

Advanced Engineering Mathematics Complex Analysis by Dennis G. Zill Problems

Chris Doble

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17 Functions of a Complex Variable

17.1 Complex Numbers

17.1.1

$$3 + 3i$$

17.1.3

$$i^8 = (i^2)^4 = (-1)^4 = 1$$

17.1.5

$$7 - 13i$$

17.1.7

$$-7 + 5i$$

17.1.9

$$11 - 10i$$

17.1.11

$$-5 + 12i$$

17.1.13

$$-2i$$

17.1.15

$$\begin{aligned}\frac{2 - 4i}{3 + 5i} &= \frac{(2 - 4i)(3 - 5i)}{34} \\ &= \frac{-14 - 22i}{34} \\ &= -\frac{7}{17} - \frac{11}{17}i\end{aligned}$$

17.1.17

$$\begin{aligned}\frac{(3 - i)(2 + 3i)}{1 + i} &= \frac{9 + 7i}{1 + i} \\ &= \frac{(9 + 7i)(1 - i)}{2} \\ &= \frac{16 - 2i}{2} \\ &= 8 - i\end{aligned}$$

17.1.27

$$\begin{aligned}\frac{1}{z} &= \frac{\bar{z}}{z\bar{z}} \\ &= \frac{x-iy}{x^2+y^2} \\ \operatorname{Re}\left(\frac{1}{z}\right) &= \frac{x}{x^2+y^2}\end{aligned}$$

17.1.29

$$\begin{aligned}2z + 4\bar{z} - 4i &= 2(x+iy) + 4(x-iy) - 4i \\ &= 6x - 2(y+2)i \\ \operatorname{Im}(2z + 4\bar{z} - 4i) &= -2y - 4\end{aligned}$$

17.1.31

$$\begin{aligned}z - 1 - 3i &= x + iy - 1 - 3i \\ &= (x-1) + (y-3)i \\ |z| &= \sqrt{(x-1)^2 + (y-3)^2}\end{aligned}$$

17.1.33

$$\begin{aligned}2z &= i(2+9i) \\ &= -9 + 2i \\ z &= -\frac{9}{2} + i\end{aligned}$$

17.1.35

$$\begin{aligned}(x+iy)^2 &= x^2 + 2xyi - y^2 \\ &= (x^2 - y^2) + 2xyi \\ x^2 &= y^2 \\ x &= y \\ 2xy &= 1 \\ x^2 &= \frac{1}{2} \\ x &= \frac{\sqrt{2}}{2} \\ z &= \frac{\sqrt{2}}{2}(1+i)\end{aligned}$$

17.1.37

$$\begin{aligned}
z + 2\bar{z} &= x + iy + 2x - 2iy \\
&= 3x - iy \\
\frac{2-i}{1+3i} &= \frac{(2-i)(1-3i)}{10} \\
&= \frac{-1-7i}{10} \\
3x - iy &= \frac{-1-7i}{10} \\
x &= -\frac{1}{30} \\
y &= \frac{7}{10} \\
z &= -\frac{1}{30} + \frac{7}{10}i
\end{aligned}$$

17.1.39

$$\begin{aligned}
|10 + 8i| &\approx 12.8 \\
|11 - 6i| &\approx 12.5
\end{aligned}$$

$11 - 6i$ is closer.

17.2 Powers and Roots**17.2.1**

$$2(\cos 0 + i \sin 0)$$

17.2.3

$$-3[\cos(-\pi/2) + i \sin(-\pi/2)]$$

17.2.5

$$\sqrt{2}[\cos(\pi/4) + i \sin(\pi/4)]$$

17.2.7

$$2[\cos(5\pi/6) + i \sin(5\pi/6)]$$

17.2.9

$$\begin{aligned}\frac{3}{-1+i} &= \frac{3(-1-i)}{2} \\ &= \frac{-3-3i}{2} \\ &= -\frac{3}{2} - \frac{3}{2}i \\ &= \frac{3\sqrt{2}}{2}[\cos(5\pi/4) + i\sin(5\pi/4)]\end{aligned}$$

17.2.11

$$-\frac{5\sqrt{3}}{2} - \frac{5}{2}i$$

17.2.13

$$5.54 + 2.30i$$

17.2.15

$$\begin{aligned}8[\cos(\pi/2) + i\sin(\pi/2)] &= 8i \\ \frac{1}{2}[\cos(-\pi/4) + i\sin(-\pi/4)] &= \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}i\end{aligned}$$

17.2.21

$$\begin{aligned}(1 + \sqrt{3}i)^9 &= \{2[\cos(\pi/3) + i\sin(\pi/3)]\}^9 \\ &= 512(\cos \pi + i\sin \pi) \\ &= -512\end{aligned}$$

17.2.23

$$\begin{aligned}\left(\frac{1}{2} + \frac{1}{2}i\right)^{10} &= \left\{\frac{\sqrt{2}}{2}[\cos(\pi/4) + i\sin(\pi/4)]\right\}^{10} \\ &= \frac{1}{32}[\cos(\pi/2) + i\sin(\pi/2)] \\ &= \frac{1}{32}i\end{aligned}$$

17.2.27

$$w_k = 2[\cos(2\pi k/3) + i \sin(2\pi k/3)]$$

$$w_0 = 2$$

$$w_1 = -1 + \sqrt{3}i$$

$$w_2 = -1 - \sqrt{3}i$$

17.2.29

$$w_k = \cos(\pi/4 + k\pi) + i \sin(\pi/4 + k\pi)$$

$$w_0 = \frac{\sqrt{2}}{2}(1 + i)$$

$$w_1 = -\frac{\sqrt{2}}{2}(1 + i)$$

17.2.31

$$w_k = \sqrt{2}[\cos(\pi/3 + k\pi) + i \sin(\pi/3 + k\pi)]$$

$$w_0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i$$

$$w_1 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i$$

17.2.33

$$z^4 + 1 = 0$$

$$z^4 = -1$$

$$w_k = \cos(\pi/4 + k\pi/2) + i \sin(\pi/4 + k\pi/2)$$

$$w_0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$$w_1 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$$w_2 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

$$w_3 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

17.3 Sets in the Complex Plane**17.3.1**

A vertical line at $\operatorname{Re}(z) = 5$.

17.3.3

A horizontal line at $\text{Im}(z) = -3$.

17.3.5

A circle of radius 2 centred at $3i$.

17.3.7

A circle of radius 5 centred at $4 - 3i$.

17.3.9

The region of the plane to the left of (but not including) $\text{Re}(z) = -1$. It is a domain.

17.3.11

The region of the plane above (but not including) $\text{Im}(z) = 3$. It is a domain.

17.3.13

The region of the plane between (but not including) $\text{Re}(z) = 3$ and $\text{Re}(z) = 5$. It is a domain.

17.3.15

$$\begin{aligned} z^2 &= (a + ib)^2 \\ &= a^2 - b^2 + 2iab \\ \text{Re}(z^2) &= a^2 - b^2 \\ \text{Re}(z^2) &> 0 \\ a^2 - b^2 &> 0 \\ a^2 &> b^2 \end{aligned}$$

The region between $y = x$ and $y = -x$. Not a domain.

17.3.17

The region between $\theta = 0$ and $\theta = 2\pi/3$. Not a domain.

17.3.19

The region outside a circle of radius 1 centred at i . It is a domain.

17.3.21

The region between the circles of radius 2 and 3 centred at i . It is a domain.

17.3.23

$$y = -x$$

17.3.25

$$\begin{aligned} z^2 + \bar{z}^2 &= (a + ib)^2 + (a - ib)^2 \\ &= a^2 + 2iab - b^2 + a^2 - 2iab - b^2 \\ &= 2(a^2 - b^2) \\ 2(a^2 - b^2) &= 2 \\ a^2 - b^2 &= 1 \\ a^2 &= b^2 + 1 \end{aligned}$$

The hyperbola $x^2 - y^2 = 1$.

17.4 Functions of a Complex Variable**17.4.1**

$$\begin{aligned} f(z) &= z^2 \\ &= (x + iy)^2 \\ &= x^2 - y^2 + 2ixy \\ u(x, y) &= x^2 - y^2 \\ &= x^2 - 4 \\ v(x, y) &= 2xy \\ &= 4x \\ x &= \frac{v}{4} \\ u &= \left(\frac{v}{4}\right)^2 - 4 \\ &= \frac{1}{16}v^2 - 4 \end{aligned}$$

17.4.3

$$\begin{aligned} u &= -y^2 \\ v &= 0 \end{aligned}$$

Line on the left half of the real axis.

17.4.5

$$\begin{aligned}u &= 0 \\v &= 2x^2\end{aligned}$$

Line on the top half of the imaginary axis.

17.4.7

$$f(x) = (6x - 5) + i(6y + 9)$$

17.4.9

$$f(z) = (x^2 - y^2 - 3x) + i(2xy - 3y + 4)$$

17.4.11

$$f(z) = (x^3 - 3xy^2 - 4x) + i(3x^2y - y^3 - 4y)$$

17.4.13

$$f(z) = \left(x + \frac{x}{x^2 + y^2}\right) i \left(y - \frac{y}{x^2 + y^2}\right)$$

17.4.15

(a) $-4 + i$

(b) $3 - 9i$

(c) $1 + 86i$

17.4.17

(a) $14 - 20i$

(b) $-13 + 43i$

(c) $3 - 26i$

17.4.19

$$6 - 5i$$

17.4.21

$$-4i$$

17.4.27

$$f'(z) = 12z^2 - 2(3+i)z - 5$$

17.4.29

$$\begin{aligned} f'(z) &= 2(z^2 - 4z + 8i) + (2z + 1)(2z - 4) \\ &= 2z^2 - 8z + 16i + 4z^2 - 8z + 2z - 4 \\ &= 6z^2 - 14z - 4 + 16i \end{aligned}$$

17.4.31

$$f'(z) = 6z(z^2 - 4i)^2$$

17.4.33

$$\begin{aligned} f'(z) &= \frac{3(2z + i) - 2(3z - 4 + 8i)}{(2z + i)^2} \\ &= \frac{6z + 3i - 6z + 8 - 16i}{(2z + i)^2} \\ &= \frac{8 - 13i}{(2z + i)^2} \end{aligned}$$

17.4.35

$$3i$$

17.4.37

$$\pm 2i$$

17.4.41

$$\begin{aligned} \frac{dx}{dt} &= 2x \\ x &= c_1 e^{2t} \\ \frac{dy}{dt} &= 2y \\ y &= c_2 e^{2t} \end{aligned}$$

17.4.43

$$\begin{aligned}
f(z) &= \frac{1}{\bar{z}} \\
&= \frac{1}{x - iy} \\
&= \frac{x + iy}{x^2 + y^2} \\
&= \frac{x}{x^2 + y^2} + i \frac{y}{x^2 + y^2} \\
\frac{dx}{dt} &= \frac{x}{x^2 + y^2} \\
\frac{dy}{dt} &= \frac{y}{x^2 + y^2} \\
\frac{dy}{dx} &= \frac{y}{x} \\
\frac{dy}{y} &= \frac{dx}{x} \\
\ln y &= \ln x + c_1 \\
y &= c_2 x
\end{aligned}$$

17.5 Cauchy-Riemann Equations**17.5.1**

$$\begin{aligned}
f(z) &= z^3 \\
&= (x + iy)^3 \\
&= (x^2 + 2ixy - y^2)(x + iy) \\
&= x^3 + ix^2y + 2ix^2y - 2xy^2 - xy^2 - iy^3 \\
&= (x^3 - 3xy^2) + i(3x^2y - y^3) \\
\frac{\partial u}{\partial x} &= 3x^2 - 3y^2 \\
&= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -6xy \\
&= -\frac{\partial v}{\partial x}
\end{aligned}$$

17.5.3

$$\begin{aligned}
f(z) &= \operatorname{Re}(z) \\
&= x \\
\frac{\partial u}{\partial x} &= 1 \\
&\neq \frac{\partial v}{\partial y}
\end{aligned}$$

17.5.5

$$\begin{aligned}
f(z) &= 4z - 6\bar{z} + 3 \\
&= 4(x + iy) - 6(x - iy) + 3 \\
&= (-2x + 3) + 10iy \\
\frac{\partial u}{\partial x} &= -2 \\
&\neq \frac{\partial v}{\partial y}
\end{aligned}$$

17.5.7

$$\begin{aligned}
f(z) &= x^2 + y^2 \\
\frac{\partial u}{\partial x} &= 2x \\
&\neq \frac{\partial v}{\partial y}
\end{aligned}$$

17.5.9

$$\begin{aligned}
f(z) &= e^x \cos y + ie^x \sin y \\
u &= e^x \cos y \\
\frac{\partial u}{\partial x} &= e^x \cos y \\
\frac{\partial u}{\partial y} &= -e^x \sin y \\
v &= e^x \sin y \\
\frac{\partial v}{\partial x} &= e^x \sin y \\
\frac{\partial v}{\partial y} &= e^x \cos y
\end{aligned}$$

Analytic everywhere.

17.5.11

$$f(z) = x + \sin x \cosh y + i(y + \cos x \sinh y)$$

$$u = x + \sin x \cosh y$$

$$\frac{\partial u}{\partial x} = 1 + \cos x \cosh y$$

$$\frac{\partial u}{\partial y} = \sin x \sinh y$$

$$v = y + \cos x \sinh y$$

$$\frac{\partial v}{\partial x} = -\sin x \sinh y$$

$$\frac{\partial v}{\partial y} = 1 + \cos x \cosh y$$

Analytic everywhere.

