

Advanced Engineering Mathematics Partial Differential Equations by Dennis G. Zill Problems

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12 Orthogonal Functions and Fourier Series

12.1 Orthogonal Functions

12.1.7

$$\begin{aligned}\int_0^{\pi/2} \sin mx \sin nx \, dx &= \frac{1}{2} \int_0^{\pi/2} [\cos(m-n)x - \cos(m+n)x] \, dx \\&= \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_0^{\pi/2} \\&= \frac{1}{2} \left(\frac{\sin(m-n)\pi/2}{m-n} - \frac{\sin(m+n)\pi/2}{m+n} \right) \\&= 0\end{aligned}$$

$$\begin{aligned}\|\sin nx\|^2 &= (\sin nx, \sin nx) \\&= \int_0^{\pi/2} \sin^2 nx \, dx \\&= \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2nx) \, dx \\&= \frac{1}{2} \left[x - \frac{1}{2n} \sin 2nx \right]_0^{\pi/2} \\&= \frac{\pi}{4} \\ \|\sin nx\| &= \frac{\sqrt{\pi}}{2}\end{aligned}$$

12.1.9

$$\begin{aligned}\int_0^\pi \sin mx \sin nx \, dx &= \frac{1}{2} \int_0^\pi [\cos(m-n)x - \cos(m+n)x] \, dx \\ &= \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_0^\pi \\ &= 0\end{aligned}$$

$$\begin{aligned}\|\sin nx\|^2 &= (\sin nx, \sin nx) \\ &= \int_0^\pi \sin^2 nx \, dx \\ &= \frac{1}{2} \int_0^\pi (1 - \cos 2nx) \, dx \\ &= \frac{1}{2} \left[x - \frac{1}{2n} \sin 2nx \right]_0^\pi \\ &= \frac{\pi}{2}\end{aligned}$$

$$\|\sin nx\| = \sqrt{\frac{\pi}{2}}$$

12.1.21

$$T = 1$$

12.1.23

$$T = 2\pi$$

12.1.25

$$T = 2\pi$$

12.2 Fourier Series

12.2.1

$$\begin{aligned}p &= \pi \\a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\&= \frac{1}{\pi} \int_0^{\pi} dx \\&= 1 \\a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\&= \frac{1}{\pi} \int_0^{\pi} \cos nx dx \\&= \frac{1}{n\pi} [\sin nx]_0^{\pi} \\&= 0 \\b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\&= \frac{1}{\pi} \int_0^{\pi} \sin nx dx \\&= -\frac{1}{n\pi} [\cos nx]_0^{\pi} \\&= -\frac{1}{n\pi} [(-1)^n - 1] \\&= \frac{1 - (-1)^n}{n\pi} \\f(x) &= \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin nx\end{aligned}$$

The series converges to $\frac{1}{2}$ at the point of discontinuity.

12.2.3

$$\begin{aligned}p &= 1 \\a_0 &= \frac{3}{2} \\a_n &= \int_{-1}^0 \cos n\pi x \, dx + \int_0^1 x \cos n\pi x \, dx \\&= \frac{1}{n\pi} [\sin n\pi x]_{-1}^0 + \frac{1}{n\pi} \left[\frac{\cos n\pi x}{n\pi} + x \sin n\pi x \right]_0^1 \\&= \frac{(-1)^n - 1}{n^2 \pi^2} \\b_n &= \int_{-1}^0 \sin n\pi x \, dx + \int_0^1 x \sin n\pi x \, dx \\&= -\frac{1}{n\pi} [\cos n\pi x]_{-1}^0 + \frac{1}{n\pi} \left[\frac{\sin n\pi x}{n\pi} - x \cos n\pi x \right]_0^1 \\&= -\frac{1}{n\pi} \\f(x) &= \frac{3}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2 \pi^2} \cos n\pi x - \frac{1}{n\pi} \sin n\pi x \right]\end{aligned}$$

The series converges to $\frac{1}{2}$ at the point of discontinuity.

12.3 Fourier Cosine and Sine Series

12.3.1

Odd

12.3.3

Neither

12.3.5

Even

12.3.7

Odd

12.3.9

Neither

12.3.11

$$\begin{aligned}
b_n &= -2\pi \int_0^1 \sin n\pi x \, dx \\
&= \frac{2}{n} [\cos n\pi x]_0^1 \\
&= \frac{2}{n} [(-1)^n - 1] \\
f &= \sum_{n=1}^{\infty} \frac{2}{n} [(-1)^n - 1] \sin n\pi x
\end{aligned}$$

12.3.13

$$\begin{aligned}
a_0 &= \frac{2}{\pi} \int_0^{\pi} x \, dx \\
&= \pi \\
a_n &= 2 \int_0^{\pi} x \cos nx \, dx \\
&= \frac{2}{n} \left[\frac{\cos nx}{n} + x \sin nx \right]_0^{\pi} \\
&= \frac{2[(-1)^n - 1]}{n^2} \\
f &= \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx
\end{aligned}$$

12.3.25

$$\begin{aligned}
 a_0 &= 2 \int_0^1 f(x) dx \\
 &= 1 \\
 a_n &= 2 \int_0^1 f(x) \cos n\pi x dx \\
 &= 2 \int_0^{1/2} \cos n\pi x dx \\
 &= \frac{2}{n\pi} [\sin n\pi x]_0^{1/2} \\
 &= \frac{2}{n\pi} \sin \frac{n\pi}{2} \\
 f &= \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n} \cos n\pi x
 \end{aligned}$$

$$\begin{aligned}
 b_n &= 2 \int_0^1 f(x) \sin n\pi x dx \\
 &= 2 \int_0^{1/2} \sin n\pi x dx \\
 &= -\frac{2}{n\pi} [\cos n\pi x]_0^{1/2} \\
 &= \frac{2}{n\pi} \left(1 - \cos \frac{n\pi}{2}\right) \\
 f &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - \cos \frac{n\pi}{2}}{n} \sin n\pi x
 \end{aligned}$$

12.3.27

$$\begin{aligned}
 a_0 &= \frac{4}{\pi} \int_0^{\pi/2} \cos x \, dx \\
 &= \frac{4}{\pi} [\sin x]_0^{\pi/2} \\
 &= \frac{4}{\pi} \\
 a_n &= \frac{4}{\pi} \int_0^{\pi/2} \cos x \cos 2nx \, dx \\
 &= \frac{2}{\pi} \int_0^{\pi/2} [\cos(1-2n)x + \cos(1+2n)x] \, dx \\
 &= \frac{2}{\pi} \left[\frac{\sin(1-2n)x}{1-2n} + \frac{\sin(1+2n)x}{1+2n} \right]_0^{\pi/2} \\
 &= \frac{2(-1)^n}{\pi} \left[\frac{1}{1-2n} + \frac{1}{1+2n} \right] \\
 &= \frac{2(-1)^n}{\pi} \frac{1+2n+1-2n}{(1-2n)(1+2n)} \\
 &= \frac{4(-1)^n}{\pi(1-2n)(1+2n)} \\
 f &= \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(1-2n)(1+2n)} \cos 2nx
 \end{aligned}$$

$$\begin{aligned}
 b_n &= \frac{4}{\pi} \int_0^{\pi/2} \cos x \sin 2nx \, dx \\
 &= \frac{2}{\pi} \int_0^{\pi/2} [\sin(2n+1)x + \sin(2n-1)x] \, dx \\
 &= -\frac{2}{\pi} \left[\frac{\cos(2n+1)x}{2n+1} + \frac{\cos(2n-1)x}{2n-1} \right]_0^{\pi/2} \\
 &= \frac{2}{\pi} \left(\frac{1}{2n+1} + \frac{1}{2n-1} \right) \\
 &= \frac{2}{\pi} \frac{4n}{4n^2-1} \\
 &= \frac{8n}{\pi(4n^2-1)} \\
 f &= \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2-1} \sin 2nx
 \end{aligned}$$

12.3.35

$$\begin{aligned}
 a_0 &= \frac{1}{\pi} \int_0^{2\pi} x^2 dx \\
 &= \frac{8}{3} \pi^2 \\
 a_n &= \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx dx \\
 &= \frac{4}{n^2} \\
 b_n &= \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nx dx \\
 &= -\frac{4\pi}{n} \\
 f &= \frac{4}{3} \pi^2 + \sum_{n=1}^{\infty} \left(\frac{4}{n^2} \cos nx - \frac{4\pi}{n} \sin nx \right)
 \end{aligned}$$

12.3.43

$$\begin{aligned}
 b_n &= \frac{10}{\pi} \int_0^{\pi} \sin nt dt \\
 &= -\frac{10}{n\pi} [\cos nt]_0^{\pi} \\
 &= \frac{10}{n\pi} [1 - (-1)^n] \\
 f &= \frac{10}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} [1 - (-1)^n] \sin nt \\
 x_p(t) &= \sum_{n=1}^{\infty} B_n \sin nt \\
 m \frac{d^2 x}{dt^2} + kx &= f(t)
 \end{aligned}$$

$$\begin{aligned}
 -mn^2 B_n + kB_n &= \frac{10}{n\pi} [1 - (-1)^n] \\
 B_n &= \frac{10}{n\pi(k - mn^2)} [1 - (-1)^n] \\
 x_p(t) &= \frac{10}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n(k - mn^2)} \sin nt \\
 &= \frac{10}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n(10 - n^2)} \sin nt
 \end{aligned}$$

12.3.45

$$a_0 = \frac{2}{\pi} \int_0^\pi (2\pi t - t^2) dt$$

$$= \frac{2}{\pi} \left[\pi t^2 - \frac{1}{3} t^3 \right]_0^\pi$$

$$= \frac{2}{\pi} \left(\pi^3 - \frac{1}{3} \pi^3 \right)$$

$$= \frac{4}{3} \pi^2$$

$$a_n = \frac{2}{\pi} \int_0^\pi (2\pi t - t^2) \cos nt dt$$

$$= -\frac{4}{n^2}$$

$$f(t) = \frac{2}{3} \pi^2 - \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nt$$

$$x_p(t) = c + \sum_{n=1}^{\infty} B_n \cos nt$$

$$\frac{1}{4} \frac{d^2 x}{dt^2} + 12x = f(t)$$

$$-\frac{n^2}{4} \sum_{n=1}^{\infty} B_n \cos nt + 12c + 12 \sum_{n=1}^{\infty} B_n \cos nt = \frac{2}{3} \pi^2 - \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nt$$

$$12c = \frac{2}{3} \pi^2$$

$$c = \frac{\pi^2}{18}$$

$$-\frac{n^2}{4} B_n + 12B_n = -\frac{4}{n^2}$$

$$\frac{48 - n^2}{4} B_n = -\frac{4}{n^2}$$

$$B_n = -\frac{16}{n^2(48 - n^2)}$$

$$x_p(t) = \frac{\pi^2}{18} - \sum_{n=1}^{\infty} \frac{16}{n^2(48 - n^2)} \cos nt$$

$$= \frac{\pi^2}{18} + 16 \sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - 48)} \cos nt$$

12.4 Complex Fourier Series

12.4.1

$$\begin{aligned}
 T &= 4 \\
 p &= 2 \\
 c_n &= \frac{1}{4} \left(\int_0^2 e^{-in\pi x/2} dx - \int_{-2}^0 e^{-in\pi x/2} dx \right) \\
 &= \frac{1}{2in\pi} ([e^{-in\pi x/2}]_{-2}^0 - [e^{-in\pi x/2}]_0^2) \\
 &= \frac{2 - e^{in\pi} - e^{-in\pi}}{2in\pi} \\
 &= \frac{2 - \cos n\pi - i \sin n\pi - \cos n\pi + i \sin n\pi}{2in\pi} \\
 &= \frac{1 - \cos n\pi}{in\pi} \\
 &= \frac{1 - (-1)^n}{in\pi} \\
 f(x) &= \sum_{n=-\infty, n \neq 0}^{\infty} \frac{1 - (-1)^n}{in\pi} e^{in\pi x/2}
 \end{aligned}$$

12.4.3

$$\begin{aligned}
 T &= 1 \\
 p &= \frac{1}{2} \\
 c_n &= \int_0^{1/4} e^{-2in\pi x} dx \\
 &= -\frac{1}{2in\pi} [e^{-2in\pi x}]_0^{1/4} \\
 &= \frac{1}{2in\pi} (1 - e^{-in\pi/2}) \\
 c_0 &= \frac{1}{4} \\
 f(x) &= \frac{1}{4} + \sum_{n=-\infty, n \neq 0}^{\infty} \frac{1 - e^{-in\pi/2}}{2in\pi} e^{2in\pi x}
 \end{aligned}$$

