Advanced Engineering Mathematics Partial Differential Equations by Dennis G. Zill Problems

Chris Doble

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Contents

12 Orthogonal Functions and Fourier Series

12.1 Orthogonal Functions

12.1.7

$$
\begin{aligned}
\int_0^{\pi/2} \sin mx \sin nx \, dx &= \frac{1}{2} \int_0^{\pi/2} [\cos(m-n)x - \cos(m+n)x] \, dx \\
&= \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_0^{\pi/2} \\
&= \frac{1}{2} \left(\frac{\sin(m-n)\pi/2}{m-n} - \frac{\sin(m+n)\pi/2}{m+n} \right) \\
&= 0 \\
||\sin nx||^2 &= (\sin nx, \sin nx) \\
&= \int_0^{\pi/2} \sin^2 nx \, dx \\
&= \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2nx) \, dx \\
&= \frac{1}{2} \left[x - \frac{1}{2n} \sin 2nx \right]_0^{\pi/2} \\
&= \frac{\pi}{4} \\
||\sin nx|| &= \frac{\sqrt{\pi}}{2}
\end{aligned}
$$

12.1.9

$$\begin{aligned}
\int_0^\pi \sin mx \sin nx \, dx &= \frac{1}{2}\int_0^\pi [\cos(m-n)x - \cos(m+n)x] \, dx \\
&= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_0^\pi \\
&= 0 \\
||\sin nx||^2 &= (\sin nx, \sin nx) \\
&= \int_0^\pi \sin^2 nx \, dx \\
&= \frac{1}{2}\int_0^\pi (1 - \cos 2nx) \, dx \\
&= \frac{1}{2}\left[x - \frac{1}{2n}\sin 2nx\right]_0^\pi \\
&= \frac{\pi}{2} \\
||\sin nx|| &= \sqrt{\frac{\pi}{2}}
\end{aligned}$$

12.1.21

$$T = 1$$

12.1.23

$$T = 2\pi$$

12.1.25

$$T = 2\pi$$

12.2 Fourier Series

12.2.1

$$
\begin{aligned}
p &= \pi \\
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} dx \\
&= 1 \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \cos nx\,dx \\
&= \frac{1}{n\pi}[\sin nx]_0^{\pi} \\
&= 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin nx\,dx \\
&= -\frac{1}{n\pi}[\cos nx]_0^{\pi} \\
&= -\frac{1}{n\pi}[(-1)^n - 1] \\
&= \frac{1-(-1)^n}{n\pi} \\
f(x) &= \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx
\end{aligned}
$$

The series converges to $\frac{1}{2}$ at the point of discontinuity.

12.2.3

$$
\begin{aligned}
p &= 1 \\
a_0 &= \frac{3}{2} \\
a_n &= \int_{-1}^{0} \cos n\pi x \, dx + \int_{0}^{1} x \cos n\pi x \, dx \\
&= \frac{1}{n\pi}[\sin n\pi x]_{-1}^{0} + \frac{1}{n\pi}\left[\frac{\cos n\pi x}{n\pi} + x \sin n\pi x\right]_0^1 \\
&= \frac{(-1)^n - 1}{n^2\pi^2} \\
b_n &= \int_{-1}^{0} \sin n\pi x \, dx + \int_{0}^{1} x \sin n\pi x \, dx \\
&= -\frac{1}{n\pi}[\cos n\pi x]_{-1}^{0} + \frac{1}{n\pi}\left[\frac{\sin n\pi x}{n\pi} - x \cos n\pi x\right]_0^1 \\
&= -\frac{1}{n\pi} \\
f(x) &= \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x - \frac{1}{n\pi}\sin n\pi x\right]
\end{aligned}
$$

The series converges to $\frac{1}{2}$ at the point of discontinuity.

12.3 Fourier Cosine and Sine Series

12.3.1

Odd

12.3.3

Neither

12.3.5

Even

12.3.7

Odd

12.3.9

Neither

12.3.11

$$\begin{aligned} b_n &= -2\pi \int_0^1 \sin n\pi x \, dx \\ &= \frac{2}{n} [\cos n\pi x]_0^1 \\ &= \frac{2}{n} [(-1)^n - 1] \\ f &= \sum_{n=1}^{\infty} \frac{2}{n} [(-1)^n - 1] \sin n\pi x \end{aligned}$$

12.3.13

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} x \, dx \\ &= \pi \\ a_n &= 2 \int_0^{\pi} x \cos nx \, dx \\ &= \frac{2}{n} \left[\frac{\cos nx}{n} + x \sin nx \right]_0^{\pi} \\ &= \frac{2[(-1)^n - 1]}{n^2} \\ f &= \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx \end{aligned}$$

12.3.25

$$
\begin{aligned}
a_0 &= 2\int_0^1 f(x)\,dx \\
&= 1 \\
a_n &= 2\int_0^1 f(x)\cos n\pi x\,dx \\
&= 2\int_0^{1/2} \cos n\pi x\,dx \\
&= \frac{2}{n\pi}[\sin n\pi x]_0^{1/2} \\
&= \frac{2}{n\pi}\sin\frac{n\pi}{2} \\
f &= \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n}\cos n\pi x
\end{aligned}
$$

$$
\begin{aligned}
b_n &= 2\int_0^1 f(x)\sin n\pi x\,dx \\
&= 2\int_0^{1/2} \sin n\pi x\,dx \\
&= -\frac{2}{n\pi}[\cos n\pi x]_0^{1/2} \\
&= \frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right) \\
f &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos\frac{n\pi}{2}}{n}\sin n\pi x
\end{aligned}
$$

12.3.27

$$
\begin{aligned}
a_0 &= \frac{4}{\pi}\int_0^{\pi/2} \cos x\,dx \\
&= \frac{4}{\pi}[\sin x]_0^{\pi/2} \\
&= \frac{4}{\pi} \\
a_n &= \frac{4}{\pi}\int_0^{\pi/2} \cos x \cos 2nx\,dx \\
&= \frac{2}{\pi}\int_0^{\pi/2} [\cos(1-2n)x + \cos(1+2n)x]\,dx \\
&= \frac{2}{\pi}\left[\frac{\sin(1-2n)x}{1-2n} + \frac{\sin(1+2n)x}{1+2n}\right]_0^{\pi/2} \\
&= \frac{2(-1)^n}{\pi}\left[\frac{1}{1-2n} + \frac{1}{1+2n}\right] \\
&= \frac{2(-1)^n}{\pi}\frac{1+2n+1-2n}{(1-2n)(1+2n)} \\
&= \frac{4(-1)^n}{\pi(1-2n)(1+2n)} \\
f &= \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(1-2n)(1+2n)}\cos 2nx
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{4}{\pi}\int_0^{\pi/2} \cos x \sin 2nx\,dx \\
&= \frac{2}{\pi}\int_0^{\pi/2} [\sin(2n+1)x + \sin(2n-1)x]\,dx \\
&= -\frac{2}{\pi}\left[\frac{\cos(2n+1)x}{2n+1} + \frac{\cos(2n-1)x}{2n-1}\right]_0^{\pi/2} \\
&= \frac{2}{\pi}\left(\frac{1}{2n+1} + \frac{1}{2n-1}\right) \\
&= \frac{2}{\pi}\frac{4n}{4n^2-1} \\
&= \frac{8n}{\pi(4n^2-1)} \\
f &= \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^1-1}\sin 2nx
\end{aligned}
$$

12.3.35

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} x^2\,dx \\
&= \frac{8}{3}\pi^2 \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx \\
&= \frac{4}{n^2} \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\,dx \\
&= -\frac{4\pi}{n} \\
f &= \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)
\end{aligned}$$

12.3.43

$$\begin{aligned}
b_n &= \frac{10}{\pi}\int_0^{\pi}\sin nt\,dt \\
&= -\frac{10}{n\pi}[\cos nt]_0^{\pi} \\
&= \frac{10}{n\pi}[1-(-1)^n] \\
f &= \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[1-(-1)^n]\sin nt \\
x_p(t) &= \sum_{n=1}^{\infty} B_n \sin nt \\
m\frac{d^2x}{dt^2} + kx &= f(t)
\end{aligned}$$

$$\begin{aligned}
-mn^2B_n + kB_n &= \frac{10}{n\pi}[1-(-1)^n] \\
B_n &= \frac{10}{n\pi(k-mn^2)}[1-(-1)^n] \\
x_p(t) &= \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n(k-mn^2)}\sin nt \\
&= \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n(10-n^2)}\sin nt
\end{aligned}$$

12.3.45

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\pi} (2\pi t - t^2)\,dt \\
&= \frac{2}{\pi}\left[\pi t^2 - \frac{1}{3}t^3\right]_0^{\pi} \\
&= \frac{2}{\pi}\left(\pi^3 - \frac{1}{3}\pi^3\right) \\
&= \frac{4}{3}\pi^2 \\
a_n &= \frac{2}{\pi}\int_0^{\pi} (2\pi t - t^2)\cos nt\,dt \\
&= -\frac{4}{n^2} \\
f(t) &= \frac{2}{3}\pi^2 - \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nt \\
x_p(t) &= c + \sum_{n=1}^{\infty} B_n \cos nt \\
\frac{1}{4}\frac{d^2x}{dt^2} + 12x &= f(t)
\end{aligned}$$

$$\begin{aligned}
-\frac{n^2}{4}\sum_{n=1}^{\infty} B_n\cos nt + 12c + 12\sum_{n=1}^{\infty} B_n\cos nt &= \frac{2}{3}\pi^2 - \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nt \\
12c &= \frac{2}{3}\pi^2 \\
c &= \frac{\pi^2}{18} \\
-\frac{n^2}{4}B_n + 12B_n &= -\frac{4}{n^2} \\
\frac{48 - n^2}{4}B_n &= -\frac{4}{n^2} \\
B_n &= -\frac{16}{n^2(48 - n^2)} \\
x_p(t) &= \frac{\pi^2}{18} - \sum_{n=1}^{\infty}\frac{16}{n^2(48 - n^2)}\cos nt \\
&= \frac{\pi^2}{18} + 16\sum_{n=1}^{\infty}\frac{1}{n^2(n^2 - 48)}\cos nt
\end{aligned}$$

12.4 Complex Fourier Series

12.4.1

$$
\begin{aligned}
T &= 4 \\
p &= 2 \\
c_n &= \frac{1}{4}\left(\int_0^2 e^{-in\pi x/2}\,dx - \int_{-2}^0 e^{-in\pi x/2}\,dx\right) \\
&= \frac{1}{2in\pi}([e^{-in\pi x/2}]_{-2}^0 - [e^{-in\pi x/2}]_0^2) \\
&= \frac{2 - e^{in\pi} - e^{-in\pi}}{2in\pi} \\
&= \frac{2 - \cos n\pi - i\sin n\pi - \cos n\pi + i\sin n\pi}{2in\pi} \\
&= \frac{1 - \cos n\pi}{in\pi} \\
&= \frac{1 - (-1)^n}{in\pi} \\
f(x) &= \sum_{n=-\infty,\ n\neq 0}^{\infty} \frac{1 - (-1)^n}{in\pi} e^{in\pi x/2}
\end{aligned}
$$

