Introduction to Electrodynamics by David J. Griffiths Problems

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December 2023

Contents

2 Electrostatics

2.1

(a) **0**

(b) The same as if only the opposite charge were present — all others are cancelled out.

2.2

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} 2 \frac{q}{\mathscr{r}^2} \cos\theta \hat{\mathbf{x}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{dq}{[(d/2)^2 + z^2]^{3/2}} \hat{\mathbf{x}}
\end{aligned}
$$

2.3

$$
\begin{aligned}
\mathbf{r} &= z\hat{\mathbf{z}} \\
\mathbf{r}' &= x\hat{\mathbf{x}} \\
\boldsymbol{\mathscr{r}} &= z\hat{\mathbf{z}} - x\hat{\mathbf{x}} \\
\mathscr{r} &= \sqrt{x^2 + z^2} \\
\hat{\boldsymbol{\mathscr{r}}} &= \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{\sqrt{x^2 + z^2}} \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_0^L \frac{\lambda}{x^2 + z^2} \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{\sqrt{x^2 + z^2}}\, dx \\
&= \frac{1}{4\pi\epsilon_0}\lambda \left(z\hat{\mathbf{z}} \int_0^L \frac{1}{(x^2 + z^2)^{3/2}}\, dx - \hat{\mathbf{x}} \int_0^L \frac{x}{(x^2 + z^2)}\, dx \right) \\
&= \frac{1}{4\pi\epsilon_0}\lambda \left[\frac{L}{z\sqrt{L^2 + z^2}}\hat{\mathbf{z}} - \left(\frac{1}{z} - \frac{1}{\sqrt{L^2 + z^2}} \right) \hat{\mathbf{x}} \right] \\
&= \frac{1}{4\pi\epsilon_0} \frac{\lambda}{z} \left[\left(-1 + \frac{z}{\sqrt{L^2 + z^2}} \right) \hat{\mathbf{x}} + \frac{L}{\sqrt{L^2 + z^2}}\hat{\mathbf{z}} \right]
\end{aligned}
$$

2.4

The electric field a distance z above the midpoint of a line segment of length $2L$ and uniform line charge λ is

$$
\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z\sqrt{z^2 + L^2}}\hat{\mathbf{z}}.
$$

Applying this to the four sides of the square, the horizontal components of opposite sides cancel leaving only the vertical component.

$$
\begin{aligned}
\cos\theta &= \frac{z}{\mathscr{r}} \\
&= \frac{z}{\sqrt{(a/2)^2 + z^2}} \\
\mathbf{E} &= 4 \left(\frac{1}{4\pi\epsilon_0} \frac{\lambda a}{\sqrt{(a/2)^2 + z^2}\sqrt{(a/2)^2 + (a/2)^2 + z^2}}\hat{\mathbf{z}} \right) \cos\theta \\
&= \frac{1}{4\pi\epsilon_0} \frac{4a\lambda z}{[(a/2)^2 + z^2]\sqrt{(a^2/2) + z^2}}\hat{\mathbf{z}}
\end{aligned}
$$

2.5

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\frac{\lambda r}{r^2+z^2}\cos\alpha\, d\theta\,\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\pi\lambda r z}{(r^2+z^2)^{3/2}}\hat{\mathbf{z}}
\end{aligned}
$$

2.6

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int\frac{dq}{\mathscr{r}^2}\cos\theta\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^R\frac{\sigma}{r^2+z^2}\frac{z}{\sqrt{r^2+z^2}}r\,dr\,d\theta\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}2\pi\sigma z\int_0^R\frac{r}{(r^2+z^2)^{3/2}}\,dr\,\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}2\pi\sigma z\left(\frac{1}{z}-\frac{1}{\sqrt{R^2+z^2}}\right)\hat{\mathbf{z}}
\end{aligned}
$$

When $R\to\infty$

$$
\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}.
$$

2.7

$$
\mathbf{E} = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}} & z>R \\ \mathbf{0} & z<R \end{cases}
$$

2.8

$$
\mathbf{E} = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}} & z>R \\ \frac{1}{4\pi\epsilon_0}\frac{qz}{R^3}\hat{\mathbf{z}} & z<R \end{cases}
$$

2.9

(a)

$$
\begin{aligned}
\rho &= \epsilon_0\nabla\cdot\mathbf{E} \\
&= \epsilon_0\frac{1}{r^2}\frac{\partial}{\partial r}(kr^5) \\
&= 5\epsilon_0 k r^2
\end{aligned}
$$

(b)

$$
\begin{aligned}
Q_{\text{enc}} &= \epsilon_0 \oint \mathbf{E} \cdot d\mathbf{a} \\
&= \epsilon_0 \int_0^{2\pi} \int_0^{\pi} kR^3 R \, d\theta R \sin\theta \, d\phi \\
&= 2\pi\epsilon_0 kR^5 [-\cos\theta]_0^{\pi} \\
&= 4\pi\epsilon_0 kR^5 \\
Q_{\text{enc}} &= \int_V \rho \, d\tau \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^{R} 5\epsilon_0 kr^2 \, dr r \, d\theta r \sin\theta \, d\phi \\
&= 10\pi\epsilon_0 k \int_0^{\pi} \int_0^{R} r^4 \sin\theta \, dr \, d\theta \\
&= 2\pi\epsilon_0 kR^5 [-\cos\theta]_0^{\pi} \\
&= 4\pi\epsilon_0 kR^5
\end{aligned}
$$

2.10

If the charge was surrounded by 8 such cubes the total flux through all the cubes would be q/ϵ_0. There are 24 outside faces to the larger cube, so the total flux through the shaded face is $q/(24\epsilon_0)$.

2.11

$$
\begin{aligned}
\int \mathbf{E}_{\text{inside}} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
&= 0 \\
\mathbf{E}_{\text{inside}} &= \mathbf{0} \\
\int \mathbf{E}_{\text{outside}} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E_{\text{outside}} &= \frac{4\pi R^2 \sigma}{\epsilon_0} \\
\mathbf{E}_{\text{outside}} &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}
\end{aligned}
$$

2.12

$$
\begin{aligned}
\int \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E &= \frac{\frac{4}{3}\pi r^3 \rho}{\epsilon_0} \\
\mathbf{E} &= \frac{r\rho}{3\epsilon_0}\hat{\mathbf{r}}
\end{aligned}
$$

2.13

$$
\begin{aligned}
\int \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
2\pi s l E &= \frac{l\lambda}{\epsilon_0} \\
\mathbf{E} &= \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{\mathbf{s}}
\end{aligned}
$$

2.14

$$
\begin{aligned}
Q_{\text{enc}} &= \int_V \rho \, d\tau \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^{r} k r'^3 \sin\theta \, dr' \, d\theta \, d\phi \\
&= 2\pi k \int_0^{\pi} \left[\frac{1}{4} r'^4 \sin\theta\right]_0^r \, d\theta \\
&= \frac{1}{2}\pi k r^4 [-\cos\theta]_0^{\pi} \\
&= \pi k r^4 \\
\int \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E &= \frac{\pi k r^4}{\epsilon_0} \\
\mathbf{E} &= \frac{k r^2}{4\epsilon_0}\hat{\mathbf{r}}
\end{aligned}
$$

2.15

(a) $\mathbf{E} = \mathbf{0}$

(b)

$$
\begin{aligned}
Q_{\text{enc}} &= \int_0^{2\pi}\int_0^{\pi}\int_a^{r} k \sin\theta \, dr' \, d\theta \, d\phi \\
&= 4\pi k(r-a) \\
4\pi r^2 E &= \frac{4\pi k(r-a)}{\epsilon_0} \\
\mathbf{E} &= \frac{k(r-a)}{\epsilon_0 r^2}\hat{\mathbf{r}}
\end{aligned}
$$

(c) $\mathbf{E} = \frac{k(b-a)}{\epsilon_0 r^2}\hat{\mathbf{r}}$

2.16

(a)

$$
\begin{aligned}
Q_{\text{enc}} &= \pi s^2 l \rho \\
2\pi s l E &= \frac{\pi s^2 l \rho}{\epsilon_0} \\
\mathbf{E} &= \frac{s\rho}{2\epsilon_0}\hat{\mathbf{s}}
\end{aligned}
$$

(b)

$$
\mathbf{E} = \frac{a^2\rho}{2\epsilon_0 s}\hat{\mathbf{s}}
$$

(c)

$$
\mathbf{E} = \mathbf{0}
$$

2.17

$$
\begin{aligned}
2AE_{\text{inside}} &= \frac{2Ay\rho}{\epsilon_0} \\
\mathbf{E}_{\text{inside}} &= \frac{y\rho}{\epsilon_0} \\
\mathbf{E} &= \begin{cases} \frac{d\rho}{\epsilon_0} & d < y \\ \frac{y\rho}{\epsilon_0} & 0 < y < d \\ -\frac{y\rho}{\epsilon_0} & -d < y < 0 \\ -\frac{d\rho}{\epsilon_0} & y < -d \end{cases}
\end{aligned}
$$

2.18

The electric field inside a uniformly charged solid sphere is

$$\mathbf{E} = \frac{r\rho}{3\epsilon_0}\hat{\mathbf{r}}.$$

$$\begin{aligned}
\mathbf{d} &= \mathbf{r}_1 - \mathbf{r}_2 \\
\mathbf{E} &= \frac{r_1\rho}{3\epsilon_0}\hat{\mathbf{r}}_1 - \frac{r_2\rho}{3\epsilon_0}\hat{\mathbf{r}}_2 \\
&= \frac{\rho}{3\epsilon_0}(\mathbf{r}_1 - \mathbf{r}_2) \\
&= \frac{\rho}{3\epsilon_0}\mathbf{d}
\end{aligned}$$

2.20

a is impossible because its curl is nonzero.

$$\begin{aligned}
V &= -\int_0^y 2kxy'\,dy' - \int_0^z 2kyz'\,dz \\
&= -2kx\left[\frac{1}{2}y'^2\right]_0^y - 2ky\left[\frac{1}{2}z'^2\right]_0^z \\
&= -k(xy^2 + yz^2) \\
-\nabla V &= k[y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}] \\
&= \mathbf{E}
\end{aligned}$$

2.21

$$
\begin{aligned}
\mathbf{E} &= \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} & r > R \\ \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3} & r < R \end{cases} \\
V_{\text{outside}}(r) &= -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\,dr' \\
&= -\frac{1}{4\pi\epsilon_0} q \left[-\frac{1}{r'}\right]_{\infty}^{r} \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{r} \\
-\nabla V_{\text{outside}} &= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \\
&= \mathbf{E}_{\text{outside}} \\
V_{\text{inside}}(r) &= -\left(\int_{\infty}^{R} \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\,dr' + \int_{R}^{r} \frac{1}{4\pi\epsilon_0}\frac{qr'}{R^3}\,dr'\right) \\
&= -\left(-\frac{1}{4\pi\epsilon_0}\frac{q}{R} + \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\left[\frac{1}{2}r'^2\right]_{R}^{r}\right) \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left[3 - \left(\frac{r}{R}\right)^2\right] \\
-\nabla V_{\text{inside}} &= \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3}\hat{\mathbf{r}} \\
&= \mathbf{E}_{\text{inside}}
\end{aligned}
$$

2.22

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{\mathbf{s}} \\
V &= -\int_{O}^{s} \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s'}\,ds' \\
&= -\frac{1}{2\pi\epsilon_0}\lambda \ln\frac{s}{O} \\
-\nabla V &= \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{\mathbf{s}}
\end{aligned}
$$

2.23

$$
\begin{aligned}
\mathbf{E} &= \begin{cases} \mathbf{0} & r < a \\ \frac{k(r-a)}{\epsilon_0 r^2}\hat{\mathbf{r}} & a < r < b \\ \frac{k(b-a)}{\epsilon_0 r^2}\hat{\mathbf{r}} & b < r \end{cases} \\
V(0) &= -\int_\infty^0 E\,dr \\
&= -\left(\int_\infty^b \frac{k(b-a)}{\epsilon_0 r^2}\,dr + \int_b^a \frac{k(r-a)}{\epsilon_0 r^2}\,dr\right) \\
&= -\left(\frac{k(b-a)}{\epsilon_0}\left[-\frac{1}{r}\right]_\infty^b + \frac{k}{\epsilon_0}\left[\ln r + \frac{a}{r}\right]_b^a\right) \\
&= -\left[-\frac{k(b-a)}{\epsilon_0 b} + \frac{k}{\epsilon_0}\left(\ln a + 1 - \ln b - \frac{a}{b}\right)\right] \\
&= -\frac{k}{\epsilon_0}\left(-1 + \frac{a}{b} + \ln\frac{a}{b} + 1 - \frac{a}{b}\right) \\
&= \frac{k}{\epsilon_0}\ln\frac{b}{a}
\end{aligned}
$$

2.24

$$
\begin{aligned}
V(b) - V(0) &= -\int_0^b E\,dr \\
&= -\left(\int_0^a \frac{s\rho}{2\epsilon_0}\,ds + \int_a^b \frac{a^2\rho}{2\epsilon_0 s}\,ds\right) \\
&= -\left(\frac{\rho}{2\epsilon_0}\left[\frac{1}{2}s^2\right]_0^a + \frac{a^2\rho}{2\epsilon_0}\ln\frac{b}{a}\right) \\
&= -\left(\frac{a^2\rho}{4\epsilon_0} + \frac{a^2\rho}{2\epsilon_0}\ln\frac{b}{a}\right) \\
&= -\frac{a^2\rho}{4\epsilon_0}\left(1 + 2\ln\frac{a}{b}\right)
\end{aligned}
$$

2.25

(a)

$$
V = \frac{1}{4\pi\epsilon_0}\frac{2q}{\sqrt{(d/2)^2 + z^2}}
$$

(b)

$$
\begin{aligned}
V &= \frac{1}{4\pi\epsilon_0}\int_{-L}^{L}\frac{\lambda}{\sqrt{x^2+z^2}}\,dx \\
&= \frac{1}{4\pi\epsilon_0}\lambda\ln\left(1+\frac{2L(L+\sqrt{L^2+z^2})}{z^2}\right)
\end{aligned}
$$

(c)

$$
\begin{aligned}
V &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{R}\frac{\sigma}{\sqrt{r^2+z^2}}r\,dr\,d\theta \\
&= \frac{1}{4\pi\epsilon_0}2\pi\sigma(\sqrt{R^2+z^2}-z)
\end{aligned}
$$

2.26

$$
\begin{aligned}
V_{\text{bottom}} &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{h}\frac{\sqrt{2}\sigma z}{\sqrt{2}z}\,d\phi\,dz \\
&= \frac{\sigma h}{2\epsilon_0} \\
V_{\text{top}} &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{h}\frac{\sqrt{2}\sigma z}{\sqrt{z^2+(h-z)^2}}\,d\phi\,dz \\
&= \frac{\sqrt{2}\sigma}{2\epsilon_0}\int_0^{h}\frac{z}{\sqrt{z^2+(h-z)^2}}\,dz \\
&= \frac{\sigma h}{4\epsilon_0}\ln(3+2\sqrt{2}) \\
V_{\text{bottom}}-V_{\text{top}} &= \frac{\sigma h}{2\epsilon_0}\left[1-\frac{1}{2}\ln(3+2\sqrt{2})\right]
\end{aligned}
$$

2.28

$$
\begin{aligned}
V(r) &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{\rho r'^2\sin\theta}{\sqrt{r^2+r'^2-2rr'\cos\theta}}\,dr'\,d\theta\,d\phi \\
&= \frac{\rho}{2\epsilon_0}\int_0^{\pi}\int_0^{R}\frac{r'^2\sin\theta}{\sqrt{r^2+r'^2-2rr'\cos\theta}}\,dr'\,d\theta \\
&= \frac{\rho}{2\epsilon_0}\left(R^2-\frac{r^2}{3}\right) \\
&= \frac{q}{8\pi\epsilon_0 R}\left(3-\frac{r^2}{R^2}\right)
\end{aligned}
$$

2.31

(a)

$$W = \frac{q^2}{4\pi\epsilon_0 a}\left(\frac{1}{\sqrt{2}} - 2\right)$$

(b)

$$\begin{aligned}
W &= \frac{1}{4\pi\epsilon_0}\left(-\frac{q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a} - \frac{q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a}\right) \\
&= \frac{q^2}{2\pi\epsilon_0 a}\left(\frac{1}{\sqrt{2}} - 2\right)
\end{aligned}$$

2.32

$$\begin{aligned}
W &= \frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{a} \\
W &= K_1 + K_2 \\
\frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{a} &= \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 \\
\frac{1}{2\pi\epsilon_0}\frac{q_A q_B}{a} &= m_A v_A^2 + m_B v_B^2 \\
0 &= m_B v_B - m_A v_A \\
v_B &= \frac{m_A}{m_B}v_A \\
\frac{1}{2\pi\epsilon_0}\frac{q_A q_B}{a} &= m_A v_A^2 + m_B\left(\frac{m_A}{m_B}v_A\right)^2 \\
&= m_A v_A^2 + \frac{m_A^2}{m_B}v_A^2 \\
&= \frac{m_A(m_A + m_B)}{m_B}v_A^2 \\
v_A &= \sqrt{\frac{1}{2\pi\epsilon_0}\frac{q_A q_B}{(m_A + m_B)a}\frac{m_B}{m_A}} \\
v_B &= \sqrt{\frac{1}{2\pi\epsilon_0}\frac{q_A q_B}{(m_A + m_B)a}\frac{m_A}{m_B}}
\end{aligned}$$

2.33

$$
\begin{aligned}
W &= \frac{1}{4\pi\epsilon_0}\left(-\frac{q^2}{a}+\frac{q^2}{2a}-\frac{q^2}{3a}+\dots\right)\\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{a}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\\
&= -\frac{1}{4\pi\epsilon_0}\frac{q^2}{a}\ln 2
\end{aligned}
$$

2.34

(a)

$$
\begin{aligned}
V &= \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left[3-\left(\frac{r}{R}\right)^2\right] & r < R \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r} & r > R \end{cases}\\
W &= \frac{1}{2}\int \rho V\, d\tau\\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\rho\frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left[3-\left(\frac{r}{R}\right)^2\right] r^2\sin\theta\, dr\, d\theta\, d\phi\\
&= \frac{q\rho}{8\epsilon_0 R}\int_0^{\pi}\int_0^{R}\left[3-\left(\frac{r}{R}\right)^2\right] r^2\sin\theta\, dr\, d\theta\\
&= \frac{q\rho R^2}{5\epsilon_0}\\
&= \frac{qR^2}{5\epsilon_0}\frac{q}{\frac{4}{3}\pi R^3}\\
&= \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathbf{E} &= \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} & r > R \\ \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3}\hat{\mathbf{r}} & r < R \end{cases} \\
E^2 &= \begin{cases} \frac{1}{16\pi^2\epsilon_0^2}\frac{q^2}{r^4} & r > R \\ \frac{1}{16\pi^2\epsilon_0^2}\frac{q^2r^2}{R^6} & r < R \end{cases} \\
W &= \frac{\epsilon_0}{2}\int E^2\,d\tau \\
&= \frac{\epsilon_0}{2}\left(\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{1}{16\pi^2\epsilon_0^2}\frac{q^2r^2}{R^6}r^2\sin\theta\,dr\,d\theta\,d\phi\right. \\
&\qquad\left.+\int_0^{2\pi}\int_0^{\pi}\int_R^{\infty}\frac{1}{16\pi^2\epsilon_0^2}\frac{q^2}{r^4}r^2\sin\theta\,dr\,d\theta\,d\phi\right) \\
&= \frac{\epsilon_0}{2}\frac{1}{16\pi^2\epsilon_0^2}2\pi q^2\left(\int_0^{\pi}\int_0^{R}\frac{r^4}{R^6}\sin\theta\,dr\,d\theta+\int_0^{\pi}\int_R^{\infty}\frac{1}{r^2}\sin\theta\,dr\,d\theta\right) \\
&= \frac{1}{16\pi\epsilon_0}q^2\left(\int_0^{\pi}\int_0^{R}\frac{r^4}{R^6}\sin\theta\,dr\,d\theta+\int_0^{\pi}\int_R^{\infty}\frac{1}{r^2}\sin\theta\,dr\,d\theta\right) \\
&= \frac{1}{16\pi\epsilon_0}q^2\left(\frac{2}{5R}+\frac{2}{R}\right) \\
&= \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}
\end{aligned}
$$

(c)

$$
\begin{aligned}
W &= \frac{\epsilon_0}{2}\left(\int_V E^2\,d\tau + \oint_S V\mathbf{E}\cdot d\mathbf{a}\right)\\
&= \frac{\epsilon_0}{2}\left(\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{1}{(4\pi\epsilon_0)^2}\frac{q^2r^2}{R^6}r^2\sin\theta\,dr\,d\theta\,d\phi\right.\\
&\qquad + \int_0^{2\pi}\int_0^{\pi}\int_R^{a}\frac{1}{(4\pi\epsilon_0)^2}\frac{q^2}{r^4}r^2\sin\theta\,dr\,d\theta\,d\phi\\
&\qquad \left. + \int_0^{2\pi}\int_0^{\pi}\frac{1}{4\pi\epsilon_0}\frac{q}{a}\frac{1}{4\pi\epsilon_0}\frac{q}{a^2}a^2\sin\theta\,d\theta\,d\phi\right)\\
&= \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}2\pi q^2\left(\int_0^{\pi}\int_0^{R}\frac{r^4}{R^6}\sin\theta\,dr\,d\theta\right.\\
&\qquad \left. + \int_0^{\pi}\int_R^{a}\frac{1}{r^2}\sin\theta\,dr\,d\theta + \int_0^{\pi}\frac{1}{a}\sin\theta\,d\theta\right)\\
&= \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}2\pi q^2\left[\frac{2}{5R} + 2\left(\frac{1}{R}-\frac{1}{a}\right)+\frac{2}{a}\right]\\
&= \frac{1}{8\pi\epsilon_0}q^2\left[\frac{1}{5R}+\frac{1}{R}\right]\\
&= \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}
\end{aligned}
$$

2.36

(a)

$$
\mathbf{E} = \begin{cases} \mathbf{0} & r < a \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} & a < r < b \\ \mathbf{0} & b < r \end{cases}
$$

$$
E^2 = \begin{cases} 0 & r < a \\ \frac{1}{(4\pi\epsilon_0)^2}\frac{q^2}{r^4} & a < r < b \\ 0 & b < r \end{cases}
$$

$$
\begin{aligned}
W &= \frac{\epsilon_0}{2}\int E^2\,d\tau\\
&= \frac{\epsilon_0}{2}\int_0^{2\pi}\int_0^{\pi}\int_a^{b}\frac{1}{(4\pi\epsilon_0)^2}\frac{q^2}{r^4}r^2\sin\theta\,dr\,d\theta\,d\phi\\
&= \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}2\pi q^2\int_0^{\pi}\int_a^{b}\frac{\sin\theta}{r^2}\,dr\,d\theta\\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)
\end{aligned}
$$

(b)

$$
\begin{aligned}
W_{\text{shell}} &= \frac{1}{8\pi\epsilon_0}\frac{q^2}{R} \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \\
\mathbf{E}_1 \cdot \mathbf{E}_2 &= -\frac{1}{(4\pi\epsilon_0)^2}\frac{q^2}{r^4} \\
W_{\text{total}} &= W_1 + W_2 + \epsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2 \, d\tau \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b}\right) - \epsilon_0 \int_0^{2\pi} \int_0^{\pi} \int_b^{\infty} \frac{1}{(4\pi\epsilon_0)^2}\frac{q^2}{r^4} r^2 \sin\theta \, dr \, d\theta \, d\phi \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b}\right) - \frac{1}{8\pi\epsilon_0} q^2 \int_0^{\pi} \int_b^{\infty} \frac{1}{r^2} \sin\theta \, dr \, d\theta \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b}\right) - \frac{1}{4\pi\epsilon_0} q^2 \int_b^{\infty} \frac{1}{r^2} \, dr \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b}\right) - \frac{1}{4\pi\epsilon_0}\frac{q^2}{b} \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{b}\right) \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)
\end{aligned}
$$

2.37

$$
\begin{aligned}
r_1 &= r \\
E_1 &= \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1^2} \\
&= \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2} \\
r_2 &= \sqrt{a^2 + r^2 - 2ar\cos\theta} \\
E_2 &= \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_2^2} \\
&= \frac{1}{4\pi\epsilon_0}\frac{q_2}{a^2 + r^2 - 2ar\cos\theta} \\
\cos\alpha &= \frac{r - a\cos\theta}{\sqrt{a^2 + r^2 - 2ar\cos\theta}} \\
\mathbf{E}_1 \cdot \mathbf{E}_2 &= E_1 E_2 \cos\alpha \\
&= \frac{1}{(4\pi\epsilon_0)^2}\frac{q_1 q_2}{r^2(a^2 + r^2 - 2ar\cos\theta)}\frac{r - a\cos\theta}{\sqrt{a^2 + r^2 - 2ar\cos\theta}} \\
&= \frac{1}{(4\pi\epsilon_0)^2}\frac{q_1 q_2(r - a\cos\theta)}{r^2(a^2 + r^2 - 2ar\cos\theta)^{3/2}} \\
\epsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2 \, d\tau &= \epsilon_0 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{(4\pi\epsilon_0)^2}\frac{q_1 q_2(r - a\cos\theta)}{r^2(a^2 + r^2 - 2ar\cos\theta)^{3/2}} r^2 \sin\theta \, dr \, d\theta \, d\phi \\
&= \frac{q_1 q_2}{8\pi\epsilon_0}\int_0^{\pi}\int_0^{\infty} \frac{(r - a\cos\theta)\sin\theta}{(a^2 + r^2 - 2ar\cos\theta)^{3/2}} \, dr \, d\theta
\end{aligned}
$$

2.38

(a)

$$
\begin{aligned}
\sigma_R &= \frac{q}{4\pi R^2} \\
\sigma_a &= -\frac{q}{4\pi a^2} \\
\sigma_b &= \frac{q}{4\pi b^2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
V &= -\int_{\infty}^{b} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \, dr - \int_a^R \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \, dr \\
&= \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{b} + \frac{1}{R} - \frac{1}{a}\right)
\end{aligned}
$$

(c)

$$\sigma_b = 0$$
$$V = \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{R} - \frac{1}{a}\right)$$

2.39

(a)

$$\sigma_a = -\frac{q_a}{4\pi a^2}$$
$$\sigma_b = -\frac{q_b}{4\pi b^2}$$
$$\sigma_R = \frac{q_a + q_b}{4\pi R^2}$$

(b)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q_a + q_b}{r^2} \hat{\mathbf{r}}$$

(c)

$$\mathbf{E}_a = \frac{1}{4\pi\epsilon_0} \frac{q_a}{r^2} \hat{\mathbf{r}}$$
$$\mathbf{E}_b = \frac{1}{4\pi\epsilon_0} \frac{q_b}{r^2} \hat{\mathbf{r}}$$

(d)

$$\mathbf{0}$$

(e) a, b

2.40

(a) No. If it's close to the wall it will induce a surface charge and be attracted.

