^2}
\end{aligned}$$

2.38

(a)

$$\begin{aligned}
 \Psi(x, t) &= \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar} \\
 \Psi(x, 0) &= \sum_{n=1}^{\infty} c_n \psi_n(x) \\
 E_n T &= \frac{n^2 \pi^2 \hbar^2}{2ma^2} \frac{4ma^2}{\pi \hbar} \\
 &= 2\pi n^2 \hbar \\
 \Psi(x, T) &= \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n T/\hbar} \\
 &= \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-2\pi i n^2} \\
 &= \sum_{n=1}^{\infty} c_n \psi_n(x) \\
 &= \Psi(x, 0)
 \end{aligned}$$

(b)

$$\begin{aligned}
 E &= \frac{1}{2} m v^2 \\
 v &= \sqrt{\frac{2E}{m}} \\
 T &= \frac{2a}{v} \\
 &= a \sqrt{\frac{2m}{E}}
 \end{aligned}$$

(c)

$$\begin{aligned}
 \frac{4ma^2}{\pi \hbar} &= a \sqrt{\frac{2m}{E}} \\
 \frac{16m^2 a^2}{\pi^2 \hbar^2} &= \frac{2m}{E} \\
 E &= \frac{\pi^2 \hbar^2}{8ma^2} \\
 &= \frac{E_1}{4}
 \end{aligned}$$

2.39

(a)

$$\Psi(x, 0) = \begin{cases} \frac{2\sqrt{3}}{a\sqrt{a}}x & 0 \leq x \leq a/2 \\ \frac{2\sqrt{3}}{a\sqrt{a}}(a-x) & a/2 \leq x \leq a \end{cases}$$

$$\frac{d}{dx}\Psi(x, 0) = \frac{2\sqrt{3}}{a\sqrt{a}} \left[1 - 2\theta\left(x - \frac{a}{2}\right) \right]$$

(b)

$$\frac{d^2}{dx^2}\Psi(x, 0) = -\frac{4\sqrt{3}}{a\sqrt{a}}\delta\left(x - \frac{a}{2}\right)$$

(c)

$$\begin{aligned} \hat{H}\Psi(x, 0) &= \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \Psi(x, 0) \\ &= \frac{\hbar^2}{2m} \frac{4\sqrt{3}}{a\sqrt{a}} \delta\left(x - \frac{a}{2}\right) + V(x)\Psi(x, 0) \\ \langle H \rangle &= \int \Psi(x, 0)^* \hat{H}\Psi(x, 0) dx \\ &= \int_0^a \Psi(x, 0)^* \left[\frac{\hbar^2}{2m} \frac{4\sqrt{3}}{a\sqrt{a}} \delta\left(x - \frac{a}{2}\right) + V(x)\Psi(x, 0) \right] dx \\ &= \Psi\left(\frac{a}{2}, 0\right)^* \frac{\hbar^2}{2m} \frac{4\sqrt{3}}{a\sqrt{a}} + \int_0^a \Psi(x, 0)^* V(x)\Psi(x, 0) dx \\ &= \frac{6\hbar^2}{ma^2} \end{aligned}$$

2.40

(a)

$$\begin{aligned}
V(x) &= \frac{1}{2}m\omega^2 x^2 \\
\xi &= \sqrt{\frac{m\omega}{\hbar}}x \\
\psi_n(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2} \\
\Psi(x, 0) &= A \left(1 - 2\sqrt{\frac{m\omega}{\hbar}}x\right)^2 e^{-\frac{m\omega}{2\hbar}x^2} \\
&= A \left(1 - 4\sqrt{\frac{m\omega}{\hbar}}x + \frac{4m\omega}{\hbar}x^2\right) e^{-\frac{m\omega}{2\hbar}x^2} \\
&= A \left(\frac{\pi\hbar}{m\omega}\right)^{1/4} \left[3\psi_0(x) - 2\sqrt{2}\psi_1(x) + 2\sqrt{2}\psi_2(x)\right] \\
1 &= A^2 \sqrt{\frac{\pi\hbar}{m\omega}} \int_{-\infty}^{\infty} (3\psi_0 - 2\sqrt{2}\psi_1 + 2\sqrt{2}\psi_2)^2 dx \\
&= A^2 \sqrt{\frac{\pi\hbar}{m\omega}} \int_{-\infty}^{\infty} (9\psi_0^2 - 12\sqrt{2}\psi_0\psi_1 + 12\sqrt{2}\psi_0\psi_2 + 8\psi_1^2 - 16\psi_1\psi_2 + 8\psi_2^2) dx \\
&= 25A^2 \sqrt{\frac{\pi\hbar}{m\omega}} \\
A &= \frac{1}{5} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \\
\Psi(x, 0) &= \frac{3}{5}\psi_0(x) - \frac{2\sqrt{2}}{5}\psi_1(x) + \frac{2\sqrt{2}}{5}\psi_2(x)
\end{aligned}$$

(b)

$$\begin{aligned}E_0 &= \frac{\hbar\omega}{2} \\P(E_0) &= \frac{9}{25} \\E_1 &= \frac{3\hbar\omega}{2} \\P(E_1) &= \frac{8}{25} \\E_2 &= \frac{5\hbar\omega}{2} \\P(E_2) &= \frac{8}{25} \\\langle E \rangle &= \frac{\hbar\omega}{2} \frac{9}{25} + \frac{3\hbar\omega}{2} \frac{8}{25} + \frac{5\hbar\omega}{2} \frac{8}{25} \\&= \frac{73}{50} \hbar\omega\end{aligned}$$

(c)

$$\begin{aligned}
\xi &= \sqrt{\frac{m\omega}{\hbar}} x \\
\psi_n(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2} \\
\Psi(x, T) &= B \left(1 + 2\sqrt{\frac{m\omega}{\hbar}} x\right)^2 e^{-\frac{m\omega}{2\hbar} x^2} \\
&= B \left(1 + 4\sqrt{\frac{m\omega}{\hbar}} x + 4\frac{m\omega}{\hbar} x^2\right) e^{-\frac{m\omega}{2\hbar} x^2} \\
&= B \left(\frac{\pi\hbar}{m\omega}\right)^{1/4} \left[3\psi_0(x) + 2\sqrt{2}\psi_1(x) + 2\sqrt{2}\psi_2(x)\right] \\
1 &= |B|^2 \sqrt{\frac{\pi\hbar}{m\omega}} \int_{-\infty}^{\infty} [3\psi_0 + 2\sqrt{2}\psi_1 + 2\sqrt{2}\psi_2]^2 dx \\
&= |B|^2 \sqrt{\frac{\pi\hbar}{m\omega}} \int_{-\infty}^{\infty} [9\psi_0^2 + 12\sqrt{2}\psi_0\psi_1 + 12\sqrt{2}\psi_0\psi_2 + 8\psi_1^2 + 16\psi_1\psi_2 + 8\psi_2^2] dx \\
&= 25|B|^2 \sqrt{\frac{\pi\hbar}{m\omega}} \\
B &= \frac{1}{5} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \\
\Psi(x, T) &= \frac{3}{5}\psi_0(x) + \frac{2\sqrt{2}}{5}\psi_1(x) + \frac{2\sqrt{2}}{5}\psi_2(x) \\
\Psi(x, t) &= \frac{3}{5}\psi_0(x)e^{-i\omega t/2} - \frac{2\sqrt{2}}{5}\psi_1(x)e^{-3i\omega t/2} + \frac{2\sqrt{2}}{5}\psi_2(x)e^{-5i\omega t/2} \\
&= e^{-i\omega t/2} \left[\frac{3}{5}\psi_0(x) - \frac{2\sqrt{2}}{5}\psi_1(x)e^{-i\omega t} + \frac{2\sqrt{2}}{5}\psi_2(x)e^{-2i\omega t} \right] \\
e^{-i\omega T} &= -1 \\
e^{-2i\omega T} &= 1 \\
T &= \frac{\pi}{\omega}
\end{aligned}$$

2.41

The argument for calculating the allowed energies and wavefunctions is the same, except there is a boundary condition $\psi(0) = 0$. This leaves only $\psi_n(x)$ for odd n .

2.43

$$\begin{aligned}
k &= \frac{\sqrt{2mE}}{\hbar} \\
\psi(x) &= \begin{cases} -A \sin kx + B \cos kx & -a < x \leq 0 \\ A \sin kx + B \cos kx & 0 \leq x < a \end{cases} \\
\Delta \left(\frac{d\psi}{dx} \right) &= 2Ak \\
\Delta \left(\frac{d\psi}{dx} \right) &= \frac{2m\alpha}{\hbar^2} \psi(0) \\
2Ak &= \frac{2m\alpha}{\hbar^2} B \\
B &= \frac{\hbar^2 k}{m\alpha} A \\
\psi(x) &= A \left(\sin kx + \frac{\hbar^2 k}{m\alpha} \cos kx \right) \\
0 &= A \sin ka + \frac{\hbar^2 k}{m\alpha} A \cos ka \\
\tan ka &= -\frac{\hbar^2 k}{m\alpha} \\
ka &\approx \frac{n\pi}{2}, \quad n = 1, 3, 5, \dots \\
E &\approx \frac{n^2 \pi^2 \hbar^2}{2m(2a)^2}, \quad n = 1, 3, 5, \dots \\
\psi(x) &= \begin{cases} A \sin kx - B \cos kx & -a < x \leq 0 \\ A \sin kx + B \cos kx & 0 \leq x < a \end{cases} \\
-B &= B \\
B &= 0 \\
\psi(x) &= A \sin kx \\
0 &= A \sin ka \\
ka &= \frac{n\pi}{2}, \quad n = 2, 4, 6, \dots \\
\psi(x) &= A \sin \left(\frac{n\pi}{2a} x \right) \\
E &= \frac{n^2 \pi^2 \hbar^2}{2m(2a)^2}, \quad n = 2, 4, 6, \dots
\end{aligned}$$

2.44

$$\begin{aligned}
-\frac{\hbar^2}{2m} \frac{d^2 \psi_1}{dx^2} \psi_2 + V \psi_1 \psi_2 &= E \psi_1 \psi_2 \\
-\frac{\hbar^2}{2m} \frac{d^2 \psi_2}{dx^2} \psi_1 + V \psi_1 \psi_2 &= E \psi_1 \psi_2 \\
\psi_2 \frac{d^2 \psi_1}{dx^2} - \psi_1 \frac{d^2 \psi_2}{dx^2} &= 0 \\
\frac{d}{dx} \left(\psi_2 \frac{d\psi_1}{dx} - \psi_1 \frac{d\psi_2}{dx} \right) &= 0 \\
\psi_2 \frac{d\psi_1}{dx} - \psi_1 \frac{d\psi_2}{dx} &= c \\
c &= 0 \\
\psi_2 \frac{d\psi_1}{dx} &= \psi_1 \frac{d\psi_2}{dx} \\
\frac{1}{\psi_1} \frac{d\psi_1}{dx} &= \frac{1}{\psi_2} \frac{d\psi_2}{dx} \\
\ln \psi_1 &= \ln \psi_2 + c \\
\psi_1 &= A \psi_2
\end{aligned}$$

2.45

(a)

$$\begin{aligned}
-\frac{\hbar^2}{2m} \frac{d^2 \psi_n}{dx^2} \psi_m + V(x) \psi_n \psi_m &= E_n \psi_n \psi_m \\
-\frac{\hbar^2}{2m} \frac{d^2 \psi_m}{dx^2} \psi_n + V(x) \psi_n \psi_m &= E_m \psi_n \psi_m \\
\frac{d^2 \psi_m}{dx^2} \psi_n - \frac{d^2 \psi_n}{dx^2} \psi_m &= \frac{2m}{\hbar^2} (E_n - E_m) \psi_n \psi_m \\
\frac{d}{dx} \left(\frac{d\psi_m}{dx} \psi_n - \frac{d\psi_n}{dx} \psi_m \right) &= \frac{2m}{\hbar^2} (E_n - E_m) \psi_n \psi_m
\end{aligned}$$

