

Quantum Computation and Quantum
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Part I

Fundamental concepts

2 Linear algebra

Exercise 2.1

$$(1, -1) + (1, 2) - (2, 1) = (0, 0)$$

Exercise 2.2

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Using the basis $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$ we get

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$|1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$a_{00} + a_{01} = 1$$

$$a_{10} + a_{11} = -1$$

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$a_{00} - a_{01} = 1$$

$$a_{10} - a_{11} = 1$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Exercise 2.5

$$\begin{aligned}
\begin{bmatrix} y_1^* & \cdots & y_n^* \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} &= \begin{bmatrix} y_1^* & \cdots & y_n^* \end{bmatrix} \left(\begin{bmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ z_2 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ z_n \end{bmatrix} \right) \\
&= \begin{bmatrix} y_1^* & \cdots & y_n^* \end{bmatrix} \begin{bmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} y_1^* & \cdots & y_n^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ z_n \end{bmatrix} \\
&= z_1 \begin{bmatrix} y_1^* & \cdots & y_n^* \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + z_n \begin{bmatrix} y_1^* & \cdots & y_n^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\
\begin{bmatrix} y_1^* & \cdots & y_n^* \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} &= y_1^* z_1 + y_2^* z_2 + \cdots + y_n^* z_n \\
&= (y_1 z_1^* + y_2 z_2^* + \cdots + y_n z_n^*)^* \\
&= \left(\begin{bmatrix} z_1^* & \cdots & z_n^* \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right)^* \\
\begin{bmatrix} v_1^* & \cdots & v_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} &= |v_1|^2 + \cdots + |v_n|^2 \\
&\geq 0
\end{aligned}$$

Exercise 2.6

$$\begin{aligned}
\left(\sum_i \lambda_i |w_i\rangle, |v\rangle \right) &= \left(|v\rangle, \sum_i \lambda_i |w_i\rangle \right)^* \\
&= \left(\sum_i \lambda_i (|v\rangle, |w_i\rangle) \right)^* \\
&= \sum_i \lambda_i^* (|v\rangle, |w_i\rangle)^* \\
&= \sum_i \lambda_i^* \langle w_i | v \rangle
\end{aligned}$$

Exercise 2.7

$$\begin{aligned}\langle w|v\rangle &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= (1)(1) + (1)(-1) \\ &= 0\end{aligned}$$

$$\begin{aligned}\frac{|w\rangle}{||w\rangle||} &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \frac{|v\rangle}{||v\rangle||} &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}\end{aligned}$$

Exercise 2.9

$$\begin{aligned}\sigma_0 &= |0\rangle\langle 0| + |1\rangle\langle 1| \\ \sigma_1 &= |1\rangle\langle 0| + |0\rangle\langle 1| \\ \sigma_2 &= i|1\rangle\langle 0| - i|0\rangle\langle 1| \\ \sigma_3 &= |0\rangle\langle 0| - |1\rangle\langle 1|\end{aligned}$$

Exercise 2.11

$$\begin{aligned}\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} &= \lambda^2 - 1 \\ \lambda &= \pm 1\end{aligned}$$

$$\begin{aligned}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} a \\ b \end{bmatrix} \\ b &= a \\ a &= b\end{aligned}$$

$$\begin{aligned}X_1 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} -a \\ -b \end{bmatrix} \\ b &= -a \\ a &= -b\end{aligned}$$

$$X_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Exercise 2.12

$$\begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$$

$$\lambda_1 = 1$$

$$\lambda_2 = 1$$

The eigenvalue 1 is degenerate. Because the matrix only has one eigenvector it can't be diagonalised.

