Advanced Engineering Mathematics Systems of Differential Equations by Dennis G. Zill Notes

Chris Doble

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10 Systems of Linear Differential Equations

10.1 Theory of Linear Systems

- A system of the form

$$\begin{aligned}
\frac{dx_1}{dt} &= g_1(t, x_1, x_2, \ldots, x_n) \\
\frac{dx_2}{dt} &= g_2(t, x_1, x_2, \ldots, x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= g_n(t, x_1, x_2, \ldots, x_n)
\end{aligned}$$

is called a **first-order system**.

- When each of the functions $g_n(t, x_1, x_2, \ldots, x_n)$ is linear in the dependent variables $x_1, x_2, \ldots, x_n$, we get the **normal form** of a first-order system of linear equations

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n + f_1(t) \\
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n + f_2(t) \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n + f_n(t).
\end{aligned}$$

  Such a system is called a **linear system**.

- When $f_i(t) = 0$ for $i = 1, 2, \ldots, n$ the linear system is said to be **homogeneous**, otherwise it's **nonhomogenous**.

- If $\mathbf{X}$, $\mathbf{A}(t)$, and $\mathbf{F}(t)$ denote the matrices

$$\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \vdots & & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{pmatrix}$$

$$\mathbf{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

  then homogeneous linear systems can be written

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

  and nonhomogeneous linear systems can be written

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}.$$

- A **solution vector** on an interval I is any column matrix

$$\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

whose entries are differentiable functions satisfying the linear system on the interval.

- The entries of a solution vector can be considered a set of parametric equations that define a curve in n-space. Such a curve is called a **trajectory**.

- The problem of solving

$$\mathbf{X}' = \mathbf{A}(t)\mathbf{X} + \mathbf{F}(t)$$

subject to

$$\mathbf{X}(t_0) = \mathbf{X}_0$$

is an **initial value problem** in matrix form.

- The **superposition principle** states that if $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are solution vectors of a homogeneous linear system on an interval I, then

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + c_n\mathbf{X}_n$$

where c_n are arbitrary constants is also a solution.

- If $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are a set of solution vectors of a homogeneous linear system on an interval I, the set is said to be **linearly dependent** if there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + x_n\mathbf{X}_n = \mathbf{0}$$

for every t in the interval. Otherwise the set is said to be **linearly independent**.

- A set of solution vectors

$$\mathbf{X}_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}, \quad \ldots, \quad \mathbf{X}_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$$

is linearly independent on an interval I if the **Wronskian**

$$W(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) = \begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{vmatrix} \neq 0$$

for every t in the interval.

- Any set of n linearly independent solution vectors of a homogeneous linear system on an interval I is said to be a **fundamental set of solutions** on that interval.

- If $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are a fundamental set of solutions of a homogeneous linear system on an interval I, then the **general solution** of the system on that interval is

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + c_n\mathbf{X}_n$$

  where c_i are arbitrary constants.

- For nonhomogenous systems, a **particular solution** $\mathbf{X}_p$ on an interval I is any vector, free from arbitrary parameters, whose entries are functions that satify the system.

- For nonhomogeneous systems, the **general solution** of the system on the interval is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$$

  where $\mathbf{X}_c$ is the general solution of the associated homogeneous system (the **complementary function**) and $\mathbf{X}_p$ is a particular solution of the nonhomogeneous system.

10.2 Homogeneous Linear Systems

10.2.1 Distinct Real Eigenvalues

- If $\mathbf{X}' = \mathbf{AX}$ is a homogeneous linear system, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are n real, distinct eigenvalues of $\mathbf{A}$, and $\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_n$ are the corresponding eigenvectors of $\mathbf{A}$, then

$$\mathbf{X} = c_1\mathbf{K}_1e^{\lambda_1 t} + c_2\mathbf{K}_2e^{\lambda_2 t} + \ldots + c_n\mathbf{K}_ne^{\lambda_n t}$$

  is the general solution of the system.

- If a system of linear equations consists of variables x and y, then the $x-y$ plane is called the **phase plane**.

