Introduction to Electrodynamics by David J. Griffiths Problems

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Contents

2 Electrostatics

2.1

(a) **0**

(b) The same as if only the opposite charge were present — all others are cancelled out.

2.2

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0} 2 \frac{q}{\mathscr{r}^2} \cos\theta \hat{\mathbf{x}} \\ &= \frac{1}{4\pi\epsilon_0} \frac{dq}{[(d/2)^2 + z^2]^{3/2}} \hat{\mathbf{x}}\end{aligned}$$

2.3

$$
\begin{aligned}
\mathbf{r} &= z\hat{\mathbf{z}} \\
\mathbf{r}' &= x\hat{\mathbf{x}} \\
\boldsymbol{\mathscr{r}} &= z\hat{\mathbf{z}} - x\hat{\mathbf{x}} \\
\mathscr{r} &= \sqrt{x^2 + z^2} \\
\hat{\boldsymbol{\mathscr{r}}} &= \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{\sqrt{x^2 + z^2}} \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_0^L \frac{\lambda}{x^2 + z^2} \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{\sqrt{x^2 + z^2}}\, dx \\
&= \frac{1}{4\pi\epsilon_0}\lambda \left(z\hat{\mathbf{z}} \int_0^L \frac{1}{(x^2 + z^2)^{3/2}}\, dx - \hat{\mathbf{x}} \int_0^L \frac{x}{(x^2 + z^2)}\, dx \right) \\
&= \frac{1}{4\pi\epsilon_0}\lambda \left[\frac{L}{z\sqrt{L^2 + z^2}}\hat{\mathbf{z}} - \left(\frac{1}{z} - \frac{1}{\sqrt{L^2 + z^2}} \right) \hat{\mathbf{x}} \right] \\
&= \frac{1}{4\pi\epsilon_0} \frac{\lambda}{z} \left[\left(-1 + \frac{z}{\sqrt{L^2 + z^2}} \right) \hat{\mathbf{x}} + \frac{L}{\sqrt{L^2 + z^2}}\hat{\mathbf{z}} \right]
\end{aligned}
$$

2.4

The electric field a distance z above the midpoint of a line segment of length $2L$ and uniform line charge λ is

$$
\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z\sqrt{z^2 + L^2}}\hat{\mathbf{z}}.
$$

Applying this to the four sides of the square, the horizontal components of opposite sides cancel leaving only the vertical component.

$$
\begin{aligned}
\cos\theta &= \frac{z}{\mathscr{r}} \\
&= \frac{z}{\sqrt{(a/2)^2 + z^2}} \\
\mathbf{E} &= 4 \left(\frac{1}{4\pi\epsilon_0} \frac{\lambda a}{\sqrt{(a/2)^2 + z^2}\sqrt{(a/2)^2 + (a/2)^2 + z^2}}\hat{\mathbf{z}} \right) \cos\theta \\
&= \frac{1}{4\pi\epsilon_0} \frac{4a\lambda z}{[(a/2)^2 + z^2]\sqrt{(a^2/2) + z^2}}\hat{\mathbf{z}}
\end{aligned}
$$

2.5

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\frac{\lambda r}{r^2+z^2}\cos\alpha\, d\theta\,\hat{\mathbf{z}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{2\pi\lambda r z}{(r^2+z^2)^{3/2}}\hat{\mathbf{z}}\end{aligned}$$

2.6

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int\frac{dq}{\eta^2}\cos\theta\hat{\mathbf{z}} \\ &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^R\frac{\sigma}{r^2+z^2}\frac{z}{\sqrt{r^2+z^2}}r\,dr\,d\theta\hat{\mathbf{z}} \\ &= \frac{1}{4\pi\epsilon_0}2\pi\sigma z\int_0^R\frac{r}{(r^2+z^2)^{3/2}}\,dr\,\hat{\mathbf{z}} \\ &= \frac{1}{4\pi\epsilon_0}2\pi\sigma z\left(\frac{1}{z}-\frac{1}{\sqrt{R^2+z^2}}\right)\hat{\mathbf{z}}\end{aligned}$$

When $R \to \infty$

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}.$$

2.7

$$\mathbf{E} = \begin{cases}\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}} & z > R \\ \mathbf{0} & z < R\end{cases}$$

2.8

$$\mathbf{E} = \begin{cases}\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}} & z > R \\ \frac{1}{4\pi\epsilon_0}\frac{qz}{R^3}\hat{\mathbf{z}} & z < R\end{cases}$$

2.9

(a)

$$\begin{aligned}\rho &= \epsilon_0\nabla\cdot\mathbf{E} \\ &= \epsilon_0\frac{1}{r^2}\frac{\partial}{\partial r}(kr^5) \\ &= 5\epsilon_0 k r^2\end{aligned}$$

(b)

$$
\begin{aligned}
Q_{\text{enc}} &= \epsilon_0 \oint \mathbf{E} \cdot d\mathbf{a} \\
&= \epsilon_0 \int_0^{2\pi} \int_0^{\pi} kR^3 R \, d\theta R \sin\theta \, d\phi \\
&= 2\pi\epsilon_0 k R^5 [-\cos\theta]_0^{\pi} \\
&= 4\pi\epsilon_0 k R^5 \\
Q_{\text{enc}} &= \int_V \rho \, d\tau \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^{R} 5\epsilon_0 k r^2 \, dr r \, d\theta r \sin\theta \, d\phi \\
&= 10\pi\epsilon_0 k \int_0^{\pi} \int_0^{R} r^4 \sin\theta \, dr \, d\theta \\
&= 2\pi\epsilon_0 k R^5 [-\cos\theta]_0^{\pi} \\
&= 4\pi\epsilon_0 k R^5
\end{aligned}
$$

2.10

If the charge was surrounded by 8 such cubes the total flux through all the cubes would be q/ϵ_0. There are 24 outside faces to the larger cube, so the total flux through the shaded face is $q/(24\epsilon_0)$.

