Classical Mechanics by John R. Taylor Problems

Chris Doble

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Contents

1 Newton's Laws of Motion

1.1

$$
\begin{aligned}
\mathbf{b} + \mathbf{c} &= 2\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}} \\
5\mathbf{b} + 2\mathbf{x} &= 7\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 2\hat{\mathbf{z}} \\
\mathbf{b} \cdot \mathbf{c} &= 1 \\
\mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \\
&= \hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}}
\end{aligned}
$$

1.5

$$\begin{aligned}
\mathbf{v}_{\text{body}} &= \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}} \\
\mathbf{v}_{\text{face}} &= \hat{\mathbf{x}} + \hat{\mathbf{z}} \\
\mathbf{v}_{\text{body}} \cdot \mathbf{v}_{\text{face}} &= v_{\text{body}} v_{\text{face}} \cos\theta \\
2 &= \sqrt{6}\cos\theta \\
\cos\theta &= \frac{2}{\sqrt{6}} \\
\theta &= \arccos\frac{2}{\sqrt{6}} \\
&= 35.26^\circ
\end{aligned}$$

1.11

The particle moves counterclockwise in an ellipse of width $2b$ and height $2c$. The angular speed is ω.

1.23

$$\begin{aligned}
\mathbf{v} &= v\cos\theta\frac{\mathbf{b}}{b} - v\sin\theta\frac{\mathbf{b}\times\mathbf{c}}{bc} \\
&= \frac{\lambda}{b}\frac{\mathbf{b}}{b} - \frac{c}{b}\frac{\mathbf{b}\times\mathbf{c}}{bc} \\
&= \frac{\lambda\mathbf{b} - \mathbf{b}\times\mathbf{c}}{b^2}
\end{aligned}$$

1.25

$$\begin{aligned}
\frac{df}{dt} &= -3f \\
\frac{1}{f}\frac{df}{dt} &= -3 \\
\ln f &= -3t + c \\
f &= ce^{-3t}
\end{aligned}$$

One constant.

1.35

$$
\begin{aligned}
F_x &= 0 \\
ma_x &= 0 \\
a_x &= 0 \\
v_x &= c_1 \\
&= v_o \cos\theta \\
r_x &= v_o \cos(\theta)t + c_2 \\
&= v_o \cos(\theta)t
\end{aligned}
$$

$$
\begin{aligned}
F_y &= 0 \\
ma_y &= 0 \\
a_y &= 0 \\
v_y &= c_3 \\
v_y &= 0 \\
r_y &= c_4 \\
r_y &= 0
\end{aligned}
$$

$$
\begin{aligned}
F_z &= -mg \\
ma_z &= -mg \\
a_z &= -g \\
v_z &= -gt + c_5 \\
&= v_o \sin\theta - gt \\
r_z &= v_o \sin(\theta)t - \frac{1}{2}gt^2 + c_6 \\
&= v_o \sin(\theta)t - \frac{1}{2}gt^2
\end{aligned}
$$

$$
\begin{aligned}
0 &= v_o \sin(\theta)t - \frac{1}{2}gt^2 \\
t &= \frac{2\sin(\theta)v_o}{g} \\
r_x &= v_o \cos(\theta)t \\
&= \frac{2\cos(\theta)\sin(\theta)v_o^2}{g} \\
&= \frac{\sin(2\theta)v_o^2}{g}
\end{aligned}
$$

1.37

(a)

$$
\begin{aligned}
F &= -mg\sin\theta \\
ma &= -mg\sin\theta \\
a &= -g\sin\theta \\
v &= c_1 - gt\sin\theta \\
&= v_o - gt\sin\theta \\
x &= v_o t - \frac{1}{2}gt^2\sin\theta
\end{aligned}
$$

(b)

$$t = \frac{2v_o}{g\sin\theta}$$

1.39

$$
\begin{aligned}
F_x &= -mg\sin\phi \\
ma_x &= -mg\sin\phi \\
a_x &= -g\sin\phi \\
v_x &= c_1 - gt\sin\phi \\
&= v_o\cos\theta - gt\sin\phi \\
r_x &= v_o t\cos\theta - \frac{1}{2}gt^2\sin\phi + c_2 \\
&= v_o t\cos\theta - \frac{1}{2}gt^2\sin\phi
\end{aligned}
$$

$$
\begin{aligned}
F_y &= -mg\cos\phi \\
ma_y &= -mg\cos\phi \\
a_y &= -g\cos\phi \\
v_y &= c_3 - gt\cos\phi \\
&= v_o\sin\theta - gt\cos\phi \\
r_y &= v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi + c_4 \\
&= v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi
\end{aligned}
$$

$$
\begin{aligned}
0 &= v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi \\
t &= \frac{2v_o\sin\theta}{g\cos\phi}
\end{aligned}
$$

