Introduction to Quantum Mechanics by David J. Griffiths Problems

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Contents

Part I
Theory

1 The Wave Function

1.1

(a)

$$\begin{aligned}
\langle j^2 \rangle &= \sum j^2 P(j) \\
&= 14^2 \frac{1}{14} + 15^2 \frac{1}{14} + 16^2 \frac{3}{14} + 22^2 \frac{2}{14} + 24^2 \frac{2}{14} + 25^2 \frac{5}{14} \\
&= \frac{3217}{7} \\
&\approx 459.571 \\
\langle j \rangle^2 &= \left(\sum j P(j) \right)^2 \\
&= 441
\end{aligned}$$

(b)

$$\begin{aligned}
\Delta j_{14} &= -7 \\
\Delta j_{15} &= -6 \\
\Delta j_{16} &= -5 \\
\Delta j_{22} &= 1 \\
\Delta j_{24} &= 3 \\
\Delta j_{25} &= 4 \\
\sigma^2 &= \sum (\Delta j)^2 P(j) \\
&= \frac{130}{7} \\
&\approx 18.571
\end{aligned}$$

(c)

$$\sigma^2 = \sqrt{\langle j^2 \rangle - \langle j \rangle^2} = 18.571$$

1.2

(a)

$$\begin{aligned}
\langle x^2 \rangle &= \int_0^h x^2 \rho(x)\, dx \\
&= \int_0^h \frac{x^{3/2}}{2\sqrt{h}}\, dx \\
&= \frac{1}{2\sqrt{h}} \left[\frac{2}{5} x^{5/2} \right]_0^h \\
&= \frac{h^2}{5} \\
\langle x \rangle^2 &= \frac{h^2}{9} \\
\sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{h^2}{5} - \frac{h^2}{9}} \\
&= h\sqrt{\frac{4}{45}} \\
&= \frac{2}{3\sqrt{5}} h
\end{aligned}$$

(b)

$$\begin{aligned}
1 - \int_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} \rho(x)\, dx &= 1 - \frac{1}{2\sqrt{h}} [2\sqrt{x}]_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} \\
&= 1 - \frac{1}{\sqrt{h}} \left(\sqrt{\frac{1}{3}h + \frac{2}{3\sqrt{5}}h} - \sqrt{\frac{1}{3}h - \frac{2}{3\sqrt{5}}h} \right) \\
&= 1 - \left(\sqrt{\frac{1}{3} + \frac{2}{3\sqrt{5}}} - \sqrt{\frac{1}{3} - \frac{2}{3\sqrt{5}}} \right) \\
&\approx 0.393
\end{aligned}$$

1.3

(a)

$$
\begin{aligned}
\rho(x) &= Ae^{-\lambda(x-a)^2} \\
1 &= \int_{-\infty}^{\infty} \rho(x)\,dx \\
&= A\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2}\,dx \\
&= A\sqrt{\frac{\pi}{\lambda}} \\
A &= \sqrt{\frac{\lambda}{\pi}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\langle x\rangle &= \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} xe^{-\lambda(x-a)^2}\,dx \\
&= a \\
\langle x^2\rangle &= \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} x^2e^{-\lambda(x-a)^2}\,dx \\
&= a^2 + \frac{1}{2\lambda} \\
\sigma &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\
&= \sqrt{a^2 + \frac{1}{2\lambda} - a^2} \\
&= \frac{1}{\sqrt{2\lambda}}
\end{aligned}
$$

1.4

(a)

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi(x,0)|^2 \, dx \\
&= \left(\frac{A}{a}\right)^2 \int_0^a x^2 \, dx + \left(\frac{A}{b-a}\right)^2 \int_a^b (b-x)^2 \, dx \\
&= \frac{1}{3}A^2 a + \left(\frac{A}{b-a}\right)^2 \left[-\frac{1}{3}(b-x)^3\right]_a^b \\
&= \frac{1}{3}A^2 a + \frac{1}{3}A^2(b-a) \\
&= \frac{1}{3}A^2 b \\
A &= \sqrt{\frac{3}{b}}
\end{aligned}
$$

(c) $x = a$

(d)

$$
\begin{aligned}
\int_0^a |\Psi(x,0)|^2 \, dx &= \frac{3}{a^2 b} \left[\frac{1}{3}x^3\right]_0^a \\
&= \frac{a}{b}
\end{aligned}
$$

(e)

$$
\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x|\Psi(x,0)|^2 \, dx \\
&= \frac{3}{a^2 b}\left[\frac{1}{4}x^4\right]_0^a + \frac{3}{b(b-a)^2}\int_a^b x(b-x)^2 \, dx \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2}\int_a^b (b^2 x - 2bx^2 + x^3) \, dx \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2}\left[\frac{1}{2}b^2x^2 - \frac{2}{3}bx^3 + \frac{1}{4}x^4\right]_a^b \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2}\left(\frac{1}{2}b^4 - \frac{2}{3}b^4 + \frac{1}{4}b^4 - \frac{1}{2}a^2b^2 + \frac{2}{3}a^3b - \frac{1}{4}a^4\right) \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2}\frac{1}{12}(b-a)^3(3a+b) \\
&= \frac{3a^2}{4b} + \frac{1}{4b}(3ab + b^2 - 3a^2 - ab) \\
&= \frac{1}{2}a + \frac{1}{4}b
\end{aligned}
$$

1.5

(a)

$$\begin{aligned}
\Psi(x,t) &= Ae^{-\lambda|x|}e^{-i\omega t} \\
\Psi(x,0) &= Ae^{-\lambda|x|} \\
1 &= A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|}\,dx \\
&= 2A^2 \int_0^{\infty} e^{-2\lambda x}\,dx \\
&= 2A^2 \left[-\frac{1}{2\lambda}e^{-2\lambda x}\right]_0^{\infty} \\
&= \frac{A^2}{\lambda} \\
A &= \sqrt{\lambda}
\end{aligned}$$

(b)

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} x\lambda e^{-2\lambda|x|}\,dx \\
&= \lambda \int_{-\infty}^{\infty} xe^{-2\lambda|x|}\,dx \\
&= 0 \\
\langle x^2\rangle &= \int_{-\infty}^{\infty} x^2\lambda e^{-2\lambda|x|}\,dx \\
&= 2\lambda \int_0^{\infty} x^2 e^{-2\lambda x}\,dx \\
&= \frac{1}{2\lambda^2}
\end{aligned}$$

(c)

$$\begin{aligned}
\sigma &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\
&= \frac{1}{\sqrt{2}\lambda} \\
1 - \int_{-\sigma}^{\sigma} \lambda e^{-2\lambda|x|}\,dx &= 1 - 2\lambda \int_0^{\sigma} e^{-2\lambda x}\,dx \\
&= 1 - 2\lambda \left[-\frac{1}{2\lambda}e^{-2\lambda x}\right]_0^{\sigma} \\
&= e^{-2\lambda\sigma} \\
&= e^{-\sqrt{2}} \\
&\approx 0.243
\end{aligned}$$

1.6

The chain rule requires that you apply it to both x and $|\Psi|^2$ which gives the same result

$$
\begin{aligned}
\frac{d\langle x\rangle}{dt} &= \frac{d}{dt}\int x|\Psi|^2\,dx \\
&= \int \frac{d}{dt}(x|\Psi|^2)\,dx \\
&= \int \left(0\cdot|\Psi|^2 + x\frac{\partial|\Psi|^2}{\partial t}\right)dx \\
&= \int x\frac{\partial|\Psi|^2}{\partial t}\,dx
\end{aligned}
$$

1.8

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\left(e^{-iV_0t/\hbar}\Psi\right) &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(e^{-iV_0t/\hbar}\Psi\right) + (V+V_0)\left(e^{-iV_0t/\hbar}\Psi\right) \\
i\hbar\left(-\frac{iV_0}{\hbar}e^{-iV_0t/\hbar}\Psi + e^{-iV_0t/\hbar}\frac{\partial\Psi}{\partial t}\right) &= -\frac{\hbar^2}{2m}e^{-iV_0t/\hbar}\frac{\partial^2\Psi}{\partial x^2} + Ve^{-iV_0t/\hbar}\Psi + V_0e^{-iV_0t/\hbar}\Psi \\
V_0\Psi + i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi + V_0\Psi \\
i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi
\end{aligned}
$$

$$
\begin{aligned}
\langle Q(x,p)\rangle &= \int \left(e^{-iV_0t/\hbar}\Psi\right)^*[Q(x,-i\hbar\partial/\partial x)]\,e^{-iV_0t/\hbar}\Psi\,dx \\
&= \int e^{iV_0t/\hbar}\Psi^*[Q(x,-i\hbar\partial/\partial x)]\,e^{-iV_0t/\hbar}\Psi\,dx \\
&= \int \Psi^*[Q(x,-i\hbar\partial/\partial x)]\Psi\,dx
\end{aligned}
$$

No effect on the expectation value.

1.9

(a)

$$
\begin{aligned}
\Psi(x,t) &= Ae^{-a[(mx^2/\hbar)+it]} \\
1 &= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)}\,dx \\
&= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)}\,dx \\
&= A^2 \sqrt{\frac{\pi\hbar}{2am}} \\
A^2 &= \sqrt{\frac{2am}{\pi\hbar}} \\
A &= \left(\frac{2am}{\pi\hbar}\right)^{1/4}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\Psi &= Ae^{-a[(mx^2/\hbar)+it]} \\
\frac{\partial\Psi}{\partial t} &= -ia\Psi \\
\frac{\partial\Psi}{\partial x} &= -\frac{2amx}{\hbar}\Psi \\
\frac{\partial^2\Psi}{\partial x^2} &= -\frac{2am}{\hbar}\left(\Psi + x\frac{\partial\Psi}{\partial x}\right) \\
&= -\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\Psi \\
V\Psi &= i\hbar\frac{\partial\Psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} \\
&= a\hbar\Psi - a\hbar\left(1 - \frac{2amx^2}{\hbar}\right)\Psi \\
V &= a\hbar - a\hbar + 2a^2mx^2 \\
&= 2a^2mx^2
\end{aligned}
$$

