Advanced Engineering Mathematics Systems of Differential Equations by Dennis G. Zill Problems

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Contents

10 Systems of Linear Differential Equations

10.1 Theory of Linear Systems

10.1.1

$$\mathbf{X}' = \begin{pmatrix} 3 & -5 \\ 4 & 8 \end{pmatrix} \mathbf{X}$$

10.1.3

$$\mathbf{X}' = \begin{pmatrix} -3 & 4 & -9 \\ 6 & -1 & 0 \\ 10 & 4 & 3 \end{pmatrix} \mathbf{X}$$

10.1.5

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} t-1 \\ -3t^2 \\ t^2 - t + 2 \end{pmatrix}$$

10.1.7

$$\frac{dx}{dt} = 4x + 2y + e^t$$
$$\frac{dy}{dt} = -x + 3y - e^t$$

10.1.9

$$\begin{aligned}\frac{dx}{dt} &= x - y + 2z + e^{-t} - 3t\\ \frac{dy}{dt} &= 3x - 4y + z + 2e^{-t} + t\\ \frac{dz}{dt} &= -2x + 5y + 6z + 2e^{-t} - t\end{aligned}$$

10.1.11

$$\begin{aligned}3(e^{-5t}) - 4(2e^{-5t}) &= -5e^{-5t}\\ &= \frac{dx}{dt}\\ 4(e^{-5t}) - 7(2e^{-5t}) &= -10e^{-5t}\\ &= \frac{dy}{dt}\end{aligned}$$

10.1.13

$$\begin{aligned}-(-e^{-3t/2}) + \frac{1}{4}(2e^{-3t/2}) &= \frac{3}{2}e^{-3t/2}\\ &= \frac{dx}{dt}\\ (-e^{-3t/2}) - (2e^{-3t/2}) &= -3e^{-3t/2}\\ &= \frac{dy}{dt}\end{aligned}$$

10.1.17

$$\begin{aligned}W(\mathbf{X}_1, \mathbf{X}_2) &= \begin{vmatrix} e^{-2t} & e^{-6t}\\ e^{-2t} & -e^{-6t}\end{vmatrix}\\ &= -e^{-8t} - e^{-8t}\\ &= -2e^{-8t}\\ &\neq 0 \text{ for } t \in (-\infty, \infty)\end{aligned}$$

Yes, they form a fundamental set.

10.1.19

$$\begin{aligned}W(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) &= \begin{vmatrix} 1+t & 1 & 3+2t\\ -2+2t & -2 & -6+4t\\ 4+2t & 4 & 12+4t\end{vmatrix}\\ &= 0\end{aligned}$$

No, they don't form a fundamental set.

10.1.21

$$\begin{aligned}
x &= 2t+5 \\
y &= -t+1 \\
\frac{dx}{dt} &= (2t+5)+4(-t+1)+2t-7 \\
&= 2 \\
\frac{dy}{dt} &= 3(2t+5)+2(-t+1)-4t-18 \\
&= -1
\end{aligned}$$

10.1.23

$$\begin{aligned}
x &= e^t+te^t \\
x' &= 2e^t+te^t \\
y &= e^t-te^t \\
y' &= -te^t \\
\frac{dx}{dt} &= 2(e^t+te^t)+(e^t-te^t)-e^t \\
&= 2e^t+te^t \\
\frac{dy}{dt} &= 3(e^t+te^t)+4(e^t-te^t)-7e^t \\
&= -te^t
\end{aligned}$$

10.2 Homogeneous Linear Systems

10.2.1

$$\mathbf{X} = c_1 \begin{pmatrix}1\\2\end{pmatrix} e^{5t} + c_2 \begin{pmatrix}-1\\1\end{pmatrix} e^{-t}$$

10.2.3

$$\mathbf{X} = c_1 \begin{pmatrix}2\\1\end{pmatrix} e^{-3t} + c_2 \begin{pmatrix}\frac{2}{5}\\1\end{pmatrix} e^{t}$$

10.2.5

$$\mathbf{X} = c_1 \begin{pmatrix}1\\4\end{pmatrix} e^{-10t} + c_2 \begin{pmatrix}5\\2\end{pmatrix} e^{8t}$$

10.2.7

$$\mathbf{X} = c_1 \begin{pmatrix}2\\3\\1\end{pmatrix} e^{2t} + c_2 \begin{pmatrix}1\\0\\2\end{pmatrix} e^{-t} + c_3 \begin{pmatrix}1\\0\\0\end{pmatrix} e^{t}$$

