Advanced Engineering Mathematics Complex Analysis by Dennis G. Zill Problems

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Contents

17 Functions of a Complex Variable

17.1 Complex Numbers

17.1.1

$$3 + 3i$$

17.1.3

$$i^8 = (i^2)^4 = (-1)^4 = 1$$

17.1.5

$$7 - 13i$$

17.1.7

$$-7 + 5i$$

17.1.9

$$11 - 10i$$

17.1.11

$$-5 + 12i$$

17.1.13

$$-2i$$

17.1.15

$$\begin{aligned} \frac{2-4i}{3+5i} &= \frac{(2-4i)(3-5i)}{34} \\ &= \frac{-14-22i}{34} \\ &= -\frac{7}{17} - \frac{11}{17}i \end{aligned}$$

17.1.17

$$\begin{aligned} \frac{(3-i)(2+3i)}{1+i} &= \frac{9+7i}{1+i} \\ &= \frac{(9+7i)(1-i)}{2} \\ &= \frac{16-2i}{2} \\ &= 8 - i \end{aligned}$$

17.1.27

$$\begin{aligned}
\frac{1}{z} &= \frac{\overline{z}}{z\overline{z}} \\
&= \frac{x - iy}{x^2 + y^2} \\
\mathrm{Re}\left(\frac{1}{z}\right) &= \frac{x}{x^2 + y^2}
\end{aligned}$$

17.1.29

$$\begin{aligned}
2z + 4\overline{z} - 4i &= 2(x + iy) + 4(x - iy) - 4i \\
&= 6x - 2(y + 2)i \\
\mathrm{Im}(2z + 4\overline{z} - 4i) &= -2y - 4
\end{aligned}$$

17.1.31

$$\begin{aligned}
z - 1 - 3i &= x + iy - 1 - 3i \\
&= (x - 1) + (y - 3)i \\
|z| &= \sqrt{(x - 1)^2 + (y - 3)^2}
\end{aligned}$$

17.1.33

$$\begin{aligned}
2z &= i(2 + 9i) \\
&= -9 + 2i \\
z &= -\frac{9}{2} + i
\end{aligned}$$

17.1.35

$$\begin{aligned}
(x + iy)^2 &= x^2 + 2xyi - y^2 \\
&= (x^2 - y^2) + 2xyi \\
x^2 &= y^2 \\
x &= y \\
2xy &= 1 \\
x^2 &= \frac{1}{2} \\
x &= \frac{\sqrt{2}}{2} \\
z &= \frac{\sqrt{2}}{2}(1 + i)
\end{aligned}$$

17.1.37

$$\begin{aligned}
z + 2\overline{z} &= x + iy + 2x - 2iy \\
&= 3x - iy \\
\frac{2-i}{1+3i} &= \frac{(2-i)(1-3i)}{10} \\
&= \frac{-1-7i}{10} \\
3x - iy &= \frac{-1-7i}{10} \\
x &= -\frac{1}{30} \\
y &= \frac{7}{10} \\
z &= -\frac{1}{30} + \frac{7}{10}i
\end{aligned}$$

17.1.39

$$\begin{aligned}
|10 + 8i| &\approx 12.8 \\
|11 - 6i| &\approx 12.5
\end{aligned}$$

$11 - 6i$ is closer.

17.2 Powers and Roots

17.2.1

$$2(\cos 0 + i \sin 0)$$

17.2.3

$$-3[\cos(-\pi/2) + i \sin(-\pi/2)]$$

17.2.5

$$\sqrt{2}[\cos(\pi/4) + i \sin(\pi/4)]$$

17.2.7

$$2[\cos(5\pi/6) + i \sin(5\pi/6)]$$

17.2.9

$$\begin{aligned}\frac{3}{-1+i} &= \frac{3(-1-i)}{2} \\ &= \frac{-3-3i}{2} \\ &= -\frac{3}{2}-\frac{3}{2}i \\ &= \frac{3\sqrt{2}}{2}[\cos(5\pi/4)+i\sin(5\pi/4)]\end{aligned}$$

17.2.11

$$-\frac{5\sqrt{3}}{2}-\frac{5}{2}i$$

17.2.13

$$5.54+2.30i$$

17.2.15

$$\begin{aligned}8[\cos(\pi/2)+i\sin(\pi/2)] &= 8i \\ \frac{1}{2}[\cos(-\pi/4)+i\sin(-\pi/4)] &= \frac{\sqrt{2}}{4}-\frac{\sqrt{2}}{4}i\end{aligned}$$

17.2.21

$$\begin{aligned}(1+\sqrt{3}i)^9 &= \{2[\cos(\pi/3)+i\sin(\pi/3)]\}^9 \\ &= 512(\cos\pi+i\sin\pi) \\ &= -512\end{aligned}$$

17.2.23

$$\begin{aligned}\left(\frac{1}{2}+\frac{1}{2}i\right)^1 0 &= \left\{\frac{\sqrt{2}}{2}[\cos(\pi/4)+i\sin(\pi/4)]\right\}^{10} \\ &= \frac{1}{32}[\cos(\pi/2)+i\sin(\pi/2)] \\ &= \frac{1}{32}i\end{aligned}$$

17.2.27

$$\begin{aligned}
w_k &= 2[\cos(2\pi k/3) + i\sin(2\pi k/3)] \\
w_0 &= 2 \\
w_1 &= -1 + \sqrt{3}i \\
w_2 &= -1 - \sqrt{3}i
\end{aligned}$$

17.2.29

$$\begin{aligned}
w_k &= \cos(\pi/4 + k\pi) + i\sin(\pi/4 + k\pi) \\
w_0 &= \frac{\sqrt{2}}{2}(1+i) \\
w_1 &= -\frac{\sqrt{2}}{2}(1+i)
\end{aligned}$$

17.2.31

$$\begin{aligned}
w_k &= \sqrt{2}[\cos(\pi/3 + k\pi) + i\sin(\pi/3 + k\pi)] \\
w_0 &= \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i \\
w_1 &= -\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i
\end{aligned}$$

17.2.33

$$\begin{aligned}
z^4 + 1 &= 0 \\
z^4 &= -1 \\
w_k &= \cos(\pi/4 + k\pi/2) + \sin(\pi/4 + k\pi/2) \\
w_0 &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \\
w_1 &= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \\
w_2 &= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \\
w_3 &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i
\end{aligned}$$

17.3 Sets in the Complex Plane

17.3.1

A vertical line at $\mathrm{Re}(z) = 5$.