(b) No. If the conductor contains a cavity containing a like charge it will be repelled.

2.41

By Gauss's law, the electric field of each plate is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$
$$2A'E = \frac{A' \frac{Q}{A}}{\epsilon_0}$$
$$\mathbf{E} = \frac{Q}{2A\epsilon_0} \hat{\mathbf{n}}$$

so the field between the plates is zero and the field outside is $Q/A\epsilon_0\hat{\mathbf{n}}$, resulting in a pressure of

$$\begin{aligned} P &= \frac{\epsilon_0}{2}E^2 \\ &= \frac{\epsilon_0}{2}\frac{Q^2}{A^2\epsilon_0^2} \\ &= \frac{Q^2}{2A^2\epsilon_0} \end{aligned}$$

2.42

$$\begin{aligned} \mathbf{E}_{\text{above}} &= \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \\ \mathbf{f} &= \frac{1}{2}\sigma\mathbf{E}_{\text{above}} \\ &= \frac{1}{2}\frac{Q}{4\pi R^2}\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\hat{\mathbf{r}} \\ &= \frac{Q^2}{32\pi^2\epsilon_0 R^4}\hat{\mathbf{r}} \\ \mathbf{F} &= \int_0^{2\pi}\int_0^{\pi/2}\frac{Q^2}{32\pi^2\epsilon_0 R^4}\cos\theta R^2\sin\theta\, d\theta\, d\phi\hat{\mathbf{z}} \\ &= \frac{Q^2}{16\pi\epsilon_0 R^2}\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta\hat{\mathbf{z}} \\ &= \frac{Q^2}{32\pi\epsilon_0 R^2}\hat{\mathbf{z}} \end{aligned}$$

2.43

$$
\begin{aligned}
\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q}{\epsilon_0} \\
2\pi s L E &= \frac{Q}{\epsilon_0} \\
\mathbf{E} &= \frac{Q}{2\pi L \epsilon_0} \frac{1}{s} \hat{\mathbf{s}} \\
V &= -\int_b^a \frac{Q}{2\pi\epsilon_0 L} \frac{1}{s}\, dr \\
&= \frac{Q}{2\pi\epsilon_0 L} \ln \frac{b}{a} \\
C &= \frac{Q}{V} \\
&= \frac{2\pi\epsilon_0 L}{\ln b/a}
\end{aligned}
$$

So the capacitance per unit length is

$$C = \frac{2\pi\epsilon_0}{\ln b/a}.$$

2.44

(a)

$$
\begin{aligned}
P &= \frac{\epsilon_0}{2} E^2 \\
W &= Fd \\
&= PA\epsilon \\
&= \frac{\epsilon_0}{2} E^2 A\epsilon
\end{aligned}
$$

(b)

$$\frac{\epsilon_0}{2} E^2 A\epsilon$$

2.46

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 3\frac{k}{r}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{k}{r}2\sin\theta\cos\theta\sin\phi\right) \\
&\quad + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{k}{r}\sin\theta\cos\phi\right) \\
&= \frac{3k}{r^2} + \frac{1}{r\sin\theta}\frac{2k}{r}\sin\phi(2\sin\theta\cos^2\theta - \sin^3\theta) - \frac{1}{r\sin\theta}\frac{k}{r}\sin\theta\sin\phi \\
&= \frac{3k}{r^2} + \frac{2k\sin\phi}{r^2}(2\cos^2\theta - \sin^2\theta) - \frac{k}{r^2}\sin\phi \\
&= \frac{k}{r^2}[3 + 2\sin\phi(2\cos^2\theta - \sin^2\theta) - \sin\phi] \\
&= \frac{k}{r^2}[3 + \sin\phi(4\cos^2\theta - 2\sin^2\theta - 1)] \\
&= \frac{k}{r^2}[3 + \sin\phi(6\cos^2\theta - 3)] \\
&= \frac{3k}{r^2}(1 + \cos 2\theta\sin\phi) \\
\rho &= \epsilon_0\nabla\cdot\mathbf{E} \\
&= \frac{3k\epsilon_0}{r^2}(1 + \cos 2\theta\sin\phi)
\end{aligned}$$

2.47

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}\hat{\mathbf{r}} \\
\rho &= \frac{Q}{\frac{4}{3}\pi R^3} \\
\rho\mathbf{E} &= \frac{3Q}{4\pi R^3}\frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}\hat{\mathbf{r}} \\
&= \frac{3r}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2\hat{\mathbf{r}} \\
F_z &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^R \frac{3r}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 \cos\theta r^2\sin\theta\, dr\, d\theta\, d\phi \\
&= \frac{3\pi}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 \int_0^{\pi/2}\int_0^R r^3\sin 2\theta\, dr\, d\theta \\
&= \frac{3\pi}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 \frac{R^4}{4} \\
&= \frac{3Q^2}{64\pi\epsilon_0 R^2}
\end{aligned}$$

2.49

$$
\begin{aligned}
Q_{\text{enc}} &= \int_0^{2\pi} \int_0^{\pi} \int_0^{r} k r'^3 \sin\theta \, dr' \, d\theta \, d\phi \\
&= 2\pi k \int_0^{\pi} \int_0^{r} r'^3 \sin\theta \, dr' \, d\theta \\
&= \pi k r^4 \\
\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E &= \frac{\pi k r^4}{\epsilon_0} \\
\mathbf{E} &= \begin{cases} \frac{kr^2}{4\epsilon_0} \hat{\mathbf{r}} & r < R \\ \frac{kR^4}{4\epsilon_0 r^2} \hat{\mathbf{r}} & r > R \end{cases} \\
W &= \frac{\epsilon_0}{2} \left(\int_0^{2\pi} \int_0^{\pi} \int_0^{R} \frac{k^2 r^4}{16\epsilon_0^2} r^2 \sin\theta \, dr \, d\theta \, d\phi \right. \\
&\qquad \left. \int_0^{2\pi} \int_0^{\pi} \int_R^{\infty} \frac{k^2 R^8}{16\epsilon_0^2 r^4} r^2 \sin\theta \, dr \, d\theta \, d\phi \right) \\
&= \frac{\epsilon_0}{2} 2\pi \frac{k^2}{16\epsilon_0^2} \left(\int_0^{\pi} \int_0^{R} r^6 \sin\theta \, dr \, d\theta + \int_0^{\pi} \int_R^{\infty} \frac{R^8 \sin\theta}{r^2} \, dr \, d\theta \right) \\
&= \frac{\pi k^2}{16\epsilon_0} \left(\frac{2R^7}{7} + 2R^7 \right) \\
&= \frac{\pi k^2 R^7}{7\epsilon_0}
\end{aligned}
$$

2.50

$$
\begin{aligned}
V(\mathbf{r}) &= A \frac{e^{-\lambda r}}{r} \\
\mathbf{E} &= -\nabla V \\
&= A e^{-\lambda r} (1 + \lambda r) \frac{\hat{\mathbf{r}}}{r^2} \\
\rho &= \epsilon_0 \nabla \cdot \mathbf{E} \\
&= \epsilon_0 \left[A e^{-\lambda r} (1 + \lambda r) \nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} + \frac{\hat{\mathbf{r}}}{r^2} \cdot \nabla \left(A e^{-\lambda r} (1 + \lambda r) \right) \right] \\
&= A\epsilon_0 \left[4\pi \delta(\mathbf{r}) + \frac{\hat{\mathbf{r}}}{r^2} \cdot (-\lambda^2 e^{-\lambda r} r \hat{\mathbf{r}}) \right] \\
&= A\epsilon_0 \left(4\pi \delta(\mathbf{r}) - \frac{\lambda^2 e^{-\lambda r}}{r} \right)
\end{aligned}
$$

2.51

$$
\begin{aligned}
V &= \int \frac{1}{4\pi\epsilon_0} \frac{\sigma}{\mathscr{r}}\, dA \\
&= \frac{\sigma}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^{R} \frac{r}{\sqrt{r^2 + R^2 - 2rR\cos\theta}}\, dr\, d\theta \\
&= \frac{R\sigma}{4\pi\epsilon_0} \int_0^{2\pi} \left[\cos\theta \ln\left(1 + \csc\frac{\theta}{2}\right) + 2\sin\frac{\theta}{2} - 1\right] d\theta \\
&= \frac{R\sigma}{\pi\epsilon_0}
\end{aligned}
$$

2.52

(a)

$$
\begin{aligned}
V_- &= \frac{1}{2\pi\epsilon_0} \lambda \ln \frac{s_-}{a} \\
&= \frac{1}{2\pi\epsilon_0} \lambda \ln \frac{\sqrt{(y+a)^2 + z^2}}{a} \\
V_+ &= -\frac{1}{2\pi\epsilon_0} \lambda \ln \frac{s_+}{a} \\
&= -\frac{1}{2\pi\epsilon_0} \lambda \ln \frac{\sqrt{(y-a)^2 + z^2}}{a} \\
V &= V_- + V_+ \\
&= \frac{1}{4\pi\epsilon_0} \lambda \ln \frac{(y+a)^2 + z^2}{(y-a)^2 + z^2}
\end{aligned}
$$

2.53

(a)

$$
\begin{aligned}
\nabla^2 V &= -\frac{\rho}{\epsilon_0} \\
\nabla \cdot \nabla V &= -\frac{\rho}{\epsilon_0} \\
\nabla \cdot \frac{dV}{dx} \hat{\mathbf{x}} &= -\frac{\rho}{\epsilon_0} \\
\frac{d^2 V}{dx^2} &= -\frac{\rho}{\epsilon_0}
\end{aligned}
$$

(b)

$$
\begin{aligned}
qV &= \frac{1}{2}mv^2 \\
v &= \sqrt{\frac{2qV}{m}}
\end{aligned}
$$

(c)

$$
I = A\rho v
$$

(d)

$$
\begin{aligned}
\frac{d^2V}{dx^2} &= -\frac{I}{Av\epsilon_0} \\
&= -\frac{I}{A\epsilon_0}\sqrt{\frac{m}{2qV}} \\
&= \beta V^{-1/2}
\end{aligned}
$$

2.55

$$
\begin{aligned}
\rho &= \epsilon_0 \nabla \cdot \mathbf{E} \\
&= a\epsilon_0
\end{aligned}
$$

2.56

$$
\begin{aligned}
E &= \frac{3GM^2}{5R} \\
E_{\text{sun}} &= 2.3 \times 10^{41}\,\text{J} \\
t &= \frac{E_{\text{sun}}}{P} \\
&= 1.89 \times 10^{7}\,\text{years}
\end{aligned}
$$

3 Potentials

3.1

$$
\begin{aligned}
V_{\text{ave}} &= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\left.\sqrt{z^2+R^2-2zR\cos\theta}\right|_0^\pi \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\left(\sqrt{z^2+R^2+2zR}-\sqrt{z^2+R^2-2zR}\right) \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\left(\sqrt{(z+R)^2}-\sqrt{(R-z)^2}\right) \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}(z+R-R+z) \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{R}
\end{aligned}
$$

The average potential due to external charges is V_{center} and the average potential due to internal charges is

$$
\frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{R}
$$

so

$$
V_{\text{ave}} = V_{\text{center}} + \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{R}.
$$

3.2

A stable equilibrium is a minimum of potential energy. Laplace's equation doesn't allow for minimums, so they must be saddle points and the charge can escape.

3.3

$$
\begin{aligned}
0 &= \nabla^2 V \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) \\
&= \frac{1}{r^2}\left(2r\frac{\partial V}{\partial r} + r^2\frac{\partial^2 V}{\partial r^2}\right) \\
&= \frac{2}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} \\
V &= \frac{c_1}{r} + c_2 \\
0 &= \nabla^2 V \\
&= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) \\
&= \frac{1}{s}\left(\frac{\partial V}{\partial s} + s\frac{\partial^2 V}{\partial s^2}\right) \\
&= \frac{1}{s}\frac{\partial V}{\partial s} + \frac{\partial^2 V}{\partial s^2} \\
V &= c_1 + c_2 \ln s
\end{aligned}
$$

3.7

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}q^2\left(-\frac{2}{(2d)^2} + \frac{2}{(4d)^2} - \frac{1}{(6d)^2}\right)\hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0}\frac{29q^2}{72d^2}\hat{\mathbf{z}}
\end{aligned}
$$

3.8

(a)

$$
\begin{aligned}
V(r,\theta) &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} + \frac{q'}{\sqrt{r^2+b^2-2rb\cos\theta}}\right] \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{Rq/a}{\sqrt{r^2+(R^2/a)^2-2r(R^2/a)\cos\theta}}\right] \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{q}{\sqrt{R^2+(ra/R)^2-2ra\cos\theta}}\right]
\end{aligned}
$$

(b)

$$
\begin{aligned}
\sigma &= -\epsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=R} \\
&= \frac{q}{4\pi R} \frac{R^2 - a^2}{(a^2 + R^2 - 2aR\cos\theta)^{3/2}} \\
Q_{\text{induced}} &= \int_0^{2\pi} \int_0^{\pi} \sigma R^2 \sin\theta \, d\theta \, d\phi \\
&= \frac{qR(R^2 - a^2)}{2} \int_0^{\pi} \frac{\sin\theta}{(a^2 + R^2 - 2aR\cos\theta)^{3/2}} \, d\theta \\
&= \frac{qR(R^2 - a^2)}{a(a^2 - R^2)} \\
&= -\frac{R}{a} q \\
&= q'
\end{aligned}
$$

(c)

$$
\begin{aligned}
W &= \frac{1}{2} qV \\
&= \frac{1}{8\pi\epsilon_0} \frac{qq'}{a - b} \\
&= -\frac{1}{8\pi\epsilon_0} \frac{q^2 R/a}{a - R^2/a} \\
&= -\frac{1}{8\pi\epsilon_0} \frac{q^2 R}{a^2 - R^2}
\end{aligned}
$$

3.9

Place the second image charge at the centre of the sphere with charge

$$q'' = 4\pi\epsilon_0 R V_0.$$

$$
\begin{aligned}
F &= \frac{1}{4\pi\epsilon_0} q \left(\frac{q'}{(a-b)^2} + \frac{q''}{a^2} \right) \\
&= \frac{qq'}{4\pi\epsilon_0} \left(\frac{1}{(a-b)^2} - \frac{1}{a^2} \right) \\
&= \frac{qq'}{4\pi\epsilon_0} \frac{a^2 - (a-b)^2}{a^2(a-b)^2} \\
&= \frac{qq'}{4\pi\epsilon_0} \frac{b(2a-b)}{a^2(a-b)^2} \\
&= \frac{q(-Rq/a)}{4\pi\epsilon_0} \frac{(R^2/a)(2a - R^2/a)}{a^2(a - R^2/a)^2} \\
&= -\frac{q^2}{4\pi\epsilon_0} \left(\frac{R}{a} \right)^3 \frac{2a^2 - R^2}{(a^2 - R^2)^2}
\end{aligned}
$$

3.10

(a)

$$
V(x, y, z) = \frac{1}{4\pi\epsilon_0} \lambda \ln \frac{y^2 + (z+d)^2}{y^2 + (z-d)^2}
$$

(b)

$$
\begin{aligned}
\sigma &= -\epsilon_0 \left. \frac{\partial V}{\partial z} \right|_{z=0} \\
&= -\frac{d\lambda}{\pi(d^2 + y^2)}
\end{aligned}
$$

3.11

You need three charges: $-q$ at $(-a, b)$, $-q$ at $(a, -b)$, and q at $(-b, -a)$. The potential is

$$
\begin{aligned}
V = \frac{1}{4\pi\epsilon_0} q \Bigg(& \frac{1}{\sqrt{(x-a)^2 + (y-b)^2}} - \frac{1}{\sqrt{(x+a)^2 + (y-b)^2}} \\
& - \frac{1}{\sqrt{(x-a)^2 + (y+b)^2}} + \frac{1}{\sqrt{(x+a)^2 + (y+b)^2}} \Bigg).
\end{aligned}
$$

The force on q is

$$
\mathbf{F} = \frac{q^2}{16\pi\epsilon_0} \left[\left(\frac{a}{(a^2 + b^2)^{3/2}} - \frac{1}{a^2} \right) \hat{\mathbf{x}} + \left(\frac{b}{(a^2 + b^2)^{3/2}} - \frac{1}{b^2} \right) \hat{\mathbf{y}} \right].
$$

The work to bring q in from infinity is

$$W = \frac{q^2}{16\pi\epsilon_0}\left(\frac{1}{\sqrt{a^2+b^2}} - \frac{1}{a} - \frac{1}{b}\right).$$

3.12

Two infinitely long wires running parallel to the x-axis a distance $2a$ apart with charge densities λ and $-\lambda$ have cylindrical equipotential surfaces with centres at

$$y_0 = \pm a \coth \frac{2\pi\epsilon_0 V_0}{\lambda}$$

radii

$$R = a \operatorname{csch} \frac{2\pi\epsilon_0 V_0}{\lambda}.$$

We know the equipotential surfaces (the pipes) and want to find the wires so we can find the potential, so

$$\begin{aligned}
d &= a \coth \frac{2\pi\epsilon_0 V_0}{\lambda} \\
R &= a \operatorname{csch} \frac{2\pi\epsilon_0 V_0}{\lambda} \\
\frac{d}{R} &= \cosh \frac{2\pi\epsilon_0 V_0}{\lambda} \\
\operatorname{arcosh} \frac{d}{R} &= \frac{2\pi\epsilon_0 V_0}{\lambda} \\
\lambda &= \frac{2\pi\epsilon_0 V_0}{\operatorname{arcosh} d/R} \\
R &= a \operatorname{csch} \operatorname{arcosh} \frac{d}{R} \\
a &= \frac{R}{\operatorname{csch} \operatorname{arcosh} d/R} \\
&= (d+R)\sqrt{\frac{2d}{d+R} - 1} \\
&= \sqrt{d^2 - R^2}
\end{aligned}$$

thus the potential is

$$V = \frac{V_0}{2 \operatorname{arcosh} d/R} \ln \frac{(y + d^2 - R^2)^2 + z^2}{(y - d^2 + R^2)^2 + z^2}.$$

3.13

$$
\begin{aligned}
V_0(y) &= \begin{cases} V_0 & 0 \le y \le \frac{a}{2} \\ -V_0 & \frac{a}{2} \le y \le a \end{cases} \\
C_n &= \frac{2}{a}\left(\int_0^{a/2} V_0 \sin\frac{n\pi y}{a}\,dy - \int_{a/2}^{a} V_0 \sin\frac{n\pi y}{a}\,dy\right) \\
&= \frac{2V_0}{n\pi}\left(\cos\frac{n\pi y}{a}\Big|_{a/2}^{a} - \cos\frac{n\pi y}{a}\Big|_{0}^{a/2}\right) \\
&= \frac{2V_0}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2} - \cos\frac{n\pi}{2} + 1\right) \\
&= \frac{2V_0}{n\pi}\left(1 + (-1)^n - 2\cos\frac{n\pi}{2}\right) \\
&= \frac{2V_0}{n\pi}\begin{cases} 0 & n \text{ is odd or divisible by 4} \\ 4 & \text{otherwise} \end{cases} \\
V &= \frac{8V_0}{\pi}\sum_{n=2,6,10,\ldots}^{\infty} \frac{1}{n} e^{-n\pi x/a} \sin\frac{n\pi y}{a}
\end{aligned}
$$

3.14

$$
\begin{aligned}
\sigma &= -\epsilon_0 \frac{\partial V}{\partial x} \\
&= \frac{4\epsilon_0 V_0 \sin\frac{\pi y}{a}}{a\left(1 - \cos\frac{2\pi y}{a}\right)} \\
&= \frac{2\epsilon_0 V_0}{a}\frac{1}{\sin \pi y/a}
\end{aligned}
$$

