# Advanced Engineering Mathematics Ordinary Differential Equations Notes

Chris Doble

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## Contents

# 1 Introduction to Differential Equations

## 1.1 Definitions and Terminology

### 1.1.1 1

2, linear

### 1.1.2 3

4, linear

### 1.1.3 5

2, nonlinear

### 1.1.4 7

3, linear

### 1.1.5 9

no; yes

### 1.1.6 15

The domain of the function is $x \in [-2, \infty)$.

$$y' = 1 + \frac{2}{\sqrt{x+2}}$$

The largest interval of definition of the solution is $x \in (-2, \infty)$.

$$\begin{aligned}(y-x)y' &= y - x + 8\\ (x + 4\sqrt{x+2} - x)(1 + \frac{2}{\sqrt{x+2}}) &= x + 4\sqrt{x+2} - x + 8\\ 4\sqrt{x+2} + 8 &= 4\sqrt{x+2} + 8\end{aligned}$$

### 1.1.7 17

The domain of the function is $x \in \mathbb{R}, x \neq \pm 2$.

$$y' = \frac{2x}{(4-x^2)^2}$$

The largest intervals of definition of the solution are $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

$$\begin{aligned}y' &= 2xy^2\\ \frac{2x}{(4-x^2)^2} &= 2x\left(\frac{1}{4-x^2}\right)^2\\ &= \frac{2x}{(4-x^2)^2}\end{aligned}$$

### 1.1.8 19

$$\begin{aligned}ln\frac{2X-1}{X-1} &= t\\ 2X - 1 &= (X-1)e^t\\ (2-e^t)X &= 1 - e^t\\ X &= \frac{e^t - 1}{e^t - 2}\end{aligned}$$

The solutions intervals of validity are $(\infty, \ln 2)$ and $(\ln 2, \infty)$.

$$
\begin{aligned}
\frac{dX}{dt} &= (X-1)(1-2X) \\
\frac{e^t}{e^t-2} - \frac{e^t(e^t-1)}{(e^t-2)^2} &= \left(\frac{e^t-1}{e^t-2} - 1\right)\left(1 - 2\frac{e^t-1}{e^t-2}\right) \\
\frac{e^t(e^t-2) - e^t(e^t-1)}{(e^t-2)^2} &= \left(\frac{e^t-1-e^t+2}{e^t-2}\right)\left(\frac{e^t-2-2e^t+2}{e^t-2}\right) \\
\frac{e^{2t}-2e^t-e^{2t}+e^t}{(e^t-2)^2} &= \left(\frac{1}{e^t-2}\right)\left(\frac{-e^t}{e^t-2}\right) \\
\frac{-e^t}{(e^t-2)^2} &= \frac{-e^t}{(e^t-2)^2}
\end{aligned}
$$

#### 1.1.9 31

$m = -2$

#### 1.1.10 33

$m = 2$ or $3$

#### 1.1.11 35

$m = -1$ or $0$

#### 1.1.12 37

$y = 2$

#### 1.1.13 39

No constant solutions

## 1.2 Initial Value Problems

#### 1.2.1 1

$$
\begin{aligned}
y(0) = -\frac{1}{3} &= \frac{1}{1 + c_1 e^{-(0)}} \\
-3 &= 1 + c_1 \\
c_1 &= -4
\end{aligned}
$$

$$
y = \frac{1}{1 - 4e^{-x}}
$$

### 1.2.2 3

$$
\begin{aligned}
y(2) = \frac{1}{3} &= \frac{1}{(2)^2 + c} \\
3 &= 4 + c \\
c &= -1
\end{aligned}
$$

$$y = \frac{1}{x^2 - 1}$$

$I = (1, \infty)$

### 1.2.3 5

$$
\begin{aligned}
y(0) = 1 &= \frac{1}{(0)^2 + c} \\
c &= 1
\end{aligned}
$$

$$y = \frac{1}{x^2 + 1}$$

$I = (-\infty, \infty)$

### 1.2.4 7

$$
\begin{aligned}
x(0) = -1 &= c_1 \cos 0 + c_2 \sin 0 \\
c_1 &= -1
\end{aligned}
$$

$$
\begin{aligned}
x'(0) = 8 &= -c_1 \sin 0 + c_2 \cos 0 \\
c_2 &= 8
\end{aligned}
$$

$$x = -\cos t + 8 \sin t$$

### 1.2.5 9

$$
\begin{aligned}
x'\left(\frac{\pi}{6}\right) = 0 &= -c_1 \sin \frac{\pi}{6} + c_2 \cos \frac{\pi}{6} \\
&= -c_1 \frac{1}{2} + c_2 \frac{\sqrt{3}}{2} \\
c_1 &= \sqrt{3} c_2
\end{aligned}
$$

$$
\begin{aligned}
x\left(\frac{\pi}{6}\right) = \frac{1}{2} &= c_1 \cos\frac{\pi}{6} + c_2 \sin\frac{\pi}{6} \\
&= \frac{3}{2}c_2 + \frac{1}{2}c_2 \\
&= 2c_2 \\
c_2 &= \frac{1}{4}
\end{aligned}
$$

$$
y = \frac{\sqrt{3}}{4}\cos t + \frac{1}{4}\sin t
$$

### 1.2.6 11

$$
\begin{aligned}
y(0) = 1 &= c_1 e^{(0)} + c_2 e^{-(0)} \\
&= c_1 + c_2 \\
c_1 &= 1 - c_2
\end{aligned}
$$

$$
\begin{aligned}
y'(0) = 2 &= c_1 e^{(0)} - c_2 e^{-(0)} \\
&= 1 - c_2 - c_2 \\
c_2 &= -\frac{1}{2}
\end{aligned}
$$

$$
y = \frac{3}{2}e^{x} - \frac{1}{2}e^{-x}
$$

### 1.2.7 13

$$
\begin{aligned}
y(-1) = 5 &= c_1 e^{(-1)} + c_2 e^{-(-1)} \\
&= c_1 e^{-1} + c_2 e \\
c_1 &= 5e - c_2 e^2
\end{aligned}
$$

$$
\begin{aligned}
y'(-1) = -5 &= c_1 e^{(-1)} - c_2 e^{-(-1)} \\
&= 5e - c_2 e^2 - c_2 e \\
c_2 e(e+1) &= 5(e+1) \\
c_2 &= \frac{5}{e}
\end{aligned}
$$

$$
y = 5e^{-x-1}
$$

### 1.2.8 15

$$y = 0$$

$$y = x^3$$

### 1.2.9 17

$$f(x, y) = y^{2/3}$$

$$\frac{\partial f}{\partial y} = \frac{2}{3y^{1/3}}$$

$y < 0$ or $y > 0$

### 1.2.10 19

$$f(x, y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$x < 0$ or $x > 0$

### 1.2.11 21

$$f(x, y) = \frac{x^2}{4 - y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2x^2 y}{(4 - y^2)^2}$$

