

Advanced Engineering Mathematics Vectors, Matrices, and Vector Calculus by Dennis G. Zill Problems

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1 Vectors

1.1 Vectors in 2-Space

1.1.1

- (a) $3\mathbf{a} = 6\mathbf{i} + 12\mathbf{j}$
- (b) $\mathbf{a} + \mathbf{b} = \mathbf{i} + 8\mathbf{j}$
- (c) $\mathbf{a} - \mathbf{b} = 3\mathbf{i}$
- (d) $\|\mathbf{a} + \mathbf{b}\| = \sqrt{1 + 8^2} = \sqrt{65}$
- (e) $\|\mathbf{a} - \mathbf{b}\| = 3$

1.1.9

- (a) $4\mathbf{a} - 2\mathbf{b} = \langle 6, -14 \rangle$
- (b) $-3\mathbf{a} - 5\mathbf{b} = \langle 2, 4 \rangle$

1.1.15

$$\overrightarrow{P_1P_2} = \langle 2, 5 \rangle$$

1.1.19

$$(1, 18)$$

1.1.21

- (a) Yes
- (b) Yes
- (c) Yes

(d) No

(e) Yes

(f) Yes

1.1.25

$$(a) \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\langle 2, 2 \rangle}{\sqrt{2^2 + 2^2}} = \frac{1}{2\sqrt{2}} \langle 2, 2 \rangle = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$(b) \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$

1.1.31

$$2 \frac{\mathbf{a}}{\|\mathbf{a}\|} = 2 \frac{\langle 3, 7 \rangle}{\sqrt{3^2 + 7^2}} = \frac{2}{\sqrt{58}} \langle 3, 7 \rangle = \left\langle \frac{6}{\sqrt{58}}, \frac{14}{\sqrt{58}} \right\rangle$$

1.1.37

$$\mathbf{x} = -(\mathbf{a} + \mathbf{b})$$

1.1.41

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$$

$$\mathbf{b} = \mathbf{i} + \mathbf{j}$$

$$\mathbf{c} = \mathbf{i} - \mathbf{j}$$

$$\mathbf{i} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$$

$$\mathbf{j} = \frac{1}{2}(\mathbf{b} - \mathbf{c})$$

$$\mathbf{a} = 2 \left(\frac{1}{2}(\mathbf{b} + \mathbf{c}) \right) + 3 \left(\frac{1}{2}(\mathbf{b} - \mathbf{c}) \right)$$

$$= \mathbf{b} + \mathbf{c} + \frac{3}{2}\mathbf{b} - \frac{3}{2}\mathbf{c}$$

$$= \frac{5}{2}\mathbf{b} - \frac{1}{2}\mathbf{c}$$

1.1.43

$$\begin{aligned}
 y &= \frac{1}{4}x^2 + 1 \\
 y(2) &= 2 \\
 y' &= \frac{1}{2}x \\
 y'(2) &= 1 \\
 \mathbf{v} &= \pm \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle
 \end{aligned}$$

1.1.45

(a)

$$\begin{aligned}
 \mathbf{F}_n &= \mathbf{F} \cos \theta \\
 \mathbf{F}_g &= \mathbf{F} \sin \theta \\
 ||\mathbf{F}_f|| &= \mu ||\mathbf{F}_n|| \\
 ||-\mathbf{F}_g|| &= \mu ||\mathbf{F}_n|| \\
 ||-\mathbf{F} \sin \theta|| &= \mu ||\mathbf{F} \cos \theta|| \\
 ||\mathbf{F}|| \sin \theta &= \mu ||\mathbf{F}|| \cos \theta \\
 \tan \theta &= \mu
 \end{aligned}$$

(b) $\theta = \arctan \mu \approx 31^\circ$

1.1.47

$$\begin{aligned}
 F_x &= \frac{qQ}{4\pi\epsilon_0} \int_{-a}^a \frac{L dy}{2a(L^2 + y^2)^{3/2}} \\
 &= \frac{LqQ}{8\pi\epsilon_0} \int_{-a}^a (L^2 + y^2)^{-3/2} dy \\
 &= \frac{LqQ}{8\pi\epsilon_0} \frac{2a}{L^2 \sqrt{a^2 + L^2}} \\
 &= \frac{aqQ}{4\pi\epsilon_0 L \sqrt{a^2 + L^2}} \\
 F_y &= -\frac{qQ}{4\pi\epsilon_0} \int_{-a}^a \frac{y dy}{2a(L^2 + y^2)^{3/2}} \\
 &= 0 \\
 \mathbf{F} &= \left\langle \frac{1}{4\pi\epsilon_0} \frac{qQ}{L \sqrt{a^2 + L^2}}, 0 \right\rangle
 \end{aligned}$$

1.1.49

Let the three sides of the triangle be vectors **a**, **b**, and **c**. The triangle is closed so it must be the case that

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0.$$

This gives

$$\mathbf{c} = -(\mathbf{a} + \mathbf{b}).$$

The vector from the midpoint of side **a** to the midpoint of side **b** is

$$\left(\mathbf{a} + \frac{1}{2}\mathbf{b}\right) - \frac{1}{2}\mathbf{a} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$$

which is parallel with **c** and half its length.

1.2 Vectors in 3-Space**1.2.7**

A plane at $z = 5$ parallel with the x - y plane.

1.2.9

A line parallel to the z axis at $x = 2$ and $y = 3$.

1.2.13

(a) $(0, 5, 4)$, $(-2, 0, 4)$, $(-2, 5, 0)$

(b) $(-2, 5, -2)$

(c) $(3, 5, 4)$

1.2.15

The planes $x = 0$, $y = 0$, and $z = 0$.

1.2.17

$(-1, 2, -3)$

1.2.19

The planes $z = \pm 5$.

1.2.21

$$\sqrt{(6-3)^2 + (4+1)^2 + (8-2)^2} = \sqrt{9+25+36} = \sqrt{70}$$

1.2.31

$$\begin{aligned}\sqrt{(2-x)^2 + (1-2)^2 + (1-3)^2} &= \sqrt{21} \\ (2-x)^2 + 1 + 4 &= 21 \\ (2-x)^2 &= 16 \\ 2-x &= \pm 4 \\ x &= 2 \pm 4 \\ &= -2 \text{ or } 6\end{aligned}$$

1.2.33

$$(4, \frac{1}{2}, \frac{3}{2})$$

1.2.37

$$(-3, -6, 1)$$

1.3 Dot Product

1.3.1

$$\mathbf{a} \cdot \mathbf{b} = 12$$

1.3.11

$$\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right) \mathbf{b} = \frac{12}{30} \mathbf{b} = \langle -\frac{2}{5}, \frac{4}{5}, 2 \rangle$$

1.3.13

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta = 25\sqrt{2}$$

1.3.17

$$\mathbf{a} \cdot \mathbf{v} = 0$$

$$3x_1 + y_1 - 1 = 0$$

$$\mathbf{b} \cdot \mathbf{v} = 0$$

$$-3x_1 + 2y_2 + 2 = 0$$

$$3y_2 + 1 = 0$$

$$y_2 = -\frac{1}{3}$$

$$3x_1 - \frac{1}{3} - 1 = 0$$

$$x_1 = \frac{4}{9}$$

$$\mathbf{v} = \left\langle \frac{4}{9}, -\frac{1}{3}, 1 \right\rangle$$

1.3.19

$$\begin{aligned}\mathbf{a} \cdot \mathbf{c} &= \mathbf{a} \cdot \left(\mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \mathbf{a} \right) \\ &= \mathbf{a} \cdot \mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \mathbf{a} \cdot \mathbf{a} \\ &= 0\end{aligned}$$

1.3.21

$$||\mathbf{a}|| = \sqrt{3^2 + 1^2}$$

$$= \sqrt{10}$$

$$||\mathbf{b}|| = \sqrt{2^2 + 2^2}$$

$$= \sqrt{8}$$

$$= 2\sqrt{2}$$

$$\mathbf{a} \cdot \mathbf{b} = 4$$

$$\theta = \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}|| ||\mathbf{b}||}$$

$$= \arccos \frac{4}{(\sqrt{10})(2\sqrt{2})}$$

$$= \arccos \frac{1}{\sqrt{5}}$$

$$\approx 63^\circ$$

1.3.25

$$||\mathbf{a}|| = \sqrt{1^2 + 2^2 + 3^2}$$

$$= \sqrt{14}$$

$$\cos \alpha = \frac{1}{\sqrt{14}}$$

$$\alpha \approx 75^\circ$$

$$\cos \beta = \frac{2}{\sqrt{14}}$$

$$\beta \approx 58^\circ$$

$$\cos \gamma = \frac{3}{\sqrt{14}}$$

$$\gamma \approx 37^\circ$$

1.3.29

$$\begin{aligned}
 \overrightarrow{AD} &= \langle s, -s, s \rangle \\
 \|\overrightarrow{AD}\| &= s\sqrt{3} \\
 \overrightarrow{AB} &= \langle s, 0, 0 \rangle \\
 \|\overrightarrow{AB}\| &= s \\
 \theta &= \arccos \frac{\overrightarrow{AD} \cdot \overrightarrow{AB}}{\|\overrightarrow{AD}\| \|\overrightarrow{AB}\|} \\
 &= \arccos \frac{s^2}{s^2\sqrt{3}} \\
 &= \arccos \frac{1}{\sqrt{3}} \\
 &\approx 55^\circ
 \end{aligned}$$

1.3.33

$$\begin{aligned}
 \text{comp}_{\mathbf{b}} \mathbf{a} &= \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} \\
 &= \frac{5}{7}
 \end{aligned}$$

1.3.37

$$\begin{aligned}
 \text{comp}_{\overrightarrow{OP}} \mathbf{a} &= \frac{\mathbf{a} \cdot \overrightarrow{OP}}{\|\overrightarrow{OP}\|} \\
 &= \frac{72}{\sqrt{109}}
 \end{aligned}$$

1.3.39

$$\begin{aligned}
 \text{proj}_{\mathbf{b}} \mathbf{a} &= \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \right) \mathbf{b} \\
 &= \frac{35}{25} \mathbf{b} \\
 &= \left\langle -\frac{21}{5}, \frac{28}{5} \right\rangle
 \end{aligned}$$

1.3.43

$$\begin{aligned}
\mathbf{a} + \mathbf{b} &= \langle 3, 4 \rangle \\
\text{proj}_{\mathbf{a}+\mathbf{b}} \mathbf{a} &= \left(\frac{\mathbf{a} \cdot (\mathbf{a} + \mathbf{b})}{(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})} \right) (\mathbf{a} + \mathbf{b}) \\
&= \frac{24}{25} (\mathbf{a} + \mathbf{b}) \\
&= \left\langle \frac{72}{25}, \frac{96}{25} \right\rangle
\end{aligned}$$

