

Quantum Computation and Quantum  
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Chuang Problems

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**Part I**

**Fundamental concepts**

**2 Linear algebra**

**Exercise 2.1**

$$(1, -1) + (1, 2) - (2, 1) = (0, 0)$$

**Exercise 2.2**

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Using the basis  $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$  and  $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$  we get

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$|1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$a_{00} + a_{01} = 1$$

$$a_{10} + a_{11} = -1$$

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$a_{00} - a_{01} = 1$$

$$a_{10} - a_{11} = 1$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

### Exercise 2.5

$$\begin{aligned}
\begin{bmatrix} y_1^* & \cdots & y_n^* \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} &= \begin{bmatrix} y_1^* & \cdots & y_n^* \end{bmatrix} \left( \begin{bmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ z_2 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ z_n \end{bmatrix} \right) \\
&= \begin{bmatrix} y_1^* & \cdots & y_n^* \end{bmatrix} \begin{bmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} y_1^* & \cdots & y_n^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ z_n \end{bmatrix} \\
&= z_1 \begin{bmatrix} y_1^* & \cdots & y_n^* \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + z_n \begin{bmatrix} y_1^* & \cdots & y_n^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\
\begin{bmatrix} y_1^* & \cdots & y_n^* \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} &= y_1^* z_1 + y_2^* z_2 + \cdots + y_n^* z_n \\
&= (y_1 z_1^* + y_2 z_2^* + \cdots + y_n z_n^*)^* \\
&= \left( \begin{bmatrix} z_1^* & \cdots & z_n^* \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right)^* \\
\begin{bmatrix} v_1^* & \cdots & v_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} &= |v_1|^2 + \cdots + |v_n|^2 \\
&\geq 0
\end{aligned}$$

### Exercise 2.6

$$\begin{aligned}
\left( \sum_i \lambda_i |w_i\rangle, |v\rangle \right) &= \left( |v\rangle, \sum_i \lambda_i |w_i\rangle \right)^* \\
&= \left( \sum_i \lambda_i (|v\rangle, |w_i\rangle) \right)^* \\
&= \sum_i \lambda_i^* (|v\rangle, |w_i\rangle)^* \\
&= \sum_i \lambda_i^* \langle w_i | v \rangle
\end{aligned}$$

**Exercise 2.7**

$$\begin{aligned}
\langle w|v\rangle &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= (1)(1) + (1)(-1) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\frac{|w\rangle}{||w\rangle||} &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\frac{|v\rangle}{||v\rangle||} &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}$$

**Exercise 2.9**

$$\begin{aligned}
\sigma_0 &= |0\rangle\langle 0| + |1\rangle\langle 1| \\
\sigma_1 &= |1\rangle\langle 0| + |0\rangle\langle 1| \\
\sigma_2 &= i|1\rangle\langle 0| - i|0\rangle\langle 1| \\
\sigma_3 &= |0\rangle\langle 0| - |1\rangle\langle 1|
\end{aligned}$$

**Exercise 2.11**

$$\begin{aligned}
\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} &= \lambda^2 - 1 \\
\lambda &= \pm 1
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} a \\ b \end{bmatrix} \\
b &= a \\
a &= b
\end{aligned}$$

$$X_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} -a \\ -b \end{bmatrix} \\
b &= -a \\
a &= -b
\end{aligned}$$

$$X_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

**Exercise 2.12**

$$\begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$$

$$\lambda_1 = 1$$

$$\lambda_2 = 1$$

The eigenvalue 1 is degenerate. Because the matrix only has one eigenvector it can't be diagonalised.

**Exercise 2.13**

$$(|w\rangle\langle v|)^\dagger = \langle v|^\dagger |w\rangle^\dagger = |v\rangle\langle w|$$

**Exercise 2.16**

$$\begin{aligned} P^2 &= \left( \sum_{i=1}^k |i\rangle\langle i| \right) \left( \sum_{j=1}^k |j\rangle\langle j| \right) \\ &= \sum_{i,j=1}^k |i\rangle\langle i|j\rangle\langle j| \\ &= \sum_{i,j=1}^k |i\rangle\delta_{ij}\langle j| \\ &= \sum_{i=1}^k |i\rangle\langle i| \\ &= P \end{aligned}$$

**Exercise 2.17**

$$\begin{aligned} A &= A^\dagger \\ \sum_i \lambda_i |i\rangle\langle i| &= \left( \sum_i \lambda_i |i\rangle\langle i| \right)^\dagger \\ &= \sum_i \lambda_i^* |i\rangle\langle i| \end{aligned}$$

$\lambda_i = \lambda_i^*$  implies the eigenvalues are real.

