

Vibrations and Waves by A. P. French Problems

Chris Doble

May 2022

Contents

1	Periodic motions	2
1.4	2
1.9	3
1.10	4
1.11	4
1.12	5
2	The superposition of periodic motions	5
2.1	5
2.2	6
2.3	7
2.4	7
3	The free vibrations of physical systems	7
3.1	7
3.2	7
3.3	8
3.4	9
3.5	9
3.6	9
3.8	9
3.9	11
3.10	12
3.11	12
3.14	12
3.15	13
3.16	14
3.17	15
3.19	17

4	Forced vibrations and resonance	19
4.3	19
4.4	19
4.6	21
4.8	22
4.9	24
4.11	24
4.12	25
4.13	26
4.14	26
4.15	27
4.17	27
5	Coupled oscillators and normal modes	28
5.2	28
5.4	29
5.5	30
5.7	31
5.8	33
5.9	36
5.10	38
5.11	39
5.13	41
5.15	42
6	Normal modes of continuous systems. Fourier analysis	42
6.1	42
6.2	43
6.5	44
6.6	44
6.9	45
6.10	45
6.11	46
6.12	47
6.14	47

1 Periodic motions

1.4

(a)

$$\begin{aligned}
 z &= Ae^{j\theta} \\
 dz &= jAe^{j\theta} d\theta \\
 &= jz d\theta
 \end{aligned}$$

The motion of the point is always perpendicular to its position.

(b)

$$\begin{aligned}
 |2 + j\sqrt{3}| &= \sqrt{2^2 + \sqrt{3}^2} \\
 &= \sqrt{7} \\
 \arg(2 + j\sqrt{3}) &= \arctan \frac{\sqrt{3}}{2} \\
 &= 41^\circ \\
 (2 - j\sqrt{3})^2 &= 4 - j4\sqrt{3} - 3 \\
 &= 1 - j4\sqrt{3} \\
 |1 - j4\sqrt{3}| &= \sqrt{1^2 + (4\sqrt{3})^2} \\
 &= 7 \\
 \arg(1 - j4\sqrt{3}) &= -\arctan 4\sqrt{3}
 \end{aligned}$$

1.9

$$\begin{aligned}
 \cos \theta + j \sin \theta &= e^{j\theta} \\
 \cos \frac{\pi}{2} + j \sin \frac{\pi}{2} &= e^{j\frac{\pi}{2}} \\
 j &= e^{j\frac{\pi}{2}} \\
 j^j &= (e^{j\frac{\pi}{2}})^j \\
 &= e^{-\frac{\pi}{2}} \\
 &\approx 0.208
 \end{aligned}$$

Yes, I would be willing to pay 20 cents because I could sell it to the mathematician and gain 0.8 cents.

1.10

$$\begin{aligned}
 y &= A \cos kx + B \sin kx \\
 \frac{dy}{dx} &= -Ak \sin kx + Bk \cos kx \\
 \frac{d^2y}{dx^2} &= -Ak^2 \cos kx - Bk^2 \sin kx \\
 &= -k^2 y
 \end{aligned}$$

$$\begin{aligned}
 C &= \sqrt{A^2 + B^2} \\
 \alpha &= \arctan\left(-\frac{B}{A}\right) \\
 y &= C \cos(kx + \alpha) \\
 &= C \operatorname{Re}[e^{j(kx + \alpha)}] \\
 &= \operatorname{Re}[(Ce^{j\alpha})e^{jkx}]
 \end{aligned}$$

1.11

(a)

$$\begin{aligned}
 x &= A \cos(\omega t + \alpha) \\
 A &= 5 \text{ cm} \\
 f &= 1 \text{ Hz} \\
 \omega &= 2\pi f \\
 &= 2\pi \text{ rad/s} \\
 \alpha &= \pm \frac{\pi}{2}
 \end{aligned}$$

(b)

$$\begin{aligned}
 x\left(\frac{8}{3}\right) &= 5 \cos\left(2\pi \frac{8}{3} + \alpha\right) \\
 &= \pm 4.33 \text{ cm} \\
 \frac{dx}{dt} &= -A\omega \sin(\omega t + \alpha) \\
 \frac{dx}{dt}\left(\frac{8}{3}\right) &= \pm 15.7 \text{ cm/s} \\
 \frac{d^2x}{dt^2} &= -A\omega^2 \cos(\omega t + \alpha) \\
 \frac{d^2x}{dt^2}\left(\frac{8}{3}\right) &= \mp 171 \text{ cm/s}^2
 \end{aligned}$$

1.12

(a)

$$v = 50 \text{ cm/s}$$

$$T = 6 \text{ s}$$

$$\theta_0 = 30^\circ$$

$$c = 300 \text{ cm}$$

$$A = \frac{c}{2\pi}$$
$$= \frac{150}{\pi} \text{ cm}$$

$$\omega = \frac{2\pi}{T}$$
$$= \frac{\pi}{3} \text{ rad/s}$$

$$\alpha = \frac{\pi}{6} \text{ rad}$$

$$x = \frac{150}{\pi} \cos\left(\frac{\pi}{3}t + \frac{\pi}{6}\right)$$

(b)

$$x(2 \text{ s}) = -41.3 \text{ cm}$$

$$\frac{dx}{dt} = -50 \sin\left(\frac{\pi}{3}t + \frac{\pi}{6}\right)$$

$$\frac{dx}{dt}(2 \text{ s}) = -25 \text{ cm/s}$$

$$\frac{d^2x}{dt^2} = -\frac{50\pi}{3} \cos\left(\frac{\pi}{3}t + \frac{\pi}{6}\right)$$

$$\frac{d^2x}{dt^2}(2 \text{ s}) = 45 \text{ cm/s}^2$$

2 The superposition of periodic motions

2.1

(a)

$$z = \sin \omega t + \cos \omega t$$
$$= \sqrt{2} \cos\left(\omega t - \frac{\pi}{4}\right)$$
$$= \sqrt{2} e^{j(\omega t - \frac{\pi}{4})}$$

(b)

$$\begin{aligned} z &= \cos(\omega t - \pi/3) - \cos \omega t \\ &= \cos \omega t \cos \frac{\pi}{3} + \sin \omega t \sin \frac{\pi}{3} - \cos \omega t \\ &= -\frac{1}{2} \cos \omega t + \frac{\sqrt{3}}{2} \sin \omega t \\ &= \cos(\omega t + 2\pi/3) \\ &= e^{j(\omega t + 2\pi/3)} \end{aligned}$$

(c)

$$\begin{aligned} z &= 3 \cos \omega t + 2 \sin \omega t \\ &= \sqrt{13} \cos(\omega t + \arctan -2/3) \end{aligned}$$

(d)

$$\begin{aligned} z &= \sin \omega t - 2 \cos(\omega t - \pi/4) + \cos \omega t \\ &= \sin \omega t - 2(\cos \omega t \cos \pi/4 + \sin \omega t \sin \pi/4) + \cos \omega t \\ &= \sin \omega t - \sqrt{2} \cos \omega t - \sqrt{2} \sin \omega t + \cos \omega t \\ &= (1 - \sqrt{2}) \cos \omega t + (1 - \sqrt{2}) \sin \omega t \\ &= (1 - \sqrt{2}) \sqrt{2} \cos(\omega t - \pi/4) \\ &= (\sqrt{2} - 2) \cos(\omega t - \pi/4) \\ &= (2 - \sqrt{2}) \cos(\omega t + 3\pi/4) \end{aligned}$$

2.2

$$\begin{aligned} x &= A_1 \cos \omega t + A_2 \cos(\omega t + \alpha_1) + A_3 \cos(\omega t + \alpha_1 + \alpha_2) \\ &= A_1 \cos \omega t + A_2(\cos \omega t \cos \alpha_1 - \sin \omega t \sin \alpha_1) \\ &\quad + A_3(\cos \omega t \cos(\alpha_1 + \alpha_2) - \sin \omega t \sin(\alpha_1 + \alpha_2)) \\ &= (A_1 + A_2 \cos \alpha_1 + A_3 \cos(\alpha_1 + \alpha_2)) \cos \omega t \\ &\quad - (A_2 \sin \alpha_1 + A_3 \sin(\alpha_1 + \alpha_2)) \sin \omega t \\ A &= \sqrt{(A_1 + A_2 \cos \alpha_1 + A_3 \cos(\alpha_1 + \alpha_2))^2 + (A_2 \sin \alpha_1 + A_3 \sin(\alpha_1 + \alpha_2))^2} \\ &\approx 0.52 \text{ mm} \\ \alpha &= \arctan \frac{A_2 \sin \alpha_1 + A_3 \sin(\alpha_1 + \alpha_2)}{A_1 + A_2 \cos \alpha_1 + A_3 \cos(\alpha_1 + \alpha_2)} \\ &\approx 0.59 \text{ rad} \\ &\approx 34^\circ \end{aligned}$$

2.3

The equation of motion is

$$x = 2A \cos\left(\frac{12\pi - 10\pi}{2}t\right) \cos\left(\frac{12\pi + 10\pi}{2}t\right)$$

with the variation in amplitude given by the term

$$2A \cos \pi t$$

so the beat period is 1 s.

