

Advanced Engineering Mathematics Ordinary Differential Equations Notes

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1 Introduction to Differential Equations

1.1 Definitions and Terminology

1.1.1 1

2, linear

1.1.2 3

4, linear

1.1.3 5

2, nonlinear

1.1.4 7

3, linear

1.1.5 9

no; yes

1.1.6 15

The domain of the function is $x \in [-2, \infty)$.

$$y' = 1 + \frac{2}{\sqrt{x+2}}$$

The largest interval of definition of the solution is $x \in (-2, \infty)$.

$$\begin{aligned}(y-x)y' &= y-x+8 \\ (x+4\sqrt{x+2}-x)(1+\frac{2}{\sqrt{x+2}}) &= x+4\sqrt{x+2}-x+8 \\ 4\sqrt{x+2}+8 &= 4\sqrt{x+2}+8\end{aligned}$$

1.1.7 17

The domain of the function is $x \in \mathbb{R}, x \neq \pm 2$.

$$y' = \frac{2x}{(4-x^2)^2}$$

The largest intervals of definition of the solution are $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

$$\begin{aligned}
y' &= 2xy^2 \\
\frac{2x}{(4-x^2)^2} &= 2x \left(\frac{1}{4-x^2} \right)^2 \\
&= \frac{2x}{(4-x^2)^2}
\end{aligned}$$

1.1.8 19

$$\begin{aligned}
\ln \frac{2X-1}{X-1} &= t \\
2X-1 &= (X-1)e^t \\
(2-e^t)X &= 1-e^t \\
X &= \frac{e^t-1}{e^t-2}
\end{aligned}$$

The solutions intervals of validity are $(\infty, \ln 2)$ and $(\ln 2, \infty)$.

$$\begin{aligned}
\frac{dX}{dt} &= (X-1)(1-2X) \\
\frac{e^t}{e^t-2} - \frac{e^t(e^t-1)}{(e^t-2)^2} &= \left(\frac{e^t-1}{e^t-2} - 1 \right) \left(1 - 2\frac{e^t-1}{e^t-2} \right) \\
\frac{e^t(e^t-2) - e^t(e^t-1)}{(e^t-2)^2} &= \left(\frac{e^t-1-e^t+2}{e^t-2} \right) \left(\frac{e^t-2-2e^t+2}{e^t-2} \right) \\
\frac{e^{2t}-2e^t-e^{2t}+e^t}{(e^t-2)^2} &= \left(\frac{1}{e^t-2} \right) \left(\frac{-e^t}{e^t-2} \right) \\
\frac{-e^t}{(e^t-2)^2} &= \frac{-e^t}{(e^t-2)^2}
\end{aligned}$$

1.1.9 31

$$m = -2$$

1.1.10 33

$$m = 2 \text{ or } 3$$

1.1.11 35

$$m = -1 \text{ or } 0$$

1.1.12 37

$$y = 2$$

1.1.13 39

No constant solutions

1.2 Initial Value Problems

1.2.1 1

$$\begin{aligned}y(0) &= -\frac{1}{3} = \frac{1}{1 + c_1 e^{-(0)}} \\-3 &= 1 + c_1 \\c_1 &= -4\end{aligned}$$

$$y = \frac{1}{1 - 4e^{-x}}$$

1.2.2 3

$$\begin{aligned}y(2) &= \frac{1}{3} = \frac{1}{(2)^2 + c} \\3 &= 4 + c \\c &= -1\end{aligned}$$

$$y = \frac{1}{x^2 - 1}$$

$$I = (1, \infty)$$

1.2.3 5

$$\begin{aligned}y(0) &= 1 = \frac{1}{(0)^2 + c} \\c &= 1\end{aligned}$$

$$y = \frac{1}{x^2 + 1}$$

$$I = (-\infty, \infty)$$

1.2.4 7

$$\begin{aligned}x(0) &= -1 = c_1 \cos 0 + c_2 \sin 0 \\c_1 &= -1\end{aligned}$$

$$\begin{aligned}x'(0) &= 8 = -c_1 \sin 0 + c_2 \cos 0 \\c_2 &= 8\end{aligned}$$

$$x = -\cos t + 8 \sin t$$

1.2.5 9

$$\begin{aligned}x'\left(\frac{\pi}{6}\right) &= 0 = -c_1 \sin \frac{\pi}{6} + c_2 \cos \frac{\pi}{6} \\&= -c_1 \frac{1}{2} + c_2 \frac{\sqrt{3}}{2} \\c_1 &= \sqrt{3}c_2\end{aligned}$$

$$\begin{aligned}x\left(\frac{\pi}{6}\right) &= \frac{1}{2} = c_1 \cos \frac{\pi}{6} + c_2 \sin \frac{\pi}{6} \\&= \frac{3}{2}c_2 + \frac{1}{2}c_2 \\&= 2c_2 \\c_2 &= \frac{1}{4}\end{aligned}$$

$$y = \frac{\sqrt{3}}{4} \cos t + \frac{1}{4} \sin t$$

1.2.6 11

$$\begin{aligned}y(0) &= 1 = c_1 e^{(0)} + c_2 e^{-(0)} \\&= c_1 + c_2 \\c_1 &= 1 - c_2\end{aligned}$$

$$\begin{aligned}y'(0) &= 2 = c_1 e^{(0)} - c_2 e^{-(0)} \\&= 1 - c_2 - c_2 \\c_2 &= -\frac{1}{2}\end{aligned}$$

$$y = \frac{3}{2}e^x - \frac{1}{2}e^{-x}$$

1.2.7 13

$$\begin{aligned}
y(-1) &= 5 = c_1 e^{(-1)} + c_2 e^{-(-1)} \\
&= c_1 e^{-1} + c_2 e \\
c_1 &= 5e - c_2 e^2
\end{aligned}$$

$$\begin{aligned}
y'(-1) &= -5 = c_1 e^{(-1)} - c_2 e^{-(-1)} \\
&= 5e - c_2 e^2 - c_2 e \\
c_2 e(e+1) &= 5(e+1) \\
c_2 &= \frac{5}{e}
\end{aligned}$$

$$y = 5e^{-x-1}$$

1.2.8 15

$$y = 0$$

$$y = x^3$$

1.2.9 17

$$f(x, y) = y^{2/3}$$

$$\frac{\partial f}{\partial y} = \frac{2}{3y^{1/3}}$$

$$y < 0 \text{ or } y > 0$$

1.2.10 19

$$f(x, y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$$x < 0 \text{ or } x > 0$$

1.2.11 21

$$f(x, y) = \frac{x^2}{4 - y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2x^2 y}{(4 - y^2)^2}$$

$$y < -2, -2 < y < 2, \text{ or } y > 2$$

1.2.12 23

$$f(x, y) = \frac{y^2}{x^2 + y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2} - \frac{2y^3}{(x^2 + y^2)^2}$$

$x \neq 0$ and $y \neq 0$

1.2.13 25

$$f(x, y) = \sqrt{y^2 - 9}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{y^2 - 9}}$$

Yes

1.2.14 27

No

1.2.15 29

(a) $y = cx$

(b)

$$f(x, y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$x \neq 0$

(c) No, the function is not differentiable at $x = 0$

1.2.16 31

(a)

$$y' = \frac{1}{(x + c)^2} = y^2$$

(b)

$$y(0) = 1 = -\frac{1}{(0) + c} \Rightarrow c = -1 \Rightarrow y = \frac{1}{1 - x}$$

