Classical Mechanics by John R. Taylor Notes

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August 2023

Contents

1 Newton's Laws of Motion

1.2 Space and Time

- In cartesian coordinates the basis vectors don't depend on time so their derivatives are $\mathbf{0}$. This means that

$$\begin{aligned}\frac{d}{dt}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) &= \frac{dx}{dt}\hat{\mathbf{x}} + x\frac{d\hat{\mathbf{x}}}{dt} + \frac{dy}{dt}\hat{\mathbf{y}} + y\frac{d\hat{\mathbf{y}}}{dt} + \frac{dz}{dt}\hat{\mathbf{z}} + z\frac{d\hat{\mathbf{z}}}{dt} \\ &= \frac{dx}{dt}\hat{\mathbf{x}} + \frac{dy}{dt}\hat{\mathbf{y}} + \frac{dz}{dt}\hat{\mathbf{z}}\end{aligned}$$

  as expected. However, in order coordinate systems (e.g. polar, spherical) the basis vectors may depend on time and their derivatives aren't $\mathbf{0}$.

1.4 Newton's First and Second Laws; Inertial Frames

- Newton's second law $\mathbf{F} = m\mathbf{a}$ can be restated as $\mathbf{F} = \dot{\mathbf{p}}$.
- An inertial frame is one where Newton's first law holds. Typically this means the frame isn't accelerating or rotating.

1.5 The Third Law and Conservation of Momentum

- Forces that act along the line joining two objects are called **central forces**.
- The **principle of conservation of momentum** states that if the net external force $\mathbf{F}_{\text{ext}}$ on an N-particle system is zero, the system's total momentum $\mathbf{P}$ is constant.

1.7 Two-Dimensional Polar Coordinates

- In two-dimensional polar coordinates, the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ depend on position and thus time. Their derivatives are

$$\begin{aligned}\frac{d\hat{\mathbf{r}}}{dt} &= \dot{\phi}\hat{\boldsymbol{\phi}} \\ \frac{d\hat{\boldsymbol{\phi}}}{dt} &= -\dot{\phi}\hat{\mathbf{r}}.\end{aligned}$$

  Consequently, the derivatives of the position vector $\mathbf{r} = r\hat{\mathbf{r}}$ are

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= \frac{d}{dt}(r\hat{\mathbf{r}}) \\ &= \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \\ &= \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}\end{aligned}$$

  and

$$\begin{aligned}\frac{d^2\mathbf{r}}{dt^2} &= \frac{d}{dt}(\dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}) \\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt} + \dot{r}\dot{\phi}\hat{\boldsymbol{\phi}} + r\ddot{\phi}\hat{\boldsymbol{\phi}} + r\dot{\phi}\frac{d\hat{\boldsymbol{\phi}}}{dt} \\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{r}\dot{\phi}\hat{\boldsymbol{\phi}} + r\ddot{\phi}\hat{\boldsymbol{\phi}} - r\dot{\phi}^2\hat{\mathbf{r}} \\ &= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\boldsymbol{\phi}}.\end{aligned}$$

- In light of the above, Newton's second law in polar coordinates can be written

$$\begin{aligned}F_r &= m(\ddot{r} - r\dot{\phi}^2) \\ F_\phi &= m(r\ddot{\phi} + 2\dot{r}\dot{\phi}).\end{aligned}$$

2 Projectiles and Charged Particles

2.1 Air Resistance

- Air resistance depends on the speed v of the moving object. For many objects the direction of the air resistance force $\mathbf{f}$ is opposite to $\mathbf{v}$, but not always. For example, the air resistance force on an airplane causes lift.

- An air resistance force can be described by the equation

$$\mathbf{f} = -f(v)\hat{\mathbf{v}}$$

where $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$ gives the direction and $f(v)$ gives the magnitude.

- $f(v)$ can be approximated as

$$f(v) = f_{\text{lin}} + f_{\text{quad}} = bv + cv^2.$$

- The linear term f_{lin} arises from the viscous drag of the medium and is generally proportional to the projectile's linear size.

- The quadratic term f_{quad} arises from the fact that the projectile must accelerate the air with which it is continually colliding and it is proportional to the density of the medium and the cross-sectional area of the projectile.