(b)

$$\begin{aligned}
\int_{x_1}^{x_2} \frac{d}{dx} (\psi'_m \psi_n - \psi'_n \psi_m) dx &= \frac{2m}{\hbar^2} (E_n - E_m) \int_{x_1}^{x_2} \psi_n \psi_m dx \\
\psi'_m(x_2) \psi_n(x_2) - \psi'_m(x_1) \psi_n(x_1) &= \frac{2m}{\hbar^2} (E_n - E_m) \int_{x_1}^{x_2} \psi_n \psi_m dx
\end{aligned}$$

2.53

(a)

$$\frac{1}{1 - i\beta} \begin{pmatrix} i\beta & 1 \\ 1 & i\beta \end{pmatrix}$$

(b)

$$\frac{e^{-2ika}}{\cos(2la) - i \frac{(k^2 + l^2)}{2kl} \sin(2la)} \begin{pmatrix} i \frac{\sin(2la)}{2kl} (l^2 - k^2) & 1 \\ 1 & i \frac{\sin(2la)}{2kl} (l^2 - k^2) \end{pmatrix}$$

3 Formalism

3.1

(a)

$$\begin{aligned} \left| \int_a^b (f^* + g^*)(f + g) dx \right| &= \left| \int_a^b (f^* f + f^* g + g^* f + g^* g) dx \right| \\ &\leq \int_a^b |f|^2 dx + \left| \int_a^b f^* g dx \right| + \left| \int_a^b g^* f dx \right| + \int_a^b |g|^2 dx \\ &\leq \int_a^b |f|^2 dx + 2 \sqrt{\int_a^b |f|^2 dx \int_a^b |g|^2 dx} + \int_a^b |g|^2 dx \end{aligned}$$

The set of all normalised functions isn't a vector space because e.g. multiplying a function by a constant also multiplies its integral by that constant meaning it's no longer a member of the vector space.

(b)

$$\begin{aligned} \langle \beta | \alpha \rangle &= \int_a^b \beta^* \alpha dx \\ &= \left(\int_a^b \alpha^* \beta dx \right)^* \\ &= \langle \alpha | \beta \rangle^* \\ \langle a | a \rangle &= \int_a^b |a|^2 dx \\ &\geq 0 \end{aligned}$$

If $\langle \alpha | \alpha \rangle = 0$ that implies $|\alpha|^2 = 0$ everywhere in the interval and thus $|\alpha\rangle = |0\rangle$.

$$\begin{aligned} \langle \alpha | (b|\beta\rangle + c|\gamma\rangle) &= \int_{x_1}^{x_2} \alpha^* (b\beta) dx + \int_{x_1}^{x_2} \alpha^* (c\gamma) dx \\ &= b \int_{x_1}^{x_2} \alpha^* \beta dx + c \int_{x_1}^{x_2} \alpha^* \gamma dx \\ &= b \langle \alpha | \beta \rangle + c \langle \alpha | \gamma \rangle \end{aligned}$$

3.2

(a)

$$\int_0^1 x^{2\nu} dx = \frac{1}{2\nu+1} [x^{2\nu+1}]_0^1$$

The integral is defined for $\nu > -1/2$. For the case $\nu = -1/2$

$$\int_0^1 x^{-1} dx = [\ln x]_0^1 = \ln 1 - \ln 0 = 0 - \infty.$$

So $f(x) = x^\nu$ is in Hilbert space for $\nu > -1/2$.

(b)

$$\begin{aligned} \int_0^1 x dx &= \frac{1}{2} \\ \int_0^1 x^3 dx &= \frac{1}{4} \\ \int_0^1 x^{-1} dx &= [\ln x]_0^1 \\ &= 0 - \infty \end{aligned}$$

$f(x)$ and $xf(x)$ are in Hilbert space, but not $(d/dx)f(x)$.

3.4

(a)

$$\begin{aligned} \langle f | (\hat{Q} + \hat{R})f \rangle &= \langle f | \hat{Q}f \rangle + \langle f | \hat{R}f \rangle \\ &= \langle \hat{Q}f | f \rangle + \langle \hat{R}f | f \rangle \\ &= \langle (\hat{Q} + \hat{R})f | f \rangle \end{aligned}$$

(b)

$$\begin{aligned} \langle f | \alpha \hat{Q}g \rangle &= \alpha \langle f | \hat{Q}g \rangle \\ &= \alpha \langle \hat{Q}f | g \rangle \\ \langle \alpha \hat{Q}f | g \rangle &= \alpha^* \langle \hat{Q}f | g \rangle \\ \alpha &= \alpha^* \end{aligned}$$

α is real.

(c)

$$\begin{aligned}\langle f|\hat{Q}\hat{R}g\rangle &= \langle \hat{Q}f|\hat{R}g\rangle \\ &= \langle \hat{R}\hat{Q}f|g\rangle\end{aligned}$$

The product of the operators is hermitian when $\hat{Q}\hat{R} = \hat{R}\hat{Q}$ i.e. $[\hat{Q}, \hat{R}] = 0$.

(d)

$$\begin{aligned}\langle \Psi|\hat{x}\Psi\rangle &= \int \Psi^* \hat{x} \Psi dx \\ &= \int \Psi^* \hat{x}^* \Psi dx \\ &= \int (\hat{x}\Psi)^* \Psi dx \\ &= \langle \hat{x}\Psi|\Psi\rangle \\ \langle \Psi|\hat{H}\Psi\rangle &= \int \Psi^* \hat{H} \Psi dx \\ &= \int \Psi^* \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \Psi dx \\ &= -\frac{\hbar^2}{2m} \int \Psi^* \frac{d^2 \Psi}{dx^2} dx + \int \Psi^* V(x) \Psi dx \\ &= -\frac{\hbar^2}{2m} \left[\Psi^* \frac{d\Psi}{dx} \Big|_{-\infty}^{\infty} - \int \frac{d\Psi^*}{dx} \frac{d\Psi}{dx} dx \right] + \langle V(x)\Psi|\Psi\rangle \\ &= \frac{\hbar^2}{2m} \left[\frac{d\Psi^*}{dx} \Psi \Big|_{-\infty}^{\infty} - \int \frac{d^2 \Psi^*}{dx^2} \Psi dx \right] + \langle V(x)\Psi|\Psi\rangle \\ &= -\frac{\hbar^2}{2m} \int \frac{d^2}{dx^2} \Psi^* \Psi dx + \langle V(x)\Psi|\Psi\rangle \\ &= \left\langle -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \Psi \middle| \Psi \right\rangle + \langle V(x)\Psi|\Psi\rangle \\ &= \langle \hat{H}\Psi|\Psi\rangle\end{aligned}$$

3.5

(a)

$$\begin{aligned}x^\dagger &= x \\ i^\dagger &= -i \\ \left(\frac{d}{dx} \right)^\dagger &= -\frac{d}{dx}\end{aligned}$$

(b)

$$\begin{aligned}
\langle f|\hat{Q}\hat{R}g\rangle &= \int f^\dagger \hat{Q}\hat{R}g \, dx \\
&= \int (\hat{Q}^\dagger f)^\dagger \hat{R}g \, dx \\
&= \int (\hat{R}^\dagger \hat{Q}^\dagger f)^\dagger g \, dx \\
&= \langle \hat{R}^\dagger \hat{Q}^\dagger f|g\rangle
\end{aligned}$$

(c)

$$\begin{aligned}
\hat{a}_+ &= \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega x) \\
\langle f|\hat{a}g\rangle &= \left\langle f \left| \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega x)g \right. \right\rangle \\
&= \frac{1}{\sqrt{2\hbar m\omega}} \langle f|(-i\hat{p} + m\omega x)g\rangle \\
&= \frac{1}{\sqrt{2\hbar m\omega}} (\langle f|-i\hat{p}g\rangle + \langle f|m\omega xg\rangle) \\
&= \frac{1}{\sqrt{2\hbar m\omega}} (\langle f|-i\hat{p}g\rangle + \langle m\omega x f|g\rangle) \\
&= \frac{1}{\sqrt{2\hbar m\omega}} (\langle i\hat{p}f|g\rangle + \langle m\omega x f|g\rangle) \\
&= \left\langle \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega x)f \left| g \right. \right\rangle \\
&= \langle \hat{a}_- f|g\rangle
\end{aligned}$$

3.6

$$\begin{aligned}
\langle f|\hat{Q}g\rangle &= \int_0^{2\pi} f^* \frac{d^2 g}{d\phi^2} d\phi \\
&= f^* \frac{dg}{d\phi} \Big|_0^{2\pi} - \int_0^{2\pi} \left(\frac{df}{d\phi} \right)^* \frac{dg}{d\phi} d\phi \\
&= - \left(\frac{df}{d\phi} \right)^* g \Big|_0^{2\pi} + \int_0^{2\pi} \left(\frac{d^2 f}{d\phi^2} \right)^* g d\phi \\
&= \langle \hat{Q}f|g\rangle
\end{aligned}$$

Yes, the operator is hermitian.

$$\begin{aligned}
\hat{Q}f &= qf \\
\frac{d^2 f}{d\phi^2} &= qf \\
\frac{d^2 f}{d\phi^2} - qf &= 0 \\
f &= Ae^{\sqrt{q}\phi} + Be^{-\sqrt{q}\phi} \\
f(\phi + 2\pi) &= Ae^{\sqrt{q}(\phi+2\pi)} + Be^{\sqrt{q}(\phi+2\pi)} \\
&= Ae^{\sqrt{q}\phi} e^{2\pi\sqrt{q}} + Be^{\sqrt{q}\phi} e^{2\pi\sqrt{q}} \\
2\pi\sqrt{q} &= 1 \\
q &= -n^2, \quad n = 0, 1, 2, \dots
\end{aligned}$$

The eigenfunctions are $f = Ae^{\pm\sqrt{q}\phi}$ and the eigenvalues are $q = 0, 1, 2, \dots$. The spectrum is degenerate as there are two eigenfunctions associated with each eigenvalue $q > 0$.

