Advanced Engineering Mathematics Vectors, Matrices, and Vector Calculus by Dennis G. Zill Problems

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Contents

1 Vectors

1.1 Vectors in 2-Space

1.1.1

(a) $3\mathbf{a} = 6\mathbf{i} + 12\mathbf{j}$

(b) $\mathbf{a} + \mathbf{b} = \mathbf{i} + 8\mathbf{j}$

(c) $\mathbf{a} - \mathbf{b} = 3\mathbf{i}$

(d) $||\mathbf{a} + \mathbf{b}|| = \sqrt{1 + 8^2} = \sqrt{65}$

(e) $||\mathbf{a} - \mathbf{b}|| = 3$

1.1.9

(a) $4\mathbf{a} - 2\mathbf{b} = \langle 6, -14 \rangle$

(b) $-3\mathbf{a} - 5\mathbf{b} = \langle 2, 4 \rangle$

1.1.15

$\overrightarrow{P_1P_2} = \langle 2, 5 \rangle$

1.1.19

$(1, 18)$

1.1.21

(a) Yes

(b) Yes

(c) Yes

(d) No

(e) Yes

(f) Yes

1.1.25

(a) $\frac{\mathbf{a}}{||\mathbf{a}||} = \frac{\langle 2,2\rangle}{\sqrt{2^2+2^2}} = \frac{1}{2\sqrt{2}}\langle 2, 2\rangle = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$

(b) $\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle$

1.1.31

$2\frac{\mathbf{a}}{||\mathbf{a}||} = 2\frac{\langle 3,7\rangle}{\sqrt{3^2+7^2}} = \frac{2}{\sqrt{58}}\langle 3, 7\rangle = \langle \frac{6}{\sqrt{58}}, \frac{14}{\sqrt{58}}\rangle$

1.1.37

$\mathbf{x} = -(\mathbf{a} + \mathbf{b})$

1.1.41

$$\begin{aligned}
\mathbf{a} &= 2\mathbf{i} + 3\mathbf{j} \\
\mathbf{b} &= \mathbf{i} + \mathbf{j} \\
\mathbf{c} &= \mathbf{i} - \mathbf{j}
\end{aligned}$$

$$\begin{aligned}
\mathbf{i} &= \frac{1}{2}(\mathbf{b} + \mathbf{c}) \\
\mathbf{j} &= \frac{1}{2}(\mathbf{b} - \mathbf{c}) \\
\mathbf{a} &= 2\left(\frac{1}{2}(\mathbf{b} + \mathbf{c})\right) + 3\left(\frac{1}{2}(\mathbf{b} - \mathbf{c})\right) \\
&= \mathbf{b} + \mathbf{c} + \frac{3}{2}\mathbf{b} - \frac{3}{2}\mathbf{c} \\
&= \frac{5}{2}\mathbf{b} - \frac{1}{2}\mathbf{c}
\end{aligned}$$

1.1.43

$$\begin{aligned}
y &= \frac{1}{4}x^2 + 1 \\
y(2) &= 2 \\
y' &= \frac{1}{2}x \\
y'(2) &= 1 \\
\mathbf{v} &= \pm\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle
\end{aligned}$$

1.1.45

(a)

$$\begin{aligned}
\mathbf{F}_n &= \mathbf{F}\cos\theta \\
\mathbf{F}_g &= \mathbf{F}\sin\theta \\
||\mathbf{F}_f|| &= \mu||\mathbf{F}_n|| \\
||-\mathbf{F}_g|| &= \mu||\mathbf{F}_n|| \\
||-\mathbf{F}\sin\theta|| &= \mu||\mathbf{F}\cos\theta|| \\
||\mathbf{F}||\sin\theta &= \mu||\mathbf{F}||\cos\theta \\
\tan\theta &= \mu
\end{aligned}$$

(b) $\theta = \arctan\mu \approx 31^\circ$

1.1.47

$$\begin{aligned}
F_x &= \frac{qQ}{4\pi\epsilon_0}\int_{-a}^{a}\frac{L\,dy}{2a(L^2+y^2)^{3/2}} \\
&= \frac{LqQ}{8\pi\epsilon_0}\int_{-a}^{a}(L^2+y^2)^{-3/2}\,dy \\
&= \frac{LqQ}{8\pi\epsilon_0}\frac{2a}{L^2\sqrt{a^2+L^2}} \\
&= \frac{aqQ}{4\pi\epsilon_0 L\sqrt{a^2+L^2}} \\
F_y &= -\frac{qQ}{4\pi\epsilon_0}\int_{-a}^{a}\frac{y\,dy}{2a(L^2+y^2)^{3/2}} \\
&= 0 \\
\mathbf{F} &= \langle\frac{1}{4\pi\epsilon_0}\frac{qQ}{L\sqrt{a^2+L^2}}, 0\rangle
\end{aligned}$$

1.1.49

Let the three sides of the triangle be vectors **a**, **b**, and **c**. The triangle is closed so it must be the case that

$$\mathbf{a}+\mathbf{b}+\mathbf{c} = 0.$$

This gives

$$\mathbf{c} = -(\mathbf{a}+\mathbf{b}).$$

The vector from the midpoint of side **a** to the midpoint of side **b** is

$$\left(\mathbf{a}+\frac{1}{2}\mathbf{b}\right) - \frac{1}{2}\mathbf{a} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$$

which is parallel with **c** and half its length.

1.2 Vectors in 3-Space

1.2.7

A plane at $z = 5$ parellel with the x-y plane.

1.2.9

A line parallel to the z axis at $x = 2$ and $y = 3$.

1.2.13

(a) $(0, 5, 4)$, $(-2, 0, 4)$, $(-2, 5, 0)$

(b) $(-2, 5, -2)$

(c) $(3, 5, 4)$

1.2.15

The planes $x = 0$, $y = 0$, and $z = 0$.

1.2.17

$(-1, 2, -3)$

1.2.19

The planes $z = \pm 5$.

1.2.21

$\sqrt{(6-3)^2 + (4+1)^2 + (8-2)^2} = \sqrt{9 + 25 + 36} = \sqrt{70}$

1.2.31

$$\begin{aligned}
\sqrt{(2-x)^2 + (1-2)^2 + (1-3)^2} &= \sqrt{21} \\
(2-x)^2 + 1 + 4 &= 21 \\
(2-x)^2 &= 16 \\
2 - x &= \pm 4 \\
x &= 2 \pm 4 \\
&= -2 \text{ or } 6
\end{aligned}$$

1.2.33

$\left(4, \frac{1}{2}, \frac{3}{2}\right)$

1.2.37

$(-3, -6, 1)$

1.3 Dot Product

1.3.1

$\mathbf{a} \cdot \mathbf{b} = 12$

1.3.11

$\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} = \frac{12}{30}\mathbf{b} = \langle -\frac{2}{5}, \frac{4}{5}, 2\rangle$

1.3.13

$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||||\mathbf{b}|| \cos\theta = 25\sqrt{2}$

1.3.17

$$
\begin{aligned}
\mathbf{a} \cdot \mathbf{v} &= 0 \\
3x_1 + y_1 - 1 &= 0 \\
\\
\mathbf{b} \cdot \mathbf{v} &= 0 \\
-3x_1 + 2y_2 + 2 &= 0 \\
\\
3y_2 + 1 &= 0 \\
y_2 &= -\frac{1}{3} \\
\\
3x_1 - \frac{1}{3} - 1 &= 0 \\
x_1 &= \frac{4}{9} \\
\\
\mathbf{v} &= \langle \frac{4}{9}, -\frac{1}{3}, 1\rangle
\end{aligned}
$$

1.3.19

$$\begin{aligned}
\mathbf{a} \cdot \mathbf{c} &= \mathbf{a} \cdot \left(\mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||^2} \mathbf{a} \right) \\
&= \mathbf{a} \cdot \mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||^2} \mathbf{a} \cdot \mathbf{a} \\
&= 0
\end{aligned}$$

1.3.21

$$\begin{aligned}
||\mathbf{a}|| &= \sqrt{3^2 + 1^2} \\
&= \sqrt{10} \\
||\mathbf{b}|| &= \sqrt{2^2 + 2^2} \\
&= \sqrt{8} \\
&= 2\sqrt{2} \\
\mathbf{a} \cdot \mathbf{b} &= 4 \\
\theta &= \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} \\
&= \arccos \frac{4}{(\sqrt{10})(2\sqrt{2})} \\
&= \arccos \frac{1}{\sqrt{5}} \\
&\approx 63^\circ
\end{aligned}$$

1.3.25

$$\begin{aligned}
||\mathbf{a}|| &= \sqrt{1^2 + 2^2 + 3^3} \\
&= \sqrt{14} \\
\cos \alpha &= \frac{1}{\sqrt{14}} \\
\alpha &\approx 75^\circ \\
\cos \beta &= \frac{2}{\sqrt{14}} \\
\beta &\approx 58^\circ \\
\cos \gamma &= \frac{3}{\sqrt{14}} \\
\gamma &\approx 37^\circ
\end{aligned}$$

1.3.29

$$
\begin{aligned}
\overrightarrow{AD} &= \langle s, -s, s\rangle \\
||\overrightarrow{AD}|| &= s\sqrt{3} \\
\overrightarrow{AB} &= \langle s, 0, 0\rangle \\
||\overrightarrow{AB}|| &= s \\
\theta &= \arccos \frac{\overrightarrow{AD}\cdot\overrightarrow{AB}}{||\overrightarrow{AD}||||\overrightarrow{AB}||} \\
&= \arccos \frac{s^2}{s^2\sqrt{3}} \\
&= \arccos \frac{1}{\sqrt{3}} \\
&\approx 55^\circ
\end{aligned}
$$

1.3.33

$$
\begin{aligned}
\text{comp}_{\mathbf{b}}\mathbf{a} &= \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{b}||} \\
&= \frac{5}{7}
\end{aligned}
$$

1.3.37

$$
\begin{aligned}
\text{comp}_{\overrightarrow{OP}}\mathbf{a} &= \frac{\mathbf{a}\cdot\overrightarrow{OP}}{||\overrightarrow{OP}||} \\
&= \frac{72}{\sqrt{109}}
\end{aligned}
$$

1.3.39

$$
\begin{aligned}
\text{proj}_{\mathbf{b}}\mathbf{a} &= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} \\
&= \frac{35}{25}\mathbf{b} \\
&= \langle -\frac{21}{5}, \frac{28}{5}\rangle
\end{aligned}
$$

1.3.43

$$
\begin{aligned}
\mathbf{a}+\mathbf{b} &= \langle 3,4\rangle \\
\operatorname{proj}_{\mathbf{a}+\mathbf{b}}\mathbf{a} &= \left(\frac{\mathbf{a}\cdot(\mathbf{a}+\mathbf{b})}{(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})}\right)(\mathbf{a}+\mathbf{b}) \\
&= \frac{24}{25}(\mathbf{a}+\mathbf{b}) \\
&= \langle\frac{72}{25},\frac{96}{25}\rangle
\end{aligned}
$$

1.3.45

$W = \mathbf{F}\cdot\mathbf{d} = Fd\cos\theta = 1000$

1.3.47

(a) $W = 0$

(b)

$$
\begin{aligned}
||\mathbf{d}|| &= \sqrt{4^2+3^2} \\
&= 5 \\
\mathbf{F} &= F\hat{\mathbf{d}} \\
&= F\frac{\mathbf{d}}{||\mathbf{d}||} \\
&= F\langle\frac{4}{5},\frac{3}{5}\rangle \\
&= \langle 24,18\rangle \\
W &= \mathbf{F}\cdot\mathbf{d} \\
&= 150\,\mathrm{J}
\end{aligned}
$$

1.4 Cross Product

1.4.1

$$
\begin{aligned}
\mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 0 & 3 & 5 \end{vmatrix} \\
&= -5\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}
\end{aligned}
$$

1.4.11

$$\begin{aligned}\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & -4 \\ -3 & 1 & 1 \end{vmatrix} \\ &= 6\mathbf{i} + 14\mathbf{j} + 4\mathbf{k}\end{aligned}$$

1.4.17

(a)

$$\begin{aligned}\mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} \\ &= \mathbf{j} - \mathbf{k} \\ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 0 & 1 & -1 \end{vmatrix} \\ &= -\mathbf{i} + \mathbf{j} + \mathbf{k}\end{aligned}$$

1.4.19

$2\mathbf{k}$

1.4.21

$$\begin{aligned}\mathbf{k} \times (2\mathbf{i} - \mathbf{j}) &= (\mathbf{k} \times 2\mathbf{i}) - (\mathbf{k} \times \mathbf{j}) \\ &= \mathbf{i} + 2\mathbf{j}\end{aligned}$$

1.4.23

$$\begin{aligned}[(2\mathbf{k}) \times (3\mathbf{j})] \times (4\mathbf{j}) &= (-6\mathbf{i}) \times (4\mathbf{j}) \\ &= -24\mathbf{k}\end{aligned}$$

1.4.37

$12\mathbf{i} - 9\mathbf{j} + 18\mathbf{k}$

1.4.53

$$\begin{aligned}
\mathbf{b}\times\mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 6 & -6 \\ \frac{5}{2} & 3 & \frac{1}{2} \end{vmatrix} \\
&= 21\mathbf{i} - 14\mathbf{j} - 21\mathbf{k} \\
\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= 4\times 21 + 6\times(-14) \\
&= 0
\end{aligned}$$

They are coplanar.

1.5 Lines and Planes in 3-Space

1.5.1

$\mathbf{r} = \langle 1, 2, 1\rangle + t\langle 2, 3, -3\rangle$

1.5.7

$$\begin{aligned}
x &= 2 + 4t \\
y &= 3 - 4t \\
z &= 5 + 3t
\end{aligned}$$

1.5.13

$$\begin{aligned}
x &= 1 + 9t \\
y &= 4 + 10t \\
z &= -9 + 7t \\
\frac{x-1}{9} = \frac{y-4}{10} &= \frac{z+9}{7}
\end{aligned}$$

1.5.19

$$\begin{aligned}
x &= 4 + 3t \\
y &= 6 + \frac{1}{2}t \\
z &= -7 - \frac{3}{2}t \\
\frac{x-4}{3} = \frac{y-6}{1/2} &= -\frac{z+7}{3/2}
\end{aligned}$$

1.5.23

$$\begin{aligned}
x &= 6 + 2t \\
y &= 4 - 3t \\
z &= -2 + 6t
\end{aligned}$$

1.5.25

$$\begin{aligned}
x &= 2 + t \\
y &= -2 \\
z &= 15
\end{aligned}$$

1.5.29

$(0, 5, 15)$, $(5, 0, \frac{15}{2})$, $(10, -5, 0)$

1.5.31

$$\begin{aligned}
4 + t_x &= 6 + 2t_x \\
t_x &= -2 \\
\\
5 + t_y &= 11 + 4t_y \\
t_y &= -2 \\
\\
-1 + 2t_z &= -3 + t_z \\
t_z &= -2
\end{aligned}$$

$(2, 3, -5)$

1.5.35

$$\begin{aligned}
\mathbf{a} &= \langle -1, 2, -2 \rangle \\
||\mathbf{a}|| &= 3 \\
\mathbf{b} &= \langle 2, 3, -6 \rangle \\
||\mathbf{b}|| &= 7 \\
\theta &= \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} \\
&\approx 40.37^\circ
\end{aligned}$$

1.5.37

$$\begin{aligned}
\mathbf{a} &= \langle 1,1,1\rangle \\
\mathbf{b} &= \langle -2,1,-5\rangle \\
\mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ -2 & 1 & -5 \end{vmatrix} \\
&= \langle -6,3,3\rangle \\
x &= 4-6t \\
y &= 1+3t \\
z &= 6+3t
\end{aligned}$$

1.5.39

$$\begin{aligned}
\langle 2,-3,4\rangle\cdot(\mathbf{r}-\langle 5,1,3\rangle) &= 0 \\
2(x-5)-3(y-1)+4(z-3) &= 0 \\
2x-3y+4z-19 &= 0
\end{aligned}$$

1.5.45

$$\begin{aligned}
\mathbf{a} &= \langle 3,5,2\rangle \\
\mathbf{b} &= \langle 2,3,1\rangle \\
\mathbf{c} &= \langle -1,-1,4\rangle \\
\mathbf{a}-\mathbf{c} &= \langle 4,6,-2\rangle \\
\mathbf{b}-\mathbf{c} &= \langle 3,4,-3\rangle \\
(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 6 & -2 \\ 3 & 4 & -3 \end{vmatrix} \\
&= \langle -10,6,-2\rangle \\
\mathbf{n}\cdot(\mathbf{r}-\mathbf{c}) &= 0 \\
\langle -10,6,-2\rangle\cdot(\langle x,y,z\rangle-\langle -1,-1,4\rangle) &= 0 \\
-10(x+1)+6(y+1)-2(z-4) &= 0 \\
-10x+6y-2z+4 &= 0
\end{aligned}$$

1.5.51

$$\begin{aligned}
\langle 1,1,-4\rangle\cdot(\mathbf{r}-\langle 2,3,-5\rangle) &= 0 \\
(x-2)+(y-3)-4(z+5) &= 0 \\
x+y-4z &= 25
\end{aligned}$$

1.5.63

(a) Not perpendicular

(b) Not perpendicular

(c) Perpendicular

(d) Perpendicular

1.5.65

$$\begin{aligned} 5x - 4y - 9t &= 8 \\ x + 4y + 3t &= 4 \end{aligned}$$

$$\begin{aligned} 6x - 6t &= 12 \\ x &= 2 + t \end{aligned}$$

$$y = \frac{1}{2} - t$$

$$z = t$$

1.5.69

$$\begin{aligned} 2(1+2t) - 3(2-t) + 2(-3t) &= -7 \\ t &= -3 \\ x &= -5 \\ y &= 5 \\ z &= 9 \end{aligned}$$

1.5.73

$$\begin{aligned} x + y - 4t &= 2 \\ 2x - y + t &= 10 \end{aligned}$$

$$\begin{aligned} 3x - 3t &= 12 \\ x &= 4 + t \end{aligned}$$

$$\begin{aligned} 2(4 + t) - y + t &= 10 \\ 8 + 2t - y + t &= 10 \\ y &= -2 + 3t \end{aligned}$$

$$z = t$$

$$\begin{aligned} x &= 5 + t \\ y &= 6 + 3t \\ z &= -12 + t \end{aligned}$$

1.5.75

$$\begin{aligned} \mathbf{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 5 \\ 1 & 1 & 1 \end{vmatrix} \\ &= \langle -6, 2, 4 \rangle \\ \mathbf{n} \cdot (\mathbf{r} - \langle 4, 0, 1 \rangle) &= 0 \\ -6(x - 4) + 2y + 4(z - 1) &= 0 \\ -6x + 2y + 4z &= -20 \\ 3x - y - 2z &= 10 \end{aligned}$$

1.6 Vector Spaces

1.6.1

Violates axiom 6

1.6.3

Violates axiom 10

1.6.5

Vector space

1.6.7

Violates axiom 2

1.6.9

Vector space

1.6.11

Subspace

1.6.13

Not a subspace

1.6.15

Subspace

1.6.17

Subspace

1.6.19

Not a subspace

1.6.23

(a)

$$\begin{aligned} k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + k_3\mathbf{u}_3 &= \mathbf{0} \\ k_1\langle 1,0,0\rangle + k_2\langle 1,1,0\rangle + k_3\langle 1,1,1\rangle &= \mathbf{0} \\ \\ k_3 &= 0 \\ \\ k_2 + k_3 &= 0 \\ k_2 &= 0 \\ \\ k_1 + k_2 + k_3 &= 0 \\ k_1 &= 0 \end{aligned}$$

(b)

$$\mathbf{a} = 7\mathbf{u}_1 - 12\mathbf{u}_2 + 8\mathbf{u}_3$$

1.6.25

Dependent

1.6.27

Independent

1.6.29

$f(x)$ is undefined at $x = -3$ and $x = -1$.