- For a spherical projectile the coefficients b and c above have the form

$$b = \beta D \text{ and } c = \gamma D^2$$

where D is the diameter of the sphere and the coefficients β and γ depend on the nature of the medium. In air at STP they have approximate values

$$\beta = 1.6 \times 10^{-4}\,\mathrm{N\,s/m^2}$$

and

$$\gamma = 0.25\,\mathrm{N\,s^2/m^4}.$$

- Depending on the natures of the medium and projectile it's often possible to neglect one of the terms in $f(v)$. To determine if this is the case we can calculate their ratio. For example, for a spherical projectile at STP

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} = \frac{cv^2}{bv} = \frac{\gamma D}{\beta}v = (1.6 \times 10^3\,\mathrm{s/m^2})Dv.$$

If the ratio is large f_{lin} can be ignored. If it's small f_{quad} can be ignored.

- The **Reynolds number** can be used to characterise the behaviour of an object in a fluid

$$R = \frac{\rho}{\mu}Dv$$

where ρ is the medium's density, μ is its viscosity, D is the linear dimension of the projectile (diameter for spherical projectiles), and v is the projectile's speed. The quadratic force f_{quad} is dominant when the Reynolds number R is large and the linear force f_{linear} is dominant when it is small.

2.2 Linear Air Resistance

- When the quadratic drag force is negligible the equation of motion becomes

$$\begin{aligned}\mathbf{F} &= \mathbf{W} - \mathbf{f} \\ m\mathbf{a} &= m\mathbf{g} - b\mathbf{v} \\ m\dot{\mathbf{v}} &= m\mathbf{g} - b\mathbf{v}.\end{aligned}$$

  This is a first-order differential equation for $\mathbf{v}$ where the horizontal and vertical components can be separated to

$$\begin{aligned}m\dot{v}_x &= -bv_x \\ m\dot{v}_y &= mg - bv_y,\end{aligned}$$

  each of which is easily solvable.

- The **terminal speed** of an object undergoing freefall and experiencing only linear drag is
$$v_{\text{ter}} = \frac{mg}{b}.$$

- The **characteristic time**

$$\tau = \frac{1}{k} = \frac{1}{b/m} = \frac{m}{b}$$

  is a measure of the importance of air resistance.

  - For horizontal motion with drag it's a measure of the time it takes for the projectile to reach $1/e$ of its initial velocity.
  - For freefall with drag it's a measure of the time it would take the projectile to reach its terminal velocity if it didn't experience drag

$$v_{\text{ter}} = g\tau.$$

  - For freefall with drag it can also be used to gauge what percentage of its terminal velocity a projectile will reach after a certain time:

Time t	Percent of v_{ter}
0	0
τ	63%
2τ	86%
3τ	95%

    From this it can be seen that after $t = 3\tau$ the projectile has effectively reached its terminal velocity.

2.4 Quadratic Air Resistance

- Equations of motion for quadratic air resistance can be solved analytically when the projectile moves in one dimension, but can only be solved numerically when it moves in multiple dimensions.
- When a projectile moves in one dimension and only experiences the force of air resistance (i.e. there are no other forces), the equation of motion is

  $$m\dot{v} = -cv^2.$$

  Using separation of variables the solution can be found to be

  $$v(t) = \frac{v_0}{1 + t/\tau}$$

  where

  $$\tau = \frac{m}{cv_0}.$$

- As in the linear case, τ is a measure of how long it takes for air resistance to slow down the projectile ($v = v_0/2$ at $t = \tau$).
- Integrating the equation for $v(t)$ gives

  $$x(t) = v_0 \tau \ln\left(1 + \frac{t}{\tau}\right).$$

- When a projectile moves in one dimension and experiences the forces of air resistance and weight, the equation of motion (with y down) is

  $$m\dot{v} = mg - cv^2.$$

  Using separation of variables the solution can be found to be

  $$v(t) = v_{\text{ter}} \tanh \frac{gt}{v_{\text{ter}}}$$

  where

  $$v_{\text{ter}} = \sqrt{\frac{mg}{c}}.$$

- Integrating the equation for $v(t)$ gives

  $$y = \frac{v_{\text{ter}}^2}{g} \ln\left(\cosh \frac{gt}{v_{\text{ter}}}\right).$$

2.5 Motion of a Charge in a Uniform Magnetic Field

- When a particle of charge q moves in a magnetic field $\mathbf{B} = (0, 0, B_z)$ with velocity $\mathbf{v} = (v_x, v_y, v_z)$ it experiences a force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = q(v_y B, -b_x B, 0).$$

  This gives the coupled equations of motion

$$\begin{aligned} m\dot{v}_x &= qBv_y \\ m\dot{v}_y &= -qBv_x \\ m\dot{v}_z &= 0 \end{aligned}$$

  or

$$\begin{aligned} \dot{v}_x &= \omega v_y \\ \dot{v}_y &= -\omega v_x \\ \dot{v}_z &= 0 \end{aligned}$$

  where $\omega = qB/m$ is called the **cyclotron frequency**.