$I = (-\infty, 1)$

$$y(0) = -1 = -\frac{1}{(0) + c} \Rightarrow c = 1 \Rightarrow y = -\frac{1}{x + 1}$$

$$I = (-1, \infty)$$

1.2.17 39

$$y(0) = 0 = c_1 \cos 3(0) + c_2 \sin 3(0)$$

$$c_1 = 0$$

$$y\left(\frac{\pi}{6}\right) = -1 = c_2 \sin 3\left(\frac{\pi}{6}\right)$$

$$c_2 = -1$$

$$y = -\sin 3x$$

1.2.18 41

$$y'(0) = 0 = -3c_1 \sin 3(0) + 3c_2 \cos 3(0)$$

$$c_2 = 0$$

$$y'\left(\frac{\pi}{4}\right) = 0 = -3c_1 \sin 3\left(\frac{\pi}{4}\right)$$

$$= -\frac{3}{\sqrt{2}}c_1$$

$$c_1 = 0$$

$$y = 0$$

1.2.19 43

$$y(0) = 0 = c_1 \cos 3(0) + c_2 \sin 3(0)$$

$$c_1 = 0$$

$$y(\pi) = 4 = c_2 \sin 3(\pi)$$

$$4 = 0$$

No solution

1.3 Differential Equations as Mathematical Models

1.3.1 1

$$\frac{dP}{dt} = kP + r$$

$$\frac{dP}{dt} = kP - r$$

1.3.2 3

$$\frac{dP}{dt} = k_b P - k_d P^2$$

1.3.3 7

$$\frac{dx}{dt} = kx(1000 - x)$$

1.3.4 9

$$\frac{dA}{dt} = -\frac{A}{100}$$

$$A(0) = 50 \text{ lb}$$

1.3.5 11

$$\frac{dA}{dt} + \frac{7}{600 - t} A = 6$$

1.3.6 13

$$\begin{aligned}\frac{dV}{dt} &= -cA_h \sqrt{2gh} \\ A_w \frac{dh}{dt} &= -cA_h \sqrt{2gh} \\ \frac{dh}{dt} &= -\frac{cA_h \sqrt{2g}}{A_w} \sqrt{h} \\ &= -\frac{c\pi r_h^2 \sqrt{2g}}{A_w} \sqrt{h} \\ &= -\frac{c\pi}{430} \sqrt{h}\end{aligned}$$

1.3.7 15

$$L \frac{di}{dt} + Ri = E$$

1.3.8 17

$$m \frac{dv}{dt} = mg - kv^2$$

1.3.9 19

$$m \frac{d^2x}{dt^2} = -kx$$

1.3.10 21

$$\begin{aligned} \frac{d}{dt}(mv) &= R - kv \\ \frac{dm}{dt}v + m \frac{dv}{dt} &= R - kv - mg \end{aligned}$$

1.3.11 23

$$g = \frac{k}{R^2} \Rightarrow k = gR^2$$

$$\frac{d^2r}{dt^2} = -\frac{gR^2}{r^2}$$

1.3.12 25

$$\frac{dA}{dt} = k(M - A)$$

1.3.13 27

$$\frac{dx}{dt} = r - kx$$

1.3.14 29

$$\begin{aligned} \frac{dy}{dx} &= \tan \theta \\ &= \tan \frac{\phi}{2} \\ &= \frac{1 - \cos \phi}{\sin \phi} \\ &= \frac{1 - x/r}{y/r} \\ &= \frac{r - x}{y} \\ &= \frac{\sqrt{x^2 + y^2} - x}{y} \end{aligned}$$

1.4 Chapter in Review

1.4.1 1

$$\frac{dy}{dx} = ky$$

1.4.2 3

$$y'' + k^2y = 0$$

1.4.3 5

$$y = c_1e^x + c_2xe^x$$

$$\begin{aligned}y' &= c_1e^x + c_2e^x + c_2xe^x \\ &= y + c_2e^x\end{aligned}$$

$$\begin{aligned}y'' &= c_1e^x + c_2e^x + c_2e^x + c_2xe^x \\ &= c_1e^x + 2c_2e^x + c_2xe^x \\ &= y' + c_2e^x\end{aligned}$$

$$y'' - 2y' + y = 0$$

1.4.4 7

a, d

1.4.5 9

b

1.4.6 11

b

1.4.7 13

$$y = ce^x$$

1.4.8 15

$$\frac{dy}{dx} = x^2 + y^2$$

1.4.9 17(a) $(-\infty, \infty)$ (b) $(-\infty, 0)$ or $(0, \infty)$ **1.4.10 19** $x_0 = -1$ and $I = (-\infty, 0)$ or $x_0 = 2$ and $I = (0, \infty)$ **1.4.11 23**

$$y = x \sin x + x \cos x$$

$$y' = \sin x + x \cos x + \cos x - x \sin x$$

$$\begin{aligned} y'' &= \cos x + \cos x - x \sin x - \sin x - \sin x - x \cos x \\ &= 2 \cos x - 2 \sin x - x \sin x - x \cos x \end{aligned}$$

$$\begin{aligned} y'' + y &= 2 \cos x - 2 \sin x - x \sin x - x \cos x + x \sin x + x \cos x \\ &= 2 \cos x - 2 \sin x \end{aligned}$$

$$I = (-\infty, \infty)$$

1.4.12 25

$$y = \sin(\ln x)$$

$$y' = \frac{1}{x} \cos(\ln x)$$

$$y'' = -\frac{1}{x^2} \cos(\ln x) - \frac{1}{x^2} \sin(\ln x)$$

$$\begin{aligned} x^2 y'' + x y' + y &= -\cos(\ln x) - \sin(\ln x) + \cos(\ln x) + \sin(\ln x) \\ &= 0 \end{aligned}$$

$$I = (0, \infty)$$

1.4.13 35

$$\begin{aligned}
y(0) = 0 &= c_1 e^{-3(0)} + c_2 e^{(0)} + 4(0) \\
&= c_1 + c_2 \\
c_1 &= -c_2
\end{aligned}$$

$$\begin{aligned}
y'(0) = 0 &= -3c_1 e^{-3(0)} + c_2 e^{(0)} + 4 \\
&= -3c_1 + c_2 + 4 \\
c_2 &= 3c_1 - 4
\end{aligned}$$

$$c_1 = -(3c_1 - 4) \Rightarrow c_1 = 1 \Rightarrow c_2 = -1$$

$$y = e^{-3x} - e^x + 4x$$

1.4.14 37

$$\begin{aligned}
y(1) = -2 &= c_1 e^{-3(1)} + c_2 e^{(1)} + 4(1) \\
&= c_1 e^{-3} + c_2 e + 4 \\
c_1 &= -e^3(c_2 e + 6)
\end{aligned}$$

$$\begin{aligned}
y'(1) = 4 &= -3c_1 e^{-3(1)} + c_2 e^{(1)} + 4 \\
&= -3c_1 e^{-3} + c_2 e + 4 \\
c_2 e &= 3c_1 e^{-3}
\end{aligned}$$

$$c_1 = -e^3(3c_1 e^{-3} + 6) = -3c_1 - 6e^3 \Rightarrow c_1 = -\frac{3}{2}e^3 \Rightarrow c_2 = -\frac{9}{2}e^{-1}$$

$$y = -\frac{3}{2}e^{3(1-x)} - \frac{9}{2}e^{x-1} + 4x$$

1.4.15 41

$$y_0 = -3, y_1 = 0$$

1.4.16 43

$$\begin{aligned}\frac{d}{dt}(mv) &= F - mg \\ \frac{d}{dt}\left(\lambda x \frac{dx}{dt}\right) &= F - \lambda xg \\ x \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + gx &= \frac{F}{\lambda} \\ x \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 32x &= 5\end{aligned}$$

2 First-Order Differential Equations

2.1 Solution Curves Without a Solution

2.1.1 21

0 is stable, 3 is unstable

2.1.2 23

2 is semi-stable

2.1.3 25

-2 is unstable, 0 is semi-stable, 2 is stable

2.1.4 27

-1 is stable, 0 is unstable

2.1.5 39

$$P_0 < h/k$$

2.1.6 41

$$g - \frac{k}{m}v^2 = 0 \Rightarrow v = \sqrt{\frac{gm}{k}}$$

2.2 Separable Equations

2.2.1 1

$$\begin{aligned}\frac{dy}{dx} &= \sin 5x \\ y &= -\frac{1}{5} \cos 5x + c\end{aligned}$$

2.2.2 3

$$\begin{aligned}
 dx + e^{3x} dy &= 0 \\
 e^{-3x} dx + dy &= 0 \\
 -\frac{1}{3}e^{-3x} + y &= c \\
 y &= \frac{1}{3}e^{-3x} + c
 \end{aligned}$$

