

# Classical Mechanics by John R. Taylor Notes

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## Contents

<b>1</b>	<b>Newton's Laws of Motion</b>	<b>2</b>
1.2	Space and Time . . . . .	2
1.4	Newton's First and Second Laws; Inertial Frames . . . . .	2
1.5	The Third Law and Conservation of Momentum . . . . .	3
1.7	Two-Dimensional Polar Coordinates . . . . .	3
<b>2</b>	<b>Projectiles and Charged Particles</b>	<b>3</b>
2.1	Air Resistance . . . . .	3
2.2	Linear Air Resistance . . . . .	5
2.4	Quadratic Air Resistance . . . . .	6
2.5	Motion of a Charge in a Uniform Magnetic Field . . . . .	7
<b>3</b>	<b>Momentum and Angular Momentum</b>	<b>7</b>
3.1	Conservation of Momentum . . . . .	7
3.2	Rockets . . . . .	7
3.3	The Center of Mass . . . . .	8
3.4	Angular Momentum for a Single Particle . . . . .	8
3.5	Angular Momentum for Several Particles . . . . .	9
<b>4</b>	<b>Energy</b>	<b>9</b>
4.1	Kinetic Energy and Work . . . . .	9
4.2	Potential Energy and Conservative Forces . . . . .	10
4.3	Force as the Gradient of Potential Energy . . . . .	10
4.4	The Second Condition that F be Conservative . . . . .	10
4.5	Time-Dependent Potential Energy . . . . .	10
4.8	Central Forces . . . . .	11
<b>5</b>	<b>Oscillations</b>	<b>11</b>
5.2	Simple Harmonic Motion . . . . .	11
5.3	Two-Dimensional Oscillators . . . . .	11
5.7	Fourier Series . . . . .	12
<b>6</b>	<b>Calculus of Variations</b>	<b>12</b>

<b>7</b>	<b>Lagrange's Equations</b>	<b>13</b>
7.1	Lagrange's Equations for Unconstrained Motion . . . . .	13
7.6	Generalized Momenta and Ignorable Coordinates . . . . .	15
7.8	More about Conservation Laws . . . . .	15
7.9	Lagrange's Equations for Magnetic Forces . . . . .	15
<b>8</b>	<b>Two-Body Central Force Problems</b>	<b>16</b>
8.1	The Problem . . . . .	16
8.2	CM and Relative Coordinates; Reduced Mass . . . . .	16
8.3	The Equations of Motion . . . . .	17
8.4	The Equivalent One-Dimensional Problem . . . . .	17
<b>9</b>	<b>Mechanics in Noninertial Frames</b>	<b>18</b>
9.1	Acceleration without Rotation . . . . .	18
9.3	The Angular Velocity Vector . . . . .	18
9.4	Time Derivatives in a Rotating Frame . . . . .	19
9.5	Newton's Second Law in a Rotating Frame . . . . .	19
<b>10</b>	<b>Rotational Motion of Rigid Bodies</b>	<b>19</b>
10.1	Properties of the Centre of Mass . . . . .	19
10.2	Rotation about a Fixed Axis . . . . .	20
10.3	Rotation about Any Axis; the Inertia Tensor . . . . .	20
10.4	Principle Axes of Inertia . . . . .	21
10.5	Finding the Principle Axes; Eigenvalue Equations . . . . .	21

# 1 Newton's Laws of Motion

## 1.2 Space and Time

- In cartesian coordinates the basis vectors don't depend on time so their derivatives are  $\mathbf{0}$ . This means that

$$\begin{aligned}\frac{d}{dt}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) &= \frac{dx}{dt}\hat{\mathbf{x}} + x\frac{d\hat{\mathbf{x}}}{dt} + \frac{dy}{dt}\hat{\mathbf{y}} + y\frac{d\hat{\mathbf{y}}}{dt} + \frac{dz}{dt}\hat{\mathbf{z}} + z\frac{d\hat{\mathbf{z}}}{dt} \\ &= \frac{dx}{dt}\hat{\mathbf{x}} + \frac{dy}{dt}\hat{\mathbf{y}} + \frac{dz}{dt}\hat{\mathbf{z}}\end{aligned}$$

as expected. However, in order coordinate systems (e.g. polar, spherical) the basis vectors may depend on time and their derivatives aren't  $\mathbf{0}$ .

## 1.4 Newton's First and Second Laws; Inertial Frames

- Newton's second law  $\mathbf{F} = m\mathbf{a}$  can be restated as  $\mathbf{F} = \dot{\mathbf{p}}$ .
- An inertial frame is one where Newton's first law holds. Typically this means the frame isn't accelerating or rotating.

## 1.5 The Third Law and Conservation of Momentum

- Forces that act along the line joining two objects are called **central forces**.
- The **principle of conservation of momentum** states that if the net external force  $\mathbf{F}_{\text{ext}}$  on an  $N$ -particle system is zero, the system's total momentum  $\mathbf{P}$  is constant.

