Advanced Engineering Mathematics Vectors, Matrices, and Vector Calculus by Dennis G. Zill Notes

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Contents

1 Vectors

1.1 Vectors in 2-Space

- The zero vector can be assigned any direction
- The vectors $\mathbf{i}$ and $\mathbf{j}$ are known as the **standard basis vectors** for $\mathbb{R}^2$

1.2 Vectors in 3-Space

- In $\mathbb{R}^3$ the octant in which all coordinates are positive is known as the **first octant**. There is no agreement for naming the other seven octants.

1.3 Dot Product

- The **dot product** is also known as the **inner product** or the **scalar product** and is denoted $\mathbf{a} \cdot \mathbf{b}$
- Two non-zero vectors are orthogonal iff their dot product is 0
- The zero vector is considered orthogonal to all vectors

- The angles α, β, and γ between a vector and the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively are called the **direction angles** of the vector
- The cosines of a vectors direction angles (the **direction cosines**) can be calculated as

$$\begin{aligned}
\cos\alpha &= \frac{\mathbf{a}\cdot\mathbf{i}}{||\mathbf{a}||||\mathbf{i}||} \\
&= \frac{a_1}{||\mathbf{a}||} \\
\cos\beta &= \frac{\mathbf{a}\cdot\mathbf{j}}{||\mathbf{a}||||\mathbf{j}||} \\
&= \frac{a_2}{||\mathbf{a}||} \\
\cos\gamma &= \frac{\mathbf{a}\cdot\mathbf{k}}{||\mathbf{a}||||\mathbf{k}||} \\
&= \frac{a_3}{||\mathbf{a}||}
\end{aligned}$$

  Equivalently, these can be calculated as the components of the unit vector $\mathbf{a}/||\mathbf{a}||$.

- To find the component of a vector $\mathbf{a}$ in the direction of a vector $\mathbf{b}$

$$\mathrm{comp}_{\mathbf{b}}\mathbf{a} = ||\mathbf{a}||\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{b}||}$$

- To project a vector $\mathbf{a}$ onto a vector $\mathbf{b}$

$$\mathrm{proj}_{\mathbf{b}}\mathbf{a} = (\mathrm{comp}_{\mathbf{b}}\mathbf{a})\frac{\mathbf{b}}{||\mathbf{b}||} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b}$$

1.4 Cross Product

- The cross product is only defined in $\mathbb{R}^3$
- The **scalar triple product** of vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is defined as

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- The area of a parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$ is $||\mathbf{a}\times\mathbf{b}||$
- The area of a triangle with sides $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{2}||\mathbf{a}\times\mathbf{b}||$
- The volume of a paralleleipied with sides $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$
- $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0$ iff $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar

1.5 Lines and Planes in 3-Space

- There is a unique line between any two points $\mathbf{r_1}$ and $\mathbf{r_2}$ in 3-space. The equation for that line is

$$\mathbf{r} = \mathbf{r_1} + t(\mathbf{r_2} - \mathbf{r_1}) = \mathbf{r_1} + t\mathbf{a}$$

  where t is called a **parameter**, the nonzero vector $\mathbf{a}$ is called a **direction vector**, and its components are called **direction numbers**.

- Equating the components of the equation above we find

$$\begin{aligned} x &= r_1 + ta_1 \\ y &= r_2 + ta_2 \\ z &= r_3 + ta_3. \end{aligned}$$

  These are the **parametric equations** for the line through $\mathbf{r_1}$ and $\mathbf{r_2}$.

- By solving the parametric equations for t and equating the results we find the **symmetric equations** for the line

$$t = \frac{x - r_1}{a_1} = \frac{y - r_2}{a_2} = \frac{z - r_3}{a_3}.$$

- Given a point P_1 and a vector $\mathbf{n}$, there exists only one plane containing P_1 with $\mathbf{n}$ normal. The vector from P_1 to another point P on that plane will be perpendicular to $\mathbf{n}$, so the equation for the plane is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$

  where $\mathbf{r} = \overrightarrow{OP}$ and $\mathbf{r_1} = \overrightarrow{OP_1}$. If

$$\mathbf{n} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$$

  the cartesian form of this equation is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

  and is called the **point-normal form**.

- The graph of any equation $ax + by + cz + d = 0$, where a, b, and c are not all zero, is a plane with the normal vector $\mathbf{n} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$.

- Given three noncollinear points, a normal vector can be found by forming two vectors from two pairs of points and take their cross product.

- A line and a plane that aren't parellel intersect at a single point.

- Two planes that aren't parallel must intersect in a line.