

Advanced Engineering Mathematics Complex Analysis by Dennis G. Zill Notes

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Contents

17 Functions of a Complex Variable	1
17.1 Complex Numbers	1
17.2 Powers and Roots	3
17.3 Sets in the Complex Plane	4
17.4 Functions of a Complex Variable	5
17.5 Cauchy-Riemann Equations	7
17.6 Exponential and Logarithmic Functions	8
17.7 Trigonometric and Hyperbolic Functions	10
17.8 Inverse Trigonometric and Hyperbolic Functions	11
18 Integration in the Complex Plane	12
18.1 Contour Integrals	12
18.2 Cauchy-Goursat Theorem	13
18.3 Independence of the Path	14
18.4 Cauchy's Integral Formulas	15
19 Series and Residues	16
19.1 Sequences and Series	16
19.2 Taylor Series	17
19.3 Laurent Series	18

17 Functions of a Complex Variable

17.1 Complex Numbers

- A **complex number** is any number of the form

$$z = a + ib$$

where a and b are real numbers and i is the imaginary unit such that $i^2 = -1$.

- The real number a in the above complex number z is called the **real part** of z and the real number b (not ib) is called the **imaginary part** of z .
- The real and imaginary parts of a complex number z are denoted $\text{Re}(z)$ and $\text{Im}(z)$, respectively.
- A real constant multiple of the imaginary unit, e.g. $6i$ is called a **pure imaginary number**.
- Two complex numbers are equal if their real and imaginary parts are equal.
- The addition and subtraction of complex numbers occur between the real and imaginary parts, e.g.

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

- The multiplication of complex numbers occurs elementwise as normal, e.g.

$$(a + bi)(c + di) = ac + adi + bci - bd.$$

- The **conjugate** of a complex number $z = a + ib$ is

$$\bar{z} = a - ib.$$

- The division of complex numbers occurs by multiplying the numerator and denominator by the conjugate of the denominator, e.g.

$$\begin{aligned} \frac{a + bi}{c + di} &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{ac - adi + bci + bd}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2}. \end{aligned}$$

- Conjugates have several interesting properties:

$$\begin{aligned} \overline{z_1 + z_2} &= \bar{z}_1 + \bar{z}_2 \\ \overline{z_1 - z_2} &= \bar{z}_1 - \bar{z}_2 \\ \overline{z_1 z_2} &= \bar{z}_1 \bar{z}_2 \\ \frac{z_1}{z_2} &= \frac{\bar{z}_1}{\bar{z}_2}. \end{aligned}$$

- The sum and product of a complex number $z = x + iy$ with its conjugate are real numbers

$$\begin{aligned} z + \bar{z} &= 2x \\ z\bar{z} &= x^2 + y^2 \end{aligned}$$

while the difference between a complex number and its conjugate is a pure imaginary number

$$z - \bar{z} = 2iy.$$

- The above properties let us define

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2} \text{ and } \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}.$$

- The **complex plane** or **z -plane** is a coordinate system where the horizontal or x -axis is called the **real axis** and the vertical or y -axis is called the **imaginary axis**. Complex numbers can be plotted in this coordinate system by considering their real and imaginary parts an ordered pair corresponding their position.
- The **modulus** or **absolute value** of a complex number $z = x + iy$ denoted by $|z|$ is the real number

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$$

This is the distance between z and the origin in the complex plane.

- If you consider two numbers in the complex plane as vectors, the length of their sum can't be longer than their individual lengths combined

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

This extends to any finite sum

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

and is known as the **triangle inequality**.

17.2 Powers and Roots

- A complex number can be expressed in **polar form**

$$z = (r \cos \theta) + i(r \sin \theta)$$

where $r = |z|$ is the nonnegative modulus of z and $\theta = \arg z$ is the **argument** of z — the angle between z and the positive real axis measured in the counterclockwise direction.

