Advanced Engineering Mathematics Systems of Differential Equations by Dennis G. Zill Problems

Chris Doble

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Contents

10 Systems of Linear Differential Equations

10.1 Theory of Linear Systems

10.1.1

$$\mathbf{X}' = \begin{pmatrix} 3 & -5 \\ 4 & 8 \end{pmatrix} \mathbf{X}$$

10.1.3

$$\mathbf{X}' = \begin{pmatrix} -3 & 4 & -9 \\ 6 & -1 & 0 \\ 10 & 4 & 3 \end{pmatrix} \mathbf{X}$$

10.1.5

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} t-1 \\ -3t^2 \\ t^2 - t + 2 \end{pmatrix}$$

10.1.7

$$\frac{dx}{dt} = 4x + 2y + e^t$$
$$\frac{dy}{dt} = -x + 3y - e^t$$

10.1.9

$$\frac{dx}{dt} = x - y + 2z + e^{-t} - 3t$$
$$\frac{dy}{dt} = 3x - 4y + z + 2e^{-t} + t$$
$$\frac{dz}{dt} = -2x + 5y + 6z + 2e^{-t} - t$$

10.1.11

$$
\begin{aligned}
3(e^{-5t}) - 4(2e^{-5t}) &= -5e^{-5t} \\
&= \frac{dx}{dt} \\
4(e^{-5t}) - 7(2e^{-5t}) &= -10e^{-5t} \\
&= \frac{dy}{dt}
\end{aligned}
$$

10.1.13

$$
\begin{aligned}
-(-e^{-3t/2}) + \frac{1}{4}(2e^{-3t/2}) &= \frac{3}{2}e^{-3t/2} \\
&= \frac{dx}{dt} \\
(-e^{-3t/2}) - (2e^{-3t/2}) &= -3e^{-3t/2} \\
&= \frac{dy}{dt}
\end{aligned}
$$

10.1.17

$$
\begin{aligned}
W(\mathbf{X}_1, \mathbf{X}_2) &= \begin{vmatrix} e^{-2t} & e^{-6t} \\ e^{-2t} & -e^{-6t} \end{vmatrix} \\
&= -e^{-8t} - e^{-8t} \\
&= -2e^{-8t} \\
&\neq 0 \text{ for } t \in (-\infty, \infty)
\end{aligned}
$$

Yes, they form a fundamental set.

10.1.19

$$
\begin{aligned}
W(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) &= \begin{vmatrix} 1+t & 1 & 3+2t \\ -2+2t & -2 & -6+4t \\ 4+2t & 4 & 12+4t \end{vmatrix} \\
&= 0
\end{aligned}
$$

No, they don't form a fundamental set.

10.1.21

$$\begin{aligned}
x &= 2t + 5 \\
y &= -t + 1 \\
\frac{dx}{dt} &= (2t + 5) + 4(-t + 1) + 2t - 7 \\
&= 2 \\
\frac{dy}{dt} &= 3(2t + 5) + 2(-t + 1) - 4t - 18 \\
&= -1
\end{aligned}$$

10.1.23

$$\begin{aligned}
x &= e^t + te^t \\
x' &= 2e^t + te^t \\
y &= e^t - te^t \\
y' &= -te^t \\
\frac{dx}{dt} &= 2(e^t + te^t) + (e^t - te^t) - e^t \\
&= 2e^t + te^t \\
\frac{dy}{dt} &= 3(e^t + te^t) + 4(e^t - te^t) - 7e^t \\
&= -te^t
\end{aligned}$$

10.2 Homogeneous Linear Systems

10.2.1

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

10.2.3

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} e^{t}$$

10.2.5

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-10t} + c_2 \begin{pmatrix} 5 \\ 2 \end{pmatrix} e^{8t}$$

10.2.7

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t}$$

10.2.13

$$\mathbf{X} = 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/2} + 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t/2}$$

10.2.15

(a)

$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{3}{100}x_1 + \frac{1}{100}x_2 \\ \frac{dx_2}{dt} &= \frac{2}{100}x_1 - \frac{2}{100}x_2 \\ \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} &= \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{2}{100} & -\frac{2}{100} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

(b)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\frac{35}{3}\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t/25} + \frac{50}{3}\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} e^{-t/100}$$

10.2.21

$$\mathbf{X} = c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2\left[\begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]$$

10.2.23

$$\mathbf{X} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix}\right] e^{2t}$$

10.2.25

$$\mathbf{X} = c_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{t}$$

10.2.31

$$\mathbf{X} = -\frac{1}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + 13\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}\right] e^{4t}$$

10.2.33

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{K}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

10.2.35

$$\begin{aligned}
\mathbf{X} &= c_1 \begin{pmatrix} 1 \\ 2-i \end{pmatrix} e^{(4+i)t} + c_2 \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{(4-i)t} \\
&= c_1 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin t \right] e^{4t} + c_2 \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t \right] e^{4t} \\
&= c_1 \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix} e^{4t}
\end{aligned}$$

10.2.37

$$\begin{aligned}
\mathbf{X} &= c_1 \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{(4+i)t} + c_2 \begin{pmatrix} 1 \\ -1-i \end{pmatrix} e^{(4-i)t} \\
&= c_1 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t \right] e^{4t} + c_2 \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin t \right] e^{4t} \\
&= c_1 \begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix} e^{4t}
\end{aligned}$$

10.2.39

$$\begin{aligned}
\mathbf{X} &= c_1 \begin{pmatrix} 5 \\ 4-3i \end{pmatrix} e^{3i} + c_2 \begin{pmatrix} 5 \\ 4+3i \end{pmatrix} e^{-3i} \\
&= c_1 \left[\begin{pmatrix} 5 \\ 4 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ -3 \end{pmatrix} \sin 3t \right] + c_2 \left[\begin{pmatrix} 0 \\ -3 \end{pmatrix} \cos 3t + \begin{pmatrix} 5 \\ 4 \end{pmatrix} \sin 3t \right] \\
&= c_1 \begin{pmatrix} 5\cos 3t \\ 4\cos 3t + 3\sin 3t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin 3t \\ 4\sin 3t - 3\cos 3t \end{pmatrix}
\end{aligned}$$

