Advanced Engineering Mathematics Ordinary Differential Equations Notes

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Contents

1 Introduction to Differential Equations

1.1 Definitions and Terminology

1.1.1 1

2, linear

1.1.2 3

4, linear

1.1.3 5

2, nonlinear

1.1.4 7

3, linear

1.1.5 9

no; yes

1.1.6 15

The domain of the function is $x \in [-2, \infty)$.

$$y' = 1 + \frac{2}{\sqrt{x+2}}$$

The largest interval of definition of the solution is $x \in (-2, \infty)$.

$$(y-x)y' = y - x + 8$$
$$(x + 4\sqrt{x+2} - x)(1 + \frac{2}{\sqrt{x+2}}) = x + 4\sqrt{x+2} - x + 8$$
$$4\sqrt{x+2} + 8 = 4\sqrt{x+2} + 8$$

1.1.7   17

The domain of the function is $x \in \mathbb{R}, x \neq \pm 2$.

$$y' = \frac{2x}{(4 - x^2)^2}$$

The largest intervals of definition of the solution are $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

$$y' = 2xy^2$$

$$\frac{2x}{(4 - x^2)^2} = 2x \left(\frac{1}{4 - x^2} \right)^2$$

$$= \frac{2x}{(4 - x^2)^2}$$

1.1.8   19

$$ln\frac{2X - 1}{X - 1} = t$$

$$2X - 1 = (X - 1)e^t$$

$$(2 - e^t)X = 1 - e^t$$

$$X = \frac{e^t - 1}{e^t - 2}$$

The solutions intervals of validity are $(\infty, \ln 2)$ and $(\ln 2, \infty)$.

$$\frac{dX}{dt} = (X - 1)(1 - 2X)$$

$$\frac{e^t}{e^t - 2} - \frac{e^t(e^t - 1)}{(e^t - 2)^2} = \left(\frac{e^t - 1}{e^t - 2} - 1 \right) \left(1 - 2\frac{e^t - 1}{e^t - 2} \right)$$

$$\frac{e^t(e^t - 2) - e^t(e^t - 1)}{(e^t - 2)^2} = \left(\frac{e^t - 1 - e^t + 2}{e^t - 2} \right) \left(\frac{e^t - 2 - 2e^t + 2}{e^t - 2} \right)$$

$$\frac{e^{2t} - 2e^t - e^{2t} + e^t}{(e^t - 2)^2} = \left(\frac{1}{e^t - 2} \right) \left(\frac{-e^t}{e^t - 2} \right)$$

$$\frac{-e^t}{(e^t - 2)^2} = \frac{-e^t}{(e^t - 2)^2}$$

1.1.9   31

$m = -2$

1.1.10   33

$m = 2$ or 3

1.1.11 35

$m = -1$ or 0

1.1.12 37

$y = 2$

1.1.13 39

No constant solutions

1.2 Initial Value Problems

1.2.1 1

$$y(0) = -\frac{1}{3} = \frac{1}{1 + c_1 e^{-(0)}}$$
$$-3 = 1 + c_1$$
$$c_1 = -4$$

$$y = \frac{1}{1 - 4e^{-x}}$$

1.2.2 3

$$y(2) = \frac{1}{3} = \frac{1}{(2)^2 + c}$$
$$3 = 4 + c$$
$$c = -1$$

$$y = \frac{1}{x^2 - 1}$$

$I = (1, \infty)$

1.2.3 5

$$y(0) = 1 = \frac{1}{(0)^2 + c}$$
$$c = 1$$

$$y = \frac{1}{x^2 + 1}$$

$I = (-\infty, \infty)$

1.2.4   7

$$x(0) = -1 = c_1 \cos 0 + c_2 \sin 0$$
$$c_1 = -1$$

$$x'(0) = 8 = -c_1 \sin 0 + c_2 \cos 0$$
$$c_2 = 8$$

$$x = -\cos t + 8 \sin t$$

1.2.5   9

$$x'\left(\frac{\pi}{6}\right) = 0 = -c_1 \sin \frac{\pi}{6} + c_2 \cos \frac{\pi}{6}$$
$$= -c_1 \frac{1}{2} + c_2 \frac{\sqrt{3}}{2}$$
$$c_1 = \sqrt{3} c_2$$

$$x\left(\frac{\pi}{6}\right) = \frac{1}{2} = c_1 \cos \frac{\pi}{6} + c_2 \sin \frac{\pi}{6}$$
$$= \frac{3}{2} c_2 + \frac{1}{2} c_2$$
$$= 2 c_2$$
$$c_2 = \frac{1}{4}$$

