Advanced Engineering Mathematics Systems of Differential Equations by Dennis G. Zill Problems

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Contents

10 Systems of Linear Differential Equations

10.1 Theory of Linear Systems

10.1.1

$$\mathbf{X}' = \begin{pmatrix} 3 & -5 \\ 4 & 8 \end{pmatrix} \mathbf{X}$$

10.1.3

$$\mathbf{X}' = \begin{pmatrix} -3 & 4 & -9 \\ 6 & -1 & 0 \\ 10 & 4 & 3 \end{pmatrix} \mathbf{X}$$

10.1.5

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} t-1 \\ -3t^2 \\ t^2 - t + 2 \end{pmatrix}$$

10.1.7

$$\begin{aligned} \frac{dx}{dt} &= 4x + 2y + e^t \\ \frac{dy}{dt} &= -x + 3y - e^t \end{aligned}$$

10.1.9

$$\begin{aligned} \frac{dx}{dt} &= x - y + 2z + e^{-t} - 3t \\ \frac{dy}{dt} &= 3x - 4y + z + 2e^{-t} + t \\ \frac{dz}{dt} &= -2x + 5y + 6z + 2e^{-t} - t \end{aligned}$$

10.1.11

$$\begin{aligned} 3(e^{-5t}) - 4(2e^{-5t}) &= -5e^{-5t} \\ &= \frac{dx}{dt} \\ 4(e^{-5t}) - 7(2e^{-5t}) &= -10e^{-5t} \\ &= \frac{dy}{dt} \end{aligned}$$

10.1.13

$$\begin{aligned} -(-e^{-3t/2}) + \frac{1}{4}(2e^{-3t/2}) &= \frac{3}{2}e^{-3t/2} \\ &= \frac{dx}{dt} \\ (-e^{-3t/2}) - (2e^{-3t/2}) &= -3e^{-3t/2} \\ &= \frac{dy}{dt} \end{aligned}$$

10.1.17

$$\begin{aligned} W(\mathbf{X}_1, \mathbf{X}_2) &= \begin{vmatrix} e^{-2t} & e^{-6t} \\ e^{-2t} & -e^{-6t} \end{vmatrix} \\ &= -e^{-8t} - e^{-8t} \\ &= -2e^{-8t} \\ &\neq 0 \text{ for } t \in (-\infty, \infty) \end{aligned}$$

Yes, they form a fundamental set.

10.1.19

$$\begin{aligned} W(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) &= \begin{vmatrix} 1+t & 1 & 3+2t \\ -2+2t & -2 & -6+4t \\ 4+2t & 4 & 12+4t \end{vmatrix} \\ &= 0 \end{aligned}$$

No, they don't form a fundamental set.

10.1.21

$$\begin{aligned} x &= 2t + 5 \\ y &= -t + 1 \\ \frac{dx}{dt} &= (2t+5) + 4(-t+1) + 2t - 7 \\ &= 2 \\ \frac{dy}{dt} &= 3(2t+5) + 2(-t+1) - 4t - 18 \\ &= -1 \end{aligned}$$

10.1.23

$$\begin{aligned} x &= e^t + te^t \\ x' &= 2e^t + te^t \\ y &= e^t - te^t \\ y' &= -te^t \\ \frac{dx}{dt} &= 2(e^t + te^t) + (e^t - te^t) - e^t \\ &= 2e^t + te^t \\ \frac{dy}{dt} &= 3(e^t + te^t) + 4(e^t - te^t) - 7e^t \\ &= -te^t \end{aligned}$$

10.2 Homogeneous Linear Systems

10.2.1

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

10.2.3

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} e^{t}$$

10.2.5

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-10t} + c_2 \begin{pmatrix} 5 \\ 2 \end{pmatrix} e^{8t}$$

10.2.7

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t}$$

10.2.13

$$\mathbf{X} = 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/2} + 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t/2}$$

10.2.15

(a)

$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{3}{100}x_1 + \frac{1}{100}x_2 \\ \frac{dx_2}{dt} &= \frac{2}{100}x_1 - \frac{2}{100}x_2 \\ \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} &= \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{2}{100} & -\frac{2}{100} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

(b)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\frac{35}{3} \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t/25} + \frac{50}{3} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} e^{-t/100}$$

10.2.21

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \left[\begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right]$$

