Introduction to Quantum Mechanics by David J. Griffiths Problems

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Contents

Part I
Theory

1 The Wave Function

1.1

(a)

$$
\begin{aligned}
\langle j^2 \rangle &= \sum j^2 P(j) \\
&= 14^2 \frac{1}{14} + 15^2 \frac{1}{14} + 16^2 \frac{3}{14} + 22^2 \frac{2}{14} + 24^2 \frac{2}{14} + 25^2 \frac{5}{14} \\
&= \frac{3217}{7} \\
&\approx 459.571 \\
\langle j \rangle^2 &= \left(\sum j P(j) \right)^2 \\
&= 441
\end{aligned}
$$

(b)

$$
\begin{aligned}
\Delta j_{14} &= -7 \\
\Delta j_{15} &= -6 \\
\Delta j_{16} &= -5 \\
\Delta j_{22} &= 1 \\
\Delta j_{24} &= 3 \\
\Delta j_{25} &= 4 \\
\sigma^2 &= \sum (\Delta j)^2 P(j) \\
&= \frac{130}{7} \\
&\approx 18.571
\end{aligned}
$$

(c)

$$\sigma^2 = \sqrt{\langle j^2 \rangle - \langle j \rangle^2} = 18.571$$

1.2

(a)

$$
\begin{aligned}
\langle x^2 \rangle &= \int_0^h x^2 \rho(x)\, dx \\
&= \int_0^h \frac{x^{3/2}}{2\sqrt{h}}\, dx \\
&= \frac{1}{2\sqrt{h}} \left[\frac{2}{5} x^{5/2} \right]_0^h \\
&= \frac{h^2}{5} \\
\langle x \rangle^2 &= \frac{h^2}{9} \\
\sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{h^2}{5} - \frac{h^2}{9}} \\
&= h\sqrt{\frac{4}{45}} \\
&= \frac{2}{3\sqrt{5}} h
\end{aligned}
$$

(b)

$$
\begin{aligned}
1 - \int_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} \rho(x)\, dx &= 1 - \frac{1}{2\sqrt{h}} [2\sqrt{x}]_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} \\
&= 1 - \frac{1}{\sqrt{h}} \left(\sqrt{\frac{1}{3}h + \frac{2}{3\sqrt{5}}h} - \sqrt{\frac{1}{3}h - \frac{2}{3\sqrt{5}}h} \right) \\
&= 1 - \left(\sqrt{\frac{1}{3} + \frac{2}{3\sqrt{5}}} - \sqrt{\frac{1}{3} - \frac{2}{3\sqrt{5}}} \right) \\
&\approx 0.393
\end{aligned}
$$

1.3

(a)

$$\begin{aligned}
\rho(x) &= Ae^{-\lambda(x-a)^2} \\
1 &= \int_{-\infty}^{\infty} \rho(x)\,dx \\
&= A\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2}\,dx \\
&= A\sqrt{\frac{\pi}{\lambda}} \\
A &= \sqrt{\frac{\lambda}{\pi}}
\end{aligned}$$

(b)

$$\begin{aligned}
\langle x\rangle &= \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} xe^{-\lambda(x-a)^2}\,dx \\
&= a \\
\langle x^2\rangle &= \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} x^2e^{-\lambda(x-a)^2}\,dx \\
&= a^2 + \frac{1}{2\lambda} \\
\sigma &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\
&= \sqrt{a^2 + \frac{1}{2\lambda} - a^2} \\
&= \frac{1}{\sqrt{2\lambda}}
\end{aligned}$$

1.4

(a)

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi(x,0)|^2 \, dx \\
&= \left(\frac{A}{a}\right)^2 \int_0^a x^2 \, dx + \left(\frac{A}{b-a}\right)^2 \int_a^b (b-x)^2 \, dx \\
&= \frac{1}{3}A^2 a + \left(\frac{A}{b-a}\right)^2 \left[-\frac{1}{3}(b-x)^3\right]_a^b \\
&= \frac{1}{3}A^2 a + \frac{1}{3}A^2(b-a) \\
&= \frac{1}{3}A^2 b \\
A &= \sqrt{\frac{3}{b}}
\end{aligned}$$

