

Advanced Engineering Mathematics Ordinary Differential Equations Notes

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Contents

1	Introduction to Differential Equations	2
1.1	Definitions and Terminology	2
1.2	Initial Value Problems	4
1.3	Differential Equations as Mathematical Models	4
2	First-Order Differential Equations	5
2.1	Solution Curves Without a Solution	5
2.2	Separable Equations	6
2.3	Linear Equations	7
2.4	Exact Equations	8
2.5	Solutions by Substitution	9
2.6	A Numerical Method	10
2.9	Modeling with Systems of First-Order DEs	11
3	Higher-Order Differential Equations	11
3.1	Theory of Linear Equations	11
3.2	Reduction of Order	13
3.3	Homogeneous Linear Equations with Constant Coefficients	14
3.4	Undetermined Coefficients	15
3.5	Variation of Parameters	15
3.6	Cauchy-Euler Equations	16
3.7	Nonlinear Equations	17
3.10	Green's Functions	18
3.10.1	Initial-Value Problems	18
3.10.2	Boundary Value Problems	18
3.12	Solving Systems of Linear Equations	19
4	The Laplace Transform	19
4.1	Definition of the Laplace Transform	19
4.2	The Inverse Transform and Transforms of Derivatives	20
4.3	Translation Theorems	20

4.4	Additional Operational Properties	21
4.5	The Dirac Delta Function	22
5	Series Solutions of Linear Differential Equations	23
5.1	Solutions about Ordinary Points	23
5.2	Solutions about Singular Points	24
5.3	Special Functions	26
5.3.1	Bessel Functions	26
6	Numerical Solutions of Ordinary Differential Equations	27
6.1	Euler Methods and Error Analysis	27
6.2	Runge-Kutta Methods	28
6.3	Multistep Methods	29

1 Introduction to Differential Equations

1.1 Definitions and Terminology

- An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation** (DE)
- An **ordinary DE** (ODE) is a DE that contains only ordinary (i.e. non-partial) derivatives of one or more functions with respect to a single independent variable
- A **partial DE** is a DE that contains only partial derivatives of one or more functions of two or more independent variables
- The **order** of a DE is the order of the highest derivative in the equation
- First order ODEs are sometimes written in the **differential form**

$$M(x, y) dx + N(x, y) dy = 0$$

- n -th order ODEs in one dependent variable can be expressed by the **general form**

$$F(x, y, y', \dots, y^{(n)}) = 0$$

- It's possible to solve ODEs in the general form uniquely for the highest derivative $y^{(n)}$ in terms of the other $n + 1$ variables, allowing them to be expressed in the **normal form**

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)})$$

- An n -th order ODE is said be **linear** in the variable y if it can be expressed in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - g(x) = 0$$

i.e. the dependent variable y and all of its derivatives aren't raised to a power or used in nonlinear functions like e^y or $\sin y$, and the coefficients a_0, a_1, \dots, a_n depend at most on the independent variable x

- A **nonlinear** ODE is one that is not linear
- A **solution** to an ODE is a function ϕ , defined on an interval I and possessing at least n derivatives that are continuous on I , such that

$$F(x, \phi(x), \phi'(x), \dots, \phi^n(x)) = 0 \text{ for all } x \text{ in } I.$$

- The **interval of definition**, **interval of validity**, or the **domain** of a solution is the interval over which the solution is valid
- A solution of a DE that is 0 on an interval I is said to be a **trivial solution**
- Because solutions to DEs must be differentiable over their interval of validity, discontinuities, etc. must be excluded from the interval
- An **explicit solution** to an ODE is one where the dependent variable is expressed solely in terms of the independent variable and constants
- An **implicit solution** to an ODE is a relation $G(x, y) = 0$ over an interval I provided there exists at least one function ϕ that satisfies the relation as well as the ODE on I
- When solving a first-order ODE we usually obtain a solution containing a single arbitrary constant or parameter c . A solution containing an arbitrary constant represents a set of solution called a **one-parameter family of solutions**
- When solving an n -th order DE we usually obtain an **n -parameter family of solutions**
- A solution of a DE that is free from arbitrary parameters is called a **particular solution**
- A **singular solution** is a solution to a DE that isn't a member of a family of solutions
- A **system of ODEs** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable. A solution of such a system is a differentiable function for each equation defined on a common interval I that satisfy each equation of the system on that interval

1.2 Initial Value Problems

- An **initial value problem** is the problem of solving a DE with some given **initial conditions**, e.g. solve

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)})$$

subject to

$$y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$$

- The domain of $y = f(x)$ differs depending on how it's considered:
 - As a function its domain is all real numbers for which it's defined
 - As a solution of a DE its domain is a single interval over which it's defined and differentiable
 - As a solution of an initial value problem its domain is a single interval over which it's defined, differentiable, and contains the initial conditions
- An initial value problem may not have any solutions. If it does it may have multiple.
- First-order initial value problems of the form

