Advanced Engineering Mathematics Complex Analysis by Dennis G. Zill Problems

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17 Functions of a Complex Variable

17.1 Complex Numbers

17.1.1

$$3 + 3i$$

17.1.3

$$i^8 = (i^2)^4 = (-1)^4 = 1$$

17.1.5

$$7 - 13i$$

17.1.7

$$-7 + 5i$$

17.1.9

$$11 - 10i$$

17.1.11

$$-5 + 12i$$

17.1.13

$$-2i$$

17.1.15

$$\begin{aligned}\frac{2-4i}{3+5i} &= \frac{(2-4i)(3-5i)}{34} \\ &= \frac{-14-22i}{34} \\ &= -\frac{7}{17} - \frac{11}{17}i\end{aligned}$$

17.1.17

$$\begin{aligned}\frac{(3-i)(2+3i)}{1+i} &= \frac{9+7i}{1+i} \\ &= \frac{(9+7i)(1-i)}{2} \\ &= \frac{16-2i}{2} \\ &= 8-i\end{aligned}$$

17.1.27

$$\begin{aligned}\frac{1}{z} &= \frac{\overline{z}}{z\overline{z}} \\ &= \frac{x-iy}{x^2+y^2} \\ \operatorname{Re}\left(\frac{1}{z}\right) &= \frac{x}{x^2+y^2}\end{aligned}$$

17.1.29

$$\begin{aligned}2z+4\overline{z}-4i &= 2(x+iy)+4(x-iy)-4i \\ &= 6x-2(y+2)i \\ \operatorname{Im}(2z+4\overline{z}-4i) &= -2y-4\end{aligned}$$

17.1.31

$$\begin{aligned}z-1-3i &= x+iy-1-3i \\ &= (x-1)+(y-3)i \\ |z| &= \sqrt{(x-1)^2+(y-3)^2}\end{aligned}$$

17.1.33

$$\begin{aligned} 2z &= i(2+9i) \\ &= -9+2i \\ z &= -\frac{9}{2}+i \end{aligned}$$

17.1.35

$$\begin{aligned} (x+iy)^2 &= x^2+2xyi-y^2 \\ &= (x^2-y^2)+2xyi \\ x^2 &= y^2 \\ x &= y \\ 2xy &= 1 \\ x^2 &= \frac{1}{2} \\ x &= \frac{\sqrt{2}}{2} \\ z &= \frac{\sqrt{2}}{2}(1+i) \end{aligned}$$

17.1.37

$$\begin{aligned} z+2\overline{z} &= x+iy+2x-2iy \\ &= 3x-iy \\ \frac{2-i}{1+3i} &= \frac{(2-i)(1-3i)}{10} \\ &= \frac{-1-7i}{10} \\ 3x-iy &= \frac{-1-7i}{10} \\ x &= -\frac{1}{30} \\ y &= \frac{7}{10} \\ z &= -\frac{1}{30}+\frac{7}{10}i \end{aligned}$$

17.1.39

$$\begin{aligned} |10+8i| &\approx 12.8 \\ |11-6i| &\approx 12.5 \end{aligned}$$

$11-6i$ is closer.

17.2 Powers and Roots

17.2.1

$$2(\cos 0 + i \sin 0)$$

17.2.3

$$-3[\cos(-\pi/2) + i \sin(-\pi/2)]$$

17.2.5

$$\sqrt{2}[\cos(\pi/4) + i \sin(\pi/4)]$$

17.2.7

$$2[\cos(5\pi/6) + i \sin(5\pi/6)]$$

17.2.9

$$\begin{aligned}
\frac{3}{-1+i} &= \frac{3(-1-i)}{2} \\
&= \frac{-3-3i}{2} \\
&= -\frac{3}{2} - \frac{3}{2}i \\
&= \frac{3\sqrt{2}}{2}[\cos(5\pi/4) + i \sin(5\pi/4)]
\end{aligned}$$

17.2.11

$$-\frac{5\sqrt{3}}{2} - \frac{5}{2}i$$

17.2.13

$$5.54 + 2.30i$$

17.2.15

$$\begin{gathered}
8[\cos(\pi/2) + i \sin(\pi/2)] = 8i \\
\frac{1}{2}[\cos(-\pi/4) + i \sin(-\pi/4)] = \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}i
\end{gathered}$$

17.2.21

$$\begin{aligned}(1+\sqrt{3}i)^9 &= \{2[\cos(\pi/3)+i\sin(\pi/3)]\}^9\\ &= 512(\cos\pi + i\sin\pi)\\ &= -512\end{aligned}$$

17.2.23

$$\begin{aligned}\left(\frac{1}{2}+\frac{1}{2}i\right)^1 0 &= \left\{\frac{\sqrt{2}}{2}[\cos(\pi/4)+i\sin(\pi/4)]\right\}^{10}\\ &= \frac{1}{32}[\cos(\pi/2)+i\sin(\pi/2)]\\ &= \frac{1}{32}i\end{aligned}$$

17.2.27

$$\begin{aligned}w_k &= 2[\cos(2\pi k/3)+i\sin(2\pi k/3)]\\ w_0 &= 2\\ w_1 &= -1+\sqrt{3}i\\ w_2 &= -1-\sqrt{3}i\end{aligned}$$

17.2.29

$$\begin{aligned}w_k &= \cos(\pi/4+k\pi)+i\sin(\pi/4+k\pi)\\ w_0 &= \frac{\sqrt{2}}{2}(1+i)\\ w_1 &= -\frac{\sqrt{2}}{2}(1+i)\end{aligned}$$

17.2.31

$$\begin{aligned}w_k &= \sqrt{2}[\cos(\pi/3+k\pi)+i\sin(\pi/3+k\pi)]\\ w_0 &= \frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}i\\ w_1 &= -\frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}i\end{aligned}$$

17.2.33

$$
\begin{aligned}
z^4 + 1 &= 0 \\
z^4 &= -1 \\
w_k &= \cos(\pi/4 + k\pi/2) + \sin(\pi/4 + k\pi/2) \\
w_0 &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \\
w_1 &= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \\
w_2 &= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \\
w_3 &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i
\end{aligned}
$$

17.3 Sets in the Complex Plane

17.3.1

A vertical line at $\mathrm{Re}(z) = 5$.

17.3.3

A horizontal line at $\mathrm{Im}(z) = -3$.

17.3.5

A circle of radius 2 centred at $3i$.

17.3.7

A circle of radius 5 centred at $4 - 3i$.

17.3.9

The region of the plane to the left of (but not including) $\mathrm{Re}(z) = -1$. It is a domain.

17.3.11

The region of the plane above (but not including) $\mathrm{Im}(z) = 3$. It is a domain.

17.3.13

The region of the plane between (but not including) $\mathrm{Re}(z) = 3$ and $\mathrm{Re}(z) = 5$. It is a domain.

17.3.15

$$\begin{aligned}
z^2 &= (a+ib)^2 \\
&= a^2 - b^2 + 2iab \\
\mathrm{Re}(z^2) &= a^2 - b^2 \\
\mathrm{Re}(z^2) &> 0 \\
a^2 - b^2 &> 0 \\
a^2 &> b^2
\end{aligned}$$

The region between $y = x$ and $y = -x$. Not a domain.

17.3.17

The region between $\theta = 0$ and $\theta = 2\pi/3$. Not a domain.

17.3.19

The region outside a circle of radius 1 centred at i. It is a domain.

17.3.21

The region between the circles of radius 2 and 3 centred at i. It is a domain.

17.3.23

$$y = -x$$

17.3.25

$$\begin{aligned}
z^2 + \overline{z}^2 &= (a+ib)^2 + (a-ib)^2 \\
&= a^2 + 2iab - b^2 + a^2 - 2iab - b^2 \\
&= 2(a^2 - b^2) \\
2(a^2 - b^2) &= 2 \\
a^2 - b^2 &= 1 \\
a^2 &= b^2 + 1
\end{aligned}$$

The hyperbola $x^2 - y^2 = 1$.

17.4 Functions of a Complex Variable

17.4.1

$$
\begin{aligned}
f(z) &= z^2 \\
&= (x+iy)^2 \\
&= x^2 - y^2 + 2ixy \\
u(x,y) &= x^2 - y^2 \\
&= x^2 - 4 \\
v(x,y) &= 2xy \\
&= 4x \\
x &= \frac{v}{4} \\
u &= \left(\frac{v}{4}\right)^2 - 4 \\
&= \frac{1}{16}v^2 - 4
\end{aligned}
$$

17.4.3

$$
\begin{aligned}
u &= -y^2 \\
v &= 0
\end{aligned}
$$

Line on the left half of the real axis.

17.4.5

$$
\begin{aligned}
u &= 0 \\
v &= 2x^2
\end{aligned}
$$

Line on the top half of the imaginary axis.

17.4.7

$$f(x) = (6x-5) + i(6y+9)$$

17.4.9

$$f(z) = (x^2 - y^2 - 3x) + i(2xy - 3y + 4)$$

17.4.11

$$f(z) = (x^3 - 3xy^2 - 4x) + i(3x^2y - y^3 - 4y)$$

17.4.13

$$f(z) = \left(x + \frac{x}{x^2 + y^2}\right) i \left(y - \frac{y}{x^2 + y^2}\right)$$

17.4.15

(a) $-4 + i$

(b) $3 - 9i$

(c) $1 + 86i$

17.4.17

(a) $14 - 20i$

(b) $-13 + 43i$

(c) $3 - 26i$

17.4.19

$$6 - 5i$$

17.4.21

$$-4i$$

17.4.27

$$f'(z) = 12z^2 - 2(3 + i)z - 5$$

17.4.29

$$\begin{aligned} f'(z) &= 2(z^2 - 4z + 8i) + (2z + 1)(2z - 4) \\ &= 2z^2 - 8z + 16i + 4z^2 - 8z + 2z - 4 \\ &= 6z^2 - 14z - 4 + 16i \end{aligned}$$

17.4.31

$$f'(z) = 6z(z^2 - 4i)^2$$

17.4.33

$$\begin{aligned} f'(z) &= \frac{3(2z+i)-2(3z-4+8i)}{(2z+i)^2} \\ &= \frac{6z+3i-6z+8-16i}{(2z+i)^2} \\ &= \frac{8-13i}{(2z+i)^2} \end{aligned}$$

17.4.35

$3i$

17.4.37

$\pm 2i$

17.4.41

$$\begin{aligned} \frac{dx}{dt} &= 2x \\ x &= c_1 e^{2t} \\ \frac{dy}{dt} &= 2y \\ y &= c_2 e^{2t} \end{aligned}$$

