Advanced Engineering Mathematics Partial Differential Equations by Dennis G. Zill Notes

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Contents

12 Orthogonal Functions and Fourier Series

12.1 Orthogonal Functions

- The **inner product** of two functions f_1 and f_2 on an interval $[a, b]$ is the number
$$(f_1, f_2) = \int_a^b f_1(x) f_2(x)\, dx.$$
- Two functions f_1 and f_2 are said to be orthogonal on an interval if $(f_1, f_2) = 0$.
- A set of real-valued functions $\{\phi_1(x), \phi_2(x), \ldots, \phi_n(x)\}$ is said to be **orthogonal** on an interval if
$$(\phi_i, \phi_j) = 0 \text{ for } i \neq j.$$
- The **square norm** of a function is
$$||\phi_n(x)||^2 = (\phi_n, \phi_n)$$
and thus its **norm** is
$$||\phi_n(x)|| = \sqrt{(\phi_n, \phi_n)}.$$

- An **orthonormal set** of functions is an orthogonal set of functions that all have a norm of 1.
- An orthogonal set can be made into an orthonormal set by dividing each member by its norm.
- If $\{\phi_n(x)\}$ is an infinite orthogonal set of functions on an interval $[a, b]$ and $f(x)$ is an arbitrary function, then it's possible to determine a set of coefficients $c_n, n = 0, 1, 2, \ldots$ such that

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) = c_0\phi_0(x) + c_1\phi_1(x) + \ldots + c_n\phi_n(x) + \ldots$$

  This is called an **orthogonal series expansion** of f or a **generalized Fourier series** where the coefficients are given by

$$c_n = \frac{(f, \phi_n)}{||\phi_n||^2}.$$

- A set of real-valued functions $\{\phi_n(x)\}$ is said to be **orthogonal with respect to a weight function** $w(x)$ on the interval $[a, b]$ if

$$\int_a^b w(x)\phi_m(x)\phi_n(x)\,dx = 0,\ m \neq n.$$

12.2 Fourier Series

- The **Fourier series** of a function f defined on the interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

  where

$$\begin{aligned} a_0 &= \frac{1}{p} \int_{-p}^{p} f(x)\,dx \\ a_n &= \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi}{p} x\,dx \\ b_n &= \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi}{p} x\,dx \end{aligned}$$

- At points of discontinuity in f, the Fourier series takes on the average of the values either side of it.
- The Fourier series of a function f gives a **periodic extension** of the function outside the interval $(-p, p)$.

12.3 Fourier Cosine and Sine Series

- A function f is said to be **even** if

  $$f(-x) = f(x)$$

  and **odd** if

  $$f(-x) = -f(x).$$

- Even and odd functions have some interesting properties:
  - The product of two even functions is even.
  - The product of two odd functions is even.
  - The product of an even function and an odd function is odd.
  - If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
  - If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

- In light of this, if a function f is even its Fourier coefficients are

  $$\begin{aligned} a_0 &= \frac{2}{p}\int_0^p f(x)\,dx \\ a_n &= \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx \\ b_n &= 0. \end{aligned}$$

  The series consists of cosine terms and is called the **Fourier cosine series**.

- Similarly, if f is odd then

  $$\begin{aligned} a_n &= 0, \;\; n = 0, 1, 2, \ldots \\ b_n &= \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx. \end{aligned}$$

  The series consists of sine terms and is called the **Fourier sine series**.

- Sometimes a Fourier series "overshoots" the original value of the function near discontinuities. This is called the **Gibbs phenomenon**.

- Taking the Fourier cosine series of a function f over the interval $[0, L]$ effectively mirrors the function around the vertical axis.

- Taking the Fourier sine series of a function f over the interval $[0, L]$ effectively rotates it 180° around the origin.

- A particular solution for a nonhomogeneous differential equation with a periodic driving force can be found by taking the Fourier transform of the driving force then using the method of undetermined coefficients to determine the coefficients.

12.4 Complex Fourier Series

- The **complex Fourier series** of a function f defined on an interval $(-p, p)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p}$$

where

$$c_n = \frac{1}{2p} \int_{-p}^{p} f(x) e^{-in\pi x/p}\, dx, \;\; n = 0, \pm 1, \pm 2, \ldots$$

- The **fundamental period** of a Fourier series is $T = 2p$.

- The **fundamental angular frequency** of a Fourier series is $\omega = \frac{2\pi}{T}$.

- A **frequency spectrum** is a plot of the points $(n\omega, |c_n|)$ where ω is the fundamental angular frequency and c_n are the coefficients of the complex Fourier series. This can be useful to see how each harmonic contributes.

12.5 Sturm-Liouville Problem

- If a boundary value problem contains an arbitrary parameter λ, the values of λ for which the problem has nontrivial solutions are called the **eigenvalues** of the problem and the associated solutions are called the **eigenfunctions** of the problem.

- An orthogonal set of functions can be generated by solving a two-point boundary-value problem involing a linear second-order differential equation containing a parameter λ.