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

are guaranteed to have a unique solution over an interval I containing x_0 if $f(x, y)$ and $\partial f / \partial y$ are continuous

1.3 Differential Equations as Mathematical Models

- A **mathematical model** is a mathematical description of a system or phenomenon
- The **level of resolution** of a model determines how many variables are included in the model
- A simple model of the growth of a population P is

$$\frac{dP}{dt} = kP$$

where $k > 0$

- A simple model of radioactive decay of an amount of substance A is

$$\frac{dA}{dt} = kA$$

where $k < 0$

- Newton's empirical law of cooling/warming states that the rate of change of the temperature of a body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium

$$\frac{dT}{dt} = k(T - T_m)$$

2 First-Order Differential Equations

2.1 Solution Curves Without a Solution

- An ODE in which the independent variable doesn't appear is said to be **autonomous**, e.g.

$$\frac{dy}{dx} = f(y)$$

- A real number c is a **critical/equilibrium/stationary point** of an autonomous DE if it is a zero of f
- If c is a critical point of an autonomous DE, then $y(x) = c$ is a solution
- A solution of the form $y(x) = c$ is called an **equilibrium solution**
- We can draw several conclusions about the solutions of an autonomous DE with n critical points and $n + 1$ subregions bounded by the critical points:
 - If (x_0, y_0) is in a subregion, it remains in that subregion for all x
 - By continuity, $f(y) < 0$ or $f(y) > 0$ for all y in a subregion and thus $y(x)$ can't have maximum/minimum points or oscillate
 - If $y(x)$ is bounded above by a critical point c_1 , it must approach $y(x) = c_1$ as $x \rightarrow -\infty$ or $x \rightarrow \infty$
 - If $y(x)$ is bounded above and below by critical points c_1 and c_2 , it must approach $y(x) = c_1$ as $x \rightarrow -\infty$ and $y(x) = c_2$ as $x \rightarrow \infty$ or vice versa
 - If $y(x)$ is bounded below by a critical point c_1 , it must approach $y(x) = c_1$ as $x \rightarrow -\infty$ or $x \rightarrow \infty$

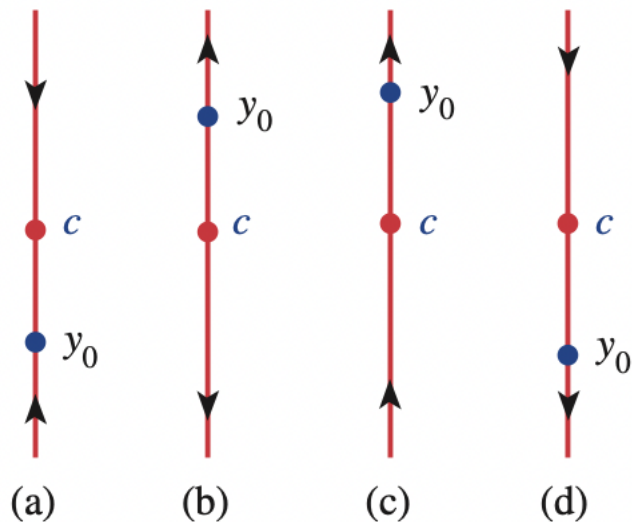


FIGURE 2.1.8 Critical point c is an attractor in (a), a repeller in (b), and semi-stable in (c) and (d)

- If $y(x)$ is a solution of an autonomous differential equation $dy/dx = f(y)$, then $y_1(x) = y(x - k)$, where k is a constant, is also a solution

2.2 Separable Equations

- A first-order ODE of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be **separable** or to have **separate variables**

- A separable first-order ODE can be solved by dividing both sides by $h(y)$ then integrating both sides with respect to x

$$\begin{aligned}
\frac{dy}{dx} &= g(x)h(y) \\
\frac{1}{h(y)} \frac{dy}{dx} &= g(x) \\
\int \frac{1}{h(y)} \frac{dy}{dx} dx &= \int g(x) dx \\
\int \frac{1}{h(y)} dy &= \int g(x) dx \\
H(y) &= G(x) + c
\end{aligned}$$

- Care should be taken when dividing by $h(y)$ as it removes constant solutions $y = r$ where $h(r) = 0$

2.3 Linear Equations

- A first-order DE of the form

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

or in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

is said to be a **linear equation** in the dependent variable y

- When $g(x) = 0$ or $f(x) = 0$ the linear equation is said to be **homogeneous** and is solvable via separation of variables, otherwise it is **nonhomogeneous**
- The nonhomogeneous linear equation's solution is the sum of two solutions $y = y_c + y_p$ where y_c is a solution of the associated homogeneous equation

$$\frac{dy}{dx} + P(x)y = 0$$

and y_p is a particular solution of the nonhomogeneous equation

- Nonhomogeneous linear equations can be solved via **variation of parameters**:
 1. Put it into standard form
 2. Determine the **integrating factor** $e^{\int P(x) dx}$
 3. Multiply by the integrating factor
 4. Recognise that the left hand side of the equation is the derivative of the product of the integrating factor and y