10.2.23

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix} \right] e^{2t}$$

10.2.25

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{t}$$

10.2.31

$$\mathbf{X} = -\frac{1}{2} \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + 13 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \right] e^{4t}$$

10.2.33

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{K}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

10.2.35

$$\begin{aligned} \mathbf{X} &= c_1 \begin{pmatrix} 1 \\ 2-i \end{pmatrix} e^{(4+i)t} + c_2 \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{(4-i)t} \\ &= c_1 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin t \right] e^{4t} + c_2 \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t \right] e^{4t} \\ &= c_1 \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix} e^{4t} \end{aligned}$$

10.2.37

$$\begin{aligned}\mathbf{X} &= c_1\begin{pmatrix}1\\-1+i\end{pmatrix}e^{(4+i)t}+c_2\begin{pmatrix}1\\-1-i\end{pmatrix}e^{(4-i)t}\\ &= c_1\left[\begin{pmatrix}1\\-1\end{pmatrix}\cos t-\begin{pmatrix}0\\1\end{pmatrix}\sin t\right]e^{4t}+c_2\left[\begin{pmatrix}0\\1\end{pmatrix}\cos t+\begin{pmatrix}1\\-1\end{pmatrix}\sin t\right]e^{4t}\\ &= c_1\begin{pmatrix}\cos t\\-\cos t-\sin t\end{pmatrix}e^{4t}+c_2\begin{pmatrix}\sin t\\\cos t-\sin t\end{pmatrix}e^{4t}\end{aligned}$$

10.2.39

$$\begin{aligned}\mathbf{X} &= c_1\begin{pmatrix}5\\4-3i\end{pmatrix}e^{3i}+c_2\begin{pmatrix}5\\4+3i\end{pmatrix}e^{-3i}\\ &= c_1\left[\begin{pmatrix}5\\4\end{pmatrix}\cos 3t-\begin{pmatrix}0\\-3\end{pmatrix}\sin 3t\right]+c_2\left[\begin{pmatrix}0\\-3\end{pmatrix}\cos 3t+\begin{pmatrix}5\\4\end{pmatrix}\sin 3t\right]\\ &= c_1\begin{pmatrix}5\cos 3t\\4\cos 3t+3\sin 3t\end{pmatrix}+c_2\begin{pmatrix}5\sin 3t\\4\sin 3t-3\cos 3t\end{pmatrix}\end{aligned}$$

10.2.47

$$\begin{aligned}\mathbf{X} &= c_1\begin{pmatrix}25\\-7\\6\end{pmatrix}e^{t}+c_2\begin{pmatrix}1+5i\\1\\1\end{pmatrix}e^{5it}+c_3\begin{pmatrix}1-5i\\1\\1\end{pmatrix}e^{-5it}\\ &= c_1\begin{pmatrix}25\\-7\\6\end{pmatrix}e^{t}+c_2\left[\begin{pmatrix}1\\1\\1\end{pmatrix}\cos 5t-\begin{pmatrix}5\\0\\0\end{pmatrix}\sin 5t\right]\\ &\quad +c_3\left[\begin{pmatrix}5\\0\\0\end{pmatrix}\cos 5t+\begin{pmatrix}1\\1\\1\end{pmatrix}\sin 5t\right]\\ &= c_1\begin{pmatrix}25\\-7\\6\end{pmatrix}e^{t}+c_2\begin{pmatrix}\cos 5t-5\sin 5t\\\cos 5t\\\cos 5t\end{pmatrix}+c_3\begin{pmatrix}5\cos 5t+\sin 5t\\\sin 5t\\\sin 5t\end{pmatrix}\\ &= -\begin{pmatrix}25\\-7\\6\end{pmatrix}e^{t}-\begin{pmatrix}\cos 5t-5\sin 5t\\\cos 5t\\\cos 5t\end{pmatrix}+6\begin{pmatrix}5\cos 5t+\sin 5t\\\sin 5t\\\sin 5t\end{pmatrix}\end{aligned}$$

10.2.49

(a)

$$\begin{pmatrix}x_1'\\x_2'\\x_3'\end{pmatrix}=\begin{pmatrix}-\frac{1}{20}&0&\frac{1}{10}\\\frac{1}{20}&-\frac{1}{20}&0\\0&\frac{1}{20}&-\frac{1}{10}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

