# Advanced Engineering Mathematics Systems of Differential Equations by Dennis G. Zill Notes

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## Contents



# 10 Systems of Linear Differential Equations

## 10.1 Theory of Linear Systems

- A system of the form

$$\begin{aligned}
\frac{dx_1}{dt} &= g_1(t, x_1, x_2, \ldots, x_n) \\
\frac{dx_2}{dt} &= g_2(t, x_1, x_2, \ldots, x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= g_n(t, x_1, x_2, \ldots, x_n)
\end{aligned}$$

  is called a **first-order system**.

- When each of the functions $g_n(t, x_1, x_2, \ldots, x_n)$ is linear in the dependent variables $x_1, x_2, \ldots, x_n$, we get the **normal form** of a first-order system of linear equations

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n + f_1(t) \\
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n + f_2(t) \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n + f_n(t).
\end{aligned}$$

Such a system is called a **linear system**.

- When $f_i(t) = 0$ for $i = 1, 2, \ldots, n$ the linear system is said to be **homogeneous**, otherwise it's **nonhomogenous**.
- If $\mathbf{X}$, $\mathbf{A}(t)$, and $\mathbf{F}(t)$ denote the matrices

$$\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \vdots & & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{pmatrix}$$

$$\mathbf{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

then homogeneous linear systems can be written

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

and nonhomogeneous linear systems can be written

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}.$$

- A **solution vector** on an interval $I$ is any column matrix

$$\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

whose entries are differentiable functions satisfying the linear system on the interval.

- The entries of a solution vector can be considered a set of parametric equations that define a curve in $n$-space. Such a curve is called a **trajectory**.

- The problem of solving

$$\mathbf{X}' = \mathbf{A}(t)\mathbf{X} + \mathbf{F}(t)$$

subject to

$$\mathbf{X}(t_0) = \mathbf{X}_0$$

is an **initial value problem** in matrix form.

- The **superposition principle** states that if $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are solution vectors of a homogeneous linear system on an interval $I$, then

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + c_n\mathbf{X}_n$$

where $c_n$ are arbitrary constants is also a solution.

- If $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are a set of solution vectors of a homogeneous linear system on an interval $I$, the set is said to be **linearly dependent** if there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + x_n\mathbf{X}_n = \mathbf{0}$$

for every $t$ in the interval. Otherwise the set is said to be **linearly independent**.

- A set of solution vectors

$$\mathbf{X}_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}, \quad \ldots, \quad \mathbf{X}_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$$

is linearly independent on an interval $I$ if the **Wronskian**

$$W(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) = \begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{vmatrix} \neq 0$$

for every $t$ in the interval.

- Any set of $n$ linearly independent solution vectors of a homogeneous linear system on an interval $I$ is said to be a **fundamental set of solutions** on that interval.

- If $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are a fundamental set of solutions of a homogeneous linear system on an interval $I$, then the **general solution** of the system on that interval is

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + c_n\mathbf{X}_n$$

  where $c_i$ are arbitrary constants.

- For nonhomogenous systems, a **particular solution** $\mathbf{X}_p$ on an interval $I$ is any vector, free from arbitrary parameters, whose entries are functions that satify the system.

- For nonhomogeneous systems, the **general solution** of the system on the interval is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$$

  where $\mathbf{X}_c$ is the general solution of the associated homogeneous system (the **complementary function**) and $\mathbf{X}_p$ is a particular solution of the nonhomogeneous system.

## 10.2 Homogeneous Linear Systems

### 10.2.1 Distinct Real Eigenvalues

- If $\mathbf{X}' = \mathbf{A}\mathbf{X}$ is a homogeneous linear system, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are $n$ real, distinct eigenvalues of $\mathbf{A}$, and $\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_n$ are the corresponding eigenvectors of $\mathbf{A}$, then

$$\mathbf{X} = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t} + \ldots + c_n\mathbf{K}_n e^{\lambda_n t}$$

  is the general solution of the system.

- If a system of linear equations consists of variables $x$ and $y$, then the $x - y$ plane is called the **phase plane**.

- Solution vectors of a linear system can be considered parametric equations and plotted on the phase plane. These are called trajectories.

- When multiple trajectories are plotted in the phase plane, it's called a **phase portrait**.

