Vibrations and Waves by A. P. French Problems

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1 Periodic motions

1.4

(a)

$$\begin{aligned} z &= Ae^{j\theta} \\ dz &= jAe^{j\theta}\,d\theta \\ &= jz\,d\theta \end{aligned}$$

The motion of the point is always perpendicular to its position.

(b)

$$\begin{aligned} |2+j\sqrt{3}| &= \sqrt{2^2+\sqrt{3}^2} \\ &= \sqrt{7} \\ \arg(2+j\sqrt{3}) &= \arctan\frac{\sqrt{3}}{2} \\ &= 41^\circ \end{aligned}$$

$$\begin{aligned} (2-j\sqrt{3})^2 &= 4-j4\sqrt{3}-3 \\ &= 1-j4\sqrt{3} \\ |1-j4\sqrt{3}| &= \sqrt{1^2+(4\sqrt{3})^2} \\ &= 7 \\ \arg(1-j4\sqrt{3}) &= -\arctan 4\sqrt{3} \end{aligned}$$

1.9

$$\begin{aligned} \cos\theta + j\sin\theta &= e^{j\theta} \\ \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} &= e^{j\frac{\pi}{2}} \\ j &= e^{j\frac{\pi}{2}} \\ j^j &= (e^{j\frac{\pi}{2}})^j \\ &= e^{-\frac{\pi}{2}} \\ &\approx 0.208 \end{aligned}$$

Yes, I would be willing to pay 20 cents because I could sell it to the mathematician and gain 0.8 cents.

1.10

$$
\begin{aligned}
y &= A\cos kx + B\sin kx \\
\frac{dy}{dx} &= -Ak\sin kx + Bk\cos kx \\
\frac{d^2y}{dx^2} &= -Ak^2\cos kx - Bk^2\sin kx \\
&= -k^2 y
\end{aligned}
$$

$$
\begin{aligned}
C &= \sqrt{A^2+B^2} \\
\alpha &= \arctan\left(-\frac{B}{A}\right) \\
y &= C\cos(kx+\alpha) \\
&= C\,\mathrm{Re}[e^{j(kx+\alpha)}] \\
&= Re[(Ce^{j\alpha})e^{jkx}]
\end{aligned}
$$

1.11

(a)

$$
\begin{aligned}
x &= A\cos(\omega t+\alpha) \\
A &= 5\,\mathrm{cm} \\
f &= 1\,\mathrm{Hz} \\
\omega &= 2\pi f \\
&= 2\pi\,\mathrm{rad/s} \\
\alpha &= \pm\frac{\pi}{2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
x\left(\frac{8}{3}\right) &= 5\cos\left(2\pi\frac{8}{3}+\alpha\right) \\
&= \pm 4.33\,\mathrm{cm} \\
\frac{dx}{dt} &= -A\omega\sin(\omega t+\alpha) \\
\frac{dx}{dt}\left(\frac{8}{3}\right) &= \pm 15.7\,\mathrm{cm/s} \\
\frac{d^2x}{dt^2} &= -A\omega^2\cos(\omega t+\alpha) \\
\frac{d^2x}{dt^2}\left(\frac{8}{3}\right) &= \mp 171\,\mathrm{cm/s^2}
\end{aligned}
$$

1.12

(a)

$$
\begin{aligned}
v &= 50\,\mathrm{cm/s} \\
T &= 6\,\mathrm{s} \\
\theta_0 &= 30^\circ \\
c &= 300\,\mathrm{cm} \\
A &= \frac{c}{2\pi} \\
&= \frac{150}{\pi}\,\mathrm{cm} \\
\omega &= \frac{2\pi}{T} \\
&= \frac{\pi}{3}\,\mathrm{rad/s} \\
\alpha &= \frac{\pi}{6}\,\mathrm{rad} \\
x &= \frac{150}{\pi}\cos\left(\frac{\pi}{3}t + \frac{\pi}{6}\right)
\end{aligned}
$$

