Quantum Computation and Quantum Information by Michael A. Nielsen and Isaac L. Chuang Problems

Chris Doble

June 2024

Contents

Part I
Fundamental concepts

2 Linear algebra

Exercise 2.1

$$(1, -1) + (1, 2) - (2, 1) = (0, 0)$$

Exercise 2.2

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Using the basis $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$ we get

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$|1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$a_{00} + a_{01} = 1$$

$$a_{10} + a_{11} = -1$$

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$a_{00} - a_{01} = 1$$

$$a_{10} - a_{11} = 1$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Exercise 2.5

$$
\begin{aligned}
\begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} &= \begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \left(\begin{bmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ z_2 \\ \vdots \\ 0 \end{bmatrix} + \dots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ z_n \end{bmatrix} \right) \\
&= \begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \begin{bmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + \begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ z_n \end{bmatrix} \\
&= z_1 \begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + z_n \begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} &= y_1^* z_1 + y_2^* z_2 + \ldots + y_n^* z_n \\
&= (y_1 z_1^* + y_2 z_2^* + \ldots + y_n z_n^*)^* \\
&= \left(\begin{bmatrix} z_1^* & \cdots & z_n^* \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right)^*
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} v_1^* & \cdots & v_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} &= |v_1|^2 + \dots + |v_n|^2 \\
&\geq 0
\end{aligned}
$$

Exercise 2.6

$$
\begin{aligned}
\left(\sum_i \lambda_i \left| w_i \right\rangle , \left| v \right\rangle \right) &= \left(\left| v \right\rangle , \sum_i \lambda_i \left| w_i \right\rangle \right)^* \\
&= \left(\sum_i \lambda_i (\left| v \right\rangle , \left| w_i \right\rangle) \right)^* \\
&= \sum_i \lambda_i^* (\left| v \right\rangle , \left| w_i \right\rangle)^* \\
&= \sum_i \lambda_i^* (\left| w_i \right\rangle , \left| v \right\rangle)
\end{aligned}
$$

Exercise 2.7

$$
\begin{aligned}
\langle w|v\rangle &= \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}\\
&= (1)(1) + (1)(-1)\\
&= 0\\
\frac{|w\rangle}{||\,|w\rangle\,||} &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\\
\frac{|v\rangle}{||\,|v\rangle\,||} &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}
\end{aligned}
$$

Exercise 2.9

$$
\begin{aligned}
\sigma_0 &= |0\rangle\langle 0| + |1\rangle\langle 1|\\
\sigma_1 &= |1\rangle\langle 0| + |0\rangle\langle 1|\\
\sigma_2 &= i\,|1\rangle\langle 0| - i\,|0\rangle\langle 1|\\
\sigma_3 &= |0\rangle\langle 0| - |1\rangle\langle 1|
\end{aligned}
$$

Exercise 2.11

$$
\begin{aligned}
\begin{vmatrix}-\lambda & 1\\ 1 & -\lambda\end{vmatrix} &= \lambda^2 - 1\\
\lambda &= \pm 1\\
\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} &= \begin{bmatrix}a\\b\end{bmatrix}\\
b &= a\\
a &= b\\
X_1 &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\\
\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} &= \begin{bmatrix}-a\\-b\end{bmatrix}\\
b &= -a\\
a &= -b\\
X_2 &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}
\end{aligned}
$$

Exercise 2.12

$$
\begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2
$$

$$
\lambda_1 = 1
$$

$$
\lambda_2 = 1
$$

The eigenvalue 1 is degenerate. Because the matrix only has one eigenvector it can't diagonalised.

Exercise 2.13

$$
(|w\rangle\langle v|)^\dagger = \langle v|^\dagger\,|w\rangle^\dagger = |v\rangle\langle w|
$$

Exercise 2.16

$$
\begin{aligned}
P^2 &= \left(\sum_{i=1}^{k} |i\rangle\langle i|\right)\left(\sum_{j=1}^{k} |j\rangle\langle j|\right) \\
&= \sum_{i=j=1}^{k} |i\rangle\langle i|j\rangle\langle j| \\
&= \sum_{i=j=1}^{k} |i\rangle\,\delta_{ij}\,\langle j| \\
&= \sum_{i=1}^{k} |i\rangle\langle i| \\
&= P
\end{aligned}
$$

Exercise 2.17

$$
\begin{aligned}
A &= A^\dagger \\
\sum_i \lambda_i |i\rangle\langle i| &= \left(\sum_i \lambda_i |i\rangle\langle i|\right)^\dagger \\
&= \sum_i \lambda_i^* |i\rangle\langle i|
\end{aligned}
$$

$\lambda_i = \lambda_i^*$ implies the eigenvalues are real.

