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## 1 First-order ODEs

**Form:** IVP

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

**Test:**  $f(x, y)$  and  $\partial f / \partial y$  are continuous over  $I$

**Property:** A unique solution is guaranteed over  $I$

### 1.1 Separable Equations

**Form:**

$$\frac{dy}{dx} = g(x)h(y)$$

**Solution:** Divide by  $h(y)$  then integrate with respect to  $x$ .

$$\begin{aligned}
\frac{dy}{dx} &= g(x)h(y) \\
\frac{1}{h(y)} \frac{dy}{dx} &= g(x) \\
\int \frac{1}{h(y)} \frac{dy}{dx} dx &= \int g(x) dx \\
\int \frac{1}{h(y)} dy &= \int g(x) dx \\
H(y) &= G(x) + c
\end{aligned}$$

## 1.2 Linear Equations

**Form:**

$$\frac{dy}{dx} + P(x)y = f(x)$$

**Solution:**

1. Determine the integrating factor  $e^{\int P(x) dx}$
2. Multiply by the integrating factor
3. Recognise that the left hand side of the equation is the derivative of the product of the integrating factor and  $y$
4. Integrate both sides of the equation
5. Solve for  $y$

## 1.3 Exact Equations

**Form:**

$$\begin{aligned}
z &= f(x, y) = c \\
dz &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = M(x, y) dx + N(x, y) dy = 0
\end{aligned}$$

**Test:**

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

**Solution:**

1. Integrate  $M(x, y)$  with respect to  $x$  to find an expression for  $z = f(x, y)$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= M(x, y) \\
f(x, y) &= \int M(x, y) dx + g(y)
\end{aligned}$$

2. Differentiate  $f(x, y)$  with respect to  $y$  and equate it to  $N(x, y)$  to find  $g'(y)$

$$\begin{aligned}\frac{\partial f}{\partial y} &= N(x, y) = \frac{\partial}{\partial y} \int M(x, y) dx + g'(y) \\ g'(y) &= N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx\end{aligned}$$

3. Integrate  $g'(y)$  with respect to  $y$  to find  $g(y)$  and substitute it into  $f(x, y)$
4. Equate  $f(x, y)$  with an unknown constant  $c$

**Note:** The steps can be performed with  $x$  and  $y$  reversed, i.e. start by integrating  $N(x, y)$  with respect to  $y$ , etc.

## 1.4 Exact Equations with Integration Constant

**Form:**

$$M(x, y) dx + N(x, y) dy = 0$$

**Test:**  $(M_y - N_x)/N$  is a function of  $x$  alone or  $(N_x - M_y)/M$  is a function of  $y$  alone

**Solution:**

1. Compute the integrating factor

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

or

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

as appropriate

2. Multiple the equation by this factor
3. The equation is now exact and can be solved as above

## 1.5 Homogeneous Equations

**Form:**

$$M(x, y) dx + N(x, y) dy = 0$$

**Test:**  $M$  and  $N$  are homogeneous functions of the same degree

**Solution:**

1. Rewrite as

$$M(x, y) = x^\alpha M(1, u) \text{ and } N(x, y) = x^\alpha N(1, u) \text{ where } u = y/x$$

or

$$M(x, y) = y^\alpha M(v, 1) \text{ and } N(x, y) = y^\alpha N(v, 1) \text{ where } v = x/y$$

2. Substitute  $y = ux$  and  $dy = u dx + x du$  or  $x = vy$  and  $dx = v dy + y dv$  as appropriate
3. Solve the resulting first-order separable DE
4. Substitute  $u = y/x$  or  $v = x/y$  as appropriate

## 1.6 Bernoulli's Equation

**Form:**

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

**Test:**  $n \neq 0$  and  $n \neq 1$

**Solution:**

1. Substitute  $y = u^{1/(1-n)}$  and  $\frac{dy}{dx} = \frac{d}{dx}(u^{1/(1-n)})$
2. Solve the resulting linear equation
3. Substitute  $u = y^{1-n}$

## 1.7 Reduction to Separation of Variables

**Form:**

$$\frac{dy}{dx} = f(Ax + By + C), B \neq 0$$

**Solution:**

1. Substitute

$$Ax + By + C = u$$

2. Solve the resulting separable equation
3. Substitute

$$u = Ax + By + C$$

## 1.8 Riccati's Equation

**Form:**

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$$

**Test:** You know a particular solution  $y_1$  of the equation

**Solution:**

1. Substitute  $y = y_1 + u$  and  $y' = y_1' + u'$
2. Solve the resulting Bernoulli equation
3. Substitute  $u = y - y_1$

## 2 Higher-order ODEs

### 2.1 Initial Value Problems

**Form:**  $n$ -th order IVP

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

subject to

$$y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$$

**Test:**  $a_n(x)$ ,  $a_{n-1}(x)$ ,  $\dots$ ,  $a_0(x)$ , and  $g(x)$  are continuous on an interval  $I$  and  $a_n(x) \neq 0$  for every  $x$  in  $I$

**Property:** A unique solution exists for every  $x = x_0$  in  $I$

### 2.2 Linear Independence

**Form:** A set of functions  $f_1, f_2, \dots, f_n$

**Test:** The Wronskian  $W(f_1, f_2, \dots, f_n) \neq 0$  for every  $x$  in an interval  $I$  where

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

**Property:** The functions are linearly independent in  $I$

### 2.3 Homogeneous Linear $n$ th-Order Equations

The general solution is of the form

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

where  $c_i$  are arbitrary constants and  $y_i$  are a fundamental set of solutions (i.e. a set of  $n$  linearly independent solutions).

## 2.4 Nonhomogeneous Linear $n$ th-Order Equations

The general solution is of the form

$$y = y_c + y_p = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

where  $y_c$  is the complementary function (i.e. the general solution of the associated homogeneous equation) and  $y_p$  is a particular solution.

## 2.5 Homogeneous Linear Second-Order Equations

**Form:**

$$y'' + P(x)y' + Q(x)y = 0$$

**Test:** A non-trivial solution  $y_1(x)$  is known

**Solution:**

1. Recognise that the ratio of two linearly independent functions isn't constant, i.e.

$$u(x) = \frac{y_1(x)}{y_2(x)} \text{ or } y_2(x) = u(x)y_1(x)$$

2. Substitute  $y_2(x) = u(x)y_1(x)$  into the DE — this will result in a DE involving only  $u''$  and  $u'$  which can be treated as a linear first-order DE in  $u' = w$
3. Solve for  $w$
4. Substitute  $w = u'$
5. Integrate to find  $u$
6. Multiply by  $y_1$  to find  $y_2$

or equivalently

$$y_2 = y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx$$

## 2.6 Homogeneous Linear Equations with Constant Coefficients

**Form:**

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

**Solution:**

1. Assume the equation has a solution of the form  $y = e^{mx}$ , giving

$$a_n m^n e^{mx} + a_{n-1} m^{n-1} e^{mx} + \cdots + a_1 m e^{mx} + a_0 e^{mx} = 0$$

2. Divide by  $e^{mx}$ , giving the auxiliary/characteristic equation

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

3. Solve for  $m$ , where

- A real root  $m$  corresponds to a solution

$$y = ce^{mx}$$

- A complex root  $\alpha + i\beta$  corresponds to a solution

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

- A root  $m$  of multiplicity  $k$  corresponds to the solutions

$$e^{mx}, xe^{mx}, x^2 e^{mx}, \dots, x^{k-1} e^{mx}$$