## 1.7 Two-Dimensional Polar Coordinates

- In two-dimensional polar coordinates, the unit vectors  $\hat{\mathbf{r}}$  and  $\hat{\phi}$  depend on position and thus time. Their derivatives are

$$\begin{aligned}\frac{d\hat{\mathbf{r}}}{dt} &= \dot{\phi}\hat{\phi} \\ \frac{d\hat{\phi}}{dt} &= -\dot{\phi}\hat{\mathbf{r}}.\end{aligned}$$

Consequently, the derivatives of the position vector  $\mathbf{r} = r\hat{\mathbf{r}}$  are

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= \frac{d}{dt}(r\hat{\mathbf{r}}) \\ &= \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \\ &= \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\phi}\end{aligned}$$

and

$$\begin{aligned}\frac{d^2\mathbf{r}}{dt^2} &= \frac{d}{dt}(\dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\phi}) \\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\frac{d\hat{\phi}}{dt} \\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\phi}\hat{\phi} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} - r\dot{\phi}^2\hat{\mathbf{r}} \\ &= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi}.\end{aligned}$$

- In light of the above, Newton's second law in polar coordinates can be written

$$\begin{aligned}F_r &= m(\ddot{r} - r\dot{\phi}^2) \\ F_\phi &= m(r\ddot{\phi} + 2\dot{r}\dot{\phi}).\end{aligned}$$

## 2 Projectiles and Charged Particles

### 2.1 Air Resistance

- Air resistance depends on the speed  $v$  of the moving object. For many objects the direction of the air resistance force  $\mathbf{f}$  is opposite to  $\mathbf{v}$ , but not always. For example, the air resistance force on an airplane causes lift.

- An air resistance force can be described by the equation

$$\mathbf{f} = -f(v)\hat{\mathbf{v}}$$

where  $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$  gives the direction and  $f(v)$  gives the magnitude.

- $f(v)$  can be approximated as

$$f(v) = f_{\text{lin}} + f_{\text{quad}} = bv + cv^2.$$

- The linear term  $f_{\text{lin}}$  arises from the viscous drag of the medium and is generally proportional to the projectile's linear size.
- The quadratic term  $f_{\text{quad}}$  arises from the fact that the projectile must accelerate the air with which it is continually colliding and it is proportional to the density of the medium and the cross-sectional area of the projectile.
- For a spherical projectile the coefficients  $b$  and  $c$  above have the form

$$b = \beta D \text{ and } c = \gamma D^2$$

where  $D$  is the diameter of the sphere and the coefficients  $\beta$  and  $\gamma$  depend on the nature of the medium. In air at STP they have approximate values

$$\beta = 1.6 \times 10^{-4} \text{ N s/m}^2$$

and

$$\gamma = 0.25 \text{ N s}^2/\text{m}^4.$$

- Depending on the natures of the medium and projectile it's often possible to neglect one of the terms in  $f(v)$ . To determine if this is the case we can calculate their ratio. For example, for a spherical projectile at STP

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} = \frac{cv^2}{bv} = \frac{\gamma D}{\beta} v = (1.6 \times 10^3 \text{ s/m}^2) D v.$$

If the ratio is large  $f_{\text{lin}}$  can be ignored. If it's small  $f_{\text{quad}}$  can be ignored.

- The **Reynolds number** can be used to characterise the behaviour of an object in a fluid

$$R = \frac{\rho}{\mu} D v$$

where  $\rho$  is the medium's density,  $\mu$  is its viscosity,  $D$  is the linear dimension of the projectile (diameter for spherical projectiles), and  $v$  is the projectile's speed. The quadratic force  $f_{\text{quad}}$  is dominant when the Reynolds number  $R$  is large and the linear force  $f_{\text{linear}}$  is dominant when it is small.

## 2.2 Linear Air Resistance

- When the quadratic drag force is negligible the equation of motion becomes

$$\begin{aligned}\mathbf{F} &= \mathbf{W} - \mathbf{f} \\ m\mathbf{a} &= m\mathbf{g} - b\mathbf{v} \\ m\dot{\mathbf{v}} &= m\mathbf{g} - b\mathbf{v}.\end{aligned}$$

This is a first-order differential equation for  $\mathbf{v}$  where the horizontal and vertical components can be separated to

$$\begin{aligned}m\dot{v}_x &= -bv_x \\ m\dot{v}_y &= mg - bv_y,\end{aligned}$$

each of which is easily solvable.

- The **terminal speed** of an object undergoing freefall and experiencing only linear drag is

$$v_{\text{ter}} = \frac{mg}{b}.$$

- The **characteristic time**

$$\tau = \frac{1}{k} = \frac{1}{b/m} = \frac{m}{b}$$

is a measure of the importance of air resistance.

- For horizontal motion with drag it's a measure of the time it takes for the projectile to reach  $1/e$  of its initial velocity.
- For freefall with drag it's a measure of the time it would take the projectile to reach its terminal velocity if it didn't experience drag

$$v_{\text{ter}} = g\tau.$$

- For freefall with drag it can also be used to gauge what percentage of its terminal velocity a projectile will reach after a certain time:

Time $t$	Percent of $v_{\text{ter}}$
0	0
$\tau$	63%
$2\tau$	86%
$3\tau$	95%

From this it can be seen that after  $t = 3\tau$  the projectile has effectively reached its terminal velocity.