1.3.45

$$W = \mathbf{F} \cdot \mathbf{d} = Fd \cos \theta = 1000$$

1.3.47

(a) $W = 0$

(b)

$$\begin{aligned}
\|\mathbf{d}\| &= \sqrt{4^2 + 3^2} \\
&= 5
\end{aligned}$$

$$\mathbf{F} = F \hat{\mathbf{d}}$$

$$= F \frac{\mathbf{d}}{\|\mathbf{d}\|}$$

$$= F \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$$

$$= \langle 24, 18 \rangle$$

$$W = \mathbf{F} \cdot \mathbf{d}$$

$$= 150 \text{ J}$$

1.4 Cross Product**1.4.1**

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 0 & 3 & 5 \end{vmatrix} \\
&= -5\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}
\end{aligned}$$

1.4.11

$$\begin{aligned}\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & -4 \\ -3 & 1 & 1 \end{vmatrix} \\ &= 6\mathbf{i} + 14\mathbf{j} + 4\mathbf{k}\end{aligned}$$

1.4.17

(a)

$$\begin{aligned}\mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} \\ &= \mathbf{j} - \mathbf{k} \\ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 0 & 1 & -1 \end{vmatrix} \\ &= -\mathbf{i} + \mathbf{j} + \mathbf{k}\end{aligned}$$

1.4.19

$2\mathbf{k}$

1.4.21

$$\begin{aligned}\mathbf{k} \times (2\mathbf{i} - \mathbf{j}) &= (\mathbf{k} \times 2\mathbf{i}) - (\mathbf{k} \times \mathbf{j}) \\ &= \mathbf{i} + 2\mathbf{j}\end{aligned}$$

1.4.23

$$\begin{aligned}[(2\mathbf{k}) \times (3\mathbf{j})] \times (4\mathbf{j}) &= (-6\mathbf{i}) \times (4\mathbf{j}) \\ &= -24\mathbf{k}\end{aligned}$$

1.4.37

$12\mathbf{i} - 9\mathbf{j} + 18\mathbf{k}$

1.4.53

$$\begin{aligned}
\mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 6 & -6 \\ \frac{5}{2} & 3 & \frac{1}{2} \end{vmatrix} \\
&= 21\mathbf{i} - 14\mathbf{j} - 21\mathbf{k} \\
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= 4 \times 21 + 6 \times (-14) \\
&= 0
\end{aligned}$$

They are coplanar.

1.5 Lines and Planes in 3-Space**1.5.1**

$$\mathbf{r} = \langle 1, 2, 1 \rangle + t\langle 2, 3, -3 \rangle$$

1.5.7

$$\begin{aligned}
x &= 2 + 4t \\
y &= 3 - 4t \\
z &= 5 + 3t
\end{aligned}$$

1.5.13

$$\begin{aligned}
x &= 1 + 9t \\
y &= 4 + 10t \\
z &= -9 + 7t \\
\frac{x-1}{9} &= \frac{y-4}{10} = \frac{z+9}{7}
\end{aligned}$$

1.5.19

$$\begin{aligned}
x &= 4 + 3t \\
y &= 6 + \frac{1}{2}t \\
z &= -7 - \frac{3}{2}t \\
\frac{x-4}{3} &= \frac{y-6}{1/2} = -\frac{z+7}{3/2}
\end{aligned}$$

1.5.23

$$\begin{aligned}x &= 6 + 2t \\y &= 4 - 3t \\z &= -2 + 6t\end{aligned}$$

1.5.25

$$\begin{aligned}x &= 2 + t \\y &= -2 \\z &= 15\end{aligned}$$

1.5.29

$$(0, 5, 15), (5, 0, \frac{15}{2}), (10, -5, 0)$$

1.5.31

$$\begin{aligned}4 + t_x &= 6 + 2t_x \\t_x &= -2\end{aligned}$$

$$\begin{aligned}5 + t_y &= 11 + 4t_y \\t_y &= -2\end{aligned}$$

$$\begin{aligned}-1 + 2t_z &= -3 + t_z \\t_z &= -2\end{aligned}$$

$$(2, 3, -5)$$

1.5.35

$$\begin{aligned}\mathbf{a} &= \langle -1, 2, -2 \rangle \\||\mathbf{a}|| &= 3 \\ \mathbf{b} &= \langle 2, 3, -6 \rangle \\||\mathbf{b}|| &= 7 \\ \theta &= \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}|| ||\mathbf{b}||} \\ &\approx 40.37^\circ\end{aligned}$$

1.5.37

$$\begin{aligned}
 \mathbf{a} &= \langle 1, 1, 1 \rangle \\
 \mathbf{b} &= \langle -2, 1, -5 \rangle \\
 \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ -2 & 1 & -5 \end{vmatrix} \\
 &= \langle -6, 3, 3 \rangle \\
 x &= 4 - 6t \\
 y &= 1 + 3t \\
 z &= 6 + 3t
 \end{aligned}$$

1.5.39

$$\begin{aligned}
 \langle 2, -3, 4 \rangle \cdot (\mathbf{r} - \langle 5, 1, 3 \rangle) &= 0 \\
 2(x - 5) - 3(y - 1) + 4(z - 3) &= 0 \\
 2x - 3y + 4z - 19 &= 0
 \end{aligned}$$

1.5.45

$$\begin{aligned}
 \mathbf{a} &= \langle 3, 5, 2 \rangle \\
 \mathbf{b} &= \langle 2, 3, 1 \rangle \\
 \mathbf{c} &= \langle -1, -1, 4 \rangle \\
 \mathbf{a} - \mathbf{c} &= \langle 4, 6, -2 \rangle \\
 \mathbf{b} - \mathbf{c} &= \langle 3, 4, -3 \rangle \\
 (\mathbf{a} - \mathbf{c}) \times (\mathbf{b} - \mathbf{c}) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 4 & 6 & -2 \\ 3 & 4 & -3 \end{vmatrix} \\
 &= \langle -10, 6, -2 \rangle \\
 \mathbf{n} \cdot (\mathbf{r} - \mathbf{c}) &= 0 \\
 \langle -10, 6, -2 \rangle \cdot (\langle x, y, z \rangle - \langle -1, -1, 4 \rangle) &= 0 \\
 -10(x + 1) + 6(y + 1) - 2(z - 4) &= 0 \\
 -10x + 6y - 2z + 4 &= 0
 \end{aligned}$$

1.5.51

$$\begin{aligned}
 \langle 1, 1, -4 \rangle \cdot (\mathbf{r} - \langle 2, 3, -5 \rangle) &= 0 \\
 (x - 2) + (y - 3) - 4(z + 5) &= 0 \\
 x + y - 4z &= 25
 \end{aligned}$$

1.5.63

- (a) Not perpendicular
- (b) Not perpendicular
- (c) Perpendicular
- (d) Perpendicular

1.5.65

$$5x - 4y - 9t = 8$$

$$x + 4y + 3t = 4$$

$$6x - 6t = 12$$

$$x = 2 + t$$

$$y = \frac{1}{2} - t$$

$$z = t$$

1.5.69

$$2(1 + 2t) - 3(2 - t) + 2(-3t) = -7$$

$$t = -3$$

$$x = -5$$

$$y = 5$$

$$z = 9$$

1.5.73

$$\begin{aligned}x + y - 4t &= 2 \\ 2x - y + t &= 10\end{aligned}$$

$$\begin{aligned}3x - 3t &= 12 \\ x &= 4 + t\end{aligned}$$

$$\begin{aligned}2(4 + t) - y + t &= 10 \\ 8 + 2t - y + t &= 10 \\ y &= -2 + 3t\end{aligned}$$

$$z = t$$

$$\begin{aligned}x &= 5 + t \\ y &= 6 + 3t \\ z &= -12 + t\end{aligned}$$

1.5.75

$$\begin{aligned}\mathbf{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 5 \\ 1 & 1 & 1 \end{vmatrix} \\ &= \langle -6, 2, 4 \rangle\end{aligned}$$

$$\begin{aligned}\mathbf{n} \cdot (\mathbf{r} - \langle 4, 0, 1 \rangle) &= 0 \\ -6(x - 4) + 2y + 4(z - 1) &= 0 \\ -6x + 2y + 4z &= -20 \\ 3x - y - 2z &= 10\end{aligned}$$

1.6 Vector Spaces**1.6.1**

Violates axiom 6

1.6.3

Violates axiom 10

1.6.5

Vector space

1.6.7

Violates axiom 2

1.6.9

Vector space

1.6.11

Subspace

1.6.13

Not a subspace

1.6.15

Subspace

1.6.17

Subspace

1.6.19

Not a subspace

1.6.23

(a)

$$k_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 + k_3 \mathbf{u}_3 = \mathbf{0}$$

$$k_1 \langle 1, 0, 0 \rangle + k_2 \langle 1, 1, 0 \rangle + k_3 \langle 1, 1, 1 \rangle = \mathbf{0}$$

$$k_3 = 0$$

$$k_2 + k_3 = 0$$

$$k_2 = 0$$

$$k_1 + k_2 + k_3 = 0$$

$$k_1 = 0$$

(b)

$$\mathbf{a} = 7\mathbf{u}_1 - 12\mathbf{u}_2 + 8\mathbf{u}_3$$

1.6.25

Dependent

1.6.27

Independent

1.6.29 $f(x)$ is undefined at $x = -3$ and $x = -1$.**1.6.31**

$$\begin{aligned}
||x|| &= \sqrt{(x, x)} \\
&= \sqrt{\int_0^{2\pi} x^2 dx} \\
&= \sqrt{\left[\frac{1}{3}x^3\right]_0^{2\pi}} \\
&= \sqrt{\frac{8}{3}\pi^3} \\
||\sin x|| &= \sqrt{(\sin x, \sin x)} \\
&= \sqrt{\int_0^{2\pi} \sin^2 x dx} \\
&= \sqrt{\left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_0^{2\pi}} \\
&= \sqrt{\pi}
\end{aligned}$$

1.7 Gram–Schmidt Orthogonalization Process

1.7.1

$$\begin{aligned}\left\langle \frac{12}{13}, \frac{5}{13} \right\rangle \cdot \left\langle \frac{5}{13}, -\frac{12}{13} \right\rangle &= 0 \\ \sqrt{\left(\frac{12}{13}\right)^2 + \left(\frac{5}{13}\right)^2} &= 1 \\ \mathbf{u} &= \left(\left\langle 4, 2 \right\rangle \cdot \left\langle \frac{12}{13}, \frac{5}{13} \right\rangle \right) \left\langle \frac{12}{13}, \frac{5}{13} \right\rangle \\ &\quad + \left(\left\langle 4, 2 \right\rangle \cdot \left\langle \frac{5}{13}, -\frac{12}{13} \right\rangle \right) \left\langle \frac{5}{13}, -\frac{12}{13} \right\rangle \\ &= \left(\frac{58}{13} \right) \left\langle \frac{12}{13}, \frac{5}{13} \right\rangle - \left(\frac{4}{13} \right) \left\langle \frac{5}{13}, -\frac{12}{13} \right\rangle\end{aligned}$$