(c) $x = a$

(d)

$$\begin{aligned}
\int_0^a |\Psi(x,0)|^2 \, dx &= \frac{3}{a^2 b} \left[\frac{1}{3}x^3\right]_0^a \\
&= \frac{a}{b}
\end{aligned}$$

(e)

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x |\Psi(x,0)|^2 \, dx \\
&= \frac{3}{a^2 b} \left[\frac{1}{4}x^4\right]_0^a + \frac{3}{b(b-a)^2} \int_a^b x(b-x)^2 \, dx \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \int_a^b (b^2 x - 2bx^2 + x^3) \, dx \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \left[\frac{1}{2}b^2x^2 - \frac{2}{3}bx^3 + \frac{1}{4}x^4\right]_a^b \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \left(\frac{1}{2}b^4 - \frac{2}{3}b^4 + \frac{1}{4}b^4 - \frac{1}{2}a^2b^2 + \frac{2}{3}a^3 b - \frac{1}{4}a^4\right) \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \frac{1}{12}(b-a)^3(3a+b) \\
&= \frac{3a^2}{4b} + \frac{1}{4b}(3ab + b^2 - 3a^2 - ab) \\
&= \frac{1}{2}a + \frac{1}{4}b
\end{aligned}$$

1.5

(a)

$$\begin{aligned}
\Psi(x,t) &= Ae^{-\lambda|x|}e^{-i\omega t} \\
\Psi(x,0) &= Ae^{-\lambda|x|} \\
1 &= A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|}\, dx \\
&= 2A^2 \int_0^{\infty} e^{-2\lambda x}\, dx \\
&= 2A^2 \left[-\frac{1}{2\lambda}e^{-2\lambda x}\right]_0^{\infty} \\
&= \frac{A^2}{\lambda} \\
A &= \sqrt{\lambda}
\end{aligned}$$

(b)

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} x\lambda e^{-2\lambda|x|}\, dx \\
&= \lambda \int_{-\infty}^{\infty} xe^{-2\lambda|x|}\, dx \\
&= 0 \\
\langle x^2\rangle &= \int_{-\infty}^{\infty} x^2\lambda e^{-2\lambda|x|}\, dx \\
&= 2\lambda \int_0^{\infty} x^2 e^{-2\lambda x}\, dx \\
&= \frac{1}{2\lambda^2}
\end{aligned}$$

(c)

$$\begin{aligned}
\sigma &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\
&= \frac{1}{\sqrt{2}\lambda} \\
1 - \int_{-\sigma}^{\sigma} \lambda e^{-2\lambda|x|}\, dx &= 1 - 2\lambda \int_0^{\sigma} e^{-2\lambda x}\, dx \\
&= 1 - 2\lambda \left[-\frac{1}{2\lambda}e^{-2\lambda x}\right]_0^{\sigma} \\
&= e^{-2\lambda\sigma} \\
&= e^{-\sqrt{2}} \\
&\approx 0.243
\end{aligned}$$

1.6

The chain rule requires that you apply it to both x and $|\Psi|^2$ which gives the same result

$$\begin{aligned}
\frac{d\langle x\rangle}{dt} &= \frac{d}{dt}\int x|\Psi|^2\,dx \\
&= \int \frac{d}{dt}(x|\Psi|^2)\,dx \\
&= \int \left(0\cdot|\Psi|^2 + x\frac{\partial|\Psi|^2}{\partial t}\right)\,dx \\
&= \int x\frac{\partial|\Psi|^2}{\partial t}\,dx
\end{aligned}$$