$$
\begin{aligned}
r_x &= \frac{2v_o^2\cos\theta\sec\phi\sin\theta}{g} - \frac{2v_o^2\sec\phi\sin^2\theta\tan\phi}{g} \\
&= \frac{2v_o^2\sin\theta(\cos\theta\cos\phi - \sin\theta\sin\phi)}{g\cos^2\phi} \\
&= \frac{2v_o^2\sin\theta\cos(\theta+\phi)}{g\cos^2\phi}
\end{aligned}
$$

$$
\begin{aligned}
\frac{dr_x}{d\theta} &= \frac{2v_o^2}{g\cos^2\phi}[\cos\theta\cos(\theta+\phi) - \sin\theta\sin(\theta+\phi)] \\
&= \frac{2v_o^2\cos(2\theta+\phi)}{g\cos^2\phi} \\
0 &= \frac{2v_o^2\cos(2\theta+\phi)}{g\cos^2\phi} \\
&= \cos(2\theta+\phi) \\
2\theta+\phi &= \frac{\pi}{2} \\
\theta &= \frac{\pi}{4} - \frac{\phi}{2} \\
r_{x,\max} &= \frac{2v_o^2\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\phi}{2}+\phi\right)}{g\cos^2\phi} \\
&= \frac{v_o^2(1-\sin\phi)}{g\cos^2\phi} \\
&= \frac{v_o^2}{g(1+\sin\phi)}
\end{aligned}
$$

1.41

$$
\begin{aligned}
F &= ma \\
T &= m\frac{v^2}{R} \\
&= m\frac{(\omega R)^2}{R} \\
&= m\omega^2 R
\end{aligned}
$$

1.47

(a)

$$
\begin{aligned}
\rho &= \sqrt{x^2+y^2} \\
\phi &= \arctan\frac{y}{x} \\
z &= z
\end{aligned}
$$

ρ is the distance of P from the z-axis.

The use of r may be unfortunate because it suggests it's the distance of P from the origin.

(b) $\hat{\boldsymbol{\rho}}$ points away from the z-axis, $\hat{\boldsymbol{\phi}}$ points counter-clockwise around the z-axis, and $\hat{\mathbf{z}}$ points in the positive z direction.

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}} + \sqrt{x^2 + y^2}\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$$

(c)

$$\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{r}}{dt} \\
&= \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\frac{d\hat{\boldsymbol{\rho}}}{dt} + \dot{z}\hat{\mathbf{z}} + z\frac{d\hat{\mathbf{z}}}{dt} \\
&= \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}} \\
\mathbf{a} &= \frac{d\mathbf{v}}{dt} \\
&= \ddot{\rho}\hat{\boldsymbol{\rho}} + \dot{\rho}\frac{d\hat{\boldsymbol{\rho}}}{dt} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}} + \rho\ddot{\phi}\hat{\boldsymbol{\phi}} + \rho\dot{\phi}\frac{d\hat{\boldsymbol{\phi}}}{dt} + \ddot{z}\hat{\mathbf{z}} \\
&= \ddot{\rho}\hat{\boldsymbol{\rho}} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}} + \rho\ddot{\phi}\hat{\boldsymbol{\phi}} - \rho\dot{\phi}^2\hat{\boldsymbol{\rho}} + \ddot{z}\hat{\mathbf{z}} \\
&= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\boldsymbol{\rho}} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\mathbf{z}}
\end{aligned}$$

2 Projectiles and Charged Particles

2.1

$$\begin{aligned}
1 &= (1.6 \times 10^3)Dv \\
v &= \frac{1}{(1.6 \times 10^3)D} \\
&= 8.9\,\mathrm{mm/s}
\end{aligned}$$

When $v \gg 1\,\mathrm{cm/s}$ the drag force can be treated as purely quadratic. For a beach ball this becomes $v \gg 1\,\mathrm{mm/s}$.

2.3

(a)

$$\begin{aligned}
\frac{f_{\text{quad}}}{f_{\text{lin}}} &= \frac{(1/4)\rho A v^2}{3\pi\eta D v} \\
&= \frac{\rho\pi\left(\frac{D}{2}\right)^2 v}{12\pi\eta D} \\
&= \frac{\rho D v}{48\eta} \\
&= \frac{R}{48}
\end{aligned}$$

(b)

$$R = \frac{Dv\rho}{\eta} \approx 0.01$$

2.5

$$\begin{aligned} v_y(t) &= v_{\text{ter}} + (v_{\text{yo}} - v_{\text{ter}})e^{-t/\tau} \\ &= v_{\text{ter}} + (2v_{\text{ter}} - v_{\text{ter}})e^{-t/\tau} \\ &= v_{\text{ter}}(1 + e^{-t/\tau}) \end{aligned}$$

The velocity starts at $2v_{\text{ter}}$ and asymptotically approaches v_{ter}.

