Introduction to Quantum Mechanics by David J. Griffiths Problems

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Part I
Theory

1 The Wave Function

1.1

(a)

$$\begin{aligned}
\langle j^2 \rangle &= \sum j^2 P(j) \\
&= 14^2 \frac{1}{14} + 15^2 \frac{1}{14} + 16^2 \frac{3}{14} + 22^2 \frac{2}{14} + 24^2 \frac{2}{14} + 25^2 \frac{5}{14} \\
&= \frac{3217}{7} \\
&\approx 459.571 \\
\langle j \rangle^2 &= \left(\sum j P(j) \right)^2 \\
&= 441
\end{aligned}$$

(b)

$$\begin{aligned}
\Delta j_{14} &= -7 \\
\Delta j_{15} &= -6 \\
\Delta j_{16} &= -5 \\
\Delta j_{22} &= 1 \\
\Delta j_{24} &= 3 \\
\Delta j_{25} &= 4 \\
\sigma^2 &= \sum (\Delta j)^2 P(j) \\
&= \frac{130}{7} \\
&\approx 18.571
\end{aligned}$$

(c)

$$\sigma^2 = \sqrt{\langle j^2 \rangle - \langle j \rangle^2} = 18.571$$

1.2

(a)

$$\begin{aligned}
\langle x^2 \rangle &= \int_0^h x^2 \rho(x)\, dx \\
&= \int_0^h \frac{x^{3/2}}{2\sqrt{h}}\, dx \\
&= \frac{1}{2\sqrt{h}} \left[\frac{2}{5} x^{5/2} \right]_0^h \\
&= \frac{h^2}{5} \\
\langle x \rangle^2 &= \frac{h^2}{9} \\
\sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{h^2}{5} - \frac{h^2}{9}} \\
&= h\sqrt{\frac{4}{45}} \\
&= \frac{2}{3\sqrt{5}} h
\end{aligned}$$

(b)

$$
\begin{aligned}
1 - \int_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} \rho(x)\,dx &= 1 - \frac{1}{2\sqrt{h}}[2\sqrt{x}]_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} \\
&= 1 - \frac{1}{\sqrt{h}}\left(\sqrt{\frac{1}{3}h + \frac{2}{3\sqrt{5}}h} - \sqrt{\frac{1}{3}h - \frac{2}{3\sqrt{5}}h}\right) \\
&= 1 - \left(\sqrt{\frac{1}{3} + \frac{2}{3\sqrt{5}}} - \sqrt{\frac{1}{3} - \frac{2}{3\sqrt{5}}}\right) \\
&\approx 0.393
\end{aligned}
$$

1.3

(a)

$$
\begin{aligned}
\rho(x) &= Ae^{-\lambda(x-a)^2} \\
1 &= \int_{-\infty}^{\infty} \rho(x)\,dx \\
&= A\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2}\,dx \\
&= A\sqrt{\frac{\pi}{\lambda}} \\
A &= \sqrt{\frac{\lambda}{\pi}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\langle x \rangle &= \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} xe^{-\lambda(x-a)^2}\,dx \\
&= a \\
\langle x^2 \rangle &= \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} x^2e^{-\lambda(x-a)^2}\,dx \\
&= a^2 + \frac{1}{2\lambda} \\
\sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{a^2 + \frac{1}{2\lambda} - a^2} \\
&= \frac{1}{\sqrt{2\lambda}}
\end{aligned}
$$

1.4

(a)

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi(x,0)|^2 \, dx \\
&= \left(\frac{A}{a}\right)^2 \int_0^a x^2 \, dx + \left(\frac{A}{b-a}\right)^2 \int_a^b (b-x)^2 \, dx \\
&= \frac{1}{3}A^2 a + \left(\frac{A}{b-a}\right)^2 \left[-\frac{1}{3}(b-x)^3\right]_a^b \\
&= \frac{1}{3}A^2 a + \frac{1}{3}A^2(b-a) \\
&= \frac{1}{3}A^2 b \\
A &= \sqrt{\frac{3}{b}}
\end{aligned}$$

(c) $x = a$

(d)

$$\begin{aligned}
\int_0^a |\Psi(x,0)|^2 \, dx &= \frac{3}{a^2 b}\left[\frac{1}{3}x^3\right]_0^a \\
&= \frac{a}{b}
\end{aligned}$$

