

# Classical Mechanics by John R. Taylor Problems

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# 1 Newton's Laws of Motion

## 1.1

$$\begin{aligned}\mathbf{b} + \mathbf{c} &= 2\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}} \\ 5\mathbf{b} + 2\mathbf{c} &= 7\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 2\hat{\mathbf{z}} \\ \mathbf{b} \cdot \mathbf{c} &= 1 \\ \mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \\ &= \hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}}\end{aligned}$$

## 1.5

$$\begin{aligned}\mathbf{v}_{\text{body}} &= \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}} \\ \mathbf{v}_{\text{face}} &= \hat{\mathbf{x}} + \hat{\mathbf{z}} \\ \mathbf{v}_{\text{body}} \cdot \mathbf{v}_{\text{face}} &= v_{\text{body}} v_{\text{face}} \cos \theta \\ 2 &= \sqrt{6} \cos \theta \\ \cos \theta &= \frac{2}{\sqrt{6}} \\ \theta &= \arccos \frac{2}{\sqrt{6}} \\ &= 35.26^\circ\end{aligned}$$

## 1.11

The particle moves counterclockwise in an ellipse of width  $2b$  and height  $2c$ . The angular speed is  $\omega$ .

## 1.23

$$\begin{aligned}\mathbf{v} &= v \cos \theta \frac{\mathbf{b}}{b} - v \sin \theta \frac{\mathbf{b} \times \mathbf{c}}{bc} \\ &= \frac{\lambda}{b} \frac{\mathbf{b}}{b} - \frac{c}{b} \frac{\mathbf{b} \times \mathbf{c}}{bc} \\ &= \frac{\lambda \mathbf{b} - \mathbf{b} \times \mathbf{c}}{b^2}\end{aligned}$$

1.25

$$\begin{aligned}\frac{df}{dt} &= -3f \\ \frac{1}{f} \frac{df}{dt} &= -3 \\ \ln f &= -3t + c \\ f &= ce^{-3t}\end{aligned}$$

One constant.

1.35

$$\begin{aligned}
 F_x &= 0 \\
 ma_x &= 0 \\
 a_x &= 0 \\
 v_x &= c_1 \\
 &= v_o \cos \theta \\
 r_x &= v_o \cos(\theta)t + c_2 \\
 &= v_o \cos(\theta)t
 \end{aligned}$$

$$\begin{aligned}
 F_y &= 0 \\
 ma_y &= 0 \\
 a_y &= 0 \\
 v_y &= c_3 \\
 v_y &= 0 \\
 r_y &= c_4 \\
 r_y &= 0
 \end{aligned}$$

$$\begin{aligned}
 F_z &= -mg \\
 ma_z &= -mg \\
 a_z &= -g \\
 v_z &= -gt + c_5 \\
 &= v_o \sin \theta - gt \\
 r_z &= v_o \sin(\theta)t - \frac{1}{2}gt^2 + c_6 \\
 &= v_o \sin(\theta)t - \frac{1}{2}gt^2
 \end{aligned}$$

$$\begin{aligned}
 0 &= v_o \sin(\theta)t - \frac{1}{2}gt^2 \\
 t &= \frac{2 \sin(\theta)v_o}{g} \\
 r_x &= v_o \cos(\theta)t \\
 &= \frac{2 \cos(\theta) \sin(\theta)v_o^2}{g} \\
 &= \frac{\sin(2\theta)v_o^2}{g}
 \end{aligned}$$

**1.37**

(a)

$$\begin{aligned}F &= -mg \sin \theta \\ma &= -mg \sin \theta \\a &= -g \sin \theta \\v &= c_1 - gt \sin \theta \\&= v_o - gt \sin \theta \\x &= v_o t - \frac{1}{2}gt^2 \sin \theta\end{aligned}$$

(b)

$$t = \frac{2v_o}{g \sin \theta}$$

### 1.39

$$\begin{aligned}
 F_x &= -mg \sin \phi \\
 ma_x &= -mg \sin \phi \\
 a_x &= -g \sin \phi \\
 v_x &= c_1 - gt \sin \phi \\
 &= v_o \cos \theta - gt \sin \phi \\
 r_x &= v_o t \cos \theta - \frac{1}{2}gt^2 \sin \phi + c_2 \\
 &= v_o t \cos \theta - \frac{1}{2}gt^2 \sin \phi
 \end{aligned}$$

$$\begin{aligned}
 F_y &= -mg \cos \phi \\
 ma_y &= -mg \cos \phi \\
 a_y &= -g \cos \phi \\
 v_y &= c_3 - gt \cos \phi \\
 &= v_o \sin \theta - gt \cos \phi \\
 r_y &= v_o t \sin \theta - \frac{1}{2}gt^2 \cos \phi + c_4 \\
 &= v_o t \sin \theta - \frac{1}{2}gt^2 \cos \phi
 \end{aligned}$$

$$\begin{aligned}
 0 &= v_o t \sin \theta - \frac{1}{2}gt^2 \cos \phi \\
 t &= \frac{2v_o \sin \theta}{g \cos \phi}
 \end{aligned}$$

$$\begin{aligned}
 r_x &= \frac{2v_o^2 \cos \theta \sec \phi \sin \theta}{g} - \frac{2v_o^2 \sec \phi \sin^2 \theta \tan \phi}{g} \\
 &= \frac{2v_o^2 \sin \theta (\cos \theta \cos \phi - \sin \theta \sin \phi)}{g \cos^2 \phi} \\
 &= \frac{2v_o^2 \sin \theta \cos(\theta + \phi)}{g \cos^2 \phi}
 \end{aligned}$$



$$\begin{aligned}
\frac{dr_x}{d\theta} &= \frac{2v_o^2}{g \cos^2 \phi} [\cos \theta \cos(\theta + \phi) - \sin \theta \sin(\theta + \phi)] \\
&= \frac{2v_o^2 \cos(2\theta + \phi)}{g \cos^2 \phi} \\
0 &= \frac{2v_o^2 \cos(2\theta + \phi)}{g \cos^2 \phi} \\
&= \cos(2\theta + \phi) \\
2\theta + \phi &= \frac{\pi}{2} \\
\theta &= \frac{\pi}{4} - \frac{\phi}{2} \\
r_{x,\max} &= \frac{2v_o^2 \sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right) \cos\left(\frac{\pi}{4} - \frac{\phi}{2} + \phi\right)}{g \cos^2 \phi} \\
&= \frac{v_o^2(1 - \sin \phi)}{g \cos^2 \phi} \\
&= \frac{v_o^2}{g(1 + \sin \phi)}
\end{aligned}$$

1.41

$$\begin{aligned}
F &= ma \\
T &= m \frac{v^2}{R} \\
&= m \frac{(\omega R)^2}{R} \\
&= m\omega^2 R
\end{aligned}$$

1.47

(a)

$$\begin{aligned}
\rho &= \sqrt{x^2 + y^2} \\
\phi &= \arctan \frac{y}{x} \\
z &= z
\end{aligned}$$

$\rho$  is the distance of  $P$  from the  $z$ -axis.

The use of  $r$  may be unfortunate because it suggests it's the distance of  $P$  from the origin.

- (b)  $\hat{\rho}$  points away from the  $z$ -axis,  $\hat{\phi}$  points counter-clockwise around the  $z$ -axis, and  $\hat{z}$  points in the positive  $z$  direction.

$$\mathbf{r} = \rho\hat{\rho} + z\hat{z} + \sqrt{x^2 + y^2}\hat{\rho} + z\hat{z}$$

(c)

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ &= \dot{\rho}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt} + \dot{z}\hat{z} + z\frac{d\hat{z}}{dt} \\ &= \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z} \\ \mathbf{a} &= \frac{d\mathbf{v}}{dt} \\ &= \ddot{\rho}\hat{\rho} + \dot{\rho}\frac{d\hat{\rho}}{dt} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \rho\dot{\phi}\frac{d\hat{\phi}}{dt} + \ddot{z}\hat{z} \\ &= \ddot{\rho}\hat{\rho} + \dot{\rho}\dot{\phi}\hat{\phi} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} - \rho\dot{\phi}^2\hat{\rho} + \ddot{z}\hat{z} \\ &= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\rho} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\hat{\phi} + \ddot{z}\hat{z}\end{aligned}$$

## 2 Projectiles and Charged Particles

### 2.1

$$\begin{aligned}1 &= (1.6 \times 10^3)Dv \\ v &= \frac{1}{(1.6 \times 10^3)D} \\ &= 8.9 \text{ mm/s}\end{aligned}$$

When  $v \gg 1 \text{ cm/s}$  the drag force can be treated as purely quadratic. For a beach ball this becomes  $v \gg 1 \text{ mm/s}$ .

### 2.3

(a)

$$\begin{aligned}\frac{f_{\text{quad}}}{f_{\text{lin}}} &= \frac{(1/4)\rho Av^2}{3\pi\eta Dv} \\ &= \frac{\rho\pi\left(\frac{D}{2}\right)^2 v}{12\pi\eta D} \\ &= \frac{\rho Dv}{48\eta} \\ &= \frac{R}{48}\end{aligned}$$

(b)

$$R = \frac{Dv\rho}{\eta} \approx 0.01$$

**2.5**

$$\begin{aligned} v_y(t) &= v_{\text{ter}} + (v_{y0} - v_{\text{ter}})e^{-t/\tau} \\ &= v_{\text{ter}} + (2v_{\text{ter}} - v_{\text{ter}})e^{-t/\tau} \\ &= v_{\text{ter}}(1 + e^{-t/\tau}) \end{aligned}$$

The velocity starts at  $2v_{\text{ter}}$  and asymptotically approaches  $v_{\text{ter}}$ .

**2.7**

$$\begin{aligned} F &= F(v) \\ m\dot{v} &= F(v) \\ m \frac{dv}{F(v)} &= dt \\ t &= \int_{v_0}^v m \frac{dv'}{F(v')} \end{aligned}$$

$$\begin{aligned} F &= F(v) \\ m\dot{v} &= F_0 \\ v &= \frac{F_0}{m}t + c \end{aligned}$$

## 2.11

(a)

$$m\dot{v} = -mg - bv$$

$$\dot{v} = -g - kv$$

$$\frac{1}{-g - kv}\dot{v} = 1$$

$$-\frac{1}{k}\ln(-g - kv) = t + c$$

$$\ln(-g - kv) = c - \frac{t}{\tau}$$

$$-g - kv = Ae^{-t/\tau}$$

$$v = \tau(-g - Ae^{-t/\tau})$$

$$= -v_{\text{ter}} - \tau Ae^{-t/\tau}$$

$$v_o = -v_{\text{ter}} - \tau A$$

$$A = -k(v_o + v_{\text{ter}})$$

$$v = -v_{\text{ter}} + (v_o + v_{\text{ter}})e^{-t/\tau}$$

$$y = -v_{\text{ter}}t - \tau(v_o + v_{\text{ter}})e^{-t/\tau} + c$$

$$0 = -\tau(v_o + v_{\text{ter}}) + c$$

$$c = \tau(v_o + v_{\text{ter}})$$

$$y = -v_{\text{ter}}t + \tau(v_o + v_{\text{ter}})(1 - e^{-t/\tau})$$

(b)

$$\begin{aligned}
0 &= -v_{\text{ter}} + (v_o + v_{\text{ter}})e^{-t/\tau} \\
e^{-t/\tau} &= \frac{v_{\text{ter}}}{v_o + v_{\text{ter}}} \\
-\frac{t}{\tau} &= \ln \frac{v_{\text{ter}}}{v_o + v_{\text{ter}}} \\
t &= -\tau \ln \frac{v_{\text{ter}}}{v_o + v_{\text{ter}}} \\
\\
y_{\text{max}} &= -v_{\text{ter}} \left( -\tau \ln \frac{v_{\text{ter}}}{v_o + v_{\text{ter}}} \right) + \tau(v_o + v_{\text{ter}}) \left( 1 - \frac{v_{\text{ter}}}{v_o + v_{\text{ter}}} \right) \\
&= \tau v_{\text{ter}} \ln \frac{v_{\text{ter}}}{v_o + v_{\text{ter}}} + \tau(v_o + v_{\text{ter}} - v_{\text{ter}}) \\
&= \tau \left( v_o + v_{\text{ter}} \ln \frac{v_{\text{ter}}}{v_o + v_{\text{ter}}} \right) \\
&= \tau \left[ v_o - v_{\text{ter}} \ln \left( 1 + \frac{v_o}{v_{\text{ter}}} \right) \right]
\end{aligned}$$