### Exercise 2.18

$$U^\dagger U = I$$

$$\left( \sum_i \lambda_i |i\rangle \langle i| \right)^\dagger \left( \sum_j \lambda_j |j\rangle \langle j| \right) = \sum_k |k\rangle \langle k|$$

$$\sum_{ij} \lambda_i^* \lambda_j |i\rangle \langle i|j\rangle \langle j| = \sum_k |k\rangle \langle k|$$

$$\sum_{ij} \lambda_i^* \lambda_j |i\rangle \delta_{ij} \langle j| = \sum_k |k\rangle \langle k|$$

$$\sum_i |\lambda_i|^2 |i\rangle \langle i| = \sum_k |k\rangle \langle k|$$

$$|\lambda_i|^2 = 1$$

$$\lambda_i = e^{i\theta}$$

**Exercise 2.19**

$$\begin{aligned} I^\dagger &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^\dagger \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

$$\begin{aligned} I^\dagger I &= II \\ &= I \end{aligned}$$

$$\begin{aligned} X^\dagger &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^\dagger \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= X \end{aligned}$$

$$\begin{aligned} X^\dagger X &= XX \\ &= I \end{aligned}$$

$$\begin{aligned} Y^\dagger &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^\dagger \\ &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ &= Y \end{aligned}$$

$$\begin{aligned} Y^\dagger Y &= YY \\ &= I \end{aligned}$$

$$\begin{aligned} Z^\dagger &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^\dagger \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= Z \end{aligned}$$

$$\begin{aligned} Z^\dagger Z &= ZZ \\ &= I \end{aligned}$$

### Exercise 2.22

$$\begin{aligned}\langle v_1|A|v_2\rangle &= \langle v_1|Av_2\rangle \\ &= \langle v_1|\lambda_2 v_2\rangle \\ &= \lambda_2 \langle v_1|v_2\rangle \\ \langle v_1|A|v_2\rangle &= \langle A^\dagger v_1|v_2\rangle \\ &= \langle Av_1|v_2\rangle \\ &= \langle \lambda_1 v_1|v_2\rangle \\ &= \lambda_1 \langle v_1|v_2\rangle \\ 0 &= (\lambda_2 - \lambda_1) \langle v_1|v_2\rangle \\ &= \langle v_1|v_2\rangle\end{aligned}$$

### Exercise 2.23

For each basis vector  $|i\rangle, i = 1, \dots, k$ ,  $P|i\rangle = |i\rangle$  and so they are eigenvectors of  $P$  with eigenvalue 1. For each basis vector  $|j\rangle, j = k+1, \dots, d$ ,  $P|j\rangle = 0$  and so they are eigenvectors of  $P$  with eigenvalue of 0. That is a total of  $d$  eigenvectors so all eigenvalues are either 0 or 1.



### Exercise 2.26

$$\begin{aligned}
|\psi\rangle &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\
|\psi\rangle^{\otimes 2} &= \left( \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left( \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \\
&= \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} \\
|\psi\rangle^{\otimes 3} &= \left( \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} \right) \left( \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \\
&= \frac{|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle}{2^{3/2}} \\
|\psi\rangle &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
|\psi\rangle^{\otimes 2} &= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
|\psi\rangle^{\otimes 3} &= \frac{1}{2^{3/2}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}$$

### Exercise 2.27

(a)

$$\begin{aligned}
X &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
Z &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
X \otimes Z &= \begin{bmatrix} (0)Z & (1)Z \\ (1)Z & (0)Z \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}
\end{aligned}$$

(b)

$$I \otimes X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(c)

$$X \otimes I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

No, the tensor product is not commutative.

### Exercise 2.28

$$\begin{aligned} (A \otimes B)^* &= \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{bmatrix}^* \\ &= \begin{bmatrix} A_{11}^*B^* & A_{12}^*B^* & \cdots & A_{1n}^*B^* \\ A_{21}^*B^* & A_{22}^*B^* & \cdots & A_{2n}^*B^* \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}^*B^* & A_{m2}^*B^* & \cdots & A_{mn}^*B^* \end{bmatrix} \\ &= A^* \otimes B^* \\ (A \otimes B)^T &= \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{bmatrix}^T \\ &= \begin{bmatrix} A_{11}B^T & A_{21}B^T & \cdots & A_{m1}B^T \\ A_{12}B^T & A_{22}B^T & \cdots & A_{m2}B^T \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n}B^T & A_{2n}B^T & \cdots & A_{mn}B^T \end{bmatrix} \\ &= A^T \otimes B^T \\ (A \otimes B)^\dagger &= [(A \otimes B)^*]^T \\ &= (A^* \otimes B^*)^T \\ &= (A^*)^T \otimes (B^*)^T \\ &= A^\dagger \otimes B^\dagger \end{aligned}$$

**Exercise 2.29**

$$\begin{aligned}
(A \otimes B)^\dagger (A \otimes B)(|a\rangle \otimes |b\rangle) &= (A^\dagger \otimes B^\dagger)(A |a\rangle \otimes B |b\rangle) \\
&= A^\dagger A |a\rangle \otimes B^\dagger B |b\rangle \\
&= |a\rangle \otimes |b\rangle
\end{aligned}$$

**Exercise 2.30**

$$\begin{aligned}
(A \otimes B)^\dagger (|a\rangle \otimes |b\rangle) &= (A^\dagger \otimes B^\dagger)(|a\rangle \otimes |b\rangle) \\
&= (A \otimes B)(|a\rangle \otimes |b\rangle)
\end{aligned}$$