17.3.3

A horizontal line at $\mathrm{Im}(z) = -3$.

17.3.5

A circle of radius 2 centred at $3i$.

17.3.7

A circle of radius 5 centred at $4 - 3i$.

17.3.9

The region of the plane to the left of (but not including) $\mathrm{Re}(z) = -1$. It is a domain.

17.3.11

The region of the plane above (but not including) $\mathrm{Im}(z) = 3$. It is a domain.

17.3.13

The region of the plane between (but not including) $\mathrm{Re}(z) = 3$ and $\mathrm{Re}(z) = 5$. It is a domain.

17.3.15

$$\begin{aligned}
z^2 &= (a + ib)^2 \\
&= a^2 - b^2 + 2iab \\
\mathrm{Re}(z^2) &= a^2 - b^2 \\
\mathrm{Re}(z^2) &> 0 \\
a^2 - b^2 &> 0 \\
a^2 &> b^2
\end{aligned}$$

The region between $y = x$ and $y = -x$. Not a domain.

17.3.17

The region between $\theta = 0$ and $\theta = 2\pi/3$. Not a domain.

17.3.19

The region outside a circle of radius 1 centred at i. It is a domain.

17.3.21

The region between the circles of radius 2 and 3 centred at i. It is a domain.

17.3.23

$$y = -x$$

17.3.25

$$\begin{aligned}
z^2 + \overline{z}^2 &= (a+ib)^2 + (a-ib)^2 \\
&= a^2 + 2iab - b^2 + a^2 - 2iab - b^2 \\
&= 2(a^2 - b^2) \\
2(a^2 - b^2) &= 2 \\
a^2 - b^2 &= 1 \\
a^2 &= b^2 + 1
\end{aligned}$$

The hyperbola $x^2 - y^2 = 1$.

17.4 Functions of a Complex Variable

17.4.1

$$\begin{aligned}
f(z) &= z^2 \\
&= (x+iy)^2 \\
&= x^2 - y^2 + 2ixy \\
u(x,y) &= x^2 - y^2 \\
&= x^2 - 4 \\
v(x,y) &= 2xy \\
&= 4x \\
x &= \frac{v}{4} \\
u &= \left(\frac{v}{4}\right)^2 - 4 \\
&= \frac{1}{16}v^2 - 4
\end{aligned}$$

17.4.3

$$\begin{aligned}
u &= -y^2 \\
v &= 0
\end{aligned}$$

Line on the left half of the real axis.

17.4.5

$$u = 0$$
$$v = 2x^2$$

Line on the top half of the imaginary axis.

17.4.7

$$f(x) = (6x - 5) + i(6y + 9)$$

17.4.9

$$f(z) = (x^2 - y^2 - 3x) + i(2xy - 3y + 4)$$

17.4.11

$$f(z) = (x^3 - 3xy^2 - 4x) + i(3x^2y - y^3 - 4y)$$

17.4.13

$$f(z) = \left(x + \frac{x}{x^2 + y^2}\right) i \left(y - \frac{y}{x^2 + y^2}\right)$$

17.4.15

(a) $-4 + i$

(b) $3 - 9i$

(c) $1 + 86i$

17.4.17

(a) $14 - 20i$

(b) $-13 + 43i$

(c) $3 - 26i$

17.4.19

$$6 - 5i$$

17.4.21

$$-4i$$

17.4.27

$$f'(z) = 12z^2 - 2(3+i)z - 5$$

17.4.29

$$\begin{aligned} f'(z) &= 2(z^2 - 4z + 8i) + (2z+1)(2z-4) \\ &= 2z^2 - 8z + 16i + 4z^2 - 8z + 2z - 4 \\ &= 6z^2 - 14z - 4 + 16i \end{aligned}$$

17.4.31

$$f'(z) = 6z(z^2 - 4i)^2$$

17.4.33

$$\begin{aligned} f'(z) &= \frac{3(2z+i) - 2(3z - 4 + 8i)}{(2z+i)^2} \\ &= \frac{6z + 3i - 6z + 8 - 16i}{(2z+i)^2} \\ &= \frac{8 - 13i}{(2z+i)^2} \end{aligned}$$

17.4.35

$3i$

17.4.37

$\pm 2i$

17.4.41

$$\begin{aligned} \frac{dx}{dt} &= 2x \\ x &= c_1 e^{2t} \\ \frac{dy}{dt} &= 2y \\ y &= c_2 e^{2t} \end{aligned}$$

17.4.43

$$
\begin{aligned}
f(z) &= \frac{1}{\bar{z}} \\
&= \frac{1}{x - iy} \\
&= \frac{x + iy}{x^2 + y^2} \\
&= \frac{x}{x^2 + y^2} + i\frac{y}{x^2 + y^2} \\
\frac{dx}{dt} &= \frac{x}{x^2 + y^2} \\
\frac{dy}{dt} &= \frac{y}{x^2 + y^2} \\
\frac{dy}{dx} &= \frac{y}{x} \\
\frac{dy}{y} &= \frac{dx}{x} \\
\ln y &= \ln x + c_1 \\
y &= c_2 x
\end{aligned}
$$