2.2.3 5

$$\begin{aligned}
 x \frac{dy}{dx} &= 4y \\
 \frac{1}{4y} dy &= \frac{1}{x} dx \\
 \frac{1}{4} \ln |4y| &= \ln |x| + c \\
 \ln |4y| &= 4 \ln |x| + c \\
 4y &= e^{4 \ln |x| + c} \\
 &= c \left(e^{\ln |x|} \right)^4 \\
 y &= cx^4
 \end{aligned}$$

2.2.4 7

$$\begin{aligned}
 \frac{dy}{dx} &= e^{3x+2y} \\
 &= e^{3x} e^{2y} \\
 e^{-2y} dy &= e^{3x} dx \\
 -\frac{1}{2}e^{-2y} &= \frac{1}{3}e^{3x} + c \\
 -3e^{-2y} &= 2e^{3x} + c
 \end{aligned}$$

2.2.5 9

$$y \ln x \frac{dx}{dy} = \left(\frac{y+1}{x} \right)^2$$

$$x^2 \ln x \, dx = \frac{(y+1)^2}{y} \, dy$$

$$x^3 \left(\frac{\ln x}{3} - \frac{1}{9} \right) = \frac{1}{2} y(y+4) + \ln y + c$$

$$\frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 = \frac{1}{2} y^2 + 2y + \ln y + c$$

2.2.6 11

$$\csc y \, dx + \sec^2 x \, dy = 0$$

$$\frac{1}{\sin y} \, dx + \frac{1}{\cos^2 x} \, dy = 0$$

$$\cos^2 x \, dx + \sin y \, dy = 0$$

$$\frac{1}{2} (1 + \cos 2x) \, dx + \sin y \, dy = 0$$

$$\frac{1}{2} \left(x + \frac{1}{2} \sin 2x \right) - \cos y + c = 0$$

$$4 \cos y = 2x + \sin 2x + c$$

2.2.7 13

$$(e^y + 1)^2 e^{-y} \, dx + (e^x + 1)^3 e^{-x} \, dy = 0$$

$$\frac{e^x}{(e^x + 1)^3} \, dx + \frac{e^y}{(e^y + 1)^2} \, dy = 0$$

$$-\frac{1}{2(e^x + 1)^2} - \frac{1}{e^y + 1} = c$$

$$(e^x + 1)^{-2} + 2(e^y + 1)^{-1} = c$$

2.2.8 15

$$\frac{dS}{dr} = kS$$

$$\frac{1}{S} \, dS = k \, dr$$

$$\ln |S| = kr + c$$

$$S = c e^{kr}$$

2.2.9 17

$$\begin{aligned}\frac{dP}{dt} &= P - P^2 \\ \frac{1}{P(1-P)} dP &= dt \\ \ln \frac{P}{1-P} &= t + c \\ \frac{P}{1-P} &= ce^t \\ P &= ce^t(1-P) \\ P &= \frac{ce^t}{1+ce^t}\end{aligned}$$

2.2.10 19

$$\begin{aligned}\frac{dy}{dx} &= \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8} \\ &= \frac{(x-1)(y+3)}{(x+4)(y-2)} \\ \frac{y-2}{y+3} dt &= \frac{x-1}{x+4} dx \\ y - 5 \ln |y+3| &= x - 5 \ln |x+4| + c \\ e^{y-5 \ln |y+3|} &= e^{x-5 \ln |x+4|+c} \\ \frac{e^y}{(y+3)^5} &= \frac{ce^x}{(x+4)^5} \\ c(x+4)^5 e^y &= (y+3)^5 e^x\end{aligned}$$

2.2.11 21

$$\begin{aligned}\frac{dy}{dx} &= x\sqrt{1-y^2} \\ (1-y^2)^{-1/2} dy &= x dx \\ \arcsin y &= \frac{1}{2}x^2 + c \\ y &= \sin \left(\frac{1}{2}x^2 + c \right)\end{aligned}$$

2.2.12 23

$$\begin{aligned}\frac{dx}{dt} &= 4(x^2 + 1) \\ \frac{1}{x^2 + 1} dx &= 4 dt \\ \arctan x &= 4t + c \\ x &= \tan(4t + c)\end{aligned}$$

$$\begin{aligned}x\left(\frac{\pi}{4}\right) &= 1 = \tan\left(4\left(\frac{\pi}{4}\right) + c\right) \\ &= \tan(\pi + c) \\ c &= \arctan(1) - \pi \\ &= -\frac{3}{4}\pi\end{aligned}$$

$$x = \tan\left(4t - \frac{3}{4}\pi\right)$$

2.2.13 25

$$\begin{aligned}x^2 \frac{dy}{dx} &= y - xy \\ &= y(1 - x) \\ \frac{1}{y} dy &= \left(\frac{1}{x^2} - \frac{1}{x}\right) dx \\ \ln |y| &= -\frac{1}{x} - \ln |x| + c \\ y &= e^{-\frac{1}{x} - \ln |x| + c} \\ &= \frac{c}{xe^{1/x}}\end{aligned}$$

$$\begin{aligned}y(-1) &= -1 = \frac{c}{(-1)e^{1/(-1)}} \\ &= -ce \\ c &= e^{-1}\end{aligned}$$

$$y = \frac{1}{xe^{1+1/x}}$$

2.2.14 29

$$\begin{aligned}\frac{dy}{dx} &= ye^{-x^2} \\ \frac{1}{y} \frac{dy}{dx} &= e^{-x^2} \\ \int_4^x \frac{1}{y} \frac{dy}{dx'} dx' &= \int_4^x e^{-x'^2} dx' \\ \ln |y|_4^x &= \int_4^x e^{-x'^2} dx' \\ \ln |y(x)| - \ln |y(4)| &= \int_4^x e^{-x'^2} dx' \\ \ln |y(x)| &= \ln |y(4)| + \int_4^x e^{-x'^2} dx' \\ y(x) &= e^{\int_4^x e^{-x'^2} dx'}\end{aligned}$$

2.2.15 31

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x+1}{2y} \\ 2y dy &= (2x+1) dx \\ y^2 &= x^2 + x + c \\ y &= \pm \sqrt{x^2 + x + c} \\ y(-2) &= -1 = -\sqrt{(-2)^2 + (-2) + c} \\ &= -\sqrt{2+c} \\ c &= -1 \\ y &= -\sqrt{x^2 + x - 1} \\ I &= \left(-\infty, -\frac{1-\sqrt{5}}{2} \right)\end{aligned}$$

2.2.16 33

$$e^y dx - e^{-x} dy = 0$$

$$e^x dx - e^{-y} dy = 0$$

$$e^x + e^{-y} = c$$

$$\ln |e^{-y}| = \ln |c - e^x|$$

$$y = -\ln |c - e^x|$$

$$y(0) = 0 = -\ln |c - e^{(0)}|$$

$$1 = c - 1$$

$$c = 2$$

$$y = -\ln |2 - e^x|$$

$$I = (-\infty, \ln 2)$$

2.3 Linear Equations

2.3.1 1

$$\frac{dy}{dx} = 5y$$

$$\ln |y| = 5x + c$$

$$y = ce^{5x}$$

$$I = (-\infty, \infty)$$

2.3.2 3

$$\frac{dy}{dx} + y = e^{3x}$$

$$e^x \frac{dy}{dx} + e^x y = e^{4x}$$

$$\frac{d}{dx}(e^x y) = e^{4x}$$

$$e^x y = \frac{1}{4} e^{4x} + c$$

$$y = \frac{1}{4} e^{3x} + ce^{-x}$$

$$I = (-\infty, \infty)$$

2.3.3 5

$$\begin{aligned}
 y' + 3x^2y &= x^2 \\
 e^{x^3}y' + 3x^2e^{x^3}y &= e^{x^3}x^2 \\
 e^{x^3}y &= \frac{1}{3}e^{x^3} + c \\
 y &= \frac{1}{3} + ce^{-x^3}
 \end{aligned}$$