2.4

(a)

$$\omega = 2\pi, \text{ rad/s} \Rightarrow f = 1 \text{ Hz}$$

(b)

$$\omega = \frac{25\pi}{2} \text{ rad/s} \Rightarrow f = \frac{25}{4} \text{ Hz}$$

(c)

$$\omega = \frac{3 + \pi}{2} \text{ rad/s} \Rightarrow f = \frac{3 + \pi}{4\pi} \text{ Hz}$$

3 The free vibrations of physical systems

3.1

$$\begin{aligned} F &= -kx \\ ma &= -kx \\ k &= -\frac{ma}{x} \\ &= 4.0 \times 10^{-5} \text{ N/m} \end{aligned}$$

3.2

(a)

$$T_0 = 2\pi \sqrt{\frac{m}{k}}$$

(b) (i)

$$\begin{aligned} mx'' &= -2kx \\ x'' &= -\frac{2k}{m}x \\ T &= 2\pi\sqrt{\frac{m}{2k}} \\ &= \frac{T_0}{\sqrt{2}} \end{aligned}$$

(ii)

$$\begin{aligned} mx'' &= -k\frac{x}{2} \\ x'' &= -\frac{k}{2m}x \\ T &= 2\pi\sqrt{\frac{2m}{k}} \\ &= \sqrt{2}T_0 \end{aligned}$$

3.3

(a)

$$\begin{aligned} y &= A \cos \omega t \\ y' &= -\omega A \sin \omega t \\ y'' &= -\omega^2 A \cos \omega t \\ g &= \omega^2 A \cos \omega t \\ \omega t &= \arccos \frac{g}{\omega^2 A} \\ t &= \frac{1}{\omega} \arccos \frac{g}{\omega^2 A} \\ y &= A \cos \arccos \frac{g}{\omega^2 A} \\ &= \frac{g}{\omega^2} \\ &= 2.5 \text{ cm} \end{aligned}$$

(b)

$$\begin{aligned}v &= -\omega A \sin \omega t \\&= -\omega A \sin \arccos \frac{g}{\omega^2 A} \\&\approx 0.87 \text{ m/s}\end{aligned}$$

$$\frac{1}{2}mv^2 = mgh$$

$$\begin{aligned}h &= \frac{v^2}{2g} \\&\approx 3.8 \text{ cm}\end{aligned}$$

$$\Delta h \approx 1.3 \text{ cm}$$

3.4

(a)

$$\begin{aligned}my'' &= -g\rho Ay \\y'' &= -\frac{g\rho A}{m}y \\ \omega &= \sqrt{\frac{g\rho A}{m}} \\&= \sqrt{\frac{g}{l}}\end{aligned}$$

3.5

$$T = 2\pi\sqrt{\frac{2L}{3g}}$$

3.6

$$T = 2\pi\sqrt{\frac{d}{g}}$$

3.8

(a)

$$\begin{aligned}mg &= \frac{AY}{l_0}x \\x &= \frac{mgl_0}{AY} \\&= \frac{mgl_0}{\pi(d/2)^2Y} \\&= 0.25 \text{ mm}\end{aligned}$$

(b)

$$F_u = u\pi(d/2)^2$$

$$\approx 215.98 \text{ N}$$

$$k = \frac{AY}{L}$$

$$= \frac{\pi(d/2)^2 Y}{L}$$

$$= \frac{\pi d^2 Y}{4L}$$

$$\approx 19\,634.95 \text{ N/m}$$

$$F_u = kx_u$$

$$x_u = \frac{F_u}{k}$$

$$\approx 1.1 \text{ cm}$$

$$mgh = \frac{1}{2} \frac{AY}{L} x_u^2 - mgx_u$$

$$h = \frac{\pi(d/2)^2 Y x_u^2}{2mgL} - x_u$$

$$= \frac{\pi d^2 Y x_u^2}{8mgL} - x_u$$

$$= 0.23 \text{ m}$$

3.9

(a)

$$\begin{aligned}
 \rho_{\text{steel}} &= 7850 \text{ kg/m}^3 \\
 V_{\text{sphere}} &= \frac{4}{3}\pi r^3 \\
 F_u &= Au \\
 &= \pi r^2 u \\
 &\approx 3455.75 \text{ N} \\
 mg &= F_u \\
 m &= \frac{F_u}{g} \\
 &\approx 352.3 \text{ kg} \\
 \rho V &= m \\
 \rho \frac{4}{3}\pi r^3 &= m \\
 r &= \sqrt[3]{\frac{3m}{4\pi\rho}} \\
 &= 22 \text{ cm}
 \end{aligned}$$

(b)

$$\begin{aligned}
 M &= -\frac{\pi n r^4}{2l}\theta \\
 c &= \frac{\pi n r^4}{2l} \\
 T &= 2\pi\sqrt{\frac{I}{c}} \\
 &= 2\pi\sqrt{\frac{2MR^2/5}{\pi n r^4/2l}} \\
 &= 2\pi\sqrt{\frac{4lMR^2}{5\pi n r^4}} \\
 &= 66 \text{ s}
 \end{aligned}$$

3.10

(a)

$$\begin{aligned}
 Y &= \frac{\text{stress}}{\text{strain}} \\
 &= \frac{F/A}{\Delta l/l_0} \\
 &= \frac{mg/A}{\Delta l/l_0} \\
 &= \frac{mgl_0}{\Delta l A} \\
 &= 5.9 \times 10^{11} \text{ N/m}^2
 \end{aligned}$$

(b)

$$\begin{aligned}
 y &= \frac{4L^3}{Yab^3} F \\
 F &= \frac{Yab^3}{4L^3} y \\
 k &= \frac{Yab^3}{4L^3} \\
 \omega_y &= \sqrt{\frac{k}{m}} \\
 &= \sqrt{\frac{Yab^3}{4L^3 m}} \\
 \omega_x &= \sqrt{\frac{Ya^3b}{4L^3 m}} \\
 \frac{\omega_y}{\omega_x} &= \sqrt{\frac{ab^3}{a^3b}} \\
 &= \frac{b}{a}
 \end{aligned}$$

(c) 3/2

3.11

(a)

$$\omega = \sqrt{\frac{A\gamma p}{lm}}$$

3.14

(a)

$$m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = 0$$

(b)

$$\begin{aligned}\omega &= \frac{\sqrt{3}}{2} \omega_0 \\ \omega^2 &= \frac{3}{4} \omega_0^2 \\ \omega_0^2 - \frac{\gamma^2}{4} &= \frac{3}{4} \omega_0^2 \\ \frac{1}{4} \omega_0^2 &= \frac{\gamma^2}{4} \\ \omega_0^2 &= \gamma^2 \\ \omega_0 &= \gamma \\ &= \frac{b}{m} \\ b &= m \omega_0 \\ &= m \sqrt{\frac{k}{m}} \\ &= 4 \text{ N}/(\text{m/s})\end{aligned}$$

3.15

(a)

$$\begin{aligned}\overline{E}_0 e^{-\gamma} &= \frac{1}{2} \overline{E}_0 \\ e^{-\gamma} &= \frac{1}{2} \\ -\gamma &= \ln \frac{1}{2} \\ \gamma &= \ln 2 \\ Q_0 &= \frac{\omega_0}{\gamma} \\ &= \frac{2\pi f}{\gamma} \\ &= \frac{512\pi}{\ln 2} \\ &\approx 2321\end{aligned}$$

(b)

$$Q = 2Q_0$$

(c)

$$\begin{aligned}\gamma &= \frac{1}{4} \\ Q &= \frac{\omega_0}{\gamma} \\ &= 4\sqrt{\frac{k}{m}} \\ &= 12 \\ \gamma &= \frac{b}{m} \\ b &= \gamma m \\ &= 0.025 \text{ N/(m/s)}\end{aligned}$$

3.16

(a)

$$\begin{aligned}x &= A \sin \omega t \\ v &= \omega A \cos \omega t \\ a &= -\omega^2 A \sin \omega t \\ E &= \int_0^{1/f} \frac{Ke^2}{c^3} (-\omega^2 A \sin \omega t)^2 dt \\ &= \frac{Ke^2 \omega^4 A^2}{c^3} \int_0^{1/f} \sin^2 \omega t dt \\ &= \frac{Ke^2 \omega^4 A^2}{c^3} \left[\frac{t}{2} - \frac{1}{4\omega} \sin 2\omega t \right]_0^{1/f} \\ &= \frac{Ke^2 \omega^4 A^2}{c^3} \left(\frac{1}{2f} - \frac{1}{4\omega} \sin 2\omega \frac{1}{f} \right) \\ &= \frac{Ke^2 (2\pi f)^4 A^2}{2fc^3} \\ &= \frac{8\pi^4 Ke^2 f^3 A^2}{c^3}\end{aligned}$$