Exercise 2.18

$$
\begin{aligned}
U^\dagger U &= I \\
\left(\sum_i \lambda_i \left|i\right\rangle \left\langle i\right|\right)^\dagger \left(\sum_i \lambda_j \left|j\right\rangle \left\langle j\right|\right) &= \sum_k \left|k\right\rangle \left\langle k\right| \\
\sum_{ij} \lambda_i^* \lambda_j \left|i\right\rangle \left\langle i|j\right\rangle \left\langle j\right| &= \sum_k \left|k\right\rangle \left\langle k\right| \\
\sum_{ij} \lambda_i^* \lambda_j \left|i\right\rangle \delta_{ij} \left\langle j\right| &= \sum_k \left|k\right\rangle \left\langle k\right| \\
\sum_i \left|\lambda_i\right|^2 \left|i\right\rangle \left\langle i\right| &= \sum_k \left|k\right\rangle \left\langle k\right| \\
\left|\lambda_i\right|^2 &= 1 \\
\lambda_i &= e^{i\theta}
\end{aligned}
$$

Exercise 2.19

$$
\begin{aligned}
I^\dagger &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^\dagger \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I \\
I^\dagger I &= II \\
&= I \\
X^\dagger &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^\dagger \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= X \\
X^\dagger X &= XX \\
&= I \\
Y^\dagger &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^\dagger \\
&= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= Y \\
Y^\dagger Y &= YY \\
&= I \\
Z^\dagger &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^\dagger \\
&= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= Z \\
Z^\dagger Z &= ZZ \\
&= I
\end{aligned}
$$

Exercise 2.22

$$\begin{aligned}
\langle v_1|A|v_2\rangle &= \langle v_1|Av_2\rangle \\
&= \langle v_1|\lambda_2 v_2\rangle \\
&= \lambda_2 \langle v_1|v_2\rangle \\
\langle v_1|A|v_2\rangle &= \langle A^\dagger v_1|v_2\rangle \\
&= \langle Av_1|v_2\rangle \\
&= \langle \lambda_1 v_1|v_2\rangle \\
&= \lambda_1 \langle v_1|v_2\rangle \\
0 &= (\lambda_2 - \lambda_1) \langle v_1|v_2\rangle \\
&= \langle v_1|v_2\rangle
\end{aligned}$$

Exercise 2.23

For each basis vector $|i\rangle, i = 1, \ldots k$, $P|i\rangle = |i\rangle$ and so they are eigenvectorrs of P with eigenvalue 1. For each basis vector $|j\rangle, j = k+1, \ldots, d$, $P|j\rangle = 0$ and so they are eigenvectors of P with eigenvalue of 0. That is a total of d eigenvectors so all eigenvalues are either 0 or 1.

Exercise 2.26

$$
\begin{aligned}
|\psi\rangle &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\
|\psi\rangle^{\otimes 2} &= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \\
&= \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} \\
|\psi\rangle^{\otimes 3} &= \left(\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \\
&= \frac{|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle}{2^{3/2}} \\
|\psi\rangle &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \\
|\psi\rangle^{\otimes 2} &= \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \\
|\psi\rangle^{\otimes 3} &= \frac{1}{2^{3/2}}\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}
\end{aligned}
$$

Exercise 2.27

(a)

$$
\begin{aligned}
X &= \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \\
Z &= \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} \\
X \otimes Z &= \begin{bmatrix}(0)Z & (1)Z\\(1)Z & (0)Z\end{bmatrix} \\
&= \begin{bmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & -1\\1 & 0 & 0 & 0\\0 & -1 & 0 & 0\end{bmatrix}
\end{aligned}
$$

(b)

$$I \otimes X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(c)

$$X \otimes I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

No, the tensor product is not commutative.

