Introduction to Quantum Mechanics by David J. Griffiths Problems

Chris Doble

March 2023

Contents

Part I
Theory

1 The Wave Function

1.1

(a)

$$\begin{aligned}
\langle j^2 \rangle &= \sum j^2 P(j) \\
&= 14^2 \frac{1}{14} + 15^2 \frac{1}{14} + 16^2 \frac{3}{14} + 22^2 \frac{2}{14} + 24^2 \frac{2}{14} + 25^2 \frac{5}{14} \\
&= \frac{3217}{7} \\
&\approx 459.571 \\
\langle j \rangle^2 &= \left(\sum j P(j) \right)^2 \\
&= 441
\end{aligned}$$

(b)

$$\begin{aligned}
\Delta j_{14} &= -7 \\
\Delta j_{15} &= -6 \\
\Delta j_{16} &= -5 \\
\Delta j_{22} &= 1 \\
\Delta j_{24} &= 3 \\
\Delta j_{25} &= 4 \\
\sigma^2 &= \sum (\Delta j)^2 P(j) \\
&= \frac{130}{7} \\
&\approx 18.571
\end{aligned}$$

(c)

$$\sigma^2 = \sqrt{\langle j^2 \rangle - \langle j \rangle^2} = 18.571$$

1.2

(a)

$$
\begin{aligned}
\langle x^2 \rangle &= \int_0^h x^2 \rho(x)\, dx \\
&= \int_0^h \frac{x^{3/2}}{2\sqrt{h}}\, dx \\
&= \frac{1}{2\sqrt{h}} \left[\frac{2}{5} x^{5/2} \right]_0^h \\
&= \frac{h^2}{5} \\
\langle x \rangle^2 &= \frac{h^2}{9} \\
\sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{h^2}{5} - \frac{h^2}{9}} \\
&= h\sqrt{\frac{4}{45}} \\
&= \frac{2}{3\sqrt{5}} h
\end{aligned}
$$

(b)

$$
\begin{aligned}
1 - \int_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} \rho(x)\, dx &= 1 - \frac{1}{2\sqrt{h}} [2\sqrt{x}]_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} \\
&= 1 - \frac{1}{\sqrt{h}} \left(\sqrt{\frac{1}{3}h + \frac{2}{3\sqrt{5}}h} - \sqrt{\frac{1}{3}h - \frac{2}{3\sqrt{5}}h} \right) \\
&= 1 - \left(\sqrt{\frac{1}{3} + \frac{2}{3\sqrt{5}}} - \sqrt{\frac{1}{3} - \frac{2}{3\sqrt{5}}} \right) \\
&\approx 0.393
\end{aligned}
$$

1.3

(a)

$$\begin{aligned}
\rho(x) &= Ae^{-\lambda(x-a)^2} \\
1 &= \int_{-\infty}^{\infty} \rho(x)\, dx \\
&= A\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2}\, dx \\
&= A\sqrt{\frac{\pi}{\lambda}} \\
A &= \sqrt{\frac{\lambda}{\pi}}
\end{aligned}$$

(b)

$$\begin{aligned}
\langle x\rangle &= \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} xe^{-\lambda(x-a)^2}\, dx \\
&= a \\
\langle x^2\rangle &= \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} x^2e^{-\lambda(x-a)^2}\, dx \\
&= a^2 + \frac{1}{2\lambda} \\
\sigma &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\
&= \sqrt{a^2 + \frac{1}{2\lambda} - a^2} \\
&= \frac{1}{\sqrt{2\lambda}}
\end{aligned}$$

1.4

(a)

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi(x,0)|^2 \, dx \\
&= \left(\frac{A}{a}\right)^2 \int_0^a x^2 \, dx + \left(\frac{A}{b-a}\right)^2 \int_a^b (b-x)^2 \, dx \\
&= \frac{1}{3} A^2 a + \left(\frac{A}{b-a}\right)^2 \left[-\frac{1}{3}(b-x)^3\right]_a^b \\
&= \frac{1}{3} A^2 a + \frac{1}{3} A^2 (b-a) \\
&= \frac{1}{3} A^2 b \\
A &= \sqrt{\frac{3}{b}}
\end{aligned}$$

(c) $x = a$

(d)

$$\begin{aligned}
\int_0^a |\Psi(x,0)|^2 \, dx &= \frac{3}{a^2 b} \left[\frac{1}{3} x^3\right]_0^a \\
&= \frac{a}{b}
\end{aligned}$$

