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1 Flow Chart

- Ordinary
  - First order
    - Linear
      - Homogeneous
        - Separation of variables
      - Nonhomogeneous
        - Bernoulli

        - Exact
        - Exact with integration constant
        - Homogeneous substitution
        - Reduction to separation of variables
        - Riccati
        - Variation of parameters
    - Nonlinear
      - Separable
        - Separation of variables
  - Second order
    - Linear
      - Homogeneous
        - Auxiliary/characteristic equation
        - Cauchy/Euler
        - Reduction of order
      - Nonhomogeneous
        - Cauchy/Euler
        - Green's function
        - Undetermined coefficients
        - Variation of parameters
    - Nonlinear
      - Reduction of order
      - Taylor series
  - Higher order
    - Linear
      - Homogeneous
        - Auxiliary/characteristic equation
        - Cauchy/Euler
      - Nonhomogeneous
        - Cauchy/Euler
        - Undetermined coefficients
        - Variation of parameters
    - Nonlinear
      - Taylor series
- Partial

2 First-order ODEs

Form: IVP

$$\frac{dy}{dx} = f(x, y)$$
$$y(x_0) = y_0$$

Test: $f(x, y)$ and $\partial f/\partial y$ are continuous over I
Property: A unique solution is guaranteed over I

2.1 Separable Equations

Form:

$$\frac{dy}{dx} = g(x)h(y)$$

Solution: Divide by $h(y)$ then integrate with respect to x.

$$\begin{aligned}
\frac{dy}{dx} &= g(x)h(y) \\
\frac{1}{h(y)}\frac{dy}{dx} &= g(x) \\
\int \frac{1}{h(y)}\frac{dy}{dx}\,dx &= \int g(x)\,dx \\
\int \frac{1}{h(y)}\,dy &= \int g(x)\,dx \\
H(y) &= G(x) + c
\end{aligned}$$

2.2 Linear Equations

Form:

$$\frac{dy}{dx} + P(x)y = f(x)$$

Solution:

1. Determine the integrating factor $e^{\int P(x)\,dx}$
2. Multiply by the integrating factor
3. Recognise that the left hand side of the equation is the derivative of the product of the integrating factor and y
4. Integrate both sides of the equation
5. Solve for y

2.3 Exact Equations

Form:

$$z = f(x, y) = c$$

$$dz = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = M(x, y)\, dx + N(x, y)\, dy = 0$$

Test:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution:

1. Integrate $M(x, y)$ with respect to x to find an expression for $z = f(x, y)$

$$\frac{\partial f}{\partial x} = M(x, y)$$
$$f(x, y) = \int M(x, y)\, dx + g(y)$$

2. Differentiate $f(x, y)$ with respect to y and equate it to $N(x, y)$ to find $g'(y)$

$$\frac{\partial f}{\partial y} = N(x, y) = \frac{\partial}{\partial y} \int M(x, y)\, dx + g'(y)$$
$$g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, dx$$

3. Integrate $g'(y)$ with respect to y to find $g(y)$ and substitute it into $f(x, y)$
4. Equate $f(x, y)$ with an unknown constant c

Note: The steps can be performed with x and y reversed, i.e. start by integrating $N(x, y)$ with respect to y, etc.

2.4 Exact Equations with Integration Constant

Form:

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

Test: $(M_y - N_x)/N$ is a function of x alone or $(N_x - M_y)/M$ is a function of y alone
Solution:

1. Compute the integrating factor

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\,dx}$$

or

$$\mu(y) = e^{\int \frac{N_x - M_y}{M}\,dy}$$

as appropriate

2. Multiple the equation by this factor
3. The equation is now exact and can be solved as above

2.5 Homogeneous Equations

Form:

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

Test: M and N are homogeneous functions of the same degree
Solution:

1. Rewrite as

$$M(x, y) = x^{\alpha} M(1, u) \text{ and } N(x, y) = x^{\alpha} N(1, u) \text{ where } u = y/x$$

or

$$M(x, y) = y^{\alpha} M(v, 1) \text{ and } N(x, y) = y^{\alpha} N(v, 1) \text{ where } v = x/y$$

2. Substitute $y = ux$ and $dy = u\,dx + x\,du$ or $x = vy$ and $dx = v\,dy + y\,dv$ as appropriate
3. Solve the resulting first-order separable DE
4. Substitude $u = y/x$ or $v = x/y$ as appropriate

2.6 Bernoulli's Equation

Form:

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

Test: $n \neq 0$ and $n \neq 1$
Solution:

1. Substitude $y = u^{1/(1-n)}$ and $\frac{dy}{dx} = \frac{d}{dx}(u^{1/(1-n)})$
2. Solve the resulting linear equation
3. Substitude $u = y^{1-n}$

