Vibrations and Waves by A. P. French Notes

Chris Doble

May 2022

Contents

1 Periodic motions **1**

2 The superposition of periodic motions **2**

3 The free vibrations of physical systems **3**

4 Forced vibrations and resonance **3**

5 Coupled oscillators and normal modes **3**

6 Normal modes of continuous systems. Fourier analysis **3**

7 Progressive waves **4**

1 Periodic motions

- Fouriers theorem states that any repeating signal of period T can be expressed as a sum of sin waves with periods T, $T/2$, etc.
- It's important to define the domain of a SHM equation, e.g. for what values of t is the motion defined?
- SHM can be considered a projection of uniform circular motion
- That uniform circular motion can be represented by a number in the complex plane, with the projection being its real part
- Multiplication by j can be considered a counter-clockwise rotation of $90°$ in the complex plane
- Euler's formula states
$$e^{j\theta} = \cos\theta + j\sin\theta$$
- Multiplication of a complex number z by $e^{j\theta}$ is equivalent to a counter-clockwise rotation of z by an angle of θ

2 The superposition of periodic motions

- The combination of two SHM's of the same period

$$x_1 = A_1 \cos(\omega t + \alpha_1)$$
$$x_2 = A_2 \cos(\omega t + \alpha_2)$$

is given by

$$x = A \cos(\omega t + \alpha)$$

where

$$A^2 = A_1^2 + A_2^2 + 2A_1 A_2 \cos(\alpha_2 - \alpha_1),$$
$$A \sin\beta = A_2 \sin(\alpha_2 - \alpha_1),$$

and

$$\alpha = \alpha_1 + \beta.$$

- The combination in complex representation

$$z_1 = A_1 e^{j(\omega t + \alpha_1)}$$
$$z_2 = A_2 e^{j(\omega t + \alpha_2)}$$

is given by

$$z = e^{j(\omega t + \alpha_1)}[A_1 + A_2 e^{j(\alpha_2 - \alpha_1)}]$$

- In the case where $A_1 = A_2$ if we denote $\delta = \alpha_2 - \alpha_1$ then

$$\beta = \frac{\delta}{2}$$

and

$$A = 2A_1 \cos\beta = 2A_1 \cos\frac{\delta}{2}$$

- The superposition of two sinusoids with different periods will itself be periodic if there exist integers n_1 and n_2 such that

$$T = n_1 T_1 = n_2 T_2$$

where T_1 and T_2 are the periods of the two sinusoids

- Periodic motion in two or more dimensions can be represented by extending the "projection of a rotating vector" approach, with one vector for each axis, e.g.

$$x = A_1 \cos\omega t$$
$$y = A_2 \cos\omega t$$

where differing amplitudes, frequencies, and phase differences product different curves called **Lissajous curves**

3 The free vibrations of physical systems

- When a tensile force is applied to a material it elongates. The ratio of the elongation to the original length x/l_0 is known as the **tensile strain**
- The ratio of the tensile force to the cross sectional area of the material F/A is known as the **tensile stress**
- The ratio of stress and strain is a constant known as **Young's modulus** Y
- The force exerted by the stretched material on another object is given by

$$\frac{F/A}{x/l_0} = -Y \Rightarrow F = -\frac{AY}{l_0}x$$

  which is in the form of Hooke's law with $k = -\frac{AY}{l_0}$

4 Forced vibrations and resonance

- Periodic motion that isn't simple harmonic is **anharmonic**

5 Coupled oscillators and normal modes

- A property of a normal mode is that all objects oscillate at the same frequency

6 Normal modes of continuous systems. Fourier analysis

- If a medium is vibrating at a natural frequency with only one end fixed (e.g. the pressure in a tube with one end open), the length of the medium must be an integer multiple of quarter wavelengths
- In one-dimensional systems, the frequency of a normal mode f_n is proportional to the mode number n for small n
- In higher-dimensional systems, the frequency of a normal mode f_n is not proportional to the mode number n
- In higher-dimensional systems, one frequency may correspond to multiple normal modes and is said to be **degenerate**
- The process of determining the coefficients of a Fourier series is called **harmonic analysis**