$$y = \frac{\sqrt{3}}{4} \cos t + \frac{1}{4} \sin t$$

1.2.6   11

$$y(0) = 1 = c_1 e^{(0)} + c_2 e^{-(0)}$$
$$= c_1 + c_2$$
$$c_1 = 1 - c_2$$

$$y'(0) = 2 = c_1 e^{(0)} - c_2 e^{-(0)}$$
$$= 1 - c_2 - c_2$$
$$c_2 = -\frac{1}{2}$$

$$y = \frac{3}{2} e^x - \frac{1}{2} e^{-x}$$

1.2.7   13

$$y(-1) = 5 = c_1 e^{(-1)} + c_2 e^{-(-1)}$$
$$= c_1 e^{-1} + c_2 e$$
$$c_1 = 5e - c_2 e^2$$

$$y'(-1) = -5 = c_1 e^{(-1)} - c_2 e^{-(-1)}$$
$$= 5e - c_2 e^2 - c_2 e$$
$$c_2 e(e + 1) = 5(e + 1)$$
$$c_2 = \frac{5}{e}$$

$$y = 5e^{-x-1}$$

1.2.8   15

$$y = 0$$

$$y = x^3$$

1.2.9   17

$$f(x, y) = y^{2/3}$$

$$\frac{\partial f}{\partial y} = \frac{2}{3y^{1/3}}$$

$y < 0$ or $y > 0$

1.2.10   19

$$f(x, y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$x < 0$ or $x > 0$

1.2.11   21

$$f(x, y) = \frac{x^2}{4 - y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2x^2 y}{(4 - y^2)^2}$$

$y < -2$, $-2 < y < 2$, or $y > 2$

1.2.12   23

$$f(x, y) = \frac{y^2}{x^2 + y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2} - \frac{2y^3}{(x^2 + y^2)^2}$$

$x \neq 0$ and $y \neq 0$

1.2.13   25

$$f(x, y) = \sqrt{y^2 - 9}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{y^2 - 9}}$$

Yes

1.2.14   27

No

1.2.15   29

(a) $y = cx$

(b)

$$f(x, y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$x \neq 0$

(c) No, the function is not differentiable at $x = 0$

1.2.16   31

(a)

$$y' = \frac{1}{(x + c)^2} = y^2$$

(b)

$$y(0) = 1 = -\frac{1}{(0) + c} \Rightarrow c = -1 \Rightarrow y = \frac{1}{1 - x}$$

$I = (-\infty, 1)$

9

$$y(0) = -1 = -\frac{1}{(0) + c} \Rightarrow c = 1 \Rightarrow y = -\frac{1}{x+1}$$

$$I = (-1, \infty)$$

1.2.17   39

$$y(0) = 0 = c_1 \cos 3(0) + c_2 \sin 3(0)$$
$$c_1 = 0$$

$$y\left(\frac{\pi}{6}\right) = -1 = c_2 \sin 3\left(\frac{\pi}{6}\right)$$
$$c_2 = -1$$

$$y = -\sin 3x$$

1.2.18   41

$$y'(0) = 0 = -3c_1 \sin 3(0) + 3c_2 \cos 3(0)$$
$$c_2 = 0$$

$$y'\left(\frac{\pi}{4}\right) = 0 = -3c_1 \sin 3\left(\frac{\pi}{4}\right)$$
$$= -\frac{3}{\sqrt{2}}c_1$$
$$c_1 = 0$$

