Vibrations and Waves by George C. King Problems

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1 Simple Harmonic Motion

1.1

(a) (i) $T = 4\,\mathrm{s}$

(ii) $\omega = \frac{\pi}{2}\,\mathrm{rad/s}$

(iii) $\omega^2 = \frac{k}{m} \Rightarrow k = m\omega^2 = \frac{\pi^2}{8}\,\mathrm{N/m}$

1.2

(a)

$$\begin{aligned} x &= A\cos\omega t \\ &= A\cos 2\pi f t \\ v &= -2\pi f A \sin 2\pi f t \\ v_{\max} &= 2\pi f A \\ &= 1.38\,\mathrm{m/s} \end{aligned}$$

(b)

$$\begin{aligned} a &= -4\pi^2 f^2 A \cos 2\pi f t \\ a_{\max} &= 4\pi^2 f^2 A \\ &= 3.82\times 10^3\,\mathrm{m/s^2} \end{aligned}$$

1.3

$$\begin{aligned} a_{\max} &\le g \\ 4\pi^2 f^2 A &\le g \\ f &\le \sqrt{\frac{g}{4\pi^2 A}} \\ &\le 1.11\,\mathrm{Hz} \end{aligned}$$

1.4

(a)

$$\frac{U}{E} = \frac{\frac{1}{2}k\left(\frac{1}{2}A\right)^2}{\frac{1}{2}kA^2} = \frac{1}{4} \Rightarrow \frac{K}{E} = \frac{3}{4}$$

(b) (i) The total energy will increase by a factor of 4

(ii) The maximum velocity will increase by a factor of 2

(iii) The maximum acceleration will increase by a factor of 2 and the period won't change

1.5

(a) $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = 0.41\,\mathrm{J}$

(b)

$$
\begin{aligned}
E &= \frac{1}{2}kA^2 \\
A &= \sqrt{\frac{2E}{k}} \\
&= 4.5\,\text{cm} \\
\omega &= \sqrt{\frac{k}{m}} \\
&= \sqrt{\frac{1600}{3}} \\
&= \frac{40}{\sqrt{3}} \\
&= 23\,\text{rad/s} \\
x &= A\cos(\omega t + \phi) \\
\phi &= \arccos\left(\frac{x}{A}\right) - \omega t \\
&= 2.7\,\text{rad} \\
x &= 0.045\cos(23t + 2.7)\,\text{m}
\end{aligned}
$$

1.6

Using the angular frequency of system (b) ω_b as the baseline, the angular frequency of system (a) ω_a is

$$
\begin{aligned}
F = ma &= -2kx \\
a &= -\frac{2k}{m}x \\
\omega_a &= \sqrt{\frac{2k}{m}} \\
&= \sqrt{2}\omega_b
\end{aligned}
$$

and the angular frequency of system (c) ω_c is

$$
\begin{aligned}
F = ma &= -\frac{k}{2}x \\
a &= -\frac{k}{2m}x \\
\omega_c &= \sqrt{\frac{k}{2m}} \\
&= \sqrt{\frac{1}{2}}\omega_b
\end{aligned}
$$

1.7

(a) The test tube experiences a bouyancy force of $F = Ag\rho x$ so its equation of motion is

$$\begin{aligned} F = ma &= -Ag\rho x \\ a &= -\frac{Ag\rho}{m}x \\ \omega &= \sqrt{\frac{Ag\rho}{m}} \end{aligned}$$

(b) The work done by the bouyancy force when moving from equilibrium to x and thus the potential energy is

$$\begin{aligned} U &= \int_0^x Ag\rho x' \, dx' \\ &= \frac{1}{2}Ag\rho x^2 \end{aligned}$$

so the total energy of the system is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}Ag\rho x^2$$

1.8

$$\mathrm{s} \propto \mathrm{kg}^{\alpha}\,\mathrm{m}^{\beta}\,\left(\mathrm{m/s^2}\right)^{\gamma}$$

so $\alpha = 0$, $\beta = 1/2$, and $\gamma = -1/2$ meaning

$$T \propto \sqrt{\frac{l}{g}}$$

1.9

(a)

$$\begin{aligned} x &= A\cos\sqrt{\frac{g}{l}}t \\ v &= -\sqrt{\frac{g}{l}}A\sin\sqrt{\frac{g}{l}}t \\ v_{\max} &= \sqrt{\frac{g}{l}}A \\ &= 0.018\,\mathrm{m/s} \end{aligned}$$

(b) The pendulum reaches its maximum speed at the bottom of its swing which occurs after a quarter cycle

$$\frac{1}{4}T = \frac{1}{4}\frac{2\pi}{\omega} = \frac{\pi}{2\sqrt{g/l}} = 0.43\,\mathrm{s}$$

1.10

$$\begin{aligned}
I\frac{d^2\theta}{dt^2} &= \tau \\
\frac{1}{3}ML^2\frac{d^2\theta}{dt^2} &= -kL\sin\theta L\cos\theta \\
\frac{1}{3}M\frac{d^2\theta}{dt^2} &= -k\theta \\
\frac{d^2\theta}{dt^2} &= -\frac{3k}{M}\theta \\
T &= \frac{2\pi}{\omega} \\
&= 2\pi\sqrt{\frac{M}{3k}}
\end{aligned}$$

1.11

(a)

$$\begin{aligned}
F = -\frac{dU}{dx} &= -\left(\frac{6a}{x^7} - \frac{12b}{x^{13}}\right) \\
0 &= \frac{12b}{x^{13}} - \frac{6a}{x^7} \\
&= \frac{12b}{x^6} - 6a \\
6a &= \frac{12b}{x^6} \\
x^6 &= \frac{2b}{a} \\
x &= \left(\frac{2b}{a}\right)^{1/6}
\end{aligned}$$

1.12

(a)

$$
\begin{aligned}
K &= \frac{1}{2}Mv^2 + \int dK \\
&= \frac{1}{2}Mv^2 + \int_0^L \frac{1}{2}\frac{m}{L}\left(\frac{l}{L}v\right)^2 \, dl \\
&= \frac{1}{2}Mv^2 + \frac{1}{2}\frac{mv^2}{L^3}\int_0^L l^2 \, dl \\
&= \frac{1}{2}Mv^2 + \frac{1}{2}\frac{mv^2}{L^3}\frac{1}{3}L^3 \\
&= \frac{1}{2}Mv^2 + \frac{1}{6}mv^2 \\
&= \frac{1}{2}(M + m/3)v^2 \\
E &= K + U \\
&= \frac{1}{2}(M + m/3)v^2 + \frac{1}{2}kx^2
\end{aligned}
$$

(b)

$$
\omega = \sqrt{\frac{k}{M + m/3}}
$$

1.13

(a)

$$
\begin{aligned}
K &= E - U \\
\frac{1}{2}mv^2 &= U(A) - U(x) \\
v &= \sqrt{2[U(A) - U(x)]/m}
\end{aligned}
$$

(b)

$$
\begin{aligned}
T &= 4\int_0^A \frac{dx}{v} \\
&= 4\int_0^A \sqrt{\frac{m}{2[U(A) - U(x)]}}\, dx \\
&= 4\sqrt{\frac{m}{2U(A)}}\int_0^A \frac{dx}{\sqrt{1 - U(x)/U(A)}}
\end{aligned}
$$

(c)

$$
\begin{aligned}
T &= 4\sqrt{\frac{m}{2\alpha A^n}}\int_0^A \frac{dx}{\sqrt{1-(x/A)^n}} \\
&= 4\sqrt{\frac{m}{2\alpha A^n}}\int_0^1 \frac{A\,d\xi}{\sqrt{1-\xi^n}} \\
&= cA^{(n/2)-1}
\end{aligned}
$$

2 The Damped Harmonic Oscillator

2.1

$$
\begin{aligned}
\left(\frac{\gamma}{2}\right)^2 &= \omega_0^2 \\
\frac{b}{2m} &= \sqrt{\frac{k}{m}} \\
b &= 2m\sqrt{\frac{k}{m}} \\
&= 2m\sqrt{\frac{mg/x}{m}} \\
&= 2m\sqrt{\frac{g}{x}} \\
&= 64\,\mathrm{kg/s}
\end{aligned}
$$