1.6.31

$$
\begin{aligned}
||x|| &= \sqrt{(x, x)} \\
&= \sqrt{\int_0^{2\pi} x^2 \, dx} \\
&= \sqrt{\left[\frac{1}{3} x^3\right]_0^{2\pi}} \\
&= \sqrt{\frac{8}{3} \pi^3} \\
||\sin x|| &= \sqrt{(\sin x, \sin x)} \\
&= \sqrt{\int_0^{2\pi} \sin^2 x \, dx} \\
&= \sqrt{\left[\frac{x}{2} - \frac{1}{4} \sin 2x\right]_0^{2\pi}} \\
&= \sqrt{\pi}
\end{aligned}
$$

1.7 Gram–Schmidt Orthogonalization Process

1.7.1

$$\langle \frac{12}{13}, \frac{5}{13} \rangle \cdot \langle \frac{5}{13}, -\frac{12}{13} \rangle = 0$$

$$\sqrt{\left(\frac{12}{13}\right)^2 + \left(\frac{5}{13}\right)^2} = 1$$

$$\begin{aligned}\mathbf{u} &= \left(\langle 4, 2 \rangle \cdot \langle \frac{12}{13}, \frac{5}{13} \rangle\right) \langle \frac{12}{13}, \frac{5}{13} \rangle \\ &\quad + \left(\langle 4, 2 \rangle \cdot \langle \frac{5}{13}, -\frac{12}{13} \rangle\right) \langle \frac{5}{13}, -\frac{12}{13} \rangle \\ &= \left(\frac{58}{13}\right) \langle \frac{12}{13}, \frac{5}{13} \rangle - \left(\frac{4}{13}\right) \langle \frac{5}{13}, -\frac{12}{13} \rangle\end{aligned}$$

1.7.3

$$\langle 1, 0, 1 \rangle \cdot \langle 0, 1, 0 \rangle = 0$$

$$\langle 1, 0, 1 \rangle \cdot \langle -1, 0, 1 \rangle = 0$$

$$\langle 0, 1, 0 \rangle \cdot \langle -1, 0, 1 \rangle = 0$$

$$B' = \{\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle, \langle 0, 1, 0 \rangle, \langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle\}$$

$$\mathbf{u} = -\frac{3}{\sqrt{2}} \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle + 7\langle 0, 1, 0 \rangle - \frac{23}{\sqrt{2}} \langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle$$

1.7.5

(a)

$$\begin{aligned}
B &= \{\langle -3, 2\rangle, \langle -1, -1\rangle\} \\
\mathbf{v}_1 &= \mathbf{u}_1 \\
&= \langle -3, 2\rangle \\
\mathbf{v}_2 &= \mathbf{u}_2 - \operatorname{proj}_{\mathbf{v}_1}\mathbf{u}_2 \\
&= \langle -1, -1\rangle - \left(\frac{\langle -1, -1\rangle \cdot \langle -3, 2\rangle}{\langle -3, 2\rangle \cdot \langle -3, 2\rangle}\right)\langle -3, 2\rangle \\
&= \langle -1, -1\rangle - \frac{1}{13}\langle -3, 2\rangle \\
&= \langle -\frac{10}{13}, -\frac{15}{13}\rangle \\
\mathbf{w}_1 &= \langle -\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\rangle \\
\mathbf{w}_2 &= \sqrt{\frac{169}{325}}\langle -\frac{10}{13}, -\frac{15}{13}\rangle \\
&= \frac{\sqrt{13}}{5}\langle -\frac{10}{13}, -\frac{15}{13}\rangle \\
&= \langle -\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}\rangle
\end{aligned}$$

1.7.9

$$
\begin{aligned}
B &= \{\langle 1,1,0\rangle, \langle 1,2,2\rangle, \langle 2,2,1\rangle\} \\
\mathbf{v}_1 &= \langle 1,1,0\rangle \\
\mathbf{v}_2 &= \mathbf{u}_2 - \mathrm{proj}_{\mathbf{v}_1}\mathbf{u}_2 \\
&= \langle 1,2,2\rangle - \left(\frac{\langle 1,2,2\rangle \cdot \langle 1,1,0\rangle}{\langle 1,1,0\rangle \cdot \langle 1,1,0\rangle}\right)\langle 1,1,0\rangle \\
&= \langle 1,2,2\rangle - \frac{3}{2}\langle 1,1,0\rangle \\
&= \langle -\frac{1}{2}, \frac{1}{2}, 2\rangle \\
\mathbf{v}_3 &= \mathbf{u}_3 - \mathrm{proj}_{\mathbf{v}_1}\mathbf{u}_3 - \mathrm{proj}_{\mathbf{v}_2}\mathbf{u}_3 \\
&= \langle 2,2,1\rangle - \left(\frac{\langle 2,2,1\rangle \cdot \langle 1,1,0\rangle}{\langle 1,1,0\rangle \cdot \langle 1,1,0\rangle}\right)\langle 1,1,0\rangle \\
&\qquad - \left(\frac{\langle 2,2,1\rangle \cdot \langle -\frac{1}{2}, \frac{1}{2}, 2\rangle}{\langle -\frac{1}{2}, \frac{1}{2}, 2\rangle \cdot \langle -\frac{1}{2}, \frac{1}{2}, 2\rangle}\right)\langle -\frac{1}{2}, \frac{1}{2}, 2\rangle \\
&= \langle 2,2,1\rangle - 2\langle 1,1,0\rangle - \frac{4}{9}\langle -\frac{1}{2}, \frac{1}{2}, 2\rangle \\
&= \langle \frac{2}{9}, -\frac{2}{9}, \frac{1}{9}\rangle \\
\mathbf{w}_1 &= \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\rangle \\
\mathbf{w}_2 &= \langle -\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}\rangle \\
\mathbf{w}_3 &= 3\langle \frac{2}{9}, -\frac{2}{9}, \frac{1}{9}\rangle \\
&= \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\rangle
\end{aligned}
$$

1.7.17

$$
\begin{aligned}
B &= \{1, x, x^2\} \\
\mathbf{v}_1 &= \mathbf{u}_1 \\
&= 1 \\
\mathbf{v}_2 &= \mathbf{u}_2 - \mathrm{proj}_{\mathbf{v}_1}\mathbf{u}_2 \\
&= \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\
&= x - \frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1} dx} \\
&= x - \frac{\left[\frac{1}{2}x^2\right]_{-1}^{1}}{2} \\
&= x \\
\mathbf{v}_3 &= \mathbf{u}_3 - \mathrm{proj}_{\mathbf{v}_1}\mathbf{u}_3 - \mathrm{proj}_{\mathbf{v}_2}\mathbf{u}_3 \\
&= \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\
&= x^2 - \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} dx} - \frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx}x \\
&= x^2 - \frac{\left[\frac{1}{3}x^3\right]_{-1}^{1}}{2} - \frac{\left[\frac{1}{4}x^4\right]_{-1}^{1}}{\left[\frac{1}{3}x^3\right]_{-1}^{1}}x \\
&= x^2 - \frac{1}{3}
\end{aligned}
$$

1.7.19

$$\begin{aligned}
||\mathbf{v}_1||^2 &= \int_{-1}^{1} \, dx \\
&= 2 \\
\mathbf{w}_1 &= \frac{1}{\sqrt{2}} \\
||\mathbf{v}_2||^2 &= \int_{-1}^{1} x^2 \, dx \\
&= \left[\frac{1}{3}x^3\right]_{-1}^{1} \\
&= \frac{2}{3} \\
\mathbf{w}_2 &= \frac{3}{\sqrt{6}}x \\
||\mathbf{v}_3||^2 &= \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 \, dx \\
&= \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) \, dx \\
&= \left[\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x\right]_{-1}^{1} \\
&= \frac{1}{5} - \frac{2}{9} + \frac{1}{9} + \frac{1}{5} - \frac{2}{9} + \frac{1}{9} \\
&= \frac{2}{5} - \frac{2}{9} \\
&= \frac{8}{45} \\
\mathbf{w}_3 &= \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3}\right) \\
&= \frac{5}{2\sqrt{10}} \left(3x^2 - 1\right)
\end{aligned}$$

1.7.21

$$
\begin{aligned}
(\mathbf{p},\mathbf{w}_1) &= \int_{-1}^{1} \frac{1}{\sqrt{2}}(9x^2-6x+5)\,dx \\
&= \frac{1}{\sqrt{2}}\left[3x^3-3x^2+5x\right]_{-1}^{1} \\
&= \frac{1}{\sqrt{2}}(3-3+5+3+3+5) \\
&= \frac{16}{\sqrt{2}} \\
(\mathbf{p},\mathbf{w}_2) &= \int_{-1}^{1} \frac{3}{\sqrt{6}}x(9x^2-6x+5)\,dx \\
&= \frac{3}{\sqrt{6}}\left[\frac{9}{4}x^4-2x^3+\frac{5}{2}x^2\right]_{-1}^{1} \\
&= \frac{3}{\sqrt{6}}\left(\frac{9}{4}-2+\frac{5}{2}-\frac{9}{4}-2-\frac{5}{2}\right) \\
&= \frac{3}{\sqrt{6}}\left(\frac{9}{4}-\frac{8}{4}+\frac{10}{4}-\frac{9}{4}-\frac{8}{4}-\frac{10}{4}\right) \\
&= -\frac{12}{\sqrt{6}} \\
(\mathbf{p},\mathbf{w}_3) &= \int_{-1}^{1} \frac{5}{2\sqrt{10}}(3x^2-1)(9x^2-6x+5)\,dx \\
&= \frac{5}{2\sqrt{10}}\int_{-1}^{1}(27x^4-18x^3+6x^2+6x-5)\,dx \\
&= \frac{5}{2\sqrt{10}}\left[\frac{27}{5}x^5-\frac{9}{2}x^4+2x^3+3x^2-5x\right]_{-1}^{1} \\
&= \frac{5}{2\sqrt{10}}\left(\frac{27}{5}-\frac{9}{2}+2+3-5+\frac{27}{5}+\frac{9}{2}+2-3-5\right) \\
&= \frac{5}{2\sqrt{10}}\left(\frac{54}{10}-\frac{45}{10}+\frac{20}{10}+\frac{30}{10}-\frac{50}{10}+\frac{54}{10}+\frac{45}{10}+\frac{20}{10}-\frac{30}{10}-\frac{50}{10}\right) \\
&= \frac{5}{2\sqrt{10}}\frac{48}{10} \\
&= \frac{12}{\sqrt{10}} \\
\mathbf{p} &= \frac{16}{\sqrt{2}}\mathbf{w}_1-\frac{12}{\sqrt{6}}\mathbf{w}_2+\frac{12}{\sqrt{10}}\mathbf{w}_3
\end{aligned}
$$

1.8 Chapter in Review

1.8.1

True

1.8.3

$$\mathbf{u} = \langle 5, -2, 1 \rangle$$
$$\mathbf{v} = \langle 2, 3, -4 \rangle$$

False

1.8.5

True

1.8.7

True

1.8.9

True

1.8.11

$9\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

1.8.13

$$\begin{aligned}(-\mathbf{k}) \times (5\mathbf{j}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -1 \\ 0 & 5 & 0 \end{vmatrix} \\ &= 5\mathbf{i}\end{aligned}$$

1.8.15

$$||-12\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}|| = \sqrt{12^2 + 4^2 + 6^2} = 14$$

1.8.17

$\langle -6, 1, -7 \rangle$

1.8.19

$$\begin{aligned} x &= 1 + t \\ y &= -2 + 3t \\ z &= -1 + 2t \end{aligned}$$

$$\begin{aligned} x + 2y - z &= 13 \\ (1+t) + 2(-2+3t) - (-1+2t) &= 13 \\ 1 + t - 4 + 6t + 1 - 2t &= 13 \\ -2 + 5t &= 13 \\ t &= 3 \end{aligned}$$

$$\begin{aligned} x &= 4 \\ y &= 7 \\ z &= 5 \end{aligned}$$

1.8.21

$$\begin{aligned} \overrightarrow{P_1P_2} &= \overrightarrow{P_2} - \overrightarrow{P_1} \\ \overrightarrow{P_2} &= \overrightarrow{P_1P_2} + \overrightarrow{P_1} \\ &= \langle 3, 5, -4\rangle + \langle 2, 1, 7\rangle \\ &= \langle 5, 6, 3\rangle \end{aligned}$$

1.8.23

$\mathbf{a} \cdot \mathbf{b} = -36\sqrt{2}$

1.8.25

$x = 12,\ y = -8,\ z = 6$

1.8.27

$$\begin{aligned} \frac{1}{2}(\mathbf{a} \times \mathbf{b}) &= \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & -1 & 2 \end{vmatrix} \\ &= \frac{1}{2}\langle 5, -4, -7\rangle \\ &= \langle \frac{5}{2}, -2, -\frac{7}{2}\rangle \end{aligned}$$

The area is $\sqrt{\left(\frac{5}{2}\right)^2+(-2)^2+\left(-\frac{7}{2}\right)^2}=\frac{3}{2}\sqrt{10}$

1.8.29

2

1.8.31

$$\begin{aligned}
\mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -2 & 1 \end{vmatrix} \\
&= \langle 1,-1,-3\rangle \\
||\mathbf{a}\times\mathbf{b}|| &= \sqrt{11} \\
\mathrm{norm}(\mathbf{a}\times\mathbf{b}) &= \langle \frac{1}{\sqrt{11}},-\frac{1}{\sqrt{11}},-\frac{3}{\sqrt{11}}\rangle
\end{aligned}$$

1.8.33

$$\mathrm{comp}_{\mathbf{b}}\mathbf{a}=\frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{b}||}=\frac{10}{5}=2$$

1.8.35

$$\begin{aligned}
\mathbf{a} &= \langle 1,2,-2\rangle \\
\mathbf{b} &= \langle 4,3,0\rangle \\
\mathbf{a}+\mathbf{b} &= \langle 5,5,-2\rangle \\
\mathrm{proj}_{\mathbf{a}}(\mathbf{a}+\mathbf{b}) &= \left(\frac{(\mathbf{a}+\mathbf{b})\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} \\
&= \frac{19}{9}\langle 1,2,-2\rangle \\
&= \langle \frac{19}{9},\frac{38}{9},-\frac{38}{9}\rangle
\end{aligned}$$

1.8.37

(a)

(b) A plane with normal $\mathbf{a}$

1.8.39

$$\frac{x-7}{4}=\frac{y-3}{-2}=\frac{z+5}{6}$$

1.8.41

$$\begin{aligned}\langle -2,3,1\rangle \cdot \langle 2,1,1\rangle &= -4+3+1 \\ &= 0\end{aligned}$$

$$\begin{aligned}1-2t &= 1+2s \\ t &= -s\end{aligned}$$

$$\begin{aligned}3t &= -4+s \\ t &= -\frac{4}{3}+\frac{s}{3} \\ &= -\frac{4}{3}-\frac{t}{3} \\ \frac{4}{3}t &= -\frac{4}{3} \\ t &= -1\end{aligned}$$

$$s = 1$$

$$\langle 3,-3,0\rangle$$

1.8.43

$$\begin{aligned}\mathbf{u} &= \langle 1,4,-2\rangle \\ \mathbf{v} &= \langle 1,1,3\rangle \\ \mathbf{n} &= \mathbf{u}\times\mathbf{v} \\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & -2 \\ 1 & 1 & 3\end{vmatrix} \\ &= \langle 14,-5,-3\rangle\end{aligned}$$

$$\begin{aligned}\mathbf{n}\cdot(\mathbf{r}-\mathbf{v}) &= 0 \\ \langle 14,-5,-3\rangle \cdot (\langle x,y,z\rangle - \langle 1,1,3\rangle) &= 0 \\ 14(x-1)-5(y-1)-3(z-3) &= 0 \\ 14x-5y-3z &= 0\end{aligned}$$

1.8.45

$$\begin{aligned}
\mathbf{F} &= \langle \frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}}, 0 \rangle \\
\mathbf{d} &= \langle 3, 3, 0 \rangle \\
\mathbf{F} \cdot \mathbf{d} &= 30\sqrt{2}\,\mathrm{J}
\end{aligned}$$

1.8.47

$$\begin{aligned}
\mathbf{F}_1 &= \langle 200, 0, 0 \rangle \\
\mathbf{F}_2 &= \langle \frac{200}{\sqrt{2}}, \frac{200}{\sqrt{2}}, 0 \rangle \\
\mathbf{F}_2 &= \mathbf{F}_1 + \mathbf{F}_3 \\
\mathbf{F}_3 &= \mathbf{F}_2 - \mathbf{F}_1 \\
&= \langle \frac{200}{\sqrt{2}}, \frac{200}{\sqrt{2}}, 0 \rangle - \langle 200, 0, 0 \rangle \\
&= \langle \frac{200}{\sqrt{2}} - 200, \frac{200}{\sqrt{2}}, 0 \rangle \\
||\mathbf{F}_3|| &= \sqrt{\left(\frac{200}{\sqrt{2}} - 200\right)^2 + \left(\frac{200}{\sqrt{2}}\right)^2} \\
&= \sqrt{\frac{40000}{2} - \frac{80000}{\sqrt{2}} + 40000 + \frac{40000}{2}} \\
&= 200\sqrt{2\left(1 - \frac{1}{\sqrt{2}}\right)} \\
&\approx 153\,\mathrm{lb}
\end{aligned}$$

2 Matrices

2.1 Matrix Algebra

2.1.1

2×4

2.1.3

3×3

2.1.5

3×4

2.1.7

No

2.1.9

No

2.1.11

$$
\begin{aligned}
x &= y - 2 \\
3x - 2 &= y \\
\\
2x - 2 &= 2 \\
2x &= 4 \\
x &= 2 \\
\\
2 &= y - 2 \\
y &= 4
\end{aligned}
$$

2.1.13

$$
\begin{aligned}
c_{23} &= 9 \\
c_{12} &= 12
\end{aligned}
$$

2.1.15

(a) $\begin{pmatrix} 2 & 11 \\ 2 & -1 \end{pmatrix}$

(b) $\begin{pmatrix} -6 & 1 \\ 14 & -19 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 28 \\ 12 & -12 \end{pmatrix}$

2.1.17

(a) $\begin{pmatrix} -11 & 6 \\ 17 & -22 \end{pmatrix}$

(b) $\begin{pmatrix} -32 & 27 \\ -4 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 19 & -18 \\ -30 & 31 \end{pmatrix}$

(d) $\begin{pmatrix} 19 & 6 \\ 3 & 22 \end{pmatrix}$

2.1.21

(a) 180

(b) $\begin{pmatrix} 4 & 8 & 10 \\ 8 & 16 & 20 \\ 10 & 20 & 25 \end{pmatrix}$

(c) $\begin{pmatrix} 6 \\ 12 \\ -5 \end{pmatrix}$

2.1.23

(a) $\begin{pmatrix} 7 & 38 \\ 10 & 75 \end{pmatrix}$

(b) $\begin{pmatrix} 7 & 38 \\ 10 & 75 \end{pmatrix}$

2.1.25

$\begin{pmatrix} -14 \\ 1 \end{pmatrix}$

2.1.27

$\begin{pmatrix} -38 \\ -2 \end{pmatrix}$

2.1.29

4×5

2.1.41

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + x_{22}x_2 &= b_2 \end{aligned}$$

2.1.43

$$\begin{aligned} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} ax + \frac{1}{2}by \\ \frac{1}{2}bx + cy \end{pmatrix} \\ &= ax^2 + \frac{1}{2}bxy + \frac{1}{2}bxy + cy^2 \\ &= ax^2 + bxy + cy^2 \end{aligned}$$

2.1.45

$\langle -1, 1\rangle$

2.1.47

$\langle -2, 0\rangle$

2.1.49

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

2.1.51

(b)

$$\begin{pmatrix} x_S \\ y_S \\ z_S \end{pmatrix} = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

2.2 Systems of Linear Algebraic Equations

2.2.1

$$\left(\begin{array}{cc|c} 1 & -1 & 11 \\ 4 & 3 & -5 \end{array}\right)$$
$$\left(\begin{array}{cc|c} 1 & -1 & 11 \\ 0 & 7 & -49 \end{array}\right)$$
$$\left(\begin{array}{cc|c} 1 & -1 & 11 \\ 0 & 1 & -7 \end{array}\right)$$
$$\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -7 \end{array}\right)$$

2.2.5

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 2 & 3 & 5 & 7 \\ 1 & -2 & 3 & -11 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 0 & 5 & 7 & 13 \\ 0 & -1 & 4 & -8 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 0 & 1 & \frac{7}{5} & \frac{13}{5} \\ 0 & -1 & 4 & -8 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 0 & 1 & \frac{7}{5} & \frac{13}{5} \\ 0 & 0 & \frac{27}{5} & -\frac{27}{5} \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -3 \\ 0 & 1 & \frac{7}{5} & \frac{13}{5} \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

2.3 Rank of a Matrix

2.3.1

$$\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{1}{3} \\ 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & \frac{10}{3} \end{pmatrix}$$

Rank 2

2.3.3

$$\begin{pmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ -1 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}$$
$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 6 & 3 & 9 \\ -1 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}$$
$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Rank 1

2.3.5

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 4 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 3 \\ 0 & 3 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 3 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 9 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

Rank 3

2.3.7

$$\begin{pmatrix} 1 & -2 \\ 3 & -6 \\ 7 & -1 \\ 4 & 5 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -2 \\ 0 & 13 \\ 0 & 13 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -2 \\ 0 & 13 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Rank 2

2.3.11

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 5 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & -3 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -3 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

Linearly independent

2.3.15

5

2.3.17

$\text{rank}(\mathbf{A}) = 2$

2.4 Determinants

2.4.1

9

2.4.3

1

2.4.5

$$\begin{aligned} M_{33} &= \begin{vmatrix} 0 & 2 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} \\ &= -2 \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} \\ &= 2 \end{aligned}$$

2.4.7

$$\begin{aligned} C_{34} &= (-1)^{3+4} \begin{vmatrix} 0 & 2 & 4 \\ 1 & 2 & -2 \\ 1 & 1 & 1 \end{vmatrix} \\ &= -\left(-2 \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} + 4 \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \right) \\ &= 10 \end{aligned}$$

2.4.9

-7

2.4.11

17

2.4.13

$$\begin{aligned} (1-\lambda)(2-\lambda) - 6 &= 2 - \lambda - 2\lambda + \lambda^2 - 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda + 1)(\lambda - 4) \end{aligned}$$

2.4.15

-48

2.4.23

$$\begin{aligned}\begin{vmatrix}1 & 1 & 1\\ x & y & z\\ 2 & 3 & 4\end{vmatrix} &= \begin{vmatrix}y & z\\ 3 & 4\end{vmatrix} - \begin{vmatrix}x & z\\ 2 & 4\end{vmatrix} + \begin{vmatrix}x & y\\ 2 & 3\end{vmatrix}\\ &= 4y - 3z - 4x + 2z + 3x - 2y\\ &= -x + 2y - z\end{aligned}$$

2.4.29

$$\begin{aligned}\begin{vmatrix}(-3-\lambda) & 10\\ 2 & (5-\lambda)\end{vmatrix} &= 0\\ (-3-\lambda)(5-\lambda) - 20 &= 0\\ -15 + 3\lambda - 5\lambda + \lambda^2 - 20 &= 0\\ \lambda^2 - 2\lambda - 35 &= 0\\ (\lambda - 7)(\lambda + 5) &= 0\\ \lambda &= -5 \text{ or } 7\end{aligned}$$

2.5 Properties of Determinants

2.5.1

8.5.4

2.5.3

8.5.7

2.5.5

8.5.5

2.5.7

8.5.3

2.5.9

8.5.1

2.5.11

-5

2.5.13

-5

2.5.15

5

2.5.17

80

2.5.19

-105

2.5.25

$$\begin{aligned}
\mathbf{AA} &= \mathbf{I} \\
\det \mathbf{A} \cdot \det \mathbf{A} &= \det \mathbf{I} \\
(\det \mathbf{A})^2 &= 1 \\
\det \mathbf{A} &= \pm 1
\end{aligned}$$

2.5.27

$$\begin{aligned}
\begin{vmatrix} a & a+1 & a+2 \\ b & b+1 & b+2 \\ c & c+1 & c+2 \end{vmatrix} &= \begin{vmatrix} a & 1 & 2 \\ b & 1 & 2 \\ c & 1 & 2 \end{vmatrix} \\
&= \begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} \\
&= 0
\end{aligned}$$

2.5.29

$$\begin{aligned}
\begin{vmatrix} 1 & 1 & 5 \\ 4 & 3 & 6 \\ 0 & -1 & 1 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 5 \\ 0 & -1 & -14 \\ 0 & -1 & 1 \end{vmatrix} \\
&= -\begin{vmatrix} 1 & 1 & 5 \\ 0 & 1 & 14 \\ 0 & 0 & 15 \end{vmatrix} \\
&= -15
\end{aligned}$$

2.5.37

$$\begin{aligned}
\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & c^2-a^2-(b+a)(c-a) \end{vmatrix} \\
&= (b-a)(c^2-a^2-(b+a)(c-a)) \\
&= (b-a)(c^2-a^2-bc+ab-ac+a^2) \\
&= (b-a)(c^2+ab-ac-bc) \\
&= (b-a)(c-a)(c-b)
\end{aligned}$$

2.5.39

$$\begin{aligned}
a_{21}C_{11}+a_{22}C_{12}+a_{23}C_{13} &= (-1)(4)+(2)(5)+(1)(-6) \\
&= 0 \\
a_{13}C_{12}+a_{23}C_{22}+a_{33}C_{32} &= (2)(5)+(1)(-7)+(1)(-3) \\
&= 0
\end{aligned}$$

