Classical Mechanics by John R. Taylor Notes

Chris Doble

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1 Newton's Laws of Motion

1.2 Space and Time

- In cartesian coordinates the basis vectors don't depend on time so their derivatives are $\mathbf{0}$. This means that

$$\begin{aligned}\frac{d}{dt}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) &= \frac{dx}{dt}\hat{\mathbf{x}} + x\frac{d\hat{\mathbf{x}}}{dt} + \frac{dy}{dt}\hat{\mathbf{y}} + y\frac{d\hat{\mathbf{y}}}{dt} + \frac{dz}{dt}\hat{\mathbf{z}} + z\frac{d\hat{\mathbf{z}}}{dt} \\ &= \frac{dx}{dt}\hat{\mathbf{x}} + \frac{dy}{dt}\hat{\mathbf{y}} + \frac{dz}{dt}\hat{\mathbf{z}}\end{aligned}$$

  as expected. However, in order coordinate systems (e.g. polar, spherical) the basis vectors may depend on time and their derivatives aren't $\mathbf{0}$.

1.4 Newton's First and Second Laws; Inertial Frames

- Newton's second law $\mathbf{F} = m\mathbf{a}$ can be restated as $\mathbf{F} = \dot{\mathbf{p}}$.
- An inertial frame is one where Newton's first law holds. Typically this means the frame isn't accelerating or rotating.

1.5 The Third Law and Conservation of Momentum

- Forces that act along the line joining two objects are called **central forces**.
- The **principle of conservation of momentum** states that if the net external force $\mathbf{F}_{\text{ext}}$ on an N-particle system is zero, the system's total momentum $\mathbf{P}$ is constant.

1.7 Two-Dimensional Polar Coordinates

- In two-dimensional polar coordinates, the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ depend on position and thus time. Their derivatives are

$$\begin{aligned} \frac{d\hat{\mathbf{r}}}{dt} &= \dot{\phi}\hat{\boldsymbol{\phi}} \\ \frac{d\hat{\boldsymbol{\phi}}}{dt} &= -\dot{\phi}\hat{\mathbf{r}}. \end{aligned}$$

  Consequently, the derivatives of the position vector $\mathbf{r} = r\hat{\mathbf{r}}$ are

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= \frac{d}{dt}(r\hat{\mathbf{r}}) \\ &= \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \\ &= \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}} \end{aligned}$$

  and

$$\begin{aligned} \frac{d^2\mathbf{r}}{dt^2} &= \frac{d}{dt}(\dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}) \\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt} + \dot{r}\dot{\phi}\hat{\boldsymbol{\phi}} + r\ddot{\phi}\hat{\boldsymbol{\phi}} + r\dot{\phi}\frac{d\hat{\boldsymbol{\phi}}}{dt} \\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{r}\dot{\phi}\hat{\boldsymbol{\phi}} + r\ddot{\phi}\hat{\boldsymbol{\phi}} - r\dot{\phi}^2\hat{\mathbf{r}} \\ &= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\boldsymbol{\phi}}. \end{aligned}$$

- In light of the above, Newton's second law in polar coordinates can be written

$$\begin{aligned} F_r &= m(\ddot{r} - r\dot{\phi}^2) \\ F_\phi &= m(r\ddot{\phi} + 2\dot{r}\dot{\phi}). \end{aligned}$$

2 Projectiles and Charged Particles

2.1 Air Resistance

- Air resistance depends on the speed v of the moving object. For many objects the direction of the air resistance force $\mathbf{f}$ is opposite to $\mathbf{v}$, but not always. For example, the air resistance force on an airplane causes lift.

- An air resistance force can be described by the equation

$$\mathbf{f} = -f(v)\hat{\mathbf{v}}$$

where $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$ gives the direction and $f(v)$ gives the magnitude.

- $f(v)$ can be approximated as

$$f(v) = f_{\text{lin}} + f_{\text{quad}} = bv + cv^2.$$

- The linear term f_{lin} arises from the viscous drag of the medium and is generally proportional to the projectile's linear size.

- The quadratic term f_{quad} arises from the fact that the projectile must accelerate the air with which it is continually colliding and it is proportional to the density of the medium and the cross-sectional area of the projectile.