1.7.3

$$\begin{aligned}\langle 1, 0, 1 \rangle \cdot \langle 0, 1, 0 \rangle &= 0 \\ \langle 1, 0, 1 \rangle \cdot \langle -1, 0, 1 \rangle &= 0 \\ \langle 0, 1, 0 \rangle \cdot \langle -1, 0, 1 \rangle &= 0 \\ B' &= \left\{ \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle, \langle 0, 1, 0 \rangle, \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle \right\} \\ \mathbf{u} &= -\frac{3}{\sqrt{2}} \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle + 7 \langle 0, 1, 0 \rangle - \frac{23}{\sqrt{2}} \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle\end{aligned}$$

1.7.5

(a)

$$B = \{\langle -3, 2 \rangle, \langle -1, -1 \rangle\}$$

$$\mathbf{v}_1 = \mathbf{u}_1$$

$$= \langle -3, 2 \rangle$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{v}_1} \mathbf{u}_2$$

$$= \langle -1, -1 \rangle - \left(\frac{\langle -1, -1 \rangle \cdot \langle -3, 2 \rangle}{\langle -3, 2 \rangle \cdot \langle -3, 2 \rangle} \right) \langle -3, 2 \rangle$$

$$= \langle -1, -1 \rangle - \frac{1}{13} \langle -3, 2 \rangle$$

$$= \left\langle -\frac{10}{13}, -\frac{15}{13} \right\rangle$$

$$\mathbf{w}_1 = \left\langle -\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right\rangle$$

$$\mathbf{w}_2 = \sqrt{\frac{169}{325}} \left\langle -\frac{10}{13}, -\frac{15}{13} \right\rangle$$

$$= \frac{\sqrt{13}}{5} \left\langle -\frac{10}{13}, -\frac{15}{13} \right\rangle$$

$$= \left\langle -\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}} \right\rangle$$

1.7.9

$$B = \{\langle 1, 1, 0 \rangle, \langle 1, 2, 2 \rangle, \langle 2, 2, 1 \rangle\}$$

$$\mathbf{v}_1 = \langle 1, 1, 0 \rangle$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{v}_1} \mathbf{u}_2$$

$$= \langle 1, 2, 2 \rangle - \left(\frac{\langle 1, 2, 2 \rangle \cdot \langle 1, 1, 0 \rangle}{\langle 1, 1, 0 \rangle \cdot \langle 1, 1, 0 \rangle} \right) \langle 1, 1, 0 \rangle$$

$$= \langle 1, 2, 2 \rangle - \frac{3}{2} \langle 1, 1, 0 \rangle$$

$$= \langle -\frac{1}{2}, \frac{1}{2}, 2 \rangle$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{\mathbf{v}_1} \mathbf{u}_3 - \text{proj}_{\mathbf{v}_2} \mathbf{u}_3$$

$$= \langle 2, 2, 1 \rangle - \left(\frac{\langle 2, 2, 1 \rangle \cdot \langle 1, 1, 0 \rangle}{\langle 1, 1, 0 \rangle \cdot \langle 1, 1, 0 \rangle} \right) \langle 1, 1, 0 \rangle$$

$$- \left(\frac{\langle 2, 2, 1 \rangle \cdot \langle -\frac{1}{2}, \frac{1}{2}, 2 \rangle}{\langle -\frac{1}{2}, \frac{1}{2}, 2 \rangle \cdot \langle -\frac{1}{2}, \frac{1}{2}, 2 \rangle} \right) \langle -\frac{1}{2}, \frac{1}{2}, 2 \rangle$$

$$= \langle 2, 2, 1 \rangle - 2\langle 1, 1, 0 \rangle - \frac{4}{9} \langle -\frac{1}{2}, \frac{1}{2}, 2 \rangle$$

$$= \langle \frac{2}{9}, -\frac{2}{9}, \frac{1}{9} \rangle$$

$$\mathbf{w}_1 = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle$$

$$\mathbf{w}_2 = \langle -\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}} \rangle$$

$$\mathbf{w}_3 = 3 \langle \frac{2}{9}, -\frac{2}{9}, \frac{1}{9} \rangle$$

$$= \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle$$

1.7.17

$$B = \{1, x, x^2\}$$

$$\mathbf{v}_1 = \mathbf{u}_1$$

$$= 1$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{v}_1} \mathbf{u}_2$$

$$= \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1$$

$$= x - \frac{\int_{-1}^1 x \, dx}{\int_{-1}^1 dx}$$

$$= x - \frac{\left[\frac{1}{2} x^2 \right]_{-1}^1}{2}$$

$$= x$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{\mathbf{v}_1} \mathbf{u}_3 - \text{proj}_{\mathbf{v}_2} \mathbf{u}_3$$

$$= \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2$$

$$= x^2 - \frac{\int_{-1}^1 x^2 \, dx}{\int_{-1}^1 dx} - \frac{\int_{-1}^1 x^3 \, dx}{\int_{-1}^1 x^2 \, dx} x$$

$$= x^2 - \frac{\left[\frac{1}{3} x^3 \right]_{-1}^1}{2} - \frac{\left[\frac{1}{4} x^4 \right]_{-1}^1}{\left[\frac{1}{3} x^3 \right]_{-1}^1} x$$

$$= x^2 - \frac{1}{3}$$

1.7.19

$$\begin{aligned}
\|\mathbf{v}_1\|^2 &= \int_{-1}^1 dx \\
&= 2 \\
\mathbf{w}_1 &= \frac{1}{\sqrt{2}} \\
\|\mathbf{v}_2\|^2 &= \int_{-1}^1 x^2 dx \\
&= \left[\frac{1}{3} x^3 \right]_{-1}^1 \\
&= \frac{2}{3} \\
\mathbf{w}_2 &= \frac{\sqrt{3}}{\sqrt{6}} x \\
\|\mathbf{v}_3\|^2 &= \int_{-1}^1 \left(x^2 - \frac{1}{3} \right)^2 dx \\
&= \int_{-1}^1 \left(x^4 - \frac{2}{3} x^2 + \frac{1}{9} \right) dx \\
&= \left[\frac{1}{5} x^5 - \frac{2}{9} x^3 + \frac{1}{9} x \right]_{-1}^1 \\
&= \frac{1}{5} - \frac{2}{9} + \frac{1}{9} + \frac{1}{5} - \frac{2}{9} + \frac{1}{9} \\
&= \frac{2}{5} - \frac{2}{9} \\
&= \frac{8}{45} \\
\mathbf{w}_3 &= \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3} \right) \\
&= \frac{5}{2\sqrt{10}} (3x^2 - 1)
\end{aligned}$$

1.7.21

$$\begin{aligned}
(\mathbf{p}, \mathbf{w}_1) &= \int_{-1}^1 \frac{1}{\sqrt{2}} (9x^2 - 6x + 5) dx \\
&= \frac{1}{\sqrt{2}} [3x^3 - 3x^2 + 5x]_{-1}^1 \\
&= \frac{1}{\sqrt{2}} (3 - 3 + 5 + 3 + 3 + 5) \\
&= \frac{16}{\sqrt{2}} \\
(\mathbf{p}, \mathbf{w}_2) &= \int_{-1}^1 \frac{3}{\sqrt{6}} x (9x^2 - 6x + 5) dx \\
&= \frac{3}{\sqrt{6}} \left[\frac{9}{4} x^4 - 2x^3 + \frac{5}{2} x^2 \right]_{-1}^1 \\
&= \frac{3}{\sqrt{6}} \left(\frac{9}{4} - 2 + \frac{5}{2} - \frac{9}{4} - 2 - \frac{5}{2} \right) \\
&= \frac{3}{\sqrt{6}} \left(\frac{9}{4} - \frac{8}{4} + \frac{10}{4} - \frac{9}{4} - \frac{8}{4} - \frac{10}{4} \right) \\
&= -\frac{12}{\sqrt{6}} \\
(\mathbf{p}, \mathbf{w}_3) &= \int_{-1}^1 \frac{5}{2\sqrt{10}} (3x^2 - 1)(9x^2 - 6x + 5) dx \\
&= \frac{5}{2\sqrt{10}} \int_{-1}^1 (27x^4 - 18x^3 + 6x^2 + 6x - 5) dx \\
&= \frac{5}{2\sqrt{10}} \left[\frac{27}{5} x^5 - \frac{9}{2} x^4 + 2x^3 + 3x^2 - 5x \right]_{-1}^1 \\
&= \frac{5}{2\sqrt{10}} \left(\frac{27}{5} - \frac{9}{2} + 2 + 3 - 5 + \frac{27}{5} + \frac{9}{2} + 2 - 3 - 5 \right) \\
&= \frac{5}{2\sqrt{10}} \left(\frac{54}{10} - \frac{45}{10} + \frac{20}{10} + \frac{30}{10} - \frac{50}{10} + \frac{54}{10} + \frac{45}{10} + \frac{20}{10} - \frac{30}{10} - \frac{50}{10} \right) \\
&= \frac{5}{2\sqrt{10}} \frac{48}{10} \\
&= \frac{12}{\sqrt{10}} \\
\mathbf{p} &= \frac{16}{\sqrt{2}} \mathbf{w}_1 - \frac{12}{\sqrt{6}} \mathbf{w}_2 + \frac{12}{\sqrt{10}} \mathbf{w}_3
\end{aligned}$$

1.8 Chapter in Review

1.8.1

True

1.8.3

$$\mathbf{u} = \langle 5, -2, 1 \rangle$$

$$\mathbf{v} = \langle 2, 3, -4 \rangle$$

False

1.8.5

True

1.8.7

True

1.8.9

True

1.8.11

$$9\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

1.8.13

$$\begin{aligned} (-\mathbf{k}) \times (5\mathbf{j}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -1 \\ 0 & 5 & 0 \end{vmatrix} \\ &= 5\mathbf{i} \end{aligned}$$

1.8.15

$$\| -12\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} \| = \sqrt{12^2 + 4^2 + 6^2} = 14$$