(b)

$$\begin{aligned}
E_0 &= \frac{1}{2}mv^2 \\
&= \frac{m(\omega A)^2}{2} \\
&= 2\pi^2 A^2 f^2 m \\
\frac{Q}{\pi}E &= E_0 \left(1 - \frac{1}{e}\right) \\
\frac{Q}{\pi} \frac{8\pi^4 K q^2 f^3 A^2}{c^3} &= 2\pi^2 A^2 f^2 m \left(1 - \frac{1}{e}\right) \\
Q \frac{4\pi K q^2 f}{c^3} &= m \left(1 - \frac{1}{e}\right) \\
Q &= \frac{c^3 m}{4\pi f K q^2} \left(1 - \frac{1}{e}\right)
\end{aligned}$$

3.17

(a)

$$\begin{aligned}
V &= \pi r^2 y_{\text{left}} \\
V &= \pi (2r)^2 y_{\text{right}} \\
\pi r^2 y_{\text{left}} &= \pi (2r)^2 y_{\text{right}} \\
y_{\text{right}} &= \frac{1}{4} y_{\text{left}} \\
\frac{y_{\text{left}}}{2} + \frac{y_{\text{right}}}{2} &= \frac{y_{\text{left}}}{2} + \frac{y_{\text{left}}}{8} \\
&= \frac{5}{8} y_{\text{left}} \\
U &= mg \frac{5}{8} y \\
&= \frac{5}{8} \rho \pi r^2 y g y \\
&= \frac{5}{8} g \rho \pi r^2 y^2
\end{aligned}$$

(b)

$$\begin{aligned}
r(x) &= r + \frac{x}{l}r \\
&= r \left(1 + \frac{x}{l}\right) \\
\frac{dy}{dt} \pi r^2 &= v \pi r(x)^2 \\
&= v \pi \left[r \left(1 + \frac{x}{l}\right)\right]^2 \\
v &= \frac{dy}{dt} \frac{1}{\left(1 + \frac{x}{l}\right)^2} \\
m &= \rho \pi r(x)^2 dx \\
&= \rho \pi \left[r \left(1 + \frac{x}{l}\right)\right]^2 dx \\
&= \rho \pi r^2 \left(1 + \frac{x}{l}\right)^2 dx \\
dK &= \frac{1}{2} m v^2 \\
&= \frac{1}{2} \rho \pi r^2 \left(1 + \frac{x}{l}\right)^2 dx \left(\frac{dy}{dt} \frac{1}{\left(1 + \frac{x}{l}\right)^2}\right)^2 \\
&= \frac{1}{2} \rho \frac{\pi r^2 dx}{(1 + x/l)^2} \left(\frac{dy}{dt}\right)^2
\end{aligned}$$

(c)

$$\begin{aligned}
K &= \frac{1}{2} \rho \pi r^2 h \left(\frac{dy}{dt}\right)^2 + \frac{1}{2} \rho \pi (2r)^2 h \left(\frac{dy}{dt}\right)^2 + \int_0^l dK \\
&= \frac{5}{2} \rho \pi r^2 h \left(\frac{dy}{dt}\right)^2 + \int_0^l \frac{1}{2} \rho \frac{\pi r^2 dx}{(1 + x/l)^2} \left(\frac{dy}{dt}\right)^2 \\
&= \frac{5}{2} \rho \pi r^2 h \left(\frac{dy}{dt}\right)^2 + \frac{1}{2} \rho \pi r^2 \int_0^l \frac{1}{(1 + x/l)^2} dx \left(\frac{dy}{dt}\right)^2 \\
&= \frac{1}{4} \rho \pi r^2 \left(l + \frac{5}{2}h\right) \left(\frac{dy}{dt}\right)^2
\end{aligned}$$

(d)

$$K + U = E$$

$$\frac{1}{4}\rho\pi r^2 \left(l + \frac{5}{2}h\right) \left(\frac{dy}{dt}\right)^2 + \frac{5}{8}g\rho\pi r^2 y^2 = E$$

$$m = \frac{1}{2}\rho\pi r^2 \left(l + \frac{5}{2}h\right)$$

$$k = \frac{5}{4}g\rho\pi r^2$$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$= 2\pi\sqrt{\frac{\frac{1}{2}\rho\pi r^2 \left(l + \frac{5}{2}h\right)}{\frac{5}{4}g\rho\pi r^2}}$$

$$= 2\pi\sqrt{\frac{2h}{g}}$$

3.19

(a)

$$m\frac{d^2x}{dt^2} + 2k(x + l - l_0) = 0$$

(b)

$$T = k(l' - l_0)$$

$$= k(\sqrt{l^2 + y^2} - l_0)$$

$$F = 2T \sin \theta$$

$$= 2k(\sqrt{l^2 + y^2} - l_0) \frac{y}{\sqrt{l^2 + y^2}}$$

$$= 2k \left(1 - \frac{l_0}{\sqrt{l^2 + y^2}}\right) y$$

$$\approx 2k \left(1 - \frac{l_0}{l}\right) y$$

$$m\frac{d^2y}{dt^2} + 2k \left(1 - \frac{l_0}{l}\right) y = 0$$

(c)

$$\begin{aligned}
T_x &= 2\pi\sqrt{\frac{m}{2k}} \\
T_y &= 2\pi\sqrt{\frac{m}{2k(1-\frac{l_0}{l})}} \\
\frac{T_x}{T_y} &= \frac{2\pi\sqrt{m/2k}}{2\pi\sqrt{\frac{m}{2k(1-l/l_0)}}} \\
&= \sqrt{\frac{m}{2k} \frac{2k(1-l/l_0)}{m}} \\
&= \sqrt{1-l/l_0}
\end{aligned}$$

(d)

$$\begin{aligned}
x &= A_x \cos\left(\sqrt{\frac{2k}{m}}t + \phi_x\right) \\
A_0 &= A_x \cos\phi_x \\
0 &= -\sqrt{\frac{2k}{m}}A_x \sin\phi_x \\
\tan\phi_x &= 0 \\
\phi_x &= 0 \\
A_x &= A_0 \\
x &= A_0 \cos\sqrt{\frac{2k}{m}}t \\
\\
y &= A_y \cos\left(\sqrt{\frac{2k(1-l_0/l)}{m}}t + \phi_y\right) \\
A_0 &= A_y \cos\phi_y \\
0 &= -\sqrt{\frac{2k(1-l_0/l)}{m}}A_y \sin\phi_y \\
\tan\phi_y &= 0 \\
\phi_y &= 0 \\
A_y &= A_0 \\
y &= A_0 \cos\sqrt{\frac{2k(1-l_0/l)}{m}}t
\end{aligned}$$

4 Forced vibrations and resonance

4.3

(a)

$$\begin{aligned}
 m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky &= 0 \\
 \omega &= \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \\
 &= \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \\
 &= \sqrt{\frac{80}{0.2} - \frac{4^2}{4 \cdot 0.2^2}} \\
 &= \sqrt{300} \\
 &= 10\sqrt{3} \\
 T &= \frac{2\pi}{\omega} \\
 &= \frac{2\pi}{10\sqrt{3}} \\
 &= \frac{\pi}{5\sqrt{3}} \text{ s}
 \end{aligned}$$

(b)

$$\begin{aligned}
 A &= \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \\
 &= \frac{F_0}{m\sqrt{\left(\frac{k}{m} - \omega^2\right)^2 + \left(\frac{b}{m}\omega\right)^2}} \\
 &\approx 1.3 \text{ cm}
 \end{aligned}$$

4.4

$$\begin{aligned}
 mg &= kh \\
 k &= \frac{mg}{h}
 \end{aligned}$$

$$\begin{aligned}
 mg &= bu \\
 b &= \frac{mg}{u}
 \end{aligned}$$

(a)

$$m \frac{d^2 x}{dt^2} + \frac{mg}{u} \frac{dx}{dt} + \frac{mg}{h} x = 0$$

(b)

$$\begin{aligned}\omega &= \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \\ &= \sqrt{\frac{g}{h} - \frac{g^2}{4u^2}} \\ &= \sqrt{\frac{g}{h} - \frac{g^2}{36gh}} \\ &= \sqrt{\frac{g}{h} - \frac{g}{36h}} \\ &= \sqrt{\frac{35g}{36h}}\end{aligned}$$

(c)

$$\frac{1}{\gamma} = \frac{u}{g} = \frac{3\sqrt{gh}}{g} = 3\sqrt{\frac{h}{g}} \text{ s}$$

(d)

$$Q = \frac{\omega_0}{\gamma} = \sqrt{\frac{g}{h}} \cdot 3\sqrt{\frac{h}{g}} = 3$$