$$y = 0$$

1.2.19   43

$$y(0) = 0 = c_1 \cos 3(0) + c_2 \sin 3(0)$$
$$c_1 = 0$$

$$y(\pi) = 4 = c_2 \sin 3(\pi)$$
$$4 = 0$$

No solution

1.3   Differential Equations as Mathematical Models

1.3.1   1

$$\frac{dP}{dt} = kP + r$$

$$\frac{dP}{dt} = kP - r$$

1.3.2   3

$$\frac{dP}{dt} = k_b P - k_d P^2$$

1.3.3   7

$$\frac{dx}{dt} = kx(1000 - x)$$

1.3.4   9

$$\frac{dA}{dt} = -\frac{A}{100}$$

$$A(0) = 50\,\mathrm{lb}$$

1.3.5   11

$$\frac{dA}{dt} + \frac{7}{600 - t}A = 6$$

1.3.6   13

$$\frac{dV}{dt} = -cA_h\sqrt{2gh}$$

$$A_w\frac{dh}{dt} = -cA_h\sqrt{2gh}$$

$$\frac{dh}{dt} = -\frac{cA_h\sqrt{2g}}{A_w}\sqrt{h}$$

$$= -\frac{c\pi r_h^2\sqrt{2g}}{A_w}\sqrt{h}$$

$$= -\frac{c\pi}{430}\sqrt{h}$$

1.3.7   15

$$L\frac{di}{dt} + Ri = E$$

1.3.8   17

$$m\frac{dv}{dt} = mg - kv^2$$

1.3.9   19

$$m\frac{d^2x}{dt^2} = -kx$$

1.3.10   21

$$\frac{d}{dt}(mv) = R - kv$$

$$\frac{dm}{dt}v + m\frac{dv}{dt} = R - kv - mg$$

1.3.11   23

$$g = \frac{k}{R^2} \Rightarrow k = gR^2$$

$$\frac{d^2r}{dt^2} = -\frac{gR^2}{r^2}$$

1.3.12   25

$$\frac{dA}{dt} = k(M - A)$$

1.3.13   27

$$\frac{dx}{dt} = r - kx$$

1.3.14   29

$$\frac{dy}{dx} = \tan\theta$$

$$= \tan\frac{\phi}{2}$$

$$= \frac{1 - \cos\phi}{\sin\phi}$$

$$= \frac{1 - x/r}{y/r}$$

$$= \frac{r - x}{y}$$

$$= \frac{\sqrt{x^2 + y^2} - x}{y}$$

1.4 Chapter in Review

1.4.1 1

$$\frac{dy}{dx} = ky$$

1.4.2 3

$$y'' + k^2 y = 0$$

1.4.3 5

$$y = c_1 e^x + c_2 x e^x$$

$$y' = c_1 e^x + c_2 e^x + c_2 x e^x$$
$$= y + c_2 e^x$$

$$y'' = c_1 e^x + c_2 e^x + c_2 e^x + c_2 x e^x$$
$$= c_1 e^x + 2c_2 e^x + c_2 x e^x$$
$$= y' + c_2 e^x$$

$$y'' - 2y' + y = 0$$

1.4.4 7

a, d

1.4.5 9

b

1.4.6 11

b

1.4.7 13

$$y = c e^x$$

1.4.8 15

$$\frac{dy}{dx} = x^2 + y^2$$

1.4.9   17

(a) $(-\infty, \infty)$

(b) $(-\infty, 0)$ or $(0, \infty)$

1.4.10   19

$x_0 = -1$ and $I = (-\infty, 0)$ or $x_0 = 2$ and $I = (0, \infty)$

1.4.11   23

$$y = x \sin x + x \cos x$$

$$y' = \sin x + x \cos x + \cos x - x \sin x$$

$$y'' = \cos x + \cos x - x \sin x - \sin x - \sin x - x \cos x$$
$$= 2 \cos x - 2 \sin x - x \sin x - x \cos x$$

$$y'' + y = 2 \cos x - 2 \sin x - x \sin x - x \cos x + x \sin x + x \cos x$$
$$= 2 \cos x - 2 \sin x$$

$I = (-\infty, \infty)$

1.4.12   25

$$y = \sin(\ln x)$$

$$y' = \frac{1}{x} \cos(\ln x)$$

$$y'' = -\frac{1}{x^2} \cos(\ln x) - \frac{1}{x^2} \sin(\ln x)$$

$$x^2 y'' + xy' + y = -\cos(\ln x) - \sin(\ln x) + \cos(\ln x) + \sin(\ln x)$$
$$= 0$$

$I = (0, \infty)$

1.4.13 35

$$y(0) = 0 = c_1 e^{-3(0)} + c_2 e^{(0)} + 4(0)$$
$$= c_1 + c_2$$
$$c_1 = -c_2$$

$$y'(0) = 0 = -3c_1 e^{-3(0)} + c_2 e^{(0)} + 4$$
$$= -3c_1 + c_2 + 4$$
$$c_2 = 3c_1 - 4$$

$$c_1 = -(3c_1 - 4) \Rightarrow c_1 = 1 \Rightarrow c_2 = -1$$

$$y = e^{-3x} - e^x + 4x$$

1.4.14 37

$$y(1) = -2 = c_1 e^{-3(1)} + c_2 e^{(1)} + 4(1)$$
$$= c_1 e^{-3} + c_2 e + 4$$
$$c_1 = -e^3 (c_2 e + 6)$$

$$y'(1) = 4 = -3c_1 e^{-3(1)} + c_2 e^{(1)} + 4$$
$$= -3c_1 e^{-3} + c_2 e + 4$$
$$c_2 e = 3c_1 e^{-3}$$

$$c_1 = -e^3 (3c_1 e^{-3} + 6) = -3c_1 - 6e^3 \Rightarrow c_1 = -\frac{3}{2} e^3 \Rightarrow c_2 = -\frac{9}{2} e^{-1}$$

$$y = -\frac{3}{2} e^{3(1-x)} - \frac{9}{2} e^{x-1} + 4x$$

1.4.15 41

$y_0 = -3, \, y_1 = 0$

1.4.16　43

$$\frac{d}{dt}(mv) = F - mg$$

$$\frac{d}{dt}(\lambda x \frac{dx}{dt}) = F - \lambda x g$$

$$x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + gx = \frac{F}{\lambda}$$

