Advanced Engineering Mathematics Systems of Differential Equations by Dennis G. Zill Notes

Chris Doble

August 2023

Contents

10 Systems of Linear Differential Equations

10.1 Theory of Linear Systems

- A system of the form

$$\begin{aligned}
\frac{dx_1}{dt} &= g_1(t, x_1, x_2, \ldots, x_n) \\
\frac{dx_2}{dt} &= g_2(t, x_1, x_2, \ldots, x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= g_n(t, x_1, x_2, \ldots, x_n)
\end{aligned}$$

  is called a **first-order system**.

- When each of the functions $g_n(t, x_1, x_2, \ldots, x_n)$ is linear in the dependent variables $x_1, x_2, \ldots, x_n$, we get the **normal form** of a first-order system of linear equations

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n + f_1(t)\\
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n + f_2(t)\\
&\vdots\\
\frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n + f_n(t).
\end{aligned}$$

  Such a system is called a **linear system**.

- When $f_i(t) = 0$ for $i = 1, 2, \ldots, n$ the linear system is said to be **homogeneous**, otherwise it's **nonhomogenous**.

- If $\mathbf{X}$, $\mathbf{A}(t)$, and $\mathbf{F}(t)$ denote the matrices

$$\mathbf{X} = \begin{pmatrix} x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{pmatrix}$$

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t)\\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t)\\ \vdots & & & \vdots\\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t)\end{pmatrix}$$

$$\mathbf{F}(t) = \begin{pmatrix} f_1(t)\\ f_2(t)\\ \vdots\\ f_n(t)\end{pmatrix}$$

  then homogeneous linear systems can be written

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

  and nonhomogeneous linear systems can be written

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}.$$

- A **solution vector** on an interval I is any column matrix

$$\mathbf{X} = \begin{pmatrix} x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{pmatrix}$$

whose entries are differentiable functions satisfying the linear system on the interval.

- The entries of a solution vector can be considered a set of parametric equations that define a curve in n-space. Such a curve is called a **trajectory**.
- The problem of solving

$$\mathbf{X}' = \mathbf{A}(t)\mathbf{X} + \mathbf{F}(t)$$

subject to

$$\mathbf{X}(t_0) = \mathbf{X}_0$$

is an **initial value problem** in matrix form.

- The **superposition principle** states that if $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are solution vectors of a homogeneous linear system on an interval I, then

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + c_n\mathbf{X}_n$$

where c_n are arbitrary constants is also a solution.

- If $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are a set of solution vectors of a homogeneous linear system on an interval I, the set is said to be **linearly dependent** if there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + x_n\mathbf{X}_n = \mathbf{0}$$

for every t in the interval. Otherwise the set is said to be **linearly independent**.

- A set of solution vectors

$$\mathbf{X}_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}, \quad \ldots, \quad \mathbf{X}_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$$

is linearly independent on an interval I if the **Wronskian**

$$W(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) = \begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{vmatrix} \neq 0$$

for every t in the interval.

- Any set of n linearly independent solution vectors of a homogeneous linear system on an interval I is said to be a **fundamental set of solutions** on that interval.

- If $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are a fundamental set of solutions of a homogeneous linear system on an interval I, then the **general solution** of the system on that interval is

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + c_n\mathbf{X}_n$$

  where c_i are arbitrary constants.

- For nonhomogenous systems, a **particular solution** $\mathbf{X}_p$ on an interval I is any vector, free from arbitrary parameters, whose entries are functions that satify the system.

- For nonhomogeneous systems, the **general solution** of the system on the interval is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$$

  where $\mathbf{X}_c$ is the general solution of the associated homogeneous system (the **complementary function**) and $\mathbf{X}_p$ is a particular solution of the nonhomogeneous system.

10.2 Homogeneous Linear Systems

10.2.1 Distinct Real Eigenvalues

- If $\mathbf{X}' = \mathbf{AX}$ is a homogeneous linear system, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are n real, distinct eigenvalues of $\mathbf{A}$, and $\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_n$ are the corresponding eigenvectors of $\mathbf{A}$, then

$$\mathbf{X} = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t} + \ldots + c_n\mathbf{K}_n e^{\lambda_n t}$$

  is the general solution of the system.

- If a system of linear equations consists of variables x and y, then the $x - y$ plane is called the **phase plane**.

- Solution vectors of a linear system can be considered parametric equations and plotted on the phase plane. These are called trajectories.

- When multiple trajectories are plotted in the phase plane, it's called a **phase portrait**.