(b)

$$
\begin{aligned}
x(2\,\mathrm{s}) &= -41.3\,\mathrm{cm} \\
\frac{dx}{dt} &= -50\sin\left(\frac{\pi}{3}t + \frac{\pi}{6}\right) \\
\frac{dx}{dt}(2\,\mathrm{s}) &= -25\,\mathrm{cm/s} \\
\frac{d^2x}{dt^2} &= -\frac{50\pi}{3}\cos\left(\frac{\pi}{3}t + \frac{\pi}{6}\right) \\
\frac{d^2x}{dt^2}(2\,\mathrm{s}) &= 45\,\mathrm{cm/s^2}
\end{aligned}
$$

2 The superposition of periodic motions

2.1

(a)

$$
\begin{aligned}
z &= \sin\omega t + \cos\omega t \\
&= \sqrt{2}\cos\left(\omega t - \frac{\pi}{4}\right) \\
&= \sqrt{2}e^{j\left(\omega t - \frac{\pi}{4}\right)}
\end{aligned}
$$

(b)

$$
\begin{aligned}
z &= \cos(\omega t - \pi/3) - \cos\omega t \\
&= \cos\omega t \cos\frac{\pi}{3} + \sin\omega t \sin\frac{\pi}{3} - \cos\omega t \\
&= -\frac{1}{2}\cos\omega t + \frac{\sqrt{3}}{2}\sin\omega t \\
&= \cos(\omega t + 2\pi/3) \\
&= e^{j(\omega t + 2\pi/3)}
\end{aligned}
$$

(c)

$$
\begin{aligned}
z &= 3\cos\omega t + 2\sin\omega t \\
&= \sqrt{13}\cos(\omega t + \arctan -2/3)
\end{aligned}
$$

(d)

$$
\begin{aligned}
z &= \sin\omega t - 2\cos(\omega t - \pi/4) + \cos\omega t \\
&= \sin\omega t - 2(\cos\omega t \cos\pi/4 + \sin\omega t \sin\pi/4) + \cos\omega t \\
&= \sin\omega t - \sqrt{2}\cos\omega t - \sqrt{2}\sin\omega t + \cos\omega t \\
&= (1-\sqrt{2})\cos\omega t + (1-\sqrt{2})\sin\omega t \\
&= (1-\sqrt{2})\sqrt{2}\cos(\omega t - \pi/4) \\
&= (\sqrt{2}-2)\cos(\omega t - \pi/4) \\
&= (2-\sqrt{2})\cos(\omega t + 3\pi/4)
\end{aligned}
$$

2.2

$$
\begin{aligned}
x &= A_1\cos\omega t + A_2\cos(\omega t + \alpha_1) + A_3\cos(\omega t + \alpha_1 + \alpha_2) \\
&= A_1\cos\omega t + A_2(\cos\omega t\cos\alpha_1 - \sin\omega t \sin\alpha_1) \\
&\qquad + A_3(\cos\omega t\cos(\alpha_1+\alpha_2) - \sin\omega t\sin(\alpha_1+\alpha_2)) \\
&= (A_1 + A_2\cos\alpha_1 + A_3\cos(\alpha_1+\alpha_2))\cos\omega t \\
&\qquad - (A_2\sin\alpha_1 + A_3\sin(\alpha_1+\alpha_2))\sin\omega t \\
A &= \sqrt{(A_1 + A_2\cos\alpha_1 + A_3\cos(\alpha_1+\alpha_2))^2 + (A_2\sin\alpha_1 + A_3\sin(\alpha_1+\alpha_2))^2} \\
&\approx 0.52\,\mathrm{mm} \\
\alpha &= \arctan\frac{A_2\sin\alpha_1 + A_3\sin(\alpha_1+\alpha_2)}{A_1 + A_2\cos\alpha_1 + A_3\cos(\alpha_1+\alpha_2)} \\
&\approx 0.59\,\mathrm{rad} \\
&\approx 34^\circ
\end{aligned}
$$

2.3

The equation of motion is

$$x = 2A\cos\left(\frac{12\pi - 10\pi}{2}t\right)\cos\left(\frac{12\pi + 10\pi}{2}t\right)$$

with the variation in amplitude given by the term

$$2A\cos\pi t$$

so the beat period is 1 s.