17.5.15

$$f(z) = 3x - y + 5 + i(ax + by - 3)$$

$$u = 3x - y + 5$$

$$\frac{\partial u}{\partial x} = 3$$

$$\frac{\partial u}{\partial y} = -1$$

$$v = ax + by - 3$$

$$\frac{\partial v}{\partial x} = a$$

$$\frac{\partial v}{\partial y} = b$$

$$a = 1$$

$$b = 3$$

17.5.17

$$f(z) = x^2 + y^2 + 2ixy$$

$$u = x^2 + y^2$$

$$\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = 2y$$

$$v = 2xy$$

$$\frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial v}{\partial y} = 2x$$

Only differentiable when $y = 0$.

17.5.19

$$f(z) = x^3 + 3xy^2 - x + i(y^3 + 3x^2y - y)$$

$$u = x^3 + 3xy^2 - x$$

$$\frac{\partial u}{\partial x} = 3x^2 + 3y^2 - 1$$

$$\frac{\partial u}{\partial y} = 6xy$$

$$v = y^3 + 3x^2y - y$$

$$\frac{\partial v}{\partial x} = 6xy$$

$$\frac{\partial v}{\partial y} = 3y^2 + 3x^2 - 1$$

Only differentiable when $x = 0$ or $y = 0$.

17.5.21

$$f(z) = e^x \cos y + ie^x \sin y$$

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \\ &= e^x \cos y + ie^x \sin y \end{aligned}$$

17.5.23

$$\begin{aligned}u &= x \\ \frac{\partial^2 u}{\partial x^2} &= 0 \\ \frac{\partial^2 u}{\partial y^2} &= 0 \\ \frac{\partial v}{\partial y} &= 1 \\ v &= y + h(x) \\ h'(x) &= 0 \\ v &= y + c \\ f(z) &= x + i(y + c)\end{aligned}$$

17.5.25

$$\begin{aligned}u &= x^2 - y^2 \\ \frac{\partial^2 u}{\partial x^2} &= 2 \\ \frac{\partial^2 u}{\partial y^2} &= -2 \\ \frac{\partial v}{\partial y} &= 2x \\ v &= 2xy + h(x) \\ 2y &= 2y + h'(x) \\ h'(x) &= 0 \\ h(x) &= c \\ v &= 2xy + c \\ f(z) &= (x^2 - y^2) + i(2xy + c)\end{aligned}$$

17.6 Exponential and Logarithmic Functions

17.6.1

$$\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

17.6.3

$$e^{-1} \frac{\sqrt{2}}{2} (1 + i)$$

17.6.5

$$-e^{\pi}$$

17.6.7

$$e^{1.5}(\cos 2 + i \sin 2) = -1.865 + 4.075i$$

17.6.9

$$\cos 5 + i \sin 5 = 0.2836 - 0.9589i$$

17.6.11

$$\begin{aligned} e^{1+5\pi i/4} e^{-1-\pi i/3} &= e^{11\pi i/12} \\ &= \cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12} \\ &= -0.9659 + 0.2588i \end{aligned}$$

17.6.13

$$\begin{aligned} f(z) &= e^{-iz} \\ &= e^{-i(x+iy)} \\ &= e^{y-ix} \\ &= e^y(\cos x - i \sin x) \end{aligned}$$

17.6.15

$$\begin{aligned} f(z) &= e^{z^2} \\ &= e^{x^2-y^2+2ixy} \\ &= e^{x^2-y^2}[\cos(2xy) + i \sin(2xy)] \end{aligned}$$

17.6.17

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x(\cos y + i \sin y) \\ |e^z| &= \sqrt{e^{2x}[\cos^2 y + \sin^2 y]} \\ &= e^x \end{aligned}$$

17.6.19

$$\begin{aligned}e^{z+\pi i} &= e^{x+i(y+\pi)} \\&= e^x[\cos(y+\pi) + i\sin(y+\pi)] \\&= e^x[-\cos y - i\sin y] \\&= -e^x(\cos y + i\sin y) \\e^{z-\pi i} &= e^{x+i(y-\pi)} \\&= e^x[\cos(y-\pi) + i\sin(y-\pi)] \\&= e^x(-\cos y - i\sin y) \\&= -e^x(\cos y + i\sin y)\end{aligned}$$

17.6.21

$$\begin{aligned}e^{\bar{z}} &= e^{x-iy} \\&= e^x(\cos y - i\sin y) \\u &= e^x \cos y \\v &= -e^x \sin y \\\frac{\partial u}{\partial x} &= e^x \cos y \\\neq \frac{\partial v}{\partial y}\end{aligned}$$

17.6.23

$$\log_e 5 + i(\pi + 2n\pi) = 1.6094 + i(\pi + 2n\pi)$$

17.6.25

$$\log_e(2\sqrt{2}) + i\left(\frac{3}{4}\pi + 2n\pi\right) = 1.0397 + i\left(\frac{3}{4}\pi + 2n\pi\right)$$

17.6.27

$$\log_e(2\sqrt{2}) + i\left(\frac{1}{3}\pi + 2n\pi\right) = 1.0397 + i\left(\frac{1}{3}\pi + 2n\pi\right)$$

17.6.29

$$\log_e(6\sqrt{2}) - \frac{\pi}{4}i = 2.1383 - \frac{\pi}{4}i$$

17.6.31

$$\log_e 13 + 2.7468i = 2.5649 + 2.7468i$$

17.6.33

$$5 \left(\log_e 2 + \frac{\pi}{3} i \right) = 3.4657 - \frac{\pi}{3} i$$

17.6.35

$$z = \log_e 4 + i \left(\frac{\pi}{2} + 2n\pi \right) = 1.3863 + i \left(\frac{\pi}{2} + 2n\pi \right)$$

17.6.37

$$\begin{aligned} z - 1 &= 2 + i \left(-\frac{\pi}{2} + 2n\pi \right) \\ z &= 3 + i \left(-\frac{\pi}{2} + 2n\pi \right) \end{aligned}$$

17.6.39

$$\begin{aligned} \ln(-i) &= i \left(-\frac{\pi}{2} + 2n\pi \right) \\ (-i)^{4i} &= e^{4i \ln(-i)} \\ &= e^{4i \times i(-\pi/2 + 2n\pi)} \\ &= e^{2\pi(1-4n)} \end{aligned}$$

17.6.41

$$\begin{aligned} \ln(1+i) &= \log_e \sqrt{2} + i \left(\frac{\pi}{4} + 2n\pi \right) \\ (1+i)^{(1+i)} &= e^{(1+i) \ln(1+i)} \\ &= e^{(1+i)[\log_e \sqrt{2} + i(\pi/4 + 2n\pi)]} \\ &= e^{\log_e \sqrt{2} + i(\pi/4 + 2n\pi) + i \log_e \sqrt{2} - (\pi/4 + 2n\pi)} \\ &= e^{(\log_e \sqrt{2} - \pi/4 - 2n\pi) + i(\log_e \sqrt{2} + \pi/4 + 2n\pi)} \\ &= e^{-2n\pi} e^{(\log_e \sqrt{2} - \pi/4) + i(\log_e \sqrt{2} + \pi/4)} \\ &= e^{-2n\pi} e^{\log_e \sqrt{2} - \pi/4} e^{i(\log_e \sqrt{2} + \pi/4)} \\ &= e^{-2n\pi} (0.2739 + 0.5837i) \end{aligned}$$

17.6.43

$$\begin{aligned} \operatorname{Ln}(-1) &= \pi i \\ (-1)^{(-2i/\pi)} &= e^{(-2i/\pi) \operatorname{Ln}(-1)} \\ &= e^{(-2i/\pi)(\pi i)} \\ &= e^2 \end{aligned}$$

17.6.47

(a)

$$\begin{aligned}
(-1+i)^2 &= -2i \\
\operatorname{Ln}(-1+i)^2 &= \operatorname{Ln}(-2i) \\
&= \log_e 2 - \frac{\pi}{2}i \\
2 \operatorname{Ln}(-1+i) &= 2 \log_e \sqrt{2} + \frac{3\pi}{2}i \\
&\neq \operatorname{Ln}(-1+i)^2
\end{aligned}$$

Not true

(b)

$$\begin{aligned}
\operatorname{Ln} i^3 &= \operatorname{Ln}(-i) \\
&= -\frac{\pi}{2}i \\
3 \operatorname{Ln} i &= \frac{3\pi}{2}i \\
&\neq \operatorname{Ln} i^3
\end{aligned}$$

Not true

(c)

$$\begin{aligned}
\ln i^3 &= i \left(-\frac{\pi}{2} + 2n\pi \right) \\
3 \ln i &= 3i \left(\frac{\pi}{2} + 2n\pi \right) \\
&\neq \ln i^3
\end{aligned}$$

Not true

17.7 Trigonometric and Hyperbolic Functions**17.7.1**

$$\begin{aligned}
\cos(3i) &= \cos 0 \cosh 3 - i \sin 0 \sinh 3 \\
&= \cosh 3 \\
&= 10.0677
\end{aligned}$$

17.7.3

$$\begin{aligned}
\sin(\pi/4 + i) &= \sin \frac{\pi}{4} \cosh 1 + i \cos \frac{\pi}{4} \sinh 1 \\
&= 1.0911 + 0.8309i
\end{aligned}$$

17.7.5

$$\begin{aligned}
\tan i &= \frac{\sin i}{\cos i} \\
&= \frac{\sin 0 \cosh 1 + i \cos 0 \sinh 1}{\cos 0 \cosh 1 + i \sin 0 \sinh 1} \\
&= \frac{i \sinh 1}{\cosh 1} \\
&= i \tanh 1 \\
&= 0.7615i
\end{aligned}$$

17.7.7

$$\begin{aligned}
\sec(\pi + i) &= \frac{1}{\cos(\pi + i)} \\
&= \frac{1}{\cos \pi \cosh 1 + \sin \pi \sinh 1} \\
&= -\frac{1}{\cosh 1} \\
&= -0.6480
\end{aligned}$$

17.7.9

$$\begin{aligned}
\cosh(\pi i) &= \cosh 0 \cos \pi + i \sinh 0 \sin \pi \\
&= -1
\end{aligned}$$

17.7.11

$$\begin{aligned}
\sinh(1 + \pi i/3) &= \sinh 1 \cos(\pi/3) + i \cosh 1 \sin(\pi/3) \\
&= 0.5876 + 1.3363i
\end{aligned}$$

17.7.15

$$\begin{aligned}
 \sin z &= 2 \\
 \frac{e^{iz} - e^{-iz}}{2i} &= 2 \\
 e^{iz} - e^{-iz} &= 4i \\
 e^{2iz} - 1 &= 4ie^{iz} \\
 e^{2iz} - 4ie^{iz} - 1 &= 0 \\
 e^{iz} &= \frac{4i \pm \sqrt{-16 + 4}}{2} \\
 &= (2 \pm \sqrt{3})i \\
 iz &= \log_e(2 \pm \sqrt{3}) + i(\pi/2 + 2n\pi) \\
 z &= (\pi/2 + 2n\pi) - i \log_e(2 \pm \sqrt{3})
 \end{aligned}$$

17.7.17

$$\begin{aligned}
 \sinh z &= -i \\
 \frac{e^z - e^{-z}}{2} &= -i \\
 e^{2z} + 2ie^z - 1 &= 0 \\
 e^z &= \frac{-2i \pm \sqrt{-4 + 4}}{2} \\
 &= -i \\
 z &= \ln(-i) \\
 &= i \left(-\frac{\pi}{2} + 2n\pi \right)
 \end{aligned}$$

17.7.19

$$\begin{aligned}
 \cos z &= \sin z \\
 \frac{e^{iz} + e^{-iz}}{2} &= \frac{e^{iz} - e^{-iz}}{2i} \\
 e^{iz} + e^{-iz} &= \frac{e^{iz} - e^{-iz}}{i} \\
 &= -i(e^{iz} - e^{-iz}) \\
 e^{2iz} + 1 &= -i(e^{2iz} - 1) \\
 e^{2iz}(1 + i) &= -1 + i \\
 e^{2iz} &= \frac{-1 + i}{1 + i} \\
 &= \frac{(-1 + i)(1 - i)}{(1 + i)(1 - i)} \\
 &= \frac{-1 + i + i + 1}{1 - i + i + 1} \\
 &= \frac{2i}{2} \\
 &= i \\
 2iz &= \ln i \\
 &= i \left(\frac{\pi}{2} + 2n\pi \right) \\
 z &= \frac{\pi}{4} + n\pi
 \end{aligned}$$

17.7.21

$$\begin{aligned}
 \cos z &= \cosh 2 \\
 \cos x \cosh y - i \sin x \sinh y &= \cosh 2 \\
 y &= \pm 2 \\
 x &= 2n\pi \\
 z &= 2n\pi \pm 2i
 \end{aligned}$$

