Contents

1 First-order ODEs

Form: IVP

$$\frac{dy}{dx} = f(x,y)$$
$$y(x_0) = y_0$$

Test: $f(x,y)$ and $\partial f/\partial y$ are continuous over I
Property: A unique solution is guaranteed over I

1.1 Separable Equations

Form:

$$\frac{dy}{dx} = g(x)h(y)$$

Solution: Divide by $h(y)$ then integrate with respect to x.

$$\begin{aligned}
\frac{dy}{dx} &= g(x)h(y) \\
\frac{1}{h(y)}\frac{dy}{dx} &= g(x) \\
\int \frac{1}{h(y)}\frac{dy}{dx}\,dx &= \int g(x)\,dx \\
\int \frac{1}{h(y)}\,dy &= \int g(x)\,dx \\
H(y) &= G(x) + c
\end{aligned}$$

1.2 Linear Equations

Form:

$$\frac{dy}{dx} + P(x)y = f(x)$$

Solution:

1. Determine the integrating factor $e^{\int P(x)\,dx}$
2. Multiply by the integrating factor
3. Recognise that the left hand side of the equation is the derivative of the product of the integrating factor and y
4. Integrate both sides of the equation
5. Solve for y

1.3 Exact Equations

Form:

$$z = f(x, y) = c$$

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = M(x, y)\,dx + N(x, y)\,dy = 0$$

Test:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution:

1. Integrate $M(x, y)$ with respect to x to find an expression for $z = f(x, y)$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= M(x, y) \\
f(x, y) &= \int M(x, y)\,dx + g(y)
\end{aligned}$$

2. Differentiate $f(x, y)$ with respect to y and equate it to $N(x, y)$ to find $g'(y)$

$$\frac{\partial f}{\partial y} = N(x, y) = \frac{\partial}{\partial y} \int M(x, y)\, dx + g'(y)$$
$$g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, dx$$

3. Integrate $g'(y)$ with respect to y to find $g(y)$ and substitute it into $f(x, y)$

4. Equate $f(x, y)$ with an unknown constant c

Note: The steps can be performed with x and y reversed, i.e. start by integrating $N(x, y)$ with respect to y, etc.

1.4 Exact Equations with Integration Constant

Form:

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

Test: $(M_y - N_x)/N$ is a function of x alone or $(N_x - M_y)/M$ is a function of y alone
Solution:

1. Compute the integrating factor

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\, dx}$$

   or

$$\mu(y) = e^{\int \frac{N_x - M_y}{M}\, dy}$$

   as appropriate

2. Multiple the equation by this factor

3. The equation is now exact and can be solved as above

1.5 Homogeneous Equations

Form:

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

Test: M and N are homogeneous functions of the same degree
Solution:

1. Rewrite as

$$M(x,y) = x^\alpha M(1,u) \text{ and } N(x,y) = x^\alpha N(1,u) \text{ where } u = y/x$$

   or

$$M(x,y) = y^\alpha M(v,1) \text{ and } N(x,y) = y^\alpha N(v,1) \text{ where } v = x/y$$

2. Substitute $y = ux$ and $dy = u\,dx + x\,du$ or $x = vy$ and $dx = v\,dy + y\,dv$ as appropriate
3. Solve the resulting first-order separable DE
4. Substitude $u = y/x$ or $v = x/y$ as appropriate

1.6 Bernoulli's Equation

Form:

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

Test: $n \neq 0$ and $n \neq 1$
Solution:

1. Substitude $y = u^{1/(1-n)}$ and $\frac{dy}{dx} = \frac{d}{dx}(u^{1/(1-n)})$
2. Solve the resulting linear equation
3. Substitude $u = y^{1-n}$

1.7 Reduction to Separation of Variables

Form:

$$\frac{dy}{dx} = f(Ax + By + C), B \neq 0$$

Solution:

1. Substitute

$$Ax + By + C = u$$

2. Solve the resulting separable equation
3. Substitute

$$u = Ax + By + C$$

1.8 Riccati's Equation

Form:

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$$

Test: You know a particular solution y_1 of the equation
Solution:

1. Substitute $y = y_1 + u$ and $y' = y_1' + u'$
2. Solve the resulting Bernoulli equation
3. Substitude $u = y - y_1$

2 Higher-order ODEs

2.1 Initial Value Problems

Form: n-th order IVP

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx_{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

subject to

$$y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1}$$

Test: $a_n(x)$, $a_{n-1}(x)$, ..., $a_0(x)$, and $g(x)$ are continuous on an interval I and $a_n(x) \neq 0$ for every x in I
Property: A unique solution exists for every $x = x_0$ in I

2.2 Linear Independence

Form: A set of functions f_1, f_2, ..., f_n
Test: The Wronskian $W(f_1, f_2, \ldots, f_n) \neq 0$ for every x in an interval I where

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

Property: The functions are linearly independent in I

2.3 Homogeneous Linear nth-Order Equations

The general solution is of the form

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

where c_i are arbitrary constants and y_i are a fundamental set of solutions (i.e. a set of n linearly independent solutions).

2.4 Nonhomogeneous Linear nth-Order Equations

The general solution is of the form

$$y = y_c + y_p = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

where y_c is the complementary function (i.e. the general solution of the associated homogeneous equation) and y_p is a particular solution.

2.5 Reduction of Order

Form:

$$y'' + P(x)y' + Q(x)y = 0$$

Test: A non-trivial solution $y_1(x)$ is known
Solution:

1. Recognise that the ratio of two linearly independent functions isn't constant, i.e.
$$u(x) = \frac{y_1(x)}{y_2(x)} \text{ or } y_2(x) = u(x)y_1(x)$$
2. Substitute $y_2(x) = u(x)y_1(x)$ into the DE — this will result in a DE involving only u'' and u' which can be treated as a linear first-order DE in $u' = w$
3. Solve for w
4. Substitute $w = u'$
5. Integrate to find u
6. Multiply by y_1 to find y_2

or equivalently

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx$$

2.6 Homogeneous Linear Equations with Constant Coefficients

Form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Solution:

1. Assume the equation has a solution of the form $y = e^{mx}$, giving
$$a_n m^n e^{mx} + a_{n-1} m^{n-1} e^{mx} + \cdots + a_1 m e^{mx} + a_0 e^{mx} = 0$$

2. Divide by e^{mx}, giving the auxiliary/characteristic equation

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

3. Solve for m, where

   - A real root m corresponds to a solution

   $$y = ce^{mx}$$

   - Complex roots $\alpha \pm i\beta$ corresponds to a solution

   $$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

   - A root m of multiplicity k corresponds to the solutions

   $$e^{mx},\ xe^{mx},\ x^2 e^{mx},\ \ldots,\ x^{k-1} x^{mx}$$

2.7 Method of Undetermined Coefficients

Form: A nonhomogeneous linear DE where the input function ($g(x)$) is comprised of constants, polynomials, exponentials $e^{\alpha x}$, sines, and cosines
Solution:

1. Solve the associated homogeneous equation
2. Assume the particular solution has the same form as the input function
3. If a term in the proposed solution is present in the complementary function, multiply it by x^n where n is the smallest positive integer that removes the duplication
4. Substitute the proposed solution into the DE
5. Solve for the unknown constants

2.8 Variation of Parameters

Form: A nonhomogeneous linear DE
Solution:

1. Solve the homogeneous equation to find the complementary function
2. Assume the solution has the form

   $$y_p = u_1(x) y_1(x) + \cdots + u_n(x) y_n(x)$$

   where n is the order of the equation and y_i are the fundamental set of solutions from the complementary equation

3. Convert to standard form by dividing by the leading coefficient

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

4. Solve the system of linear equations

$$\begin{aligned}
y_1u_1' + \cdots + y_nu_n' &= 0\\
y_1'u_1' + \cdots + y_n'u_n' &= 0\\
&\vdots\\
y_1^{(n-1)}u_1' + \cdots + y_n^{(n-1)}y_n' &= 0\\
y_1^{(n)}u_1' + \cdots + y_n^{(n)}u_n' &= f(x)
\end{aligned}$$

   via Cramer's method:

   (a) Compute the Wronskian of y_i

   $$W = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \ddots & \vdots \\ y_1^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

   (b) Compute u_i' for $i = 1, \ldots, n$ where

   $$u_i' = \frac{W_i}{W}$$

   and W_i is the determinant of the matrix formed by replacing the ith column of the Wronskian matrix with the column vector

   $$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(x) \end{bmatrix}$$

5. Integrate each u_i' to find u_i