(e)

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x |\Psi(x,0)|^2 \, dx \\
&= \frac{3}{a^2 b} \left[\frac{1}{4} x^4\right]_0^a + \frac{3}{b(b-a)^2} \int_a^b x(b-x)^2 \, dx \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \int_a^b (b^2 x - 2bx^2 + x^3) \, dx \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \left[\frac{1}{2} b^2 x^2 - \frac{2}{3} b x^3 + \frac{1}{4} x^4\right]_a^b \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \left(\frac{1}{2} b^4 - \frac{2}{3} b^4 + \frac{1}{4} b^4 - \frac{1}{2} a^2 b^2 + \frac{2}{3} a^3 b - \frac{1}{4} a^4\right) \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \frac{1}{12} (b-a)^3 (3a+b) \\
&= \frac{3a^2}{4b} + \frac{1}{4b} (3ab + b^2 - 3a^2 - ab) \\
&= \frac{1}{2} a + \frac{1}{4} b
\end{aligned}$$

1.5

(a)

$$
\begin{aligned}
\Psi(x,t) &= Ae^{-\lambda|x|}e^{-i\omega t} \\
\Psi(x,0) &= Ae^{-\lambda|x|} \\
1 &= A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|}\,dx \\
&= 2A^2 \int_0^{\infty} e^{-2\lambda x}\,dx \\
&= 2A^2 \left[-\frac{1}{2\lambda}e^{-2\lambda x}\right]_0^{\infty} \\
&= \frac{A^2}{\lambda} \\
A &= \sqrt{\lambda}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} x\lambda e^{-2\lambda|x|}\,dx \\
&= \lambda \int_{-\infty}^{\infty} xe^{-2\lambda|x|}\,dx \\
&= 0 \\
\langle x^2\rangle &= \int_{-\infty}^{\infty} x^2\lambda e^{-2\lambda|x|}\,dx \\
&= 2\lambda \int_0^{\infty} x^2 e^{-2\lambda x}\,dx \\
&= \frac{1}{2\lambda^2}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\sigma &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\
&= \frac{1}{\sqrt{2}\lambda} \\
1 - \int_{-\sigma}^{\sigma} \lambda e^{-2\lambda|x|}\,dx &= 1 - 2\lambda \int_0^{\sigma} e^{-2\lambda x}\,dx \\
&= 1 - 2\lambda \left[-\frac{1}{2\lambda}e^{-2\lambda x}\right]_0^{\sigma} \\
&= e^{-2\lambda\sigma} \\
&= e^{-\sqrt{2}} \\
&\approx 0.243
\end{aligned}
$$

1.6

The chain rule requires that you apply it to both x and $|\Psi|^2$ which gives the same result

$$
\begin{aligned}
\frac{d\langle x\rangle}{dt} &= \frac{d}{dt}\int x|\Psi|^2\,dx \\
&= \int \frac{d}{dt}(x|\Psi|^2)\,dx \\
&= \int \left(0\cdot|\Psi|^2 + x\frac{\partial|\Psi|^2}{\partial t}\right)\,dx \\
&= \int x\frac{\partial|\Psi|^2}{\partial t}\,dx
\end{aligned}
$$

1.8

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\left(e^{-iV_0t/\hbar}\Psi\right) &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(e^{-iV_0t/\hbar}\Psi\right) + (V+V_0)\left(e^{-iV_0t/\hbar}\Psi\right) \\
i\hbar\left(-\frac{iV_0}{\hbar}e^{-iV_0t/\hbar}\Psi + e^{-iV_0t/\hbar}\frac{\partial\Psi}{\partial t}\right) &= -\frac{\hbar^2}{2m}e^{-iV_0t/\hbar}\frac{\partial^2\Psi}{\partial x^2} + Ve^{-iV_0t/\hbar}\Psi + V_0e^{-iV_0t/\hbar}\Psi \\
V_0\Psi + i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi + V_0\Psi \\
i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi
\end{aligned}
$$

$$
\begin{aligned}
\langle Q(x,p)\rangle &= \int \left(e^{-iV_0t/\hbar}\Psi\right)^*\left[Q(x,-i\hbar\partial/\partial x)\right]e^{-iV_0t/\hbar}\Psi\,dx \\
&= \int e^{iV_0t/\hbar}\Psi^*\left[Q(x,-i\hbar\partial/\partial x)\right]e^{-iV_0t/\hbar}\Psi\,dx \\
&= \int \Psi^*[Q(x,-i\hbar\partial/\partial x)]\Psi\,dx
\end{aligned}
$$

No effect on the expectation value.

