Advanced Engineering Mathematics Ordinary Differential Equations Notes

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Contents

1 Introduction to Differential Equations

1.1 Definitions and Terminology

1.1.1 1

2, linear

1.1.2 3

4, linear

1.1.3 5

2, nonlinear

1.1.4 7

3, linear

1.1.5 9

no; yes

1.1.6 15

The domain of the function is $x \in [-2, \infty)$.

$$y' = 1 + \frac{2}{\sqrt{x+2}}$$

The largest interval of definition of the solution is $x \in (-2, \infty)$.

$$(y-x)y' = y - x + 8$$
$$(x + 4\sqrt{x+2} - x)(1 + \frac{2}{\sqrt{x+2}}) = x + 4\sqrt{x+2} - x + 8$$
$$4\sqrt{x+2} + 8 = 4\sqrt{x+2} + 8$$

1.1.7 17

The domain of the function is $x \in \mathbb{R}, x \neq \pm 2$.

$$y' = \frac{2x}{(4-x^2)^2}$$

The largest intervals of definition of the solution are $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

$$\begin{aligned} y' &= 2xy^2 \\ \frac{2x}{(4-x^2)^2} &= 2x\left(\frac{1}{4-x^2}\right)^2 \\ &= \frac{2x}{(4-x^2)^2} \end{aligned}$$

1.1.8 19

$$\begin{aligned} ln\frac{2X-1}{X-1} &= t \\ 2X-1 &= (X-1)e^t \\ (2-e^t)X &= 1-e^t \\ X &= \frac{e^t-1}{e^t-2} \end{aligned}$$

The solutions intervals of validity are $(\infty, \ln 2)$ and $(\ln 2, \infty)$.

$$\begin{aligned} \frac{dX}{dt} &= (X-1)(1-2X) \\ \frac{e^t}{e^t-2} - \frac{e^t(e^t-1)}{(e^t-2)^2} &= \left(\frac{e^t-1}{e^t-2}-1\right)\left(1-2\frac{e^t-1}{e^t-2}\right) \\ \frac{e^t(e^t-2)-e^t(e^t-1)}{(e^t-2)^2} &= \left(\frac{e^t-1-e^t+2}{e^t-2}\right)\left(\frac{e^t-2-2e^t+2}{e^t-2}\right) \\ \frac{e^{2t}-2e^t-e^{2t}+e^t}{(e^t-2)^2} &= \left(\frac{1}{e^t-2}\right)\left(\frac{-e^t}{e^t-2}\right) \\ \frac{-e^t}{(e^t-2)^2} &= \frac{-e^t}{(e^t-2)^2} \end{aligned}$$

1.1.9 31

$m = -2$

1.1.10 33

$m = 2$ or 3

1.1.11 35

$m = -1$ or 0

1.1.12 37

$y = 2$

1.1.13 39

No constant solutions

1.2 Initial Value Problems

1.2.1 1

$$\begin{aligned} y(0) = -\frac{1}{3} &= \frac{1}{1 + c_1 e^{-(0)}} \\ -3 &= 1 + c_1 \\ c_1 &= -4 \end{aligned}$$

$$y = \frac{1}{1 - 4e^{-x}}$$

1.2.2 3

$$\begin{aligned} y(2) = \frac{1}{3} &= \frac{1}{(2)^2 + c} \\ 3 &= 4 + c \\ c &= -1 \end{aligned}$$

$$y = \frac{1}{x^2 - 1}$$

$I = (1, \infty)$

1.2.3 5

$$\begin{aligned} y(0) = 1 &= \frac{1}{(0)^2 + c} \\ c &= 1 \end{aligned}$$

$$y = \frac{1}{x^2 + 1}$$

$I = (-\infty, \infty)$

1.2.4 7

$$\begin{aligned} x(0) = -1 &= c_1 \cos 0 + c_2 \sin 0 \\ c_1 &= -1 \end{aligned}$$

$$\begin{aligned} x'(0) = 8 &= -c_1 \sin 0 + c_2 \cos 0 \\ c_2 &= 8 \end{aligned}$$

$$x = -\cos t + 8 \sin t$$

1.2.5 9

$$\begin{aligned} x'\left(\frac{\pi}{6}\right) = 0 &= -c_1 \sin\frac{\pi}{6} + c_2 \cos\frac{\pi}{6} \\ &= -c_1 \frac{1}{2} + c_2 \frac{\sqrt{3}}{2} \\ c_1 &= \sqrt{3} c_2 \end{aligned}$$

$$\begin{aligned} x\left(\frac{\pi}{6}\right) = \frac{1}{2} &= c_1 \cos\frac{\pi}{6} + c_2 \sin\frac{\pi}{6} \\ &= \frac{3}{2} c_2 + \frac{1}{2} c_2 \\ &= 2c_2 \\ c_2 &= \frac{1}{4} \end{aligned}$$

$$y = \frac{\sqrt{3}}{4} \cos t + \frac{1}{4} \sin t$$

1.2.6 11

$$\begin{aligned} y(0) = 1 &= c_1 e^{(0)} + c_2 e^{-(0)} \\ &= c_1 + c_2 \\ c_1 &= 1 - c_2 \end{aligned}$$

$$\begin{aligned} y'(0) = 2 &= c_1 e^{(0)} - c_2 e^{-(0)} \\ &= 1 - c_2 - c_2 \\ c_2 &= -\frac{1}{2} \end{aligned}$$

$$y = \frac{3}{2} e^{x} - \frac{1}{2} e^{-x}$$

1.2.7 13

$$
\begin{aligned}
y(-1) = 5 &= c_1 e^{(-1)} + c_2 e^{-(-1)} \\
&= c_1 e^{-1} + c_2 e \\
c_1 &= 5e - c_2 e^2
\end{aligned}
$$

$$
\begin{aligned}
y'(-1) = -5 &= c_1 e^{(-1)} - c_2 e^{-(-1)} \\
&= 5e - c_2 e^2 - c_2 e \\
c_2 e(e+1) &= 5(e+1) \\
c_2 &= \frac{5}{e}
\end{aligned}
$$

$$y = 5e^{-x-1}$$

1.2.8 15

$$y = 0$$

$$y = x^3$$

1.2.9 17

$$f(x, y) = y^{2/3}$$

$$\frac{\partial f}{\partial y} = \frac{2}{3y^{1/3}}$$

$y < 0$ or $y > 0$

1.2.10 19

$$f(x, y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$x < 0$ or $x > 0$

1.2.11 21

$$f(x, y) = \frac{x^2}{4 - y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2x^2 y}{(4 - y^2)^2}$$

$y < -2$, $-2 < y < 2$, or $y > 2$

1.2.12 23

$$f(x,y) = \frac{y^2}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2} - \frac{2y^3}{(x^2+y^2)^2}$$

$x \neq 0$ and $y \neq 0$

1.2.13 25

$$f(x,y) = \sqrt{y^2-9}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{y^2-9}}$$

Yes

1.2.14 27

No

1.2.15 29

(a) $y = cx$

(b)

$$f(x,y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$x \neq 0$

(c) No, the function is not differentiable at $x = 0$

1.2.16 31

(a)

$$y' = \frac{1}{(x+c)^2} = y^2$$

(b)

$$y(0) = 1 = -\frac{1}{(0)+c} \Rightarrow c = -1 \Rightarrow y = \frac{1}{1-x}$$

$I = (-\infty, 1)$

$$y(0) = -1 = -\frac{1}{(0)+c} \Rightarrow c = 1 \Rightarrow y = -\frac{1}{x+1}$$

$I = (-1, \infty)$

1.2.17 39

$$\begin{aligned} y(0) = 0 &= c_1 \cos 3(0) + c_2 \sin 3(0) \\ c_1 &= 0 \end{aligned}$$

$$\begin{aligned} y\left(\frac{\pi}{6}\right) = -1 &= c_2 \sin 3\left(\frac{\pi}{6}\right) \\ c_2 &= -1 \end{aligned}$$

$$y = -\sin 3x$$

1.2.18 41

$$\begin{aligned} y'(0) = 0 &= -3c_1 \sin 3(0) + 3c_2 \cos 3(0) \\ c_2 &= 0 \end{aligned}$$

$$\begin{aligned} y'\left(\frac{\pi}{4}\right) = 0 &= -3c_1 \sin 3\left(\frac{\pi}{4}\right) \\ &= -\frac{3}{\sqrt{2}}c_1 \\ c_1 &= 0 \end{aligned}$$

$$y = 0$$

1.2.19 43

$$\begin{aligned} y(0) = 0 &= c_1 \cos 3(0) + c_2 \sin 3(0) \\ c_1 &= 0 \end{aligned}$$

$$\begin{aligned} y(\pi) = 4 &= c_2 \sin 3(\pi) \\ 4 &= 0 \end{aligned}$$

No solution

1.3 Differential Equations as Mathematical Models

1.3.1 1

$$\frac{dP}{dt} = kP + r$$

$$\frac{dP}{dt} = kP - r$$

1.3.2 3

$$\frac{dP}{dt} = k_b P - k_d P^2$$

1.3.3 7

$$\frac{dx}{dt} = kx(1000 - x)$$

1.3.4 9

$$\frac{dA}{dt} = -\frac{A}{100}$$

$$A(0) = 50\,\mathrm{lb}$$

1.3.5 11

$$\frac{dA}{dt} + \frac{7}{600 - t}A = 6$$

1.3.6 13

$$\begin{aligned}
\frac{dV}{dt} &= -cA_h\sqrt{2gh} \\
A_w\frac{dh}{dt} &= -cA_h\sqrt{2gh} \\
\frac{dh}{dt} &= -\frac{cA_h\sqrt{2g}}{A_w}\sqrt{h} \\
&= -\frac{c\pi r_h^2\sqrt{2g}}{A_w}\sqrt{h} \\
&= -\frac{c\pi}{430}\sqrt{h}
\end{aligned}$$

1.3.7 15

$$L\frac{di}{dt} + Ri = E$$

1.3.8 17

$$m\frac{dv}{dt} = mg - kv^2$$

1.3.9 19

$$m\frac{d^2x}{dt^2} = -kx$$

1.3.10 21

$$\begin{aligned} \frac{d}{dt}(mv) &= R - kv \\ \frac{dm}{dt}v + m\frac{dv}{dt} &= R - kv - mg \end{aligned}$$

1.3.11 23

$$g = \frac{k}{R^2} \Rightarrow k = gR^2$$

$$\frac{d^2r}{dt^2} = -\frac{gR^2}{r^2}$$

1.3.12 25

$$\frac{dA}{dt} = k(M - A)$$

1.3.13 27

$$\frac{dx}{dt} = r - kx$$

1.3.14 29

$$\begin{aligned} \frac{dy}{dx} &= \tan\theta \\ &= \tan\frac{\phi}{2} \\ &= \frac{1-\cos\phi}{\sin\phi} \\ &= \frac{1-x/r}{y/r} \\ &= \frac{r-x}{y} \\ &= \frac{\sqrt{x^2+y^2}-x}{y} \end{aligned}$$

1.4 Chapter in Review

1.4.1 1

$$\frac{dy}{dx} = ky$$

1.4.2 3

$$y'' + k^2 y = 0$$

1.4.3 5

$$y = c_1 e^x + c_2 x e^x$$

$$\begin{aligned} y' &= c_1 e^x + c_2 e^x + c_2 x e^x \\ &= y + c_2 e^x \end{aligned}$$

$$\begin{aligned} y'' &= c_1 e^x + c_2 e^x + c_2 e^x + c_2 x e^x \\ &= c_1 e^x + 2c_2 e^x + c_2 x e^x \\ &= y' + c_2 e^x \end{aligned}$$

$$y'' - 2y' + y = 0$$

1.4.4 7

a, d

1.4.5 9

b

1.4.6 11

b

1.4.7 13

$$y = ce^x$$

1.4.8 15

$$\frac{dy}{dx} = x^2 + y^2$$

1.4.9 17

(a) $(-\infty, \infty)$

(b) $(-\infty, 0)$ or $(0, \infty)$

1.4.10 19

$x_0 = -1$ and $I = (-\infty, 0)$ or $x_0 = 2$ and $I = (0, \infty)$

1.4.11 23

$$y = x\sin x + x\cos x$$

$$y' = \sin x + x\cos x + \cos x - x\sin x$$

$$\begin{aligned} y'' &= \cos x + \cos x - x\sin x - \sin x - \sin x - x\cos x \\ &= 2\cos x - 2\sin x - x\sin x - x\cos x \end{aligned}$$

$$\begin{aligned} y'' + y &= 2\cos x - 2\sin x - x\sin x - x\cos x + x\sin x + x\cos x \\ &= 2\cos x - 2\sin x \end{aligned}$$