1.8.17

$$\langle -6, 1, -7 \rangle$$

1.8.19

$$\begin{aligned}x &= 1 + t \\y &= -2 + 3t \\z &= -1 + 2t\end{aligned}$$

$$\begin{aligned}x + 2y - z &= 13 \\(1 + t) + 2(-2 + 3t) - (-1 + 2t) &= 13 \\1 + t - 4 + 6t + 1 - 2t &= 13 \\-2 + 5t &= 13 \\t &= 3\end{aligned}$$

$$\begin{aligned}x &= 4 \\y &= 7 \\z &= 5\end{aligned}$$

1.8.21

$$\begin{aligned}\overrightarrow{P_1P_2} &= \vec{P_2} - \vec{P_1} \\ \vec{P_2} &= \overrightarrow{P_1P_2} + \vec{P_1} \\ &= \langle 3, 5, -4 \rangle + \langle 2, 1, 7 \rangle \\ &= \langle 5, 6, 3 \rangle\end{aligned}$$

1.8.23

$$\mathbf{a} \cdot \mathbf{b} = -36\sqrt{2}$$

1.8.25

$$x = 12, y = -8, z = 6$$

1.8.27

$$\begin{aligned}\frac{1}{2}(\mathbf{a} \times \mathbf{b}) &= \frac{1}{2} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & -1 & 2 \end{vmatrix} \\ &= \frac{1}{2} \langle 5, -4, -7 \rangle \\ &= \left\langle \frac{5}{2}, -2, -\frac{7}{2} \right\rangle\end{aligned}$$

The area is $\sqrt{\left(\frac{5}{2}\right)^2 + (-2)^2 + \left(-\frac{7}{2}\right)^2} = \frac{3}{2}\sqrt{10}$

1.8.29

2

1.8.31

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -2 & 1 \end{vmatrix} \\ &= \langle 1, -1, -3 \rangle \\ \|\mathbf{a} \times \mathbf{b}\| &= \sqrt{11} \\ \text{norm}(\mathbf{a} \times \mathbf{b}) &= \left\langle \frac{1}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, -\frac{3}{\sqrt{11}} \right\rangle\end{aligned}$$

1.8.33

$$\text{comp}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} = \frac{10}{5} = 2$$

1.8.35

$$\begin{aligned}\mathbf{a} &= \langle 1, 2, -2 \rangle \\ \mathbf{b} &= \langle 4, 3, 0 \rangle \\ \mathbf{a} + \mathbf{b} &= \langle 5, 5, -2 \rangle \\ \text{proj}_{\mathbf{a}}(\mathbf{a} + \mathbf{b}) &= \left(\frac{(\mathbf{a} + \mathbf{b}) \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}} \right) \mathbf{a} \\ &= \frac{19}{9} \langle 1, 2, -2 \rangle \\ &= \left\langle \frac{19}{9}, \frac{38}{9}, -\frac{38}{9} \right\rangle\end{aligned}$$

1.8.37

(a)

(b) A plane with normal \mathbf{a}

1.8.39

$$\frac{x-7}{4} = \frac{y-3}{-2} = \frac{z+5}{6}$$

1.8.41

$$\begin{aligned}\langle -2, 3, 1 \rangle \cdot \langle 2, 1, 1 \rangle &= -4 + 3 + 1 \\ &= 0\end{aligned}$$

$$\begin{aligned}1 - 2t &= 1 + 2s \\ t &= -s\end{aligned}$$

$$3t = -4 + s$$

$$\begin{aligned}t &= -\frac{4}{3} + \frac{s}{3} \\ &= -\frac{4}{3} - \frac{t}{3}\end{aligned}$$

$$\begin{aligned}\frac{4}{3}t &= -\frac{4}{3} \\ t &= -1\end{aligned}$$

$$s = 1$$

$$\langle 3, -3, 0 \rangle$$

1.8.43

$$\mathbf{u} = \langle 1, 4, -2 \rangle$$

$$\mathbf{v} = \langle 1, 1, 3 \rangle$$

$$\mathbf{n} = \mathbf{u} \times \mathbf{v}$$

$$\begin{aligned}&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & -2 \\ 1 & 1 & 3 \end{vmatrix} \\ &= \langle 14, -5, -3 \rangle\end{aligned}$$

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{v}) = 0$$

$$\langle 14, -5, -3 \rangle \cdot (\langle x, y, z \rangle - \langle 1, 1, 3 \rangle) = 0$$

$$14(x - 1) - 5(y - 1) - 3(z - 3) = 0$$

$$14x - 5y - 3z = 0$$

1.8.45

$$\begin{aligned}\mathbf{F} &= \left\langle \frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}}, 0 \right\rangle \\ \mathbf{d} &= \langle 3, 3, 0 \rangle \\ \mathbf{F} \cdot \mathbf{d} &= 30\sqrt{2} \text{ J}\end{aligned}$$

1.8.47

$$\begin{aligned}\mathbf{F}_1 &= \langle 200, 0, 0 \rangle \\ \mathbf{F}_2 &= \left\langle \frac{200}{\sqrt{2}}, \frac{200}{\sqrt{2}}, 0 \right\rangle \\ \mathbf{F}_2 &= \mathbf{F}_1 + \mathbf{F}_3 \\ \mathbf{F}_3 &= \mathbf{F}_2 - \mathbf{F}_1 \\ &= \left\langle \frac{200}{\sqrt{2}}, \frac{200}{\sqrt{2}}, 0 \right\rangle - \langle 200, 0, 0 \rangle \\ &= \left\langle \frac{200}{\sqrt{2}} - 200, \frac{200}{\sqrt{2}}, 0 \right\rangle \\ \|\mathbf{F}_3\| &= \sqrt{\left(\frac{200}{\sqrt{2}} - 200\right)^2 + \left(\frac{200}{\sqrt{2}}\right)^2} \\ &= \sqrt{\frac{40000}{2} - \frac{80000}{\sqrt{2}} + 40000 + \frac{40000}{2}} \\ &= 200\sqrt{2\left(1 - \frac{1}{\sqrt{2}}\right)} \\ &\approx 153 \text{ lb}\end{aligned}$$

2 Matrices

2.1 Matrix Algebra

2.1.1

$$2 \times 4$$

2.1.3

$$3 \times 3$$

2.1.5

$$3 \times 4$$

2.1.7

No

2.1.9

No

2.1.11

$$\begin{aligned}x &= y - 2 \\ 3x - 2 &= y\end{aligned}$$

$$\begin{aligned}2x - 2 &= 2 \\ 2x &= 4 \\ x &= 2\end{aligned}$$

$$\begin{aligned}2 &= y - 2 \\ y &= 4\end{aligned}$$

2.1.13

$$\begin{aligned}c_{23} &= 9 \\ c_{12} &= 12\end{aligned}$$

2.1.15

$$(a) \begin{pmatrix} 2 & 11 \\ 2 & -1 \end{pmatrix}$$

$$(b) \begin{pmatrix} -6 & 1 \\ 14 & -19 \end{pmatrix}$$

$$(c) \begin{pmatrix} 2 & 28 \\ 12 & -12 \end{pmatrix}$$

2.1.17

$$(a) \begin{pmatrix} -11 & 6 \\ 17 & -22 \end{pmatrix}$$

$$(b) \begin{pmatrix} -32 & 27 \\ -4 & -1 \end{pmatrix}$$

$$(c) \begin{pmatrix} 19 & -18 \\ -30 & 31 \end{pmatrix}$$

$$(d) \begin{pmatrix} 19 & 6 \\ 3 & 22 \end{pmatrix}$$

2.1.21

$$(a) 180$$

$$(b) \begin{pmatrix} 4 & 8 & 10 \\ 8 & 16 & 20 \\ 10 & 20 & 25 \end{pmatrix}$$

$$(c) \begin{pmatrix} 6 \\ 12 \\ -5 \end{pmatrix}$$

2.1.23

$$(a) \begin{pmatrix} 7 & 38 \\ 10 & 75 \end{pmatrix}$$

$$(b) \begin{pmatrix} 7 & 38 \\ 10 & 75 \end{pmatrix}$$

2.1.25

$$\begin{pmatrix} -14 \\ 1 \end{pmatrix}$$

2.1.27

$$\begin{pmatrix} -38 \\ -2 \end{pmatrix}$$

2.1.29

$$4 \times 5$$

2.1.41

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

2.1.43

$$\begin{aligned} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} ax + \frac{1}{2}by \\ \frac{1}{2}bx + cy \end{pmatrix} \\ &= ax^2 + \frac{1}{2}bxy + \frac{1}{2}bxy + cy^2 \\ &= ax^2 + bxy + cy^2 \end{aligned}$$

2.1.45

$$\langle -1, 1 \rangle$$

2.1.47

$$\langle -2, 0 \rangle$$

2.1.49

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2.1.51

(b)

$$\begin{pmatrix} x_S \\ y_S \\ z_S \end{pmatrix} = \begin{pmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

2.2 Systems of Linear Algebraic Equations

2.2.1

$$\begin{pmatrix} 1 & -1 & | & 11 \\ 4 & 3 & | & -5 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -1 & | & 11 \\ 0 & 7 & | & -49 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -1 & | & 11 \\ 0 & 1 & | & -7 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & | & 4 \\ 0 & 1 & | & -7 \end{pmatrix}$$

2.2.5

$$\begin{pmatrix} 1 & -1 & -1 & | & -3 \\ 2 & 3 & 5 & | & 7 \\ 1 & -2 & 3 & | & -11 \end{pmatrix} \\
 \begin{pmatrix} 1 & -1 & -1 & | & -3 \\ 0 & 5 & 7 & | & 13 \\ 0 & -1 & 4 & | & -8 \end{pmatrix} \\
 \begin{pmatrix} 1 & -1 & -1 & | & -3 \\ 0 & 1 & \frac{7}{5} & | & \frac{13}{5} \\ 0 & -1 & 4 & | & -8 \end{pmatrix} \\
 \begin{pmatrix} 1 & -1 & -1 & | & -3 \\ 0 & 1 & \frac{7}{5} & | & \frac{13}{5} \\ 0 & 0 & \frac{27}{5} & | & -\frac{27}{5} \end{pmatrix} \\
 \begin{pmatrix} 1 & -1 & -1 & | & -3 \\ 0 & 1 & \frac{7}{5} & | & \frac{13}{5} \\ 0 & 0 & 1 & | & -1 \end{pmatrix} \\
 \begin{pmatrix} 1 & -1 & 0 & | & -4 \\ 0 & 1 & 0 & | & 4 \\ 0 & 0 & 1 & | & -1 \end{pmatrix} \\
 \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 4 \\ 0 & 0 & 1 & | & -1 \end{pmatrix}$$

2.3 Rank of a Matrix

2.3.1

$$\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \\
 \begin{pmatrix} 1 & -\frac{1}{3} \\ 1 & 3 \end{pmatrix} \\
 \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & \frac{10}{3} \end{pmatrix}$$

