

# Advanced Engineering Mathematics Systems of Differential Equations by Dennis G. Zill Problems

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## 10 Systems of Linear Differential Equations

### 10.1 Theory of Linear Systems

#### 10.1.1

$$\mathbf{x}' = \begin{pmatrix} 3 & -5 \\ 4 & 8 \end{pmatrix} \mathbf{x}$$

#### 10.1.3

$$\mathbf{x}' = \begin{pmatrix} -3 & 4 & -9 \\ 6 & -1 & 0 \\ 10 & 4 & 3 \end{pmatrix} \mathbf{x}$$

10.1.5

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} t-1 \\ -3t^2 \\ t^2-t+2 \end{pmatrix}$$

10.1.7

$$\begin{aligned} \frac{dx}{dt} &= 4x + 2y + e^t \\ \frac{dy}{dt} &= -x + 3y - e^t \end{aligned}$$

10.1.9

$$\begin{aligned} \frac{dx}{dt} &= x - y + 2z + e^{-t} - 3t \\ \frac{dy}{dt} &= 3x - 4y + z + 2e^{-t} + t \\ \frac{dz}{dt} &= -2x + 5y + 6z + 2e^{-t} - t \end{aligned}$$

10.1.11

$$\begin{aligned} 3(e^{-5t}) - 4(2e^{-5t}) &= -5e^{-5t} \\ &= \frac{dx}{dt} \\ 4(e^{-5t}) - 7(2e^{-5t}) &= -10e^{-5t} \\ &= \frac{dy}{dt} \end{aligned}$$

10.1.13

$$\begin{aligned} -(-e^{-3t/2}) + \frac{1}{4}(2e^{-3t/2}) &= \frac{3}{2}e^{-3t/2} \\ &= \frac{dx}{dt} \\ (-e^{-3t/2}) - (2e^{-3t/2}) &= -3e^{-3t/2} \\ &= \frac{dy}{dt} \end{aligned}$$

**10.1.17**

$$\begin{aligned}
W(\mathbf{X}_1, \mathbf{X}_2) &= \begin{vmatrix} e^{-2t} & e^{-6t} \\ e^{-2t} & -e^{-6t} \end{vmatrix} \\
&= -e^{-8t} - e^{-8t} \\
&= -2e^{-8t} \\
&\neq 0 \text{ for } t \in (-\infty, \infty)
\end{aligned}$$

Yes, they form a fundamental set.

**10.1.19**

$$\begin{aligned}
W(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) &= \begin{vmatrix} 1+t & 1 & 3+2t \\ -2+2t & -2 & -6+4t \\ 4+2t & 4 & 12+4t \end{vmatrix} \\
&= 0
\end{aligned}$$

No, they don't form a fundamental set.

**10.1.21**

$$\begin{aligned}
x &= 2t + 5 \\
y &= -t + 1 \\
\frac{dx}{dt} &= (2t + 5) + 4(-t + 1) + 2t - 7 \\
&= 2 \\
\frac{dy}{dt} &= 3(2t + 5) + 2(-t + 1) - 4t - 18 \\
&= -1
\end{aligned}$$

**10.1.23**

$$\begin{aligned}
x &= e^t + te^t \\
x' &= 2e^t + te^t \\
y &= e^t - te^t \\
y' &= -te^t \\
\frac{dx}{dt} &= 2(e^t + te^t) + (e^t - te^t) - e^t \\
&= 2e^t + te^t \\
\frac{dy}{dt} &= 3(e^t + te^t) + 4(e^t - te^t) - 7e^t \\
&= -te^t
\end{aligned}$$

## 10.2 Homogeneous Linear Systems

10.2.1

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

10.2.3

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} e^t$$

10.2.5

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-10t} + c_2 \begin{pmatrix} 5 \\ 2 \end{pmatrix} e^{8t}$$

10.2.7

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t$$

10.2.13

$$\mathbf{X} = 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/2} + 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t/2}$$

10.2.15

(a)

$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{3}{100}x_1 + \frac{1}{100}x_2 \\ \frac{dx_2}{dt} &= \frac{2}{100}x_1 - \frac{2}{100}x_2 \\ \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} &= \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{2}{100} & -\frac{2}{100} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

(b)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\frac{35}{3} \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t/25} + \frac{50}{3} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} e^{-t/100}$$

10.2.21

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \left[ \begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right]$$

**10.2.23**

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix} \right] e^{2t}$$

**10.2.25**

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^t$$

**10.2.31**

$$\mathbf{X} = -\frac{1}{2} \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + 13 \left[ \begin{pmatrix} 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \right] e^{4t}$$

**10.2.33**

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{K}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

**10.2.35**

$$\begin{aligned} \mathbf{X} &= c_1 \begin{pmatrix} 1 \\ 2-i \end{pmatrix} e^{(4+i)t} + c_2 \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{(4-i)t} \\ &= c_1 \left[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin t \right] e^{4t} + c_2 \left[ \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t \right] e^{4t} \\ &= c_1 \begin{pmatrix} \cos t \\ 2 \cos t + \sin t \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} \sin t \\ 2 \sin t - \cos t \end{pmatrix} e^{4t} \end{aligned}$$

10.2.37

$$\begin{aligned}
 \mathbf{X} &= c_1 \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{(4+i)t} + c_2 \begin{pmatrix} 1 \\ -1-i \end{pmatrix} e^{(4-i)t} \\
 &= c_1 \left[ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t \right] e^{4t} + c_2 \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin t \right] e^{4t} \\
 &= c_1 \begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix} e^{4t}
 \end{aligned}$$

