

Introduction to Quantum Mechanics by David J. Griffiths Problems

Chris Doble

March 2023

Contents

I	Theory	1
1	The Wave Function	2
1.1	2
1.2	3
1.3	4
1.4	5
1.5	6
1.6	7
1.8	7

Part I

Theory

1 The Wave Function

1.1

(a)

$$\begin{aligned}\langle j^2 \rangle &= \sum j^2 P(j) \\ &= 14^2 \frac{1}{14} + 15^2 \frac{1}{14} + 16^2 \frac{3}{14} + 22^2 \frac{2}{14} + 24^2 \frac{2}{14} + 25^2 \frac{5}{14} \\ &= \frac{3217}{7} \\ &\approx 459.571 \\ \langle j \rangle^2 &= \left(\sum j P(j) \right)^2 \\ &= 441\end{aligned}$$

(b)

$$\begin{aligned}\Delta j_{14} &= -7 \\ \Delta j_{15} &= -6 \\ \Delta j_{16} &= -5 \\ \Delta j_{22} &= 1 \\ \Delta j_{24} &= 3 \\ \Delta j_{25} &= 4 \\ \sigma^2 &= \sum (\Delta j)^2 P(j) \\ &= \frac{130}{7} \\ &\approx 18.571\end{aligned}$$

(c)

$$\sigma^2 = \sqrt{\langle j^2 \rangle - \langle j \rangle^2} = 18.571$$

1.2

(a)

$$\begin{aligned}
 \langle x^2 \rangle &= \int_0^h x^2 \rho(x) dx \\
 &= \int_0^h \frac{x^{3/2}}{2\sqrt{h}} dx \\
 &= \frac{1}{2\sqrt{h}} \left[\frac{2}{5} x^{5/2} \right]_0^h \\
 &= \frac{h^2}{5} \\
 \langle x \rangle^2 &= \frac{h^2}{9} \\
 \sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
 &= \sqrt{\frac{h^2}{5} - \frac{h^2}{9}} \\
 &= h \sqrt{\frac{4}{45}} \\
 &= \frac{2}{3\sqrt{5}} h
 \end{aligned}$$

(b)

$$\begin{aligned}
 1 - \int_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} \rho(x) dx &= 1 - \frac{1}{2\sqrt{h}} [2\sqrt{x}]_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} \\
 &= 1 - \frac{1}{\sqrt{h}} \left(\sqrt{\frac{1}{3}h + \frac{2}{3\sqrt{5}}h} - \sqrt{\frac{1}{3}h - \frac{2}{3\sqrt{5}}h} \right) \\
 &= 1 - \left(\sqrt{\frac{1}{3} + \frac{2}{3\sqrt{5}}} - \sqrt{\frac{1}{3} - \frac{2}{3\sqrt{5}}} \right) \\
 &\approx 0.393
 \end{aligned}$$

1.3

(a)

$$\begin{aligned}\rho(x) &= A e^{-\lambda(x-a)^2} \\ 1 &= \int_{-\infty}^{\infty} \rho(x) dx \\ &= A \int_{-\infty}^{\infty} e^{-\lambda(x-a)^2} dx \\ &= A \sqrt{\frac{\pi}{\lambda}} \\ A &= \sqrt{\frac{\lambda}{\pi}}\end{aligned}$$

(b)

$$\begin{aligned}\langle x \rangle &= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x e^{-\lambda(x-a)^2} dx \\ &= a \\ \langle x^2 \rangle &= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2} dx \\ &= a^2 + \frac{1}{2\lambda} \\ \sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\ &= \sqrt{a^2 + \frac{1}{2\lambda} - a^2} \\ &= \frac{1}{\sqrt{2\lambda}}\end{aligned}$$

1.4

(a)

$$\begin{aligned}
 1 &= \int_{-\infty}^{\infty} |\Psi(x, 0)|^2 dx \\
 &= \left(\frac{A}{a}\right)^2 \int_0^a x^2 dx + \left(\frac{A}{b-a}\right)^2 \int_a^b (b-x)^2 dx \\
 &= \frac{1}{3}A^2a + \left(\frac{A}{b-a}\right)^2 \left[-\frac{1}{3}(b-x)^3\right]_a^b \\
 &= \frac{1}{3}A^2a + \frac{1}{3}A^2(b-a) \\
 &= \frac{1}{3}A^2b \\
 A &= \sqrt{\frac{3}{b}}
 \end{aligned}$$

(c) $x = a$

(d)

$$\begin{aligned}
 \int_0^a |\Psi(x, 0)|^2 dx &= \frac{3}{a^2b} \left[\frac{1}{3}x^3\right]_0^a \\
 &= \frac{a}{b}
 \end{aligned}$$

