

# Advanced Engineering Mathematics Ordinary Differential Equations Notes

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# 1 Introduction to Differential Equations

## 1.1 Definitions and Terminology

### 1.1.1 1

2, linear

### 1.1.2 3

4, linear

### 1.1.3 5

2, nonlinear

### 1.1.4 7

3, linear

### 1.1.5 9

no; yes

**1.1.6 15**

The domain of the function is  $x \in [-2, \infty)$ .

$$y' = 1 + \frac{2}{\sqrt{x+2}}$$

The largest interval of definition of the solution is  $x \in (-2, \infty)$ .

$$\begin{aligned}(y-x)y' &= y-x+8 \\ (x+4\sqrt{x+2}-x)(1+\frac{2}{\sqrt{x+2}}) &= x+4\sqrt{x+2}-x+8 \\ 4\sqrt{x+2}+8 &= 4\sqrt{x+2}+8\end{aligned}$$

**1.1.7 17**

The domain of the function is  $x \in \mathbb{R}, x \neq \pm 2$ .

$$y' = \frac{2x}{(4-x^2)^2}$$

The largest intervals of definition of the solution are  $(-\infty, -2)$ ,  $(-2, 2)$ , and  $(2, \infty)$ .

$$\begin{aligned}y' &= 2xy^2 \\ \frac{2x}{(4-x^2)^2} &= 2x \left( \frac{1}{4-x^2} \right)^2 \\ &= \frac{2x}{(4-x^2)^2}\end{aligned}$$

**1.1.8 19**

$$\begin{aligned}\ln \frac{2X-1}{X-1} &= t \\ 2X-1 &= (X-1)e^t \\ (2-e^t)X &= 1-e^t \\ X &= \frac{e^t-1}{e^t-2}\end{aligned}$$

The solutions intervals of validity are  $(\infty, \ln 2)$  and  $(\ln 2, \infty)$ .

$$\begin{aligned}
\frac{dX}{dt} &= (X-1)(1-2X) \\
\frac{e^t}{e^t-2} - \frac{e^t(e^t-1)}{(e^t-2)^2} &= \left(\frac{e^t-1}{e^t-2} - 1\right) \left(1 - 2\frac{e^t-1}{e^t-2}\right) \\
\frac{e^t(e^t-2) - e^t(e^t-1)}{(e^t-2)^2} &= \left(\frac{e^t-1-e^t+2}{e^t-2}\right) \left(\frac{e^t-2-2e^t+2}{e^t-2}\right) \\
\frac{e^{2t}-2e^t-e^{2t}+e^t}{(e^t-2)^2} &= \left(\frac{1}{e^t-2}\right) \left(\frac{-e^t}{e^t-2}\right) \\
\frac{-e^t}{(e^t-2)^2} &= \frac{-e^t}{(e^t-2)^2}
\end{aligned}$$

### 1.1.9 31

$$m = -2$$

### 1.1.10 33

$$m = 2 \text{ or } 3$$

### 1.1.11 35

$$m = -1 \text{ or } 0$$

### 1.1.12 37

$$y = 2$$

### 1.1.13 39

No constant solutions

## 1.2 Initial Value Problems

### 1.2.1 1

$$\begin{aligned}
y(0) &= -\frac{1}{3} = \frac{1}{1+c_1e^{-(0)}} \\
-3 &= 1+c_1 \\
c_1 &= -4
\end{aligned}$$

$$y = \frac{1}{1-4e^{-x}}$$

**1.2.2 3**

$$y(2) = \frac{1}{3} = \frac{1}{(2)^2 + c}$$

$$3 = 4 + c$$

$$c = -1$$

$$y = \frac{1}{x^2 - 1}$$

$$I = (1, \infty)$$

**1.2.3 5**

$$y(0) = 1 = \frac{1}{(0)^2 + c}$$

$$c = 1$$

$$y = \frac{1}{x^2 + 1}$$

$$I = (-\infty, \infty)$$

**1.2.4 7**

$$x(0) = -1 = c_1 \cos 0 + c_2 \sin 0$$

$$c_1 = -1$$

$$x'(0) = 8 = -c_1 \sin 0 + c_2 \cos 0$$

$$c_2 = 8$$

$$x = -\cos t + 8 \sin t$$

**1.2.5 9**

$$x'\left(\frac{\pi}{6}\right) = 0 = -c_1 \sin \frac{\pi}{6} + c_2 \cos \frac{\pi}{6}$$

$$= -c_1 \frac{1}{2} + c_2 \frac{\sqrt{3}}{2}$$

$$c_1 = \sqrt{3}c_2$$

$$\begin{aligned}
x\left(\frac{\pi}{6}\right) &= \frac{1}{2} = c_1 \cos \frac{\pi}{6} + c_2 \sin \frac{\pi}{6} \\
&= \frac{3}{2}c_2 + \frac{1}{2}c_2 \\
&= 2c_2 \\
c_2 &= \frac{1}{4}
\end{aligned}$$

$$y = \frac{\sqrt{3}}{4} \cos t + \frac{1}{4} \sin t$$

### 1.2.6 11

$$\begin{aligned}
y(0) &= 1 = c_1 e^{(0)} + c_2 e^{-(0)} \\
&= c_1 + c_2 \\
c_1 &= 1 - c_2
\end{aligned}$$

$$\begin{aligned}
y'(0) &= 2 = c_1 e^{(0)} - c_2 e^{-(0)} \\
&= 1 - c_2 - c_2 \\
c_2 &= -\frac{1}{2}
\end{aligned}$$

$$y = \frac{3}{2}e^x - \frac{1}{2}e^{-x}$$

### 1.2.7 13

$$\begin{aligned}
y(-1) &= 5 = c_1 e^{(-1)} + c_2 e^{-(-1)} \\
&= c_1 e^{-1} + c_2 e \\
c_1 &= 5e - c_2 e^2
\end{aligned}$$

$$\begin{aligned}
y'(-1) &= -5 = c_1 e^{(-1)} - c_2 e^{-(-1)} \\
&= 5e - c_2 e^2 - c_2 e \\
c_2 e(e+1) &= 5(e+1) \\
c_2 &= \frac{5}{e}
\end{aligned}$$

$$y = 5e^{-x-1}$$



**1.2.8 15**

$$y = 0$$

$$y = x^3$$

**1.2.9 17**

$$f(x, y) = y^{2/3}$$

$$\frac{\partial f}{\partial y} = \frac{2}{3y^{1/3}}$$

$$y < 0 \text{ or } y > 0$$

**1.2.10 19**

$$f(x, y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$$x < 0 \text{ or } x > 0$$