$y < -2$, $-2 < y < 2$, or $y > 2$

### 1.2.12 23

$$f(x, y) = \frac{y^2}{x^2 + y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2} - \frac{2y^3}{(x^2 + y^2)^2}$$

$x \neq 0$ and $y \neq 0$

### 1.2.13 25

$$f(x, y) = \sqrt{y^2 - 9}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{y^2 - 9}}$$

Yes

### 1.2.14 27

No

### 1.2.15 29

(a) $y = cx$

(b)

$$f(x, y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$x \neq 0$

(c) No, the function is not differentiable at $x = 0$

### 1.2.16 31

(a)

$$y' = \frac{1}{(x+c)^2} = y^2$$

(b)

$$y(0) = 1 = -\frac{1}{(0)+c} \Rightarrow c = -1 \Rightarrow y = \frac{1}{1-x}$$

$I = (-\infty, 1)$

$$y(0) = -1 = -\frac{1}{(0)+c} \Rightarrow c = 1 \Rightarrow y = -\frac{1}{x+1}$$

$I = (-1, \infty)$

### 1.2.17 39

$$\begin{aligned} y(0) = 0 &= c_1 \cos 3(0) + c_2 \sin 3(0) \\ c_1 &= 0 \end{aligned}$$

$$\begin{aligned} y\left(\frac{\pi}{6}\right) = -1 &= c_2 \sin 3\left(\frac{\pi}{6}\right) \\ c_2 &= -1 \end{aligned}$$

$$y = -\sin 3x$$

#### 1.2.18 41

$$\begin{aligned} y'(0) = 0 &= -3c_1 \sin 3(0) + 3c_2 \cos 3(0) \\ c_2 &= 0 \end{aligned}$$

$$\begin{aligned} y'\left(\frac{\pi}{4}\right) = 0 &= -3c_1 \sin 3\left(\frac{\pi}{4}\right) \\ &= -\frac{3}{\sqrt{2}}c_1 \\ c_1 &= 0 \end{aligned}$$

$$y = 0$$

#### 1.2.19 43

$$\begin{aligned} y(0) = 0 &= c_1 \cos 3(0) + c_2 \sin 3(0) \\ c_1 &= 0 \end{aligned}$$

$$\begin{aligned} y(\pi) = 4 &= c_2 \sin 3(\pi) \\ 4 &= 0 \end{aligned}$$

No solution

### 1.3 Differential Equations as Mathematical Models

#### 1.3.1 1

$$\frac{dP}{dt} = kP + r$$

$$\frac{dP}{dt} = kP - r$$

#### 1.3.2 3

$$\frac{dP}{dt} = k_b P - k_d P^2$$

#### 1.3.3 7

$$\frac{dx}{dt} = kx(1000 - x)$$

### 1.3.4 9

$$\frac{dA}{dt} = -\frac{A}{100}$$

$$A(0) = 50\,\text{lb}$$

### 1.3.5 11

$$\frac{dA}{dt} + \frac{7}{600-t}A = 6$$

### 1.3.6 13

$$\begin{aligned}
\frac{dV}{dt} &= -cA_h\sqrt{2gh} \\
A_w\frac{dh}{dt} &= -cA_h\sqrt{2gh} \\
\frac{dh}{dt} &= -\frac{cA_h\sqrt{2g}}{A_w}\sqrt{h} \\
&= -\frac{c\pi r_h^2\sqrt{2g}}{A_w}\sqrt{h} \\
&= -\frac{c\pi}{430}\sqrt{h}
\end{aligned}$$

### 1.3.7 15

$$L\frac{di}{dt} + Ri = E$$

### 1.3.8 17

$$m\frac{dv}{dt} = mg - kv^2$$

### 1.3.9 19

$$m\frac{d^2x}{dt^2} = -kx$$

### 1.3.10 21

$$\begin{aligned}
\frac{d}{dt}(mv) &= R - kv \\
\frac{dm}{dt}v + m\frac{dv}{dt} &= R - kv - mg
\end{aligned}$$

### 1.3.11 23

$$g = \frac{k}{R^2} \Rightarrow k = gR^2$$

$$\frac{d^2r}{dt^2} = -\frac{gR^2}{r^2}$$

### 1.3.12 25

$$\frac{dA}{dt} = k(M - A)$$

### 1.3.13 27

$$\frac{dx}{dt} = r - kx$$

### 1.3.14 29

$$\begin{aligned}
\frac{dy}{dx} &= \tan\theta \\
&= \tan\frac{\phi}{2} \\
&= \frac{1 - \cos\phi}{\sin\phi} \\
&= \frac{1 - x/r}{y/r} \\
&= \frac{r - x}{y} \\
&= \frac{\sqrt{x^2 + y^2} - x}{y}
\end{aligned}$$

## 1.4 Chapter in Review

### 1.4.1 1

$$\frac{dy}{dx} = ky$$

### 1.4.2 3

$$y'' + k^2y = 0$$

### 1.4.3 5

$$y = c_1 e^x + c_2 x e^x$$

$$\begin{aligned} y' &= c_1 e^x + c_2 e^x + c_2 x e^x \\ &= y + c_2 e^x \end{aligned}$$

$$\begin{aligned} y'' &= c_1 e^x + c_2 e^x + c_2 e^x + c_2 x e^x \\ &= c_1 e^x + 2c_2 e^x + c_2 x e^x \\ &= y' + c_2 e^x \end{aligned}$$

$$y'' - 2y' + y = 0$$

### 1.4.4 7

a, d

### 1.4.5 9

b

### 1.4.6 11

b

### 1.4.7 13

$$y = ce^x$$

### 1.4.8 15

$$\frac{dy}{dx} = x^2 + y^2$$

### 1.4.9 17

(a) $(-\infty, \infty)$

(b) $(-\infty, 0)$ or $(0, \infty)$

### 1.4.10 19

$x_0 = -1$ and $I = (-\infty, 0)$ or $x_0 = 2$ and $I = (0, \infty)$

### 1.4.11 23

$$y = x\sin x + x\cos x$$

$$y' = \sin x + x\cos x + \cos x - x\sin x$$

$$\begin{aligned} y'' &= \cos x + \cos x - x\sin x - \sin x - \sin x - x\cos x \\ &= 2\cos x - 2\sin x - x\sin x - x\cos x \end{aligned}$$

$$\begin{aligned} y'' + y &= 2\cos x - 2\sin x - x\sin x - x\cos x + x\sin x + x\cos x \\ &= 2\cos x - 2\sin x \end{aligned}$$