(e)

$$\delta = \frac{\pi}{2}$$

(f)

$$\begin{aligned}
A &= \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \\
&= \frac{F_0}{m\sqrt{\left(\frac{g}{h} - \omega^2\right) + \left(\frac{g}{u}\omega\right)^2}} \\
&= \frac{F_0}{m\sqrt{\left(\frac{g}{h} - \frac{2g}{h}\right)^2 + \left(\frac{g}{3\sqrt{gh}}\sqrt{\frac{2g}{h}}\right)^2}} \\
&= \frac{F_0}{m\sqrt{\left(\frac{g}{h}\right)^2 + \left(\frac{\sqrt{2}}{3}\frac{g}{h}\right)^2}} \\
&= \frac{F_0}{m\sqrt{\left(\frac{g}{h}\right)^2 + \frac{2}{9}\left(\frac{g}{h}\right)^2}} \\
&= \frac{F_0}{m\sqrt{\frac{11}{9}\left(\frac{g}{h}\right)^2}} \\
&= \sqrt{\frac{9}{11}} \frac{F_0 h}{gm} \\
&= \sqrt{\frac{9}{11}} h \\
&\approx 0.9h
\end{aligned}$$

4.6

(a)

$$\begin{aligned}
m\left(\frac{d^2y}{dt^2} + \frac{d^2\eta}{dt^2}\right) &= -kx - b\frac{dy}{dt} \\
\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y &= -\frac{d^2\eta}{dt^2}
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y &= C\omega^2 \cos \omega t \\
y &= A(\omega) \cos(\omega t + \delta(\omega)) \\
A(\omega) &= \frac{C\omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\omega_0/Q)^2}} \\
\tan \delta(\omega) &= \frac{\omega\omega_0/Q}{\omega_0^2 - \omega^2}
\end{aligned}$$

(d)

$$\begin{aligned}T_0 &= \frac{2\pi}{\omega_0} \\ \omega_0 &= \frac{2\pi}{T_0} \\ &= \frac{\pi}{15} \text{ s}\end{aligned}$$

$$Q = 2$$

$$\begin{aligned}T &= \frac{2\pi}{\omega} \\ \omega &= \frac{2\pi}{T} \\ &= \frac{\pi}{600} \text{ s}\end{aligned}$$

$$\begin{aligned}A &= C\omega^2 \\ C &= \frac{A}{\omega^2} \\ &= A \left(\frac{600}{\pi} \right)^2 \\ &\approx 3.65 \times 10^{-5}\end{aligned}$$

$$A(\omega) \approx 2.28 \times 10^{-8} \text{ m}$$

4.8

(a)

$$\begin{aligned}m \frac{d^2x}{dt^2} &= -b \frac{dx}{dt} \\ m \frac{dv}{dt} &= -bv \\ \frac{1}{v} \frac{dv}{dt} &= -\frac{b}{m} \\ \ln v &= -\frac{b}{m}t + c \\ v &= e^{(-bt/m)+c} \\ &= v_0 e^{-\gamma t} \\ x &= C - \frac{v_0}{\gamma} e^{-\gamma t}\end{aligned}$$

(b)

$$\begin{aligned}
\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} &= \frac{F_0}{m} \cos \omega t \\
x &= A \cos(\omega t - \delta) \\
\frac{dx}{dt} &= -A\omega \sin(\omega t - \delta) \\
\frac{d^2x}{dt^2} &= -A\omega^2 \cos(\omega t - \delta) \\
-A\omega^2 \cos(\omega t - \delta) - \gamma A\omega \sin(\omega t - \delta) &= \frac{F_0}{m} \cos \omega t \\
-A\omega^2 (\cos \omega t \cos \delta + \sin \omega t \sin \delta) \\
-\gamma A\omega (\sin \omega t \cos \delta - \cos \omega t \sin \delta) &= \frac{F_0}{m} \cos \omega t \\
A\omega (\gamma \sin \delta - \omega \cos \delta) \cos \omega t \\
-A\omega (\gamma \cos \delta + \omega \sin \delta) \sin \omega t &= \frac{F_0}{m} \cos \omega t \\
A\omega (\gamma \cos \delta + \omega \sin \delta) &= 0 \\
\gamma \cos \delta + \omega \sin \delta &= 0 \\
\tan \delta
\end{aligned}$$

$$\begin{aligned}
A\omega (\gamma \sin \delta - \omega \cos \delta) &= \frac{F_0}{m} \\
\gamma \sin \delta - \omega \cos \delta &= -\sqrt{\gamma^2 + \omega^2} \cos \left(\delta + \arctan \frac{\gamma}{\omega} \right) \\
&= -\sqrt{\gamma^2 + \omega^2} \cos(\delta + \arctan(-\tan \delta)) \\
&= -\sqrt{\gamma^2 + \omega^2} \cos(\delta - \arctan(\tan \delta)) \\
&= -\sqrt{\gamma^2 + \omega^2} \\
-A\omega \sqrt{\gamma^2 + \omega^2} &= \frac{F_0}{m} \\
|A| &= \frac{F_0}{m\omega \sqrt{\gamma^2 + \omega^2}}
\end{aligned}$$

4.9

(a)

$$\begin{aligned}
 x &= A \sin \omega t \\
 \frac{dx}{dt} &= A \omega \cos \omega t \\
 W &= \int dW \\
 &= \int F dx \\
 &= \int b \frac{dx}{dt} dx \\
 &= b \int \left(\frac{dx}{dt} \right)^2 dt \\
 &= b \int_0^T (A \omega \cos \omega t)^2 dt \\
 &= A^2 b \omega^2 \int_0^{2\pi/\omega} \cos^2 \omega t dt \\
 &= A^2 b \omega^2 \left[\frac{t}{2} + \frac{1}{4\omega} \sin 2\omega t \right]_0^{2\pi/\omega} \\
 &= A^2 b \omega \pi
 \end{aligned}$$

4.11

(a)

$$\begin{aligned}
 A &= \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \\
 &= \frac{2/0.2}{\sqrt{\left(\frac{80}{0.2} - 30^2\right)^2 + \left(\frac{4}{0.2} 30\right)^2}} \\
 &\approx 1.3 \text{ cm} \\
 \tan \delta &= \frac{\gamma\omega}{\omega_0^2 - \omega^2} \\
 \delta &= \arctan \frac{\frac{b}{m}\omega}{\frac{k}{m} - \omega^2} \\
 &\approx 130^\circ
 \end{aligned}$$

(b)

$$\begin{aligned}x &= A \cos(\omega t - \delta) \\ \frac{dx}{dt} &= -A\omega \sin(\omega t - \delta) \\ W &= \int dW \\ &= \int F dx \\ &= \int b \frac{dx}{dt} dx \\ &= b \int_0^T \left(\frac{dx}{dt} \right)^2 dt \\ &= b \int_0^T (-A\omega \sin(\omega t - \delta))^2 dt \\ &= A^2 b \omega^2 \int_0^{2\pi/\omega} \sin^2(\omega t - \delta) dt \\ &= A^2 b \omega \pi \\ &\approx 6.4 \times 10^{-2} \text{ J}\end{aligned}$$

(c)

$$P = \frac{W}{t} = 0.31 \text{ W}$$

4.12

(a)

$$\begin{aligned}mg &= kx \\ k &= \frac{mg}{x} \\ &\approx 785 \text{ N/m}\end{aligned}$$

$$\begin{aligned}\omega_0 &= \sqrt{\frac{k}{m}} \\ &\approx 19.8 \text{ rad/s}\end{aligned}$$

(b)

$$\begin{aligned}A &= BQ \\ &= 1.5 \text{ cm}\end{aligned}$$

(c)

$$\begin{aligned}\overline{P}(\omega) &= \frac{1.082B^2m\omega_0^5}{2Q} \frac{1}{0.0016\omega_0^2 + \omega_0^2/Q^2} \\ &\approx 0.093 \text{ W}\end{aligned}$$

4.13

(a)

$$\omega_0 = 40 \text{ rad/s}$$

$$\begin{aligned}2 &= \frac{\omega_0}{Q} \\ Q &= \frac{\omega_0}{2} \\ &= 20\end{aligned}$$

(b)

$$\begin{aligned}E_0 e^{-2\pi n/Q} &= E_0 e^{-5} \\ -\frac{2\pi n}{Q} &= -5 \\ n &= \frac{5Q}{2\pi} \\ &\approx 16\end{aligned}$$

4.14

(a)

$$\begin{aligned}1.02\omega_0 - 0.98\omega_0 &= \frac{\omega_0}{Q} \\ 0.04\omega_0 &= \frac{\omega_0}{Q} \\ Q &= 25\end{aligned}$$

(b)

$$\gamma = 0.04\omega_0$$

(c)

$$\begin{aligned}\frac{E_0 - E(2\pi/\omega_0)}{E_0} &= 1 - \frac{E_0 e^{-0.04\omega_0(2\pi/\omega_0)}}{E_0} \\ &= 1 - e^{-2\pi \cdot 0.04} \\ &\approx 22\%\end{aligned}$$

