Advanced Engineering Mathematics Complex Analysis by Dennis G. Zill Notes

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Contents

17 Functions of a Complex Variable

17.1 Complex Numbers

- A **complex number** is any number of the form

$$z = a + ib$$

where a and b are real numbers and i is the imaginary unit such that $i^2 = -1$.

- The real number a in the above complex number z is called the **real part** of z and the real number b (not ib) is called the **imaginary part** of z.

- The real and imaginary parts of a complex number z are denoted $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively.

- A real constant multiple of the imaginary unit, e.g. $6i$ is called a **pure imaginary number**.

- Two complex numbers are equal if their real and imaginary parts are equal.

- The addition and subtraction of complex numbers occur between the real and imaginary parts, e.g.

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

- The multiplication of complex numbers occurs elementwise as normal, e.g.

$$(a + bi)(c + di) = ac + adi + bci - bd.$$

- The **conjugate** of a complex number $z = a + ib$ is
$$\overline{z} = a - ib.$$

- The division of complex numbers occurs by multiplying the numerator and denominator by the conjugate of the denominator, e.g.
$$\begin{aligned}\frac{a+bi}{c+di} &= \frac{(a+bi)(c-di)}{(c+di)(c-di)} \\ &= \frac{ac - adi + bci + bd}{c^2+d^2} \\ &= \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}.\end{aligned}$$

- Conjugates have several interesting properties:
$$\begin{aligned}\overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2} \\ \overline{z_1 - z_2} &= \overline{z_1} - \overline{z_2} \\ \overline{z_1 z_2} &= \overline{z_1}\,\overline{z_2} \\ \frac{z_1}{z_2} &= \frac{\overline{z_1}}{\overline{z_2}}.\end{aligned}$$

- The sum and product of a complex number $z = x + iy$ with its conjugate are real numbers
$$\begin{aligned}z + \overline{z} &= 2x \\ z\overline{z} &= x^2 + y^2\end{aligned}$$
while the difference between a complex number and its conjugate is a purre imaginary number
$$z - \overline{z} = 2iy.$$

- The above properties let us define
$$\mathrm{Re}(z) = \frac{z+\overline{z}}{2} \text{ and } \mathrm{Im}(z) = \frac{z-\overline{z}}{2i}.$$

- The **complex plane** or **z-plane** is a coordinate system where the horizontal or x-axis is called the **real axis** and the vertical or y-axis is called the **imaginary axis**. Complex numbers can be plotted in this coordinate system by considering their real and imaginary parts an ordered pair corresponding their position.

- The **modulus** or **absolute value** of a complex number $z = x+iy$ denoted by $|z|$ is the real number
$$|z| = \sqrt{x^2+y^2} = \sqrt{z\overline{z}}.$$
This is the distance between z and the origin in the complex plane.

- If you consider two numbers in the complex plane as vectors, the length of their sum can't be longer than their individual lengths combined

  $$|z_1 + z_2| \leq |z_1| + |z_2|.$$

  This extends to any finite sum

  $$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

  and is known as the **triangle inequality**.