Introduction to Electrodynamics by David J. Griffiths Problems

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2 Electrostatics

2.1

(a) **0**

(b) The same as if only the opposite charge were present — all others are cancelled out.

2.2

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} 2 \frac{q}{\imath^2} \cos\theta \hat{\mathbf{x}}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{dq}{[(d/2)^2 + z^2]^{3/2}} \hat{\mathbf{x}}$$

4

2.3

$$\mathbf{r} = z\hat{\mathbf{z}}$$

$$\mathbf{r}' = x\hat{\mathbf{x}}$$

$$\boldsymbol{\imath} = z\hat{\mathbf{z}} - x\hat{\mathbf{x}}$$

$$\boldsymbol{\imath} = \sqrt{x^2 + z^2}$$

$$\hat{\boldsymbol{\imath}} = \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{\sqrt{x^2 + z^2}}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_0^L \frac{\lambda}{x^2 + z^2} \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{\sqrt{x^2 + z^2}} \, dx$$

$$= \frac{1}{4\pi\epsilon_0} \lambda \left(z\hat{\mathbf{z}} \int_0^L \frac{1}{(x^2 + z^2)^{3/2}} \, dx - \hat{\mathbf{x}} \int_0^L \frac{x}{(x^2 + z^2)} \, dx \right)$$

$$= \frac{1}{4\pi\epsilon_0} \lambda \left[\frac{L}{z\sqrt{L^2 + z^2}} \hat{\mathbf{z}} - \left(\frac{1}{z} - \frac{1}{\sqrt{L^2 + z^2}} \right) \hat{\mathbf{x}} \right]$$

$$= \frac{1}{4\pi\epsilon_0} \frac{\lambda}{z} \left[\left(-1 + \frac{z}{\sqrt{L^2 + z^2}} \right) \hat{\mathbf{x}} + \frac{L}{\sqrt{L^2 + z^2}} \hat{\mathbf{z}} \right]$$

2.4

The electric field a distance z above the midpoint of a line segment of length $2L$ and uniform line charge λ is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z\sqrt{z^2 + L^2}} \hat{\mathbf{z}}.$$

Applying this to the four sides of the square, the horizontal components of opposite sides cancel leaving only the vertical component.

$$\cos\theta = \frac{z}{\boldsymbol{\imath}}$$

$$= \frac{z}{\sqrt{(a/2)^2 + z^2}}$$

$$\mathbf{E} = 4 \left(\frac{1}{4\pi\epsilon_0} \frac{\lambda a}{\sqrt{(a/2)^2 + z^2}\sqrt{(a/2)^2 + (a/2)^2 + z^2}} \hat{\mathbf{z}} \right) \cos\theta$$

$$= \frac{1}{4\pi\epsilon_0} \frac{4a\lambda z}{[(a/2)^2 + z^2]\sqrt{(a^2/2) + z^2}} \hat{\mathbf{z}}$$

2.5

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \frac{\lambda r}{r^2 + z^2} \cos\alpha \, d\theta \, \hat{\mathbf{z}}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{2\pi\lambda r z}{(r^2 + z^2)^{3/2}} \hat{\mathbf{z}}$$

2.6

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{\imath^2} \cos\theta \hat{\mathbf{z}}$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^R \frac{\sigma}{r^2 + z^2} \frac{z}{\sqrt{r^2 + z^2}} r \, dr \, d\theta \hat{\mathbf{z}}$$

$$= \frac{1}{4\pi\epsilon_0} 2\pi\sigma z \int_0^R \frac{r}{(r^2 + z^2)^{3/2}} \, dr \, \hat{\mathbf{z}}$$

$$= \frac{1}{4\pi\epsilon_0} 2\pi\sigma z \left(\frac{1}{z} - \frac{1}{\sqrt{R^2 + z^2}} \right) \hat{\mathbf{z}}$$

When $R \to \infty$

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0} \hat{\mathbf{z}}.$$

2.7

$$\mathbf{E} = \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q}{z^2} \hat{\mathbf{z}} & z > R \\ \mathbf{0} & z < R \end{cases}$$

2.8

$$\mathbf{E} = \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q}{z^2} \hat{\mathbf{z}} & z > R \\ \frac{1}{4\pi\epsilon_0} \frac{qz}{R^3} \hat{\mathbf{z}} & z < R \end{cases}$$

2.9

(a)

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E}$$

$$= \epsilon_0 \frac{1}{r^2} \frac{\partial}{\partial r} (kr^5)$$

$$= 5\epsilon_0 kr^2$$

(b)

$$Q_{\text{enc}} = \epsilon_0 \oint \mathbf{E} \cdot d\mathbf{a}$$

$$= \epsilon_0 \int_0^{2\pi} \int_0^{\pi} kR^3 R \, d\theta \, R \sin\theta \, d\phi$$

$$= 2\pi\epsilon_0 kR^5 [-\cos\theta]_0^{\pi}$$

$$= 4\pi\epsilon_0 kR^5$$

$$Q_{\text{enc}} = \int_V \rho \, d\tau$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^R 5\epsilon_0 kr^2 \, dr \, r \, d\theta \, r \sin\theta \, d\phi$$

$$= 10\pi\epsilon_0 k \int_0^{\pi} \int_0^R r^4 \sin\theta \, dr \, d\theta$$

$$= 2\pi\epsilon_0 kR^5 [-\cos\theta]_0^{\pi}$$

$$= 4\pi\epsilon_0 kR^5$$

2.10

If the charge was surrounded by 8 such cubes the total flux through all the cubes would be q/ϵ_0. There are 24 outside faces to the larger cube, so the total flux through the shaded face is $q/(24\epsilon_0)$.

2.11

$$\int \mathbf{E}_{\text{inside}} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$= 0$$

$$\mathbf{E}_{\text{inside}} = \mathbf{0}$$

$$\int \mathbf{E}_{\text{outside}} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$4\pi r^2 E_{\text{outside}} = \frac{4\pi R^2 \sigma}{\epsilon_0}$$

$$\mathbf{E}_{\text{outside}} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

2.12

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$4\pi r^2 E = \frac{\frac{4}{3}\pi r^3 \rho}{\epsilon_0}$$

$$\mathbf{E} = \frac{r\rho}{3\epsilon_0}\hat{\mathbf{r}}$$

2.13

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$2\pi s l E = \frac{l\lambda}{\epsilon_0}$$

$$\mathbf{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{\mathbf{s}}$$

2.14

$$Q_{\text{enc}} = \int_V \rho \, d\tau$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^r kr'^3 \sin\theta \, dr' \, d\theta \, d\phi$$

$$= 2\pi k \int_0^{\pi} \left[\frac{1}{4}r'^4 \sin\theta\right]_0^r d\theta$$

$$= \frac{1}{2}\pi kr^4 [-\cos\theta]_0^{\pi}$$

$$= \pi kr^4$$

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$4\pi r^2 E = \frac{\pi kr^4}{\epsilon_0}$$

$$\mathbf{E} = \frac{kr^2}{4\epsilon_0}\hat{\mathbf{r}}$$

2.15

(a) $\mathbf{E} = \mathbf{0}$

(b)

$$Q_{\text{enc}} = \int_0^{2\pi} \int_0^{\pi} \int_a^r k \sin\theta \, dr' \, d\theta \, d\phi$$
$$= 4\pi k(r-a)$$
$$4\pi r^2 E = \frac{4\pi k(r-a)}{\epsilon_0}$$
$$\mathbf{E} = \frac{k(r-a)}{\epsilon_0 r^2}\hat{\mathbf{r}}$$

(c) $\mathbf{E} = \frac{k(b-a)}{\epsilon_0 r^2}\hat{\mathbf{r}}$

2.16

(a)

$$Q_{\text{enc}} = \pi s^2 l \rho$$
$$2\pi s l E = \frac{\pi s^2 l \rho}{\epsilon_0}$$
$$\mathbf{E} = \frac{s\rho}{2\epsilon_0}\hat{\mathbf{s}}$$

(b)

$$\mathbf{E} = \frac{a^2 \rho}{2\epsilon_0 s}\hat{\mathbf{s}}$$

(c)

$$\mathbf{E} = \mathbf{0}$$

2.17

$$2AE_{\text{inside}} = \frac{2Ay\rho}{\epsilon_0}$$
$$\mathbf{E}_{\text{inside}} = \frac{y\rho}{\epsilon_0}$$
$$\mathbf{E} = \begin{cases} \frac{d\rho}{\epsilon_0} & d < y \\ \frac{y\rho}{\epsilon_0} & 0 < y < d \\ -\frac{y\rho}{\epsilon_0} & -d < y < 0 \\ -\frac{d\rho}{\epsilon_0} & y < -d \end{cases}$$

2.18

The electric field inside a uniformly charged solid sphere is

$$\mathbf{E} = \frac{r\rho}{3\epsilon_0}\hat{\mathbf{r}}.$$

$$\mathbf{d} = \mathbf{r}_1 - \mathbf{r}_2$$
$$\mathbf{E} = \frac{r_1\rho}{3\epsilon_0}\hat{\mathbf{r}}_1 - \frac{r_2\rho}{3\epsilon_0}\hat{\mathbf{r}}_2$$
$$= \frac{\rho}{3\epsilon_0}(\mathbf{r}_1 - \mathbf{r}_2)$$
$$= \frac{\rho}{3\epsilon_0}\mathbf{d}$$

2.20

a is impossible because its curl is nonzero.

$$V = -\int_0^y 2kxy'\,dy' - \int_0^z 2kyz'\,dz$$
$$= -2kx\left[\frac{1}{2}y'^2\right]_0^y - 2ky\left[\frac{1}{2}z'^2\right]_0^z$$
$$= -k(xy^2 + yz^2)$$
$$-\nabla V = k[y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}]$$
$$= \mathbf{E}$$

2.21

$$\mathbf{E} = \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} & r > R \\ \frac{1}{4\pi\epsilon_0} \frac{qr}{R^3} & r < R \end{cases}$$

$$V_{\text{outside}}(r) = -\int_\infty^r \frac{1}{4\pi\epsilon_0} \frac{q}{r'^2} \, dr'$$

$$= -\frac{1}{4\pi\epsilon_0} q \left[-\frac{1}{r'} \right]_\infty^r$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

$$-\nabla V_{\text{outside}} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

$$= \mathbf{E}_{\text{outside}}$$

$$V_{\text{inside}}(r) = -\left(\int_\infty^R \frac{1}{4\pi\epsilon_0} \frac{q}{r'^2} \, dr' + \int_R^r \frac{1}{4\pi\epsilon_0} \frac{qr'}{R^3} \, dr' \right)$$

$$= -\left(-\frac{1}{4\pi\epsilon_0} \frac{q}{R} + \frac{1}{4\pi\epsilon_0} \frac{q}{R^3} \left[\frac{1}{2} r'^2 \right]_R^r \right)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{2R} \left[3 - \left(\frac{r}{R} \right)^2 \right]$$

$$-\nabla V_{\text{inside}} = \frac{1}{4\pi\epsilon_0} \frac{qr}{R^3} \hat{\mathbf{r}}$$

$$= \mathbf{E}_{\text{inside}}$$

2.22

$$\mathbf{E} = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{s} \hat{\mathbf{s}}$$

$$V = -\int_O^s \frac{1}{2\pi\epsilon_0} \frac{\lambda}{s'} \, ds'$$

$$= -\frac{1}{2\pi\epsilon_0} \lambda \ln \frac{s}{O}$$

$$-\nabla V = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{s} \hat{\mathbf{s}}$$

2.23

$$\mathbf{E} = \begin{cases} \mathbf{0} & r < a \\ \frac{k(r-a)}{\epsilon_0 r^2}\hat{\mathbf{r}} & a < r < b \\ \frac{k(b-a)}{\epsilon_0 r^2}\hat{\mathbf{r}} & b < r \end{cases}$$

$$V(0) = -\int_\infty^0 E\, dr$$

$$= -\left(\int_\infty^b \frac{k(b-a)}{\epsilon_0 r^2}\, dr + \int_b^a \frac{k(r-a)}{\epsilon_0 r^2}\, dr \right)$$

$$= -\left(\frac{k(b-a)}{\epsilon_0}\left[-\frac{1}{r} \right]_\infty^b + \frac{k}{\epsilon_0}\left[\ln r + \frac{a}{r} \right]_b^a \right)$$

$$= -\left[-\frac{k(b-a)}{\epsilon_0 b} + \frac{k}{\epsilon_0}\left(\ln a + 1 - \ln b - \frac{a}{b} \right) \right]$$

$$= -\frac{k}{\epsilon_0}\left(-1 + \frac{a}{b} + \ln \frac{a}{b} + 1 - \frac{a}{b} \right)$$

$$= \frac{k}{\epsilon_0} \ln \frac{b}{a}$$

2.24

$$V(b) - V(0) = -\int_0^b E\, dr$$

$$= -\left(\int_0^a \frac{s\rho}{2\epsilon_0}\, ds + \int_a^b \frac{a^2\rho}{2\epsilon_0 s}\, ds \right)$$

$$= -\left(\frac{\rho}{2\epsilon_0}\left[\frac{1}{2}s^2 \right]_0^a + \frac{a^2\rho}{2\epsilon_0} \ln \frac{b}{a} \right)$$

$$= -\left(\frac{a^2\rho}{4\epsilon_0} + \frac{a^2\rho}{2\epsilon_0} \ln \frac{b}{a} \right)$$

$$= -\frac{a^2\rho}{4\epsilon_0}\left(1 + 2\ln \frac{a}{b} \right)$$

2.25

(a)

$$V = \frac{1}{4\pi\epsilon_0} \frac{2q}{\sqrt{(d/2)^2 + z^2}}$$

(b)

$$V = \frac{1}{4\pi\epsilon_0} \int_{-L}^{L} \frac{\lambda}{\sqrt{x^2 + z^2}} \, dx$$

$$= \frac{1}{4\pi\epsilon_0} \lambda \ln\left(1 + \frac{2L(L + \sqrt{L^2 + z^2})}{z^2}\right)$$

(c)

$$V = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^R \frac{\sigma}{\sqrt{r^2 + z^2}} r \, dr \, d\theta$$

$$= \frac{1}{4\pi\epsilon_0} 2\pi\sigma(\sqrt{R^2 + z^2} - z)$$

2.26

$$V_{\text{bottom}} = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^h \frac{\sqrt{2}\sigma z}{\sqrt{2}z} \, d\phi \, dz$$

$$= \frac{\sigma h}{2\epsilon_0}$$

$$V_{\text{top}} = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^h \frac{\sqrt{2}\sigma z}{\sqrt{z^2 + (h-z)^2}} \, d\phi \, dz$$

$$= \frac{\sqrt{2}\sigma}{2\epsilon_0} \int_0^h \frac{z}{\sqrt{z^2 + (h-z)^2}} \, dz$$

$$= \frac{\sigma h}{4\epsilon_0} \ln(3 + 2\sqrt{2})$$

$$V_{\text{bottom}} - V_{\text{top}} = \frac{\sigma h}{2\epsilon_0} \left[1 - \frac{1}{2} \ln(3 + 2\sqrt{2})\right]$$

2.28

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^{\pi} \int_0^R \frac{\rho r'^2 \sin\theta}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} \, dr' \, d\theta \, d\phi$$

$$= \frac{\rho}{2\epsilon_0} \int_0^{\pi} \int_0^R \frac{r'^2 \sin\theta}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} \, dr' \, d\theta$$

$$= \frac{\rho}{2\epsilon_0} \left(R^2 - \frac{r^2}{3}\right)$$

$$= \frac{q}{8\pi\epsilon_0 R} \left(3 - \frac{r^2}{R^2}\right)$$

13

2.31

(a)

$$W = \frac{q^2}{4\pi\epsilon_0 a}\left(\frac{1}{\sqrt{2}} - 2\right)$$

(b)

$$W = \frac{1}{4\pi\epsilon_0}\left(-\frac{q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a} - \frac{q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a}\right)$$

$$= \frac{q^2}{2\pi\epsilon_0 a}\left(\frac{1}{\sqrt{2}} - 2\right)$$

2.32

$$W = \frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{a}$$

$$W = K_1 + K_2$$

$$\frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{a} = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2$$

$$\frac{1}{2\pi\epsilon_0}\frac{q_A q_B}{a} = m_A v_A^2 + m_B v_B^2$$

$$0 = m_B v_B - m_A v_A$$

$$v_B = \frac{m_A}{m_B}v_A$$

$$\frac{1}{2\pi\epsilon_0}\frac{q_A q_B}{a} = m_A v_A^2 + m_B\left(\frac{m_A}{m_B}v_A\right)^2$$

$$= m_A v_A^2 + \frac{m_A^2}{m_B}v_A^2$$

$$= \frac{m_A(m_A + m_B)}{m_B}v_A^2$$

$$v_A = \sqrt{\frac{1}{2\pi\epsilon_0}\frac{q_A q_B}{(m_A + m_B)a}\frac{m_B}{m_A}}$$

$$v_B = \sqrt{\frac{1}{2\pi\epsilon_0}\frac{q_A q_B}{(m_A + m_B)a}\frac{m_A}{m_B}}$$

2.33

$$W = \frac{1}{4\pi\epsilon_0} \left(-\frac{q^2}{a} + \frac{q^2}{2a} - \frac{q^2}{3a} + \cdots \right)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q^2}{a} \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$= -\frac{1}{4\pi\epsilon_0} \frac{q^2}{a} \ln 2$$

2.34

(a)

$$V = \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q}{2R} \left[3 - \left(\frac{r}{R}\right)^2 \right] & r < R \\ \frac{1}{4\pi\epsilon_0} \frac{q}{r} & r > R \end{cases}$$

$$W = \frac{1}{2} \int \rho V \, d\tau$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho \frac{1}{4\pi\epsilon_0} \frac{q}{2R} \left[3 - \left(\frac{r}{R}\right)^2 \right] r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{q\rho}{8\epsilon_0 R} \int_0^{\pi} \int_0^{R} \left[3 - \left(\frac{r}{R}\right)^2 \right] r^2 \sin\theta \, dr \, d\theta$$

$$= \frac{q\rho R^2}{5\epsilon_0}$$

$$= \frac{qR^2}{5\epsilon_0} \frac{q}{\frac{4}{3}\pi R^3}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{3q^2}{5R}$$