17.4.43

$$\begin{aligned}
f(z) &= \frac{1}{\bar{z}} \\
&= \frac{1}{x - iy} \\
&= \frac{x + iy}{x^2 + y^2} \\
&= \frac{x}{x^2 + y^2} + i\frac{y}{x^2 + y^2} \\
\frac{dx}{dt} &= \frac{x}{x^2 + y^2} \\
\frac{dy}{dt} &= \frac{y}{x^2 + y^2} \\
\frac{dy}{dx} &= \frac{y}{x} \\
\frac{dy}{y} &= \frac{dx}{x} \\
\ln y &= \ln x + c_1 \\
y &= c_2 x
\end{aligned}$$

17.5 Cauchy-Riemann Equations

17.5.1

$$\begin{aligned}
f(z) &= z^3 \\
&= (x + iy)^3 \\
&= (x^2 + 2ixy - y^2)(x + iy) \\
&= x^3 + ix^2y + 2ix^2y - 2xy^2 - xy^2 - iy^3 \\
&= (x^3 - 3xy^2) + i(3x^2y - y^3) \\
\frac{\partial u}{\partial x} &= 3x^2 - 3y^2 \\
&= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -6xy \\
&= -\frac{\partial v}{\partial x}
\end{aligned}$$

17.5.3

$$\begin{aligned} f(z) &= \mathrm{Re}(z) \\ &= x \\ \frac{\partial u}{\partial x} &= 1 \\ &\neq \frac{\partial v}{\partial y} \end{aligned}$$

17.5.5

$$\begin{aligned} f(z) &= 4z - 6\overline{z} + 3 \\ &= 4(x+iy) - 6(x-iy) + 3 \\ &= (-2x+3) + 10iy \\ \frac{\partial u}{\partial x} &= -2 \\ &\neq \frac{\partial v}{\partial y} \end{aligned}$$

17.5.7

$$\begin{aligned} f(z) &= x^2 + y^2 \\ \frac{\partial u}{\partial x} &= 2x \\ &\neq \frac{\partial v}{\partial y} \end{aligned}$$

17.5.9

$$\begin{aligned} f(z) &= e^x \cos y + ie^x \sin y \\ u &= e^x \cos y \\ \frac{\partial u}{\partial x} &= e^x \cos y \\ \frac{\partial u}{\partial y} &= -e^x \sin y \\ v &= e^x \sin y \\ \frac{\partial v}{\partial x} &= e^x \sin y \\ \frac{\partial v}{\partial y} &= e^x \cos y \end{aligned}$$

Analytic everywhere.

17.5.11

$$\begin{aligned}
f(z) &= x + \sin x \cosh y + i(y + \cos x \sinh y) \\
u &= x + \sin x \cosh y \\
\frac{\partial u}{\partial x} &= 1 + \cos x \cosh y \\
\frac{\partial u}{\partial y} &= \sin x \sinh y \\
v &= y + \cos x \sinh y \\
\frac{\partial v}{\partial x} &= -\sin x \sinh y \\
\frac{\partial v}{\partial y} &= 1 + \cos x \cosh y
\end{aligned}$$

Analytic everywhere.

17.5.15

$$\begin{aligned}
f(z) &= 3x - y + 5 + i(ax + by - 3) \\
u &= 3x - y + 5 \\
\frac{\partial u}{\partial x} &= 3 \\
\frac{\partial u}{\partial y} &= -1 \\
v &= ax + by - 3 \\
\frac{\partial v}{\partial x} &= a \\
\frac{\partial v}{\partial y} &= b \\
a &= 1 \\
b &= 3
\end{aligned}$$

17.5.17

$$
\begin{aligned}
f(z) &= x^2 + y^2 + 2ixy \\
u &= x^2 + y^2 \\
\frac{\partial u}{\partial x} &= 2x \\
\frac{\partial u}{\partial y} &= 2y \\
v &= 2xy \\
\frac{\partial v}{\partial x} &= 2y \\
\frac{\partial v}{\partial y} &= 2x
\end{aligned}
$$

Only differentiable when $y = 0$.

17.5.19

$$
\begin{aligned}
f(z) &= x^3 + 3xy^2 - x + i(y^3 + 3x^2y - y) \\
u &= x^3 + 3xy^2 - x \\
\frac{\partial u}{\partial x} &= 3x^2 + 3y^2 - 1 \\
\frac{\partial u}{\partial y} &= 6xy \\
v &= y^3 + 3x^2y - y \\
\frac{\partial v}{\partial x} &= 6xy \\
\frac{\partial v}{\partial y} &= 3y^2 + 3x^2 - 1
\end{aligned}
$$

Only differentiable when $x = 0$ or $y = 0$.

17.5.21

$$
\begin{aligned}
f(z) &= e^x \cos y + ie^x \sin y \\
f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\
&= e^x \cos y + ie^x \sin y
\end{aligned}
$$

17.5.23

$$\begin{aligned}
u &= x \\
\frac{\partial^2 u}{\partial x^2} &= 0 \\
\frac{\partial^2 u}{\partial y^2} &= 0 \\
\frac{\partial v}{\partial y} &= 1 \\
v &= y + h(x) \\
h'(x) &= 0 \\
v &= y + c \\
f(z) &= x + i(y + c)
\end{aligned}$$

17.5.25

$$\begin{aligned}
u &= x^2 - y^2 \\
\frac{\partial^2 u}{\partial x^2} &= 2 \\
\frac{\partial^2 u}{\partial y^2} &= -2 \\
\frac{\partial v}{\partial y} &= 2x \\
v &= 2xy + h(x) \\
2y &= 2y + h'(x) \\
h'(x) &= 0 \\
h(x) &= c \\
v &= 2xy + c \\
f(z) &= (x^2 - y^2) + i(2xy + c)
\end{aligned}$$

17.6 Exponential and Logarithmic Functions

17.6.1

$$\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

17.6.3

$$e^{-1}\frac{\sqrt{2}}{2}(1 + i)$$

17.6.5

$$-e^{\pi}$$

17.6.7

$$e^{1.5}(\cos 2 + i\sin 2) = -1.865 + 4.075i$$

17.6.9

$$\cos 5 + i\sin 5 = 0.2836 - 0.9589i$$

17.6.11

$$\begin{aligned} e^{1+5\pi i/4}e^{-1-\pi i/3} &= e^{11\pi i/12} \\ &= \cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12} \\ &= -0.9659 + 0.2588i \end{aligned}$$

17.6.13

$$\begin{aligned} f(z) &= e^{-iz} \\ &= e^{-i(x+iy)} \\ &= e^{y-ix} \\ &= e^{y}(\cos x - i\sin x) \end{aligned}$$

17.6.15

$$\begin{aligned} f(z) &= e^{z^2} \\ &= e^{x^2-y^2+2ixy} \\ &= e^{x^2-y^2}[\cos(2xy) + i\sin(2xy)] \end{aligned}$$

17.6.17

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x(\cos y + i\sin y) \\ |e^z| &= \sqrt{e^{2x}[\cos^2 y + \sin^2 y]} \\ &= e^x \end{aligned}$$

17.6.19

$$\begin{aligned}
e^{z+\pi i} &= e^{x+i(y+\pi)} \\
&= e^x[\cos(y+\pi) + i\sin(y+\pi)] \\
&= e^x[-\cos y - i\sin y] \\
&= -e^x(\cos y + i\sin y) \\
e^{z-\pi i} &= e^{x+i(y-\pi)} \\
&= e^x[\cos(y-\pi) + i\sin(y-\pi)] \\
&= e^x(-\cos y - i\sin y) \\
&= -e^x(\cos y + i\sin y)
\end{aligned}$$

17.6.21

$$\begin{aligned}
e^{\overline{z}} &= e^{x-iy} \\
&= e^x(\cos y - i\sin y) \\
u &= e^x\cos y \\
v &= -e^x\sin y \\
\frac{\partial u}{\partial x} &= e^x\cos y \\
&\neq \frac{\partial v}{\partial y}
\end{aligned}$$

17.6.23

$$\log_e 5 + i(\pi + 2n\pi) = 1.6094 + i(\pi + 2n\pi)$$

17.6.25

$$\log_e(2\sqrt{2}) + i\left(\frac{3}{4}\pi + 2n\pi\right) = 1.0397 + i\left(\frac{3}{4}\pi + 2n\pi\right)$$

17.6.27

$$\log_e(2\sqrt{2}) + i\left(\frac{1}{3}\pi + 2n\pi\right) = 1.0397 + i\left(\frac{1}{3}\pi + 2n\pi\right)$$

17.6.29

$$\log_e(6\sqrt{2}) - \frac{\pi}{4}i = 2.1383 - \frac{\pi}{4}i$$

17.6.31

$$\log_e 13 + 2.7468i = 2.5649 + 2.7468i$$

17.6.33

$$5\left(\log_e 2+\frac{\pi}{3}i\right)=3.4657-\frac{\pi}{3}i$$

17.6.35

$$z=\log_e 4+i\left(\frac{\pi}{2}+2n\pi\right)=1.3863+i\left(\frac{\pi}{2}+2n\pi\right)$$

17.6.37

$$\begin{aligned}
z-1&=2+i\left(-\frac{\pi}{2}+2n\pi\right)\\
z&=3+i\left(-\frac{\pi}{2}+2n\pi\right)
\end{aligned}$$

17.6.39

$$\begin{aligned}
\ln(-i)&=i\left(-\frac{\pi}{2}+2n\pi\right)\\
(-i)^{4i}&=e^{4i\ln(-i)}\\
&=e^{4i\times i(-\pi/2+2n\pi)}\\
&=e^{2\pi(1-4n)}
\end{aligned}$$

17.6.41

$$\begin{aligned}
\ln(1+i)&=\log_e\sqrt{2}+i\left(\frac{\pi}{4}+2n\pi\right)\\
(1+i)^{(1+i)}&=e^{(1+i)\ln(1+i)}\\
&=e^{(1+i)[\log_e\sqrt{2}+i(\pi/4+2n\pi)]}\\
&=e^{\log_e\sqrt{2}+i(\pi/4+2n\pi)+i\log_e\sqrt{2}-(\pi/4+2n\pi)}\\
&=e^{(\log_e\sqrt{2}-\pi/4-2n\pi)+i(\log_e\sqrt{2}+\pi/4+2n\pi)}\\
&=e^{-2n\pi}e^{(\log_e\sqrt{2}-\pi/4)+i(\log_e\sqrt{2}+\pi/4)}\\
&=e^{-2n\pi}e^{\log_e\sqrt{2}-\pi/4}e^{i(\log_e\sqrt{2}+\pi/4)}\\
&=e^{-2n\pi}(0.2739+0.5837i)
\end{aligned}$$

17.6.43

$$\begin{aligned}
\mathrm{Ln}(-1)&=\pi i\\
(-1)^{(-2i/\pi)}&=e^{(-2i/\pi)\,\mathrm{Ln}(-1)}\\
&=e^{(-2i/\pi)(\pi i)}\\
&=e^{2}
\end{aligned}$$