10.2.13

$$\mathbf{X} = 2\begin{pmatrix}0\\1\end{pmatrix}e^{-t/2} + 3\begin{pmatrix}1\\1\end{pmatrix}e^{t/2}$$

10.2.15

(a)

$$\frac{dx_1}{dt} = -\frac{3}{100}x_1 + \frac{1}{100}x_2$$
$$\frac{dx_2}{dt} = \frac{2}{100}x_1 - \frac{2}{100}x_2$$
$$\begin{pmatrix}\frac{dx_1}{dt}\\\frac{dx_2}{dt}\end{pmatrix} = \begin{pmatrix}-\frac{3}{100} & \frac{1}{100}\\\frac{2}{100} & -\frac{2}{100}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

(b)

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = -\frac{35}{3}\begin{pmatrix}-1\\1\end{pmatrix}e^{-t/25} + \frac{50}{3}\begin{pmatrix}\frac{1}{2}\\1\end{pmatrix}e^{-t/100}$$

10.2.21

$$\mathbf{X} = c_1\begin{pmatrix}1\\3\end{pmatrix} + c_2\left[\begin{pmatrix}1\\3\end{pmatrix}t + \begin{pmatrix}0\\-1\end{pmatrix}\right]$$

10.2.23

$$\mathbf{X} = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix}1\\1\end{pmatrix}t + \begin{pmatrix}0\\\frac{1}{3}\end{pmatrix}\right]e^{2t}$$

10.2.25

$$\mathbf{X} = c_1\begin{pmatrix}1\\0\\1\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix}e^{2t} + c_3\begin{pmatrix}1\\1\\1\end{pmatrix}e^{t}$$

10.2.31

$$\mathbf{X} = -\frac{1}{2}\begin{pmatrix}2\\1\end{pmatrix}e^{4t} + 13\left[\begin{pmatrix}2\\1\end{pmatrix}t + \begin{pmatrix}0\\\frac{1}{2}\end{pmatrix}\right]e^{4t}$$

10.2.33

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{K}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

10.2.35

$$\begin{aligned}
\mathbf{X} &= c_1 \begin{pmatrix} 1 \\ 2-i \end{pmatrix} e^{(4+i)t} + c_2 \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{(4-i)t} \\
&= c_1 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin t \right] e^{4t} + c_2 \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t \right] e^{4t} \\
&= c_1 \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix} e^{4t}
\end{aligned}$$

10.2.37

$$\begin{aligned}
\mathbf{X} &= c_1 \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{(4+i)t} + c_2 \begin{pmatrix} 1 \\ -1-i \end{pmatrix} e^{(4-i)t} \\
&= c_1 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t \right] e^{4t} + c_2 \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin t \right] e^{4t} \\
&= c_1 \begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix} e^{4t}
\end{aligned}$$

10.2.39

$$\begin{aligned}
\mathbf{X} &= c_1 \begin{pmatrix} 5 \\ 4-3i \end{pmatrix} e^{3i} + c_2 \begin{pmatrix} 5 \\ 4+3i \end{pmatrix} e^{-3i} \\
&= c_1 \left[\begin{pmatrix} 5 \\ 4 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ -3 \end{pmatrix} \sin 3t \right] + c_2 \left[\begin{pmatrix} 0 \\ -3 \end{pmatrix} \cos 3t + \begin{pmatrix} 5 \\ 4 \end{pmatrix} \sin 3t \right] \\
&= c_1 \begin{pmatrix} 5\cos 3t \\ 4\cos 3t + 3\sin 3t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin 3t \\ 4\sin 3t - 3\cos 3t \end{pmatrix}
\end{aligned}$$

10.2.47

$$\begin{aligned}
\mathbf{X} &= c_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1+5i \\ 1 \\ 1 \end{pmatrix} e^{5it} + c_3 \begin{pmatrix} 1-5i \\ 1 \\ 1 \end{pmatrix} e^{-5it} \\
&= c_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^{t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cos 5t - \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \sin 5t \right] \\
&\quad + c_3 \left[\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \cos 5t + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \sin 5t \right] \\
&= c_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} \cos 5t - 5\sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + c_3 \begin{pmatrix} 5\cos 5t + \sin 5t \\ \sin 5t \\ \sin 5t \end{pmatrix} \\
&= - \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^{t} - \begin{pmatrix} \cos 5t - 5\sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + 6 \begin{pmatrix} 5\cos 5t + \sin 5t \\ \sin 5t \\ \sin 5t \end{pmatrix}
\end{aligned}$$

