

# Advanced Engineering Mathematics Complex Analysis by Dennis G. Zill Notes

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## 17 Functions of a Complex Variable

### 17.1 Complex Numbers

- A **complex number** is any number of the form

$$z = a + ib$$

where  $a$  and  $b$  are real numbers and  $i$  is the imaginary unit such that  $i^2 = -1$ .

- The real number  $a$  in the above complex number  $z$  is called the **real part** of  $z$  and the real number  $b$  (not  $ib$ ) is called the **imaginary part** of  $z$ .
- The real and imaginary parts of a complex number  $z$  are denoted  $\text{Re}(z)$  and  $\text{Im}(z)$ , respectively.
- A real constant multiple of the imaginary unit, e.g.  $6i$  is called a **pure imaginary number**.
- Two complex numbers are equal if their real and imaginary parts are equal.
- The addition and subtraction of complex numbers occur between the real and imaginary parts, e.g.

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

- The multiplication of complex numbers occurs elementwise as normal, e.g.

$$(a + bi)(c + di) = ac + adi + bci - bd.$$

- The **conjugate** of a complex number  $z = a + ib$  is

$$\bar{z} = a - ib.$$

- The division of complex numbers occurs by multiplying the numerator and denominator by the conjugate of the denominator, e.g.

$$\begin{aligned} \frac{a + bi}{c + di} &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{ac - adi + bci + bd}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2}. \end{aligned}$$

- Conjugates have several interesting properties:

$$\begin{aligned} \overline{z_1 + z_2} &= \bar{z}_1 + \bar{z}_2 \\ \overline{z_1 - z_2} &= \bar{z}_1 - \bar{z}_2 \\ \overline{z_1 z_2} &= \bar{z}_1 \bar{z}_2 \\ \frac{z_1}{z_2} &= \frac{\bar{z}_1}{\bar{z}_2}. \end{aligned}$$

- The sum and product of a complex number  $z = x + iy$  with its conjugate are real numbers

$$\begin{aligned}z + \bar{z} &= 2x \\ z\bar{z} &= x^2 + y^2\end{aligned}$$

while the difference between a complex number and its conjugate is a pure imaginary number

$$z - \bar{z} = 2iy.$$

- The above properties let us define

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2} \text{ and } \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}.$$

- The **complex plane** or  **$z$ -plane** is a coordinate system where the horizontal or  $x$ -axis is called the **real axis** and the vertical or  $y$ -axis is called the **imaginary axis**. Complex numbers can be plotted in this coordinate system by considering their real and imaginary parts an ordered pair corresponding their position.
- The **modulus** or **absolute value** of a complex number  $z = x + iy$  denoted by  $|z|$  is the real number

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$$

This is the distance between  $z$  and the origin in the complex plane.

- If you consider two numbers in the complex plane as vectors, the length of their sum can't be longer than their individual lengths combined

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

This extends to any finite sum

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

and is known as the **triangle inequality**.

## 17.2 Powers and Roots

- A complex number can be expressed in **polar form**

$$z = (r \cos \theta) + i(r \sin \theta)$$

where  $r = |z|$  is the nonnegative modulus of  $z$  and  $\theta = \arg z$  is the **argument** of  $z$  — the angle between  $z$  and the positive real axis measured in the counterclockwise direction.

- The argument of a complex number  $z$  isn't unique as any multiple of  $2\pi$  can be added to it. The **principle argument** of  $z$  denoted  $\text{Arg } z$  is the argument of  $z$  restricted to the interval  $-\pi \leq \text{Arg } z \leq \pi$ .
- Multiplication and division of complex numbers is simpler in polar form. For two complex numbers  $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$  and  $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$  we get

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)].$$

- The above formulas can be used to find integer powers of a complex number  $z$

$$z^n = r^n (\cos n\theta + i \sin n\theta)$$

where  $n$  is an integer (including negative integers).

- **DeMoivre's formula** is a special case of the above where  $r = 1$  so

$$z^n = (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

- A number  $w$  is said to be an  **$n$ th root** of a nonzero complex number  $z$  if  $w^n = z$ . The  $n$ th roots of a nonzero complex number  $z = r(\cos \theta + i \sin \theta)$  are

$$w_k = r^{1/n} \left[ \cos \left( \frac{\theta + 2k\pi}{n} \right) + i \sin \left( \frac{\theta + 2k\pi}{n} \right) \right]$$

where  $k = 0, 1, 2, \dots, n-1$ .

- The root  $w$  of a complex number  $z$  obtained by using the principle argument of  $z$  with  $k = 0$  is called the **principle  $n$ th root** of  $z$ .
- Since the  $n$ th roots of a complex number have the same modulus they lie on a circle of radius  $r^{1/n}$ . The arguments of subsequent roots differ by  $2\pi/n$  so they're also equally spaced around the circle.