10.2.47

$$\begin{aligned}
\mathbf{X} &= c_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1+5i \\ 1 \\ 1 \end{pmatrix} e^{5it} + c_3 \begin{pmatrix} 1-5i \\ 1 \\ 1 \end{pmatrix} e^{-5it} \\
&= c_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^{t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cos 5t - \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \sin 5t \right] \\
&\quad + c_3 \left[\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \cos 5t + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \sin 5t \right] \\
&= c_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} \cos 5t - 5\sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + c_3 \begin{pmatrix} 5\cos 5t + \sin 5t \\ \sin 5t \\ \sin 5t \end{pmatrix} \\
&= - \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^{t} - \begin{pmatrix} \cos 5t - 5\sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + 6 \begin{pmatrix} 5\cos 5t + \sin 5t \\ \sin 5t \\ \sin 5t \end{pmatrix}
\end{aligned}$$

10.2.49

(a)

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} -\frac{1}{20} & 0 & \frac{1}{10} \\ \frac{1}{20} & -\frac{1}{20} & 0 \\ 0 & \frac{1}{20} & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(b)

$$\begin{aligned}
\mathbf{X} &= c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1-i \\ i \\ 1 \end{pmatrix} e^{\left(-\frac{1}{10}+\frac{1}{20}i\right)t} + c_3 \begin{pmatrix} -1+i \\ -i \\ 1 \end{pmatrix} e^{\left(-\frac{1}{10}-\frac{1}{20}i\right)t} \\
&= c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \left[\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cos \frac{1}{20}t - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \sin \frac{1}{20}t \right] e^{-t/10} \\
&\quad + c_3 \left[\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cos \frac{1}{20}t + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \sin \frac{1}{20}t \right] e^{-t/10} \\
&= c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \sin \frac{t}{20} - \cos \frac{t}{20} \\ -\sin \frac{t}{20} \\ \cos \frac{t}{20} \end{pmatrix} e^{-t/10} \\
&\quad + c_3 \begin{pmatrix} -\cos \frac{t}{20} - \sin \frac{t}{20} \\ \cos \frac{t}{20} \\ \sin \frac{t}{20} \end{pmatrix} e^{-t/10} \\
&= 11 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - 6 \begin{pmatrix} \sin \frac{t}{20} - \cos \frac{t}{20} \\ -\sin \frac{t}{20} \\ \cos \frac{t}{20} \end{pmatrix} e^{-t/10} \\
&\quad - 2 \begin{pmatrix} -\cos \frac{t}{20} - \sin \frac{t}{20} \\ \cos \frac{t}{20} \\ \sin \frac{t}{20} \end{pmatrix} e^{-t/10}
\end{aligned}$$

10.3 Solution by Diagonalization

10.3.1

$$\mathbf{X} = \begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 e^{7t} \\ c_2 e^{-4t} \end{pmatrix} = \begin{pmatrix} 3c_1 e^{7t} - 2c_2 e^{-4t} \\ c_1 e^{7t} + 3c_2 e^{-4t} \end{pmatrix}$$

10.3.3

$$\mathbf{X} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_1 e^{3t/2} \\ c_2 e^{t/2} \end{pmatrix} = \begin{pmatrix} c_1 e^{3t/2} + c_2 e^{t/2} \\ 2c_1 e^{3t/2} - 2c_2 e^{t/2} \end{pmatrix}$$

10.3.5

$$\mathbf{X} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{6t} \\ c_2 e^{-4t} \\ c_3 e^{2t} \end{pmatrix} = \begin{pmatrix} -c_2 e^{-4t} + c_3 e^{2t} \\ c_2 e^{-4t} + c_3 e^{2t} \\ c_1 e^{6t} + c_3 e^{2t} \end{pmatrix}$$

10.3.11

(a)

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\mathbf{M}$ has an inverse because it has a nonzero determinant (the product of the diagonal entries).

(b)

$$\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & \frac{k_2}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

(c)

$$\mathbf{X} = \mathbf{PY}$$

$$\begin{aligned} \mathbf{PY}'' + \mathbf{BPY} &= \mathbf{0} \\ \mathbf{Y}'' + \mathbf{P^{-1}BPY} &= \mathbf{0} \\ \mathbf{Y}'' + \mathbf{DY} &= \mathbf{0} \\ \begin{pmatrix} y_1'' \\ y_2'' \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \mathbf{0} \\ y_1'' + 6y_1 &= 0 \\ y_1 &= c_1 \cos \sqrt{6}t + c_2 \sin \sqrt{6}t \\ y_2'' + y_2 &= 0 \\ y_2 &= c_3 \cos t + c_4 \sin t \end{aligned}$$

$$\begin{aligned} \mathbf{X} &= \mathbf{PY} \\ &= \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \cos \sqrt{6}t + c_2 \sin \sqrt{6}t \\ c_3 \cos t + c_4 \sin t \end{pmatrix} \\ &= \begin{pmatrix} -2c_1 \cos \sqrt{6}t - 2c_2 \sin \sqrt{6}t + c_3 \cos t + c_4 \sin t \\ c_1 \cos \sqrt{6}t + c_2 \sin \sqrt{6}t + 2c_3 \cos t + 2c_4 \sin t \end{pmatrix} \\ &= c_3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos t + c_4 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t + c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cos \sqrt{6}t + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \sin \sqrt{6}t \end{aligned}$$

10.4 Nonhomogeneous Linear Systems

10.4.1

$$\begin{aligned}
\begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -7 \\ 5 \end{pmatrix} \\
\mathbf{X}_c &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} \\
\mathbf{X}_p &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} -7 \\ 5 \end{pmatrix} \\
&= \begin{pmatrix} 2a_1 + 3a_2 - 7 \\ -a_1 - 2a_2 + 5 \end{pmatrix} \\
\mathbf{X}_p &= \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\
\mathbf{X} &= \mathbf{X}_c + \mathbf{X}_p \\
&= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}
\end{aligned}$$

10.4.3

$$
\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} \\
\mathbf{X}_p &= \begin{pmatrix} a_3t^2 + a_2t + a_1 \\ b_3t^2 + b_2t + b_1 \end{pmatrix} \\
\begin{pmatrix} 2a_3t + a_2 \\ 2b_3t + b_2 \end{pmatrix} &= \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a_3t^2 + a_2t + a_1 \\ b_3t^2 + b_2t + b_1 \end{pmatrix} + \begin{pmatrix} -2t^2 \\ t + 5 \end{pmatrix} \\
&= \begin{pmatrix} (a_3 + 3b_3 - 2)t^2 + (a_2 + 3b_2)t + (a_1 + 3b_1) \\ (3a_3 + b_3)t^2 + (3a_2 + b_2 + 1)t + (3a_1 + b_1 + 5) \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} (a_3 + 3b_3 - 2)t^2 + (a_2 - 2a_3 + 3b_2)t + (a_1 - a_2 + 3b_1) \\ (3a_3 + b_3)t^2 + (3a_2 + b_2 - 2b_3 + 1)t + (3a_1 + b_1 - b_2 + 5) \end{pmatrix} \\
a_3 &= -\frac{1}{4} \\
b_3 &= \frac{3}{4} \\
a_2 &= \frac{1}{4} \\
b_2 &= -\frac{1}{4} \\
a_1 &= -2 \\
b_1 &= \frac{3}{4} \\
\mathbf{X}_p &= \begin{pmatrix} -\frac{1}{4}t^2 + \frac{1}{4}t - 2 \\ \frac{3}{4}t^2 - \frac{1}{4}t + \frac{3}{4} \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} + \begin{pmatrix} -\frac{1}{4}t^2 + \frac{1}{4}t - 2 \\ \frac{3}{4}t^2 - \frac{1}{4}t + \frac{3}{4} \end{pmatrix}
\end{aligned}
$$

