Advanced Engineering Mathematics Partial Differential Equations by Dennis G. Zill Problems

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Contents

12   Orthogonal Functions and Fourier Series

12.1   Orthogonal Functions

12.1.7

$$\int_0^{\pi/2} \sin mx \sin nx \, dx = \frac{1}{2} \int_0^{\pi/2} [\cos(m-n)x - \cos(m+n)x] \, dx$$

$$= \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_0^{\pi/2}$$

$$= \frac{1}{2} \left(\frac{\sin(m-n)\pi/2}{m-n} - \frac{\sin(m+n)\pi/2}{m+n} \right)$$

$$= 0$$

$$|| \sin nx ||^2 = (\sin nx, \sin nx)$$

$$= \int_0^{\pi/2} \sin^2 nx \, dx$$

$$= \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2nx) \, dx$$

$$= \frac{1}{2} \left[x - \frac{1}{2n} \sin 2nx \right]_0^{\pi/2}$$

$$= \frac{\pi}{4}$$

$$|| \sin nx || = \frac{\sqrt{\pi}}{2}$$

4

12.1.9

$$\int_0^\pi \sin mx \sin nx \, dx = \frac{1}{2} \int_0^\pi [\cos(m-n)x - \cos(m+n)x] \, dx$$

$$= \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_0^\pi$$

$$= 0$$

$$||\sin nx||^2 = (\sin nx, \sin nx)$$

$$= \int_0^\pi \sin^2 nx \, dx$$

$$= \frac{1}{2} \int_0^\pi (1 - \cos 2nx) \, dx$$

$$= \frac{1}{2} \left[x - \frac{1}{2n} \sin 2nx \right]_0^\pi$$

$$= \frac{\pi}{2}$$

$$||\sin nx|| = \sqrt{\frac{\pi}{2}}$$

12.1.21

$$T = 1$$

12.1.23

$$T = 2\pi$$

12.1.25

$$T = 2\pi$$

12.2   Fourier Series

12.2.1

$$p = \pi$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} dx$$

$$= 1$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \cos nx\,dx$$

$$= \frac{1}{n\pi} [\sin nx]_0^{\pi}$$

$$= 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \sin nx\,dx$$

$$= -\frac{1}{n\pi} [\cos nx]_0^{\pi}$$

$$= -\frac{1}{n\pi} [(-1)^n - 1]$$

$$= \frac{1 - (-1)^n}{n\pi}$$

$$f(x) = \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin nx$$

The series converges to $\frac{1}{2}$ at the point of discontinuity.

12.2.3

$$p = 1$$
$$a_0 = \frac{3}{2}$$
$$a_n = \int_{-1}^{0} \cos n\pi x\, dx + \int_{0}^{1} x \cos n\pi x\, dx$$
$$= \frac{1}{n\pi}[\sin n\pi x]_{-1}^{0} + \frac{1}{n\pi}\left[\frac{\cos n\pi x}{n\pi} + x \sin n\pi x\right]_{0}^{1}$$
$$= \frac{(-1)^n - 1}{n^2\pi^2}$$
$$b_n = \int_{-1}^{0} \sin n\pi x\, dx + \int_{0}^{1} x \sin n\pi x\, dx$$
$$= -\frac{1}{n\pi}[\cos n\pi x]_{-1}^{0} + \frac{1}{n\pi}\left[\frac{\sin n\pi x}{n\pi} - x \cos n\pi x\right]_{0}^{1}$$
$$= -\frac{1}{n\pi}$$
$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x - \frac{1}{n\pi}\sin n\pi x\right]$$

The series converges to $\frac{1}{2}$ at the point of discontinuity.

12.3   Fourier Cosine and Sine Series

12.3.1

Odd

12.3.3

Neither

12.3.5

Even

12.3.7

Odd

12.3.9

Neither

12.3.11

$$b_n = -2\pi \int_0^1 \sin n\pi x \, dx$$
$$= \frac{2}{n}[\cos n\pi x]_0^1$$
$$= \frac{2}{n}[(-1)^n - 1]$$
$$f = \sum_{n=1}^{\infty} \frac{2}{n}[(-1)^n - 1] \sin n\pi x$$

12.3.13

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \, dx$$
$$= \pi$$
$$a_n = 2 \int_0^{\pi} x \cos nx \, dx$$
$$= \frac{2}{n}\left[\frac{\cos nx}{n} + x \sin nx\right]_0^{\pi}$$
$$= \frac{2[(-1)^n - 1]}{n^2}$$
$$f = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx$$

12.3.25

$$a_0 = 2 \int_0^1 f(x)\, dx$$

$$= 1$$

$$a_n = 2 \int_0^1 f(x) \cos n\pi x\, dx$$

$$= 2 \int_0^{1/2} \cos n\pi x\, dx$$

$$= \frac{2}{n\pi} [\sin n\pi x]_0^{1/2}$$

$$= \frac{2}{n\pi} \sin \frac{n\pi}{2}$$

$$f = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^\infty \frac{\sin \frac{n\pi}{2}}{n} \cos n\pi x$$

$$b_n = 2 \int_0^1 f(x) \sin n\pi x\, dx$$

$$= 2 \int_0^{1/2} \sin n\pi x\, dx$$

$$= -\frac{2}{n\pi} [\cos n\pi x]_0^{1/2}$$

$$= \frac{2}{n\pi} \left(1 - \cos \frac{n\pi}{2} \right)$$

$$f = \frac{2}{\pi} \sum_{n=1}^\infty \frac{1 - \cos \frac{n\pi}{2}}{n} \sin n\pi x$$

12.3.27

$$a_0 = \frac{4}{\pi} \int_0^{\pi/2} \cos x \, dx$$

$$= \frac{4}{\pi} [\sin x]_0^{\pi/2}$$

$$= \frac{4}{\pi}$$

$$a_n = \frac{4}{\pi} \int_0^{\pi/2} \cos x \cos 2nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi/2} [\cos(1 - 2n)x + \cos(1 + 2n)x] \, dx$$

$$= \frac{2}{\pi} \left[\frac{\sin(1 - 2n)x}{1 - 2n} + \frac{\sin(1 + 2n)x}{1 + 2n} \right]_0^{\pi/2}$$

$$= \frac{2(-1)^n}{\pi} \left[\frac{1}{1 - 2n} + \frac{1}{1 + 2n} \right]$$

$$= \frac{2(-1)^n}{\pi} \frac{1 + 2n + 1 - 2n}{(1 - 2n)(1 + 2n)}$$

$$= \frac{4(-1)^n}{\pi(1 - 2n)(1 + 2n)}$$

$$f = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(1 - 2n)(1 + 2n)} \cos 2nx$$

$$b_n = \frac{4}{\pi} \int_0^{\pi/2} \cos x \sin 2nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi/2} [\sin(2n + 1)x + \sin(2n - 1)x] \, dx$$

$$= -\frac{2}{\pi} \left[\frac{\cos(2n + 1)x}{2n + 1} + \frac{\cos(2n - 1)x}{2n - 1} \right]_0^{\pi/2}$$

$$= \frac{2}{\pi} \left(\frac{1}{2n + 1} + \frac{1}{2n - 1} \right)$$

$$= \frac{2}{\pi} \frac{4n}{4n^2 - 1}$$

$$= \frac{8n}{\pi(4n^2 - 1)}$$

$$f = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^1 - 1} \sin 2nx$$

12.3.35

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} x^2 \, dx$$

$$= \frac{8}{3}\pi^2$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx \, dx$$

$$= \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nx \, dx$$

$$= -\frac{4\pi}{n}$$

$$f = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty} \left(\frac{4}{n^2} \cos nx - \frac{4\pi}{n} \sin nx \right)$$

12.3.43

$$b_n = \frac{10}{\pi} \int_0^{\pi} \sin nt \, dt$$

$$= -\frac{10}{n\pi} [\cos nt]_0^{\pi}$$

$$= \frac{10}{n\pi} [1 - (-1)^n]$$

$$f = \frac{10}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} [1 - (-1)^n] \sin nt$$

$$x_p(t) = \sum_{n=1}^{\infty} B_n \sin nt$$

$$m\frac{d^2x}{dt^2} + kx = f(t)$$

$$-mn^2 B_n + kB_n = \frac{10}{n\pi} [1 - (-1)^n]$$

$$B_n = \frac{10}{n\pi(k - mn^2)} [1 - (-1)^n]$$

$$x_p(t) = \frac{10}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n(k - mn^2)} \sin nt$$

$$= \frac{10}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n(10 - n^2)} \sin nt$$

12.3.45

$$a_0 = \frac{2}{\pi} \int_0^\pi \left(2\pi t - t^2\right) dt$$

$$= \frac{2}{\pi} \left[\pi t^2 - \frac{1}{3}t^3\right]_0^\pi$$

$$= \frac{2}{\pi} \left(\pi^3 - \frac{1}{3}\pi^3\right)$$

$$= \frac{4}{3}\pi^2$$

$$a_n = \frac{2}{\pi} \int_0^\pi \left(2\pi t - t^2\right) \cos nt \, dt$$

$$= -\frac{4}{n^2}$$

$$f(t) = \frac{2}{3}\pi^2 - \sum_{n=1}^\infty \frac{4}{n^2} \cos nt$$

$$x_p(t) = c + \sum_{n=1}^\infty B_n \cos nt$$

$$\frac{1}{4}\frac{d^2x}{dt^2} + 12x = f(t)$$

$$-\frac{n^2}{4}\sum_{n=1}^\infty B_n \cos nt + 12c + 12\sum_{n=1}^\infty B_n \cos nt = \frac{2}{3}\pi^2 - \sum_{n=1}^\infty \frac{4}{n^2} \cos nt$$

$$12c = \frac{2}{3}\pi^2$$

$$c = \frac{\pi^2}{18}$$

$$-\frac{n^2}{4}B_n + 12B_n = -\frac{4}{n^2}$$

$$\frac{48 - n^2}{4}B_n = -\frac{4}{n^2}$$

$$B_n = -\frac{16}{n^2(48 - n^2)}$$

$$x_p(t) = \frac{\pi^2}{18} - \sum_{n=1}^\infty \frac{16}{n^2(48 - n^2)} \cos nt$$

$$= \frac{\pi^2}{18} + 16\sum_{n=1}^\infty \frac{1}{n^2(n^2 - 48)} \cos nt$$

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12.4 Complex Fourier Series

12.4.1

$$T = 4$$
$$p = 2$$
$$c_n = \frac{1}{4}\left(\int_0^2 e^{-in\pi x/2}\,dx - \int_{-2}^0 e^{-in\pi x/2}\,dx\right)$$
$$= \frac{1}{2in\pi}([e^{-in\pi x/2}]_{-2}^0 - [e^{-in\pi x/2}]_0^2)$$
$$= \frac{2 - e^{in\pi} - e^{-in\pi}}{2in\pi}$$
$$= \frac{2 - \cos n\pi - i\sin n\pi - \cos n\pi + i\sin n\pi}{2in\pi}$$
$$= \frac{1 - \cos n\pi}{in\pi}$$
$$= \frac{1 - (-1)^n}{in\pi}$$
$$f(x) = \sum_{n=-\infty,\ n\neq 0}^{\infty} \frac{1 - (-1)^n}{in\pi} e^{in\pi x/2}$$