12.4.5

$$T = 2\pi$$

$$p = \pi$$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} dx$$

$$= \frac{i}{n}$$

$$c_0 = \frac{1}{2\pi} \int_0^{2\pi} x dx$$

$$= \pi$$

$$f(x) = \pi + \sum_{n=-\infty, n \neq 0}^{n=\infty} \frac{i}{n} e^{inx}$$

12.5 Sturm-Liouville Problem

12.5.1

$$y'' + \lambda y = 0$$

$$y'(0) = 0$$

$$y(1) + y'(1) = 0$$

$$\lambda = \alpha^2$$

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$y' = -\alpha c_1 \sin \alpha x + \alpha c_2 \cos \alpha x$$

$$c_2 = 0$$

$$y = c_1 \cos \alpha x$$

$$c_1 \cos \alpha - \alpha c_1 \sin \alpha = 0$$

$$c_1 \cos \alpha = \alpha c_1 \sin \alpha$$

$$\alpha \tan \alpha = 0$$

$$\alpha = \cot \alpha$$

$$\lambda_1 = 0.740174$$

$$y_1 = \cos 0.860334x$$

$$\lambda_2 = 11.734872$$

$$y_2 = \cos 3.42562x$$

$$\lambda_3 = 41.438831$$

$$y_3 = \cos 6.4373x$$

$$\lambda_4 = 90.808130$$

$$y_4 = \cos 9.52933x$$

12.5.5

$$\begin{aligned}(y_n, y_n) &= \int_0^1 \cos^2 \alpha_n x \, dx \\&= \frac{1}{2} \int_0^1 (1 + \cos 2\alpha_n x) \, dx \\&= \frac{1}{2} \left[x + \frac{1}{2\alpha_n} \sin 2\alpha_n x \right]_0^1 \\&= \frac{1}{2} \left(1 + \frac{1}{2\alpha_n} \sin 2\alpha_n \right) \\&= \frac{1}{2} \left(1 + \frac{1}{\alpha_n} \sin \alpha_n \cos \alpha_n \right) \\&= \frac{1}{2} (1 + \tan \alpha_n \sin \alpha_n \cos \alpha_n) \\&= \frac{1}{2} (1 + \sin^2 \alpha_n)\end{aligned}$$

12.5.7

(a)

$$x^2 y'' + xy' + \lambda y = 0$$

$$y(1) = 0$$

$$y(5) = 0$$

$$\lambda = \alpha^2$$

$$y = x^m$$

$$y' = mx^{m-1}$$

$$y'' = m(m-1)x^{m-2}$$

$$x^2 m(m-1)x^{m-2} + mx^{m-1} + \alpha^2 x^m = 0$$

$$m(m-1) + m + \alpha^2 = 0$$

$$m^2 + \alpha^2 = 0$$

$$m = \pm i\alpha$$

$$y = c_1 \cos(\alpha \ln x) + c_2 \sin(\alpha \ln x)$$

$$0 = c_1$$

$$0 = c_2 \sin(\alpha \ln 5)$$

$$\alpha = \frac{n\pi}{\ln 5}$$

$$\lambda = \left(\frac{n\pi}{\ln 5} \right)^2$$

$$y_n = \sin \left(\frac{n\pi}{\ln 5} \ln x \right)$$

(b)

$$x^2 y'' + xy' + \lambda y = 0$$

$$y'' + \frac{1}{x} y' + \lambda \frac{1}{x^2} y = 0$$

$$e^{\ln x} y'' + \frac{1}{x} e^{\ln x} y' + \lambda e^{\ln x} \frac{1}{x^2} y = 0$$

$$\frac{d}{dx} (e^{\ln x} y') + \lambda e^{\ln x} \frac{1}{x^2} y = 0$$

$$\frac{d}{dx} (xy') + \lambda \frac{1}{x} y = 0$$

(c)

$$\int_1^5 \frac{1}{x} \sin \left(\frac{m\pi}{\ln 5} \ln x \right) \sin \left(\frac{n\pi}{\ln 5} \ln x \right) dx = 0, \quad m \neq n$$

12.5.9

$$\begin{aligned}
 xy'' + (1-x)y' + ny &= 0 \\
 y'' + \left(\frac{1}{x} - 1\right)y' + n\frac{1}{x}y &= 0 \\
 e^{\int\left(\frac{1}{x}-1\right)dx} &= e^{\ln(x)-x} \\
 &= xe^{-x} \\
 xe^{-x}y'' + \left(\frac{1}{x} - 1\right)xe^{-x}y' + n\frac{1}{x}xe^{-x}y &= 0 \\
 \frac{d}{dx}(xe^{-x}y') + ne^{-x}y &= 0 \\
 \int_0^\infty e^{-x}L_m(x)L_n(x)dx &= 0, \quad m \neq n
 \end{aligned}$$

12.6 Bessel and Legendre Series

12.6.1

$$\begin{aligned}
 J_1(3\alpha) &= 0 \\
 \alpha_1 &= 1.277 \\
 \alpha_2 &= 2.338 \\
 \alpha_3 &= 3.391 \\
 \alpha_4 &= 4.441
 \end{aligned}$$

12.6.3

$$\begin{aligned}
 J_0(2\alpha) &= 0 \\
 c_i &= \frac{1}{2J_1^2(2\alpha_i)} \int_0^2 xJ_0(\alpha_i x) dx \\
 &= \frac{1}{2J_1^2(2\alpha_i)} \int_0^2 \frac{d}{dx} \left[\frac{1}{\alpha_i} xJ_1(\alpha_i x) \right] dx \\
 &= \frac{1}{\alpha_i J_1(2\alpha_i)} \\
 f(x) &= \sum_{i=1}^{\infty} \frac{J_0(\alpha_i x)}{\alpha_i J_1(2\alpha_i)}
 \end{aligned}$$

12.6.5

$$J_0(2\alpha) + 2\alpha J_0'(2\alpha) = 0$$

$$h = 1$$

$$b = 2$$

$$\begin{aligned} c_i &= \frac{2\alpha_i^2}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)} \int_0^2 x J_0(\alpha_i x) dx \\ &= \frac{2\alpha_i^2}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)} \int_0^2 \frac{d}{dx} \left[\frac{1}{\alpha_i} x J_1(\alpha_i x) \right] dx \\ &= \frac{4\alpha_i J_1(2\alpha_i)}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)} \end{aligned}$$

$$f(x) = 4 \sum_{i=1}^{\infty} \frac{\alpha_i J_1(2\alpha_i)}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)} J_0(\alpha_i x)$$

12.6.7

$$f(x) = 5x, \quad 0 < x < 4$$

$$4J_1(4\alpha) + 4\alpha J_1'(4\alpha) = 0$$

$$h = 3$$

$$n = 1$$

$$b = 4$$

$$\begin{aligned} c_i &= \frac{2\alpha_i^2}{(16\alpha_i^2 + 8)J_1^2(4\alpha_i)} \int_0^4 5x^2 J_1(\alpha_i x) dx \\ &= \frac{10\alpha_i^2}{(16\alpha_i^2 + 8)J_1^2(4\alpha_i)} \int_0^4 \frac{d}{dx} \left[\frac{1}{\alpha_i} x^2 J_2(\alpha_i x) \right] dx \\ &= \frac{160\alpha_i J_2(4\alpha_i)}{(16\alpha_i^2 + 8)J_1^2(4\alpha_i)} \end{aligned}$$

$$f(x) = \sum_{i=1}^{\infty} \frac{20\alpha_i J_2(4\alpha_i)}{(2\alpha_i^2 + 1)J_1^2(4\alpha_i)} J_1(\alpha_i x)$$

12.6.15

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ x & 0 < x < 1 \end{cases}$$

$$c_n = \frac{2n+1}{2} \int_0^1 x P_n(x) dx$$

$$c_0 = \frac{1}{4}$$

$$c_1 = \frac{1}{2}$$

$$c_2 = \frac{5}{16}$$

$$c_3 = 0$$

$$c_4 = -\frac{3}{32}$$

$$c_5 = 0$$

$$c_6 = \frac{13}{256}$$

12.6.21

$$c_0 = \frac{1}{2}$$

$$c_1 = \frac{5}{8}$$

$$c_2 = -\frac{3}{16}$$

$$c_3 = \frac{13}{128}$$

12.7 Chapter in Review**12.7.1**

$$(x^2 - 1, x^5) = \int_{-\pi}^{\pi} (x^2 - 1)x^5 dx$$

$$= \int_{-\pi}^{\pi} (x^7 - x^5) dx$$

$$= \left[\frac{1}{8}x^8 - \frac{1}{6}x^6 \right]_{-\pi}^{\pi}$$

$$= \frac{1}{8}\pi^8 - \frac{1}{6}\pi^6 - \frac{1}{8}\pi^8 + \frac{1}{6}\pi^6$$

$$= 0$$

True

12.7.3

Fourier cosine

12.7.5

False

12.7.7

5.5, 1, 0

12.7.9

True

12.7.13

$$f(x) = |x| - x, \quad -1 < x < 1$$

$$L = 2$$

$$p = 1$$

$$a_0 = \int_{-1}^1 (|x| - x) dx$$

$$= \int_{-1}^0 -2x dx$$

$$= -[x^2]_{-1}^0$$

$$= -(0 - 1)$$

$$= 1$$

$$a_n = \int_{-1}^1 (|x| - x) \cos n\pi x dx$$

$$= -2 \int_{-1}^0 x \cos n\pi x dx$$

$$= \frac{2[(-1)^n - 1]}{n^2 \pi^2}$$

$$b_n = \int_{-1}^1 (|x| - x) \sin n\pi x dx$$

$$= -2 \int_{-1}^0 x \sin n\pi x dx$$

$$= \frac{2(-1)^n}{n\pi}$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{n^2 \pi^2} \cos n\pi x + \frac{2(-1)^n}{n\pi} \sin n\pi x$$

12.7.17

$$x^2 y'' + x y' + 9 \lambda y = 0$$

$$y'(1) = 0$$

$$y(e) = 0$$

$$y = x^m$$

$$y' = m x^{m-1}$$

$$y'' = m(m-1)x^{m-2}$$

$$x^2 m(m-1)x^{m-2} + x m x^{m-1} + 9 \lambda x^m = 0$$

$$m(m-1) + m + 9 \lambda = 0$$

$$m^2 + 9 \lambda = 0$$

$$m = \pm 3\sqrt{\lambda}i$$

$$y = c_1 \cos(3\sqrt{\lambda} \ln x) + c_2 \sin(3\sqrt{\lambda} \ln x)$$

$$y' = \frac{3\sqrt{\lambda}}{x} [c_2 \cos(3\sqrt{\lambda} \ln x) - c_1 \sin(3\sqrt{\lambda} \ln x)]$$

$$y'(1) = 0$$

$$0 = 3\sqrt{\lambda}c_2$$

$$c_2 = 0$$

$$y(e) = 0$$

$$0 = c_1 \cos 3\sqrt{\lambda}$$

$$= \cos 3\sqrt{\lambda}$$

$$3\sqrt{\lambda} = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}$$

$$= \frac{2n+1}{2}\pi$$

$$\lambda_n = \left(\frac{2n+1}{6}\pi \right)^2$$

$$y_n = \cos \left(\frac{2n+1}{2}\pi \ln x \right)$$

13 Boundary-Value Problems in Rectangular Coordinates

13.1 Separable Partial Differential Equations

13.1.1

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$$

$$u = X(x)Y(y)$$

$$X'Y = XY'$$

$$\frac{X'}{X} = \frac{Y'}{Y}$$

$$\frac{X'}{X} = \lambda$$

$$\ln X = \lambda x + c_1$$

$$X = c_1 e^{\lambda x}$$

$$\frac{Y'}{Y} = \lambda$$

$$\ln Y = \lambda y + c_2$$

$$Y = c_2 e^{\lambda y}$$

$$u = XY$$

$$= c_1 c_2 e^{\lambda(x+y)}$$

$$= c_3 e^{\lambda(x+y)}$$

13.1.3

$$u_x + u_y = u$$
$$X'Y + XY' = XY$$

$$\frac{X'}{X}Y + Y' = Y$$

$$\frac{X'}{X} + \frac{Y'}{Y} = 1$$

$$\frac{X'}{X} = 1 - \frac{Y'}{Y}$$

$$\frac{X'}{X} = \lambda$$

$$X = c_1 e^{\lambda x}$$

$$1 - \frac{Y'}{Y} = \lambda$$

$$Y' + (\lambda - 1)Y = 0$$

$$Y = c_2 e^{-(\lambda-1)y}$$

$$u = c_3 e^{\lambda x - (\lambda-1)y}$$

13.1.5

$$\begin{aligned}
x \frac{\partial u}{\partial x} &= y \frac{\partial u}{\partial y} \\
xX'Y &= yXY' \\
x \frac{X'}{X} &= y \frac{Y'}{Y}
\end{aligned}$$

$$\begin{aligned}
x \frac{X'}{X} &= \lambda \\
\frac{X'}{X} &= \frac{\lambda}{x} \\
\ln X &= \lambda \ln x + c_1 \\
X &= c_1 x^\lambda
\end{aligned}$$

$$Y = c_2 y^\lambda$$

$$\begin{aligned}
u &= XY \\
&= c_3 (xy)^\lambda
\end{aligned}$$

13.1.7

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} &= 0 \\
X''Y + X'Y' + XY'' &= 0
\end{aligned}$$

Not separable.