$$x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 32x = 5$$

2　First-Order Differential Equations

2.1　Solution Curves Without a Solution

2.1.1　21

0 is stable, 3 is unstable

2.1.2　23

2 is semi-stable

2.1.3　25

-2 is unstable, 0 is semi-stable, 2 is stable

2.1.4　27

-1 is stable, 0 is unstable

2.1.5　39

$P_0 < h/k$

2.1.6　41

$$g - \frac{k}{m}v^2 = 0 \Rightarrow v = \sqrt{\frac{gm}{k}}$$

2.2　Separable Equations

2.2.1　1

$$\frac{dy}{dx} = \sin 5x$$

$$y = -\frac{1}{5}\cos 5x + c$$

2.2.2   3

$$dx + e^{3x}\,dy = 0$$
$$e^{-3x}\,dx + dy = 0$$
$$-\frac{1}{3}e^{-3x} + y = c$$
$$y = \frac{1}{3}e^{-3x} + c$$

2.2.3   5

$$x\frac{dy}{dx} = 4y$$
$$\frac{1}{4y}\,dy = \frac{1}{x}\,dx$$
$$\frac{1}{4}\ln|4y| = \ln|x| + c$$
$$\ln|4y| = 4\ln|x| + c$$
$$4y = e^{4\ln|x|+c}$$
$$= c\left(e^{\ln|x|}\right)^4$$
$$y = cx^4$$

2.2.4   7

$$\frac{dy}{dx} = e^{3x+2y}$$
$$= e^{3x}e^{2y}$$
$$e^{-2y}\,dy = e^{3x}\,dx$$
$$-\frac{1}{2}e^{-2y} = \frac{1}{3}e^{3x} + c$$
$$-3e^{-2y} = 2e^{3x} + c$$

17

2.2.5 9

$$y \ln x \frac{dx}{dy} = \left(\frac{y+1}{x} \right)^2$$

$$x^2 \ln x \, dx = \frac{(y+1)^2}{y} \, dy$$

$$x^3 \left(\frac{\ln x}{3} - \frac{1}{9} \right) = \frac{1}{2} y(y+4) + \ln y + c$$

$$\frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 = \frac{1}{2} y^2 + 2y + \ln y + c$$

2.2.6 11

$$\csc y \, dx + \sec^2 x \, dy = 0$$

$$\frac{1}{\sin y} \, dx + \frac{1}{\cos^2 x} \, dy = 0$$

$$\cos^2 x \, dx + \sin y \, dy = 0$$

$$\frac{1}{2}(1 + \cos 2x) \, dx + \sin y \, dy = 0$$

$$\frac{1}{2} \left(x + \frac{1}{2} \sin 2x \right) - \cos y + c = 0$$

$$4 \cos y = 2x + \sin 2x + c$$

2.2.7 13

$$(e^y + 1)^2 e^{-y} \, dx + (e^x + 1)^3 e^{-x} \, dy = 0$$

$$\frac{e^x}{(e^x + 1)^3} \, dx + \frac{e^y}{(e^y + 1)^2} = 0$$

$$-\frac{1}{2(e^x + 1)^2} - \frac{1}{e^y + 1} = c$$

$$(e^x + 1)^{-2} + 2(e^y + 1)^{-1} = c$$

2.2.8 15

$$\frac{dS}{dr} = kS$$

$$\frac{1}{S} \, dS = k \, dr$$

$$\ln |S| = kr + c$$

$$S = ce^{kr}$$

18

2.2.9 17

$$\frac{dP}{dt} = P - P^2$$

$$\frac{1}{P(1-P)}\, dP = dt$$

$$\ln\frac{P}{1-P} = t + c$$

$$\frac{P}{1-P} = ce^t$$

$$P = ce^t(1-P)$$

$$P = \frac{ce^t}{1 + ce^t}$$

2.2.10 19

$$\frac{dy}{dx} = \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$$

$$= \frac{(x-1)(y+3)}{(x+4)(y-2)}$$

$$\frac{y-2}{y+3}\, dt = \frac{x-1}{x+4}\, dx$$

$$y - 5\ln|y+3| = x - 5\ln|x+4| + c$$

$$e^{y - 5\ln|y+3|} = e^{x - 5\ln|x+4| + c}$$

$$\frac{e^y}{(y+3)^5} = \frac{ce^x}{(x+4)^5}$$

$$c(x+4)^5 e^y = (y+3)^5 e^x$$

2.2.11 21

$$\frac{dy}{dx} = x\sqrt{1-y^2}$$

$$(1-y^2)^{-1/2}\, dy = x\, dx$$

$$\arcsin y = \frac{1}{2}x^2 + c$$

$$y = \sin\left(\frac{1}{2}x^2 + c\right)$$

2.2.12   23

$$\frac{dx}{dt} = 4(x^2 + 1)$$

$$\frac{1}{x^2 + 1} \, dx = 4 \, dt$$

$$\arctan x = 4t + c$$

$$x = \tan(4t + c)$$

$$x\left(\frac{\pi}{4}\right) = 1 = \tan\left(4\left(\frac{\pi}{4}\right) + c\right)$$