(c)

$$
\begin{aligned}
\langle x\rangle &= A^2\int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)}x\,dx \\
&= 0 \\
\left\langle x^2\right\rangle &= A^2\int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)}x^2\,dx \\
&= 2A^2\int_{0}^{\infty} e^{-2a(mx^2/\hbar)}x^2\,dx \\
&= \frac{\hbar}{4am} \\
\langle p\rangle &= \int_{-\infty}^{\infty} \Psi^*\left[-i\hbar\frac{\partial}{\partial x}\right]\Psi\,dx \\
&= -i\hbar\int_{-\infty}^{\infty} Ae^{-a[(mx^2/\hbar)-it]}\left(-\frac{2amx}{\hbar}Ae^{-a[(mx^2/\hbar)+it]}\right)\,dx \\
&= 2iA^2am\int_{-\infty}^{\infty} xe^{-2amx^2/\hbar}\,dx \\
&= 0 \\
\left\langle p^2\right\rangle &= \int_{-\infty}^{\infty} \Psi^*\left[-\hbar^2\frac{\partial^2}{\partial x^2}\right]\Psi\,dx \\
&= -\hbar^2\int_{-\infty}^{\infty} Ae^{-a[(mx^2/\hbar)-it]}\left[-\frac{2am}{\hbar}\left(1-\frac{2amx^2}{\hbar}\right)Ae^{-a[(mx^2/\hbar)+it]}\right]\,dx \\
&= 2A^2am\hbar\int_{-\infty}^{\infty} e^{-2amx^2/\hbar}\left(1-\frac{2amx^2}{\hbar}\right)\,dx \\
&= am\hbar
\end{aligned}
$$

(d)

$$
\begin{aligned}
\sigma_x &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\
&= \sqrt{\frac{\hbar}{4am}} \\
\sigma_p &= \sqrt{am\hbar} \\
\sigma_x\sigma_p &= \sqrt{\frac{1}{4}\hbar^2} \\
&= \frac{1}{2}\hbar \\
&\geq \frac{1}{2}\hbar
\end{aligned}
$$

Yes, this is consistent with Heisenberg's uncertainty principle.

1.10

(a)

$$
\begin{aligned}
P(0) &= 0 \\
P(1) &= \frac{2}{25} \\
&= 0.08 \\
P(2) &= \frac{3}{25} \\
&= 0.12 \\
P(3) &= \frac{1}{5} \\
&= 0.2 \\
P(4) &= \frac{3}{25} \\
&= 0.12 \\
P(5) &= \frac{3}{25} \\
&= 0.2 \\
P(6) &= \frac{3}{25} \\
&= 0.2 \\
P(7) &= \frac{1}{25} \\
&= 0.04 \\
P(8) &= \frac{2}{25} \\
&= 0.08 \\
P(9) &= \frac{3}{25} \\
&= 0.12
\end{aligned}
$$

(b) The most probable digit is 3, the median digit is 4, and the average value is $\frac{118}{25} = 4.72$.

(c) $\sigma = 2.474$

1.14

(a)

$$
\begin{aligned}
P_{ab}(t) &= \int_a^b |\Psi(x,t)|^2\,dx \\
\frac{dP_{ab}}{dt} &= \frac{d}{dt}\int_a^b |\Psi(x,t)|^2\,dx \\
&= \int_a^b \frac{d}{dt}\left(|\Psi(x,t)|^2\right)\,dx \\
&= \int_a^b \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\right]\,dx \\
&= J(a,t) - J(b,t)
\end{aligned}
$$

The units are s^{-1}.

(b)

$$
\begin{aligned}
\Psi(x,t) &= Ae^{-a[(mx^2/\hbar)+it]} \\
\frac{\partial\Psi}{\partial x} &= -\frac{2amx}{\hbar}\Psi \\
\Psi^*(x,t) &= Ae^{-a[(mx^2/\hbar)-it]} \\
\frac{\partial\Psi^*}{\partial x} &= -\frac{2amx}{\hbar}\Psi^* \\
J(x,t) &= \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right) \\
&= \frac{i\hbar}{2m}\left[\Psi\left(-\frac{2amx}{\hbar}\Psi^*\right) - \Psi^*\left(-\frac{2amx}{\hbar}\Psi\right)\right] \\
&= 0
\end{aligned}
$$

1.15

$$
\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2\,dx &= \int_{-\infty}^{\infty}\left(\frac{\partial\Psi_1^*}{\partial t}\Psi_2 + \Psi_1^*\frac{\partial\Psi_2}{\partial t}\right)dx \\
&= \int_{-\infty}^{\infty}\left[\left(-i\frac{\hbar}{2m}\frac{\partial^2\Psi_1^*}{\partial x^2} + i\frac{V}{\hbar}\Psi_1^*\right)\Psi_2\right. \\
&\qquad \left. + \Psi_1^*\left(i\frac{\hbar}{2m}\frac{\partial^2\Psi_2}{\partial x^2} - i\frac{V}{\hbar}\Psi_2\right)\right]dx \\
&= i\frac{\hbar}{2m}\int_{-\infty}^{\infty}\left(\Psi_1^*\frac{\partial^2\Psi_2}{\partial x^2} - \frac{\partial^2\Psi_1^*}{\partial x^2}\Psi_2\right)dx \\
&= i\frac{\hbar}{2m}\left[\Psi_1^*\frac{\partial\Psi_2}{\partial x}\bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial}{\partial x}(\Psi_1^*\Psi_2)\,dx\right. \\
&\qquad \left. \frac{\partial\Psi_1^*}{\partial x}\Psi_2\bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial}{\partial x}(\Psi_1^*\Psi_2)\,dx\right] \\
&= 0
\end{aligned}
$$

1.16

(a)

$$
\begin{aligned}
1 &= \int_{-a}^{a} A^2(a^2-x^2)^2\,dx \\
&= A^2\int_0^a (a^2-x^2)^2\,dx \\
&= \frac{16}{15}A^2a^5 \\
A &= \sqrt{\frac{15}{16a^5}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\langle x\rangle &= \int_{-a}^{a} xA(a^2-x^2)\,dx \\
&= 0
\end{aligned}
$$

(c)

$$
\begin{aligned}
\langle p\rangle &= \int_{-a}^{a}\Psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi\,dx \\
&= 2iA^2\hbar\int_{-a}^{a} x(a^2-x^2)\,dx \\
&= 0
\end{aligned}
$$

(d)

$$\begin{aligned}\left\langle x^2\right\rangle &= \int_{-a}^{a} \Psi^* x^2 \Psi \, dx \\ &= A^2 \int_{-a}^{a} x^2 (a^2 - x^2)^2 \, dx \\ &= A^2 \frac{16}{105} a^7 \\ &= \frac{a^2}{7}\end{aligned}$$

(e)

$$\begin{aligned}\left\langle p^2\right\rangle &= \int_{-a}^{a} \Psi^* \left(-\hbar^2 \frac{\partial^2}{\partial x^2}\right) \Psi \, dx \\ &= -\hbar^2 \int_{-a}^{a} A(a^2 - x^2)(-2A) \, dx \\ &= 4A^2\hbar^2 \int_{0}^{a} (a^2 - x^2) \, dx \\ &= 4A^2\hbar^2 \left[a^2 x - \frac{1}{3} x^3\right]_0^a \\ &= 4A^2\hbar^2 \left(a^3 - \frac{1}{3} a^3\right) \\ &= \frac{8}{3} A^2 a^3 \hbar^2 \\ &= \frac{8}{3} \frac{15}{16 a^5} a^3 \hbar^2 \\ &= \frac{5}{2} \frac{\hbar^2}{a^2}\end{aligned}$$

(f)

$$\begin{aligned}\sigma_x &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\ &= \sqrt{\frac{a^2}{7}} \\ &= \frac{a}{\sqrt{7}}\end{aligned}$$

(g)

$$\begin{aligned}\sigma_p &= \sqrt{\langle p^2\rangle - \langle p\rangle^2} \\ &= \sqrt{\frac{5}{2}} \frac{\hbar}{a}\end{aligned}$$

(h)

$$\begin{aligned}
\sigma_x \sigma_p &= \sqrt{\frac{5}{14}}\hbar \\
&\geq \frac{1}{2}\hbar
\end{aligned}$$

1.18

(a)

$$\begin{aligned}
\frac{h}{\sqrt{3mk_B T}} &> d \\
\frac{\sqrt{3mk_B T}}{h} &< \frac{1}{d} \\
T_{\text{electron}} &< \frac{h^2}{3d^2 m k_B} \\
&< 1.3 \times 10^5\,\mathrm{K} \\
T_{\text{nuclei}} &< 2.5\,\mathrm{K}
\end{aligned}$$

(b)

$$\begin{aligned}
PV &= Nk_B T \\
\frac{V}{N} &= \frac{k_B T}{P} \\
d &= \left(\frac{k_B T}{P}\right)^{1/3} \\
\frac{h}{\sqrt{3mk_B t}} &> \left(\frac{k_B T}{P}\right)^{1/3} \\
T &< \frac{1}{k_B}\left(\frac{h^2}{3m}\right)^{3/5} P^{2/5}
\end{aligned}$$

2 Time-Independent Schrödinger Equation

2.1

(a)

$$\begin{aligned}
\int_{-\infty}^{\infty} |\Psi|^2\,dx &= \int_{-\infty}^{\infty} \Psi^* \Psi\,dx \\
&= \int_{-\infty}^{\infty} \psi^* e^{i(E_0 - i\Gamma)t/\hbar} \psi e^{-i(E_0 + i\Gamma)t/\hbar}\,dx \\
&= e^{2\Gamma t/\hbar} \int_{-\infty}^{\infty} |\psi|^2\,dx
\end{aligned}$$

In order for this to equal 1 for all t, Γ must be 0.

(b) If $\psi(x)$ is a complex solution to the time-independent Schrödinger equation then so is $\psi^*(x)$ and $\psi(x) + \psi^*(x)$ which is real.

2.2

If ψ and its second derivative always have the same sign, ψ will increase or decrease without bound forever. This means there is no non-zero choice of constant A such that

$$\int_{-\infty}^{\infty} |A\Psi|^2 \, dx = 1$$

and thus the equation can't be normalised.

The classical analog of this is statements is that the potential energy of a system can't exceed its total energy.

2.3

The time-independent Schrödinger equation in an infinite square well is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi.$$

If $E = 0$ then $\psi = Ax + B$ which isn't normalisable.

If $E < 0$ then $\psi = Ae^{kt} + Be^{-kt}$ where $k \in \mathbb{R}$ which also isn't normalisable.