Exercise 2.13

$$(|w\rangle\langle v|)^\dagger = \langle v|^\dagger |w\rangle^\dagger = |v\rangle\langle w|$$

Exercise 2.16

$$\begin{aligned} P^2 &= \left(\sum_{i=1}^k |i\rangle\langle i| \right) \left(\sum_{j=1}^k |j\rangle\langle j| \right) \\ &= \sum_{i,j=1}^k |i\rangle\langle i|j\rangle\langle j| \\ &= \sum_{i,j=1}^k |i\rangle\delta_{ij}\langle j| \\ &= \sum_{i=1}^k |i\rangle\langle i| \\ &= P \end{aligned}$$

Exercise 2.17

$$\begin{aligned} A &= A^\dagger \\ \sum_i \lambda_i |i\rangle\langle i| &= \left(\sum_i \lambda_i |i\rangle\langle i| \right)^\dagger \\ &= \sum_i \lambda_i^* |i\rangle\langle i| \end{aligned}$$

$\lambda_i = \lambda_i^*$ implies the eigenvalues are real.

Exercise 2.18

$$\begin{aligned}
 U^\dagger U &= I \\
 \left(\sum_i \lambda_i |i\rangle \langle i| \right)^\dagger \left(\sum_j \lambda_j |j\rangle \langle j| \right) &= \sum_k |k\rangle \langle k| \\
 \sum_{ij} \lambda_i^* \lambda_j |i\rangle \langle i|j\rangle \langle j| &= \sum_k |k\rangle \langle k| \\
 \sum_{ij} \lambda_i^* \lambda_j |i\rangle \delta_{ij} \langle j| &= \sum_k |k\rangle \langle k| \\
 \sum_i |\lambda_i|^2 |i\rangle \langle i| &= \sum_k |k\rangle \langle k| \\
 |\lambda_i|^2 &= 1 \\
 \lambda_i &= e^{i\theta}
 \end{aligned}$$

Exercise 2.19

$$\begin{aligned} I^\dagger &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^\dagger \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

$$\begin{aligned} I^\dagger I &= II \\ &= I \end{aligned}$$

$$\begin{aligned} X^\dagger &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^\dagger \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= X \end{aligned}$$

$$\begin{aligned} X^\dagger X &= XX \\ &= I \end{aligned}$$

$$\begin{aligned} Y^\dagger &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^\dagger \\ &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ &= Y \end{aligned}$$

$$\begin{aligned} Y^\dagger Y &= YY \\ &= I \end{aligned}$$

$$\begin{aligned} Z^\dagger &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^\dagger \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= Z \end{aligned}$$

$$\begin{aligned} Z^\dagger Z &= ZZ \\ &= I \end{aligned}$$

Exercise 2.22

$$\begin{aligned}\langle v_1|A|v_2\rangle &= \langle v_1|Av_2\rangle \\ &= \langle v_1|\lambda_2 v_2\rangle \\ &= \lambda_2 \langle v_1|v_2\rangle \\ \langle v_1|A|v_2\rangle &= \langle A^\dagger v_1|v_2\rangle \\ &= \langle Av_1|v_2\rangle \\ &= \langle \lambda_1 v_1|v_2\rangle \\ &= \lambda_1 \langle v_1|v_2\rangle \\ 0 &= (\lambda_2 - \lambda_1) \langle v_1|v_2\rangle \\ &= \langle v_1|v_2\rangle\end{aligned}$$

Exercise 2.23

For each basis vector $|i\rangle, i = 1, \dots, k$, $P|i\rangle = |i\rangle$ and so they are eigenvectors of P with eigenvalue 1. For each basis vector $|j\rangle, j = k+1, \dots, d$, $P|j\rangle = 0$ and so they are eigenvectors of P with eigenvalue of 0. That is a total of d eigenvectors so all eigenvalues are either 0 or 1.

Exercise 2.26

$$\begin{aligned}
|\psi\rangle &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\
|\psi\rangle^{\otimes 2} &= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \\
&= \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} \\
|\psi\rangle^{\otimes 3} &= \left(\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} \right) \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \\
&= \frac{|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle}{2^{3/2}} \\
|\psi\rangle &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
|\psi\rangle^{\otimes 2} &= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
|\psi\rangle^{\otimes 3} &= \frac{1}{2^{3/2}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}$$

Exercise 2.27

(a)

$$\begin{aligned}
X &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
Z &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
X \otimes Z &= \begin{bmatrix} (0)Z & (1)Z \\ (1)Z & (0)Z \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}
\end{aligned}$$

(b)

$$I \otimes X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(c)

$$X \otimes I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

No, the tensor product is not commutative.