## 2.4 Quadratic Air Resistance

- Equations of motion for quadratic air resistance can be solved analytically when the projectile moves in one dimension, but can only be solved numerically when it moves in multiple dimensions.
- When a projectile moves in one dimension and only experiences the force of air resistance (i.e. there are no other forces), the equation of motion is

$$m\dot{v} = -cv^2.$$

Using separation of variables the solution can be found to be

$$v(t) = \frac{v_0}{1 + t/\tau}$$

where

$$\tau = \frac{m}{cv_0}.$$

- As in the linear case,  $\tau$  is a measure of how long it takes for air resistance to slow down the projectile ( $v = v_0/2$  at  $t = \tau$ ).
- Integrating the equation for  $v(t)$  gives

$$x(t) = v_0\tau \ln\left(1 + \frac{t}{\tau}\right).$$

- When a projectile moves in one dimension and experiences the forces of air resistance and weight, the equation of motion (with  $y$  down) is

$$m\dot{v} = mg - cv^2.$$

Using separation of variables the solution can be found to be

$$v(t) = v_{\text{ter}} \tanh \frac{gt}{v_{\text{ter}}}$$

where

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}}.$$

- Integrating the equation for  $v(t)$  gives

$$y = \frac{v_{\text{ter}}^2}{g} \ln\left(\cosh \frac{gt}{v_{\text{ter}}}\right).$$

## 2.5 Motion of a Charge in a Uniform Magnetic Field

- When a particle of charge  $q$  moves in a magnetic field  $\mathbf{B} = (0, 0, B_z)$  with velocity  $\mathbf{v} = (v_x, v_y, v_z)$  it experiences a force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = q(v_y B, -v_x B, 0).$$

This gives the coupled equations of motion

$$\begin{aligned} m\dot{v}_x &= qBv_y \\ m\dot{v}_y &= -qBv_x \\ m\dot{v}_z &= 0 \end{aligned}$$

or

$$\begin{aligned} \dot{v}_x &= \omega v_y \\ \dot{v}_y &= -\omega v_x \\ \dot{v}_z &= 0 \end{aligned}$$

where  $\omega = qB/m$  is called the **cyclotron frequency**.

- If we define a complex value

$$\eta = v_x + iv_y,$$

its derivative is

$$\begin{aligned} \dot{\eta} &= \dot{v}_x + i\dot{v}_y \\ &= \omega v_y - i\omega v_x \\ &= -i\omega\eta \end{aligned}$$

which has the solution

$$\eta = Ae^{-i\omega t}.$$

## 3 Momentum and Angular Momentum

### 3.1 Conservation of Momentum

- The **principle of conservation of momentum** states that if the net external force  $\mathbf{F}_{\text{ext}}$  on an  $N$ -particle system is zero, the system's total mechanical momentum  $\mathbf{P} = \sum m_\alpha \mathbf{v}_\alpha$  is constant.

### 3.2 Rockets

- Newton's second law for a rocket is

$$m\dot{\mathbf{v}} = -\dot{m}\mathbf{v}_{\text{ex}}$$

where  $\dot{m}$  is the (negative) rate of change of the mass of the rocket and  $\mathbf{v}_{\text{ex}}$  is the velocity of the exhaust. The quantity on the right hand side of the equation is called the **thrust**.

- The equation above can be solved by separation of variables giving

$$v - v_0 = v_{\text{ex}} \ln \frac{m_0}{m}$$

which is often called the **rocket equation**.

### 3.3 The Center of Mass

- The **centre of mass** of a system is defined to be

$$\mathbf{R} = \frac{1}{M} \sum_{\alpha=1}^N m_{\alpha} \mathbf{r}_{\alpha}$$

where  $M$  is the total mass of all particles in the system,  $m_{\alpha}$  is the mass of particle  $\alpha$ , and  $\mathbf{r}_{\alpha}$  is the vector from the origin to particle  $\alpha$ .

- The total momentum of a system can be written in terms of its centre of mass

$$\mathbf{P} = \sum_{\alpha} \mathbf{p}_{\alpha} = \sum_{\alpha} m_{\alpha} \dot{\mathbf{r}}_{\alpha} = M \dot{\mathbf{R}}$$

i.e. the total momentum of  $N$  particles is equivalent to that of a single particle of mass  $M$  with velocity equal to that of the centre of mass.

- Differentiating the above we find

$$\begin{aligned} \frac{d}{dt} \mathbf{P} &= \frac{d}{dt} (M \dot{\mathbf{R}}) \\ \mathbf{F}_{\text{ext}} &= M \ddot{\mathbf{R}} \end{aligned}$$

i.e. the centre of mass moves as if it was a single particle of mass  $M$  subject to the net external force on the system.