2.2

$$
\begin{aligned}
\frac{A_{n+1}}{A_n} &= 0.90 \\
e^{-2.5\gamma/2} &= 0.90 \\
e^{2.5\gamma/2} &= \frac{1}{0.90} \\
\frac{2.5\gamma}{2} &= \ln\frac{1}{0.90} \\
\gamma &= \frac{2}{2.5}\ln\frac{1}{0.90} \\
&= 8.43\times 10^{-2}\,\mathrm{s}^{-1} \\
F &= -bv \\
&= -(4.21\times 10^{-2})v
\end{aligned}
$$

2.3

After 10 cycles the amplitude has decreased by a factor of 5/3. The energy of the system is proportional to the amplitude squared, so

$$\begin{aligned}
E(300) &= E(0)e^{-300/\tau} \\
e^{300/\tau} &= \frac{E(0)}{E(300)} \\
\tau &= \frac{300}{\ln[E(0)/E(300)]} \\
&= \frac{300}{\ln \frac{25}{9}} \\
&= 294\,\mathrm{s} \\
Q &= \omega_0 \tau \\
&= \frac{2\pi\tau}{T} \\
&= 61.5
\end{aligned}$$

2.4

$$\begin{aligned}
\frac{E(10T)}{E_0} &= \frac{E_0 e^{-\gamma 10T}}{E_0} \\
\frac{1}{2} &= e^{-\gamma 10T} \\
\frac{E(50T)}{E_0} &= \frac{E_0 e^{-\gamma 50T}}{E_0} \\
&= (e^{-\gamma 10T})^5 \\
&= \left(\frac{1}{2}\right)^5 \\
&= \frac{1}{32}
\end{aligned}$$

2.5

(a)

$$\begin{aligned}
Q_{0.01} &= 310 \\
\omega_{0.01} &= 3.14\,\mathrm{rad/s} \\
Q_{0.30} &= 10.5 \\
\omega_{0.30} &= 3.14\,\mathrm{rad/s} \\
Q_{1.00} &= 3.14 \\
\omega_{1.00} &= 3.10\,\mathrm{rad/s}
\end{aligned}$$

(c)

$$
\begin{aligned}
\gamma^2/4 &= \pi^2 \\
\gamma &= 2\pi \\
x &= Ae^{-\pi t} + Bte^{-\pi t} \\
A &= 10\,\mathrm{mm} \\
v &= -10\pi e^{-\pi t} + Be^{-\pi t} - \pi Bte^{-\pi t} \\
0 &= -10\pi + B \\
B &= 10\pi \\
x &= 10e^{-\pi t} + 10\pi te^{-\pi t}
\end{aligned}
$$

2.6

$$
\begin{aligned}
\frac{\omega}{\omega_0} &= \frac{\omega_0\sqrt{1-1/4Q^2}}{\omega_0} \\
&= \sqrt{1-1/4Q^2} \\
&= 1 - \frac{1/4Q^2}{2} + \cdots \\
&\approx 1 - \frac{Q^2}{8}
\end{aligned}
$$

2.7

The amplitude of each pendulum decreases over time by a factor of

$$
\begin{aligned}
\exp\left(-\frac{\gamma t}{2}\right) &= \exp\left(-\frac{bt}{2m}\right) \\
&= \exp\left(-\frac{bt}{2\cdot\frac{4}{3}\pi r^3\rho}\right) \\
&= \exp\left(-\frac{3bt}{8\pi r^3\rho}\right) \\
&= \exp\left(-\frac{3bt}{8\pi r^3}\right)^{1/\rho}.
\end{aligned}
$$

After 10 minutes the amplitude of oscillation of the aluminium pendulum has decreased to half of its initial value

$$
\begin{aligned}
\exp\left(-\frac{225b}{\pi r^3}\right)^{1/\rho_a} &= \frac{1}{2} \\
\exp\left(-\frac{225b}{\pi r^3}\right) &= \left(\frac{1}{2}\right)^{\rho_a}
\end{aligned}
$$

so the brass pendulum's amplitude of oscillation has decreased by a factor of

$$\begin{aligned}
\exp\left(-\frac{225b}{\pi r^3}\right)^{1/\rho_b} &= \left(\frac{1}{2}\right)^{\rho_a/\rho_b} \\
&= 0.802
\end{aligned}$$

2.8

(a)

$$\begin{aligned}
x &= A\sin\omega t \\
v &= \omega A\cos\omega t \\
a &= -\omega^2 A\sin\omega t \\
E &= \int_0^T \frac{Ke^2a^2}{c^3}\,dt \\
&= \int_0^T \frac{Ke^2\omega^4A^2\sin^2\omega t}{c^3}\,dt \\
&= \frac{Ke^2\omega^4A^2}{c^3}\int_0^{2\pi/\omega}\sin^2\omega t\,dt \\
&= \frac{Ke^2\omega^4A^2}{c^3}\left[\frac{t}{2}-\frac{1}{4\omega}\sin 2\omega t\right]_0^{2\pi/\omega} \\
&= \frac{Ke^2\omega^3A^2\pi}{c^3}
\end{aligned}$$

(b)

$$\begin{aligned}
Q &= \frac{\frac{1}{2}m\omega^2A^2}{\frac{Ke^2\omega^3A^2\pi}{2\pi c^3}} \\
&= \frac{c^3m}{e^2K\omega}
\end{aligned}$$

(c)

$$
\begin{aligned}
\tau &= \frac{1}{\gamma} \\
&= \frac{Q}{\omega} \\
&= \frac{c^3 m}{e^2 K \omega^2} \\
&= \frac{c^3 m}{e^2 K \left(2\pi(c/\lambda)\right)^2} \\
&= \frac{\lambda^2 c m}{4\pi^2 e^2 K} \\
&\approx 1.13 \times 10^{-8}\,\mathrm{s}
\end{aligned}
$$

3 Forced Oscillations

3.1

$$
\begin{aligned}
A(2\,\mathrm{rad/s}) &= 1.3 \times 10^{-2}\,\mathrm{m} \\
\delta(2\,\mathrm{rad/s}) &= 0.58^\circ \\
A(20\,\mathrm{rad/s}) &= 0.13\,\mathrm{m} \\
\delta(20\,\mathrm{rad/s}) &= 90^\circ \\
A(100\,\mathrm{rad/s}) &= 5.2 \times 10^{-4}\,\mathrm{m} \\
\delta(100\,\mathrm{rad/s}) &= 179^\circ
\end{aligned}
$$

3.2

$$
\begin{aligned}
A(\omega) &= \frac{a\omega_0/\omega}{\sqrt{(\omega_0/\omega - \omega/\omega_0)^2 + 1/Q^2}} \\
&= \frac{au}{\sqrt{(u - 1/u)^2 + 1/Q^2}} \\
&= \frac{a}{\sqrt{(1 - 1/u^2)^2 + 1/u^2 Q^2}}
\end{aligned}
$$

$A(\omega)$ is maximised when the denominator is minimised which occurs when

$$
\begin{aligned}
\frac{d}{du}((1-u^{-2})^2 + Q^{-2}u^{-2}) &= 0 \\
2(1-u^{-2})2u^{-3} - 2Q^{-2}u^{-3} &= 0 \\
4(1-u^{-2}) - 2Q^{-2} &= 0 \\
\frac{4Q^2-2}{Q^2} &= \frac{4}{u^2} \\
\frac{Q^2}{4Q^2-2} &= \frac{u^2}{4} \\
\frac{4Q^2}{4Q^2-2} &= u^2 \\
\frac{1}{1-1/2Q^2} &= u^2 \\
\frac{1}{\sqrt{1-1/2Q^2}} &= \frac{\omega_0}{\omega_{\max}} \\
\omega_{\max} &= \omega_0\sqrt{1-1/2Q^2}
\end{aligned}
$$