10.2.49

(a)

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} -\frac{1}{20} & 0 & \frac{1}{10} \\ \frac{1}{20} & -\frac{1}{20} & 0 \\ 0 & \frac{1}{20} & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(b)

$$
\begin{aligned}
\mathbf{X} &= c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1-i \\ i \\ 1 \end{pmatrix} e^{\left(-\frac{1}{10}+\frac{1}{20}i\right)t} + c_3 \begin{pmatrix} -1+i \\ -i \\ 1 \end{pmatrix} e^{\left(-\frac{1}{10}-\frac{1}{20}i\right)t} \\
&= c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \left[\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cos \frac{1}{20}t - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \sin \frac{1}{20}t \right] e^{-t/10} \\
&\quad + c_3 \left[\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cos \frac{1}{20}t + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \sin \frac{1}{20}t \right] e^{-t/10} \\
&= c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \sin \frac{t}{20} - \cos \frac{t}{20} \\ -\sin \frac{t}{20} \\ \cos \frac{t}{20} \end{pmatrix} e^{-t/10} \\
&\quad + c_3 \begin{pmatrix} -\cos \frac{t}{20} - \sin \frac{t}{20} \\ \cos \frac{t}{20} \\ \sin \frac{t}{20} \end{pmatrix} e^{-t/10} \\
&= 11 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - 6 \begin{pmatrix} \sin \frac{t}{20} - \cos \frac{t}{20} \\ -\sin \frac{t}{20} \\ \cos \frac{t}{20} \end{pmatrix} e^{-t/10} \\
&\quad - 2 \begin{pmatrix} -\cos \frac{t}{20} - \sin \frac{t}{20} \\ \cos \frac{t}{20} \\ \sin \frac{t}{20} \end{pmatrix} e^{-t/10}
\end{aligned}
$$

10.3 Solution by Diagonalization

10.3.1

$$
\mathbf{X} = \begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 e^{7t} \\ c_2 e^{-4t} \end{pmatrix} = \begin{pmatrix} 3c_1 e^{7t} - 2c_2 e^{-4t} \\ c_1 e^{7t} + 3c_2 e^{-4t} \end{pmatrix}
$$

10.3.3

$$
\mathbf{X} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_1 e^{3t/2} \\ c_2 e^{t/2} \end{pmatrix} = \begin{pmatrix} c_1 e^{3t/2} + c_2 e^{t/2} \\ 2c_1 e^{3t/2} - 2c_2 e^{t/2} \end{pmatrix}
$$

10.3.5

$$
\mathbf{X} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{6t} \\ c_2 e^{-4t} \\ c_3 e^{2t} \end{pmatrix} = \begin{pmatrix} -c_2 e^{-4t} + c_3 e^{2t} \\ c_2 e^{-4t} + c_3 e^{2t} \\ c_1 e^{6t} + c_3 e^{2t} \end{pmatrix}
$$

10.3.11

(a)

$$
\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
$$

$\mathbf{M}$ has an inverse because it has a nonzero determinant (the product of the diagonal entries).

(b)

$$\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & \frac{k_2}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

(c)

$$\mathbf{X} = \mathbf{PY}$$

$$\begin{aligned} \mathbf{PY}'' + \mathbf{BPY} &= \mathbf{0} \\ \mathbf{Y}'' + \mathbf{P^{-1}BPY} &= \mathbf{0} \\ \mathbf{Y}'' + \mathbf{DY} &= \mathbf{0} \\ \begin{pmatrix} y_1'' \\ y_2'' \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \mathbf{0} \\ y_1'' + 6y_1 &= 0 \\ y_1 &= c_1 \cos\sqrt{6}t + c_2 \sin\sqrt{6}t \\ y_2'' + y_2 &= 0 \\ y_2 &= c_3 \cos t + c_4 \sin t \end{aligned}$$

$$\begin{aligned} \mathbf{X} &= \mathbf{PY} \\ &= \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \cos\sqrt{6}t + c_2 \sin\sqrt{6}t \\ c_3 \cos t + c_4 \sin t \end{pmatrix} \\ &= \begin{pmatrix} -2c_1 \cos\sqrt{6}t - 2c_2 \sin\sqrt{6}t + c_3 \cos t + c_4 \sin t \\ c_1 \cos\sqrt{6}t + c_2 \sin\sqrt{6}t + 2c_3 \cos t + 2c_4 \sin t \end{pmatrix} \\ &= c_3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos t + c_4 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t + c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cos\sqrt{6}t + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \sin\sqrt{6}t \end{aligned}$$