17.8 Inverse Trigonometric and Hyperbolic Functions

17.8.1

$$\begin{aligned}\arcsin z &= -i \ln[iz + (1 - z^2)^{1/2}] \\ \arcsin(-i) &= -i \ln[i(-i) + (1 - (-i)^2)^{1/2}] \\ &= -i \ln[1 \pm \sqrt{2}] \\ \ln(1 + \sqrt{2}) &= \log_e(1 + \sqrt{2}) + 2n\pi i \\ \ln(1 - \sqrt{2}) &= \ln\left(-\frac{1}{1 + \sqrt{2}}\right) \\ &= -\ln[-(1 + \sqrt{2})] \\ &= -[\log_e(1 + \sqrt{2}) + i(\pi + 2n\pi)] \\ &= -\log_e(1 + \sqrt{2}) + i(\pi + 2n\pi) \\ \ln(1 \pm \sqrt{2}) &= (-1)^n \log_e(1 + \sqrt{2}) + n\pi i \\ \arcsin(-i) &= -i[(-1)^n \log_e(1 + \sqrt{2}) + n\pi i] \\ &= n\pi - (-1)^n i \log_e(1 + \sqrt{2}) \\ &= n\pi + (-1)^{n+1} i \log_e(1 + \sqrt{2})\end{aligned}$$

17.8.3

$$\begin{aligned}\arcsin 0 &= -i \ln(\pm 1) \\ &= -i(n\pi i) \\ &= n\pi\end{aligned}$$

17.8.5

$$\begin{aligned}\arccos 2 &= -i \ln[2 + i(1 - 2^2)^{1/2}] \\ &= -i \ln[2 \pm \sqrt{3}] \\ \ln(2 + \sqrt{3}) &= \log_e(2 + \sqrt{3}) + 2n\pi i \\ \ln(2 - \sqrt{3}) &= \log_e(2 - \sqrt{3}) + 2n\pi i \\ &= -\log_e(2 + \sqrt{3}) + 2n\pi i \\ \ln(2 \pm \sqrt{3}) &= \pm \log_e(2 + \sqrt{3}) + 2n\pi i \\ \arccos 2 &= 2n\pi \pm i \log_e(2 + \sqrt{3})\end{aligned}$$

17.8.7

$$\begin{aligned}
 \arccos \frac{1}{2} &= -i \ln \left\{ \frac{1}{2} + i \left[1 - \left(\frac{1}{2} \right)^2 \right]^{1/2} \right\} \\
 &= -i \ln \left(\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \right) \\
 \ln \left(\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \right) &= i \left(\pm \frac{\pi}{3} + 2n\pi \right) \\
 \arccos \frac{1}{2} &= \pm \frac{\pi}{3} + 2n\pi
 \end{aligned}$$

17.8.9

$$\begin{aligned}
 \arctan 1 &= \frac{i}{2} \ln \frac{1+i}{-1+i} \\
 \frac{1+i}{-1+i} &= \frac{(1+i)(-1-i)}{(-1+i)(-1-i)} \\
 &= -i \\
 \ln(-i) &= i \left(-\frac{\pi}{2} + 2n\pi \right) \\
 \arctan 1 &= \frac{\pi}{4} + n\pi
 \end{aligned}$$

17.8.11

$$\begin{aligned}
\operatorname{arcsinh} \frac{4}{3} &= \ln \left\{ \frac{4}{3} + \left[\left(\frac{4}{3} \right)^2 + 1 \right]^{1/2} \right\} \\
&= \ln \left(\frac{4}{3} \pm \frac{5}{3} \right) \\
\ln \left(\frac{4}{3} + \frac{5}{3} \right) &= \ln \frac{9}{3} \\
&= \ln 3 \\
&= \log_e 3 + 2n\pi i \\
\ln \left(\frac{4}{3} - \frac{5}{3} \right) &= \ln \left(-\frac{1}{3} \right) \\
&= \log_e \frac{1}{3} + i(\pi + 2n\pi) \\
&= -\log_e 3 + i(\pi + 2n\pi) \\
\operatorname{arcsinh} \frac{4}{3} &= (-1)^n \log_e 3 + n\pi i
\end{aligned}$$

17.9 Chapter in Review**17.9.1**

0, 32

17.9.3

$$\begin{aligned}
\frac{3+4i}{3-4i} &= \frac{(3+4i)^2}{(3-4i)(3+4i)} \\
&= \frac{-7+24i}{25} \\
&= -\frac{7}{25} + \frac{24}{25}i \\
\operatorname{Re} \left(\frac{z}{\bar{z}} \right) &= -\frac{7}{25}
\end{aligned}$$

17.9.5

$$\begin{aligned}\frac{4i}{-3-4i} &= \frac{(4i)(-3+4i)}{(-3-4i)(-3+4i)} \\ &= \frac{-16-12i}{25} \\ &= -\frac{16}{25} - \frac{12}{25}i \\ |z| &= \sqrt{\left(\frac{16}{25}\right)^2 + \left(\frac{12}{25}\right)^2} \\ &= \frac{4}{5}\end{aligned}$$

17.9.7

False

17.9.9

$$\begin{aligned}e^z &= 2i \\ z &= \ln(2i) \\ &= \log_e 2 + i\left(\frac{\pi}{2} + 2n\pi\right)\end{aligned}$$

17.9.11

$$\begin{aligned}(1+i)^{(2+i)} &= e^{(2+i)\ln(1+i)} \\ \ln(1+i) &= \log_e \sqrt{2} + \frac{\pi}{4}i \\ (2+i)\left(\log_e \sqrt{2} + \frac{\pi}{4}i\right) &= 2\log_e \sqrt{2} + \frac{\pi}{2}i + i\log_e \sqrt{2} - \frac{\pi}{4} \\ &= \left(2\log_e \sqrt{2} - \frac{\pi}{4}\right) + i\left(\log_e \sqrt{2} + \frac{\pi}{2}\right) \\ (1+i)^{(2+i)} &= e^{2\log_e \sqrt{2} - \pi/4} \left[\cos\left(\log_e \sqrt{2} + \frac{\pi}{2}\right) + i\sin\left(\log_e \sqrt{2} + \frac{\pi}{2}\right) \right] \\ &\approx -0.3097 + 0.8576i\end{aligned}$$

17.9.13

False

17.9.15

$$\operatorname{Ln}(-ie^3) = 3 - \frac{\pi}{2}i$$

17.9.21

$$\begin{aligned}z^2 &= x^2 - y^2 + 2ixy \\ \operatorname{Im}(z^2) &\leq 2 \\ 2xy &\leq 2\end{aligned}$$

17.9.23

$$\frac{1}{\sqrt{x^2 + y^2}} \leq 1$$

17.9.27

$$\begin{aligned}z^4 &= 1 - i \\ z_k &= 2^{1/8} e^{(-\pi/4 + 2k\pi)i/4} \\ &= 2^{1/8} e^{i(k\pi/2 - \pi/16)} \\ z_0 &= 1.0695 - 0.2127i \\ z_1 &= 0.2127 + 1.0695i \\ z_2 &= -1.0695 + 0.2127i \\ x_3 &= -0.2127 - 1.0695i\end{aligned}$$

18 Integration in the Complex Plane

18.1 Contour Integrals

18.1.1

$$\begin{aligned}z(t) &= 2t + i(4t - 1) \\ z'(t) &= 2 + 4i \\ f(z(t)) &= (2t + 3) + i(4t - 1) \\ f(z(t))z'(t) &= [(2t + 3) + i(4t - 1)](2 + 4i) \\ &= (2t + 3)(2) + (2t + 3)(4i) + i(4t - 1)(2) + i(4t - 1)(4i) \\ &= 4t + 6 + 8it + 12i + 8it - 2i - 16t + 4 \\ &= (-12t + 10) + i(16t + 10) \\ \int_C f(z) dz &= \int_1^3 f(z(t))z'(t) dt \\ &= \int_1^3 (-12t + 10) dt + i \int_1^3 (16t + 10) dt \\ &= -28 + 84i\end{aligned}$$

18.1.3

$$\begin{aligned}
 z(t) &= 3t + 2it \\
 z'(t) &= 3 + 2i \\
 \int_C f(z) dz &= \int_{-2}^2 (3t + 2it)^2 (3 + 2i) dt \\
 &= \int_{-2}^2 [(3 + 2i)t]^2 (3 + 2i) dt \\
 &= (3 + 2i)^3 \int_{-2}^2 t^2 dt \\
 &= (-9 + 46i) \frac{16}{3} \\
 &= -48 + \frac{736}{3}i
 \end{aligned}$$

18.1.5

$$\begin{aligned}
 z(t) &= e^{it} \\
 z'(t) &= ie^{it} \\
 \int_C f(z) dz &= \int_{-\pi/2}^{\pi/2} \frac{1 + e^{it}}{e^{it}} ie^{it} dt \\
 &= i \int_{-\pi/2}^{\pi/2} (1 + e^{it}) dt \\
 &= i \left[t + \frac{1}{i} e^{it} \right]_{-\pi/2}^{\pi/2} \\
 &= i [t - ie^{it}]_{-\pi/2}^{\pi/2} \\
 &= i \left(\frac{\pi}{2} - ie^{\pi i/2} + \frac{\pi}{2} + ie^{-\pi i/2} \right) \\
 &= i(\pi + 2)
 \end{aligned}$$

18.1.7

$$\begin{aligned}
 z(t) &= \cos t + i \sin t \\
 z'(t) &= -\sin t + i \cos t \\
 \int_C f(z) dz &= \int_0^{2\pi} \cos t (-\sin t + i \cos t) dt \\
 &= \int_0^{2\pi} \left(-\frac{1}{2} \sin 2t + i \cos^2 t \right) dt \\
 &= \pi i
 \end{aligned}$$

18.1.9

$$\begin{aligned}
z(t) &= (1-t) + it \\
z'(t) &= -1 + i \\
\int_C f(z) dz &= \int_0^1 [(1-t)^2 + it^3](-1+i) dt \\
&= \int_0^1 (1-2t+t^2+it^3)(-1+i) dt \\
&= \int_0^1 (-1+i+2t-2it-t^2+it^2-it^3-t^3) dt \\
&= \int_0^1 (-1+2t-t^2-t^3) dt + i \int_0^1 (1-2t+t^2-t^3) dt \\
&= -\frac{7}{12} + \frac{1}{12}i
\end{aligned}$$

18.1.17

$$\begin{aligned}
z(t) &= 1 + it \\
z'(t) &= i \\
\int_{C_1} f(z) dz &= \int_0^1 i dt \\
&= i \\
z(t) &= (1-t) + i(1-t) \\
z'(t) &= -(1+i) \\
\int_{C_2} f(z) dz &= -\int_0^1 (1-t)(1+i) dt \\
&= -\int_0^1 (1+i-t-it) dt \\
&= -\frac{1}{2} - \frac{1}{2}i \\
z(t) &= t \\
z'(t) &= 1 \\
\int_{C_3} f(z) dz &= \int_0^1 t dt \\
&= \frac{1}{2} \\
\int_C f(z) dz &= \frac{1}{2}i
\end{aligned}$$

18.1.19

$$z(t) = 1 + it$$

$$z'(t) = i$$

$$\begin{aligned}\int_{C_1} f(z) dz &= \int_0^1 (1 + it)^2 i dt \\ &= i \int_0^1 (1 + 2it - t^2) dt \\ &= -1 + \frac{2}{3}i\end{aligned}$$

$$z(t) = (1 + i)(1 - t)$$

$$z'(t) = -(1 + i)$$

$$\begin{aligned}\int_{C_2} f(z) dz &= - \int_0^1 [(1 + i)(1 - t)]^2 (1 + i) dt \\ &= - \int_0^1 (1 - t + i - it)^2 (1 + i) dt \\ &= \frac{2}{3} - \frac{2}{3}i\end{aligned}$$

$$z(t) = t$$

$$z'(t) = 1$$

$$\begin{aligned}\int_{C_3} f(z) dz &= \int_0^1 t^2 dt \\ &= \frac{1}{3}\end{aligned}$$

$$\int_C f(z) dz = 0$$

18.1.23

$$z(t) = t + i(1 - t^2)$$

$$z'(t) = 1 - 2it$$

$$\int_C f(z) dz = \frac{4}{3} - \frac{5}{3}i$$

18.1.25

$$\begin{aligned}
 L &= 10\pi \\
 |z^2 + 1| &\geq |z^2| - 1 \\
 \left| \frac{e^z}{z^2 + 1} \right| &\leq \frac{|e^z|}{|z^2| - 1} \\
 &= \frac{e^5}{24} \\
 &= M \\
 \left| \oint \frac{e^z}{z^2 + 1} dz \right| &\leq ML \\
 &= \frac{5\pi e^5}{12}
 \end{aligned}$$

18.1.27

$$\begin{aligned}
 z(t) &= (1 + i)t, \quad 0 \leq t \leq 1 \\
 L &= \sqrt{2} \\
 |z^2 + 4| &= |2it^2 + 4| \\
 &\leq |2i + 4| \\
 &= \sqrt{20} \\
 &= 2\sqrt{5} \\
 &= M \\
 \left| \oint (z^2 + 4) dz \right| &\leq ML \\
 &= 2\sqrt{10}
 \end{aligned}$$

18.1.33

$$\begin{aligned}
 z(t) &= e^{it} \\
 z'(t) &= ie^{it} \\
 \oint \overline{f(z)} dz &= \int_0^{2\pi} 2e^{-it} ie^{it} dt \\
 &= 4\pi i
 \end{aligned}$$

The circulation is 0 and the flux is 4π .