10.4.5

$$\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} \frac{1}{9} \\ 1 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} e^{3t} \\
\mathbf{X}_p &= \begin{pmatrix} a \\ b \end{pmatrix} e^{t} \\
\begin{pmatrix} a \\ b \end{pmatrix} e^{t} &= \begin{pmatrix} 4 & \frac{1}{3} \\ 9 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} e^{t} + \begin{pmatrix} -3 \\ 10 \end{pmatrix} e^{t} \\
\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 4a + \frac{1}{3}b - 3 \\ 9a + 6b + 10 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 3a + \frac{1}{3}b - 3 \\ 9a + 5b + 10 \end{pmatrix} \\
\mathbf{X}_p &= \begin{pmatrix} \frac{55}{36} \\ -\frac{19}{4} \end{pmatrix} e^{t} \\
\mathbf{X} &= c_1 \begin{pmatrix} \frac{1}{9} \\ 1 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} \frac{55}{36} \\ -\frac{19}{4} \end{pmatrix} e^{t}
\end{aligned}$$

10.4.9

$$\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t} \\
\mathbf{X}_p &= \begin{pmatrix} a \\ b \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} -a - 2b + 3 \\ 3a + 4b + 3 \end{pmatrix} \\
\mathbf{X}_p &= \begin{pmatrix} -9 \\ 6 \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} -9 \\ 6 \end{pmatrix} \\
\begin{pmatrix} -4 \\ 5 \end{pmatrix} &= c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -9 \\ 6 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -2c_1 - c_2 - 5 \\ 3c_1 + c_2 + 1 \end{pmatrix} \\
\mathbf{X} &= 4 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{2t} - 13 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} -9 \\ 6 \end{pmatrix}
\end{aligned}$$

10.4.11

(a)

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{1}{50} & -\frac{1}{25} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(b)

$$\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} e^{-t/20} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/50} \\
\mathbf{X}_p &= \begin{pmatrix} a \\ b \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{1}{50} & -\frac{1}{25} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{3}{100}a + \frac{1}{100}b \\ \frac{1}{50}a - \frac{1}{25}b + 1 \end{pmatrix} \\
\mathbf{X}_p &= \begin{pmatrix} 10 \\ 30 \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} e^{-t/20} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/50} + \begin{pmatrix} 10 \\ 30 \end{pmatrix} \\
\begin{pmatrix} 60 \\ 10 \end{pmatrix} &= c_1 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 10 \\ 30 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -\frac{1}{2}c_1 + c_2 - 50 \\ c_1 + c_2 + 20 \end{pmatrix} \\
\mathbf{X} &= -\frac{140}{3} \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} e^{-t/20} + \frac{80}{3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/50} + \begin{pmatrix} 10 \\ 30 \end{pmatrix}
\end{aligned}$$

(c)

$$\begin{aligned}
\lim_{t\to\infty} x_1(t) &= \lim_{t\to\infty} \frac{70}{3} e^{-t/20} + \frac{80}{3} e^{-t/50} + 10 \\
&= 10 \\
\lim_{t\to\infty} x_1(t) &= \lim_{t\to\infty} -\frac{140}{3} e^{-t/20} + \frac{80}{3} e^{-t/50} + 30 \\
&= 30
\end{aligned}$$

10.4.13

$$\begin{aligned}
\begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} \\
\mathbf{X}_c &= c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathbf{X}_p &= \mathbf{\Phi}(t) \int \mathbf{\Phi}^{-1}(t)\mathbf{F}\, dt \\
&= \begin{pmatrix} 3e^t & 1 \\ 2e^t & 1 \end{pmatrix} \int \begin{pmatrix} e^{-t} & -e^{-t} \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \end{pmatrix} dt \\
&= \begin{pmatrix} 3e^t & 1 \\ 2e^t & 1 \end{pmatrix} \int \begin{pmatrix} 5e^{-t} \\ -11 \end{pmatrix} dt \\
&= \begin{pmatrix} 3e^t & 1 \\ 2e^t & 1 \end{pmatrix} \begin{pmatrix} -5e^{-t} \\ -11t \end{pmatrix} \\
&= \begin{pmatrix} -15 - 11t \\ -10 - 11t \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t + c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -15 - 11t \\ -10 - 11t \end{pmatrix}
\end{aligned}$$

10.4.15

$$\begin{aligned}
\mathbf{X}' &= \begin{pmatrix} 3 & -5 \\ \frac{3}{4} & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2} \\
\mathbf{X}_c &= c_1 \begin{pmatrix} \frac{10}{3} \\ 1 \end{pmatrix} e^{3t/2} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t/2} \\
\mathbf{X}_p &= \begin{pmatrix} \frac{1}{2}(-15 - 13t) \\ \frac{1}{4}(-9 - 13t) \end{pmatrix} e^{t/2} \\
\mathbf{X} &= c_1 \begin{pmatrix} \frac{10}{3} \\ 1 \end{pmatrix} e^{3t/2} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t/2} + \frac{1}{4} \begin{pmatrix} -30 - 26t \\ -9 - 13t \end{pmatrix} e^{t/2}
\end{aligned}$$

10.4.33

$$
\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} \\
\mathbf{X}_p &= \begin{pmatrix} 2e^{2t}t - 2e^{4t}t - e^{2t} + e^{4t} \\ 2e^{2t}t + 2e^{4t}t + e^{2t} + e^{4t} \end{pmatrix} \\
&= \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} \\
\mathbf{X} &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} \\
&\quad + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} \\
\begin{pmatrix} 1 \\ 1 \end{pmatrix} &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -c_1 + c_2 - 1 \\ c_1 + c_2 + 1 \end{pmatrix} \\
\mathbf{X} &= \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{4t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t}
\end{aligned}
$$

