

Vibrations and Waves by George C. King Problems

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1 Simple Harmonic Motion

1.1

- (a) (i) $T = 4 \text{ s}$
(ii) $\omega = \frac{\pi}{2} \text{ rad/s}$
(iii) $\omega^2 = \frac{k}{m} \Rightarrow k = m\omega^2 = \frac{\pi^2}{8} \text{ N/m}$

1.2

- (a)

$$\begin{aligned}
x &= A \cos \omega t \\
&= A \cos 2\pi f t \\
v &= -2\pi f A \sin 2\pi f t \\
v_{\max} &= 2\pi f A \\
&= 1.38 \text{ m/s}
\end{aligned}$$

- (b)

$$\begin{aligned}
a &= -4\pi^2 f^2 A \cos 2\pi f t \\
a_{\max} &= 4\pi^2 f^2 A \\
&= 3.82 \times 10^3 \text{ m/s}^2
\end{aligned}$$

1.3

$$\begin{aligned}
a_{\max} &\leq g \\
4\pi^2 f^2 A &\leq g \\
f &\leq \sqrt{\frac{g}{4\pi^2 A}} \\
&\leq 1.11 \text{ Hz}
\end{aligned}$$

1.4

(a)

$$\frac{U}{E} = \frac{\frac{1}{2}k\left(\frac{1}{2}A\right)^2}{\frac{1}{2}kA^2} = \frac{1}{4} \Rightarrow \frac{K}{E} = \frac{3}{4}$$

(b) (i) The total energy will increase by a factor of 4

(ii) The maximum velocity will increase by a factor of 2

(iii) The maximum acceleration will increase by a factor of 2 and the period won't change

1.5

(a) $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = 0.41 \text{ J}$

(b)

$$E = \frac{1}{2}kA^2$$

$$A = \sqrt{\frac{2E}{k}}$$

$$= 4.5 \text{ cm}$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{1600}{3}}$$

$$= \frac{40}{\sqrt{3}}$$

$$= 23 \text{ rad/s}$$

$$x = A \cos(\omega t + \phi)$$

$$\phi = \arccos\left(\frac{x}{A}\right) - \omega t$$

$$= 2.7 \text{ rad}$$

$$x = 0.045 \cos(23t + 2.7) \text{ m}$$

1.6

Using the angular frequency of system (b) ω_b as the baseline, the angular frequency of system (a) ω_a is

$$\begin{aligned}
F &= ma = -2kx \\
a &= -\frac{2k}{m}x \\
\omega_a &= \sqrt{\frac{2k}{m}} \\
&= \sqrt{2}\omega_b
\end{aligned}$$

and the angular frequency of system (c) ω_c is

$$\begin{aligned}
F &= ma = -\frac{k}{2}x \\
a &= -\frac{k}{2m}x \\
\omega_c &= \sqrt{\frac{k}{2m}} \\
&= \sqrt{\frac{1}{2}}\omega_b
\end{aligned}$$

1.7

- (a) The test tube experiences a bouyancy force of $F = Ag\rho x$ so its equation of motion is

$$\begin{aligned}
F &= ma = -Ag\rho x \\
a &= -\frac{Ag\rho}{m}x \\
\omega &= \sqrt{\frac{Ag\rho}{m}}
\end{aligned}$$

- (b) The work done by the bouyancy force when moving from equilibrium to x and thus the potential energy is

$$\begin{aligned}
U &= \int_0^x Ag\rho x' dx' \\
&= \frac{1}{2}Ag\rho x^2
\end{aligned}$$

so the total energy of the system is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}Ag\rho x^2$$

1.8

$$s \propto \text{kg}^\alpha \text{m}^\beta (\text{m/s}^2)^\gamma$$

so $\alpha = 0$, $\beta = 1/2$, and $\gamma = -1/2$ meaning

$$T \propto \sqrt{\frac{l}{g}}$$

1.9

(a)

$$x = A \cos \sqrt{\frac{g}{l}} t$$

$$v = -\sqrt{\frac{g}{l}} A \sin \sqrt{\frac{g}{l}} t$$

$$\begin{aligned} v_{\max} &= \sqrt{\frac{g}{l}} A \\ &= 0.018 \text{ m/s} \end{aligned}$$

