

Advanced Engineering Mathematics Complex Analysis by Dennis G. Zill Notes

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17 Functions of a Complex Variable

17.1 Complex Numbers

- A **complex number** is any number of the form

$$z = a + ib$$

where a and b are real numbers and i is the imaginary unit such that $i^2 = -1$.

- The real number a in the above complex number z is called the **real part** of z and the real number b (not ib) is called the **imaginary part** of z .
- The real and imaginary parts of a complex number z are denoted $\text{Re}(z)$ and $\text{Im}(z)$, respectively.
- A real constant multiple of the imaginary unit, e.g. $6i$ is called a **pure imaginary number**.
- Two complex numbers are equal if their real and imaginary parts are equal.
- The addition and subtraction of complex numbers occur between the real and imaginary parts, e.g.

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

- The multiplication of complex numbers occurs elementwise as normal, e.g.

$$(a + bi)(c + di) = ac + adi + bci - bd.$$

- The **conjugate** of a complex number $z = a + ib$ is

$$\bar{z} = a - ib.$$

- The division of complex numbers occurs by multiplying the numerator and denominator by the conjugate of the denominator, e.g.

$$\begin{aligned} \frac{a + bi}{c + di} &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{ac - adi + bci + bd}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2}. \end{aligned}$$

- Conjugates have several interesting properties:

$$\begin{aligned} \overline{z_1 + z_2} &= \bar{z}_1 + \bar{z}_2 \\ \overline{z_1 - z_2} &= \bar{z}_1 - \bar{z}_2 \\ \overline{z_1 z_2} &= \bar{z}_1 \bar{z}_2 \\ \frac{z_1}{z_2} &= \frac{\bar{z}_1}{\bar{z}_2}. \end{aligned}$$

- The sum and product of a complex number $z = x + iy$ with its conjugate are real numbers

$$\begin{aligned}z + \bar{z} &= 2x \\ z\bar{z} &= x^2 + y^2\end{aligned}$$

while the difference between a complex number and its conjugate is a pure imaginary number

$$z - \bar{z} = 2iy.$$

- The above properties let us define

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2} \text{ and } \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}.$$

- The **complex plane** or **z -plane** is a coordinate system where the horizontal or x -axis is called the **real axis** and the vertical or y -axis is called the **imaginary axis**. Complex numbers can be plotted in this coordinate system by considering their real and imaginary parts an ordered pair corresponding their position.
- The **modulus** or **absolute value** of a complex number $z = x + iy$ denoted by $|z|$ is the real number

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$$

This is the distance between z and the origin in the complex plane.

- If you consider two numbers in the complex plane as vectors, the length of their sum can't be longer than their individual lengths combined

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

This extends to any finite sum

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

and is known as the **triangle inequality**.

17.2 Powers and Roots

- A complex number can be expressed in **polar form**

$$z = (r \cos \theta) + i(r \sin \theta)$$

where $r = |z|$ is the nonnegative modulus of z and $\theta = \arg z$ is the **argument** of z — the angle between z and the positive real axis measured in the counterclockwise direction.

- The argument of a complex number z isn't unique as any multiple of 2π can be added to it. The **principle argument** of z denoted $\text{Arg } z$ is the argument of z restricted to the interval $-\pi \leq \text{Arg } z \leq \pi$.
- Multiplication and division of complex numbers is simpler in polar form. For two complex numbers $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ we get

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)].$$

- The above formulas can be used to find integer powers of a complex number z

$$z^n = r^n (\cos n\theta + i \sin n\theta)$$

where n is an integer (including negative integers).

- **DeMoivre's formula** is a special case of the above where $r = 1$ so

$$z^n = (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

- A number w is said to be an **n th root** of a nonzero complex number z if $w^n = z$. The n th roots of a nonzero complex number $z = r(\cos \theta + i \sin \theta)$ are

$$w_k = r^{1/n} \left[\cos \left(\frac{\theta + 2k\pi}{n} \right) + i \sin \left(\frac{\theta + 2k\pi}{n} \right) \right]$$

where $k = 0, 1, 2, \dots, n-1$.

- The root w of a complex number z obtained by using the principle argument of z with $k = 0$ is called the **principle n th root** of z .
- Since the n th roots of a complex number have the same modulus they lie on a circle of radius $r^{1/n}$. The arguments of subsequent roots differ by $2\pi/n$ so they're also equally spaced around the circle.

17.3 Sets in the Complex Plane

- The points $z = x + iy$ that satisfy the equation

$$|z - z_0| = \rho$$

for $\rho > 0$ lie on a circle of radius ρ centred at the point z_0 .

- The points z satisfying the inequality $|z - z_0| < \rho$ for $\rho > 0$ lie within, but not on, a circle of radius ρ centered at the point z_0 . This set is called a **neighborhood** of z_0 or an **open disk**.
- A point z_0 is said to be an **interior point** of a set S of the complex plane if there exists some neighborhood of z_0 that lies entirely within S .

