Advanced Engineering Mathematics Ordinary Differential Equations Notes

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Contents

1 Introduction to Differential Equations

1.1 Definitions and Terminology

1.1.1 1

2, linear

1.1.2 3

4, linear

1.1.3 5

2, nonlinear

1.1.4 7

3, linear

1.1.5 9

no; yes

1.1.6 15

The domain of the function is $x \in [-2, \infty)$.

$$y' = 1 + \frac{2}{\sqrt{x+2}}$$

The largest interval of definition of the solution is $x \in (-2, \infty)$.

$$\begin{aligned}
(y-x)y' &= y - x + 8 \\
(x + 4\sqrt{x+2} - x)(1 + \frac{2}{\sqrt{x+2}}) &= x + 4\sqrt{x+2} - x + 8 \\
4\sqrt{x+2} + 8 &= 4\sqrt{x+2} + 8
\end{aligned}$$

1.1.7 17

The domain of the function is $x \in \mathbb{R}, x \neq \pm 2$.

$$y' = \frac{2x}{(4-x^2)^2}$$

The largest intervals of definition of the solution are $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

$$
\begin{aligned}
y' &= 2xy^2 \\
\frac{2x}{(4-x^2)^2} &= 2x\left(\frac{1}{4-x^2}\right)^2 \\
&= \frac{2x}{(4-x^2)^2}
\end{aligned}
$$

1.1.8 19

$$
\begin{aligned}
ln\frac{2X-1}{X-1} &= t \\
2X-1 &= (X-1)e^t \\
(2-e^t)X &= 1-e^t \\
X &= \frac{e^t-1}{e^t-2}
\end{aligned}
$$

The solutions intervals of validity are $(\infty, \ln 2)$ and $(\ln 2, \infty)$.

$$
\begin{aligned}
\frac{dX}{dt} &= (X-1)(1-2X) \\
\frac{e^t}{e^t-2} - \frac{e^t(e^t-1)}{(e^t-2)^2} &= \left(\frac{e^t-1}{e^t-2}-1\right)\left(1-2\frac{e^t-1}{e^t-2}\right) \\
\frac{e^t(e^t-2)-e^t(e^t-1)}{(e^t-2)^2} &= \left(\frac{e^t-1-e^t+2}{e^t-2}\right)\left(\frac{e^t-2-2e^t+2}{e^t-2}\right) \\
\frac{e^{2t}-2e^t-e^{2t}+e^t}{(e^t-2)^2} &= \left(\frac{1}{e^t-2}\right)\left(\frac{-e^t}{e^t-2}\right) \\
\frac{-e^t}{(e^t-2)^2} &= \frac{-e^t}{(e^t-2)^2}
\end{aligned}
$$

1.1.9 31

$m = -2$

1.1.10 33

$m = 2$ or 3

1.1.11 35

$m = -1$ or 0

1.1.12 37

$y = 2$

1.1.13 39

No constant solutions

1.2 Initial Value Problems

1.2.1 1

$$
\begin{aligned}
y(0) = -\frac{1}{3} &= \frac{1}{1 + c_1 e^{-(0)}} \\
-3 &= 1 + c_1 \\
c_1 &= -4
\end{aligned}
$$

$$y = \frac{1}{1 - 4e^{-x}}$$

1.2.2 3

$$
\begin{aligned}
y(2) = \frac{1}{3} &= \frac{1}{(2)^2 + c} \\
3 &= 4 + c \\
c &= -1
\end{aligned}
$$

$$y = \frac{1}{x^2 - 1}$$

$I = (1, \infty)$

1.2.3 5

$$
\begin{aligned}
y(0) = 1 &= \frac{1}{(0)^2 + c} \\
c &= 1
\end{aligned}
$$

$$y = \frac{1}{x^2 + 1}$$

$I = (-\infty, \infty)$

1.2.4 7

$$\begin{aligned} x(0) = -1 &= c_1 \cos 0 + c_2 \sin 0 \\ c_1 &= -1 \end{aligned}$$