$$I = (-\infty, \infty)$$

2.3.4 7

$$\begin{aligned}
 x^2y' + xy &= 1 \\
 y' + x^{-1}y &= x^{-2} \\
 e^{\ln x}y' + x^{-1}e^{\ln x}y &= e^{\ln x}x^{-2} \\
 \frac{d}{dx}(e^{\ln x}y) &= x^{-1} \\
 \frac{d}{dx}(xy) &= x^{-1} \\
 xy &= \ln x + c \\
 y &= \frac{\ln x + c}{x}
 \end{aligned}$$

$$I = (0, \infty)$$

2.3.5 9

$$\begin{aligned}
 x \frac{dy}{dx} - y &= x^2 \sin x \\
 \frac{dy}{dx} - x^{-1}y &= x \sin x \\
 e^{-\ln x} \frac{dy}{dx} - x^{-1}e^{-\ln x}y &= e^{-\ln x}x \sin x \\
 \frac{d}{dx}(e^{-\ln x}y) &= \sin x \\
 x^{-1}y &= -\cos x + c \\
 y &= cx - x \cos x
 \end{aligned}$$

$$I = (0, \infty)$$

2.3.6 11

$$\begin{aligned}
 x \frac{dy}{dx} + 4y &= x^3 - x \\
 \frac{dy}{dx} + 4x^{-1}y &= x^2 - 1 \\
 e^{4 \ln x} \frac{dy}{dx} + 4x^{-1}e^{4 \ln x}y &= e^{4 \ln x}(x^2 - 1) \\
 \frac{d}{dx}(e^{4 \ln x}y) &= x^6 - x^4 \\
 x^4y &= \frac{1}{7}x^7 - \frac{1}{5}x^5 + c \\
 y &= \frac{1}{7}x^3 - \frac{1}{5}x^2 + cx^{-4}
 \end{aligned}$$

$$I = (0, \infty)$$

2.3.7 13

$$\begin{aligned}
 x^2y' + x(x+2)y &= e^x \\
 y' + x^{-1}(x+2)y &= x^{-2}e^x \\
 e^{x+2 \ln x}y' + x^{-1}(x+2)e^{x+2 \ln x}y &= e^{x+2 \ln x}x^{-2}e^x \\
 \frac{d}{dx}(e^x x^2 y) &= e^{2x} \\
 e^x x^2 y &= \frac{1}{2}e^{2x} + c \\
 y &= \frac{e^x}{2x^2} + \frac{c}{e^x x^2}
 \end{aligned}$$

$$I = (0, \infty)$$

2.3.8 15

$$\begin{aligned}
 y \, dx - 4(x + y^6) \, dy &= 0 \\
 y \frac{dx}{dy} - 4x - 4y^6 &= 0 \\
 \frac{dx}{dy} - \frac{4}{y}x &= 4y^5 \\
 e^{-4 \ln y} \frac{dx}{dy} - \frac{4}{y} e^{-4 \ln y} x &= 4e^{-4 \ln y} y^5 \\
 \frac{d}{dy} (e^{-4 \ln y} x) &= 4y \\
 y^{-4} x &= 2y^2 + c \\
 x &= 2y^6 + cy^4
 \end{aligned}$$

$$I = (0, \infty)$$

2.3.9 17

$$\begin{aligned}
 \cos x \frac{dy}{dx} + (\sin x)y &= 1 \\
 \frac{dy}{dx} + (\tan x)y &= \sec x \\
 e^{\ln(\sec x)} \frac{dy}{dx} + (\tan x)e^{\ln(\sec x)}y &= e^{\ln(\sec x)} \sec x \\
 \frac{d}{dx} (e^{\ln(\sec x)}y) &= \sec^2 x \\
 y \sec x &= \tan x + c \\
 y &= \sin x + c \cos x
 \end{aligned}$$

$$I = (-\pi/2, \pi/2)$$

2.3.10 19

$$\begin{aligned}
 (x+1)\frac{dy}{dx} + (x+2)y &= 2xe^{-x} \\
 \frac{dy}{dx} + \frac{x+2}{x+1}y &= \frac{2xe^{-x}}{x+1} \\
 e^{x+\ln|x+1|}\frac{dy}{dx} + \frac{x+2}{x+1}e^{x+\ln|x+1|}y &= e^{x+\ln|x+1|}\frac{2xe^{-x}}{x+1} \\
 \frac{d}{dx}(e^{x+\ln|x+1|}y) &= 2x \\
 e^x(x+1)y &= x^2 + c \\
 y &= \frac{x^2 + c}{e^x(x+1)}
 \end{aligned}$$

$$I = (-1, \infty)$$

2.3.11 21

$$\begin{aligned}
 \frac{dr}{d\theta} + r \sec \theta &= \cos \theta \\
 e^{\ln|\sec \theta + \tan \theta|}\frac{dr}{d\theta} + e^{\ln|\sec \theta + \tan \theta|}r \sec \theta &= e^{\ln|\sec \theta + \tan \theta|}\cos \theta \\
 \frac{d}{d\theta}(e^{\ln|\sec \theta + \tan \theta|}r) &= 1 + \sin \theta \\
 (\sec \theta + \tan \theta)r &= \theta - \cos \theta + c \\
 r &= \frac{\theta - \cos \theta + c}{\sec \theta + \tan \theta}
 \end{aligned}$$

$$I = (-\pi/2, \pi/2)$$

2.3.12 23

$$\begin{aligned}
 x\frac{dy}{dx} + (3x+1)y &= e^{-3x} \\
 \frac{dy}{dx} + (3+x^{-1})y &= e^{-3x}x^{-1} \\
 e^{3x+\ln|x|}\frac{dy}{dx} + (3+x^{-1})e^{3x+\ln|x|}y &= 1 \\
 \frac{d}{dx}(e^{3x+\ln|x|}y) &= 1 \\
 e^{3x}xy &= x + c \\
 y &= \frac{x+c}{e^{3x}x}
 \end{aligned}$$

$$I = (0, \infty)$$

2.3.13 25

$$\begin{aligned}
 xy' + y &= e^x \\
 y' + x^{-1}y &= e^x x^{-1} \\
 e^{\ln|x|}y' + x^{-1}e^{\ln|x|}y &= e^x \\
 \frac{d}{dx}(e^{\ln|x|}y) &= e^x \\
 xy &= e^x + c \\
 y &= \frac{e^x + c}{x}
 \end{aligned}$$

$$\begin{aligned}
 y(1) = 2 &= \frac{e^{(1)} + c}{(1)} \\
 c &= 2 - e
 \end{aligned}$$

$$y = \frac{e^x + 2 - e}{x}$$

$$I = (0, \infty)$$

2.3.14 27

$$\begin{aligned}
 L\frac{di}{dt} + Ri &= E \\
 \frac{di}{dt} + \frac{R}{L}i &= \frac{E}{L} \\
 e^{Rt/L}\frac{di}{dt} + \frac{R}{L}e^{Rt/L}i &= \frac{E}{L}e^{Rt/L} \\
 \frac{d}{dt}(e^{Rt/L}i) &= \frac{E}{L}e^{Rt/L} \\
 e^{Rt/L}i &= \frac{E}{R}e^{Rt/L} + c \\
 i &= \frac{E}{R} + ce^{-Rt/L}
 \end{aligned}$$

$$\begin{aligned}
 i(0) = i_0 &= \frac{E}{R} + ce^{-R(0)/L} \\
 &= \frac{E}{R} + c \\
 c &= i_0 - \frac{E}{R}
 \end{aligned}$$

$$i = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right) e^{-Rt/L}$$

$$I = (-\infty, \infty)$$

2.3.15 53

$$\begin{aligned}\frac{dE}{dt} &= -\frac{1}{RC}E \\ \frac{1}{E} \frac{dE}{dt} &= -\frac{1}{RC} \\ \ln|E| &= -\frac{1}{RC}t + c \\ E &= ce^{-t/RC}\end{aligned}$$