- Solution vectors of a linear system can be considered parametric equations and plotted on the phase plane. These are called trajectories.

- When multiple trajectories are plotted in the phase plane, it's called a **phase portrait**.

10.2.2 Repeated Eigenvalues

- If the coefficient matrix $\mathbf{A}$ of a linear system has an eigenvalue λ of multiplicity m, it may be possible to find m linearly independent eigenvectors

$\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_m$ associated with the eigenvalue in which case the m solution vectors associated with the eigenvalue are

$$\begin{aligned}\mathbf{X}_1 &= \mathbf{K}_1 e^{\lambda t} \\ \mathbf{X}_2 &= \mathbf{K}_2 e^{\lambda t} \\ &\vdots \\ \mathbf{X}_m &= \mathbf{K}_m e^{\lambda t}.\end{aligned}$$

- If the coefficient matrix $\mathbf{A}$ of a linear system has an eigenvalue λ of multiplicity m and it's not possible to find m linearly independent eigenvectors associated with the eigenvalue, then the m solution vectors associated with the eigenvalue are

$$\begin{aligned}\mathbf{X}_1 &= \mathbf{K}_1 e^{\lambda t} \\ \mathbf{X}_2 &= \mathbf{K}_1 t e^{\lambda t} + \mathbf{K_2} e^{\lambda t} \\ &\vdots \\ \mathbf{X}_m &= \mathbf{K}_1 \frac{t^{m-1}}{(m-1)!} e^{\lambda t} + \mathbf{K}_2 \frac{t^{m-2}}{(m-2)!} e^{\lambda t} + \ldots + \mathbf{K}_m e^{\lambda t}\end{aligned}$$

  where $\mathbf{K}_i$ are the solutions to the equations

$$\begin{aligned}(\mathbf{A} - \lambda \mathbf{I})\mathbf{K}_1 &= \mathbf{0} \\ (\mathbf{A} - \lambda \mathbf{I})\mathbf{K}_2 &= \mathbf{K}_1 \\ &\vdots \\ (\mathbf{A} - \lambda \mathbf{I})\mathbf{K}_m &= \mathbf{K}_{m-1}.\end{aligned}$$

10.2.3 Complex Eigenvalues

- If $\mathbf{A}$ is the coefficient matrix of a homogeneous linear system and it has a complex eigenvalue $\lambda = \alpha + i\beta$ and associated eigenvector $\mathbf{K}_1$, then

$$\mathbf{X}_1 = \mathbf{K}_1 e^{\lambda t} \text{ and } \mathbf{X}_2 = \overline{\mathbf{K}}_1 e^{\overline{\lambda} t}$$

  are solutions of the system.

- The solutions above can be made real by writing them as

$$\begin{aligned}\mathbf{X}_1 &= [\mathbf{B}_1 \cos \beta t - \mathbf{B}_2 \sin \beta t] e^{\alpha t} \\ \mathbf{X}_2 &= [\mathbf{B}_2 \cos \beta t + \mathbf{B}_1 \sin \beta t] e^{\alpha t}\end{aligned}$$

  where $\mathbf{B}_1 = \text{Re}(\mathbf{K}_1)$ and $\mathbf{B}_2 = \text{Im}(\mathbf{K}_1)$.

10.3 Solution by Diagonalization

- A homogeneous linear system $\mathbf{X}' = \mathbf{AX}$ in which each x_i' is expressed as a linear combination of $x_1, x_2, \ldots, x_n$ is said to be **coupled**. If each x_i' is expressed solely in terms of x_i the system is said to be **uncoupled**.