(c)

$$\begin{aligned}
y_{\text{max}} &= \tau \left[ v_o - v_{\text{ter}} \ln \left( 1 + \frac{v_o}{v_{\text{ter}}} \right) \right] \\
&= \tau \left[ v_o - g\tau \ln \left( 1 + \frac{v_o}{g\tau} \right) \right] \\
&\approx \tau \left\{ v_o - g\tau \left[ \frac{v_o}{g\tau} - \frac{1}{2} \left( \frac{v_o}{g\tau} \right)^2 \right] \right\} \\
&= \tau \left( v_o - v_o + \frac{1}{2} \frac{v_o^2}{g\tau} \right) \\
&= \frac{1}{2} \frac{v_o^2}{g}
\end{aligned}$$

## 2.13

$$\begin{aligned}
 v^2 &= \frac{2}{m} \int_{x_0}^x -kx' dx' \\
 &= -\frac{2k}{m} \left( \frac{1}{2}x^2 - \frac{1}{2}x_0^2 \right) \\
 &= -\frac{k}{m}(x^2 - x_0^2) \\
 v &= \sqrt{\frac{k}{m}(x_0^2 - x^2)} \\
 &= \omega \sqrt{x_0^2 - x^2}
 \end{aligned}$$

$$\begin{aligned}
 \int_{x_0}^x \frac{1}{\sqrt{x_0^2 - x'^2}} dx' &= \int_0^t \omega dt \\
 \arctan \frac{x}{\sqrt{x_0^2 - x^2}} - \arctan \frac{x_0}{\sqrt{x_0^2 - x_0^2}} &= \omega t \\
 \arctan \frac{x}{\sqrt{x_0^2 - x^2}} &= \omega t + \frac{\pi}{2} \\
 \frac{x}{\sqrt{x_0^2 - x^2}} &= \tan \left( \omega t + \frac{\pi}{2} \right) \\
 &= -\cot \omega t \\
 \frac{\sqrt{x_0^2 - x^2}}{x} &= -\tan \omega t \\
 \sqrt{x_0^2 - x^2} &= -x \tan \omega t \\
 x_0^2 - x^2 &= x^2 \tan^2 \omega t \\
 x^2 &= \frac{x_0^2}{1 + \tan^2 \omega t} \\
 &= \frac{x_0^2 \cos^2 \omega t}{\cos^2 \omega t + \sin^2 \omega t} \\
 &= x_0^2 \cos^2 \omega t \\
 x &= x_0 \cos \omega t
 \end{aligned}$$

2.15

$$\begin{aligned}
a_y &= -g \\
v_y &= v_{y0} - gt \\
y &= v_{y0}t - \frac{1}{2}gt^2 \\
0 &= v_{y0}t - \frac{1}{2}gt^2 \\
t &= \frac{2v_{y0}}{g} \\
x &= v_{x0}t \\
R &= \frac{2v_{x0}v_{y0}}{g}
\end{aligned}$$

2.19

(a)

$$\begin{aligned}
x &= v_{x0}t \\
y &= v_{y0}t - \frac{1}{2}gt^2 \\
&= \frac{v_{y0}}{v_{x0}}x - \frac{1}{2}g\left(\frac{x}{v_{x0}}\right)^2
\end{aligned}$$

(b)

$$\begin{aligned}
y &= \frac{v_{y0} + v_{\text{ter}}}{v_{x0}}x + v_{\text{ter}}\tau \ln\left(1 - \frac{x}{v_{x0}\tau}\right) \\
&\approx \frac{v_{y0}}{v_{x0}}x + \frac{g\tau}{v_{x0}}x - g\tau^2\left[\frac{x}{v_{x0}\tau} + \frac{1}{2}\left(\frac{x}{v_{x0}\tau}\right)^2\right] \\
&= \frac{v_{y0}}{v_{x0}}x - \frac{1}{2}g\left(\frac{x}{v_{x0}}\right)^2
\end{aligned}$$

## 2.23

(a)

$$\begin{aligned}
 v_{\text{ter}} &= \sqrt{\frac{mg}{c}} \\
 &= \sqrt{\frac{mg}{\gamma D^2}} \\
 &= \sqrt{\frac{mg}{0.25D^2}} \\
 &= \sqrt{\frac{\rho \frac{4}{3}\pi \left(\frac{D}{2}\right)^3 g}{0.25D^2}} \\
 &= \sqrt{\frac{4\pi\rho Dg}{6}} \\
 &= 22 \text{ m/s}
 \end{aligned}$$

(b)

$$\begin{aligned}
 m &= \rho V \\
 &= \rho \frac{4}{3}\pi \left(\frac{D}{2}\right)^3 \\
 &= \frac{\pi\rho D^3}{6} \\
 D^2 &= \left(\frac{6m}{\pi\rho}\right)^{2/3} \\
 v_{\text{ter}} &= \sqrt{\frac{mg}{0.25D^2}} \\
 &= \sqrt{\frac{mg}{0.25(6m/\pi\rho)^{2/3}}} \\
 &= 140 \text{ m/s}
 \end{aligned}$$

(c)

$$v_{\text{ter}} = 107 \text{ m/s}$$



## 2.27

$$\begin{aligned}
m\dot{v} &= -mg \sin \theta - cv^2 \\
-\frac{\sqrt{m} \arctan \frac{\sqrt{cv}}{\sqrt{gm \sin \theta}}}{\sqrt{cg \sin \theta}} &= t + c_1 \\
\arctan \frac{\sqrt{cv}}{\sqrt{gm \sin \theta}} &= \sqrt{\frac{cg \sin \theta}{m}}(c_1 - t) \\
\frac{\sqrt{cv}}{\sqrt{gm \sin \theta}} &= \tan \left[ \sqrt{\frac{cg \sin \theta}{m}}(c_1 - t) \right] \\
v &= \sqrt{\frac{gm \sin \theta}{c}} \tan \left[ \sqrt{\frac{cg \sin \theta}{m}}(c_1 - t) \right] \\
v_0 &= \sqrt{\frac{gm \sin \theta}{c}} \tan \left( \sqrt{\frac{cg \sin \theta}{m}}c_1 \right) \\
c_1 &= \sqrt{\frac{m}{cg \sin \theta}} \arctan \left( \sqrt{\frac{c}{gm \sin \theta}}v_0 \right) \\
v &= \sqrt{\frac{gm \sin \theta}{c}} \tan \left[ \arctan \left( \sqrt{\frac{c}{gm \sin \theta}}v_0 \right) - \sqrt{\frac{cg \sin \theta}{m}}t \right] \\
0 &= \sqrt{\frac{gm \sin \theta}{c}} \tan \left[ \arctan \left( \sqrt{\frac{c}{gm \sin \theta}}v_0 \right) - \sqrt{\frac{cg \sin \theta}{m}}t \right] \\
\sqrt{\frac{cg \sin \theta}{m}}t &= \arctan \left( \sqrt{\frac{c}{gm \sin \theta}}v_0 \right) \\
t &= \sqrt{\frac{m}{cg \sin \theta}} \arctan \left( \sqrt{\frac{c}{gm \sin \theta}}v_0 \right)
\end{aligned}$$

## 2.29

$$\begin{aligned}
v(t) &= v_{\text{ter}} \tanh \frac{gt}{v_{\text{ter}}} \\
v(1) &= 9.6 \text{ m/s} \\
v(5) &= 38 \text{ m/s} \\
v(10) &= 48 \text{ m/s} \\
v(20) &= 50 \text{ m/s} \\
v(30) &= 50 \text{ m/s}
\end{aligned}$$

**2.31**

(a)

$$\begin{aligned}
v_{\text{ter}} &= \sqrt{\frac{mg}{c}} \\
&= \sqrt{\frac{mg}{0.25D^2}} \\
&= 20.2 \text{ m/s}
\end{aligned}$$

(b)

$$\begin{aligned}
y &= -30 + \frac{v_{\text{ter}}^2}{g} \ln \left( \cosh \frac{gt}{v_{\text{ter}}} \right) \\
0 &= -30 + \frac{v_{\text{ter}}^2}{g} \ln \left( \cosh \frac{gt}{v_{\text{ter}}} \right) \\
t &= 2.78 \text{ s} \\
v(2.78) &= 17.6 \text{ m/s}
\end{aligned}$$

**2.33**

(b)

$$\begin{aligned}
\cosh z &= \frac{e^z + e^{-z}}{2} \\
&= \frac{1}{2} \left[ \left( 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots \right) + \left( 1 - z + \frac{z^2}{2} - \frac{z^3}{6} + \cdots \right) \right] \\
&= 1 + \frac{z^2}{2} + \frac{z^4}{24} + \frac{z^6}{720} + \cdots \\
\cos iz &= 1 - \frac{(iz)^2}{2} + \frac{(iz)^4}{24} - \frac{(iz)^6}{720} + \cdots \\
&= 1 + \frac{z^2}{2} + \frac{z^4}{24} + \frac{z^6}{720} + \cdots \\
&= \cosh z \\
\sinh z &= -i \sin iz
\end{aligned}$$

(c)

$$\begin{aligned}\frac{d}{dz} \cosh z &= \frac{d}{dz} \left( \frac{e^z + e^{-z}}{2} \right) \\ &= \frac{e^z - e^{-z}}{2} \\ &= \sinh z \\ \frac{d}{dz} \sinh z &= \frac{d}{dz} \left( \frac{e^z - e^{-z}}{2} \right) \\ &= \frac{e^z + e^{-z}}{2} \\ &= \cosh z\end{aligned}$$

(d)

$$\begin{aligned}\cosh^2 z - \sinh^2 z &= \left( \frac{e^z + e^{-z}}{2} \right)^2 - \left( \frac{e^z - e^{-z}}{2} \right)^2 \\ &= \frac{1}{4} (e^{2z} + 2 + e^{-2z} - e^{2z} + 2 - e^{-2z}) \\ &= 1\end{aligned}$$

(e)

$$\begin{aligned}\int \frac{1}{\sqrt{1+x^2}} dx &= \int \frac{\cosh z}{\sqrt{1+\sinh^2 z}} dz \\ &= \int 1 dz \\ &= z \\ &= \operatorname{arcsinh} x\end{aligned}$$

## 2.35

(a)

$$\begin{aligned}
 m\dot{v} &= mg - cv^2 \\
 \dot{v} &= g \left( 1 - \frac{v^2}{v_{\text{ter}}^2} \right) \\
 \int_0^v \frac{1}{1 - v'^2/v_{\text{ter}}^2} dv' &= \int_0^t g dt \\
 v_{\text{ter}} \operatorname{arctanh} \frac{v}{v_{\text{ter}}} &= gt \\
 v &= v_{\text{ter}} \tanh \frac{gt}{v_{\text{ter}}} \\
 y &= \int_0^t v_{\text{ter}} \tanh \frac{gt'}{v_{\text{ter}}} dt' \\
 &= \frac{v_{\text{ter}}^2}{g} \ln \left[ \cosh \left( \frac{gt}{v_{\text{ter}}} \right) \right]
 \end{aligned}$$

(b)

$$\begin{aligned}
 v &= g\tau \tanh \frac{t}{\tau} \\
 y &= g\tau^2 \ln \left[ \cosh \left( \frac{t}{\tau} \right) \right] \\
 v(\tau) &= g\tau \tanh 1 \\
 &= 0.76v_{\text{ter}} \\
 v(2\tau) &= 0.96v_{\text{ter}} \\
 v(3\tau) &= 0.99v_{\text{ter}}
 \end{aligned}$$

(c)

$$\begin{aligned}
 y &= g\tau^2 \ln \left[ \cosh \left( \frac{t}{\tau} \right) \right] \\
 &= g\tau^2 \ln \left( \frac{e^{t/\tau} + e^{-t/\tau}}{2} \right) \\
 &= g\tau^2 \ln \left( \frac{e^{t/\tau}}{2} \right) \\
 &= g\tau^2 (\ln e^{t/\tau} - \ln 2) \\
 &= g\tau t - g\tau^2 \ln 2 \\
 &= v_{\text{ter}} t - g\tau^2 \ln 2
 \end{aligned}$$

