Vibrations and Waves by A. P. French Problems

Chris Doble

May 2023

Contents

1 Periodic motions

1.4

(a)

$$\begin{aligned} z &= Ae^{j\theta} \\ dz &= jAe^{j\theta}\, d\theta \\ &= jz\, d\theta \end{aligned}$$

The motion of the point is always perpendicular to its position.

(b)

$$\begin{aligned} |2 + j\sqrt{3}| &= \sqrt{2^2 + \sqrt{3}^2} \\ &= \sqrt{7} \\ \arg(2 + j\sqrt{3}) &= \arctan \frac{\sqrt{3}}{2} \\ &= 41^\circ \end{aligned}$$

$$\begin{aligned} (2 - j\sqrt{3})^2 &= 4 - j4\sqrt{3} - 3 \\ &= 1 - j4\sqrt{3} \\ |1 - j4\sqrt{3}| &= \sqrt{1^2 + (4\sqrt{3})^2} \\ &= 7 \\ \arg(1 - j4\sqrt{3}) &= -\arctan 4\sqrt{3} \end{aligned}$$

1.9

$$
\begin{aligned}
\cos\theta + j\sin\theta &= e^{j\theta} \\
\cos\frac{\pi}{2} + j\sin\frac{\pi}{2} &= e^{j\frac{\pi}{2}} \\
j &= e^{j\frac{\pi}{2}} \\
j^j &= (e^{j\frac{\pi}{2}})^j \\
&= e^{-\frac{\pi}{2}} \\
&\approx 0.208
\end{aligned}
$$

Yes, I would be willing to pay 20 cents because I could sell it to the mathematician and gain 0.8 cents.

1.10

$$
\begin{aligned}
y &= A\cos kx + B\sin kx \\
\frac{dy}{dx} &= -Ak\sin kx + Bk\cos kx \\
\frac{d^2y}{dx^2} &= -Ak^2\cos kx - Bk^2\sin kx \\
&= -k^2 y
\end{aligned}
$$

$$
\begin{aligned}
C &= \sqrt{A^2 + B^2} \\
\alpha &= \arctan\left(-\frac{B}{A}\right) \\
y &= C\cos(kx + \alpha) \\
&= C\,\mathrm{Re}[e^{j(kx+\alpha)}] \\
&= Re[(Ce^{j\alpha})e^{jkx}]
\end{aligned}
$$

1.11

(a)

$$
\begin{aligned}
x &= A\cos(\omega t + \alpha) \\
A &= 5\,\mathrm{cm} \\
f &= 1\,\mathrm{Hz} \\
\omega &= 2\pi f \\
&= 2\pi\,\mathrm{rad/s} \\
\alpha &= \pm\frac{\pi}{2}
\end{aligned}
$$

(b)

$$\begin{aligned}
x\left(\frac{8}{3}\right) &= 5\cos\left(2\pi\frac{8}{3}+\alpha\right)\\
&= \pm 4.33\,\mathrm{cm}\\
\frac{dx}{dt} &= -A\omega\sin(\omega t+\alpha)\\
\frac{dx}{dt}\left(\frac{8}{3}\right) &= \pm 15.7\,\mathrm{cm/s}\\
\frac{d^2x}{dt^2} &= -A\omega^2\cos(\omega t+\alpha)\\
\frac{d^2x}{dt^2}\left(\frac{8}{3}\right) &= \mp 171\,\mathrm{cm/s^2}
\end{aligned}$$

1.12

(a)

$$\begin{aligned}
v &= 50\,\mathrm{cm/s}\\
T &= 6\,\mathrm{s}\\
\theta_0 &= 30^\circ\\
c &= 300\,\mathrm{cm}\\
A &= \frac{c}{2\pi}\\
&= \frac{150}{\pi}\,\mathrm{cm}\\
\omega &= \frac{2\pi}{T}\\
&= \frac{\pi}{3}\,\mathrm{rad/s}\\
\alpha &= \frac{\pi}{6}\,\mathrm{rad}\\
x &= \frac{150}{\pi}\cos\left(\frac{\pi}{3}t+\frac{\pi}{6}\right)
\end{aligned}$$

(b)

$$\begin{aligned}
x(2\,\mathrm{s}) &= -41.3\,\mathrm{cm}\\
\frac{dx}{dt} &= -50\sin\left(\frac{\pi}{3}t+\frac{\pi}{6}\right)\\
\frac{dx}{dt}(2\,\mathrm{s}) &= -25\,\mathrm{cm/s}\\
\frac{d^2x}{dt^2} &= -\frac{50\pi}{3}\cos\left(\frac{\pi}{3}t+\frac{\pi}{6}\right)\\
\frac{d^2x}{dt^2}(2\,\mathrm{s}) &= 45\,\mathrm{cm/s^2}
\end{aligned}$$

2 The superposition of periodic motions

2.1

(a)

$$\begin{aligned} z &= \sin \omega t + \cos \omega t \\ &= \sqrt{2} \cos\left(\omega t - \frac{\pi}{4}\right) \\ &= \sqrt{2} e^{j\left(\omega t - \frac{\pi}{4}\right)} \end{aligned}$$

(b)

$$\begin{aligned} z &= \cos(\omega t - \pi/3) - \cos \omega t \\ &= \cos \omega t \cos \frac{\pi}{3} + \sin \omega t \sin \frac{\pi}{3} - \cos \omega t \\ &= -\frac{1}{2} \cos \omega t + \frac{\sqrt{3}}{2} \sin \omega t \\ &= \cos(\omega t + 2\pi/3) \\ &= e^{j(\omega t + 2\pi/3)} \end{aligned}$$

(c)

$$\begin{aligned} z &= 3 \cos \omega t + 2 \sin \omega t \\ &= \sqrt{13} \cos(\omega t + \arctan -2/3) \end{aligned}$$

(d)

$$\begin{aligned} z &= \sin \omega t - 2 \cos(\omega t - \pi/4) + \cos \omega t \\ &= \sin \omega t - 2(\cos \omega t \cos \pi/4 + \sin \omega t \sin \pi/4) + \cos \omega t \\ &= \sin \omega t - \sqrt{2} \cos \omega t - \sqrt{2} \sin \omega t + \cos \omega t \\ &= (1 - \sqrt{2}) \cos \omega t + (1 - \sqrt{2}) \sin \omega t \\ &= (1 - \sqrt{2}) \sqrt{2} \cos(\omega t - \pi/4) \\ &= (\sqrt{2} - 2) \cos(\omega t - \pi/4) \\ &= (2 - \sqrt{2}) \cos(\omega t + 3\pi/4) \end{aligned}$$

2.2

$$
\begin{aligned}
x &= A_1\cos\omega t + A_2\cos(\omega t+\alpha_1) + A_3\cos(\omega t+\alpha_1+\alpha_2)\\
&= A_1\cos\omega t + A_2(\cos\omega t\cos\alpha_1 - \sin\omega t\sin\alpha_1)\\
&\qquad + A_3(\cos\omega t\cos(\alpha_1+\alpha_2) - \sin\omega t\sin(\alpha_1+\alpha_2))\\
&= (A_1 + A_2\cos\alpha_1 + A_3\cos(\alpha_1+\alpha_2))\cos\omega t\\
&\qquad - (A_2\sin\alpha_1 + A_3\sin(\alpha_1+\alpha_2))\sin\omega t\\
A &= \sqrt{(A_1 + A_2\cos\alpha_1 + A_3\cos(\alpha_1+\alpha_2))^2 + (A_2\sin\alpha_1 + A_3\sin(\alpha_1+\alpha_2))^2}\\
&\approx 0.52\,\mathrm{mm}\\
\alpha &= \arctan\frac{A_2\sin\alpha_1 + A_3\sin(\alpha_1+\alpha_2)}{A_1 + A_2\cos\alpha_1 + A_3\cos(\alpha_1+\alpha_2)}\\
&\approx 0.59\,\mathrm{rad}\\
&\approx 34^\circ
\end{aligned}
$$

2.3

The equation of motion is

$$
x = 2A\cos\left(\frac{12\pi - 10\pi}{2}t\right)\cos\left(\frac{12\pi + 10\pi}{2}t\right)
$$

with the variation in amplitude given by the term

$$
2A\cos\pi t
$$

so the beat period is 1 s.

2.4

(a)

$$
\omega = 2\pi, \mathrm{rad/s} \Rightarrow f = 1\,\mathrm{Hz}
$$

(b)

$$
\omega = \frac{25\pi}{2}\,\mathrm{rad/s} \Rightarrow f = \frac{25}{4}\,\mathrm{Hz}
$$

(c)

$$
\omega = \frac{3+\pi}{2}\,\mathrm{rad/s} \Rightarrow f = \frac{3+\pi}{4\pi}\,\mathrm{Hz}
$$

3 The free vibrations of physical systems

3.1

$$
\begin{aligned}
F &= -kx \\
ma &= -kx \\
k &= -\frac{ma}{x} \\
&= 4.0 \times 10^{-5}\,\mathrm{N/m}
\end{aligned}
$$

3.2

(a)

$$T_0 = 2\pi\sqrt{\frac{m}{k}}$$

(b) (i)

$$
\begin{aligned}
mx'' &= -2kx \\
x'' &= -\frac{2k}{m}x \\
T &= 2\pi\sqrt{\frac{m}{2k}} \\
&= \frac{T_0}{\sqrt{2}}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
mx'' &= -k\frac{x}{2} \\
x'' &= -\frac{k}{2m}x \\
T &= 2\pi\sqrt{\frac{2m}{k}} \\
&= \sqrt{2}T_0
\end{aligned}
$$

3.3

(a)

$$\begin{aligned}
y &= A\cos\omega t \\
y' &= -\omega A\sin\omega t \\
y'' &= -\omega^2 A\cos\omega t \\
g &= \omega^2 A\cos\omega t \\
\omega t &= \arccos\frac{g}{\omega^2 A} \\
t &= \frac{1}{\omega}\arccos\frac{g}{\omega^2 A} \\
y &= A\cos\arccos\frac{g}{\omega^2 A} \\
&= \frac{g}{\omega^2} \\
&= 2.5\,\mathrm{cm}
\end{aligned}$$

(b)

$$\begin{aligned}
v &= -\omega A\sin\omega t \\
&= -\omega A\sin\arccos\frac{g}{\omega^2 A} \\
&\approx 0.87\,\mathrm{m/s} \\
\frac{1}{2}mv^2 &= mgh \\
h &= \frac{v^2}{2g} \\
&\approx 3.8\,\mathrm{cm} \\
\Delta h &\approx 1.3\,\mathrm{cm}
\end{aligned}$$

3.4

(a)

$$\begin{aligned}
my'' &= -g\rho Ay \\
y'' &= -\frac{g\rho A}{m}y \\
\omega &= \sqrt{\frac{g\rho A}{m}} \\
&= \sqrt{\frac{g}{l}}
\end{aligned}$$

3.5

$$T = 2\pi\sqrt{\frac{2L}{3g}}$$

3.6

$$T = 2\pi\sqrt{\frac{d}{g}}$$

3.8

(a)

$$\begin{aligned} mg &= \frac{AY}{l_0}x \\ x &= \frac{mgl_0}{AY} \\ &= \frac{mgl_0}{\pi(d/2)^2 Y} \\ &= 0.25\,\text{mm} \end{aligned}$$

(b)

$$\begin{aligned} F_u &= u\pi(d/2)^2 \\ &\approx 215.98\,\text{N} \\ k &= \frac{AY}{L} \\ &= \frac{\pi(d/2)^2 Y}{L} \\ &= \frac{\pi d^2 Y}{4L} \\ &\approx 19\,634.95\,\text{N/m} \\ F_u &= kx_u \\ x_u &= \frac{F_u}{k} \\ &\approx 1.1\,\text{cm} \\ mgh &= \frac{1}{2}\frac{AY}{L}x_u^2 - mgx_u \\ h &= \frac{\pi(d/2)^2 Y x_u^2}{2mgL} - x_u \\ &= \frac{\pi d^2 Y x_u^2}{8mgL} - x_u \\ &= 0.23\,\text{m} \end{aligned}$$