- Given a linear system $\mathbf{X}' = \mathbf{AX}$, if the coefficient matrix $\mathbf{A}$ is diagonalisable such that $\mathbf{P^{-1}AP} = \mathbf{D}$ then the system can be solved by:

  1. Substituting $\mathbf{X} = \mathbf{PY}$ which gives $\mathbf{PY}' = \mathbf{APY}$ or $\mathbf{Y}' = \mathbf{P^{-1}APY} = \mathbf{DY}$
  2. Because $\mathbf{D}$ is a diagonal matrix with $\mathbf{A}$'s eigenvalues along the diagonal, this means the solutions to $\mathbf{Y}' = \mathbf{DY}$ are

$$\mathbf{Y} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

  3. These solutions can then be substituted into $\mathbf{X} = \mathbf{PY}$ to solve for $\mathbf{X}$

10.4 Nonhomogeneous Linear Systems

10.4.1 Undetermined Coefficients

- The **method of undetermined coefficients** can be applied to a linear system $\mathbf{X}' = \mathbf{AX} + \mathbf{F(t)}$ when the entries of $\mathbf{A}$ are constants and the entries of $\mathbf{F}(t)$ are constants, polynomials, exponential functions, sines and cosines, or finite sums and products of these functions.

- To apply the method of undetermined coefficients:

  1. Solve the associated homogeneous linear system to find the complementary function $\mathbf{X}_c$.
  2. Assume the particular solution $\mathbf{X}_p$ has the same form as $\mathbf{F}(t)$.
  3. Substitute the trial solution into the system and solve for the unknowns.
  4. The general solution is $\mathbf{X} = \mathbf{X_c} + \mathbf{X_p}$.

- If $\mathbf{F}(t)$ contains a term that's present in the complementary function, that term needs to be adjusted (similar to how you multiply by x^n in the method of undetermined coefficients for ODEs). The textbook doesn't cover the rules for this.

10.4.2 Variation of Parameters

- If $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ is a fundamental set of solutions of the homogeneous linear system $\mathbf{X}' = \mathbf{AX}$ on an interval I, then the general solution is

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + c_n\mathbf{X}_n$$

  which can also be written

$$\mathbf{X} = \mathbf{\Phi}(t)\mathbf{C} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \ldots & \mathbf{X}_n \end{pmatrix} \mathbf{C}$$

  where $\mathbf{\Phi}(t)$ is called a **fundamental matrix** and $\mathbf{C}$ is a column vector containing the arbitrary constants $c_1, c_2, \ldots, c_n$.

- A fundamental matrix:
  - always has an inverse, and
  - has the property that $\mathbf{\Phi}'(t) = \mathbf{A}\mathbf{\Phi}(t)$.

- The **method of variation of parameters** finds a particular solution to a nonhomogeneous linear system by replacing column vector of unknown constants $\mathbf{C}$ with a column vector of functions

$$\mathbf{U}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}$$

  such that $\mathbf{X}_p = \mathbf{\Phi}(t)\mathbf{U}(t)$ is a particular solution to the system.

- $\mathbf{U}(t)$ can be calculated as

$$\mathbf{U}(t) = \int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,dt$$

  so

$$\mathbf{X}_p = \mathbf{\Phi}(t) \int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,dt$$

  and

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = \mathbf{\Phi}(t)\mathbf{C} + \mathbf{\Phi}(t) \int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,dt.$$

- When solving initial value problems via the method of variation of parameters where you're given $\mathbf{X}(t_0) = \mathbf{X}_0$, the column vector of arbitrary constants $\mathbf{C}$ can be calculated as

$$\mathbf{C} = \mathbf{\Phi}^{-1}(t_0)\mathbf{X}_0.$$

10.4.3 Diagonalization

- If the coefficient matrix $\mathbf{A}$ in a nonhomogeneous linear system $\mathbf{X}' = \mathbf{AX} + \mathbf{F}(t)$ is diagonalizable, the system can be solved by:

  1. Substituting $\mathbf{X} = \mathbf{PY}$ which gives $\mathbf{PY}' = \mathbf{APY} + \mathbf{F}(t)$ or $\mathbf{Y}' = \mathbf{P^{-1}APY} + \mathbf{P^{-1}F}(t)$ or $\mathbf{Y}' = \mathbf{DY} + \mathbf{G}$
  2. Because $\mathbf{D}$ is a diagonal matrix with $\mathbf{A}$'s eigenvalues along the diagonal and $\mathbf{G} = \mathbf{P^{-1}F}(t)$ this means $\mathbf{Y}' = \mathbf{DY} + \mathbf{G(t)}$ is a set of n uncoupled equations of the form