12.4.3

$$T = 1$$
$$p = \frac{1}{2}$$
$$c_n = \int_0^{1/4} e^{-2in\pi x}\,dx$$
$$= -\frac{1}{2in\pi}[e^{-2in\pi x}]_0^{1/4}$$
$$= \frac{1}{2in\pi}(1 - e^{-in\pi/2})$$
$$c_0 = \frac{1}{4}$$
$$f(x) = \frac{1}{4} + \sum_{n=-\infty,\ n\neq 0}^{\infty} \frac{1 - e^{-in\pi/2}}{2in\pi} e^{2in\pi x}$$

12.4.5

$$T = 2\pi$$

$$p = \pi$$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} \, dx$$

$$= \frac{i}{n}$$

$$c_0 = \frac{1}{2\pi} \int_0^{2\pi} x \, dx$$

$$= \pi$$

$$f(x) = \pi + \sum_{n=-\infty,\ n\neq 0}^{n=\infty} \frac{i}{n} e^{inx}$$

12.5 Sturm-Liouville Problem

12.5.1

$$y'' + \lambda y = 0$$
$$y'(0) = 0$$
$$y(1) + y'(1) = 0$$
$$\lambda = \alpha^2$$
$$y = c_1 \cos \alpha x + c_2 \sin \alpha x$$
$$y' = -\alpha c_1 \sin \alpha x + \alpha c_2 \cos \alpha x$$
$$c_2 = 0$$
$$y = c_1 \cos \alpha x$$

$$c_1 \cos \alpha - \alpha c_1 \sin \alpha = 0$$
$$c_1 \cos \alpha = \alpha c_1 \sin \alpha$$
$$\alpha \tan \alpha = 0$$
$$\alpha = \cot \alpha$$
$$\lambda_1 = 0.740174$$
$$y_1 = \cos 0.860334x$$
$$\lambda_2 = 11.734872$$
$$y_2 = \cos 3.42562x$$
$$\lambda_3 = 41.438831$$
$$y_3 = \cos 6.4373x$$
$$\lambda_4 = 90.808130$$
$$y_4 = \cos 9.52933x$$

12.5.5

$$(y_n, y_n) = \int_0^1 \cos^2 \alpha_n x \, dx$$
$$= \frac{1}{2} \int_0^1 (1 + \cos 2\alpha_n x) \, dx$$
$$= \frac{1}{2} \left[x + \frac{1}{2\alpha_n} \sin 2\alpha_n x \right]_0^1$$
$$= \frac{1}{2} \left(1 + \frac{1}{2\alpha_n} \sin 2\alpha_n \right)$$
$$= \frac{1}{2} \left(1 + \frac{1}{\alpha_n} \sin \alpha_n \cos \alpha_n \right)$$
$$= \frac{1}{2} \left(1 + \tan \alpha_n \sin \alpha_n \cos \alpha_n \right)$$
$$= \frac{1}{2} (1 + \sin^2 \alpha_n)$$

12.5.7

(a)

$$x^2 y'' + xy' + \lambda y = 0$$
$$y(1) = 0$$
$$y(5) = 0$$

$$\lambda = \alpha^2$$
$$y = x^m$$
$$y' = mx^{m-1}$$
$$y'' = m(m-1)x^{m-2}$$
$$x^2 m(m-1)x^{m-2} + xmx^{m-1} + \alpha^2 x^m = 0$$
$$m(m-1) + m + \alpha^2 = 0$$
$$m^2 + \alpha^2 = 0$$
$$m = \pm i\alpha$$
$$y = c_1 \cos(\alpha \ln x) + c_2 \sin(\alpha \ln x)$$

$$0 = c_1$$
$$0 = c_2 \sin(\alpha \ln 5)$$
$$\alpha = \frac{n\pi}{\ln 5}$$
$$\lambda = \left(\frac{n\pi}{\ln 5}\right)^2$$

$$y_n = \sin\left(\frac{n\pi}{\ln 5} \ln x\right)$$

(b)

$$x^2 y'' + xy' + \lambda y = 0$$
$$y'' + \frac{1}{x}y' + \lambda \frac{1}{x^2}y = 0$$
$$e^{\ln x} y'' + \frac{1}{x}e^{\ln x}y' + \lambda e^{\ln x}\frac{1}{x^2}y = 0$$
$$\frac{d}{dx}(e^{\ln x}y') + \lambda e^{\ln x}\frac{1}{x^2}y = 0$$
$$\frac{d}{dx}(xy') + \lambda \frac{1}{x}y = 0$$

(c)

$$\int_1^5 \frac{1}{x} \sin\left(\frac{m\pi}{\ln 5} \ln x\right) \sin\left(\frac{n\pi}{\ln 5} \ln x\right) = 0, \ m \neq n$$

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12.5.9

$$xy'' + (1-x)y' + ny = 0$$

$$y'' + \left(\frac{1}{x} - 1\right)y' + n\frac{1}{x}y = 0$$

$$e^{\int \left(\frac{1}{x} - 1\right) dx} = e^{\ln(x) - x}$$

$$= xe^{-x}$$

$$xe^{-x}y'' + \left(\frac{1}{x} - 1\right)xe^{-x}y' + n\frac{1}{x}xe^{-x}y = 0$$

$$\frac{d}{dx}(xe^{-x}y') + ne^{-x}y = 0$$

$$\int_0^\infty e^{-x}L_m(x)L_n(x)\,dx = 0,\ m \neq n$$

12.6   Bessel and Legendre Series

12.6.1

$$J_1(3\alpha) = 0$$

$$\alpha_1 = 1.277$$

$$\alpha_2 = 2.338$$

$$\alpha_3 = 3.391$$

$$\alpha_4 = 4.441$$

12.6.3

$$J_0(2\alpha) = 0$$

$$c_i = \frac{1}{2J_1^2(2\alpha_i)} \int_0^2 xJ_0(\alpha_i x)\,dx$$

$$= \frac{1}{2J_1^2(2\alpha_i)} \int_0^2 \frac{d}{dx}\left[\frac{1}{\alpha_i}xJ_1(\alpha_i x)\right]dx$$

$$= \frac{1}{\alpha_i J_1(2\alpha_i)}$$

$$f(x) = \sum_{i=1}^\infty \frac{J_0(\alpha_i x)}{\alpha_i J_1(2\alpha_i)}$$

12.6.5

$$J_0(2\alpha) + 2\alpha J_0'(2\alpha) = 0$$
$$h = 1$$
$$b = 2$$
$$c_i = \frac{2\alpha_i^2}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)} \int_0^2 x J_0(\alpha_i x)\, dx$$
$$= \frac{2\alpha_i^2}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)} \int_0^2 \frac{d}{dx}\left[\frac{1}{\alpha_i} x J_1(\alpha_i x)\right] dx$$
$$= \frac{4\alpha_i J_1(2\alpha_i)}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)}$$
$$f(x) = 4\sum_{i=1}^{\infty} \frac{\alpha_i J_1(2\alpha_i)}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)} J_0(\alpha_i x)$$

12.6.7

$$f(x) = 5x,\ 0 < x < 4$$
$$4J_1(4\alpha) + 4\alpha J_1'(4\alpha) = 0$$
$$h = 3$$
$$n = 1$$
$$b = 4$$
$$c_i = \frac{2\alpha_i^2}{(16\alpha_i^2 + 8)J_1^2(4\alpha_i)} \int_0^4 5x^2 J_1(\alpha_i x)\, dx$$
$$= \frac{10\alpha_i^2}{(16\alpha_i^2 + 8)J_1^2(4\alpha_i)} \int_0^4 \frac{d}{dx}\left[\frac{1}{\alpha_i} x^2 J_2(\alpha_i x)\right] dx$$
$$= \frac{160\alpha_i J_2(4\alpha_i)}{(16\alpha_i^2 + 8)J_1^2(4\alpha_i)}$$
$$f(x) = \sum_{i=1}^{\infty} \frac{20\alpha_i J_2(4\alpha_i)}{(2\alpha_i^2 + 1)J_1^2(4\alpha_i)} J_1(\alpha_i x)$$

12.6.15

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ x & 0 < x < 1 \end{cases}$$

$$c_n = \frac{2n+1}{2} \int_0^1 x P_n(x)\, dx$$

$$c_0 = \frac{1}{4}$$

$$c_1 = \frac{1}{2}$$

$$c_2 = \frac{5}{16}$$

$$c_3 = 0$$

$$c_4 = -\frac{3}{32}$$

$$c_5 = 0$$

$$c_6 = \frac{13}{256}$$

12.6.21

$$c_0 = \frac{1}{2}$$

$$c_1 = \frac{5}{8}$$

$$c_2 = -\frac{3}{16}$$

$$c_3 = \frac{13}{128}$$

12.7   Chapter in Review

12.7.1

$$(x^2 - 1, x^5) = \int_{-\pi}^{\pi} (x^2 - 1)x^5 \, dx$$

$$= \int_{-\pi}^{\pi} (x^7 - x^5) \, dx$$

$$= \left[\frac{1}{8}x^8 - \frac{1}{6}x^6 \right]_{-\pi}^{\pi}$$

$$= \frac{1}{8}\pi^8 - \frac{1}{6}\pi^6 - \frac{1}{8}\pi^8 + \frac{1}{6}\pi^6$$

$$= 0$$

True

12.7.3

Fourier cosine

12.7.5

False

12.7.7

5.5, 1, 0

12.7.9

True

12.7.13

$$f(x) = |x| - x, \ -1 < x < 1$$
$$L = 2$$
$$p = 1$$
$$a_0 = \int_{-1}^{1} (|x| - x) \, dx$$
$$= \int_{-1}^{0} -2x \, dx$$
$$= -[x^2]_{-1}^{0}$$
$$= -(0 - 1)$$
$$= 1$$
$$a_n = \int_{-1}^{1} (|x| - x) \cos n\pi x \, dx$$
$$= -2 \int_{-1}^{0} x \cos n\pi x \, dx$$
$$= \frac{2[(-1)^n - 1]}{n^2 \pi^2}$$
$$b_n = \int_{-1}^{1} (|x| - x) \sin n\pi x \, dx$$
$$= -2 \int_{-1}^{0} x \sin n\pi x \, dx$$
$$= \frac{2(-1)^n}{n\pi}$$
$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{n^2 \pi^2} \cos n\pi x + \frac{2(-1)^n}{n\pi} \sin n\pi x$$