### 17.3 Sets in the Complex Plane

- The points  $z = x + iy$  that satisfy the equation

$$|z - z_0| = \rho$$

for  $\rho > 0$  lie on a circle of radius  $\rho$  centred at the point  $z_0$ .

- The points  $z$  satisfying the inequality  $|z - z_0| < \rho$  for  $\rho > 0$  lie within, but not on, a circle of radius  $\rho$  centered at the point  $z_0$ . This set is called a **neighborhood** of  $z_0$  or an **open disk**.
- A point  $z_0$  is said to be an **interior point** of a set  $S$  of the complex plane if there exists some neighborhood of  $z_0$  that lies entirely within  $S$ .

- If every point  $z$  of a set  $S$  is an interior point, then  $S$  is said to be an **open set**. An example of a set that isn't open is the set of points satisfying the inequality  $\operatorname{Re}(z) \geq 0$ . This isn't open because it includes the line  $\operatorname{Re}(z) = 0$  and no points on that line are interior to the set because, no matter what  $\rho$  you choose, some points in the neighborhood have  $\operatorname{Re}(z) < 0$ .
- If every neighborhood of a point  $z_0$  contains at least one point that is in a set  $S$  and at least one point that is not in  $S$ , then  $z_0$  is said to be a **boundary point** of  $S$ .
- The **boundary** of a set  $S$  in the complex plane is the set of all boundary points of  $S$ .
- If any pair of points in a set  $S$  can be connected by a polygonal line that lies entirely within the set, then  $S$  is said to be **connected**.
- An open connected set is called a **domain**.
- A **region** is a set in the complex plane with all, some, or none of its boundary points. A region containing all of its boundary points is said to be **closed**.

## 17.4 Functions of a Complex Variable

- A **function**  $f$  from a set  $A$  to a set  $B$  is a rule of correspondence that assigns to each element of  $A$  one and only one element of  $B$ .
- If  $b$  is the element of  $B$  assigned to the element  $a$  of  $A$ ,  $b$  is said to be the **image** of  $a$  and is denoted  $b = f(a)$ .
- The set  $A$  is called the **domain** of  $f$ .
- The set of all images in  $B$  is called the **range** of  $f$ .
- If  $A$  is a set of real numbers,  $f$  is said to be a **function of a real variable**  $x$ .
- If  $A$  is a set of complex numbers,  $f$  is said to be a **function of a complex variable**  $z$  or a **complex function**.
- The image  $w$  of a complex number  $z$  is

$$w = f(z) = u(x, y) + iv(x, y)$$

where  $u$  and  $v$  are the real and imaginary parts of  $w$  and are real-valued functions.

- Although we cannot draw a graph of a complex function  $w = f(z)$  (because it would require a four-dimensional coordinate system), it can be interpreted as a **mapping** or **transformation** from the  $z$  plane to the  $w$  plane.

- A complex function may be interpreted as a two-dimensional fluid flow by considering  $w = f(z)$  as the fluid velocity vector at the point  $z$ . In that case, if  $x(t) + iy(t)$  is a parametric representation of a particle's position over time then

$$\begin{aligned}\frac{dx}{dt} &= u(x, y) \\ \frac{dy}{dt} &= v(x, y)\end{aligned}$$

and the family of solutions to this system of differential equations are called the **streamlines** of the flow associated with  $f(z)$ .

#### Definition 17.4.1 Limit of a Function

Suppose the function  $f$  is defined in some neighborhood of  $z_0$ , except possibly at  $z_0$  itself. Then  $f$  is said to possess a **limit** at  $z_0$ , written

$$\lim_{z \rightarrow z_0} f(z) = L$$

if, for each  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $|f(z) - L| < \varepsilon$  whenever  $0 < |z - z_0| < \delta$ .

- For a function  $f$  of a real variable  $x$ , the limit  $\lim_{x \rightarrow x_0} f(x) = L$  means  $f$  approaches  $L$  as you approach from both the left and right. If however  $f$  is a function of a complex variable it means  $f$  approaches  $L$  as you approach from any direction in the complex plane.

#### Theorem 17.4.1 Limit of Sum, Product, Quotient

Suppose  $\lim_{z \rightarrow z_0} f(z) = L_1$  and  $\lim_{z \rightarrow z_0} g(z) = L_2$ . Then

$$(i) \quad \lim_{z \rightarrow z_0} [f(z) + g(z)] = L_1 + L_2$$

$$(ii) \quad \lim_{z \rightarrow z_0} f(z)g(z) = L_1L_2$$

$$(iii) \quad \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{L_1}{L_2}, \quad L_2 \neq 0.$$

- A function  $f$  is continuous at a point  $z_0$  if

$$\lim_{z \rightarrow z_0} f(z) = f(z_0).$$

- A function  $f$  defined by

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0, \quad a_n \neq 0$$

where  $n$  is a nonnegative integer and the coefficients  $a_i$ ,  $i = 0, 1, \dots, n$ , are complex constants is called a **polynomial** of degree  $n$ .