2.4

$$
\begin{aligned}
\Psi_n(x,t) &= \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)e^{-i(n^2\pi^2\hbar/2ma^2)t} \\
\langle x\rangle &= \int_0^a \Psi_n^* x \Psi_n \, dx \\
&= \frac{2}{a}\int_0^a x\sin^2\left(\frac{n\pi}{a}x\right)\, dx \\
&= \frac{a}{2} \\
\left\langle x^2\right\rangle &= \int_0^a \Psi_n^* x^2 \Psi_n \, dx \\
&= \frac{2}{a}\int_0^a x^2\sin^2\left(\frac{n\pi}{a}x\right)\, dx \\
&= a^2\left(\frac{1}{3}-\frac{1}{2n^2\pi^2}\right) \\
\langle p\rangle &= \int_0^a \Psi_n^*\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi_n \, dx \\
&= -i\frac{2\hbar n\pi}{a^2}\int_0^a \sin\left(\frac{n\pi}{a}x\right)\cos\left(\frac{n\pi}{a}x\right)\, dx \\
&= 0 \\
\left\langle p^2\right\rangle &= \int_0^a \Psi_n^*\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)\Psi_n \, dx \\
&= \frac{2\hbar^2 n^2\pi^2}{a^3}\int_0^a \sin^2\left(\frac{n\pi}{a}x\right)\, dx \\
&= \left(\frac{n\pi\hbar}{a}\right)^2 \\
\sigma_x &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\
&= \frac{a}{2}\sqrt{\frac{1}{3}-\frac{2}{n^2\pi^2}} \\
\sigma_p &= \sqrt{\langle p^2\rangle - \langle p\rangle^2} \\
&= \frac{n\pi\hbar}{a}
\end{aligned}
$$

2.5

(a)

$$
\begin{aligned}
1 &= \int_0^a A^2(\psi_1 + \psi_2)^2\,dx \\
&= A^2 \int_0^a (\psi_1^2 + 2\psi_1\psi_2 + \psi_2^2)\,dx \\
&= \frac{2A^2}{a}\left[\int_0^a \sin^2\left(\frac{\pi}{a}x\right)\,dx + \int_0^a \sin^2\left(\frac{2\pi}{a}x\right)\,dx\right] \\
&= 2A^2 \\
A &= \frac{1}{\sqrt{2}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi}{a}x\right)e^{-4i\omega t}\right] \\
|\Psi(x,t)|^2 &= \Psi^*\Psi \\
&= \frac{1}{a}\left[\sin\left(\frac{\pi}{a}x\right)e^{i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{4i\omega t}\right] \\
&\quad \left[\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{-4i\omega t}\right] \\
&= \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)e^{-3i\omega t}\right. \\
&\quad \left. + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)e^{3i\omega t} + \sin^2\left(\frac{2\pi}{a}x\right)\right] \\
&= \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right)\right. \\
&\quad \left. +2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right]
\end{aligned}
$$

(c)

$$
\begin{aligned}
\langle x \rangle &= \int_0^a \Psi^* x \Psi\,dx \\
&= \int_0^a x|\Psi|^2\,dx \\
&= \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t)\right]
\end{aligned}
$$

(d)

$$
\begin{aligned}
\langle p \rangle &= m\frac{d\langle x \rangle}{dt} \\
&= \frac{16am\omega}{3\pi^2}\sin(3\omega t) \\
&= \frac{8\hbar}{3a}\sin(3\omega t)
\end{aligned}
$$

(e) You can get E_1 or E_2 and the probability of getting each is 1/2. $H = \frac{1}{2}(E_1 + E_2)$ is the mean of the two possible energy values.

2.6

$$
\begin{aligned}
\Psi(x,0) &= A[\psi_1 + e^{i\phi}\psi_2] \\
1 &= \int_0^a |\Psi|^2\,dx \\
&= \int_0^a \Psi^*\Psi\,dx \\
&= A^2\int_0^a (\psi_1 + e^{-i\phi}\psi_2)(\psi_1 + e^{i\phi}\psi_2)\,dx \\
&= A^2\int_0^a (\psi_1^2 + e^{i\phi}\psi_1\psi_2 + e^{-i\phi}\psi_1\psi_2 + \psi_2^2)\,dx \\
&= \frac{2A^2}{a}\int_0^a \left[\sin^2\left(\frac{\pi}{a}x\right) + e^{i\phi}\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\right. \\
&\qquad \left. e^{-i\phi}\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right)\right]\,dx \\
&= \frac{2A^2}{a}\int_0^a \left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos\phi\right. \\
&\qquad \left. +\sin^2\left(\frac{2\pi}{a}x\right)\right]\,dx \\
&= 2A^2 \\
A &= \frac{1}{\sqrt{2}} \\
\Psi(x,t) &= \frac{1}{\sqrt{a}}\left[\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{i(\phi-4\omega t)}\right]
\end{aligned}
$$

$$
\begin{aligned}
|\Psi|^2 &= \Psi^*\Psi \\
&= \frac{1}{a}\left[\sin\left(\frac{\pi}{a}x\right)e^{i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{-i(\phi-4\omega t)}\right] \\
&\quad \left[\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{i(\phi-4\omega t)}\right] \\
&= \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)e^{i(\phi-3\omega t)}\right. \\
&\quad \left.\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)e^{-i(\phi-3\omega t)} + \sin^2\left(\frac{2\pi}{a}x\right)\right] \\
&= \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right)\right. \\
&\quad \left.+2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(\phi-3\omega t)\right] \\
\langle x\rangle &= \int_0^a \Psi^* x\Psi\,dx \\
&= \int_0^a x|\Psi|^2\,dx \\
&= \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t-\phi)\right]
\end{aligned}
$$

2.7

(a)

$$
\begin{aligned}
1 &= \int_0^a |\Psi|^2\,dx \\
&= A^2\left[\int_0^{a/2} x^2\,dx + \int_{a/2}^a (a-x)^2\,dx\right] \\
&= A^2\left\{\frac{1}{3}\left[\frac{a}{2}\right]^3 + \left[-\frac{1}{3}(a-x)^3\right]_{a/2}^a\right\} \\
&= A^2\left(\frac{a^3}{24} + \frac{a^3}{24}\right) \\
&= \frac{A^2a^3}{12} \\
A &= \frac{2\sqrt{3}}{\sqrt{a^3}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
c_n &= \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \Psi(x,0)\, dx \\
&= \sqrt{\frac{2}{a}} \left[\int_0^{a/2} \sin\left(\frac{n\pi}{a}x\right) Ax\, dx + \int_{a/2}^{a} \sin\left(\frac{n\pi}{a}x\right) A(a-x)\, dx \right] \\
&= \frac{2\sqrt{6}}{a^2} \left[\int_0^{a/2} x \sin\left(\frac{n\pi}{a}x\right)\, dx + \int_{a/2}^{a} (a-x) \sin\left(\frac{n\pi}{a}x\right)\, dx \right] \\
&= \frac{8\sqrt{6}}{n^2\pi^2} \sin^2\left(\frac{n\pi}{4}\right) \sin\left(\frac{n\pi}{2}\right) \\
&= \begin{cases} 0 & n \text{ even} \\ (-1)^{(n-1)/2} \frac{4\sqrt{6}}{n^2\pi^2} & n \text{ odd} \end{cases} \\
\Psi(x,t) &= \frac{4\sqrt{6}}{\pi^2} \sqrt{\frac{2}{a}} \sum_{n=1,3,5,\ldots}^{\infty} (-1)^{(n-1)/2} \frac{1}{n^2} \sin\left(\frac{n\pi}{a}x\right) e^{-i(n^2\pi^2\hbar/2ma^2)t}
\end{aligned}
$$

(c)

$$
\begin{aligned}
|c_1|^2 &= \left(\frac{4\sqrt{6}}{\pi^2}\right)^2 \\
&\approx 0.985
\end{aligned}
$$

(d)

$$
\begin{aligned}
E_n &= \frac{n^2\pi^2\hbar^2}{2ma^2} \\
\langle H \rangle &= \sum_{n=0}^{\infty} |c_{2n+1}|^2 E_{2n+1} \\
&= \sum_{n=0}^{\infty} \left(\frac{4\sqrt{6}}{(2n+1)^2\pi^2}\right)^2 \frac{(2n+1)^2\pi^2\hbar^2}{2ma^2} \\
&= \sum_{n=0}^{\infty} \frac{48\hbar^2}{(2n+1)^2 ma^2\pi^2} \\
&= \frac{48\hbar^2}{ma^2\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\
&= \frac{6\hbar^2}{ma^2}
\end{aligned}
$$

2.8

$$
\begin{aligned}
1 &= \int_0^{a/2} |\Psi|^2 \, dx \\
&= A^2 \int_0^{a/2} \, dx \\
&= \frac{aA^2}{2} \\
A &= \sqrt{\frac{2}{a}} \\
c_n &= \frac{2}{a} \int_0^{a/2} \sin\left(\frac{n\pi}{a} x\right) \, dx \\
|c_1|^2 &= \left(\frac{2}{\pi}\right)^2 \\
&\approx 0.405
\end{aligned}
$$

2.9

$$
\begin{aligned}
\Psi(x,0) &= Ax(a-x) \\
\langle H \rangle &= \int_0^a \Psi(x,0)^* \hat{H} \Psi(x,0) \, dx \\
&= \int_0^a \Psi(x,0)^* \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}\right) \Psi(x,0) \, dx \\
&= \frac{A^2 \hbar^2}{m} \int_0^a x(a-x) \, dx \\
&= \frac{30\hbar^2}{ma^5} \frac{a^3}{6} \\
&= \frac{5\hbar^2}{ma^2}
\end{aligned}
$$

2.10

(a)

$$
\begin{aligned}
\psi_2(x) &= \frac{1}{\sqrt{2!}}(\hat{a}_+)\psi_1 \\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx}+m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}xe^{-\frac{m\omega}{2\hbar}x^2} \\
&= \frac{1}{\sqrt{2}\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(-\hbar\frac{d}{dx}+m\omega x\right)xe^{-\frac{m\omega}{2\hbar}x^2} \\
&= \frac{1}{\sqrt{2}\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-\hbar\left(e^{-\frac{m\omega}{2\hbar}x^2}-\frac{m\omega}{\hbar}x^2e^{-\frac{m\omega}{2\hbar}x^2}\right)+m\omega x^2e^{-\frac{m\omega}{2\hbar}x^2}\right] \\
&= \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2-1\right)e^{-\frac{m\omega}{2\hbar}x^2}
\end{aligned}
$$