- If every point z of a set S is an interior point, then S is said to be an **open set**. An example of a set that isn't open is the set of points satisfying the inequality $\operatorname{Re}(z) \geq 0$. This isn't open because it includes the line $\operatorname{Re}(z) = 0$ and no points on that line are interior to the set because, no matter what ρ you choose, some points in the neighborhood have $\operatorname{Re}(z) < 0$.
- If every neighborhood of a point z_0 contains at least one point that is in a set S and at least one point that is not in S , then z_0 is said to be a **boundary point** of S .
- The **boundary** of a set S in the complex plane is the set of all boundary points of S .
- If any pair of points in a set S can be connected by a polygonal line that lies entirely within the set, then S is said to be **connected**.
- An open connected set is called a **domain**.
- A **region** is a set in the complex plane with all, some, or none of its boundary points. A region containing all of its boundary points is said to be **closed**.

17.4 Functions of a Complex Variable

- A **function** f from a set A to a set B is a rule of correspondence that assigns to each element of A one and only one element of B .
- If b is the element of B assigned to the element a of A , b is said to be the **image** of a and is denoted $b = f(a)$.
- The set A is called the **domain** of f .
- The set of all images in B is called the **range** of f .
- If A is a set of real numbers, f is said to be a **function of a real variable** x .
- If A is a set of complex numbers, f is said to be a **function of a complex variable** z or a **complex function**.
- The image w of a complex number z is

$$w = f(z) = u(x, y) + iv(x, y)$$

where u and v are the real and imaginary parts of w and are real-valued functions.

- Although we cannot draw a graph of a complex function $w = f(z)$ (because it would require a four-dimensional coordinate system), it can be interpreted as a **mapping** or **transformation** from the z plane to the w plane.

- A complex function may be interpreted as a two-dimensional fluid flow by considering $w = f(z)$ as the fluid velocity vector at the point z . In that case, if $x(t) + iy(t)$ is a parametric representation of a particle's position over time then

$$\begin{aligned}\frac{dx}{dt} &= u(x, y) \\ \frac{dy}{dt} &= v(x, y)\end{aligned}$$

and the family of solutions to this system of differential equations are called the **streamlines** of the flow associated with $f(z)$.

Definition 17.4.1 Limit of a Function

Suppose the function f is defined in some neighborhood of z_0 , except possibly at z_0 itself. Then f is said to possess a **limit** at z_0 , written

$$\lim_{z \rightarrow z_0} f(z) = L$$

if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

- For a function f of a real variable x , the limit $\lim_{x \rightarrow x_0} f(x) = L$ means f approaches L as you approach from both the left and right. If however f is a function of a complex variable it means f approaches L as you approach from any direction in the complex plane.

Theorem 17.4.1 Limit of Sum, Product, Quotient

Suppose $\lim_{z \rightarrow z_0} f(z) = L_1$ and $\lim_{z \rightarrow z_0} g(z) = L_2$. Then

$$(i) \quad \lim_{z \rightarrow z_0} [f(z) + g(z)] = L_1 + L_2$$

$$(ii) \quad \lim_{z \rightarrow z_0} f(z)g(z) = L_1L_2$$

$$(iii) \quad \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{L_1}{L_2}, \quad L_2 \neq 0.$$

- A function f is continuous at a point z_0 if

$$\lim_{z \rightarrow z_0} f(z) = f(z_0).$$

- A function f defined by

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0, \quad a_n \neq 0$$

where n is a nonnegative integer and the coefficients a_i , $i = 0, 1, \dots, n$, are complex constants is called a **polynomial** of degree n .

- Polynomials are continuous on the entire complex plane.

- A **rational function**

$$f(z) = \frac{g(z)}{h(z)}$$

is continuous everywhere $h(z) \neq 0$.

Definition 17.4.3 Derivative

Suppose the complex function f is defined in a neighborhood of a point z_0 . The **derivative** of f at z_0 is

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad (3)$$

provided this limit exists.

- In order for a complex function to be differentiable, the limit must approach the same value from every direction. This is a greater demand than in real variables. If you take an arbitrary complex function, there's a good chance it isn't differentiable.

Definition 17.4.4 Analyticity at a Point

A complex function $w = f(z)$ is said to be **analytic at a point** z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0 .

- Analyticity at a point is a neighborhood property. A function can be differentiable at a point but if the neighboring points aren't also differentiable, it's not analytic at that point.
- A function is analytic in a domain D if it is analytic at every point in D .
- A function that is analytic everywhere is called an **entire function**.

17.5 Cauchy-Riemann Equations

Theorem 17.5.1 Cauchy–Riemann Equations

Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivatives of u and v exist and satisfy the **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad (1)$$

- If a complex function $f(z) = u(x, y) + iv(x, y)$ is analytic throughout a domain D , then the real functions u and v must satisfy the Cauchy-Riemann equations at every point in D .