10.4.35

$$
\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-12t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} \\
\mathbf{X}_p &= \frac{1}{29} \begin{pmatrix} -76\cos t + 332\sin t \\ -168\cos t + 276\sin t \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-12t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} - \frac{4}{29} \begin{pmatrix} 19 \\ 42 \end{pmatrix} \cos t + \frac{4}{29} \begin{pmatrix} 83 \\ 69 \end{pmatrix} \sin t \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{4}{29} \begin{pmatrix} 19 \\ 42 \end{pmatrix} \\
&= \begin{pmatrix} -3c_1 + c_2 - \frac{76}{29} \\ c_1 + 3c_2 - \frac{168}{29} \end{pmatrix} \\
\mathbf{X} &= -\frac{6}{29} \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-12t} + 2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} - \frac{4}{29} \begin{pmatrix} 19 \\ 42 \end{pmatrix} \cos t + \frac{4}{29} \begin{pmatrix} 83 \\ 69 \end{pmatrix} \sin t
\end{aligned}
$$

10.4.37

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 5 & -2 \\ 21 & -8 \end{pmatrix} \\
\mathbf{F} &= \begin{pmatrix} 6 \\ 4 \end{pmatrix} \\
\mathbf{P} &= \begin{pmatrix} 2 & 1 \\ 7 & 3 \end{pmatrix} \\
\mathbf{G} &= \begin{pmatrix} -14 \\ 34 \end{pmatrix} \\
\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} &= \begin{pmatrix} -2y_1 - 14 \\ -y_2 + 34 \end{pmatrix} \\
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} c_1 e^{-2t} - 7 \\ c_2 e^{-t} + 34 \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} 2 \\ 7 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + \begin{pmatrix} 20 \\ 53 \end{pmatrix}
\end{aligned}$$

10.4.39

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \\
\mathbf{P} &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\
\mathbf{G} &= \begin{pmatrix} 4 + t \\ 4 - t \end{pmatrix} \\
\mathbf{Y}' &= \mathbf{DY} + \mathbf{G} \\
&= \begin{pmatrix} 10y_1 + 4 + t \\ 4 - t \end{pmatrix} \\
\mathbf{Y} &= \begin{pmatrix} c_1 e^{10t} - \frac{1}{10}t - \frac{41}{100} \\ -\frac{1}{2}t^2 + 4t + c_2 \end{pmatrix} \\
\mathbf{X} &= \mathbf{PY} \\
&= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{10t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} t^2 + \begin{pmatrix} -\frac{41}{10} \\ \frac{39}{10} \end{pmatrix} t - \frac{41}{100} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{aligned}$$

10.5 Matrix Exponential

10.5.1

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

10.5.3

$$\begin{pmatrix} t+1 & t & t \\ t & t+1 & t \\ -2t & -2t & -2t+1 \end{pmatrix}$$

10.5.5

$$\begin{pmatrix} c_1 e^{t} \\ c_2 e^{2t} \end{pmatrix}$$

10.5.7

$$\mathbf{X} = c_1 \begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} + c_2 \begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + c_3 \begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix}$$

10.5.9

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -3 \\ \frac{1}{2} \end{pmatrix}$$

10.5.11

$$\mathbf{X} = c_1 \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2 \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

10.5.13

$$\mathbf{X} = c_1 \begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} + c_2 \begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + c_3 \begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -4 \\ 6 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} - 4 \begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + 6 \begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix}$$

10.5.15

$$e^{\mathbf{A}t} = \begin{pmatrix} \frac{1}{2}e^{-2t}(3e^{4t}-1) & \frac{3}{4}e^{-2t}(e^{4t}-1) \\ e^{-2t}-e^{2t} & -\frac{1}{2}e^{-2t}(e^{4t}-3) \end{pmatrix}$$

10.5.17

$$e^{\mathbf{A}t} = \begin{pmatrix} e^{2t}(1+3t) & -9e^{2t}t \\ e^{2t}t & e^{2t}(1-3t) \end{pmatrix}$$

10.5.25

$$
\begin{aligned}
\mathbf{X} &= e^{\mathbf{A}t}\mathbf{C} \\
&= \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}\mathbf{C} \\
&= \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} e^{5t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \mathbf{C} \\
&= \begin{pmatrix} -\frac{1}{2}e^{3t}(-3+e^{2t}) & \frac{1}{2}e^{3t}(-1+e^{2t}) \\ -\frac{3}{2}e^{3t}(-1+e^{2t}) & \frac{1}{2}e^{3t}(-1+3e^{2t}) \end{pmatrix} \mathbf{C}
\end{aligned}
$$

10.6 Chapter in Review

10.6.1

$\frac{1}{3}$

10.6.5

$$
\mathbf{X} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^t \right]
$$

10.6.7

$$
\begin{aligned}
\mathbf{X} &= c_1 \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{1+2i} + c_2 \begin{pmatrix} i \\ 1 \end{pmatrix} e^{1-2i} \\
&= c_3 \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin 2t \right] e^t + c_4 \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 2t \right] e^t \\
&= \begin{pmatrix} c_3 e^t \sin 2t - c_4 e^t \cos 2t \\ c_3 e^t \cos 2t + c_4 e^t \sin 2t \end{pmatrix} \\
&= c_3 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^t + c_4 \begin{pmatrix} -\cos 2t \\ \sin 2t \end{pmatrix} e^t
\end{aligned}
$$

10.6.9

$$
\mathbf{X} = c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -7 \\ -12 \\ 16 \end{pmatrix} e^{-3t} + c_3 \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} e^{2t}
$$

11 Systems of Nonlinear Differential Equations

11.1 Autonomous Systems

11.1.1

$$
\begin{aligned}
x' &= y \\
y' &= -9 \sin x
\end{aligned}
$$

Critical points at $(n\pi, 0), n \in \mathbb{Z}$.

11.1.3

$$\begin{aligned} x' &= y \\ y' &= x^2 - y(1 - x^3) \end{aligned}$$

$$\begin{aligned} 0 &= y \\ 0 &= x^2 - y(1 - x^3) \\ &= x^2 \\ 0 &= x \end{aligned}$$

Critical point at $(0, 0)$.

11.1.5

$$\begin{aligned} x' &= y \\ y' &= \epsilon x^3 - x \end{aligned}$$

$$\begin{aligned} 0 &= y \\ 0 &= \epsilon x^3 - x \\ &= \epsilon x^2 - 1 \\ x &= \sqrt{\frac{1}{\epsilon}} \end{aligned}$$

Critical points at $(0, 0)$ and $\left(\pm\sqrt{\frac{1}{\epsilon}}, 0\right)$.