(b)

$$\mathbf{E} = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} & r > R \\ \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3}\hat{\mathbf{r}} & r < R \end{cases}$$

$$E^2 = \begin{cases} \frac{1}{16\pi^2\epsilon_0^2}\frac{q^2}{r^4} & r > R \\ \frac{1}{16\pi^2\epsilon_0^2}\frac{q^2 r^2}{R^6} & r < R \end{cases}$$

$$W = \frac{\epsilon_0}{2}\int E^2\, d\tau$$

$$= \frac{\epsilon_0}{2}\left(\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{1}{16\pi^2\epsilon_0^2}\frac{q^2 r^2}{R^6}r^2\sin\theta\, dr\, d\theta\, d\phi\right.$$

$$\left. + \int_0^{2\pi}\int_0^{\pi}\int_R^{\infty}\frac{1}{16\pi^2\epsilon_0^2}\frac{q^2}{r^4}r^2\sin\theta\, dr\, d\theta\, d\phi\right)$$

$$= \frac{\epsilon_0}{2}\frac{1}{16\pi^2\epsilon_0^2}2\pi q^2\left(\int_0^{\pi}\int_0^{R}\frac{r^4}{R^6}\sin\theta\, dr\, d\theta + \int_0^{\pi}\int_R^{\infty}\frac{1}{r^2}\sin\theta\, dr\, d\theta\right)$$

$$= \frac{1}{16\pi\epsilon_0}q^2\left(\int_0^{\pi}\int_0^{R}\frac{r^4}{R^6}\sin\theta\, dr\, d\theta + \int_0^{\pi}\int_R^{\infty}\frac{1}{r^2}\sin\theta\, dr\, d\theta\right)$$

$$= \frac{1}{16\pi\epsilon_0}q^2\left(\frac{2}{5R} + \frac{2}{R}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}$$

(c)

$$W = \frac{\epsilon_0}{2} \left(\int_V E^2 \, d\tau + \oint_S V\mathbf{E} \cdot d\mathbf{a} \right)$$

$$= \frac{\epsilon_0}{2} \left(\int_0^{2\pi} \int_0^{\pi} \int_0^R \frac{1}{(4\pi\epsilon_0)^2} \frac{q^2 r^2}{R^6} r^2 \sin\theta \, dr \, d\theta \, d\phi \right.$$

$$+ \int_0^{2\pi} \int_0^{\pi} \int_R^a \frac{1}{(4\pi\epsilon_0)^2} \frac{q^2}{r^4} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\left. + \int_0^{2\pi} \int_0^{\pi} \frac{1}{4\pi\epsilon_0} \frac{q}{a} \frac{1}{4\pi\epsilon_0} \frac{q}{a^2} a^2 \sin\theta \, d\theta \, d\phi \right)$$

$$= \frac{\epsilon_0}{2} \frac{1}{(4\pi\epsilon_0)^2} 2\pi q^2 \left(\int_0^{\pi} \int_0^R \frac{r^4}{R^6} \sin\theta \, dr \, d\theta \right.$$

$$\left. + \int_0^{\pi} \int_R^a \frac{1}{r^2} \sin\theta \, dr \, d\theta + \int_0^{\pi} \frac{1}{a} \sin\theta \, d\theta \right)$$

$$= \frac{\epsilon_0}{2} \frac{1}{(4\pi\epsilon_0)^2} 2\pi q^2 \left[\frac{2}{5R} + 2\left(\frac{1}{R} - \frac{1}{a}\right) + \frac{2}{a} \right]$$

$$= \frac{1}{8\pi\epsilon_0} q^2 \left[\frac{1}{5R} + \frac{1}{R} \right]$$

$$= \frac{1}{4\pi\epsilon_0} \frac{3q^2}{5R}$$

2.36

(a)

$$\mathbf{E} = \begin{cases} \mathbf{0} & r < a \\ \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} & a < r < b \\ \mathbf{0} & b < r \end{cases}$$

$$E^2 = \begin{cases} 0 & r < a \\ \frac{1}{(4\pi\epsilon_0)^2} \frac{q^2}{r^4} & a < r < b \\ 0 & b < r \end{cases}$$

$$W = \frac{\epsilon_0}{2} \int E^2 \, d\tau$$

$$= \frac{\epsilon_0}{2} \int_0^{2\pi} \int_0^{\pi} \int_a^b \frac{1}{(4\pi\epsilon_0)^2} \frac{q^2}{r^4} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{\epsilon_0}{2} \frac{1}{(4\pi\epsilon_0)^2} 2\pi q^2 \int_0^{\pi} \int_a^b \frac{\sin\theta}{r^2} \, dr \, d\theta$$

$$= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)$$

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(b)

$$W_{\text{shell}} = \frac{1}{8\pi\epsilon_0} \frac{q^2}{R}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

$$\mathbf{E}_1 \cdot \mathbf{E}_2 = -\frac{1}{(4\pi\epsilon_0)^2} \frac{q^2}{r^4}$$

$$
\begin{aligned}
W_{\text{total}} &= W_1 + W_2 + \epsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2 \, d\tau \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} + \frac{1}{b} \right) - \epsilon_0 \int_0^{2\pi} \int_0^{\pi} \int_b^{\infty} \frac{1}{(4\pi\epsilon_0)^2} \frac{q^2}{r^4} r^2 \sin\theta \, dr \, d\theta \, d\phi \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} + \frac{1}{b} \right) - \frac{1}{8\pi\epsilon_0} q^2 \int_0^{\pi} \int_b^{\infty} \frac{1}{r^2} \sin\theta \, dr \, d\theta \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} + \frac{1}{b} \right) - \frac{1}{4\pi\epsilon_0} q^2 \int_b^{\infty} \frac{1}{r^2} \, dr \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} + \frac{1}{b} \right) - \frac{1}{4\pi\epsilon_0} \frac{q^2}{b} \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} + \frac{1}{b} - \frac{2}{b} \right) \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)
\end{aligned}
$$

2.37

$$r_1 = r$$

$$E_1 = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1^2}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}$$

$$r_2 = \sqrt{a^2 + r^2 - 2ar\cos\theta}$$

$$E_2 = \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_2^2}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q_2}{a^2 + r^2 - 2ar\cos\theta}$$

$$\cos\alpha = \frac{r - a\cos\theta}{\sqrt{a^2 + r^2 - 2ar\cos\theta}}$$

$$\mathbf{E}_1 \cdot \mathbf{E}_2 = E_1 E_2 \cos\alpha$$

$$= \frac{1}{(4\pi\epsilon_0)^2}\frac{q_1 q_2}{r^2(a^2 + r^2 - 2ar\cos\theta)}\frac{r - a\cos\theta}{\sqrt{a^2 + r^2 - 2ar\cos\theta}}$$

$$= \frac{1}{(4\pi\epsilon_0)^2}\frac{q_1 q_2(r - a\cos\theta)}{r^2(a^2 + r^2 - 2ar\cos\theta)^{3/2}}$$

$$\epsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2 \, d\tau = \epsilon_0 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{(4\pi\epsilon_0)^2}\frac{q_1 q_2(r - a\cos\theta)}{r^2(a^2 + r^2 - 2ar\cos\theta)^{3/2}}r^2\sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{q_1 q_2}{8\pi\epsilon_0}\int_0^{\pi}\int_0^{\infty}\frac{(r - a\cos\theta)\sin\theta}{(a^2 + r^2 - 2ar\cos\theta)^{3/2}}\, dr \, d\theta$$

2.38

(a)

$$\sigma_R = \frac{q}{4\pi R^2}$$

$$\sigma_a = -\frac{q}{4\pi a^2}$$

$$\sigma_b = \frac{q}{4\pi b^2}$$

(b)

$$V = -\int_\infty^b \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\, dr - \int_a^R \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\, dr$$

$$= \frac{1}{4\pi\epsilon_0}q\left(\frac{1}{b} + \frac{1}{R} - \frac{1}{a}\right)$$

19

(c)

$$\sigma_b = 0$$

$$V = \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{R} - \frac{1}{a} \right)$$

2.39

(a)

$$\sigma_a = -\frac{q_a}{4\pi a^2}$$

$$\sigma_b = -\frac{q_b}{4\pi b^2}$$

$$\sigma_R = \frac{q_a + q_b}{4\pi R^2}$$

(b)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q_a + q_b}{r^2} \hat{\mathbf{r}}$$

(c)

$$\mathbf{E}_a = \frac{1}{4\pi\epsilon_0} \frac{q_a}{r^2} \hat{\mathbf{r}}$$

$$\mathbf{E}_b = \frac{1}{4\pi\epsilon_0} \frac{q_b}{r^2} \hat{\mathbf{r}}$$

(d)

$$\mathbf{0}$$

(e) a, b

2.40

(a) No. If it's close to the wall it will induce a surface charge and be attracted.

(b) No. If the conductor contains a cavity containing a like charge it will be repelled.

2.41

By Gauss's law, the electric field of each plate is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$2A'E = \frac{A'\frac{Q}{A}}{\epsilon_0}$$

$$\mathbf{E} = \frac{Q}{2A\epsilon_0} \hat{\mathbf{n}}$$

so the field between the plates is zero and the field outside is $Q/A\epsilon_0 \hat{\mathbf{n}}$, resulting in a pressure of

$$P = \frac{\epsilon_0}{2} E^2$$

$$= \frac{\epsilon_0}{2} \frac{Q^2}{A^2 \epsilon_0^2}$$

$$= \frac{Q^2}{2 A^2 \epsilon_0}$$

2.42

$$\mathbf{E}_{\text{above}} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}}$$

$$\mathbf{f} = \frac{1}{2} \sigma \mathbf{E}_{\text{above}}$$

$$= \frac{1}{2} \frac{Q}{4\pi R^2} \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \hat{\mathbf{r}}$$

$$= \frac{Q^2}{32\pi^2 \epsilon_0 R^4} \hat{\mathbf{r}}$$

$$\mathbf{F} = \int_0^{2\pi} \int_0^{\pi/2} \frac{Q^2}{32\pi^2 \epsilon_0 R^4} \cos\theta R^2 \sin\theta \, d\theta \, d\phi \hat{\mathbf{z}}$$

$$= \frac{Q^2}{16\pi\epsilon_0 R^2} \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \hat{\mathbf{z}}$$

$$= \frac{Q^2}{32\pi\epsilon_0 R^2} \hat{\mathbf{z}}$$

2.43

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q}{\epsilon_0}$$

$$2\pi s L E = \frac{Q}{\epsilon_0}$$

$$\mathbf{E} = \frac{Q}{2\pi L \epsilon_0} \frac{1}{s} \hat{\mathbf{s}}$$

$$V = -\int_b^a \frac{Q}{2\pi \epsilon_0 L} \frac{1}{s} \, dr$$

$$= \frac{Q}{2\pi \epsilon_0 L} \ln \frac{b}{a}$$

$$C = \frac{Q}{V}$$

$$= \frac{2\pi \epsilon_0 L}{\ln b/a}$$

So the capacitance per unit length is

$$C = \frac{2\pi \epsilon_0}{\ln b/a}.$$

2.44

(a)

$$P = \frac{\epsilon_0}{2} E^2$$

$$W = Fd$$

$$= P A \epsilon$$

$$= \frac{\epsilon_0}{2} E^2 A \epsilon$$

(b)

$$\frac{\epsilon_0}{2} E^2 A \epsilon$$

2.46

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 3\frac{k}{r}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{k}{r}2\sin\theta\cos\theta\sin\phi\right)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{k}{r}\sin\theta\cos\phi\right)$$

$$= \frac{3k}{r^2} + \frac{1}{r\sin\theta}\frac{2k}{r}\sin\phi(2\sin\theta\cos^2\theta - \sin^3\theta) - \frac{1}{r\sin\theta}\frac{k}{r}\sin\theta\sin\phi$$

$$= \frac{3k}{r^2} + \frac{2k\sin\phi}{r^2}(2\cos^2\theta - \sin^2\theta) - \frac{k}{r^2}\sin\phi$$

$$= \frac{k}{r^2}[3 + 2\sin\phi(2\cos^2\theta - \sin^2\theta) - \sin\phi]$$

$$= \frac{k}{r^2}[3 + \sin\phi(4\cos^2\theta - 2\sin^2\theta - 1)]$$

$$= \frac{k}{r^2}[3 + \sin\phi(6\cos^2\theta - 3)]$$

$$= \frac{3k}{r^2}(1 + \cos 2\theta\sin\phi)$$

$$\rho = \epsilon_0\nabla \cdot \mathbf{E}$$

$$= \frac{3k\epsilon_0}{r^2}(1 + \cos 2\theta\sin\phi)$$

2.47

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}\hat{\mathbf{r}}$$

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3}$$

$$\rho\mathbf{E} = \frac{3Q}{4\pi R^3}\frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}\hat{\mathbf{r}}$$

$$= \frac{3r}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2\hat{\mathbf{r}}$$

$$F_z = \int_0^{2\pi}\int_0^{\pi/2}\int_0^R \frac{3r}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2\cos\theta r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{3\pi}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2\int_0^{\pi/2}\int_0^R r^3\sin 2\theta\,dr\,d\theta$$

$$= \frac{3\pi}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2\frac{R^4}{4}$$

$$= \frac{3Q^2}{64\pi\epsilon_0 R^2}$$

2.49

$$Q_{\text{enc}} = \int_0^{2\pi} \int_0^\pi \int_0^r kr'^3 \sin\theta \, dr' \, d\theta \, d\phi$$

$$= 2\pi k \int_0^\pi \int_0^r r'^3 \sin\theta \, dr' \, d\theta$$

$$= \pi k r^4$$

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$4\pi r^2 E = \frac{\pi k r^4}{\epsilon_0}$$

$$\mathbf{E} = \begin{cases} \frac{kr^2}{4\epsilon_0}\hat{\mathbf{r}} & r < R \\ \frac{kR^4}{4\epsilon_0 r^2}\hat{\mathbf{r}} & r > R \end{cases}$$

$$W = \frac{\epsilon_0}{2}\left(\int_0^{2\pi}\int_0^\pi\int_0^R \frac{k^2 r^4}{16\epsilon_0^2} r^2 \sin\theta \, dr \, d\theta \, d\phi \right.$$

$$\left. \int_0^{2\pi}\int_0^\pi\int_R^\infty \frac{k^2 R^8}{16\epsilon_0^2 r^4} r^2 \sin\theta \, dr \, d\theta \, d\phi\right)$$

$$= \frac{\epsilon_0}{2} 2\pi \frac{k^2}{16\epsilon_0^2}\left(\int_0^\pi\int_0^R r^6 \sin\theta \, dr \, d\theta + \int_0^\pi\int_R^\infty \frac{R^8 \sin\theta}{r^2} \, dr \, d\theta\right)$$

$$= \frac{\pi k^2}{16\epsilon_0}\left(\frac{2R^7}{7} + 2R^7\right)$$

$$= \frac{\pi k^2 R^7}{7\epsilon_0}$$

2.50

$$V(\mathbf{r}) = A\frac{e^{-\lambda r}}{r}$$

$$\mathbf{E} = -\nabla V$$

$$= Ae^{-\lambda r}(1 + \lambda r)\frac{\hat{\mathbf{r}}}{r^2}$$

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E}$$

$$= \epsilon_0\left[Ae^{-\lambda r}(1+\lambda r)\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} + \frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla\left(Ae^{-\lambda r}(1+\lambda r)\right)\right]$$

$$= A\epsilon_0\left[4\pi\delta(\mathbf{r}) + \frac{\hat{\mathbf{r}}}{r^2}\cdot(-\lambda^2 e^{-\lambda r}r\hat{\mathbf{r}})\right]$$

$$= A\epsilon_0\left(4\pi\delta(\mathbf{r}) - \frac{\lambda^2 e^{-\lambda r}}{r}\right)$$

2.51

$$V = \int \frac{1}{4\pi\epsilon_0} \frac{\sigma}{\imath} \, dA$$

$$= \frac{\sigma}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^R \frac{r}{\sqrt{r^2 + R^2 - 2rR\cos\theta}} \, dr \, d\theta$$

$$= \frac{R\sigma}{4\pi\epsilon_0} \int_0^{2\pi} \left[\cos\theta \ln\left(1 + \csc\frac{\theta}{2}\right) + 2\sin\frac{\theta}{2} - 1\right] d\theta$$

$$= \frac{R\sigma}{\pi\epsilon_0}$$

2.52

(a)

$$V_- = \frac{1}{2\pi\epsilon_0} \lambda \ln \frac{s_-}{a}$$

$$= \frac{1}{2\pi\epsilon_0} \lambda \ln \frac{\sqrt{(y+a)^2 + z^2}}{a}$$

$$V_+ = -\frac{1}{2\pi\epsilon_0} \lambda \ln \frac{s_+}{a}$$

$$= -\frac{1}{2\pi\epsilon_0} \lambda \ln \frac{\sqrt{(y-a)^2 + z^2}}{a}$$

$$V = V_- + V_+$$

$$= \frac{1}{4\pi\epsilon_0} \lambda \ln \frac{(y+a)^2 + z^2}{(y-a)^2 + z^2}$$

2.53

(a)

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \nabla V = -\frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \frac{dV}{dx}\hat{\mathbf{x}} = -\frac{\rho}{\epsilon_0}$$

$$\frac{d^2 V}{dx^2} = -\frac{\rho}{\epsilon_0}$$

(b)

$$qV = \frac{1}{2}mv^2$$

$$v = \sqrt{\frac{2qV}{m}}$$

(c)

$$I = A\rho v$$

(d)

$$\frac{d^2V}{dx^2} = -\frac{I}{Av\epsilon_0}$$

$$= -\frac{I}{A\epsilon_0}\sqrt{\frac{m}{2qV}}$$

$$= \beta V^{-1/2}$$

2.55

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E}$$

$$= a\epsilon_0$$

2.56

$$E = \frac{3GM^2}{5R}$$

$$E_{\text{sun}} = 2.3 \times 10^{41} \text{ J}$$

$$t = \frac{E_{\text{sun}}}{P}$$

$$= 1.89 \times 10^7 \text{ years}$$

3 Potentials

3.1

$$V_{\text{ave}} = \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\left.\sqrt{z^2 + R^2 - 2zR\cos\theta}\right|_0^\pi$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\left(\sqrt{z^2 + R^2 + 2zR} - \sqrt{z^2 + R^2 - 2zR}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\left(\sqrt{(z+R)^2} - \sqrt{(R-z)^2}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}(z + R - R + z)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{R}$$

The average potential due to external charges is V_{center} and the average potential due to internal charges is

$$\frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{R}$$

so

$$V_{\text{ave}} = V_{\text{center}} + \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{R}.$$

3.2

A stable equilibrium is a minimum of potential energy. Laplace's equation doesn't allow for minimums, so they must be saddle points and the charge can escape.