3.15

(a)

$$\begin{aligned}
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} &= 0 \\
V(0,y) &= 0 \\
V(b,y) &= V_0(y) \\
V(x,0) &= 0 \\
V(x,a) &= 0 \\
V &= X(x)Y(y) \\
X''Y + XY'' &= 0 \\
\frac{X''}{X} + \frac{Y''}{Y} &= 0 \\
\frac{Y''}{Y} &= -\alpha^2 \\
Y'' + \alpha^2 Y &= 0 \\
Y &= c_1 \cos \alpha y + c_2 \sin \alpha y \\
Y &= c_2 \sin \alpha y \\
Y &= c_2 \sin \frac{n\pi y}{a}, n \in \mathbb{R} \\
\frac{X''}{X} &= \alpha^2 \\
X'' - \alpha^2 X &= 0 \\
X &= c_3 \cosh \alpha x + c_4 \sinh \alpha x \\
X &= c_4 \sinh \alpha x \\
&= c_4 \sinh \frac{n\pi x}{a}, n \in \mathbb{R} \\
V &= \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi x}{a} \sin \frac{n\pi y}{a} \\
V_0(y) &= V(b,y) \\
&= \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi y}{a} \\
C_n \sinh \frac{n\pi b}{a} &= \frac{2}{a} \int_0^a V_0(y) \sin \frac{n\pi y}{a} \, dy \\
C_n &= \frac{2}{a \sinh n\pi b/a} \int_0^a V_0(y) \sin \frac{n\pi y}{a} \, dy
\end{aligned}$$

(b)

$$
\begin{aligned}
C_n &= \frac{2V_0}{a\sinh n\pi b/a}\int_0^a \sin\frac{n\pi y}{a}\,dy \\
&= \frac{2V_0}{a\sinh n\pi b/a}\,\frac{a[1-(-1)^n]}{n\pi} \\
&= \frac{2V_0[1-(-1)^n]}{n\pi\sinh n\pi b/a} \\
V &= \frac{2V_0}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n\sinh n\pi b/a}\sinh\frac{n\pi x}{a}\sin\frac{n\pi y}{a}
\end{aligned}
$$

3.16

$$
\begin{aligned}
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} &= 0 \\
V(0, y, z) &= 0 \\
V(a, y, z) &= 0 \\
V(x, 0, z) &= 0 \\
V(x, a, z) &= 0 \\
V(x, y, 0) &= 0 \\
V(x, y, a) &= V_0 \\
V &= X(x)Y(y)Z(z) \\
X''YZ + XY''Z + XYZ'' &= 0 \\
\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} &= 0 \\
\frac{X''}{X} &= -\alpha^2 \\
\frac{Y''}{Y} &= -\beta^2 \\
\frac{Z''}{Z} &= \alpha^2 + \beta^2 \\
X'' + \alpha^2 X &= 0 \\
X &= c_1 \cos \alpha x + c_2 \sin \alpha x \\
X &= c_2 \sin \alpha x \\
X &= c_2 \sin \frac{n\pi x}{a}, n \in \mathbb{R} \\
\frac{Y''}{Y} &= -\beta^2 \\
Y'' + \beta^2 Y &= 0 \\
Y &= c_3 \cos \beta y + c_4 \sin \beta y \\
Y &= c_4 \sin \beta y \\
Y &= c_4 \sin \frac{m\pi y}{a}, m \in \mathbb{R} \\
\frac{Z''}{Z} &= \alpha^2 + \beta^2 \\
Z'' - (\alpha^2 + \beta^2)Z &= 0 \\
Z &= c_5 \cosh \sqrt{\alpha^2 + \beta^2} z + c_6 \sinh \sqrt{\alpha^2 + \beta^2} z \\
&= c_5 \cosh \pi \sqrt{(n/a)^2 + (m/a)^2} z \\
&\quad + c_6 \sinh \pi \sqrt{(n/a)^2 + (m/a)^2} z \\
Z &= c_6 \sinh \pi \sqrt{(n/a)^2 + (m/a)^2} z
\end{aligned}
$$

$$
\begin{aligned}
V &= \sum_{n=1}^{\infty} \sum_{m=1^{\infty}} C_{n,m} \sinh\left(\pi\sqrt{(n/a)^2 + (m/a)^2} z\right) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a} \\
V_0 &= V(x, y, a) \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{n,m} \sinh\left(\pi\sqrt{n^2 + m^2}\right) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a} \\
C_{n,m} &= \frac{4V_0}{a^2 \sinh\left(\pi\sqrt{n^2 + m^2}\right)} \int_0^a \int_0^a \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a}\, dy\, dx \\
&= \frac{4V_0}{a^2 \sinh\left(\pi\sqrt{n^2 + m^2}\right)} \frac{a^2[-1 + (-1)^m][-1 + (-1)^n]}{nm\pi^2} \\
&= \frac{4V_0[-1 + (-1)^n][-1 + (-1)^m]}{nm\pi^2 \sinh\left(\pi\sqrt{n^2 + m^2}\right)} \\
&= \begin{cases} 0 & n \text{ even or } m \text{ even} \\ \frac{16V_0}{nm\pi^2 \sinh\left(\pi\sqrt{n^2+m^2}\right)} & \text{otherwise} \end{cases} \\
V &= \frac{16V_0}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{nm} \frac{\sinh\left(\pi\sqrt{n^2 + m^2} z/a\right)}{\sinh\left(\pi\sqrt{n^2 + m^2}\right)} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a}
\end{aligned}
$$

3.17

$$
\begin{aligned}
P_3(x) &= \frac{1}{2^3 3!}\left(\frac{d}{dx}\right)^3 (x^2-1)^3 \\
&= \frac{1}{48}\frac{d^3}{dx^3}\left[(x^2-1)^3\right] \\
&= \frac{1}{48}\frac{d^2}{dx^2}\left[6x(x^2-1)^2\right] \\
&= \frac{1}{48}\frac{d}{dx}\left[6(x^2-1)^2 + 24x^2(x^2-1)\right] \\
&= \frac{1}{48}\left[24x(x^2-1) + 48x(x^2-1) + 48x^3\right] \\
&= \frac{1}{48}\left(24x^3 - 24x + 48x^3 - 48x + 48x^3\right) \\
&= \frac{120}{48}x^3 - \frac{72}{48}x \\
&= \frac{5}{2}x^3 - \frac{3}{2}x \\
\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) &= -12\sin\theta\Theta \\
\Theta &= \frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta \\
\frac{d\Theta}{d\theta} &= -\frac{15}{2}\cos^2\theta\sin\theta + \frac{3}{2}\sin\theta \\
\sin\theta\frac{d\Theta}{d\theta} &= -\frac{15}{2}\cos^2\theta\sin^2\theta + \frac{3}{2}\sin^2\theta \\
\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) &= \frac{3}{2}(1-5\cos 2\theta)\sin 2\theta \\
\frac{3}{2}(1-5\cos 2\theta)\sin 2\theta &= -12\sin\theta\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) \\
3(1-5\cos 2\theta)\sin 2\theta &= 12\sin\theta\cos\theta(3-5\cos^2\theta) \\
&= 6(3-5\cos^2\theta)\sin 2\theta \\
&= 6\left(3 - 5\frac{1+\cos 2\theta}{2}\right)\sin 2\theta \\
&= 3(6-5-5\cos 2\theta)\sin 2\theta \\
&= 3(1-5\cos 2\theta)\sin 2\theta \\
\int_{-1}^{1} P_1(x)P_3(x)\,dx &= \int_{-1}^{1} x\left(\frac{5}{2}x^3 - \frac{3}{2}x\right)\,dx \\
&= \left[\frac{1}{2}x^5 - \frac{1}{2}x^3\right]_{-1}^{1} \\
&= \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \\
&= 0
\end{aligned}
$$

3.18

(a)

$$\begin{aligned}
A_l &= \frac{V_0(2l+1)}{2R^l}\int_0^\pi P_l(\cos\theta)\sin\theta\,d\theta \\
&= \begin{cases} V_0 & l = 0 \\ 0 & l \neq 0 \end{cases} \\
V(r,\theta) &= \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \\
&= V_0 \\
B_l &= \frac{V_0(2l+1)}{2} R^{l+1}\int_0^\pi P_l(\cos\theta)\sin\theta\,d\theta \\
&= \begin{cases} V_0 R & l = 0 \\ 0 & l \neq 0 \end{cases} \\
V(r,\theta) &= \frac{V_0 R}{r}
\end{aligned}$$

(b)

$$\begin{aligned}
A_l &= \frac{\sigma_0}{2\epsilon_0 R^{l-1}}\int_0^\pi P_l(\cos\theta)\sin\theta\,d\theta \\
&= \begin{cases} \frac{R\sigma_0}{\epsilon_0} & l = 0 \\ 0 & l \neq 0 \end{cases} \\
V(r,\theta) &= \frac{R\sigma_0}{\epsilon_0} \\
B_l &= A_l R^{2l+1} \\
&= \begin{cases} \frac{R^2\sigma_0}{\epsilon_0} & l = 0 \\ 0 & l \neq 0 \end{cases} \\
V(r,\theta) &= \frac{R^2\sigma_0}{\epsilon_0 r}
\end{aligned}$$

3.19

$$
\begin{aligned}
V_0 &= k\cos 3\theta \\
A_l &= \frac{k(2l+1)}{2R^l}\int_0^\pi \cos 3\theta P_l(\cos\theta)\sin\theta\,d\theta \\
&= \begin{cases} -\frac{3k}{5R} & l=1 \\ \frac{8k}{5R^3} & l=3 \\ 0 & \text{otherwise} \end{cases} \\
V(r,\theta) &= -\frac{3k}{5R} r P_1(\cos\theta) + \frac{8k}{5R^3} r^3 P_3(\cos\theta) \\
&= \frac{kr}{5R}\left[-3P_1(\cos\theta) + \frac{8}{R^2} r^2 P_3(\cos\theta)\right] \\
B_l &= \frac{k(2l+1)}{2} R^{l+1}\int_0^\pi \cos 3\theta P_l(\cos\theta)\sin\theta\,d\theta \\
&= \begin{cases} -\frac{3kR^2}{5} & l=1 \\ \frac{8kR^4}{5} & l=3 \\ 0 & \text{otherwise} \end{cases} \\
V(r,\theta) &= -\frac{3kR^2}{5r^2} P_1(\cos\theta) + \frac{8kR^4}{5r^4} P_3(\cos\theta) \\
&= \frac{kR^2}{5r^2}\left[\frac{8R^2}{r^2} P_3(\cos\theta) - 3P_1(\cos\theta)\right] \\
\sigma(\theta) &= -\epsilon_0\left(\frac{\partial V_{\text{above}}}{\partial r} - \frac{\partial V_{\text{below}}}{\partial r}\right) \\
&= \frac{\epsilon_0 k(12\cos\theta + 35\cos 3\theta)}{5R}
\end{aligned}
$$

3.20

$$
\begin{aligned}
V(r,\theta) &= \begin{cases} \sum_{l=0}^\infty \frac{2l+1}{2}\frac{r^l}{R^l}\left(\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta\right) P_l(\cos\theta) & r <= R \\ \sum_{l=0}^\infty \frac{2l+1}{2}\frac{R^{l+1}}{r^{l+1}}\left(\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta\right) P_l(\cos\theta) & r >= R \end{cases} \\
\sigma_0 &= -\epsilon_0\left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\right)\bigg|_{r=R} \\
&= \frac{\epsilon_0}{2R}\sum_{l=0}^\infty (2l+1)^2 C_l P_l(\cos\theta)
\end{aligned}
$$

3.21

$$
V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} - E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta
$$

3.22

(a)

$$\begin{aligned}
\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} &= \frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2}-r\right) \\
&= \frac{\sigma r}{2\epsilon_0}\left(\sqrt{1+(R/r)^2}-1\right) \\
&= \frac{\sigma r}{2\epsilon_0}\left[\left(1+\frac{(R/r)^2}{2}-\frac{(R/r)^4}{8}+\dots\right)-1\right] \\
&= \frac{\sigma}{2\epsilon_0}\left(\frac{R^2}{2r}-\frac{R^4}{8r^3}+\dots\right) \\
B_0 &= \frac{\sigma R^2}{4\epsilon_0} \\
B_1 &= 0 \\
B_2 &= -\frac{\sigma R^4}{16\epsilon_0}
\end{aligned}$$

(b)

$$\begin{aligned}
\sum_{l=0}^{\infty} A_l r^l &= \frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2}-r\right) \\
&= \frac{\sigma}{2\epsilon_0}\left(R\sqrt{1+(r/R)^2}-r\right) \\
&= \frac{\sigma}{2\epsilon_0}\left[R\left(1+\frac{(r/R)^2}{2}-\frac{(r/R)^4}{8}+\dots\right)-r\right] \\
&= \frac{\sigma}{2\epsilon_0}\left(R-r+\frac{r^2}{2R}-\frac{r^4}{8R^3}+\dots\right) \\
A_0 &= \frac{\sigma R}{2\epsilon_0} \\
A_1 &= -\frac{\sigma}{2\epsilon_0} \\
A_2 &= \frac{\sigma}{4\epsilon_0 R} \\
A_0' &= A_0 \\
A_1' &= -A_1 \\
A_2' &= A_2
\end{aligned}$$

3.23

$$
\begin{aligned}
V(r,\theta) &= \sum_{l=1}^{\infty} A_l r^l P_l(\cos\theta) \\
A_l &= \frac{\sigma_0}{2\epsilon_0 R^{l-1}} \left(\int_0^{\pi/2} P_l(\cos\theta)\sin\theta\, d\theta - \int_{\pi/2}^{\pi} P_l(\cos\theta)\sin\theta\, d\theta \right) \\
A_0 &= 0 \\
A_1 &= \frac{\sigma_0}{2\epsilon_0} \\
A_2 &= 0 \\
A_3 &= -\frac{\sigma_0}{8\epsilon_0 R^2} \\
A_4 &= 0 \\
A_5 &= \frac{\sigma_0}{16\epsilon_0 R^4} \\
A_6 &= 0 \\
V(r,\theta) &= \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \\
B_l &= A_l R^{2l+1} \\
B_0 &= 0 \\
B_1 &= \frac{\sigma_0 R^3}{2\epsilon_0} \\
B_2 &= 0 \\
B_3 &= -\frac{\sigma_0 R^5}{8\epsilon_0} \\
B_4 &= 0 \\
B_5 &= \frac{\sigma_0 R^7}{16\epsilon_0} \\
B_6 &= 0
\end{aligned}
$$

3.24

$$
\begin{aligned}
0 &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \\
&= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} \\
V(s,\phi) &= S(s)\Phi(\phi) \\
0 &= \frac{1}{s}\frac{\partial}{\partial s}(sS')\Phi + \frac{1}{s^2}S\Phi'' \\
&= \frac{1}{s}(S' + sS'')\Phi + \frac{1}{s^2}S\Phi'' \\
&= \frac{S'}{sS} + \frac{S''}{S} + \frac{\Phi''}{s^2\Phi} \\
&= \frac{s^2S'' + sS'}{S} + \frac{\Phi''}{\Phi} \\
\frac{\Phi''}{\Phi} &= 0 \\
\Phi'' &= 0 \\
\Phi &= c_1 + c_2\phi \\
\frac{\Phi''}{\Phi} &= -n^2 \\
\Phi'' + \alpha^2\Phi &= 0 \\
\Phi &= c_3\cos\alpha\phi + c_4\sin\alpha\phi \\
\Phi(0) &= \Phi(2\pi) \\
c_1 &= c_3\cos 2\pi\alpha + c_4\sin 2\pi\alpha \\
\alpha &= n, n \in \mathbb{R} \\
\Phi &= c_3\cos n\phi + c_4\sin n\pi \\
\frac{s^2S'' + sS'}{S} &= 0 \\
s^2S'' + sS' &= 0 \\
sS'' + S' &= 0 \\
S &= c_5 + c_6\ln s
\end{aligned}
$$

$$
\begin{aligned}
\frac{s^2S'' + sS'}{S} &= n^2 \\
s^2S'' + sS' - n^2S &= 0 \\
S &= s^m \\
S' &= ms^{m-1} \\
S'' &= m(m-1)s^{m-2} \\
m(m-1)s^m + ms^m - n^2s^m &= 0 \\
m^2 - m + m - n^2 &= 0 \\
m^2 - n^2 &= 0 \\
(m+n)(m-n) &= 0 \\
S &= c_7s^n + c_8s^{-n} \\
V &= S(s)\Phi(\phi) \\
&= (c_1 + c_2\phi)(c_5 + c_6 \ln s) \\
&\quad + \sum_{n=1}^{\infty}(c_7s^n + c_8s^{-n})(c_3\cos n\phi + c_4 \sin n\phi) \\
&= c_1 + c_2 \ln s \\
&\quad + \sum_{n=1}^{\infty}[s^n(A_n\cos n\phi + B_n \sin n\phi) + s^{-n}(C_n \cos n\phi + D_n \sin n\phi)]
\end{aligned}
$$

3.25

$$
\begin{aligned}
V &= 0 \text{ at } s = R \\
V &\to -E_0 s\cos\phi \text{ as } s \to \infty \\
V &= \left(A_1 s + \frac{C_1}{s}\right)\cos\phi \\
0 &= A_1R + \frac{C_1}{R} \\
C_2 &= -A_1R^2 \\
A_1 &= -E_0 \\
V &= E_0 s\left(\frac{R^2}{s^2} - 1\right)\cos\phi \\
\sigma &= -\epsilon_0\left(\frac{\partial V_{\text{out}}}{\partial s} - \frac{\partial V_{\text{in}}}{\partial s}\right)\bigg|_{s=R} \\
&= -\epsilon_0 \left.\frac{\partial V_{\text{out}}}{\partial s}\right|_{s=R} \\
&= 2\epsilon_0 E_0 \cos\phi
\end{aligned}
$$

3.27

$$
\begin{aligned}
V(z) &= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{z^{(n+1)}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')\,d\tau' \\
&= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{z^{(n+1)}}\int_0^R\int_0^\pi\int_0^{2\pi}(r')^n P_n(\cos\theta)k\frac{R}{(r')^2}(R-2r')\sin\theta(r')^2\sin\theta\,dr'\,d\theta\,d\phi \\
&= \frac{kR}{2\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{z^{(n+1)}}\int_0^R\int_0^\pi (r')^n(R-2r')\sin^2\theta P_n(\cos\theta)\,dr'\,d\theta \\
&\approx \frac{1}{4\pi\epsilon_0}\frac{k\pi^2R^5}{48z^3}
\end{aligned}
$$

3.28

$$
\begin{aligned}
V(r,\theta) &= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')\,d\tau' \\
&= \frac{\lambda R}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int_0^{2\pi} R^n P_n(\sin\phi\sin\theta)\,d\phi \\
V_0 &= \frac{\lambda R}{4\pi\epsilon_0}\frac{2\pi}{r} \\
&= \frac{\lambda R}{2\epsilon_0 r} \\
V_1 &= 0 \\
V_2 &= -\frac{\lambda R}{4\pi\epsilon_0}\frac{1}{4r^3}\pi R^2(1+3\cos 2\theta) \\
&= -\frac{\lambda R^3}{8\epsilon_0 r^3}(3\cos^2\theta - 1)
\end{aligned}
$$

3.29

$$
\begin{aligned}
\mathbf{p} &= \sum^{n} q_i\mathbf{r}_i' \\
&= 2aq\hat{\mathbf{z}} \\
V_{\text{dip}}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\frac{2aq\cos\theta}{r^2}
\end{aligned}
$$

3.30

(a)

$$\begin{aligned}
\sigma &= k\cos\theta \\
\mathbf{p} &= \int \mathbf{r}'\,\rho(\mathbf{r}')\,d\tau' \\
&= \int_0^{2\pi}\int_0^{\pi} R(\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}})k\cos\theta R^2\sin\theta\,d\theta\,d\phi \\
&= \frac{1}{2}kR^3\int_0^{2\pi}\int_0^{\pi}(\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}})\sin 2\theta\,d\theta\,d\phi \\
&= \frac{4}{3}\pi R^3 k\hat{\mathbf{z}}
\end{aligned}$$

(b)

$$\begin{aligned}
V_{\text{dip}}(\mathbf{r}) &\approx \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \\
&= \frac{1}{4\pi\epsilon_0}\frac{4\pi R^3 k\cos\theta}{3r^2} \\
&= \frac{kR^3}{3\epsilon_0}\frac{1}{r^2}\cos\theta \\
V_{\text{dip}}(r,\theta) &= \frac{kR^3}{3\epsilon_0}\frac{1}{r^2}\cos\theta
\end{aligned}$$

Higher multipoles are all 0.