10.4 Nonhomogeneous Linear Systems

10.4.1

$$\begin{aligned}
\begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -7 \\ 5 \end{pmatrix} \\
\mathbf{X}_c &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} \\
\mathbf{X}_p &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} -7 \\ 5 \end{pmatrix} \\
&= \begin{pmatrix} 2a_1 + 3a_2 - 7 \\ -a_1 - 2a_2 + 5 \end{pmatrix} \\
\mathbf{X}_p &= \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\
\mathbf{X} &= \mathbf{X}_c + \mathbf{X}_p \\
&= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}
\end{aligned}$$

10.4.3

$$
\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} \\
\mathbf{X}_p &= \begin{pmatrix} a_3 t^2 + a_2 t + a_1 \\ b_3 t^2 + b_2 t + b_1 \end{pmatrix} \\
\begin{pmatrix} 2a_3 t + a_2 \\ 2b_3 t + b_2 \end{pmatrix} &= \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a_3 t^2 + a_2 t + a_1 \\ b_3 t^2 + b_2 t + b_1 \end{pmatrix} + \begin{pmatrix} -2t^2 \\ t + 5 \end{pmatrix} \\
&= \begin{pmatrix} (a_3 + 3b_3 - 2)t^2 + (a_2 + 3b_2)t + (a_1 + 3b_1) \\ (3a_3 + b_3)t^2 + (3a_2 + b_2 + 1)t + (3a_1 + b_1 + 5) \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} (a_3 + 3b_3 - 2)t^2 + (a_2 - 2a_3 + 3b_2)t + (a_1 - a_2 + 3b_1) \\ (3a_3 + b_3)t^2 + (3a_2 + b_2 - 2b_3 + 1)t + (3a_1 + b_1 - b_2 + 5) \end{pmatrix} \\
a_3 &= -\frac{1}{4} \\
b_3 &= \frac{3}{4} \\
a_2 &= \frac{1}{4} \\
b_2 &= -\frac{1}{4} \\
a_1 &= -2 \\
b_1 &= \frac{3}{4} \\
\mathbf{X}_p &= \begin{pmatrix} -\frac{1}{4}t^2 + \frac{1}{4}t - 2 \\ \frac{3}{4}t^2 - \frac{1}{4}t + \frac{3}{4} \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} + \begin{pmatrix} -\frac{1}{4}t^2 + \frac{1}{4}t - 2 \\ \frac{3}{4}t^2 - \frac{1}{4}t + \frac{3}{4} \end{pmatrix}
\end{aligned}
$$

10.4.5

$$
\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} \frac{1}{9} \\ 1 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} e^{3t} \\
\mathbf{X}_p &= \begin{pmatrix} a \\ b \end{pmatrix} e^{t} \\
\begin{pmatrix} a \\ b \end{pmatrix} e^{t} &= \begin{pmatrix} 4 & \frac{1}{3} \\ 9 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} e^{t} + \begin{pmatrix} -3 \\ 10 \end{pmatrix} e^{t} \\
\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 4a + \frac{1}{3}b - 3 \\ 9a + 6b + 10 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 3a + \frac{1}{3}b - 3 \\ 9a + 5b + 10 \end{pmatrix} \\
\mathbf{X}_p &= \begin{pmatrix} \frac{55}{36} \\ -\frac{19}{4} \end{pmatrix} e^{t} \\
\mathbf{X} &= c_1 \begin{pmatrix} \frac{1}{9} \\ 1 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} \frac{55}{36} \\ -\frac{19}{4} \end{pmatrix} e^{t}
\end{aligned}
$$

10.4.9

$$
\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t} \\
\mathbf{X}_p &= \begin{pmatrix} a \\ b \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} -a - 2b + 3 \\ 3a + 4b + 3 \end{pmatrix} \\
\mathbf{X}_p &= \begin{pmatrix} -9 \\ 6 \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} -9 \\ 6 \end{pmatrix} \\
\begin{pmatrix} -4 \\ 5 \end{pmatrix} &= c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -9 \\ 6 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -2c_1 - c_2 - 5 \\ 3c_1 + c_2 + 1 \end{pmatrix} \\
\mathbf{X} &= 4 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{2t} - 13 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} -9 \\ 6 \end{pmatrix}
\end{aligned}
$$