(b)

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1-i \\ i \\ 1 \end{pmatrix} e^{\left(-\frac{1}{10}+\frac{1}{20}i\right)t} + c_3 \begin{pmatrix} -1+i \\ -i \\ 1 \end{pmatrix} e^{\left(-\frac{1}{10}-\frac{1}{20}i\right)t}$$

$$= c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \left[\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cos \frac{1}{20}t - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \sin \frac{1}{20}t \right] e^{-t/10}$$

$$+ c_3 \left[\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cos \frac{1}{20}t + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \sin \frac{1}{20}t \right] e^{-t/10}$$

$$= c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \sin \frac{t}{20} - \cos \frac{t}{20} \\ -\sin \frac{t}{20} \\ \cos \frac{t}{20} \end{pmatrix} e^{-t/10}$$

$$+ c_3 \begin{pmatrix} -\cos \frac{t}{20} - \sin \frac{t}{20} \\ \cos \frac{t}{20} \\ \sin \frac{t}{20} \end{pmatrix} e^{-t/10}$$

$$= 11 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - 6 \begin{pmatrix} \sin \frac{t}{20} - \cos \frac{t}{20} \\ -\sin \frac{t}{20} \\ \cos \frac{t}{20} \end{pmatrix} e^{-t/10}$$

$$- 2 \begin{pmatrix} -\cos \frac{t}{20} - \sin \frac{t}{20} \\ \cos \frac{t}{20} \\ \sin \frac{t}{20} \end{pmatrix} e^{-t/10}$$

10.3 Solution by Diagonalization

10.3.1

$$\mathbf{X} = \begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 e^{7t} \\ c_2 e^{-4t} \end{pmatrix} = \begin{pmatrix} 3c_1 e^{7t} - 2c_2 e^{-4t} \\ c_1 e^{7t} + 3c_2 e^{-4t} \end{pmatrix}$$

10.3.3

$$\mathbf{X} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_1 e^{3t/2} \\ c_2 e^{t/2} \end{pmatrix} = \begin{pmatrix} c_1 e^{3t/2} + c_2 e^{t/2} \\ 2c_1 e^{3t/2} - 2c_2 e^{t/2} \end{pmatrix}$$

10.3.5

$$\mathbf{X} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{6t} \\ c_2 e^{-4t} \\ c_3 e^{2t} \end{pmatrix} = \begin{pmatrix} -c_2 e^{-4t} + c_3 e^{2t} \\ c_2 e^{-4t} + c_3 e^{2t} \\ c_1 e^{6t} + c_3 e^{2t} \end{pmatrix}$$

10.3.11

(a)

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\mathbf{M}$ has an inverse because it has a nonzero determinant (the product of the diagonal entries).

(b)

$$\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & \frac{k_2}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

(c)

$$\mathbf{X} = \mathbf{PY}$$

$$\begin{aligned} \mathbf{PY}'' + \mathbf{BPY} &= \mathbf{0} \\ \mathbf{Y}'' + \mathbf{P}^{-1}\mathbf{BPY} &= \mathbf{0} \\ \mathbf{Y}'' + \mathbf{DY} &= \mathbf{0} \\ \begin{pmatrix} y_1'' \\ y_2'' \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \mathbf{0} \\ y_1'' + 6y_1 &= 0 \\ y_1 &= c_1 \cos\sqrt{6}t + c_2 \sin\sqrt{6}t \\ y_2'' + y_2 &= 0 \\ y_2 &= c_3 \cos t + c_4 \sin t \end{aligned}$$

$$\begin{aligned} \mathbf{X} &= \mathbf{PY} \\ &= \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \cos\sqrt{6}t + c_2 \sin\sqrt{6}t \\ c_3 \cos t + c_4 \sin t \end{pmatrix} \\ &= \begin{pmatrix} -2c_1 \cos\sqrt{6}t - 2c_2 \sin\sqrt{6}t + c_3 \cos t + c_4 \sin t \\ c_1 \cos\sqrt{6}t + c_2 \sin\sqrt{6}t + 2c_3 \cos t + 2c_4 \sin t \end{pmatrix} \\ &= c_3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos t + c_4 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t + c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cos\sqrt{6}t + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \sin\sqrt{6}t \end{aligned}$$