5. Integrate both sides of the equation

6. Solve for y

- The **general solution** of a DE is a family of solutions that contains all possible solutions (except singular solutions)
- A term $y = f(x)$ in a solution is called a **transient term** if $f(x) \rightarrow 0$ as $x \rightarrow \infty$
- When either $P(x)$ or $f(x)$ is a piecewise-defined function the equation is then referred to as a **piecewise-linear differential equation** that can be solved by solving each interval in isolation then choosing appropriate constants to ensure the overall solution is continuous
- The **error function** and **complementary error function** are defined

$$\operatorname{erf} x + \operatorname{erfc} x = 1$$
$$\left(\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \right) + \left(\frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \right) = 1$$

2.4 Exact Equations

- The **differential** of a function $z = f(x, y)$ is

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

- A differential expression $M(x, y) dx + N(x, y) dy$ is an **exact differential** in the region R of the xy -plane if it corresponds to the differential of some function $f(x, y)$
- A first-order DE of the form

$$M(x, y) dx + N(x, y) dy = 0$$

is said to be an **exact equation** if the expression on the left side is an exact differential

- A necessary and sufficient condition that $M(x, y) dx + N(x, y) dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- Exact differentials can be solved by

1. Integrating $M(x, y)$ with respect to x to find an expression for $f(x, y)$

$$\frac{\partial f}{\partial x} = M(x, y)$$

$$f(x, y) = \int M(x, y) dx + g(y)$$

2. Differentiating $f(x, y)$ with respect to y and equating it to $N(x, y)$ to find $g'(y)$

$$\frac{\partial f}{\partial y} = N(x, y) = \frac{\partial}{\partial y} \int M(x, y) dx + g'(y)$$

$$g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx$$

3. Integrating $g'(y)$ with respect to y to find $g(y)$ and substituting it into $f(x, y)$
4. Equating $f(x, y)$ with an unknown constant c

- x and y can be swapped in the steps above (i.e. you can start by integrating $N(x, y)$ with respect to y , etc.)
- A nonexact DE $M(x, y) dx + N(x, y) dy = 0$ can sometimes be transformed into an exact DE by finding an appropriate integrating factor

- If $(M_y - N_x)/N$ is a function of x alone, then an integrating factor is

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

- If $(N_x - M_y)/M$ is a function of y alone, then an integrating factor is

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

2.5 Solutions by Substitution

- A function $f(x, y)$ is said to be a **homogeneous function** of degree α if

$$f(tx, ty) = t^\alpha f(x, y)$$

- A first-order DE of the form

$$M(x, y) dx + N(x, y) dy = 0$$

is said to be **homogeneous** if both M and N are homogeneous functions of the same degree

- To solve a homogeneous first-order DE:

1. Rewrite it as

$$M(x, y) = x^\alpha M(1, u) \text{ and } N(x, y) = x^\alpha N(1, u) \text{ where } u = y/x$$

or

$$M(x, y) = y^\alpha M(v, 1) \text{ and } N(x, y) = y^\alpha N(v, 1) \text{ where } v = x/y$$

2. Substitute $y = ux$ and $dy = u dx + x du$ or $x = vy$ and $dx = v dy + y dv$ as appropriate
3. Solve the resulting first-order separable DE
4. Substitute $u = y/x$ or $v = x/y$ as appropriate

- The DE

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

where n is any real number is called **Bernoulli's equation**

- For $n = 0$ and $n = 1$ Bernoulli's equation is linear
- To solve Bernoulli's equation for $n \neq 0$ and $n \neq 1$:

1. Substitute $y = u^{1/(1-n)}$ and $\frac{dy}{dx} = \frac{d}{dx}(u^{1/(1-n)})$
2. Solve the resulting linear equation
3. Substitute $u = y^{n-1}$

- A DE of the form

$$\frac{dy}{dx} = f(Ax + By + C)$$

can always be reduced to an equation with separable variables by means of the substitution

$$u = Ax + By + C, B \neq 0$$

2.6 A Numerical Method

- Approximate values for points on a solution curve near an initial point can be calculated via a **linearization** of the solution curve — a straight line that has the same slope as the initial point and passes through it
- **Euler's method** approximates a solution curve by iteratively stepping along its linearizations

$$y_{n+1} = y_n + hf(x_n, y_n)$$

where h is the **step size**

2.9 Modeling with Systems of First-Order DEs

- In a system of DEs

$$\frac{dx}{dt} = g_1(t, x, y)$$

and

$$\frac{dy}{dt} = g_2(t, x, y)$$

if g_1 and g_2 are linear in x and y , i.e.