Rank 2

2.3.3

$$\begin{pmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ -1 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}$$
$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 6 & 3 & 9 \\ -1 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}$$
$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Rank 1

2.3.5

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 4 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 3 \\ 0 & 3 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 3 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 9 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

Rank 3

2.3.7

$$\begin{pmatrix} 1 & -2 \\ 3 & -6 \\ 7 & -1 \\ 4 & 5 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -2 \\ 0 & 13 \\ 0 & 13 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -2 \\ 0 & 13 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Rank 2

2.3.11

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 5 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & -3 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -3 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

Linearly independent

2.3.15

5

2.3.17

$\text{rank}(\mathbf{A}) = 2$

2.4 Determinants

2.4.1

9

2.4.3

1

2.4.5

$$\begin{aligned} M_{33} &= \begin{vmatrix} 0 & 2 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} \\ &= -2 \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} \\ &= 2 \end{aligned}$$

2.4.7

$$\begin{aligned} C_{34} &= (-1)^{3+4} \begin{vmatrix} 0 & 2 & 4 \\ 1 & 2 & -2 \\ 1 & 1 & 1 \end{vmatrix} \\ &= - \left(-2 \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} + 4 \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \right) \\ &= 10 \end{aligned}$$

2.4.9

-7

2.4.11

17

2.4.13

$$\begin{aligned} (1 - \lambda)(2 - \lambda) - 6 &= 2 - \lambda - 2\lambda + \lambda^2 - 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda + 1)(\lambda - 4) \end{aligned}$$

2.4.15

-48

2.4.23

$$\begin{aligned}\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 2 & 3 & 4 \end{vmatrix} &= \begin{vmatrix} y & z \\ 3 & 4 \end{vmatrix} - \begin{vmatrix} x & z \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} x & y \\ 2 & 3 \end{vmatrix} \\ &= 4y - 3z - 4x + 2z + 3x - 2y \\ &= -x + 2y - z\end{aligned}$$

2.4.29

$$\begin{aligned}\begin{vmatrix} (-3 - \lambda) & 10 \\ 2 & (5 - \lambda) \end{vmatrix} &= 0 \\ (-3 - \lambda)(5 - \lambda) - 20 &= 0 \\ -15 + 3\lambda - 5\lambda + \lambda^2 - 20 &= 0 \\ \lambda^2 - 2\lambda - 35 &= 0 \\ (\lambda - 7)(\lambda + 5) &= 0 \\ \lambda &= -5 \text{ or } 7\end{aligned}$$

2.5 Properties of Determinants

2.5.1

8.5.4

2.5.3

8.5.7

2.5.5

8.5.5

2.5.7

8.5.3

2.5.9

8.5.1

2.5.11

-5

2.5.13

-5

2.5.15

5

2.5.17

80

2.5.19

-105

2.5.25

$$\begin{aligned}\mathbf{A}\mathbf{A} &= \mathbf{I} \\ \det \mathbf{A} \cdot \det \mathbf{A} &= \det \mathbf{I} \\ (\det \mathbf{A})^2 &= 1 \\ \det \mathbf{A} &= \pm 1\end{aligned}$$

2.5.27

$$\begin{aligned}\begin{vmatrix} a & a+1 & a+2 \\ b & b+1 & b+2 \\ c & c+1 & c+2 \end{vmatrix} &= \begin{vmatrix} a & 1 & 2 \\ b & 1 & 2 \\ c & 1 & 2 \end{vmatrix} \\ &= \begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} \\ &= 0\end{aligned}$$

2.5.29

$$\begin{aligned}\begin{vmatrix} 1 & 1 & 5 \\ 4 & 3 & 6 \\ 0 & -1 & 1 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 5 \\ 0 & -1 & -14 \\ 0 & -1 & 1 \end{vmatrix} \\ &= - \begin{vmatrix} 1 & 1 & 5 \\ 0 & 1 & 14 \\ 0 & 0 & 15 \end{vmatrix} \\ &= -15\end{aligned}$$

2.5.37

$$\begin{aligned}
 \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix} \\
 &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & c^2-a^2-(b+a)(c-a) \end{vmatrix} \\
 &= (b-a)(c^2-a^2-(b+a)(c-a)) \\
 &= (b-a)(c^2-a^2-bc+ab-ac+a^2) \\
 &= (b-a)(c^2+ab-ac-bc) \\
 &= (b-a)(c-a)(c-b)
 \end{aligned}$$

2.5.39

$$\begin{aligned}
 a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13} &= (-1)(4) + (2)(5) + (1)(-6) \\
 &= 0 \\
 a_{13}C_{12} + a_{23}C_{22} + a_{33}C_{32} &= (2)(5) + (1)(-7) + (1)(-3) \\
 &= 0
 \end{aligned}$$

2.5.41

$$\begin{aligned}
 \mathbf{A} + \mathbf{B} &= \begin{pmatrix} 10 & 0 \\ 0 & -3 \end{pmatrix} \\
 \det(\mathbf{A} + \mathbf{B}) &= -30 \\
 \det \mathbf{A} &= 10 \\
 \det \mathbf{B} &= -31 \\
 -30 &\neq 10 - 31
 \end{aligned}$$

2.6 Inverse of a Matrix

2.6.3

$$\begin{aligned}
 \det \mathbf{A} &= 9 \\
 \mathbf{A}^{-1} &= \frac{1}{9} \begin{pmatrix} 1 & 1 \\ -4 & 5 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{9} & \frac{1}{9} \\ -\frac{4}{9} & \frac{5}{9} \end{pmatrix}
 \end{aligned}$$

2.6.5

$$\det \mathbf{A} = 12$$

$$\begin{aligned}\mathbf{A}^{-1} &= \frac{1}{12} \begin{pmatrix} 2 & 0 \\ 3 & 6 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{6} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}\end{aligned}$$

2.6.7

$$\begin{aligned}\det \mathbf{A} &= (1) \begin{vmatrix} 4 & 4 \\ -1 & 1 \end{vmatrix} - (3) \begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} + (5) \begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} \\ &= 8 + 6 - 30 \\ &= -16\end{aligned}$$

$$\begin{aligned}\mathbf{A}^{-1} &= -\frac{1}{16} \begin{pmatrix} 8 & 2 & -6 \\ -8 & -4 & 4 \\ -8 & 6 & -2 \end{pmatrix}^T \\ &= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{8} & \frac{1}{4} & -\frac{3}{8} \\ \frac{3}{8} & -\frac{1}{4} & \frac{1}{8} \end{pmatrix}\end{aligned}$$

2.6.15

$$\begin{aligned}&\left(\begin{array}{cc|cc} 6 & -2 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{array} \right) \\ &\left(\begin{array}{cc|cc} 1 & -\frac{1}{3} & \frac{1}{6} & 0 \\ 0 & 1 & 0 & \frac{1}{4} \end{array} \right) \\ &\left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{6} & \frac{1}{12} \\ 0 & 1 & 0 & \frac{1}{4} \end{array} \right)\end{aligned}$$

2.6.17

$$\begin{pmatrix} 1 & 3 & | & 1 & 0 \\ 5 & 3 & | & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & | & 1 & 0 \\ 0 & -12 & | & -5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & | & 1 & 0 \\ 0 & 1 & | & \frac{5}{12} & -\frac{1}{12} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & | & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & | & \frac{5}{12} & -\frac{1}{12} \end{pmatrix}$$

2.6.27

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$= \begin{pmatrix} \frac{2}{3} & \frac{4}{3} \\ -\frac{1}{3} & \frac{5}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{5}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ -\frac{17}{12} & \frac{55}{12} \end{pmatrix}$$

2.6.29

$$\begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$$

2.6.31

$$\begin{pmatrix} 4 & -3 \\ x & -4 \end{pmatrix} = \frac{1}{3x-16} \begin{pmatrix} -4 & -x \\ 3 & 4 \end{pmatrix}^T$$

$$= \frac{1}{3x-16} \begin{pmatrix} -4 & 3 \\ -x & 4 \end{pmatrix}$$

$$-1 = \frac{1}{3x-16}$$

$$16-3x=1$$

$$3x=15$$

$$x=5$$

2.6.45

$$\begin{aligned}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 4 \\ 14 \end{pmatrix} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= -\frac{1}{3} \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 14 \end{pmatrix} \\ &= -\frac{1}{3} \begin{pmatrix} -18 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ -2 \end{pmatrix}\end{aligned}$$

2.6.49

$$\begin{aligned}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix} \\ \det \mathbf{A} &= \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} \\ &= 1 - 6 \\ &= -5 \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= -\frac{1}{5} \begin{pmatrix} 1 & 5 & -6 \\ -1 & -5 & 1 \\ -1 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix} \\ &= -\frac{1}{5} \begin{pmatrix} 1 & -1 & -1 \\ 5 & -5 & 0 \\ -6 & 1 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix} \\ &= -\frac{1}{5} \begin{pmatrix} -10 \\ -20 \\ 30 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix}\end{aligned}$$

2.6.55

$$\begin{aligned}\det \begin{pmatrix} 1 & 2 & -1 \\ 4 & -1 & 1 \\ 5 & 1 & -2 \end{pmatrix} &= (1) \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} - (2) \begin{vmatrix} 4 & 1 \\ 5 & -2 \end{vmatrix} + (-1) \begin{vmatrix} 4 & -1 \\ 5 & 1 \end{vmatrix} \\ &= 1 + 26 - 9 \\ &= 18\end{aligned}$$

Only trivial solution

2.7 Cramer's Rule

2.7.1

$$\mathbf{A} = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$$

$$\det \mathbf{A} = 10$$

$$\det \mathbf{A}_1 = -6$$

$$\det \mathbf{A}_2 = 12$$

$$x_1 = -\frac{3}{5}$$

$$x_2 = \frac{6}{5}$$

2.7.11

$$\mathbf{A} = \begin{pmatrix} 2-k & k \\ k & 3-k \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\begin{aligned} \det \mathbf{A} &= (2-k)(3-k) - k^2 \\ &= 6 - 5k \end{aligned}$$

$$\begin{aligned} \det \mathbf{A}_1 &= 4(3-k) - 3k \\ &= 12 - 7k \end{aligned}$$

$$\begin{aligned} \det \mathbf{A}_2 &= 3(2-k) - 4k \\ &= 6 - 7k \end{aligned}$$

$$x_1 = \frac{12 - 7k}{6 - 5k}$$

$$x_2 = \frac{6 - 7k}{6 - 5k}$$

The system is inconsistent when $k = \frac{6}{5}$

2.7.13

$$\mathbf{A} = \begin{pmatrix} \cos 25^\circ & -\cos 15^\circ \\ \sin 25^\circ & \sin 15^\circ \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 300 \end{pmatrix}$$

$$\begin{aligned} \det \mathbf{A} &= \cos 25^\circ \sin 15^\circ + \cos 15^\circ \sin 25^\circ \\ &= \sin 40^\circ \end{aligned}$$

$$\det \mathbf{A}_1 = 300 \cos 15^\circ$$

$$\det \mathbf{A}_2 = 300 \cos 25^\circ$$

$$\begin{aligned} T_1 &= \frac{300 \cos 15^\circ}{\sin 40^\circ} \\ &\approx 451 \text{ lb} \end{aligned}$$

$$\begin{aligned} T_2 &= \frac{300 \cos 25^\circ}{\sin 40^\circ} \\ &\approx 423 \text{ lb} \end{aligned}$$