(d)

$$\omega'_0 = \sqrt{2}\omega_0$$

(e)

$$Q' = \sqrt{2}Q$$

(f)

$$\overline{P}'_m = \overline{P}_m$$

(g)

$$E'_0 = E_0$$

4.15

(b)

$$\frac{\omega_1}{Q} = \frac{\omega_1}{5}$$

$$Q = 5$$

(c)

$$Q = \frac{\omega_0}{\gamma}$$

$$= \sqrt{\frac{k}{m} \frac{m}{b}}$$

$$b = \frac{1}{5} \sqrt{km}$$

4.17

(a)

$$W = PT = 10 \cdot \frac{2\pi}{10^6} = \frac{2\pi}{10^5} = 6.28 \times 10^{-5} \text{ J}$$

(b)

$$W = E_0(1 - e^{-\gamma(2\pi/\omega)})$$

$$E_0 = \frac{W}{1 - e^{-\gamma(2\pi/\omega)}}$$

$$\approx 1.03 \times 10^{-3} \text{ J}$$

(c)

$$\frac{1}{\gamma} = \frac{1}{(1.005 - 0.995) \times 10^6} \approx 1 \times 10^{-4} \text{ s}$$

5 Coupled oscillators and normal modes

5.2

(a)

$$T_{\text{clamped}} = \frac{2\pi}{\sqrt{\omega_0^2 + \omega_c^2}}$$

$$T_{\text{clamped}}^2 = \frac{4\pi^2}{\omega_0^2 + \omega_c^2}$$

$$\omega_0^2 + \omega_c^2 = \frac{4\pi^2}{T_{\text{clamped}}^2}$$

$$\omega_c = \sqrt{\frac{4\pi^2}{T_{\text{clamped}}^2} - \frac{g}{l}}$$

$$T_1 = \frac{2\pi}{\omega_0}$$

$$= \frac{2\pi}{\sqrt{g/l}}$$

$$\approx 1.27 \text{ s}$$

$$T_2 = \frac{2\pi}{\omega'}$$

$$= \frac{2\pi}{\sqrt{\omega_0^2 + 2\omega_c^2}}$$

$$= \frac{2\pi}{\sqrt{\frac{g}{l} + 2\sqrt{\frac{4\pi^2}{T_{\text{clamped}}^2} - \frac{g}{l}}}}$$

$$\approx 1.23 \text{ s}$$

(b)

$$T_{\text{beat}} = \frac{2\pi}{\omega' - \omega_0}$$

$$= \frac{2\pi}{\frac{2\pi}{T_2} - \frac{2\pi}{T_1}}$$

$$\approx 40 \text{ s}$$

5.4

$$\begin{aligned}\frac{d^2x_A}{dt^2} + \frac{k_A + k_C}{m}x_A - \frac{k_C}{m}x_B &= 0 \\ \frac{d^2x_B}{dt^2} - \frac{k_C}{m}x_A + \frac{k_B + k_C}{m}x_B &= 0\end{aligned}$$

$$\begin{aligned}x_A &= A \cos \omega t \\ x'_A &= -A\omega \sin \omega t \\ x''_A &= -A\omega^2 \cos \omega t\end{aligned}$$

$$\begin{aligned}x_B &= A' \cos \omega t \\ x'_B &= -A'\omega \sin \omega t \\ x''_B &= -A'\omega^2 \cos \omega t\end{aligned}$$

$$\begin{aligned}-A\omega^2 \cos \omega t + \frac{k_A + k_C}{m}A \cos \omega t - \frac{k_C}{m}A' \cos \omega t &= 0 \\ -A\omega^2 + \frac{k_A + k_C}{m}A - \frac{k_C}{m}A' &= 0 \\ \left(\frac{k_A + k_C}{m} - \omega^2\right)A - \frac{k_C}{m}A' &= 0 \\ \frac{k_C}{k_A + k_C - m\omega^2} &= \frac{A}{A'}\end{aligned}$$

$$\begin{aligned}-A'\omega^2 \cos \omega t - \frac{k_C}{m}A \cos \omega t + \frac{k_B + k_C}{m}A' \cos \omega t &= 0 \\ -A'\omega^2 - \frac{k_C}{m}A + \frac{k_B + k_C}{m}A' &= 0 \\ -\frac{k_C}{m}A + \left(\frac{k_B + k_C}{m} - \omega^2\right)A' &= 0 \\ \frac{k_B + k_C - m\omega^2}{k_C} &= \frac{A}{A'}\end{aligned}$$

$$\begin{aligned}\frac{k_C}{k_A + k_C - m\omega^2} &= \frac{k_B + k_C - m\omega^2}{k_C} \\ k_C^2 &= (k_A + k_C - m\omega^2)(k_B + k_C - m\omega^2) \\ &= k_A k_B + k_A k_C - k_A m\omega^2 + k_B k_C + k_C^2 - k_C m\omega^2 - k_B m\omega^2 \\ &\quad - k_C m\omega^2 + m^2 \omega^4 \\ 0 &= m^2(\omega^2)^2 - (k_A + k_B + 2k_C)m\omega^2 + k_A k_B + k_A k_C + k_B k_C\end{aligned}$$

$$\begin{aligned}
m\omega^2 &= \frac{(k_A + k_B + 2k_C) \pm \sqrt{(k_A + k_B + 2k_C)^2 - 4(k_A k_B + k_A k_C + k_B k_C)}}{2} \\
&= \frac{(k_A + k_B + 2k_C) \pm \sqrt{(k_A - k_B)^2 + 4k_C^2}}{2} \\
&= \left(\frac{k_A + k_B}{2} + k_C \right) \pm \sqrt{\left(\frac{k_A - k_B}{2} \right)^2 + k_C^2}
\end{aligned}$$

5.5

$$\begin{aligned}
\frac{d^2 x_A}{dt^2} - \alpha \frac{d^2 x_B}{dt^2} + \omega_0^2 x_A &= 0 \\
\frac{d^2 x_B}{dt^2} - \alpha \frac{d^2 x_A}{dt^2} + \omega_0^2 x_B &= 0
\end{aligned}$$

$$\begin{aligned}
x_A &= A \cos \omega t \\
x_A'' &= -A\omega^2 \cos \omega t
\end{aligned}$$

$$\begin{aligned}
x_B &= B \cos \omega t \\
x_B'' &= -B\omega^2 \cos \omega t
\end{aligned}$$

$$\begin{aligned}
-A\omega^2 \cos \omega t + \alpha B\omega^2 \cos \omega t + A\omega_0^2 \cos \omega t &= 0 \\
-A\omega^2 + \alpha B\omega^2 + A\omega_0^2 &= 0 \\
(\omega_0^2 - \omega^2)A + \alpha\omega^2 B &= 0 \\
\frac{\alpha\omega^2}{\omega^2 - \omega_0^2} &= \frac{A}{B}
\end{aligned}$$

$$\begin{aligned}
-B\omega^2 \cos \omega t + \alpha A\omega^2 \cos \omega t + B\omega_0^2 \cos \omega t &= 0 \\
\alpha\omega^2 A + (\omega_0^2 - \omega^2)B &= 0 \\
\frac{\omega^2 - \omega_0^2}{\alpha\omega^2} &= \frac{A}{B}
\end{aligned}$$

$$\begin{aligned}
\frac{\alpha\omega^2}{\omega^2 - \omega_0^2} &= \frac{\omega^2 - \omega_0^2}{\alpha\omega^2} \\
(\alpha\omega^2)^2 &= (\omega^2 - \omega_0^2)^2 \\
\alpha\omega^2 &= \omega^2 - \omega_0^2 \\
(\alpha - 1)\omega^2 &= -\omega_0^2 \\
\omega^2 &= \frac{\omega_0^2}{1 - \alpha} \\
\omega &= \frac{\omega_0}{\sqrt{1 - \alpha}}
\end{aligned}$$

$$\begin{aligned}
\alpha\omega^2 &= \omega_0^2 - \omega^2 \\
(\alpha + 1)\omega^2 &= \omega_0^2 \\
\omega &= \frac{\omega_0}{\sqrt{\alpha + 1}}
\end{aligned}$$