(d)

$$\begin{aligned}
 y &= \frac{(v_{\text{ter}})^2}{g} \ln \left[ \cosh \left( \frac{gt}{v_{\text{ter}}} \right) \right] \\
 &\approx \frac{(v_{\text{ter}})^2}{g} \ln \left[ 1 + \frac{1}{2} \left( \frac{gt}{v_{\text{ter}}} \right)^2 \right] \\
 &\approx \frac{(v_{\text{ter}})^2}{g} \frac{1}{2} \left( \frac{gt}{v_{\text{ter}}} \right)^2 \\
 &= \frac{1}{2} gt^2
 \end{aligned}$$

## 2.39

(a)

$$\begin{aligned}
 m\dot{v} &= -cv^2 - 3 \\
 \int_{v_0}^v \frac{m}{-cv'^2 - 3} dv' &= \int_0^t dt' \\
 \frac{m}{\sqrt{3}c} \left[ \arctan \left( \sqrt{\frac{c}{3}} v_0 \right) - \arctan \left( \sqrt{\frac{c}{3}} v \right) \right] &= t
 \end{aligned}$$

	Speed	Time
	15 m/s	6.34 s
(b)	10 m/s	18.4 s
	5 m/s	48.3 s
	0 m/s	142 s

## 2.41

$$\begin{aligned}
m\dot{v} &= -mg - cv^2 \\
\dot{v} &= -g \left[ 1 + \left( \frac{v}{v_{\text{ter}}} \right)^2 \right] \\
v \frac{dv}{dy} &= -g \left[ 1 + \left( \frac{v}{v_{\text{ter}}} \right)^2 \right] \\
\int_{v_0}^v \frac{v'}{1 + (v'/v_{\text{ter}})^2} dv' &= \int_0^y -g dy' \\
\frac{1}{2} v_{\text{ter}}^2 [\ln(v_{\text{ter}}^2 + v^2) - \ln(v_{\text{ter}}^2 + v_0^2)] &= -gy \\
\ln \frac{v_{\text{ter}}^2 + v^2}{v_{\text{ter}}^2 + v_0^2} &= -\frac{2gy}{v_{\text{ter}}^2} \\
\frac{v_{\text{ter}}^2 + v^2}{v_{\text{ter}}^2 + v_0^2} &= e^{-2gy/v_{\text{ter}}^2} \\
v &= \sqrt{(v_{\text{ter}}^2 + v_0^2)e^{-2gy/v_{\text{ter}}^2} - v_{\text{ter}}^2} \\
0 &= \sqrt{(v_{\text{ter}}^2 + v_0^2)e^{-2gy/v_{\text{ter}}^2} - v_{\text{ter}}^2} \\
v_{\text{ter}}^2 &= (v_{\text{ter}}^2 + v_0^2)e^{-2gy/v_{\text{ter}}^2} \\
\frac{v_{\text{ter}}^2}{v_{\text{ter}}^2 + v_0^2} &= e^{-2gy/v_{\text{ter}}^2} \\
-\frac{2gy}{v_{\text{ter}}^2} &= \ln \frac{v_{\text{ter}}^2}{v_{\text{ter}}^2 + v_0^2} \\
y &= -\frac{v_{\text{ter}}^2}{2g} \ln \frac{v_{\text{ter}}^2}{v_{\text{ter}}^2 + v_0^2} \\
&= \frac{v_{\text{ter}}^2}{2g} \ln \frac{v_{\text{ter}}^2 + v_0^2}{v_{\text{ter}}^2} \\
y_{\text{max}} &= 17.6 \text{ m}
\end{aligned}$$

## 2.45

(a)

$$\begin{aligned}
 z &= re^{i\theta} \\
 &= r(\cos \theta + i \sin \theta) \\
 &= \sqrt{x^2 + y^2} \left[ \cos \left( \arctan \frac{y}{x} \right) + i \sin \left( \arctan \frac{y}{x} \right) \right] \\
 &= \sqrt{x^2 + y^2} \left( \frac{1}{\sqrt{1 + \frac{y^2}{x^2}}} + i \frac{y}{x\sqrt{1 + \frac{y^2}{x^2}}} \right) \\
 &= \sqrt{x^2 + y^2} \left( \frac{x}{\sqrt{x^2 + y^2}} + i \frac{y}{\sqrt{x^2 + y^2}} \right) \\
 &= x + iy
 \end{aligned}$$

$r$  is the distance between  $z$  and the origin,  $\theta$  is the angle between the positive real axis and  $z$ .

(b)

$$z = \sqrt{3^2 + 4^2} e^{i \arctan 4/3} = 5e^{0.927i}$$

(c)

$$z = 2 \cos -\frac{\pi}{3} + i 2 \sin -\frac{\pi}{3} = 1 - \sqrt{3}i$$

## 2.47

(a)

$$\begin{aligned}
 z + w &= 9 + 4i \\
 z - w &= 3 + 12i \\
 zw &= (6 + 8i)(3 - 4i) \\
 &= 18 - 24i + 24i + 32 \\
 &= 50 \\
 \frac{z}{w} &= \frac{zw^*}{ww^*} \\
 &= \frac{(6 + 8i)(3 + 4i)}{(3 - 4i)(3 + 4i)} \\
 &= \frac{18 + 24i + 24i - 32}{9 + 12i - 12i + 16} \\
 &= \frac{-14 + 48i}{25} \\
 &= -\frac{14}{25} + \frac{48}{25}i
 \end{aligned}$$

(b)

$$\begin{aligned}z + w &= \left(8 \cos \frac{\pi}{3} + i8 \sin \frac{\pi}{3}\right) + \left(4 \cos \frac{\pi}{6} + i4 \sin \frac{\pi}{6}\right) \\&= (4 + 2\sqrt{3}) + i(4\sqrt{3} + 2) \\z - w &= (4 - 2\sqrt{3}) + i(4\sqrt{3} - 2) \\zw &= 32e^{i\pi/2} \\&= 32i \\\frac{z}{w} &= 2e^{i\pi/6} \\&= \sqrt{3} + i\end{aligned}$$

## 2.49

(a)

$$\begin{aligned}z^2 &= (e^{i\theta})^2 \\&= e^{i2\theta} \\&= \cos 2\theta + i \sin 2\theta \\z^2 &= (\cos \theta + i \sin \theta)^2 \\&= \cos^2 \theta + 2i \cos \theta \sin \theta - \sin^2 \theta \\&= \cos^2 \theta + i \sin 2\theta - \sin^2 \theta\end{aligned}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\begin{aligned}\cos 2\theta + i \sin 2\theta &= \cos^2 \theta + 2i \sin \theta \cos \theta - \sin^2 \theta \\i \sin 2\theta &= 2i \sin \theta \cos \theta \\\sin 2\theta &= 2 \sin \theta \cos \theta\end{aligned}$$



(b)

$$\begin{aligned}z^3 &= (e^{i\theta})^3 \\&= e^{i3\theta} \\&= \cos 3\theta + i \sin 3\theta \\z^3 &= (\cos \theta + i \sin \theta)^3 \\&= \cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta\end{aligned}$$

$$\begin{aligned}\cos 3\theta + i \sin 3\theta &= \cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta \\&= \cos \theta (\cos^2 \theta - 3 \sin^2 \theta) + i (3 \cos^2 \theta \sin \theta - \sin^3 \theta) \\&= \cos \theta (\cos^2 \theta - 3 \sin^2 \theta) + i [3(1 - \sin^2 \theta) \sin \theta - \sin^3 \theta] \\&= \cos \theta (\cos^2 \theta - 3 \sin^2 \theta) + i (3 \sin \theta - 4 \sin^3 \theta) \\\cos 3\theta &= \cos \theta (\cos^2 \theta - 3 \sin^2 \theta)\end{aligned}$$

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$$

### 2.53

$$\begin{aligned}
\mathbf{B} &= B_z \hat{\mathbf{z}} \\
\mathbf{E} &= E_z \hat{\mathbf{z}} \\
\mathbf{v} \times \mathbf{B} &= B_z v_y \hat{\mathbf{x}} - B_z v_x \hat{\mathbf{y}} \\
\mathbf{F} &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
&= qB_z v_y \hat{\mathbf{x}} - qB_z v_x \hat{\mathbf{y}} + qE_z \hat{\mathbf{z}}
\end{aligned}$$

$$\begin{aligned}
m\dot{v}_x &= qB_z v_y \\
\dot{v}_x &= \omega v_y
\end{aligned}$$

$$\begin{aligned}
m\dot{v}_y &= -qB_z v_x \\
\dot{v}_y &= -\omega v_x
\end{aligned}$$

$$\begin{aligned}
m\dot{v}_z &= qE_z \\
\dot{v}_z &= \frac{q}{m} E_z
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \begin{pmatrix} v_x \\ v_y \end{pmatrix} &= \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} \\
\begin{pmatrix} v_x \\ v_y \end{pmatrix} &= c_1 \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos \omega t - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin \omega t \right] + c_2 \left[ \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos \omega t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin \omega t \right] \\
&= \begin{pmatrix} c_1 \sin \omega t - c_2 \cos \omega t \\ c_1 \cos \omega t + c_2 \sin \omega t \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
x &= -\frac{c_1}{\omega} \cos \omega t - \frac{c_2}{\omega} \sin \omega t + x_0 \\
y &= \frac{c_1}{\omega} \sin \omega t - \frac{c_2}{\omega} \cos \omega t + y_0
\end{aligned}$$

$$\begin{aligned}
v_z &= \frac{q}{m} E_z t + v_{z0} \\
z &= \frac{q}{2m} E_z t^2 + v_{z0} t + z_0
\end{aligned}$$

The particle moves in a helix oriented along the  $z$ -axis.

2.55

(a)

$$\begin{aligned}
 \mathbf{B} &= B\hat{\mathbf{z}} \\
 \mathbf{E} &= E\hat{\mathbf{y}} \\
 \mathbf{F} &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
 &= Bqv_y\hat{\mathbf{x}} + q(E - Bv_x)\hat{\mathbf{y}} \\
 \dot{v}_x &= \frac{Bq}{m}v_y \\
 &= \omega v_y \\
 \dot{v}_y &= \frac{Eq}{m} - \frac{Bq}{m}v_x \\
 &= \frac{Eq}{m} - \omega v_x \\
 \dot{v}_z &= 0
 \end{aligned}$$

The net force has no  $\hat{\mathbf{z}}$  component, so the motion stays in the  $xy$ -plane.

(b)

$$\begin{aligned}
 0 &= \frac{Eq}{m} - \omega v_x \\
 v_x &= \frac{Eq}{\omega m} \\
 &= \frac{E}{B}
 \end{aligned}$$

(c)

$$\begin{aligned}
\begin{pmatrix} \dot{v}_x \\ \dot{v}_y \end{pmatrix} &= \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{Eq}{m} \end{pmatrix} \\
\mathbf{V}_c &= \begin{pmatrix} c_1 \sin \omega t - c_2 \cos \omega t \\ c_1 \cos \omega t + c_2 \sin \omega t \end{pmatrix} \\
\mathbf{V}_p &= \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{Eq}{m} \end{pmatrix} \\
&= \begin{pmatrix} c_4 \omega \\ -c_3 \omega + \frac{Eq}{m} \end{pmatrix} \\
c_3 &= \frac{Eq}{m\omega} \\
&= \frac{E}{B} \\
&= v_{\text{dr}} \\
c_4 &= 0 \\
\mathbf{V}_p &= \begin{pmatrix} v_{\text{dr}} \\ 0 \end{pmatrix} \\
\mathbf{V} &= \begin{pmatrix} c_1 \sin \omega t - c_2 \cos \omega t + v_{\text{dr}} \\ c_1 \cos \omega t + c_2 \sin \omega t \end{pmatrix} \\
\begin{pmatrix} v_{x0} \\ 0 \end{pmatrix} &= \begin{pmatrix} -c_2 + v_{\text{dr}} \\ c_1 \end{pmatrix} \\
c_1 &= 0 \\
c_2 &= v_{\text{dr}} - v_{x0} \\
\mathbf{V} &= \begin{pmatrix} v_{\text{dr}} + (v_{x0} - v_{\text{dr}}) \cos \omega t \\ -(v_{x0} - v_{\text{dr}}) \sin \omega t \end{pmatrix}
\end{aligned}$$

(d)

$$\begin{aligned}
x &= v_{\text{dr}} t + \frac{v_{x0} - v_{\text{dr}}}{\omega} \sin \omega t + x_0 \\
y &= \frac{v_{x0} - v_{\text{dr}}}{\omega} \cos \omega t + y_0 \\
z &= z_0
\end{aligned}$$

## 3 Momentum and Angular Momentum

### 3.3

$$mv_0 = \frac{m}{3}(v_1 + v_2 \cos \theta + v_3 \cos \theta)$$

$$3v_0 = v_0 + \sqrt{2}v_2$$

$$2v_0 = \sqrt{2}v_2$$

$$v_2 = \sqrt{2}v_0$$

$$\mathbf{v}_2 = \sqrt{2}v_0 \left( \cos \frac{\pi}{4} \hat{\mathbf{x}} + \sin \frac{\pi}{4} \hat{\mathbf{y}} \right)$$

$$= v_0(\hat{\mathbf{x}} + \hat{\mathbf{y}})$$

$$\mathbf{v}_3 = v_0(\hat{\mathbf{x}} - \hat{\mathbf{y}})$$

### 3.7

$$v = v_{\text{ex}} \ln \frac{m_0}{m}$$

$$= 2079 \text{ m/s}$$

$$F = 25 \text{ MN}$$

$$W = 19.6 \text{ MN}$$

The thrust is 1.28 times the weight on Earth.