- A **regular Sturm-Liouville problem** is a boundary value problem

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

subject to

$$A_1 y(a) + B_1 y'(a) = 0$$
$$A_2 y(b) + B_2 y'(b) = 0$$

where p, q, r, and r' are real-valued functions continuous on an interval $[a, b]$, $r(x) > 0$ and $p(x) > 0$ for every x in that interval, the coefficients in the boundary conditions are real and independent of λ, A_1 and B_1 are not both zero, and A_2 and B_2 are not both zero.

- A boundary condition

$$A_1 y(a) + B_1 y'(a) = C$$

is said to be **homogeneous** if $C = 0$ and **nonhomogeneous** otherwise.

- A boundary-value problem consisting of a homogeneous differential equation and a homogeneous boundary condition is said to be homogeneous, otherwise it's nonhomogeneous.

- Multiple boundary conditions are said to be **separated** if each deals with values at a single point $x = a$ and **mixed** if each deals with values at multiple points $x = a, b, \ldots$

- If a boundary-value problem can be identified as a Sturm-Liouville problem we know it has several properties:

  - There exists an infinite number of real eigenvalues that can be arranged in increasing order $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$ such that $\lambda_n \to \infty$ as $n \to \infty$.
  - For each eigenvalue there is only one eigenfunction.
  - Eigenfunctions corresponding to different eigenvalues are linearly independent.
  - The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on the interval $[a, b]$.

- If a Sturm-Liouville problem has $r(a) = 0$ and boundary conditions are specified at $x = b$, or $r(b) = 0$ and boundary conditions are specified at $x = a$, then it is called a **singular boundary-value problem**.

- If a Sturm-Liouville problem has $r(a) = r(b)$ and boundary conditions $y(a) = y(b)$, $y'(a) = y'(b)$, then it is called a **periodic boundary-value problem**.

- If the solutions to a singular or periodic boundary-value problem are bounded on the interval $[a, b]$ then the orthogonality relation holds.

- Any second-order linear differential equation

  $$a(x)y'' + b(x)y' + [c(x) + \lambda d(x)]y = 0$$

  can be transformed into a Sturm-Liouville problem providing the coefficients are continuous and $a(x) \neq 0$ on the interval of interest. This can be done by:

  1. dividing by a,
  2. multiplying by the integrating factor $e^{\int (b/a)\, dx}$,
  3. recognising that

  $$e^{\int (b/a)\, dx} y'' + \frac{b}{a} e^{\int (b/a)\, dx} y' = \frac{d}{dx}\left[e^{\int (b/a)\, dx} y'\right],$$

4. and rewriting the equation as

$$\frac{d}{dx}\left[e^{\int (b/a)\,dx}y'\right]+\left(\frac{c}{a}e^{\int (b/a)\,dx}+\lambda\frac{d}{a}e^{\int (b/a)\,dx}\right)\lambda=0$$

which is the desired form and lets us recognise

$$\begin{aligned}r(x)&=e^{\int (b/a)\,dx}\\ q(x)&=\frac{c}{a}e^{\int (b/a)\,dx}\\ p(x)&=\frac{d}{a}e^{\int (b/a)\,dx}.\end{aligned}$$

12.6 Bessel and Legendre Series

12.6.1 Fourier-Bessel Series

Definition 12.6.1 Fourier–Bessel Series

The **Fourier–Bessel series** of a function f defined on the interval $(0, b)$ is given by

(*i*)

$$f(x)=\sum_{i=1}^{\infty}c_iJ_n(\alpha_i x) \tag{15}$$

$$c_i=\frac{2}{b^2J_{n+1}^2(\alpha_i b)}\int_0^b xJ_n(\alpha_i x)f(x)\,dx, \tag{16}$$

where the α_i are defined by $J_n(\alpha b)=0$.

(*ii*)

$$f(x)=\sum_{i=1}^{\infty}c_iJ_n(\alpha_i x) \tag{17}$$

$$c_i=\frac{2\alpha_i^2}{(\alpha_i^2b^2-n^2+h^2)J_n^2(\alpha_i b)}\int_0^b xJ_n(\alpha_i x)f(x)\,dx, \tag{18}$$

where the α_i are defined by $hJ_n(\alpha b)+\alpha bJ_n'(\alpha b)=0$.

(*iii*)

$$f(x)=c_1+\sum_{i=2}^{\infty}c_iJ_0(\alpha_i x) \tag{19}$$

$$c_1=\frac{2}{b^2}\int_0^b xf(x)\,dx,\quad c_i=\frac{2}{b^2J_0^2(\alpha_i b)}\int_0^b xJ_0(\alpha_i x)f(x)\,dx, \tag{20}$$

where the α_i are defined by $J_0'(\alpha b)=0$.

- The Fourier-Bessel series converges to f where it is continuous and

  $$\frac{f(x+)+f(x-)}{2}$$

  where it is discontinuous.

12.6.2 Fourier-Legendre Series

Definition 12.6.2 Fourier–Legendre Series

The **Fourier–Legendre series** of a function f defined on the interval $(-1, 1)$ is given by

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \tag{21}$$

where

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx. \tag{22}$$

- The Fourier-Legendre series converges to f where it is continuous and

$$\frac{f(x+) + f(x-)}{2}$$

where it is discontinuous.