3.7

(a)

$$\begin{aligned}
h &= af + bg \\
\hat{Q}h &= \hat{Q}(af + bg) \\
&= \hat{Q}(af) + \hat{Q}(bg) \\
&= a\hat{Q}f + b\hat{Q}g \\
&= aqf + bqg \\
&= q(af + bg) \\
&= qh
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{d^2}{dx^2} e^x &= e^x \\
\frac{d^2}{dx^2} e^{-x} &= e^{-x} \\
f &= e^x + e^{-x} \\
g &= e^x - e^{-x}
\end{aligned}$$

3.8

(a)

$$\begin{aligned}
 \hat{Q} &= i \frac{d}{d\phi} \\
 \hat{Q}f &= qf \\
 i \frac{df}{d\phi} &= qf \\
 \frac{df}{d\phi} + iqf &= 0 \\
 f &= Ae^{-iq\phi} \\
 e^{-2\pi iq} &= 1 \\
 q &= 0, \pm 1, \pm 2, \dots \\
 \int_0^{2\pi} Ae^{-iq\phi} Be^{-iq'\phi} d\phi &= AB \int_0^{2\pi} e^{-i(q+q')\phi} d\phi \\
 &= 0
 \end{aligned}$$

(b)

$$\begin{aligned}
 \hat{Q} &= \frac{d^2}{d\phi^2} \\
 \hat{Q}f &= qf \\
 \frac{d^2 f}{d\phi^2} - qf &= 0 \\
 f &= Ae^{\pm\sqrt{q}\phi} \\
 q &= -n^2, n = 0, 1, 2, \dots \\
 \int_0^{2\pi} Ae^{\pm\sqrt{q}\phi} Be^{\pm\sqrt{q'}\phi} d\phi &= AB \int_0^{2\pi} e^{\pm in\phi} e^{\pm in'\phi} d\phi \\
 &= AB \int_0^{2\pi} e^{i(\pm n \pm n')\phi} d\phi \\
 &= AB \left[\frac{1}{i(\pm n \pm n')} e^{i(\pm n \pm n')\phi} \right]_0^{2\pi} \\
 &= AB \frac{1}{i(\pm n \pm n')} [e^{i(\pm n \pm n')2\pi} - 1] \\
 &= 0
 \end{aligned}$$

3.9

(a) Infinite square well

(b) Delta function barrier

(c) Delta function well

3.11

$$\begin{aligned}
\Psi_0(x, t) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} e^{-i\omega t/2} \\
\Phi_0(p, t) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} e^{-i\omega t/2} dx \\
&= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-i\omega t/2} \int_{-\infty}^{\infty} e^{-ipx/\hbar} e^{-m\omega x^2/2\hbar} dx \\
&= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-i\omega t/2} \sqrt{\frac{2\pi\hbar}{m\omega}} e^{-p^2/2\hbar m\omega} \\
&= \frac{1}{(\pi\hbar m\omega)^{1/4}} e^{-p^2/2\hbar m\omega} e^{-i\omega t/2} \\
\frac{p^2}{2m} &= \frac{\hbar\omega}{2} \\
p &= \pm\sqrt{\hbar m\omega} \\
1 - \int_{-\sqrt{\hbar m\omega}}^{\sqrt{\hbar m\omega}} |\Phi_0|^2 dp &= 1 - \frac{1}{(\pi\hbar m\omega)^{1/4}} \int_{-\sqrt{\hbar m\omega}}^{\sqrt{\hbar m\omega}} e^{-p^2/\hbar m\omega} dp \\
&= 1 - \frac{1}{(\pi\hbar m\omega)^{1/2}} \sqrt{\pi\hbar m\omega} \operatorname{erf} 1 \\
&= 0.16
\end{aligned}$$

3.12

$$\begin{aligned}
\Psi(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m}t)} dk \\
\Phi(p, t) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m}t)} dk \right) dx \\
&= \frac{1}{2\pi\sqrt{\hbar}} \int_{-\infty}^{\infty} \phi(k) e^{-i\frac{\hbar k^2}{2m}t} \left(\int_{-\infty}^{\infty} e^{i(k - p/\hbar)x} dx \right) dk \\
&= \frac{1}{2\pi\sqrt{\hbar}} \int_{-\infty}^{\infty} \phi(k) e^{-i\frac{\hbar k^2}{2m}t} 2\pi\delta(k - p/\hbar) dk \\
&= \frac{1}{\sqrt{\hbar}} \int_{-\infty}^{\infty} \delta(k - p/\hbar) \phi(k) e^{-i\frac{\hbar k^2}{2m}t} dk \\
&= \frac{1}{\sqrt{\hbar}} \phi(p/\hbar) e^{-i\frac{p^2}{2\hbar m}t} \\
|\Phi(p, t)|^2 &= \frac{1}{\hbar} |\phi(p/\hbar)|^2
\end{aligned}$$

3.14

(a)

$$\begin{aligned}
 [\hat{A} + \hat{B}, \hat{C}] &= (\hat{A} + \hat{B})\hat{C} - \hat{C}(\hat{A} + \hat{B}) \\
 &= \hat{A}\hat{C} + \hat{B}\hat{C} - \hat{C}\hat{A} - \hat{C}\hat{B} \\
 &= \hat{A}\hat{C} - \hat{C}\hat{A} + \hat{B}\hat{C} - \hat{C}\hat{B} \\
 &= [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}] \\
 \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B} &= \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} + \hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B} \\
 &= \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} \\
 &= [\hat{A}\hat{B}, \hat{C}]
 \end{aligned}$$

(b)

$$\begin{aligned}
 [x^n, \hat{p}] &= \left[x^n, -i\hbar \frac{d}{dx} \right] \\
 &= x^n \left(-i\hbar \frac{d}{dx} \right) - \left(-i\hbar \frac{d}{dx} \right) x^n \\
 &= x^n \left(-i\hbar \frac{d}{dx} \right) (1) + i\hbar n x^{n-1} \\
 &= i\hbar n x^{n-1}
 \end{aligned}$$

(c)

$$\begin{aligned}
 [f(x), \hat{p}]g(x) &= f(x) \left(-i\hbar \frac{d}{dx} \right) g(x) - \left(-i\hbar \frac{d}{dx} \right) [f(x)g(x)] \\
 &= -i\hbar f(x) \frac{dg}{dx} + i\hbar \left[\frac{df}{dx} g(x) + f(x) \frac{dg}{dx} \right] \\
 &= i\hbar \frac{df}{dx} g(x) \\
 [f(x), \hat{p}] &= i\hbar \frac{df}{dx}
 \end{aligned}$$

(d)

$$\begin{aligned}
[\hat{H}, \hat{a}_-]g &= \hat{H}\hat{a}_-g - \hat{a}_-\hat{H}g \\
&= \hbar\omega \left(\hat{a}_-\hat{a}_+ - \frac{1}{2} \right) \hat{a}_-g - \hat{a}_-\hbar\omega \left(\hat{a}_-\hat{a}_+ - \frac{1}{2} \right) g \\
&= \hbar\omega \hat{a}_-\hat{a}_+\hat{a}_-g - \frac{1}{2}\hbar\omega \hat{a}_-g - \hbar\omega \hat{a}_-^2\hat{a}_+g + \frac{1}{2}\hbar\omega \hat{a}_-g \\
&= \hbar\omega \hat{a}_-\hat{a}_+\hat{a}_-g - \hbar\omega \hat{a}_-^2\hat{a}_+g \\
&= \hbar\omega \hat{a}_-(\hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+)g \\
&= -\hbar\omega \hat{a}_-g \\
[\hat{H}, \hat{a}_-] &= -\hbar\omega \hat{a}_- \\
[\hat{H}, \hat{a}_+]g &= \hat{H}\hat{a}_+g - \hat{a}_+\hat{H}g \\
&= \hbar\omega \left(\hat{a}_-\hat{a}_+ - \frac{1}{2} \right) \hat{a}_+g - \hat{a}_+\hbar\omega \left(\hat{a}_-\hat{a}_+ - \frac{1}{2} \right) g \\
&= \hbar\omega \hat{a}_-\hat{a}_+^2g - \frac{1}{2}\hbar\omega \hat{a}_+g - \hbar\omega \hat{a}_+\hat{a}_-\hat{a}_+g + \frac{1}{2}\hbar\omega \hat{a}_+g \\
&= \hbar\omega \hat{a}_-\hat{a}_+^2g - \hbar\omega \hat{a}_+\hat{a}_-\hat{a}_+g \\
&= \hbar\omega (\hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_-)\hat{a}_+g \\
&= \hbar\omega \hat{a}_+g \\
[\hat{H}, \hat{a}_+] &= \hbar\omega \hat{a}_+
\end{aligned}$$

3.15

$$\begin{aligned}
\left[x, \frac{p^2}{2m} + V \right] g &= x \left(\frac{p^2}{2m} + V \right) g - \left(\frac{p^2}{2m} + V \right) xg \\
&= x \frac{p^2}{2m} g + xVg - \frac{p^2}{2m} xg - Vxg \\
&= \frac{1}{2m} (xp^2g - p^2xg) \\
&= \frac{1}{2m} \left[-\hbar^2 x \frac{d^2g}{dx^2} + \hbar^2 \frac{d}{dx} \left(g + x \frac{dg}{dx} \right) \right] \\
&= \frac{1}{2m} \left[-\hbar^2 x \frac{d^2g}{dx^2} + \hbar^2 \left(\frac{dg}{dx} + \frac{dg}{dx} + x \frac{d^2g}{dx^2} \right) \right] \\
&= \frac{\hbar^2}{m} \frac{dg}{dx} \\
\left[x, \frac{p^2}{2m} + V \right] &= \frac{\hbar^2}{m} \frac{d}{dx} \\
&= -\frac{\hbar}{im} \langle p \rangle \\
\sigma_x^2 \sigma_H^2 &\geq \left(\frac{1}{2i} \left\langle \left[x, \frac{p^2}{2m} + V \right] \right\rangle \right)^2 \\
&= \frac{\hbar^2}{4m^2} |\langle p \rangle|^2 \\
\sigma_x \sigma_H &\geq \frac{\hbar}{2m} |\langle p \rangle|
\end{aligned}$$

This doesn't tell us much because for stationary states $\sigma_H = 0$ and $\langle p \rangle = 0$ so this says $0 \geq 0$.

3.17

$$\begin{aligned}
\left(-i\hbar \frac{d}{dx} - \langle p \rangle \right) \Psi &= ia(x - \langle x \rangle) \Psi \\
-i\hbar \frac{d\Psi}{dx} - \langle p \rangle \Psi &= ia(x - \langle x \rangle) \Psi \\
\frac{d\Psi}{dx} + \frac{\langle p \rangle + ia(x - \langle x \rangle)}{i\hbar} \Psi &= 0 \\
\frac{d\Psi}{dx} + \left[\frac{a}{\hbar} (x - \langle x \rangle) - i \frac{\langle p \rangle}{\hbar} \right] \Psi &= 0 \\
\frac{d\Psi}{dx} + \frac{a}{\hbar} x \Psi - \frac{a}{\hbar} \langle x \rangle \Psi - i \frac{\langle p \rangle}{\hbar} \Psi &= 0 \\
\Psi &= Ae^{-ax^2/2\hbar} e^{a\langle x \rangle x/\hbar} e^{i\langle p \rangle x/\hbar} \\
&= Be^{-a(x - \langle x \rangle)^2/2\hbar} e^{i\langle p \rangle x/\hbar}
\end{aligned}$$

3.18

(a)

$$\begin{aligned} Q &= 1 \\ \hat{Q} &= 1 \\ [\hat{H}, \hat{Q}] &= 0 \\ \frac{d}{dt} \langle Q \rangle &= 0 \end{aligned}$$

(b)

$$\begin{aligned} Q &= H \\ \hat{Q} &= \hat{H} \\ [\hat{H}, \hat{Q}] &= [\hat{H}, \hat{H}] \\ &= 0 \\ \frac{d}{dt} \langle H \rangle &= 0 \end{aligned}$$

(c)

$$\begin{aligned} Q &= x \\ \hat{Q} &= x \\ [\hat{H}, \hat{Q}] &= [\hat{H}, x] \\ &= -i \frac{\hbar}{m} \hat{p} \\ \frac{d}{dt} \langle x \rangle &= \frac{i}{\hbar} \left\langle -i \frac{\hbar}{m} p \right\rangle \\ &= \frac{\langle p \rangle}{m} \end{aligned}$$

(d)

$$\begin{aligned} Q &= p \\ \hat{Q} &= \hat{p} \\ [\hat{H}, \hat{Q}] &= [\hat{H}, \hat{p}] \\ &= i\hbar \frac{\partial V}{\partial x} \\ \frac{d}{dt} \langle p \rangle &= \frac{i}{\hbar} \left\langle i\hbar \frac{\partial V}{\partial x} \right\rangle \\ &= - \left\langle \frac{\partial V}{\partial x} \right\rangle \end{aligned}$$

3.19

(a)

$$\begin{aligned}\frac{d^2}{dt^2} \langle x \rangle &= \frac{d}{dx} \left(\frac{\langle p \rangle}{m} \right) \\ &= -\frac{1}{m} \left\langle \frac{\partial V}{\partial x} \right\rangle \\ &= 0\end{aligned}$$