3.3

$$0 = \nabla^2 V$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right)$$

$$= \frac{1}{r^2} \left(2r \frac{\partial V}{\partial r} + r^2 \frac{\partial^2 V}{\partial r^2} \right)$$

$$= \frac{2}{r} \frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2}$$

$$V = \frac{c_1}{r} + c_2$$

$$0 = \nabla^2 V$$

$$= \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial V}{\partial s} \right)$$

$$= \frac{1}{s} \left(\frac{\partial V}{\partial s} + s \frac{\partial^2 V}{\partial s^2} \right)$$

$$= \frac{1}{s} \frac{\partial V}{\partial s} + \frac{\partial^2 V}{\partial s^2}$$

$$V = c_1 + c_2 \ln s$$

3.7

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} q^2 \left(-\frac{2}{(2d)^2} + \frac{2}{(4d)^2} - \frac{1}{(6d)^2} \right) \hat{\mathbf{z}}$$

$$= -\frac{1}{4\pi\epsilon_0} \frac{29q^2}{72d^2} \hat{\mathbf{z}}$$

3.8

(a)

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} + \frac{q'}{\sqrt{r^2 + b^2 - 2rb\cos\theta}} \right]$$

$$= \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{Rq/a}{\sqrt{r^2 + (R^2/a)^2 - 2r(R^2/a)\cos\theta}} \right]$$

$$= \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{q}{\sqrt{R^2 + (ra/R)^2 - 2ra\cos\theta}} \right]$$

(b)

$$\sigma = -\epsilon_0 \left. \frac{\partial V}{\partial r} \right|_{r=R}$$

$$= \frac{q}{4\pi R} \frac{R^2 - a^2}{(a^2 + R^2 - 2aR\cos\theta)^{3/2}}$$

$$Q_{\text{induced}} = \int_0^{2\pi} \int_0^{\pi} \sigma R^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{qR(R^2 - a^2)}{2} \int_0^{\pi} \frac{\sin\theta}{(a^2 + R^2 - 2aR\cos\theta)^{3/2}} \, d\theta$$

$$= \frac{qR(R^2 - a^2)}{a(a^2 - R^2)}$$

$$= -\frac{R}{a} q$$

$$= q'$$

(c)

$$W = \frac{1}{2} qV$$

$$= \frac{1}{8\pi\epsilon_0} \frac{qq'}{a - b}$$

$$= -\frac{1}{8\pi\epsilon_0} \frac{q^2 R/a}{a - R^2/a}$$

$$= -\frac{1}{8\pi\epsilon_0} \frac{q^2 R}{a^2 - R^2}$$

3.9

Place the second image charge at the centre of the sphere with charge

$$q'' = 4\pi\epsilon_0 R V_0.$$

$$F = \frac{1}{4\pi\epsilon_0} q \left(\frac{q'}{(a-b)^2} + \frac{q''}{a^2} \right)$$

$$= \frac{qq'}{4\pi\epsilon_0} \left(\frac{1}{(a-b)^2} - \frac{1}{a^2} \right)$$

$$= \frac{qq'}{4\pi\epsilon_0} \frac{a^2 - (a-b)^2}{a^2(a-b)^2}$$

$$= \frac{qq'}{4\pi\epsilon_0} \frac{b(2a-b)}{a^2(a-b)^2}$$

$$= \frac{q(-Rq/a)}{4\pi\epsilon_0} \frac{(R^2/a)(2a - R^2/a)}{a^2(a - R^2/a)^2}$$

$$= -\frac{q^2}{4\pi\epsilon_0} \left(\frac{R}{a} \right)^3 \frac{2a^2 - R^2}{(a^2 - R^2)^2}$$

3.10

(a)
$$V(x,y,z) = \frac{1}{4\pi\epsilon_0} \lambda \ln \frac{y^2 + (z+d)^2}{y^2 + (z-d)^2}$$

(b)
$$\sigma = -\epsilon_0 \left. \frac{\partial V}{\partial z} \right|_{z=0}$$

$$= -\frac{d\lambda}{\pi(d^2 + y^2)}$$

3.11

You need three charges: $-q$ at $(-a, b)$, $-q$ at $(a, -b)$, and q at $(-b, -a)$. The potential is

$$V = \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{\sqrt{(x-a)^2 + (y-b)^2}} - \frac{1}{\sqrt{(x+a)^2 + (y-b)^2}} \right.$$

$$\left. - \frac{1}{\sqrt{(x-a)^2 + (y+b)^2}} + \frac{1}{\sqrt{(x+a)^2 + (y+b)^2}} \right).$$

The force on q is

$$\mathbf{F} = \frac{q^2}{16\pi\epsilon_0} \left[\left(\frac{a}{(a^2+b^2)^{3/2}} - \frac{1}{a^2} \right) \hat{\mathbf{x}} + \left(\frac{b}{(a^2+b^2)^{3/2}} - \frac{1}{b^2} \right) \hat{\mathbf{y}} \right].$$

The work to bring q in from infinity is

$$W = \frac{q^2}{16\pi\epsilon_0}\left(\frac{1}{\sqrt{a^2+b^2}} - \frac{1}{a} - \frac{1}{b}\right).$$

3.12

Two infinitely long wires running parallel to the x-axis a distance $2a$ apart with charge densities λ and $-\lambda$ have cylindrical equipotential surfaces with centres at

$$y_0 = \pm a\coth\frac{2\pi\epsilon_0 V_0}{\lambda}$$

radii

$$R = a\operatorname{csch}\frac{2\pi\epsilon_0 V_0}{\lambda}.$$

We know the equipotential surfaces (the pipes) and want to find the wires so we can find the potential, so

$$d = a\coth\frac{2\pi\epsilon_0 V_0}{\lambda}$$

$$R = a\operatorname{csch}\frac{2\pi\epsilon_0 V_0}{\lambda}$$

$$\frac{d}{R} = \cosh\frac{2\pi\epsilon_0 V_0}{\lambda}$$

$$\operatorname{arcosh}\frac{d}{R} = \frac{2\pi\epsilon_0 V_0}{\lambda}$$

$$\lambda = \frac{2\pi\epsilon_0 V_0}{\operatorname{arcosh} d/R}$$

$$R = a\operatorname{csch}\operatorname{arcosh}\frac{d}{R}$$

$$a = \frac{R}{\operatorname{csch}\operatorname{arcosh} d/R}$$

$$= (d+R)\sqrt{\frac{2d}{d+R} - 1}$$

$$= \sqrt{d^2 - R^2}$$

thus the potential is

$$V = \frac{V_0}{2\operatorname{arcosh} d/R}\ln\frac{(y + d^2 - R^2)^2 + z^2}{(y - d^2 + R^2)^2 + z^2}.$$

3.13

$$V_0(y) = \begin{cases} V_0 & 0 \le y \le \frac{a}{2} \\ -V_0 & \frac{a}{2} \le y \le a \end{cases}$$

$$C_n = \frac{2}{a} \left(\int_0^{a/2} V_0 \sin \frac{n\pi y}{a} \, dy - \int_{a/2}^a V_0 \sin \frac{n\pi y}{a} \, dy \right)$$

$$= \frac{2V_0}{n\pi} \left(\cos \frac{n\pi y}{a} \Big|_{a/2}^a - \cos \frac{n\pi y}{a} \Big|_0^{a/2} \right)$$

$$= \frac{2V_0}{n\pi} \left(\cos n\pi - \cos \frac{n\pi}{2} - \cos \frac{n\pi}{2} + 1 \right)$$

$$= \frac{2V_0}{n\pi} \left(1 + (-1)^n - 2\cos \frac{n\pi}{2} \right)$$

$$= \frac{2V_0}{n\pi} \begin{cases} 0 & n \text{ is odd or divisible by 4} \\ 4 & \text{otherwise} \end{cases}$$

$$V = \frac{8V_0}{\pi} \sum_{n=2,6,10,\dots}^{\infty} \frac{1}{n} e^{-n\pi x/a} \sin \frac{n\pi y}{a}$$

3.14

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial x}$$

$$= \frac{4\epsilon_0 V_0 \sin \frac{\pi y}{a}}{a \left(1 - \cos \frac{2\pi y}{a} \right)}$$

$$= \frac{2\epsilon_0 V_0}{a} \frac{1}{\sin \pi y/a}$$

3.15

(a)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$V(0, y) = 0$$

$$V(b, y) = V_0(y)$$

$$V(x, 0) = 0$$

$$V(x, a) = 0$$

$$V = X(x)Y(y)$$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\frac{Y''}{Y} = -\alpha^2$$

$$Y'' + \alpha^2 Y = 0$$

$$Y = c_1 \cos \alpha y + c_2 \sin \alpha y$$

$$Y = c_2 \sin \alpha y$$

$$Y = c_2 \sin \frac{n\pi y}{a}, n \in \mathbb{R}$$

$$\frac{X''}{X} = \alpha^2$$

$$X'' - \alpha^2 X = 0$$

$$X = c_3 \cosh \alpha x + c_4 \sinh \alpha x$$

$$X = c_4 \sinh \alpha x$$

$$= c_4 \sinh \frac{n\pi x}{a}, n \in \mathbb{R}$$

$$V = \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi x}{a} \sin \frac{n\pi y}{a}$$

$$V_0(y) = V(b, y)$$

$$= \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi y}{a}$$

$$C_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a V_0(y) \sin \frac{n\pi y}{a} \, dy$$

$$C_n = \frac{2}{a \sinh n\pi b/a} \int_0^a V_0(y) \sin \frac{n\pi y}{a} \, dy$$

(b)

$$C_n = \frac{2V_0}{a \sinh n\pi b/a} \int_0^a \sin \frac{n\pi y}{a} \, dy$$

$$= \frac{2V_0}{a \sinh n\pi b/a} \frac{a[1 - (-1)^n]}{n\pi}$$

$$= \frac{2V_0[1 - (-1)^n]}{n\pi \sinh n\pi b/a}$$

$$V = \frac{2V_0}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n \sinh n\pi b/a} \sinh \frac{n\pi x}{a} \sin \frac{n\pi y}{a}$$

3.16

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V(0, y, z) = 0$$
$$V(a, y, z) = 0$$
$$V(x, 0, z) = 0$$
$$V(x, a, z) = 0$$
$$V(x, y, 0) = 0$$
$$V(x, y, a) = V_0$$
$$V = X(x)Y(y)Z(z)$$
$$X''YZ + XY''Z + XYZ'' = 0$$
$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$
$$\frac{X''}{X} = -\alpha^2$$
$$\frac{Y''}{Y} = -\beta^2$$
$$\frac{Z''}{Z} = \alpha^2 + \beta^2$$
$$X'' + \alpha^2 X = 0$$
$$X = c_1 \cos \alpha x + c_2 \sin \alpha x$$
$$X = c_2 \sin \alpha x$$
$$X = c_2 \sin \frac{n\pi x}{a}, n \in \mathbb{R}$$
$$\frac{Y''}{Y} = -\beta^2$$
$$Y'' + \beta^2 Y = 0$$
$$Y = c_3 \cos \beta y + c_4 \sin \beta y$$
$$Y = c_4 \sin \beta y$$
$$Y = c_4 \sin \frac{m\pi y}{a}, m \in \mathbb{R}$$
$$\frac{Z''}{Z} = \alpha^2 + \beta^2$$
$$Z'' - (\alpha^2 + \beta^2)Z = 0$$
$$Z = c_5 \cosh \sqrt{\alpha^2 + \beta^2}\, z + c_6 \sinh \sqrt{\alpha^2 + \beta^2}\, z$$
$$= c_5 \cosh \pi \sqrt{(n/a)^2 + (m/a)^2}\, z$$
$$+ c_6 \sinh \pi \sqrt{(n/a)^2 + (m/a)^2}\, z$$
$$Z = c_6 \sinh \pi \sqrt{(n/a)^2 + (m/a)^2}\, z$$

$$V = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{n,m} \sinh\left(\pi\sqrt{(n/a)^2 + (m/a)^2}\, z\right) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a}$$

$$V_0 = V(x,y,a)$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{n,m} \sinh\left(\pi\sqrt{n^2 + m^2}\right) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a}$$

$$C_{n,m} = \frac{4V_0}{a^2 \sinh\left(\pi\sqrt{n^2 + m^2}\right)} \int_0^a \int_0^a \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a}\, dy\, dx$$

$$= \frac{4V_0}{a^2 \sinh\left(\pi\sqrt{n^2 + m^2}\right)} \frac{a^2[-1 + (-1)^m][-1 + (-1)^n]}{nm\pi^2}$$

$$= \frac{4V_0[-1 + (-1)^n][-1 + (-1)^m]}{nm\pi^2 \sinh\left(\pi\sqrt{n^2 + m^2}\right)}$$

$$= \begin{cases} 0 & n \text{ even or } m \text{ even} \\ \frac{16V_0}{nm\pi^2 \sinh\left(\pi\sqrt{n^2+m^2}\right)} & \text{otherwise} \end{cases}$$

$$V = \frac{16V_0}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{nm} \frac{\sinh\left(\pi\sqrt{n^2 + m^2}\, z/a\right)}{\sinh\left(\pi\sqrt{n^2 + m^2}\right)} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a}$$

3.17

$$P_3(x) = \frac{1}{2^3 3!} \left(\frac{d}{dx} \right)^3 (x^2 - 1)^3$$

$$= \frac{1}{48} \frac{d^3}{dx^3} \left[(x^2 - 1)^3 \right]$$

$$= \frac{1}{48} \frac{d^2}{dx^2} \left[6x(x^2 - 1)^2 \right]$$

$$= \frac{1}{48} \frac{d}{dx} \left[6(x^2 - 1)^2 + 24x^2(x^2 - 1) \right]$$

$$= \frac{1}{48} \left[24x(x^2 - 1) + 48x(x^2 - 1) + 48x^3 \right]$$

$$= \frac{1}{48} \left(24x^3 - 24x + 48x^3 - 48x + 48x^3 \right)$$

$$= \frac{120}{48} x^3 - \frac{72}{48} x$$

$$= \frac{5}{2} x^3 - \frac{3}{2} x$$

$$\frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) = -12 \sin\theta \Theta$$

$$\Theta = \frac{5}{2} \cos^3\theta - \frac{3}{2} \cos\theta$$

$$\frac{d\Theta}{d\theta} = -\frac{15}{2} \cos^2\theta \sin\theta + \frac{3}{2} \sin\theta$$

$$\sin\theta \frac{d\Theta}{d\theta} = -\frac{15}{2} \cos^2\theta \sin^2\theta + \frac{3}{2} \sin^2\theta$$

$$\frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) = \frac{3}{2} (1 - 5\cos 2\theta) \sin 2\theta$$

$$\frac{3}{2} (1 - 5\cos 2\theta) \sin 2\theta = -12 \sin\theta \left(\frac{5}{2} \cos^3\theta - \frac{3}{2} \cos\theta \right)$$

$$3(1 - 5\cos 2\theta) \sin 2\theta = 12 \sin\theta \cos\theta (3 - 5\cos^2\theta)$$

$$= 6(3 - 5\cos^2\theta) \sin 2\theta$$

$$= 6 \left(3 - 5 \frac{1 + \cos 2\theta}{2} \right) \sin 2\theta$$

$$= 3(6 - 5 - 5\cos 2\theta) \sin 2\theta$$

$$= 3(1 - 5\cos 2\theta) \sin 2\theta$$

$$\int_{-1}^{1} P_1(x) P_3(x) \, dx = \int_{-1}^{1} x \left(\frac{5}{2} x^3 - \frac{3}{2} x \right) dx$$

$$= \left[\frac{1}{2} x^5 - \frac{1}{2} x^3 \right]_{-1}^{1}$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2}$$

$$= 0$$

37

3.18

(a)

$$A_l = \frac{V_0(2l+1)}{2R^l} \int_0^\pi P_l(\cos\theta)\sin\theta\, d\theta$$

$$= \begin{cases} V_0 & l = 0 \\ 0 & l \neq 0 \end{cases}$$

$$V(r,\theta) = \sum_{l=0}^\infty A_l r^l P_l(\cos\theta)$$

$$= V_0$$

$$B_l = \frac{V_0(2l+1)}{2} R^{l+1} \int_0^\pi P_l(\cos\theta)\sin\theta\, d\theta$$

$$= \begin{cases} V_0 R & l = 0 \\ 0 & l \neq 0 \end{cases}$$

$$V(r,\theta) = \frac{V_0 R}{r}$$

(b)

$$A_l = \frac{\sigma_0}{2\epsilon_0 R^{l-1}} \int_0^\pi P_l(\cos\theta)\sin\theta\, d\theta$$

$$= \begin{cases} \frac{R\sigma_0}{\epsilon_0} & l = 0 \\ 0 & l \neq 0 \end{cases}$$