### 3.9

$$-\dot{m}v_{\text{ex}} = m_0g$$

$$v_{\text{ex}} = -\frac{m_0g}{\dot{m}}$$

$$= 2352 \text{ m/s}$$

### 3.11

(a)

$$m\dot{v} = -\dot{m}v_{\text{ex}} + F_{\text{ext}}$$

(b)

$$\begin{aligned}
 m\dot{v} &= -\dot{m}v_{\text{ex}} - mg \\
 \dot{v} &= -\frac{v_{\text{ex}}}{m}\dot{m} - g \\
 \int_0^t \dot{v} dt &= \int_0^t \left( -\frac{v_{\text{ex}}}{m}\dot{m} - g \right) dt \\
 \int_0^v dv' &= -v_{\text{ex}} \int_{m_0}^m \frac{1}{m'} dm' - \int_0^t g dt' \\
 v &= -v_{\text{ex}} \ln \frac{m}{m_0} - gt \\
 &= v_{\text{ex}} \ln \frac{m_0}{m} - gt
 \end{aligned}$$

(c)

$$v = 903 \text{ m/s}$$

It would be 2079 m/s without gravity (2.3 times larger).

(d) The rocket wouldn't take off until it was light enough (from burning fuel) that its thrust was greater than its weight.

### 3.13

$$\begin{aligned}
 v &= v_{\text{ex}} \ln \frac{m_0}{m} - gt \\
 &= v_{\text{ex}} \ln \frac{m_0}{m_0 - kt} - gt \\
 y &= v_{\text{ex}} t - \frac{mv_{\text{ex}}}{k} \ln \frac{m_0}{m} - \frac{1}{2}gt^2 \\
 y(2 \text{ min}) &= 40 \text{ km}
 \end{aligned}$$

### 3.15

$$\begin{aligned}M &= m_1 + m_2 + m_3 \\&= m_1 + m_1 + 10m_1 \\&= 12m_1 \\ \mathbf{R} &= \frac{1}{12m_1} \left[ m_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + m_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + m_3 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right] \\&= \frac{1}{12m_1} \begin{pmatrix} 2m_1 \\ 0 \\ 0 \end{pmatrix} \\&= \begin{pmatrix} \frac{1}{6} \\ 0 \\ 0 \end{pmatrix}\end{aligned}$$

### 3.17

If we let Earth be at the origin, then

$$\begin{aligned}R &= \frac{dM_m}{M_e + M_m} \\&= 4630 \text{ km}\end{aligned}$$

The centre of mass is inside Earth.

### 3.19

- (a) No external forces apply during the explosion so the path of the centre of mass would be unchanged.
- (b) 100 m before the target.
- (c) No.

### 3.21

$$\begin{aligned}
\mathbf{R} &= \frac{1}{M} \int \mathbf{r} \, dm \\
&= \frac{2}{\sigma \pi R^2} \int \mathbf{r} \sigma \, dA \\
&= \frac{2}{\pi R^2} \int_0^R \int_0^\pi \mathbf{r} r \, d\phi \, dr \\
&= \frac{2}{\pi R^2} \int_0^R \int_0^\pi r (\cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}}) r \, d\phi \, dr \\
&= \frac{2}{\pi R^2} \int_0^R r^2 \int_0^\pi (\cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}}) \, d\phi \, dr \\
&= \frac{2}{\pi R^2} \int_0^R r^2 [\sin \phi \hat{\mathbf{x}} - \cos \phi \hat{\mathbf{y}}]_0^\pi \, dr \\
&= \frac{4}{\pi R^2} \int_0^R r^2 \, dr \hat{\mathbf{y}} \\
&= \frac{4R}{3\pi} \hat{\mathbf{y}}
\end{aligned}$$

### 3.25

$$\begin{aligned}
L &= L_0 \\
I\omega &= I_0\omega_0 \\
mr^2\omega &= mr_0^2\omega_0 \\
\omega &= \left(\frac{r_0}{r}\right)^2 \omega_0
\end{aligned}$$

### 3.29

$$\begin{aligned}
I\omega &= I_0\omega_0 \\
\frac{2}{5} \left(\frac{4}{3}\pi R^3\rho\right) R^2\omega &= \frac{2}{5} \left(\frac{4}{3}\pi R_0^3\rho\right) R_0^2\omega_0 \\
\omega &= \left(\frac{R_0}{R}\right)^5 \omega_0
\end{aligned}$$

If the radius doubles the angular velocity is  $\omega_0/32$ .



**3.31**

$$\begin{aligned}
I &= \int r^2 dm \\
&= \int r^2 \sigma dA \\
&= \frac{M}{\pi R^2} \int_0^R \int_0^{2\pi} r^3 d\phi dr \\
&= \frac{1}{2} MR^2
\end{aligned}$$

**3.33**

$$\begin{aligned}
I &= \int r^2 dm \\
&= \int r^2 \sigma dA \\
&= \frac{M}{(2b)^2} \int_{-b}^b \int_{-b}^b (x^2 + y^2) dx dy \\
&= \frac{M}{4b^2} \int_{-b}^b \left[ \frac{1}{3} x^3 + xy^2 \right]_{-b}^b dy \\
&= \frac{M}{4b} \int_{-b}^b \left( \frac{2}{3} b^2 + 2y^2 \right) dy \\
&= \frac{M}{4b} \left[ \frac{2}{3} b^2 y + \frac{2}{3} y^3 \right]_{-b}^b \\
&= \frac{2}{3} Mb^2
\end{aligned}$$

### 3.35

(b)

$$\Gamma_{\text{ext}} = RMg \sin \gamma$$

$$\dot{L} = \Gamma_{\text{ext}}$$

$$I\dot{\omega} = RMg \sin \gamma$$

$$\frac{3}{2}MR^2\dot{\omega} = RMg \sin \gamma$$

$$\dot{\omega} = \frac{2g \sin \gamma}{3R}$$

$$\begin{aligned}\dot{v} &= R\dot{\omega} \\ &= \frac{2}{3}g \sin \gamma\end{aligned}$$

(c)

$$\begin{aligned}M\dot{v} &= Mg \sin \gamma - f \\ f &= M(g \sin \gamma - \dot{v})\end{aligned}$$

$$\begin{aligned}\Gamma_{\text{ext}} &= Rf \\ &= RM(g \sin \gamma - \dot{v})\end{aligned}$$

$$\dot{L} = \Gamma_{\text{ext}}$$

$$I\dot{\omega} = RM(g \sin \gamma - \dot{v})$$

$$\frac{1}{2}MR^2\dot{\omega} = RM(g \sin \gamma - \dot{v})$$

$$\dot{\omega} = \frac{2(g \sin \gamma - \dot{v})}{R}$$

$$\begin{aligned}\dot{v} &= R\dot{\omega} \\ &= 2(g \sin \gamma - \dot{v}) \\ &= \frac{2}{3}g \sin \gamma\end{aligned}$$

**3.37**

(b)

$$\begin{aligned}\sum m_{\alpha} r'_{\alpha} &= \sum m_{\alpha} (r_{\alpha} - R) \\ &= \sum m_{\alpha} r_{\alpha} - \sum m_{\alpha} R \\ &= MR - MR \\ &= 0\end{aligned}$$

## 4 Energy

**4.3**

(a)

$$\begin{aligned}\int_P^Q \mathbf{F} \cdot d\mathbf{r} &= \int_P^O \mathbf{F} \cdot d\mathbf{r} + \int_O^Q \mathbf{F} \cdot d\mathbf{r} \\ &= 0\end{aligned}$$

(b)

$$\begin{aligned}x &= 1 - t \\ y &= t \\ \mathbf{r} &= (1 - t)\hat{\mathbf{x}} + t\hat{\mathbf{y}} \\ d\mathbf{r} &= (-\hat{\mathbf{x}} + \hat{\mathbf{y}}) dt \\ \mathbf{F} \cdot d\mathbf{r} &= (x + y) dt \\ &= [(1 - t) + 1] dt \\ &= dt \\ \int_P^Q \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 dt \\ &= 1\end{aligned}$$

(c)

$$\begin{aligned}\mathbf{r} &= \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}} \\ d\mathbf{r} &= (-\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}) d\phi \\ \mathbf{F} \cdot d\mathbf{r} &= (\sin^2 \phi + \cos^2 \phi) d\phi \\ &= d\phi \\ \int_P^Q \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} d\phi \\ &= \frac{\pi}{2}\end{aligned}$$

## 4.7

(a)

$$\begin{aligned}
 \mathbf{F} &= -m\gamma y^2 \hat{\mathbf{y}} \\
 W &= \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} \\
 &= -m\gamma \int_{y_1}^{y_2} y^2 dy \\
 &= -\frac{1}{3}m\gamma(y_2^3 - y_1^3) \\
 U(\mathbf{r}) &= \frac{1}{3}m\gamma y^3
 \end{aligned}$$

(b) Assuming no friction

$$\begin{aligned}
 \frac{1}{2}mv^2 &= \frac{1}{3}m\gamma h^3 \\
 v &= \sqrt{\frac{2}{3}\gamma h^3}
 \end{aligned}$$

## 4.9

(a)

$$\begin{aligned}
 U(x) &= -\int_0^x -kx' dx' \\
 &= \frac{1}{2}kx^2
 \end{aligned}$$

## 4.11

(a)

$$\begin{aligned}
 \frac{\partial f}{\partial x} &= 0 \\
 \frac{\partial f}{\partial y} &= 2ay + 2bz \\
 \frac{\partial f}{\partial z} &= 2by + 2cz
 \end{aligned}$$

(b)

$$\begin{aligned}
 \frac{\partial g}{\partial x} &= -ay^2z^3 \sin(axy^2z^3) \\
 \frac{\partial g}{\partial y} &= -2axyz^3 \sin(axy^2z^3) \\
 \frac{\partial g}{\partial z} &= -3axy^2z^2 \sin(axy^2z^3)
 \end{aligned}$$

(c)

$$\begin{aligned}\frac{\partial h}{\partial x} &= \frac{ax}{r} \\ \frac{\partial h}{\partial y} &= \frac{ay}{r} \\ \frac{\partial h}{\partial z} &= \frac{az}{r}\end{aligned}$$

#### 4.13

(a)

$$\nabla f = \frac{1}{r^2}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})$$

(b)

$$\nabla f = nr^{n-2}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})$$

(c)

$$\nabla f = \frac{g'(r)}{r}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})$$

#### 4.15

$$\begin{aligned}df &= \nabla f \cdot d\mathbf{r} \\ &= (2, 4, 6) \cdot (0.01, 0.03, 0.05) \\ &= 0.44 \\ f(1.01, 1.03, 1.05) - f(1, 1, 1) &= 0.4494\end{aligned}$$

#### 4.19

(a) An ellipse that is two times wider than it is tall.