**1.2.11 21**

$$f(x, y) = \frac{x^2}{4 - y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2x^2y}{(4 - y^2)^2}$$

$$y < -2, -2 < y < 2, \text{ or } y > 2$$

**1.2.12 23**

$$f(x, y) = \frac{y^2}{x^2 + y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2} - \frac{2y^3}{(x^2 + y^2)^2}$$

$$x \neq 0 \text{ and } y \neq 0$$

**1.2.13 25**

$$f(x, y) = \sqrt{y^2 - 9}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{y^2 - 9}}$$

Yes

**1.2.14 27**

No

**1.2.15 29**

(a)  $y = cx$

(b)

$$f(x, y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$$x \neq 0$$

(c) No, the function is not differentiable at  $x = 0$ **1.2.16 31**

(a)

$$y' = \frac{1}{(x+c)^2} = y^2$$

(b)

$$y(0) = 1 = -\frac{1}{(0)+c} \Rightarrow c = -1 \Rightarrow y = \frac{1}{1-x}$$

$$I = (-\infty, 1)$$

$$y(0) = -1 = -\frac{1}{(0)+c} \Rightarrow c = 1 \Rightarrow y = -\frac{1}{x+1}$$

$$I = (-1, \infty)$$

**1.2.17 39**

$$y(0) = 0 = c_1 \cos 3(0) + c_2 \sin 3(0)$$

$$c_1 = 0$$

$$y\left(\frac{\pi}{6}\right) = -1 = c_2 \sin 3\left(\frac{\pi}{6}\right)$$

$$c_2 = -1$$

$$y = -\sin 3x$$

**1.2.18 41**

$$\begin{aligned}y'(0) = 0 &= -3c_1 \sin 3(0) + 3c_2 \cos 3(0) \\c_2 &= 0\end{aligned}$$

$$\begin{aligned}y'\left(\frac{\pi}{4}\right) = 0 &= -3c_1 \sin 3\left(\frac{\pi}{4}\right) \\&= -\frac{3}{\sqrt{2}}c_1 \\c_1 &= 0\end{aligned}$$

$$y = 0$$

**1.2.19 43**

$$\begin{aligned}y(0) = 0 &= c_1 \cos 3(0) + c_2 \sin 3(0) \\c_1 &= 0\end{aligned}$$

$$\begin{aligned}y(\pi) = 4 &= c_2 \sin 3(\pi) \\4 &= 0\end{aligned}$$

No solution

**1.3 Differential Equations as Mathematical Models**

**1.3.1 1**

$$\frac{dP}{dt} = kP + r$$

$$\frac{dP}{dt} = kP - r$$

**1.3.2 3**

$$\frac{dP}{dt} = k_b P - k_d P^2$$

**1.3.3 7**

$$\frac{dx}{dt} = kx(1000 - x)$$

**1.3.4 9**

$$\frac{dA}{dt} = -\frac{A}{100}$$

$$A(0) = 50 \text{ lb}$$

**1.3.5 11**

$$\frac{dA}{dt} + \frac{7}{600-t}A = 6$$

**1.3.6 13**

$$\begin{aligned}\frac{dV}{dt} &= -cA_h\sqrt{2gh} \\ A_w\frac{dh}{dt} &= -cA_h\sqrt{2gh} \\ \frac{dh}{dt} &= -\frac{cA_h\sqrt{2g}}{A_w}\sqrt{h} \\ &= -\frac{c\pi r_h^2\sqrt{2g}}{A_w}\sqrt{h} \\ &= -\frac{c\pi}{430}\sqrt{h}\end{aligned}$$

**1.3.7 15**

$$L\frac{di}{dt} + Ri = E$$

**1.3.8 17**

$$m\frac{dv}{dt} = mg - kv^2$$

**1.3.9 19**

$$m\frac{d^2x}{dt^2} = -kx$$

**1.3.10 21**

$$\begin{aligned}\frac{d}{dt}(mv) &= R - kv \\ \frac{dm}{dt}v + m\frac{dv}{dt} &= R - kv - mg\end{aligned}$$

**1.3.11 23**

$$g = \frac{k}{R^2} \Rightarrow k = gR^2$$

$$\frac{d^2r}{dt^2} = -\frac{gR^2}{r^2}$$

**1.3.12 25**

$$\frac{dA}{dt} = k(M - A)$$

**1.3.13 27**

$$\frac{dx}{dt} = r - kx$$

**1.3.14 29**

$$\begin{aligned}\frac{dy}{dx} &= \tan \theta \\ &= \tan \frac{\phi}{2} \\ &= \frac{1 - \cos \phi}{\sin \phi} \\ &= \frac{1 - x/r}{y/r} \\ &= \frac{r - x}{y} \\ &= \frac{\sqrt{x^2 + y^2} - x}{y}\end{aligned}$$

**1.4 Chapter in Review**

**1.4.1 1**

$$\frac{dy}{dx} = ky$$

**1.4.2 3**

$$y'' + k^2y = 0$$

**1.4.3 5**

$$y = c_1 e^x + c_2 x e^x$$

$$\begin{aligned} y' &= c_1 e^x + c_2 e^x + c_2 x e^x \\ &= y + c_2 e^x \end{aligned}$$

$$\begin{aligned} y'' &= c_1 e^x + c_2 e^x + c_2 e^x + c_2 x e^x \\ &= c_1 e^x + 2c_2 e^x + c_2 x e^x \\ &= y' + c_2 e^x \end{aligned}$$

$$y'' - 2y' + y = 0$$

**1.4.4 7**

a, d

**1.4.5 9**

b

**1.4.6 11**

b

**1.4.7 13**

$$y = c e^x$$

**1.4.8 15**

$$\frac{dy}{dx} = x^2 + y^2$$

**1.4.9 17**

(a)  $(-\infty, \infty)$

(b)  $(-\infty, 0)$  or  $(0, \infty)$

**1.4.10 19**

$x_0 = -1$  and  $I = (-\infty, 0)$  or  $x_0 = 2$  and  $I = (0, \infty)$

**1.4.11 23**

$$y = x \sin x + x \cos x$$

$$y' = \sin x + x \cos x + \cos x - x \sin x$$

$$\begin{aligned} y'' &= \cos x + \cos x - x \sin x - \sin x - \sin x - x \cos x \\ &= 2 \cos x - 2 \sin x - x \sin x - x \cos x \end{aligned}$$

$$\begin{aligned} y'' + y &= 2 \cos x - 2 \sin x - x \sin x - x \cos x + x \sin x + x \cos x \\ &= 2 \cos x - 2 \sin x \end{aligned}$$

$$I = (-\infty, \infty)$$

**1.4.12 25**

$$y = \sin(\ln x)$$

$$y' = \frac{1}{x} \cos(\ln x)$$

$$y'' = -\frac{1}{x^2} \cos(\ln x) - \frac{1}{x^2} \sin(\ln x)$$

$$\begin{aligned} x^2 y'' + x y' + y &= -\cos(\ln x) - \sin(\ln x) + \cos(\ln x) + \sin(\ln x) \\ &= 0 \end{aligned}$$