18.2 Cauchy-Goursat Theorem

18.2.1

$$\begin{aligned}z &= e^{it}, \quad 0 \leq t \leq 2\pi \\z' &= ie^{it} \\ \int (z^3 - 1 + 3i) dz &= \int_0^{2\pi} [(e^{it})^3 - 1 + 3i] ie^{it} dt \\ &= i \int_0^{2\pi} (e^{4it} - e^{it} + 3ie^{it}) dt \\ &= \left[\frac{1}{4} e^{4it} - e^{it} + 3ie^{it} \right]_0^{2\pi} \\ &= \frac{1}{4} e^{8\pi i} - e^{2\pi i} + 3ie^{2\pi i} - \frac{1}{4} + 1 - 3i \\ &= \frac{1}{4} - 1 + 3i - \frac{1}{4} + 1 - 3i \\ &= 0\end{aligned}$$

18.2.9

$$\begin{aligned}\int_C \frac{1}{z} dz &= \int_0^{2\pi} \frac{1}{e^{it}} ie^{it} dt \\ &= 2\pi i\end{aligned}$$

18.2.11

$$\begin{aligned}\oint_C \left(z + \frac{1}{z} \right) dz &= \oint_C \frac{1}{z^{-1}} dz + \oint_C \frac{1}{z} dz \\ &= 0 + 2\pi i \\ &= 2\pi i\end{aligned}$$

18.2.13

$$\begin{aligned}z^2 - \pi^2 &= (z + \pi)(z - \pi) \\ \frac{z}{z^2 - \pi^2} &= \frac{1/2}{z + \pi} + \frac{1/2}{z - \pi} \\ \oint_C \frac{z}{z^2 - \pi^2} dz &= \frac{1}{2} \oint_C \left(\frac{1}{z + \pi} + \frac{1}{z - \pi} \right) dz \\ &= 0\end{aligned}$$

18.2.15

(a)

$$\begin{aligned}
\frac{2z+1}{z^2+z} &= \frac{2z+1}{z(z+1)} \\
&= \frac{1}{z} + \frac{1}{z+1} \\
\oint_C \frac{2z+1}{z^2+z} dz &= \oint_C \frac{2z+1}{z(z+1)} dz \\
&= \oint_C \left(\frac{1}{z} + \frac{1}{z+1} \right) dz \\
&= 2\pi i
\end{aligned}$$

(b) $4\pi i$

(c) 0

18.2.17

(a)

$$\begin{aligned}
\frac{-3z+2}{z^2-8z+12} &= \frac{1}{z-1} - \frac{4}{z-6} \\
\oint_C \frac{-3z+2}{z^2-8z+12} dz &= \oint_C \left(\frac{1}{z-1} - \frac{4}{z-6} \right) dz \\
&= -8\pi i
\end{aligned}$$

(b) $-6\pi i$ **18.2.19**

$$\begin{aligned}
\frac{z-1}{z(z-i)(z-3i)} &= \frac{1}{3z} - \frac{1/2-i/2}{z-i} + \frac{1/6-i/2}{z-3i} \\
\oint_C \frac{z-1}{z(z-i)(z-3i)} dz &= -\left(\frac{1}{2} - \frac{i}{2} \right) 2\pi i \\
&= -\pi(1+i)
\end{aligned}$$

18.2.21

$$\begin{aligned}
 \frac{8z-3}{z^2-z} &= \frac{3}{z} + \frac{5}{z-1} \\
 \oint_C f(z) dz &= \oint_{C_1} f(z) dz - \oint_{C_2} f(z) dz \\
 &= \oint_{C_1} \left(\frac{3}{z} + \frac{5}{z-1} \right) dz - \oint_{C_2} \left(\frac{3}{z} + \frac{5}{z-1} \right) dz \\
 &= 6\pi i - 10\pi i \\
 &= -4\pi i
 \end{aligned}$$

18.2.23

$$\begin{aligned}
 \oint_C \left(\frac{e^z}{z+3} - 3\bar{z} \right) dz &= \oint_C \frac{e^z}{z+3} dz - 3 \oint_C \bar{z} dz \\
 &= -3 \oint_0^{2\pi} e^{-it} i e^{it} dt \\
 &= -6\pi i
 \end{aligned}$$

18.3 Independence of the Path

18.3.1

(a)

$$\begin{aligned}
 z(t) &= i(t-1), \quad 0 \leq t \leq 2 \\
 z'(t) &= i \\
 \int_C (4z-1) dz &= \int_0^2 \{4[i(t-1)]-1\} i dt \\
 &= \int_0^2 [4(1-t)-i] dt \\
 &= \left[4\left(t - \frac{1}{2}t^2\right) - it \right]_0^2 \\
 &= -2i
 \end{aligned}$$

(b)

$$\begin{aligned}
 F(z) &= 2z^2 - z \\
 \int_C (4z-1) dz &= F(i) - F(-i) \\
 &= [2(i)^2 - (i)] - [2(-i)^2 - (-i)] \\
 &= -2 - i + 2 - i \\
 &= -2i
 \end{aligned}$$

18.3.3

$$\begin{aligned}
z(-1) &= -2 + 7i \\
z(1) &= 2 - i \\
\int_C 2z \, dz &= z^2 \Big|_{-2+7i}^{2-i} \\
&= (2-i)^2 - (-2+7i)^2 \\
&= 48 + 24i
\end{aligned}$$

18.3.5

$$\begin{aligned}
\int_0^{3+i} z^2 \, dz &= \frac{1}{3} z^3 \Big|_0^{3+i} \\
&= \frac{1}{3} (3+i)^3 \\
&= 6 + \frac{26}{3}i
\end{aligned}$$

18.3.7

$$\begin{aligned}
\int_{1-i}^{1+i} z^3 \, dz &= \frac{1}{4} z^4 \Big|_{1-i}^{1+i} \\
&= \frac{1}{4} [(1+i)^4 - (1-i)^4] \\
&= 0
\end{aligned}$$

18.3.9

$$\begin{aligned}
\int_{-i/2}^{1-i} (2z+1)^2 \, dz &= z + 2z^2 + \frac{4}{3}z^3 \Big|_{-i/2}^{1-i} \\
&= -\frac{7}{6} - \frac{22}{3}i
\end{aligned}$$

18.3.11

$$\begin{aligned}
\int_{i/2}^i e^{\pi z} \, dz &= \frac{1}{\pi} e^{\pi z} \Big|_{i/2}^i \\
&= -\frac{1}{\pi} (1+i)
\end{aligned}$$

18.3.13

$$\begin{aligned}\int_{\pi}^{\pi+2i} \sin \frac{z}{2} dz &= -2 \cos \frac{z}{2} \Big|_{\pi}^{\pi+2i} \\ &= -2 \cos \left(\frac{\pi}{2} + i \right) \\ &= 2i \sinh 1 \\ &\approx 2.3504i\end{aligned}$$

18.3.15

$$\begin{aligned}\int_{\pi i}^{2\pi i} \cosh z dz &= \sinh(2\pi i) - \sinh(\pi i) \\ &= 0\end{aligned}$$

18.3.17

$$\begin{aligned}\int_C \frac{1}{z} dz &= \operatorname{Ln} 4e^{\pi i/2} - \operatorname{Ln} 4e^{-\pi i/2} \\ &= \log_e 4 + \frac{\pi}{2}i - \log_e 4 + \frac{\pi}{2}i \\ &= \pi i\end{aligned}$$

18.3.19

$$\begin{aligned}\int_{-4i}^{4i} \frac{1}{z^2} dz &= -\frac{1}{z} \Big|_{-4i}^{4i} \\ &= -\frac{1}{4i} - \frac{1}{4i} \\ &= -\frac{1}{2i} \\ &= \frac{i}{2}\end{aligned}$$

18.3.21

$$\begin{aligned}\int_{\pi}^i e^z \cos z dz &= \frac{1}{2} \int_{\pi}^i [e^{z(1+i)} + e^{z(1-i)}] dz \\ &= \frac{1}{2} \left(\frac{e^{z(1+i)}}{1+i} + \frac{e^{z(1-i)}}{1-i} \right) \Big|_{\pi}^i \\ &\approx 11.4928 + 0.9667i\end{aligned}$$

18.3.23

$$\begin{aligned}
 \int_i^{1+i} z e^z dz &= z e^z \Big|_i^{1+i} - \int_i^{1+i} e^z dz \\
 &= (1+i)e^{1+i} - i e^i - [e^z]_i^{1+i} \\
 &= (1+i)e^{1+i} - i e^i - e^{1+i} + e^i \\
 &\approx -0.9055 + 1.7698i
 \end{aligned}$$

18.4 Cauchy's Integral Formulas

18.4.1

$$8\pi i$$

18.4.3

$$2\pi i e^{\pi i} = -2\pi i$$

18.4.5

$$2\pi i [(-2i)^2 - 3(-2i) + 4i] = 2\pi i (-4 + 10i) = -20\pi - 8\pi i$$

18.4.7

(a)

$$\begin{aligned}
 \oint_C \frac{z^2}{z^2 + 4} dz &= \oint_C \frac{\frac{z^2}{z+2i}}{z-2i} dz \\
 &= 2\pi i \frac{(2i)^2}{(2i) + 2i} \\
 &= -2\pi
 \end{aligned}$$

(b)

$$\begin{aligned}
 \oint_C \frac{z^2}{z^2 + 4} dz &= \oint_C \frac{\frac{z^2}{z-2i}}{z+2i} dz \\
 &= 2\pi i \frac{(-2i)^2}{(-2i) - 2i} \\
 &= 2\pi i \frac{-4}{-4i} \\
 &= 2\pi
 \end{aligned}$$

18.4.9

$$\begin{aligned}\oint_C \frac{z^2 + 4}{z^2 - 5iz - 4} dz &= \oint_C \frac{z^2 + 4}{(z - i)(z - 4i)} dz \\ &= \oint_C \frac{\frac{z^2 + 4}{z - i}}{z - 4i} dz \\ &= -8\pi\end{aligned}$$

18.4.11

$$\begin{aligned}\frac{2\pi i}{2!} \frac{d^2}{dz^2}(e^{z^2}) \Big|_{z=i} &= \pi i \frac{d}{dz}(2ze^{z^2}) \Big|_{z=i} \\ &= \pi i (2e^{z^2} + 4z^2 e^{z^2}) \Big|_{z=i} \\ &= \pi i (2e^{-1} - 4e^{-1}) \\ &= -\frac{2\pi}{e} i\end{aligned}$$

18.4.13

$$\begin{aligned}\oint_C \frac{\cos 2z}{z^5} dz &= \frac{2\pi i}{4!} \frac{d^4}{dz^4}(\cos 2z) \Big|_{z=0} \\ &= \frac{\pi}{12} i (16 \cos 2z) \Big|_{z=0} \\ &= \frac{4}{3} \pi i\end{aligned}$$

18.4.19

$$\begin{aligned}\oint_C \left(\frac{e^{2iz}}{z^4} - \frac{z^4}{(z - i)^3} \right) dz &= \frac{2\pi i}{3!} \frac{d^3}{dz^3}(e^{2iz}) \Big|_{z=0} - \frac{2\pi i}{2!} \frac{d^2}{dz^2}(z^4) \Big|_{z=i} \\ &= \frac{\pi}{3} i (-8ie^{2iz}) \Big|_{z=0} - \pi i (12z^2) \Big|_{z=i} \\ &= \frac{8}{3} \pi + 12\pi i\end{aligned}$$

18.4.21

$$\begin{aligned}
 \oint_C \frac{1}{z^3(z-1)^2} dz &= \oint_{C_1} \frac{\frac{1}{(z-1)^2}}{z^3} dz + \oint_{C_2} \frac{\frac{1}{z^3}}{(z-1)^2} dz \\
 &= \frac{2\pi i}{2!} \frac{d^2}{dz^2} \left(\frac{1}{(z-1)^2} \right) \Big|_{z=0} \\
 &\quad + 2\pi i \frac{d}{dz} \left(\frac{1}{z^3} \right) \Big|_{z=1} \\
 &= 6\pi i - 6\pi i \\
 &= 0
 \end{aligned}$$

18.4.23

$$\begin{aligned}
 \oint_C \frac{3z+1}{z(z-2)^2} dz &= \oint_{C_1} \frac{\frac{3z+1}{z}}{(z-2)^2} dz - \oint_{C_2} \frac{\frac{3z+1}{(z-2)^2}}{z} dz \\
 &= 2\pi i \frac{d}{dz} \left(\frac{3z+1}{z} \right) \Big|_{z=2} - 2\pi i \frac{3z+1}{(z-2)^2} \Big|_{z=0} \\
 &= -\frac{1}{2}\pi i - \frac{1}{2}\pi i \\
 &= -\pi i
 \end{aligned}$$

18.5 Chapter in Review

18.5.1

True

18.5.3

True

18.5.5

0

18.5.7

$$\begin{aligned}
 \oint_C \frac{z^3 + e^z}{(z + \pi i)^3} dz &= \frac{2\pi i}{2!} \frac{d^2}{dz^2} (z^3 + e^z) \Big|_{z=-\pi i} \\
 &= \pi i (6z + e^z) \Big|_{z=-\pi i} \\
 &= \pi i (-6\pi i - 1) \\
 &= 6\pi^2 - \pi i
 \end{aligned}$$

18.5.9

$$\begin{aligned}\oint_C \frac{1}{(z-z_0)(z-z_1)} dz &= \oint_{C_1} \frac{\frac{1}{z-z_0}}{z-z_1} dz + \oint_{C_2} \frac{\frac{1}{z-z_1}}{z-z_0} dz \\ &= 2\pi i \frac{1}{z_1-z_0} + 2\pi i \frac{1}{z_0-z_1} \\ &= 0\end{aligned}$$

True

18.5.11

$2\pi i$ if $n = -1$, 0 otherwise.