(b) The pendulum reaches its maximum speed at the bottom of its swing which occurs after a quarter cycle

$$\frac{1}{4}T = \frac{1}{4} \frac{2\pi}{\omega} = \frac{\pi}{2\sqrt{g/l}} = 0.43 \text{ s}$$

1.10

$$I \frac{d^2\theta}{dt^2} = \tau$$

$$\frac{1}{3}ML^2 \frac{d^2\theta}{dt^2} = -kL \sin \theta L \cos \theta$$

$$\frac{1}{3}M \frac{d^2\theta}{dt^2} = -k\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{3k}{M}\theta$$

$$T = \frac{2\pi}{\omega}$$

$$= 2\pi \sqrt{\frac{M}{3k}}$$

1.11

(a)

$$\begin{aligned}
 F &= -\frac{dU}{dx} = -\left(\frac{6a}{x^7} - \frac{12b}{x^{13}}\right) \\
 0 &= \frac{12b}{x^{13}} - \frac{6a}{x^7} \\
 &= \frac{12b}{x^6} - 6a \\
 6a &= \frac{12b}{x^6} \\
 x^6 &= \frac{2b}{a} \\
 x &= \left(\frac{2b}{a}\right)^{1/6}
 \end{aligned}$$

1.12

(a)

$$\begin{aligned}
 K &= \frac{1}{2}Mv^2 + \int dK \\
 &= \frac{1}{2}Mv^2 + \int_0^L \frac{1}{2} \frac{m}{L} \left(\frac{l}{L}v\right)^2 dl \\
 &= \frac{1}{2}Mv^2 + \frac{1}{2} \frac{mv^2}{L^3} \int_0^L l^2 dl \\
 &= \frac{1}{2}Mv^2 + \frac{1}{2} \frac{mv^2}{L^3} \frac{1}{3}L^3 \\
 &= \frac{1}{2}Mv^2 + \frac{1}{6}mv^2 \\
 &= \frac{1}{2}(M + m/3)v^2 \\
 E &= K + U \\
 &= \frac{1}{2}(M + m/3)v^2 + \frac{1}{2}kx^2
 \end{aligned}$$

(b)

$$\omega = \sqrt{\frac{k}{M + m/3}}$$

1.13

(a)

$$\begin{aligned} K &= E - U \\ \frac{1}{2}mv^2 &= U(A) - U(x) \\ v &= \sqrt{2[U(A) - U(x)]/m} \end{aligned}$$

(b)

$$\begin{aligned} T &= 4 \int_0^A \frac{dx}{v} \\ &= 4 \int_0^A \sqrt{\frac{m}{2[U(A) - U(x)]}} dx \\ &= 4 \sqrt{\frac{m}{2U(A)}} \int_0^A \frac{dx}{\sqrt{1 - U(x)/U(A)}} \end{aligned}$$

(c)

$$\begin{aligned} T &= 4 \sqrt{\frac{m}{2\alpha A^n}} \int_0^A \frac{dx}{\sqrt{1 - (x/A)^n}} \\ &= 4 \sqrt{\frac{m}{2\alpha A^n}} \int_0^1 \frac{A d\xi}{\sqrt{1 - \xi^n}} \\ &= cA^{(n/2)-1} \end{aligned}$$

2 The Damped Harmonic Oscillator

2.1

$$\begin{aligned} \left(\frac{\gamma}{2}\right)^2 &= \omega_0^2 \\ \frac{b}{2m} &= \sqrt{\frac{k}{m}} \\ b &= 2m\sqrt{\frac{k}{m}} \\ &= 2m\sqrt{\frac{mg/x}{m}} \\ &= 2m\sqrt{\frac{g}{x}} \\ &= 64 \text{ kg/s} \end{aligned}$$

2.2

$$\begin{aligned}
 \frac{A_{n+1}}{A_n} &= 0.90 \\
 e^{-2.5\gamma/2} &= 0.90 \\
 e^{2.5\gamma/2} &= \frac{1}{0.90} \\
 \frac{2.5\gamma}{2} &= \ln \frac{1}{0.90} \\
 \gamma &= \frac{2}{2.5} \ln \frac{1}{0.90} \\
 &= 8.43 \times 10^{-2} \text{ s}^{-1} \\
 F &= -bv \\
 &= -(4.21 \times 10^{-2})v
 \end{aligned}$$