### 10.2.2 Repeated Eigenvalues

- If the coefficient matrix $\mathbf{A}$ of a linear system has an eigenvalue $\lambda$ of multiplicity $m$, it may be possible to find $m$ linearly independent eigenvectors

$\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_m$ associated with the eigenvalue in which case the $m$ solution vectors associated with the eigenvalue are

$$\begin{aligned}
\mathbf{X}_1 &= \mathbf{K}_1 e^{\lambda t} \\
\mathbf{X}_2 &= \mathbf{K}_2 e^{\lambda t} \\
&\vdots \\
\mathbf{X}_m &= \mathbf{K}_m e^{\lambda t}.
\end{aligned}$$

- If the coefficient matrix $\mathbf{A}$ of a linear system has an eigenvalue $\lambda$ of multiplicity $m$ and it's not possible to find $m$ linearly independent eigenvectors associated with the eigenvalue, then the $m$ solution vectors associated with the eigenvalue are

$$\begin{aligned}
\mathbf{X}_1 &= \mathbf{K}_1 e^{\lambda t} \\
\mathbf{X}_2 &= \mathbf{K}_1 t e^{\lambda t} + \mathbf{K_2} e^{\lambda t} \\
&\vdots \\
\mathbf{X}_m &= \mathbf{K}_1 \frac{t^{m-1}}{(m-1)!} e^{\lambda t} + \mathbf{K}_2 \frac{t^{m-2}}{(m-2)!} e^{\lambda t} + \ldots + \mathbf{K}_m e^{\lambda t}
\end{aligned}$$

  where $\mathbf{K}_i$ are the solutions to the equations

$$\begin{aligned}
(\mathbf{A} - \lambda \mathbf{I})\mathbf{K}_1 &= \mathbf{0} \\
(\mathbf{A} - \lambda \mathbf{I})\mathbf{K}_2 &= \mathbf{K}_1 \\
&\vdots \\
(\mathbf{A} - \lambda \mathbf{I})\mathbf{K}_m &= \mathbf{K}_{m-1}.
\end{aligned}$$

### 10.2.3 Complex Eigenvalues

- If $\mathbf{A}$ is the coefficient matrix of a homogeneous linear system and it has a complex eigenvalue $\lambda = \alpha + i\beta$ and associated eigenvector $\mathbf{K}_1$, then

$$\mathbf{X}_1 = \mathbf{K}_1 e^{\lambda t} \text{ and } \mathbf{X}_2 = \overline{\mathbf{K}}_1 e^{\overline{\lambda} t}$$

  are solutions of the system.

- The solutions above can be made real by writing them as

$$\begin{aligned}
\mathbf{X}_1 &= [\mathbf{B}_1 \cos \beta t - \mathbf{B}_2 \sin \beta t] e^{\alpha t} \\
\mathbf{X}_2 &= [\mathbf{B}_2 \cos \beta t + \mathbf{B}_1 \sin \beta t] e^{\alpha t}
\end{aligned}$$

  where $\mathbf{B}_1 = \mathrm{Re}(\mathbf{K}_1)$ and $\mathbf{B}_2 = \mathrm{Im}(\mathbf{K}_1)$.

## 10.3 Solution by Diagonalization

- A homogeneous linear system $\mathbf{X}' = \mathbf{A}\mathbf{X}$ in which each $x_i'$ is expressed as a linear combination of $x_1, x_2, \ldots, x_n$ is said to be **coupled**. If each $x_i'$ is expressed solely in terms of $x_i$ the system is said to be **uncoupled**.

- Given a linear system $\mathbf{X}' = \mathbf{A}\mathbf{X}$, if the coefficient matrix $\mathbf{A}$ is diagonalisable such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ then the system can be solved by:

  1. Substituting $\mathbf{X} = \mathbf{P}\mathbf{Y}$ which gives $\mathbf{P}\mathbf{Y}' = \mathbf{A}\mathbf{P}\mathbf{Y}$ or $\mathbf{Y}' = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{Y} = \mathbf{D}\mathbf{Y}$
  2. Because $\mathbf{D}$ is a diagonal matrix with $\mathbf{A}$'s eigenvalues along the diagonal, this means the solutions to $\mathbf{Y}' = \mathbf{D}\mathbf{Y}$ are

  $$\mathbf{Y} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

  3. These solutions can then be substituted into $\mathbf{X} = \mathbf{P}\mathbf{Y}$ to solve for $\mathbf{X}$