- The argument of a complex number z isn't unique as any multiply of 2π can be added to it. The **principle argument** of z denoted $\operatorname{Arg} z$ is the argument of z restricted to the intercal $-\pi \leq \operatorname{Arg} z \leq \pi$.
- Multiplication and division of complex numbers is simpler in polar form. For two complex numbers $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ we get

$$\begin{aligned} z_1 z_2 &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]. \end{aligned}$$

- The above formulas can be used to find integer powers of a complex number z

$$z^n = r^n(\cos n\theta + i \sin n\theta)$$

where n is an integer (including negative integers).

- **DeMoivre's formula** is a special case of the above where $r = 1$ so

$$z^n = (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

- A number w is said to be an **n th root** of a nonzero complex number z if $w^n = z$. The n th roots of a nonzero complex number $z = r(\cos \theta + i \sin \theta)$ are

$$w_k = r^{1/n} \left[\cos \left(\frac{\theta + 2k\pi}{n} \right) + i \sin \left(\frac{\theta + 2k\pi}{n} \right) \right]$$

where $k = 0, 1, 2, \dots, n-1$.

- The root w of a complex number z obtained by using the principle argument of z with $k = 0$ is called the **principle n th root** of z .
- Since the n th roots of a complex number have the same modulus they lie on a circle of radius $r^{1/n}$. The arguments of subsequent roots differ by $2\pi/n$ so they're also equally spaced around the circle.

17.3 Sets in the Complex Plane

- The points $z = x + iy$ that satisfy the equation

$$|z - z_0| = \rho$$

for $\rho > 0$ lie on a circle of radius ρ centred at the point z_0 .

- The points z satisfying the inequality $|z - z_0| < \rho$ for $\rho > 0$ lie within, but not on, a circle of radius ρ centered at the point z_0 . This set is called a **neighborhood** of z_0 or an **open disk**.
- A point z_0 is said to be an **interior point** of a set S of the complex plane if there exists some neighborhood of z_0 that lies entirely within S .
- If every point z of a set S is an interior point, then S is said to be an **open set**. An example of a set that isn't open is the set of points satisfying the inequality $\operatorname{Re}(z) \geq 0$. This isn't open because it includes the line $\operatorname{Re}(z) = 0$ and no points on that line are interior to the set because, no matter what ρ you choose, some points in the neighborhood have $\operatorname{Re}(z) < 0$.
- If every neighborhood of a point z_0 contains at least one point that is in a set S and at least one point that is not in S , then z_0 is said to be a **boundary point** of S .

- The **boundary** of a set S in the complex plane is the set of all boundary points of S .
- If any pair of points in a set S can be connected by a polygonal line that lies entirely within the set, then S is said to be **connected**.
- An open connected set is called a **domain**.
- A **region** is a set in the complex plane with all, some, or none of its boundary points. A region containing all of its boundary points is said to be **closed**.

17.4 Functions of a Complex Variable

- A **function** f from a set A to a set B is a rule of correspondence that assigns to each element of A one and only one element of B .
- If b is the element of B assigned to the element a of A , b is said to be the **image** of a and is denoted $b = f(a)$.
- The set A is called the **domain** of f .
- The set of all images in B is called the **range** of f .
- If A is a set of real numbers, f is said to be a **function of a real variable** x .
- If A is a set of complex numbers, f is said to be a **function of a complex variable** z or a **complex function**.
- The image w of a complex number z is

$$w = f(z) = u(x, y) + iv(x, y)$$

where u and v are the real and imaginary parts of w and are real-valued functions.

- Although we cannot draw a graph of a complex function $w = f(z)$ (because it would require a four-dimensional coordinate system), it can be interpreted as a **mapping** or **transformation** from the z plane to the w plane.
- A complex function may be interpreted as a two-dimensional fluid flow by considering $w = f(z)$ as the fluid velocity vector at the point z . In that case, if $x(t) + iy(t)$ is a parametric representation of a particle's position over time then

$$\begin{aligned}\frac{dx}{dt} &= u(x, y) \\ \frac{dy}{dt} &= v(x, y)\end{aligned}$$

and the family of solutions to this system of differential equations are called the **streamlines** of the flow associated with $f(z)$.