$$I = (0, \infty)$$

**1.4.13 35**

$$\begin{aligned} y(0) = 0 &= c_1 e^{-3(0)} + c_2 e^{(0)} + 4(0) \\ &= c_1 + c_2 \\ c_1 &= -c_2 \end{aligned}$$

$$\begin{aligned} y'(0) = 0 &= -3c_1 e^{-3(0)} + c_2 e^{(0)} + 4 \\ &= -3c_1 + c_2 + 4 \\ c_2 &= 3c_1 - 4 \end{aligned}$$

$$c_1 = -(3c_1 - 4) \Rightarrow c_1 = 1 \Rightarrow c_2 = -1$$

$$y = e^{-3x} - e^x + 4x$$

**1.4.14 37**

$$\begin{aligned}
y(1) &= -2 = c_1 e^{-3(1)} + c_2 e^{(1)} + 4(1) \\
&= c_1 e^{-3} + c_2 e + 4 \\
c_1 &= -e^3(c_2 e + 6)
\end{aligned}$$

$$\begin{aligned}
y'(1) &= 4 = -3c_1 e^{-3(1)} + c_2 e^{(1)} + 4 \\
&= -3c_1 e^{-3} + c_2 e + 4 \\
c_2 e &= 3c_1 e^{-3}
\end{aligned}$$

$$c_1 = -e^3(3c_1 e^{-3} + 6) = -3c_1 - 6e^3 \Rightarrow c_1 = -\frac{3}{2}e^3 \Rightarrow c_2 = -\frac{9}{2}e^{-1}$$

$$y = -\frac{3}{2}e^{3(1-x)} - \frac{9}{2}e^{x-1} + 4x$$

**1.4.15 41**

$$y_0 = -3, y_1 = 0$$

**1.4.16 43**

$$\begin{aligned}
\frac{d}{dt}(mv) &= F - mg \\
\frac{d}{dt}\left(\lambda x \frac{dx}{dt}\right) &= F - \lambda xg \\
x \frac{d^2 x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + gx &= \frac{F}{\lambda} \\
x \frac{d^2 x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 32x &= 5
\end{aligned}$$

**2 First-Order Differential Equations****2.1 Solution Curves Without a Solution****2.1.1 21**

0 is stable, 3 is unstable

**2.1.2 23**

2 is semi-stable



**2.1.3 25**

−2 is unstable, 0 is semi-stable, 2 is stable

**2.1.4 27**

−1 is stable, 0 is unstable

**2.1.5 39**

$$P_0 < h/k$$

**2.1.6 41**

$$g - \frac{k}{m}v^2 = 0 \Rightarrow v = \sqrt{\frac{gm}{k}}$$

**2.2 Separable Equations****2.2.1 1**

$$\begin{aligned}\frac{dy}{dx} &= \sin 5x \\ y &= -\frac{1}{5} \cos 5x + c\end{aligned}$$

**2.2.2 3**

$$\begin{aligned}dx + e^{3x} dy &= 0 \\ e^{-3x} dx + dy &= 0 \\ -\frac{1}{3}e^{-3x} + y &= c \\ y &= \frac{1}{3}e^{-3x} + c\end{aligned}$$

### 2.2.3 5

$$\begin{aligned}
 x \frac{dy}{dx} &= 4y \\
 \frac{1}{4y} dy &= \frac{1}{x} dx \\
 \frac{1}{4} \ln |4y| &= \ln |x| + c \\
 \ln |4y| &= 4 \ln |x| + c \\
 4y &= e^{4 \ln |x| + c} \\
 &= c \left( e^{\ln |x|} \right)^4 \\
 y &= cx^4
 \end{aligned}$$

### 2.2.4 7

$$\begin{aligned}
 \frac{dy}{dx} &= e^{3x+2y} \\
 &= e^{3x} e^{2y} \\
 e^{-2y} dy &= e^{3x} dx \\
 -\frac{1}{2} e^{-2y} &= \frac{1}{3} e^{3x} + c \\
 -3e^{-2y} &= 2e^{3x} + c
 \end{aligned}$$

### 2.2.5 9

$$\begin{aligned}
 y \ln x \frac{dx}{dy} &= \left( \frac{y+1}{x} \right)^2 \\
 x^2 \ln x dx &= \frac{(y+1)^2}{y} dy \\
 x^3 \left( \frac{\ln x}{3} - \frac{1}{9} \right) &= \frac{1}{2} y(y+4) + \ln y + c \\
 \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 &= \frac{1}{2} y^2 + 2y + \ln y + c
 \end{aligned}$$

**2.2.6 11**

$$\begin{aligned}
 \csc y \, dx + \sec^2 x \, dy &= 0 \\
 \frac{1}{\sin y} \, dx + \frac{1}{\cos^2 x} \, dy &= 0 \\
 \cos^2 x \, dx + \sin y \, dy &= 0 \\
 \frac{1}{2}(1 + \cos 2x) \, dx + \sin y \, dy &= 0 \\
 \frac{1}{2} \left( x + \frac{1}{2} \sin 2x \right) - \cos y + c &= 0 \\
 4 \cos y &= 2x + \sin 2x + c
 \end{aligned}$$

**2.2.7 13**

$$\begin{aligned}
 (e^y + 1)^2 e^{-y} \, dx + (e^x + 1)^3 e^{-x} \, dy &= 0 \\
 \frac{e^x}{(e^x + 1)^3} \, dx + \frac{e^y}{(e^y + 1)^2} \, dy &= 0 \\
 -\frac{1}{2(e^x + 1)^2} - \frac{1}{e^y + 1} &= c \\
 (e^x + 1)^{-2} + 2(e^y + 1)^{-1} &= c
 \end{aligned}$$

**2.2.8 15**

$$\begin{aligned}
 \frac{dS}{dr} &= kS \\
 \frac{1}{S} \, dS &= k \, dr \\
 \ln |S| &= kr + c \\
 S &= ce^{kr}
 \end{aligned}$$

**2.2.9 17**

$$\begin{aligned}\frac{dP}{dt} &= P - P^2 \\ \frac{1}{P(1-P)} dP &= dt \\ \ln \frac{P}{1-P} &= t + c \\ \frac{P}{1-P} &= ce^t \\ P &= ce^t(1-P) \\ P &= \frac{ce^t}{1+ce^t}\end{aligned}$$

**2.2.10 19**

$$\begin{aligned}\frac{dy}{dx} &= \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8} \\ &= \frac{(x-1)(y+3)}{(x+4)(y-2)} \\ \frac{y-2}{y+3} dt &= \frac{x-1}{x+4} dx \\ y - 5 \ln |y+3| &= x - 5 \ln |x+4| + c \\ e^{y-5 \ln |y+3|} &= e^{x-5 \ln |x+4|+c} \\ \frac{e^y}{(y+3)^5} &= \frac{ce^x}{(x+4)^5} \\ c(x+4)^5 e^y &= (y+3)^5 e^x\end{aligned}$$

