Advanced Engineering Mathematics Ordinary Differential Equations Notes

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Contents

1 Introduction to Differential Equations

1.1 Definitions and Terminology

1.1.1 1

2, linear

1.1.2 3

4, linear

1.1.3 5

2, nonlinear

1.1.4 7

3, linear

1.1.5 9

no; yes

1.1.6 15

The domain of the function is $x \in [-2, \infty)$.

$$y' = 1 + \frac{2}{\sqrt{x+2}}$$

The largest interval of definition of the solution is $x \in (-2, \infty)$.

$$\begin{aligned}
(y-x)y' &= y - x + 8 \\
(x + 4\sqrt{x+2} - x)(1 + \frac{2}{\sqrt{x+2}}) &= x + 4\sqrt{x+2} - x + 8 \\
4\sqrt{x+2} + 8 &= 4\sqrt{x+2} + 8
\end{aligned}$$

1.1.7 17

The domain of the function is $x \in \mathbb{R}, x \neq \pm 2$.

$$y' = \frac{2x}{(4-x^2)^2}$$

The largest intervals of definition of the solution are $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

$$\begin{aligned}
y' &= 2xy^2 \\
\frac{2x}{(4-x^2)^2} &= 2x\left(\frac{1}{4-x^2}\right)^2 \\
&= \frac{2x}{(4-x^2)^2}
\end{aligned}$$

1.1.8 19

$$\begin{aligned}
ln\frac{2X-1}{X-1} &= t \\
2X - 1 &= (X-1)e^t \\
(2-e^t)X &= 1 - e^t \\
X &= \frac{e^t - 1}{e^t - 2}
\end{aligned}$$

The solutions intervals of validity are $(\infty, \ln 2)$ and $(\ln 2, \infty)$.

$$
\begin{aligned}
\frac{dX}{dt} &= (X-1)(1-2X) \\
\frac{e^t}{e^t-2} - \frac{e^t(e^t-1)}{(e^t-2)^2} &= \left(\frac{e^t-1}{e^t-2} - 1\right)\left(1 - 2\frac{e^t-1}{e^t-2}\right) \\
\frac{e^t(e^t-2) - e^t(e^t-1)}{(e^t-2)^2} &= \left(\frac{e^t-1-e^t+2}{e^t-2}\right)\left(\frac{e^t-2-2e^t+2}{e^t-2}\right) \\
\frac{e^{2t}-2e^t-e^{2t}+e^t}{(e^t-2)^2} &= \left(\frac{1}{e^t-2}\right)\left(\frac{-e^t}{e^t-2}\right) \\
\frac{-e^t}{(e^t-2)^2} &= \frac{-e^t}{(e^t-2)^2}
\end{aligned}
$$

1.1.9 31

$m = -2$

1.1.10 33

$m = 2$ or 3

1.1.11 35

$m = -1$ or 0

1.1.12 37

$y = 2$

1.1.13 39

No constant solutions

1.2 Initial Value Problems

1.2.1 1

$$
\begin{aligned}
y(0) = -\frac{1}{3} &= \frac{1}{1 + c_1 e^{-(0)}} \\
-3 &= 1 + c_1 \\
c_1 &= -4
\end{aligned}
$$

$$
y = \frac{1}{1 - 4e^{-x}}
$$

1.2.2 3

$$
\begin{aligned}
y(2) = \frac{1}{3} &= \frac{1}{(2)^2 + c} \\
3 &= 4 + c \\
c &= -1
\end{aligned}
$$

$$
y = \frac{1}{x^2 - 1}
$$

$I = (1, \infty)$

1.2.3 5

$$
\begin{aligned}
y(0) = 1 &= \frac{1}{(0)^2 + c} \\
c &= 1
\end{aligned}
$$

$$
y = \frac{1}{x^2 + 1}
$$

$I = (-\infty, \infty)$

1.2.4 7

$$
\begin{aligned}
x(0) = -1 &= c_1 \cos 0 + c_2 \sin 0 \\
c_1 &= -1
\end{aligned}
$$

$$
\begin{aligned}
x'(0) = 8 &= -c_1 \sin 0 + c_2 \cos 0 \\
c_2 &= 8
\end{aligned}
$$

$$
x = -\cos t + 8 \sin t
$$

1.2.5 9

$$
\begin{aligned}
x'\left(\frac{\pi}{6}\right) = 0 &= -c_1 \sin \frac{\pi}{6} + c_2 \cos \frac{\pi}{6} \\
&= -c_1 \frac{1}{2} + c_2 \frac{\sqrt{3}}{2} \\
c_1 &= \sqrt{3} c_2
\end{aligned}
$$

$$
\begin{aligned}
x\left(\frac{\pi}{6}\right) = \frac{1}{2} &= c_1 \cos\frac{\pi}{6} + c_2 \sin\frac{\pi}{6} \\
&= \frac{3}{2}c_2 + \frac{1}{2}c_2 \\
&= 2c_2 \\
c_2 &= \frac{1}{4}
\end{aligned}
$$

$$
y = \frac{\sqrt{3}}{4}\cos t + \frac{1}{4}\sin t
$$

1.2.6 11

$$
\begin{aligned}
y(0) = 1 &= c_1 e^{(0)} + c_2 e^{-(0)} \\
&= c_1 + c_2 \\
c_1 &= 1 - c_2
\end{aligned}
$$

$$
\begin{aligned}
y'(0) = 2 &= c_1 e^{(0)} - c_2 e^{-(0)} \\
&= 1 - c_2 - c_2 \\
c_2 &= -\frac{1}{2}
\end{aligned}
$$