Definition 17.4.1 Limit of a Function

Suppose the function f is defined in some neighborhood of z_0 , except possibly at z_0 itself. Then f is said to possess a **limit** at z_0 , written

$$\lim_{z \rightarrow z_0} f(z) = L$$

if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

- For a function f of a real variable x , the limit $\lim_{x \rightarrow x_0} f(x) = L$ means f approaches L as you approach from both the left and right. If however f is a function of a complex variable it means f approaches L as you approach from any direction in the complex plane.

Theorem 17.4.1 Limit of Sum, Product, Quotient

Suppose $\lim_{z \rightarrow z_0} f(z) = L_1$ and $\lim_{z \rightarrow z_0} g(z) = L_2$. Then

$$(i) \quad \lim_{z \rightarrow z_0} [f(z) + g(z)] = L_1 + L_2$$

$$(ii) \quad \lim_{z \rightarrow z_0} f(z)g(z) = L_1L_2$$

$$(iii) \quad \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{L_1}{L_2}, \quad L_2 \neq 0.$$

- A function f is continuous at a point z_0 if

$$\lim_{z \rightarrow z_0} f(z) = f(z_0).$$

- A function f defined by

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0, \quad a_n \neq 0$$

where n is a nonnegative integer and the coefficients a_i , $i = 0, 1, \dots, n$, are complex constants is called a **polynomial** of degree n .

- Polynomials are continuous on the entire complex plane.

- A **rational function**

$$f(z) = \frac{g(z)}{h(z)}$$

is continuous everywhere $h(z) \neq 0$.

Definition 17.4.3 Derivative

Suppose the complex function f is defined in a neighborhood of a point z_0 . The **derivative** of f at z_0 is

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad (3)$$

provided this limit exists.

- In order for a complex function to be differentiable, the limit must approach the same value from every direction. This is a greater demand than in real variables. If you take an arbitrary complex function, there's a good chance it isn't differentiable.

Definition 17.4.4 Analyticity at a Point

A complex function $w = f(z)$ is said to be **analytic at a point** z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0 .

- Analyticity at a point is a neighborhood property. A function can be differentiable at a point but if the neighboring points aren't also differentiable, it's not analytic at that point.
- A function is analytic in a domain D if it is analytic at every point in D .
- A function that is analytic everywhere is called an **entire function**.

17.5 Cauchy-Riemann Equations

Theorem 17.5.1 Cauchy–Riemann Equations

Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivatives of u and v exist and satisfy the **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad (1)$$

- If a complex function $f(z) = u(x, y) + iv(x, y)$ is analytic throughout a domain D , then the real functions u and v must satisfy the Cauchy–Riemann equations at every point in D .

Theorem 17.5.2 Criterion for Analyticity

Suppose the real-valued functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a domain D . If u and v satisfy the Cauchy–Riemann equations at all points of D , then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D .

- The Cauchy-Riemann equations are derived assuming the function is differentiable at a particular point. That being the case, they can also be used as a formula for the derivative of the function