**2.2.11 21**

$$\begin{aligned}\frac{dy}{dx} &= x\sqrt{1-y^2} \\ (1-y^2)^{-1/2} dy &= x dx \\ \arcsin y &= \frac{1}{2}x^2 + c \\ y &= \sin \left( \frac{1}{2}x^2 + c \right)\end{aligned}$$

**2.2.12 23**

$$\begin{aligned}\frac{dx}{dt} &= 4(x^2 + 1) \\ \frac{1}{x^2 + 1} dx &= 4 dt \\ \arctan x &= 4t + c \\ x &= \tan(4t + c)\end{aligned}$$

$$\begin{aligned}x\left(\frac{\pi}{4}\right) &= 1 = \tan\left(4\left(\frac{\pi}{4}\right) + c\right) \\ &= \tan(\pi + c) \\ c &= \arctan(1) - \pi \\ &= -\frac{3}{4}\pi\end{aligned}$$

$$x = \tan\left(4t - \frac{3}{4}\pi\right)$$

**2.2.13 25**

$$\begin{aligned}x^2 \frac{dy}{dx} &= y - xy \\ &= y(1 - x) \\ \frac{1}{y} dy &= \left(\frac{1}{x^2} - \frac{1}{x}\right) dx \\ \ln |y| &= -\frac{1}{x} - \ln |x| + c \\ y &= e^{-\frac{1}{x} - \ln |x| + c} \\ &= \frac{c}{xe^{1/x}}\end{aligned}$$

$$\begin{aligned}y(-1) &= -1 = \frac{c}{(-1)e^{1/(-1)}} \\ &= -ce \\ c &= e^{-1}\end{aligned}$$

$$y = \frac{1}{xe^{1+1/x}}$$

**2.2.14 29**

$$\begin{aligned}\frac{dy}{dx} &= ye^{-x^2} \\ \frac{1}{y} \frac{dy}{dx} &= e^{-x^2} \\ \int_4^x \frac{1}{y} \frac{dy}{dx'} dx' &= \int_4^x e^{-x'^2} dx' \\ \ln |y|_4^x &= \int_4^x e^{-x'^2} dx' \\ \ln |y(x)| - \ln |y(4)| &= \int_4^x e^{-x'^2} dx' \\ \ln |y(x)| &= \ln |y(4)| + \int_4^x e^{-x'^2} dx' \\ y(x) &= e^{\int_4^x e^{-x'^2} dx'}\end{aligned}$$

**2.2.15 31**

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x+1}{2y} \\ 2y dy &= (2x+1) dx \\ y^2 &= x^2 + x + c \\ y &= \pm \sqrt{x^2 + x + c} \\ y(-2) &= -1 = -\sqrt{(-2)^2 + (-2) + c} \\ &= -\sqrt{2+c} \\ c &= -1 \\ y &= -\sqrt{x^2 + x - 1} \\ I &= \left( -\infty, -\frac{1-\sqrt{5}}{2} \right)\end{aligned}$$

### 2.2.16 33

$$e^y dx - e^{-x} dy = 0$$

$$e^x dx - e^{-y} dy = 0$$

$$e^x + e^{-y} = c$$

$$\ln |e^{-y}| = \ln |c - e^x|$$

$$y = -\ln |c - e^x|$$

$$y(0) = 0 = -\ln |c - e^{(0)}|$$

$$1 = c - 1$$

$$c = 2$$

$$y = -\ln |2 - e^x|$$

$$I = (-\infty, \ln 2)$$

## 2.3 Linear Equations

### 2.3.1 1

$$\frac{dy}{dx} = 5y$$

$$\ln |y| = 5x + c$$

$$y = ce^{5x}$$

$$I = (-\infty, \infty)$$

### 2.3.2 3

$$\frac{dy}{dx} + y = e^{3x}$$

$$e^x \frac{dy}{dx} + e^x y = e^{4x}$$

$$\frac{d}{dx}(e^x y) = e^{4x}$$

$$e^x y = \frac{1}{4} e^{4x} + c$$

$$y = \frac{1}{4} e^{3x} + ce^{-x}$$

$$I = (-\infty, \infty)$$

2.3.3 5

$$\begin{aligned}
 y' + 3x^2y &= x^2 \\
 e^{x^3}y' + 3x^2e^{x^3}y &= e^{x^3}x^2 \\
 e^{x^3}y &= \frac{1}{3}e^{x^3} + c \\
 y &= \frac{1}{3} + ce^{-x^3}
 \end{aligned}$$

$$I = (-\infty, \infty)$$

2.3.4 7

$$\begin{aligned}
 x^2y' + xy &= 1 \\
 y' + x^{-1}y &= x^{-2} \\
 e^{\ln x}y' + x^{-1}e^{\ln x}y &= e^{\ln x}x^{-2} \\
 \frac{d}{dx}(e^{\ln x}y) &= x^{-1} \\
 \frac{d}{dx}(xy) &= x^{-1} \\
 xy &= \ln x + c \\
 y &= \frac{\ln x + c}{x}
 \end{aligned}$$

$$I = (0, \infty)$$

2.3.5 9

$$\begin{aligned}
 x \frac{dy}{dx} - y &= x^2 \sin x \\
 \frac{dy}{dx} - x^{-1}y &= x \sin x \\
 e^{-\ln x} \frac{dy}{dx} - x^{-1}e^{-\ln x}y &= e^{-\ln x}x \sin x \\
 \frac{d}{dx}(e^{-\ln x}y) &= \sin x \\
 x^{-1}y &= -\cos x + c \\
 y &= cx - x \cos x
 \end{aligned}$$

$$I = (0, \infty)$$



**2.3.6 11**

$$\begin{aligned}
 x \frac{dy}{dx} + 4y &= x^3 - x \\
 \frac{dy}{dx} + 4x^{-1}y &= x^2 - 1 \\
 e^{4 \ln x} \frac{dy}{dx} + 4x^{-1}e^{4 \ln x}y &= e^{4 \ln x}(x^2 - 1) \\
 \frac{d}{dx}(e^{4 \ln x}y) &= x^6 - x^4 \\
 x^4y &= \frac{1}{7}x^7 - \frac{1}{5}x^5 + c \\
 y &= \frac{1}{7}x^3 - \frac{1}{5}x^2 + cx^{-4}
 \end{aligned}$$