$$V(r,\theta) = \frac{R\sigma_0}{\epsilon_0}$$

$$B_l = A_l R^{2l+1}$$

$$= \begin{cases} \frac{R^2\sigma_0}{\epsilon_0} & l = 0 \\ 0 & l \neq 0 \end{cases}$$

$$V(r,\theta) = \frac{R^2\sigma_0}{\epsilon_0 r}$$

3.19

$$V_0 = k\cos 3\theta$$

$$A_l = \frac{k(2l+1)}{2R^l} \int_0^\pi \cos 3\theta P_l(\cos\theta)\sin\theta\,d\theta$$

$$= \begin{cases} -\frac{3k}{5R} & l=1 \\ \frac{8k}{5R^3} & l=3 \\ 0 & \text{otherwise} \end{cases}$$

$$V(r,\theta) = -\frac{3k}{5R}rP_1(\cos\theta) + \frac{8k}{5R^3}r^3 P_3(\cos\theta)$$

$$= \frac{kr}{5R}\left[-3P_1(\cos\theta) + \frac{8}{R^2}r^2 P_3(\cos\theta)\right]$$

$$B_l = \frac{k(2l+1)}{2}R^{l+1}\int_0^\pi \cos 3\theta P_l(\cos\theta)\sin\theta\,d\theta$$

$$= \begin{cases} -\frac{3kR^2}{5} & l=1 \\ \frac{8kR^4}{5} & l=3 \\ 0 & \text{otherwise} \end{cases}$$

$$V(r,\theta) = -\frac{3kR^2}{5r^2}P_1(\cos\theta) + \frac{8kR^4}{5r^4}P_3(\cos\theta)$$

$$= \frac{kR^2}{5r^2}\left[\frac{8R^2}{r^2}P_3(\cos\theta) - 3P_1(\cos\theta)\right]$$

$$\sigma(\theta) = -\epsilon_0\left(\frac{\partial V_{\text{above}}}{\partial r} - \frac{\partial V_{\text{below}}}{\partial r}\right)$$

$$= \frac{\epsilon_0 k(12\cos\theta + 35\cos 3\theta)}{5R}$$

3.20

$$V(r,\theta) = \begin{cases} \sum_{l=0}^\infty \frac{2l+1}{2}\frac{r^l}{R^l}\left(\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta\right)P_l(\cos\theta) & r <= R \\ \sum_{l=0}^\infty \frac{2l+1}{2}\frac{R^{l+1}}{r^{l+1}}\left(\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta\right)P_l(\cos\theta) & r >= R \end{cases}$$

$$\sigma_0 = -\epsilon_0\left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\right)\bigg|_{r=R}$$

$$= \frac{\epsilon_0}{2R}\sum_{l=0}^\infty (2l+1)^2 C_l P_l(\cos\theta)$$

3.21

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} - E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta$$

3.22

(a)

$$\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} = \frac{\sigma}{2\epsilon_0} \left(\sqrt{r^2 + R^2} - r \right)$$

$$= \frac{\sigma r}{2\epsilon_0} \left(\sqrt{1 + (R/r)^2} - 1 \right)$$

$$= \frac{\sigma r}{2\epsilon_0} \left[\left(1 + \frac{(R/r)^2}{2} - \frac{(R/r)^4}{8} + \dots \right) - 1 \right]$$

$$= \frac{\sigma}{2\epsilon_0} \left(\frac{R^2}{2r} - \frac{R^4}{8r^3} + \dots \right)$$

$$B_0 = \frac{\sigma R^2}{4\epsilon_0}$$

$$B_1 = 0$$

$$B_2 = -\frac{\sigma R^4}{16\epsilon_0}$$

(b)

$$\sum_{l=0}^{\infty} A_l r^l = \frac{\sigma}{2\epsilon_0} \left(\sqrt{r^2 + R^2} - r \right)$$

$$= \frac{\sigma}{2\epsilon_0} \left(R\sqrt{1 + (r/R)^2} - r \right)$$

$$= \frac{\sigma}{2\epsilon_0} \left[R \left(1 + \frac{(r/R)^2}{2} - \frac{(r/R)^4}{8} + \dots \right) - r \right]$$

$$= \frac{\sigma}{2\epsilon_0} \left(R - r + \frac{r^2}{2R} - \frac{r^4}{8R^3} + \dots \right)$$

$$A_0 = \frac{\sigma R}{2\epsilon_0}$$

$$A_1 = -\frac{\sigma}{2\epsilon_0}$$

$$A_2 = \frac{\sigma}{4\epsilon_0 R}$$

$$A_0' = A_0$$

$$A_1' = -A_1$$

$$A_2' = A_2$$

3.23

$$V(r,\theta) = \sum_{l=1}^{\infty} A_l r^l P_l(\cos\theta)$$

$$A_l = \frac{\sigma_0}{2\epsilon_0 R^{l-1}} \left(\int_0^{\pi/2} P_l(\cos\theta)\sin\theta\, d\theta - \int_{\pi/2}^{\pi} P_l(\cos\theta)\sin\theta\, d\theta \right)$$

$$A_0 = 0$$

$$A_1 = \frac{\sigma_0}{2\epsilon_0}$$

$$A_2 = 0$$

$$A_3 = -\frac{\sigma_0}{8\epsilon_0 R^2}$$

$$A_4 = 0$$

$$A_5 = \frac{\sigma_0}{16\epsilon_0 R^4}$$

$$A_6 = 0$$

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$B_l = A_l R^{2l+1}$$

$$B_0 = 0$$

$$B_1 = \frac{\sigma_0 R^3}{2\epsilon_0}$$

$$B_2 = 0$$

$$B_3 = -\frac{\sigma_0 R^5}{8\epsilon_0}$$

$$B_4 = 0$$

$$B_5 = \frac{\sigma_0 R^7}{16\epsilon_0}$$

$$B_6 = 0$$

3.24

$$0 = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2}$$

$$V(s,\phi) = S(s)\Phi(\phi)$$

$$0 = \frac{1}{s}\frac{\partial}{\partial s}(sS')\Phi + \frac{1}{s^2}S\Phi''$$

$$= \frac{1}{s}(S' + sS'')\Phi + \frac{1}{s^2}S\Phi''$$

$$= \frac{S'}{sS} + \frac{S''}{S} + \frac{\Phi''}{s^2\Phi}$$

$$= \frac{s^2 S'' + sS'}{S} + \frac{\Phi''}{\Phi}$$

$$\frac{\Phi''}{\Phi} = 0$$

$$\Phi'' = 0$$

$$\Phi = c_1 + c_2\phi$$

$$\frac{\Phi''}{\Phi} = -n^2$$

$$\Phi'' + \alpha^2\Phi = 0$$

$$\Phi = c_3\cos\alpha\phi + c_4\sin\alpha\phi$$

$$\Phi(0) = \Phi(2\pi)$$

$$c_1 = c_3\cos 2\pi\alpha + c_4\sin 2\pi\alpha$$

$$\alpha = n, n \in \mathbb{R}$$

$$\Phi = c_3\cos n\phi + c_4\sin n\pi$$

$$\frac{s^2 S'' + sS'}{S} = 0$$

$$s^2 S'' + sS' = 0$$

$$sS'' + S' = 0$$

$$S = c_5 + c_6\ln s$$

$$\frac{s^2 S'' + sS'}{S} = n^2$$

$$s^2 S'' + sS' - n^2 S = 0$$

$$S = s^m$$

$$S' = ms^{m-1}$$

$$S'' = m(m-1)s^{m-2}$$

$$m(m-1)s^m + ms^m - n^2 s^m = 0$$

$$m^2 - m + m - n^2 = 0$$

$$m^2 - n^2 = 0$$

$$(m+n)(m-n) = 0$$

$$S = c_7 s^n + c_8 s^{-n}$$

$$V = S(s)\Phi(\phi)$$

$$= (c_1 + c_2\phi)(c_5 + c_6 \ln s)$$

$$+ \sum_{n=1}^{\infty} (c_7 s^n + c_8 s^{-n})(c_3 \cos n\phi + c_4 \sin n\phi)$$

$$= c_1 + c_2 \ln s$$

$$+ \sum_{n=1}^{\infty} [s^n(A_n \cos n\phi + B_n \sin n\phi) + s^{-n}(C_n \cos n\phi + D_n \sin n\phi)]$$

3.25

$$V = 0 \text{ at } s = R$$

$$V \to -E_0 s \cos\phi \text{ as } s \to \infty$$

$$V = \left(A_1 s + \frac{C_1}{s}\right)\cos\phi$$

$$0 = A_1 R + \frac{C_1}{R}$$

$$C_2 = -A_1 R^2$$

$$A_1 = -E_0$$

$$V = E_0 s \left(\frac{R^2}{s^2} - 1\right)\cos\phi$$

$$\sigma = -\epsilon_0 \left(\frac{\partial V_{\text{out}}}{\partial s} - \frac{\partial V_{\text{in}}}{\partial s}\right)\Bigg|_{s=R}$$

$$= -\epsilon_0 \frac{\partial V_{\text{out}}}{\partial s}\Bigg|_{s=R}$$

$$= 2\epsilon_0 E_0 \cos\phi$$

43

3.27

$$V(z) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{z^{(n+1)}} \int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}') \, d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{z^{(n+1)}} \int_0^R \int_0^\pi \int_0^{2\pi} (r')^n P_n(\cos\theta) k \frac{R}{(r')^2}(R - 2r') \sin\theta (r')^2 \sin\theta \, dr' \, d\theta \, d\phi$$

$$= \frac{kR}{2\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{z^{(n+1)}} \int_0^R \int_0^\pi (r')^n (R - 2r') \sin^2\theta P_n(\cos\theta) \, dr' \, d\theta$$

$$\approx \frac{1}{4\pi\epsilon_0} \frac{k\pi^2 R^5}{48z^3}$$

3.28

$$V(r, \theta) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}') \, d\tau'$$

$$= \frac{\lambda R}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int_0^{2\pi} R^n P_n(\sin\phi\sin\theta) \, d\phi$$

$$V_0 = \frac{\lambda R}{4\pi\epsilon_0} \frac{2\pi}{r}$$

$$= \frac{\lambda R}{2\epsilon_0 r}$$

$$V_1 = 0$$

$$V_2 = -\frac{\lambda R}{4\pi\epsilon_0} \frac{1}{4r^3} \pi R^2 (1 + 3\cos 2\theta)$$

$$= -\frac{\lambda R^3}{8\epsilon_0 r^3}(3\cos^2\theta - 1)$$

3.29

$$\mathbf{p} = \sum_{i}^{n} q_i \mathbf{r}'_i$$

$$= 2aq\hat{\mathbf{z}}$$

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{2aq\cos\theta}{r^2}$$

3.30

(a)

$$\sigma = k\cos\theta$$

$$\mathbf{p} = \int \mathbf{r}'\,\rho(\mathbf{r}')\,d\tau'$$

$$= \int_0^{2\pi}\int_0^{\pi} R(\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}})k\cos\theta R^2\sin\theta\,d\theta\,d\phi$$

$$= \frac{1}{2}kR^3\int_0^{2\pi}\int_0^{\pi}(\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}})\sin 2\theta\,d\theta\,d\phi$$

$$= \frac{4}{3}\pi R^3 k\hat{\mathbf{z}}$$

(b)

$$V_{\text{dip}}(\mathbf{r}) \approx \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{4\pi R^3 k\cos\theta}{3r^2}$$

$$= \frac{kR^3}{3\epsilon_0}\frac{1}{r^2}\cos\theta$$

$$V_{\text{dip}}(r,\theta) = \frac{kR^3}{3\epsilon_0}\frac{1}{r^2}\cos\theta$$

Higher multipoles are all 0.

3.32

(a)

$$Q = 2q$$

$$\mathbf{p} = 3aq\hat{\mathbf{z}}$$

$$V \approx \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{r} + \frac{3aq\cos\theta}{r^2}\right)$$

(b)

$$Q = 2q$$

$$\mathbf{p} = aq\hat{\mathbf{z}}$$

$$V \approx \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{r} + \frac{aq\cos\theta}{r^2}\right)$$

(c)

$$Q = 2q$$
$$\mathbf{p} = 3aq\hat{\mathbf{y}}$$
$$V \approx \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{r} + \frac{3aq\sin\theta\sin\phi}{r^2}\right)$$

3.33

(a)

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\frac{p}{a^3}\hat{\mathbf{z}}$$
$$\mathbf{F} = -\frac{1}{4\pi\epsilon_0}\frac{pq}{a^3}\hat{\mathbf{z}}$$

(b)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2p}{a^3}\hat{\mathbf{z}}$$
$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{2pq}{a^3}\hat{\mathbf{z}}$$

(c)

$$W = \int \mathbf{F} \cdot d\mathbf{l}$$
$$= \int_0^{\pi/2} aq\mathbf{E} \cdot d\boldsymbol{\theta}$$
$$= \frac{1}{4\pi\epsilon_0}\frac{pq}{a^2}\int_0^{\pi/2}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) \cdot d\boldsymbol{\theta}$$
$$= \frac{1}{4\pi\epsilon_0}\frac{pq}{a^2}\int_0^{\pi/2}\sin\theta\, d\theta$$
$$= \frac{1}{4\pi\epsilon_0}\frac{pq}{a^2}$$

3.34

$$Q = -q$$
$$\mathbf{p} = qa\hat{\mathbf{z}}$$
$$V = \frac{1}{4\pi\epsilon_0}q\left(-\frac{1}{r} + \frac{a\cos\theta}{r^2}\right)$$
$$\mathbf{E} = -\nabla V$$
$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}[(2a\cos\theta - r)\hat{\mathbf{r}} + a\sin\theta\hat{\boldsymbol{\theta}}]$$

3.35

$$\mathbf{p} = \int \mathbf{r}' \rho(\mathbf{r}') \, d\tau'$$

$$= \left(\int_0^{2\pi} \int_0^{\pi/2} \int_0^R r \cos\theta \rho_0 r^2 \sin\theta \, dr \, d\theta \, d\phi \right.$$

$$\left. - \int_0^{2\pi} \int_{\pi/2}^{\pi} \int_0^R r \cos\theta \rho_0 r^2 \sin\theta \, dr \, d\theta \, d\phi \right) \hat{\mathbf{z}}$$

$$= \pi\rho_0 \left(\int_0^{\pi/2} \int_0^R r^3 \sin 2\theta \, dr \, d\theta - \int_{\pi/2}^{\pi} \int_0^R r^3 \sin 2\theta \, dr \, d\theta \right) \hat{\mathbf{z}}$$

$$= \frac{1}{2} \pi\rho_0 R^4 \hat{\mathbf{z}}$$

$$\mathbf{E}_{\text{dip}}(r,\theta) = \frac{1}{4\pi\epsilon_0} \frac{\pi\rho_0 R^4}{2r^3} (2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})$$

3.36

The factor of $1/4\pi\epsilon_0 r^3$ is the common, so the goal is to show that

$$p(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) = 2(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} + p\sin\theta\hat{\boldsymbol{\theta}}$$

$$= 2(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - (\mathbf{p}\cdot\hat{\boldsymbol{\theta}})\hat{\boldsymbol{\theta}}$$

$$= 3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{p}}.$$

3.37

$$V_{\text{ave}} = \frac{1}{4\pi R^2} \oint V \, da$$

$$\frac{dV_{\text{ave}}}{dR} = \frac{1}{4\pi R^2} \oint \nabla V \cdot d\mathbf{a}$$

$$= \frac{1}{4\pi R^2} \int \nabla^2 V \, d\tau$$

$$= 0$$

3.38

$$E_{qz} = \frac{1}{4\pi\epsilon_0} \frac{q}{\imath^2} \cos\theta$$

$$= \frac{1}{4\pi\epsilon_0} \frac{qd}{(x^2 + y^2 + d^2)^{3/2}}$$

$$0 = E_{qz} + E_{\sigma z}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{qd}{(x^2 + y^2 + d^2)^{3/2}} - \frac{\sigma}{2\epsilon_0}$$

$$\sigma = \frac{qd}{2\pi(x^2 + y^2 + d^2)^{3/2}}$$

3.39

$$E = \frac{q^2}{4\pi\epsilon_0}\left[\left(\sum_{n=1}^{\infty} \frac{1}{(2an - 2x)^2}\right) - \left(\sum_{n=0}^{\infty} \frac{1}{(2an + 2x)^2}\right)\right]$$

3.40

Set $V = 0$ at $x = 0$. The cylinder is a conductor and is thus an equipotential, so $V = 0$ at the surface. Place two infinite line charges within the cylinder at $x = \pm R^2/a$, giving

$$V = \frac{\lambda}{2\pi\epsilon_0}\left(\ln\frac{a}{\sqrt{s^2 + a^2 - 2sa\cos\phi}} - \ln\frac{a}{\sqrt{s^2 + a^2 + 2sa\cos\phi}}\right.$$

$$+ \ln\frac{R^2/a}{\sqrt{s^2 + (R^2/a)^2 + 2s(R^2/a)\cos\phi}}$$

$$\left. - \ln\frac{R^2/a}{\sqrt{s^2 + (R^2/a)^2 - 2s(R^2/a)\cos\phi}}\right)$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left(\ln\frac{s^2 + a^2 + 2sa\cos\phi}{s^2 + a^2 - 2sa\cos\phi} + \ln\frac{(sa/R)^2 + R^2 - 2sa\cos\phi}{(sa/R)^2 + R^2 + 2sa\cos\phi}\right)$$

$$= \frac{\lambda}{4\pi\epsilon_0}\ln\frac{(s^2 + a^2 + 2sa\cos\phi)[(sa/R)^2 + R^2 - 2sa\cos\phi]}{(s^2 + a^2 - 2sa\cos\phi)[(sa/R)^2 + R^2 + 2sa\cos\phi]}$$

3.41

(a) For a sphere of charge q, $q' + q'' = q \Rightarrow q'' = q - q'$ so

$$F = \frac{q}{4\pi\epsilon_0} \left(\frac{q''}{a^2} + \frac{q'}{(a-b)^2} \right)$$

$$= \frac{q}{4\pi\epsilon_0} \left(\frac{q}{a^2} - \frac{q'}{a^2} + \frac{q'}{(a-b)^2} \right)$$

$$= \frac{q^2}{4\pi\epsilon_0 a^3} \left[a - R^3 \frac{2a^2 - R^2}{(a^2 - R^2)^2} \right]$$

and solving for $F = 0$ gives $r = 5.663\,12$ Å.