3.32

(a)

$$\begin{aligned}
Q &= 2q \\
\mathbf{p} &= 3aq\hat{\mathbf{z}} \\
V &\approx \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{r} + \frac{3aq\cos\theta}{r^2}\right)
\end{aligned}$$

(b)

$$\begin{aligned}
Q &= 2q \\
\mathbf{p} &= aq\hat{\mathbf{z}} \\
V &\approx \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{r} + \frac{aq\cos\theta}{r^2}\right)
\end{aligned}$$

(c)

$$
\begin{aligned}
Q &= 2q \\
\mathbf{p} &= 3aq\hat{\mathbf{y}} \\
V &\approx \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{r} + \frac{3aq\sin\theta\sin\phi}{r^2}\right)
\end{aligned}
$$

3.33

(a)

$$
\begin{aligned}
\mathbf{E} &= -\frac{1}{4\pi\epsilon_0}\frac{p}{a^3}\hat{\mathbf{z}} \\
\mathbf{F} &= -\frac{1}{4\pi\epsilon_0}\frac{pq}{a^3}\hat{\mathbf{z}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{2p}{a^3}\hat{\mathbf{z}} \\
\mathbf{F} &= \frac{1}{4\pi\epsilon_0}\frac{2pq}{a^3}\hat{\mathbf{z}}
\end{aligned}
$$

(c)

$$
\begin{aligned}
W &= \int \mathbf{F}\cdot d\mathbf{l} \\
&= \int_0^{\pi/2} aq\mathbf{E}\cdot d\boldsymbol{\theta} \\
&= \frac{1}{4\pi\epsilon_0}\frac{pq}{a^2}\int_0^{\pi/2}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})\cdot d\boldsymbol{\theta} \\
&= \frac{1}{4\pi\epsilon_0}\frac{pq}{a^2}\int_0^{\pi/2}\sin\theta\, d\theta \\
&= \frac{1}{4\pi\epsilon_0}\frac{pq}{a^2}
\end{aligned}
$$

3.34

$$
\begin{aligned}
Q &= -q \\
\mathbf{p} &= qa\hat{\mathbf{z}} \\
V &= \frac{1}{4\pi\epsilon_0}q\left(-\frac{1}{r} + \frac{a\cos\theta}{r^2}\right) \\
\mathbf{E} &= -\nabla V \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}[(2a\cos\theta - r)\hat{\mathbf{r}} + a\sin\theta\hat{\boldsymbol{\theta}}]
\end{aligned}
$$

3.35

$$
\begin{aligned}
\mathbf{p} &= \int \mathbf{r}' \rho(\mathbf{r}')\, d\tau' \\
&= \left(\int_0^{2\pi} \int_0^{\pi/2} \int_0^R r\cos\theta \rho_0 r^2 \sin\theta\, dr\, d\theta\, d\phi \right. \\
&\qquad \left. - \int_0^{2\pi} \int_{\pi/2}^{\pi} \int_0^R r\cos\theta \rho_0 r^2 \sin\theta\, dr\, d\theta\, d\phi \right) \hat{\mathbf{z}} \\
&= \pi\rho_0 \left(\int_0^{\pi/2} \int_0^R r^3 \sin 2\theta\, dr\, d\theta - \int_{\pi/2}^{\pi} \int_0^R r^3 \sin 2\theta\, dr\, d\theta \right) \hat{\mathbf{z}} \\
&= \frac{1}{2}\pi\rho_0 R^4 \hat{\mathbf{z}} \\
\mathbf{E}_{\text{dip}}(r,\theta) &= \frac{1}{4\pi\epsilon_0} \frac{\pi\rho_0 R^4}{2r^3} (2\cos\theta \hat{\mathbf{r}} + \sin\theta \hat{\boldsymbol{\theta}})
\end{aligned}
$$

3.36

The factor of $1/4\pi\epsilon_0 r^3$ is the common, so the goal is to show that

$$
\begin{aligned}
p(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) &= 2(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} + p\sin\theta\hat{\boldsymbol{\theta}} \\
&= 2(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - (\mathbf{p}\cdot\hat{\boldsymbol{\theta}})\hat{\boldsymbol{\theta}} \\
&= 3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{p}}.
\end{aligned}
$$

3.37

$$
\begin{aligned}
V_{\text{ave}} &= \frac{1}{4\pi R^2} \oint V\, da \\
\frac{dV_{\text{ave}}}{dR} &= \frac{1}{4\pi R^2} \oint \nabla V \cdot d\mathbf{a} \\
&= \frac{1}{4\pi R^2} \int \nabla^2 V\, d\tau \\
&= 0
\end{aligned}
$$

3.38

$$
\begin{aligned}
E_{qz} &= \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\cos\theta \\
&= \frac{1}{4\pi\epsilon_0}\frac{qd}{(x^2+y^2+d^2)^{3/2}} \\
0 &= E_{qz} + E_{\sigma z} \\
&= \frac{1}{4\pi\epsilon_0}\frac{qd}{(x^2+y^2+d^2)^{3/2}} - \frac{\sigma}{2\epsilon_0} \\
\sigma &= \frac{qd}{2\pi(x^2+y^2+d^2)^{3/2}}
\end{aligned}
$$

3.39

$$
E = \frac{q^2}{4\pi\epsilon_0}\left[\left(\sum_{n=1}^{\infty}\frac{1}{(2an-2x)^2}\right) - \left(\sum_{n=0}^{\infty}\frac{1}{(2an+2x)^2}\right)\right]
$$

3.40

Set $V = 0$ at $x = 0$. The cylinder is a conductor and is thus an equipotential, so $V = 0$ at the surface. Place two infinite line charges within the cylinder at $x = \pm R^2/a$, giving

$$
\begin{aligned}
V = \frac{\lambda}{2\pi\epsilon_0}\Bigg(&\ln\frac{a}{\sqrt{s^2+a^2-2sa\cos\phi}} - \ln\frac{a}{\sqrt{s^2+a^2+2sa\cos\phi}} \\
&+ \ln\frac{R^2/a}{\sqrt{s^2+(R^2/a)^2+2s(R^2/a)\cos\phi}} \\
&- \ln\frac{R^2/a}{\sqrt{s^2+(R^2/a)^2-2s(R^2/a)\cos\phi}}\Bigg) \\
= \frac{\lambda}{4\pi\epsilon_0}&\left(\ln\frac{s^2+a^2+2sa\cos\phi}{s^2+a^2-2sa\cos\phi} + \ln\frac{(sa/R)^2+R^2-2sa\cos\phi}{(sa/R)^2+R^2+2sa\cos\phi}\right) \\
= \frac{\lambda}{4\pi\epsilon_0}&\ln\frac{(s^2+a^2+2sa\cos\phi)[(sa/R)^2+R^2-2sa\cos\phi]}{(s^2+a^2-2sa\cos\phi)[(sa/R)^2+R^2+2sa\cos\phi]}
\end{aligned}
$$

3.41

(a) For a sphere of charge q, $q' + q'' = q \Rightarrow q'' = q - q'$ so

$$\begin{aligned}
F &= \frac{q}{4\pi\epsilon_0}\left(\frac{q''}{a^2} + \frac{q'}{(a-b)^2}\right) \\
&= \frac{q}{4\pi\epsilon_0}\left(\frac{q}{a^2} - \frac{q'}{a^2} + \frac{q'}{(a-b)^2}\right) \\
&= \frac{q^2}{4\pi\epsilon_0 a^3}\left[a - R^3\,\frac{2a^2 - R^2}{(a^2 - R^2)^2}\right]
\end{aligned}$$

and solving for $F = 0$ gives $r = 5.663\,12\,\text{Å}$.

3.43

(a)

$$
\begin{aligned}
\lim_{r\to\infty} V_{\text{above}}(r,\theta) &= 0 \\
V_{\text{below}}(a,\theta) &= V_0 \\
V_{\text{above}}(b,\theta) &= V_{\text{below}}(b,\theta) \\
\left.\frac{\partial V_{\text{above}}}{\partial r}\right|_{r=b} - \left.\frac{\partial V_{\text{below}}}{\partial r}\right|_{r=b} &= -\frac{k\cos\theta}{\epsilon_0} \\
V_{\text{above}}(r,\theta) &= \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \\
\left.\frac{\partial V_{\text{above}}}{\partial r}\right|_{r=b} &= \sum_{l=0}^{\infty} -(l+1)\frac{B_l}{b^{l+2}} P_l(\cos\theta) \\
V_{\text{below}}(r,\theta) &= \sum_{l=0}^{\infty} \left(C_l r^l + \frac{D_l}{r^{l+1}}\right) P_l(\cos\theta) \\
V_0 &= V_{\text{below}}(a,\theta) \\
&= \sum_{l=0}^{\infty} \left(C_l a^l + \frac{D_l}{a^{l+1}}\right) P_l(\cos\theta) \\
V_0 &= C_0 + \frac{D_0}{a} \\
0 &= C_l a^l + \frac{D_l}{a^{l+1}}, l \neq 0 \\
\left.\frac{\partial V_{\text{below}}}{\partial r}\right|_{r=b} &= \sum_{l=0}^{\infty} \left(C_l l b^{l-1} - (l+1)\frac{D_l}{b^{l+2}}\right) P_l(\cos\theta) \\
V_{\text{above}}(b,\theta) &= V_{\text{below}}(b,\theta) \\
\sum_{l=0}^{\infty} \frac{B_l}{b^{l+1}} P_l(\cos\theta) &= \sum_{l=0}^{\infty} \left(C_l b^l + \frac{D_l}{b^{l+1}}\right) P_l(\cos\theta) \\
\frac{B_l}{b^{l+1}} &= C_l b^l + \frac{D_l}{b^{l+1}} \\
-\frac{k\cos\theta}{\epsilon_0} &= \sum_{l=0}^{\infty} \left[-(l+1)\frac{B_l}{b^{l+2}} - C_l l b^{l-1} + (l+1)\frac{D_l}{b^{l+2}}\right] P_l(\cos\theta) \\
-\frac{k}{\epsilon_0} &= -2\frac{B_1}{b^3} - C_1 + 2\frac{D_1}{b^3} \\
0 &= -(l+1)\frac{B_l}{b^{l+2}} - C_l l b^{l-1} + (l+1)\frac{D_l}{b^{l+2}}, l \neq 1 \\
B_0 &= aV_0 \\
C_0 &= 0 \\
D_0 &= aV_0
\end{aligned}
$$

$$
\begin{aligned}
B_1 &= \frac{(b^3 - a^3)k}{3\epsilon_0} \\
C_1 &= \frac{k}{3\epsilon_0} \\
D_1 &= -\frac{a^3 k}{3\epsilon_0} \\
B_l &= 0 \\
C_l &= 0 \\
D_l &= 0 \\
V &= \begin{cases} \frac{aV_0}{r} + \frac{(r^3 - a^3)k\cos\theta}{3\epsilon_0 r^2} & a \le r \le b \\ \frac{aV_0}{r} + \frac{(b^3 - a^3)k\cos\theta}{3\epsilon_0 r^2} & r \ge b \end{cases}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\sigma &= -\epsilon_0 \left. \frac{\partial V_{\text{below}}}{\partial r} \right|_{r=a} \\
&= \frac{\epsilon_0 V_0}{a} - k\cos\theta
\end{aligned}
$$

(c)

$$
\begin{aligned}
Q &= \oint \sigma_i \, dA \\
&= \int_0^{2\pi} \int_0^{\pi} \left(\frac{\epsilon_0 V_0}{a} - k\cos\theta \right) a^2 \sin\theta \, d\theta \, d\phi \\
&= 2\pi \int_0^{\pi} \left(a\epsilon_0 V_0 \sin\theta - \frac{1}{2} a^2 k \sin 2\theta \right) \, d\theta \\
&= 4\pi\epsilon_0 a V_0 \\
V &\approx \frac{aV_0}{r} \\
\frac{1}{4\pi\epsilon_0} \frac{Q}{r} &= \frac{aV_0}{r} \\
Q &= 4\pi\epsilon_0 a V_0
\end{aligned}
$$

3.44

$$
\begin{aligned}
V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')\,d\tau' \\
&= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int_{-a}^{a} z^n P_n(\cos\theta)\frac{Q}{2a}\,dz \\
&= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{Q}{2ar^{(n+1)}}P_n(\cos\theta)\left[\frac{1}{n+1}z^{n+1}\right]_{-a}^{a} \\
&= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{Q}{2a(n+1)r^{(n+1)}}P_n(\cos\theta)[a^{n+1}-(-1)^{n+1}a^{n+1}] \\
&= \frac{Q}{4\pi\epsilon_0}\frac{1}{r}\left[1+\frac{1}{3}\left(\frac{a}{r}\right)^2 P_2(\cos\theta)+\frac{1}{5}\left(\frac{a}{r}\right)^4 P_4(\cos\theta)+\dots\right]
\end{aligned}
$$

3.45

$$
V = a_0 + b_0\ln s + \sum_{k=1}^{\infty}[s^k(a_k\cos k\phi + b_k\sin k\phi) + s^{-k}(c_k\cos k\phi + d_k\sin k\phi)]
$$

$$
\begin{aligned}
\lim_{s\to\infty} V_{\text{above}}(s,\phi) &= 0 \\
V_{\text{above}}(R,\phi) &= V_{\text{below}}(R,\phi) \\
&= \begin{cases} -\sigma_0/\epsilon_0 & 0 \le \phi \le \pi \\ \sigma_0/\epsilon_0 & \pi \le \phi \le 2\pi \end{cases} \\
\left.\frac{\partial V_{\text{above}}}{\partial s}\right|_{s=R} - \left.\frac{\partial V}{\partial s}\right|_{s=R} &= -\frac{1}{\epsilon_0}\sigma \\
V_{\text{above}}(s,\phi) &= \sum_{k=1}^{\infty} s^{-k}(c_k \cos k\phi + d_k \sin k\phi) \\
\left.\frac{\partial V_{\text{above}}}{\partial s}\right|_{s=R} &= \sum_{k=1}^{\infty} -kR^{-(k+1)}(c_k \cos k\phi + d_k \sin k\phi) \\
V_{\text{below}}(s,\phi) &= e_0 + \sum_{k=1}^{\infty} s^{k}(e_k \cos k\phi + f_k \sin k\phi) \\
\left.\frac{\partial V_{\text{below}}}{\partial s}\right|_{s=R} &= \sum_{k=1}^{\infty} kR^{k-1}(e_k \cos k\phi + f_k \sin k\phi) \\
\sum_{k=1}^{\infty} R^{-k}(c_k \cos k\phi + d_k \sin k\phi) &= e_0 + \sum_{k=1}^{\infty} R^{k}(e_k \cos k\phi + f_k \sin k\phi) \\
e_0 &= 0 \\
R^{-k}c_k &= R^k e_k \\
R^{-k}d_k &= R^k f_k
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\sigma}{\epsilon_0} &= \sum_{k=1}^{\infty} \left[-kR^{-(k+1)}(c_k \cos k\phi + d_k \sin k\phi) - kR^{k-1}(e_k \cos k\phi + f_k \sin k\phi)\right] \\
&= \sum_{k=1}^{\infty} -k \left[\left(R^{-(k+1)}c_k + R^{k-1}e_k\right)\cos k\phi + \left(R^{-(k+1)}d_k + R^{k-1}f_k\right)\sin k\phi\right] \\
\frac{1}{\pi}\int_0^{2\pi} -\frac{\sigma}{\epsilon_0}\cos k\phi \, d\phi &= -k(R^{-(k+1)}c_k + R^{k-1}e_k) \\
\frac{\sigma_0(\sin 2k\pi - 2\sin k\pi)}{k\pi\epsilon_0} &= -k(R^{-(k+1)}c_k + R^{k-1}e_k) \\
\frac{1}{\pi}\int_0^{2\pi} -\frac{\sigma}{\epsilon_0}\sin k\phi \, d\phi &= -k(R^{-(k+1)}d_k + R^{k-1}f_k) \\
\frac{4\sigma_0 \cos k\pi \sin^2 k\pi/2}{k\pi\epsilon_0} &= -k(R^{-(k+1)}d_k + R^{k-1}f_k)
\end{aligned}
$$

$$
\begin{aligned}
c_k &= 0 \\
d_k &= \frac{2R^{k+1}\sigma_0 \cos k\pi \sin^2 k\pi/2}{k^2\pi\epsilon_0} \\
e_k &= 0 \\
f_k &= -\frac{2R^{-(k-1)}\sigma_0 \cos k\pi \sin^2 k\pi/2}{k^2\pi\epsilon_0} \\
V_{\text{above}} &= \frac{2\sigma_0}{\pi\epsilon_0}\sum_{k=1}^{\infty} s^{-k}\frac{R^{k+1}\cos k\pi \sin^2 k\pi/2}{k^2}\sin k\phi \\
&= -\frac{2\sigma_0}{\pi\epsilon_0}\sum_{k=1,3,5,\ldots}^{\infty}\frac{1}{k^2}R^{k+1}s^{-k}\sin k\phi \\
V_{\text{below}} &= -\frac{2\sigma_0}{\pi\epsilon_0}\sum_{k=1}^{\infty} s^{k}\frac{R^{-(k-1)}\cos k\pi \sin^2 k\pi/2}{k^2}\sin k\phi \\
&= \frac{2\sigma_0}{\pi\epsilon_0}\sum_{k=1,3,5,\ldots}^{\infty}\frac{1}{k^2}R^{-(k-1)}s^{k}\sin k\phi \\
V &= \frac{2R\sigma_0}{\pi\epsilon_0}\begin{cases}\sum_{k=1,3,5,\ldots}^{\infty}\frac{1}{k^2}(s/R)^k\sin k\phi & s\le R\\ -\sum_{k=1,3,5,\ldots}^{\infty}\frac{1}{k^2}(R/s)^k\sin k\phi & s\ge R\end{cases}
\end{aligned}
$$

3.46

(a)

$$\frac{1}{4\pi\epsilon_0}\frac{1}{r}\int_{-a}^{a} k\cos\frac{\pi z}{2a}\,dz = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\frac{4ak}{\pi}$$

(b)

$$\frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int_{-a}^{a} z\cos\theta k\sin\frac{\pi z}{a}\,dz = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\frac{2a^2k\cos\theta}{\pi}$$

(c)

$$\frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int_{-a}^{a} z^2P_2(\cos\theta)k\cos\frac{\pi z}{a}\,dz = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left(-\frac{4a^3k}{\pi^2}\right)P_2(\cos\theta)$$

3.47

(a)

$$
\begin{aligned}
\mathbf{E}_{\text{ave}} &= \frac{1}{\frac{4}{3}\pi R^3} \int \mathbf{E}\, d\tau \\
&= \frac{1}{4\pi\epsilon_0} \frac{1}{\frac{4}{3}\pi R^3} \int \frac{q}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}\, d\tau' \\
\mathbf{E}_{\text{ave}} &= \int \frac{1}{4\pi\epsilon_0} \frac{\rho}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}\, d\tau' \\
&= \frac{1}{4\pi\epsilon_0} \frac{1}{\frac{4}{3}\pi R^3} \int \frac{q}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}\, d\tau'
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathbf{p} &= q\mathbf{r} \\
\oint \mathbf{E} \cdot d\mathbf{A} &= \frac{Q_{\text{encl}}}{\epsilon_0} \\
4\pi r^2 E &= \frac{\frac{4}{3}\pi r^3 \rho}{\epsilon_0} \\
\mathbf{E} &= \frac{r\rho}{3\epsilon_0} \hat{\mathbf{r}} \\
&= -\frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}}{R^3}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 + \ldots \\
&= -\frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}_1}{R^3} - \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}_2}{R^3} + \ldots \\
&= -\frac{1}{4\pi\epsilon_0} \frac{1}{R^3} (\mathbf{p}_1 + \mathbf{p}_2 + \ldots) \\
&= -\frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}}{R^3}
\end{aligned}
$$

(d)

$$
\begin{aligned}
\mathbf{E}_{\text{ave}} &= \frac{1}{\frac{4}{3}\pi R^3}\int \mathbf{E}\,d\tau' \\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{\frac{4}{3}\pi R^3}\int \frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau' \\
\mathbf{E}_r &= -\frac{1}{4\pi\epsilon_0}\int \frac{\rho}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau' \\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{\frac{4}{3}\pi R^3}\int \frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau' \\
\rho &= -\frac{q}{\frac{4}{3}\pi R^3} \\
Q &= \frac{4}{3}\pi R^3\rho \\
&= -q \\
\mathbf{E}_r &= \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{z}}
\end{aligned}
$$

This is the electric field at the origin.

3.48

(a)

$$
\begin{aligned}
\mathbf{E}_{\text{dip}}(r,\theta) &= \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) \\
&= \frac{p}{4\pi\epsilon_0 r^3}[2\cos\theta(\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}) \\
&\quad + \sin\theta(\cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}})] \\
&= \frac{p}{4\pi\epsilon_0 r^3}[3\cos\theta\sin\theta\cos\phi\,\hat{\mathbf{x}} + 3\cos\theta\sin\theta\sin\phi\,\hat{\mathbf{y}} \\
&\quad + (2\cos^2\theta - \sin^2\theta)\hat{\mathbf{z}}] \\
\mathbf{E}_{\text{ave}} &= \frac{1}{\frac{4}{3}\pi R^3}\int \mathbf{E}_{\text{dip}}\,d\tau' \\
&= \frac{3p}{16\pi^2\epsilon_0 R^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{1}{r^3}[3\cos\theta\sin\theta\cos\phi\,\hat{\mathbf{x}} \\
&\quad + 3\cos\theta\sin\theta\sin\phi\,\hat{\mathbf{y}} + (2\cos^2\theta - \sin^2\theta)\hat{\mathbf{z}}]r^2\sin\theta\,dr\,d\theta\,d\phi \\
&= \frac{3p}{16\pi^2\epsilon_0 R^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{1}{r}[3\cos\theta\sin^2\theta\cos\phi\,\hat{\mathbf{x}} \\
&\quad + 3\cos\theta\sin^2\theta\sin\phi\,\hat{\mathbf{y}} + (2\cos^2\theta - \sin^2\theta)\sin\theta\,\hat{\mathbf{z}}]\,dr\,d\theta\,d\phi \\
&= \mathbf{0}
\end{aligned}
$$

(b)

$$-\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3} = \frac{1}{\frac{4}{3}\pi R^3}\int \mathbf{E}\,d\tau'$$
$$\mathbf{E} = -\frac{\mathbf{p}}{3\epsilon_0}\delta^3(\mathbf{r})$$

3.50

(a)

$$\begin{aligned}
\int \mathbf{E}_1 \cdot \mathbf{E}_2\,d\tau &= \int (-\nabla V_1)\cdot \mathbf{E}_2\,d\tau \\
&= \int [V_1(\nabla\cdot\mathbf{E}_2) - \nabla\cdot(V_1\mathbf{E}_2)]\,d\tau \\
&= \int \frac{\rho_2 V_1}{\epsilon_0}\,d\tau - \int \nabla\cdot(V_1\mathbf{E}_2)\,d\tau \\
&= \int \frac{\rho_2 V_1}{\epsilon_0}\,d\tau - \oint V_1\mathbf{E}_2\cdot d\mathbf{a} \\
&= \int \frac{\rho_2 V_1}{\epsilon_0}\,d\tau \\
\int \mathbf{E}_1 \cdot \mathbf{E}_2\,d\tau &= \int \mathbf{E}_1\cdot(-\nabla V_2)\,d\tau \\
&= \int [V_2(\nabla\cdot\mathbf{E}_1) - \nabla\cdot(V_2\mathbf{E}_1)]\,d\tau \\
&= \int \frac{\rho_1 V_2}{\epsilon_0}\,d\tau - \int \nabla\cdot(V_2\mathbf{E}_1)\,d\tau \\
&= \int \frac{\rho_1 V_2}{\epsilon_0}\,d\tau - \oint V_2\mathbf{E}_1\cdot\,d\mathbf{a} \\
&= \int \frac{\rho_1 V_2}{\epsilon_0}\,d\tau \\
\int \frac{\rho_1 V_2}{\epsilon_0}\,d\tau &= \int \frac{\rho_2 V_1}{\epsilon_0}\,d\tau \\
\int \rho_1 V_2\,d\tau &= \int \rho_2 V_1\,d\tau
\end{aligned}$$

(b)

$$
\begin{aligned}
Q_a &= \int_a \rho_1 \, d\tau \\
&= Q \\
Q_b &= \int_b \rho_1 \, d\tau \\
&= 0 \\
V_{1b} &= V_{ab} \\
Q_a &= \int_a \rho_2 \, d\tau \\
&= 0 \\
Q_b &= \int_b \rho_2 \, d\tau \\
&= Q \\
V_{2a} &= V_{ba} \\
\int \rho_1 V_2 \, d\tau &= \int_a \rho_1 V_2 \, d\tau + \int_b \rho_1 V_2 \, d\tau \\
&= V_{2a} \int_a \rho_1 \, d\tau + V_{2b} \int \rho_1 \, d\tau \\
&= V_{ba} Q \\
\int \rho_2 V_1 \, d\tau &= \int_a \rho_2 V_1 \, d\tau + \int_b \rho_2 V_1 \, d\tau \\
&= V_{1a} \int_a \rho_2 \, d\tau + V_{1b} \int \rho_2 \, d\tau \\
&= V_{ab} Q \\
V_{ba} Q &= V_{ab} Q \\
V_{ba} &= V_{ab}
\end{aligned}
$$

3.51

(a)

$$\begin{aligned}
\int \rho_2 V_1 \, d\tau &= Q_{l2}V_{l1} + Q_{x2}V_{x1} + Q_{r2}V_{r1} \\
&= 0 \\
\int \rho_1 V_2 \, d\tau &= Q_{l1}V_{l2} + Q_{x1}V_{x2} + Q_{r1}V_{r2} \\
&= q\frac{x}{d}V_0 + Q_2 V_0 \\
Q_2 &= -\frac{qx}{d}
\end{aligned}$$

$$\begin{aligned}
\int \rho_2 V_1 \, d\tau &= Q_{l2}V_{l1} + Q_{x2}V_{x1} + Q_{r2}V_{r1} \\
&= 0 \\
\int \rho_1 V_2 \, d\tau &= Q_{l1}V_{l2} + Q_{x1}V_{x2} + Q_{r1}V_{r2} \\
&= Q_1 V_0 + q\left(1 - \frac{x}{d}\right) V_0 \\
Q_1 &= q\left(\frac{x}{d} - 1\right)
\end{aligned}$$