Exercise 2.28

$$\begin{aligned}
(A \otimes B)^* &= \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{bmatrix}^* \\
&= \begin{bmatrix} A_{11}^*B^* & A_{12}^*B^* & \cdots & A_{1n}^*B^* \\ A_{21}^*B^* & A_{22}^*B^* & \cdots & A_{2n}^*B^* \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}^*B^* & A_{m2}^*B^* & \cdots & A_{mn}^*B^* \end{bmatrix} \\
&= A^* \otimes B^* \\
(A \otimes B)^T &= \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{bmatrix}^T \\
&= \begin{bmatrix} A_{11}B^T & A_{21}B^T & \cdots & A_{m1}B^T \\ A_{12}B^T & A_{22}B^T & \cdots & A_{m2}B^T \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n}B^T & A_{2n}B^T & \cdots & A_{mn}B^T \end{bmatrix} \\
&= A^T \otimes B^T \\
(A \otimes B)^\dagger &= [(A \otimes B)^*]^T \\
&= (A^* \otimes B^*)^T \\
&= (A^*)^T \otimes (B^*)^T \\
&= A^\dagger \otimes B^\dagger
\end{aligned}$$

Exercise 2.29

$$\begin{aligned}(A \otimes B)^\dagger (A \otimes B)(|a\rangle \otimes |b\rangle) &= (A^\dagger \otimes B^\dagger)(A|a\rangle \otimes B|b\rangle) \\ &= A^\dagger A|a\rangle \otimes B^\dagger B|b\rangle \\ &= |a\rangle \otimes |b\rangle\end{aligned}$$

Exercise 2.30

$$\begin{aligned}(A \otimes B)^\dagger (|a\rangle \otimes |b\rangle) &= (A^\dagger \otimes B^\dagger)(|a\rangle \otimes |b\rangle) \\ &= (A \otimes B)(|a\rangle \otimes |b\rangle)\end{aligned}$$

Exercise 2.34

$$\begin{aligned}A &= \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \\ \lambda_1 &= 1 \\ \mathbf{x}_1 &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ \lambda_2 &= 7 \\ \mathbf{x}_2 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ P &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ A &= PDP^{-1} \\ D &= P^{-1}AP \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \\ \sqrt{A} &= P\sqrt{D}P^{-1} \\ &= \frac{1}{2}\begin{bmatrix} 1+\sqrt{7} & -1+\sqrt{7} \\ -1+\sqrt{7} & 1+\sqrt{7} \end{bmatrix} \\ \ln A &= P\ln(D)P^{-1} \\ &= \frac{\ln 7}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\end{aligned}$$

Exercise 2.35

$$
\begin{aligned}
\mathbf{v} &= (v_x, v_y, v_z) \\
\mathbf{v} \cdot \boldsymbol{\sigma} &= v_x\sigma_1 + v_y\sigma_2 + v_z\sigma_3 \\
&= \begin{bmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{bmatrix} \\
A &= i\theta\mathbf{v} \cdot \boldsymbol{\sigma} \\
\lambda_1 &= -i\theta v \\
\mathbf{x}_1 &= \begin{bmatrix} (v_z - v)/(v_x + iv_y) \\ 1 \end{bmatrix} \\
\lambda_2 &= i\theta v \\
\mathbf{x}_2 &= \begin{bmatrix} (v_z + v)/(v_x + iv_y) \\ 1 \end{bmatrix} \\
P &= \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} \\
\exp(i\theta\mathbf{v} \cdot \boldsymbol{\sigma}) &= P \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix} P^{-1} \\
&= \begin{bmatrix} \cos\theta + iv_z \sin\theta & i(v_x - iv_y)\sin\theta \\ i(v_x + iv_y)\sin\theta & \cos\theta - iv_z\sin\theta \end{bmatrix} \\
&= \cos\theta I + i\sin\theta \begin{bmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{bmatrix} \\
&= \cos\theta I + i\sin\theta\mathbf{v} \cdot \boldsymbol{\sigma}
\end{aligned}
$$

Exercise 2.36

$$
\begin{aligned}
\operatorname{tr} I &= 1 + 1 \\
&= 2 \\
\operatorname{tr} X &= 0 + 0 \\
&= 0 \\
\operatorname{tr} Y &= 0 + 0 \\
&= 0 \\
\operatorname{tr} Z &= 1 - 1 \\
&= 0
\end{aligned}
$$

Exercise 2.37

$$
\begin{aligned}
\operatorname{tr}(AB) &= \sum_i \langle i|AB|i\rangle \\
&= \sum_i \langle i|AIB|i\rangle \\
&= \sum_i \langle i|Ai\rangle \langle i|B|i\rangle \\
&= \sum_i \langle i|B|i\rangle \langle i|A|i\rangle \\
&= \sum_i \langle i|BIA|i\rangle \\
&= \sum_i \langle i|BA|i\rangle \\
&= \operatorname{tr}(BA)
\end{aligned}
$$

