

Advanced Engineering Mathematics Ordinary Differential Equations Notes

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1 Introduction to Differential Equations

1.1 Definitions and Terminology

- An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation**
- An **ordinary differential equation** (ODE) is a differential equation that contains only ordinary (i.e. non-partial) derivatives of one or more functions with respect to a single independent variable
- A **partial differential equation** is a differential equation that contains only partial derivatives of one or more functions of two or more independent variables
- The **order** of a differential equation is the order of the highest derivative in the equation
- First order ODEs are sometimes written in the **differential form**

$$M(x, y) dx + N(x, y) dy = 0$$

- n -th order ODEs in one dependent variable can be expressed by the **general form**

$$F(x, y, y', \dots, y^{(n)}) = 0$$

- It's possible to solve ODEs in the general form uniquely for the highest derivative $y^{(n)}$ in terms of the other $n + 1$ variables, allowing them to be expressed in the **normal form**

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)})$$

- An n -th order ODE is said be **linear** in the variable y if it can be expressed in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y - g(x) = 0$$

i.e. the dependent variable y and all of its derivatives aren't raised to a power or used in nonlinear functions like e^y or $\sin y$, and the coefficients a_0, a_1, \dots, a_n depend at most on the independent variable x

- A **nonlinear** ODE is one that is not linear
- A **solution** to an ODE is a function ϕ , defined on an interval I and possessing at least n derivatives that are continuous on I , such that

$$F(x, \phi(x), \phi'(x), \dots, \phi^n(x)) = 0 \text{ for all } x \text{ in } I.$$

- The **interval of definition**, **interval of validity**, or the **domain** of a solution is the interval over which the solution is valid
- A solution of a differential equation that is 0 on an interval I is said to be a **trivial solution**
- Because solutions to differential equations must be differentiable over their interval of validity, discontinuities, etc. must be excluded from the interval
- An **explicit solution** to an ODE is one where the dependent variable is expressed solely in terms of the independent variable and constants
- An **implicit solution** to an ODE is a relation $G(x, y) = 0$ over an interval I provided there exists at least one function ϕ that satisfies the relation as well as the ODE on I
- When solving a first-order ODE we usually obtain a solution containing a single arbitrary constant or parameter c . A solution containing an arbitrary constant represents a set of solution called a **one-parameter family of solutions**
- When solving an n -th order differential equation we usually obtain an **n -parameter family of solutions**
- A solution of a differential equation that is free from arbitrary parameters is called a **particular solution**

- A **singular solution** is a solution to a differential equation that isn't a member of a family of solutions
- A **system of ODEs** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable. A solution of such a system is a differentiable function for each equation defined on a common interval I that satisfy each equation of the system on that interval

1.2 Initial Value Problems

- An **initial value problem** is the problem of solving a differential equation with some given **initial conditions**, e.g. solve

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)})$$

subject to

$$y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$$

- The domain of $y = f(x)$ differs depending on how it's considered:
 - As a function its domain is all real numbers for which it's defined
 - As a solution of a differential equation its domain is a single interval over which it's defined and differentiable
 - As a solution of an initial value problem its domain is a single interval over which it's defined, differentiable, and contains the initial conditions
- An initial value problem may not have any solutions. If it does it may have multiple.
- First-order initial value problems of the form

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

are guaranteed to have a unique solution over an interval I containing x_0 if $f(x, y)$ and $\partial f / \partial y$ are continuous

1.3 Differential Equations as Mathematical Models

- A **mathematical model** is a mathematical description of a system or phenomenon
- The **level of resolution** of a model determines how many variables are included in the model

- A simple model of the growth of a population P is

$$\frac{dP}{dt} = kP$$

where $k > 0$

- A simple model of radioactive decay of an amount of substance A is

$$\frac{dA}{dt} = kA$$

where $k < 0$

- Newton's empirical law of cooling/warming states that the rate of change of the temperature of a body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium

$$\frac{dT}{dt} = k(T - T_m)$$