2.3

After 10 cycles the amplitude has decreased by a factor of 5/3. The energy of the system is proportional to the amplitude squared, so

$$\begin{aligned}
 E(300) &= E(0)e^{-300/\tau} \\
 e^{300/\tau} &= \frac{E(0)}{E(300)} \\
 \tau &= \frac{300}{\ln[E(0)/E(300)]} \\
 &= \frac{300}{\ln \frac{25}{9}} \\
 &= 294 \text{ s} \\
 Q &= \omega_0 \tau \\
 &= \frac{2\pi\tau}{T} \\
 &= 61.5
 \end{aligned}$$

2.4

$$\begin{aligned}
 \frac{E(10T)}{E_0} &= \frac{E_0 e^{-\gamma 10T}}{E_0} \\
 \frac{1}{2} &= e^{-\gamma 10T} \\
 \frac{E(50T)}{E_0} &= \frac{E_0 e^{-\gamma 50T}}{E_0} \\
 &= (e^{-\gamma 10T})^5 \\
 &= \left(\frac{1}{2}\right)^5 \\
 &= \frac{1}{32}
 \end{aligned}$$

2.5

(a)

$$\begin{aligned}
 Q_{0.01} &= 310 \\
 \omega_{0.01} &= 3.14 \text{ rad/s} \\
 Q_{0.30} &= 10.5 \\
 \omega_{0.30} &= 3.14 \text{ rad/s} \\
 Q_{1.00} &= 3.14 \\
 \omega_{1.00} &= 3.10 \text{ rad/s}
 \end{aligned}$$

(c)

$$\begin{aligned}
 \gamma^2/4 &= \pi^2 \\
 \gamma &= 2\pi \\
 x &= Ae^{-\pi t} + Bte^{-\pi t} \\
 A &= 10 \text{ mm} \\
 v &= -10\pi e^{-\pi t} + Be^{-\pi t} - \pi Bte^{-\pi t} \\
 0 &= -10\pi + B \\
 B &= 10\pi \\
 x &= 10e^{-\pi t} + 10\pi te^{-\pi t}
 \end{aligned}$$

2.6

$$\begin{aligned}
 \frac{\omega}{\omega_0} &= \frac{\omega_0 \sqrt{1 - 1/4Q^2}}{\omega_0} \\
 &= \sqrt{1 - 1/4Q^2} \\
 &= 1 - \frac{1/4Q^2}{2} + \dots \\
 &\approx 1 - \frac{Q^2}{8}
 \end{aligned}$$

2.7

The amplitude of each pendulum decreases over time by a factor of

$$\begin{aligned}
 \exp\left(-\frac{\gamma t}{2}\right) &= \exp\left(-\frac{bt}{2m}\right) \\
 &= \exp\left(-\frac{bt}{2 \cdot \frac{4}{3}\pi r^3 \rho}\right) \\
 &= \exp\left(-\frac{3bt}{8\pi r^3 \rho}\right) \\
 &= \exp\left(-\frac{3bt}{8\pi r^3}\right)^{1/\rho}.
 \end{aligned}$$

After 10 minutes the amplitude of oscillation of the aluminium pendulum has decreased to half of its initial value

$$\begin{aligned}
 \exp\left(-\frac{225b}{\pi r^3}\right)^{1/\rho_a} &= \frac{1}{2} \\
 \exp\left(-\frac{225b}{\pi r^3}\right) &= \left(\frac{1}{2}\right)^{\rho_a}
 \end{aligned}$$

so the brass pendulum's amplitude of oscillation has decreased by a factor of

$$\begin{aligned}
 \exp\left(-\frac{225b}{\pi r^3}\right)^{1/\rho_b} &= \left(\frac{1}{2}\right)^{\rho_a/\rho_b} \\
 &= 0.802
 \end{aligned}$$

