

Advanced Engineering Mathematics Vectors, Matrices, and Vector Calculus by Dennis G. Zill

Notes

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1 Vectors

1.1 Vectors in 2-Space

- The zero vector can be assigned any direction
- The vectors \mathbf{i} and \mathbf{j} are known as the **standard basis vectors** for \mathbb{R}^2

1.2 Vectors in 3-Space

- In \mathbb{R}^3 the octant in which all coordinates are positive is known as the **first octant**. There is no agreement for naming the other seven octants.

1.3 Dot Product

- The **dot product** is also known as the **inner product** or the **scalar product** and is denoted $\mathbf{a} \cdot \mathbf{b}$
- Two non-zero vectors are orthogonal iff their dot product is 0
- The zero vector is considered orthogonal to all vectors
- The angles α , β , and γ between a vector and the unit vectors \mathbf{i} , \mathbf{j} , and \mathbf{k} , respectively are called the **direction angles** of the vector
- The cosines of a vectors direction angles (the **direction cosines**) can be calculated as

$$\begin{aligned}\cos \alpha &= \frac{\mathbf{a} \cdot \mathbf{i}}{\|\mathbf{a}\| \|\mathbf{i}\|} \\ &= \frac{a_1}{\|\mathbf{a}\|} \\ \cos \beta &= \frac{\mathbf{a} \cdot \mathbf{j}}{\|\mathbf{a}\| \|\mathbf{j}\|} \\ &= \frac{a_2}{\|\mathbf{a}\|} \\ \cos \gamma &= \frac{\mathbf{a} \cdot \mathbf{k}}{\|\mathbf{a}\| \|\mathbf{k}\|} \\ &= \frac{a_3}{\|\mathbf{a}\|}\end{aligned}$$

Equivalently, these can be calculated as the components of the unit vector $\mathbf{a}/\|\mathbf{a}\|$.

- To find the component of a vector \mathbf{a} in the direction of a vector \mathbf{b}

$$\text{comp}_{\mathbf{b}} \mathbf{a} = \|\mathbf{a}\| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}$$

- To project a vector \mathbf{a} onto a vector \mathbf{b}

$$\text{proj}_{\mathbf{b}} \mathbf{a} = (\text{comp}_{\mathbf{b}} \mathbf{a}) \frac{\mathbf{b}}{\|\mathbf{b}\|} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \right) \mathbf{b}$$

1.4 Cross Product

- The cross product is only defined in \mathbb{R}^3
- The **scalar triple product** of vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} is defined as

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- The area of a parallelogram with sides \mathbf{a} and \mathbf{b} is $\|\mathbf{a} \times \mathbf{b}\|$
- The area of a triangle with sides \mathbf{a} and \mathbf{b} is $\frac{1}{2}\|\mathbf{a} \times \mathbf{b}\|$
- The volume of a parallelepiped with sides \mathbf{a} , \mathbf{b} , and \mathbf{c} is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ iff \mathbf{a} , \mathbf{b} , and \mathbf{c} are coplanar

1.5 Lines and Planes in 3-Space

- There is a unique line between any two points \mathbf{r}_1 and \mathbf{r}_2 in 3-space. The equation for that line is

$$\mathbf{r} = \mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1) = \mathbf{r}_1 + t\mathbf{a}$$

where t is called a **parameter**, the nonzero vector \mathbf{a} is called a **direction vector**, and its components are called **direction numbers**.

- Equating the components of the equation above we find

$$\begin{aligned}x &= r_1 + ta_1 \\y &= r_2 + ta_2 \\z &= r_3 + ta_3.\end{aligned}$$

These are the **parametric equations** for the line through \mathbf{r}_1 and \mathbf{r}_2 .

- By solving the parametric equations for t and equating the results we find the **symmetric equations** for the line

$$t = \frac{x - r_1}{a_1} = \frac{y - r_2}{a_2} = \frac{z - r_3}{a_3}.$$

- Given a point P_1 and a vector \mathbf{n} , there exists only one plane containing P_1 with \mathbf{n} normal. The vector from P_1 to another point P on that plane will be perpendicular to \mathbf{n} , so the equation for the plane is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$

where $\mathbf{r} = \overrightarrow{OP}$ and $\mathbf{r}_1 = \overrightarrow{OP_1}$. If

$$\mathbf{n} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$$

the cartesian form of this equation is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

and is called the **point-normal form**.