- Polynomials are continuous on the entire complex plane.

- A **rational function**

$$f(z) = \frac{g(z)}{h(z)}$$

is continuous everywhere  $h(z) \neq 0$ .

#### Definition 17.4.3 Derivative

Suppose the complex function  $f$  is defined in a neighborhood of a point  $z_0$ . The **derivative** of  $f$  at  $z_0$  is

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad (3)$$

provided this limit exists.

- In order for a complex function to be differentiable, the limit must approach the same value from every direction. This is a greater demand than in real variables. If you take an arbitrary complex function, there's a good chance it isn't differentiable.

#### Definition 17.4.4 Analyticity at a Point

A complex function  $w = f(z)$  is said to be **analytic at a point**  $z_0$  if  $f$  is differentiable at  $z_0$  and at every point in some neighborhood of  $z_0$ .

- Analyticity at a point is a neighborhood property. A function can be differentiable at a point but if the neighboring points aren't also differentiable, it's not analytic at that point.
- A function is analytic in a domain  $D$  if it is analytic at every point in  $D$ .
- A function that is analytic everywhere is called an **entire function**.

## 17.5 Cauchy-Riemann Equations

#### Theorem 17.5.1 Cauchy–Riemann Equations

Suppose  $f(z) = u(x, y) + iv(x, y)$  is differentiable at a point  $z = x + iy$ . Then at  $z$  the first-order partial derivatives of  $u$  and  $v$  exist and satisfy the **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad (1)$$

- If a complex function  $f(z) = u(x, y) + iv(x, y)$  is analytic throughout a domain  $D$ , then the real functions  $u$  and  $v$  must satisfy the Cauchy-Riemann equations at every point in  $D$ .

**Theorem 17.5.2    Criterion for Analyticity**

Suppose the real-valued functions  $u(x, y)$  and  $v(x, y)$  are continuous and have continuous first-order partial derivatives in a domain  $D$ . If  $u$  and  $v$  satisfy the Cauchy-Riemann equations at all points of  $D$ , then the complex function  $f(z) = u(x, y) + iv(x, y)$  is analytic in  $D$ .

- The Cauchy-Riemann equations are derived assuming the function is differentiable at a particular point. That being the case, they can also be used as a formula for the derivative of the function

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

- Because analyticity implies differentiability, theorem 17.5.2 can also be used to determine if a function is differentiable at a point.
- A real-valued function  $\phi(x, y)$  that has continuous second-order partial derivatives in a domain  $D$  and satisfies Laplace's equation is said to be **harmonic** in  $D$ .
- If a function  $f(z) = u(x, y) + iv(x, y)$  is analytic in a domain  $D$  then the functions  $u(x, y)$  and  $v(x, y)$  are harmonic functions.
- If a given function  $u(x, y)$  is harmonic in a domain  $D$  it is sometimes possible to find another function  $v(x, y)$  that is harmonic in  $D$  such that  $u(x, y) + iv(x, y)$  is analytic in  $D$ . The function  $v$  is called the **harmonic conjugate function** of  $u$ .
- To find the harmonic conjugate function of a given function  $u$ :
  1. Take the first-order partial derivatives of  $u$  with respect to  $x$  and  $y$ .
  2. If  $u(x, y) + iv(x, y)$  is analytic in a domain  $D$  then  $u$  and  $v$  must satisfy the Cauchy-Riemann equations in  $D$  from which we can find expressions for  $\partial v / \partial x$  and  $\partial v / \partial y$ .
  3. Integrate  $\partial v / \partial x$  with respect to  $x$  to get an expression for  $v$  with an unknown constant  $h(y)$ .
  4. Take the first-order partial derivative of  $v$  with respect to  $y$ , equate it with the other expression for  $\partial v / \partial y$ , and solve for  $h'(y)$ .
  5. Integrate  $h'(y)$  and substitute the result to find  $v$ .



## 17.6 Exponential and Logarithmic Functions

- The exponential function for complex numbers is defined as

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

- $e^z$  is analytic for all  $z$ , i.e. it's an entire function.
- Like its real-valued counterpart,

$$\begin{aligned}\frac{d}{dz}e^z &= e^z, \\ e^{z_1}e^{z_2} &= e^{z_1+z_2},\end{aligned}$$

and

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}.$$

- Since

$$e^{2\pi i} = \cos(2\pi) + i \sin(2\pi) = 1$$

and

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z$$

the complex function  $f(z) = e^z$  is **periodic** with complex period  $2\pi i$ . Because of this complex periodicity an infinite horizontal strip of height  $2\pi$  contains all possible values for the function. The strip  $-\pi < y \leq \pi$  is called the **fundamental region**.