12.7.17

$$x^2 y'' + xy' + 9\lambda y = 0$$
$$y'(1) = 0$$
$$y(e) = 0$$

$$y = x^m$$
$$y' = mx^{m-1}$$
$$y'' = m(m-1)x^{m-2}$$
$$x^2 m(m-1)x^{m-2} + xmx^{m-1} + 9\lambda x^m = 0$$
$$m(m-1) + m + 9\lambda = 0$$
$$m^2 + 9\lambda = 0$$
$$m = \pm 3\sqrt{\lambda}i$$
$$y = c_1 \cos(3\sqrt{\lambda}\ln x) + c_2 \sin(3\sqrt{\lambda}\ln x)$$
$$y' = \frac{3\sqrt{\lambda}}{x}[c_2 \cos(3\sqrt{\lambda}\ln x) - c_1 \sin(3\sqrt{\lambda}\ln x)]$$
$$y'(1) = 0$$
$$0 = 3\sqrt{\lambda}c_2$$
$$c_2 = 0$$
$$y(e) = 0$$
$$0 = c_1 \cos 3\sqrt{\lambda}$$
$$= \cos 3\sqrt{\lambda}$$
$$3\sqrt{\lambda} = \frac{\pi}{2} + n\pi, \ n \in \mathbb{Z}$$
$$= \frac{2n+1}{2}\pi$$
$$\lambda_n = \left(\frac{2n+1}{6}\pi\right)^2$$
$$y_n = \cos\left(\frac{2n+1}{2}\pi \ln x\right)$$

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13 Boundary-Value Problems in Rectangular Coordinates

13.1 Separable Partial Differential Equations

13.1.1

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$$

$$u = X(x)Y(y)$$

$$X'Y = XY'$$

$$\frac{X'}{X} = \frac{Y'}{Y}$$

$$\frac{X'}{X} = \lambda$$

$$\ln X = \lambda x + c_1$$

$$X = c_1 e^{\lambda x}$$

$$\frac{Y'}{Y} = \lambda$$

$$\ln Y = \lambda y + c_2$$

$$Y = c_2 e^{\lambda y}$$

$$u = XY$$

$$= c_1 c_2 e^{\lambda(x+y)}$$

$$= c_3 e^{\lambda(x+y)}$$

13.1.3

$$u_x + u_y = u$$
$$X'Y + XY' = XY$$
$$\frac{X'}{X}Y + Y' = Y$$
$$\frac{X'}{X} + \frac{Y'}{Y} = 1$$
$$\frac{X'}{X} = 1 - \frac{Y'}{Y}$$

$$\frac{X'}{X} = \lambda$$
$$X = c_1 e^{\lambda x}$$

$$1 - \frac{Y'}{Y} = \lambda$$
$$Y' + (\lambda - 1)Y = 0$$
$$Y = c_2 e^{-(\lambda - 1)y}$$

$$u = c_3 e^{\lambda x - (\lambda - 1)y}$$

13.1.5

$$x\frac{\partial u}{\partial x} = y\frac{\partial u}{\partial y}$$
$$xX'Y = yXY'$$
$$x\frac{X'}{X} = y\frac{Y'}{Y}$$

$$x\frac{X'}{X} = \lambda$$
$$\frac{X'}{X} = \frac{\lambda}{x}$$
$$\ln X = \lambda \ln x + c_1$$
$$X = c_1 x^\lambda$$

$$Y = c_2 y^\lambda$$

$$u = XY$$
$$= c_3(xy)^\lambda$$

13.1.7

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$
$$X''Y + X'Y' + XY'' = 0$$

Not separable.

13.1.9

$$k\frac{\partial^2 u}{\partial x^2} - u = \frac{\partial u}{\partial t}, \; k > 0$$
$$kTX'' - TX = T'X$$
$$k\frac{X''}{X} - 1 = \frac{T'}{T}$$

$$\frac{T'}{T} = \lambda$$
$$T' - \lambda T = 0$$
$$T = c_1 e^{\lambda t}$$

$$k\frac{X''}{X} - 1 = \lambda$$
$$X'' - \frac{\lambda + 1}{k}X = 0$$
$$X = \begin{cases} c_1 \cos\sqrt{\frac{\lambda+1}{k}}x + c_2 \sin\sqrt{\frac{\lambda+1}{k}}x & \lambda < -1 \\ c_1 x + c_2 & \lambda = -1 \\ c_1 \cosh\sqrt{\frac{\lambda+1}{k}}x + c_2 \sinh\sqrt{\frac{\lambda+1}{k}}x & \lambda > -1 \end{cases}$$

$$u = TX$$
$$= \begin{cases} e^{\lambda t}\left(c_1 \cos\sqrt{\frac{\lambda+1}{k}}x + c_2 \sin\sqrt{\frac{\lambda+1}{k}}x\right) & \lambda < -1 \\ e^{\lambda t}\left(c_1 x + c_2\right) & \lambda = -1 \\ e^{\lambda t}\left(c_1 \cosh\sqrt{\frac{\lambda+1}{k}}x + c_2 \sinh\sqrt{\frac{\lambda+1}{k}}x\right) & \lambda > -1 \end{cases}$$

13.1.11

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$a^2 TX'' = T''X$$

$$a^2\frac{X''}{X} = \frac{T''}{T}$$

$$\frac{T''}{T} = \lambda$$

$$T'' - \lambda T = 0$$

$$T = \begin{cases} c_1 \cos\sqrt{\lambda}t + c_2 \sin\sqrt{\lambda}t & \lambda < 0 \\ c_1 t + c_2 & \lambda = 0 \\ c_1 \cosh\sqrt{\lambda}t + c_2 \sinh\sqrt{\lambda}t & \lambda > 0 \end{cases}$$

$$a^2\frac{X''}{X} = \lambda$$

$$X'' - \frac{\lambda}{a^2}X = 0$$

$$X = \begin{cases} c_1 \cos\frac{\sqrt{\lambda}}{a}x + c_2 \sin\frac{\sqrt{\lambda}}{a}x & \lambda < 0 \\ c_1 x + c_2 & \lambda = 0 \\ c_1 \cosh\frac{\sqrt{\lambda}}{a}x + c_2 \sinh\frac{\sqrt{\lambda}}{a}x & \lambda > 0 \end{cases}$$

$$u = TX$$

$$= \begin{cases} (c_1 \cos\sqrt{\lambda}t + c_2 \sin\sqrt{\lambda}t)(c_3 \cos\frac{\sqrt{\lambda}}{a}x + c_4 \sin\frac{\sqrt{\lambda}}{a}x) & \lambda < 0 \\ (c_1 t + c_2)(c_3 x + c_4) & \lambda = 0 \\ (c_1 \cosh\sqrt{\lambda}t + c_2 \sinh\sqrt{\lambda}t)(c_3 \cosh\frac{\sqrt{\lambda}}{a}x + c_4 \sinh\frac{\sqrt{\lambda}}{a}x) & \lambda > 0 \end{cases}$$

13.1.17

Elliptic

13.1.19

Parabolic

13.1.21

Hyperbolic

13.1.23

Parabolic

13.1.25

Hyperbolic

13.2 Classical PDEs and Boundary-Value Problems

13.2.1

$$k^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$
$$u(0,t) = 0$$
$$\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$$
$$u(x,0) = f(x)$$

13.2.3

$$k^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$
$$u(0,t) = 100$$
$$\left.\frac{\partial u}{\partial x}\right|_{x=L} = -hu(L,t)$$
$$u(x,0) = f(x)$$

13.2.5

$$k^2\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t}$$
$$u(0,t) = \sin\frac{\pi}{L}t$$
$$u(L,t) = 0$$
$$u(x,0) = f(x)$$

13.2.7

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$
$$u(0, t) = 0$$
$$u(L, t) = 0$$
$$u(x, 0) = x(L - x)$$
$$\frac{\partial u}{\partial t}\bigg|_{t=0} = 0$$

13.2.9

$$a^2 \frac{\partial^2 u}{\partial x^2} - c \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2}$$
$$u(0, t) = 0$$
$$u(L, t) = \sin \pi t$$
$$u(x, 0) = f(x)$$
$$\frac{\partial u}{\partial t}\bigg|_{t=0} = 0$$

13.2.11

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
$$\frac{\partial u}{\partial x}\bigg|_{x=0} = 0$$
$$\frac{\partial u}{\partial y}\bigg|_{y=0} = 0$$
$$u(x, 2) = 0$$
$$u(4, y) = f(y)$$

13.3 Heat Equation

13.3.1

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

$$u(0,t) = 0$$

$$u(L,t) = 0$$

$$u(x,0) = \begin{cases} 1 & 0 < x < L/2 \\ 0 & L/2 < x < L \end{cases}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

$$= \frac{2}{L}\int_0^{L/2}\sin\frac{n\pi}{L}x\,dx$$

$$= -\frac{2}{n\pi}\left[\cos\frac{n\pi}{L}x\right]_0^{L/2}$$

$$= \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)e^{-k(n^2\pi^2/L^2)t}\sin\frac{n\pi}{L}x$$

13.3.3

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

$$u(x,0) = f(x)$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$$

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$
$$X'(x) = -\alpha c_1 \sin \alpha x + \alpha c_2 \cos \alpha x$$
$$0 = X'(0)$$
$$= \alpha c_2$$
$$c_2 = 0$$
$$0 = X'(L)$$
$$= -\alpha c_1 \sin \alpha L$$
$$\alpha L = n\pi$$
$$\alpha = \frac{n\pi}{L}$$
$$X(x) = c_1 \cos \frac{n\pi}{L} x$$

$$T(t) = c_3 e^{-k(n^2\pi^2/L^2)t}$$

$$u_n = A_n e^{-k(n^2\pi^2/L^2)t} \cos \frac{n\pi}{L} x$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x \, dx$$

$$u_n = \frac{1}{L} \int_0^L f(x) \, dx$$
$$+ \frac{2}{L} \sum_{n=1}^{\infty} \left(\int_0^L f(x) \cos \frac{n\pi}{L} x \, dx \right) e^{-k(n^2\pi^2/L^2)t} \cos \frac{n\pi}{L} x$$

13.3.5

$$k\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t}$$

$$u(x,0) = f(x)$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$$

$$kX''T - hXT = XT'$$

$$k\frac{X''}{X} - h = \frac{T'}{T}$$

$$k\frac{X''}{X} - h = -\lambda$$

$$X'' + \frac{\lambda - h}{k}X = 0$$

$$X = c_1 \cos \omega x + c_2 \sin \omega x$$

$$X' = -\omega c_1 \sin \omega x + \omega c_2 \cos \omega x$$

$$0 = X'(0)$$

$$= \omega c_2$$

$$c_2 = 0$$

$$0 = X'(L)$$

$$= -\omega c_2 \sin \omega L$$

$$\omega L = n\pi$$

$$\omega = \frac{n\pi}{L}$$

$$X_n = c_1 \cos \frac{n\pi}{L}x$$

$$T_n = c_3 e^{-\lambda t}$$

$$= c_3 e^{-(h+kn^2\pi^2/L^2)t}$$

$$u_n = X_n T_n$$

$$= A_n e^{-(h+kn^2\pi^2/L^2)t} \cos \frac{n\pi}{L}x$$

$$= e^{-ht} A_n e^{-(kn^2\pi^2/L^2)t} \cos \frac{n\pi}{L}x$$

$$A_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi}{L}x\,dx$$

$$u = e^{-ht}\left[\frac{1}{L}\int_0^L f(x)\,dx + \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx\right)e^{-(kn^2\pi^2/L^2)t}\cos\frac{n\pi}{L}x\right]$$