3.9

(a)

$$\begin{aligned}
\rho_{\text{steel}} &= 7850\,\text{kg/m}^3 \\
V_{\text{sphere}} &= \frac{4}{3}\pi r^3 \\
F_u &= Au \\
&= \pi r^2 u \\
&\approx 3455.75\,\text{N} \\
mg &= F_u \\
m &= \frac{F_u}{g} \\
&\approx 352.3\,\text{kg} \\
\rho V &= m \\
\rho\frac{4}{3}\pi r^3 &= m \\
r &= \sqrt[3]{\frac{3m}{4\pi\rho}} \\
&= 22\,\text{cm}
\end{aligned}$$

(b)

$$\begin{aligned}
M &= -\frac{\pi n r^4}{2l}\theta \\
c &= \frac{\pi n r^4}{2l} \\
T &= 2\pi\sqrt{\frac{I}{c}} \\
&= 2\pi\sqrt{\frac{2MR^2/5}{\pi n r^4/2l}} \\
&= 2\pi\sqrt{\frac{4lMR^2}{5\pi n r^4}} \\
&= 66\,\text{s}
\end{aligned}$$

3.10

(a)

$$\begin{aligned} Y &= \frac{\text{stress}}{\text{strain}} \\ &= \frac{F/A}{\Delta l/l_0} \\ &= \frac{mg/A}{\Delta l/l_0} \\ &= \frac{mgl_0}{\Delta l A} \\ &= 5.9 \times 10^{11}\,\mathrm{N/m^2} \end{aligned}$$

(b)

$$\begin{aligned} y &= \frac{4L^3}{Yab^3}F \\ F &= \frac{Yab^3}{4L^3}y \\ k &= \frac{Yab^3}{4L^3} \\ \omega_y &= \sqrt{\frac{k}{m}} \\ &= \sqrt{\frac{Yab^3}{4L^3m}} \\ \omega_x &= \sqrt{\frac{Ya^3b}{4L^3m}} \\ \frac{\omega_y}{\omega_x} &= \sqrt{\frac{ab^3}{a^3b}} \\ &= \frac{b}{a} \end{aligned}$$

(c) 3/2

3.11

(a)

$$\omega = \sqrt{\frac{A\gamma p}{lm}}$$

3.14

(a)

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

(b)

$$
\begin{aligned}
\omega &= \frac{\sqrt{3}}{2}\omega_0 \\
\omega^2 &= \frac{3}{4}\omega_0^2 \\
\omega_0^2 - \frac{\gamma^2}{4} &= \frac{3}{4}\omega_0^2 \\
\frac{1}{4}\omega_0^2 &= \frac{\gamma^2}{4} \\
\omega_0^2 &= \gamma^2 \\
\omega_0 &= \gamma \\
&= \frac{b}{m} \\
b &= m\omega_0 \\
&= m\sqrt{\frac{k}{m}} \\
&= 4\,\mathrm{N/(m/s)}
\end{aligned}
$$

3.15

(a)

$$
\begin{aligned}
\overline{E}_0 e^{-\gamma} &= \frac{1}{2}\overline{E}_0 \\
e^{-\gamma} &= \frac{1}{2} \\
-\gamma &= \ln\frac{1}{2} \\
\gamma &= \ln 2 \\
Q_0 &= \frac{\omega_0}{\gamma} \\
&= \frac{2\pi f}{\gamma} \\
&= \frac{512\pi}{\ln 2} \\
&\approx 2321
\end{aligned}
$$

(b)

$$Q = 2Q_0$$

(c)

$$\begin{aligned}
\gamma &= \frac{1}{4} \\
Q &= \frac{\omega_0}{\gamma} \\
&= 4\sqrt{\frac{k}{m}} \\
&= 12 \\
\gamma &= \frac{b}{m} \\
b &= \gamma m \\
&= 0.025\,\mathrm{N/(m/s)}
\end{aligned}$$

3.16

(a)

$$\begin{aligned}
x &= A\sin\omega t \\
v &= \omega A\cos\omega t \\
a &= -\omega^2 A\sin\omega t \\
E &= \int_0^{1/f} \frac{Ke^2}{c^3}(-\omega^2 A\sin\omega t)^2\,dt \\
&= \frac{Ke^2\omega^4 A^2}{c^3}\int_0^{1/f} \sin^2\omega t\,dt \\
&= \frac{Ke^2\omega^4 A^2}{c^3}\left[\frac{t}{2} - \frac{1}{4\omega}\sin 2\omega t\right]_0^{1/f} \\
&= \frac{Ke^2\omega^4 A^2}{c^3}\left(\frac{1}{2f} - \frac{1}{4\omega}\sin 2\omega\frac{1}{f}\right) \\
&= \frac{Ke^2(2\pi f)^4 A^2}{2fc^3} \\
&= \frac{8\pi^4 Ke^2 f^3 A^2}{c^3}
\end{aligned}$$

(b)

$$
\begin{aligned}
E_0 &= \frac{1}{2}mv^2 \\
&= \frac{m(\omega A)^2}{2} \\
&= 2\pi^2 A^2 f^2 m \\
\frac{Q}{\pi}E &= E_0\left(1-\frac{1}{e}\right) \\
\frac{Q}{\pi}\frac{8\pi^4 K q^2 f^3 A^2}{c^3} &= 2\pi^2 A^2 f^2 m\left(1-\frac{1}{e}\right) \\
Q\frac{4\pi K q^2 f}{c^3} &= m\left(1-\frac{1}{e}\right) \\
Q &= \frac{c^3 m}{4\pi f K q^2}\left(1-\frac{1}{e}\right)
\end{aligned}
$$

3.17

(a)

$$
\begin{aligned}
V &= \pi r^2 y_{\text{left}} \\
V &= \pi(2r)^2 y_{\text{right}} \\
\pi r^2 y_{\text{left}} &= \pi(2r)^2 y_{\text{right}} \\
y_{\text{right}} &= \frac{1}{4}y_{\text{left}} \\
\frac{y_{\text{left}}}{2}+\frac{y_{\text{right}}}{2} &= \frac{y_{\text{left}}}{2}+\frac{y_{\text{left}}}{8} \\
&= \frac{5}{8}y_{\text{left}} \\
U &= mg\frac{5}{8}y \\
&= \frac{5}{8}\rho\pi r^2 y g y \\
&= \frac{5}{8}g\rho\pi r^2 y^2
\end{aligned}
$$

(b)

$$
\begin{aligned}
r(x) &= r + \frac{x}{l} r \\
&= r\left(1 + \frac{x}{l}\right) \\
\frac{dy}{dt}\pi r^2 &= v\pi r(x)^2 \\
&= v\pi\left[r\left(1 + \frac{x}{l}\right)\right]^2 \\
v &= \frac{dy}{dt}\frac{1}{\left(1 + \frac{x}{l}\right)^2} \\
m &= \rho\pi r(x)^2\,dx \\
&= \rho\pi\left[r\left(1 + \frac{x}{l}\right)\right]^2\,dx \\
&= \rho\pi r^2\left(1 + \frac{x}{l}\right)^2\,dx \\
dK &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}\rho\pi r^2\left(1 + \frac{x}{l}\right)^2\,dx\left(\frac{dy}{dt}\frac{1}{\left(1 + \frac{x}{l}\right)^2}\right)^2 \\
&= \frac{1}{2}\rho\frac{\pi r^2\,dx}{(1 + x/l)^2}\left(\frac{dy}{dt}\right)^2
\end{aligned}
$$

(c)

$$
\begin{aligned}
K &= \frac{1}{2}\rho\pi r^2 h\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}\rho\pi(2r)^2 h\left(\frac{dy}{dt}\right)^2 + \int_0^l dK \\
&= \frac{5}{2}\rho\pi r^2 h\left(\frac{dy}{dt}\right)^2 + \int_0^l \frac{1}{2}\rho\frac{\pi r^2\,dx}{(1 + x/l)^2}\left(\frac{dy}{dt}\right)^2 \\
&= \frac{5}{2}\rho\pi r^2 h\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}\rho\pi r^2\int_0^l \frac{1}{(1 + x/l)^2}\,dx\left(\frac{dy}{dt}\right)^2 \\
&= \frac{1}{4}\rho\pi r^2\left(l + \frac{5}{2}h\right)\left(\frac{dy}{dt}\right)^2
\end{aligned}
$$

(d)

$$
\begin{aligned}
K + U &= E \\
\frac{1}{4}\rho\pi r^2 \left(l + \frac{5}{2}h\right)\left(\frac{dy}{dt}\right)^2 + \frac{5}{8}g\rho\pi r^2 y^2 &= E \\
m &= \frac{1}{2}\rho\pi r^2 \left(l + \frac{5}{2}h\right) \\
k &= \frac{5}{4}g\rho\pi r^2 \\
T &= 2\pi\sqrt{\frac{m}{k}} \\
&= 2\pi\sqrt{\frac{\frac{1}{2}\rho\pi r^2 \left(l + \frac{5}{2}h\right)}{\frac{5}{4}g\rho\pi r^2}} \\
&= 2\pi\sqrt{\frac{2h}{g}}
\end{aligned}
$$

3.19

(a)

$$
m\frac{d^2x}{dt^2} + 2k(x + l - l_0) = 0
$$

(b)

$$
\begin{aligned}
T &= k(l' - l_0) \\
&= k(\sqrt{l^2 + y^2} - l_0) \\
F &= 2T\sin\theta \\
&= 2k(\sqrt{l^2 + y^2} - l_0)\frac{y}{\sqrt{l^2 + y^2}} \\
&= 2k\left(1 - \frac{l_0}{\sqrt{l^2 + y^2}}\right)y \\
&\approx 2k\left(1 - \frac{l_0}{l}\right)y
\end{aligned}
$$

$$
m\frac{d^2y}{dt^2} + 2k\left(1 - \frac{l_0}{l}\right)y = 0
$$

(c)

$$
\begin{aligned}
T_x &= 2\pi\sqrt{\frac{m}{2k}} \\
T_y &= 2\pi\sqrt{\frac{m}{2k\left(1-\frac{l_0}{l}\right)}} \\
\frac{T_x}{T_y} &= \frac{2\pi\sqrt{m/2k}}{2\pi\sqrt{\frac{m}{2k(1-l/l_0)}}} \\
&= \sqrt{\frac{m}{2k}\frac{2k(1-l/l_0)}{m}} \\
&= \sqrt{1-l/l_0}
\end{aligned}
$$

(d)

$$
\begin{aligned}
x &= A_x\cos\left(\sqrt{\frac{2k}{m}}t+\phi_x\right) \\
A_0 &= A_x\cos\phi_x \\
0 &= -\sqrt{\frac{2k}{m}}A_x\sin\phi_x \\
\tan\phi_x &= 0 \\
\phi_x &= 0 \\
A_x &= A_0 \\
x &= A_0\cos\sqrt{\frac{2k}{m}}t
\end{aligned}
$$

$$
\begin{aligned}
y &= A_y\cos\left(\sqrt{\frac{2k(1-l_0/l)}{m}}t+\phi_y\right) \\
A_0 &= A_y\cos\phi_y \\
0 &= -\sqrt{\frac{2k(1-l_0/l)}{m}}A_y\sin\phi_y \\
\tan\phi_y &= 0 \\
\phi_y &= 0 \\
A_y &= A_0 \\
y &= A_0\cos\sqrt{\frac{2k(1-l_0/l)}{m}}t
\end{aligned}
$$