12.4.3

$$
\begin{aligned}
T &= 1 \\
p &= \frac{1}{2} \\
c_n &= \int_0^{1/4} e^{-2in\pi x}\,dx \\
&= -\frac{1}{2in\pi}[e^{-2in\pi x}]_0^{1/4} \\
&= \frac{1}{2in\pi}(1 - e^{-in\pi/2}) \\
c_0 &= \frac{1}{4} \\
f(x) &= \frac{1}{4} + \sum_{n=-\infty,\ n\neq 0}^{\infty} \frac{1 - e^{-in\pi/2}}{2in\pi} e^{2in\pi x}
\end{aligned}
$$

12.4.5

$$\begin{aligned}
T &= 2\pi \\
p &= \pi \\
c_n &= \frac{1}{2\pi}\int_0^{2\pi} xe^{-inx}\,dx \\
&= \frac{i}{n} \\
c_0 &= \frac{1}{2\pi}\int_0^{2\pi} x\,dx \\
&= \pi \\
f(x) &= \pi + \sum_{n=-\infty,\ n\neq 0}^{n=\infty} \frac{i}{n}e^{inx}
\end{aligned}$$

12.5 Sturm-Liouville Problem

12.5.1

$$
\begin{aligned}
y'' + \lambda y &= 0 \\
y'(0) &= 0 \\
y(1) + y'(1) &= 0 \\
\lambda &= \alpha^2 \\
y &= c_1 \cos \alpha x + c_2 \sin \alpha x \\
y' &= -\alpha c_1 \sin \alpha x + \alpha c_2 \cos \alpha x \\
c_2 &= 0 \\
y &= c_1 \cos \alpha x
\end{aligned}
$$

$$
\begin{aligned}
c_1 \cos \alpha - \alpha c_1 \sin \alpha &= 0 \\
c_1 \cos \alpha &= \alpha c_1 \sin \alpha \\
\alpha \tan \alpha &= 0 \\
\alpha &= \cot \alpha \\
\lambda_1 &= 0.740174 \\
y_1 &= \cos 0.860334x \\
\lambda_2 &= 11.734872 \\
y_2 &= \cos 3.42562x \\
\lambda_3 &= 41.438831 \\
y_3 &= \cos 6.4373x \\
\lambda_4 &= 90.808130 \\
y_4 &= \cos 9.52933x
\end{aligned}
$$

12.5.5

$$
\begin{aligned}
(y_n, y_n) &= \int_0^1 \cos^2 \alpha_n x \, dx \\
&= \frac{1}{2} \int_0^1 (1 + \cos 2\alpha_n x) \, dx \\
&= \frac{1}{2} \left[x + \frac{1}{2\alpha_n} \sin 2\alpha_n x \right]_0^1 \\
&= \frac{1}{2} \left(1 + \frac{1}{2\alpha_n} \sin 2\alpha_n \right) \\
&= \frac{1}{2} \left(1 + \frac{1}{\alpha_n} \sin \alpha_n \cos \alpha_n \right) \\
&= \frac{1}{2} \left(1 + \tan \alpha_n \sin \alpha_n \cos \alpha_n \right) \\
&= \frac{1}{2} (1 + \sin^2 \alpha_n)
\end{aligned}
$$

12.5.7

(a)

$$\begin{aligned} x^2y'' + xy' + \lambda y &= 0 \\ y(1) &= 0 \\ y(5) &= 0 \end{aligned}$$

$$\begin{aligned} \lambda &= \alpha^2 \\ y &= x^m \\ y' &= mx^{m-1} \\ y'' &= m(m-1)x^{m-2} \\ x^2m(m-1)x^{m-2} + xmx^{m-1} + \alpha^2x^m &= 0 \\ m(m-1) + m + \alpha^2 &= 0 \\ m^2 + \alpha^2 &= 0 \\ m &= \pm i\alpha \\ y &= c_1\cos(\alpha \ln x) + c_2 \sin(\alpha \ln x) \end{aligned}$$

$$\begin{aligned} 0 &= c_1 \\ 0 &= c_2 \sin(\alpha \ln 5) \\ \alpha &= \frac{n\pi}{\ln 5} \\ \lambda &= \left(\frac{n\pi}{\ln 5}\right)^2 \end{aligned}$$

$$y_n = \sin\left(\frac{n\pi}{\ln 5}\ln x\right)$$

(b)

$$\begin{aligned} x^2y'' + xy' + \lambda y &= 0 \\ y'' + \frac{1}{x}y' + \lambda\frac{1}{x^2}y &= 0 \\ e^{\ln x}y'' + \frac{1}{x}e^{\ln x}y' + \lambda e^{\ln x}\frac{1}{x^2}y &= 0 \\ \frac{d}{dx}(e^{\ln x}y') + \lambda e^{\ln x}\frac{1}{x^2}y &= 0 \\ \frac{d}{dx}(xy') + \lambda\frac{1}{x}y &= 0 \end{aligned}$$

(c)

$$\int_1^5 \frac{1}{x}\sin\left(\frac{m\pi}{\ln 5}\ln x\right)\sin\left(\frac{n\pi}{\ln 5}\ln x\right) = 0,\ m \neq n$$

12.5.9

$$\begin{aligned}
xy'' + (1-x)y' + ny &= 0 \\
y'' + \left(\frac{1}{x} - 1\right) y' + n\frac{1}{x} y &= 0 \\
e^{\int \left(\frac{1}{x}-1\right) dx} &= e^{\ln(x)-x} \\
&= xe^{-x} \\
xe^{-x}y'' + \left(\frac{1}{x} - 1\right) xe^{-x}y' + n\frac{1}{x}xe^{-x}y &= 0 \\
\frac{d}{dx}(xe^{-x}y') + ne^{-x}y &= 0 \\
\int_0^\infty e^{-x} L_m(x) L_n(x)\, dx &= 0,\ m \neq n
\end{aligned}$$

12.6 Bessel and Legendre Series

12.6.1

$$\begin{aligned}
J_1(3\alpha) &= 0 \\
\alpha_1 &= 1.277 \\
\alpha_2 &= 2.338 \\
\alpha_3 &= 3.391 \\
\alpha_4 &= 4.441
\end{aligned}$$

12.6.3

$$\begin{aligned}
J_0(2\alpha) &= 0 \\
c_i &= \frac{1}{2J_1^2(2\alpha_i)} \int_0^2 xJ_0(\alpha_i x)\, dx \\
&= \frac{1}{2J_1^2(2\alpha_i)} \int_0^2 \frac{d}{dx}\left[\frac{1}{\alpha_i} xJ_1(\alpha_i x)\right] dx \\
&= \frac{1}{\alpha_i J_1(2\alpha_i)} \\
f(x) &= \sum_{i=1}^{\infty} \frac{J_0(\alpha_i x)}{\alpha_i J_1(2\alpha_i)}
\end{aligned}$$

12.6.5

$$\begin{aligned}
J_0(2\alpha) + 2\alpha J_0'(2\alpha) &= 0 \\
h &= 1 \\
b &= 2 \\
c_i &= \frac{2\alpha_i^2}{(4\alpha_i^2+1)J_0^2(2\alpha_i)} \int_0^2 xJ_0(\alpha_i x)\,dx \\
&= \frac{2\alpha_i^2}{(4\alpha_i^2+1)J_0^2(2\alpha_i)} \int_0^2 \frac{d}{dx}\left[\frac{1}{\alpha_i}xJ_1(\alpha_i x)\right]\,dx \\
&= \frac{4\alpha_i J_1(2\alpha_i)}{(4\alpha_i^2+1)J_0^2(2\alpha_i)} \\
f(x) &= 4\sum_{i=1}^{\infty} \frac{\alpha_i J_1(2\alpha_i)}{(4\alpha_i^2+1)J_0^2(2\alpha_i)} J_0(\alpha_i x)
\end{aligned}$$

12.6.7

$$\begin{aligned}
f(x) &= 5x,\ 0 < x < 4 \\
4J_1(4\alpha) + 4\alpha J_1'(4\alpha) &= 0 \\
h &= 3 \\
n &= 1 \\
b &= 4 \\
c_i &= \frac{2\alpha_i^2}{(16\alpha_i^2+8)J_1^2(4\alpha_i)} \int_0^4 5x^2 J_1(\alpha_i x)\,dx \\
&= \frac{10\alpha_i^2}{(16\alpha_i^2+8)J_1^2(4\alpha_i)} \int_0^4 \frac{d}{dx}\left[\frac{1}{\alpha_i}x^2 J_2(\alpha_i x)\right]\,dx \\
&= \frac{160\alpha_i J_2(4\alpha_i)}{(16\alpha_i^2+8)J_1^2(4\alpha_i)} \\
f(x) &= \sum_{i=1}^{\infty} \frac{20\alpha_i J_2(4\alpha_i)}{(2\alpha_i^2+1)J_1^2(4\alpha_i)} J_1(\alpha_i x)
\end{aligned}$$

12.6.15

$$
\begin{aligned}
f(x) &= \begin{cases} 0 & -1 < x < 0 \\ x & 0 < x < 1 \end{cases} \\
c_n &= \frac{2n+1}{2} \int_0^1 x P_n(x)\, dx \\
c_0 &= \frac{1}{4} \\
c_1 &= \frac{1}{2} \\
c_2 &= \frac{5}{16} \\
c_3 &= 0 \\
c_4 &= -\frac{3}{32} \\
c_5 &= 0 \\
c_6 &= \frac{13}{256}
\end{aligned}
$$

12.6.21

$$
\begin{aligned}
c_0 &= \frac{1}{2} \\
c_1 &= \frac{5}{8} \\
c_2 &= -\frac{3}{16} \\
c_3 &= \frac{13}{128}
\end{aligned}
$$

12.7 Chapter in Review

12.7.1

$$
\begin{aligned}
(x^2 - 1, x^5) &= \int_{-\pi}^{\pi} (x^2 - 1) x^5 \, dx \\
&= \int_{-\pi}^{\pi} (x^7 - x^5)\, dx \\
&= \left[\frac{1}{8} x^8 - \frac{1}{6} x^6 \right]_{-\pi}^{\pi} \\
&= \frac{1}{8}\pi^8 - \frac{1}{6}\pi^6 - \frac{1}{8}\pi^8 + \frac{1}{6}\pi^6 \\
&= 0
\end{aligned}
$$

True

12.7.3

Fourier cosine

12.7.5

False

12.7.7

5.5, 1, 0

12.7.9

True

12.7.13

$$
\begin{aligned}
f(x) &= |x| - x,\ \ -1 < x < 1 \\
L &= 2 \\
p &= 1 \\
a_0 &= \int_{-1}^{1} (|x| - x)\, dx \\
&= \int_{-1}^{0} -2x\, dx \\
&= -[x^2]_{-1}^{0} \\
&= -(0 - 1) \\
&= 1 \\
a_n &= \int_{-1}^{1} (|x| - x) \cos n\pi x\, dx \\
&= -2 \int_{-1}^{0} x \cos n\pi x\, dx \\
&= \frac{2[(-1)^n - 1]}{n^2\pi^2} \\
b_n &= \int_{-1}^{1} (|x| - x) \sin n\pi x\, dx \\
&= -2 \int_{-1}^{0} x \sin n\pi x\, dx \\
&= \frac{2(-1)^n}{n\pi} \\
f(x) &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{n^2\pi^2} \cos n\pi x + \frac{2(-1)^n}{n\pi} \sin n\pi x
\end{aligned}
$$