2.4

(a)

$$\omega = 2\pi, \mathrm{rad/s} \Rightarrow f = 1\,\mathrm{Hz}$$

(b)

$$\omega = \frac{25\pi}{2}\,\mathrm{rad/s} \Rightarrow f = \frac{25}{4}\,\mathrm{Hz}$$

(c)

$$\omega = \frac{3+\pi}{2}\,\mathrm{rad/s} \Rightarrow f = \frac{3+\pi}{4\pi}\,\mathrm{Hz}$$

3 The free vibrations of physical systems

3.1

$$\begin{aligned} F &= -kx \\ ma &= -kx \\ k &= -\frac{ma}{x} \\ &= 4.0 \times 10^{-5}\,\mathrm{N/m} \end{aligned}$$

3.2

(a)

$$T_0 = 2\pi\sqrt{\frac{m}{k}}$$

(b) (i)

$$
\begin{aligned}
mx'' &= -2kx \\
x'' &= -\frac{2k}{m}x \\
T &= 2\pi\sqrt{\frac{m}{2k}} \\
&= \frac{T_0}{\sqrt{2}}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
mx'' &= -k\frac{x}{2} \\
x'' &= -\frac{k}{2m}x \\
T &= 2\pi\sqrt{\frac{2m}{k}} \\
&= \sqrt{2}T_0
\end{aligned}
$$

3.3

(a)

$$
\begin{aligned}
y &= A\cos\omega t \\
y' &= -\omega A\sin\omega t \\
y'' &= -\omega^2 A\cos\omega t \\
g &= \omega^2 A\cos\omega t \\
\omega t &= \arccos\frac{g}{\omega^2 A} \\
t &= \frac{1}{\omega}\arccos\frac{g}{\omega^2 A} \\
y &= A\cos\arccos\frac{g}{\omega^2 A} \\
&= \frac{g}{\omega^2} \\
&= 2.5\,\mathrm{cm}
\end{aligned}
$$

(b)

$$\begin{aligned}
v &= -\omega A \sin \omega t \\
&= -\omega A \sin \arccos \frac{g}{\omega^2 A} \\
&\approx 0.87\,\mathrm{m/s} \\
\frac{1}{2} m v^2 &= mgh \\
h &= \frac{v^2}{2g} \\
&\approx 3.8\,\mathrm{cm} \\
\Delta h &\approx 1.3\,\mathrm{cm}
\end{aligned}$$

3.4

(a)

$$\begin{aligned}
my'' &= -g\rho A y \\
y'' &= -\frac{g\rho A}{m} y \\
\omega &= \sqrt{\frac{g\rho A}{m}} \\
&= \sqrt{\frac{g}{l}}
\end{aligned}$$

3.5

$$T = 2\pi \sqrt{\frac{2L}{3g}}$$

3.6

$$T = 2\pi \sqrt{\frac{d}{g}}$$

3.8

(a)

$$\begin{aligned}
mg &= \frac{AY}{l_0} x \\
x &= \frac{mgl_0}{AY} \\
&= \frac{mgl_0}{\pi (d/2)^2 Y} \\
&= 0.25\,\mathrm{mm}
\end{aligned}$$

(b)

$$
\begin{aligned}
F_u &= u\pi(d/2)^2 \\
&\approx 215.98\,\mathrm{N} \\
k &= \frac{AY}{L} \\
&= \frac{\pi(d/2)^2 Y}{L} \\
&= \frac{\pi d^2 Y}{4L} \\
&\approx 19\,634.95\,\mathrm{N/m} \\
F_u &= kx_u \\
x_u &= \frac{F_u}{k} \\
&\approx 1.1\,\mathrm{cm} \\
mgh &= \frac{1}{2}\frac{AY}{L}x_u^2 - mgx_u \\
h &= \frac{\pi(d/2)^2 Y x_u^2}{2mgL} - x_u \\
&= \frac{\pi d^2 Y x_u^2}{8mgL} - x_u \\
&= 0.23\,\mathrm{m}
\end{aligned}
$$