13.1.9

$$k \frac{\partial^2 u}{\partial x^2} - u = \frac{\partial u}{\partial t}, \quad k > 0$$

$$kTX'' - TX = T'X$$

$$k \frac{X''}{X} - 1 = \frac{T'}{T}$$

$$\frac{T'}{T} = \lambda$$

$$T' - \lambda T = 0$$

$$T = c_1 e^{\lambda t}$$

$$k \frac{X''}{X} - 1 = \lambda$$

$$X'' - \frac{\lambda+1}{k} X = 0$$

$$X = \begin{cases} c_1 \cos \sqrt{\frac{\lambda+1}{k}} x + c_2 \sin \sqrt{\frac{\lambda+1}{k}} x & \lambda < -1 \\ c_1 x + c_2 & \lambda = -1 \\ c_1 \cosh \sqrt{\frac{\lambda+1}{k}} x + c_2 \sinh \sqrt{\frac{\lambda+1}{k}} x & \lambda > -1 \end{cases}$$

$$u = TX$$

$$= \begin{cases} e^{\lambda t} \left(c_1 \cos \sqrt{\frac{\lambda+1}{k}} x + c_2 \sin \sqrt{\frac{\lambda+1}{k}} x \right) & \lambda < -1 \\ e^{\lambda t} (c_1 x + c_2) & \lambda = -1 \\ e^{\lambda t} \left(c_1 \cosh \sqrt{\frac{\lambda+1}{k}} x + c_2 \sinh \sqrt{\frac{\lambda+1}{k}} x \right) & \lambda > -1 \end{cases}$$

13.1.11

$$\begin{aligned} a^2 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t^2} \\ a^2 T X'' &= T'' X \\ a^2 \frac{X''}{X} &= \frac{T''}{T} \end{aligned}$$

$$\begin{aligned} \frac{T''}{T} &= \lambda \\ T'' - \lambda T &= 0 \end{aligned}$$

$$T = \begin{cases} c_1 \cos \sqrt{\lambda} t + c_2 \sin \sqrt{\lambda} t & \lambda < 0 \\ c_1 t + c_2 & \lambda = 0 \\ c_1 \cosh \sqrt{\lambda} t + c_2 \sinh \sqrt{\lambda} t & \lambda > 0 \end{cases}$$

$$\begin{aligned} a^2 \frac{X''}{X} &= \lambda \\ X'' - \frac{\lambda}{a^2} X &= 0 \end{aligned}$$

$$X = \begin{cases} c_1 \cos \frac{\sqrt{\lambda}}{a} x + c_2 \sin \frac{\sqrt{\lambda}}{a} x & \lambda < 0 \\ c_1 x + c_2 & \lambda = 0 \\ c_1 \cosh \frac{\sqrt{\lambda}}{a} x + c_2 \sinh \frac{\sqrt{\lambda}}{a} x & \lambda > 0 \end{cases}$$

$$\begin{aligned} u &= T X \\ &= \begin{cases} (c_1 \cos \sqrt{\lambda} t + c_2 \sin \sqrt{\lambda} t)(c_3 \cos \frac{\sqrt{\lambda}}{a} x + c_4 \sin \frac{\sqrt{\lambda}}{a} x) & \lambda < 0 \\ (c_1 t + c_2)(c_3 x + c_4) & \lambda = 0 \\ (c_1 \cosh \sqrt{\lambda} t + c_2 \sinh \sqrt{\lambda} t)(c_3 \cosh \frac{\sqrt{\lambda}}{a} x + c_4 \sinh \frac{\sqrt{\lambda}}{a} x) & \lambda > 0 \end{cases} \end{aligned}$$

13.1.17

Elliptic

13.1.19

Parabolic

13.1.21

Hyperbolic

13.1.23

Parabolic

13.1.25

Hyperbolic

13.2 Classical PDEs and Boundary-Value Problems**13.2.1**

$$\begin{aligned}
 k^2 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t} \\
 u(0, t) &= 0 \\
 \left. \frac{\partial u}{\partial x} \right|_{x=L} &= 0 \\
 u(x, 0) &= f(x)
 \end{aligned}$$

13.2.3

$$\begin{aligned}
 k^2 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t} \\
 u(0, t) &= 100 \\
 \left. \frac{\partial u}{\partial x} \right|_{x=L} &= -hu(L, t) \\
 u(x, 0) &= f(x)
 \end{aligned}$$

13.2.5

$$\begin{aligned}
 k^2 \frac{\partial^2 u}{\partial x^2} - hu &= \frac{\partial u}{\partial t} \\
 u(0, t) &= \sin \frac{\pi}{L} t \\
 u(L, t) &= 0 \\
 u(x, 0) &= f(x)
 \end{aligned}$$

13.2.7

$$\begin{aligned}
a^2 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t^2} \\
u(0, t) &= 0 \\
u(L, t) &= 0 \\
u(x, 0) &= x(L - x) \\
\left. \frac{\partial u}{\partial t} \right|_{t=0} &= 0
\end{aligned}$$

13.2.9

$$\begin{aligned}
a^2 \frac{\partial^2 u}{\partial x^2} - c \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial t^2} \\
u(0, t) &= 0 \\
u(L, t) &= \sin \pi t \\
u(x, 0) &= f(x) \\
\left. \frac{\partial u}{\partial t} \right|_{t=0} &= 0
\end{aligned}$$

13.2.11

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\
\left. \frac{\partial u}{\partial x} \right|_{x=0} &= 0 \\
\left. \frac{\partial u}{\partial y} \right|_{y=0} &= 0 \\
u(x, 2) &= 0 \\
u(4, y) &= f(y)
\end{aligned}$$

13.3 Heat Equation

13.3.1

$$\begin{aligned}k \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t} \\u(0, t) &= 0 \\u(L, t) &= 0 \\u(x, 0) &= \begin{cases} 1 & 0 < x < L/2 \\ 0 & L/2 < x < L \end{cases} \\A_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx \\&= \frac{2}{L} \int_0^{L/2} \sin \frac{n\pi}{L} x \, dx \\&= -\frac{2}{n\pi} \left[\cos \frac{n\pi}{L} x \right]_0^{L/2} \\&= \frac{2}{n\pi} \left(1 - \cos \frac{n\pi}{2} \right) \\u(x, t) &= \sum_{n=1}^{\infty} \frac{2}{n\pi} \left(1 - \cos \frac{n\pi}{2} \right) e^{-k(n^2 \pi^2 / L^2)t} \sin \frac{n\pi}{L} x\end{aligned}$$

13.3.3

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

$$u(x, 0) = f(x)$$

$$\left. \frac{\partial u}{\partial x} \right|_{x=0} = 0$$

$$\left. \frac{\partial u}{\partial x} \right|_{x=L} = 0$$

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$X'(x) = -\alpha c_1 \sin \alpha x + \alpha c_2 \cos \alpha x$$

$$0 = X'(0)$$

$$= \alpha c_2$$

$$c_2 = 0$$

$$0 = X'(L)$$

$$= -\alpha c_1 \sin \alpha L$$

$$\alpha L = n\pi$$

$$\alpha = \frac{n\pi}{L}$$

$$X(x) = c_1 \cos \frac{n\pi}{L} x$$

$$T(t) = c_3 e^{-k(n^2 \pi^2 / L^2) t}$$

$$u_n = A_n e^{-k(n^2 \pi^2 / L^2) t} \cos \frac{n\pi}{L} x$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x \, dx$$

$$u_n = \frac{1}{L} \int_0^L f(x) \, dx$$

$$+ \frac{2}{L} \sum_{n=1}^{\infty} \left(\int_0^L f(x) \cos \frac{n\pi}{L} x \, dx \right) e^{-k(n^2 \pi^2 / L^2) t} \cos \frac{n\pi}{L} x$$

13.3.5

$$\begin{aligned}
k \frac{\partial^2 u}{\partial x^2} - hu &= \frac{\partial u}{\partial t} \\
u(x, 0) &= f(x) \\
\left. \frac{\partial u}{\partial x} \right|_{x=0} &= 0 \\
\left. \frac{\partial u}{\partial x} \right|_{x=L} &= 0 \\
kX''T - hXT &= XT' \\
k \frac{X''}{X} - h &= \frac{T'}{T} \\
k \frac{X''}{X} - h &= -\lambda \\
X'' + \frac{\lambda - h}{k} X &= 0 \\
X &= c_1 \cos \omega x + c_2 \sin \omega x \\
X' &= -\omega c_1 \sin \omega x + \omega c_2 \cos \omega x \\
0 &= X'(0) \\
&= \omega c_2 \\
c_2 &= 0 \\
0 &= X'(L) \\
&= -\omega c_1 \sin \omega L \\
\omega L &= n\pi \\
\omega &= \frac{n\pi}{L} \\
X_n &= c_1 \cos \frac{n\pi}{L} x \\
T_n &= c_3 e^{-\lambda t} \\
&= c_3 e^{-(h + kn^2 \pi^2 / L^2)t} \\
u_n &= X_n T_n \\
&= A_n e^{-(h + kn^2 \pi^2 / L^2)t} \cos \frac{n\pi}{L} x \\
&= e^{-ht} A_n e^{-(kn^2 \pi^2 / L^2)t} \cos \frac{n\pi}{L} x \\
A_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x \, dx \\
u &= e^{-ht} \left[\frac{1}{L} \int_0^L f(x) \, dx + \frac{2}{L} \sum_{n=1}^{\infty} \left(\int_0^L f(x) \cos \frac{n\pi}{L} x \, dx \right) e^{-(kn^2 \pi^2 / L^2)t} \cos \frac{n\pi}{L} x \right]
\end{aligned}$$

13.4 Wave Equation

13.4.1

$$\begin{aligned}A_n &= \frac{2}{L} \int_0^L \frac{1}{4} x(L-x) \sin \frac{n\pi}{L} x \, dx \\&= -\frac{[-1 + (-1)^n] L^2}{n^3 \pi^3} \\u(x, t) &= \sum_{n=1}^{\infty} -\frac{[-1 + (-1)^n] L^2}{n^3 \pi^3} \cos \frac{n\pi a}{L} t \sin \frac{n\pi}{L} x\end{aligned}$$

13.4.3

$$\begin{aligned}a^2 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t^2} \\u(0, t) &= 0 \\u(\pi, t) &= 0 \\u(x, 0) &= 0 \\\left. \frac{\partial u}{\partial t} \right|_{t=0} &= \sin x \\X(x) &= c_1 \cos \alpha x + c_2 \sin \alpha x \\0 &= X(0) \\&= c_1 \\0 &= X(\pi) \\&= c_2 \sin \alpha \pi \\\alpha &= n \\X(x) &= c_2 \sin nx \\T(t) &= c_3 \cos ant + c_4 \sin ant \\u_n &= (A_n \cos ant + B_n \sin ant) \sin nx \\A_n &= 0 \\u_n &= B_n \sin ant \sin nx \\\sin x &= a \sum_{n=1}^{\infty} n B_n \sin nx \\B_1 &= \frac{1}{a} \\B_n &= 0 \\u &= \frac{1}{a} \sin at \sin x\end{aligned}$$

13.4.5

$$\begin{aligned}
L &= 1 \\
f(x) &= x(1-x) \\
g(x) &= x(1-x) \\
A_n &= 2 \int_0^1 x(1-x) \sin n\pi x \, dx \\
&= \frac{4[1 - (-1)^n]}{n^3\pi^3} \\
B_n &= \frac{2}{n\pi a} \int_0^1 x(1-x) \sin n\pi x \, dx \\
&= \frac{4[1 - (-1)^n]}{an^4\pi^4} \\
u(x, t) &= \frac{4}{\pi^3} \sum_{n=1}^{\infty} \left(\frac{1 - (-1)^n}{n^3} \cos n\pi at + \frac{1 - (-1)^n}{an^4\pi} \sin n\pi at \right) \sin n\pi x
\end{aligned}$$

13.4.7

$$\begin{aligned}
f(x) &= \begin{cases} \frac{2h}{L}x & 0 < x < L/2 \\ 2h - \frac{2h}{L}x & L/2 < x < L \end{cases} \\
A_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L}x \, dx \\
&= \frac{2}{L} \left[\int_0^{L/2} \frac{2h}{L}x \sin \frac{n\pi}{L}x \, dx + \int_{L/2}^L \left(2h - \frac{2h}{L}x \right) \sin \frac{n\pi}{L}x \, dx \right] \\
&= \frac{4h}{L} \left[\frac{1}{L} \int_0^{L/2} x \sin \frac{n\pi}{L}x \, dx + \int_{L/2}^L \left(1 - \frac{1}{L}x \right) \sin \frac{n\pi}{L}x \, dx \right] \\
&= \frac{8h \sin \frac{n\pi}{2}}{n^2\pi^2} \\
u(x, t) &= \frac{8h}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \cos \frac{n\pi a}{L}t \sin \frac{n\pi}{L}x
\end{aligned}$$