2.8

(a)

$$\begin{aligned}
 x &= A \sin \omega t \\
 v &= \omega A \cos \omega t \\
 a &= -\omega^2 A \sin \omega t \\
 E &= \int_0^T \frac{K e^2 a^2}{c^3} dt \\
 &= \int_0^T \frac{K e^2 \omega^4 A^2 \sin^2 \omega t}{c^3} dt \\
 &= \frac{K e^2 \omega^4 A^2}{c^3} \int_0^{2\pi/\omega} \sin^2 \omega t dt \\
 &= \frac{K e^2 \omega^4 A^2}{c^3} \left[\frac{t}{2} - \frac{1}{4\omega} \sin 2\omega t \right]_0^{2\pi/\omega} \\
 &= \frac{K e^2 \omega^3 A^2 \pi}{c^3}
 \end{aligned}$$

(b)

$$\begin{aligned}
 Q &= \frac{\frac{1}{2} m \omega^2 A^2}{\frac{K e^2 \omega^3 A^2 \pi}{2\pi c^3}} \\
 &= \frac{c^3 m}{e^2 K \omega}
 \end{aligned}$$

(c)

$$\begin{aligned}
 \tau &= \frac{1}{\gamma} \\
 &= \frac{Q}{\omega} \\
 &= \frac{c^3 m}{e^2 K \omega^2} \\
 &= \frac{c^3 m}{e^2 K (2\pi(c/\lambda))^2} \\
 &= \frac{\lambda^2 c m}{4\pi^2 e^2 K} \\
 &\approx 1.13 \times 10^{-8} \text{ s}
 \end{aligned}$$

3 Forced Oscillations

3.1

$$\begin{aligned}
 A(2 \text{ rad/s}) &= 1.3 \times 10^{-2} \text{ m} \\
 \delta(2 \text{ rad/s}) &= 0.58^\circ \\
 A(20 \text{ rad/s}) &= 0.13 \text{ m} \\
 \delta(20 \text{ rad/s}) &= 90^\circ \\
 A(100 \text{ rad/s}) &= 5.2 \times 10^{-4} \text{ m} \\
 \delta(100 \text{ rad/s}) &= 179^\circ
 \end{aligned}$$

3.2

$$\begin{aligned}
 A(\omega) &= \frac{a\omega_0/\omega}{\sqrt{(\omega_0/\omega - \omega/\omega_0)^2 + 1/Q^2}} \\
 &= \frac{au}{\sqrt{(u - 1/u)^2 + 1/Q^2}} \\
 &= \frac{a}{\sqrt{(1 - 1/u^2)^2 + 1/u^2 Q^2}}
 \end{aligned}$$

$A(\omega)$ is maximised when the denominator is minimised which occurs when

$$\begin{aligned}
 \frac{d}{du}((1 - u^{-2})^2 + Q^{-2}u^{-2}) &= 0 \\
 2(1 - u^{-2})2u^{-3} - 2Q^{-2}u^{-3} &= 0 \\
 4(1 - u^{-2}) - 2Q^{-2} &= 0 \\
 \frac{4Q^2 - 2}{Q^2} &= \frac{4}{u^2} \\
 \frac{Q^2}{4Q^2 - 2} &= \frac{u^2}{4} \\
 \frac{4Q^2}{4Q^2 - 2} &= u^2 \\
 \frac{1}{1 - 1/2Q^2} &= u^2 \\
 \frac{1}{\sqrt{1 - 1/2Q^2}} &= \frac{\omega_0}{\omega_{\max}} \\
 \omega_{\max} &= \omega_0 \sqrt{1 - 1/2Q^2}
 \end{aligned}$$