2.11

$$
\begin{aligned}
\int \mathbf{E}_{\text{inside}} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
&= 0 \\
\mathbf{E}_{\text{inside}} &= \mathbf{0} \\
\int \mathbf{E}_{\text{outside}} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E_{\text{outside}} &= \frac{4\pi R^2 \sigma}{\epsilon_0} \\
\mathbf{E}_{\text{outside}} &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}
\end{aligned}
$$

2.12

$$\begin{aligned}
\int \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E &= \frac{\frac{4}{3}\pi r^3 \rho}{\epsilon_0} \\
\mathbf{E} &= \frac{r\rho}{3\epsilon_0}\hat{\mathbf{r}}
\end{aligned}$$

2.13

$$\begin{aligned}
\int \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
2\pi s l E &= \frac{l\lambda}{\epsilon_0} \\
\mathbf{E} &= \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{\mathbf{s}}
\end{aligned}$$

2.14

$$\begin{aligned}
Q_{\text{enc}} &= \int_V \rho \, d\tau \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^{r} k r'^3 \sin\theta \, dr' \, d\theta \, d\phi \\
&= 2\pi k \int_0^{\pi} \left[\frac{1}{4} r'^4 \sin\theta \right]_0^r d\theta \\
&= \frac{1}{2}\pi k r^4 [-\cos\theta]_0^{\pi} \\
&= \pi k r^4 \\
\int \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E &= \frac{\pi k r^4}{\epsilon_0} \\
\mathbf{E} &= \frac{k r^2}{4\epsilon_0}\hat{\mathbf{r}}
\end{aligned}$$

2.15

(a) $\mathbf{E} = \mathbf{0}$

(b)

$$\begin{aligned}
Q_{\text{enc}} &= \int_0^{2\pi}\int_0^{\pi}\int_a^{r} k\sin\theta\,dr'\,d\theta\,d\phi \\
&= 4\pi k(r-a) \\
4\pi r^2 E &= \frac{4\pi k(r-a)}{\epsilon_0} \\
\mathbf{E} &= \frac{k(r-a)}{\epsilon_0 r^2}\hat{\mathbf{r}}
\end{aligned}$$

(c) $\mathbf{E} = \frac{k(b-a)}{\epsilon_0 r^2}\hat{\mathbf{r}}$

2.16

(a)

$$\begin{aligned}
Q_{\text{enc}} &= \pi s^2 l\rho \\
2\pi s l E &= \frac{\pi s^2 l\rho}{\epsilon_0} \\
\mathbf{E} &= \frac{s\rho}{2\epsilon_0}\hat{\mathbf{s}}
\end{aligned}$$

(b)

$$\mathbf{E} = \frac{a^2\rho}{2\epsilon_0 s}\hat{\mathbf{s}}$$

(c)

$$\mathbf{E} = \mathbf{0}$$

2.17

$$\begin{aligned}
2AE_{\text{inside}} &= \frac{2Ay\rho}{\epsilon_0} \\
\mathbf{E}_{\text{inside}} &= \frac{y\rho}{\epsilon_0} \\
\mathbf{E} &= \begin{cases} \frac{d\rho}{\epsilon_0} & d < y \\ \frac{y\rho}{\epsilon_0} & 0 < y < d \\ -\frac{y\rho}{\epsilon_0} & -d < y < 0 \\ -\frac{d\rho}{\epsilon_0} & y < -d \end{cases}
\end{aligned}$$

2.18

The electric field inside a uniformly charged solid sphere is

$$\mathbf{E} = \frac{r\rho}{3\epsilon_0}\hat{\mathbf{r}}.$$

$$\begin{aligned}
\mathbf{d} &= \mathbf{r}_1 - \mathbf{r}_2 \\
\mathbf{E} &= \frac{r_1\rho}{3\epsilon_0}\hat{\mathbf{r}}_1 - \frac{r_2\rho}{3\epsilon_0}\hat{\mathbf{r}}_2 \\
&= \frac{\rho}{3\epsilon_0}(\mathbf{r}_1 - \mathbf{r}_2) \\
&= \frac{\rho}{3\epsilon_0}\mathbf{d}
\end{aligned}$$

2.20

a is impossible because its curl is nonzero.

$$\begin{aligned}
V &= -\int_0^y 2kxy'\,dy' - \int_0^z 2kyz'\,dz \\
&= -2kx\left[\frac{1}{2}y'^2\right]_0^y - 2ky\left[\frac{1}{2}z'^2\right]_0^z \\
&= -k(xy^2 + yz^2) \\
-\nabla V &= k[y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}] \\
&= \mathbf{E}
\end{aligned}$$

2.21

$$
\begin{aligned}
\mathbf{E} &= \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} & r > R \\ \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3} & r < R \end{cases} \\
V_{\text{outside}}(r) &= -\int_\infty^r \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\,dr' \\
&= -\frac{1}{4\pi\epsilon_0} q \left[-\frac{1}{r'}\right]_\infty^r \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{r} \\
-\nabla V_{\text{outside}} &= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \\
&= \mathbf{E}_{\text{outside}} \\
V_{\text{inside}}(r) &= -\left(\int_\infty^R \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\,dr' + \int_R^r \frac{1}{4\pi\epsilon_0}\frac{qr'}{R^3}\,dr'\right) \\
&= -\left(-\frac{1}{4\pi\epsilon_0}\frac{q}{R} + \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\left[\frac{1}{2}r'^2\right]_R^r\right) \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left[3 - \left(\frac{r}{R}\right)^2\right] \\
-\nabla V_{\text{inside}} &= \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3}\hat{\mathbf{r}} \\
&= \mathbf{E}_{\text{inside}}
\end{aligned}
$$