Theorem 17.5.2 Criterion for Analyticity

Suppose the real-valued functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a domain D . If u and v satisfy the Cauchy-Riemann equations at all points of D , then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D .

- The Cauchy-Riemann equations are derived assuming the function is differentiable at a particular point. That being the case, they can also be used as a formula for the derivative of the function

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

- Because analyticity implies differentiability, theorem 17.5.2 can also be used to determine if a function is differentiable at a point.
- A real-valued function $\phi(x, y)$ that has continuous second-order partial derivatives in a domain D and satisfies Laplace's equation is said to be **harmonic** in D .
- If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D then the functions $u(x, y)$ and $v(x, y)$ are harmonic functions.
- If a given function $u(x, y)$ is harmonic in a domain D it is sometimes possible to find another function $v(x, y)$ that is harmonic in D such that $u(x, y) + iv(x, y)$ is analytic in D . The function v is called the **harmonic conjugate function** of u .
- To find the harmonic conjugate function of a given function u :
 1. Take the first-order partial derivatives of u with respect to x and y .
 2. If $u(x, y) + iv(x, y)$ is analytic in a domain D then u and v must satisfy the Cauchy-Riemann equations in D from which we can find expressions for $\partial v / \partial x$ and $\partial v / \partial y$.
 3. Integrate $\partial v / \partial x$ with respect to x to get an expression for v with an unknown constant $h(y)$.
 4. Take the first-order partial derivative of v with respect to y , equate it with the other expression for $\partial v / \partial y$, and solve for $h'(y)$.
 5. Integrate $h'(y)$ and substitute the result to find v .

17.6 Exponential and Logarithmic Functions

- The exponential function for complex numbers is defined as

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

- e^z is analytic for all z , i.e. it's an entire function.
- Like its real-valued counterpart,

$$\begin{aligned}\frac{d}{dz}e^z &= e^z, \\ e^{z_1}e^{z_2} &= e^{z_1+z_2},\end{aligned}$$

and

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}.$$

- Since

$$e^{2\pi i} = \cos(2\pi) + i \sin(2\pi) = 1$$

and

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z$$

the complex function $f(z) = e^z$ is **periodic** with complex period $2\pi i$. Because of this complex periodicity an infinite horizontal strip of height 2π contains all possible values for the function. The strip $-\pi < y \leq \pi$ is called the **fundamental region**.

- For $z \neq 0$ and $\theta = \arg z$,

$$\ln z = \log_e |z| + i(\theta + 2n\pi), \quad n = 0, \pm 1, \pm 2, \dots$$

This means there are infinitely many values of the logarithm of a complex number z . This makes sense as the complex exponential is periodic.

- The **principal value** of $\ln z$ is the complex logarithm corresponding to $n = 0$ and $\theta = \text{Arg } z$. It is denoted $\text{Ln } z$.
- Some familiar properties of the real-valued logarithm hold for the complex-valued logarithm, e.g.

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$

and

$$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2,$$

however they don't necessarily hold for the principal value.

- $\text{Ln } z$ is discontinuous and thus not analytic at $z = 0$ because $\ln z$ is undefined at $z = 0$ and on the negative real axis because $\text{Arg } z$ is discontinuous there.

- The derivative of $\text{Ln } z$ is

$$\frac{d}{dz} \text{Ln } z = \frac{1}{z}.$$

- The complex power of a complex number is defined as

$$z^\alpha = e^{\alpha \text{Ln } z}, \quad z \neq 0.$$

In general this is multiple-valued because $\text{Ln } z$ is multiple-valued — only if $\alpha = n$, $n = 0, \pm 1, \pm 2, \dots$ is it single-valued. If $\text{Ln } z$ is replaced with $\ln z$ then we get the **principle value** of z^α .

17.7 Trigonometric and Hyperbolic Functions

Definition 17.7.1 Trigonometric Sine and Cosine

For any complex number $z = x + iy$,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \quad (2)$$

- The other trigonometric functions ($\tan z$, etc.) are defined as usual.
- Because e^{iz} and e^{-iz} are entire functions, $\sin z$ and $\cos z$ are also entire functions.
- $\sin z = 0$ for the real numbers $z = n\pi$, $n \in \mathbb{Z}$ and $\cos z = 0$ for the real numbers $z = (2n+1)\pi/2$, $n \in \mathbb{Z}$. This means that $\tan z$ and $\sec z$ are analytic except at the points where $\cos z = 0$ and $\cot z$ and $\csc z$ are analytic except at the points where $\sin z = 0$.
- The usual derivatives and trigonometric functions are still valid in the complex case.
- $\sin z$ can be expressed as

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

and $\cos z$ can be expressed as

$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$

- The only zeroes of $\sin z$ are the real numbers $z = n\pi$, $n \in \mathbb{Z}$ and the only zeroes of $\cos z$ are the real numbers $z = (2n+1)\pi/2$, $n \in \mathbb{Z}$.