- For a spherical projectile the coefficients b and c above have the form

$$b = \beta D \text{ and } c = \gamma D^2$$

where D is the diameter of the sphere and the coefficients β and γ depend on the nature of the medium. In air at STP they have approximate values

$$\beta = 1.6 \times 10^{-4}\,\mathrm{N\,s/m^2}$$

and

$$\gamma = 0.25\,\mathrm{N\,s^2/m^4}.$$

- Depending on the natures of the medium and projectile it's often possible to neglect one of the terms in $f(v)$. To determine if this is the case we can calculate their ratio. For example, for a spherical projectile at STP

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} = \frac{cv^2}{bv} = \frac{\gamma D}{\beta} v = (1.6 \times 10^3\,\mathrm{s/m^2})Dv.$$

If the ratio is large f_{lin} can be ignored. If it's small f_{quad} can be ignored.

- The **Reynolds number** can be used to characterise the behaviour of an object in a fluid

$$R = \frac{\rho}{\mu} D v$$

where ρ is the medium's density, μ is its viscosity, D is the linear dimension of the projectile (diameter for spherical projectiles), and v is the projectile's speed. The quadratic force f_{quad} is dominant when the Reynolds number R is large and the linear force f_{linear} is dominant when it is small.

2.2 Linear Air Resistance

- When the quadratic drag force is negligible the equation of motion becomes

$$\begin{aligned}\mathbf{F} &= \mathbf{W} - \mathbf{f}\\ m\mathbf{a} &= m\mathbf{g} - b\mathbf{v}\\ m\dot{\mathbf{v}} &= m\mathbf{g} - b\mathbf{v}.\end{aligned}$$

  This is a first-order differential equation for $\mathbf{v}$ where the horizontal and vertical components can be separated to

$$\begin{aligned}m\dot{v}_x &= -bv_x\\ m\dot{v}_y &= mg - bv_y,\end{aligned}$$

  each of which is easily solvable.

- The **terminal speed** of an object undergoing freefall and experiencing only linear drag is

$$v_{\text{ter}} = \frac{mg}{b}.$$

- The **characteristic time**

$$\tau = \frac{1}{k} = \frac{1}{b/m} = \frac{m}{b}$$

  is a measure of the importance of air resistance.

  - For horizontal motion with drag it's a measure of the time it takes for the projectile to reach $1/e$ of its initial velocity.
  - For freefall with drag it's a measure of the time it would take the projectile to reach its terminal velocity if it didn't experience drag

$$v_{\text{ter}} = g\tau.$$

  - For freefall with drag it can also be used to gauge what percentage of its terminal velocity a projectile will reach after a certain time:

Time t	Percent of v_{ter}
0	0
τ	63%
2τ	86%
3τ	95%

    From this it can be seen that after $t = 3\tau$ the projectile has effectively reached its terminal velocity.

2.4 Quadratic Air Resistance

- Equations of motion for quadratic air resistance can be solved analytically when the projectile moves in one dimension, but can only be solved numerically when it moves in multiple dimensions.
- When a projectile moves in one dimension and only experiences the force of air resistance (i.e. there are no other forces), the equation of motion is

  $$m\dot{v} = -cv^2.$$

  Using separation of variables the solution can be found to be

  $$v(t) = \frac{v_0}{1 + t/\tau}$$

  where

  $$\tau = \frac{m}{cv_0}.$$

- As in the linear case, τ is a measure of how long it takes for air resistance to slow down the projectile ($v = v_0/2$ at $t = \tau$).
- Integrating the equation for $v(t)$ gives

  $$x(t) = v_0 \tau \ln\left(1 + \frac{t}{\tau}\right).$$

- When a projectile moves in one dimension and experiences the forces of air resistance and weight, the equation of motion (with y down) is

  $$m\dot{v} = mg - cv^2.$$

  Using separation of variables the solution can be found to be

  $$v(t) = v_{\text{ter}} \tanh \frac{gt}{v_{\text{ter}}}$$

  where

  $$v_{\text{ter}} = \sqrt{\frac{mg}{c}}.$$

- Integrating the equation for $v(t)$ gives

  $$y = \frac{v_{\text{ter}}^2}{g} \ln\left(\cosh \frac{gt}{v_{\text{ter}}}\right).$$