(b)

$$\begin{aligned}\nabla f &= (2x, 8y, 0) \\ \nabla f|_{(1,1,1)} &= (2, 8, 0) \\ \mathbf{n} &= (1, 4, 0)/\sqrt{17}\end{aligned}$$

#### 4.21

$$\begin{aligned}
\mathbf{F} &= -\frac{GMm}{r^2} \hat{\mathbf{r}} \\
\nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{GMm}{r^3}x & -\frac{GMm}{r^3}y & -\frac{GMm}{r^3}z \end{vmatrix} \\
&= \left[ \frac{\partial}{\partial y} \left( -\frac{GMm}{r^3}z \right) - \frac{\partial}{\partial z} \left( -\frac{GMm}{r^3}y \right) \right] \hat{\mathbf{x}} \\
&\quad - \left[ \frac{\partial}{\partial x} \left( -\frac{GMm}{r^3}z \right) - \frac{\partial}{\partial z} \left( -\frac{GMm}{r^3}x \right) \right] \hat{\mathbf{y}} \\
&\quad + \left[ \frac{\partial}{\partial x} \left( -\frac{GMm}{r^3}y \right) - \frac{\partial}{\partial y} \left( -\frac{GMm}{r^3}x \right) \right] \hat{\mathbf{z}} \\
&= -GMm \left[ \left( -\frac{3yz}{r^5} + \frac{3yz}{r^5} \right) \hat{\mathbf{x}} + \left( -\frac{3xz}{r^5} + \frac{3xz}{r^5} \right) \hat{\mathbf{y}} \right. \\
&\quad \left. \left( -\frac{3xy}{r^5} + \frac{3xy}{r^4} \right) \hat{\mathbf{z}} \right] \\
&= \mathbf{0} \\
U(\mathbf{r}) &= -\int_{\infty}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}' \\
&= -\int_{\infty}^r -\frac{GMm}{r'^2} dr' \\
&= GMm \left[ -\frac{1}{r'} \right]_{\infty}^r \\
&= GMm \left( -\frac{1}{r} + \frac{1}{\infty} \right) \\
&= -\frac{GMm}{r}
\end{aligned}$$

#### 4.23

(a)

$$\begin{aligned}
\nabla \times \mathbf{F} &= \mathbf{0} \\
U &= -\int_{\mathbf{0}}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} \\
&= -\left( \int_0^x kx \, dx + \int_0^y 2ky \, dy + \int_0^z 3kz \, dz \right) \\
&= -\frac{1}{2}k(x^2 + 2y^2 + 3z^2)
\end{aligned}$$

(b)

$$\nabla \times \mathbf{F} = \mathbf{0}$$

$$\begin{aligned} U &= - \int_0^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} \\ &= - \left( \int_0^x ky \, dx + \int_0^y kx \, dy \right) \\ &= -kxy \end{aligned}$$

(c)

$$\nabla \times \mathbf{F} = 2k\hat{\mathbf{z}}$$

Not conservative

#### 4.29

(a) The mass will oscillate around  $x = 0$ .

(b)

$$\begin{aligned} t &= \sqrt{\frac{m}{2}} \int_0^A \frac{dx}{\sqrt{kA^4 - kx^4}} \\ &= \sqrt{\frac{m}{2k}} \int_0^A \frac{dx}{\sqrt{A^4 - x^4}} \\ \tau &= 4t \end{aligned}$$

(d)  $\tau \approx 3.71 \text{ s}$

#### 4.31

(a)

$$E = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_2 - m_1)gx$$

(b)

$$(m_1 + m_2)\ddot{x} = (m_1 - m_2)g$$

$$c = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_2 - m_1)gx$$

$$0 = (m_1 + m_2)\dot{x}\ddot{x} + (m_2 - m_1)g\dot{x}$$

$$(m_1 + m_2)\ddot{x} = (m_1 - m_2)g$$

**4.35**

(a)

$$E = \frac{1}{2} \left( m_1 + m_2 + \frac{I}{R^2} \right) \dot{x}^2 + (m_2 - m_1)gx$$

(b)

$$0 = \left( m_1 + m_2 + \frac{I}{R^2} \right) \dot{x}\ddot{x} + (m_2 - m_1)g\dot{x}$$

$$\left( m_1 + m_2 + \frac{I}{R^2} \right) \ddot{x} = (m_1 - m_2)g$$

$$m_1\ddot{x} = m_1g - T_2$$

$$T_2 = m_1g - m_1\ddot{x}$$

$$m_2\ddot{x} = T_1 - m_2g$$

$$T_1 = m_2\ddot{x} + m_2g$$

$$\omega = -\frac{\dot{x}}{R}$$

$$\dot{\omega} = -\frac{\ddot{x}}{R}$$

$$I\dot{\omega} = (T_1 - T_2)R$$

$$-I\frac{\ddot{x}}{R} = (m_2\ddot{x} + m_2g - m_1g - m_1\ddot{x})R$$

$$\left( m_1 + m_2 + \frac{I}{R^2} \right) \ddot{x} = (m_1 - m_2)g$$

**4.37**

(a)

$$U(\phi) = MgR(1 - \cos \phi) - mgR\phi$$



(b)

$$\begin{aligned}
\frac{dU(\phi)}{d\phi} &= MgR \sin \phi - mgR \\
&= gR(M \sin \phi - m) \\
0 &= gR(M \sin \phi - m) \\
m &= M \sin \phi
\end{aligned}$$

$$\frac{d^2U(\phi)}{d\phi^2} = MgR \cos \phi$$

There is a position of equilibrium at  $\phi = \arcsin \frac{m}{M}$ . It is stable if  $\phi < \frac{\pi}{2}$ , i.e.  $m < M$ .

#### 4.51

$$\begin{aligned}
U(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) &= U_{\text{int}} + U_{\text{ext}} \\
&= [U_{12}(\mathbf{r}_1 - \mathbf{r}_2) + U_{13}(\mathbf{r}_1 - \mathbf{r}_3) + U_{14}(\mathbf{r}_1 - \mathbf{r}_4) + U_{23}(\mathbf{r}_2 - \mathbf{r}_3) \\
&\quad + U_{24}(\mathbf{r}_2 - \mathbf{r}_4) + U_{34}(\mathbf{r}_3 - \mathbf{r}_4)] + [U_1(\mathbf{r}_1) + U_2(\mathbf{r}_2) + U_3(\mathbf{r}_3) \\
&\quad + U_4(\mathbf{r}_4)] \\
\mathbf{F}_3 &= \mathbf{F}_{3,\text{int}} + \mathbf{F}_{3,\text{ext}} \\
&= [\mathbf{F}_{13} + \mathbf{F}_{23} + \mathbf{F}_{34}] + \mathbf{F}_{3,\text{ext}} \\
&= -\nabla_3 U_{13} - \nabla_3 U_{23} - \nabla_3 U_{34} - \nabla_3 U_{3,\text{ext}} \\
&= -\nabla_3 U
\end{aligned}$$

#### 4.53

(a)

$$\begin{aligned}
F &= ma \\
\frac{ke^2}{r^2} &= m \frac{v^2}{r} \\
v^2 &= \frac{ke^2}{mr} \\
K &= \frac{1}{2}mv^2 \\
&= \frac{ke^2}{2r} \\
U &= -\frac{ke^2}{r} \\
K &= -\frac{1}{2}U
\end{aligned}$$

(b)

$$\begin{aligned} E &= K_1 + K_2 + U_{12} + U_{1p} + U_{2p} \\ &= \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 - ke^2 \left( \frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_{12}} \right) \end{aligned}$$

(c)

$$\begin{aligned} E_{\text{before}} &= K_1 + K_2 + U_1 + U_2 + U_{12} \\ &= T_2 + \frac{ke^2}{2r} - \frac{ke^2}{r} \\ &= T_2 - \frac{ke^2}{2r} \end{aligned}$$

$$\begin{aligned} E_{\text{after}} &= K'_1 + K'_2 + U_1 + U_2 + U_{12} \\ &= T'_1 + \frac{ke^2}{2r'} - \frac{ke^2}{r'} \\ &= T'_1 - \frac{ke^2}{2r'} \\ T_2 - \frac{ke^2}{2r} &= T'_1 - \frac{ke^2}{2r'} \\ T'_1 &= T_2 + \frac{ke^2}{2} \left( \frac{1}{r'} - \frac{1}{r} \right) \end{aligned}$$

## 5 Oscillations

### 5.3

$$\begin{aligned} U(\phi) &= mgl(1 - \cos \phi) \\ &\approx mgl \left( 1 - 1 + \frac{1}{2}\phi^2 \right) \\ &= \frac{1}{2}mgl\phi^2 \\ k &= mgl \end{aligned}$$

## 5.5

$$\begin{aligned}
 x(t) &= C_1 e^{i\omega t} + C_2 e^{-i\omega t} \\
 &= C_1 (\cos \omega t + i \sin \omega t) + C_2 (\cos \omega t - i \sin \omega t) \\
 &= (C_1 + C_2) \cos \omega t + i(C_1 - C_2) \sin \omega t \\
 &= B_1 \cos \omega t + B_2 \sin \omega t \\
 B_1 &= C_1 + C_2 \\
 B_2 &= i(C_1 - C_2)
 \end{aligned}$$

$$\begin{aligned}
 x(t) &= A \cos(\omega t - \delta) \\
 A &= \sqrt{B_1^2 + B_2^2} \\
 \delta &= \arctan \frac{B_2}{B_1}
 \end{aligned}$$

$$\begin{aligned}
 x(t) &= C \operatorname{Re} e^{\omega t} \\
 C &= A e^{-i\delta}
 \end{aligned}$$

## 5.7

(a)  $B_1 = x_0, B_2 = \frac{v_0}{\omega}$

(b)

$$\begin{aligned}
 \omega &= \sqrt{\frac{k}{m}} \\
 &= 10 \text{ rad/s} \\
 B_1 &= 3.0 \text{ m} \\
 B_2 &= 5.0 \text{ m}
 \end{aligned}$$

(c)  $x = 0 \text{ m}$  at  $t = 0.26 \text{ s}$ ,  $\dot{x} = 0 \text{ m/s}$  at  $t = 0.10 \text{ s}$ .

## 5.9

$$\begin{aligned}
 \frac{1}{2}kA^2 &= \frac{1}{2}mv^2 \\
 \frac{k}{m} &= \left(\frac{v}{A}\right)^2 \\
 \tau &= \frac{2\pi}{\omega} \\
 &= \frac{2\pi}{\sqrt{k/m}} \\
 &= \frac{2\pi}{v/A} \\
 &= 1.05 \text{ s}
 \end{aligned}$$

## 5.11

$$\begin{aligned}
 \frac{1}{2}kx_1^2 + \frac{1}{2}mv_1^2 &= \frac{1}{2}kx_2^2 + \frac{1}{2}mv_2^2 \\
 kx_1^2 + mv_1^2 &= kx_2^2 + mv_2^2 \\
 \frac{k}{m}x_1^2 + v_1^2 &= \frac{k}{m}x_2^2 + v_2^2 \\
 \omega^2(x_1^2 - x_2^2) &= v_2^2 - v_1^2 \\
 \omega &= \sqrt{\frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{2}kA^2 &= \frac{1}{2}kx_1^2 + \frac{1}{2}mv_1^2 \\
 A^2 &= x_1^2 + \frac{m}{k}v_1^2 \\
 A &= \sqrt{x_1^2 + \frac{v_1^2}{\omega^2}} \\
 &= \sqrt{x_1^2 + v_1^2 \frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}} \\
 &= \sqrt{\frac{x_1^2(v_2^2 - v_1^2) + v_1^2(x_1^2 - x_2^2)}{v_2^2 - v_1^2}} \\
 &= \sqrt{\frac{x_2^2v_1^2 - x_1^2v_2^2}{v_1^2 - v_2^2}}
 \end{aligned}$$

