Vibrations and Waves by George C. King Notes

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Contents

1 Simple Harmonic Motion

- The equation of motion for a simple harmonic oscillator is

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

  where

$$\omega^2 = \frac{k}{m}$$

- The general solution of the equation of motion for a simple harmonic oscillator is

$$x = A\cos(\omega t + \phi)$$

  or equivalently

$$x = a\cos\omega t + b\sin\omega t$$

- The angular frequency ω is determined entirely by properties of the oscillator, e.g. its mass and spring coefficient

- The total energy of a harmonic oscillator is

$$E = \frac{1}{2}kA^2$$

- Nearly all potential wells have a shape that is parabolic when sufficiently close to the equilibrium position, so most oscillating systems will oscillate with SHM when the amplitude of oscillation is small

- The vibrations of nuclei in a molecule can be modeled by SHM, but only a discrete set of vibrational energies is possible, namely

$$\frac{1}{2}\hbar\omega,\ \frac{3}{2}\hbar\omega,\ \frac{5}{2}\hbar\omega,\ \ldots$$

  where $\hbar$ is Planck's constant divided by 2π

- The total energy of a system undergoing SHM is always given by an expression of the form

$$E = \frac{1}{2}\alpha v^2 + \frac{1}{2}\beta x^2$$

  where α and β are physical constants — if we obtain this equation during the analysis of a system we know we have SHM

- The equation of motion for a system described by the energy equation above is

$$\frac{d^2x}{dt^2} = -\frac{\beta}{\alpha}x$$

2 The Damped Harmonic Oscillator

- The equation of motion of a damped harmonic oscillator is

$$\begin{aligned} F = ma &= -kx - bv \\ m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx &= 0 \\ \frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x &= 0 \end{aligned}$$

  where $\gamma = b/m$ and $\omega_0^2 = k/m$

- ω_0 is known as the **natural frequency of oscillation**, i.e. the oscillation frequency if there were no damping

- **Light damping / underdamped**

  - The motion is still oscillatory but the amplitude decreases expontentially

- This occurs when $\gamma^2/4 < \omega_0^2$
- The general solution is

$$x = A_0 e^{-\gamma t/2} \cos(\omega t + \phi)$$

where A_0 is the initial amplitude
- Successive maxima decrease by the same fractional amount

$$\frac{A_n}{A_{n+1}} = e^{\gamma T/2}$$

- The natural logarithm of A_n/A_{n+1} is called the **logarithmic decrement**

$$\ln\left(\frac{A_n}{A_{n+1}}\right) = \frac{\gamma T}{2}$$

- **Heavy damping / overdamped**
  - The motion is not oscillatory and returns sluggishly to the equilibrium position
  - This occurs when $\gamma^2/4 > \omega_0^2$
  - The general solution is

$$\begin{aligned} x &= e^{-\gamma t/2}[Ae^{\alpha t} + Be^{-\alpha t}] \\ &= Ae^{(\alpha-\gamma/2)t} + Be^{-(\alpha+\gamma/2)t} \end{aligned}$$

where $\alpha = \sqrt{\gamma^2/4 - \omega_0^2}$

- **Critical damping**
  - The motion is not oscillatory and returns as quickly as possible to the equilibrium position
  - This occurs when $\gamma^2/4 = \omega_0^2$
  - The general solution is

$$x = Ae^{-\gamma t/2} + Bte^{-\gamma t/2}$$

- The total energy of an underdamped system decreases over time

$$E = E_0 e^{-\gamma t}$$

where E_0 is the initial energy of the system

- The **decay time** or **time constant** of the system $\tau = 1/\gamma$ is the time it takes for its energy to decrease by a factor of e

- The **quality factor** of a harmonic oscillator is a dimensionless value that gives a measure of the degree of damping

$$Q = \frac{\omega_0}{\gamma}$$

where large values indicate little damping and small values indicate more damping

- The quality factor can also be used as a measure of fraction of energy lost (i.e. $\Delta E/E$) per cycle $2\pi/Q$ or per radian $1/Q$