1.9

(a)

$$
\begin{aligned}
\Psi(x,t) &= Ae^{-a[(mx^2/\hbar)+it]} \\
1 &= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)}\,dx \\
&= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)}\,dx \\
&= A^2 \sqrt{\frac{\pi\hbar}{2am}} \\
A^2 &= \sqrt{\frac{2am}{\pi\hbar}} \\
A &= \left(\frac{2am}{\pi\hbar}\right)^{1/4}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\Psi &= Ae^{-a[(mx^2/\hbar)+it]} \\
\frac{\partial\Psi}{\partial t} &= -ia\Psi \\
\frac{\partial\Psi}{\partial x} &= -\frac{2amx}{\hbar}\Psi \\
\frac{\partial^2\Psi}{\partial x^2} &= -\frac{2am}{\hbar}\left(\Psi + x\frac{\partial\Psi}{\partial x}\right) \\
&= -\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\Psi \\
V\Psi &= i\hbar\frac{\partial\Psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} \\
&= a\hbar\Psi - a\hbar\left(1 - \frac{2amx^2}{\hbar}\right)\Psi \\
V &= a\hbar - a\hbar + 2a^2mx^2 \\
&= 2a^2mx^2
\end{aligned}
$$

(c)

$$
\begin{aligned}
\langle x\rangle &= A^2\int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)}x\,dx \\
&= 0 \\
\langle x^2\rangle &= A^2\int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)}x^2\,dx \\
&= 2A^2\int_{0}^{\infty} e^{-2a(mx^2/\hbar)}x^2\,dx \\
&= \frac{\hbar}{4am} \\
\langle p\rangle &= \int_{-\infty}^{\infty}\Psi^*\left[-i\hbar\frac{\partial}{\partial x}\right]\Psi\,dx \\
&= -i\hbar\int_{-\infty}^{\infty} Ae^{-a[(mx^2/\hbar)-it]}\left(-\frac{2amx}{\hbar}Ae^{-a[(mx^2/\hbar)+it]}\right)\,dx \\
&= 2iA^2am\int_{-\infty}^{\infty} xe^{-2amx^2/\hbar}\,dx \\
&= 0 \\
\langle p^2\rangle &= \int_{-\infty}^{\infty}\Psi^*\left[-\hbar^2\frac{\partial^2}{\partial x^2}\right]\Psi\,dx \\
&= -\hbar^2\int_{-\infty}^{\infty} Ae^{-a[(mx^2/\hbar)-it]}\left[-\frac{2am}{\hbar}\left(1-\frac{2amx^2}{\hbar}\right)Ae^{-a[(mx^2/\hbar)+it]}\right]\,dx \\
&= 2A^2am\hbar\int_{-\infty}^{\infty} e^{-2amx^2/\hbar}\left(1-\frac{2amx^2}{\hbar}\right)\,dx \\
&= am\hbar
\end{aligned}
$$

(d)

$$
\begin{aligned}
\sigma_x &= \sqrt{\langle x^2\rangle-\langle x\rangle^2} \\
&= \sqrt{\frac{\hbar}{4am}} \\
\sigma_p &= \sqrt{am\hbar} \\
\sigma_x\sigma_p &= \sqrt{\frac{1}{4}\hbar^2} \\
&= \frac{1}{2}\hbar \\
&\geq \frac{1}{2}\hbar
\end{aligned}
$$

Yes, this is consistent with Heisenberg's uncertainty principle.

1.10

(a)

$$
\begin{aligned}
P(0) &= 0 \\
P(1) &= \frac{2}{25} \\
&= 0.08 \\
P(2) &= \frac{3}{25} \\
&= 0.12 \\
P(3) &= \frac{1}{5} \\
&= 0.2 \\
P(4) &= \frac{3}{25} \\
&= 0.12 \\
P(5) &= \frac{3}{25} \\
&= 0.2 \\
P(6) &= \frac{3}{25} \\
&= 0.2 \\
P(7) &= \frac{1}{25} \\
&= 0.04 \\
P(8) &= \frac{2}{25} \\
&= 0.08 \\
P(9) &= \frac{3}{25} \\
&= 0.12
\end{aligned}
$$

(b) The most probable digit is 3, the median digit is 4, and the average value is $\frac{118}{25} = 4.72$.