3.43

(a)

$$\lim_{r \to \infty} V_{\text{above}}(r, \theta) = 0$$

$$V_{\text{below}}(a, \theta) = V_0$$

$$V_{\text{above}}(b, \theta) = V_{\text{below}}(b, \theta)$$

$$\frac{\partial V_{\text{above}}}{\partial r}\bigg|_{r=b} - \frac{\partial V_{\text{below}}}{\partial r}\bigg|_{r=b} = -\frac{k \cos \theta}{\epsilon_0}$$

$$V_{\text{above}}(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

$$\frac{\partial V_{\text{above}}}{\partial r}\bigg|_{r=b} = \sum_{l=0}^{\infty} -(l+1)\frac{B_l}{b^{l+2}} P_l(\cos \theta)$$

$$V_{\text{below}}(r, \theta) = \sum_{l=0}^{\infty} \left(C_l r^l + \frac{D_l}{r^{l+1}} \right) P_l(\cos \theta)$$

$$V_0 = V_{\text{below}}(a, \theta)$$

$$= \sum_{l=0}^{\infty} \left(C_l a^l + \frac{D_l}{a^{l+1}} \right) P_l(\cos \theta)$$

$$V_0 = C_0 + \frac{D_0}{a}$$

$$0 = C_l a^l + \frac{D_l}{a^{l+1}}, l \neq 0$$

$$\frac{\partial V_{\text{below}}}{\partial r}\bigg|_{r=b} = \sum_{l=0}^{\infty} \left(C_l l b^{l-1} - (l+1)\frac{D_l}{b^{l+2}} \right) P_l(\cos \theta)$$

$$V_{\text{above}}(b, \theta) = V_{\text{below}}(b, \theta)$$

$$\sum_{l=0}^{\infty} \frac{B_l}{b^{l+1}} P_l(\cos \theta) = \sum_{l=0}^{\infty} \left(C_l b^l + \frac{D_l}{b^{l+1}} \right) P_l(\cos \theta)$$

$$\frac{B_l}{b^{l+1}} = C_l b^l + \frac{D_l}{b^{l+1}}$$

$$-\frac{k \cos \theta}{\epsilon_0} = \sum_{l=0}^{\infty} \left[-(l+1)\frac{B_l}{b^{l+2}} - C_l l b^{l-1} + (l+1)\frac{D_l}{b^{l+2}} \right] P_l(\cos \theta)$$

$$-\frac{k}{\epsilon_0} = -2\frac{B_1}{b^3} - C_1 + 2\frac{D_1}{b^3}$$

$$0 = -(l+1)\frac{B_l}{b^{l+2}} - C_l l b^{l-1} + (l+1)\frac{D_l}{b^{l+2}}, l \neq 1$$

$$B_0 = aV_0$$

$$C_0 = 0$$

$$D_0 = aV_0$$

$$B_1 = \frac{(b^3 - a^3)k}{3\epsilon_0}$$

$$C_1 = \frac{k}{3\epsilon_0}$$

$$D_1 = -\frac{a^3 k}{3\epsilon_0}$$

$$B_l = 0$$

$$C_l = 0$$

$$D_l = 0$$

$$V = \begin{cases} \frac{aV_0}{r} + \frac{(r^3 - a^3)k \cos\theta}{3\epsilon_0 r^2} & a \leq r \leq b \\ \frac{aV_0}{r} + \frac{(b^3 - a^3)k \cos\theta}{3\epsilon_0 r^2} & r \geq b \end{cases}$$

(b)

$$\sigma = -\epsilon_0 \left.\frac{\partial V_{\text{below}}}{\partial r}\right|_{r=a}$$

$$= \frac{\epsilon_0 V_0}{a} - k \cos\theta$$

(c)

$$Q = \oint \sigma_i \, dA$$

$$= \int_0^{2\pi} \int_0^\pi \left(\frac{\epsilon_0 V_0}{a} - k \cos\theta\right) a^2 \sin\theta \, d\theta \, d\phi$$

$$= 2\pi \int_0^\pi \left(a\epsilon_0 V_0 \sin\theta - \frac{1}{2} a^2 k \sin 2\theta\right) d\theta$$

$$= 4\pi\epsilon_0 a V_0$$

$$V \approx \frac{aV_0}{r}$$

$$\frac{1}{4\pi\epsilon_0} \frac{Q}{r} = \frac{aV_0}{r}$$

$$Q = 4\pi\epsilon_0 a V_0$$

3.44

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')\, d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int_{-a}^{a} z^n P_n(\cos\theta)\frac{Q}{2a}\, dz$$

$$= \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{Q}{2ar^{(n+1)}} P_n(\cos\theta) \left[\frac{1}{n+1} z^{n+1}\right]_{-a}^{a}$$

$$= \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{Q}{2a(n+1)r^{(n+1)}} P_n(\cos\theta)[a^{n+1} - (-1)^{n+1}a^{n+1}]$$

$$= \frac{Q}{4\pi\epsilon_0} \frac{1}{r} \left[1 + \frac{1}{3}\left(\frac{a}{r}\right)^2 P_2(\cos\theta) + \frac{1}{5}\left(\frac{a}{r}\right)^4 P_4(\cos\theta) + \ldots\right]$$

3.45

$$V = a_0 + b_0 \ln s + \sum_{k=1}^{\infty} [s^k(a_k\cos k\phi + b_k\sin k\phi) + s^{-k}(c_k\cos k\phi + d_k\sin k\phi)]$$

$$\lim_{s\to\infty} V_{\text{above}}(s,\phi) = 0$$

$$V_{\text{above}}(R,\phi) = V_{\text{below}}(R,\phi)$$

$$= \begin{cases} -\sigma_0/\epsilon_0 & 0 \le \phi \le \pi \\ \sigma_0/\epsilon_0 & \pi \le \phi \le 2\pi \end{cases}$$

$$\left.\frac{\partial V_{\text{above}}}{\partial s}\right|_{s=R} - \left.\frac{\partial V}{\partial s}\right|_{s=R} = -\frac{1}{\epsilon_0}\sigma$$

$$V_{\text{above}}(s,\phi) = \sum_{k=1}^{\infty} s^{-k}(c_k \cos k\phi + d_k \sin k\phi)$$

$$\left.\frac{\partial V_{\text{above}}}{\partial s}\right|_{s=R} = \sum_{k=1}^{\infty} -kR^{-(k+1)}(c_k \cos k\phi + d_k \sin k\phi)$$

$$V_{\text{below}}(s,\phi) = e_0 + \sum_{k=1}^{\infty} s^{k}(e_k \cos k\phi + f_k \sin k\phi)$$

$$\left.\frac{\partial V_{\text{below}}}{\partial s}\right|_{s=R} = \sum_{k=1}^{\infty} kR^{k-1}(e_k \cos k\phi + f_k \sin k\phi)$$

$$\sum_{k=1}^{\infty} R^{-k}(c_k \cos k\phi + d_k \sin k\phi) = e_0 + \sum_{k=1}^{\infty} R^{k}(e_k \cos k\phi + f_k \sin k\phi)$$

$$e_0 = 0$$

$$R^{-k}c_k = R^k e_k$$

$$R^{-k}d_k = R^k f_k$$

$$-\frac{\sigma}{\epsilon_0} = \sum_{k=1}^{\infty} \left[-kR^{-(k+1)}(c_k \cos k\phi + d_k \sin k\phi) - kR^{k-1}(e_k \cos k\phi + f_k \sin k\phi) \right]$$

$$= \sum_{k=1}^{\infty} -k\left[\left(R^{-(k+1)}c_k + R^{k-1}e_k\right)\cos k\phi + \left(R^{-(k+1)}d_k + R^{k-1}f_k\right)\sin k\phi \right]$$

$$\frac{1}{\pi}\int_0^{2\pi} -\frac{\sigma}{\epsilon_0}\cos k\phi\, d\phi = -k(R^{-(k+1)}c_k + R^{k-1}e_k)$$

$$\frac{\sigma_0(\sin 2k\pi - 2\sin k\pi)}{k\pi\epsilon_0} = -k(R^{-(k+1)}c_k + R^{k-1}e_k)$$

$$\frac{1}{\pi}\int_0^{2\pi} -\frac{\sigma}{\epsilon_0}\sin k\phi\, d\phi = -k(R^{-(k+1)}d_k + R^{k-1}f_k)$$

$$\frac{4\sigma_0 \cos k\pi \sin^2 k\pi/2}{k\pi\epsilon_0} = -k(R^{-(k+1)}d_k + R^{k-1}f_k)$$

$$c_k = 0$$

$$d_k = \frac{2R^{k+1}\sigma_0 \cos k\pi \sin^2 k\pi/2}{k^2 \pi \epsilon_0}$$

$$e_k = 0$$

$$f_k = -\frac{2R^{-(k-1)}\sigma_0 \cos k\pi \sin^2 k\pi/2}{k^2 \pi \epsilon_0}$$

$$V_{\text{above}} = \frac{2\sigma_0}{\pi \epsilon_0} \sum_{k=1}^{\infty} s^{-k} \frac{R^{k+1} \cos k\pi \sin^2 k\pi/2}{k^2} \sin k\phi$$

$$= -\frac{2\sigma_0}{\pi \epsilon_0} \sum_{k=1,3,5,\ldots}^{\infty} \frac{1}{k^2} R^{k+1} s^{-k} \sin k\phi$$

$$V_{\text{below}} = -\frac{2\sigma_0}{\pi \epsilon_0} \sum_{k=1}^{\infty} s^k \frac{R^{-(k-1)} \cos k\pi \sin^2 k\pi/2}{k^2} \sin k\phi$$

$$= \frac{2\sigma_0}{\pi \epsilon_0} \sum_{k=1,3,5,\ldots}^{\infty} \frac{1}{k^2} R^{-(k-1)} s^k \sin k\phi$$

$$V = \frac{2R\sigma_0}{\pi \epsilon_0} \begin{cases} \sum_{k=1,3,5,\ldots}^{\infty} \frac{1}{k^2}(s/R)^k \sin k\phi & s \leq R \\ -\sum_{k=1,3,5,\ldots}^{\infty} \frac{1}{k^2}(R/s)^k \sin k\phi & s \geq R \end{cases}$$

3.46

(a)

$$\frac{1}{4\pi\epsilon_0} \frac{1}{r} \int_{-a}^{a} k \cos \frac{\pi z}{2a} \, dz = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \frac{4ak}{\pi}$$

(b)

$$\frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \int_{-a}^{a} z \cos \theta k \sin \frac{\pi z}{a} \, dz = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \frac{2a^2 k \cos \theta}{\pi}$$

(c)

$$\frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int_{-a}^{a} z^2 P_2(\cos \theta) k \cos \frac{\pi z}{a} \, dz = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \left(-\frac{4a^3 k}{\pi^2} \right) P_2(\cos \theta)$$

54

3.47

(a)

$$\mathbf{E}_{\text{ave}} = \frac{1}{\frac{4}{3}\pi R^3} \int \mathbf{E}\, d\tau$$

$$= \frac{1}{4\pi\epsilon_0} \frac{1}{\frac{4}{3}\pi R^3} \int \frac{q}{\imath^2} \hat{\boldsymbol{\imath}}\, d\tau'$$

$$\mathbf{E}_{\text{ave}} = \int \frac{1}{4\pi\epsilon_0} \frac{\rho}{\imath^2} \hat{\boldsymbol{\imath}}\, d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \frac{1}{\frac{4}{3}\pi R^3} \int \frac{q}{\imath^2} \hat{\boldsymbol{\imath}}\, d\tau'$$

(b)

$$\mathbf{p} = q\mathbf{r}$$

$$\oint \mathbf{E}\cdot d\mathbf{A} = \frac{Q_{\text{encl}}}{\epsilon_0}$$

$$4\pi r^2 E = \frac{\frac{4}{3}\pi r^3 \rho}{\epsilon_0}$$

$$\mathbf{E} = \frac{r\rho}{3\epsilon_0}\hat{\mathbf{r}}$$

$$= -\frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}}{R^3}$$

(c)

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \dots$$

$$= -\frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}_1}{R^3} - \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}_2}{R^3} + \dots$$

$$= -\frac{1}{4\pi\epsilon_0} \frac{1}{R^3}(\mathbf{p}_1 + \mathbf{p}_2 + \dots)$$

$$= -\frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}}{R^3}$$

(d)

$$\mathbf{E}_{\text{ave}} = \frac{1}{\frac{4}{3}\pi R^3} \int \mathbf{E} \, d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \frac{1}{\frac{4}{3}\pi R^3} \int \frac{q}{\imath^2} \hat{\boldsymbol{\imath}} \, d\tau'$$

$$\mathbf{E}_r = -\frac{1}{4\pi\epsilon_0} \int \frac{\rho}{\imath^2} \hat{\boldsymbol{\imath}} \, d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \frac{1}{\frac{4}{3}\pi R^3} \int \frac{q}{\imath^2} \hat{\boldsymbol{\imath}} \, d\tau'$$

$$\rho = -\frac{q}{\frac{4}{3}\pi R^3}$$

$$Q = \frac{4}{3}\pi R^3 \rho$$

$$= -q$$

$$\mathbf{E}_r = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{z}}$$

$$= -\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{z}}$$

This is the electric field at the origin.

3.48

(a)

$$\mathbf{E}_{\text{dip}}(r, \theta) = \frac{p}{4\pi\epsilon_0 r^3} (2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$= \frac{p}{4\pi\epsilon_0 r^3} [2\cos\theta(\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}})$$

$$+ \sin\theta(\cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}})]$$

$$= \frac{p}{4\pi\epsilon_0 r^3} [3\cos\theta\sin\theta\cos\phi\,\hat{\mathbf{x}} + 3\cos\theta\sin\theta\sin\phi\,\hat{\mathbf{y}}$$

$$+ (2\cos^2\theta - \sin^2\theta)\hat{\mathbf{z}}]$$

$$\mathbf{E}_{\text{ave}} = \frac{1}{\frac{4}{3}\pi R^3} \int \mathbf{E}_{\text{dip}} \, d\tau'$$

$$= \frac{3p}{16\pi^2\epsilon_0 R^3} \int_0^{2\pi} \int_0^\pi \int_0^R \frac{1}{r^3} [3\cos\theta\sin\theta\cos\phi\,\hat{\mathbf{x}}$$

$$+ 3\cos\theta\sin\theta\sin\phi\,\hat{\mathbf{y}} + (2\cos^2\theta - \sin^2\theta)\hat{\mathbf{z}}]r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{3p}{16\pi^2\epsilon_0 R^3} \int_0^{2\pi} \int_0^\pi \int_0^R \frac{1}{r} [3\cos\theta\sin^2\theta\cos\phi\,\hat{\mathbf{x}}$$

$$+ 3\cos\theta\sin^2\theta\sin\phi\,\hat{\mathbf{y}} + (2\cos^2\theta - \sin^2\theta)\sin\theta\,\hat{\mathbf{z}}]\,dr\,d\theta\,d\phi$$

$$= \mathbf{0}$$

(b)

$$-\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3} = \frac{1}{\frac{4}{3}\pi R^3}\int \mathbf{E}\, d\tau'$$

$$\mathbf{E} = -\frac{\mathbf{p}}{3\epsilon_0}\delta^3(\mathbf{r})$$

3.50

(a)

$$\int \mathbf{E}_1 \cdot \mathbf{E}_2\, d\tau = \int (-\nabla V_1) \cdot \mathbf{E}_2\, d\tau$$

$$= \int [V_1(\nabla \cdot \mathbf{E}_2) - \nabla \cdot (V_1\mathbf{E}_2)]\, d\tau$$

$$= \int \frac{\rho_2 V_1}{\epsilon_0}\, d\tau - \int \nabla \cdot (V_1\mathbf{E}_2)\, d\tau$$

$$= \int \frac{\rho_2 V_1}{\epsilon_0}\, d\tau - \oint V_1\mathbf{E}_2 \cdot d\mathbf{a}$$

$$= \int \frac{\rho_2 V_1}{\epsilon_0}\, d\tau$$

$$\int \mathbf{E}_1 \cdot \mathbf{E}_2\, d\tau = \int \mathbf{E}_1 \cdot (-\nabla V_2)\, d\tau$$

$$= \int [V_2(\nabla \cdot \mathbf{E}_1) - \nabla \cdot (V_2\mathbf{E}_1)]\, d\tau$$

$$= \int \frac{\rho_1 V_2}{\epsilon_0}\, d\tau - \int \nabla \cdot (V_2\mathbf{E}_1)\, d\tau$$

$$= \int \frac{\rho_1 V_2}{\epsilon_0}\, d\tau - \oint V_2\mathbf{E}_1 \cdot d\mathbf{a}$$

$$= \int \frac{\rho_1 V_2}{\epsilon_0}\, d\tau$$

$$\int \frac{\rho_1 V_2}{\epsilon_0}\, d\tau = \int \frac{\rho_2 V_1}{\epsilon_0}\, d\tau$$

$$\int \rho_1 V_2\, d\tau = \int \rho_2 V_1\, d\tau$$

(b)

$$Q_a = \int_a \rho_1 \, d\tau$$
$$= Q$$
$$Q_b = \int_b \rho_1 \, d\tau$$
$$= 0$$
$$V_{1b} = V_{ab}$$
$$Q_a = \int_a \rho_2 \, d\tau$$
$$= 0$$
$$Q_b = \int_b \rho_2 \, d\tau$$
$$= Q$$
$$V_{2a} = V_{ba}$$
$$\int \rho_1 V_2 \, d\tau = \int_a \rho_1 V_2 \, d\tau + \int_b \rho_1 V_2 \, d\tau$$
$$= V_{2a} \int_a \rho_1 \, d\tau + V_{2b} \int \rho_1 \, d\tau$$
$$= V_{ba} Q$$
$$\int \rho_2 V_1 \, d\tau = \int_a \rho_2 V_1 \, d\tau + \int_b \rho_2 V_1 \, d\tau$$
$$= V_{1a} \int_a \rho_2 \, d\tau + V_{1b} \int \rho_2 \, d\tau$$
$$= V_{ab} Q$$
$$V_{ba} Q = V_{ab} Q$$
$$V_{ba} = V_{ab}$$

