Quantum Computation and Quantum Information by Michael A. Nielsen and Isaac L. Chuang Problems

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Part I
Fundamental concepts

2 Linear algebra

Exercise 2.1

$$(1,-1)+(1,2)-(2,1)=(0,0)$$

Exercise 2.2

$$A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

Using the basis $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$ we get

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$|1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$a_{00} + a_{01} = 1$$

$$a_{10} + a_{11} = -1$$

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$a_{00} - a_{01} = 1$$

$$a_{10} - a_{11} = 1$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Exercise 2.5

$$\begin{aligned}
\begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} &= \begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \left(\begin{bmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ z_2 \\ \vdots \\ 0 \end{bmatrix} + \dots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ z_n \end{bmatrix} \right) \\
&= \begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \begin{bmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + \begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ z_n \end{bmatrix} \\
&= z_1 \begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + z_n \begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} y_1^* & \dots & y_n^* \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} &= y_1^* z_1 + y_2^* z_2 + \dots + y_n^* z_n \\
&= (y_1 z_1^* + y_2 z_2^* + \dots + y_n z_n^*)^* \\
&= \left(\begin{bmatrix} z_1^* & \cdots & z_n^* \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right)^*
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} v_1^* & \cdots & v_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} &= |v_1|^2 + \dots + |v_n|^2 \\
&\geq 0
\end{aligned}$$

Exercise 2.6

$$\begin{aligned}
\left(\sum_i \lambda_i \left| w_i \right\rangle, \left| v \right\rangle \right) &= \left(\left| v \right\rangle, \sum_i \lambda_i \left| w_i \right\rangle \right)^* \\
&= \left(\sum_i \lambda_i (\left| v \right\rangle, \left| w_i \right\rangle) \right)^* \\
&= \sum_i \lambda_i^* (\left| v \right\rangle, \left| w_i \right\rangle)^* \\
&= \sum_i \lambda_i^* (\left| w_i \right\rangle, \left| v \right\rangle)
\end{aligned}$$

Exercise 2.7

$$\begin{aligned}
\langle w|v\rangle &= \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}\\
&= (1)(1) + (1)(-1)\\
&= 0\\
\frac{|w\rangle}{||\,|w\rangle\,||} &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\\
\frac{|v\rangle}{||\,|v\rangle\,||} &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}
\end{aligned}$$

Exercise 2.9

$$\begin{aligned}
\sigma_0 &= |0\rangle\langle 0| + |1\rangle\langle 1|\\
\sigma_1 &= |1\rangle\langle 0| + |0\rangle\langle 1|\\
\sigma_2 &= i\,|1\rangle\langle 0| - i\,|0\rangle\langle 1|\\
\sigma_3 &= |0\rangle\langle 0| - |1\rangle\langle 1|
\end{aligned}$$

Exercise 2.11

$$\begin{aligned}
\begin{vmatrix}-\lambda & 1\\ 1 & -\lambda\end{vmatrix} &= \lambda^2 - 1\\
\lambda &= \pm 1\\
\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} &= \begin{bmatrix}a\\b\end{bmatrix}\\
b &= a\\
a &= b\\
X_1 &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\\
\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} &= \begin{bmatrix}-a\\-b\end{bmatrix}\\
b &= -a\\
a &= -b\\
X_2 &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}
\end{aligned}$$

Exercise 2.12

$$\begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$$
$$\lambda_1 = 1$$
$$\lambda_2 = 1$$

The eigenvalue 1 is degenerate. Because the matrix only has one eigenvector it can't diagonalised.

Exercise 2.13

$$(|w\rangle \langle v|)^\dagger = \langle v|^\dagger \, |w\rangle^\dagger = |v\rangle \langle w|$$

Exercise 2.16

$$\begin{aligned} P^2 &= \left(\sum_{i=1}^{k} |i\rangle \langle i|\right) \left(\sum_{j=1}^{k} |j\rangle \langle j|\right) \\ &= \sum_{i=j=1}^{k} |i\rangle \langle i|j\rangle \langle j| \\ &= \sum_{i=j=1}^{k} |i\rangle \delta_{ij} \langle j| \\ &= \sum_{i=1}^{k} |i\rangle \langle i| \\ &= P \end{aligned}$$

Exercise 2.17

$$\begin{aligned} A &= A^\dagger \\ \sum_i \lambda_i |i\rangle \langle i| &= \left(\sum_i \lambda_i |i\rangle \langle i|\right)^\dagger \\ &= \sum_i \lambda_i^* |i\rangle \langle i| \end{aligned}$$

$\lambda_i = \lambda_i^*$ implies the eigenvalues are real.