(b)

$$
\begin{aligned}
\int \rho_2 V_1 \, d\tau &= Q_{a2} V_{a1} + Q_{r2} V_{r1} + Q_{b2} V_{b1} \\
&= 0 \\
V(a,\theta) &= 0 \\
V(b,\theta) &= V_0 \\
V(r,\theta) &= \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \\
0 &= \sum_{l=0}^{\infty} \left(A_l a^l + \frac{B_l}{a^{l+1}} \right) P_l(\cos\theta) \\
0 &= A_l a^l + \frac{B_l}{a^{l+1}} \\
B_l &= -A_l a^{2l+1} \\
V(r,\theta) &= \sum_{l=0}^{\infty} A_l \left(r^l - \frac{a^{2l+1}}{r^{l+1}} \right) P_l(\cos\theta) \\
V_0 &= \sum_{l=0}^{\infty} A_l \left(b^l - \frac{a^{2l+1}}{b^{l+1}} \right) P_l(\cos\theta) \\
&= A_0 \left(1 - \frac{a}{b} \right) \\
A_0 &= \frac{b}{b-a} V_0 \\
A_n &= 0, \; n \neq 0 \\
V(r,\theta) &= V_0 \frac{b}{b-a} \left(1 - \frac{a}{r} \right) \\
\int \rho_1 V_2 \, d\tau &= Q_{r1} V_{r2} + Q_{b1} V_{b2} \\
&= q V_0 \frac{b}{b-a} \left(1 - \frac{a}{r} \right) + Q_2 V_0 \\
Q_2 &= -\frac{qb}{b-a} \left(1 - \frac{a}{r} \right)
\end{aligned}
$$

$$
\begin{aligned}
\int \rho_2 V_1 \, d\tau &= Q_{a2} V_{a1} + Q_{r2} V_{r1} + Q_{b2} V_{b1} \\
&= 0 \\
V(a, \theta) &= V_0 \\
V(b, \theta) &= 0 \\
0 &= \sum_{l=0}^{\infty} \left(A_l b^l + \frac{B_l}{b^{l+1}} \right) P_l(\cos\theta) \\
0 &= A_l b^l \frac{B_l}{b^{l+1}} \\
B_l &= -A_l b^{2l+1} \\
V(r, \theta) &= \sum_{l=0}^{\infty} A_l \left(r^l - \frac{b^{2l+1}}{r^{l+1}} \right) P_l(\cos\theta) \\
V_0 &= \sum_{l=0}^{\infty} A_l \left(a^l - \frac{b^{2l+1}}{a^{l+1}} \right) P_l(\cos\theta) \\
V_0 &= A_0 \left(1 - \frac{b}{a} \right) \\
A_0 &= V_0 \frac{a}{a-b} \\
V(r, \theta) &= V_0 \frac{a}{a-b} \left(1 - \frac{b}{r} \right) \\
\int \rho_1 V_2 \, d\tau &= Q_{a1} V_{a2} + Q_{r1} V_{r2} + Q_{b1} V_{b2} \\
&= Q_1 V_0 + q V_0 \frac{a}{a-b} \left(1 - \frac{b}{r} \right) \\
Q_1 &= -\frac{qa}{a-b} \left(1 - \frac{b}{r} \right)
\end{aligned}
$$

3.52

(a)

$$
\begin{aligned}
V_{\text{quad}}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int (r')^2 P_2(\cos\alpha)\rho(\mathbf{r}')\,d\tau' \\
\int (r')^2 P_2(\cos\alpha)\rho(\mathbf{r}')\,d\tau' &= \int (r')^2\left[\frac{1}{2}(3\cos^2\alpha - 1)\right]\rho(\mathbf{r}')\,d\tau' \\
&= \frac{1}{2}\int (r')^2[3(\hat{\mathbf{r}}'\cdot\hat{\mathbf{r}})^2 - 1]\rho(\mathbf{r}')\,d\tau' \\
&= \frac{1}{2}\int [3(\mathbf{r}'\cdot\hat{\mathbf{r}})^2 - (r')^2]\rho(\mathbf{r}')\,d\tau' \\
&= \frac{1}{2}\int [3(\mathbf{r}'\cdot\hat{\mathbf{r}})^2 - (r')^2(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})]\rho(\mathbf{r}')\,d\tau' \\
&= \frac{1}{2}\int \left[3\sum_{i,j=1}^{3} r'_i r'_j \hat{r}_i\hat{r}_j - (r')^2\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j\delta_{ij}\right]\rho(\mathbf{r}')\,d\tau' \\
&= \sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j\frac{1}{2}\int [3r'_i r'_j - (r')^2\delta_{ij}]\rho(\mathbf{r}')\,d\tau' \\
&= \sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j Q_{ij}
\end{aligned}
$$

(b)

$$
\begin{aligned}
Q_{11} &= 0 \\
Q_{12} &= \frac{3a^2q}{2} \\
Q_{13} &= 0 \\
Q_{21} &= \frac{3a^2q}{2} \\
Q_{22} &= 0 \\
Q_{23} &= 0 \\
Q_{31} &= 0 \\
Q_{32} &= 0 \\
Q_{33} &= 0
\end{aligned}
$$

4 Electric Fields in Matter

4.1

$$
\begin{aligned}
V(x) &= 500\frac{x}{d} = 500\,000x \\
\mathbf{E} &= -\nabla V = -500\,000\,\mathrm{N/C} \\
\alpha &= 4\pi\epsilon_0(0.667 \times 10^{-30}) \\
&= 7.42 \times 10^{-41}\,\mathrm{C^2\,m/N} \\
\alpha E &= qd \\
d &= \frac{\alpha E}{q} \\
&= 2.32 \times 10^{-16}\,\mathrm{m} \\
\frac{d}{R} &= 4.6 \times 10^{-6} \\
V &= 1.88 \times 10^{8}\,\mathrm{V}
\end{aligned}
$$

4.2

$$
\begin{aligned}
\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E &= \frac{1}{\epsilon_0} \int_0^{2\pi} \int_0^{\pi} \int_0^{r} \rho(r') r'^2 \sin\theta \, dr' \, d\theta \, d\phi \\
&= \frac{4\pi}{\epsilon_0} \int_0^{r} \frac{q}{\pi a^3} e^{-2r'/a} r'^2 \, dr' \\
&= \frac{4q}{\epsilon_0 a^3} \int_0^{r} e^{-2r'/a} r'^2 \, dr' \\
&= \frac{4q}{\epsilon_0 a^3} \frac{1}{4} a \left[a^2 - e^{-2r/a} (a^2 + 2ar + 2r^2) \right] \\
&= \frac{q}{\epsilon_0 a^2} \left[a^2 - e^{-2r/a} (a^2 + 2ar + 2r^2) \right] \\
\mathbf{E}_e(r) &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \left[1 - e^{-2r/a} \left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2} \right) \right] \hat{\mathbf{r}} \\
E &= \frac{1}{4\pi\epsilon_0} \frac{q}{d^2} \left(\frac{4}{3} \frac{d^3}{a^3} \right) \\
&= \frac{1}{4\pi\epsilon_0} \frac{4}{3a^3} (qd) \\
&= \frac{1}{4\pi\epsilon_0} \frac{4}{3a^3} p \\
&= \frac{1}{4\pi\epsilon_0} \frac{4}{3a^3} \alpha E \\
\alpha &= 3\pi\epsilon_0 a^3
\end{aligned}
$$

4.3

$$\begin{aligned}
\rho(r) &= kr \\
Q_{\text{enc}} &= \int_0^{2\pi}\int_0^{\pi}\int_0^{r} kr'^3 \sin\theta \, dr' \, d\theta \, d\phi \\
&= \pi k r^4 \\
\oint \mathbf{E}\cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E &= \frac{\pi k r^4}{\epsilon_0} \\
\mathbf{E} &= \frac{kr^2}{4\epsilon_0} \\
E &= \frac{kd^2}{4\epsilon_0} \\
d &= \sqrt{\frac{4\epsilon_0 E}{k}} \\
p &= ed \\
&= 2e\sqrt{\frac{\epsilon_0}{k}}\sqrt{E}
\end{aligned}$$

p is proportional to $\sqrt{E}$.

4.4

$$\begin{aligned}
E_q &= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \\
p &= \alpha E_q \\
&= \frac{1}{4\pi\epsilon_0}\frac{\alpha q}{r^2} \\
\mathbf{E}_{\text{dip}} &= \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) \\
E_{\text{dip}} &= \frac{1}{4\pi\epsilon_0}\frac{2p}{r^3} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2}{r^3}\left(\frac{1}{4\pi\epsilon_0}\frac{\alpha q}{r^2}\right) \\
&= \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{2\alpha q}{r^5} \\
F &= \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{2\alpha q^2}{r^5}
\end{aligned}$$

4.5

$$\begin{aligned}
\mathbf{E}_{p_1} &= -\frac{1}{4\pi\epsilon_0}\frac{p_1}{r^3}\hat{\mathbf{z}} \\
\boldsymbol{\tau}_{p_2} &= \mathbf{p}_2 \times \mathbf{E}_{p_1} \\
&= p_2\hat{\mathbf{x}} \times -\frac{1}{4\pi\epsilon_0}\frac{p_1}{r^3}\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{p_1p_2}{r^3}\hat{\mathbf{y}} \\
\mathbf{E}_{p_2} &= \frac{1}{4\pi\epsilon_0}\frac{2p_2}{r^3}\hat{\mathbf{x}} \\
\boldsymbol{\tau}_{p_1} &= \mathbf{p}_1 \times \mathbf{E}_{p_2} \\
&= p_1\hat{\mathbf{z}} \times -\frac{1}{4\pi\epsilon_0}\frac{2p_2}{r^3}\hat{\mathbf{x}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2p_1p_2}{r^3}\hat{\mathbf{y}}
\end{aligned}$$

4.6

$$\begin{aligned}
\mathbf{E}_i &= \frac{1}{4\pi\epsilon_0}\frac{p}{8z^3}(\sin\theta\hat{\mathbf{x}} + 2\cos\theta\hat{\mathbf{y}}) \\
\boldsymbol{\tau} &= \mathbf{p} \times \mathbf{E}_i \\
&= p\begin{pmatrix}\sin\theta \\ \cos\theta \\ 0\end{pmatrix} \times \frac{1}{4\pi\epsilon_0}\frac{p}{8z^3}\begin{pmatrix}\sin\theta \\ 2\cos\theta \\ 0\end{pmatrix} \\
&= \frac{1}{4\pi\epsilon_0}\frac{p^2\sin 2\theta}{16z^3}\hat{\mathbf{z}}
\end{aligned}$$

The dipole will come to rest at $\theta = n\pi, n \in \mathbb{Z}$.

4.8

$$\begin{aligned}
U &= -\mathbf{p}_1 \cdot \mathbf{E} \\
&= -\mathbf{p}_1 \cdot \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}_2\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_2] \\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[\mathbf{p}_1\cdot\mathbf{p}_2 - 3(\mathbf{p}_1\cdot\hat{\mathbf{r}})(\mathbf{p}_2\cdot\hat{\mathbf{r}})]
\end{aligned}$$

4.9

(a)

$$
\begin{aligned}
\mathbf{E} &= \frac{q}{4\pi\epsilon_0}\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{3/2}} \\
F_x &= (\mathbf{p}\cdot\nabla)\mathbf{E} \\
&= \left(p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z}\right)\frac{q}{4\pi\epsilon_0}\frac{x}{(x^2+y^2+z^2)^{3/2}} \\
&= \frac{q}{4\pi\epsilon_0}\left[p_x\left(\frac{1}{(x^2+y^2+z^2)^{3/2}} - 3x\frac{x}{(x^2+y^2+z^2)^{5/2}}\right)\right. \\
&\qquad \left. + p_y\left(-3x\frac{y}{(x^2+y^2+z^2)^{5/2}}\right) + p_z\left(-3x\frac{z}{(x^2+y^2+z^2)^{5/2}}\right)\right] \\
&= \frac{q}{4\pi\epsilon_0}\left(\frac{p_x}{r^3} - 3x\frac{p_x x + p_y y + p_z z}{r^5}\right) \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{p}}{r^3} - \frac{3\mathbf{r}(\mathbf{p}\cdot\mathbf{r})}{r^5}\right]_x \\
\mathbf{F} &= \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}[\mathbf{p} - 3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]
\end{aligned}
$$

(b)

$$
\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]
$$

4.10

(a)

$$
\begin{aligned}
\mathbf{P} &= k\mathbf{r} \\
\sigma_b &= \mathbf{P}\cdot\hat{\mathbf{n}} \\
&= kR \\
\rho_b &= -\nabla\cdot\mathbf{P} \\
&= -\frac{1}{r^2}\frac{\partial}{\partial r}(kr^3) \\
&= -3k
\end{aligned}
$$

(b)

$$
\begin{aligned}
\oint \mathbf{E}_{\text{inside}} \cdot d\mathbf{a} &= \frac{Q_{\text{encl}}}{\epsilon_0} \\
4\pi r^2 E_{\text{inside}} &= \frac{1}{\epsilon_0} \int_0^{2\pi} \int_0^{\pi} \int_0^{r} -3kr'^2 \sin\theta \, dr' \, d\theta \, d\phi \\
&= -\frac{12\pi k}{\epsilon_0} \int_0^{r} r'^2 \, dr' \\
&= -\frac{4\pi k r^3}{\epsilon_0} \\
\mathbf{E}_{\text{inside}} &= -\frac{kr}{\epsilon_0}\hat{\mathbf{r}} \\
&= -\frac{k}{\epsilon_0}\mathbf{r} \\
\oint \mathbf{E}_{\text{outside}} \cdot d\mathbf{a} &= \frac{Q_{\text{encl}}}{\epsilon_0} \\
4\pi r^2 E_{\text{outside}} &= \frac{1}{\epsilon_0} \left(4\pi k R^3 - \int_0^{2\pi} \int_0^{\pi} \int_0^{R} 3kr^2 \sin\theta \, dr \, d\theta \, d\phi \right) \\
&= \frac{1}{\epsilon_0} \left(4\pi k R^3 - 4\pi k R^3 \right) \\
\mathbf{E}_{\text{outside}} &= \mathbf{0}
\end{aligned}
$$

4.11

$$
\begin{aligned}
\sigma_b &= \mathbf{P} \cdot \hat{\mathbf{n}} \\
&= \begin{cases} 0 & \text{on the side} \\ P & \text{on one end} \\ -P & \text{on the other end} \end{cases} \\
\rho_b &= -\nabla \cdot \mathbf{P} \\
&= 0
\end{aligned}
$$

4.13

$$
\begin{aligned}
\oint \mathbf{E}_{\text{inside}} \cdot d\mathbf{a} &= \frac{Q_{\text{encl}}}{\epsilon_0} \\
2\pi s L E_{\text{inside}} &= \frac{\pi s^2 L \rho}{\epsilon_0} \\
\mathbf{E}_{\text{inside}} &= \frac{\rho s}{2\epsilon_0}\hat{\mathbf{s}} \\
\mathbf{E}_{\text{inside}} &= \frac{\rho}{2\epsilon_0}(\mathbf{s}_+ - \mathbf{s}_-) \\
&= -\frac{\rho \mathbf{d}}{2\epsilon_0} \\
&= -\frac{\mathbf{P}}{2\epsilon_0} \\
\oint \mathbf{E}_{\text{outside}} \cdot d\mathbf{a} &= \frac{Q_{\text{encl}}}{\epsilon_0} \\
2\pi s L E_{\text{outisde}} &= \frac{\pi a^2 L \rho}{\epsilon_0} \\
\mathbf{E}_{\text{outside}} &= \frac{a^2 \rho}{2\epsilon_0 s}\hat{\mathbf{s}} \\
\mathbf{E}_{\text{outside}} &= \frac{a^2 \rho}{2\epsilon_0}\left(\frac{\hat{\mathbf{s}}_+}{s_+} - \frac{\hat{\mathbf{s}}_-}{s_-}\right)
\end{aligned}
$$

4.14

$$
\begin{aligned}
\oint_{\mathcal{S}} \mathbf{P} \cdot \hat{\mathbf{n}} - \int_{\mathcal{V}} \nabla \cdot \mathbf{P}\, d\tau' &= \oint_{\mathcal{S}} \mathbf{P} \cdot \hat{\mathbf{n}} - \oint_{\mathcal{S}} \mathbf{P} \cdot \hat{\mathbf{n}} \\
&= 0
\end{aligned}
$$

4.15

(a)

$$\begin{aligned}
\sigma_{b,a} &= \mathbf{P} \cdot \hat{\mathbf{n}} \\
&= -\frac{k}{a} \\
\sigma_{b,b} &= \frac{k}{b} \\
\rho_b &= -\nabla \cdot \mathbf{P} \\
&= -\frac{1}{r^2}\frac{\partial}{\partial r}(kr) \\
&= -\frac{k}{r^2} \\
\mathbf{E} &= \mathbf{0}, r < a \\
\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E &= \frac{1}{\epsilon_0}\left(-4\pi a^2 \frac{k}{a} - \int_0^{2\pi}\int_0^{\pi}\int_a^{r} \frac{k}{r'^2} r'^2 \sin\theta \, dr' \, d\theta \, d\phi\right) \\
&= \frac{1}{\epsilon_0}[-4\pi a k - 4\pi k(r-a)] \\
\mathbf{E} &= -\frac{k}{\epsilon_0 r}\hat{\mathbf{r}}, a < r < b
\end{aligned}$$

$$\mathbf{E} = \mathbf{0}$$

(b)

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$
$$\mathbf{D} = \mathbf{0}$$
$$\epsilon_0 \mathbf{E} + \mathbf{P} = \mathbf{0}$$
$$\mathbf{E} = \mathbf{0}$$
$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$
$$\mathbf{D} = \mathbf{0}$$
$$\epsilon_0 \mathbf{E} + \mathbf{P} = \mathbf{0}$$
$$\mathbf{E} = -\frac{\mathbf{P}}{\epsilon_0}$$
$$= -\frac{k}{\epsilon_0 r}\hat{\mathbf{r}}$$
$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$
$$\mathbf{D} = \mathbf{0}$$
$$\epsilon_0 \mathbf{E} + \mathbf{P} = \mathbf{0}$$
$$\mathbf{E} = \mathbf{0}$$

4.16

(a)

$$\mathbf{E} = \mathbf{E}_0 + \frac{1}{3\epsilon_0}\mathbf{P}$$
$$\mathbf{D} = \mathbf{D}_0 - \frac{2}{3}\mathbf{P}$$

4.18

(a)

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$
$$2AE = \frac{A\sigma}{\epsilon_0}$$
$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$$
$$\mathbf{E}_{\text{vac}} = -\frac{\sigma}{\epsilon_0}\hat{\mathbf{y}}$$
$$\mathbf{D} = \epsilon_0 \mathbf{E}_{\text{vac}}$$
$$= -\sigma\hat{\mathbf{y}}$$

(b)

$$\begin{aligned}
\mathbf{E}_1 &= \frac{1}{\epsilon_1}\mathbf{D} \\
&= -\frac{\sigma}{2\epsilon_0}\hat{\mathbf{y}} \\
\mathbf{E}_2 &= \frac{1}{\epsilon_2}\mathbf{D} \\
&= -\frac{\sigma}{1.5\epsilon_0}\hat{\mathbf{y}}
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{P}_1 &= \epsilon_0\chi_{e1}\mathbf{E}_1 \\
&= -\frac{\sigma}{2}\hat{\mathbf{y}} \\
\mathbf{P}_2 &= \epsilon_0\chi_{e2}\mathbf{E}_2 \\
&= -\frac{\sigma}{3}\hat{\mathbf{y}}
\end{aligned}$$

(d)

$$\begin{aligned}
V &= a\frac{\sigma}{2\epsilon_0} + a\frac{2\sigma}{3\epsilon_0} \\
&= \frac{a\sigma}{\epsilon_0}\left(\frac{3}{6} + \frac{4}{6}\right) \\
&= \frac{7a\sigma}{6\epsilon_0}
\end{aligned}$$

(e)

$$\begin{aligned}
\sigma_{b1,\text{top}} &= \mathbf{P}_1 \cdot \hat{\mathbf{y}} \\
&= -\frac{\sigma}{2} \\
\sigma_{b1,\text{bottom}} &= \mathbf{P}_1 \cdot -\hat{\mathbf{y}} \\
&= \frac{\sigma}{2} \\
\sigma_{b2,\text{top}} &= \mathbf{P}_2 \cdot \hat{\mathbf{y}} \\
&= -\frac{\sigma}{3} \\
\sigma_{b2,\text{bottom}} &= \mathbf{P}_2 \cdot -\hat{\mathbf{y}} \\
&= \frac{\sigma}{3}
\end{aligned}$$

(f)

$$\begin{aligned}
\mathbf{E}_1 &= -\frac{\sigma}{\epsilon_0}\hat{\mathbf{y}} + \frac{\sigma}{2\epsilon_0}\hat{\mathbf{y}} \\
&= -\frac{\sigma}{2\epsilon_0}\hat{\mathbf{y}} \\
\mathbf{E}_2 &= -\frac{\sigma}{\epsilon_0}\hat{\mathbf{y}} + \frac{\sigma}{3\epsilon_0}\hat{\mathbf{y}} \\
&= -\frac{2\sigma}{3\epsilon_0}\hat{\mathbf{y}}
\end{aligned}$$

4.19

$$\begin{aligned}
Q &= A\sigma \\
E_{\text{vac}} &= \frac{\sigma}{\epsilon_0} \\
V_{\text{vac}} &= \frac{d\sigma}{\epsilon_0} \\
C_{\text{vac}} &= \frac{Q}{V_{\text{vac}}} \\
&= \frac{A\epsilon_0}{d} \\
E' &= \frac{1}{\epsilon_r}E_{\text{vac}} \\
&= \frac{\sigma}{\epsilon_0\epsilon_r} \\
V' &= \frac{d\sigma}{2\epsilon_0} + \frac{d\sigma}{2\epsilon_0\epsilon_r} \\
&= \frac{d\sigma}{2\epsilon_0}\left(1 + \frac{1}{\epsilon_r}\right) \\
&= \frac{d\sigma(1+\epsilon_r)}{2\epsilon_0\epsilon_r} \\
C' &= \frac{Q}{V'} \\
&= \frac{2A\epsilon_0\epsilon_r}{d(1+\epsilon_r)} \\
\frac{C'}{C_{\text{vac}}} &= \frac{2\epsilon_r}{1+\epsilon_r}
\end{aligned}$$

$$
\begin{aligned}
E &= \frac{\sigma}{\epsilon_0} \\
\sigma &= \epsilon_0 E \\
&= \epsilon_0 \frac{V}{d} \\
P &= \epsilon_0 \chi_e E \\
&= \epsilon_0 \chi_e \frac{V}{d} \\
\sigma_b &= -\epsilon_0 \chi_e \frac{V}{d} \\
\sigma_{\text{total}} &= \sigma_f + \sigma_b \\
\epsilon_0 \frac{V}{d} &= \sigma_f - \epsilon_0 \chi_e \frac{V}{d} \\
\sigma_f &= \epsilon_0 \frac{V}{d} + \epsilon_0 \chi_e \frac{V}{d} \\
&= \epsilon_0 \frac{V}{d} (1 + \chi_e) \\
&= \frac{\epsilon_0 \epsilon_r V}{d} \\
C' &= \frac{Q'}{V} \\
&= \frac{1}{V} \left(\frac{A\sigma}{2} + \frac{A\sigma_f}{2} \right) \\
&= \frac{A}{2V} \left(\frac{\epsilon_0 V}{d} + \frac{\epsilon_0 \epsilon_r V}{d} \right) \\
&= \frac{A\epsilon_0 (1 + \epsilon_r)}{2d} \\
\frac{C'}{C} &= \frac{1 + \epsilon_r}{2}
\end{aligned}
$$

