Contents

1 First-order ODEs

Form: IVP

$$\frac{dy}{dx} = f(x,y)$$
$$y(x_0) = y_0$$

Test: $f(x,y)$ and $\partial f/\partial y$ are continuous over I
Property: A unique solution is guaranteed over I

1.1 Separable Equations

Form:

$$\frac{dy}{dx} = g(x)h(y)$$

Solution: Divide by $h(y)$ then integrate with respect to x.

$$\begin{aligned} \frac{dy}{dx} &= g(x)h(y) \\ \frac{1}{h(y)}\frac{dy}{dx} &= g(x) \\ \int \frac{1}{h(y)}\frac{dy}{dx}\,dx &= \int g(x)\,dx \\ \int \frac{1}{h(y)}\,dy &= \int g(x)\,dx \\ H(y) &= G(x) + c \end{aligned}$$

1.2 Linear Equations

Form:

$$\frac{dy}{dx} + P(x)y = f(x)$$

Solution:

1. Determine the integrating factor $e^{\int P(x)\,dx}$
2. Multiply by the integrating factor
3. Recognise that the left hand side of the equation is the derivative of the product of the integrating factor and y
4. Integrate both sides of the equation
5. Solve for y

1.3 Exact Equations

Form:

$$z = f(x,y) = c$$

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = M(x,y)\,dx + N(x,y)\,dy = 0$$

Test:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution:

1. Integrate $M(x,y)$ with respect to x to find an expression for $z = f(x,y)$

$$\begin{aligned} \frac{\partial f}{\partial x} &= M(x,y) \\ f(x,y) &= \int M(x,y)\,dx + g(y) \end{aligned}$$

2. Differentiate $f(x,y)$ with respect to y and equate it to $N(x,y)$ to find $g'(y)$

$$\begin{aligned} \frac{\partial f}{\partial y} = N(x,y) &= \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y) \\ g'(y) &= N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx \end{aligned}$$

3. Integrate $g'(y)$ with respect to y to find $g(y)$ and substitute it into $f(x,y)$
4. Equate $f(x,y)$ with an unknown constant c

Note: The steps can be performed with x and y reversed, i.e. start by integrating $N(x,y)$ with respect to y, etc.

1.4 Exact Equations with Integration Constant

Form:

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Test: $(M_y - N_x)/N$ is a function of x alone or $(N_x - M_y)/M$ is a function of y alone
Solution:

1. Compute the integrating factor
$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\,dx}$$
or
$$\mu(y) = e^{\int \frac{N_x - M_y}{M}\,dy}$$
as appropriate

2. Multiple the equation by this factor

3. The equation is now exact and can be solved as above

1.5 Homogeneous Equations

Form:

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Test: M and N are homogeneous functions of the same degree
Solution:

1. Rewrite as
$$M(x,y) = x^\alpha M(1,u) \text{ and } N(x,y) = x^\alpha N(1,u) \text{ where } u = y/x$$
or
$$M(x,y) = y^\alpha M(v,1) \text{ and } N(x,y) = y^\alpha N(v,1) \text{ where } v = x/y$$

2. Substitute $y = ux$ and $dy = u\,dx + x\,du$ or $x = vy$ and $dx = v\,dy + y\,dv$ as appropriate

3. Solve the resulting first-order separable DE

4. Substitude $u = y/x$ or $v = x/y$ as appropriate

1.6 Bernoulli's Equation

Form:

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

Test: $n \neq 0$ and $n \neq 1$
Solution:

1. Substitude $y = u^{1/(1-n)}$ and $\frac{dy}{dx} = \frac{d}{dx}(u^{1/(1-n)})$
2. Solve the resulting linear equation
3. Substitude $u = y^{n-1}$

1.7 Reduction to Separation of Variables

Form:

$$\frac{dy}{dx} = f(Ax + By + C), B \neq 0$$

Solution:

1. Substitute
$$Ax + By + C = u$$
2. Solve the resulting separable equation
3. Substitute
$$u = Ax + By + C$$