11.1.7

$x' = x + xy$ can only be 0 if $x = 0$ or $y = -1$. If $x = 0$, $y' = -y - xy$ is 0 if $y = 0$. If $y = -1$, it's 0 if $x = -1$. Therefore the critical points are $(0, 0)$ and $(-1, -1)$.

11.1.9

$$\begin{aligned} x' &= 3x^2 - 4y \\ 3x^2 = 4y \\ x &= \sqrt{\frac{4}{3}y} \end{aligned}$$

$$\begin{aligned} y' &= x - y \\ 0 &= \sqrt{\frac{4}{3}y} - y \\ y^2 &= \frac{4}{3}y \\ y &= \frac{4}{3} \end{aligned}$$

The critical points are $(0,0)$ and $\left(\frac{4}{3}, \frac{4}{3}\right)$.

11.1.11

$$\begin{aligned} x' &= x\left(10 - x - \frac{1}{2}y\right) \\ y' &= y(16 - y - x) \end{aligned}$$

$(0,0)$, $(0,16)$, $(10,0)$, $(4,12)$

11.1.13

$$\begin{aligned} x' &= x^2 e^y \\ y' &= y(e^x - 1) \end{aligned}$$

All points on the line $x = 0$.

11.1.15

$$\begin{aligned} x' &= x(1 - x^2 - 3y^2) \\ y' &= y(3 - x^2 - 3y^2) \end{aligned}$$

$(0,0)$, $(0,\pm 1)$, $(\pm 1, 0)$

11.1.17

(a)

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

(b)

$$\begin{aligned} \begin{pmatrix} 2 \\ -2 \end{pmatrix} &= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} c_1 - c_2 - 2 \\ 2c_1 + c_2 + 2 \end{pmatrix} \\ \mathbf{X} &= -2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} \end{aligned}$$

11.1.19

(a)

$$\begin{aligned} x &= c_1(4\cos 3t - 3\sin 3t) + c_2(3\cos 3t + 4\sin 3t) \\ y &= c_1(5\cos 3t) + c_2(5\sin 3t) \end{aligned}$$

(b)

$$\begin{aligned} 4 &= 4c_1 + 3c_2 \\ 5 &= 5c_1 \\ c_1 &= 1 \\ c_2 &= 0 \end{aligned}$$

$$\begin{aligned} x &= 4\cos 3t - 3\sin 3t \\ y &= 5\cos 3t \end{aligned}$$

11.1.21

(a)

$$\begin{aligned} x &= c_1(-\cos t + \sin t)e^{4t} + c_2(-\cos t - \sin t)e^{4t} \\ y &= c_1(2\cos t)e^{4t} + c_2(2\sin t)e^{4t} \end{aligned}$$

(b)

$$
\begin{aligned}
-1 &= -c_1 - c_2 \\
2 &= 2c_1 \\
c_1 &= 1 \\
c_2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
x &= (\sin t - \cos t)e^{4t} \\
y &= 2\cos t e^{4t}
\end{aligned}
$$

11.1.23

$$
\begin{aligned}
\frac{dr}{dt} &= \frac{1}{r}\{x[-y - x(x^2+y^2)^2] + y[x - y(x^2+y^2)^2]\} \\
&= \frac{1}{r}[-xy - x^2r^4 + xy - y^2r^4] \\
&= -r^5 \\
\frac{1}{r^5}\frac{dr}{dt} &= -1 \\
-\frac{1}{4}\frac{1}{r^4} &= c_1 - t \\
\frac{1}{r^4} &= 4t + c_1 \\
r &= \frac{1}{\sqrt[4]{4t+c_1}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\theta}{dt} &= \frac{1}{r^2}\{-y[-y - x(x^2+y^2)^2] + x[x - y(x^2+y^2)^2]\} \\
&= \frac{1}{r^2}[y^2 + xyr^2 + x^2 - xyr^2] \\
&= 1 \\
\theta &= t + c_2
\end{aligned}
$$

$$
\begin{aligned}
4 &= \frac{1}{\sqrt[4]{c_1}} \\
c_1 &= \frac{1}{256}
\end{aligned}
$$

$$
0 = c_2
$$

$$
\begin{aligned}
r &= \frac{1}{\sqrt[4]{4t + \frac{1}{256}}} \\
&= \frac{4}{\sqrt[4]{1024t+1}} \\
\theta &= t
\end{aligned}
$$

11.1.25

$$
\begin{aligned}
\frac{dr}{dt} &= \frac{1}{r}\{x[-y + x(1 - x^2 - y^2)] + y[x + y(1 - x^2 - y^2)]\} \\
&= \frac{1}{r}[-xy + x^2(1 - r^2) + xy + y^2(1 - r^2)] \\
&= r(1 - r^2) \\
&= -r^3 + r \\
r &= \pm\frac{e^t}{\sqrt{e^{2t} + c_1}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\theta}{dt} &= \frac{1}{r^2}\{-y[-y + x(1 - x^2 - y^2)] + x[x + y(1 - x^2 - y^2)]\} \\
&= \frac{1}{r^2}[y^2 - xy(1 - r^2) + x^2 + xy(1 - r^2)] \\
&= 1 \\
\theta &= t + c_2
\end{aligned}
$$

$$
\begin{aligned}
1 &= \pm\frac{1}{\sqrt{1 + c_1}} \\
c_1 &= 0 \\
c_2 &= 0 \\
r &= 1 \\
\theta &= t
\end{aligned}
$$

$$
\begin{aligned}
2 &= \pm\frac{1}{\sqrt{1 + c_1}} \\
c_1 &= -\frac{3}{4} \\
c_2 &= 0 \\
r &= \frac{e^t}{\sqrt{e^{2t} - \frac{3}{4}}} \\
\theta &= t
\end{aligned}
$$

11.1.27

No periodic solutions.

11.2 Stability of Linear Systems

11.2.1

Stable node

11.2.3

Unstable spiral

11.2.5

Degenerate stable node

11.2.7

Saddle point

11.2.9

Saddle point

11.2.11

Saddle point

11.2.13

Degenerate stable node

11.2.15

Stable spiral

11.2.17

$$-1 + \mu^2 < 0 \Rightarrow |\mu| < 1$$

11.2.19

Saddle point when $\mu < -1$, unstable spiral when $-1 < \mu < 3$, unstable node when $\mu \geq 3$.

