Quantum Computation and Quantum Information by Michael A. Nielsen and Isaac L. Chuang

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Part I
Fundamental concepts

1 Introduction and overview

1.2 Quantum bits

- The special states $|0\rangle$ and $|1\rangle$ form an orthonormal basis and are known as **computational basis states**.

- A quantum bit (**qubit**) is a linear combination of the computational basis states

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

where α and β are complex numbers.

- When we measure a qubit we either get $|0\rangle$ with probability $|\alpha|^2$ or $|1\rangle$ with probability $|\beta|^2$. Thus, $|\alpha|^2 + |\beta|^2 = 1$ and a qubit can be thought of as a unit vector in a two-dimensional complex vector space.

- If a qubit is in the state

  $$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

  there's a 50/50 chance of measuring $|0\rangle$ or $|1\rangle$.

- If we let

  $$\alpha = e^{i\gamma} \cos\frac{\theta}{2}$$

  and

  $$\beta = e^{i\gamma} e^{i\varphi} \sin\frac{\theta}{2}$$

  then

  $$\begin{aligned} |\alpha|^2 + |\beta|^2 &= \alpha^*\alpha + \beta^*\beta \\ &= \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} \\ &= 1 \end{aligned}$$

  so the qubit is still normalised and it can be written

  $$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right).$$

  It turns out that $e^{i\gamma}$ has no observable effects and we can effectively write

  $$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle.$$

  This defines a point on a three-dimensional sphere known as the **Bloch sphere** where θ and φ take on their usual roles in a spherical coordinate system.

- Before measurement a qubit is in a linear combination of $|0\rangle$ and $|1\rangle$ but when measured you get one or the other and the state of the system changes to match the measured result.

1.2.1 Multiple Bits

- A two qubit system has four computational basis state $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$ so the general expression for the state of such a system is

  $$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle.$$

- If you were to measure the first qubit, you would get $|0\rangle$ with probability $|\alpha_{00}|^2 + |\alpha_{01}|^2$ and the system would be left in the state

$$|\psi\rangle = \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}},$$

  i.e. it is renormalised such that the normalisation condition still holds.

1.3 Quantum Computation

1.3.1 Single Qubit Gates

- The quantum NOT changes $|0\rangle$ to $|1\rangle$ and $|1\rangle$ to $|0\rangle$. It acts linearly on superpositions of those states, i.e. it turns $\alpha|0\rangle + \beta|1\rangle$ into $\beta|0\rangle + \alpha|1\rangle$. If a quantum state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is written in vector notation as

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

  then the quantum NOT gate can be expressed in matrix form as

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

- In order to preserve the normalisation condition, matrix representations of quantum gates must be unitary, i.e. $M^\dagger M = I$ where I is the identity matrix.

- An arbitrary unitary 2x2 matrix can be decomposed into a finite set of other 2x2 matrices. This means an arbitrary single qubit gate can be generated by a finite set of other gates.

1.3.2 Multiply Qubit Gates

- The controlled-NOT or CNOT gate is a multi-qubit gate that has two input qubits known as the control qubit and the target qubit. If the control qubit is set to $|0\rangle$ the target qubit is left alone, but if it's set to $|1\rangle$ the target qubit is flipped. Another way of writing this is $|A, B\rangle \rightarrow |A, A \oplus B\rangle$ where $\oplus$ is modulo-two addition. Yet another way of writing this is in matrix form

$$U_{CN} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

  where the first column describes what happens to the $|00\rangle$ basis state, etc.

- Other classical gates like NAND or XOR can't be represented as quantum gates as they're irreversibile. For example, given the output $A \oplus B$ from an XOR gate it's not possible to determine what the inputs A and B were.

- Any multiple qubit logic gate may be composed from CNOT and single qubit gates.

1.3.3 Measurements in bases other than the computational basis

- Given any basis states $|a\rangle$ and $|b\rangle$ it is possible to express an arbitrary state as a linear combination $\alpha |a\rangle + \beta |b\rangle$ of those states. For example, if $|a\rangle = |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|b\rangle = |-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$ then

$$|\phi\rangle = \alpha |0\rangle + \beta |1\rangle = \alpha \frac{|+\rangle + |-\rangle}{\sqrt{2}} + \beta \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{\alpha + \beta}{\sqrt{2}} |+\rangle + \frac{\alpha - \beta}{\sqrt{2}} |-\rangle .$$

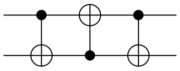
- If $|a\rangle$ and $|b\rangle$ are orthonormal, it's also possible to perform a measurement with respect to that basis. In the example above you would measure $|+\rangle$ with probability $|\alpha + \beta|^2/2$ and $|-\rangle$ with probability $|\alpha - \beta|^2/2$.

1.3.4 Quantum circuits

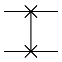
- Applying a CNOT gate three times swaps the state of two qubits:

$$\begin{aligned} |a, b\rangle &\to |a, a \oplus b\rangle \\ &\to |a \oplus (a \oplus b), a \oplus b\rangle = |b, a \oplus b\rangle \\ &\to |b, (a \oplus b) \oplus b\rangle = |b, a\rangle . \end{aligned}$$

  This can be represented in a quantum circuit diagram

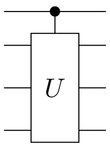


  which is equivalent to

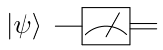


- Quantum circuits don't allow loops (they are acyclic), they don't allow multiple wires to be joined into one, and they don't allow one wire to be split into multiple wires.

- If U is a unitary matrix operating on n qubits then U can be regarded as a quantum gate. Then we can define a controlled-U gate which takes a single control qubit and n target qubits. If the control qubit is set to 0 then nothing happens to the target qubits. If it's set to 1 then the U gate is applied to the target qubits.



- Measurement is represented as

  where the double lines represent a classical bit.