(e)

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x|\Psi(x,0)|^2 \, dx \\
&= \frac{3}{a^2 b}\left[\frac{1}{4}x^4\right]_0^a + \frac{3}{b(b-a)^2}\int_a^b x(b-x)^2 \, dx \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2}\int_a^b (b^2 x - 2bx^2 + x^3) \, dx \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2}\left[\frac{1}{2}b^2x^2 - \frac{2}{3}bx^3 + \frac{1}{4}x^4\right]_a^b \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2}\left(\frac{1}{2}b^4 - \frac{2}{3}b^4 + \frac{1}{4}b^4 - \frac{1}{2}a^2b^2 + \frac{2}{3}a^3b - \frac{1}{4}a^4\right) \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2}\frac{1}{12}(b-a)^3(3a+b) \\
&= \frac{3a^2}{4b} + \frac{1}{4b}(3ab + b^2 - 3a^2 - ab) \\
&= \frac{1}{2}a + \frac{1}{4}b
\end{aligned}$$

1.5

(a)

$$\begin{aligned}
\Psi(x,t) &= Ae^{-\lambda|x|}e^{-i\omega t} \\
\Psi(x,0) &= Ae^{-\lambda|x|} \\
1 &= A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|}\,dx \\
&= 2A^2 \int_0^{\infty} e^{-2\lambda x}\,dx \\
&= 2A^2 \left[-\frac{1}{2\lambda}e^{-2\lambda x}\right]_0^{\infty} \\
&= \frac{A^2}{\lambda} \\
A &= \sqrt{\lambda}
\end{aligned}$$

(b)

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x\lambda e^{-2\lambda|x|}\,dx \\
&= \lambda \int_{-\infty}^{\infty} xe^{-2\lambda|x|}\,dx \\
&= 0 \\
\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2\lambda e^{-2\lambda|x|}\,dx \\
&= 2\lambda \int_0^{\infty} x^2 e^{-2\lambda x}\,dx \\
&= \frac{1}{2\lambda^2}
\end{aligned}$$

(c)

$$\begin{aligned}
\sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \frac{1}{\sqrt{2}\lambda} \\
1 - \int_{-\sigma}^{\sigma} \lambda e^{-2\lambda|x|}\,dx &= 1 - 2\lambda \int_0^{\sigma} e^{-2\lambda x}\,dx \\
&= 1 - 2\lambda \left[-\frac{1}{2\lambda}e^{-2\lambda x}\right]_0^{\sigma} \\
&= e^{-2\lambda\sigma} \\
&= e^{-\sqrt{2}} \\
&\approx 0.243
\end{aligned}$$

1.6

The chain rule requires that you apply it to both x and $|\Psi|^2$ which gives the same result

$$\begin{aligned}
\frac{d\langle x\rangle}{dt} &= \frac{d}{dt}\int x|\Psi|^2\,dx \\
&= \int \frac{d}{dt}(x|\Psi|^2)\,dx \\
&= \int \left(0\cdot|\Psi|^2 + x\frac{\partial|\Psi|^2}{\partial t}\right)dx \\
&= \int x\frac{\partial|\Psi|^2}{\partial t}\,dx
\end{aligned}$$

1.8

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\left(e^{-iV_0t/\hbar}\Psi\right) &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(e^{-iV_0t/\hbar}\Psi\right) + (V+V_0)\left(e^{-iV_0t/\hbar}\Psi\right) \\
i\hbar\left(-\frac{iV_0}{\hbar}e^{-iV_0t/\hbar}\Psi + e^{-iV_0t/\hbar}\frac{\partial\Psi}{\partial t}\right) &= -\frac{\hbar^2}{2m}e^{-iV_0t/\hbar}\frac{\partial^2\Psi}{\partial x^2} + Ve^{-iV_0t/\hbar}\Psi + V_0e^{-iV_0t/\hbar}\Psi \\
V_0\Psi + i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi + V_0\Psi \\
i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi
\end{aligned}$$

$$\begin{aligned}
\langle Q(x,p)\rangle &= \int \left(e^{-iV_0t/\hbar}\Psi\right)^*\left[Q(x,-i\hbar\partial/\partial x)\right]e^{-iV_0t/\hbar}\Psi\,dx \\
&= \int e^{iV_0t/\hbar}\Psi^*\left[Q(x,-i\hbar\partial/\partial x)\right]e^{-iV_0t/\hbar}\Psi\,dx \\
&= \int \Psi^*[Q(x,-i\hbar\partial/\partial x)]\Psi\,dx
\end{aligned}$$

No effect on the expectation value.