18.5.13

$$\begin{aligned}z_1 &= -4 + iy, \quad 0 \leq y \leq 2 \\ z'_1 &= i \\ z_2 &= x + 2i, \quad -4 \leq x \leq 3 \\ z'_2 &= 1 \\ z_3 &= 3 + i(2-y), \quad 0 \leq y \leq 2 \\ z'_3 &= -i \\ \int_C (x+iy) dz &= \int_0^2 (-4+iy)i dy + \int_{-4}^3 (x+2i) dx + \int_0^2 [3+i(2-y)](-i) dy \\ &= \int_0^2 (-y-4i) dy + \int_{-4}^3 (x+2i) dx - \int_0^2 [(y-2)+3i] dy \\ &= \left[-\frac{1}{2}y^2 - 4iy \right]_0^2 + \left[\frac{1}{2}x^2 + 2ix \right]_{-4}^3 - \left[\frac{1}{2}y^2 - 2y + 3iy \right]_0^2 \\ &= -2 - 8i + \frac{9}{2} + 6i - 8 + 8i - 2 + 4 - 6i \\ &= -\frac{7}{2}\end{aligned}$$

18.5.15

$$\begin{aligned}\int_C |z^2| dz &= \int_0^2 |(t+it^2)^2|(1+2it) dt \\ &= \int_0^2 (t^2+t^4)(1+2it) dt \\ &= \frac{136}{15} + \frac{88}{3}i\end{aligned}$$

18.5.17

0

18.5.19

$$\begin{aligned}z(-1) &= 1 \\z(1) &= 1 + 4i \\ \int_C \sin z \, dz &= -\cos z \Big|_1^{1+4i} \\ &= \cos 1 - \cos(1 + 4i) \\ &\approx -14.2144 + 22.9637i\end{aligned}$$

18.5.21

$2\pi i$

18.5.23

$$\begin{aligned}\oint_C \frac{e^{-2z}}{z^4} \, dz &= \frac{2\pi i}{3!} \frac{d^3}{dz^3} (e^{-2z}) \Big|_{z=0} \\ &= -\frac{8}{3}\pi i\end{aligned}$$

18.5.25

$$\begin{aligned}\oint_C \frac{1}{2z^2 + 7z + 3} \, dz &= \oint_C \frac{\frac{1}{z+3}}{2z+1} \, dz \\ &= \oint_C \frac{\frac{1}{2(z+3)}}{z + \frac{1}{2}} \, dz \\ &= 2\pi i \frac{1}{2\left(-\frac{1}{2} + 3\right)} \\ &= \frac{2}{5}\pi i\end{aligned}$$

18.5.27

2π

19 Series and Residues

19.1 Sequences and Series

19.1.1

$$5i, -5, -5i, 5, 5i$$

19.1.3

$$0, 2, 0, 2, 0$$

19.1.5

$$\lim_{n \rightarrow \infty} \frac{3ni + 2}{n + ni} = \frac{3i}{1 + i}$$

Converges

19.1.7

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(ni + 2)^2}{n^2 i} &= \lim_{n \rightarrow \infty} \frac{-n^2 + 4ni + 4}{n^2 i} \\ &= \lim_{n \rightarrow \infty} \frac{-1 + 4i/n + 4/n^2}{i} \\ &= -\frac{1}{i} \end{aligned}$$

Converges

19.1.9

Diverges

19.1.11

$$\begin{aligned} \frac{4n + 3in}{2n + i} &= \frac{(4n + 3in)(2n - i)}{(2n + i)(2n - i)} \\ &= \frac{8n^2 - 4in + 6in^2 + 3n}{4n^2 + 1} \\ &= \frac{8n^2 + 3n}{4n^2 + 1} + i \frac{6n^2 - 4n}{4n^2 + 1} \\ \lim_{n \rightarrow \infty} \frac{8n^2 + 3n}{4n^2 + 1} &= 2 \\ \lim_{n \rightarrow \infty} \frac{6n^2 - 4n}{4n^2 + 1} &= \frac{3}{2} \end{aligned}$$

19.1.13

$$\begin{aligned}S_1 &= \frac{1}{1+2i} - \frac{1}{2+2i} \\S_2 &= \frac{1}{1+2i} - \frac{1}{2+2i} + \frac{1}{2+2i} - \frac{1}{3+2i} \\S_n &= \frac{1}{1+2i} - \frac{1}{n+1+2i} \\ \lim_{n \rightarrow \infty} S_n &= \frac{1}{1+2i}\end{aligned}$$

19.1.15

$$|1-i| = \sqrt{2}$$

Divergent

19.1.17

$$\begin{aligned}\left| \frac{i}{2} \right| &= \frac{1}{2} \\ \frac{i/2}{1-i/2} &= \frac{i}{2-i} \\ &= \frac{i(2+i)}{(2-i)(2+i)} \\ &= \frac{-1+2i}{5}\end{aligned}$$

19.1.19

$$\begin{aligned}
 \frac{2}{1+2i} &= \frac{2(1-2i)}{(1+2i)(1-2i)} \\
 &= \frac{2-4i}{5} \\
 \left| \frac{2}{1+2i} \right| &= \sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{4}{5}\right)^2} \\
 &= \sqrt{\frac{4}{25} + \frac{16}{25}} \\
 &= \sqrt{\frac{20}{25}} \\
 &= \sqrt{\frac{4}{5}} \\
 &= \frac{2}{\sqrt{5}} \\
 &= \frac{2\sqrt{5}}{5} \\
 \frac{3}{1 - \frac{2}{1+2i}} &= \frac{3(1+2i)}{1+2i-2} \\
 &= \frac{3+6i}{-1+2i} \\
 &= \frac{(3+6i)(-1-2i)}{(-1+2i)(-1-2i)} \\
 &= \frac{-3-6i-6i+12}{5} \\
 &= \frac{9-12i}{5}
 \end{aligned}$$

19.1.21

$$\begin{aligned}
 a_k &= \frac{1}{(1-2i)^{k+1}} \\
 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(1-2i)^{n+1}}{(1-2i)^{n+2}} \right| \\
 &= \left| \frac{1}{1-2i} \right| \\
 &= \left| \frac{1+2i}{(1-2i)(1+2i)} \right| \\
 &= \left| \frac{1+2i}{5} \right| \\
 &= \frac{\sqrt{5}}{5} \\
 R &= \sqrt{5}
 \end{aligned}$$

The circle of convergence is $|z-2i| = \sqrt{5}$.

19.1.23

$$\begin{aligned}
 a_k &= \frac{(-1)^k}{k2^k} \\
 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}/[(n+1)2^{(n+1)}]}{(-1)^n/(n2^n)} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{n2^n}{(n+1)2^{(n+1)}} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{n}{2(n+1)} \right| \\
 &= \frac{1}{2} \\
 R &= 2
 \end{aligned}$$

The circle of convergence is $|z-1-i| = 2$.

19.1.25

$$\begin{aligned}
 a_k &= (1+3i)^k \\
 \lim_{n \rightarrow \infty} \left| \frac{(1+3i)^{n+1}}{(1+3i)^n} \right| &= \lim_{n \rightarrow \infty} |1+3i| \\
 &= \sqrt{10} \\
 R &= \frac{1}{\sqrt{10}}
 \end{aligned}$$

The circle of convergence is $|z - i| = \frac{1}{\sqrt{10}}$.

19.1.27

$$\begin{aligned} a_k &= \frac{1}{5^{2k}} \\ \lim_{n \rightarrow \infty} \left| \frac{5^{2n}}{5^{2(n+1)}} \right| &= \frac{1}{25} \\ R &= 25 \end{aligned}$$

The circle of convergence is $|z - 4 - 3i| = 25$.

19.2 Taylor Series

19.2.1

$$\begin{aligned} f(z) &= \frac{z}{1+z} \\ &= z(1 - z + z^2 - z^3 + \dots) \\ &= z - z^2 + z^3 - z^4 + \dots \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} z^k \\ R &= 1 \end{aligned}$$

19.2.3

$$\begin{aligned} f(z) &= \frac{1}{(1+2z)^2} \\ \frac{1}{1+2z} &= 1 - 2z + 4z^2 - 8z^3 + 16z^4 - \dots \\ &= \sum_{k=0}^{\infty} (-1)^k (2z)^k \\ \frac{d}{dz} \frac{1}{1+2z} &= -\frac{2}{(1+2z)^2} \\ &= \sum_{k=1}^{\infty} (-1)^k 2k (2z)^{k-1} \\ f(z) &= \sum_{k=1}^{\infty} (-1)^{k-1} k (2z)^{k-1} \\ R &= \frac{1}{2} \end{aligned}$$

19.2.5

$$\begin{aligned}
f(z) &= e^{-2z} \\
&= \sum_{k=0}^{\infty} \frac{(-2z)^k}{k!} \\
R &= \infty
\end{aligned}$$

19.2.7

$$\begin{aligned}
f(z) &= \sinh z \\
&= \frac{e^z - e^{-z}}{2} \\
&= \frac{1}{2} \sum_{k=0}^{\infty} \frac{z^k - (-z)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} \\
R &= \infty
\end{aligned}$$

19.2.9

$$\begin{aligned}
f(z) &= \cos \frac{z}{2} \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{(z/2)^{2k}}{(2k)!} \\
R &= \infty
\end{aligned}$$

19.2.11

$$\begin{aligned}
f(z) &= \sin z^2 \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{z^{4k+2}}{(2k+1)!} \\
R &= \infty
\end{aligned}$$

19.2.13

$$\begin{aligned}
f(z) &= \frac{1}{z} \\
f'(z) &= -\frac{1}{z^2} \\
f''(z) &= \frac{2}{z^3} \\
f'''(z) &= -\frac{6}{z^4} \\
f^{(n)}(z) &= (-1)^n \frac{n!}{z^{n+1}} \\
f(z) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!} (z-1)^k \\
&= \sum_{k=0}^{\infty} (-1)^k (z-1)^k \\
R &= 1
\end{aligned}$$

19.2.15

$$\begin{aligned}
f(z) &= \frac{1}{3-z} \\
f'(z) &= \frac{1}{(3-z)^2} \\
f''(z) &= \frac{2}{(3-z)^3} \\
f^{(n)}(z) &= \frac{n!}{(3-z)^{n+1}} \\
f(z) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(2i)}{k!} (z-2i)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{(3-2i)^{k+1}} (z-2i)^k \\
R &= \sqrt{13}
\end{aligned}$$

19.2.17

$$\begin{aligned}
 f(z) &= \frac{z-1}{3-z} \\
 f^{(n)}(z) &= (-1)^{n+1} \frac{2n!}{(z-3)^{n+1}} \\
 f(z) &= \sum_{k=1}^{\infty} \frac{f^{(k)}(1)}{k!} (z-1)^k \\
 &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{(-2)^{k+1}} (z-1)^k \\
 &= \sum_{k=1}^{\infty} \frac{(z-1)^k}{2^k} \\
 R &= 2
 \end{aligned}$$

19.2.19

$$\begin{aligned}
 f(z) &= \cos z \\
 f'(z) &= -\sin z \\
 f''(z) &= -\cos z \\
 f'''(z) &= \sin z \\
 f^{(4)}(z) &= \cos z \\
 f(z) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(\pi/4)}{k!} (z-\pi/4)^k \\
 &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2(1!)}(z-\pi/4) - \frac{\sqrt{2}}{2(2!)}(z-\pi/4)^2 + \frac{\sqrt{2}}{2(3!)}(z-\pi/4)^3 + \cdots \\
 R &= \infty
 \end{aligned}$$

19.2.21

$$\begin{aligned}
 f(z) &= e^z \\
 &= \sum_{k=0}^{\infty} \frac{f^{(k)}(3i)}{k!} (z-3i)^k \\
 &= e^{3i} \sum_{k=0}^{\infty} \frac{(z-3i)^k}{k!} \\
 R &= \infty
 \end{aligned}$$

19.2.23

$$f(z) = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots$$

19.2.25

$$\begin{aligned} f(z) &= \frac{i}{(z-i)(z-2i)} \\ &= \frac{1}{z-2i} - \frac{1}{z-i} \\ f^{(n)}(z) &= (-1)^n n! \left(\frac{1}{(z-2i)^{n+1}} - \frac{1}{(z-i)^{n+1}} \right) \\ f^{(n)}(0) &= (-1)^n n! \left(\frac{1}{(-2i)^{n+1}} - \frac{1}{(-i)^{n+1}} \right) \\ &= n! \left(\frac{1}{i^{n+1}} - \frac{1}{(2i)^{n+1}} \right) \\ &= \frac{n!}{i^{n+1}} \left(1 - \frac{1}{2^{n+1}} \right) \\ f(z) &= \sum_{k=0}^{\infty} \frac{1 - \frac{1}{2^{k+1}}}{i^{k+1}} z^k \\ R &= 1 \end{aligned}$$

19.2.27

$$R = \sqrt{20} = 2\sqrt{5}$$

19.2.29

$$\begin{aligned} f(z) &= \frac{1}{z+2} \\ f^{(n)}(z) &= (-1)^n \frac{n!}{(z+2)^{n+1}} \\ f_1(z) &= \sum_{k=0}^{\infty} (-1)^k (z+1)^k \\ R_1 &= 1 \\ f_2(z) &= \sum_{k=0}^{\infty} (-1)^k \frac{(z-i)^k}{(2+i)^{k+1}} \\ R_2 &= \sqrt{5} \end{aligned}$$

