Advanced Engineering Mathematics Ordinary Differential Equations Notes

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Contents

1 Introduction to Differential Equations

1.1 Definitions and Terminology

1.1.1 1

2, linear

1.1.2 3

4, linear

1.1.3 5

2, nonlinear

1.1.4 7

3, linear

1.1.5 9

no; yes

1.1.6 15

The domain of the function is $x \in [-2, \infty)$.

$$y' = 1 + \frac{2}{\sqrt{x+2}}$$

The largest interval of definition of the solution is $x \in (-2, \infty)$.

$$\begin{aligned}
(y-x)y' &= y - x + 8 \\
(x + 4\sqrt{x+2} - x)(1 + \frac{2}{\sqrt{x+2}}) &= x + 4\sqrt{x+2} - x + 8 \\
4\sqrt{x+2} + 8 &= 4\sqrt{x+2} + 8
\end{aligned}$$

1.1.7 17

The domain of the function is $x \in \mathbb{R}, x \neq \pm 2$.

$$y' = \frac{2x}{(4-x^2)^2}$$

The largest intervals of definition of the solution are $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

$$\begin{aligned}
y' &= 2xy^2 \\
\frac{2x}{(4-x^2)^2} &= 2x\left(\frac{1}{4-x^2}\right)^2 \\
&= \frac{2x}{(4-x^2)^2}
\end{aligned}$$

1.1.8 19

$$\begin{aligned}
ln\frac{2X-1}{X-1} &= t \\
2X - 1 &= (X-1)e^t \\
(2-e^t)X &= 1 - e^t \\
X &= \frac{e^t - 1}{e^t - 2}
\end{aligned}$$

The solutions intervals of validity are $(\infty, \ln 2)$ and $(\ln 2, \infty)$.

$$
\begin{aligned}
\frac{dX}{dt} &= (X-1)(1-2X) \\
\frac{e^t}{e^t-2} - \frac{e^t(e^t-1)}{(e^t-2)^2} &= \left(\frac{e^t-1}{e^t-2} - 1\right)\left(1 - 2\frac{e^t-1}{e^t-2}\right) \\
\frac{e^t(e^t-2) - e^t(e^t-1)}{(e^t-2)^2} &= \left(\frac{e^t-1-e^t+2}{e^t-2}\right)\left(\frac{e^t-2-2e^t+2}{e^t-2}\right) \\
\frac{e^{2t}-2e^t-e^{2t}+e^t}{(e^t-2)^2} &= \left(\frac{1}{e^t-2}\right)\left(\frac{-e^t}{e^t-2}\right) \\
\frac{-e^t}{(e^t-2)^2} &= \frac{-e^t}{(e^t-2)^2}
\end{aligned}
$$

1.1.9 31

$m = -2$

1.1.10 33

$m = 2$ or 3

1.1.11 35

$m = -1$ or 0

1.1.12 37

$y = 2$

1.1.13 39

No constant solutions

1.2 Initial Value Problems

1.2.1 1

$$
\begin{aligned}
y(0) = -\frac{1}{3} &= \frac{1}{1 + c_1 e^{-(0)}} \\
-3 &= 1 + c_1 \\
c_1 &= -4
\end{aligned}
$$

$$
y = \frac{1}{1 - 4e^{-x}}
$$

1.2.2 3

$$\begin{aligned} y(2) = \frac{1}{3} &= \frac{1}{(2)^2 + c} \\ 3 &= 4 + c \\ c &= -1 \end{aligned}$$

$$y = \frac{1}{x^2 - 1}$$

$I = (1, \infty)$

1.2.3 5

$$\begin{aligned} y(0) = 1 &= \frac{1}{(0)^2 + c} \\ c &= 1 \end{aligned}$$

$$y = \frac{1}{x^2 + 1}$$

$I = (-\infty, \infty)$

1.2.4 7

$$\begin{aligned} x(0) = -1 &= c_1 \cos 0 + c_2 \sin 0 \\ c_1 &= -1 \end{aligned}$$