3.51

(a)

$$\int \rho_2 V_1 \, d\tau = Q_{l2}V_{l1} + Q_{x2}V_{x1} + Q_{r2}V_{r1}$$

$$= 0$$

$$\int \rho_1 V_2 \, d\tau = Q_{l1}V_{l2} + Q_{x1}V_{x2} + Q_{r1}V_{r2}$$

$$= q\frac{x}{d}V_0 + Q_2 V_0$$

$$Q_2 = -\frac{qx}{d}$$

$$\int \rho_2 V_1 \, d\tau = Q_{l2}V_{l1} + Q_{x2}V_{x1} + Q_{r2}V_{r1}$$

$$= 0$$

$$\int \rho_1 V_2 \, d\tau = Q_{l1}V_{l2} + Q_{x1}V_{x2} + Q_{r1}V_{r2}$$

$$= Q_1 V_0 + q\left(1 - \frac{x}{d}\right)V_0$$

$$Q_1 = q\left(\frac{x}{d} - 1\right)$$

(b)

$$\int \rho_2 V_1 \, d\tau = Q_{a2} V_{a1} + Q_{r2} V_{r1} + Q_{b2} V_{b1}$$

$$= 0$$

$$V(a, \theta) = 0$$

$$V(b, \theta) = V_0$$

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

$$0 = \sum_{l=0}^{\infty} \left(A_l a^l + \frac{B_l}{a^{l+1}} \right) P_l(\cos \theta)$$

$$0 = A_l a^l + \frac{B_l}{a^{l+1}}$$

$$B_l = -A_l a^{2l+1}$$

$$V(r, \theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{a^{2l+1}}{r^{l+1}} \right) P_l(\cos \theta)$$

$$V_0 = \sum_{l=0}^{\infty} A_l \left(b^l - \frac{a^{2l+1}}{b^{l+1}} \right) P_l(\cos \theta)$$

$$= A_0 \left(1 - \frac{a}{b} \right)$$

$$A_0 = \frac{b}{b-a} V_0$$

$$A_n = 0, \, n \neq 0$$

$$V(r, \theta) = V_0 \frac{b}{b-a} \left(1 - \frac{a}{r} \right)$$

$$\int \rho_1 V_2 \, d\tau = Q_{r1} V_{r2} + Q_{b1} V_{b2}$$

$$= q V_0 \frac{b}{b-a} \left(1 - \frac{a}{r} \right) + Q_2 V_0$$

$$Q_2 = -\frac{qb}{b-a} \left(1 - \frac{a}{r} \right)$$

$$\int \rho_2 V_1 \, d\tau = Q_{a2}V_{a1} + Q_{r2}V_{r1} + Q_{b2}V_{b1}$$

$$= 0$$

$$V(a, \theta) = V_0$$

$$V(b, \theta) = 0$$

$$0 = \sum_{l=0}^{\infty} \left(A_l b^l + \frac{B_l}{b^{l+1}} \right) P_l(\cos\theta)$$

$$0 = A_l b^l \frac{B_l}{b^{l+1}}$$

$$B_l = -A_l b^{2l+1}$$

$$V(r, \theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{b^{2l+1}}{r^{l+1}} \right) P_l(\cos\theta)$$

$$V_0 = \sum_{l=0}^{\infty} A_l \left(a^l - \frac{b^{2l+1}}{a^{l+1}} \right) P_l(\cos\theta)$$

$$V_0 = A_0 \left(1 - \frac{b}{a} \right)$$

$$A_0 = V_0 \frac{a}{a-b}$$

$$V(r, \theta) = V_0 \frac{a}{a-b} \left(1 - \frac{b}{r} \right)$$

$$\int \rho_1 V_2 \, d\tau = Q_{a1}V_{a2} + Q_{r1}V_{r2} + Q_{b1}V_{b2}$$

$$= Q_1 V_0 + q V_0 \frac{a}{a-b} \left(1 - \frac{b}{r} \right)$$

$$Q_1 = -\frac{qa}{a-b} \left(1 - \frac{b}{r} \right)$$

3.52

(a)

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int (r')^2 P_2(\cos\alpha)\rho(\mathbf{r}')\,d\tau'$$

$$\int (r')^2 P_2(\cos\alpha)\rho(\mathbf{r}')\,d\tau' = \int (r')^2 \left[\frac{1}{2}(3\cos^2\alpha - 1)\right]\rho(\mathbf{r}')\,d\tau'$$

$$= \frac{1}{2}\int (r')^2[3(\hat{\mathbf{r}}'\cdot\hat{\mathbf{r}})^2 - 1]\rho(\mathbf{r}')\,d\tau'$$

$$= \frac{1}{2}\int [3(\mathbf{r}'\cdot\hat{\mathbf{r}})^2 - (r')^2]\rho(\mathbf{r}')\,d\tau'$$

$$= \frac{1}{2}\int [3(\mathbf{r}'\cdot\hat{\mathbf{r}})^2 - (r')^2(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})]\rho(\mathbf{r}')\,d\tau'$$

$$= \frac{1}{2}\int \left[3\sum_{i,j=1}^{3} r_i' r_j'\hat{r}_i\hat{r}_j - (r')^2\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j\delta_{ij}\right]\rho(\mathbf{r}')\,d\tau'$$

$$= \sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j\frac{1}{2}\int [3r_i'r_j' - (r')^2\delta_{ij}]\rho(\mathbf{r}')\,d\tau'$$

$$= \sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j Q_{ij}$$

(b)

$$Q_{11} = 0$$
$$Q_{12} = \frac{3a^2q}{2}$$
$$Q_{13} = 0$$
$$Q_{21} = \frac{3a^2q}{2}$$
$$Q_{22} = 0$$
$$Q_{23} = 0$$
$$Q_{31} = 0$$
$$Q_{32} = 0$$
$$Q_{33} = 0$$

4 Electric Fields in Matter

4.1

$$V(x) = 500\frac{x}{d} = 500\,000x$$
$$\mathbf{E} = -\nabla V = -500\,000\,\text{N/C}$$
$$\alpha = 4\pi\epsilon_0(0.667 \times 10^{-30})$$
$$= 7.42 \times 10^{-41}\,\text{C}^2\,\text{m/N}$$
$$\alpha E = qd$$
$$d = \frac{\alpha E}{q}$$
$$= 2.32 \times 10^{-16}\,\text{m}$$
$$\frac{d}{R} = 4.6 \times 10^{-6}$$
$$V = 1.88 \times 10^8\,\text{V}$$

4.2

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$4\pi r^2 E = \frac{1}{\epsilon_0} \int_0^{2\pi} \int_0^{\pi} \int_0^r \rho(r') r'^2 \sin\theta \, dr' \, d\theta \, d\phi$$

$$= \frac{4\pi}{\epsilon_0} \int_0^r \frac{q}{\pi a^3} e^{-2r'/a} r'^2 \, dr'$$

$$= \frac{4q}{\epsilon_0 a^3} \int_0^r e^{-2r'/a} r'^2 \, dr'$$

$$= \frac{4q}{\epsilon_0 a^3} \frac{1}{4} a \left[a^2 - e^{-2r/a}(a^2 + 2ar + 2r^2) \right]$$

$$= \frac{q}{\epsilon_0 a^2} \left[a^2 - e^{-2r/a}(a^2 + 2ar + 2r^2) \right]$$

$$\mathbf{E}_e(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \left[1 - e^{-2r/a} \left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2} \right) \right] \hat{\mathbf{r}}$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{d^2} \left(\frac{4}{3} \frac{d^3}{a^3} \right)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{4}{3a^3} (qd)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{4}{3a^3} p$$

$$= \frac{1}{4\pi\epsilon_0} \frac{4}{3a^3} \alpha E$$

$$\alpha = 3\pi\epsilon_0 a^3$$

4.3

$$\rho(r) = kr$$

$$Q_{\text{enc}} = \int_0^{2\pi} \int_0^{\pi} \int_0^r kr'^3 \sin\theta \, dr' \, d\theta \, d\phi$$

$$= \pi k r^4$$

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$4\pi r^2 E = \frac{\pi k r^4}{\epsilon_0}$$

$$\mathbf{E} = \frac{kr^2}{4\epsilon_0}$$

$$E = \frac{kd^2}{4\epsilon_0}$$

$$d = \sqrt{\frac{4\epsilon_0 E}{k}}$$

$$p = ed$$

$$= 2e\sqrt{\frac{\epsilon_0}{k}}\sqrt{E}$$

p is proportional to $\sqrt{E}$.

4.4

$$E_q = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

$$p = \alpha E_q$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\alpha q}{r^2}$$

$$\mathbf{E}_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})$$

$$E_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{2p}{r^3}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{2}{r^3}\left(\frac{1}{4\pi\epsilon_0}\frac{\alpha q}{r^2}\right)$$

$$= \left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{2\alpha q}{r^5}$$

$$F = \left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{2\alpha q^2}{r^5}$$

4.5

$$\mathbf{E}_{p_1} = -\frac{1}{4\pi\epsilon_0}\frac{p_1}{r^3}\hat{\mathbf{z}}$$

$$\boldsymbol{\tau}_{p_2} = \mathbf{p}_2 \times \mathbf{E}_{p_1}$$

$$= p_2\hat{\mathbf{x}} \times -\frac{1}{4\pi\epsilon_0}\frac{p_1}{r^3}\hat{\mathbf{z}}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{p_1 p_2}{r^3}\hat{\mathbf{y}}$$

$$\mathbf{E}_{p_2} = \frac{1}{4\pi\epsilon_0}\frac{2p_2}{r^3}\hat{\mathbf{x}}$$

$$\boldsymbol{\tau}_{p_1} = \mathbf{p}_1 \times \mathbf{E}_{p_2}$$

$$= p_1\hat{\mathbf{z}} \times -\frac{1}{4\pi\epsilon_0}\frac{2p_2}{r^3}\hat{\mathbf{x}}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{2p_1 p_2}{r^3}\hat{\mathbf{y}}$$

4.6

$$\mathbf{E}_i = \frac{1}{4\pi\epsilon_0}\frac{p}{8z^3}(\sin\theta\hat{\mathbf{x}} + 2\cos\theta\hat{\mathbf{y}})$$

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}_i$$

$$= p\begin{pmatrix}\sin\theta \\ \cos\theta \\ 0\end{pmatrix} \times \frac{1}{4\pi\epsilon_0}\frac{p}{8z^3}\begin{pmatrix}\sin\theta \\ 2\cos\theta \\ 0\end{pmatrix}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{p^2\sin 2\theta}{16z^3}\hat{\mathbf{z}}$$

The dipole will come to rest at $\theta = n\pi, n \in \mathbb{Z}$.

4.8

$$U = -\mathbf{p}_1 \cdot \mathbf{E}$$

$$= -\mathbf{p}_1 \cdot \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_2]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[\mathbf{p}_1 \cdot \mathbf{p}_2 - 3(\mathbf{p}_1 \cdot \hat{\mathbf{r}})(\mathbf{p}_2 \cdot \hat{\mathbf{r}})]$$

4.9

(a)

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{(x^2 + y^2 + z^2)^{3/2}}$$

$$F_x = (\mathbf{p} \cdot \nabla)\mathbf{E}$$

$$= \left(p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z}\right)\frac{q}{4\pi\epsilon_0}\frac{x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= \frac{q}{4\pi\epsilon_0}\left[p_x\left(\frac{1}{(x^2 + y^2 + z^2)^{3/2}} - 3x\frac{x}{(x^2 + y^2 + z^2)^{5/2}}\right)\right.$$

$$\left. + p_y\left(-3x\frac{y}{(x^2 + y^2 + z^2)^{5/2}}\right) + p_z\left(-3x\frac{z}{(x^2 + y^2 + z^2)^{5/2}}\right)\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{p_x}{r^3} - 3x\frac{p_x x + p_y y + p_z z}{r^5}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{p}}{r^3} - \frac{3\mathbf{r}(\mathbf{p} \cdot \mathbf{r})}{r^5}\right]_x$$

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}[\mathbf{p} - 3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}]$$

(b)

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}[3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]$$

4.10

(a)

$$\mathbf{P} = k\mathbf{r}$$

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

$$= kR$$

$$\rho_b = -\nabla \cdot \mathbf{P}$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}(kr^3)$$

$$= -3k$$

(b)

$$\oint \mathbf{E}_{\text{inside}} \cdot d\mathbf{a} = \frac{Q_{\text{encl}}}{\epsilon_0}$$

$$4\pi r^2 E_{\text{inside}} = \frac{1}{\epsilon_0} \int_0^{2\pi} \int_0^{\pi} \int_0^r -3kr'^2 \sin\theta \, dr' \, d\theta \, d\phi$$

$$= -\frac{12\pi k}{\epsilon_0} \int_0^r r'^2 \, dr'$$

$$= -\frac{4\pi k r^3}{\epsilon_0}$$

$$\mathbf{E}_{\text{inside}} = -\frac{kr}{\epsilon_0}\hat{\mathbf{r}}$$

$$= -\frac{k}{\epsilon_0}\mathbf{r}$$

$$\oint \mathbf{E}_{\text{outside}} \cdot d\mathbf{a} = \frac{Q_{\text{encl}}}{\epsilon_0}$$

$$4\pi r^2 E_{\text{outside}} = \frac{1}{\epsilon_0}\left(4\pi k R^3 - \int_0^{2\pi} \int_0^{\pi} \int_0^R 3kr^2 \sin\theta \, dr \, d\theta \, d\phi\right)$$

$$= \frac{1}{\epsilon_0}\left(4\pi k R^3 - 4\pi k R^3\right)$$

$$\mathbf{E}_{\text{outside}} = \mathbf{0}$$

4.11

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

$$= \begin{cases} 0 & \text{on the side} \\ P & \text{on one end} \\ -P & \text{on the other end} \end{cases}$$

$$\rho_b = -\nabla \cdot \mathbf{P}$$

$$= 0$$

4.13

$$\oint \mathbf{E}_{\text{inside}} \cdot d\mathbf{a} = \frac{Q_{\text{encl}}}{\epsilon_0}$$

$$2\pi s L E_{\text{inside}} = \frac{\pi s^2 L \rho}{\epsilon_0}$$

$$\mathbf{E}_{\text{inside}} = \frac{\rho s}{2\epsilon_0} \hat{\mathbf{s}}$$

$$\mathbf{E}_{\text{inside}} = \frac{\rho}{2\epsilon_0}(\mathbf{s}_+ - \mathbf{s}_-)$$

$$= -\frac{\rho \mathbf{d}}{2\epsilon_0}$$

$$= -\frac{\mathbf{P}}{2\epsilon_0}$$

$$\oint \mathbf{E}_{\text{outside}} \cdot d\mathbf{a} = \frac{Q_{\text{encl}}}{\epsilon_0}$$

$$2\pi s L E_{\text{outisde}} = \frac{\pi a^2 L \rho}{\epsilon_0}$$

$$\mathbf{E}_{\text{outside}} = \frac{a^2 \rho}{2\epsilon_0 s} \hat{\mathbf{s}}$$

$$\mathbf{E}_{\text{outside}} = \frac{a^2 \rho}{2\epsilon_0}\left(\frac{\hat{\mathbf{s}}_+}{s_+} - \frac{\hat{\mathbf{s}}_-}{s_-}\right)$$

4.14

$$\oint_{\mathcal{S}} \mathbf{P} \cdot \hat{\mathbf{n}} - \int_{\mathcal{V}} \nabla \cdot \mathbf{P} \, d\tau' = \oint_{\mathcal{S}} \mathbf{P} \cdot \hat{\mathbf{n}} - \oint_{\mathcal{S}} \mathbf{P} \cdot \hat{\mathbf{n}}$$

$$= 0$$

4.15

(a)

$$\sigma_{b,a} = \mathbf{P} \cdot \hat{\mathbf{n}}$$

$$= -\frac{k}{a}$$

$$\sigma_{b,b} = \frac{k}{b}$$

$$\rho_b = -\nabla \cdot \mathbf{P}$$

$$= -\frac{1}{r^2} \frac{\partial}{\partial r} (kr)$$

$$= -\frac{k}{r^2}$$

$$\mathbf{E} = \mathbf{0}, r < a$$

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$4\pi r^2 E = \frac{1}{\epsilon_0} \left(-4\pi a^2 \frac{k}{a} - \int_0^{2\pi} \int_0^\pi \int_a^r \frac{k}{r'^2} r'^2 \sin\theta \, dr' \, d\theta \, d\phi \right)$$

$$= \frac{1}{\epsilon_0} [-4\pi ak - 4\pi k (r - a)]$$

$$\mathbf{E} = -\frac{k}{\epsilon_0 r} \hat{\mathbf{r}}, a < r < b$$

$$\mathbf{E} = \mathbf{0}$$

(b)

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$
$$\mathbf{D} = \mathbf{0}$$
$$\epsilon_0 \mathbf{E} + \mathbf{P} = \mathbf{0}$$
$$\mathbf{E} = \mathbf{0}$$

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$
$$\mathbf{D} = \mathbf{0}$$
$$\epsilon_0 \mathbf{E} + \mathbf{P} = \mathbf{0}$$
$$\mathbf{E} = -\frac{\mathbf{P}}{\epsilon_0}$$
$$= -\frac{k}{\epsilon_0 r}\hat{\mathbf{r}}$$