$$g_1(t, x, y) = c_1x + c_2y + f_1(t)$$

and

$$g_2(t, x, y) = c_3x + c_4y + f_2(t)$$

it is said to be a **linear system**

3 Higher-Order Differential Equations

3.1 Theory of Linear Equations

- An **n th-order initial-value problem (IVP)** is to solve

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

subject to

$$y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$$

- If $a_n(x)$, $a_{n-1}(x)$, \dots , $a_1(x)$, $a_0(x)$, and $g(x)$ are continuous on an interval I and $a_n(x) \neq 0$ for every x in the interval, then a unique solution exists for the above IVP for every $x = x_0$ within the interval
- An **initial value problem** is when all of the constraints are located at the same point while a **boundary value problem** is when they're at different points
- Boundary value problems may have many, one, or no solutions
- When $g(x) = 0$ the DE is said to be **homogeneous**, otherwise it's **non-homogeneous**
- The symbol D is called a **differential operator** because it transforms a differentiable function into another function

$$Dy = \frac{dy}{dx}$$

- Higher-order derivatives can be expressed as

$$D^n = \frac{d^n y}{dx^n}$$

- An **n th-order differential operator** is defined to be

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)$$

- As a consequence of the properties of differentiation

$$D(cf(x)) = cDf(x)$$

and

$$D\{f(x) + g(x)\} = Df(x) + Dg(x)$$

- The superposition principle for homogeneous linear n th-order differential equation states that if y_1, y_2, \dots, y_k are solutions of the equation on an interval I then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

where c_i are arbitrary constants is also a solution on the interval

- A set of functions $f_1(x), f_2(x), \dots, f_n(x)$ is said to be **linearly dependent** on an interval I if there exists constants c_1, c_2, \dots, c_n , not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every x in the interval. Otherwise it is said to be **linearly independent**

- The **Wronskian** of a set of n functions that are $n - 1$ times differentiable is defined as

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

- If y_1, y_2, \dots, y_n are n solutions to a homogeneous linear n th-order differential equation on an interval I then the set of solutions is **linearly independent** on I iff $W(y_1, y_2, \dots, y_n) \neq 0$ for every x in the interval
- Any set of n linearly independent solutions of a homogeneous linear n th-order differential equation on an interval I is said to be a **fundamental set of solutions** on the interval

- If y_1, y_2, \dots, y_n are a fundamental set of solutions of a homogeneous linear n th-order DE on an interval I then the **general solution** of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x)$$

where c_i are arbitrary constants

- Another way of saying the above is that any solution on the interval can be expressed as a linear combination of the fundamental set of solutions
- A linear combination of a fundamental set of solutions of a homogeneous linear n th-order DE

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x)$$

is called the **complementary function** of associated nonhomogeneous DEs

- If y_p is any particular solution to a nonhomogeneous linear n th-order DE on an interval I and y_1, y_2, \dots, y_n are a fundamental set of solutions of the associated homogeneous DE on I , then the **general solution** of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x) + y_p(x)$$

where c_i are arbitrary constants

- Another way of saying the above is that any solution on the interval can be expressed as $y = y_c + y_p$
- The superposition for nonhomogeneous linear n th-order differential equations states that if $y_{p_1}, y_{p_2}, \dots, y_{p_k}$ are k particular solutions of a nonhomogeneous linear n th-order differential equation on an interval I corresponding, in turn, to k distinct functions g_1, g_2, \dots, g_k , then

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x) + \dots + y_{p_k}(x)$$

is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_0(x)y = g_1(x) + g_2(x) + \dots + g_k(x)$$

3.2 Reduction of Order

- The **reduction of order** method requires knowledge of one non-trivial solution and comprises the following steps:

1. Recognise that the ratio of two linearly independent functions isn't constant, i.e.

$$u(x) = \frac{y_1(x)}{y_2(x)} \text{ or } y_2(x) = u(x)y_1(x)$$

2. Substitute $y_2(x) = u(x)y_1(x)$ into the DE — this will result in a DE involving only u'' and u' which can be treated as a linear first-order DE in $u' = w$
 3. Solve for w
 4. Substitute $w = u'$
 5. Integrate to find u
 6. Multiply by y_1 to find y_2
- A formula for the above on a DE in standard form

$$y'' + P(x)y' + Q(x)y = 0$$

is

$$y_2 = y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx$$

3.3 Homogeneous Linear Equations with Constant Coefficients

- All solutions to homogenous linear DEs

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where a_i are real constants and $a_n \neq 0$ are either exponential functions or constructed from exponential functions

- Substituting a solution $y = e^{mx}$ we find

$$e^{mx}(a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0) = 0$$

where the term in brackets is called the **auxiliary equation** of the DE

- Thus, the solution $y = e^{mx}$ is valid if m is a root of the auxiliary equation
- Real roots correspond to solutions of the form

$$y = ce^{mx}$$

- Complex roots $\alpha \pm i\beta$ correspond to solutions of the form

$$y_1 = c_1 e^{\alpha x} \cos \beta x \text{ and } y_2 = c_2 e^{\alpha x} \sin \beta x$$