10.2.39

$$\begin{aligned}
 \mathbf{X} &= c_1 \begin{pmatrix} 5 \\ 4-3i \end{pmatrix} e^{3i} + c_2 \begin{pmatrix} 5 \\ 4+3i \end{pmatrix} e^{-3i} \\
 &= c_1 \left[ \begin{pmatrix} 5 \\ 4 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ -3 \end{pmatrix} \sin 3t \right] + c_2 \left[ \begin{pmatrix} 0 \\ -3 \end{pmatrix} \cos 3t + \begin{pmatrix} 5 \\ 4 \end{pmatrix} \sin 3t \right] \\
 &= c_1 \begin{pmatrix} 5 \cos 3t \\ 4 \cos 3t + 3 \sin 3t \end{pmatrix} + c_2 \begin{pmatrix} 5 \sin 3t \\ 4 \sin 3t - 3 \cos 3t \end{pmatrix}
 \end{aligned}$$

10.2.47

$$\begin{aligned}
 \mathbf{X} &= c_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1+5i \\ 1 \\ 1 \end{pmatrix} e^{5it} + c_3 \begin{pmatrix} 1-5i \\ 1 \\ 1 \end{pmatrix} e^{-5it} \\
 &= c_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t + c_2 \left[ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cos 5t - \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \sin 5t \right] \\
 &\quad + c_3 \left[ \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \cos 5t + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \sin 5t \right] \\
 &= c_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t + c_2 \begin{pmatrix} \cos 5t - 5 \sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + c_3 \begin{pmatrix} 5 \cos 5t + \sin 5t \\ \sin 5t \\ \sin 5t \end{pmatrix} \\
 &= - \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t - \begin{pmatrix} \cos 5t - 5 \sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + 6 \begin{pmatrix} 5 \cos 5t + \sin 5t \\ \sin 5t \\ \sin 5t \end{pmatrix}
 \end{aligned}$$

10.2.49

(a)

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{20} & 0 & \frac{1}{10} \\ \frac{1}{20} & -\frac{1}{20} & 0 \\ 0 & \frac{1}{20} & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$



(b)

$$\begin{aligned}
\mathbf{X} &= c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1-i \\ i \\ 1 \end{pmatrix} e^{(-\frac{1}{10} + \frac{1}{20}i)t} + c_3 \begin{pmatrix} -1+i \\ -i \\ 1 \end{pmatrix} e^{(-\frac{1}{10} - \frac{1}{20}i)t} \\
&= c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \left[ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cos \frac{1}{20}t - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \sin \frac{1}{20}t \right] e^{-t/10} \\
&\quad + c_3 \left[ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cos \frac{1}{20}t + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \sin \frac{1}{20}t \right] e^{-t/10} \\
&= c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \sin \frac{t}{20} - \cos \frac{t}{20} \\ -\sin \frac{t}{20} \\ \cos \frac{t}{20} \end{pmatrix} e^{-t/10} \\
&\quad + c_3 \begin{pmatrix} -\cos \frac{t}{20} - \sin \frac{t}{20} \\ \cos \frac{t}{20} \\ \sin \frac{t}{20} \end{pmatrix} e^{-t/10} \\
&= 11 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - 6 \begin{pmatrix} \sin \frac{t}{20} - \cos \frac{t}{20} \\ -\sin \frac{t}{20} \\ \cos \frac{t}{20} \end{pmatrix} e^{-t/10} \\
&\quad - 2 \begin{pmatrix} -\cos \frac{t}{20} - \sin \frac{t}{20} \\ \cos \frac{t}{20} \\ \sin \frac{t}{20} \end{pmatrix} e^{-t/10}
\end{aligned}$$

### 10.3 Solution by Diagonalization

10.3.1

$$\mathbf{X} = \begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 e^{7t} \\ c_2 e^{-4t} \end{pmatrix} = \begin{pmatrix} 3c_1 e^{7t} - 2c_2 e^{-4t} \\ c_1 e^{7t} + 3c_2 e^{-4t} \end{pmatrix}$$

10.3.3

$$\mathbf{X} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_1 e^{3t/2} \\ c_2 e^{t/2} \end{pmatrix} = \begin{pmatrix} c_1 e^{3t/2} + c_2 e^{t/2} \\ 2c_1 e^{3t/2} - 2c_2 e^{t/2} \end{pmatrix}$$

10.3.5

$$\mathbf{X} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{6t} \\ c_2 e^{-4t} \\ c_3 e^{2t} \end{pmatrix} = \begin{pmatrix} -c_2 e^{-4t} + c_3 e^{2t} \\ c_2 e^{-4t} + c_3 e^{2t} \\ c_1 e^{6t} + c_3 e^{2t} \end{pmatrix}$$

10.3.11

(a)

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\mathbf{M}$  has an inverse because it has a nonzero determinant (the product of the diagonal entries).

(b)

$$\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & \frac{k_2}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

(c)

$$\mathbf{X} = \mathbf{P}\mathbf{Y}$$

$$\mathbf{P}\mathbf{Y}'' + \mathbf{B}\mathbf{P}\mathbf{Y} = \mathbf{0}$$

$$\mathbf{Y}'' + \mathbf{P}^{-1}\mathbf{B}\mathbf{P}\mathbf{Y} = \mathbf{0}$$

$$\mathbf{Y}'' + \mathbf{D}\mathbf{Y} = \mathbf{0}$$

$$\begin{pmatrix} y_1'' \\ y_2'' \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{0}$$

$$y_1'' + 6y_1 = 0$$

$$y_1 = c_1 \cos \sqrt{6}t + c_2 \sin \sqrt{6}t$$

$$y_2'' + y_2 = 0$$

$$y_2 = c_3 \cos t + c_4 \sin t$$

$$\mathbf{X} = \mathbf{P}\mathbf{Y}$$

$$= \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \cos \sqrt{6}t + c_2 \sin \sqrt{6}t \\ c_3 \cos t + c_4 \sin t \end{pmatrix}$$

$$= \begin{pmatrix} -2c_1 \cos \sqrt{6}t - 2c_2 \sin \sqrt{6}t + c_3 \cos t + c_4 \sin t \\ c_1 \cos \sqrt{6}t + c_2 \sin \sqrt{6}t + 2c_3 \cos t + 2c_4 \sin t \end{pmatrix}$$

$$= c_3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos t + c_4 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t + c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cos \sqrt{6}t + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \sin \sqrt{6}t$$