2.7 Reduction to Separation of Variables

Form:

$$\frac{dy}{dx} = f(Ax + By + C), B \neq 0$$

Solution:

1. Substitute
$$Ax + By + C = u$$
2. Solve the resulting separable equation
3. Substitute
$$u = Ax + By + C$$

2.8 Riccati's Equation

Form:

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$$

Test: You know a particular solution y_1 of the equation
Solution:

1. Substitute $y = y_1 + u$ and $y' = y_1' + u'$
2. Solve the resulting Bernoulli equation
3. Substitude $u = y - y_1$

3 Higher-order ODEs

3.1 Initial Value Problems

Form: n-th order IVP

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx_{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

subject to

$$y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1}$$

Test: $a_n(x)$, $a_{n-1}(x)$, …, $a_0(x)$, and $g(x)$ are continuous on an interval I and $a_n(x) \neq 0$ for every x in I
Property: A unique solution exists for every $x = x_0$ in I

3.2 Linear Independence

Form: A set of functions f_1, f_2, ..., f_n
Test: The Wronskian $W(f_1, f_2, \ldots, f_n) \neq 0$ for every x in an interval I where

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

Property: The functions are linearly independent in I

3.3 Linear Equations

3.3.1 Homogeneous Linear nth-Order Equations

The general solution is of the form

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

where c_i are arbitrary constants and y_i are a fundamental set of solutions (i.e. a set of n linearly independent solutions).

3.3.2 Nonhomogeneous Linear nth-Order Equations

The general solution is of the form

$$y = y_c + y_p = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

where y_c is the complementary function (i.e. the general solution of the associated homogeneous equation) and y_p is a particular solution.

3.3.3 Reduction of Order

Form:

$$y'' + P(x)y' + Q(x)y = 0$$

Test: A non-trivial solution $y_1(x)$ is known
Solution:

1. Recognise that the ratio of two linearly independent functions isn't constant, i.e.
$$u(x) = \frac{y_1(x)}{y_2(x)} \text{ or } y_2(x) = u(x)y_1(x)$$
2. Substitute $y_2(x) = u(x)y_1(x)$ into the DE — this will result in a DE involving only u'' and u' which can be treated as a linear first-order DE in $u' = w$

3. Solve for w
4. Substitute $w = u'$
5. Integrate to find u
6. Multiply by y_1 to find y_2

or equivalently

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx$$

3.3.4 Homogeneous Linear Equations with Constant Coefficients

Form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Solution:

1. Assume the equation has a solution of the form $y = e^{mx}$, giving

$$a_n m^n e^{mx} + a_{n-1} m^{n-1} e^{mx} + \cdots + a_1 m e^{mx} + a_0 e^{mx} = 0$$

2. Divide by e^{mx}, giving the auxiliary/characteristic equation

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

3. Solve for m, where
   - A real root m corresponds to a solution

$$y = ce^{mx}$$

   - Complex roots $\alpha \pm i\beta$ correspond to solutions

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

   - A root m of multiplicity k corresponds to the solutions

$$e^{mx},\ xe^{mx},\ x^2 e^{mx},\ \ldots,\ x^{k-1} x^{mx}$$

3.3.5 Method of Undetermined Coefficients

Form: A nonhomogeneous linear DE where the input function ($g(x)$) is comprised of constants, polynomials, exponentials $e^{\alpha x}$, sines, and cosines
Solution:

1. Solve the associated homogeneous equation
2. Assume the particular solution has the same form as the input function

3. If a term in the proposed solution is present in the complementary function, multiply it by x^n where n is the smallest positive integer that removes the duplication

4. Substitute the proposed solution into the DE

5. Solve for the unknown constants

3.3.6 Variation of Parameters

Form: A nonhomogeneous linear DE
Solution:

1. Solve the homogeneous equation to find the complementary function

2. Assume the solution has the form

$$y_p = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x)$$

   where n is the order of the equation and y_i are the fundamental set of solutions from the complementary equation

3. Convert to standard form by dividing by the leading coefficient

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

4. Solve the system of linear equations

$$\begin{aligned} y_1u_1' + \cdots + y_nu_n' &= 0 \\ y_1'u_1' + \cdots + y_n'u_n' &= 0 \\ &\vdots \\ y_1^{(n-1)}u_1' + \cdots + y_n^{(n-1)}y_n' &= 0 \\ y_1^{(n)}u_1' + \cdots + y_n^{(n)}u_n' &= f(x) \end{aligned}$$

   via Cramer's method:

   (a) Compute the Wronskian of y_i

$$W = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \ddots & \vdots \\ y_1^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