1.9

(a)

$$
\begin{aligned}
\Psi(x,t) &= Ae^{-a[(mx^2/\hbar)+it]} \\
1 &= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)}\,dx \\
&= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)}\,dx \\
&= A^2 \sqrt{\frac{\pi\hbar}{2am}} \\
A^2 &= \sqrt{\frac{2am}{\pi\hbar}} \\
A &= \left(\frac{2am}{\pi\hbar}\right)^{1/4}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\Psi &= Ae^{-a[(mx^2/\hbar)+it]} \\
\frac{\partial\Psi}{\partial t} &= -ia\Psi \\
\frac{\partial\Psi}{\partial x} &= -\frac{2amx}{\hbar}\Psi \\
\frac{\partial^2\Psi}{\partial x^2} &= -\frac{2am}{\hbar}\left(\Psi + x\frac{\partial\Psi}{\partial x}\right) \\
&= -\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\Psi \\
V\Psi &= i\hbar\frac{\partial\Psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} \\
&= a\hbar\Psi - a\hbar\left(1 - \frac{2amx^2}{\hbar}\right)\Psi \\
V &= a\hbar - a\hbar + 2a^2mx^2 \\
&= 2a^2mx^2
\end{aligned}
$$

(c)

$$
\begin{aligned}
\langle x \rangle &= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)} x \, dx \\
&= 0 \\
\left\langle x^2 \right\rangle &= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)} x^2 \, dx \\
&= 2A^2 \int_{0}^{\infty} e^{-2a(mx^2/\hbar)} x^2 \, dx \\
&= \frac{\hbar}{4am} \\
\langle p \rangle &= \int_{-\infty}^{\infty} \Psi^* \left[-i\hbar \frac{\partial}{\partial x} \right] \Psi \, dx \\
&= -i\hbar \int_{-\infty}^{\infty} A e^{-a[(mx^2/\hbar)-it]} \left(-\frac{2amx}{\hbar} A e^{-a[(mx^2/\hbar)+it]} \right) dx \\
&= 2iA^2 am \int_{-\infty}^{\infty} x e^{-2amx^2/\hbar} \, dx \\
&= 0 \\
\left\langle p^2 \right\rangle &= \int_{-\infty}^{\infty} \Psi^* \left[-\hbar^2 \frac{\partial^2}{\partial x^2} \right] \Psi \, dx \\
&= -\hbar^2 \int_{-\infty}^{\infty} A e^{-a[(mx^2/\hbar)-it]} \left[-\frac{2am}{\hbar} \left(1 - \frac{2amx^2}{\hbar} \right) A e^{-a[(mx^2/\hbar)+it]} \right] dx \\
&= 2A^2 am\hbar \int_{-\infty}^{\infty} e^{-2amx^2/\hbar} \left(1 - \frac{2amx^2}{\hbar} \right) dx \\
&= am\hbar
\end{aligned}
$$

(d)

$$
\begin{aligned}
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{\hbar}{4am}} \\
\sigma_p &= \sqrt{am\hbar} \\
\sigma_x \sigma_p &= \sqrt{\frac{1}{4}\hbar^2} \\
&= \frac{1}{2}\hbar \\
&\geq \frac{1}{2}\hbar
\end{aligned}
$$

Yes, this is consistent with Heisenberg's uncertainty principle.