13.4.11

$$\begin{aligned}
X &= c_1 \cos \omega x + c_2 \sin \omega x \\
X' &= -\omega c_1 \sin \omega x + \omega c_2 \cos \omega x \\
0 &= X'(0) \\
&= \omega c_2 \\
&= c_2 \\
0 &= X'(L) \\
&= -\omega c_1 \sin \omega L \\
\omega &= \frac{n\pi}{L} \\
X &= c_1 \cos \frac{n\pi}{L} x \\
T &= c_3 \cos \frac{n\pi a}{L} t + c_4 \sin \frac{n\pi a}{L} t \\
T' &= \frac{n\pi a}{L} \left(-c_3 \sin \frac{n\pi a}{L} t + c_4 \cos \frac{n\pi a}{L} t \right) \\
0 &= T'(0) \\
&= \frac{n\pi a}{L} c_4 \\
&= c_4 \\
u_n &= B_n \cos \frac{n\pi a}{L} t \cos \frac{n\pi}{L} x \\
f(x) &= u(x, 0) \\
x &= \sum_{n=1}^{\infty} B_n \cos \frac{n\pi}{L} x \\
B_0 &= \frac{2}{L} \int_0^L x \, dx \\
&= L \\
B_n &= \frac{2}{L} \int_0^L x \cos \frac{n\pi}{L} x \, dx \\
&= \frac{2L[-1 + (-1)^n]}{n^2 \pi^2} \\
u &= \frac{L}{2} + \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{-1 + (-1)^n}{n^2} \cos \frac{n\pi a}{L} t \cos \frac{n\pi}{L} x
\end{aligned}$$

13.4.15

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t^2} + 2\beta \frac{\partial u}{\partial t} \\
X''T &= XT'' + 2\beta XT' \\
\frac{X''}{X} &= \frac{T''}{T'} + 2\beta \frac{T'}{T} \\
\frac{X''}{X} &= -\lambda \\
X'' + \lambda X &= 0 \\
X(x) &= c_1 \cos \omega x + c_2 \sin \omega x \\
0 &= X(0) \\
&= c_1 \\
0 &= X(\pi) \\
&= c_2 \sin \omega \pi \\
\omega &= n \\
X &= c_2 \sin nx \\
\frac{T''}{T} + 2\beta \frac{T'}{T} &= -n^2 \\
T'' + 2\beta T' + n^2 T &= 0 \\
m^2 + 2\beta m + n^2 &= 0 \\
m &= \frac{-2\beta \pm \sqrt{4\beta^2 - 4n^2}}{2} \\
&= -\beta \pm i\sqrt{n^2 - \beta^2} \\
T &= e^{-\beta t} (c_1 \cos \sqrt{n^2 - \beta^2} t + c_2 \sin \sqrt{n^2 - \beta^2} t) \\
u_n &= (A_n \cos \sqrt{n^2 - \beta^2} t + B_n \sin \sqrt{n^2 - \beta^2} t) e^{-\beta t} \sin nx \\
f(x) &= \sum_{n=1}^{\infty} A_n \sin nx \\
A_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \\
\frac{\partial u_n}{\partial t} &= (-\sqrt{n^2 - \beta^2} A_n \sin \sqrt{n^2 - \beta^2} t + \sqrt{n^2 - \beta^2} B_n \cos \sqrt{n^2 - \beta^2} t) e^{-\beta t} \sin nx \\
&\quad - \beta (A_n \cos \sqrt{n^2 - \beta^2} t + B_n \sin \sqrt{n^2 - \beta^2} t) e^{-\beta t} \sin nx \\
0 &= \left. \frac{\partial u_n}{\partial t} \right|_{t=0} \\
&= \sqrt{n^2 - \beta^2} B_n \sin nx - \beta A_n \sin nx \\
B_n &= \frac{\beta A_n}{\sqrt{n^2 - \beta^2}}
\end{aligned}$$

$$u(x, t) = \frac{2}{\pi} e^{-\beta t} \sum_{n=1}^{\infty} \left(\int_0^{\pi} f(x) \sin nx \, dx \right) \left(\cos \sqrt{n^2 - \beta^2} t + \frac{\beta}{\sqrt{n^2 - \beta^2}} \sin \sqrt{n^2 - \beta^2} t \right) \sin nx$$

13.5 Laplace's Equation

13.5.1

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{a} y \sin \frac{n\pi}{a} x$$

$$A_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx$$

13.5.5

$$Y'' - \lambda Y = 0$$

$$Y'(0) = 0$$

$$Y'(1) = 0$$

$$Y = c_1 \cos \omega y + c_2 \sin \omega y$$

$$Y' = -\omega c_1 \sin \omega y + \omega c_2 \cos \omega y$$

$$0 = \omega c_2$$

$$= c_2$$

$$0 = -\omega c_1 \sin \omega$$

$$\omega = n\pi$$

$$Y = c_1 \cos n\pi y$$

$$Y = c_1 + c_2 y$$

$$Y' = c_2$$

$$Y = c_1$$

$$X'' + \lambda X = 0$$

$$X = c_3 \sinh n\pi x + c_4 \cosh n\pi x$$

$$0 = X(0)$$

$$= c_4$$

$$X = c_3 \sinh n\pi x$$

$$X = c_3 + c_4 x$$

$$0 = X(0)$$

$$= c_3$$

$$X = c_4 x$$

$$u = A_0 x$$

$$u_n = A_n \sinh n\pi x \cos n\pi y$$

$$u = A_0 x + \sum_{n=1}^{\infty} A_n \sinh n\pi x \cos n\pi y$$

$$f(y) = u(1, y)$$

$$1 - y = A_0 + \sum_{n=1}^{\infty} A_n \sinh n\pi \cos n\pi y$$

$$A_0 = \frac{a_0}{2}$$

$$= \int_0^1 (1 - y) dy$$

$$= \frac{1}{2}$$

$$\begin{aligned}
A_n \sinh n\pi &= 2 \int_0^1 (1-y) \cos n\pi y \, dy \\
A_n &= \frac{2[1 - (-1)^n]}{n^2 \pi^2 \sinh n\pi} \\
u &= \frac{1}{2}x + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2 \sinh n\pi} \sinh n\pi x \cos n\pi y
\end{aligned}$$

13.5.7

$$Y'' - \lambda Y = 0$$

$$Y(0) = 0$$

$$Y(\pi) = 0$$

$$Y = c_1 \cos \omega y + c_2 \sin \omega y$$

$$0 = Y(0)$$

$$= c_1$$

$$0 = Y(\pi)$$

$$= c_2 \sin \omega \pi$$

$$\omega = n$$

$$Y = c_2 \sin ny$$

$$X'' + \lambda X = 0$$

$$X = c_3 \sinh nx + c_4 \cosh nx$$

$$X' = nc_3 \cosh nx + nc_4 \sinh nx$$

$$X'(0)Y(y) = X(0)Y(y)$$

$$X'(0) = X(0)$$

$$nc_3 = c_4$$

$$X = c_3(\sinh nx + n \cosh nx)$$

$$u = \sum_{n=1}^{\infty} A_n(\sinh nx + n \cosh nx) \sin ny$$

$$1 = u(\pi, y)$$

$$= \sum_{n=1}^{\infty} A_n(\sinh n\pi + n \cosh n\pi) \sin ny$$

$$A_n(\sinh n\pi + n \cosh n\pi) = \frac{2}{\pi} \int_0^{\pi} \sin ny \, dx$$

$$A_n = \frac{2[1 - (-1)^n]}{n\pi(\sinh n\pi + n \cosh n\pi)}$$

$$u = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \frac{\sinh nx + n \cosh nx}{\sinh n\pi + n \cosh n\pi} \sin ny$$

13.5.11

$$X'' + \lambda X = 0$$

$$X = c_1 \cos \omega x + c_2 \sin \omega y$$

$$0 = X(0)$$

$$= c_1$$

$$0 = X(\pi)$$

$$= c_2 \sin \omega \pi$$

$$\omega = n$$

$$X = c_2 \sin nx$$

$$Y'' - \lambda Y = 0$$

$$Y = c_3 e^{ny} + c_4 e^{-ny}$$

$$0 = \lim_{y \rightarrow \infty} Y$$

$$= \lim_{y \rightarrow \infty} (c_3 e^{ny} + c_4 e^{-ny})$$

$$0 = c_3$$

$$Y = c_4 e^{-ny}$$

$$u = \sum_{n=1}^{\infty} A_n e^{-ny} \sin nx$$

$$f(x) = u(x, 0)$$

$$= \sum_{n=1}^{\infty} A_n \sin nx$$

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$u = \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\int_0^{\pi} f(x) \sin nx \, dx \right) e^{-ny} \sin nx$$

13.5.13

$$X'' + \lambda X = 0$$

$$X = c_1 \sin \frac{n\pi}{a} x$$

$$Y'' - \lambda Y = 0$$

$$Y = c_2 \sinh \frac{n\pi}{a} y + c_3 \cosh \frac{n\pi}{a} y$$

$$u = \sum_{n=1}^{\infty} \left(A_n \sinh \frac{n\pi}{a} y + B_n \cosh \frac{n\pi}{a} y \right) \sin \frac{n\pi}{a} x$$

$$f(x) = u(x, 0)$$

$$= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{a} x$$

$$B_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx$$

$$g(x) = u(x, b)$$

$$= \sum_{n=1}^{\infty} \left(A_n \sinh \frac{n\pi b}{a} + B_n \cosh \frac{n\pi b}{a} \right) \sin \frac{n\pi}{a} x$$

$$A_n \sinh \frac{n\pi b}{a} + B_n \cosh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x \, dx$$

$$A_n = \frac{1}{\sinh \frac{n\pi b}{a}} \left(\frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x \, dx - B_n \cosh \frac{n\pi b}{a} \right)$$

13.6 Nonhomogeneous Boundary-Value Problems

13.6.1

$$\begin{aligned}ku_{xx} &= u_t \\u(0, t) &= 100 \\u(1, t) &= 100 \\u(x, 0) &= 0 \\\psi &= 100 \\kv_{xx} &= v_t \\v(0, t) &= 0 \\v(1, t) &= 0 \\v(x, 0) &= -100 \\kX''T &= XT' \\\frac{X''}{X} &= \frac{T'}{kT} \\X &= c_1 \sin n\pi x, \quad n = 1, 2, 3, \dots \\T' + k\lambda T &= 0 \\T &= c_2 e^{-kn^2\pi^2 t} \\v(x, t) &= \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x \\-100 &= v(x, 0) \\&= \sum_{n=1}^{\infty} A_n \sin n\pi x \\A_n &= -200 \int_0^1 \sin n\pi x \, dx \\&= \frac{200[-1 + (-1)^n]}{n\pi} \\v(x, t) &= \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{-1 + (-1)^n}{n} e^{-kn^2\pi^2 t} \sin n\pi x \\u(x, t) &= 100 + \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{-1 + (-1)^n}{n} e^{-kn^2\pi^2 t} \sin n\pi x\end{aligned}$$

13.6.3

$$ku_{xx} + r = u_t$$

$$u(0, t) = u_0$$

$$u(1, t) = u_0$$

$$u(x, 0) = 0$$

$$u(x, t) = v(x, t) + \psi(x)$$

$$k\psi'' + r = 0$$

$$\psi'' = -\frac{r}{k}$$

$$\psi = -\frac{r}{2k}x^2 + c_1x + c_2$$

$$\psi(0) = u_0$$

$$c_2 = u_0$$

$$u_0 = \psi(1)$$

$$= -\frac{r}{2k} + c_1 + u_0$$

$$c_1 = \frac{r}{2k}$$

$$\psi(x) = \frac{r}{2k}x(1-x) + u_0$$

$$kv_{xx} = v_t$$

$$v(0, t) = 0$$

$$v(L, t) = 0$$

$$v(x, 0) = \frac{r}{2k}x(x-1) - u_0$$

$$X = c_1 \sin n\pi x, \quad n = 1, 2, 3, \dots$$

$$T = c_2 e^{-kn^2\pi^2 t}$$

$$v = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x$$

$$A_n = 2 \int_0^1 \left[\frac{r}{2k}x(x-1) - u_0 \right] \sin n\pi x \, dx$$

$$= \frac{2[-1 + (-1)^n](r + kn^2\pi^2 u_0)}{kn^3\pi^3}$$

$$v = \frac{2}{k\pi^3} \sum_{n=1}^{\infty} \frac{[-1 + (-1)^n](r + kn^2\pi^2 u_0)}{n^3} \sin n\pi x$$

$$u = \frac{r}{2k}x(1-x) + u_0 + \frac{2}{k\pi^3} \sum_{n=1}^{\infty} \frac{[-1 + (-1)^n](r + kn^2\pi^2 u_0)}{n^3} e^{-kn^2\pi^2 t} \sin n\pi x$$