13.4   Wave Equation

13.4.1

$$A_n = \frac{2}{L} \int_0^L \frac{1}{4} x(L-x) \sin \frac{n\pi}{L} x \, dx$$

$$= -\frac{[-1+(-1)^n]L^2}{n^3\pi^3}$$

$$u(x,t) = \sum_{n=1}^{\infty} -\frac{[-1+(-1)^n]L^2}{n^3\pi^3} \cos \frac{n\pi a}{L} t \sin \frac{n\pi}{L} x$$

13.4.3

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$u(0,t) = 0$$

$$u(\pi,t) = 0$$

$$u(x,0) = 0$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin x$$

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$0 = X(0)$$

$$= c_1$$

$$0 = X(\pi)$$

$$= c_2 \sin \alpha \pi$$

$$\alpha = n$$

$$X(x) = c_2 \sin nx$$

$$T(t) = c_3 \cos ant + c_4 \sin ant$$

$$u_n = (A_n \cos ant + B_n \sin ant) \sin nx$$

$$A_n = 0$$

$$u_n = B_n \sin ant \sin nx$$

$$\sin x = a \sum_{n=1}^{\infty} nB_n \sin nx$$

$$B_1 = \frac{1}{a}$$

$$B_n = 0$$

$$u = \frac{1}{a} \sin at \sin x$$

13.4.5

$$L = 1$$
$$f(x) = x(1 - x)$$
$$g(x) = x(1 - x)$$
$$A_n = 2 \int_0^1 x(1 - x) \sin n\pi x \, dx$$
$$= \frac{4[1 - (-1)^n]}{n^3 \pi^3}$$
$$B_n = \frac{2}{n\pi a} \int_0^1 x(1 - x) \sin n\pi x \, dx$$
$$= \frac{4[1 - (-1)^n]}{a n^4 \pi^4}$$
$$u(x, t) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \left(\frac{1 - (-1)^n}{n^3} \cos n\pi a t + \frac{1 - (-1)^n}{a n^4 \pi} \sin n\pi a t \right) \sin n\pi x$$

13.4.7

$$f(x) = \begin{cases} \frac{2h}{L} x & 0 < x < L/2 \\ 2h - \frac{2h}{L} x & L/2 < x < L \end{cases}$$
$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$
$$= \frac{2}{L} \left[\int_0^{L/2} \frac{2h}{L} x \sin \frac{n\pi}{L} x \, dx + \int_{L/2}^L \left(2h - \frac{2h}{L} x \right) \sin \frac{n\pi}{L} x \, dx \right]$$
$$= \frac{4h}{L} \left[\frac{1}{L} \int_0^{L/2} x \sin \frac{n\pi}{L} x \, dx + \int_{L/2}^L \left(1 - \frac{1}{L} x \right) \sin \frac{n\pi}{L} x \, dx \right]$$
$$= \frac{8h \sin \frac{n\pi}{2}}{n^2 \pi^2}$$
$$u(x, t) = \frac{8h}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \cos \frac{n\pi a}{L} t \sin \frac{n\pi}{L} x$$

13.4.11

$$X = c_1 \cos \omega x + c_2 \sin \omega x$$

$$X' = -\omega c_1 \sin \omega x + \omega c_2 \cos \omega x$$

$$0 = X'(0)$$

$$= \omega c_2$$

$$= c_2$$

$$0 = X'(L)$$

$$= -\omega c_1 \sin \omega L$$

$$\omega = \frac{n\pi}{L}$$

$$X = c_1 \cos \frac{n\pi}{L} x$$

$$T = c_3 \cos \frac{n\pi a}{L} t + c_4 \sin \frac{n\pi a}{L} t$$

$$T' = \frac{n\pi a}{L} \left(-c_3 \sin \frac{n\pi a}{L} t + c_4 \cos \frac{n\pi a}{L} t \right)$$

$$0 = T'(0)$$

$$= \frac{n\pi a}{L} c_4$$

$$= c_4$$

$$u_n = B_n \cos \frac{n\pi a}{L} t \cos \frac{n\pi}{L} x$$

$$f(x) = u(x,0)$$

$$x = \sum_{n=1}^{\infty} B_n \cos \frac{n\pi}{L} x$$

$$B_0 = \frac{2}{L} \int_0^L x \, dx$$

$$= L$$

$$B_n = \frac{2}{L} \int_0^L x \cos \frac{n\pi}{L} x \, dx$$

$$= \frac{2L[-1 + (-1)^n]}{n^2 \pi^2}$$

$$u = \frac{L}{2} + \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{-1 + (-1)^n}{n^2} \cos \frac{n\pi a}{L} t \cos \frac{n\pi}{L} x$$

13.4.15

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2\beta \frac{\partial u}{\partial t}$$

$$X''T = XT'' + 2\beta XT'$$

$$\frac{X''}{X} = \frac{T''}{T'} + 2\beta \frac{T'}{T}$$

$$\frac{X''}{X} = -\lambda$$

$$X'' + \lambda X = 0$$

$$X(x) = c_1 \cos \omega x + c_2 \sin \omega x$$

$$0 = X(0)$$

$$= c_1$$

$$0 = X(\pi)$$

$$= c_2 \sin \omega \pi$$

$$\omega = n$$

$$X = c_2 \sin nx$$

$$\frac{T''}{T} + 2\beta \frac{T'}{T} = -n^2$$

$$T'' + 2\beta T' + n^2 T = 0$$

$$m^2 + 2\beta m + n^2 = 0$$

$$m = \frac{-2\beta \pm \sqrt{4\beta^2 - 4n^2}}{2}$$

$$= -\beta \pm i\sqrt{n^2 - \beta^2}$$

$$T = e^{-\beta t}(c_1 \cos \sqrt{n^2 - \beta^2}t + c_2 \sin \sqrt{n^2 - \beta^2}t)$$

$$u_n = (A_n \cos \sqrt{n^2 - \beta^2}t + B_n \sin \sqrt{n^2 - \beta^2}t)e^{-\beta t} \sin nx$$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin nx$$

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$\frac{\partial u_n}{\partial t} = (-\sqrt{n^2 - \beta^2}A_n \sin \sqrt{n^2 - \beta^2}t + \sqrt{n^2 - \beta^2}B_n \cos \sqrt{n^2 - \beta^2}t)e^{-\beta t} \sin nx$$

$$- \beta(A_n \cos \sqrt{n^2 - \beta^2}t + B_n \sin \sqrt{n^2 - \beta^2}t)e^{-\beta t} \sin nx$$

$$0 = \frac{\partial u_n}{\partial t}\bigg|_{t=0}$$

$$= \sqrt{n^2 - \beta^2}B_n \sin nx - \beta A_n \sin nx$$

$$B_n = \frac{\beta A_n}{\sqrt{n^2 - \beta^2}}$$

$$u(x,t) = \frac{2}{\pi} e^{-\beta t} \sum_{n=1}^{\infty} \left(\int_0^{\pi} f(x) \sin nx \, dx \right)$$

$$\left(\cos \sqrt{n^2 - \beta^2} t + \frac{\beta}{\sqrt{n^2 - \beta^2}} \sin \sqrt{n^2 - \beta^2} t \right) \sin nx$$

13.5   Laplace's Equation

13.5.1

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{a} y \sin \frac{n\pi}{a} x$$

$$A_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx$$

13.5.5

$$Y'' - \lambda Y = 0$$
$$Y'(0) = 0$$
$$Y'(1) = 0$$
$$Y = c_1 \cos \omega y + c_2 \sin \omega y$$
$$Y' = -\omega c_1 \sin \omega y + \omega c_2 \cos \omega y$$
$$0 = \omega c_2$$
$$= c_2$$
$$0 = -\omega c_1 \sin \omega$$
$$\omega = n\pi$$
$$Y = c_1 \cos n\pi y$$
$$Y = c_1 + c_2 y$$
$$Y' = c_2$$
$$Y = c_1$$
$$X'' + \lambda X = 0$$
$$X = c_3 \sinh n\pi x + c_4 \cosh n\pi x$$
$$0 = X(0)$$
$$= c_4$$
$$X = c_3 \sinh n\pi x$$
$$X = c_3 + c_4 x$$
$$0 = X(0)$$
$$= c_3$$
$$X = c_4 x$$
$$u = A_0 x$$
$$u_n = A_n \sinh n\pi x \cos n\pi y$$
$$u = A_0 x + \sum_{n=1}^{\infty} A_n \sinh n\pi x \cos n\pi y$$
$$f(y) = u(1, y)$$
$$1 - y = A_0 + \sum_{n=1}^{\infty} A_n \sinh n\pi \cos n\pi y$$
$$A_0 = \frac{a_0}{2}$$
$$= \int_0^1 (1 - y) \, dy$$
$$= \frac{1}{2}$$

$$A_n \sinh n\pi = 2 \int_0^1 (1-y) \cos n\pi y \, dy$$

$$A_n = \frac{2[1-(-1)^n]}{n^2\pi^2 \sinh n\pi}$$

$$u = \frac{1}{2}x + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2 \sinh n\pi} \sinh n\pi x \cos n\pi y$$

13.5.7

$$Y'' - \lambda Y = 0$$
$$Y(0) = 0$$
$$Y(\pi) = 0$$
$$Y = c_1 \cos \omega y + c_2 \sin \omega y$$
$$0 = Y(0)$$
$$= c_1$$
$$0 = Y(\pi)$$
$$= c_2 \sin \omega \pi$$
$$\omega = n$$
$$Y = c_2 \sin ny$$
$$X'' + \lambda X = 0$$
$$X = c_3 \sinh nx + c_4 \cosh nx$$
$$X' = nc_3 \cosh nx + nc_4 \sinh nx$$
$$X'(0)Y(y) = X(0)Y(y)$$
$$X'(0) = X(0)$$
$$nc_3 = c_4$$
$$X = c_3(\sinh nx + n \cosh nx)$$
$$u = \sum_{n=1}^{\infty} A_n(\sinh nx + n \cosh nx) \sin ny$$
$$1 = u(\pi, y)$$
$$= \sum_{n=1}^{\infty} A_n(\sinh n\pi + n \cosh n\pi) \sin ny$$
$$A_n(\sinh n\pi + n \cosh n\pi) = \frac{2}{\pi} \int_0^{\pi} \sin ny \, dx$$
$$A_n = \frac{2[1 - (-1)^n]}{n\pi(\sinh n\pi + n \cosh n\pi)}$$
$$u = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \frac{\sinh nx + n \cosh nx}{\sinh n\pi + n \cosh n\pi} \sin ny$$

13.5.11

$$X'' + \lambda X = 0$$
$$X = c_1 \cos \omega x + c_2 \sin \omega y$$
$$0 = X(0)$$
$$= c_1$$
$$0 = X(\pi)$$
$$= c_2 \sin \omega \pi$$
$$\omega = n$$
$$X = c_2 \sin nx$$
$$Y'' - \lambda Y = 0$$
$$Y = c_3 e^{ny} + c_4 e^{-ny}$$
$$0 = \lim_{y \to \infty} Y$$
$$= \lim_{y \to \infty} (c_3 e^{ny} + c_4 e^{-ny})$$
$$0 = c_3$$
$$Y = c_4 e^{-ny}$$
$$u = \sum_{n=1}^{\infty} A_n e^{-ny} \sin nx$$
$$f(x) = u(x, 0)$$
$$= \sum_{n=1}^{\infty} A_n \sin nx$$
$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$
$$u = \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\int_0^{\pi} f(x) \sin nx \, dx \right) e^{-ny} \sin nx$$