Exercise 2.38

$$
\begin{aligned}
\operatorname{tr}(A+B) &= \sum_i (A_{ii} + B_{ii}) \\
&= \sum_i A_{ii} + \sum_i B_{ii} \\
&= \operatorname{tr} A + \operatorname{tr} B \\
\operatorname{tr}(zA) &= \sum_i zA_{ii} \\
&= z\sum_i A_{ii} \\
&= z \operatorname{tr} A
\end{aligned}
$$

Exercise 2.42

$$
\begin{aligned}
\frac{[A,B] + \{A,B\}}{2} &= \frac{1}{2}(AB - BA + AB + BA) \\
&= AB
\end{aligned}
$$

Exercise 2.44

$$\begin{aligned}
[A, B] &= 0 \\
AB - BA &= 0 \\
\{A, B\} &= 0 \\
AB + BA &= 0 \\
AB - BA + AB + BA &= 0 \\
2AB &= 0 \\
AB &= 0 \\
A^{-1}AB &= 0 \\
B &= 0
\end{aligned}$$

Exercise 2.45

$$\begin{aligned}
[A, B]^\dagger &= (AB - BA)^\dagger \\
&= (AB)^\dagger - (BA)^\dagger \\
&= B^\dagger A^\dagger - A^\dagger B^\dagger \\
&= [B^\dagger, A^\dagger]
\end{aligned}$$

Exercise 2.46

$$\begin{aligned}
[A, B] &= AB - BA \\
&= -(BA - AB) \\
&= -[B, A]
\end{aligned}$$

Exercise 2.47

$$\begin{aligned}
(i[A, B])^\dagger &= -i[B^\dagger, A^\dagger] \\
&= -i[B, A] \\
&= i[A, B]
\end{aligned}$$

Exercise 2.48

$$
\begin{aligned}
J &= A \\
K &= A \\
U &= I \\
J &= I \\
K &= I \\
U &= A \\
J &= A \\
K &= A \\
U &= I
\end{aligned}
$$

Exercise 2.51

$$
\begin{aligned}
H &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
H^\dagger &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
H^\dagger H &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\
&= I
\end{aligned}
$$

Exercise 2.52

$$
\begin{aligned}
H^2 &= HH \\
&= H^\dagger H \\
I
\end{aligned}
$$

Exercise 2.53

$$
\begin{aligned}
H\mathbf{v} &= \lambda\mathbf{v} \\
\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 \\ 1 & -1\end{bmatrix}\mathbf{v} - \lambda\mathbf{v} &= 0 \\
\begin{bmatrix}\frac{1}{\sqrt{2}} - \lambda & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} - \lambda\end{bmatrix}\mathbf{v} &= 0 \\
\left(\frac{1}{\sqrt{2}} - \lambda\right)\left(-\frac{1}{\sqrt{2}} - \lambda\right) - \frac{1}{2} &= 0 \\
-\frac{1}{2} + \lambda^2 - \frac{1}{2} &= 0 \\
\lambda^2 - 1 &= 0 \\
(\lambda - 1)(\lambda + 1) &= 0
\end{aligned}
$$

The eigenvalues are ± 1. The eigenvectors are $\begin{bmatrix}1 \\ -1 \pm \sqrt{2}\end{bmatrix}$.

Exercise 2.54

$$
\begin{aligned}
\exp(A)\exp(B) &= \exp\left(\sum_i a_i \left|i\right\rangle\left\langle i\right|\right)\exp\left(\sum_i b_i \left|i\right\rangle\left\langle i\right|\right) \\
&= \exp\left(\sum_i a_i \left|i\right\rangle\left\langle i\right| + \sum_i b_i \left|i\right\rangle\left\langle i\right|\right) \\
&= \exp(A + B)
\end{aligned}
$$

Exercise 2.55

$$
\begin{aligned}
U(t_1, t_2) &= \exp\left[\frac{-iH(t_2 - t_1)}{\hbar}\right] \\
U(t_1, t_2)^\dagger &= \exp\left[\frac{iH(t_2 - t_1)}{\hbar}\right] \\
U(t_1, t_2)^\dagger U(t_1, t_2) &= 1
\end{aligned}
$$