5.7

(b)

$$\begin{aligned}
\frac{d^2x_A}{dt^2} + \frac{k_0 + k_c}{m}x_A - \frac{k_c}{m}x_B &= 0 \\
\frac{d^2x_B}{dt^2} - \frac{k_c}{m}x_A + \frac{k_0 + k_c}{m}x_B &= 0
\end{aligned}$$

$$\begin{aligned}
x_A &= A \cos \omega t \\
x_A'' &= -A\omega^2 \cos \omega t
\end{aligned}$$

$$\begin{aligned}
x_B &= B \cos \omega t \\
x_B'' &= -B\omega^2 \cos \omega t
\end{aligned}$$

$$\begin{aligned}
-A\omega^2 \cos \omega t + \frac{k_0 + k_c}{m} A \cos \omega t - \frac{k_c}{m} B \cos \omega t &= 0 \\
-A\omega^2 + \frac{k_0 + k_c}{m} A - \frac{k_c}{m} B &= 0 \\
(k_0 + k_c - m\omega^2)A - k_c B &= 0 \\
\frac{k_c}{k_0 + k_c - m\omega^2} &= \frac{A}{B}
\end{aligned}$$

$$\begin{aligned}
-B\omega^2 \cos \omega t - \frac{k_c}{m} A \cos \omega t + \frac{k_0 + k_c}{m} B \cos \omega t &= 0 \\
-B\omega^2 - \frac{k_c}{m} A + \frac{k_0 + k_c}{m} B &= 0 \\
-k_c A + (k_0 + k_c - m\omega^2)B &= 0 \\
\frac{k_0 + k_c - m\omega^2}{k_c} &= \frac{A}{B}
\end{aligned}$$

$$\begin{aligned}
\frac{k_c}{k_0 + k_c - m\omega^2} &= \frac{k_0 + k_c - m\omega^2}{k_c} \\
(k_0 + k_c - m\omega^2)^2 &= k_c^2 \\
k_0 + k_c - m\omega^2 &= \pm k_c
\end{aligned}$$

$$\begin{aligned}
m\omega_1^2 &= k_0 \\
\omega_1 &= \omega_0
\end{aligned}$$

$$\begin{aligned}
m\omega_2^2 &= k_0 + 2k_c \\
\omega_2 &= \sqrt{\omega_0^2 + \frac{2k_c}{m}}
\end{aligned}$$

(c)

$$\begin{aligned}\omega_0 &= 2\pi f_1 \\ &\approx 7.16 \text{ rad/s}\end{aligned}$$

$$\begin{aligned}\omega_A &= 2\pi f_A \\ \sqrt{\omega_0^2 + \frac{k_c}{m}} &= 2\pi f_A \\ \omega_0^2 + \frac{k_c}{m} &= (2\pi f_A)^2 \\ \frac{k_c}{m} &= (2\pi f_A)^2 - \omega_0^2 \\ &\approx 78.0 \text{ s}^{-2}\end{aligned}$$

$$\begin{aligned}2\pi f_2 &= \sqrt{\omega_0^2 + 2\frac{k_c}{m}} \\ f_2 &= \frac{1}{2\pi} \sqrt{\omega_0^2 + 2\frac{k_c}{m}} \\ &\approx 2.29 \text{ Hz}\end{aligned}$$

(d)

$$\begin{aligned}\frac{k_c}{k_0} &= \frac{k_c/m}{k_0/m} \\ &= \frac{k_c/m}{\omega_0^2} \\ &\approx 1.52\end{aligned}$$

5.8

(a)

$$\begin{aligned}Fa \cos \theta &= mgL \sin \theta \\ \frac{Fa}{mgL} &= \tan \theta \\ &\approx \theta\end{aligned}$$

$$\begin{aligned}F'L \cos \theta &= mgL \sin \theta \\ F' &= mg \tan \theta \\ &\approx mg\theta \\ &= \frac{Fa}{L}\end{aligned}$$

(b)

$$\begin{aligned}
I \frac{d^2 \theta_1}{dt^2} &= -mgL \sin \theta_1 + k(a \sin \theta_2 - a \sin \theta_1)a \cos \theta_1 \\
mL^2 \frac{d^2 \theta_1}{dt^2} &\approx -mgL \theta_1 + a^2 k(\theta_2 - \theta_1) \\
0 &= \frac{d^2 \theta_1}{dt^2} + \left(\frac{g}{L} + \frac{a^2 k}{mL^2} \right) \theta_1 - \frac{a^2 k}{mL^2} \theta_2
\end{aligned}$$

$$\begin{aligned}
I \frac{d^2 \theta_2}{dt^2} &= -mgL \sin \theta_2 - k(a \sin \theta_2 - a \sin \theta_1)a \cos \theta_2 \\
mL^2 \frac{d^2 \theta_2}{dt^2} &\approx -mgL \theta_2 - a^2 k(\theta_2 - \theta_1) \\
0 &= \frac{d^2 \theta_2}{dt^2} - \frac{a^2 k}{mL^2} \theta_1 + \left(\frac{g}{L} + \frac{a^2 k}{mL^2} \right) \theta_2
\end{aligned}$$

(d) Let $\omega_0^2 = \frac{g}{L}$ and $\omega_c = \frac{a^2 k}{mL^2}$ then

$$\begin{aligned}
\theta_1 &= A \cos \omega t \\
\theta_1'' &= -A\omega^2 \cos \omega t
\end{aligned}$$

$$\begin{aligned}
\theta_2 &= B \cos \omega t \\
\theta_2'' &= -B\omega^2 \cos \omega t
\end{aligned}$$

$$\begin{aligned}
0 &= -A\omega^2 \cos \omega t + (\omega_0^2 + \omega_c)A \cos \omega t - \omega_c B \cos \omega t \\
&= -A\omega^2 + (\omega_0^2 + \omega_c)A - \omega_c B \\
&= (\omega_0 + \omega_c - \omega^2)A - \omega_c B \\
\frac{A}{B} &= \frac{\omega_c}{\omega_0 + \omega_c - \omega^2}
\end{aligned}$$

$$\begin{aligned}
0 &= -B\omega^2 \cos \omega t - \omega_c A \cos \omega t + (\omega_0 + \omega_c)B \cos \omega t \\
&= -B\omega^2 - \omega_c A + (\omega_0 + \omega_c)B \\
&= -\omega_c A + (\omega_0 + \omega_c - \omega^2)B \\
\frac{A}{B} &= \frac{\omega_0 + \omega_c - \omega^2}{\omega_c}
\end{aligned}$$

$$\frac{\omega_c}{\omega_0 + \omega_c - \omega^2} = \frac{\omega_0 + \omega_c - \omega^2}{\omega_c}$$

$$(\omega_0 + \omega_c - \omega^2)^2 = \omega_c^2$$

$$\omega_0 + \omega_c - \omega^2 = \pm \omega_c$$

$$\omega_1 = \omega_0$$

$$= \sqrt{\frac{g}{L}}$$

$$\omega_2 = \sqrt{\omega_0 + 2\omega_c}$$

$$= \sqrt{\frac{g}{L} + 2\frac{a^2 k}{mL^2}}$$

5.9

(a)

$$\begin{aligned}\frac{d^2 x_1}{dt^2} + \frac{k}{m_1}(x_1 - x_2) &= 0 \\ \frac{d^2 x_2}{dt^2} + \frac{k}{m_2}(-x_1 + 2x_2 - x_3) &= 0 \\ \frac{d^2 x_3}{dt^2} + \frac{k}{m_1}(-x_2 + x_3) &= 0\end{aligned}$$

$$x_1 = A \cos \omega t$$

$$x_2 = B \cos \omega t$$

$$x_3 = C \cos \omega t$$

$$-A\omega^2 \cos \omega t + \frac{k}{m_1}(A \cos \omega t - B \cos \omega t) = 0$$

$$-A\omega^2 + \frac{k}{m_1}(A - B) = 0$$

$$\left(\frac{k}{m_1} - \omega^2\right)A - \frac{k}{m_1}B = 0$$

$$\frac{k/m_1}{k/m_1 - \omega^2} = \frac{A}{B}$$

$$-C\omega^2 \cos \omega t + \frac{k}{m_1}(-B \cos \omega t + C \cos \omega t) = 0$$

$$-C\omega^2 + \frac{k}{m_1}(-B + C) = 0$$

$$-\frac{k}{m_1}B + \left(\frac{k}{m_1} - \omega^2\right)C = 0$$

$$\frac{k/m_1}{k/m_1 - \omega^2} = \frac{C}{B}$$

$$\begin{aligned}
-B\omega^2 \cos \omega t + \frac{k}{m_2}(-A \cos \omega t + 2B \cos \omega t - C \cos \omega t) &= 0 \\
-B\omega^2 + \frac{k}{m_2}(-A + 2B - C) &= 0 \\
-\omega^2 + \frac{k}{m_2}\left(-\frac{A}{B} + 2 - \frac{C}{B}\right) &= 0 \\
-\omega^2 + \frac{k}{m_2}\left(-\frac{k/m_1}{k/m_1 - \omega^2} + 2 - \frac{k/m_1}{k/m_1 - \omega^2}\right) &= 0 \\
\frac{2k}{m_2}\left(1 - \frac{k}{k - m_1\omega^2}\right) &= \omega^2 \\
\frac{2k}{m_2}(k - m_1\omega^2 - k) &= \omega^2(k - m_1\omega^2) \\
-2k\omega^2 \frac{m_1}{m_2} &= k\omega^2 - m_1(\omega^2)^2 \\
2k \frac{m_1}{m_2} + k &= m_1\omega^2 \\
\frac{2k}{m_2} + \frac{k}{m_1} &= \omega^2 \\
\frac{k(2m_1 + m_2)}{m_1m_2} &= \omega^2 \\
\sqrt{\frac{k(2m_1 + m_2)}{m_1m_2}} &= \omega
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\omega_1}{\omega_2} &= \sqrt{\frac{k(2m_1 + m_2)/m_1m_2}{k/m_1}} \\
&= \sqrt{\frac{km_1(2m_1 + m_2)}{km_1m_2}} \\
&= \sqrt{\frac{2m_1 + m_2}{m_2}} \\
&= \sqrt{\frac{2 \cdot 16 + 12}{12}} \\
&= \sqrt{\frac{44}{12}} \\
&= \sqrt{\frac{11}{3}} \\
&\approx 1.91
\end{aligned}$$