3 Forced Oscillations

- The equation of motion for an undamped forced harmonic oscillator is

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos\omega t$$

the general solution of which is

$$x = A(\omega)\cos(\omega t - \delta)$$

where

$$A(\omega) = \frac{F_0}{k(1 - \omega^2/\omega_0^2)} \text{ and } \delta = 0$$

for $\omega < \omega_0$ and

$$A(\omega) = -\frac{F_0}{k(1 - \omega^2/\omega_0^2)} \text{ and } \delta = \pi$$

for $\omega > \omega_0$

- From the above it can be seen that:
  - $A(\omega) \to F_0/k$ as $\omega \to 0$
  - $A(\omega) \to \infty$ as $\omega \to \omega_0$
  - $A(\omega) \to 0$ as $\omega \to \infty$

- The equation of motion for a damped forced harmonic oscillator is

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

where $\gamma = b/m$ and $\omega_0^2 = k/m$ the general solution of which is

$$x = A(\omega)\cos(\omega t - \delta)$$

where

$$A(\omega) = \frac{F_0}{m[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2]^{1/2}}$$

and

$$\delta = \arctan\frac{\omega\gamma}{\omega_0^2 - \omega^2}$$

- From the above it can be seen that:
  - $A(\omega) \to F_0/k$ as $\omega \to 0$
  - $A(\omega) \to F_0\omega_0/k\gamma$ as $\omega \to \omega_0$
  - $A(\omega) \to 0$ as $\omega \to \infty$
- $A(\omega)$ is maximised when its denominator is minimised, leading to

  $$\omega_{\text{max}} = \omega_0(1 - \gamma^2/2\omega_0^2)^{1/2}$$

  and thus

  $$A_{\text{max}} = \frac{F_0\omega_0/\gamma}{k(1 - \gamma^2/4\omega_0^2)^{1/2}}$$

- The power absorbed by a damped oscillator to sustain its motion is exactly equal to the rate at which the energy is dissipated, i.e.

  $$\begin{aligned} P(t) &= bv(t) \times v(t) \\ &= b[v(t)]^2 \\ &= v[v_0(t)]^2 \sin^2(\omega t - \delta) \end{aligned}$$

- The average power absorbed over one cycle is

  $$\overline{P}(\omega) = \frac{b[v_0(\omega)]^2}{2} = \frac{\omega^2 F_0^2 \gamma}{2m[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2]}$$

- From the above it can be seen that:
  - $\overline{P}(\omega) \to 0$ as $\omega \to 0$
  - $\overline{P}(\omega) \to F_0^2/2m\gamma$ as $\omega \to \omega_0$
  - $\overline{P}(\omega) \to 0$ as $\omega \to \infty$
- The **power resonance curve** of an oscillating system graphs the average power absorbed by the system over a cycle to the driving frequency
- The **full width at half height** of a power resonance curve is the width of the curve at height $P_{\text{max}}/2$, is a measure of the sharpness of the system's response to an applied force, and is equal to $\omega_{\text{fwhh}} = \gamma = \omega_0/Q$
- From the above it can be seen that

  $$Q = \frac{\omega_0}{\gamma} = \frac{\omega_0}{\omega_{\text{fwhh}}}$$

- A resonance circuit can be used to amplify AC signals around a particular frequency by the Q-factor of the circuit — this makes them useful in radio receivers to tune a specific frequency

- When a driving force is first applied to a system, the system will be inclined to oscillate at its natural frequency ω_0. The behaviour of the system is described by the sum of two oscillations, one at frequency ω_0 and the other at ω. Eventually the ω_0 oscillations die out leaving the system in its **steady state** condition. The initial behaviour is reffered to as its **transient response**.
- The equation of motion for damped forced oscillations is the second-order nonhomogeneous linear differential equation

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t.$$

  The oscillations at frequency ω_0 present only during the transient response are described by the complementary function of this equation, i.e. a fundamental set of solutions of the associated homogeneous differential equation, and the oscillations at frequency ω are described by a particular solution of this equation.
- If $z = x + yi$, the **complex conjugate** of z is $z^* = x - yi$
- The product of a complex number with its conjugate is $zz^* = z^2 + y^2$
- The **modulus** of a complex number is defined as $|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}$
- Division of complex numbers can be performed like so