## 10.4 Nonhomogeneous Linear Systems

### 10.4.1

$$\begin{aligned}\begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -7 \\ 5 \end{pmatrix} \\ \mathbf{X}_c &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^t \\ \mathbf{X}_p &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} -7 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 2a_1 + 3a_2 - 7 \\ -a_1 - 2a_2 + 5 \end{pmatrix} \\ \mathbf{X}_p &= \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ \mathbf{X} &= \mathbf{X}_c + \mathbf{X}_p \\ &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} -1 \\ 3 \end{pmatrix}\end{aligned}$$

### 10.4.3

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t}$$

$$\mathbf{X}_p = \begin{pmatrix} a_3 t^2 + a_2 t + a_1 \\ b_3 t^2 + b_2 t + b_1 \end{pmatrix}$$

$$\begin{pmatrix} 2a_3 t + a_2 \\ 2b_3 t + b_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a_3 t^2 + a_2 t + a_1 \\ b_3 t^2 + b_2 t + b_1 \end{pmatrix} + \begin{pmatrix} -2t^2 \\ t + 5 \end{pmatrix}$$

$$= \begin{pmatrix} (a_3 + 3b_3 - 2)t^2 + (a_2 + 3b_2)t + (a_1 + 3b_1) \\ (3a_3 + b_3)t^2 + (3a_2 + b_2 + 1)t + (3a_1 + b_1 + 5) \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (a_3 + 3b_3 - 2)t^2 + (a_2 - 2a_3 + 3b_2)t + (a_1 - a_2 + 3b_1) \\ (3a_3 + b_3)t^2 + (3a_2 + b_2 - 2b_3 + 1)t + (3a_1 + b_1 - b_2 + 5) \end{pmatrix}$$

$$a_3 = -\frac{1}{4}$$

$$b_3 = \frac{3}{4}$$

$$a_2 = \frac{1}{4}$$

$$b_2 = -\frac{1}{4}$$

$$a_1 = -2$$

$$b_1 = \frac{3}{4}$$

$$\mathbf{X}_p = \begin{pmatrix} -\frac{1}{4}t^2 + \frac{1}{4}t - 2 \\ \frac{3}{4}t^2 - \frac{1}{4}t + \frac{3}{4} \end{pmatrix}$$

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} + \begin{pmatrix} -\frac{1}{4}t^2 + \frac{1}{4}t - 2 \\ \frac{3}{4}t^2 - \frac{1}{4}t + \frac{3}{4} \end{pmatrix}$$

### 10.4.5

$$\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} \frac{1}{9} \\ 1 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} e^{3t} \\
\mathbf{X}_p &= \begin{pmatrix} a \\ b \end{pmatrix} e^t \\
\begin{pmatrix} a \\ b \end{pmatrix} e^t &= \begin{pmatrix} 4 & \frac{1}{3} \\ 9 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} e^t + \begin{pmatrix} -3 \\ 10 \end{pmatrix} e^t \\
\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 4a + \frac{1}{3}b - 3 \\ 9a + 6b + 10 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 3a + \frac{1}{3}b - 3 \\ 9a + 5b + 10 \end{pmatrix} \\
\mathbf{X}_p &= \begin{pmatrix} \frac{55}{36} \\ -\frac{19}{4} \end{pmatrix} e^t \\
\mathbf{X} &= c_1 \begin{pmatrix} \frac{1}{9} \\ 1 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} \frac{55}{36} \\ -\frac{19}{4} \end{pmatrix} e^t
\end{aligned}$$

### 10.4.9

$$\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^t \\
\mathbf{X}_p &= \begin{pmatrix} a \\ b \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} -a - 2b + 3 \\ 3a + 4b + 3 \end{pmatrix} \\
\mathbf{X}_p &= \begin{pmatrix} -9 \\ 6 \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} -9 \\ 6 \end{pmatrix} \\
\begin{pmatrix} -4 \\ 5 \end{pmatrix} &= c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -9 \\ 6 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -2c_1 - c_2 - 5 \\ 3c_1 + c_2 + 1 \end{pmatrix} \\
\mathbf{X} &= 4 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{2t} - 13 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} -9 \\ 6 \end{pmatrix}
\end{aligned}$$

### 10.4.11

(a)

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{1}{50} & -\frac{1}{25} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(b)

$$\begin{aligned} \mathbf{X}_c &= c_1 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} e^{-t/20} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/50} \\ \mathbf{X}_p &= \begin{pmatrix} a \\ b \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{1}{50} & -\frac{1}{25} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{3}{100}a + \frac{1}{100}b \\ \frac{1}{50}a - \frac{1}{25}b + 1 \end{pmatrix} \\ \mathbf{X}_p &= \begin{pmatrix} 10 \\ 30 \end{pmatrix} \\ \mathbf{X} &= c_1 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} e^{-t/20} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/50} + \begin{pmatrix} 10 \\ 30 \end{pmatrix} \\ \begin{pmatrix} 60 \\ 10 \end{pmatrix} &= c_1 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 10 \\ 30 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -\frac{1}{2}c_1 + c_2 - 50 \\ c_1 + c_2 + 20 \end{pmatrix} \\ \mathbf{X} &= -\frac{140}{3} \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} e^{-t/20} + \frac{80}{3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/50} + \begin{pmatrix} 10 \\ 30 \end{pmatrix} \end{aligned}$$