2.22

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{\mathbf{s}} \\
V &= -\int_O^s \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s'}\,ds' \\
&= -\frac{1}{2\pi\epsilon_0}\lambda \ln\frac{s}{O} \\
-\nabla V &= \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{\mathbf{s}}
\end{aligned}
$$

2.23

$$\mathbf{E} = \begin{cases} \mathbf{0} & r < a \\ \frac{k(r-a)}{\epsilon_0 r^2}\hat{\mathbf{r}} & a < r < b \\ \frac{k(b-a)}{\epsilon_0 r^2}\hat{\mathbf{r}} & b < r \end{cases}$$

$$\begin{aligned}
V(0) &= -\int_\infty^0 E\,dr \\
&= -\left(\int_\infty^b \frac{k(b-a)}{\epsilon_0 r^2}\,dr + \int_b^a \frac{k(r-a)}{\epsilon_0 r^2}\,dr\right) \\
&= -\left(\frac{k(b-a)}{\epsilon_0}\left[-\frac{1}{r}\right]_\infty^b + \frac{k}{\epsilon_0}\left[\ln r + \frac{a}{r}\right]_b^a\right) \\
&= -\left[-\frac{k(b-a)}{\epsilon_0 b} + \frac{k}{\epsilon_0}\left(\ln a + 1 - \ln b - \frac{a}{b}\right)\right] \\
&= -\frac{k}{\epsilon_0}\left(-1 + \frac{a}{b} + \ln\frac{a}{b} + 1 - \frac{a}{b}\right) \\
&= \frac{k}{\epsilon_0}\ln\frac{b}{a}
\end{aligned}$$

2.24

$$\begin{aligned}
V(b) - V(0) &= -\int_0^b E\,dr \\
&= -\left(\int_0^a \frac{s\rho}{2\epsilon_0}\,ds + \int_a^b \frac{a^2\rho}{2\epsilon_0 s}\,ds\right) \\
&= -\left(\frac{\rho}{2\epsilon_0}\left[\frac{1}{2}s^2\right]_0^a + \frac{a^2\rho}{2\epsilon_0}\ln\frac{b}{a}\right) \\
&= -\left(\frac{a^2\rho}{4\epsilon_0} + \frac{a^2\rho}{2\epsilon_0}\ln\frac{b}{a}\right) \\
&= -\frac{a^2\rho}{4\epsilon_0}\left(1 + 2\ln\frac{a}{b}\right)
\end{aligned}$$

2.25

(a)

$$V = \frac{1}{4\pi\epsilon_0}\frac{2q}{\sqrt{(d/2)^2 + z^2}}$$

(b)

$$
\begin{aligned}
V &= \frac{1}{4\pi\epsilon_0}\int_{-L}^{L}\frac{\lambda}{\sqrt{x^2+z^2}}\,dx \\
&= \frac{1}{4\pi\epsilon_0}\lambda\ln\left(1+\frac{2L(L+\sqrt{L^2+z^2})}{z^2}\right)
\end{aligned}
$$

(c)

$$
\begin{aligned}
V &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{R}\frac{\sigma}{\sqrt{r^2+z^2}}r\,dr\,d\theta \\
&= \frac{1}{4\pi\epsilon_0}2\pi\sigma(\sqrt{R^2+z^2}-z)
\end{aligned}
$$

2.26

$$
\begin{aligned}
V_{\text{bottom}} &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{h}\frac{\sqrt{2}\sigma z}{\sqrt{2}z}\,d\phi\,dz \\
&= \frac{\sigma h}{2\epsilon_0} \\
V_{\text{top}} &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{h}\frac{\sqrt{2}\sigma z}{\sqrt{z^2+(h-z)^2}}\,d\phi\,dz \\
&= \frac{\sqrt{2}\sigma}{2\epsilon_0}\int_0^{h}\frac{z}{\sqrt{z^2+(h-z)^2}}\,dz \\
&= \frac{\sigma h}{4\epsilon_0}\ln(3+2\sqrt{2}) \\
V_{\text{bottom}}-V_{\text{top}} &= \frac{\sigma h}{2\epsilon_0}\left[1-\frac{1}{2}\ln(3+2\sqrt{2})\right]
\end{aligned}
$$

2.28

$$
\begin{aligned}
V(r) &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{\rho r'^2\sin\theta}{\sqrt{r^2+r'^2-2rr'\cos\theta}}\,dr'\,d\theta\,d\phi \\
&= \frac{\rho}{2\epsilon_0}\int_0^{\pi}\int_0^{R}\frac{r'^2\sin\theta}{\sqrt{r^2+r'^2-2rr'\cos\theta}}\,dr'\,d\theta \\
&= \frac{\rho}{2\epsilon_0}\left(R^2-\frac{r^2}{3}\right) \\
&= \frac{q}{8\pi\epsilon_0 R}\left(3-\frac{r^2}{R^2}\right)
\end{aligned}
$$

2.31

(a)

$$W = \frac{q^2}{4\pi\epsilon_0 a}\left(\frac{1}{\sqrt{2}} - 2\right)$$

(b)

$$\begin{aligned}
W &= \frac{1}{4\pi\epsilon_0}\left(-\frac{q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a} - \frac{q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a}\right) \\
&= \frac{q^2}{2\pi\epsilon_0 a}\left(\frac{1}{\sqrt{2}} - 2\right)
\end{aligned}$$