### 5.13

$$\begin{aligned}
U(r) &= U_0 \left( \frac{r}{R} + \lambda^2 \frac{R}{r} \right) \\
\frac{dU(r)}{dr} &= U_0 \left( \frac{1}{R} - \lambda^2 \frac{R}{r^2} \right) \\
0 &= \frac{dU(r_0)}{dr} \\
&= U_0 \left( \frac{1}{R} - \lambda^2 \frac{R}{r_0^2} \right) \\
\frac{1}{R} &= \lambda^2 \frac{R}{r_0^2} \\
r_0 &= \lambda R
\end{aligned}$$

$$\begin{aligned}
U(r_0 + x) - U(r_0) &= U_0 \left( \frac{r_0 + x}{R} + \lambda^2 \frac{R}{r_0 + x} \right) - U_0 \left( \frac{r_0}{R} + \lambda^2 \frac{R}{r_0} \right) \\
&= U_0 \left[ \frac{1}{R} x + \lambda^2 R \left( \frac{1}{r_0 + x} - \frac{1}{r_0} \right) \right] \\
&\approx U_0 \left[ \frac{1}{R} x + \lambda^2 R \left( \frac{1}{r_0} - \frac{x}{r_0^2} + \frac{x^2}{r_0^3} - \frac{1}{r_0} \right) \right] \\
&= U_0 \left[ \frac{1}{R} x + \lambda^2 R \left( \frac{x^2}{r_0^3} - \frac{x}{r_0^2} \right) \right] \\
&= U_0 \left[ \frac{1}{R} x + \lambda^2 R \left( \frac{x^2}{(\lambda R)^3} - \frac{x}{(\lambda R)^2} \right) \right] \\
&= \frac{U_0 x^2}{\lambda R^2} \\
&= \frac{1}{2} \left( \frac{2U_0}{\lambda R^2} \right) x^2
\end{aligned}$$

$$\begin{aligned}
\omega &= \sqrt{\frac{k}{m}} \\
&= \sqrt{\frac{2U_0}{\lambda m R^2}}
\end{aligned}$$

## 5.17

(a)

$$\begin{aligned}
 x(t) &= A_x \cos \omega_x t \\
 y(t) &= A_y \cos(\omega_y t - \delta) \\
 \frac{\omega_x}{\omega_y} &= \frac{p}{q} \\
 \omega_x \tau &= 2\pi p \\
 \omega_y \tau &= 2\pi q \\
 (\omega_x + \omega_y) \tau &= 2\pi(p + q) \\
 \tau &= \frac{2\pi(p + q)}{\omega_x + \omega_y} \\
 x(\tau) &= A_x \cos \left( \omega_x \frac{2\pi(p + q)}{\omega_x + \omega_y} \right) \\
 &= A_x \cos \left( \frac{2\pi(p + q)}{1 + \omega_y/\omega_x} \right) \\
 &= A_x \cos \left( \frac{2\pi(p + q)}{1 + q/p} \right) \\
 &= A_x \cos \left( 2\pi \frac{p(p + q)}{p + q} \right) \\
 y(\tau) &= A_y \cos \left( \omega_y \frac{2\pi(p + q)}{\omega_x + \omega_y} - \delta \right) \\
 &= A_y \cos \left( 2\pi \frac{p + q}{1 + \omega_x/\omega_y} - \delta \right) \\
 &= A_y \cos \left( 2\pi \frac{q(p + q)}{p + q} - \delta \right)
 \end{aligned}$$

## 5.23

$$\begin{aligned}
 \frac{dE}{dt} &= m\dot{x}\ddot{x} + kx\dot{x} \\
 &= \dot{x}(m\ddot{x} + kx) \\
 &= -b\dot{x}^2
 \end{aligned}$$

## 5.25

(a)

$$\tau = \frac{2\pi}{\omega_1}$$

(b)

$$\begin{aligned}0 &= Ae^{-\beta t} \cos \omega_1 t \\ &= \cos \omega_1 t \\ t &= \frac{\frac{\pi}{2} + n\pi}{\omega_1}, n \in \mathbb{Z}\end{aligned}$$

The time between successive zeroes is  $\pi/\omega_1$ . The period  $\tau$  is twice this.

(c)

$$\begin{aligned}e^{-\beta\tau} &= e^{-(\omega_0/2)(2\pi/\omega_1)} \\ &= e^{-\pi\omega_0/\omega_1} \\ &= e^{-\pi\omega_0/\sqrt{\omega_0^2-(\omega_0/2)^2}} \\ &= e^{-\pi\omega_0/\sqrt{3\omega_0^2/4}} \\ &= e^{-\pi\sqrt{4/3}} \\ &\approx 0.027\end{aligned}$$

## 5.29

$$\begin{aligned}
\tau_0 &= 1 \text{ s} \\
\omega_0 &= 2\pi f \\
&= \frac{2\pi}{\tau} \\
&= 2\pi \text{ rad/s} \\
\frac{1}{2} &= e^{-\beta\tau_1} \\
&= e^{-2\pi\beta/\omega_1} \\
&= e^{-2\pi\beta/\sqrt{\omega_0^2 - \beta^2}} \\
\ln \frac{1}{2} &= -\frac{2\pi\beta}{\sqrt{\omega_0^2 - \beta^2}} \\
\sqrt{\omega_0^2 - \beta^2} \ln \frac{1}{2} &= -2\pi\beta \\
(\omega_0^2 - \beta^2) \ln^2 \frac{1}{2} &= 4\pi^2 \beta^2 \\
\omega_0^2 \ln^2 \frac{1}{2} &= \left(4\pi^2 + \ln^2 \frac{1}{2}\right) \beta^2 \\
\beta &= \pm \frac{\ln \frac{1}{2}}{\sqrt{4\pi^2 + \ln^2 \frac{1}{2}}} \omega_0 \\
&\approx 0.11\omega_0 \\
\tau_1 &= \frac{2\pi}{\omega_1} \\
&= \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}} \\
&= \frac{2\pi}{\sqrt{\omega_0^2 - 0.0121\omega_0^2}} \\
&\approx 1.006 \text{ s}
\end{aligned}$$

## 5.43

(a)

$$\begin{aligned}
4mg &= 4kx \\
k &= \frac{mg}{x} \\
&\approx 4 \times 10^4 \text{ N/m}
\end{aligned}$$

(b)

$$\omega_0 = \sqrt{\frac{2k}{m}} = 40 \text{ rad/s} \approx 6 \text{ Hz}$$



(c)

$$5 \text{ m/s} \approx 18 \text{ km/h}$$

## 6 Calculus of Variations

### 6.5

$$\begin{aligned}|AP| &= \frac{|AB|}{2} \operatorname{crd} \left( \frac{\pi}{2} - \theta \right) \\ &= \frac{|AB|}{2} 2 \sin \left( \frac{\frac{\pi}{2} - \theta}{2} \right) \\ &= |AB| \sin \left( \frac{\pi}{4} - \frac{\theta}{2} \right)\end{aligned}$$

$$\begin{aligned}|PB| &= \frac{|AB|}{2} \operatorname{crd} \left( \frac{\pi}{2} + \theta \right) \\ &= \frac{|AB|}{2} 2 \sin \left( \frac{\frac{\pi}{2} + \theta}{2} \right) \\ &= |AB| \sin \left( \frac{\pi}{4} + \frac{\theta}{2} \right)\end{aligned}$$

$$\begin{aligned}S &= |AP| + |PB| \\ &= |AB| \left[ \sin \left( \frac{\pi}{4} - \frac{\theta}{2} \right) + \sin \left( \frac{\pi}{4} + \frac{\theta}{2} \right) \right]\end{aligned}$$

$$\begin{aligned}\frac{dS}{d\theta} &= |AB| \left[ -\frac{1}{2} \cos \left( \frac{\pi}{4} - \frac{\theta}{2} \right) + \frac{1}{2} \cos \left( \frac{\pi}{4} + \frac{\theta}{2} \right) \right] \\ &= \frac{1}{2} |AB| \left[ \cos \left( \frac{\pi}{4} + \frac{\theta}{2} \right) - \cos \left( \frac{\pi}{4} - \frac{\theta}{2} \right) \right]\end{aligned}$$

$$\begin{aligned}\frac{dS}{d\theta} &= 0 \\ \frac{\pi}{4} + \frac{\theta}{2} &= \frac{\pi}{4} - \frac{\theta}{2} \\ \theta &= 0\end{aligned}$$

$$\begin{aligned}\frac{d^2S}{d\theta^2} &= -\frac{1}{4} |AB| \left[ \sin \left( \frac{\pi}{4} - \frac{\theta}{2} \right) + \sin \left( \frac{\pi}{4} + \frac{\theta}{2} \right) \right] \\ \left. \frac{d^2S}{d\theta^2} \right|_{\theta=0} &= -\frac{\sqrt{2}}{4} |AB|\end{aligned}$$

$\frac{dS}{d\theta} = 0$  and  $\frac{d^2S}{d\theta^2} < 0$  at  $\theta = 0$  so it is a maximum.

## 6.7

$$\begin{aligned}
S &= \int_P^Q dS \\
&= \int_P^Q \sqrt{(R d\phi)^2 + (dz)^2} \\
&= \int_{z_1}^{z_2} \sqrt{1 + \left(R \frac{d\phi}{dz}\right)^2} dz \\
f\left(z, \phi, \frac{d\phi}{dz}\right) &= \sqrt{1 + \left(R \frac{d\phi}{dz}\right)^2} \\
\frac{\partial f}{\partial \phi} &= 0 \\
\frac{\partial f}{\partial \phi'} &= \frac{R^2 \frac{d\phi}{dz}}{\sqrt{1 + \left(R \frac{d\phi}{dz}\right)^2}} \\
\frac{\partial f}{\partial \phi} - \frac{d}{dz} \frac{\partial f}{\partial \phi'} &= 0 \\
\frac{R^2 \frac{d\phi}{dz}}{\sqrt{1 + \left(R \frac{d\phi}{dz}\right)^2}} &= c_1 \\
R^4 \left(\frac{d\phi}{dz}\right)^2 &= c_1 \left[1 + \left(R \frac{d\phi}{dz}\right)^2\right] \\
(R^4 - c_1 R^2) \left(\frac{d\phi}{dz}\right)^2 &= c_1 \\
\frac{d\phi}{dz} &= \sqrt{\frac{c_1}{R^4 - c_1 R^2}} \\
&= c_2 \\
\phi &= c_2 z + c_3
\end{aligned}$$

## 6.9

$$\begin{aligned}
 f(x, y, y') &= y'^2 + yy' + y^2 \\
 \frac{\partial f}{\partial y} &= y' + 2y \\
 \frac{\partial f}{\partial y'} &= 2y' + y \\
 \frac{d}{dx} \frac{\partial f}{\partial y'} &= 2y'' + y' \\
 \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} &= 0 \\
 y' + 2y - 2y'' - y' &= 0 \\
 y'' &= y \\
 y &= c_1 e^x + c_2 e^{-x} \\
 y(0) &= 0 \\
 &= c_1 + c_2 \\
 y(1) &= 1 \\
 &= c_1 e + c_2 e^{-1} \\
 1 &= -c_2 e + c_2 e^{-1} \\
 &= c_2 (e^{-1} - e) \\
 c_2 &= \frac{1}{e^{-1} - e} \\
 c_1 &= \frac{1}{e - e^{-1}} \\
 y &= \frac{e^x}{e - e^{-1}} + \frac{e^{-x}}{e^{-1} - e} \\
 &= \frac{e^x - e^{-x}}{e - e^{-1}} \\
 &= \frac{\sinh x}{\sinh 1}
 \end{aligned}$$

## 6.11

$$\begin{aligned}
 f(x, y, y') &= \sqrt{x(1 + y'^2)} \\
 \frac{\partial f}{\partial y} &= 0 \\
 \frac{\partial f}{\partial y'} &= \frac{\sqrt{xy'}}{\sqrt{1 + y'^2}} \\
 \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} &= 0 \\
 \frac{\sqrt{xy'}}{\sqrt{1 + y'^2}} &= c_1 \\
 xy'^2 &= c_1(1 + y'^2) \\
 (x - c_1)y'^2 &= c_1 \\
 y' &= \sqrt{\frac{c_1}{x - c_1}} \\
 y &= 2c_1 \sqrt{\frac{x - c_1}{c_1}} + c_2
 \end{aligned}$$

## 7 Lagrange's Equations

### 7.1

$$\begin{aligned}
 K &= \frac{1}{2}m\mathbf{v}^2 \\
 &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\
 U &= mgz \\
 \mathcal{L} &= K - U \\
 &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \\
 F_x &= \frac{\partial \mathcal{L}}{\partial x} \\
 &= 0 \\
 F_y &= \frac{\partial \mathcal{L}}{\partial y} \\
 &= 0 \\
 F_z &= \frac{\partial \mathcal{L}}{\partial z} \\
 &= -mg
 \end{aligned}$$

### 7.3

$$\begin{aligned}\mathcal{L} &= K - U \\ &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2)\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \\ m\ddot{x} &= -kx\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial y} &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} \\ m\ddot{y} &= -ky\end{aligned}$$

The mass is a harmonic oscillator in each dimension.