17.5 Cauchy-Riemann Equations

17.5.1

$$
\begin{aligned}
f(z) &= z^3 \\
&= (x + iy)^3 \\
&= (x^2 + 2ixy - y^2)(x + iy) \\
&= x^3 + ix^2y + 2ix^2y - 2xy^2 - xy^2 - iy^3 \\
&= (x^3 - 3xy^2) + i(3x^2y - y^3) \\
\frac{\partial u}{\partial x} &= 3x^2 - 3y^2 \\
&= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -6xy \\
&= -\frac{\partial v}{\partial x}
\end{aligned}
$$

17.5.3

$$\begin{aligned} f(z) &= \operatorname{Re}(z) \\ &= x \\ \frac{\partial u}{\partial x} &= 1 \\ &\neq \frac{\partial v}{\partial y} \end{aligned}$$

17.5.5

$$\begin{aligned} f(z) &= 4z - 6\overline{z} + 3 \\ &= 4(x+iy) - 6(x-iy) + 3 \\ &= (-2x+3) + 10iy \\ \frac{\partial u}{\partial x} &= -2 \\ &\neq \frac{\partial v}{\partial y} \end{aligned}$$

17.5.7

$$\begin{aligned} f(z) &= x^2 + y^2 \\ \frac{\partial u}{\partial x} &= 2x \\ &\neq \frac{\partial v}{\partial y} \end{aligned}$$

17.5.9

$$\begin{aligned} f(z) &= e^x \cos y + ie^x \sin y \\ u &= e^x \cos y \\ \frac{\partial u}{\partial x} &= e^x \cos y \\ \frac{\partial u}{\partial y} &= -e^x \sin y \\ v &= e^x \sin y \\ \frac{\partial v}{\partial x} &= e^x \sin y \\ \frac{\partial v}{\partial y} &= e^x \cos y \end{aligned}$$

Analytic everywhere.

17.5.11

$$\begin{aligned}
f(z) &= x + \sin x \cosh y + i(y + \cos x \sinh y) \\
u &= x + \sin x \cosh y \\
\frac{\partial u}{\partial x} &= 1 + \cos x \cosh y \\
\frac{\partial u}{\partial y} &= \sin x \sinh y \\
v &= y + \cos x \sinh y \\
\frac{\partial v}{\partial x} &= -\sin x \sinh y \\
\frac{\partial v}{\partial y} &= 1 + \cos x \cosh y
\end{aligned}$$

Analytic everywhere.

17.5.15

$$\begin{aligned}
f(z) &= 3x - y + 5 + i(ax + by - 3) \\
u &= 3x - y + 5 \\
\frac{\partial u}{\partial x} &= 3 \\
\frac{\partial u}{\partial y} &= -1 \\
v &= ax + by - 3 \\
\frac{\partial v}{\partial x} &= a \\
\frac{\partial v}{\partial y} &= b \\
a &= 1 \\
b &= 3
\end{aligned}$$

17.5.17

$$\begin{aligned}
f(z) &= x^2 + y^2 + 2ixy \\
u &= x^2 + y^2 \\
\frac{\partial u}{\partial x} &= 2x \\
\frac{\partial u}{\partial y} &= 2y \\
v &= 2xy \\
\frac{\partial v}{\partial x} &= 2y \\
\frac{\partial v}{\partial y} &= 2x
\end{aligned}$$

Only differentiable when $y = 0$.

17.5.19

$$\begin{aligned}
f(z) &= x^3 + 3xy^2 - x + i(y^3 + 3x^2y - y) \\
u &= x^3 + 3xy^2 - x \\
\frac{\partial u}{\partial x} &= 3x^2 + 3y^2 - 1 \\
\frac{\partial u}{\partial y} &= 6xy \\
v &= y^3 + 3x^2y - y \\
\frac{\partial v}{\partial x} &= 6xy \\
\frac{\partial v}{\partial y} &= 3y^2 + 3x^2 - 1
\end{aligned}$$

Only differentiable when $x = 0$ or $y = 0$.

17.5.21

$$\begin{aligned}
f(z) &= e^x \cos y + ie^x \sin y \\
f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\
&= e^x \cos y + ie^x \sin y
\end{aligned}$$

17.5.23

$$
\begin{aligned}
u &= x \\
\frac{\partial^2 u}{\partial x^2} &= 0 \\
\frac{\partial^2 u}{\partial y^2} &= 0 \\
\frac{\partial v}{\partial y} &= 1 \\
v &= y + h(x) \\
h'(x) &= 0 \\
v &= y + c \\
f(z) &= x + i(y + c)
\end{aligned}
$$

17.5.25

$$
\begin{aligned}
u &= x^2 - y^2 \\
\frac{\partial^2 u}{\partial x^2} &= 2 \\
\frac{\partial^2 u}{\partial y^2} &= -2 \\
\frac{\partial v}{\partial y} &= 2x \\
v &= 2xy + h(x) \\
2y &= 2y + h'(x) \\
h'(x) &= 0 \\
h(x) &= c \\
v &= 2xy + c \\
f(z) &= (x^2 - y^2) + i(2xy + c)
\end{aligned}
$$