- A root m of multiplicity k corresponds to the solutions

$$e^{mx}, xe^{mx}, x^2 e^{mx}, \dots, x^{k-1} e^{mx}$$

3.4 Undetermined Coefficients

- The **method of undetermined coefficients** may be used to find a particular solution to nonhomogenous linear differential equations where the input function is comprised of constants, polynomials, exponentials $e^{\alpha x}$, sines, and cosines
- To apply the method you:
 1. Solve the associated homogeneous equation
 2. Assume the particular solution has the same form as the input function
 3. If a term in the proposed solution is present in the complementary function, multiply it by x^n where n is the smallest positive integer that removes the duplication
 4. Substitute the proposed solution into the DE
 5. Solve for the unknown constants

TABLE 3.4.1 Trial Particular Solutions

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A \cos 4x + B \sin 4x$
6. $\cos 4x$	$A \cos 4x + B \sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. $x^2 e^{5x}$	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$
11. $5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
12. $xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$

3.5 Variation of Parameters

- The **method of variation of parameters** can be used to find a particular solution of a nonhomogeneous linear n th-order DE
- To apply the method you:
 1. Solve the homogeneous equation to find the complementary function
 2. Assume the solution has the form

$$y_p = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x)$$

where n is the order of the equation and y_i are the fundamental set of solutions from the complementary equation

3. Convert to standard form by dividing by the leading coefficient

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

4. Solve the system of linear equations

$$\begin{aligned}
 y_1 u'_1 + \cdots + y_n u'_n &= 0 \\
 y'_1 u'_1 + \cdots + y'_n u'_n &= 0 \\
 &\vdots \\
 y_1^{(n-1)} u'_1 + \cdots + y_n^{(n-1)} u'_n &= 0 \\
 y_1^{(n)} u'_1 + \cdots + y_n^{(n)} u'_n &= f(x)
 \end{aligned}$$

via Cramer's method:

(a) Compute the Wronskian of y_i

$$W = \begin{vmatrix} y_1 & \cdots & y_n \\ y'_1 & \cdots & y'_n \\ \vdots & \ddots & \vdots \\ y_1^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

(b) Compute u'_i for $i = 1, \dots, n$ where

$$u'_i = \frac{W_i}{W}$$

and W_i is the determinant of the matrix formed by replacing the i th column of the Wronskian matrix with the column vector

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(x) \end{bmatrix}$$

5. Integrate each u'_i to find u_i

3.6 Cauchy-Euler Equations

- A **Cauchy-Euler equation** is a linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

- To solve a homogeneous Cauchy-Euler equation you:

1. Assume the equation has a solution of the form $y = x^m$, giving

$$\begin{aligned}
 a_n x^n \frac{d^n y}{dx^n} &= a_n x^n m(m-1)(m-2) \cdots (m-n+1) x^{m-n} \\
 &= a_n m(m-1)(m-2) \cdots (m-n+1) x^m
 \end{aligned}$$

and the equation then becomes

$$f(m)x^m = 0$$

where $f(m)$ is a polynomial in m known as the auxiliary or characteristic equation, the roots of which form the general solution

2. Solve the auxiliary equation where

– A real root m corresponds to a solution

$$y = cx^m$$

– Complex roots $\alpha \pm i\beta$ correspond to solutions

$$x^\alpha (c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x))$$

– A root m of multiplicity k corresponds to solutions

$$x^m, x^m \ln x, x^m (\ln x)^2 \dots, x^m (\ln x)^{k-1}$$

- To solve a nonhomogeneous Cauchy-Euler equation you:
 1. Solve the associated homogeneous equation
 2. Find a particular solution via variation of parameters

3.7 Nonlinear Equations

- The superposition principle does not hold for nonlinear equations
- Nonlinear second order DEs of the form $F(x, y', y'') = 0$ where y is missing can sometimes be solved by:
 1. Substitute $u = y'$ (and thus $u' = y''$)
 2. Solve the resulting DE for u
 3. Integrate to find y
- Nonlinear second order DEs of the form $F(y, y', y'') = 0$ where x is missing can sometimes be solved by:

1. Substitute $u = y'$ and

$$y'' = \frac{du}{dy} \frac{dy}{dx} = u \frac{du}{dy}$$

2. Solve the resulting DE for u

3. Integrate to find y

- Nonlinear initial-value problems can sometimes be solved by substituting the initial conditions into a Taylor series centred at x_0 . The initial conditions can also be substituted into subsequent derivatives to add further terms to the series

3.10 Green's Functions

- Green's functions are useful because they allow you to express the solution of a DE in terms of the input function $g(x)$, making it easy to see how different input functions change the solution