$$\begin{aligned}E(4) &= E_0 = ce^{-(4)/RC} \\ c &= E_0 e^{4/RC}\end{aligned}$$

$$E = E_0 e^{(4-t)/RC}$$

2.4 Exact Equations

2.4.1 1

$$f(x, y) = x^2 - x + g(y)$$

$$\frac{\partial f}{\partial y} = g'(y) = 3y + 7$$

$$g(y) = \frac{3}{2}y^2 + 7y$$

$$x^2 - x + \frac{3}{2}y^2 + 7y = c$$

2.4.2 3

$$f(x, y) = \frac{5}{2}x^2 + 4xy + g(y)$$

$$4x + g'(y) = 4x - 8y^3 \Rightarrow g'(y) = -8y^3$$

$$g(y) = -2y^4$$

$$\frac{5}{2}x^2 + 4xy - 2y^4 = c$$

2.4.3 5

$$f(x, y) = x^2 y^2 - 3x + g(y)$$

$$2x^2 y + g'(y) = 2x^2 y + 4 \Rightarrow g'(y) = 4$$

$$g(y) = 4y$$

$$x^2 y^2 - 3x + 4y = c$$

2.4.4 7

Not exact

2.4.5 9

$$f(x, y) = \frac{1}{2}x^2 - xy^3 - y^2 \cos x + g(y)$$

$$-3xy^2 - 2y \cos x + g'(y) = -3xy^2 - 2y \cos x \Rightarrow g'(y) = 0$$

$$\frac{1}{2}x^2 - xy^3 - y^2 \cos x = c$$

2.4.6 11

Not exact

2.4.7 13

$$f(x, y) = xy + g(x)$$

$$y + g'(x) = -2xe^x + y - 6x^2 \Rightarrow g'(x) = -2xe^x - 6x^2$$

$$g(x) = -2e^x(x - 1) - 2x^3$$

$$xy - 2e^x(x - 1) - 2x^3 = c$$

2.4.8 21

$$f(x, y) = \frac{1}{3}(x + y)^3 + g(y)$$

$$(x + y)^2 + g'(y) = 2xy + x^2 - 1 \Rightarrow g'(y) = -y^2 - 1$$

$$g(y) = -\frac{1}{3}y^3 - y$$

$$\frac{1}{3}(x + y)^3 - \frac{1}{3}y^3 - y = c$$

$$\frac{1}{3}(1 + 1)^3 - \frac{1}{3}1^3 - 1 = c \Rightarrow c = \frac{4}{3}$$

$$x^3 + 3x^2y + 3xy^2 - 3y = 4$$

2.4.9 23

$$f(x, y) = 4ty + t^2 - 5t + g(y)$$

$$4t + g'(y) = 6y + 4t - 1 \Rightarrow g'(y) = 6y - 1$$

$$g(y) = 3y^2 - y$$

$$4ty + t^2 - 5t + 3y^2 - y = c$$

$$4(-1)(2) + (-1)^2 - 5(-1) + 3(2)^2 - (2) = c \Rightarrow c = 8$$

$$4ty + t^2 - 5t + 3y^2 - y = 8$$

2.4.10 27

$$3y^2 + 4kxy^3 = 3y^2 + 40xy^3 \Rightarrow k = 10$$

2.4.11 31

$$M_y = 4y$$

$$N_x = 2y$$

$$\frac{M_y - N_x}{N} = \frac{4y - 2y}{2xy} = \frac{1}{x}$$

$$\mu(x) = e^{\ln x} = x$$

$$(2xy^2 + 3x^2) dx + 2x^2y dy = 0$$

$$f(x, y) = x^2y^2 + x^3 + g(y)$$

$$2x^2y + g'(y) = 2x^2y \Rightarrow g'(y) = 0$$

$$x^2y^2 + x^3 = c$$

2.4.12 33

$$M_y = 6x$$

$$N_x = 18x$$

$$\frac{M_y - N_x}{N} = \frac{6x - 18x}{4y + 9x^2}$$

$$\frac{N_x - M_y}{M} = \frac{18x - 6x}{6xy} = \frac{2}{y}$$

$$\mu(y) = e^{2 \ln y} = y^2$$

$$6xy^3 dx + (4y^3 + 9x^2y^2) dy = 0$$

$$f(x, y) = 3x^2y^3 + g(y)$$

$$9x^2y^2 + g'(y) = 4y^3 + 9x^2y^2 \Rightarrow g'(y) = 4y^3$$

$$g(y) = y^4$$

$$3x^2y^3 + y^4 = c$$

2.4.13 37

$$M_y = 0$$

$$N_x = 2xy$$

$$\frac{N_x - M_y}{M} = \frac{2xy - 0}{x} = 2y$$

$$\mu(y) = e^{y^2}$$

$$e^{y^2} x \, dx + e^{y^2} (x^2 y + 4y) \, dy = 0$$

$$f(x, y) = \frac{1}{2} e^{y^2} x^2 + g(y)$$

$$y e^{y^2} x^2 + g'(y) = e^{y^2} (x^2 y + 4y) \Rightarrow g'(y) = 4e^{y^2} y$$

$$g(y) = 2e^{y^2}$$

$$\frac{1}{2} e^{y^2} x^2 + 2e^{y^2} = c$$

$$\frac{1}{2} e^{(0)^2} (4)^2 + 2e^{(0)^2} = c \Rightarrow c = 10$$

$$\frac{1}{2} e^{y^2} x^2 + 2e^{y^2} = 10$$

2.4.14 39

(c)

$$(0)^3 + 2(0)^2(-2) + (-2)^2 = c \Rightarrow c = 4$$

$$y^2 + 2x^2 y + x^3 - 4 = 0$$

$$\begin{aligned} y &= \frac{-(2x^2) \pm \sqrt{(2x^2)^2 - 4(1)(x^3 - 4)}}{2(1)} \\ &= \frac{-2x^2 \pm \sqrt{4x^4 - 4(x^3 - 4)}}{2} \\ &= -x^2 \pm \sqrt{x^4 - x^3 + 4} \end{aligned}$$

2.4.15 45

(a)

$$xv \frac{dv}{dx} + v^2 = 32x \Rightarrow xv \, dv + (v^2 - 32x) \, dx = 0$$

$$M_x = v$$

$$N_v = 2v$$

$$\frac{M_x - N_v}{N} = \frac{v - 2v}{v^2 - 32x}$$

$$\frac{N_v - M_x}{M} = \frac{2v - v}{xv} = \frac{1}{x}$$

$$\mu(x) = e^{\ln x} = x$$

$$x^2 v \, dv + (xv^2 - 32x^2) \, dx = 0$$

$$f(x, v) = \frac{1}{2}x^2v^2 + g(x)$$

$$xv^2 + g'(x) = xv^2 - 32x^2 \Rightarrow g'(x) = -32x^2$$

$$g(x) = -\frac{32}{3}x^3$$

$$\frac{1}{2}(3)^2(0)^2 - \frac{32}{3}(3)^3 = c \Rightarrow c = -288$$

$$\frac{1}{2}x^2v^2 - \frac{32}{3}x^3 = -288 \Rightarrow v = 8\sqrt{\frac{x}{3} - \frac{9}{x^2}}$$

(b) $v = 12.7 \text{ ft/s}$

2.5 Solutions by Substitution

2.5.1 1

$$\begin{aligned}(x - y) dx + x dy &= 0 \\(x - ux) dx + x(u dx + x du) &= 0 \\x dx + x^2 du &= 0 \\x^{-1} dx + du &= 0 \\\ln |x| + u &= c \\\ln |x| + \frac{y}{x} &= c \\y &= cx - x \ln |x|\end{aligned}$$

2.5.2 3

$$\begin{aligned}x dx + (y - 2x) dy &= 0 \\vy(v dy + y dv) + (y - 2vy) dy &= 0 \\(v^2 y + y - 2vy) dy + vy^2 dv &= 0 \\y(v^2 - 2v + 1) dy + vy^2 dv &= 0 \\(v - 1)^2 dy + vy dv &= 0 \\\frac{1}{y} dy + \frac{v}{(v - 1)^2} dv &= 0 \\\ln |y| + \frac{1}{1 - v} + \ln |v - 1| &= c \\\ln |y| + \frac{1}{1 - x/y} + \ln \left| \frac{x}{y} - 1 \right| &= c \\\ln |x - y| + \frac{y}{y - x} &= c \\(y - x) \ln |x - y| + y &= c(y - x) \\(x - y) \ln |x - y| &= y + c(x - y)\end{aligned}$$

