

Advanced Engineering Mathematics Partial Differential Equations by Dennis G. Zill Problems

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12 Orthogonal Functions and Fourier Series

12.1 Orthogonal Functions

12.1.7

$$\begin{aligned}
\int_0^{\pi/2} \sin mx \sin nx \, dx &= \frac{1}{2} \int_0^{\pi/2} [\cos(m-n)x - \cos(m+n)x] \, dx \\
&= \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_0^{\pi/2} \\
&= \frac{1}{2} \left(\frac{\sin(m-n)\pi/2}{m-n} - \frac{\sin(m+n)\pi/2}{m+n} \right) \\
&= 0 \\
\|\sin nx\|^2 &= (\sin nx, \sin nx) \\
&= \int_0^{\pi/2} \sin^2 nx \, dx \\
&= \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2nx) \, dx \\
&= \frac{1}{2} \left[x - \frac{1}{2n} \sin 2nx \right]_0^{\pi/2} \\
&= \frac{\pi}{4} \\
\|\sin nx\| &= \frac{\sqrt{\pi}}{2}
\end{aligned}$$

12.1.9

$$\begin{aligned}\int_0^\pi \sin mx \sin nx \, dx &= \frac{1}{2} \int_0^\pi [\cos(m-n)x - \cos(m+n)x] \, dx \\ &= \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_0^\pi \\ &= 0\end{aligned}$$

$$\begin{aligned}\|\sin nx\|^2 &= (\sin nx, \sin nx) \\ &= \int_0^\pi \sin^2 nx \, dx \\ &= \frac{1}{2} \int_0^\pi (1 - \cos 2nx) \, dx \\ &= \frac{1}{2} \left[x - \frac{1}{2n} \sin 2nx \right]_0^\pi \\ &= \frac{\pi}{2}\end{aligned}$$

$$\|\sin nx\| = \sqrt{\frac{\pi}{2}}$$

12.1.21

$$T = 1$$

12.1.23

$$T = 2\pi$$

12.1.25

$$T = 2\pi$$

12.2 Fourier Series

12.2.1

$$\begin{aligned}p &= \pi \\a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\&= \frac{1}{\pi} \int_0^{\pi} dx \\&= 1 \\a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\&= \frac{1}{\pi} \int_0^{\pi} \cos nx dx \\&= \frac{1}{n\pi} [\sin nx]_0^{\pi} \\&= 0 \\b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\&= \frac{1}{\pi} \int_0^{\pi} \sin nx dx \\&= -\frac{1}{n\pi} [\cos nx]_0^{\pi} \\&= -\frac{1}{n\pi} [(-1)^n - 1] \\&= \frac{1 - (-1)^n}{n\pi} \\f(x) &= \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin nx\end{aligned}$$

The series converges to $\frac{1}{2}$ at the point of discontinuity.

12.2.3

$$\begin{aligned}p &= 1 \\a_0 &= \frac{3}{2} \\a_n &= \int_{-1}^0 \cos n\pi x \, dx + \int_0^1 x \cos n\pi x \, dx \\&= \frac{1}{n\pi} [\sin n\pi x]_{-1}^0 + \frac{1}{n\pi} \left[\frac{\cos n\pi x}{n\pi} + x \sin n\pi x \right]_0^1 \\&= \frac{(-1)^n - 1}{n^2 \pi^2} \\b_n &= \int_{-1}^0 \sin n\pi x \, dx + \int_0^1 x \sin n\pi x \, dx \\&= -\frac{1}{n\pi} [\cos n\pi x]_{-1}^0 + \frac{1}{n\pi} \left[\frac{\sin n\pi x}{n\pi} - x \cos n\pi x \right]_0^1 \\&= -\frac{1}{n\pi} \\f(x) &= \frac{3}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2 \pi^2} \cos n\pi x - \frac{1}{n\pi} \sin n\pi x \right]\end{aligned}$$

The series converges to $\frac{1}{2}$ at the point of discontinuity.