- When a body is continuous the expression for its centre of mass becomes an integral

$$\mathbf{R} = \frac{1}{M} \int \mathbf{r} dm = \frac{1}{M} \int \rho \mathbf{r} dV.$$

### 3.4 Angular Momentum for a Single Particle

- The **angular momentum** of a particle relative to an origin  $O$  is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

where  $\mathbf{r}$  is measured relative to  $O$ .



- Taking the derivative of angular momentum gives

$$\begin{aligned}\frac{d}{dt}\mathbf{L} &= \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) \\ \dot{\mathbf{L}} &= \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} \\ &= \mathbf{r} \times \mathbf{F} \\ &= \boldsymbol{\tau}.\end{aligned}$$

In other words, the rate of change in angular momentum about an origin  $O$  is equal to the net torque about that origin.

- We can simplify some one-particle problems by choosing the origin such that the net torque is 0 and thus angular momentum is constant.

### 3.5 Angular Momentum for Several Particles

- The **total angular momentum** of a system is

$$\mathbf{L} = \sum_{\alpha=1}^N \mathbf{L}_{\alpha} = \sum_{\alpha=1}^N \mathbf{r}_{\alpha} \times \mathbf{p}_{\alpha}.$$

- Differentiating the above

$$\dot{\mathbf{L}} = \sum_{\alpha} \dot{\mathbf{L}}_{\alpha} = \sum_{\alpha} \mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha} = \boldsymbol{\tau}_{\text{ext}}$$

we find that the rate of change of the total angular momentum of the system is equal to the net torque on the system.

- The **principle of conservation of angular momentum** states that if the net external torque on a system is 0, the system's total angular momentum is constant. This assumes that all internal forces are central and obey Newton's third law.
- The principle of conservation of momentum and the result  $\dot{\mathbf{L}} = \boldsymbol{\tau}_{\text{ext}}$  also hold if  $\mathbf{L}$  and  $\boldsymbol{\tau}_{\text{ext}}$  are measured about the centre of mass, even if the centre of mass is being accelerated and is thus not an inertial frame.

## 4 Energy

### 4.1 Kinetic Energy and Work

- The **work-kinetic-energy theorem** states that the change in a particle's kinetic energy between two points is equal to the work done by the net force on the particle between those two points

$$\Delta K = \int_1^2 \mathbf{F} \cdot d\mathbf{r}.$$

## 4.2 Potential Energy and Conservative Forces

- A force  $\mathbf{F}$  acting on a particle is considered **conservative** if:
  - $\mathbf{F}$  depends only on the particle's position  $\mathbf{r}$  (and not on its velocity  $\mathbf{v}$ , time  $t$ , or any other variable), and
  - for any two points 1 and 2, the work done by  $\mathbf{F}$  is the same for all paths between 1 and 2.
- Only conservative forces have associated **potential energy** functions.
- The potential energy function  $U(\mathbf{r})$  of a conservative force  $\mathbf{F}$  is defined as

$$U(\mathbf{r}) = - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') d\mathbf{r}'$$

where  $\mathbf{r}_0$  is an arbitrary point at which  $U(\mathbf{r}_0)$  is defined to be 0.

- The **principle of conservation of energy** states that if all the forces acting on a particle are conservative, each with its corresponding potential energy function  $U_i(\mathbf{r})$ , the **total mechanical energy**

$$E = K + U = K + U_1(\mathbf{r}) + \cdots + U_n(\mathbf{r}),$$

is constant in time.

- If nonconservative forces do work then the total energy of the system changes by that amount

$$\Delta E = W_{\text{nc}}.$$

## 4.3 Force as the Gradient of Potential Energy

- A conservative force  $\mathbf{F}$  can be expressed as the negative gradient of its potential energy function  $U$

$$\mathbf{F} = -\nabla U.$$

## 4.4 The Second Condition that $\mathbf{F}$ be Conservative

- A force  $\mathbf{F}$  is conservative if  $\nabla \times \mathbf{F} = \mathbf{0}$ .

## 4.5 Time-Dependent Potential Energy

- If a time-dependent force  $\mathbf{F}(t)$  has the property  $\nabla \times \mathbf{F}(t) = \mathbf{0}$  it's still possible to define an associated potential energy function  $U(\mathbf{r}, t)$  where  $\mathbf{F}(t) = -\nabla U(t)$  but it's no longer guaranteed that total mechanical energy is conserved over time.

## 4.8 Central Forces

- A central force is conservative if and only if it's spherically symmetric.