2.5.41

$$\begin{aligned}
\mathbf{A}+\mathbf{B} &= \begin{pmatrix} 10 & 0 \\ 0 & -3 \end{pmatrix} \\
\det(\mathbf{A}+\mathbf{B}) &= -30 \\
\det\mathbf{A} &= 10 \\
\det\mathbf{B} &= -31 \\
-30 &\neq 10-31
\end{aligned}$$

2.6 Inverse of a Matrix

2.6.3

$$\begin{aligned}
\det\mathbf{A} &= 9 \\
\mathbf{A}^{-1} &= \frac{1}{9}\begin{pmatrix} 1 & 1 \\ -4 & 5 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{9} & \frac{1}{9} \\ -\frac{4}{9} & \frac{5}{9} \end{pmatrix}
\end{aligned}$$

2.6.5

$$
\begin{aligned}
\det \mathbf{A} &= 12 \\
\mathbf{A}^{-1} &= \frac{1}{12}\begin{pmatrix} 2 & 0 \\ 3 & 6 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{6} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}
\end{aligned}
$$

2.6.7

$$
\begin{aligned}
\det \mathbf{A} &= (1)\begin{vmatrix} 4 & 4 \\ -1 & 1 \end{vmatrix} - (3)\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} + (5)\begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} \\
&= 8 + 6 - 30 \\
&= -16 \\
\mathbf{A}^{-1} &= -\frac{1}{16}\begin{pmatrix} 8 & 2 & -6 \\ -8 & -4 & 4 \\ -8 & 6 & -2 \end{pmatrix}^T \\
&= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{8} & \frac{1}{4} & -\frac{3}{8} \\ \frac{3}{8} & -\frac{1}{4} & \frac{1}{8} \end{pmatrix}
\end{aligned}
$$

2.6.15

$$
\left(\begin{array}{cc|cc} 6 & -2 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{array}\right)
$$

$$
\left(\begin{array}{cc|cc} 1 & -\frac{1}{3} & \frac{1}{6} & 0 \\ 0 & 1 & 0 & \frac{1}{4} \end{array}\right)
$$

$$
\left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{6} & \frac{1}{12} \\ 0 & 1 & 0 & \frac{1}{4} \end{array}\right)
$$

2.6.17

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right)$$
$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -12 & -5 & 1 \end{array}\right)$$
$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{5}{12} & -\frac{1}{12} \end{array}\right)$$
$$\left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & \frac{5}{12} & -\frac{1}{12} \end{array}\right)$$

2.6.27

$$\begin{aligned} (\mathbf{AB})^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} \\ &= \begin{pmatrix} \frac{2}{3} & \frac{4}{3} \\ -\frac{1}{3} & \frac{5}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{5}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ -\frac{17}{12} & \frac{55}{12} \end{pmatrix} \end{aligned}$$

2.6.29

$$\begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$$

2.6.31

$$\begin{aligned} \begin{pmatrix} 4 & -3 \\ x & -4 \end{pmatrix} &= \frac{1}{3x-16} \begin{pmatrix} -4 & -x \\ 3 & 4 \end{pmatrix}^T \\ &= \frac{1}{3x-16} \begin{pmatrix} -4 & 3 \\ -x & 4 \end{pmatrix} \\ -1 &= \frac{1}{3x-16} \\ 16 - 3x &= 1 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

2.6.45

$$\begin{aligned}
\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 4 \\ 14 \end{pmatrix} \\
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= -\frac{1}{3} \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 14 \end{pmatrix} \\
&= -\frac{1}{3} \begin{pmatrix} -18 \\ 6 \end{pmatrix} \\
&= \begin{pmatrix} 6 \\ -2 \end{pmatrix}
\end{aligned}$$

2.6.49

$$\begin{aligned}
\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix} \\
\det \mathbf{A} &= \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} \\
&= 1 - 6 \\
&= -5 \\
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= -\frac{1}{5} \begin{pmatrix} 1 & 5 & -6 \\ -1 & -5 & 1 \\ -1 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix} \\
&= -\frac{1}{5} \begin{pmatrix} 1 & -1 & -1 \\ 5 & -5 & 0 \\ -6 & 1 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix} \\
&= -\frac{1}{5} \begin{pmatrix} -10 \\ -20 \\ 30 \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix}
\end{aligned}$$

2.6.55

$$\begin{aligned}
\det \begin{pmatrix} 1 & 2 & -1 \\ 4 & -1 & 1 \\ 5 & 1 & -2 \end{pmatrix} &= (1) \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} - (2) \begin{vmatrix} 4 & 1 \\ 5 & -2 \end{vmatrix} + (-1) \begin{vmatrix} 4 & -1 \\ 5 & 1 \end{vmatrix} \\
&= 1 + 26 - 9 \\
&= 18
\end{aligned}$$

Only trivial solution

2.7 Cramer's Rule

2.7.1

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix} \\
\mathbf{B} &= \begin{pmatrix} 3 \\ -6 \end{pmatrix} \\
\det \mathbf{A} &= 10 \\
\det \mathbf{A}_1 &= -6 \\
\det \mathbf{A}_2 &= 12 \\
x_1 &= -\frac{3}{5} \\
x_2 &= \frac{6}{5}
\end{aligned}
$$

2.7.11

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 2-k & k \\ k & 3-k \end{pmatrix} \\
\mathbf{B} &= \begin{pmatrix} 4 \\ 3 \end{pmatrix} \\
\det \mathbf{A} &= (2-k)(3-k) - k^2 \\
&= 6 - 5k \\
\det \mathbf{A}_1 &= 4(3-k) - 3k \\
&= 12 - 7k \\
\det \mathbf{A}_2 &= 3(2-k) - 4k \\
&= 6 - 7k \\
x_1 &= \frac{12-7k}{6-5k} \\
x_2 &= \frac{6-7k}{6-5k}
\end{aligned}
$$

The system is inconsistent when $k = \frac{6}{5}$

2.7.13

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} \cos 25^\circ & -\cos 15^\circ \\ \sin 25^\circ & \sin 15^\circ \end{pmatrix} \\
\mathbf{B} &= \begin{pmatrix} 0 \\ 300 \end{pmatrix} \\
\det \mathbf{A} &= \cos 25^\circ \sin 15^\circ + \cos 15^\circ \sin 25^\circ \\
&= \sin 40^\circ \\
\det \mathbf{A}_1 &= 300 \cos 15^\circ \\
\det \mathbf{A}_2 &= 300 \cos 25^\circ \\
T_1 &= \frac{300 \cos 15^\circ}{\sin 40^\circ} \\
&\approx 451\,\mathrm{lb} \\
T_2 &= \frac{300 \cos 25^\circ}{\sin 40^\circ} \\
&\approx 423\,\mathrm{lb}
\end{aligned}$$

2.8 The Eigenvalue Problem

2.8.1

$\mathbf{K}_3$ with $\lambda = -1$

2.8.3

$\mathbf{K}_3$ with $\lambda = 0$

2.8.5

$\mathbf{K}_2$ with $\lambda = 3$
$\mathbf{K}_3$ with $\lambda = 1$

2.8.7

$$\begin{aligned}
\det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} -1-\lambda & 2 \\ -7 & 8-\lambda \end{vmatrix} \\
&= (-1-\lambda)(8-\lambda) + 14 \\
&= -8 + \lambda - 8\lambda + \lambda^2 + 14 \\
&= \lambda^2 - 7\lambda + 6 \\
&= (\lambda - 1)(\lambda - 6) \\
\lambda_1 &= 1 \\
\lambda_2 &= 6
\end{aligned}$$

$$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ -7 & 7 & 0 \end{array}\right)$$

$x_1 = x_2$

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -7 & 2 & 0 \\ -7 & 2 & 0 \end{array}\right)$$

$x_1 = \frac{2}{7}x_2$

$$\mathbf{X}_2 = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

Nonsingular

2.8.9

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} -8-\lambda & -1 \\ 16 & -\lambda \end{vmatrix} \\ &= -\lambda(-8-\lambda) + 16 \\ &= 8\lambda + \lambda^2 + 16 \\ &= (\lambda+4)^2 \\ \lambda_1 = \lambda_2 &= -4 \end{aligned}$$

$$\left(\begin{array}{cc|c} -4 & -1 & 0 \\ 16 & 4 & 0 \end{array}\right)$$

$x_1 = -\frac{1}{4}x_2$

$$\mathbf{X}_1 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

Nonsingular

2.8.11

$$\begin{aligned}\det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} -1-\lambda & 2 \\ -5 & 1-\lambda \end{vmatrix} \\ &= (-1-\lambda)(1-\lambda) + 10 \\ &= -1 + \lambda - \lambda + \lambda^2 + 10 \\ &= \lambda^2 + 9 \\ &= (\lambda - 3i)(\lambda + 3i) \\ \lambda_1 &= 3i \\ \lambda_2 &= -3i \end{aligned}$$

$$\left(\begin{array}{cc|c} -1-3i & 2 & 0 \\ -5 & 1-3i & 0 \end{array}\right)$$

$$\mathbf{X}_1 = \begin{pmatrix} 1-3i \\ 5 \end{pmatrix}$$

$$\mathbf{X}_2 = \begin{pmatrix} 1+3i \\ 5 \end{pmatrix}$$

Nonsingular

2.8.23

$$\begin{aligned}\det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{vmatrix} \\ &= (5-\lambda)^2 - 1 \\ &= 25 - 10\lambda + \lambda^2 - 1 \\ &= \lambda^2 - 10\lambda + 24 \\ &= (\lambda - 4)(\lambda - 6) \\ \lambda_1 &= 4 \\ \lambda_2 &= 6 \end{aligned}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\mathbf{X_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{aligned}
\lambda_1' &= \frac{1}{4} \\
\lambda_2' &= \frac{1}{6} \\
\mathbf{X}_1' &= \mathbf{X}_1 \\
\mathbf{X}_2' &= \mathbf{X}_2
\end{aligned}$$

2.9 Powers of Matrices

2.9.1

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= (1-\lambda)(5-\lambda) + 8 \\
&= 5 - \lambda - 5\lambda + \lambda^2 + 8 \\
&= \lambda^2 - 6\lambda + 13 \\
\mathbf{A}^2 &= 6\mathbf{A} - 13\mathbf{I} \\
\begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix} &= 6\begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix} - 13\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\begin{pmatrix} -7 & -12 \\ 24 & 17 \end{pmatrix} &= \begin{pmatrix} -7 & -12 \\ 24 & 17 \end{pmatrix}
\end{aligned}$$

2.9.3

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix} \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= (-1-\lambda)(4-\lambda) - 6 \\
&= -4 + \lambda - 4\lambda + \lambda^2 - 6 \\
&= \lambda^2 - 3\lambda - 10 \\
&= (\lambda - 5)(\lambda + 2)
\end{aligned}
$$

$$
\begin{aligned}
\lambda^m &= c_0 + c_1\lambda \\
(-2)^m &= c_0 - 2c_1 \\
(5)^m &= c_0 + 5c_1
\end{aligned}
$$

$$
\begin{aligned}
(-2)^m + \frac{2}{5}(5)^m &= \frac{7}{5}c_0 \\
\frac{5}{7}(-2)^m + \frac{2}{7}(5)^m &= c_0
\end{aligned}
$$

$$
\begin{aligned}
(5)^m - (-2)^m &= 7c_1 \\
\frac{1}{7}(5)^m - \frac{1}{7}(-2)^m &= c_1
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A}^m &= \frac{1}{7}\begin{pmatrix} 5^m + 6(-2)^m & 3(5)^m - 3(-2)^m \\ 2(5)^m - 2(-2)^m & 6(5)^m + (-2)^m \end{pmatrix} \\
\mathbf{A}^3 &= \frac{1}{7}\begin{pmatrix} 5^3 + 6(-2)^3 & 3(5)^3 - 3(-2)^3 \\ 2(5)^3 - 2(-2)^3 & 6(5)^3 + (-2)^3 \end{pmatrix} \\
&= \begin{pmatrix} 11 & 57 \\ 38 & 106 \end{pmatrix}
\end{aligned}
$$

2.9.5

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 8 & 5 \\ 4 & 0 \end{pmatrix} \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= -\lambda(8-\lambda) - 20 \\
&= -8\lambda + \lambda^2 - 20 \\
&= \lambda^2 - 8\lambda - 20 \\
&= (\lambda - 10)(\lambda + 2)
\end{aligned}
$$

$$
\begin{aligned}
\lambda^m &= c_0 + c_1\lambda \\
(-2)^m &= c_0 - 2c_1 \\
10^m &= c_0 + 10c_1
\end{aligned}
$$

$$
\begin{aligned}
10^m + 5(-2)^m &= 6c_0 \\
\frac{1}{6}(10^m + 5(-2)^m) &= c_0 \\
\frac{1}{12}(2 \cdot 10^m + 10(-2)^m) &= c_0
\end{aligned}
$$

$$
\begin{aligned}
10^m - (-2)^m &= 12c_1 \\
\frac{1}{12}(10^m - (-2)^m) &= c_1
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A}^m &= \frac{1}{12}\begin{pmatrix} 10 \cdot 10^m + 2(-2)^m & 5 \cdot 10^m - 5(-2)^m \\ 4 \cdot 10^m - 4(-2)^m & 2 \cdot 10^m + 10(-2)^m \end{pmatrix} \\
\mathbf{A}^5 &= \begin{pmatrix} 83328 & 41680 \\ 33344 & 16640 \end{pmatrix}
\end{aligned}
$$

2.9.11

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 7 & 3 \\ -3 & 1 \end{pmatrix} \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= (7-\lambda)(1-\lambda) + 9 \\
&= 7 - 7\lambda - \lambda + \lambda^2 + 9 \\
&= \lambda^2 - 8\lambda + 16 \\
&= (\lambda - 4)^2
\end{aligned}$$

$$\begin{aligned}
\lambda^m &= c_0 + c_1 \lambda \\
m\lambda^{m-1} &= c_1
\end{aligned}$$

$$4^{m-1} m = c_1$$

$$\begin{aligned}
4^m &= c_0 + 4^m m \\
4^m(1-m) &= c_0
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}^m &= \begin{pmatrix} 4^m(1-m) + 7 \cdot 4^{m-1} m & 3 \cdot 4^{m-1} m \\ -3 \cdot 4^{m-1} m & 4^m(1-m) + 4^{m-1} m \end{pmatrix} \\
&= 4^m \begin{pmatrix} 1 - m + \frac{7}{4} m & \frac{3}{4} m \\ -\frac{3}{4} m & 1 - m + \frac{1}{4} m \end{pmatrix} \\
&= 4^m \begin{pmatrix} 1 + \frac{3}{4} m & \frac{3}{4} m \\ -\frac{3}{4} m & 1 - \frac{3}{4} m \end{pmatrix} \\
\mathbf{A}^6 &= 4^6 \begin{pmatrix} 1 + \frac{3}{4} 6 & \frac{3}{4} 6 \\ -\frac{3}{4} 6 & 1 - \frac{3}{4} 6 \end{pmatrix} \\
&= 4^5 \begin{pmatrix} 4 + 18 & 18 \\ -18 & 4 - 18 \end{pmatrix} \\
&= 4^5 \begin{pmatrix} 22 & 18 \\ -18 & -14 \end{pmatrix} \\
&= \begin{pmatrix} 22528 & 18432 \\ -18432 & -14336 \end{pmatrix}
\end{aligned}$$

2.9.13

(a)

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \\
\det(\mathbf{A} - \lambda\mathbf{I}) &= (1-\lambda)(3-\lambda) - 3 \\
&= 3 - \lambda - 3\lambda + \lambda^2 - 3 \\
&= \lambda^2 - 4\lambda \\
&= \lambda(\lambda - 4)
\end{aligned}
$$

$$
\begin{aligned}
\lambda^m &= c_1\lambda \\
4^m &= 4c_1 \\
c_1 &= 4^{m-1}
\end{aligned}
$$

$$
\mathbf{A}^m = 4^{m-1}\mathbf{A}
$$

2.9.15

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix} \\
\det(\mathbf{A} - \lambda\mathbf{I}) &= (2-\lambda)(3-\lambda) + 4 \\
&= 6 - 2\lambda - 3\lambda + \lambda^2 + 4 \\
&= \lambda^2 - 5\lambda + 10 \\
10\mathbf{I} &= 5\mathbf{A} - \mathbf{A}^2 \\
\mathbf{I} &= \frac{1}{2}\mathbf{A} - \frac{1}{10}\mathbf{A}^2 \\
\mathbf{A}^{-1} &= \frac{1}{2}\mathbf{I} - \frac{1}{10}\mathbf{A} \\
&= \begin{pmatrix} \frac{3}{10} & \frac{2}{5} \\ -\frac{1}{10} & \frac{1}{5} \end{pmatrix}
\end{aligned}
$$

2.10 Orthogonal Matrices

2.10.5

Orthogonal

2.10.7

Orthogonal

2.10.9

Not orthogonal

2.10.11

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 9 \\ 9 & 1 \end{pmatrix} \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= (1-\lambda)^2 - 81 \\
&= \lambda^2 - 2\lambda + 1 - 81 \\
&= \lambda^2 - 2\lambda - 80 \\
&= (\lambda - 10)(\lambda + 8)
\end{aligned}
$$

$$\begin{pmatrix} 9 & 9 \\ 9 & 9 \end{pmatrix}$$

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -9 & 9 \\ 9 & -9 \end{pmatrix}$$

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

2.10.13

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= (1-\lambda)(9-\lambda) - 9 \\
&= 9 - \lambda - 9\lambda + \lambda^2 - 9 \\
&= \lambda^2 - 10\lambda \\
&= \lambda(\lambda - 10)
\end{aligned}
$$

$$\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$$

$$\mathbf{X}_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -9 & 3 \\ 3 & -1 \end{pmatrix}$$

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}$$

2.10.19

$$\begin{aligned} \frac{3}{5}a + \frac{4}{5}b &= 0 \\ 3a &= -4b \\ a &= -\frac{4}{3}b \\ a &= -\frac{4}{5} \\ b &= \frac{3}{5} \end{aligned}$$

2.10.21

(b) $\lambda_1 = -2, \lambda_2 = -2, \lambda_3 = 4$

(c)

$$
\begin{aligned}
\mathbf{W}_1 &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \\
\mathbf{V}_2 &= \mathbf{K}_2 - (\mathbf{K}_2 \cdot \mathbf{W}_1)\mathbf{W}_1 \\
&= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix} \\
|\mathbf{V}_2| &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (-1)^2} \\
&= \sqrt{\frac{3}{2}} \\
\mathbf{W}_2 &= \sqrt{\frac{2}{3}} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix}
\end{aligned}
$$

$$
\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}
$$

2.11 Approximation of Eigenvalues

2.11.1

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \\
\mathbf{X}_0 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathbf{A}^5\mathbf{X}_0 &= 32\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathbf{K}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\lambda_1 &= \frac{\mathbf{A}\mathbf{X}_5 \cdot \mathbf{X}_5}{\mathbf{X}_5 \cdot \mathbf{X}_5} \\
&= 2
\end{aligned}$$

2.11.3

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 2 & 4 \\ 3 & 13 \end{pmatrix} \\
\mathbf{X}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathbf{X}_2 &= \begin{pmatrix} \frac{3}{8} \\ 1 \end{pmatrix} \\
\mathbf{X}_3 &= \begin{pmatrix} 0.3363 \\ 1 \end{pmatrix} \\
\mathbf{X}_4 &= \begin{pmatrix} 0.3335 \\ 1 \end{pmatrix} \\
\mathbf{X}_5 &= \begin{pmatrix} 0.3333 \\ 1 \end{pmatrix} \\
\mathbf{K}_1 &= \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \\
\lambda_1 &= \frac{\mathbf{A}\mathbf{K}_1 \cdot \mathbf{K}_1}{\mathbf{K}_1 \cdot \mathbf{K}_1} \\
&= 14
\end{aligned}$$

2.11.7

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} \\
\mathbf{X}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathbf{A}^5\mathbf{X}_1 &= \begin{pmatrix} 0.5008 \\ 1 \end{pmatrix} \\
\lambda_1 &= 7 \\
\mathbf{K}_1 &= \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \\
\mathbf{B} &= \mathbf{A} - \lambda_1 \mathbf{K}_1 \mathbf{K}_1^T \\
&= \begin{pmatrix} \frac{8}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{2}{5} \end{pmatrix} \\
\mathbf{B}^5\mathbf{X}_1 &= \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \\
\lambda_2 &= 2
\end{aligned}$$

2.11.11

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} \\
\det \mathbf{A} &= 1 \\
\mathbf{A}^{-1} &= \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix} \\
\mathbf{X}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
(\mathbf{A}^{-1})^5\mathbf{X}_1 &= \begin{pmatrix} 1 \\ -0.7913 \end{pmatrix} \\
\lambda_1' &\approx 4.78 \\
\lambda_1 &\approx 0.21
\end{aligned}$$

2.11.13

(a)

$$\begin{aligned}
EI\frac{d^2y}{dx^2} + Py &= 0 \\
EI\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + Py_i &= 0 \\
EI(y_{i+1} - 2y_i + y_{i-1}) + Ph^2 y_i &= 0
\end{aligned}$$

(b)

$$\begin{aligned}EI(y_2-2y_1)+Ph^2y_1&=0\\EI(2y_1-y_2)&=Ph^2y_1\end{aligned}$$

$$\begin{aligned}EI(y_3-2y_2+y_1)+Ph^2y_2&=0\\EI(-y_1+2y_2-y_3)&=Ph^2y_2\end{aligned}$$

$$\begin{aligned}EI(-2y_3+y_2)+Ph^2y_3&=0\\EI(-y_2+2y_3)&=Ph^2y_3\end{aligned}$$

$$\begin{pmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=\frac{PL^2}{16EI}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$$

(c)

$$\begin{aligned}\det\mathbf{A}&=2\begin{vmatrix}2&-1\\-1&2\end{vmatrix}+\begin{vmatrix}-1&-1\\0&2\end{vmatrix}\\&=4\\\mathbf{A}^{-1}&=\frac{1}{\det\mathbf{A}}\operatorname{adj}\mathbf{A}\\&=\frac{1}{4}\begin{pmatrix}3&2&1\\2&4&2\\1&2&3\end{pmatrix}^T\\&=\begin{pmatrix}\frac{3}{4}&\frac{1}{2}&\frac{1}{4}\\\frac{1}{2}&1&\frac{1}{2}\\\frac{1}{4}&\frac{1}{2}&\frac{3}{4}\end{pmatrix}\end{aligned}$$

(d)

$$\begin{aligned}\mathbf{X}_1&=\begin{pmatrix}1\\1\\1\end{pmatrix}\\(\mathbf{A}^{-1})^6\mathbf{X}_1&=\begin{pmatrix}0.7071\\1\\0.7071\end{pmatrix}\\\lambda_1'&\approx1.7071\\\lambda_1&\approx0.59\end{aligned}$$