$$I = (0, \infty)$$

**2.3.7 13**

$$\begin{aligned}
 x^2y' + x(x+2)y &= e^x \\
 y' + x^{-1}(x+2)y &= x^{-2}e^x \\
 e^{x+2 \ln x}y' + x^{-1}(x+2)e^{x+2 \ln x}y &= e^{x+2 \ln x}x^{-2}e^x \\
 \frac{d}{dx}(e^x x^2 y) &= e^{2x} \\
 e^x x^2 y &= \frac{1}{2}e^{2x} + c \\
 y &= \frac{e^x}{2x^2} + \frac{c}{e^x x^2}
 \end{aligned}$$

$$I = (0, \infty)$$

2.3.8 15

$$\begin{aligned}
 y \, dx - 4(x + y^6) \, dy &= 0 \\
 y \frac{dx}{dy} - 4x - 4y^6 &= 0 \\
 \frac{dx}{dy} - \frac{4}{y}x &= 4y^5 \\
 e^{-4 \ln y} \frac{dx}{dy} - \frac{4}{y} e^{-4 \ln y} x &= 4e^{-4 \ln y} y^5 \\
 \frac{d}{dy} (e^{-4 \ln y} x) &= 4y \\
 y^{-4} x &= 2y^2 + c \\
 x &= 2y^6 + cy^4
 \end{aligned}$$

$$I = (0, \infty)$$

2.3.9 17

$$\begin{aligned}
 \cos x \frac{dy}{dx} + (\sin x)y &= 1 \\
 \frac{dy}{dx} + (\tan x)y &= \sec x \\
 e^{\ln(\sec x)} \frac{dy}{dx} + (\tan x)e^{\ln(\sec x)}y &= e^{\ln(\sec x)} \sec x \\
 \frac{d}{dx} (e^{\ln(\sec x)}y) &= \sec^2 x \\
 y \sec x &= \tan x + c \\
 y &= \sin x + c \cos x
 \end{aligned}$$

$$I = (-\pi/2, \pi/2)$$

**2.3.10 19**

$$\begin{aligned}
 (x+1)\frac{dy}{dx} + (x+2)y &= 2xe^{-x} \\
 \frac{dy}{dx} + \frac{x+2}{x+1}y &= \frac{2xe^{-x}}{x+1} \\
 e^{x+\ln|x+1|}\frac{dy}{dx} + \frac{x+2}{x+1}e^{x+\ln|x+1|}y &= e^{x+\ln|x+1|}\frac{2xe^{-x}}{x+1} \\
 \frac{d}{dx}(e^{x+\ln|x+1|}y) &= 2x \\
 e^x(x+1)y &= x^2 + c \\
 y &= \frac{x^2 + c}{e^x(x+1)}
 \end{aligned}$$

$$I = (-1, \infty)$$

**2.3.11 21**

$$\begin{aligned}
 \frac{dr}{d\theta} + r \sec \theta &= \cos \theta \\
 e^{\ln|\sec \theta + \tan \theta|}\frac{dr}{d\theta} + e^{\ln|\sec \theta + \tan \theta|}r \sec \theta &= e^{\ln|\sec \theta + \tan \theta|}\cos \theta \\
 \frac{d}{d\theta}(e^{\ln|\sec \theta + \tan \theta|}r) &= 1 + \sin \theta \\
 (\sec \theta + \tan \theta)r &= \theta - \cos \theta + c \\
 r &= \frac{\theta - \cos \theta + c}{\sec \theta + \tan \theta}
 \end{aligned}$$

$$I = (-\pi/2, \pi/2)$$

**2.3.12 23**

$$\begin{aligned}
 x\frac{dy}{dx} + (3x+1)y &= e^{-3x} \\
 \frac{dy}{dx} + (3+x^{-1})y &= e^{-3x}x^{-1} \\
 e^{3x+\ln|x|}\frac{dy}{dx} + (3+x^{-1})e^{3x+\ln|x|}y &= 1 \\
 \frac{d}{dx}(e^{3x+\ln|x|}y) &= 1 \\
 e^{3x}xy &= x + c \\
 y &= \frac{x+c}{e^{3x}x}
 \end{aligned}$$

$$I = (0, \infty)$$

2.3.13 25

$$\begin{aligned}
 xy' + y &= e^x \\
 y' + x^{-1}y &= e^x x^{-1} \\
 e^{\ln|x|}y' + x^{-1}e^{\ln|x|}y &= e^x \\
 \frac{d}{dx}(e^{\ln|x|}y) &= e^x \\
 xy &= e^x + c \\
 y &= \frac{e^x + c}{x}
 \end{aligned}$$

$$\begin{aligned}
 y(1) &= 2 = \frac{e^{(1)} + c}{(1)} \\
 c &= 2 - e
 \end{aligned}$$

$$y = \frac{e^x + 2 - e}{x}$$

$$I = (0, \infty)$$

2.3.14 27

$$\begin{aligned}
 L \frac{di}{dt} + Ri &= E \\
 \frac{di}{dt} + \frac{R}{L}i &= \frac{E}{L} \\
 e^{Rt/L} \frac{di}{dt} + \frac{R}{L}e^{Rt/L}i &= \frac{E}{L}e^{Rt/L} \\
 \frac{d}{dt}(e^{Rt/L}i) &= \frac{E}{L}e^{Rt/L} \\
 e^{Rt/L}i &= \frac{E}{R}e^{Rt/L} + c \\
 i &= \frac{E}{R} + ce^{-Rt/L}
 \end{aligned}$$

$$\begin{aligned}
 i(0) = i_0 &= \frac{E}{R} + ce^{-R(0)/L} \\
 &= \frac{E}{R} + c \\
 c &= i_0 - \frac{E}{R}
 \end{aligned}$$

$$i = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right) e^{-Rt/L}$$

$$I = (-\infty, \infty)$$

**2.3.15 53**

$$\begin{aligned}\frac{dE}{dt} &= -\frac{1}{RC}E \\ \frac{1}{E} \frac{dE}{dt} &= -\frac{1}{RC} \\ \ln|E| &= -\frac{1}{RC}t + c \\ E &= ce^{-t/RC}\end{aligned}$$