10.4 Nonhomogeneous Linear Systems

10.4.1

$$\begin{aligned}
\begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -7 \\ 5 \end{pmatrix} \\
\mathbf{X}_c &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} \\
\mathbf{X}_p &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} -7 \\ 5 \end{pmatrix} \\
&= \begin{pmatrix} 2a_1 + 3a_2 - 7 \\ -a_1 - 2a_2 + 5 \end{pmatrix} \\
\mathbf{X}_p &= \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\
\mathbf{X} &= \mathbf{X}_c + \mathbf{X}_p \\
&= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}
\end{aligned}$$

10.4.3

$$\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} \\
\mathbf{X}_p &= \begin{pmatrix} a_3t^2 + a_2t + a_1 \\ b_3t^2 + b_2t + b_1 \end{pmatrix} \\
\begin{pmatrix} 2a_3t + a_2 \\ 2b_3t + b_2 \end{pmatrix} &= \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a_3t^2 + a_2t + a_1 \\ b_3t^2 + b_2t + b_1 \end{pmatrix} + \begin{pmatrix} -2t^2 \\ t+5 \end{pmatrix} \\
&= \begin{pmatrix} (a_3 + 3b_3 - 2)t^2 + (a_2 + 3b_2)t + (a_1 + 3b_1) \\ (3a_3 + b_3)t^2 + (3a_2 + b_2 + 1)t + (3a_1 + b_1 + 5) \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} (a_3 + 3b_3 - 2)t^2 + (a_2 - 2a_3 + 3b_2)t + (a_1 - a_2 + 3b_1) \\ (3a_3 + b_3)t^2 + (3a_2 + b_2 - 2b_3 + 1)t + (3a_1 + b_1 - b_2 + 5) \end{pmatrix} \\
a_3 &= -\frac{1}{4} \\
b_3 &= \frac{3}{4} \\
a_2 &= \frac{1}{4} \\
b_2 &= -\frac{1}{4} \\
a_1 &= -2 \\
b_1 &= \frac{3}{4} \\
\mathbf{X}_p &= \begin{pmatrix} -\frac{1}{4}t^2 + \frac{1}{4}t - 2 \\ \frac{3}{4}t^2 - \frac{1}{4}t + \frac{3}{4} \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} + \begin{pmatrix} -\frac{1}{4}t^2 + \frac{1}{4}t - 2 \\ \frac{3}{4}t^2 - \frac{1}{4}t + \frac{3}{4} \end{pmatrix}
\end{aligned}$$

10.4.5

$$
\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} \frac{1}{9} \\ 1 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} e^{3t} \\
\mathbf{X}_p &= \begin{pmatrix} a \\ b \end{pmatrix} e^{t} \\
\begin{pmatrix} a \\ b \end{pmatrix} e^{t} &= \begin{pmatrix} 4 & \frac{1}{3} \\ 9 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} e^{t} + \begin{pmatrix} -3 \\ 10 \end{pmatrix} e^{t} \\
\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 4a + \frac{1}{3}b - 3 \\ 9a + 6b + 10 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 3a + \frac{1}{3}b - 3 \\ 9a + 5b + 10 \end{pmatrix} \\
\mathbf{X}_p &= \begin{pmatrix} \frac{55}{36} \\ -\frac{19}{4} \end{pmatrix} e^{t} \\
\mathbf{X} &= c_1 \begin{pmatrix} \frac{1}{9} \\ 1 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} \frac{55}{36} \\ -\frac{19}{4} \end{pmatrix} e^{t}
\end{aligned}
$$

10.4.9

$$
\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t} \\
\mathbf{X}_p &= \begin{pmatrix} a \\ b \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} -a - 2b + 3 \\ 3a + 4b + 3 \end{pmatrix} \\
\mathbf{X}_p &= \begin{pmatrix} -9 \\ 6 \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} -9 \\ 6 \end{pmatrix} \\
\begin{pmatrix} -4 \\ 5 \end{pmatrix} &= c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -9 \\ 6 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -2c_1 - c_2 - 5 \\ 3c_1 + c_2 + 1 \end{pmatrix} \\
\mathbf{X} &= 4 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{2t} - 13 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} -9 \\ 6 \end{pmatrix}
\end{aligned}
$$

10.4.11

(a)