- If we define a complex value

$$\eta = v_x + iv_y,$$

  its derivative is

$$\begin{aligned} \dot{\eta} &= \dot{v}_x + i\dot{v}_y \\ &= \omega v_y - i\omega v_x \\ &= -i\omega\eta \end{aligned}$$

  which has the solution

$$\eta = Ae^{-i\omega t}.$$

3 Momentum and Angular Momentum

3.1 Conservation of Momentum

- The **principle of conservation of momentum** states that if the net external force $\mathbf{F}_{\text{ext}}$ on an N-particle system is zero, the system's total mechanical momentum $\mathbf{P} = \sum m_\alpha v_\alpha$ is constant.

3.2 Rockets

- Newton's second law for a rocket is

$$m\dot{v} = -\dot{m}v_{\text{ex}}$$

  where $\dot{m}$ is the (negative) rate of change of the mass of the rocket and v_{ex} is the velocity of the exhaust. The quantity on the right hand side of the equation is called the **thrust**.

- The equation above can be solved by separation of variables giving

$$v - v_0 = v_{\text{ex}} \ln \frac{m_0}{m}$$

which is often called the **rocket equation**.

3.3 The Center of Mass

- The **centre of mass** of a system is defined to be

$$\mathbf{R} = \frac{1}{M} \sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha$$

where M is the total mass of all particles in the system, m_α is the mass of particle α, and $\mathbf{r}_\alpha$ is the vector from the origin to particle α.

- The total momentum of a system can be written in terms of its centre of mass

$$\mathbf{P} = \sum_\alpha \mathbf{p}_\alpha = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha = M\dot{\mathbf{R}}$$

i.e. the total momentum of N particles is equivalent to that of a single particle of mass M with velocity equal to that of the centre of mass.

- Differentiating the above we find

$$\begin{aligned} \frac{d}{dt}\mathbf{P} &= \frac{d}{dt}(M\dot{\mathbf{R}}) \\ \mathbf{F}_{\text{ext}} &= M\ddot{\mathbf{R}} \end{aligned}$$

i.e. the centre of mass moves as if it was a single particle of mass M subject to the net external force on the system.

- When a body is continuous the expression for its centre of mass becomes an integral

$$\mathbf{R} = \frac{1}{M} \int \mathbf{r}\, dm = \frac{1}{M} \int \rho \mathbf{r}\, dV.$$

3.4 Angular Momentum for a Single Particle

- The **angular momentum** of a particle relative to an origin O is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

where $\mathbf{r}$ is measured relative to O.

- Taking the derivative of angular momentum gives

$$\begin{aligned}\frac{d}{dt}\mathbf{L} &= \frac{d}{dt}(\mathbf{r}\times\mathbf{p})\\ \dot{\mathbf{L}} &= \dot{\mathbf{r}}\times\mathbf{p}+\mathbf{r}\times\dot{\mathbf{p}}\\ &= \mathbf{r}\times\mathbf{F}\\ &= \boldsymbol{\tau}.\end{aligned}$$

  In other words, the rate of change in angular momentum about an origin O is equal to the net torque about that origin.

- We can simplify some one-particle problems by choosing the origin such that the net torque is 0 and thus angular momentum is constant.

3.5 Angular Momentum for Several Particles

- The **total angular momentum** of a system is

$$\mathbf{L} = \sum_{\alpha=1}^{N}\mathbf{L}_\alpha = \sum_{\alpha=1}^{N}\mathbf{r}_\alpha\times\mathbf{p}_\alpha.$$

- Differentiating the above

$$\dot{\mathbf{L}} = \sum_\alpha \dot{\mathbf{L}}_\alpha = \sum_\alpha \mathbf{r}_\alpha\times\mathbf{F}_\alpha = \boldsymbol{\tau}_{\text{ext}}$$

  we find that the rate of change of the total angular momentum of the system is equal to the net torque on the system.

- The **principle of conservation of angular momentum** states that if the net external torque on a system is 0, the system's total angular momentum is constant. This assumes that all internal forces are central and obey Newton's third law.

- The principle of conservation of momentum and the result $\dot{\mathbf{L}} = \boldsymbol{\tau}_{\text{ext}}$ also hold if $\mathbf{L}$ and $\boldsymbol{\tau}_{\text{ext}}$ are measured about the centre of mass, even if the centre of mass is being accelerated and is thus not an interial frame.