at which point the amplitude will be

$$
\begin{aligned}
A_{\max} &= \frac{a\omega_0/\omega_0\sqrt{1-1/2Q^2}}{\sqrt{\left(\frac{\omega_0}{\omega_0\sqrt{1-1/2Q^2}} - \frac{\omega_0\sqrt{1-1/2Q^2}}{\omega_0}\right)^2 + 1/Q^2}} \\
&= \frac{a/\sqrt{1-1/2Q^2}}{\sqrt{\left(\frac{1}{\sqrt{1-1/2Q^2}} - \sqrt{1-1/2Q^2}\right)^2 + 1/Q^2}} \\
&= \frac{a/\sqrt{1-1/2Q^2}}{\sqrt{\left(\frac{1-1+1/2Q^2}{\sqrt{1-1/2Q^2}}\right)^2 + 1/Q^2}} \\
&= \frac{a/\sqrt{1-1/2Q^2}}{\sqrt{\frac{1}{4Q^4(1-1/2Q^2)} + 1/Q^2}} \\
&= \frac{a/\sqrt{1-1/2Q^2}}{\sqrt{\frac{1+4Q^2(1-1/2Q^2)}{4Q^2(1-1/2Q^2)}}} \\
&= \frac{a}{\sqrt{1-1/2Q^2}}\sqrt{\frac{4Q^2(1-1/2Q^2)}{1+4Q^2(1-1/2Q^2)}} \\
&= a\sqrt{\frac{4Q^2}{1+4Q^2(1-1/2Q^2)}} \\
&= aQ\sqrt{\frac{4}{4(1-1/2Q^2)+1/Q^2}} \\
&= \frac{aQ}{\sqrt{1-1/2Q^2+1/4Q^2}} \\
&= \frac{aQ}{\sqrt{1-1/4Q^2}}
\end{aligned}
$$

3.3

(a)

$$
\begin{aligned}
\frac{\omega_0 - \omega_{\max}}{\omega_0} &= 1 - \sqrt{1-1/2Q^2} \\
&\approx 0.25\,\%
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{A_{\max} - A_0}{A_0} &= \frac{1}{\sqrt{1-1/4Q^2}} - 1 \\
&\approx 0.13\,\%
\end{aligned}
$$

3.4

$$\begin{aligned}
\overline{P}_{\max} &= \frac{F_0^2}{2m\gamma} \\
&= 50\,\mathrm{W} \\
\overline{P}(\omega) &= \frac{F_0^2}{2m\omega_0 Q[4(\Delta\omega/\omega_0)^2 + 1/Q^2]} \\
&= \frac{\overline{P}_{\max}}{Q^2[4(\Delta\omega/\omega_0)^2 + 1/Q^2]} \\
&= \frac{50}{625[4(\Delta\omega/100)^2 + 1/625]} \\
&= \frac{50}{\frac{1}{4}\Delta\omega^2 + 1}
\end{aligned}$$

3.5

(a)

$$\frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \approx 398\,\mathrm{Hz}$$

(b)

$$I = \frac{V_0}{R} = \frac{15}{75} = 0.2\,\mathrm{A}$$

3.6

$$i^i = (e^{i\pi/2})^i = e^{-\pi/2} = 0.208$$

3.7

$$\begin{aligned}
z &= Ae^{i(\omega t + \phi)} \\
x &= \mathrm{Re}\, z \\
&= A\cos(\omega t + \phi) \\
\frac{dz}{dt} &= i\omega A e^{i(\omega t + \phi)} \\
\frac{dx}{dt} &= \mathrm{Re}\,\frac{dz}{dt} \\
&= -\omega A \sin(\omega t + \phi) \\
&= \omega A \cos(\omega t + \phi + \pi/2)
\end{aligned}$$

$\frac{dx}{dt}$ is in advance of x by 90°

3.8

(a)

$$
\begin{aligned}
m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + mg\frac{x-\xi}{l} &= 0 \\
m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + m\frac{g}{l}x &= m\frac{g}{l}a\cos\omega t \\
m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + m\omega_0^2 x &= m\omega_0^2 a\cos\omega t \\
\mathrm{Re}\left(m\frac{d^2z}{dt^2} + b\frac{dz}{dt} + m\omega_0^2 z\right) &= \mathrm{Re}(m\omega_0^2 a e^{i\omega t})
\end{aligned}
$$

(b)

$$
\begin{aligned}
m\frac{d^2}{dt^2}(Ae^{i(\omega t-\delta)}) + b\frac{d}{dt}(Ae^{i(\omega t-\delta)}) + m\omega_0^2 Ae^{i(\omega t-\delta)} &= m\omega_0^2 a e^{i\omega t} \\
-m\omega^2 Ae^{i(\omega t-\delta)} + i\omega bAe^{i(\omega t-\delta)} + m\omega_0^2 Ae^{i(\omega t-\delta)} &= m\omega_0^2 a e^{i\omega t} \\
-m\omega^2 A + i\omega bA + m\omega_0^2 A &= m\omega_0^2 a e^{i\delta}
\end{aligned}
$$

$$
\begin{aligned}
-m\omega^2 A + m\omega_0^2 A &= m\omega_0^2 a\cos\delta \\
-\omega^2 A + \omega_0^2 A &= \omega_0^2 a\cos\delta \\
\frac{\omega_0^2-\omega^2}{\omega_0^2 a}A &= \cos\delta
\end{aligned}
$$

$$
\begin{aligned}
\omega bA &= m\omega_0^2 a\sin\delta \\
\frac{\omega b}{m\omega_0^2 a}A &= \sin\delta
\end{aligned}
$$

$$
\begin{aligned}
\frac{\omega b}{m(\omega_0^2-\omega^2)} &= \tan\delta \\
\frac{\omega\gamma}{\omega_0^2-\omega^2} &= \tan\delta
\end{aligned}
$$

$$
\begin{aligned}
A &= \frac{\omega_0^2 a\cos\delta}{\omega_0^2-\omega^2} \\
&= \frac{\omega_0^2 a}{\omega_0^2-\omega^2}\,\frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} \\
&= \frac{a\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}
\end{aligned}
$$

3.9

(a)

$$\begin{aligned}
A(t_{75}) &= A_0 e^{-\gamma t_{75}/2} \\
\frac{A(t_{75})}{A_0} &= e^{-\gamma t_{75}/2} \\
\ln \frac{A_0/e}{A_0} &= -\frac{\gamma t_{75}}{2} \\
-1 &= -\frac{\gamma t_{75}}{2} \\
\frac{2}{t_{75}} &= \gamma
\end{aligned}$$

$$\begin{aligned}
Q &= \frac{\omega_0}{\gamma} \\
&= \frac{75 \cdot 2\pi}{t_{75}} \frac{t_{75}}{2} \\
&= 75\pi
\end{aligned}$$

(b)

$$\begin{aligned}
A(\omega_0) &= a \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega_0^2)^2 + \omega_0^2 \gamma^2}} \\
&= a \frac{\omega_0^2}{\sqrt{\omega_0^2 \gamma^2}} \\
&= a \frac{\omega_0}{\gamma} \\
&= aQ \\
&= (0.5\,\mathrm{mm}) 75\pi \\
&= 0.12\,\mathrm{m}
\end{aligned}$$

(c)

$$
\begin{aligned}
\frac{aQ}{2} &= a\frac{\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} \\
&\approx a\frac{\omega_0^2}{\sqrt{(2\omega_0(-\Delta\omega))^2+\omega_0^2\gamma^2}} \\
\frac{2\omega_0^2}{Q} &= \sqrt{4\omega_0^2\Delta\omega^2+\omega_0^2\gamma^2} \\
\frac{4\omega_0^4}{Q^2} &= 4\omega_0^2\Delta\omega^2+\omega_0^2\gamma^2 \\
4\omega_0^2\Delta\omega^2 &= \frac{4\omega_0^4}{(\omega_0/\gamma)^2}-\omega_0^2\gamma^2 \\
(2\Delta\omega)^2 &= 4\gamma^2-\gamma^2 \\
2\Delta\omega &= \gamma\sqrt{3} \\
&= \frac{\omega_0}{Q}\sqrt{3} \\
&= \frac{\sqrt{g/l}}{Q}\sqrt{3} \\
&= 0.019\,\mathrm{rad/s}
\end{aligned}
$$

3.10

(a) (i)

$$
\begin{aligned}
K &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}m\left(-\omega A\sin(\omega t-\delta)\right)^2 \\
&= \frac{1}{2}m\omega^2A^2\sin^2(\omega t-\delta)
\end{aligned}
$$

(ii)

$$
\begin{aligned}
U &= \frac{1}{2}kx^2 \\
&= \frac{1}{2}k(A\cos(\omega t-\delta))^2 \\
&= \frac{1}{2}kA^2\cos^2(\omega t-\delta)
\end{aligned}
$$

