Quantum Computation and Quantum Information by Michael A. Nielsen and Isaac L. Chuang Problems

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Part I
Fundamental concepts

2 Linear algebra

Exercise 2.1

$$(1, -1) + (1, 2) - (2, 1) = (0, 0)$$

Exercise 2.2

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Using the basis $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$ we get

$$
\begin{aligned}
|0\rangle &= \frac{|+\rangle + |-\rangle}{\sqrt{2}} \\
|1\rangle &= \frac{|+\rangle - |-\rangle}{\sqrt{2}} \\
\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
a_{00} + a_{01} &= 1 \\
a_{10} + a_{11} &= -1 \\
\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
a_{00} - a_{01} &= 1 \\
a_{10} - a_{11} &= 1 \\
A &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\end{aligned}
$$

Exercise 2.5

$$\begin{aligned}
\begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} &= \begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \left(\begin{bmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ z_2 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ z_n \end{bmatrix} \right) \\
&= \begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \begin{bmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ z_n \end{bmatrix} \\
&= z_1 \begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + z_n \begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\
\begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} &= y_1^* z_1 + y_2^* z_2 + \ldots + y_n^* z_n \\
&= (y_1 z_1^* + y_2 z_2^* + \ldots + y_n z_n^*)^* \\
&= \left(\begin{bmatrix} z_1^* & \cdots & z_n^* \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right)^* \\
\begin{bmatrix} v_1^* & \cdots & v_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} &= |v_1|^2 + \cdots + |v_n|^2 \\
&\geq 0
\end{aligned}$$

Exercise 2.6

$$\begin{aligned}
\left(\sum_i \lambda_i \left| w_i \right\rangle, \left| v \right\rangle \right) &= \left(\left| v \right\rangle, \sum_i \lambda_i \left| w_i \right\rangle \right)^* \\
&= \left(\sum_i \lambda_i (\left| v \right\rangle, \left| w_i \right\rangle) \right)^* \\
&= \sum_i \lambda_i^* (\left| v \right\rangle, \left| w_i \right\rangle)^* \\
&= \sum_i \lambda_i^* (\left| w_i \right\rangle, \left| v \right\rangle)
\end{aligned}$$

Exercise 2.7

$$
\begin{aligned}
\langle w|v\rangle &= \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}\\
&= (1)(1) + (1)(-1)\\
&= 0\\
\frac{|w\rangle}{||\,|w\rangle\,||} &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\\
\frac{|v\rangle}{||\,|v\rangle\,||} &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}
\end{aligned}
$$

Exercise 2.9

$$
\begin{aligned}
\sigma_0 &= |0\rangle\langle 0| + |1\rangle\langle 1|\\
\sigma_1 &= |1\rangle\langle 0| + |0\rangle\langle 1|\\
\sigma_2 &= i\,|1\rangle\langle 0| - i\,|0\rangle\langle 1|\\
\sigma_3 &= |0\rangle\langle 0| - |1\rangle\langle 1|
\end{aligned}
$$