13.5.13

$$X'' + \lambda X = 0$$

$$X = c_1 \sin \frac{n\pi}{a} x$$

$$Y'' - \lambda Y = 0$$

$$Y = c_2 \sinh \frac{n\pi}{a} y + c_3 \cosh \frac{n\pi}{a} y$$

$$u = \sum_{n=1}^{\infty} \left(A_n \sinh \frac{n\pi}{a} y + B_n \cosh \frac{n\pi}{a} y \right) \sin \frac{n\pi}{a} x$$

$$f(x) = u(x, 0)$$

$$= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{a} x$$

$$B_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx$$

$$g(x) = u(x, b)$$

$$= \sum_{n=1}^{\infty} \left(A_n \sinh \frac{n\pi b}{a} + B_n \cosh \frac{n\pi b}{a} \right) \sin \frac{n\pi}{a} x$$

$$A_n \sinh \frac{n\pi b}{a} + B_n \cosh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x \, dx$$

$$A_n = \frac{1}{\sinh \frac{n\pi b}{a}} \left(\frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x \, dx - B_n \cosh \frac{n\pi b}{a} \right)$$

13.6 Nonhomogeneous Boundary-Value Problems

13.6.1

$$ku_{xx} = u_t$$
$$u(0, t) = 100$$
$$u(1, t) = 100$$
$$u(x, 0) = 0$$
$$\psi = 100$$
$$kv_{xx} = v_t$$
$$v(0, t) = 0$$
$$v(1, t) = 0$$
$$v(x, 0) = -100$$
$$kX''T = XT'$$
$$\frac{X''}{X} = \frac{T'}{kT}$$
$$X = c_1 \sin n\pi x, \ \ n = 1, 2, 3, \ldots$$
$$T' + k\lambda T = 0$$
$$T = c_2 e^{-kn^2\pi^2 t}$$
$$v(x, t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x$$
$$-100 = v(x, 0)$$
$$= \sum_{n=1}^{\infty} A_n \sin n\pi x$$
$$A_n = -200 \int_0^1 \sin n\pi x \, dx$$
$$= \frac{200[-1 + (-1)^n]}{n\pi}$$
$$v(x, t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{-1 + (-1)^n}{n} e^{-kn^2\pi^2 t} \sin n\pi x$$
$$u(x, t) = 100 + \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{-1 + (-1)^n}{n} e^{-kn^2\pi^2 t} \sin n\pi x$$

13.6.3

$$ku_{xx} + r = u_t$$
$$u(0,t) = u_0$$
$$u(1,t) = u_0$$
$$u(x,0) = 0$$
$$u(x,t) = v(x,t) + \psi(x)$$
$$k\psi'' + r = 0$$
$$\psi'' = -\frac{r}{k}$$
$$\psi = -\frac{r}{2k}x^2 + c_1 x + c_2$$
$$\psi(0) = u_0$$
$$c_2 = u_0$$
$$u_0 = \psi(1)$$
$$= -\frac{r}{2k} + c_1 + u_0$$
$$c_1 = \frac{r}{2k}$$
$$\psi(x) = \frac{r}{2k}x(1-x) + u_0$$
$$kv_{xx} = v_t$$
$$v(0,t) = 0$$
$$v(L,t) = 0$$
$$v(x,0) = \frac{r}{2k}x(x-1) - u_0$$
$$X = c_1 \sin n\pi x, \ n = 1,2,3,\ldots$$
$$T = c_2 e^{-kn^2\pi^2 t}$$
$$v = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x$$
$$A_n = 2\int_0^1 \left[\frac{r}{2k}x(x-1) - u_0\right] \sin n\pi x \, dx$$
$$= \frac{2[-1 + (-1)^n](r + kn^2\pi^2 u_0)}{kn^3\pi^3}$$
$$v = \frac{2}{k\pi^3} \sum_{n=1}^{\infty} \frac{[-1 + (-1)^n](r + kn^2\pi^2 u_0)}{n^3} \sin n\pi x$$
$$u = \frac{r}{2k}x(1-x) + u_0 + \frac{2}{k\pi^3} \sum_{n=1}^{\infty} \frac{[-1 + (-1)^n](r + kn^2\pi^2 u_0)}{n^3} e^{-kn^2\pi^2 t} \sin n\pi x$$

13.6.7

$$ku_{xx} - h(u - u_0) = u_t$$
$$u(0, t) = u_0$$
$$u(1, t) = 0$$
$$u(x, 0) = f(x)$$
$$u(x, t) = v(x, t) + \psi(x)$$
$$u_{xx} = v_{xx} + \psi''$$
$$u_t = v_t$$
$$k(v_{xx} + \psi'') - h[(v + \psi) - u_0] = v_t$$
$$kv_{xx} + k\psi'' - hv - h\psi + hu_0 = v_t$$
$$\psi'' - \frac{h}{k}\psi = -\frac{h}{k}u_0$$
$$\psi_c = c_1 \sinh\sqrt{\frac{h}{k}}x + c_2 \cosh\sqrt{\frac{h}{k}}x$$
$$\psi_p = c_3$$
$$\psi_p'' = 0$$
$$-\frac{h}{k}c_3 = -\frac{h}{k}u_0$$
$$c_3 = u_0$$
$$\psi = \psi_c + \psi_p$$
$$= u_0 + c_1 \sinh\sqrt{\frac{h}{k}}x + c_2 \cosh\sqrt{\frac{h}{k}}x$$
$$\psi(0) = u_0$$
$$c_2 = 0$$
$$0 = \psi(1)$$
$$= u_0 + c_1 \sinh\sqrt{\frac{h}{k}}$$
$$c_1 = -\frac{u_0}{\sinh\sqrt{h/k}}$$
$$\psi = u_0\left(1 - \frac{\sinh\sqrt{h/k}x}{\sinh\sqrt{h/k}}\right)$$

13.6.13

$$u_{xx} = u_t$$

$$u(0, t) = \sin t$$

$$u(1, t) = 0$$

$$u(x, 0) = 0$$

$$u(x, t) = v(x, t) + (1 - x) \sin t$$

$$v_{xx} + (x - 1) \cos t = v_t$$

$$(x - 1) \cos t = \sum_{n=1}^{\infty} A_n \sin n\pi x$$

$$A_n = 2 \cos t \int_0^1 (x - 1) \sin n\pi x \, dx$$

$$= -\frac{2 \cos t}{n\pi}$$

$$(x - 1) \cos t = -\frac{2 \cos t}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi x$$

$$v = \sum_{n=1}^{\infty} v_n(t) \sin n\pi x$$

$$v_{xx} = \sum_{n=1}^{\infty} -n^2 \pi^2 v_n(t) \sin n\pi x$$

$$v_t = \sum_{n=1}^{\infty} v_n'(t) \sin n\pi x$$

$$\sum_{n=1}^{\infty} [-n^2 \pi^2 v_n(t) - v_n'(t)] \sin n\pi x = \frac{2 \cos t}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi x$$

$$v_n'(t) + n^2 \pi^2 v_n(t) = -\frac{2 \cos t}{\pi n}$$

$$\frac{d}{dt}[e^{n^2 \pi^2 t} v_n(t)] = -\frac{2 e^{n^2 \pi^2 t} \cos t}{\pi n}$$

$$v_n(t) = c_n e^{-n^2 \pi^2 t} - \frac{2[n^2 \pi^2 \cos t + \sin t]}{n\pi + n^5 \pi^5}$$

$$0 = \sum_{n=1}^{\infty} \left(c_n - \frac{2n^2 \pi^2}{n\pi + n^5 \pi^5} \right) \sin n\pi x$$

$$c_n = \frac{2n^2 \pi^2}{n\pi + n^5 \pi^5}$$

$$u(x, t) = (1 - x) \sin t + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n^2 \pi^2 e^{-n^2 \pi^2 t} - n^2 \pi^2 \cos t - \sin t}{n(1 + n^4 \pi^4)} \sin n\pi x$$

13.6.17

$$u_{xx} + xe^{-3t} = u_t$$
$$u(0,t) = 0$$
$$u(\pi, t) = 0$$
$$u(x, 0) = 0$$
$$xe^{-3t} = \sum_{n=1}^{\infty} \left(\frac{2}{\pi} \int_0^{\pi} xe^{-3t} \sin nx \, dx \right) \sin nx$$
$$= -2e^{-3t} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$
$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin nx$$
$$u_{xx} = \sum_{n=1}^{\infty} -n^2 u_n(t) \sin nx$$
$$u_t = \sum_{n=1}^{\infty} u_n'(t) \sin nx$$
$$\sum_{n=1}^{\infty} [-n^2 u_n(t) - u_n'(t)] \sin nx = 2e^{-3t} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$
$$-n^2 u_n - u_n' = 2e^{-3t} \frac{(-1)^n}{n}$$
$$u_n' + n^2 u_n = -2e^{-3t} \frac{(-1)^n}{n}$$
$$\frac{d}{dt}[e^{n^2 t} u_n] = -2e^{(n^2-3)t} \frac{(-1)^n}{n}$$
$$u_n = c_n e^{-n^2 t} - 2\frac{(-1)^n}{n(n^2 - 3)} e^{-3t}$$
$$0 = u(x, 0)$$
$$= \sum_{n=1}^{\infty} \left[c_n - 2\frac{(-1)^n}{n(n^2 - 3)} \right] \sin nx$$
$$c_n = 2\frac{(-1)^n}{n(n^2 - 3)}$$
$$u(x,t) = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n^2 - 3)} \left(e^{-n^2 t} - e^{-3t} \right) \sin nx$$

13.7 Orthogonal Series Expansion

13.7.1

$$ku_{xx} = u_t$$

$$u_x|_{x=0} = 0$$

$$u_x|_{x=1} = -hu(1,t)$$

$$u(x,0) = 1$$

$$u(x,t) = X(x)T(t)$$

$$kX''(x)T(t) = X(x)T'(t)$$

$$\frac{X''}{X} = \frac{T'}{kT}$$

$$X'' + \lambda X = 0$$

$$X'(0) = 0$$

$$X'(1) = -hX(1)$$

$$X = c_1 \cos \omega x + c_2 \sin \omega x$$

$$X' = -\omega c_1 \sin \omega x + \omega c_2 \cos \omega x$$

$$0 = X'(0)$$

$$= \omega c_2$$

$$= c_2$$

$$X'(1) = -hX(1)$$

$$-\omega c_1 \sin \omega = -hc_1 \cos \omega$$

$$\tan \omega = \frac{h}{\omega}$$

$$X_n = c_1 \cos \omega_n x$$

$$T' + k\lambda T = 0$$

$$T_n = c_3 e^{-k\omega_n^2 t}$$

$$u = \sum_{n=1}^{\infty} A_n e^{-k\omega_n^2 t} \cos \omega_n x$$

$$1 = u(x,0)$$

$$= \sum_{n=1}^{\infty} A_n \cos \omega_n x$$

$$A_n = \frac{\int_0^1 \cos \omega_n x \, dx}{\int_0^1 \cos^2 \omega_n x \, dx}$$

$$= \frac{2 \sin \omega_n}{\omega_n + \cos \omega_n \sin \omega_n}$$

$$= \frac{2h \sin \omega_n}{\omega_n (h + \sin^2 \omega_n)}$$

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$$u(x,t) = 2h \sum_{n=1}^{\infty} \frac{\sin \omega_n}{\omega_n (h + \sin^2 \omega_n)} e^{-k\omega_n^2 t} \cos \omega_n x$$

