

# Advanced Engineering Mathematics Systems of Differential Equations by Dennis G. Zill Notes

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## 10 Systems of Linear Differential Equations

### 10.1 Theory of Linear Systems

- A system of the form

$$\begin{aligned}\frac{dx_1}{dt} &= g_1(t, x_1, x_2, \dots, x_n) \\ \frac{dx_2}{dt} &= g_2(t, x_1, x_2, \dots, x_n) \\ &\vdots \\ \frac{dx_n}{dt} &= g_n(t, x_1, x_2, \dots, x_n)\end{aligned}$$

is called a **first-order system**.

- When each of the functions  $g_n(t, x_1, x_2, \dots, x_n)$  is linear in the dependent variables  $x_1, x_2, \dots, x_n$ , we get the **normal form** of a first-order system

of linear equations

$$\begin{aligned}\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n + f_1(t) \\ \frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \dots + a_{2n}(t)x_n + f_2(t) \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \dots + a_{nn}(t)x_n + f_n(t).\end{aligned}$$

Such a system is called a **linear system**.

- When  $f_i(t) = 0$  for  $i = 1, 2, \dots, n$  the linear system is said to be **homogeneous**, otherwise it's **nonhomogeneous**.
- If  $\mathbf{X}$ ,  $\mathbf{A}(t)$ , and  $\mathbf{F}(t)$  denote the matrices

$$\begin{aligned}\mathbf{X} &= \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \\ \mathbf{A}(t) &= \begin{pmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{pmatrix} \\ \mathbf{F}(t) &= \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}\end{aligned}$$

then homogeneous linear systems can be written

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

and nonhomogeneous linear systems can be written

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}.$$

- A **solution vector** on an interval  $I$  is any column matrix

$$\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

whose entries are differentiable functions satisfying the linear system on the interval.

- The entries of a solution vector can be considered a set of parametric equations that define a curve in  $n$ -space. Such a curve is called a **trajectory**.
- The problem of solving

$$\mathbf{X}' = \mathbf{A}(t)\mathbf{X} + \mathbf{F}(t)$$

subject to

$$\mathbf{X}(t_0) = \mathbf{X}_0$$

is an **initial value problem** in matrix form.

- The **superposition principle** states that if  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$  are solution vectors of a homogeneous linear system on an interval  $I$ , then

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \dots + c_n\mathbf{X}_n$$

where  $c_n$  are arbitrary constants is also a solution.

- If  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$  are a set of solution vectors of a homogeneous linear system on an interval  $I$ , the set is said to be **linearly dependent** if there exist constants  $c_1, c_2, \dots, c_n$  not all zero such that

$$c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \dots + c_n\mathbf{X}_n = \mathbf{0}$$

for every  $t$  in the interval. Otherwise the set is said to be **linearly independent**.

- A set of solution vectors

$$\mathbf{X}_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}, \quad \dots, \quad \mathbf{X}_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$$

is linearly independent on an interval  $I$  if the **Wronskian**

$$W(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n) = \begin{vmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{vmatrix} \neq 0$$

for every  $t$  in the interval.

- Any set of  $n$  linearly independent solution vectors of a homogeneous linear system on an interval  $I$  is said to be a **fundamental set of solutions** on that interval.

- If  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$  are a fundamental set of solutions of a homogeneous linear system on an interval  $I$ , then the **general solution** of the system on that interval is

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \dots + c_n\mathbf{X}_n$$

where  $c_i$  are arbitrary constants.

- For nonhomogeneous systems, a **particular solution**  $\mathbf{X}_p$  on an interval  $I$  is any vector, free from arbitrary parameters, whose entries are functions that satisfy the system.
- For nonhomogeneous systems, the **general solution** of the system on the interval is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$$

where  $\mathbf{X}_c$  is the general solution of the associated homogeneous system (the **complementary function**) and  $\mathbf{X}_p$  is a particular solution of the nonhomogeneous system.