10.4.11

(a)

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{1}{50} & -\frac{1}{25} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(b)

$$\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} e^{-t/20} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/50} \\
\mathbf{X}_p &= \begin{pmatrix} a \\ b \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{1}{50} & -\frac{1}{25} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{3}{100}a + \frac{1}{100}b \\ \frac{1}{50}a - \frac{1}{25}b + 1 \end{pmatrix} \\
\mathbf{X}_p &= \begin{pmatrix} 10 \\ 30 \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} e^{-t/20} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/50} + \begin{pmatrix} 10 \\ 30 \end{pmatrix} \\
\begin{pmatrix} 60 \\ 10 \end{pmatrix} &= c_1 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 10 \\ 30 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -\frac{1}{2}c_1 + c_2 - 50 \\ c_1 + c_2 + 20 \end{pmatrix} \\
\mathbf{X} &= -\frac{140}{3} \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} e^{-t/20} + \frac{80}{3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/50} + \begin{pmatrix} 10 \\ 30 \end{pmatrix}
\end{aligned}$$

(c)

$$\begin{aligned}
\lim_{t\to\infty} x_1(t) &= \lim_{t\to\infty} \frac{70}{3} e^{-t/20} + \frac{80}{3} e^{-t/50} + 10 \\
&= 10 \\
\lim_{t\to\infty} x_1(t) &= \lim_{t\to\infty} -\frac{140}{3} e^{-t/20} + \frac{80}{3} e^{-t/50} + 30 \\
&= 30
\end{aligned}$$

10.4.13

$$
\begin{aligned}
\begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} \\
\mathbf{X}_c &= c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathbf{X}_p &= \mathbf{\Phi}(t) \int \mathbf{\Phi}^{-1}(t) \mathbf{F}\, dt \\
&= \begin{pmatrix} 3e^t & 1 \\ 2e^t & 1 \end{pmatrix} \int \begin{pmatrix} e^{-t} & -e^{-t} \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \end{pmatrix} dt \\
&= \begin{pmatrix} 3e^t & 1 \\ 2e^t & 1 \end{pmatrix} \int \begin{pmatrix} 5e^{-t} \\ -11 \end{pmatrix} dt \\
&= \begin{pmatrix} 3e^t & 1 \\ 2e^t & 1 \end{pmatrix} \begin{pmatrix} -5e^{-t} \\ -11t \end{pmatrix} \\
&= \begin{pmatrix} -15 - 11t \\ -10 - 11t \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t + c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -15 - 11t \\ -10 - 11t \end{pmatrix}
\end{aligned}
$$

10.4.15

$$
\begin{aligned}
\mathbf{X}' &= \begin{pmatrix} 3 & -5 \\ \frac{3}{4} & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2} \\
\mathbf{X}_c &= c_1 \begin{pmatrix} \frac{10}{3} \\ 1 \end{pmatrix} e^{3t/2} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t/2} \\
\mathbf{X}_p &= \begin{pmatrix} \frac{1}{2}(-15 - 13t) \\ \frac{1}{4}(-9 - 13t) \end{pmatrix} e^{t/2} \\
\mathbf{X} &= c_1 \begin{pmatrix} \frac{10}{3} \\ 1 \end{pmatrix} e^{3t/2} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t/2} + \frac{1}{4} \begin{pmatrix} -30 - 26t \\ -9 - 13t \end{pmatrix} e^{t/2}
\end{aligned}
$$

10.4.33

$$
\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} \\
\mathbf{X}_p &= \begin{pmatrix} 2e^{2t}t - 2e^{4t}t - e^{2t} + e^{4t} \\ 2e^{2t}t + 2e^{4t}t + e^{2t} + e^{4t} \end{pmatrix} \\
&= \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} \\
\mathbf{X} &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} \\
&\quad + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} \\
\begin{pmatrix} 1 \\ 1 \end{pmatrix} &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -c_1 + c_2 - 1 \\ c_1 + c_2 + 1 \end{pmatrix} \\
\mathbf{X} &= \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{4t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t}
\end{aligned}
$$