(c) $\sigma = 2.474$

1.14

(a)

$$
\begin{aligned}
P_{ab}(t) &= \int_a^b |\Psi(x,t)|^2 \, dx \\
\frac{dP_{ab}}{dt} &= \frac{d}{dt}\int_a^b |\Psi(x,t)|^2 \, dx \\
&= \int_a^b \frac{d}{dt}\left(|\Psi(x,t)|^2\right) \, dx \\
&= \int_a^b \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\right] \, dx \\
&= J(a,t) - J(b,t)
\end{aligned}
$$

The units are s^{-1}.

(b)

$$
\begin{aligned}
\Psi(x,t) &= Ae^{-a[(mx^2/\hbar)+it]} \\
\frac{\partial \Psi}{\partial x} &= -\frac{2amx}{\hbar}\Psi \\
\Psi^*(x,t) &= Ae^{-a[(mx^2/\hbar)-it]} \\
\frac{\partial \Psi^*}{\partial x} &= -\frac{2amx}{\hbar}\Psi^* \\
J(x,t) &= \frac{i\hbar}{2m}\left(\Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi}{\partial x}\right) \\
&= \frac{i\hbar}{2m}\left[\Psi\left(-\frac{2amx}{\hbar}\Psi^*\right) - \Psi^*\left(-\frac{2amx}{\hbar}\Psi\right)\right] \\
&= 0
\end{aligned}
$$

1.15

$$\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2\,dx &= \int_{-\infty}^{\infty}\left(\frac{\partial\Psi_1^*}{\partial t}\Psi_2+\Psi_1^*\frac{\partial\Psi_2}{\partial t}\right)dx \\
&= \int_{-\infty}^{\infty}\left[\left(-i\frac{\hbar}{2m}\frac{\partial^2\Psi_1^*}{\partial x^2}+i\frac{V}{\hbar}\Psi_1^*\right)\Psi_2\right. \\
&\qquad \left.+\Psi_1^*\left(i\frac{\hbar}{2m}\frac{\partial^2\Psi_2}{\partial x^2}-i\frac{V}{\hbar}\Psi_2\right)\right]dx \\
&= i\frac{\hbar}{2m}\int_{-\infty}^{\infty}\left(\Psi_1^*\frac{\partial^2\Psi_2}{\partial x^2}-\frac{\partial^2\Psi_1^*}{\partial x^2}\Psi_2\right)dx \\
&= i\frac{\hbar}{2m}\left[\Psi_1^*\frac{\partial\Psi_2}{\partial x}\bigg|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{\partial}{\partial x}(\Psi_1^*\Psi_2)\,dx\right. \\
&\qquad \left.\frac{\partial\Psi_1^*}{\partial x}\Psi_2\bigg|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{\partial}{\partial x}(\Psi_1^*\Psi_2)\,dx\right] \\
&= 0
\end{aligned}$$

1.16

(a)

$$\begin{aligned}
1 &= \int_{-a}^{a}A^2(a^2-x^2)^2\,dx \\
&= A^2\int_0^a(a^2-x^2)^2\,dx \\
&= \frac{16}{15}A^2a^5 \\
A &= \sqrt{\frac{15}{16a^5}}
\end{aligned}$$

(b)

$$\begin{aligned}
\langle x\rangle &= \int_{-a}^{a}xA(a^2-x^2)\,dx \\
&= 0
\end{aligned}$$

(c)

$$\begin{aligned}
\langle p\rangle &= \int_{-a}^{a}\Psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi\,dx \\
&= 2iA^2\hbar\int_{-a}^{a}x(a^2-x^2)\,dx \\
&= 0
\end{aligned}$$

(d)

$$
\begin{aligned}
\langle x^2 \rangle &= \int_{-a}^{a} \Psi^* x^2 \Psi \, dx \\
&= A^2 \int_{-a}^{a} x^2 (a^2 - x^2)^2 \, dx \\
&= A^2 \frac{16}{105} a^7 \\
&= \frac{a^2}{7}
\end{aligned}
$$