$$= \tan(\pi + c)$$

$$c = \arctan(1) - \pi$$

$$= -\frac{3}{4}\pi$$

$$x = \tan\left(4t - \frac{3}{4}\pi\right)$$

2.2.13   25

$$x^2 \frac{dy}{dx} = y - xy$$

$$= y(1 - x)$$

$$\frac{1}{y} \, dy = \left(\frac{1}{x^2} - \frac{1}{x}\right) dx$$

$$\ln|y| = -\frac{1}{x} - \ln|x| + c$$

$$y = e^{-\frac{1}{x} - \ln|x| + c}$$

$$= \frac{c}{xe^{1/x}}$$

$$y(-1) = -1 = \frac{c}{(-1)e^{1/(-1)}}$$

$$= -ce$$

$$c = e^{-1}$$

$$y = \frac{1}{xe^{1+1/x}}$$

2.2.14 29

$$\frac{dy}{dx} = ye^{-x^2}$$

$$\frac{1}{y}\frac{dy}{dx} = e^{-x^2}$$

$$\int_4^x \frac{1}{y}\frac{dy}{dx'}\,dx' = \int_4^x e^{-x'^2}\,dx'$$

$$\ln|y|\big|_4^x = \int_4^x e^{-x'^2}\,dx'$$

$$\ln|y(x)| - \ln|y(4)| = \int_4^x e^{-x'^2}\,dx'$$

$$\ln|y(x)| = \ln|y(4)| + \int_4^x e^{-x'^2}\,dx'$$

$$y(x) = e^{\int_4^x e^{-x'^2}\,dx'}$$

2.2.15 31

$$\frac{dy}{dx} = \frac{2x+1}{2y}$$

$$2y\,dy = (2x+1)\,dx$$

$$y^2 = x^2 + x + c$$

$$y = \pm\sqrt{x^2 + x + c}$$

$$y(-2) = -1 = -\sqrt{(-2)^2 + (-2) + c}$$

$$= -\sqrt{2 + c}$$

$$c = -1$$

$$y = -\sqrt{x^2 + x - 1}$$

$$I = \left(-\infty, -\frac{1-\sqrt{5}}{2}\right)$$

2.2.16   33

$$e^y \, dx - e^{-x} \, dy = 0$$
$$e^x \, dx - e^{-y} \, dy = 0$$
$$e^x + e^{-y} = c$$
$$\ln |e^{-y}| = \ln |c - e^x|$$
$$y = -\ln |c - e^x|$$

$$y(0) = 0 = -\ln |c - e^{(0)}|$$
$$1 = c - 1$$
$$c = 2$$

$$y = -\ln |2 - e^x|$$

$$I = (-\infty, \ln 2)$$

2.3   Linear Equations

2.3.1   1

$$\frac{dy}{dx} = 5y$$
$$\ln |y| = 5x + c$$
$$y = ce^{5x}$$

$$I = (-\infty, \infty)$$

2.3.2   3

$$\frac{dy}{dx} + y = e^{3x}$$
$$e^x \frac{dy}{dx} + e^x y = e^{4x}$$
$$\frac{d}{dx}(e^x y) = e^{4x}$$
$$e^x y = \frac{1}{4} e^{4x} + c$$
$$y = \frac{1}{4} e^{3x} + ce^{-x}$$

$$I = (-\infty, \infty)$$

2.3.3   5

$$y' + 3x^2 y = x^2$$
$$e^{x^3} y' + 3x^2 e^{x^3} y = e^{x^3} x^2$$
$$e^{x^3} y = \frac{1}{3} e^{x^3} + c$$
$$y = \frac{1}{3} + ce^{-x^3}$$

$$I = (-\infty, \infty)$$

2.3.4   7

$$x^2 y' + xy = 1$$
$$y' + x^{-1} y = x^{-2}$$
$$e^{\ln x} y' + x^{-1} e^{\ln x} y = e^{\ln x} x^{-2}$$
$$\frac{d}{dx}\left(e^{\ln x} y\right) = x^{-1}$$
$$\frac{d}{dx}(xy) = x^{-1}$$
$$xy = \ln x + c$$
$$y = \frac{\ln x + c}{x}$$