13.7.3

$$u_{xx} + u_{yy} = 0$$

$$u(0, y) = 0$$

$$u_x|_{x=a} = -hu(a, y)$$

$$u(x, 0) = 0$$

$$u(x, b) = f(x)$$

$$u(x, y) = X(x)Y(y)$$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

$$\frac{X''}{X} = -\lambda$$

$$X'' + \lambda X = 0$$

$$X(0) = 0$$

$$X'(a) = -hX(a)$$

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$X' = -\alpha c_1 \sin \alpha x + \alpha c_2 \cos \alpha x$$

$$0 = X(0)$$

$$= c_1$$

$$X'(a) = -hX(a)$$

$$\alpha c_2 \cos \alpha a = -hc_2 \sin \alpha a$$

$$\tan \alpha a = -\frac{\alpha}{h}$$

$$X = c_2 \sin \alpha_n x$$

$$Y'' - \lambda Y = 0$$

$$Y(0) = 0$$

$$Y = c_3 \sinh \alpha_n y + c_4 \cosh \alpha_n y$$

$$0 = Y(0)$$

$$= c_4$$

$$Y = c_3 \sinh \alpha_n y$$

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sinh \alpha_n y \sin \alpha_n x$$

$$f(x) = u(x, b)$$

$$= \sum_{n=1}^{\infty} A_n \sinh \alpha_n b \sin \alpha_n x$$

$$A_n \sinh \alpha_n b = \frac{\int_0^a f(x) \sin \alpha_n x \, dx}{\int_0^a \sin^2 \alpha_n x \, dx}$$

$$A_n = \frac{\int_0^a f(x) \sin \alpha_n x \, dx}{\left(\frac{a}{2} - \frac{\sin 2a\alpha_n}{4\alpha_n}\right) \sinh \alpha_n b}$$

$$= \frac{2h \int_0^a f(x) \sin \alpha_n x \, dx}{(ah + \cos^2 \alpha_n a) \sinh \alpha_n b}$$

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2h \int_0^a f(x) \sin \alpha_n x \, dx}{(ah + \cos^2 \alpha_n a) \sinh \alpha_n b} \sinh \alpha_n y \sin \alpha_n x$$

13.7.5

$$ku_{xx} = u_t$$

$$u(x,0) = f(x)$$

$$u(0,t) = 0$$

$$u_x|_{x=L} = 0$$

$$\frac{X''}{X} = \frac{T'}{kT}$$

$$X'' + \lambda X = 0$$

$$X(0) = 0$$

$$X'(L) = 0$$

$$X = c_1 \cos \omega x + c_2 \sin \omega x$$

$$X' = -\omega c_1 \sin \omega x + \omega c_2 \cos \omega x$$

$$0 = X(0)$$

$$= c_1$$

$$0 = X'(L)$$

$$= \omega c_2 \cos \omega L$$

$$\omega L = \frac{(2n-1)\pi}{2}$$

$$\omega = \frac{(2n-1)\pi}{2L}, \ \ n = 1,2,3,\ldots$$

$$X = c_2 \sin \frac{(2n-1)\pi}{2L} x$$

$$T' + k\lambda T = 0$$

$$T = c_3 e^{-k\lambda t}$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k(2n-1)^2 \pi^2 t / 4L^2} \sin \frac{(2n-1)\pi}{2L} x$$

$$f(x) = u(x,0)$$

$$= \sum_{n=1}^{\infty} A_n \sin \frac{(2n-1)\pi}{2L} x$$

$$A_n = \frac{\int_0^L f(x) \sin \frac{(2n-1)\pi}{2L} x \, dx}{\int_0^L \sin^2 \frac{(2n-1)\pi}{2L} x \, dx}$$

$$= \frac{2}{L} \int_0^L f(x) \sin \frac{(2n-1)\pi}{2L} x \, dx$$

$$u(x,t) = \frac{2}{L} \sum_{n=1}^{\infty} \left(\int_0^L f(x) \sin \frac{(2n-1)\pi}{2L} x \, dx \right) e^{-k(2n-1)^2 \pi^2 t / 4L^2} \sin \frac{(2n-1)\pi}{2L} x$$

13.7.7

$$u_{xx} + u_{yy} = 0$$
$$u_x|_{x=0} = 0$$
$$u(1, y) = u_0$$
$$u(x, 0) = 0$$
$$u_y|_{y=1} = 0$$
$$X''Y + XY'' = 0$$
$$\frac{Y''}{Y} = -\frac{X''}{X}$$
$$Y'' + \lambda Y = 0$$
$$Y(0) = 0$$
$$Y'(1) = 0$$
$$Y = c_1 \cos \omega y + c_2 \sin \omega y$$
$$Y' = -\omega c_1 \sin \omega y + \omega c_2 \cos \omega y$$
$$0 = Y(0)$$
$$= c_1$$
$$0 = Y'(1)$$
$$= \omega c_2 \cos \omega$$
$$\omega = \frac{2n-1}{2}\pi, \ n = 1, 2, 3, \dots$$
$$Y = c_2 \sin \frac{2n-1}{2}\pi y$$
$$X'' - \lambda X = 0$$
$$X'(0) = 0$$
$$X = c_1 \cosh \frac{2n-1}{2}\pi x + c_2 \sinh \frac{2n-1}{2}\pi x$$
$$X' = \frac{2n-1}{2}\pi \left(c_1 \sinh \frac{2n-1}{2}\pi x + c_2 \cosh \frac{2n-1}{2}\pi x \right)$$
$$0 = X'(0)$$
$$= \frac{2n-1}{2}\pi c_2$$
$$= c_2$$
$$X = c_1 \cosh \frac{2n-1}{2}\pi x$$
$$u(x, y) = \sum_{n=1}^{\infty} A_n \cosh \left(\frac{2n-1}{2}\pi x \right) \sin \left(\frac{2n-1}{2}\pi y \right)$$

$$u_0 = u(1, y)$$

$$= \sum_{n=1}^{\infty} A_n \cosh\left(\frac{2n-1}{2}\pi\right) \sin\left(\frac{2n-1}{2}\pi y\right)$$

$$A_n \cosh\left(\frac{2n-1}{2}\pi\right) = \frac{\int_0^1 u_0 \sin\left(\frac{2n-1}{2}\pi y\right) \, dy}{\int_0^1 \sin^2\left(\frac{2n-1}{2}\pi y\right) \, dy}$$

$$= \frac{4u_0}{(2n-1)\pi}$$

$$A_n = \frac{4u_0}{(2n-1)\pi \cosh\left(\frac{2n-1}{2}\pi\right)}$$

$$u(x, y) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)\cosh\left(\frac{2n-1}{2}\pi\right)} \cosh\left(\frac{2n-1}{2}\pi x\right) \sin\left(\frac{2n-1}{2}\pi y\right)$$

13.8   Fourier Series in Two Variables
13.8.1

$$k(u_{xx} + u_{yy}) = u_t$$
$$u(0, y, t) = 0$$
$$u(\pi, y, t) = 0$$
$$u(x, 0, t) = 0$$
$$u(x, \pi, t) = 0$$
$$u(x, y, 0) = u_0$$
$$X'' + \lambda X = 0$$
$$X(0) = 0$$
$$X(\pi) = 0$$
$$X = c_1 \cos \alpha x + c_2 \sin \alpha x$$
$$0 = X(0)$$
$$= c_1$$
$$0 = X(\pi)$$
$$= c_2 \sin \alpha \pi$$
$$X = c_2 \sin mx, \; m = 1, 2, 3, \ldots$$
$$Y'' + \mu Y = 0$$
$$Y(0) = 0$$
$$Y(\pi) = 0$$
$$Y = c_3 \cos \beta y + c_4 \sin \beta y$$
$$0 = Y(0)$$
$$= c_3$$
$$0 = Y(\pi)$$
$$= c_4 \sin \beta \pi$$
$$Y = c_4 \sin ny, \; n = 1, 2, 3, \ldots$$
$$T' + k(\lambda + \mu)T = 0$$
$$T = c_5 e^{-k(m^2+n^2)t}$$
$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{-k(m^2+n^2)t} \sin mx \sin ny$$
$$u_0 = u(x, y, 0)$$
$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin mx \sin ny$$

$$A_{mn} = \frac{4u_0}{\pi^2} \int_0^\pi \int_0^\pi \sin mx \sin ny \, dx \, dy$$

$$= \frac{4u_0[1-(-1)^m][1-(-1)^n]}{\pi^2 mn}$$

$$u(x,y,t) = \frac{4u_0}{\pi^2} \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{[1-(-1)^m][1-(-1)^n]}{mn} e^{-k(m^2+n^2)t} \sin mx \sin ny$$

13.8.3

$$a^2(u_{xx} + u_{yy}) = u_{tt}$$
$$u(0, y, t) = 0$$
$$u(\pi, y, t) = 0$$
$$u(x, 0, t) = 0$$
$$u(x, \pi, t) = 0$$
$$u(x, y, 0) = xy(x - \pi)(y - \pi)$$
$$u_t|_{t=0} = 0$$
$$a^2(X''YT + XY''T) = XYT''$$
$$a^2\left(\frac{X''}{X} + \frac{Y''}{Y}\right) = \frac{T''}{T}$$
$$\frac{X''}{X} = -\frac{Y''}{Y} + \frac{T''}{a^2T}$$
$$X'' + \lambda X = 0$$
$$X(0) = 0$$
$$X(\pi) = 0$$
$$X = c_1 \cos \alpha x + c_2 \sin \alpha x$$
$$0 = X(0)$$
$$= c_1$$
$$0 = X(\pi)$$
$$= c_2 \sin \alpha \pi$$
$$X = c_2 \sin mx, \ m = 1, 2, 3, \ldots$$
$$Y'' + \mu Y = 0$$
$$Y(0) = 0$$
$$Y(\pi) = 0$$
$$Y = c_3 \cos \beta y + c_4 \sin \beta y$$
$$0 = Y(0)$$
$$= c_3$$
$$0 = Y(\pi)$$
$$= c_4 \sin \beta \pi$$
$$Y = c_4 \sin ny, \ n = 1, 2, 3, \ldots$$

$$T'' + a^2(\lambda + \mu)T = 0$$
$$T'(0) = 0$$
$$T = c_5 \cos a\sqrt{m^2 + n^2}t + c_6 \sin a\sqrt{m^2 + n^2}t$$
$$T' = a\sqrt{m^2 + n^2}(-c_5 \sin a\sqrt{m^2 + n^2}t + c_6 \cos a\sqrt{m^2 + n^2}t)$$
$$0 = T'(0)$$
$$= c_6$$
$$T = c_5 \cos a\sqrt{m^2 + n^2}t$$
$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin mx \sin ny \cos a\sqrt{m^2 + n^2}t$$
$$xy(x-\pi)(y-\pi) = u(x,y,0)$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin mx \sin ny$$
$$A_{mn} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} xy(x-\pi)(y-\pi)\sin mx \sin ny \, dy \, dx$$
$$= \frac{16[1-(-1)^m][1-(-1)^n]}{m^3 n^3 \pi^2}$$
$$u(x,y,t) = \frac{16}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{[1-(-1)^m][1-(-1)^n]}{m^3 n^3}\sin mx \sin ny \cos a\sqrt{m^2 + n^2}t$$