$$\begin{aligned}E(4) &= E_0 = ce^{-(4)/RC} \\ c &= E_0 e^{4/RC}\end{aligned}$$

$$E = E_0 e^{(4-t)/RC}$$

## **2.4 Exact Equations**

**2.4.1 1**

$$f(x, y) = x^2 - x + g(y)$$

$$\frac{\partial f}{\partial y} = g'(y) = 3y + 7$$

$$g(y) = \frac{3}{2}y^2 + 7y$$

$$x^2 - x + \frac{3}{2}y^2 + 7y = c$$

**2.4.2 3**

$$f(x, y) = \frac{5}{2}x^2 + 4xy + g(y)$$

$$4x + g'(y) = 4x - 8y^3 \Rightarrow g'(y) = -8y^3$$

$$g(y) = -2y^4$$

$$\frac{5}{2}x^2 + 4xy - 2y^4 = c$$

**2.4.3 5**

$$f(x, y) = x^2y^2 - 3x + g(y)$$

$$2x^2y + g'(y) = 2x^2y + 4 \Rightarrow g'(y) = 4$$

$$g(y) = 4y$$

$$x^2y^2 - 3x + 4y = c$$

**2.4.4 7**

Not exact

**2.4.5 9**

$$f(x, y) = \frac{1}{2}x^2 - xy^3 - y^2 \cos x + g(y)$$

$$-3xy^2 - 2y \cos x + g'(y) = -3xy^2 - 2y \cos x \Rightarrow g'(y) = 0$$

$$\frac{1}{2}x^2 - xy^3 - y^2 \cos x = c$$

**2.4.6 11**

Not exact

**2.4.7 13**

$$f(x, y) = xy + g(x)$$

$$y + g'(x) = -2xe^x + y - 6x^2 \Rightarrow g'(x) = -2xe^x - 6x^2$$

$$g(x) = -2e^x(x - 1) - 2x^3$$

$$xy - 2e^x(x - 1) - 2x^3 = c$$

**2.4.8 21**

$$f(x, y) = \frac{1}{3}(x + y)^3 + g(y)$$

$$(x + y)^2 + g'(y) = 2xy + x^2 - 1 \Rightarrow g'(y) = -y^2 - 1$$

$$g(y) = -\frac{1}{3}y^3 - y$$

$$\frac{1}{3}(x + y)^3 - \frac{1}{3}y^3 - y = c$$

$$\frac{1}{3}(1 + 1)^3 - \frac{1}{3}1^3 - 1 = c \Rightarrow c = \frac{4}{3}$$

$$x^3 + 3x^2y + 3xy^2 - 3y = 4$$

**2.4.9 23**

$$f(x, y) = 4ty + t^2 - 5t + g(y)$$

$$4t + g'(y) = 6y + 4t - 1 \Rightarrow g'(y) = 6y - 1$$

$$g(y) = 3y^2 - y$$

$$4ty + t^2 - 5t + 3y^2 - y = c$$

$$4(-1)(2) + (-1)^2 - 5(-1) + 3(2)^2 - (2) = c \Rightarrow c = 8$$

$$4ty + t^2 - 5t + 3y^2 - y = 8$$

**2.4.10 27**

$$3y^2 + 4kxy^3 = 3y^2 + 40xy^3 \Rightarrow k = 10$$

**2.4.11 31**

$$M_y = 4y$$

$$N_x = 2y$$

$$\frac{M_y - N_x}{N} = \frac{4y - 2y}{2xy} = \frac{1}{x}$$

$$\mu(x) = e^{\ln x} = x$$

$$(2xy^2 + 3x^2) dx + 2x^2y dy = 0$$

$$f(x, y) = x^2y^2 + x^3 + g(y)$$

$$2x^2y + g'(y) = 2x^2y \Rightarrow g'(y) = 0$$

$$x^2y^2 + x^3 = c$$

**2.4.12 33**

$$M_y = 6x$$

$$N_x = 18x$$

$$\frac{M_y - N_x}{N} = \frac{6x - 18x}{4y + 9x^2}$$

$$\frac{N_x - M_y}{M} = \frac{18x - 6x}{6xy} = \frac{2}{y}$$

$$\mu(y) = e^{2 \ln y} = y^2$$

$$6xy^3 dx + (4y^3 + 9x^2y^2) dy = 0$$

$$f(x, y) = 3x^2y^3 + g(y)$$

$$9x^2y^2 + g'(y) = 4y^3 + 9x^2y^2 \Rightarrow g'(y) = 4y^3$$

$$g(y) = y^4$$

$$3x^2y^3 + y^4 = c$$



**2.4.13 37**

$$M_y = 0$$

$$N_x = 2xy$$

$$\frac{N_x - M_y}{M} = \frac{2xy - 0}{x} = 2y$$

$$\mu(y) = e^{y^2}$$

$$e^{y^2} x \, dx + e^{y^2} (x^2 y + 4y) \, dy = 0$$

$$f(x, y) = \frac{1}{2} e^{y^2} x^2 + g(y)$$

$$y e^{y^2} x^2 + g'(y) = e^{y^2} (x^2 y + 4y) \Rightarrow g'(y) = 4e^{y^2} y$$

$$g(y) = 2e^{y^2}$$

$$\frac{1}{2} e^{y^2} x^2 + 2e^{y^2} = c$$

$$\frac{1}{2} e^{(0)^2} (4)^2 + 2e^{(0)^2} = c \Rightarrow c = 10$$

$$\frac{1}{2} e^{y^2} x^2 + 2e^{y^2} = 10$$

**2.4.14 39**

(c)

$$(0)^3 + 2(0)^2(-2) + (-2)^2 = c \Rightarrow c = 4$$

$$y^2 + 2x^2 y + x^3 - 4 = 0$$

$$\begin{aligned} y &= \frac{-(2x^2) \pm \sqrt{(2x^2)^2 - 4(1)(x^3 - 4)}}{2(1)} \\ &= \frac{-2x^2 \pm \sqrt{4x^4 - 4(x^3 - 4)}}{2} \\ &= -x^2 \pm \sqrt{x^4 - x^3 + 4} \end{aligned}$$

**2.4.15 45**

(a)

$$xv \frac{dv}{dx} + v^2 = 32x \Rightarrow xv \, dv + (v^2 - 32x) \, dx = 0$$

$$M_x = v$$

$$N_v = 2v$$

$$\frac{M_x - N_v}{N} = \frac{v - 2v}{v^2 - 32x}$$

$$\frac{N_v - M_x}{M} = \frac{2v - v}{xv} = \frac{1}{x}$$

$$\mu(x) = e^{\ln x} = x$$

$$x^2 v \, dv + (xv^2 - 32x^2) \, dx = 0$$

$$f(x, v) = \frac{1}{2}x^2v^2 + g(x)$$

$$xv^2 + g'(x) = xv^2 - 32x^2 \Rightarrow g'(x) = -32x^2$$

$$g(x) = -\frac{32}{3}x^3$$

$$\frac{1}{2}(3)^2(0)^2 - \frac{32}{3}(3)^3 = c \Rightarrow c = -288$$

$$\frac{1}{2}x^2v^2 - \frac{32}{3}x^3 = -288 \Rightarrow v = 8\sqrt{\frac{x}{3} - \frac{9}{x^2}}$$

(b)  $v = 12.7 \text{ ft/s}$

## 2.5 Solutions by Substitution

### 2.5.1 1

$$\begin{aligned}(x - y) dx + x dy &= 0 \\(x - ux) dx + x(u dx + x du) &= 0 \\x dx + x^2 du &= 0 \\x^{-1} dx + du &= 0 \\\ln |x| + u &= c \\\ln |x| + \frac{y}{x} &= c \\y &= cx - x \ln |x|\end{aligned}$$

