

# Advanced Engineering Mathematics Systems of Differential Equations by Dennis G. Zill Problems

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## 10 Systems of Linear Differential Equations

### 10.1 Theory of Linear Systems

#### 10.1.1

$$\mathbf{x}' = \begin{pmatrix} 3 & -5 \\ 4 & 8 \end{pmatrix} \mathbf{x}$$

10.1.3

$$\mathbf{X}' = \begin{pmatrix} -3 & 4 & -9 \\ 6 & -1 & 0 \\ 10 & 4 & 3 \end{pmatrix} \mathbf{X}$$

10.1.5

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} t-1 \\ -3t^2 \\ t^2-t+2 \end{pmatrix}$$

10.1.7

$$\begin{aligned} \frac{dx}{dt} &= 4x + 2y + e^t \\ \frac{dy}{dt} &= -x + 3y - e^t \end{aligned}$$

10.1.9

$$\begin{aligned} \frac{dx}{dt} &= x - y + 2z + e^{-t} - 3t \\ \frac{dy}{dt} &= 3x - 4y + z + 2e^{-t} + t \\ \frac{dz}{dt} &= -2x + 5y + 6z + 2e^{-t} - t \end{aligned}$$

10.1.11

$$\begin{aligned} 3(e^{-5t}) - 4(2e^{-5t}) &= -5e^{-5t} \\ &= \frac{dx}{dt} \\ 4(e^{-5t}) - 7(2e^{-5t}) &= -10e^{-5t} \\ &= \frac{dy}{dt} \end{aligned}$$

10.1.13

$$\begin{aligned} -(-e^{-3t/2}) + \frac{1}{4}(2e^{-3t/2}) &= \frac{3}{2}e^{-3t/2} \\ &= \frac{dx}{dt} \\ (-e^{-3t/2}) - (2e^{-3t/2}) &= -3e^{-3t/2} \\ &= \frac{dy}{dt} \end{aligned}$$

**10.1.17**

$$\begin{aligned}
W(\mathbf{X}_1, \mathbf{X}_2) &= \begin{vmatrix} e^{-2t} & e^{-6t} \\ e^{-2t} & -e^{-6t} \end{vmatrix} \\
&= -e^{-8t} - e^{-8t} \\
&= -2e^{-8t} \\
&\neq 0 \text{ for } t \in (-\infty, \infty)
\end{aligned}$$

Yes, they form a fundamental set.

**10.1.19**

$$\begin{aligned}
W(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) &= \begin{vmatrix} 1+t & 1 & 3+2t \\ -2+2t & -2 & -6+4t \\ 4+2t & 4 & 12+4t \end{vmatrix} \\
&= 0
\end{aligned}$$

No, they don't form a fundamental set.

**10.1.21**

$$\begin{aligned}
x &= 2t + 5 \\
y &= -t + 1 \\
\frac{dx}{dt} &= (2t + 5) + 4(-t + 1) + 2t - 7 \\
&= 2 \\
\frac{dy}{dt} &= 3(2t + 5) + 2(-t + 1) - 4t - 18 \\
&= -1
\end{aligned}$$

**10.1.23**

$$\begin{aligned}
x &= e^t + te^t \\
x' &= 2e^t + te^t \\
y &= e^t - te^t \\
y' &= -te^t \\
\frac{dx}{dt} &= 2(e^t + te^t) + (e^t - te^t) - e^t \\
&= 2e^t + te^t \\
\frac{dy}{dt} &= 3(e^t + te^t) + 4(e^t - te^t) - 7e^t \\
&= -te^t
\end{aligned}$$

## 10.2 Homogeneous Linear Systems

10.2.1

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

10.2.3

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} e^t$$

10.2.5

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-10t} + c_2 \begin{pmatrix} 5 \\ 2 \end{pmatrix} e^{8t}$$

10.2.7

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t$$

10.2.13

$$\mathbf{X} = 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/2} + 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t/2}$$

10.2.15

(a)

$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{3}{100}x_1 + \frac{1}{100}x_2 \\ \frac{dx_2}{dt} &= \frac{2}{100}x_1 - \frac{2}{100}x_2 \\ \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} &= \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{2}{100} & -\frac{2}{100} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

(b)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\frac{35}{3} \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t/25} + \frac{50}{3} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} e^{-t/100}$$

10.2.21

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \left[ \begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right]$$