5.10

$$\begin{aligned}\frac{d^2 y_1}{dt^2} + \frac{k}{m}(2y_1 - y_2) &= 0 \\ \frac{d^2 y_2}{dt^2} + \frac{k}{m}(y_2 - y_1) &= 0\end{aligned}$$

$$y_1 = A \cos \omega t$$

$$y_2 = B \cos \omega t$$

$$-A\omega^2 \cos \omega t + \frac{k}{m}(2A \cos \omega t - B \cos \omega t) = 0$$

$$-A\omega^2 + \frac{k}{m}(2A - B) = 0$$

$$\left(2\frac{k}{m} - \omega^2\right)A - \frac{k}{m}B = 0$$

$$\frac{k/m}{2k/m - \omega^2} = \frac{A}{B}$$

$$-B\omega^2 \cos \omega t + \frac{k}{m}(B \cos \omega t - A \cos \omega t) = 0$$

$$-B\omega^2 + \frac{k}{m}(B - A) = 0$$

$$-\frac{k}{m}A + \left(\frac{k}{m} - \omega^2\right)B = 0$$

$$\frac{k/m - \omega^2}{k/m} = \frac{A}{B}$$

$$\begin{aligned}
\frac{k/m}{2k/m - \omega^2} &= \frac{k/m - \omega^2}{k/m} \\
\left(\frac{k}{m}\right)^2 &= \left(\frac{2k}{m} - \omega^2\right) \left(\frac{k}{m} - \omega^2\right) \\
&= 2\left(\frac{k}{m}\right)^2 - 2\frac{k}{m}\omega^2 - \frac{k}{m}\omega^2 + (\omega^2)^2 \\
0 &= (\omega^2)^2 - 3\frac{k}{m}\omega^2 + \left(\frac{k}{m}\right)^2
\end{aligned}$$

$$\begin{aligned}
\omega^2 &= \frac{3(k/m) \pm \sqrt{9(k/m)^2 - 4(k/m)^2}}{2} \\
&= (3 \pm \sqrt{5}) \frac{k}{2m}
\end{aligned}$$

$$\begin{aligned}
\frac{A}{B} &= \frac{k/m}{2k/m - (3 \pm \sqrt{5})k/2m} \\
&= \frac{1}{2 - (3 \pm \sqrt{5})/2} \\
&= \frac{2}{4 - 3 \pm \sqrt{5}} \\
&= \frac{2}{1 \pm \sqrt{5}}
\end{aligned}$$

5.11

(a)

$$\begin{aligned}
M_1 \frac{d^2 x_1}{dt^2} &= -kx_1 + M_2 g \sin \theta \\
&\approx -kx_1 + M_2 \frac{g}{l}(x_2 - x_1)
\end{aligned}$$

$$\begin{aligned}
M_2 \frac{d^2 x_2}{dt^2} &= -M_2 g \sin \theta \\
&\approx -M_2 \frac{g}{l}(x_2 - x_1)
\end{aligned}$$

(b)

$$\begin{aligned}\frac{d^2 x_1}{dt^2} + \frac{k}{M} x_1 - \frac{g}{l} (x_2 - x_1) &= 0 \\ \frac{d^2 x_2}{dt^2} + \frac{g}{l} (x_2 - x_1) &= 0\end{aligned}$$

$$\begin{aligned}-A\omega^2 \cos \omega t + \frac{k}{M} A \cos \omega t - \frac{g}{l} (B \cos \omega t - A \cos \omega t) &= 0 \\ -A\omega^2 + \frac{k}{M} A - \frac{g}{l} (B - A) &= 0 \\ \left(\frac{k}{M} + \frac{g}{l} - \omega^2 \right) A - \frac{g}{l} B &= 0 \\ \frac{g/l}{k/M + g/l - \omega^2} &= \frac{A}{B}\end{aligned}$$

$$\begin{aligned}-B\omega^2 \cos \omega t + \frac{g}{l} (B \cos \omega t - A \cos \omega t) &= 0 \\ -B\omega^2 + \frac{g}{l} (B - A) &= 0 \\ -\frac{g}{l} A + \left(\frac{g}{l} - \omega^2 \right) B &= 0 \\ \frac{g/l - \omega^2}{g/l} &= \frac{A}{B}\end{aligned}$$

$$\begin{aligned}\frac{g/l}{k/M + g/l - \omega^2} &= \frac{g/l - \omega^2}{g/l} \\ \left(\frac{g}{l} \right)^2 &= \left(\frac{g}{l} - \omega^2 \right) \left(\frac{k}{M} + \frac{g}{l} - \omega^2 \right) \\ &= \frac{k}{M} \frac{g}{l} + \left(\frac{g}{l} \right)^2 - \frac{g}{l} \omega^2 - \frac{k}{M} \omega^2 - \frac{g}{l} \omega^2 + (\omega^2)^2 \\ 0 &= (\omega^2)^2 - \left(2\frac{g}{l} + \frac{k}{M} \right) \omega^2 + \frac{k}{M} \frac{g}{l}\end{aligned}$$

$$\begin{aligned}
\omega^2 &= \frac{\left(2\frac{g}{l} + \frac{k}{M}\right) \pm \sqrt{\left(2\frac{g}{l} + \frac{k}{M}\right)^2 - 4\frac{k}{M}\frac{g}{l}}}{2} \\
&= \frac{\left(2\frac{g}{l} + \frac{k}{M}\right) \pm \sqrt{4\left(\frac{g}{l}\right)^2 + 4\frac{k}{M}\frac{g}{l} + \left(\frac{k}{M}\right)^2 - 4\frac{k}{M}\frac{g}{l}}}{2} \\
&= \frac{\left(2\frac{g}{l} + \frac{k}{M}\right) \pm \sqrt{4\left(\frac{g}{l}\right)^2 + \left(\frac{k}{M}\right)^2}}{2} \\
&= \frac{g}{l} + \frac{k}{2M} \pm \sqrt{\left(\frac{g}{l}\right)^2 + \left(\frac{k}{2M}\right)^2}
\end{aligned}$$

5.13

(a)

$$\begin{aligned}
\frac{d^2y}{dt^2} + \frac{3T}{2lm}y &= 0 \\
T &= \frac{2\pi}{\omega} \\
&= \frac{2\pi}{\sqrt{3T/2lm}} \\
&= 2\pi\sqrt{\frac{2lm}{3T}}
\end{aligned}$$

(c)

$$\begin{aligned}
\omega_0 &= \sqrt{\frac{T}{ml}} \\
\omega_n &= 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right] \\
\omega_1 &= 2\omega_0 \sin\left(\frac{\pi}{6}\right) \\
&= \omega_0 \\
&= \sqrt{\frac{T}{ml}} \\
\omega_2 &= 2\omega_0 \sin\left(\frac{\pi}{3}\right) \\
&= \sqrt{3}\omega_0 \\
&= \sqrt{\frac{3T}{ml}}
\end{aligned}$$

5.15

(c)

$$\begin{aligned}\omega_0 &= \sqrt{\frac{T}{ml}} \\ \omega_n &= 2\omega_0 \sin \frac{n\pi}{8} \\ \omega_1 &= 2\omega_0 \sin \frac{\pi}{8} \\ &= \omega_0 \sqrt{2 - \sqrt{2}} \\ &\approx 0.765\omega_0 \\ \omega_2 &= 2\omega_0 \sin \frac{\pi}{4} \\ &= \omega_0 \sqrt{2} \\ &\approx 1.42\omega_0 \\ \omega_3 &= 2\omega_0 \sin \frac{3\pi}{8} \\ &= \omega_0 \sqrt{2 + \sqrt{2}} \\ &\approx 1.85\omega_0\end{aligned}$$