$I = (-\infty, \infty)$

### 1.4.12 25

$$y = \sin(\ln x)$$

$$y' = \frac{1}{x}\cos(\ln x)$$

$$y'' = -\frac{1}{x^2}\cos(\ln x) - \frac{1}{x^2}\sin(\ln x)$$

$$\begin{aligned} x^2y'' + xy' + y &= -\cos(\ln x) - \sin(\ln x) + \cos(\ln x) + \sin(\ln x) \\ &= 0 \end{aligned}$$

$I = (0, \infty)$

### 1.4.13 35

$$\begin{aligned} y(0) = 0 &= c_1e^{-3(0)} + c_2e^{(0)} + 4(0) \\ &= c_1 + c_2 \\ c_1 &= -c_2 \end{aligned}$$

$$\begin{aligned} y'(0) = 0 &= -3c_1e^{-3(0)} + c_2e^{(0)} + 4 \\ &= -3c_1 + c_2 + 4 \\ c_2 &= 3c_1 - 4 \end{aligned}$$

$$c_1 = -(3c_1 - 4) \Rightarrow c_1 = 1 \Rightarrow c_2 = -1$$

$$y = e^{-3x} - e^{x} + 4x$$

### 1.4.14 37

$$\begin{aligned} y(1) = -2 &= c_1 e^{-3(1)} + c_2 e^{(1)} + 4(1) \\ &= c_1 e^{-3} + c_2 e + 4 \\ c_1 &= -e^3(c_2 e + 6) \end{aligned}$$

$$\begin{aligned} y'(1) = 4 &= -3c_1 e^{-3(1)} + c_2 e^{(1)} + 4 \\ &= -3c_1 e^{-3} + c_2 e + 4 \\ c_2 e &= 3c_1 e^{-3} \end{aligned}$$

$$c_1 = -e^3(3c_1 e^{-3} + 6) = -3c_1 - 6e^3 \Rightarrow c_1 = -\frac{3}{2}e^3 \Rightarrow c_2 = -\frac{9}{2}e^{-1}$$

$$y = -\frac{3}{2}e^{3(1-x)} - \frac{9}{2}e^{x-1} + 4x$$

### 1.4.15 41

$y_0 = -3$, $y_1 = 0$

### 1.4.16 43

$$\begin{aligned} \frac{d}{dt}(mv) &= F - mg \\ \frac{d}{dt}(\lambda x \frac{dx}{dt}) &= F - \lambda x g \\ x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + gx &= \frac{F}{\lambda} \\ x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 32x &= 5 \end{aligned}$$

# 2 First-Order Differential Equations

## 2.1 Solution Curves Without a Solution

### 2.1.1 21

0 is stable, 3 is unstable

### 2.1.2 23

2 is semi-stable

### 2.1.3 25

$-2$ is unstable, 0 is semi-stable, 2 is stable

### 2.1.4 27

$-1$ is stable, 0 is unstable

### 2.1.5 39

$P_0 < h/k$

### 2.1.6 41

$$g - \frac{k}{m}v^2 = 0 \Rightarrow v = \sqrt{\frac{gm}{k}}$$

## 2.2 Separable Equations

### 2.2.1 1

$$\begin{aligned}
\frac{dy}{dx} &= \sin 5x \\
y &= -\frac{1}{5}\cos 5x + c
\end{aligned}$$

### 2.2.2 3

$$\begin{aligned}
dx + e^{3x}\, dy &= 0 \\
e^{-3x}\, dx + \, dy &= 0 \\
-\frac{1}{3}e^{-3x} + y &= c \\
y = \frac{1}{3}e^{-3x} + c
\end{aligned}$$

### 2.2.3 5

$$\begin{aligned}
x\frac{dy}{dx} &= 4y \\
\frac{1}{4y}\,dy &= \frac{1}{x}\,dx \\
\frac{1}{4}\ln|4y| &= \ln|x| + c \\
\ln|4y| &= 4\ln|x| + c \\
4y &= e^{4\ln|x|+c} \\
&= c\left(e^{\ln|x|}\right)^4 \\
y &= cx^4
\end{aligned}$$

### 2.2.4 7

$$\begin{aligned}
\frac{dy}{dx} &= e^{3x+2y} \\
&= e^{3x}e^{2y} \\
e^{-2y}\,dy &= e^{3x}\,dx \\
-\frac{1}{2}e^{-2y} &= \frac{1}{3}e^{3x} + c \\
-3e^{-2y} &= 2e^{3x} + c
\end{aligned}$$

### 2.2.5 9

$$\begin{aligned}
y\ln x\frac{dx}{dy} &= \left(\frac{y+1}{x}\right)^2 \\
x^2\ln x\,dx &= \frac{(y+1)^2}{y}\,dy \\
x^3\left(\frac{\ln x}{3} - \frac{1}{9}\right) &= \frac{1}{2}y(y+4) + \ln y + c \\
\frac{1}{3}x^3\ln x - \frac{1}{9}x^3 &= \frac{1}{2}y^2 + 2y + \ln y + c
\end{aligned}$$

### 2.2.6 11

$$\begin{aligned}
\csc y\, dx + \sec^2 x\, dy &= 0 \\
\frac{1}{\sin y}\, dx + \frac{1}{\cos^2 x}\, dy &= 0 \\
\cos^2 x\, dx + \sin y\, dy &= 0 \\
\frac{1}{2}(1 + \cos 2x)\, dx + \sin y\, dy &= 0 \\
\frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) - \cos y + c &= 0 \\
4\cos y &= 2x + \sin 2x + c
\end{aligned}$$

### 2.2.7 13

$$\begin{aligned}
(e^y + 1)^2 e^{-y}\, dx + (e^x + 1)^3 e^{-x}\, dy &= 0 \\
\frac{e^x}{(e^x + 1)^3}\, dx + \frac{e^y}{(e^y + 1)^2} &= 0 \\
-\frac{1}{2(e^x + 1)^2} - \frac{1}{e^y + 1} &= c \\
(e^x + 1)^{-2} + 2(e^y + 1)^{-1} &= c
\end{aligned}$$

### 2.2.8 15

$$\begin{aligned}
\frac{dS}{dr} &= kS \\
\frac{1}{S}\, dS &= k\, dr \\
\ln |S| &= kr + c \\
S &= ce^{kr}
\end{aligned}$$

### 2.2.9 17

$$
\begin{aligned}
\frac{dP}{dt} &= P - P^2 \\
\frac{1}{P(1-P)}\, dP &= dt \\
\ln \frac{P}{1-P} &= t + c \\
\frac{P}{1-P} &= ce^t \\
P &= ce^t(1-P) \\
P &= \frac{ce^t}{1+ce^t}
\end{aligned}
$$