- The graph of any equation $ax + by + cz + d = 0$, where a , b , and c are not all zero, is a plane with the normal vector $\mathbf{n} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$.
- Given three noncollinear points, a normal vector can be found by forming two vectors from two pairs of points and take their cross product.
- A line and a plane that aren't parallel intersect at a single point.
- Two planes that aren't parallel must intersect in a line.

1.6 Vector Spaces

- The length of a vector is called its **norm**
- The process of multiplying a vector by the reciprocal of its norm is called **normalizing** the vector
- Two nonzero vectors \mathbf{a} and \mathbf{b} in \mathbb{R}^n are said to be orthogonal if $\mathbf{a} \cdot \mathbf{b} = 0$

Definition 7.6.1 Vector Space

Let V be a set of elements on which two operations called **vector addition** and **scalar multiplication** are defined. Then V is said to be a **vector space** if the following 10 properties are satisfied.

Axioms for Vector Addition:

- (i) If \mathbf{x} and \mathbf{y} are in V , then $\mathbf{x} + \mathbf{y}$ is in V .
- (ii) For all \mathbf{x}, \mathbf{y} in V , $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$. ← commutative law
- (iii) For all $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in V , $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$. ← associative law
- (iv) There is a unique vector $\mathbf{0}$ in V such that
 $\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x}$. ← zero vector
- (v) For each \mathbf{x} in V , there exists a vector $-\mathbf{x}$ such that
 $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$. ← negative of a vector

Axioms for Scalar Multiplication:

- (vi) If k is any scalar and \mathbf{x} is in V , then $k\mathbf{x}$ is in V .
- (vii) $k(\mathbf{x} + \mathbf{y}) = k\mathbf{x} + k\mathbf{y}$ ← distributive law
- (viii) $(k_1 + k_2)\mathbf{x} = k_1\mathbf{x} + k_2\mathbf{x}$ ← distributive law
- (ix) $k_1(k_2\mathbf{x}) = (k_1k_2)\mathbf{x}$
- (x) $1\mathbf{x} = \mathbf{x}$

- If a subset W of a vector space V is itself a vector space under the operations of vector addition and scalar multiplication defined on V , then W is called a **subspace** of V
- Every vector space has at least two subspaces: itself and the zero subspace $\{\mathbf{0}\}$
- A set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is said to be **linearly independent** if the only constants satisfying the equation

$$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \dots + k_n\mathbf{x}_n = \mathbf{0}$$

are $k_1 = k_2 = \dots = k_n = 0$. If the set of vectors is not linearly independent it is said to be **linearly dependent**.

- If a set of vectors $B = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ in a vector space V is linearly independent and every vector in V can be expressed as a linear combination of vectors in B then B is said to be a **basis** for V .
- The number of vectors in a basis B for a vector space V is said to be the **dimension** of the space.

- If the basis of a vector space contains a finite number of vectors, then the space is **finite dimensional**; otherwise it is **infinite dimensional**.
- If S denotes any set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ in a vector space V , then the set of all linear combinations of the vectors in S

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots + c_n\mathbf{x}_n$$

is called the **span** of the vectors and is denoted $\text{Span}(S)$.

- $\text{Span}(S)$ is a subspace of V and is said to be a subspace spanned by its vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$.
- If $V = \text{Span}(S)$ then S is said to be a **spanning set** for the vector space V or that S **spans** V .