6 Normal modes of continuous systems. Fourier analysis

6.1

(a)

$$f_1 = \frac{1}{2L} \sqrt{\frac{T}{\mu}} = 10 \text{ Hz}$$

(b) $5f_1$ (50 Hz) and multiples thereof

6.2

$$\begin{aligned}
 f_1 &= \frac{1}{2L} \sqrt{\frac{T}{\mu}} \\
 &= \frac{1}{2L} \sqrt{\frac{LT}{M}} \\
 &= \sqrt{\frac{T}{4LM}} \\
 f_2 &= 2f_1 \\
 f_3 &= 3f_1
 \end{aligned}$$

$$\begin{aligned}
 \omega_0 &= 2\sqrt{\frac{3T}{LM}} \\
 \omega_n &= 4\sqrt{\frac{3T}{LM}} \sin \frac{n\pi}{8} \\
 f'_n &= \frac{2}{\pi} \sqrt{\frac{3T}{LM}} \sin \frac{n\pi}{8} \\
 f'_1 &= \frac{2}{\pi} \sqrt{\frac{3T}{LM}} \frac{\sqrt{2-\sqrt{2}}}{2} \\
 &= \frac{2\sqrt{3(2-\sqrt{2})}}{\pi} \sqrt{\frac{T}{4LM}} \\
 &\approx 0.84f_1 \\
 f_2 &= \frac{2\sqrt{6}}{\pi} \sqrt{\frac{T}{4LM}} \\
 &\approx 1.56f_1 \\
 f_3 &= \frac{2\sqrt{3(2+\sqrt{2})}}{\pi} \sqrt{\frac{T}{4LM}} \\
 &\approx 2.04f_1
 \end{aligned}$$

6.5

$$\begin{aligned}
 \omega &= 2\pi f \\
 &= 2\pi \frac{v}{\lambda} \\
 &= 2\pi \frac{\sqrt{T/\mu}}{2L} \\
 &= \pi \frac{\sqrt{LT/m}}{L} \\
 &= \pi \sqrt{\frac{T}{Lm}}
 \end{aligned}$$

6.6

(a)

$$\xi(x, t) = A \sin\left(\frac{\omega x}{v}\right) \cos \omega t$$

$$\begin{aligned}
 \cos \frac{\omega L}{2v} &= 0 \\
 \frac{\omega L}{2v} &= \left(n - \frac{1}{2}\right) \pi \\
 \omega_n &= \left(n - \frac{1}{2}\right) \pi \frac{2v}{L} \\
 &= \frac{(2n-1)\pi}{L} \sqrt{\frac{Y}{\rho}}
 \end{aligned}$$

(b)

$$\begin{aligned}
 \lambda_n &= \frac{v}{f_n} \\
 &= \frac{2\pi v}{\omega_n} \\
 &= 2\pi v \frac{L}{(2n-1)\pi v} \\
 &= \frac{2L}{2n-1}
 \end{aligned}$$

(c) The nodes are at $x = 0$ and at each multiple of a half wavelength, i.e.

$$\frac{L}{2n-1}$$

6.9

(b)

$$\begin{aligned}\xi(x, t) &= A_1 \sin \frac{\pi x}{L} \cos 100\pi t + A_2 \sin \frac{2\pi x}{L} \cos 200\pi t \\ \xi_{\max}(L/4) &= A_1 \frac{\sqrt{2}}{2} \cos 100\pi t_1 + A_2 \cos 200\pi t_1 \\ \xi_{\max}(L/2) &= A_1 \cos 100\pi t_2 \\ \xi_{\max}(3L/4) &= A_1 \frac{\sqrt{2}}{2} \cos 100\pi t_3 - A_2 \cos 200\pi t_3\end{aligned}$$

$$\begin{aligned}A_1 &= \xi_{\max}(L/2) \\ &= 10 \mu\text{m}\end{aligned}$$

$$\begin{aligned}\xi_{\max}(L/4) &= A_1 \frac{\sqrt{2}}{2} + A_2 \\ &= \xi_{\max}(L/2) \frac{\sqrt{2}}{2} + A_2 \\ A_2 &= \xi_{\max}(L/4) - \xi_{\max}(L/2) \frac{\sqrt{2}}{2} \\ &= (10 \mu\text{m}) - (10 \mu\text{m}) \frac{\sqrt{2}}{2} \\ &= (10 \text{ mm}) \left(1 - \frac{\sqrt{2}}{2}\right) \\ &\approx 2.93 \mu\text{m}\end{aligned}$$

6.10

(a)

$$f_n = \frac{nc}{2L}$$

(b) (i)

$$\begin{aligned}n_{\min} &= \frac{2f_{\min}L}{c} \\ &= 4\,999\,990 \\ n_{\max} &= \frac{2f_{\max}L}{c} \\ &= 5\,000\,010 \\ n_{\max} - n_{\min} + 1 &= 21\end{aligned}$$

(ii)

$$\begin{aligned}n_{\max} - n_{\min} &< 2 \\ \frac{2f_{\max}L}{c} - \frac{2f_{\min}L}{c} &< 2 \\ L &< \frac{c}{f_{\max} - f_{\min}} \\ &= 15 \text{ cm}\end{aligned}$$

6.11

(a)

$$\begin{aligned}y_n(x, t) &= A_n \sin \frac{n\pi x}{L} \sin \left(\frac{n\pi}{L} \sqrt{\frac{LT}{M}} t \right) \\ y'_n(x, t) &= A_n \frac{n\pi}{L} \sqrt{\frac{LT}{M}} \sin \frac{n\pi x}{L} \cos \left(\frac{n\pi}{L} \sqrt{\frac{LT}{M}} t \right) \\ y'_n(x, 0) &= A_n \frac{n\pi}{L} \sqrt{\frac{LT}{M}} \sin \frac{n\pi x}{L} \\ dE &= \frac{1}{2} y'_n(x, 0)^2 dm \\ &= \frac{1}{2} \left(A_n \frac{n\pi}{L} \sqrt{\frac{LT}{M}} \sin \frac{n\pi x}{L} \right)^2 \frac{M}{L} dx \\ E &= \int dE \\ &= \frac{1}{2} \left(A_n \frac{n\pi}{L} \sqrt{\frac{LT}{M}} \right)^2 \frac{M}{L} \int_0^L \sin^2 \frac{n\pi x}{L} dx \\ &= \frac{A_n^2 n^2 \pi^2 T}{2L^2} \left[\frac{x}{2} - \frac{L}{4n\pi} \sin \frac{2n\pi x}{L} \right]_0^L \\ &= \frac{A_n^2 n^2 \pi^2 T}{4L}\end{aligned}$$

(b)

$$\begin{aligned}E &= \frac{A_1^2 1^2 \pi^2 T}{4L} + \frac{A_3^2 3^2 \pi^2 T}{4L} \\ &= \frac{\pi^2 T}{4L} (A_1^2 + 9A_3^2)\end{aligned}$$

6.12

(a)

$$\begin{aligned}
 W &= \int_0^h 2T \sin \theta \, dy \\
 &= 2T \int_0^h \frac{y}{L/2} \, dy \\
 &= \frac{4T}{L} \int_0^h y \, dy \\
 &= \frac{4T}{L} \left[\frac{1}{2} y^2 \right]_0^h \\
 &= \frac{2h^2 T}{L}
 \end{aligned}$$

(b)

$$\begin{aligned}
 T &= \frac{2\pi}{\omega} \\
 &= \frac{2L}{v} \\
 &= 2\sqrt{\frac{LM}{T}}
 \end{aligned}$$

6.14

(a)

$$\begin{aligned}
 y(x) &= Ax(L-x) \\
 B_n &= \frac{4AL^2}{\pi^3 n^3} (1 - \cos(\pi n)) \\
 &= \begin{cases} 0 & \text{even } n \\ \frac{8AL^2}{\pi^3 n^3} & \text{odd } n \end{cases} \\
 y(x) &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \\
 &= \frac{8AL^2}{\pi^3} \left[\sin\left(\frac{\pi x}{L}\right) + \frac{1}{27} \sin\left(\frac{3\pi x}{L}\right) + \frac{1}{225} \sin\left(\frac{5\pi x}{L}\right) + \cdots \right]
 \end{aligned}$$

(b)

$$\begin{aligned}y(x) &= A \sin \frac{\pi x}{L} \\B_n &= \frac{2}{L} \int_0^L A \sin \frac{\pi x}{L} \sin \frac{n\pi x}{L} dx \\&= \begin{cases} A & n = 1 \\ 0 & n \neq 1 \end{cases} \\y(x) &= A \sin \frac{\pi x}{L}\end{aligned}$$

(c)

$$\begin{aligned}y(x) &= \begin{cases} A \sin \frac{2\pi x}{L} & 0 \leq x \leq \frac{L}{2} \\ 0 & \frac{L}{2} \leq x \leq L \end{cases} \\B_n &= \frac{2}{L} \int_0^L y(x) \sin \frac{n\pi x}{L} dx \\&= \frac{2}{L} \int_0^{L/2} A \sin \frac{2\pi x}{L} \sin \frac{n\pi x}{L} dx \\&= \frac{4A \sin \frac{n\pi}{2}}{\pi(4 - n^2)} \\&= \begin{cases} \frac{A}{2} & n = 2 \\ 0 & \text{even } n \\ (-1)^{(n-1)/2} \frac{4A}{\pi(4 - n^2)} & \text{odd } n \end{cases}\end{aligned}$$