13.9 Chapter in Review

13.9.1

$$u_{xy} = u$$
$$X'Y' = XY$$
$$\frac{X'}{X} = \frac{Y}{Y'}$$
$$X' + \lambda X = 0$$
$$X = c_1 e^{-\lambda x}$$
$$Y' + \frac{1}{\lambda}Y = 0$$
$$Y = c_2 e^{-y/\lambda}$$
$$u = c_3 e^{-\lambda x - y/\lambda}$$

13.9.3

$$ku_{xx} = u_t$$
$$u(0,t) = u_0$$
$$-u_x|_{x=\pi} = u(\pi, t) - u_1$$
$$u(x,0) = 0$$
$$u(x,t) = v(x,t) + \psi(x)$$
$$u_{xx} = v_{xx} + \psi''$$
$$u_t = v_t$$
$$k(v_{xx} + \psi'') = v_t$$
$$kv_{xx} = v_t$$
$$k\psi'' = 0$$
$$\psi(0) = u_0$$
$$-\psi'(\pi) = \psi(\pi) - u_1$$
$$\psi = c_1 x + c_2$$
$$\psi' = c_1$$
$$u_0 = \psi(0)$$
$$= c_2$$
$$-c_1 = c_1\pi + u_0 - u_1$$
$$c_1 = \frac{u_1 - u_0}{\pi + 1}$$
$$\psi = \frac{u_1 - u_0}{\pi + 1}x + u_0$$

13.9.7

$$u_{xx} + u_{yy} = 0$$

$$u(0, y) = 0$$

$$u(\pi, y) = 50$$

$$u(x, 0) = 0$$

$$u(x, \pi) = 0$$

$$X''Y + XY'' = 0$$

$$-\frac{X''}{X} = \frac{Y''}{Y}$$

$$Y'' + \lambda Y = 0$$

$$Y = c_1 \cos \omega y + c_2 \sin \omega y$$

$$0 = Y(0)$$

$$= c_1$$

$$0 = Y(\pi)$$

$$= c_2 \sin \omega \pi$$

$$Y = c_2 \sin ny, \ \ n = 1, 2, 3, \ldots$$

$$X'' - n^2 X = 0$$

$$X = c_3 \cosh nx + c_4 \sinh nx$$

$$0 = X(0)$$

$$= c_3$$

$$X = c_4 \sinh nx$$

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sinh nx \sin ny$$

$$50 = u(\pi, y)$$

$$= \sum_{n=1}^{\infty} A_n \sinh n\pi \sin ny$$

$$A_n \sinh n\pi = \frac{100}{\pi} \int_0^{\pi} \sin ny \, dy$$

$$= \frac{100}{\pi} \frac{1 - (-1)^n}{n}$$

$$A_n = \frac{100}{\pi} \frac{1 - (-1)^n}{n \sinh n\pi}$$

$$u(x, y) = \frac{100}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n \sinh n\pi} \sinh nx \sin ny$$

14 Boundary-Value Problems in Other Coordinate Systems

14.1 Polar Coordinates

14.1.1

$$u(1,\theta) = \begin{cases} u_0 & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi \end{cases}$$

$$\Theta(\theta) = \Theta(\theta + 2\pi)$$

$$\Theta'' + \lambda\Theta = 0$$

$$\Theta = c_1$$

$$\Theta = c_1 \cos n\theta + c_2 \sin n\theta, \ n = 1, 2, 3, \ldots$$

$$r^2 R'' + rR' - \lambda R = 0$$

$$R'' + \frac{1}{r}R' = 0$$

$$R = c_3 + c_4 \ln r$$

$$r^2 R'' + rR' - n^2 R = 0$$

$$R = c_3 r^n + c_4 r^{-n}$$

$$u_0 = A_0, \ n = 0$$

$$u_n = r^n(A_n \cos n\theta + B_n \sin n\theta), \ n = 1, 2, 3, \ldots$$

$$u = A_0 + \sum_{n=1}^{\infty} r^n(A_n \cos n\theta + B_n \sin n\theta)$$

$$u(1,\theta) = A_0 + \sum_{n=1}^{\infty}(A_n \cos n\theta + B_n \sin n\theta)$$

$$A_0 = \frac{a_0}{2}$$

$$= \frac{1}{2\pi}\int_0^\pi u_0 \, d\theta$$

$$= \frac{u_0}{2}$$

$$A_n = \frac{1}{\pi}\int_0^\pi u_0 \cos n\theta \, d\theta$$

$$= 0$$

$$B_n = \frac{1}{\pi}\int_0^\pi u_0 \sin n\theta \, d\theta$$

$$= \frac{\mu_0}{\pi}\frac{1 - (-1)^n}{n}$$

$$u(r,\theta) = \frac{\mu_0}{2} + \frac{\mu_0}{\pi}\sum_{n=1}^{\infty} r^n \frac{1 - (-1)^n}{n}\sin n\theta$$

14.1.5

$$\frac{\partial u}{\partial \theta}\bigg|_{\theta=0} = 0$$

$$\frac{\partial u}{\partial \theta}\bigg|_{\theta=\pi} = 0$$

$$u(2,\theta) = \begin{cases} u_0 & 0 < \theta < \pi/2 \\ 0 & \pi/2 < \theta < \pi \end{cases}$$

$$\Theta'' + \lambda\Theta = 0$$

$$\Theta'(0) = 0$$

$$\Theta'(\pi) = 0$$

$$\Theta = c_1$$

$$\Theta = c_1 \cos\omega\theta + c_2 \sin\omega\theta$$

$$\Theta' = -\omega c_1 \sin\omega\theta + \omega c_2 \cos\omega\theta$$

$$0 = \Theta'(0)$$

$$= \omega c_2$$

$$= c_2$$

$$0 = \Theta'(\pi)$$

$$= -\omega c_1 \sin\omega\pi$$

$$\omega = n, \ n = 1, 2, 3, \ldots$$

$$\Theta = c_1, \ n = 0$$

$$\Theta = c_1 \cos n\theta, \ n = 1, 2, 3, \ldots$$

$$r^2 R'' + rR' - \lambda R = 0$$

$$R = c_3, \ n = 0$$

$$R = c_3 r^n, \ n = 1, 2, 3, \ldots$$

$$u = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\theta, \ n = 1, 2, 3, \ldots$$

$$A_0 = \frac{a_0}{2}$$

$$= \frac{1}{\pi} \int_0^{\pi/2} u_0 \, d\theta$$

$$= \frac{u_0}{2}$$

$$2^n A_n = \frac{2}{\pi} \int_0^{\pi/2} u_0 \cos n\theta \, d\theta$$

$$A_n = \frac{2u_0 \sin\frac{n\pi}{2}}{2^n \pi n}$$

$$u = \frac{u_0}{2} + \frac{2u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{n\pi}{2} \left(\frac{r}{2}\right)^n \cos n\theta$$

14.1.9

$$u(r, 0) = 0$$
$$u(r, \beta) = 0$$
$$u(c, \theta) = f(\theta)$$
$$\Theta'' + \lambda\Theta = 0$$
$$\Theta(0) = 0$$
$$\Theta(\beta) = 0$$
$$\Theta = c_1 \cos\omega\theta + c_2 \sin\omega\theta$$
$$0 = \Theta(0)$$
$$= c_1$$
$$0 = \Theta(\beta)$$
$$= c_2 \sin\omega\beta$$
$$\omega\beta = n\pi$$
$$\omega = \frac{n\pi}{\beta}, \ n = 1, 2, 3, \ldots$$
$$\Theta = c_2 \sin\frac{n\pi}{\beta}\theta, \ n = 1, 2, 3, \ldots$$
$$R = c_3 r^{n\pi/\beta}, \ n = 1, 2, 3, \ldots$$
$$u = \sum_{n=1}^{\infty} A_n r^{n\pi/\beta} \sin\frac{n\pi}{\beta}\theta$$
$$A_n = \frac{2}{\beta c^{n\pi/\beta}} \int_0^\beta f(\theta) \sin\frac{n\pi}{\beta}\theta \, d\theta$$
$$u = \frac{2}{\beta} \sum_{n=1}^{\infty} \left(\frac{r}{c}\right)^{n\pi/\beta} \left(\int_0^\beta f(\theta) \sin\frac{n\pi}{\beta}\theta \, d\theta\right) \sin\frac{n\pi}{\beta}\theta$$

14.1.11

$$u(a, \theta) = f(\theta)$$
$$u(b, \theta) = 0$$
$$\Theta'' + \lambda \Theta' = 0$$
$$\Theta(\theta) = \Theta(\theta + 2\pi)$$
$$\Theta = c_1$$
$$\Theta = c_1 \cos \omega\theta + c_2 \sin \omega\theta$$
$$\omega = n, \ \ n = 1, 2, 3, \ldots$$
$$R = c_3 + c_4 \ln r$$
$$0 = R(b)$$
$$= c_3 + c_4 \ln b$$
$$c_3 = -c_4 \ln b$$
$$R = c_4 \ln \frac{r}{b}$$
$$R = c_3 r^n + c_4 r^{-n}$$
$$0 = R(b)$$
$$= c_3 b^n + c_4 b^{-n}$$
$$c_4 = -c_3 b^{2n}$$
$$R = c_3 (r^n - b^{2n} r^{-n})$$
$$u = A_0 \ln \frac{r}{b} + \sum_{n=1}^{\infty} (r^n - b^{2n} r^{-n})(A_n \cos n\theta + B_n \sin n\theta)$$
$$u(a, \theta) = A_0 \ln \frac{a}{b} + \sum_{n=1}^{\infty} (a^n - b^{2n} a^{-n})(A_n \cos n\theta + B_n \sin n\theta)$$
$$A_0 = \frac{1}{2\pi \ln a/b} \int_0^{2\pi} f(\theta) \, d\theta$$
$$A_n = \frac{1}{\pi(a^n - b^{2n} a^{-n})} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta$$
$$B_n = \frac{1}{\pi(a^n - b^{2n} a^{-n})} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$$