**10.2.23**

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix} \right] e^{2t}$$

**10.2.25**

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^t$$

**10.2.31**

$$\mathbf{X} = -\frac{1}{2} \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + 13 \left[ \begin{pmatrix} 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \right] e^{4t}$$

**10.2.33**

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{K}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

**10.2.35**

$$\begin{aligned} \mathbf{X} &= c_1 \begin{pmatrix} 1 \\ 2-i \end{pmatrix} e^{(4+i)t} + c_2 \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{(4-i)t} \\ &= c_1 \left[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin t \right] e^{4t} + c_2 \left[ \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t \right] e^{4t} \\ &= c_1 \begin{pmatrix} \cos t \\ 2 \cos t + \sin t \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} \sin t \\ 2 \sin t - \cos t \end{pmatrix} e^{4t} \end{aligned}$$

10.2.37

$$\begin{aligned}
 \mathbf{X} &= c_1 \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{(4+i)t} + c_2 \begin{pmatrix} 1 \\ -1-i \end{pmatrix} e^{(4-i)t} \\
 &= c_1 \left[ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t \right] e^{4t} + c_2 \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin t \right] e^{4t} \\
 &= c_1 \begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix} e^{4t}
 \end{aligned}$$

10.2.39

$$\begin{aligned}
 \mathbf{X} &= c_1 \begin{pmatrix} 5 \\ 4-3i \end{pmatrix} e^{3i} + c_2 \begin{pmatrix} 5 \\ 4+3i \end{pmatrix} e^{-3i} \\
 &= c_1 \left[ \begin{pmatrix} 5 \\ 4 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ -3 \end{pmatrix} \sin 3t \right] + c_2 \left[ \begin{pmatrix} 0 \\ -3 \end{pmatrix} \cos 3t + \begin{pmatrix} 5 \\ 4 \end{pmatrix} \sin 3t \right] \\
 &= c_1 \begin{pmatrix} 5 \cos 3t \\ 4 \cos 3t + 3 \sin 3t \end{pmatrix} + c_2 \begin{pmatrix} 5 \sin 3t \\ 4 \sin 3t - 3 \cos 3t \end{pmatrix}
 \end{aligned}$$

10.2.47

$$\begin{aligned}
 \mathbf{X} &= c_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1+5i \\ 1 \\ 1 \end{pmatrix} e^{5it} + c_3 \begin{pmatrix} 1-5i \\ 1 \\ 1 \end{pmatrix} e^{-5it} \\
 &= c_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t + c_2 \left[ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cos 5t - \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \sin 5t \right] \\
 &\quad + c_3 \left[ \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \cos 5t + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \sin 5t \right] \\
 &= c_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t + c_2 \begin{pmatrix} \cos 5t - 5 \sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + c_3 \begin{pmatrix} 5 \cos 5t + \sin 5t \\ \sin 5t \\ \sin 5t \end{pmatrix} \\
 &= - \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t - \begin{pmatrix} \cos 5t - 5 \sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + 6 \begin{pmatrix} 5 \cos 5t + \sin 5t \\ \sin 5t \\ \sin 5t \end{pmatrix}
 \end{aligned}$$

10.2.49

(a)

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{20} & 0 & \frac{1}{10} \\ \frac{1}{20} & -\frac{1}{20} & 0 \\ 0 & \frac{1}{20} & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(b)

$$\begin{aligned}
\mathbf{X} &= c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1-i \\ i \\ 1 \end{pmatrix} e^{(-\frac{1}{10} + \frac{1}{20}i)t} + c_3 \begin{pmatrix} -1+i \\ -i \\ 1 \end{pmatrix} e^{(-\frac{1}{10} - \frac{1}{20}i)t} \\
&= c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \left[ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cos \frac{1}{20}t - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \sin \frac{1}{20}t \right] e^{-t/10} \\
&\quad + c_3 \left[ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cos \frac{1}{20}t + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \sin \frac{1}{20}t \right] e^{-t/10} \\
&= c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \sin \frac{t}{20} - \cos \frac{t}{20} \\ -\sin \frac{t}{20} \\ \cos \frac{t}{20} \end{pmatrix} e^{-t/10} \\
&\quad + c_3 \begin{pmatrix} -\cos \frac{t}{20} - \sin \frac{t}{20} \\ \cos \frac{t}{20} \\ \sin \frac{t}{20} \end{pmatrix} e^{-t/10} \\
&= 11 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - 6 \begin{pmatrix} \sin \frac{t}{20} - \cos \frac{t}{20} \\ -\sin \frac{t}{20} \\ \cos \frac{t}{20} \end{pmatrix} e^{-t/10} \\
&\quad - 2 \begin{pmatrix} -\cos \frac{t}{20} - \sin \frac{t}{20} \\ \cos \frac{t}{20} \\ \sin \frac{t}{20} \end{pmatrix} e^{-t/10}
\end{aligned}$$