$$\begin{aligned} x'(0) = 8 &= -c_1 \sin 0 + c_2 \cos 0 \\ c_2 &= 8 \end{aligned}$$

$$x = -\cos t + 8 \sin t$$

1.2.5 9

$$\begin{aligned} x'\left(\frac{\pi}{6}\right) = 0 &= -c_1 \sin \frac{\pi}{6} + c_2 \cos \frac{\pi}{6} \\ &= -c_1 \frac{1}{2} + c_2 \frac{\sqrt{3}}{2} \\ c_1 &= \sqrt{3} c_2 \end{aligned}$$

$$\begin{aligned} x\left(\frac{\pi}{6}\right) = \frac{1}{2} &= c_1 \cos \frac{\pi}{6} + c_2 \sin \frac{\pi}{6} \\ &= \frac{3}{2} c_2 + \frac{1}{2} c_2 \\ &= 2c_2 \\ c_2 &= \frac{1}{4} \end{aligned}$$

$$y = \frac{\sqrt{3}}{4} \cos t + \frac{1}{4} \sin t$$

1.2.6 11

$$\begin{aligned} y(0) = 1 &= c_1 e^{(0)} + c_2 e^{-(0)} \\ &= c_1 + c_2 \\ c_1 &= 1 - c_2 \end{aligned}$$

$$\begin{aligned} y'(0) = 2 &= c_1 e^{(0)} - c_2 e^{-(0)} \\ &= 1 - c_2 - c_2 \\ c_2 &= -\frac{1}{2} \end{aligned}$$

$$y = \frac{3}{2} e^{x} - \frac{1}{2} e^{-x}$$

1.2.7 13

$$\begin{aligned} y(-1) = 5 &= c_1 e^{(-1)} + c_2 e^{-(-1)} \\ &= c_1 e^{-1} + c_2 e \\ c_1 &= 5e - c_2 e^2 \end{aligned}$$

$$\begin{aligned} y'(-1) = -5 &= c_1 e^{(-1)} - c_2 e^{-(-1)} \\ &= 5e - c_2 e^2 - c_2 e \\ c_2 e(e+1) &= 5(e+1) \\ c_2 &= \frac{5}{e} \end{aligned}$$

$$y = 5e^{-x-1}$$

1.2.8 15

$$y = 0$$

$$y = x^3$$

1.2.9 17

$$f(x, y) = y^{2/3}$$

$$\frac{\partial f}{\partial y} = \frac{2}{3y^{1/3}}$$

$y < 0$ or $y > 0$

1.2.10 19

$$f(x, y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$x < 0$ or $x > 0$

1.2.11 21

$$f(x, y) = \frac{x^2}{4 - y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2x^2 y}{(4 - y^2)^2}$$

$y < -2$, $-2 < y < 2$, or $y > 2$

1.2.12 23

$$f(x, y) = \frac{y^2}{x^2 + y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2} - \frac{2y^3}{(x^2 + y^2)^2}$$

$x \neq 0$ and $y \neq 0$

1.2.13 25

$$f(x, y) = \sqrt{y^2 - 9}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{y^2 - 9}}$$

Yes

1.2.14 27

No

1.2.15 29

(a) $y = cx$

(b)

$$f(x, y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$x \neq 0$

(c) No, the function is not differentiable at $x = 0$

1.2.16 31

(a)

$$y' = \frac{1}{(x + c)^2} = y^2$$

(b)

$$y(0) = 1 = -\frac{1}{(0) + c} \Rightarrow c = -1 \Rightarrow y = \frac{1}{1 - x}$$

$I = (-\infty, 1)$

$$y(0) = -1 = -\frac{1}{(0)+c} \Rightarrow c = 1 \Rightarrow y = -\frac{1}{x+1}$$