### 2.2.10 19

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8} \\
&= \frac{(x-1)(y+3)}{(x+4)(y-2)} \\
\frac{y-2}{y+3}\, dt &= \frac{x-1}{x+4}\, dx \\
y - 5\ln|y+3| &= x - 5\ln|x+4| + c \\
e^{y-5\ln|y+3|} &= e^{x-5\ln|x+4|+c} \\
\frac{e^y}{(y+3)^5} &= \frac{ce^x}{(x+4)^5} \\
c(x+4)^5 e^y &= (y+3)^5 e^x
\end{aligned}
$$

### 2.2.11 21

$$
\begin{aligned}
\frac{dy}{dx} &= x\sqrt{1-y^2} \\
(1-y^2)^{-1/2}\, dy &= x\, dx \\
\arcsin y &= \frac{1}{2}x^2 + c \\
y &= \sin\left(\frac{1}{2}x^2 + c\right)
\end{aligned}
$$

### 2.2.12 23

$$\begin{aligned}
\frac{dx}{dt} &= 4(x^2+1) \\
\frac{1}{x^2+1}\,dx &= 4\,dt \\
\arctan x &= 4t + c \\
x &= \tan(4t+c)
\end{aligned}$$

$$\begin{aligned}
x\left(\frac{\pi}{4}\right) = 1 &= \tan\left(4\left(\frac{\pi}{4}\right)+c\right) \\
&= \tan(\pi + c) \\
c &= \arctan(1) - \pi \\
&= -\frac{3}{4}\pi
\end{aligned}$$

$$x = \tan\left(4t - \frac{3}{4}\pi\right)$$

### 2.2.13 25

$$\begin{aligned}
x^2\frac{dy}{dx} &= y - xy \\
&= y(1-x) \\
\frac{1}{y}\,dy &= \left(\frac{1}{x^2} - \frac{1}{x}\right)\,dx \\
\ln|y| &= -\frac{1}{x} - \ln|x| + c \\
y &= e^{-\frac{1}{x} - \ln|x| + c} \\
&= \frac{c}{xe^{1/x}}
\end{aligned}$$

$$\begin{aligned}
y(-1) = -1 &= \frac{c}{(-1)e^{1/(-1)}} \\
&= -ce \\
c &= e^{-1}
\end{aligned}$$

$$y = \frac{1}{xe^{1+1/x}}$$

### 2.2.14 29

$$\begin{aligned}
\frac{dy}{dx} &= ye^{-x^2} \\
\frac{1}{y}\frac{dy}{dx} &= e^{-x^2} \\
\int_4^x \frac{1}{y}\frac{dy}{dx'}\,dx' &= \int_4^x e^{-x'^2}\,dx' \\
\ln|y|\big|_4^x &= \int_4^x e^{-x'^2}\,dx' \\
\ln|y(x)| - \ln|y(4)| &= \int_4^x e^{-x'^2}\,dx' \\
\ln|y(x)| &= \ln|y(4)| + \int_4^x e^{-x'^2}\,dx' \\
y(x) &= e^{\int_4^x e^{-x'^2}\,dx'}
\end{aligned}$$

### 2.2.15 31

$$\begin{aligned}
\frac{dy}{dx} &= \frac{2x+1}{2y} \\
2y\,dy &= (2x+1)\,dx \\
y^2 &= x^2 + x + c \\
y &= \pm\sqrt{x^2+x+c}
\end{aligned}$$

$$\begin{aligned}
y(-2) = -1 &= -\sqrt{(-2)^2 + (-2) + c} \\
&= -\sqrt{2+c} \\
c &= -1
\end{aligned}$$

$$y = -\sqrt{x^2+x-1}$$

$$I = \left(-\infty, -\frac{1-\sqrt{5}}{2}\right)$$

### 2.2.16 33

$$\begin{aligned}
e^y\,dx - e^{-x}\,dy &= 0\\
e^x\,dx - e^{-y}\,dy &= 0\\
e^x + e^{-y} &= c\\
\ln|e^{-y}| &= \ln|c - e^x|\\
y &= -\ln|c - e^x|
\end{aligned}$$

$$\begin{aligned}
y(0) = 0 &= -\ln|c - e^{(0)}|\\
1 &= c - 1\\
c &= 2
\end{aligned}$$

$$y = -\ln|2 - e^x|$$

$$I = (-\infty, \ln 2)$$

## 2.3 Linear Equations

### 2.3.1 1

$$\begin{aligned}
\frac{dy}{dx} &= 5y\\
\ln|y| &= 5x + c\\
y &= ce^{5x}
\end{aligned}$$

$$I = (-\infty, \infty)$$

### 2.3.2 3

$$\begin{aligned}
\frac{dy}{dx} + y &= e^{3x}\\
e^x\frac{dy}{dx} + e^x y &= e^{4x}\\
\frac{d}{dx}(e^x y) &= e^{4x}\\
e^x y &= \frac{1}{4}e^{4x} + c\\
y &= \frac{1}{4}e^{3x} + ce^{-x}
\end{aligned}$$

$$I = (-\infty, \infty)$$

### 2.3.3 5

$$\begin{aligned} y' + 3x^2 y &= x^2 \\ e^{x^3} y' + 3x^2 e^{x^3} y &= e^{x^3} x^2 \\ e^{x^3} y &= \frac{1}{3} e^{x^3} + c \\ y &= \frac{1}{3} + ce^{-x^3} \end{aligned}$$

$$I = (-\infty, \infty)$$

### 2.3.4 7

$$\begin{aligned} x^2 y' + xy &= 1 \\ y' + x^{-1} y &= x^{-2} \\ e^{\ln x} y' + x^{-1} e^{\ln x} y &= e^{\ln x} x^{-2} \\ \frac{d}{dx}(e^{\ln x} y) &= x^{-1} \\ \frac{d}{dx}(xy) &= x^{-1} \\ xy &= \ln x + c \\ y &= \frac{\ln x + c}{x} \end{aligned}$$

$$I = (0, \infty)$$

### 2.3.5 9

$$\begin{aligned} x\frac{dy}{dx} - y &= x^2 \sin x \\ \frac{dy}{dx} - x^{-1} y &= x \sin x \\ e^{-\ln x} \frac{dy}{dx} - x^{-1} e^{-\ln x} y &= e^{-\ln x} x \sin x \\ \frac{d}{dx}(e^{-\ln x} y) &= \sin x \\ x^{-1} y &= -\cos x + c \\ y &= cx - x \cos x \end{aligned}$$

$$I = (0, \infty)$$

### 2.3.6 11

$$
\begin{aligned}
x\frac{dy}{dx} + 4y &= x^3 - x \\
\frac{dy}{dx} + 4x^{-1}y &= x^2 - 1 \\
e^{4\ln x}\frac{dy}{dx} + 4x^{-1}e^{4\ln x}y &= e^{4\ln x}(x^2 - 1) \\
\frac{d}{dx}(e^{4\ln x}y) &= x^6 - x^4 \\
x^4 y &= \frac{1}{7}x^7 - \frac{1}{5}x^5 + c \\
y &= \frac{1}{7}x^3 - \frac{1}{5}x^2 + cx^{-4}
\end{aligned}
$$