12.7.17

$$\begin{aligned}
x^2y'' + xy' + 9\lambda y &= 0 \\
y'(1) &= 0 \\
y(e) &= 0
\end{aligned}$$

$$\begin{aligned}
y &= x^m \\
y' &= mx^{m-1} \\
y'' &= m(m-1)x^{m-2} \\
x^2m(m-1)x^{m-2} + xmx^{m-1} + 9\lambda x^m &= 0 \\
m(m-1) + m + 9\lambda &= 0 \\
m^2 + 9\lambda &= 0 \\
m &= \pm 3\sqrt{\lambda} i \\
y &= c_1 \cos(3\sqrt{\lambda}\ln x) + c_2 \sin(3\sqrt{\lambda}\ln x) \\
y' &= \frac{3\sqrt{\lambda}}{x}[c_2 \cos(3\sqrt{\lambda}\ln x) - c_1 \sin(3\sqrt{\lambda}\ln x)] \\
y'(1) &= 0 \\
0 &= 3\sqrt{\lambda} c_2 \\
c_2 &= 0 \\
y(e) &= 0 \\
0 &= c_1 \cos 3\sqrt{\lambda} \\
&= \cos 3\sqrt{\lambda} \\
3\sqrt{\lambda} &= \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z} \\
&= \frac{2n+1}{2}\pi \\
\lambda_n &= \left(\frac{2n+1}{6}\pi\right)^2 \\
y_n &= \cos\left(\frac{2n+1}{2}\pi \ln x\right)
\end{aligned}$$

13 Boundary-Value Problems in Rectangular Coordinates

13.1 Separable Partial Differential Equations

13.1.1

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial y} \\
u &= X(x)Y(y) \\
X'Y &= XY' \\
\frac{X'}{X} &= \frac{Y'}{Y}
\end{aligned}$$

$$\begin{aligned}
\frac{X'}{X} &= \lambda \\
\ln X &= \lambda x + c_1 \\
X &= c_1 e^{\lambda x}
\end{aligned}$$

$$\begin{aligned}
\frac{Y'}{Y} &= \lambda \\
\ln Y &= \lambda y + c_2 \\
Y &= c_2 e^{\lambda y} \\
u &= XY \\
&= c_1 c_2 e^{\lambda(x+y)} \\
&= c_3 e^{\lambda(x+y)}
\end{aligned}$$

13.1.3

$$\begin{aligned}
u_x + u_y &= u \\
X'Y + XY' &= XY \\
\frac{X'}{X}Y + Y' &= Y \\
\frac{X'}{X} + \frac{Y'}{Y} &= 1 \\
\frac{X'}{X} &= 1 - \frac{Y'}{Y}
\end{aligned}$$

$$\begin{aligned}
\frac{X'}{X} &= \lambda \\
X &= c_1 e^{\lambda x}
\end{aligned}$$

$$\begin{aligned}
1 - \frac{Y'}{Y} &= \lambda \\
Y' + (\lambda - 1)Y &= 0 \\
Y &= c_2 e^{-(\lambda-1)y}
\end{aligned}$$

$$u = c_3 e^{\lambda x - (\lambda - 1)y}$$

13.1.5

$$\begin{aligned}
x\frac{\partial u}{\partial x} &= y\frac{\partial u}{\partial y} \\
xX'Y &= yXY' \\
x\frac{X'}{X} &= y\frac{Y'}{Y}
\end{aligned}$$

$$\begin{aligned}
x\frac{X'}{X} &= \lambda \\
\frac{X'}{X} &= \frac{\lambda}{x} \\
\ln X &= \lambda \ln x + c_1 \\
X &= c_1 x^\lambda
\end{aligned}$$

$$Y = c_2 y^\lambda$$

$$\begin{aligned}
u &= XY \\
&= c_3 (xy)^\lambda
\end{aligned}$$

13.1.7

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} &= 0 \\
X''Y + X'Y' + XY'' &= 0
\end{aligned}$$

Not separable.

13.1.9

$$\begin{aligned}
k\frac{\partial^2 u}{\partial x^2} - u &= \frac{\partial u}{\partial t},\ k > 0 \\
kTX'' - TX &= T'X \\
k\frac{X''}{X} - 1 &= \frac{T'}{T}
\end{aligned}$$

$$\begin{aligned}
\frac{T'}{T} &= \lambda \\
T' - \lambda T &= 0 \\
T &= c_1 e^{\lambda t}
\end{aligned}$$

$$\begin{aligned}
k\frac{X''}{X} - 1 &= \lambda \\
X'' - \frac{\lambda+1}{k}X &= 0 \\
X &= \begin{cases} c_1 \cos\sqrt{\frac{\lambda+1}{k}}x + c_2 \sin\sqrt{\frac{\lambda+1}{k}}x & \lambda < -1 \\ c_1 x + c_2 & \lambda = -1 \\ c_1 \cosh\sqrt{\frac{\lambda+1}{k}}x + c_2 \sinh\sqrt{\frac{\lambda+1}{k}}x & \lambda > -1 \end{cases}
\end{aligned}$$

$$\begin{aligned}
u &= TX \\
&= \begin{cases} e^{\lambda t}\left(c_1 \cos\sqrt{\frac{\lambda+1}{k}}x + c_2 \sin\sqrt{\frac{\lambda+1}{k}}x\right) & \lambda < -1 \\ e^{\lambda t}\left(c_1 x + c_2\right) & \lambda = -1 \\ e^{\lambda t}\left(c_1 \cosh\sqrt{\frac{\lambda+1}{k}}x + c_2 \sinh\sqrt{\frac{\lambda+1}{k}}x\right) & \lambda > -1 \end{cases}
\end{aligned}$$

13.1.11

$$
\begin{aligned}
a^2\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t^2} \\
a^2TX'' &= T''X \\
a^2\frac{X''}{X} &= \frac{T''}{T}
\end{aligned}
$$

$$
\begin{aligned}
\frac{T''}{T} &= \lambda \\
T'' - \lambda T &= 0 \\
T &= \begin{cases} c_1\cos\sqrt{\lambda}t + c_2\sin\sqrt{\lambda}t & \lambda < 0 \\ c_1t + c_2 & \lambda = 0 \\ c_1\cosh\sqrt{\lambda}t + c_2\sinh\sqrt{\lambda}t & \lambda > 0 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
a^2\frac{X''}{X} &= \lambda \\
X'' - \frac{\lambda}{a^2}X &= 0 \\
X &= \begin{cases} c_1\cos\frac{\sqrt{\lambda}}{a}x + c_2\sin\frac{\sqrt{\lambda}}{a}x & \lambda < 0 \\ c_1x + c_2 & \lambda = 0 \\ c_1\cosh\frac{\sqrt{\lambda}}{a}x + c_2\sinh\frac{\sqrt{\lambda}}{a}x & \lambda > 0 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
u &= TX \\
&= \begin{cases} (c_1\cos\sqrt{\lambda}t + c_2\sin\sqrt{\lambda}t)(c_3\cos\frac{\sqrt{\lambda}}{a}x + c_4\sin\frac{\sqrt{\lambda}}{a}x) & \lambda < 0 \\ (c_1t + c_2)(c_3x + c_4) & \lambda = 0 \\ (c_1\cosh\sqrt{\lambda}t + c_2\sinh\sqrt{\lambda}t)(c_3\cosh\frac{\sqrt{\lambda}}{a}x + c_4\sinh\frac{\sqrt{\lambda}}{a}x) & \lambda > 0 \end{cases}
\end{aligned}
$$

13.1.17

Elliptic

13.1.19

Parabolic

13.1.21

Hyperbolic

13.1.23

Parabolic

13.1.25

Hyperbolic

13.2 Classical PDEs and Boundary-Value Problems

13.2.1

$$
\begin{aligned}
k^2 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t} \\
u(0,t) &= 0 \\
\left.\frac{\partial u}{\partial x}\right|_{x=L} &= 0 \\
u(x,0) &= f(x)
\end{aligned}
$$

13.2.3

$$
\begin{aligned}
k^2 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t} \\
u(0,t) &= 100 \\
\left.\frac{\partial u}{\partial x}\right|_{x=L} &= -hu(L,t) \\
u(x,0) &= f(x)
\end{aligned}
$$

13.2.5

$$
\begin{aligned}
k^2 \frac{\partial^2 u}{\partial x^2} - hu &= \frac{\partial u}{\partial t} \\
u(0,t) &= \sin \frac{\pi}{L} t \\
u(L,t) &= 0 \\
u(x,0) &= f(x)
\end{aligned}
$$

13.2.7

$$
\begin{aligned}
a^2\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t^2}\\
u(0,t) &= 0\\
u(L,t) &= 0\\
u(x,0) &= x(L-x)\\
\left.\frac{\partial u}{\partial t}\right|_{t=0} &= 0
\end{aligned}
$$

13.2.9

$$
\begin{aligned}
a^2\frac{\partial^2 u}{\partial x^2} - c\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial t^2}\\
u(0,t) &= 0\\
u(L,t) &= \sin \pi t\\
u(x,0) &= f(x)\\
\left.\frac{\partial u}{\partial t}\right|_{t=0} &= 0
\end{aligned}
$$

13.2.11

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0\\
\left.\frac{\partial u}{\partial x}\right|_{x=0} &= 0\\
\left.\frac{\partial u}{\partial y}\right|_{y=0} &= 0\\
u(x,2) &= 0\\
u(4,y) &= f(y)
\end{aligned}
$$

13.3 Heat Equation

13.3.1

$$
\begin{aligned}
k\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t} \\
u(0,t) &= 0 \\
u(L,t) &= 0 \\
u(x,0) &= \begin{cases} 1 & 0 < x < L/2 \\ 0 & L/2 < x < L \end{cases} \\
A_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx \\
&= \frac{2}{L}\int_0^{L/2} \sin\frac{n\pi}{L}x\,dx \\
&= -\frac{2}{n\pi}\left[\cos\frac{n\pi}{L}x\right]_0^{L/2} \\
&= \frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right) \\
u(x,t) &= \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)e^{-k(n^2\pi^2/L^2)t}\sin\frac{n\pi}{L}x
\end{aligned}
$$

13.3.3

$$\begin{aligned}
k\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t} \\
u(x,0) &= f(x) \\
\left.\frac{\partial u}{\partial x}\right|_{x=0} &= 0 \\
\left.\frac{\partial u}{\partial x}\right|_{x=L} &= 0
\end{aligned}$$

$$\begin{aligned}
X(x) &= c_1 \cos \alpha x + c_2 \sin \alpha x \\
X'(x) &= -\alpha c_1 \sin \alpha x + \alpha c_2 \cos \alpha x \\
0 &= X'(0) \\
&= \alpha c_2 \\
c_2 &= 0 \\
0 &= X'(L) \\
&= -\alpha c_1 \sin \alpha L \\
\alpha L &= n\pi \\
\alpha &= \frac{n\pi}{L} \\
X(x) &= c_1 \cos \frac{n\pi}{L} x
\end{aligned}$$

$$T(t) = c_3 e^{-k(n^2\pi^2/L^2)t}$$

$$\begin{aligned}
u_n &= A_n e^{-k(n^2\pi^2/L^2)t} \cos \frac{n\pi}{L} x \\
A_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x \, dx \\
u_n &= \frac{1}{L} \int_0^L f(x) \, dx \\
&\qquad + \frac{2}{L} \sum_{n=1}^{\infty} \left(\int_0^L f(x) \cos \frac{n\pi}{L} x \, dx \right) e^{-k(n^2\pi^2/L^2)t} \cos \frac{n\pi}{L} x
\end{aligned}$$