10.4.35

$$
\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-12t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} \\
\mathbf{X}_p &= \frac{1}{29} \begin{pmatrix} -76\cos t + 332\sin t \\ -168\cos t + 276\sin t \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-12t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} - \frac{4}{29} \begin{pmatrix} 19 \\ 42 \end{pmatrix} \cos t + \frac{4}{29} \begin{pmatrix} 83 \\ 69 \end{pmatrix} \sin t \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{4}{29} \begin{pmatrix} 19 \\ 42 \end{pmatrix} \\
&= \begin{pmatrix} -3c_1 + c_2 - \frac{76}{29} \\ c_1 + 3c_2 - \frac{168}{29} \end{pmatrix} \\
\mathbf{X} &= -\frac{6}{29} \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-12t} + 2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} - \frac{4}{29} \begin{pmatrix} 19 \\ 42 \end{pmatrix} \cos t + \frac{4}{29} \begin{pmatrix} 83 \\ 69 \end{pmatrix} \sin t
\end{aligned}
$$

10.4.37

$$\mathbf{A} = \begin{pmatrix} 5 & -2 \\ 21 & -8 \end{pmatrix}$$
$$\mathbf{F} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$
$$\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 7 & 3 \end{pmatrix}$$
$$\mathbf{G} = \begin{pmatrix} -14 \\ 34 \end{pmatrix}$$
$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -2y_1 - 14 \\ -y_2 + 34 \end{pmatrix}$$
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-2t} - 7 \\ c_2 e^{-t} + 34 \end{pmatrix}$$
$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 7 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + \begin{pmatrix} 20 \\ 53 \end{pmatrix}$$

10.4.39

$$\mathbf{A} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$$
$$\mathbf{P} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$\mathbf{G} = \begin{pmatrix} 4 + t \\ 4 - t \end{pmatrix}$$
$$\mathbf{Y}' = \mathbf{DY} + \mathbf{G}$$
$$= \begin{pmatrix} 10y_1 + 4 + t \\ 4 - t \end{pmatrix}$$
$$\mathbf{Y} = \begin{pmatrix} c_1 e^{10t} - \frac{1}{10}t - \frac{41}{100} \\ -\frac{1}{2}t^2 + 4t + c_2 \end{pmatrix}$$
$$\mathbf{X} = \mathbf{PY}$$
$$= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{10t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} t^2 + \begin{pmatrix} -\frac{41}{10} \\ \frac{39}{10} \end{pmatrix} t - \frac{41}{100} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

10.5 Matrix Exponential

10.5.1

$$\begin{pmatrix} e^{t} & 0 \\ 0 & e^{2t} \end{pmatrix}$$
$$\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

10.5.3

$$\begin{pmatrix} t+1 & t & t \\ t & t+1 & t \\ -2t & -2t & -2t+1 \end{pmatrix}$$

10.5.5

$$\begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix}$$

10.5.7

$$\mathbf{X} = c_1 \begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} + c_2 \begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + c_3 \begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix}$$

10.5.9

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -3 \\ \frac{1}{2} \end{pmatrix}$$

10.5.11

$$\mathbf{X} = c_1 \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2 \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

10.5.13

$$\mathbf{X} = c_1 \begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} + c_2 \begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + c_3 \begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -4 \\ 6 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} - 4 \begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + 6 \begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix}$$

10.5.15

$$e^{\mathbf{A}t} = \begin{pmatrix} \frac{1}{2}e^{-2t}(3e^{4t}-1) & \frac{3}{4}e^{-2t}(e^{4t}-1) \\ e^{-2t} - e^{2t} & -\frac{1}{2}e^{-2t}(e^{4t}-3) \end{pmatrix}$$

10.5.17

$$e^{\mathbf{A}t} = \begin{pmatrix} e^{2t}(1+3t) & -9e^{2t}t \\ e^{2t}t & e^{2t}(1-3t) \end{pmatrix}$$

10.5.25

$$
\begin{aligned}
\mathbf{X} &= e^{\mathbf{A}t}\mathbf{C} \\
&= \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}\mathbf{C} \\
&= \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} e^{5t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \mathbf{C} \\
&= \begin{pmatrix} -\frac{1}{2}e^{3t}(-3+e^{2t}) & \frac{1}{2}e^{3t}(-1+e^{2t}) \\ -\frac{3}{2}e^{3t}(-1+e^{2t}) & \frac{1}{2}e^{3t}(-1+3e^{2t}) \end{pmatrix} \mathbf{C}
\end{aligned}
$$