$I = (-\infty, \infty)$

1.4.12 25

$$y = \sin(\ln x)$$

$$y' = \frac{1}{x}\cos(\ln x)$$

$$y'' = -\frac{1}{x^2}\cos(\ln x) - \frac{1}{x^2}\sin(\ln x)$$

$$\begin{aligned} x^2y'' + xy' + y &= -\cos(\ln x) - \sin(\ln x) + \cos(\ln x) + \sin(\ln x) \\ &= 0 \end{aligned}$$

$I = (0, \infty)$

1.4.13 35

$$\begin{aligned} y(0) = 0 &= c_1 e^{-3(0)} + c_2 e^{(0)} + 4(0) \\ &= c_1 + c_2 \\ c_1 &= -c_2 \end{aligned}$$

$$\begin{aligned} y'(0) = 0 &= -3c_1 e^{-3(0)} + c_2 e^{(0)} + 4 \\ &= -3c_1 + c_2 + 4 \\ c_2 &= 3c_1 - 4 \end{aligned}$$

$$c_1 = -(3c_1 - 4) \Rightarrow c_1 = 1 \Rightarrow c_2 = -1$$

$$y = e^{-3x} - e^{x} + 4x$$

1.4.14 37

$$\begin{aligned} y(1) = -2 &= c_1 e^{-3(1)} + c_2 e^{(1)} + 4(1) \\ &= c_1 e^{-3} + c_2 e + 4 \\ c_1 &= -e^{3}(c_2 e + 6) \end{aligned}$$

$$\begin{aligned} y'(1) = 4 &= -3c_1 e^{-3(1)} + c_2 e^{(1)} + 4 \\ &= -3c_1 e^{-3} + c_2 e + 4 \\ c_2 e &= 3c_1 e^{-3} \end{aligned}$$

$$c_1 = -e^{3}(3c_1 e^{-3} + 6) = -3c_1 - 6e^{3} \Rightarrow c_1 = -\frac{3}{2}e^{3} \Rightarrow c_2 = -\frac{9}{2}e^{-1}$$

$$y = -\frac{3}{2}e^{3(1-x)} - \frac{9}{2}e^{x-1} + 4x$$

1.4.15 41

$y_0 = -3,\ y_1 = 0$

1.4.16 43

$$
\begin{aligned}
\frac{d}{dt}(mv) &= F - mg \\
\frac{d}{dt}(\lambda x \frac{dx}{dt}) &= F - \lambda x g \\
x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + gx &= \frac{F}{\lambda} \\
x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 32x &= 5
\end{aligned}
$$

2 First-Order Differential Equations

2.1 Solution Curves Without a Solution

2.1.1 21

0 is stable, 3 is unstable

2.1.2 23

2 is semi-stable

2.1.3 25

-2 is unstable, 0 is semi-stable, 2 is stable

2.1.4 27

-1 is stable, 0 is unstable

2.1.5 39

$P_0 < h/k$

2.1.6 41

$$g - \frac{k}{m}v^2 = 0 \Rightarrow v = \sqrt{\frac{gm}{k}}$$

2.2 Separable Equations

2.2.1 1

$$
\begin{aligned}
\frac{dy}{dx} &= \sin 5x \\
y &= -\frac{1}{5}\cos 5x + c
\end{aligned}
$$

2.2.2 3

$$
\begin{aligned}
dx + e^{3x}\, dy &= 0 \\
e^{-3x}\, dx + \, dy &= 0 \\
-\frac{1}{3}e^{-3x} + y &= c \\
y &= \frac{1}{3}e^{-3x} + c
\end{aligned}
$$

2.2.3 5

$$
\begin{aligned}
x\frac{dy}{dx} &= 4y \\
\frac{1}{4y}\, dy &= \frac{1}{x}\, dx \\
\frac{1}{4}\ln|4y| &= \ln|x| + c \\
\ln|4y| &= 4\ln|x| + c \\
4y &= e^{4\ln|x|+c} \\
&= c\left(e^{\ln|x|}\right)^4 \\
y &= cx^4
\end{aligned}
$$

2.2.4 7

$$
\begin{aligned}
\frac{dy}{dx} &= e^{3x+2y} \\
&= e^{3x}e^{2y} \\
e^{-2y}\, dy &= e^{3x}\, dx \\
-\frac{1}{2}e^{-2y} &= \frac{1}{3}e^{3x} + c \\
-3e^{-2y} &= 2e^{3x} + c
\end{aligned}
$$

2.2.5 9

$$\begin{aligned}
y \ln x \frac{dx}{dy} &= \left(\frac{y+1}{x}\right)^2 \\
x^2 \ln x \, dx &= \frac{(y+1)^2}{y} \, dy \\
x^3 \left(\frac{\ln x}{3} - \frac{1}{9}\right) &= \frac{1}{2} y(y+4) + \ln y + c \\
\frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 &= \frac{1}{2} y^2 + 2y + \ln y + c
\end{aligned}$$

2.2.6 11

$$\begin{aligned}
\csc y \, dx + \sec^2 x \, dy &= 0 \\
\frac{1}{\sin y} \, dx + \frac{1}{\cos^2 x} \, dy &= 0 \\
\cos^2 x \, dx + \sin y \, dy &= 0 \\
\frac{1}{2}(1 + \cos 2x) \, dx + \sin y \, dy &= 0 \\
\frac{1}{2}\left(x + \frac{1}{2} \sin 2x\right) - \cos y + c &= 0 \\
4 \cos y &= 2x + \sin 2x + c
\end{aligned}$$

2.2.7 13

$$\begin{aligned}
(e^y + 1)^2 e^{-y} \, dx + (e^x + 1)^3 e^{-x} \, dy &= 0 \\
\frac{e^x}{(e^x + 1)^3} \, dx + \frac{e^y}{(e^y + 1)^2} &= 0 \\
-\frac{1}{2(e^x + 1)^2} - \frac{1}{e^y + 1} &= c \\
(e^x + 1)^{-2} + 2(e^y + 1)^{-1} &= c
\end{aligned}$$

2.2.8 15

$$\begin{aligned}
\frac{dS}{dr} &= kS \\
\frac{1}{S} \, dS &= k \, dr \\
\ln |S| &= kr + c \\
S &= ce^{kr}
\end{aligned}$$

2.2.9 17

$$
\begin{aligned}
\frac{dP}{dt} &= P - P^2 \\
\frac{1}{P(1-P)}\,dP &= dt \\
\ln\frac{P}{1-P} &= t + c \\
\frac{P}{1-P} &= ce^t \\
P &= ce^t(1-P) \\
P &= \frac{ce^t}{1+ce^t}
\end{aligned}
$$

2.2.10 19

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{xy+3x-y-3}{xy-2x+4y-8} \\
&= \frac{(x-1)(y+3)}{(x+4)(y-2)} \\
\frac{y-2}{y+3}\,dt &= \frac{x-1}{x+4}\,dx \\
y - 5\ln|y+3| &= x - 5\ln|x+4| + c \\
e^{y-5\ln|y+3|} &= e^{x-5\ln|x+4|+c} \\
\frac{e^y}{(y+3)^5} &= \frac{ce^x}{(x+4)^5} \\
c(x+4)^5e^y &= (y+3)^5e^x
\end{aligned}
$$

2.2.11 21

$$
\begin{aligned}
\frac{dy}{dx} &= x\sqrt{1-y^2} \\
(1-y^2)^{-1/2}\,dy &= x\,dx \\
\arcsin y &= \frac{1}{2}x^2 + c \\
y &= \sin\left(\frac{1}{2}x^2 + c\right)
\end{aligned}
$$

2.2.12 23

$$\begin{aligned}
\frac{dx}{dt} &= 4(x^2+1) \\
\frac{1}{x^2+1}\,dx &= 4\,dt \\
\arctan x &= 4t + c \\
x &= \tan(4t+c)
\end{aligned}$$

$$\begin{aligned}
x\left(\frac{\pi}{4}\right) = 1 &= \tan\left(4\left(\frac{\pi}{4}\right)+c\right) \\
&= \tan(\pi + c) \\
c &= \arctan(1) - \pi \\
&= -\frac{3}{4}\pi
\end{aligned}$$

$$x = \tan\left(4t - \frac{3}{4}\pi\right)$$

2.2.13 25

$$\begin{aligned}
x^2\frac{dy}{dx} &= y - xy \\
&= y(1-x) \\
\frac{1}{y}\,dy &= \left(\frac{1}{x^2} - \frac{1}{x}\right)\,dx \\
\ln|y| &= -\frac{1}{x} - \ln|x| + c \\
y &= e^{-\frac{1}{x} - \ln|x| + c} \\
&= \frac{c}{xe^{1/x}}
\end{aligned}$$

$$\begin{aligned}
y(-1) = -1 &= \frac{c}{(-1)e^{1/(-1)}} \\
&= -ce \\
c &= e^{-1}
\end{aligned}$$

$$y = \frac{1}{xe^{1+1/x}}$$

2.2.14 29

$$\begin{aligned}
\frac{dy}{dx} &= ye^{-x^2} \\
\frac{1}{y}\frac{dy}{dx} &= e^{-x^2} \\
\int_4^x \frac{1}{y}\frac{dy}{dx'}\,dx' &= \int_4^x e^{-x'^2}\,dx' \\
\ln|y|\big|_4^x &= \int_4^x e^{-x'^2}\,dx' \\
\ln|y(x)| - \ln|y(4)| &= \int_4^x e^{-x'^2}\,dx' \\
\ln|y(x)| &= \ln|y(4)| + \int_4^x e^{-x'^2}\,dx' \\
y(x) &= e^{\int_4^x e^{-x'^2}\,dx'}
\end{aligned}$$

2.2.15 31

$$\begin{aligned}
\frac{dy}{dx} &= \frac{2x+1}{2y} \\
2y\,dy &= (2x+1)\,dx \\
y^2 &= x^2 + x + c \\
y &= \pm\sqrt{x^2+x+c}
\end{aligned}$$

$$\begin{aligned}
y(-2) = -1 &= -\sqrt{(-2)^2 + (-2) + c} \\
&= -\sqrt{2+c} \\
c &= -1
\end{aligned}$$

$$y = -\sqrt{x^2+x-1}$$

$$I = \left(-\infty, -\frac{1-\sqrt{5}}{2}\right)$$

2.2.16 33

$$
\begin{aligned}
e^y\,dx - e^{-x}\,dy &= 0\\
e^x\,dx - e^{-y}\,dy &= 0\\
e^x + e^{-y} &= c\\
\ln|e^{-y}| &= \ln|c - e^x|\\
y &= -\ln|c - e^x|
\end{aligned}
$$

$$
\begin{aligned}
y(0) = 0 &= -\ln|c - e^{(0)}|\\
1 &= c - 1\\
c &= 2
\end{aligned}
$$

$$y = -\ln|2 - e^x|$$

$$I = (-\infty, \ln 2)$$

2.3 Linear Equations

2.3.1 1

$$
\begin{aligned}
\frac{dy}{dx} &= 5y\\
\ln|y| &= 5x + c\\
y &= ce^{5x}
\end{aligned}
$$

$$I = (-\infty, \infty)$$

2.3.2 3

$$
\begin{aligned}
\frac{dy}{dx} + y &= e^{3x}\\
e^x\frac{dy}{dx} + e^x y &= e^{4x}\\
\frac{d}{dx}(e^x y) &= e^{4x}\\
e^x y &= \frac{1}{4}e^{4x} + c\\
y &= \frac{1}{4}e^{3x} + ce^{-x}
\end{aligned}
$$

$$I = (-\infty, \infty)$$

2.3.3 5

$$\begin{aligned}
y' + 3x^2 y &= x^2 \\
e^{x^3} y' + 3x^2 e^{x^3} y &= e^{x^3} x^2 \\
e^{x^3} y &= \frac{1}{3} e^{x^3} + c \\
y &= \frac{1}{3} + c e^{-x^3}
\end{aligned}$$

$$I = (-\infty, \infty)$$

2.3.4 7

$$\begin{aligned}
x^2 y' + xy &= 1 \\
y' + x^{-1} y &= x^{-2} \\
e^{\ln x} y' + x^{-1} e^{\ln x} y &= e^{\ln x} x^{-2} \\
\frac{d}{dx}(e^{\ln x} y) &= x^{-1} \\
\frac{d}{dx}(xy) &= x^{-1} \\
xy &= \ln x + c \\
y &= \frac{\ln x + c}{x}
\end{aligned}$$