- For  $z \neq 0$  and  $\theta = \arg z$ ,

$$\ln z = \log_e |z| + i(\theta + 2n\pi), \quad n = 0, \pm 1, \pm 2, \dots$$

This means there are infinitely many values of the logarithm of a complex number  $z$ . This makes sense as the complex exponential is periodic.

- The **principal value** of  $\ln z$  is the complex logarithm corresponding to  $n = 0$  and  $\theta = \text{Arg } z$ . It is denoted  $\text{Ln } z$ .
- Some familiar properties of the real-valued logarithm hold for the complex-valued logarithm, e.g.

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$

and

$$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2,$$

however they don't necessarily hold for the principal value.

- $\text{Ln } z$  is discontinuous and thus not analytic at  $z = 0$  because  $\ln z$  is undefined at  $z = 0$  and on the negative real axis because  $\text{Arg } z$  is discontinuous there.

- The derivative of  $\text{Ln } z$  is

$$\frac{d}{dz} \text{Ln } z = \frac{1}{z}.$$

- The complex power of a complex number is defined as

$$z^\alpha = e^{\alpha \text{Ln } z}, \quad z \neq 0.$$

In general this is multiple-valued because  $\text{Ln } z$  is multiple-valued — only if  $\alpha = n$ ,  $n = 0, \pm 1, \pm 2, \dots$  is it single-valued. If  $\text{Ln } z$  is replaced with  $\ln z$  then we get the **principle value** of  $z^\alpha$ .

## 17.7 Trigonometric and Hyperbolic Functions

### Definition 17.7.1 Trigonometric Sine and Cosine

For any complex number  $z = x + iy$ ,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \quad (2)$$

- The other trigonometric functions ( $\tan z$ , etc.) are defined as usual.
- Because  $e^{iz}$  and  $e^{-iz}$  are entire functions,  $\sin z$  and  $\cos z$  are also entire functions.
- $\sin z = 0$  for the real numbers  $z = n\pi$ ,  $n \in \mathbb{Z}$  and  $\cos z = 0$  for the real numbers  $z = (2n+1)\pi/2$ ,  $n \in \mathbb{Z}$ . This means that  $\tan z$  and  $\sec z$  are analytic except at the points where  $\cos z = 0$  and  $\cot z$  and  $\csc z$  are analytic except at the points where  $\sin z = 0$ .
- The usual derivatives and trigonometric functions are still valid in the complex case.
- $\sin z$  can be expressed as

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

and  $\cos z$  can be expressed as

$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$

- The only zeroes of  $\sin z$  are the real numbers  $z = n\pi$ ,  $n \in \mathbb{Z}$  and the only zeroes of  $\cos z$  are the real numbers  $z = (2n+1)\pi/2$ ,  $n \in \mathbb{Z}$ .

### Definition 17.7.2 Hyperbolic Sine and Cosine

For any complex number  $z = x + iy$ ,

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2}. \quad (10)$$

- The complex trigonometric functions can be expressed in terms of the complex hyperbolic functions and vice versa

$$\begin{aligned}\sin z &= -i \sinh(iz), & \cos z &= \cosh(iz) \\ \sinh z &= -i \sin(iz), & \cosh z &= \cos(iz).\end{aligned}$$

- $\sinh z$  can be expressed as

$$\sinh z = \sinh x \cos y + i \cosh x \sin y$$

and  $\cosh z$  can be expressed as

$$\cosh z = \cosh x \cos y + i \sinh x \sin y.$$

- The zeroes of  $\sinh z$  are  $z = n\pi i$ ,  $n \in \mathbb{Z}$  and the zeroes of  $\cosh z$  are  $z = (2n+1)\pi i/2$ ,  $n \in \mathbb{Z}$ .
- $\sin z$  and  $\cos z$  are  $2\pi$  periodic while  $\sinh z$  and  $\cosh z$  are  $2\pi i$  periodic.

## 17.8 Inverse Trigonometric and Hyperbolic Functions

- Because the complex trigonometric functions are multi-valued, their inverse functions are also multi-valued.
- The definitions of those inverse functions are

$$\begin{aligned}\arcsin z &= -i \ln[iz + (1 - z^2)^{1/2}], \\ \arccos z &= -i \ln[z + i(1 - z^2)^{1/2}], \text{ and} \\ \arctan z &= \frac{i}{2} \ln \frac{i+z}{i-z}.\end{aligned}$$

- The derivatives of the inverse trigonometric functions are

$$\begin{aligned}\frac{d}{dz} \arcsin z &= \frac{1}{(1 - z^2)^{1/2}}, \\ \frac{d}{dz} \arccos z &= \frac{-1}{(1 - z^2)^{1/2}}, \text{ and} \\ \frac{d}{dz} \arctan z &= \frac{1}{1 + z^2}.\end{aligned}$$