at which point the amplitude will be

$$\begin{aligned}
A_{\max} &= \frac{a\omega_0/\omega_0\sqrt{1-1/2Q^2}}{\sqrt{\left(\frac{\omega_0}{\omega_0\sqrt{1-1/2Q^2}} - \frac{\omega_0\sqrt{1-1/2Q^2}}{\omega_0}\right)^2 + 1/Q^2}} \\
&= \frac{a/\sqrt{1-1/2Q^2}}{\sqrt{\left(\frac{1}{\sqrt{1-1/2Q^2}} - \sqrt{1-1/2Q^2}\right)^2 + 1/Q^2}} \\
&= \frac{a/\sqrt{1-1/2Q^2}}{\sqrt{\left(\frac{1-1+1/2Q^2}{\sqrt{1-1/2Q^2}}\right)^2 + 1/Q^2}} \\
&= \frac{a/\sqrt{1-1/2Q^2}}{\sqrt{\frac{1}{4Q^4(1-1/2Q^2)} + 1/Q^2}} \\
&= \frac{a/\sqrt{1-1/2Q^2}}{\sqrt{\frac{1+4Q^2(1-1/2Q^2)}{4Q^2(1-1/2Q^2)}}} \\
&= \frac{a}{\sqrt{1-1/2Q^2}} \sqrt{\frac{4Q^2(1-1/2Q^2)}{1+4Q^2(1-1/2Q^2)}} \\
&= a\sqrt{\frac{4Q^2}{1+4Q^2(1-1/2Q^2)}} \\
&= aQ\sqrt{\frac{4}{4(1-1/2Q^2)+1/Q^2}} \\
&= \frac{aQ}{\sqrt{1-1/2Q^2+1/4Q^2}} \\
&= \frac{aQ}{\sqrt{1-1/4Q^2}}
\end{aligned}$$

3.3

(a)

$$\begin{aligned}
\frac{\omega_0 - \omega_{\max}}{\omega_0} &= 1 - \sqrt{1-1/2Q^2} \\
&\approx 0.25 \%
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{A_{\max} - A_0}{A_0} &= \frac{1}{\sqrt{1-1/4Q^2}} - 1 \\
&\approx 0.13 \%
\end{aligned}$$

3.4

$$\begin{aligned}
 \overline{P}_{\max} &= \frac{F_0^2}{2m\gamma} \\
 &= 50 \text{ W} \\
 \overline{P}(\omega) &= \frac{F_0^2}{2m\omega_0 Q [4(\Delta\omega/\omega_0)^2 + 1/Q^2]} \\
 &= \frac{\overline{P}_{\max}}{Q^2 [4(\Delta\omega/\omega_0)^2 + 1/Q^2]} \\
 &= \frac{50}{625 [4(\Delta\omega/100)^2 + 1/625]} \\
 &= \frac{50}{\frac{1}{4}\Delta\omega^2 + 1}
 \end{aligned}$$

3.5

(a)

$$\frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \approx 398 \text{ Hz}$$

(b)

$$I = \frac{V_0}{R} = \frac{15}{75} = 0.2 \text{ A}$$

3.6

$$i^i = (e^{i\pi/2})^i = e^{-\pi/2} = 0.208$$

3.7

$$\begin{aligned}
 z &= Ae^{i(\omega t + \phi)} \\
 x &= \text{Re } z \\
 &= A \cos(\omega t + \phi) \\
 \frac{dz}{dt} &= i\omega Ae^{i(\omega t + \phi)} \\
 \frac{dx}{dt} &= \text{Re } \frac{dz}{dt} \\
 &= -\omega A \sin(\omega t + \phi) \\
 &= \omega A \cos(\omega t + \phi + \pi/2)
 \end{aligned}$$

$\frac{dx}{dt}$ is in advance of x by 90°

3.8

(a)

$$\begin{aligned}
m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + mg \frac{x - \xi}{l} &= 0 \\
m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + m \frac{g}{l} x &= m \frac{g}{l} a \cos \omega t \\
m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + m \omega_0^2 x &= m \omega_0^2 a \cos \omega t \\
\operatorname{Re} \left(m \frac{d^2 z}{dt^2} + b \frac{dz}{dt} + m \omega_0^2 z \right) &= \operatorname{Re}(m \omega_0^2 a e^{i \omega t})
\end{aligned}$$

(b)

$$\begin{aligned}
m \frac{d^2}{dt^2} (A e^{i(\omega t - \delta)}) + b \frac{d}{dt} (A e^{i(\omega t - \delta)}) + m \omega_0^2 A e^{i(\omega t - \delta)} &= m \omega_0^2 a e^{i \omega t} \\
-m \omega^2 A e^{i(\omega t - \delta)} + i \omega b A e^{i(\omega t - \delta)} + m \omega_0^2 A e^{i(\omega t - \delta)} &= m \omega_0^2 a e^{i \omega t} \\
-m \omega^2 A + i \omega b A + m \omega_0^2 A &= m \omega_0^2 a e^{i \delta}
\end{aligned}$$