Exercise 2.28

$$\begin{aligned} (A \otimes B)^* &= \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{bmatrix}^* \\ &= \begin{bmatrix} A_{11}^*B^* & A_{12}^*B^* & \cdots & A_{1n}^*B^* \\ A_{21}^*B^* & A_{22}^*B^* & \cdots & A_{2n}^*B^* \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}^*B^* & A_{m2}^*B^* & \cdots & A_{mn}^*B^* \end{bmatrix} \\ &= A^* \otimes B^* \\ (A \otimes B)^T &= \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{bmatrix}^T \\ &= \begin{bmatrix} A_{11}B^T & A_{21}B^T & \cdots & A_{m1}B^T \\ A_{12}B^T & A_{22}B^T & \cdots & A_{m2}B^T \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n}B^T & A_{2n}B^T & \cdots & A_{mn}B^T \end{bmatrix} \\ &= A^T \otimes B^T \\ (A \otimes B)^\dagger &= [(A \otimes B)^*]^T \\ &= (A^* \otimes B^*)^T \\ &= (A^*)^T \otimes (B^*)^T \\ &= A^\dagger \otimes B^\dagger \end{aligned}$$

Exercise 2.29

$$\begin{aligned}
(A \otimes B)^\dagger (A \otimes B)(|a\rangle \otimes |b\rangle) &= (A^\dagger \otimes B^\dagger)(A |a\rangle \otimes B |b\rangle) \\
&= A^\dagger A |a\rangle \otimes B^\dagger B |b\rangle \\
&= |a\rangle \otimes |b\rangle
\end{aligned}$$

Exercise 2.30

$$\begin{aligned}
(A \otimes B)^\dagger (|a\rangle \otimes |b\rangle) &= (A^\dagger \otimes B^\dagger)(|a\rangle \otimes |b\rangle) \\
&= (A \otimes B)(|a\rangle \otimes |b\rangle)
\end{aligned}$$

Exercise 2.34

$$\begin{aligned}
A &= \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \\
\lambda_1 &= 1 \\
\mathbf{x}_1 &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
\lambda_2 &= 7 \\
\mathbf{x}_2 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\
A &= PDP^{-1} \\
D &= P^{-1}AP \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \\
\sqrt{A} &= P\sqrt{D}P^{-1} \\
&= \frac{1}{2} \begin{bmatrix} 1 + \sqrt{7} & -1 + \sqrt{7} \\ -1 + \sqrt{7} & 1 + \sqrt{7} \end{bmatrix} \\
\ln A &= P \ln(D) P^{-1} \\
&= \frac{\ln 7}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\end{aligned}$$

Exercise 2.35

$$\begin{aligned}
\mathbf{v} &= (v_x, v_y, v_z) \\
\mathbf{v} \cdot \boldsymbol{\sigma} &= v_x \sigma_1 + v_y \sigma_2 + v_z \sigma_3 \\
&= \begin{bmatrix} v_z & v_x - i v_y \\ v_x + i v_y & -v_z \end{bmatrix} \\
A &= i\theta \mathbf{v} \cdot \boldsymbol{\sigma} \\
\lambda_1 &= -i\theta v \\
\mathbf{x}_1 &= \begin{bmatrix} (v_z - v)/(v_x + i v_y) \\ 1 \end{bmatrix} \\
\lambda_2 &= i\theta v \\
\mathbf{x}_2 &= \begin{bmatrix} (v_z + v)/(v_x + i v_y) \\ 1 \end{bmatrix} \\
P &= [\mathbf{x}_1 \quad \mathbf{x}_2] \\
\exp(i\theta \mathbf{v} \cdot \boldsymbol{\sigma}) &= P \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix} P^{-1} \\
&= \begin{bmatrix} \cos \theta + i v_z \sin \theta & i(v_x - i v_y) \sin \theta \\ i(v_x + i v_y) \sin \theta & \cos \theta - i v_z \sin \theta \end{bmatrix} \\
&= \cos \theta I + i \sin \theta \begin{bmatrix} v_z & v_x - i v_y \\ v_x + i v_y & -v_z \end{bmatrix} \\
&= \cos \theta I + i \sin \theta \mathbf{v} \cdot \boldsymbol{\sigma}
\end{aligned}$$