2.5.3 5

$$\begin{aligned}
 (y^2 + yx) dx - x^2 dy &= 0 \\
 ((ux)^2 + ux^2) dx - x^2(u dx + x du) &= 0 \\
 u^2 x^2 dx - x^3 du &= 0 \\
 \frac{1}{x} dx - \frac{1}{u^2} du &= 0 \\
 \ln |x| + \frac{1}{u} &= c \\
 \ln |x| + \frac{x}{y} &= c \\
 y &= \frac{x}{c - \ln |x|}
 \end{aligned}$$

2.5.4 7

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{y - x}{y + x} \\
 (y + x) dy + (x - y) dx &= 0 \\
 (ux + x)(u dx + x du) + (x - ux) dx &= 0 \\
 (u^2 x + x) dx + (ux^2 + x^2) du &= 0 \\
 x(u^2 + 1) dx + x^2(u + 1) du &= 0 \\
 \frac{1}{x} dx + \frac{u + 1}{u^2 + 1} du &= 0 \\
 \ln |x| + \frac{1}{2} \ln |u^2 + 1| + \arctan u &= c \\
 \ln |x^2 + y^2| + 2 \arctan \frac{y}{x} &= c
 \end{aligned}$$

2.5.5 9

$$\begin{aligned}
-y \, dx + (x + \sqrt{xy}) \, dy &= 0 \\
-ux \, dx + (x + \sqrt{ux^2})(u \, dx + x \, du) &= 0 \\
u\sqrt{ux^2} \, dx + (x^2 + x\sqrt{ux^2}) \, du &= 0 \\
u^{3/2}x \, dx + x^2(1 + \sqrt{u}) \, du &= 0 \\
\frac{1}{x} \, dx + \frac{1 + \sqrt{u}}{u^{3/2}} \, du &= 0 \\
\frac{1}{x} \, dx + (u^{-3/2} + u^{-1}) \, du &= 0 \\
\ln |x| - 2u^{-1/2} + \ln |u| &= c \\
\ln |x| - 2(y/x)^{-1/2} + \ln |y/x| &= c \\
\ln |y| - 2\sqrt{\frac{x}{y}} &= c \\
4\frac{x}{y} &= (\ln |y| - c)^2 \\
4x &= y(\ln |y| - c)^2
\end{aligned}$$

2.5.6 11

$$\begin{aligned}
xy^2 \frac{dy}{dx} &= y^3 - x^3 \\
xy^2 \, dy + (x^3 - y^3) \, dx &= 0 \\
x(ux)^2(u \, dx + x \, du) + (x^3 - (ux)^3) \, dx &= 0 \\
x^3 \, dx + u^2 x^4 \, du &= 0 \\
x^{-1} \, dx + u^2 \, du &= 0 \\
\ln |x| + \frac{1}{3} u^3 &= c \\
\ln |x| + \frac{1}{3} \left(\frac{y}{x}\right)^3 &= c \\
\ln |1| + \frac{1}{3} \left(\frac{2}{1}\right)^3 = c \Rightarrow c &= \frac{8}{3} + \ln 1 \\
\ln |x| + \frac{1}{3} \left(\frac{y}{x}\right)^3 &= \frac{8}{3} \\
y^3 + 3x^3 \ln |x| &= 8x^3
\end{aligned}$$

2.5.7 13

$$\begin{aligned}
 (x + ye^{y/x}) dx - xe^{y/x} dy &= 0 \\
 (x + uxe^u) dx - xe^u(u dx + x du) &= 0 \\
 x dx - x^2 e^u du &= 0 \\
 x^{-1} dx - e^u du &= 0 \\
 \ln |x| - e^u &= c \\
 \ln |x| - e^{y/x} &= c
 \end{aligned}$$

$$\ln |1| - e^{0/1} = c \Rightarrow c = -1$$

$$\ln |x| = e^{y/x} - 1$$

2.5.8 15

$$\begin{aligned}
 x \frac{dy}{dx} + y &= \frac{1}{y^2} \\
 \frac{dy}{dx} + x^{-1}y &= x^{-1}y^{-2} \\
 u = y^{1-n} = y^3 &\Rightarrow y = u^{1/3} \Rightarrow \frac{dy}{dx} = \frac{1}{3}u^{-2/3} \frac{du}{dx}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{3}u^{-2/3} \frac{du}{dx} + x^{-1}u^{1/3} &= x^{-1}u^{-2/3} \\
 \frac{du}{dx} + 3x^{-1}u &= 3x^{-1} \\
 e^{3 \ln |x|} \frac{du}{dx} + 3x^{-1}e^{3 \ln |x|}u &= 3x^2 \\
 \frac{d}{dx}(x^3u) &= 3x^2 \\
 x^3u &= x^3 + c \\
 y^3 &= 1 + cx^{-3}
 \end{aligned}$$

2.5.9 17

$$\begin{aligned}
 \frac{dy}{dx} &= y(xy^3 - 1) \\
 \frac{dy}{dx} + y &= xy^4
 \end{aligned}$$

$$u = y^{1-n} = y^{-3} \Rightarrow y = u^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}u^{-4/3}\frac{du}{dx}$$

$$-\frac{1}{3}u^{-4/3}\frac{du}{dx} + u^{-1/3} = xu^{-4/3}$$

$$\frac{du}{dx} - 3u = -3x$$

$$e^{-3x}\frac{du}{dx} - 3e^{-3x}u = -3e^{-3x}x$$

$$\frac{d}{dt}(e^{-3x}u) = -3e^{-3x}x$$

$$e^{-3x}u = e^{-3x}x + \frac{1}{3}e^{-3x} + c$$

$$u = x + \frac{1}{3} + ce^{3x}$$

$$y^{-3} = x + \frac{1}{3} + ce^{3x}$$

2.5.10 21

$$x^2\frac{dy}{dx} - 2xy = 3y^4$$

$$\frac{dy}{dx} - 2x^{-1}y = 3x^{-2}y^4$$

$$u = y^{1-n} = y^{-3} \Rightarrow y = u^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}u^{-4/3}\frac{du}{dx}$$

$$-\frac{1}{3}u^{-4/3}\frac{du}{dx} - 2x^{-1}u^{-1/3} = 3x^{-2}u^{-4/3}$$

$$\frac{du}{dx} + 6x^{-1}u = -9x^{-2}$$

$$e^{6\ln|x|}\frac{du}{dx} + 6x^{-1}e^{6\ln|x|}u = -9e^{6\ln|x|}x^{-2}$$

$$\frac{d}{dx}(x^6u) = -9x^4$$

$$x^6u = -\frac{9}{5}x^5 + c$$

$$u = -\frac{9}{5}x^{-1} + cx^{-6}$$

$$y^{-3} = -\frac{9}{5}x^{-1} + cx^{-6}$$

$$\left(\frac{1}{2}\right)^{-3} = -\frac{9}{5}(1)^{-1} + c(1)^{-6} \Rightarrow c = \frac{49}{5}$$

$$y^{-3} = -\frac{9}{5}x^{-1} + \frac{49}{5}x^{-6}$$

2.5.11 23

Let $u = x + y + 1$ so $\frac{du}{dx} = 1 + \frac{dy}{dx}$ and

$$\begin{aligned}\frac{du}{dx} - 1 &= u^2 \\ \frac{1}{u^2 + 1} du &= dx \\ \arctan u &= x + c \\ \arctan(x + y + 1) &= x + c \\ x + y + 1 &= \tan(x + c) \\ y &= -x - 1 + \tan(x + c)\end{aligned}$$