14.1.19

$$u(c, \theta) = f(\theta)$$
$$u(\infty, \theta) = 0$$
$$u(r, \theta) = u(r, \theta + 2\pi)$$
$$\Theta = c_1$$
$$\Theta = c_1 \cos n\theta + c_2 \sin n\theta$$
$$R = c_4 \ln r$$
$$R = c_4 r^{-n}$$
$$u = A_0 \ln r + \sum_{n=1}^{\infty} r^{-n}(A_n \cos n\theta + B_n \sin n\theta)$$
$$u(c, \theta) = A_0 \ln c + \sum_{n=1}^{\infty} c^{-n}(A_n \cos n\theta + B_n \sin n\theta)$$
$$A_0 = \frac{1}{2\pi \ln c} \int_0^{2\pi} f(\theta)\, d\theta$$
$$A_n = \frac{c^n}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta\, d\theta$$
$$B_n = \frac{c^n}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta\, d\theta$$

14.2 Cylindrical Coordinates

14.2.1

$$A_n = 0$$
$$B_n = \frac{2}{a\alpha_n c^2 J_1^2(\alpha_n c)} \int_0^c r J_0(\alpha_n r)\, dr$$
$$= \frac{2}{a\alpha_n^2 c J_1(\alpha_n c)}$$
$$u(r, t) = \frac{2}{ac} \sum_{n=1}^{\infty} \frac{\sin a\alpha_n t}{\alpha_n^2 J_1(\alpha_n c)} J_0(\alpha_n r)$$

14.2.3

$$u(r, z) = u_0 \sum_{n=1}^{\infty} \frac{\sinh[\alpha_n(4 - z)]}{\alpha_n \sinh 4\alpha_n J_1(2\alpha_n)} J_0(\alpha_n r)$$

14.2.5

$$u(1, z) = z$$

$$\left.\frac{\partial u}{\partial z}\right|_{z=0} = 0$$

$$\left.\frac{\partial u}{\partial z}\right|_{z=1} = 0$$

$$Z'' + \lambda Z = 0$$

$$Z = c_1$$

$$Z = c_1 \cos \alpha z + c_2 \sin \alpha z$$

$$Z' = -\alpha c_1 \sin \alpha z + \alpha c_2 \cos \alpha z$$

$$0 = Z'(0)$$

$$= c_2$$

$$0 = Z'(1)$$

$$= -\alpha c_1 \sin \alpha$$

$$\alpha = n\pi$$

$$rR'' + R' - \alpha^2 rR = 0$$

$$R = c_3$$

$$R = c_3 I_0(n\pi r)$$

$$u(r, t) = A_0 + \sum_{n=1}^{\infty} A_n I_0(n\pi r) \cos n\pi z$$

$$z = u(1, z)$$

$$= A_0 + \sum_{n=1}^{\infty} A_n I_0(n\pi) \cos n\pi z$$

$$A_0 = \frac{1}{2}$$

$$A_n I_0(n\pi) = 2 \int_0^1 z \cos n\pi z \, dz$$

$$A_n = \frac{2[-1 + (-1)^n]}{n^2 \pi^2 I_0(n\pi)}$$

$$u(r, t) = \frac{1}{2} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{-1 + (-1)^n}{n^2 I_0(n\pi)} I_0(n\pi r) \cos n\pi z$$

14.2.9

$$k \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) = \frac{\partial u}{\partial t}$$

$$u(c, t) = 0$$

$$u(r, 0) = f(r)$$

$$kR''T + \frac{k}{r} R'T = RT'$$

$$\frac{rR'' + R'}{rR} = \frac{T'}{kT}$$

$$rR'' + R' + \lambda rR = 0$$

$$R = c_1 J_0(\alpha r)$$

$$0 = R(c)$$

$$= c_1 J_0(\alpha c)$$

$$\alpha_n = \frac{x_n}{c}$$

$$R = c_1 J_0(\alpha_n r)$$

$$T' + k\lambda T = 0$$

$$T = c_3 e^{-k\alpha_n^2 t}$$

$$u(r, t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n t} J_0(\alpha_n r)$$

$$f(r) = u(r, 0)$$

$$= \sum_{n=1}^{\infty} A_n J_0(\alpha_n r)$$

$$A_n = \frac{2}{c^2 J_1^2(\alpha_n c)} \int_0^c r J_0(\alpha_n r) f(r) \, dr$$

14.2.11

$$\frac{rR'' + R'}{rR} = \frac{T'}{kT}$$

$$rR'' + R' + \lambda rR = 0$$

$$R = c_1 J_0(\alpha r)$$

$$R' = -\alpha c_1 J_1(\alpha r)$$

$$R'(1) = -hR(1)$$

$$\alpha c_1 J_0'(\alpha) = -hc_1 J_0(\alpha)$$

$$hJ_0(\alpha) + \alpha J_0'(\alpha) = 0$$

$$R = c_1 J_0(\alpha_n r)$$

$$T' + \lambda kT = 0$$

$$T = c_3 e^{-\alpha_n^2 kt}$$

$$u(r,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha_n^2 kt} J_0(\alpha_n r)$$

$$f(r) = u(r,0)$$

$$= \sum_{n=1}^{\infty} A_n J_0(\alpha_n r)$$

$$A_n = \frac{2\alpha_n^2}{(\alpha_n^2 + h^2)J_0^2(\alpha_n)} \int_0^1 r J_0(\alpha_n r) f(r)\, dr$$

14.2.13

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t}$$

$$u(r,t) = v(r,t) + \psi(r)$$

$$R''T + \psi'' + \frac{1}{r}(R'T + \psi') = RT'$$

$$\psi'' + \frac{1}{r}\psi' = 0$$

$$\psi = 100$$

$$\frac{rR'' + R'}{rR} = \frac{T'}{T}$$

$$rR'' + R' + \lambda rR = 0$$

$$R = c_1 J_0(\alpha r)$$

$$0 = R(2)$$

$$= c_1 J_0(2\alpha)$$

$$\alpha_n = \frac{x_n}{2}$$

$$R = c_1 J_0(\alpha_n r)$$

$$T = c_2 e^{-\alpha_n^2 t}$$

$$v(r,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha_n^2 t} J_0(\alpha_n r)$$

$$f(r) - \psi(r) = v(r,0)$$

$$f(r) - 100 = \sum_{n=1}^{\infty} c_n J_0(\alpha_n r)$$

$$c_n = \frac{2}{4J_1^2(2\alpha_n)} \int_0^2 r J_0(\alpha_n r)[f(r) - 100]\, dr$$

$$= \frac{50}{J_1^2(2\alpha_n)} \int_0^1 r J_0(\alpha_n r)\, dr$$

$$= \frac{50 J_1(\alpha_n)}{\alpha_n J_1^2(2\alpha_n)}$$

$$u(r,t) = 100 + 50 \sum_{n=1}^{\infty} \frac{J_1(\alpha_n)}{\alpha_n J_1^2(2\alpha_n)} e^{-\alpha_n^2 t} J_0(\alpha_n r)$$

14.3 Spherical Coordinates

14.3.1

$$u(r,\theta) = 25 P_0(\cos\theta) + \frac{75}{2}\frac{r}{c}P_1(\cos\theta) - \frac{175}{8}\left(\frac{r}{c}\right)^3 P_3(\cos\theta) + \frac{275}{16}\left(\frac{r}{c}\right)^5 P_5(\cos\theta)$$

14.3.5

$$u(a, \theta) = f(\theta)$$

$$u(b, \theta) = 0$$

$$R(r) = c_1 r^n + c_2 r^{-(n+1)}$$

$$0 = R(b)$$

$$= c_1 b^n + c_2 b^{-(n+1)}$$

$$c_2 = -c_1 b^{2n+1}$$

$$R = c_1 (r^n - b^{2n+1} r^{-(n+1)})$$

$$= c_1 \left(r^n - \frac{b^{2n+1}}{r^{n+1}} \right)$$

$$= c_1 \left[\left(\frac{r}{b} \right)^n - \left(\frac{b}{r} \right)^{n+1} \right]$$

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n \left[\left(\frac{r}{b} \right)^n - \left(\frac{b}{r} \right)^{n+1} \right] P_n(\cos\theta)$$

$$f(\theta) = u(a, \theta)$$

$$= \sum_{n=1}^{\infty} A_n \left[\left(\frac{a}{b} \right)^n - \left(\frac{b}{a} \right)^{n+1} \right] P_n(\cos\theta)$$

$$A_n = \frac{1}{(a/b)^n - (b/a)^{n+1}} \frac{2n+1}{2} \int_{-1}^{1} f(\theta) P_n(\cos\theta) \, d\theta$$

14.4 Chapter in Review

14.4.1

$$R = c_1$$
$$R = c_1 r^\alpha$$
$$\Theta = c_2$$
$$\Theta = c_2 \cos \alpha\theta + c_3 \sin \alpha\theta$$
$$\Theta(\theta) = \Theta(\theta + 2\pi)$$
$$\alpha = n, \ n = 1, 2, 3, \ldots$$
$$\Theta = c_2 \cos n\theta + c_3 \sin n\theta$$
$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$
$$f(\theta) = u(c,\theta)$$
$$= A_0 + \sum_{n=1}^{\infty} c^n (A_n \cos n\theta + B_n \sin n\theta)$$
$$A_0 = 0$$
$$A_n = 0$$
$$B_n = \frac{2u_0[1 - (-1)^n]}{c^n n\pi}$$
$$u(r,\theta) = \frac{2u_0}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{c^n n} r^n \sin n\theta$$

15 Integral Transform Method

15.1 Error Function

15.1.1

(a)

$$\text{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2} \, du$$
$$\tau = u^2$$
$$d\tau = 2u \, du$$
$$du = \frac{d\tau}{2\sqrt{\tau}}$$
$$\text{erf}(\sqrt{t}) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{e^{-\tau}}{\sqrt{\tau}} \, d\tau$$

(b)

$$\mathcal{L}\{t^{-1/2}\} = \frac{\sqrt{\pi}}{s^{1/2}}$$

$$\mathcal{L}\{\text{erf}(\sqrt{t})\} = \mathcal{L}\left\{\frac{1}{\sqrt{\pi}}\int_0^t \frac{e^{-\tau}}{\sqrt{\tau}}\,d\tau\right\}$$

$$= \frac{1}{\sqrt{\pi}}\frac{1}{s}\frac{\Gamma(1/2)}{\sqrt{s+1}}$$

$$= \frac{1}{s\sqrt{s+1}}$$

15.1.9

$$y(t) = 1 - \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}}\,d\tau$$

$$Y(s) = \frac{1}{s} - Y(s)\sqrt{\frac{\pi}{s}}$$

$$\left(1 + \sqrt{\frac{\pi}{s}}\right)Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s(1 + \sqrt{\pi/s})}$$

$$= \frac{1}{\sqrt{s}(\sqrt{s} + \sqrt{\pi})}$$

$$y(t) = e^{\pi t}\,\text{erfc}(\sqrt{\pi t})$$

15.1.11

$$\int_a^b e^{-u^2}\,du = \int_0^b e^{-u^2}\,du - \int_0^a e^{-u^2}\,du$$

$$= \frac{\sqrt{\pi}}{2}\left(\frac{2}{\sqrt{\pi}}\int_0^b e^{-u^2}\,du - \frac{2}{\sqrt{\pi}}\int_0^a e^{-u^2}\,du\right)$$

$$= \frac{\sqrt{\pi}}{2}[\text{erf}(b) - \text{erf}(a)]$$