(c)

$$\begin{aligned} \lim_{t \rightarrow \infty} x_1(t) &= \lim_{t \rightarrow \infty} \frac{70}{3} e^{-t/20} + \frac{80}{3} e^{-t/50} + 10 \\ &= 10 \\ \lim_{t \rightarrow \infty} x_2(t) &= \lim_{t \rightarrow \infty} -\frac{140}{3} e^{-t/20} + \frac{80}{3} e^{-t/50} + 30 \\ &= 30 \end{aligned}$$

10.4.13

$$\begin{aligned}
 \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} \\
 \mathbf{X}_c &= c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
 \mathbf{X}_p &= \Phi(t) \int \Phi^{-1}(t) \mathbf{F} dt \\
 &= \begin{pmatrix} 3e^t & 1 \\ 2e^t & 1 \end{pmatrix} \int \begin{pmatrix} e^{-t} & -e^{-t} \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \end{pmatrix} dt \\
 &= \begin{pmatrix} 3e^t & 1 \\ 2e^t & 1 \end{pmatrix} \int \begin{pmatrix} 5e^{-t} \\ -11 \end{pmatrix} dt \\
 &= \begin{pmatrix} 3e^t & 1 \\ 2e^t & 1 \end{pmatrix} \begin{pmatrix} -5e^{-t} \\ -11t \end{pmatrix} \\
 &= \begin{pmatrix} -15 - 11t \\ -10 - 11t \end{pmatrix} \\
 \mathbf{X} &= c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -15 - 11t \\ -10 - 11t \end{pmatrix}
 \end{aligned}$$

10.4.15

$$\begin{aligned}
 \mathbf{X}' &= \begin{pmatrix} 3 & -5 \\ \frac{3}{4} & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2} \\
 \mathbf{X}_c &= c_1 \begin{pmatrix} \frac{10}{3} \\ 1 \end{pmatrix} e^{3t/2} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t/2} \\
 \mathbf{X}_p &= \begin{pmatrix} \frac{1}{2}(-15 - 13t) \\ \frac{1}{4}(-9 - 13t) \end{pmatrix} e^{t/2} \\
 \mathbf{X} &= c_1 \begin{pmatrix} \frac{10}{3} \\ 1 \end{pmatrix} e^{3t/2} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t/2} + \frac{1}{4} \begin{pmatrix} -30 - 26t \\ -9 - 13t \end{pmatrix} e^{t/2}
 \end{aligned}$$

**10.4.33**

$$\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} \\
\mathbf{X}_p &= \begin{pmatrix} 2e^{2t}t - 2e^{4t}t - e^{2t} + e^{4t} \\ 2e^{2t}t + 2e^{4t}t + e^{2t} + e^{4t} \end{pmatrix} \\
&= \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} \\
\mathbf{X} &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} \\
&\quad + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} \\
\begin{pmatrix} 1 \\ 1 \end{pmatrix} &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -c_1 + c_2 - 1 \\ c_1 + c_2 + 1 \end{pmatrix} \\
\mathbf{X} &= \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{4t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t}
\end{aligned}$$

**10.4.35**

$$\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-12t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} \\
\mathbf{X}_p &= \frac{1}{29} \begin{pmatrix} -76 \cos t + 332 \sin t \\ -168 \cos t + 276 \sin t \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-12t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} - \frac{4}{29} \begin{pmatrix} 19 \\ 42 \end{pmatrix} \cos t + \frac{4}{29} \begin{pmatrix} 83 \\ 69 \end{pmatrix} \sin t \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{4}{29} \begin{pmatrix} 19 \\ 42 \end{pmatrix} \\
&= \begin{pmatrix} -3c_1 + c_2 - \frac{76}{29} \\ c_1 + 3c_2 - \frac{168}{29} \end{pmatrix} \\
\mathbf{X} &= -\frac{6}{29} \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-12t} + 2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} - \frac{4}{29} \begin{pmatrix} 19 \\ 42 \end{pmatrix} \cos t + \frac{4}{29} \begin{pmatrix} 83 \\ 69 \end{pmatrix} \sin t
\end{aligned}$$



### 10.4.37

$$\begin{aligned}
 \mathbf{A} &= \begin{pmatrix} 5 & -2 \\ 21 & -8 \end{pmatrix} \\
 \mathbf{F} &= \begin{pmatrix} 6 \\ 4 \end{pmatrix} \\
 \mathbf{P} &= \begin{pmatrix} 2 & 1 \\ 7 & 3 \end{pmatrix} \\
 \mathbf{G} &= \begin{pmatrix} -14 \\ 34 \end{pmatrix} \\
 \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} &= \begin{pmatrix} -2y_1 - 14 \\ -y_2 + 34 \end{pmatrix} \\
 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} c_1 e^{-2t} - 7 \\ c_2 e^{-t} + 34 \end{pmatrix} \\
 \mathbf{X} &= c_1 \begin{pmatrix} 2 \\ 7 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + \begin{pmatrix} 20 \\ 53 \end{pmatrix}
 \end{aligned}$$

### 10.4.39

$$\begin{aligned}
 \mathbf{A} &= \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \\
 \mathbf{P} &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\
 \mathbf{G} &= \begin{pmatrix} 4+t \\ 4-t \end{pmatrix} \\
 \mathbf{Y}' &= \mathbf{D}\mathbf{Y} + \mathbf{G} \\
 &= \begin{pmatrix} 10y_1 + 4 + t \\ 4 - t \end{pmatrix} \\
 \mathbf{Y} &= \begin{pmatrix} c_1 e^{10t} - \frac{1}{10}t - \frac{41}{100} \\ -\frac{1}{2}t^2 + 4t + c_2 \end{pmatrix} \\
 \mathbf{X} &= \mathbf{P}\mathbf{Y} \\
 &= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{10t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} t^2 + \begin{pmatrix} -\frac{41}{39} \\ \frac{10}{10} \end{pmatrix} t - \frac{41}{100} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
 \end{aligned}$$

