

Contents

1	First-order ODEs	1
1.1	Separable Equations	1
1.2	Linear Equations	2
1.3	Exact Equations	2
1.4	Exact Equations with Integration Constant	3
1.5	Homogeneous Equations	3
1.6	Bernoulli's Equation	4
1.7	Reduction to Separation of Variables	4
1.8	Riccati's Equation	4

1 First-order ODEs

Form: IVP

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

Test: $f(x, y)$ and $\partial f / \partial y$ are continuous over I

Property: A unique solution is guaranteed over I

1.1 Separable Equations

Form:

$$\frac{dy}{dx} = g(x)h(y)$$

Solution: Divide by $h(y)$ then integrate with respect to x .

$$\frac{dy}{dx} = g(x)h(y)$$

$$\frac{1}{h(y)} \frac{dy}{dx} = g(x)$$

$$\int \frac{1}{h(y)} \frac{dy}{dx} dx = \int g(x) dx$$

$$\int \frac{1}{h(y)} dy = \int g(x) dx$$

$$H(y) = G(x) + c$$

1.2 Linear Equations

Form:

$$\frac{dy}{dx} + P(x)y = f(x)$$

Solution:

1. Determine the integrating factor $e^{\int P(x) dx}$
2. Multiply by the integrating factor
3. Recognise that the left hand side of the equation is the derivative of the product of the integrating factor and y
4. Integrate both sides of the equation
5. Solve for y

1.3 Exact Equations

Form:

$$z = f(x, y) = c$$
$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = M(x, y) dx + N(x, y) dy = 0$$

Test:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution:

1. Integrate $M(x, y)$ with respect to x to find an expression for $z = f(x, y)$

$$\frac{\partial f}{\partial x} = M(x, y)$$
$$f(x, y) = \int M(x, y) dx + g(y)$$

2. Differentiate $f(x, y)$ with respect to y and equate it to $N(x, y)$ to find $g'(y)$

$$\frac{\partial f}{\partial y} = N(x, y) = \frac{\partial}{\partial y} \int M(x, y) dx + g'(y)$$
$$g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx$$

3. Integrate $g'(y)$ with respect to y to find $g(y)$ and substitute it into $f(x, y)$
4. Equate $f(x, y)$ with an unknown constant c

Note: The steps can be performed with x and y reversed, i.e. start by integrating $N(x, y)$ with respect to y , etc.

1.4 Exact Equations with Integration Constant

Form:

$$M(x, y) dx + N(x, y) dy = 0$$

Test: $(M_y - N_x)/N$ is a function of x alone or $(N_x - M_y)/M$ is a function of y alone

Solution:

1. Compute the integrating factor

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

or

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

as appropriate

2. Multiple the equation by this factor
3. The equation is now exact and can be solved as above

1.5 Homogeneous Equations

Form:

$$M(x, y) dx + N(x, y) dy = 0$$

Test: M and N are homogeneous functions of the same degree

Solution:

1. Rewrite as

$$M(x, y) = x^\alpha M(1, u) \text{ and } N(x, y) = x^\alpha N(1, u) \text{ where } u = y/x$$

or

$$M(x, y) = y^\alpha M(v, 1) \text{ and } N(x, y) = y^\alpha N(v, 1) \text{ where } v = x/y$$

2. Substitute $y = ux$ and $dy = u dx + x du$ or $x = vy$ and $dx = v dy + y dv$ as appropriate
3. Solve the resulting first-order separable DE
4. Substitute $u = y/x$ or $v = x/y$ as appropriate

1.6 Bernoulli's Equation

Form:

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

Test: $n \neq 0$ and $n \neq 1$

Solution:

1. Substitute $y = u^{1/(1-n)}$ and $\frac{dy}{dx} = \frac{d}{dx}(u^{1/(1-n)})$
2. Solve the resulting linear equation
3. Substitute $u = y^{1-n}$

1.7 Reduction to Separation of Variables

Form:

$$\frac{dy}{dx} = f(Ax + By + C), B \neq 0$$

Solution:

1. Substitute

$$Ax + By + C = u$$

2. Solve the resulting separable equation
3. Substitute

$$u = Ax + By + C$$

1.8 Riccati's Equation

Form:

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$$

Test: You know a particular solution y_1 of the equation

Solution:

1. Substitute $y = y_1 + u$ and $y' = y_1' + u'$
2. Solve the resulting Bernoulli equation
3. Substitute $u = y - y_1$