(b)

$$\begin{aligned}\frac{d^2}{dt^2} \langle x \rangle &= \frac{d}{dx} \left(\frac{\langle p \rangle}{m} \right) \\ &= -\frac{1}{m} \left\langle \frac{\partial V}{\partial x} \right\rangle \\ &= -\omega^2 \langle x \rangle \\ \frac{d^2}{dt^2} \langle x \rangle + \omega^2 \langle x \rangle &= 0 \\ \langle x \rangle &= A \sin \omega t + B \cos \omega t\end{aligned}$$

3.20

$$\begin{aligned}\Psi &= \frac{1}{\sqrt{2}}(\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar}) \\ \hat{H}\Psi &= \frac{1}{\sqrt{2}}(E_1 \psi_1 e^{-iE_1 t/\hbar} + E_2 \psi_2 e^{-iE_2 t/\hbar}) \\ \hat{H}^2\Psi &= \frac{1}{\sqrt{2}}(E_1^2 \psi_1 e^{-iE_1 t/\hbar} + E_2^2 \psi_2 e^{-iE_2 t/\hbar}) \\ \langle H^2 \rangle &= \langle \Psi | \hat{H}^2 | \Psi \rangle \\ &= \frac{1}{2} \int_0^a (\psi_1^* e^{iE_1 t/\hbar} + \psi_2^* e^{iE_2 t/\hbar})(E_1^2 \psi_1 e^{-iE_1 t/\hbar} + E_2^2 \psi_2 e^{-iE_2 t/\hbar}) dx \\ &= \frac{1}{2} \int_0^a (E_1^2 |\psi_1|^2 + E_2^2 \psi_1^* \psi_2^* e^{i(E_1 - E_2)t/\hbar} + E_1^2 \psi_2^* \psi_1 e^{i(E_2 - E_1)t/\hbar} \\ &\quad + E_2^2 |\psi_2|^2) dx \\ &= \frac{1}{2}(E_1^2 + E_2^2) \\ \langle H \rangle &= \frac{1}{2}(E_1 + E_2) \\ \sigma_H^2 &= \langle H^2 \rangle - \langle H \rangle^2 \\ &= \frac{1}{2}(E_1^2 + E_2^2) - \frac{1}{4}(E_1 + E_2)^2 \\ &= \frac{1}{4}(E_2 - E_1)^2\end{aligned}$$

$$\begin{aligned}
\omega &= \frac{\pi^2 \hbar}{2ma^2} \\
\langle x \rangle &= \frac{a}{2} \left[1 - \frac{32}{9\pi^2} \cos(3\omega t) \right] \\
\langle x^2 \rangle &= \frac{1}{2} \int_0^a (\psi_1^* e^{iE_1 t/\hbar} + \psi_2^* e^{iE_2 t/\hbar}) x^2 (\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar}) dx \\
&= \frac{1}{2} \int_0^a x^2 (|\psi_1|^2 + \psi_1^* \psi_2 e^{i(E_1 - E_2)t/\hbar} + \psi_2^* \psi_1 e^{i(E_2 - E_1)t/\hbar} + |\psi_2|^2) dx \\
&= \frac{1}{a} \int_0^a x^2 \left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right) \right. \\
&\quad \left. + 2 \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) \cos(3\omega t) \right] dx \\
&= \frac{a^2}{144\pi^2} [-45 + 48\pi^2 - 256 \cos(3\omega t)] \\
\sigma_x^2 &= \langle x^2 \rangle - \langle x \rangle^2 \\
&= \frac{a^2}{4} \left[\frac{1}{3} - \frac{5}{4\pi^2} - \left(\frac{32}{9\pi^2} \right)^2 \cos^2(3\omega t) \right] \\
\frac{d\langle x \rangle}{dt} &= \frac{16a\omega}{3\pi^2} \sin(3\omega t) \\
&= \frac{8\hbar}{3ma} \sin(3\omega t)
\end{aligned}$$

$$\begin{aligned}
\sigma_H^2 \sigma_x^2 &\geq \left(\frac{\hbar}{2} \right)^2 \left| \frac{d\langle x \rangle}{dt} \right|^2 \\
\frac{1}{4} \left(\frac{3\pi^2 \hbar^2}{2ma^2} \right)^2 \frac{a^2}{4} \left[\frac{1}{3} - \frac{5}{4\pi^2} - \left(\frac{32}{9\pi^2} \right)^2 \cos^2(3\omega t) \right] &\geq \left(\frac{\hbar}{2} \right)^2 \left[\frac{8\hbar}{3ma} \sin(3\omega t) \right]^2 \\
\left(\frac{3}{4} \right)^2 \left[\frac{1}{3} - \frac{5}{4\pi^2} - \left(\frac{32}{9\pi^2} \right)^2 \cos^2(3\omega t) \right] &\geq \left(\frac{8}{3\pi^2} \right)^2 \sin^2(3\omega t) \\
\frac{1}{3} - \frac{5}{4\pi^2} &\geq \left(\frac{32}{9\pi^2} \right)^2
\end{aligned}$$

3.23

$$\begin{aligned}
\hat{P}^2 |\beta\rangle &= \hat{P}(\hat{P} |\beta\rangle) \\
&= \hat{P}(\langle\alpha|\beta\rangle |\alpha\rangle) \\
&= \langle\alpha|\beta\rangle \hat{P} |\alpha\rangle \\
&= \langle\alpha|\beta\rangle (\langle\alpha|\alpha\rangle |\alpha\rangle) \\
&= \langle\alpha|\beta\rangle |\alpha\rangle \\
&= \hat{P} |\beta\rangle
\end{aligned}$$

So, $\hat{P}^2 = \hat{P}$.

$$\begin{aligned}
\hat{P} |\beta\rangle &= \lambda |\beta\rangle \\
\langle\alpha|\beta\rangle |\alpha\rangle &= \lambda |\beta\rangle
\end{aligned}$$

If $|\beta\rangle$ is a constant multiple of $|\alpha\rangle$, then

$$\begin{aligned}
\langle\alpha|c\alpha\rangle |\alpha\rangle &= \lambda c |\alpha\rangle \\
c \langle\alpha|\alpha\rangle |\alpha\rangle &= \lambda c |\alpha\rangle \\
c |\alpha\rangle &= \lambda c |\alpha\rangle
\end{aligned}$$

thus $\lambda = 1$.

If $|\beta\rangle$ is orthogonal to $|\alpha\rangle$, then

$$\begin{aligned}
\langle\alpha|\beta\rangle |\alpha\rangle &= \lambda |\beta\rangle \\
0 &= \lambda |\beta\rangle
\end{aligned}$$

thus $\lambda = 0$.

3.24

$$\begin{aligned}
\hat{Q} &= \hat{Q}^\dagger \\
Q_{mn} &= \langle e_m | \hat{Q} | e_n \rangle \\
Q_{nm} &= \langle e_n | \hat{Q} | e_m \rangle \\
Q_{nm}^* &= \langle e_n | \hat{Q} | e_m \rangle^* \\
&= \langle e_m | \hat{Q}^\dagger | e_n \rangle \\
&= \langle e_m | \hat{Q} | e_n \rangle \\
&= Q_{mn}
\end{aligned}$$

3.25

$$\hat{H}|\alpha\rangle = \lambda|\alpha\rangle$$

$$\epsilon[(\langle 1|\alpha\rangle + \langle 2|\alpha\rangle)|1\rangle + (\langle 1|\alpha\rangle - \langle 2|\alpha\rangle)|2\rangle] = \lambda(\langle 1|\alpha\rangle|1\rangle + \langle 2|\alpha\rangle|2\rangle)$$

From this we get two equations

$$\epsilon(\langle 1|\alpha\rangle + \langle 2|\alpha\rangle) = \lambda\langle 1|\alpha\rangle$$

$$\epsilon(\langle 1|\alpha\rangle - \langle 2|\alpha\rangle) = \lambda\langle 2|\alpha\rangle$$

If we let $|\alpha\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ then this becomes

$$\epsilon(a + b) = \lambda a$$

$$\epsilon(a - b) = \lambda b$$

or in matrix form

$$\epsilon \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda = \pm\sqrt{2}\epsilon$ and the eigenvectors are $|1\rangle + (\sqrt{2} \pm 1)|2\rangle$.

3.26

(a)

$$\langle\alpha| = -i\langle 1| - 2\langle 2| + i\langle 3|$$

$$\langle\beta| = -i\langle 1| + 2\langle 3|$$

(b)

$$\begin{aligned}\langle\alpha|\beta\rangle &= (-i\langle 1| - 2\langle 2| + i\langle 3|)(i|1\rangle + 2|3\rangle) \\ &= \langle 1|1\rangle - 2i\langle 1|3\rangle - 2i\langle 2|1\rangle - 4\langle 2|3\rangle - \langle 3|1\rangle + 2i\langle 3|3\rangle \\ &= 1 + 2i \\ \langle\beta|\alpha\rangle &= (-i\langle 1| + 2\langle 3|)(i|1\rangle - 2|2\rangle - i|3\rangle) \\ &= \langle 1|1\rangle + 2i\langle 1|2\rangle - \langle 1|3\rangle + 2i\langle 3|1\rangle - 4\langle 3|2\rangle - 2i\langle 3|3\rangle \\ &= 1 - 2i \\ &= \langle\alpha|\beta\rangle^*\end{aligned}$$

(c)

$$\begin{aligned}
\hat{A} &= |\alpha\rangle \langle \beta| \\
\hat{A} |1\rangle &= |\alpha\rangle \langle \beta|1\rangle \\
&= |\alpha\rangle \langle 1|\beta\rangle^* \\
&= -i |\alpha\rangle \\
&= |1\rangle + 2i |2\rangle - |3\rangle \\
A_{11} &= \langle 1|\hat{A}|1\rangle \\
&= 1 \\
A_{21} &= 2i \\
A_{31} &= -1 \\
\hat{A} |2\rangle &= |\alpha\rangle \langle \beta|2\rangle \\
&= |\alpha\rangle \langle 2|\beta\rangle^* \\
&= 0 \\
A_{12} &= 0 \\
A_{22} &= 0 \\
A_{32} &= 0 \\
\hat{A} |3\rangle &= |\alpha\rangle \langle \beta|3\rangle \\
&= |\alpha\rangle \langle 3|\beta\rangle^* \\
&= 2 |\alpha\rangle \\
&= 2i |1\rangle - 4 |2\rangle - 2i |3\rangle \\
A_{13} &= 2i \\
A_{23} &= -4 \\
A_{33} &= -2i \\
A &= \begin{pmatrix} 1 & 0 & 2i \\ 2i & 0 & -4 \\ -1 & 0 & -2i \end{pmatrix}
\end{aligned}$$

It's not hermitian.