(iii)

$$
\begin{aligned}
E &= \frac{1}{2}m\omega^2A^2\sin^2(\omega t-\delta)+\frac{1}{2}kA^2\cos^2(\omega t-\delta) \\
&= \frac{1}{2}mA^2[\omega^2\sin^2(\omega t-\delta)+\omega_0^2\cos^2(\omega t-\delta)]
\end{aligned}
$$

(b)

$$
\begin{aligned}
0 &= \frac{dE}{dt} \\
&= \frac{1}{2}mA^2[2\omega^3 \sin(\omega t - \delta)\cos(\omega t - \delta) - 2\omega_0^3 \cos(\omega t - \delta)\sin(\omega t - \delta)] \\
&= \frac{1}{2}mA^2 \sin(2(\omega t - \delta))(\omega^3 - \omega_0^3) \\
\omega &= \omega_0
\end{aligned}
$$

$$
\begin{aligned}
E &= \frac{1}{2}mA^2[\omega_0^2 \sin^2(\omega_0 t - \delta) + \omega_0^2 \cos^2(\omega_0 t - \delta)] \\
&= \frac{1}{2}mA^2\omega_0^2
\end{aligned}
$$

(c)

$$
\begin{aligned}
\overline{K} &= \frac{1}{T}\int_{t_0}^{t_0+T} \frac{1}{2}m\omega^2A^2\sin^2(\omega t - \delta)\,dt \\
&= \frac{m\omega^2A^2}{2T}\int_{t_0}^{t_0+T} \sin^2(\omega t - \delta)\,dt \\
&= \frac{m\omega^2A^2}{4} \\
\overline{E} &= \frac{1}{T}\int_{t_0}^{t_0+T} \frac{1}{2}mA^2[\omega^2\sin^2(\omega t - \delta) + \omega_0^2\cos^2(\omega t - \delta)]\,dt \\
&= \frac{mA^2}{2T}\left(\int_{t_0}^{t_0+T} \omega^2\sin^2(\omega t - \delta)\,dt + \int_{t_0}^{t_0+T} \omega_0^2\cos^2(\omega t - \delta)\,dt\right) \\
&= \frac{mA^2}{4}(\omega^2 + \omega_0^2) \\
\frac{\overline{K}}{\overline{E}} &= \frac{m\omega^2A^2}{4}\frac{4}{mA^2(\omega^2 + \omega_0^2)} \\
&= \frac{\omega^2}{\omega^2 + \omega_0^2} \\
&= \frac{1}{1 + (\omega_0/\omega)^2}
\end{aligned}
$$

(d)

$$
\begin{aligned}
\overline{E} &= \overline{K} + \overline{U} \\
&= \frac{m\omega^2 A^2}{4} + \frac{kA^2}{4} \\
&= \frac{1}{4} m A^2 (\omega^2 + \omega_0^2) \\
&= \frac{1}{4} m (\omega^2 + \omega_0^2) \left(\frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} \right)^2 \\
&= \frac{1}{4} m (\omega^2 + \omega_0^2) \frac{F_0^2}{m^2 [(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2]} \\
&= \frac{F_0^2 (\omega_0^2 + \omega^2)}{4m[(\omega_0^2 - \omega^2)^2 + \omega^2 b^2/m^2]}
\end{aligned}
$$

3.11

(a)

$$
\begin{aligned}
x &= A \cos \omega t \\
v &= -\omega A \sin \omega t \\
W &= \int_0^T b v^2 \, dt \\
&= b \int_0^{2\pi/\omega} \omega^2 A^2 \sin^2 \omega t \, dt \\
&= \pi b \omega A^2
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{W}{E} &= \frac{\pi b \omega A^2}{\frac{1}{2} m \omega^2 A^2} \\
&= \frac{2\pi b}{m\omega}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\frac{W}{E} &= \frac{2\pi b}{m\omega_0} \\
&= \frac{2\pi \gamma}{\omega_0} \\
&= \frac{2\pi}{Q}
\end{aligned}
$$

3.12

Let T' be the number of seconds in 8 days and $T = 2\pi\sqrt{l/g}$ by the period of the pendulum, then

$$\begin{aligned}
Q &= 2\pi \frac{\text{stored energy}}{\text{energy dissipated/cycle}} \\
&= 2\pi \frac{1}{2} m_1 \omega^2 A^2 \frac{\text{cycles in 8 days}}{\text{energy dissipated in 8 days}} \\
&= \pi m_1 \frac{g}{l} A^2 \frac{T'/T}{m_2 g h} \\
&= \frac{\pi m_1 A^2 T'}{m_2 l h T} \\
&= \frac{m_1 A^2 \cdot 8 \cdot 24 \cdot 60 \cdot 60}{2 m_2 l h \sqrt{l/g}} \\
&\approx 70
\end{aligned}$$

4 Coupled Oscillators

4.1

(a)

$$\begin{aligned}
\omega_1 &= \sqrt{\frac{g}{l}} \\
&= 5.7\,\text{rad/s} \\
\omega_2 &= \sqrt{\frac{g}{l} + \frac{2k}{m}} \\
&= 6.0\,\text{rad/s}
\end{aligned}$$

(b) The oscillation of the first pendulum is described by the equation

$$x_a = A \cos \frac{(\omega_2 - \omega_1)t}{2} \cos \frac{(\omega_2 + \omega_1)t}{2},$$

the amplitude of which temporarily becomes 0 at

$$\frac{(\omega_2 - \omega_1)t}{2} = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{\omega_2 - \omega_1} = 11.6\,\text{s}.$$

4.2

(a)

$$
\begin{aligned}
5.0\,\text{mm} &= \frac{1}{2}(C_1 + C_2) \\
5.0\,\text{mm} &= \frac{1}{2}(C_1 - C_2) \\
10\,\text{mm} &= C_1 \\
0.0\,\text{mm} &= C_2
\end{aligned}
$$

(b) (i)

$$
\begin{aligned}
5.0\,\text{mm} &= \frac{1}{2}(C_1 + C_2) \\
-5.0\,\text{mm} &= \frac{1}{2}(C_1 - C_2) \\
0.0\,\text{mm} &= C_1 \\
10\,\text{mm} &= C_2
\end{aligned}
$$

(ii)

$$
\begin{aligned}
10\,\text{mm} &= \frac{1}{2}(C_1 + C_2) \\
0\,\text{mm} &= \frac{1}{2}(C_1 - C_2) \\
10\,\text{mm} &= C_1 \\
10\,\text{mm} &= C_2
\end{aligned}
$$

(iii)

$$
\begin{aligned}
10\,\text{mm} &= \frac{1}{2}(C_1 + C_2) \\
5.0\,\text{mm} &= \frac{1}{2}(C_1 - C_2) \\
15\,\text{mm} &= C_1 \\
5.0\,\text{mm} &= C_2
\end{aligned}
$$

4.3

$$\begin{aligned} m\frac{d^2x_a}{dt^2} &= kx_b - 2kx_a \\ m\frac{d^2x_b}{dt^2} &= kx_a - 2kx_b \end{aligned}$$

$$\begin{aligned} m\frac{d^2(x_a + x_b)}{dt^2} &= -k(x_a + x_b) \\ m\frac{d^2q_1}{dt^2} &= -kq_1 \\ \omega_1 &= \sqrt{\frac{k}{m}} \end{aligned}$$

$$\begin{aligned} m\frac{d^2(x_a - x_b)}{dt^2} &= -3k(x_a - x_b) \\ m\frac{d^2q_2}{dt^2} &= -3kq_2 \\ \omega_2 &= \sqrt{\frac{3k}{m}} \end{aligned}$$

4.4

(a)

$$\begin{aligned} E_a &= \frac{1}{2}m\omega^2A^2 \\ &= \frac{1}{2}mA^2\left(\frac{\omega_2+\omega_1}{2}\right)^2\cos^2\frac{(\omega_2-\omega_1)t}{2} \\ E_b &= \frac{1}{2}m\omega^2A^2 \\ &= \frac{1}{2}m\left(\frac{\omega_2+\omega_1}{2}\right)^2A^2\sin^2\frac{(\omega_2-\omega_1)t}{2} \end{aligned}$$