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

- Because analyticity implies differentiability, theorem 17.5.2 can also be used to determine if a function is differentiable at a point.
- A real-valued function $\phi(x, y)$ that has continuous second-order partial derivatives in a domain D and satisfies Laplace's equation is said to be **harmonic** in D .
- If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D then the functions $u(x, y)$ and $v(x, y)$ are harmonic functions.
- If a given function $u(x, y)$ is harmonic in a domain D it is sometimes possible to find another function $v(x, y)$ that is harmonic in D such that $u(x, y) + iv(x, y)$ is analytic in D . The function v is called the **harmonic conjugate function** of u .
- To find the harmonic conjugate function of a given function u :
 1. Take the first-order partial derivatives of u with respect to x and y .
 2. If $u(x, y) + iv(x, y)$ is analytic in a domain D then u and v must satisfy the Cauchy-Riemann equations in D from which we can find expressions for $\partial v / \partial x$ and $\partial v / \partial y$.
 3. Integrate $\partial v / \partial x$ with respect to x to get an expression for v with an unknown constant $h(y)$.
 4. Take the first-order partial derivative of v with respect to y , equate it with the other expression for $\partial v / \partial y$, and solve for $h'(y)$.
 5. Integrate $h'(y)$ and substitute the result to find v .

17.6 Exponential and Logarithmic Functions

- The exponential function for complex numbers is defined as

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

- e^z is analytic for all z , i.e. it's an entire function.
- Like its real-valued counterpart,

$$\begin{aligned} \frac{d}{dz} e^z &= e^z, \\ e^{z_1} e^{z_2} &= e^{z_1+z_2}, \end{aligned}$$

and

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}.$$

- Since

$$e^{2\pi i} = \cos(2\pi) + i \sin(2\pi) = 1$$

and

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z$$

the complex function $f(z) = e^z$ is **periodic** with complex period $2\pi i$. Because of this complex periodicity an infinite horizontal strip of height 2π contains all possible values for the function. The strip $-\pi < y \leq \pi$ is called the **fundamental region**.

- For $z \neq 0$ and $\theta = \arg z$,

$$\ln z = \log_e |z| + i(\theta + 2n\pi), \quad n = 0, \pm 1, \pm 2, \dots$$

This means there are infinitely many values of the logarithm of a complex number z . This makes sense as the complex exponential is periodic.

- The **principal value** of $\ln z$ is the complex logarithm corresponding to $n = 0$ and $\theta = \text{Arg } z$. It is denoted $\text{Ln } z$.
- Some familiar properties of the real-valued logarithm hold for the complex-valued logarithm, e.g.

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$

and

$$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2,$$

however they don't necessarily hold for the principal value.

- $\text{Ln } z$ is discontinuous and thus not analytic at $z = 0$ because $\ln z$ is undefined at $z = 0$ and on the negative real axis because $\text{Arg } z$ is discontinuous there.
- The derivative of $\text{Ln } z$ is

$$\frac{d}{dz} \text{Ln } z = \frac{1}{z}.$$

- The complex power of a complex number is defined as

$$z^\alpha = e^{\alpha \ln z}, \quad z \neq 0.$$

In general this is multiple-valued because $\ln z$ is multiple-valued — only if $\alpha = n$, $n = 0, \pm 1, \pm 2, \dots$ is it single-valued. If $\ln z$ is replaced with $\text{Ln } z$ then we get the **principle value** of z^α .

17.7 Trigonometric and Hyperbolic Functions

Definition 17.7.1 Trigonometric Sine and Cosine

For any complex number $z = x + iy$,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \quad (2)$$

- The other trigonometric functions ($\tan z$, etc.) are defined as usual.
- Because e^{iz} and e^{-iz} are entire functions, $\sin z$ and $\cos z$ are also entire functions.
- $\sin z = 0$ for the real numbers $z = n\pi$, $n \in \mathbb{Z}$ and $\cos z = 0$ for the real numbers $z = (2n+1)\pi/2$, $n \in \mathbb{Z}$. This means that $\tan z$ and $\sec z$ are analytic except at the points where $\cos z = 0$ and $\cot z$ and $\csc z$ are analytic except at the points where $\sin z = 0$.
- The usual derivatives and trigonometric functions are still valid in the complex case.
- $\sin z$ can be expressed as

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

and $\cos z$ can be expressed as

$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$

- The only zeroes of $\sin z$ are the real numbers $z = n\pi$, $n \in \mathbb{Z}$ and the only zeroes of $\cos z$ are the real numbers $z = (2n+1)\pi/2$, $n \in \mathbb{Z}$.