# 5 Oscillations

## 5.2 Simple Harmonic Motion

- The equation of motion for a harmonic oscillator

$$\ddot{x} = -\frac{k}{m}x = -\omega^2 x$$

can be solved in multiple ways:

- the exponential solution

$$x = c_1 e^{i\omega t} + c_2 e^{-i\omega t},$$

- the sine and cosine solutions

$$x = c_1 \cos \omega t + c_2 \sin \omega t,$$

and

- the phase shifted cosine solution

$$x = A \cos(\omega t - \delta)$$

where

$$A = \sqrt{c_1^2 + c_2^2}$$

with  $c_1$  and  $c_2$  coming from the sine and cosine solutions above and

$$\delta = \arctan -\frac{c_1}{c_2}.$$

## 5.3 Two-Dimensional Oscillators

- An **isotropic harmonic oscillator** in  $n > 1$  dimensional space experiences a restoring force directed towards the equilibrium position and with magnitude  $kr$  where  $r$  is the object's distance from equilibrium.
- In two dimensions an isotropic harmonic oscillator has general solutions

$$x(t) = A_x \cos \omega t$$

$$y(t) = A_y \cos(\omega t - \delta).$$

It was possible to eliminate the phase from  $x(t)$  by redefining the origin of time but in general it isn't possible to also eliminate the phase from  $y(t)$ .

- An **anisotropic harmonic oscillator** is similar to an isotropic harmonic oscillator but the spring constants are different in different directions.

## 5.7 Fourier Series

- Any periodic function with period  $T$  can be represented as a Fourier series

$$f(t) = \sum_{n=0}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

where

$$\begin{aligned}a_0 &= \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt \\a_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(n\omega t) dt \\b_0 &= 0 \\b_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(n\omega t) dt \\\omega &= \frac{2\pi}{T}.\end{aligned}$$

## 6 Calculus of Variations

- A **functional** is a mapping from a space  $X$  to the real or complex numbers. When  $X$  is the space of functions a functional is a “function of a function”, i.e. it takes a function as an argument.
- The goal of the **calculus of variations** is to find maxima and minima of functionals, i.e. functions that maximise or minimise the value of the functional. This is analogous to finding real numbers that maximise or minimise a function in single-variable calculus.
- A functional of the form

$$S = \int_{x_1}^{x_2} f[x, y(x), y'(x)] dx$$

can be solved using the **Euler-Lagrange equation**

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0.$$

- A solution to the Euler-Lagrange equation isn't guaranteed to be a minimum — it could be a maximum or an inflection point, as in single-variable calculus. In general it's difficult to determine the nature of a given solution so other methods (e.g. inspection) must be used.

- A functional with multiple functions as arguments, e.g.

$$S = \int_{t_1}^{t_2} f[t, x(t), x'(t), y(t), y'(t)] dt,$$

results in a Euler-Lagrange equation for each function, e.g.

$$\begin{aligned} \frac{\partial f}{\partial x} - \frac{d}{dt} \frac{\partial f}{\partial x'} &= 0 \\ \frac{\partial f}{\partial y} - \frac{d}{dt} \frac{\partial f}{\partial y'} &= 0. \end{aligned}$$

These can then be solved as above.

- Under Lagrangian mechanics, the independent variable is time  $t$  and the dependent variable(s) depend on the system under consideration. In general they're denoted  $q_1, q_2, \dots, q_n$  and are called **generalized coordinates**.

## 7 Lagrange's Equations

- Lagrangian mechanics has two advantages over Newtonian mechanics:
  - Lagrange's equations have the same form in all coordinate systems, and
  - Lagrange's equations omit the forces of constraint (e.g. the normal force that keeps a bead on a wire), simplifying calculations.

### 7.1 Lagrange's Equations for Unconstrained Motion

- The **Lagrangian function** or **Lagrangian** is defined as

$$\mathcal{L} = K - U,$$

i.e. the kinetic energy minus the potential energy.

- **Hamilton's principle** states that the actual path taken by a particle between points 1 and 2 in a given time interval  $t_1$  to  $t_2$  is such that the action integral

$$S = \int_{t_1}^{t_2} \mathcal{L} dt$$

is stationary when taken along the actual path, i.e. the actual path is the solution of the Euler-Lagrange equation when applied to the Lagrangian.

- A Lagrangian can be written in terms of any **generalized coordinates**  $q_1, q_2, q_3$  providing each position  $\mathbf{r}$  corresponds to a unique value  $(q_1, q_2, q_3)$  and vice versa.

- The derivative of the Lagrangian with respect to  $x$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial}{\partial x}(K - U) = \frac{\partial}{\partial x} \left( \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(x, y) \right) = -\frac{\partial U(x, y)}{\partial x} = F_x$$

is the  $x$  component of the force while the derivative with respect to  $\dot{x}$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}(K - U) = \frac{\partial}{\partial \dot{x}} \left( \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(x, y) \right) = m\dot{x} = p_x$$

is the  $x$  component of the momentum. The same applies to the  $y$  and  $z$  dimensions. When generalized coordinates  $q_1, q_2, q_3$  are used the corresponding values behave like forces and momenta and are called **generalized forces** and **generalized momenta**, respectively.

- Another way of stating the above is

$$\frac{\partial \mathcal{L}}{\partial q_i} = (\textit{ith component of generalized force})$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = (\textit{ith component of generalized momentum}).$$

Using this terminology, the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

takes the form

$$(\textit{generalized force}) = (\textit{rate of change of generalized momentum}).$$

- For example, in 2D polar coordinates  $(r, \phi)$  the generalized force for the  $\phi$  coordinate is the torque on the particle and the generalized momentum is the angular momentum.
- Conservation laws can be derived from the Euler-Lagrange equations in generalized coordinates. For example, if the  $i$ th component of the generalized force is zero

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0$$

then the rate of change of the  $i$ th component of the generalized momentum is also zero and thus it doesn't change.