2.8 The Eigenvalue Problem

2.8.1

\mathbf{K}_3 with $\lambda = -1$

2.8.3

\mathbf{K}_3 with $\lambda = 0$

2.8.5

\mathbf{K}_2 with $\lambda = 3$

\mathbf{K}_3 with $\lambda = 1$

2.8.7

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} -1 - \lambda & 2 \\ -7 & 8 - \lambda \end{vmatrix} \\ &= (-1 - \lambda)(8 - \lambda) + 14 \\ &= -8 + \lambda - 8\lambda + \lambda^2 + 14 \\ &= \lambda^2 - 7\lambda + 6 \\ &= (\lambda - 1)(\lambda - 6) \\ \lambda_1 &= 1 \\ \lambda_2 &= 6 \end{aligned}$$

$$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ -7 & 7 & 0 \end{array} \right)$$

$$x_1 = x_2$$

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -7 & 2 & 0 \\ -7 & 2 & 0 \end{array} \right)$$

$$x_1 = \frac{2}{7}x_2$$

$$\mathbf{X}_2 = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

Nonsingular

2.8.9

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} -8 - \lambda & -1 \\ 16 & -\lambda \end{vmatrix} \\ &= -\lambda(-8 - \lambda) + 16 \\ &= 8\lambda + \lambda^2 + 16 \\ &= (\lambda + 4)^2 \end{aligned}$$

$$\lambda_1 = \lambda_2 = -4$$

$$\left(\begin{array}{cc|c} -4 & -1 & 0 \\ 16 & 4 & 0 \end{array} \right)$$

$$x_1 = -\frac{1}{4}x_2$$

$$\mathbf{X}_1 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

Nonsingular

2.8.11

$$\begin{aligned}
\det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} -1 - \lambda & 2 \\ -5 & 1 - \lambda \end{vmatrix} \\
&= (-1 - \lambda)(1 - \lambda) + 10 \\
&= -1 + \lambda - \lambda + \lambda^2 + 10 \\
&= \lambda^2 + 9 \\
&= (\lambda - 3i)(\lambda + 3i) \\
\lambda_1 &= 3i \\
\lambda_2 &= -3i
\end{aligned}$$

$$\left(\begin{array}{cc|c} -1 - 3i & 2 & 0 \\ -5 & 1 - 3i & 0 \end{array} \right)$$

$$\mathbf{X}_1 = \begin{pmatrix} 1 - 3i \\ 5 \end{pmatrix}$$

$$\mathbf{X}_2 = \begin{pmatrix} 1 + 3i \\ 5 \end{pmatrix}$$

Nonsingular

2.8.23

$$\begin{aligned}
\det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} 5 - \lambda & 1 \\ 1 & 5 - \lambda \end{vmatrix} \\
&= (5 - \lambda)^2 - 1 \\
&= 25 - 10\lambda + \lambda^2 - 1 \\
&= \lambda^2 - 10\lambda + 24 \\
&= (\lambda - 4)(\lambda - 6) \\
\lambda_1 &= 4 \\
\lambda_2 &= 6
\end{aligned}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda'_1 = \frac{1}{4}$$

$$\lambda'_2 = \frac{1}{6}$$

$$\mathbf{X}'_1 = \mathbf{X}_1$$

$$\mathbf{X}'_2 = \mathbf{X}_2$$

2.9 Powers of Matrices

2.9.1

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= (1 - \lambda)(5 - \lambda) + 8 \\ &= 5 - \lambda - 5\lambda + \lambda^2 + 8 \\ &= \lambda^2 - 6\lambda + 13 \end{aligned}$$

$$\mathbf{A}^2 = 6\mathbf{A} - 13\mathbf{I}$$

$$\begin{aligned} \begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix} &= 6 \begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix} - 13 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} -7 & -12 \\ 24 & 17 \end{pmatrix} &= \begin{pmatrix} -7 & -12 \\ 24 & 17 \end{pmatrix} \end{aligned}$$

2.9.3

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix} \\ \det(\mathbf{A} - \lambda \mathbf{I}) &= (-1 - \lambda)(4 - \lambda) - 6 \\ &= -4 + \lambda - 4\lambda + \lambda^2 - 6 \\ &= \lambda^2 - 3\lambda - 10 \\ &= (\lambda - 5)(\lambda + 2)\end{aligned}$$

$$\begin{aligned}\lambda^m &= c_0 + c_1 \lambda \\ (-2)^m &= c_0 - 2c_1 \\ (5)^m &= c_0 + 5c_1\end{aligned}$$

$$\begin{aligned}(-2)^m + \frac{2}{5}(5)^m &= \frac{7}{5}c_0 \\ \frac{5}{7}(-2)^m + \frac{2}{7}(5)^m &= c_0\end{aligned}$$

$$\begin{aligned}(5)^m - (-2)^m &= 7c_1 \\ \frac{1}{7}(5)^m - \frac{1}{7}(-2)^m &= c_1\end{aligned}$$

$$\begin{aligned}\mathbf{A}^m &= \frac{1}{7} \begin{pmatrix} 5^m + 6(-2)^m & 3(5)^m - 3(-2)^m \\ 2(5)^m - 2(-2)^m & 6(5)^m + (-2)^m \end{pmatrix} \\ \mathbf{A}^3 &= \frac{1}{7} \begin{pmatrix} 5^3 + 6(-2)^3 & 3(5)^3 - 3(-2)^3 \\ 2(5)^3 - 2(-2)^3 & 6(5)^3 + (-2)^3 \end{pmatrix} \\ &= \begin{pmatrix} 11 & 57 \\ 38 & 106 \end{pmatrix}\end{aligned}$$

2.9.5

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} 8 & 5 \\ 4 & 0 \end{pmatrix} \\ \det(\mathbf{A} - \lambda \mathbf{I}) &= -\lambda(8 - \lambda) - 20 \\ &= -8\lambda + \lambda^2 - 20 \\ &= \lambda^2 - 8\lambda - 20 \\ &= (\lambda - 10)(\lambda + 2)\end{aligned}$$

$$\begin{aligned}\lambda^m &= c_0 + c_1\lambda \\ (-2)^m &= c_0 - 2c_1 \\ 10^m &= c_0 + 10c_1\end{aligned}$$

$$\begin{aligned}10^m + 5(-2)^m &= 6c_0 \\ \frac{1}{6}(10^m + 5(-2)^m) &= c_0 \\ \frac{1}{12}(2 \cdot 10^m + 10(-2)^m) &= c_0\end{aligned}$$

$$\begin{aligned}10^m - (-2)^m &= 12c_1 \\ \frac{1}{12}(10^m - (-2)^m) &= c_1\end{aligned}$$

$$\begin{aligned}\mathbf{A}^m &= \frac{1}{12} \begin{pmatrix} 10 \cdot 10^m + 2(-2)^m & 5 \cdot 10^m - 5(-2)^m \\ 4 \cdot 10^m - 4(-2)^m & 2 \cdot 10^m + 10(-2)^m \end{pmatrix} \\ \mathbf{A}^5 &= \begin{pmatrix} 83328 & 41680 \\ 33344 & 16640 \end{pmatrix}\end{aligned}$$

2.9.11

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 7 & 3 \\ -3 & 1 \end{pmatrix} \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= (7 - \lambda)(1 - \lambda) + 9 \\
&= 7 - 7\lambda - \lambda + \lambda^2 + 9 \\
&= \lambda^2 - 8\lambda + 16 \\
&= (\lambda - 4)^2
\end{aligned}$$

$$\begin{aligned}
\lambda^m &= c_0 + c_1 \lambda \\
m\lambda^{m-1} &= c_1
\end{aligned}$$

$$4^{m-1}m = c_1$$

$$\begin{aligned}
4^m &= c_0 + 4^m m \\
4^m(1 - m) &= c_0
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}^m &= \begin{pmatrix} 4^m(1 - m) + 7 \cdot 4^{m-1}m & 3 \cdot 4^{m-1}m \\ -3 \cdot 4^{m-1}m & 4^m(1 - m) + 4^{m-1}m \end{pmatrix} \\
&= 4^m \begin{pmatrix} 1 - m + \frac{7}{4}m & \frac{3}{4}m \\ -\frac{3}{4}m & 1 - m + \frac{1}{4}m \end{pmatrix} \\
&= 4^m \begin{pmatrix} 1 + \frac{3}{4}m & \frac{3}{4}m \\ -\frac{3}{4}m & 1 - \frac{3}{4}m \end{pmatrix} \\
\mathbf{A}^6 &= 4^6 \begin{pmatrix} 1 + \frac{3}{4}6 & \frac{3}{4}6 \\ -\frac{3}{4}6 & 1 - \frac{3}{4}6 \end{pmatrix} \\
&= 4^5 \begin{pmatrix} 4 + 18 & 18 \\ -18 & 4 - 18 \end{pmatrix} \\
&= 4^5 \begin{pmatrix} 22 & 18 \\ -18 & -14 \end{pmatrix} \\
&= \begin{pmatrix} 22528 & 18432 \\ -18432 & -14336 \end{pmatrix}
\end{aligned}$$

2.9.13

(a)

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= (1 - \lambda)(3 - \lambda) - 3 \\
&= 3 - \lambda - 3\lambda + \lambda^2 - 3 \\
&= \lambda^2 - 4\lambda \\
&= \lambda(\lambda - 4)
\end{aligned}$$

$$\begin{aligned}
\lambda^m &= c_1 \lambda \\
4^m &= 4c_1 \\
c_1 &= 4^{m-1}
\end{aligned}$$

$$\mathbf{A}^m = 4^{m-1} \mathbf{A}$$

2.9.15

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix} \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= (2 - \lambda)(3 - \lambda) + 4 \\
&= 6 - 2\lambda - 3\lambda + \lambda^2 + 4 \\
&= \lambda^2 - 5\lambda + 10 \\
10\mathbf{I} &= 5\mathbf{A} - \mathbf{A}^2 \\
\mathbf{I} &= \frac{1}{2}\mathbf{A} - \frac{1}{10}\mathbf{A}^2 \\
\mathbf{A}^{-1} &= \frac{1}{2}\mathbf{I} - \frac{1}{10}\mathbf{A} \\
&= \begin{pmatrix} \frac{3}{10} & \frac{2}{5} \\ -\frac{1}{10} & \frac{1}{5} \end{pmatrix}
\end{aligned}$$

