Advanced Engineering Mathematics Complex Analysis by Dennis G. Zill Notes

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Contents

17 Functions of a Complex Variable

17.1 Complex Numbers

- A **complex number** is any number of the form

$$z = a + ib$$

  where a and b are real numbers and i is the imaginary unit such that $i^2 = -1$.

- The real number a in the above complex number z is called the **real part** of z and the real number b (not ib) is called the **imaginary part** of z.

- The real and imaginary parts of a complex number z are denoted $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively.

- A real constant multiple of the imaginary unit, e.g. $6i$ is called a **pure imaginary number**.

- Two complex numbers are equal if their real and imaginary parts are equal.

- The addition and subtraction of complex numbers occur between the real and imaginary parts, e.g.

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

- The multiplication of complex numbers occurs elementwise as normal, e.g.

$$(a + bi)(c + di) = ac + adi + bci - bd.$$

- The **conjugate** of a complex number $z = a + ib$ is

$$\overline{z} = a - ib.$$

- The division of complex numbers occurs by multiplying the numerator and denominator by the conjugate of the denominator, e.g.

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{ac - adi + bci + bd}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}.\end{aligned}$$

- Conjugates have several interesting properties:

$$\begin{aligned}\overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2} \\ \overline{z_1 - z_2} &= \overline{z_1} - \overline{z_2} \\ \overline{z_1 z_2} &= \overline{z_1}\,\overline{z_2} \\ \overline{\frac{z_1}{z_2}} &= \frac{\overline{z_1}}{\overline{z_2}}.\end{aligned}$$

- The sum and product of a complex number $z = x + iy$ with its conjugate are real numbers

$$\begin{aligned}z + \overline{z} &= 2x \\ z\overline{z} &= x^2 + y^2\end{aligned}$$

  while the difference between a complex number and its conjugate is a purre imaginary number

$$z - \overline{z} = 2iy.$$

- The above properties let us define

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2} \text{ and } \mathrm{Im}(z) = \frac{z - \overline{z}}{2i}.$$

- The **complex plane** or **z-plane** is a coordinate system where the horizontal or x-axis is called the **real axis** and the vertical or y-axis is called the **imaginary axis**. Complex numbers can be plotted in this coordinate system by considering their real and imaginary parts an ordered pair corresponding their position.

- The **modulus** or **absolute value** of a complex number $z = x+iy$ denoted by $|z|$ is the real number

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$$

  This is the distance between z and the origin in the complex plane.

- If you consider two numbers in the complex plane as vectors, the length of their sum can't be longer than their individual lengths combined

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

  This extends to any finite sum

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

  and is known as the **triangle inequality**.

17.2 Powers and Roots

- A complex number can be expressed in **polar form**

$$z = (r\cos\theta) + i(r\sin\theta)$$

  where $r = |z|$ is the nonnegative modulus of z and $\theta = \arg z$ is the **argument** of z — the angle between z and the positive real axis measured in the counterclockwise direction.

- The argument of a complex number z isn't unique as any multiply of 2π can be added to it. The **principle argument** of z denoted $\operatorname{Arg} z$ is the argument of z restricted to the intercal $-\pi \leq \operatorname{Arg} z \leq \pi$.

- Multiplication and division of complex numbers is simpler in polar form. For two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ we get

$$\begin{aligned} z_1 z_2 &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]. \end{aligned}$$

- The above formulas can be used to find integer powers of a complex number z

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

  where n is an integer (including negative integers).

- **DeMoivre's formula** is a special case of the above where $r = 1$ so

$$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

- A number w is said to be an ***n*th root** of a nonzero complex number z if $w^n = z$. The nth roots of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ are

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

  where $k = 0, 1, 2, \ldots, n-1$.

- The root w of a complex number z obtained by using the principle argument of z with $k = 0$ is called the **principle *n*th root** of z.

- Since the nth roots of a complex number have the same modulus they lie on a circle of radius $r^{1/n}$. The arguments of subsequent roots differ by $2\pi/n$ so they're also equally spaced around the circle.

17.3 Sets in the Complex Plane

- The points $z = x + iy$ that satisfy the equation

$$|z - z_0| = \rho$$

  for $\rho > 0$ lie on a circle of radius ρ centred at the point z_0.

- The points z satisfying the inequality $|z - z_0| < \rho$ for $\rho > 0$ lie within, but not on, a circle of radius ρ centered at the point z_0. This set is called a **neighborhood** of z_0 or an **open disk**.

- A point z_0 is said to be an **interior point** of a set S of the complex plane if there exists some neighborhood of z_0 that lies entirely within S.

- If every point z of a set S is an interior point, then S is said to be an **open set**. An example of a set that isn't open is the set of points satisfying the inequality $\mathrm{Re}(z) \geq 0$. This isn't open because it includes the line $\mathrm{Re}(z) = 0$ and no points on that line are interior to the set because, no matter what ρ you choose, some points in the neighborhood have $\mathrm{Re}(z) < 0$.

- If every neighborhood of a point z_0 contains at least one point that is in a set S and at least one point that is not in S, then z_0 is said to be a **boundary point** of S.

- The **boundary** of a set S in the complex plane is the set of all boundary points of S.

- If any pair of points in a set S can be connected by a polygonal line that lies entirely within the set, then S is said to be **connected**.

- An open connected set is called a **domain**.

- A **region** is a set in the complex plane with all, some, or none of its boundary points. A region containing all of its boundary points is said to be **closed**.