4.20

$$
\begin{aligned}
\oint \mathbf{D} \cdot d\mathbf{a} &= Q_{\mathrm{f}} \\
4\pi r^2 D &= \frac{4}{3}\pi r^3 \rho \\
\mathbf{D} &= \frac{r\rho}{3}\hat{\mathbf{r}} \\
\mathbf{D} &= \epsilon \mathbf{E} \\
\mathbf{E} &= \frac{1}{\epsilon}\mathbf{D} \\
&= \frac{r\rho}{3\epsilon_0\epsilon_r}\hat{\mathbf{r}},\ r < R \\
\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\mathrm{enc}}}{\epsilon_0} \\
4\pi r^2 E &= \frac{1}{\epsilon_0}\frac{4}{3}\pi R^3 \rho \\
\mathbf{E} &= \frac{R^3\rho}{3\epsilon_0 r^2}\hat{\mathbf{r}},\ r > R \\
V &= -\int_{\infty}^{0} \mathbf{E} \cdot d\mathbf{l} \\
&= -\left(\int_{\infty}^{R} \frac{R^3\rho}{3\epsilon_0 r^2}\,dr + \int_{R}^{0} \frac{r\rho}{3\epsilon_0\epsilon_r}\,dr\right) \\
&= -\left(\frac{R^3\rho}{3\epsilon_0}\left[-\frac{1}{r}\right]_{\infty}^{R} + \frac{\rho}{3\epsilon_0\epsilon_r}\left[\frac{1}{2}r^2\right]_{R}^{0}\right) \\
&= -\left(-\frac{R^2\rho}{3\epsilon_0} - \frac{R^2\rho}{6\epsilon_0\epsilon_r}\right) \\
&= \frac{R^2\rho}{3\epsilon_0}\left(1 + \frac{1}{2\epsilon_r}\right)
\end{aligned}
$$

4.21

$$
\begin{aligned}
\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
2\pi s L E &= \frac{\pi a^2 L \rho}{\epsilon_0} \\
\mathbf{E} &= \frac{a^2 \rho}{2\epsilon_0 s}\hat{\mathbf{s}},\ a < s < b \\
\mathbf{E} &= \frac{\mathbf{E}_{\text{vac}}}{\epsilon_r} \\
&= \frac{a^2 \rho}{2\epsilon_0 \epsilon_r s}\hat{\mathbf{s}},\ b < s < c \\
V &= -\int_c^a \mathbf{E} \cdot d\mathbf{l} \\
&= -\left(\int_c^b \frac{a^2 \rho}{2\epsilon_0 \epsilon_r s}\, ds + \int_b^a \frac{a^2 \rho}{2\epsilon_0 s}\, ds \right) \\
&= -\left(\frac{a^2 \rho}{2\epsilon_0 \epsilon_r} \ln \frac{b}{c} + \frac{a^2 \rho}{2\epsilon_0} \ln \frac{a}{b} \right) \\
&= \frac{a^2 \rho}{2\epsilon_0} \left(\ln \frac{b}{a} + \frac{1}{\epsilon_r} \ln \frac{c}{b} \right) \\
C &= \frac{Q}{V} \\
&= \frac{\pi a^2 \rho}{\frac{a^2 \rho}{2\epsilon_0} \left(\ln \frac{b}{a} + \frac{1}{\epsilon_r} \ln \frac{c}{b} \right)} \\
&= \frac{2\pi \epsilon_0}{\ln \frac{b}{a} + \frac{1}{\epsilon_r} \ln \frac{c}{b}}
\end{aligned}
$$

4.22

$$
\begin{aligned}
V_{\text{in}}(a,\phi) &= V_{\text{out}}(a,\phi) \\
\epsilon \left.\frac{\partial V_{\text{in}}}{\partial s}\right|_{s=a} &= \epsilon_0 \left.\frac{\partial V_{\text{out}}}{\partial s}\right|_{s=a} \\
V_{\text{out}} &\to -E_0 s\cos\phi,\ \text{ for } s \gg a \\
V_{\text{in}} &= \sum_{k=1}^{\infty} s^k (a_k \cos k\phi + b_k \sin k\phi) \\
V_{\text{out}} &= -E_0 s\cos\phi + \sum_{k=1}^{\infty} s^{-k}(c_k\cos k\phi + d_k \sin k\phi) \\
\sum_{k=1}^{\infty} a^k (a_k \cos k\phi + b_k \sin k\phi) &= -E_0 a\cos\phi + \sum_{k=1}^{\infty} a^{-k}(c_k\cos k\phi + d_k \sin k\phi) \\
a a_1 &= \frac{c_1}{a} - E_0 a \\
a_1 &= \frac{c_1}{a^2} - E_0 \\
a^k a_k &= a^{-k} c_k \\
a_k &= a^{-2k} c_k,\ \text{ for } k \neq 1 \\
a^k b_k &= a^{-k} d_k \\
b_k &= a^{-2k} d_k \\
\epsilon_0(1+\chi_e)\sum_{k=1}^{\infty} k a^{k-1}(a_k \cos k\phi + b_k \sin k\phi) &= \epsilon_0\left(-E_0\cos\phi - \sum_{k=1}^{\infty} k a^{-(k+1)}(c_k\cos k\phi + d_k\sin k\phi)\right) \\
(1+\chi_e)a_1 &= -\left(E_0 + \frac{c_1}{a^2}\right) \\
a_1 &= -\frac{1}{1+\chi_e}\left(E_0 + \frac{c_1}{a^2}\right) \\
\frac{c_1}{a^2} - E_0 &= -\frac{1}{1+\chi_e}\left(E_0 + \frac{c_1}{a^2}\right) \\
c_1 - a^2 E_0 &= -\frac{1}{1+\chi_e}(a^2 E_0 + c_1) \\
c_1 \frac{2+\chi_e}{1+\chi_e} &= a^2 E_0 \frac{\chi_e}{1+\chi_e} \\
c_1 &= a^2 E_0 \frac{\chi_e}{2+\chi_e} \\
a_1 &= E_0\left(\frac{\chi_e}{2+\chi_e} - 1\right) \\
&= -\frac{E_0}{1+\chi_e/2}
\end{aligned}
$$

$$
\begin{aligned}
(1+\chi_e)a^{k-1}a_k &= -a^{-(k+1)}c_k \\
a_k &= -\frac{1}{1+\chi_e}a^{-2k}c_k \\
a^{-2k}c_k &= -\frac{1}{1+\chi_e}a^{-2k}c_k \\
c_k &= 0 \\
a_k &= 0 \\
(1+\chi_e)a^{k-1}b_k &= -a^{-(k+1)}d_k \\
b_k &= -\frac{1}{1+\chi_e}a^{-2k}d_k \\
a^{-2k}d_k &= -\frac{1}{1+\chi_e}a^{-2k}d_k \\
d_k &= 0 \\
b_k &= 0 \\
V_{\text{in}} &= -\frac{E_0}{1+\chi_e/2}s\cos\phi \\
&= -\frac{E_0}{1+\chi_e/2}x \\
\mathbf{E}_{\text{in}} &= -\frac{\partial V_{\text{in}}}{\partial x}\hat{\mathbf{x}} \\
&= \frac{\mathbf{E}_0}{1+\chi_e/2}
\end{aligned}
$$

4.23

$$
\begin{aligned}
\mathbf{P}_0 &= \epsilon_0 \chi_e \mathbf{E}_0 \\
\mathbf{E}_1 &= -\frac{1}{3\epsilon_0}\mathbf{P}_0 \\
&= -\frac{\chi_e}{3}\mathbf{E}_0 \\
\mathbf{P}_1 &= \epsilon_0 \chi_e \mathbf{E}_1 \\
&= -\frac{\epsilon_0 \chi_e^2}{3}\mathbf{E}_0 \\
\mathbf{E}_2 &= -\frac{1}{3\epsilon_0}\mathbf{P}_1 \\
&= \frac{\chi_e^2}{9}\mathbf{E}_0 \\
\mathbf{E}_n &= \left(-\frac{\chi_e}{3}\right)^n \mathbf{E}_0 \\
\mathbf{E} &= \sum_{n=0}^{\infty} \mathbf{E}_n \\
&= \mathbf{E}_0 \sum_{n=0}^{\infty} \left(-\frac{\chi_e}{3}\right)^n \\
&= \frac{3}{3+\chi_e}\mathbf{E}_0 \\
&= \frac{3}{2+\epsilon_r}\mathbf{E}_0
\end{aligned}
$$

4.26

$$\mathbf{D} = \begin{cases} \mathbf{0} & r < a \\ \frac{Q}{4\pi r^2}\hat{\mathbf{r}} & a < r < b \\ \frac{Q}{4\pi r^2}\hat{\mathbf{r}} & b < r \end{cases}$$

$$\mathbf{E} = \begin{cases} \mathbf{0} & r < a \\ \frac{1}{4\pi\epsilon_0(1+\chi_e)}\frac{Q}{r^2}\hat{\mathbf{r}} & a < r < b \\ \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} & b < r \end{cases}$$

$$\mathbf{D}\cdot\mathbf{E} = \begin{cases} 0 & r < a \\ \frac{Q^2}{16\pi^2\epsilon_0(1+\chi_e)r^4} & a < r < b \\ \frac{Q^2}{16\pi^2\epsilon_0 r^4} & b < r \end{cases}$$

$$\begin{aligned} W &= \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,d\tau \\ &= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \mathbf{D}\cdot\mathbf{E}\; r^2\sin\theta\,dr\,d\theta\,d\phi \\ &= \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{1+\chi_e}\int_a^b \frac{1}{r^2}\,dr + \int_b^{\infty}\frac{1}{r^2}\,dr\right) \\ &= \frac{Q^2}{8\pi\epsilon_0}\left[\frac{1}{1+\chi_e}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{b}\right] \\ &= \frac{Q^2}{8\pi\epsilon_0(1+\chi_e)}\left(\frac{1}{a}+\frac{\chi_e}{b}\right) \end{aligned}$$

4.27

$$\mathbf{E} = \begin{cases} -\frac{\mathbf{P}}{3\epsilon_0} & r < R \\ \frac{R^3 P}{3\epsilon_0 r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) & r > R \end{cases}$$

$$E^2 = \begin{cases} \frac{P^2}{9\epsilon_0^2} & r < R \\ \frac{R^6 P^2}{9\epsilon_0^2 r^6}(1 + 3\cos^2\theta) & r > R \end{cases}$$

$$\begin{aligned}
W &= \frac{\epsilon_0}{2}\int E^2\,d\tau \\
&= \frac{\epsilon_0}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} E^2 r^2 \sin\theta\,dr\,d\theta\,d\phi \\
&= \epsilon_0\pi\left(\frac{P^2}{9\epsilon_0^2}\int_0^{\pi}\int_0^{R} r^2\sin\theta\,dr\,d\theta + \frac{R^6P^2}{9\epsilon_0^2}\int_0^{\pi}\int_R^{\infty}\frac{1+3\cos^2\theta}{r^4}\sin\theta\,dr\,d\theta\right) \\
&= \epsilon_0\pi\left(\frac{2P^2R^3}{27\epsilon_0^2} + \frac{4P^2R^3}{27\epsilon_0^2}\right) \\
&= \frac{2\pi P^2 R^3}{9\epsilon_0}
\end{aligned}$$

$$W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,d\tau$$

4.28

$$\begin{aligned}
\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{encl}}}{\epsilon_0} \\
2\pi s L E &= \frac{\lambda L}{\epsilon_0} \\
\mathbf{E} &= \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}} \\
V &= \int_a^b \mathbf{E} \cdot d\mathbf{l} \\
&= \frac{\lambda}{2\pi\epsilon_0} \ln\frac{b}{a} \\
\oint \mathbf{D} \cdot d\mathbf{a} &= Q_{\text{encl,f}} \\
2\pi s L D &= \lambda' L \\
\mathbf{D} &= \frac{\lambda'}{2\pi s}\hat{\mathbf{s}} \\
\mathbf{E} &= \frac{\lambda'}{2\pi\epsilon s}\hat{\mathbf{s}} \\
V &= \int_a^b \mathbf{E} \cdot d\mathbf{l} \\
&= \frac{\lambda'}{2\pi\epsilon} \ln\frac{b}{a} \\
\frac{\lambda'}{2\pi\epsilon} \ln\frac{b}{a} &= \frac{\lambda}{2\pi\epsilon_0} \ln\frac{b}{a} \\
\lambda' &= \epsilon_r \lambda \\
Q &= \lambda' h + \lambda(\ell - h) \\
&= (\epsilon_r h + \ell - h)\lambda \\
&= (\chi_e h + \ell)\lambda \\
C &= \frac{Q}{V} \\
&= (\chi_e h + \ell)\lambda \frac{2\pi\epsilon_0}{\lambda \ln(b/a)} \\
&= 2\pi\epsilon_0 \frac{\chi_e h + \ell}{\ln(b/a)} \\
\frac{dC}{dh} &= \frac{2\pi\epsilon_0 \chi_e}{\ln(b/a)}
\end{aligned}$$

$$
\begin{aligned}
F_{\text{up}} &= \frac{1}{2}V^2\frac{dC}{dh} \\
&= \frac{\pi\epsilon_0\chi_e V^2}{\ln(b/a)} \\
F_{\text{down}} &= mg \\
&= \pi(b^2-a^2)h\rho g \\
F_{\text{down}} &= F_{\text{up}} \\
\pi(b^2-a^2)h\rho g &= \frac{\pi\epsilon_0\chi_e V^2}{\ln(b/a)} \\
h &= \frac{\epsilon_0\chi_e V^2}{\rho g(b^2-a^2)\ln(b/a)}
\end{aligned}
$$

4.29

(a)

$$
\begin{aligned}
\mathbf{E}_1 &= -\frac{1}{4\pi\epsilon_0}\frac{p_1}{r^3}\hat{\mathbf{z}} \\
\mathbf{F}_2 &= \frac{1}{4\pi\epsilon_0}\frac{3p_1p_2}{r^4}\hat{\mathbf{z}} \\
\mathbf{E}_2 &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p_2}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p_2}] \\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[\frac{3(\mathbf{p_2}\cdot\mathbf{r})\mathbf{r}}{r^2}-\mathbf{p_2}\right] \\
&= \frac{p_2}{4\pi\epsilon_0}\frac{3y(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}})-(x^2+y^2+z^2)\hat{\mathbf{y}}}{(x^2+y^2+z^2)^{5/2}} \\
&= \frac{p_2}{4\pi\epsilon_0}\frac{3xy\hat{\mathbf{x}}+(-x^2+2y^2-z^2)\hat{\mathbf{y}}+3yz\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{5/2}} \\
E_{2z} &= \frac{p_2}{4\pi\epsilon_0}\frac{3yz}{(x^2+y^2+z^2)^{5/2}} \\
\nabla E_{2z}|_{x=0,y=-r,z=0} &= -\frac{3p_2}{4\pi\epsilon_0 r^4}\hat{\mathbf{z}} \\
\mathbf{F}_1 &= -\frac{1}{4\pi\epsilon_0}\frac{3p_1p_2}{r^4}\hat{\mathbf{z}}
\end{aligned}
$$

Newton's third law is obeyed.

(b)

$$
\begin{aligned}
\boldsymbol{\tau}_2 &= \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{p}_1 \times \mathbf{E}_1 \\
&= r\hat{\mathbf{y}} \times \frac{1}{4\pi\epsilon_0}\frac{3p_1p_2}{r^4}\hat{\mathbf{z}} - \frac{1}{4\pi\epsilon_0}\frac{p_1p_2}{r^3}\hat{\mathbf{x}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2p_1p_2}{r^3}\hat{\mathbf{x}}
\end{aligned}
$$

This is opposite to the torque experienced by $\mathbf{p}_1$.

4.30

The electric field is perpendicular to each plate and "curves" towards the other. At $y = 0$ the electric field is purely vertical. $\mathbf{p}$ only has a y component so $(\mathbf{p} \cdot \nabla)\mathbf{E} = p_y \nabla E_y$. The potential difference between the two plates is constant and as x increases the distance between them also increases, meaning the magnitude of the electric field decreases. E_y is negative at $y = 0$ so this means it increases with increasing x and thus that the x component of ∇E_y is positive. E_y is constant for small changes in y around $y = 0$ so the y component of ∇E_y is 0. We assume the plates are very long in the z direction so the z component of ∇E_y is also 0. This means the dipole experiences a force in the x direction.

4.31

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{Q(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})}{(x^2 + y^2 + z^2)^{3/2}} \\
E_y &= \frac{1}{4\pi\epsilon_0}\frac{Qy}{(x^2 + y^2 + z^2)^{3/2}} \\
\nabla E_y\big|_{x=R,y=z=0} &= \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\hat{\mathbf{y}} \\
\mathbf{F} &= (\mathbf{p} \cdot \nabla)\mathbf{E} \\
&= p_y \nabla E_y \\
&= \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}
\end{aligned}
$$

4.32

(a)

$$
\begin{aligned}
\mathbf{p} &= \alpha\mathbf{E} \\
\mathbf{F} &= (\mathbf{p}\cdot\nabla)\mathbf{E} \\
&= (\alpha\mathbf{E}\cdot\nabla)\mathbf{E} \\
&= \alpha(\mathbf{E}\cdot\nabla)\mathbf{E} \\
\nabla(E^2) &= 2\mathbf{E}\times(\nabla\times\mathbf{E}) + 2(\mathbf{E}\cdot\nabla)\mathbf{E} \\
(\mathbf{E}\cdot\nabla)\mathbf{E} &= \frac{1}{2}\nabla(E^2) \\
\mathbf{F} &= \frac{1}{2}\alpha\nabla(E^2)
\end{aligned}
$$

4.33

$$
\begin{aligned}
\mathbf{P} &= k\mathbf{r} \\
&= k(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \\
\sigma_b &= \mathbf{P}\cdot\hat{\mathbf{n}} \\
Q_{\text{face}} &= \int \sigma_b \, da \\
&= \int_{-a/2}^{a/2}\int_{-a/2}^{a/2} k\left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + \frac{a}{2}\hat{\mathbf{z}}\right)\cdot\hat{\mathbf{z}}\, dx\, dy \\
&= \frac{ak}{2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} dx\, dy \\
&= \frac{a^3k}{2} \\
Q_{\text{surface}} &= 3a^3k \\
\rho_b &= -\nabla\cdot\mathbf{P} \\
&= -3k \\
Q_{\text{volume}} &= \rho V \\
&= -3a^3k \\
Q_{\text{total}} &= 0
\end{aligned}
$$

4.34

$$
\begin{aligned}
\epsilon &= \epsilon_0 \left(1 + \frac{x}{d}\right) \\
\oint \mathbf{D} \cdot d\mathbf{a} &= Q_{\text{enc,f}} \\
AD &= A\sigma_f \\
\mathbf{D} &= \sigma_f \hat{\mathbf{x}} \\
\mathbf{E} &= \frac{\mathbf{D}}{\epsilon} \\
&= \frac{\sigma_f}{\epsilon_0(1 + x/d)}\hat{\mathbf{x}} \\
V &= \int_0^d \mathbf{E} \cdot d\mathbf{l} \\
&= \frac{\sigma_f}{\epsilon_0} \int_0^f \frac{1}{1 + x/d}\, dx \\
&= \frac{\sigma_f}{\epsilon_0} d \ln 2 \\
\sigma_f &= \frac{\epsilon_0 V}{d \ln 2} \\
\mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} \\
\mathbf{P} &= \mathbf{D} - \epsilon_0 \mathbf{E} \\
&= \sigma_f \hat{\mathbf{x}} - \frac{\sigma_f}{1 + x/d}\hat{\mathbf{x}} \\
&= \sigma_f \left(1 - \frac{1}{1 + x/d}\right) \hat{\mathbf{x}} \\
&= \frac{\epsilon_0 V x}{d(d + x) \ln 2}\hat{\mathbf{x}} \\
\sigma_b|_{x=0} &= (\mathbf{P} \cdot -\hat{\mathbf{x}})|_{x=0} \\
&= 0 \\
\sigma_b|_{x=d} &= (\mathbf{P} \cdot \hat{\mathbf{x}})|_{x=d} \\
&= \frac{\epsilon_0 V}{2d \ln 2} \\
Q_{\text{surface}} &= A\frac{\epsilon_0 V}{2d \ln 2} \\
\rho_b &= -\nabla \cdot \mathbf{P} \\
&= -\frac{\epsilon_0 V}{(d + x)^2 \ln 2}
\end{aligned}
$$

$$
\begin{aligned}
Q_{\text{volume}} &= \int_0^d A\rho_b\,dx \\
&= -A\frac{\epsilon_0 V}{\ln 2}\int_0^d \frac{1}{(d+x)^2}\,dx \\
&= -A\frac{\epsilon_0 V}{2d\ln 2}
\end{aligned}
$$

4.35

$$
\begin{aligned}
\oint \mathbf{D}\cdot d\mathbf{a} &= Q_{\text{encl,f}} \\
4\pi r^2 D &= q \\
\mathbf{D} &= \frac{q}{4\pi r^2}\hat{\mathbf{r}} \\
\mathbf{E} &= \frac{\mathbf{D}}{\epsilon} \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{(1+\chi_e)r^2}\hat{\mathbf{r}} \\
\mathbf{D} &= \epsilon_0\mathbf{E}+\mathbf{P} \\
\mathbf{P} &= \mathbf{D}-\epsilon_0\mathbf{E} \\
&= \frac{q}{4\pi r^2}\hat{\mathbf{r}} - \frac{q}{4\pi(1+\chi_e)r^2}\hat{\mathbf{r}} \\
&= \frac{q}{4\pi r^2}\left(1-\frac{1}{1+\chi_e}\right)\hat{\mathbf{r}} \\
&= \frac{q\chi_e}{4\pi(1+\chi_e)r^2}\hat{\mathbf{r}} \\
\rho_b &= -\nabla\cdot\mathbf{P} \\
&= -\frac{q\chi_e}{4\pi(1+\chi_e)}\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) \\
&= -\frac{q\chi_e}{1+\chi_e}\delta^3(\mathbf{r}) \\
\sigma_b &= \mathbf{P}\cdot\hat{\mathbf{n}} \\
&= \frac{q\chi_e}{4\pi(1+\chi_e)R^2} \\
Q_{\text{surface}} &= A\sigma_b \\
&= \frac{q\chi_e}{1+\chi_e}
\end{aligned}
$$

4.36

$$\begin{aligned}
D_1^{\perp} - D_2^{\perp} &= \sigma_f \\
\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} &= 0 \\
\epsilon_1 E_1^{\perp} &= \epsilon_2 E_2^{\perp} \\
E_1^{\parallel} - E_2^{\parallel} &= 0 \\
E_1^{\parallel} &= E_2^{\parallel} \\
\tan\theta_2 &= 1 \\
\tan\theta_2 &= \frac{1}{\epsilon_2/\epsilon_1} \\
&= \frac{\epsilon_1}{\epsilon_2} \\
\frac{\tan\theta_2}{\tan\theta_1} &= \frac{\epsilon_2}{\epsilon_1}
\end{aligned}$$

If an electric field moved from air to a convex dielectric lens, $\epsilon_2/\epsilon_0 > 1$ meaning it would bend away from the normal. The lens would defocus the field.