2.10 Orthogonal Matrices**2.10.5**

Orthogonal

2.10.7

Orthogonal

2.10.9

Not orthogonal

2.10.11

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 9 \\ 9 & 1 \end{pmatrix} \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= (1 - \lambda)^2 - 81 \\
&= \lambda^2 - 2\lambda + 1 - 81 \\
&= \lambda^2 - 2\lambda - 80 \\
&= (\lambda - 10)(\lambda + 8)
\end{aligned}$$

$$\begin{aligned}
&\begin{pmatrix} 9 & 9 \\ 9 & 9 \end{pmatrix} \\
\mathbf{X}_1 &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&\begin{pmatrix} -9 & 9 \\ 9 & -9 \end{pmatrix} \\
\mathbf{X}_2 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}
\end{aligned}$$

2.10.13

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= (1 - \lambda)(9 - \lambda) - 9 \\
&= 9 - \lambda - 9\lambda + \lambda^2 - 9 \\
&= \lambda^2 - 10\lambda \\
&= \lambda(\lambda - 10)
\end{aligned}$$

$$\begin{aligned}
&\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \\
\mathbf{X}_1 &= \begin{pmatrix} 3 \\ -1 \end{pmatrix}
\end{aligned}$$

$$\begin{pmatrix} -9 & 3 \\ 3 & -1 \end{pmatrix}$$

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}$$

2.10.19

$$\frac{3}{5}a + \frac{4}{5}b = 0$$

$$3a = -4b$$

$$a = -\frac{4}{3}b$$

$$a = -\frac{4}{5}$$

$$b = \frac{3}{5}$$

2.10.21

$$(b) \ \lambda_1 = -2, \lambda_2 = -2, \lambda_3 = 4$$

(c)

$$\mathbf{W}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\begin{aligned} \mathbf{V}_2 &= \mathbf{K}_2 - (\mathbf{K}_2 \cdot \mathbf{W}_1) \mathbf{W}_1 \\ &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} |\mathbf{V}_2| &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (-1)^2} \\ &= \sqrt{\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} \mathbf{W}_2 &= \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

2.11 Approximation of Eigenvalues

2.11.1

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \\ \mathbf{X}_0 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \mathbf{A}^5 \mathbf{X}_0 &= 32 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \mathbf{K}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \lambda_1 &= \frac{\mathbf{A} \mathbf{X}_5 \cdot \mathbf{X}_5}{\mathbf{X}_5 \cdot \mathbf{X}_5} \\ &= 2\end{aligned}$$

2.11.3

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} 2 & 4 \\ 3 & 13 \end{pmatrix} \\ \mathbf{X}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \mathbf{X}_2 &= \begin{pmatrix} \frac{3}{8} \\ 1 \end{pmatrix} \\ \mathbf{X}_3 &= \begin{pmatrix} 0.3363 \\ 1 \end{pmatrix} \\ \mathbf{X}_4 &= \begin{pmatrix} 0.3335 \\ 1 \end{pmatrix} \\ \mathbf{X}_5 &= \begin{pmatrix} 0.3333 \\ 1 \end{pmatrix} \\ \mathbf{K}_1 &= \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \\ \lambda_1 &= \frac{\mathbf{A} \mathbf{K}_1 \cdot \mathbf{K}_1}{\mathbf{K}_1 \cdot \mathbf{K}_1} \\ &= 14\end{aligned}$$

2.11.7

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} \\
\mathbf{X}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathbf{A}^5 \mathbf{X}_1 &= \begin{pmatrix} 0.5008 \\ 1 \end{pmatrix} \\
\lambda_1 &= 7 \\
\mathbf{K}_1 &= \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \\
\mathbf{B} &= \mathbf{A} - \lambda_1 \mathbf{K}_1 \mathbf{K}_1^T \\
&= \begin{pmatrix} \frac{8}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{2}{5} \end{pmatrix} \\
\mathbf{B}^5 \mathbf{X}_1 &= \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \\
\lambda_2 &= 2
\end{aligned}$$

2.11.11

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} \\
\det \mathbf{A} &= 1 \\
\mathbf{A}^{-1} &= \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix} \\
\mathbf{X}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
(\mathbf{A}^{-1})^5 \mathbf{X}_1 &= \begin{pmatrix} 1 \\ -0.7913 \end{pmatrix} \\
\lambda'_1 &\approx 4.78 \\
\lambda_1 &\approx 0.21
\end{aligned}$$

2.11.13

(a)

$$\begin{aligned}
EI \frac{d^2 y}{dx^2} + Py &= 0 \\
EI \left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right) + Py_i &= 0 \\
EI(y_{i+1} - 2y_i + y_{i-1}) + Ph^2 y_i &= 0
\end{aligned}$$

(b)

$$\begin{aligned}EI(y_2 - 2y_1) + Ph^2y_1 &= 0 \\EI(2y_1 - y_2) &= Ph^2y_1\end{aligned}$$

$$\begin{aligned}EI(y_3 - 2y_2 + y_1) + Ph^2y_2 &= 0 \\EI(-y_1 + 2y_2 - y_3) &= Ph^2y_2\end{aligned}$$

$$\begin{aligned}EI(-2y_3 + y_2) + Ph^2y_3 &= 0 \\EI(-y_2 + 2y_3) &= Ph^2y_3\end{aligned}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \frac{PL^2}{16EI} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

(c)

$$\begin{aligned}\det \mathbf{A} &= 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} \\ &= 4\end{aligned}$$

$$\begin{aligned}\mathbf{A}^{-1} &= \frac{1}{\det \mathbf{A}} \text{adj } \mathbf{A} \\ &= \frac{1}{4} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}^T \\ &= \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}\end{aligned}$$

(d)

$$\begin{aligned}\mathbf{X}_1 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ (\mathbf{A}^{-1})^6 \mathbf{X}_1 &= \begin{pmatrix} 0.7071 \\ 1 \\ 0.7071 \end{pmatrix} \\ \lambda'_1 &\approx 1.7071 \\ \lambda_1 &\approx 0.59\end{aligned}$$

(e)

$$\begin{aligned}\lambda_1 &= \frac{PL^2}{16EI} \\ P &= \frac{16EI\lambda_1}{L^2} \\ &\approx \frac{9.44EI}{L^2}\end{aligned}$$

2.12 Diagonalization

2.12.1

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \\ \det(\mathbf{A} - \lambda\mathbf{I}) &= (2 - \lambda)(4 - \lambda) - 3 \\ &= 8 - 2\lambda - 4\lambda + \lambda^2 - 3 \\ &= \lambda^2 - 6\lambda + 5 \\ &= (\lambda - 5)(\lambda - 1) \\ \lambda_1 &= 1 \\ \lambda_2 &= 5\end{aligned}$$

$$\begin{aligned}&\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \\ \mathbf{X}_1 &= \begin{pmatrix} -3 \\ 1 \end{pmatrix} \\ &\begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix} \\ \mathbf{X}_2 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \\ \mathbf{P} &= \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \\ \mathbf{P}^{-1} &= \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \\ \mathbf{D} &= \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}\end{aligned}$$

2.12.3

Not diagonalisable

2.12.5

$$\mathbf{P} = \begin{pmatrix} 13 & 1 \\ 2 & 1 \end{pmatrix}$$
$$\mathbf{D} = \begin{pmatrix} -7 & 0 \\ 0 & 4 \end{pmatrix}$$

2.12.7

$$\mathbf{P} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$\mathbf{D} = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

2.12.21

$$\mathbf{P} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

2.12.23

$$\mathbf{P} = \begin{pmatrix} \sqrt{\frac{2}{5}} & -\sqrt{\frac{5}{2}} \\ 1 & 1 \end{pmatrix}$$
$$\mathbf{D} = \begin{pmatrix} 10 & 0 \\ 0 & 3 \end{pmatrix}$$

2.12.35

$$\mathbf{A} = \mathbf{PDP}^{-1}$$
$$= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$$

2.12.39

$$\begin{aligned}\mathbf{D} &= \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \\ \mathbf{P} &= \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \\ \mathbf{A}^5 &= \mathbf{PD}^5\mathbf{P}^{-1} \\ &= \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 32 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} 21 & 11 \\ 22 & 10 \end{pmatrix}\end{aligned}$$

2.13 LU-Factorisation

2.13.1

$$\begin{aligned}\mathbf{A} &= \mathbf{LU} \\ \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} \\ &= \begin{pmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \end{pmatrix} \\ u_{11} &= 2 \\ u_{12} &= -2 \\ l_{21} &= \frac{1}{2} \\ u_{22} &= 3 \\ \mathbf{L} &= \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \\ \mathbf{U} &= \begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix}\end{aligned}$$

2.13.3

$$\begin{pmatrix} -1 & 4 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \end{pmatrix}$$

$$u_{11} = -1$$

$$u_{12} = 4$$

$$l_{21} = -2$$

$$u_{22} = 10$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} -1 & 4 \\ 0 & 10 \end{pmatrix}$$

2.13.11

$$\begin{pmatrix} 3 & 9 \\ 1 & 11 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & 9 \\ 0 & 8 \end{pmatrix}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} 3 & 9 \\ 0 & 8 \end{pmatrix}$$

2.13.13

$$\begin{pmatrix} -4 & -2 \\ -1 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} -4 & -2 \\ 0 & -\frac{5}{2} \end{pmatrix}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{1}{4} & 1 \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} -4 & -2 \\ 0 & -\frac{5}{2} \end{pmatrix}$$

2.13.21

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix}$$

$$\mathbf{LY} = \mathbf{B}$$

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$y_1 = 1$$

$$y_2 = -\frac{5}{2}$$

$$\mathbf{UX} = \mathbf{Y}$$

$$\begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{5}{2} \end{pmatrix}$$

$$x_2 = -\frac{5}{6}$$

$$2x_1 - 2\left(-\frac{5}{6}\right) = 1$$

$$2x_1 = -\frac{2}{3}$$

$$x_1 = -\frac{1}{3}$$

$$\mathbf{X} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{5}{6} \end{pmatrix}$$