(e)

$$\begin{aligned}
\lambda_1 &= \frac{PL^2}{16EI} \\
P &= \frac{16EI\lambda_1}{L^2} \\
&\approx \frac{9.44EI}{L^2}
\end{aligned}$$

2.12 Diagonalization

2.12.1

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \\
\det(\mathbf{A} - \lambda\mathbf{I}) &= (2-\lambda)(4-\lambda) - 3 \\
&= 8 - 2\lambda - 4\lambda + \lambda^2 - 3 \\
&= \lambda^2 - 6\lambda + 5 \\
&= (\lambda - 5)(\lambda - 1) \\
\lambda_1 &= 1 \\
\lambda_2 &= 5
\end{aligned}$$

$$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$$

$$\mathbf{X}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}$$

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{aligned}
\mathbf{P} &= \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \\
\mathbf{P}^{-1} &= \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \\
\mathbf{D} &= \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}
\end{aligned}$$

2.12.3

Not diagonalisable

2.12.5

$$\mathbf{P} = \begin{pmatrix} 13 & 1 \\ 2 & 1 \end{pmatrix}$$
$$\mathbf{D} = \begin{pmatrix} -7 & 0 \\ 0 & 4 \end{pmatrix}$$

2.12.7

$$\mathbf{P} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$\mathbf{D} = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

2.12.21

$$\mathbf{P} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

2.12.23

$$\mathbf{P} = \begin{pmatrix} \sqrt{\frac{2}{5}} & -\sqrt{\frac{5}{2}} \\ 1 & 1 \end{pmatrix}$$
$$\mathbf{D} = \begin{pmatrix} 10 & 0 \\ 0 & 3 \end{pmatrix}$$

2.12.35

$$\begin{aligned} \mathbf{A} &= \mathbf{PDP}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \end{aligned}$$

2.12.39

$$\begin{aligned}
\mathbf{D} &= \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \\
\mathbf{P} &= \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \\
\mathbf{A}^5 &= \mathbf{P}\mathbf{D}^5\mathbf{P}^{-1} \\
&= \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 32 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \\
&= \begin{pmatrix} 21 & 11 \\ 22 & 10 \end{pmatrix}
\end{aligned}$$

2.13 LU-Factorisation

2.13.1

$$\begin{aligned}
\mathbf{A} &= \mathbf{L}\mathbf{U} \\
\begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} \\
&= \begin{pmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \end{pmatrix} \\
u_{11} &= 2 \\
u_{12} &= -2 \\
l_{21} &= \frac{1}{2} \\
u_{22} &= 3 \\
\mathbf{L} &= \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \\
\mathbf{U} &= \begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix}
\end{aligned}$$

2.13.3

$$\begin{pmatrix} -1 & 4 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \end{pmatrix}$$
$$u_{11} = -1$$
$$u_{12} = 4$$
$$l_{21} = -2$$
$$u_{22} = 10$$
$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$
$$\mathbf{U} = \begin{pmatrix} -1 & 4 \\ 0 & 10 \end{pmatrix}$$

2.13.11

$$\begin{pmatrix} 3 & 9 \\ 1 & 11 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & 9 \\ 0 & 8 \end{pmatrix}$$
$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{pmatrix}$$
$$\mathbf{U} = \begin{pmatrix} 3 & 9 \\ 0 & 8 \end{pmatrix}$$

2.13.13

$$\begin{pmatrix} -4 & -2 \\ -1 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} -4 & -2 \\ 0 & -\frac{5}{2} \end{pmatrix}$$
$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{1}{4} & 1 \end{pmatrix}$$
$$\mathbf{U} = \begin{pmatrix} -4 & -2 \\ 0 & -\frac{5}{2} \end{pmatrix}$$

2.13.21

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix}$$

$$\mathbf{LY} = \mathbf{B}$$

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$y_1 = 1$$

$$y_2 = -\frac{5}{2}$$

$$\mathbf{UX} = \mathbf{Y}$$

$$\begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{5}{2} \end{pmatrix}$$

$$x_2 = -\frac{5}{6}$$

$$2x_1 - 2\left(-\frac{5}{6}\right) = 1$$

$$2x_1 = -\frac{2}{3}$$

$$x_1 = -\frac{1}{3}$$

$$\mathbf{X} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{5}{6} \end{pmatrix}$$

2.13.23

$$
\begin{aligned}
\mathbf{L} &= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \\
\mathbf{U} &= \begin{pmatrix} -1 & 4 \\ 0 & 10 \end{pmatrix} \\
\mathbf{LY} &= \mathbf{B} \\
\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 15 \\ 5 \end{pmatrix} \\
y_1 &= 15 \\
y_2 &= 35 \\
\mathbf{UX} &= \mathbf{Y} \\
\begin{pmatrix} -1 & 4 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 15 \\ 35 \end{pmatrix} \\
x_2 &= \frac{7}{2} \\
-x_1 + 4\frac{7}{2} &= 15 \\
x_1 &= -1 \\
\mathbf{X} &= \begin{pmatrix} -1 \\ \frac{7}{2} \end{pmatrix}
\end{aligned}
$$

2.13.31

$$\begin{aligned}
\mathbf{LY} &= \mathbf{B} \\
\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &= \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \\
y_1 &= 2 \\
y_2 &= 2 \\
y_3 &= -3 \\
\mathbf{UX} &= \mathbf{Y} \\
\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} \\
x_3 &= -3 \\
x_2 &= 5 \\
x_1 &= 0 \\
\mathbf{X} &= \begin{pmatrix} 0 \\ 5 \\ -3 \end{pmatrix}
\end{aligned}$$

2.14 Cryptography

2.14.1

$$\begin{aligned}
\mathbf{B} &= \mathbf{AM} \\
&= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 19 & 5 & 14 & 4 & 0 \\ 8 & 5 & 12 & 16 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 35 & 15 & 38 & 36 & 0 \\ 27 & 10 & 26 & 20 & 0 \end{pmatrix}
\end{aligned}$$

2.14.7

$$\begin{aligned}
\mathbf{A}^{-1} &= \begin{pmatrix} 2 & -3 \\ -5 & 8 \end{pmatrix} \\
\mathbf{M} &= \mathbf{A^{-1}B} \\
&= \begin{pmatrix} 19 & 20 & 21 & 4 & 25 \\ 0 & 8 & 1 & 18 & 4 \end{pmatrix} \\
&= \text{STUDY HARD}
\end{aligned}$$

2.14.11

$$\mathbf{A^{-1}B} = \mathbf{M}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 17 & 16 \\ -30 & -31 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ m_{21} & m_{22} \end{pmatrix}$$

$$17a_{11} - 30a_{12} = 4$$

$$16a_{11} - 31a_{12} = 1$$

$$17a_{11} - \frac{30}{31}16a_{11} = 4 - \frac{30}{31}$$

$$527a_{11} - 480a_{11} = 124 - 30$$

$$47a_{11} = 94$$

$$a_{11} = 2$$

$$-30a_{12} + \frac{17}{16}31a_{12} = 4 - \frac{17}{16}$$

$$-480a_{12} + 527a_{12} = 64 - 17$$

$$47a_{12} = 47$$

$$a_{12} = 1$$

$$\begin{pmatrix} 2 & 1 \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 5 & 25 \\ -6 & -50 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ 1 & 25 \end{pmatrix}$$

$$5a_{21} - 6a_{22} = 1$$

$$25a_{21} - 50a_{22} = 25$$

$$5a_{21} - \frac{6}{50}25a_{21} = 1 - \frac{6}{50}25$$

$$250a_{21} - 150a_{21} = 50 - 150$$

$$100a_{21} = -100$$

$$a_{21} = -1$$

$$4a_{22} = -4$$

$$a_{22} = -1$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\mathbf{A^{-1}B} = \text{DAD I NEED MONEY TODAY}$$

2.15 Chapter in Review

2.15.1

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{pmatrix}$$

2.15.3

$$\mathbf{AB} = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$$
$$\mathbf{BA} = \begin{pmatrix} 11 \end{pmatrix}$$

2.15.5

False

2.15.7

$$\det\left(\frac{1}{2}\mathbf{A}\right) = \frac{5}{8}$$
$$\det -\mathbf{A}^T = -5$$

2.15.9

0

2.15.11

False

2.15.13

True

2.15.15

False

2.15.17

True

2.15.19

False

2.15.23

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}^2$$

2.15.25

$$\left(\begin{array}{ccc|c} 5 & -1 & 1 & -9 \\ 2 & 4 & 0 & 27 \\ 1 & 1 & 5 & 9 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{5} & \frac{1}{5} & -\frac{9}{5} \\ 1 & 2 & 0 & \frac{27}{2} \\ 1 & 1 & 5 & 9 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{5} & \frac{1}{5} & -\frac{9}{5} \\ 0 & \frac{11}{5} & -\frac{1}{5} & \frac{153}{10} \\ 0 & \frac{6}{5} & \frac{24}{5} & \frac{54}{5} \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{5} & \frac{1}{5} & -\frac{9}{5} \\ 0 & 1 & -\frac{1}{11} & \frac{153}{22} \\ 0 & 1 & 4 & 9 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{5} & \frac{1}{5} & -\frac{9}{5} \\ 0 & 1 & -\frac{1}{11} & \frac{153}{22} \\ 0 & 0 & \frac{45}{11} & \frac{45}{22} \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{5} & \frac{1}{5} & -\frac{9}{5} \\ 0 & 1 & -\frac{1}{11} & \frac{153}{22} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{5} & 0 & -\frac{19}{10} \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$$

$$\mathbf{X} = \begin{pmatrix} -\frac{1}{2} \\ 7 \\ \frac{1}{2} \end{pmatrix}$$

2.15.29

240

2.15.31

$$\begin{vmatrix} 1 & -1 & 1 \\ 5 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = (1)\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + (1)\begin{vmatrix} 5 & -1 \\ 1 & 1 \end{vmatrix} + (1)\begin{vmatrix} 5 & 1 \\ 1 & 2 \end{vmatrix}$$
$$= 18$$

The matrix is nonsingular so the system only has the trivial solution.

2.15.35

$$\begin{aligned}
\det \mathbf{A} &= \begin{vmatrix} 1 & 2 & -3 \\ 2 & -4 & 3 \\ 0 & 4 & 6 \end{vmatrix} \\
&= (1)\begin{vmatrix} -4 & 3 \\ 4 & 6 \end{vmatrix} - (2)\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} + (-3)\begin{vmatrix} 2 & -4 \\ 0 & 4 \end{vmatrix} \\
&= -36 - 24 - 24 \\
&= -84 \\
x_1 &= \frac{\det \mathbf{A}_1}{\det \mathbf{A}} \\
&= -\frac{1}{2} \\
x_2 &= \frac{\det \mathbf{A}_2}{\det \mathbf{A}} \\
&= \frac{1}{4} \\
x_3 &= \frac{\det \mathbf{A}_3}{\det \mathbf{A}} \\
&= \frac{2}{3}
\end{aligned}$$

2.15.37

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\begin{aligned} x &= \frac{\begin{vmatrix} X & \sin\theta \\ Y & \cos\theta \end{vmatrix}}{\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}} \\ &= X\cos\theta - Y\sin\theta \\ y &= \begin{vmatrix} \cos\theta & X \\ -\sin\theta & Y \end{vmatrix} \\ &= X\sin\theta + Y\cos\theta \end{aligned}$$

2.15.39

$$\begin{pmatrix} 2 & 3 & -1 \\ 1 & -2 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 9 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \frac{1}{3} \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 1 \\ 4 & 6 & 7 \end{pmatrix} \begin{pmatrix} 6 \\ -3 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} 7 \\ 5 \\ 23 \end{pmatrix} \end{aligned}$$

2.15.41

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= (1-\lambda)(3-\lambda) - 8 \\
&= 3 - \lambda - 3\lambda + \lambda^2 - 8 \\
&= \lambda^2 - 4\lambda - 5 \\
&= (\lambda - 5)(\lambda + 1) \\
\lambda_1 &= -1 \\
\lambda_2 &= 5
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \\
\mathbf{K}_1 &= \begin{pmatrix} 1 & -1 \end{pmatrix} \\
\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \\
\mathbf{K}_2 &= \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}
\end{aligned}$$

2.15.43

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} \\
&= (3-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & 3-\lambda \end{vmatrix} + 4\begin{vmatrix} 2 & -\lambda \\ 4 & 2 \end{vmatrix} \\
&= (3-\lambda)(-\lambda(3-\lambda)-4) - 2(2(3-\lambda)-8) + 4(4+4\lambda) \\
&= (3-\lambda)(\lambda^2 - 3\lambda - 4) - 2(-2-2\lambda) + 16 + 16\lambda \\
&= 3\lambda^2 - 9\lambda - 12 - \lambda^3 + 3\lambda^2 + 4\lambda + 4 + 4\lambda + 16 + 16\lambda \\
&= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 \\
&= -(\lambda-8)(\lambda+1)^2
\end{aligned}$$

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{2}{5} & -\frac{4}{5} \\ 1 & -4 & 1 \\ 1 & \frac{1}{2} & -\frac{5}{4} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & -\frac{18}{5} & \frac{9}{5} \\ 0 & \frac{9}{10} & -\frac{9}{20} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{X}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{X}_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

2.15.47

$$a^2 + b^2 + c^2 = 1$$

$$-a\frac{1}{\sqrt{2}} + c\frac{1}{\sqrt{2}} = 0$$

$$a = c$$

$$a\frac{1}{\sqrt{3}} + b\frac{1}{\sqrt{3}} + c\frac{1}{\sqrt{3}} = 0$$

$$a + b + c = 0$$

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

2.15.57

$$\begin{pmatrix}1 & 1 & 1\\ 1 & -2 & 3\\ 2 & 0 & -3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}6\\ 2\\ 3\end{pmatrix}$$

$$\mathbf{A} = \mathbf{LU}$$

$$\begin{pmatrix}1 & 1 & 1\\ 1 & -2 & 3\\ 2 & 0 & -3\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{pmatrix}\begin{pmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{pmatrix}$$

$$= \begin{pmatrix}u_{11} & u_{12} & u_{13}\\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23}\\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{pmatrix}$$

$$u_{11} = 1$$

$$u_{12} = 1$$

$$u_{13} = 1$$

$$l_{21} = 1$$

$$u_{22} = -3$$

$$u_{23} = 2$$

$$l_{31} = 2$$

$$l_{32} = \frac{2}{3}$$

$$u_{33} = -\frac{19}{3}$$

$$\mathbf{L} = \begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 2 & \frac{2}{3} & 1\end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix}1 & 1 & 1\\ 0 & -3 & 2\\ 0 & 0 & -\frac{19}{3}\end{pmatrix}$$

$$
\begin{aligned}
\mathbf{AX} &= \mathbf{B} \\
\mathbf{LUX} &= \mathbf{B} \\
\mathbf{LY} &= \mathbf{B} \\
\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & \frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &= \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} \\
\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &= \begin{pmatrix} 6 \\ -4 \\ -\frac{19}{3} \end{pmatrix} \\
\mathbf{UX} &= \mathbf{Y} \\
\begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & -\frac{19}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 6 \\ -4 \\ -\frac{19}{3} \end{pmatrix} \\
\mathbf{X} &= \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}
\end{aligned}
$$

3 Vector Calculus

3.1 Vector Functions

3.1.11

$$
\begin{aligned}
x &= t \\
y &= t \\
z &= 2t^2 \\
\mathbf{r} &= \langle t, t, 2t^2 \rangle
\end{aligned}
$$

3.1.13

$$
\begin{aligned}
x &= 3\cos t \\
z &= 9 - 9\cos^2 t \\
&= 9(1 - \cos^2 t) \\
&= 9\sin^2 t \\
9\cos^2 t + y^2 &= 9 \\
y^2 &= 9\sin^2 t \\
y &= 3\sin t \\
\mathbf{r} &= \langle 3\cos t, 3\sin t, 9\sin^2 t \rangle
\end{aligned}
$$

3.1.15

$$\begin{aligned}\mathbf{r}(t) &= \langle \frac{\sin 2t}{t}, (t-2)^5, t\ln t\rangle \\ \lim_{t\to 0^+}\mathbf{r}(t) &= \langle 2, -32, 0\rangle\end{aligned}$$

3.1.17

$$\begin{aligned}\mathbf{r}(t) &= \langle \ln t, 1, 0\rangle \\ \mathbf{r}'(t) &= \langle \frac{1}{t}, 0, 0\rangle \\ \mathbf{r}''(t) &= \langle -\frac{1}{t^2}, 0, 0\rangle\end{aligned}$$

3.1.19

$$\begin{aligned}\mathbf{r}(t) &= \langle te^{2t}, t^3, 4t^2 - t\rangle \\ \mathbf{r}'(t) &= \langle e^{2t} + 2te^{2t}, 3t^2, 8t-1\rangle \\ \mathbf{r}''(t) &= \langle 4e^{2t} + 4te^{2t}, 6t, 8\rangle\end{aligned}$$

3.1.25

$$\begin{aligned}x &= 2 + t \\ y &= 2 + 2t \\ z &= \frac{8}{3} + 4t\end{aligned}$$

3.1.27

$$\frac{d}{dt}[\mathbf{r}(t)\times\mathbf{r}'(t)] = \mathbf{r}(t)\times\mathbf{r}''(t)$$

3.1.29

$$\begin{aligned}\frac{d}{dt}[\mathbf{r}(t)\cdot(\mathbf{r}'(t)\times\mathbf{r}''(t))] &= \mathbf{r}(t)\cdot\frac{d}{dt}(\mathbf{r}'(t)\times\mathbf{r}''(t)) \\ &= \mathbf{r}(t)\cdot(\mathbf{r}'(t)\times\mathbf{r}'''(t))\end{aligned}$$

3.1.31

$$\frac{d}{dt}\left[\mathbf{r}_1(2t) + \mathbf{r}_2\left(\frac{1}{t}\right)\right] = 2\mathbf{r}_1'(2t) - \frac{1}{t^2}\mathbf{r}_2'\left(\frac{1}{t}\right)$$

3.1.33

$$\int_{-1}^{2} \langle t, 3t^2, 4t^3 \rangle \, dt = \langle \frac{3}{2}, 9, 15 \rangle$$

3.1.35

$$\int \langle te^t, -e^{-2t}, te^{t^2} \rangle \, dt = \langle e^t(t-1), \frac{1}{2}e^{-2t}, \frac{1}{2}e^{t^2} \rangle + \mathbf{c}$$

3.1.37

$$\begin{aligned}
\mathbf{r}(t) &= \langle 6t, 3t^2, t^3 \rangle + \mathbf{c} \\
\mathbf{r}(0) &= \mathbf{r}_0 \\
\mathbf{c} &= \mathbf{r}_0 \\
\mathbf{r}(t) &= \langle 6t+1, 3t^2-2, t^3+1 \rangle
\end{aligned}$$

3.1.39

$$\begin{aligned}
\mathbf{r}'(t) &= \langle 6(t^2-1), 7-6\sqrt{t}, 2(t-1) \rangle \\
\mathbf{r}(t) &= \langle 6\left(\frac{1}{3}t^3 - t + 1\right), 7t - 4t^{3/2} - 3, 2\left(\frac{1}{2}t^2 - t\right) \rangle
\end{aligned}$$

3.1.41

$$\begin{aligned}
\mathbf{r}'(t) &= \langle -a\sin t, a\cos t, c \rangle \\
s &= \int_0^{2\pi} ||\mathbf{r}'(t)|| \, dt \\
&= \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t + c^2} \, dt \\
&= \int_0^{2\pi} \sqrt{a^2 + c^2} \, dt \\
&= 2\pi\sqrt{a^2+c^2}
\end{aligned}$$

3.1.43

$$\begin{aligned}
\mathbf{r}'(t) &= \langle e^t(\cos 2t - 2\sin 2t), e^t(\sin 2t + 2\cos 2t), e^t\rangle \\
s &= \int_0^{3\pi} ||\mathbf{r}'(t)||\,dt \\
&= \int_0^{3\pi} \sqrt{(e^t(\cos 2t - 2\sin 2t))^2 + (e^t(\sin 2t + 2\cos 2t))^2 + (e^t)^2}\,dt \\
&= \int_0^{3\pi} e^t\sqrt{(\cos 2t - 2\sin 2t)^2 + (\sin 2t + 2\cos 2t)^2 + 1}\,dt \\
&= \int_0^{3\pi} e^t\sqrt{5\cos^2 2t + 5\sin^2 2t + 1}\,dt \\
&= \sqrt{6}\int_0^{3\pi} e^t\,dt \\
&= \sqrt{6}(e^{3\pi} - 1)
\end{aligned}$$

3.1.45

$$\begin{aligned}
\mathbf{r}(t) &= \langle a\cos t, a\sin t\rangle \\
\mathbf{r}'(t) &= \langle -a\sin t, a\cos t\rangle \\
||\mathbf{r}'(t)|| &= \sqrt{a^2\sin^2 t + a^2\cos^2 t} \\
&= a \\
s &= \int_0^t a\,du \\
&= at \\
t &= \frac{s}{a} \\
\mathbf{r}(s) &= \langle a\cos\frac{s}{a}, a\sin\frac{s}{a}\rangle \\
\mathbf{r}'(s) &= \langle -\sin\frac{s}{a}, \cos\frac{s}{a}\rangle \\
||\mathbf{r}'(s)|| &= \sqrt{\sin^2\frac{s}{a} + \cos^2\frac{s}{a}} \\
&= 1
\end{aligned}$$

3.1.47

$$\begin{aligned}
||\mathbf{r}(t)|| &= c \\
\sqrt{f(t)^2 + g(t)^2 + h(t)^2} &= c \\
\frac{1}{\sqrt{f(t)^2 + g(t)^2 + h(t)^2}}(2f(t)f'(t) + 2g(t)g'(t) + 2h(t)h'(t)) &= 0 \\
\frac{2(f(t)f'(t) + g(t)g'(t) + h(t)h'(t))}{||\mathbf{r}(t)||} &= 0 \\
\mathbf{r}(t) \cdot \mathbf{r}'(t) &= 0
\end{aligned}$$

Either $||\mathbf{r}'(t)|| = 0$ or they're perpendicular.