(b)

$$\begin{aligned} \frac{(\omega_2-\omega_1)T}{2} &= \pi \\ T &= \frac{2\pi}{\omega_2-\omega_1} \\ \omega &= \frac{2\pi}{T} \\ &= \omega_2-\omega_1 \end{aligned}$$

4.5

(a)

$$m\frac{d^2x_1}{dt^2} = -2mg\frac{x_1}{l} + mg\frac{x_2 - x_1}{l}$$
$$\frac{d^2x_1}{dt^2} + \frac{3g}{l}x_1 - \frac{g}{l}x_2 = 0$$

$$m\frac{d^2x_2}{dt^2} = -mg\frac{x_2 - x_1}{l}$$
$$\frac{d^2x_2}{dt^2} - \frac{g}{l}x_1 + \frac{g}{l}x_2 = 0$$

(b)

$$-\omega^2 A\cos\omega t + \frac{3g}{l}A\cos\omega t - \frac{g}{l}B\cos\omega t = 0$$
$$A\left(\frac{3g}{l} - \omega^2\right) = B\left(\frac{g}{l}\right)$$

$$-\omega^2 B\cos\omega t - \frac{g}{l}A\cos\omega t + \frac{g}{l}B\cos\omega t = 0$$
$$A\left(\frac{g}{l}\right) = B\left(\frac{g}{l} - \omega^2\right)$$

$$\frac{3g/l - \omega^2}{g/l} = \frac{g/l}{g/l - \omega^2}$$
$$\left(\frac{3g}{l} - \omega^2\right)\left(\frac{g}{l} - \omega^2\right) = \left(\frac{g}{l}\right)^2$$
$$3\left(\frac{g}{l}\right)^2 - \frac{3g}{l}\omega^2 - \frac{g}{l}\omega^2 + \omega^4 = \left(\frac{g}{l}\right)^2$$
$$(\omega^2)^2 - \frac{4g}{l}\omega^2 + 2\left(\frac{g}{l}\right)^2 = 0$$

$$\omega^2 = \frac{\frac{4g}{l} \pm \sqrt{\frac{16g^2}{l^2} - 8\left(\frac{g}{l}\right)^2}}{2}$$
$$= (2 \pm \sqrt{2})\frac{g}{l}$$
$$\omega = \sqrt{(2 \pm \sqrt{2})\frac{g}{l}}$$

$$
\begin{aligned}
\frac{A}{B} &= \frac{g/l}{3g/l - \omega^2} \\
&= \frac{g}{3g - l\omega^2} \\
&= \frac{g}{3g - l(2 \pm \sqrt{2})\frac{g}{l}} \\
&= \frac{1}{3 - (2 \pm \sqrt{2})} \\
&= \frac{1}{1 \pm \sqrt{2}}
\end{aligned}
$$

(c)

$$
\begin{aligned}
T &= \frac{2\pi}{\omega} \\
&= \frac{2\pi}{\sqrt{(2 \pm \sqrt{2})\frac{g}{l}}} \\
&= 1.09\,\mathrm{s} \text{ or } 2.62\,\mathrm{s}
\end{aligned}
$$

4.6

(b)

$$
\begin{aligned}
m\frac{d^2x_1}{dt^2} &= k(x_2 - x_1) \\
\frac{d^2x_1}{dt^2} + \omega_1^2 x_1 - \omega_1^2 x_2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
M\frac{d^2x_2}{dt^2} &= -k(x_2 - x_1) + k(x_3 - x_2) \\
&= kx_1 - 2kx_2 + kx_3 \\
\frac{d^2x_2}{dt^2} - \omega_2^2 x_1 + 2\omega_2^2 x_2 - \omega_2^2 x_3 &= 0
\end{aligned}
$$

$$
\begin{aligned}
m\frac{d^2x_3}{dt^2} &= -k(x_3 - x_2) \\
\frac{d^2x_3}{dt^2} - \omega_1^2 x_2 + \omega_1^2 x_3 &= 0
\end{aligned}
$$

(c)

$$\begin{aligned} x_1 &= A\cos\omega t \\ x_2 &= B\cos\omega t \\ x_3 &= C\cos\omega t \end{aligned}$$

$$\begin{aligned} -\omega^2 A\cos\omega t + \omega_1^2 A\cos\omega t - \omega_1^2 B\cos\omega t &= 0 \\ A(\omega_1^2 - \omega^2) &= B(\omega_1^2) \end{aligned}$$

$$\begin{aligned} -\omega^2 B\cos\omega t - \omega_2^2 A\cos\omega t + 2\omega_2^2 B\cos\omega t - \omega_2^2 C\cos\omega t &= 0 \\ (A+C)(\omega_2^2) &= B(2\omega_2^2 - \omega^2) \end{aligned}$$

$$\begin{aligned} -\omega^2 C\cos\omega t - \omega_1^2 B\cos\omega t + \omega_1^2 C\cos\omega t &= 0 \\ C(\omega_1^2 - \omega^2) &= B(\omega_1^2) \end{aligned}$$

$$(A+C)(\omega_1^2 - \omega^2) = B(2\omega_1^2)$$

$$\begin{aligned} \frac{\omega_2^2}{\omega_1^2 - \omega^2} &= \frac{2\omega_2^2 - \omega^2}{2\omega_1^2} \\ 2\omega_1^2\omega_2^2 &= (2\omega_2^2 - \omega^2)(\omega_1^2 - \omega^2) \\ &= 2\omega_1^2\omega_2^2 - 2\omega_2^2\omega^2 - \omega_1^2\omega^2 + \omega^4 \\ 0 &= (\omega^2)^2 - (\omega_1^2 + 2\omega_2^2)\omega^2 \\ &= \omega^2 - \omega_1^2 - 2\omega_2^2 \\ \omega^2 &= \omega_1^2 + 2\omega_2^2 \\ &= \frac{k}{m} + 2\frac{k}{M} \\ \omega &= \sqrt{\frac{k(M+2m)}{Mm}} \end{aligned}$$

(d)

$$\begin{aligned} \frac{\sqrt{\frac{k(M+2m)}{Mm}}}{\sqrt{\frac{k}{m}}} &= \sqrt{\frac{M+2m}{M}} \\ &= \sqrt{1 + 2m/M} \\ &= \sqrt{1 + 16/6} \\ &\approx 1.91 \end{aligned}$$

4.7

(a) Let x_1 be the top mass's displacement from equilibrium and x_2 be the bottom mass's, then the equations of motion are

$$
\begin{aligned}
3m\frac{d^2x_1}{dt^2} &= -4kx_1 + k(x_2 - x_1) \\
&= -5kx_1 + kx_2 \\
\frac{d^2x_1}{dt^2} + \frac{5k}{3m}x_1 - \frac{k}{3m}x_2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
m\frac{d^2x_2}{dt^2} &= -k(x_2 - x_1) \\
&= kx_1 - kx_2 \\
\frac{d^2x_2}{dt^2} - \frac{k}{m}x_1 + \frac{k}{m}x_2 &= 0
\end{aligned}
$$

Assuming solutions of the form $x_1 = A\cos\omega t$ and $x_2 = B\cos\omega t$ and substituting into the above gives

$$
\begin{aligned}
-\omega^2 A\cos\omega t + \frac{5k}{3m}A\cos\omega t - \frac{k}{3m}B\cos\omega t &= 0 \\
A\left(\frac{5k}{3m} - \omega^2\right) &= B\left(\frac{k}{3m}\right)
\end{aligned}
$$

$$
\begin{aligned}
-\omega^2 B\cos\omega t - \frac{k}{m}A\cos\omega t + \frac{k}{m}B\cos\omega t &= 0 \\
A\left(\frac{k}{m}\right) &= B\left(\frac{k}{m} - \omega^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\frac{5k}{3m} - \omega^2}{\frac{k}{m}} &= \frac{\frac{k}{3m}}{\frac{k}{m} - \omega^2} \\
\left(\frac{5k}{3m} - \omega^2\right)\left(\frac{k}{m} - \omega^2\right) &= \frac{1}{3}\left(\frac{k}{m}\right)^2 \\
\frac{5}{3}\left(\frac{k}{m}\right)^2 - \frac{5k}{3m}\omega^2 - \frac{k}{m}\omega^2 + \omega^4 &= \frac{1}{3}\left(\frac{k}{m}\right)^2 \\
(\omega^2)^2 - \frac{8}{3}\frac{k}{m}\omega^2 + \frac{4}{3}\left(\frac{k}{m}\right)^2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
\omega^2 &= \frac{\frac{8}{3}\frac{k}{m} \pm \sqrt{\left(\frac{8}{3}\frac{k}{m}\right)^2 - \frac{16}{3}\left(\frac{k}{m}\right)^2}}{2} \\
&= \frac{\frac{8}{3}\frac{k}{m} \pm \sqrt{\frac{64}{9}\left(\frac{k}{m}\right)^2 - \frac{16}{3}\left(\frac{k}{m}\right)^2}}{2} \\
&= \frac{\frac{8}{3}\frac{k}{m} \pm \sqrt{\frac{16}{9}\left(\frac{k}{m}\right)^2}}{2} \\
&= \frac{\frac{8}{3}\frac{k}{m} \pm \frac{4}{3}\frac{k}{m}}{2} \\
&= \left(\frac{4}{3} \pm \frac{2}{3}\right)\frac{k}{m} \\
\omega &= \sqrt{\frac{2k}{3m}} \text{ or } \sqrt{\frac{2k}{m}}
\end{aligned}
$$