1.8

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\left(e^{-iV_0t/\hbar}\Psi\right) &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(e^{-iV_0t/\hbar}\Psi\right) + (V+V_0)\left(e^{-iV_0t/\hbar}\Psi\right) \\
i\hbar\left(-\frac{iV_0}{\hbar}e^{-iV_0t/\hbar}\Psi + e^{-iV_0t/\hbar}\frac{\partial\Psi}{\partial t}\right) &= -\frac{\hbar^2}{2m}e^{-iV_0t/\hbar}\frac{\partial^2\Psi}{\partial x^2} + Ve^{-iV_0t/\hbar}\Psi + V_0e^{-iV_0t/\hbar}\Psi \\
V_0\Psi + i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi + V_0\Psi \\
i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi
\end{aligned}$$

$$\begin{aligned}
\langle Q(x,p)\rangle &= \int \left(e^{-iV_0t/\hbar}\Psi\right)^*[Q(x,-i\hbar\partial/\partial x)]\,e^{-iV_0t/\hbar}\Psi\,dx \\
&= \int e^{iV_0t/\hbar}\Psi^*\,[Q(x,-i\hbar\partial/\partial x)]\,e^{-iV_0t/\hbar}\Psi\,dx \\
&= \int \Psi^*[Q(x,-i\hbar\partial/\partial x)]\Psi\,dx
\end{aligned}$$

No effect on the expectation value.

1.9

(a)

$$
\begin{aligned}
\Psi(x,t) &= Ae^{-a[(mx^2/\hbar)+it]} \\
1 &= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)}\,dx \\
&= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)}\,dx \\
&= A^2 \sqrt{\frac{\pi\hbar}{2am}} \\
A^2 &= \sqrt{\frac{2am}{\pi\hbar}} \\
A &= \left(\frac{2am}{\pi\hbar}\right)^{1/4}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\Psi &= Ae^{-a[(mx^2/\hbar)+it]} \\
\frac{\partial\Psi}{\partial t} &= -ia\Psi \\
\frac{\partial\Psi}{\partial x} &= -\frac{2amx}{\hbar}\Psi \\
\frac{\partial^2\Psi}{\partial x^2} &= -\frac{2am}{\hbar}\left(\Psi + x\frac{\partial\Psi}{\partial x}\right) \\
&= -\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\Psi \\
V\Psi &= i\hbar\frac{\partial\Psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} \\
&= a\hbar\Psi - a\hbar\left(1 - \frac{2amx^2}{\hbar}\right)\Psi \\
V &= a\hbar - a\hbar + 2a^2mx^2 \\
&= 2a^2mx^2
\end{aligned}
$$

(c)

$$
\begin{aligned}
\langle x \rangle &= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)} x \, dx \\
&= 0 \\
\langle x^2 \rangle &= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)} x^2 \, dx \\
&= 2A^2 \int_{0}^{\infty} e^{-2a(mx^2/\hbar)} x^2 \, dx \\
&= \frac{\hbar}{4am} \\
\langle p \rangle &= \int_{-\infty}^{\infty} \Psi^* \left[-i\hbar \frac{\partial}{\partial x} \right] \Psi \, dx \\
&= -i\hbar \int_{-\infty}^{\infty} A e^{-a[(mx^2/\hbar)-it]} \left(-\frac{2amx}{\hbar} A e^{-a[(mx^2/\hbar)+it]} \right) dx \\
&= 2iA^2 am \int_{-\infty}^{\infty} x e^{-2amx^2/\hbar} \, dx \\
&= 0 \\
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \Psi^* \left[-\hbar^2 \frac{\partial^2}{\partial x^2} \right] \Psi \, dx \\
&= -\hbar^2 \int_{-\infty}^{\infty} A e^{-a[(mx^2/\hbar)-it]} \left[-\frac{2am}{\hbar} \left(1 - \frac{2amx^2}{\hbar} \right) A e^{-a[(mx^2/\hbar)+it]} \right] dx \\
&= 2A^2 am\hbar \int_{-\infty}^{\infty} e^{-2amx^2/\hbar} \left(1 - \frac{2amx^2}{\hbar} \right) dx \\
&= am\hbar
\end{aligned}
$$

(d)

$$
\begin{aligned}
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{\hbar}{4am}} \\
\sigma_p &= \sqrt{am\hbar} \\
\sigma_x \sigma_p &= \sqrt{\frac{1}{4}\hbar^2} \\
&= \frac{1}{2}\hbar \\
&\geq \frac{1}{2}\hbar
\end{aligned}
$$

Yes, this is consistent with Heisenberg's uncertainty principle.