$$\begin{aligned} x'(0) = 8 &= -c_1 \sin 0 + c_2 \cos 0 \\ c_2 &= 8 \end{aligned}$$

$$x = -\cos t + 8 \sin t$$

1.2.5 9

$$\begin{aligned} x'\left(\frac{\pi}{6}\right) = 0 &= -c_1 \sin \frac{\pi}{6} + c_2 \cos \frac{\pi}{6} \\ &= -c_1 \frac{1}{2} + c_2 \frac{\sqrt{3}}{2} \\ c_1 &= \sqrt{3} c_2 \end{aligned}$$

$$
\begin{aligned}
x\left(\frac{\pi}{6}\right) = \frac{1}{2} &= c_1 \cos\frac{\pi}{6} + c_2 \sin\frac{\pi}{6} \\
&= \frac{3}{2}c_2 + \frac{1}{2}c_2 \\
&= 2c_2 \\
c_2 &= \frac{1}{4}
\end{aligned}
$$

$$
y = \frac{\sqrt{3}}{4}\cos t + \frac{1}{4}\sin t
$$

1.2.6 11

$$
\begin{aligned}
y(0) = 1 &= c_1 e^{(0)} + c_2 e^{-(0)} \\
&= c_1 + c_2 \\
c_1 &= 1 - c_2
\end{aligned}
$$

$$
\begin{aligned}
y'(0) = 2 &= c_1 e^{(0)} - c_2 e^{-(0)} \\
&= 1 - c_2 - c_2 \\
c_2 &= -\frac{1}{2}
\end{aligned}
$$

$$
y = \frac{3}{2}e^{x} - \frac{1}{2}e^{-x}
$$

1.2.7 13

$$
\begin{aligned}
y(-1) = 5 &= c_1 e^{(-1)} + c_2 e^{-(-1)} \\
&= c_1 e^{-1} + c_2 e \\
c_1 &= 5e - c_2 e^2
\end{aligned}
$$

$$
\begin{aligned}
y'(-1) = -5 &= c_1 e^{(-1)} - c_2 e^{-(-1)} \\
&= 5e - c_2 e^2 - c_2 e \\
c_2 e(e+1) &= 5(e+1) \\
c_2 &= \frac{5}{e}
\end{aligned}
$$

$$
y = 5e^{-x-1}
$$

1.2.8 15

$$y = 0$$

$$y = x^3$$

1.2.9 17

$$f(x, y) = y^{2/3}$$

$$\frac{\partial f}{\partial y} = \frac{2}{3y^{1/3}}$$

$y < 0$ or $y > 0$

1.2.10 19

$$f(x, y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$x < 0$ or $x > 0$

1.2.11 21

$$f(x, y) = \frac{x^2}{4 - y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2x^2 y}{(4 - y^2)^2}$$

$y < -2$, $-2 < y < 2$, or $y > 2$

1.2.12 23

$$f(x, y) = \frac{y^2}{x^2 + y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2} - \frac{2y^3}{(x^2 + y^2)^2}$$

$x \neq 0$ and $y \neq 0$

1.2.13 25

$$f(x, y) = \sqrt{y^2 - 9}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{y^2 - 9}}$$

Yes

1.2.14 27

No

1.2.15 29

(a) $y = cx$

(b)

$$f(x,y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$x \neq 0$

(c) No, the function is not differentiable at $x = 0$

1.2.16 31

(a)

$$y' = \frac{1}{(x+c)^2} = y^2$$

(b)

$$y(0) = 1 = -\frac{1}{(0)+c} \Rightarrow c = -1 \Rightarrow y = \frac{1}{1-x}$$

$I = (-\infty, 1)$

$$y(0) = -1 = -\frac{1}{(0)+c} \Rightarrow c = 1 \Rightarrow y = -\frac{1}{x+1}$$