- The definitions of the hyperbolic inverse functions and their derivatives are

$$\sinh^{-1}z = \ln[z + (z^2 + 1)^{1/2}]$$

$$\cosh^{-1}z = \ln[z + (z^2 - 1)^{1/2}]$$

$$\tanh^{-1}z = \frac{1}{2} \ln \frac{1+z}{1-z}$$

$$\frac{d}{dz} \sinh^{-1}z = \frac{1}{(z^2 + 1)^{1/2}}$$

$$\frac{d}{dz} \cosh^{-1}z = \frac{1}{(z^2 - 1)^{1/2}}$$

$$\frac{d}{dz} \tanh^{-1}z = \frac{1}{1 - z^2}.$$

## 18 Integration in the Complex Plane

### 18.1 Contour Integrals

- In complex variables, a piecewise smooth curve  $C$  is called a **contour** or **path**. An integral of a complex function  $f(z)$  on  $C$  is denoted  $\int_C f(z) dz$  or  $\oint_C f(z) dz$  if  $C$  is closed — this is called a **contour integral** or a **complex integral**.

1. Let  $f(z) = u(x, y) + iv(x, y)$  be defined at all points on a smooth curve  $C$  defined by  $x = x(t)$ ,  $y = y(t)$ ,  $a \leq t \leq b$ .
2. Divide  $C$  into  $n$  subarcs according to the partition  $a = t_0 < t_1 < \dots < t_n = b$  of  $[a, b]$ . The corresponding points on the curve  $C$  are  $z_0 = x_0 + iy_0 = x(t_0) + iy(t_0)$ ,  $z_1 = x_1 + iy_1 = x(t_1) + iy(t_1)$ ,  $\dots$ ,  $z_n = x_n + iy_n = x(t_n) + iy(t_n)$ . Let  $\Delta z_k = z_k - z_{k-1}$ ,  $k = 1, 2, \dots, n$ .
3. Let  $\|P\|$  be the **norm** of the partition, that is, the maximum value of  $|\Delta z_k|$ .
4. Choose a sample point  $z_k^* = x_k^* + iy_k^*$  on each subarc. See [FIGURE 18.1.1](#).
5. Form the sum  $\sum_{k=1}^n f(z_k^*) \Delta z_k$ .

#### Definition 18.1.1 Contour Integral

Let  $f$  be defined at points of a smooth curve  $C$  defined by  $x = x(t)$ ,  $y = y(t)$ ,  $a \leq t \leq b$ . The **contour integral** of  $f$  along  $C$  is

$$\int_C f(z) dz = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(z_k^*) \Delta z_k. \quad (1)$$

**Theorem 18.1.1 Evaluation of a Contour Integral**

If  $f$  is continuous on a smooth curve  $C$  given by  $z(t) = x(t) + iy(t)$ ,  $a \leq t \leq b$ , then

$$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt. \quad (3)$$

- If a complex function  $f$  is continuous on a smooth curve  $C$  and if  $|f(z)| \leq M$  for all  $z$  on  $C$ , then

$$\left| \int_C f(z) dz \right| \leq ML,$$

where

$$L = \int_a^b |z'(t)| dt$$

is the length of  $C$ . This is sometimes called the **ML-inequality**.

- If  $\mathbf{T}$  is the unit tangent vector to a positively oriented simple closed curve  $C$  then

$$\oint_C f \cdot \mathbf{T} ds = \operatorname{Re} \left( \oint_C \overline{f(z)} dz \right)$$

is called the **circulation** around  $C$  and measures the tendency of the flow to rotate the curve  $C$ .

- If  $\mathbf{N}$  is the normal vector to a positive oriented simple closed curve  $C$  then

$$\oint_C f \cdot \mathbf{N} ds = \operatorname{Im} \left( \oint_C \overline{f(z)} dz \right)$$

is called the **net flux** across  $C$  and measures the difference between the rates at which fluid enters and exits the region bounded by  $C$ .

**18.2 Cauchy-Goursat Theorem**

- A domain  $D$  is said to be **simply connected** if every simple closed contour  $C$  lying entirely in  $D$  can be shrunk to a point without leaving  $D$ , i.e. the domain has no holes in it.
- A domain that is not simply connected is called a **multiply connected domain**. A domain with one hole is called **doubly connected**, a domain with two holes **triply connected**, etc.

**Theorem 18.2.1 Cauchy-Goursat Theorem**

Suppose a function  $f$  is analytic in a simply connected domain  $D$ . Then for every simple closed contour  $C$  in  $D$ ,  $\oint_C f(z) dz = 0$ .

- An alternative way of stating the Cauchy-Goursat Theorem is: if  $f$  is analytic at all points on and within a simple closed contour  $C$ , then  $\oint_C f(z) dz = 0$ .
- If  $D$  is a double connected domain and  $C$  and  $C_1$  are simple closed contours such that  $C_1$  surrounds the “hole” in the domain and is interior to  $C$ , then

$$\oint_C f(z) dz = \oint_{C_1} f(z) dz.$$

This is called the principle of **deformation of contours** since  $C_1$  can be considered a continuous deformation of the contour  $C$  (or vice versa) under which the value of the integral doesn’t change.