19.3 Laurent Series

19.3.1

$$\begin{aligned}f(z) &= \frac{\cos z}{z} \\&= \frac{1}{z} \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \right) \\&= \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} + \cdots\end{aligned}$$

19.3.3

$$\begin{aligned}f(z) &= e^{-1/z^2} \\&= 1 - \frac{1}{1!z^2} + \frac{1}{2!z^4} - \frac{1}{3!z^6} + \cdots\end{aligned}$$

19.3.5

$$\begin{aligned}f(z) &= \frac{e^z}{z-1} \\&= \frac{e^{1+z-1}}{z-1} \\&= \frac{e}{z-1} \left(1 + \frac{z-1}{1!} + \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} + \cdots \right) \\&= \frac{e}{z-1} + e + \frac{e(z-1)}{2!} + \frac{e(z-1)^2}{3!} + \cdots\end{aligned}$$

19.3.7

$$\begin{aligned}f(z) &= \frac{1}{z(z-3)} \\&= -\frac{1}{3z} \frac{1}{1-\frac{z}{3}} \\&= -\frac{1}{3z} \left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \cdots \right) \\&= -\frac{1}{3z} - \frac{1}{9} - \frac{z}{27} - \frac{z^2}{81} - \cdots\end{aligned}$$

19.3.9

$$\begin{aligned}
 f(z) &= \frac{1}{z(z-3)} \\
 &= \frac{1}{z-3} \frac{1}{3+z-3} \\
 &= \frac{1}{3(z-3)} \frac{1}{1+\frac{z-3}{3}} \\
 &= \frac{1}{3(z-3)} \left[1 - \frac{z-3}{3} + \left(\frac{z-3}{3} \right)^2 - \left(\frac{z-3}{3} \right)^3 + \dots \right] \\
 &= \frac{1}{3(z-3)} - \frac{1}{3^2} + \frac{z-3}{3^3} - \frac{(z-3)^2}{3^4} + \dots
 \end{aligned}$$

19.3.11

$$\begin{aligned}
 f(z) &= \frac{1}{z(z-3)} \\
 &= \frac{1}{3(z-3)} - \frac{1}{3z} \\
 &= \frac{1}{3(1+z-4)} - \frac{1}{3(4+z-4)} \\
 &= \frac{1}{3(z-4)} \frac{1}{1+\frac{1}{z-4}} - \frac{1}{12} \frac{1}{1+\frac{z-4}{4}} \\
 &= \frac{1}{3(z-4)} \left(1 - \frac{1}{z-4} + \frac{1}{(z-4)^2} - \frac{1}{(z-4)^3} + \dots \right) \\
 &\quad - \frac{1}{12} \left[1 - \frac{z-4}{4} + \left(\frac{z-4}{4} \right)^2 - \left(\frac{z-4}{4} \right)^3 + \dots \right] \\
 &= \dots - \frac{1}{3(z-4)^2} + \frac{1}{3(z-4)} - \frac{1}{12} + \frac{z-4}{3 \cdot 4^2} - \frac{(z-4)^2}{3 \cdot 4^3} + \dots
 \end{aligned}$$

19.3.13

$$\begin{aligned}
 f(z) &= \frac{1}{(z-1)(z-2)} \\
 &= \frac{1}{1-z} - \frac{1}{2-z} \\
 &= -\frac{1}{z} \frac{1}{1-\frac{1}{z}} - \frac{1}{2} \frac{1}{1-\frac{z}{2}} \\
 &= -\frac{1}{z} \left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots \right) \\
 &\quad - \frac{1}{2} \left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots \right) \\
 &= \cdots - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{z}{2^2} - \frac{z^2}{2^3} - \cdots
 \end{aligned}$$

19.3.15

$$\begin{aligned}
 f(z) &= \frac{1}{(z-1)(z-2)} \\
 &= -\frac{1}{z-1} + \frac{1}{z-2} \\
 &= -\frac{1}{z-1} + \frac{1}{-1+z-1} \\
 &= -\frac{1}{z-1} - \frac{1}{1-(z-1)} \\
 &= -\frac{1}{z-1} - 1 - (z-1) - (z-1)^2 - \cdots
 \end{aligned}$$

19.3.17

$$\begin{aligned}
 f(z) &= \frac{z}{(z+1)(z-2)} \\
 &= \frac{1}{3(z+1)} + \frac{2}{3(z-2)} \\
 &= \frac{1}{3(z+1)} + \frac{2}{3(-3+z+1)} \\
 &= \frac{1}{3(z+1)} - \frac{2}{9\left(1-\frac{z+1}{3}\right)} \\
 &= \frac{1}{3(z+1)} - \frac{2}{9} \left[1 + \frac{z+1}{3} + \left(\frac{z+1}{3}\right)^2 + \left(\frac{z+1}{3}\right)^3 + \cdots \right] \\
 &= \frac{1}{3(z+1)} - \frac{2}{3^2} - \frac{2(z+1)}{3^3} - \frac{2(z+1)^2}{3^4} - \cdots
 \end{aligned}$$

19.3.19

$$\begin{aligned}
f(z) &= \frac{z}{(z+1)(z-2)}, \quad 1 < |z| < 2 \\
&= \frac{2}{3(z-2)} + \frac{1}{3(z+1)} \\
&= -\frac{1}{3\left(1-\frac{z}{2}\right)} + \frac{1}{3z\left(1+\frac{1}{z}\right)} \\
&= -\frac{1}{3} \left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots \right] \\
&\quad + \frac{1}{3z} \left(1 - \frac{1}{z} + \frac{1}{z^2} - \cdots \right) \\
&= \cdots - \frac{1}{3z^2} + \frac{1}{3z} - \frac{1}{3} - \frac{z}{3 \cdot 2} - \frac{z^2}{3 \cdot 2^2} - \cdots
\end{aligned}$$

19.4 Zeroes and Poles**19.4.1**

$$\begin{aligned}
f(z) &= \frac{e^{2z} - 1}{z} \\
&= -\frac{1}{z} + \frac{1}{z} \left(1 + \frac{2z}{1!} + \frac{4z^2}{2!} + \cdots \right) \\
&= \frac{2}{1!} + \frac{4z}{2!} + \frac{8z^2}{3!} + \cdots \\
f(0) &= 2
\end{aligned}$$

19.4.3

$z = -2 + i$, order 2.

19.4.5

$$\begin{aligned}
f(z) &= z^4 + z^2 \\
&= z^2(z^2 + 1) \\
&= z^2(z+i)(z-i)
\end{aligned}$$

Zeroes are: $z = 0$ order 2, $z = \pm i$ order 1.

19.4.7

$$f(z) = e^{2z} - e^z$$

Zeroes are $2\pi ni$, $n \in \mathbb{R}$ order 1.

19.4.9

$$\begin{aligned}
f(z) &= z(1 - \cos z^2) \\
&= z \left[1 - \left(1 - \frac{z^4}{2!} + \frac{z^8}{4!} - \cdots \right) \right] \\
&= z \left(\frac{z^4}{2!} - \frac{z^8}{4!} + \frac{z^{12}}{6!} - \cdots \right) \\
&= z^5 \left(\frac{1}{2!} - \frac{z^4}{4!} + \frac{z^8}{6!} - \cdots \right)
\end{aligned}$$

Order 5.

19.4.11

$$\begin{aligned}
f(z) &= 1 - e^{z-1} \\
&= 1 - \left[1 + \frac{z-1}{1!} + \frac{(z-1)^2}{2!} + \cdots \right] \\
&= -\frac{z-1}{1!} - \frac{(z-1)^2}{2!} - \cdots \\
&= -(z-1) \left[1 + \frac{z-1}{2!} + \frac{(z-1)^2}{3!} + \cdots \right]
\end{aligned}$$

Order 1.

19.4.13

$$\begin{aligned}
f(z) &= \frac{3z-1}{z^2+2z+5} \\
&= \frac{3z-1}{(z+1+2i)(z+1-2i)}
\end{aligned}$$

Poles are $z = -1 \pm 2i$ both of order 1.

19.4.15

$$f(z) = \frac{1+4i}{(z+2)(z+i)^4}$$

Poles are $z = -2$ order 1, and $z = -i$ order 4.

19.4.17

$$\begin{aligned}
 f(z) &= \tan z \\
 &= \frac{\sin z}{\cos z}
 \end{aligned}$$

Poles are $z = \frac{\pi}{2} + n\pi$, $n \in \mathbb{R}$ order 1.

19.5 Residues and Residue Theorem**19.5.1**

$$\begin{aligned}
 f(z) &= \frac{2}{(z-1)(z+4)} \\
 &= \frac{2}{z-1} \frac{1}{5+z-1} \\
 &= \frac{2}{5(z-1)} \frac{1}{1+\frac{z-1}{5}} \\
 &= \frac{2}{5(z-1)} \left[1 - \frac{z-1}{5} + \left(\frac{z-1}{5} \right)^2 - \dots \right] \\
 &= \frac{2}{5(z-1)} - \frac{2}{5^2} + \frac{2(z-1)}{5^3} - \dots \\
 \text{Res}(f(z), 1) &= \frac{2}{5}
 \end{aligned}$$

19.5.3

$$\begin{aligned}
 f(z) &= \frac{4z-6}{z(2-z)} \\
 &= \frac{4z-6}{2z} \frac{1}{1-\frac{z}{2}} \\
 &= \left(2 - \frac{3}{z} \right) \left[1 + \frac{z}{2} + \left(\frac{z}{2} \right)^2 + \dots \right] \\
 \text{Res}(f(z), 0) &= -3
 \end{aligned}$$

19.5.5

$$\begin{aligned}
 f(z) &= e^{-2/z^2} \\
 &= 1 - \frac{2}{z^2} + \frac{4}{2! \cdot z^4} + \dots \\
 \text{Res}(f(z), 0) &= 0
 \end{aligned}$$

19.5.7

$$\begin{aligned}
f(z) &= \frac{z}{z^2 + 16} \\
&= \frac{z}{(z + 4i)(z - 4i)} \\
\text{Res}(f(z), -4i) &= \lim_{z \rightarrow -4i} (z + 4i) \frac{z}{(z + 4i)(z - 4i)} \\
&= \frac{-4i}{-8i} \\
&= \frac{1}{2} \\
\text{Res}(f(z), 4i) &= \frac{1}{2}
\end{aligned}$$

19.5.9

$$\begin{aligned}
f(z) &= \frac{1}{z^4 + z^3 - 2z^2} \\
&= \frac{1}{z^2(z^2 + z - 2)} \\
&= \frac{1}{z^2(z - 1)(z + 2)} \\
\text{Res}(f(z), 1) &= \frac{1}{3} \\
\text{Res}(f(z), -2) &= -\frac{1}{12} \\
\text{Res}(f(z), 0) &= \lim_{z \rightarrow 0} \frac{d}{dz} \frac{1}{(z - 1)(z + 2)} \\
&= -\frac{1}{4}
\end{aligned}$$

19.5.11

$$\begin{aligned}
f(z) &= \frac{5z^2 - 4z + 3}{(z + 1)(z + 2)(z + 3)} \\
\text{Res}(f(z), -1) &= 6 \\
\text{Res}(f(z), -2) &= -31 \\
\text{Res}(f(z), -3) &= 30
\end{aligned}$$

19.5.13

$$f(z) = \frac{\cos z}{z^2(z - \pi)^3}$$

$$\operatorname{Res}(f(z), 0) = -\frac{3}{\pi^4}$$

$$\operatorname{Res}(f(z), \pi) = \frac{1}{2} \left(\frac{1}{\pi^2} - \frac{6}{\pi^4} \right)$$

19.5.15

$$f(z) = \sec z$$

$$= \frac{1}{\cos z}$$

$$\operatorname{Res} \left(f(z), \frac{\pi}{2} + n\pi \right) = \frac{1}{-\sin \left(\frac{\pi}{2} + n\pi \right)}$$

$$= (-1)^{n+1}$$

19.5.17

(a) 0

(b)

$$\operatorname{Res}(f(z), 1) = \frac{1}{9}$$

$$\oint_C \frac{1}{(z-1)(z+2)^2} dz = \frac{2}{9}\pi i$$

(c)

$$\operatorname{Res}(f(z), -2) = -\frac{1}{9}$$

$$\oint_C \frac{1}{(z-1)(z+2)^2} dz = 0$$

19.5.19

(a)

$$\begin{aligned}
f(z) &= z^3 e^{-1/z^2} \\
&= z^3 \left(1 - \frac{1}{1! \cdot z^2} + \frac{1}{2! \cdot z^4} - \frac{1}{3! \cdot z^6} + \cdots \right) \\
&= z^3 - \frac{z}{1!} + \frac{1}{2! \cdot z} - \frac{1}{3! \cdot z^3} + \cdots \\
\text{Res}(f(z), 0) &= \frac{1}{2} \\
\oint_C f(z) dz &= \pi i
\end{aligned}$$

(b) πi

(c) 0

19.5.21

$$\begin{aligned}
f(z) &= \frac{1}{z^2 + 4z + 13} \\
&= \frac{1}{(z + 2 + 3i)(z + 2 - 3i)} \\
\text{Res}(f(z), -2 + 3i) &= \lim_{z \rightarrow -2 + 3i} (z + 2 - 3i) f(z) \\
&= \frac{1}{6i} \\
&= -\frac{1}{6}i \\
\oint_C f(z) dz &= \frac{\pi}{3}
\end{aligned}$$

