Advanced Engineering Mathematics Systems of Differential Equations by Dennis G. Zill Problems

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Contents

10 Systems of Linear Differential Equations

10.1 Theory of Linear Systems

10.1.1

$$\mathbf{X}' = \begin{pmatrix} 3 & -5 \\ 4 & 8 \end{pmatrix} \mathbf{X}$$

10.1.3

$$\mathbf{X}' = \begin{pmatrix} -3 & 4 & -9 \\ 6 & -1 & 0 \\ 10 & 4 & 3 \end{pmatrix} \mathbf{X}$$

10.1.5

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} t-1 \\ -3t^2 \\ t^2 - t + 2 \end{pmatrix}$$

10.1.7

$$\begin{aligned} \frac{dx}{dt} &= 4x + 2y + e^t \\ \frac{dy}{dt} &= -x + 3y - e^t \end{aligned}$$

10.1.9

$$\begin{aligned} \frac{dx}{dt} &= x - y + 2z + e^{-t} - 3t \\ \frac{dy}{dt} &= 3x - 4y + z + 2e^{-t} + t \\ \frac{dz}{dt} &= -2x + 5y + 6z + 2e^{-t} - t \end{aligned}$$

10.1.11

$$\begin{aligned}
3(e^{-5t}) - 4(2e^{-5t}) &= -5e^{-5t} \\
&= \frac{dx}{dt} \\
4(e^{-5t}) - 7(2e^{-5t}) &= -10e^{-5t} \\
&= \frac{dy}{dt}
\end{aligned}$$

10.1.13

$$\begin{aligned}
-(-e^{-3t/2}) + \frac{1}{4}(2e^{-3t/2}) &= \frac{3}{2}e^{-3t/2} \\
&= \frac{dx}{dt} \\
(-e^{-3t/2}) - (2e^{-3t/2}) &= -3e^{-3t/2} \\
&= \frac{dy}{dt}
\end{aligned}$$

10.1.17

$$\begin{aligned}
W(\mathbf{X}_1, \mathbf{X}_2) &= \begin{vmatrix} e^{-2t} & e^{-6t} \\ e^{-2t} & -e^{-6t} \end{vmatrix} \\
&= -e^{-8t} - e^{-8t} \\
&= -2e^{-8t} \\
&\neq 0 \text{ for } t \in (-\infty, \infty)
\end{aligned}$$

Yes, they form a fundamental set.

10.1.19

$$\begin{aligned}
W(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) &= \begin{vmatrix} 1+t & 1 & 3+2t \\ -2+2t & -2 & -6+4t \\ 4+2t & 4 & 12+4t \end{vmatrix} \\
&= 0
\end{aligned}$$

No, they don't form a fundamental set.

10.1.21

$$\begin{aligned}
x &= 2t + 5 \\
y &= -t + 1 \\
\frac{dx}{dt} &= (2t + 5) + 4(-t + 1) + 2t - 7 \\
&= 2 \\
\frac{dy}{dt} &= 3(2t + 5) + 2(-t + 1) - 4t - 18 \\
&= -1
\end{aligned}$$

10.1.23

$$\begin{aligned}
x &= e^t + te^t \\
x' &= 2e^t + te^t \\
y &= e^t - te^t \\
y' &= -te^t \\
\frac{dx}{dt} &= 2(e^t + te^t) + (e^t - te^t) - e^t \\
&= 2e^t + te^t \\
\frac{dy}{dt} &= 3(e^t + te^t) + 4(e^t - te^t) - 7e^t \\
&= -te^t
\end{aligned}$$

10.2 Homogeneous Linear Systems

10.2.1

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

10.2.3

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} e^{t}$$

10.2.5

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-10t} + c_2 \begin{pmatrix} 5 \\ 2 \end{pmatrix} e^{8t}$$

10.2.7

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t}$$

10.2.13

$$\mathbf{X} = 2\begin{pmatrix}0\\1\end{pmatrix}e^{-t/2} + 3\begin{pmatrix}1\\1\end{pmatrix}e^{t/2}$$