(e)

$$
\begin{aligned}
\langle p^2 \rangle &= \int_{-a}^{a} \Psi^* \left(-\hbar^2 \frac{\partial^2}{\partial x^2} \right) \Psi \, dx \\
&= -\hbar^2 \int_{-a}^{a} A(a^2 - x^2)(-2A) \, dx \\
&= 4A^2 \hbar^2 \int_0^a (a^2 - x^2) \, dx \\
&= 4A^2 \hbar^2 \left[a^2 x - \frac{1}{3} x^3 \right]_0^a \\
&= 4A^2 \hbar^2 \left(a^3 - \frac{1}{3} a^3 \right) \\
&= \frac{8}{3} A^2 a^3 \hbar^2 \\
&= \frac{8}{3} \frac{15}{16a^5} a^3 \hbar^2 \\
&= \frac{5}{2} \frac{\hbar^2}{a^2}
\end{aligned}
$$

(f)

$$
\begin{aligned}
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{a^2}{7}} \\
&= \frac{a}{\sqrt{7}}
\end{aligned}
$$

(g)

$$
\begin{aligned}
\sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\
&= \sqrt{\frac{5}{2}} \frac{\hbar}{a}
\end{aligned}
$$

(h)

$$
\begin{aligned}
\sigma_x \sigma_p &= \sqrt{\frac{5}{14}}\hbar \\
&\geq \frac{1}{2}\hbar
\end{aligned}
$$

1.18

(a)

$$
\begin{aligned}
\frac{h}{\sqrt{3mk_BT}} &> d \\
\frac{\sqrt{3mk_BT}}{h} &< \frac{1}{d} \\
T_{\text{electron}} &< \frac{h^2}{3d^2mk_B} \\
&< 1.3 \times 10^5\,\text{K} \\
T_{\text{nuclei}} &< 2.5\,\text{K}
\end{aligned}
$$

(b)

$$
\begin{aligned}
PV &= Nk_BT \\
\frac{V}{N} &= \frac{k_BT}{P} \\
d &= \left(\frac{k_BT}{P}\right)^{1/3} \\
\frac{h}{\sqrt{3mk_Bt}} &> \left(\frac{k_BT}{P}\right)^{1/3} \\
T &< \frac{1}{k_B}\left(\frac{h^2}{3m}\right)^{3/5} P^{2/5}
\end{aligned}
$$

2 Time-Independent Schrödinger Equation

2.1

(a)

$$
\begin{aligned}
\int_{-\infty}^{\infty} |\Psi|^2\,dx &= \int_{-\infty}^{\infty} \Psi^*\Psi\,dx \\
&= \int_{-\infty}^{\infty} \psi^* e^{i(E_0-i\Gamma)t/\hbar}\psi e^{-i(E_0+i\Gamma)t/\hbar}\,dx \\
&= e^{2\Gamma t/\hbar}\int_{-\infty}^{\infty} |\psi|^2\,dx
\end{aligned}
$$

In order for this to equal 1 for all t, Γ must be 0.

(b) If $\psi(x)$ is a complex solution to the time-independent Schrödinger equation then so is $\psi^*(x)$ and $\psi(x) + \psi^*(x)$ which is real.

2.2

If ψ and its second derivative always have the same sign, ψ will increase or decrease without bound forever. This means there is no non-zero choice of constant A such that

$$\int_{-\infty}^{\infty} |A\Psi|^2 \, dx = 1$$

and thus the equation can't be normalised.

The classical analog of this is statements is that the potential energy of a system can't exceed its total energy.

2.3

The time-independent Schrödinger equation in an infinite square well is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi.$$

If $E = 0$ then $\psi = Ax + B$ which isn't normalisable.

If $E < 0$ then $\psi = Ae^{kt} + Be^{-kt}$ where $k \in \mathbb{R}$ which also isn't normalisable.

2.4

$$
\begin{aligned}
\Psi_n(x,t) &= \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-i(n^2\pi^2\hbar/2ma^2)t} \\
\langle x \rangle &= \int_0^a \Psi_n^* x \Psi_n \, dx \\
&= \frac{2}{a} \int_0^a x \sin^2\left(\frac{n\pi}{a}x\right) \, dx \\
&= \frac{a}{2} \\
\langle x^2 \rangle &= \int_0^a \Psi_n^* x^2 \Psi_n \, dx \\
&= \frac{2}{a} \int_0^a x^2 \sin^2\left(\frac{n\pi}{a}x\right) \, dx \\
&= a^2 \left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right) \\
\langle p \rangle &= \int_0^a \Psi_n^* \left(-i\hbar\frac{\partial}{\partial x}\right) \Psi_n \, dx \\
&= -i\frac{2\hbar n\pi}{a^2} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \cos\left(\frac{n\pi}{a}x\right) \, dx \\
&= 0 \\
\langle p^2 \rangle &= \int_0^a \Psi_n^* \left(-\hbar^2\frac{\partial^2}{\partial x^2}\right) \Psi_n \, dx \\
&= \frac{2\hbar^2 n^2\pi^2}{a^3} \int_0^a \sin^2\left(\frac{n\pi}{a}x\right) \, dx \\
&= \left(\frac{n\pi\hbar}{a}\right)^2 \\
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \frac{a}{2}\sqrt{\frac{1}{3} - \frac{2}{n^2\pi^2}} \\
\sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\
&= \frac{n\pi\hbar}{a}
\end{aligned}
$$