$I = (-1, \infty)$

1.2.17 39

$$\begin{aligned} y(0) = 0 &= c_1 \cos 3(0) + c_2 \sin 3(0) \\ c_1 &= 0 \end{aligned}$$

$$y\left(\frac{\pi}{6}\right) = -1 = c_2 \sin 3\left(\frac{\pi}{6}\right)$$
$$c_2 = -1$$

$$y = -\sin 3x$$

1.2.18 41

$$
\begin{aligned}
y'(0) = 0 &= -3c_1 \sin 3(0) + 3c_2 \cos 3(0) \\
c_2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
y'\left(\frac{\pi}{4}\right) = 0 &= -3c_1 \sin 3\left(\frac{\pi}{4}\right) \\
&= -\frac{3}{\sqrt{2}} c_1 \\
c_1 &= 0
\end{aligned}
$$

$$y = 0$$

1.2.19 43

$$
\begin{aligned}
y(0) = 0 &= c_1 \cos 3(0) + c_2 \sin 3(0) \\
c_1 &= 0
\end{aligned}
$$

$$
\begin{aligned}
y(\pi) = 4 &= c_2 \sin 3(\pi) \\
4 &= 0
\end{aligned}
$$

No solution

1.3 Differential Equations as Mathematical Models

1.3.1 1

$$\frac{dP}{dt} = kP + r$$

$$\frac{dP}{dt} = kP - r$$

1.3.2 3

$$\frac{dP}{dt} = k_b P - k_d P^2$$

1.3.3 7

$$\frac{dx}{dt} = kx(1000 - x)$$

1.3.4 9

$$\frac{dA}{dt} = -\frac{A}{100}$$

$$A(0) = 50\,\mathrm{lb}$$

1.3.5 11

$$\frac{dA}{dt} + \frac{7}{600-t}A = 6$$

1.3.6 13

$$\begin{aligned}
\frac{dV}{dt} &= -cA_h\sqrt{2gh} \\
A_w\frac{dh}{dt} &= -cA_h\sqrt{2gh} \\
\frac{dh}{dt} &= -\frac{cA_h\sqrt{2g}}{A_w}\sqrt{h} \\
&= -\frac{c\pi r_h^2\sqrt{2g}}{A_w}\sqrt{h} \\
&= -\frac{c\pi}{430}\sqrt{h}
\end{aligned}$$

1.3.7 15

$$L\frac{di}{dt} + Ri = E$$

1.3.8 17

$$m\frac{dv}{dt} = mg - kv^2$$

1.3.9 19

$$m\frac{d^2x}{dt^2} = -kx$$

1.3.10 21

$$\begin{aligned}
\frac{d}{dt}(mv) &= R - kv \\
\frac{dm}{dt}v + m\frac{dv}{dt} &= R - kv - mg
\end{aligned}$$

1.3.11 23

$$g = \frac{k}{R^2} \Rightarrow k = gR^2$$

$$\frac{d^2r}{dt^2} = -\frac{gR^2}{r^2}$$

1.3.12 25

$$\frac{dA}{dt} = k(M - A)$$

1.3.13 27

$$\frac{dx}{dt} = r - kx$$

1.3.14 29

$$\begin{aligned}\frac{dy}{dx} &= \tan\theta \\ &= \tan\frac{\phi}{2} \\ &= \frac{1-\cos\phi}{\sin\phi} \\ &= \frac{1-x/r}{y/r} \\ &= \frac{r-x}{y} \\ &= \frac{\sqrt{x^2+y^2}-x}{y}\end{aligned}$$

1.4 Chapter in Review

1.4.1 1

$$\frac{dy}{dx} = ky$$

1.4.2 3

$$y'' + k^2y = 0$$

1.4.3 5

$$y = c_1 e^x + c_2 x e^x$$

$$\begin{aligned} y' &= c_1 e^x + c_2 e^x + c_2 x e^x \\ &= y + c_2 e^x \end{aligned}$$

$$\begin{aligned} y'' &= c_1 e^x + c_2 e^x + c_2 e^x + c_2 x e^x \\ &= c_1 e^x + 2c_2 e^x + c_2 x e^x \\ &= y' + c_2 e^x \end{aligned}$$