1.10

(a)

$$\begin{aligned}
P(0) &= 0\\
P(1) &= \frac{2}{25}\\
&= 0.08\\
P(2) &= \frac{3}{25}\\
&= 0.12\\
P(3) &= \frac{1}{5}\\
&= 0.2\\
P(4) &= \frac{3}{25}\\
&= 0.12\\
P(5) &= \frac{3}{25}\\
&= 0.2\\
P(6) &= \frac{3}{25}\\
&= 0.2\\
P(7) &= \frac{1}{25}\\
&= 0.04\\
P(8) &= \frac{2}{25}\\
&= 0.08\\
P(9) &= \frac{3}{25}\\
&= 0.12
\end{aligned}$$

(b) The most probable digit is 3, the median digit is 4, and the average value is $\frac{118}{25} = 4.72$.

(c) $\sigma = 2.474$

1.14

(a)

$$\begin{aligned}
P_{ab}(t) &= \int_a^b |\Psi(x,t)|^2 \, dx \\
\frac{dP_{ab}}{dt} &= \frac{d}{dt}\int_a^b |\Psi(x,t)|^2 \, dx \\
&= \int_a^b \frac{d}{dt}\left(|\Psi(x,t)|^2\right) \, dx \\
&= \int_a^b \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\right] \, dx \\
&= J(a,t) - J(b,t)
\end{aligned}$$

The units are s^{-1}.

(b)

$$\begin{aligned}
\Psi(x,t) &= Ae^{-a[(mx^2/\hbar)+it]} \\
\frac{\partial \Psi}{\partial x} &= -\frac{2amx}{\hbar}\Psi \\
\Psi^*(x,t) &= Ae^{-a[(mx^2/\hbar)-it]} \\
\frac{\partial \Psi^*}{\partial x} &= -\frac{2amx}{\hbar}\Psi^* \\
J(x,t) &= \frac{i\hbar}{2m}\left(\Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi}{\partial x}\right) \\
&= \frac{i\hbar}{2m}\left[\Psi\left(-\frac{2amx}{\hbar}\Psi^*\right) - \Psi^*\left(-\frac{2amx}{\hbar}\Psi\right)\right] \\
&= 0
\end{aligned}$$

1.15

$$
\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2\,dx &= \int_{-\infty}^{\infty}\left(\frac{\partial\Psi_1^*}{\partial t}\Psi_2+\Psi_1^*\frac{\partial\Psi_2}{\partial t}\right)dx \\
&= \int_{-\infty}^{\infty}\left[\left(-i\frac{\hbar}{2m}\frac{\partial^2\Psi_1^*}{\partial x^2}+i\frac{V}{\hbar}\Psi_1^*\right)\Psi_2\right. \\
&\qquad \left.+\Psi_1^*\left(i\frac{\hbar}{2m}\frac{\partial^2\Psi_2}{\partial x^2}-i\frac{V}{\hbar}\Psi_2\right)\right]dx \\
&= i\frac{\hbar}{2m}\int_{-\infty}^{\infty}\left(\Psi_1^*\frac{\partial^2\Psi_2}{\partial x^2}-\frac{\partial^2\Psi_1^*}{\partial x^2}\Psi_2\right)dx \\
&= i\frac{\hbar}{2m}\left[\Psi_1^*\frac{\partial\Psi_2}{\partial x}\bigg|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{\partial}{\partial x}(\Psi_1^*\Psi_2)\,dx\right. \\
&\qquad \left.\frac{\partial\Psi_1^*}{\partial x}\Psi_2\bigg|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{\partial}{\partial x}(\Psi_1^*\Psi_2)\,dx\right] \\
&= 0
\end{aligned}
$$

1.16

(a)

$$
\begin{aligned}
1 &= \int_{-a}^{a}A^2(a^2-x^2)^2\,dx \\
&= A^2\int_0^a(a^2-x^2)^2\,dx \\
&= \frac{16}{15}A^2a^5 \\
A &= \sqrt{\frac{15}{16a^5}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\langle x\rangle &= \int_{-a}^{a}xA(a^2-x^2)\,dx \\
&= 0
\end{aligned}
$$