(b)

$$
A\left(\frac{k}{m}\right) = B\left(\frac{k}{m} - \omega^2\right)
$$

$$
\begin{aligned}
A\left(\frac{k}{m}\right) &= B\left(\frac{k}{m} - \frac{2k}{3m}\right) \\
&= B\left(\frac{k}{3m}\right) \\
A &= \frac{1}{3}B
\end{aligned}
$$

$$
\begin{aligned}
A\left(\frac{k}{m}\right) &= B\left(\frac{k}{m} - \frac{2k}{m}\right) \\
&= B\left(-\frac{k}{m}\right) \\
A &= -B
\end{aligned}
$$

So the first normal mode is

$$
\begin{aligned}
x_1 &= A\cos\sqrt{\frac{2k}{3m}}t \\
x_2 &= 3A\cos\sqrt{\frac{2k}{3m}}t
\end{aligned}
$$

where the masses oscillate in phase and the lower mass has an amplitude 3 times greater than the upper mass.

The second normal mode is

$$x_1 = A\cos\sqrt{\frac{2k}{m}}t$$
$$x_2 = -A\cos\sqrt{\frac{2k}{m}}t$$

where the masses oscillate 180° out of phase with equal amplitude.

4.8

There are 5 normal modes in the transverse direction.

4.9

(a)

$$M\frac{d^2x_1}{dt^2} + (k_1+k_2)x_1 - k_2x_2 = F_0\cos\omega t$$
$$m\frac{d^2x_2}{dt^2} - k_2x_1 + k_2x_2 = 0$$

(b)

$$\begin{aligned} -M\omega^2 A\cos\omega t + (k_1+k_2)A\cos\omega t - k_2B\cos\omega t &= F_0\cos\omega t \\ A(k_1+k_2-M\omega^2) &= Bk_2 + F_0 \end{aligned}$$

$$\begin{aligned} -m\omega^2 B\cos\omega t - k_2A\cos\omega t + k_2B\cos\omega t &= 0 \\ Ak_2 &= B(k_2-m\omega^2) \end{aligned}$$

$$\begin{aligned} B &= A\frac{k_2}{k_2-m\omega^2} \\ A(k_1+k_2-M\omega^2) &= A\frac{k_2^2}{k_2-m\omega^2} + F_0 \\ A\left(k_1+k_2-M\omega^2 - \frac{k_2^2}{k_2-m\omega^2}\right) &= F_0 \\ A\left(\frac{(k_1+k_2-M\omega^2)(k_2-m\omega^2)-k_2^2}{k_2-m\omega^2}\right) &= F_0 \\ \frac{F_0(k_2-m\omega^2)}{(k_1+k_2-M\omega^2)(k_2-m\omega^2)-k_2^2} &= A \end{aligned}$$

$$
\begin{aligned}
A &= \frac{Bk_2 + F_0}{k_1 + k_2 - M\omega^2} \\
B &= \frac{Bk_2 + F_0}{k_1 + k_2 - M\omega^2} \frac{k_2}{k_2 - m\omega^2} \\
&= \frac{Bk_2^2 + F_0 k_2}{(k_1 + k_2 - M\omega^2)(k_2 - m\omega^2)} \\
B\left(1 - \frac{k_2^2}{(k_1 + k_2 - M\omega^2)(k_2 - m\omega^2)}\right) &= \frac{F_0 k_2}{(k_1 + k_2 - M\omega^2)(k_2 - m\omega^2)} \\
B[(k_1 + k_2 - M\omega^2)(k_2 - m\omega^2) - k_2^2] &= F_0 k_2 \\
\frac{F_0 k_2}{(k_1 + k_2 - M\omega^2)(k_2 - m\omega^2) - k_2^2} &= B
\end{aligned}
$$

(c)

$$
\begin{aligned}
A &= \frac{F_0\left(k_2 - m\frac{k_1}{M}\right)}{\left(k_1 + k_2 - M\frac{k_1}{M}\right)\left(k_2 - m\frac{k_1}{M}\right) - k_2^2} \\
&= \frac{F_0\left(k_2 - \frac{k_2}{k_1}k_1\right)}{(k_1 + k_2 - k_1)\left(k_2 - \frac{k_2}{k_1}k_1\right) - k_2^2} \\
&= 0
\end{aligned}
$$

4.10

(a)

$$
\begin{aligned}
m\frac{d^2x_1}{dt^2} &= -kx_1 + k(x_2 - x_1) \\
m\frac{d^2x_1}{dt^2} + 2kx_1 - kx_2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
m\frac{d^2x_2}{dt^2} &= -k(x_2 - x_1) + k(x_3 - x_2) \\
m\frac{d^2x_2}{dt^2} - kx_1 + 2kx_2 - kx_3 &= 0
\end{aligned}
$$

$$
\begin{aligned}
m\frac{d^2x_3}{dt^2} &= -k(x_3 - x_2) - kx_3 \\
m\frac{d^2x_3}{dt^2} - kx_2 + 2kx_3 &= 0
\end{aligned}
$$

$$
\begin{aligned}
-m\omega^2 A\cos\omega t + 2kA\cos\omega t - kB\cos\omega t &= 0\\
A(2k - m\omega^2) &= B(k)
\end{aligned}
$$

$$
\begin{aligned}
-m\omega^2 B\cos\omega t - kA\cos\omega t + 2kB\cos\omega t - kC\cos\omega t &= 0\\
(A+C)(k) &= B(2k - m\omega^2)
\end{aligned}
$$

$$
\begin{aligned}
-m\omega^2 C\cos\omega t - kB\cos\omega t + 2kC\cos\omega t &= 0\\
C(2k - m\omega^2) &= B(k)
\end{aligned}
$$

$$
(A+C)(2k - m\omega^2) = B(2k)
$$

$$
\begin{aligned}
\frac{k}{2k - m\omega^2} &= \frac{2k - m\omega^2}{2k}\\
(2k - m\omega^2)^2 &= 2k^2\\
4k^2 - 4km\omega^2 + m^2\omega^4 &= 2k^2\\
(\omega^2)^2 - 4\frac{k}{m}\omega^2 + 2\left(\frac{k}{m}\right)^2 &= 0\\
\omega^2 &= \frac{4\frac{k}{m} \pm \sqrt{\left(4\frac{k}{m}\right)^2 - 8\left(\frac{k}{m}\right)^2}}{2}\\
&= \frac{4\frac{k}{m} \pm \sqrt{16\left(\frac{k}{m}\right)^2 - 8\left(\frac{k}{m}\right)^2}}{2}\\
\omega &= \sqrt{(2 \pm \sqrt{2})\frac{k}{m}}
\end{aligned}
$$