4 Energy

4.1 Kinetic Energy and Work

- The **work-kinetic-energy theorem** states that the change in a particle's kinetic energy between two points is equal to the work done by the net force on the particle between those two points

$$\Delta K = \int_1^2 \mathbf{F}\cdot d\mathbf{r}.$$

4.2 Potential Energy and Conservative Forces

- A force $\mathbf{F}$ acting on a particle is considered **conservative** if:
  - $\mathbf{F}$ depends only on the particle's position $\mathbf{r}$ (and not on its velocity $\mathbf{v}$, time t, or any other variable), and
  - for any two points 1 and 2, the work done by $\mathbf{F}$ is the same for all paths between 1 and 2.
- Only conservative forces have associated **potential energy** functions.
- The potential energy function $U(\mathbf{r})$ of a conservative force $\mathbf{F}$ is defined as

$$U(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}')\, d\mathbf{r}'$$

  where $\mathbf{r}_0$ is an arbitrary point at which $U(\mathbf{r}_0)$ is defined to be 0.
- The **principle of conservation of energy** states that if all the forces acting on a particle are conservative, each with its corresponding potential energy function $U_i(\mathbf{r})$, the **total mechanical energy**

$$E = K + U = K + U_1(\mathbf{r}) + \cdots + U_n(\mathbf{r}),$$

  is constant in time.
- If nonconservative forces do work then the total energy of the system changes by that amount

$$\Delta E = W_{\mathrm{nc}}.$$

4.3 Force as the Gradient of Potential Energy

- A conservative force $\mathbf{F}$ can be expressed as the negative gradient of its potential energy function U

$$\mathbf{F} = -\nabla U.$$

4.4 The Second Condition that F be Conservative

- A force $\mathbf{F}$ is conservative if $\nabla \times \mathbf{F} = \mathbf{0}$.

4.5 Time-Dependent Potential Energy

- If a time-dependent force $\mathbf{F}(t)$ has the property $\nabla \times \mathbf{F}(t) = \mathbf{0}$ it's still possible to define an associated potential energy function $U(\mathbf{r}, t)$ where $\mathbf{F}(t) = -\nabla U(t)$ but it's no longer guaranteed that total mechanical energy is conserved over time.

4.8 Central Forces

- A central force is conservative if and only if it's spherically symmetric.

5 Oscillations

5.2 Simple Harmonic Motion

- The equation of motion for a harmonic oscillator
$$\ddot{x} = -\frac{k}{m}x = -\omega^2 x$$
can be solved in multiple ways:
  - the exponential solution
  $$x = c_1 e^{i\omega t} + c_2 e^{-i\omega t},$$
  - the sine and cosine solutions
  $$x = c_1 \cos \omega t + c_2 \sin \omega t,$$
  and
  - the phase shifted cosine solution
  $$x = A \cos(\omega t - \delta)$$
  where
  $$A = \sqrt{c_1^2 + c_2^2}$$
  with c_1 and c_2 coming from the sine and cosine solutions above and
  $$\delta = \arctan -\frac{c_1}{c_2}.$$

5.3 Two-Dimensional Oscillators

- An **isotropic harmonic oscillator** in $n > 1$ dimensional space experiences a restoring force directed towards the equilibrium position and with magnitude kr where r is the object's distance from equilibrium.
- In two dimensions an isotropic harmonic oscillator has general solutions
$$x(t) = A_x \cos \omega t$$
$$y(t) = A_y \cos(\omega t - \delta).$$
It was possible to eliminate the phase from $x(t)$ by redefining the origin of time but in general it isn't possible to also eliminate the phase from $y(t)$.
- An **anisotropic harmonic oscillator** is similar to an isotropic harmonic oscillator but the spring constants are different in different directions.

5.7 Fourier Series

- Any periodic function with period T can be represented as a Fourier series

$$f(t) = \sum_{n=0}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

where

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, dt \\ a_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(n\omega t)\, dt \\ b_0 &= 0 \\ b_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(n\omega t)\, dt \\ \omega &= \frac{2\pi}{T}. \end{aligned}$$

6 Calculus of Variations

- A **functional** is a mapping from a space X to the real or complex numbers. When X is the space of functions a functional is a "function of a function", i.e. it takes a function as an argument.

- The goal of the **calculus of variations** is to find maxima and minima of functionals, i.e. functions that maximise or minimise the value of the functional. This is analogous to finding real numbers that maximise or minimise a function in single-variable calculus.