4 Forced vibrations and resonance

4.3

(a)

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

$$\begin{aligned}
\omega &= \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \\
&= \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \\
&= \sqrt{\frac{80}{0.2} - \frac{4^2}{4 \cdot 0.2^2}} \\
&= \sqrt{300} \\
&= 10\sqrt{3} \\
T &= \frac{2\pi}{\omega} \\
&= \frac{2\pi}{10\sqrt{3}} \\
&= \frac{\pi}{5\sqrt{3}}\,\mathrm{s}
\end{aligned}$$

(b)

$$\begin{aligned}
A &= \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \\
&= \frac{F_0}{m\sqrt{\left(\frac{k}{m} - \omega^2\right)^2 + \left(\frac{b}{m}\omega\right)^2}} \\
&\approx 1.3\,\mathrm{cm}
\end{aligned}$$

4.4

$$\begin{aligned}
mg &= kh \\
k &= \frac{mg}{h}
\end{aligned}$$

$$\begin{aligned}
mg &= bu \\
b &= \frac{mg}{u}
\end{aligned}$$

(a)

$$m\frac{d^2x}{dt^2} + \frac{mg}{u}\frac{dx}{dt} + \frac{mg}{h}x = 0$$

(b)

$$
\begin{aligned}
\omega &= \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \\
&= \sqrt{\frac{g}{h} - \frac{g^2}{4u^2}} \\
&= \sqrt{\frac{g}{h} - \frac{g^2}{36gh}} \\
&= \sqrt{\frac{g}{h} - \frac{g}{36h}} \\
&= \sqrt{\frac{35g}{36h}}
\end{aligned}
$$

(c)

$$\frac{1}{\gamma} = \frac{u}{g} = \frac{3\sqrt{gh}}{g} = 3\sqrt{\frac{h}{g}}\,\mathrm{s}$$

(d)

$$Q = \frac{\omega_0}{\gamma} = \sqrt{\frac{g}{h}} \cdot 3\sqrt{\frac{h}{g}} = 3$$

(e)

$$\delta = \frac{\pi}{2}$$

(f)

$$
\begin{aligned}
A &= \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}} \\
&= \frac{F_0}{m\sqrt{\left(\frac{g}{h}-\omega^2\right)+\left(\frac{g}{u}\omega\right)^2}} \\
&= \frac{F_0}{m\sqrt{\left(\frac{g}{h}-\frac{2g}{h}\right)^2+\left(\frac{g}{3\sqrt{gh}}\sqrt{\frac{2g}{h}}\right)^2}} \\
&= \frac{F_0}{m\sqrt{\left(\frac{g}{h}\right)^2+\left(\frac{\sqrt{2}}{3}\frac{g}{h}\right)^2}} \\
&= \frac{F_0}{m\sqrt{\left(\frac{g}{h}\right)^2+\frac{2}{9}\left(\frac{g}{h}\right)^2}} \\
&= \frac{F_0}{m\sqrt{\frac{11}{9}\left(\frac{g}{h}\right)^2}} \\
&= \sqrt{\frac{9}{11}}\frac{F_0 h}{gm} \\
&= \sqrt{\frac{9}{11}}h \\
&\approx 0.9h
\end{aligned}
$$

4.6

(a)

$$
\begin{aligned}
m\left(\frac{d^2y}{dt^2}+\frac{d^2\eta}{dt^2}\right) &= -kx-b\frac{dy}{dt} \\
\frac{d^2y}{dt^2}+\gamma\frac{dy}{dt}+\omega_0^2 y &= -\frac{d^2\eta}{dt^2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{d^2y}{dt^2}+\gamma\frac{dy}{dt}+\omega_0^2 y &= C\omega^2\cos\omega t \\
y &= A(\omega)\cos(\omega t+\delta(\omega)) \\
A(\omega) &= \frac{C\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+(\omega\omega_0/Q)^2}} \\
\tan\delta(\omega) &= \frac{\omega\omega_0/Q}{\omega_0^2-\omega^2}
\end{aligned}
$$

(d)

$$
\begin{aligned}
T_0 &= \frac{2\pi}{\omega_0} \\
\omega_0 &= \frac{2\pi}{T_0} \\
&= \frac{\pi}{15}\,\mathrm{s}
\end{aligned}
$$

$$Q = 2$$

$$
\begin{aligned}
T &= \frac{2\pi}{\omega} \\
\omega &= \frac{2\pi}{T} \\
&= \frac{\pi}{600}\,\mathrm{s}
\end{aligned}
$$

$$
\begin{aligned}
A &= C\omega^2 \\
C &= \frac{A}{\omega^2} \\
&= A\left(\frac{600}{\pi}\right)^2 \\
&\approx 3.65 \times 10^{-5}
\end{aligned}
$$

$$A(\omega) \approx 2.28 \times 10^{-8}\,\mathrm{m}$$

4.8

(a)

$$
\begin{aligned}
m\frac{d^2x}{dt^2} &= -b\frac{dx}{dt} \\
m\frac{dv}{dt} &= -bv \\
\frac{1}{v}\frac{dv}{dt} &= -\frac{b}{m} \\
\ln v &= -\frac{b}{m}t + c \\
v &= e^{(-bt/m)+c} \\
&= v_0 e^{-\gamma t} \\
x &= C - \frac{v_0}{\gamma}e^{-\gamma t}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} &= \frac{F_0}{m}\cos\omega t \\
x &= A\cos(\omega t - \delta) \\
\frac{dx}{dt} &= -A\omega\sin(\omega t - \delta) \\
\frac{d^2x}{dt^2} &= -A\omega^2\cos(\omega t - \delta) \\
-A\omega^2\cos(\omega t - \delta) - \gamma A\omega\sin(\omega t - \delta) &= \frac{F_0}{m}\cos\omega t \\
-A\omega^2(\cos\omega t\cos\delta + \sin\omega t\sin\delta) & \\
-\gamma A\omega(\sin\omega t\cos\delta - \cos\omega t\sin\delta) &= \frac{F_0}{m}\cos\omega t \\
A\omega(\gamma\sin\delta - \omega\cos\delta)\cos\omega t & \\
-A\omega(\gamma\cos\delta + \omega\sin\delta)\sin\omega t &= \frac{F_0}{m}\cos\omega t
\end{aligned}
$$

$$
\begin{aligned}
A\omega(\gamma\cos\delta + \omega\sin\delta) &= 0 \\
\gamma\cos\delta + \omega\sin\delta &= 0 \\
\tan\delta &
\end{aligned}
$$

$$
\begin{aligned}
A\omega(\gamma\sin\delta - \omega\cos\delta) &= \frac{F_0}{m} \\
\gamma\sin\delta - \omega\cos\delta &= -\sqrt{\gamma^2+\omega^2}\cos\left(\delta + \arctan\frac{\gamma}{\omega}\right) \\
&= -\sqrt{\gamma^2+\omega^2}\cos(\delta + \arctan(-\tan\delta)) \\
&= -\sqrt{\gamma^2+\omega^2}\cos(\delta - \arctan(\tan\delta)) \\
&= -\sqrt{\gamma^2+\omega^2} \\
-A\omega\sqrt{\gamma^2+\omega^2} &= \frac{F_0}{m} \\
|A| &= \frac{F_0}{m\omega\sqrt{\gamma^2+\omega^2}}
\end{aligned}
$$

4.9

(a)

$$
\begin{aligned}
x &= A \sin \omega t \\
\frac{dx}{dt} &= A\omega \cos \omega t \\
W &= \int dW \\
&= \int F \, dx \\
&= \int b \frac{dx}{dt} \, dx \\
&= b \int \left(\frac{dx}{dt} \right)^2 \, dt \\
&= b \int_0^T (A\omega \cos \omega t)^2 \, dt \\
&= A^2 b \omega^2 \int_0^{2\pi/\omega} \cos^2 \omega t \, dt \\
&= A^2 b \omega^2 \left[\frac{t}{2} + \frac{1}{4\omega} \sin 2\omega t \right]_0^{2\pi/\omega} \\
&= A^2 b \omega \pi
\end{aligned}
$$

4.11

(a)

$$
\begin{aligned}
A &= \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \\
&= \frac{2/0.2}{\sqrt{\left(\frac{80}{0.2} - 30^2\right)^2 + \left(\frac{4}{0.2} 30\right)^2}} \\
&\approx 1.3 \, \mathrm{cm}
\end{aligned}
$$

$$
\begin{aligned}
\tan \delta &= \frac{\gamma\omega}{\omega_0^2 - \omega^2} \\
\delta &= \arctan \frac{\frac{b}{m}\omega}{\frac{k}{m} - \omega^2} \\
&\approx 130^\circ
\end{aligned}
$$

(b)

$$
\begin{aligned}
x &= A\cos(\omega t - \delta) \\
\frac{dx}{dt} &= -A\omega \sin(\omega t - \delta) \\
W &= \int dW \\
&= \int F\,dx \\
&= \int b\frac{dx}{dt}\,dx \\
&= b\int_0^T \left(\frac{dx}{dt}\right)^2\,dt \\
&= b\int_0^T (-A\omega\sin(\omega t - \delta))^2\,dt \\
&= A^2 b\omega^2 \int_0^{2\pi/\omega} \sin^2(\omega t - \delta)\,dt \\
&= A^2 b\omega\pi \\
&\approx 6.4 \times 10^{-2}\,\mathrm{J}
\end{aligned}
$$

(c)

$$
P = \frac{W}{t} = 0.31\,\mathrm{W}
$$

4.12

(a)

$$
\begin{aligned}
mg &= kx \\
k &= \frac{mg}{x} \\
&\approx 785\,\mathrm{N/m}
\end{aligned}
$$

$$
\begin{aligned}
\omega_0 &= \sqrt{\frac{k}{m}} \\
&\approx 19.8\,\mathrm{rad/s}
\end{aligned}
$$

(b)

$$
\begin{aligned}
A &= BQ \\
&= 1.5\,\mathrm{cm}
\end{aligned}
$$

(c)

$$\begin{aligned}\overline{P}(\omega) &= \frac{1.082B^2 m\omega_0^5}{2Q}\frac{1}{0.0016\omega_0^2 + \omega_0^2/Q^2} \\ &\approx 0.093\,\mathrm{W}\end{aligned}$$

4.13

(a)

$$\omega_0 = 40\,\mathrm{rad/s}$$

$$\begin{aligned}2 &= \frac{\omega_0}{Q} \\ Q &= \frac{\omega_0}{2} \\ &= 20\end{aligned}$$

(b)

$$\begin{aligned}E_0 e^{-2\pi n/Q} &= E_0 e^{-5} \\ -\frac{2\pi n}{Q} &= -5 \\ n &= \frac{5Q}{2\pi} \\ &\approx 16\end{aligned}$$

4.14

(a)

$$\begin{aligned}1.02\omega_0 - 0.98\omega_0 &= \frac{\omega_0}{Q} \\ 0.04\omega_0 &= \frac{\omega_0}{Q} \\ Q &= 25\end{aligned}$$

(b)

$$\gamma = 0.04\omega_0$$

(c)

$$\begin{aligned}\frac{E_0 - E(2\pi/\omega_0)}{E_0} &= 1 - \frac{E_0 e^{-0.04\omega_0(2\pi/\omega_0)}}{E_0} \\ &= 1 - e^{-2\pi\cdot 0.04} \\ &\approx 22\%\end{aligned}$$

(d)

$$\omega_0' = \sqrt{2}\omega_0$$

(e)

$$Q' = \sqrt{2}Q$$

(f)

$$\overline{P}_m' = \overline{P}_m$$

(g)

$$E_0' = E_0$$

4.15

(b)

$$\begin{aligned}\frac{\omega_1}{Q} &= \frac{\omega_1}{5} \\ Q &= 5\end{aligned}$$