Definition 17.7.2 Hyperbolic Sine and Cosine

For any complex number $z = x + iy$,

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2}. \quad (10)$$

- The complex trigonometric functions can be expressed in terms of the complex hyperbolic functions and vice versa

$$\begin{aligned}\sin z &= -i \sinh(iz), & \cos z &= \cosh(iz) \\ \sinh z &= -i \sin(iz), & \cosh z &= \cos(iz).\end{aligned}$$

- $\sinh z$ can be expressed as

$$\sinh z = \sinh x \cos y + i \cosh x \sin y$$

and $\cosh z$ can be expressed as

$$\cosh z = \cosh x \cos y + i \sinh x \sin y.$$

- The zeroes of $\sinh z$ are $z = n\pi i$, $n \in \mathbb{Z}$ and the zeroes of $\cosh z$ are $z = (2n+1)\pi i/2$, $n \in \mathbb{Z}$.
- $\sin z$ and $\cos z$ are 2π periodic while $\sinh z$ and $\cosh z$ are $2\pi i$ periodic.

17.8 Inverse Trigonometric and Hyperbolic Functions

- Because the complex trigonometric functions are multi-valued, their inverse functions are also multi-valued.
- The definitions of those inverse functions are

$$\begin{aligned}\arcsin z &= -i \ln[iz + (1 - z^2)^{1/2}], \\ \arccos z &= -i \ln[z + i(1 - z^2)^{1/2}], \text{ and} \\ \arctan z &= \frac{i}{2} \ln \frac{i+z}{i-z}.\end{aligned}$$

- The derivatives of the inverse trigonometric functions are

$$\begin{aligned}\frac{d}{dz} \arcsin z &= \frac{1}{(1 - z^2)^{1/2}}, \\ \frac{d}{dz} \arccos z &= \frac{-1}{(1 - z^2)^{1/2}}, \text{ and} \\ \frac{d}{dz} \arctan z &= \frac{1}{1 + z^2}.\end{aligned}$$

- The definitions of the hyperbolic inverse functions and their derivatives are

$$\sinh^{-1} z = \ln [z + (z^2 + 1)^{1/2}]$$

$$\cosh^{-1} z = \ln [z + (z^2 - 1)^{1/2}]$$

$$\tanh^{-1} z = \frac{1}{2} \ln \frac{1+z}{1-z}$$

$$\frac{d}{dz} \sinh^{-1} z = \frac{1}{(z^2 + 1)^{1/2}}$$

$$\frac{d}{dz} \cosh^{-1} z = \frac{1}{(z^2 - 1)^{1/2}}$$

$$\frac{d}{dz} \tanh^{-1} z = \frac{1}{1 - z^2}.$$

18 Integration in the Complex Plane

18.1 Contour Integrals

- In complex variables, a piecewise smooth curve C is called a **contour** or **path**. An integral of a complex function $f(z)$ on C is denoted $\int_C f(z) dz$ or $\oint_C f(z) dz$ if C is closed — this is called a **contour integral** or a **complex integral**.

1. Let $f(z) = u(x, y) + iv(x, y)$ be defined at all points on a smooth curve C defined by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$.
2. Divide C into n subarcs according to the partition $a = t_0 < t_1 < \dots < t_n = b$ of $[a, b]$. The corresponding points on the curve C are $z_0 = x_0 + iy_0 = x(t_0) + iy(t_0)$, $z_1 = x_1 + iy_1 = x(t_1) + iy(t_1)$, \dots , $z_n = x_n + iy_n = x(t_n) + iy(t_n)$. Let $\Delta z_k = z_k - z_{k-1}$, $k = 1, 2, \dots, n$.
3. Let $\|P\|$ be the **norm** of the partition, that is, the maximum value of $|\Delta z_k|$.
4. Choose a sample point $z_k^* = x_k^* + iy_k^*$ on each subarc. See [FIGURE 18.1.1](#).
5. Form the sum $\sum_{k=1}^n f(z_k^*) \Delta z_k$.

Definition 18.1.1 Contour Integral

Let f be defined at points of a smooth curve C defined by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$. The **contour integral** of f along C is

$$\int_C f(z) dz = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(z_k^*) \Delta z_k. \quad (1)$$

Theorem 18.1.1 Evaluation of a Contour Integral

If f is continuous on a smooth curve C given by $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, then

$$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt. \quad (3)$$

- If a complex function f is continuous on a smooth curve C and if $|f(z)| \leq M$ for all z on C , then

$$\left| \int_C f(z) dz \right| \leq ML,$$

where

$$L = \int_a^b |z'(t)| dt$$

is the length of C . This is sometimes called the **ML-inequality**.