- A functional of the form

$$S = \int_{x_1}^{x_2} f[x, y(x), y'(x)]\, dx$$

can be solved using the **Euler-Lagrange equation**

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0.$$

- A solution to the Euler-Lagrange equation isn't guaranteed to be a minimum — it could be a maximum or an inflection point, as in single-variable calculus. In general it's difficult to determine the nature of a given solution so other methods (e.g. inspection) must be used.

- A functional with multiple functions as arguments, e.g.

$$S = \int_{t_1}^{t_2} f[t, x(t), x'(t), y(t), y'(t)]\, dt,$$

results in a Euler-Lagrange equation for each function, e.g.

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial x'} = 0$$
$$\frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial y'} = 0.$$

These can then be solved as above.

- Under Lagrangian mechanics, the independent variable is time t and the dependent variable(s) depend on the system under consideration. In general they're denoted $q_1, q_2, \dots, q_n$ and are called **generalized coordinates**.

7 Lagrange's Equations

- Lagrangian mechanics has two advantages over Newtonian mechanics:
  - Lagrange's equations have the same form in all coordinate systems, and
  - Lagrange's equations omit the forces of constraint (e.g. the normal force that keeps a bead on a wire), simplifying calculations.

7.1 Lagrange's Equations for Unconstrained Motion

- The **Lagrangian function** or **Lagrangian** is defined as

$$\mathcal{L} = K - U,$$

i.e. the kinetic energy minus the potential energy.

- **Hamilton's principle** states that the actual path taken by a particle between points 1 and 2 in a given time interval t_1 to t_2 is such that the action integral

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

is stationary when taken along the actual path, i.e. the actual path is the solution of the Euler-Lagrange equation when applied to the Lagrangian.

- A Lagrangian can be written in terms of any **generalized coordinates** q_1, q_2, q_3 providing each position $\mathbf{r}$ corresponds to a unique value (q_1, q_2, q_3) and vice versa.

- The derivative of the Lagrangian with respect to x

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial}{\partial x}(K - U) = \frac{\partial}{\partial x}\left(\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(x, y)\right) = -\frac{\partial U(x, y)}{\partial x} = F_x$$

is the x component of the force while the derivative with respect to $\dot{x}$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}(K - U) = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(x, y)\right) = m\dot{x} = p_x$$

is the x component of the momentum. The same applies to the y and z dimensions. When generalized coordinates q_1, q_2, q_3 are used the corresponding values behave like forces and momenta and are called **generalized forces** and **generalized momenta**, respectively.

- Another way of stating the above is

$$\frac{\partial \mathcal{L}}{\partial q_i} = (i\text{th component of generalized force})$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = (i\text{th component of generalized momentum}).$$

Using this terminology, the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

takes the form

$$(\text{generalized force}) = (\text{rate of change of generalized momentum}).$$

- For example, in 2D polar coordinates (r, ϕ) the generalized force for the ϕ coordinate is the torque on the particle and the generalized momentum is the angular momentum.

- Conservation laws can be derived from the Euler-Lagrange equations in generalized coordinates. For example, if the ith component of the generalized force is zero

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0$$

then the rate of change of the ith component of the generalized momentum is also zero and thus it doesn't change.

- If the relationship between $\mathbf{r}$ and the generalized coordinates $q_1, q_2, \ldots, q_n$ doesn't involve t the generalized coordinates are called **natural** and have some additional properties.

- The number of **degrees of freedom** of a system is the number of coordinates that can be independently varied in a small displacement, i.e. the number of independent "directions" in which the system can move from any given initial configuration.
- When the number of degrees of freedom of an N particle system is less than $3N$ (or $2N$ in two dimensions), the system is said to be **constrained**.
- When the number of degrees of freedom of a system matches the number of generalized coordinates required to model the system, it is said to be **holonomic**.
- In order to apply Lagrange's equations to a system its constraints must be holonomic, i.e. they must be expressible in the form

$$f(q_1, q_2, \ldots, q_n, t) = 0.$$

- The generalized coordinates can be measured relative to a non-inertial reference frame providing the Lagrangian $\mathcal{L} = K - U$ is originally written as inertial.

7.6 Generalized Momenta and Ignorable Coordinates

- When the Lagrangian is independent of a coordinate q_i, that coordinate is said to be **ignorable** or **cyclic**. When choosing coordinates, it is desirable to make as many ignorable as possible.

7.8 More about Conservation Laws

- If the Lagrangian is unchanged by spacial translation, the total momentum of the system is conserved.
- If the Lagrangian is unchanged by time translation, the total energy of the system is conserved.