2.32

$$\begin{aligned}
W &= \frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{a} \\
W &= K_1 + K_2 \\
\frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{a} &= \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 \\
\frac{1}{2\pi\epsilon_0}\frac{q_A q_B}{a} &= m_A v_A^2 + m_B v_B^2 \\
0 &= m_B v_B - m_A v_A \\
v_B &= \frac{m_A}{m_B} v_A \\
\frac{1}{2\pi\epsilon_0}\frac{q_A q_B}{a} &= m_A v_A^2 + m_B\left(\frac{m_A}{m_B} v_A\right)^2 \\
&= m_A v_A^2 + \frac{m_A^2}{m_B} v_A^2 \\
&= \frac{m_A(m_A + m_B)}{m_B} v_A^2 \\
v_A &= \sqrt{\frac{1}{2\pi\epsilon_0}\frac{q_A q_B}{(m_A + m_B)a}\frac{m_B}{m_A}} \\
v_B &= \sqrt{\frac{1}{2\pi\epsilon_0}\frac{q_A q_B}{(m_A + m_B)a}\frac{m_A}{m_B}}
\end{aligned}$$

2.33

$$
\begin{aligned}
W &= \frac{1}{4\pi\epsilon_0}\left(-\frac{q^2}{a}+\frac{q^2}{2a}-\frac{q^2}{3a}+\dots\right)\\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{a}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\\
&= -\frac{1}{4\pi\epsilon_0}\frac{q^2}{a}\ln 2
\end{aligned}
$$

2.34

(a)

$$
\begin{aligned}
V &= \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left[3-\left(\frac{r}{R}\right)^2\right] & r<R\\ \frac{1}{4\pi\epsilon_0}\frac{q}{r} & r>R \end{cases}\\
W &= \frac{1}{2}\int \rho V\,d\tau\\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\rho\frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left[3-\left(\frac{r}{R}\right)^2\right]r^2\sin\theta\,dr\,d\theta\,d\phi\\
&= \frac{q\rho}{8\epsilon_0 R}\int_0^{\pi}\int_0^{R}\left[3-\left(\frac{r}{R}\right)^2\right]r^2\sin\theta\,dr\,d\theta\\
&= \frac{q\rho R^2}{5\epsilon_0}\\
&= \frac{qR^2}{5\epsilon_0}\frac{q}{\frac{4}{3}\pi R^3}\\
&= \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathbf{E} &= \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} & r > R \\ \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3}\hat{\mathbf{r}} & r < R \end{cases} \\
E^2 &= \begin{cases} \frac{1}{16\pi^2\epsilon_0^2}\frac{q^2}{r^4} & r > R \\ \frac{1}{16\pi^2\epsilon_0^2}\frac{q^2r^2}{R^6} & r < R \end{cases} \\
W &= \frac{\epsilon_0}{2}\int E^2\,d\tau \\
&= \frac{\epsilon_0}{2}\left(\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{1}{16\pi^2\epsilon_0^2}\frac{q^2r^2}{R^6}r^2\sin\theta\,dr\,d\theta\,d\phi\right. \\
&\qquad\left.+\int_0^{2\pi}\int_0^{\pi}\int_R^{\infty}\frac{1}{16\pi^2\epsilon_0^2}\frac{q^2}{r^4}r^2\sin\theta\,dr\,d\theta\,d\phi\right) \\
&= \frac{\epsilon_0}{2}\frac{1}{16\pi^2\epsilon_0^2}2\pi q^2\left(\int_0^{\pi}\int_0^{R}\frac{r^4}{R^6}\sin\theta\,dr\,d\theta+\int_0^{\pi}\int_R^{\infty}\frac{1}{r^2}\sin\theta\,dr\,d\theta\right) \\
&= \frac{1}{16\pi\epsilon_0}q^2\left(\int_0^{\pi}\int_0^{R}\frac{r^4}{R^6}\sin\theta\,dr\,d\theta+\int_0^{\pi}\int_R^{\infty}\frac{1}{r^2}\sin\theta\,dr\,d\theta\right) \\
&= \frac{1}{16\pi\epsilon_0}q^2\left(\frac{2}{5R}+\frac{2}{R}\right) \\
&= \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}
\end{aligned}
$$

(c)

$$
\begin{aligned}
W &= \frac{\epsilon_0}{2}\left(\int_V E^2\,d\tau + \oint_S V\mathbf{E}\cdot d\mathbf{a}\right)\\
&= \frac{\epsilon_0}{2}\Bigg(\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{1}{(4\pi\epsilon_0)^2}\frac{q^2r^2}{R^6}r^2\sin\theta\,dr\,d\theta\,d\phi\\
&\qquad+\int_0^{2\pi}\int_0^{\pi}\int_R^{a}\frac{1}{(4\pi\epsilon_0)^2}\frac{q^2}{r^4}r^2\sin\theta\,dr\,d\theta\,d\phi\\
&\qquad+\int_0^{2\pi}\int_0^{\pi}\frac{1}{4\pi\epsilon_0}\frac{q}{a}\frac{1}{4\pi\epsilon_0}\frac{q}{a^2}a^2\sin\theta\,d\theta\,d\phi\Bigg)\\
&= \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}2\pi q^2\Bigg(\int_0^{\pi}\int_0^{R}\frac{r^4}{R^6}\sin\theta\,dr\,d\theta\\
&\qquad+\int_0^{\pi}\int_R^{a}\frac{1}{r^2}\sin\theta\,dr\,d\theta+\int_0^{\pi}\frac{1}{a}\sin\theta\,d\theta\Bigg)\\
&= \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}2\pi q^2\left[\frac{2}{5R}+2\left(\frac{1}{R}-\frac{1}{a}\right)+\frac{2}{a}\right]\\
&= \frac{1}{8\pi\epsilon_0}q^2\left[\frac{1}{5R}+\frac{1}{R}\right]\\
&= \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}
\end{aligned}
$$