17.6 Exponential and Logarithmic Functions

17.6.1

$$\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

17.6.3

$$e^{-1}\frac{\sqrt{2}}{2}(1 + i)$$

17.6.5

$$-e^{\pi}$$

17.6.7

$$e^{1.5}(\cos 2 + i\sin 2) = -1.865 + 4.075i$$

17.6.9

$$\cos 5 + i\sin 5 = 0.2836 - 0.9589i$$

17.6.11

$$\begin{aligned} e^{1+5\pi i/4}e^{-1-\pi i/3} &= e^{11\pi i/12} \\ &= \cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12} \\ &= -0.9659 + 0.2588i \end{aligned}$$

17.6.13

$$\begin{aligned} f(z) &= e^{-iz} \\ &= e^{-i(x+iy)} \\ &= e^{y-ix} \\ &= e^{y}(\cos x - i\sin x) \end{aligned}$$

17.6.15

$$\begin{aligned} f(z) &= e^{z^2} \\ &= e^{x^2-y^2+2ixy} \\ &= e^{x^2-y^2}[\cos(2xy) + i\sin(2xy)] \end{aligned}$$

17.6.17

$$\begin{aligned} e^{z} &= e^{x+iy} \\ &= e^{x}(\cos y + i\sin y) \\ |e^{z}| &= \sqrt{e^{2x}[\cos^2 y + \sin^2 y]} \\ &= e^{x} \end{aligned}$$

17.6.19

$$\begin{aligned}
e^{z+\pi i} &= e^{x+i(y+\pi)} \\
&= e^x[\cos(y+\pi) + i\sin(y+\pi)] \\
&= e^x[-\cos y - i\sin y] \\
&= -e^x(\cos y + i\sin y) \\
e^{z-\pi i} &= e^{x+i(y-\pi)} \\
&= e^x[\cos(y-\pi) + i\sin(y-\pi)] \\
&= e^x(-\cos y - i\sin y) \\
&= -e^x(\cos y + i\sin y)
\end{aligned}$$

17.6.21

$$\begin{aligned}
e^{\overline{z}} &= e^{x-iy} \\
&= e^x(\cos y - i\sin y) \\
u &= e^x \cos y \\
v &= -e^x \sin y \\
\frac{\partial u}{\partial x} &= e^x \cos y \\
&\neq \frac{\partial v}{\partial y}
\end{aligned}$$

17.6.23

$$\log_e 5 + i(\pi + 2n\pi) = 1.6094 + i(\pi + 2n\pi)$$

17.6.25

$$\log_e(2\sqrt{2}) + i\left(\frac{3}{4}\pi + 2n\pi\right) = 1.0397 + i\left(\frac{3}{4}\pi + 2n\pi\right)$$

17.6.27

$$\log_e(2\sqrt{2}) + i\left(\frac{1}{3}\pi + 2n\pi\right) = 1.0397 + i\left(\frac{1}{3}\pi + 2n\pi\right)$$

17.6.29

$$\log_e(6\sqrt{2}) - \frac{\pi}{4}i = 2.1383 - \frac{\pi}{4}i$$

17.6.31

$$\log_e 13 + 2.7468i = 2.5649 + 2.7468i$$

17.6.33

$$5\left(\log_e 2+\frac{\pi}{3}i\right)=3.4657-\frac{\pi}{3}i$$

17.6.35

$$z=\log_e 4+i\left(\frac{\pi}{2}+2n\pi\right)=1.3863+i\left(\frac{\pi}{2}+2n\pi\right)$$

17.6.37

$$\begin{aligned}z-1&=2+i\left(-\frac{\pi}{2}+2n\pi\right)\\ z&=3+i\left(-\frac{\pi}{2}+2n\pi\right)\end{aligned}$$

17.6.39

$$\begin{aligned}\ln(-i)&=i\left(-\frac{\pi}{2}+2n\pi\right)\\ (-i)^{4i}&=e^{4i\ln(-i)}\\ &=e^{4i\times i(-\pi/2+2n\pi)}\\ &=e^{2\pi(1-4n)}\end{aligned}$$

17.6.41

$$\begin{aligned}\ln(1+i)&=\log_e\sqrt{2}+i\left(\frac{\pi}{4}+2n\pi\right)\\ (1+i)^{(1+i)}&=e^{(1+i)\ln(1+i)}\\ &=e^{(1+i)[\log_e\sqrt{2}+i(\pi/4+2n\pi)]}\\ &=e^{\log_e\sqrt{2}+i(\pi/4+2n\pi)+i\log_e\sqrt{2}-(\pi/4+2n\pi)}\\ &=e^{(\log_e\sqrt{2}-\pi/4-2n\pi)+i(\log_e\sqrt{2}+\pi/4+2n\pi)}\\ &=e^{-2n\pi}e^{(\log_e\sqrt{2}-\pi/4)+i(\log_e\sqrt{2}+\pi/4)}\\ &=e^{-2n\pi}e^{\log_e\sqrt{2}-\pi/4}e^{i(\log_e\sqrt{2}+\pi/4)}\\ &=e^{-2n\pi}(0.2739+0.5837i)\end{aligned}$$

17.6.43

$$\begin{aligned}\mathrm{Ln}(-1)&=\pi i\\ (-1)^{(-2i/\pi)}&=e^{(-2i/\pi)\,\mathrm{Ln}(-1)}\\ &=e^{(-2i/\pi)(\pi i)}\\ &=e^{2}\end{aligned}$$

17.6.47

(a)

$$\begin{aligned}
(-1+i)^2 &= -2i \\
\mathrm{Ln}(-1+i)^2 &= \mathrm{Ln}(-2i) \\
&= \log_e 2 - \frac{\pi}{2}i \\
2\,\mathrm{Ln}(-1+i) &= 2\log_e \sqrt{2} + \frac{3\pi}{2}i \\
&\neq \mathrm{Ln}(-1+i)^2
\end{aligned}$$