- If  $z_0$  is a constant complex number interior to a simple closed contour  $C$ , then

$$\oint_C \frac{dz}{(z - z_0)^n} = \begin{cases} 2\pi i & n = 1 \\ 0 & n \text{ an integer} \neq 1 \end{cases}.$$

#### Theorem 18.2.2 Cauchy–Goursat Theorem for Multiply Connected Domains

Suppose  $C, C_1, \dots, C_n$  are simple closed curves with a positive orientation such that  $C_1, C_2, \dots, C_n$  are interior to  $C$  but the regions interior to each  $C_k, k = 1, 2, \dots, n$ , have no points in common. If  $f$  is analytic on each contour and at each point interior to  $C$  but exterior to all the  $C_k, k = 1, 2, \dots, n$ , then

$$\oint_C f(z) dz = \sum_{k=1}^n \oint_{C_k} f(z) dz. \quad (6)$$

### 18.3 Independence of the Path

- Let  $z_0$  and  $z_1$  be points in a domain  $D$ . A contour integral  $\int_C f(z) dz$  is said to be **independent of the path** if its value is the same for all contours  $C$  in  $D$  with an initial point  $z_0$  and a terminal point  $z_1$ .
- If  $f$  is an analytic function in a simply connected domain  $D$ , then  $\int_C f(z) dz$  is independent of path  $C$ .
- Suppose  $f$  is continuous in a domain  $D$ . If there exists a function  $F$  such that  $F'(z) = f(z)$  for each  $z$  in  $D$ , then  $F$  is called the **antiderivative** of  $f$ .
- The general antiderivative of a complex function includes a complex integration constant.
- Suppose  $f$  is continuous in a domain  $D$  and  $F$  is an antiderivative of  $f$  in  $D$ . Then for any contour  $C$  in  $D$  with initial point  $z_0$  and terminal point  $z_1$ ,

$$\int_C f(z) dz = F(z_1) - F(z_0).$$

- A consequence of the above is that if  $C$  is closed, then

$$\oint_C f(z) dz = 0.$$

- If  $f$  is analytic in a simply connected domain  $D$ , then  $f$  has an antiderivative in  $D$ ; this, there exists a function  $F$  such that  $F'(z) = f(z)$  for all  $z$  in  $D$ .
- Suppose  $f$  and  $g$  are analytic in a simply connected domain  $D$  that contains the contour  $C$ . If  $z_0$  and  $z_1$  are the initial and terminal points of  $C$ , then the **integration by parts** formula is valid in  $D$ :

$$\int_{z_0}^{z_1} f(z)g'(z) dz = f(z)g(z)|_{z_0}^{z_1} - \int_{z_0}^{z_1} f'(z)g(z) dz.$$

## 18.4 Cauchy's Integral Formulas

### Theorem 18.4.1 Cauchy's Integral Formula

Let  $f$  be analytic in a simply connected domain  $D$ , and let  $C$  be a simple closed contour lying entirely within  $D$ . If  $z_0$  is any point within  $C$ , then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz. \quad (1)$$

- Cauchy's integral formula is useful when a contour integral has the form

$$\oint \frac{f(z)}{z - z_0} dz$$

in which case you know its value is  $2\pi i f(z_0)$ .

### Theorem 18.4.2 Cauchy's Integral Formula for Derivatives

Let  $f$  be analytic in a simply connected domain  $D$ , and let  $C$  be a simple closed contour lying entirely within  $D$ . If  $z_0$  is any point interior to  $C$ , then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz. \quad (6)$$

- Cauchy's integral formula for derivatives is useful when a contour integral has the form

$$\oint \frac{f(z)}{(z - z_0)^{n+1}} dz$$

in which case you know its value is  $\frac{2\pi i}{n!} f^{(n)}(z_0)$ .

- **Liouville's theorem** states that the only bounded entire functions are constants.

## 19 Series and Residues

### 19.1 Sequences and Series

- A **sequence** is a function whose domain is the set of positive integers, i.e. for each integer  $n = 1, 2, 3, \dots$  we assign a complex number  $z_n$ .
- If  $\lim_{n \rightarrow \infty} z_n = L$  we say the sequence  $\{z_n\}$  is **convergent**. In other words,  $\{z_n\}$  converges to the number  $L$  if, for every positive number  $\varepsilon$ , an  $N$  can be found such that  $|z_n - L| < \varepsilon$  whenever  $n > N$ .
- A sequence  $\{z_n\}$  converges to a complex number  $L$  if and only if  $\operatorname{Re}(z_n)$  converges to  $\operatorname{Re}(L)$  and  $\operatorname{Im}(z_n)$  converges to  $\operatorname{Im}(L)$ .
- An **infinite series** of complex numbers

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + \dots$$

is **convergent** if the sequence of partial sums  $\{S_n\}$ , where

$$S_n = z_1 + z_2 + \dots + z_n$$

converges. If  $S_n \rightarrow L$  as  $n \rightarrow \infty$ , we say that the **sum** of the series is  $L$ .

