Advanced Engineering Mathematics Partial Differential Equations by Dennis G. Zill Problems

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Contents

12 Orthogonal Functions and Fourier Series

12.1 Orthogonal Functions

12.1.7

$$
\begin{aligned}
\int_0^{\pi/2} \sin mx \sin nx \, dx &= \frac{1}{2} \int_0^{\pi/2} [\cos(m-n)x - \cos(m+n)x] \, dx \\
&= \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_0^{\pi/2} \\
&= \frac{1}{2} \left(\frac{\sin(m-n)\pi/2}{m-n} - \frac{\sin(m+n)\pi/2}{m+n} \right) \\
&= 0 \\
||\sin nx||^2 &= (\sin nx, \sin nx) \\
&= \int_0^{\pi/2} \sin^2 nx \, dx \\
&= \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2nx) \, dx \\
&= \frac{1}{2} \left[x - \frac{1}{2n} \sin 2nx \right]_0^{\pi/2} \\
&= \frac{\pi}{4} \\
||\sin nx|| &= \frac{\sqrt{\pi}}{2}
\end{aligned}
$$

12.1.9

$$\begin{aligned}
\int_0^\pi \sin mx \sin nx \, dx &= \frac{1}{2}\int_0^\pi [\cos(m-n)x - \cos(m+n)x] \, dx \\
&= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_0^\pi \\
&= 0 \\
||\sin nx||^2 &= (\sin nx, \sin nx) \\
&= \int_0^\pi \sin^2 nx \, dx \\
&= \frac{1}{2}\int_0^\pi (1-\cos 2nx) \, dx \\
&= \frac{1}{2}\left[x - \frac{1}{2n}\sin 2nx\right]_0^\pi \\
&= \frac{\pi}{2} \\
||\sin nx|| &= \sqrt{\frac{\pi}{2}}
\end{aligned}$$

12.1.21

$$T = 1$$

12.1.23

$$T = 2\pi$$

12.1.25

$$T = 2\pi$$

12.2 Fourier Series

12.2.1

$$
\begin{aligned}
p &= \pi \\
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} dx \\
&= 1 \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \cos nx\,dx \\
&= \frac{1}{n\pi}[\sin nx]_0^{\pi} \\
&= 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin nx\,dx \\
&= -\frac{1}{n\pi}[\cos nx]_0^{\pi} \\
&= -\frac{1}{n\pi}[(-1)^n - 1] \\
&= \frac{1-(-1)^n}{n\pi} \\
f(x) &= \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx
\end{aligned}
$$

The series converges to $\frac{1}{2}$ at the point of discontinuity.

12.2.3

$$\begin{aligned}
p &= 1\\
a_0 &= \frac{3}{2}\\
a_n &= \int_{-1}^{0} \cos n\pi x\,dx + \int_0^1 x\cos n\pi x\,dx\\
&= \frac{1}{n\pi}[\sin n\pi x]_{-1}^{0} + \frac{1}{n\pi}\left[\frac{\cos n\pi x}{n\pi} + x\sin n\pi x\right]_0^1\\
&= \frac{(-1)^n - 1}{n^2\pi^2}\\
b_n &= \int_{-1}^{0} \sin n\pi x\,dx + \int_0^1 x\sin n\pi x\,dx\\
&= -\frac{1}{n\pi}[\cos n\pi x]_{-1}^{0} + \frac{1}{n\pi}\left[\frac{\sin n\pi x}{n\pi} - x\cos n\pi x\right]_0^1\\
&= -\frac{1}{n\pi}\\
f(x) &= \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x - \frac{1}{n\pi}\sin n\pi x\right]
\end{aligned}$$

The series converges to $\frac{1}{2}$ at the point of discontinuity.