(e)

$$\begin{aligned}
 \langle x \rangle &= \int_{-\infty}^{\infty} x |\Psi(x, 0)|^2 dx \\
 &= \frac{3}{a^2b} \left[\frac{1}{4}x^4\right]_0^a + \frac{3}{b(b-a)^2} \int_a^b x(b-x)^2 dx \\
 &= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \int_a^b (b^2x - 2bx^2 + x^3) dx \\
 &= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \left[\frac{1}{2}b^2x^2 - \frac{2}{3}bx^3 + \frac{1}{4}x^4\right]_a^b \\
 &= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \left(\frac{1}{2}b^4 - \frac{2}{3}b^4 + \frac{1}{4}b^4 - \frac{1}{2}a^2b^2 + \frac{2}{3}a^3b - \frac{1}{4}a^4\right) \\
 &= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \frac{1}{12}(b-a)^3(3a+b) \\
 &= \frac{3a^2}{4b} + \frac{1}{4b}(3ab + b^2 - 3a^2 - ab) \\
 &= \frac{1}{2}a + \frac{1}{4}b
 \end{aligned}$$

1.5

(a)

$$\Psi(x, t) = Ae^{-\lambda|x|}e^{-i\omega t}$$

$$\Psi(x, 0) = Ae^{-\lambda|x|}$$

$$\begin{aligned} 1 &= A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|} dx \\ &= 2A^2 \int_0^{\infty} e^{-2\lambda x} dx \\ &= 2A^2 \left[-\frac{1}{2\lambda} e^{-2\lambda x} \right]_0^{\infty} \\ &= \frac{A^2}{\lambda} \\ A &= \sqrt{\lambda} \end{aligned}$$

(b)

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{\infty} x \lambda e^{-2\lambda|x|} dx \\ &= \lambda \int_{-\infty}^{\infty} x e^{-2\lambda|x|} dx \\ &= 0 \\ \langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 \lambda e^{-2\lambda|x|} dx \\ &= 2\lambda \int_0^{\infty} x^2 e^{-2\lambda x} dx \\ &= \frac{1}{2\lambda^2} \end{aligned}$$

(c)

$$\begin{aligned} \sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\ &= \frac{1}{\sqrt{2}\lambda} \\ 1 - \int_{-\sigma}^{\sigma} \lambda e^{-2\lambda|x|} dx &= 1 - 2\lambda \int_0^{\sigma} e^{-2\lambda x} dx \\ &= 1 - 2\lambda \left[-\frac{1}{2\lambda} e^{-2\lambda x} \right]_0^{\sigma} \\ &= e^{-2\lambda\sigma} \\ &= e^{-\sqrt{2}} \\ &\approx 0.243 \end{aligned}$$

1.6

The chain rule requires that you apply it to both x and $|\Psi|^2$ which gives the same result

$$\begin{aligned}
 \frac{d\langle x \rangle}{dt} &= \frac{d}{dt} \int x |\Psi|^2 dx \\
 &= \int \frac{d}{dt} (x |\Psi|^2) dx \\
 &= \int \left(0 \cdot |\Psi|^2 + x \frac{\partial |\Psi|^2}{\partial t} \right) dx \\
 &= \int x \frac{\partial |\Psi|^2}{\partial t} dx
 \end{aligned}$$

1.8

$$\begin{aligned}
 i\hbar \frac{\partial}{\partial t} \left(e^{-iV_0 t/\hbar} \Psi \right) &= -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \left(e^{-iV_0 t/\hbar} \Psi \right) + (V + V_0) \left(e^{-iV_0 t/\hbar} \Psi \right) \\
 i\hbar \left(-\frac{iV_0}{\hbar} e^{-iV_0 t/\hbar} \Psi + e^{-iV_0 t/\hbar} \frac{\partial \Psi}{\partial t} \right) &= -\frac{\hbar^2}{2m} e^{-iV_0 t/\hbar} \frac{\partial^2 \Psi}{\partial x^2} + V e^{-iV_0 t/\hbar} \Psi + V_0 e^{-iV_0 t/\hbar} \Psi \\
 V_0 \Psi + i\hbar \frac{\partial \Psi}{\partial t} &= -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V \Psi + V_0 \Psi \\
 i\hbar \frac{\partial \Psi}{\partial t} &= -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V \Psi
 \end{aligned}$$

$$\begin{aligned}
 \langle Q(x, p) \rangle &= \int \left(e^{-iV_0 t/\hbar} \Psi \right)^* [Q(x, -i\hbar \partial/\partial x)] e^{-iV_0 t/\hbar} \Psi dx \\
 &= \int e^{iV_0 t/\hbar} \Psi^* [Q(x, -i\hbar \partial/\partial x)] e^{-iV_0 t/\hbar} \Psi dx \\
 &= \int \Psi^* [Q(x, -i\hbar \partial/\partial x)] \Psi dx
 \end{aligned}$$

No effect on the expectation value.