3.27

(a)

$$\begin{aligned}
 \hat{Q}|\alpha\rangle &= \hat{Q} \sum_{n=1}^{\infty} \langle e_n|\alpha\rangle |e_n\rangle \\
 &= \hat{Q} \left(\sum_{n=1}^{\infty} |e_n\rangle \langle e_n| \right) |\alpha\rangle \\
 &= \left(\sum_{n=1}^{\infty} \hat{Q} |e_n\rangle \langle e_n| \right) |\alpha\rangle \\
 &= \left(\sum_{n=1}^{\infty} q_n |e_n\rangle \langle e_n| \right) |\alpha\rangle \\
 \hat{Q} &= \sum_{n=1}^{\infty} q_n |e_n\rangle \langle e_n|
 \end{aligned}$$

(b)

$$\begin{aligned}
 \hat{Q} &= \sum_{n=1}^{\infty} q_n |e_n\rangle \langle e_n| \\
 \hat{Q}^2 &= \left(\sum_{n=1}^{\infty} q_n |e_n\rangle \langle e_n| \right) \left(\sum_{l=1}^{\infty} q_l |e_l\rangle \langle e_l| \right) \\
 &= \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} q_n q_l |e_n\rangle \langle e_n| e_l\rangle \langle e_l| \\
 &= \sum_{n=1}^{\infty} q_n^2 |e_n\rangle \langle e_n| \\
 e^{\hat{Q}} &= \sum_{n=1}^{\infty} e^{q_n} |e_n\rangle \langle e_n| \\
 &= \sum_{n=1}^{\infty} \left(\sum_{k=0}^{\infty} \frac{q_n^k}{k!} \right) |e_n\rangle \langle e_n| \\
 &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{n=1}^{\infty} q_n^k |e_n\rangle \langle e_n| \right) \\
 &= \sum_{k=0}^{\infty} \frac{1}{k!} \hat{Q}^k \\
 &= 1 + \hat{Q} + \frac{1}{2} \hat{Q}^2 + \frac{1}{3!} \hat{Q}^3 + \dots
 \end{aligned}$$

3.28

(a)

$$\begin{aligned}\sin \hat{D} &= \hat{D} - \frac{D^3}{3!} + \frac{\hat{D}^5}{5!} - \frac{\hat{D}^7}{7!} + \cdots \\ (\sin \hat{D})x^5 &= 5x^4 - 10x^2 + 1\end{aligned}$$

(b)

$$\begin{aligned}\frac{1}{1 - \hat{D}/2} &= 1 + \frac{\hat{D}}{2} + \left(\frac{\hat{D}}{2}\right)^2 + \left(\frac{\hat{D}}{2}\right)^3 + \cdots \\ &= 1 + \frac{\hat{D}}{2} + \frac{\hat{D}^2}{4} + \frac{\hat{D}^3}{8} + \cdots \\ \frac{1}{1 - \hat{D}/2} \cos x &= \cos x - \frac{1}{2} \sin x - \frac{1}{4} \cos x + \frac{1}{8} \sin x + \cdots \\ &= \left(-\frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \cdots\right) \sin x + \left(1 - \frac{1}{4} + \frac{1}{16} - \cdots\right) \cos x \\ &= -\frac{2}{5} \sin x + \frac{4}{5} \cos x\end{aligned}$$

3.30

$$\begin{aligned}c_n(t) &= \langle n | S(t) \rangle \\ &= \left\langle n \left| \int dx |x\rangle \langle x| \right| S(t) \right\rangle \\ &= \int \langle n | x \rangle \langle x | S(t) \rangle dx \\ &= \int \langle x | n \rangle^* \Psi(x, t) dx \\ &= \int \psi_n(x)^* \Psi(x, t) dx\end{aligned}$$

3.31

$$\begin{aligned}
|e_1\rangle &= 1 \\
\langle e_1|e_1\rangle &= \int_{-1}^1 dx \\
&= 2 \\
|e'_1\rangle &= \frac{1}{\sqrt{2}} \\
|e_2\rangle &= x \\
\langle e'_1|e_2\rangle &= \int_{-1}^1 \frac{1}{\sqrt{2}} x dx \\
&= \frac{1}{\sqrt{2}} \left[\frac{1}{2} x^2 \right]_{-1}^1 \\
&= 0 \\
\langle e_2|e_2\rangle &= \int_{-1}^1 x^2 dx \\
&= \left[\frac{1}{3} x^3 \right]_{-1}^1 \\
&= \frac{2}{3} \\
|e'_2\rangle &= \sqrt{\frac{3}{2}} x \\
|e_3\rangle &= x^2 \\
\langle e'_1|e_3\rangle &= \int_{-1}^1 \frac{1}{\sqrt{2}} x^2 dx \\
&= \frac{1}{\sqrt{2}} \left[\frac{1}{3} x^3 \right]_{-1}^1 \\
&= \frac{\sqrt{2}}{3} \\
\langle e'_2|e_3\rangle &= \int_{-1}^1 \sqrt{\frac{3}{2}} x^3 dx \\
&= 0 \\
|e'_3\rangle &= x^2 - \frac{\sqrt{2}}{3} |e'_1\rangle \\
&= x^2 - \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
\langle e'_3 | e'_3 \rangle &= \int_{-1}^1 \left(x^2 - \frac{1}{3} \right)^2 dx \\
&= \int_{-1}^1 \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9} \right) dx \\
&= \left[\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x \right]_{-1}^1 \\
&= \frac{2}{5} - \frac{4}{9} + \frac{2}{9} \\
&= \frac{18}{45} - \frac{20}{45} + \frac{10}{45} \\
&= \frac{8}{45}
\end{aligned}$$

$$\begin{aligned}
|e''_3\rangle &= \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3} \right) \\
&= \sqrt{\frac{5}{2}} \left(\frac{3}{2}x^2 - \frac{1}{2} \right)
\end{aligned}$$

$$|e_4\rangle = x^3$$

$$\begin{aligned}
\langle e'_1 | e_4 \rangle &= \int_{-1}^1 \frac{1}{\sqrt{2}} x^3 dx \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\langle e'_2 | e_4 \rangle &= \int_{-1}^1 \sqrt{\frac{3}{2}} x^4 dx \\
&= \sqrt{\frac{3}{2}} \left[\frac{1}{5} x^5 \right] \\
&= \frac{\sqrt{6}}{5}
\end{aligned}$$

$$\begin{aligned}
\langle e''_3 | e_4 \rangle &= \int_{-1}^1 \sqrt{\frac{5}{2}} \left(\frac{3}{2}x^2 - \frac{1}{2} \right) x^3 dx \\
&= \sqrt{\frac{5}{2}} \int_{-1}^1 \left(\frac{3}{2}x^5 - \frac{1}{2}x^3 \right) dx \\
&= 0
\end{aligned}$$

$$\begin{aligned}
|e'_4\rangle &= |e_4\rangle - \frac{\sqrt{6}}{5} |e'_2\rangle \\
&= x^3 - \frac{3}{5}x
\end{aligned}$$

$$\begin{aligned}
\langle e'_4 | e'_4 \rangle &= \int_{-1}^1 \left(x^3 - \frac{3}{5}x \right)^2 dx \\
&= \int_{-1}^1 \left(x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2 \right) dx \\
&= \left[\frac{1}{7}x^7 - \frac{6}{25}x^5 + \frac{3}{25}x^3 \right]_{-1}^1 \\
&= \frac{2}{7} - \frac{12}{25} + \frac{6}{25} \\
&= \frac{50}{175} - \frac{84}{175} + \frac{42}{175} \\
&= \frac{8}{175} \\
|e''_4\rangle &= \sqrt{\frac{175}{8}} \left(x^3 - \frac{3}{5}x \right) \\
&= \sqrt{\frac{7}{2}} \left(\frac{5}{2}x^3 - \frac{3}{2}x \right)
\end{aligned}$$

3.32

(a)

$$\begin{aligned}
\langle \hat{Q} \rangle &= \langle \Psi | \hat{Q} | \Psi \rangle \\
&= \langle \Psi | \hat{Q}^\dagger | \Psi \rangle^* \\
&= -\langle \Psi | \hat{Q} | \Psi \rangle^* \\
&= -\langle \hat{Q} \rangle^*
\end{aligned}$$

(b)

$$\begin{aligned}
\hat{Q} |\psi\rangle &= \lambda |\psi\rangle \\
\langle \psi | \hat{Q} | \psi \rangle &= \langle \psi | \lambda | \psi \rangle \\
&= \lambda \langle \psi | \psi \rangle \\
\langle \psi | \hat{Q} | \psi \rangle^* &= \lambda^* \langle \psi | \psi \rangle^* \\
\langle \psi | \hat{Q}^\dagger | \psi \rangle &= \lambda^* \langle \psi | \psi \rangle \\
-\langle \psi | \hat{Q} | \psi \rangle &= \lambda^* \langle \psi | \psi \rangle \\
\langle \psi | \hat{Q} | \psi \rangle &= -\lambda^* \langle \psi | \psi \rangle \\
\hat{Q} |\psi\rangle &= -\lambda^* |\psi\rangle
\end{aligned}$$

(c)

$$\begin{aligned}
\hat{Q}|f\rangle &= \lambda_1|f\rangle \\
\hat{Q}|g\rangle &= \lambda_2|g\rangle \\
\langle g|\hat{Q}|f\rangle &= \langle g|\lambda_1|f\rangle \\
&= \lambda_1\langle g|f\rangle \\
\langle f|\hat{Q}^\dagger|g\rangle &= \lambda_1^*\langle f|g\rangle \\
-\langle f|\hat{Q}|g\rangle &= \lambda_1^*\langle f|g\rangle \\
\langle f|\lambda_2|g\rangle &= -\lambda_1^*\langle f|g\rangle \\
\lambda_2\langle f|g\rangle &= -\lambda_1^*\langle f|g\rangle \\
\lambda_2 &= -\lambda_1^* \\
\lambda_2 &= \lambda_1 \\
\langle f|g\rangle &= 0
\end{aligned}$$

(d)

$$\begin{aligned}
[\hat{A}, \hat{B}] &= \hat{A}\hat{B} - \hat{B}\hat{A} \\
[\hat{A}, \hat{B}]^\dagger &= (\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger \\
&= \hat{A}^\dagger\hat{B}^\dagger - \hat{B}^\dagger\hat{A}^\dagger \\
&= -(\hat{A}\hat{B} - \hat{B}\hat{A}) \\
&= -[\hat{A}, \hat{B}] \\
[\hat{A}, \hat{B}]^\dagger &= \hat{A}^\dagger\hat{B}^\dagger - \hat{B}^\dagger\hat{A}^\dagger \\
&= -(\hat{A}\hat{B} - \hat{B}\hat{A}) \\
&= -[\hat{A}, \hat{B}]
\end{aligned}$$

(e)

$$\begin{aligned}
\hat{Q}|q_n\rangle &= \lambda_n|q_n\rangle \\
&= (x + iy)|q_n\rangle \\
&= x|q_n\rangle + iy|q_n\rangle \\
&= \hat{X}|q_n\rangle + \hat{Y}|q_n\rangle \\
&= (\hat{X} + \hat{Y})|q_n\rangle
\end{aligned}$$

3.33

(a) ψ_1

(b) b_1 and b_2 with $P(b_1) = \frac{9}{25}$ and $P(b_2) = \frac{16}{25}$.