3.9

(a)

$$
\begin{aligned}
\rho_{\text{steel}} &= 7850\,\mathrm{kg/m^3} \\
V_{\text{sphere}} &= \frac{4}{3}\pi r^3 \\
F_u &= Au \\
&= \pi r^2 u \\
&\approx 3455.75\,\mathrm{N} \\
mg &= F_u \\
m &= \frac{F_u}{g} \\
&\approx 352.3\,\mathrm{kg} \\
\rho V &= m \\
\rho \frac{4}{3}\pi r^3 &= m \\
r &= \sqrt[3]{\frac{3m}{4\pi\rho}} \\
&= 22\,\mathrm{cm}
\end{aligned}
$$

(b)

$$
\begin{aligned}
M &= -\frac{\pi n r^4}{2l}\theta \\
c &= \frac{\pi n r^4}{2l} \\
T &= 2\pi\sqrt{\frac{I}{c}} \\
&= 2\pi\sqrt{\frac{2MR^2/5}{\pi n r^4/2l}} \\
&= 2\pi\sqrt{\frac{4lMR^2}{5\pi n r^4}} \\
&= 66\,\mathrm{s}
\end{aligned}
$$

3.10

(a)

$$\begin{aligned}
Y &= \frac{\text{stress}}{\text{strain}} \\
&= \frac{F/A}{\Delta l/l_0} \\
&= \frac{mg/A}{\Delta l/l_0} \\
&= \frac{mgl_0}{\Delta l A} \\
&= 5.9 \times 10^{11}\,\text{N/m}^2
\end{aligned}$$

(b)

$$\begin{aligned}
y &= \frac{4L^3}{Yab^3}F \\
F &= \frac{Yab^3}{4L^3}y \\
k &= \frac{Yab^3}{4L^3} \\
\omega_y &= \sqrt{\frac{k}{m}} \\
&= \sqrt{\frac{Yab^3}{4L^3m}} \\
\omega_x &= \sqrt{\frac{Ya^3b}{4L^3m}} \\
\frac{\omega_y}{\omega_x} &= \sqrt{\frac{ab^3}{a^3b}} \\
&= \frac{b}{a}
\end{aligned}$$

(c) 3/2

3.11

(a)

$$\omega = \sqrt{\frac{A\gamma p}{lm}}$$

3.14

(a)

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

(b)

$$
\begin{aligned}
\omega &= \frac{\sqrt{3}}{2}\omega_0 \\
\omega^2 &= \frac{3}{4}\omega_0^2 \\
\omega_0^2 - \frac{\gamma^2}{4} &= \frac{3}{4}\omega_0^2 \\
\frac{1}{4}\omega_0^2 &= \frac{\gamma^2}{4} \\
\omega_0^2 &= \gamma^2 \\
\omega_0 &= \gamma \\
&= \frac{b}{m} \\
b &= m\omega_0 \\
&= m\sqrt{\frac{k}{m}} \\
&= 4\,\mathrm{N/(m/s)}
\end{aligned}
$$

3.15

(a)

$$
\begin{aligned}
\overline{E}_0 e^{-\gamma} &= \frac{1}{2}\overline{E}_0 \\
e^{-\gamma} &= \frac{1}{2} \\
-\gamma &= \ln\frac{1}{2} \\
\gamma &= \ln 2 \\
Q_0 &= \frac{\omega_0}{\gamma} \\
&= \frac{2\pi f}{\gamma} \\
&= \frac{512\pi}{\ln 2} \\
&\approx 2321
\end{aligned}
$$