Not true

(b)

$$\begin{aligned}
\mathrm{Ln}\, i^3 &= \mathrm{Ln}(-i) \\
&= -\frac{\pi}{2}i \\
3\,\mathrm{Ln}\, i &= \frac{3\pi}{2}i \\
&\neq \mathrm{Ln}\, i^3
\end{aligned}$$

Not true

(c)

$$\begin{aligned}
\ln i^3 &= i\left(-\frac{\pi}{2} + 2n\pi\right) \\
3\ln i &= 3i\left(\frac{\pi}{2} + 2n\pi\right) \\
&\neq \ln i^3
\end{aligned}$$

Not true

17.7 Trigonometric and Hyperbolic Functions

17.7.1

$$\begin{aligned}
\cos(3i) &= \cos 0 \cosh 3 - i \sin 0 \sinh 3 \\
&= \cosh 3 \\
&= 10.0677
\end{aligned}$$

17.7.3

$$\begin{aligned}
\sin(\pi/4 + i) &= \sin\frac{\pi}{4}\cosh 1 + i\cos\frac{\pi}{4}\sinh 1 \\
&= 1.0911 + 0.8309i
\end{aligned}$$

17.7.5

$$\begin{aligned}
\tan i &= \frac{\sin i}{\cos i} \\
&= \frac{\sin 0 \cosh 1 + i \cos 0 \sinh 1}{\cos 0 \cosh 1 + i \sin 0 \sinh 1} \\
&= \frac{i \sinh 1}{\cosh 1} \\
&= i \tanh 1 \\
&= 0.7615i
\end{aligned}$$

17.7.7

$$\begin{aligned}
\sec(\pi + i) &= \frac{1}{\cos(\pi + i)} \\
&= \frac{1}{\cos \pi \cosh 1 + \sin \pi \sinh 1} \\
&= -\frac{1}{\cosh 1} \\
&= -0.6480
\end{aligned}$$

17.7.9

$$\begin{aligned}
\cosh(\pi i) &= \cosh 0 \cos \pi + i \sinh 0 \sin \pi \\
&= -1
\end{aligned}$$

17.7.11

$$\begin{aligned}
\sinh(1 + \pi i/3) &= \sinh 1 \cos(\pi/3) + i \cosh 1 \sin(\pi/3) \\
&= 0.5876 + 1.3363i
\end{aligned}$$

17.7.15

$$\begin{aligned}
\sin z &= 2 \\
\frac{e^{iz} - e^{-iz}}{2i} &= 2 \\
e^{iz} - e^{-iz} &= 4i \\
e^{2iz} - 1 &= 4ie^{iz} \\
e^{2iz} - 4ie^{iz} - 1 &= 0 \\
e^{iz} &= \frac{4i \pm \sqrt{-16+4}}{2} \\
&= (2 \pm \sqrt{3})i \\
iz &= \log_e(2 \pm \sqrt{3}) + i(\pi/2 + 2n\pi) \\
z &= (\pi/2 + 2n\pi) - i\log_e(2 \pm \sqrt{3})
\end{aligned}$$

17.7.17

$$\begin{aligned}
\sinh z &= -i \\
\frac{e^{z} - e^{-z}}{2} &= -i \\
e^{2z} + 2ie^{z} - 1 &= 0 \\
e^{z} &= \frac{-2i \pm \sqrt{-4+4}}{2} \\
&= -i \\
z &= \ln(-i) \\
&= i\left(-\frac{\pi}{2} + 2n\pi\right)
\end{aligned}$$

17.7.19

$$\begin{aligned}
\cos z &= \sin z \\
\frac{e^{iz} + e^{-iz}}{2} &= \frac{e^{iz} - e^{-iz}}{2i} \\
e^{iz} + e^{-iz} &= \frac{e^{iz} - e^{-iz}}{i} \\
&= -i(e^{iz} - e^{-iz}) \\
e^{2iz} + 1 &= -i(e^{2iz} - 1) \\
e^{2iz}(1+i) &= -1 + i \\
e^{2iz} &= \frac{-1+i}{1+i} \\
&= \frac{(-1+i)(1-i)}{(1+i)(1-i)} \\
&= \frac{-1+i+i+1}{1-i+i+1} \\
&= \frac{2i}{2} \\
&= i \\
2iz &= \ln i \\
&= i\left(\frac{\pi}{2} + 2n\pi\right) \\
z &= \frac{\pi}{4} + n\pi
\end{aligned}$$

17.7.21

$$\begin{aligned}
\cos z &= \cosh 2 \\
\cos x \cosh y - i \sin x \sinh y &= \cosh 2 \\
y &= \pm 2 \\
x &= 2n\pi \\
z &= 2n\pi \pm 2i
\end{aligned}$$

17.8 Inverse Trigonometric and Hyperbolic Functions

17.8.1

$$
\begin{aligned}
\arcsin z &= -i \ln[iz + (1 - z^2)^{1/2}] \\
\arcsin(-i) &= -i \ln[i(-i) + (1 - (-i)^2)^{1/2}] \\
&= -i \ln[1 \pm \sqrt{2}] \\
\ln(1 + \sqrt{2}) &= \log_e(1 + \sqrt{2}) + 2n\pi i \\
\ln(1 - \sqrt{2}) &= \ln\left(-\frac{1}{1 + \sqrt{2}}\right) \\
&= -\ln[-(1 + \sqrt{2})] \\
&= -[\log_e(1 + \sqrt{2}) + i(\pi + 2n\pi)] \\
&= -\log_e(1 + \sqrt{2}) + i(\pi + 2n\pi) \\
\ln(1 \pm \sqrt{2}) &= (-1)^n \log_e(1 + \sqrt{2}) + n\pi i \\
\arcsin(-i) &= -i[(-1)^n \log_e(1 + \sqrt{2}) + n\pi i] \\
&= n\pi - (-1)^n i \log_e(1 + \sqrt{2}) \\
&= n\pi + (-1)^{n+1} i \log_e(1 + \sqrt{2})
\end{aligned}
$$