1.7 Gram–Schmidt Orthogonalization Process

- An **orthonormal basis** is a basis whose vectors are mutually orthogonal and are unit vectors.
- If $B = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ is an orthonormal basis for \mathbb{R}^n then an arbitrary vector \mathbf{u} can be expressed as

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{u} \cdot \mathbf{w}_2)\mathbf{w}_2 + \dots + (\mathbf{u} \cdot \mathbf{w}_n)\mathbf{w}_n$$

- The **Gram-Schmidt Orthogonalization Process** is a process for converting any basis of a vector space into an orthonormal basis. First the basis vectors are made orthogonal to each other, then they are normalized. More specifically, to convert a basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ into an orthogonal basis $B' = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$

1. Let $\mathbf{v}_1 = \mathbf{u}_1$
2. Let $\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{v}_1} \mathbf{u}_2$
3. ...
4. Let $\mathbf{v}_n = \mathbf{u}_n - \text{proj}_{\mathbf{v}_1} \mathbf{u}_n - \text{proj}_{\mathbf{v}_2} \mathbf{u}_n - \dots - \text{proj}_{\mathbf{v}_{n-1}} \mathbf{u}_n$

and to convert B' into an orthonormal basis $B'' = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$, normalize each \mathbf{v}_i , $i = 1, 2, \dots, n$.

2 Matrices

2.1 Matrix Algebra

- Vectors can be written as horizontal or vertical arrays of numbers

- A **matrix** is any rectangular array of numbers or functions

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- The numbers or functions in the array are called the **elements** or **entries** of the matrix
- If a matrix has m rows and n columns we say that its **size** is m by n or $m \times n$
- An $n \times n$ matrix is called a **square** matrix of **order** n
- The entry in the i th row and the j th column of an $m \times n$ matrix **A** is written a_{ij}
- An $m \times 1$ matrix

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

is called a **column vector**

- A $1 \times n$ matrix

$$(a_1 \quad a_2 \quad \cdots \quad a_n)$$

is called a **row vector**

Definition 8.1.6 Matrix Multiplication

Let **A** be a matrix having m rows and p columns, and let **B** be a matrix having p rows and n columns. The **product** **AB** is the $m \times n$ matrix

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1p}b_{p1} & \cdots & a_{11}b_{1n} + a_{12}b_{2n} + \cdots + a_{1p}b_{pn} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2p}b_{p1} & \cdots & a_{21}b_{1n} + a_{22}b_{2n} + \cdots + a_{2p}b_{pn} \\ \vdots & & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mp}b_{p1} & \cdots & a_{m1}b_{1n} + a_{m2}b_{2n} + \cdots + a_{mp}b_{pn} \end{pmatrix} \\ &= \left(\sum_{k=1}^p a_{ik}b_{kj} \right)_{m \times n}. \end{aligned}$$

- Matrix multiplication is associative, i.e. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
- Matrix multiplication is distributive, i.e. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ and $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$
- The **transpose** of an $m \times n$ matrix \mathbf{A} is an $n \times m$ matrix \mathbf{A}^T

$$\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

i.e. the matrix is flipped along the main diagonal

Theorem 8.1.2 Properties of Transpose

Suppose \mathbf{A} and \mathbf{B} are matrices and k a scalar. Then

- | | |
|--|----------------------------------|
| (i) $(\mathbf{A}^T)^T = \mathbf{A}$ | ← transpose of a transpose |
| (ii) $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$ | ← transpose of a sum |
| (iii) $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$ | ← transpose of a product |
| (iv) $(k\mathbf{A})^T = k\mathbf{A}^T$ | ← transpose of a scalar multiple |

- A matrix that consists of all zero entries is called a **zero matrix**
- A square matrix is said to be a **triangular matrix** if all of its entries above or below the main diagonal are zeroes. More specifically they are called **lower triangular** and **upper triangular** matrices, respectively.
- A square matrix is called a **diagonal matrix** if all entries not on the main diagonal are 0.
- A square matrix whose entries on the main diagonal are all equal is called a **scalar matrix**
- A square matrix that has the property $\mathbf{A} = \mathbf{A}^T$ is called a **symmetric matrix**

2.2 Systems of Linear Algebraic Equations

- In a linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned}$$

the values a_{ij} are called the **coefficients** and the values b_n are called the **constants**

- If all the constants are zero the system is said to be **homogeneous**, otherwise it is **nonhomogeneous**
- A linear system is said to be **consistent** if it has at least one solution, otherwise it's **inconsistent**
- A linear system can be transformed into an equivalent system (i.e. one that has the same solutions) via three elementary operations:
 1. Multiply an equation by a nonzero constant
 2. Interchange the positions of equations in the system
 3. Add a multiple of one equation to any other equation

- A linear system can be represented by an **augmented matrix**, e.g.