(c)

$$\begin{aligned}Q &= \frac{\omega_0}{\gamma} \\ &= \sqrt{\frac{k}{m}}\frac{m}{b} \\ b &= \frac{1}{5}\sqrt{km}\end{aligned}$$

4.17

(a)

$$W = PT = 10 \cdot \frac{2\pi}{10^6} = \frac{2\pi}{10^5} = 6.28 \times 10^{-5}\,\mathrm{J}$$

(b)

$$\begin{aligned}W &= E_0(1 - e^{-\gamma(2\pi/\omega)}) \\ E_0 &= \frac{W}{1 - e^{-\gamma(2\pi/\omega)}} \\ &\approx 1.03 \times 10^{-3}\,\mathrm{J}\end{aligned}$$

(c)

$$\frac{1}{\gamma} = \frac{1}{(1.005 - 0.995) \times 10^6} \approx 1 \times 10^{-4}\,\mathrm{s}$$

5 Coupled oscillators and normal modes

5.2

(a)

$$\begin{aligned}
T_{\text{clamped}} &= \frac{2\pi}{\sqrt{\omega_0^2 + \omega_c^2}} \\
T_{\text{clamped}}^2 &= \frac{4\pi^2}{\omega_0^2 + \omega_c^2} \\
\omega_0^2 + \omega_c^2 &= \frac{4\pi^2}{T_{\text{clamped}}^2} \\
\omega_c &= \sqrt{\frac{4\pi^2}{T_{\text{clamped}}^2} - \frac{g}{l}}
\end{aligned}$$

$$\begin{aligned}
T_1 &= \frac{2\pi}{\omega_0} \\
&= \frac{2\pi}{\sqrt{g/l}} \\
&\approx 1.27\,\text{s}
\end{aligned}$$

$$\begin{aligned}
T_2 &= \frac{2\pi}{\omega'} \\
&= \frac{2\pi}{\sqrt{\omega_0^2 + 2\omega_c^2}} \\
&= \frac{2\pi}{\sqrt{\frac{g}{l} + 2\sqrt{\frac{4\pi^2}{T_{\text{clamped}}^2} - \frac{g}{l}}}} \\
&\approx 1.23\,\text{s}
\end{aligned}$$

(b)

$$\begin{aligned}
T_{\text{beat}} &= \frac{2\pi}{\omega' - \omega_0} \\
&= \frac{2\pi}{\frac{2\pi}{T_2} - \frac{2\pi}{T_1}} \\
&\approx 40\,\text{s}
\end{aligned}$$

5.4

$$\begin{aligned}
\frac{d^2x_A}{dt^2} + \frac{k_A + k_C}{m}x_A - \frac{k_C}{m}x_B = 0 \\
\frac{d^2x_B}{dt^2} - \frac{k_C}{m}x_A + \frac{k_B + k_C}{m}x_B = 0
\end{aligned}$$

$$\begin{aligned}
x_A &= A\cos\omega t \\
x'_A &= -A\omega\sin\omega t \\
x''_A &= -A\omega^2\cos\omega t
\end{aligned}$$

$$\begin{aligned}
x_B &= A'\cos\omega t \\
x'_B &= -A'\omega\sin\omega t \\
x''_B &= -A'\omega^2\cos\omega t
\end{aligned}$$

$$\begin{aligned}
-A\omega^2\cos\omega t + \frac{k_A + k_C}{m}A\cos\omega t - \frac{k_C}{m}A'\cos\omega t &= 0 \\
-A\omega^2 + \frac{k_A + k_C}{m}A - \frac{k_C}{m}A' &= 0 \\
\left(\frac{k_A + k_C}{m} - \omega^2\right)A - \frac{k_C}{m}A' &= 0 \\
\frac{k_C}{k_A + k_C - m\omega^2} &= \frac{A}{A'}
\end{aligned}$$

$$\begin{aligned}
-A'\omega^2\cos\omega t - \frac{k_C}{m}A\cos\omega t + \frac{k_B + k_C}{m}A'\cos\omega t &= 0 \\
-A'\omega^2 - \frac{k_C}{m}A + \frac{k_B + k_C}{m}A' &= 0 \\
-\frac{k_C}{m}A + \left(\frac{k_B + k_C}{m} - \omega^2\right)A' &= 0 \\
\frac{k_B + k_C - m\omega^2}{k_C} &= \frac{A}{A'}
\end{aligned}$$

$$\begin{aligned}
\frac{k_C}{k_A + k_C - m\omega^2} &= \frac{k_B + k_C - m\omega^2}{k_C} \\
k_C^2 &= (k_A + k_C - m\omega^2)(k_B + k_C - m\omega^2) \\
&= k_Ak_B + k_Ak_C - k_Am\omega^2 + k_Bk_C + k_C^2 - k_Cm\omega^2 - k_Bm\omega^2 \\
&\quad - k_Cm\omega^2 + m^2\omega^4 \\
0 &= m^2(\omega^2)^2 - (k_A + k_B + 2k_C)m\omega^2 + k_Ak_B + k_Ak_C + k_Bk_C
\end{aligned}$$

$$
\begin{aligned}
m\omega^2 &= \frac{(k_A + k_B + 2k_C) \pm \sqrt{(k_A + k_B + 2k_C)^2 - 4(k_A k_B + k_A k_C + k_B k_C)}}{2} \\
&= \frac{(k_A + k_B + 2k_C) \pm \sqrt{(k_A - k_B)^2 + 4k_C^2}}{2} \\
&= \left(\frac{k_A + k_B}{2} + k_C\right) \pm \sqrt{\left(\frac{k_A - k_B}{2}\right)^2 + k_C^2}
\end{aligned}
$$

5.5

$$
\begin{aligned}
\frac{d^2 x_A}{dt^2} - \alpha \frac{d^2 x_B}{dt^2} + \omega_0^2 x_A = 0 \\
\frac{d^2 x_B}{dt^2} - \alpha \frac{d^2 x_A}{dt^2} + \omega_0^2 x_B = 0
\end{aligned}
$$

$$
\begin{aligned}
x_A &= A \cos \omega t \\
x_A'' &= -A\omega^2 \cos \omega t
\end{aligned}
$$

$$
\begin{aligned}
x_B &= B \cos \omega t \\
x_B'' &= -B\omega^2 \cos \omega t
\end{aligned}
$$

$$
\begin{aligned}
-A\omega^2 \cos \omega t + \alpha B \omega^2 \cos \omega t + A\omega_0^2 \cos \omega t &= 0 \\
-A\omega^2 + \alpha B \omega^2 + A\omega_0^2 &= 0 \\
(\omega_0^2 - \omega^2) A + \alpha \omega^2 B &= 0 \\
\frac{\alpha \omega^2}{\omega^2 - \omega_0^2} &= \frac{A}{B}
\end{aligned}
$$

$$
\begin{aligned}
-B\omega^2 \cos \omega t + \alpha A \omega^2 \cos \omega t + B\omega_0^2 \cos \omega t &= 0 \\
\alpha \omega^2 A + (\omega_0^2 - \omega^2) B &= 0 \\
\frac{\omega^2 - \omega_0^2}{\alpha \omega^2} &= \frac{A}{B}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\alpha\omega^2}{\omega^2-\omega_0^2} &= \frac{\omega^2-\omega_0^2}{\alpha\omega^2} \\
(\alpha\omega^2)^2 &= (\omega^2-\omega_0^2)^2 \\
\alpha\omega^2 &= \omega^2-\omega_0^2 \\
(\alpha-1)\omega^2 &= -\omega_0^2 \\
\omega^2 &= \frac{\omega_0^2}{1-\alpha} \\
\omega &= \frac{\omega_0}{\sqrt{1-\alpha}}
\end{aligned}
$$

$$
\begin{aligned}
\alpha\omega^2 &= \omega_0^2-\omega^2 \\
(\alpha+1)\omega^2 &= \omega_0^2 \\
\omega &= \frac{\omega_0}{\sqrt{\alpha+1}}
\end{aligned}
$$

5.7

(b)

$$
\begin{aligned}
\frac{d^2x_A}{dt^2} + \frac{k_0+k_c}{m}x_A - \frac{k_c}{m}x_B = 0 \\
\frac{d^2x_B}{dt^2} - \frac{k_c}{m}x_A + \frac{k_0+k_c}{m}x_B = 0
\end{aligned}
$$

$$
\begin{aligned}
x_A &= A\cos\omega t \\
x_A'' &= -A\omega^2\cos\omega t
\end{aligned}
$$

$$
\begin{aligned}
x_B &= B\cos\omega t \\
x_B'' &= -B\omega^2\cos\omega t
\end{aligned}
$$

$$
\begin{aligned}
-A\omega^2\cos\omega t + \frac{k_0+k_c}{m}A\cos\omega t - \frac{k_c}{m}B\cos\omega t &= 0\\
-A\omega^2 + \frac{k_0+k_c}{m}A - \frac{k_c}{m}B &= 0\\
(k_0+k_c-m\omega^2)A - k_cB &= 0\\
\frac{k_c}{k_0+k_c-m\omega^2} &= \frac{A}{B}
\end{aligned}
$$

$$
\begin{aligned}
-B\omega^2\cos\omega t - \frac{k_c}{m}A\cos\omega t + \frac{k_0+k_c}{m}B\cos\omega t &= 0\\
-B\omega^2 - \frac{k_c}{m}A + \frac{k_0+k_c}{m}B &= 0\\
-k_cA + (k_0+k_c-m\omega^2)B &= 0\\
\frac{k_0+k_c-m\omega^2}{k_c} &= \frac{A}{B}
\end{aligned}
$$

$$
\begin{aligned}
\frac{k_c}{k_0+k_c-m\omega^2} &= \frac{k_0+k_c-m\omega^2}{k_c}\\
(k_0+k_c-m\omega^2)^2 &= k_c^2\\
k_0+k_c-m\omega^2 &= \pm k_c
\end{aligned}
$$

$$
\begin{aligned}
m\omega_1^2 &= k_0\\
\omega_1 &= \omega_0
\end{aligned}
$$

$$
\begin{aligned}
m\omega_2^2 &= k_0+2k_c\\
\omega_2 &= \sqrt{\omega_0^2+\frac{2k_c}{m}}
\end{aligned}
$$

(c)

$$\begin{aligned}\omega_0 &= 2\pi f_1\\ &\approx 7.16\,\mathrm{rad/s}\end{aligned}$$

$$\begin{aligned}\omega_A &= 2\pi f_A\\ \sqrt{\omega_0^2+\frac{k_c}{m}} &= 2\pi f_A\\ \omega_0^2+\frac{k_c}{m} &= (2\pi f_A)^2\\ \frac{k_c}{m} &= (2\pi f_A)^2-\omega_0^2\\ &\approx 78.0\,\mathrm{s}^{-2}\end{aligned}$$

$$\begin{aligned}2\pi f_2 &= \sqrt{\omega_0^2+2\frac{k_c}{m}}\\ f_2 &= \frac{1}{2\pi}\sqrt{\omega_0^2+2\frac{k_c}{m}}\\ &\approx 2.29\,\mathrm{Hz}\end{aligned}$$

(d)

$$\begin{aligned}\frac{k_c}{k_0} &= \frac{k_c/m}{k_0/m}\\ &= \frac{k_c/m}{\omega_0^2}\\ &\approx 1.52\end{aligned}$$

5.8

(a)

$$\begin{aligned}Fa\cos\theta &= mgL\sin\theta\\ \frac{Fa}{mgL} &= \tan\theta\\ &\approx \theta\end{aligned}$$

$$\begin{aligned}F'L\cos\theta &= mgL\sin\theta\\ F' &= mg\tan\theta\\ &\approx mg\theta\\ &= \frac{Fa}{L}\end{aligned}$$