$$y'' - 2y' + y = 0$$

1.4.4 7

a, d

1.4.5 9

b

1.4.6 11

b

1.4.7 13

$$y = ce^x$$

1.4.8 15

$$\frac{dy}{dx} = x^2 + y^2$$

1.4.9 17

(a) $(-\infty, \infty)$

(b) $(-\infty, 0)$ or $(0, \infty)$

1.4.10 19

$x_0 = -1$ and $I = (-\infty, 0)$ or $x_0 = 2$ and $I = (0, \infty)$

1.4.11 23

$$y = x \sin x + x \cos x$$

$$y' = \sin x + x \cos x + \cos x - x \sin x$$

$$\begin{aligned} y'' &= \cos x + \cos x - x \sin x - \sin x - \sin x - x \cos x \\ &= 2 \cos x - 2 \sin x - x \sin x - x \cos x \end{aligned}$$

$$\begin{aligned} y'' + y &= 2 \cos x - 2 \sin x - x \sin x - x \cos x + x \sin x + x \cos x \\ &= 2 \cos x - 2 \sin x \end{aligned}$$

$I = (-\infty, \infty)$

1.4.12 25

$$y = \sin(\ln x)$$

$$y' = \frac{1}{x} \cos(\ln x)$$

$$y'' = -\frac{1}{x^2} \cos(\ln x) - \frac{1}{x^2} \sin(\ln x)$$

$$\begin{aligned} x^2 y'' + x y' + y &= -\cos(\ln x) - \sin(\ln x) + \cos(\ln x) + \sin(\ln x) \\ &= 0 \end{aligned}$$

$I = (0, \infty)$

1.4.13 35

$$\begin{aligned} y(0) = 0 &= c_1 e^{-3(0)} + c_2 e^{(0)} + 4(0) \\ &= c_1 + c_2 \\ c_1 &= -c_2 \end{aligned}$$

$$\begin{aligned} y'(0) = 0 &= -3 c_1 e^{-3(0)} + c_2 e^{(0)} + 4 \\ &= -3 c_1 + c_2 + 4 \\ c_2 &= 3 c_1 - 4 \end{aligned}$$

$$c_1 = -(3 c_1 - 4) \Rightarrow c_1 = 1 \Rightarrow c_2 = -1$$

$$y = e^{-3x} - e^{x} + 4x$$

1.4.14 37

$$
\begin{aligned}
y(1) = -2 &= c_1 e^{-3(1)} + c_2 e^{(1)} + 4(1) \\
&= c_1 e^{-3} + c_2 e + 4 \\
c_1 &= -e^3(c_2 e + 6)
\end{aligned}
$$

$$
\begin{aligned}
y'(1) = 4 &= -3c_1 e^{-3(1)} + c_2 e^{(1)} + 4 \\
&= -3c_1 e^{-3} + c_2 e + 4 \\
c_2 e &= 3c_1 e^{-3}
\end{aligned}
$$

$$c_1 = -e^3(3c_1 e^{-3} + 6) = -3c_1 - 6e^3 \Rightarrow c_1 = -\frac{3}{2}e^3 \Rightarrow c_2 = -\frac{9}{2}e^{-1}$$

$$y = -\frac{3}{2}e^{3(1-x)} - \frac{9}{2}e^{x-1} + 4x$$

1.4.15 41

$y_0 = -3$, $y_1 = 0$

1.4.16 43

$$
\begin{aligned}
\frac{d}{dt}(mv) &= F - mg \\
\frac{d}{dt}(\lambda x \frac{dx}{dt}) &= F - \lambda x g \\
x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + gx &= \frac{F}{\lambda} \\
x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 32x &= 5
\end{aligned}
$$

2 First-Order Differential Equations

2.1 Solution Curves Without a Solution

2.1.1 21

0 is stable, 3 is unstable

2.1.2 23

2 is semi-stable

2.1.3 25

-2 is unstable, 0 is semi-stable, 2 is stable

2.1.4 27

-1 is stable, 0 is unstable

2.1.5 39

$P_0 < h/k$

2.1.6 41

$$g - \frac{k}{m}v^2 = 0 \Rightarrow v = \sqrt{\frac{gm}{k}}$$

2.2 Separable Equations

2.2.1 1

$$\begin{aligned}\frac{dy}{dx} &= \sin 5x \\ y &= -\frac{1}{5}\cos 5x + c\end{aligned}$$

