Vibrations and Waves by A. P. French Notes

Chris Doble

May 2022

Contents

1 Periodic motions

- Fouriers theorem states that any repeating signal of period T can be expressed as a sum of sin waves with periods T, $T/2$, etc.
- It's important to define the domain of a SHM equation, e.g. for what values of t is the motion defined?
- SHM can be considered a projection of uniform circular motion
- That uniform circular motion can be represented by a number in the complex plane, with the projection being its real part
- Multiplication by j can be considered a counter-clockwise rotation of 90° in the complex plane
- Euler's formula states
$$e^{j\theta} = \cos\theta + j\sin\theta$$
- Multiplication of a complex number z by $e^{j\theta}$ is equivalent to a counter-clockwise rotation of z by an angle of θ

2 The superposition of periodic motions

- The combination of two SHM's of the same period

$$x_1 = A_1 \cos(\omega t + \alpha_1)$$
$$x_2 = A_2 \cos(\omega t + \alpha_2)$$

is given by

$$x = A\cos(\omega t + \alpha)$$

where

$$A^2 = A_1^2 + A_2^2 + 2A_1 A_2 \cos(\alpha_2 - \alpha_1),$$
$$A \sin\beta = A_2 \sin(\alpha_2 - \alpha_1),$$

and

$$\alpha = \alpha_1 + \beta.$$

- The combination in complex representation

$$z_1 = A_1 e^{j(\omega t + \alpha_1)}$$
$$z_2 = A_2 e^{j(\omega t + \alpha_2)}$$

is given by

$$z = e^{j(\omega t + \alpha_1)}[A_1 + A_2 e^{j(\alpha_2 - \alpha_1)}]$$

- In the case where $A_1 = A_2$ if we denote $\delta = \alpha_2 - \alpha_1$ then

$$\beta = \frac{\delta}{2}$$

and

$$A = 2A_1 \cos\beta = 2A_1 \cos\frac{\delta}{2}$$

- The superposition of two sinusoids with different periods will itself be periodic if there exist integers n_1 and n_2 such that

$$T = n_1 T_1 = n_2 T_2$$

where T_1 and T_2 are the periods of the two sinusoids

- Periodic motion in two or more dimensions can be represented by extending the "projection of a rotating vector" approach, with one vector for each axis, e.g.

$$x = A_1 \cos\omega t$$
$$y = A_2 \cos\omega t$$

where differing amplitudes, frequencies, and phase differences product different curves called **Lissajous curves**

3 The free vibrations of physical systems

- When a tensile force is applied to a material it elongates. The ratio of the elongation to the original length x/l_0 is known as the **tensile strain**
- The ratio of the tensile force to the cross sectional area of the material F/A is known as the **tensile stress**
- The ratio of stress and strain is a constant known as **Young's modulus** Y
- The force exerted by the stretched material on another object is given by

$$\frac{F/A}{x/l_0} = -Y \Rightarrow F = -\frac{AY}{l_0}x$$

  which is in the form of Hooke's law with $k = -\frac{AY}{l_0}$

4 Forced vibrations and resonance

- Periodic motion that isn't simple harmonic is **anharmonic**

5 Coupled oscillators and normal modes

- A property of a normal mode is that all objects oscillate at the same frequency

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6 Normal modes of continuous systems. Fourier analysis

- If a medium is vibrating at a natural frequency with only one end fixed (e.g. the pressure in a tube with one end open), the length of the medium must be an integer multiple of quarter wavelengths
- In one-dimensional systems, the frequency of a normal mode f_n is proportional to the mode number n for small n
- In higher-dimensional systems, the frequency of a normal mode f_n is not proportional to the mode number n
- In higher-dimensional systems, one frequency may correspond to multiple normal modes and is said to be **degenerate**
- The process of determining the coefficients of a Fourier series is called **harmonic analysis**

- One way to think of orthogonal functions is as vectors of infinite dimension. Two n-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if their scalar product is 0, i.e.

  $$\mathbf{a} \cdot \mathbf{b} = 0 \text{ if } \sum_{0}^{n} a_n b_n = 0.$$

  If two functions $f(x)$ and $g(x)$ are considered vectors of infinite dimension then the expression is similar

  $$\int_0^L f(x) g(x)\, dx = 0 \text{ is approximately } \sum_{n=0}^{\infty} f(x_n) g(x_n) = 0$$