(b)

$$\begin{aligned}
I\frac{d^2\theta_1}{dt^2} &= -mgL\sin\theta_1 + k(a\sin\theta_2 - a\sin\theta_1)a\cos\theta_1 \\
mL^2\frac{d^2\theta_1}{dt^2} &\approx -mgL\theta_1 + a^2k(\theta_2-\theta_1) \\
0 &= \frac{d^2\theta_1}{dt^2} + \left(\frac{g}{L} + \frac{a^2k}{mL^2}\right)\theta_1 - \frac{a^2k}{mL^2}\theta_2
\end{aligned}$$

$$\begin{aligned}
I\frac{d^2\theta_2}{dt^2} &= -mgL\sin\theta_2 - k(a\sin\theta_2 - a\sin\theta_1)a\cos\theta_2 \\
mL^2\frac{d^2\theta_2}{dt^2} &\approx -mgL\theta_2 - a^2k(\theta_2-\theta_1) \\
0 &= \frac{d^2\theta_2}{dt^2} - \frac{a^2k}{mL^2}\theta_1 + \left(\frac{g}{L} + \frac{a^2k}{mL^2}\right)\theta_2
\end{aligned}$$

(d) Let $\omega_0^2 = \frac{g}{L}$ and $\omega_c = \frac{a^2k}{mL^2}$ then

$$\begin{aligned}
\theta_1 &= A\cos\omega t \\
\theta_1'' &= -A\omega^2\cos\omega t
\end{aligned}$$

$$\begin{aligned}
\theta_2 &= B\cos\omega t \\
\theta_2'' &= -B\omega^2\cos\omega t
\end{aligned}$$

$$\begin{aligned}
0 &= -A\omega^2\cos\omega t + (\omega_0^2+\omega_c)A\cos\omega t - \omega_c B\cos\omega t \\
&= -A\omega^2 + (\omega_0^2+\omega_c)A - \omega_c B \\
&= (\omega_0+\omega_c-\omega^2)A - \omega_c B \\
\frac{A}{B} &= \frac{\omega_c}{\omega_0+\omega_c-\omega^2}
\end{aligned}$$

$$\begin{aligned}
0 &= -B\omega^2\cos\omega t - \omega_c A\cos\omega t + (\omega_0+\omega_c)B\cos\omega t \\
&= -B\omega^2 - \omega_c A + (\omega_0+\omega_c)B \\
&= -\omega_c A + (\omega_0+\omega_c-\omega^2)B \\
\frac{A}{B} &= \frac{\omega_0+\omega_c-\omega^2}{\omega_c}
\end{aligned}$$

$$
\begin{aligned}
\frac{\omega_c}{\omega_0 + \omega_c - \omega^2} &= \frac{\omega_0 + \omega_c - \omega^2}{\omega_c} \\
(\omega_0 + \omega_c - \omega^2)^2 &= \omega_c^2 \\
\omega_0 + \omega_c - \omega^2 &= \pm\omega_c
\end{aligned}
$$

$$
\begin{aligned}
\omega_1 &= \omega_0 \\
&= \sqrt{\frac{g}{L}}
\end{aligned}
$$

$$
\begin{aligned}
\omega_2 &= \sqrt{\omega_0 + 2\omega_c} \\
&= \sqrt{\frac{g}{L} + 2\frac{a^2 k}{mL^2}}
\end{aligned}
$$

5.9

(a)

$$\frac{d^2x_1}{dt^2} + \frac{k}{m_1}(x_1 - x_2) = 0$$
$$\frac{d^2x_2}{dt^2} + \frac{k}{m_2}(-x_1 + 2x_2 - x_3) = 0$$
$$\frac{d^2x_3}{dt^2} + \frac{k}{m_1}(-x_2 + x_3) = 0$$

$$x_1 = A\cos\omega t$$
$$x_2 = B\cos\omega t$$
$$x_3 = C\cos\omega t$$

$$-A\omega^2\cos\omega t + \frac{k}{m_1}(A\cos\omega t - B\cos\omega t) = 0$$
$$-A\omega^2 + \frac{k}{m_1}(A - B) = 0$$
$$\left(\frac{k}{m_1} - \omega^2\right)A - \frac{k}{m_1}B = 0$$
$$\frac{k/m_1}{k/m_1 - \omega^2} = \frac{A}{B}$$

$$-C\omega^2\cos\omega t + \frac{k}{m_1}(-B\cos\omega t + C\cos\omega t) = 0$$
$$-C\omega^2 + \frac{k}{m_1}(-B + C) = 0$$
$$-\frac{k}{m_1}B + \left(\frac{k}{m_1} - \omega^2\right)C = 0$$
$$\frac{k/m_1}{k/m_1 - \omega^2} = \frac{C}{B}$$

$$-B\omega^2\cos\omega t + \frac{k}{m_2}(-A\cos\omega t + 2B\cos\omega t - C\cos\omega t) = 0$$

$$-B\omega^2 + \frac{k}{m_2}(-A + 2B - C) = 0$$

$$-\omega^2 + \frac{k}{m_2}\left(-\frac{A}{B} + 2 - \frac{C}{B}\right) = 0$$

$$-\omega^2 + \frac{k}{m_2}\left(-\frac{k/m_1}{k/m_1 - \omega^2} + 2 - \frac{k/m_1}{k/m_1 - \omega^2}\right) = 0$$

$$\frac{2k}{m_2}\left(1 - \frac{k}{k - m_1\omega^2}\right) = \omega^2$$

$$\frac{2k}{m_2}(k - m_1\omega^2 - k) = \omega^2(k - m_1\omega^2)$$

$$-2k\omega^2\frac{m_1}{m_2} = k\omega^2 - m_1(\omega^2)^2$$

$$2k\frac{m_1}{m_2} + k = m_1\omega^2$$

$$\frac{2k}{m_2} + \frac{k}{m_1} = \omega^2$$

$$\frac{k(2m_1 + m_2)}{m_1 m_2} = \omega^2$$

$$\sqrt{\frac{k(2m_1 + m_2)}{m_1 m_2}} = \omega$$

(b)

$$\begin{aligned}
\frac{\omega_1}{\omega_2} &= \sqrt{\frac{k(2m_1 + m_2)/m_1 m_2}{k/m_1}} \\
&= \sqrt{\frac{km_1(2m_1 + m_2)}{km_1 m_2}} \\
&= \sqrt{\frac{2m_1 + m_2}{m_2}} \\
&= \sqrt{\frac{2\cdot 16 + 12}{12}} \\
&= \sqrt{\frac{44}{12}} \\
&= \sqrt{\frac{11}{3}} \\
&\approx 1.91
\end{aligned}$$

5.10

$$\frac{d^2y_1}{dt^2} + \frac{k}{m}(2y_1 - y_2) = 0$$
$$\frac{d^2y_2}{dt^2} + \frac{k}{m}(y_2 - y_1) = 0$$

$$y_1 = A\cos\omega t$$
$$y_2 = B\cos\omega t$$

$$-A\omega^2\cos\omega t + \frac{k}{m}(2A\cos\omega t - B\cos\omega t) = 0$$
$$-A\omega^2 + \frac{k}{m}(2A - B) = 0$$
$$\left(2\frac{k}{m} - \omega^2\right)A - \frac{k}{m}B = 0$$
$$\frac{k/m}{2k/m - \omega^2} = \frac{A}{B}$$

$$-B\omega^2\cos\omega t + \frac{k}{m}(B\cos\omega t - A\cos\omega t) = 0$$
$$-B\omega^2 + \frac{k}{m}(B - A) = 0$$
$$-\frac{k}{m}A + \left(\frac{k}{m} - \omega^2\right)B = 0$$
$$\frac{k/m - \omega^2}{k/m} = \frac{A}{B}$$

$$
\begin{aligned}
\frac{k/m}{2k/m-\omega^2} &= \frac{k/m-\omega^2}{k/m} \\
\left(\frac{k}{m}\right)^2 &= \left(\frac{2k}{m}-\omega^2\right)\left(\frac{k}{m}-\omega^2\right) \\
&= 2\left(\frac{k}{m}\right)^2 - 2\frac{k}{m}\omega^2 - \frac{k}{m}\omega^2 + (\omega^2)^2 \\
0 &= (\omega^2)^2 - 3\frac{k}{m}\omega^2 + \left(\frac{k}{m}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
\omega^2 &= \frac{3(k/m) \pm \sqrt{9(k/m)^2 - 4(k/m)^2}}{2} \\
&= (3 \pm \sqrt{5})\frac{k}{2m}
\end{aligned}
$$

$$
\begin{aligned}
\frac{A}{B} &= \frac{k/m}{2k/m - (3\pm\sqrt{5})k/2m} \\
&= \frac{1}{2-(3\pm\sqrt{5})/2} \\
&= \frac{2}{4-3\pm\sqrt{5}} \\
&= \frac{2}{1\pm\sqrt{5}}
\end{aligned}
$$

5.11

(a)

$$
\begin{aligned}
M_1\frac{d^2x_1}{dt^2} &= -kx_1 + M_2 g\sin\theta \\
&\approx -kx_1 + M_2\frac{g}{l}(x_2-x_1)
\end{aligned}
$$

$$
\begin{aligned}
M_2\frac{d^2x_2}{dt^2} &= -M_2 g\sin\theta \\
&\approx -M_2\frac{g}{l}(x_2-x_1)
\end{aligned}
$$

(b)

$$\begin{aligned}
\frac{d^2x_1}{dt^2} + \frac{k}{M}x_1 - \frac{g}{l}(x_2 - x_1) &= 0 \\
\frac{d^2x_2}{dt^2} + \frac{g}{l}(x_2 - x_1) &= 0
\end{aligned}$$

$$\begin{aligned}
-A\omega^2\cos\omega t + \frac{k}{M}A\cos\omega t - \frac{g}{l}(B\cos\omega t - A\cos\omega t) &= 0 \\
-A\omega^2 + \frac{k}{M}A - \frac{g}{l}(B - A) &= 0 \\
\left(\frac{k}{M} + \frac{g}{l} - \omega^2\right)A - \frac{g}{l}B &= 0 \\
\frac{g/l}{k/M + g/l - \omega^2} &= \frac{A}{B}
\end{aligned}$$

$$\begin{aligned}
-B\omega^2\cos\omega t + \frac{g}{l}(B\cos\omega t - A\cos\omega t) &= 0 \\
-B\omega^2 + \frac{g}{l}(B - A) &= 0 \\
-\frac{g}{l}A + \left(\frac{g}{l} - \omega^2\right)B &= 0 \\
\frac{g/l - \omega^2}{g/l} &= \frac{A}{B}
\end{aligned}$$

$$\begin{aligned}
\frac{g/l}{k/M + g/l - \omega^2} &= \frac{g/l - \omega^2}{g/l} \\
\left(\frac{g}{l}\right)^2 &= \left(\frac{g}{l} - \omega^2\right)\left(\frac{k}{M} + \frac{g}{l} - \omega^2\right) \\
&= \frac{k}{M}\frac{g}{l} + \left(\frac{g}{l}\right)^2 - \frac{g}{l}\omega^2 - \frac{k}{M}\omega^2 - \frac{g}{l}\omega^2 + (\omega^2)^2 \\
0 &= (\omega^2)^2 - \left(2\frac{g}{l} + \frac{k}{M}\right)\omega^2 + \frac{k}{M}\frac{g}{l}
\end{aligned}$$

$$
\begin{aligned}
\omega^2 &= \frac{\left(2\frac{g}{l} + \frac{k}{M}\right) \pm \sqrt{\left(2\frac{g}{l} + \frac{k}{M}\right)^2 - 4\frac{k}{M}\frac{g}{l}}}{2} \\
&= \frac{\left(2\frac{g}{l} + \frac{k}{M}\right) \pm \sqrt{4\left(\frac{g}{l}\right)^2 + 4\frac{k}{M}\frac{g}{l} + \left(\frac{k}{M}\right)^2 - 4\frac{k}{M}\frac{g}{l}}}{2} \\
&= \frac{\left(2\frac{g}{l} + \frac{k}{M}\right) \pm \sqrt{4\left(\frac{g}{l}\right)^2 + \left(\frac{k}{M}\right)^2}}{2} \\
&= \frac{g}{l} + \frac{k}{2M} \pm \sqrt{\left(\frac{g}{l}\right)^2 + \left(\frac{k}{2M}\right)^2}
\end{aligned}
$$