(b)

$$Q = 2Q_0$$

(c)

$$
\begin{aligned}
\gamma &= \frac{1}{4} \\
Q &= \frac{\omega_0}{\gamma} \\
&= 4\sqrt{\frac{k}{m}} \\
&= 12 \\
\gamma &= \frac{b}{m} \\
b &= \gamma m \\
&= 0.025\,\mathrm{N/(m/s)}
\end{aligned}
$$

3.16

(a)

$$
\begin{aligned}
x &= A\sin\omega t \\
v &= \omega A\cos\omega t \\
a &= -\omega^2 A\sin\omega t \\
E &= \int_0^{1/f} \frac{Ke^2}{c^3}(-\omega^2 A\sin\omega t)^2\,dt \\
&= \frac{Ke^2\omega^4A^2}{c^3}\int_0^{1/f}\sin^2\omega t\,dt \\
&= \frac{Ke^2\omega^4A^2}{c^3}\left[\frac{t}{2} - \frac{1}{4\omega}\sin 2\omega t\right]_0^{1/f} \\
&= \frac{Ke^2\omega^4A^2}{c^3}\left(\frac{1}{2f} - \frac{1}{4\omega}\sin 2\omega\frac{1}{f}\right) \\
&= \frac{Ke^2(2\pi f)^4A^2}{2fc^3} \\
&= \frac{8\pi^4Ke^2f^3A^2}{c^3}
\end{aligned}
$$

(b)

$$
\begin{aligned}
E_0 &= \frac{1}{2}mv^2 \\
&= \frac{m(\omega A)^2}{2} \\
&= 2\pi^2 A^2 f^2 m \\
\frac{Q}{\pi}E &= E_0\left(1-\frac{1}{e}\right) \\
\frac{Q}{\pi}\frac{8\pi^4 K q^2 f^3 A^2}{c^3} &= 2\pi^2 A^2 f^2 m\left(1-\frac{1}{e}\right) \\
Q\frac{4\pi K q^2 f}{c^3} &= m\left(1-\frac{1}{e}\right) \\
Q &= \frac{c^3 m}{4\pi f K q^2}\left(1-\frac{1}{e}\right)
\end{aligned}
$$

3.17

(a)

$$
\begin{aligned}
V &= \pi r^2 y_{\text{left}} \\
V &= \pi(2r)^2 y_{\text{right}} \\
\pi r^2 y_{\text{left}} &= \pi(2r)^2 y_{\text{right}} \\
y_{\text{right}} &= \frac{1}{4}y_{\text{left}} \\
\frac{y_{\text{left}}}{2} + \frac{y_{\text{right}}}{2} &= \frac{y_{\text{left}}}{2} + \frac{y_{\text{left}}}{8} \\
&= \frac{5}{8}y_{\text{left}} \\
U &= mg\frac{5}{8}y \\
&= \frac{5}{8}\rho\pi r^2 ygy \\
&= \frac{5}{8}g\rho\pi r^2 y^2
\end{aligned}
$$

(b)

$$
\begin{aligned}
r(x) &= r + \frac{x}{l} r \\
&= r\left(1 + \frac{x}{l}\right) \\
\frac{dy}{dt}\pi r^2 &= v\pi r(x)^2 \\
&= v\pi\left[r\left(1 + \frac{x}{l}\right)\right]^2 \\
v &= \frac{dy}{dt}\frac{1}{\left(1 + \frac{x}{l}\right)^2} \\
m &= \rho\pi r(x)^2\,dx \\
&= \rho\pi\left[r\left(1 + \frac{x}{l}\right)\right]^2\,dx \\
&= \rho\pi r^2\left(1 + \frac{x}{l}\right)^2\,dx \\
dK &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}\rho\pi r^2\left(1 + \frac{x}{l}\right)^2\,dx\left(\frac{dy}{dt}\frac{1}{\left(1 + \frac{x}{l}\right)^2}\right)^2 \\
&= \frac{1}{2}\rho\frac{\pi r^2\,dx}{(1 + x/l)^2}\left(\frac{dy}{dt}\right)^2
\end{aligned}
$$