10.2.15

(a)

$$\frac{dx_1}{dt} = -\frac{3}{100}x_1 + \frac{1}{100}x_2$$
$$\frac{dx_2}{dt} = \frac{2}{100}x_1 - \frac{2}{100}x_2$$
$$\begin{pmatrix}\frac{dx_1}{dt}\\ \frac{dx_2}{dt}\end{pmatrix} = \begin{pmatrix}-\frac{3}{100} & \frac{1}{100}\\ \frac{2}{100} & -\frac{2}{100}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

(b)

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = -\frac{35}{3}\begin{pmatrix}-1\\1\end{pmatrix}e^{-t/25} + \frac{50}{3}\begin{pmatrix}\frac{1}{2}\\1\end{pmatrix}e^{-t/100}$$

10.2.21

$$\mathbf{X} = c_1\begin{pmatrix}1\\3\end{pmatrix} + c_2\left[\begin{pmatrix}1\\3\end{pmatrix}t + \begin{pmatrix}0\\-1\end{pmatrix}\right]$$

10.2.23

$$\mathbf{X} = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix}1\\1\end{pmatrix}t + \begin{pmatrix}0\\\frac{1}{3}\end{pmatrix}\right]e^{2t}$$

10.2.25

$$\mathbf{X} = c_1\begin{pmatrix}1\\0\\1\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix}e^{2t} + c_3\begin{pmatrix}1\\1\\1\end{pmatrix}e^{t}$$

10.2.31

$$\mathbf{X} = -\frac{1}{2}\begin{pmatrix}2\\1\end{pmatrix}e^{4t} + 13\left[\begin{pmatrix}2\\1\end{pmatrix}t + \begin{pmatrix}0\\\frac{1}{2}\end{pmatrix}\right]e^{4t}$$

10.2.33

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{K}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

10.2.35

$$\begin{aligned}
\mathbf{X} &= c_1 \begin{pmatrix} 1 \\ 2-i \end{pmatrix} e^{(4+i)t} + c_2 \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{(4-i)t} \\
&= c_1 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin t \right] e^{4t} + c_2 \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t \right] e^{4t} \\
&= c_1 \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix} e^{4t}
\end{aligned}$$

10.2.37

$$\begin{aligned}
\mathbf{X} &= c_1 \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{(4+i)t} + c_2 \begin{pmatrix} 1 \\ -1-i \end{pmatrix} e^{(4-i)t} \\
&= c_1 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t \right] e^{4t} + c_2 \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin t \right] e^{4t} \\
&= c_1 \begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix} e^{4t}
\end{aligned}$$

10.2.39

$$\begin{aligned}\mathbf{X} &= c_1 \begin{pmatrix} 5 \\ 4-3i \end{pmatrix} e^{3i} + c_2 \begin{pmatrix} 5 \\ 4+3i \end{pmatrix} e^{-3i} \\ &= c_1 \left[\begin{pmatrix} 5 \\ 4 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ -3 \end{pmatrix} \sin 3t \right] + c_2 \left[\begin{pmatrix} 0 \\ -3 \end{pmatrix} \cos 3t + \begin{pmatrix} 5 \\ 4 \end{pmatrix} \sin 3t \right] \\ &= c_1 \begin{pmatrix} 5\cos 3t \\ 4\cos 3t + 3\sin 3t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin 3t \\ 4\sin 3t - 3\cos 3t \end{pmatrix}\end{aligned}$$

10.2.47

$$\begin{aligned}\mathbf{X} &= c_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1+5i \\ 1 \\ 1 \end{pmatrix} e^{5it} + c_3 \begin{pmatrix} 1-5i \\ 1 \\ 1 \end{pmatrix} e^{-5it} \\ &= c_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cos 5t - \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \sin 5t \right] \\ &\quad + c_3 \left[\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \cos 5t + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \sin 5t \right] \\ &= c_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t + c_2 \begin{pmatrix} \cos 5t - 5\sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + c_3 \begin{pmatrix} 5\cos 5t + \sin 5t \\ \sin 5t \\ \sin 5t \end{pmatrix} \\ &= - \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t - \begin{pmatrix} \cos 5t - 5\sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + 6 \begin{pmatrix} 5\cos 5t + \sin 5t \\ \sin 5t \\ \sin 5t \end{pmatrix}\end{aligned}$$

10.2.49

(a)

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} -\frac{1}{20} & 0 & \frac{1}{10} \\ \frac{1}{20} & -\frac{1}{20} & 0 \\ 0 & \frac{1}{20} & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(b)

