

# Advanced Engineering Mathematics Vectors, Matrices, and Vector Calculus by Dennis G. Zill

## Notes

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June 2023

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## 1 Vectors

### 1.1 Vectors in 2-Space

- The zero vector can be assigned any direction
- The vectors  $\mathbf{i}$  and  $\mathbf{j}$  are known as the **standard basis vectors** for  $\mathbb{R}^2$

### 1.2 Vectors in 3-Space

- In  $\mathbb{R}^3$  the octant in which all coordinates are positive is known as the **first octant**. There is no agreement for naming the other seven octants.

### 1.3 Dot Product

- The **dot product** is also known as the **inner product** or the **scalar product** and is denoted  $\mathbf{a} \cdot \mathbf{b}$
- Two non-zero vectors are orthogonal iff their dot product is 0
- The zero vector is considered orthogonal to all vectors

- The angles  $\alpha$ ,  $\beta$ , and  $\gamma$  between a vector and the unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$ , respectively are called the **direction angles** of the vector
- The cosines of a vectors direction angles (the **direction cosines**) can be calculated as

$$\begin{aligned}\cos \alpha &= \frac{\mathbf{a} \cdot \mathbf{i}}{||\mathbf{a}|| ||\mathbf{i}||} \\ &= \frac{a_1}{||\mathbf{a}||} \\ \cos \beta &= \frac{\mathbf{a} \cdot \mathbf{j}}{||\mathbf{a}|| ||\mathbf{j}||} \\ &= \frac{a_2}{||\mathbf{a}||} \\ \cos \gamma &= \frac{\mathbf{a} \cdot \mathbf{k}}{||\mathbf{a}|| ||\mathbf{k}||} \\ &= \frac{a_3}{||\mathbf{a}||}\end{aligned}$$

Equivalently, these can be calculated as the components of the unit vector  $\mathbf{a}/||\mathbf{a}||$ .

- To find the component of a vector  $\mathbf{a}$  in the direction of a vector  $\mathbf{b}$

$$\text{comp}_{\mathbf{b}} \mathbf{a} = ||\mathbf{a}|| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||}$$

- To project a vector  $\mathbf{a}$  onto a vector  $\mathbf{b}$

$$\text{proj}_{\mathbf{b}} \mathbf{a} = (\text{comp}_{\mathbf{b}} \mathbf{a}) \frac{\mathbf{b}}{||\mathbf{b}||} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \right) \mathbf{b}$$

## 1.4 Cross Product

- The cross product is only defined in  $\mathbb{R}^3$
- The **scalar triple product** of vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  is defined as

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- The area of a parallelogram with sides  $\mathbf{a}$  and  $\mathbf{b}$  is  $||\mathbf{a} \times \mathbf{b}||$
- The area of a triangle with sides  $\mathbf{a}$  and  $\mathbf{b}$  is  $\frac{1}{2}||\mathbf{a} \times \mathbf{b}||$
- The volume of a parallelepiped with sides  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  is  $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$  iff  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are coplanar

## 1.5 Lines and Planes in 3-Space

- There is a unique line between any two points  $\mathbf{r}_1$  and  $\mathbf{r}_2$  in 3-space. The equation for that line is

$$\mathbf{r} = \mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1) = \mathbf{r}_1 + t\mathbf{a}$$

where  $t$  is called a **parameter**, the nonzero vector  $\mathbf{a}$  is called a **direction vector**, and its components are called **direction numbers**.

- Equating the components of the equation above we find

$$\begin{aligned}x &= r_1 + ta_1 \\y &= r_2 + ta_2 \\z &= r_3 + ta_3.\end{aligned}$$

These are the **parametric equations** for the line through  $\mathbf{r}_1$  and  $\mathbf{r}_2$ .

- By solving the parametric equations for  $t$  and equating the results we find the **symmetric equations** for the line

$$t = \frac{x - r_1}{a_1} = \frac{y - r_2}{a_2} = \frac{z - r_3}{a_3}.$$

- Given a point  $P_1$  and a vector  $\mathbf{n}$ , there exists only one plane containing  $P_1$  with  $\mathbf{n}$  normal. The vector from  $P_1$  to another point  $P$  on that plane will be perpendicular to  $\mathbf{n}$ , so the equation for the plane is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$

where  $\mathbf{r} = \overrightarrow{OP}$  and  $\mathbf{r}_1 = \overrightarrow{OP_1}$ . If

$$\mathbf{n} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$$

the cartesian form of this equation is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

and is called the **point-normal form**.