## 10.2 Homogeneous Linear Systems

### 10.2.1 Distinct Real Eigenvalues

- If  $\mathbf{X}' = \mathbf{A}\mathbf{X}$  is a homogeneous linear system,  $\lambda_1, \lambda_2, \dots, \lambda_n$  are  $n$  real, distinct eigenvalues of  $\mathbf{A}$ , and  $\mathbf{K}_1, \mathbf{K}_2, \dots, \mathbf{K}_n$  are the corresponding eigenvectors of  $\mathbf{A}$ , then

$$\mathbf{X} = c_1\mathbf{K}_1e^{\lambda_1 t} + c_2\mathbf{K}_2e^{\lambda_2 t} + \dots + c_n\mathbf{K}_ne^{\lambda_n t}$$

is the general solution of the system.

- If a system of linear equations consists of variables  $x$  and  $y$ , then the  $x - y$  plane is called the **phase plane**.
- Solution vectors of a linear system can be considered parametric equations and plotted on the phase plane. These are called trajectories.
- When multiple trajectories are plotted in the phase plane, it's called a **phase portrait**.

### 10.2.2 Repeated Eigenvalues

- If the coefficient matrix  $\mathbf{A}$  of a linear system has an eigenvalue  $\lambda$  of multiplicity  $m$ , it may be possible to find  $m$  linearly independent eigenvectors

$\mathbf{K}_1, \mathbf{K}_2, \dots, \mathbf{K}_m$  associated with the eigenvalue in which case the  $m$  solution vectors associated with the eigenvalue are

$$\begin{aligned}\mathbf{X}_1 &= \mathbf{K}_1 e^{\lambda t} \\ \mathbf{X}_2 &= \mathbf{K}_2 e^{\lambda t} \\ &\vdots \\ \mathbf{X}_m &= \mathbf{K}_m e^{\lambda t}.\end{aligned}$$

- If the coefficient matrix  $\mathbf{A}$  of a linear system has an eigenvalue  $\lambda$  of multiplicity  $m$  and it's not possible to find  $m$  linearly independent eigenvectors associated with the eigenvalue, then the  $m$  solution vectors associated with the eigenvalue are

$$\begin{aligned}\mathbf{X}_1 &= \mathbf{K}_1 e^{\lambda t} \\ \mathbf{X}_2 &= \mathbf{K}_1 t e^{\lambda t} + \mathbf{K}_2 e^{\lambda t} \\ &\vdots \\ \mathbf{X}_m &= \mathbf{K}_1 \frac{t^{m-1}}{(m-1)!} e^{\lambda t} + \mathbf{K}_2 \frac{t^{m-2}}{(m-2)!} e^{\lambda t} + \dots + \mathbf{K}_m e^{\lambda t}\end{aligned}$$

where  $\mathbf{K}_i$  are the solutions to the equations

$$\begin{aligned}(\mathbf{A} - \lambda \mathbf{I})\mathbf{K}_1 &= \mathbf{0} \\ (\mathbf{A} - \lambda \mathbf{I})\mathbf{K}_2 &= \mathbf{K}_1 \\ &\vdots \\ (\mathbf{A} - \lambda \mathbf{I})\mathbf{K}_m &= \mathbf{K}_{m-1}.\end{aligned}$$

### 10.2.3 Complex Eigenvalues

- If  $\mathbf{A}$  is the coefficient matrix of a homogeneous linear system and it has a complex eigenvalue  $\lambda = \alpha + i\beta$  and associated eigenvector  $\mathbf{K}_1$ , then

$$\mathbf{X}_1 = \mathbf{K}_1 e^{\lambda t} \text{ and } \mathbf{X}_2 = \overline{\mathbf{K}_1} e^{\bar{\lambda} t}$$

are solutions of the system.

- The solutions above can be made real by writing them as

$$\begin{aligned}\mathbf{X}_1 &= [\mathbf{B}_1 \cos \beta t - \mathbf{B}_2 \sin \beta t] e^{\alpha t} \\ \mathbf{X}_2 &= [\mathbf{B}_2 \cos \beta t + \mathbf{B}_1 \sin \beta t] e^{\alpha t}\end{aligned}$$

where  $\mathbf{B}_1 = \text{Re}(\mathbf{K}_1)$  and  $\mathbf{B}_2 = \text{Im}(\mathbf{K}_1)$ .