### 7.5

$$\begin{aligned}df &= \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta \\ d\mathbf{r} &= dr + r d\theta \\ df &= \nabla f \cdot d\mathbf{r} \\ &= (\nabla f)_r dr + (\nabla f)_\theta r d\theta \\ (\nabla f)_r &= \frac{\partial f}{\partial r} \\ (\nabla f)_\theta &= \frac{1}{r} \frac{\partial f}{\partial \theta} \\ \nabla f &= \frac{\partial f}{\partial r} dr + \frac{1}{r} \frac{\partial f}{\partial \theta} d\theta\end{aligned}$$

### 7.7

(a)

$$m_i \ddot{\mathbf{r}}_i = -\nabla U_i$$

(b)

$$\mathcal{L}(\mathbf{r}_1, \dots, \mathbf{r}_n, \dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_n) = \frac{1}{2}m_1 \dot{r}_1^2 + \dots + \frac{1}{2}m_n \dot{r}_n^2 - U(\mathbf{r}_1, \dots, \mathbf{r}_n)$$

### 7.9

$$\begin{aligned}x &= R \cos \phi \\ y &= R \sin \phi \\ \phi &= \arctan \frac{y}{x}\end{aligned}$$

7.11

$$\begin{aligned}x &= A \cos \omega t + l \sin \phi \\y &= l \cos \phi \\ \phi &= \arctan \frac{x - A \cos \omega t}{y}\end{aligned}$$

7.15

$$\begin{aligned}\mathcal{L} &= K - U \\ &= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_2gx \\ \frac{\partial \mathcal{L}}{\partial x} &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \\ (m_1 + m_2)\ddot{x} &= m_2g \\ \ddot{x} &= \frac{m_2}{m_1 + m_2}g\end{aligned}$$

7.17

$$\begin{aligned}\mathcal{L} &= K - U \\ &= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}m_2\dot{x}^2 - (m_2gx - m_1gx) \\ &= \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}^2 + (m_1 - m_2)gx \\ \frac{\partial \mathcal{L}}{\partial x} &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \\ (m_1 - m_2)g &= \left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} \\ \ddot{x} &= \frac{(m_1 - m_2)g}{m_1 + m_2 + \frac{I}{R^2}}\end{aligned}$$

## 7.21

$$\begin{aligned}
 \mathcal{L} &= K - U \\
 &= \frac{1}{2}mv^2 \\
 &= \frac{1}{2}m[\dot{r}^2 + (r\dot{\phi})^2] \\
 &= \frac{1}{2}m(\dot{r}^2 + r^2\omega^2) \\
 \frac{\partial \mathcal{L}}{\partial r} &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} \\
 m\ddot{r} &= mr\omega^2 \\
 \ddot{r} &= r\omega^2 \\
 r &= c_1e^{\omega t} + c_2e^{-\omega t}
 \end{aligned}$$

If the bead is initially at rest at the origin

$$\begin{aligned}
 0 &= c_1 + c_2 \\
 0 &= c_1\omega - c_2\omega \\
 &= c_1 - c_2 \\
 c_1 &= 0 \\
 c_2 &= 0
 \end{aligned}$$

thus the bead stays at the origin.

If it is released from a point  $r_0 > 0$

$$\begin{aligned}
 r_0 &= c_1 + c_2 \\
 0 &= c_1 - c_2 \\
 c_1 &= c_2 \\
 &= \frac{r_0}{2} \\
 r &= \frac{r_0}{2}(e^{\omega t} + e^{-\omega t})
 \end{aligned}$$

$r$  eventually grows as  $\frac{r_0}{2}e^{\omega t}$ .



## 7.23

$$\begin{aligned}
 K &= \frac{1}{2}mv^2 \\
 &= \frac{1}{2}m(\dot{x} + \dot{X})^2 \\
 &= \frac{1}{2}m(\dot{x} - A\omega \sin \omega t)^2 \\
 &= \frac{1}{2}m(\dot{x}^2 - 2\dot{x}A\omega \sin \omega t + A^2\omega^2 \sin^2 \omega t)
 \end{aligned}$$

$$U = \frac{1}{2}kx^2$$

$$\mathcal{L} = K - U$$

$$= \frac{1}{2}m(\dot{x}^2 - 2\dot{x}A\omega \sin \omega t + A^2\omega^2 \sin^2 \omega t) - \frac{1}{2}kx^2$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$-kx = \frac{d}{dt} \left[ \frac{1}{2}m(2\dot{x} - 2A\omega \sin \omega t) \right]$$

$$= m\ddot{x} - Am\omega^2 \cos \omega t$$

$$\ddot{x} + \frac{k}{m}x = A\omega^2 \cos \omega t$$

$$\ddot{x} + \omega_0^2 x = B \cos \omega t$$

## 7.27

$$\begin{aligned}
K &= \frac{1}{2}(4m)\dot{x}_1^2 + \frac{1}{2}(3m)(\dot{x}_1 - \dot{x}_2)^2 + \frac{1}{2}m(\dot{x}_1 + \dot{x}_2)^2 \\
&= 2m\dot{x}_1^2 + \frac{3}{2}m(\dot{x}_1^2 - 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2) + \frac{1}{2}m(\dot{x}_1^2 + 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2) \\
&= \frac{1}{2}m(4\dot{x}_1^2 + 3\dot{x}_1^2 - 6\dot{x}_1\dot{x}_2 + 3\dot{x}_2^2 + \dot{x}_1^2 + 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2) \\
&= 2m(2\dot{x}_1^2 - \dot{x}_1\dot{x}_2 + \dot{x}_2^2) \\
U &= 4mgx_1 + 3mg(x_2 - x_1) - mg(x_1 + x_2) \\
&= mg(4x_1 + 3x_2 - 3x_1 - x_1 - x_2) \\
&= 2mgx_2 \\
\mathcal{L} &= K - U \\
&= 2m(2\dot{x}_1^2 - \dot{x}_1\dot{x}_2 + \dot{x}_2^2) - 2mgx_2
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_1} \\
0 &= \frac{d}{dt}(8m\dot{x}_1 - 2m\dot{x}_2) \\
&= 8m\ddot{x}_1 - 2m\ddot{x}_2 \\
\ddot{x}_1 &= \frac{1}{4}\ddot{x}_2
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_2} &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_2} \\
-2mg &= \frac{d}{dt}(-2m\dot{x}_1 + 4m\dot{x}_2) \\
&= -2m\ddot{x}_1 + 4m\ddot{x}_2 \\
\ddot{x}_2 &= \frac{1}{2}\ddot{x}_1 - \frac{1}{2}g
\end{aligned}$$

$$\begin{aligned}
\ddot{x}_1 &= \frac{1}{4} \left( \frac{1}{2}\ddot{x}_1 - \frac{1}{2}g \right) \\
&= \frac{1}{8}\ddot{x}_1 - \frac{1}{8}g \\
\frac{7}{8}\ddot{x}_1 &= -\frac{1}{8}g \\
\ddot{x}_1 &= -\frac{1}{7}g
\end{aligned}$$

The acceleration of the mass  $4m$  is  $\frac{1}{7}g$  downwards.

## 7.29

$$x = R \cos \omega t + l \sin \phi$$

$$\dot{x} = -R\omega \sin \omega t + l\dot{\phi} \cos \phi$$

$$y = R \sin \omega t - l \cos \phi$$

$$\dot{y} = R\omega \cos \omega t + l\dot{\phi} \sin \phi$$

$$K = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m[R^2\omega^2 + l^2\dot{\phi}^2 + 2R\omega l\dot{\phi} \sin(\phi - \omega t)]$$

$$U = mgy$$

$$= mg(R \sin \omega t - l \cos \phi)$$

$$\mathcal{L} = K - U$$

$$= \frac{1}{2}m[R^2\omega^2 + l^2\dot{\phi}^2 + 2R\omega l\dot{\phi} \sin(\phi - \omega t)] - mg(R \sin \omega t - l \cos \phi)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$mR\omega l\dot{\phi} \cos(\phi - \omega t) - mgl \sin \phi = \frac{d}{dt}[ml^2\dot{\phi} + mR\omega l \sin(\phi - \omega t)]$$

$$= ml^2\ddot{\phi} + mR\omega l \cos(\phi - \omega t)(\dot{\phi} - \omega)$$

$$l\ddot{\phi} = R\omega^2 \cos(\phi - \omega t) - g \sin \phi$$

### 7.31

(a)

$$X = x + L \sin \phi$$

$$\dot{X} = \dot{x} + L\dot{\phi} \cos \phi$$

$$y = -L \cos \phi$$

$$\dot{y} = L\dot{\phi} \sin \phi$$

$$\begin{aligned} K &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Mv^2 \\ &= \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}LM\dot{\phi}(2\dot{x} \cos \phi + L\dot{\phi}) \end{aligned}$$

$$U = \frac{1}{2}kx^2 - MgL \cos \phi$$

$$\mathcal{L} = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}LM\dot{\phi}(2\dot{x} \cos \phi + L\dot{\phi}) - \frac{1}{2}kx^2 + MgL \cos \phi$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$-kx = (m+M)\ddot{x} + LM(\ddot{\phi} \cos \phi - \dot{\phi}^2 \sin \phi)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$M(L\ddot{\phi} + \ddot{x} \cos \phi) = -Mg \sin \phi$$

(b)

$$-kx = (m+M)\ddot{x} + LM(\ddot{\phi} - \phi\dot{\phi}^2)$$

$$\approx (m+M)\ddot{x} + LM\ddot{\phi}$$

$$M(L\ddot{\phi} + \ddot{x}) = -Mg\phi$$

### 7.33

$$\begin{aligned}
X &= -x \cos \omega t \\
\dot{X} &= -\dot{x} \cos \omega t + \omega x \sin \omega t \\
y &= x \sin \omega t \\
\dot{y} &= \dot{x} \sin \omega t + \omega x \cos \omega t \\
v^2 &= \dot{X}^2 + \dot{y}^2 \\
&= (-\dot{x} \cos \omega t + \omega x \sin \omega t)^2 + (\dot{x} \sin \omega t + \omega x \cos \omega t)^2 \\
&= \dot{x}^2 + \omega^2 x^2 \\
K &= \frac{1}{2} m v^2 \\
&= \frac{1}{2} m (\dot{x}^2 + \omega^2 x^2) \\
U &= mgy \\
&= mgx \sin \omega t \\
\mathcal{L} &= K - U \\
&= \frac{1}{2} m (\dot{x}^2 + \omega^2 x^2) - mgx \sin \omega t \\
\frac{\partial \mathcal{L}}{\partial x} &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \\
\ddot{x} - \omega^2 x &= -g \sin \omega t \\
x &= x_0 \cosh \omega t + \frac{g}{2\omega^2} (\sin \omega - \sinh \omega t)
\end{aligned}$$

### 7.35

$$\begin{aligned}
x &= R \cos \omega t + R \cos(\omega t + \phi) \\
\dot{x} &= -R[\omega \sin \omega t + (\omega + \dot{\phi}) \sin(\omega t + \phi)] \\
y &= R[\sin \omega t + \sin(\omega t + \phi)] \\
\dot{y} &= R[\omega \cos \omega t + (\omega + \dot{\phi}) \cos(\omega t + \phi)] \\
v^2 &= R^2[\omega^2 + (\omega + \dot{\phi})^2 + 2\omega(\omega + \dot{\phi}) \cos \phi] \\
\mathcal{L} &= K - U \\
&= \frac{1}{2} m v^2 \\
&= \frac{1}{2} m R^2[\omega^2 + (\omega + \dot{\phi})^2 + 2\omega(\omega + \dot{\phi}) \cos \phi]
\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \phi} &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\ -m\omega(\omega + \dot{\phi})R^2 \sin \phi &= mR^2(\ddot{\phi} - \omega\dot{\phi} \sin \phi) \\ -\omega^2 \sin \phi &= \ddot{\phi}\end{aligned}$$

This is the equation of motion for a simple pendulum. For small  $\phi$  the angular frequency is  $\omega$ .