10.2.2 Repeated Eigenvalues

- If the coefficient matrix $\mathbf{A}$ of a linear system has an eigenvalue λ of multiplicity m, it may be possible to find m linearly independent eigenvectors

$\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_m$ associated with the eigenvalue in which case the m solution vectors associated with the eigenvalue are

$$
\begin{aligned}
\mathbf{X}_1 &= \mathbf{K}_1 e^{\lambda t} \\
\mathbf{X}_2 &= \mathbf{K}_2 e^{\lambda t} \\
&\vdots \\
\mathbf{X}_m &= \mathbf{K}_m e^{\lambda t}.
\end{aligned}
$$

- If the coefficient matrix $\mathbf{A}$ of a linear system has an eigenvalue λ of multiplicity m and it's not possible to find m linearly independent eigenvectors associated with the eigenvalue, then the m solution vectors associated with the eigenvalue are

$$
\begin{aligned}
\mathbf{X}_1 &= \mathbf{K}_1 e^{\lambda t} \\
\mathbf{X}_2 &= \mathbf{K}_1 t e^{\lambda t} + \mathbf{K_2} e^{\lambda t} \\
&\vdots \\
\mathbf{X}_m &= \mathbf{K}_1 \frac{t^{m-1}}{(m-1)!} e^{\lambda t} + \mathbf{K}_2 \frac{t^{m-2}}{(m-2)!} e^{\lambda t} + \ldots + \mathbf{K}_m e^{\lambda t}
\end{aligned}
$$

where $\mathbf{K}_i$ are the solutions to the equations

$$
\begin{aligned}
(\mathbf{A} - \lambda \mathbf{I}) \mathbf{K}_1 &= \mathbf{0} \\
(\mathbf{A} - \lambda \mathbf{I}) \mathbf{K}_2 &= \mathbf{K}_1 \\
&\vdots \\
(\mathbf{A} - \lambda \mathbf{I}) \mathbf{K}_m &= \mathbf{K}_{m-1}.
\end{aligned}
$$

10.2.3 Complex Eigenvalues

- If $\mathbf{A}$ is the coefficient matrix of a homogeneous linear system and it has a complex eigenvalue $\lambda = \alpha + i\beta$ and associated eigenvector $\mathbf{K}_1$, then

$$\mathbf{X}_1 = \mathbf{K}_1 e^{\lambda t} \text{ and } \mathbf{X}_2 = \overline{\mathbf{K}}_1 e^{\overline{\lambda} t}$$

are solutions of the system.

- The solutions above can be made real by writing them as

$$
\begin{aligned}
\mathbf{X}_1 &= [\mathbf{B}_1 \cos \beta t - \mathbf{B}_2 \sin \beta t] e^{\alpha t} \\
\mathbf{X}_2 &= [\mathbf{B}_2 \cos \beta t + \mathbf{B}_1 \sin \beta t] e^{\alpha t}
\end{aligned}
$$

where $\mathbf{B}_1 = \mathrm{Re}(\mathbf{K}_1)$ and $\mathbf{B}_2 = \mathrm{Im}(\mathbf{K}_1)$.

10.3 Solution by Diagonalization

- A homogeneous linear system $\mathbf{X}' = \mathbf{AX}$ in which each x_i' is expressed as a linear combination of $x_1, x_2, \ldots, x_n$ is said to be **coupled**. If each x_i' is expressed solely in terms of x_i the system is said to be **uncoupled**.

- Given a linear system $\mathbf{X}' = \mathbf{AX}$, if the coefficient matrix $\mathbf{A}$ is diagonalisable such that $\mathbf{P^{-1}AP} = \mathbf{D}$ then the system can be solved by:

  1. Substituting $\mathbf{X} = \mathbf{PY}$ which gives $\mathbf{PY}' = \mathbf{APY}$ or $\mathbf{Y}' = \mathbf{P^{-1}APY} = \mathbf{DY}$
  2. Because $\mathbf{D}$ is a diagonal matrix with $\mathbf{A}$'s eigenvalues along the diagonal, this means the solutions to $\mathbf{Y}' = \mathbf{DY}$ are

  $$\mathbf{Y} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

  3. These solutions can then be substituted into $\mathbf{X} = \mathbf{PY}$ to solve for $\mathbf{X}$

10.4 Nonhomogeneous Linear Systems

10.4.1 Undetermined Coefficients

- The **method of undetermined coefficients** can be applied to a linear system $\mathbf{X}' = \mathbf{AX} + \mathbf{F}(\mathbf{t})$ when the entries of $\mathbf{A}$ are constants and the entries of $\mathbf{F}(t)$ are constants, polynomials, exponential functions, sines and cosines, or finite sums and products of these functions.

- To apply the method of undetermined coefficients:

  1. Solve the associated homogeneous linear system to find the complementary function $\mathbf{X}_c$.
  2. Assume the particular solution $\mathbf{X}_p$ has the same form as $\mathbf{F}(t)$.
  3. Substitute the trial solution into the system and solve for the unknowns.
  4. The general solution is $\mathbf{X} = \mathbf{X_c} + \mathbf{X_p}$.

- If $\mathbf{F}(t)$ contains a term that's present in the complementary function, that term needs to be adjusted (similar to how you multiply by x^n in the method of undetermined coefficients for ODEs). The textbook doesn't cover the rules for this.