3.10.1 Initial-Value Problems

- The solution of a second-order IVP

$$y'' + P(x)y' + Q(x)y = f(x), y(x_0) = y_0, y'(x_0) = y_1$$

can be expressed as

$$y = y_h + y_p$$

where y_h is the solution to the associated homogeneous equation with nonhomogeneous initial conditions

$$y'' + P(x)y' + Q(x)y = 0, y(x_0) = y_0, y'(x_0) = y_1$$

and y_p is the solution to the nonhomogeneous equation with homogeneous initial conditions

$$y'' + P(x)y' + Q(x)y = f(x), y(x_0) = 0, y'(x_0) = 0$$

- If $P(x)$ and $Q(x)$ are constant y_h can be found via the auxiliary / characteristic equation
- If y_1 and y_2 form a fundamental set of solutions to the associated homogeneous equation, then y_p is given by

$$y_p(x) = \int_{x_0}^x G(x, t) f(t) dt$$

where $G(x, y)$ is the Green's function for the differential equation

$$G(x, t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}$$

and $W(t)$ is the Wronskian

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

3.10.2 Boundary Value Problems

- If y_1 and y_2 are linearly independent solutions of

$$y'' + P(x)y' + Q(x)y = 0$$

on $[a, b]$ and satisfy the boundary conditions

$$A_1 y_1(a) + B_1 y_1(b) = 0$$

and

$$A_2 y_2(a) + B_2 y_2(b) = 0$$

then the BVP

$$y'' + P(x)y' + Q(x)y = f(x)$$

subject to the same boundary conditions has a particular solution

$$y_p(x) = \int_a^b G(x, t) f(t) dt$$

where $G(x, t)$ is the Green's function for the differential equation

$$G(x, y) = \begin{cases} \frac{y_1(t)y_2(x)}{W(t)} & a \leq t \leq x \\ \frac{y_1(x)y_2(t)}{W(t)} & x \leq t \leq b \end{cases}$$

and $W(t)$ is the Wronskian

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

3.12 Solving Systems of Linear Equations

- Systems of linear differential equations can be solved in a similar manner to systems of equations, namely by adding and subtracting multiples of different equations to eliminate particular variables
- We can also apply the differential operator D as part of the elimination process
- Once you have an equation for each dependent variable it's important to substitute them back into the original differential equation to determine the constraints on the parameters — not all of them can be chosen arbitrarily

4 The Laplace Transform

4.1 Definition of the Laplace Transform

- If a function $f(t)$ is defined for $t \geq 0$ and the limit

$$\int_0^\infty K(s, t) f(t) dt = \lim_{b \rightarrow \infty} \int_0^b K(s, t) f(t) dt$$

exist, the integral is said to **exist** or be **convergent**, otherwise it does not exist or is **divergent**

- If a function $f(t)$ is defined for $t \geq 0$ then the limit

$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t) dt$$

is called the **Laplace transform** of f providing the integral converges

- \mathcal{L} is a linear transform, i.e.

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$$

- A function is said to be **piecewise continuous** on $[0, \infty)$ if, in any interval defined by $0 \leq a \leq t \leq b$, there are at most a finite number of points t_k , $k = 1, 2, \dots, n$ ($t_{k-1} < t_k$), at which f has finite discontinuities and is continuous on each open interval defined by $t_{k-1} < t < t_k$
- A function is said to be of **exponential order** if there exists constants c , $M > 0$, and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$
- If $f(t)$ is piecewise continuous on the interval $[0, \infty)$ and of exponential order, then $\mathcal{L}\{f(t)\}$ exists for $s > c$

4.2 The Inverse Transform and Transforms of Derivatives

- \mathcal{L}^{-1} is a linear transform, i.e.

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathcal{L}^{-1}\{F(s)\} + \beta \mathcal{L}^{-1}\{G(s)\}$$

- If $f, f', \dots, f^{(n-1)}$ are continuous on $[0, \infty)$, are of exponential order, and $f^{(n)}$ is piecewise continuous on $[0, \infty)$, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$

where $F(s) = \mathcal{L}\{f(t)\}$

- The Laplace transform can be used to solve linear IVPs:
 1. Take the Laplace transform of the DE, resulting in an algebraic equation in $F(s) = \mathcal{L}\{f(s)\}$ where $f(s)$ is the goal
 2. Solve the equation for $F(s)$
 3. Apply the inverse Laplace transform to find $f(s)$

4.3 Translation Theorems

- The **first translation theorem** states that if

$$\mathcal{L}\{f(t)\} = F(s)$$

then

$$\mathcal{L}\{e^{at} f(t)\} = F(s - a)$$

and

$$\mathcal{L}^{-1}\{F(s - a)\} = \mathcal{L}^{-1}\{F(s)|_{s \rightarrow s-a}\} = e^{at} f(t)$$

- The **unit step function** or **Heaviside function** is defined to be

$$\mathcal{U}(t-a) = \begin{cases} 0 & 0 \leq t < a \\ 1 & t \geq a \end{cases}$$

