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2 First-order ODEs

Form: IVP

$$\frac{dy}{dx} = f(x, y)$$
$$y(x_0) = y_0$$

Test: $f(x, y)$ and $\partial f/\partial y$ are continuous over I
Property: A unique solution is guaranteed over I

2.1 Separable Equations

Form:

$$\frac{dy}{dx} = g(x)h(y)$$

Solution: Divide by $h(y)$ then integrate with respect to x.

$$\begin{aligned}
\frac{dy}{dx} &= g(x)h(y) \\
\frac{1}{h(y)}\frac{dy}{dx} &= g(x) \\
\int \frac{1}{h(y)}\frac{dy}{dx}\,dx &= \int g(x)\,dx \\
\int \frac{1}{h(y)}\,dy &= \int g(x)\,dx \\
H(y) &= G(x) + c
\end{aligned}$$

2.2 Linear Equations

Form:

$$\frac{dy}{dx} + P(x)y = f(x)$$

Solution:

1. Determine the integrating factor $e^{\int P(x)\,dx}$
2. Multiply by the integrating factor
3. Recognise that the left hand side of the equation is the derivative of the product of the integrating factor and y
4. Integrate both sides of the equation
5. Solve for y

2.3 Exact Equations

Form:

$$z = f(x, y) = c$$

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = M(x, y)\,dx + N(x, y)\,dy = 0$$

Test:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution:

1. Integrate $M(x, y)$ with respect to x to find an expression for $z = f(x, y)$

$$\frac{\partial f}{\partial x} = M(x, y)$$
$$f(x, y) = \int M(x, y)\, dx + g(y)$$

2. Differentiate $f(x, y)$ with respect to y and equate it to $N(x, y)$ to find $g'(y)$

$$\frac{\partial f}{\partial y} = N(x, y) = \frac{\partial}{\partial y} \int M(x, y)\, dx + g'(y)$$
$$g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, dx$$

3. Integrate $g'(y)$ with respect to y to find $g(y)$ and substitute it into $f(x, y)$

4. Equate $f(x, y)$ with an unknown constant c

Note: The steps can be performed with x and y reversed, i.e. start by integrating $N(x, y)$ with respect to y, etc.

2.4 Exact Equations with Integration Constant

Form:

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

Test: $(M_y - N_x)/N$ is a function of x alone or $(N_x - M_y)/M$ is a function of y alone
Solution:

1. Compute the integrating factor

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\, dx}$$

or

$$\mu(y) = e^{\int \frac{N_x - M_y}{M}\, dy}$$

as appropriate

2. Multiple the equation by this factor

3. The equation is now exact and can be solved as above

2.5 Homogeneous Equations

Form:

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Test: M and N are homogeneous functions of the same degree
Solution:

1. Rewrite as

$$M(x,y) = x^\alpha M(1,u) \text{ and } N(x,y) = x^\alpha N(1,u) \text{ where } u = y/x$$

   or

$$M(x,y) = y^\alpha M(v,1) \text{ and } N(x,y) = y^\alpha N(v,1) \text{ where } v = x/y$$

2. Substitute $y = ux$ and $dy = u\,dx + x\,du$ or $x = vy$ and $dx = v\,dy + y\,dv$ as appropriate
3. Solve the resulting first-order separable DE
4. Substitude $u = y/x$ or $v = x/y$ as appropriate

2.6 Bernoulli's Equation

Form:

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

Test: $n \neq 0$ and $n \neq 1$
Solution:

1. Substitude $y = u^{1/(1-n)}$ and $\frac{dy}{dx} = \frac{d}{dx}(u^{1/(1-n)})$
2. Solve the resulting linear equation
3. Substitude $u = y^{1-n}$

2.7 Reduction to Separation of Variables

Form:

$$\frac{dy}{dx} = f(Ax + By + C), B \neq 0$$

Solution:

1. Substitute

$$Ax + By + C = u$$

2. Solve the resulting separable equation
3. Substitute

$$u = Ax + By + C$$

2.8 Riccati's Equation

Form:

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$$

Test: You know a particular solution y_1 of the equation
Solution:

1. Substitute $y = y_1 + u$ and $y' = y_1' + u'$
2. Solve the resulting Bernoulli equation
3. Substitude $u = y - y_1$

3 Higher-order ODEs

3.1 Initial Value Problems

Form: n-th order IVP

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx_{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

subject to

$$y(x_0) = y_0,\, y'(x_0) = y_1,\, \ldots,\, y^{(n-1)}(x_0) = y_{n-1}$$

Test: $a_n(x)$, $a_{n-1}(x)$, ..., $a_0(x)$, and $g(x)$ are continuous on an interval I and $a_n(x) \neq 0$ for every x in I
Property: A unique solution exists for every $x = x_0$ in I

3.2 Linear Independence

Form: A set of functions f_1, f_2, ..., f_n
Test: The Wronskian $W(f_1, f_2, \ldots, f_n) \neq 0$ for every x in an interval I where

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

Property: The functions are linearly independent in I

3.3 Linear Equations

3.3.1 Homogeneous Linear nth-Order Equations

The general solution is of the form

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

where c_i are arbitrary constants and y_i are a fundamental set of solutions (i.e. a set of n linearly independent solutions).

3.3.2 Nonhomogeneous Linear nth-Order Equations

The general solution is of the form

$$y = y_c + y_p = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

where y_c is the complementary function (i.e. the general solution of the associated homogeneous equation) and y_p is a particular solution.

3.3.3 Reduction of Order

Form:

$$y'' + P(x)y' + Q(x)y = 0$$

Test: A non-trivial solution $y_1(x)$ is known
Solution:

1. Recognise that the ratio of two linearly independent functions isn't constant, i.e.
$$u(x) = \frac{y_1(x)}{y_2(x)} \text{ or } y_2(x) = u(x)y_1(x)$$
2. Substitute $y_2(x) = u(x)y_1(x)$ into the DE — this will result in a DE involving only u'' and u' which can be treated as a linear first-order DE in $u' = w$
3. Solve for w
4. Substitute $w = u'$
5. Integrate to find u
6. Multiply by y_1 to find y_2

or equivalently

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx$$

3.3.4 Homogeneous Linear Equations with Constant Coefficients

Form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Solution:

1. Assume the equation has a solution of the form $y = e^{mx}$, giving

$$a_n m^n e^{mx} + a_{n-1} m^{n-1} e^{mx} + \cdots + a_1 m e^{mx} + a_0 e^{mx} = 0$$

2. Divide by e^{mx}, giving the auxiliary/characteristic equation

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

3. Solve for m, where

   - A real root m corresponds to a solution

$$y = ce^{mx}$$

   - Complex roots $\alpha \pm i\beta$ correspond to solutions

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

   - A root m of multiplicity k corresponds to the solutions

$$e^{mx},\ xe^{mx},\ x^2 e^{mx},\ \ldots,\ x^{k-1} x^{mx}$$

3.3.5 Method of Undetermined Coefficients

Form: A nonhomogeneous linear DE where the input function ($g(x)$) is comprised of constants, polynomials, exponentials $e^{\alpha x}$, sines, and cosines
Solution:

1. Solve the associated homogeneous equation
2. Assume the particular solution has the same form as the input function
3. If a term in the proposed solution is present in the complementary function, multiply it by x^n where n is the smallest positive integer that removes the duplication
4. Substitute the proposed solution into the DE
5. Solve for the unknown constants

3.3.6 Variation of Parameters

Form: A nonhomogeneous linear DE
Solution:

1. Solve the homogeneous equation to find the complementary function
2. Assume the solution has the form
$$y_p = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x)$$
where n is the order of the equation and y_i are the fundamental set of solutions from the complementary equation
3. Convert to standard form by dividing by the leading coefficient
$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$
4. Solve the system of linear equations
$$\begin{aligned} y_1u_1' + \cdots + y_nu_n' &= 0 \\ y_1'u_1' + \cdots + y_n'u_n' &= 0 \\ &\vdots \\ y_1^{(n-1)}u_1' + \cdots + y_n^{(n-1)}y_n' &= 0 \\ y_1^{(n)}u_1' + \cdots + y_n^{(n)}u_n' &= f(x) \end{aligned}$$
via Cramer's method:
   (a) Compute the Wronskian of y_i
   $$W = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \ddots & \vdots \\ y_1^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$
   (b) Compute u_i' for $i = 1, \ldots, n$ where
   $$u_i' = \frac{W_i}{W}$$
   and W_i is the determinant of the matrix formed by replacing the ith column of the Wronskian matrix with the column vector
   $$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(x) \end{bmatrix}$$
5. Integrate each u_i' to find u_i