13.6.7

$$\begin{aligned}
ku_{xx} - h(u - u_0) &= u_t \\
u(0, t) &= u_0 \\
u(1, t) &= 0 \\
u(x, 0) &= f(x) \\
u(x, t) &= v(x, t) + \psi(x) \\
u_{xx} &= v_{xx} + \psi'' \\
u_t &= v_t \\
k(v_{xx} + \psi'') - h[(v + \psi) - u_0] &= v_t \\
kv_{xx} + k\psi'' - hv - h\psi + hu_0 &= v_t \\
\psi'' - \frac{h}{k}\psi &= -\frac{h}{k}u_0 \\
\psi_c &= c_1 \sinh \sqrt{\frac{h}{k}}x + c_2 \cosh \sqrt{\frac{h}{k}}x \\
\psi_p &= c_3 \\
\psi_p'' &= 0 \\
-\frac{h}{k}c_3 &= -\frac{h}{k}u_0 \\
c_3 &= u_0 \\
\psi &= \psi_c + \psi_p \\
&= u_0 + c_1 \sinh \sqrt{\frac{h}{k}}x + c_2 \cosh \sqrt{\frac{h}{k}}x \\
\psi(0) &= u_0 \\
c_2 &= 0 \\
0 &= \psi(1) \\
&= u_0 + c_1 \sinh \sqrt{\frac{h}{k}} \\
c_1 &= -\frac{u_0}{\sinh \sqrt{h/k}} \\
\psi &= u_0 \left(1 - \frac{\sinh \sqrt{h/k}x}{\sinh \sqrt{h/k}} \right)
\end{aligned}$$

13.6.13

$$\begin{aligned}
u_{xx} &= u_t \\
u(0, t) &= \sin t \\
u(1, t) &= 0 \\
u(x, 0) &= 0 \\
u(x, t) &= v(x, t) + (1 - x) \sin t \\
v_{xx} + (x - 1) \cos t &= v_t \\
(x - 1) \cos t &= \sum_{n=1}^{\infty} A_n \sin n\pi x \\
A_n &= 2 \cos t \int_0^1 (x - 1) \sin n\pi x \, dx \\
&= -\frac{2 \cos t}{n\pi} \\
(x - 1) \cos t &= -\frac{2 \cos t}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi x \\
v &= \sum_{n=1}^{\infty} v_n(t) \sin n\pi x \\
v_{xx} &= \sum_{n=1}^{\infty} -n^2 \pi^2 v_n(t) \sin n\pi x \\
v_t &= \sum_{n=1}^{\infty} v'_n(t) \sin n\pi x \\
\sum_{n=1}^{\infty} [-n^2 \pi^2 v_n(t) - v'_n(t)] \sin n\pi x &= \frac{2 \cos t}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi x \\
v'_n(t) + n^2 \pi^2 v_n(t) &= -\frac{2 \cos t}{\pi n} \\
\frac{d}{dt} [e^{n^2 \pi^2 t} v_n(t)] &= -\frac{2 e^{n^2 \pi^2 t} \cos t}{\pi n} \\
v_n(t) &= c_n e^{-n^2 \pi^2 t} - \frac{2 [n^2 \pi^2 \cos t + \sin t]}{n\pi + n^5 \pi^5} \\
0 &= \sum_{n=1}^{\infty} \left(c_n - \frac{2 n^2 \pi^2}{n\pi + n^5 \pi^5} \right) \sin n\pi x \\
c_n &= \frac{2 n^2 \pi^2}{n\pi + n^5 \pi^5} \\
u(x, t) &= (1 - x) \sin t + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n^2 \pi^2 e^{-n^2 \pi^2 t} - n^2 \pi^2 \cos t - \sin t}{n(1 + n^4 \pi^4)} \sin n\pi x
\end{aligned}$$

13.6.17

$$u_{xx} + xe^{-3t} = u_t$$

$$u(0, t) = 0$$

$$u(\pi, t) = 0$$

$$u(x, 0) = 0$$

$$xe^{-3t} = \sum_{n=1}^{\infty} \left(\frac{2}{\pi} \int_0^{\pi} xe^{-3t} \sin nx \, dx \right) \sin nx$$

$$= -2e^{-3t} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin nx$$

$$u_{xx} = \sum_{n=1}^{\infty} -n^2 u_n(t) \sin nx$$

$$u_t = \sum_{n=1}^{\infty} u'_n(t) \sin nx$$

$$\sum_{n=1}^{\infty} [-n^2 u_n(t) - u'_n(t)] \sin nx = 2e^{-3t} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$

$$-n^2 u_n - u'_n = 2e^{-3t} \frac{(-1)^n}{n}$$

$$u'_n + n^2 u_n = -2e^{-3t} \frac{(-1)^n}{n}$$

$$\frac{d}{dt} [e^{n^2 t} u_n] = -2e^{(n^2-3)t} \frac{(-1)^n}{n}$$

$$u_n = c_n e^{-n^2 t} - 2 \frac{(-1)^n}{n(n^2-3)} e^{-3t}$$

$$0 = u(x, 0)$$

$$= \sum_{n=1}^{\infty} \left[c_n - 2 \frac{(-1)^n}{n(n^2-3)} \right] \sin nx$$

$$c_n = 2 \frac{(-1)^n}{n(n^2-3)}$$

$$u(x, t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n^2-3)} (e^{-n^2 t} - e^{-3t}) \sin nx$$

13.7 Orthogonal Series Expansion

13.7.1

$$\begin{aligned}ku_{xx} &= u_t \\ u_x|_{x=0} &= 0 \\ u_x|_{x=1} &= -hu(1, t) \\ u(x, 0) &= 1 \\ u(x, t) &= X(x)T(t) \\ kX''(x)T(t) &= X(x)T'(t) \\ \frac{X''}{X} &= \frac{T'}{kT} \\ X'' + \lambda X &= 0 \\ X'(0) &= 0 \\ X'(1) &= -hX(1) \\ X &= c_1 \cos \omega x + c_2 \sin \omega x \\ X' &= -\omega c_1 \sin \omega x + \omega c_2 \cos \omega x \\ 0 &= X'(0) \\ &= \omega c_2 \\ &= c_2 \\ X'(1) &= -hX(1) \\ -\omega c_1 \sin \omega &= -hc_1 \cos \omega \\ \tan \omega &= \frac{h}{\omega} \\ X_n &= c_1 \cos \omega_n x \\ T' + k\lambda T &= 0 \\ T_n &= c_3 e^{-k\omega_n^2 t} \\ u &= \sum_{n=1}^{\infty} A_n e^{-k\omega_n^2 t} \cos \omega_n x \\ 1 &= u(x, 0) \\ &= \sum_{n=1}^{\infty} A_n \cos \omega_n x \\ A_n &= \frac{\int_0^1 \cos \omega_n x \, dx}{\int_0^1 \cos^2 \omega_n x \, dx} \\ &= \frac{2 \sin \omega_n}{\omega_n + \cos \omega_n \sin \omega_n} \\ &= \frac{2h \sin \omega_n}{\omega_n (h + \sin^2 \omega_n)}\end{aligned}$$

$$u(x, t) = 2h \sum_{n=1}^{\infty} \frac{\sin \omega_n}{\omega_n (h + \sin^2 \omega_n)} e^{-k\omega_n^2 t} \cos \omega_n x$$

13.7.3

$$u_{xx} + u_{yy} = 0$$

$$u(0, y) = 0$$

$$u_x|_{x=a} = -hu(a, y)$$

$$u(x, 0) = 0$$

$$u(x, b) = f(x)$$

$$u(x, y) = X(x)Y(y)$$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

$$\frac{X''}{X} = -\lambda$$

$$X'' + \lambda X = 0$$

$$X(0) = 0$$

$$X'(a) = -hX(a)$$

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$X' = -\alpha c_1 \sin \alpha x + \alpha c_2 \cos \alpha x$$

$$0 = X(0)$$

$$= c_1$$

$$X'(a) = -hX(a)$$

$$\alpha c_2 \cos \alpha a = -hc_2 \sin \alpha a$$

$$\tan \alpha a = -\frac{\alpha}{h}$$

$$X = c_2 \sin \alpha_n x$$

$$Y'' - \lambda Y = 0$$

$$Y(0) = 0$$

$$Y = c_3 \sinh \alpha_n y + c_4 \cosh \alpha_n y$$

$$0 = Y(0)$$

$$= c_4$$

$$Y = c_3 \sinh \alpha_n y$$

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sinh \alpha_n y \sin \alpha_n x$$

$$f(x) = u(x, b)$$

$$= \sum_{n=1}^{\infty} A_n \sinh \alpha_n b \sin \alpha_n x$$

$$\begin{aligned}
A_n \sinh \alpha_n b &= \frac{\int_0^a f(x) \sin \alpha_n x \, dx}{\int_0^a \sin^2 \alpha_n x \, dx} \\
A_n &= \frac{\int_0^a f(x) \sin \alpha_n x \, dx}{\left(\frac{a}{2} - \frac{\sin 2a\alpha_n}{4\alpha_n}\right) \sinh \alpha_n b} \\
&= \frac{2h \int_0^a f(x) \sin \alpha_n x \, dx}{(ah + \cos^2 \alpha_n a) \sinh \alpha_n b} \\
u(x, y) &= \sum_{n=1}^{\infty} \frac{2h \int_0^a f(x) \sin \alpha_n x \, dx}{(ah + \cos^2 \alpha_n a) \sinh \alpha_n b} \sinh \alpha_n y \sin \alpha_n x
\end{aligned}$$

13.7.5

$$\begin{aligned}
ku_{xx} &= u_t \\
u(x, 0) &= f(x) \\
u(0, t) &= 0 \\
u_x|_{x=L} &= 0 \\
\frac{X''}{X} &= \frac{T'}{kT} \\
X'' + \lambda X &= 0 \\
X(0) &= 0 \\
X'(L) &= 0 \\
X &= c_1 \cos \omega x + c_2 \sin \omega x \\
X' &= -\omega c_1 \sin \omega x + \omega c_2 \cos \omega x \\
0 &= X(0) \\
&= c_1 \\
0 &= X'(L) \\
&= \omega c_2 \cos \omega L \\
\omega L &= \frac{(2n-1)\pi}{2} \\
\omega &= \frac{(2n-1)\pi}{2L}, \quad n = 1, 2, 3, \dots \\
X &= c_2 \sin \frac{(2n-1)\pi}{2L} x \\
T' + k\lambda T &= 0 \\
T &= c_3 e^{-k\lambda t} \\
u(x, t) &= \sum_{n=1}^{\infty} A_n e^{-k(2n-1)^2 \pi^2 t / 4L^2} \sin \frac{(2n-1)\pi}{2L} x \\
f(x) &= u(x, 0) \\
&= \sum_{n=1}^{\infty} A_n \sin \frac{(2n-1)\pi}{2L} x \\
A_n &= \frac{\int_0^L f(x) \sin \frac{(2n-1)\pi}{2L} x \, dx}{\int_0^L \sin^2 \frac{(2n-1)\pi}{2L} x \, dx} \\
&= \frac{2}{L} \int_0^L f(x) \sin \frac{(2n-1)\pi}{2L} x \, dx \\
u(x, t) &= \frac{2}{L} \sum_{n=1}^{\infty} \left(\int_0^L f(x) \sin \frac{(2n-1)\pi}{2L} x \, dx \right) e^{-k(2n-1)^2 \pi^2 t / 4L^2} \sin \frac{(2n-1)\pi}{2L} x
\end{aligned}$$

13.7.7

$$u_{xx} + u_{yy} = 0$$

$$u_x|_{x=0} = 0$$

$$u(1, y) = u_0$$

$$u(x, 0) = 0$$

$$u_y|_{y=1} = 0$$

$$X''Y + XY'' = 0$$

$$\frac{Y''}{Y} = -\frac{X''}{X}$$

$$Y'' + \lambda Y = 0$$

$$Y(0) = 0$$

$$Y'(1) = 0$$

$$Y = c_1 \cos \omega y + c_2 \sin \omega y$$

$$Y' = -\omega c_1 \sin \omega y + \omega c_2 \cos \omega y$$

$$0 = Y(0)$$

$$= c_1$$

$$0 = Y'(1)$$

$$= \omega c_2 \cos \omega$$

$$\omega = \frac{2n-1}{2}\pi, \quad n = 1, 2, 3, \dots$$

$$Y = c_2 \sin \frac{2n-1}{2}\pi y$$

$$X'' - \lambda X = 0$$

$$X'(0) = 0$$

$$X = c_1 \cosh \frac{2n-1}{2}\pi x + c_2 \sinh \frac{2n-1}{2}\pi x$$

$$X' = \frac{2n-1}{2}\pi \left(c_1 \sinh \frac{2n-1}{2}\pi x + c_2 \cosh \frac{2n-1}{2}\pi x \right)$$

$$0 = X'(0)$$

$$= \frac{2n-1}{2}\pi c_2$$

$$= c_2$$

$$X = c_1 \cosh \frac{2n-1}{2}\pi x$$

$$u(x, y) = \sum_{n=1}^{\infty} A_n \cosh \left(\frac{2n-1}{2}\pi x \right) \sin \left(\frac{2n-1}{2}\pi y \right)$$

$$\begin{aligned}
u_0 &= u(1, y) \\
&= \sum_{n=1}^{\infty} A_n \cosh\left(\frac{2n-1}{2}\pi\right) \sin\left(\frac{2n-1}{2}\pi y\right) \\
A_n \cosh\left(\frac{2n-1}{2}\pi\right) &= \frac{\int_0^1 u_0 \sin\left(\frac{2n-1}{2}\pi y\right) dy}{\int_0^1 \sin^2\left(\frac{2n-1}{2}\pi y\right) dy} \\
&= \frac{4u_0}{(2n-1)\pi} \\
A_n &= \frac{4u_0}{(2n-1)\pi \cosh\left(\frac{2n-1}{2}\pi\right)} \\
u(x, y) &= \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1) \cosh\left(\frac{2n-1}{2}\pi\right)} \cosh\left(\frac{2n-1}{2}\pi x\right) \sin\left(\frac{2n-1}{2}\pi y\right)
\end{aligned}$$