Exercise 2.56

$$\begin{aligned}
K &= -i\ln(U) \\
\ln U &= \ln\left(\sum_u u\,|u\rangle\langle u|\right) \\
&= \sum_u \ln(u)\,|u\rangle\langle u| \\
&= \sum_u i\theta_u\,|u\rangle\langle u| \\
K &= -i\ln(U) \\
&= \sum_u \theta_u\,|u\rangle\langle u| \\
K^\dagger &= \left(\sum_u \theta_u\,|u\rangle\langle u|\right)^\dagger \\
&= \sum_u \theta_u\,|u\rangle\langle u| \\
&= K
\end{aligned}$$

Exercise 2.57

Assuming the state to be measured is $|\psi\rangle$, then the state of the system after the first measurement is

$$|\psi'\rangle = \frac{L_l\,|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}.$$

The state of the system after the second measurement is

$$\begin{aligned}
|\psi''\rangle &= \frac{M_m\,|\psi'\rangle}{\sqrt{\langle\psi'|M_m^\dagger M_m|\psi'\rangle}} \\
&= \frac{1}{\sqrt{\langle\frac{L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}|M_m^\dagger M_m|\frac{L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}\rangle}}\frac{M_m L_l\,|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}} \\
&= \frac{M_m L_l\,|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}} \\
&= \frac{N_{lm}\psi}{\sqrt{\langle\psi|N_{lm}^\dagger N_{lm}|\psi\rangle}}
\end{aligned}$$

Exercise 2.58

$$\begin{aligned}\langle M \rangle &= m \\ \Delta(M) &= 0\end{aligned}$$

Exercise 2.59

$$\begin{aligned}\langle X \rangle &= \langle 0|X|0 \rangle \\ &= \begin{bmatrix}1 & 0\end{bmatrix} \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix} \begin{bmatrix}1 \\ 0\end{bmatrix} \\ &= 0 \\ \langle X^2 \rangle &= \langle 0|X^2|0 \rangle \\ &= \begin{bmatrix}1 & 0\end{bmatrix} \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}^2 \begin{bmatrix}1 \\ 0\end{bmatrix} \\ &= \begin{bmatrix}1 & 0\end{bmatrix} \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} \begin{bmatrix}1 \\ 0\end{bmatrix} \\ &= 1 \\ \Delta(X) &= \sqrt{\langle X^2 \rangle - \langle X \rangle^2} \\ &= 1\end{aligned}$$

Exercise 2.60

$$\begin{aligned}\mathbf{v} \cdot \boldsymbol{\sigma} &= v_1\sigma_x + v_2\sigma_2 + v_3\sigma_3 \\ &= v_1 \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix} + v_2 \begin{bmatrix}0 & -i \\ i & 0\end{bmatrix} + v_3 \begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix} \\ &= \begin{bmatrix}v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3\end{bmatrix} \\ 0 &= (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) \\ &= \lambda^2 - v_1^2 - v_2^2 - v_3^2 \\ &= \lambda^2 - 1 \\ &= (\lambda - 1)(\lambda + 1) \\ \lambda &= \pm 1\end{aligned}$$

Exercise 2.62

If the measurement operators and the POVM elements coincide then

$$M_m = E_m = M_m^\dagger M_m$$

and

$$M_m^\dagger = (M_m^\dagger M_m)^\dagger = M_m^\dagger M_m = M_m,$$

i.e. M_m are Hermitian. Then

$$E_m = M_m^\dagger M_m = M_m^2 = M_m,$$

i.e. M_m are projectors.

Exercise 2.63

$$\begin{aligned} M_m^\dagger M_m &= \sqrt{E_m} U_m^\dagger U_m \sqrt{E_m} \\ &= \sqrt{E_m} I \sqrt{E_m} \\ &= E_m \end{aligned}$$

So U_m can be arbitrary unitary operators.

Exercise 2.65

$$\begin{aligned} \frac{|0\rangle + |1\rangle}{\sqrt{2}} &= |+\rangle \\ \frac{|0\rangle - |1\rangle}{\sqrt{2}} &= |-\rangle \end{aligned}$$

Exercise 2.66

$$\begin{aligned} X &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ Z &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ |\psi\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}} \\ Z_2 |\psi\rangle &= \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\ X_1 Z_2 |\psi\rangle &= \frac{|10\rangle - |01\rangle}{\sqrt{2}} \\ \langle\psi|X_1 Z_2|\psi\rangle &= \left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right)\left(\frac{|10\rangle - |01\rangle}{\sqrt{2}}\right) \\ &= \frac{1}{2}(\langle 00|10\rangle - \langle 00|01\rangle + \langle 11|10\rangle - \langle 11|01\rangle) \\ &= 0 \end{aligned}$$