13.3.5

$$
\begin{aligned}
k\frac{\partial^2 u}{\partial x^2} - hu &= \frac{\partial u}{\partial t} \\
u(x,0) &= f(x) \\
\left.\frac{\partial u}{\partial x}\right|_{x=0} &= 0 \\
\left.\frac{\partial u}{\partial x}\right|_{x=L} &= 0 \\
kX''T - hXT &= XT' \\
k\frac{X''}{X} - h &= \frac{T'}{T} \\
k\frac{X''}{X} - h &= -\lambda \\
X'' + \frac{\lambda - h}{k}X &= 0 \\
X &= c_1 \cos \omega x + c_2 \sin \omega x \\
X' &= -\omega c_1 \sin \omega x + \omega c_2 \cos \omega x \\
0 &= X'(0) \\
&= \omega c_2 \\
c_2 &= 0 \\
0 &= X'(L) \\
&= -\omega c_2 \sin \omega L \\
\omega L &= n\pi \\
\omega &= \frac{n\pi}{L} \\
X_n &= c_1 \cos \frac{n\pi}{L} x \\
T_n &= c_3 e^{-\lambda t} \\
&= c_3 e^{-(h + kn^2\pi^2/L^2)t} \\
u_n &= X_n T_n \\
&= A_n e^{-(h + kn^2\pi^2/L^2)t} \cos \frac{n\pi}{L} x \\
&= e^{-ht} A_n e^{-(kn^2\pi^2/L^2)t} \cos \frac{n\pi}{L} x \\
A_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x \, dx \\
u &= e^{-ht} \left[\frac{1}{L} \int_0^L f(x)\, dx + \frac{2}{L} \sum_{n=1}^{\infty} \left(\int_0^L f(x) \cos \frac{n\pi}{L} x \, dx \right) e^{-(kn^2\pi^2/L^2)t} \cos \frac{n\pi}{L} x \right]
\end{aligned}
$$

13.4 Wave Equation

13.4.1

$$
\begin{aligned}
A_n &= \frac{2}{L}\int_0^L \frac{1}{4}x(L-x)\sin\frac{n\pi}{L}x\,dx \\
&= -\frac{[-1+(-1)^n]L^2}{n^3\pi^3} \\
u(x,t) &= \sum_{n=1}^{\infty} -\frac{[-1+(-1)^n]L^2}{n^3\pi^3}\cos\frac{n\pi a}{L}t\sin\frac{n\pi}{L}x
\end{aligned}
$$

13.4.3

$$
\begin{aligned}
a^2\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t^2} \\
u(0,t) &= 0 \\
u(\pi,t) &= 0 \\
u(x,0) &= 0 \\
\left.\frac{\partial u}{\partial t}\right|_{t=0} &= \sin x \\
X(x) &= c_1\cos\alpha x + c_2\sin\alpha x \\
0 &= X(0) \\
&= c_1 \\
0 &= X(\pi) \\
&= c_2\sin\alpha\pi \\
\alpha &= n \\
X(x) &= c_2\sin nx \\
T(t) &= c_3\cos ant + c_4\sin ant \\
u_n &= (A_n\cos ant + B_n\sin ant)\sin nx \\
A_n &= 0 \\
u_n &= B_n\sin ant\sin nx \\
\sin x &= a\sum_{n=1}^{\infty} nB_n\sin nx \\
B_1 &= \frac{1}{a} \\
B_n &= 0 \\
u &= \frac{1}{a}\sin at\sin x
\end{aligned}
$$

13.4.5

$$\begin{aligned}
L &= 1 \\
f(x) &= x(1-x) \\
g(x) &= x(1-x) \\
A_n &= 2\int_0^1 x(1-x)\sin n\pi x\,dx \\
&= \frac{4[1-(-1)^n]}{n^3\pi^3} \\
B_n &= \frac{2}{n\pi a}\int_0^1 x(1-x)\sin n\pi x\,dx \\
&= \frac{4[1-(-1)^n]}{an^4\pi^4} \\
u(x,t) &= \frac{4}{\pi^3}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^3}\cos n\pi at + \frac{1-(-1)^n}{an^4\pi}\sin n\pi at\right)\sin n\pi x
\end{aligned}$$

13.4.7

$$\begin{aligned}
f(x) &= \begin{cases} \frac{2h}{L}x & 0 < x < L/2 \\ 2h - \frac{2h}{L}x & L/2 < x < L \end{cases} \\
A_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx \\
&= \frac{2}{L}\left[\int_0^{L/2}\frac{2h}{L}x\sin\frac{n\pi}{L}x\,dx + \int_{L/2}^{L}\left(2h-\frac{2h}{L}x\right)\sin\frac{n\pi}{L}x\,dx\right] \\
&= \frac{4h}{L}\left[\frac{1}{L}\int_0^{L/2} x\sin\frac{n\pi}{L}x\,dx + \int_{L/2}^{L}\left(1-\frac{1}{L}x\right)\sin\frac{n\pi}{L}x\,dx\right] \\
&= \frac{8h\sin\frac{n\pi}{2}}{n^2\pi^2} \\
u(x,t) &= \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\cos\frac{n\pi a}{L}t\sin\frac{n\pi}{L}x
\end{aligned}$$

13.4.11

$$\begin{aligned}
X &= c_1 \cos \omega x + c_2 \sin \omega x \\
X' &= -\omega c_1 \sin \omega x + \omega c_2 \cos \omega x \\
0 &= X'(0) \\
&= \omega c_2 \\
&= c_2 \\
0 &= X'(L) \\
&= -\omega c_1 \sin \omega L \\
\omega &= \frac{n\pi}{L} \\
X &= c_1 \cos \frac{n\pi}{L} x \\
T &= c_3 \cos \frac{n\pi a}{L} t + c_4 \sin \frac{n\pi a}{L} t \\
T' &= \frac{n\pi a}{L} \left(-c_3 \sin \frac{n\pi a}{L} t + c_4 \cos \frac{n\pi a}{L} t \right) \\
0 &= T'(0) \\
&= \frac{n\pi a}{L} c_4 \\
&= c_4 \\
u_n &= B_n \cos \frac{n\pi a}{L} t \cos \frac{n\pi}{L} x \\
f(x) &= u(x, 0) \\
x &= \sum_{n=1}^{\infty} B_n \cos \frac{n\pi}{L} x \\
B_0 &= \frac{2}{L} \int_0^L x \, dx \\
&= L \\
B_n &= \frac{2}{L} \int_0^L x \cos \frac{n\pi}{L} x \, dx \\
&= \frac{2L[-1 + (-1)^n]}{n^2 \pi^2} \\
u &= \frac{L}{2} + \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{-1 + (-1)^n}{n^2} \cos \frac{n\pi a}{L} t \cos \frac{n\pi}{L} x
\end{aligned}$$

13.4.15

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t^2} + 2\beta\frac{\partial u}{\partial t} \\
X''T &= XT'' + 2\beta XT' \\
\frac{X''}{X} &= \frac{T''}{T'} + 2\beta\frac{T'}{T} \\
\frac{X''}{X} &= -\lambda \\
X'' + \lambda X &= 0 \\
X(x) &= c_1\cos\omega x + c_2\sin\omega x \\
0 &= X(0) \\
&= c_1 \\
0 &= X(\pi) \\
&= c_2\sin\omega\pi \\
\omega &= n \\
X &= c_2\sin nx \\
\frac{T''}{T} + 2\beta\frac{T'}{T} &= -n^2 \\
T'' + 2\beta T' + n^2 T &= 0 \\
m^2 + 2\beta m + n^2 &= 0 \\
m &= \frac{-2\beta \pm \sqrt{4\beta^2 - 4n^2}}{2} \\
&= -\beta \pm i\sqrt{n^2-\beta^2} \\
T &= e^{-\beta t}(c_1\cos\sqrt{n^2-\beta^2}t + c_2\sin\sqrt{n^2-\beta^2}t) \\
u_n &= (A_n\cos\sqrt{n^2-\beta^2}t + B_n\sin\sqrt{n^2-\beta^2}t)e^{-\beta t}\sin nx \\
f(x) &= \sum_{n=1}^{\infty} A_n\sin nx \\
A_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \\
\frac{\partial u_n}{\partial t} &= (-\sqrt{n^2-\beta^2}A_n\sin\sqrt{n^2-\beta^2}t + \sqrt{n^2-\beta^2}B_n\cos\sqrt{n^2-\beta^2}t)e^{-\beta t}\sin nx \\
&\quad - \beta(A_n\cos\sqrt{n^2-\beta^2}t + B_n\sin\sqrt{n^2-\beta^2}t)e^{-\beta t}\sin nx \\
0 &= \left.\frac{\partial u_n}{\partial t}\right|_{t=0} \\
&= \sqrt{n^2-\beta^2}B_n\sin nx - \beta A_n\sin nx \\
B_n &= \frac{\beta A_n}{\sqrt{n^2-\beta^2}}
\end{aligned}
$$

$$u(x,t) = \frac{2}{\pi} e^{-\beta t} \sum_{n=1}^{\infty} \left(\int_0^{\pi} f(x) \sin nx \, dx \right)$$
$$\left(\cos \sqrt{n^2 - \beta^2} t + \frac{\beta}{\sqrt{n^2 - \beta^2}} \sin \sqrt{n^2 - \beta^2} t \right) \sin nx$$

13.5 Laplace's Equation

13.5.1

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{a} y \sin \frac{n\pi}{a} x$$
$$A_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx$$

13.5.5

$$
\begin{aligned}
Y'' - \lambda Y &= 0 \\
Y'(0) &= 0 \\
Y'(1) &= 0 \\
Y &= c_1 \cos \omega y + c_2 \sin \omega y \\
Y' &= -\omega c_1 \sin \omega y + \omega c_2 \cos \omega y \\
0 &= \omega c_2 \\
&= c_2 \\
0 &= -\omega c_1 \sin \omega \\
\omega &= n\pi \\
Y &= c_1 \cos n\pi y \\
Y &= c_1 + c_2 y \\
Y' &= c_2 \\
Y &= c_1 \\
X'' + \lambda X &= 0 \\
X &= c_3 \sinh n\pi x + c_4 \cosh n\pi x \\
0 &= X(0) \\
&= c_4 \\
X &= c_3 \sinh n\pi x \\
X &= c_3 + c_4 x \\
0 &= X(0) \\
&= c_3 \\
X &= c_4 x \\
u &= A_0 x \\
u_n &= A_n \sinh n\pi x \cos n\pi y \\
u &= A_0 x + \sum_{n=1}^{\infty} A_n \sinh n\pi x \cos n\pi y \\
f(y) &= u(1, y) \\
1 - y &= A_0 + \sum_{n=1}^{\infty} A_n \sinh n\pi \cos n\pi y \\
A_0 &= \frac{a_0}{2} \\
&= \int_0^1 (1 - y)\, dy \\
&= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
A_n \sinh n\pi &= 2\int_0^1 (1-y)\cos n\pi y\, dy \\
A_n &= \frac{2[1-(-1)^n]}{n^2\pi^2 \sinh n\pi} \\
u &= \frac{1}{2}x + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2 \sinh n\pi} \sinh n\pi x \cos n\pi y
\end{aligned}
$$

