Introduction to Electrodynamics by David J. Griffiths Notes

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1 Vector Algebra

1.6 The Theory of Vector Fields

1.6.1 The Helmholtz Theorem

- The **Helmholtz theorem** states that a vector field $\mathbf{F}$ is uniquely determined if you're given its divergence $\nabla \cdot \mathbf{F}$, curl $\nabla \times \mathbf{F}$, and sufficient boundary conditions.

1.6.2 Potentials

- If the curl of a vector field vanishes everywhere, then it can be expressed as the gradient of a **scalar potential**
$$\nabla \times \mathbf{F} = \mathbf{0} \Leftrightarrow \mathbf{F} = -\nabla V.$$
- If the divergence of a vector field vanishes everywhere, then it can be expressed as the curl of a **vector potential**
$$\nabla \cdot \mathbf{F} = 0 \Leftrightarrow \mathbf{F} = \nabla \times \mathbf{A}.$$

2 Electrostatics

2.1 The Electric Field

2.1.2 Coulomb's Law

- **Couloumb's law** gives the force between two point charges q and Q
$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathscr{r}} \hat{\boldsymbol{\mathscr{r}}}$$
where
$$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C}^2/(\mathrm{N\,m}^2)$$
is the **permittivity of free space** and $\boldsymbol{\mathscr{r}}$ is the separation vector between the two charges.

2.1.3 The Electric Field

- The **electric field $\mathbf{E}$** is a vector field that varies from point to point and gives the force per unit charge that would be exerted on a test charge if placed at a particular point.
- For a collection of n source charges q_i at displacements $\boldsymbol{\mathscr{r}}_i$ from a test charge, the electric field is
$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{\mathscr{r}_i^2} \hat{\boldsymbol{\mathscr{r}}}.$$

2.1.4 Continuous Charge Distributions

- Couloumb's law for a continuous charge distribution is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{1}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}\, dq.$$

2.2 Divergence and Curl of Electrostatic Fields

2.2.1 Field Lines, Flux, and Gauss's Law

- **Gauss's law** states that the electric field flux through a closed surface is proportional to the amount of charge within that surface

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$$

or

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho.$$

2.2.4 The Curl of E

- The curl of an electric field is $\mathbf{0}$

$$\nabla \times \mathbf{E} = \mathbf{0}.$$

2.3 Electric Potential

2.3.1 Introduction to Potential

- The **electric potential** at a point $\mathbf{r}$ is defined as

$$V(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$$

where $\mathcal{O}$ is an agreed origin.

- The potential difference between two points $\mathbf{a}$ and $\mathbf{b}$ is

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}.$$

- The electric field and potential are also related by the equation

$$\mathbf{E} = -\nabla V.$$

2.3.2 Comments on Potential

- The choice of origin $\mathcal{O}$ in the definiton of vector potential only affects the absolute potential values, not potential differences. Typically it is chosen to be "at infinity" unless the charge distribution itself extends to infinity.
- Electric potential obeys the superposition principle.
- The units of electric potential is $\mathrm{N\,m/C} = \mathrm{J/C} = \mathrm{V}$.

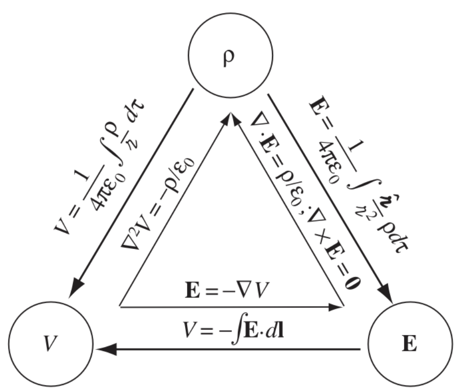
2.3.3 Poisson's Equation and Laplace's Equation

- If
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$
and
$$\mathbf{E} = -\nabla V$$
then
$$\begin{aligned} \nabla \cdot (-\nabla V) &= \frac{\rho}{\epsilon_0} \\ \nabla^2 V &= -\frac{\rho}{\epsilon_0}. \end{aligned}$$
This is known as **Poisson's equation**. In regions where $\rho = 0$ it reduces to **Laplace's equation**
$$\nabla^2 V = 0.$$

2.3.4 The Potential of a Localized Charge Distribution

- The potential of a continuous charge distribution is
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\imath} \, d\tau'$$
where the reference is point is set to infinity.

2.3.5 Boundary Conditions



- The normal component of the electric field is discontinuous by an amount σ/ϵ_0 at any boundary, i.e.
$$E_{\text{above}} - E_{\text{below}} = \frac{\sigma}{\epsilon_0}.$$

- The tangential component of the electric field is always continuous at any boundary.
- The electric potential is always continuous at any boundary, however because $\mathbf{E} = -\nabla V$, the gradient of the electric potential inherits the discontinuity at boundaries with surface charge, i.e.

$$\nabla V_{\text{above}} - \nabla V_{\text{below}} = -\frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

or

$$\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{\sigma}{\epsilon_0}$$

where

$$\frac{\partial V}{\partial n} = \nabla V \cdot \hat{\mathbf{n}}.$$

2.4 Work and Energy in Electrostatics

2.4.1 The Work It Takes to Move a Charge

- The work required to move a charge Q from infinity to a point $\mathbf{r}$ is

$$W = Q[V(\mathbf{r}) - V(\infty)] = QV(\mathbf{r}).$$

In that sense, the electric potential is the energy per unit charge required to assemble a system of point charges.

2.4.2 The Energy of a Point Charge Distribution

- If you bring a first charge in from infinity you do no work because there are no other charges. If you bring a second charge in from inifnity you do work against the electric field of the first charge. The third does work against the first and second, and so on. Thus the total work required to assemble a collection of charges is

$$\begin{aligned} W &= \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \ldots + \frac{q_1 q_n}{\mathscr{r}1n} + \frac{q_2 q_3}{\mathscr{r}_{23}} + \ldots\right) \\ &= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}} \end{aligned}$$

or if we count each pair of charges twice and divide by two

$$W = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}.$$

If we pull q_i out the front we get

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i\left(\sum_{j\neq i}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_j}{\mathscr{r}ij}\right) = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i).$$

2.4.3 The Energy of a Continuous Charge Distribution

- For a volume charge density ρ the work to assemble a continuous charge distribution is

$$W = \frac{1}{2} \int \rho V \, d\tau$$

  or equivalently

$$W = \frac{\epsilon_0}{2} \int E^2 \, d\tau$$

  where the integral is taken over all space.

2.4.4 Comments on Electrostatic Energy

- The energy of an electrostatic field does not obey the superposition principle.