(c)

$$\begin{aligned}
\phi_1 &= \frac{3}{5}\psi_1 + \frac{4}{5}\psi_2 \\
\phi_2 &= \frac{4}{5}\psi_1 - \frac{3}{5}\psi_2 \\
P(a_1) &= P(b_1) \left(\frac{3}{5}\right)^2 + P(b_2) \left(\frac{4}{5}\right)^2 \\
&= \left(\frac{9}{25}\right)^2 + \left(\frac{16}{25}\right)^2 \\
&= \frac{337}{625} \\
&\approx 53.9\%
\end{aligned}$$

3.34

(a)

$$\begin{aligned}
\Phi_n(p, t) &= \frac{1}{\sqrt{2\pi\hbar}} \int_0^a e^{-ipx/\hbar} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-iE_n t/\hbar} dx \\
&= \frac{1}{\sqrt{\pi\hbar a}} e^{-iE_n t/\hbar} \int_0^a e^{-ipx/\hbar} \sin\left(\frac{n\pi}{a}x\right) dx \\
&= \sqrt{\frac{a\pi}{\hbar}} \frac{ne^{-iE_n t/\hbar}}{(n\pi)^2 - (ap/\hbar)^2} [1 - (-1)^n e^{-ipa/\hbar}]
\end{aligned}$$

(b)

$$|\Phi_n(p, t)|^2 = \frac{a\pi}{\hbar} \frac{4n^2}{[(n\pi)^2 - (ap/\hbar)^2]^2} \begin{cases} \cos^2\left(\frac{a}{2\hbar}p\right) & n \text{ odd} \\ \sin^2\left(\frac{a}{2\hbar}p\right) & n \text{ even} \end{cases}$$

3.35

$$\begin{aligned}
\Psi(x, 0) &= \begin{cases} \frac{1}{\sqrt{2n\lambda}} e^{i2\pi x/\lambda} & -n\lambda < x < n\lambda \\ 0 & \text{otherwise} \end{cases} \\
\Phi(p, 0) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \Psi(x, 0) dx \\
&= \frac{1}{2\sqrt{\pi\hbar n\lambda}} \int_{-n\lambda}^{n\lambda} e^{i(2\pi/\lambda - p/\hbar)x} dx \\
&= \frac{1}{2\sqrt{\pi\hbar n\lambda}} \frac{1}{i(2\pi/\lambda - p/\hbar)} \left[e^{i(2\pi/\lambda - p/\hbar)x} \right]_{-n\lambda}^{n\lambda} \\
&= \frac{1}{2\sqrt{\pi\hbar n\lambda}} \frac{1}{i(2\pi/\lambda - p/\hbar)} e^{i(2\pi/\lambda - p/\hbar)n\lambda} - e^{-i(2\pi/\lambda - p/\hbar)n\lambda} \\
&= \frac{1}{\sqrt{\pi\hbar n\lambda}} \frac{1}{2\pi/\lambda - p/\hbar} \sin \left[\left(\frac{2\pi}{\lambda} - \frac{p}{\hbar} \right) n\lambda \right] \\
&= \sqrt{\frac{\lambda\hbar}{n\pi}} \frac{1}{\lambda p - 2\pi\hbar} \sin \left(\frac{n\lambda}{\hbar} p \right) \\
w_x &= 2n\lambda \\
w_p &= \frac{2\pi\hbar}{n\lambda}
\end{aligned}$$

As $n \rightarrow \infty$, $w_x \rightarrow \infty$ and $w_p \rightarrow 0$.

$$\begin{aligned}
w_x w_p &= 2n\lambda \frac{2\pi\hbar}{n\lambda} \\
&= 4\pi\hbar \\
&\geq \frac{\hbar}{2}
\end{aligned}$$

3.36

(a)

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \left(\frac{A}{x^2 + a^2} \right)^2 dx \\
&= |A|^2 \int_{-\infty}^{\infty} \frac{1}{(x^2 + a^2)^2} dx \\
&= \frac{\pi |A|^2}{2a^3} \\
A &= a \sqrt{\frac{2a}{\pi}}
\end{aligned}$$

(b)

$$\begin{aligned}
\langle x \rangle &= \langle \Psi | x | \Psi \rangle \\
&= \int_{-\infty}^{\infty} \Psi^* x \Psi \, dx \\
&= \frac{a^3}{\pi} \int_{-\infty}^{\infty} \frac{2x}{(x^2 + a^2)^2} \, dx \\
u &= x^2 + a^2 \\
du &= 2x \, dx \\
\langle x \rangle &= \frac{a^3}{\pi} \int_{-\infty}^{\infty} \frac{1}{u^2} \\
&= \frac{a^3}{\pi} \left[-\frac{1}{u} \right]_{-\infty}^{\infty} \\
&= 0 \\
\langle x^2 \rangle &= \langle \Psi | x^2 | \Psi \rangle \\
&= \int_{-\infty}^{\infty} \Psi^* x^2 \Psi \, dx \\
&= \frac{2a^3}{\pi} \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)^2} \, dx \\
&= a^2 \\
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= a
\end{aligned}$$

(c)

$$\begin{aligned}
\Phi(x, 0) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} a \sqrt{\frac{2a}{\pi}} \frac{1}{x^2 + a^2} \, dx \\
&= \frac{a\sqrt{a}}{\pi\sqrt{\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \frac{1}{x^2 + a^2} \, dx \\
&= \sqrt{\frac{a}{\hbar}} e^{-|p|a/\hbar} \\
\frac{a}{\hbar} \int_{-\infty}^{\infty} e^{-2|p|a/\hbar} \, dp &= 1
\end{aligned}$$

(d)

$$\begin{aligned}
\langle p \rangle &= \langle \Phi | p | \Phi \rangle \\
&= \int_{-\infty}^{\infty} \Phi^* p \Psi dp \\
&= \frac{a}{\hbar} \int_{-\infty}^{\infty} p e^{-2|p|a/\hbar} dp \\
&= 0 \\
\langle p^2 \rangle &= \frac{a}{\hbar} \int_{-\infty}^{\infty} p^2 e^{-2|p|a/\hbar} dp \\
&= \frac{\hbar^2}{2a^2} \\
\sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\
&= \frac{\hbar}{\sqrt{2}a}
\end{aligned}$$

(e)

$$\begin{aligned}
\sigma_x \sigma_p &= \frac{\hbar}{\sqrt{2}} \\
&\geq \frac{\hbar}{2}
\end{aligned}$$

3.37

$$\begin{aligned}
[\hat{H}, xp] &= x[\hat{H}, p] + [\hat{H}, x]p \\
&= i\hbar x \frac{dV}{dx} - \frac{i\hbar p^2}{m} \\
\frac{d}{dt} \langle xp \rangle &= \frac{i}{\hbar} \langle [\hat{H}, xp] \rangle \\
&= \frac{i}{\hbar} \left\langle i\hbar x \frac{dV}{dx} - \frac{i\hbar p^2}{m} \right\rangle \\
&= \left\langle \frac{p^2}{m} \right\rangle - \left\langle x \frac{dV}{dx} \right\rangle \\
&= 2 \langle T \rangle - \left\langle x \frac{dV}{dx} \right\rangle
\end{aligned}$$

The left hand side is 0 because expectation values are constant in stationary states.

$$\begin{aligned}
V &= \frac{1}{2}m\omega^2 x^2 \\
\frac{dV}{dx} &= m\omega^2 x \\
x \frac{dV}{dx} &= m\omega^2 x^2 \\
&= 2V \\
2 \langle T \rangle &= \left\langle x \frac{dV}{dx} \right\rangle \\
&= \langle 2V \rangle \\
&= 2 \langle V \rangle \\
\langle T \rangle &= \langle V \rangle
\end{aligned}$$

3.38

$$\begin{aligned}
\Psi(x, 0) &= \frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \\
\Psi(x, t) &= \frac{1}{\sqrt{2}}(\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar}) \\
0 &= \int_{-\infty}^{\infty} \Psi(x, 0) \Psi(x, t) dx \\
&= \frac{1}{2} \int_{-\infty}^{\infty} (\psi_1 + \psi_2)(\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar}) dx \\
&= \frac{1}{2} \int_{-\infty}^{\infty} (\psi_1^2 e^{-iE_1 t/\hbar} + \psi_1 \psi_2 e^{-iE_2 t/\hbar} + \psi_1 \psi_2 e^{-iE_1 t/\hbar} + \psi_2^2 e^{-iE_2 t/\hbar}) dx \\
&= \frac{1}{2} (e^{-iE_1 t/\hbar} + e^{-iE_2 t/\hbar}) \\
&= e^{-iE_1 t/\hbar} + e^{-iE_2 t/\hbar} \\
e^{-iE_1 t/\hbar} &= -e^{-iE_2 t/\hbar} \\
&= e^{\pi i} e^{-iE_2 t/\hbar} \\
-\frac{iE_1 t}{\hbar} &= i \left(\pi - \frac{E_2 t}{\hbar} \right) \\
-\frac{E_1 t}{\hbar} &= \pi - \frac{E_2 t}{\hbar} \\
t &= \frac{\pi \hbar}{E_2 - E_1} \\
\Delta t &= \frac{t}{\pi} \\
&= \frac{\hbar}{E_2 - E_1} \\
\Delta E &= \frac{1}{2} (E_2 - E_1) \\
\Delta E \Delta t &= \frac{\hbar}{2}
\end{aligned}$$

3.41

$$\begin{aligned}
\Psi &= \frac{e^{i\theta_0}}{\sqrt{2}} \psi_0 e^{-i\omega t/2} + \frac{e^{i\theta_1}}{\sqrt{2}} \psi_1 e^{-3i\omega t/2} \\
\langle p \rangle &= \langle \Psi | p | \Psi \rangle \\
&= \frac{1}{2} \langle \psi_0 | p | \psi_0 \rangle + \frac{1}{2} e^{i(\theta_1 - \theta_0)} e^{-i\omega t} \langle \psi_0 | p | \psi_1 \rangle \\
&\quad + \frac{1}{2} e^{i(\theta_0 - \theta_1)} e^{i\omega t} \langle \psi_1 | p | \psi_0 \rangle + \frac{1}{2} \langle \psi_1 | p | \psi_1 \rangle \\
\langle \psi_0 | p | \psi_0 \rangle &= \frac{d}{dt} \langle \psi_0 | x | \psi_0 \rangle \\
&= 0 \\
\langle \psi_1 | p | \psi_1 \rangle &= \frac{d}{dt} \langle \psi_1 | x | \psi_1 \rangle \\
&= 0 \\
\langle p \rangle &= \frac{1}{2} e^{i(\theta_1 - \theta_0 - \omega t)} \langle \psi_0 | p | \psi_1 \rangle + \frac{1}{2} e^{-i(\theta_1 - \theta_0 - \omega t)} \langle \psi_1 | p | \psi_0 \rangle \\
\langle \psi_0 | p | \psi_1 \rangle &= -i \sqrt{\frac{\hbar m \omega}{2}} \\
\langle \psi_1 | p | \psi_0 \rangle &= i \sqrt{\frac{\hbar m \omega}{2}} \\
\langle p \rangle &= -\frac{1}{2} i e^{i(\theta_1 - \theta_0 - \omega t)} \sqrt{\frac{\hbar m \omega}{2}} + \frac{1}{2} i e^{-i(\theta_1 - \theta_0 - \omega t)} \sqrt{\frac{\hbar m \omega}{2}} \\
&= -\frac{1}{2} i \sqrt{\frac{\hbar m \omega}{2}} (e^{i(\theta_1 - \theta_0 - \omega t)} - e^{-i(\theta_1 - \theta_0 - \omega t)}) \\
&= \sqrt{\frac{\hbar m \omega}{2}} \sin(\theta_1 - \theta_0 - \omega t)
\end{aligned}$$

The largest possible value of $\langle p \rangle$ is $\sqrt{\hbar m \omega / 2}$. If it takes on this value at $t = 0$ then

$$\Psi(x, t) = \frac{1}{\sqrt{2}} e^{-i\omega t/2} (\psi_0 + i\psi_1 e^{-i\omega t}).$$

3.43

(a)

$$\begin{aligned}
|z|^2 &= \text{Re}(z)^2 + \text{Im}(z)^2 \\
&= \left(\frac{z+z^*}{2}\right)^2 + \left(\frac{z-z^*}{2i}\right)^2 \\
\sigma_A^2 \sigma_B^2 &\geq \left(\frac{\langle f|g\rangle + \langle g|f\rangle}{2}\right)^2 + \left(\frac{\langle f|g\rangle - \langle g|f\rangle}{2i}\right)^2 \\
&= \left(\frac{\langle \hat{A}\hat{B}\rangle + \langle \hat{B}\hat{A}\rangle - 2\langle A\rangle\langle B\rangle}{2}\right)^2 + \left(\frac{\langle [\hat{A}, \hat{B}]\rangle}{2i}\right)^2 \\
&= \frac{1}{4}[\langle -i[\hat{A}, \hat{B}]\rangle^2 + (\langle \hat{A}\hat{B}\rangle + \langle \hat{B}\hat{A}\rangle - 2\langle A\rangle\langle B\rangle)^2] \\
&= \frac{1}{4}(\langle C\rangle^2 + \langle D\rangle^2)
\end{aligned}$$