$$I = (0, \infty)$$

2.3.5   9

$$x\frac{dy}{dx} - y = x^2 \sin x$$
$$\frac{dy}{dx} - x^{-1} y = x \sin x$$
$$e^{-\ln x}\frac{dy}{dx} - x^{-1} e^{-\ln x} y = e^{-\ln x} x \sin x$$
$$\frac{d}{dx}\left(e^{-\ln x} y\right) = \sin x$$
$$x^{-1} y = -\cos x + c$$
$$y = cx - x\cos x$$

$$I = (0, \infty)$$

2.3.6 11

$$x\frac{dy}{dx} + 4y = x^3 - x$$

$$\frac{dy}{dx} + 4x^{-1}y = x^2 - 1$$

$$e^{4\ln x}\frac{dy}{dx} + 4x^{-1}e^{4\ln x}y = e^{4\ln x}(x^2 - 1)$$

$$\frac{d}{dx}(e^{4\ln x}y) = x^6 - x^4$$

$$x^4 y = \frac{1}{7}x^7 - \frac{1}{5}x^5 + c$$

$$y = \frac{1}{7}x^3 - \frac{1}{5}x^2 + cx^{-4}$$

$$I = (0, \infty)$$

2.3.7 13

$$x^2 y' + x(x+2)y = e^x$$

$$y' + x^{-1}(x+2)y = x^{-2}e^x$$

$$e^{x+2\ln x}y' + x^{-1}(x+2)e^{x+2\ln x}y = e^{x+2\ln x}x^{-2}e^x$$

$$\frac{d}{dx}(e^x x^2 y) = e^{2x}$$

$$e^x x^2 y = \frac{1}{2}e^{2x} + c$$

$$y = \frac{e^x}{2x^2} + \frac{c}{e^x x^2}$$

$$I = (0, \infty)$$

2.3.8 15

$$y\,dx - 4(x + y^6)\,dy = 0$$

$$y\frac{dx}{dy} - 4x - 4y^6 = 0$$

$$\frac{dx}{dy} - \frac{4}{y}x = 4y^5$$

$$e^{-4\ln y}\frac{dx}{dy} - \frac{4}{y}e^{-4\ln y}x = 4e^{-4\ln y}y^5$$

$$\frac{d}{dy}\left(e^{-4\ln y}x\right) = 4y$$

$$y^{-4}x = 2y^2 + c$$

$$x = 2y^6 + cy^4$$

$$I = (0, \infty)$$

2.3.9 17

$$\cos x\frac{dy}{dx} + (\sin x)y = 1$$

$$\frac{dy}{dx} + (\tan x)y = \sec x$$

$$e^{\ln(\sec x)}\frac{dy}{dx} + (\tan x)e^{\ln(\sec x)}y = e^{\ln(\sec x)}\sec x$$

$$\frac{d}{dx}\left(e^{\ln(\sec x)}y\right) = \sec^2 x$$

$$y\sec x = \tan x + c$$

$$y = \sin x + c\cos x$$

$$I = (-\pi/2, \pi/2)$$

2.3.10   19

$$(x+1)\frac{dy}{dx} + (x+2)y = 2xe^{-x}$$

$$\frac{dy}{dx} + \frac{x+2}{x+1}y = \frac{2xe^{-x}}{x+1}$$

$$e^{x+\ln|x+1|}\frac{dy}{dx} + \frac{x+2}{x+1}e^{x+\ln|x+1|}y = e^{x+\ln|x+1|}\frac{2xe^{-x}}{x+1}$$

$$\frac{d}{dx}(e^{x+\ln|x+1|}y) = 2x$$

$$e^x(x+1)y = x^2 + c$$

$$y = \frac{x^2 + c}{e^x(x+1)}$$

$$I = (-1, \infty)$$

2.3.11   21

$$\frac{dr}{d\theta} + r\sec\theta = \cos\theta$$

$$e^{\ln|\sec\theta+\tan\theta|}\frac{dr}{d\theta} + e^{\ln|\sec\theta+\tan\theta|}r\sec\theta = e^{\ln|\sec\theta+\tan\theta|}\cos\theta$$