$$
y = \frac{3}{2}e^{x} - \frac{1}{2}e^{-x}
$$

1.2.7 13

$$
\begin{aligned}
y(-1) = 5 &= c_1 e^{(-1)} + c_2 e^{-(-1)} \\
&= c_1 e^{-1} + c_2 e \\
c_1 &= 5e - c_2 e^2
\end{aligned}
$$

$$
\begin{aligned}
y'(-1) = -5 &= c_1 e^{(-1)} - c_2 e^{-(-1)} \\
&= 5e - c_2 e^2 - c_2 e \\
c_2 e(e+1) &= 5(e+1) \\
c_2 &= \frac{5}{e}
\end{aligned}
$$

$$
y = 5e^{-x-1}
$$

1.2.8 15

$$y = 0$$

$$y = x^3$$

1.2.9 17

$$f(x, y) = y^{2/3}$$

$$\frac{\partial f}{\partial y} = \frac{2}{3y^{1/3}}$$

$y < 0$ or $y > 0$

1.2.10 19

$$f(x, y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$x < 0$ or $x > 0$

1.2.11 21

$$f(x, y) = \frac{x^2}{4 - y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2x^2 y}{(4 - y^2)^2}$$

$y < -2$, $-2 < y < 2$, or $y > 2$

1.2.12 23

$$f(x, y) = \frac{y^2}{x^2 + y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2} - \frac{2y^3}{(x^2 + y^2)^2}$$

$x \neq 0$ and $y \neq 0$

1.2.13 25

$$f(x, y) = \sqrt{y^2 - 9}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{y^2 - 9}}$$

Yes

1.2.14 27

No

1.2.15 29

(a) $y = cx$

(b)

$$f(x,y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$x \neq 0$

(c) No, the function is not differentiable at $x = 0$

1.2.16 31

(a)

$$y' = \frac{1}{(x+c)^2} = y^2$$

(b)

$$y(0) = 1 = -\frac{1}{(0)+c} \Rightarrow c = -1 \Rightarrow y = \frac{1}{1-x}$$

$I = (-\infty, 1)$

$$y(0) = -1 = -\frac{1}{(0)+c} \Rightarrow c = 1 \Rightarrow y = -\frac{1}{x+1}$$

$I = (-1, \infty)$

1.2.17 39

$$\begin{aligned} y(0) = 0 &= c_1 \cos 3(0) + c_2 \sin 3(0) \\ c_1 &= 0 \end{aligned}$$

$$\begin{gathered} y\left(\frac{\pi}{6}\right) = -1 = c_2 \sin 3\left(\frac{\pi}{6}\right) \\ c_2 = -1 \end{gathered}$$

$$y = -\sin 3x$$

1.2.18 41

$$
\begin{aligned}
y'(0) = 0 &= -3c_1 \sin 3(0) + 3c_2 \cos 3(0) \\
c_2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
y'\left(\frac{\pi}{4}\right) = 0 &= -3c_1 \sin 3\left(\frac{\pi}{4}\right) \\
&= -\frac{3}{\sqrt{2}} c_1 \\
c_1 &= 0
\end{aligned}
$$

$$y = 0$$

1.2.19 43

$$
\begin{aligned}
y(0) = 0 &= c_1 \cos 3(0) + c_2 \sin 3(0) \\
c_1 &= 0
\end{aligned}
$$

$$
\begin{aligned}
y(\pi) = 4 &= c_2 \sin 3(\pi) \\
4 &= 0
\end{aligned}
$$

No solution

1.3 Differential Equations as Mathematical Models

1.3.1 1

$$\frac{dP}{dt} = kP + r$$

$$\frac{dP}{dt} = kP - r$$

1.3.2 3

$$\frac{dP}{dt} = k_b P - k_d P^2$$

1.3.3 7

$$\frac{dx}{dt} = kx(1000 - x)$$

1.3.4 9

$$\frac{dA}{dt} = -\frac{A}{100}$$

$$A(0) = 50\,\mathrm{lb}$$

1.3.5 11

$$\frac{dA}{dt} + \frac{7}{600 - t}A = 6$$

1.3.6 13

$$\begin{aligned}
\frac{dV}{dt} &= -cA_h\sqrt{2gh} \\
A_w\frac{dh}{dt} &= -cA_h\sqrt{2gh} \\
\frac{dh}{dt} &= -\frac{cA_h\sqrt{2g}}{A_w}\sqrt{h} \\
&= -\frac{c\pi r_h^2\sqrt{2g}}{A_w}\sqrt{h} \\
&= -\frac{c\pi}{430}\sqrt{h}
\end{aligned}$$

1.3.7 15

$$L\frac{di}{dt} + Ri = E$$

1.3.8 17

$$m\frac{dv}{dt} = mg - kv^2$$

1.3.9 19

$$m\frac{d^2x}{dt^2} = -kx$$

1.3.10 21

$$\begin{aligned}
\frac{d}{dt}(mv) &= R - kv \\
\frac{dm}{dt}v + m\frac{dv}{dt} &= R - kv - mg
\end{aligned}$$

1.3.11 23

$$g = \frac{k}{R^2} \Rightarrow k = gR^2$$

$$\frac{d^2r}{dt^2} = -\frac{gR^2}{r^2}$$

1.3.12 25

$$\frac{dA}{dt} = k(M - A)$$

1.3.13 27

$$\frac{dx}{dt} = r - kx$$

1.3.14 29

$$\begin{aligned}
\frac{dy}{dx} &= \tan\theta \\
&= \tan\frac{\phi}{2} \\
&= \frac{1-\cos\phi}{\sin\phi} \\
&= \frac{1-x/r}{y/r} \\
&= \frac{r-x}{y} \\
&= \frac{\sqrt{x^2+y^2}-x}{y}
\end{aligned}$$