## 10.5 Matrix Exponential

### 10.5.1

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

10.5.3

$$\begin{pmatrix} t+1 & t & t \\ t & t+1 & t \\ -2t & -2t & -2t+1 \end{pmatrix}$$

10.5.5

$$\begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix}$$

10.5.7

$$\mathbf{X} = c_1 \begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} + c_2 \begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + c_3 \begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix}$$

10.5.9

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -3 \\ \frac{1}{2} \end{pmatrix}$$

10.5.11

$$\mathbf{X} = c_1 \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2 \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

10.5.13

$$\begin{aligned} \mathbf{X} &= c_1 \begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} + c_2 \begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + c_3 \begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ -4 \\ 6 \end{pmatrix} &= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \mathbf{X} &= \begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} - 4 \begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + 6 \begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix} \end{aligned}$$

10.5.15

$$e^{\mathbf{A}t} = \begin{pmatrix} \frac{1}{2}e^{-2t}(3e^{4t}-1) & \frac{3}{4}e^{-2t}(e^{4t}-1) \\ e^{-2t}-e^{2t} & -\frac{1}{2}e^{-2t}(e^{4t}-3) \end{pmatrix}$$

10.5.17

$$e^{\mathbf{A}t} = \begin{pmatrix} e^{2t}(1+3t) & -9e^{2t}t \\ e^{2t}t & e^{2t}(1-3t) \end{pmatrix}$$

### 10.5.25

$$\begin{aligned}
\mathbf{X} &= e^{\mathbf{A}t} \mathbf{C} \\
&= \mathbf{P} e^{\mathbf{D}t} \mathbf{P}^{-1} \mathbf{C} \\
&= \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} e^{5t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \mathbf{C} \\
&= \begin{pmatrix} -\frac{1}{2}e^{3t}(-3+e^{2t}) & \frac{1}{2}e^{3t}(-1+e^{2t}) \\ -\frac{3}{2}e^{3t}(-1+e^{2t}) & \frac{1}{2}e^{3t}(-1+3e^{2t}) \end{pmatrix} \mathbf{C}
\end{aligned}$$

## 10.6 Chapter in Review

### 10.6.1

$$\frac{1}{3}$$

### 10.6.5

$$\mathbf{X} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^t + c_2 \left[ \begin{pmatrix} -1 \\ 1 \end{pmatrix} t e^t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^t \right]$$

### 10.6.7

$$\begin{aligned}
\mathbf{X} &= c_1 \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{1+2i} + c_2 \begin{pmatrix} i \\ 1 \end{pmatrix} e^{1-2i} \\
&= c_3 \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin 2t \right] e^t + c_4 \left[ \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 2t \right] e^t \\
&= \begin{pmatrix} c_3 e^t \sin 2t - c_4 e^t \cos 2t \\ c_3 e^t \cos 2t + c_4 e^t \sin 2t \end{pmatrix} \\
&= c_3 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^t + c_4 \begin{pmatrix} -\cos 2t \\ \sin 2t \end{pmatrix} e^t
\end{aligned}$$

### 10.6.9

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -7 \\ -12 \\ 16 \end{pmatrix} e^{-3t} + c_3 \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} e^{2t}$$

## 11 Systems of Nonlinear Differential Equations

### 11.1 Autonomous Systems

#### 11.1.1

$$\begin{aligned}
x' &= y \\
y' &= -9 \sin x
\end{aligned}$$

Critical points at  $(n\pi, 0), n \in \mathbb{Z}$ .

### 11.1.3

$$\begin{aligned}x' &= y \\y' &= x^2 - y(1 - x^3)\end{aligned}$$

$$\begin{aligned}0 &= y \\0 &= x^2 - y(1 - x^3) \\&= x^2 \\0 &= x\end{aligned}$$

Critical point at  $(0, 0)$ .

### 11.1.5

$$\begin{aligned}x' &= y \\y' &= \epsilon x^3 - x\end{aligned}$$

$$\begin{aligned}0 &= y \\0 &= \epsilon x^3 - x \\&= \epsilon x^2 - 1 \\x &= \sqrt{\frac{1}{\epsilon}}\end{aligned}$$

Critical points at  $(0, 0)$  and  $(\pm\sqrt{\frac{1}{\epsilon}}, 0)$ .

### 11.1.7

$x' = x + xy$  can only be 0 if  $x = 0$  or  $y = -1$ . If  $x = 0$ ,  $y' = -y - xy$  is 0 if  $y = 0$ . If  $y = -1$ , it's 0 if  $x = -1$ . Therefore the critical points are  $(0, 0)$  and  $(-1, -1)$ .

**11.1.9**

$$\begin{aligned}
 x' &= 3x^2 - 4y \\
 3x^2 &= 4y \\
 x &= \sqrt{\frac{4}{3}y}
 \end{aligned}$$

$$\begin{aligned}
 y' &= x - y \\
 0 &= \sqrt{\frac{4}{3}y} - y \\
 y^2 &= \frac{4}{3}y \\
 y &= \frac{4}{3}
 \end{aligned}$$

The critical points are  $(0, 0)$  and  $(\frac{4}{3}, \frac{4}{3})$ .