2.5.12 25

Let $u = x + y$ so $\frac{du}{dx} = 1 + \frac{dy}{dx}$ and

$$\begin{aligned}\frac{du}{dx} - 1 &= \tan^2 u \\ \frac{1}{1 + \tan^2 u} du &= dx \\ \frac{1}{2}(u + \sin u \cos u) &= x + c \\ x + y + \sin(x + y) \cos(x + y) &= 2(x + c) \\ x + y + \frac{1}{2} \sin(2(x + y)) &= 2(x + c) \\ 2x + 2y + \sin(2(x + y)) &= 4(x + c) \\ 2y - 2x + \sin(2(x + y)) &= c\end{aligned}$$

2.5.13 35

(a) Let $y = y_1 + u$ so $\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{du}{dx}$ but $\frac{dy_1}{dx} = P(x) + Q(x)y_1 + R(x)y_1^2$ so

$$\begin{aligned}
\frac{dy}{dx} &= P(x) + Q(x)y + R(x)y^2 \\
P(x) + Q(x)y_1 + R(x)y_1^2 + \frac{du}{dx} &= P(x) + Q(x)(y_1 + u) + R(x)(y_1 + u)^2 \\
\frac{du}{dx} &= Q(x)u + R(x)(2y_1u + u^2) \\
\frac{du}{dx} - (Q(x) + 2R(x)y_1)u &= R(x)u^2
\end{aligned}$$

(b) Let $y = 2x^{-1} + u$ so $\frac{dy}{dx} = -2x^{-2} + \frac{du}{dx}$ and

$$\begin{aligned}
-\frac{2}{x^2} + \frac{du}{dx} &= -\frac{4}{x^2} - \frac{1}{x} \left(\frac{2}{x} + u \right) + \left(\frac{2}{x} + u \right)^2 \\
\frac{du}{dx} &= \frac{2}{x^2} - \frac{4}{x^2} - \frac{2}{x^2} - \frac{u}{x} + \frac{4}{x^2} + \frac{4u}{x} + u^2 \\
\frac{du}{dx} - \frac{3}{x}u &= u^2
\end{aligned}$$

Let $v = u^{1-n} = u^{-1}$ so $u = v^{-1}$ and $\frac{du}{dx} = -v^{-2} \frac{dv}{dx}$

$$\begin{aligned}
-v^{-2} \frac{dv}{dx} - \frac{3}{x}v^{-1} &= v^{-2} \\
\frac{dv}{dx} + \frac{3}{x}v &= -1 \\
e^{3 \ln |x|} \frac{dv}{dx} + \frac{3}{x}e^{3 \ln |x|}v &= -e^{3 \ln |x|} \\
\frac{d}{dt}(x^3v) &= -x^3 \\
x^3v &= -\frac{1}{4}x^4 + c \\
\frac{1}{y - y_1} &= -\frac{1}{4}x + cx^{-3} \\
y &= y_1 + \left(-\frac{1}{4}x + cx^{-3} \right)^{-1} \\
&= \frac{2}{x} + \left(-\frac{1}{4}x + cx^{-3} \right)^{-1}
\end{aligned}$$

2.5.14 37

$$\begin{aligned}
\frac{dP}{dt} &= P(a - bP) \\
\frac{dP}{dt} - aP &= -bP^2
\end{aligned}$$

Let $u = P^{1-n} = P^{-1}$ so $P = u^{-1}$ and $\frac{dP}{dt} = -u^{-2} \frac{du}{dt}$

$$-u^{-2} \frac{du}{dt} - au^{-1} = -bu^{-2}$$

$$\frac{du}{dt} + au = b$$

$$e^{at} \frac{du}{dt} + ae^{at} u = be^{at}$$

$$\frac{d}{dt}(e^{at} u) = be^{at}$$

$$e^{at} u = \frac{b}{a} e^{at} + c$$

$$P^{-1} = \frac{b}{a} + ce^{-at}$$

$$= \frac{b + ce^{-at}}{a}$$

$$P = \frac{a}{b + ce^{-at}}$$

2.6 A Numerical Method

2.6.1 1

$x_0 = 1$	$y_0 = 5$
$x_1 = 1.1$	$y_1 = y_0 + hf(x_0, y_0) = 3.8000$
$x_2 = 1.2$	$y_2 = y_1 + hf(x_1, y_1) = 2.9800$

$x_0 = 1$	$y_0 = 5$
$x_1 = 1.05$	$y_1 = y_0 + hf(x_0, y_0) = 4.4000$
$x_2 = 1.1$	$y_2 = y_1 + hf(x_1, y_1) = 3.8950$
$x_3 = 1.15$	$y_3 = y_2 + hf(x_2, y_2) = 3.4708$
$x_4 = 1.2$	$y_4 = y_3 + hf(x_3, y_3) = 3.1152$

2.7 Linear Models

2.7.1 1

$$P(t) = P_0 e^{kt}$$

$$P(5) = 2P_0 = P_0 e^{5k} \Rightarrow k = \frac{\ln 2}{5} = 0.139$$

$$P(t) = P_0 e^{0.139t}$$

$$3P_0 = P_0 e^{0.139t} \Rightarrow t = 7.9 \text{ years}$$

$$4P_0 = P_0 e^{0.139t} \Rightarrow t = 10 \text{ years}$$

2.7.2 5

$$A(t) = A_0 e^{kt}$$

$$A(3.3) = \frac{1}{2}A_0 = A_0 e^{3.3k} \Rightarrow k = -0.21$$

$$0.1A_0 = A_0 e^{-0.21t} \Rightarrow t = 11 \text{ hours}$$

2.7.3 9

$$\frac{dI}{dt} = kI \Rightarrow I(t) = ce^{kt}$$

$$I(3) = 0.25I_0 = I_0 e^{3k} \Rightarrow k = -0.462$$

$$I(15) = I_0 e^{-0.462(15)} = 0.001I_0$$

2.7.4 11

$$0.145A_0 = A_0 e^{-0.00012097t} \Rightarrow t = 15\,963 \text{ years}$$

2.7.5 13

$$\frac{dT}{dt} = k(T - T_m)$$

$$\frac{1}{T - T_m} \frac{dT}{dt} = k$$

$$\ln(T - T_m) = kt + c$$

$$T - T_m = ce^{kt}$$

$$T = T_m + ce^{kt}$$

$$= 10 + 60e^{kt}$$

$$T(0.5) = 50 = 10 + 60e^{0.5k} \Rightarrow k = -0.811$$

$$T(1) = 36.7$$

$$15 = 10 + 60e^{-0.811t} \Rightarrow t = 3.06 \text{ min}$$

2.7.6 21

$$\begin{aligned}\frac{dA}{dt} &= 4 - \frac{A}{50} \\ \frac{dA}{dt} + \frac{A}{50} &= 4 \\ \frac{d}{dt}(e^{t/50}A) &= 4e^{t/50} \\ e^{t/50}A &= 200e^{t/50} + c \\ A &= 200 + ce^{-t/50}\end{aligned}$$

$$A(0) = 30 = 200 + ce^{-(0)/50} \Rightarrow c = -170$$

$$A(t) = 200 - 170e^{-t/50}$$

2.7.7 25

$$V(t) = 500 - 5t$$

$$\begin{aligned}\frac{dA}{dt} &= 10 - \frac{10}{500 - 5t}A \\ \frac{dA}{dt} + \frac{10}{500 - 5t}A &= 10 \\ \frac{dA}{dt} - 2\frac{-5}{500 - 5t}A &= 10 \\ e^{-2 \ln(500-5t)} \frac{dA}{dt} - 2\frac{-5}{500 - 5t}e^{-2 \ln(500-5t)}A &= e^{-2 \ln(500-5t)}10 \\ \frac{d}{dt}(A(500 - 5t)^{-2}) &= 10(500 - 5t)^{-2} \\ A(500 - 5t)^{-2} &= \frac{2}{500 - 5t} + c \\ A &= 2(500 - 5t) + c(500 - 5t)^2 \\ &= 1000 - 10t + c(500 - 5t)^2\end{aligned}$$

$$A(0) = 0 \Rightarrow c = -0.004$$

$$A(t) = 1000 - 10t - 0.004(500 - 5t)^2 = 1000 - 10t - \frac{1}{10}(100 - t)^2$$

The tank is empty at $t = 100$

2.7.8 29

$$i(t) = \frac{3}{5} - \frac{3}{5}e^{-500t}$$

$$i \rightarrow \frac{3}{5} \text{ as } t \rightarrow \infty$$

2.7.9 33

$$i(t) = \begin{cases} 60(1 - e^{-t/10}), & 0 \leq t \leq 20 \\ 383e^{-t/10}, & t > 20 \end{cases}$$