2.13.23

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} -1 & 4 \\ 0 & 10 \end{pmatrix}$$

$$\mathbf{LY} = \mathbf{B}$$

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix}$$

$$y_1 = 15$$

$$y_2 = 35$$

$$\mathbf{UX} = \mathbf{Y}$$

$$\begin{pmatrix} -1 & 4 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 35 \end{pmatrix}$$

$$x_2 = \frac{7}{2}$$

$$-x_1 + 4\frac{7}{2} = 15$$

$$x_1 = -1$$

$$\mathbf{X} = \begin{pmatrix} -1 \\ \frac{7}{2} \end{pmatrix}$$

2.13.31

$$\mathbf{LY} = \mathbf{B}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

$$y_1 = 2$$

$$y_2 = 2$$

$$y_3 = -3$$

$$\mathbf{UX} = \mathbf{Y}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

$$x_3 = -3$$

$$x_2 = 5$$

$$x_1 = 0$$

$$\mathbf{X} = \begin{pmatrix} 0 \\ 5 \\ -3 \end{pmatrix}$$

2.14 Cryptography

2.14.1

$$\mathbf{B} = \mathbf{AM}$$

$$= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 19 & 5 & 14 & 4 & 0 \\ 8 & 5 & 12 & 16 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 35 & 15 & 38 & 36 & 0 \\ 27 & 10 & 26 & 20 & 0 \end{pmatrix}$$

2.14.7

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & -3 \\ -5 & 8 \end{pmatrix}$$

$$\mathbf{M} = \mathbf{A}^{-1}\mathbf{B}$$

$$= \begin{pmatrix} 19 & 20 & 21 & 4 & 25 \\ 0 & 8 & 1 & 18 & 4 \end{pmatrix}$$

$$= \text{STUDY HARD}$$

2.14.11

$$\mathbf{A}^{-1}\mathbf{B} = \mathbf{M}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 17 & 16 \\ -30 & -31 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ m_{21} & m_{22} \end{pmatrix}$$

$$17a_{11} - 30a_{12} = 4$$

$$16a_{11} - 31a_{12} = 1$$

$$17a_{11} - \frac{30}{31}16a_{11} = 4 - \frac{30}{31}$$

$$527a_{11} - 480a_{11} = 124 - 30$$

$$47a_{11} = 94$$

$$a_{11} = 2$$

$$-30a_{12} + \frac{17}{16}31a_{12} = 4 - \frac{17}{16}$$

$$-480a_{12} + 527a_{12} = 64 - 17$$

$$47a_{12} = 47$$

$$a_{12} = 1$$

$$\begin{pmatrix} 2 & 1 \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 5 & 25 \\ -6 & -50 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ 1 & 25 \end{pmatrix}$$

$$5a_{21} - 6a_{22} = 1$$

$$25a_{21} - 50a_{22} = 25$$

$$5a_{21} - \frac{6}{50}25a_{21} = 1 - \frac{6}{50}25$$

$$250a_{21} - 150a_{21} = 50 - 150$$

$$100a_{21} = -100$$

$$a_{21} = -1$$

$$4a_{22} = -4$$

$$a_{22} = -1$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{B} = \text{DAD I NEED MONEY TODAY}$$

2.15 Chapter in Review

2.15.1

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{pmatrix}$$

2.15.3

$$\mathbf{AB} = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$$
$$\mathbf{BA} = (11)$$

2.15.5

False

2.15.7

$$\det\left(\frac{1}{2}\mathbf{A}\right) = \frac{5}{8}$$
$$\det -\mathbf{A}^T = -5$$

2.15.9

0

2.15.11

False

2.15.13

True

2.15.15

False

2.15.17

True

2.15.19

False

2.15.23

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}^2$$

2.15.25

$$\begin{pmatrix} 5 & -1 & 1 & | & -9 \\ 2 & 4 & 0 & | & 27 \\ 1 & 1 & 5 & | & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{1}{5} & \frac{1}{5} & | & -\frac{9}{5} \\ 1 & 2 & 0 & | & \frac{27}{2} \\ 1 & 1 & 5 & | & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{1}{5} & \frac{1}{5} & | & -\frac{9}{5} \\ 0 & \frac{11}{5} & -\frac{1}{5} & | & \frac{153}{5} \\ 0 & \frac{6}{5} & \frac{24}{5} & | & \frac{10}{5} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{1}{5} & \frac{1}{5} & | & -\frac{9}{5} \\ 0 & 1 & -\frac{1}{11} & | & \frac{153}{22} \\ 0 & 1 & 4 & | & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{1}{5} & \frac{1}{5} & | & -\frac{9}{5} \\ 0 & 1 & -\frac{1}{11} & | & \frac{153}{22} \\ 0 & 0 & \frac{45}{11} & | & \frac{45}{22} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{1}{5} & \frac{1}{5} & | & -\frac{9}{5} \\ 0 & 1 & -\frac{1}{11} & | & \frac{153}{22} \\ 0 & 0 & 1 & | & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{1}{5} & 0 & | & -\frac{19}{10} \\ 0 & 1 & 0 & | & 7 \\ 0 & 0 & 1 & | & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & | & -\frac{1}{2} \\ 0 & 1 & 0 & | & 7 \\ 0 & 0 & 1 & | & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} -\frac{1}{2} \\ 7 \\ \frac{1}{2} \end{pmatrix}$$

2.15.29

240

2.15.31

$$\begin{vmatrix} 1 & -1 & 1 \\ 5 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = (1) \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + (1) \begin{vmatrix} 5 & -1 \\ 1 & 1 \end{vmatrix} + (1) \begin{vmatrix} 5 & 1 \\ 1 & 2 \end{vmatrix} \\
= 18$$

The matrix is nonsingular so the system only has the trivial solution.

2.15.35

$$\begin{aligned}
\det \mathbf{A} &= \begin{vmatrix} 1 & 2 & -3 \\ 2 & -4 & 3 \\ 0 & 4 & 6 \end{vmatrix} \\
&= (1) \begin{vmatrix} -4 & 3 \\ 4 & 6 \end{vmatrix} - (2) \begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} + (-3) \begin{vmatrix} 2 & -4 \\ 0 & 4 \end{vmatrix} \\
&= -36 - 24 - 24 \\
&= -84 \\
x_1 &= \frac{\det \mathbf{A}_1}{\det \mathbf{A}} \\
&= -\frac{1}{2} \\
x_2 &= \frac{\det \mathbf{A}_2}{\det \mathbf{A}} \\
&= \frac{1}{4} \\
x_3 &= \frac{\det \mathbf{A}_3}{\det \mathbf{A}} \\
&= \frac{2}{3}
\end{aligned}$$

2.15.37

$$\begin{aligned}
\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} X \\ Y \end{pmatrix} \\
x &= \frac{\begin{vmatrix} X & \sin \theta \\ Y & \cos \theta \end{vmatrix}}{\begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix}} \\
&= X \cos \theta - Y \sin \theta \\
y &= \frac{\begin{vmatrix} \cos \theta & X \\ -\sin \theta & Y \end{vmatrix}}{\begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix}} \\
&= X \sin \theta + Y \cos \theta
\end{aligned}$$

2.15.39

$$\begin{aligned}
\begin{pmatrix} 2 & 3 & -1 \\ 1 & -2 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 6 \\ -3 \\ 9 \end{pmatrix} \\
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \frac{1}{3} \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 1 \\ 4 & 6 & 7 \end{pmatrix} \begin{pmatrix} 6 \\ -3 \\ 9 \end{pmatrix} \\
&= \begin{pmatrix} 7 \\ 5 \\ 23 \end{pmatrix}
\end{aligned}$$

2.15.41

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \\ \det(\mathbf{A} - \lambda \mathbf{I}) &= (1 - \lambda)(3 - \lambda) - 8 \\ &= 3 - \lambda - 3\lambda + \lambda^2 - 8 \\ &= \lambda^2 - 4\lambda - 5 \\ &= (\lambda - 5)(\lambda + 1) \\ \lambda_1 &= -1 \\ \lambda_2 &= 5\end{aligned}$$

$$\begin{aligned}&\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \\ \mathbf{K}_1 &= \begin{pmatrix} 1 & -1 \end{pmatrix} \\ &\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \\ \mathbf{K}_2 &= \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}\end{aligned}$$

2.15.43

$$\begin{aligned}
 \mathbf{A} &= \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \\
 \det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} \\
 &= (3-\lambda) \begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 4 & 3-\lambda \end{vmatrix} + 4 \begin{vmatrix} 2 & -\lambda \\ 4 & 2 \end{vmatrix} \\
 &= (3-\lambda)(-\lambda(3-\lambda) - 4) - 2(2(3-\lambda) - 8) + 4(4 + 4\lambda) \\
 &= (3-\lambda)(\lambda^2 - 3\lambda - 4) - 2(-2 - 2\lambda) + 16 + 16\lambda \\
 &= 3\lambda^2 - 9\lambda - 12 - \lambda^3 + 3\lambda^2 + 4\lambda + 4 + 4\lambda + 16 + 16\lambda \\
 &= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 \\
 &= -(\lambda - 8)(\lambda + 1)^2 \\
 &\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \\
 &\begin{pmatrix} 1 & -\frac{2}{5} & -\frac{4}{5} \\ 1 & -4 & 1 \\ 1 & \frac{1}{2} & -\frac{5}{4} \end{pmatrix} \\
 &\begin{pmatrix} 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & -\frac{11}{5} & \frac{9}{5} \\ 0 & \frac{9}{10} & -\frac{9}{20} \end{pmatrix} \\
 &\begin{pmatrix} 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{pmatrix} \\
 &\begin{pmatrix} 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \\
 \mathbf{X}_1 &= \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}
 \end{aligned}$$

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{X}_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

2.15.47

$$a^2 + b^2 + c^2 = 1$$

$$-a \frac{1}{\sqrt{2}} + c \frac{1}{\sqrt{2}} = 0$$

$$a = c$$

$$a \frac{1}{\sqrt{3}} + b \frac{1}{\sqrt{3}} + c \frac{1}{\sqrt{3}} = 0$$

$$a + b + c = 0$$

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

2.15.57

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 2 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$$

$$\mathbf{A} = \mathbf{LU}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 2 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$$= \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{pmatrix}$$

$$u_{11} = 1$$

$$u_{12} = 1$$

$$u_{13} = 1$$

$$l_{21} = 1$$

$$u_{22} = -3$$

$$u_{23} = 2$$

$$l_{31} = 2$$

$$l_{32} = \frac{2}{3}$$

$$u_{33} = -\frac{19}{3}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & \frac{2}{3} & 1 \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & -\frac{19}{3} \end{pmatrix}$$

$$\mathbf{AX} = \mathbf{B}$$

$$\mathbf{LUX} = \mathbf{B}$$

$$\mathbf{LY} = \mathbf{B}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & \frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ -\frac{19}{3} \end{pmatrix}$$

$$\mathbf{UX} = \mathbf{Y}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & -\frac{19}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ -\frac{19}{3} \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$