- If \mathbf{T} is the unit tangent vector to a positively oriented simple closed curve C then

$$\oint_C f \cdot \mathbf{T} ds = \operatorname{Re} \left(\oint_C \overline{f(z)} dz \right)$$

is called the **circulation** around C and measures the tendency of the flow to rotate the curve C .

- If \mathbf{N} is the normal vector to a positive oriented simple closed curve C then

$$\oint_C f \cdot \mathbf{N} ds = \operatorname{Im} \left(\oint_C \overline{f(z)} dz \right)$$

is called the **net flux** across C and measures the difference between the rates at which fluid enters and exits the region bounded by C .

18.2 Cauchy-Goursat Theorem

- A domain D is said to be **simply connected** if every simple closed contour C lying entirely in D can be shrunk to a point without leaving D , i.e. the domain has no holes in it.
- A domain that is not simply connected is called a **multiply connected domain**. A domain with one hole is called **doubly connected**, a domain with two holes **triply connected**, etc.

Theorem 18.2.1 Cauchy-Goursat Theorem

Suppose a function f is analytic in a simply connected domain D . Then for every simple closed contour C in D , $\oint_C f(z) dz = 0$.

- An alternative way of stating the Cauchy-Goursat Theorem is: if f is analytic at all points on and within a simple closed contour C , then $\oint_C f(z) dz = 0$.
- If D is a double connected domain and C and C_1 are simple closed contours such that C_1 surrounds the “hole” in the domain and is interior to C , then

$$\oint_C f(z) dz = \oint_{C_1} f(z) dz.$$

This is called the principle of **deformation of contours** since C_1 can be considered a continuous deformation of the contour C (or vice versa) under which the value of the integral doesn’t change.

- If z_0 is a constant complex number interior to a simple closed contour C , then

$$\oint_C \frac{dz}{(z - z_0)^n} = \begin{cases} 2\pi i & n = 1 \\ 0 & n \text{ an integer} \neq 1 \end{cases}.$$

Theorem 18.2.2 Cauchy–Goursat Theorem for Multiply Connected Domains

Suppose C, C_1, \dots, C_n are simple closed curves with a positive orientation such that C_1, C_2, \dots, C_n are interior to C but the regions interior to each $C_k, k = 1, 2, \dots, n$, have no points in common. If f is analytic on each contour and at each point interior to C but exterior to all the $C_k, k = 1, 2, \dots, n$, then

$$\oint_C f(z) dz = \sum_{k=1}^n \oint_{C_k} f(z) dz. \quad (6)$$

18.3 Independence of the Path

- Let z_0 and z_1 be points in a domain D . A contour integral $\int_C f(z) dz$ is said to be **independent of the path** if its value is the same for all contours C in D with an initial point z_0 and a terminal point z_1 .
- If f is an analytic function in a simply connected domain D , then $\int_C f(z) dz$ is independent of path C .
- Suppose f is continuous in a domain D . If there exists a function F such that $F'(z) = f(z)$ for each z in D , then F is called the **antiderivative** of f .
- The general antiderivative of a complex function includes a complex integration constant.
- Suppose f is continuous in a domain D and F is an antiderivative of f in D . Then for any contour C in D with initial point z_0 and terminal point z_1 ,

$$\int_C f(z) dz = F(z_1) - F(z_0).$$

- A consequence of the above is that if C is closed, then

$$\oint_C f(z) dz = 0.$$

- If f is analytic in a simply connected domain D , then f has an antiderivative in D ; this, there exists a function F such that $F'(z) = f(z)$ for all z in D .
- Suppose f and g are analytic in a simply connected domain D that contains the contour C . If z_0 and z_1 are the initial and terminal points of C , then the **integration by parts** formula is valid in D :

$$\int_{z_0}^{z_1} f(z)g'(z) dz = f(z)g(z)|_{z_0}^{z_1} - \int_{z_0}^{z_1} f'(z)g(z) dz.$$

18.4 Cauchy's Integral Formulas

Theorem 18.4.1 Cauchy's Integral Formula

Let f be analytic in a simply connected domain D , and let C be a simple closed contour lying entirely within D . If z_0 is any point within C , then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz. \quad (1)$$

- Cauchy's integral formula is useful when a contour integral has the form

$$\oint \frac{f(z)}{z - z_0} dz$$

in which case you know its value is $2\pi i f(z_0)$.

Theorem 18.4.2 Cauchy's Integral Formula for Derivatives

Let f be analytic in a simply connected domain D , and let C be a simple closed contour lying entirely within D . If z_0 is any point interior to C , then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz. \quad (6)$$

- Cauchy's integral formula for derivatives is useful when a contour integral has the form

$$\oint \frac{f(z)}{(z - z_0)^{n+1}} dz$$

in which case you know its value is $\frac{2\pi i}{n!} f^{(n)}(z_0)$.

- **Liouville's theorem** states that the only bounded entire functions are constants.