$$I = (0, \infty)$$

2.3.5 9

$$\begin{aligned}
x \frac{dy}{dx} - y &= x^2 \sin x \\
\frac{dy}{dx} - x^{-1} y &= x \sin x \\
e^{-\ln x} \frac{dy}{dx} - x^{-1} e^{-\ln x} y &= e^{-\ln x} x \sin x \\
\frac{d}{dx}(e^{-\ln x} y) &= \sin x \\
x^{-1} y &= -\cos x + c \\
y &= cx - x \cos x
\end{aligned}$$

$$I = (0, \infty)$$

2.3.6 11

$$
\begin{aligned}
x\frac{dy}{dx} + 4y &= x^3 - x \\
\frac{dy}{dx} + 4x^{-1}y &= x^2 - 1 \\
e^{4\ln x}\frac{dy}{dx} + 4x^{-1}e^{4\ln x}y &= e^{4\ln x}(x^2 - 1) \\
\frac{d}{dx}(e^{4\ln x}y) &= x^6 - x^4 \\
x^4 y &= \frac{1}{7}x^7 - \frac{1}{5}x^5 + c \\
y &= \frac{1}{7}x^3 - \frac{1}{5}x^2 + cx^{-4}
\end{aligned}
$$

$$I = (0, \infty)$$

2.3.7 13

$$
\begin{aligned}
x^2y' + x(x+2)y &= e^x \\
y' + x^{-1}(x+2)y &= x^{-2}e^x \\
e^{x+2\ln x}y' + x^{-1}(x+2)e^{x+2\ln x}y &= e^{x+2\ln x}x^{-2}e^x \\
\frac{d}{dx}(e^x x^2 y) &= e^{2x} \\
e^x x^2 y &= \frac{1}{2}e^{2x} + c \\
y &= \frac{e^x}{2x^2} + \frac{c}{e^x x^2}
\end{aligned}
$$

$$I = (0, \infty)$$

2.3.8 15

$$\begin{aligned}
y\,dx - 4(x+y^6)\,dy &= 0\\
y\frac{dx}{dy} - 4x - 4y^6 &= 0\\
\frac{dx}{dy} - \frac{4}{y}x &= 4y^5\\
e^{-4\ln y}\frac{dx}{dy} - \frac{4}{y}e^{-4\ln y}x &= 4e^{-4\ln y}y^5\\
\frac{d}{dy}(e^{-4\ln y}x) &= 4y\\
y^{-4}x &= 2y^2 + c\\
x &= 2y^6 + cy^4
\end{aligned}$$

$$I = (0, \infty)$$

2.3.9 17

$$\begin{aligned}
\cos x\frac{dy}{dx} + (\sin x)y &= 1\\
\frac{dy}{dx} + (\tan x)y &= \sec x\\
e^{\ln(\sec x)}\frac{dy}{dx} + (\tan x)e^{\ln(\sec x)}y &= e^{\ln(\sec x)}\sec x\\
\frac{d}{dx}(e^{\ln(\sec x)}y) &= \sec^2 x\\
y\sec x &= \tan x + c\\
y &= \sin x + c\cos x
\end{aligned}$$

$$I = (-\pi/2, \pi/2)$$

2.3.10 19

$$\begin{aligned}
(x+1)\frac{dy}{dx}+(x+2)y &= 2xe^{-x}\\
\frac{dy}{dx}+\frac{x+2}{x+1}y &= \frac{2xe^{-x}}{x+1}\\
e^{x+\ln|x+1|}\frac{dy}{dx}+\frac{x+2}{x+1}e^{x+\ln|x+1|}y &= e^{x+\ln|x+1|}\frac{2xe^{-x}}{x+1}\\
\frac{d}{dx}(e^{x+\ln|x+1|}y) &= 2x\\
e^x(x+1)y &= x^2+c\\
y &= \frac{x^2+c}{e^x(x+1)}
\end{aligned}$$

$$I=(-1,\infty)$$

2.3.11 21

$$\begin{aligned}
\frac{dr}{d\theta}+r\sec\theta &= \cos\theta\\
e^{\ln|\sec\theta+\tan\theta|}\frac{dr}{d\theta}+e^{\ln|\sec\theta+\tan\theta|}r\sec\theta &= e^{\ln|\sec\theta+\tan\theta|}\cos\theta\\
\frac{d}{d\theta}(e^{\ln|\sec\theta+\tan\theta|}r) &= 1+\sin\theta\\
(\sec\theta+\tan\theta)r &= \theta-\cos\theta+c\\
r &= \frac{\theta-\cos\theta+c}{\sec\theta+\tan\theta}
\end{aligned}$$

$$I=(-\pi/2,\pi/2)$$

2.3.12 23

$$\begin{aligned}
x\frac{dy}{dx}+(3x+1)y &= e^{-3x}\\
\frac{dy}{dx}+(3+x^{-1})y &= e^{-3x}x^{-1}\\
e^{3x+\ln|x|}\frac{dy}{dx}+(3+x^{-1})e^{3x+\ln|x|}y &= 1\\
\frac{d}{dx}(e^{3x+\ln|x|}y) &= 1\\
e^{3x}xy &= x+c\\
y &= \frac{x+c}{e^{3x}x}
\end{aligned}$$

$$I=(0,\infty)$$

2.3.13 25

$$\begin{aligned}
xy' + y &= e^x \\
y' + x^{-1}y &= e^x x^{-1} \\
e^{\ln|x|}y' + x^{-1}e^{\ln|x|}y &= e^x \\
\frac{d}{dx}(e^{\ln|x|}y) &= e^x \\
xy &= e^x + c \\
y &= \frac{e^x + c}{x}
\end{aligned}$$

$$\begin{aligned}
y(1) = 2 &= \frac{e^{(1)} + c}{(1)} \\
c &= 2 - e
\end{aligned}$$

$$y = \frac{e^x + 2 - e}{x}$$

$$I = (0, \infty)$$

2.3.14 27

$$\begin{aligned}
L\frac{di}{dt} + Ri &= E \\
\frac{di}{dt} + \frac{R}{L}i &= \frac{E}{L} \\
e^{Rt/L}\frac{di}{dt} + \frac{R}{L}e^{Rt/L}i &= \frac{E}{L}e^{Rt/L} \\
\frac{d}{dt}(e^{Rt/L}i) &= \frac{E}{L}e^{Rt/L} \\
e^{Rt/L}i &= \frac{E}{R}e^{Rt/L} + c \\
i &= \frac{E}{R} + ce^{-Rt/L}
\end{aligned}$$

$$\begin{aligned}
i(0) = i_0 &= \frac{E}{R} + ce^{-R(0)/L} \\
&= \frac{E}{R} + c \\
c &= i_0 - \frac{E}{R}
\end{aligned}$$

$$i = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right) e^{-Rt/L}$$

$$I = (-\infty, \infty)$$

2.3.15 53

$$\begin{aligned} \frac{dE}{dt} &= -\frac{1}{RC}E \\ \frac{1}{E}\frac{dE}{dt} &= -\frac{1}{RC} \\ \ln|E| &= -\frac{1}{RC}t + c \\ E &= ce^{-t/RC} \end{aligned}$$

$$\begin{aligned} E(4) = E_0 &= ce^{-(4)/RC} \\ c &= E_0 e^{4/RC} \end{aligned}$$

$$E = E_0 e^{(4-t)/RC}$$

2.4 Exact Equations

2.4.1 1

$$f(x,y) = x^2 - x + g(y)$$

$$\frac{\partial f}{\partial y} = g'(y) = 3y + 7$$

$$g(y) = \frac{3}{2}y^2 + 7y$$

$$x^2 - x + \frac{3}{2}y^2 + 7y = c$$

2.4.2 3

$$f(x,y) = \frac{5}{2}x^2 + 4xy + g(y)$$

$$4x + g'(y) = 4x - 8y^3 \Rightarrow g'(y) = -8y^3$$

$$g(y) = -2y^4$$

$$\frac{5}{2}x^2 + 4xy - 2y^4 = c$$

2.4.3 5

$$f(x,y) = x^2y^2 - 3x + g(y)$$

$$2x^2y + g'(y) = 2x^2y + 4 \Rightarrow g'(y) = 4$$

$$g(y) = 4y$$

$$x^2y^2 - 3x + 4y = c$$

2.4.4 7

Not exact

2.4.5 9

$$f(x,y) = \frac{1}{2}x^2 - xy^3 - y^2\cos x + g(y)$$

$$-3xy^2 - 2y\cos x + g'(y) = -3xy^2 - 2y\cos x \Rightarrow g'(y) = 0$$

$$\frac{1}{2}x^2 - xy^3 - y^2\cos x = c$$

2.4.6 11

Not exact

2.4.7 13

$$f(x,y) = xy + g(x)$$

$$y + g'(x) = -2xe^x + y - 6x^2 \Rightarrow g'(x) = -2xe^x - 6x^2$$

$$g(x) = -2e^x(x-1) - 2x^3$$

$$xy - 2e^x(x-1) - 2x^3 = c$$

2.4.8 21

$$f(x,y) = \frac{1}{3}(x+y)^3 + g(y)$$

$$(x+y)^2 + g'(y) = 2xy + x^2 - 1 \Rightarrow g'(y) = -y^2 - 1$$

$$g(y) = -\frac{1}{3}y^3 - y$$

$$\frac{1}{3}(x+y)^3 - \frac{1}{3}y^3 - y = c$$

$$\frac{1}{3}(1+1)^3 - \frac{1}{3}1^3 - 1 = c \Rightarrow c = \frac{4}{3}$$

$$x^3 + 3x^2y + 3xy^2 - 3y = 4$$

2.4.9 23

$$f(x,y) = 4ty + t^2 - 5t + g(y)$$

$$4t + g'(y) = 6y + 4t - 1 \Rightarrow g'(y) = 6y - 1$$

$$g(y) = 3y^2 - y$$

$$4ty + t^2 - 5t + 3y^2 - y = c$$

$$4(-1)(2) + (-1)^2 - 5(-1) + 3(2)^2 - (2) = c \Rightarrow c = 8$$

$$4ty + t^2 - 5t + 3y^2 - y = 8$$

2.4.10 27

$$3y^2 + 4kxy^3 = 3y^2 + 40xy^3 \Rightarrow k = 10$$

2.4.11 31

$$M_y = 4y$$

$$N_x = 2y$$

$$\frac{M_y - N_x}{N} = \frac{4y - 2y}{2xy} = \frac{1}{x}$$

$$\mu(x) = e^{\ln x} = x$$

$$(2xy^2 + 3x^2)\,dx + 2x^2y\,dy = 0$$

$$f(x,y) = x^2y^2 + x^3 + g(y)$$

$$2x^2y + g'(y) = 2x^2y \Rightarrow g'(y) = 0$$

$$x^2y^2 + x^3 = c$$

2.4.12 33

$$M_y = 6x$$

$$N_x = 18x$$

$$\frac{M_y - N_x}{N} = \frac{6x - 18x}{4y + 9x^2}$$

$$\frac{N_x - M_y}{M} = \frac{18x - 6x}{6xy} = \frac{2}{y}$$

$$\mu(y) = e^{2\ln y} = y^2$$

$$6xy^3\,dx + (4y^3 + 9x^2y^2)\,dy = 0$$

$$f(x,y) = 3x^2y^3 + g(y)$$

$$9x^2y^2 + g'(y) = 4y^3 + 9x^2y^2 \Rightarrow g'(y) = 4y^3$$

$$g(y) = y^4$$

$$3x^2y^3 + y^4 = c$$

2.4.13 37

$$M_y = 0$$

$$N_x = 2xy$$

$$\frac{N_x - M_y}{M} = \frac{2xy - 0}{x} = 2y$$

$$\mu(y) = e^{y^2}$$

$$e^{y^2} x\,dx + e^{y^2}(x^2y + 4y)\,dy = 0$$

$$f(x,y) = \frac{1}{2}e^{y^2}x^2 + g(y)$$

$$ye^{y^2}x^2 + g'(y) = e^{y^2}(x^2y + 4y) \Rightarrow g'(y) = 4e^{y^2}y$$

$$g(y) = 2e^{y^2}$$

$$\frac{1}{2}e^{y^2}x^2 + 2e^{y^2} = c$$

$$\frac{1}{2}e^{(0)^2}(4)^2 + 2e^{(0)^2} = c \Rightarrow c = 10$$

$$\frac{1}{2}e^{y^2}x^2 + 2e^{y^2} = 10$$

2.4.14 39

(c)

$$(0)^3 + 2(0)^2(-2) + (-2)^2 = c \Rightarrow c = 4$$

$$y^2 + 2x^2y + x^3 - 4 = 0$$

$$\begin{aligned} y &= \frac{-(2x^2) \pm \sqrt{(2x^2)^2 - 4(1)(x^3-4)}}{2(1)} \\ &= \frac{-2x^2 \pm \sqrt{4x^4 - 4(x^3-4)}}{2} \\ &= -x^2 \pm \sqrt{x^4 - x^3 + 4} \end{aligned}$$