2.11

(a)

$$
\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} \psi_0^* x \psi_0 \, dx \\
&= \alpha^2 \int_{-\infty}^{\infty} x e^{-\frac{m\omega}{\hbar}x^2} \, dx \\
&= 0 \\
\langle p \rangle &= m \frac{d \langle x \rangle}{dt} \\
&= 0 \\
\langle x^2 \rangle &= \int_{-\infty}^{\infty} \psi_0^* x^2 \psi_0 \, dx \\
&= \alpha^2 \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} \, dx \\
&= \frac{\hbar}{2m\omega} \\
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \psi_0^* \left(-\hbar^2 \frac{d^2}{dx^2} \right) \psi_0 \, dx \\
&= -\hbar^2 \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \int_{-\infty}^{\infty} e^{-\frac{m\omega}{2\hbar}x^2} \frac{d}{dx} \left(-\frac{m\omega}{\hbar} x e^{-\frac{m\omega}{2\hbar}x^2} \right) \, dx \\
&= \hbar^2 \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \frac{m\omega}{\hbar} \int_{-\infty}^{\infty} e^{-\frac{m\omega}{2\hbar}x^2} \left(e^{-\frac{m\omega}{2\hbar}x^2} - \frac{m\omega}{\hbar} x^2 e^{-\frac{m\omega}{2\hbar}x^2} \right) \, dx \\
&= \hbar^2 \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \frac{m\omega}{\hbar} \int_{-\infty}^{\infty} \left(1 - \frac{m\omega}{\hbar} x^2 \right) e^{-\frac{m\omega}{\hbar}x^2} \, dx \\
&= \hbar^2 \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \frac{m\omega}{\hbar} \frac{\hbar\sqrt{\pi}}{2\sqrt{\hbar m\omega}} \\
&= \frac{1}{2} m\hbar\omega
\end{aligned}
$$

$$
\begin{aligned}
\psi_1(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}} x e^{-\frac{m\omega}{2\hbar}x^2} \\
\langle x \rangle &= 0 \\
\langle p \rangle &= m \frac{d \langle x \rangle}{dt} \\
&= 0 \\
\left\langle x^2 \right\rangle &= \int_{-\infty}^{\infty} \psi_1^* x^2 \psi_1 \, dx \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{2m\omega}{\hbar} \int_{-\infty}^{\infty} x^4 e^{-\frac{m\omega}{\hbar}x^2} \, dx \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{2m\omega}{\hbar} \frac{3}{4} \sqrt{\pi} \left(\frac{\hbar}{m\omega}\right)^{5/2} \\
&= \frac{3}{2} \frac{\hbar}{m\omega} \\
\left\langle p^2 \right\rangle &= \int_{-\infty}^{\infty} \psi_1^* \left(-\hbar^2 \frac{d^2}{dx^2}\right) \psi_1 \, dx \\
&= -\hbar^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{2m\omega}{\hbar} \int_{-\infty}^{\infty} x e^{-\frac{m\omega}{2\hbar}x^2} \frac{d}{dx} \left(e^{-\frac{m\omega}{2\hbar}x^2} - \frac{m\omega}{\hbar} x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right) \, dx \\
&= -\hbar^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{2m\omega}{\hbar} \int_{-\infty}^{\infty} x e^{-\frac{m\omega}{2\hbar}x^2} \left[-\frac{m\omega}{\hbar} x e^{-\frac{m\omega}{2\hbar}x^2} \right. \\
&\qquad \left. - \frac{2m\omega}{\hbar} x e^{-\frac{m\omega}{2\hbar}x^2} + \left(\frac{m\omega}{\hbar}\right)^2 x^3 e^{-\frac{m\omega}{2\hbar}x^2}\right] \, dx \\
&= 2\hbar^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right)^2 \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} \left(3 - \frac{m\omega}{\hbar} x^2\right) \, dx \\
&= 2\hbar^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right)^2 \frac{3}{4} \sqrt{\pi} \left(\frac{h}{m\omega}\right)^{3/2} \\
&= \frac{3}{2} m\hbar\omega
\end{aligned}
$$

(b)

$$
\begin{aligned}
\sigma_x &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\
&= \sqrt{\frac{\hbar}{2m\omega}} \\
\sigma_p &= \sqrt{\langle p^2\rangle - \langle p\rangle^2} \\
&= \sqrt{\frac{m\hbar\omega}{2}} \\
\sigma_x\sigma_p &= \frac{\hbar}{2} \\
\sigma_x &= \sqrt{\frac{3\hbar}{2m\omega}} \\
\sigma_p &= \sqrt{\frac{3m\hbar\omega}{2}} \\
\sigma_x\sigma_p &= \frac{3}{2}\hbar
\end{aligned}
$$

(c)

$$
\begin{aligned}
\langle T\rangle &= \frac{\left\langle p^2\right\rangle}{2m} \\
&= \frac{\hbar\omega}{4} \\
\langle V\rangle &= \frac{1}{2}m\omega^2\left\langle x^2\right\rangle \\
&= \frac{1}{4}\hbar\omega \\
\langle T\rangle &= \frac{\left\langle p^2\right\rangle}{2m} \\
&= \frac{3}{4}\hbar\omega \\
\langle V\rangle &= \frac{1}{2}m\omega^2\left\langle x^2\right\rangle \\
&= \frac{3}{4}\hbar\omega
\end{aligned}
$$

2.12

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} \psi_n^* x \psi_n \, dx \\
&= \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}_+ + \hat{a}_-) \psi_n \, dx \\
&= \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{\infty} \psi_n^* (\sqrt{n+1}\psi_{n+1} + \sqrt{n}\psi_{n-1}) \, dx \\
&= 0 \\
\langle p\rangle &= \int_{-\infty}^{\infty} \psi_n^* p \psi_n \, dx \\
&= i\sqrt{\frac{\hbar m\omega}{2}} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}_+ - \hat{a}_-) \psi_n \, dx \\
&= i\sqrt{\frac{\hbar m\omega}{2}} \int_{-\infty}^{\infty} \psi_n^* (\sqrt{n+1}\psi_{n+1} - \sqrt{n}\psi_{n-1}) \, dx \\
&= 0 \\
\left\langle x^2\right\rangle &= \int_{-\infty}^{\infty} \psi_n^* x^2 \psi_n \, dx \\
&= \frac{\hbar}{2m\omega} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}_+^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-^2) \psi_n \, dx \\
&= \frac{\hbar}{2m\omega} (2n+1) \int_{-\infty}^{\infty} |\psi_n|^2 \, dx \\
&= \frac{\hbar}{m\omega} \left(n + \frac{1}{2}\right) \\
\left\langle p^2\right\rangle &= \int_{-\infty}^{\infty} \psi_n^* p^2 \psi_n \, dx \\
&= -\frac{\hbar m\omega}{2} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}_+^2 - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a} - +\hat{a}_-^2) \psi_n \, dx \\
&= \frac{\hbar m\omega}{2} (2n+1) \int_{-\infty}^{\infty} |\psi_n|^2 \, dx \\
&= \hbar m\omega \left(n + \frac{1}{2}\right) \\
\langle T\rangle &= \left\langle \frac{p^2}{2m} \right\rangle \\
&= \frac{1}{2}\hbar\omega \left(n + \frac{1}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
\sigma_x &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\
&= \sqrt{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)} \\
\sigma_p &= \sqrt{\hbar m\omega\left(n+\frac{1}{2}\right)} \\
\sigma_x\sigma_p &= (2n+1)\frac{\hbar}{2} \\
&\geq \frac{\hbar}{2}
\end{aligned}
$$

2.13

(a)

$$
\begin{aligned}
\Psi(x,0) &= A[3\psi_0(x) + 4\psi_1(x)] \\
1 &= \int_{-\infty}^{\infty} |\Psi(x,0)|^2\,dx \\
&= A^2\int_{-\infty}^{\infty}[9\psi_0(x)^2 + 24\psi_0(x)\psi_1(x) + 16\psi_1(x)^2]\,dx \\
&= 25A^2 \\
A &= \frac{1}{5}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\Psi(x,t) &= \frac{1}{5}[3\psi_0(x)e^{-i\omega t/2} + 4\psi_1(x)e^{-3i\omega t/2}] \\
|\Psi(x,t)|^2 &= \Psi(x,t)^*\Psi(x,t) \\
&= \frac{1}{25}[3\psi_0(x)e^{i\omega t/2} + 4\psi_1(x)e^{3i\omega t/2}][3\psi_0(x)e^{-i\omega t/2} + 4\psi_1(x)e^{-3i\omega t/2}] \\
&= \frac{1}{25}[9\psi_0(x)^2 + 12\psi_0(x)\psi_1(x)e^{-i\omega t} + 12\psi_0(x)\psi_1(x)e^{i\omega t} + 16\psi_1(x)^2] \\
&= \frac{1}{25}[9\psi_0(x)^2 + 16\psi_1(x)^2 + 24\psi_0(x)\psi_1(x)\cos\omega t]
\end{aligned}
$$

(c)

$$
\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} \Psi^* x \Psi \, dx \\
&= \frac{1}{25}\int_{-\infty}^{\infty} x(9\psi_0^2 + 16\psi_1^2 + 24\psi_0\psi_1 \cos\omega t)\, dx \\
&= \frac{24}{25}\int_{-\infty}^{\infty} x\psi_0\psi_1 \cos(\omega t)\, dx \\
&= \frac{24}{25}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\sqrt{\frac{2m\omega}{\hbar}}\cos(\omega t)\int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2}\, dx \\
&= \frac{24}{25}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\sqrt{\frac{2m\omega}{\hbar}}\cos(\omega t)\frac{1}{2}\sqrt{\pi}\left(\frac{\hbar}{m\omega}\right)^{3/2} \\
&= \frac{24}{25}\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t) \\
\langle p\rangle &= m\frac{d\langle x\rangle}{dt} \\
&= -\frac{24}{25}\sqrt{\frac{\hbar m\omega}{2}}\sin(\omega t) \\
\frac{d\langle p\rangle}{dt} &= -\frac{24}{25}\sqrt{\frac{\hbar m\omega}{2}}\omega\cos(\omega t) \\
V &= \frac{1}{2}m\omega^2 x^2 \\
\frac{\partial V}{d\partial} &= m\omega^2 x \\
\left\langle -\frac{\partial V}{\partial x}\right\rangle &= -m\omega^2\langle x\rangle \\
&= -\frac{24}{25}\sqrt{\frac{\hbar m\omega}{2}}\omega\cos(\omega t) \\
&= \frac{d\langle p\rangle}{dt}
\end{aligned}
$$

