

# Advanced Engineering Mathematics Complex Analysis by Dennis G. Zill Notes

Chris Doble

February 2024

## Contents

<b>17 Functions of a Complex Variable</b>	<b>1</b>
17.1 Complex Numbers . . . . .	1
17.2 Powers and Roots . . . . .	3
17.3 Sets in the Complex Plane . . . . .	4
17.4 Functions of a Complex Variable . . . . .	5
17.5 Cauchy-Riemann Equations . . . . .	7
17.6 Exponential and Logarithmic Functions . . . . .	8
17.7 Trigonometric and Hyperbolic Functions . . . . .	10

## 17 Functions of a Complex Variable

### 17.1 Complex Numbers

- A **complex number** is any number of the form

$$z = a + ib$$

where  $a$  and  $b$  are real numbers and  $i$  is the imaginary unit such that  $i^2 = -1$ .

- The real number  $a$  in the above complex number  $z$  is called the **real part** of  $z$  and the real number  $b$  (not  $ib$ ) is called the **imaginary part** of  $z$ .
- The real and imaginary parts of a complex number  $z$  are denoted  $\text{Re}(z)$  and  $\text{Im}(z)$ , respectively.
- A real constant multiple of the imaginary unit, e.g.  $6i$  is called a **pure imaginary number**.
- Two complex numbers are equal if their real and imaginary parts are equal.

- The addition and subtraction of complex numbers occur between the real and imaginary parts, e.g.

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

- The multiplication of complex numbers occurs elementwise as normal, e.g.

$$(a + bi)(c + di) = ac + adi + bci - bd.$$

- The **conjugate** of a complex number  $z = a + ib$  is

$$\bar{z} = a - ib.$$

- The division of complex numbers occurs by multiplying the numerator and denominator by the conjugate of the denominator, e.g.

$$\begin{aligned} \frac{a + bi}{c + di} &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{ac - adi + bci + bd}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2}. \end{aligned}$$

- Conjugates have several interesting properties:

$$\begin{aligned} \overline{z_1 + z_2} &= \bar{z}_1 + \bar{z}_2 \\ \overline{z_1 - z_2} &= \bar{z}_1 - \bar{z}_2 \\ \overline{z_1 z_2} &= \bar{z}_1 \bar{z}_2 \\ \frac{z_1}{z_2} &= \frac{\bar{z}_1}{\bar{z}_2}. \end{aligned}$$

- The sum and product of a complex number  $z = x + iy$  with its conjugate are real numbers

$$\begin{aligned} z + \bar{z} &= 2x \\ z\bar{z} &= x^2 + y^2 \end{aligned}$$

while the difference between a complex number and its conjugate is a pure imaginary number

$$z - \bar{z} = 2iy.$$

- The above properties let us define

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2} \text{ and } \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}.$$

- The **complex plane** or  **$z$ -plane** is a coordinate system where the horizontal or  $x$ -axis is called the **real axis** and the vertical or  $y$ -axis is called the **imaginary axis**. Complex numbers can be plotted in this coordinate system by considering their real and imaginary parts an ordered pair corresponding their position.
- The **modulus** or **absolute value** of a complex number  $z = x + iy$  denoted by  $|z|$  is the real number

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$$

This is the distance between  $z$  and the origin in the complex plane.

- If you consider two numbers in the complex plane as vectors, the length of their sum can't be longer than their individual lengths combined

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

This extends to any finite sum

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

and is known as the **triangle inequality**.

## 17.2 Powers and Roots

- A complex number can be expressed in **polar form**

$$z = (r \cos \theta) + i(r \sin \theta)$$

where  $r = |z|$  is the nonnegative modulus of  $z$  and  $\theta = \arg z$  is the **argument** of  $z$  — the angle between  $z$  and the positive real axis measured in the counterclockwise direction.

- The argument of a complex number  $z$  isn't unique as any multiply of  $2\pi$  can be added to it. The **principle argument** of  $z$  denoted  $\text{Arg } z$  is the argument of  $z$  restricted to the interval  $-\pi \leq \text{Arg } z \leq \pi$ .
- Multiplication and division of complex numbers is simpler in polar form. For two complex numbers  $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$  and  $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$  we get

$$\begin{aligned} z_1 z_2 &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]. \end{aligned}$$

- The above formulas can be used to find integer powers of a complex number  $z$

$$z^n = r^n (\cos n\theta + i \sin n\theta)$$

where  $n$  is an integer (including negative integers).

