Advanced Engineering Mathematics Partial Differential Equations by Dennis G. Zill Notes

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12 Orthogonal Functions and Fourier Series

12.1 Orthogonal Functions

- The **inner product** of two functions f_1 and f_2 on an interval $[a, b]$ is the number
$$(f_1, f_2) = \int_a^b f_1(x) f_2(x)\, dx.$$
- Two functions f_1 and f_2 are said to be orthogonal on an interval if $(f_1, f_2) = 0$.
- A set of real-valued functions $\{\phi_1(x), \phi_2(x), \ldots, \phi_n(x)\}$ is said to be **orthogonal** on an interval if
$$(\phi_i, \phi_j) = 0 \text{ for } i \neq j.$$
- The **square norm** of a function is
$$||\phi_n(x)||^2 = (\phi_n, \phi_n)$$
and thus its **norm** is
$$||\phi_n(x)|| = \sqrt{(\phi_n, \phi_n)}.$$
- An **orthonormal set** of functions is an orthogonal set of functions that all have a norm of 1.

- An orthogonal set can be made into an orthonormal set by dividing each member by its norm.
- If $\{\phi_n(x)\}$ is an infinite orthogonal set of functions on an interval $[a, b]$ and $f(x)$ is an arbitrary function, then it's possible to determine a set of coefficients $c_n, n = 0, 1, 2, \ldots$ such that

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) = c_0\phi_0(x) + c_1\phi_1(x) + \ldots + c_n\phi_n(x) + \ldots$$

  This is called an **orthogonal series expansion** of f or a **generalized Fourier series** where the coefficients are given by

$$c_n = \frac{(f, \phi_n)}{||\phi_n||^2}.$$

- A set of real-valued functions $\{\phi_n(x)\}$ is said to be **orthogonal with respect to a weight function** $w(x)$ on the interval $[a, b]$ if

$$\int_a^b w(x)\phi_m(x)\phi_n(x)\,dx = 0,\ \ m \neq n.$$

12.2 Fourier Series

- The **Fourier series** of a function f defined on the interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$$

  where

$$\begin{aligned} a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\,dx \\ a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx \\ b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx \end{aligned}$$

- At points of discontinuity in f, the Fourier series takes on the average of the values either side of it.
- The Fourier series of a function f gives a **periodic extension** of the function outside the interval $(-p, p)$.