$$\begin{aligned}
-m \omega^2 A + m \omega_0^2 A &= m \omega_0^2 a \cos \delta \\
-\omega^2 A + \omega_0^2 A &= \omega_0^2 a \cos \delta \\
\frac{\omega_0^2 - \omega^2}{\omega_0^2 a} A &= \cos \delta
\end{aligned}$$

$$\begin{aligned}
\omega b A &= m \omega_0^2 a \sin \delta \\
\frac{\omega b}{m \omega_0^2 a} A &= \sin \delta
\end{aligned}$$

$$\begin{aligned}
\frac{\omega b}{m(\omega_0^2 - \omega^2)} &= \tan \delta \\
\frac{\omega \gamma}{\omega_0^2 - \omega^2} &= \tan \delta
\end{aligned}$$

$$\begin{aligned}
A &= \frac{\omega_0^2 a \cos \delta}{\omega_0^2 - \omega^2} \\
&= \frac{\omega_0^2 a}{\omega_0^2 - \omega^2} \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} \\
&= \frac{a \omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}}
\end{aligned}$$

3.9

(a)

$$\begin{aligned}
 A(t_{75}) &= A_0 e^{-\gamma t_{75}/2} \\
 \frac{A(t_{75})}{A_0} &= e^{-\gamma t_{75}/2} \\
 \ln \frac{A_0/e}{A_0} &= -\frac{\gamma t_{75}}{2} \\
 -1 &= -\frac{\gamma t_{75}}{2} \\
 \frac{2}{t_{75}} &= \gamma
 \end{aligned}$$

$$\begin{aligned}
 Q &= \frac{\omega_0}{\gamma} \\
 &= \frac{75 \cdot 2\pi}{t_{75}} \frac{t_{75}}{2} \\
 &= 75\pi
 \end{aligned}$$

(b)

$$\begin{aligned}
 A(\omega_0) &= a \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega_0^2)^2 + \omega_0^2 \gamma^2}} \\
 &= a \frac{\omega_0^2}{\sqrt{\omega_0^2 \gamma^2}} \\
 &= a \frac{\omega_0}{\gamma} \\
 &= aQ \\
 &= (0.5 \text{ mm}) 75\pi \\
 &= 0.12 \text{ m}
 \end{aligned}$$

(c)

$$\begin{aligned}
\frac{aQ}{2} &= a \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} \\
&\approx a \frac{\omega_0^2}{\sqrt{(2\omega_0(-\Delta\omega))^2 + \omega_0^2 \gamma^2}} \\
\frac{2\omega_0^2}{Q} &= \sqrt{4\omega_0^2 \Delta\omega^2 + \omega_0^2 \gamma^2} \\
\frac{4\omega_0^4}{Q^2} &= 4\omega_0^2 \Delta\omega^2 + \omega_0^2 \gamma^2 \\
4\omega_0^2 \Delta\omega^2 &= \frac{4\omega_0^4}{(\omega_0/\gamma)^2} - \omega_0^2 \gamma^2 \\
(2\Delta\omega)^2 &= 4\gamma^2 - \gamma^2 \\
2\Delta\omega &= \gamma\sqrt{3} \\
&= \frac{\omega_0}{Q} \sqrt{3} \\
&= \frac{\sqrt{g/l}}{Q} \sqrt{3} \\
&= 0.019 \text{ rad/s}
\end{aligned}$$

3.10

(a) (i)

$$\begin{aligned}
K &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}m(-\omega A \sin(\omega t - \delta))^2 \\
&= \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t - \delta)
\end{aligned}$$

(ii)

$$\begin{aligned}
U &= \frac{1}{2}kx^2 \\
&= \frac{1}{2}k(A \cos(\omega t - \delta))^2 \\
&= \frac{1}{2}kA^2 \cos^2(\omega t - \delta)
\end{aligned}$$

(iii)

$$\begin{aligned}
E &= \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t - \delta) + \frac{1}{2}kA^2 \cos^2(\omega t - \delta) \\
&= \frac{1}{2}mA^2[\omega^2 \sin^2(\omega t - \delta) + \omega_0^2 \cos^2(\omega t - \delta)]
\end{aligned}$$