17.8.3

$$
\begin{aligned}
\arcsin 0 &= -i \ln(\pm 1) \\
&= -i(n\pi i) \\
&= n\pi
\end{aligned}
$$

17.8.5

$$
\begin{aligned}
\arccos 2 &= -i \ln[2 + i(1 - 2^2)^{1/2}] \\
&= -i \ln[2 \pm \sqrt{3}] \\
\ln(2 + \sqrt{3}) &= \log_e(2 + \sqrt{3}) + 2n\pi i \\
\ln(2 - \sqrt{3}) &= \log_e(2 - \sqrt{3}) + 2n\pi i \\
&= -\log_e(2 + \sqrt{3}) + 2n\pi i \\
\ln(2 \pm \sqrt{3}) &= \pm \log_e(2 + \sqrt{3}) + 2n\pi i \\
\arccos 2 &= 2n\pi \pm i \log_e(2 + \sqrt{3})
\end{aligned}
$$

17.8.7

$$\begin{aligned}
\arccos\frac{1}{2} &= -i\ln\left\{\frac{1}{2} + i\left[1 - \left(\frac{1}{2}\right)^2\right]^{1/2}\right\} \\
&= -i\ln\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right) \\
\ln\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right) &= i\left(\pm\frac{\pi}{3} + 2n\pi\right) \\
\arccos\frac{1}{2} &= \pm\frac{\pi}{3} + 2n\pi
\end{aligned}$$

17.8.9

$$\begin{aligned}
\arctan 1 &= \frac{i}{2}\ln\frac{1+i}{-1+i} \\
\frac{1+i}{-1+i} &= \frac{(1+i)(-1-i)}{(-1+i)(-1-i)} \\
&= -i \\
\ln(-i) &= i\left(-\frac{\pi}{2} + 2n\pi\right) \\
\arctan 1 &= \frac{\pi}{4} + n\pi
\end{aligned}$$

17.8.11

$$
\begin{aligned}
\operatorname{arcsinh}\frac{4}{3} &= \ln\left\{\frac{4}{3} + \left[\left(\frac{4}{3}\right)^2 + 1\right]^{1/2}\right\} \\
&= \ln\left(\frac{4}{3} \pm \frac{5}{3}\right) \\
\ln\left(\frac{4}{3} + \frac{5}{3}\right) &= \ln\frac{9}{3} \\
&= \ln 3 \\
&= \log_e 3 + 2n\pi i \\
\ln\left(\frac{4}{3} - \frac{5}{3}\right) &= \ln\left(-\frac{1}{3}\right) \\
&= \log_e \frac{1}{3} + i(\pi + 2n\pi) \\
&= -\log_e 3 + i(\pi + 2n\pi) \\
\operatorname{arcsinh}\frac{4}{3} &= (-1)^n \log_e 3 + n\pi i
\end{aligned}
$$

17.9 Chapter in Review

17.9.1

0, 32

17.9.3

$$
\begin{aligned}
\frac{3+4i}{3-4i} &= \frac{(3+4i)^2}{(3-4i)(3+4i)} \\
&= \frac{-7+24i}{25} \\
&= -\frac{7}{25} + \frac{24}{25}i \\
\operatorname{Re}\left(\frac{z}{\bar{z}}\right) &= -\frac{7}{25}
\end{aligned}
$$

17.9.5

$$\begin{aligned}\frac{4i}{-3-4i} &= \frac{(4i)(-3+4i)}{(-3-4i)(-3+4i)}\\ &= \frac{-16-12i}{25}\\ &= -\frac{16}{25}-\frac{12}{25}i\\ |z| &= \sqrt{\left(\frac{16}{25}\right)^2+\left(\frac{12}{25}\right)^2}\\ &= \frac{4}{5}\end{aligned}$$

17.9.7

False

17.9.9

$$\begin{aligned}e^z &= 2i\\ z &= \ln(2i)\\ &= \log_e 2 + i\left(\frac{\pi}{2}+2n\pi\right)\end{aligned}$$

17.9.11

$$\begin{aligned}(1+i)^{(2+i)} &= e^{(2+i)\ln(1+i)}\\ ln(1+i) &= \log_e\sqrt{2}+\frac{\pi}{4}i\\ (2+i)\left(\log_e\sqrt{2}+\frac{\pi}{4}i\right) &= 2\log_e\sqrt{2}+\frac{\pi}{2}i+i\log_e\sqrt{2}-\frac{\pi}{4}\\ &= \left(2\log_e\sqrt{2}-\frac{\pi}{4}\right)+i\left(\log_e\sqrt{2}+\frac{\pi}{2}\right)\\ (1+i)^{(2+i)} &= e^{2\log_e\sqrt{2}-\pi/4}\left[\cos\left(\log_e\sqrt{2}+\frac{\pi}{2}\right)+i\sin\left(\log_e\sqrt{2}+\frac{\pi}{2}\right)\right]\\ &\approx -0.3097+0.8576i\end{aligned}$$

17.9.13

False

17.9.15

$$\mathrm{Ln}(-ie^3) = 3-\frac{\pi}{2}i$$

17.9.21

$$\begin{aligned}
z^2 &= x^2 - y^2 + 2ixy \\
\operatorname{Im}(z^2) &\leq 2 \\
2xy &\leq 2
\end{aligned}$$