(d)

$$
\begin{aligned}
E_0 &= \frac{\hbar\omega}{2} \\
P(E_0) &= \frac{9}{25} \\
E_1 &= \frac{3\hbar\omega}{2} \\
P(E_1) &= \frac{16}{25}
\end{aligned}
$$

2.14

$$
\begin{aligned}
1-\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_{-\sqrt{\hbar/m\omega}}^{\sqrt{\hbar/m\omega}} e^{-m\omega x^2/\hbar}\,dx &= 1-\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\sqrt{\frac{\pi\hbar}{m\omega}}\,\mathrm{erf}\,1 \\
&= 0.157
\end{aligned}
$$

2.15

$$
\begin{aligned}
a_{j+2} &= \frac{-2(n-j)}{(j+1)(j+2)}a_j \\
a_3 &= -\frac{4}{3}a_1 \\
a_5 &= \frac{4}{15}a_1 \\
H_5(\xi) &= a_1\left(\xi-\frac{4}{3}\xi^3+\frac{4}{15}\xi^5\right) \\
&= \frac{1}{120}a_1(120\xi-160\xi^3+32\xi^5) \\
&= 32\xi^5-160\xi^3+120\xi \\
a_2 &= -6a_0 \\
a_4 &= \frac{-8}{12}a_2 \\
&= 4a_0 \\
a_6 &= \frac{-4}{30}a_4 \\
&= -\frac{8}{15}a_0 \\
H_6(\xi) &= a_0\left(1-6\xi^2+4\xi^4-\frac{8}{15}\xi^6\right) \\
&= \frac{1}{120}a_0(120-720\xi^2+480\xi^4-64\xi^6) \\
&= 64\xi^6-480\xi^4+720\xi^2-120
\end{aligned}
$$

2.17

$$
\begin{aligned}
Ae^{ikx} + Be^{-ikx} &= A[\cos(kx) + i\sin(kx)] + B[\cos(kx) - i\sin(kx)] \\
&= (A+B)\cos(kx) + i(A-B)\sin(kx) \\
C &= A + B \\
D &= i(A - B) \\
-iD &= A - B \\
A &= \frac{C - iD}{2} \\
B &= \frac{C + iD}{2}
\end{aligned}
$$

2.18

$$
\begin{aligned}
\Psi_k(x,t) &= Ae^{i\left(kx - \frac{\hbar k^2}{2m}t\right)} \\
J(x,t) &= \frac{i\hbar}{2m}\left(\Psi\frac{\partial \Psi^*}{\partial x} - \Psi^*\frac{\partial \Psi}{\partial x}\right) \\
&= \frac{\hbar k|A|^2}{m}
\end{aligned}
$$

The probability travels in the same direction as the wave.

2.20

(a)

$$
\begin{aligned}
\Psi(x,0) &= Ae^{-a|x|} \\
1 &= \int_{-\infty}^{\infty} \Psi^*\Psi \, dx \\
&= |A|^2 \int_{-\infty}^{\infty} e^{-2a|x|} \, dx \\
&= 2|A|^2 \int_{0}^{\infty} e^{-2ax} \, dx \\
&= \frac{|A|^2}{a} \\
A &= \sqrt{a}
\end{aligned}
$$

(b)

$$\begin{aligned}
\phi(k) &= \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|-ikx}\, dx \\
&= \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} [\cos(kx) - i\sin(kx)]\, dx \\
&= \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} \cos(kx)\, dx \\
&= \sqrt{\frac{a}{2\pi}} 2 \int_{0}^{\infty} e^{-ax} \cos(kx)\, dx \\
&= \sqrt{\frac{a}{2\pi}} \frac{2a}{a^2 + k^2}
\end{aligned}$$

(c)

$$\Psi(x,t) = \frac{a^{3/2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{a^2+k^2} e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)}\, dk$$

2.21

(a)

$$\begin{aligned}
\Psi(x,0) &= Ae^{-ax^2} \\
1 &= A^2 \int_{-\infty}^{\infty} e^{-2ax^2}\, dx \\
&= \sqrt{\frac{\pi}{2a}} A^2 \\
A &= \left(\frac{2a}{\pi}\right)^{1/4}
\end{aligned}$$

(b)

$$\begin{aligned}
\phi(k) &= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-(ax^2+ikx)}\, dx \\
&= \frac{1}{(2\pi a)^{1/4}} e^{-k^2/4a} \\
\Psi(x,t) &= \frac{1}{\sqrt{2\pi}} \frac{1}{(2\pi a)^{1/4}} \int_{-\infty}^{\infty} e^{-\frac{k^2}{4a} + i\left(kx - \frac{\hbar k^2}{2m} t\right)}\, dk \\
\Psi(x,t) &= \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\gamma} e^{-ax^2/\gamma^2}
\end{aligned}$$

(c)

$$
\begin{aligned}
|\Psi(x,t)|^2 &= \Psi^*\Psi \\
&= \left(\frac{2a}{\pi}\right)^{1/2} \frac{1}{\gamma^*} e^{-ax^2/(\gamma^*)^2} \frac{1}{\gamma} e^{-ax^2/\gamma^2} \\
&= \left(\frac{2a}{\pi}\right)^{1/2} \frac{1}{\sqrt{1-2i\hbar at/m}} e^{-ax^2/(1-2i\hbar at/m)} \\
&\quad \frac{1}{\sqrt{1+2i\hbar at/m}} e^{-ax^2/(1+2i\hbar at/m)} \\
&= \left(\frac{2a}{\pi}\right)^{1/2} \frac{1}{\sqrt{1+(2\hbar at/m)^2}} e^{-2ax^2/[1+(2a\hbar t/m)^2]} \\
&= \sqrt{\frac{2}{\pi}} w e^{-2w^2x^2}
\end{aligned}
$$

As t increases $|\Psi|^2$ flattens out and broadens.

(d)

$$
\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} \Psi^* x \Psi \, dx \\
&= 0 \\
\langle p \rangle &= m \frac{d\langle x \rangle}{dt} \\
&= 0 \\
\langle x^2 \rangle &= \int_{-\infty}^{\infty} \Psi^* x^2 \Psi \, dx \\
&= \sqrt{\frac{2}{\pi}} w \int_{-\infty}^{\infty} x^2 e^{-2w^2x^2} \, dx \\
&= \frac{1}{4w^2}
\end{aligned}
$$

2.22

(a) -25

(b) 1

(c) 0

2.26

$$
\begin{aligned}
F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-ikx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \\
f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dk
\end{aligned}
$$

2.29

$$
\begin{aligned}
\psi(x) &= \begin{cases} Fe^{-\kappa x} & x > a \\ C\sin(lx) & 0 < x < a \\ -\psi(-x) & x < 0 \end{cases} \\
Fe^{-\kappa a} &= C\sin(la) \\
-\kappa Fe^{-\kappa a} &= lC\cos(la) \\
-\frac{1}{\kappa} &= \frac{1}{l}\tan(la) \\
\tan z &= -\frac{l}{\kappa} \\
&= -\frac{la}{\kappa a} \\
&= -\frac{z}{\sqrt{z_0^2 - z^2}}
\end{aligned}
$$

For large z_0 the intersections occur just below $z_n = n\pi$ so

$$
\begin{aligned}
z &= la \\
n\pi &\approx \frac{\sqrt{2m(E+V_0)}}{\hbar} a \\
E + V_0 &\approx \frac{n^2\pi^2\hbar^2}{2ma^2}.
\end{aligned}
$$

As z_0 decreases there are fewer and fewer bound states. When $z_0 < \pi/2$ there are no odd bound states.

2.30

$$
\begin{aligned}
\psi(x) &= \begin{cases} Fe^{-\kappa x} & x > a \\ D\cos(lx) & 0 < x < a \\ \psi(-x) & x < 0 \end{cases} \\
1 &= \int_{-\infty}^{\infty} |\psi|^2 \, dx \\
&= 2\left(|D|^2 \int_0^a \cos^2(lx)\, dx + |F|^2 \int_a^\infty e^{-2\kappa x}\, dx \right) \\
&= 2\left[|D|^2 \frac{2al + \sin(2al)}{4l} + \frac{|F|^2}{2\kappa} e^{-2\kappa a} \right] \\
&= |D|^2 \left[a + \frac{\sin(2al)}{2l} + \frac{\cos^2(al)}{\kappa} \right] \\
&= |D|^2 \left[a + \frac{2\sin(al)\cos(al)}{2l} + \frac{\cos^3(al)}{l\sin(al)} \right] \\
&= |D|^2 \left\{ a + \frac{\cos(al)}{l\sin(al)} [\sin^2(al) + \cos^2(al)] \right\} \\
&= |D|^2 \left[a + \frac{\cos(al)}{l\sin(al)} \right] \\
&= |D|^2 \left[a + \frac{1}{l\tan(al)} \right] \\
&= |D|^2 \left[a + \frac{1}{\kappa} \right] \\
D &= \frac{1}{\sqrt{a + 1/\kappa}}
\end{aligned}
$$

$$
\begin{aligned}
1 &= \left\{ \frac{1}{a + 1/\kappa} \left[a + \frac{\sin(2al)}{2l} \right] + |F|^2 \frac{e^{-2\kappa a}}{\kappa} \right\} \\
\frac{(Fe^{-\kappa a})^2}{\kappa} &= 1 - \frac{1}{a + 1/\kappa} \left[a + \frac{\sin(2al)}{2l} \right] \\
(Fe^{-\kappa a})^2 &= \kappa - \frac{\kappa}{a + 1/\kappa} \left[a + \frac{\sin(2al)}{2l} \right] \\
&= \frac{\kappa a + 1 - \kappa a - \kappa \sin(al)\cos(al)/l}{a + 1/\kappa} \\
&= \frac{1 - \sin^2(al)}{a + 1/\kappa} \\
F &= \frac{e^{\kappa a}\cos(al)}{\sqrt{a + 1/\kappa}}
\end{aligned}
$$

2.31

$$
\begin{aligned}
1 &= 2aV_0 \\
V_0 &= \frac{1}{2a} \\
z_0 &= \frac{a}{\hbar}\sqrt{2mV_0} \\
&= \frac{a}{\hbar}\sqrt{\frac{m}{a}} \\
&= \frac{\sqrt{am}}{\hbar} \\
\lim_{a\to 0} z_0 &= 0
\end{aligned}
$$