3.3.7 Cauchy-Euler Equations

Form:

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

Solution:

- If the equation is homogeneous:

  1. Assume the equation has a solution of the form $y = x^m$, giving

     $$\begin{aligned} a_n x^n \frac{d^n y}{dx^n} &= a_n x^n m(m-1)(m-2)\cdots(m-n+1)x^{m-n} \\ &= a_n m(m-1)(m-2)\cdots(m-n+1)x^m \end{aligned}$$

     and the equation then becomes

     $$f(m)x^m = 0$$

     where $f(m)$ is a polynomial in m known as the auxiliary or characteristic equation, the roots of which form the general solution
  2. Solve the auxiliary equation where
     - A real root m corresponds to a solution

       $$y = cx^m$$

     - Complex roots $\alpha \pm i\beta$ correspond to solutions

       $$x^\alpha (c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x))$$

     - A root m of multiplicity k corresponds to solutions

       $$x^m,\ x^m \ln x,\ x^m (\ln x)^2\ \ \ldots,\ x^m (\ln x)^{k-1}$$

- If the equation is nonhomogeneous:

  1. Solve the associated homogeneous equation
  2. Find a particular solution via variation of parameters

3.3.8 Green's Functions for IVPs

Form: The IVP

$$y'' + P(x)y' + Q(x) = f(x)$$

subject to $y(x_0) = y_0$ and $y'(x_0) = y_1$
Solution:

1. Solve the homogeneous equation with nonhomogeneous conditions

$$y'' + P(x)y' + Q(x)y = 0,\ y(x_0) = y_0,\ y'(x_0) = y_1$$

giving the solution y_h and the fundamental set of solutions y_1 and y_2

2. Solve the nonhomogeneous equation with homogeneous conditions

$$y'' + P(x)y' + Q(x)y = f(x),\ y(x_0) = 0,\ y'(x_0) = 0$$

using the formula

$$y_p(x) = \int_{x_0}^{x} G(x,t)f(t)\,dt$$

where $G(x,t)$ is the Green's function for the differential equation

$$G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}$$

and $W(t)$ is the Wronskian

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

3. The solution is $y = y_h + y_p$

3.3.9 Green's Functions for BVPs

Form: The BVP

$$y'' + P(x)y' + Q(x)y = f(x)$$

subject to

$$A_1y(a) + B_1y(a) = 0$$

and

$$A_2y(b) + B_2y(b) = 0$$

Solution:

1. Solve the associated homogeneous equation to find the fundamental set of solution y_1 and y_2 valid on $[a,b]$

2. Ensure y_1 and y_2 satisfy the boundary conditions

$$A_1y_1(a) + B_1y_1(a) = 0$$

and

$$A_2y_2(b) + B_2y_2(b) = 0$$

   - It's important that y_1 satisfies the starting boundary condition and y_2 satisfies the ending!

3. Then a particular solution is

$$y_p(x) = \int_a^b G(x,t) f(t)\, dt$$

where $G(x,t)$ is the Green's function for the differential equation

$$G(x,t) = \begin{cases} \frac{y_1(t)y_2(x)}{W(t)} & a \le t \le x \\ \frac{y_1(x)y_2(t)}{W(t)} & x \le t \le b \end{cases}$$

and $W(t)$ is the Wronskian

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

3.4 Nonlinear Equations

3.4.1 Reducation of Order

Form: Nonlinear second-order DE

$$F(x, y', y'') = 0$$

i.e. y is missing
Solution:

1. Substitute $u = y'$ (and thus $u' = y''$)
2. Solve the resulting DE for u
3. Integrate to find y

Form: Nonlinear second-order DE

$$F(y, y', y'') = 0$$

i.e. x is missing
Solution:

1. Substitute $u = y'$ and
$$y'' = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy}$$
2. Solve the resulting DE for u
3. Integrate to find y

3.4.2 Taylor Series

Form: Nonlinear initial value problem
Solution:

1. Substitute the initial conditions into a Taylor series centred at x_0
2. Take further derivatives of the equation and substitute the initial conditions in to find additional terms for the Taylor series