2.5

(a)

$$
\begin{aligned}
1 &= \int_0^a A^2(\psi_1+\psi_2)^2\,dx \\
&= A^2\int_0^a (\psi_1^2+2\psi_1\psi_2+\psi_2^2)\,dx \\
&= \frac{2A^2}{a}\left[\int_0^a \sin^2\left(\frac{\pi}{a}x\right)\,dx + \int_0^a \sin^2\left(\frac{2\pi}{a}x\right)\,dx\right] \\
&= 2A^2 \\
A &= \frac{1}{\sqrt{2}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi}{a}x\right)e^{-4i\omega t}\right] \\
|\Psi(x,t)|^2 &= \Psi^*\Psi \\
&= \frac{1}{a}\left[\sin\left(\frac{\pi}{a}x\right)e^{i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{4i\omega t}\right] \\
&\quad\ \left[\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{-4i\omega t}\right] \\
&= \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)e^{-3i\omega t}\right. \\
&\quad\ \left. + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)e^{3i\omega t} + \sin^2\left(\frac{2\pi}{a}x\right)\right] \\
&= \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right)\right. \\
&\quad\ \left. + 2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right]
\end{aligned}
$$

(c)

$$
\begin{aligned}
\langle x\rangle &= \int_0^a \Psi^* x\Psi\,dx \\
&= \int_0^a x|\Psi|^2\,dx \\
&= \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t)\right]
\end{aligned}
$$

(d)

$$
\begin{aligned}
\langle p\rangle &= m\frac{d\langle x\rangle}{dt} \\
&= \frac{16am\omega}{3\pi^2}\sin(3\omega t) \\
&= \frac{8\hbar}{3a}\sin(3\omega t)
\end{aligned}
$$

(e) You can get E_1 or E_2 and the probability of getting each is 1/2. $H = \frac{1}{2}(E_1 + E_2)$ is the mean of the two possible energy values.

2.6

$$
\begin{aligned}
\Psi(x,0) &= A[\psi_1 + e^{i\phi}\psi_2] \\
1 &= \int_0^a |\Psi|^2\,dx \\
&= \int_0^a \Psi^*\Psi\,dx \\
&= A^2\int_0^a (\psi_1 + e^{-i\phi}\psi_2)(\psi_1 + e^{i\phi}\psi_2)\,dx \\
&= A^2\int_0^a (\psi_1^2 + e^{i\phi}\psi_1\psi_2 + e^{-i\phi}\psi_1\psi_2 + \psi_2^2)\,dx \\
&= \frac{2A^2}{a}\int_0^a \left[\sin^2\left(\frac{\pi}{a}x\right) + e^{i\phi}\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\right. \\
&\qquad \left. e^{-i\phi}\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right)\right]\,dx \\
&= \frac{2A^2}{a}\int_0^a \left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos\phi\right. \\
&\qquad \left. + \sin^2\left(\frac{2\pi}{a}x\right)\right]\,dx \\
&= 2A^2 \\
A &= \frac{1}{\sqrt{2}} \\
\Psi(x,t) &= \frac{1}{\sqrt{a}}\left[\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{i(\phi-4\omega t)}\right]
\end{aligned}
$$

$$
\begin{aligned}
|\Psi|^2 &= \Psi^*\Psi \\
&= \frac{1}{a}\left[\sin\left(\frac{\pi}{a}x\right)e^{i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{-i(\phi-4\omega t)}\right] \\
&\quad \left[\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{i(\phi-4\omega t)}\right] \\
&= \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)e^{i(\phi-3\omega t)}\right. \\
&\quad \left.\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)e^{-i(\phi-3\omega t)} + \sin^2\left(\frac{2\pi}{a}x\right)\right] \\
&= \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right)\right. \\
&\quad \left. +2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(\phi-3\omega t)\right] \\
\langle x\rangle &= \int_0^a \Psi^* x\Psi\,dx \\
&= \int_0^a x|\Psi|^2\,dx \\
&= \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t-\phi)\right]
\end{aligned}
$$