10.6 Chapter in Review

10.6.1

$\frac{1}{3}$

10.6.5

$$\mathbf{X} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^t \right]$$

10.6.7

$$
\begin{aligned}
\mathbf{X} &= c_1 \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{1+2i} + c_2 \begin{pmatrix} i \\ 1 \end{pmatrix} e^{1-2i} \\
&= c_3 \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin 2t \right] e^t + c_4 \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 2t \right] e^t \\
&= \begin{pmatrix} c_3 e^t \sin 2t - c_4 e^t \cos 2t \\ c_3 e^t \cos 2t + c_4 e^t \sin 2t \end{pmatrix} \\
&= c_3 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^t + c_4 \begin{pmatrix} -\cos 2t \\ \sin 2t \end{pmatrix} e^t
\end{aligned}
$$

10.6.9

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -7 \\ -12 \\ 16 \end{pmatrix} e^{-3t} + c_3 \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} e^{2t}$$

11 Systems of Nonlinear Differential Equations

11.1 Autonomous Systems

11.1.1

$$
\begin{aligned}
x' &= y \\
y' &= -9 \sin x
\end{aligned}
$$

Critical points at $(n\pi, 0), n \in \mathbb{Z}$.

11.1.3

$$\begin{aligned} x' &= y \\ y' &= x^2 - y(1 - x^3) \end{aligned}$$

$$\begin{aligned} 0 &= y \\ 0 &= x^2 - y(1 - x^3) \\ &= x^2 \\ 0 &= x \end{aligned}$$

Critical point at $(0, 0)$.

11.1.5

$$\begin{aligned} x' &= y \\ y' &= \epsilon x^3 - x \end{aligned}$$

$$\begin{aligned} 0 &= y \\ 0 &= \epsilon x^3 - x \\ &= \epsilon x^2 - 1 \\ x &= \sqrt{\frac{1}{\epsilon}} \end{aligned}$$

Critical points at $(0, 0)$ and $\left(\pm\sqrt{\frac{1}{\epsilon}}, 0\right)$.

11.1.7

$x' = x + xy$ can only be 0 if $x = 0$ or $y = -1$. If $x = 0$, $y' = -y - xy$ is 0 if $y = 0$. If $y = -1$, it's 0 if $x = -1$. Therefore the critical points are $(0, 0)$ and $(-1, -1)$.

11.1.9

$$\begin{aligned} x' &= 3x^2 - 4y \\ 3x^2 &= 4y \\ x &= \sqrt{\frac{4}{3}y} \end{aligned}$$

$$\begin{aligned} y' &= x - y \\ 0 &= \sqrt{\frac{4}{3}y} - y \\ y^2 &= \frac{4}{3}y \\ y &= \frac{4}{3} \end{aligned}$$

The critical points are $(0, 0)$ and $\left(\frac{4}{3}, \frac{4}{3}\right)$.

11.1.11

$$\begin{aligned} x' &= x\left(10 - x - \frac{1}{2}y\right) \\ y' &= y(16 - y - x) \end{aligned}$$

$(0, 0)$, $(0, 16)$, $(10, 0)$, $(4, 12)$

11.1.13

$$\begin{aligned} x' &= x^2 e^y \\ y' &= y(e^x - 1) \end{aligned}$$

All points on the line $x = 0$.

11.1.15

$$\begin{aligned} x' &= x(1 - x^2 - 3y^2) \\ y' &= y(3 - x^2 - 3y^2) \end{aligned}$$

$(0, 0)$, $(0, \pm 1)$, $(\pm 1, 0)$

11.1.17

(a)

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

(b)

$$\begin{aligned} \begin{pmatrix} 2 \\ -2 \end{pmatrix} &= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} c_1 - c_2 - 2 \\ 2c_1 + c_2 + 2 \end{pmatrix} \\ \mathbf{X} &= -2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} \end{aligned}$$

11.1.19

(a)

$$\begin{aligned} x &= c_1(4\cos 3t - 3\sin 3t) + c_2(3\cos 3t + 4\sin 3t) \\ y &= c_1(5\cos 3t) + c_2(5\sin 3t) \end{aligned}$$