(b) Compute u_i' for $i = 1, \ldots, n$ where

$$u_i' = \frac{W_i}{W}$$

and W_i is the determinant of the matrix formed by replacing the ith column of the Wronskian matrix with the column vector

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(x) \end{bmatrix}$$

5. Integrate each u_i' to find u_i

3.3.7 Cauchy-Euler Equations

Form:

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

Solution:

- If the equation is homogeneous:

  1. Assume the equation has a solution of the form $y = x^m$, giving

  $$\begin{aligned} a_n x^n \frac{d^n y}{dx^n} &= a_n x^n m(m-1)(m-2)\cdots(m-n+1)x^{m-n} \\ &= a_n m(m-1)(m-2)\cdots(m-n+1)x^m \end{aligned}$$

     and the equation then becomes

  $$f(m)x^m = 0$$

     where $f(m)$ is a polynomial in m known as the auxiliary or characteristic equation, the roots of which form the general solution
  2. Solve the auxiliary equation where
     - A real root m corresponds to a solution

     $$y = cx^m$$

     - Complex roots $\alpha \pm i\beta$ correspond to solutions

     $$x^\alpha(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x))$$

     - A root m of multiplicity k corresponds to solutions

     $$x^m,\ x^m \ln x,\ x^m (\ln x)^2\ \ \ldots,\ x^m (\ln x)^{k-1}$$

- If the equation is nonhomogeneous:

  1. Solve the associated homogeneous equation
  2. Find a particular solution via variation of parameters

3.3.8 Green's Functions for IVPs

Form: The IVP

$$y'' + P(x)y' + Q(x) = f(x)$$

subject to $y(x_0) = y_0$ and $y'(x_0) = y_1$
Solution:

1. Solve the homogeneous equation with nonhomogeneous conditions

$$y'' + P(x)y' + Q(x)y = 0,\ y(x_0) = y_0,\ y'(x_0) = y_1$$

giving the solution y_h and the fundamental set of solutions y_1 and y_2

2. Solve the nonhomogeneous equation with homogeneous conditions

$$y'' + P(x)y' + Q(x)y = f(x),\ y(x_0) = 0,\ y'(x_0) = 0$$

using the formula

$$y_p(x) = \int_{x_0}^{x} G(x,t)f(t)\,dt$$

where $G(x,t)$ is the Green's function for the differential equation

$$G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}$$

and $W(t)$ is the Wronskian

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

3. The solution is $y = y_h + y_p$

3.3.9 Green's Functions for BVPs

Form: The BVP

$$y'' + P(x)y' + Q(x)y = f(x)$$

subject to

$$A_1 y(a) + B_1 y(a) = 0$$

and

$$A_2 y(b) + B_2 y(b) = 0$$

Solution:

1. Solve the associated homogeneous equation to find the fundamental set of solution y_1 and y_2 valid on $[a, b]$

2. Ensure y_1 and y_2 satisfy the boundary conditions

$$A_1 y_1(a) + B_1 y_1(a) = 0$$

and

$$A_2 y_2(b) + B_2 y_2(b) = 0$$

   - It's important that y_1 satisfies the starting boundary condition and y_2 satisfies the ending!

3. Then a particular solution is

$$y_p(x) = \int_a^b G(x,t) f(t)\, dt$$

where $G(x,t)$ is the Green's function for the differential equation

$$G(x,t) = \begin{cases} \frac{y_1(t)y_2(x)}{W(t)} & a \leq t \leq x \\ \frac{y_1(x)y_2(t)}{W(t)} & x \leq t \leq b \end{cases}$$

and $W(t)$ is the Wronskian

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

3.4 Nonlinear Equations

3.4.1 Reducation of Order

Form: Nonlinear second-order DE

$$F(x, y', y'') = 0$$

i.e. y is missing
Solution:

1. Substitute $u = y'$ (and thus $u' = y''$)
2. Solve the resulting DE for u
3. Integrate to find y

Form: Nonlinear second-order DE

$$F(y, y', y'') = 0$$

i.e. x is missing
Solution:

1. Substitute $u = y'$ and
$$y'' = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy}$$
2. Solve the resulting DE for u
3. Integrate to find y

3.4.2 Taylor Series

Form: Nonlinear initial value problem
Solution:

1. Substitute the initial conditions into a Taylor series centred at x_0
2. Take further derivatives of the equation and substitute the initial conditions in to find additional terms for the Taylor series

4 Systems of ODEs

4.1 Linear Equations with Constant Coefficients

Form: n linear equations with constant coefficients
Solution:

1. Apply the differential operator D and add/subtract multiples of the equations to each other to eliminate variables until you're left with a single dependent variable
2. Repeat the process for each dependent variable
3. Substitute the resulting equations into the original DE to determine the constraints on the parameters as not all of them can be chosen arbitrarily