13.5.7

$$
\begin{aligned}
Y'' - \lambda Y &= 0 \\
Y(0) &= 0 \\
Y(\pi) &= 0 \\
Y &= c_1 \cos \omega y + c_2 \sin \omega y \\
0 &= Y(0) \\
&= c_1 \\
0 &= Y(\pi) \\
&= c_2 \sin \omega \pi \\
\omega &= n \\
Y &= c_2 \sin ny \\
X'' + \lambda X &= 0 \\
X &= c_3 \sinh nx + c_4 \cosh nx \\
X' &= nc_3 \cosh nx + nc_4 \sinh nx \\
X'(0)Y(y) &= X(0)Y(y) \\
X'(0) &= X(0) \\
nc_3 &= c_4 \\
X &= c_3(\sinh nx + n \cosh nx) \\
u &= \sum_{n=1}^{\infty} A_n(\sinh nx + n \cosh nx) \sin ny \\
1 &= u(\pi, y) \\
&= \sum_{n=1}^{\infty} A_n(\sinh n\pi + n \cosh n\pi) \sin ny \\
A_n(\sinh n\pi + n \cosh n\pi) &= \frac{2}{\pi} \int_0^{\pi} \sin ny \, dx \\
A_n &= \frac{2[1 - (-1)^n]}{n\pi(\sinh n\pi + n \cosh n\pi)} \\
u &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \frac{\sinh nx + n \cosh nx}{\sinh n\pi + n \cosh n\pi} \sin ny
\end{aligned}
$$

13.5.11

$$
\begin{aligned}
X'' + \lambda X &= 0 \\
X &= c_1 \cos \omega x + c_2 \sin \omega y \\
0 &= X(0) \\
&= c_1 \\
0 &= X(\pi) \\
&= c_2 \sin \omega \pi \\
\omega &= n \\
X &= c_2 \sin nx \\
Y'' - \lambda Y &= 0 \\
Y &= c_3 e^{ny} + c_4 e^{-ny} \\
0 &= \lim_{y \to \infty} Y \\
&= \lim_{y \to \infty} (c_3 e^{ny} + c_4 e^{-ny}) \\
0 &= c_3 \\
Y &= c_4 e^{-ny} \\
u &= \sum_{n=1}^{\infty} A_n e^{-ny} \sin nx \\
f(x) &= u(x, 0) \\
&= \sum_{n=1}^{\infty} A_n \sin nx \\
A_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \\
u &= \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\int_0^{\pi} f(x) \sin nx \, dx \right) e^{-ny} \sin nx
\end{aligned}
$$

13.5.13

$$
\begin{aligned}
X'' + \lambda X &= 0 \\
X &= c_1 \sin \frac{n\pi}{a} x \\
Y'' - \lambda Y &= 0 \\
Y &= c_2 \sinh \frac{n\pi}{a} y + c_3 \cosh \frac{n\pi}{a} y \\
u &= \sum_{n=1}^{\infty} \left(A_n \sinh \frac{n\pi}{a} y + B_n \cosh \frac{n\pi}{a} y \right) \sin \frac{n\pi}{a} x \\
f(x) &= u(x, 0) \\
&= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{a} x \\
B_n &= \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx \\
g(x) &= u(x, b) \\
&= \sum_{n=1}^{\infty} \left(A_n \sinh \frac{n\pi b}{a} + B_n \cosh \frac{n\pi b}{a} \right) \sin \frac{n\pi}{a} x \\
A_n \sinh \frac{n\pi b}{a} + B_n \cosh \frac{n\pi b}{a} &= \frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x \, dx \\
A_n &= \frac{1}{\sinh \frac{n\pi b}{a}} \left(\frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x \, dx - B_n \cosh \frac{n\pi b}{a} \right)
\end{aligned}
$$

13.6 Nonhomogeneous Boundary-Value Problems

13.6.1

$$\begin{aligned}
ku_{xx} &= u_t \\
u(0,t) &= 100 \\
u(1,t) &= 100 \\
u(x,0) &= 0 \\
\psi &= 100 \\
kv_{xx} &= v_t \\
v(0,t) &= 0 \\
v(1,t) &= 0 \\
v(x,0) &= -100 \\
kX''T &= XT' \\
\frac{X''}{X} &= \frac{T'}{kT} \\
X &= c_1 \sin n\pi x,\ \ n = 1, 2, 3, \ldots \\
T' + k\lambda T &= 0 \\
T &= c_2 e^{-kn^2\pi^2 t} \\
v(x,t) &= \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x \\
-100 &= v(x,0) \\
&= \sum_{n=1}^{\infty} A_n \sin n\pi x \\
A_n &= -200 \int_0^1 \sin n\pi x \, dx \\
&= \frac{200[-1 + (-1)^n]}{n\pi} \\
v(x,t) &= \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{-1 + (-1)^n}{n} e^{-kn^2\pi^2 t} \sin n\pi x \\
u(x,t) &= 100 + \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{-1 + (-1)^n}{n} e^{-kn^2\pi^2 t} \sin n\pi x
\end{aligned}$$

13.6.3

$$
\begin{aligned}
ku_{xx} + r &= u_t \\
u(0,t) &= u_0 \\
u(1,t) &= u_0 \\
u(x,0) &= 0 \\
u(x,t) &= v(x,t) + \psi(x) \\
k\psi'' + r &= 0 \\
\psi'' &= -\frac{r}{k} \\
\psi &= -\frac{r}{2k}x^2 + c_1 x + c_2 \\
\psi(0) &= u_0 \\
c_2 &= u_0 \\
u_0 &= \psi(1) \\
&= -\frac{r}{2k} + c_1 + u_0 \\
c_1 &= \frac{r}{2k} \\
\psi(x) &= \frac{r}{2k}x(1-x) + u_0 \\
kv_{xx} &= v_t \\
v(0,t) &= 0 \\
v(L,t) &= 0 \\
v(x,0) &= \frac{r}{2k}x(x-1) - u_0 \\
X &= c_1 \sin n\pi x, \;\; n = 1, 2, 3, \dots \\
T &= c_2 e^{-kn^2\pi^2 t} \\
v &= \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x \\
A_n &= 2\int_0^1 \left[\frac{r}{2k}x(x-1) - u_0\right] \sin n\pi x \, dx \\
&= \frac{2[-1+(-1)^n](r + kn^2\pi^2 u_0)}{kn^3\pi^3} \\
v &= \frac{2}{k\pi^3} \sum_{n=1}^{\infty} \frac{[-1+(-1)^n](r + kn^2\pi^2 u_0)}{n^3} \sin n\pi x \\
u &= \frac{r}{2k}x(1-x) + u_0 + \frac{2}{k\pi^3} \sum_{n=1}^{\infty} \frac{[-1+(-1)^n](r + kn^2\pi^2 u_0)}{n^3} e^{-kn^2\pi^2 t} \sin n\pi x
\end{aligned}
$$

13.6.7

$$\begin{aligned}
ku_{xx} - h(u - u_0) &= u_t \\
u(0,t) &= u_0 \\
u(1,t) &= 0 \\
u(x,0) &= f(x) \\
u(x,t) &= v(x,t) + \psi(x) \\
u_{xx} &= v_{xx} + \psi'' \\
u_t &= v_t \\
k(v_{xx} + \psi'') - h[(v + \psi) - u_0] &= v_t \\
kv_{xx} + k\psi'' - hv - h\psi + hu_0 &= v_t \\
\psi'' - \frac{h}{k}\psi &= -\frac{h}{k}u_0 \\
\psi_c &= c_1 \sinh \sqrt{\frac{h}{k}}x + c_2 \cosh \sqrt{\frac{h}{k}}x \\
\psi_p &= c_3 \\
\psi_p'' &= 0 \\
-\frac{h}{k}c_3 &= -\frac{h}{k}u_0 \\
c_3 &= u_0 \\
\psi &= \psi_c + \psi_p \\
&= u_0 + c_1 \sinh \sqrt{\frac{h}{k}}x + c_2 \cosh \sqrt{\frac{h}{k}}x \\
\psi(0) &= u_0 \\
c_2 &= 0 \\
0 &= \psi(1) \\
&= u_0 + c_1 \sinh \sqrt{\frac{h}{k}} \\
c_1 &= -\frac{u_0}{\sinh \sqrt{h/k}} \\
\psi &= u_0 \left(1 - \frac{\sinh \sqrt{h/k}x}{\sinh \sqrt{h/k}}\right)
\end{aligned}$$

13.6.13

$$\begin{aligned}
u_{xx} &= u_t \\
u(0,t) &= \sin t \\
u(1,t) &= 0 \\
u(x,0) &= 0 \\
u(x,t) &= v(x,t) + (1-x)\sin t \\
v_{xx} + (x-1)\cos t &= v_t \\
(x-1)\cos t &= \sum_{n=1}^{\infty} A_n \sin n\pi x \\
A_n &= 2\cos t \int_0^1 (x-1)\sin n\pi x\, dx \\
&= -\frac{2\cos t}{n\pi} \\
(x-1)\cos t &= -\frac{2\cos t}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi x \\
v &= \sum_{n=1}^{\infty} v_n(t) \sin n\pi x \\
v_{xx} &= \sum_{n=1}^{\infty} -n^2\pi^2 v_n(t) \sin n\pi x \\
v_t &= \sum_{n=1}^{\infty} v_n'(t) \sin n\pi x \\
\sum_{n=1}^{\infty} [-n^2\pi^2 v_n(t) - v_n'(t)] \sin n\pi x &= \frac{2\cos t}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi x \\
v_n'(t) + n^2\pi^2 v_n(t) &= -\frac{2\cos t}{\pi n} \\
\frac{d}{dt}[e^{n^2\pi^2 t} v_n(t)] &= -\frac{2e^{n^2\pi^2 t}\cos t}{\pi n} \\
v_n(t) &= c_n e^{-n^2\pi^2 t} - \frac{2[n^2\pi^2 \cos t + \sin t]}{n\pi + n^5\pi^5} \\
0 &= \sum_{n=1}^{\infty} \left(c_n - \frac{2n^2\pi^2}{n\pi + n^5\pi^5} \right) \sin n\pi x \\
c_n &= \frac{2n^2\pi^2}{n\pi + n^5\pi^5} \\
u(x,t) &= (1-x)\sin t + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n^2\pi^2 e^{-n^2\pi^2 t} - n^2\pi^2\cos t - \sin t}{n(1+n^4\pi^4)} \sin n\pi x
\end{aligned}$$