3.2 Motion on a Curve

3.2.1

$$\begin{aligned}
\mathbf{r}(t) &= \langle t^2, \frac{1}{4}t^4, 0\rangle \\
\mathbf{r}'(t) &= \langle 2t, t^3, 0\rangle \\
||\mathbf{r}'(1)|| &= ||\langle 2, 1, 0\rangle|| \\
&= \sqrt{2^2 + 1^2 + 0^2} \\
&= \sqrt{5}
\end{aligned}$$

3.2.9

$$\begin{aligned}
\mathbf{r}(t) &= \langle t^2, t^3 - 2t, t^2 - 5t\rangle \\
z &= 0 \\
t^2 - 5t &= 0 \\
t(t-5) &= 0 \\
t_1 &= 0 \\
t_2 &= 5 \\
\mathbf{r}(0) &= \langle 0, 0, 0\rangle \\
\mathbf{r}(5) &= \langle 25, 115, 0\rangle \\
\mathbf{r}'(t) &= \langle 2t, 3t^2 - 2, 2t - 5\rangle \\
\mathbf{r}'(0) &= \langle 0, -2, -5\rangle \\
\mathbf{r}'(5) &= \langle 10, 73, 5\rangle \\
\mathbf{r}''(t) &= \langle 2, 6t, 2\rangle \\
\mathbf{r}''(0) &= \langle 2, 0, 2\rangle \\
\mathbf{r}''(5) &= \langle 2, 30, 2\rangle
\end{aligned}$$

3.2.11

(a)

$$\begin{aligned}
\mathbf{r}''(t) &= \langle 0, -g\rangle \\
\mathbf{r}'(t) &= \langle 240\sqrt{3}, 240 - gt\rangle \\
\mathbf{r}(t) &= \langle 240\sqrt{3}t, 240t - \frac{1}{2}gt^2\rangle \\
x(t) &= 240\sqrt{3}t \\
y(t) &= 240t - \frac{1}{2}gt^2 \\
&= 240t - 16t^2
\end{aligned}$$

(b)

$$\begin{aligned}
240 - 32t &= 0 \\
t &= \frac{15}{2} \\
y\left(\frac{15}{2}\right) &= 240\frac{15}{2} - 16\left(\frac{15}{2}\right)^2 \\
&= 1800 - 900 \\
&= 900\,\mathrm{ft}
\end{aligned}$$

(c)

$$\begin{aligned}
y(t) &= 0 \\
240t - 16t^2 &= 0 \\
t(240 - 16t) &= 0 \\
t_1 &= 0 \\
t_2 &= 15 \\
x(15) &= 240\sqrt{3}(15) \\
&= 3600\sqrt{3} \\
&\approx 6235\,\mathrm{ft}
\end{aligned}$$

(d)

$$\begin{aligned}
||\mathbf{v}(15)|| &= \sqrt{(240\sqrt{3})^2 + (240 - 32(15))^2} \\
&= 480\,\mathrm{ft/s}
\end{aligned}$$

3.2.23

$$\begin{aligned}
\mathbf{r}(t) &= \langle r_0 \cos\omega t, r_0 \sin\omega t\rangle \\
\mathbf{v}(t) &= \langle -\omega r_0 \sin\omega t, \omega r_0 \cos\omega t\rangle \\
v &= ||\mathbf{v}(t)|| \\
&= \sqrt{(-\omega r_0 \sin\omega t)^2 + (\omega r_0 \cos\omega t)^2} \\
&= \omega r_0 \\
\mathbf{a}(t) &= \langle -\omega^2 r_0 \cos\omega t, -\omega^2 r_0 \sin\omega t\rangle \\
&= -\omega^2 \mathbf{r}(t) \\
a &= ||\mathbf{a}(t)|| \\
&= || -\omega^2 \mathbf{r}(t)|| \\
&= \omega^2 r_0 \\
&= \frac{v^2}{r_0}
\end{aligned}$$

3.2.25

$$\begin{aligned}
m'g &= mg - ma \\
m' &= m\left(1 - \frac{v^2}{gr}\right) \\
&\approx 191.3\,\mathrm{lb}
\end{aligned}$$

3.2.27

$$\mathbf{v}(t) = \langle 6t^2x, -4ty^2, 2t(z+1)\rangle$$

$$\begin{aligned}
\frac{dx}{dt} &= 6t^2x \\
\frac{1}{x}\frac{dx}{dt} &= 6t^2 \\
\ln x &= 2t^3 + c_1 \\
x &= c_1e^{2t^3}
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dt} &= -4ty^2 \\
\frac{1}{y^2}\frac{dy}{dt} &= -4t \\
-\frac{1}{y} &= -2t^2 + c_2 \\
y &= \frac{1}{2t^2 + c_2}
\end{aligned}$$

$$\begin{aligned}
\frac{dz}{dt} &= 2t(z+1) \\
\frac{1}{z+1}\frac{dz}{dt} &= 2t \\
\ln(z+1) &= t^2 + c_3 \\
z+1 &= c_3e^{t^2} \\
z &= c_3e^{t^2} - 1
\end{aligned}$$

$$\mathbf{r}(t) = \langle c_1e^{2t^3}, \frac{1}{2t^2+c_2}, c_3e^{t^2} - 1\rangle$$

3.2.29

(a)

$$\begin{aligned}
\boldsymbol{\tau} &= \mathbf{r} \times \mathbf{F} \\
&= \mathbf{r} \times \left(-k\frac{Mm}{r^2}\frac{\mathbf{r}}{r}\right) \\
&= -\frac{kMm}{r^3}(\mathbf{r} \times \mathbf{r}) \\
&= \mathbf{0}
\end{aligned}$$

(b) Torque is the derivative of angular momentum with respect to time. If there's no torque angular momentum doesn't change.

3.3 Curvature and Components of Acceleration

3.3.1

$$
\begin{aligned}
\mathbf{r}(t) &= \langle t\cos t - \sin t, t\sin t + \cos t, t^2\rangle \\
\mathbf{r}'(t) &= \langle -t\sin t, t\cos t, 2t\rangle \\
||\mathbf{r}'(t)|| &= \sqrt{(-t\sin t)^2 + (t\cos t)^2 + (2t)^2} \\
&= t\sqrt{\sin^2 t + \cos^2 t + 4} \\
&= \sqrt{5}t \\
\mathbf{T} &= \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} \\
&= \frac{1}{\sqrt{5}}\langle -\sin t, \cos t, 2\rangle
\end{aligned}
$$

3.3.3

$$\begin{aligned}
\mathbf{r}(t) &= \langle a\cos t, a\sin t, ct\rangle \\
\mathbf{r}'(t) &= \langle -a\sin t, a\cos t, c\rangle \\
||\mathbf{r}'(t)|| &= \sqrt{(-a\sin t)^2 + (a\cos t)^2 + (c)^2} \\
&= \sqrt{a^2 + c^2} \\
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} \\
&= \frac{1}{\sqrt{a^2 + c^2}}\langle -a\sin t, a\cos t, c\rangle \\
\frac{d\mathbf{T}}{dt} &= \frac{1}{\sqrt{a^2 + c^2}}\langle -a\cos t, -a\sin t, 0\rangle \\
\left|\left|\frac{d\mathbf{T}}{dt}\right|\right| &= \frac{a}{\sqrt{a^2 + c^2}} \\
\mathbf{N} &= \frac{d\mathbf{T}/dt}{||d\mathbf{T}/dt||} \\
&= \langle -\cos t, -\sin t, 0\rangle \\
\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\
&= -\frac{1}{\sqrt{a^2 + c^2}}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -a\sin t & a\cos t & c \\ \cos t & \sin t & 0 \end{vmatrix} \\
&= \frac{1}{\sqrt{a^2 + c^2}}\langle c\sin t, -c\cos t, a\rangle \\
\kappa &= \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||} \\
&= \frac{a}{a^2 + c^2}
\end{aligned}$$

3.3.5

$$\begin{aligned}
\mathbf{r}(t) &= \langle 2\cos t, 2\sin t, 3t\rangle \\
\mathbf{r}(\pi/4) &= \langle \sqrt{2}, \sqrt{2}, \frac{3\pi}{4}\rangle \\
\mathbf{B}(t) &= \frac{1}{\sqrt{13}}\langle 3\sin t, -3\cos t, 2\rangle \\
\mathbf{B}(\pi/4) &= \frac{1}{\sqrt{13}}\langle \frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 2\rangle
\end{aligned}$$

$$\begin{aligned}
0 &= \mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0) \\
&= \mathbf{B}(\pi/4)\cdot(\mathbf{r}-\mathbf{r}(\pi/4)) \\
&= \frac{1}{\sqrt{13}}\langle \frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 2\rangle \cdot \left(\langle x, y, z\rangle - \langle \sqrt{2}, \sqrt{2}, \frac{3\pi}{4}\rangle\right) \\
&= \frac{1}{\sqrt{13}}\langle \frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 2\rangle \cdot \langle x-\sqrt{2}, y-\sqrt{2}, z-\frac{3\pi}{4}\rangle \\
&= \frac{3}{\sqrt{26}}(x-\sqrt{2}) - \frac{3}{\sqrt{26}}(y-\sqrt{2}) + \frac{2}{\sqrt{13}}\left(z-\frac{3\pi}{4}\right) \\
&= \frac{3}{\sqrt{2}}x - 3 - \frac{3}{\sqrt{2}}y + 3 + 2z - \frac{3\pi}{2} \\
\frac{3\pi}{2} &= \frac{3}{\sqrt{2}}x - \frac{3}{\sqrt{2}}y + 2z \\
3\pi &= 3\sqrt{2}x - 3\sqrt{2}y + 4z
\end{aligned}$$

3.3.7

$$\begin{aligned}
\mathbf{r}(t) &= \langle 1, t, t^2 \rangle \\
\mathbf{r}'(t) &= \langle 0, 1, 2t \rangle \\
||\mathbf{r}'(t)|| &= \sqrt{1+4t^2} \\
\mathbf{r}''(t) &= \langle 0, 0, 2 \rangle \\
a_T &= \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{||\mathbf{r}'(t)||} \\
&= \frac{4t}{\sqrt{1+4t^2}} \\
\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} \\
&= \langle 2, 0, 0 \rangle \\
a_N &= \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||} \\
&= \frac{2}{\sqrt{1+4t^2}}
\end{aligned}$$

3.3.17

$$\begin{aligned}
\mathbf{r}(t) &= \langle a\cos t, b\sin t, ct \rangle \\
\mathbf{r}'(t) &= \langle -a\sin t, b\cos t, c \rangle \\
||\mathbf{r}'(t)|| &= \sqrt{(-a\sin t)^2 + (b\cos t)^2 + (c)^2} \\
&= \sqrt{a^2\sin^2 t + b^2\cos^2 t + c^2} \\
\mathbf{r}''(t) &= \langle -a\cos t, -b\sin t, 0 \rangle \\
\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -a\sin t & b\cos t & c \\ -a\cos t & -b\sin t & 0 \end{vmatrix} \\
&= \langle bc\sin t, -ac\cos t, ab \rangle \\
||\mathbf{r}'(t) \times \mathbf{r}''(t)|| &= \sqrt{(bc\sin t)^2 + (-ac\cos t)^2 + (ab)^2} \\
&= \sqrt{b^2c^2\sin^2 t + a^2c^2\cos^2 t + a^2b^2} \\
\kappa &= \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)}{||\mathbf{r}'(t)||^3} \\
&= \frac{\sqrt{b^2c^2\sin^2 t + a^2c^2\cos^2 t + a^2b^2}}{(a^2\sin^2 t + b^2\cos^2 t + c^2)^{3/2}}
\end{aligned}$$

3.3.23

$$
\begin{aligned}
y &= x^2 \\
\kappa &= \frac{2}{(1+4x^2)^{3/2}} \\
\rho &= \frac{1}{\kappa} \\
&= \frac{(1+4x^2)^{3/2}}{2} \\
\kappa(0) &= 2 \\
\kappa(1) &= \frac{2}{5\sqrt{5}}
\end{aligned}
$$

The curve is sharper at $(0,0)$.

3.4 Partial Derivatives

3.4.13

$$
\begin{aligned}
z &= x^2 - xy^2 + 4y^5 \\
\frac{\partial z}{\partial x} &= 2x - y^2 \\
\frac{\partial z}{\partial y} &= -2xy + 20y^4
\end{aligned}
$$

3.4.15

$$
\begin{aligned}
z &= 5x^4y^3 - x^2y^6 + 6x^5 - 4y \\
\frac{\partial z}{\partial x} &= 20x^3y^3 - 2xy^6 + 30x^4 \\
\frac{\partial z}{\partial y} &= 15x^4y^2 - 6x^2y^5 - 4
\end{aligned}
$$

3.4.33

$$\begin{aligned}
z &= \ln(x^2+y^2) \\
\frac{\partial z}{\partial x} &= \frac{2x}{x^2+y^2} \\
\frac{\partial^2 z}{\partial x^2} &= \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} \\
\frac{\partial z}{\partial y} &= \frac{2y}{x^2+y^2} \\
\frac{\partial^2 z}{\partial y^2} &= \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} \\
\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} &= \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} + \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} \\
&= \frac{4}{x^2+y^2} - \frac{4(x^2+y^2)}{(x^2+y^2)^2} \\
&= 0
\end{aligned}$$

3.4.39

$$\begin{aligned}
z &= e^{uv^2} \\
u &= x^3 \\
v &= x - y^2 \\
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \\
&= (v^2 e^{uv^2})(3x^2) + (2uve^{uv^2})(1) \\
&= 3e^{uv^2} v^2 x^2 + 2e^{uv^2} uv \\
\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \\
&= (v^2 e^{uv^2})(0) + (2uve^{uv^2})(-2y) \\
&= -4uvye^{uv^2}
\end{aligned}$$

3.4.49

$$\begin{aligned}
z &= \ln(u^2+v^2)\\
u &= t^2\\
v &= t^{-2}\\
\frac{dz}{dt} &= \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}\\
&= \frac{2u}{u^2+v^2}2t + \frac{2v}{u^2+v^2} - 2t^{-3}\\
&= \frac{4(ut - vt^{-3})}{u^2+v^2}
\end{aligned}$$

3.4.57

$$\begin{aligned}
A(x,y,\theta) &= \frac{1}{2}xy\sin\theta\\
\frac{dA}{dt} &= \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} + \frac{\partial A}{\partial \theta}\frac{d\theta}{dt}\\
&= \frac{1}{2}y\sin\theta\frac{dx}{dt} + \frac{1}{2}x\sin\theta\frac{dy}{dt} + \frac{1}{2}xy\cos\theta\frac{d\theta}{dt}\\
&\approx 5.31\,\mathrm{cm}^2/\mathrm{s}
\end{aligned}$$

3.5 Directional Derivative

3.5.1

$$\nabla f = \langle 2x - 3x^2y^2, -2x^3y + 4y^3\rangle$$

3.5.3

$$\nabla F = \langle \frac{y^2}{z^3}, \frac{2xy}{z^3}, -\frac{3xy^2}{z^4}\rangle$$

3.5.5

$$\begin{aligned}
\nabla f &= \langle 2x, -8y\rangle\\
\nabla f(2,4) &= \langle 4, -32\rangle
\end{aligned}$$

3.5.7

$$\begin{aligned}
\nabla F &= \langle 2xz^2\sin 4y, 4x^2z^2\cos 4y, 2x^2z\sin 4y\rangle\\
\nabla F(-2,\pi/3,1) &= \langle 2\sqrt{3}, -8, -4\sqrt{3}\rangle
\end{aligned}$$

3.5.9

$$\begin{aligned} D_{\mathbf{u}}f &= \langle 2x, 2y\rangle \cdot \langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle \\ &= \sqrt{3}x + y \end{aligned}$$

3.5.11

$$\begin{aligned} D_{\mathbf{u}}f &= \langle 15x^2y^6, 30x^3y^5\rangle \cdot \langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle \\ &= \frac{15\sqrt{3}}{2}x^2y^6 + 15x^3y^5 \\ D_{\mathbf{u}}f(-1,1) &= \frac{15\sqrt{3}}{2} - 15 \\ &= 15\left(\frac{\sqrt{3}}{2} - 1\right) \\ &= \frac{15}{2}(\sqrt{3} - 2) \end{aligned}$$

3.5.21

$$\begin{aligned} \mathbf{u} &= \frac{\langle -4, -1\rangle}{||\langle -4, -1\rangle||} \\ &= \langle -\frac{4}{\sqrt{17}}, -\frac{1}{\sqrt{17}}\rangle \\ D_{\mathbf{u}}f &= \langle 2(x-y), -2(x-y)\rangle \cdot \langle -\frac{4}{\sqrt{17}}, -\frac{1}{\sqrt{17}}\rangle \\ &= \frac{2}{\sqrt{17}}(-4(x-y) + (x-y)) \\ &= \frac{6}{\sqrt{17}}(y-x) \\ D_{\mathbf{u}}f(4,2,4) &= \frac{6}{\sqrt{17}}(2-4) \\ &= -\frac{12}{\sqrt{17}} \end{aligned}$$

3.5.23

$$\begin{aligned} \nabla f &= \langle 2e^{2x}\sin y, e^{2x}\cos y\rangle \\ \nabla f(0, \pi/4) &= \langle \sqrt{2}, \frac{1}{\sqrt{2}}\rangle \\ \sqrt{2 + \frac{1}{2}} &= \sqrt{\frac{5}{2}} \end{aligned}$$

3.5.27

$$\begin{aligned}\nabla f &= \langle 2x\sec^2(x^2+y^2), 2y\sec^2(x^2+y^2)\rangle \\ -\nabla f(\sqrt{\pi/6}, \sqrt{\pi/6}) &= -\langle 2\sqrt{\frac{\pi}{6}}\sec^2\left(\frac{\pi}{6}+\frac{\pi}{6}\right), 2\sqrt{\frac{\pi}{6}}\sec^2\left(\frac{\pi}{6}+\frac{\pi}{6}\right)\rangle \\ &= -8\sqrt{\frac{\pi}{6}}\langle 1,1\rangle\end{aligned}$$

3.5.33

(a)

$$\mathbf{u}_0 = \frac{\langle 3,-4\rangle}{||\langle 3,-4\rangle||} = \langle \frac{3}{5}, -\frac{4}{5}\rangle$$

(b)

$$\mathbf{u}_{\max} = \frac{\langle 4,3\rangle}{||\langle 4,3\rangle||} = \langle \frac{4}{5}, \frac{3}{5}\rangle$$

(c)

$$\mathbf{u}_{\min} = \langle -\frac{4}{5}, -\frac{3}{5}\rangle$$

3.5.37

$$\nabla f = \langle 3x^2-12, 2y-10\rangle$$

$$\begin{aligned}3x^2-12 &= 0 \\ x^2 &= 4 \\ x &= \pm 2\end{aligned}$$

$$\begin{aligned}2y-10 &= 0 \\ y &= 5\end{aligned}$$

3.5.43

$$\begin{aligned}\nabla f &= \langle 3x^2+y^3+ye^{xy}, -2y^2+3xy^2+xe^{xy}\rangle \\ f &= x^3+xy^3+e^{xy}-\frac{2}{3}y^3\end{aligned}$$

3.6 Tangent Planes and Normal Lines

3.6.13

$$\begin{aligned} F(x,y,z) &= z - x^2 - y^2 \\ \nabla F &= \langle -2x, -2y, 1\rangle \\ \langle -4, -1, 17\rangle & \end{aligned}$$

3.6.15

$$\begin{aligned} x^2+y^2+z^2 &= 9 \\ F(x,y,z) &= x^2+y^2+z^2 \\ \nabla F &= \langle 2x, 2y, 2z\rangle \\ \nabla F(-2,2,1) &= \langle -4, 4, 2\rangle \\ \mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0) &= 0 \\ \langle -4,4,2\rangle \cdot (\langle x,y,z\rangle - \langle -2,2,1\rangle) &= 0 \\ -4(x+2)+4(y-2)+2(z-1) &= 0 \\ -4x-8+4y-8+2z-2 &= 0 \\ -2x+2y+z &= 9 \end{aligned}$$

3.6.17

$$\begin{aligned} x^2-y^2-3z^2 &= 5 \\ F(x,y,z) &= x^2-y^2-3z^2 \\ \nabla F &= \langle 2x, -2y, -6z\rangle \\ \nabla F(6,2,3) &= \langle 12, -4, -18\rangle \\ 12(x-6)-4(y-2)-18(z-3) &= 0 \\ 12x-72-4y+8-18z+54 &= 0 \\ 6x-2y-9z &= 5 \end{aligned}$$

3.6.21

$$
\begin{aligned}
z &= \cos(2x+y) \\
F(x,y,z) &= \cos(2x+y) - z \\
\nabla F &= \langle -2\sin(2x+y), -\sin(2x+y), -1 \rangle \\
\nabla F(\pi/2, \pi/4, -1/\sqrt{2}) &= \langle \sqrt{2}, 1/\sqrt{2}, -1 \rangle \\
0 &= \sqrt{2}\left(x - \frac{\pi}{2}\right) + \frac{1}{\sqrt{2}}\left(y - \frac{\pi}{4}\right) - \left(z + \frac{1}{\sqrt{2}}\right) \\
&= \sqrt{2}x - \frac{1}{\sqrt{2}}\pi + \frac{1}{\sqrt{2}}y - \frac{1}{\sqrt{2}}\frac{\pi}{4} - z - \frac{1}{\sqrt{2}} \\
&= 2x - \pi + y - \frac{\pi}{4} - \sqrt{2}z - 1 \\
1 + \frac{5\pi}{4} &= 2x + y - \sqrt{2}z
\end{aligned}
$$

3.6.25

$$
\begin{aligned}
x^2 + y^2 + z^2 &= 7 \\
F(x,y,z) &= x^2 + y^2 + z^2 \\
\nabla F &= \langle 2x, 2y, 2z \rangle \\
\langle 2x, 2y, 2z \rangle &= k\langle 2, 4, 6 \rangle \\
x^2 + (2x)^2 + (3x)^2 &= 7 \\
x^2 + 4x^2 + 9x^2 &= 7 \\
14x^2 &= 7 \\
x^2 &= \frac{1}{2} \\
x &= \pm\frac{1}{\sqrt{2}} \\
\langle \frac{1}{\sqrt{2}}, \sqrt{2}, \frac{3}{\sqrt{2}} \rangle & \\
\langle -\frac{1}{\sqrt{2}}, -\sqrt{2}, -\frac{3}{\sqrt{2}} \rangle &
\end{aligned}
$$

3.6.27

$$\begin{aligned}
x^2+4x+y^2+z^2-2z &= 11 \\
F(x,y,z) &= x^2+4x+y^2+z^2-2z \\
\nabla F &= \langle 2x+4, 2y, 2z-2 \rangle
\end{aligned}$$

$$\begin{aligned}
2x+4 &= 0 \\
x &= -2
\end{aligned}$$

$$\begin{aligned}
2y &= 0 \\
y &= 0
\end{aligned}$$

$$\begin{aligned}
2z-2 &\neq 0 \\
z &\neq 1
\end{aligned}$$

$$\begin{aligned}
(-2)^2+4(-2)+z^2-2z &= 11 \\
4-8+z^2-2z &= 11 \\
z^2-2z-15 &= 0 \\
(z-5)(z+3) &= 0 \\
(-2,0,-3)& \\
(-2,0,5)&
\end{aligned}$$

3.6.33

$$\begin{aligned}
x^2+2y^2+z^2 &= 4 \\
F(x,y,z) &= x^2+2y^2+z^2 \\
\nabla F &= \langle 2x, 4y, 2z \rangle \\
\nabla F(1,-1,1) &= \langle 2,-4,2 \rangle \\
\mathbf{n}(t) &= \langle 1+2t, -1-4t, 1+2t \rangle
\end{aligned}$$

3.6.35

$$
\begin{aligned}
z &= 4x^2 + 9y^2 + 1 \\
F(x, y, z) &= 4x^2 + 9y^2 - z + 1 \\
\nabla F &= \langle 8x, 18y, -1 \rangle \\
\nabla F\left(\frac{1}{2}, \frac{1}{3}, 3\right) &= \langle 4, 6, -1 \rangle \\
\mathbf{n}(t) &= \langle \frac{1}{2} + 4t, \frac{1}{3} + 6t, 3 - t \rangle
\end{aligned}
$$

$$
\frac{x - \frac{1}{2}}{4} = \frac{y - \frac{1}{3}}{6} = 3 - z
$$

3.7 Curl and Divergence

3.7.7

$$
\begin{aligned}
\mathbf{F} &= \langle xz, yz, xy \rangle \\
\nabla \times \mathbf{F} &= \langle x - y, x - y, 0 \rangle \\
\nabla \cdot \mathbf{F} &= 2z
\end{aligned}
$$

3.7.35

$$
\begin{aligned}
\mathbf{F} &= \langle xy, 4yz^2, 2xz \rangle \\
\nabla \times \mathbf{F} &= \langle -8yz, -2z, -x \rangle \\
\nabla \times (\nabla \times \mathbf{F}) &= \langle 2, 1 - 8y, 8z \rangle
\end{aligned}
$$

3.7.45

$$
\begin{aligned}
\nabla \cdot \mathbf{F} &= 2xyz - 2xyz + 1 \\
&= 1
\end{aligned}
$$

If $\mathbf{F}$ where the curl of another vector field then $\nabla \cdot \mathbf{F}$ would be 0 but it's not.