- If the relationship between  $\mathbf{r}$  and the generalized coordinates  $q_1, q_2, \dots, q_n$  doesn't involve  $t$  the generalized coordinates are called **natural** and have some additional properties.

- The number of **degrees of freedom** of a system is the number of coordinates that can be independently varied in a small displacement, i.e. the number of independent “directions” in which the system can move from any given initial configuration.
- When the number of degrees of freedom of an  $N$  particle system is less than  $3N$  (or  $2N$  in two dimensions), the system is said to be **constrained**.
- When the number of degrees of freedom of a system matches the number of generalized coordinates required to model the system, it is said to be **holonomic**.
- In order to apply Lagrange’s equations to a system its constraints must be holonomic, i.e. they must be expressible in the form

$$f(q_1, q_2, \dots, q_n, t) = 0.$$

- The generalized coordinates can be measured relative to a non-inertial reference frame providing the Lagrangian  $\mathcal{L} = K - U$  is originally written as inertial.

## 7.6 Generalized Momenta and Ignorable Coordinates

- When the Lagrangian is independent of a coordinate  $q_i$ , that coordinate is said to be **ignorable** or **cyclic**. When choosing coordinates, it is desirable to make as many ignorable as possible.

## 7.8 More about Conservation Laws

- If the Lagrangian is unchanged by spacial translation, the total momentum of the system is conserved.
- If the Lagrangian is unchanged by time translation, the total energy of the system is conserved.

## 7.9 Lagrange’s Equations for Magnetic Forces

- For a given mechanical system with generalized coordinates  $q = (q_1, \dots, q_n)$ , a **Lagrangian**  $\mathcal{L}$  is a function  $\mathcal{L}(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t)$  of the coordinates and velocities, such that the correct equations of motion for the system are the Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \text{ for } i = 1, \dots, n.$$

- It’s important to note that the above does not define a unique Lagrangian function — any function  $\mathcal{L}$  that gives the correct equations of motion is valid and has all the correct properties.

- The Lagrangian for a particle of charge  $q$  and mass  $m$  moving in electric and magnetic fields  $\mathbf{E}$  and  $\mathbf{B}$  is

$$\mathcal{L} = \frac{1}{2}m\dot{\mathbf{r}}^2 - q(V - \dot{\mathbf{r}} \cdot \mathbf{A}).$$

## 8 Two-Body Central Force Problems

### 8.1 The Problem

- If two objects that experience a conservative central force, their potential energy depends only on the distance between them

$$U(\mathbf{r}_1, \mathbf{r}_2) = U(|\mathbf{r}_1 - \mathbf{r}_2|) = U(r)$$

and thus the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2 - U(r).$$

### 8.2 CM and Relative Coordinates; Reduced Mass

- It is simplest if the generalized coordinates are chosen to be the position of the centre of mass of the system

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}$$

and the relative position of the two bodies

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2.$$

- This results in a kinetic energy

$$K = \frac{1}{2}(M\dot{\mathbf{R}}^2 + \mu\dot{\mathbf{r}}^2)$$

where

$$\mu = \frac{m_1m_2}{M} = \frac{m_1m_2}{m_1 + m_2}$$

is the **reduced mass** of the system.

- The Lagrangian is then

$$\mathcal{L} = K - U = \frac{1}{2}M\dot{\mathbf{R}}^2 + \left[ \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r) \right] = \mathcal{L}_{\text{cm}} + \mathcal{L}_{\text{rel}}$$

where each generalized coordinate only appears in one “sub-Lagrangian” and can be solved separately.



### 8.3 The Equations of Motion

- Because  $\mathcal{L}_{\text{cm}}$  doesn't include  $\mathbf{R}$  the equation of motion for the centre of mass is

$$M\ddot{\mathbf{R}} = \mathbf{0},$$

i.e. the centre of mass moves with constant velocity.

- The equation of relative motion is

$$\mu\ddot{\mathbf{r}} = -\nabla U(r),$$

i.e. the two bodies move as if they were a single particle of mass  $\mu$  with potential energy  $U(r)$ .

- If we choose to use the inertial centre-of-mass reference frame,  $\mathcal{L}_{\text{cm}} = 0$  and  $\mathcal{L} = \mathcal{L}_{\text{rel}}$  becomes a one-body problem.
- The total angular momentum in the centre-of-mass frame is

$$\mathbf{L} = \mathbf{r} \times \mu\dot{\mathbf{r}}.$$

Because the total angular momentum is conserved — including its direction — this means that  $\mathbf{r}$  and  $\dot{\mathbf{r}}$  are confined to a plane that we can choose to be the  $xy$  plane. The three-dimensional two-body problem has been turned into a two-dimensional one-body problem.