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 + y_2^2} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

- An **Argand diagram** is two-dimensional graph where the x-axis is used as the real axis and the y-axis is used as the imaginary axis
- Using **Euler's formula**

$$e^{ix} = \cos x + i\sin x$$

  a complex number can be represented as

$$z = x + iy = r(\cos\theta + i\sin\theta) = ze^{i\theta}$$

  where r is the modulus $|z|$ and θ is the angle of z from the positive x-axis known as its **argument**
- Multiplication of complex numbers is equivalent to rotation and scaling in the complex plane

$$r_1e^{i\theta} \times r_2e^{i\phi} = r_1r_2e^{i(\theta+\phi)}$$

- Phasor diagrams can be represented on the complex plane with phasors as complex numbers $z = Ae^{i(\omega t+\phi)}$ and their projection onto the x-axis as their real components
- Differentiation with respect to time of a complex phasor is equivalent to multiplication by $i\omega$

4 Coupled Oscillators

- Systems of two or more coupled oscillators can oscillate in multiple ways called **normal modes**, each with its own frequency called the **normal frequency**
- In a normal mode, each oscillator oscillates at the same frequency
- Without damping, once a system is in a normal mode it stays there
- The equations of motion of a system of coupled oscillators are a system of differential equations and thus the movements of the oscillators are described by a linear combination of the solutions of that system
- Those equations of motion are often intertwined and involve multiple variables, e.g. the positions of two pendulums x_1 and x_2. It's possible to introduce new variables called **normal coordinates** that result in independent solutions in one variable, e.g. $q_1 = x_a + x_b$ and $q_2 = x_a - x_b$
- Energy never flows from one normal mode to another
- In general it's difficult to determine the normal modes of the system a priori. A more general approach is to take advantage of the knowledge that in a normal mode all oscillators will oscillate at the same frequency and:
  1. assume solutions of the form $A\cos\omega t$, $B\cos\omega t$, etc.,
  2. subtitute them into the equations of motion, and
  3. rearrange to remove the constants A, B, etc. and solve for ω
- There are as many normal modes as there are degrees of freedom in the system, e.g. two coupled oscillators moving in one dimension have 2 normal modes, three coupled oscillators moving on two dimensions have 6 normal modes, etc.
- Coupled oscillators experience large amplitude oscillations when the driving frequency is close to the normal frequency
- The motion of driven coupled oscillators may be solved in a similar fashion to their free moving counterparts:
  1. Determine the equations of motion for the oscillators
  2. Combine the equations in such a way that the normal coordinates are evident
  3. Conver the equations to use normal coordinates
  4. Solve the resulting second-order nonhomogeneous linear differential equations by assuming solutions of the form $C_1\cos\omega_1 t$, etc.

5. Convert the solutions back from normal coordinates

- Oscillations that occur along the line connecting oscillators are called **longitudinal oscillations**
- Oscillations that occur perpendicular to the line connecting oscillators are called **transverse oscillations**

5 Travelling Waves

- The equation of a wave moving at velocity v in the positive x direction is of the form
$$y(x,t) = f(x - vt)$$
- The equation of a wave moving at velocity v in the negative x direction is of the form
$$y(x,t) = g(x + vt)$$
- The general form of any transverse wave motion can be written as
$$y = f(x - vt) + g(x + vt)$$
- The **wavelength** λ of a wave is the length of one complete pattern of the wave, e.g. between two maxima
- When the displacements of a wave lie in a single plane, e.g. the x-y plane, it is said to be **linearly polarised**
- The frequency f, wavelength λ, and velocity v of a wave are related by the expression
$$f\lambda = v$$
- The frequency f and period T of a wave are related by the expression
$$f = \frac{1}{T}$$
- The **wavenumber** of a wave
$$k = \frac{2\pi}{\lambda}$$
is a measure of radians per unit distance
- The **wave equation** in one dimension is
$$\frac{\partial^2\psi}{\partial t^2} = v^2\frac{\partial^2\psi}{\partial x^2}$$
and its general solution is
$$\psi = f(x - vt) + g(x - vt)$$