$I = (-1, \infty)$

1.2.17 39

$$\begin{aligned} y(0) = 0 &= c_1 \cos 3(0) + c_2 \sin 3(0) \\ c_1 &= 0 \end{aligned}$$

$$\begin{aligned} y\left(\frac{\pi}{6}\right) = -1 &= c_2 \sin 3\left(\frac{\pi}{6}\right) \\ c_2 &= -1 \end{aligned}$$

$$y = -\sin 3x$$

1.2.18 41

$$\begin{aligned} y'(0) = 0 &= -3c_1 \sin 3(0) + 3c_2 \cos 3(0) \\ c_2 &= 0 \end{aligned}$$

$$\begin{aligned} y'\left(\frac{\pi}{4}\right) = 0 &= -3c_1 \sin 3\left(\frac{\pi}{4}\right) \\ &= -\frac{3}{\sqrt{2}}c_1 \\ c_1 &= 0 \end{aligned}$$

$$y = 0$$

1.2.19 43

$$\begin{aligned} y(0) = 0 &= c_1 \cos 3(0) + c_2 \sin 3(0) \\ c_1 &= 0 \end{aligned}$$

$$\begin{aligned} y(\pi) = 4 &= c_2 \sin 3(\pi) \\ 4 &= 0 \end{aligned}$$

No solution

1.3 Differential Equations as Mathematical Models

1.3.1 1

$$\frac{dP}{dt} = kP + r$$

$$\frac{dP}{dt} = kP - r$$

1.3.2 3

$$\frac{dP}{dt} = k_b P - k_d P^2$$

1.3.3 7

$$\frac{dx}{dt} = kx(1000 - x)$$

1.3.4 9

$$\frac{dA}{dt} = -\frac{A}{100}$$

$$A(0) = 50\,\mathrm{lb}$$

1.3.5 11

$$\frac{dA}{dt} + \frac{7}{600 - t}A = 6$$

1.3.6 13

$$\begin{aligned}
\frac{dV}{dt} &= -cA_h\sqrt{2gh} \\
A_w\frac{dh}{dt} &= -cA_h\sqrt{2gh} \\
\frac{dh}{dt} &= -\frac{cA_h\sqrt{2g}}{A_w}\sqrt{h} \\
&= -\frac{c\pi r_h^2\sqrt{2g}}{A_w}\sqrt{h} \\
&= -\frac{c\pi}{430}\sqrt{h}
\end{aligned}$$

1.3.7 15

$$L\frac{di}{dt} + Ri = E$$

1.3.8 17

$$m\frac{dv}{dt} = mg - kv^2$$

1.3.9 19

$$m\frac{d^2x}{dt^2} = -kx$$

1.3.10 21

$$\begin{aligned}\frac{d}{dt}(mv) &= R - kv \\ \frac{dm}{dt}v + m\frac{dv}{dt} &= R - kv - mg\end{aligned}$$

1.3.11 23

$$g = \frac{k}{R^2} \Rightarrow k = gR^2$$

$$\frac{d^2r}{dt^2} = -\frac{gR^2}{r^2}$$

1.3.12 25

$$\frac{dA}{dt} = k(M - A)$$

1.3.13 27

$$\frac{dx}{dt} = r - kx$$

1.3.14 29

$$\begin{aligned}\frac{dy}{dx} &= \tan\theta \\ &= \tan\frac{\phi}{2} \\ &= \frac{1-\cos\phi}{\sin\phi} \\ &= \frac{1-x/r}{y/r} \\ &= \frac{r-x}{y} \\ &= \frac{\sqrt{x^2+y^2}-x}{y}\end{aligned}$$

1.4 Chapter in Review

1.4.1 1

$$\frac{dy}{dx} = ky$$

1.4.2 3

$$y'' + k^2 y = 0$$

1.4.3 5

$$y = c_1 e^x + c_2 x e^x$$

$$\begin{aligned} y' &= c_1 e^x + c_2 e^x + c_2 x e^x \\ &= y + c_2 e^x \end{aligned}$$

$$\begin{aligned} y'' &= c_1 e^x + c_2 e^x + c_2 e^x + c_2 x e^x \\ &= c_1 e^x + 2c_2 e^x + c_2 x e^x \\ &= y' + c_2 e^x \end{aligned}$$