13.8 Fourier Series in Two Variables

13.8.1

$$k(u_{xx} + u_{yy}) = u_t$$

$$u(0, y, t) = 0$$

$$u(\pi, y, t) = 0$$

$$u(x, 0, t) = 0$$

$$u(x, \pi, t) = 0$$

$$u(x, y, 0) = u_0$$

$$X'' + \lambda X = 0$$

$$X(0) = 0$$

$$X(\pi) = 0$$

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$0 = X(0)$$

$$= c_1$$

$$0 = X(\pi)$$

$$= c_2 \sin \alpha \pi$$

$$X = c_2 \sin mx, \quad m = 1, 2, 3, \dots$$

$$Y'' + \mu Y = 0$$

$$Y(0) = 0$$

$$Y(\pi) = 0$$

$$Y = c_3 \cos \beta y + c_4 \sin \beta y$$

$$0 = Y(0)$$

$$= c_3$$

$$0 = Y(\pi)$$

$$= c_4 \sin \beta \pi$$

$$Y = c_4 \sin ny, \quad n = 1, 2, 3, \dots$$

$$T' + k(\lambda + \mu)T = 0$$

$$T = c_5 e^{-k(m^2 + n^2)t}$$

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{-k(m^2 + n^2)t} \sin mx \sin ny$$

$$u_0 = u(x, y, 0)$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin mx \sin ny$$

$$\begin{aligned}
A_{mn} &= \frac{4u_0}{\pi^2} \int_0^\pi \int_0^\pi \sin mx \sin ny \, dx \, dy \\
&= \frac{4u_0[1 - (-1)^m][1 - (-1)^n]}{\pi^2 mn} \\
u(x, y, t) &= \frac{4u_0}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{[1 - (-1)^m][1 - (-1)^n]}{mn} e^{-k(m^2+n^2)t} \sin mx \sin ny
\end{aligned}$$

13.8.3

$$a^2(u_{xx} + u_{yy}) = u_{tt}$$

$$u(0, y, t) = 0$$

$$u(\pi, y, t) = 0$$

$$u(x, 0, t) = 0$$

$$u(x, \pi, t) = 0$$

$$u(x, y, 0) = xy(x - \pi)(y - \pi)$$

$$u_t|_{t=0} = 0$$

$$a^2(X''YT + XY''T) = XYT''$$

$$a^2\left(\frac{X''}{X} + \frac{Y''}{Y}\right) = \frac{T''}{T}$$

$$\frac{X''}{X} = -\frac{Y''}{Y} + \frac{T''}{a^2T}$$

$$X'' + \lambda X = 0$$

$$X(0) = 0$$

$$X(\pi) = 0$$

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$0 = X(0)$$

$$= c_1$$

$$0 = X(\pi)$$

$$= c_2 \sin \alpha \pi$$

$$X = c_2 \sin mx, \quad m = 1, 2, 3, \dots$$

$$Y'' + \mu Y = 0$$

$$Y(0) = 0$$

$$Y(\pi) = 0$$

$$Y = c_3 \cos \beta y + c_4 \sin \beta y$$

$$0 = Y(0)$$

$$= c_3$$

$$0 = Y(\pi)$$

$$= c_4 \sin \beta \pi$$

$$Y = c_4 \sin ny, \quad n = 1, 2, 3, \dots$$

$$T'' + a^2(\lambda + \mu)T = 0$$

$$T'(0) = 0$$

$$T = c_5 \cos a\sqrt{m^2 + n^2}t + c_6 \sin a\sqrt{m^2 + n^2}t$$

$$T' = a\sqrt{m^2 + n^2}(-c_5 \sin a\sqrt{m^2 + n^2}t + c_6 \cos a\sqrt{m^2 + n^2}t)$$

$$0 = T'(0)$$

$$= c_6$$

$$T = c_5 \cos a\sqrt{m^2 + n^2}t$$

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin mx \sin ny \cos a\sqrt{m^2 + n^2}t$$

$$xy(x - \pi)(y - \pi) = u(x, y, 0)$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin mx \sin ny$$

$$A_{mn} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} xy(x - \pi)(y - \pi) \sin mx \sin ny \, dy \, dx$$

$$= \frac{16[1 - (-1)^m][1 - (-1)^n]}{m^3 n^3 \pi^2}$$

$$u(x, y, t) = \frac{16}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{[1 - (-1)^m][1 - (-1)^n]}{m^3 n^3} \sin mx \sin ny \cos a\sqrt{m^2 + n^2}t$$

13.9 Chapter in Review

13.9.1

$$u_{xy} = u$$

$$X'Y' = XY$$

$$\frac{X'}{X} = \frac{Y'}{Y}$$

$$X' + \lambda X = 0$$

$$X = c_1 e^{-\lambda x}$$

$$Y' + \frac{1}{\lambda} Y = 0$$

$$Y = c_2 e^{-y/\lambda}$$

$$u = c_3 e^{-\lambda x - y/\lambda}$$

13.9.3

$$ku_{xx} = u_t$$

$$u(0, t) = u_0$$

$$-u_x|_{x=\pi} = u(\pi, t) - u_1$$

$$u(x, 0) = 0$$

$$u(x, t) = v(x, t) + \psi(x)$$

$$u_{xx} = v_{xx} + \psi''$$

$$u_t = v_t$$

$$k(v_{xx} + \psi'') = v_t$$

$$kv_{xx} = v_t$$

$$k\psi'' = 0$$

$$\psi(0) = u_0$$

$$-\psi'(\pi) = \psi(\pi) - u_1$$

$$\psi = c_1x + c_2$$

$$\psi' = c_1$$

$$u_0 = \psi(0)$$

$$= c_2$$

$$-c_1 = c_1\pi + u_0 - u_1$$

$$c_1 = \frac{u_1 - u_0}{\pi + 1}$$

$$\psi = \frac{u_1 - u_0}{\pi + 1}x + u_0$$

13.9.7

$$u_{xx} + u_{yy} = 0$$

$$u(0, y) = 0$$

$$u(\pi, y) = 50$$

$$u(x, 0) = 0$$

$$u(x, \pi) = 0$$

$$X''Y + XY'' = 0$$

$$-\frac{X''}{X} = \frac{Y''}{Y}$$

$$Y'' + \lambda Y = 0$$

$$Y = c_1 \cos \omega y + c_2 \sin \omega y$$

$$0 = Y(0)$$

$$= c_1$$

$$0 = Y(\pi)$$

$$= c_2 \sin \omega \pi$$

$$Y = c_2 \sin ny, \quad n = 1, 2, 3, \dots$$

$$X'' - n^2 X = 0$$

$$X = c_3 \cosh nx + c_4 \sinh nx$$

$$0 = X(0)$$

$$= c_3$$

$$X = c_4 \sinh nx$$

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sinh nx \sin ny$$

$$50 = u(\pi, y)$$

$$= \sum_{n=1}^{\infty} A_n \sinh n\pi \sin ny$$

$$A_n \sinh n\pi = \frac{100}{\pi} \int_0^{\pi} \sin ny \, dy$$

$$= \frac{100}{\pi} \frac{1 - (-1)^n}{n}$$

$$A_n = \frac{100}{\pi} \frac{1 - (-1)^n}{n \sinh n\pi}$$

$$u(x, y) = \frac{100}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n \sinh n\pi} \sinh nx \sin ny$$

14 Boundary-Value Problems in Other Coordinate Systems

14.1 Polar Coordinates

14.1.1

$$u(1, \theta) = \begin{cases} u_0 & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi \end{cases}$$

$$\Theta(\theta) = \Theta(\theta + 2\pi)$$

$$\Theta'' + \lambda\Theta = 0$$

$$\Theta = c_1$$

$$\Theta = c_1 \cos n\theta + c_2 \sin n\theta, \quad n = 1, 2, 3, \dots$$

$$r^2 R'' + rR' - \lambda R = 0$$

$$R'' + \frac{1}{r}R' = 0$$

$$R = c_3 + c_4 \ln r$$

$$r^2 R'' + rR' - n^2 R = 0$$

$$R = c_3 r^n + c_4 r^{-n}$$

$$u_0 = A_0, \quad n = 0$$

$$u_n = r^n (A_n \cos n\theta + B_n \sin n\theta), \quad n = 1, 2, 3, \dots$$

$$u = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$u(1, \theta) = A_0 + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)$$

$$A_0 = \frac{a_0}{2}$$

$$= \frac{1}{2\pi} \int_0^\pi u_0 d\theta$$

$$= \frac{u_0}{2}$$

$$A_n = \frac{1}{\pi} \int_0^\pi u_0 \cos n\theta d\theta$$

$$= 0$$

$$B_n = \frac{1}{\pi} \int_0^\pi u_0 \sin n\theta d\theta$$

$$= \frac{\mu_0}{\pi} \frac{1 - (-1)^n}{n}$$

$$u(r, \theta) = \frac{\mu_0}{2} + \frac{\mu_0}{\pi} \sum_{n=1}^{\infty} r^n \frac{1 - (-1)^n}{n} \sin n\theta$$

14.1.5

$$\left. \frac{\partial u}{\partial \theta} \right|_{\theta=0} = 0$$

$$\left. \frac{\partial u}{\partial \theta} \right|_{\theta=\pi} = 0$$

$$u(2, \theta) = \begin{cases} u_0 & 0 < \theta < \pi/2 \\ 0 & \pi/2 < \theta < \pi \end{cases}$$

$$\Theta'' + \lambda \Theta = 0$$

$$\Theta'(0) = 0$$

$$\Theta'(\pi) = 0$$

$$\Theta = c_1$$

$$\Theta = c_1 \cos \omega \theta + c_2 \sin \omega \theta$$

$$\Theta' = -\omega c_1 \sin \omega \theta + \omega c_2 \cos \omega \theta$$

$$0 = \Theta'(0)$$

$$= \omega c_2$$

$$= c_2$$

$$0 = \Theta'(\pi)$$

$$= -\omega c_1 \sin \omega \pi$$

$$\omega = n, \quad n = 1, 2, 3, \dots$$

$$\Theta = c_1, \quad n = 0$$

$$\Theta = c_1 \cos n\theta, \quad n = 1, 2, 3, \dots$$

$$r^2 R'' + rR' - \lambda R = 0$$

$$R = c_3, \quad n = 0$$

$$R = c_3 r^n, \quad n = 1, 2, 3, \dots$$

$$u = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\theta, \quad n = 1, 2, 3, \dots$$

$$A_0 = \frac{a_0}{2}$$

$$= \frac{1}{\pi} \int_0^{\pi/2} u_0 d\theta$$

$$= \frac{u_0}{2}$$

$$2^n A_n = \frac{2}{\pi} \int_0^{\pi/2} u_0 \cos n\theta d\theta$$

$$A_n = \frac{2u_0 \sin \frac{n\pi}{2}}{2^n \pi n}$$

$$u = \frac{u_0}{2} + \frac{2u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \left(\frac{r}{2}\right)^n \cos n\theta$$

14.1.9

$$u(r, 0) = 0$$

$$u(r, \beta) = 0$$

$$u(c, \theta) = f(\theta)$$

$$\Theta'' + \lambda\Theta = 0$$

$$\Theta(0) = 0$$

$$\Theta(\beta) = 0$$

$$\Theta = c_1 \cos \omega\theta + c_2 \sin \omega\theta$$

$$0 = \Theta(0)$$

$$= c_1$$

$$0 = \Theta(\beta)$$

$$= c_2 \sin \omega\beta$$

$$\omega\beta = n\pi$$

$$\omega = \frac{n\pi}{\beta}, \quad n = 1, 2, 3, \dots$$

$$\Theta = c_2 \sin \frac{n\pi}{\beta} \theta, \quad n = 1, 2, 3, \dots$$

$$R = c_3 r^{n\pi/\beta}, \quad n = 1, 2, 3, \dots$$

$$u = \sum_{n=1}^{\infty} A_n r^{n\pi/\beta} \sin \frac{n\pi}{\beta} \theta$$

$$A_n = \frac{2}{\beta c^{n\pi/\beta}} \int_0^{\beta} f(\theta) \sin \frac{n\pi}{\beta} \theta \, d\theta$$

$$u = \frac{2}{\beta} \sum_{n=1}^{\infty} \left(\frac{r}{c}\right)^{n\pi/\beta} \left(\int_0^{\beta} f(\theta) \sin \frac{n\pi}{\beta} \theta \, d\theta \right) \sin \frac{n\pi}{\beta} \theta$$