### 2.5.2 3

$$\begin{aligned}x dx + (y - 2x) dy &= 0 \\vy(v dy + y dv) + (y - 2vy) dy &= 0 \\(v^2 y + y - 2vy) dy + vy^2 dv &= 0 \\y(v^2 - 2v + 1) dy + vy^2 dv &= 0 \\(v - 1)^2 dy + vy dv &= 0 \\\frac{1}{y} dy + \frac{v}{(v - 1)^2} dv &= 0 \\\ln |y| + \frac{1}{1 - v} + \ln |v - 1| &= c \\\ln |y| + \frac{1}{1 - x/y} + \ln \left| \frac{x}{y} - 1 \right| &= c \\\ln |x - y| + \frac{y}{y - x} &= c \\(y - x) \ln |x - y| + y &= c(y - x) \\(x - y) \ln |x - y| &= y + c(x - y)\end{aligned}$$

2.5.3 5

$$\begin{aligned}
 (y^2 + yx) dx - x^2 dy &= 0 \\
 ((ux)^2 + ux^2) dx - x^2(u dx + x du) &= 0 \\
 u^2 x^2 dx - x^3 du &= 0 \\
 \frac{1}{x} dx - \frac{1}{u^2} du &= 0 \\
 \ln |x| + \frac{1}{u} &= c \\
 \ln |x| + \frac{x}{y} &= c \\
 y &= \frac{x}{c - \ln |x|}
 \end{aligned}$$

2.5.4 7

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{y - x}{y + x} \\
 (y + x) dy + (x - y) dx &= 0 \\
 (ux + x)(u dx + x du) + (x - ux) dx &= 0 \\
 (u^2 x + x) dx + (ux^2 + x^2) du &= 0 \\
 x(u^2 + 1) dx + x^2(u + 1) du &= 0 \\
 \frac{1}{x} dx + \frac{u + 1}{u^2 + 1} du &= 0 \\
 \ln |x| + \frac{1}{2} \ln |u^2 + 1| + \arctan u &= c \\
 \ln |x^2 + y^2| + 2 \arctan \frac{y}{x} &= c
 \end{aligned}$$

2.5.5 9

$$\begin{aligned}
 -y \, dx + (x + \sqrt{xy}) \, dy &= 0 \\
 -ux \, dx + (x + \sqrt{ux^2})(u \, dx + x \, du) &= 0 \\
 u\sqrt{ux^2} \, dx + (x^2 + x\sqrt{ux^2}) \, du &= 0 \\
 u^{3/2}x \, dx + x^2(1 + \sqrt{u}) \, du &= 0 \\
 \frac{1}{x} \, dx + \frac{1 + \sqrt{u}}{u^{3/2}} \, du &= 0 \\
 \frac{1}{x} \, dx + (u^{-3/2} + u^{-1}) \, du &= 0 \\
 \ln |x| - 2u^{-1/2} + \ln |u| &= c \\
 \ln |x| - 2(y/x)^{-1/2} + \ln |y/x| &= c \\
 \ln |y| - 2\sqrt{\frac{x}{y}} &= c \\
 4\frac{x}{y} &= (\ln |y| - c)^2 \\
 4x &= y(\ln |y| - c)^2
 \end{aligned}$$

2.5.6 11

$$\begin{aligned}
 xy^2 \frac{dy}{dx} &= y^3 - x^3 \\
 xy^2 \, dy + (x^3 - y^3) \, dx &= 0 \\
 x(ux)^2(u \, dx + x \, du) + (x^3 - (ux)^3) \, dx &= 0 \\
 x^3 \, dx + u^2 x^4 \, du &= 0 \\
 x^{-1} \, dx + u^2 \, du &= 0 \\
 \ln |x| + \frac{1}{3} u^3 &= c \\
 \ln |x| + \frac{1}{3} \left(\frac{y}{x}\right)^3 &= c \\
 \ln |1| + \frac{1}{3} \left(\frac{2}{1}\right)^3 = c \Rightarrow c &= \frac{8}{3} + \ln 1 \\
 \ln |x| + \frac{1}{3} \left(\frac{y}{x}\right)^3 &= \frac{8}{3} \\
 y^3 + 3x^3 \ln |x| &= 8x^3
 \end{aligned}$$

2.5.7 13

$$\begin{aligned}
 (x + ye^{y/x}) dx - xe^{y/x} dy &= 0 \\
 (x + uxe^u) dx - xe^u(u dx + x du) &= 0 \\
 x dx - x^2 e^u du &= 0 \\
 x^{-1} dx - e^u du &= 0 \\
 \ln |x| - e^u &= c \\
 \ln |x| - e^{y/x} &= c
 \end{aligned}$$

$$\ln |1| - e^{0/1} = c \Rightarrow c = -1$$

$$\ln |x| = e^{y/x} - 1$$

2.5.8 15

$$\begin{aligned}
 x \frac{dy}{dx} + y &= \frac{1}{y^2} \\
 \frac{dy}{dx} + x^{-1}y &= x^{-1}y^{-2} \\
 u = y^{1-n} = y^3 \Rightarrow y &= u^{1/3} \Rightarrow \frac{dy}{dx} = \frac{1}{3}u^{-2/3} \frac{du}{dx}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{3}u^{-2/3} \frac{du}{dx} + x^{-1}u^{1/3} &= x^{-1}u^{-2/3} \\
 \frac{du}{dx} + 3x^{-1}u &= 3x^{-1} \\
 e^{3 \ln |x|} \frac{du}{dx} + 3x^{-1}e^{3 \ln |x|}u &= 3x^2 \\
 \frac{d}{dx}(x^3u) &= 3x^2 \\
 x^3u &= x^3 + c \\
 y^3 &= 1 + cx^{-3}
 \end{aligned}$$