2.2.2 3

$$\begin{aligned}dx + e^{3x}\,dy &= 0 \\ e^{-3x}\,dx + \,dy &= 0 \\ -\frac{1}{3}e^{-3x} + y &= c \\ y &= \frac{1}{3}e^{-3x} + c\end{aligned}$$

2.2.3 5

$$\begin{aligned}
x\frac{dy}{dx} &= 4y \\
\frac{1}{4y}\,dy &= \frac{1}{x}\,dx \\
\frac{1}{4}\ln|4y| &= \ln|x| + c \\
\ln|4y| &= 4\ln|x| + c \\
4y &= e^{4\ln|x|+c} \\
&= c\left(e^{\ln|x|}\right)^4 \\
y &= cx^4
\end{aligned}$$

2.2.4 7

$$\begin{aligned}
\frac{dy}{dx} &= e^{3x+2y} \\
&= e^{3x}e^{2y} \\
e^{-2y}\,dy &= e^{3x}\,dx \\
-\frac{1}{2}e^{-2y} &= \frac{1}{3}e^{3x} + c \\
-3e^{-2y} &= 2e^{3x} + c
\end{aligned}$$

2.2.5 9

$$\begin{aligned}
y\ln x\frac{dx}{dy} &= \left(\frac{y+1}{x}\right)^2 \\
x^2\ln x\,dx &= \frac{(y+1)^2}{y}\,dy \\
x^3\left(\frac{\ln x}{3} - \frac{1}{9}\right) &= \frac{1}{2}y(y+4) + \ln y + c \\
\frac{1}{3}x^3\ln x - \frac{1}{9}x^3 &= \frac{1}{2}y^2 + 2y + \ln y + c
\end{aligned}$$

2.2.6 11

$$
\begin{aligned}
\csc y\,dx + \sec^2 x\,dy &= 0 \\
\frac{1}{\sin y}\,dx + \frac{1}{\cos^2 x}\,dy &= 0 \\
\cos^2 x\,dx + \sin y\,dy &= 0 \\
\frac{1}{2}(1 + \cos 2x)\,dx + \sin y\,dy &= 0 \\
\frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) - \cos y + c &= 0 \\
4\cos y &= 2x + \sin 2x + c
\end{aligned}
$$

2.2.7 13

$$
\begin{aligned}
(e^y + 1)^2 e^{-y}\,dx + (e^x + 1)^3 e^{-x}\,dy &= 0 \\
\frac{e^x}{(e^x + 1)^3}\,dx + \frac{e^y}{(e^y + 1)^2} &= 0 \\
-\frac{1}{2(e^x + 1)^2} - \frac{1}{e^y + 1} &= c \\
(e^x + 1)^{-2} + 2(e^y + 1)^{-1} &= c
\end{aligned}
$$

2.2.8 15

$$
\begin{aligned}
\frac{dS}{dr} &= kS \\
\frac{1}{S}\,dS &= k\,dr \\
\ln|S| &= kr + c \\
S &= ce^{kr}
\end{aligned}
$$

2.2.9 17

$$
\begin{aligned}
\frac{dP}{dt} &= P - P^2 \\
\frac{1}{P(1-P)}\,dP &= dt \\
\ln\frac{P}{1-P} &= t + c \\
\frac{P}{1-P} &= ce^t \\
P &= ce^t(1-P) \\
P &= \frac{ce^t}{1+ce^t}
\end{aligned}
$$

2.2.10 19

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8} \\
&= \frac{(x-1)(y+3)}{(x+4)(y-2)} \\
\frac{y-2}{y+3}\,dt &= \frac{x-1}{x+4}\,dx \\
y - 5\ln|y+3| &= x - 5\ln|x+4| + c \\
e^{y-5\ln|y+3|} &= e^{x-5\ln|x+4|+c} \\
\frac{e^y}{(y+3)^5} &= \frac{ce^x}{(x+4)^5} \\
c(x+4)^5e^y &= (y+3)^5e^x
\end{aligned}
$$