- The **second translation theorem** states that if $a > 0$ and

$$\mathcal{L}\{f(s)\} = F(s)$$

then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

and

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$$

- If f and \mathcal{U} aren't shifted by the same amount when applying the second translation theorem, an alternate form can be applied

$$\mathcal{L}\{f(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$$

4.4 Additional Operational Properties

- If $F(s) = \mathcal{L}\{f(t)\}$ and $n = 1, 2, 3, \dots$, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

- If functions f and g are piecewise continuous on the interval $[0, \infty)$ then the **convolution** of f and g , denoted $f * g$, is a function defined by the integral

$$f * g = \int_0^t f(\tau)g(t-\tau) d\tau$$

- The **convolution theorem** states that if f and g are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$

and

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g$$

- Under the convolution theorem if $g(t) = 1$ then $\mathcal{L}\{g(t)\} = G(s) = 1/s$,

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{F(s)}{s},$$

and

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau) d\tau$$

- **Volterra integral equations** have the form

$$f(t) = g(t) + \int_0^t f(\tau)g(t-\tau) d\tau$$

and can be solved by using the convolution theorem while taking the Laplace transform

- An **integro-differential equation** is an equation that involves both integrals and derivatives of a function
- If $f(t)$ is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period T , then

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt$$

4.5 The Dirac Delta Function

- A **unit impulse** function is defined as

$$\delta_a(t - t_0) = \begin{cases} 0 & 0 \leq t < t_0 - a \\ \frac{1}{2a} & t_0 - a \leq t < t_0 + a \\ 0 & t_0 + a \leq t \end{cases}$$

and it possesses the property

$$\int_0^\infty \delta_a(t - t_0) dt = 1$$

- The **Dirac delta function** is defined as

$$\delta(t - t_0) = \lim_{a \rightarrow 0} \delta_a(t - t_0)$$

and has the properties

$$\delta(t - t_0) = \begin{cases} \infty & t = t_0 \\ 0 & t \neq t_0 \end{cases}$$

and

$$\int_0^\infty \delta(t - t_0) dt = 1$$

- For $t_0 > 0$

$$\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}$$

5 Series Solutions of Linear Differential Equations

5.1 Solutions about Ordinary Points

- A **power series** is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

- A point x_0 is said to be an **ordinary point** of the differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

if both $P(x)$ and $Q(x)$ are analytic at x_0 . A point that is not an ordinary point is said to be a **singular point** of the equation.

- If $x = x_0$ is an ordinary point of the differential equation above, we can always find two linearly independent solutions in the form of a power series centred at x_0 . Such a solution is said to be a **solution about the ordinary point** x_0
- A series solution converges at least on some interval $|x - x_0| < R$ where R is the distance from x_0 to the closest singular point
- A series solution can be found for a homogeneous linear second-order differential equation by

1. Assume the solution is of the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

and thus

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

2. Substitute the assumed solution into the DE
3. Group the summations
4. Find a recurrence relation for the coefficients which will result in all coefficients being expressed in terms of c_0 or c_1
5. Group terms by c_0 and c_1 , giving

$$y(x) = c_0 y_1(x) + c_1 y_2(x)$$

where $y_1(x)$ and $y_2(x)$ are the two linearly independent solutions

5.2 Solutions about Singular Points

- A singular point x_0 is said to be a **regular singular point** of the differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

if the functions $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2Q(x)$ are both analytic at x_0 . A singular point that is not regular is said to be an **irregular singular point** of the equation.

- This, if $x - x_0$ appears at most to the first power in the denominator of $P(x)$ and at most to the second power of the denominator of $Q(x)$, then $x = x_0$ is a regular singular point
- **Frobenius' theorem** states that if $x = x_0$ is a regular singular point of the differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

then there exists at least one nonzero solution of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

where r is a constant to be determined

- When applying Frobenius' theorem, r can be determined by equating the total coefficient of the lowest power of x to 0 and solving for r . This coefficient is called the **indicial equation** and its solutions the **indicial roots** or **exponents**
- Frobenius' theorem can be applied like so:

1. Assume the solution is of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

where $x = x_0$ is a regular singular point and thus

$$y' = \sum_{n=0}^{\infty} (n+r)c_n (x - x_0)^{n+r-1}$$

and

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n (x - x_0)^{n+r-2}$$

2. Substitute the assumed solution into the DE
3. Group the summations

4. Solve the indicial equation to determine the value(s) of r
 5. Solve the recurrence relation(s) given by the value(s) of r to determine constants
 6. Use the constants to determine the solution(s)
- Assuming the indicial roots are real and $r_1 > r_2$, there are three cases to consider:
 1. If r_1 and r_2 are distinct and don't differ by an integer, then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}$$

and

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

2. If $r_1 - r_2 = N$ where N is a positive integer, then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0$$

and

$$y_2(x) = C y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, \quad b_0 \neq 0$$

where C is a constant that may be zero

3. If $r_1 = r_2$, then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0$$

and

$$y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

- In cases 2 and 3 above it may not be possible to find a second solution. Instead a second solution can be found using the first solution and reduction of order