Definition 17.7.2 Hyperbolic Sine and Cosine

For any complex number $z = x + iy$,

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2}. \quad (10)$$

- The complex trigonometric functions can be expressed in terms of the complex hyperbolic functions and vice versa

$$\begin{aligned} \sin z &= -i \sinh(iz), & \cos z &= \cosh(iz) \\ \sinh z &= -i \sin(iz), & \cosh z &= \cos(iz). \end{aligned}$$

- $\sinh z$ can be expressed as

$$\sinh z = \sinh x \cos y + i \cosh x \sin y$$

and $\cosh z$ can be expressed as

$$\cosh z = \cosh x \cos y + i \sinh x \sin y.$$

- The zeroes of $\sinh z$ are $z = n\pi i$, $n \in \mathbb{Z}$ and the zeroes of $\cosh z$ are $z = (2n+1)\pi i/2$, $n \in \mathbb{Z}$.
- $\sin z$ and $\cos z$ are 2π periodic while $\sinh z$ and $\cosh z$ are $2\pi i$ periodic.

17.8 Inverse Trigonometric and Hyperbolic Functions

- Because the complex trigonometric functions are multi-valued, their inverse functions are also multi-valued.
- The definitions of those inverse functions are

$$\begin{aligned}\arcsin z &= -i \ln[iz + (1 - z^2)^{1/2}], \\ \arccos z &= -i \ln[z + i(1 - z^2)^{1/2}], \text{ and} \\ \arctan z &= \frac{i}{2} \ln \frac{i+z}{i-z}.\end{aligned}$$

- The derivatives of the inverse trigonometric functions are

$$\begin{aligned}\frac{d}{dz} \arcsin z &= \frac{1}{(1 - z^2)^{1/2}}, \\ \frac{d}{dz} \arccos z &= \frac{-1}{(1 - z^2)^{1/2}}, \text{ and} \\ \frac{d}{dz} \arctan z &= \frac{1}{1 + z^2}.\end{aligned}$$

- The definitions of the hyperbolic inverse functions and their derivatives are

$$\sinh^{-1}z = \ln[z + (z^2 + 1)^{1/2}]$$

$$\cosh^{-1}z = \ln[z + (z^2 - 1)^{1/2}]$$

$$\tanh^{-1}z = \frac{1}{2} \ln \frac{1+z}{1-z}$$

$$\frac{d}{dz} \sinh^{-1}z = \frac{1}{(z^2 + 1)^{1/2}}$$

$$\frac{d}{dz} \cosh^{-1}z = \frac{1}{(z^2 - 1)^{1/2}}$$

$$\frac{d}{dz} \tanh^{-1}z = \frac{1}{1 - z^2}.$$

18 Integration in the Complex Plane

18.1 Contour Integrals

- In complex variables, a piecewise smooth curve C is called a **contour** or **path**. An integral of a complex function $f(z)$ on C is denoted $\int_C f(z) dz$ or $\oint_C f(z) dz$ if C is closed — this is called a **contour integral** or a **complex integral**.

1. Let $f(z) = u(x, y) + iv(x, y)$ be defined at all points on a smooth curve C defined by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$.
2. Divide C into n subarcs according to the partition $a = t_0 < t_1 < \dots < t_n = b$ of $[a, b]$. The corresponding points on the curve C are $z_0 = x_0 + iy_0 = x(t_0) + iy(t_0)$, $z_1 = x_1 + iy_1 = x(t_1) + iy(t_1)$, \dots , $z_n = x_n + iy_n = x(t_n) + iy(t_n)$. Let $\Delta z_k = z_k - z_{k-1}$, $k = 1, 2, \dots, n$.
3. Let $\|P\|$ be the **norm** of the partition, that is, the maximum value of $|\Delta z_k|$.
4. Choose a sample point $z_k^* = x_k^* + iy_k^*$ on each subarc. See [FIGURE 18.1.1](#).
5. Form the sum $\sum_{k=1}^n f(z_k^*) \Delta z_k$.