12.3 Fourier Cosine and Sine Series

12.3.1

Odd

12.3.3

Neither

12.3.5

Even

12.3.7

Odd

12.3.9

Neither

12.3.11

$$\begin{aligned}
b_n &= -2\pi \int_0^1 \sin n\pi x \, dx \\
&= \frac{2}{n} [\cos n\pi x]_0^1 \\
&= \frac{2}{n} [(-1)^n - 1] \\
f &= \sum_{n=1}^{\infty} \frac{2}{n} [(-1)^n - 1] \sin n\pi x
\end{aligned}$$

12.3.13

$$\begin{aligned}
a_0 &= \frac{2}{\pi} \int_0^{\pi} x \, dx \\
&= \pi \\
a_n &= 2 \int_0^{\pi} x \cos nx \, dx \\
&= \frac{2}{n} \left[\frac{\cos nx}{n} + x \sin nx \right]_0^{\pi} \\
&= \frac{2[(-1)^n - 1]}{n^2} \\
f &= \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx
\end{aligned}$$

12.3.25

$$\begin{aligned}
 a_0 &= 2 \int_0^1 f(x) dx \\
 &= 1 \\
 a_n &= 2 \int_0^1 f(x) \cos n\pi x dx \\
 &= 2 \int_0^{1/2} \cos n\pi x dx \\
 &= \frac{2}{n\pi} [\sin n\pi x]_0^{1/2} \\
 &= \frac{2}{n\pi} \sin \frac{n\pi}{2} \\
 f &= \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n} \cos n\pi x
 \end{aligned}$$

$$\begin{aligned}
 b_n &= 2 \int_0^1 f(x) \sin n\pi x dx \\
 &= 2 \int_0^{1/2} \sin n\pi x dx \\
 &= -\frac{2}{n\pi} [\cos n\pi x]_0^{1/2} \\
 &= \frac{2}{n\pi} \left(1 - \cos \frac{n\pi}{2}\right) \\
 f &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - \cos \frac{n\pi}{2}}{n} \sin n\pi x
 \end{aligned}$$

12.3.27

$$\begin{aligned}
 a_0 &= \frac{4}{\pi} \int_0^{\pi/2} \cos x \, dx \\
 &= \frac{4}{\pi} [\sin x]_0^{\pi/2} \\
 &= \frac{4}{\pi} \\
 a_n &= \frac{4}{\pi} \int_0^{\pi/2} \cos x \cos 2nx \, dx \\
 &= \frac{2}{\pi} \int_0^{\pi/2} [\cos(1-2n)x + \cos(1+2n)x] \, dx \\
 &= \frac{2}{\pi} \left[\frac{\sin(1-2n)x}{1-2n} + \frac{\sin(1+2n)x}{1+2n} \right]_0^{\pi/2} \\
 &= \frac{2(-1)^n}{\pi} \left[\frac{1}{1-2n} + \frac{1}{1+2n} \right] \\
 &= \frac{2(-1)^n}{\pi} \frac{1+2n+1-2n}{(1-2n)(1+2n)} \\
 &= \frac{4(-1)^n}{\pi(1-2n)(1+2n)} \\
 f &= \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(1-2n)(1+2n)} \cos 2nx
 \end{aligned}$$

$$\begin{aligned}
 b_n &= \frac{4}{\pi} \int_0^{\pi/2} \cos x \sin 2nx \, dx \\
 &= \frac{2}{\pi} \int_0^{\pi/2} [\sin(2n+1)x + \sin(2n-1)x] \, dx \\
 &= -\frac{2}{\pi} \left[\frac{\cos(2n+1)x}{2n+1} + \frac{\cos(2n-1)x}{2n-1} \right]_0^{\pi/2} \\
 &= \frac{2}{\pi} \left(\frac{1}{2n+1} + \frac{1}{2n-1} \right) \\
 &= \frac{2}{\pi} \frac{4n}{4n^2-1} \\
 &= \frac{8n}{\pi(4n^2-1)} \\
 f &= \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2-1} \sin 2nx
 \end{aligned}$$

12.3.35

$$\begin{aligned}
 a_0 &= \frac{1}{\pi} \int_0^{2\pi} x^2 dx \\
 &= \frac{8}{3} \pi^2 \\
 a_n &= \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx dx \\
 &= \frac{4}{n^2} \\
 b_n &= \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nx dx \\
 &= -\frac{4\pi}{n} \\
 f &= \frac{4}{3} \pi^2 + \sum_{n=1}^{\infty} \left(\frac{4}{n^2} \cos nx - \frac{4\pi}{n} \sin nx \right)
 \end{aligned}$$