- The sum of the geometric series

$$\sum_{k=1}^{\infty} az^{k-1} = a + az + az^2 + \dots$$

converges to

$$\frac{a}{1 - z}$$

when  $|z| < 1$  and diverges otherwise.

- If  $\sum_{k=1}^{\infty} z_k$  converges, then  $\lim_{n \rightarrow \infty} z_n = 0$ .
- If  $\lim_{n \rightarrow \infty} z_n \neq 0$  then the series  $\sum_{k=1}^{\infty} z_k$  diverges.
- An infinite series  $\sum_{k=1}^{\infty} z_k$  is said to be **absolutely convergent** if  $\sum_{k=1}^{\infty} |z_k|$  converges. Absolute convergence implies convergence.

#### Theorem 19.1.4 Ratio Test

Suppose  $\sum_{k=1}^{\infty} z_k$  is a series of nonzero complex terms such that

$$\lim_{n \rightarrow \infty} \left| \frac{z_{n+1}}{z_n} \right| = L. \quad (9)$$

- (i) If  $L < 1$ , then the series converges absolutely.
- (ii) If  $L > 1$  or  $L = \infty$ , then the series diverges.
- (iii) If  $L = 1$ , the test is inconclusive.



**Theorem 19.1.5    Root Test**

Suppose  $\sum_{k=1}^{\infty} z_k$  is a series of complex terms such that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|z_n|} = L. \quad (10)$$

- (i) If  $L < 1$ , then the series converges absolutely.
- (ii) If  $L > 1$  or  $L = \infty$ , then the series diverges.
- (iii) If  $L = 1$ , the test is inconclusive.

- An infinite series of the form

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$$

where the coefficients  $a_k$  are complex constants is called a **power series** in  $z - z_0$ . The power series is said to be **centred at  $z_0$** , and the complex point  $z_0$  is referred to as the **centre** of the series.

- Every complex power series has a **radius of convergence  $R$**  where  $R$  is a real number. The power series converges for all  $z$  within the **circle of convergence**  $|z - z_0| < R$  and diverges for  $|z - z_0| > R$ . The series may converge at some, all, or none of the points on the actual circle of convergence.
- For a power series

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

the ratio test depends only on the coefficients  $a_k$ . If

1.  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \neq 0$ , the radius of convergence is  $R = 1/L$ ;
2.  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$ , the radius of convergence is  $\infty$ ;
3.  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$ , the radius of convergence is  $R = 0$ .

**19.2    Taylor Series**

- A power series  $\sum_{k=1}^{\infty} a_k (z - z_0)^k$  has a radius of convergence  $R$ . For each complex number  $z$  within the circle of convergence, when substituted into the power series it converges to a unique value  $L$ . This defines a function  $f$  that maps each  $z$  to its corresponding  $L$ .

**Theorem 19.2.1 Continuity**

A power series  $\sum_{k=0}^{\infty} a_k(z - z_0)^k$  represents a continuous function  $f$  within its circle of convergence  $|z - z_0| = R, R \neq 0$ .

**Theorem 19.2.2 Term-by-Term Integration**

A power series  $\sum_{k=0}^{\infty} a_k(z - z_0)^k$  can be integrated term by term within its circle of convergence  $|z - z_0| = R, R \neq 0$ , for every contour  $C$  lying entirely within the circle of convergence.

**Theorem 19.2.3 Term-by-Term Differentiation**

A power series  $\sum_{k=0}^{\infty} a_k(z - z_0)^k$  can be differentiated term by term within its circle of convergence  $|z - z_0| = R, R \neq 0$ .

**Theorem 19.2.4 Taylor's Theorem**

Let  $f$  be analytic within a domain  $D$  and let  $z_0$  be a point in  $D$ . Then  $f$  has the series representation

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \quad (8)$$

valid for the largest circle  $C$  with center at  $z_0$  and radius  $R$  that lies entirely within  $D$ .

- The radius of convergence of a Taylor series is the distance from the centre  $z_0$  to the nearest isolated singularity: a point at which the series fails to be analytic but is analytic at all points in some neighborhood of the point.

**19.3 Laurent Series**

- If a complex function  $f$  fails to be analytic at a point  $z = z_0$ , then this point is said to be a **singularity** or a **singular point** of the function.
- Suppose  $z = z_0$  is a singularity of a complex function  $f$ . It is said to be an **isolated singularity** if there exists some **deleted neighborhood**, or **punctured open disk**,  $0 < |z - z_0| < R$  of  $z_0$  in which  $f$  is analytic.
- A singular point  $z = z_0$  of a complex function  $f$  is said to be **nonisolated** if every neighborhood of  $z_0$  contains at least one singularity of  $f$  other than  $z_0$ .