$$y'' - 2y' + y = 0$$

1.4.4 7

a, d

1.4.5 9

b

1.4.6 11

b

1.4.7 13

$$y = ce^x$$

1.4.8 15

$$\frac{dy}{dx} = x^2 + y^2$$

1.4.9 17

(a) $(-\infty, \infty)$

(b) $(-\infty, 0)$ or $(0, \infty)$

1.4.10 19

$x_0 = -1$ and $I = (-\infty, 0)$ or $x_0 = 2$ and $I = (0, \infty)$

1.4.11 23

$$y = x \sin x + x \cos x$$

$$y' = \sin x + x \cos x + \cos x - x \sin x$$

$$\begin{aligned} y'' &= \cos x + \cos x - x \sin x - \sin x - \sin x - x \cos x \\ &= 2 \cos x - 2 \sin x - x \sin x - x \cos x \end{aligned}$$

$$\begin{aligned} y'' + y &= 2 \cos x - 2 \sin x - x \sin x - x \cos x + x \sin x + x \cos x \\ &= 2 \cos x - 2 \sin x \end{aligned}$$

$I = (-\infty, \infty)$

1.4.12 25

$$y = \sin(\ln x)$$

$$y' = \frac{1}{x} \cos(\ln x)$$

$$y'' = -\frac{1}{x^2} \cos(\ln x) - \frac{1}{x^2} \sin(\ln x)$$

$$\begin{aligned} x^2 y'' + xy' + y &= -\cos(\ln x) - \sin(\ln x) + \cos(\ln x) + \sin(\ln x) \\ &= 0 \end{aligned}$$

$I = (0, \infty)$

1.4.13 35

$$
\begin{aligned}
y(0) = 0 &= c_1e^{-3(0)} + c_2e^{(0)} + 4(0) \\
&= c_1 + c_2 \\
c_1 &= -c_2
\end{aligned}
$$

$$
\begin{aligned}
y'(0) = 0 &= -3c_1e^{-3(0)} + c_2e^{(0)} + 4 \\
&= -3c_1 + c_2 + 4 \\
c_2 &= 3c_1 - 4
\end{aligned}
$$

$$c_1 = -(3c_1 - 4) \Rightarrow c_1 = 1 \Rightarrow c_2 = -1$$

$$y = e^{-3x} - e^x + 4x$$

1.4.14 37

$$
\begin{aligned}
y(1) = -2 &= c_1e^{-3(1)} + c_2e^{(1)} + 4(1) \\
&= c_1e^{-3} + c_2e + 4 \\
c_1 &= -e^3(c_2e + 6)
\end{aligned}
$$

$$
\begin{aligned}
y'(1) = 4 &= -3c_1e^{-3(1)} + c_2e^{(1)} + 4 \\
&= -3c_1e^{-3} + c_2e + 4 \\
c_2e &= 3c_1e^{-3}
\end{aligned}
$$

$$c_1 = -e^3(3c_1e^{-3} + 6) = -3c_1 - 6e^3 \Rightarrow c_1 = -\frac{3}{2}e^3 \Rightarrow c_2 = -\frac{9}{2}e^{-1}$$

$$y = -\frac{3}{2}e^{3(1-x)} - \frac{9}{2}e^{x-1} + 4x$$

1.4.15 41

$y_0 = -3$, $y_1 = 0$

1.4.16 43

$$\frac{d}{dt}(mv) = F - mg$$
$$\frac{d}{dt}(\lambda x \frac{dx}{dt}) = F - \lambda x g$$
$$x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + gx = \frac{F}{\lambda}$$
$$x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 32x = 5$$

2 First-Order Differential Equations

2.1 Solution Curves Without a Solution

2.1.1 21

0 is stable, 3 is unstable

2.1.2 23

2 is semi-stable

2.1.3 25

-2 is unstable, 0 is semi-stable, 2 is stable

2.1.4 27

-1 is stable, 0 is unstable

2.1.5 39

$P_0 < h/k$

2.1.6 41

$$g - \frac{k}{m}v^2 = 0 \Rightarrow v = \sqrt{\frac{gm}{k}}$$