$$\frac{d}{d\theta}(e^{\ln|\sec\theta+\tan\theta|}r) = 1 + \sin\theta$$

$$(\sec\theta + \tan\theta)r = \theta - \cos\theta + c$$

$$r = \frac{\theta - \cos\theta + c}{\sec\theta + \tan\theta}$$

$$I = (-\pi/2, \pi/2)$$

2.3.12   23

$$x\frac{dy}{dx} + (3x+1)y = e^{-3x}$$

$$\frac{dy}{dx} + (3 + x^{-1})y = e^{-3x}x^{-1}$$

$$e^{3x+\ln|x|}\frac{dy}{dx} + (3 + x^{-1})e^{3x+\ln|x|}y = 1$$

$$\frac{d}{dx}(e^{3x+\ln|x|}y) = 1$$

$$e^{3x}xy = x + c$$

$$y = \frac{x+c}{e^{3x}x}$$

$$I = (0, \infty)$$

26

2.3.13 25

$$xy' + y = e^x$$

$$y' + x^{-1}y = e^x x^{-1}$$

$$e^{\ln|x|}y' + x^{-1}e^{\ln|x|}y = e^x$$

$$\frac{d}{dx}(e^{\ln|x|}y) = e^x$$

$$xy = e^x + c$$

$$y = \frac{e^x + c}{x}$$

$$y(1) = 2 = \frac{e^{(1)} + c}{(1)}$$

$$c = 2 - e$$

$$y = \frac{e^x + 2 - e}{x}$$

$$I = (0, \infty)$$

2.3.14 27

$$L\frac{di}{dt} + Ri = E$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

$$e^{Rt/L}\frac{di}{dt} + \frac{R}{L}e^{Rt/L}i = \frac{E}{L}e^{Rt/L}$$

$$\frac{d}{dt}(e^{Rt/L}i) = \frac{E}{L}e^{Rt/L}$$

$$e^{Rt/L}i = \frac{E}{R}e^{Rt/L} + c$$

$$i = \frac{E}{R} + ce^{-Rt/L}$$

$$i(0) = i_0 = \frac{E}{R} + ce^{-R(0)/L}$$

$$= \frac{E}{R} + c$$

$$c = i_0 - \frac{E}{R}$$

$$i = \frac{E}{R} + \left(i_0 - \frac{E}{R} \right) e^{-Rt/L}$$

$$I = (-\infty, \infty)$$

2.3.15   53

$$\frac{dE}{dt} = -\frac{1}{RC}E$$
$$\frac{1}{E}\frac{dE}{dt} = -\frac{1}{RC}$$
$$\ln |E| = -\frac{1}{RC}t + c$$
$$E = ce^{-t/RC}$$