5.13

(a)

$$
\begin{aligned}
\frac{d^2y}{dt^2} + \frac{3T}{2lm}y = 0 \\
T &= \frac{2\pi}{\omega} \\
&= \frac{2\pi}{\sqrt{3T/2lm}} \\
&= 2\pi\sqrt{\frac{2lm}{3T}}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\omega_0 &= \sqrt{\frac{T}{ml}} \\
\omega_n &= 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right] \\
\omega_1 &= 2\omega_0 \sin\left(\frac{\pi}{6}\right) \\
&= \omega_0 \\
&= \sqrt{\frac{T}{ml}} \\
\omega_2 &= 2\omega_0 \sin\left(\frac{\pi}{3}\right) \\
&= \sqrt{3}\omega_0 \\
&= \sqrt{\frac{3T}{ml}}
\end{aligned}
$$

5.15

(c)

$$
\begin{aligned}
\omega_0 &= \sqrt{\frac{T}{ml}} \\
\omega_n &= 2\omega_0 \sin\frac{n\pi}{8} \\
\omega_1 &= 2\omega_0 \sin\frac{\pi}{8} \\
&= \omega_0\sqrt{2-\sqrt{2}} \\
&\approx 0.765\omega_0 \\
\omega_2 &= 2\omega_0 \sin\frac{\pi}{4} \\
&= \omega_0\sqrt{2} \\
&\approx 1.42\omega_0 \\
\omega_3 &= 2\omega_0 \sin\frac{3\pi}{8} \\
&= \omega_0\sqrt{2+\sqrt{2}} \\
&\approx 1.85\omega_0
\end{aligned}
$$

6 Normal modes of continuous systems. Fourier analysis

6.1

(a)

$$f_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}} = 10\,\mathrm{Hz}$$

(b) $5f_1$ (50 Hz) and multiples thereof

6.2

$$
\begin{aligned}
f_1 &= \frac{1}{2L}\sqrt{\frac{T}{\mu}} \\
&= \frac{1}{2L}\sqrt{\frac{LT}{M}} \\
&= \sqrt{\frac{T}{4LM}} \\
f_2 &= 2f_1 \\
f_3 &= 3f_1
\end{aligned}
$$

$$
\begin{aligned}
\omega_0 &= 2\sqrt{\frac{3T}{LM}} \\
\omega_n &= 4\sqrt{\frac{3T}{LM}}\sin\frac{n\pi}{8} \\
f'_n &= \frac{2}{\pi}\sqrt{\frac{3T}{LM}}\sin\frac{n\pi}{8} \\
f'_1 &= \frac{2}{\pi}\sqrt{\frac{3T}{LM}}\frac{\sqrt{2-\sqrt{2}}}{2} \\
&= \frac{2\sqrt{3(2-\sqrt{2})}}{\pi}\sqrt{\frac{T}{4LM}} \\
&\approx 0.84f_1 \\
f_2 &= \frac{2\sqrt{6}}{\pi}\sqrt{\frac{T}{4LM}} \\
&\approx 1.56f_1 \\
f_3 &= \frac{2\sqrt{3(2+\sqrt{2})}}{\pi}\sqrt{\frac{T}{4LM}} \\
&\approx 2.04f_1
\end{aligned}
$$

6.5

$$\begin{aligned}\omega &= 2\pi f\\ &= 2\pi\frac{v}{\lambda}\\ &= 2\pi\frac{\sqrt{T/\mu}}{2L}\\ &= \pi\frac{\sqrt{LT/m}}{L}\\ &= \pi\sqrt{\frac{T}{Lm}}\end{aligned}$$

6.6

(a)

$$\xi(x,t) = A\sin\left(\frac{\omega x}{v}\right)\cos\omega t$$

$$\begin{aligned}\cos\frac{\omega L}{2v} &= 0\\ \frac{\omega L}{2v} &= \left(n-\frac{1}{2}\right)\pi\\ \omega_n &= \left(n-\frac{1}{2}\right)\pi\frac{2v}{L}\\ &= \frac{(2n-1)\pi}{L}\sqrt{\frac{Y}{\rho}}\end{aligned}$$

(b)

$$\begin{aligned}\lambda_n &= \frac{v}{f_n}\\ &= \frac{2\pi v}{\omega_n}\\ &= 2\pi v\frac{L}{(2n-1)\pi v}\\ &= \frac{2L}{2n-1}\end{aligned}$$

(c) The nodes are at $x = 0$ and at each multiple of a half wavelength, i.e.

$$\frac{L}{2n-1}$$

6.9

(b)

$$\begin{aligned}
\xi(x,t) &= A_1 \sin\frac{\pi x}{L}\cos 100\pi t + A_2 \sin\frac{2\pi x}{L}\cos 200\pi t \\
\xi_{\max}(L/4) &= A_1\frac{\sqrt{2}}{2}\cos 100\pi t_1 + A_2\cos 200\pi t_1 \\
\xi_{\max}(L/2) &= A_1\cos 100\pi t_2 \\
\xi_{\max}(3L/4) &= A_1\frac{\sqrt{2}}{2}\cos 100\pi t_3 - A_2\cos 200\pi t_3
\end{aligned}$$

$$\begin{aligned}
A_1 &= \xi_{\max}(L/2) \\
&= 10\,\mu\text{m}
\end{aligned}$$

$$\begin{aligned}
\xi_{\max}(L/4) &= A_1\frac{\sqrt{2}}{2} + A_2 \\
&= \xi_{\max}(L/2)\frac{\sqrt{2}}{2} + A_2 \\
A_2 &= \xi_{\max}(L/4) - \xi_{\max}(L/2)\frac{\sqrt{2}}{2} \\
&= (10\,\mu\text{m}) - (10\,\mu\text{m})\frac{\sqrt{2}}{2} \\
&= (10\,\text{mm})\left(1 - \frac{\sqrt{2}}{2}\right) \\
&\approx 2.93\,\mu\text{m}
\end{aligned}$$

6.10

(a)

$$f_n = \frac{nc}{2L}$$

(b) (i)

$$\begin{aligned}
n_{\min} &= \frac{2f_{\min}L}{c} \\
&= 4\,999\,990 \\
n_{\max} &= \frac{2f_{\max}L}{c} \\
&= 5\,000\,010 \\
n_{\max} - n_{\min} + 1 &= 21
\end{aligned}$$

(ii)

$$\begin{aligned}
n_{\max} - n_{\min} &< 2 \\
\frac{2f_{\max}L}{c} - \frac{2f_{\min}L}{c} &< 2 \\
L &< \frac{c}{f_{\max} - f_{\min}} \\
&= 15\,\mathrm{cm}
\end{aligned}$$

6.11

(a)

$$\begin{aligned}
y_n(x,t) &= A_n \sin\frac{n\pi x}{L} \sin\left(\frac{n\pi}{L}\sqrt{\frac{LT}{M}}t\right) \\
y_n'(x,t) &= A_n \frac{n\pi}{L}\sqrt{\frac{LT}{M}} \sin\frac{n\pi x}{L} \cos\left(\frac{n\pi}{L}\sqrt{\frac{LT}{M}}t\right) \\
y_n'(x,0) &= A_n \frac{n\pi}{L}\sqrt{\frac{LT}{M}} \sin\frac{n\pi x}{L} \\
dE &= \frac{1}{2} y_n'(x,0)^2\, dm \\
&= \frac{1}{2}\left(A_n \frac{n\pi}{L}\sqrt{\frac{LT}{M}} \sin\frac{n\pi x}{L}\right)^2 \frac{M}{L}\, dx \\
E &= \int dE \\
&= \frac{1}{2}\left(A_n \frac{n\pi}{L}\sqrt{\frac{LT}{M}}\right)^2 \frac{M}{L} \int_0^L \sin^2\frac{n\pi x}{L}\, dx \\
&= \frac{A_n^2 n^2 \pi^2 T}{2L^2}\left[\frac{x}{2} - \frac{L}{4n\pi}\sin\frac{2n\pi x}{L}\right]_0^L \\
&= \frac{A_n^2 n^2 \pi^2 T}{4L}
\end{aligned}$$

(b)

$$\begin{aligned}
E &= \frac{A_1^2 1^2 \pi^2 T}{4L} + \frac{A_3^2 3^2 \pi^2 T}{4L} \\
&= \frac{\pi^2 T}{4L}(A_1^2 + 9A_3^2)
\end{aligned}$$

6.12

(a)

$$\begin{aligned} W &= \int_0^h 2T\sin\theta\,dy \\ &= 2T\int_0^h \frac{y}{L/2}\,dy \\ &= \frac{4T}{L}\int_0^h y\,dy \\ &= \frac{4T}{L}\left[\frac{1}{2}y^2\right]_0^h \\ &= \frac{2h^2T}{L} \end{aligned}$$

(b)

$$\begin{aligned} T &= \frac{2\pi}{\omega} \\ &= \frac{2L}{v} \\ &= 2\sqrt{\frac{LM}{T}} \end{aligned}$$

6.14

(a)

$$\begin{aligned} y(x) &= Ax(L-x) \\ B_n &= \frac{4AL^2}{\pi^3n^3}(1-\cos(\pi n)) \\ &= \begin{cases} 0 & \text{even } n \\ \frac{8AL^2}{\pi^3n^3} & \text{odd } n \end{cases} \\ y(x) &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \\ &= \frac{8AL^2}{\pi^3}\left[\sin\left(\frac{\pi x}{L}\right) + \frac{1}{27}\sin\left(\frac{3\pi x}{L}\right) + \frac{1}{225}\sin\left(\frac{5\pi x}{L}\right) + \cdots\right] \end{aligned}$$

(b)

$$
\begin{aligned}
y(x) &= A \sin \frac{\pi x}{L} \\
B_n &= \frac{2}{L} \int_0^L A \sin \frac{\pi x}{L} \sin \frac{n \pi x}{L} \, dx \\
&= \begin{cases} A & n = 1 \\ 0 & n \neq 1 \end{cases} \\
y(x) &= A \sin \frac{\pi x}{L}
\end{aligned}
$$

(c)

$$
\begin{aligned}
y(x) &= \begin{cases} A \sin \frac{2\pi x}{L} & 0 \leq x \leq \frac{L}{2} \\ 0 & \frac{L}{2} \leq x \leq L \end{cases} \\
B_n &= \frac{2}{L} \int_0^L y(x) \sin \frac{n \pi x}{L} \, dx \\
&= \frac{2}{L} \int_0^{L/2} A \sin \frac{2\pi x}{L} \sin \frac{n \pi x}{L} \, dx \\
&= \frac{4A \sin \frac{n\pi}{2}}{\pi (4 - n^2)} \\
&= \begin{cases} \frac{A}{2} & n = 2 \\ 0 & \text{even } n \\ (-1)^{(n-1)/2} \frac{4A}{\pi(4-n^2)} & \text{odd } n \end{cases}
\end{aligned}
$$