17.6.47

(a)

$$
\begin{aligned}
(-1+i)^2 &= -2i \\
\operatorname{Ln}(-1+i)^2 &= \operatorname{Ln}(-2i) \\
&= \log_e 2 - \frac{\pi}{2} i \\
2\operatorname{Ln}(-1+i) &= 2\log_e \sqrt{2} + \frac{3\pi}{2} i \\
&\neq \operatorname{Ln}(-1+i)^2
\end{aligned}
$$

Not true

(b)

$$
\begin{aligned}
\operatorname{Ln} i^3 &= \operatorname{Ln}(-i) \\
&= -\frac{\pi}{2} i \\
3\operatorname{Ln} i &= \frac{3\pi}{2} i \\
&\neq \operatorname{Ln} i^3
\end{aligned}
$$

Not true

(c)

$$
\begin{aligned}
\ln i^3 &= i\left(-\frac{\pi}{2} + 2n\pi\right) \\
3\ln i &= 3i\left(\frac{\pi}{2} + 2n\pi\right) \\
&\neq \ln i^3
\end{aligned}
$$

Not true

17.7 Trigonometric and Hyperbolic Functions

17.7.1

$$
\begin{aligned}
\cos(3i) &= \cos 0 \cosh 3 - i \sin 0 \sinh 3 \\
&= \cosh 3 \\
&= 10.0677
\end{aligned}
$$

17.7.3

$$
\begin{aligned}
\sin(\pi/4 + i) &= \sin\frac{\pi}{4}\cosh 1 + i\cos\frac{\pi}{4}\sinh 1 \\
&= 1.0911 + 0.8309i
\end{aligned}
$$

17.7.5

$$\begin{aligned}
\tan i &= \frac{\sin i}{\cos i} \\
&= \frac{\sin 0 \cosh 1 + i \cos 0 \sinh 1}{\cos 0 \cosh 1 + i \sin 0 \sinh 1} \\
&= \frac{i \sinh 1}{\cosh 1} \\
&= i \tanh 1 \\
&= 0.7615i
\end{aligned}$$

17.7.7

$$\begin{aligned}
\sec(\pi + i) &= \frac{1}{\cos(\pi + i)} \\
&= \frac{1}{\cos \pi \cosh 1 + \sin \pi \sinh 1} \\
&= -\frac{1}{\cosh 1} \\
&= -0.6480
\end{aligned}$$

17.7.9

$$\begin{aligned}
\cosh(\pi i) &= \cosh 0 \cos \pi + i \sinh 0 \sin \pi \\
&= -1
\end{aligned}$$

17.7.11

$$\begin{aligned}
\sinh(1 + \pi i/3) &= \sinh 1 \cos(\pi/3) + i \cosh 1 \sin(\pi/3) \\
&= 0.5876 + 1.3363i
\end{aligned}$$

17.7.15

$$\begin{aligned}
\sin z &= 2 \\
\frac{e^{iz} - e^{-iz}}{2i} &= 2 \\
e^{iz} - e^{-iz} &= 4i \\
e^{2iz} - 1 &= 4ie^{iz} \\
e^{2iz} - 4ie^{iz} - 1 &= 0 \\
e^{iz} &= \frac{4i \pm \sqrt{-16+4}}{2} \\
&= (2 \pm \sqrt{3})i \\
iz &= \log_e(2 \pm \sqrt{3}) + i(\pi/2 + 2n\pi) \\
z &= (\pi/2 + 2n\pi) - i\log_e(2 \pm \sqrt{3})
\end{aligned}$$

17.7.17

$$\begin{aligned}
\sinh z &= -i \\
\frac{e^{z} - e^{-z}}{2} &= -i \\
e^{2z} + 2ie^{z} - 1 &= 0 \\
e^{z} &= \frac{-2i \pm \sqrt{-4+4}}{2} \\
&= -i \\
z &= \ln(-i) \\
&= i\left(-\frac{\pi}{2} + 2n\pi\right)
\end{aligned}$$

17.7.19

$$\begin{aligned}
\cos z &= \sin z \\
\frac{e^{iz}+e^{-iz}}{2} &= \frac{e^{iz}-e^{-iz}}{2i} \\
e^{iz}+e^{-iz} &= \frac{e^{iz}-e^{-iz}}{i} \\
&= -i(e^{iz}-e^{-iz}) \\
e^{2iz}+1 &= -i(e^{2iz}-1) \\
e^{2iz}(1+i) &= -1+i \\
e^{2iz} &= \frac{-1+i}{1+i} \\
&= \frac{(-1+i)(1-i)}{(1+i)(1-i)} \\
&= \frac{-1+i+i+1}{1-i+i+1} \\
&= \frac{2i}{2} \\
&= i \\
2iz &= \ln i \\
&= i\left(\frac{\pi}{2}+2n\pi\right) \\
z &= \frac{\pi}{4}+n\pi
\end{aligned}$$

17.7.21

$$\begin{aligned}
\cos z &= \cosh 2 \\
\cos x \cosh y - i \sin x \sinh y &= \cosh 2 \\
y &= \pm 2 \\
x &= 2n\pi \\
z &= 2n\pi \pm 2i
\end{aligned}$$

17.8 Inverse Trigonometric and Hyperbolic Functions

17.8.1

$$
\begin{aligned}
\arcsin z &= -i\ln[iz + (1-z^2)^{1/2}] \\
\arcsin(-i) &= -i\ln[i(-i) + (1-(-i)^2)^{1/2}] \\
&= -i\ln[1 \pm \sqrt{2}] \\
\ln(1+\sqrt{2}) &= \log_e(1+\sqrt{2}) + 2n\pi i \\
\ln(1-\sqrt{2}) &= \ln\left(-\frac{1}{1+\sqrt{2}}\right) \\
&= -\ln[-(1+\sqrt{2})] \\
&= -[\log_e(1+\sqrt{2}) + i(\pi + 2n\pi)] \\
&= -\log_e(1+\sqrt{2}) + i(\pi + 2n\pi) \\
\ln(1\pm\sqrt{2}) &= (-1)^n\log_e(1+\sqrt{2}) + n\pi i \\
\arcsin(-i) &= -i[(-1)^n\log_e(1+\sqrt{2}) + n\pi i] \\
&= n\pi - (-1)^n i\log_e(1+\sqrt{2}) \\
&= n\pi + (-1)^{n+1} i\log_e(1+\sqrt{2})
\end{aligned}
$$

17.8.3

$$
\begin{aligned}
\arcsin 0 &= -i\ln(\pm 1) \\
&= -i(n\pi i) \\
&= n\pi
\end{aligned}
$$

17.8.5

$$
\begin{aligned}
\arccos 2 &= -i\ln[2 + i(1-2^2)^{1/2}] \\
&= -i\ln[2\pm\sqrt{3}] \\
\ln(2+\sqrt{3}) &= \log_e(2+\sqrt{3}) + 2n\pi i \\
\ln(2-\sqrt{3}) &= \log_e(2-\sqrt{3}) + 2n\pi i \\
&= -\log_e(2+\sqrt{3}) + 2n\pi i \\
\ln(2\pm\sqrt{3}) &= \pm\log_e(2+\sqrt{3}) + 2n\pi i \\
\arccos 2 &= 2n\pi \pm i\log_e(2+\sqrt{3})
\end{aligned}
$$

17.8.7

$$\begin{aligned}
\arccos\frac{1}{2} &= -i\ln\left\{\frac{1}{2}+i\left[1-\left(\frac{1}{2}\right)^2\right]^{1/2}\right\} \\
&= -i\ln\left(\frac{1}{2}\pm\frac{\sqrt{3}}{2}i\right) \\
\ln\left(\frac{1}{2}\pm\frac{\sqrt{3}}{2}i\right) &= i\left(\pm\frac{\pi}{3}+2n\pi\right) \\
\arccos\frac{1}{2} &= \pm\frac{\pi}{3}+2n\pi
\end{aligned}$$

17.8.9

$$\begin{aligned}
\arctan 1 &= \frac{i}{2}\ln\frac{1+i}{-1+i} \\
\frac{1+i}{-1+i} &= \frac{(1+i)(-1-i)}{(-1+i)(-1-i)} \\
&= -i \\
\ln(-i) &= i\left(-\frac{\pi}{2}+2n\pi\right) \\
\arctan 1 &= \frac{\pi}{4}+n\pi
\end{aligned}$$

17.8.11

$$
\begin{aligned}
\operatorname{arcsinh}\frac{4}{3} &= \ln\left\{\frac{4}{3}+\left[\left(\frac{4}{3}\right)^2+1\right]^{1/2}\right\}\\
&= \ln\left(\frac{4}{3}\pm\frac{5}{3}\right)\\
\ln\left(\frac{4}{3}+\frac{5}{3}\right) &= \ln\frac{9}{3}\\
&= \ln 3\\
&= \log_e 3+2n\pi i\\
\ln\left(\frac{4}{3}-\frac{5}{3}\right) &= \ln\left(-\frac{1}{3}\right)\\
&= \log_e\frac{1}{3}+i(\pi+2n\pi)\\
&= -\log_e 3+i(\pi+2n\pi)\\
\operatorname{arcsinh}\frac{4}{3} &= (-1)^n\log_e 3+n\pi i
\end{aligned}
$$

17.9 Chapter in Review

17.9.1

0, 32

17.9.3

$$
\begin{aligned}
\frac{3+4i}{3-4i} &= \frac{(3+4i)^2}{(3-4i)(3+4i)}\\
&= \frac{-7+24i}{25}\\
&= -\frac{7}{25}+\frac{24}{25}i\\
\operatorname{Re}\left(\frac{z}{\bar{z}}\right) &= -\frac{7}{25}
\end{aligned}
$$

17.9.5

$$\begin{aligned}\frac{4i}{-3-4i} &= \frac{(4i)(-3+4i)}{(-3-4i)(-3+4i)}\\ &= \frac{-16-12i}{25}\\ &= -\frac{16}{25}-\frac{12}{25}i\\ |z| &= \sqrt{\left(\frac{16}{25}\right)^2+\left(\frac{12}{25}\right)^2}\\ &= \frac{4}{5}\end{aligned}$$

17.9.7

False

17.9.9

$$\begin{aligned}e^z &= 2i\\ z &= \ln(2i)\\ &= \log_e 2 + i\left(\frac{\pi}{2}+2n\pi\right)\end{aligned}$$