**Exercise 2.34**

$$\begin{aligned}
A &= \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \\
\lambda_1 &= 1 \\
\mathbf{x}_1 &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
\lambda_2 &= 7 \\
\mathbf{x}_2 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\
A &= PDP^{-1} \\
D &= P^{-1}AP \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \\
\sqrt{A} &= P\sqrt{D}P^{-1} \\
&= \frac{1}{2} \begin{bmatrix} 1 + \sqrt{7} & -1 + \sqrt{7} \\ -1 + \sqrt{7} & 1 + \sqrt{7} \end{bmatrix} \\
\ln A &= P \ln(D) P^{-1} \\
&= \frac{\ln 7}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\end{aligned}$$

### Exercise 2.35

$$\begin{aligned}
\mathbf{v} &= (v_x, v_y, v_z) \\
\mathbf{v} \cdot \boldsymbol{\sigma} &= v_x \sigma_1 + v_y \sigma_2 + v_z \sigma_3 \\
&= \begin{bmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{bmatrix} \\
A &= i\theta \mathbf{v} \cdot \boldsymbol{\sigma} \\
\lambda_1 &= -i\theta v \\
\mathbf{x}_1 &= \begin{bmatrix} (v_z - v)/(v_x + iv_y) \\ 1 \end{bmatrix} \\
\lambda_2 &= i\theta v \\
\mathbf{x}_2 &= \begin{bmatrix} (v_z + v)/(v_x + iv_y) \\ 1 \end{bmatrix} \\
P &= [\mathbf{x}_1 \quad \mathbf{x}_2] \\
\exp(i\theta \mathbf{v} \cdot \boldsymbol{\sigma}) &= P \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix} P^{-1} \\
&= \begin{bmatrix} \cos \theta + iv_z \sin \theta & i(v_x - iv_y) \sin \theta \\ i(v_x + iv_y) \sin \theta & \cos \theta - iv_z \sin \theta \end{bmatrix} \\
&= \cos \theta I + i \sin \theta \begin{bmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{bmatrix} \\
&= \cos \theta I + i \sin \theta \mathbf{v} \cdot \boldsymbol{\sigma}
\end{aligned}$$

### Exercise 2.36

$$\begin{aligned}
\text{tr } I &= 1 + 1 \\
&= 2 \\
\text{tr } X &= 0 + 0 \\
&= 0 \\
\text{tr } Y &= 0 + 0 \\
&= 0 \\
\text{tr } Z &= 1 - 1 \\
&= 0
\end{aligned}$$

**Exercise 2.37**

$$\begin{aligned}
\operatorname{tr}(AB) &= \sum_i \langle i|AB|i\rangle \\
&= \sum_i \langle i|AIB|i\rangle \\
&= \sum_i \langle i|Ai\rangle \langle i|B|i\rangle \\
&= \sum_i \langle i|B|i\rangle \langle i|A|i\rangle \\
&= \sum_i \langle i|BIA|i\rangle \\
&= \sum_i \langle i|BA|i\rangle \\
&= \operatorname{tr}(BA)
\end{aligned}$$

**Exercise 2.38**

$$\begin{aligned}
\operatorname{tr}(A+B) &= \sum_i (A_{ii} + B_{ii}) \\
&= \sum_i A_{ii} + \sum_i B_{ii} \\
&= \operatorname{tr} A + \operatorname{tr} B \\
\operatorname{tr}(zA) &= \sum_i zA_{ii} \\
&= z \sum_i A_{ii} \\
&= z \operatorname{tr} A
\end{aligned}$$

**Exercise 2.42**

$$\begin{aligned}
\frac{[A, B] + \{A, B\}}{2} &= \frac{1}{2}(AB - BA + AB + BA) \\
&= AB
\end{aligned}$$

**Exercise 2.44**

$$\begin{aligned}[A, B] &= 0 \\ AB - BA &= 0 \\ \{A, B\} &= 0 \\ AB + BA &= 0 \\ AB - BA + AB + BA &= 0 \\ 2AB &= 0 \\ AB &= 0 \\ A^{-1}AB &= 0 \\ B &= 0\end{aligned}$$

**Exercise 2.45**

$$\begin{aligned}[A, B]^{\dagger} &= (AB - BA)^{\dagger} \\ &= (AB)^{\dagger} - (BA)^{\dagger} \\ &= B^{\dagger}A^{\dagger} - A^{\dagger}B^{\dagger} \\ &= [B^{\dagger}, A^{\dagger}]\end{aligned}$$

**Exercise 2.46**

$$\begin{aligned}[A, B] &= AB - BA \\ &= -(BA - AB) \\ &= -[B, A]\end{aligned}$$

**Exercise 2.47**

$$\begin{aligned}(i[A, B])^{\dagger} &= -i[B^{\dagger}, A^{\dagger}] \\ &= -i[B, A] \\ &= i[A, B]\end{aligned}$$