- **DeMoivre's formula** is a special case of the above where  $r = 1$  so

$$z^n = (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

- A number  $w$  is said to be an  **$n$ th root** of a nonzero complex number  $z$  if  $w^n = z$ . The  $n$ th roots of a nonzero complex number  $z = r(\cos \theta + i \sin \theta)$  are

$$w_k = r^{1/n} \left[ \cos \left( \frac{\theta + 2k\pi}{n} \right) + i \sin \left( \frac{\theta + 2k\pi}{n} \right) \right]$$

where  $k = 0, 1, 2, \dots, n - 1$ .

- The root  $w$  of a complex number  $z$  obtained by using the principle argument of  $z$  with  $k = 0$  is called the **principle  $n$ th root** of  $z$ .
- Since the  $n$ th roots of a complex number have the same modulus they lie on a circle of radius  $r^{1/n}$ . The arguments of subsequent roots differ by  $2\pi/n$  so they're also equally spaced around the circle.

### 17.3 Sets in the Complex Plane

- The points  $z = x + iy$  that satisfy the equation

$$|z - z_0| = \rho$$

for  $\rho > 0$  lie on a circle of radius  $\rho$  centred at the point  $z_0$ .

- The points  $z$  satisfying the inequality  $|z - z_0| < \rho$  for  $\rho > 0$  lie within, but not on, a circle of radius  $\rho$  centered at the point  $z_0$ . This set is called a **neighborhood** of  $z_0$  or an **open disk**.
- A point  $z_0$  is said to be an **interior point** of a set  $S$  of the complex plane if there exists some neighborhood of  $z_0$  that lies entirely within  $S$ .
- If every point  $z$  of a set  $S$  is an interior point, then  $S$  is said to be an **open set**. An example of a set that isn't open is the set of points satisfying the inequality  $\operatorname{Re}(z) \geq 0$ . This isn't open because it includes the line  $\operatorname{Re}(z) = 0$  and no points on that line are interior to the set because, no matter what  $\rho$  you choose, some points in the neighborhood have  $\operatorname{Re}(z) < 0$ .
- If every neighborhood of a point  $z_0$  contains at least one point that is in a set  $S$  and at least one point that is not in  $S$ , then  $z_0$  is said to be a **boundary point** of  $S$ .
- The **boundary** of a set  $S$  in the complex plane is the set of all boundary points of  $S$ .
- If any pair of points in a set  $S$  can be connected by a polygonal line that lies entirely within the set, then  $S$  is said to be **connected**.
- An open connected set is called a **domain**.

- A **region** is a set in the complex plane with all, some, or none of its boundary points. A region containing all of its boundary points is said to be **closed**.

## 17.4 Functions of a Complex Variable

- A **function**  $f$  from a set  $A$  to a set  $B$  is a rule of correspondence that assigns to each element of  $A$  one and only one element of  $B$ .
- If  $b$  is the element of  $B$  assigned to the element  $a$  of  $A$ ,  $b$  is said to be the **image** of  $a$  and is denoted  $b = f(a)$ .
- The set  $A$  is called the **domain** of  $f$ .
- The set of all images in  $B$  is called the **range** of  $f$ .
- If  $A$  is a set of real numbers,  $f$  is said to be a **function of a real variable**  $x$ .
- If  $A$  is a set of complex numbers,  $f$  is said to be a **function of a complex variable**  $z$  or a **complex function**.
- The image  $w$  of a complex number  $z$  is

$$w = f(z) = u(x, y) + iv(x, y)$$

where  $u$  and  $v$  are the real and imaginary parts of  $w$  and are real-valued functions.

- Although we cannot draw a graph of a complex function  $w = f(z)$  (because it would require a four-dimensional coordinate system), it can be interpreted as a **mapping** or **transformation** from the  $z$  plane to the  $w$  plane.
- A complex function may be interpreted as a two-dimensional fluid flow by considering  $w = f(z)$  as the fluid velocity vector at the point  $z$ . In that case, if  $x(t) + iy(t)$  is a parametric representation of a particle's position over time then

$$\begin{aligned}\frac{dx}{dt} &= u(x, y) \\ \frac{dy}{dt} &= v(x, y)\end{aligned}$$

and the family of solutions to this system of differential equations are called the **streamlines** of the flow associated with  $f(z)$ .

**Definition 17.4.1 Limit of a Function**

Suppose the function  $f$  is defined in some neighborhood of  $z_0$ , except possibly at  $z_0$  itself. Then  $f$  is said to possess a **limit** at  $z_0$ , written

$$\lim_{z \rightarrow z_0} f(z) = L$$

if, for each  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $|f(z) - L| < \varepsilon$  whenever  $0 < |z - z_0| < \delta$ .

- For a function  $f$  of a real variable  $x$ , the limit  $\lim_{x \rightarrow x_0} f(x) = L$  means  $f$  approaches  $L$  as you approach from both the left and right. If however  $f$  is a function of a complex variable it means  $f$  approaches  $L$  as you approach from any direction in the complex plane.