2.36

(a)

$$
\mathbf{E} = \begin{cases}\mathbf{0} & r<a\\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} & a<r<b\\ \mathbf{0} & b<r\end{cases}
$$

$$
E^2 = \begin{cases}0 & r<a\\ \frac{1}{(4\pi\epsilon_0)^2}\frac{q^2}{r^4} & a<r<b\\ 0 & b<r\end{cases}
$$

$$
\begin{aligned}
W &= \frac{\epsilon_0}{2}\int E^2\,d\tau\\
&= \frac{\epsilon_0}{2}\int_0^{2\pi}\int_0^{\pi}\int_a^{b}\frac{1}{(4\pi\epsilon_0)^2}\frac{q^2}{r^4}r^2\sin\theta\,dr\,d\theta\,d\phi\\
&= \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}2\pi q^2\int_0^{\pi}\int_a^{b}\frac{\sin\theta}{r^2}\,dr\,d\theta\\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)
\end{aligned}
$$

(b)

$$
\begin{aligned}
W_{\text{shell}} &= \frac{1}{8\pi\epsilon_0}\frac{q^2}{R} \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \\
\mathbf{E}_1 \cdot \mathbf{E}_2 &= -\frac{1}{(4\pi\epsilon_0)^2}\frac{q^2}{r^4} \\
W_{\text{total}} &= W_1 + W_2 + \epsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2 \, d\tau \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b}\right) - \epsilon_0 \int_0^{2\pi}\int_0^{\pi}\int_b^{\infty} \frac{1}{(4\pi\epsilon_0)^2}\frac{q^2}{r^4} r^2 \sin\theta \, dr \, d\theta \, d\phi \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b}\right) - \frac{1}{8\pi\epsilon_0} q^2 \int_0^{\pi}\int_b^{\infty} \frac{1}{r^2}\sin\theta \, dr \, d\theta \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b}\right) - \frac{1}{4\pi\epsilon_0} q^2 \int_b^{\infty} \frac{1}{r^2} \, dr \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b}\right) - \frac{1}{4\pi\epsilon_0}\frac{q^2}{b} \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{b}\right) \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)
\end{aligned}
$$

2.37

$$
\begin{aligned}
r_1 &= r \\
E_1 &= \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1^2} \\
&= \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2} \\
r_2 &= \sqrt{a^2 + r^2 - 2ar\cos\theta} \\
E_2 &= \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_2^2} \\
&= \frac{1}{4\pi\epsilon_0}\frac{q_2}{a^2 + r^2 - 2ar\cos\theta} \\
\cos\alpha &= \frac{r - a\cos\theta}{\sqrt{a^2 + r^2 - 2ar\cos\theta}} \\
\mathbf{E}_1 \cdot \mathbf{E}_2 &= E_1 E_2 \cos\alpha \\
&= \frac{1}{(4\pi\epsilon_0)^2}\frac{q_1 q_2}{r^2(a^2 + r^2 - 2ar\cos\theta)}\frac{r - a\cos\theta}{\sqrt{a^2 + r^2 - 2ar\cos\theta}} \\
&= \frac{1}{(4\pi\epsilon_0)^2}\frac{q_1 q_2 (r - a\cos\theta)}{r^2(a^2 + r^2 - 2ar\cos\theta)^{3/2}} \\
\epsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2 \, d\tau &= \epsilon_0 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{(4\pi\epsilon_0)^2}\frac{q_1 q_2 (r - a\cos\theta)}{r^2(a^2 + r^2 - 2ar\cos\theta)^{3/2}} r^2 \sin\theta \, dr \, d\theta \, d\phi \\
&= \frac{q_1 q_2}{8\pi\epsilon_0}\int_0^{\pi}\int_0^{\infty} \frac{(r - a\cos\theta)\sin\theta}{(a^2 + r^2 - 2ar\cos\theta)^{3/2}} \, dr \, d\theta
\end{aligned}
$$

2.38

(a)

$$
\begin{aligned}
\sigma_R &= \frac{q}{4\pi R^2} \\
\sigma_a &= -\frac{q}{4\pi a^2} \\
\sigma_b &= \frac{q}{4\pi b^2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
V &= -\int_{\infty}^{b} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \, dr - \int_a^R \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \, dr \\
&= \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{b} + \frac{1}{R} - \frac{1}{a}\right)
\end{aligned}
$$

(c)

$$\sigma_b = 0$$
$$V = \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{R} - \frac{1}{a}\right)$$

2.39

(a)

$$\sigma_a = -\frac{q_a}{4\pi a^2}$$
$$\sigma_b = -\frac{q_b}{4\pi b^2}$$
$$\sigma_R = \frac{q_a + q_b}{4\pi R^2}$$

(b)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q_a + q_b}{r^2}\hat{\mathbf{r}}$$

(c)

$$\mathbf{E}_a = \frac{1}{4\pi\epsilon_0}\frac{q_a}{r^2}\hat{\mathbf{r}}$$
$$\mathbf{E}_b = \frac{1}{4\pi\epsilon_0}\frac{q_b}{r^2}\hat{\mathbf{r}}$$

(d)

$$\mathbf{0}$$

(e) a, b

2.40

(a) No. If it's close to the wall it will induce a surface charge and be attracted.

(b) No. If the conductor contains a cavity containing a like charge it will be repelled.