### 10.3 Solution by Diagonalization

10.3.1

$$\mathbf{X} = \begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 e^{7t} \\ c_2 e^{-4t} \end{pmatrix} = \begin{pmatrix} 3c_1 e^{7t} - 2c_2 e^{-4t} \\ c_1 e^{7t} + 3c_2 e^{-4t} \end{pmatrix}$$

10.3.3

$$\mathbf{X} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_1 e^{3t/2} \\ c_2 e^{t/2} \end{pmatrix} = \begin{pmatrix} c_1 e^{3t/2} + c_2 e^{t/2} \\ 2c_1 e^{3t/2} - 2c_2 e^{t/2} \end{pmatrix}$$

10.3.5

$$\mathbf{X} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{6t} \\ c_2 e^{-4t} \\ c_3 e^{2t} \end{pmatrix} = \begin{pmatrix} -c_2 e^{-4t} + c_3 e^{2t} \\ c_2 e^{-4t} + c_3 e^{2t} \\ c_1 e^{6t} + c_3 e^{2t} \end{pmatrix}$$

10.3.11

(a)

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



$\mathbf{M}$  has an inverse because it has a nonzero determinant (the product of the diagonal entries).

(b)

$$\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & \frac{k_2}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

(c)

$$\mathbf{X} = \mathbf{P}\mathbf{Y}$$

$$\mathbf{P}\mathbf{Y}'' + \mathbf{B}\mathbf{P}\mathbf{Y} = \mathbf{0}$$

$$\mathbf{Y}'' + \mathbf{P}^{-1}\mathbf{B}\mathbf{P}\mathbf{Y} = \mathbf{0}$$

$$\mathbf{Y}'' + \mathbf{D}\mathbf{Y} = \mathbf{0}$$

$$\begin{pmatrix} y_1'' \\ y_2'' \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{0}$$

$$y_1'' + 6y_1 = 0$$

$$y_1 = c_1 \cos \sqrt{6}t + c_2 \sin \sqrt{6}t$$

$$y_2'' + y_2 = 0$$

$$y_2 = c_3 \cos t + c_4 \sin t$$

$$\mathbf{X} = \mathbf{P}\mathbf{Y}$$

$$= \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \cos \sqrt{6}t + c_2 \sin \sqrt{6}t \\ c_3 \cos t + c_4 \sin t \end{pmatrix}$$

$$= \begin{pmatrix} -2c_1 \cos \sqrt{6}t - 2c_2 \sin \sqrt{6}t + c_3 \cos t + c_4 \sin t \\ c_1 \cos \sqrt{6}t + c_2 \sin \sqrt{6}t + 2c_3 \cos t + 2c_4 \sin t \end{pmatrix}$$

$$= c_3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos t + c_4 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t + c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cos \sqrt{6}t + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \sin \sqrt{6}t$$

## 10.4 Nonhomogeneous Linear Systems

### 10.4.1

$$\begin{aligned}\begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -7 \\ 5 \end{pmatrix} \\ \mathbf{X}_c &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^t \\ \mathbf{X}_p &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} -7 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 2a_1 + 3a_2 - 7 \\ -a_1 - 2a_2 + 5 \end{pmatrix} \\ \mathbf{X}_p &= \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ \mathbf{X} &= \mathbf{X}_c + \mathbf{X}_p \\ &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} -1 \\ 3 \end{pmatrix}\end{aligned}$$

### 10.4.3

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t}$$

$$\mathbf{X}_p = \begin{pmatrix} a_3 t^2 + a_2 t + a_1 \\ b_3 t^2 + b_2 t + b_1 \end{pmatrix}$$

$$\begin{pmatrix} 2a_3 t + a_2 \\ 2b_3 t + b_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a_3 t^2 + a_2 t + a_1 \\ b_3 t^2 + b_2 t + b_1 \end{pmatrix} + \begin{pmatrix} -2t^2 \\ t + 5 \end{pmatrix}$$