2.7

(a)

$$
\begin{aligned}
1 &= \int_0^a |\Psi|^2\,dx \\
&= A^2\left[\int_0^{a/2} x^2\,dx + \int_{a/2}^a (a-x)^2\,dx\right] \\
&= A^2\left\{\frac{1}{3}\left[\frac{a}{2}\right]^3 + \left[-\frac{1}{3}(a-x)^3\right]_{a/2}^a\right\} \\
&= A^2\left(\frac{a^3}{24} + \frac{a^3}{24}\right) \\
&= \frac{A^2a^3}{12} \\
A &= \frac{2\sqrt{3}}{\sqrt{a^3}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
c_n &= \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \Psi(x,0)\, dx \\
&= \sqrt{\frac{2}{a}} \left[\int_0^{a/2} \sin\left(\frac{n\pi}{a}x\right) Ax\, dx + \int_{a/2}^a \sin\left(\frac{n\pi}{a}x\right) A(a-x)\, dx \right] \\
&= \frac{2\sqrt{6}}{a^2} \left[\int_0^{a/2} x \sin\left(\frac{n\pi}{a}x\right)\, dx + \int_{a/2}^a (a-x) \sin\left(\frac{n\pi}{a}x\right)\, dx \right] \\
&= \frac{8\sqrt{6}}{n^2\pi^2} \sin^2\left(\frac{n\pi}{4}\right) \sin\left(\frac{n\pi}{2}\right) \\
&= \begin{cases} 0 & n \text{ even} \\ (-1)^{(n-1)/2} \frac{4\sqrt{6}}{n^2\pi^2} & n \text{ odd} \end{cases} \\
\Psi(x,t) &= \frac{4\sqrt{6}}{\pi^2} \sqrt{\frac{2}{a}} \sum_{n=1,3,5,\ldots}^{\infty} (-1)^{(n-1)/2} \frac{1}{n^2} \sin\left(\frac{n\pi}{a}x\right) e^{-i(n^2\pi^2\hbar/2ma^2)t}
\end{aligned}
$$

(c)

$$
\begin{aligned}
|c_1|^2 &= \left(\frac{4\sqrt{6}}{\pi^2}\right)^2 \\
&\approx 0.985
\end{aligned}
$$

(d)

$$
\begin{aligned}
E_n &= \frac{n^2\pi^2\hbar^2}{2ma^2} \\
\langle H \rangle &= \sum_{n=0}^{\infty} |c_{2n+1}|^2 E_{2n+1} \\
&= \sum_{n=0}^{\infty} \left(\frac{4\sqrt{6}}{(2n+1)^2\pi^2}\right)^2 \frac{(2n+1)^2\pi^2\hbar^2}{2ma^2} \\
&= \sum_{n=0}^{\infty} \frac{48\hbar^2}{(2n+1)^2 ma^2\pi^2} \\
&= \frac{48\hbar^2}{ma^2\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\
&= \frac{6\hbar^2}{ma^2}
\end{aligned}
$$

2.8

$$
\begin{aligned}
1 &= \int_0^{a/2} |\Psi|^2 \, dx \\
&= A^2 \int_0^{a/2} \; dx \\
&= \frac{aA^2}{2} \\
A &= \sqrt{\frac{2}{a}} \\
c_n &= \frac{2}{a} \int_0^{a/2} \sin\left(\frac{n\pi}{a} x\right) \, dx \\
|c_1|^2 &= \left(\frac{2}{\pi}\right)^2 \\
&\approx 0.405
\end{aligned}
$$

2.9

$$
\begin{aligned}
\Psi(x,0) &= Ax(a-x) \\
\langle H \rangle &= \int_0^a \Psi(x,0)^* \hat{H} \Psi(x,0) \, dx \\
&= \int_0^a \Psi(x,0)^* \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}\right) \Psi(x,0) \, dx \\
&= \frac{A^2 \hbar^2}{m} \int_0^a x(a-x) \, dx \\
&= \frac{30\hbar^2}{ma^5} \frac{a^3}{6} \\
&= \frac{5\hbar^2}{ma^2}
\end{aligned}
$$