7.9 Lagrange's Equations for Magnetic Forces

- For a given mechanical system with generalized coordinates $q = (q_1, \ldots, q_n)$, a **Lagrangian** $\mathcal{L}$ is a function $\mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$ of the coordinates and velocities, such that the correct equations of motion for the system are the Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \text{ for } i = 1, \ldots, n.$$

- It's important to note that the above does not define a unique Lagrangian function — any function $\mathcal{L}$ that gives the correct equations of motion is valid and has all the correct properties.

- The Lagrangian for a particle of charge q and mass m moving in electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ is

$$\mathcal{L} = \frac{1}{2}m\dot{\mathbf{r}}^2 - q(V - \dot{\mathbf{r}} \cdot \mathbf{A}).$$

8 Two-Body Central Force Problems

8.1 The Problem

- If two objects that experience a conservative central force, their potential energy depends only on the distance between them

$$U(\mathbf{r}_1, \mathbf{r}_2) = U(|\mathbf{r}_1 - \mathbf{r}_2|) = U(r)$$

and thus the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2 - U(r).$$

8.2 CM and Relative Coordinates; Reduced Mass

- It is simplest if the generalized coordinates are chosen to be the position of the centre of mass of the system

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}$$

and the relative position of the two bodies

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2.$$

- This results in a kinetic energy

$$K = \frac{1}{2}(M\dot{\mathbf{R}}^2 + \mu\dot{\mathbf{r}}^2)$$

where

$$\mu = \frac{m_1 m_2}{M} = \frac{m_1 m_2}{m_1 + m_2}$$

is the **reduced mass** of the system.

- The Lagrangian is then

$$\mathcal{L} = K - U = \frac{1}{2}M\dot{\mathbf{R}}^2 + \left[\frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)\right] = \mathcal{L}_{\text{cm}} + \mathcal{L}_{\text{rel}}$$

where each generalized coordinate only appears in one “sub-Lagrangian” and can be solved separately.

8.3 The Equations of Motion

- Because $\mathcal{L}_{\text{cm}}$ doesn't include $\mathbf{R}$ the equation of motion for the centre of mass is

$$M\ddot{\mathbf{R}} = \mathbf{0},$$

i.e. the centre of mass moves with constant velocity.

- The equation of relative motion is

$$\mu\ddot{\mathbf{r}} = -\nabla U(r),$$

i.e. the two bodies move as if they were a single particle of mass μ with potential energy $U(r)$.

- If we choose to use the inertial centre-of-mass reference frame, $\mathcal{L}_{\text{cm}} = 0$ and $\mathcal{L} = \mathcal{L}_{\text{rel}}$ becomes a one-body problem.

- The total angular momentum in the centre-of-mass frame is

$$\mathbf{L} = \mathbf{r} \times \mu\dot{\mathbf{r}}.$$

Because the total angular momentum is conserved — including its direction — this means that $\mathbf{r}$ and $\dot{\mathbf{r}}$ are confined to a plane that we can choose to be the xy plane. The three-dimensional two-body problem has been turned into a two-dimensional one-body problem.

- The Lagrangian for this two-dimensional problem in polar coordinates is

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r).$$

Because this doesn't involve ϕ the Lagrange equation corresponding to ϕ is

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2\dot{\phi} = \text{const} = \ell$$

which is simply a statement of the conservation of angular momentum. The Lagrange equation corresponding to r is

$$\mu r\dot{\phi}^2 - \frac{dU}{dr} = \mu\ddot{r}.$$

8.4 The Equivalent One-Dimensional Problem

- Rearranging the ϕ equation we find

$$\dot{\phi} = \frac{\ell}{\mu r^2}$$

where ℓ is determined by initial conditions.

- The radial equation can be rewritten as

$$\begin{aligned}\mu\ddot{r} &= -\frac{dU}{dr} + \mu r\dot{\phi}^2 \\ &= -\frac{dU}{dr} + F_{\text{cf}}\end{aligned}$$

where F_{cf} is the fictitious centifugal force

$$F_{\text{cf}} = \mu r\dot{\phi}^2 = \frac{\ell^2}{\mu r^3} = -\frac{d}{dr}\left(\frac{\ell^2}{2\mu r^2}\right) = -\frac{dU_{\text{cf}}}{dr}.$$

- The radial equation can now be written in terms of the **effective potential energy**

$$\mu\ddot{r} = -\frac{d}{dr}[U(r) + U_{\text{cf}}(r)] = -\frac{d}{dr}U_{\text{eff}}(r).$$

- The total energy of the one-body system is

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + U(r)$$

and this value is conserved.