2.4.15 45

(a)

$$xv\frac{dv}{dx} + v^2 = 32x \Rightarrow xv\,dv + (v^2 - 32x)\,dx = 0$$

$$M_x = v$$

$$N_v = 2v$$

$$\frac{M_x - N_v}{N} = \frac{v - 2v}{v^2 - 32x}$$

$$\frac{N_v - M_x}{M} = \frac{2v - v}{xv} = \frac{1}{x}$$

$$\mu(x) = e^{\ln x} = x$$

$$x^2v\,dv + (xv^2 - 32x^2)\,dx = 0$$

$$f(x, v) = \frac{1}{2}x^2v^2 + g(x)$$

$$xv^2 + g'(x) = xv^2 - 32x^2 \Rightarrow g'(x) = -32x^2$$

$$g(x) = -\frac{32}{3}x^3$$

$$\frac{1}{2}(3)^2(0)^2 - \frac{32}{3}(3)^3 = c \Rightarrow c = -288$$

$$\frac{1}{2}x^2v^2 - \frac{32}{3}x^3 = -288 \Rightarrow v = 8\sqrt{\frac{x}{3} - \frac{9}{x^2}}$$

(b) $v = 12.7\,\text{ft/s}$

2.5 Solutions by Substitution

2.5.1 1

$$
\begin{aligned}
(x - y)\,dx + x\,dy &= 0 \\
(x - ux)\,dx + x(u\,dx + x\,du) &= 0 \\
x\,dx + x^2\,du &= 0 \\
x^{-1}\,dx + du &= 0 \\
\ln|x| + u &= c \\
\ln|x| + \frac{y}{x} &= c \\
y &= cx - x\ln|x|
\end{aligned}
$$

2.5.2 3

$$
\begin{aligned}
x\,dx + (y - 2x)\,dy &= 0 \\
vy(v\,dy + y\,dv) + (y - 2vy)\,dy &= 0 \\
(v^2y + y - 2vy)\,dy + vy^2\,dv &= 0 \\
y(v^2 - 2v + 1)\,dy + vy^2\,dv &= 0 \\
(v - 1)^2\,dy + vy\,dv &= 0 \\
\frac{1}{y}\,dy + \frac{v}{(v-1)^2}\,dv &= 0 \\
\ln|y| + \frac{1}{1 - v} + \ln|v - 1| &= c \\
\ln|y| + \frac{1}{1 - x/y} + \ln\left|\frac{x}{y} - 1\right| &= c \\
\ln|x - y| + \frac{y}{y - x} &= c \\
(y - x)\ln|x - y| + y &= c(y - x) \\
(x - y)\ln|x - y| &= y + c(x - y)
\end{aligned}
$$

2.5.3 5

$$
\begin{aligned}
(y^2 + yx)\,dx - x^2\,dy &= 0 \\
((ux)^2 + ux^2)\,dx - x^2(u\,dx + x\,du) &= 0 \\
u^2x^2\,dx - x^3\,du &= 0 \\
\frac{1}{x}\,dx - \frac{1}{u^2}\,du &= 0 \\
\ln|x| + \frac{1}{u} &= c \\
\ln|x| + \frac{x}{y} &= c \\
y &= \frac{x}{c - \ln|x|}
\end{aligned}
$$

2.5.4 7

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{y - x}{y + x} \\
(y + x)\,dy + (x - y)\,dx &= 0 \\
(ux + x)(u\,dx + x\,du) + (x - ux)\,dx &= 0 \\
(u^2x + x)\,dx + (ux^2 + x^2)\,du &= 0 \\
x(u^2 + 1)\,dx + x^2(u + 1)\,du &= 0 \\
\frac{1}{x}\,dx + \frac{u + 1}{u^2 + 1}\,du &= 0 \\
\ln|x| + \frac{1}{2}\ln|u^2 + 1| + \arctan u &= c \\
\ln|x^2 + y^2| + 2\arctan\frac{y}{x} &= c
\end{aligned}
$$

2.5.5 9

$$\begin{aligned}
-y\,dx + (x + \sqrt{xy})\,dy &= 0\\
-ux\,dx + (x + \sqrt{ux^2})(u\,dx + x\,du) &= 0\\
u\sqrt{ux^2}\,dx + (x^2 + x\sqrt{ux^2})\,du &= 0\\
u^{3/2}x\,dx + x^2(1+\sqrt{u})\,du &= 0\\
\frac{1}{x}\,dx + \frac{1+\sqrt{u}}{u^{3/2}}\,du &= 0\\
\frac{1}{x}\,dx + (u^{-3/2} + u^{-1})\,du &= 0\\
\ln|x| - 2u^{-1/2} + \ln|u| &= c\\
\ln|x| - 2(y/x)^{-1/2} + \ln|y/x| &= c\\
\ln|y| - 2\sqrt{\frac{x}{y}} &= c\\
4\frac{x}{y} &= (\ln|y| - c)^2\\
4x &= y(\ln|y| - c)^2
\end{aligned}$$

2.5.6 11

$$\begin{aligned}
xy^2\frac{dy}{dx} &= y^3 - x^3\\
xy^2\,dy + (x^3 - y^3)\,dx &= 0\\
x(ux)^2(u\,dx + x\,du) + (x^3 - (ux)^3)\,dx &= 0\\
x^3\,dx + u^2x^4\,du &= 0\\
x^{-1}\,dx + u^2\,du &= 0\\
\ln|x| + \frac{1}{3}u^3 &= c\\
\ln|x| + \frac{1}{3}\left(\frac{y}{x}\right)^3 &= c
\end{aligned}$$

$$\ln|1| + \frac{1}{3}\left(\frac{2}{1}\right)^3 = c \Rightarrow c = \frac{8}{3} + \ln 1$$

$$\begin{aligned}
\ln|x| + \frac{1}{3}\left(\frac{y}{x}\right)^3 &= \frac{8}{3}\\
y^3 + 3x^3\ln|x| &= 8x^3
\end{aligned}$$

2.5.7 13

$$\begin{aligned}(x + ye^{y/x})\,dx - xe^{y/x}\,dy &= 0\\(x + uxe^{u})\,dx - xe^{u}(u\,dx + x\,du) &= 0\\x\,dx - x^2e^{u}\,du &= 0\\x^{-1}\,dx - e^{u}\,du &= 0\\\ln|x| - e^{u} &= c\\\ln|x| - e^{y/x} &= c\end{aligned}$$

$$\ln|1| - e^{0/1} = c \Rightarrow c = -1$$

$$\ln|x| = e^{y/x} - 1$$

2.5.8 15

$$\begin{aligned}x\frac{dy}{dx} + y &= \frac{1}{y^2}\\\frac{dy}{dx} + x^{-1}y &= x^{-1}y^{-2}\end{aligned}$$

$$u = y^{1-n} = y^3 \Rightarrow y = u^{1/3} \Rightarrow \frac{dy}{dx} = \frac{1}{3}u^{-2/3}\frac{du}{dx}$$

$$\begin{aligned}\frac{1}{3}u^{-2/3}\frac{du}{dx} + x^{-1}u^{1/3} &= x^{-1}u^{-2/3}\\\frac{du}{dx} + 3x^{-1}u &= 3x^{-1}\\e^{3\ln|x|}\frac{du}{dx} + 3x^{-1}e^{3\ln|x|}u &= 3x^2\\\frac{d}{dx}(x^3u) &= 3x^2\\x^3u &= x^3 + c\\y^3 &= 1 + cx^{-3}\end{aligned}$$

2.5.9 17

$$\begin{aligned}\frac{dy}{dx} &= y(xy^3 - 1)\\\frac{dy}{dx} + y &= xy^4\end{aligned}$$

$$u = y^{1-n} = y^{-3} \Rightarrow y = u^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}u^{-4/3}\frac{du}{dx}$$

$$\begin{aligned}
-\frac{1}{3}u^{-4/3}\frac{du}{dx} + u^{-1/3} &= xu^{-4/3} \\
\frac{du}{dx} - 3u &= -3x \\
e^{-3x}\frac{du}{dx} - 3e^{-3x}u &= -3e^{-3x}x \\
\frac{d}{dt}(e^{-3x}u) &= -3e^{-3x}x \\
e^{-3x}u &= e^{-3x}x + \frac{1}{3}e^{-3x} + c \\
u &= x + \frac{1}{3} + ce^{3x} \\
y^{-3} &= x + \frac{1}{3} + ce^{3x}
\end{aligned}$$

2.5.10 21

$$\begin{aligned}
x^2\frac{dy}{dx} - 2xy &= 3y^4 \\
\frac{dy}{dx} - 2x^{-1}y &= 3x^{-2}y^4
\end{aligned}$$

$$u = y^{1-n} = y^{-3} \Rightarrow y = u^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}u^{-4/3}\frac{du}{dx}$$

$$\begin{aligned}
-\frac{1}{3}u^{-4/3}\frac{du}{dx} - 2x^{-1}u^{-1/3} &= 3x^{-2}u^{-4/3} \\
\frac{du}{dx} + 6x^{-1}u &= -9x^{-2} \\
e^{6\ln|x|}\frac{du}{dx} + 6x^{-1}e^{6\ln|x|}u &= -9e^{6\ln|x|}x^{-2} \\
\frac{d}{dx}(x^6u) &= -9x^4 \\
x^6u &= -\frac{9}{5}x^5 + c \\
u &= -\frac{9}{5}x^{-1} + cx^{-6} \\
y^{-3} &= -\frac{9}{5}x^{-1} + cx^{-6}
\end{aligned}$$

$$\left(\frac{1}{2}\right)^{-3} = -\frac{9}{5}(1)^{-1} + c(1)^{-6} \Rightarrow c = \frac{49}{5}$$

$$y^{-3} = -\frac{9}{5}x^{-1} + \frac{49}{5}x^{-6}$$

2.5.11 23

Let $u = x + y + 1$ so $\frac{du}{dx} = 1 + \frac{dy}{dx}$ and

$$\begin{aligned}
\frac{du}{dx} - 1 &= u^2 \\
\frac{1}{u^2+1}\,du &= dx \\
\arctan u &= x + c \\
\arctan(x+y+1) &= x + c \\
x + y + 1 &= \tan(x+c) \\
y &= -x - 1 + \tan(x+c)
\end{aligned}$$

2.5.12 25

Let $u = x + y$ so $\frac{du}{dx} = 1 + \frac{dy}{dx}$ and

$$\begin{aligned}
\frac{du}{dx} - 1 &= \tan^2 u \\
\frac{1}{1+\tan^2 u}\,du &= dx \\
\frac{1}{2}(u + \sin u \cos u) &= x + c \\
x + y + \sin(x+y)\cos(x+y) &= 2(x+c) \\
x + y + \frac{1}{2}\sin(2(x+y)) &= 2(x+c) \\
2x + 2y + \sin(2(x+y)) &= 4(x+c) \\
2y - 2x + \sin(2(x+y)) &= c
\end{aligned}$$

2.5.13 35

(a) Let $y = y_1 + u$ so $\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{du}{dx}$ but $\frac{dy_1}{dx} = P(x) + Q(x)y_1 + R(x)y_1^2$ so

$$\begin{aligned}
\frac{dy}{dx} &= P(x) + Q(x)y + R(x)y^2 \\
P(x) + Q(x)y_1 + R(x)y_1^2 + \frac{du}{dx} &= P(x) + Q(x)(y_1 + u) + R(x)(y_1 + u)^2 \\
\frac{du}{dx} &= Q(x)u + R(x)(2y_1 u + u^2)
\end{aligned}$$

$$\frac{du}{dx} - (Q(x) + 2R(x)y_1)u = R(x)u^2$$

(b) Let $y = 2x^{-1} + u$ so $\frac{dy}{dx} = -2x^{-2} + \frac{du}{dx}$ and

$$\begin{aligned}
-\frac{2}{x^2} + \frac{du}{dx} &= -\frac{4}{x^2} - \frac{1}{x}\left(\frac{2}{x} + u\right) + \left(\frac{2}{x} + u\right)^2 \\
\frac{du}{dx} &= \frac{2}{x^2} - \frac{4}{x^2} - \frac{2}{x^2} - \frac{u}{x} + \frac{4}{x^2} + \frac{4u}{x} + u^2
\end{aligned}$$

$$\frac{du}{dx} - \frac{3}{x}u = u^2$$

Let $v = u^{1-n} = u^{-1}$ so $u = v^{-1}$ and $\frac{du}{dx} = -v^{-2}\frac{dv}{dx}$

$$\begin{aligned}
-v^{-2}\frac{dv}{dx} - \frac{3}{x}v^{-1} &= v^{-2} \\
\frac{dv}{dx} + \frac{3}{x}v &= -1 \\
e^{3\ln|x|}\frac{dv}{dx} + \frac{3}{x}e^{3\ln|x|}v &= -e^{3\ln|x|} \\
\frac{d}{dt}(x^3 v) &= -x^3 \\
x^3 v &= -\frac{1}{4}x^4 + c \\
\frac{1}{y - y_1} &= -\frac{1}{4}x + cx^{-3} \\
y &= y_1 + \left(-\frac{1}{4}x + cx^{-3}\right)^{-1} \\
&= \frac{2}{x} + \left(-\frac{1}{4}x + cx^{-3}\right)^{-1}
\end{aligned}$$