17.9.23

$$\frac{1}{\sqrt{x^2+y^2}} \leq 1$$

17.9.27

$$\begin{aligned}
z^4 &= 1 - i \\
z_k &= 2^{1/8} e^{(-\pi/4 + 2k\pi)i/4} \\
&= 2^{1/8} e^{i(k\pi/2 - \pi/16)} \\
z_0 &= 1.0695 - 0.2127i \\
z_1 &= 0.2127 + 1.0695i \\
z_2 &= -1.0695 + 0.2127i \\
x_3 &= -0.2127 - 1.0695i
\end{aligned}$$

18 Integration in the Complex Plane

18.1 Contour Integrals

18.1.1

$$\begin{aligned}
z(t) &= 2t + i(4t-1) \\
z'(t) &= 2 + 4i \\
f(z(t)) &= (2t+3) + i(4t-1) \\
f(z(t))z'(t) &= [(2t+3) + i(4t-1)](2+4i) \\
&= (2t+3)(2) + (2t+3)(4i) + i(4t-1)(2) + i(4t-1)(4i) \\
&= 4t + 6 + 8it + 12i + 8it - 2i - 16t + 4 \\
&= (-12t+10) + i(16t+10) \\
\int_C f(z)\,dz &= \int_1^3 f(z(t))z'(t)\,dt \\
&= \int_1^3 (-12t+10)\,dt + i\int_1^3 (16t+10)\,dt \\
&= -28 + 84i
\end{aligned}$$

18.1.3

$$\begin{aligned}
z(t) &= 3t + 2it \\
z'(t) &= 3 + 2i \\
\int_C f(z)\,dz &= \int_{-2}^{2} (3t+2it)^2(3+2i)\,dt \\
&= \int_{-2}^{2} [(3+2i)t]^2(3+2i)\,dt \\
&= (3+2i)^3 \int_{-2}^{2} t^2\,dt \\
&= (-9+46i)\frac{16}{3} \\
&= -48 + \frac{736}{3}i
\end{aligned}$$

18.1.5

$$\begin{aligned}
z(t) &= e^{it} \\
z'(t) &= ie^{it} \\
\int_C f(z)\,dz &= \int_{-\pi/2}^{\pi/2} \frac{1+e^{it}}{e^{it}} ie^{it}\,dt \\
&= i\int_{-\pi/2}^{\pi/2} (1+e^{it})\,dt \\
&= i\left[t + \frac{1}{i}e^{it}\right]_{-\pi/2}^{\pi/2} \\
&= i[t - ie^{it}]_{-\pi/2}^{\pi/2} \\
&= i\left(\frac{\pi}{2} - ie^{\pi i/2} + \frac{\pi}{2} + ie^{-\pi i/2}\right) \\
&= i(\pi+2)
\end{aligned}$$

18.1.7

$$\begin{aligned}
z(t) &= \cos t + i\sin t \\
z'(t) &= -\sin t + i\cos t \\
\int_C f(z)\,dz &= \int_0^{2\pi} \cos t(-\sin t + i\cos t)\,dt \\
&= \int_0^{2\pi} \left(-\frac{1}{2}\sin 2t + i\cos^2 t\right)\,dt \\
&= \pi i
\end{aligned}$$

18.1.9

$$
\begin{aligned}
z(t) &= (1-t)+it \\
z'(t) &= -1+i \\
\int_C f(z)\,dz &= \int_0^1 [(1-t)^2+it^3](-1+i)\,dt \\
&= \int_0^1 (1-2t+t^2+it^3)(-1+i)\,dt \\
&= \int_0^1 (-1+i+2t-2it-t^2+it^2-it^3-t^3)\,dt \\
&= \int_0^1 (-1+2t-t^2-t^3)\,dt + i\int_0^1 (1-2t+t^2-t^3)\,dt \\
&= -\frac{7}{12}+\frac{1}{12}i
\end{aligned}
$$

18.1.17

$$
\begin{aligned}
z(t) &= 1+it \\
z'(t) &= i \\
\int_{C_1} f(z)\,dz &= \int_0^1 i\,dt \\
&= i \\
z(t) &= (1-t)+i(1-t) \\
z'(t) &= -(1+i) \\
\int_{C_2} f(z)\,dz &= -\int_0^1 (1-t)(1+i)\,dt \\
&= -\int_0^1 (1+i-t-it)\,dt \\
&= -\frac{1}{2}-\frac{1}{2}i \\
z(t) &= t \\
z'(t) &= 1 \\
\int_{C_3} f(z)\,dz &= \int_0^1 t\,dt \\
&= \frac{1}{2} \\
\int_C f(z)\,dz &= \frac{1}{2}i
\end{aligned}
$$

18.1.19

$$
\begin{aligned}
z(t) &= 1 + it \\
z'(t) &= i \\
\int_{C_1} f(z)\,dz &= \int_0^1 (1+it)^2 i\,dt \\
&= i\int_0^1 (1 + 2it - t^2)\,dt \\
&= -1 + \frac{2}{3}i \\
z(t) &= (1+i)(1-t) \\
z'(t) &= -(1+i) \\
\int_{C_2} f(z)\,dz &= -\int_0^1 [(1+i)(1-t)]^2(1+i)\,dt \\
&= -\int_0^1 (1 - t + i - it)^2(1+i)\,dt \\
&= \frac{2}{3} - \frac{2}{3}i \\
z(t) &= t \\
z'(t) &= 1 \\
\int_{C_3} f(z)\,dz &= \int_0^1 t^2\,dt \\
&= \frac{1}{3} \\
\int_C f(z)\,dz &= 0
\end{aligned}
$$