$$I = (0, \infty)$$

### 2.3.7 13

$$
\begin{aligned}
x^2y' + x(x+2)y &= e^x \\
y' + x^{-1}(x+2)y &= x^{-2}e^x \\
e^{x+2\ln x}y' + x^{-1}(x+2)e^{x+2\ln x}y &= e^{x+2\ln x}x^{-2}e^x \\
\frac{d}{dx}(e^x x^2 y) &= e^{2x} \\
e^x x^2 y &= \frac{1}{2}e^{2x} + c \\
y &= \frac{e^x}{2x^2} + \frac{c}{e^x x^2}
\end{aligned}
$$

$$I = (0, \infty)$$

### 2.3.8 15

$$
\begin{aligned}
y\,dx - 4(x + y^6)\,dy &= 0 \\
y\frac{dx}{dy} - 4x - 4y^6 &= 0 \\
\frac{dx}{dy} - \frac{4}{y}x &= 4y^5 \\
e^{-4\ln y}\frac{dx}{dy} - \frac{4}{y}e^{-4\ln y}x &= 4e^{-4\ln y}y^5 \\
\frac{d}{dy}(e^{-4\ln y}x) &= 4y \\
y^{-4}x &= 2y^2 + c \\
x &= 2y^6 + cy^4
\end{aligned}
$$

$$I = (0, \infty)$$

### 2.3.9 17

$$
\begin{aligned}
\cos x\frac{dy}{dx} + (\sin x)y &= 1 \\
\frac{dy}{dx} + (\tan x)y &= \sec x \\
e^{\ln(\sec x)}\frac{dy}{dx} + (\tan x)e^{\ln(\sec x)}y &= e^{\ln(\sec x)}\sec x \\
\frac{d}{dx}(e^{\ln(\sec x)}y) &= \sec^2 x \\
y\sec x &= \tan x + c \\
y &= \sin x + c\cos x
\end{aligned}
$$

$$I = (-\pi/2, \pi/2)$$

### 2.3.10 19

$$
\begin{aligned}
(x+1)\frac{dy}{dx} + (x+2)y &= 2xe^{-x} \\
\frac{dy}{dx} + \frac{x+2}{x+1}y &= \frac{2xe^{-x}}{x+1} \\
e^{x+\ln|x+1|}\frac{dy}{dx} + \frac{x+2}{x+1}e^{x+\ln|x+1|}y &= e^{x+\ln|x+1|}\frac{2xe^{-x}}{x+1} \\
\frac{d}{dx}(e^{x+\ln|x+1|}y) &= 2x \\
e^x(x+1)y &= x^2+c \\
y &= \frac{x^2+c}{e^x(x+1)}
\end{aligned}
$$

$$I = (-1, \infty)$$

### 2.3.11 21

$$
\begin{aligned}
\frac{dr}{d\theta} + r\sec\theta &= \cos\theta \\
e^{\ln|\sec\theta+\tan\theta|}\frac{dr}{d\theta} + e^{\ln|\sec\theta+\tan\theta|}r\sec\theta &= e^{\ln|\sec\theta+\tan\theta|}\cos\theta \\
\frac{d}{d\theta}(e^{\ln|\sec\theta+\tan\theta|}r) &= 1+\sin\theta \\
(\sec\theta+\tan\theta)r &= \theta-\cos\theta+c \\
r &= \frac{\theta-\cos\theta+c}{\sec\theta+\tan\theta}
\end{aligned}
$$

$$I = (-\pi/2, \pi/2)$$

### 2.3.12 23

$$
\begin{aligned}
x\frac{dy}{dx} + (3x+1)y &= e^{-3x} \\
\frac{dy}{dx} + (3+x^{-1})y &= e^{-3x}x^{-1} \\
e^{3x+\ln|x|}\frac{dy}{dx} + (3+x^{-1})e^{3x+\ln|x|}y &= 1 \\
\frac{d}{dx}(e^{3x+\ln|x|}y) &= 1 \\
e^{3x}xy &= x+c \\
y &= \frac{x+c}{e^{3x}x}
\end{aligned}
$$

$$I = (0, \infty)$$

### 2.3.13 25

$$\begin{aligned} xy' + y &= e^x \\ y' + x^{-1}y &= e^x x^{-1} \\ e^{\ln|x|}y' + x^{-1}e^{\ln|x|}y &= e^x \\ \frac{d}{dx}(e^{\ln|x|}y) &= e^x \\ xy &= e^x + c \\ y &= \frac{e^x + c}{x} \end{aligned}$$

$$\begin{aligned} y(1) = 2 &= \frac{e^{(1)} + c}{(1)} \\ c &= 2 - e \end{aligned}$$

$$y = \frac{e^x + 2 - e}{x}$$

$$I = (0, \infty)$$

### 2.3.14 27

$$\begin{aligned} L\frac{di}{dt} + Ri &= E \\ \frac{di}{dt} + \frac{R}{L}i &= \frac{E}{L} \\ e^{Rt/L}\frac{di}{dt} + \frac{R}{L}e^{Rt/L}i &= \frac{E}{L}e^{Rt/L} \\ \frac{d}{dt}(e^{Rt/L}i) &= \frac{E}{L}e^{Rt/L} \\ e^{Rt/L}i &= \frac{E}{R}e^{Rt/L} + c \\ i &= \frac{E}{R} + ce^{-Rt/L} \end{aligned}$$

$$\begin{aligned} i(0) = i_0 &= \frac{E}{R} + ce^{-R(0)/L} \\ &= \frac{E}{R} + c \\ c &= i_0 - \frac{E}{R} \end{aligned}$$

$$i = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right) e^{-Rt/L}$$

$$I = (-\infty, \infty)$$

### 2.3.15 53

$$\begin{aligned}\frac{dE}{dt} &= -\frac{1}{RC}E \\ \frac{1}{E}\frac{dE}{dt} &= -\frac{1}{RC} \\ \ln|E| &= -\frac{1}{RC}t + c \\ E &= ce^{-t/RC}\end{aligned}$$

