

Advanced Engineering Mathematics Complex Analysis by Dennis G. Zill Notes

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February 2024

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17 Functions of a Complex Variable

17.1 Complex Numbers

- A **complex number** is any number of the form

$$z = a + ib$$

where a and b are real numbers and i is the imaginary unit such that $i^2 = -1$.

- The real number a in the above complex number z is called the **real part** of z and the real number b (not ib) is called the **imaginary part** of z .
- The real and imaginary parts of a complex number z are denoted $\text{Re}(z)$ and $\text{Im}(z)$, respectively.
- A real constant multiple of the imaginary unit, e.g. $6i$ is called a **pure imaginary number**.
- Two complex numbers are equal if their real and imaginary parts are equal.
- The addition and subtraction of complex numbers occur between the real and imaginary parts, e.g.

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

- The multiplication of complex numbers occurs elementwise as normal, e.g.

$$(a + bi)(c + di) = ac + adi + bci - bd.$$

- The **conjugate** of a complex number $z = a + ib$ is

$$\bar{z} = a - ib.$$

- The division of complex numbers occurs by multiplying the numerator and denominator by the conjugate of the denominator, e.g.

$$\begin{aligned} \frac{a + bi}{c + di} &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{ac - adi + bci + bd}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2}. \end{aligned}$$

- Conjugates have several interesting properties:

$$\begin{aligned} \overline{z_1 + z_2} &= \bar{z}_1 + \bar{z}_2 \\ \overline{z_1 - z_2} &= \bar{z}_1 - \bar{z}_2 \\ \overline{z_1 z_2} &= \bar{z}_1 \bar{z}_2 \\ \frac{z_1}{z_2} &= \frac{\bar{z}_1}{\bar{z}_2}. \end{aligned}$$

- The sum and product of a complex number $z = x + iy$ with its conjugate are real numbers

$$\begin{aligned} z + \bar{z} &= 2x \\ z\bar{z} &= x^2 + y^2 \end{aligned}$$

while the difference between a complex number and its conjugate is a pure imaginary number

$$z - \bar{z} = 2iy.$$

- The above properties let us define

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2} \text{ and } \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}.$$

- The **complex plane** or **z -plane** is a coordinate system where the horizontal or x -axis is called the **real axis** and the vertical or y -axis is called the **imaginary axis**. Complex numbers can be plotted in this coordinate system by considering their real and imaginary parts an ordered pair corresponding their position.

- The **modulus** or **absolute value** of a complex number $z = x + iy$ denoted by $|z|$ is the real number

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$$

This is the distance between z and the origin in the complex plane.

- If you consider two numbers in the complex plane as vectors, the length of their sum can't be longer than their individual lengths combined

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

This extends to any finite sum

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

and is known as the **triangle inequality**.

17.2 Powers and Roots

- A complex number can be expressed in **polar form**

$$z = (r \cos \theta) + i(r \sin \theta)$$

where $r = |z|$ is the nonnegative modulus of z and $\theta = \arg z$ is the **argument** of z — the angle between z and the positive real axis measured in the counterclockwise direction.

- The argument of a complex number z isn't unique as any multiply of 2π can be added to it. The **principle argument** of z denoted $\text{Arg } z$ is the argument of z restricted to the interval $-\pi \leq \text{Arg } z \leq \pi$.
- Multiplication and division of complex numbers is simpler in polar form. For two complex numbers $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ we get

$$\begin{aligned} z_1 z_2 &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]. \end{aligned}$$

- The above formulas can be used to find integer powers of a complex number z

$$z^n = r^n (\cos n\theta + i \sin n\theta)$$

where n is an integer (including negative integers).

- **DeMoivre's formula** is a special case of the above where $r = 1$ so

$$z^n = (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

- A number w is said to be an **n th root** of a nonzero complex number z if $w^n = z$. The n th roots of a nonzero complex number $z = r(\cos \theta + i \sin \theta)$ are

$$w_k = r^{1/n} \left[\cos \left(\frac{\theta + 2k\pi}{n} \right) + i \sin \left(\frac{\theta + 2k\pi}{n} \right) \right]$$

where $k = 0, 1, 2, \dots, n - 1$.

- The root w of a complex number z obtained by using the principle argument of z with $k = 0$ is called the **principle n th root** of z .
- Since the n th roots of a complex number have the same modulus they lie on a circle of radius $r^{1/n}$. The arguments of subsequent roots differ by $2\pi/n$ so they're also equally spaced around the circle.

17.3 Sets in the Complex Plane

- The points $z = x + iy$ that satisfy the equation

$$|z - z_0| = \rho$$

for $\rho > 0$ lie on a circle of radius ρ centred at the point z_0 .

- The points z satisfying the inequality $|z - z_0| < \rho$ for $\rho > 0$ lie within, but not on, a circle of radius ρ centered at the point z_0 . This set is called a **neighborhood** of z_0 or an **open disk**.
- A point z_0 is said to be an **interior point** of a set S of the complex plane if there exists some neighborhood of z_0 that lies entirely within S .
- If every point z of a set S is an interior point, then S is said to be an **open set**. An example of a set that isn't open is the set of points satisfying the inequality $\operatorname{Re}(z) \geq 0$. This isn't open because it includes the line $\operatorname{Re}(z) = 0$ and no points on that line are interior to the set because, no matter what ρ you choose, some points in the neighborhood have $\operatorname{Re}(z) < 0$.
- If every neighborhood of a point z_0 contains at least one point that is in a set S and at least one point that is not in S , then z_0 is said to be a **boundary point** of S .
- The **boundary** of a set S in the complex plane is the set of all boundary points of S .
- If any pair of points in a set S can be connected by a polygonal line that lies entirely within the set, then S is said to be **connected**.
- An open connected set is called a **domain**.
- A **region** is a set in the complex plane with all, some, or none of its boundary points. A region containing all of its boundary points is said to be **closed**.