- An intuition for the wave equation is "the acceleration experienced by a point on the wave at a particular time is a constant multiple of the curvature of the wave at that point"

- The velocity of a wave in a taut string v is

$$v = \sqrt{\frac{T}{\mu}}$$

  where T is the tension in the string and μ is its mass per unit length

- The total kinetic and potential energies contained within one wavelength of a sinusoidal wave

$$y = A\sin(kx - \omega t)$$

  are equal and have the value

$$\frac{1}{4}\mu\omega^2 A^2 \lambda$$

  meaning the total energy is

$$\frac{1}{2}\mu\omega^2 A^2 \lambda$$

- The power of a sinusoidal wave, i.e. the energy carried by the wave past a point per unit time, is

$$P = \frac{1}{2}\mu\omega^2 A^2 v$$

- When a wave encounters a discontinuity, some fraction of the wave is transmitted and the remaining fraction is reflected

- The **incident wave** is the original wave

- On either side of a discontinuity, the displacement and the gradient must be the same at all times

- The ratio of the transmitted amplitude to the incident amplitude is

$$\frac{A_2}{A_1} = \frac{2k_1}{k_1 + k_2} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}} = T_{12}$$

  where k_n is the wave number of each medium, μ_n is the mass per unit length of each medium, and T_{12} is called the **transmission coefficient of amplitude**

- The transmission coefficient of amplitude is a positive value in the range $(0, 2)$, i.e. the transmitted wave is always in phase with the incident wave

- The ratio of the reflected amplitude to the incident amplitude is

$$\frac{B_1}{A_1} = \frac{k_1 - k_2}{k_1 + k_2} = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}} = R_{12}$$

where R_{12} is called the **reflection coefficient of amplitude**

- The reflection coefficient of amplitude is a value in the range $(-1, 1)$, i.e. if $\mu_1 > \mu_2$ the reflected wave will be in phase with the incident wave and if $\mu_1 < \mu_2$ it will be $90°$ out of phase

- The wave equation in two dimensions is

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right)$$

- For a sinusoidal wave travelling in two dimensions, the solution to the wave equation is

$$z(x, y, t) = A \cos(k_1 x + k_2 y - \omega t)$$

which is a planar wave with velocity

$$v = \sqrt{\frac{S}{\sigma}} = \frac{\omega}{\sqrt{k_1^2 + k_2^2}} = \frac{\omega}{k}$$

and angle from the positive x-axis ϕ where

$$\tan \phi = -\frac{k_1}{k_2}$$

- The wave equation for two-dimensional waves of circular symmetry is

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \frac{\partial z}{\partial r} = \frac{1}{v^2} \frac{\partial^2 z}{\partial t^2}$$

and its solutions involve Bessel functions but for large r it can be simplified to

$$\frac{\partial^2 z}{\partial r^2} = \frac{1}{v^2} \frac{\partial^2 z}{\partial t^2}$$

which has the same form as the one-dimensional wave equation with solutions of the form

$$z(r, t) = A \cos(kr - \omega t)$$

6 Standing Waves

- In a standing wave, a **node** is a point whose displacement is 0 at all times and an **antinode** is a point taht experiences that maximum displacement

- In a standing wave, each individual particle undergoes SHM about its equilibrium position but different particules have different amplitudes
- The general equation for a standing wave on a string is

$$y_n(x,t) = A_n \sin\left(\frac{n\pi}{L}x\right)\cos\omega_n t$$

  where

$$\omega_n = \frac{n\pi v}{L}$$

  and $n = 1, 2, 3, \ldots$ corresponds to a different standing wave pattern or **mode**
- The first normal mode can also be called the **fundamental mode**
- The period of a standing wave is

$$T = \frac{2\pi}{\omega_n} = \frac{2L}{nv}$$

- The wavelength of a standing wave is

$$\lambda_n = \frac{2L}{n}$$

- The wavenumber of a standing wave is

$$k_n = \frac{n\pi}{L}$$

- Using the definition of the wavenumber of a standing wave, the general equation can be rewritten