2.34

(a)

$$
\begin{aligned}
V(x) &= \begin{cases} 0 & x \leq 0 \\ V_0 & x > 0 \end{cases} \\
-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} &= E\psi \\
\frac{d^2\psi}{dx^2} &= -\frac{2mE}{\hbar^2}\psi \\
&= -k^2\psi \\
k &= \frac{\sqrt{2mE}}{\hbar} \\
\psi &= Ae^{ikx} + Be^{-ikx} \\
-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi &= E\psi \\
\frac{d^2\psi}{dx^2} &= -\frac{2m(E-V_0)}{\hbar^2}\psi \\
&= \kappa^2\psi \\
\kappa &= \frac{\sqrt{2m(V_0-E)}}{\hbar} \\
\psi &= Fe^{-\kappa x} \\
A + B &= F \\
ik(A-B) &= -\kappa F \\
F &= -i\frac{k}{\kappa}(A-B) \\
A + B &= -i\frac{k}{\kappa}(A-B) \\
\left(1 - i\frac{k}{\kappa}\right)B &= -\left(1 + i\frac{k}{\kappa}\right)A \\
B &= -\frac{1+ik/\kappa}{1-ik/\kappa}A \\
R &= \frac{|B|^2}{|A|^2} \\
&= \left(-\frac{1+ik/\kappa}{1-ik/\kappa}\right)\left(-\frac{1-ik/\kappa}{1+ik/\kappa}\right) \\
&= 1
\end{aligned}
$$

(b)

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \\ Fe^{ilx} & x > 0 \end{cases}$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$$l = \frac{\sqrt{2m(E - V_0)}}{\hbar}$$

$$A + B = F$$

$$ik(A - B) = ilF$$

$$F = \frac{k}{l}(A - B)$$

$$A + B = \frac{k}{l}(A - B)$$

$$\left(\frac{k}{l} + 1\right) B = \left(\frac{k}{l} - 1\right) A$$

$$B = \frac{k/l - 1}{k/l + 1} A$$

$$R = \frac{|B|^2}{|A|^2}$$

$$= \left(\frac{k/l - 1}{k/l + 1}\right)^2$$

$$= \left(\frac{k - l}{k + l}\right)^2$$

$$= \frac{(k - l)^4}{(k^2 - l^2)^2}$$

$$k^2 - l^2 = \frac{2mE}{\hbar^2} - \frac{2m(E - V_0)}{\hbar^2}$$

$$= \frac{2m}{\hbar^2} V_0$$

$$k - l = \frac{\sqrt{2m}}{\hbar}(\sqrt{E} - \sqrt{E - V_0})$$

$$R = \frac{(\sqrt{E} - \sqrt{E - V_0})^4}{V_0^2}$$

(d)

$$
\begin{aligned}
B &= F - A \\
F &= \frac{k}{l}(A - F + A) \\
\left(1 + \frac{k}{l}\right) F &= \frac{2k}{l} A \\
F &= \frac{2k}{k+l} A \\
\frac{l}{k} &= \sqrt{\frac{E - V_0}{E}} \\
T &= \left|\frac{F}{A}\right|^2 \frac{l}{k} \\
&= \left(\frac{2k}{k+l}\right)^2 \frac{l}{k} \\
&= \frac{4kl}{(k+l)^2} \\
&= \frac{4kl(k-l)^2}{(k^2-l^2)^2} \\
&= \frac{4\sqrt{E}\sqrt{E-V_0}(\sqrt{E} - \sqrt{E-V_0})^2}{V_0^2} \\
T + R &= \frac{4kl}{(k+l)^2} + \frac{(k-l)^2}{(k+l)^2} \\
&= \frac{4kl + k^2 - 2kl + l^2}{(k+l)^2} \\
&= \frac{k^2 + 2kl + l^2}{(k+l)^2} \\
&= \frac{(k+l)^2}{(k+l)^2} \\
&= 1
\end{aligned}
$$

2.35

(a)

$$
\begin{aligned}
V(x) &= \begin{cases} 0 & x < 0 \\ -V_0 & x > 0 \end{cases} \\
-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} &= E\psi \\
\frac{d^2\psi}{dx^2} &= -\frac{2mE}{\hbar^2}\psi \\
&= -k^2\psi \\
k &= \frac{\sqrt{2mE}}{\hbar} \\
\psi &= Ae^{ikx} + Be^{-ikx} \\
-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi &= E\psi \\
\frac{d^2\psi}{dx^2} &= -\frac{2m}{\hbar^2}(E+V_0)\psi \\
&= -l^2\psi \\
l &= \frac{\sqrt{2m(E+V_0)}}{\hbar} \\
\psi &= Fe^{ilx} \\
A + B &= F \\
ik(A-B) &= ilF \\
k(A-B) &= l(A+B) \\
(k+l)B &= (k-l)A \\
B &= \frac{k-l}{k+l}A \\
R &= \left|\frac{B}{A}\right|^2 \\
&= \left(\frac{k-l}{k+l}\right)^2 \\
&= \left(\frac{\sqrt{E}-\sqrt{E+V_0}}{\sqrt{E}+\sqrt{E+V_0}}\right)^2 \\
&= \frac{1}{9}
\end{aligned}
$$

(c)

$$T = 1 - R = \frac{8}{9}$$

2.36

$$
\begin{aligned}
V(x) &= \begin{cases} 0 & |x| < a \\ \infty & |x| > a \end{cases} \\
-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} &= E\psi \\
\frac{d^2\psi}{dx^2} &= -\frac{2mE}{\hbar^2}\psi \\
&= -k^2\psi \\
k &= \frac{\sqrt{2mE}}{\hbar} \\
\psi &= A\sin kx + B\cos kx \\
0 &= -A\sin ka + B\cos ka \\
0 &= A\sin ka + B\cos ka \\
B\cos ka &= 0 \\
k &= \frac{n\pi}{2a},\ n = 1, 3, 5, \ldots \\
E &= \frac{n^2\pi^2\hbar^2}{2m(2a)^2} \\
\psi &= B\cos\left(\frac{n\pi}{2a}x\right),\ n = 1, 3, 5, \ldots \\
1 &= |B|^2\int_{-a}^{a}\cos^2\left(\frac{n\pi}{2a}x\right)\,dx \\
B &= \frac{1}{\sqrt{a}} \\
\psi &= \frac{1}{\sqrt{a}}\cos\left(\frac{n\pi}{2a}x\right),\ n = 1, 3, 5, \ldots \\
A\sin ka &= 0 \\
k &= \frac{n\pi}{2a},\ n = 2, 4, 6, \ldots \\
E &= \frac{n^2\pi^2\hbar^2}{2m(2a)^2} \\
\psi &= A\sin\left(\frac{n\pi}{2a}x\right),\ n = 2, 4, 6, \ldots \\
1 &= |A|\int_{-a}^{a}\sin^2\left(\frac{n\pi}{2a}x\right)\,dx \\
A &= \frac{1}{\sqrt{a}} \\
\psi &= \frac{1}{\sqrt{a}}\sin\left(\frac{n\pi}{2a}x\right),\ n = 2, 4, 6, \ldots
\end{aligned}
$$

2.37

$$
\begin{aligned}
\Psi(x,0) &= A\sin^3\left(\frac{\pi}{a}x\right)\\
&= A\left[\frac{3}{4}\sin\left(\frac{\pi}{a}x\right) - \frac{1}{4}\sin\left(\frac{3\pi}{a}x\right)\right]\\
&= A\sqrt{\frac{a}{2}}\left[\frac{3}{4}\psi_1(x) - \frac{1}{4}\psi_3(x)\right]\\
1 &= |A|^2\frac{a}{2}\int_0^a\left[\frac{3}{4}\psi_1(x) - \frac{1}{4}\psi_3(x)\right]^2\,dx\\
&= |A|^2\frac{a}{2}\int_0^a\left[\frac{9}{16}\psi_1(x)^2 - \frac{3}{8}\psi_1(x)\psi_3(x) + \frac{1}{16}\psi_3(x)^2\right]\,dx\\
&= \frac{5}{16}a|A|^2\\
A &= \frac{4}{\sqrt{5a}}\\
\Psi(x,0) &= \frac{1}{\sqrt{10}}[3\psi_1(x) - \psi_3(x)]\\
\Psi(x,t) &= \frac{1}{\sqrt{10}}[3\psi_1(x)e^{-iE_1t/\hbar} - \psi_3(x)e^{-iE_3t/\hbar}]\\
\langle x\rangle &= \int_0^a \Psi^* x\Psi\,dx\\
&= \frac{1}{10}\int_0^a x\left(9\psi_1^2 + \psi_3^2 - 3\psi_1\psi_3e^{-i(E_3-E_1)t/\hbar} - 3\psi_1\psi_3e^{-i(E_1-E_3)t/\hbar}\right)\,dx\\
&= \frac{1}{10}\int_0^a x\left[9\psi_1^2 + \psi_3^2 - 6\psi_1\psi_3\cos\left(\frac{E_3-E_1}{\hbar}t\right)\right]\,dx\\
&= \frac{1}{10}\left[9\langle x\rangle_1 + \langle x\rangle_3 - 6\cos\left(\frac{E_3-E_1}{\hbar}t\right)\int_0^a x\psi_1\psi_3\,dx\right]\\
&= \frac{1}{10}\left[\frac{9}{2}a + \frac{1}{2}a\right]\\
&= \frac{a}{2}\\
P(E_1) &= \frac{9}{10}\\
P(E_3) &= \frac{1}{10}\\
\langle E\rangle &= E_1P(E_1) + E_3P(E_3)\\
&= \frac{9\pi^2\hbar^2}{20ma^2} + \frac{9\pi^2\hbar^2}{20ma^2}\\
&= \frac{9\pi^2\hbar^2}{10ma^2}
\end{aligned}
$$

2.38

(a)

$$
\begin{aligned}
\Psi(x,t) &= \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar} \\
\Psi(x,0) &= \sum_{n=1}^{\infty} c_n \psi_n(x) \\
E_n T &= \frac{n^2\pi^2\hbar^2}{2ma^2}\frac{4ma^2}{\pi\hbar} \\
&= 2\pi n^2 \hbar \\
\Psi(x,T) &= \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n T/\hbar} \\
&= \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-2\pi i n^2} \\
&= \sum_{n=1}^{\infty} c_n \psi_n(x) \\
&= \Psi(x,0)
\end{aligned}
$$