17.4 Functions of a Complex Variable

- A **function** f from a set A to a set B is a rule of correspondence that assigns to each element of A one and only one element of B .
- If b is the element of B assigned to the element a of A , b is said to be the **image** of a and is denoted $b = f(a)$.
- The set A is called the **domain** of f .
- The set of all images in B is called the **range** of f .
- If A is a set of real numbers, f is said to be a **function of a real variable** x .
- If A is a set of complex numbers, f is said to be a **function of a complex variable** z or a **complex function**.
- The image w of a complex number z is

$$w = f(z) = u(x, y) + iv(x, y)$$

where u and v are the real and imaginary parts of w and are real-valued functions.

- Although we cannot draw a graph of a complex function $w = f(z)$ (because it would require a four-dimensional coordinate system), it can be interpreted as a **mapping** or **transformation** from the z plane to the w plane.
- A complex function may be interpreted as a two-dimensional fluid flow by considering $w = f(z)$ as the fluid velocity vector at the point z . In that case, if $x(t) + iy(t)$ is a parametric representation of a particle's position over time then

$$\begin{aligned}\frac{dx}{dt} &= u(x, y) \\ \frac{dy}{dt} &= v(x, y)\end{aligned}$$

and the family of solutions to this system of differential equations are called the **streamlines** of the flow associated with $f(z)$.

Definition 17.4.1 Limit of a Function

Suppose the function f is defined in some neighborhood of z_0 , except possibly at z_0 itself. Then f is said to possess a **limit** at z_0 , written

$$\lim_{z \rightarrow z_0} f(z) = L$$

if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

- For a function f of a real variable x , the limit $\lim_{x \rightarrow x_0} f(x) = L$ means f approaches L as you approach from both the left and right. If however f is a function of a complex variable it means f approaches L as you approach from any direction in the complex plane.

Theorem 17.4.1 Limit of Sum, Product, Quotient

Suppose $\lim_{z \rightarrow z_0} f(z) = L_1$ and $\lim_{z \rightarrow z_0} g(z) = L_2$. Then

(i) $\lim_{z \rightarrow z_0} [f(z) + g(z)] = L_1 + L_2$

(ii) $\lim_{z \rightarrow z_0} f(z)g(z) = L_1L_2$

(iii) $\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{L_1}{L_2}, \quad L_2 \neq 0.$

- A function f is continuous at a point z_0 if

$$\lim_{z \rightarrow z_0} f(z) = f(z_0).$$

- A function f defined by

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0, \quad a_n \neq 0$$

where n is a nonnegative integer and the coefficients a_i , $i = 0, 1, \dots, n$, are complex constants is called a **polynomial** of degree n .

- Polynomials are continuous on the entire complex plane.
- A **rational function**

$$f(z) = \frac{g(z)}{h(z)}$$

is continuous everywhere $h(z) \neq 0$.

Definition 17.4.3 Derivative

Suppose the complex function f is defined in a neighborhood of a point z_0 . The **derivative** of f at z_0 is

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad (3)$$

provided this limit exists.

- In order for a complex function to be differentiable, the limit must approach the same value from every direction. This is a greater demand than in real variables. If you take an arbitrary complex function, there's a good chance it isn't differentiable.

Definition 17.4.4 Analyticity at a Point

A complex function $w = f(z)$ is said to be **analytic at a point** z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0 .

- Analyticity at a point is a neighborhood property. A function can be differentiable at a point but if the neighboring points aren't also differentiable, it's not analytic at that point.
- A function is analytic in a domain D if it is analytic at every point in D .
- A function that is analytic everywhere is called an **entire function**.

17.5 Cauchy-Riemann Equations

Theorem 17.5.1 Cauchy–Riemann Equations

Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivatives of u and v exist and satisfy the **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad (1)$$

- If a complex function $f(z) = u(x, y) + iv(x, y)$ is analytic throughout a domain D , then the real functions u and v must satisfy the Cauchy–Riemann equations at every point in D .

Theorem 17.5.2 Criterion for Analyticity

Suppose the real-valued functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a domain D . If u and v satisfy the Cauchy–Riemann equations at all points of D , then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D .

- The Cauchy–Riemann equations are derived assuming the function is differentiable at a particular point. That being the case, they can also be used as a formula for the derivative of the function

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

- Because analyticity implies differentiability, theorem 17.5.2 can also be used to determine if a function is differentiable at a point.
- A real-valued function $\phi(x, y)$ that has continuous second-order partial derivatives in a domain D and satisfies Laplace's equation is said to be **harmonic** in D .

- If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D then the functions $u(x, y)$ and $v(x, y)$ are harmonic functions.
- If a given function $u(x, y)$ is harmonic in a domain D it is sometimes possible to find another function $v(x, y)$ that is harmonic in D such that $u(x, y) + iv(x, y)$ is analytic in D . The function v is called the **harmonic conjugate function** of u .
- To find the harmonic conjugate function of a given function u :
 1. Take the first-order partial derivatives of u with respect to x and y .
 2. If $u(x, y) + iv(x, y)$ is analytic in a domain D then u and v must satisfy the Cauchy-Riemann equations in D from which we can find expressions for $\partial v/\partial x$ and $\partial v/\partial y$.
 3. Integrate $\partial v/\partial x$ with respect to x to get an expression for v with an unknown constant $h(y)$.
 4. Take the first-order partial derivative of v with respect to y , equate it with the other expression for $\partial v/\partial y$, and solve for $h'(y)$.
 5. Integrate $h'(y)$ and substitute the result to find v .