$$y_n(x,t) = A_n \sin k_n x \cos\omega t$$

- A standing wave is a superposition of two travelling waves of equal frequency and amplitude travelling in opposite directions
- The total energy of a string vibrating in the n-th mode is

$$E_n = \frac{1}{4}\mu L A_n^2 \omega_n^2$$

- Standing waves are the normal modes of a vibrating string
- The superposition principle states that if $y_1(x,t)$ and $y_2(x,t)$ are both solutions to the wave equation, then so is $c_1 y_1(x,t) + c_2 y_2(x,t)$
- In general, the motion of a vibrating string can be expressed as a superposition of normal modes of the string, i.e.

$$y(x,t) = \sum_n y_n(x,t) = \sum_n A_n \sin\left(\frac{n\pi}{L}x\right)\cos\omega_n t$$

- Any function $f(x)$ where $f(0) = f(L) = 0$ can be written as a superposition of sine waves
$$f(x) = \sum_n A_n \sin\left(\frac{n\pi}{L}x\right)$$
where
$$A_n = \frac{2}{L}\int_0^L dx \sin\left(\frac{n\pi}{L}x\right) f(x),\ n = 1,\ 2,\ \ldots$$

- This works because $\sin\frac{m\pi}{L}x$ and $\sin\frac{n\pi}{L}x$ are orthogonal when $m \neq n$, i.e.
$$\int_0^L dx \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right) = 0,\ m \neq n$$
but
$$\int_0^L dx \sin^2\left(\frac{n\pi}{L}x\right) = \frac{L}{2}$$
so integrating the product of $\sin\left(\frac{n\pi}{L}x\right)$ and $f(x)$ in A_n effectively "picks out" the amplitude of the n-th normal mode

- The total energy of a vibrating string expressed as a superposition of normal modes is given by
$$E = \sum_n E_n = \sum_n \frac{1}{4}\mu L A_n^2\omega_n^2 = \frac{1}{4}\mu L \sum_n A_n^2\omega_n^2$$

7 Interference and Diffraction of Waves

- If two waves are emitted from the same source, travel different distances, and recombine, the difference in distance travelled s determines the shape of the resulting wave:
  - If s is an integral multiple of λ the waves are said to be **in phase** and there is **constructive interference** and the amplitude of the resulting wave is double the amplitude of the original waves
$$s = n\lambda,\ n = 0,\ \pm1,\ \pm2,\ \ldots$$
or
$$\phi = 2n\pi,\ n = 0,\ \pm1,\ \pm2,\ \ldots$$
  - If s is an odd integral multiple of $\lambda/2$ the waves are said to be **out of phase** and there is **destructive interference** and the amplitude of the resulting wave is 0
$$s = \left(n + \frac{1}{2}\right)\lambda,\ n = 0,\ \pm1,\ \pm2,\ \ldots$$
or
$$\phi = (2n+1)\pi,\ n = 0,\ \pm1,\ \pm2,\ \ldots$$

  - Other values of s will result in behaviour somewhere between these two extremes

- **Huygen's principle** states that each point on a primary wavefront acts as a source of secondary wavelets such that the wavefront at some later time is the envelope of these wavelets, i.e.

  1. For each point on a primary wavefront draw an arc of radius $v\Delta t$ in the direction of travel where v is the velocity of the wave
  2. The envelope of the resulting arcs is the shape of the wave at time $t + \Delta t$

- Two waves are said to be **coherent** if their frequency and waveform are identical

- **Young's double-slit experiment** works via **division of wavefront** where a single source wave is sent through two narrow slits and the resulting waves interfere with each other on a far away detector

- Interference can also occur via **division of amplitude** where a single source wave is split via e.g. a semi-silvered mirror

- The bending or spreading of waves around obstances is called **diffraction**

- If a test point P is sufficiently far from a slit causing diffraction that the secondary wavelets have become plane waves we have **Fraunhofer diffraction**

- If the source of the primary waves or a test point P is too close to a slit causing diffraction that the curvature of the incoming or outgoing wavefronts must be taken into consideration we have **Fresnel diffraction**