4.39

(a)

$$\begin{aligned}
V &= \begin{cases} V_0 & r \le R \\ V_0 \frac{R}{r} & r \ge R \end{cases} \\
\mathbf{E} &= -\nabla V \\
&= \begin{cases} \mathbf{0} & r \le R \\ V_0 \frac{R}{r^2}\hat{\mathbf{r}} & r \ge R \end{cases} \\
\mathbf{P} &= \epsilon_0 \chi_e V_0 \frac{R}{r^2}\hat{\mathbf{r}},\ z < 0 \\
\sigma_b &= \mathbf{P}\cdot\hat{\mathbf{n}} \\
&= -\frac{\epsilon_0 \chi_e V_0}{R},\ \text{on the surface that touches the sphere} \\
\oint \sigma_f \, da &= Q_{\text{total}} \\
4\pi R^2 \sigma_f &= 4\pi\epsilon_0 V_0 R \\
\sigma_f &= \begin{cases} \epsilon_0 V_0/R & \text{northern hemisphere} \\ \epsilon_0 V_0 (1+\chi_e)/R & \text{southern hemisphere} \end{cases}
\end{aligned}$$

(b)

$$
\begin{aligned}
\sigma &= \sigma_b + \sigma_f \\
&= \frac{\epsilon_0 V_0}{R} \\
Q_{\text{total}} &= 4\pi\epsilon_0 V_0 R \\
V &= V_0 \frac{R}{r}
\end{aligned}
$$

5 Magnetostatics

5.1

The charge is positive.

$$
\begin{aligned}
a^2 + (R-d)^2 &= R^2 \\
a^2 + R^2 - 2dR + d^2 &= R^2 \\
2dR &= a^2 + d^2 \\
R &= \frac{a^2 + d^2}{2d} \\
p &= qBR \\
&= \frac{qB(a^2 + d^2)}{2d}
\end{aligned}
$$

5.2

$$
\begin{aligned}
y(t) &= C_1 \cos\omega t + C_2 \sin\omega t + \frac{E}{B} t + C_3 \\
y'(t) &= -C_1 \omega \sin\omega t + C_2 \omega \cos\omega t + \frac{E}{B} \\
z(t) &= C_2 \cos\omega t - C_1 \sin\omega t + C_4 \\
z'(t) &= -C_2 \omega \sin\omega t - C_1 \omega \cos\omega t
\end{aligned}
$$

(a)

$$
\begin{aligned}
0 &= C_1 + C_3 \\
\frac{E}{B} &= C_2\omega + \frac{E}{B} \\
0 &= C_2 + C_4 \\
0 &= -C_1\omega \\
C_1 &= 0 \\
C_2 &= 0 \\
C_3 &= 0 \\
C_4 &= 0 \\
y(t) &= \frac{E}{B}t \\
z(t) &= 0
\end{aligned}
$$

(b)

$$
\begin{aligned}
0 &= C_1 + C_3 \\
\frac{E}{2B} &= C_2\omega + \frac{E}{B} \\
0 &= C_2 + C_4 \\
0 &= -C_1\omega \\
C_1 &= 0 \\
C_2 &= -\frac{E}{2B\omega} \\
C_3 &= 0 \\
C_4 &= \frac{E}{2B\omega} \\
y(t) &= -\frac{E}{2B\omega}\sin\omega t + \frac{E}{B}t \\
z(t) &= -\frac{E}{2B\omega}\cos\omega t + \frac{E}{2B\omega}
\end{aligned}
$$

(c)

$$\begin{aligned}
0 &= C_1 + C_3 \\
\frac{E}{B} &= C_2\omega + \frac{E}{B} \\
0 &= C_2 + C_4 \\
\frac{E}{B} &= -C_1\omega \\
C_1 &= -\frac{E}{B\omega} \\
C_2 &= 0 \\
C_3 &= \frac{E}{B\omega} \\
C_4 &= 0 \\
y(t) &= -\frac{E}{B\omega}\cos\omega t + \frac{E}{B}t + \frac{E}{B\omega} \\
z(t) &= \frac{E}{B\omega}\sin\omega t
\end{aligned}$$

5.3

(a)

$$\begin{aligned}
\mathbf{0} &= q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \\
0 &= E + vB \\
v &= \frac{E}{B}
\end{aligned}$$

(b)

$$\begin{aligned}
qBR &= p \\
&= mv \\
&= m\frac{E}{B} \\
\frac{q}{m} &= \frac{E}{B^2R}
\end{aligned}$$

5.4

The forces on the sides at $y = \pm a/2$ cancel, leaving the sides at $z = \pm a/2$ which both experience an upwards force of $F = IB = a^2Ik/2$ meaning the total force is $F_{\text{total}} = a^2Ik$ in the positive z direction.

5.5

(a)

$$K = \frac{I}{2\pi a}$$

(b)

$$\begin{aligned}
J(s) &= \frac{k}{s} \\
I &= \int J(s)\,da \\
&= \int_0^a \int_0^{2\pi} \frac{k}{s} s\,d\phi\,ds \\
&= 2\pi a k \\
k &= \frac{I}{2\pi a} \\
J(s) &= \frac{I}{2\pi a s}
\end{aligned}$$

5.6

(a)

$$K = \sigma\omega r$$

(b)

$$\mathbf{J} = \rho\omega r \sin\theta \hat{\boldsymbol{\phi}} = \frac{3Q\omega r \sin\theta}{4\pi R^3}\hat{\boldsymbol{\phi}}$$

5.8

(a)

$$\begin{aligned}
\mathbf{B}_{\text{side}} &= \frac{\mu_0 I}{4\pi R}\left[\sin\left(\frac{\pi}{4}\right) - \sin\left(-\frac{\pi}{4}\right)\right]\hat{\mathbf{z}} \\
&= \sqrt{2}\frac{\mu_0 I}{4\pi R}\hat{\mathbf{z}} \\
\mathbf{B}_{\text{total}} &= \sqrt{2}\frac{\mu_0 I}{\pi R}\hat{\mathbf{z}}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{B}_{\text{side}} &= \frac{\mu_0 I}{4\pi R}\left[\sin\left(\frac{\pi}{n}\right) - \sin\left(-\frac{\pi}{n}\right)\right]\hat{\mathbf{z}} \\
&= \frac{\mu_0 I}{2\pi R}\sin\frac{\pi}{n}\hat{\mathbf{z}} \\
\mathbf{B}_{\text{total}} &= \frac{n\mu_0 I}{2\pi R}\sin\frac{\pi}{n}\hat{\mathbf{z}}
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{B}_{\text{circle}} &= \frac{\mu_0 I}{2R}\hat{\mathbf{z}} \\
\mathbf{B}_{\text{total}} &= \lim_{n\to\infty} \frac{n\mu_0 I}{2\pi R} \sin\frac{\pi}{n}\hat{\mathbf{z}} \\
&= \frac{\mu_0 I}{2\pi R} \lim_{n\to\infty} n \sin\frac{\pi}{n}\hat{\mathbf{z}} \\
&= \frac{\mu_0 I}{2R}\hat{\mathbf{z}}
\end{aligned}$$

5.9

(a)

$$\mathbf{B} = \frac{\mu_0 I}{8}\left(\frac{1}{a} - \frac{1}{b}\right)\hat{\mathbf{z}}$$

(b)

$$B = \frac{\mu_0 I}{2R}\left(\frac{1}{\pi} + \frac{1}{2}\right) \text{ into the page}$$

5.10

(a)

$$\begin{aligned}
F &= aIB_{\text{bottom}} - aIB_{\text{top}} \\
&= \frac{\mu_0 a I^2}{2\pi}\left(\frac{1}{s} - \frac{1}{s+a}\right) \\
&= \frac{\mu_0 a I^2}{2\pi}\frac{a}{s(s+a)} \\
&= \frac{\mu_0 a^2 I^2}{2\pi s(s+a)} \text{ upwards}
\end{aligned}$$

(b)

$$
\begin{aligned}
F_{\text{bottom}} &= \frac{\mu_0 a I^2}{2\pi s} \text{ upwards} \\
y &= s + x \sin\frac{\pi}{3} \\
&= s + \frac{\sqrt{3}}{2}x \\
B &= \frac{\mu_0 I}{2\pi(s + \sqrt{3}x/2)} \text{ out of page} \\
F_{\text{side}} &= I \int d\mathbf{l} \times \mathbf{B} \\
&= \frac{\mu_0 I^2}{2\pi} \int_0^a \frac{1}{s + \sqrt{3}x/2}\, dx \\
&= \frac{\mu_0 I^2}{\sqrt{3}\pi} \ln\left(1 + \frac{\sqrt{3}a}{2s}\right) \\
F_{\text{total}} &= \frac{\mu_0 a I^2}{2\pi s} - \frac{\mu_0 I^2}{\sqrt{3}\pi} \ln\left(1 + \frac{\sqrt{3}a}{2s}\right) \\
&= \frac{\mu_0 I^2}{2\pi}\left[\frac{a}{s} - \frac{2}{\sqrt{3}} \ln\left(1 + \frac{\sqrt{3}a}{2s}\right)\right]
\end{aligned}
$$

5.14

(a)

$$
\mathbf{B} = \begin{cases} \mathbf{0} & s < a \\ \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}} & s > a \end{cases}
$$

(b)

$$\begin{aligned}
\mathbf{J}(s) &= ks\hat{\mathbf{z}} \\
I &= \int_0^a 2\pi s' J(s')\, ds' \\
&= 2\pi k \int_0^a s'^2\, ds' \\
&= \frac{2}{3}\pi k a^3 \\
k &= \frac{3I}{2\pi a^3} \\
\mathbf{J}(s) &= \frac{3Is}{2\pi a^3}\hat{\mathbf{z}} \\
I_{\text{enc}} &= \int_0^s 2\pi s' J(s')\, ds' \\
&= \frac{3I}{a^3}\int_0^s s'^2\, ds' \\
&= \frac{Is^3}{a^3} \\
\mathbf{B} &= \begin{cases} \frac{\mu_0 I s^2}{2\pi a^3}\hat{\boldsymbol{\phi}} & s < a \\ \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}} & s > a \end{cases}
\end{aligned}$$

5.15

$$\begin{aligned}
\oint \mathbf{B}\cdot d\mathbf{l} &= \mu_0 I_{\text{enc}} \\
2BL &= \mu_0(2JLz) \\
\mathbf{B} &= -\mu_0 Jz\hat{\mathbf{y}},\ |z| < a \\
\oint \mathbf{B}\cdot d\mathbf{l} &= m_0 I_{\text{enc}} \\
2BL &= \mu_0(2aJL) \\
\mathbf{B} &= \begin{cases} -\mu_0 Ja\hat{\mathbf{y}} & z > a \\ \mu_0 Ja\hat{\mathbf{y}} & z < -a \end{cases}
\end{aligned}$$

5.16

$$\mathbf{B}_{\text{inner}} = \begin{cases} -\mu_0 n_1 I\hat{\mathbf{z}} & s < a \\ \mathbf{0} & s > a \end{cases}$$

$$\mathbf{B}_{\text{outer}} = \begin{cases} \mu_0 n_2 I\hat{\mathbf{z}} & s < b \\ \mathbf{0} & s > b \end{cases}$$

$$\mathbf{B} = \begin{cases} \mu_0 I(n_2 - n_1)\hat{\mathbf{z}} & s < a \\ \mu_0 n_2 I\hat{\mathbf{z}} & a < s < b \\ \mathbf{0} & b < s \end{cases}$$

5.17

(a)

$$\begin{aligned} \mathbf{K}_{\text{top}} &= \sigma v\hat{\mathbf{x}} \\ \mathbf{B}_{\text{top}} &= \begin{cases} -\frac{\mu_0}{2}K\hat{\mathbf{y}} & \text{above} \\ \frac{\mu_0}{2}K\hat{\mathbf{y}} & \text{below} \end{cases} \\ &= \begin{cases} -\frac{\mu_0 \sigma v}{2}\hat{\mathbf{y}} & \text{above} \\ \frac{\mu_0 \sigma v}{2}\hat{\mathbf{y}} & \text{below} \end{cases} \\ \mathbf{K}_{\text{bottom}} &= -\sigma v\hat{\mathbf{x}} \\ \mathbf{B}_{\text{bottom}} &= \begin{cases} \frac{\mu_0 \sigma v}{2}\hat{\mathbf{y}} & \text{above} \\ -\frac{\mu_0 \sigma v}{2}\hat{\mathbf{y}} & \text{below} \end{cases} \\ \mathbf{B} &= \begin{cases} \mu_0 \sigma v\hat{\mathbf{y}} & \text{between} \\ \mathbf{0} & \text{otherwise} \end{cases} \end{aligned}$$

The magnetic field between the plates points into the page.

(b)

$$\begin{aligned} \mathbf{f}_{\text{magnetic}} &= \sigma(\mathbf{v} \times \mathbf{B}) \\ &= \frac{1}{2}\mu_0 \sigma^2 v^2 \hat{\mathbf{z}} \end{aligned}$$

(c)

$$
\begin{aligned}
\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
2AE &= \frac{\sigma A}{\epsilon_0} \\
\mathbf{E} &= \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}} \\
f_{\text{electric}} &= \frac{\sigma^2}{2\epsilon_0} \\
f_{\text{electric}} &= f_{\text{magnetic}} \\
\frac{\sigma^2}{2\epsilon_0} &= \frac{1}{2}\mu_0\sigma^2 v^2 \\
v &= \frac{1}{\sqrt{\epsilon_0\mu_0}} \\
&= c
\end{aligned}
$$

5.19

It doesn't matter.

5.20

(a)

$$
\begin{aligned}
\rho_{\text{copper}} &= 8960\,\text{kg/m}^3 \\
m_{\text{copper}} &= 1.0552 \times 10^{-25}\,\text{kg} \\
\rho &= e\frac{\rho_{\text{copper}}}{m_{\text{copper}}} \\
&= 1.36 \times 10^4\,\text{C/cm}^3
\end{aligned}
$$

(b)

$$
v = 9.1 \times 10^{-3}\,\text{cm/s} = 9.1 \times 10^{-5}\,\text{m/s}
$$

(c)

$$
\begin{aligned}
\mathbf{B} &= \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}} \\
&= \frac{50\mu_0}{\pi}\hat{\boldsymbol{\phi}}\,\text{T} \\
\mathbf{f}_{\text{magnetic}} &= \frac{50\mu_0}{\pi}\text{N} \\
&\approx 2.0 \times 10^{-5}\,\text{N/m} \\
&\approx 2.0 \times 10^{-7}\,\text{N/cm}
\end{aligned}
$$

(d)

$$
\begin{aligned}
\lambda &= 1.07 \times 10^{4}\,\mathrm{C/m} \\
\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{encl}}}{\epsilon_0} \\
2\pi s L E &= \frac{\rho A L}{\epsilon_0} \\
\mathbf{E} &= \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{\mathbf{s}} \\
&= 1.92 \times 10^{16}\,\mathrm{N/C} \\
\mathbf{f}_{\text{electric}} &= \lambda \mathbf{E} \\
&= 2 \times 10^{20}\,\mathrm{N/m} \\
&= 2 \times 10^{18}\,\mathrm{N/cm} \\
\frac{f_{\text{electric}}}{f_{\text{magnetic}}} &= 10^{25}
\end{aligned}
$$

5.23

$$
\begin{aligned}
\mathbf{A} &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{\mathscr{r}}\, dl' \\
&= \frac{\mu_0 I}{4\pi}\hat{\mathbf{z}} \int_{z_1}^{z_2} \frac{1}{\sqrt{s^2 + z^2}}\, dz \\
&= \frac{\mu_0 I}{4\pi}\hat{\mathbf{z}} \left[\ln\left(z + \sqrt{s^2 + z^2}\right)\right]_{z_1}^{z_2} \\
&= \frac{\mu_0 I}{4\pi} \ln\left(\frac{z_2 + \sqrt{s^2 + z_2^2}}{z_1 + \sqrt{s^2 + z_1^2}}\right)\hat{\mathbf{z}}
\end{aligned}
$$

5.24

$$
\begin{aligned}
\mathbf{A} &= k\hat{\boldsymbol{\phi}} \\
\mathbf{B} &= \nabla \times \mathbf{A} \\
&= \frac{k}{s}\hat{\mathbf{z}} \\
\mathbf{J} &= \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \\
&= \frac{k}{\mu_0 s^2}\hat{\boldsymbol{\phi}}
\end{aligned}
$$

5.25

$$
\begin{aligned}
\nabla \cdot \mathbf{A} &= \nabla \cdot \left[-\frac{1}{2}(\mathbf{r} \times \mathbf{B})\right] \\
&= -\frac{1}{2}\nabla \cdot (\mathbf{r} \times \mathbf{B}) \\
&= -\frac{1}{2}\left[\mathbf{B} \cdot (\nabla \times \mathbf{r}) - \mathbf{r} \cdot (\nabla \times \mathbf{B})\right] \\
&= 0 \\
\nabla \times \mathbf{A} &= \nabla \times \left[-\frac{1}{2}(\mathbf{r} \times \mathbf{B})\right] \\
&= -\frac{1}{2}\nabla \times (\mathbf{r} \times \mathbf{B}) \\
&= -\frac{1}{2}[(\mathbf{B} \cdot \nabla)\mathbf{r} - (\mathbf{r} \cdot \nabla)\mathbf{B} + \mathbf{r}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})] \\
&= -\frac{1}{2}[\mathbf{B} - 3\mathbf{B}] \\
&= \mathbf{B}
\end{aligned}
$$

You could add the gradient of any scalar function and this would still hold.

5.26

(a)

$$
\begin{aligned}
\mathbf{A} &= A(s)\hat{\mathbf{z}} \\
\mathbf{B} &= \nabla \times \mathbf{A} \\
\frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}} &= -\frac{\partial A_z}{\partial s}\hat{\boldsymbol{\phi}} \\
\frac{\mu_0 I}{2\pi s} &= -\frac{dA}{ds} \\
A &= -\frac{\mu_0 I}{2\pi}\ln s\,\hat{\mathbf{z}} \\
\nabla \cdot \mathbf{A} &= \frac{\partial A_z}{\partial z} \\
&= 0 \\
\nabla \times \mathbf{A} &= -\frac{\partial A_z}{\partial s}\hat{\boldsymbol{\phi}} \\
&= \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}
\end{aligned}
$$

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$
$$2\pi s B = \mu_0 I \left(\frac{s}{R}\right)^2$$
$$\mathbf{B} = \frac{\mu_0 I s}{2\pi R^2}\hat{\boldsymbol{\phi}}$$
$$\mathbf{A} = A(s)\hat{\mathbf{z}}$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$
$$\frac{\mu_0 I s}{2\pi R^2}\hat{\boldsymbol{\phi}} = -\frac{dA}{ds}\hat{\boldsymbol{\phi}}$$
$$A(s) = -\frac{\mu_0 I s^2}{4\pi R^2}$$
$$\mathbf{A} = -\frac{\mu_0 I s^2}{4\pi R^2}\hat{\mathbf{z}}$$

5.27

$$\mathbf{A} = A(z)\hat{\mathbf{x}}$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$
$$-\frac{\mu_0 K}{2}\hat{\mathbf{y}} = \frac{dA}{dz}\hat{\mathbf{y}}$$
$$A(z) = -\frac{\mu_0 K z}{2}$$
$$\mathbf{A} = -\frac{\mu_0 K z}{2}\hat{\mathbf{x}}$$

5.30

$$
\begin{aligned}
\mathbf{B}_{\text{in}} &= \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}) \\
\mathbf{B}_{\text{out}} &= \frac{\mu_0 R^4\omega\sigma}{3r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) \\
\mathbf{B} &= \int_0^r \mathbf{B}_{\text{out}}\, dR + \int_r^R \mathbf{B}_{\text{in}}\, dR \\
&= \int_0^r \frac{\mu_0 R'^4\omega\rho}{3r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})\, dR' + \int_r^R \frac{2\mu_0 R'\omega\rho}{3}(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}})\, dR' \\
&= \frac{\mu_0\omega\rho}{3r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})\left[\frac{1}{5}R^5\right]_0^r + \frac{2\mu_0\omega\rho}{3}(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}})\left[\frac{1}{2}R'^2\right]_r^R \\
&= \frac{\mu_0\omega\rho r^2}{15}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) + \frac{\mu_0\omega\rho(R^2 - r^2)}{3}(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}) \\
&= \frac{\mu_0\omega\rho}{3}\left[\left(R^2 - \frac{3r^2}{5}\right)\cos\theta\hat{\mathbf{r}} + \left(\frac{6r^2}{5} - R^2\right)\sin\theta\hat{\boldsymbol{\theta}}\right]
\end{aligned}
$$

5.32

(a)

$$
\begin{aligned}
\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} &= -\mu_0 nI\hat{\mathbf{z}} \\
\mu_0(\mathbf{K} \times \hat{\mathbf{n}}) &= \mu_0(nI\hat{\boldsymbol{\phi}} \times \hat{\mathbf{s}}) \\
&= -\mu_0 nI\hat{\mathbf{z}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathbf{A}_{\text{above}}(R) &= \frac{\mu_0 R\sigma}{3}(\boldsymbol{\omega} \times \mathbf{r}) \\
\mathbf{A}_{\text{below}}(R) &= \frac{\mu_0 R\sigma}{3}(\boldsymbol{\omega} \times \mathbf{r}) \\
\mathbf{A}_{\text{above}} &= \mathbf{A}_{\text{below}} \\
\frac{\partial \mathbf{A}_{\text{above}}}{\partial r} &= -\frac{2\mu_0 R^4\omega\sigma}{3}\frac{\sin\theta}{r^3}\hat{\boldsymbol{\phi}} \\
\frac{\partial \mathbf{A}_{\text{below}}}{\partial r} &= \frac{\mu_0 R\omega\sigma}{3}\sin\theta\hat{\boldsymbol{\phi}} \\
\left.\frac{\partial \mathbf{A}_{\text{above}}}{\partial r}\right|_{r=R} - \left.\frac{\partial \mathbf{A}_{\text{below}}}{\partial r}\right|_{r=R} &= -\mu_0 R\omega\sigma\sin\theta\hat{\boldsymbol{\phi}} \\
&= -\mu_0\sigma\mathbf{v} \\
&= -\mu_0\mathbf{K}
\end{aligned}
$$

5.34

$$
\begin{aligned}
\hat{\mathbf{m}} &= \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}} \\
\mathbf{B}_{\text{dip}}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] \\
&= \frac{\mu_0 m}{4\pi r^3}[3(\hat{\mathbf{m}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{m}}] \\
&= \frac{\mu_0 m}{4\pi r^3}[3\cos\theta\hat{\mathbf{r}} - (\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}})] \\
&= \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})
\end{aligned}
$$

5.35

(a)

$$\mathbf{m} = I\mathbf{a} = I\pi R^2\hat{\mathbf{z}}$$

(b)

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) \approx \frac{\mu_0 I\pi R^2}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})$$

(c)

$$
\begin{aligned}
\mathbf{B}(z) &= \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}\hat{\mathbf{z}} \\
&\approx \frac{\mu_0 I}{2}\frac{R^2}{z^3}\hat{\mathbf{z}}, z \gg R \\
\mathbf{B}_{\text{dip}}(z) &= \frac{\mu_0 I}{2}\frac{R^2}{z^3}\hat{\mathbf{z}}
\end{aligned}
$$

5.36

$$
\begin{aligned}
s &= \sqrt{(w/2)^2 + z^2} \\
\sin\theta_1 &= -\frac{w/2}{\sqrt{(w/2)^2 + s^2}} \\
&= -\frac{w/2}{\sqrt{w^2/2 + z^2}} \\
\sin\theta_2 &= \frac{w/2}{\sqrt{w^2/2 + z^2}} \\
B_{\text{side}} &= \frac{\mu_0 I}{4\pi} \frac{w}{\sqrt{(w^2/2 + z^2)(w^2/4 + z^2)}} \\
\mathbf{B}_{\text{total}} &= 4B_{\text{side}} \cos\theta \hat{\mathbf{z}} \\
&= 4B_{\text{side}} \frac{w/2}{\sqrt{w^2/4 + z^2}} \hat{\mathbf{z}} \\
&= \frac{\mu_0 I}{\pi} \frac{w^2}{2(w^2/4 + z^2)\sqrt{w^2/2 + z^2}} \hat{\mathbf{z}} \\
&= \frac{\mu_0 I w^2}{2\pi z^3} \text{ for } z \gg w \\
\mathbf{m} &= I w^2 \hat{\mathbf{z}} \\
\mathbf{B}_{\text{dip}}(\mathbf{r}) &= \frac{\mu_0 m}{4\pi r^3} (2\cos\theta \hat{\mathbf{r}} + \sin\theta \hat{\boldsymbol{\theta}}) \\
&= \frac{\mu_0 I w^2}{2\pi z^3} \hat{\mathbf{z}}
\end{aligned}
$$

5.37

(a)

$$
\begin{aligned}
\mathbf{m} &= \int I \, d\mathbf{a} \\
&= \int_0^R \sigma r \omega \pi r^2 \, dr \hat{\mathbf{z}} \\
&= \pi\sigma\omega \left[\frac{1}{4} r^4\right]_0^R \hat{\mathbf{z}} \\
&= \frac{1}{4}\pi\sigma\omega R^4 \hat{\mathbf{z}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathbf{m} &= \int I \, d\mathbf{a} \\
&= \int_0^\pi \sigma\omega R \sin\theta R \pi (R\sin\theta)^2 \, d\theta \, \hat{\mathbf{z}} \\
&= \pi\sigma\omega R^4 \int_0^\pi \sin^3\theta \, d\theta \, \hat{\mathbf{z}} \\
&= \frac{4}{3}\pi\sigma\omega R^4 \hat{\mathbf{z}} \\
\mathbf{A}(\mathbf{r}) &= \frac{\mu_0 R^4 \omega\sigma}{3} \frac{\sin\theta}{r^2} \hat{\boldsymbol{\phi}} \\
\mathbf{A}_{\text{dip}}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} \\
&= \frac{\mu_0 R^4 \omega\sigma}{3} \frac{\sin\theta}{r^2} \hat{\boldsymbol{\phi}}
\end{aligned}
$$

5.39

(a) Yes, because magnetic fields do no work.