17.9.11

$$\begin{aligned}(1+i)^{(2+i)} &= e^{(2+i)\ln(1+i)}\\ ln(1+i) &= \log_e\sqrt{2}+\frac{\pi}{4}i\\ (2+i)\left(\log_e\sqrt{2}+\frac{\pi}{4}i\right) &= 2\log_e\sqrt{2}+\frac{\pi}{2}i+i\log_e\sqrt{2}-\frac{\pi}{4}\\ &= \left(2\log_e\sqrt{2}-\frac{\pi}{4}\right)+i\left(\log_e\sqrt{2}+\frac{\pi}{2}\right)\\ (1+i)^{(2+i)} &= e^{2\log_e\sqrt{2}-\pi/4}\left[\cos\left(\log_e\sqrt{2}+\frac{\pi}{2}\right)+i\sin\left(\log_e\sqrt{2}+\frac{\pi}{2}\right)\right]\\ &\approx -0.3097+0.8576i\end{aligned}$$

17.9.13

False

17.9.15

$$\mathrm{Ln}(-ie^3) = 3-\frac{\pi}{2}i$$

17.9.21

$$\begin{aligned} z^2 &= x^2 - y^2 + 2ixy \\ \mathrm{Im}(z^2) &\le 2 \\ 2xy &\le 2 \end{aligned}$$

17.9.23

$$\frac{1}{\sqrt{x^2+y^2}} \le 1$$

17.9.27

$$\begin{aligned} z^4 &= 1 - i \\ z_k &= 2^{1/8} e^{(-\pi/4 + 2k\pi)i/4} \\ &= 2^{1/8} e^{i(k\pi/2 - \pi/16)} \\ z_0 &= 1.0695 - 0.2127i \\ z_1 &= 0.2127 + 1.0695i \\ z_2 &= -1.0695 + 0.2127i \\ x_3 &= -0.2127 - 1.0695i \end{aligned}$$

18 Integration in the Complex Plane

18.1 Contour Integrals

18.1.1

$$\begin{aligned} z(t) &= 2t + i(4t-1) \\ z'(t) &= 2 + 4i \\ f(z(t)) &= (2t+3) + i(4t-1) \\ f(z(t))z'(t) &= [(2t+3) + i(4t-1)](2+4i) \\ &= (2t+3)(2) + (2t+3)(4i) + i(4t-1)(2) + i(4t-1)(4i) \\ &= 4t + 6 + 8it + 12i + 8it - 2i - 16t + 4 \\ &= (-12t + 10) + i(16t + 10) \\ \int_C f(z)\,dz &= \int_1^3 f(z(t))z'(t)\,dt \\ &= \int_1^3 (-12t+10)\,dt + i\int_1^3 (16t+10)\,dt \\ &= -28 + 84i \end{aligned}$$

18.1.3

$$\begin{aligned}
z(t) &= 3t + 2it \\
z'(t) &= 3 + 2i \\
\int_C f(z)\,dz &= \int_{-2}^{2} (3t+2it)^2(3+2i)\,dt \\
&= \int_{-2}^{2} [(3+2i)t]^2(3+2i)\,dt \\
&= (3+2i)^3 \int_{-2}^{2} t^2\,dt \\
&= (-9+46i)\frac{16}{3} \\
&= -48 + \frac{736}{3}i
\end{aligned}$$

18.1.5

$$\begin{aligned}
z(t) &= e^{it} \\
z'(t) &= ie^{it} \\
\int_C f(z)\,dz &= \int_{-\pi/2}^{\pi/2} \frac{1+e^{it}}{e^{it}} ie^{it}\,dt \\
&= i\int_{-\pi/2}^{\pi/2} (1+e^{it})\,dt \\
&= i\left[t + \frac{1}{i}e^{it}\right]_{-\pi/2}^{\pi/2} \\
&= i[t - ie^{it}]_{-\pi/2}^{\pi/2} \\
&= i\left(\frac{\pi}{2} - ie^{\pi i/2} + \frac{\pi}{2} + ie^{-\pi i/2}\right) \\
&= i(\pi+2)
\end{aligned}$$

18.1.7

$$\begin{aligned}
z(t) &= \cos t + i\sin t \\
z'(t) &= -\sin t + i\cos t \\
\int_C f(z)\,dz &= \int_0^{2\pi} \cos t(-\sin t + i\cos t)\,dt \\
&= \int_0^{2\pi} \left(-\frac{1}{2}\sin 2t + i\cos^2 t\right)\,dt \\
&= \pi i
\end{aligned}$$

18.1.9

$$\begin{aligned}
z(t) &= (1-t) + it \\
z'(t) &= -1 + i \\
\int_C f(z)\,dz &= \int_0^1 [(1-t)^2 + it^3](-1+i)\,dt \\
&= \int_0^1 (1 - 2t + t^2 + it^3)(-1+i)\,dt \\
&= \int_0^1 (-1 + i + 2t - 2it - t^2 + it^2 - it^3 - t^3)\,dt \\
&= \int_0^1 (-1 + 2t - t^2 - t^3)\,dt + i\int_0^1 (1 - 2t + t^2 - t^3)\,dt \\
&= -\frac{7}{12} + \frac{1}{12}i
\end{aligned}$$

18.1.17

$$\begin{aligned}
z(t) &= 1 + it \\
z'(t) &= i \\
\int_{C_1} f(z)\,dz &= \int_0^1 i\,dt \\
&= i \\
z(t) &= (1-t) + i(1-t) \\
z'(t) &= -(1+i) \\
\int_{C_2} f(z)\,dz &= -\int_0^1 (1-t)(1+i)\,dt \\
&= -\int_0^1 (1 + i - t - it)\,dt \\
&= -\frac{1}{2} - \frac{1}{2}i \\
z(t) &= t \\
z'(t) &= 1 \\
\int_{C_3} f(z)\,dz &= \int_0^1 t\,dt \\
&= \frac{1}{2} \\
\int_C f(z)\,dz &= \frac{1}{2}i
\end{aligned}$$

18.1.19

$$
\begin{aligned}
z(t) &= 1 + it \\
z'(t) &= i \\
\int_{C_1} f(z)\,dz &= \int_0^1 (1+it)^2 i\,dt \\
&= i\int_0^1 (1 + 2it - t^2)\,dt \\
&= -1 + \frac{2}{3}i \\
z(t) &= (1+i)(1-t) \\
z'(t) &= -(1+i) \\
\int_{C_2} f(z)\,dz &= -\int_0^1 [(1+i)(1-t)]^2(1+i)\,dt \\
&= -\int_0^1 (1 - t + i - it)^2(1+i)\,dt \\
&= \frac{2}{3} - \frac{2}{3}i \\
z(t) &= t \\
z'(t) &= 1 \\
\int_{C_3} f(z)\,dz &= \int_0^1 t^2\,dt \\
&= \frac{1}{3} \\
\int_C f(z)\,dz &= 0
\end{aligned}
$$

18.1.23

$$
\begin{aligned}
z(t) &= t + i(1-t^2) \\
z'(t) &= 1 - 2it \\
\int_C f(z)\,dz &= \frac{4}{3} - \frac{5}{3}i
\end{aligned}
$$

18.1.25

$$\begin{aligned}
L &= 10\pi \\
|z^2+1| &\geq |z^2|-1 \\
\left|\frac{e^z}{z^2+1}\right| &\leq \frac{|e^z|}{|z^2|-1} \\
&= \frac{e^5}{24} \\
&= M \\
\left|\oint \frac{e^z}{z^2+1}\,dz\right| &\leq ML \\
&= \frac{5\pi e^5}{12}
\end{aligned}$$

18.1.27

$$\begin{aligned}
z(t) &= (1+i)t,\ 0 \leq t \leq 1 \\
L &= \sqrt{2} \\
|z^2+4| &= |2it^2+4| \\
&\leq |2i+4| \\
&= \sqrt{20} \\
&= 2\sqrt{5} \\
&= M \\
\left|\oint (z^2+4)\,dz\right| &\leq ML \\
&= 2\sqrt{10}
\end{aligned}$$

18.1.33

$$\begin{aligned}
z(t) &= e^{it} \\
z'(t) &= ie^{it} \\
\oint \overline{f(z)}\,dz &= \int_0^{2\pi} 2e^{-it}ie^{it}\,dt \\
&= 4\pi i
\end{aligned}$$

The circulation is 0 and the flux is 4π.

18.2 Cauchy-Goursat Theorem

18.2.1

$$
\begin{aligned}
z &= e^{it},\ 0 \le t \le 2\pi \\
z' &= ie^{it} \\
\int (z^3 - 1 + 3i)\, dz &= \int_0^{2\pi} [(e^{it})^3 - 1 + 3i] ie^{it}\, dt \\
&= i \int_0^{2\pi} (e^{4it} - e^{it} + 3ie^{it})\, dt \\
&= \left[\frac{1}{4} e^{4it} - e^{it} + 3ie^{it} \right]_0^{2\pi} \\
&= \frac{1}{4} e^{8\pi i} - e^{2\pi i} + 3ie^{2\pi i} - \frac{1}{4} + 1 - 3i \\
&= \frac{1}{4} - 1 + 3i - \frac{1}{4} + 1 - 3i \\
&= 0
\end{aligned}
$$

18.2.9

$$
\begin{aligned}
\int_C \frac{1}{z}\, dz &= \int_0^{2\pi} \frac{1}{e^{it}} ie^{it}\, dt \\
&= 2\pi i
\end{aligned}
$$

18.2.11

$$
\begin{aligned}
\oint_C \left(z + \frac{1}{z} \right) dz &= \oint_C \frac{1}{z^{-1}}\, dz + \oint_C \frac{1}{z}\, dz \\
&= 0 + 2\pi i \\
&= 2\pi i
\end{aligned}
$$

18.2.13

$$
\begin{aligned}
z^2 - \pi^2 &= (z + \pi)(z - \pi) \\
\frac{z}{z^2 - \pi^2} &= \frac{1/2}{z + \pi} + \frac{1/2}{z - \pi} \\
\oint_C \frac{z}{z^2 - \pi^2}\, dz &= \frac{1}{2} \oint \left(\frac{1}{z + \pi} + \frac{1}{z - \pi} \right) dz \\
&= 0
\end{aligned}
$$

18.2.15

(a)

$$\begin{aligned}
\frac{2z+1}{z^2+z} &= \frac{2z+1}{z(z+1)} \\
&= \frac{1}{z} + \frac{1}{z+1} \\
\oint_C \frac{2z+1}{z^2+z}\,dz &= \oint_C \frac{2z+1}{z(z+1)}\,dz \\
&= \oint_C \left(\frac{1}{z} + \frac{1}{z+1}\right)\,dz \\
&= 2\pi i
\end{aligned}$$

(b) $4\pi i$

(c) 0

18.2.17

(a)

$$\begin{aligned}
\frac{-3z+2}{z^2-8z+12} &= \frac{1}{z-1} - \frac{4}{z-6} \\
\oint_C \frac{-3z+2}{z^2-8z+12}\,dz &= \oint \left(\frac{1}{z-1} - \frac{4}{z-6}\right)\,dz \\
&= -8\pi i
\end{aligned}$$