7 Progressive waves

7.2

(a)

$$
\begin{aligned}
y &= 0.3 \sin \pi(0.5x - 50t) \\
A &= 0.3\,\mathrm{cm}
\end{aligned}
$$

$$
\begin{aligned}
\frac{2\pi}{\lambda} &= \frac{\pi}{2} \\
\lambda &= 4\,\mathrm{cm}
\end{aligned}
$$

$$
k = \frac{1}{4}\,\mathrm{rad/cm}
$$

$$
\begin{aligned}
2\pi f &= 50\pi \\
f &= 25\,\mathrm{Hz}
\end{aligned}
$$

$$
\begin{aligned}
T &= \frac{1}{25}\,\mathrm{s} \\
v &= f\lambda \\
&= 1\,\mathrm{m/s}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{\partial y}{\partial t} &= -15\pi \cos \pi(0.5s - 50t) \\
\left.\frac{\partial y}{\partial t}\right|_{\max} &= 15\pi\,\mathrm{cm/s}
\end{aligned}
$$

7.3

$$
\xi = 0.003 \sin 10\pi \left(\frac{x}{3000} + t\right)
$$

7.4

(a)

$$
\begin{aligned}
f &= 20\,\mathrm{Hz} \\
v &= 80\,\mathrm{m/s} \\
\lambda &= \frac{v}{f} \\
&= 4\,\mathrm{m} \\
k &= \frac{1}{\lambda} \\
&= \frac{1}{4}\,\mathrm{rad/m} \\
\frac{\pi}{6} &= 2\pi k x \\
x &= \frac{1}{12k} \\
&= \frac{1}{3}\,\mathrm{m}
\end{aligned}
$$

(b)

$$\omega \Delta t = 2\pi f \Delta t = 72^\circ$$

7.5

(a)

$$
\begin{aligned}
\mu &= 0.1\,\mathrm{kg/m} \\
F &= 50\,\mathrm{N} \\
A &= 0.02\,\mathrm{m} \\
T &= 0.1\,\mathrm{s} \\
v &= \sqrt{\frac{F}{\mu}} \\
&\approx 22.4\,\mathrm{m/s}
\end{aligned}
$$

(b)

$$\lambda = \frac{v}{f} = vT = 2.24\,\mathrm{m}$$

(c)

$$y(x,t) = 0.02 \sin\left(2.81x - 20\pi t + \frac{1}{6}\pi\right)$$

7.6

(a)

$$\begin{aligned}
\frac{L}{v} &= 0.1\,\mathrm{s} \\
\frac{L}{\sqrt{T/\mu}} &= 0.1\,\mathrm{s} \\
L\sqrt{\frac{\mu}{100gL\mu}} &= 0.1\,\mathrm{s} \\
\sqrt{\frac{L}{100g}} &= 0.1\,\mathrm{s} \\
L &= g(1\,\mathrm{s}^2) \\
&\approx 10\,\mathrm{m}
\end{aligned}$$

(b)

$$y(x,t) = A\sin\left(\frac{3\pi}{L}x\right)\cos 30\pi t$$

7.7

$$\begin{aligned}
y(x,t) &= 0.02\sin\pi(x - vt) \\
&= 0.02\sin\pi(x - 100t) \\
y(5, 0.1) &= 0\,\mathrm{cm} \\
\frac{\partial y}{\partial t} &= -2\pi\cos\pi(x - 100t) \\
\left.\frac{\partial y}{\partial t}\right|_{x=5\,\mathrm{m}, t=0.1\,\mathrm{s}} &= 6.28\,\mathrm{m/s}
\end{aligned}$$

7.8

(a)

$$f = \frac{3}{2}\,\mathrm{Hz}$$

(b) For a wave moving in the positive x direction:

$$\begin{aligned}
-\frac{2\pi}{\lambda}x_2 &= -2n\pi + \frac{\pi}{8} \\
&= \frac{-16n\pi + \pi}{8} \\
\lambda &= \frac{16}{16n - 1},\ n = 1,\ 2,\ 3,\ \ldots
\end{aligned}$$

For a wave moving in the negative x direction:

$$\begin{aligned}\frac{2\pi}{\lambda} &= 2n\pi + \frac{\pi}{8}\\ &= \frac{16n\pi + \pi}{8}\\ \lambda &= \frac{16}{16n+1},\, n = 0,\, 1,\, 2,\, \ldots\end{aligned}$$

(c)

$$\begin{aligned}v_{+} &= f\lambda\\ &= \frac{3}{2}\frac{16}{16n-1}\\ &= \frac{24}{16n-1}\\ &= \frac{8}{5}\,\mathrm{m/s},\, \frac{24}{31}\,\mathrm{m/s},\, \ldots\\ v_{-} &= -f\lambda\\ &= -\frac{3}{2}\frac{16}{16n+1}\\ &= -\frac{24}{16n+1}\\ &= -24\,\mathrm{m/s},\, -\frac{24}{17},\mathrm{m/s},\, \ldots\end{aligned}$$

(d) We can't tell

7.12

(b)

$$v_{\max} \approx 4\,\mathrm{m/s}$$

(c)

$$\begin{aligned}
v &= \sqrt{\frac{T}{\mu}} \\
&= \sqrt{\frac{T}{m/L}} \\
&= \sqrt{\frac{LT}{m}} \\
v^2 &= \frac{LT}{m} \\
T &= \frac{mv^2}{L} \\
&= \frac{2 \cdot 40^2}{100} \\
&= 32\,\mathrm{N}
\end{aligned}$$

(d)

$$y(x,t) = 0.2 \sin\left(16\pi t + \frac{2\pi}{5} x\right)$$

7.13

(b) $v = \frac{u}{2}$ m/s in the positive x direction

(c)

$$\begin{aligned}
y(x,t) &= \frac{b^3}{b^2 + (2x - ut)^2} \\
\frac{\partial y}{\partial t} &= \frac{2b^3 u(2x - ut)}{(b^2 + (2x - ut)^2)^2} \\
\left.\frac{\partial y}{\partial t}\right|_{t=0} &= \frac{4b^3 ux}{(b^2 + 4x^2)^2}
\end{aligned}$$

7.16

(a)

$$\begin{aligned} \mu &= \frac{m}{L} \\ &= 4 \times 10^{-4}\,\mathrm{kg/m} \\ T &= mg \\ &= 100\,\mathrm{N} \\ v &= \sqrt{\frac{T}{\mu}} \\ &= 500\,\mathrm{m/s} \end{aligned}$$

The horizontal portion of the string corresponds to the time when the switch was completely closed. It is 40 cm long meaning the switch was closed for

$$\frac{0.4\,\mathrm{m}}{500\,\mathrm{m/s}} = 0.8 \times 10^{-4}\,\mathrm{s}$$

(c) The portion of the string corresponding to the opening of the contact has a steeper slope meaning it was moving faster. It moved 5 mm over $\frac{0.2\,\mathrm{m}}{500\,\mathrm{m/s}} = 4 \times 10^{-4}\,\mathrm{s}$ meaning its speed was $\frac{0.005\,\mathrm{m}}{4\times 10^{-4}\,\mathrm{s}} = 12.5\,\mathrm{m/s}$

(d) If the contact started moving at $t = 0$ then the photo was taken at $\frac{1\,\mathrm{m}}{500\,\mathrm{m/s}} = 2 \times 10^{-3}\,\mathrm{s}$

7.17

(a)

$$\begin{aligned} y &= A\sin(5x - 10t) + A\sin(4x - 9t) \\ &= 2A\cos\pi\left[\left(\frac{5}{2\pi} - \frac{4}{2\pi}\right)x - \left(\frac{10}{2\pi} - \frac{9}{2\pi}\right)t\right] \\ &\quad \times \sin 2\pi\left[\frac{\frac{5}{2\pi} + \frac{4}{2\pi}}{2}x - \frac{\frac{10}{2\pi} + \frac{9}{2\pi}}{2}t\right] \\ &= 2A\cos\left(\frac{x}{2} - \frac{t}{2}\right)\sin\left(\frac{9}{2}x - \frac{19}{2}t\right) \end{aligned}$$

(b)

$$
\begin{aligned}
2\pi f &= \frac{1}{2} \\
f &= \frac{1}{4\pi}\,\mathrm{Hz}
\end{aligned}
$$

$$
\begin{aligned}
\frac{2\pi}{\lambda} &= \frac{1}{2} \\
\lambda &= 4\pi\,\mathrm{m}
\end{aligned}
$$

$$
\begin{aligned}
v &= f\lambda \\
&= 1\,\mathrm{m/s}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\frac{2\pi}{\lambda} &= \frac{1}{2} \\
\frac{\lambda}{2} &= 2\pi\,\mathrm{m}
\end{aligned}
$$

7.18

(a)

$$
\begin{aligned}
v_p &= \sqrt{\frac{2\pi S}{\rho\lambda}} \\
\frac{f}{k} &= \sqrt{\frac{2\pi k S}{\rho}} \\
f &= \sqrt{\frac{2\pi k^3 S}{\rho}} \\
v_g = \frac{df}{dk} &= \frac{1}{2}\left(\frac{2\pi k^3 S}{\rho}\right)^{-1/2} \frac{6\pi k^2 S}{\rho} \\
&= \frac{3}{2\sqrt{k}}\sqrt{\frac{2\pi k^2 S}{\rho}} \\
&= \frac{3}{2} v_p
\end{aligned}
$$

(b) The group moves faster than the individual waves within the group. Waves will appear and disappear at the edges of the group.

(c)

$$\begin{aligned}
\frac{2\pi}{\lambda} &= \pi\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) \\
&= \frac{\pi(\lambda_2 - \lambda_1)}{\lambda_1\lambda_2} \\
\frac{\lambda}{2} &= \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1} \\
&\approx 50\,\mathrm{cm}
\end{aligned}$$

7.20

(a)

$$\begin{aligned}
K &= \frac{1}{2}\rho Al\left(\frac{dy}{dt}\right)^2 \\
U &= g\rho Ay^2 \\
\omega^2 &= \frac{2g\rho A}{\rho Al} \\
&= \frac{2g}{l}
\end{aligned}$$

$$\begin{aligned}
T &= \frac{1}{f} \\
&= \frac{2\pi}{\omega} \\
&= 2\pi\sqrt{\frac{l}{2g}} \\
&= \pi\sqrt{\frac{2l}{g}}
\end{aligned}$$

(b)

$$\begin{aligned}
v &= f\lambda \\
&= \frac{\lambda}{T} \\
&= \frac{\lambda}{\pi\sqrt{\lambda/g}} \\
&= \frac{\lambda}{\pi}\sqrt{\frac{g}{\lambda}} \\
&= \frac{\sqrt{g\lambda}}{\pi}
\end{aligned}$$

(c)

$$v(\lambda) = \sqrt{\frac{g\lambda}{2\pi}}$$
$$v(500\,\mathrm{m}) \approx 28.2\,\mathrm{m/s}$$

7.21

(a)

$$\lambda_n = \frac{2(N+1)l}{n}$$
$$\omega_n = 2\omega_0 \sin\frac{n\pi}{2(N+1)}$$