- The Lagrangian for this two-dimensional problem in polar coordinates is

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r).$$

Because this doesn't involve  $\phi$  the Lagrange equation corresponding to  $\phi$  is

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} = \text{const} = \ell$$

which is simply a statement of the conservation of angular momentum. The Lagrange equation corresponding to  $r$  is

$$\mu r \dot{\phi}^2 - \frac{dU}{dr} = \mu \ddot{r}.$$

### 8.4 The Equivalent One-Dimensional Problem

- Rearranging the  $\phi$  equation we find

$$\dot{\phi} = \frac{\ell}{\mu r^2}$$

where  $\ell$  is determined by initial conditions.

- The radial equation can be rewritten as

$$\begin{aligned}\mu\ddot{r} &= -\frac{dU}{dr} + \mu r\dot{\phi}^2 \\ &= -\frac{dU}{dr} + F_{\text{cf}}\end{aligned}$$

where  $F_{\text{cf}}$  is the fictitious centrifugal force

$$F_{\text{cf}} = \mu r\dot{\phi}^2 = \frac{\ell^2}{\mu r^3} = -\frac{d}{dr} \left( \frac{\ell^2}{2\mu r^2} \right) = -\frac{dU_{\text{cf}}}{dr}.$$

- The radial equation can now be written in terms of the **effective potential energy**

$$\mu\ddot{r} = -\frac{d}{dr}[U(r) + U_{\text{cf}}(r)] = -\frac{d}{dr}U_{\text{eff}}(r).$$

- The total energy of the one-body system is

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + U(r)$$

and this value is conserved.

## 9 Mechanics in Noninertial Frames

### 9.1 Acceleration without Rotation

- A noninertial frame of reference  $\mathcal{S}$  has acceleration  $\mathbf{A}$  relative to an inertial frame of reference  $\mathcal{S}_0$ . Newton's second law can be used in  $\mathcal{S}$  providing we add an extra force-like term called the **inertial force**

$$m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{A}.$$

### 9.3 The Angular Velocity Vector

- The angular velocity vector  $\boldsymbol{\omega}$  has direction equal to that of the axis of rotation (using the right-hand rule to disambiguate direction) and magnitude equal to the rate of rotation.
- If a body is rotating with angular velocity  $\boldsymbol{\omega}$  about an axis through  $O$ , the velocity of a point  $P$  (position  $\mathbf{r}$ ) fixed on the body is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}.$$

- Suppose there are two frames of reference 2 and 1. Frame 2 is rotating with angular velocity  $\boldsymbol{\omega}_{21}$  relative to frame 1. A body 3 is rotating with angular velocities  $\boldsymbol{\omega}_{31}$  and  $\boldsymbol{\omega}_{32}$  relative to frames 1 and 2, respectively. These angular velocity vectors add such that

$$\boldsymbol{\omega}_{31} = \boldsymbol{\omega}_{32} + \boldsymbol{\omega}_{21}.$$

## 9.4 Time Derivatives in a Rotating Frame

- Given an inertial reference frame  $\mathcal{S}_0$  and a noninertial reference frame  $\mathcal{S}$  that is rotating with angular velocity  $\boldsymbol{\Omega}$  relative to  $\mathcal{S}_0$ , the time derivative of a vector  $\mathbf{Q}$  differs between the two with the relation

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}_0} = \left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}} + \boldsymbol{\Omega} \times \mathbf{Q}.$$

## 9.5 Newton's Second Law in a Rotating Frame

- Newton's second law in a noninertial frame rotating with angular velocity  $\boldsymbol{\Omega}$  is

$$\begin{aligned} m\ddot{\mathbf{r}} &= \mathbf{F} + \mathbf{F}_{\text{cor}} + \mathbf{F}_{\text{cf}} \\ &= \mathbf{F} + 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega} \end{aligned}$$

where  $\mathbf{F}_{\text{cor}}$  is the **Coriolis force** and  $\mathbf{F}_{\text{cf}}$  is the **centrifugal force**.

# 10 Rotational Motion of Rigid Bodies

## 10.1 Properties of the Centre of Mass

- The total angular momentum of a system of  $N$  particles is

$$\begin{aligned} \mathbf{L} &= \mathbf{L}_O + \mathbf{L}_{\text{CM}} \\ &= \mathbf{R} \times \mathbf{P} + \sum \mathbf{r}'_{\alpha} \times m_{\alpha} \dot{\mathbf{r}}'_{\alpha} \end{aligned}$$

where  $\mathbf{L}_O$  is the angular momentum of the centre of mass of the system relative to the origin,  $\mathbf{L}_{\text{CM}}$  is the angular momentum of the particles of the system relative to its centre of mass,  $\mathbf{R}$  is the centre of mass of the system,  $\mathbf{P}$  is the linear momentum of the system,  $\mathbf{r}'_{\alpha}$  is the position of particle  $\alpha$  relative to the centre of mass of the system,  $m_{\alpha}$  is the mass of particle  $\alpha$ , and  $\dot{\mathbf{r}}'_{\alpha}$  is the velocity of particle  $\alpha$  relative to the centre of mass of the system.