19.5.23

$$\begin{aligned}
f(z) &= \frac{z}{z^4 - 1} \\
&= \frac{z}{(z^2 + 1)(z^2 - 1)} \\
&= \frac{z}{(z + i)(z - i)(z + 1)(z - 1)} \\
\oint_C f(z) dz &= 2\pi i \left(-\frac{1}{4} - \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \right) \\
&= 0
\end{aligned}$$

19.5.25

$$\begin{aligned}
 f(z) &= \frac{ze^z}{z^2 - 1} \\
 &= \frac{ze^z}{(z+1)(z-1)} \\
 \oint_C f(z) dz &= 2\pi i \left(\frac{1}{2e} + \frac{e}{2} \right) \\
 &= 2\pi i \frac{e + e^{-1}}{2} \\
 &= 2\pi i \cosh 1
 \end{aligned}$$

19.6 Evaluation of Real Integrals

19.6.1

$$\begin{aligned}
 \int_0^{2\pi} \frac{1}{1 + 0.5 \sin \theta} d\theta &= \oint_C \frac{1}{1 + \frac{1}{2} \frac{1}{2i}(z - z^{-1})} \frac{dz}{iz} \\
 &= \oint_C \frac{1}{iz + \frac{1}{4}z^2 - \frac{1}{4}} dz \\
 &= 4 \oint_C \frac{1}{z^2 + 4iz - 1} dz \\
 &= 4 \oint_C \frac{1}{[z + (2 + \sqrt{3})i][z + (2 - \sqrt{3})i]} dz \\
 &= 4 \cdot 2\pi i \operatorname{Res}(f(z), (-2 + \sqrt{3})i) \\
 &= \frac{4\pi}{\sqrt{3}}
 \end{aligned}$$

19.6.3

$$\begin{aligned}
 \int_0^{2\pi} \frac{\cos \theta}{3 + \sin \theta} d\theta &= \oint_C \frac{\frac{1}{2}(z + z^{-1})}{3 + \frac{1}{2i}(z - z^{-1})} \frac{dz}{iz} \\
 &= \oint_C \frac{z + z^{-1}}{z^2 + 6iz - 1} dz \\
 &= \oint_C \frac{z^2 + 1}{z(z^2 + 6iz - 1)} dz \\
 &= \oint_C \frac{z^2 + 1}{z[z + (3 + 2\sqrt{2})i][z + (3 - 2\sqrt{2})i]} dz \\
 &= 2\pi i [\operatorname{Res}(f(z), 0) + \operatorname{Res}(f(z), (-3 + 2\sqrt{2})i)] \\
 &= 2\pi i(-1 + 1) \\
 &= 0
 \end{aligned}$$

19.6.5

$$\begin{aligned}
 \int_0^\pi \frac{1}{2 - \cos \theta} d\theta &= \frac{1}{2} \int_{-\pi}^\pi \frac{1}{2 - \cos \theta} d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \frac{1}{2 + \cos t} dt \\
 &= \frac{1}{2} \oint_C \frac{1}{2 + \frac{1}{2}(z + z^{-1})} \frac{dz}{iz} \\
 &= \frac{1}{i} \oint_C \frac{1}{z^2 + 4z + 1} dz \\
 &= \frac{1}{i} \oint_C \frac{1}{(z + 2 + \sqrt{3})(z + 2 - \sqrt{3})} dz \\
 &= \frac{1}{i} 2\pi i \operatorname{Res}(f(z), -2 + \sqrt{3}) \\
 &= \frac{\pi}{\sqrt{3}}
 \end{aligned}$$

19.6.11

$$\begin{aligned}
 \int_{-\infty}^\infty \frac{1}{x^2 - 2x + 2} dx &= \oint_C \frac{1}{(z - 1 - i)(z - 1 + i)} dz \\
 &= 2\pi i \operatorname{Res}(f(z), 1 + i) \\
 &= \pi
 \end{aligned}$$

19.6.13

$$\begin{aligned}
 \int_{-\infty}^\infty \frac{1}{(x^2 + 4)^2} dx &= \oint_C \frac{1}{(z + 2i)^2(z - 2i)^2} dz \\
 &= 2\pi i \operatorname{Res}(f(z), 2i) \\
 &= 2\pi i \lim_{z \rightarrow 2i} \frac{d}{dz} \frac{1}{(z + 2i)^2} \\
 &= \frac{\pi}{16}
 \end{aligned}$$

19.6.15

$$\begin{aligned}
 \int_{-\infty}^\infty \frac{1}{(x^2 + 1)^3} dx &= \oint_C \frac{1}{(z + i)^3(z - i)^3} dz \\
 &= 2\pi i \operatorname{Res}(f(z), i) \\
 &= 2\pi i \frac{1}{2} \lim_{z \rightarrow i} \frac{d^2}{dz^2} \frac{1}{(z + i)^3} \\
 &= \frac{3\pi}{8}
 \end{aligned}$$

19.6.21

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} dx &= \operatorname{Re} \left(\oint_C \frac{1}{z^2 + 1} e^{iz} dz \right) \\
&= \operatorname{Re} \left(\oint_C \frac{1}{(z + i)(z - i)} e^{iz} dz \right) \\
&= \operatorname{Re}[2\pi i \operatorname{Res}(f(z), i)] \\
&= \frac{\pi}{e}
\end{aligned}$$

19.6.23

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 1} dx &= \operatorname{Im} \left(\oint_C \frac{z}{z^2 + 1} e^{iz} dz \right) \\
&= \operatorname{Im} \left(\frac{z}{(z + i)(z - i)} e^{iz} dz \right) \\
&= \operatorname{Im}[2\pi i \operatorname{Res}(f(z), i)] \\
&= \frac{\pi}{e}
\end{aligned}$$

19.6.31

$$\begin{aligned}
\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin x}{x} dx &= \operatorname{Im} \left(\oint_C \frac{e^{iz}}{z} dz \right) \\
&= \operatorname{Im}[\pi i \operatorname{Res}(f(z), 0)] \\
&= \pi
\end{aligned}$$

19.7 Chapter in Review**19.7.1**

True

19.7.3

$$\begin{aligned}
f(z) &= \frac{1}{z - 3} \\
&= -\frac{1}{3} \frac{1}{1 - \frac{z}{3}} \\
&= -\frac{1}{3} \left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \cdots \right) \\
&= -\frac{1}{3} - \frac{z}{9} - \frac{z^2}{27} - \frac{z^3}{81} - \cdots
\end{aligned}$$

False

19.7.5

$$\begin{aligned}f(z) &= e^{1/(z-1)} \\&= 1 + \frac{1}{1! \cdot (z-1)} + \frac{1}{2! \cdot (z-1)^2} + \frac{1}{3! \cdot (z-1)^3} + \cdots\end{aligned}$$

True

19.7.7

True

19.7.9

$$\begin{aligned}f(z) &= \cot \pi z \\&= \frac{\cos \pi z}{\sin \pi z} \\ \operatorname{Res}(f(z), 0) &= \frac{\cos 0}{\pi \cos 0} \\&= \frac{1}{\pi}\end{aligned}$$

19.7.11

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{\frac{(z-i)^{n+1}}{(2+i)^{n+2}}}{\frac{(z-i)^n}{(2+i)^{n+1}}} \right| &= \left| \frac{z-i}{2+i} \right| \\ \left| \frac{z-i}{2+i} \right| &= 1 \\ \frac{|z-i|}{\sqrt{5}} &= 1 \\ |z-i| &= \sqrt{5}\end{aligned}$$

19.7.13

$$\begin{aligned}
 f(z) &= e^z \cos z \\
 &= e^z \frac{e^{iz} + e^{-iz}}{2} \\
 &= \frac{e^{z(1+i)} + e^{z(1-i)}}{2} \\
 f^{(n)}(z) &= \frac{(1+i)^n e^{z(1+i)} + (1-i)^n e^{z(1-i)}}{2} \\
 f^{(n)}(0) &= \frac{(1+i)^n + (1-i)^n}{2} \\
 &= (\sqrt{2})^n \cos \frac{n\pi}{4} \\
 f(z) &= 1 + \sum_{k=1}^{\infty} \frac{(\sqrt{2})^k \cos(k\pi/4)}{k!} z^k
 \end{aligned}$$

19.7.15

$$\begin{aligned}
 f(z) &= \frac{1 - e^{iz}}{z^4} \\
 &= \frac{1}{z^4} - \frac{1}{z^4} \left(1 + \frac{iz}{1!} - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \cdots \right) \\
 &= -\frac{i}{1! \cdot z^3} + \frac{1}{2! \cdot z^2} + \frac{i}{3! \cdot z} - \frac{1}{4!} - \frac{iz}{5!} + \cdots
 \end{aligned}$$

19.7.17

$$\begin{aligned}
 f(z) &= (z-i)^2 \sin \frac{1}{z-i} \\
 &= (z-i)^2 \left[\frac{1}{z-i} - \frac{1}{3! \cdot (z-i)^3} + \frac{1}{5! \cdot (z-i)^5} - \cdots \right] \\
 &= (z-i) - \frac{1}{3! \cdot (z-i)} + \frac{1}{5! \cdot (z-i)^3} - \cdots
 \end{aligned}$$

19.7.19

(a)

$$\begin{aligned}
 f(z) &= \frac{2}{z^2 - 4z + 3} \\
 &= \frac{1}{1-z} - \frac{1}{3} \frac{1}{1-\frac{z}{3}} \\
 &= (1+z+z^2+\cdots) - \frac{1}{3} \left(1 + \frac{z}{3} + \frac{z^2}{9} + \cdots\right) \\
 &= \frac{2}{3} + \frac{8}{9}z + \frac{26}{27}z^2 + \cdots
 \end{aligned}$$

(b)

$$\begin{aligned}
 f(z) &= \frac{2}{z^2 - 4z + 3} \\
 &= -\frac{1}{3} \frac{1}{1-\frac{z}{3}} - \frac{1}{z} \frac{1}{1-\frac{1}{z}} \\
 &= -\frac{1}{3} \left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \cdots\right) - \frac{1}{z} \left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) \\
 &= \cdots - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{3} - \frac{z}{3^2} - \frac{z^2}{3^3} - \cdots
 \end{aligned}$$

(c)

$$\begin{aligned}
 f(z) &= \frac{2}{z^2 - 4z + 3} \\
 &= \frac{1}{z-3} - \frac{1}{z-1} \\
 &= \frac{1}{z} \frac{1}{1-\frac{3}{z}} - \frac{1}{z} \frac{1}{1-\frac{1}{z}} \\
 &= \frac{1}{z} \left(1 + \frac{3}{z} + \frac{3^2}{z^2} + \cdots\right) - \frac{1}{z} \left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) \\
 &= \frac{1}{z} \left(\frac{2}{z} + \frac{8}{z^2} + \frac{26}{z^3} + \cdots\right) \\
 &= \frac{2}{z^2} + \frac{8}{z^3} + \frac{26}{z^4} + \cdots
 \end{aligned}$$

(d)

$$\begin{aligned} f(z) &= \frac{2}{z^2 - 4z + 3} \\ &= \frac{2}{(z-1)(z-3)} \\ &= \frac{2}{(z-1)(-2+z-1)} \\ &= -\frac{1}{z-1} \frac{1}{1-\frac{z-1}{2}} \\ &= -\frac{1}{z-1} \left[1 + \frac{z-1}{2} + \frac{(z-1)^2}{2^2} + \dots \right] \\ &= -\frac{1}{z-1} - \frac{1}{2} - \frac{z-1}{2^2} - \dots \end{aligned}$$

19.7.21

$$\begin{aligned} \oint_C \frac{2z+5}{z(z+2)(z-1)^4} dz &= 2\pi i [\text{Res}(f(z), 0) + \text{Res}(f(z), -2)] \\ &= 2\pi i \left(\frac{5}{2} - \frac{1}{162} \right) \\ &= \frac{404}{81} \pi i \end{aligned}$$

19.7.23

$$\begin{aligned} 2 \sin z - 1 &= 0 \\ \sin z &= \frac{1}{2} \\ z &= \frac{\pi}{6} \\ \oint_C \frac{1}{2 \sin z - 1} dz &= 2\pi i \text{Res} \left(f(z), \frac{\pi}{6} \right) \\ &= 2\pi i \frac{1}{2 \cos \frac{\pi}{6}} \\ &= \frac{2}{\sqrt{3}} \pi i \end{aligned}$$

19.7.25

$$\begin{aligned}
\oint_C \frac{e^{2z}}{z^4 + 2z^3 + 2z^2} dz &= \oint_C \frac{e^{2z}}{z^2(z^2 + 2z + 2)} dz \\
&= \oint_C \frac{e^{2z}}{z^2(z + 1 + i)(z + 1 - i)} dz \\
&= 2\pi i [\text{Res}(f(z), 0) + \text{Res}(f(z), -1 - i) + \text{Res}(f(z), -1 + i)] \\
&= 2\pi i \left(\frac{1}{2} + \frac{1}{4}e^{-2(1+i)} + \frac{1}{4}e^{-2(1-i)} \right) \\
&= (\pi + \pi e^{-2} \cos 2)i
\end{aligned}$$

19.7.31

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 2x + 2)(x^2 + 1)^2} dx &= \oint_C \frac{z^2}{(z + 1 + i)(z + 1 - i)(z + i)^2(z - i)^2} dz \\
&= 2\pi i [\text{Res}(f(z), -1 + i) + \text{Res}(f(z), i)] \\
&= 2\pi i \left(\frac{3}{25} - \frac{4}{25}i - \frac{3}{25} + \frac{9}{100}i \right) \\
&= \frac{7}{50}\pi
\end{aligned}$$

20 Conformal Mappings**20.1 Complex Functions as Mappings****20.1.1**

$$\begin{aligned}
z &= x + ix \\
f(z) &= \frac{1}{z} \\
&= \frac{1}{x + ix} \\
&= \frac{x - ix}{2x^2} \\
&= \frac{1 - i}{2x} \\
&= \frac{1}{2x}(1 - i) \\
v &= -u
\end{aligned}$$

20.1.3

$$\begin{aligned}
z &= x + i\frac{1}{x} \\
f(z) &= z^2 \\
&= x^2 - \frac{1}{x^2} + 2i \\
v &= 2
\end{aligned}$$

20.1.5

$$\begin{aligned}
|z| &= 1, y > 0 \\
f(z) &= \operatorname{Ln} z \\
&= \log_e 1 + i\theta \\
v &= t, 0 < t < \pi
\end{aligned}$$

20.1.7

$$\theta = \frac{\theta_0}{2}$$

20.1.11

$f(z) = z^{-1}$ so $w = r^{-1}e^{-i\theta}$ which is the fourth quadrant.