Exercise 2.36

$$\begin{aligned}
\text{tr } I &= 1 + 1 \\
&= 2 \\
\text{tr } X &= 0 + 0 \\
&= 0 \\
\text{tr } Y &= 0 + 0 \\
&= 0 \\
\text{tr } Z &= 1 - 1 \\
&= 0
\end{aligned}$$

Exercise 2.37

$$\begin{aligned}
\operatorname{tr}(AB) &= \sum_i \langle i|AB|i\rangle \\
&= \sum_i \langle i|AIB|i\rangle \\
&= \sum_i \langle i|Ai\rangle \langle i|B|i\rangle \\
&= \sum_i \langle i|B|i\rangle \langle i|A|i\rangle \\
&= \sum_i \langle i|BIA|i\rangle \\
&= \sum_i \langle i|BA|i\rangle \\
&= \operatorname{tr}(BA)
\end{aligned}$$

Exercise 2.38

$$\begin{aligned}
\operatorname{tr}(A+B) &= \sum_i (A_{ii} + B_{ii}) \\
&= \sum_i A_{ii} + \sum_i B_{ii} \\
&= \operatorname{tr} A + \operatorname{tr} B \\
\operatorname{tr}(zA) &= \sum_i zA_{ii} \\
&= z \sum_i A_{ii} \\
&= z \operatorname{tr} A
\end{aligned}$$

Exercise 2.42

$$\begin{aligned}
\frac{[A, B] + \{A, B\}}{2} &= \frac{1}{2}(AB - BA + AB + BA) \\
&= AB
\end{aligned}$$

Exercise 2.44

$$\begin{aligned}[A, B] &= 0 \\ AB - BA &= 0 \\ \{A, B\} &= 0 \\ AB + BA &= 0 \\ AB - BA + AB + BA &= 0 \\ 2AB &= 0 \\ AB &= 0 \\ A^{-1}AB &= 0 \\ B &= 0\end{aligned}$$

Exercise 2.45

$$\begin{aligned}[A, B]^{\dagger} &= (AB - BA)^{\dagger} \\ &= (AB)^{\dagger} - (BA)^{\dagger} \\ &= B^{\dagger}A^{\dagger} - A^{\dagger}B^{\dagger} \\ &= [B^{\dagger}, A^{\dagger}]\end{aligned}$$

Exercise 2.46

$$\begin{aligned}[A, B] &= AB - BA \\ &= -(BA - AB) \\ &= -[B, A]\end{aligned}$$

Exercise 2.47

$$\begin{aligned}(i[A, B])^{\dagger} &= -i[B^{\dagger}, A^{\dagger}] \\ &= -i[B, A] \\ &= i[A, B]\end{aligned}$$

Exercise 2.48

$$J = A$$

$$K = A$$

$$U = I$$

$$J = I$$

$$K = I$$

$$U = A$$

$$J = A$$

$$K = A$$

$$U = I$$

Exercise 2.51

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H^\dagger = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H^\dagger H = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= I$$

Exercise 2.52

$$H^2 = HH$$

$$= H^\dagger H$$

$$I$$

Exercise 2.53

$$\begin{aligned}
H\mathbf{v} &= \lambda\mathbf{v} \\
\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{v} - \lambda\mathbf{v} &= 0 \\
\begin{bmatrix} \frac{1}{\sqrt{2}} - \lambda & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} - \lambda \end{bmatrix} \mathbf{v} &= 0 \\
\left(\frac{1}{\sqrt{2}} - \lambda\right) \left(-\frac{1}{\sqrt{2}} - \lambda\right) - \frac{1}{2} &= 0 \\
-\frac{1}{2} + \lambda^2 - \frac{1}{2} &= 0 \\
\lambda^2 - 1 &= 0 \\
(\lambda - 1)(\lambda + 1) &= 0
\end{aligned}$$

The eigenvalues are ± 1 . The eigenvectors are $\begin{bmatrix} 1 \\ -1 \pm \sqrt{2} \end{bmatrix}$.