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$
$$\mathbf{D} = \mathbf{0}$$
$$\epsilon_0 \mathbf{E} + \mathbf{P} = \mathbf{0}$$
$$\mathbf{E} = \mathbf{0}$$

4.16

(a)

$$\mathbf{E} = \mathbf{E}_0 + \frac{1}{3\epsilon_0}\mathbf{P}$$
$$\mathbf{D} = \mathbf{D}_0 - \frac{2}{3}\mathbf{P}$$

4.18

(a)

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$
$$2AE = \frac{A\sigma}{\epsilon_0}$$
$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$$
$$\mathbf{E}_{\text{vac}} = -\frac{\sigma}{\epsilon_0}\hat{\mathbf{y}}$$
$$\mathbf{D} = \epsilon_0 \mathbf{E}_{\text{vac}}$$
$$= -\sigma\hat{\mathbf{y}}$$

(b)

$$\mathbf{E}_1 = \frac{1}{\epsilon_1}\mathbf{D}$$

$$= -\frac{\sigma}{2\epsilon_0}\hat{\mathbf{y}}$$

$$\mathbf{E}_2 = \frac{1}{\epsilon_2}\mathbf{D}$$

$$= -\frac{\sigma}{1.5\epsilon_0}\hat{\mathbf{y}}$$

(c)

$$\mathbf{P}_1 = \epsilon_0\chi_{e1}\mathbf{E}_1$$

$$= -\frac{\sigma}{2}\hat{\mathbf{y}}$$

$$\mathbf{P}_2 = \epsilon_0\chi_{e2}\mathbf{E}_2$$

$$= -\frac{\sigma}{3}\hat{\mathbf{y}}$$

(d)

$$V = a\frac{\sigma}{2\epsilon_0} + a\frac{2\sigma}{3\epsilon_0}$$

$$= \frac{a\sigma}{\epsilon_0}\left(\frac{3}{6} + \frac{4}{6}\right)$$

$$= \frac{7a\sigma}{6\epsilon_0}$$

(e)

$$\sigma_{b1,\text{top}} = \mathbf{P}_1 \cdot \hat{\mathbf{y}}$$

$$= -\frac{\sigma}{2}$$

$$\sigma_{b1,\text{bottom}} = \mathbf{P}_1 \cdot -\hat{\mathbf{y}}$$

$$= \frac{\sigma}{2}$$

$$\sigma_{b2,\text{top}} = \mathbf{P}_2 \cdot \hat{\mathbf{y}}$$

$$= -\frac{\sigma}{3}$$

$$\sigma_{b2,\text{bottom}} = \mathbf{P}_2 \cdot -\hat{\mathbf{y}}$$

$$= \frac{\sigma}{3}$$

(f)

$$\mathbf{E}_1 = -\frac{\sigma}{\epsilon_0}\hat{\mathbf{y}} + \frac{\sigma}{2\epsilon_0}\hat{\mathbf{y}}$$

$$= -\frac{\sigma}{2\epsilon_0}\hat{\mathbf{y}}$$

$$\mathbf{E}_2 = -\frac{\sigma}{\epsilon_0}\hat{\mathbf{y}} + \frac{\sigma}{3\epsilon_0}\hat{\mathbf{y}}$$

$$= -\frac{2\sigma}{3\epsilon_0}\hat{\mathbf{y}}$$

4.19

$$Q = A\sigma$$

$$E_{\text{vac}} = \frac{\sigma}{\epsilon_0}$$

$$V_{\text{vac}} = \frac{d\sigma}{\epsilon_0}$$

$$C_{\text{vac}} = \frac{Q}{V_{\text{vac}}}$$

$$= \frac{A\epsilon_0}{d}$$

$$E' = \frac{1}{\epsilon_r}E_{\text{vac}}$$

$$= \frac{\sigma}{\epsilon_0\epsilon_r}$$

$$V' = \frac{d\sigma}{2\epsilon_0} + \frac{d\sigma}{2\epsilon_0\epsilon_r}$$

$$= \frac{d\sigma}{2\epsilon_0}\left(1 + \frac{1}{\epsilon_r}\right)$$

$$= \frac{d\sigma(1 + \epsilon_r)}{2\epsilon_0\epsilon_r}$$

$$C' = \frac{Q}{V'}$$

$$= \frac{2A\epsilon_0\epsilon_r}{d(1 + \epsilon_r)}$$

$$\frac{C'}{C_{\text{vac}}} = \frac{2\epsilon_r}{1 + \epsilon_r}$$

$$E = \frac{\sigma}{\epsilon_0}$$

$$\sigma = \epsilon_0 E$$

$$= \epsilon_0 \frac{V}{d}$$

$$P = \epsilon_0 \chi_e E$$

$$= \epsilon_0 \chi_e \frac{V}{d}$$

$$\sigma_b = -\epsilon_0 \chi_e \frac{V}{d}$$

$$\sigma_{\text{total}} = \sigma_f + \sigma_b$$

$$\epsilon_0 \frac{V}{d} = \sigma_f - \epsilon_0 \chi_e \frac{V}{d}$$

$$\sigma_f = \epsilon_0 \frac{V}{d} + \epsilon_0 \chi_e \frac{V}{d}$$

$$= \epsilon_0 \frac{V}{d}(1 + \chi_e)$$

$$= \frac{\epsilon_0 \epsilon_r V}{d}$$

$$C' = \frac{Q'}{V}$$

$$= \frac{1}{V}\left(\frac{A\sigma}{2} + \frac{A\sigma_f}{2}\right)$$

$$= \frac{A}{2V}\left(\frac{\epsilon_0 V}{d} + \frac{\epsilon_0 \epsilon_r V}{d}\right)$$

$$= \frac{A\epsilon_0(1 + \epsilon_r)}{2d}$$

$$\frac{C'}{C} = \frac{1 + \epsilon_r}{2}$$

4.20

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{\mathrm{f}}$$

$$4\pi r^2 D = \frac{4}{3}\pi r^3 \rho$$

$$\mathbf{D} = \frac{r\rho}{3}\hat{\mathbf{r}}$$

$$\mathbf{D} = \epsilon \mathbf{E}$$

$$\mathbf{E} = \frac{1}{\epsilon}\mathbf{D}$$

$$= \frac{r\rho}{3\epsilon_0 \epsilon_r}\hat{\mathbf{r}}, \, r < R$$

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\mathrm{enc}}}{\epsilon_0}$$

$$4\pi r^2 E = \frac{1}{\epsilon_0}\frac{4}{3}\pi R^3 \rho$$

$$\mathbf{E} = \frac{R^3 \rho}{3\epsilon_0 r^2}\hat{\mathbf{r}}, \, r > R$$

$$V = -\int_{\infty}^{0} \mathbf{E} \cdot d\mathbf{l}$$

$$= -\left(\int_{\infty}^{R} \frac{R^3 \rho}{3\epsilon_0 r^2}\, dr + \int_{R}^{0} \frac{r\rho}{3\epsilon_0 \epsilon_r}\, dr \right)$$

$$= -\left(\frac{R^3 \rho}{3\epsilon_0}\left[-\frac{1}{r} \right]_{\infty}^{R} + \frac{\rho}{3\epsilon_0 \epsilon_r}\left[\frac{1}{2}r^2 \right]_{R}^{0} \right)$$

$$= -\left(-\frac{R^2 \rho}{3\epsilon_0} - \frac{R^2 \rho}{6\epsilon_0 \epsilon_r} \right)$$

$$= \frac{R^2 \rho}{3\epsilon_0}\left(1 + \frac{1}{2\epsilon_r} \right)$$

4.21

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$2\pi s L E = \frac{\pi a^2 L \rho}{\epsilon_0}$$

$$\mathbf{E} = \frac{a^2 \rho}{2\epsilon_0 s} \hat{\mathbf{s}}, \ a < s < b$$

$$\mathbf{E} = \frac{\mathbf{E}_{\text{vac}}}{\epsilon_r}$$

$$= \frac{a^2 \rho}{2\epsilon_0 \epsilon_r s} \hat{\mathbf{s}}, \ b < s < c$$

$$V = -\int_c^a \mathbf{E} \cdot d\mathbf{l}$$

$$= -\left(\int_c^b \frac{a^2 \rho}{2\epsilon_0 \epsilon_r s} \, ds + \int_b^a \frac{a^2 \rho}{2\epsilon_0 s} \, ds \right)$$

$$= -\left(\frac{a^2 \rho}{2\epsilon_0 \epsilon_r} \ln \frac{b}{c} + \frac{a^2 \rho}{2\epsilon_0} \ln \frac{a}{b} \right)$$

$$= \frac{a^2 \rho}{2\epsilon_0} \left(\ln \frac{b}{a} + \frac{1}{\epsilon_r} \ln \frac{c}{b} \right)$$

$$C = \frac{Q}{V}$$

$$= \frac{\pi a^2 \rho}{\frac{a^2 \rho}{2\epsilon_0} \left(\ln \frac{b}{a} + \frac{1}{\epsilon_r} \ln \frac{c}{b} \right)}$$

$$= \frac{2\pi \epsilon_0}{\ln \frac{b}{a} + \frac{1}{\epsilon_r} \ln \frac{c}{b}}$$

4.22

$$V_{\text{in}}(a, \phi) = V_{\text{out}}(a, \phi)$$

$$\epsilon \left.\frac{\partial V_{\text{in}}}{\partial s}\right|_{s=a} = \epsilon_0 \left.\frac{\partial V_{\text{out}}}{\partial s}\right|_{s=a}$$

$$V_{\text{out}} \to -E_0 s \cos\phi, \text{ for } s \gg a$$

$$V_{\text{in}} = \sum_{k=1}^{\infty} s^k (a_k \cos k\phi + b_k \sin k\phi)$$

$$V_{\text{out}} = -E_0 s \cos\phi + \sum_{k=1}^{\infty} s^{-k}(c_k \cos k\phi + d_k \sin k\phi)$$

$$\sum_{k=1}^{\infty} a^k(a_k \cos k\phi + b_k \sin k\phi) = -E_0 a \cos\phi + \sum_{k=1}^{\infty} a^{-k}(c_k \cos k\phi + d_k \sin k\phi)$$

$$a a_1 = \frac{c_1}{a} - E_0 a$$

$$a_1 = \frac{c_1}{a^2} - E_0$$

$$a^k a_k = a^{-k} c_k$$

$$a_k = a^{-2k} c_k, \text{ for } k \neq 1$$

$$a^k b_k = a^{-k} d_k$$

$$b_k = a^{-2k} d_k$$

$$\epsilon_0(1 + \chi_e) \sum_{k=1}^{\infty} k a^{k-1}(a_k \cos k\phi + b_k \sin k\phi) = \epsilon_0 \left(-E_0 \cos\phi - \sum_{k=1}^{\infty} k a^{-(k+1)}(c_k \cos k\phi + d_k \sin k\phi)\right)$$

$$(1 + \chi_e) a_1 = -\left(E_0 + \frac{c_1}{a^2}\right)$$

$$a_1 = -\frac{1}{1 + \chi_e}\left(E_0 + \frac{c_1}{a^2}\right)$$

$$\frac{c_1}{a^2} - E_0 = -\frac{1}{1 + \chi_e}\left(E_0 + \frac{c_1}{a^2}\right)$$

$$c_1 - a^2 E_0 = -\frac{1}{1 + \chi_e}(a^2 E_0 + c_1)$$

$$c_1 \frac{2 + \chi_e}{1 + \chi_e} = a^2 E_0 \frac{\chi_e}{1 + \chi_e}$$

$$c_1 = a^2 E_0 \frac{\chi_e}{2 + \chi_e}$$

$$a_1 = E_0\left(\frac{\chi_e}{2 + \chi_e} - 1\right)$$

$$= -\frac{E_0}{1 + \chi_e/2}$$

$$(1 + \chi_e) a^{k-1} a_k = -a^{-(k+1)} c_k$$

$$a_k = -\frac{1}{1 + \chi_e} a^{-2k} c_k$$

$$a^{-2k} c_k = -\frac{1}{1 + \chi_e} a^{-2k} c_k$$

$$c_k = 0$$

$$a_k = 0$$

$$(1 + \chi_e) a^{k-1} b_k = -a^{-(k+1)} d_k$$

$$b_k = -\frac{1}{1 + \chi_e} a^{-2k} d_k$$

$$a^{-2k} d_k = -\frac{1}{1 + \chi_e} a^{-2k} d_k$$

$$d_k = 0$$

$$b_k = 0$$

$$V_{\text{in}} = -\frac{E_0}{1 + \chi_e/2} s \cos \phi$$

$$= -\frac{E_0}{1 + \chi_e/2} x$$

$$\mathbf{E}_{\text{in}} = -\frac{\partial V_{\text{in}}}{\partial x} \hat{\mathbf{x}}$$

$$= \frac{\mathbf{E}_0}{1 + \chi_e/2}$$

4.23

$$\mathbf{P}_0 = \epsilon_0 \chi_e \mathbf{E}_0$$

$$\mathbf{E}_1 = -\frac{1}{3\epsilon_0} \mathbf{P}_0$$

$$= -\frac{\chi_e}{3} \mathbf{E}_0$$

$$\mathbf{P}_1 = \epsilon_0 \chi_e \mathbf{E}_1$$

$$= -\frac{\epsilon_0 \chi_e^2}{3} \mathbf{E}_0$$

$$\mathbf{E}_2 = -\frac{1}{3\epsilon_0} \mathbf{P}_1$$

$$= \frac{\chi_e^2}{9} \mathbf{E}_0$$

$$\mathbf{E}_n = \left(-\frac{\chi_e}{3}\right)^n \mathbf{E}_0$$

$$\mathbf{E} = \sum_{n=0}^{\infty} \mathbf{E}_n$$

$$= \mathbf{E}_0 \sum_{n=0}^{\infty} \left(-\frac{\chi_e}{3}\right)^n$$

$$= \frac{3}{3 + \chi_e} \mathbf{E}_0$$

$$= \frac{3}{2 + \epsilon_r} \mathbf{E}_0$$

4.26

$$\mathbf{D} = \begin{cases} \mathbf{0} & r < a \\ \frac{Q}{4\pi r^2}\hat{\mathbf{r}} & a < r < b \\ \frac{Q}{4\pi r^2}\hat{\mathbf{r}} & b < r \end{cases}$$

$$\mathbf{E} = \begin{cases} \mathbf{0} & r < a \\ \frac{1}{4\pi\epsilon_0(1+\chi_e)}\frac{Q}{r^2}\hat{\mathbf{r}} & a < r < b \\ \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} & b < r \end{cases}$$

$$\mathbf{D}\cdot\mathbf{E} = \begin{cases} 0 & r < a \\ \frac{Q^2}{16\pi^2\epsilon_0(1+\chi_e)r^4} & a < r < b \\ \frac{Q^2}{16\pi^2\epsilon_0 r^4} & b < r \end{cases}$$

$$W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\, d\tau$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\mathbf{D}\cdot\mathbf{E}\; r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{1+\chi_e}\int_a^b\frac{1}{r^2}\,dr + \int_b^{\infty}\frac{1}{r^2}\,dr\right)$$

$$= \frac{Q^2}{8\pi\epsilon_0}\left[\frac{1}{1+\chi_e}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{b}\right]$$

$$= \frac{Q^2}{8\pi\epsilon_0(1+\chi_e)}\left(\frac{1}{a}+\frac{\chi_e}{b}\right)$$

4.27

$$\mathbf{E} = \begin{cases} -\dfrac{\mathbf{P}}{3\epsilon_0} & r < R \\ \dfrac{R^3 P}{3\epsilon_0 r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right) & r > R \end{cases}$$

$$E^2 = \begin{cases} \dfrac{P^2}{9\epsilon_0^2} & r < R \\ \dfrac{R^6 P^2}{9\epsilon_0^2 r^6}\left(1 + 3\cos^2\theta\right) & r > R \end{cases}$$

$$W = \frac{\epsilon_0}{2}\int E^2\,d\tau$$

$$= \frac{\epsilon_0}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} E^2 r^2 \sin\theta\,dr\,d\theta\,d\phi$$

$$= \epsilon_0\pi\left(\frac{P^2}{9\epsilon_0^2}\int_0^{\pi}\int_0^{R} r^2\sin\theta\,dr\,d\theta + \frac{R^6 P^2}{9\epsilon_0^2}\int_0^{\pi}\int_R^{\infty}\frac{1+3\cos^2\theta}{r^4}\sin\theta\,dr\,d\theta\right)$$

$$= \epsilon_0\pi\left(\frac{2P^2 R^3}{27\epsilon_0^2} + \frac{4P^2 R^3}{27\epsilon_0^2}\right)$$

$$= \frac{2\pi P^2 R^3}{9\epsilon_0}$$

$$W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,d\tau$$

4.28

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{encl}}}{\epsilon_0}$$

$$2\pi s L E = \frac{\lambda L}{\epsilon_0}$$

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}$$

$$V = \int_a^b \mathbf{E} \cdot d\mathbf{l}$$

$$= \frac{\lambda}{2\pi\epsilon_0} \ln\frac{b}{a}$$

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{\text{encl,f}}$$

$$2\pi s L D = \lambda' L$$

$$\mathbf{D} = \frac{\lambda'}{2\pi s}\hat{\mathbf{s}}$$

$$\mathbf{E} = \frac{\lambda'}{2\pi\epsilon s}\hat{\mathbf{s}}$$

$$V = \int_a^b \mathbf{E} \cdot d\mathbf{l}$$

$$= \frac{\lambda'}{2\pi\epsilon} \ln\frac{b}{a}$$

$$\frac{\lambda'}{2\pi\epsilon} \ln\frac{b}{a} = \frac{\lambda}{2\pi\epsilon_0} \ln\frac{b}{a}$$

$$\lambda' = \epsilon_r \lambda$$

$$Q = \lambda' h + \lambda(\ell - h)$$

$$= (\epsilon_r h + \ell - h)\lambda$$

$$= (\chi_e h + \ell)\lambda$$

$$C = \frac{Q}{V}$$

$$= (\chi_e h + \ell)\lambda \frac{2\pi\epsilon_0}{\lambda \ln(b/a)}$$

$$= 2\pi\epsilon_0 \frac{\chi_e h + \ell}{\ln(b/a)}$$

$$\frac{dC}{dh} = \frac{2\pi\epsilon_0 \chi_e}{\ln(b/a)}$$

$$F_{\text{up}} = \frac{1}{2}V^2\frac{dC}{dh}$$

$$= \frac{\pi\epsilon_0\chi_e V^2}{\ln(b/a)}$$

$$F_{\text{down}} = mg$$

$$= \pi(b^2 - a^2)h\rho g$$

$$F_{\text{down}} = F_{\text{up}}$$

$$\pi(b^2 - a^2)h\rho g = \frac{\pi\epsilon_0\chi_e V^2}{\ln(b/a)}$$

$$h = \frac{\epsilon_0\chi_e V^2}{\rho g(b^2 - a^2)\ln(b/a)}$$

4.29

(a)

$$\mathbf{E}_1 = -\frac{1}{4\pi\epsilon_0}\frac{p_1}{r^3}\hat{\mathbf{z}}$$

$$\mathbf{F}_2 = \frac{1}{4\pi\epsilon_0}\frac{3p_1 p_2}{r^4}\hat{\mathbf{z}}$$

$$\mathbf{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p_2}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p_2}]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[\frac{3(\mathbf{p_2}\cdot\mathbf{r})\mathbf{r}}{r^2} - \mathbf{p_2}\right]$$

$$= \frac{p_2}{4\pi\epsilon_0}\frac{3y(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) - (x^2 + y^2 + z^2)\hat{\mathbf{y}}}{(x^2 + y^2 + z^2)^{5/2}}$$

$$= \frac{p_2}{4\pi\epsilon_0}\frac{3xy\hat{\mathbf{x}} + (-x^2 + 2y^2 - z^2)\hat{\mathbf{y}} + 3yz\hat{\mathbf{z}}}{(x^2 + y^2 + z^2)^{5/2}}$$

$$E_{2z} = \frac{p_2}{4\pi\epsilon_0}\frac{3yz}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\nabla E_{2z}\big|_{x=0,y=-r,z=0} = -\frac{3p_2}{4\pi\epsilon_0 r^4}\hat{\mathbf{z}}$$

$$\mathbf{F}_1 = -\frac{1}{4\pi\epsilon_0}\frac{3p_1 p_2}{r^4}\hat{\mathbf{z}}$$

Newton's third law is obeyed.