(b) (i) For $\omega = \sqrt{\frac{2k}{m}}$

$$
\begin{aligned}
A\left(2k - m\frac{2k}{m}\right) &= B(k)\\
0 &= B(k)
\end{aligned}
$$

so $B = 0$ and

$$
\begin{aligned}
(A+C)(k) &= 0\\
A &= -C
\end{aligned}
$$

(ii) For $\omega = \sqrt{(2+\sqrt{2})\frac{k}{m}}$

$$\begin{aligned} A\left(2k - m(2+\sqrt{2})\frac{k}{m}\right) &= B(k) \\ A(2k - 2k - \sqrt{2}k) &= B(k) \\ A &= -\frac{1}{\sqrt{2}}B \end{aligned}$$

and

$$\begin{aligned} C\left(2k - m(2+\sqrt{2})\frac{k}{m}\right) &= B(k) \\ C(2k - 2k - \sqrt{2}k) &= B(k) \\ C &= -\frac{1}{\sqrt{2}}B \\ &= A \end{aligned}$$

(iii) For $\omega = \sqrt{(2-\sqrt{2})\frac{k}{m}}$

$$\begin{aligned} A\left(2k - m(2-\sqrt{2})\frac{k}{m}\right) &= B(k) \\ A &= \frac{1}{\sqrt{2}}B \end{aligned}$$

and

$$\begin{aligned} C\left(2k - m(2-\sqrt{2})\frac{k}{m}\right) &= B(k) \\ C &= \frac{1}{\sqrt{2}}B \\ &= A \end{aligned}$$

5 Travelling Waves

5.1

$$\begin{aligned}
y &= A\cos\left(\frac{2\pi}{\lambda}x + 2\pi f t\right) \\
&= 15\cos(0.25x + 75t) \\
A &= 15\,\mathrm{mm} \\
\lambda &= 25\,\mathrm{mm} \\
f &= 12\,\mathrm{Hz} \\
v &= 30\,\mathrm{cm/s}
\end{aligned}$$

The wave is travelling in the negative x direction

5.2

$$\begin{aligned}
A &= 0.15\,\mathrm{m} \\
f &= 10\,\mathrm{Hz} \\
v &= 50\,\mathrm{m/s} \\
y &= A\cos\left(\frac{2\pi}{v/f}x - 2\pi f t\right) \\
&= 0.15\cos(1.3x - 63t)
\end{aligned}$$

5.3

(a) (i)

$$\begin{aligned}
y &= A\sin 2\pi f(t - x/v) \\
\frac{\partial y}{\partial t} &= 2\pi f A\cos 2\pi f(t - x/v) \\
\frac{\partial^2 y}{\partial t^2} &= -(2\pi f)^2 A\sin 2\pi f(t - x/v) \\
\frac{\partial y}{\partial x} &= -\frac{2\pi f}{v}A\cos 2\pi f(t - x/v) \\
\frac{\partial^2 y}{\partial x^2} &= -\left(\frac{2\pi f}{v}\right)^2 A\sin 2\pi f(t - x/v) \\
-(2\pi f)^2 A\sin 2\pi f(t - x/v) &= v^2\left(-\left(\frac{2\pi f}{v}\right)^2 A\sin 2\pi f(t - x/v)\right)
\end{aligned}$$

(ii)

$$\begin{aligned}
y &= A\sin\frac{2\pi}{\lambda}(x+vt) \\
\frac{\partial y}{\partial t} &= \frac{2\pi v}{\lambda}A\cos\frac{2\pi}{\lambda}(x+vt) \\
\frac{\partial^2 y}{\partial t^2} &= -\left(\frac{2\pi v}{\lambda}\right)^2 A\sin\frac{2\pi}{\lambda}(x+vt) \\
\frac{\partial y}{\partial x} &= \frac{2\pi}{\lambda}A\cos\frac{2\pi}{\lambda}(x+vt) \\
\frac{\partial^2 y}{\partial x^2} &= -\left(\frac{2\pi}{\lambda}\right)^2 A\sin\frac{2\pi}{\lambda}(x+vt) \\
-\left(\frac{2\pi v}{\lambda}\right)^2 A\sin\frac{2\pi}{\lambda}(x+vt) &= v^2\left(-\left(\frac{2\pi}{\lambda}\right)^2 A\sin\frac{2\pi}{\lambda}(x+vt)\right)
\end{aligned}$$

(iii)

$$\begin{aligned}
y &= A\sin 2\pi(x/\lambda - t/T) \\
\frac{\partial y}{\partial t} &= -\frac{2\pi}{T}A\cos 2\pi(x/\lambda - t/T) \\
\frac{\partial^2 y}{\partial t^2} &= -\left(\frac{2\pi}{T}\right)^2 A\sin 2\pi(x/\lambda - t/T) \\
\frac{\partial y}{\partial x} &= \frac{2\pi}{\lambda}A\cos 2\pi(x/\lambda - t/T) \\
\frac{\partial^2 y}{\partial x^2} &= -\left(\frac{2\pi}{\lambda}\right)^2 A\sin 2\pi(x/\lambda - t/T) \\
-\left(\frac{2\pi}{T}\right)^2 A\sin 2\pi\left(\frac{x}{\lambda}-\frac{t}{T}\right) &= v^2\left(-\left(\frac{2\pi}{\lambda}\right)^2 A\sin 2\pi\left(\frac{x}{\lambda}-\frac{t}{T}\right)\right) \\
\frac{1}{T^2} &= \frac{v^2}{\lambda^2} \\
f^2 &= f^2
\end{aligned}$$

(iv)

$$
\begin{aligned}
y &= Ae^{i(\omega t+kx)} \\
\frac{\partial y}{\partial t} &= i\omega Ae^{i(\omega t+kx)} \\
\frac{\partial^2 y}{\partial t^2} &= -\omega^2 Ae^{i(\omega t+kx)} \\
\frac{\partial y}{\partial x} &= ikAe^{i(\omega t+kx)} \\
\frac{\partial^2 y}{\partial x^2} &= -k^2 Ae^{i(\omega t+kx)} \\
-\omega^2 Ae^{i(\omega t+kx)} &= v^2 \left(-k^2 Ae^{i(\omega t+kx)}\right) \\
\omega^2 &= v^2 k^2 \\
(2\pi f)^2 &= v^2 \left(\frac{2\pi}{\lambda}\right)^2 \\
f^2 &= \left(\frac{v}{\lambda}\right)^2 \\
&= f^2
\end{aligned}
$$

(v)

$$
\begin{aligned}
y &= A\cos(\omega_1 t - k_1 x) + B\cos(\omega_2 t - k_2 x) \\
\frac{\partial y}{\partial t} &= -\omega_1 A\sin(\omega_1 t - k_1 x) - \omega_2 B\sin(\omega_2 t - k_2 x) \\
\frac{\partial^2 y}{\partial t^2} &= -\omega_1^2 A\cos(\omega_1 t - k_1 x) - \omega_2^2 B\cos(\omega_1 t - k_2 x) \\
\frac{\partial y}{\partial x} &= k_1 A\sin(\omega_1 t - k_1 x) + k_2 B\sin(\omega_2 t - k_2 x) \\
\frac{\partial^2 y}{\partial x^2} &= -k_1^2 A\cos(\omega_1 t - k_1 x) - k_2^2 B\cos(\omega_2 t - k_2 x) \\
\omega_1^2 A + \omega_2^2 B &= v^2 \left(k_1^2 A + k_2^2 B\right) \\
\omega_1^2 A + \omega_2^2 B &= \omega_1^2 A + \omega_2^2 B
\end{aligned}
$$

(b)

$$
\begin{aligned}
\psi &= A\sin(k_1x + k_2y + k_3z - \omega t)\\
\frac{\partial^2\psi}{\partial t^2} &= -\omega^2 A\sin(k_1x + k_2y + k_3z - \omega t)\\
\frac{\partial^2\psi}{\partial x^2} &= -k_1^2 A\sin(k_1x + k_2y + k_3z - \omega t)\\
\frac{\partial^2\psi}{\partial y^2} &= -k_2^2 A\sin(k_1x + k_2y + k_3z - \omega t)\\
\frac{\partial^2\psi}{\partial z^2} &= -k_3^2 A\sin(k_1x + k_2y + k_3z - \omega t)\\
-\frac{\omega^2}{v^2} &= -k_1^2 - k_2^2 - k_3^2\\
v &= \frac{\omega}{\sqrt{k_1^2 + k_2^2 + k_3^2}}
\end{aligned}
$$

5.4

(a) No difference

(b) 180° out of phase

5.5

$$
\begin{aligned}
y(x,t) &= A\exp - \frac{(x^2 - vt)^2}{a^2}\\
y(x+\delta x, t+\delta t) &= A\exp - \frac{(x + \delta x - v(t+\delta t))^2}{a^2}\\
&= A\exp - \frac{(x - vt)^2}{a^2}
\end{aligned}
$$