Definition 18.1.1 Contour Integral

Let f be defined at points of a smooth curve C defined by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$. The **contour integral** of f along C is

$$\int_C f(z) dz = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(z_k^*) \Delta z_k. \quad (1)$$

Theorem 18.1.1 Evaluation of a Contour Integral

If f is continuous on a smooth curve C given by $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, then

$$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt. \quad (3)$$

- If a complex function f is continuous on a smooth curve C and if $|f(z)| \leq M$ for all z on C , then

$$\left| \int_C f(z) dz \right| \leq ML,$$

where

$$L = \int_a^b |z'(t)| dt$$

is the length of C . This is sometimes called the **ML-inequality**.

- If \mathbf{T} is the unit tangent vector to a positively oriented simple closed curve C then

$$\oint_C f \cdot \mathbf{T} ds = \operatorname{Re} \left(\oint_C \overline{f(z)} dz \right)$$

is called the **circulation** around C and measures the tendency of the flow to rotate the curve C .

- If \mathbf{N} is the normal vector to a positive oriented simple closed curve C then

$$\oint_C f \cdot \mathbf{N} ds = \operatorname{Im} \left(\oint_C \overline{f(z)} dz \right)$$

is called the **net flux** across C and measures the difference between the rates at which fluid enters and exits the region bounded by C .

18.2 Cauchy-Goursat Theorem

- A domain D is said to be **simply connected** if every simple closed contour C lying entirely in D can be shrunk to a point without leaving D , i.e. the domain has no holes in it.
- A domain that is not simply connected is called a **multiply connected domain**. A domain with one hole is called **doubly connected**, a domain with two holes **triply connected**, etc.

Theorem 18.2.1 Cauchy-Goursat Theorem

Suppose a function f is analytic in a simply connected domain D . Then for every simple closed contour C in D , $\oint_C f(z) dz = 0$.

- An alternative way of stating the Cauchy-Goursat Theorem is: if f is analytic at all points on and within a simple closed contour C , then $\oint_C f(z) dz = 0$.
- If D is a double connected domain and C and C_1 are simple closed contours such that C_1 surrounds the “hole” in the domain and is interior to C , then

$$\oint_C f(z) dz = \oint_{C_1} f(z) dz.$$

This is called the principle of **deformation of contours** since C_1 can be considered a continuous deformation of the contour C (or vice versa) under which the value of the integral doesn’t change.

- If z_0 is a constant complex number interior to a simple closed contour C , then

$$\oint_C \frac{dz}{(z - z_0)^n} = \begin{cases} 2\pi i & n = 1 \\ 0 & n \text{ an integer} \neq 1 \end{cases}.$$

Theorem 18.2.2 Cauchy–Goursat Theorem for Multiply Connected Domains

Suppose C, C_1, \dots, C_n are simple closed curves with a positive orientation such that C_1, C_2, \dots, C_n are interior to C but the regions interior to each $C_k, k = 1, 2, \dots, n$, have no points in common. If f is analytic on each contour and at each point interior to C but exterior to all the $C_k, k = 1, 2, \dots, n$, then

$$\oint_C f(z) dz = \sum_{k=1}^n \oint_{C_k} f(z) dz. \quad (6)$$

18.3 Independence of the Path

- Let z_0 and z_1 be points in a domain D . A contour integral $\int_C f(z) dz$ is said to be **independent of the path** if its value is the same for all contours C in D with an initial point z_0 and a terminal point z_1 .
- If f is an analytic function in a simply connected domain D , then $\int_C f(z) dz$ is independent of path C .
- Suppose f is continuous in a domain D . If there exists a function F such that $F'(z) = f(z)$ for each z in D , then F is called the **antiderivative** of f .
- The general antiderivative of a complex function includes a complex integration constant.
- Suppose f is continuous in a domain D and F is an antiderivative of f in D . Then for any contour C in D with initial point z_0 and terminal point z_1 ,

