Advanced Engineering Mathematics Systems of Differential Equations by Dennis G. Zill Notes

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August 2023

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10 Systems of Linear Differential Equations

10.1 Theory of Linear Systems

- A system of the form

$$\begin{aligned}
\frac{dx_1}{dt} &= g_1(t, x_1, x_2, \ldots, x_n) \\
\frac{dx_2}{dt} &= g_2(t, x_1, x_2, \ldots, x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= g_n(t, x_1, x_2, \ldots, x_n)
\end{aligned}$$

  is called a **first-order system**.

- When each of the functions $g_n(t, x_1, x_2, \ldots, x_n)$ is linear in the dependent variables $x_1, x_2, \ldots, x_n$, we get the **normal form** of a first-order system of linear equations

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n + f_1(t) \\
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n + f_2(t) \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n + f_n(t).
\end{aligned}$$

Such a system is called a **linear system**.

- When $f_i(t) = 0$ for $i = 1, 2, \ldots, n$ the linear system is said to be **homogeneous**, otherwise it's **nonhomogenous**.

- If $\mathbf{X}$, $\mathbf{A}(t)$, and $\mathbf{F}(t)$ denote the matrices

$$\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \vdots & & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{pmatrix}$$

$$\mathbf{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

then homogeneous linear systems can be written

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

and nonhomogeneous linear systems can be written

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}.$$

- A **solution vector** on an interval I is any column matrix

$$\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

whose entries are differentiable functions satisfying the linear system on the interval.

- The entries of a solution vector can be considered a set of parametric equations that define a curve in n-space. Such a curve is called a **trajectory**.

- The problem of solving

$$\mathbf{X}' = \mathbf{A}(t)\mathbf{X} + \mathbf{F}(t)$$

subject to

$$\mathbf{X}(t_0) = \mathbf{X}_0$$

is an **initial value problem** in matrix form.

- The **superposition principle** states that if $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are solution vectors of a homogeneous linear system on an interval I, then

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + c_n\mathbf{X}_n$$

  where c_n are arbitrary constants is also a solution.

- If $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are a set of solution vectors of a homogeneous linear system on an interval I, the set is said to be **linearly dependent** if there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + x_n\mathbf{X}_n = \mathbf{0}$$

  for every t in the interval. Otherwise the set is said to be **linearly independent**.

- A set of solution vectors

$$\mathbf{X}_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}, \quad \ldots, \quad \mathbf{X}_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$$

  is linearly independent on an interval I if the **Wronskian**

$$W(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) = \begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{vmatrix} \neq 0$$

  for every t in the interval.

- Any set of n linearly independent solution vectors of a homogeneous linear system on an interval I is said to be a **fundamental set of solutions** on that interval.

- If $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are a fundamental set of solutions of a homogeneous linear system on an interval I, then the **general solution** of the system on that interval is

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + c_n\mathbf{X}_n$$

  where c_i are arbitrary constants.

- For nonhomogenous systems, a **particular solution** $\mathbf{X}_p$ on an interval I is any vector, free from arbitrary parameters, whose entries are functions that satify the system.

- For nonhomogeneous systems, the **general solution** of the system on the interval is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$$

  where $\mathbf{X}_c$ is the general solution of the associated homogeneous system (the **complementary function**) and $\mathbf{X}_p$ is a particular solution of the nonhomogeneous system.