Advanced Engineering Mathematics Ordinary Differential Equations Notes

Chris Doble

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1 Introduction to Differential Equations

1.1 Definitions and Terminology

- An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation**
- An **ordinary differential equation** (ODE) is a differential equation that contains only ordinary (i.e. non-partial) derivatives of one or more functions with respect to a single independent variable
- A **partial differential equation** is a differential equation that contains only partial derivatives of one or more functions of two or more independent variables
- The **order** of a differential equation is the order of the highest derivative in the equation
- First order ODEs are sometimes written in the **differential form**

  $$M(x, y)\,dx + N(x, y)\,dy = 0$$

- n-th order ODEs in one dependent variable can be expressed by the **general form**

  $$F(x, y, y', \ldots, y^{(n)}) = 0$$

- It's possible to solve ODEs in the general form uniquely for the highest derivative $y^{(n)}$ in terms of the other $n+1$ variables, allowing them to be expressed in the **normal form**

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

- An n-th order ODE is said be **linear** in the variable y if it can be expressed in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - g(x) = 0$$

  i.e. the dependent variable y and all of its derivatives aren't raised to a power or used in nonlinear functions like e^y or $\sin y$, and the coefficients a_0, a_1, ..., a_n depend at most on the independent variable x

- A **nonlinear** ODE is one that is not linear

- A **solution** to an ODE is a function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, such that

$$F(x, \phi(x), \phi'(x), \ldots, \phi^n(x)) = 0 \text{ for all } x \text{ in } I.$$

- The **interval of definition**, **interval of validity**, or the **domain** of a solution is the interval over which the solution is valid

- A solution of a differential equation that is 0 on an interval I is said to be a **trivial solution**

- Because solutions to differential equations must be differentiable over their interval of validity, discontinuities, etc. must be excluded from the interval

- An **explicit solution** to an ODE is one where the dependent variable is expressed solely in terms of the independent variable and constants

- An **implicit solution** to an ODE is a relation $G(x, y) = 0$ over an interval I provided there exists at least one function ϕ that satisfies the relation as well as the ODE on I

- When solving a first-order ODE we usually obtain a solution containing a single arbitrary constant or parameter c. A solution containing an arbitrary constant represents a set of solution called a **one-parameter family of solutions**

- When solving an n-th order differential equation we usually obtain an **n-parameter family of solutions**

- A solution of a differential equation that is free from arbitrary parameters is called a **particular solution**

- A **singular solution** is a solution to a differential equation that isn't a member of a family of solutions
- A **system of ODEs** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable. A solution of such a system is a differentiable function for each equation defined on a common interval I that satisfy each equation of the system on that interval

1.2 Initial Value Problems

- An **initial value problem** is the problem of solving a differential equation with some given **initial conditions**, e.g. solve

  $$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

  subject to

  $$y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1}$$

- The domain of $y = f(x)$ differs depending on how it's considered:
  - As a function its domain is all real numbers for which it's defined
  - As a solution of a differential equation its domain is a single interval over which it's defined an differentiable
  - As a solution of an initial value problem its domain is a single interval over which it's defined, differentiable, and contains the initial conditions
- An initial value problem may not have any solutions. If it does it may have multiple.
- First-order initial value problems of the form

  $$\frac{dy}{dx} = f(x, y)$$

  $$y(x_0) = y_0$$

  are guaranteed to have a unique solution over an interval I containing x_0 if $f(x, y)$ and $\partial f / \partial y$ are continuous