(b)

$$\begin{aligned}
B &= A \\
\hat{C} &= -i[\hat{A}, \hat{B}] \\
&= 0 \\
\hat{D} &= 2(\hat{A}^2 - \langle A\rangle^2) \\
\langle D\rangle &= \langle \Psi|D|\Psi\rangle \\
&= \langle \Psi|2(\hat{A}^2 - \langle A\rangle^2)|\Psi\rangle \\
&= 2(\langle \Psi|\hat{A}^2|\Psi\rangle - \langle A\rangle^2) \\
&= 2(\langle A^2\rangle - \langle A\rangle^2) \\
&= 2\sigma_A^2 \\
\sigma_A^2 \sigma_B^2 &= \frac{1}{4}(\langle C\rangle^2 + \langle D\rangle^2) \\
\sigma_A^4 &\geq \sigma_A^4
\end{aligned}$$

3.44

(a) The eigenvalues of \mathbf{H} are $a-b$, $a+b$, and c and the associated eigenvectors

are $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, respectively.

$$|S(t)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-ict/\hbar}$$

(b)

$$\begin{aligned}
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= \frac{1}{2} \left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right] \\
|S(t)\rangle &= \frac{1}{2} \left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-i(a+b)t/\hbar} - \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-i(a-b)t/\hbar} \right] \\
&= \frac{1}{2} e^{-iat/\hbar} \begin{pmatrix} e^{-ibt/\hbar} + e^{ibt/\hbar} \\ 0 \\ e^{-ibt/\hbar} - e^{ibt/\hbar} \end{pmatrix} \\
&= e^{-iat/\hbar} \begin{pmatrix} \cos(bt/\hbar) \\ 0 \\ -i \sin(bt/\hbar) \end{pmatrix}
\end{aligned}$$

3.45

$$\begin{aligned}
\langle n|\hat{x}|S(t)\rangle &= \left\langle n \left| \hat{x} \sum_{n'=0}^{\infty} |n'\rangle \langle n'| \right| S(t) \right\rangle \\
&= \sum_{n'=0}^{\infty} \langle n|\hat{x}|n'\rangle \langle n'|S(t)\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}} \sum_{n'=0}^{\infty} (\sqrt{n'}\delta_{n,n'-1} + \sqrt{n}\delta_{n',n-1})c_{n'}(t) \\
&= \sqrt{\frac{\hbar}{2m\omega}} [\sqrt{n+1}c_{n+1}(t) + \sqrt{n}c_{n-1}(t)]
\end{aligned}$$

3.46

(a)

$$h_1 = \hbar\omega$$

$$h_2 = 2\hbar\omega$$

$$h_3 = 2\hbar\omega$$

$$|h_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|h_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|h_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$a_1 = 2\lambda$$

$$a_2 = \lambda$$

$$a_3 = -\lambda$$

$$|a_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|a_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$|a_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$b_1 = 2\mu$$

$$b_2 = \mu$$

$$b_3 = -\mu$$

$$|b_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|b_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$|b_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

(b)

$$\begin{aligned}
\langle H \rangle &= \langle S(0) | H | S(0) \rangle \\
&= \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\
&= \hbar\omega \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} c_1 \\ 2c_2 \\ 2c_3 \end{pmatrix} \\
&= \hbar\omega (|c_1|^2 + 2|c_2|^2 + 2|c_3|^2) \\
\langle A \rangle &= \lambda(c_1^*c_2 + c_2^*c_1 + 2|c_3|^2) \\
\langle B \rangle &= \mu(2|c_1|^2 + c_2^*c_3 + c_3^*c_2)
\end{aligned}$$

(c)

$$|S(t)\rangle = c_1 |h_1\rangle e^{-i\omega t} + c_2 |h_2\rangle e^{-2i\omega t} + c_3 |h_3\rangle e^{-2i\omega t}$$

You could measure H as $\hbar\omega$ with probability $|c_1|^2$ or $2\hbar\omega$ with probability $|c_2|^2 + |c_3|^2$.

4 Quantum Mechanics in Three Dimensions

4.1

(a)

$$\begin{aligned}[r_i, r_j] &= 0 \\ [p_i, p_j] &= 0 \\ [x, p_x]f &= x \left(-i\hbar \frac{\partial}{\partial x} \right) f - \left(-i\hbar \frac{\partial}{\partial x} \right) (xf) \\ &= -i\hbar x \frac{\partial f}{\partial x} + i\hbar \left(f + x \frac{\partial f}{\partial x} \right) \\ &= i\hbar f \\ [p_x, x] &= \left(-i\hbar \frac{\partial}{\partial x} \right) (xf) - x \left(-i\hbar \frac{\partial}{\partial x} \right) f \\ &= -i\hbar \left(f + x \frac{\partial f}{\partial x} \right) + i\hbar x \frac{\partial f}{\partial x} \\ &= -i\hbar f \\ [x, p_y] &= x \left(-i\hbar \frac{\partial}{\partial y} \right) f - \left(-i\hbar \frac{\partial}{\partial y} \right) (xf) \\ &= -i\hbar x \frac{\partial f}{\partial y} + i\hbar x \frac{\partial f}{\partial y} \\ &= 0 \\ [r_i, p_j] &= i\hbar \delta_{ij} \\ [p_i, r_j] &= -i\hbar \delta_{ij}\end{aligned}$$

(b)

$$\begin{aligned}
[\hat{H}, x]f &= \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)(xf) - x\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)f \\
&= -\frac{\hbar^2}{2m}\left[\frac{\partial}{\partial x}\left(f + x\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(x\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(x\frac{\partial f}{\partial z}\right)\right] \\
&\quad + Vxf + \frac{\hbar^2}{2m}x\nabla^2 f - Vxf \\
&= -\frac{\hbar^2}{2m}\left(2\frac{\partial f}{\partial x} + x\nabla^2 f\right) + \frac{\hbar^2}{2m}x\nabla^2 f \\
&= -\frac{\hbar^2}{m}\frac{\partial f}{\partial x} \\
[\hat{H}, x] &= -\frac{\hbar^2}{m}\frac{\partial}{\partial x} \\
[\hat{H}, \mathbf{r}] &= -\frac{\hbar^2}{m}\nabla \\
\frac{d}{dt}\langle \mathbf{r} \rangle &= \frac{i}{\hbar}\langle [\hat{H}, \mathbf{r}] \rangle \\
&= \frac{1}{m}\langle -i\hbar\nabla \rangle \\
&= \frac{1}{m}\langle \mathbf{p} \rangle \\
[\hat{H}, \hat{p}_x]f &= \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\left(-i\hbar\frac{\partial}{\partial x}\right)f - \left(-i\hbar\frac{\partial}{\partial x}\right)\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)f \\
&= -i\hbar\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\frac{\partial f}{\partial x} + i\hbar\left(\frac{\partial}{\partial x}\right)\left(-\frac{\hbar^2}{2m}\nabla^2 f + Vf\right) \\
&= i\frac{\hbar^3}{2m}\nabla^2\frac{\partial f}{\partial x} - i\hbar V\frac{\partial f}{\partial x} - i\frac{\hbar^3}{2m}\nabla^2\frac{\partial f}{\partial x} + i\hbar\left(\frac{\partial V}{\partial x}f + V\frac{\partial f}{\partial x}\right) \\
&= i\hbar\frac{\partial V}{\partial x}f \\
[\hat{H}, \hat{p}_x] &= i\hbar\frac{\partial V}{\partial x} \\
[\hat{H}, \mathbf{p}] &= i\hbar\nabla V \\
\frac{d}{dt}\langle \mathbf{p} \rangle &= \frac{i}{\hbar}\langle [\hat{H}, \mathbf{p}] \rangle \\
&= \langle -\nabla V \rangle
\end{aligned}$$

(c)

$$\begin{aligned}\sigma_{r_i}\sigma_{p_j} &\geq \frac{1}{2i}\langle [r_i, p_j] \rangle \\ &= \frac{1}{2i}i\hbar\delta_{ij} \\ &= \frac{\hbar}{2}\delta_{ij}\end{aligned}$$

4.2

(a)

$$\begin{aligned}\psi(\mathbf{r}) &= X(x)Y(y)Z(z) \\ -\frac{\hbar^2}{2m}\nabla^2\psi &= E\psi \\ -\frac{\hbar^2}{2m}\left(\frac{\partial^2 X}{\partial x^2}YZ + X\frac{\partial^2 Y}{\partial y^2}Z + XY\frac{\partial^2 Z}{\partial z^2}\right) &= EXYZ \\ \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} &= -\frac{2m}{\hbar^2}E\end{aligned}$$

The terms on the left hand side are each functions of a different variable, so they must be constant. Starting with the X term:

$$\begin{aligned}\frac{X''}{X} &= -\alpha \\ X'' + \alpha X &= 0\end{aligned}$$

If $\alpha < 0$

$$X = A_x e^{\sqrt{-\alpha}x} + B_x e^{-\sqrt{-\alpha}x}$$

If $\alpha = 0$

$$X = A_x x + B_x$$

If $\alpha > 0$

$$X = A_x \sin(\sqrt{\alpha}x) + B_x \cos(\sqrt{\alpha}x)$$

Boundary conditions require that $\alpha > 0$. $X(0) = 0$ so $B_x = 0$. $X(a) = 0$ so $\sqrt{\alpha} = n_x \pi / a \Rightarrow \alpha = n_x^2 \pi^2 / a^2$,

$$X = A_x \sin\left(\frac{n_x \pi}{a}x\right).$$

Repeating the above for Y and Z finds

$$Y = A_y \sin\left(\frac{n_y \pi}{a} y\right)$$

$$Z = A_z \sin\left(\frac{n_z \pi}{a} z\right)$$

so

$$\psi(\mathbf{r}) = A_x A_y A_z \sin\left(\frac{n_x \pi}{a} x\right) \sin\left(\frac{n_y \pi}{a} y\right) \sin\left(\frac{n_z \pi}{a} z\right).$$

Assuming $A_x = A_y = A_z = A$ and normalising finds

$$1 = \int_0^a \int_0^a \int_0^a A^6 \sin^2\left(\frac{n_x \pi}{a} x\right) \sin^2\left(\frac{n_y \pi}{a} y\right) \sin^2\left(\frac{n_z \pi}{a} z\right) d^3 \mathbf{r}$$

$$= A^6 \frac{a^3}{8}$$

$$A^6 = \frac{8}{a^3}$$

$$= \left(\frac{2}{a}\right)^3$$

$$A = \sqrt{\frac{2}{a}}$$

so

$$\psi(\mathbf{r}) = \left(\frac{2}{a}\right)^{3/2} \sin\left(\frac{n_x \pi}{a} x\right) \sin\left(\frac{n_y \pi}{a} y\right) \sin\left(\frac{n_z \pi}{a} z\right).$$

Finally

$$-\frac{2m}{\hbar^2} E = -\alpha - \beta - \gamma$$

$$= -\frac{\pi^2 n_x^2}{a^2} - \frac{\pi^2 n_y^2}{a^2} - \frac{\pi^2 n_z^2}{a^2}$$

$$E = \frac{\pi^2 \hbar^2}{2ma^2} (n_x^2 + n_y^2 + n_z^2), \quad n_x, n_y, n_z = 1, 2, 3, \dots$$