2.41

By Gauss's law, the electric field of each plate is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$
$$2A'E = \frac{A'\frac{Q}{A}}{\epsilon_0}$$
$$\mathbf{E} = \frac{Q}{2A\epsilon_0}\hat{\mathbf{n}}$$

so the field between the plates is zero and the field outside is $Q/A\epsilon_0\hat{\mathbf{n}}$, resulting in a pressure of

$$\begin{aligned}
P &= \frac{\epsilon_0}{2}E^2 \\
&= \frac{\epsilon_0}{2}\frac{Q^2}{A^2\epsilon_0^2} \\
&= \frac{Q^2}{2A^2\epsilon_0}
\end{aligned}$$

2.42

$$\begin{aligned}
\mathbf{E}_{\text{above}} &= \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \\
\mathbf{f} &= \frac{1}{2}\sigma\mathbf{E}_{\text{above}} \\
&= \frac{1}{2}\frac{Q}{4\pi R^2}\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\hat{\mathbf{r}} \\
&= \frac{Q^2}{32\pi^2\epsilon_0 R^4}\hat{\mathbf{r}} \\
\mathbf{F} &= \int_0^{2\pi}\int_0^{\pi/2}\frac{Q^2}{32\pi^2\epsilon_0 R^4}\cos\theta R^2\sin\theta\, d\theta\, d\phi\hat{\mathbf{z}} \\
&= \frac{Q^2}{16\pi\epsilon_0 R^2}\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta\hat{\mathbf{z}} \\
&= \frac{Q^2}{32\pi\epsilon_0 R^2}\hat{\mathbf{z}}
\end{aligned}$$

2.43

$$\begin{aligned}
\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q}{\epsilon_0} \\
2\pi s L E &= \frac{Q}{\epsilon_0} \\
\mathbf{E} &= \frac{Q}{2\pi L \epsilon_0} \frac{1}{s} \hat{\mathbf{s}} \\
V &= -\int_b^a \frac{Q}{2\pi\epsilon_0 L} \frac{1}{s}\, dr \\
&= \frac{Q}{2\pi\epsilon_0 L} \ln \frac{b}{a} \\
C &= \frac{Q}{V} \\
&= \frac{2\pi\epsilon_0 L}{\ln b/a}
\end{aligned}$$

So the capacitance per unit length is

$$C = \frac{2\pi\epsilon_0}{\ln b/a}.$$

2.44

(a)

$$\begin{aligned}
P &= \frac{\epsilon_0}{2} E^2 \\
W &= Fd \\
&= PA\epsilon \\
&= \frac{\epsilon_0}{2} E^2 A\epsilon
\end{aligned}$$

(b)

$$\frac{\epsilon_0}{2} E^2 A\epsilon$$

2.46

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 3\frac{k}{r}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{k}{r}2\sin\theta\cos\theta\sin\phi\right) \\
&\quad + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{k}{r}\sin\theta\cos\phi\right) \\
&= \frac{3k}{r^2} + \frac{1}{r\sin\theta}\frac{2k}{r}\sin\phi(2\sin\theta\cos^2\theta - \sin^3\theta) - \frac{1}{r\sin\theta}\frac{k}{r}\sin\theta\sin\phi \\
&= \frac{3k}{r^2} + \frac{2k\sin\phi}{r^2}(2\cos^2\theta - \sin^2\theta) - \frac{k}{r^2}\sin\phi \\
&= \frac{k}{r^2}[3 + 2\sin\phi(2\cos^2\theta - \sin^2\theta) - \sin\phi] \\
&= \frac{k}{r^2}[3 + \sin\phi(4\cos^2\theta - 2\sin^2\theta - 1)] \\
&= \frac{k}{r^2}[3 + \sin\phi(6\cos^2\theta - 3)] \\
&= \frac{3k}{r^2}(1 + \cos 2\theta\sin\phi) \\
\rho &= \epsilon_0 \nabla \cdot \mathbf{E} \\
&= \frac{3k\epsilon_0}{r^2}(1 + \cos 2\theta\sin\phi)
\end{aligned}
$$

2.47

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}\hat{\mathbf{r}} \\
\rho &= \frac{Q}{\frac{4}{3}\pi R^3} \\
\rho\mathbf{E} &= \frac{3Q}{4\pi R^3}\frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}\hat{\mathbf{r}} \\
&= \frac{3r}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2\hat{\mathbf{r}} \\
F_z &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^R \frac{3r}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 \cos\theta r^2\sin\theta\, dr\, d\theta\, d\phi \\
&= \frac{3\pi}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 \int_0^{\pi/2}\int_0^R r^3\sin 2\theta\, dr\, d\theta \\
&= \frac{3\pi}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 \frac{R^4}{4} \\
&= \frac{3Q^2}{64\pi\epsilon_0 R^2}
\end{aligned}
$$

2.49

$$\begin{aligned}
Q_{\text{enc}} &= \int_0^{2\pi}\int_0^{\pi}\int_0^{r} kr'^3 \sin\theta \, dr' \, d\theta \, d\phi \\
&= 2\pi k \int_0^{\pi}\int_0^{r} r'^3 \sin\theta \, dr' \, d\theta \\
&= \pi k r^4 \\
\oint \mathbf{E}\cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E &= \frac{\pi k r^4}{\epsilon_0} \\
\mathbf{E} &= \begin{cases} \frac{kr^2}{4\epsilon_0}\hat{\mathbf{r}} & r < R \\ \frac{kR^4}{4\epsilon_0 r^2}\hat{\mathbf{r}} & r > R \end{cases} \\
W &= \frac{\epsilon_0}{2}\left(\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{k^2 r^4}{16\epsilon_0^2} r^2 \sin\theta \, dr \, d\theta \, d\phi \right. \\
&\qquad \left. \int_0^{2\pi}\int_0^{\pi}\int_R^{\infty} \frac{k^2 R^8}{16\epsilon_0^2 r^4} r^2 \sin\theta \, dr \, d\theta \, d\phi \right) \\
&= \frac{\epsilon_0}{2} 2\pi \frac{k^2}{16\epsilon_0^2}\left(\int_0^{\pi}\int_0^{R} r^6 \sin\theta \, dr \, d\theta + \int_0^{\pi}\int_R^{\infty} \frac{R^8 \sin\theta}{r^2} \, dr \, d\theta\right) \\
&= \frac{\pi k^2}{16\epsilon_0}\left(\frac{2R^7}{7} + 2R^7\right) \\
&= \frac{\pi k^2 R^7}{7\epsilon_0}
\end{aligned}$$