$$\begin{aligned}E(4) = E_0 &= ce^{-(4)/RC} \\ c &= E_0 e^{4/RC}\end{aligned}$$

$$E = E_0 e^{(4-t)/RC}$$

## 2.4 Exact Equations

### 2.4.1 1

$$f(x, y) = x^2 - x + g(y)$$

$$\frac{\partial f}{\partial y} = g'(y) = 3y + 7$$

$$g(y) = \frac{3}{2}y^2 + 7y$$

$$x^2 - x + \frac{3}{2}y^2 + 7y = c$$

### 2.4.2 3

$$f(x, y) = \frac{5}{2}x^2 + 4xy + g(y)$$

$$4x + g'(y) = 4x - 8y^3 \Rightarrow g'(y) = -8y^3$$

$$g(y) = -2y^4$$

$$\frac{5}{2}x^2 + 4xy - 2y^4 = c$$

### 2.4.3 5

$$f(x,y) = x^2y^2 - 3x + g(y)$$

$$2x^2y + g'(y) = 2x^2y + 4 \Rightarrow g'(y) = 4$$

$$g(y) = 4y$$

$$x^2y^2 - 3x + 4y = c$$

### 2.4.4 7

Not exact

### 2.4.5 9

$$f(x,y) = \frac{1}{2}x^2 - xy^3 - y^2\cos x + g(y)$$

$$-3xy^2 - 2y\cos x + g'(y) = -3xy^2 - 2y\cos x \Rightarrow g'(y) = 0$$

$$\frac{1}{2}x^2 - xy^3 - y^2\cos x = c$$

### 2.4.6 11

Not exact

### 2.4.7 13

$$f(x,y) = xy + g(x)$$

$$y + g'(x) = -2xe^x + y - 6x^2 \Rightarrow g'(x) = -2xe^x - 6x^2$$

$$g(x) = -2e^x(x-1) - 2x^3$$

$$xy - 2e^x(x-1) - 2x^3 = c$$

### 2.4.8 21

$$f(x,y) = \frac{1}{3}(x+y)^3 + g(y)$$

$$(x+y)^2 + g'(y) = 2xy + x^2 - 1 \Rightarrow g'(y) = -y^2 - 1$$

$$g(y) = -\frac{1}{3}y^3 - y$$

$$\frac{1}{3}(x+y)^3 - \frac{1}{3}y^3 - y = c$$

$$\frac{1}{3}(1+1)^3 - \frac{1}{3}1^3 - 1 = c \Rightarrow c = \frac{4}{3}$$

$$x^3 + 3x^2y + 3xy^2 - 3y = 4$$

### 2.4.9 23

$$f(x,y) = 4ty + t^2 - 5t + g(y)$$

$$4t + g'(y) = 6y + 4t - 1 \Rightarrow g'(y) = 6y - 1$$

$$g(y) = 3y^2 - y$$

$$4ty + t^2 - 5t + 3y^2 - y = c$$

$$4(-1)(2) + (-1)^2 - 5(-1) + 3(2)^2 - (2) = c \Rightarrow c = 8$$

$$4ty + t^2 - 5t + 3y^2 - y = 8$$

### 2.4.10 27

$$3y^2 + 4kxy^3 = 3y^2 + 40xy^3 \Rightarrow k = 10$$

### 2.4.11 31

$$M_y = 4y$$

$$N_x = 2y$$

$$\frac{M_y - N_x}{N} = \frac{4y - 2y}{2xy} = \frac{1}{x}$$

$$\mu(x) = e^{\ln x} = x$$

$$(2xy^2 + 3x^2)\,dx + 2x^2y\,dy = 0$$

$$f(x,y) = x^2y^2 + x^3 + g(y)$$

$$2x^2y + g'(y) = 2x^2y \Rightarrow g'(y) = 0$$

$$x^2y^2 + x^3 = c$$

### 2.4.12 33

$$M_y = 6x$$

$$N_x = 18x$$

$$\frac{M_y - N_x}{N} = \frac{6x - 18x}{4y + 9x^2}$$

$$\frac{N_x - M_y}{M} = \frac{18x - 6x}{6xy} = \frac{2}{y}$$

$$\mu(y) = e^{2\ln y} = y^2$$

$$6xy^3\,dx + (4y^3 + 9x^2y^2)\,dy = 0$$

$$f(x,y) = 3x^2y^3 + g(y)$$

$$9x^2y^2 + g'(y) = 4y^3 + 9x^2y^2 \Rightarrow g'(y) = 4y^3$$

$$g(y) = y^4$$

$$3x^2y^3 + y^4 = c$$

### 2.4.13 37

$$M_y = 0$$

$$N_x = 2xy$$

$$\frac{N_x - M_y}{M} = \frac{2xy - 0}{x} = 2y$$

$$\mu(y) = e^{y^2}$$

$$e^{y^2} x\,dx + e^{y^2}(x^2 y + 4y)\,dy = 0$$

$$f(x,y) = \frac{1}{2}e^{y^2}x^2 + g(y)$$

$$ye^{y^2}x^2 + g'(y) = e^{y^2}(x^2y + 4y) \Rightarrow g'(y) = 4e^{y^2}y$$

$$g(y) = 2e^{y^2}$$

$$\frac{1}{2}e^{y^2}x^2 + 2e^{y^2} = c$$

$$\frac{1}{2}e^{(0)^2}(4)^2 + 2e^{(0)^2} = c \Rightarrow c = 10$$

$$\frac{1}{2}e^{y^2}x^2 + 2e^{y^2} = 10$$

### 2.4.14 39

(c)

$$(0)^3 + 2(0)^2(-2) + (-2)^2 = c \Rightarrow c = 4$$

$$y^2 + 2x^2y + x^3 - 4 = 0$$

$$\begin{aligned} y &= \frac{-(2x^2) \pm \sqrt{(2x^2)^2 - 4(1)(x^3-4)}}{2(1)} \\ &= \frac{-2x^2 \pm \sqrt{4x^4 - 4(x^3-4)}}{2} \\ &= -x^2 \pm \sqrt{x^4 - x^3 + 4} \end{aligned}$$

**2.4.15 45**

(a)

$$xv\frac{dv}{dx} + v^2 = 32x \Rightarrow xv\,dv + (v^2 - 32x)\,dx = 0$$

$$M_x = v$$

$$N_v = 2v$$

$$\frac{M_x - N_v}{N} = \frac{v - 2v}{v^2 - 32x}$$

$$\frac{N_v - M_x}{M} = \frac{2v - v}{xv} = \frac{1}{x}$$

$$\mu(x) = e^{\ln x} = x$$

$$x^2v\,dv + (xv^2 - 32x^2)\,dx = 0$$

$$f(x, v) = \frac{1}{2}x^2v^2 + g(x)$$

$$xv^2 + g'(x) = xv^2 - 32x^2 \Rightarrow g'(x) = -32x^2$$

$$g(x) = -\frac{32}{3}x^3$$

$$\frac{1}{2}(3)^2(0)^2 - \frac{32}{3}(3)^3 = c \Rightarrow c = -288$$

$$\frac{1}{2}x^2v^2 - \frac{32}{3}x^3 = -288 \Rightarrow v = 8\sqrt{\frac{x}{3} - \frac{9}{x^2}}$$