20.1.13

The segment $\pi/4 \leq \theta \leq \pi/2$.

20.1.15

The circle $|w - 4i| = 1$.

20.1.17

The strip $-1 \leq u \leq 0$.

20.1.19

Wedge $0 \leq \operatorname{Arg} w \leq 3\pi/4$.

20.1.21

$$w = -iz - 1$$

20.1.23

$$w = 2z - 2$$

20.1.25

$$w = (e^{-\pi i/4} z)^4 = e^{-\pi i} z^4 = -z^4$$

20.1.27

$$w = (e^z)^{3/2} = e^{3z/2}$$

20.1.29

$$w = -z + i$$

20.2 Conformal Mappings

20.2.1

$$\begin{aligned} f(z) &= z^3 - 3z + 1 \\ f'(z) &= 3z^2 - 3 \end{aligned}$$

Conformal except at $z = \pm 1$.

20.2.3

$$\begin{aligned} f(z) &= z + e^z + 1 \\ f'(z) &= 1 + e^z \end{aligned}$$

Conformal except at $z = (\pi + 2\pi n)i$, $n \in \mathbb{R}$.

20.2.7

$$\begin{aligned} w &= \cos z \\ &= \sin(\pi/2 - z) \end{aligned}$$

The image is the same as in example 2 except rotated 180° .

A horizontal line $z(t) = t + ib$ is mapped onto either the upper or lower portion of the ellipse

$$\frac{u^2}{\cosh^2 b} + \frac{v^2}{\sinh^2 b} = 1$$

according to whether $b < 0$ or $b > 0$, respectively.

20.2.9

The image is the wedge $0 \leq \text{Arg } z \leq \frac{\pi}{4}$. The line segment $[-\pi/2, 0]$ maps to the line from the origin to $(\sqrt{2}/2, \sqrt{2}/2)$ and the line segment $[0, \pi/2]$ maps to the line from the origin to $(1, 0)$.

20.2.11

$$w = \cos\left(\frac{\pi z}{2}\right)$$

BA maps to the range $[1, \infty)$ on the u -axis.

20.2.13

$$w = \left(\frac{1+z}{1-z}\right)^{1/2}$$

AB maps to the line from $e^{\pi i/4}$ out to infinite on the same ray from the origin.

20.2.15

$$w = \left(\frac{e^{\pi/z} + e^{-\pi/z}}{e^{\pi/z} - e^{-\pi/z}}\right)^{1/2}$$

BC is mapped to a vertical line on the v -axis from the origin to i .

20.2.17

$$w = \sin\left(-i \text{Ln } z - \frac{\pi}{2}\right)$$

AB maps to the line $(-\infty, -1]$ on the u -axis.

20.2.19

$$\begin{aligned} w &= z^4 \\ U &= \frac{1}{\pi} \text{Arg } w \\ u &= \frac{1}{\pi} \text{Arg}(z^4) \end{aligned}$$

20.2.21

$$\begin{aligned} w &= i \frac{1-z}{1+z} \\ U &= \frac{1}{\pi} \text{Arg } w \\ u &= \frac{1}{\pi} \text{Arg}\left(i \frac{1-z}{1+z}\right) \end{aligned}$$

20.2.23

$$w = z^2$$

$$U = \frac{1}{\pi} [\text{Arg}(w - 1) - \text{Arg}(w + 1)]$$

$$u = \frac{1}{\pi} [\text{Arg}(z^2 - 1) - \text{Arg}(z^2 + 1)]$$

20.2.25

$$w = e^{\pi z}$$

$$U = \frac{10}{\pi} [\text{Arg}(w - 1) - \text{Arg}(w + 1)]$$

$$u = \frac{10}{\pi} [\text{Arg}(e^{\pi z} - 1) - \text{Arg}(e^{\pi z} + 1)]$$

20.3 Linear Fractional Transformations**20.3.1**

(a) $\infty, i, 0$

(b) $T(i) = 1, |z| = 1; T(2) = \frac{i}{2}, T(1+i) = \frac{1}{2} + \frac{i}{2}, v = \frac{1}{2}$

(c) $|w| \geq 1$

20.3.3

(a) $-1, \infty, 1$

(b) $T(i) = -i, T(-1) = 0, u = 0; T(2) = 3, T(1+i) = 1 - 2i, |w - 1| = 2$

(c) $u \leq 0$

20.3.5

$$\begin{aligned}
 S^{-1} &= \text{adj} \begin{pmatrix} i & 1 \\ 1 & -1 \end{pmatrix} \\
 &= \begin{pmatrix} -1 & -1 \\ -1 & i \end{pmatrix} \\
 S^{-1}(w) &= \frac{-w-1}{-w+i} \\
 T &= \begin{pmatrix} 1 & 0 \\ i & -1 \end{pmatrix} \\
 S^{-1}T &= \begin{pmatrix} -1 & -1 \\ -1 & i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i & -1 \end{pmatrix} \\
 &= \begin{pmatrix} -1-i & 1 \\ -2 & -i \end{pmatrix} \\
 S^{-1}(T(z)) &= \frac{(-1-i)z+1}{-2z-i}
 \end{aligned}$$

20.3.7

$$\begin{aligned}
 S^{-1} &= \text{adj} \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \\
 &= \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \\
 S^{-1}(w) &= \frac{-w+2}{-w+1} \\
 T &= \begin{pmatrix} 2 & -3 \\ 1 & -3 \end{pmatrix} \\
 S^{-1}T &= \begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix} \\
 S^{-1}(T(z)) &= \frac{3}{z}
 \end{aligned}$$

20.3.9

$$\begin{aligned}
 \frac{w-0}{w-\infty} \frac{1-\infty}{1-0} &= \frac{z+1}{z-2} \frac{0-2}{0+1} \\
 w &= -2 \frac{z+1}{z-2}
 \end{aligned}$$

20.3.11

$$\frac{w-0}{w-2} \frac{i-2}{i-0} = \frac{z-0}{z-\infty} \frac{1-\infty}{1-0}$$

$$\frac{w}{w-2} \frac{i-2}{i} = z$$

$$w = \frac{2z}{z-1-2i}$$

20.3.17

$$U = \frac{1}{\log_e 2} \log_e \left| \frac{z+2}{z-1} \right|$$

20.3.19

1, i , $-i$ to 0, 1, -1

$$\frac{w-0}{w+1} \frac{1+1}{1-0} = \frac{z-1}{z+i} \frac{i+i}{i-1}$$

$$\frac{2w}{w+1} = (1-i) \frac{z-1}{z+i}$$

$$w = i \frac{1-z}{1+z}$$

20.4 Schwarz-Christoffel Transformations**20.4.1**

First quadrant

20.4.3

$$\arg f'(t) = \text{Arg } 1 - \frac{1}{2} \text{Arg}(z+1) + \frac{1}{2} \text{Arg}(z-1)$$

$$= -\frac{1}{2} \text{Arg}(z+1) + \frac{1}{2} \text{Arg}(z-1)$$

$$= \begin{cases} 0 & z < -1 \\ \frac{\pi}{2} & -1 < z < 1 \\ 0 & z > 1 \end{cases}$$

$(-\infty, -1]$ maps to $(-\infty, 0]$, $[-1, 1]$ maps to $[0, ai]$, and $[1, \infty)$ maps to $[ai, \infty + ai)$.

20.4.5

$$f'(t) = A(z+1)^{-1/2} z^{-1/2} (z-1)^{-1/2}$$

20.4.7

$$f'(t) = A(z+1)^{-1/3}z^{-1/3}$$

20.4.9

$$\begin{aligned} f'(z) &= A(z+1)^{-1/2}(z-1)^{-1/2} \\ &= \frac{A}{\sqrt{(z+1)(z-1)}} \\ &= \frac{A}{\sqrt{z^2-1}} \\ f(z) &= \int \frac{A}{\sqrt{z^2-1}} dz \\ &= A \operatorname{arccosh} z + B \\ \pi i &= A \operatorname{arccosh}(-1) + B \\ &= \pi i A + B \\ 0 &= A \operatorname{arccosh}(1) + B \\ &= B \\ A &= 1 \\ f(z) &= \operatorname{arccosh} z \end{aligned}$$

20.4.11

$$\begin{aligned} f'(z) &= (z+1)^{(\alpha_1/\pi)-1} z^{(\alpha_2/\pi)-1} (z-1)^{(\alpha_1/\pi)-1} \\ \lim_{z_1 \rightarrow \infty} f'(z) &= z^{-1} \\ f(z) &= \operatorname{Ln} z + B \\ \pi i &= \operatorname{Ln}(-1) + B \\ &= \pi i + B \\ B &= 0 \\ f(z) &= \operatorname{Ln} z \end{aligned}$$

20.4.13

$$\begin{aligned}
f'(z) &= A(z+1)^{(\alpha_1/\pi)-1} z^{(\alpha_2/\pi)-1} (z-1)^{(\alpha_1/\pi)-1} \\
\lim_{u_1 \rightarrow 0} f'(z) &= A(z+1)^{-1/2} z(z-1)^{-1/2} \\
&= \frac{Az}{\sqrt{z^2-1}} \\
f(z) &= A\sqrt{z^2-1} + B \\
0 &= B \\
ai &= Ai \\
A &= a \\
f(z) &= a\sqrt{z^2-1}
\end{aligned}$$

20.5 Poisson Integral Formulas

20.5.1

$$\begin{aligned}
u(x, y) &= -\frac{1}{\pi} [\text{Arg } z - \text{Arg}(z+1)] + \frac{1}{\pi} [\text{Arg}(z-1) - \text{Arg } z] \\
&= \frac{1}{\pi} [\text{Arg}(z-1) + \text{Arg}(z+1) - 2 \text{Arg } z] \\
&= \frac{1}{\pi} \text{Arg} \left[\frac{(z-1)(z+1)}{z^2} \right] \\
&= \frac{1}{\pi} \text{Arg} \left(\frac{z^2-1}{z^2} \right) \\
&= \frac{1}{\pi} \text{Arg} \left(1 - \frac{1}{z^2} \right)
\end{aligned}$$

20.5.3

$$\begin{aligned}
u(x, y) &= -\frac{1}{\pi} [\text{Arg}(z+1) - \text{Arg}(z+2)] + \frac{1}{\pi} [\text{Arg } z - \text{Arg}(z+1)] \\
&\quad + \frac{5}{\pi} [\pi - \text{Arg}(z-1)] \\
&= 5 + \frac{1}{\pi} [\text{Arg } z + \text{Arg}(z+2) - 2 \text{Arg}(z+1) - \text{Arg}(z-1)] \\
&= 5 + \frac{1}{\pi} \text{Arg} \left[\frac{z(z+2)}{(z-1)(z+1)^2} \right]
\end{aligned}$$

20.5.5

$$\begin{aligned}
 u(x, 0) &= \begin{cases} x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \\
 u(x, y) &= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{u(t, 0)}{(x-t)^2 + y^2} dt \\
 &= \frac{y}{\pi} \int_0^1 \frac{t^2}{(x-t)^2 + y^2} dt \\
 &= \frac{y}{\pi} \left\{ 1 + \frac{y^2 - x^2}{y} \left[\arctan \left(\frac{x-1}{y} \right) - \arctan \frac{x}{y} \right] + x \log_e \left[\frac{(x-1)^2 + y^2}{x^2 + y^2} \right] \right\}
 \end{aligned}$$

20.5.7

$$\begin{aligned}
 f(z) &= z^2 \\
 U(w) &= \frac{5}{\pi} \operatorname{Arg}(w+1) + \frac{1}{\pi} [\operatorname{Arg}(w-1) - \operatorname{Arg} w] \\
 &= \frac{1}{\pi} [5 \operatorname{Arg}(w+1) + \operatorname{Arg}(w-1) - \operatorname{Arg} w] \\
 &= \frac{1}{\pi} \operatorname{Arg} \left[\frac{(w+1)^5(w-1)}{w} \right] \\
 u(x, y) &= U(f(z)) \\
 &= \frac{1}{\pi} \operatorname{Arg} \left[\frac{(z^2+1)^5(z^2-1)}{z^2} \right]
 \end{aligned}$$