2.50

$$\begin{aligned}
V(\mathbf{r}) &= A\frac{e^{-\lambda r}}{r} \\
\mathbf{E} &= -\nabla V \\
&= Ae^{-\lambda r}(1+\lambda r)\frac{\hat{\mathbf{r}}}{r^2} \\
\rho &= \epsilon_0 \nabla \cdot \mathbf{E} \\
&= \epsilon_0\left[Ae^{-\lambda r}(1+\lambda r)\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2} + \frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla\left(Ae^{-\lambda r}(1+\lambda r)\right)\right] \\
&= A\epsilon_0\left[4\pi\delta(\mathbf{r}) + \frac{\hat{\mathbf{r}}}{r^2}\cdot(-\lambda^2 e^{-\lambda r} r\hat{\mathbf{r}})\right] \\
&= A\epsilon_0\left(4\pi\delta(\mathbf{r}) - \frac{\lambda^2 e^{-\lambda r}}{r}\right)
\end{aligned}$$

2.51

$$
\begin{aligned}
V &= \int \frac{1}{4\pi\epsilon_0} \frac{\sigma}{\imath} \, dA \\
&= \frac{\sigma}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^R \frac{r}{\sqrt{r^2 + R^2 - 2rR\cos\theta}} \, dr \, d\theta \\
&= \frac{R\sigma}{4\pi\epsilon_0} \int_0^{2\pi} \left[\cos\theta \ln\left(1 + \csc\frac{\theta}{2}\right) + 2\sin\frac{\theta}{2} - 1 \right] d\theta \\
&= \frac{R\sigma}{\pi\epsilon_0}
\end{aligned}
$$

2.52

(a)

$$
\begin{aligned}
V_- &= \frac{1}{2\pi\epsilon_0} \lambda \ln \frac{s_-}{a} \\
&= \frac{1}{2\pi\epsilon_0} \lambda \ln \frac{\sqrt{(y+a)^2 + z^2}}{a} \\
V_+ &= -\frac{1}{2\pi\epsilon_0} \lambda \ln \frac{s_+}{a} \\
&= -\frac{1}{2\pi\epsilon_0} \lambda \ln \frac{\sqrt{(y-a)^2 + z^2}}{a} \\
V &= V_- + V_+ \\
&= \frac{1}{4\pi\epsilon_0} \lambda \ln \frac{(y+a)^2 + z^2}{(y-a)^2 + z^2}
\end{aligned}
$$

2.53

(a)

$$
\begin{aligned}
\nabla^2 V &= -\frac{\rho}{\epsilon_0} \\
\nabla \cdot \nabla V &= -\frac{\rho}{\epsilon_0} \\
\nabla \cdot \frac{dV}{dx} \hat{\mathbf{x}} &= -\frac{\rho}{\epsilon_0} \\
\frac{d^2 V}{dx^2} &= -\frac{\rho}{\epsilon_0}
\end{aligned}
$$

(b)

$$qV = \frac{1}{2}mv^2$$
$$v = \sqrt{\frac{2qV}{m}}$$

(c)

$$I = A\rho v$$

(d)

$$\begin{aligned} \frac{d^2V}{dx^2} &= -\frac{I}{Av\epsilon_0} \\ &= -\frac{I}{A\epsilon_0}\sqrt{\frac{m}{2qV}} \\ &= \beta V^{-1/2} \end{aligned}$$

2.55

$$\begin{aligned} \rho &= \epsilon_0 \nabla \cdot \mathbf{E} \\ &= a\epsilon_0 \end{aligned}$$

2.56

$$\begin{aligned} E &= \frac{3GM^2}{5R} \\ E_{\text{sun}} &= 2.3 \times 10^{41}\,\text{J} \\ t &= \frac{E_{\text{sun}}}{P} \\ &= 1.89 \times 10^{7}\,\text{years} \end{aligned}$$

3 Potentials

3.1

$$
\begin{aligned}
V_{\text{ave}} &= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\ \sqrt{z^2+R^2-2zR\cos\theta}\Big|_0^\pi \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\left(\sqrt{z^2+R^2+2zR}-\sqrt{z^2+R^2-2zR}\right) \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\left(\sqrt{(z+R)^2}-\sqrt{(R-z)^2}\right) \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}(z+R-R+z) \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{R}
\end{aligned}
$$

The average potential due to external charges is V_{center} and the average potential due to internal charges is

$$\frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{R}$$

so

$$V_{\text{ave}} = V_{\text{center}} + \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{R}.$$

3.2

A stable equilibrium is a minimum of potential energy. Laplace's equation doesn't allow for minimums, so they must be saddle points and the charge can escape.

3.3

$$
\begin{aligned}
0 &= \nabla^2 V \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) \\
&= \frac{1}{r^2}\left(2r\frac{\partial V}{\partial r} + r^2\frac{\partial^2 V}{\partial r^2}\right) \\
&= \frac{2}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} \\
V &= \frac{c_1}{r} + c_2 \\
0 &= \nabla^2 V \\
&= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) \\
&= \frac{1}{s}\left(\frac{\partial V}{\partial s} + s\frac{\partial^2 V}{\partial s^2}\right) \\
&= \frac{1}{s}\frac{\partial V}{\partial s} + \frac{\partial^2 V}{\partial s^2} \\
V &= c_1 + c_2 \ln s
\end{aligned}
$$

3.7

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}q^2\left(-\frac{2}{(2d)^2} + \frac{2}{(4d)^2} - \frac{1}{(6d)^2}\right)\hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0}\frac{29q^2}{72d^2}\hat{\mathbf{z}}
\end{aligned}
$$

3.8

(a)

$$
\begin{aligned}
V(r,\theta) &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} + \frac{q'}{\sqrt{r^2+b^2-2rb\cos\theta}}\right] \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{Rq/a}{\sqrt{r^2+(R^2/a)^2-2r(R^2/a)\cos\theta}}\right] \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{q}{\sqrt{R^2+(ra/R)^2-2ra\cos\theta}}\right]
\end{aligned}
$$