(b) $-6\pi i$

18.2.19

$$\begin{aligned}
\frac{z-1}{z(z-i)(z-3i)} &= \frac{1}{3z} - \frac{1/2-i/2}{z-i} + \frac{1/6-i/2}{z-3i} \\
\oint_C \frac{z-1}{z(z-i)(z-3i)}\,dz &= -\left(\frac{1}{2} - \frac{i}{2}\right)2\pi i \\
&= -\pi(1+i)
\end{aligned}$$

18.2.21

$$\begin{aligned}
\frac{8z-3}{z^2-z} &= \frac{3}{z} + \frac{5}{z-1} \\
\oint_C f(z)\,dz &= \oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz \\
&= \oint_{C_1} \left(\frac{3}{z} + \frac{5}{z-1}\right)\,dz - \oint_{C_2} \left(\frac{3}{z} + \frac{5}{z-1}\right)\,dz \\
&= 6\pi i - 10\pi i \\
&= -4\pi i
\end{aligned}$$

18.2.23

$$\begin{aligned}
\oint_C \left(\frac{e^z}{z+3} - 3\overline{z}\right)\,dz &= \oint_C \frac{e^z}{z+3}\,dz - 3\oint_C \overline{z}\,dz \\
&= -3\oint_0^{2\pi} e^{-it} i e^{it}\,dt \\
&= -6\pi i
\end{aligned}$$

18.3 Independence of the Path

18.3.1

(a)

$$\begin{aligned}
z(t) &= i(t-1),\ 0 \le t \le 2 \\
z'(t) &= i \\
\int_C (4z-1)\,dz &= \int_0^2 \{4[i(t-1)] - 1\} i\,dt \\
&= \int_0^2 [4(1-t) - i]\,dt \\
&= \left[4\left(t - \frac{1}{2}t^2\right) - it\right]_0^2 \\
&= -2i
\end{aligned}$$

(b)

$$\begin{aligned}
F(z) &= 2z^2 - z \\
\int_C (4z-1)\,dz &= F(i) - F(-i) \\
&= [2(i)^2 - (i)] - [2(-i)^2 - (-i)] \\
&= -2 - i + 2 - i \\
&= -2i
\end{aligned}$$

18.3.3

$$\begin{aligned}
z(-1) &= -2+7i \\
z(1) &= 2-i \\
\int_C 2z\,dz &= z^2|_{-2+7i}^{2-i} \\
&= (2-i)^2 - (-2+7i)^2 \\
&= 48+24i
\end{aligned}$$

18.3.5

$$\begin{aligned}
\int_0^{3+i} z^2\,dz &= \left. \frac{1}{3}z^3 \right|_0^{3+i} \\
&= \frac{1}{3}(3+i)^3 \\
&= 6+\frac{26}{3}i
\end{aligned}$$

18.3.7

$$\begin{aligned}
\int_{1-i}^{1+i} z^3\,dz &= \left. \frac{1}{4}z^4 \right|_{1-i}^{1+i} \\
&= \frac{1}{4}[(1+i)^4 - (1-i)^4] \\
&= 0
\end{aligned}$$

18.3.9

$$\begin{aligned}
\int_{-i/2}^{1-i} (2z+1)^2\,dz &= \left. z+2z^2+\frac{4}{3}z^3 \right|_{-i/2}^{1-i} \\
&= -\frac{7}{6} - \frac{22}{3}i
\end{aligned}$$

18.3.11

$$\begin{aligned}
\int_{i/2}^{i} e^{\pi z}\,dz &= \left. \frac{1}{\pi}e^{\pi z} \right|_{i/2}^{i} \\
&= -\frac{1}{\pi}(1+i)
\end{aligned}$$

18.3.13

$$\begin{aligned}\int_{\pi}^{\pi+2i} \sin\frac{z}{2}\,dz &= -2\cos\frac{z}{2}\Big|_{\pi}^{\pi+2i}\\ &= -2\cos\left(\frac{\pi}{2}+i\right)\\ &= 2i\sinh 1\\ &\approx 2.3504i\end{aligned}$$

18.3.15

$$\begin{aligned}\int_{\pi i}^{2\pi i} \cosh z\,dz &= \sinh(2\pi i) - \sinh(\pi i)\\ &= 0\end{aligned}$$

18.3.17

$$\begin{aligned}\int_C \frac{1}{z}\,dz &= \mathrm{Ln}\,4e^{\pi i/2} - \mathrm{Ln}\,4e^{-\pi i/2}\\ &= \log_e 4 + \frac{\pi}{2}i - \log_e 4 + \frac{\pi}{2}i\\ &= \pi i\end{aligned}$$

18.3.19

$$\begin{aligned}\int_{-4i}^{4i} \frac{1}{z^2}\,dz &= -\frac{1}{z}\Big|_{-4i}^{4i}\\ &= -\frac{1}{4i} - \frac{1}{4i}\\ &= -\frac{1}{2i}\\ &= \frac{i}{2}\end{aligned}$$

18.3.21

$$\begin{aligned}\int_{\pi}^{i} e^z\cos z\,dz &= \frac{1}{2}\int_{\pi}^{i}[e^{z(1+i)} + e^{z(1-i)}]\,dz\\ &= \frac{1}{2}\left(\frac{e^{z(1+i)}}{1+i} + \frac{e^{z(1-i)}}{1-i}\Big|_{\pi}^{i}\right)\\ &\approx 11.4928 + 0.9667i\end{aligned}$$

18.3.23

$$\begin{aligned}\int_i^{1+i} ze^z\,dz &= ze^z|_i^{1+i} - \int_i^{1+i} e^z\,dz \\ &= (1+i)e^{1+i} - ie^i - [e^z]_i^{1+i} \\ &= (1+i)e^{1+i} - ie^i - e^{1+i} + e^i \\ &\approx -0.9055 + 1.7698i\end{aligned}$$

18.4 Cauchy's Integral Formulas

18.4.1

$8\pi i$

18.4.3

$2\pi i e^{\pi i} = -2\pi i$

18.4.5

$2\pi i[(-2i)^2 - 3(-2i) + 4i] = 2\pi i(-4 + 10i) = -20\pi - 8\pi i$

18.4.7

(a)

$$\begin{aligned}\oint_C \frac{z^2}{z^2+4}\,dz &= \oint_C \frac{\frac{z^2}{z+2i}}{z-2i}\,dz \\ &= 2\pi i \frac{(2i)^2}{(2i)+2i} \\ &= -2\pi\end{aligned}$$

(b)

$$\begin{aligned}\oint_C \frac{z^2}{z^2+4}\,dz &= \oint_C \frac{\frac{z^2}{z-2i}}{z+2i}\,dz \\ &= 2\pi i \frac{(-2i)^2}{(-2i)-2i} \\ &= 2\pi i \frac{-4}{-4i} \\ &= 2\pi\end{aligned}$$

18.4.9

$$\begin{aligned}
\oint_C \frac{z^2+4}{z^2-5iz-4}\,dz &= \oint_C \frac{z^2+4}{(z-i)(z-4i)}\,dz \\
&= \oint_C \frac{\frac{z^2+4}{z-i}}{z-4i}\,dz \\
&= -8\pi
\end{aligned}$$

18.4.11

$$\begin{aligned}
\frac{2\pi i}{2!}\ \frac{d^2}{dz^2}(e^{z^2})\Big|_{z=i} &= \pi i\ \frac{d}{dz}(2ze^{z^2})\Big|_{z=i} \\
&= \pi i\ (2e^{z^2}+4z^2e^{z^2})\Big|_{z=i} \\
&= \pi i(2e^{-1}-4e^{-1}) \\
&= -\frac{2\pi}{e}i
\end{aligned}$$

18.4.13

$$\begin{aligned}
\oint_C \frac{\cos 2z}{z^5}\,dz &= \frac{2\pi i}{4!}\ \frac{d^4}{dz^4}(\cos 2z)\Big|_{z=0} \\
&= \frac{\pi}{12}i(16\cos 2z)|_{z=0} \\
&= \frac{4}{3}\pi i
\end{aligned}$$

18.4.19

$$\begin{aligned}
\oint_C \left(\frac{e^{2iz}}{z^4}-\frac{z^4}{(z-i)^3}\right)dz &= \frac{2\pi i}{3!}\ \frac{d^3}{dz^3}(e^{2iz})\Big|_{z=0} - \frac{2\pi i}{2!}\ \frac{d^2}{dz^2}(z^4)\Big|_{z=i} \\
&= \frac{\pi}{3}i(-8ie^{2iz})|_{z=0} - \pi i(12z^2)|_{z=i} \\
&= \frac{8}{3}\pi + 12\pi i
\end{aligned}$$

18.4.21

$$\begin{aligned}
\oint_C \frac{1}{z^3(z-1)^2}\,dz &= \oint_{C_1} \frac{\frac{1}{(z-1)^2}}{z^3}\,dz + \oint_{C_2} \frac{\frac{1}{z^3}}{(z-1)^2}\,dz \\
&= \frac{2\pi i}{2!}\;\frac{d^2}{dz^2}\left(\frac{1}{(z-1)^2}\right)\bigg|_{z=0} \\
&\quad + 2\pi i\;\frac{d}{dz}\left(\frac{1}{z^3}\right)\bigg|_{z=1} \\
&= 6\pi i - 6\pi i \\
&= 0
\end{aligned}$$

18.4.23

$$\begin{aligned}
\oint_C \frac{3z+1}{z(z-2)^2}\,dz &= \oint_{C_1} \frac{\frac{3z+1}{z}}{(z-2)^2}\,dz - \oint_{C_2} \frac{\frac{3z+1}{(z-2)^2}}{z}\,dz \\
&= 2\pi i\;\frac{d}{dz}\left(\frac{3z+1}{z}\right)\bigg|_{z=2} - 2\pi i\;\frac{3z+1}{(z-2)^2}\bigg|_{z=0} \\
&= -\frac{1}{2}pii - \frac{1}{2}\pi i \\
&= -\pi i
\end{aligned}$$

18.5 Chapter in Review

18.5.1

True

18.5.3

True

18.5.5

0

18.5.7

$$\begin{aligned}
\oint_C \frac{z^3+e^z}{(z+\pi i)^3}\,dz &= \frac{2\pi i}{2!}\;\frac{d^2}{dz^2}(z^3+e^z)\bigg|_{z=-\pi i} \\
&= \pi i(6z+e^z)|_{z=-\pi i} \\
&= \pi i(-6\pi i - 1) \\
&= 6\pi^2 - \pi i
\end{aligned}$$

18.5.9

$$
\begin{aligned}
\oint_C \frac{1}{(z-z_0)(z-z_1)}\,dz &= \oint_{C_1} \frac{\frac{1}{z-z_0}}{z-z_1}\,dz + \oint_{C_2} \frac{\frac{1}{z-z_1}}{z-z_0}\,dz \\
&= 2\pi i \frac{1}{z_1-z_0} + 2\pi i \frac{1}{z_0-z_1} \\
&= 0
\end{aligned}
$$

True

18.5.11

$2\pi i$ if $n=-1$, 0 otherwise.