$$= \begin{pmatrix} (a_3 + 3b_3 - 2)t^2 + (a_2 + 3b_2)t + (a_1 + 3b_1) \\ (3a_3 + b_3)t^2 + (3a_2 + b_2 + 1)t + (3a_1 + b_1 + 5) \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (a_3 + 3b_3 - 2)t^2 + (a_2 - 2a_3 + 3b_2)t + (a_1 - a_2 + 3b_1) \\ (3a_3 + b_3)t^2 + (3a_2 + b_2 - 2b_3 + 1)t + (3a_1 + b_1 - b_2 + 5) \end{pmatrix}$$

$$a_3 = -\frac{1}{4}$$

$$b_3 = \frac{3}{4}$$

$$a_2 = \frac{1}{4}$$

$$b_2 = -\frac{1}{4}$$

$$a_1 = -2$$

$$b_1 = \frac{3}{4}$$

$$\mathbf{X}_p = \begin{pmatrix} -\frac{1}{4}t^2 + \frac{1}{4}t - 2 \\ \frac{3}{4}t^2 - \frac{1}{4}t + \frac{3}{4} \end{pmatrix}$$

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} + \begin{pmatrix} -\frac{1}{4}t^2 + \frac{1}{4}t - 2 \\ \frac{3}{4}t^2 - \frac{1}{4}t + \frac{3}{4} \end{pmatrix}$$

### 10.4.5

$$\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} \frac{1}{9} \\ 1 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} e^{3t} \\
\mathbf{X}_p &= \begin{pmatrix} a \\ b \end{pmatrix} e^t \\
\begin{pmatrix} a \\ b \end{pmatrix} e^t &= \begin{pmatrix} 4 & \frac{1}{3} \\ 9 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} e^t + \begin{pmatrix} -3 \\ 10 \end{pmatrix} e^t \\
\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 4a + \frac{1}{3}b - 3 \\ 9a + 6b + 10 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 3a + \frac{1}{3}b - 3 \\ 9a + 5b + 10 \end{pmatrix} \\
\mathbf{X}_p &= \begin{pmatrix} \frac{55}{36} \\ -\frac{19}{4} \end{pmatrix} e^t \\
\mathbf{X} &= c_1 \begin{pmatrix} \frac{1}{9} \\ 1 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} \frac{55}{36} \\ -\frac{19}{4} \end{pmatrix} e^t
\end{aligned}$$

### 10.4.9

$$\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^t \\
\mathbf{X}_p &= \begin{pmatrix} a \\ b \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} -a - 2b + 3 \\ 3a + 4b + 3 \end{pmatrix} \\
\mathbf{X}_p &= \begin{pmatrix} -9 \\ 6 \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} -9 \\ 6 \end{pmatrix} \\
\begin{pmatrix} -4 \\ 5 \end{pmatrix} &= c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -9 \\ 6 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -2c_1 - c_2 - 5 \\ 3c_1 + c_2 + 1 \end{pmatrix} \\
\mathbf{X} &= 4 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{2t} - 13 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} -9 \\ 6 \end{pmatrix}
\end{aligned}$$

### 10.4.11

(a)

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{1}{50} & -\frac{1}{25} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(b)

$$\begin{aligned} \mathbf{X}_c &= c_1 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} e^{-t/20} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/50} \\ \mathbf{X}_p &= \begin{pmatrix} a \\ b \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{1}{50} & -\frac{1}{25} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{3}{100}a + \frac{1}{100}b \\ \frac{1}{50}a - \frac{1}{25}b + 1 \end{pmatrix} \\ \mathbf{X}_p &= \begin{pmatrix} 10 \\ 30 \end{pmatrix} \\ \mathbf{X} &= c_1 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} e^{-t/20} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/50} + \begin{pmatrix} 10 \\ 30 \end{pmatrix} \\ \begin{pmatrix} 60 \\ 10 \end{pmatrix} &= c_1 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 10 \\ 30 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -\frac{1}{2}c_1 + c_2 - 50 \\ c_1 + c_2 + 20 \end{pmatrix} \\ \mathbf{X} &= -\frac{140}{3} \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} e^{-t/20} + \frac{80}{3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/50} + \begin{pmatrix} 10 \\ 30 \end{pmatrix} \end{aligned}$$

(c)

$$\begin{aligned} \lim_{t \rightarrow \infty} x_1(t) &= \lim_{t \rightarrow \infty} \frac{70}{3} e^{-t/20} + \frac{80}{3} e^{-t/50} + 10 \\ &= 10 \\ \lim_{t \rightarrow \infty} x_2(t) &= \lim_{t \rightarrow \infty} -\frac{140}{3} e^{-t/20} + \frac{80}{3} e^{-t/50} + 30 \\ &= 30 \end{aligned}$$