19 Series and Residues

19.1 Sequences and Series

- A **sequence** is a function whose domain is the set of positive integers, i.e. for each integer $n = 1, 2, 3, \dots$ we assign a complex number z_n .
- If $\lim_{n \rightarrow \infty} z_n = L$ we say the sequence $\{z_n\}$ is **convergent**. In other words, $\{z_n\}$ converges to the number L if, for every positive number ε , an N can be found such that $|z_n - L| < \varepsilon$ whenever $n > N$.
- A sequence $\{z_n\}$ converges to a complex number L if and only if $\operatorname{Re}(z_n)$ converges to $\operatorname{Re}(L)$ and $\operatorname{Im}(z_n)$ converges to $\operatorname{Im}(L)$.
- An **infinite series** of complex numbers

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + \dots$$

is **convergent** if the sequence of partial sums $\{S_n\}$, where

$$S_n = z_1 + z_2 + \dots + z_n$$

converges. If $S_n \rightarrow L$ as $n \rightarrow \infty$, we say that the **sum** of the series is L .

- The sum of the geometric series

$$\sum_{k=1}^{\infty} az^{k-1} = a + az + az^2 + \dots$$

converges to

$$\frac{a}{1 - z}$$

when $|z| < 1$ and diverges otherwise.

- If $\sum_{k=1}^{\infty} z_k$ converges, then $\lim_{n \rightarrow \infty} z_n = 0$.
- If $\lim_{n \rightarrow \infty} z_n \neq 0$ then the series $\sum_{k=1}^{\infty} z_k$ diverges.
- An infinite series $\sum_{k=1}^{\infty} z_k$ is said to be **absolutely convergent** if $\sum_{k=1}^{\infty} |z_k|$ converges. Absolute convergence implies convergence.

Theorem 19.1.4 Ratio Test

Suppose $\sum_{k=1}^{\infty} z_k$ is a series of nonzero complex terms such that

$$\lim_{n \rightarrow \infty} \left| \frac{z_{n+1}}{z_n} \right| = L. \quad (9)$$

- (i) If $L < 1$, then the series converges absolutely.
- (ii) If $L > 1$ or $L = \infty$, then the series diverges.
- (iii) If $L = 1$, the test is inconclusive.

Theorem 19.1.5 Root Test

Suppose $\sum_{k=1}^{\infty} z_k$ is a series of complex terms such that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|z_n|} = L. \quad (10)$$

- (i) If $L < 1$, then the series converges absolutely.
- (ii) If $L > 1$ or $L = \infty$, then the series diverges.
- (iii) If $L = 1$, the test is inconclusive.

- An infinite series of the form

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$$

where the coefficients a_k are complex constants is called a **power series** in $z - z_0$. The power series is said to be **centred at z_0** , and the complex point z_0 is referred to as the **centre** of the series.

- Every complex power series has a **radius of convergence R** where R is a real number. The power series converges for all z within the **circle of convergence** $|z - z_0| < R$ and diverges for $|z - z_0| > R$. The series may converge at some, all, or none of the points on the actual circle of convergence.
- For a power series

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

the ratio test depends only on the coefficients a_k . If

1. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \neq 0$, the radius of convergence is $R = 1/L$;
2. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$, the radius of convergence is ∞ ;
3. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, the radius of convergence is $R = 0$.

19.2 Taylor Series

- A power series $\sum_{k=1}^{\infty} a_k (z - z_0)^k$ has a radius of convergence R . For each complex number z within the circle of convergence, when substituted into the power series it converges to a unique value L . This defines a function f that maps each z to its corresponding L .

Theorem 19.2.1 Continuity

A power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ represents a continuous function f within its circle of convergence $|z - z_0| = R, R \neq 0$.

Theorem 19.2.2 Term-by-Term Integration

A power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ can be integrated term by term within its circle of convergence $|z - z_0| = R, R \neq 0$, for every contour C lying entirely within the circle of convergence.

Theorem 19.2.3 Term-by-Term Differentiation

A power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ can be differentiated term by term within its circle of convergence $|z - z_0| = R, R \neq 0$.

Theorem 19.2.4 Taylor's Theorem

Let f be analytic within a domain D and let z_0 be a point in D . Then f has the series representation

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \quad (8)$$

valid for the largest circle C with center at z_0 and radius R that lies entirely within D .

- The radius of convergence of a Taylor series is the distance from the centre z_0 to the nearest isolated singularity: a point at which the series fails to be analytic but is analytic at all points in some neighborhood of the point.

19.3 Laurent Series

- If a complex function f fails to be analytic at a point $z = z_0$, then this point is said to be a **singularity** or a **singular point** of the function.
- Suppose $z = z_0$ is a singularity of a complex function f . It is said to be an **isolated singularity** if there exists some **deleted neighborhood**, or **punctured open disk**, $0 < |z - z_0| < R$ of z_0 in which f is analytic.
- A singular point $z = z_0$ of a complex function f is said to be **nonisolated** if every neighborhood of z_0 contains at least one singularity of f other than z_0 .