2.5.14 37

$$\frac{dP}{dt} = P(a - bP)$$

$$\frac{dP}{dt} - aP = -bP^2$$

Let $u = P^{1-n} = P^{-1}$ so $P = u^{-1}$ and $\frac{dP}{dt} = -u^{-2}\frac{du}{dt}$

$$\begin{aligned}
-u^{-2}\frac{du}{dt} - au^{-1} &= -bu^{-2} \\
\frac{du}{dt} + au &= b \\
e^{at}\frac{du}{dt} + ae^{at}u &= be^{at} \\
\frac{d}{dt}(e^{at}u) &= be^{at} \\
e^{at}u &= \frac{b}{a}e^{at} + c \\
P^{-1} &= \frac{b}{a} + ce^{-at} \\
&= \frac{b + ce^{-at}}{a} \\
P &= \frac{a}{b + ce^{-at}}
\end{aligned}$$

2.6 A Numerical Method

2.6.1 1

$$\begin{aligned}
x_0 &= 1 & y_0 &= 5 \\
x_1 &= 1.1 & y_1 &= y_0 + hf(x_0, y_0) = 3.8000 \\
x_2 &= 1.2 & y_2 &= y_1 + hf(x_1, y_1) = 2.9800
\end{aligned}$$

$$\begin{aligned}
x_0 &= 1 & y_0 &= 5 \\
x_1 &= 1.05 & y_1 &= y_0 + hf(x_0, y_0) = 4.4000 \\
x_2 &= 1.1 & y_2 &= y_1 + hf(x_1, y_1) = 3.8950 \\
x_3 &= 1.15 & y_3 &= y_2 + hf(x_2, y_2) = 3.4708 \\
x_4 &= 1.2 & y_4 &= y_3 + hf(x_3, y_3) = 3.1152
\end{aligned}$$

2.7 Linear Models

2.7.1 1

$$P(t) = P_0 e^{kt}$$

$$P(5) = 2P_0 = P_0 e^{5k} \Rightarrow k = \frac{\ln 2}{5} = 0.139$$

$$P(t) = P_0 e^{0.139t}$$

$$3P_0 = P_0 e^{0.139t} \Rightarrow t = 7.9\,\text{years}$$

$$4P_0 = P_0 e^{0.139t} \Rightarrow t = 10\,\text{years}$$

2.7.2 5

$$A(t) = A_0 e^{kt}$$

$$A(3.3) = \frac{1}{2}A_0 = A_0 e^{3.3k} \Rightarrow k = -0.21$$

$$0.1A_0 = A_0 e^{-0.21t} \Rightarrow t = 11\,\text{hours}$$

2.7.3 9

$$\frac{dI}{dt} = kI \Rightarrow I(t) = ce^{kt}$$

$$I(3) = 0.25I_0 = I_0 e^{3k} \Rightarrow k = -0.462$$

$$I(15) = I_0 e^{-0.462(15)} = 0.001I_0$$

2.7.4 11

$$0.145A_0 = A_0 e^{-0.00012097t} \Rightarrow t = 15\,963\,\text{years}$$

2.7.5 13

$$\begin{aligned}
\frac{dT}{dt} &= k(T - T_m) \\
\frac{1}{T - T_m}\frac{dT}{dt} &= k \\
\ln(T - T_m) &= kt + c \\
T - T_m &= ce^{kt} \\
T &= T_m + ce^{kt} \\
&= 10 + 60e^{kt}
\end{aligned}$$

$$T(0.5) = 50 = 10 + 60e^{0.5k} \Rightarrow k = -0.811$$

$$T(1) = 36.7$$

$$15 = 10 + 60e^{-0.811t} \Rightarrow t = 3.06\,\text{min}$$

2.7.6 21

$$\begin{aligned}\frac{dA}{dt} &= 4 - \frac{A}{50}\\ \frac{dA}{dt} + \frac{A}{50} &= 4\\ \frac{d}{dt}(e^{t/50}A) &= 4e^{t/50}\\ e^{t/50}A &= 200e^{t/50} + c\\ A &= 200 + ce^{-t/50}\end{aligned}$$

$$A(0) = 30 = 200 + ce^{-(0)/50} \Rightarrow c = -170$$

$$A(t) = 200 - 170e^{-t/50}$$

2.7.7 25

$$V(t) = 500 - 5t$$

$$\begin{aligned}\frac{dA}{dt} &= 10 - \frac{10}{500-5t}A\\ \frac{dA}{dt} + \frac{10}{500-5t}A &= 10\\ \frac{dA}{dt} - 2\frac{-5}{500-5t}A &= 10\\ e^{-2\ln(500-5t)}\frac{dA}{dt} - 2\frac{-5}{500-5t}e^{-2\ln(500-5t)}A &= e^{-2\ln(500-5t)}10\\ \frac{d}{dt}(A(500-5t)^{-2}) &= 10(500-5t)^{-2}\\ A(500-5t)^{-2} &= \frac{2}{500-5t} + c\\ A &= 2(500-5t) + c(500-5t)^2\\ &= 1000 - 10t + c(500-5t)^2\end{aligned}$$

$$A(0) = 0 \Rightarrow c = -0.004$$

$$A(t) = 1000 - 10t - 0.004(500-5t)^2 = 1000 - 10t - \frac{1}{10}(100-t)^2$$

The tank is empty at $t = 100$

2.7.8 29

$$i(t) = \frac{3}{5} - \frac{3}{5}e^{-500t}$$

$i \to \frac{3}{5}$ as $t \to \infty$

2.7.9 33

$$i(t) = \begin{cases} 60(1 - e^{-t/10}), & 0 \le t \le 20 \\ 383e^{-t/10}, & t > 20 \end{cases}$$

2.7.10 35

(a)

$$\begin{aligned} m\frac{dv}{dt} &= mg - kv \\ \frac{dv}{dt} + \frac{k}{m}v &= g \\ \frac{d}{dt}(e^{kt/m}v) &= e^{kt/m}g \\ e^{kt/m}v &= \frac{gm}{k}e^{kt/m} + c \\ v &= \frac{gm}{k} + ce^{-kt/m} \end{aligned}$$

$$v(0) = v_0 = \frac{gm}{k} + ce^{-k(0)/m} \Rightarrow c = v_0 - \frac{gm}{k}$$

$$v(t) = \frac{gm}{k} + \left(v_0 - \frac{gm}{k}\right)e^{-kt/m}$$

(b)

$$v_t = \frac{gm}{k}$$

(c)

$$s(t) = \frac{gm}{k}t - \frac{m}{k}\left(v_0 - \frac{gm}{k}\right)e^{-kt/m} + c$$

$$s(0) = 0 = -\frac{m}{k}\left(v_0 - \frac{gm}{k}\right) + c \Rightarrow c = \frac{m}{k}\left(v_0 - \frac{gm}{k}\right)$$

$$\begin{aligned} s(t) &= \frac{m}{k}\left(gt - \left(v_0 + \frac{gm}{k}\right)e^{-kt/m} + v_0 - \frac{gm}{k}\right) \\ &= \frac{m}{k}\left(gt + \left(v_0 - \frac{gm}{k}\right)\left(1 - e^{-kt/m}\right)\right) \end{aligned}$$

2.7.11 41

(a)

$$\begin{aligned}\frac{dP}{dt} &= k_1 P - k_2 P \\ &= (k_1 - k_2)P \\ P &= ce^{(k_1-k_2)t}\end{aligned}$$

2.7.12 43

(a) $x = r/k$

2.8 Nonlinear Models

2.8.1 1

(a) $N = 2000$

(b)

$$N = \frac{1}{0.0005 + (1 - 0.0005)e^{-t}}$$

$$N(10) = 1834$$

2.8.2 3

$P = 1.0 \times 10^6$

$$P = \frac{500}{0.0005 + (0.1 - 0.0005)e^{-0.1t}}$$

$$\begin{aligned}500000 &= \frac{500}{0.0005 + (0.1 - 0.0005)e^{-0.1t}} \\ 500000(0.0005 + (0.1 - 0.0005)e^{-0.1t}) &= 500 \\ e^{-0.1t} &= \frac{0.001 - 0.0005}{0.1 - 0.0005} \\ t &= 52.9\,\text{months}\end{aligned}$$

2.8.3 11

29.3 g; 60 g; 0 g; 30 g

2.8.4 13

(a)

$$\begin{aligned}
\frac{dh}{dt} &= -\frac{A_h}{A_w}\sqrt{2gh} \\
\frac{1}{\sqrt{h}}\frac{dh}{dt} &= -\frac{A_h}{A_w}\sqrt{2g} \\
2\sqrt{h} &= -\frac{A_h}{A_w}\sqrt{2g}t + c \\
\sqrt{h} &= c - \frac{A_h}{A_w}\sqrt{\frac{g}{2}}t \\
h &= \left(c - \frac{A_h}{A_w}\sqrt{\frac{g}{2}}t\right)^2
\end{aligned}$$

$$h(0) = H = c^2 \Rightarrow c = \sqrt{H}$$

$$h = \left(\sqrt{H} - \frac{A_h}{A_w}\sqrt{\frac{g}{2}}t\right)^2 = \left(\sqrt{H} - 4\frac{A_h}{A_w}t\right)^2$$

Interval of definition is $[0, \frac{A_w\sqrt{H}}{4A_h}]$

(b) $1821\,\mathrm{s} = 30\,\mathrm{min}$

2.8.5 15

(a)

$$\begin{aligned}
\frac{dh}{dt} &= -\frac{5}{6h^{3/2}} \\
h^{3/2}\frac{dh}{dt} &= -\frac{5}{6} \\
\frac{2}{5}h^{5/2} &= -\frac{5}{6}t + c \\
h &= \left(c - \frac{25}{12}t\right)^{2/5}
\end{aligned}$$

$$h(0) = H = c^{2/5} \Rightarrow c = H^{5/2}$$

$$h = \left(H^{5/2} - \frac{25}{12}t\right)^{2/5}$$

$$0 = \left(H^{5/2} - \frac{25}{12}t\right)^{2/5} \Rightarrow t = \frac{12}{25}H^{5/2} = 858\,\mathrm{s}$$

(b)

$$
\begin{aligned}
V(h) &= \pi r^2 \frac{h}{3} \\
&= \pi \left(h \tan \frac{\pi}{6} \right)^2 \frac{h}{3} \\
&= \pi \left(\frac{h}{\sqrt{3}} \right)^2 \frac{h}{3} \\
&= \frac{1}{9} \pi h^3
\end{aligned}
$$

$$
\begin{aligned}
\frac{dV}{dt} &= -cA_h \sqrt{2gh} \\
\frac{d}{dt} \left(\frac{1}{9} \pi h^3 \right) &= -cA_h \sqrt{2gh} \\
\frac{1}{3} \pi h^2 \frac{dh}{dt} &= -cA_h \sqrt{2gh} \\
h^{3/2} \frac{dh}{dt} &= -\frac{24}{\pi} cA_h \\
\frac{2}{5} h^{5/2} &= c_1 - \frac{24}{\pi} cA_h t \\
h &= \left(c_1 - \frac{60}{\pi} cA_h t \right)^{2/5}
\end{aligned}
$$

$$
h(0) = H = c_1^{2/5} \Rightarrow c_1 = H^{5/2}
$$

$$
h = \left(H^{5/2} - \frac{60}{\pi} cA_h t \right)^{2/5}
$$

$$
\begin{aligned}
0 &= \left(H^{5/2} - \frac{60}{\pi} cA_h t \right)^{2/5} \\
t &= \frac{\pi H^{5/2}}{60 cA_h} \\
&= 243 \, \mathrm{s}
\end{aligned}
$$