18.5.13

$$
\begin{aligned}
z_1 &= -4+iy,\ 0 \le y \le 2 \\
z_1' &= i \\
z_2 &= x+2i,\ -4 \le x \le 3 \\
z_2' &= 1 \\
z_3 &= 3+i(2-y),\ 0 \le y \le 2 \\
z_3' &= -i
\end{aligned}
$$

$$
\begin{aligned}
\int_C (x+iy)\,dz &= \int_0^2 (-4+iy)i\,dy + \int_{-4}^3 (x+2i)\,dx + \int_0^2 [3+i(2-y)](-i)\,dy \\
&= \int_0^2 (-y-4i)\,dy + \int_{-4}^3 (x+2i)\,dx - \int_0^2 [(y-2)+3i]\,dy \\
&= \left[-\frac{1}{2}y^2 - 4iy\right]_0^2 + \left[\frac{1}{2}x^2 + 2ix\right]_{-4}^3 - \left[\frac{1}{2}y^2 - 2y + 3iy\right]_0^2 \\
&= -2 - 8i + \frac{9}{2} + 6i - 8 + 8i - 2 + 4 - 6i \\
&= -\frac{7}{2}
\end{aligned}
$$

18.5.15

$$
\begin{aligned}
\int_C |z^2|\,dz &= \int_0^2 |(t+it^2)^2|(1+2it)\,dt \\
&= \int_0^2 (t^2+t^4)(1+2it)\,dt \\
&= \frac{136}{15} + \frac{88}{3}i
\end{aligned}
$$

18.5.17

0

18.5.19

$$\begin{aligned} z(-1) &= 1 \\ z(1) &= 1+4i \\ \int_C \sin z \, dz &= -\cos z|_1^{1+4i} \\ &= \cos 1 - \cos(1+4i) \\ &\approx -14.2144 + 22.9637i \end{aligned}$$

18.5.21

$2\pi i$

18.5.23

$$\begin{aligned} \oint_C \frac{e^{-2z}}{z^4} \, dz &= \frac{2\pi i}{3!} \, \frac{d^3}{dz^3}(e^{-2z})\bigg|_{z=0} \\ &= -\frac{8}{3}\pi i \end{aligned}$$

18.5.25

$$\begin{aligned} \oint_C \frac{1}{2z^2+7z+3} \, dz &= \oint_C \frac{\frac{1}{z+3}}{2z+1} \, dz \\ &= \oint_C \frac{\frac{1}{2(z+3)}}{z+\frac{1}{2}} \, dz \\ &= 2\pi i \frac{1}{2\left(-\frac{1}{2}+3\right)} \\ &= \frac{2}{5}\pi i \end{aligned}$$

18.5.27

2π

19 Series and Residues

19.1 Sequences and Series

19.1.1

$5i, -5, -5i, 5, 5i$

19.1.3

$0, 2, 0, 2, 0$

19.1.5

$$\lim_{n\to\infty} \frac{3ni+2}{n+ni} = \frac{3i}{1+i}$$

Converges

19.1.7

$$\begin{aligned}
\lim_{n\to\infty} \frac{(ni+2)^2}{n^2 i} &= \lim_{n\to\infty} \frac{-n^2+4ni+4}{n^2 i} \\
&= \lim_{n\to\infty} \frac{-1+4i/n+4/n^2}{i} \\
&= -\frac{1}{i}
\end{aligned}$$

Converges

19.1.9

Diverges

19.1.11

$$\begin{aligned}
\frac{4n+3in}{2n+i} &= \frac{(4n+3in)(2n-i)}{(2n+i)(2n-i)} \\
&= \frac{8n^2-4in+6in^2+3n}{4n^2+1} \\
&= \frac{8n^2+3n}{4n^2+1} + i\frac{6n^2-4n}{4n^2+1}
\end{aligned}$$

$$\lim_{n\to\infty} \frac{8n^2+3n}{4n^2+1} = 2$$

$$\lim_{n\to\infty} \frac{6n^2-4n}{4n^2+1} = \frac{3}{2}$$

19.1.13

$$\begin{aligned}
S_1 &= \frac{1}{1+2i} - \frac{1}{2+2i} \\
S_2 &= \frac{1}{1+2i} - \frac{1}{2+2i} + \frac{1}{2+2i} - \frac{1}{3+2i} \\
S_n &= \frac{1}{1+2i} - \frac{1}{n+1+2i} \\
\lim_{n\to\infty} S_n &= \frac{1}{1+2i}
\end{aligned}$$

19.1.15

$$|1-i| = \sqrt{2}$$

Divergent

19.1.17

$$\begin{aligned}
\left|\frac{i}{2}\right| &= \frac{1}{2} \\
\frac{i/2}{1-i/2} &= \frac{i}{2-i} \\
&= \frac{i(2+i)}{(2-i)(2+i)} \\
&= \frac{-1+2i}{5}
\end{aligned}$$

19.1.19

$$\begin{aligned}
\frac{2}{1+2i} &= \frac{2(1-2i)}{(1+2i)(1-2i)} \\
&= \frac{2-4i}{5} \\
\left|\frac{2}{1+2i}\right| &= \sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{4}{5}\right)^2} \\
&= \sqrt{\frac{4}{25} + \frac{16}{25}} \\
&= \sqrt{\frac{20}{25}} \\
&= \sqrt{\frac{4}{5}} \\
&= \frac{2}{\sqrt{5}} \\
&= \frac{2\sqrt{5}}{5} \\
\frac{3}{1-\frac{2}{1+2i}} &= \frac{3(1+2i)}{1+2i-2} \\
&= \frac{3+6i}{-1+2i} \\
&= \frac{(3+6i)(-1-2i)}{(-1+2i)(-1-2i)} \\
&= \frac{-3-6i-6i+12}{5} \\
&= \frac{9-12i}{5}
\end{aligned}$$

19.1.21

$$\begin{aligned}
a_k &= \frac{1}{(1-2i)^{k+1}} \\
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(1-2i)^{n+1}}{(1-2i)^{n+2}}\right| \\
&= \left|\frac{1}{1-2i}\right| \\
&= \left|\frac{1+2i}{(1-2i)(1+2i)}\right| \\
&= \left|\frac{1+2i}{5}\right| \\
&= \frac{\sqrt{5}}{5} \\
R &= \sqrt{5}
\end{aligned}$$

The circle of convergence is $|z-2i| = \sqrt{5}$.

19.1.23

$$\begin{aligned}
a_k &= \frac{(-1)^k}{k2^k} \\
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}/[(n+1)2^{(n+1)}]}{(-1)^n/(n2^n)}\right| \\
&= \lim_{n\to\infty}\left|\frac{n2^n}{(n+1)2^{(n+1)}}\right| \\
&= \lim_{n\to\infty}\left|\frac{n}{2(n+1)}\right| \\
&= \frac{1}{2} \\
R &= 2
\end{aligned}$$

The circle of convergence is $|z-1-i| = 2$.

19.1.25

$$\begin{aligned}
a_k &= (1+3i)^k \\
\lim_{n\to\infty}\left|\frac{(1+3i)^{n+1}}{(1+3i)^n}\right| &= \lim_{n\to\infty}|1+3i| \\
&= \sqrt{10} \\
R &= \frac{1}{\sqrt{10}}
\end{aligned}$$

The circle of convergence is $|z-i| = \frac{1}{\sqrt{10}}$.

19.1.27

$$\begin{aligned} a_k &= \frac{1}{5^{2k}} \\ \lim_{n\to\infty}\left|\frac{5^{2n}}{5^{2(n+1)}}\right| &= \frac{1}{25} \\ R &= 25 \end{aligned}$$

The circle of convergence is $|z-4-3i| = 25$.

19.2 Taylor Series

19.2.1

$$\begin{aligned} f(z) &= \frac{z}{1+z} \\ &= z(1-z+z^2-z^3+\cdots) \\ &= z-z^2+z^3-z^4+\cdots \\ &= \sum_{k=1}^{\infty}(-1)^{k+1}z^k \\ R &= 1 \end{aligned}$$

19.2.3

$$\begin{aligned} f(z) &= \frac{1}{(1+2z)^2} \\ \frac{1}{1+2z} &= 1-2z+4z^2-8z^3+16z^4-\cdots \\ &= \sum_{k=0}^{\infty}(-1)^k(2z)^k \\ \frac{d}{dz}\frac{1}{1+2z} &= -\frac{2}{(1+2z)^2} \\ &= \sum_{k=1}^{\infty}(-1)^k 2k(2z)^{k-1} \\ f(z) &= \sum_{k=1}^{\infty}(-1)^{k-1}k(2z)^{k-1} \\ R &= \frac{1}{2} \end{aligned}$$

19.2.5

$$\begin{aligned} f(z) &= e^{-2z} \\ &= \sum_{k=0}^{\infty} \frac{(-2z)^k}{k!} \\ R &= \infty \end{aligned}$$

19.2.7

$$\begin{aligned} f(z) &= \sinh z \\ &= \frac{e^z - e^{-z}}{2} \\ &= \frac{1}{2} \sum_{k=0}^{\infty} \frac{z^k - (-z)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} \\ R &= \infty \end{aligned}$$

19.2.9

$$\begin{aligned} f(z) &= \cos \frac{z}{2} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(z/2)^{2k}}{(2k)!} \\ R &= \infty \end{aligned}$$

19.2.11

$$\begin{aligned} f(z) &= \sin z^2 \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{z^{4k+2}}{(2k+1)!} \\ R &= \infty \end{aligned}$$

19.2.13

$$
\begin{aligned}
f(z) &= \frac{1}{z} \\
f'(z) &= -\frac{1}{z^2} \\
f''(z) &= \frac{2}{z^3} \\
f'''(z) &= -\frac{6}{z^4} \\
f^{(n)}(z) &= (-1)^n \frac{n!}{z^{n+1}} \\
f(z) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(z-1)^k \\
&= \sum_{k=0}^{\infty} (-1)^k (z-1)^k \\
R &= 1
\end{aligned}
$$

19.2.15

$$
\begin{aligned}
f(z) &= \frac{1}{3-z} \\
f'(z) &= \frac{1}{(3-z)^2} \\
f''(z) &= \frac{2}{(3-z)^3} \\
f^{(n)}(z) &= \frac{n!}{(3-z)^{n+1}} \\
f(z) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(2i)}{k!}(z-2i)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{(3-2i)^{k+1}}(z-2i)^k \\
R &= \sqrt{13}
\end{aligned}
$$