(b)

$$\begin{aligned} 0 &= \frac{dE}{dt} \\ &= \frac{1}{2}mA^2[2\omega^3 \sin(\omega t - \delta) \cos(\omega t - \delta) - 2\omega_0^3 \cos(\omega t - \delta) \sin(\omega t - \delta)] \\ &= \frac{1}{2}mA^2 \sin(2(\omega t - \delta))(\omega^3 - \omega_0^3) \\ \omega &= \omega_0 \end{aligned}$$

$$\begin{aligned} E &= \frac{1}{2}mA^2[\omega_0^2 \sin^2(\omega_0 t - \delta) + \omega_0^2 \cos^2(\omega_0 t - \delta)] \\ &= \frac{1}{2}mA^2\omega_0^2 \end{aligned}$$

(c)

$$\begin{aligned} \overline{K} &= \frac{1}{T} \int_{t_0}^{t_0+T} \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t - \delta) dt \\ &= \frac{m\omega^2 A^2}{2T} \int_{t_0}^{t_0+T} \sin^2(\omega t - \delta) dt \\ &= \frac{m\omega^2 A^2}{4} \\ \overline{E} &= \frac{1}{T} \int_{t_0}^{t_0+T} \frac{1}{2}mA^2[\omega^2 \sin^2(\omega t - \delta) + \omega_0^2 \cos^2(\omega t - \delta)] dt \\ &= \frac{mA^2}{2T} \left(\int_{t_0}^{t_0+T} \omega^2 \sin^2(\omega t - \delta) dt + \int_{t_0}^{t_0+T} \omega_0^2 \cos^2(\omega t - \delta) dt \right) \\ &= \frac{mA^2}{4}(\omega^2 + \omega_0^2) \\ \frac{\overline{K}}{\overline{E}} &= \frac{m\omega^2 A^2}{4} \frac{4}{mA^2(\omega^2 + \omega_0^2)} \\ &= \frac{\omega^2}{\omega^2 + \omega_0^2} \\ &= \frac{1}{1 + (\omega_0/\omega)^2} \end{aligned}$$

(d)

$$\begin{aligned}
\overline{E} &= \overline{K} + \overline{U} \\
&= \frac{m\omega^2 A^2}{4} + \frac{kA^2}{4} \\
&= \frac{1}{4}mA^2(\omega^2 + \omega_0^2) \\
&= \frac{1}{4}m(\omega^2 + \omega_0^2) \left(\frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \right)^2 \\
&= \frac{1}{4}m(\omega^2 + \omega_0^2) \frac{F_0^2}{m^2[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2]} \\
&= \frac{F_0^2(\omega_0^2 + \omega^2)}{4m[(\omega_0^2 - \omega^2)^2 + \omega^2b^2/m^2]}
\end{aligned}$$

3.11

(a)

$$\begin{aligned}
x &= A \cos \omega t \\
v &= -\omega A \sin \omega t \\
W &= \int_0^T bv^2 dt \\
&= b \int_0^{2\pi/\omega} \omega^2 A^2 \sin^2 \omega t dt \\
&= \pi b \omega A^2
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{W}{E} &= \frac{\pi b \omega A^2}{\frac{1}{2}m\omega^2 A^2} \\
&= \frac{2\pi b}{m\omega}
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{W}{E} &= \frac{2\pi b}{m\omega_0} \\
&= \frac{2\pi\gamma}{\omega_0} \\
&= \frac{2\pi}{Q}
\end{aligned}$$

3.12

Let T' be the number of seconds in 8 days and $T = 2\pi\sqrt{l/g}$ be the period of the pendulum, then

$$\begin{aligned} Q &= 2\pi \frac{\text{stored energy}}{\text{energy dissipated/cycle}} \\ &= 2\pi \frac{1}{2} m_1 \omega^2 A^2 \frac{\text{cycles in 8 days}}{\text{energy dissipated in 8 days}} \\ &= \pi m_1 \frac{g}{l} A^2 \frac{T'/T}{m_2 g h} \\ &= \frac{\pi m_1 A^2 T'}{m_2 l h T} \\ &= \frac{m_1 A^2 \cdot 8 \cdot 24 \cdot 60 \cdot 60}{2 m_2 l h \sqrt{l/g}} \\ &\approx 70 \end{aligned}$$