10.4.2 Variation of Parameters

- If $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ is a fundamental set of solutions of the homogeneous linear system $\mathbf{X}' = \mathbf{A}\mathbf{X}$ on an interval I, then the general solution is

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + c_n\mathbf{X}_n$$

  which can also be written

$$\mathbf{X} = \boldsymbol{\Phi}(t)\mathbf{C} = \begin{pmatrix}\mathbf{X}_1 & \mathbf{X}_2 & \ldots & \mathbf{X}_n\end{pmatrix}\mathbf{C}$$

  where $\boldsymbol{\Phi}(t)$ is called a **fundamental matrix** and $\mathbf{C}$ is a column vector containing the arbitrary constants $c_1, c_2, \ldots, c_n$.

- A fundamental matrix:
  - always has an inverse, and
  - has the property that $\boldsymbol{\Phi}'(t) = \mathbf{A}\boldsymbol{\Phi}(t)$.

- The **method of variation of parameters** finds a particular solution to a nonhomogeneous linear system by replacing column vector of unknown constants $\mathbf{C}$ with a column vector of functions

$$\mathbf{U}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}$$

  such that $\mathbf{X}_p = \boldsymbol{\Phi}(t)\mathbf{U}(t)$ is a particular solution to the system.

- $\mathbf{U}(t)$ can be calculated as

$$\mathbf{U}(t) = \int \boldsymbol{\Phi}^{-1}(t)\mathbf{F}(t)\,dt$$

  so

$$\mathbf{X}_p = \boldsymbol{\Phi}(t)\int \boldsymbol{\Phi}^{-1}(t)\mathbf{F}(t)\,dt$$

  and

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = \boldsymbol{\Phi}(t)\mathbf{C} + \boldsymbol{\Phi}(t)\int \boldsymbol{\Phi}^{-1}(t)\mathbf{F}(t)\,dt.$$

- When solving initial value problems via the method of variation of parameters where you're given $\mathbf{X}(t_0) = \mathbf{X}_0$, the column vector of arbitrary constants $\mathbf{C}$ can be calculated as

$$\mathbf{C} = \boldsymbol{\Phi}^{-1}(t_0)\mathbf{X}_0.$$

10.4.3 Diagonalization

- If the coefficient matrix $\mathbf{A}$ in a nonhomogeneous linear system $\mathbf{X}' = \mathbf{AX} + \mathbf{F}(t)$ is diagonalizable, the system can be solved by:

  1. Substituting $\mathbf{X} = \mathbf{PY}$ which gives $\mathbf{PY}' = \mathbf{APY} + \mathbf{F}(t)$ or $\mathbf{Y}' = \mathbf{P^{-1}APY} + \mathbf{P^{-1}F}(t)$ or $\mathbf{Y}' = \mathbf{DY} + \mathbf{G}$
  2. Because $\mathbf{D}$ is a diagonal matrix with $\mathbf{A}$'s eigenvalues along the diagonal and $\mathbf{G} = \mathbf{P^{-1}F}(t)$ this means $\mathbf{Y}' = \mathbf{DY} + \mathbf{G(t)}$ is a set of n uncoupled equations of the form

  $$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ v_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 y_1 + g_1(t) \\ \lambda_2 y_2 + g_2(t) \\ \vdots \\ \lambda_n y_n + g_n(t) \end{pmatrix}$$

  3. These equations can be solved and substituted into $\mathbf{X} = \mathbf{PY}$ to solve for $\mathbf{X}$.

10.5 Matrix Exponential

- The linear first-order differential equation $x' = ax$ has a general solution $x = ce^{at}$. Similarly, the system $\mathbf{X}' = \mathbf{AX}$ has a general solution $\mathbf{X} = e^{\mathbf{A}t}\mathbf{C}$ where $e^{\mathbf{A}t}$ is an $n \times n$ matrix given by the **matrix exponential** and $\mathbf{C}$ is a $n \times 1$ matrix of arbitrary constants.

- The matrix exponential is defined as

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \mathbf{A}^k \frac{t^k}{k!}.$$

- The exponential of a diagonal matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

  is

$$e^{\mathbf{A}t} = \begin{pmatrix} e^{a_{11}t} & 0 & \dots & 0 \\ 0 & e^{a_{22}t} & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & e^{a_{nn}t} \end{pmatrix}.$$

- The nonhomogeneous system $\mathbf{X}' = \mathbf{AX} + \mathbf{F}$ has a general solution

$$\mathbf{X} = \mathbf{X_c} + \mathbf{X}_p = e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t} \int e^{-\mathbf{A}t}\mathbf{F}\, dt$$

where

$$e^{-\mathbf{A}t} = (e^{\mathbf{A}t})^{-1}$$

is the inverse of $e^{\mathbf{A}t}$.

- A matrix exponential can be calculated with the inverse Laplace transform

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}.$$

- A matrix exponential or that of one of its eigenvalues can be calculated as

$$e^{\mathbf{A}t} = \sum_{j=0}^{n-1} \mathbf{A}^j b_j(t) \text{ or } e^{\lambda t} = \sum_{j=0}^{n-1} \lambda^j b_j(t)$$

where $b_j(t)$ are the same for both expressions. This means that for an nxn matrix with n distinct eigenvalues the expressions for $e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}$ give n equations with n unknowns $(b_j(t))$. Solving for the $b_j(t)$ and substituting them into the expression for $e^{\mathbf{A}t}$ allows us to calculate the matrix exponential.