(b)

$$
\begin{aligned}
E &= \frac{1}{2}mv^2 \\
v &= \sqrt{\frac{2E}{m}} \\
T &= \frac{2a}{v} \\
&= a\sqrt{\frac{2m}{E}}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\frac{4ma^2}{\pi\hbar} &= a\sqrt{\frac{2m}{E}} \\
\frac{16m^2a^2}{\pi^2\hbar^2} &= \frac{2m}{E} \\
E &= \frac{\pi^2\hbar^2}{8ma^2} \\
&= \frac{E_1}{4}
\end{aligned}
$$

2.39

(a)

$$\Psi(x,0)=\begin{cases}\frac{2\sqrt{3}}{a\sqrt{a}}x & 0\le x\le a/2\\ \frac{2\sqrt{3}}{a\sqrt{a}}(a-x) & a/2\le x\le a\end{cases}$$

$$\frac{d}{dx}\Psi(x,0)=\frac{2\sqrt{3}}{a\sqrt{a}}\left[1-2\theta\left(x-\frac{a}{2}\right)\right]$$

(b)

$$\frac{d^2}{dx^2}\Psi(x,0)=-\frac{4\sqrt{3}}{a\sqrt{a}}\delta\left(x-\frac{a}{2}\right)$$

(c)

$$\begin{aligned}\hat{H}\Psi(x,0)&=\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]\Psi(x,0)\\&=\frac{\hbar^2}{2m}\frac{4\sqrt{3}}{a\sqrt{a}}\delta\left(x-\frac{a}{2}\right)+V(x)\Psi(x,0)\\ \langle H\rangle&=\int\Psi(x,0)^*\hat{H}\Psi(x,0)\,dx\\&=\int_0^a\Psi(x,0)^*\left[\frac{\hbar^2}{2m}\frac{4\sqrt{3}}{a\sqrt{a}}\delta\left(x-\frac{a}{2}\right)+V(x)\Psi(x,0)\right]dx\\&=\Psi\left(\frac{a}{2},0\right)^*\frac{\hbar^2}{2m}\frac{4\sqrt{3}}{a\sqrt{a}}+\int_0^a\Psi(x,0)^*V(x)\Psi(x,0)\,dx\\&=\frac{6\hbar^2}{ma^2}\end{aligned}$$

2.40

(a)

$$
\begin{aligned}
V(x) &= \frac{1}{2}m\omega^2x^2 \\
\xi &= \sqrt{\frac{m\omega}{\hbar}}x \\
\psi_n(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\xi^2/2} \\
\Psi(x,0) &= A\left(1-2\sqrt{\frac{m\omega}{\hbar}}x\right)^2 e^{-\frac{m\omega}{2\hbar}x^2} \\
&= A\left(1-4\sqrt{\frac{m\omega}{\hbar}}x+\frac{4m\omega}{\hbar}x^2\right)e^{-\frac{m\omega}{2\hbar}x^2} \\
&= A\left(\frac{\pi\hbar}{m\omega}\right)^{1/4}\left[3\psi_0(x)-2\sqrt{2}\psi_1(x)+2\sqrt{2}\psi_2(x)\right] \\
1 &= A^2\sqrt{\frac{\pi\hbar}{m\omega}}\int_{-\infty}^{\infty}(3\psi_0-2\sqrt{2}\psi_1+2\sqrt{2}\psi_2)^2\,dx \\
&= A^2\sqrt{\frac{\pi\hbar}{m\omega}}\int_{-\infty}^{\infty}(9\psi_0^2-12\sqrt{2}\psi_0\psi_1+12\sqrt{2}\psi_0\psi_2+8\psi_1^2-16\psi_1\psi_2+8\psi_2^2)\,dx \\
&= 25A^2\sqrt{\frac{\pi\hbar}{m\omega}} \\
A &= \frac{1}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \\
\Psi(x,0) &= \frac{3}{5}\psi_0(x)-\frac{2\sqrt{2}}{5}\psi_1(x)+\frac{2\sqrt{2}}{5}\psi_2(x)
\end{aligned}
$$

(b)

$$
\begin{aligned}
E_0 &= \frac{\hbar\omega}{2} \\
P(E_0) &= \frac{9}{25} \\
E_1 &= \frac{3\hbar\omega}{2} \\
P(E_1) &= \frac{8}{25} \\
E_2 &= \frac{5\hbar\omega}{2} \\
P(E_2) &= \frac{8}{25} \\
\langle E \rangle &= \frac{\hbar\omega}{2}\frac{9}{25} + \frac{3\hbar\omega}{2}\frac{8}{25} + \frac{5\hbar\omega}{2}\frac{8}{25} \\
&= \frac{73}{50}\hbar\omega
\end{aligned}
$$

(c)

$$
\begin{aligned}
\xi &= \sqrt{\frac{m\omega}{\hbar}}x \\
\psi_n(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2} \\
\Psi(x,T) &= B\left(1 + 2\sqrt{\frac{m\omega}{\hbar}}x\right)^2 e^{-\frac{m\omega}{2\hbar}x^2} \\
&= B\left(1 + 4\sqrt{\frac{m\omega}{\hbar}}x + 4\frac{m\omega}{\hbar}x^2\right) e^{-\frac{m\omega}{2\hbar}x^2} \\
&= B\left(\frac{\pi\hbar}{m\omega}\right)^{1/4}\left[3\psi_0(x) + 2\sqrt{2}\psi_1(x) + 2\sqrt{2}\psi_2(x)\right] \\
1 &= |B|^2\sqrt{\frac{\pi\hbar}{m\omega}}\int_{-\infty}^{\infty}[3\psi_0 + 2\sqrt{2}\psi_1 + 2\sqrt{2}\psi_2]^2\,dx \\
&= |B|^2\sqrt{\frac{\pi\hbar}{m\omega}}\int_{-\infty}^{\infty}[9\psi_0^2 + 12\sqrt{2}\psi_0\psi_1 + 12\sqrt{2}\psi_0\psi_2 + 8\psi_1^2 + 16\psi_1\psi_2 + 8\psi_2^2]\,dx \\
&= 25|B|^2\sqrt{\frac{\pi\hbar}{m\omega}} \\
B &= \frac{1}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \\
\Psi(x,T) &= \frac{3}{5}\psi_0(x) + \frac{2\sqrt{2}}{5}\psi_1(x) + \frac{2\sqrt{2}}{5}\psi_2(x) \\
\Psi(x,t) &= \frac{3}{5}\psi_0(x)e^{-i\omega t/2} - \frac{2\sqrt{2}}{5}\psi_1(x)e^{-3i\omega t/2} + \frac{2\sqrt{2}}{5}\psi_2(x)e^{-5i\omega t/2} \\
&= e^{-i\omega t/2}\left[\frac{3}{5}\psi_0(x) - \frac{2\sqrt{2}}{5}\psi_1(x)e^{-i\omega t} + \frac{2\sqrt{2}}{5}e^{-2i\omega t}\right] \\
e^{-i\omega T} &= -1 \\
e^{-2i\omega T} &= 1 \\
T &= \frac{\pi}{\omega}
\end{aligned}
$$

2.41

The argument for calculating the allowed energies and wavefunctions is the same, except there is a boundary condition $\psi(0) = 0$. This leaves only $\psi_n(x)$ for odd n.

2.43

$$
\begin{aligned}
k &= \frac{\sqrt{2mE}}{\hbar} \\
\psi(x) &= \begin{cases} -A\sin kx + B\cos kx & -a < x \le 0 \\ A\sin kx + B\cos kx & 0 \le x < a \end{cases} \\
\Delta\left(\frac{d\psi}{dx}\right) &= 2Ak \\
\Delta\left(\frac{d\psi}{dx}\right) &= \frac{2m\alpha}{\hbar^2}\psi(0) \\
2Ak &= \frac{2m\alpha}{\hbar^2}B \\
B &= \frac{\hbar^2 k}{m\alpha}A \\
\psi(x) &= A\left(\sin kx + \frac{\hbar^2 k}{m\alpha}\cos kx\right) \\
0 &= A\sin ka + \frac{\hbar^2 k}{m\alpha}A\cos ka \\
\tan ka &= -\frac{\hbar^2 k}{m\alpha} \\
ka &\approx \frac{n\pi}{2},\ n = 1, 3, 5, \ldots \\
E &\approx \frac{n^2\pi^2\hbar^2}{2m(2a)^2},\ n = 1, 3, 5, \ldots \\
\psi(x) &= \begin{cases} A\sin kx - B\cos kx & -a < x \le 0 \\ A\sin kx + B\cos kx & 0 \le x < a \end{cases} \\
-B &= B \\
B &= 0 \\
\psi(x) &= A\sin kx \\
0 &= A\sin ka \\
ka &= \frac{n\pi}{2},\ n = 2, 4, 6, \ldots \\
\psi(x) &= A\sin\left(\frac{n\pi}{2a}x\right) \\
E &= \frac{n^2\pi^2\hbar^2}{2m(2a)^2},\ n = 2, 4, 6, \ldots
\end{aligned}
$$

2.44

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2}\psi_2 + V\psi_1\psi_2 &= E\psi_1\psi_2 \\
-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2}\psi_1 + V\psi_1\psi_2 &= E\psi_1\psi_2 \\
\psi_2\frac{d^2\psi_1}{dx^2} - \psi_1\frac{d^2\psi_2}{dx^2} &= 0 \\
\frac{d}{dx}\left(\psi_2\frac{d\psi_1}{dx} - \psi_1\frac{d\psi_2}{dx}\right) &= 0 \\
\psi_2\frac{d\psi_1}{dx} - \psi_1\frac{d\psi_2}{dx} &= c \\
c &= 0 \\
\psi_2\frac{d\psi_1}{dx} &= \psi_1\frac{d\psi_2}{dx} \\
\frac{1}{\psi_1}\frac{d\psi_1}{dx} &= \frac{1}{\psi_2}\frac{d\psi_2}{dx} \\
\ln\psi_1 &= \ln\psi_2 + c \\
\psi_1 &= A\psi_2
\end{aligned}
$$

2.45

(a)

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2}\psi_m + V(x)\psi_n\psi_m &= E_n\psi_n\psi_m \\
-\frac{\hbar^2}{2m}\frac{d^2\psi_m}{dx^2}\psi_n + V(x)\psi_n\psi_m &= E_m\psi_n\psi_m \\
\frac{d^2\psi_m}{dx^2}\psi_n - \frac{d^2\psi_n}{dx^2}\psi_m &= \frac{2m}{\hbar^2}(E_n - E_m)\psi_n\psi_m \\
\frac{d}{dx}\left(\frac{d\psi_m}{dx}\psi_n - \frac{d\psi_n}{dx}\psi_m\right) &= \frac{2m}{\hbar^2}(E_n - E_m)\psi_n\psi_m
\end{aligned}
$$