3.8 Line Integrals

3.8.1

$$
\begin{aligned}
G(x,y) &= 2xy \\
x &= 5\cos t \\
y &= 5\sin t \\
\int_C G(x,y)\,dx &= \int_0^{\pi/4} 2(5\cos t)(5\sin t)(-5\sin t)\,dt \\
&= -250\int_0^{\pi/4} \cos t \sin^2 t\,dt \\
&= -125\int_0^{\pi/4} (\cos t - \cos t\cos 2t)\,dt \\
&= -125\int_0^{\pi/4} \left[\cos t - \frac{1}{2}(\cos t + \cos 3t)\right]\,dt \\
&= -125\left[\sin t - \frac{1}{2}\left(\sin t + \frac{1}{3}\sin 3t\right)\right]_0^{\pi/4} \\
&= -125\left[\frac{1}{\sqrt{2}} - \frac{1}{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{3}\frac{1}{\sqrt{2}}\right)\right] \\
&= -125\left(\frac{1}{\sqrt{2}} - \frac{4}{6\sqrt{2}}\right) \\
&= -\frac{125}{3\sqrt{2}}
\end{aligned}
$$

$$
\begin{aligned}
\int_C G(x,y)\,dy &= \int_0^{\pi/4} 2(5\cos t)(5\sin t)(5\cos t)\,dt \\
&= 250\int_0^{\pi/4} \cos^2 t\sin t\,dt \\
&= 125\int_0^{\pi/4} (\sin t + \cos 2t\sin t)\,dt \\
&= 125\int_0^{\pi/4} \left[\sin t + \frac{1}{2}(\sin 3t - \sin t)\right]\,dt \\
&= 125\left[-\cos t + \frac{1}{2}\left(-\frac{1}{3}\cos 3t + \cos t\right)\right]_0^{\pi/4} \\
&= 125\left[-\frac{1}{\sqrt{2}} + \frac{1}{2}\left(\frac{1}{3}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + 1 - \frac{1}{2}\left(-\frac{1}{3} + 1\right)\right] \\
&= 125\left[-\frac{1}{\sqrt{2}} + \frac{4}{6\sqrt{2}} + 1 + \frac{1}{6} - \frac{1}{2}\right] \\
&= 125\left[\frac{2}{3} - \frac{2}{6\sqrt{2}}\right] \\
&= 125\left[\frac{4}{6} - \frac{\sqrt{2}}{6}\right] \\
&= \frac{125}{6}(4 - \sqrt{2})
\end{aligned}
$$

$$
\begin{aligned}
\int_C G(x,y)\,ds &= \int_0^{\pi/4} 2(5\cos t)(5\sin t)\sqrt{[-5\sin t]^2 + [5\cos t]^2}\,dt \\
&= 50\int_0^{\pi/4} \cos t\sin t\sqrt{25\sin^2 t + 25\cos^2 t}\,dt \\
&= 250\int_0^{\pi/4} \cos t\sin t\,dt \\
&= 125\int_0^{\pi/4} \sin 2t\,dt \\
&= 125\left[-\frac{1}{2}\cos 2t\right]_0^{\pi/4} \\
&= \frac{125}{2}
\end{aligned}
$$

3.8.5

$$\begin{aligned}
G(x,y,z) &= z \\
x &= \cos t \\
y &= \sin t \\
z &= t \\
\int_C G(x,y,z)\,dx &= \int_0^{\pi/2} (t)(-\sin t)\,dt \\
&= [t\cos t - \sin t]_0^{\pi/2} \\
&= -1
\end{aligned}$$

$$\begin{aligned}
\int_C G(x,y,z)\,dy &= \int_0^{\pi/2} (t)(\cos t)\,dt \\
&= [t\sin t + \cos t]_0^{\pi/2} \\
&= \frac{\pi}{2} - 1
\end{aligned}$$

$$\begin{aligned}
\int_C G(x,y,z)\,dz &= \int_0^{\pi/2} t\,dt \\
&= \left[\frac{1}{2}t^2\right]_0^{\pi/2} \\
&= \frac{\pi^2}{8}
\end{aligned}$$

$$\begin{aligned}
\int_C G(x,y,z)\,ds &= \int_0^{\pi/2} t\sqrt{\sin^2 t + \cos^2 t + 1}\,dt \\
&= \sqrt{2}\left[\frac{1}{2}t^2\right]_0^{\pi/2} \\
&= \frac{\sqrt{2}\pi^2}{8}
\end{aligned}$$

3.8.7

$$\begin{aligned}
\int_C (2x+y)\,dx + xy\,dy &= \int_{-1}^{2} (2x+x+3)\,dx + x(x+3)\,dx \\
&= \int_{-1}^{2} (x^2+6x+3)\,dx \\
&= \left[\frac{1}{3}x^3 + 3x^2 + 3x\right]_{-1}^{2} \\
&= \left(\frac{8}{3} + 12 + 6\right) - \left(-\frac{1}{3} + 3 - 3\right) \\
&= 21
\end{aligned}$$

3.8.9

$$\begin{aligned}
\int_C (2x+y)\,dx + xy\,dy &= \int_{-1}^{2} (2x+2)\,dx + \int_{2}^{5} 2y\,dy \\
&= [x^2+2x]_{-1}^{2} + [y^2]_2^5 \\
&= (4+4) - (1-2) + (25) - (4) \\
&= 30
\end{aligned}$$

3.8.11

$$\begin{aligned}
\int_0^1 x^2\,dx + x(2x)\,dx &= \int_0^1 3x^2\,dx \\
&= [x^3]_0^1 \\
&= 1
\end{aligned}$$

3.8.13

$$\int_0^1 dx = 1$$

3.8.19

$$\begin{aligned}
\int_{-2}^{2} x^2\,dx &= \left[\frac{1}{3}x^3\right]_{-2}^{2} \\
&= \frac{16}{3} \\
\int_{2}^{-2} (x^2+4-x^2)\,dx - 2x\sqrt{4-x^2}\,\frac{-x}{\sqrt{4-x^2}}\,dx &= \int_{2}^{-2} (2x^2+4)\,dx \\
&= \left[\frac{2}{3}x^3+4x\right]_{2}^{-2} \\
&= \left(-\frac{16}{3}-8\right)-\left(\frac{16}{3}+8\right) \\
&= -\frac{32}{3}-16 \\
&= -\frac{80}{3}
\end{aligned}$$

$$\oint (x^2+y^2)\,dx - 2xy\,dy = -\frac{64}{3}$$

3.8.21

$$\begin{aligned}\int_{-1}^{1} -x^2\,dx &= \left[-\frac{1}{3}x^3\right]_{-1}^{1}\\ &= -\frac{2}{3}\end{aligned}$$

$$\begin{aligned}\int_{-1}^{1} -y^2\,dy &= \left[-\frac{1}{3}y^3\right]_{-1}^{1}\\ &= -\frac{2}{3}\end{aligned}$$

$$\begin{aligned}\int_{1}^{-1} x^2\,dx &= \left[\frac{1}{3}x^3\right]_{1}^{-1}\\ &= -\frac{2}{3}\end{aligned}$$

$$\begin{aligned}\int_{1}^{-1} y^2\,dy &= \left[\frac{1}{3}y^3\right]_{1}^{-1}\\ &= -\frac{2}{3}\end{aligned}$$

$$\oint_C x^2y^3\,dx - xy^2\,dy = -\frac{8}{3}$$

3.8.29

$$\begin{aligned}\mathbf{r}'(t) &= \langle -2e^{-2t}, e^t\rangle\\ \int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{\ln 2} \langle e^{3t}, -e^{-3t}\rangle \cdot \langle -2e^{-2t}, e^t\rangle\,dt\\ &= \int_0^{\ln 2} (-2e^t - e^{-2t})\,dt\\ &= \left[-2e^t + \frac{1}{2}e^{-2t}\right]_0^{\ln 2}\\ &= -4 + \frac{1}{8} + 2 - \frac{1}{2}\\ &= -\frac{19}{8}\end{aligned}$$

3.8.31

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_1^e \langle \ln x, x\rangle \cdot \langle 1, \frac{1}{x}\rangle\, dx\\ &= \int_1^e (\ln x + 1)\, dx\\ &= [x(\ln x - 1) + x]_1^e\\ &= e\end{aligned}$$

3.8.35

$$\begin{aligned}\mathbf{r}(t) &= \langle 3\cos\theta, 3\sin\theta\rangle\\ \mathbf{r}'(t) &= \langle -3\sin\theta, 3\cos\theta\rangle\\ \int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} (-3a\sin\theta + 3b\cos\theta)\, d\theta\\ &= [3a\cos\theta + 3b\sin\theta]_0^{2\pi}\\ &= 0\end{aligned}$$

3.8.40

$$\begin{aligned}\rho(x,y) &= kx\\ m &= \int_C \rho(x,y)\, ds\\ &= \int_0^{\pi} kx\sqrt{[-\sin t]^2 + [\cos t]^2}\, dt\\ &= k\int_0^{\pi} (1+\cos t)\, dt\\ &= k[t + \sin t]_0^{\pi}\\ &= k\pi\end{aligned}$$

3.8.41

$$\begin{aligned}
M_y &= \int_C x\rho(x,y)\,ds \\
&= \int_0^\pi k(1+\cos t)^2\sqrt{[-\sin t]^2+[\cos t]^2}\,dt \\
&= k\int_0^\pi (1+\cos t)^2\,dt \\
&= k\left[\frac{1}{4}(6t+8\sin t+\sin 2t)\right]_0^\pi \\
&= \frac{3}{2}k\pi
\end{aligned}$$

$$\begin{aligned}
\overline{x} &= \frac{M_y}{m} \\
&= \frac{3}{2}
\end{aligned}$$

$$\begin{aligned}
M_x &= \int_C y\rho(x,y)\,ds \\
&= \int_0^\pi (\sin t)k(1+\cos t)\sqrt{[-\sin t]^2+[\cos t]^2}\,dt \\
&= k\int_0^\pi (\sin t+\cos t\sin t)\,dt \\
&= k\int_0^\pi \left(\sin t+\frac{1}{2}\sin 2t\right)\,dt \\
&= k\left[-\cos t-\frac{1}{4}\cos 2t\right]_0^\pi \\
&= k\left(1-\frac{1}{4}+1+\frac{1}{4}\right) \\
&= 2k
\end{aligned}$$

$$\begin{aligned}
\overline{y} &= \frac{M_x}{m} \\
&= \frac{2}{\pi}
\end{aligned}$$

3.9 Independence of the Path

3.9.1

$$\begin{aligned}
\phi &= \frac{1}{3}x^3 + g(y) \\
g'(y) &= y^2 \\
\phi &= \frac{1}{3}x^3 + \frac{1}{3}y^3 \\
\phi(2,2) - \phi(0,0) &= \frac{16}{3}
\end{aligned}$$

3.9.3

$$\begin{aligned}
\phi &= \frac{1}{2}x^2 + 2xy + g(y) \\
2x + g'(y) &= 2x - y \\
g'(y) &= -y \\
g(y) &= -\frac{1}{2}y^2 \\
\phi &= \frac{1}{2}x^2 + 2xy - \frac{1}{2}y^2 \\
\phi(3,2) - \phi(1,0) &= \left(\frac{1}{2}9 + 12 - \frac{1}{2}4\right) - \left(\frac{1}{2}\right) \\
&= 14
\end{aligned}$$

3.9.11

$$\begin{aligned}
\frac{\partial P}{\partial y} &= 12x^3y^2 \\
\frac{\partial Q}{\partial x} &= 12x^3y^2 \\
\phi &= x^4y^3 + 3x + g(y) \\
3x^4y^2 + g'(y) &= 3x^4y^2 + 1 \\
g'(y) &= 1 \\
g(y) &= y \\
\phi &= x^4y^3 + 3x + y
\end{aligned}$$

3.9.13

$$\begin{aligned}
\frac{\partial P}{\partial y} &= 2y\cos xy^2 - 2xy^3\sin xy^2 \\
\frac{\partial Q}{\partial x} &= -2y\sin xy^2 - 2xy^3\cos xy^2
\end{aligned}$$

Not conservative

3.9.15

$$\begin{aligned}
\frac{\partial P}{\partial y} &= 1 \\
\frac{\partial Q}{\partial x} &= 1 \\
\phi &= \frac{1}{4}x^4 + xy + g(y) \\
x + g'(y) &= x + y^3 \\
g'(y) &= y^3 \\
g(y) &= \frac{1}{4}y^4 \\
\phi &= \frac{1}{4}x^4 + xy + \frac{1}{4}y^4
\end{aligned}$$

3.9.17

$$\begin{aligned}
\phi &= x^2 + e^{-y}x + g(y) \\
-e^{-y}x + g'(y) &= 4y - xe^{-y} \\
g'(y) &= 4y \\
g(y) &= 2y^2 \\
\phi &= x^2 + e^{-y}x + 2y^2 \\
\phi(1,1) - \phi(0,0) &= (1 + e^{-1} + 2) - (0) \\
&= 3 + e^{-1}
\end{aligned}$$

3.9.19

$$\begin{aligned}
\phi &= xyz + g(y,z) \\
xz + g'(y,z) &= xz \\
\phi &= xyz \\
\phi(2,4,8) - \phi(1,1,1) &= 63
\end{aligned}$$

3.9.21

$$
\begin{aligned}
\phi &= x^2\sin y + xe^{3z} + g(y,z)\\
x^2\cos y + g'(y,z) &= x^2\cos y\\
g'(y,z) &= 0\\
\phi &= x^2\sin y + xe^{3z} + g(z)\\
3xe^{3z} + g'(z) &= 3xe^{3z} + 5\\
g'(z) &= 5\\
g(z) &= 5z\\
\phi &= x^2\sin y + xe^{3z} + 5z\\
\phi(2,\pi/2,1) - \phi(1,0,0) &= \left((2)^2\sin\frac{\pi}{2} + (2)e^{3(1)} + 5(1)\right) - (1)\\
&= 8 + 2e^3
\end{aligned}
$$

3.9.25

$$
\begin{aligned}
\phi &= xy + yz\cos x + g(y,z)\\
x + z\cos x + g'(y,z) &= x + z\cos x\\
g'(y,z) &= 0\\
g'(y,z) &= h(z)\\
y\cos x + h'(z) &= y\cos z\\
h'(z) &= 0\\
\phi &= xy + yz\cos x\\
\phi(\pi,1,4) - \phi(0,4,0) &= \pi - 4
\end{aligned}
$$

3.9.27

$$
\begin{aligned}
P &= -Gm_1m_2\frac{x}{(x^2+y^2+z^3)^{3/2}}\\
Q &= -Gm_1m_2\frac{y}{(x^2+y^2+z^3)^{3/2}}\\
R &= -Gm_1m_2\frac{z}{(x^2+y^2+z^3)^{3/2}}\\
\phi &= \frac{Gm_1m_2}{\sqrt{x^2+y^2+z^2}}\\
&= \frac{Gm_1m_2}{||\mathbf{r}||}
\end{aligned}
$$

3.10 Double Integrals

3.10.1

$$
\begin{aligned}
\int_{-1}^{3}(6xy-5e^y)\,dx &= [3x^2y-5xe^y]_{-1}^{3} \\
&= (3(3)^2y-5(3)e^y)-(3(-1)^2y-5(-1)e^y) \\
&= 27y-15e^y-3y-5e^y \\
&= 24y-20e^y
\end{aligned}
$$

3.10.3

$$
\begin{aligned}
\int_{1}^{3x} x^3e^{xy}\,dy &= [x^2e^{xy}]_{1}^{3x} \\
&= x^2e^{3x^2}-x^2e^x
\end{aligned}
$$

3.10.13

$$
\begin{aligned}
\int_0^1\int_0^x x^3y^2\,dy\,dx &= \int_0^1\left[\frac{1}{3}x^3y^3\right]_0^x\,dx \\
&= \int_0^1\frac{1}{3}x^6\,dx \\
&= \frac{1}{21}
\end{aligned}
$$

3.10.15

$$
\begin{aligned}
\int_0^1\int_{x^3}^{x^2}(2x+4y+1)\,dy\,dx &= \int_0^1[2xy+2y^2+y]_{x^3}^{x^2}\,dx \\
&= \int_0^1(2x^3+2x^4+x^2-2x^4-2x^6-x^3)\,dx \\
&= \int_0^1(-2x^6+x^3+x^2)\,dx \\
&= \left[-\frac{2}{7}x^7+\frac{1}{4}x^4+\frac{1}{3}x^3\right]_0^1 \\
&= -\frac{2}{7}+\frac{1}{4}+\frac{1}{3} \\
&= -\frac{24}{84}+\frac{21}{84}+\frac{28}{84} \\
&= \frac{25}{84}
\end{aligned}
$$

3.10.23

$$\begin{aligned}2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}(4-y)\,dx\,dy &= 2\int_{-2}^{2}[4x-xy]_0^{\sqrt{4-y^2}}\,dy\\ &= 2\int_{-2}^{2}(4-y)\sqrt{4-y^2}\,dy\\ &= 16\pi\end{aligned}$$

3.10.25

$$\begin{aligned}z &= 6-2x-y\\ 0 &= 6-2x-y\\ y &= 6-2x\\ \int_0^3\int_0^{6-2x}(6-2x-y)\,dy\,dx &= \int_0^3\left[(6-2x)y-\frac{1}{2}y^2\right]_0^{6-2x}\,dx\\ &= \int_0^3\frac{1}{2}(6-2x)^2\,dx\\ &= \frac{1}{2}\left[-\frac{1}{6}(6-2x)^3\right]_0^3\\ &= -\frac{1}{12}(0-216)\\ &= 18\end{aligned}$$

3.10.27

$$\begin{aligned}\int_0^2\int_0^{\sqrt{4-x^2}}\frac{1}{2}(4-x+y)\,dy\,dx &= \frac{1}{2}\int_0^2\left[(4-x)y+\frac{1}{2}y^2\right]_0^{\sqrt{4-x^2}}\,dx\\ &= \frac{1}{2}\int_0^2\left[(4-x)\sqrt{4-x^2}+\frac{1}{2}(4-x^2)\right]\,dx\\ &= 2\pi\end{aligned}$$

3.10.35

$$\begin{aligned}
\int_0^1 \int_x^1 x^2\sqrt{1+y^4}\,dy\,dx &= \int_0^1 \int_0^y x^2\sqrt{1+y^4}\,dx\,dy \\
&= \int_0^1 \sqrt{1+y^4}\left[\frac{1}{3}x^3\right]_0^y \, dy \\
&= \frac{1}{3}\int_0^1 y^3\sqrt{1+y^4}\,dy \\
&= \frac{1}{18}[(1+y^4)^{3/2}]_0^1 \\
&= \frac{1}{18}(2\sqrt{2}-1)
\end{aligned}$$

3.11 Double Integrals in Polar Coordinates

3.11.1

$$\begin{aligned}
\int_0^{2\pi} \int_0^{3+3\sin\theta} r\,dr\,d\theta &= \int_0^{2\pi} \left[\frac{1}{2}r^2\right]_0^{3+3\sin\theta} \, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} (3+3\sin\theta)^2\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi} (9+18\sin\theta+9\sin^2\theta)\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left[9+18\sin\theta+\frac{9}{2}(1-\cos 2\theta)\right] \, d\theta \\
&= \frac{1}{2}\left[9\theta-18\cos\theta+\frac{9}{2}\left(\theta-\frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} \\
&= \frac{1}{2}[(18\pi-18+9\pi)-(-18)] \\
&= \frac{1}{2}27\pi
\end{aligned}$$

3.11.5

$$\begin{aligned}
4\int_{-\pi/6}^{\pi/6}\int_{0}^{5\cos 3\theta} r\,dr\,d\theta &= 2\int_{-\pi/6}^{\pi/6}[r^2]_0^{5\cos 3\theta}\,d\theta \\
&= 50\int_{-\pi/6}^{\pi/6}\cos^2 3\theta\,d\theta \\
&= 25\int_{-\pi/6}^{\pi/6}(1+\cos 6\theta)\,d\theta \\
&= 25\left[\theta+\frac{1}{6}\sin 6\theta\right]_{-\pi/6}^{\pi/6} \\
&= 25\left[\left(\frac{\pi}{6}\right)-\left(-\frac{\pi}{6}\right)\right] \\
&= \frac{25\pi}{3}
\end{aligned}$$

3.11.25

$$\begin{aligned}
\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\sqrt{x^2+y^2}\,dy\,dx &= \int_0^{\pi}\int_0^3 r^2\,dr\,d\theta \\
&= \int_0^{\pi}\left[\frac{1}{3}r^3\right]_0^3\,d\theta \\
&= 9\int_0^{\pi}d\theta \\
&= 9\pi
\end{aligned}$$