- The graph of any equation  $ax + by + cz + d = 0$ , where  $a$ ,  $b$ , and  $c$  are not all zero, is a plane with the normal vector  $\mathbf{n} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ .
- Given three noncollinear points, a normal vector can be found by forming two vectors from two pairs of points and take their cross product.
- A line and a plane that aren't parallel intersect at a single point.
- Two planes that aren't parallel must intersect in a line.

## 1.6 Vector Spaces

- The length of a vector is called its **norm**
- The process of multiplying a vector by the reciprocal of its norm is called **normalizing** the vector
- Two nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$  in  $\mathbb{R}^n$  are said to be orthogonal if  $\mathbf{a} \cdot \mathbf{b} = 0$

### Definition 7.6.1 Vector Space

Let  $V$  be a set of elements on which two operations called **vector addition** and **scalar multiplication** are defined. Then  $V$  is said to be a **vector space** if the following 10 properties are satisfied.

#### Axioms for Vector Addition:

- (i) If  $\mathbf{x}$  and  $\mathbf{y}$  are in  $V$ , then  $\mathbf{x} + \mathbf{y}$  is in  $V$ .
- (ii) For all  $\mathbf{x}, \mathbf{y}$  in  $V$ ,  $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ . ← commutative law
- (iii) For all  $\mathbf{x}, \mathbf{y}, \mathbf{z}$  in  $V$ ,  $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$ . ← associative law
- (iv) There is a unique vector  $\mathbf{0}$  in  $V$  such that  
 $\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x}$ . ← zero vector
- (v) For each  $\mathbf{x}$  in  $V$ , there exists a vector  $-\mathbf{x}$  such that  
 $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$ . ← negative of a vector

#### Axioms for Scalar Multiplication:

- (vi) If  $k$  is any scalar and  $\mathbf{x}$  is in  $V$ , then  $k\mathbf{x}$  is in  $V$ .
- (vii)  $k(\mathbf{x} + \mathbf{y}) = k\mathbf{x} + k\mathbf{y}$  ← distributive law
- (viii)  $(k_1 + k_2)\mathbf{x} = k_1\mathbf{x} + k_2\mathbf{x}$  ← distributive law
- (ix)  $k_1(k_2\mathbf{x}) = (k_1k_2)\mathbf{x}$
- (x)  $1\mathbf{x} = \mathbf{x}$

- If a subset  $W$  of a vector space  $V$  is itself a vector space under the operations of vector addition and scalar multiplication defined on  $V$ , then  $W$  is called a **subspace** of  $V$
- Every vector space has at least two subspaces: itself and the zero subspace  $\{\mathbf{0}\}$
- A set of vectors  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$  is said to be **linearly independent** if the only constants satisfying the equation

$$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \dots + k_n\mathbf{x}_n = \mathbf{0}$$

are  $k_1 = k_2 = \dots = k_n = 0$ . If the set of vectors is not linearly independent it is said to be **linearly dependent**.

- If a set of vectors  $B = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$  in a vector space  $V$  is linearly independent and every vector in  $V$  can be expressed as a linear combination of vectors in  $B$  then  $B$  is said to be a **basis** for  $V$ .
- The number of vectors in a basis  $B$  for a vector space  $V$  is said to be the **dimension** of the space.

- If the basis of a vector space contains a finite number of vectors, then the space is **finite dimensional**; otherwise it is **infinite dimensional**.
- If  $S$  denotes any set of vectors  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$  in a vector space  $V$ , then the set of all linear combinations of the vectors in  $S$

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots + c_n\mathbf{x}_n$$

is called the **span** of the vectors and is denoted  $\text{Span}(S)$ .

- $\text{Span}(S)$  is a subspace of  $V$  and is said to be a subspace spanned by its vectors  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ .
- If  $V = \text{Span}(S)$  then  $S$  is said to be a **spanning set** for the vector space  $V$  or that  $S$  **spans**  $V$ .