**11.1.11**

$$\begin{aligned}
 x' &= x \left( 10 - x - \frac{1}{2}y \right) \\
 y' &= y(16 - y - x)
 \end{aligned}$$

$(0, 0)$ ,  $(0, 16)$ ,  $(10, 0)$ ,  $(4, 12)$

**11.1.13**

$$\begin{aligned}
 x' &= x^2 e^y \\
 y' &= y(e^x - 1)
 \end{aligned}$$

All points on the line  $x = 0$ .

**11.1.15**

$$\begin{aligned}
 x' &= x(1 - x^2 - 3y^2) \\
 y' &= y(3 - x^2 - 3y^2)
 \end{aligned}$$

$(0, 0)$ ,  $(0, \pm 1)$ ,  $(\pm 1, 0)$

**11.1.17**

(a)

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

(b)

$$\begin{pmatrix} 2 \\ -2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 - c_2 - 2 \\ 2c_1 + c_2 + 2 \end{pmatrix}$$

$$\mathbf{X} = -2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

**11.1.19**

(a)

$$x = c_1(4 \cos 3t - 3 \sin 3t) + c_2(3 \cos 3t + 4 \sin 3t)$$

$$y = c_1(5 \cos 3t) + c_2(5 \sin 3t)$$

(b)

$$4 = 4c_1 + 3c_2$$

$$5 = 5c_1$$

$$c_1 = 1$$

$$c_2 = 0$$

$$x = 4 \cos 3t - 3 \sin 3t$$

$$y = 5 \cos 3t$$

**11.1.21**

(a)

$$x = c_1(-\cos t + \sin t)e^{4t} + c_2(-\cos t - \sin t)e^{4t}$$

$$y = c_1(2 \cos t)e^{4t} + c_2(2 \sin t)e^{4t}$$

(b)

$$-1 = -c_1 - c_2$$

$$2 = 2c_1$$

$$c_1 = 1$$

$$c_2 = 0$$

$$x = (\sin t - \cos t)e^{4t}$$

$$y = 2 \cos t e^{4t}$$

### 11.1.23

$$\begin{aligned}\frac{dr}{dt} &= \frac{1}{r} \{x[-y - x(x^2 + y^2)^2] + y[x - y(x^2 + y^2)^2]\} \\ &= \frac{1}{r} [-xy - x^2r^4 + xy - y^2r^4] \\ &= -r^5\end{aligned}$$

$$\begin{aligned}\frac{1}{r^5} \frac{dr}{dt} &= -1 \\ -\frac{1}{4} \frac{1}{r^4} &= c_1 - t \\ \frac{1}{r^4} &= 4t + c_1 \\ r &= \frac{1}{\sqrt[4]{4t + c_1}}\end{aligned}$$

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{1}{r^2} \{-y[-y - x(x^2 + y^2)^2] + x[x - y(x^2 + y^2)^2]\} \\ &= \frac{1}{r^2} [y^2 + xyr^2 + x^2 - xyr^2] \\ &= 1 \\ \theta &= t + c_2\end{aligned}$$

$$\begin{aligned}4 &= \frac{1}{\sqrt[4]{c_1}} \\ c_1 &= \frac{1}{256}\end{aligned}$$

$$0 = c_2$$

$$\begin{aligned}r &= \frac{1}{\sqrt[4]{4t + \frac{1}{256}}} \\ &= \frac{4}{\sqrt[4]{1024t + 1}} \\ \theta &= t\end{aligned}$$



**11.1.25**

$$\begin{aligned}
 \frac{dr}{dt} &= \frac{1}{r} \{x[-y + x(1 - x^2 - y^2)] + y[x + y(1 - x^2 - y^2)]\} \\
 &= \frac{1}{r} [-xy + x^2(1 - r^2) + xy + y^2(1 - r^2)] \\
 &= r(1 - r^2) \\
 &= -r^3 + r \\
 r &= \pm \frac{e^t}{\sqrt{e^{2t} + c_1}}
 \end{aligned}$$

$$\begin{aligned}
 \frac{d\theta}{dt} &= \frac{1}{r^2} \{-y[-y + x(1 - x^2 - y^2)] + x[x + y(1 - x^2 - y^2)]\} \\
 &= \frac{1}{r^2} [y^2 - xy(1 - r^2) + x^2 + xy(1 - r^2)] \\
 &= 1 \\
 \theta &= t + c_2
 \end{aligned}$$

$$\begin{aligned}
 1 &= \pm \frac{1}{\sqrt{1 + c_1}} \\
 c_1 &= 0 \\
 c_2 &= 0 \\
 r &= 1 \\
 \theta &= t
 \end{aligned}$$

$$\begin{aligned}
 2 &= \pm \frac{1}{\sqrt{1 + c_1}} \\
 c_1 &= -\frac{3}{4} \\
 c_2 &= 0 \\
 r &= \frac{e^t}{\sqrt{e^{2t} - \frac{3}{4}}} \\
 \theta &= t
 \end{aligned}$$

**11.1.27**

No periodic solutions.

## 11.2 Stability of Linear Systems

### 11.2.1

Stable node

### 11.2.3

Unstable spiral

### 11.2.5

Degenerate stable node

### 11.2.7

Saddle point

### 11.2.9

Saddle point

### 11.2.11

Saddle point

### 11.2.13

Degenerate stable node

### 11.2.15

Stable spiral

### 11.2.17

$$-1 + \mu^2 < 0 \Rightarrow |\mu| < 1$$

### 11.2.19

Saddle point when  $\mu < -1$ , unstable spiral when  $-1 < \mu < 3$ , unstable node when  $\mu \geq 3$ .