3.12 Green's Theorem

3.12.1

$$
\begin{aligned}
\int_0^1 x\,dx &= \left[\frac{1}{2}x^2\right]_0^1 \\
&= \frac{1}{2} \\
\int_0^3 y\,dy &= \left[\frac{1}{2}y^2\right]_0^3 \\
&= \frac{9}{2} \\
\int_1^0 (x-3x)\,dx + x(3x)3\,dx &= \int_1^0 (9x^2-2x)\,dx \\
&= \left[3x^3 - x^2\right]_1^0 \\
&= -2 \\
\oint_C (x-y)\,dx + xy\,dy &= 3
\end{aligned}
$$

$$
\begin{aligned}
\iint_R (y+1)\,dA &= \int_0^1 \int_0^{3x} (y+1)\,dy\,dx \\
&= \int_0^1 \left[\frac{1}{2}y^2 + y\right]_0^{3x} dx \\
&= \int_0^1 \left(\frac{9}{2}x^2 + 3x\right) dx \\
&= \left[\frac{3}{2}x^3 + \frac{3}{2}x^2\right]_0^1 \\
&= 3
\end{aligned}
$$

3.12.5

$$
\begin{aligned}
\oint_C 2y\,dx + 5x\,dy &= \iint_R 3\,dA \\
&= 75\pi
\end{aligned}
$$

3.12.7

$$\begin{aligned}
\oint_C (x^4 - 2y^3)\,dx + (2x^3 - y^4)\,dy &= \iint_R (6x^2 + 6y^2)\,dA \\
&= \int_0^{2\pi} \int_0^2 6r^3\,dr\,d\theta \\
&= \frac{3}{2} \int_0^{2\pi} [r^4]_0^2\,d\theta \\
&= 24 \int_0^{2\pi} d\theta \\
&= 48\pi
\end{aligned}$$

3.12.9

$$\begin{aligned}
\oint_C 2xy\,dx + 3xy^2\,dy &= \iint_R (3y^2 - 2x)\,dA \\
&= \int_1^2 \int_2^{2x} (3y^2 - 2x)\,dy\,dx \\
&= \int_1^2 [y^3 - 2xy]_2^{2x}\,dx \\
&= \int_1^2 (8x^3 - 4x^2 + 4x - 8)\,dx \\
&= \left[2x^4 - \frac{4}{3}x^3 + 2x^2 - 8x\right]_1^2 \\
&= \left(32 - \frac{32}{3} + 8 - 16\right) - \left(2 - \frac{4}{3} + 2 - 8\right) \\
&= \frac{96}{3} - \frac{32}{3} + \frac{24}{3} - \frac{48}{3} - \frac{6}{3} + \frac{4}{3} - \frac{6}{3} + \frac{24}{3} \\
&= \frac{56}{3}
\end{aligned}$$

3.12.15

$$\begin{aligned}
\oint_C ay\,dx + bx\,dy &= \iint_R (b - a)\,dA \\
&= (b - a)A
\end{aligned}$$

3.12.19

$$\begin{aligned}
A &= \oint_C x\,dy \\
&= \int_0^{2\pi} (a\cos^3 t)(3a\sin^2 t\cos t)\,dt \\
&= 3a^2 \int_0^{2\pi} \cos^4 t \sin^2 t\,dt \\
&= \frac{3a^2\pi}{8}
\end{aligned}$$

3.12.23

$$\begin{aligned}
\oint_C (4x^2 - y^3)\,dx + (x^3 + y^2)\,dy &= \iint_R (3x^2 + 3y^2)\,dA \\
&= \iint_R 3r^2\,dA \\
&= 3\int_0^{2\pi} \int_1^2 r^3\,dr\,d\theta \\
&= 3\int_0^{2\pi} \left[\frac{1}{4}r^4\right]_1^2 d\theta \\
&= \frac{45}{4}\int_0^{2\pi} d\theta \\
&= \frac{45}{2}\pi
\end{aligned}$$

3.12.25

$$\begin{aligned}
\frac{\partial P}{\partial y} &= \frac{-3y^2}{(x^2+y^2)^2} + \frac{4y^4}{(x^2+y^2)^3} \\
&= \frac{-3y^2(x^2+y^2)+4y^4}{(x^2+y^2)^3} \\
&= \frac{y^4-3x^2y^2}{(x^2+y^2)^3} \\
\frac{\partial Q}{\partial x} &= \frac{y^2}{(x^2+y^2)^2} - \frac{4x^2y^2}{(x^2+y^2)^3} \\
&= \frac{y^2(x^2+y^2)-4x^2y^2}{(x^2+y^2)^3} \\
&= \frac{y^4-3x^2y^2}{(x^2+y^2)^3} \\
\oint_C \frac{-y^3}{(x^2+y^2)^2}\,dx + \frac{xy^2}{(x^2+y^2)^2}\,dy &= \int_0^{2\pi} (\sin^4\theta + \cos^2\theta\sin^2\theta)\,dt \\
&= \pi
\end{aligned}$$

3.12.27

$$\begin{aligned}
\iint_R x^2\,dA &= \oint_C \frac{1}{3}x^3\,dy \\
&= \frac{1}{3}\int_0^{2\pi} (3\cos\theta)^3(2\cos\theta)\,d\theta \\
&= 18\int_0^{2\pi} \cos^4\theta\,d\theta \\
&= \frac{27\pi}{2}
\end{aligned}$$

3.12.29

$$\begin{aligned}
\oint_C (x-y)\,dx + (x+y)\,dy &= \iint_R 2\,dA \\
&= \frac{3}{2}\pi
\end{aligned}$$

3.12.33

$$
\begin{aligned}
\mathbf{F} &= \langle -y, x \rangle \\
&= \langle -r\sin\theta, r\cos\theta \rangle \\
\oint -y\,dx + x\,dy &= \int_0^{2\pi} [(-r\sin\theta)(-r\sin\theta) + (r\cos\theta)(r\cos\theta)]\,d\theta \\
&= \int_0^{2\pi} [r^2\sin^2\theta + r^2\cos^2\theta]\,d\theta \\
&= \int_0^{2\pi} r^2\,d\theta \\
&= \int_0^{2\pi} (1+\cos\theta)^2\,d\theta \\
&= 3\pi
\end{aligned}
$$

3.13 Surface Integrals

3.13.1

$$
\begin{aligned}
2x + 3y + 4z &= 12 \\
z &= 3 - \frac{1}{2}x - \frac{3}{4}y \\
y &= 4 - \frac{2}{3}x \\
A &= \iint_S dS \\
&= \int_0^6 \int_0^{4-\frac{2}{3}x} \sqrt{1 + \left[-\frac{1}{2}\right]^2 + \left[-\frac{3}{4}\right]^2}\,dy\,dx \\
&= \frac{\sqrt{29}}{4} \int_0^6 \left(4 - \frac{2}{3}x\right)\,dx \\
&= \frac{\sqrt{29}}{4} \left[4x - \frac{1}{3}x^2\right]_0^6 \\
&= 3\sqrt{29}
\end{aligned}
$$

3.13.3

$$\begin{aligned}
x^2 + z^2 &= 16 \\
z &= \sqrt{16 - x^2} \\
A &= \int_0^2 \int_0^5 \sqrt{1 + \left[-\frac{x}{\sqrt{16 - x^2}}\right]^2}\, dy\, dx \\
&= 5 \int_0^2 \sqrt{1 + \frac{x^2}{16 - x^2}}\, dx \\
&= 5 \int_0^2 \sqrt{\frac{16}{16 - x^2}}\, dx \\
&= 20 \int_0^2 \frac{1}{\sqrt{16 - x^2}}\, dx \\
&= 20 \left[\arcsin \frac{x}{4}\right]_0^2 \\
&= \frac{10}{3}\pi
\end{aligned}$$

3.13.15

$$\begin{aligned}
\iint_S G(x, y, z)\, dS &= \int_0^{\sqrt{2}} \int_0^4 x\sqrt{1 + [-2x]^2}\, dy\, dx \\
&= \int_0^{\sqrt{2}} \int_0^4 x\sqrt{1 + 4x^2}\, dy\, dx \\
&= 4 \int_0^{\sqrt{2}} x\sqrt{1 + 4x^2}\, dx \\
&= 4 \left[\frac{1}{12}(4x^2 + 1)^{3/2}\right]_0^{\sqrt{2}} \\
&= \frac{26}{3}
\end{aligned}$$

3.13.17

$$\begin{aligned}\iint_S G(x,y,z)\,dS &= 2\int_0^{\sqrt{1-x^2}} \int_{-1}^{1} xz^3 \sqrt{1+\left[\frac{x}{\sqrt{x^2+y^2}}\right]^2+\left[\frac{y}{\sqrt{x^2+y^2}}\right]^2}\,dA \\ &= 2\sqrt{2}\int_0^{\sqrt{1-x^2}} \int_{-1}^{1} x(x^2+y^2)^{3/2}\,dx\,dy \\ &= 2\sqrt{2}\int_0^{\sqrt{1-x^2}} \left[\frac{1}{5}(x^2+y^2)^{5/2}\right]_{-1}^{1} dy \\ &= 0\end{aligned}$$

3.13.19

$$\begin{aligned} x^2+y^2+z^2 &= 36 \\ z &= \sqrt{36-x^2-y^2}\end{aligned}$$

$$\begin{aligned}\iint_R (x^2+y^2)z\sqrt{1+\left[-\frac{x}{\sqrt{36-x^2-y^2}}\right]^2+\left[-\frac{y}{\sqrt{36-x^2-y^2}}\right]^2}\,dA \\ \int_0^6 \int_0^{\pi/2} r^2\sqrt{36-r^2}\sqrt{1+\frac{r^2}{36-r^2}}\,r\,d\theta\,dr \\ 6\int_0^6 \int_0^{\pi/2} r^3\,d\theta\,dr \\ 3\pi\int_0^6 r^3\,dr \\ 3\pi\left[\frac{1}{4}r^4\right]_0^6 \\ 972\pi\end{aligned}$$

3.13.25

$$\begin{aligned}
x + 2y + 3z &= 6 \\
y &= 3 - \frac{1}{2}x - \frac{3}{2}z \\
0 &= 3 - \frac{1}{2}x - \frac{3}{2}z \\
\frac{3}{2}z &= 3 - \frac{1}{2}x \\
z &= 2 - \frac{1}{3}x \\
0 &= 2 - \frac{1}{3}x \\
x &= 6
\end{aligned}$$

$$\begin{aligned}
\iint_S (3z^2 + 4yz)\, dS &= \int_0^6 \int_0^{2-\frac{1}{3}x} (3z^2 + 4yz) \sqrt{1 + \left[-\frac{1}{2}\right]^2 + \left[-\frac{3}{2}\right]^2}\, dz\, dx \\
&= \frac{\sqrt{14}}{2} \int_0^6 \int_0^{2-\frac{1}{3}x} \left[3z^2 + 4\left(3 - \frac{1}{2}x - \frac{3}{2}z\right) z\right] dz\, dx \\
&= \frac{\sqrt{14}}{2} \int_0^6 \int_0^{2-\frac{1}{3}x} \left(-3z^2 - 2xz + 12z\right) dz\, dx \\
&= \frac{\sqrt{14}}{2} \int_0^6 [-z^3 - xz^2 + 6z^2]_0^{2-\frac{1}{3}x}\, dx \\
&= \frac{\sqrt{14}}{2} \int_0^6 \left[-\left(2 - \frac{1}{3}x\right)^3 + (6 - x)\left(2 - \frac{1}{3}x\right)^2\right] dx \\
&= \frac{\sqrt{14}}{2} \int_0^6 \left(-\frac{2}{27}x^3 + \frac{4}{3}x^2 - 8x + 16\right) dx \\
&= \frac{\sqrt{14}}{2} \left[-\frac{1}{54}x^4 + \frac{4}{9}x^3 - 4x^2 + 16x\right]_0^6 \\
&= 12\sqrt{14}
\end{aligned}$$

3.13.27

$$\begin{aligned} x+y+z &= 1 \\ z &= 1-x-y \end{aligned}$$

$$\begin{aligned} 0 &= 1-x-y \\ y &= 1-x \end{aligned}$$

$$\begin{aligned} \iint_S kx^2\,dS &= \int_0^1 \int_0^{1-x} kx^2\sqrt{1+[-1]^2+[-1]^2}\,dy\,dx \\ &= \sqrt{3}k\int_0^1 (x^2-x^3)\,dx \\ &= \sqrt{3}k\left[\frac{1}{3}x^3-\frac{1}{4}x^4\right]_0^1 \\ &= \frac{\sqrt{3}}{12}k \end{aligned}$$

3.13.29

$$
\begin{aligned}
\mathbf{F} &= \langle x, 2z, y\rangle \\
g(x,y,z) &= y^2 + z^2 - 4 \\
\nabla g &= \langle 0, 2y, 2z\rangle \\
||\nabla g|| &= \sqrt{4y^2 + 4z^2} \\
&= 2\sqrt{y^2 + z^2} \\
&= 4 \\
\mathbf{n} &= \frac{\nabla g}{||\nabla g||} \\
&= \langle 0, \frac{y}{2}, \frac{z}{2}\rangle \\
z &= \sqrt{4 - y^2} \\
\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS &= \int_0^3 \int_0^2 \left(yz + \frac{1}{2}yz\right)\sqrt{1 + \left[-\frac{y}{\sqrt{4-y^2}}\right]^2}\, dy\, dx \\
&= \int_0^3 \int_0^2 \frac{3}{2}yz\sqrt{1 + \frac{y^2}{4-y^2}}\, dy\, dx \\
&= \int_0^3 \int_0^2 \frac{3}{2}yz\sqrt{\frac{4}{4-y^2}}\, dy\, dx \\
&= 3\int_0^3 \int_0^2 y\, dy\, dx \\
&= 3\int_0^3 \left[\frac{1}{2}y^2\right]_0^2 dx \\
&= 6\int_0^3 dx \\
&= 18
\end{aligned}
$$

3.13.31

$$\begin{aligned}
\mathbf{F} &= \langle x, y, z\rangle \\
g(x,y,z) &= 5 - x^2 - y^2 - z \\
\nabla g &= \langle -2x, -2y, -1\rangle \\
||\nabla g|| &= \sqrt{4x^2+4y^2+1} \\
\mathbf{n} &= \langle -2x, -2y, -1\rangle / \sqrt{4x^2+4y^2+1} \\
z &= 5 - x^2 - y^2 \\
0 &= 5 - x^2 - y^2 \\
y^2 &= 5 - x^2 \\
\mathbf{F}\cdot\mathbf{n} &= \frac{-2x^2-2y^2-z}{\sqrt{4x^2+4y^2+1}} \\
y &= \sqrt{5-x^2} \\
\iint_S (\mathbf{F}\cdot\mathbf{n})\,dS &= 2\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} \frac{-2x^2-2y^2-z}{\sqrt{4x^2+4y^2+1}}\sqrt{1+[-2x]^2+[-2y]^2}\,dy\,dx \\
&= 2\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} [-2x^2-2y^2-(5-x^2-y^2)]\,dy\,dx \\
&= 2\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (-x^2-y^2-5)\,dy\,dx \\
&= 2\int_{-2}^{2}\left[-\frac{1}{3}y^3-(x^2+5)y\right]_0^{\sqrt{4-x^2}}\,dx \\
&= 2\int_{-2}^{2}\left(-\frac{1}{3}(4-x^2)^{3/2}-(x^2+5)\sqrt{4-x^2}\right)\,dx \\
&= -28\pi
\end{aligned}$$

3.13.35

$$\begin{aligned}
\mathbf{F} &= \langle y^2, x^2, 5z \rangle \\
\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\, dS &= 5\pi \\
z &= x^2 + y^2 \\
g(x,y,z) &= x^2 + y^2 - z \\
\nabla g &= \langle 2x, 2y, -1 \rangle \\
||\nabla g|| &= \sqrt{4x^2 + 4y^2 + 1} \\
\mathbf{n} &= \frac{\langle 2x, 2y, -1 \rangle}{\sqrt{4x^2 + 4y^2 + 1}} \\
\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\, dS &= \iint_R \frac{2xy^2 + 2x^2y - 5z}{\sqrt{4x^2+4y^2+1}} \sqrt{1 + [2x]^2 + [2y]^2}\, dA \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [2xy^2 + 2x^2y - 5(x^2+y^2)]\, dy\, dx \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2xy^2 + 2x^2y - 5x^2 - 5y^2)\, dy\, dx \\
&= \int_{-1}^{1} \left[\frac{2}{3}xy^3 + x^2y^2 - 5x^2y - \frac{5}{3}y^3\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \\
&= \int_{-1}^{1} \left[2\left(\frac{2}{3}x - \frac{5}{3}\right)(1-x^2)^{3/2} - 10x^2\sqrt{1-x^2}\right] dx \\
&= -\frac{5}{2}\pi \\
\iint_{S} \mathbf{F}\cdot\mathbf{n}\, dS &= 5\pi - \frac{5}{2}\pi \\
&= \frac{5}{2}\pi
\end{aligned}$$

3.13.37

$$
\begin{aligned}
T(x,y,z) &= x^2+y^2+z^2 \\
&= r^2 \\
\mathbf{F} &= -\nabla T \\
&= \langle -2r, 0, 0\rangle \\
\iint_R \mathbf{F}\cdot\mathbf{n}\,dA &= \int_0^{2\pi}\int_0^{\pi} -2a^3\sin\theta\,d\theta\,d\phi \\
&= -4\pi a^3\int_0^{\pi}\sin\theta\,d\theta \\
&= -4\pi a^3[-\cos\theta]_0^{\pi} \\
&= -8\pi a^3
\end{aligned}
$$

3.13.39

$$
\begin{aligned}
\mathbf{E} &= \frac{kq}{r^2}\hat{\mathbf{r}} \\
\iint_S \mathbf{E}\cdot\mathbf{n}\,dA &= \int_0^{2\pi}\int_0^{\pi}\frac{kq}{a^2}a^2\sin\theta\,d\theta\,d\phi \\
&= 2\pi kq[-\cos\theta]_0^{\pi} \\
&= 4\pi kq
\end{aligned}
$$

3.14 Stokes' Theorem

3.14.1

$$
\begin{aligned}
\mathbf{F} &= \langle 5y, -5x, 3\rangle \\
\mathbf{r} &= \langle 2\cos t, 2\sin t, 1\rangle \\
d\mathbf{r} &= \langle -2\sin t, 2\cos t, 0\rangle\,dt \\
\oint \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi}(-20\sin^2 t - 20\cos^2 t)\,dt \\
&= -40\pi
\end{aligned}
$$

$$
\begin{aligned}
\nabla\times\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 5y & -5x & 3 \end{vmatrix} \\
&= \langle 0, 0, -10\rangle \\
\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS &= -40\pi
\end{aligned}
$$

3.14.5

$$\begin{aligned}
\mathbf{F} &= \langle 2z + x, y - z, x + y \rangle \\
\nabla \times \mathbf{F} &= \langle 2, 1, 0 \rangle \\
x + y + z &= 1 \\
z &= 1 - x - y \\
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \\
&= 3 \int_0^1 \int_0^{1-x} \, dy \, dx \\
&= 3 \int_0^1 (1 - x) \, dx \\
&= 3 \left[x - \frac{1}{2} x^2 \right]_0^1 \\
&= \frac{3}{2}
\end{aligned}$$

3.14.7

$$\begin{aligned}
\mathbf{F} &= \langle xy, 2yz, xz \rangle \\
\nabla \times \mathbf{F} &= \langle -2y, -z, -x \rangle \\
z &= 1 - y \\
\mathbf{n} &= \langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \\
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \\
&= \int_0^2 \int_0^1 \left(-\frac{z}{\sqrt{2}} - \frac{x}{\sqrt{2}} \right) \sqrt{2} \, dy \, dx \\
&= \int_0^2 \int_0^1 (y - x - 1) \, dy \, dx \\
&= \int_0^2 \left[\frac{1}{2} y^2 - (x + 1) y \right]_0^1 \, dx \\
&= \int_0^2 \left(-\frac{1}{2} - x \right) \, dx \\
&= \left[-\frac{1}{2} x - \frac{1}{2} x^2 \right]_0^2 \\
&= -3
\end{aligned}$$

3.14.9

$$
\begin{aligned}
\mathbf{F} &= \langle y^3, -x^3, z^3 \rangle \\
\nabla \times \mathbf{F} &= \langle 0, 0, -3x^2 - 3y^2 \rangle \\
\mathbf{n} &= \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle \\
\oint \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \\
&= \frac{1}{\sqrt{3}} \iint_S (-3x^2 - 3y^2)\, dS \\
&= \iint_S (-3x^2 - 3y^2)\, dA \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (-3x^2 - 3y^2)\, dy\, dx \\
&= \int_{-1}^{1} \left[-3x^2 y - y^3\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\, dx \\
&= \int_{-1}^{1} [-6x^2\sqrt{1-x^2} - 2(1-x^2)^{3/2}]\, dx \\
&= -\frac{3\pi}{2}
\end{aligned}
$$

3.14.11

$$
\begin{aligned}
\mathbf{F} &= \langle x, x^3y^2, z\rangle \\
\nabla \times \mathbf{F} &= \langle 0, 0, 3x^2y^2\rangle
\end{aligned}
$$

$$
\begin{aligned}
z &= \sqrt{4-4x^2-y^2} \\
z^2 &= 4-4x^2-y^2 \\
0 &= 4-4x^2-y^2 \\
y &= 2\sqrt{1-x^2} \\
g(x,y,z) &= 4x^2+y^2+z^2-4 \\
\nabla g &= \langle 8x, 2y, 2z\rangle \\
||\nabla g|| &= \sqrt{64x^2+4y^2+4z^2} \\
&= 2\sqrt{16x^2+y^2+z^2} \\
\mathbf{n} &= \frac{\nabla g}{||\nabla g||} \\
&= \frac{\langle 4x, y, z\rangle}{\sqrt{16x^2+y^2+z^2}}
\end{aligned}
$$

$$
\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} \\
&= \iint_S \frac{3x^2y^2z}{\sqrt{16x^2+y^2+z^2}}\,dS \\
&= \iint_S \frac{3x^2y^2\sqrt{4-4x^2-y^2}}{\sqrt{16x^2+y^2+4-4x^2-y^2}}\,dS \\
&= \iint_S 3x^2y^2\,dA \\
&= \int_{-1}^{1}\int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} 3x^2y^2\,dy\,dx \\
&= \int_{-1}^{1}[x^2y^3]_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}}\,dx \\
&= 16\int_{-1}^{1} x^2(1-x^2)^{3/2}\,dx \\
&= \pi
\end{aligned}
$$