(c)

$$
\begin{aligned}
K &= \frac{1}{2}\rho\pi r^2 h\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}\rho\pi(2r)^2 h\left(\frac{dy}{dt}\right)^2 + \int_0^l dK \\
&= \frac{5}{2}\rho\pi r^2 h\left(\frac{dy}{dt}\right)^2 + \int_0^l \frac{1}{2}\rho\frac{\pi r^2\,dx}{(1 + x/l)^2}\left(\frac{dy}{dt}\right)^2 \\
&= \frac{5}{2}\rho\pi r^2 h\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}\rho\pi r^2\int_0^l \frac{1}{(1 + x/l)^2}\,dx\left(\frac{dy}{dt}\right)^2 \\
&= \frac{1}{4}\rho\pi r^2\left(l + \frac{5}{2}h\right)\left(\frac{dy}{dt}\right)^2
\end{aligned}
$$

(d)

$$K + U = E$$

$$\frac{1}{4}\rho\pi r^2\left(l+\frac{5}{2}h\right)\left(\frac{dy}{dt}\right)^2+\frac{5}{8}g\rho\pi r^2y^2 = E$$

$$m = \frac{1}{2}\rho\pi r^2\left(l+\frac{5}{2}h\right)$$

$$k = \frac{5}{4}g\rho\pi r^2$$

$$\begin{aligned} T &= 2\pi\sqrt{\frac{m}{k}} \\ &= 2\pi\sqrt{\frac{\frac{1}{2}\rho\pi r^2\left(l+\frac{5}{2}h\right)}{\frac{5}{4}g\rho\pi r^2}} \\ &= 2\pi\sqrt{\frac{2h}{g}} \end{aligned}$$

3.19

(a)

$$m\frac{d^2x}{dt^2}+2k(x+l-l_0) = 0$$

(b)

$$\begin{aligned} T &= k(l'-l_0) \\ &= k(\sqrt{l^2+y^2}-l_0) \\ F &= 2T\sin\theta \\ &= 2k(\sqrt{l^2+y^2}-l_0)\frac{y}{\sqrt{l^2+y^2}} \\ &= 2k\left(1-\frac{l_0}{\sqrt{l^2+y^2}}\right)y \\ &\approx 2k\left(1-\frac{l_0}{l}\right)y \end{aligned}$$

$$m\frac{d^2y}{dt^2}+2k\left(1-\frac{l_0}{l}\right)y = 0$$

(c)

$$
\begin{aligned}
T_x &= 2\pi\sqrt{\frac{m}{2k}} \\
T_y &= 2\pi\sqrt{\frac{m}{2k\left(1-\frac{l_0}{l}\right)}} \\
\frac{T_x}{T_y} &= \frac{2\pi\sqrt{m/2k}}{2\pi\sqrt{\frac{m}{2k(1-l/l_0)}}} \\
&= \sqrt{\frac{m}{2k}\frac{2k(1-l/l_0)}{m}} \\
&= \sqrt{1-l/l_0}
\end{aligned}
$$

(d)

$$
\begin{aligned}
x &= A_x\cos\left(\sqrt{\frac{2k}{m}}t+\phi_x\right) \\
A_0 &= A_x\cos\phi_x \\
0 &= -\sqrt{\frac{2k}{m}}A_x\sin\phi_x \\
\tan\phi_x &= 0 \\
\phi_x &= 0 \\
A_x &= A_0 \\
x &= A_0\cos\sqrt{\frac{2k}{m}}t
\end{aligned}
$$

$$
\begin{aligned}
y &= A_y\cos\left(\sqrt{\frac{2k(1-l_0/l)}{m}}t+\phi_y\right) \\
A_0 &= A_y\cos\phi_y \\
0 &= -\sqrt{\frac{2k(1-l_0/l)}{m}}A_y\sin\phi_y \\
\tan\phi_y &= 0 \\
\phi_y &= 0 \\
A_y &= A_0 \\
y &= A_0\cos\sqrt{\frac{2k(1-l_0/l)}{m}}t
\end{aligned}
$$