(b)

$$
\begin{aligned}
\int_{x_1}^{x_2}\frac{d}{dx}(\psi_m'\psi_n - \psi_n'\psi_m)\,dx &= \frac{2m}{\hbar^2}(E_n - E_m)\int_{x_1}^{x_2}\psi_n\psi_m\,dx \\
\psi_m'(x_2)\psi_n(x_2) - \psi_m'(x_1)\psi_n(x_1) &= \frac{2m}{\hbar^2}(E_n - E_m)\int_{x_1}^{x_2}\psi_n\psi_m\,dx
\end{aligned}
$$

2.53

(a)

$$
\frac{1}{1-i\beta}\begin{pmatrix} i\beta & 1 \\ 1 & i\beta \end{pmatrix}
$$

(b)

$$\frac{e^{-2ika}}{\cos(2la) - i\frac{(k^2+l^2)}{2kl}\sin(2la)}\begin{pmatrix} i\frac{\sin(2la)}{2kl}(l^2-k^2) & 1 \\ 1 & i\frac{\sin(2la)}{2kl}(l^2-k^2)\end{pmatrix}$$

3 Formalism

3.1

(a)

$$\begin{aligned}\left|\int_a^b (f^*+g^*)(f+g)\,dx\right| &= \left|\int_a^b (f^*f + f^*g + g^*f + g^*g)\,dx\right| \\ &\leq \int_a^b |f|^2\,dx + \left|\int_a^b f^*g\,dx\right| + \left|\int_a^b g^*f\,dx\right| + \int_a^b |g|^2\,dx \\ &\leq \int_a^b |f|^2\,dx + 2\sqrt{\int_a^b |f|^2\,dx\int_a^b |g|^2\,dx} + \int_a^b |g|^2\,dx\end{aligned}$$

The set of all normalised functions isn't a vector space because e.g. multiplying a function by a constant also multiplies its integral by that constant meaning it's no longer a member of the vector space.

(b)

$$\begin{aligned}\langle\beta|\alpha\rangle &= \int_a^b \beta^*\alpha\,dx \\ &= \left(\int_a^b \alpha^*\beta\,dx\right)^* \\ &= \langle\alpha|\beta\rangle^* \\ \langle a|a\rangle &= \int_a^b |a|^2\,dx \\ &\geq 0\end{aligned}$$

If $\langle\alpha|\alpha\rangle = 0$ that implies $|\alpha|^2 = 0$ everywhere in the interval and thus $|\alpha\rangle = |0\rangle$.

$$\begin{aligned}\langle\alpha|\left(b\,|\beta\rangle + c\,|\gamma\rangle\right) &= \int_{x_1}^{x_2}\alpha^*(b\beta)\,dx + \int_{x_1}^{x_2}\alpha^*(c\gamma)\,dx \\ &= b\int_{x_1}^{x_2}\alpha^*\beta\,dx + c\int_{x_1}^{x_2}\alpha^*\gamma\,dx \\ &= b\,\langle\alpha|\beta\rangle + c\,\langle\alpha|\gamma\rangle\end{aligned}$$

3.2

(a)

$$\int_0^1 x^{2\nu}\,dx = \frac{1}{2\nu+1}\left[x^{2\nu+1}\right]_0^1$$

The integral is defined for $\nu > -1/2$. For the case $\nu = -1/2$

$$\int_0^1 x^{-1}\,dx = [\ln x]_0^1 = \ln 1 - \ln 0 = 0 - \infty.$$

So $f(x) = x^\nu$ is in Hilbert space for $\nu > -1/2$.

(b)

$$\begin{aligned}
\int_0^1 x\,dx &= \frac{1}{2}\\
\int_0^1 x^3\,dx &= \frac{1}{4}\\
\int_0^1 x^{-1}\,dx &= [\ln x]_0^1\\
&= 0 - \infty
\end{aligned}$$

$f(x)$ and $xf(x)$ are in Hilbert space, but not $(d/dx)f(x)$.

3.4

(a)

$$\begin{aligned}
\langle f|(\hat{Q}+\hat{R})f\rangle &= \langle f|\hat{Q}f\rangle + \langle f|\hat{R}f\rangle\\
&= \langle \hat{Q}f|f\rangle + \langle \hat{R}f|f\rangle\\
&= \langle (\hat{Q}+\hat{R})f|f\rangle
\end{aligned}$$

(b)

$$\begin{aligned}
\langle f|\alpha\hat{Q}g\rangle &= \alpha\,\langle f|\hat{Q}g\rangle\\
&= \alpha\,\langle \hat{Q}f|g\rangle\\
\langle \alpha\hat{Q}f|g\rangle &= \alpha^*\,\langle \hat{Q}f|g\rangle\\
\alpha &= \alpha^*
\end{aligned}$$

α is real.

(c)

$$\begin{aligned}\langle f|\hat{Q}\hat{R}g\rangle &= \langle\hat{Q}f|\hat{R}g\rangle \\ &= \langle\hat{R}\hat{Q}f|g\rangle\end{aligned}$$

The product of the operators is hermitian when $\hat{Q}\hat{R} = \hat{R}\hat{Q}$ i.e. $[\hat{Q}, \hat{R}] = 0$.

(d)

$$\begin{aligned}\langle\Psi|\hat{x}\Psi\rangle &= \int \Psi^*\hat{x}\Psi\,dx \\ &= \int \Psi^*\hat{x}^*\Psi\,dx \\ &= \int (\hat{x}\Psi)^*\Psi\,dx \\ &= \langle\hat{x}\Psi|\Psi\rangle \\ \langle\Psi|\hat{H}\Psi\rangle &= \int \Psi^*\hat{H}\Psi\,dx \\ &= \int \Psi^*\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\Psi\,dx \\ &= -\frac{\hbar^2}{2m}\int \Psi^*\frac{d^2\Psi}{dx^2}\,dx + \int \Psi^*V(x)\Psi\,dx \\ &= -\frac{\hbar^2}{2m}\left[\Psi^*\frac{d\Psi}{dx}\bigg|_{-\infty}^{\infty} - \int \frac{d\Psi^*}{dx}\frac{d\Psi}{dx}\,dx\right] + \langle V(x)\Psi|\Psi\rangle \\ &= \frac{\hbar^2}{2m}\left[\frac{d\Psi^*}{dx}\Psi\bigg|_{-\infty}^{\infty} - \int \frac{d^2\Psi^*}{dx}\Psi\,dx\right] + \langle V(x)\Psi|\Psi\rangle \\ &= -\frac{\hbar^2}{2m}\int \frac{d^2}{dx^2}\Psi^*\Psi\,dx + \langle V(x)\Psi|\Psi\rangle \\ &= \left\langle -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi\,\middle|\,\Psi\right\rangle + \langle V(x)\Psi|\Psi\rangle \\ &= \langle\hat{H}\Psi|\Psi\rangle\end{aligned}$$

3.5

(a)

$$\begin{aligned}x^\dagger &= x \\ i^\dagger &= -i \\ \left(\frac{d}{dx}\right)^\dagger &= -\frac{d}{dx}\end{aligned}$$

(b)

$$\begin{aligned}
\langle f|\hat{Q}\hat{R}g\rangle &= \int f^\dagger \hat{Q}\hat{R}g\,dx \\
&= \int (\hat{Q}^\dagger f)^\dagger \hat{R}g\,dx \\
&= \int (\hat{R}^\dagger \hat{Q}^\dagger f)^\dagger g\,dx \\
&= \langle \hat{R}^\dagger \hat{Q}^\dagger f|g\rangle
\end{aligned}$$

(c)

$$\begin{aligned}
\hat{a}_+ &= \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega x) \\
\langle f|\hat{a}g\rangle &= \left\langle f \,\middle|\, \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega x)g \right\rangle \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\,\langle f|(-i\hat{p} + m\omega x)g\rangle \\
&= \frac{1}{\sqrt{2\hbar m\omega}}(\langle f| - i\hat{p}g\rangle + \langle f|m\omega x g\rangle) \\
&= \frac{1}{\sqrt{2\hbar m\omega}}(\langle f| - i\hat{p}g\rangle + \langle m\omega x f|g\rangle) \\
&= \frac{1}{\sqrt{2\hbar m\omega}}(\langle i\hat{p}f|g\rangle + \langle m\omega x f|g\rangle) \\
&= \left\langle \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega x)f \,\middle|\, g \right\rangle \\
&= \langle \hat{a}_- f|g\rangle
\end{aligned}$$

3.6

$$\begin{aligned}
\langle f|\hat{Q}g\rangle &= \int_0^{2\pi} f^* \frac{d^2 g}{d\phi^2}\,d\phi \\
&= f^* \frac{dg}{d\phi}\bigg|_0^{2\pi} - \int_0^{2\pi} \left(\frac{df}{d\phi}\right)^* \frac{dg}{d\phi}\,d\phi \\
&= -\left(\frac{df}{d\phi}\right)^* g\bigg|_0^{2\pi} + \int_0^{2\pi} \left(\frac{d^2 f}{d\phi}\right)^* g\,d\phi \\
&= \langle \hat{Q}f|g\rangle
\end{aligned}$$

Yes, the operator is hermitian.

$$
\begin{aligned}
\hat{Q}f &= qf \\
\frac{d^2 f}{d\phi^2} &= qf \\
\frac{d^2 f}{d\phi^2} - qf &= 0 \\
f &= Ae^{\sqrt{q}\phi} + Be^{-\sqrt{q}\phi} \\
f(\phi + 2\pi) &= Ae^{\sqrt{q}(\phi+2\pi)} + Be^{\sqrt{q}(\phi+2\pi)} \\
&= Ae^{\sqrt{q}\phi}e^{2\pi\sqrt{q}} + Be^{\sqrt{q}\phi}e^{2\pi\sqrt{q}} \\
2\pi\sqrt{q} &= 1 \\
q &= -n^2,\, n = 0, 1, 2, \ldots
\end{aligned}
$$

The eigenfunctions are $f = Ae^{\pm\sqrt{q}\phi}$ and the eigenvalues are $q = 0, 1, 2, \ldots$ The spectrum is degenerate as there are two eigenfunctions associated with each eigenvalue $q > 0$.