3.14.13

$$\mathbf{F} = \langle 6yz, 5x, yze^{x^2} \rangle$$

$$\begin{aligned}
z &= \frac{1}{4}x^2 + y^2 \\
4 &= \frac{1}{4}x^2 + y^2 \\
y &= \sqrt{4 - \frac{1}{4}x^2}
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}(t) &= \langle 4\cos t, 2\sin t, 4 \rangle \\
d\mathbf{r} &= \langle -4\sin t, 2\cos t, 0 \rangle \, dt
\end{aligned}$$

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^{2\pi} [6(2\sin t)(4)(-4\sin t) + 5(4\cos t)(2\cos t)] \, dt \\
&= \int_0^{2\pi} (-192\sin^2 t + 40\cos^2 t) \, dt \\
&= 8\int_0^{2\pi} \left[-12(1 - \cos 2t) + \frac{5}{2}(1 + \cos 2t) \right] \, dt \\
&= 8\left[-12t + 6\sin 2t + \frac{5}{2}t + \frac{5}{4}\sin 2t \right] \, dt_0^{2\pi} \\
&= -152\pi
\end{aligned}$$

3.14.17

$$
\begin{aligned}
\mathbf{F} &= \langle z^2 e^{x^2}, xy^2, \arctan y\rangle \\
\nabla \times \mathbf{F} &= \langle \frac{1}{y^2+1}, 2ze^{x^2}, y^2\rangle
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 \\
g(x,y,z) &= z \\
\nabla g &= \langle 0,0,1\rangle \\
||\nabla g|| &= 1 \\
\mathbf{n} &= \langle 0,0,1\rangle
\end{aligned}
$$

$$
\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} \\
&= \iint_S y^2\,dS \\
&= \iint_S y^2\,dA \\
&= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} y^2\,dy\,dx \\
&= \int_{-3}^{3}\left[\frac{1}{3}y^3\right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\,dx \\
&= \frac{2}{3}\int_{-3}^{3}(9-x^2)^{3/2}\,dx \\
&= \frac{81}{4}\pi
\end{aligned}
$$

3.15 Triple Integrals

3.15.1

$$\begin{aligned}
\int_2^4 \int_{-2}^2 \int_{-1}^1 (x+y+z)\,dx\,dy\,dz &= \int_2^4 \int_{-2}^2 \left[\frac{1}{2}x^2 + (y+z)x\right]_{-1}^1 \\
&= \int_2^4 \int_{-2}^2 (2y+2z)\,dy\,dz \\
&= \int_2^4 [y^2 + 2zy]_{-2}^2\,dz \\
&= \int_2^4 8z\,dz \\
&= [4z^2]_2^4 \\
&= 48
\end{aligned}$$

3.15.9

$$\begin{aligned}
\iiint_D z\,dV &= \int_0^5 \int_1^3 \int_y^{y+2} z\,dx\,dy\,dz \\
&= \int_0^5 \int_1^3 2z\,dy\,dz \\
&= \int_0^5 4z\,dz \\
&= [2z^2]_0^5 \\
&= 50
\end{aligned}$$

3.15.11

$$\int_0^2 \int_0^{4-2y} \int_{x+2y}^4 F(x,y,z)\,dz\,dx\,dy$$

$$\int_0^4 \int_0^{z/2} \int_0^{z-2y} F(x,y,z)\,dx\,dy\,dz$$

$$\int_0^2 \int_{2y}^4 \int_0^{z-2y} F(x,y,z)\,dx\,dz\,dy$$

$$\int_0^4 \int_0^z \int_0^{(z-x)/2} F(x,y,z)\,dy\,dx\,dz$$

$$\int_0^4 \int_x^4 \int_0^{(z-x)/2} F(x,y,z)\,dy\,dz\,dx$$

$$\int_0^4 \int_0^{2-x/2} \int_{x+2y}^4 F(x,y,z)\,dz\,dy\,dx$$

3.15.13

(a)

$$\int_0^2 \int_{x^3}^8 \int_0^4 dz\, dy\, dx$$

(b)

$$\int_0^8 \int_0^4 \int_0^{\sqrt[3]{y}} dx\, dz\, dy$$

(c)

$$\int_0^4 \int_0^2 \int_{x^3}^8 dy\, dx\, dz$$

3.15.21

$$\begin{aligned}
\int_0^3 \int_{-\sqrt{2}}^{\sqrt{2}} \int_{y^2}^{4-y^2} dx\, dy\, dz &= \int_0^3 \int_{-\sqrt{2}}^{\sqrt{2}} (4-2y^2)\, dy\, dz \\
&= \int_0^3 \left[4y - \frac{2}{3}y^3\right]_{-\sqrt{2}}^{\sqrt{2}} dz \\
&= \int_0^3 \left[\left(4\sqrt{2} - \frac{2}{3}2^{3/2}\right) - \left(-4\sqrt{2} + \frac{2}{3}2^{3/2}\right)\right] dz \\
&= \int_0^3 \left(8\sqrt{2} - \frac{4}{3}2^{3/2}\right) dz \\
&= \frac{16}{3}\sqrt{2} \int_0^3 dz \\
&= 16\sqrt{2}
\end{aligned}$$

3.15.25

$$\begin{aligned}
m &= \int_0^4 \int_0^2 \int_{x^3}^8 kz\,dy\,dx\,dz \\
&= \int_0^4 kz \int_0^2 (8 - x^3)\,dx\,dz \\
&= \int_0^4 kz \left[8x - \frac{1}{4}x^4\right]_0^2 dz \\
&= 12k \int_0^4 z\,dz \\
&= 12k \left[\frac{1}{2}z^2\right]_0^4 \\
&= 96k \\
M_{xy} &= \int_0^4 \int_0^2 \int_{x^3}^8 kz^2\,dy\,dx\,dz \\
&= k \int_0^4 z^2 \int_0^2 (8 - x^3)\,dx\,dz \\
&= k \int_0^4 z^2 \left[8x - \frac{1}{4}x^4\right]_0^2 dz \\
&= 12k \int_0^4 z^2\,dz \\
&= 12k \left[\frac{1}{3}z^3\right]_0^4 \\
&= 256k \\
M_{xz} &= \int_0^4 \int_0^2 \int_{x^3}^8 kyz\,dy\,dx\,dz \\
&= k \int_0^4 z \int_0^2 \left[\frac{1}{2}y^2\right]_{x^3}^8 dx\,dz \\
&= \frac{k}{2} \int_0^4 z \int_0^2 (64 - x^6)\,dx\,dz \\
&= \frac{k}{2} \int_0^4 z \left[64x - \frac{1}{7}x^7\right]_0^2 dz \\
&= \frac{384}{7}k \int_0^4 z\,dz \\
&= \frac{384}{7}k \left[\frac{1}{2}z^2\right]_0^4 \\
&= \frac{3072}{7}k
\end{aligned}$$

$$
\begin{aligned}
M_{yz} &= \int_0^4 \int_0^2 \int_{x^3}^8 kxz\,dy\,dx\,dz \\
&= k\int_0^4 z\int_0^2 x(8-x^3)\,dx\,dz \\
&= k\int_0^4 z\left[4x^2-\frac{1}{5}x^5\right]_0^2 dz \\
&= \frac{48}{5}k\int_0^4 z\,dz \\
&= \frac{48}{5}k\left[\frac{1}{2}z^2\right]_0^4 \\
&= \frac{384}{5}k \\
\overline{x} &= \frac{M_{yz}}{m} \\
&= \frac{384k/5}{96k} \\
&= \frac{4}{5} \\
\overline{y} &= \frac{M_{xz}}{m} \\
&= \frac{3072k/7}{96k} \\
&= \frac{32}{7} \\
\overline{z} &= \frac{M_{xy}}{m} \\
&= \frac{256k}{96k} \\
&= \frac{8}{3}
\end{aligned}
$$

3.15.29

$$
\begin{aligned}
m &= \iiint \rho\,dV \\
&= \int_{-1}^1 \int_{2y+2}^{8-y} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x+y+4)\,dx\,dz\,dy
\end{aligned}
$$

3.15.35

$$
\left(10\cos\frac{3\pi}{4}, 10\sin\frac{3\pi}{4}, 5\right) = \left(-5\sqrt{2}, 5\sqrt{2}, 5\right)
$$

3.15.39

$$\left(\sqrt{2}, -\frac{\pi}{4}, -9\right)$$

3.15.43

$$r^2 + z^2 = 25$$

3.15.45

$$r^2 - z^2 = 1$$

3.15.47

$$z = x^2 + y^2$$

3.15.49

$$x = 5$$

3.15.51

$$\begin{aligned} r^2 &= 4 \\ r^2 + z^2 &= 16 \\ z &= 0 \\ \iiint dV &= \int_0^{2\pi} \int_0^2 \int_0^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta \\ &= 2\pi \int_0^2 r\sqrt{16-r^2}\,dr \end{aligned}$$

Let $u = 16 - r^2$ so $du = -2r\,dr$

$$\begin{aligned} -\pi \int_{16}^{12} \sqrt{u}\,du &= -\pi \left[\frac{2}{3}u^{3/2}\right]_{16}^{12} \\ &= -\frac{2}{3}\pi(12^{3/2} - 16^{3/2}) \\ &= \frac{2}{3}\pi(64 - 8 \cdot 3^{3/2}) \end{aligned}$$

3.15.53

$$\begin{aligned} z &= r^2 \\ r^2 &= 25 \\ z &= 0 \\ \iiint dV &= \int_0^{2\pi}\int_0^5\int_0^{r^2} r\,dz\,dr\,d\theta \\ &= 2\pi\int_0^5 r^3\,dr \\ &= 2\pi\left[\frac{1}{4}r^4\right]_0^5 \\ &= \frac{625}{2}\pi \end{aligned}$$

3.15.59

(a)

$$\begin{aligned} (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \\ \left(\frac{2}{3}\sin\frac{\pi}{2}\cos\frac{\pi}{6}, \frac{2}{3}\sin\frac{\pi}{2}\sin\frac{\pi}{6}, \frac{2}{3}\cos\frac{\pi}{2}\right) \\ \left(\frac{\sqrt{3}}{3}, \frac{1}{3}, 0\right) \end{aligned}$$

(b)

$$\begin{aligned} (\rho\sin\phi, \theta, \rho\cos\phi) \\ \left(\frac{2}{3}\sin\frac{\pi}{2}, \frac{\pi}{6}, \frac{2}{3}\cos\frac{\pi}{2}\right) \\ \left(\frac{2}{3}, \frac{\pi}{6}, 0\right) \end{aligned}$$

3.15.63

$$\begin{aligned} \left(\sqrt{x^2+y^2+z^2}, \arccos\frac{z}{\sqrt{x^2+y^2+z^2}}, \arctan\frac{y}{x}\right) \\ \left(\sqrt{50}, \frac{\pi}{2}, -\frac{3\pi}{4}\right) \end{aligned}$$

3.15.67

$$\rho = 8$$

3.15.69

$$\begin{aligned}
(\rho\cos\phi)^2 &= 3(\rho\sin\phi\cos\theta)^2 + 3(\rho\sin\phi\sin\theta)^2 \\
\rho^2\cos^2\phi &= 3\rho^2\sin^2\phi\cos^2\theta + 3\rho^2\sin^2\phi\sin^2\theta \\
\cos^2\phi &= 3sin^2\phi(\cos^2\theta + \sin^2\theta) \\
\cos^2\phi &= 3\sin^2\phi \\
\tan\phi &= \frac{1}{\sqrt{3}} \\
\phi &= \pm\frac{\pi}{6}
\end{aligned}$$

3.15.71

$$x^2 + y^2 + z^2 = 100$$

3.15.73

$$z = 2$$

3.15.75

$$\begin{aligned}
\rho\cos\phi &= \sqrt{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2} \\
\cos\phi &= \sin\phi \\
\phi &= \frac{\pi}{4}
\end{aligned}$$

$$\rho = 3$$

$$\begin{aligned}
\iiint dV &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^3 r^2\sin\phi\,dr\,d\phi\,d\theta \\
&= 2\pi\int_0^{\pi/4}\sin\phi\left[\frac{1}{3}r^3\right]_0^3 d\phi \\
&= 18\pi\int_0^{\pi/4}\sin\phi\,d\phi \\
&= 18\pi[-\cos\phi]_0^{\pi/4} \\
&= 18\pi\left(1 - \frac{1}{\sqrt{2}}\right) \\
&= 9\pi(2 - \sqrt{2})
\end{aligned}$$

3.15.77

$$\begin{aligned}
\phi &= \frac{\pi}{6} \\
\rho\cos\phi &= 2 \\
\iiint dV &= \int_0^{\pi/2}\int_0^{\pi/6}\int_0^{2\sec\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \frac{\pi}{2}\int_0^{\pi/6}\sin\phi\left[\frac{1}{3}\rho^3\right]_0^{2\sec\phi}\,d\phi \\
&= \frac{4}{3}\pi\int_0^{\pi/6}\sec^2\phi\tan\phi\,d\phi
\end{aligned}$$

Let $u = \tan\phi$ so $du = \sec^2\phi\,d\phi$

$$\begin{aligned}
\frac{4}{3}\pi\int_0^{1/\sqrt{3}} u\,du &= \frac{4}{3}\pi\left[\frac{1}{2}u^2\right]_0^{1/\sqrt{3}} \\
&= \frac{2}{9}\pi
\end{aligned}$$

3.15.81

$$\begin{aligned}
\rho\cos\phi &= 4\\
\rho &= 4\sec\phi\\
\\
\rho &= 5\\
\\
5\cos\phi &= 4\\
\phi &= \arccos\frac{4}{5}
\end{aligned}$$

$$\begin{aligned}
\iiint dV &= \int_0^{2\pi}\int_0^{\arccos 4/5}\int_{4\sec\phi}^{5} k\rho\sin\phi\,d\rho\,d\phi\,d\theta\\
&= 2\pi k\int_0^{\arccos 4/5}\sin\phi\left[\frac{1}{2}\rho^2\right]_{4\sec\phi}^{5}\,d\phi\\
&= \pi k\int_0^{\arccos 4/5}\sin\phi(25-16\sec^2\phi)\,d\phi\\
&= \pi k\int_0^{\arccos 4/5}(25\sin\phi-16\sec\phi\tan\phi)\,d\phi\\
&= \pi k[-25\cos\phi-16\sec\phi]_0^{\arccos 4/5}\\
&= \pi k\left[\left(-25\frac{4}{5}-16\frac{5}{4}\right)-(-25-16)\right]\\
&= \pi k\,(-20-20+25+16)\\
&= \pi k
\end{aligned}$$

3.16 Divergence Theorem

3.16.1

$$
\begin{aligned}
\mathbf{F} &= \langle xy, yz, xz \rangle \\
\int_0^1 \int_0^1 x(1)\,dx\,dy &= \frac{1}{2} \\
\int_0^1 \int_0^1 -(0)z\,dx\,dz &= 0 \\
\int_0^1 \int_0^1 -(1)z\,dx\,dz &= \frac{1}{2} \\
\int_0^1 \int_0^1 (1)y\,dy\,dz &= \frac{1}{2} \\
\int_0^1 \int_0^1 -(0)y\,dy\,dz &= 0 \\
\int_0^1 \int_0^1 -x(0)\,dx\,dy &= 0 \\
\iint_S (\mathbf{F}\cdot\mathbf{n})\,dS &= \frac{3}{2} \\
\nabla\cdot\mathbf{F} &= x+y+z \\
\iiint_D (\nabla\cdot\mathbf{F})\,dV &= \int_0^1 \int_0^1 \int_0^1 (x+y+z)\,dx\,dy\,dz \\
&= \int_0^1 \int_0^1 \left[\frac{1}{2}x^2 + (y+z)x\right]_0^1 dy\,dz \\
&= \int_0^1 \int_0^1 \left(\frac{1}{2} + y + z\right) dy\,dz \\
&= \int_0^1 \left[\frac{1}{2}y^2 + \left(\frac{1}{2} + z\right) y\right]_0^1 dz \\
&= \int_0^1 (1+z)\,dz \\
&= \left[z + \frac{1}{2}z^2\right]_0^1 \\
&= \frac{3}{2}
\end{aligned}
$$

3.16.3

$$
\begin{aligned}
\mathbf{F} &= \langle x^3, y^3, z^3 \rangle \\
\nabla \cdot \mathbf{F} &= 3x^2 + 3y^2 + 3z^2 \\
&= 3r^2 \\
\iint_S (\mathbf{F} \cdot \mathbf{n})\, dS &= \iiint_D (\nabla \cdot \mathbf{F})\, dV \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^{a} 3r^4 \sin\phi \, dr\, d\phi\, d\theta \\
&= 6\pi \int_0^{\pi} \sin\phi \left[\frac{1}{5} r^5\right]_0^a d\phi \\
&= \frac{6}{5} a^5 \pi \int_0^{\pi} \sin\phi \, d\phi \\
&= \frac{6}{5} a^5 \pi [-\cos\phi]_0^{\pi} \\
&= \frac{12}{5} a^5 \pi
\end{aligned}
$$

3.16.5

$$
\begin{aligned}
\mathbf{F} &= \langle y^2, xz^3, (z-1)^2 \rangle \\
\nabla \cdot \mathbf{F} &= 2z - 2 \\
\iint_S (\mathbf{F} \cdot \mathbf{n})\, dS &= \iiint_D (\nabla \cdot \mathbf{F})\, dV \\
&= \int_1^{5} \int_0^{2\pi} \int_0^{4} (2z-2) r \, dr\, d\theta\, dz \\
&= 2\pi \int_1^{5} (2z-2) \left[\frac{1}{2} r^2\right]_0^4 dz \\
&= 16\pi [z^2 - 2z]_1^5 \\
&= 16\pi [(25 - 10) - (1 - 2)] \\
&= 256\pi
\end{aligned}
$$

3.16.9

$$
\begin{aligned}
\mathbf{F} &= \frac{\langle x, y, z\rangle}{x^2+y^2+z^2} \\
\nabla \cdot \mathbf{F} &= \frac{1}{x^2+y^2+z^2} \\
&= \frac{1}{r^2} \\
\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS &= \iiint_D (\nabla\cdot\mathbf{F})\, dV \\
&= \int_0^{2\pi}\int_0^{\pi}\int_a^b \sin\phi\, dr\, d\phi\, d\theta \\
&= 2\pi(b-a)[-\cos\phi]_0^{\pi} \\
&= 4\pi(b-a)
\end{aligned}
$$

3.17 Change of Variables in Multiple Integrals

3.17.1

$$
\begin{aligned}
&(0,0) \\
&(-2,8) \\
&(16,20) \\
&(14,28)
\end{aligned}
$$

3.17.7

$$
\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -ve^{-u} & e^{-u} \\ ve^{u} & e^{u} \end{vmatrix}
$$
$$
= -2v
$$

3.17.9

$$
\begin{aligned}
\frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} -\frac{2y}{x^3} & \frac{1}{x^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} \\
&= -\frac{3y^2}{x^4} \\
\frac{\partial(x,y)}{\partial(u,v)} &= \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} \\
&= -\frac{x^4}{3y^2} \\
&= -\frac{1}{3u^2}
\end{aligned}
$$

3.17.13

$$\begin{aligned}
\left(-\frac{8}{3}, \frac{5}{3}\right) &\to (-6, -1) \\
(0, 3) &\to (-6, 3) \\
(4, -1) &\to (6, 3) \\
\left(\frac{4}{3}, -\frac{7}{3}\right) &\to (6, -1) \\
\frac{\partial(u, v)}{\partial(x, y)} &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \\
&= \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} \\
&= 3 \\
\iint_R (x + y)\, dA &= \int_{-1}^{3} \int_{-6}^{6} \frac{1}{3} v \, du \, dv \\
&= 4 \left[\frac{1}{2} v^2\right]_{-1}^{3} \\
&= 16
\end{aligned}$$

3.17.15

$$\begin{aligned}
y &= \left(\frac{1}{2}y^2\right)\\
&= \frac{1}{4}y^4\\
y &= \sqrt[3]{4}\\
x &= \frac{1}{2}4^{2/3}\\
\left(\frac{1}{2}4^{2/3}, \sqrt[3]{4}\right) &\to (1,2)\\
(2,2) &\to (2,2)\\
\left(\sqrt[3]{4}, \frac{1}{2}4^{2/3}\right) &\to (2,1)\\
(1,1) &\to (1,1)\\
\frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2}\\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix}\\
&= 3\\
\iint_R \frac{y^2}{x}\,dA &= \int_1^2\int_1^2 \frac{1}{3}v\,du\,dv\\
&= \frac{1}{3}\left[\frac{1}{2}v^2\right]_1^2\\
&= \frac{1}{2}
\end{aligned}$$

3.17.23

$$\begin{aligned}
u &= y - x \\
v &= y + x \\
(0,0) &\to (0,0) \\
(0,1) &\to (1,1) \\
(1,0) &\to (-1,1) \\
\frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} \\
&= -2 \\
\int_0^1 \int_0^{1-x} e^{(y-x)/(y+x)}\, dy\, dx &= \int_0^1 \int_{-v}^{v} -\frac{1}{2} e^{u/v}\, du\, dv \\
&= -\frac{1}{2} \int_0^1 \left[v e^{u/v} \right]_{-v}^{v} dv \\
&= -\frac{1}{2} \int_0^1 v(e - e^{-1})\, dv \\
&= -\frac{1}{2}(e - e^{-1}) \left[\frac{1}{2} v^2 \right]_0^1 \\
&= -\frac{1}{4}(e - e^{-1})
\end{aligned}$$

3.17.27

$$\begin{aligned}
u &= xy \\
v &= xy^{1.4} \\
\frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} y & x \\ y^{1.4} & 1.4xy^{0.4} \end{vmatrix} \\
&= 1.4xy^{1.4} - xy^{1.4} \\
&= \frac{1}{0.4} xy^{1.4} \\
\iint_R dA &= \int_a^b \int_c^d \frac{1}{0.4v}\, dv\, du \\
&= \frac{1}{0.4} \int_a^b [\ln v]_c^d\, du \\
&= \frac{1}{0.4} \ln \frac{d}{c} \int_a^b du \\
&= \frac{1}{0.4}(b - a) \ln \frac{d}{c}
\end{aligned}$$

3.17.29

$$
\begin{aligned}
u &= \frac{x}{5} \\
v &= \frac{y}{3} \\
u^2 + v^2 &= 1 \\
\frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{3} \end{vmatrix} \\
&= \frac{1}{15} \\
\iint_R \left(\frac{x^2}{25} + \frac{y^2}{9} \right) \, dA &= \iint_{R'} (u^2 + v^2) 15 \, dA' \\
&= 15 \iint_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta \\
&= 30\pi \left[\frac{1}{4} r^4 \right] \\
&= \frac{15}{2}\pi
\end{aligned}
$$