Exercise 2.18

$$
\begin{aligned}
U^\dagger U &= I \\
\left(\sum_i \lambda_i \left|i\right\rangle \left\langle i\right|\right)^\dagger \left(\sum_i \lambda_j \left|j\right\rangle \left\langle j\right|\right) &= \sum_k \left|k\right\rangle \left\langle k\right| \\
\sum_{ij} \lambda_i^* \lambda_j \left|i\right\rangle \left\langle i|j\right\rangle \left\langle j\right| &= \sum_k \left|k\right\rangle \left\langle k\right| \\
\sum_{ij} \lambda_i^* \lambda_j \left|i\right\rangle \delta_{ij} \left\langle j\right| &= \sum_k \left|k\right\rangle \left\langle k\right| \\
\sum_i \left|\lambda_i\right|^2 \left|i\right\rangle \left\langle i\right| &= \sum_k \left|k\right\rangle \left\langle k\right| \\
\left|\lambda_i\right|^2 &= 1 \\
\lambda_i &= e^{i\theta}
\end{aligned}
$$

Exercise 2.19

$$\begin{aligned}
I^\dagger &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^\dagger \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I \\
I^\dagger I &= II \\
&= I \\
X^\dagger &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^\dagger \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= X \\
X^\dagger X &= XX \\
&= I \\
Y^\dagger &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^\dagger \\
&= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= Y \\
Y^\dagger Y &= YY \\
&= I \\
Z^\dagger &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^\dagger \\
&= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= Z \\
Z^\dagger Z &= ZZ \\
&= I
\end{aligned}$$

Exercise 2.22

$$
\begin{aligned}
\langle v_1|A|v_2\rangle &= \langle v_1|Av_2\rangle \\
&= \langle v_1|\lambda_2 v_2\rangle \\
&= \lambda_2 \langle v_1|v_2\rangle \\
\langle v_1|A|v_2\rangle &= \langle A^\dagger v_1|v_2\rangle \\
&= \langle Av_1|v_2\rangle \\
&= \langle \lambda_1 v_1|v_2\rangle \\
&= \lambda_1 \langle v_1|v_2\rangle \\
0 &= (\lambda_2 - \lambda_1) \langle v_1|v_2\rangle \\
&= \langle v_1|v_2\rangle
\end{aligned}
$$

Exercise 2.23

For each basis vector $|i\rangle, i = 1, \ldots k$, $P|i\rangle = |i\rangle$ and so they are eigenvectorrs of P with eigenvalue 1. For each basis vector $|j\rangle, j = k+1, \ldots, d$, $P|j\rangle = 0$ and so they are eigenvectors of P with eigenvalue of 0. That is a total of d eigenvectors so all eigenvalues are either 0 or 1.

Exercise 2.26

$$
\begin{aligned}
|\psi\rangle &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\
|\psi\rangle^{\otimes 2} &= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \\
&= \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} \\
|\psi\rangle^{\otimes 3} &= \left(\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \\
&= \frac{|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle}{2^{3/2}} \\
|\psi\rangle &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
|\psi\rangle^{\otimes 2} &= \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
|\psi\rangle^{\otimes 3} &= \frac{1}{2^{3/2}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

Exercise 2.27

(a)

$$
\begin{aligned}
X &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
Z &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
X \otimes Z &= \begin{bmatrix} (0)Z & (1)Z \\ (1)Z & (0)Z \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

(b)

$$I \otimes X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(c)

$$X \otimes I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

No, the tensor product is not commutative.

Exercise 2.28

$$\begin{aligned}
(A \otimes B)^* &= \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{bmatrix}^* \\
&= \begin{bmatrix} A_{11}^*B^* & A_{12}^*B^* & \cdots & A_{1n}^*B^* \\ A_{21}^*B^* & A_{22}^*B^* & \cdots & A_{2n}^*B^* \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}^*B^* & A_{m2}^*B^* & \cdots & A_{mn}^*B^* \end{bmatrix} \\
&= A^* \otimes B^* \\
(A \otimes B)^T &= \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{bmatrix}^T \\
&= \begin{bmatrix} A_{11}B^T & A_{21}B^T & \cdots & A_{m1}B^T \\ A_{12}B^T & A_{22}B^T & \cdots & A_{m2}B^T \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n}B^T & A_{2n}B^T & \cdots & A_{mn}B^T \end{bmatrix} \\
&= A^T \otimes B^T \\
(A \otimes B)^\dagger &= [(A \otimes B)^*]^T \\
&= (A^* \otimes B^*)^T \\
&= (A^*)^T \otimes (B^*)^T \\
&= A^\dagger \otimes B^\dagger
\end{aligned}$$