11.2.23

(a) $(3, 4)$

(b) Saddle point

11.2.25

(a) $(0.5, 2)$

(b) Unstable spiral

11.3 Linearization and Local Stability

11.3.1

The Jacobian is
$$\begin{pmatrix} \alpha & -\beta + 2y \\ \beta - y & \alpha - x \end{pmatrix}.$$

At $(0,0)$ this is
$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

The eigenvalues of this matrix are $\alpha \pm i\beta$ so if $\alpha > 0$ then $(0,0)$ is an unstable critical point and if $\alpha < 0$ it is a stable critical point.

11.3.3

$$\begin{aligned} g(x) &= kx(n+1-x) \\ g'(x) &= k(n+1-x) - kx \\ &= k(n+1-2x) \\ g'(0) &= k(n+1) \\ g'(n+1) &= k(n+1-2(n+1)) \\ &= -k(n+1) \end{aligned}$$

$x = 0$ is unstable, $x = n+1$ is stable.

11.3.5

$$\begin{aligned} g(T) &= k(T - T_0) \\ g'(T) &= k \\ g'(T_0) &= k \end{aligned}$$

$T = T_0$ is unstable.

11.3.7

$$\begin{aligned}
g(x) &= k(\alpha - x)(\beta - x), \alpha > \beta \\
g'(x) &= -k(\beta - x) - k(\alpha - x) \\
g'(\alpha) &= -k(\beta - \alpha) - k(\alpha - \alpha) \\
&= k(\alpha - \beta) \\
g'(\beta) &= -k(\beta - \beta) - k(\alpha - \beta) \\
&= -k(\alpha - \beta)
\end{aligned}$$

$x = \alpha$ is unstable, $x = \beta$ is stable.

11.3.9

$$\begin{aligned}
g(P) &= P(a - bP)(1 - cP^{-1}), P > 0, a < bc \\
g'(P) &= (a - bP)(1 - cP^{-1}) - bP(1 - cP^{-1}) + cP^{-1}(a - bP) \\
g'\left(\frac{a}{b}\right) &= \left(a - b\frac{a}{b}\right)\left(1 - c\frac{b}{a}\right) - b\frac{a}{b}\left(1 - c\frac{b}{a}\right) + c\frac{b}{a}\left(a - b\frac{a}{b}\right) \\
&= -a\left(1 - \frac{bc}{a}\right) \\
g'(c) &= (a - bc)\left(1 - c\frac{1}{c}\right) - bc\left(1 - c\frac{1}{c}\right) + c\frac{1}{c}(a - bc) \\
&= a - bc
\end{aligned}$$

$P = \frac{a}{b}$ is unstable, $P = c$ is stable.

11.3.11

$$\begin{aligned} x' &= 1 - 2xy \\ y' &= 2xy - y \end{aligned}$$

$$\begin{aligned} 0 &= 1 - 2xy \\ 2xy &= 1 \end{aligned}$$

$$\begin{aligned} 0 &= 1 - y \\ y &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} -2y & -2x \\ 2y & 2x - 1 \end{pmatrix} \\ \mathbf{A}|_{\left(\frac{1}{2},1\right)} &= \begin{pmatrix} -2 & -1 \\ 2 & 0 \end{pmatrix} \end{aligned}$$

$\left(\frac{1}{2}, 1\right)$ is a stable spiral.

11.3.13

$$\begin{aligned} x' &= y - x^2 + 2 \\ y' &= 2xy - y \end{aligned}$$

$$\begin{aligned} 0 &= y - x^2 + 2 \\ y &= x^2 - 2 \end{aligned}$$

$$\begin{aligned} 0 &= 2x(x^2 - 2) - (x^2 - 2) \\ &= 2x^3 - x^2 - 4x + 2 \\ &= (x^2 - 2)(2x - 1) \\ &= (x + \sqrt{2})(x - \sqrt{2})(2x - 1) \\ x &= \frac{1}{2} \text{ or } \pm\sqrt{2} \\ y &= -\frac{7}{4} \text{ or } 0 \end{aligned}$$

Critical points are $\left(\frac{1}{2}, -\frac{7}{4}\right)$ and $(\pm\sqrt{2}, 0)$.

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} -2x & 1 \\ 2y & 2x-1 \end{pmatrix} \\
\mathbf{A}|_{(\frac{1}{2},-\frac{7}{4})} &= \begin{pmatrix} -1 & 1 \\ -\frac{7}{2} & 0 \end{pmatrix} \\
\mathbf{A}|_{(-\sqrt{2},0)} &= \begin{pmatrix} 2\sqrt{2} & 1 \\ 0 & -2\sqrt{2}-1 \end{pmatrix} \\
\mathbf{A}|_{(\sqrt{2},0)} &= \begin{pmatrix} -2\sqrt{2} & 1 \\ 0 & 2\sqrt{2}-1 \end{pmatrix}
\end{aligned}
$$

$\left(\frac{1}{2},-\frac{7}{4}\right)$ is a stable spiral, $(\pm\sqrt{2},0)$ are saddle points.

11.3.15

$$
\begin{aligned}
x' &= -3x + y^2 + 2 \\
y' &= x^2 - y^2
\end{aligned}
$$

$$
\begin{aligned}
0 &= -3x + y^2 + 2 \\
y^2 &= 3x - 2
\end{aligned}
$$

$$
\begin{aligned}
0 &= x^2 - 3x + 2 \\
&= (x-2)(x-1) \\
x &= 1 \text{ or } 2
\end{aligned}
$$

$$
y = \pm 1 \text{ or } \pm 2
$$

Critical points are $(1,\pm 1)$ and $(2,\pm 2)$.

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} -3 & 2y \\ 2x & -2y \end{pmatrix} \\
\mathbf{A}|_{(1,-1)} &= \begin{pmatrix} -3 & -2 \\ 2 & 2 \end{pmatrix} \\
\mathbf{A}|_{(1,1)} &= \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix} \\
\mathbf{A}|_{(2,-2)} &= \begin{pmatrix} -3 & -4 \\ 4 & 4 \end{pmatrix} \\
\mathbf{A}|_{(2,2)} &= \begin{pmatrix} -3 & 4 \\ 4 & -4 \end{pmatrix}
\end{aligned}
$$

$(1, -1)$ is a saddle point, $(1, 1)$ is a stable node, $(2, -2)$ is an unstable spiral, $(2, 2)$ is a saddle point.

11.3.23

It's not possible to classify $x = 0$.

11.3.25

It's not possible to classify $x = 0$ but $x = \pm\sqrt{\frac{1}{\epsilon}}$ are saddle points.

11.3.29

(a) The critical point at $(0, 0)$ is a stable spiral.