Exercise 2.54

$$\begin{aligned}
\exp(A) \exp(B) &= \exp\left(\sum_i a_i |i\rangle \langle i|\right) \exp\left(\sum_i b_i |i\rangle \langle i|\right) \\
&= \exp\left(\sum_i a_i |i\rangle \langle i| + \sum_i b_i |i\rangle \langle i|\right) \\
&= \exp(A + B)
\end{aligned}$$

Exercise 2.55

$$\begin{aligned}
U(t_1, t_2) &= \exp\left[\frac{-iH(t_2 - t_1)}{\hbar}\right] \\
U(t_1, t_2)^\dagger &= \exp\left[\frac{iH(t_2 - t_1)}{\hbar}\right] \\
U(t_1, t_2)^\dagger U(t_1, t_2) &= 1
\end{aligned}$$

Exercise 2.56

$$\begin{aligned}
K &= -i \ln(U) \\
\ln U &= \ln \left(\sum_u u |u\rangle \langle u| \right) \\
&= \sum_u \ln(u) |u\rangle \langle u| \\
&= \sum_u i\theta_u |u\rangle \langle u| \\
K &= -i \ln(U) \\
&= \sum_u \theta_u |u\rangle \langle u| \\
K^\dagger &= \left(\sum_u \theta_u |u\rangle \langle u| \right)^\dagger \\
&= \sum_u \theta_u |u\rangle \langle u| \\
&= K
\end{aligned}$$

Exercise 2.57

Assuming the state to be measured is $|\psi\rangle$, then the state of the system after the first measurement is

$$|\psi'\rangle = \frac{L_l |\psi\rangle}{\sqrt{\langle \psi | L_l^\dagger L_l | \psi \rangle}}.$$

The state of the system after the second measurement is

$$\begin{aligned}
|\psi''\rangle &= \frac{M_m |\psi'\rangle}{\sqrt{\langle \psi' | M_m^\dagger M_m | \psi' \rangle}} \\
&= \frac{1}{\sqrt{\langle \frac{L_l |\psi\rangle}{\sqrt{\langle \psi | L_l^\dagger L_l | \psi \rangle}} | M_m^\dagger M_m | \frac{L_l |\psi\rangle}{\sqrt{\langle \psi | L_l^\dagger L_l | \psi \rangle}} \rangle}} \frac{M_m L_l |\psi\rangle}{\sqrt{\langle \psi | L_l^\dagger L_l | \psi \rangle}} \\
&= \frac{M_m L_l |\psi\rangle}{\sqrt{\langle \psi | L_l^\dagger M_m^\dagger M_m L_l | \psi \rangle}} \\
&= \frac{N_{lm} \psi}{\sqrt{\langle \psi | N_{lm}^\dagger N_{lm} | \psi \rangle}}
\end{aligned}$$

Exercise 2.58

$$\begin{aligned}\langle M \rangle &= m \\ \Delta(M) &= 0\end{aligned}$$

Exercise 2.59

$$\begin{aligned}\langle X \rangle &= \langle 0|X|0 \rangle \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= 0 \\ \langle X^2 \rangle &= \langle 0|X^2|0 \rangle \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= 1 \\ \Delta(X) &= \sqrt{\langle X^2 \rangle - \langle X \rangle^2} \\ &= 1\end{aligned}$$

Exercise 2.60

$$\begin{aligned}\mathbf{v} \cdot \boldsymbol{\sigma} &= v_1 \sigma_x + v_2 \sigma_y + v_3 \sigma_z \\ &= v_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix} \\ 0 &= (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) \\ &= \lambda^2 - v_1^2 - v_2^2 - v_3^2 \\ &= \lambda^2 - 1 \\ &= (\lambda - 1)(\lambda + 1) \\ \lambda &= \pm 1\end{aligned}$$