### 11.2.23

(a)  $(3, 4)$

(b) Saddle point

**11.2.25**

- (a)  $(0.5, 2)$   
 (b) Unstable spiral

**11.3 Linearization and Local Stability****11.3.1**

The Jacobian is

$$\begin{pmatrix} \alpha & -\beta + 2y \\ \beta - y & \alpha - x \end{pmatrix}.$$

At  $(0, 0)$  this is

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

The eigenvalues of this matrix are  $\alpha \pm i\beta$  so if  $\alpha > 0$  then  $(0, 0)$  is an unstable critical point and if  $\alpha < 0$  it is a stable critical point.

**11.3.3**

$$\begin{aligned} g(x) &= kx(n+1-x) \\ g'(x) &= k(n+1-x) - kx \\ &= k(n+1-2x) \\ g'(0) &= k(n+1) \\ g'(n+1) &= k(n+1-2(n+1)) \\ &= -k(n+1) \end{aligned}$$

$x = 0$  is unstable,  $x = n+1$  is stable.

**11.3.5**

$$\begin{aligned} g(T) &= k(T - T_0) \\ g'(T) &= k \\ g'(T_0) &= k \end{aligned}$$

$T = T_0$  is unstable.

**11.3.7**

$$\begin{aligned}
g(x) &= k(\alpha - x)(\beta - x), \alpha > \beta \\
g'(x) &= -k(\beta - x) - k(\alpha - x) \\
g'(\alpha) &= -k(\beta - \alpha) - k(\alpha - \alpha) \\
&= k(\alpha - \beta) \\
g'(\beta) &= -k(\beta - \beta) - k(\alpha - \beta) \\
&= -k(\alpha - \beta)
\end{aligned}$$

$x = \alpha$  is unstable,  $x = \beta$  is stable.

**11.3.9**

$$\begin{aligned}
g(P) &= P(a - bP)(1 - cP^{-1}), P > 0, a < bc \\
g'(P) &= (a - bP)(1 - cP^{-1}) - bP(1 - cP^{-1}) + cP^{-1}(a - bP) \\
g'\left(\frac{a}{b}\right) &= \left(a - b\frac{a}{b}\right)\left(1 - c\frac{b}{a}\right) - b\frac{a}{b}\left(1 - c\frac{b}{a}\right) + c\frac{b}{a}\left(a - b\frac{a}{b}\right) \\
&= -a\left(1 - \frac{bc}{a}\right) \\
g'(c) &= (a - bc)\left(1 - c\frac{1}{c}\right) - bc\left(1 - c\frac{1}{c}\right) + c\frac{1}{c}(a - bc) \\
&= a - bc
\end{aligned}$$

$P = \frac{a}{b}$  is unstable,  $P = c$  is stable.

**11.3.11**

$$\begin{aligned}x' &= 1 - 2xy \\y' &= 2xy - y\end{aligned}$$

$$\begin{aligned}0 &= 1 - 2xy \\2xy &= 1\end{aligned}$$

$$\begin{aligned}0 &= 1 - y \\y &= 1 \\x &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} -2y & -2x \\ 2y & 2x - 1 \end{pmatrix} \\ \mathbf{A}|_{(\frac{1}{2}, 1)} &= \begin{pmatrix} -2 & -1 \\ 2 & 0 \end{pmatrix}\end{aligned}$$

$(\frac{1}{2}, 1)$  is a stable spiral.

**11.3.13**

$$\begin{aligned}x' &= y - x^2 + 2 \\y' &= 2xy - y\end{aligned}$$

$$\begin{aligned}0 &= y - x^2 + 2 \\y &= x^2 - 2\end{aligned}$$

$$\begin{aligned}0 &= 2x(x^2 - 2) - (x^2 - 2) \\&= 2x^3 - x^2 - 4x + 2 \\&= (x^2 - 2)(2x - 1) \\&= (x + \sqrt{2})(x - \sqrt{2})(2x - 1)\end{aligned}$$

$$x = \frac{1}{2} \text{ or } \pm\sqrt{2}$$

$$y = -\frac{7}{4} \text{ or } 0$$

Critical points are  $(\frac{1}{2}, -\frac{7}{4})$  and  $(\pm\sqrt{2}, 0)$ .

$$\mathbf{A} = \begin{pmatrix} -2x & 1 \\ 2y & 2x-1 \end{pmatrix}$$

$$\mathbf{A}|_{(\frac{1}{2}, -\frac{7}{4})} = \begin{pmatrix} -1 & 1 \\ -\frac{7}{2} & 0 \end{pmatrix}$$

$$\mathbf{A}|_{(-\sqrt{2}, 0)} = \begin{pmatrix} 2\sqrt{2} & 1 \\ 0 & -2\sqrt{2}-1 \end{pmatrix}$$

$$\mathbf{A}|_{(\sqrt{2}, 0)} = \begin{pmatrix} -2\sqrt{2} & 1 \\ 0 & 2\sqrt{2}-1 \end{pmatrix}$$

$(\frac{1}{2}, -\frac{7}{4})$  is a stable spiral,  $(\pm\sqrt{2}, 0)$  are saddle points.

### 11.3.15

$$x' = -3x + y^2 + 2$$

$$y' = x^2 - y^2$$

$$0 = -3x + y^2 + 2$$

$$y^2 = 3x - 2$$

$$0 = x^2 - 3x + 2$$

$$= (x-2)(x-1)$$

$$x = 1 \text{ or } 2$$

$$y = \pm 1 \text{ or } \pm 2$$

Critical points are  $(1, \pm 1)$  and  $(2, \pm 2)$ .

$$\mathbf{A} = \begin{pmatrix} -3 & 2y \\ 2x & -2y \end{pmatrix}$$

$$\mathbf{A}|_{(1, -1)} = \begin{pmatrix} -3 & -2 \\ 2 & 2 \end{pmatrix}$$

$$\mathbf{A}|_{(1, 1)} = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}$$

$$\mathbf{A}|_{(2, -2)} = \begin{pmatrix} -3 & -4 \\ 4 & 4 \end{pmatrix}$$

$$\mathbf{A}|_{(2, 2)} = \begin{pmatrix} -3 & 4 \\ 4 & -4 \end{pmatrix}$$

$(1, -1)$  is a saddle point,  $(1, 1)$  is a stable node,  $(2, -2)$  is an unstable spiral,  $(2, 2)$  is a saddle point.