2.7.10 35

(a)

$$m \frac{dv}{dt} = mg - kv$$

$$\frac{dv}{dt} + \frac{k}{m}v = g$$

$$\frac{d}{dt}(e^{kt/m}v) = e^{kt/m}g$$

$$e^{kt/m}v = \frac{gm}{k}e^{kt/m} + c$$

$$v = \frac{gm}{k} + ce^{-kt/m}$$

$$v(0) = v_0 = \frac{gm}{k} + ce^{-k(0)/m} \Rightarrow c = v_0 - \frac{gm}{k}$$

$$v(t) = \frac{gm}{k} + \left(v_0 - \frac{gm}{k}\right)e^{-kt/m}$$

(b)

$$v_t = \frac{gm}{k}$$

(c)

$$s(t) = \frac{gm}{k}t - \frac{m}{k}\left(v_0 - \frac{gm}{k}\right)e^{-kt/m} + c$$

$$s(0) = 0 = -\frac{m}{k}\left(v_0 - \frac{gm}{k}\right) + c \Rightarrow c = \frac{m}{k}\left(v_0 - \frac{gm}{k}\right)$$

$$\begin{aligned} s(t) &= \frac{m}{k}\left(gt - \left(v_0 + \frac{gm}{k}\right)e^{-kt/m} + v_0 - \frac{gm}{k}\right) \\ &= \frac{m}{k}\left(gt + \left(v_0 - \frac{gm}{k}\right)\left(1 - e^{-kt/m}\right)\right) \end{aligned}$$

2.7.11 41

(a)

$$\begin{aligned}\frac{dP}{dt} &= k_1P - k_2P \\ &= (k_1 - k_2)P \\ P &= ce^{(k_1 - k_2)t}\end{aligned}$$

2.7.12 43(a) $x = r/k$ **2.8 Nonlinear Models****2.8.1 1**(a) $N = 2000$

(b)

$$N = \frac{1}{0.0005 + (1 - 0.0005)e^{-t}}$$

$$N(10) = 1834$$

2.8.2 3

$$P = 1.0 \times 10^6$$

$$P = \frac{500}{0.0005 + (0.1 - 0.0005)e^{-0.1t}}$$

$$500000 = \frac{500}{0.0005 + (0.1 - 0.0005)e^{-0.1t}}$$

$$500000(0.0005 + (0.1 - 0.0005)e^{-0.1t}) = 500$$

$$e^{-0.1t} = \frac{0.001 - 0.0005}{0.1 - 0.0005}$$

$$t = 52.9 \text{ months}$$

2.8.3 11

29.3 g; 60 g; 0 g; 30 g

2.8.4 13

(a)

$$\begin{aligned}\frac{dh}{dt} &= -\frac{A_h}{A_w} \sqrt{2gh} \\ \frac{1}{\sqrt{h}} \frac{dh}{dt} &= -\frac{A_h}{A_w} \sqrt{2g} \\ 2\sqrt{h} &= -\frac{A_h}{A_w} \sqrt{2g}t + c \\ \sqrt{h} &= c - \frac{A_h}{A_w} \sqrt{\frac{g}{2}}t \\ h &= \left(c - \frac{A_h}{A_w} \sqrt{\frac{g}{2}}t\right)^2\end{aligned}$$

$$h(0) = H = c^2 \Rightarrow c = \sqrt{H}$$

$$h = \left(\sqrt{H} - \frac{A_h}{A_w} \sqrt{\frac{g}{2}}t\right)^2 = \left(\sqrt{H} - 4\frac{A_h}{A_w}t\right)^2$$

Interval of definition is $[0, \frac{A_w \sqrt{H}}{4A_h}]$

(b) 1821 s = 30 min

2.8.5 15

(a)

$$\begin{aligned}\frac{dh}{dt} &= -\frac{5}{6h^{3/2}} \\ h^{3/2} \frac{dh}{dt} &= -\frac{5}{6} \\ \frac{2}{5} h^{5/2} &= -\frac{5}{6}t + c \\ h &= \left(c - \frac{25}{12}t\right)^{2/5}\end{aligned}$$

$$h(0) = H = c^{2/5} \Rightarrow c = H^{5/2}$$

$$h = \left(H^{5/2} - \frac{25}{12}t\right)^{2/5}$$

$$0 = \left(H^{5/2} - \frac{25}{12}t\right)^{2/5} \Rightarrow t = \frac{12}{25}H^{5/2} = 858 \text{ s}$$

(b)

$$\begin{aligned}
V(h) &= \pi r^2 \frac{h}{3} \\
&= \pi \left(h \tan \frac{\pi}{6} \right)^2 \frac{h}{3} \\
&= \pi \left(\frac{h}{\sqrt{3}} \right)^2 \frac{h}{3} \\
&= \frac{1}{9} \pi h^3
\end{aligned}$$

$$\begin{aligned}
\frac{dV}{dt} &= -cA_h \sqrt{2gh} \\
\frac{d}{dt} \left(\frac{1}{9} \pi h^3 \right) &= -cA_h \sqrt{2gh} \\
\frac{1}{3} \pi h^2 \frac{dh}{dt} &= -cA_h \sqrt{2gh} \\
h^{3/2} \frac{dh}{dt} &= -\frac{24}{\pi} cA_h \\
\frac{2}{5} h^{5/2} &= c_1 - \frac{24}{\pi} cA_h t \\
h &= \left(c_1 - \frac{60}{\pi} cA_h t \right)^{2/5}
\end{aligned}$$

$$h(0) = H = c_1^{2/5} \Rightarrow c_1 = H^{5/2}$$

$$h = \left(H^{5/2} - \frac{60}{\pi} cA_h t \right)^{2/5}$$

$$0 = \left(H^{5/2} - \frac{60}{\pi} cA_h t \right)^{2/5}$$

$$\begin{aligned}
t &= \frac{\pi H^{5/2}}{60 cA_h} \\
&= 243 \text{ s}
\end{aligned}$$

2.8.6 17

(a)

$$\begin{aligned}
 m \frac{dv}{dt} &= mg - kv^2 \\
 \frac{m}{mg - kv^2} \frac{dv}{dt} &= 1 \\
 \sqrt{\frac{m}{gk}} \operatorname{arctanh} \left(\sqrt{\frac{k}{gm}} v \right) &= t + c_1 \\
 v &= \sqrt{\frac{gm}{k}} \tanh \left(\sqrt{\frac{gk}{m}} (t + c_1) \right)
 \end{aligned}$$

$$\begin{aligned}
 v(0) = v_0 &= \sqrt{\frac{gm}{k}} \tanh c_1 \\
 c_1 &= \operatorname{arctanh} \sqrt{\frac{k}{gm}} v_0
 \end{aligned}$$

(b) $v_t = \sqrt{gm/k}$

(c)

$$\begin{aligned}
 s &= \frac{m}{k} \ln \cosh \left(\sqrt{\frac{gk}{m}} t + c_1 \right) + c_2 \\
 c_2 &= -\frac{m}{k} \ln \cosh c_1
 \end{aligned}$$

2.8.7 21

(a) $W = 0, W = 2$

(b)

$$\begin{aligned}
 \frac{dW}{dt} &= W\sqrt{4 - 2W} \\
 \frac{1}{W\sqrt{4 - 2W}} \frac{dW}{dt} &= 1 \\
 -\operatorname{arctanh} \left(\frac{1}{2} \sqrt{4 - 2W} \right) &= t + c \\
 \frac{1}{2} \sqrt{4 - 2W} &= \tanh(c - t) \\
 W &= 2 - 2 \tanh^2(c - t) \\
 &= 2(1 - \tanh^2(c - t)) \\
 &= 2 \operatorname{sech}^2(c - t)
 \end{aligned}$$