13.6.17

$$
\begin{aligned}
u_{xx} + xe^{-3t} &= u_t \\
u(0,t) &= 0 \\
u(\pi,t) &= 0 \\
u(x,0) &= 0 \\
xe^{-3t} &= \sum_{n=1}^{\infty} \left(\frac{2}{\pi} \int_0^{\pi} xe^{-3t} \sin nx \, dx \right) \sin nx \\
&= -2e^{-3t} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx \\
u(x,t) &= \sum_{n=1}^{\infty} u_n(t) \sin nx \\
u_{xx} &= \sum_{n=1}^{\infty} -n^2 u_n(t) \sin nx \\
u_t &= \sum_{n=1}^{\infty} u_n'(t) \sin nx \\
\sum_{n=1}^{\infty} [-n^2 u_n(t) - u_n'(t)] \sin nx &= 2e^{-3t} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx \\
-n^2 u_n - u_n' &= 2e^{-3t} \frac{(-1)^n}{n} \\
u_n' + n^2 u_n &= -2e^{-3t} \frac{(-1)^n}{n} \\
\frac{d}{dt}[e^{n^2 t} u_n] &= -2e^{(n^2-3)t} \frac{(-1)^n}{n} \\
u_n &= c_n e^{-n^2 t} - 2 \frac{(-1)^n}{n(n^2-3)} e^{-3t} \\
0 &= u(x,0) \\
&= \sum_{n=1}^{\infty} \left[c_n - 2 \frac{(-1)^n}{n(n^2-3)} \right] \sin nx \\
c_n &= 2 \frac{(-1)^n}{n(n^2-3)} \\
u(x,t) &= 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n^2-3)} \left(e^{-n^2 t} - e^{-3t} \right) \sin nx
\end{aligned}
$$

13.7 Orthogonal Series Expansion

13.7.1

$$
\begin{aligned}
ku_{xx} &= u_t \\
u_x|_{x=0} &= 0 \\
u_x|_{x=1} &= -hu(1,t) \\
u(x,0) &= 1 \\
u(x,t) &= X(x)T(t) \\
kX''(x)T(t) &= X(x)T'(t) \\
\frac{X''}{X} &= \frac{T'}{kT} \\
X'' + \lambda X &= 0 \\
X'(0) &= 0 \\
X'(1) &= -hX(1) \\
X &= c_1 \cos \omega x + c_2 \sin \omega x \\
X' &= -\omega c_1 \sin \omega x + \omega c_2 \cos \omega x \\
0 &= X'(0) \\
&= \omega c_2 \\
&= c_2 \\
X'(1) &= -hX(1) \\
-\omega c_1 \sin \omega &= -hc_1 \cos \omega \\
\tan \omega &= \frac{h}{\omega} \\
X_n &= c_1 \cos \omega_n x \\
T' + k\lambda T &= 0 \\
T_n &= c_3 e^{-k\omega_n^2 t} \\
u &= \sum_{n=1}^{\infty} A_n e^{-k\omega_n^2 t} \cos \omega_n x \\
1 &= u(x,0) \\
&= \sum_{n=1}^{\infty} A_n \cos \omega_n x \\
A_n &= \frac{\int_0^1 \cos \omega_n x \, dx}{\int_0^1 \cos^2 \omega_n x \, dx} \\
&= \frac{2 \sin \omega_n}{\omega_n + \cos \omega_n \sin \omega_n} \\
&= \frac{2h \sin \omega_n}{\omega_n (h + \sin^2 \omega_n)}
\end{aligned}
$$

$$u(x,t) = 2h \sum_{n=1}^{\infty} \frac{\sin \omega_n}{\omega_n (h + \sin^2 \omega_n)} e^{-k\omega_n^2 t} \cos \omega_n x$$

13.7.3

$$\begin{aligned}
u_{xx} + u_{yy} &= 0 \\
u(0, y) &= 0 \\
u_x|_{x=a} &= -hu(a, y) \\
u(x, 0) &= 0 \\
u(x, b) &= f(x) \\
u(x, y) &= X(x)Y(y) \\
X''Y + XY'' &= 0 \\
\frac{X''}{X} &= -\frac{Y''}{Y} \\
\frac{X''}{X} &= -\lambda \\
X'' + \lambda X &= 0 \\
X(0) &= 0 \\
X'(a) &= -hX(a) \\
X &= c_1 \cos \alpha x + c_2 \sin \alpha x \\
X' &= -\alpha c_1 \sin \alpha x + \alpha c_2 \cos \alpha x \\
0 &= X(0) \\
&= c_1 \\
X'(a) &= -hX(a) \\
\alpha c_2 \cos \alpha a &= -hc_2 \sin \alpha a \\
\tan \alpha a &= -\frac{\alpha}{h} \\
X &= c_2 \sin \alpha_n x \\
Y'' - \lambda Y &= 0 \\
Y(0) &= 0 \\
Y &= c_3 \sinh \alpha_n y + c_4 \cosh \alpha_n y \\
0 &= Y(0) \\
&= c_4 \\
Y &= c_3 \sinh \alpha_n y \\
u(x, y) &= \sum_{n=1}^{\infty} A_n \sinh \alpha_n y \sin \alpha_n x \\
f(x) &= u(x, b) \\
&= \sum_{n=1}^{\infty} A_n \sinh \alpha_n b \sin \alpha_n x
\end{aligned}$$

$$
\begin{aligned}
A_n \sinh \alpha_n b &= \frac{\int_0^a f(x) \sin \alpha_n x \, dx}{\int_0^a \sin^2 \alpha_n x \, dx} \\
A_n &= \frac{\int_0^a f(x) \sin \alpha_n x \, dx}{\left(\frac{a}{2} - \frac{\sin 2a\alpha_n}{4\alpha_n} \right) \sinh \alpha_n b} \\
&= \frac{2h \int_0^a f(x) \sin \alpha_n x \, dx}{(ah + \cos^2 \alpha_n a) \sinh \alpha_n b} \\
u(x, y) &= \sum_{n=1}^{\infty} \frac{2h \int_0^a f(x) \sin \alpha_n x \, dx}{(ah + \cos^2 \alpha_n a) \sinh \alpha_n b} \sinh \alpha_n y \sin \alpha_n x
\end{aligned}
$$

13.7.5

$$
\begin{aligned}
ku_{xx} &= u_t \\
u(x,0) &= f(x) \\
u(0,t) &= 0 \\
u_x|_{x=L} &= 0 \\
\frac{X''}{X} &= \frac{T'}{kT} \\
X'' + \lambda X &= 0 \\
X(0) &= 0 \\
X'(L) &= 0 \\
X &= c_1 \cos \omega x + c_2 \sin \omega x \\
X' &= -\omega c_1 \sin \omega x + \omega c_2 \cos \omega x \\
0 &= X(0) \\
&= c_1 \\
0 &= X'(L) \\
&= \omega c_2 \cos \omega L \\
\omega L &= \frac{(2n-1)\pi}{2} \\
\omega &= \frac{(2n-1)\pi}{2L},\ n = 1, 2, 3, \ldots \\
X &= c_2 \sin \frac{(2n-1)\pi}{2L} x \\
T' + k\lambda T &= 0 \\
T &= c_3 e^{-k\lambda t} \\
u(x,t) &= \sum_{n=1}^{\infty} A_n e^{-k(2n-1)^2\pi^2 t/4L^2} \sin \frac{(2n-1)\pi}{2L} x \\
f(x) &= u(x,0) \\
&= \sum_{n=1}^{\infty} A_n \sin \frac{(2n-1)\pi}{2L} x \\
A_n &= \frac{\int_0^L f(x) \sin \frac{(2n-1)\pi}{2L} x \, dx}{\int_0^L \sin^2 \frac{(2n-1)\pi}{2L} x \, dx} \\
&= \frac{2}{L} \int_0^L f(x) \sin \frac{(2n-1)\pi}{2L} x \, dx \\
u(x,t) &= \frac{2}{L} \sum_{n=1}^{\infty} \left(\int_0^L f(x) \sin \frac{(2n-1)\pi}{2L} x \, dx \right) e^{-k(2n-1)^2\pi^2 t/4L^2} \sin \frac{(2n-1)\pi}{2L} x
\end{aligned}
$$

13.7.7

$$
\begin{aligned}
u_{xx} + u_{yy} &= 0 \\
u_x|_{x=0} &= 0 \\
u(1, y) &= u_0 \\
u(x, 0) &= 0 \\
u_y|_{y=1} &= 0 \\
X''Y + XY'' &= 0 \\
\frac{Y''}{Y} &= -\frac{X''}{X} \\
Y'' + \lambda Y &= 0 \\
Y(0) &= 0 \\
Y'(1) &= 0 \\
Y &= c_1 \cos \omega y + c_2 \sin \omega y \\
Y' &= -\omega c_1 \sin \omega y + \omega c_2 \cos \omega y \\
0 &= Y(0) \\
&= c_1 \\
0 &= Y'(1) \\
&= \omega c_2 \cos \omega \\
\omega &= \frac{2n-1}{2}\pi,\ n = 1, 2, 3, \dots \\
Y &= c_2 \sin \frac{2n-1}{2}\pi y \\
X'' - \lambda X &= 0 \\
X'(0) &= 0 \\
X &= c_1 \cosh \frac{2n-1}{2}\pi x + c_2 \sinh \frac{2n-1}{2}\pi x \\
X' &= \frac{2n-1}{2}\pi \left(c_1 \sinh \frac{2n-1}{2}\pi x + c_2 \cosh \frac{2n-1}{2}\pi x \right) \\
0 &= X'(0) \\
&= \frac{2n-1}{2}\pi c_2 \\
&= c_2 \\
X &= c_1 \cosh \frac{2n-1}{2}\pi x \\
u(x, y) &= \sum_{n=1}^{\infty} A_n \cosh \left(\frac{2n-1}{2}\pi x \right) \sin \left(\frac{2n-1}{2}\pi y \right)
\end{aligned}
$$

$$
\begin{aligned}
u_0 &= u(1,y) \\
&= \sum_{n=1}^{\infty} A_n \cosh\left(\frac{2n-1}{2}\pi\right) \sin\left(\frac{2n-1}{2}\pi y\right) \\
A_n \cosh\left(\frac{2n-1}{2}\pi\right) &= \frac{\int_0^1 u_0 \sin\left(\frac{2n-1}{2}\pi y\right)\, dy}{\int_0^1 \sin^2\left(\frac{2n-1}{2}\pi y\right)\, dy} \\
&= \frac{4u_0}{(2n-1)\pi} \\
A_n &= \frac{4u_0}{(2n-1)\pi \cosh\left(\frac{2n-1}{2}\pi\right)} \\
u(x,y) &= \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)\cosh\left(\frac{2n-1}{2}\pi\right)} \cosh\left(\frac{2n-1}{2}\pi x\right) \sin\left(\frac{2n-1}{2}\pi y\right)
\end{aligned}
$$

13.8 Fourier Series in Two Variables

13.8.1

$$
\begin{aligned}
k(u_{xx} + u_{yy}) &= u_t \\
u(0, y, t) &= 0 \\
u(\pi, y, t) &= 0 \\
u(x, 0, t) &= 0 \\
u(x, \pi, t) &= 0 \\
u(x, y, 0) &= u_0 \\
X'' + \lambda X &= 0 \\
X(0) &= 0 \\
X(\pi) &= 0 \\
X &= c_1 \cos \alpha x + c_2 \sin \alpha x \\
0 &= X(0) \\
&= c_1 \\
0 &= X(\pi) \\
&= c_2 \sin \alpha \pi \\
X &= c_2 \sin mx, \ m = 1, 2, 3, \ldots \\
Y'' + \mu Y &= 0 \\
Y(0) &= 0 \\
Y(\pi) &= 0 \\
Y &= c_3 \cos \beta y + c_4 \sin \beta y \\
0 &= Y(0) \\
&= c_3 \\
0 &= Y(\pi) \\
&= c_4 \sin \beta \pi \\
Y &= c_4 \sin ny, \ n = 1, 2, 3, \ldots \\
T' + k(\lambda + \mu)T &= 0 \\
T &= c_5 e^{-k(m^2+n^2)t} \\
u(x, y, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{-k(m^2+n^2)t} \sin mx \sin ny \\
u_0 &= u(x, y, 0) \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin mx \sin ny
\end{aligned}
$$

$$
\begin{aligned}
A_{mn} &= \frac{4u_0}{\pi^2} \int_0^{\pi} \int_0^{\pi} \sin mx \sin ny \, dx \, dy \\
&= \frac{4u_0[1-(-1)^m][1-(-1)^n]}{\pi^2 mn} \\
u(x,y,t) &= \frac{4u_0}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{[1-(-1)^m][1-(-1)^n]}{mn} e^{-k(m^2+n^2)t} \sin mx \sin ny
\end{aligned}
$$