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{1}{50} & -\frac{1}{25} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(b)

$$\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} e^{-t/20} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/50} \\
\mathbf{X}_p &= \begin{pmatrix} a \\ b \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{1}{50} & -\frac{1}{25} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{3}{100}a + \frac{1}{100}b \\ \frac{1}{50}a - \frac{1}{25}b + 1 \end{pmatrix} \\
\mathbf{X}_p &= \begin{pmatrix} 10 \\ 30 \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} e^{-t/20} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/50} + \begin{pmatrix} 10 \\ 30 \end{pmatrix} \\
\begin{pmatrix} 60 \\ 10 \end{pmatrix} &= c_1 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 10 \\ 30 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -\frac{1}{2}c_1 + c_2 - 50 \\ c_1 + c_2 + 20 \end{pmatrix} \\
\mathbf{X} &= -\frac{140}{3} \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} e^{-t/20} + \frac{80}{3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/50} + \begin{pmatrix} 10 \\ 30 \end{pmatrix}
\end{aligned}$$

(c)

$$\begin{aligned}
\lim_{t\to\infty} x_1(t) &= \lim_{t\to\infty} \frac{70}{3} e^{-t/20} + \frac{80}{3} e^{-t/50} + 10 \\
&= 10 \\
\lim_{t\to\infty} x_1(t) &= \lim_{t\to\infty} -\frac{140}{3} e^{-t/20} + \frac{80}{3} e^{-t/50} + 30 \\
&= 30
\end{aligned}$$

10.4.13

$$
\begin{aligned}
\begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} \\
\mathbf{X}_c &= c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathbf{X}_p &= \mathbf{\Phi}(t) \int \mathbf{\Phi}^{-1}(t) \mathbf{F} \, dt \\
&= \begin{pmatrix} 3e^t & 1 \\ 2e^t & 1 \end{pmatrix} \int \begin{pmatrix} e^{-t} & -e^{-t} \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \end{pmatrix} \, dt \\
&= \begin{pmatrix} 3e^t & 1 \\ 2e^t & 1 \end{pmatrix} \int \begin{pmatrix} 5e^{-t} \\ -11 \end{pmatrix} \, dt \\
&= \begin{pmatrix} 3e^t & 1 \\ 2e^t & 1 \end{pmatrix} \begin{pmatrix} -5e^{-t} \\ -11t \end{pmatrix} \\
&= \begin{pmatrix} -15 - 11t \\ -10 - 11t \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t + c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -15 - 11t \\ -10 - 11t \end{pmatrix}
\end{aligned}
$$

10.4.15

$$
\begin{aligned}
\mathbf{X}' &= \begin{pmatrix} 3 & -5 \\ \frac{3}{4} & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2} \\
\mathbf{X}_c &= c_1 \begin{pmatrix} \frac{10}{3} \\ 1 \end{pmatrix} e^{3t/2} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t/2} \\
\mathbf{X}_p &= \begin{pmatrix} \frac{1}{2}(-15 - 13t) \\ \frac{1}{4}(-9 - 13t) \end{pmatrix} e^{t/2} \\
\mathbf{X} &= c_1 \begin{pmatrix} \frac{10}{3} \\ 1 \end{pmatrix} e^{3t/2} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t/2} + \frac{1}{4} \begin{pmatrix} -30 - 26t \\ -9 - 13t \end{pmatrix} e^{t/2}
\end{aligned}
$$

10.4.33

$$
\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} \\
\mathbf{X}_p &= \begin{pmatrix} 2e^{2t}t - 2e^{4t}t - e^{2t} + e^{4t} \\ 2e^{2t}t + 2e^{4t}t + e^{2t} + e^{4t} \end{pmatrix} \\
&= \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} \\
\mathbf{X} &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} \\
&\quad + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} \\
\begin{pmatrix} 1 \\ 1 \end{pmatrix} &= c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -c_1 + c_2 - 1 \\ c_1 + c_2 + 1 \end{pmatrix} \\
\mathbf{X} &= \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{4t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t}
\end{aligned}
$$

