Contents

1 First-order ODEs

- **Form:** IVP

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

Test: $f(x, y)$ and $\partial f/\partial y$ are continuous over I
Property: A unique solution is guaranteed over I

1.1 Separable ODEs

- **Form:**

$$\frac{dy}{dx} = g(x)h(y)$$

Solution: Divide by $h(y)$ then integrate with respect to x.

$$\begin{aligned}
\frac{dy}{dx} &= g(x)h(y) \\
\frac{1}{h(y)}\frac{dy}{dx} &= g(x) \\
\int \frac{1}{h(y)}\frac{dy}{dx}\,dx &= \int g(x)\,dx \\
\int \frac{1}{h(y)}\,dy &= \int g(x)\,dx \\
H(y) &= G(x) + c
\end{aligned}$$

1.2 Linear Equations

- **Form:**

$$\frac{dy}{dx} + P(x)y = f(x)$$

Solution:

1. Determine the integrating factor $e^{\int P(x)\,dx}$

2. Multiply by the integrating factor
3. Recognise that the left hand side of the equation is the derivative of the product of the integrating factor and y
4. Integrate both sides of the equation
5. Solve for y

1.3 Exact equations

- **Form:**

$$z = f(x,y) = c$$

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = M(x,y)\,dx + N(x,y)\,dy = 0$$

Test:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution:

1. Integrate $M(x,y)$ with respect to x to find an expression for $z = f(x,y)$

$$\begin{aligned}\frac{\partial f}{\partial x} &= M(x,y)\\ f(x,y) &= \int M(x,y)\,dx + g(y)\end{aligned}$$

2. Differentiate $f(x,y)$ with respect to y and equate it to $N(x,y)$ to find $g'(y)$

$$\begin{aligned}\frac{\partial f}{\partial y} = N(x,y) &= \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y)\\ g'(y) &= N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx\end{aligned}$$

3. Integrate $g'(y)$ with respect to y to find $g(y)$ and substitute it into $f(x,y)$
4. Equate $f(x,y)$ with an unknown constant c

Note: The steps can be performed with x and y reversed, i.e. start by integrating $N(x,y)$ with respect to y, etc.

- **Form:**
$$M(x, y)\, dx + N(x, y)\, dy = 0$$

  Test: $(M_y - N_x)/N$ is a function of x alone or $(N_x - M_y)/M$ is a function of y alone
  Solution:

  1. Compute the integrating factor
  $$\mu(x) = e^{\int \frac{M_y - N_x}{N}\, dx}$$
  or
  $$\mu(y) = e^{\int \frac{N_x - M_y}{M}\, dy}$$
  as appropriate
  2. Multiple the equation by this factor
  3. The equation is now exact and can be solved as above