2.8.6 17

(a)

$$\begin{aligned}
m\frac{dv}{dt} &= mg - kv^2 \\
\frac{m}{mg - kv^2}\frac{dv}{dt} &= 1 \\
\sqrt{\frac{m}{gk}}\operatorname{arctanh}\left(\sqrt{\frac{k}{gm}}v\right) &= t + c_1 \\
v &= \sqrt{\frac{gm}{k}}\tanh\left(\sqrt{\frac{gk}{m}}(t + c_1)\right)
\end{aligned}$$

$$\begin{aligned}
v(0) = v_0 &= \sqrt{\frac{gm}{k}}\tanh c_1 \\
c_1 &= \operatorname{arctanh}\sqrt{\frac{k}{gm}}v_0
\end{aligned}$$

(b) $v_t = \sqrt{gm/k}$

(c)

$$\begin{aligned}
s &= \frac{m}{k}\ln\cosh\left(\sqrt{\frac{gk}{m}}t + c_1\right) + c_2 \\
c_2 &= -\frac{m}{k}\ln\cosh c_1
\end{aligned}$$

2.8.7 21

(a) $W = 0$, $W = 2$

(b)

$$\begin{aligned}
\frac{dW}{dt} &= W\sqrt{4 - 2W} \\
\frac{1}{W\sqrt{4 - 2W}}\frac{dW}{dt} &= 1 \\
-\operatorname{arctanh}\left(\frac{1}{2}\sqrt{4 - 2W}\right) &= t + c \\
\frac{1}{2}\sqrt{4 - 2W} &= \tanh(c - t) \\
W &= 2 - 2\tanh^2(c - t) \\
&= 2(1 - \tanh^2(c - t)) \\
&= 2\operatorname{sech}^2(c - t)
\end{aligned}$$

2.9 Modeling with Systems of First-Order DEs

2.9.1 1

$$\begin{aligned}\frac{dx}{dt} &= -\lambda_1 x\\ \ln|x| &= -\lambda_1 t + c_1\\ x &= c_1 e^{-\lambda_1 t}\end{aligned}$$

$$x(0) = x_0 = c_1 e^{-\lambda_1(0)} \Rightarrow c_1 = x_0$$

$$x = x_0 e^{-\lambda_1 t}$$

$$\begin{aligned}\frac{dy}{dt} &= \lambda_1 x - \lambda_2 y\\ &= \lambda_1 x_0 e^{-\lambda_1 t} - \lambda_2 y\\ \frac{dy}{dt} + \lambda_2 y &= \lambda_1 x_0 e^{-\lambda_1 t}\\ \frac{d}{dt}(e^{\lambda_2 t} y) &= \lambda_1 x_0 e^{(\lambda_2 - \lambda_1)t}\\ e^{\lambda_2 t} y &= \frac{\lambda_1}{\lambda_2 - \lambda_1} x_0 e^{(\lambda_2 - \lambda_1)t} + c_2\\ y &= \frac{\lambda_1}{\lambda_2 - \lambda_1} x_0 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}\end{aligned}$$

$$y(0) = 0 = \frac{\lambda_1}{\lambda_2 - \lambda_1} x_0 e^{-\lambda_1(0)} + c_2 e^{-\lambda_2(0)} \Rightarrow c_2 = -\frac{\lambda_1}{\lambda_2 - \lambda_1} x_0$$

$$y = \frac{\lambda_1}{\lambda_2 - \lambda_1} x_0 (e^{-\lambda_1 t} - e^{-\lambda_2 t})$$

$$\begin{aligned}\frac{dz}{dt} &= \lambda_2 y\\ &= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} x_0 (e^{-\lambda_1 t} - e^{-\lambda_2 t})\\ z &= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} x_0 (-\frac{1}{\lambda_1} e^{-\lambda_1 t} + \frac{1}{\lambda_2} e^{-\lambda_2 t}) + c_3\\ &= \frac{\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t}}{\lambda_2 - \lambda_1} x_0 + c_3\end{aligned}$$

$$\begin{aligned} z(0) = 0 &= \frac{\lambda_1 e^{-\lambda_2(0)} - \lambda_2 e^{-\lambda_1(0)}}{\lambda_2 - \lambda_1} x_0 + c_3 \\ &= \frac{\lambda_1 - \lambda_2}{\lambda_2 - \lambda_1} x_0 + c_3 \\ c_3 &= x_0 \end{aligned}$$

$$z = \frac{\lambda_1(e^{-\lambda_2 t} - 1) + \lambda_2(1 - e^{-\lambda_1 t})}{\lambda_2 - \lambda_1} x_0$$

2.9.2 3

$5\,\text{days},\ 20\,\text{days},\ 147\,\text{days}$

2.9.3 5

(a)

$$\begin{aligned} \frac{dP}{dt} &= -(\lambda_A + \lambda_C)P \\ P &= ce^{-(\lambda_A + \lambda_C)t} \end{aligned}$$

$$P(0) = P_0 = ce^{-(\lambda_A + \lambda_C)(0)} \Rightarrow c = P_0$$

$$P = P_0 e^{-(\lambda_A + \lambda_C)t}$$

(b)

$$\frac{1}{2} P_0 = P_0 e^{-(\lambda_A + \lambda_C)t} \Rightarrow t = \frac{\ln 1/2}{-(\lambda_A + \lambda_C)} = 1.25 \times 10^9\,\text{years}$$

(c)

$$\begin{aligned} \frac{dA}{dt} &= \lambda_A P \\ &= \lambda_A P_0 e^{-(\lambda_A + \lambda_C)t} \\ A &= -\frac{\lambda_A}{\lambda_A + \lambda_C} P_0 e^{-(\lambda_A + \lambda_C)t} + c \end{aligned}$$

$$A(0) = 0 = -\frac{\lambda_A}{\lambda_A + \lambda_C} P_0 e^{-(\lambda_A + \lambda_C)(0)} + c \Rightarrow c = \frac{\lambda_A}{\lambda_A + \lambda_C} P_0$$

$$A = \frac{\lambda_A}{\lambda_A + \lambda_C} P_0 (1 - e^{-(\lambda_A + \lambda_C)t})$$

$$\begin{aligned}\frac{dC}{dt} &= \lambda_C P \\ &= \lambda_C P_0 e^{-(\lambda_A+\lambda_C)t} \\ C &= -\frac{\lambda_C}{\lambda_A+\lambda_C} P_0 e^{-(\lambda_A+\lambda_C)t} + c\end{aligned}$$

$$C(0) = 0 = -\frac{\lambda_C}{\lambda_A+\lambda_C} P_0 e^{-(\lambda_A+\lambda_C)(0)} + c \Rightarrow c = \frac{\lambda_C}{\lambda_A+\lambda_C} P_0$$

$$C = \frac{\lambda_C}{\lambda_A+\lambda_C} P_0 (1 - e^{-(\lambda_A+\lambda_C)t})$$

(d)

$$\frac{\lambda_A}{\lambda_A+\lambda_C} = 10.5\%$$

$$\frac{\lambda_C}{\lambda_A+\lambda_C} = 89.5\%$$

2.9.4 7

$$\frac{dx_1}{dt} = 6 - \frac{2}{25}x_1 + \frac{1}{50}x_2$$

$$\frac{dx_2}{dt} = \frac{2}{25}x_1 - \frac{2}{25}x_2$$

2.9.5 9

(a)

$$V_1 = 100 + t$$

$$V_2 = 100 - t$$

$$\frac{dx_1}{dt} = \frac{3}{100-t}x_2 - \frac{2}{100+t}x_1$$

$$\frac{dx_2}{dt} = \frac{2}{100+t}x_1 - \frac{3}{100-t}x_2$$

(b)

$$\frac{dx_1}{dt} = -\frac{dx_2}{dt}$$

This makes sense because it's a closed system. Salt is moving from tank B to tank A.

$$x_1 = c - x_2$$

$$x_1(0) = c - x_2(0) \Rightarrow 100 = c - 50 \Rightarrow c = 150$$

$$\begin{aligned}
\frac{dx_2}{dt} &= \frac{2}{100+t}(150 - x_2) - \frac{3}{100-t}x_2 \\
&= \frac{300}{100+t} - \frac{2}{100+t}x_2 - \frac{3}{100-t}x_2 \\
\frac{dx_2}{dt} + \left(\frac{2}{100+t} + \frac{3}{100-t}\right)x_2 &= \frac{300}{100+t} \\
\frac{d}{dt}(e^{2\ln|100+t|-3ln|100-t|}x_2) &= \frac{300}{100+t}e^{2\ln|100+t|-3ln|100-t|} \\
\frac{d}{dt}\left(\frac{(100+t)^2}{(100-t)^3}x_2\right) &= \frac{300(100+t)}{(100-t)^3} \\
&= \frac{30000}{(100-t)^3} + \frac{300t}{(100-t)^3} \\
\frac{(100+t)^2}{(100-t)^3}x_2 &= \frac{15000}{(100-t)^2} + \frac{300(t-50)}{(100-t)^2} + c
\end{aligned}$$

$$\begin{aligned}
x_2 &= \frac{(100-t)^3}{(100+t)^2}\left(\frac{15000}{(100-t)^2} + \frac{300(t-50)}{(100-t)^2} + c\right) \\
x_2(0) = 50 &= \frac{100^3}{100^2}\left(\frac{15000}{100^2} + \frac{300(-50)}{100^2} + c\right) \\
&= 100c \\
c &= \frac{1}{2}
\end{aligned}$$

$$x_2(30) = 47.4\,\text{lb}$$

2.9.6 15

$$i_1 = i_2 + i_3$$

$$\begin{aligned} i_1 R_1 + \frac{di_2}{dt} L_1 + i_2 R_2 &= E(t) \\ (i_2 + i_3) R_1 + \frac{di_2}{dt} L_1 + i_2 R_2 &= E(t) \end{aligned}$$

$$\begin{aligned} i_1 R_1 + \frac{di_3}{dt} L_2 + i_3 R_3 &= E(t) \\ (i_2 + i_3) R_1 + \frac{di_3}{dt} L_2 + i_3 R_3 &= E(t) \end{aligned}$$

2.9.7 17

$i(0) = i_0$, $s(0) = n - i_0$, $r(0) = 0$; It's consistent because no one leaves the community

2.10 Chapter 2 in Review

2.10.1 1

$y = -A/k$; repeller; attractor

2.10.2 3

$$\frac{dy}{dx} = (y-1)^2(y-3)^2$$

2.10.3 5

$\frac{dy}{dx} = x^n$ is semi-stable for even n, unstable for odd n
$\frac{dy}{dx} = -x^n$ is semi-stable for even n, stable for odd n

2.10.4 9

$$\begin{aligned} (y^2+1)\, dx &= y \sec^2 x \, dy \\ \cos^2 x \, dx &= \frac{y}{y^2+1} \, dy \\ \frac{1}{2}(1+\cos 2x)\, dx &= \frac{1}{2} \frac{2y}{y^2+1} \, dy \\ x + \frac{1}{2} \sin 2x &= \ln |y^2+1| + c \\ 2x + \sin 2x &= 2 \ln |y^2+1| + c \end{aligned}$$

2.10.5 11

$$
\begin{aligned}
(6x+1)y^2\frac{dy}{dx} + 3x^2 + 2y^3 &= 0 \\
(6x+1)y^2\,dy + (3x^2+2y^3)\,dx &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= 3x^2 + 2y^3 \\
f &= x^3 + 2xy^3 + g(y)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial y} = 6xy^2 + g'(y) &= 6xy^2 + y^2 \\
g'(y) &= y^2 \\
g(y) &= \frac{1}{3}y^3
\end{aligned}
$$

$$
\begin{aligned}
f(x,y) &= x^3 + 2xy^3 + \frac{1}{3}y^3 \\
c &= x^3 + 2xy^3 + \frac{1}{3}y^3
\end{aligned}
$$

2.10.6 13

$$
\begin{aligned}
t\frac{dQ}{dt} + Q &= t^4 \ln t \\
\frac{dQ}{dt} + \frac{1}{t}Q &= t^3 \ln t \\
\frac{d}{dt}(tQ) &= t^4 \ln t \\
tQ &= \frac{1}{25}t^5(5\ln t - 1) + c \\
Q &= \frac{1}{25}t^4(5\ln t - 1) + ct^{-1}
\end{aligned}
$$

2.10.7 15

$$
(8xy - 2x)\,dx + (x^2+4)\,dy = 0
$$

$$
M_y = 8x
$$

$$
N_x = 2x
$$

$$\frac{M_y - N_x}{N} = \frac{6x}{x^2+4}$$

$$\mu(x) = e^{3\ln|x^2+4|} = (x^2+4)^3$$

$$(x^2+4)^3(8xy-2x)\,dx + (x^2+4)^4\,dy = 0$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= (x^2+4)^4\\ f(x,y) &= (x^2+4)^4 y + g(x)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial x} = 8x(x^2+4)^3 y + g'(x) &= (8xy-2x)(x^2+4)^3\\ g'(x) &= -2x(x^2+4)^3\\ g(x) &= -\frac{1}{4}(x^2+4)^4\end{aligned}$$

$$\begin{aligned}c &= (y - \frac{1}{4})(x^2+4)^4\\ y &= \frac{1}{4} + c(x^2+4)^{-4}\end{aligned}$$