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ v_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 y_1 + g_1(t) \\ \lambda_2 y_2 + g_2(t) \\ \vdots \\ \lambda_n y_n + g_n(t) \end{pmatrix}$$

  3. These equations can be solved and substituted into $\mathbf{X} = \mathbf{PY}$ to solve for $\mathbf{X}$.

10.5 Matrix Exponential

- The linear first-order differential equation $x' = ax$ has a general solution $x = ce^{at}$. Similarly, the system $\mathbf{X}' = \mathbf{AX}$ has a general solution $\mathbf{X} = e^{\mathbf{A}t}\mathbf{C}$ where $e^{\mathbf{A}t}$ is an $n \times n$ matrix given by the **matrix exponential** and $\mathbf{C}$ is a $n \times 1$ matrix of arbitrary constants.

- The matrix exponential is defined as

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \mathbf{A}^k \frac{t^k}{k!}.$$

- The exponential of a diagonal matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

  is

$$e^{\mathbf{A}t} = \begin{pmatrix} e^{a_{11}t} & 0 & \dots & 0 \\ 0 & e^{a_{22}t} & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & e^{a_{nn}t} \end{pmatrix}.$$

- The nonhomogeneous system $\mathbf{X}' = \mathbf{AX} + \mathbf{F}$ has a general solution

$$\mathbf{X} = \mathbf{X_c} + \mathbf{X}_p = e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t} \int e^{-\mathbf{A}t}\mathbf{F}\, dt$$

where

$$e^{-\mathbf{A}t} = (e^{\mathbf{A}t})^{-1}$$

is the inverse of $e^{\mathbf{A}t}$.

- A matrix exponential can be calculated with the inverse Laplace transform

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}.$$

- A matrix exponential or that of one of its eigenvalues can be calculated as

$$e^{\mathbf{A}t} = \sum_{j=0}^{n-1} \mathbf{A}^j b_j(t) \text{ or } e^{\lambda t} = \sum_{j=0}^{n-1} \lambda^j b_j(t)$$

where $b_j(t)$ are the same for both expressions. This means that for an nxn matrix with n distinct eigenvalues the expressions for $e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}$ give n equations with n unknowns ($b_j(t)$). Solving for the $b_j(t)$ and substituting them into the expression for $e^{\mathbf{A}t}$ allows us to calculate the matrix exponential.

11 Systems of Nonlinear Differential Equations

11.1 Autonomous Systems

- A system of first-order differential equations is called **autonomous** when the system can be written in the form

$$\begin{aligned}
\frac{dx_1}{dt} &= g_1(x_1, x_2, \ldots, x_n) \\
\frac{dx_2}{dt} &= g_2(x_1, x_2, \ldots, x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= g_n(x_1, x_2, \ldots, x_n),
\end{aligned}$$

i.e. the independent variable t doesn't appear on the right-hand side of each equation.

- Any autonomous second-order differential equation $x'' = g(x, x')$ can be written as the autonomous system

$$\begin{aligned}
x' &= y \\
y' &= g(x, y).
\end{aligned}$$

- A system with $n = 2$ is called a **plane autonomous system** and its equations

$$\frac{dx}{dt} = P(x, y)$$
$$\frac{dy}{dt} = Q(x, y)$$

define a vector field in a region of the plane that represents a particle's velocity at a given point (x, y).

- If $P(x, y)$, $Q(x, y)$, and the first-order partial derivatives $\partial P/\partial x$, $\partial P/\partial y$, $\partial Q/\partial x$, and $\partial Q/\partial y$ are continuous in the region R of the plane, then a solution to the plane autonomous system may be one of three types:

  1. A **constant solution** $x(t) = x_0, y(t) = y_0$ (or $\mathbf{X} = \mathbf{X}_0$ for all t). Also known as a **critical/stationary point** or an **equilibrium solution**.
  2. A solution $x = x(t), y = y(t)$ defines an **arc** — a plane curve that does not cross itself. Crossing itself would imply that the system has two solutions at a single point.
  3. A periodic solution $x = x(t), y = y(t)$. A periodic solution is called a **cycle**. If p is the period of the solution, then $\mathbf{X}(t + p) = \mathbf{X}(t)$.