(b)

$$
\begin{aligned}
\mathbf{B} &= \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}} \\
\mathbf{v} &= \dot{s}\hat{\mathbf{s}} + s\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}} \\
\mathbf{F} &= q\mathbf{v} \times \mathbf{B} \\
&= q \begin{vmatrix} \hat{\mathbf{s}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \dot{s} & s\dot{\phi} & \dot{z} \\ 0 & \frac{\mu_0 I}{2\pi s} & 0 \end{vmatrix} \\
&= \frac{\mu_0 I q}{2\pi s} (-\dot{z}\hat{\mathbf{s}} + \dot{s}\hat{\mathbf{z}})
\end{aligned}
$$

(c)

$$
\begin{aligned}
\mathbf{a} &= \frac{d\mathbf{v}}{dt} \\
&= (\ddot{s} - s\dot{\phi}^2)\hat{\mathbf{s}} + (2\dot{s}\dot{\phi} + s\ddot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\mathbf{z}} \\
\mathbf{F} &= m\mathbf{a} \\
\ddot{s} - s\dot{\phi}^2 &= -\alpha\frac{\dot{z}}{s} \\
2\dot{s}\dot{\phi} + s\ddot{\phi} &= 0 \\
\ddot{z} &= \alpha\frac{\dot{s}}{s}
\end{aligned}
$$

(d)

$$
\begin{aligned}
\ddot{s} - s\dot{\phi}^2 &= -\alpha \frac{\dot{z}}{s} \\
2\dot{s}\dot{\phi} + s\ddot{\phi} &= 0 \\
0 &= \alpha \frac{\dot{s}}{s} \\
\dot{s} &= 0 \\
\ddot{s} &= 0 \\
s &= c_1 \\
s\ddot{\phi} &= 0 \\
\dot{\phi} &= c_2 \\
\phi &= c_2 t + c_3 \\
z &= c_4 t + c_5
\end{aligned}
$$

Helix

5.41

(a) Downwards

(b)

$$
\begin{aligned}
qE &= qvB \\
E &= vB \\
V &= Et \\
&= vBt
\end{aligned}
$$

(c) The voltage would be reversed (higher potential at the top).

5.42

$$
\begin{aligned}
\mathbf{F} &= I \int d\mathbf{l} \times \mathbf{B} \\
&= I \left(\int d\mathbf{l} \right) \times \mathbf{B} \\
F &= IBw
\end{aligned}
$$

5.43

$$
\begin{aligned}
\mathbf{B} &= B(s)\,\hat{\mathbf{z}} \\
0 &= \int \mathbf{B}\cdot d\mathbf{a} \\
&= \int (B(s)\,\hat{\mathbf{z}})\cdot(s\,ds\,d\phi\,\hat{\mathbf{z}}) \\
&= \int_0^R B(s)s\,ds\,d\phi \\
&= 2\pi\int_0^R B(s)s\,ds \\
&= \int_0^R B(s)s\,ds \\
\mathbf{v} &= \dot{s}\hat{\mathbf{s}} + s\dot{\phi}\hat{\boldsymbol{\phi}} \\
\mathbf{F} &= q\mathbf{v}\times\mathbf{B} \\
&= q\begin{vmatrix} \hat{\mathbf{s}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \dot{s} & s\dot{\phi} & 0 \\ 0 & 0 & B(s) \end{vmatrix} \\
&= B(s)q(s\dot{\phi}\hat{\mathbf{s}} - \dot{s}\hat{\boldsymbol{\phi}}) \\
\mathbf{r} &= s\hat{\mathbf{s}} \\
\boldsymbol{\tau} &= \mathbf{r}\times\mathbf{F} \\
&= B(s)qs\begin{vmatrix} \hat{\mathbf{s}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ s & 0 & 0 \\ s\dot{\phi} & -\dot{s} & 0 \end{vmatrix} \\
&= -B(s)qs\dot{s}\hat{\mathbf{z}} \\
\Delta L &= \int_{t_0}^{t} \tau\,dt \\
&= -\int_{t_0}^{t} B(s)qs\dot{s}\,dt \\
&= -q\int_0^R B(s)s\,ds \\
&= 0
\end{aligned}
$$

Initially the particle has no angular momentum and its change in angular momentum when it leaves is zero, so it still has no angular momentum and must be moving radially.

5.47

(a)

$$\mathbf{B} = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}\hat{\mathbf{z}}$$
$$\mathbf{B} = \frac{\mu_0 I R^2}{2}\left(\frac{1}{[R^2+(d/2+z)^2]^{3/2}} + \frac{1}{[R^2+(d/2-z)^2]^{3/2}}\right)\hat{\mathbf{z}}$$
$$\left.\frac{\partial B}{\partial z}\right|_{z=0} = 0$$

(b)

$$d = R$$
$$B = \frac{8\mu_0 I}{5\sqrt{5}R}$$

5.48

$$\begin{aligned}
B &= \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}} \\
B &= \int B\, drs \\
&= \frac{\mu_0\sigma\omega}{2}\int_0^R \frac{r^3}{(r^2+z^2)^{3/2}}\, dr \\
&= \frac{\mu_0\sigma\omega}{2}\frac{(z-\sqrt{R^2+z^2})^2}{\sqrt{R^2+z^2}} \\
&= \frac{\mu_0\sigma\omega}{2}\left(\frac{R^2+2z^2}{\sqrt{R^2+z^2}} - 2z\right) \\
\mathbf{m} &= \frac{1}{4}\pi\sigma\omega R^4\hat{\mathbf{z}} \\
\mathbf{B}_{\text{dip}}(\mathbf{r}) &= \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) \\
&= \frac{\mu_0\sigma\omega R^4}{8z^3}\hat{\mathbf{z}}
\end{aligned}$$

5.49

$$
\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \\
\mathbf{r} &= y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \\
\mathbf{r}' &= R(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}) \\
\boldsymbol{\mathscr{r}} &= \mathbf{r} - \mathbf{r}' \\
&= -R\cos\phi\hat{\mathbf{x}} + (y - R\sin\phi)\hat{\mathbf{y}} + z\hat{\mathbf{z}} \\
\mathscr{r}^2 &= R^2\cos^2\phi + (y - R\sin\phi)^2 + z^2 \\
&= R^2 + y^2 + z^2 - 2yR\sin\phi \\
x &= R\cos\phi \\
dx &= -R\sin\phi\, d\phi \\
y &= R\sin\phi \\
dy &= R\cos\phi\, d\phi \\
d\mathbf{l} &= -R\sin\phi\, d\phi\, \hat{\mathbf{x}} + R\cos\phi\, d\phi\, \hat{\mathbf{y}} \\
d\mathbf{l} \times \boldsymbol{\mathscr{r}} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -R\sin\phi\, d\phi & R\cos\phi\, d\phi & 0 \\ -R\cos\phi & y - R\sin\phi & z \end{vmatrix} \\
&= zR\cos\phi\, d\phi\, \hat{\mathbf{x}} + zR\sin\phi\, d\phi\, \hat{\mathbf{y}} \\
&\qquad + [R\sin\phi\, d\phi(R\sin\phi - y) + R^2\cos^2\phi\, d\phi]\hat{\mathbf{z}} \\
&= zR\cos\phi\, d\phi\, \hat{\mathbf{x}} + zR\sin\phi\, d\phi\, \hat{\mathbf{y}} \\
&\qquad + [R^2\, d\phi - yR\sin\phi\, d\phi]\hat{\mathbf{z}} \\
B_x &= \frac{\mu_0 IRz}{4\pi} \int_0^{2\pi} \frac{\cos\phi}{(R^2 + y^2 + z^2 - 2yR\sin\phi)^{3/2}}\, d\phi \\
&= 0 \\
B_y &= \frac{\mu_0 IRz}{4\pi} \int_0^{2\pi} \frac{\sin\phi}{(R^2 + y^2 + z^2 - 2yR\sin\phi)^{3/2}}\, d\phi \\
B_z &= \frac{\mu_0 IR}{4\pi} \int_0^{2\pi} \frac{R - y\sin\phi}{(R^2 + y^2 + z^2 - 2yR\sin\phi)^{3/2}}\, d\phi
\end{aligned}
$$

5.50

$$\begin{aligned}
\mathbf{F}_2 &= I_2 \oint d\mathbf{l}_2 \times \mathbf{B}_1 \\
&= I_2 \oint d\mathbf{l}_2 \times \left(\frac{\mu_0}{4\pi} I_1 \oint \frac{d\mathbf{l}_1 \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \\
&= \frac{\mu_0}{4\pi} I_1 I_2 \oint \oint d\mathbf{l}_2 \times \left(d\mathbf{l}_1 \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \\
&= \frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \left[d\mathbf{l}_1 \left(d\mathbf{l}_2 \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) - \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} (d\mathbf{l}_2 \cdot d\mathbf{l}_1) \right] \\
&= -\frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} (d\mathbf{l}_2 \cdot d\mathbf{l}_1)
\end{aligned}$$

5.51

(a)

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0 I}{4\pi} \oint \frac{d\mathbf{l} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \\
&= -\frac{\mu_0 I}{4\pi} \oint \frac{d\mathbf{l} \times \hat{\mathbf{r}}}{r^2} \\
&= \frac{\mu_0 I}{4\pi} \oint \frac{\sin\phi \, dl \, \hat{\mathbf{z}}}{r^2} \\
&= \frac{\mu_0 I}{4\pi} \oint \frac{r \, d\theta \, \hat{\mathbf{z}}}{r^2} \\
&= \frac{\mu_0 I}{4\pi} \oint \frac{d\theta}{r} \hat{\mathbf{z}}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0 I}{4\pi} \oint_0^{2\pi} \frac{d\theta}{R} \hat{\mathbf{z}} \\
&= \frac{\mu_0 I}{2R} \hat{\mathbf{z}}
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0 I}{4\pi} \oint_0^{2\pi} \frac{\sqrt{\theta}}{a} \, d\theta \, \hat{\mathbf{z}} \\
&= \frac{\mu_0 I}{4\pi a} \left[\frac{2}{3} \theta^{3/2} \right]_0^{2\pi} \hat{\mathbf{z}} \\
&= \frac{\mu_0 I \sqrt{2\pi}}{3a} \, \hat{\mathbf{z}}
\end{aligned}$$

(d)

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0 I}{4\pi} \oint_0^{2\pi} \frac{1 + e\cos\theta}{p}\, d\theta\, \hat{\mathbf{z}} \\
&= \frac{\mu_0 I}{4\pi p} [\theta + e\sin\theta]_0^{2\pi}\, \hat{\mathbf{z}} \\
&= \frac{\mu_0 I}{2p} \hat{\mathbf{z}}
\end{aligned}$$

5.52

(a)

$$\mathbf{A} = \frac{1}{4\pi} \int \frac{\mathbf{B} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau$$

5.57

$$\begin{aligned}
\mathbf{B}_0 &= -B_0 \hat{\mathbf{z}} \\
&= -B_0 (\cos\theta\, \hat{\mathbf{r}} - \sin\theta\, \hat{\boldsymbol{\theta}}) \\
\mathbf{m} &= m_0 \hat{\mathbf{z}} \\
\mathbf{B}_m &= \frac{\mu_0 m_0}{4\pi r^3} (2\cos\theta\, \hat{\mathbf{r}} + \sin\theta\, \hat{\boldsymbol{\theta}}) \\
\mathbf{B} &= \mathbf{B}_0 + \mathbf{B}_m \\
&= \left(\frac{\mu_0 m_0}{2\pi r^3} - B_0\right) \cos\theta\, \hat{\mathbf{r}} + \left(\frac{\mu_0 m_0}{4\pi r^3} + B_0\right) \sin\theta\, \hat{\boldsymbol{\theta}} \\
\mathbf{B} \cdot \hat{\mathbf{r}} &= \left(\frac{\mu_0 m_0}{2\pi r^3} - B_0\right) \cos\theta \\
0 &= \frac{\mu_0 m_0}{2\pi R^3} - B_0 \\
R &= \left(\frac{\mu_0 m_0}{2\pi B_0}\right)^{1/3}
\end{aligned}$$

5.58

(a)

$$
\begin{aligned}
\mathbf{m} &= \frac{Q}{2\pi R}\omega R \pi R^2\,\hat{\mathbf{z}} \\
&= \frac{1}{2}\omega Q R^2\,\hat{\mathbf{z}} \\
J &= \int_0^{2\pi} \frac{M}{2\pi R} R^3\,d\theta \\
&= MR^2 \\
\mathbf{L} &= J\boldsymbol{\omega} \\
&= MR^2\omega\,\hat{\mathbf{z}} \\
\frac{\mathbf{m}}{\mathbf{L}} &= \frac{\omega Q R^2/2}{MR^2\omega} \\
&= \frac{Q}{2M}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathbf{m} &= \int_0^{\pi} \frac{1}{2}\omega 2\pi (R\sin\theta) R \frac{Q}{4\pi R^2}(R\sin\theta)^2\,d\theta\,\hat{\mathbf{z}} \\
&= \frac{1}{4}\omega Q R^2 \int_0^{\pi} \sin^3\theta\,d\theta\,\hat{\mathbf{z}} \\
&= \frac{1}{3}\omega Q R^2\,\hat{\mathbf{z}} \\
\mathbf{J} &= \frac{2}{3} MR^2 \\
\mathbf{L} &= J\boldsymbol{\omega} \\
&= \frac{2}{3} MR^2\omega \\
\frac{\mathbf{m}}{\mathbf{L}} &= \frac{\omega Q R^2/3}{2MR^2\omega/3} \\
&= \frac{Q}{2M}
\end{aligned}
$$

5.60

(a)

$$
\begin{aligned}
\mathbf{m}_{\text{plate}} &= \int I\, d\mathbf{a} \\
&= \int_0^R \omega r \sigma \pi r^2\, dr\, \hat{\mathbf{z}} \\
&= \frac{1}{4}\pi\omega\sigma R^4\, \hat{\mathbf{z}} \\
\mathbf{m}_{\text{sphere}} &= \int_0^\pi \mathbf{m}_{\text{plate}}\, d\theta \\
&= \int_0^\pi \frac{1}{4}\pi\omega\rho R\sin\theta(R\sin\theta)^4\, d\theta\, \hat{\mathbf{z}} \\
&= \frac{1}{4}\pi\omega\rho R^5 \int_0^\pi \sin^5\theta\, d\theta\, \hat{\mathbf{z}} \\
&= \frac{4}{15}\pi\omega\rho R^5\, \hat{\mathbf{z}} \\
&= \frac{1}{5}\omega Q R^2\, \hat{\mathbf{z}}
\end{aligned}
$$

(b)

$$
\mathbf{B}_{\text{ave}} = \frac{\mu_0 \omega Q}{10\pi R}\, \hat{\mathbf{z}}
$$

(c)

$$
\begin{aligned}
\mathbf{A}_{\text{dip}}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \\
&= \frac{\mu_0}{4\pi}\frac{\omega Q R^2}{5}\frac{\sin\theta}{r^2}\hat{\boldsymbol{\phi}}
\end{aligned}
$$

6 Magnetic Fields in Matter

6.1

$$
\begin{aligned}
\mathbf{m}_{\text{circle}} &= I\pi a^2 \hat{\mathbf{z}} \\
\mathbf{B}_{\text{circle}}(\mathbf{r}) &= -\frac{\mu_0 I \pi a^2}{4\pi r^3}\hat{\mathbf{z}} \\
\mathbf{m}_{\text{square}} &= I b^2\, \hat{\mathbf{y}} \\
\boldsymbol{\tau} &= \mathbf{m}_{\text{square}} \times \mathbf{B}_{\text{circle}} \\
&= -\frac{\mu_0}{4}\frac{(abI)^2}{r^3}\hat{\mathbf{x}}
\end{aligned}
$$

Its equilibrium orientation is down.

6.3

(b)

$$\begin{aligned}
\mathbf{m}_1 &= m_1\hat{\mathbf{x}} \\
\mathbf{B} &= \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}_1\cdot\hat{\mathbf{x}})\hat{\mathbf{x}} - \mathbf{m}_1] \\
&= \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(m_1\hat{\mathbf{x}}\cdot\hat{\mathbf{x}})\hat{\mathbf{x}} - m_1\hat{\mathbf{x}}] \\
&= \frac{\mu_0}{4\pi}\frac{2m_1}{r^3}\hat{\mathbf{x}} \\
\mathbf{m}_2 &= m_2\hat{\mathbf{x}} \\
\mathbf{m}_2\cdot\mathbf{B} &= m_2\hat{\mathbf{x}}\cdot\frac{\mu_0}{4\pi}\frac{2m_1}{r^3}\hat{\mathbf{x}} \\
&= \frac{\mu_0}{4\pi}\frac{2m_1m_2}{r^3} \\
\mathbf{F} &= \nabla(\mathbf{m}\cdot\mathbf{B}) \\
&= -\frac{\mu_0}{4\pi}\frac{6m_1m_2}{r^4}\hat{\mathbf{x}}
\end{aligned}$$

6.5

(a)

$$\begin{aligned}
\mathbf{B} &= \begin{cases} \mu_0 J_0 x\,\hat{\mathbf{y}} & |x| \le a \\ \mu_0 J_0 a\,\hat{\mathbf{y}} & |x| \ge a \end{cases} \\
\mathbf{F} &= \nabla(\mathbf{m}\cdot\mathbf{B}) \\
&= \nabla(m_0\,\hat{\mathbf{x}}\cdot\mu_0 J_0 x\,\hat{\mathbf{y}}) \\
&= \mathbf{0}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{F} &= \nabla(\mathbf{m}\cdot\mathbf{B}) \\
&= \nabla(m_0\,\hat{\mathbf{y}}\cdot\mu_0 J_0 x\,\hat{\mathbf{y}}) \\
&= \mu_0 J_0 m_0\,\hat{\mathbf{x}}
\end{aligned}$$

6.7

$$\begin{aligned}
\mathbf{J}_b &= \nabla\times\mathbf{M} \\
&= \mathbf{0} \\
\mathbf{K}_b &= \mathbf{M}\times\hat{\mathbf{n}} \\
&= M\hat{\boldsymbol{\phi}} \\
\mathbf{B} &= \begin{cases} \mu_0\mathbf{M} & \text{inside} \\ \mathbf{0} & \text{outside} \end{cases}
\end{aligned}$$

6.8

$$
\begin{aligned}
\mathbf{M} &= ks^2\,\hat{\boldsymbol{\phi}} \\
\mathbf{J}_b &= \nabla \times \mathbf{M} \\
&= \frac{1}{s}\frac{\partial}{\partial s}(ks^3)\,\hat{\mathbf{z}} \\
&= 3ks\,\hat{\mathbf{z}} \\
\mathbf{K}_b &= \mathbf{M} \times \hat{\mathbf{n}} \\
&= -kR^2\,\hat{\mathbf{z}} \\
\oint \mathbf{B} \cdot d\mathbf{l} &= \mu_0 Q_{\text{enc}} \\
2\pi s B &= \mu_0 \int_0^{2\pi} \int_0^{s} 3ks'^2\,ds'\,d\theta \\
&= 2\pi\mu_0 ks^3 \\
\mathbf{B} &= \mu_0 ks^2\,\hat{\boldsymbol{\phi}} \\
&= \mu_0 \mathbf{M} \\
\oint \mathbf{B} \cdot d\mathbf{l} &= \mu_0 Q_{\text{enc}} \\
2\pi s B &= \mu_0 \left(\int_0^{2\pi} \int_0^{R} 3ks^2\,ds\,d\theta - 2\pi k R^3 \right) \\
&= 0 \\
\mathbf{B} &= \mathbf{0}
\end{aligned}
$$

6.12

(a)

$$\begin{aligned}
\mathbf{M} &= ks\,\hat{\mathbf{z}} \\
\mathbf{J}_b &= \nabla \times \mathbf{M} \\
&= -k\,\hat{\boldsymbol{\phi}} \\
\mathbf{K}_b &= \mathbf{M} \times \hat{\mathbf{n}} \\
&= kR\,\hat{\boldsymbol{\phi}} \\
\mathbf{B}_J &= \int_s^R -\mu_0 k\,ds\,\hat{\mathbf{z}} \\
&= -\mu_0 k(R-s)\,\hat{\mathbf{z}} \\
\mathbf{B}_K &= \mu_0 kR\,\hat{\mathbf{z}} \\
\mathbf{B}_{\text{in}} &= \mu_0 k[R-(R-s)]\,\hat{\mathbf{z}} \\
&= \mu_0 ks\,\hat{\mathbf{z}} \\
\mathbf{B} &= \begin{cases} \mu_0 ks\,\hat{\mathbf{z}} & s < R \\ \mathbf{0} & s > R \end{cases}
\end{aligned}$$

(b)

$$\begin{aligned}
\oint \mathbf{H} \cdot d\mathbf{l} &= I_{f_{\text{enc}}} \\
\mathbf{H} &= \mathbf{0} \\
\mathbf{0} &= \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \\
\mathbf{B} &= \mu_0 \mathbf{M} \\
&= \begin{cases} \mu_0 ks\,\hat{\mathbf{z}} & s < R \\ \mathbf{0} & s > R \end{cases}
\end{aligned}$$

6.13

(a)

$$\begin{aligned}
\mathbf{B} &= \mathbf{B}_0 - \frac{2}{3}\mu_0 \mathbf{M} \\
\mathbf{H} &= \frac{1}{\mu_0}\mathbf{B} \\
&= \frac{1}{\mu_0}\mathbf{B}_0 + \frac{1}{3}\mathbf{M} \\
&= \mathbf{H}_0 + \frac{1}{3}\mathbf{M}
\end{aligned}$$

6.14

6.15