- The values  $\mathbf{L}_O$  and  $\mathbf{L}_{\text{CM}}$  are independently conserved, with

$$\dot{\mathbf{L}}_O = \mathbf{R} \times \mathbf{F}^{\text{ext}} = \boldsymbol{\tau}^{\text{ext}} \text{ (about origin)}$$

and

$$\dot{\mathbf{L}}_{\text{CM}} = \sum \mathbf{r}'_{\alpha} \times \mathbf{F}_{\alpha}^{\text{ext}} = \boldsymbol{\tau}^{\text{ext}} \text{ (about centre of mass)}.$$

- The total kinetic energy of a system of  $N$  particles is

$$\begin{aligned} T &= T_{\text{CM}} + T_{\text{relative to CM}} \\ &= \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \sum m_{\alpha} \dot{\mathbf{r}}_{\alpha}^2 \end{aligned}$$

- The total potential energy of a system of  $N$  particles is

$$U = U^{\text{ext}} + U^{\text{int}}$$

where  $U^{\text{ext}}$  is the total potential energy due to external forces and  $U^{\text{int}}$  is the total potential energies for all pairs of particles

$$U^{\text{int}} = \sum_{\alpha < \beta} U_{\alpha\beta}(r_{\alpha\beta}).$$

However, in a rigid body the distances between all particles are fixed so  $U^{\text{int}}$  is constant and can be ignored.

## 10.2 Rotation about a Fixed Axis

- For a system of  $N$  particles rotating around the  $z$  axis, the angular momentum of the system is

$$\begin{aligned} \mathbf{L} &= \sum_{\alpha} \boldsymbol{\ell}_{\alpha} \\ &= \sum_{\alpha} m_{\alpha} \mathbf{r}_{\alpha} \times \mathbf{v}_{\alpha} \\ &= \sum_{\alpha} m_{\alpha} \omega (-z_{\alpha} x_{\alpha}, -z_{\alpha} y_{\alpha}, x_{\alpha}^2 + y_{\alpha}^2). \end{aligned}$$

Thus,  $\mathbf{L}$  may have  $\mathbf{x}$  and/or  $\mathbf{y}$  components and  $\mathbf{L} = I\boldsymbol{\omega}$  doesn't hold.

- The angular momentum can be expressed as

$$\mathbf{L} = (I_{xz}\omega, I_{yz}\omega, I_{zz}\omega)$$

where

$$\begin{aligned} I_{xz} &= - \sum_{\alpha} m_{\alpha} x_{\alpha} z_{\alpha}, \\ I_{yz} &= - \sum_{\alpha} m_{\alpha} y_{\alpha} z_{\alpha}, \text{ and} \\ I_{zz} &= \sum_{\alpha} m_{\alpha} (x_{\alpha}^2 + y_{\alpha}^2) \end{aligned}$$

are called the **products of inertia** of the body. The notation means that e.g.  $I_{xz}$  is the  $x$  component of the angular momentum for a body rotating around the  $z$  axis.

## 10.3 Rotation about Any Axis; the Inertia Tensor

- Continuing from the above, if a body has angular velocity

$$\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$$

its angular momentum is

$$\begin{aligned}L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z\end{aligned}$$

where

$$\begin{aligned}I_{xx} &= \sum_{\alpha} m_{\alpha}(y_{\alpha}^2 + z_{\alpha}^2), \\I_{xy} &= - \sum_{\alpha} m_{\alpha}x_{\alpha}y_{\alpha},\end{aligned}$$

and the other values follow the same pattern.

- The above can also be written

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$$

where  $\mathbf{I}$  is the **moment of inertia tensor**

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}.$$

## 10.4 Principle Axes of Inertia

- In general, the angular momentum  $\mathbf{L}$  of a spinning object is not parallel to  $\boldsymbol{\omega}$ . However for any rigid body and any point  $O$  there are three perpendicular axes through  $O$  such that  $\mathbf{I}$  is diagonal when  $\boldsymbol{\omega}$  is parallel to one of these axes, so is  $\mathbf{L}$ . These are called the **principle axes**.
- The three moments of inertia about the three principle axes are called the **principle moments**.
- If we choose the principle axes of a rigid body as our frame of reference rotation calculations are simplified but we're using a noninertial frame of reference.

## 10.5 Finding the Principle Axes; Eigenvalue Equations

- If  $\boldsymbol{\omega}$  is parallel to a principle axis, then  $\mathbf{L} = \mathbf{I}\boldsymbol{\omega} = \lambda\boldsymbol{\omega}$ . This can be converted to an eigenvalue equation

$$(\mathbf{I} - \lambda\mathbf{1})\boldsymbol{\omega} = 0$$

where the eigenvectors are the principle axes. Thus to find the principle axes of a rigid body about a point we must calculate an initial inertia tensor then calculate its eigenvectors. The updated inertia tensor will be

$$\mathbf{I} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$