$$E(4) = E_0 = ce^{-(4)/RC}$$
$$c = E_0 e^{4/RC}$$

$$E = E_0 e^{(4-t)/RC}$$

2.4   Exact Equations

2.4.1   1

$$f(x,y) = x^2 - x + g(y)$$

$$\frac{\partial f}{\partial y} = g'(y) = 3y + 7$$

$$g(y) = \frac{3}{2}y^2 + 7y$$

$$x^2 - x + \frac{3}{2}y^2 + 7y = c$$

2.4.2   3

$$f(x,y) = \frac{5}{2}x^2 + 4xy + g(y)$$

$$4x + g'(y) = 4x - 8y^3 \Rightarrow g'(y) = -8y^3$$

$$g(y) = -2y^4$$

$$\frac{5}{2}x^2 + 4xy - 2y^4 = c$$

2.4.3 5

$$f(x, y) = x^2 y^2 - 3x + g(y)$$

$$2x^2 y + g'(y) = 2x^2 y + 4 \Rightarrow g'(y) = 4$$

$$g(y) = 4y$$

$$x^2 y^2 - 3x + 4y = c$$

2.4.4 7

Not exact

2.4.5 9

$$f(x, y) = \frac{1}{2}x^2 - xy^3 - y^2 \cos x + g(y)$$

$$-3xy^2 - 2y \cos x + g'(y) = -3xy^2 - 2y \cos x \Rightarrow g'(y) = 0$$

$$\frac{1}{2}x^2 - xy^3 - y^2 \cos x = c$$

2.4.6 11

Not exact

2.4.7 13

$$f(x, y) = xy + g(x)$$

$$y + g'(x) = -2xe^x + y - 6x^2 \Rightarrow g'(x) = -2xe^x - 6x^2$$

$$g(x) = -2e^x(x - 1) - 2x^3$$

$$xy - 2e^x(x - 1) - 2x^3 = c$$

2.4.8 21

$$f(x, y) = \frac{1}{3}(x+y)^3 + g(y)$$

$$(x+y)^2 + g'(y) = 2xy + x^2 - 1 \Rightarrow g'(y) = -y^2 - 1$$

$$g(y) = -\frac{1}{3}y^3 - y$$

$$\frac{1}{3}(x+y)^3 - \frac{1}{3}y^3 - y = c$$

$$\frac{1}{3}(1+1)^3 - \frac{1}{3}1^3 - 1 = c \Rightarrow c = \frac{4}{3}$$

$$x^3 + 3x^2y + 3xy^2 - 3y = 4$$

2.4.9 23

$$f(x, y) = 4ty + t^2 - 5t + g(y)$$

$$4t + g'(y) = 6y + 4t - 1 \Rightarrow g'(y) = 6y - 1$$

$$g(y) = 3y^2 - y$$

$$4ty + t^2 - 5t + 3y^2 - y = c$$

$$4(-1)(2) + (-1)^2 - 5(-1) + 3(2)^2 - (2) = c \Rightarrow c = 8$$

$$4ty + t^2 - 5t + 3y^2 - y = 8$$

2.4.10 27

$$3y^2 + 4kxy^3 = 3y^2 + 40xy^3 \Rightarrow k = 10$$

2.4.11   31

$$M_y = 4y$$

$$N_x = 2y$$

$$\frac{M_y - N_x}{N} = \frac{4y - 2y}{2xy} = \frac{1}{x}$$

$$\mu(x) = e^{\ln x} = x$$

$$(2xy^2 + 3x^2)\,dx + 2x^2y\,dy = 0$$

$$f(x, y) = x^2y^2 + x^3 + g(y)$$

$$2x^2y + g'(y) = 2x^2y \Rightarrow g'(y) = 0$$

$$x^2y^2 + x^3 = c$$

2.4.12   33

$$M_y = 6x$$

$$N_x = 18x$$

$$\frac{M_y - N_x}{N} = \frac{6x - 18x}{4y + 9x^2}$$

$$\frac{N_x - M_y}{M} = \frac{18x - 6x}{6xy} = \frac{2}{y}$$

$$\mu(y) = e^{2\ln y} = y^2$$

$$6xy^3\,dx + (4y^3 + 9x^2y^2)\,dy = 0$$

$$f(x, y) = 3x^2y^3 + g(y)$$

$$9x^2y^2 + g'(y) = 4y^3 + 9x^2y^2 \Rightarrow g'(y) = 4y^3$$

$$g(y) = y^4$$

$$3x^2y^3 + y^4 = c$$

2.4.13 37

$$M_y = 0$$

$$N_x = 2xy$$

$$\frac{N_x - M_y}{M} = \frac{2xy - 0}{x} = 2y$$

$$\mu(y) = e^{y^2}$$

$$e^{y^2} x \, dx + e^{y^2}(x^2 y + 4y) \, dy = 0$$

$$f(x, y) = \frac{1}{2} e^{y^2} x^2 + g(y)$$

$$y e^{y^2} x^2 + g'(y) = e^{y^2}(x^2 y + 4y) \Rightarrow g'(y) = 4 e^{y^2} y$$

$$g(y) = 2 e^{y^2}$$

$$\frac{1}{2} e^{y^2} x^2 + 2 e^{y^2} = c$$

$$\frac{1}{2} e^{(0)^2}(4)^2 + 2 e^{(0)^2} = c \Rightarrow c = 10$$

$$\frac{1}{2} e^{y^2} x^2 + 2 e^{y^2} = 10$$

2.4.14 39

(c)

$$(0)^3 + 2(0)^2(-2) + (-2)^2 = c \Rightarrow c = 4$$

$$y^2 + 2x^2 y + x^3 - 4 = 0$$

$$y = \frac{-(2x^2) \pm \sqrt{(2x^2)^2 - 4(1)(x^3 - 4)}}{2(1)}$$

$$= \frac{-2x^2 \pm \sqrt{4x^4 - 4(x^3 - 4)}}{2}$$

$$= -x^2 \pm \sqrt{x^4 - x^3 + 4}$$

2.4.15   45

(a)

$$xv\frac{dv}{dx} + v^2 = 32x \Rightarrow xv\,dv + (v^2 - 32x)\,dx = 0$$

$$M_x = v$$

$$N_v = 2v$$

$$\frac{M_x - N_v}{N} = \frac{v - 2v}{v^2 - 32x}$$

$$\frac{N_v - M_x}{M} = \frac{2v - v}{xv} = \frac{1}{x}$$

$$\mu(x) = e^{\ln x} = x$$

$$x^2v\,dv + (xv^2 - 32x^2)\,dx = 0$$

$$f(x,v) = \frac{1}{2}x^2v^2 + g(x)$$

$$xv^2 + g'(x) = xv^2 - 32x^2 \Rightarrow g'(x) = -32x^2$$

$$g(x) = -\frac{32}{3}x^3$$

$$\frac{1}{2}(3)^2(0)^2 - \frac{32}{3}(3)^3 = c \Rightarrow c = -288$$

$$\frac{1}{2}x^2v^2 - \frac{32}{3}x^3 = -288 \Rightarrow v = 8\sqrt{\frac{x}{3} - \frac{9}{x^2}}$$

(b) $v = 12.7\,\text{ft/s}$