8 Boundary effects and interference

8.1

$$
\begin{aligned}
\frac{A_{r,0}}{A} &= 1 \\
\frac{A_{r,0.25}}{A} &= \frac{v_2 - v_1}{v_2 + v_1} \\
&= \frac{\sqrt{\frac{T}{\mu_2}} - \sqrt{\frac{T}{\mu_1}}}{\sqrt{\frac{T}{\mu_2}} + \sqrt{\frac{T}{\mu_1}}} \\
&= \frac{\frac{1}{\sqrt{\mu_2}} - \frac{1}{\sqrt{\mu_1}}}{\frac{1}{\sqrt{\mu_1}} + \frac{1}{\sqrt{\mu_1}}} \\
&= \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1 \mu_2}} \frac{\sqrt{\mu_1 \mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}} \\
&= \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}} \\
&= \frac{1 - \sqrt{\frac{\mu_2}{\mu_1}}}{1 + \sqrt{\frac{\mu_2}{\mu_1}}} \\
&= \frac{1 - \frac{1}{2}}{1 + \frac{1}{2}} \\
&= \frac{1}{2} \frac{2}{3} \\
&= \frac{1}{3} \\
\frac{A_{r,1}}{A} &= 0 \\
\frac{A_{r,4}}{A} &= \frac{1 - 2}{1 + 2} \\
&= -\frac{1}{3} \\
\frac{A_{r,\infty}}{A} &= -1
\end{aligned}
$$

$$
\begin{aligned}
\frac{A_{t,0}}{A} &= \frac{2v_2}{v_1 + v_2} \\
&= \frac{2\sqrt{\frac{T}{\mu_2}}}{\sqrt{\frac{T}{\mu_1}} + \sqrt{\frac{T}{\mu_2}}} \\
&= \frac{2\frac{1}{\sqrt{\mu_2}}}{\frac{1}{\sqrt{\mu_1}} + \frac{1}{\sqrt{\mu_2}}} \\
&= \frac{2}{\sqrt{\mu_2}} \frac{\sqrt{\mu_1 \mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}} \\
&= \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}} \\
&= \frac{2}{1 + \sqrt{\frac{\mu_2}{\mu_1}}} \\
&= 2 \\
\frac{A_{t,0.25}}{A} &= \frac{2}{1 + \frac{1}{2}} \\
&= \frac{4}{3} \\
\frac{A_{t,1}}{A} &= 1 \\
\frac{A_{t,4}}{A} &= \frac{2}{3} \\
\frac{A_{t,\infty}}{A} &= 0
\end{aligned}
$$

8.3

$$\begin{aligned}
P &= IV_X \\
&= I^2 X \\
&= \left(\frac{V}{R+X}\right)^2 X \\
\frac{dP}{dX} &= \left(\frac{V}{R+X}\right)^2 - \frac{2V^2X}{(R+X)^3} \\
&= \frac{V^2(R+X) - 2V^2X}{(R+X)^3} \\
&= \frac{V^2(R-X)}{(R+X)^3} \\
0 &= \frac{V^2(R-X)}{(R+X)^3} \\
X &= R
\end{aligned}$$

8.4

$$\begin{aligned}
Z_C &= \frac{1}{j\omega C} \\
Z_L &= j\omega L \\
Z_R &= R \\
Z &= Z_C + Z_L + Z_R \\
&= R + \left(L\omega - \frac{1}{C\omega}\right) j \\
&= \sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}\, e^{j \arctan \frac{L\omega - 1/C\omega}{R}} \\
I &= \frac{V}{Z} \\
&= \frac{V_0 e^{j\omega t}}{|Z| e^{j \arg Z}} \\
&= \frac{V_0}{|Z|} e^{j(\omega t - \arg Z)} \\
P_R &= I^2 R \\
&= \frac{RV_0^2}{|Z|^2} e^{j2(\omega t - \arg Z)} \\
\operatorname{Re} P_R &= \frac{RV_0^2}{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2} \cos 2 \left[\omega t - \arctan \frac{L\omega - 1/C\omega}{R}\right]
\end{aligned}$$

Power dissipation is maximised when

$$
\begin{aligned}
L\omega - \frac{1}{C\omega} &= 0 \\
L\omega &= \frac{1}{C\omega} \\
CL\omega^2 &= 1 \\
\omega &= \frac{1}{\sqrt{CL}}
\end{aligned}
$$

8.5

(a)

$$
\begin{aligned}
y_1(x,t) &= f_1\left(t - \frac{x}{v_1}\right) + g_1\left(t + \frac{x}{v_1}\right) \\
y_2(x,t) &= f_2\left(t - \frac{x}{v_2}\right)
\end{aligned}
$$

$$
\begin{aligned}
y_1(0,t) &= y_2(0,t) \\
f_1(t) + g_1(t) &= f_2(t)
\end{aligned}
$$

$$
\begin{aligned}
K_1 \frac{\partial y_1}{\partial x}(0,t) &= K_2 \frac{\partial y_2}{\partial x}(0,t) \\
\rho_1 v_1 (f_1'(t) - g_1'(t)) &= \rho_2 v_2 f_2'(t) \\
\rho_1 v_1 (f_1(t) - g_1(t)) &= \rho_2 v_2 f_2(t)
\end{aligned}
$$

$$
\begin{aligned}
\rho_1 v_1 (f_1(t) + f_1(t) - f_2(t)) &= \rho_2 v_2 f_2(t) \\
2\rho_1 v_1 f_1(t) &= (\rho_1 v_1 + \rho_2 v_2) f_2(t) \\
f_2(t) &= \frac{2\rho_1 v_1}{\rho_1 v_1 + \rho_2 v_2} f_1(t) \\
&\approx 5.85 \times 10^{-4} f_1(t)
\end{aligned}
$$

(b)

$$\begin{aligned}
\frac{E_2}{E_1} &= \frac{\frac{1}{2}\frac{v_2}{f}\rho_2(2\pi f A_2)^2}{\frac{1}{2}\frac{v_1}{f}\rho_1(2\pi f A_1)^2} \\
&= \frac{A_2^2\rho_2 v_2}{A_1^2\rho_1 v_1} \\
&= \frac{(5.85\times 10^{-4}A_1)^2\rho_2 v_2}{A_1^2\rho_1 v_1} \\
&= (5.85\times 10^{-4})^2\frac{\rho_2 v_2}{\rho_1 v_1} \\
&\approx 1.17\times 10^{-3}
\end{aligned}$$

8.7

(a)

$$\begin{aligned}
v &= \sqrt{\frac{\gamma R T(z)}{M}} \\
\frac{dv}{dz} &= \frac{1}{2}\sqrt{\frac{\gamma R}{M T(z)}}T'(z) \\
R &= \frac{v}{dv/dz} \\
&= \frac{\sqrt{\gamma R T(z)/M}}{\frac{1}{2}T'(z)\sqrt{\gamma R/MT(z)}} \\
&= \frac{2T(z)}{T'(z)}
\end{aligned}$$

8.8

(a)

$$\begin{aligned}
f_{\max} - f_{\min} &= \frac{f}{1-\frac{u}{v}} - \frac{f}{1+\frac{u}{v}} \\
&\approx 315\,\mathrm{Hz}
\end{aligned}$$

8.9

$$
\begin{aligned}
\lambda_{\max} - \lambda_{\min} &= \lambda_0 \left(1 + \frac{u}{v} - 1 + \frac{u}{v}\right) \\
&= \frac{2\lambda_0 u}{v} \\
u &= \frac{v(\lambda_{\max} - \lambda_{\min})}{2\lambda_0} \\
&= 1000\,\mathrm{m/s} \\
u &= \sqrt{\frac{3KT}{M}} \\
T &= \frac{Mu^2}{3K} \\
&\approx 920\,\mathrm{K}
\end{aligned}
$$

8.13

(a)

$$
\begin{aligned}
t_R - t_S &= \frac{\sqrt{h^2 + (ut_S)^2}}{v} \\
v(t_R - t_S) &= \sqrt{h^2 + (ut_S)^2} \\
v^2(t_R - t_S)^2 &= h^2 + (ut_S)^2 \\
v^2(t_R^2 - 2t_R t_S + t_S^2) &= h^2 + (ut_S)^2 \\
(v^2 - u^2)t_S^2 - 2t_R v^2 t_S + t_R^2 v^2 - h^2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
t_S &= \frac{2t_R v^2 \pm \sqrt{4t_R^2 v^4 - 4(v^2 - u^2)(t_R^2 v^2 - h^2)}}{2(v^2 - u^2)} \\
&= \frac{t_R \pm \sqrt{t_R^2 - \left(1 - \frac{u^2}{v^2}\right)\left(t_R^2 - \frac{h^2}{v^2}\right)}}{1 - \frac{u^2}{v^2}} \\
\left(1 - \frac{u^2}{v^2}\right) t_S &= t_R \pm \sqrt{t_R^2 - \left(1 - \frac{u^2}{v^2}\right)\left(t_R^2 - \frac{h^2}{v^2}\right)} \\
&= t_R \pm \sqrt{t_R^2 - \left(t_R^2 - \frac{h^2}{v^2} - t_R^2 \frac{u^2}{v^2} + \frac{h^2 u^2}{v^4}\right)} \\
&= t_R \pm \frac{1}{v}\sqrt{h^2\left(1 - \frac{u^2}{v^2}\right) + u^2 t_R^2}
\end{aligned}
$$

(b)

$$\begin{aligned}
\cos\theta &= \frac{ut_S}{\sqrt{h^2 + (ut_S)^2}} \\
f(t_S) &= \frac{f_0}{1 - \frac{u\cos\theta}{v}} \\
&= \frac{f_0}{1 - \frac{u}{v}\frac{ut_S}{\sqrt{h^2+(ut_S)^2}}}
\end{aligned}$$

(c) When the source is far away the h^2 term can be dropped giving

$$\begin{aligned}
f_{\mathrm{max}} &= \frac{f_0}{1 - \frac{u}{v}} \\
f_{\mathrm{max}}\left(1 - \frac{u}{v}\right) &= f_0 \\
f_{\mathrm{min}} &= \frac{f_0}{1 + \frac{u}{v}} \\
f_{\mathrm{min}}\left(1 + \frac{u}{v}\right) &= f_0 \\
f_{\mathrm{max}}\left(1 - \frac{u}{v}\right) &= f_{\mathrm{min}}\left(1 + \frac{u}{v}\right) \\
f_{\mathrm{max}}(v - u) &= f_{\mathrm{min}}(v + u) \\
(f_{\mathrm{min}} + f_{\mathrm{max}})u &= (f_{\mathrm{max}} - f_{\mathrm{min}})v \\
u &= \frac{f_{\mathrm{max}} - f_{\mathrm{min}}}{f_{\mathrm{max}} + f_{\mathrm{min}}}v \\
&\approx 34.3\,\mathrm{m/s}
\end{aligned}$$

$$
\begin{aligned}
f_{\max} &= \frac{f_0}{1 + \frac{u^2 t_{S,\max}}{v\sqrt{h^2 + (u t_{S,\max})^2}}} \\
&= \frac{f_0}{1 - \frac{2353}{v\sqrt{h^2+4706}}} \\
f_{\max}\left(1 - \frac{2353}{v\sqrt{h^2+4706}}\right) &= f_0 \\
f_{\min} &= \frac{f_0}{1 + \frac{u^2 t_{S,\min}}{v\sqrt{h^2 + (u t_{S,\min})^2}}} \\
&= \frac{f_0}{1 + \frac{2353}{v\sqrt{h^2+4706}}} \\
f_{\min}\left(1 + \frac{2353}{v\sqrt{h^2+4706}}\right) &= f_0 \\
f_{\max}\left(1 - \frac{2353}{v\sqrt{h^2+4706}}\right) &= f_{\min}\left(1 + \frac{2353}{v\sqrt{h^2+4706}}\right) \\
f_{\max}\left(\sqrt{h^2+4706} - \frac{2353}{v}\right) &= f_{\min}\left(\sqrt{h^2+4706} + \frac{2353}{v}\right) \\
(f_{\max} - f_{\min})\sqrt{h^2 - 4706} &= (f_{\max} + f_{\min})\frac{2353}{v} \\
\sqrt{h^2 - 4706} &= \frac{f_{\max} + f_{\min}}{f_{\max} - f_{\min}}\frac{2353}{v} \\
h^2 &= 4706 + \left(\frac{f_{\max} + f_{\min}}{f_{\max} - f_{\min}}\frac{2353}{v}\right)^2 \\
h &= \sqrt{4706 + \left(\frac{f_{\max} + f_{\min}}{f_{\max} - f_{\min}}\frac{2353}{v}\right)^2} \\
&\approx 350\,\mathrm{m}
\end{aligned}
$$