$$\int_C f(z) dz = F(z_1) - F(z_0).$$

- A consequence of the above is that if C is closed, then

$$\oint_C f(z) dz = 0.$$

- If f is analytic in a simply connected domain D , then f has an antiderivative in D ; this, there exists a function F such that $F'(z) = f(z)$ for all z in D .
- Suppose f and g are analytic in a simply connected domain D that contains the contour C . If z_0 and z_1 are the initial and terminal points of C , then the **integration by parts** formula is valid in D :

$$\int_{z_0}^{z_1} f(z)g'(z) dz = f(z)g(z)|_{z_0}^{z_1} - \int_{z_0}^{z_1} f'(z)g(z) dz.$$

18.4 Cauchy's Integral Formulas

Theorem 18.4.1 Cauchy's Integral Formula

Let f be analytic in a simply connected domain D , and let C be a simple closed contour lying entirely within D . If z_0 is any point within C , then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz. \quad (1)$$

- Cauchy's integral formula is useful when a contour integral has the form

$$\oint \frac{f(z)}{z - z_0} dz$$

in which case you know its value is $2\pi i f(z_0)$.

Theorem 18.4.2 Cauchy's Integral Formula for Derivatives

Let f be analytic in a simply connected domain D , and let C be a simple closed contour lying entirely within D . If z_0 is any point interior to C , then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz. \quad (6)$$

- Cauchy's integral formula for derivatives is useful when a contour integral has the form

$$\oint \frac{f(z)}{(z - z_0)^{n+1}} dz$$

in which case you know its value is $\frac{2\pi i}{n!} f^{(n)}(z_0)$.

- **Liouville's theorem** states that the only bounded entire functions are constants.

19 Series and Residues

19.1 Sequences and Series

- A **sequence** is a function whose domain is the set of positive integers, i.e. for each integer $n = 1, 2, 3, \dots$ we assign a complex number z_n .
- If $\lim_{n \rightarrow \infty} z_n = L$ we say the sequence $\{z_n\}$ is **convergent**. In other words, $\{z_n\}$ converges to the number L if, for every positive number ε , an N can be found such that $|z_n - L| < \varepsilon$ whenever $n > N$.
- A sequence $\{z_n\}$ converges to a complex number L if and only if $\operatorname{Re}(z_n)$ converges to $\operatorname{Re}(L)$ and $\operatorname{Im}(z_n)$ converges to $\operatorname{Im}(L)$.
- An **infinite series** of complex numbers

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + \dots$$

is **convergent** if the sequence of partial sums $\{S_n\}$, where

$$S_n = z_1 + z_2 + \dots + z_n$$

converges. If $S_n \rightarrow L$ as $n \rightarrow \infty$, we say that the **sum** of the series is L .

- The sum of the geometric series

$$\sum_{k=1}^{\infty} az^{k-1} = a + az + az^2 + \dots$$

converges to

$$\frac{a}{1 - z}$$

when $|z| < 1$ and diverges otherwise.

- If $\sum_{k=1}^{\infty} z_k$ converges, then $\lim_{n \rightarrow \infty} z_n = 0$.
- If $\lim_{n \rightarrow \infty} z_n \neq 0$ then the series $\sum_{k=1}^{\infty} z_k$ diverges.
- An infinite series $\sum_{k=1}^{\infty} z_k$ is said to be **absolutely convergent** if $\sum_{k=1}^{\infty} |z_k|$ converges. Absolute convergence implies convergence.