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1-i \\ i \\ 1 \end{pmatrix} e^{\left(-\frac{1}{10}+\frac{1}{20}i\right)t} + c_3 \begin{pmatrix} -1+i \\ -i \\ 1 \end{pmatrix} e^{\left(-\frac{1}{10}-\frac{1}{20}i\right)t}$$

$$= c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \left[\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cos \frac{1}{20}t - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \sin \frac{1}{20}t \right] e^{-t/10}$$

$$+ c_3 \left[\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cos \frac{1}{20}t + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \sin \frac{1}{20}t \right] e^{-t/10}$$

$$= c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \sin \frac{t}{20} - \cos \frac{t}{20} \\ -\sin \frac{t}{20} \\ \cos \frac{t}{20} \end{pmatrix} e^{-t/10}$$

$$+ c_3 \begin{pmatrix} -\cos \frac{t}{20} - \sin \frac{t}{20} \\ \cos \frac{t}{20} \\ \sin \frac{t}{20} \end{pmatrix} e^{-t/10}$$

$$= 11 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - 6 \begin{pmatrix} \sin \frac{t}{20} - \cos \frac{t}{20} \\ -\sin \frac{t}{20} \\ \cos \frac{t}{20} \end{pmatrix} e^{-t/10}$$

$$- 2 \begin{pmatrix} -\cos \frac{t}{20} - \sin \frac{t}{20} \\ \cos \frac{t}{20} \\ \sin \frac{t}{20} \end{pmatrix} e^{-t/10}$$

10.3 Solution by Diagonalization

10.3.1

$$\mathbf{X} = \begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 e^{7t} \\ c_2 e^{-4t} \end{pmatrix} = \begin{pmatrix} 3c_1 e^{7t} - 2c_2 e^{-4t} \\ c_1 e^{7t} + 3c_2 e^{-4t} \end{pmatrix}$$

10.3.3

$$\mathbf{X} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_1 e^{3t/2} \\ c_2 e^{t/2} \end{pmatrix} = \begin{pmatrix} c_1 e^{3t/2} + c_2 e^{t/2} \\ 2c_1 e^{3t/2} - 2c_2 e^{t/2} \end{pmatrix}$$

10.3.5

$$\mathbf{X} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{6t} \\ c_2 e^{-4t} \\ c_3 e^{2t} \end{pmatrix} = \begin{pmatrix} -c_2 e^{-4t} + c_3 e^{2t} \\ c_2 e^{-4t} + c_3 e^{2t} \\ c_1 e^{6t} + c_3 e^{2t} \end{pmatrix}$$

10.3.11

(a)

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\mathbf{M}$ has an inverse because it has a nonzero determinant (the product of the diagonal entries).

(b)

$$\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & \frac{k_2}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

(c)

$$\mathbf{X} = \mathbf{PY}$$

$$\begin{aligned} \mathbf{PY}'' + \mathbf{BPY} &= \mathbf{0} \\ \mathbf{Y}'' + \mathbf{P}^{-1}\mathbf{BPY} &= \mathbf{0} \\ \mathbf{Y}'' + \mathbf{DY} &= \mathbf{0} \\ \begin{pmatrix} y_1'' \\ y_2'' \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \mathbf{0} \\ y_1'' + 6y_1 &= 0 \\ y_1 &= c_1 \cos\sqrt{6}t + c_2 \sin\sqrt{6}t \\ y_2'' + y_2 &= 0 \\ y_2 &= c_3 \cos t + c_4 \sin t \end{aligned}$$

$$\begin{aligned} \mathbf{X} &= \mathbf{PY} \\ &= \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \cos\sqrt{6}t + c_2 \sin\sqrt{6}t \\ c_3 \cos t + c_4 \sin t \end{pmatrix} \\ &= \begin{pmatrix} -2c_1 \cos\sqrt{6}t - 2c_2 \sin\sqrt{6}t + c_3 \cos t + c_4 \sin t \\ c_1 \cos\sqrt{6}t + c_2 \sin\sqrt{6}t + 2c_3 \cos t + 2c_4 \sin t \end{pmatrix} \\ &= c_3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos t + c_4 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t + c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cos\sqrt{6}t + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \sin\sqrt{6}t \end{aligned}$$