9 Mechanics in Noninertial Frames

9.1 Acceleration without Rotation

- A noninertial frame of reference $\mathcal{S}$ has acceleration $\mathbf{A}$ relative to an inertial frame of reference $\mathcal{S}_0$. Newton's second law can be used in $\mathcal{S}$ providing we add an extra force-like term called the **inertial force**

$$m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{A}.$$

9.3 The Angular Velocity Vector

- The angular velocity vector $\boldsymbol{\omega}$ has direction equal to that of the axis of rotation (using the right-hand rule to disambiguate direction) and magnitude equal to the rate of rotation.

- If a body is rotating with angular velocity $\boldsymbol{\omega}$ about an axis through O, the velocity of a point P (position $\mathbf{r}$) fixed on the body is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}.$$

- Suppose there are two frames of reference 2 and 1. Frame 2 is rotating with angular velocity $\boldsymbol{\omega}_{21}$ relative to frame 1. A body 3 is rotating with angular velocities $\boldsymbol{\omega}_{31}$ and $\boldsymbol{\omega}_{32}$ relative to frames 1 and 2, respectively. These angular velocity vectors add such that

$$\boldsymbol{\omega}_{31} = \boldsymbol{\omega}_{32} + \boldsymbol{\omega}_{21}.$$

9.4 Time Derivatives in a Rotating Frame

- Given an inertial reference frame $\mathcal{S}_0$ and a noninertial reference frame $\mathcal{S}$ that is rotating with angular velocity $\mathbf{\Omega}$ relative to $\mathcal{S}_0$, the time derivative of a vector $\mathbf{Q}$ differs between the two with the relation

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}_0} = \left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}} + \mathbf{\Omega} \times \mathbf{Q}.$$

9.5 Newton's Second Law in a Rotating Frame

- Newton's second law in a noninertial frame rotating with angular velocity $\mathbf{\Omega}$ is

$$\begin{aligned} m\ddot{\mathbf{r}} &= \mathbf{F} + \mathbf{F}_{\text{cor}} + \mathbf{F}_{cf} \\ &= \mathbf{F} + 2m\dot{\mathbf{r}} \times \mathbf{\Omega} + m(\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{\Omega} \end{aligned}$$

  where $\mathbf{F}_{\text{cor}}$ is the **Coriolis force** and $\mathbf{F}_{\text{cf}}$ is the **centrifugal force**.

10 Rotational Motion of Rigid Bodies

10.1 Properties of the Centre of Mass

- The total angular momentum of a system of N particles is

$$\begin{aligned} \mathbf{L} &= \mathbf{L}_{\text{O}} + \mathbf{L}_{\text{CM}} \\ &= \mathbf{R} \times \mathbf{P} + \sum \mathbf{r}'_\alpha \times m_\alpha \dot{\mathbf{r}}'_\alpha \end{aligned}$$

  where $\mathbf{L}_{\text{O}}$ is the angular momentum of the centre of mass of the system relative to the origin, $\mathbf{L}_{\text{CM}}$ is the angular momentum of the particles of the system relative to its centre of mass, $\mathbf{R}$ is the centre of mass of the system, $\mathbf{P}$ is the linear momentum of the system, $\mathbf{r}'_\alpha$ is the position of partical α relative to the centre of mass of the system, m_α is the mass of particle α, and $\dot{\mathbf{r}}'_\alpha$ is the velocity of particle α relative to the centre of mass of the system.

- The values $\mathbf{L}_{\text{O}}$ and $\mathbf{L}_{\text{CM}}$ are independently conserved, with

$$\dot{\mathbf{L}}_{\text{O}} = \mathbf{R} \times \mathbf{F}^{\text{ext}} = \boldsymbol{\tau}^{\text{ext}} \text{ (about origin)}$$

  and

$$\dot{\mathbf{L}}_{\text{CM}} = \sum \mathbf{r}'_\alpha \times \mathbf{F}^{\text{ext}}_\alpha = \boldsymbol{\tau}^{\text{ext}} \text{ (about centre of mass).}$$

- The total kinetic energy of a system of N particles is

$$\begin{aligned} T &= T_{\text{CM}} + T_{\text{relative to CM}} \\ &= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\sum m_\alpha \dot{\mathbf{r}}_\alpha^2 \end{aligned}$$

- The total potential energy of a system of N particles is

$$U = U^{\text{ext}} + U^{\text{int}}$$

where U^{ext} is the total potential energy due to external forces and U^{int} is the total potential energies for all pairs of particles

$$U^{\text{int}} = \sum_{\alpha<\beta} U_{\alpha\beta}(r_{\alpha\beta}).$$

However, in a rigid body the distances between all particles are fixed so U^{int} is constant and can be ignored.