Theorem 19.1.4 Ratio Test

Suppose $\sum_{k=1}^{\infty} z_k$ is a series of nonzero complex terms such that

$$\lim_{n \rightarrow \infty} \left| \frac{z_{n+1}}{z_n} \right| = L. \quad (9)$$

- (i) If $L < 1$, then the series converges absolutely.
- (ii) If $L > 1$ or $L = \infty$, then the series diverges.
- (iii) If $L = 1$, the test is inconclusive.

Theorem 19.1.5 Root Test

Suppose $\sum_{k=1}^{\infty} z_k$ is a series of complex terms such that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|z_n|} = L. \quad (10)$$

- (i) If $L < 1$, then the series converges absolutely.
- (ii) If $L > 1$ or $L = \infty$, then the series diverges.
- (iii) If $L = 1$, the test is inconclusive.

- An infinite series of the form

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$$

where the coefficients a_k are complex constants is called a **power series** in $z - z_0$. The power series is said to be **centred at z_0** , and the complex point z_0 is referred to as the **centre** of the series.

- Every complex power series has a **radius of convergence R** where R is a real number. The power series converges for all z within the **circle of convergence** $|z - z_0| < R$ and diverges for $|z - z_0| > R$. The series may converge at some, all, or none of the points on the actual circle of convergence.
- For a power series

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

the ratio test depends only on the coefficients a_k . If

1. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \neq 0$, the radius of convergence is $R = 1/L$;
2. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$, the radius of convergence is ∞ ;
3. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, the radius of convergence is $R = 0$.

19.2 Taylor Series

- A power series $\sum_{k=1}^{\infty} a_k (z - z_0)^k$ has a radius of convergence R . For each complex number z within the circle of convergence, when substituted into the power series it converges to a unique value L . This defines a function f that maps each z to its corresponding L .

Theorem 19.2.1 Continuity

A power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ represents a continuous function f within its circle of convergence $|z - z_0| = R, R \neq 0$.

Theorem 19.2.2 Term-by-Term Integration

A power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ can be integrated term by term within its circle of convergence $|z - z_0| = R, R \neq 0$, for every contour C lying entirely within the circle of convergence.

Theorem 19.2.3 Term-by-Term Differentiation

A power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ can be differentiated term by term within its circle of convergence $|z - z_0| = R, R \neq 0$.

Theorem 19.2.4 Taylor's Theorem

Let f be analytic within a domain D and let z_0 be a point in D . Then f has the series representation

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \quad (8)$$

valid for the largest circle C with center at z_0 and radius R that lies entirely within D .

- The radius of convergence of a Taylor series is the distance from the centre z_0 to the nearest isolated singularity: a point at which the series fails to be analytic but is analytic at all points in some neighborhood of the point.

19.3 Laurent Series

- If a complex function f fails to be analytic at a point $z = z_0$, then this point is said to be a **singularity** or a **singular point** of the function.
- Suppose $z = z_0$ is a singularity of a complex function f . It is said to be an **isolated singularity** if there exists some **deleted neighborhood**, or **punctured open disk**, $0 < |z - z_0| < R$ of z_0 in which f is analytic.
- A singular point $z = z_0$ of a complex function f is said to be **nonisolated** if every neighborhood of z_0 contains at least one singularity of f other than z_0 .

Theorem 19.3.1 Laurent's Theorem

Let f be analytic within the annular domain D defined by $r < |z - z_0| < R$. Then f has the series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \quad (3)$$

valid for $r < |z - z_0| < R$. The coefficients a_k are given by

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} ds, \quad k = 0, \pm 1, \pm 2, \dots, \quad (4)$$

where C is a simple closed curve that lies entirely within D and has z_0 in its interior (see [FIGURE 19.3.1](#)).

- Under Laurent's theorem, the part of $f(z)$ with negative powers of $z - z_0$ is called the **principle part** and the part with positive powers is called the **analytic part**.
- The coefficient formula of theorem 19.3.1 isn't used often. Generally f is decomposed into functions for which the series are known (e.g. $\cos z$, e^z , etc.), and those series are combined to find the Laurent series.