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx$$

5.3 Special Functions

- The equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

is called **Bessel's equation of order ν** where $\nu \geq 0$

- The equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$

is called **Legendre's equation of order n** where n is a nonnegative integer

5.3.1 Bessel Functions

- The indicial roots are $r_1 = \nu$ and $r_2 = -\nu$
- $\Gamma(x)$ is the gamma function and it has the property that

$$\Gamma(1 + \alpha) = \alpha\Gamma(\alpha)$$

- The first solution is

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1 + \nu + n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

and it converges on $[0, \infty)$ if $\nu \geq 0$

- The second solution is

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1 - \nu + n)} \left(\frac{x}{2}\right)^{2n-\nu}$$

and, depending on the value of ν , may contain negative powers of x and thus it converges on the interval $(0, \infty)$

- These solutions are known as **Bessel functions of the first kind** of order ν and $-\nu$
- The general solution to a Bessel equation of order ν is

$$y = c_1 J_\nu(x) + c_2 J_{-\nu}(x), \nu \neq \text{integer}$$

- The function

$$Y_\nu(x) = \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$$

is called the **Bessel function of the second kind** of order ν

- A general solution to a Bessel function of order ν is

$$y = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

- Sometimes it's possible to transform a DE into a Bessel function via a change of variable, e.g. by substituting $t = \alpha x$ in the **parametric Bessel function of order ν**

$$x^2 y'' + xy' + (\alpha^2 x^2 - \nu^2)y = 0$$

it can be transformed into

$$t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + (t^2 - \nu^2)y = 0$$

which has the general solution

$$y = c_1 J_\nu(t) + c_2 Y_\nu(t)$$

or

$$y = c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x)$$

6 Numerical Solutions of Ordinary Differential Equations

6.1 Euler Methods and Error Analysis

- **Round-off error** occurs when calculators or computers round off numbers to fit within the limits of what they can represent
- If we assume that y_n is accurate, then the difference between the computed and actual values of y_{n+1} is called the **local truncation error**, **formula error**, or **discretization error**
- The upper bound on the absolute error of the local truncation error for Euler's formula is

$$M \frac{h^2}{2!}$$

where

$$M = \max_{x_n < x < x_{n+1}} |y''(x)|$$

- The local truncation error for Euler's method is $O(h^2)$
- If $e(h)$ denotes the error in a numerical calculation depending on h , then $e(h)$ is said to be $O(h^n)$ if there is a constant C and a positive integer n such that $|e(h)| \leq Ch^n$
- If y_n isn't necessarily accurate, i.e. it contains its own local truncation error, the difference between the computer and actual values of y_{n+1} is called the **global truncation error** (this may be greater than the local truncation error as it's affected by the truncation errors of previous values)

- The global truncation error for Euler's method is $O(h)$
- If a method for the numerical solution of a differential equation has local truncation error $O(h^{\alpha+1})$ then the global truncation error is $O(h^\alpha)$
- The **improved Euler's method** uses an average of the gradients at the original point and the point predicted by Euler's method

$$y_{n+1} = y_n + h \frac{f(x_n, y_n) + f(x_n, y_{n+1}^*)}{2}$$

where

$$y_{n+1}^* = y_n + hf(x_n, y_n)$$

- The local truncation error for the improved Euler's method is $O(h^3)$ and the global truncation error is $O(h^2)$

6.2 Runge-Kutta Methods

- Runge-Kutta methods are methods for obtaining approximate solutions to first-order initial value problems
- There are Runge-Kutta methods of different orders
- Each Runge-Kutta method is a weighted average of slopes over the interval $x_n < x < x_{n+1}$

$$y_{n+1} = y_n + h(w_1 k_1 + w_2 k_2 + \cdots + w_m k_m)$$

where m is the order of the method

- Euler's method is said to be a first-order Runge-Kutta method
- The improved Euler's method is said to be a second-order Runge-Kutta method
- The local truncation error for RK4 is $y^{(5)}(c)/5!$ or $O(h^5)$ and the global truncation error is $O(h^4)$
- Numerical methods that use a variable step size are called **adaptive methods**
- One of the more popular adaptive methods is the **Runge-Kutta-Fehlberg method** or the RKF45 method

6.3 Multistep Methods

- Methods that compute successive values based only on information from the immediately preceding value are called **single step** or **starting methods**
- Methods that compute successive values based on information from multiple previous values are called **multistep** or **continuing methods**
- The **Adams-Bashforth-Moulton method** is a fourth-order multistep method
- A numerical method is said to be **stable** if small changes in the initial condition result in only small changes in the computed solution
- A numerical method is said to be **unstable** if it's not stable
- Sometimes multistep methods are less computationally intensive because you evaluate the function fewer times