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array} \right)$$

- We say that two matrices are **row equivalent** if one can be obtained from the other via a series of elementary row operations
- **Gaussian elimination** is the process of applying elementary row operations to a matrix to put it into **row-echelon form** where:
 1. The first nonzero entry in a row is a 1
 2. In subsequent rows, the first 1 entry appears to the right of the 1 entry in earlier rows
 3. Rows consisting of all zeroes are at the bottom of the matrix
- **Gauss-Jordan elimination** is the same as Gaussian elimination with an additional constraint that puts the matrix into **reduced row-echelon form** where a column containing a first entry 1 has zeroes everywhere else
- A homogeneous linear system always has a trivial solution where all variables are equal to zero and will have an infinite number of nontrivial solutions if the number of equations m is less than the number of variables n , i.e. $m < n$
- If \mathbf{X}_1 is a solution to $\mathbf{AX} = \mathbf{0}$, then so is $c\mathbf{X}_1$ for any constant c
- If \mathbf{X}_1 and \mathbf{X}_2 are solutions of $\mathbf{AX} = \mathbf{0}$, then so is $\mathbf{X}_1 + \mathbf{X}_2$
- If a linear system contains more equations than variables it is said to be **overdetermined**; if it contains fewer equations than variables it is said to be **underdetermined**

2.3 Rank of a Matrix

- The **rank** of a matrix \mathbf{A} denoted $\text{rank}(\mathbf{A})$ is the number of linearly independent row vectors in \mathbf{A}
- The row vectors of an $m \times n$ matrix \mathbf{A} span a subspace of \mathbb{R}^n . This is called the **row space** of \mathbf{A} . The set of linearly independent row vectors in \mathbf{A} are a basis for that subspace

Theorem 8.3.1 Rank of a Matrix by Row Reduction

If a matrix \mathbf{A} is row equivalent to a row-echelon form \mathbf{B} , then

- (i) the row space of \mathbf{A} = the row space of \mathbf{B} ,
- (ii) the nonzero rows of \mathbf{B} form a basis for the row space of \mathbf{A} , and
- (iii) $\text{rank}(\mathbf{A})$ = the number of nonzero rows in \mathbf{B} .

- A linear system of equations $\mathbf{AX} = \mathbf{B}$ is consistent iff the rank of the coefficient matrix \mathbf{A} is equal to the rank of the augmented matrix of the system $(\mathbf{A}|\mathbf{B})$
- Suppose a linear system $\mathbf{AX} = \mathbf{B}$ with m equations and n variables is consistent. If $\text{rank}(\mathbf{A}) = r$ then the solution of the system contains $n - r$ variables

2.4 Determinants

- Suppose \mathbf{A} is an $n \times n$ matrix. Associated with \mathbf{A} is a number called the **determinant of \mathbf{A}** and is denoted by

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

- A determinant of an $n \times n$ matrix is called a **determinant of order n**
- The determinant of a 1×1 matrix is the element of the matrix
- Each element in an $n \times n$ matrix has an associated **cofactor** defined as

$$a_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix produced by deleting row i and column j from \mathbf{A}

- The determinant of an arbitrary $n \times n$ matrix \mathbf{A} can be calculated by choosing an arbitrary row or column and summing the products of each element in that column/row with their cofactors, e.g. if we choose the first row of a 3×3 matrix then

$$\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11}M_{11} + a_{12}M_{12} + a_{13}M_{13} \\
&= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}(a_{22}|a_{33}| - a_{23}|a_{32}|) - a_{12}(a_{21}|a_{33}| - a_{23}|a_{31}|) \\
&\quad + a_{13}(a_{21}|a_{32}| - a_{22}|a_{31}|)
\end{aligned}$$