2.10.8 17

$$\begin{aligned}2\frac{dy}{dx} + (4\cos x)y &= x\\ \frac{dy}{dx} + (2\cos x)y &= \frac{1}{2}x\\ e^{2\sin x}\frac{dy}{dx} + (2\cos x)e^{2\sin x}y &= \frac{1}{2}xe^{2\sin x}\\ \frac{d}{dx}(e^{2\sin x}y) &= \frac{1}{2}xe^{2\sin x}\\ \int_0^x \frac{d}{dx}(e^{2\sin x'}y)\,dx' &= \int_0^x \frac{1}{2}x'e^{2\sin x'}\,dx'\\ e^{2\sin x}y - e^{2\sin 0} &= \int_0^x \frac{1}{2}x'e^{2\sin x'}\,dx'\\ y &= \frac{1}{e^{2\sin x}}\left(1 + \int_0^x \frac{1}{2}x'e^{2\sin x'}\,dx'\right)\end{aligned}$$

2.10.9 19

$$
\begin{aligned}
x\frac{dy}{dx} + 2y &= xe^{x^2} \\
\frac{dy}{dx} + \frac{2}{x}y &= e^{x^2} \\
\frac{d}{dt}(x^2 y) &= x^2 e^{x^2} \\
\int_1^x \frac{d}{dt}(x'^2 y)\, dx' &= \int_1^x x'^2 e^{x'^2}\, dx' \\
x^2 y - 3 &= \int_1^x x'^2 e^{x'^2}\, dx' \\
y &= \frac{3}{x^2} + \frac{1}{x^2}\int_1^x x'^2 e^{x'^2}\, dx'
\end{aligned}
$$

2.10.10 21

$$
\begin{aligned}
\frac{dy}{dx} + y &= e^{-x} \\
\frac{d}{dx}(e^x y) &= 1 \\
e^x y &= x + c_1 \\
y &= (x + c_1)e^{-x}
\end{aligned}
$$

$$y(0) = 5 = c_1$$

$$y = (x+5)e^{-x}$$

$$
\begin{aligned}
\frac{dy}{dx} + y &= 0 \\
\frac{d}{dt}(e^x y) &= 0 \\
e^x y &= c_2 \\
y &= c_2 e^{-x}
\end{aligned}
$$

$$(1+5)e^{-1} = c_2 e^{-1} \Rightarrow c_2 = 6$$

$$y = \begin{cases} (x+5)e^{-x} & 0 \le x < 1 \\ 6e^{-x} & x \ge 1 \end{cases}$$

2.10.11 23

$$\begin{aligned}\sin x\frac{dy}{dx} + (\cos x)y &= 0\\ \frac{dy}{dx} + (\cot x)y &= 0\\ \frac{d}{dx}(y\sin x) &= 0\\ y\sin x &= c\\ y &= c\csc x\end{aligned}$$

$$y(7\pi/6) = -2 = c\csc\frac{7\pi}{6} \Rightarrow c = 1$$

$$y = \csc x$$

$$I = (\pi, 2\pi)$$

2.10.12 25

(a) Because $\sqrt{y}$ isn't defined for $y < 0$

(b)

$$\begin{aligned}\frac{dy}{dx} &= \sqrt{y}\\ y^{-1/2}\frac{dy}{dx} &= 1\\ 2\sqrt{y} &= x + c\\ y &= \frac{1}{4}(x+c)^2\end{aligned}$$

$$y(x_0) = y_0 = \frac{1}{4}(x_0 + c)^2 \Rightarrow c = \sqrt{4y_0} - x_0$$

$$y = \frac{1}{4}(x + \sqrt{4y_0} - x_0)^2$$

2.10.13 29

$$\begin{aligned}\frac{dP}{dt} &= kP\\ P &= P_0e^{kt}\\ P(45) &= 8.99\times 10^9 \text{ people}\end{aligned}$$

2.10.14 31

(a)

$$0.53A_0 = A_0e^{-0.00012097t} \Rightarrow t = 5248 \text{ years ago}$$

(b) 3257 BC

2.10.15 35

(a)

$$\begin{aligned}
k(T - T_m) &= 0 \\
T &= T_m \\
&= T_2 + B(T_1 - T) \\
&= \frac{BT_1 + T_2}{1 + B}
\end{aligned}$$

T_m is the same

(b)

$$\begin{aligned}
\frac{dT}{dt} &= k(T - T_m) \\
&= k(T - (T_2 + B(T_1 - T))) \\
&= k((1 + B)T - BT_1 - T_2) \\
\frac{dT}{dt} - k(1 + B)T &= -k(BT_1 + T_2) \\
\frac{d}{dt}(e^{-k(1+B)t}T) &= -k(BT_1 + T_2)e^{-k(1+B)t} \\
e^{-k(1+B)t}T &= \frac{BT_1 + T_2}{1 + B}e^{-k(1+B)t} + c \\
T &= \frac{BT_1 + T_2}{1 + B} + ce^{k(1+B)t}
\end{aligned}$$

$$\begin{aligned}
T(0) = T_1 &= \frac{BT_1 + T_2}{1 + B} + ce^{k(1+B)(0)} \\
c &= T_1 - \frac{BT_1 + T_2}{1 + B} \\
&= \frac{T_1(1 + B) - BT_1 - T_2}{1 + B} \\
&= \frac{T_1 - T_2}{1 + B}
\end{aligned}$$

$$T = \frac{BT_1 + T_2 + (T_1 - T_2)e^{k(1+B)t}}{1 + B}$$

2.10.16 37

$$
\begin{aligned}
(k_1 + k_2 t)\frac{dq}{dt} + \frac{1}{C}q &= E_0 \\
\frac{dq}{dt} + \frac{1}{C(k_1 + k_2 t)}q &= \frac{E_0}{k_1 + k_2 t} \\
\frac{d}{dt}(e^{\frac{\ln|C(k_1+k_2t)|}{Ck_2}}q) &= \frac{E_0}{k_1 + k_2 t}e^{\frac{\ln|C(k_1+k_2t)|}{Ck_2}} \\
\frac{d}{dt}((C(k_1 + k_2 t))^{1/Ck_2}q) &= \frac{E_0}{k_1 + k_2 t}(C(k_1 + k_2 t))^{1/Ck_2} \\
(C(k_1 + k_2 t))^{1/Ck_2}q &= E_0 C(C(k_1 + k_2 t))^{1/Ck_2} + c \\
q &= E_0 C + c(C(k_1 + k_2 t))^{-1/Ck_2}
\end{aligned}
$$

$$
\begin{aligned}
q(0) = q_0 &= E_0 C + c(C(k_1 + k_2(0)))^{-1/Ck_2} \\
q_0 &= E_0 C + c(Ck_1)^{-1/Ck_2} \\
c &= (q_0 - E_0 C)(Ck_1)^{1/Ck_2}
\end{aligned}
$$

$$
\begin{aligned}
q &= E_0 C + (q_0 - E_0 C)(Ck_1)^{1/Ck_2}(C(k_1 + k_2 t))^{-1/Ck_2} \\
&= E_0 C + (q_0 - E_0 C)\left(\frac{k_1}{k_1 + k_2 t}\right)^{1/Ck_2}
\end{aligned}
$$

2.10.17 39

$$
\begin{aligned}
\frac{dh}{dt} &= -c\frac{\pi r_h^2}{\pi r_w^2}\sqrt{2gh} \\
\frac{1}{\sqrt{h}}\frac{dh}{dt} &= -8c(r_h/r_w)^2 \\
2\sqrt{h} &= c_1 - 8c(r_h/r_w)^2 t \\
h &= (c_1 - 4c(r_h/r_w)^2 t)^2
\end{aligned}
$$

$$
h(0) = 2 = (c_1 - 4c(r_h/r_w)^2(0))^2 \Rightarrow c_1 = \sqrt{2}
$$

$$
h = (\sqrt{2} - 4c(r_h/r_w)^2 t)^2 = (\sqrt{2} - (1.63 \times 10^{-5})t)^2
$$

2.10.18 43

$$
\begin{aligned}
\frac{dx}{dt} &= k_1 x(\alpha - x) \\
\frac{1}{x(\alpha - x)}\frac{dx}{dt} &= k_1 \\
\left(\frac{1}{x} + \frac{1}{\alpha - x}\right)\frac{dx}{dt} &= \alpha k_1 \\
\ln|x| - \ln|\alpha - x| &= \alpha k_1 t + c_1 \\
\ln\left|\frac{x}{\alpha - x}\right| &= \alpha k_1 t + c_1 \\
\frac{x}{\alpha - x} &= c_1 e^{\alpha k_1 t} \\
x &= (\alpha - x)c_1 e^{\alpha k_1 t} \\
(c_1 e^{\alpha k_1 t} + 1)x &= \alpha c_1 e^{\alpha k_1 t} \\
x &= \frac{\alpha c_1 e^{\alpha k_1 t}}{c_1 e^{\alpha k_1 t} + 1}
\end{aligned}
$$

$$
\begin{aligned}
\frac{dy}{dt} &= k_2 x y \\
&= k_2 \frac{\alpha c_1 e^{\alpha k_1 t}}{c_1 e^{\alpha k_1 t} + 1} y \\
\frac{1}{y}\frac{dy}{dt} &= k_2 \frac{\alpha c_1 e^{\alpha k_1 t}}{c_1 e^{\alpha k_1 t} + 1} \\
\ln|y| &= \frac{k_2}{k_1}\ln|c_1 e^{\alpha k_1 t} + 1| + c_2 \\
y &= c_2(c_1 e^{\alpha k_1 t} + 1)^{k_2/k_1}
\end{aligned}
$$

2.10.19 45

$$
\begin{aligned}
\frac{dP}{dt} &= kP \ln\frac{450}{P} \\
\frac{1}{P\ln(450/P)}\frac{dP}{dt} &= k \\
-\ln(\ln\frac{450}{P}) &= kt + c \\
\ln\frac{450}{P} &= ce^{-kt} \\
\frac{450}{P} &= e^{ce^{-kt}} \\
P &= \frac{450}{e^{ce^{-kt}}}
\end{aligned}
$$

$$P(0) = 40 = \frac{450}{e^{ce^{-k(0)}}} \Rightarrow c = \ln\frac{450}{40} = 2.42$$

$$\begin{aligned}
P(15) = 95 &= \frac{450}{e^{2.42e^{-k(15)}}} \\
2.42e^{-15k} &= \ln\frac{450}{95} \\
k &= -\frac{\ln(\ln(450/95)/2.42)}{15} \\
&= 0.0295
\end{aligned}$$

$$P(30) = \frac{450}{e^{2.42e^{-0.0295(30)}}} = 166$$

2.10.20 47

$$\begin{aligned}
y &= c_1 x \\
\frac{dy}{dx} &= c_1
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{1}{c_1} \\
y &= -\frac{1}{c_1}x + c_2
\end{aligned}$$

2.10.21 49

$$\begin{aligned}
y &= -x - 1 + c_1 e^x \\
\frac{dy}{dx} &= c_1 e^x - 1
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{1}{c_1 e^x - 1} \\
y &= x - \ln(1 - c_1 e^x) + c_2
\end{aligned}$$

