Vibrations and Waves by George C. King Problems

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Contents

1 Simple Harmonic Motion

1.1

(a) (i) $T = 4\,\mathrm{s}$

(ii) $\omega = \frac{\pi}{2}\,\mathrm{rad/s}$

(iii) $\omega^2 = \frac{k}{m} \Rightarrow k = m\omega^2 = \frac{\pi^2}{8}\,\mathrm{N/m}$

1.2

(a)

$$\begin{aligned} x &= A\cos\omega t \\ &= A\cos 2\pi f t \\ v &= -2\pi f A \sin 2\pi f t \\ v_{\max} &= 2\pi f A \\ &= 1.38\,\mathrm{m/s} \end{aligned}$$

(b)

$$\begin{aligned} a &= -4\pi^2 f^2 A \cos 2\pi f t \\ a_{\max} &= 4\pi^2 f^2 A \\ &= 3.82\times 10^3\,\mathrm{m/s^2} \end{aligned}$$

1.3

$$\begin{aligned} a_{\max} &\le g \\ 4\pi^2 f^2 A &\le g \\ f &\le \sqrt{\frac{g}{4\pi^2 A}} \\ &\le 1.11\,\mathrm{Hz} \end{aligned}$$

1.4

(a)

$$\frac{U}{E} = \frac{\frac{1}{2}k\left(\frac{1}{2}A\right)^2}{\frac{1}{2}kA^2} = \frac{1}{4} \Rightarrow \frac{K}{E} = \frac{3}{4}$$

(b) (i) The total energy will increase by a factor of 4

(ii) The maximum velocity will increase by a factor of 2

(iii) The maximum acceleration will increase by a factor of 2 and the period won't change

1.5

(a) $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = 0.41\,\mathrm{J}$

(b)

$$
\begin{aligned}
E &= \frac{1}{2}kA^2 \\
A &= \sqrt{\frac{2E}{k}} \\
&= 4.5\,\mathrm{cm} \\
\omega &= \sqrt{\frac{k}{m}} \\
&= \sqrt{\frac{1600}{3}} \\
&= \frac{40}{\sqrt{3}} \\
&= 23\,\mathrm{rad/s} \\
x &= A\cos(\omega t + \phi) \\
\phi &= \arccos\left(\frac{x}{A}\right) - \omega t \\
&= 2.7\,\mathrm{rad} \\
x &= 0.045\cos(23t + 2.7)\,\mathrm{m}
\end{aligned}
$$

1.6

Using the angular frequency of system (b) ω_b as the baseline, the angular frequency of system (a) ω_a is

$$
\begin{aligned}
F = ma &= -2kx \\
a &= -\frac{2k}{m}x \\
\omega_a &= \sqrt{\frac{2k}{m}} \\
&= \sqrt{2}\omega_b
\end{aligned}
$$

and the angular frequency of system (c) ω_c is

$$
\begin{aligned}
F = ma &= -\frac{k}{2}x \\
a &= -\frac{k}{2m}x \\
\omega_c &= \sqrt{\frac{k}{2m}} \\
&= \sqrt{\frac{1}{2}}\omega_b
\end{aligned}
$$

1.7

(a) The test tube experiences a bouyancy force of $F = Ag\rho x$ so its equation of motion is

$$\begin{aligned} F = ma &= -Ag\rho x \\ a &= -\frac{Ag\rho}{m}x \\ \omega &= \sqrt{\frac{Ag\rho}{m}} \end{aligned}$$

(b) The work done by the bouyancy force when moving from equilibrium to x and thus the potential energy is

$$\begin{aligned} U &= \int_0^x Ag\rho x'\, dx' \\ &= \frac{1}{2}Ag\rho x^2 \end{aligned}$$

so the total energy of the system is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}Ag\rho x^2$$