2.7

$$\begin{aligned} F &= F(v) \\ m\dot{v} &= F(v) \\ m\frac{dv}{F(v)} &= dt \\ t &= \int_{v_{\text{o}}}^{v} m\frac{dv'}{F(v')} \end{aligned}$$

$$\begin{aligned} F &= F(v) \\ m\dot{v} &= F_{\text{o}} \\ v &= \frac{F_{\text{o}}}{m}t + c \end{aligned}$$

2.11

(a)

$$\begin{aligned}
m\dot{v} &= -mg - bv \\
\dot{v} &= -g - kv \\
\frac{1}{-g-kv}\dot{v} &= 1 \\
-\frac{1}{k}\ln(-g-kv) &= t + c \\
\ln(-g-kv) &= c - \frac{t}{\tau} \\
-g - kv &= Ae^{-t/\tau} \\
v &= \tau(-g - Ae^{-t/\tau}) \\
&= -v_{\text{ter}} - \tau A e^{-t/\tau}
\end{aligned}$$

$$\begin{aligned}
v_{\text{o}} &= -v_{\text{ter}} - \tau A \\
A &= -k(v_{\text{o}} + v_{\text{ter}})
\end{aligned}$$

$$v = -v_{\text{ter}} + (v_{\text{o}} + v_{\text{ter}})e^{-t/\tau}$$

$$y = -v_{\text{ter}}t - \tau(v_{\text{o}} + v_{\text{ter}})e^{-t/\tau} + c$$

$$\begin{aligned}
0 &= -\tau(v_{\text{o}} + v_{\text{ter}}) + c \\
c &= \tau(v_{\text{o}} + v_{\text{ter}})
\end{aligned}$$

$$y = -v_{\text{ter}}t + \tau(v_{\text{o}} + v_{\text{ter}})(1 - e^{-t/\tau})$$

(b)

$$\begin{aligned}
0 &= -v_{\text{ter}} + (v_{\text{o}} + v_{\text{ter}})e^{-t/\tau} \\
e^{-t/\tau} &= \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} \\
-\frac{t}{\tau} &= \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} \\
t &= -\tau \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}}
\end{aligned}$$

$$\begin{aligned}
y_{\max} &= -v_{\text{ter}} \left(-\tau \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} \right) + \tau (v_{\text{o}} + v_{\text{ter}}) \left(1 - \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} \right) \\
&= \tau v_{\text{ter}} \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} + \tau (v_{\text{o}} + v_{\text{ter}} - v_{\text{ter}}) \\
&= \tau \left(v_{\text{o}} + v_{\text{ter}} \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} \right) \\
&= \tau \left[v_{\text{o}} - v_{\text{ter}} \ln \left(1 + \frac{v_{\text{o}}}{v_{\text{ter}}} \right) \right]
\end{aligned}$$

(c)

$$\begin{aligned}
y_{\max} &= \tau \left[v_{\text{o}} - v_{\text{ter}} \ln \left(1 + \frac{v_{\text{o}}}{v_{\text{ter}}} \right) \right] \\
&= \tau \left[v_{\text{o}} - g\tau \ln \left(1 + \frac{v_{\text{o}}}{g\tau} \right) \right] \\
&\approx \tau \left\{ v_{\text{o}} - g\tau \left[\frac{v_{\text{o}}}{g\tau} - \frac{1}{2} \left(\frac{v_{\text{o}}}{g\tau} \right)^2 \right] \right\} \\
&= \tau \left(v_{\text{o}} - v_{\text{o}} + \frac{1}{2} \frac{v_{\text{o}}^2}{g\tau} \right) \\
&= \frac{1}{2} \frac{v_{\text{o}}^2}{g}
\end{aligned}$$

2.13

$$\begin{aligned}
v^2 &= \frac{2}{m}\int_{x_0}^{x} -kx'\,dx' \\
&= -\frac{2k}{m}\left(\frac{1}{2}x^2 - \frac{1}{2}x_0^2\right) \\
&= -\frac{k}{m}(x^2 - x_0^2) \\
v &= \sqrt{\frac{k}{m}(x_0^2 - x^2)} \\
&= \omega\sqrt{x_0^2 - x^2}
\end{aligned}$$

$$\begin{aligned}
\int_{x_0}^{x} \frac{1}{\sqrt{x_0^2 - x'^2}}\,dx' &= \int_0^t \omega\,dt \\
\arctan\frac{x}{\sqrt{x_0^2 - x^2}} - \arctan\frac{x_0}{\sqrt{x_0^2 - x_0^2}} &= \omega t \\
\arctan\frac{x}{\sqrt{x_0^2 - x^2}} &= \omega t + \frac{\pi}{2} \\
\frac{x}{\sqrt{x_0^2 - x^2}} &= \tan\left(\omega t + \frac{\pi}{2}\right) \\
&= -\cot\omega t \\
\frac{\sqrt{x_0^2 - x^2}}{x} &= -\tan\omega t \\
\sqrt{x_0^2 - x^2} &= -x\tan\omega t \\
x_0^2 - x^2 &= x^2\tan^2\omega t \\
x^2 &= \frac{x_0^2}{1 + \tan^2\omega t} \\
&= \frac{x_0^2\cos^2\omega t}{\cos^2\omega t + \sin^2\omega t} \\
&= x_0^2\cos^2\omega t \\
x &= x_0\cos\omega t
\end{aligned}$$