19.2.17

$$\begin{aligned}
f(z) &= \frac{z-1}{3-z} \\
f^{(n)}(z) &= (-1)^{n+1}\frac{2n!}{(z-3)^{n+1}} \\
f(z) &= \sum_{k=1}^{\infty}\frac{f^{(k)}(1)}{k!}(z-1)^k \\
&= \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2}{(-2)^{k+1}}(z-1)^k \\
&= \sum_{k=1}^{\infty}\frac{(z-1)^k}{2^k} \\
R &= 2
\end{aligned}$$

19.2.19

$$\begin{aligned}
f(z) &= \cos z \\
f'(z) &= -\sin z \\
f''(z) &= -\cos z \\
f'''(z) &= \sin z \\
f^{(4)}(z) &= \cos z \\
f(z) &= \sum_{k=0}^{\infty}\frac{f^{(k)}(\pi/4)}{k!}(z-\pi/4)^k \\
&= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2(1!)}(z-\pi/4) - \frac{\sqrt{2}}{2(2!)}(z-\pi/4)^2 + \frac{\sqrt{2}}{2(3!)}(z-\pi/4)^3 + \cdots \\
R &= \infty
\end{aligned}$$

19.2.21

$$\begin{aligned}
f(z) &= e^z \\
&= \sum_{k=0}^{\infty}\frac{f^{(k)}(3i)}{k!}(z-3i)^k \\
&= e^{3i}\sum_{k=0}^{\infty}\frac{(z-3i)^k}{k!} \\
R &= \infty
\end{aligned}$$

19.2.23

$$f(z) = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots$$

19.2.25

$$\begin{aligned}
f(z) &= \frac{i}{(z-i)(z-2i)} \\
&= \frac{1}{z-2i} - \frac{1}{z-i} \\
f^{(n)}(z) &= (-1)^n n! \left(\frac{1}{(z-2i)^{n+1}} - \frac{1}{(z-i)^{n+1}} \right) \\
f^{(n)}(0) &= (-1)^n n! \left(\frac{1}{(-2i)^{n+1}} - \frac{1}{(-i)^{n+1}} \right) \\
&= n! \left(\frac{1}{i^{n+1}} - \frac{1}{(2i)^{n+1}} \right) \\
&= \frac{n!}{i^{n+1}} \left(1 - \frac{1}{2^{n+1}} \right) \\
f(z) &= \sum_{k=0}^{\infty} \frac{1 - \frac{1}{2^{k+1}}}{i^{k+1}} z^k \\
R &= 1
\end{aligned}$$

19.2.27

$R = \sqrt{20} = 2\sqrt{5}$

19.2.29

$$\begin{aligned}
f(z) &= \frac{1}{z+2} \\
f^{(n)}(z) &= (-1)^n \frac{n!}{(z+2)^{n+1}} \\
f_1(z) &= \sum_{k=0}^{\infty} (-1)^k (z+1)^k \\
R_1 &= 1 \\
f_2(z) &= \sum_{k=0}^{\infty} (-1)^k \frac{(z-i)^k}{(2+i)^{k+1}} \\
R_2 &= \sqrt{5}
\end{aligned}$$

19.3 Laurent Series

19.3.1

$$\begin{aligned} f(z) &= \frac{\cos z}{z} \\ &= \frac{1}{z}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right) \\ &= \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} + \cdots \end{aligned}$$

19.3.3

$$\begin{aligned} f(z) &= e^{-1/z^2} \\ &= 1 - \frac{1}{1!z^2} + \frac{1}{2!z^4} - \frac{1}{3!z^6} + \cdots \end{aligned}$$

19.3.5

$$\begin{aligned} f(z) &= \frac{e^z}{z-1} \\ &= \frac{e^{1+z-1}}{z-1} \\ &= \frac{e}{z-1}\left(1 + \frac{z-1}{1!} + \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} + \cdots\right) \\ &= \frac{e}{z-1} + e + \frac{e(z-1)}{2!} + \frac{e(z-1)^2}{3!} + \cdots \end{aligned}$$

19.3.7

$$\begin{aligned} f(z) &= \frac{1}{z(z-3)} \\ &= -\frac{1}{3z}\frac{1}{1-\frac{z}{3}} \\ &= -\frac{1}{3z}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \cdots\right) \\ &= -\frac{1}{3z} - \frac{1}{9} - \frac{z}{27} - \frac{z^2}{81} - \cdots \end{aligned}$$

19.3.9

$$\begin{aligned}
f(z) &= \frac{1}{z(z-3)} \\
&= \frac{1}{z-3}\frac{1}{3+z-3} \\
&= \frac{1}{3(z-3)}\frac{1}{1+\frac{z-3}{3}} \\
&= \frac{1}{3(z-3)}\left[1-\frac{z-3}{3}+\left(\frac{z-3}{3}\right)^2-\left(\frac{z-3}{3}\right)^3+\cdots\right] \\
&= \frac{1}{3(z-3)}-\frac{1}{3^2}+\frac{z-3}{3^3}-\frac{(z-3)^2}{3^4}+\cdots
\end{aligned}$$

19.3.11

$$\begin{aligned}
f(z) &= \frac{1}{z(z-3)} \\
&= \frac{1}{3(z-3)}-\frac{1}{3z} \\
&= \frac{1}{3(1+z-4)}-\frac{1}{3(4+z-4)} \\
&= \frac{1}{3(z-4)}\frac{1}{1+\frac{1}{z-4}}-\frac{1}{12}\frac{1}{1+\frac{z-4}{4}} \\
&= \frac{1}{3(z-4)}\left(1-\frac{1}{z-4}+\frac{1}{(z-4)^2}-\frac{1}{(z-4)^3}+\cdots\right) \\
&\quad -\frac{1}{12}\left[1-\frac{z-4}{4}+\left(\frac{z-4}{4}\right)^2-\left(\frac{z-4}{4}\right)^3+\cdots\right] \\
&= \cdots-\frac{1}{3(z-4)^2}+\frac{1}{3(z-4)}-\frac{1}{12}+\frac{z-4}{3\cdot 4^2}-\frac{(z-4)^2}{3\cdot 4^3}+\cdots
\end{aligned}$$

19.3.13

$$\begin{aligned}
f(z) &= \frac{1}{(z-1)(z-2)} \\
&= \frac{1}{1-z} - \frac{1}{2-z} \\
&= -\frac{1}{z}\frac{1}{1-\frac{1}{z}} - \frac{1}{2}\frac{1}{1-\frac{z}{2}} \\
&= -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) \\
&\quad -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots\right) \\
&= \cdots - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{z}{2^2} - \frac{z^2}{2^3} - \cdots
\end{aligned}$$

19.3.15

$$\begin{aligned}
f(z) &= \frac{1}{(z-1)(z-2)} \\
&= -\frac{1}{z-1} + \frac{1}{z-2} \\
&= -\frac{1}{z-1} + \frac{1}{-1+z-1} \\
&= -\frac{1}{z-1} - \frac{1}{1-(z-1)} \\
&= -\frac{1}{z-1} - 1 - (z-1) - (z-1)^2 - \cdots
\end{aligned}$$

19.3.17

$$\begin{aligned}
f(z) &= \frac{z}{(z+1)(z-2)} \\
&= \frac{1}{3(z+1)} + \frac{2}{3(z-2)} \\
&= \frac{1}{3(z+1)} + \frac{2}{3(-3+z+1)} \\
&= \frac{1}{3(z+1)} - \frac{2}{9\left(1-\frac{z+1}{3}\right)} \\
&= \frac{1}{3(z+1)} - \frac{2}{9}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\left(\frac{z+1}{3}\right)^3+\cdots\right] \\
&= \frac{1}{3(z+1)} - \frac{2}{3^2} - \frac{2(z+1)}{3^3} - \frac{2(z+1)^2}{3^4} - \cdots
\end{aligned}$$

19.3.19

$$
\begin{aligned}
f(z) &= \frac{z}{(z+1)(z-2)},\ 1 < |z| < 2 \\
&= \frac{2}{3(z-2)} + \frac{1}{3(z+1)} \\
&= -\frac{1}{3\left(1-\frac{z}{2}\right)} + \frac{1}{3z\left(1+\frac{1}{z}\right)} \\
&= -\frac{1}{3}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right] \\
&\quad + \frac{1}{3z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \cdots\right) \\
&= \cdots - \frac{1}{3z^2} + \frac{1}{3z} - \frac{1}{3} - \frac{z}{3\cdot 2} - \frac{z^2}{3\cdot 2^2} - \cdots
\end{aligned}
$$

19.4 Zeroes and Poles

19.4.1

$$
\begin{aligned}
f(z) &= \frac{e^{2z}-1}{z} \\
&= -\frac{1}{z} + \frac{1}{z}\left(1 + \frac{2z}{1!} + \frac{4z^2}{2!} + \cdots\right) \\
&= \frac{2}{1!} + \frac{4z}{2!} + \frac{8z^2}{3!} + \cdots \\
f(0) &= 2
\end{aligned}
$$

19.4.3

$z = -2 + i$, order 2.

19.4.5

$$
\begin{aligned}
f(z) &= z^4 + z^2 \\
&= z^2(z^2+1) \\
&= z^2(z+i)(z-i)
\end{aligned}
$$

Zeroes are: $z = 0$ order 2, $z = \pm i$ order 1.

19.4.7

$$f(z) = e^{2z} - e^z$$

Zeroes are $2\pi ni$, $n \in \mathbb{R}$ order 1.

19.4.9

$$\begin{aligned}
f(z) &= z(1-\cos z^2) \\
&= z\left[1-\left(1-\frac{z^4}{2!}+\frac{z^8}{4!}-\cdots\right)\right] \\
&= z\left(\frac{z^4}{2!}-\frac{z^8}{4!}+\frac{z^{12}}{6!}-\cdots\right) \\
&= z^5\left(\frac{1}{2!}-\frac{z^4}{4!}+\frac{z^8}{6!}-\cdots\right)
\end{aligned}$$

Order 5.

19.4.11

$$\begin{aligned}
f(z) &= 1-e^{z-1} \\
&= 1-\left[1+\frac{z-1}{1!}+\frac{(z-1)^2}{2!}+\cdots\right] \\
&= -\frac{z-1}{1!}-\frac{(z-1)^2}{2!}-\cdots \\
&= -(z-1)\left[1+\frac{z-1}{2!}+\frac{(z-1)^2}{3!}+\cdots\right]
\end{aligned}$$

Order 1.

19.4.13

$$\begin{aligned}
f(z) &= \frac{3z-1}{z^2+2z+5} \\
&= \frac{3z-1}{(z+1+2i)(z+1-2i)}
\end{aligned}$$

Poles are $z=-1\pm 2i$ both of order 1.

19.4.15

$$f(z)=\frac{1+4i}{(z+2)(z+i)^4}$$

Poles are $z=-2$ order 1, and $z=-i$ order 4.