3 Higher-Order Differential Equations

3.1 Theory of Linear Equations

3.1.1 1

$$\begin{aligned} y &= c_1 e^{x} + c_2 e^{-x} \\ 0 &= c_1 e^{(0)} + c_2 e^{-(0)} \\ &= c_1 + c_2 \end{aligned}$$

$$\begin{aligned} y' &= c_1 e^{x} - c_2 e^{-x} \\ 1 &= c_1 e^{(0)} - c_2 e^{-(0)} \\ &= c_1 - c_2 \end{aligned}$$

$$c_2 = c_1 - 1 \Rightarrow 0 = c_1 + c_1 - 1 \Rightarrow c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}$$

$$y = \frac{1}{2}(e^{x} - x^{-x})$$

3.1.2 3

$$\begin{aligned} y &= c_1 x + c_2 x \ln x \\ 3 &= c_1(1) + c_2(1)\ln(1) \\ &= c_1 \end{aligned}$$

$$\begin{aligned} y' &= 3 + c_2(1 + \ln x) \\ -1 &= 3 + c_2(1 + \ln(1)) \\ c_2 &= -4 \end{aligned}$$

$$y = 3x - 4x \ln x$$

3.1.3 9

$(-\infty, 2)$

3.1.4 11

(a)

$$
\begin{aligned}
y &= c_1 e^{x} + c_2 e^{-x} \\
0 &= c_1 e^{(0)} + c_2 e^{-(0)} \\
&= c_1 + c_2
\end{aligned}
$$

$$
\begin{aligned}
1 &= c_1 e^{(1)} + c_2 e^{-(1)} \\
&= c_1 e + c_2 e^{-1} \\
&= c_1 e - c_1 e^{-1} \\
&= c_1 (e - e^{-1}) \\
c_1 &= \frac{1}{e - e^{-1}} \\
c_2 &= -\frac{1}{e - e^{-1}}
\end{aligned}
$$

$$
y = \frac{e^{x} - e^{-x}}{e - e^{-1}}
$$

(b)

$$
\begin{aligned}
y &= c_3 \cosh x + c_4 \sinh x \\
0 &= c_3 \cosh 0 + c_4 \sinh 0 \\
&= c_3
\end{aligned}
$$

$$
\begin{aligned}
y &= c_4 \sinh x \\
1 &= c_4 \sinh 1 \\
c_4 &= \operatorname{csch} 1
\end{aligned}
$$

$$
y = (\operatorname{csch} 1) \sinh x
$$

(c)

$$
(\operatorname{csch} 1) \sinh x = \frac{2}{e - e^{-1}} \frac{e^{x} - e^{-x}}{2} = \frac{e^{x} - e^{-x}}{e - e^{-1}}
$$

3.1.5 13

(a)

$$\begin{aligned}
y &= c_1 e^x \cos x + c_2 e^x \sin x \\
1 &= c_1 e^{(0)} \cos(0) + c_2 e^{(0)} \sin(0) \\
&= c_1
\end{aligned}$$

$$\begin{aligned}
y' &= e^x \cos x - e^x \sin x + c_2 e^x \sin x + c_2 e^x \cos x \\
0 &= e^{(\pi)} \cos(\pi) - e^{(\pi)} \sin(\pi) + c_2 e^{(\pi)} \sin(\pi) + c_2 e^{(\pi)} \cos(\pi) \\
&= -e^\pi - c_2 e^\pi \\
c_2 &= -1
\end{aligned}$$

$$y = e^x \cos x - e^x \sin x$$

(b)

$$\begin{aligned}
y &= c_1 e^x \cos x + c_2 e^x \sin x \\
1 &= c_1 e^{(0)} \cos(0) + c_2 e^{(0)} \sin(0) \\
&= c_1
\end{aligned}$$

$$\begin{aligned}
y &= c_1 e^x \cos x + c_2 e^x \sin x \\
-1 &= e^{(\pi)} \cos(\pi) + c_2 e^{(\pi)} \sin(\pi) \\
&= -e^\pi
\end{aligned}$$

No solution

(c)

$$c_1 = 1$$

$$\begin{aligned}
y &= c_1 e^x \cos x + c_2 e^x \sin x \\
1 &= e^{(\pi/2)} \cos\left(\frac{\pi}{2}\right) + c_2 e^{(\pi/2)} \sin\left(\frac{\pi}{2}\right) \\
&= c_2 e^{\pi/2} \\
c_2 &= e^{-\pi/2}
\end{aligned}$$

$$y = e^x \cos x + e^{x-\pi/2} \sin x$$

(d)

$$
\begin{aligned}
y &= c_1 e^x \cos x + c_2 e^x \sin x \\
0 &= c_1 e^{(0)} \cos(0) + c_2 e^{(0)} \sin(0) \\
&= c_1 e \\
&= c_1
\end{aligned}
$$

$$
\begin{aligned}
y &= c_1 e^x \cos x + c_2 e^x \sin x \\
0 &= c_2 e^{(\pi)} \sin(\pi)
\end{aligned}
$$

$$y = c_2 e^x \sin x$$

3.1.6 15

Dependent

3.1.7 17

Dependent

3.1.8 19

Dependent

3.1.9 21

Independent

3.1.10 23

$$y'' - y' - 12y = 9e^{-3x} + 3e^{-3x} - 12e^{-3x} = 0$$

$$y'' - y' - 12y = 16e^{4x} - 4e^{4x} - 12e^{4x} = 0$$

Both functions are solutions of the differential equation and are linearly independent, so they form a fundamental set of solutions.

$$y = c_1 e^{-3x} + c_2 e^{4x}$$

3.1.11 25

$$\begin{aligned} y'' - 2y' + 5y &= e^x \cos 2x - 2e^x \sin 2x - 2e^x \sin 2x - 4e^x \cos 2x \\ &\quad - 2(e^x \cos 2x - 2e^x \sin 2x) + 5e^x \cos 2x \\ &= 0 \end{aligned}$$

$$\begin{aligned} y'' - 2y' + 5y &= e^x \sin 2x + 2e^x \cos 2x + 2e^x \cos 2x - 4e^x \sin 2x \\ &\quad - 2(e^x \sin 2x + 2e^x \cos 2x) + 5e^x \sin 2x \\ &= 0 \end{aligned}$$

$$\begin{aligned} W(e^x \cos 2x, e^x \sin 2x) &= \begin{vmatrix} e^x \cos 2x & e^x \sin 2x \\ e^x \cos 2x - 2e^x \sin 2x & e^x \sin 2x + 2e^x \cos 2x \end{vmatrix} \\ &= e^x \cos 2x(e^x \sin 2x + 2e^x \cos 2x) \\ &\quad - e^x \sin 2x(e^x \cos 2x - 2e^x \sin 2x) \\ &= e^{2x}(\sin 2x \cos 2x + 2\cos^2 2x - \sin 2x \cos 2x + 2\sin^2 2x) \\ &= 2e^{2x} \end{aligned}$$

Both functions are solutions to the differential equation and the Wronskian does not equal 0 for all x in the interval.

$$y = c_1 e^x \cos 2x + c_2 e^x \sin 2x$$

3.1.12 27

$$\begin{aligned} x^2 y'' - 6xy' + 12y &= x^2(6x) - 6x(3x^2) + 12(x^3) \\ &= 6x^3 - 18x^3 + 12x^3 \\ &= 0 \end{aligned}$$

$$\begin{aligned} x^2 y'' - 6xy' + 12y &= x^2(12x^2) - 6x(4x^3) + 12(x^4) \\ &= 12x^4 - 24x^4 + 12x^4 \\ &= 0 \end{aligned}$$

$$\begin{aligned} W(x^3, x^4) &= \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix} \\ &= (x^3)(4x^3) - (x^4)(3x^2) \\ &= 4x^6 - 3x^6 \\ &= x^6 \end{aligned}$$

Both functions are solutions to the differential equation and, because 0 isn't included in the interval, the Wronskian does not equal 0 for all x in the interval.

$$y = c_1x^3 + c_2x^4$$

3.1.13 35

(a)

$$y'' - 6y' + 5y = 12e^{2x} - 6(6e^{2x}) + 5(3e^{2x}) = -9e^{2x}$$

$$y'' - 6y' + 5y = 2 - 6(2x+3) + 5(x^2+3x) = 5x^2 + 3x - 16$$

(b)

$$y = 3e^{2x} + x^2 + 3x$$

$$y = -\frac{1}{9}(3e^{2x}) - 2(x^2+3x) = -\frac{1}{3}e^{2x} - 2(x^2+3x)$$

3.2 Reduction of Order

3.2.1 1

$$\begin{aligned}
y_2(x) &= u(x)e^{2x} \\
y_2'(x) &= u'(x)e^{2x} + 2u(x)e^{2x} \\
&= (u'(x) + 2u(x))e^{2x} \\
y_2''(x) &= u''(x)e^{2x} + 2u'(x)e^{2x} + 2u'(x)e^{2x} + 4u(x)e^{2x} \\
&= (u''(x) + 4u'(x) + 4u(x))e^{2x}
\end{aligned}$$

$$\begin{aligned}
y'' - 4y' + 4y &= 0 \\
(u''(x) + 4u'(x) + 4u(x))e^{2x} - 4(u'(x) + 2u(x))e^{2x} + 4u(x)e^{2x} &= 0 \\
u''(x) &= 0 \\
u'(x) &= c_1 \\
u(x) &= c_1x + c_2
\end{aligned}$$

$$y_2(x) = xe^{2x}$$

3.2.2 3

$$\begin{aligned}
y_2(x) &= u(x)y_1(x) \\
&= u(x)\cos 4x \\
y_2'(x) &= u'(x)\cos 4x - 4u(x)\sin 4x \\
y_2''(x) &= u''(x)\cos 4x - 4u'(x)\sin 4x - 4u'(x)\sin 4x - 16u(x)\cos 4x \\
&= u''(x)\cos 4x - 8u'(x)\sin 4x - 16u(x)\cos 4x
\end{aligned}$$

$$\begin{aligned}
y'' + 16y &= 0 \\
u''(x)\cos 4x - 8u'(x)\sin 4x - 16u(x)\cos 4x + 16u(x)\cos 4x &= 0 \\
u''(x)\cos 4x - 8u'(x)\sin 4x &= 0 \\
u''(x) - 8(\tan 4x)u'(x) &= 0
\end{aligned}$$

$$\begin{aligned}
e^{\int -8\tan 4x\,dx}u''(x) - 8(\tan 4x)e^{\int -8\tan 4x\,dx}u'(x) &= 0 \\
\frac{d}{dx}(u'(x)\cos^2 4x) &= 0 \\
u'(x)\cos^2 4x &= c_1 \\
u'(x) &= c_1\sec^2 4x \\
u(x) &= c_1\tan 4x + c_2
\end{aligned}$$

$$y_2 = \sin 4x$$

3.2.3 5

$$\begin{aligned}
y_2(x) &= u(x)y_1(x) \\
&= u(x)\cosh x \\
y_2'(x) &= u'(x)\cosh x + u(x)\sinh x \\
y_2''(x) &= u''(x)\cosh x + u'(x)\sinh x + u'(x)\sinh x + u(x)\cosh(x) \\
&= u''(x)\cosh x + 2u'(x)\sinh x + u(x)\cosh x
\end{aligned}$$

$$\begin{aligned}
y'' - y &= 0 \\
u''(x)\cosh x + 2u'(x)\sinh x + u(x)\cosh x - u(x)\cosh x &= 0 \\
u''(x)\cosh x + 2u'(x)\sinh x &= 0 \\
u''(x) + 2(\tanh x)u'(x) &= 0
\end{aligned}$$

$$
\begin{aligned}
e^{\int 2\tanh x\,dx}u''(x) + 2(\tanh x)e^{\int 2\tanh x\,dx}u'(x) &= 0 \\
\frac{d}{dx}(u'(x)\cosh^2 x) &= 0 \\
u'(x)\cosh^2 x &= c_1 \\
u'(x) &= c_1 \operatorname{sech}^2 x \\
u(x) &= c_1 \tanh x + c_2
\end{aligned}
$$

$$y_2 = \sinh x$$

3.2.4 7

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \\
&= e^{2x/3} \int \frac{e^{4x/3}}{e^{4x/3}}\,dx \\
&= xe^{2x/3}
\end{aligned}
$$

3.2.5 9

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \\
&= x^4 \int \frac{x^7}{x^8}\,dx \\
&= x^4 \ln|x|
\end{aligned}
$$

3.2.6 11

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \\
&= \ln x \int \frac{1}{x \ln^2 x}\,dx \\
&= -1
\end{aligned}
$$

3.2.7 13

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \\
&= x\sin(\ln x) \int \frac{1}{x\sin^2(\ln x)}\,dx \\
&= x\cos(\ln x)
\end{aligned}
$$

3.2.8 15

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \\
&= (x+1) \int \frac{1-2x-x^2}{(x+1)^2}\,dx \\
&= (x+1)\left(-x-\frac{2}{x+1}\right) \\
&= x^2+x+2
\end{aligned}
$$

3.2.9 17

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \\
&= e^{-2x} \int \frac{e^{-\int 0\,dx}}{(e^{-2x})^2}\,dx \\
&= e^{-2x} \int e^{4x}\,dx \\
&= e^{2x}
\end{aligned}
$$

$$
y_p(x) = -\frac{1}{2}
$$

3.2.10 19

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \\
&= e^{x} \int \frac{e^{3x}}{e^{2x}}\,dx \\
&= e^{2x}
\end{aligned}
$$

$$
y_p(x) = \frac{5}{2}e^{3x}
$$

3.2.11 21

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \\
&= x \int \frac{e^{-\int (1-x^{-1})\,dx}}{x^2}\,dx \\
&= x \int \frac{e^{\ln|x|-x}}{x^2}\,dx \\
&= x \int \frac{1}{xe^x}\,dx \\
&= x \int_{x_0}^{x} \frac{1}{te^t}\,dt
\end{aligned}
$$