13.8.3

$$\begin{aligned}
a^2(u_{xx}+u_{yy}) &= u_{tt}\\
u(0,y,t) &= 0\\
u(\pi,y,t) &= 0\\
u(x,0,t) &= 0\\
u(x,\pi,t) &= 0\\
u(x,y,0) &= xy(x-\pi)(y-\pi)\\
u_t|_{t=0} &= 0\\
a^2(X''YT+XY''T) &= XYT''\\
a^2\left(\frac{X''}{X}+\frac{Y''}{Y}\right) &= \frac{T''}{T}\\
\frac{X''}{X} &= -\frac{Y''}{Y}+\frac{T''}{a^2T}\\
X''+\lambda X &= 0\\
X(0) &= 0\\
X(\pi) &= 0\\
X &= c_1\cos\alpha x + c_2\sin\alpha x\\
0 &= X(0)\\
&= c_1\\
0 &= X(\pi)\\
&= c_2\sin\alpha\pi\\
X &= c_2\sin mx,\ m=1,2,3,\ldots\\
Y''+\mu Y &= 0\\
Y(0) &= 0\\
Y(\pi) &= 0\\
Y &= c_3\cos\beta y + c_4\sin\beta y\\
0 &= Y(0)\\
&= c_3\\
0 &= Y(\pi)\\
&= c_4\sin\beta\pi\\
Y &= c_4\sin ny,\ n=1,2,3,\ldots
\end{aligned}$$

$$
\begin{aligned}
T'' + a^2(\lambda+\mu)T &= 0 \\
T'(0) &= 0 \\
T &= c_5 \cos a\sqrt{m^2+n^2}t + c_6 \sin a\sqrt{m^2+n^2}t \\
T' &= a\sqrt{m^2+n^2}(-c_5 \sin a\sqrt{m^2+n^2}t + c_6 \cos a\sqrt{m^2+n^2}t) \\
0 &= T'(0) \\
&= c_6 \\
T &= c_5 \cos a\sqrt{m^2+n^2}t \\
u(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin mx \sin ny \cos a\sqrt{m^2+n^2}t \\
xy(x-\pi)(y-\pi) &= u(x,y,0) \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin mx \sin ny \\
A_{mn} &= \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} xy(x-\pi)(y-\pi) \sin mx \sin ny \, dy \, dx \\
&= \frac{16[1-(-1)^m][1-(-1)^n]}{m^3 n^3 \pi^2} \\
u(x,y,t) &= \frac{16}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{[1-(-1)^m][1-(-1)^n]}{m^3 n^3} \sin mx \sin ny \cos a\sqrt{m^2+n^2}t
\end{aligned}
$$

13.9 Chapter in Review

13.9.1

$$
\begin{aligned}
u_{xy} &= u \\
X'Y' &= XY \\
\frac{X'}{X} &= \frac{Y}{Y'} \\
X' + \lambda X &= 0 \\
X &= c_1 e^{-\lambda x} \\
Y' + \frac{1}{\lambda}Y &= 0 \\
Y &= c_2 e^{-y/\lambda} \\
u &= c_3 e^{-\lambda x - y/\lambda}
\end{aligned}
$$

13.9.3

$$\begin{aligned}
ku_{xx} &= u_t \\
u(0,t) &= u_0 \\
-u_x|_{x=\pi} &= u(\pi,t) - u_1 \\
u(x,0) &= 0 \\
u(x,t) &= v(x,t) + \psi(x) \\
u_{xx} &= v_{xx} + \psi'' \\
u_t &= v_t \\
k(v_{xx} + \psi'') &= v_t \\
kv_{xx} &= v_t \\
k\psi'' &= 0 \\
\psi(0) &= u_0 \\
-\psi'(\pi) &= \psi(\pi) - u_1 \\
\psi &= c_1 x + c_2 \\
\psi' &= c_1 \\
u_0 &= \psi(0) \\
&= c_2 \\
-c_1 &= c_1\pi + u_0 - u_1 \\
c_1 &= \frac{u_1 - u_0}{\pi + 1} \\
\psi &= \frac{u_1 - u_0}{\pi + 1} x + u_0
\end{aligned}$$

13.9.7

$$
\begin{aligned}
u_{xx} + u_{yy} &= 0 \\
u(0, y) &= 0 \\
u(\pi, y) &= 50 \\
u(x, 0) &= 0 \\
u(x, \pi) &= 0 \\
X''Y + XY'' &= 0 \\
-\frac{X''}{X} &= \frac{Y''}{Y} \\
Y'' + \lambda Y &= 0 \\
Y &= c_1 \cos \omega y + c_2 \sin \omega y \\
0 &= Y(0) \\
&= c_1 \\
0 &= Y(\pi) \\
&= c_2 \sin \omega \pi \\
Y &= c_2 \sin ny, \;\; n = 1, 2, 3, \ldots \\
X'' - n^2 X &= 0 \\
X &= c_3 \cosh nx + c_4 \sinh nx \\
0 &= X(0) \\
&= c_3 \\
X &= c_4 \sinh nx \\
u(x, y) &= \sum_{n=1}^{\infty} A_n \sinh nx \sin ny \\
50 &= u(\pi, y) \\
&= \sum_{n=1}^{\infty} A_n \sinh n\pi \sin ny \\
A_n \sinh n\pi &= \frac{100}{\pi} \int_0^{\pi} \sin ny \, dy \\
&= \frac{100}{\pi} \frac{1 - (-1)^n}{n} \\
A_n &= \frac{100}{\pi} \frac{1 - (-1)^n}{n \sinh n\pi} \\
u(x, y) &= \frac{100}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n \sinh n\pi} \sinh nx \sin ny
\end{aligned}
$$

14 Boundary-Value Problems in Other Coordinate Systems

14.1 Polar Coordinates

14.1.1

$$
\begin{aligned}
u(1,\theta) &= \begin{cases} u_0 & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi \end{cases} \\
\Theta(\theta) &= \Theta(\theta + 2\pi) \\
\Theta'' + \lambda\Theta &= 0 \\
\Theta &= c_1 \\
\Theta &= c_1 \cos n\theta + c_2 \sin n\theta,\ n = 1, 2, 3, \ldots \\
r^2 R'' + rR' - \lambda R &= 0 \\
R'' + \frac{1}{r} R' &= 0 \\
R &= c_3 + c_4 \ln r \\
r^2 R'' + rR' - n^2 R &= 0 \\
R &= c_3 r^n + c_4 r^{-n} \\
u_0 &= A_0,\ n = 0 \\
u_n &= r^n (A_n \cos n\theta + B_n \sin n\theta),\ n = 1, 2, 3, \ldots \\
u &= A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \\
u(1,\theta) &= A_0 + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) \\
A_0 &= \frac{a_0}{2} \\
&= \frac{1}{2\pi} \int_0^{\pi} u_0 \, d\theta \\
&= \frac{u_0}{2} \\
A_n &= \frac{1}{\pi} \int_0^{\pi} u_0 \cos n\theta \, d\theta \\
&= 0 \\
B_n &= \frac{1}{\pi} \int_0^{\pi} u_0 \sin n\theta \, d\theta \\
&= \frac{\mu_0}{\pi} \frac{1 - (-1)^n}{n} \\
u(r,\theta) &= \frac{\mu_0}{2} + \frac{\mu_0}{\pi} \sum_{n=1}^{\infty} r^n \frac{1 - (-1)^n}{n} \sin n\theta
\end{aligned}
$$

14.1.5

$$\begin{aligned}
\left.\frac{\partial u}{\partial \theta}\right|_{\theta=0} &= 0 \\
\left.\frac{\partial u}{\partial \theta}\right|_{\theta=\pi} &= 0 \\
u(2,\theta) &= \begin{cases} u_0 & 0<\theta<\pi/2 \\ 0 & \pi/2<\theta<\pi \end{cases} \\
\Theta'' + \lambda\Theta &= 0 \\
\Theta'(0) &= 0 \\
\Theta'(\pi) &= 0 \\
\Theta &= c_1 \\
\Theta &= c_1 \cos\omega\theta + c_2 \sin\omega\theta \\
\Theta' &= -\omega c_1 \sin\omega\theta + \omega c_2 \cos\omega\theta \\
0 &= \Theta'(0) \\
&= \omega c_2 \\
&= c_2 \\
0 &= \Theta'(\pi) \\
&= -\omega c_1 \sin\omega\pi \\
\omega &= n,\ n = 1,2,3,\ldots \\
\Theta &= c_1,\ n = 0 \\
\Theta &= c_1 \cos n\theta,\ n = 1,2,3,\ldots \\
r^2R'' + rR' - \lambda R &= 0 \\
R &= c_3,\ n = 0 \\
R &= c_3 r^n,\ n = 1,2,3,\ldots \\
u &= A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\theta,\ n = 1,2,3,\ldots \\
A_0 &= \frac{a_0}{2} \\
&= \frac{1}{\pi}\int_0^{\pi/2} u_0\, d\theta \\
&= \frac{u_0}{2} \\
2^n A_n &= \frac{2}{\pi}\int_0^{\pi/2} u_0 \cos n\theta\, d\theta \\
A_n &= \frac{2u_0 \sin\frac{n\pi}{2}}{2^n \pi n} \\
u &= \frac{u_0}{2} + \frac{2u_0}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin\frac{n\pi}{2}\left(\frac{r}{2}\right)^n \cos n\theta
\end{aligned}$$

14.1.9

$$
\begin{aligned}
u(r,0) &= 0\\
u(r,\beta) &= 0\\
u(c,\theta) &= f(\theta)\\
\Theta'' + \lambda\Theta &= 0\\
\Theta(0) &= 0\\
\Theta(\beta) &= 0\\
\Theta &= c_1 \cos\omega\theta + c_2 \sin\omega\theta\\
0 &= \Theta(0)\\
&= c_1\\
0 &= \Theta(\beta)\\
&= c_2 \sin\omega\beta\\
\omega\beta &= n\pi\\
\omega &= \frac{n\pi}{\beta},\ n = 1,2,3,\ldots\\
\Theta &= c_2 \sin\frac{n\pi}{\beta}\theta,\ n = 1,2,3,\ldots\\
R &= c_3 r^{n\pi/\beta},\ n = 1,2,3,\ldots\\
u &= \sum_{n=1}^{\infty} A_n r^{n\pi/\beta} \sin\frac{n\pi}{\beta}\theta\\
A_n &= \frac{2}{\beta c^{n\pi/\beta}} \int_0^{\beta} f(\theta) \sin\frac{n\pi}{\beta}\theta\, d\theta\\
u &= \frac{2}{\beta} \sum_{n=1}^{\infty} \left(\frac{r}{c}\right)^{n\pi/\beta} \left(\int_0^{\beta} f(\theta)\sin\frac{n\pi}{\beta}\theta\, d\theta\right) \sin\frac{n\pi}{\beta}\theta
\end{aligned}
$$