18.1.23

$$
\begin{aligned}
z(t) &= t + i(1-t^2) \\
z'(t) &= 1 - 2it \\
\int_C f(z)\,dz &= \frac{4}{3} - \frac{5}{3}i
\end{aligned}
$$

18.1.25

$$\begin{aligned}
L &= 10\pi \\
|z^2+1| &\geq |z^2| - 1 \\
\left|\frac{e^z}{z^2+1}\right| &\leq \frac{|e^z|}{|z^2|-1} \\
&= \frac{e^5}{24} \\
&= M \\
\left|\oint \frac{e^z}{z^2+1}\,dz\right| &\leq ML \\
&= \frac{5\pi e^5}{12}
\end{aligned}$$

18.1.27

$$\begin{aligned}
z(t) &= (1+i)t,\ 0 \leq t \leq 1 \\
L &= \sqrt{2} \\
|z^2+4| &= |2it^2+4| \\
&\leq |2i+4| \\
&= \sqrt{20} \\
&= 2\sqrt{5} \\
&= M \\
\left|\oint (z^2+4)\,dz\right| &\leq ML \\
&= 2\sqrt{10}
\end{aligned}$$

18.1.33

$$\begin{aligned}
z(t) &= e^{it} \\
z'(t) &= ie^{it} \\
\oint \overline{f(z)}\,dz &= \int_0^{2\pi} 2e^{-it} ie^{it}\,dt \\
&= 4\pi i
\end{aligned}$$

The circulation is 0 and the flux is 4π.

18.2 Cauchy-Goursat Theorem

18.2.1

$$
\begin{aligned}
z &= e^{it},\ 0 \le t \le 2\pi \\
z' &= ie^{it} \\
\int (z^3 - 1 + 3i)\, dz &= \int_0^{2\pi} [(e^{it})^3 - 1 + 3i] ie^{it}\, dt \\
&= i \int_0^{2\pi} (e^{4it} - e^{it} + 3ie^{it})\, dt \\
&= \left[\frac{1}{4} e^{4it} - e^{it} + 3ie^{it} \right]_0^{2\pi} \\
&= \frac{1}{4} e^{8\pi i} - e^{2\pi i} + 3ie^{2\pi i} - \frac{1}{4} + 1 - 3i \\
&= \frac{1}{4} - 1 + 3i - \frac{1}{4} + 1 - 3i \\
&= 0
\end{aligned}
$$

18.2.9

$$
\begin{aligned}
\int_C \frac{1}{z}\, dz &= \int_0^{2\pi} \frac{1}{e^{it}} ie^{it}\, dt \\
&= 2\pi i
\end{aligned}
$$

18.2.11

$$
\begin{aligned}
\oint_C \left(z + \frac{1}{z} \right) dz &= \oint_C \frac{1}{z^{-1}}\, dz + \oint_C \frac{1}{z}\, dz \\
&= 0 + 2\pi i \\
&= 2\pi i
\end{aligned}
$$

18.2.13

$$
\begin{aligned}
z^2 - \pi^2 &= (z + \pi)(z - \pi) \\
\frac{z}{z^2 - \pi^2} &= \frac{1/2}{z + \pi} + \frac{1/2}{z - \pi} \\
\oint_C \frac{z}{z^2 - \pi^2}\, dz &= \frac{1}{2} \oint \left(\frac{1}{z + \pi} + \frac{1}{z - \pi} \right) dz \\
&= 0
\end{aligned}
$$

18.2.15

(a)

$$
\begin{aligned}
\frac{2z+1}{z^2+z} &= \frac{2z+1}{z(z+1)} \\
&= \frac{1}{z} + \frac{1}{z+1} \\
\oint_C \frac{2z+1}{z^2+z}\,dz &= \oint_C \frac{2z+1}{z(z+1)}\,dz \\
&= \oint_C \left(\frac{1}{z} + \frac{1}{z+1}\right)\,dz \\
&= 2\pi i
\end{aligned}
$$

(b) $4\pi i$

(c) 0

18.2.17

(a)

$$
\begin{aligned}
\frac{-3z+2}{z^2-8z+12} &= \frac{1}{z-1} - \frac{4}{z-6} \\
\oint_C \frac{-3z+2}{z^2-8z+12}\,dz &= \oint \left(\frac{1}{z-1} - \frac{4}{z-6}\right)\,dz \\
&= -8\pi i
\end{aligned}
$$

(b) $-6\pi i$

18.2.19

$$
\begin{aligned}
\frac{z-1}{z(z-i)(z-3i)} &= \frac{1}{3z} - \frac{1/2-i/2}{z-i} + \frac{1/6-i/2}{z-3i} \\
\oint_C \frac{z-1}{z(z-i)(z-3i)}\,dz &= -\left(\frac{1}{2} - \frac{i}{2}\right)2\pi i \\
&= -\pi(1+i)
\end{aligned}
$$

18.2.21

$$
\begin{aligned}
\frac{8z-3}{z^2-z} &= \frac{3}{z} + \frac{5}{z-1} \\
\oint_C f(z)\,dz &= \oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz \\
&= \oint_{C_1} \left(\frac{3}{z} + \frac{5}{z-1}\right)\,dz - \oint_{C_2} \left(\frac{3}{z} + \frac{5}{z-1}\right)\,dz \\
&= 6\pi i - 10\pi i \\
&= -4\pi i
\end{aligned}
$$

18.2.23

$$
\begin{aligned}
\oint_C \left(\frac{e^z}{z+3} - 3\overline{z}\right)\,dz &= \oint_C \frac{e^z}{z+3}\,dz - 3\oint_C \overline{z}\,dz \\
&= -3\oint_0^{2\pi} e^{-it} i e^{it}\,dt \\
&= -6\pi i
\end{aligned}
$$