2.2.11 21

$$
\begin{aligned}
\frac{dy}{dx} &= x\sqrt{1-y^2} \\
(1-y^2)^{-1/2}\,dy &= x\,dx \\
\arcsin y &= \frac{1}{2}x^2 + c \\
y &= \sin\left(\frac{1}{2}x^2 + c\right)
\end{aligned}
$$

2.2.12 23

$$\begin{aligned}\frac{dx}{dt} &= 4(x^2+1)\\ \frac{1}{x^2+1}\,dx &= 4\,dt\\ \arctan x &= 4t + c\\ x &= \tan(4t+c)\end{aligned}$$

$$\begin{aligned}x\left(\frac{\pi}{4}\right) = 1 &= \tan\left(4\left(\frac{\pi}{4}\right)+c\right)\\ &= \tan(\pi + c)\\ c &= \arctan(1) - \pi\\ &= -\frac{3}{4}\pi\end{aligned}$$

$$x = \tan\left(4t - \frac{3}{4}\pi\right)$$

2.2.13 25

$$\begin{aligned}x^2\frac{dy}{dx} &= y - xy\\ &= y(1-x)\\ \frac{1}{y}\,dy &= \left(\frac{1}{x^2} - \frac{1}{x}\right)\,dx\\ \ln|y| &= -\frac{1}{x} - \ln|x| + c\\ y &= e^{-\frac{1}{x}-\ln|x|+c}\\ &= \frac{c}{xe^{1/x}}\end{aligned}$$

$$\begin{aligned}y(-1) = -1 &= \frac{c}{(-1)e^{1/(-1)}}\\ &= -ce\\ c &= e^{-1}\end{aligned}$$

$$y = \frac{1}{xe^{1+1/x}}$$

2.2.14 29

$$\begin{aligned}
\frac{dy}{dx} &= ye^{-x^2} \\
\frac{1}{y}\frac{dy}{dx} &= e^{-x^2} \\
\int_4^x \frac{1}{y}\frac{dy}{dx'}\,dx' &= \int_4^x e^{-x'^2}\,dx' \\
\ln|y|\big|_4^x &= \int_4^x e^{-x'^2}\,dx' \\
\ln|y(x)| - \ln|y(4)| &= \int_4^x e^{-x'^2}\,dx' \\
\ln|y(x)| &= \ln|y(4)| + \int_4^x e^{-x'^2}\,dx' \\
y(x) &= e^{\int_4^x e^{-x'^2}\,dx'}
\end{aligned}$$

2.2.15 31

$$\begin{aligned}
\frac{dy}{dx} &= \frac{2x+1}{2y} \\
2y\,dy &= (2x+1)\,dx \\
y^2 &= x^2 + x + c \\
y &= \pm\sqrt{x^2+x+c}
\end{aligned}$$

$$\begin{aligned}
y(-2) = -1 &= -\sqrt{(-2)^2 + (-2) + c} \\
&= -\sqrt{2+c} \\
c &= -1
\end{aligned}$$

$$y = -\sqrt{x^2+x-1}$$

$$I = \left(-\infty, -\frac{1-\sqrt{5}}{2}\right)$$

2.2.16 33

$$
\begin{aligned}
e^y\,dx - e^{-x}\,dy &= 0\\
e^x\,dx - e^{-y}\,dy &= 0\\
e^x + e^{-y} &= c\\
\ln|e^{-y}| &= \ln|c - e^x|\\
y &= -\ln|c - e^x|
\end{aligned}
$$

$$
\begin{aligned}
y(0) = 0 &= -\ln|c - e^{(0)}|\\
1 &= c - 1\\
c &= 2
\end{aligned}
$$

$$y = -\ln|2 - e^x|$$

$$I = (-\infty, \ln 2)$$