### 11.3.23

It's not possible to classify  $x = 0$ .

### 11.3.25

It's not possible to classify  $x = 0$  but  $x = \pm\sqrt{\frac{1}{\epsilon}}$  are saddle points.

### 11.3.29

(a) The critical point at  $(0, 0)$  is a stable spiral.

### 11.3.33

(a)

$$\begin{aligned}x' &= 2xy \\y' &= 1 - x^2 + y^2\end{aligned}$$

$$\begin{aligned}0 &= 1 - x^2 + y^2 \\x &= \sqrt{y^2 + 1}\end{aligned}$$

$$\begin{aligned}0 &= 2\sqrt{y^2 + 1}y \\&= 4(y^2 + 1)y^2 \\&= 4y^4 + 4y^2 \\y &= 0\end{aligned}$$

$$x = \pm 1$$

Critical points are  $(\pm 1, 0)$ .

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} 2y & 2x \\ -2x & 2y \end{pmatrix} \\ \mathbf{A}|_{(-1,0)} &= \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \\ \mathbf{A}|_{(1,0)} &= \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\end{aligned}$$

The trace of both the matrices is 0 and the determinant is 4, so we know the eigenvalues are pure imaginary but don't know the nature of the critical points.

(b)

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\
 &= \frac{1 - x^2 + y^2}{2xy} \\
 y &= \pm \sqrt{-x^2 + c_1x - 1} \\
 y^2 &= -x^2 + c_1x - 1 \\
 &= -x^2 + 2c_2x - 1 \\
 &= -(x^2 - 2c_2x + c^2) + c^2 - 1 \\
 &= -(x - c)^2 + c^2 - 1 \\
 (x - c)^2 + y^2 &= c^2 - 1
 \end{aligned}$$

### 11.3.37

$$\begin{aligned}
 Lq'' + Rq' + \alpha q + \beta q^3 &= 0 \\
 q' &= r \\
 r' &= -\frac{1}{L}(Rr + \alpha q + \beta q^3)
 \end{aligned}$$

When  $\beta > 0$  the only critical point is  $(q, r) = (0, 0)$ . When  $\beta < 0$  the critical points are  $(0, 0)$  and  $(\pm\sqrt{-\frac{\alpha}{\beta}}, 0)$ .

$$\begin{aligned}
 \mathbf{A} &= \begin{pmatrix} 0 & 1 \\ -\frac{1}{L}(\alpha + 3\beta q^2) & -\frac{R}{L} \end{pmatrix} \\
 \mathbf{A}|_{(0,0)} &= \begin{pmatrix} 0 & 1 \\ -\frac{\alpha}{L} & -\frac{R}{L} \end{pmatrix} \\
 \mathbf{A}|_{(\pm\sqrt{-\frac{\alpha}{\beta}}, 0)} &= \begin{pmatrix} 0 & 1 \\ \frac{2\alpha}{L} & -\frac{R}{L} \end{pmatrix}
 \end{aligned}$$

The eigenvalues of  $\mathbf{A}|_{(0,0)}$  are

$$\frac{-R \pm \sqrt{R^2 - 4L\alpha}}{2L}$$

both of which are negative, so  $(0, 0)$  is stable.

The eigenvalues of  $\mathbf{A}|_{(\pm\sqrt{-\frac{\alpha}{\beta}}, 0)}$  are

$$\frac{-R \pm \sqrt{R^2 + 8L\alpha}}{2L}$$



of which one is negative and the other positive meaning  $\left(\pm\sqrt{-\frac{\alpha}{\beta}}, 0\right)$  are saddle points.

### 11.3.39

(a)

$$\theta'' + \sin \theta = \frac{1}{2}$$

$$\begin{aligned}\theta' &= r \\ r' &= \frac{1}{2} - \sin \theta\end{aligned}$$

$$\begin{aligned}\theta'(\pi/6, 0) &= 0 \\ r'(\pi/6, 0) &= \frac{1}{2} - \sin \frac{\pi}{6} \\ &= 0 \\ \theta'(5\pi/6, 0) &= 0 \\ r'(5\pi/6, 0) &= \frac{1}{2} - \sin \frac{5\pi}{6} \\ &= 0\end{aligned}$$

(b)

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} 0 & 1 \\ -\cos \theta & 0 \end{pmatrix} \\ \mathbf{A}|_{(\frac{\pi}{6}, 0)} &= \begin{pmatrix} 0 & 1 \\ -\frac{\sqrt{3}}{2} & 0 \end{pmatrix} \\ \mathbf{A}|_{(\frac{5\pi}{6}, 0)} &= \begin{pmatrix} 0 & 1 \\ \frac{\sqrt{3}}{2} & 0 \end{pmatrix}\end{aligned}$$

The eigenvalues of  $\mathbf{A}|_{(\frac{\pi}{6}, 0)}$  are  $\pm \frac{3^{1/4}}{\sqrt{2}}i$  so  $(\frac{\pi}{6}, 0)$  is a center, a stable spiral, or an unstable spiral.

The eigenvalues of  $\mathbf{A}|_{(\frac{5\pi}{6}, 0)}$  are  $\pm \frac{3^{1/4}}{\sqrt{2}}$  so  $(\frac{5\pi}{6}, 0)$  is a saddle point.