10.4.35

$$
\begin{aligned}
\mathbf{X}_c &= c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-12t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} \\
\mathbf{X}_p &= \frac{1}{29} \begin{pmatrix} -76\cos t + 332\sin t \\ -168\cos t + 276\sin t \end{pmatrix} \\
\mathbf{X} &= c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-12t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} - \frac{4}{29} \begin{pmatrix} 19 \\ 42 \end{pmatrix} \cos t + \frac{4}{29} \begin{pmatrix} 83 \\ 69 \end{pmatrix} \sin t \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{4}{29} \begin{pmatrix} 19 \\ 42 \end{pmatrix} \\
&= \begin{pmatrix} -3c_1 + c_2 - \frac{76}{29} \\ c_1 + 3c_2 - \frac{168}{29} \end{pmatrix} \\
\mathbf{X} &= -\frac{6}{29} \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-12t} + 2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} - \frac{4}{29} \begin{pmatrix} 19 \\ 42 \end{pmatrix} \cos t + \frac{4}{29} \begin{pmatrix} 83 \\ 69 \end{pmatrix} \sin t
\end{aligned}
$$

10.4.37

$$\mathbf{A} = \begin{pmatrix} 5 & -2 \\ 21 & -8 \end{pmatrix}$$
$$\mathbf{F} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$
$$\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 7 & 3 \end{pmatrix}$$
$$\mathbf{G} = \begin{pmatrix} -14 \\ 34 \end{pmatrix}$$
$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -2y_1 - 14 \\ -y_2 + 34 \end{pmatrix}$$
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-2t} - 7 \\ c_2 e^{-t} + 34 \end{pmatrix}$$
$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 7 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + \begin{pmatrix} 20 \\ 53 \end{pmatrix}$$

10.4.39

$$\mathbf{A} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$$
$$\mathbf{P} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$\mathbf{G} = \begin{pmatrix} 4 + t \\ 4 - t \end{pmatrix}$$
$$\begin{aligned} \mathbf{Y}' &= \mathbf{DY} + \mathbf{G} \\ &= \begin{pmatrix} 10y_1 + 4 + t \\ 4 - t \end{pmatrix} \\ \mathbf{Y} &= \begin{pmatrix} c_1 e^{10t} - \frac{1}{10}t - \frac{41}{100} \\ -\frac{1}{2}t^2 + 4t + c_2 \end{pmatrix} \\ \mathbf{X} &= \mathbf{PY} \\ &= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{10t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} t^2 + \begin{pmatrix} -\frac{41}{10} \\ \frac{39}{10} \end{pmatrix} t - \frac{41}{100} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

10.5 Matrix Exponential

10.5.1

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$$
$$\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

10.5.3

$$\begin{pmatrix} t+1 & t & t \\ t & t+1 & t \\ -2t & -2t & -2t+1 \end{pmatrix}$$

10.5.5

$$\begin{pmatrix} c_1 e^{t} \\ c_2 e^{2t} \end{pmatrix}$$

10.5.7

$$\mathbf{X} = c_1 \begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} + c_2 \begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + c_3 \begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix}$$

10.5.9

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -3 \\ \frac{1}{2} \end{pmatrix}$$

10.5.11

$$\mathbf{X} = c_1 \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2 \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

10.5.13

$$\mathbf{X} = c_1 \begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} + c_2 \begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + c_3 \begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -4 \\ 6 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} - 4 \begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + 6 \begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix}$$

10.5.15

$$e^{\mathbf{A}t} = \begin{pmatrix} \frac{1}{2}e^{-2t}(3e^{4t}-1) & \frac{3}{4}e^{-2t}(e^{4t}-1) \\ e^{-2t}-e^{2t} & -\frac{1}{2}e^{-2t}(e^{4t}-3) \end{pmatrix}$$

10.5.17

$$e^{\mathbf{A}t} = \begin{pmatrix} e^{2t}(1+3t) & -9e^{2t}t \\ e^{2t}t & e^{2t}(1-3t) \end{pmatrix}$$

10.5.25

$$\begin{aligned}
\mathbf{X} &= e^{\mathbf{A}t}\mathbf{C} \\
&= \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}\mathbf{C} \\
&= \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} e^{5t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \mathbf{C} \\
&= \begin{pmatrix} -\frac{1}{2}e^{3t}(-3+e^{2t}) & \frac{1}{2}e^{3t}(-1+e^{2t}) \\ -\frac{3}{2}e^{3t}(-1+e^{2t}) & \frac{1}{2}e^{3t}(-1+3e^{2t}) \end{pmatrix} \mathbf{C}
\end{aligned}$$