1.8

$$\mathrm{s} \propto \mathrm{kg}^\alpha\, \mathrm{m}^\beta\, \left(\mathrm{m/s^2}\right)^\gamma$$

so $\alpha = 0$, $\beta = 1/2$, and $\gamma = -1/2$ meaning

$$T \propto \sqrt{\frac{l}{g}}$$

1.9

(a)

$$\begin{aligned} x &= A\cos\sqrt{\frac{g}{l}}t \\ v &= -\sqrt{\frac{g}{l}}A\sin\sqrt{\frac{g}{l}}t \\ v_{\max} &= \sqrt{\frac{g}{l}}A \\ &= 0.018\,\mathrm{m/s} \end{aligned}$$

(b) The pendulum reaches its maximum speed at the bottom of its swing which occurs after a quarter cycle

$$\frac{1}{4}T = \frac{1}{4}\frac{2\pi}{\omega} = \frac{\pi}{2\sqrt{g/l}} = 0.43\,\mathrm{s}$$

1.10

$$\begin{aligned}
I\frac{d^2\theta}{dt^2} &= \tau \\
\frac{1}{3}ML^2\frac{d^2\theta}{dt^2} &= -kL\sin\theta L\cos\theta \\
\frac{1}{3}M\frac{d^2\theta}{dt^2} &= -k\theta \\
\frac{d^2\theta}{dt^2} &= -\frac{3k}{M}\theta \\
T &= \frac{2\pi}{\omega} \\
&= 2\pi\sqrt{\frac{M}{3k}}
\end{aligned}$$

1.11

(a)

$$\begin{aligned}
F = -\frac{dU}{dx} &= -\left(\frac{6a}{x^7} - \frac{12b}{x^{13}}\right) \\
0 &= \frac{12b}{x^{13}} - \frac{6a}{x^7} \\
&= \frac{12b}{x^6} - 6a \\
6a &= \frac{12b}{x^6} \\
x^6 &= \frac{2b}{a} \\
x &= \left(\frac{2b}{a}\right)^{1/6}
\end{aligned}$$

1.12

(a)

$$\begin{aligned}
K &= \frac{1}{2}Mv^2 + \int dK \\
&= \frac{1}{2}Mv^2 + \int_0^L \frac{1}{2}\frac{m}{L}\left(\frac{l}{L}v\right)^2 dl \\
&= \frac{1}{2}Mv^2 + \frac{1}{2}\frac{mv^2}{L^3}\int_0^L l^2\,dl \\
&= \frac{1}{2}Mv^2 + \frac{1}{2}\frac{mv^2}{L^3}\frac{1}{3}L^3 \\
&= \frac{1}{2}Mv^2 + \frac{1}{6}mv^2 \\
&= \frac{1}{2}(M + m/3)v^2 \\
E &= K + U \\
&= \frac{1}{2}(M + m/3)v^2 + \frac{1}{2}kx^2
\end{aligned}$$

(b)

$$\omega = \sqrt{\frac{k}{M + m/3}}$$

1.13

(a)

$$\begin{aligned}
K &= E - U \\
\frac{1}{2}mv^2 &= U(A) - U(x) \\
v &= \sqrt{2[U(A) - U(x)]/m}
\end{aligned}$$

(b)

$$\begin{aligned}
T &= 4\int_0^A \frac{dx}{v} \\
&= 4\int_0^A \sqrt{\frac{m}{2[U(A) - U(x)]}}\,dx \\
&= 4\sqrt{\frac{m}{2U(A)}}\int_0^A \frac{dx}{\sqrt{1 - U(x)/U(A)}}
\end{aligned}$$

(c)

$$
\begin{aligned}
T &= 4\sqrt{\frac{m}{2\alpha A^n}} \int_0^A \frac{dx}{\sqrt{1-(x/A)^n}} \\
&= 4\sqrt{\frac{m}{2\alpha A^n}} \int_0^1 \frac{A\,d\xi}{\sqrt{1-\xi^n}} \\
&= cA^{(n/2)-1}
\end{aligned}
$$