Exercise 2.29

$$\begin{aligned}(A \otimes B)^\dagger (A \otimes B)(|a\rangle \otimes |b\rangle) &= (A^\dagger \otimes B^\dagger)(A|a\rangle \otimes B|b\rangle) \\ &= A^\dagger A|a\rangle \otimes B^\dagger B|b\rangle \\ &= |a\rangle \otimes |b\rangle\end{aligned}$$

Exercise 2.30

$$\begin{aligned}(A \otimes B)^\dagger (|a\rangle \otimes |b\rangle) &= (A^\dagger \otimes B^\dagger)(|a\rangle \otimes |b\rangle) \\ &= (A \otimes B)(|a\rangle \otimes |b\rangle)\end{aligned}$$

Exercise 2.34

$$\begin{aligned}A &= \begin{bmatrix}4 & 3 \\ 3 & 4\end{bmatrix} \\ \lambda_1 &= 1 \\ \mathbf{x}_1 &= \begin{bmatrix}-1 \\ 1\end{bmatrix} \\ \lambda_2 &= 7 \\ \mathbf{x}_2 &= \begin{bmatrix}1 \\ 1\end{bmatrix} \\ P &= \begin{bmatrix}-1 & 1 \\ 1 & 1\end{bmatrix} \\ A &= PDP^{-1} \\ D &= P^{-1}AP \\ &= \begin{bmatrix}1 & 0 \\ 0 & 7\end{bmatrix} \\ \sqrt{A} &= P\sqrt{D}P^{-1} \\ &= \frac{1}{2}\begin{bmatrix}1+\sqrt{7} & -1+\sqrt{7} \\ -1+\sqrt{7} & 1+\sqrt{7}\end{bmatrix} \\ \ln A &= P\ln(D)P^{-1} \\ &= \frac{\ln 7}{2}\begin{bmatrix}1 & 1 \\ 1 & 1\end{bmatrix}\end{aligned}$$

Exercise 2.35

$$
\begin{aligned}
\mathbf{v} &= (v_x, v_y, v_z) \\
\mathbf{v}\cdot\boldsymbol{\sigma} &= v_x\sigma_1 + v_y\sigma_2 + v_z\sigma_3 \\
&= \begin{bmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{bmatrix} \\
A &= i\theta\mathbf{v}\cdot\boldsymbol{\sigma} \\
\lambda_1 &= -i\theta v \\
\mathbf{x}_1 &= \begin{bmatrix} (v_z - v)/(v_x + iv_y) \\ 1 \end{bmatrix} \\
\lambda_2 &= i\theta v \\
\mathbf{x}_2 &= \begin{bmatrix} (v_z + v)/(v_x + iv_y) \\ 1 \end{bmatrix} \\
P &= \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} \\
\exp(i\theta\mathbf{v}\cdot\boldsymbol{\sigma}) &= P \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix} P^{-1} \\
&= \begin{bmatrix} \cos\theta + iv_z\sin\theta & i(v_x - iv_y)\sin\theta \\ i(v_x + iv_y)\sin\theta & \cos\theta - iv_z\sin\theta \end{bmatrix} \\
&= \cos\theta I + i\sin\theta \begin{bmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{bmatrix} \\
&= \cos\theta I + i\sin\theta\mathbf{v}\cdot\boldsymbol{\sigma}
\end{aligned}
$$

Exercise 2.36

$$
\begin{aligned}
\operatorname{tr} I &= 1 + 1 \\
&= 2 \\
\operatorname{tr} X &= 0 + 0 \\
&= 0 \\
\operatorname{tr} Y &= 0 + 0 \\
&= 0 \\
\operatorname{tr} Z &= 1 - 1 \\
&= 0
\end{aligned}
$$

Exercise 2.37

$$
\begin{aligned}
\operatorname{tr}(AB) &= \sum_i \langle i|AB|i\rangle \\
&= \sum_i \langle i|AIB|i\rangle \\
&= \sum_i \langle i|Ai\rangle \langle i|B|i\rangle \\
&= \sum_i \langle i|B|i\rangle \langle i|A|i\rangle \\
&= \sum_i \langle i|BIA|i\rangle \\
&= \sum_i \langle i|BA|i\rangle \\
&= \operatorname{tr}(BA)
\end{aligned}
$$

Exercise 2.38

$$
\begin{aligned}
\operatorname{tr}(A+B) &= \sum_i (A_{ii} + B_{ii}) \\
&= \sum_i A_{ii} + \sum_i B_{ii} \\
&= \operatorname{tr} A + \operatorname{tr} B \\
\operatorname{tr}(zA) &= \sum_i zA_{ii} \\
&= z \sum_i A_{ii} \\
&= z \operatorname{tr} A
\end{aligned}
$$