10.2 Rotation about a Fixed Axis

- For a system of N particles rotating around the z axis, the angular momentum of the system is is

$$\begin{aligned}\mathbf{L} &= \sum_\alpha \boldsymbol{\ell}_\alpha \\ &= \sum_\alpha m_\alpha \mathbf{r}_\alpha \times \mathbf{v}_\alpha \\ &= \sum_\alpha m_\alpha \omega(-z_\alpha x_\alpha, -z_\alpha y_\alpha, x_\alpha^2 + y_\alpha^2).\end{aligned}$$

Thus, $\mathbf{L}$ may have $\mathbf{x}$ and/or $\mathbf{y}$ components and $\mathbf{L} = I\boldsymbol{\omega}$ doesn't hold.

- The angular momentum can be expressed as

$$\mathbf{L} = (I_{xz}\omega, I_{yz}\omega, I_{zz}\omega)$$

where

$$\begin{aligned}I_{xz} &= -\sum_\alpha m_\alpha x_\alpha z_\alpha, \\ I_{yz} &= -\sum_\alpha m_\alpha y_\alpha z_\alpha, \text{ and} \\ I_{zz} &= \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2)\end{aligned}$$

are called the **products of inertia** of the body. The notation means that e.g. I_{xz} is the x component of the angular momentum for a body rotating around the z axis.

10.3 Rotation about Any Axis; the Inertia Tensor

- Continuing from the above, if a body has angular velocity

$$\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$$

its angular momentum is

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$
$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$
$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

where

$$I_{xx} = \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2),$$
$$I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha,$$

and the other values follow the same pattern.

- The above can also be written

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$$

where **I** is the **moment of inertia tensor**

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}.$$

10.4 Principle Axes of Inertia

- In general, the angular momentum **L** of a spinning object is not parallel to $\boldsymbol{\omega}$. However for any rigid body and any point O there are three perpendicular axes through O such that **I** is diagonal when $\boldsymbol{\omega}$ is parallel to one of these axes, so is **L**. These are called the **principle axes**.
- The three moments of inertia about the three principle axes are called the **principle moments**.
- If we choose the principle axes of a rigid body as our frame of reference rotation calculations are simplified but we're using a noninertial frame of reference.

10.5 Finding the Principle Axes; Eigenvalue Equations

- If $\boldsymbol{\omega}$ is parallel to a principle axis, then $\mathbf{L} = \mathbf{I}\boldsymbol{\omega} = \lambda\boldsymbol{\omega}$. This can be converted to an eigenvalue equation

$$(\mathbf{I} - \lambda\mathbf{1})\boldsymbol{\omega} = 0$$

where the eigenvectors are the principle axes. Thus to find the principle axes of a rigid body about a point we must calculate an initial intertia tensor then calculate its eigenvectors. The updated inertia tensor will be

$$\mathbf{I} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

10.7 Euler's Equations

- The axes x, y, and z are chosen to be in an inertial frame called the **space frame** while the axes $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are chosen to align with the body's principle axes and this is is a non-inertial frame known as the **body frame**. If the body is rotating around a fixed point that is taken to be the origin of the body frame, otherwise the body's centre of mass is used.

- Suppose a body is rotating with angular velocity $\boldsymbol{\omega}$ and has principle moments of inertia λ_1, λ_2, and λ_3, then its angular momentum in the body frame is
$$\mathbf{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3).$$
If a torque $\boldsymbol{\Gamma}$ acts on the body, then in the space frame
$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{space}} = \boldsymbol{\Gamma}.$$
In the body frame this becomes
$$\begin{aligned}\left(\frac{d\mathbf{L}}{dt}\right)_{\text{space}} &= \left(\frac{d\mathbf{L}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \mathbf{L}\\ &= \dot{\mathbf{L}} + \boldsymbol{\omega} \times \mathbf{L}.\end{aligned}$$
Equating the previous two equations gives
$$\dot{\mathbf{L}} + \boldsymbol{\omega} + \mathbf{L} = \boldsymbol{\Gamma}$$
or
$$\begin{aligned}\lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3 &= \Gamma_1\\ \lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_3\omega_1 &= \Gamma_2\\ \lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1\omega_2 &= \Gamma_3.\end{aligned}$$
where all quantities are in the body frame. This is known as **Euler's equation** and it is the equation of motion for the rotation of the body.