(b)

$$E_1 = 3 \frac{\pi^2 \hbar^2}{2ma^2}, n = 1$$

$$E_2 = 6 \frac{\pi^2 \hbar^2}{2ma^2}, n = 3$$

$$E_3 = 9 \frac{\pi^2 \hbar^2}{2ma^2}, n = 3$$

$$E_4 = 11 \frac{\pi^2 \hbar^2}{2ma^2}, n = 3$$

$$E_5 = 12 \frac{\pi^2 \hbar^2}{2ma^2}, n = 1$$

$$E_6 = 14 \frac{\pi^2 \hbar^2}{2ma^2}, n = 6$$

(c)

$$E_{14} = 27 \frac{\pi^2 \hbar^2}{2ma^2}$$

$$d = 4$$

4.3

(a)

$$\begin{aligned}
\phi(r, \theta, \phi) &= Ae^{-r/a} \\
\frac{\partial \psi}{\partial r} &= -\frac{A}{a}e^{-r/a} \\
\frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) &= \frac{\partial}{\partial r} \left(-\frac{A}{a}r^2e^{-r/a} \right) \\
&= -\frac{A}{a} \left(2re^{-r/a} - \frac{1}{a}r^2e^{-r/a} \right) \\
&= \frac{A}{a^2}r^2 \left(1 - \frac{2a}{r} \right) e^{-r/a} \\
\frac{\partial \psi}{\partial \theta} &= 0 \\
\frac{\partial^2 \psi}{\partial \phi^2} &= 0 \\
-\frac{\hbar^2}{2m} \frac{A}{a^2} \left(1 - \frac{2a}{r} \right) e^{-r/a} + V(r)Ae^{-r/a} &= EAe^{-r/a} \\
-\frac{\hbar^2}{2ma^2} \left(1 - \frac{2a}{r} \right) + V(r) &= E \\
-\frac{\hbar^2}{2ma^2} &= E \\
V(r) &= -\frac{\hbar^2}{mar}
\end{aligned}$$

(b)

$$\begin{aligned}
\phi(r, \theta, \phi) &= Ae^{-r^2/a^2} \\
\frac{\partial \psi}{\partial r} &= -\frac{2}{a^2}rAe^{-r^2/a^2} \\
\frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) &= \frac{\partial}{\partial r} \left(-\frac{2}{a^2}r^3Ae^{-r^2/a^2} \right) \\
&= -\frac{2A}{a^2} \left(3r^2e^{-r^2/a^2} - \frac{2}{a^2}r^4e^{-r^2/a^2} \right) \\
&= \frac{2A}{a^2}r^2 \left(\frac{2}{a^2}r^2 - 3 \right) e^{-r^2/a^2}
\end{aligned}$$

$$\begin{aligned}
-\frac{\hbar^2}{2m} \frac{2A}{a^2} \left(\frac{2}{a^2} r^2 - 3 \right) e^{-r^2/a^2} + V(r) A e^{-r^2/a^2} &= E A e^{-r^2/a^2} \\
-\frac{\hbar^2}{2m} \frac{2}{a^2} \left(\frac{2}{a^2} r^2 - 3 \right) + V(r) &= E \\
\frac{\hbar^2}{2m} \frac{2}{a^2} 3 &= E \\
\frac{3\hbar^2}{ma^2} &= E \\
\frac{3\hbar^2}{ma^2} + \frac{\hbar^2}{2m} \frac{2}{a^2} \left(\frac{2}{a^2} r^2 - 3 \right) &= V(r) \\
\frac{2\hbar^2}{ma^4} r^2 &= V(r)
\end{aligned}$$

4.4

$$\begin{aligned}
Y_0^0 &= \frac{1}{2\sqrt{\pi}} \\
Y_2^1 &= \frac{1}{2} \sqrt{\frac{5}{6\pi}} e^{i\phi} P_2^1(\cos \theta) \\
&= \frac{1}{2} \sqrt{\frac{5}{6\pi}} e^{i\phi} (-1)^1 (1 - \cos^2 \theta)^{1/2} \left(\frac{d}{dx} \right) P_2(\cos \theta) \\
&= -\frac{1}{2} \sqrt{\frac{5}{6\pi}} e^{i\phi} \sin \theta \left(\frac{d\theta}{dx} \frac{d}{d\theta} \right) \frac{1}{2} (3 \cos^2 \theta - 1) \\
&= -\frac{1}{4} \sqrt{\frac{5}{6\pi}} e^{i\phi} \sin \theta \left(-\frac{1}{\sin \theta} \right) (-6 \cos \theta \sin \theta) \\
&= -\frac{1}{2} \sqrt{\frac{15}{2\pi}} e^{i\phi} \sin \theta \cos \theta \\
\int |Y_0^0|^2 d\Omega &= \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \sin \theta d\theta d\phi \\
&= 1 \\
\int |Y_2^1|^2 d\Omega &= \frac{15}{8\pi} \int_0^\pi \int_0^{2\pi} \sin^3 \theta \cos^2 \theta d\theta d\phi \\
&= 1 \\
\int (Y_0^0)^* Y_2^1 d\Omega &= -\frac{1}{2\sqrt{\pi}} \frac{1}{2} \sqrt{\frac{15}{2\pi}} \int_0^\pi \int_0^{2\pi} e^{i\phi} \sin^2 \theta \cos \theta d\theta d\phi \\
&= 0
\end{aligned}$$

4.5

$$\begin{aligned}
\Theta(\theta) &= A \ln \left(\tan \frac{\theta}{2} \right) \\
0 &= \sin \theta \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) \\
&= \frac{1}{2} A \sin \theta \frac{d}{d\theta} \left(\sin \theta \csc \frac{\theta}{2} \sec \frac{\theta}{2} \right) \\
&= \frac{1}{2} A \sin \theta \left(\cos \theta \csc \frac{\theta}{2} \sec \frac{\theta}{2} - \frac{1}{2} \csc^2 \frac{\theta}{2} \sin \theta + \frac{1}{2} \sec^2 \frac{\theta}{2} \sin \theta \right) \\
&= 0
\end{aligned}$$

It's not a valid physical solution because it blows up at $\theta = 0$ and $\theta = \pi$.

4.6

$$\begin{aligned}
Y_\ell^{-m} &= \sqrt{\frac{(2\ell+1)(\ell+m)!}{4\pi(\ell-m)!}} e^{-im\phi} P_\ell^{-m}(\cos \theta) \\
&= \sqrt{\frac{(2\ell+1)(\ell+m)!}{4\pi(\ell-m)!}} e^{-im\phi} (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos \theta) \\
&= (-1)^m \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}} e^{-im\phi} P_\ell^m(\cos \theta) \\
&= (-1)^m (Y_\ell^m)^*
\end{aligned}$$

4.7

$$\begin{aligned}
Y_3^2(\theta, \phi) &= \sqrt{\frac{7}{480\pi}} e^{2i\phi} P_3^2(\cos \theta) \\
&= \frac{1}{4} \sqrt{\frac{105}{2\pi}} e^{2i\phi} \sin^2 \theta \cos \theta \\
Y_\ell^\ell(\theta, \phi) &= \sqrt{\frac{2\ell+1}{4\pi(2\ell)!}} e^{i\ell\phi} P_\ell^\ell(\cos \theta) \\
P_\ell^\ell(x) &= (-1)^\ell (1-x^2)^{\ell/2} \left(\frac{d}{dx} \right)^\ell P_\ell(x) \\
&= (-1)^\ell (1-x^2)^{\ell/2} \left(\frac{d}{dx} \right)^\ell \left[\frac{1}{2^\ell \ell!} \left(\frac{d}{dx} \right)^\ell (x^2-1)^\ell \right] \\
&= (-1)^\ell \frac{1}{2^\ell \ell!} (1-x^2)^{\ell/2} \left(\frac{d}{dx} \right)^{2\ell} (x^2-1)^\ell \\
&= (-1)^\ell \frac{(2\ell)!}{2^\ell \ell!} (1-x^2)^{\ell/2} \\
Y_\ell^\ell(\theta, \phi) &= \sqrt{\frac{2\ell+1}{4\pi(2\ell)!}} e^{i\ell\phi} (-1)^\ell \frac{(2\ell)!}{2^\ell \ell!} (1-\cos^2 \theta)^{\ell/2} \\
&= \frac{1}{\ell!} \sqrt{\frac{(2\ell+1)!}{4\pi}} \left(-\frac{1}{2} e^{i\phi} \sin \theta \right)^\ell
\end{aligned}$$

4.9

(a)

$$\begin{aligned}
 n_1(x) &= -(-x) \left(\frac{1}{x} \frac{d}{dx} \right) \frac{\cos x}{x} \\
 &= -\frac{\sin x}{x} - \frac{\cos x}{x^2} \\
 n_2(x) &= -(-x)^2 \left(\frac{1}{x} \frac{d}{dx} \right)^2 \frac{\cos x}{x} \\
 &= -x^2 \left(\frac{1}{x} \frac{d}{dx} \right) \left[\frac{1}{x} \frac{d}{dx} \left(\frac{\cos x}{x} \right) \right] \\
 &= -x^2 \left(\frac{1}{x} \frac{d}{dx} \right) \left[\frac{1}{x} \left(-\frac{\sin x}{x} - \frac{\cos x}{x^2} \right) \right] \\
 &= x \frac{d}{dx} \left(\frac{\sin x}{x^2} + \frac{\cos x}{x^3} \right) \\
 &= x \left(\frac{\cos x}{x^2} - \frac{2 \sin x}{x^3} - \frac{\sin x}{x^3} - \frac{3 \cos x}{x^4} \right) \\
 &= \left(\frac{1}{x} - \frac{3}{x^3} \right) \cos x - \frac{3 \sin x}{x^2}
 \end{aligned}$$

(b)

$$\begin{aligned}
 n_1(x) &\approx -\frac{x}{x} - \frac{1}{x^2} \\
 &= -1 - \frac{1}{x^2} \\
 n_2(x) &\approx \left(\frac{1}{x} - \frac{3}{x^3} \right) - \frac{3}{x^2} x \\
 &= -\frac{2}{x} - \frac{3}{x^3}
 \end{aligned}$$

4.10

(a)

$$\begin{aligned}
 -\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \frac{\hbar^2}{2m} \frac{2}{r^2} u &= Eu \\
 \frac{d^2 u}{dr^2} - \frac{2}{r^2} u &= -\frac{2mE}{\hbar^2} u \\
 \frac{d^2 u}{dr^2} &= \left(\frac{2}{r^2} - k^2 \right) u \\
 j_1(kr) &= \frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr} \\
 u &= Ar j_1(kr) \\
 &= Ar \left[\frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr} \right]
 \end{aligned}$$

$$\begin{aligned}
 \frac{du}{dr} &= A \left[\frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr} \right] \\
 &\quad + Ar \left[\frac{k \cos(kr)}{(kr)^2} - \frac{2 \sin(kr)}{k^2 r^3} + \frac{k \sin(kr)}{kr} + \frac{\cos(kr)}{kr^2} \right] \\
 &= \frac{A}{kr} \left[\frac{\sin(kr)}{kr} - \cos(kr) \right] \\
 &\quad + \frac{A}{k} \left[\frac{2 \cos(kr)}{r} - \frac{2 \sin(kr)}{kr^2} + k \sin(kr) \right] \\
 \frac{d^2 u}{dr^2} &= \frac{A(-2 + k^2 r^2)(kr \cos(kr) - \sin(kr))}{k^2 r^3} \\
 &= \frac{2 - k^2 r^2}{r^2} Ar \left[\frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr} \right] \\
 &= \left(\frac{2}{r^2} - k^2 \right) u
 \end{aligned}$$

(b)

$$0 = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$= \frac{\sin x}{x} - \cos x$$

$$\cos x = \frac{\sin x}{x}$$

$$x = \tan x$$

$$x \approx \left(N + \frac{1}{2}\right) \pi, n \in \mathbb{Z}$$

$$ka = x$$

$$\frac{\sqrt{2mE_{N1}}}{\hbar} a \approx \left(N + \frac{1}{2}\right) \pi$$

$$E_{N1} = \frac{\hbar^2 \pi^2}{2ma^2} \left(N + \frac{1}{2}\right)^2, N \in \mathbb{Z}$$