**Theorem 17.4.1 Limit of Sum, Product, Quotient**

Suppose  $\lim_{z \rightarrow z_0} f(z) = L_1$  and  $\lim_{z \rightarrow z_0} g(z) = L_2$ . Then

$$(i) \quad \lim_{z \rightarrow z_0} [f(z) + g(z)] = L_1 + L_2$$

$$(ii) \quad \lim_{z \rightarrow z_0} f(z)g(z) = L_1L_2$$

$$(iii) \quad \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{L_1}{L_2}, \quad L_2 \neq 0.$$

- A function  $f$  is continuous at a point  $z_0$  if

$$\lim_{z \rightarrow z_0} f(z) = f(z_0).$$

- A function  $f$  defined by

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0, \quad a_n \neq 0$$

where  $n$  is a nonnegative integer and the coefficients  $a_i$ ,  $i = 0, 1, \dots, n$ , are complex constants is called a **polynomial** of degree  $n$ .

- Polynomials are continuous on the entire complex plane.
- A **rational function**

$$f(z) = \frac{g(z)}{h(z)}$$

is continuous everywhere  $h(z) \neq 0$ .

### Definition 17.4.3 Derivative

Suppose the complex function  $f$  is defined in a neighborhood of a point  $z_0$ . The **derivative** of  $f$  at  $z_0$  is

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad (3)$$

provided this limit exists.

- In order for a complex function to be differentiable, the limit must approach the same value from every direction. This is a greater demand than in real variables. If you take an arbitrary complex function, there's a good chance it isn't differentiable.

### Definition 17.4.4 Analyticity at a Point

A complex function  $w = f(z)$  is said to be **analytic at a point**  $z_0$  if  $f$  is differentiable at  $z_0$  and at every point in some neighborhood of  $z_0$ .

- Analyticity at a point is a neighborhood property. A function can be differentiable at a point but if the neighboring points aren't also differentiable, it's not analytic at that point.
- A function is analytic in a domain  $D$  if it is analytic at every point in  $D$ .
- A function that is analytic everywhere is called an **entire function**.

## 17.5 Cauchy-Riemann Equations

### Theorem 17.5.1 Cauchy–Riemann Equations

Suppose  $f(z) = u(x, y) + iv(x, y)$  is differentiable at a point  $z = x + iy$ . Then at  $z$  the first-order partial derivatives of  $u$  and  $v$  exist and satisfy the **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad (1)$$

- If a complex function  $f(z) = u(x, y) + iv(x, y)$  is analytic throughout a domain  $D$ , then the real functions  $u$  and  $v$  must satisfy the Cauchy–Riemann equations at every point in  $D$ .

### Theorem 17.5.2 Criterion for Analyticity

Suppose the real-valued functions  $u(x, y)$  and  $v(x, y)$  are continuous and have continuous first-order partial derivatives in a domain  $D$ . If  $u$  and  $v$  satisfy the Cauchy–Riemann equations at all points of  $D$ , then the complex function  $f(z) = u(x, y) + iv(x, y)$  is analytic in  $D$ .

- The Cauchy-Riemann equations are derived assuming the function is differentiable at a particular point. That being the case, they can also be used as a formula for the derivative of the function

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

- Because analyticity implies differentiability, theorem 17.5.2 can also be used to determine if a function is differentiable at a point.
- A real-valued function  $\phi(x, y)$  that has continuous second-order partial derivatives in a domain  $D$  and satisfies Laplace's equation is said to be **harmonic** in  $D$ .
- If a function  $f(z) = u(x, y) + iv(x, y)$  is analytic in a domain  $D$  then the functions  $u(x, y)$  and  $v(x, y)$  are harmonic functions.
- If a given function  $u(x, y)$  is harmonic in a domain  $D$  it is sometimes possible to find another function  $v(x, y)$  that is harmonic in  $D$  such that  $u(x, y) + iv(x, y)$  is analytic in  $D$ . The function  $v$  is called the **harmonic conjugate function** of  $u$ .
- To find the harmonic conjugate function of a given function  $u$ :
  1. Take the first-order partial derivatives of  $u$  with respect to  $x$  and  $y$ .
  2. If  $u(x, y) + iv(x, y)$  is analytic in a domain  $D$  then  $u$  and  $v$  must satisfy the Cauchy-Riemann equations in  $D$  from which we can find expressions for  $\partial v / \partial x$  and  $\partial v / \partial y$ .
  3. Integrate  $\partial v / \partial x$  with respect to  $x$  to get an expression for  $v$  with an unknown constant  $h(y)$ .
  4. Take the first-order partial derivative of  $v$  with respect to  $y$ , equate it with the other expression for  $\partial v / \partial y$ , and solve for  $h'(y)$ .
  5. Integrate  $h'(y)$  and substitute the result to find  $v$ .