10.4.13

$$\begin{aligned}
 \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} \\
 \mathbf{X}_c &= c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
 \mathbf{X}_p &= \Phi(t) \int \Phi^{-1}(t) \mathbf{F} dt \\
 &= \begin{pmatrix} 3e^t & 1 \\ 2e^t & 1 \end{pmatrix} \int \begin{pmatrix} e^{-t} & -e^{-t} \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \end{pmatrix} dt \\
 &= \begin{pmatrix} 3e^t & 1 \\ 2e^t & 1 \end{pmatrix} \int \begin{pmatrix} 5e^{-t} \\ -11 \end{pmatrix} dt \\
 &= \begin{pmatrix} 3e^t & 1 \\ 2e^t & 1 \end{pmatrix} \begin{pmatrix} -5e^{-t} \\ -11t \end{pmatrix} \\
 &= \begin{pmatrix} -15 - 11t \\ -10 - 11t \end{pmatrix} \\
 \mathbf{X} &= c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -15 - 11t \\ -10 - 11t \end{pmatrix}
 \end{aligned}$$

10.4.15

$$\begin{aligned}
 \mathbf{X}' &= \begin{pmatrix} 3 & -5 \\ \frac{3}{4} & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2} \\
 \mathbf{X}_c &= c_1 \begin{pmatrix} \frac{10}{3} \\ 1 \end{pmatrix} e^{3t/2} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t/2} \\
 \mathbf{X}_p &= \begin{pmatrix} \frac{1}{2}(-15 - 13t) \\ \frac{1}{4}(-9 - 13t) \end{pmatrix} e^{t/2} \\
 \mathbf{X} &= c_1 \begin{pmatrix} \frac{10}{3} \\ 1 \end{pmatrix} e^{3t/2} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t/2} + \frac{1}{4} \begin{pmatrix} -30 - 26t \\ -9 - 13t \end{pmatrix} e^{t/2}
 \end{aligned}$$

**10.4.33**

$$\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} \\
\mathbf{X}_p &= \begin{pmatrix} 2e^{2t}t - 2e^{4t}t - e^{2t} + e^{4t} \\ 2e^{2t}t + 2e^{4t}t + e^{2t} + e^{4t} \end{pmatrix} \\
&= \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} \\
\mathbf{X} &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} \\
&\quad + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} \\
\begin{pmatrix} 1 \\ 1 \end{pmatrix} &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -c_1 + c_2 - 1 \\ c_1 + c_2 + 1 \end{pmatrix} \\
\mathbf{X} &= \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{4t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t}
\end{aligned}$$

**10.4.35**

$$\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-12t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} \\
\mathbf{X}_p &= \frac{1}{29} \begin{pmatrix} -76 \cos t + 332 \sin t \\ -168 \cos t + 276 \sin t \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-12t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} - \frac{4}{29} \begin{pmatrix} 19 \\ 42 \end{pmatrix} \cos t + \frac{4}{29} \begin{pmatrix} 83 \\ 69 \end{pmatrix} \sin t \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{4}{29} \begin{pmatrix} 19 \\ 42 \end{pmatrix} \\
&= \begin{pmatrix} -3c_1 + c_2 - \frac{76}{29} \\ c_1 + 3c_2 - \frac{168}{29} \end{pmatrix} \\
\mathbf{X} &= -\frac{6}{29} \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-12t} + 2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} - \frac{4}{29} \begin{pmatrix} 19 \\ 42 \end{pmatrix} \cos t + \frac{4}{29} \begin{pmatrix} 83 \\ 69 \end{pmatrix} \sin t
\end{aligned}$$

**10.4.37**

$$\begin{aligned}
 \mathbf{A} &= \begin{pmatrix} 5 & -2 \\ 21 & -8 \end{pmatrix} \\
 \mathbf{F} &= \begin{pmatrix} 6 \\ 4 \end{pmatrix} \\
 \mathbf{P} &= \begin{pmatrix} 2 & 1 \\ 7 & 3 \end{pmatrix} \\
 \mathbf{G} &= \begin{pmatrix} -14 \\ 34 \end{pmatrix} \\
 \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} &= \begin{pmatrix} -2y_1 - 14 \\ -y_2 + 34 \end{pmatrix} \\
 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} c_1 e^{-2t} - 7 \\ c_2 e^{-t} + 34 \end{pmatrix} \\
 \mathbf{X} &= c_1 \begin{pmatrix} 2 \\ 7 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + \begin{pmatrix} 20 \\ 53 \end{pmatrix}
 \end{aligned}$$