2.5.9 17

$$\begin{aligned}
 \frac{dy}{dx} &= y(xy^3 - 1) \\
 \frac{dy}{dx} + y &= xy^4
 \end{aligned}$$

$$u = y^{1-n} = y^{-3} \Rightarrow y = u^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}u^{-4/3}\frac{du}{dx}$$

$$-\frac{1}{3}u^{-4/3}\frac{du}{dx} + u^{-1/3} = xu^{-4/3}$$

$$\frac{du}{dx} - 3u = -3x$$

$$e^{-3x}\frac{du}{dx} - 3e^{-3x}u = -3e^{-3x}x$$

$$\frac{d}{dt}(e^{-3x}u) = -3e^{-3x}x$$

$$e^{-3x}u = e^{-3x}x + \frac{1}{3}e^{-3x} + c$$

$$u = x + \frac{1}{3} + ce^{3x}$$

$$y^{-3} = x + \frac{1}{3} + ce^{3x}$$

**2.5.10 21**

$$x^2\frac{dy}{dx} - 2xy = 3y^4$$

$$\frac{dy}{dx} - 2x^{-1}y = 3x^{-2}y^4$$

$$u = y^{1-n} = y^{-3} \Rightarrow y = u^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}u^{-4/3}\frac{du}{dx}$$

$$-\frac{1}{3}u^{-4/3}\frac{du}{dx} - 2x^{-1}u^{-1/3} = 3x^{-2}u^{-4/3}$$

$$\frac{du}{dx} + 6x^{-1}u = -9x^{-2}$$

$$e^{6\ln|x|}\frac{du}{dx} + 6x^{-1}e^{6\ln|x|}u = -9e^{6\ln|x|}x^{-2}$$

$$\frac{d}{dx}(x^6u) = -9x^4$$

$$x^6u = -\frac{9}{5}x^5 + c$$

$$u = -\frac{9}{5}x^{-1} + cx^{-6}$$

$$y^{-3} = -\frac{9}{5}x^{-1} + cx^{-6}$$

$$\left(\frac{1}{2}\right)^{-3} = -\frac{9}{5}(1)^{-1} + c(1)^{-6} \Rightarrow c = \frac{49}{5}$$

$$y^{-3} = -\frac{9}{5}x^{-1} + \frac{49}{5}x^{-6}$$

### 2.5.11 23

Let  $u = x + y + 1$  so  $\frac{du}{dx} = 1 + \frac{dy}{dx}$  and

$$\begin{aligned}\frac{du}{dx} - 1 &= u^2 \\ \frac{1}{u^2 + 1} du &= dx \\ \arctan u &= x + c \\ \arctan(x + y + 1) &= x + c \\ x + y + 1 &= \tan(x + c) \\ y &= -x - 1 + \tan(x + c)\end{aligned}$$

### 2.5.12 25

Let  $u = x + y$  so  $\frac{du}{dx} = 1 + \frac{dy}{dx}$  and

$$\begin{aligned}\frac{du}{dx} - 1 &= \tan^2 u \\ \frac{1}{1 + \tan^2 u} du &= dx \\ \frac{1}{2}(u + \sin u \cos u) &= x + c \\ x + y + \sin(x + y) \cos(x + y) &= 2(x + c) \\ x + y + \frac{1}{2} \sin(2(x + y)) &= 2(x + c) \\ 2x + 2y + \sin(2(x + y)) &= 4(x + c) \\ 2y - 2x + \sin(2(x + y)) &= c\end{aligned}$$

### 2.5.13 35

(a) Let  $y = y_1 + u$  so  $\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{du}{dx}$  but  $\frac{dy_1}{dx} = P(x) + Q(x)y_1 + R(x)y_1^2$  so



$$\begin{aligned}
\frac{dy}{dx} &= P(x) + Q(x)y + R(x)y^2 \\
P(x) + Q(x)y_1 + R(x)y_1^2 + \frac{du}{dx} &= P(x) + Q(x)(y_1 + u) + R(x)(y_1 + u)^2 \\
\frac{du}{dx} &= Q(x)u + R(x)(2y_1u + u^2) \\
\frac{du}{dx} - (Q(x) + 2R(x)y_1)u &= R(x)u^2
\end{aligned}$$

(b) Let  $y = 2x^{-1} + u$  so  $\frac{dy}{dx} = -2x^{-2} + \frac{du}{dx}$  and

$$\begin{aligned}
-\frac{2}{x^2} + \frac{du}{dx} &= -\frac{4}{x^2} - \frac{1}{x} \left( \frac{2}{x} + u \right) + \left( \frac{2}{x} + u \right)^2 \\
\frac{du}{dx} &= \frac{2}{x^2} - \frac{4}{x^2} - \frac{2}{x^2} - \frac{u}{x} + \frac{4}{x^2} + \frac{4u}{x} + u^2 \\
\frac{du}{dx} - \frac{3}{x}u &= u^2
\end{aligned}$$

Let  $v = u^{1-n} = u^{-1}$  so  $u = v^{-1}$  and  $\frac{du}{dx} = -v^{-2} \frac{dv}{dx}$

$$\begin{aligned}
-v^{-2} \frac{dv}{dx} - \frac{3}{x}v^{-1} &= v^{-2} \\
\frac{dv}{dx} + \frac{3}{x}v &= -1 \\
e^{3 \ln |x|} \frac{dv}{dx} + \frac{3}{x}e^{3 \ln |x|}v &= -e^{3 \ln |x|} \\
\frac{d}{dt}(x^3v) &= -x^3 \\
x^3v &= -\frac{1}{4}x^4 + c \\
\frac{1}{y - y_1} &= -\frac{1}{4}x + cx^{-3} \\
y &= y_1 + \left( -\frac{1}{4}x + cx^{-3} \right)^{-1} \\
&= \frac{2}{x} + \left( -\frac{1}{4}x + cx^{-3} \right)^{-1}
\end{aligned}$$

**2.5.14 37**

$$\begin{aligned}
\frac{dP}{dt} &= P(a - bP) \\
\frac{dP}{dt} - aP &= -bP^2
\end{aligned}$$

Let  $u = P^{1-n} = P^{-1}$  so  $P = u^{-1}$  and  $\frac{dP}{dt} = -u^{-2} \frac{du}{dt}$

$$-u^{-2} \frac{du}{dt} - au^{-1} = -bu^{-2}$$

$$\frac{du}{dt} + au = b$$

$$e^{at} \frac{du}{dt} + ae^{at} u = be^{at}$$

$$\frac{d}{dt}(e^{at} u) = be^{at}$$

$$e^{at} u = \frac{b}{a} e^{at} + c$$

$$P^{-1} = \frac{b}{a} + ce^{-at}$$

$$= \frac{b + ce^{-at}}{a}$$

$$P = \frac{a}{b + ce^{-at}}$$

## 2.6 A Numerical Method

### 2.6.1 1

$x_0 = 1$	$y_0 = 5$
$x_1 = 1.1$	$y_1 = y_0 + hf(x_0, y_0) = 3.8000$
$x_2 = 1.2$	$y_2 = y_1 + hf(x_1, y_1) = 2.9800$

$x_0 = 1$	$y_0 = 5$
$x_1 = 1.05$	$y_1 = y_0 + hf(x_0, y_0) = 4.4000$
$x_2 = 1.1$	$y_2 = y_1 + hf(x_1, y_1) = 3.8950$
$x_3 = 1.15$	$y_3 = y_2 + hf(x_2, y_2) = 3.4708$
$x_4 = 1.2$	$y_4 = y_3 + hf(x_3, y_3) = 3.1152$