19.4.17

$$\begin{aligned} f(z) &= \tan z \\ &= \frac{\sin z}{\cos z} \end{aligned}$$

Poles are $z = \frac{\pi}{2} + n\pi$, $n \in \mathbb{R}$ order 1.

19.5 Residues and Residue Theorem

19.5.1

$$\begin{aligned} f(z) &= \frac{2}{(z-1)(z+4)} \\ &= \frac{2}{z-1}\frac{1}{5+z-1} \\ &= \frac{2}{5(z-1)}\frac{1}{1+\frac{z-1}{5}} \\ &= \frac{2}{5(z-1)}\left[1 - \frac{z-1}{5} + \left(\frac{z-1}{5}\right)^2 - \cdots\right] \\ &= \frac{2}{5(z-1)} - \frac{2}{5^2} + \frac{2(z-1)}{5^3} - \cdots \\ \operatorname{Res}(f(z), 1) &= \frac{2}{5} \end{aligned}$$

19.5.3

$$\begin{aligned} f(z) &= \frac{4z-6}{z(2-z)} \\ &= \frac{4z-6}{2z}\frac{1}{1-\frac{z}{2}} \\ &= \left(2 - \frac{3}{z}\right)\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right] \\ \operatorname{Res}(f(z), 0) &= -3 \end{aligned}$$

19.5.5

$$\begin{aligned} f(z) &= e^{-2/z^2} \\ &= 1 - \frac{2}{z^2} + \frac{4}{2! \cdot z^4} + \cdots \\ \operatorname{Res}(f(z), 0) &= 0 \end{aligned}$$

19.5.7

$$\begin{aligned}
f(z) &= \frac{z}{z^2+16} \\
&= \frac{z}{(z+4i)(z-4i)} \\
\mathrm{Res}(f(z),-4i) &= \lim_{z\to -4i}(z+4i)\frac{z}{(z+4i)(z-4i)} \\
&= \frac{-4i}{-8i} \\
&= \frac{1}{2} \\
\mathrm{Res}(f(z),4i) &= \frac{1}{2}
\end{aligned}$$

19.5.9

$$\begin{aligned}
f(z) &= \frac{1}{z^4+z^3-2z^2} \\
&= \frac{1}{z^2(z^2+z-2)} \\
&= \frac{1}{z^2(z-1)(z+2)} \\
\mathrm{Res}(f(z),1) &= \frac{1}{3} \\
\mathrm{Res}(f(z),-2) &= -\frac{1}{12} \\
\mathrm{Res}(f(z),0) &= \lim_{z\to 0}\frac{d}{dz}\frac{1}{(z-1)(z+2)} \\
&= -\frac{1}{4}
\end{aligned}$$

19.5.11

$$\begin{aligned}
f(z) &= \frac{5z^2-4z+3}{(z+1)(z+2)(z+3)} \\
\mathrm{Res}(f(z),-1) &= 6 \\
\mathrm{Res}(f(z),-2) &= -31 \\
\mathrm{Res}(f(z),-3) &= 30
\end{aligned}$$

19.5.13

$$\begin{aligned} f(z) &= \frac{\cos z}{z^2(z-\pi)^3} \\ \operatorname{Res}(f(z),0) &= -\frac{3}{\pi^4} \\ \operatorname{Res}(f(z),\pi) &= \frac{1}{2}\left(\frac{1}{\pi^2}-\frac{6}{\pi^4}\right) \end{aligned}$$

19.5.15

$$\begin{aligned} f(z) &= \sec z \\ &= \frac{1}{\cos z} \\ \operatorname{Res}\left(f(z),\frac{\pi}{2}+n\pi\right) &= \frac{1}{-\sin\left(\frac{\pi}{2}+n\pi\right)} \\ &= (-1)^{n+1} \end{aligned}$$

19.5.17

(a) 0

(b)

$$\begin{aligned} \operatorname{Res}(f(z),1) &= \frac{1}{9} \\ \oint_C \frac{1}{(z-1)(z+2)^2}\,dz &= \frac{2}{9}\pi i \end{aligned}$$

(c)

$$\begin{aligned} \operatorname{Res}(f(z),-2) &= -\frac{1}{9} \\ \oint_C \frac{1}{(z-1)(z+2)^2}\,dz &= 0 \end{aligned}$$

19.5.19

(a)

$$\begin{aligned}
f(z) &= z^3 e^{-1/z^2} \\
&= z^3 \left(1 - \frac{1}{1! \cdot z^2} + \frac{1}{2! \cdot z^4} - \frac{1}{3! \cdot z^6} + \cdots\right) \\
&= z^3 - \frac{z}{1!} + \frac{1}{2! \cdot z} - \frac{1}{3! \cdot z^3} + \cdots \\
\operatorname{Res}(f(z), 0) &= \frac{1}{2} \\
\oint_C f(z)\, dz &= \pi i
\end{aligned}$$

(b) πi

(c) 0

19.5.21

$$\begin{aligned}
f(z) &= \frac{1}{z^2 + 4z + 13} \\
&= \frac{1}{(z + 2 + 3i)(z + 2 - 3i)} \\
\operatorname{Res}(f(z), -2 + 3i) &= \lim_{z \to -2+3i} (z + 2 - 3i) f(z) \\
&= \frac{1}{6i} \\
&= -\frac{1}{6} i \\
\oint_C f(z)\, dz &= \frac{\pi}{3}
\end{aligned}$$

19.5.23

$$\begin{aligned}
f(z) &= \frac{z}{z^4 - 1} \\
&= \frac{z}{(z^2 + 1)(z^2 - 1)} \\
&= \frac{z}{(z + i)(z - i)(z + 1)(z - 1)} \\
\oint_C f(z)\, dz &= 2\pi i \left(-\frac{1}{4} - \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) \\
&= 0
\end{aligned}$$

19.5.25

$$
\begin{aligned}
f(z) &= \frac{ze^z}{z^2-1} \\
&= \frac{ze^z}{(z+1)(z-1)} \\
\oint_C f(z)\,dz &= 2\pi i\left(\frac{1}{2e}+\frac{e}{2}\right) \\
&= 2\pi i\frac{e+e^{-1}}{2} \\
&= 2\pi i\cosh 1
\end{aligned}
$$

19.6 Evaluation of Real Integrals

19.6.1

$$
\begin{aligned}
\int_0^{2\pi}\frac{1}{1+0.5\sin\theta}\,d\theta &= \oint_C \frac{1}{1+\frac{1}{2}\frac{1}{2i}(z-z^{-1})}\frac{dz}{iz} \\
&= \oint_C \frac{1}{iz+\frac{1}{4}z^2-\frac{1}{4}}\,dz \\
&= 4\oint_C \frac{1}{z^2+4iz-1}\,dz \\
&= 4\oint_C \frac{1}{[z+(2+\sqrt{3})i][z+(2-\sqrt{3})i]}\,dz \\
&= 4\cdot 2\pi i\operatorname{Res}(f(z),(-2+\sqrt{3})i) \\
&= \frac{4\pi}{\sqrt{3}}
\end{aligned}
$$

19.6.3

$$
\begin{aligned}
\int_0^{2\pi}\frac{\cos\theta}{3+\sin\theta}\,d\theta &= \oint_C \frac{\frac{1}{2}(z+z^{-1})}{3+\frac{1}{2i}(z-z^{-1})}\frac{dz}{iz} \\
&= \oint_C \frac{z+z^{-1}}{z^2+6iz-1}\,dz \\
&= \oint_C \frac{z^2+1}{z(z^2+6iz-1)}\,dz \\
&= \oint_C \frac{z^2+1}{z[z+(3+2\sqrt{2})i][z+(3-2\sqrt{2})i]}\,dz \\
&= 2\pi i[\operatorname{Res}(f(z),0)+\operatorname{Res}(f(z),(-3+2\sqrt{2})i)] \\
&= 2\pi i(-1+1) \\
&= 0
\end{aligned}
$$

19.6.5

$$\begin{aligned}
\int_0^\pi \frac{1}{2-\cos\theta}\,d\theta &= \frac{1}{2}\int_{-\pi}^{\pi} \frac{1}{2-\cos\theta}\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \frac{1}{2+\cos t}\,dt \\
&= \frac{1}{2}\oint_C \frac{1}{2+\frac{1}{2}(z+z^{-1})}\,\frac{dz}{iz} \\
&= \frac{1}{i}\oint_C \frac{1}{z^2+4z+1}\,dz \\
&= \frac{1}{i}\oint_C \frac{1}{(z+2+\sqrt{3})(z+2-\sqrt{3})}\,dz \\
&= \frac{1}{i}2\pi i\,\mathrm{Res}(f(z),-2+\sqrt{3}) \\
&= \frac{\pi}{\sqrt{3}}
\end{aligned}$$

19.6.11

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{x^2-2x+2}\,dx &= \oint_C \frac{1}{(z-1-i)(z-1+i)}\,dz \\
&= 2\pi i\,\mathrm{Res}(f(z),1+i) \\
&= \pi
\end{aligned}$$

19.6.13

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2}\,dx &= \oint_C \frac{1}{(z+2i)^2(z-2i)^2}\,dz \\
&= 2\pi i\,\mathrm{Res}(f(z),2i) \\
&= 2\pi i \lim_{z\to 2i}\frac{d}{dz}\frac{1}{(z+2i)^2} \\
&= \frac{\pi}{16}
\end{aligned}$$

19.6.15

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3}\,dx &= \oint_C \frac{1}{(z+i)^3(z-i)^3}\,dz \\
&= 2\pi i\,\mathrm{Res}(f(z),i) \\
&= 2\pi i\frac{1}{2}\lim_{z\to i}\frac{d^2}{dz^2}\frac{1}{(z+i)^3} \\
&= \frac{3\pi}{8}
\end{aligned}$$

19.6.21

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx &= \mathrm{Re}\left(\oint_C \frac{1}{z^2+1} e^{iz}\,dz\right) \\
&= \mathrm{Re}\left(\oint_C \frac{1}{(z+i)(z-i)} e^{iz}\,dz\right) \\
&= \mathrm{Re}[2\pi i \,\mathrm{Res}(f(z), i)] \\
&= \frac{\pi}{e}
\end{aligned}$$

19.6.23

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+1}\,dx &= \mathrm{Im}\left(\oint_C \frac{z}{z^2+1} e^{iz}\,dz\right) \\
&= \mathrm{Im}\left(\frac{z}{(z+i)(z-i)} e^{iz}\,dz\right) \\
&= \mathrm{Im}[2\pi i \,\mathrm{Res}(f(z), i)] \\
&= \frac{\pi}{e}
\end{aligned}$$

19.6.31

$$\begin{aligned}
\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx &= \mathrm{Im}\left(\oint_C \frac{e^{iz}}{z}\,dz\right) \\
&= \mathrm{Im}[\pi i \,\mathrm{Res}(f(z), 0)] \\
&= \pi
\end{aligned}$$