## 17.6 Exponential and Logarithmic Functions

- The exponential function for complex numbers is defined as

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

- $e^z$  is analytic for all  $z$ , i.e. it's an entire function.
- Like its real-valued counterpart,

$$\begin{aligned} \frac{d}{dz} e^z &= e^z, \\ e^{z_1} e^{z_2} &= e^{z_1+z_2}, \end{aligned}$$

and

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}.$$



- Since

$$e^{2\pi i} = \cos(2\pi) + i \sin(2\pi) = 1$$

and

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z$$

the complex function  $f(z) = e^z$  is **periodic** with complex period  $2\pi i$ . Because of this complex periodicity an infinite horizontal strip of height  $2\pi$  contains all possible values for the function. The strip  $-\pi < y \leq \pi$  is called the **fundamental region**.

- For  $z \neq 0$  and  $\theta = \arg z$ ,

$$\ln z = \log_e |z| + i(\theta + 2n\pi), \quad n = 0, \pm 1, \pm 2, \dots$$

This means there are infinitely many values of the logarithm of a complex number  $z$ . This makes sense as the complex exponential is periodic.

- The **principal value** of  $\ln z$  is the complex logarithm corresponding to  $n = 0$  and  $\theta = \text{Arg } z$ . It is denoted  $\text{Ln } z$ .
- Some familiar properties of the real-valued logarithm hold for the complex-valued logarithm, e.g.

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$

and

$$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2,$$

however they don't necessarily hold for the principal value.

- $\text{Ln } z$  is discontinuous and thus not analytic at  $z = 0$  because  $\ln z$  is undefined at  $z = 0$  and on the negative real axis because  $\text{Arg } z$  is discontinuous there.
- The derivative of  $\text{Ln } z$  is

$$\frac{d}{dz} \text{Ln } z = \frac{1}{z}.$$

- The complex power of a complex number is defined as

$$z^\alpha = e^{\alpha \ln z}, \quad z \neq 0.$$

In general this is multiple-valued because  $\ln z$  is multiple-valued — only if  $\alpha = n$ ,  $n = 0, \pm 1, \pm 2, \dots$  is it single-valued. If  $\ln z$  is replaced with  $\text{Ln } z$  then we get the **principle value** of  $z^\alpha$ .

## 17.7 Trigonometric and Hyperbolic Functions

### Definition 17.7.1 Trigonometric Sine and Cosine

For any complex number  $z = x + iy$ ,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \quad (2)$$

- The other trigonometric functions ( $\tan z$ , etc.) are defined as usual.
- Because  $e^{iz}$  and  $e^{-iz}$  are entire functions,  $\sin z$  and  $\cos z$  are also entire functions.
- $\sin z = 0$  for the real numbers  $z = n\pi$ ,  $n \in \mathbb{Z}$  and  $\cos z = 0$  for the real numbers  $z = (2n+1)\pi/2$ ,  $n \in \mathbb{Z}$ . This means that  $\tan z$  and  $\sec z$  are analytic except at the points where  $\cos z = 0$  and  $\cot z$  and  $\csc z$  are analytic except at the points where  $\sin z = 0$ .
- The usual derivatives and trigonometric functions are still valid in the complex case.
- $\sin z$  can be expressed as

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

and  $\cos z$  can be expressed as

$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$

- The only zeroes of  $\sin z$  are the real numbers  $z = n\pi$ ,  $n \in \mathbb{Z}$  and the only zeroes of  $\cos z$  are the real numbers  $z = (2n+1)\pi/2$ ,  $n \in \mathbb{Z}$ .

### Definition 17.7.2 Hyperbolic Sine and Cosine

For any complex number  $z = x + iy$ ,

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2}. \quad (10)$$

- The complex trigonometric functions can be expressed in terms of the complex hyperbolic functions and vice versa

$$\begin{aligned} \sin z &= -i \sinh(iz), & \cos z &= \cosh(iz) \\ \sinh z &= -i \sin(iz), & \cosh z &= \cos(iz). \end{aligned}$$

- $\sinh z$  can be expressed as

$$\sinh z = \sinh x \cos y + i \cosh x \sin y$$

and  $\cosh z$  can be expressed as

$$\cosh z = \cosh x \cos y + i \sinh x \sin y.$$

- The zeroes of  $\sinh z$  are  $z = n\pi i, n \in \mathbb{Z}$  and the zeroes of  $\cosh z$  are  $z = (2n + 1)\pi i/2, n \in \mathbb{Z}$ .
- $\sin z$  and  $\cos z$  are  $2\pi$  periodic while  $\sinh z$  and  $\cosh z$  are  $2\pi i$  periodic.