(b)

$$\boldsymbol{\tau}_2 = \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{p}_1 \times \mathbf{E}_1$$

$$= r\hat{\mathbf{y}} \times \frac{1}{4\pi\epsilon_0} \frac{3p_1p_2}{r^4}\hat{\mathbf{z}} - \frac{1}{4\pi\epsilon_0} \frac{p_1p_2}{r^3}\hat{\mathbf{x}}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{2p_1p_2}{r^3}\hat{\mathbf{x}}$$

This is opposite to the torque experienced by $\mathbf{p}_1$.

4.30

The electric field is perpendicular to each plate and "curves" towards the other. At $y = 0$ the electric field is purely vertical. $\mathbf{p}$ only has a y component so $(\mathbf{p} \cdot \nabla)\mathbf{E} = p_y\nabla E_y$. The potential difference between the two plates is constant and as x increases the distance between them also increases, meaning the magnitude of the electric field decreases. E_y is negative at $y = 0$ so this means it increases with increasing x and thus that the x component of ∇E_y is positive. E_y is constant for small changes in y around $y = 0$ so the y component of ∇E_y is 0. We assume the plates are very long in the z direction so the z component of ∇E_y is also 0. This means the dipole experiences a force in the x direction.

4.31

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}\hat{\mathbf{r}}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})}{(x^2 + y^2 + z^2)^{3/2}}$$

$$E_y = \frac{1}{4\pi\epsilon_0} \frac{Qy}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\nabla E_y|_{x=R, y=z=0} = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^3}\hat{\mathbf{y}}$$

$$\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}$$

$$= p_y\nabla E_y$$

$$= \frac{Q}{4\pi\epsilon_0} \frac{\mathbf{p}}{R^3}$$

4.32

(a)

$$\mathbf{p} = \alpha \mathbf{E}$$
$$\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}$$
$$= (\alpha \mathbf{E} \cdot \nabla)\mathbf{E}$$
$$= \alpha(\mathbf{E} \cdot \nabla)\mathbf{E}$$
$$\nabla(E^2) = 2\mathbf{E} \times (\nabla \times \mathbf{E}) + 2(\mathbf{E} \cdot \nabla)\mathbf{E}$$
$$(\mathbf{E} \cdot \nabla)\mathbf{E} = \frac{1}{2}\nabla(E^2)$$
$$\mathbf{F} = \frac{1}{2}\alpha\nabla(E^2)$$

4.33

$$\mathbf{P} = k\mathbf{r}$$
$$= k(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})$$
$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$
$$Q_{\text{face}} = \int \sigma_b \, da$$
$$= \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} k\left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + \frac{a}{2}\hat{\mathbf{z}}\right) \cdot \hat{\mathbf{z}} \, dx \, dy$$
$$= \frac{ak}{2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} dx \, dy$$
$$= \frac{a^3 k}{2}$$
$$Q_{\text{surface}} = 3a^3 k$$
$$\rho_b = -\nabla \cdot \mathbf{P}$$
$$= -3k$$
$$Q_{\text{volume}} = \rho V$$
$$= -3a^3 k$$
$$Q_{\text{total}} = 0$$

4.34

$$\epsilon = \epsilon_0 \left(1 + \frac{x}{d}\right)$$

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{\text{enc,f}}$$

$$AD = A\sigma_f$$

$$\mathbf{D} = \sigma_f \hat{\mathbf{x}}$$

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon}$$

$$= \frac{\sigma_f}{\epsilon_0(1 + x/d)}\hat{\mathbf{x}}$$

$$V = \int_0^d \mathbf{E} \cdot d\mathbf{l}$$

$$= \frac{\sigma_f}{\epsilon_0} \int_0^f \frac{1}{1 + x/d}\, dx$$

$$= \frac{\sigma_f}{\epsilon_0} d \ln 2$$

$$\sigma_f = \frac{\epsilon_0 V}{d \ln 2}$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E}$$

$$= \sigma_f \hat{\mathbf{x}} - \frac{\sigma_f}{1 + x/d}\hat{\mathbf{x}}$$

$$= \sigma_f \left(1 - \frac{1}{1 + x/d}\right)\hat{\mathbf{x}}$$

$$= \frac{\epsilon_0 V x}{d(d + x) \ln 2}\hat{\mathbf{x}}$$

$$\sigma_b|_{x=0} = (\mathbf{P} \cdot -\hat{\mathbf{x}})|_{x=0}$$

$$= 0$$

$$\sigma_b|_{x=d} = (\mathbf{P} \cdot \hat{\mathbf{x}})|_{x=d}$$

$$= \frac{\epsilon_0 V}{2d \ln 2}$$

$$Q_{\text{surface}} = A\frac{\epsilon_0 V}{2d \ln 2}$$

$$\rho_b = -\nabla \cdot \mathbf{P}$$

$$= -\frac{\epsilon_0 V}{(d + x)^2 \ln 2}$$

$$Q_{\text{volume}} = \int_0^d A\rho_b \, dx$$

$$= -A\frac{\epsilon_0 V}{\ln 2} \int_0^d \frac{1}{(d+x)^2} \, dx$$

$$= -A\frac{\epsilon_0 V}{2d \ln 2}$$

4.35

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{\text{encl,f}}$$

$$4\pi r^2 D = q$$

$$\mathbf{D} = \frac{q}{4\pi r^2}\hat{\mathbf{r}}$$

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{(1+\chi_e)r^2}\hat{\mathbf{r}}$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E}$$

$$= \frac{q}{4\pi r^2}\hat{\mathbf{r}} - \frac{q}{4\pi(1+\chi_e)r^2}\hat{\mathbf{r}}$$

$$= \frac{q}{4\pi r^2}\left(1 - \frac{1}{1+\chi_e}\right)\hat{\mathbf{r}}$$

$$= \frac{q\chi_e}{4\pi(1+\chi_e)r^2}\hat{\mathbf{r}}$$

$$\rho_b = -\nabla \cdot \mathbf{P}$$

$$= -\frac{q\chi_e}{4\pi(1+\chi_e)}\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right)$$

$$= -\frac{q\chi_e}{1+\chi_e}\delta^3(\mathbf{r})$$

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

$$= \frac{q\chi_e}{4\pi(1+\chi_e)R^2}$$

$$Q_{\text{surface}} = A\sigma_b$$

$$= \frac{q\chi_e}{1+\chi_e}$$

4.36

$$D_1^{\perp} - D_2^{\perp} = \sigma_f$$
$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0$$
$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$$
$$E_1^{\parallel} - E_2^{\parallel} = 0$$
$$E_1^{\parallel} = E_2^{\parallel}$$
$$\tan \theta_2 = 1$$
$$\tan \theta_2 = \frac{1}{\epsilon_2/\epsilon_1}$$
$$= \frac{\epsilon_1}{\epsilon_2}$$
$$\frac{\tan \theta_2}{\tan \theta_1} = \frac{\epsilon_2}{\epsilon_1}$$

If an electric field moved from air to a convex dielectric lens, $\epsilon_2/\epsilon_0 > 1$ meaning it would bend away from the normal. The lens would defocus the field.

4.39

(a)

$$V = \begin{cases} V_0 & r \leq R \\ V_0 \frac{R}{r} & r \geq R \end{cases}$$
$$\mathbf{E} = -\nabla V$$
$$= \begin{cases} \mathbf{0} & r \leq R \\ V_0 \frac{R}{r^2} \hat{\mathbf{r}} & r \geq R \end{cases}$$
$$\mathbf{P} = \epsilon_0 \chi_e V_0 \frac{R}{r^2} \hat{\mathbf{r}}, \, z < 0$$
$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$
$$= -\frac{\epsilon_0 \chi_e V_0}{R}, \text{ on the surface that touches the sphere}$$
$$\oint \sigma_f \, da = Q_{\text{total}}$$
$$4\pi R^2 \sigma_f = 4\pi \epsilon_0 V_0 R$$
$$\sigma_f = \begin{cases} \epsilon_0 V_0/R & \text{northern hemisphere} \\ \epsilon_0 V_0 (1 + \chi_e)/R & \text{southern hemisphere} \end{cases}$$

(b)

$$\sigma = \sigma_b + \sigma_f$$
$$= \frac{\epsilon_0 V_0}{R}$$
$$Q_{\text{total}} = 4\pi\epsilon_0 V_0 R$$
$$V = V_0 \frac{R}{r}$$

5   Magnetostatics

5.1

The charge is positive.

$$a^2 + (R - d)^2 = R^2$$
$$a^2 + R^2 - 2dR + d^2 = R^2$$
$$2dR = a^2 + d^2$$
$$R = \frac{a^2 + d^2}{2d}$$
$$p = qBR$$
$$= \frac{qB(a^2 + d^2)}{2d}$$

5.2

$$y(t) = C_1 \cos\omega t + C_2 \sin\omega t + \frac{E}{B}t + C_3$$
$$y'(t) = -C_1\omega \sin\omega t + C_2\omega \cos\omega t + \frac{E}{B}$$
$$z(t) = C_2 \cos\omega t - C_1 \sin\omega t + C_4$$
$$z'(t) = -C_2\omega \sin\omega t - C_1\omega \cos\omega t$$

(a)

$$0 = C_1 + C_3$$

$$\frac{E}{B} = C_2\omega + \frac{E}{B}$$

$$0 = C_2 + C_4$$

$$0 = -C_1\omega$$

$$C_1 = 0$$

$$C_2 = 0$$

$$C_3 = 0$$

$$C_4 = 0$$

$$y(t) = \frac{E}{B}t$$

$$z(t) = 0$$

(b)

$$0 = C_1 + C_3$$

$$\frac{E}{2B} = C_2\omega + \frac{E}{B}$$

$$0 = C_2 + C_4$$

$$0 = -C_1\omega$$

$$C_1 = 0$$

$$C_2 = -\frac{E}{2B\omega}$$

$$C_3 = 0$$

$$C_4 = \frac{E}{2B\omega}$$

$$y(t) = -\frac{E}{2B\omega}\sin\omega t + \frac{E}{B}t$$

$$z(t) = -\frac{E}{2B\omega}\cos\omega t + \frac{E}{2B\omega}$$

(c)

$$0 = C_1 + C_3$$

$$\frac{E}{B} = C_2\omega + \frac{E}{B}$$

$$0 = C_2 + C_4$$

$$\frac{E}{B} = -C_1\omega$$

$$C_1 = -\frac{E}{B\omega}$$

$$C_2 = 0$$

$$C_3 = \frac{E}{B\omega}$$

$$C_4 = 0$$

$$y(t) = -\frac{E}{B\omega}\cos\omega t + \frac{E}{B}t + \frac{E}{B\omega}$$

$$z(t) = \frac{E}{B\omega}\sin\omega t$$

5.3

(a)

$$\mathbf{0} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$0 = E + vB$$

$$v = \frac{E}{B}$$

(b)

$$qBR = p$$

$$= mv$$

$$= m\frac{E}{B}$$

$$\frac{q}{m} = \frac{E}{B^2 R}$$

5.4

The forces on the sides at $y = \pm a/2$ cancel, leaving the sides at $z = \pm a/2$ which both experience an upwards force of $F = IB = a^2 Ik/2$ meaning the total force is $F_{\text{total}} = a^2 Ik$ in the positive z direction.

5.5

(a)

$$K = \frac{I}{2\pi a}$$

(b)

$$J(s) = \frac{k}{s}$$

$$I = \int J(s)\, da$$

$$= \int_0^a \int_0^{2\pi} \frac{k}{s} s\, d\phi\, ds$$

$$= 2\pi ak$$

$$k = \frac{I}{2\pi a}$$

$$J(s) = \frac{I}{2\pi as}$$

5.6

(a)

$$K = \sigma \omega r$$

(b)

$$\mathbf{J} = \rho \omega r \sin\theta\, \hat{\phi} = \frac{3Q\omega r \sin\theta}{4\pi R^3}\hat{\phi}$$

5.8

(a)

$$\mathbf{B}_{\text{side}} = \frac{\mu_0 I}{4\pi R}\left[\sin\left(\frac{\pi}{4}\right) - \sin\left(-\frac{\pi}{4}\right)\right]\hat{\mathbf{z}}$$

$$= \sqrt{2}\frac{\mu_0 I}{4\pi R}\hat{\mathbf{z}}$$

$$\mathbf{B}_{\text{total}} = \sqrt{2}\frac{\mu_0 I}{\pi R}\hat{\mathbf{z}}$$

(b)

$$\mathbf{B}_{\text{side}} = \frac{\mu_0 I}{4\pi R}\left[\sin\left(\frac{\pi}{n}\right) - \sin\left(-\frac{\pi}{n}\right)\right]\hat{\mathbf{z}}$$

$$= \frac{\mu_0 I}{2\pi R}\sin\frac{\pi}{n}\hat{\mathbf{z}}$$

$$\mathbf{B}_{\text{total}} = \frac{n\mu_0 I}{2\pi R}\sin\frac{\pi}{n}\hat{\mathbf{z}}$$

(c)

$$\mathbf{B}_{\text{circle}} = \frac{\mu_0 I}{2R} \hat{\mathbf{z}}$$

$$\mathbf{B}_{\text{total}} = \lim_{n \to \infty} \frac{n \mu_0 I}{2\pi R} \sin \frac{\pi}{n} \hat{\mathbf{z}}$$

$$= \frac{\mu_0 I}{2\pi R} \lim_{n \to \infty} n \sin \frac{\pi}{n} \hat{\mathbf{z}}$$

$$= \frac{\mu_0 I}{2R} \hat{\mathbf{z}}$$

5.9

(a)

$$\mathbf{B} = \frac{\mu_0 I}{8} \left(\frac{1}{a} - \frac{1}{b} \right) \hat{\mathbf{z}}$$

(b)

$$B = \frac{\mu_0 I}{2R} \left(\frac{1}{\pi} + \frac{1}{2} \right) \text{ into the page}$$

5.10

(a)

$$F = aIB_{\text{bottom}} - aIB_{\text{top}}$$

$$= \frac{\mu_0 a I^2}{2\pi} \left(\frac{1}{s} - \frac{1}{s+a} \right)$$

$$= \frac{\mu_0 a I^2}{2\pi} \frac{a}{s(s+a)}$$

$$= \frac{\mu_0 a^2 I^2}{2\pi s(s+a)} \text{ upwards}$$

(b)

$$F_{\text{bottom}} = \frac{\mu_0 a I^2}{2\pi s} \text{ upwards}$$

$$y = s + x \sin \frac{\pi}{3}$$

$$= s + \frac{\sqrt{3}}{2} x$$

$$B = \frac{\mu_0 I}{2\pi (s + \sqrt{3}x/2)} \text{ out of page}$$

$$F_{\text{side}} = I \int d\mathbf{l} \times \mathbf{B}$$

$$= \frac{\mu_0 I^2}{2\pi} \int_0^a \frac{1}{s + \sqrt{3}x/2} \, dx$$

$$= \frac{\mu_0 I^2}{\sqrt{3}\pi} \ln \left(1 + \frac{\sqrt{3}a}{2s} \right)$$

$$F_{\text{total}} = \frac{\mu_0 a I^2}{2\pi s} - \frac{\mu_0 I^2}{\sqrt{3}\pi} \ln \left(1 + \frac{\sqrt{3}a}{2s} \right)$$

$$= \frac{\mu_0 I^2}{2\pi} \left[\frac{a}{s} - \frac{2}{\sqrt{3}} \ln \left(1 + \frac{\sqrt{3}a}{2s} \right) \right]$$