12.3.43

$$\begin{aligned}
 b_n &= \frac{10}{\pi} \int_0^{\pi} \sin nt dt \\
 &= -\frac{10}{n\pi} [\cos nt]_0^{\pi} \\
 &= \frac{10}{n\pi} [1 - (-1)^n] \\
 f &= \frac{10}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} [1 - (-1)^n] \sin nt \\
 x_p(t) &= \sum_{n=1}^{\infty} B_n \sin nt \\
 m \frac{d^2 x}{dt^2} + kx &= f(t) \\
 -mn^2 B_n + kB_n &= \frac{10}{n\pi} [1 - (-1)^n] \\
 B_n &= \frac{10}{n\pi(k - mn^2)} [1 - (-1)^n] \\
 x_p(t) &= \frac{10}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n(k - mn^2)} \sin nt \\
 &= \frac{10}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n(10 - n^2)} \sin nt
 \end{aligned}$$

12.3.45

$$a_0 = \frac{2}{\pi} \int_0^\pi (2\pi t - t^2) dt$$

$$= \frac{2}{\pi} \left[\pi t^2 - \frac{1}{3} t^3 \right]_0^\pi$$

$$= \frac{2}{\pi} \left(\pi^3 - \frac{1}{3} \pi^3 \right)$$

$$= \frac{4}{3} \pi^2$$

$$a_n = \frac{2}{\pi} \int_0^\pi (2\pi t - t^2) \cos nt dt$$

$$= -\frac{4}{n^2}$$

$$f(t) = \frac{2}{3} \pi^2 - \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nt$$

$$x_p(t) = c + \sum_{n=1}^{\infty} B_n \cos nt$$

$$\frac{1}{4} \frac{d^2 x}{dt^2} + 12x = f(t)$$

$$-\frac{n^2}{4} \sum_{n=1}^{\infty} B_n \cos nt + 12c + 12 \sum_{n=1}^{\infty} B_n \cos nt = \frac{2}{3} \pi^2 - \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nt$$

$$12c = \frac{2}{3} \pi^2$$

$$c = \frac{\pi^2}{18}$$

$$-\frac{n^2}{4} B_n + 12B_n = -\frac{4}{n^2}$$

$$\frac{48 - n^2}{4} B_n = -\frac{4}{n^2}$$

$$B_n = -\frac{16}{n^2(48 - n^2)}$$

$$x_p(t) = \frac{\pi^2}{18} - \sum_{n=1}^{\infty} \frac{16}{n^2(48 - n^2)} \cos nt$$

$$= \frac{\pi^2}{18} + 16 \sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - 48)} \cos nt$$

12.4 Complex Fourier Series

12.4.1

$$\begin{aligned}T &= 4 \\p &= 2 \\c_n &= \frac{1}{4} \left(\int_0^2 e^{-in\pi x/2} dx - \int_{-2}^0 e^{-in\pi x/2} dx \right) \\&= \frac{1}{2in\pi} ([e^{-in\pi x/2}]_{-2}^0 - [e^{-in\pi x/2}]_0^2) \\&= \frac{2 - e^{in\pi} - e^{-in\pi}}{2in\pi} \\&= \frac{2 - \cos n\pi - i \sin n\pi - \cos n\pi + i \sin n\pi}{2in\pi} \\&= \frac{1 - \cos n\pi}{in\pi} \\&= \frac{1 - (-1)^n}{in\pi} \\f(x) &= \sum_{n=-\infty, n \neq 0}^{\infty} \frac{1 - (-1)^n}{in\pi} e^{in\pi x/2}\end{aligned}$$

12.4.3

$$\begin{aligned}T &= 1 \\p &= \frac{1}{2} \\c_n &= \int_0^{1/4} e^{-2in\pi x} dx \\&= -\frac{1}{2in\pi} [e^{-2in\pi x}]_0^{1/4} \\&= \frac{1}{2in\pi} (1 - e^{-in\pi/2}) \\c_0 &= \frac{1}{4} \\f(x) &= \frac{1}{4} + \sum_{n=-\infty, n \neq 0}^{\infty} \frac{1 - e^{-in\pi/2}}{2in\pi} e^{2in\pi x}\end{aligned}$$