5.6

(a) (i) $f = c/\lambda = 200\,\text{kHz}$

(ii) $f = 6 \times 10^{14}\,\text{Hz}$

(iii) $f = 3 \times 10^{18}\,\text{Hz}$

(iv) $f = c/\lambda = c/(2\pi/k) = ck/(2\pi) = 1 \times 10^{8}\,\text{Hz}$

(v) $f = 69\,\text{kHz}$

(b)

$$
\begin{aligned}
\lambda_{20} &= 17\,\text{m}\\
\lambda_{15} &= 2.3\,\text{cm}\\
\lambda_{440} &= 78\,\text{cm}
\end{aligned}
$$

5.7

(a)

$$\begin{aligned} \mathrm{m/s} &= \sqrt{\mathrm{N\,m/kg}} \\ &= \sqrt{kg.(m/s^2).m/kg} \\ &= \sqrt{(\mathrm{m/s})^2} \\ &= \mathrm{m/s} \end{aligned}$$

(b) The string with the smallest mass / the thinnest string, i.e. the high E string

5.8

(a) (i)

$$\begin{aligned} L &= 25\,\mathrm{m} \\ m &= 100\,\mathrm{g} \\ T &= 10\,\mathrm{N} \\ f &= 25\,\mathrm{Hz} \\ A &= 15\,\mathrm{mm} \\ v &= \sqrt{\frac{T}{\mu}} \\ &= 50\,\mathrm{m/s} \end{aligned}$$

(ii)

$$\begin{aligned} y &= A \sin\left(\frac{2\pi}{f}x - 2\pi f t\right) \\ y' &= -2\pi f A \cos\left(\frac{2\pi}{f}x - 2\pi f t\right) \\ y'_{\mathrm{max}} &= 2\pi f A \\ &= 2.4\,\mathrm{m/s} \end{aligned}$$

(b)

$$
\begin{aligned}
L &= 0.75\,\mathrm{m} \\
m &= 125\,\mathrm{g} \\
T &= 2.5\,\mathrm{N} \\
v &= \sqrt{\frac{S}{\sigma}} \\
&= \sqrt{\frac{T/L}{m/L^2}} \\
&= \sqrt{\frac{LT}{m}} \\
&= 3.9\,\mathrm{m/s}
\end{aligned}
$$

5.9

(a)

$$
\begin{aligned}
T(y) &= \mu y g \\
v &= \sqrt{\frac{T}{\mu}} \\
&= \sqrt{\frac{\mu y g}{\mu}} \\
&= \sqrt{yg}
\end{aligned}
$$

(b)

$$
\begin{aligned}
t &= 2\int_0^L \frac{1}{v}\,dy \\
&= 2\int_0^L \frac{1}{\sqrt{gy}}\,dy \\
&= \frac{2}{\sqrt{g}}[2\sqrt{y}]_0^L \\
&= 4\sqrt{\frac{L}{g}} \\
&= 2.0\,\mathrm{s}
\end{aligned}
$$

5.10

(a)

$$
\begin{aligned}
\mu &= 30\,\mathrm{g/m} \\
T &= 12\,\mathrm{N} \\
f &= 150\,\mathrm{Hz} \\
A &= 1.5\,\mathrm{cm} \\
P &= \frac{1}{2}\mu\omega^2 A^2 v \\
&= \frac{1}{2}\mu(2\pi f)^2 A^2 \sqrt{\frac{T}{\mu}} \\
&= 60\,\mathrm{W}
\end{aligned}
$$

(b) (i) $P = 240\,\mathrm{W}$

(ii) $P = 15\,\mathrm{W}$

5.11

(a)

$$
\begin{aligned}
I_1 &= \frac{k}{r_1^2} \\
I_2 &= \frac{k}{r_2^2} \\
&= I_1 \left(\frac{r_1}{r_2}\right)^2
\end{aligned}
$$

(b)

$$
\begin{aligned}
I_1 &= 25\,\mathrm{W/m^2} \\
r_1 &= 1.0\,\mathrm{m} \\
I_2 &= 1.0\,\mathrm{W/m^2} \\
r_2 &= \sqrt{\frac{I_1}{I_2}} r_1 \\
&= 5.0\,\mathrm{m}
\end{aligned}
$$

5.12

$$P_{\text{earth}} \approx 1.5\,\mathrm{kW}$$

$$P_{\text{jupiter}} \approx 57\,\mathrm{W}$$

5.13

(a)

$$\begin{aligned}
\mu_1 &= 1.0\,\mathrm{g/cm} \\
\mu_2 &= 4.0\,\mathrm{g/cm} \\
A_1 &= 3.0\,\mathrm{cm} \\
\lambda_1 &= 25\,\mathrm{cm} \\
v_1 &= \sqrt{\frac{T}{\mu_1}} \\
f\lambda_1 &= \sqrt{\frac{T}{\mu_1}} \\
\lambda_1\sqrt{\mu_1} &= \frac{\sqrt{T}}{f} \\
\lambda_1\sqrt{\mu_1} &= \lambda_2\sqrt{\mu_2} \\
\lambda_2 &= \lambda_1\sqrt{\frac{\mu_1}{\mu_2}} \\
&= 12.5\,\mathrm{cm} \\
A_2 &= A_1\frac{2\sqrt{\mu_1}}{\sqrt{\mu_1}+\sqrt{\mu_2}} \\
&= 2.0\,\mathrm{cm}
\end{aligned}$$

(b)

$$\begin{aligned}
B_1 &= A_1\frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_1}+\sqrt{\mu_2}} \\
&= -1\,\mathrm{cm}
\end{aligned}$$

1/9 of the power is reflected

5.14

(a) The refractive index and wave number are directly proportional, so

$$\begin{aligned}
B_1 &= A_1\frac{k_1 - nk_1}{k_1 + nk_1} \\
&= A_1\frac{1-n}{1+n} \\
&= -0.2A_1
\end{aligned}$$

Intensity is proportional to amplitude squared, so 0.04 of the incident light intensity is reflected.

(b)

$$\begin{aligned}
n_1 &= 1.39 \\
\lambda_1 &= 550\,\mathrm{nm} \\
f &= \frac{c}{\lambda} \\
&= 5.5 \times 10^{14}\,\mathrm{Hz} \\
n_2 &= 1.5 \\
\lambda_2 &= \frac{v}{f} \\
&= \frac{c}{fn} \\
&= 392\,\mathrm{nm} \\
L &= \frac{\lambda_2}{4} \\
&= 98\,\mathrm{nm}
\end{aligned}$$

(c)

$$2\pi \times \frac{2L}{\lambda_2} = 2\pi \Rightarrow L = \frac{\lambda_2}{2} = 198\,\mathrm{nm}$$

5.15

(a)

$$\begin{aligned}
m\frac{\partial^2 y_r}{\partial t^2} &= -T\sin\theta_1 - T\sin\theta_2 \\
\frac{\partial^2 y_r}{\partial t^2} &= -\frac{T}{m}\left[\frac{y_r - y_{r-1}}{a} + \frac{y_r - y_{r+1}}{a}\right] \\
&= \frac{T}{m}\left[\frac{y_{r+1} - y_r}{a} - \frac{y_r - y_{r-1}}{a}\right]
\end{aligned}$$

(b)

$$
\begin{aligned}
\frac{\partial^2 y_r}{\partial t^2} &= \frac{T}{\delta x m}[y(x+\delta x) - 2y(x) + y(x-\delta x)] \\
&= \frac{T}{\delta x m}\left[y(x) + \delta x \frac{\partial y}{\partial x} + \frac{1}{2}(\delta x)^2 \frac{\partial^2 y}{\partial x^2} - 2y(x) + y(x) - \delta x \frac{\partial y}{\partial x}\right. \\
&\qquad \left. + \frac{1}{2}(\delta x)^2 \frac{\partial^2 y}{\partial x^2}\right] \\
&= \frac{T}{\delta x m}\left[(\delta x)^2 \frac{\partial^2 y}{\partial x^2}\right] \\
&= \frac{T}{m/\delta x}\frac{\partial^2 y}{\partial x^2} \\
&= \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2}
\end{aligned}
$$