(c)

$$
\begin{aligned}
\langle p\rangle &= \int_{-a}^{a}\Psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi\,dx \\
&= 2iA^2\hbar\int_{-a}^{a}x(a^2-x^2)\,dx \\
&= 0
\end{aligned}
$$

(d)

$$
\begin{aligned}
\langle x^2 \rangle &= \int_{-a}^{a} \Psi^* x^2 \Psi \, dx \\
&= A^2 \int_{-a}^{a} x^2 (a^2 - x^2)^2 \, dx \\
&= A^2 \frac{16}{105} a^7 \\
&= \frac{a^2}{7}
\end{aligned}
$$

(e)

$$
\begin{aligned}
\langle p^2 \rangle &= \int_{-a}^{a} \Psi^* \left(-\hbar^2 \frac{\partial^2}{\partial x^2} \right) \Psi \, dx \\
&= -\hbar^2 \int_{-a}^{a} A(a^2 - x^2)(-2A) \, dx \\
&= 4A^2\hbar^2 \int_{0}^{a} (a^2 - x^2) \, dx \\
&= 4A^2\hbar^2 \left[a^2 x - \frac{1}{3} x^3 \right]_0^a \\
&= 4A^2\hbar^2 \left(a^3 - \frac{1}{3} a^3 \right) \\
&= \frac{8}{3} A^2 a^3 \hbar^2 \\
&= \frac{8}{3} \frac{15}{16a^5} a^3 \hbar^2 \\
&= \frac{5}{2} \frac{\hbar^2}{a^2}
\end{aligned}
$$

(f)

$$
\begin{aligned}
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{a^2}{7}} \\
&= \frac{a}{\sqrt{7}}
\end{aligned}
$$

(g)

$$
\begin{aligned}
\sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\
&= \sqrt{\frac{5}{2}} \frac{\hbar}{a}
\end{aligned}
$$

(h)

$$\sigma_x\sigma_p = \sqrt{\frac{5}{14}}\hbar$$
$$\geq \frac{1}{2}\hbar$$

1.18

(a)

$$\begin{aligned}
\frac{h}{\sqrt{3mk_BT}} &> d \\
\frac{\sqrt{3mk_BT}}{h} &< \frac{1}{d} \\
T_{\text{electron}} &< \frac{h^2}{3d^2mk_B} \\
&< 1.3\times 10^5\,\text{K} \\
T_{\text{nuclei}} &< 2.5\,\text{K}
\end{aligned}$$

(b)

$$\begin{aligned}
PV &= Nk_BT \\
\frac{V}{N} &= \frac{k_BT}{P} \\
d &= \left(\frac{k_BT}{P}\right)^{1/3} \\
\frac{h}{\sqrt{3mk_Bt}} &> \left(\frac{k_BT}{P}\right)^{1/3} \\
T &< \frac{1}{k_B}\left(\frac{h^2}{3m}\right)^{3/5}P^{2/5}
\end{aligned}$$

2 Time-Independent Schrödinger Equation

2.1

(a)

$$\begin{aligned}
\int_{-\infty}^{\infty}|\Psi|^2\,dx &= \int_{-\infty}^{\infty}\Psi^*\Psi\,dx \\
&= \int_{-\infty}^{\infty}\psi^* e^{i(E_0-i\Gamma)t/\hbar}\psi e^{-i(E_0+i\Gamma)t/\hbar}\,dx \\
&= e^{2\Gamma t/\hbar}\int_{-\infty}^{\infty}|\psi|^2\,dx
\end{aligned}$$

In order for this to equal 1 for all t, Γ must be 0.

(b) If $\psi(x)$ is a complex solution to the time-independent Schrödinger equation then so is $\psi^*(x)$ and $\psi(x) + \psi^*(x)$ which is real.

2.2

If ψ and its second derivative always have the same sign, ψ will increase or decrease without bound forever. This means there is no non-zero choice of constant A such that

$$\int_{-\infty}^{\infty} |A\Psi|^2 \, dx = 1$$

and thus the equation can't be normalised.

The classical analog of this is statements is that the potential energy of a system can't exceed its total energy.