**10.4.39**

$$\begin{aligned}
 \mathbf{A} &= \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \\
 \mathbf{P} &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\
 \mathbf{G} &= \begin{pmatrix} 4+t \\ 4-t \end{pmatrix} \\
 \mathbf{Y}' &= \mathbf{D}\mathbf{Y} + \mathbf{G} \\
 &= \begin{pmatrix} 10y_1 + 4 + t \\ 4 - t \end{pmatrix} \\
 \mathbf{Y} &= \begin{pmatrix} c_1 e^{10t} - \frac{1}{10}t - \frac{41}{100} \\ -\frac{1}{2}t^2 + 4t + c_2 \end{pmatrix} \\
 \mathbf{X} &= \mathbf{P}\mathbf{Y} \\
 &= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{10t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} t^2 + \begin{pmatrix} -\frac{41}{39} \\ \frac{10}{10} \end{pmatrix} t - \frac{41}{100} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
 \end{aligned}$$

## **10.5 Matrix Exponential**

**10.5.1**

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$$



10.5.3

$$\begin{pmatrix} t+1 & t & t \\ t & t+1 & t \\ -2t & -2t & -2t+1 \end{pmatrix}$$

10.5.5

$$\begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix}$$

10.5.7

$$\mathbf{X} = c_1 \begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} + c_2 \begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + c_3 \begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix}$$

10.5.9

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -3 \\ \frac{1}{2} \end{pmatrix}$$

10.5.11

$$\mathbf{X} = c_1 \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2 \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

10.5.13

$$\begin{aligned} \mathbf{X} &= c_1 \begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} + c_2 \begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + c_3 \begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ -4 \\ 6 \end{pmatrix} &= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \mathbf{X} &= \begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} - 4 \begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + 6 \begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix} \end{aligned}$$

10.5.15

$$e^{\mathbf{A}t} = \begin{pmatrix} \frac{1}{2}e^{-2t}(3e^{4t}-1) & \frac{3}{4}e^{-2t}(e^{4t}-1) \\ e^{-2t}-e^{2t} & -\frac{1}{2}e^{-2t}(e^{4t}-3) \end{pmatrix}$$

10.5.17

$$e^{\mathbf{A}t} = \begin{pmatrix} e^{2t}(1+3t) & -9e^{2t}t \\ e^{2t}t & e^{2t}(1-3t) \end{pmatrix}$$

### 10.5.25

$$\begin{aligned}
\mathbf{X} &= e^{\mathbf{A}t} \mathbf{C} \\
&= \mathbf{P} e^{\mathbf{D}t} \mathbf{P}^{-1} \mathbf{C} \\
&= \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} e^{5t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \mathbf{C} \\
&= \begin{pmatrix} -\frac{1}{2}e^{3t}(-3+e^{2t}) & \frac{1}{2}e^{3t}(-1+e^{2t}) \\ -\frac{3}{2}e^{3t}(-1+e^{2t}) & \frac{1}{2}e^{3t}(-1+3e^{2t}) \end{pmatrix} \mathbf{C}
\end{aligned}$$

## 10.6 Chapter in Review

### 10.6.1

$$\frac{1}{3}$$

### 10.6.5

$$\mathbf{X} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^t + c_2 \left[ \begin{pmatrix} -1 \\ 1 \end{pmatrix} t e^t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^t \right]$$

### 10.6.7

$$\begin{aligned}
\mathbf{X} &= c_1 \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{1+2i} + c_2 \begin{pmatrix} i \\ 1 \end{pmatrix} e^{1-2i} \\
&= c_3 \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin 2t \right] e^t + c_4 \left[ \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 2t \right] e^t \\
&= \begin{pmatrix} c_3 e^t \sin 2t - c_4 e^t \cos 2t \\ c_3 e^t \cos 2t + c_4 e^t \sin 2t \end{pmatrix} \\
&= c_3 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^t + c_4 \begin{pmatrix} -\cos 2t \\ \sin 2t \end{pmatrix} e^t
\end{aligned}$$

### 10.6.9

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -7 \\ -12 \\ 16 \end{pmatrix} e^{-3t} + c_3 \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} e^{2t}$$