12.4.5

$$T = 2\pi$$

$$p = \pi$$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} dx$$

$$= \frac{i}{n}$$

$$c_0 = \frac{1}{2\pi} \int_0^{2\pi} x dx$$

$$= \pi$$

$$f(x) = \pi + \sum_{n=-\infty, n \neq 0}^{n=\infty} \frac{i}{n} e^{inx}$$

12.5 Sturm-Liouville Problem

12.5.1

$$y'' + \lambda y = 0$$

$$y'(0) = 0$$

$$y(1) + y'(1) = 0$$

$$\lambda = \alpha^2$$

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$y' = -\alpha c_1 \sin \alpha x + \alpha c_2 \cos \alpha x$$

$$c_2 = 0$$

$$y = c_1 \cos \alpha x$$

$$c_1 \cos \alpha - \alpha c_1 \sin \alpha = 0$$

$$c_1 \cos \alpha = \alpha c_1 \sin \alpha$$

$$\alpha \tan \alpha = 0$$

$$\alpha = \cot \alpha$$

$$\lambda_1 = 0.740174$$

$$y_1 = \cos 0.860334x$$

$$\lambda_2 = 11.734872$$

$$y_2 = \cos 3.42562x$$

$$\lambda_3 = 41.438831$$

$$y_3 = \cos 6.4373x$$

$$\lambda_4 = 90.808130$$

$$y_4 = \cos 9.52933x$$

12.5.5

$$\begin{aligned}(y_n, y_n) &= \int_0^1 \cos^2 \alpha_n x \, dx \\&= \frac{1}{2} \int_0^1 (1 + \cos 2\alpha_n x) \, dx \\&= \frac{1}{2} \left[x + \frac{1}{2\alpha_n} \sin 2\alpha_n x \right]_0^1 \\&= \frac{1}{2} \left(1 + \frac{1}{2\alpha_n} \sin 2\alpha_n \right) \\&= \frac{1}{2} \left(1 + \frac{1}{\alpha_n} \sin \alpha_n \cos \alpha_n \right) \\&= \frac{1}{2} (1 + \tan \alpha_n \sin \alpha_n \cos \alpha_n) \\&= \frac{1}{2} (1 + \sin^2 \alpha_n)\end{aligned}$$

12.5.7

(a)

$$x^2 y'' + xy' + \lambda y = 0$$

$$y(1) = 0$$

$$y(5) = 0$$

$$\lambda = \alpha^2$$

$$y = x^m$$

$$y' = mx^{m-1}$$

$$y'' = m(m-1)x^{m-2}$$

$$x^2 m(m-1)x^{m-2} + mx^{m-1} + \alpha^2 x^m = 0$$

$$m(m-1) + m + \alpha^2 = 0$$

$$m^2 + \alpha^2 = 0$$

$$m = \pm i\alpha$$

$$y = c_1 \cos(\alpha \ln x) + c_2 \sin(\alpha \ln x)$$

$$0 = c_1$$

$$0 = c_2 \sin(\alpha \ln 5)$$

$$\alpha = \frac{n\pi}{\ln 5}$$

$$\lambda = \left(\frac{n\pi}{\ln 5} \right)^2$$

$$y_n = \sin \left(\frac{n\pi}{\ln 5} \ln x \right)$$

(b)

$$x^2 y'' + xy' + \lambda y = 0$$

$$y'' + \frac{1}{x} y' + \lambda \frac{1}{x^2} y = 0$$

$$e^{\ln x} y'' + \frac{1}{x} e^{\ln x} y' + \lambda e^{\ln x} \frac{1}{x^2} y = 0$$

$$\frac{d}{dx} (e^{\ln x} y') + \lambda e^{\ln x} \frac{1}{x^2} y = 0$$

$$\frac{d}{dx} (xy') + \lambda \frac{1}{x} y = 0$$

(c)

$$\int_1^5 \frac{1}{x} \sin \left(\frac{m\pi}{\ln 5} \ln x \right) \sin \left(\frac{n\pi}{\ln 5} \ln x \right) dx = 0, \quad m \neq n$$

12.5.9

$$xy'' + (1-x)y' + ny = 0$$

$$y'' + \left(\frac{1}{x} - 1\right)y' + n\frac{1}{x}y = 0$$

$$e^{\int (\frac{1}{x} - 1) dx} = e^{\ln(x) - x}$$

$$= xe^{-x}$$

$$xe^{-x}y'' + \left(\frac{1}{x} - 1\right)xe^{-x}y' + n\frac{1}{x}xe^{-x}y = 0$$

$$\frac{d}{dx}(xe^{-x}y') + ne^{-x}y = 0$$

$$\int_0^\infty e^{-x}L_m(x)L_n(x)dx = 0, \quad m \neq n$$