**Theorem 19.3.1    Laurent's Theorem**

Let  $f$  be analytic within the annular domain  $D$  defined by  $r < |z - z_0| < R$ . Then  $f$  has the series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \quad (3)$$

valid for  $r < |z - z_0| < R$ . The coefficients  $a_k$  are given by

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} ds, \quad k = 0, \pm 1, \pm 2, \dots, \quad (4)$$

where  $C$  is a simple closed curve that lies entirely within  $D$  and has  $z_0$  in its interior (see [FIGURE 19.3.1](#)).

- Under Laurent's theorem, the part of  $f(z)$  with negative powers of  $z - z_0$  is called the **principle part** and the part with positive powers is called the **analytic part**.
- The coefficient formula of theorem 19.3.1 isn't used often. Generally  $f$  is decomposed into functions for which the series are known (e.g.  $\cos z$ ,  $e^z$ , etc.), and those series are combined to find the Laurent series.

## 19.4 Zeroes and Poles

- An isolated singularity  $z = z_0$  can be categorised based on the number of terms contained in the principal part of its Laurent expansion (the part with negative powers).
  - If the principal part is zero, i.e. the Laurent expansion consists only of parts with nonnegative powers, then  $z = z_0$  is called a **removable singularity**.
  - If the principal part contains a finite number of nonzero terms, then  $z = z_0$  is called a **pole**. If the last nonzero coefficient of the principal part is  $a_{-n}$ ,  $n \geq 1$  then we say that  $z = z_0$  is a **pole of order  $n$** . A pole of order 1 is called a **simple pole**.
  - If the principal part contains infinitely many nonzero terms, then  $z = z_0$  is called an **essential singularity**.

| $z = z_0$             | Laurent Series                                                                                                                  |
|-----------------------|---------------------------------------------------------------------------------------------------------------------------------|
| Removable singularity | $a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$                                                                                   |
| Pole of order $n$     | $\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \dots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \dots$ |
| Simple pole           | $\frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$                                                          |
| Essential singularity | $\dots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$                     |

- A point  $z_0$  is said to be a **zero** of a function  $f$  if  $f(z_0) = 0$ .
- A point  $z_0$  is said to be a **zero of order  $n$**  of a function  $f$  if  $f(z_0) = 0$ ,  $f'(z_0) = 0$ ,  $\dots$ ,  $f^{(n-1)}(z_0) = 0$  but  $f^{(n)}(z_0) \neq 0$ .

#### Theorem 19.4.1 Pole of Order $n$

If the functions  $f$  and  $g$  are analytic at  $z = z_0$  and  $f$  has a zero of order  $n$  at  $z = z_0$  and  $g(z_0) \neq 0$ , then the function  $F(z) = g(z)/f(z)$  has a pole of order  $n$  at  $z = z_0$ .

- Theorem 19.4.1 can sometimes be used to determine the poles of a function by inspection, e.g. in

$$F(z) = \frac{2z + 5}{z - 1}$$

the denominator has a zero of order 1 at  $z = 1$  and the numerator is nonzero at that point so  $F$  has a simple pole at  $z = 1$ .

## 19.5 Residues and Residue Theorem

- If a complex function  $f$  has an isolated singularity at a point  $z_0$  then it has a Laurent series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k.$$

The coefficient  $a_{-1}$  of  $1/(z - z_0)$  is called the **residue** of  $f$  at  $z_0$  and is denoted

$$a_{-1} = \text{Res}(f(z), z_0).$$

#### Theorem 19.5.1 Residue at a Simple Pole

If  $f$  has a simple pole at  $z = z_0$ , then

$$\text{Res}(f(z), z_0) = \lim_{z \rightarrow z_0} (z - z_0)f(z). \quad (1)$$

**Theorem 19.5.2    Residue at a Pole of Order  $n$** 

If  $f$  has a pole of order  $n$  at  $z = z_0$ , then

$$\operatorname{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z). \quad (2)$$

- Suppose a complex function  $f$  can be written as a quotient  $f(z) = g(z)/h(z)$  where  $g$  and  $h$  are analytic at  $z = z_0$ . If  $g(z_0) \neq 0$  and  $h$  has a zero of order 1 at  $z_0$ , then  $f$  has a simple pole at  $z = z_0$  and

$$\operatorname{Res}(f(z), z_0) = \frac{g(z)}{h'(z)}.$$

**Theorem 19.5.3    Cauchy's Residue Theorem**

Let  $D$  be a simply connected domain and  $C$  a simple closed contour lying entirely within  $D$ . If a function  $f$  is analytic on and within  $C$ , except at a finite number of singular points  $z_1, z_2, \dots, z_n$  within  $C$ , then

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^n \operatorname{Res}(f(z), z_k). \quad (5)$$

- L'Hôpital's rule is valid for complex analysis.