Theorem 19.3.1 Laurent's Theorem

Let f be analytic within the annular domain D defined by $r < |z - z_0| < R$. Then f has the series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \quad (3)$$

valid for $r < |z - z_0| < R$. The coefficients a_k are given by

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} ds, \quad k = 0, \pm 1, \pm 2, \dots, \quad (4)$$

where C is a simple closed curve that lies entirely within D and has z_0 in its interior (see [FIGURE 19.3.1](#)).

- Under Laurent's theorem, the part of $f(z)$ with negative powers of $z - z_0$ is called the **principle part** and the part with positive powers is called the **analytic part**.
- The coefficient formula of theorem 19.3.1 isn't used often. Generally f is decomposed into functions for which the series are known (e.g. $\cos z$, e^z , etc.), and those series are combined to find the Laurent series.

19.4 Zeroes and Poles

- An isolated singularity $z = z_0$ can be categorised based on the number of terms contained in the principal part of its Laurent expansion (the part with negative powers).
 - If the principal part is zero, i.e. the Laurent expansion consists only of parts with nonnegative powers, then $z = z_0$ is called a **removable singularity**.
 - If the principal part contains a finite number of nonzero terms, then $z = z_0$ is called a **pole**. If the last nonzero coefficient of the principal part is a_{-n} , $n \geq 1$ then we say that $z = z_0$ is a **pole of order n** . A pole of order 1 is called a **simple pole**.
 - If the principal part contains infinitely many nonzero terms, then $z = z_0$ is called an **essential singularity**.

$z = z_0$	Laurent Series
Removable singularity	$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$
Pole of order n	$\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \dots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \dots$
Simple pole	$\frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$
Essential singularity	$\dots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$

- A point z_0 is said to be a **zero** of a function f if $f(z_0) = 0$.
- A point z_0 is said to be a **zero of order n** of a function f if $f(z_0) = 0$, $f'(z_0) = 0$, \dots , $f^{(n-1)}(z_0) = 0$ but $f^{(n)}(z_0) \neq 0$.

Theorem 19.4.1 Pole of Order n

If the functions f and g are analytic at $z = z_0$ and f has a zero of order n at $z = z_0$ and $g(z_0) \neq 0$, then the function $F(z) = g(z)/f(z)$ has a pole of order n at $z = z_0$.

- Theorem 19.4.1 can sometimes be used to determine the poles of a function by inspection, e.g. in

$$F(z) = \frac{2z + 5}{z - 1}$$

the denominator has a zero of order 1 at $z = 1$ and the numerator is nonzero at that point so F has a simple pole at $z = 1$.

19.5 Residues and Residue Theorem

- If a complex function f has an isolated singularity at a point z_0 then it has a Laurent series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k.$$

The coefficient a_{-1} of $1/(z - z_0)$ is called the **residue** of f at z_0 and is denoted

$$a_{-1} = \text{Res}(f(z), z_0).$$

Theorem 19.5.1 Residue at a Simple Pole

If f has a simple pole at $z = z_0$, then

$$\text{Res}(f(z), z_0) = \lim_{z \rightarrow z_0} (z - z_0)f(z). \quad (1)$$

Theorem 19.5.2 Residue at a Pole of Order n

If f has a pole of order n at $z = z_0$, then

$$\text{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z). \quad (2)$$

- Suppose a complex function f can be written as a quotient $f(z) = g(z)/h(z)$ where g and h are analytic at $z = z_0$. If $g(z_0) \neq 0$ and h has a zero of order 1 at z_0 , then f has a simple pole at $z = z_0$ and

$$\text{Res}(f(z), z_0) = \frac{g(z)}{h'(z)}.$$

Theorem 19.5.3 Cauchy's Residue Theorem

Let D be a simply connected domain and C a simple closed contour lying entirely within D . If a function f is analytic on and within C , except at a finite number of singular points z_1, z_2, \dots, z_n within C , then

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f(z), z_k). \quad (5)$$

- L'Hôpital's rule is valid for complex analysis.

19.6 Evaluation of Real Integrals

- An integral of the form

$$\int_0^{2\pi} F(\cos \theta, \sin \theta) d\theta$$

where F is a rational function can be evaluated by converting it to a complex integral where the contour is the unit circle centred at the origin

$$\oint_C F\left(\frac{1}{2}(z + z^{-1}), \frac{1}{2i}(z - z^{-1})\right) \frac{dz}{iz}$$

where C is $|z| = 1$.

- An improper integral of the form $\int_{-\infty}^{\infty} f(x) dx$ is defined in terms of two limits

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{r \rightarrow \infty} \int_{-r}^0 f(x) dx + \lim_{R \rightarrow \infty} \int_0^R f(x) dx.$$

If both limits exist, the integral is said to be **convergent**. If one or both of the limits fail to exist the integral is said to be **divergent**.