### 7.37

(a)

$$\begin{aligned}K &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}m\dot{r}^2 \\ &= m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 \\ U &= mgr \\ \mathcal{L} &= K - U \\ &= m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - mgr\end{aligned}$$

(b)

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial r} &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} \\ mr\dot{\phi}^2 - mg &= \frac{d}{dt}(2m\dot{r}) \\ &= 2m\ddot{r}\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \phi} &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\ 0 &= \frac{d}{dt}(mr^2\dot{\phi}) \\ \ell &= mr^2\dot{\phi}\end{aligned}$$

(c)

$$\dot{\phi} = \frac{\ell}{mr^2}$$

$$mr \left( \frac{\ell}{mr^2} \right)^2 - mg = 2m\ddot{r}$$

$$\frac{\ell^2}{mr^3} - mg = 2m\ddot{r}$$

$$m\ddot{r} = \frac{\ell^2}{2mr^3} - \frac{mg}{2}$$

$$\frac{\ell^2}{2mr_0^3} - \frac{mg}{2} = 0$$

$$\frac{2mr_0^3}{\ell^2} = \frac{2}{mg}$$

$$r_0 = \sqrt[3]{\frac{\ell^2}{m^2g}}$$

### 7.39

(a)

$$\begin{aligned} K &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}I_\theta\dot{\theta}^2 + \frac{1}{2}I_\phi\dot{\phi}^2 \\ &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}m(r\sin\theta)^2\dot{\phi}^2 \\ &= \frac{1}{2}m[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2] \end{aligned}$$

$$\mathcal{L} = K - U$$

$$= \frac{1}{2}m[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2] - U(r)$$

(b)

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}}$$

$$mr\dot{\theta}^2 + mr \sin^2 \theta \dot{\phi}^2 - \frac{dU(r)}{dr} = m\ddot{r}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}}$$

$$mr^2 \sin \theta \cos \theta \dot{\phi}^2 = \frac{d}{dt}(mr^2 \dot{\theta})$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$0 = \frac{d}{dt}(mr^2 \sin^2 \theta \dot{\phi})$$

(c) The motion remains in the equatorial plane.

(d) The motion remains on that line of longitude.



7.41

$$z = k\rho^2$$

$$\dot{z} = 2k\rho\dot{\rho}$$

$$\phi = \omega t$$

$$\dot{\phi} = \omega$$

$$\begin{aligned} K &= \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}I_{\phi}\omega^2 + \frac{1}{2}m\dot{z}^2 \\ &= \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\omega^2 + \frac{1}{2}m(2k\rho\dot{\rho})^2 \\ &= \frac{1}{2}m[\dot{\rho}^2 + (\rho\omega)^2 + (2k\rho\dot{\rho})^2] \end{aligned}$$

$$U = mgz$$

$$= mgk\rho^2$$

$$\mathcal{L} = K - U$$

$$= \frac{1}{2}m[\dot{\rho}^2 + (\rho\omega)^2 + (2k\rho\dot{\rho})^2] - mgk\rho^2$$

$$\frac{\partial \mathcal{L}}{\partial \rho} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\rho}}$$

$$\begin{aligned} m\rho\omega^2 + 4mk^2\rho\dot{\rho}^2 - 2mgk\rho &= \frac{d}{dt}(m\dot{\rho} + 4mk^2\rho^2\dot{\rho}) \\ &= m\ddot{\rho} + 8mk^2\rho\dot{\rho}^2 + 4mk^2\rho^2\ddot{\rho} \\ (1 + 4k^2\rho^2)\ddot{\rho} + 4k^2\rho\dot{\rho}^2 &= (\omega^2 - 2gk)\rho \end{aligned}$$

## 8 Two-Body Central Force Problems

### 8.1

$$\begin{aligned}\mathbf{r}_1 &= \mathbf{R} + \frac{m_2}{M} \mathbf{r} \\ &= \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{M} + \frac{m_2}{M} (\mathbf{r}_1 - \mathbf{r}_2) \\ &= \frac{(m_1 + m_2) \mathbf{r}_1}{M} \\ &= \mathbf{r}_1 \\ \mathbf{r}_2 &= \mathbf{R} - \frac{m_1}{M} \mathbf{r} \\ &= \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{M} - \frac{m_1}{M} (\mathbf{r}_1 - \mathbf{r}_2) \\ &= \frac{(m_1 + m_2) \mathbf{r}_2}{M} \\ &= \mathbf{r}_2 \\ K &= \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \mu \dot{\mathbf{r}}^2 \\ &= \frac{1}{2} M \left( \frac{m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2}{M} \right)^2 + \frac{1}{2} \frac{m_1 m_2}{M} (\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2)^2 \\ &= \frac{1}{2} \frac{m_1^2 \dot{\mathbf{r}}_1^2 + m_1 m_2 \dot{\mathbf{r}}_1 \dot{\mathbf{r}}_2 + m_2^2 \dot{\mathbf{r}}_2^2}{M} + \frac{1}{2} \frac{m_1 m_2}{M} (\dot{\mathbf{r}}_1^2 - 2 \dot{\mathbf{r}}_1 \dot{\mathbf{r}}_2 + \dot{\mathbf{r}}_2^2) \\ &= \frac{1}{2} \left( \frac{m_1^2 + m_1 m_2}{M} \dot{\mathbf{r}}_1^2 + \frac{m_2^2 + m_1 m_2}{M} \dot{\mathbf{r}}_2^2 \right) \\ &= \frac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{r}}_2^2\end{aligned}$$

## 8.2

(a)

$$\begin{aligned}
\mathcal{L} &= K - U \\
&= \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - m_1gz_1 - m_2gz_2 - U(r) \\
&= \frac{1}{2}m_1\left(\dot{\mathbf{R}} + \frac{m_2}{M}\dot{\mathbf{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\mathbf{R}} - \frac{m_1}{M}\dot{\mathbf{r}}\right)^2 - m_1g\left(Z + \frac{m_2}{M}z\right) \\
&\quad - m_2g\left(Z - \frac{m_1}{M}z\right) - U(r) \\
&= \frac{1}{2}m_1\left(\dot{\mathbf{R}}^2 + 2\frac{m_2}{M}\dot{\mathbf{r}}\dot{\mathbf{R}} + \frac{m_2^2}{M^2}\dot{\mathbf{r}}^2\right) + \frac{1}{2}m_2\left(\dot{\mathbf{R}}^2 - 2\frac{m_1}{M}\dot{\mathbf{r}}\dot{\mathbf{R}} + \frac{m_1^2}{M^2}\dot{\mathbf{r}}^2\right) \\
&\quad - m_1g\left(Z + \frac{m_2}{M}z\right) - m_2g\left(Z - \frac{m_1}{M}z\right) - U(r) \\
&= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - MgZ - U(r) \\
&= \left(\frac{1}{2}M\dot{\mathbf{R}}^2 - MgZ\right) + \left(\frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)\right) \\
&= \mathcal{L}_{\text{cm}} + \mathcal{L}_{\text{rel}}
\end{aligned}$$

### 8.3

$$\begin{aligned}
\dot{\mathbf{R}} &= \frac{m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2}{M} \\
\mathcal{L}_{\text{cm}} &= \frac{1}{2} M \dot{\mathbf{R}}^2 - MgZ \\
&= \frac{1}{2} M (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - MgZ \\
\frac{\partial \mathcal{L}_{\text{cm}}}{\partial Z} &= \frac{d}{dt} \frac{\partial \mathcal{L}_{\text{cm}}}{\partial \dot{Z}} \\
-Mg &= M \ddot{Z} \\
Z &= \dot{R}_0 t - \frac{1}{2} g t^2 \\
&= \frac{m_1}{M} v_0 t - \frac{1}{2} g t^2 \\
\\
\mathcal{L}_{\text{rel}} &= \frac{1}{2} \mu \dot{\mathbf{r}}^2 - \frac{1}{2} k (r - L)^2 \\
\frac{\partial \mathcal{L}_{\text{rel}}}{\partial r} &= \frac{d}{dt} \frac{\partial \mathcal{L}_{\text{rel}}}{\partial \dot{r}} \\
-k(r - L) &= \mu \ddot{r} \\
\omega &= \sqrt{\frac{k}{\mu}} \\
r &= \frac{v_0}{\omega} \sin \omega t + L \\
\\
z_1 &= Z + \frac{m_2}{M} r \\
&= \frac{m_1}{M} v_0 t - \frac{1}{2} g t^2 + \frac{m_2}{M} \left( \frac{v_0}{\omega} \sin \omega t + L \right) \\
\\
z_2 &= Z - \frac{m_1}{M} r \\
&= \frac{m_1}{M} v_0 t - \frac{1}{2} g t^2 - \frac{m_1}{M} \left( \frac{v_0}{\omega} \sin \omega t + L \right)
\end{aligned}$$

## 8.7

(a)

$$\begin{aligned}
 m_1 \frac{v^2}{r} &= \frac{Gm_1m_2}{r^2} \\
 v &= \sqrt{\frac{Gm_2}{r}} \\
 T &= \frac{2\pi r}{v} \\
 &= 2\pi r \sqrt{\frac{r}{Gm_2}}
 \end{aligned}$$

(b)

$$\begin{aligned}
 \mathcal{L} &= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{GM\mu}{r} \\
 \frac{\partial \mathcal{L}}{\partial r} &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} \\
 \mu r \dot{\phi}^2 - \frac{GM\mu}{r^2} &= \mu \ddot{r} \\
 r \left( \frac{v}{r} \right)^2 &= \frac{GM}{r} \\
 v &= \sqrt{\frac{GM}{r}} \\
 T &= \frac{2\pi r}{v} \\
 &= \frac{2\pi r^{3/2}}{\sqrt{GM}}
 \end{aligned}$$

This is equal to the period found in part (a) when  $m_2 \rightarrow \infty$ .

(c)  $T = 0.703 \text{ years}$

## 8.9

(a)

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - \frac{1}{2}k(|\mathbf{r}_1 - \mathbf{r}_2| - L)^2 \\
&= \frac{1}{2}m_1\left(\dot{\mathbf{R}} + \frac{m_2}{M}\dot{\mathbf{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\mathbf{R}} - \frac{m_1}{M}\dot{\mathbf{r}}\right)^2 - \frac{1}{2}k(r - L)^2 \\
&= \frac{1}{2}m_1\left(\dot{\mathbf{R}}^2 + 2\frac{m_2}{M}\dot{\mathbf{r}}\dot{\mathbf{R}} + \frac{m_2^2}{M^2}\dot{\mathbf{r}}^2\right) + \frac{1}{2}m_2\left(\dot{\mathbf{R}}^2 - 2\frac{m_1}{M}\dot{\mathbf{r}}\dot{\mathbf{R}} + \frac{m_1^2}{M^2}\dot{\mathbf{r}}^2\right) \\
&\quad - \frac{1}{2}k(r - L)^2 \\
&= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - \frac{1}{2}k(r - L)^2 \\
&= \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{2}k(r - L)^2
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial X} &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{X}} \\
0 &= M\ddot{X} \\
X &= \dot{X}_0 t + X_0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial Y} &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{Y}} \\
0 &= M\ddot{Y} \\
Y &= \dot{Y}_0 t + Y_0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial Z} &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{Z}} \\
0 &= M\ddot{Z} \\
Z &= \dot{Z}_0 t + Z_0
\end{aligned}$$

(c)

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}}$$

$$\mu r \dot{\phi}^2 - k(r - L) = \mu \ddot{r}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$0 = \frac{d}{dt} (\mu r^2 \dot{\phi})$$

$$\mu r^2 \dot{\phi} = \ell$$

$r$  remains constant:

$$\mu r \dot{\phi}^2 - k(r - L) = 0$$

$$\dot{\phi} = \sqrt{\frac{k(r - L)}{\mu r}}$$

$$r = \frac{\mu r \dot{\phi}^2}{k} + L$$

$\phi$  remains constant:

$$\mu \ddot{r} = -k(r - L)$$

$$\ddot{r} = -\frac{k}{\mu}(r - L)$$

$$= -\frac{2k}{m_1}(r - L)$$

$$= -\omega^2(r - L)$$

$$\omega = \sqrt{\frac{2k}{m_1}}$$

$$r = L + A \cos(\omega t - \delta)$$