11.3.33

(a)

$$\begin{aligned} x' &= 2xy \\ y' &= 1 - x^2 + y^2 \end{aligned}$$

$$\begin{aligned} 0 &= 1 - x^2 + y^2 \\ x &= \sqrt{y^2 + 1} \end{aligned}$$

$$\begin{aligned} 0 &= 2\sqrt{y^2 + 1y} \\ &= 4(y^2 + 1)y^2 \\ &= 4y^4 + 4y^2 \\ y &= 0 \end{aligned}$$

$$x = \pm 1$$

Critical points are $(\pm 1, 0)$.

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 2y & 2x \\ -2x & 2y \end{pmatrix} \\ \mathbf{A}|_{(-1,0)} &= \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \\ \mathbf{A}|_{(1,0)} &= \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \end{aligned}$$

The trace of both the matrices is 0 and the determinant is 4, so we know the eigenvalues are pure imaginary but don't know the nature of the critical points.

(b)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\
&= \frac{1 - x^2 + y^2}{2xy} \\
y &= \pm\sqrt{-x^2 + c_1 x - 1} \\
y^2 &= -x^2 + c_1 x - 1 \\
&= -x^2 + 2c_2 x - 1 \\
&= -(x^2 - 2c_2 x + c^2) + c^2 - 1 \\
&= -(x - c)^2 + c^2 - 1 \\
(x - c)^2 + y^2 &= c^2 - 1
\end{aligned}$$

11.3.37

$$\begin{aligned}
Lq'' + Rq' + \alpha q + \beta q^3 &= 0 \\
q' &= r \\
r' &= -\frac{1}{L}(Rr + \alpha q + \beta q^3)
\end{aligned}$$

When $\beta > 0$ the only critical point is $(q, r) = (0, 0)$. When $\beta < 0$ the critical points are $(0, 0)$ and $\left(\pm\sqrt{-\frac{\alpha}{\beta}}, 0\right)$.

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 0 & 1 \\ -\frac{1}{L}(\alpha + 3\beta q^2) & -\frac{R}{L} \end{pmatrix} \\
\mathbf{A}|_{(0,0)} &= \begin{pmatrix} 0 & 1 \\ -\frac{\alpha}{L} & -\frac{R}{L} \end{pmatrix} \\
\mathbf{A}|_{\left(\pm\sqrt{-\frac{\alpha}{\beta}}, 0\right)} &= \begin{pmatrix} 0 & 1 \\ \frac{2\alpha}{L} & -\frac{R}{L} \end{pmatrix}
\end{aligned}$$

The eigenvalues of $\mathbf{A}|_{(0,0)}$ are

$$\frac{-R \pm \sqrt{R^2 - 4L\alpha}}{2L}$$

both of which are negative, so $(0, 0)$ is stable.

The eigenvalues of $\mathbf{A}|_{\left(\pm\sqrt{-\frac{\alpha}{\beta}}, 0\right)}$ are

$$\frac{-R \pm \sqrt{R^2 + 8L\alpha}}{2L}$$

of which one is negative and the other positive meaning $\left(\pm\sqrt{-\frac{\alpha}{\beta}}, 0\right)$ are saddle points.

11.3.39

(a)

$$\theta'' + \sin\theta = \frac{1}{2}$$

$$\begin{aligned}\theta' &= r\\ r' &= \frac{1}{2} - \sin\theta\end{aligned}$$

$$\begin{aligned}\theta'(\pi/6, 0) &= 0\\ r'(\pi/6, 0) &= \frac{1}{2} - \sin\frac{\pi}{6}\\ &= 0\\ \theta'(5\pi/6, 0) &= 0\\ r'(5\pi/6, 0) &= \frac{1}{2} - \sin\frac{5\pi}{6}\\ &= 0\end{aligned}$$

(b)

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} 0 & 1\\ -\cos\theta & 0\end{pmatrix}\\ \mathbf{A}|_{\left(\frac{\pi}{6}, 0\right)} &= \begin{pmatrix} 0 & 1\\ -\frac{\sqrt{3}}{2} & 0\end{pmatrix}\\ \mathbf{A}|_{\left(\frac{5\pi}{6}, 0\right)} &= \begin{pmatrix} 0 & 1\\ \frac{\sqrt{3}}{2} & 0\end{pmatrix}\end{aligned}$$

The eigenvalues of $\mathbf{A}|_{\left(\frac{\pi}{6},0\right)}$ are $\pm\frac{3^{1/4}}{\sqrt{2}}i$ so $\left(\frac{\pi}{6}, 0\right)$ is a center, a stable spiral, or an unstable spiral.

The eigenvalues of $\mathbf{A}|_{\left(\frac{5\pi}{6}\right)}$ are $\pm\frac{3^{1/4}}{\sqrt{2}}$ so $\left(\frac{5\pi}{6}, 0\right)$ is a saddle point.

11.4 Autonomous Systems as Mathematical Models

11.4.1

$$\begin{aligned}
y^2 &= \frac{2g}{l}\cos x + c \\
\omega_0^2 &= \frac{2g}{l}\cos\frac{\pi}{3} + c \\
c &= \omega_0^2 - \frac{g}{l}
\end{aligned}$$

$$\begin{aligned}
y^2 &= \frac{2g}{l}\left(\cos x + \frac{l}{2g}\omega_0^2 - \frac{1}{2}\right) \\
0 &= \frac{2g}{l}\left(\cos x + \frac{l}{2g}\omega_0^2 - \frac{1}{2}\right) \\
\cos x &= \frac{1}{2} - \frac{l}{2g}\omega_0^2
\end{aligned}$$

This equation has two solutions providing

$$\begin{aligned}
\frac{1}{2} - \frac{l}{2g}\omega_0^2 &> -1 \\
\frac{l}{2g}\omega_0^2 &< \frac{3}{2} \\
|\omega_0| &< \sqrt{\frac{3g}{l}}.
\end{aligned}$$

11.4.5

(a)

$$\begin{aligned}
x(0) &= x_0 \\
x'(0) &= v_0 \\
z &= \frac{1}{2}x^2
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\
&= \frac{-g\frac{x}{1+x^2}}{y} \\
\int y\,dy &= \int -g\frac{x}{1+x^2}\,dx \\
\frac{1}{2}y^2 &= c - \frac{g}{2}\ln(1+x^2) \\
y^2 &= c - g\ln(1+x^2)
\end{aligned}$$

$$\begin{aligned}
v_0^2 &= c - g\ln(1+x_0^2) \\
c &= v_0^2 + g\ln(1+x_0^2)
\end{aligned}$$

$$\begin{aligned}
y^2 &= v_0^2 + g\ln(1+x_0^2) - g\ln(1+x^2) \\
&= v_0^2 + g\ln\left(\frac{1+x_0^2}{1+x^2}\right)
\end{aligned}$$