14.1.11

$$u(a, \theta) = f(\theta)$$

$$u(b, \theta) = 0$$

$$\Theta'' + \lambda\Theta' = 0$$

$$\Theta(\theta) = \Theta(\theta + 2\pi)$$

$$\Theta = c_1$$

$$\Theta = c_1 \cos \omega\theta + c_2 \sin \omega\theta$$

$$\omega = n, \quad n = 1, 2, 3, \dots$$

$$R = c_3 + c_4 \ln r$$

$$0 = R(b)$$

$$= c_3 + c_4 \ln b$$

$$c_3 = -c_4 \ln b$$

$$R = c_4 \ln \frac{r}{b}$$

$$R = c_3 r^n + c_4 r^{-n}$$

$$0 = R(b)$$

$$= c_3 b^n + c_4 b^{-n}$$

$$c_4 = -c_3 b^{2n}$$

$$R = c_3 (r^n - b^{2n} r^{-n})$$

$$u = A_0 \ln \frac{r}{b} + \sum_{n=1}^{\infty} (r^n - b^{2n} r^{-n}) (A_n \cos n\theta + B_n \sin n\theta)$$

$$u(a, \theta) = A_0 \ln \frac{a}{b} + \sum_{n=1}^{\infty} (a^n - b^{2n} a^{-n}) (A_n \cos n\theta + B_n \sin n\theta)$$

$$A_0 = \frac{1}{2\pi \ln a/b} \int_0^{2\pi} f(\theta) d\theta$$

$$A_n = \frac{1}{\pi(a^n - b^{2n} a^{-n})} \int_0^{2\pi} f(\theta) \cos n\theta d\theta$$

$$B_n = \frac{1}{\pi(a^n - b^{2n} a^{-n})} \int_0^{2\pi} f(\theta) \sin n\theta d\theta$$

14.1.19

$$\begin{aligned}u(c, \theta) &= f(\theta) \\u(\infty, \theta) &= 0 \\u(r, \theta) &= u(r, \theta + 2\pi) \\\Theta &= c_1 \\\Theta &= c_1 \cos n\theta + c_2 \sin n\theta \\R &= c_4 \ln r \\R &= c_4 r^{-n} \\u &= A_0 \ln r + \sum_{n=1}^{\infty} r^{-n} (A_n \cos n\theta + B_n \sin n\theta) \\u(c, \theta) &= A_0 \ln c + \sum_{n=1}^{\infty} c^{-n} (A_n \cos n\theta + B_n \sin n\theta) \\A_0 &= \frac{1}{2\pi \ln c} \int_0^{2\pi} f(\theta) d\theta \\A_n &= \frac{c^n}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta \\B_n &= \frac{c^n}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta\end{aligned}$$

14.2 Cylindrical Coordinates

14.2.1

$$\begin{aligned}A_n &= 0 \\B_n &= \frac{2}{a\alpha_n c^2 J_1^2(\alpha_n c)} \int_0^c r J_0(\alpha_n r) dr \\&= \frac{2}{a\alpha_n^2 c J_1(\alpha_n c)} \\u(r, t) &= \frac{2}{ac} \sum_{n=1}^{\infty} \frac{\sin a\alpha_n t}{\alpha_n^2 J_1(\alpha_n c)} J_0(\alpha_n r)\end{aligned}$$

14.2.3

$$u(r, z) = u_0 \sum_{n=1}^{\infty} \frac{\sinh[\alpha_n(4-z)]}{\alpha_n \sinh 4\alpha_n J_1(2\alpha_n)} J_0(\alpha_n r)$$

14.2.5

$$u(1, z) = z$$

$$\left. \frac{\partial u}{\partial z} \right|_{z=0} = 0$$

$$\left. \frac{\partial u}{\partial z} \right|_{z=1} = 0$$

$$Z'' + \lambda Z = 0$$

$$Z = c_1$$

$$Z = c_1 \cos \alpha z + c_2 \sin \alpha z$$

$$Z' = -\alpha c_1 \sin \alpha z + \alpha c_2 \cos \alpha z$$

$$0 = Z'(0)$$

$$= c_2$$

$$0 = Z'(1)$$

$$= -\alpha c_1 \sin \alpha$$

$$\alpha = n\pi$$

$$rR'' + R' - \alpha^2 rR = 0$$

$$R = c_3$$

$$R = c_3 I_0(n\pi r)$$

$$u(r, t) = A_0 + \sum_{n=1}^{\infty} A_n I_0(n\pi r) \cos n\pi z$$

$$z = u(1, z)$$

$$= A_0 + \sum_{n=1}^{\infty} A_n I_0(n\pi) \cos n\pi z$$

$$A_0 = \frac{1}{2}$$

$$A_n I_0(n\pi) = 2 \int_0^1 z \cos n\pi z \, dz$$

$$A_n = \frac{2[-1 + (-1)^n]}{n^2 \pi^2 I_0(n\pi)}$$

$$u(r, t) = \frac{1}{2} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{-1 + (-1)^n}{n^2 I_0(n\pi)} I_0(n\pi r) \cos n\pi z$$

14.2.9

$$k \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) = \frac{\partial u}{\partial t}$$

$$u(c, t) = 0$$

$$u(r, 0) = f(r)$$

$$kR''T + \frac{k}{r}R'T = RT'$$

$$\frac{rR'' + R'}{rR} = \frac{T'}{kT}$$

$$rR'' + R' + \lambda rR = 0$$

$$R = c_1 J_0(\alpha r)$$

$$0 = R(c)$$

$$= c_1 J_0(\alpha c)$$

$$\alpha_n = \frac{x_n}{c}$$

$$R = c_1 J_0(\alpha_n r)$$

$$T' + k\lambda T = 0$$

$$T = c_3 e^{-k\alpha_n^2 t}$$

$$u(r, t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} J_0(\alpha_n r)$$

$$f(r) = u(r, 0)$$

$$= \sum_{n=1}^{\infty} A_n J_0(\alpha_n r)$$

$$A_n = \frac{2}{c^2 J_1^2(\alpha_n c)} \int_0^c r J_0(\alpha_n r) f(r) dr$$

14.2.11

$$\begin{aligned}
\frac{rR'' + R'}{rR} &= \frac{T'}{kT} \\
rR'' + R' + \lambda rR &= 0 \\
R &= c_1 J_0(\alpha r) \\
R' &= -\alpha c_1 J_1(\alpha r) \\
R'(1) &= -hR(1) \\
\alpha c_1 J_0'(\alpha) &= -h c_1 J_0(\alpha) \\
hJ_0(\alpha) + \alpha J_0'(\alpha) &= 0 \\
R &= c_1 J_0(\alpha_n r) \\
T' + \lambda kT &= 0 \\
T &= c_3 e^{-\alpha_n^2 k t} \\
u(r, t) &= \sum_{n=1}^{\infty} A_n e^{-\alpha_n^2 k t} J_0(\alpha_n r) \\
f(r) &= u(r, 0) \\
&= \sum_{n=1}^{\infty} A_n J_0(\alpha_n r) \\
A_n &= \frac{2\alpha_n^2}{(\alpha_n^2 + h^2) J_0^2(\alpha_n)} \int_0^1 r J_0(\alpha_n r) f(r) dr
\end{aligned}$$

14.2.13

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial t} \\
u(r, t) &= v(r, t) + \psi(r) \\
R''T + \psi'' + \frac{1}{r}(R'T + \psi') &= RT' \\
\psi'' + \frac{1}{r}\psi' &= 0 \\
\psi &= 100 \\
\frac{rR'' + R'}{rR} &= \frac{T'}{T} \\
rR'' + R' + \lambda rR &= 0 \\
R &= c_1 J_0(\alpha r) \\
0 &= R(2) \\
&= c_1 J_0(2\alpha) \\
\alpha_n &= \frac{x_n}{2} \\
R &= c_1 J_0(\alpha_n r) \\
T &= c_2 e^{-\alpha_n^2 t} \\
v(r, t) &= \sum_{n=1}^{\infty} c_n e^{-\alpha_n^2 t} J_0(\alpha_n r) \\
f(r) - \psi(r) &= v(r, 0) \\
f(r) - 100 &= \sum_{n=1}^{\infty} c_n J_0(\alpha_n r) \\
c_n &= \frac{2}{4J_1^2(2\alpha_n)} \int_0^2 r J_0(\alpha_n r) [f(r) - 100] dr \\
&= \frac{50}{J_1^2(2\alpha_n)} \int_0^1 r J_0(\alpha_n r) dr \\
&= \frac{50J_1(\alpha_n)}{\alpha_n J_1^2(2\alpha_n)} \\
u(r, t) &= 100 + 50 \sum_{n=1}^{\infty} \frac{J_1(\alpha_n)}{\alpha_n J_1^2(2\alpha_n)} e^{-\alpha_n^2 t} J_0(\alpha_n r)
\end{aligned}$$

14.3 Spherical Coordinates

14.3.1

$$u(r, \theta) = 25P_0(\cos \theta) + \frac{75}{2} \frac{r}{c} P_1(\cos \theta) - \frac{175}{8} \left(\frac{r}{c}\right)^3 P_3(\cos \theta) + \frac{275}{16} \left(\frac{r}{c}\right)^5 P_5(\cos \theta)$$

14.3.5

$$u(a, \theta) = f(\theta)$$

$$u(b, \theta) = 0$$

$$R(r) = c_1 r^n + c_2 r^{-(n+1)}$$

$$0 = R(b)$$

$$= c_1 b^n + c_2 b^{-(n+1)}$$

$$c_2 = -c_1 b^{2n+1}$$

$$R = c_1 (r^n - b^{2n+1} r^{-(n+1)})$$

$$= c_1 \left(r^n - \frac{b^{2n+1}}{r^{n+1}} \right)$$

$$= c_1 \left[\left(\frac{r}{b} \right)^n - \left(\frac{b}{r} \right)^{n+1} \right]$$

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n \left[\left(\frac{r}{b} \right)^n - \left(\frac{b}{r} \right)^{n+1} \right] P_n(\cos \theta)$$

$$f(\theta) = u(a, \theta)$$

$$= \sum_{n=1}^{\infty} A_n \left[\left(\frac{a}{b} \right)^n - \left(\frac{b}{a} \right)^{n+1} \right] P_n(\cos \theta)$$

$$A_n = \frac{1}{(a/b)^n - (b/a)^{n+1}} \frac{2n+1}{2} \int_{-1}^1 f(\theta) P_n(\cos \theta) d\theta$$

14.4 Chapter in Review

14.4.1

$$\begin{aligned}R &= c_1 \\R &= c_1 r^\alpha \\ \Theta &= c_2 \\ \Theta &= c_2 \cos \alpha \theta + c_3 \sin \alpha \theta \\ \Theta(\theta) &= \Theta(\theta + 2\pi) \\ \alpha &= n, \quad n = 1, 2, 3, \dots \\ \Theta &= c_2 \cos n\theta + c_3 \sin n\theta \\ u(r, \theta) &= A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \\ f(\theta) &= u(c, \theta) \\ &= A_0 + \sum_{n=1}^{\infty} c^n (A_n \cos n\theta + B_n \sin n\theta) \\ A_0 &= 0 \\ A_n &= 0 \\ B_n &= \frac{2u_0[1 - (-1)^n]}{c^n n\pi} \\ u(r, \theta) &= \frac{2u_0}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{c^n n} r^n \sin n\theta\end{aligned}$$

15 Integral Transform Method

15.1 Error Function

15.1.1

(a)

$$\begin{aligned}\operatorname{erf}(\sqrt{t}) &= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2} du \\ \tau &= u^2 \\ d\tau &= 2u du \\ du &= \frac{d\tau}{2\sqrt{\tau}} \\ \operatorname{erf}(\sqrt{t}) &= \frac{1}{\sqrt{\pi}} \int_0^t \frac{e^{-\tau}}{\sqrt{\tau}} d\tau\end{aligned}$$

(b)

$$\begin{aligned}
 \mathcal{L}\{t^{-1/2}\} &= \frac{\sqrt{\pi}}{s^{1/2}} \\
 \mathcal{L}\{\operatorname{erf}(\sqrt{t})\} &= \mathcal{L}\left\{\frac{1}{\sqrt{\pi}} \int_0^t \frac{e^{-\tau}}{\sqrt{\tau}} d\tau\right\} \\
 &= \frac{1}{\sqrt{\pi}} \frac{1}{s} \frac{\Gamma(1/2)}{\sqrt{s+1}} \\
 &= \frac{1}{s\sqrt{s+1}}
 \end{aligned}$$