(b)

$$\begin{aligned} 4 &= 4c_1 + 3c_2 \\ 5 &= 5c_1 \\ c_1 &= 1 \\ c_2 &= 0 \end{aligned}$$

$$\begin{aligned} x &= 4\cos 3t - 3\sin 3t \\ y &= 5\cos 3t \end{aligned}$$

11.1.21

(a)

$$\begin{aligned} x &= c_1(-\cos t + \sin t)e^{4t} + c_2(-\cos t - \sin t)e^{4t} \\ y &= c_1(2\cos t)e^{4t} + c_2(2\sin t)e^{4t} \end{aligned}$$

(b)

$$\begin{aligned}
-1 &= -c_1 - c_2 \\
2 &= 2c_1 \\
c_1 &= 1 \\
c_2 &= 0
\end{aligned}$$

$$\begin{aligned}
x &= (\sin t - \cos t)e^{4t} \\
y &= 2\cos t e^{4t}
\end{aligned}$$

11.1.23

$$\begin{aligned}
\frac{dr}{dt} &= \frac{1}{r}\{x[-y - x(x^2+y^2)^2] + y[x - y(x^2+y^2)^2]\} \\
&= \frac{1}{r}[-xy - x^2r^4 + xy - y^2r^4] \\
&= -r^5 \\
\frac{1}{r^5}\frac{dr}{dt} &= -1 \\
-\frac{1}{4}\frac{1}{r^4} &= c_1 - t \\
\frac{1}{r^4} &= 4t + c_1 \\
r &= \frac{1}{\sqrt[4]{4t + c_1}}
\end{aligned}$$

$$\begin{aligned}
\frac{d\theta}{dt} &= \frac{1}{r^2}\{-y[-y - x(x^2+y^2)^2] + x[x - y(x^2+y^2)^2]\} \\
&= \frac{1}{r^2}[y^2 + xyr^2 + x^2 - xyr^2] \\
&= 1 \\
\theta &= t + c_2
\end{aligned}$$

$$\begin{aligned}
4 &= \frac{1}{\sqrt[4]{c_1}} \\
c_1 &= \frac{1}{256}
\end{aligned}$$

$$0 = c_2$$

$$\begin{aligned}
r &= \frac{1}{\sqrt[4]{4t + \frac{1}{256}}} \\
&= \frac{4}{\sqrt[4]{1024t + 1}} \\
\theta &= t
\end{aligned}$$

11.1.25

$$
\begin{aligned}
\frac{dr}{dt} &= \frac{1}{r}\{x[-y + x(1 - x^2 - y^2)] + y[x + y(1 - x^2 - y^2)]\} \\
&= \frac{1}{r}[-xy + x^2(1 - r^2) + xy + y^2(1 - r^2)] \\
&= r(1 - r^2) \\
&= -r^3 + r \\
r &= \pm\frac{e^t}{\sqrt{e^{2t} + c_1}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\theta}{dt} &= \frac{1}{r^2}\{-y[-y + x(1 - x^2 - y^2)] + x[x + y(1 - x^2 - y^2)]\} \\
&= \frac{1}{r^2}[y^2 - xy(1 - r^2) + x^2 + xy(1 - r^2)] \\
&= 1 \\
\theta &= t + c_2
\end{aligned}
$$

$$
\begin{aligned}
1 &= \pm\frac{1}{\sqrt{1 + c_1}} \\
c_1 &= 0 \\
c_2 &= 0 \\
r &= 1 \\
\theta &= t
\end{aligned}
$$

$$
\begin{aligned}
2 &= \pm\frac{1}{\sqrt{1 + c_1}} \\
c_1 &= -\frac{3}{4} \\
c_2 &= 0 \\
r &= \frac{e^t}{\sqrt{e^{2t} - \frac{3}{4}}} \\
\theta &= t
\end{aligned}
$$

11.1.27

No periodic solutions.

11.2 Stability of Linear Systems

11.2.1

Stable node

11.2.3

Unstable spiral

11.2.5

Degenerate stable node

11.2.7

Saddle point

11.2.9

Saddle point

11.2.11

Saddle point

11.2.13

Degenerate stable node

11.2.15

Stable spiral

11.2.17

$$-1 + \mu^2 < 0 \Rightarrow |\mu| < 1$$

11.2.19

Saddle point when $\mu < -1$, unstable spiral when $-1 < \mu < 3$, unstable node when $\mu \geq 3$.

11.2.23

(a) $(3, 4)$

(b) Saddle point

11.2.25

(a) $(0.5, 2)$

(b) Unstable spiral