14.1.11

$$
\begin{aligned}
u(a,\theta) &= f(\theta) \\
u(b,\theta) &= 0 \\
\Theta'' + \lambda\Theta' &= 0 \\
\Theta(\theta) &= \Theta(\theta + 2\pi) \\
\Theta &= c_1 \\
\Theta &= c_1 \cos\omega\theta + c_2 \sin\omega\theta \\
\omega &= n,\ \ n = 1, 2, 3, \ldots \\
R &= c_3 + c_4 \ln r \\
0 &= R(b) \\
&= c_3 + c_4 \ln b \\
c_3 &= -c_4 \ln b \\
R &= c_4 \ln \frac{r}{b} \\
R &= c_3 r^n + c_4 r^{-n} \\
0 &= R(b) \\
&= c_3 b^n + c_4 b^{-n} \\
c_4 &= -c_3 b^{2n} \\
R &= c_3 (r^n - b^{2n} r^{-n}) \\
u &= A_0 \ln \frac{r}{b} + \sum_{n=1}^{\infty} (r^n - b^{2n} r^{-n})(A_n \cos n\theta + B_n \sin n\theta) \\
u(a,\theta) &= A_0 \ln \frac{a}{b} + \sum_{n=1}^{\infty} (a^n - b^{2n} a^{-n})(A_n \cos n\theta + B_n \sin n\theta) \\
A_0 &= \frac{1}{2\pi \ln a/b} \int_0^{2\pi} f(\theta)\, d\theta \\
A_n &= \frac{1}{\pi(a^n - b^{2n} a^{-n})} \int_0^{2\pi} f(\theta) \cos n\theta\, d\theta \\
B_n &= \frac{1}{\pi(a^n - b^{2n} a^{-n})} \int_0^{2\pi} f(\theta) \sin n\theta\, d\theta
\end{aligned}
$$

14.1.19

$$\begin{aligned}
u(c,\theta) &= f(\theta) \\
u(\infty,\theta) &= 0 \\
u(r,\theta) &= u(r,\theta+2\pi) \\
\Theta &= c_1 \\
\Theta &= c_1\cos n\theta + c_2\sin n\theta \\
R &= c_4\ln r \\
R &= c_4 r^{-n} \\
u &= A_0\ln r + \sum_{n=1}^{\infty} r^{-n}(A_n\cos n\theta + B_n\sin n\theta) \\
u(c,\theta) &= A_0\ln c + \sum_{n=1}^{\infty} c^{-n}(A_n\cos n\theta + B_n\sin n\theta) \\
A_0 &= \frac{1}{2\pi\ln c}\int_0^{2\pi} f(\theta)\,d\theta \\
A_n &= \frac{c^n}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta \\
B_n &= \frac{c^n}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta
\end{aligned}$$

14.2 Cylindrical Coordinates

14.2.1

$$\begin{aligned}
A_n &= 0 \\
B_n &= \frac{2}{a\alpha_n c^2 J_1^2(\alpha_n c)}\int_0^c rJ_0(\alpha_n r)\,dr \\
&= \frac{2}{a\alpha_n^2 cJ_1(\alpha_n c)} \\
u(r,t) &= \frac{2}{ac}\sum_{n=1}^{\infty}\frac{\sin a\alpha_n t}{\alpha_n^2 J_1(\alpha_n c)}J_0(\alpha_n r)
\end{aligned}$$

14.2.3

$$u(r,z) = u_0\sum_{n=1}^{\infty}\frac{\sinh[\alpha_n(4-z)]}{\alpha_n\sinh 4\alpha_n J_1(2\alpha_n)}J_0(\alpha_n r)$$

14.2.5

$$\begin{aligned}
u(1,z) &= z \\
\left.\frac{\partial u}{\partial z}\right|_{z=0} &= 0 \\
\left.\frac{\partial u}{\partial z}\right|_{z=1} &= 0 \\
Z'' + \lambda Z &= 0 \\
Z &= c_1 \\
Z &= c_1 \cos \alpha z + c_2 \sin \alpha z \\
Z' &= -\alpha c_1 \sin \alpha z + \alpha c_2 \cos \alpha z \\
0 &= Z'(0) \\
&= c_2 \\
0 &= Z'(1) \\
&= -\alpha c_1 \sin \alpha \\
\alpha &= n\pi \\
rR'' + R' - \alpha^2 rR &= 0 \\
R &= c_3 \\
R &= c_3 I_0(n\pi r) \\
u(r,t) &= A_0 + \sum_{n=1}^{\infty} A_n I_0(n\pi r) \cos n\pi z \\
z &= u(1,z) \\
&= A_0 + \sum_{n=1}^{\infty} A_n I_0(n\pi) \cos n\pi z \\
A_0 &= \frac{1}{2} \\
A_n I_0(n\pi) &= 2\int_0^1 z \cos n\pi z \, dz \\
A_n &= \frac{2[-1+(-1)^n]}{n^2\pi^2 I_0(n\pi)} \\
u(r,t) &= \frac{1}{2} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{-1+(-1)^n}{n^2 I_0(n\pi)} I_0(n\pi r) \cos n\pi z
\end{aligned}$$

14.2.9

$$
\begin{aligned}
k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) &= \frac{\partial u}{\partial t} \\
u(c,t) &= 0 \\
u(r,0) &= f(r) \\
kR''T + \frac{k}{r}R'T &= RT' \\
\frac{rR'' + R'}{rR} &= \frac{T'}{kT} \\
rR'' + R' + \lambda rR &= 0 \\
R &= c_1 J_0(\alpha r) \\
0 &= R(c) \\
&= c_1 J_0(\alpha c) \\
\alpha_n &= \frac{x_n}{c} \\
R &= c_1 J_0(\alpha_n r) \\
T' + k\lambda T &= 0 \\
T &= c_3 e^{-k\alpha_n^2 t} \\
u(r,t) &= \sum_{n=1}^{\infty} A_n e^{-k\alpha_n t} J_0(\alpha_n r) \\
f(r) &= u(r,0) \\
&= \sum_{n=1}^{\infty} A_n J_0(\alpha_n r) \\
A_n &= \frac{2}{c^2 J_1^2(\alpha_n c)} \int_0^c r J_0(\alpha_n r) f(r)\, dr
\end{aligned}
$$

14.2.11

$$\begin{aligned}
\frac{rR'' + R'}{rR} &= \frac{T'}{kT} \\
rR'' + R' + \lambda rR &= 0 \\
R &= c_1 J_0(\alpha r) \\
R' &= -\alpha c_1 J_1(\alpha r) \\
R'(1) &= -hR(1) \\
\alpha c_1 J_0'(\alpha) &= -hc_1 J_0(\alpha) \\
hJ_0(\alpha) + \alpha J_0'(\alpha) &= 0 \\
R &= c_1 J_0(\alpha_n r) \\
T' + \lambda kT &= 0 \\
T &= c_3 e^{-\alpha_n^2 kt} \\
u(r,t) &= \sum_{n=1}^{\infty} A_n e^{-\alpha_n^2 kt} J_0(\alpha_n r) \\
f(r) &= u(r,0) \\
&= \sum_{n=1}^{\infty} A_n J_0(\alpha_n r) \\
A_n &= \frac{2\alpha_n^2}{(\alpha_n^2 + h^2) J_0^2(\alpha_n)} \int_0^1 r J_0(\alpha_n r) f(r)\, dr
\end{aligned}$$

14.2.13

$$
\begin{aligned}
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial t} \\
u(r,t) &= v(r,t) + \psi(r) \\
R''T + \psi'' + \frac{1}{r}(R'T + \psi') &= RT' \\
\psi'' + \frac{1}{r}\psi' &= 0 \\
\psi &= 100 \\
\frac{rR'' + R'}{rR} &= \frac{T'}{T} \\
rR'' + R' + \lambda r R &= 0 \\
R &= c_1 J_0(\alpha r) \\
0 &= R(2) \\
&= c_1 J_0(2\alpha) \\
\alpha_n &= \frac{x_n}{2} \\
R &= c_1 J_0(\alpha_n r) \\
T &= c_2 e^{-\alpha_n^2 t} \\
v(r,t) &= \sum_{n=1}^{\infty} c_n e^{-\alpha_n^2 t} J_0(\alpha_n r) \\
f(r) - \psi(r) &= v(r,0) \\
f(r) - 100 &= \sum_{n=1}^{\infty} c_n J_0(\alpha_n r) \\
c_n &= \frac{2}{4J_1^2(2\alpha_n)} \int_0^2 r J_0(\alpha_n r)[f(r) - 100]\, dr \\
&= \frac{50}{J_1^2(2\alpha_n)} \int_0^1 r J_0(\alpha_n r)\, dr \\
&= \frac{50 J_1(\alpha_n)}{\alpha_n J_1^2(2\alpha_n)} \\
u(r,t) &= 100 + 50 \sum_{n=1}^{\infty} \frac{J_1(\alpha_n)}{\alpha_n J_1^2(2\alpha_n)} e^{-\alpha_n^2 t} J_0(\alpha_n r)
\end{aligned}
$$

14.3 Spherical Coordinates

14.3.1

$$u(r,\theta) = 25P_0(\cos\theta) + \frac{75}{2}\frac{r}{c}P_1(\cos\theta) - \frac{175}{8}\left(\frac{r}{c}\right)^3 P_3(\cos\theta) + \frac{275}{16}\left(\frac{r}{c}\right)^5 P_5(\cos\theta)$$

14.3.5

$$
\begin{aligned}
u(a,\theta) &= f(\theta) \\
u(b,\theta) &= 0 \\
R(r) &= c_1 r^n + c_2 r^{-(n+1)} \\
0 &= R(b) \\
&= c_1 b^n + c_2 b^{-(n+1)} \\
c_2 &= -c_1 b^{2n+1} \\
R &= c_1 (r^n - b^{2n+1} r^{-(n+1)}) \\
&= c_1 \left(r^n - \frac{b^{2n+1}}{r^{n+1}} \right) \\
&= c_1 \left[\left(\frac{r}{b}\right)^n - \left(\frac{b}{r}\right)^{n+1} \right] \\
u(r,\theta) &= \sum_{n=1}^{\infty} A_n \left[\left(\frac{r}{b}\right)^n - \left(\frac{b}{r}\right)^{n+1} \right] P_n(\cos\theta) \\
f(\theta) &= u(a,\theta) \\
&= \sum_{n=1}^{\infty} A_n \left[\left(\frac{a}{b}\right)^n - \left(\frac{b}{a}\right)^{n+1} \right] P_n(\cos\theta) \\
A_n &= \frac{1}{(a/b)^n - (b/a)^{n+1}} \frac{2n+1}{2} \int_{-1}^{1} f(\theta) P_n(\cos\theta)\, d\theta
\end{aligned}
$$