- Sometimes it's not possible to find explicit expressions for the solutions of nonlinear systems in one coordinate system, but it is in another. If we take the expressions for polar coordinates

$$r^2 = x^2 + y^2$$
$$\theta = \arctan \frac{y}{x}$$

and differentiate them with respect to t

$$\frac{dr}{dt} = \frac{1}{r}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$
$$\frac{d\theta}{dt} = \frac{1}{r^2}\left(-y\frac{dx}{dt} + x\frac{dy}{dt}\right)$$

sometimes substituting a system's expressions for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ into these equations allow them to be solved.

11.2 Stability of Linear Systems

- Considering a plane system as the velocity field of a particle, if a particle is placed at $\mathbf{X}_0$ near a critical point $\mathbf{X}_1$ and the particle eventually moves to $\mathbf{X}_1$ or remains near it (e.g. circles), then $\mathbf{X}_1$ is said to be **locally stable**. If it's not possible to find a neighborhood in which the particle remains near $\mathbf{X}_1$ then it is said to be **unstable**.

- If $\mathbf{A}$ is an $n \times n$ matrix, the **trace** of $\mathbf{A}$ is the sum of the main diagonal entries.

- If $\mathbf{A}$ is the 2×2 coefficient matrix of a homogeneous linear system then its characteristic equation can be written as

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \lambda^2 - \tau\lambda - \Delta = 0$$

  where τ is its trace and Δ is its determinant. Using the quadractic equation we find that

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

  and thus the nature of $\mathbf{A}$'s eigenvalues are determined by the expression

$$\tau^2 - 4\Delta.$$

- The phase portrait of a system of two linear differential equations is determined by the eigenvalues and eigenvectors. There are three cases:

  1. The eigenvalues are real and distinct ($\tau^2 - 4\Delta > 0$) so the general solution has the form $\mathbf{X} = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t}$.
     - If both eigenvalues are negative $\lim_{t\to\infty} \mathbf{X} = \mathbf{0}$ and $\mathbf{X}$ will approach $\mathbf{0}$ from one of two directions determined by $\mathbf{K}_1$ and $\mathbf{K}_2$. In this case a critical point is called a **stable node**.
     - If both eigenvalues are positive $\mathbf{X}$ becomes unbounded as t increases, moving in one of two directions determined by $\mathbf{K}_1$ and $\mathbf{K}_2$. In this case a critical point is called a **unstable node**.
     - If the eigenvalues have opposite signs $\mathbf{X}$ approaches $\mathbf{0}$ from the direction of the eigenvector associated with the negative eigenvalue, then becomes unbounded in the direction of the other eigenvector. In this case a critical point is called a **saddle point**.
  2. There is a single real and repeated eigenvalue ($\tau^2 - 4\Delta = 0$). If $\lambda < 0$ then $\mathbf{X}$ approaches $\mathbf{0}$ from a direction determined by the eigenvector(s) and the critical point is called a **degenerate stable node**. If $\lambda > 0$ then $\mathbf{X}$ moves away from $\mathbf{0}$ in the same direction and the critical point is called a **degenerate unstable node**.
  3. The eigenvalues are complex ($\tau^2 - 4\Delta < 0$).
     - If the eigenvalues are pure imaginary $\mathbf{X}$ moves in an ellipse with centre at the origin. The ellipses are all traversed in the same direction. In this case a critical point is called a **centre**.
     - When there is a nonzero real part the $e^{\alpha t}$ changes the radius over time. If $\alpha < 0$ the ellipse spirals towards $\mathbf{0}$ and is called a **stable spiral point**. If $\alpha > 0$ the ellipse spirals away from $\mathbf{0}$ and is called an **unstable spiral point**.