(b) $v = 12.7\,\mathrm{ft/s}$

## 2.5 Solutions by Substitution

### 2.5.1 1

$$
\begin{aligned}
(x-y)\,dx + x\,dy &= 0\\
(x-ux)\,dx + x(u\,dx + x\,du) &= 0\\
x\,dx + x^2\,du &= 0\\
x^{-1}\,dx + du &= 0\\
\ln|x| + u &= c\\
\ln|x| + \frac{y}{x} &= c\\
y &= cx - x\ln|x|
\end{aligned}
$$

### 2.5.2 3

$$
\begin{aligned}
x\,dx + (y-2x)\,dy &= 0\\
vy(v\,dy + y\,dv) + (y-2vy)\,dy &= 0\\
(v^2y + y - 2vy)\,dy + vy^2\,dv &= 0\\
y(v^2 - 2v + 1)\,dy + vy^2\,dv &= 0\\
(v-1)^2\,dy + vy\,dv &= 0\\
\frac{1}{y}\,dy + \frac{v}{(v-1)^2}\,dv &= 0\\
\ln|y| + \frac{1}{1-v} + \ln|v-1| &= c\\
\ln|y| + \frac{1}{1-x/y} + \ln\left|\frac{x}{y} - 1\right| &= c\\
\ln|x-y| + \frac{y}{y-x} &= c\\
(y-x)\ln|x-y| + y &= c(y-x)\\
(x-y)\ln|x-y| &= y + c(x-y)
\end{aligned}
$$

### 2.5.3 5

$$
\begin{aligned}
(y^2 + yx)\,dx - x^2\,dy &= 0 \\
((ux)^2 + ux^2)\,dx - x^2(u\,dx + x\,du) &= 0 \\
u^2x^2\,dx - x^3\,du &= 0 \\
\frac{1}{x}\,dx - \frac{1}{u^2}\,du &= 0 \\
\ln|x| + \frac{1}{u} &= c \\
\ln|x| + \frac{x}{y} &= c \\
y &= \frac{x}{c - \ln|x|}
\end{aligned}
$$

### 2.5.4 7

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{y-x}{y+x} \\
(y+x)\,dy + (x-y)\,dx &= 0 \\
(ux+x)(u\,dx + x\,du) + (x - ux)\,dx &= 0 \\
(u^2x + x)\,dx + (ux^2 + x^2)\,du &= 0 \\
x(u^2+1)\,dx + x^2(u+1)\,du &= 0 \\
\frac{1}{x}\,dx + \frac{u+1}{u^2+1}\,du &= 0 \\
\ln|x| + \frac{1}{2}\ln|u^2+1| + \arctan u &= c \\
\ln|x^2+y^2| + 2\arctan\frac{y}{x} &= c
\end{aligned}
$$

### 2.5.5 9

$$
\begin{aligned}
-y\,dx + (x + \sqrt{xy})\,dy &= 0 \\
-ux\,dx + (x + \sqrt{ux^2})(u\,dx + x\,du) &= 0 \\
u\sqrt{ux^2}\,dx + (x^2 + x\sqrt{ux^2})\,du &= 0 \\
u^{3/2}x\,dx + x^2(1 + \sqrt{u})\,du &= 0 \\
\frac{1}{x}\,dx + \frac{1 + \sqrt{u}}{u^{3/2}}\,du &= 0 \\
\frac{1}{x}\,dx + (u^{-3/2} + u^{-1})\,du &= 0 \\
\ln|x| - 2u^{-1/2} + \ln|u| &= c \\
\ln|x| - 2(y/x)^{-1/2} + \ln|y/x| &= c \\
\ln|y| - 2\sqrt{\frac{x}{y}} &= c \\
4\frac{x}{y} &= (\ln|y| - c)^2 \\
4x &= y(\ln|y| - c)^2
\end{aligned}
$$

### 2.5.6 11

$$
\begin{aligned}
xy^2\frac{dy}{dx} &= y^3 - x^3 \\
xy^2\,dy + (x^3 - y^3)\,dx &= 0 \\
x(ux)^2(u\,dx + x\,du) + (x^3 - (ux)^3)\,dx &= 0 \\
x^3\,dx + u^2x^4\,du &= 0 \\
x^{-1}\,dx + u^2\,du &= 0 \\
\ln|x| + \frac{1}{3}u^3 &= c \\
\ln|x| + \frac{1}{3}\left(\frac{y}{x}\right)^3 &= c
\end{aligned}
$$

$$
\ln|1| + \frac{1}{3}\left(\frac{2}{1}\right)^3 = c \Rightarrow c = \frac{8}{3} + \ln 1
$$

$$
\begin{aligned}
\ln|x| + \frac{1}{3}\left(\frac{y}{x}\right)^3 &= \frac{8}{3} \\
y^3 + 3x^3\ln|x| &= 8x^3
\end{aligned}
$$

### 2.5.7 13

$$
\begin{aligned}
(x + ye^{y/x})\,dx - xe^{y/x}\,dy &= 0 \\
(x + uxe^{u})\,dx - xe^{u}(u\,dx + x\,du) &= 0 \\
x\,dx - x^2e^{u}\,du &= 0 \\
x^{-1}\,dx - e^{u}\,du &= 0 \\
\ln|x| - e^{u} &= c \\
\ln|x| - e^{y/x} &= c
\end{aligned}
$$

$$\ln|1| - e^{0/1} = c \Rightarrow c = -1$$

$$\ln|x| = e^{y/x} - 1$$

### 2.5.8 15

$$
\begin{aligned}
x\frac{dy}{dx} + y &= \frac{1}{y^2} \\
\frac{dy}{dx} + x^{-1}y &= x^{-1}y^{-2}
\end{aligned}
$$

$$u = y^{1-n} = y^3 \Rightarrow y = u^{1/3} \Rightarrow \frac{dy}{dx} = \frac{1}{3}u^{-2/3}\frac{du}{dx}$$

$$
\begin{aligned}
\frac{1}{3}u^{-2/3}\frac{du}{dx} + x^{-1}u^{1/3} &= x^{-1}u^{-2/3} \\
\frac{du}{dx} + 3x^{-1}u &= 3x^{-1} \\
e^{3\ln|x|}\frac{du}{dx} + 3x^{-1}e^{3\ln|x|}u &= 3x^2 \\
\frac{d}{dx}(x^3u) &= 3x^2 \\
x^3u &= x^3 + c \\
y^3 &= 1 + cx^{-3}
\end{aligned}
$$