15.1.9

$$\begin{aligned}
 y(t) &= 1 - \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}} d\tau \\
 Y(s) &= \frac{1}{s} - Y(s) \sqrt{\frac{\pi}{s}} \\
 \left(1 + \sqrt{\frac{\pi}{s}}\right) Y(s) &= \frac{1}{s} \\
 Y(s) &= \frac{1}{s(1 + \sqrt{\pi/s})} \\
 &= \frac{1}{\sqrt{s}(\sqrt{s} + \sqrt{\pi})} \\
 y(t) &= e^{\pi t} \operatorname{erfc}(\sqrt{\pi t})
 \end{aligned}$$

15.1.11

$$\begin{aligned}
 \int_a^b e^{-u^2} du &= \int_0^b e^{-u^2} du - \int_0^a e^{-u^2} du \\
 &= \frac{\sqrt{\pi}}{2} \left(\frac{2}{\sqrt{\pi}} \int_0^b e^{-u^2} du - \frac{2}{\sqrt{\pi}} \int_0^a e^{-u^2} du \right) \\
 &= \frac{\sqrt{\pi}}{2} [\operatorname{erf}(b) - \operatorname{erf}(a)]
 \end{aligned}$$

15.2 Applications of the Laplace Transform

15.2.1

$$\begin{aligned}
a^2 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t^2} \\
u(0, t) &= 0 \\
u(L, t) &= 0 \\
u(x, 0) &= A \sin \frac{\pi x}{L} \\
\left. \frac{\partial u}{\partial t} \right|_{t=0} &= 0 \\
a^2 \frac{\partial^2 U}{\partial x^2} &= s^2 U(x, s) - su(x, 0) - u_t(x, 0) \\
\frac{\partial^2 U}{\partial x^2} - \frac{s^2}{a^2} U(x, s) &= -\frac{As}{a^2} \sin \frac{\pi x}{L} \\
U(0, s) &= 0 \\
U(L, s) &= 0 \\
U_c &= c_1 \cosh \frac{s}{a} x + c_2 \sinh \frac{s}{a} x \\
U_p &= c_3 \cos \frac{\pi x}{L} + c_4 \sin \frac{\pi x}{L} \\
\frac{\partial U_p}{\partial x} &= -c_3 \frac{\pi}{L} \sin \frac{\pi x}{L} + c_4 \frac{\pi}{L} \cos \frac{\pi x}{L} \\
\frac{\partial^2 U_p}{\partial x^2} &= -c_3 \left(\frac{\pi}{L} \right)^2 \cos \frac{\pi x}{L} - c_4 \left(\frac{\pi}{L} \right)^2 \sin \frac{\pi x}{L} \\
-\frac{As}{a^2} \sin \frac{\pi x}{L} &= -\left(\frac{\pi}{L} \right)^2 \left(c_3 \cos \frac{\pi x}{L} + c_4 \sin \frac{\pi x}{L} \right) \\
&\quad - \left(\frac{s}{a} \right)^2 \left(c_3 \cos \frac{\pi x}{L} + c_4 \sin \frac{\pi x}{L} \right) \\
\frac{As}{a^2} \sin \frac{\pi x}{L} &= \left[\left(\frac{\pi}{L} \right)^2 + \left(\frac{s}{a} \right)^2 \right] c_3 \cos \frac{\pi x}{L} \\
&\quad + \left[\left(\frac{\pi}{L} \right)^2 + \left(\frac{s}{a} \right)^2 \right] c_4 \sin \frac{\pi x}{L} \\
c_3 &= 0 \\
c_4 &= \frac{As/a^2}{(\pi/L)^2 + (s/a)^2} \\
U_p &= \frac{As/a^2}{(\pi/L)^2 + (s/a)^2} \sin \frac{\pi x}{L} \\
U &= c_1 \cosh \frac{s}{a} x + c_2 \sinh \frac{s}{a} x + \frac{As/a^2}{(\pi/L)^2 + (s/a)^2} \sin \frac{\pi x}{L} \\
0 &= U(0, s) \\
&= c_1
\end{aligned}$$

$$\begin{aligned}
0 &= U(L, s) \\
&= c_2 \sinh \frac{s}{a} L \\
&= c_2 \\
U(x, s) &= \frac{As/a^2}{(\pi/L)^2 + (s/a)^2} \sin \frac{\pi x}{L} \\
&= \frac{As}{(\pi a/L)^2 + s^2} \sin \frac{\pi x}{L} \\
u(x, t) &= \mathcal{L}^{-1}\{U(x, s)\} \\
&= A \cos \frac{\pi a}{L} t \sin \frac{\pi x}{L}
\end{aligned}$$

15.2.3

$$\begin{aligned}
U_{xx} - \left(\frac{s}{a}\right)^2 U &= 0 \\
U &= c_1 e^{-(s/a)x} \\
U(0, s) &= F(s) \\
U &= F(s) e^{-(s/a)x} \\
u(x, t) &= f(t - x/a) \mathcal{U}(t - x/a)
\end{aligned}$$

15.2.5

$$\begin{aligned}
U(x, s) &= c_1 e^{-(x/a)s} - \frac{g}{s^3} \\
F(s) &= U(0, s) \\
&= c_1 - \frac{g}{s^3} \\
c_1 &= F(s) + \frac{g}{s^3} \\
U(x, s) &= \left[F(s) + \frac{g}{s^3}\right] e^{-(x/a)s} - \frac{g}{s^3} \\
u(x, t) &= \left[A \sin \omega \left(t - \frac{x}{a}\right) + \frac{1}{2}g \left(t - \frac{x}{a}\right)^2\right] \mathcal{U}\left(t - \frac{x}{a}\right) - \frac{1}{2}gt^2
\end{aligned}$$

15.2.11

$$\begin{aligned}
u_{xx} &= u_t \\
U_{xx} - sU &= -u_1 \\
U_c &= c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} \\
U_p &= c_3 \\
U_{pxx} &= 0 \\
0 - sc_3 &= -u_1 \\
c_3 &= \frac{u_1}{s} \\
U &= c_2 e^{-\sqrt{s}x} + \frac{u_1}{s} \\
\frac{u_0}{s} &= U(0, s) \\
&= c_2 + \frac{u_1}{s} \\
c_2 &= \frac{u_0 - u_1}{s} \\
U &= \frac{u_0 - u_1}{s} e^{-\sqrt{s}x} + \frac{u_1}{s} \\
u(x, t) &= (u_0 - u_1) \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) + u_1
\end{aligned}$$

15.3 Fourier Integral

15.3.1

$$\begin{aligned}
A(\alpha) &= \int_{-1}^0 -\cos \alpha x \, dx + \int_0^1 2 \cos \alpha x \, dx \\
&= \frac{\sin \alpha}{\alpha} \\
B(\alpha) &= \int_{-1}^0 -\sin \alpha x \, dx + \int_0^1 2 \sin \alpha x \, dx \\
&= \frac{3(1 - \cos \alpha)}{\alpha} \\
f(x) &= \frac{1}{\pi} \int_0^\infty \frac{\sin \alpha \cos \alpha x + 3(1 - \cos \alpha) \sin \alpha x}{\alpha} \, d\alpha
\end{aligned}$$

15.3.7

$$\begin{aligned}
A(\alpha) &= \int_0^1 5 \sin \alpha x \, dx \\
&= \frac{5(1 - \cos \alpha)}{\alpha} \\
f(x) &= \frac{10}{\pi} \int_0^\infty \frac{(1 - \cos \alpha) \sin \alpha x}{\alpha} \, d\alpha
\end{aligned}$$

15.3.13

$$\begin{aligned}
A(\alpha) &= \int_0^\infty e^{-kx} \cos \alpha x \, dx \\
&= \frac{k}{\alpha^2 + k^2} \\
f(x) &= \frac{2k}{\pi} \int_0^\infty \frac{\cos \alpha x}{\alpha^2 + k^2} \, d\alpha \\
A(\alpha) &= \int_0^\infty e^{-kx} \sin \alpha x \, dx \\
&= \frac{\alpha}{\alpha^2 + k^2} \\
f(x) &= \frac{2}{\pi} \int_0^\infty \frac{\alpha}{\alpha^2 + k^2} \sin \alpha x \, d\alpha
\end{aligned}$$

15.3.17

$$f(x) = \frac{2}{\pi} \int_0^\infty e^{-\alpha} \cos \alpha x \, d\alpha = \frac{2}{\pi} \frac{1}{1 + x^2}$$

15.4 Fourier Transform

15.4.1

$$\begin{aligned}ku_{xx} &= u_t \\ u(x, 0) &= e^{-|x|} \\ -\alpha^2 kU &= U_t \\ U(s, 0) &= \int_{-\infty}^{\infty} e^{-|x|+i\alpha x} dx \\ &= \frac{2}{1+\alpha^2} \\ U &= ce^{-\alpha^2 kt} \\ \frac{2}{1+\alpha^2} &= c \\ U(s, t) &= \frac{2}{1+\alpha^2} e^{-\alpha^2 kt} \\ u(x, t) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-\alpha^2 kt - i\alpha x}}{1+\alpha^2} d\alpha\end{aligned}$$

15.4.3

$$\begin{aligned}
ku_{xx} &= u_t \\
u(0, t) &= u_0 \\
u(x, 0) &= 0 \\
k[-\alpha^2 U + \alpha u(0, t)] &= U_t \\
U_t + k\alpha^2 U &= k\alpha u_0 \\
U_c &= c_1 e^{-k\alpha^2 t} \\
U_p &= c_2 \\
-k\alpha^2 c_2 + k\alpha u_0 &= 0 \\
c_2 &= \frac{u_0}{\alpha} \\
U &= c_1 e^{-k\alpha^2 t} + \frac{u_0}{\alpha} \\
0 &= U(\alpha, 0) \\
&= c_1 + \frac{u_0}{\alpha} \\
c_1 &= -\frac{u_0}{\alpha} \\
U &= \frac{u_0}{\alpha} (1 - e^{-k\alpha^2 t}) \\
u &= \frac{2u_0}{\pi} \int_0^\infty \frac{1 - e^{-k\alpha^2 t}}{\alpha} \sin \alpha x \, d\alpha
\end{aligned}$$

15.4.5

$$\begin{aligned}
U_t + k\alpha^2 U &= 0 \\
U &= ce^{-k\alpha^2 t} \\
U(\alpha, 0) &= \int_0^1 \sin \alpha x \, dx \\
&= \frac{1}{\alpha} [-\cos \alpha x]_0^1 \\
&= \frac{1 - \cos \alpha}{\alpha} \\
\frac{1 - \cos \alpha}{\alpha} &= c \\
U &= \frac{1 - \cos \alpha}{\alpha} e^{-k\alpha^2 t} \\
u &= \frac{2}{\pi} \int_0^\infty \frac{1 - \cos \alpha}{\alpha} e^{-k\alpha^2 t} \sin \alpha x \, d\alpha
\end{aligned}$$

15.4.9

(a)

$$a^2 u_{xx} = u_{tt}$$

$$u(x, 0) = f(x)$$

$$u_t|_{t=0} = g(x)$$

$$U_{tt} + a^2 \alpha^2 U = 0$$

$$U = c_1 \cos a\alpha t + c_2 \sin a\alpha t$$

$$F(\alpha) = U(\alpha, 0)$$

$$= c_1$$

$$G(\alpha) = U_t|_{t=0}$$

$$= a\alpha c_2$$

$$c_2 = \frac{G(\alpha)}{a\alpha}$$

$$U = F(\alpha) \cos a\alpha t + G(\alpha) \frac{\sin a\alpha t}{a\alpha}$$

$$u = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[F(\alpha) \cos a\alpha t + G(\alpha) \frac{\sin a\alpha t}{a\alpha} \right] e^{-i\alpha x} d\alpha$$

(b)

$$\begin{aligned} u &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) \cos a\alpha t e^{-i\alpha x} d\alpha \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) \frac{1}{2} (e^{i\alpha at} + e^{-i\alpha at}) e^{-i\alpha x} d\alpha \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2} [F(\alpha) e^{i\alpha at} + F(\alpha) e^{-i\alpha at}] e^{-i\alpha x} d\alpha \\ &= \frac{1}{2} [F(x + at) + F(x - at)] \end{aligned}$$

15.4.13

$$u_{xx} + u_{yy} = 0$$

$$u_x|_{x=0} = 0$$

$$u(x, 0) = \begin{cases} 50 & 0 < x < 1 \\ 0 & x > 1 \end{cases}$$

$$U_{yy} - \alpha^2 U = 0$$

$$U = c_1 e^{-\alpha y}$$

$$\begin{aligned} U(\alpha, 0) &= \int_0^1 50 \cos \alpha x \, dx \\ &= \frac{50 \sin \alpha}{\alpha} \end{aligned}$$

$$\frac{50 \sin \alpha}{\alpha} = c_1$$

$$U = \frac{50 \sin \alpha}{\alpha} e^{-\alpha y}$$

$$u = \frac{100}{\pi} \int_0^\infty \frac{\sin \alpha}{\alpha} e^{-\alpha y} \cos \alpha x \, d\alpha$$