- If we know a priori that an improper integral converges we can evaluate it with a single limit

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx.$$

However, this limit may exist even if the improper integral is divergent in which case it is called the **Cauchy principal value** and is denoted

$$\text{P.V.} \int_{-\infty}^{\infty} f(x) dx.$$

- An integral of the form

$$\int_{-\infty}^{\infty} f(x) dx$$

where $f(x) = P(x)/Q(x)$ is continuous on $(-\infty, \infty)$ can be evaluated by replacing x with the complex variable z and integrating over a closed contour C consisting of the interval $[-R, R]$ on the real axis and a semicircle C_R of radius large enough to enclose all the poles of $f(z) = P(z)/Q(z)$ in the upper half-plane $\text{Re}(z) > 0$. By Cauchy's residue theorem we have

$$\oint_C f(z) dz = \int_{C_R} f(z) dz + \int_{-R}^R f(x) dx = 2\pi i \sum_{k=1}^n \text{Res}(f(z), z_k)$$

and if we assume $\int_{C_R} f(z) dz \rightarrow 0$ as $R \rightarrow \infty$ we get

$$\text{P.V.} \int_{-\infty}^{\infty} f(x) dx = 2\pi i \sum_{k=1}^n \text{Res}(f(z), z_k).$$

Theorem 19.6.1 Behavior of Integral as $R \rightarrow \infty$

Suppose $f(z) = P(z)/Q(z)$, where the degree of $P(z)$ is n and the degree of $Q(z)$ is $m \geq n + 2$. If C_R is a semicircular contour $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, then $\int_{C_R} f(z) dz \rightarrow 0$ as $R \rightarrow \infty$.

- Integrals of the form $\int_{-\infty}^{\infty} f(x) \cos \alpha x dx$ and $\int_{-\infty}^{\infty} f(x) \sin \alpha x dx$, $\alpha > 0$ are referred to as **Fourier integrals**. They appear as the real and imaginary parts in the improper integral $\int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$.
- When $f(x) = P(x)/Q(x)$ is continuous on $(-\infty, \infty)$ we can evaluate both forms of Fourier integrals at the same time by considering the integral $\int_C f(z) e^{i\alpha z} dz$ where $\alpha > 0$ and the contour C consists of the interval $[-R, R]$ on the real axis and a semicircular contour C_R with radius large enough to enclose the poles of $f(z)$ in the upper half-plane.

Theorem 19.6.2 Behavior of Integral as $R \rightarrow \infty$

Suppose $f(z) = P(z)/Q(z)$, where the degree of $P(z)$ is n and the degree of $Q(z)$ is $m \geq n + 1$. If C_R is a semicircular contour $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, and $\alpha > 0$, then $\int_{C_R} (P(z)/Q(z))e^{i\alpha z} dz \rightarrow 0$ as $R \rightarrow \infty$.

- The above approaches to evaluating integrals of the form $\int_{-\infty}^{\infty} f(x) dx$, $\int_{-\infty}^{\infty} f(x) \cos \alpha x dx$, and $\int_{-\infty}^{\infty} f(x) \sin \alpha x dx$ all assume $f(x)$ is continuous on $(-\infty, \infty)$. If that's not the case and $f(x)$ has a pole at $z = c$ we instead use an **indented contour** where a semicircular contour centred at $z = c$ is included to bypass the pole.

Theorem 19.6.3 Behavior of Integral as $r \rightarrow 0$

Suppose f has a simple pole $z = c$ on the real axis. If C_r is the contour defined by $z = c + re^{i\theta}$, $0 \leq \theta \leq \pi$, then

$$\lim_{r \rightarrow 0} \int_{C_r} f(z) dz = \pi i \operatorname{Res}(f(z), c).$$

- Using the above theorem we can evaluate an integral where $f(x)$ has a pole on the real axis at $z = c$ by replacing x with the complex variable z and integrating over a closed contour C consisting of the interval $[-R, c - r]$, a positively-oriented semicircle C_r of radius r centred at $z = c$, the interval $[c + r, R]$, and a semicircle C_R of radius R centred at $z = 0$. By Cauchy's residue theorem we have

$$\oint_C = \int_{-R}^{c-r} + \int_{C_r} + \int_{c+r}^R + \int_{C_R} = 2\pi i \sum_{k=1}^n \operatorname{Res}(f(z), z_k)$$

and by theorem 19.6.3 as we take the limit $R \rightarrow \infty$ and $r \rightarrow 0$ we get

$$\text{P.V.} \int_{-\infty}^{\infty} f(x) dx = \pi i \operatorname{Res}(f(z), c) + 2\pi i \sum_{k=1}^n \operatorname{Res}(f(z), z_k).$$