### 2.5.9 17

$$
\begin{aligned}
\frac{dy}{dx} &= y(xy^3 - 1) \\
\frac{dy}{dx} + y &= xy^4
\end{aligned}
$$

$$u = y^{1-n} = y^{-3} \Rightarrow y = u^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}u^{-4/3}\frac{du}{dx}$$

$$\begin{aligned}
-\frac{1}{3}u^{-4/3}\frac{du}{dx} + u^{-1/3} &= xu^{-4/3} \\
\frac{du}{dx} - 3u &= -3x \\
e^{-3x}\frac{du}{dx} - 3e^{-3x}u &= -3e^{-3x}x \\
\frac{d}{dt}(e^{-3x}u) &= -3e^{-3x}x \\
e^{-3x}u &= e^{-3x}x + \frac{1}{3}e^{-3x} + c \\
u &= x + \frac{1}{3} + ce^{3x} \\
y^{-3} &= x + \frac{1}{3} + ce^{3x}
\end{aligned}$$

### 2.5.10 21

$$\begin{aligned}
x^2\frac{dy}{dx} - 2xy &= 3y^4 \\
\frac{dy}{dx} - 2x^{-1}y &= 3x^{-2}y^4
\end{aligned}$$

$$u = y^{1-n} = y^{-3} \Rightarrow y = u^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}u^{-4/3}\frac{du}{dx}$$

$$\begin{aligned}
-\frac{1}{3}u^{-4/3}\frac{du}{dx} - 2x^{-1}u^{-1/3} &= 3x^{-2}u^{-4/3} \\
\frac{du}{dx} + 6x^{-1}u &= -9x^{-2} \\
e^{6\ln|x|}\frac{du}{dx} + 6x^{-1}e^{6\ln|x|}u &= -9e^{6\ln|x|}x^{-2} \\
\frac{d}{dx}(x^6u) &= -9x^4 \\
x^6u &= -\frac{9}{5}x^5 + c \\
u &= -\frac{9}{5}x^{-1} + cx^{-6} \\
y^{-3} &= -\frac{9}{5}x^{-1} + cx^{-6}
\end{aligned}$$

$$\left(\frac{1}{2}\right)^{-3} = -\frac{9}{5}(1)^{-1} + c(1)^{-6} \Rightarrow c = \frac{49}{5}$$

$$y^{-3} = -\frac{9}{5}x^{-1} + \frac{49}{5}x^{-6}$$

### 2.5.11 23

Let $u = x + y + 1$ so $\frac{du}{dx} = 1 + \frac{dy}{dx}$ and

$$\begin{aligned}
\frac{du}{dx} - 1 &= u^2 \\
\frac{1}{u^2+1}\,du &= dx \\
\arctan u &= x + c \\
\arctan(x+y+1) &= x + c \\
x + y + 1 &= \tan(x+c) \\
y &= -x - 1 + \tan(x+c)
\end{aligned}$$

### 2.5.12 25

Let $u = x + y$ so $\frac{du}{dx} = 1 + \frac{dy}{dx}$ and

$$\begin{aligned}
\frac{du}{dx} - 1 &= \tan^2 u \\
\frac{1}{1+\tan^2 u}\,du &= dx \\
\frac{1}{2}(u + \sin u \cos u) &= x + c \\
x + y + \sin(x+y)\cos(x+y) &= 2(x+c) \\
x + y + \frac{1}{2}\sin(2(x+y)) &= 2(x+c) \\
2x + 2y + \sin(2(x+y)) &= 4(x+c) \\
2y - 2x + \sin(2(x+y)) &= c
\end{aligned}$$

### 2.5.13 35

(a) Let $y = y_1 + u$ so $\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{du}{dx}$ but $\frac{dy_1}{dx} = P(x) + Q(x)y_1 + R(x)y_1^2$ so

$$\begin{aligned}
\frac{dy}{dx} &= P(x) + Q(x)y + R(x)y^2 \\
P(x) + Q(x)y_1 + R(x)y_1^2 + \frac{du}{dx} &= P(x) + Q(x)(y_1 + u) + R(x)(y_1 + u)^2 \\
\frac{du}{dx} &= Q(x)u + R(x)(2y_1 u + u^2) \\
\frac{du}{dx} - (Q(x) + 2R(x)y_1)u &= R(x)u^2
\end{aligned}$$

(b) Let $y = 2x^{-1} + u$ so $\frac{dy}{dx} = -2x^{-2} + \frac{du}{dx}$ and

$$\begin{aligned}
-\frac{2}{x^2} + \frac{du}{dx} &= -\frac{4}{x^2} - \frac{1}{x}\left(\frac{2}{x} + u\right) + \left(\frac{2}{x} + u\right)^2 \\
\frac{du}{dx} &= \frac{2}{x^2} - \frac{4}{x^2} - \frac{2}{x^2} - \frac{u}{x} + \frac{4}{x^2} + \frac{4u}{x} + u^2 \\
\frac{du}{dx} - \frac{3}{x}u &= u^2
\end{aligned}$$

Let $v = u^{1-n} = u^{-1}$ so $u = v^{-1}$ and $\frac{du}{dx} = -v^{-2}\frac{dv}{dx}$

$$\begin{aligned}
-v^{-2}\frac{dv}{dx} - \frac{3}{x}v^{-1} &= v^{-2} \\
\frac{dv}{dx} + \frac{3}{x}v &= -1 \\
e^{3\ln|x|}\frac{dv}{dx} + \frac{3}{x}e^{3\ln|x|}v &= -e^{3\ln|x|} \\
\frac{d}{dt}(x^3 v) &= -x^3 \\
x^3 v &= -\frac{1}{4}x^4 + c \\
\frac{1}{y - y_1} &= -\frac{1}{4}x + cx^{-3} \\
y &= y_1 + \left(-\frac{1}{4}x + cx^{-3}\right)^{-1} \\
&= \frac{2}{x} + \left(-\frac{1}{4}x + cx^{-3}\right)^{-1}
\end{aligned}$$

**2.5.14 37**

$$\begin{aligned}
\frac{dP}{dt} &= P(a - bP) \\
\frac{dP}{dt} - aP &= -bP^2
\end{aligned}$$

Let $u = P^{1-n} = P^{-1}$ so $P = u^{-1}$ and $\frac{dP}{dt} = -u^{-2}\frac{du}{dt}$

$$\begin{aligned}
-u^{-2}\frac{du}{dt} - au^{-1} &= -bu^{-2} \\
\frac{du}{dt} + au &= b \\
e^{at}\frac{du}{dt} + ae^{at}u &= be^{at} \\
\frac{d}{dt}(e^{at}u) &= be^{at} \\
e^{at}u &= \frac{b}{a}e^{at} + c \\
P^{-1} &= \frac{b}{a} + ce^{-at} \\
&= \frac{b + ce^{-at}}{a} \\
P &= \frac{a}{b + ce^{-at}}
\end{aligned}$$