(b) z is maximised when x is maximised. That occurs when $y = x' = 0$, so

$$\begin{aligned}
0 &= v_0^2 + g\ln\left(\frac{1+x_0^2}{1+x^2}\right) \\
\frac{1+x_0^2}{1+x^2} &= e^{-v_0^2/g} \\
\frac{1+x^2}{1+x_0^2} &= e^{v_0^2/g} \\
1+x^2 &= e^{v_0^2/g}(1+x_0^2) \\
x^2 &= e^{v_0^2/g}(1+x_0^2) - 1
\end{aligned}$$

and thus

$$z_{\max} = \frac{1}{2}x^2 = \frac{1}{2}\left[e^{v_0^2/g}(1+x_0^2) - 1\right].$$

11.4.9

(a) $(0,0)$ remains a critical point. Assuming $x \neq 0$ and $y \neq 0$ the coordinates of the new critical point are:

$$\begin{aligned} 0 &= x(-a + by - \epsilon_1) \\ &= -a + by - \epsilon_1 \\ y &= \frac{a + \epsilon_1}{b} \end{aligned}$$

$$\begin{aligned} 0 &= y(-cx + d - \epsilon_2) \\ &= -cx + d - \epsilon_2 \\ x &= \frac{d - \epsilon_2}{c}. \end{aligned}$$

The Jacobian matrix is

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} -a + by - \epsilon_1 & bx \\ -cy & -cx + d - \epsilon_2 \end{pmatrix} \\ \mathbf{A}\big|_{\left(\frac{d-\epsilon_2}{c}, \frac{a+\epsilon_1}{b}\right)} &= \begin{pmatrix} 0 & \frac{b}{c}(d - \epsilon_2) \\ -\frac{c}{b}(a + \epsilon_1) & 0 \end{pmatrix} \end{aligned}$$

the trace of which is $\tau = 0$ and the determinant is $\Delta = (a + \epsilon_1)(d - \epsilon_2)$. If $\epsilon_2 < d$ then $\Delta > 0$ and the eigenvalues of the Jacobian are pure imaginary, meaning the critical point is a centre, a stable spiral, or an unstable spiral.

(b) Yes, because increasing ϵ_1 increases the y-coordinate of the critical point and increasing ϵ_2 decreases its x-coordinate.

11.4.11

- $(0, 100)$ is a stable critical point.
- $(20, 40)$ is a saddle point.
- $(50, 0)$ is a stable critical point.
- $(0, 0)$ is an unstable point.

11.4.17

(a)

$$x' = y$$
$$y' = -\frac{\beta}{m}y|y| - \frac{k}{m}x$$

$$0 = y$$
$$0 = -\frac{k}{m}x$$
$$x = 0$$

The only critical point is $(0,0)$.

(b)

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\beta}{m}2|y| \end{pmatrix}$$
$$\mathbf{A}|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix}$$

The trace of the Jacobian is $\tau = 0$ and its determinant is $\Delta = \frac{k}{m}$ so $(0,0)$ is a centre, a stable spiral point or an unstable spiral point. Physical considerations show that the system must be a stable spiral point.

11.5 Periodic Solutions, Limit Cycles, and Global Stability

11.5.1

$$0 = x - y$$
$$x = y$$

$$0 = 2 + xy$$
$$= 2x^2$$

$$0 = 2y^2$$

x and y can't be complex numbers, so the system has no critical points.

11.5.3

$$\nabla \cdot \mathbf{V} = -1 - 1 = -2$$

The $\nabla \cdot \mathbf{V}$ doesn't change sign so the system has no periodic solutions.

11.5.5

$$\nabla \cdot \mathbf{V} = -\mu + 3y^2$$

If $\mu < 0$ then $\nabla \cdot \mathbf{V}$ doesn't change sign so the system has no periodic solutions.

11.5.7

The only critical point is at $(0,0)$ and the eigenvalues of the Jacobian at that point have opposite signs, so it's a saddle point. By the Corollary to Theorem 11.5.1 there is no periodic solution.

11.5.11

$\nabla \cdot \mathbf{V} = 4(1 - x^2 - 3y^2) > 0$ within the region defined by the ellipse $x^2 + 3y^2 = 1$.

11.5.15

$$\begin{aligned}
\mathbf{V} \cdot \mathbf{n} &= \begin{pmatrix} -x + y + xy \\ x - y - x^2 - y^3 \end{pmatrix} \cdot \begin{pmatrix} -x \\ -y \end{pmatrix} \\
&= -x(-x + y + xy) - y(x - y - x^2 - y^3) \\
&= x^2 - xy - x^2 y - xy + y^2 + x^2 y + y^4 \\
&= x^2 - 2xy + y^2 + y^4 \\
&= (x - y)^2 + y^4
\end{aligned}$$

$(x - y)^2 + y^4$ is always positive, so any circle centred at the origin is an invariant region.

11.5.17

Yes, by Theorem 11.5.5 b.

11.5.19

$$\begin{aligned}
\mathbf{V} \cdot \mathbf{n} &= -x(y) - y(-x - (1 - x^2)y) \\
&= -xy + xy + y^2(1 - x^2) \\
&= y^2(1 - x^2)
\end{aligned}$$

Thus $\mathbf{V} \cdot \mathbf{n} \geq 0$ if $x^2 \leq 1$. If the region R is defined by $x^2 + y^2 \leq r^2$ where $r < 1$ then $x^2 < 1$ and R is an invariant region.

Because $\nabla \cdot \mathbf{V} = -(1 - x^2)$ doesn't change sign in R, there are no periodic solutions in R (Theorem 11.5.2).

The only critical point in R is $(0,0)$ and it is stable and thus by Theorem 11.5.6 b $\lim_{t \to \infty} \mathbf{X}(t) = (0,0)$ for all initial points.

11.5.21

(a)

$$\begin{aligned}\nabla \cdot \mathbf{V} &= 2xy - 1 - x^2\\ &= -x^2 + 2xy - 1\end{aligned}$$

$\nabla \cdot \mathbf{V} < 0$ in R so there are no periodic solutions in R.

(b) R contains a single stable node at $\left(\frac{3}{2}, \frac{2}{9}\right)$ so by Theorem 11.5.6 b $\lim_{t\to\infty} \mathbf{X}(t) = \left(\frac{3}{2}, \frac{2}{9}\right)$.