(b)

$$
\begin{aligned}
\sigma &= -\epsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=R} \\
&= \frac{q}{4\pi R}\frac{R^2 - a^2}{(a^2 + R^2 - 2aR\cos\theta)^{3/2}} \\
Q_{\text{induced}} &= \int_0^{2\pi}\int_0^{\pi} \sigma R^2 \sin\theta\, d\theta\, d\phi \\
&= \frac{qR(R^2 - a^2)}{2}\int_0^{\pi} \frac{\sin\theta}{(a^2 + R^2 - 2aR\cos\theta)^{3/2}}\, d\theta \\
&= \frac{qR(R^2 - a^2)}{a(a^2 - R^2)} \\
&= -\frac{R}{a}q \\
&= q'
\end{aligned}
$$

(c)

$$
\begin{aligned}
W &= \frac{1}{2}qV \\
&= \frac{1}{8\pi\epsilon_0}\frac{qq'}{a-b} \\
&= -\frac{1}{8\pi\epsilon_0}\frac{q^2R/a}{a - R^2/a} \\
&= -\frac{1}{8\pi\epsilon_0}\frac{q^2R}{a^2 - R^2}
\end{aligned}
$$

3.9

Place the second image charge at the centre of the sphere with charge

$$q'' = 4\pi\epsilon_0 R V_0.$$

$$
\begin{aligned}
F &= \frac{1}{4\pi\epsilon_0} q \left(\frac{q'}{(a-b)^2} + \frac{q''}{a^2} \right) \\
&= \frac{qq'}{4\pi\epsilon_0} \left(\frac{1}{(a-b)^2} - \frac{1}{a^2} \right) \\
&= \frac{qq'}{4\pi\epsilon_0} \frac{a^2 - (a-b)^2}{a^2(a-b)^2} \\
&= \frac{qq'}{4\pi\epsilon_0} \frac{b(2a-b)}{a^2(a-b)^2} \\
&= \frac{q(-Rq/a)}{4\pi\epsilon_0} \frac{(R^2/a)(2a - R^2/a)}{a^2(a - R^2/a)^2} \\
&= -\frac{q^2}{4\pi\epsilon_0} \left(\frac{R}{a} \right)^3 \frac{2a^2 - R^2}{(a^2 - R^2)^2}
\end{aligned}
$$

3.10

(a)

$$
V(x,y,z) = \frac{1}{4\pi\epsilon_0} \lambda \ln \frac{y^2 + (z+d)^2}{y^2 + (z-d)^2}
$$

(b)

$$
\begin{aligned}
\sigma &= -\epsilon_0 \left. \frac{\partial V}{\partial z} \right|_{z=0} \\
&= -\frac{d\lambda}{\pi(d^2 + y^2)}
\end{aligned}
$$

3.11

You need three charges: $-q$ at $(-a,b)$, $-q$ at $(a,-b)$, and q at $(-b,-a)$. The potential is

$$
\begin{aligned}
V = \frac{1}{4\pi\epsilon_0} q \Bigg(& \frac{1}{\sqrt{(x-a)^2 + (y-b)^2}} - \frac{1}{\sqrt{(x+a)^2 + (y-b)^2}} \\
& - \frac{1}{\sqrt{(x-a)^2 + (y+b)^2}} + \frac{1}{\sqrt{(x+a)^2 + (y+b)^2}} \Bigg).
\end{aligned}
$$

The force on q is

$$
\mathbf{F} = \frac{q^2}{16\pi\epsilon_0} \left[\left(\frac{a}{(a^2+b^2)^{3/2}} - \frac{1}{a^2} \right) \hat{\mathbf{x}} + \left(\frac{b}{(a^2+b^2)^{3/2}} - \frac{1}{b^2} \right) \hat{\mathbf{y}} \right].
$$

The work to bring q in from infinity is

$$W = \frac{q^2}{16\pi\epsilon_0}\left(\frac{1}{\sqrt{a^2+b^2}} - \frac{1}{a} - \frac{1}{b}\right).$$

3.12

Two infinitely long wires running parallel to the x-axis a distance $2a$ apart with charge densities λ and $-\lambda$ have cylindrical equipotential surfaces with centres at

$$y_0 = \pm a \coth \frac{2\pi\epsilon_0 V_0}{\lambda}$$

radii

$$R = a \operatorname{csch} \frac{2\pi\epsilon_0 V_0}{\lambda}.$$

We know the equipotential surfaces (the pipes) and want to find the wires so we can find the potential, so

$$\begin{aligned}
d &= a \coth \frac{2\pi\epsilon_0 V_0}{\lambda} \\
R &= a \operatorname{csch} \frac{2\pi\epsilon_0 V_0}{\lambda} \\
\frac{d}{R} &= \cosh \frac{2\pi\epsilon_0 V_0}{\lambda} \\
\operatorname{arcosh} \frac{d}{R} &= \frac{2\pi\epsilon_0 V_0}{\lambda} \\
\lambda &= \frac{2\pi\epsilon_0 V_0}{\operatorname{arcosh} d/R} \\
R &= a \operatorname{csch} \operatorname{arcosh} \frac{d}{R} \\
a &= \frac{R}{\operatorname{csch} \operatorname{arcosh} d/R} \\
&= (d+R)\sqrt{\frac{2d}{d+R} - 1} \\
&= \sqrt{d^2 - R^2}
\end{aligned}$$

thus the potential is

$$V = \frac{V_0}{2 \operatorname{arcosh} d/R} \ln \frac{(y + d^2 - R^2)^2 + z^2}{(y - d^2 + R^2)^2 + z^2}.$$