- One way to think of orthogonal functions is as vectors of infinite dimension. Two n-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if their scalar product is 0, i.e.
$$\mathbf{a}\cdot\mathbf{b} = 0 \text{ if } \sum_0^n a_n b_n = 0.$$
If two functions $f(x)$ and $g(x)$ are considered vectors of infinite dimension then the expression is similar
$$\int_0^L f(x)g(x)\,dx = 0 \text{ is approximately } \sum_{n=0}^{\infty} f(x_n)g(x_n) = 0$$

7 Progressive waves

- A normal mode of vibration of a stretched string can be described as the superposition of two sine waves, identical to one another, traveling in opposite directions
$$\begin{aligned} y_n(x,t) &= A\sin\left(\frac{n\pi x}{L}\right)\cos\omega_n t \\ &= \frac{1}{2}A_n\left[\sin\left(\frac{n\pi x}{L} - \omega_n t\right) + \sin\left(\frac{n\pi x}{L} + \omega_n t\right)\right] \\ &= \frac{1}{2}A_n\left[\sin\left(\frac{2\pi}{\lambda}(x - vt)\right) + \sin\left(\frac{2\pi}{\lambda}(x + vt)\right)\right] \end{aligned}$$
- In reality the wave velocity v is typically a function of the frequency f / the wavelength λ
- When deriving the wave equation it's possible to deal only with first derivatives, i.e.
$$\begin{aligned} y(x,t) &= A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right] \\ \frac{\partial y}{\partial x} &= \frac{2\pi}{\lambda}A\cos\left[\frac{2\pi}{\lambda}(x - vt)\right] \\ \frac{\partial y}{\partial t} &= -\frac{2\pi v}{\lambda}A\cos\left[\frac{2\pi}{\lambda}(x - vt)\right] \\ \frac{\partial y}{\partial x} &= -\frac{1}{v}\frac{\partial y}{\partial t} \end{aligned}$$
however this only applies to waves travelling in the positive x direction. By taking the second derivative we arrive at a relation that also applies to waves travelling in he negative x direction
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

- If a function $y(t)$ is even with respect to its midpoint in time, i.e. $y(-t) = +y(t)$, then it can be represented by a Fourier series of cosine functions alone. If it is odd, i.e. $y(-t) = -y(t)$ then it can be represented by sine functions alone. Otherwise its Fourier series contains both sine and cosine functions.

- In performing the frequency analysis of a short pulse of a particular frequency f, we find that the longer the pulse the better it is represented by a single sinusoidal wave of frequency f — the width of its frequency spectrum narrows. Inversely, as the pulse shortens the width of its frequency spectrum broadens.

- **Cut-off** is the inability of a dispersive medium to transmit waves above (or possibly below) a certain critical frequency. The rate at which waves above the maximum frequency attenuate is proportional to the frequency.

- Energy per unit length is also known as energy density

- The kinetic energy per unit length of a sinusoidal wave on a stretched string given by $y(x,t) = f(x \pm vt) = f(z)$ is

  $$\frac{dK}{dx} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\mu v^2[f'(z)]^2$$

  and the potential energy per unit length is

  $$\frac{dU}{dx} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 = \frac{1}{2}T[f'(z)]^2.$$

  Given that

  $$v = \sqrt{\frac{T}{\mu}} \Rightarrow \mu v^2 = T$$

  these values are equal.

- The total energy in a wavelength of a sinusoidal wave on a stretched string is given by

  $$E = \frac{1}{2}(\lambda\mu)\mu_0^2$$

  where

  $$\mu_0 = 2\pi f A.$$

  This is also the amount of work that must be done on the string to establish that wavelength

- The rate at which work is done on a stretched string to establish a sinusoidal wave is

  $$P = \frac{1}{2}\mu\mu_0^2 v$$

  which is the amount of energy in the wave per unit length $\frac{1}{2}\mu\mu_0^2$ times the velocity at which the wave propagates v