1.10

(a)

$$
\begin{aligned}
P(0) &= 0 \\
P(1) &= \frac{2}{25} \\
&= 0.08 \\
P(2) &= \frac{3}{25} \\
&= 0.12 \\
P(3) &= \frac{1}{5} \\
&= 0.2 \\
P(4) &= \frac{3}{25} \\
&= 0.12 \\
P(5) &= \frac{3}{25} \\
&= 0.2 \\
P(6) &= \frac{3}{25} \\
&= 0.2 \\
P(7) &= \frac{1}{25} \\
&= 0.04 \\
P(8) &= \frac{2}{25} \\
&= 0.08 \\
P(9) &= \frac{3}{25} \\
&= 0.12
\end{aligned}
$$

(b) The most probable digit is 3, the median digit is 4, and the average value is $\frac{118}{25} = 4.72$.

(c) $\sigma = 2.474$

1.14

(a)

$$\begin{aligned}
P_{ab}(t) &= \int_a^b |\Psi(x,t)|^2\,dx \\
\frac{dP_{ab}}{dt} &= \frac{d}{dt}\int_a^b |\Psi(x,t)|^2\,dx \\
&= \int_a^b \frac{d}{dt}\left(|\Psi(x,t)|^2\right)\,dx \\
&= \int_a^b \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\right]\,dx \\
&= J(a,t) - J(b,t)
\end{aligned}$$

The units are s^{-1}.

(b)

$$\begin{aligned}
\Psi(x,t) &= Ae^{-a[(mx^2/\hbar)+it]} \\
\frac{\partial\Psi}{\partial x} &= -\frac{2amx}{\hbar}\Psi \\
\Psi^*(x,t) &= Ae^{-a[(mx^2/\hbar)-it]} \\
\frac{\partial\Psi^*}{\partial x} &= -\frac{2amx}{\hbar}\Psi^* \\
J(x,t) &= \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right) \\
&= \frac{i\hbar}{2m}\left[\Psi\left(-\frac{2amx}{\hbar}\Psi^*\right) - \Psi^*\left(-\frac{2amx}{\hbar}\Psi\right)\right] \\
&= 0
\end{aligned}$$

1.15

$$\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2\,dx &= \int_{-\infty}^{\infty}\left(\frac{\partial\Psi_1^*}{\partial t}\Psi_2+\Psi_1^*\frac{\partial\Psi_2}{\partial t}\right)\,dx \\
&= \int_{-\infty}^{\infty}\left[\left(-i\frac{\hbar}{2m}\frac{\partial^2\Psi_1^*}{\partial x^2}+i\frac{V}{\hbar}\Psi_1^*\right)\Psi_2\right. \\
&\qquad \left.+\Psi_1^*\left(i\frac{\hbar}{2m}\frac{\partial^2\Psi_2}{\partial x^2}-i\frac{V}{\hbar}\Psi_2\right)\right]\,dx \\
&= i\frac{\hbar}{2m}\int_{-\infty}^{\infty}\left(\Psi_1^*\frac{\partial^2\Psi_2}{\partial x^2}-\frac{\partial^2\Psi_1^*}{\partial x^2}\Psi_2\right)\,dx \\
&= i\frac{\hbar}{2m}\left[\Psi_1^*\frac{\partial\Psi_2}{\partial x}\bigg|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{\partial}{\partial x}(\Psi_1^*\Psi_2)\,dx\right. \\
&\qquad \left.\frac{\partial\Psi_1^*}{\partial x}\Psi_2\bigg|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{\partial}{\partial x}(\Psi_1^*\Psi_2)\,dx\right] \\
&= 0
\end{aligned}$$

1.16

(a)

$$\begin{aligned}
1 &= \int_{-a}^{a}A^2(a^2-x^2)^2\,dx \\
&= A^2\int_0^a(a^2-x^2)^2\,dx \\
&= \frac{16}{15}A^2a^5 \\
A &= \sqrt{\frac{15}{16a^5}}
\end{aligned}$$

(b)

$$\begin{aligned}
\langle x\rangle &= \int_{-a}^{a}xA(a^2-x^2)\,dx \\
&= 0
\end{aligned}$$

(c)

$$\begin{aligned}
\langle p\rangle &= \int_{-a}^{a}\Psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi\,dx \\
&= 2iA^2\hbar\int_{-a}^{a}x(a^2-x^2)\,dx \\
&= 0
\end{aligned}$$

(d)

$$
\begin{aligned}
\left\langle x^2 \right\rangle &= \int_{-a}^{a} \Psi^* x^2 \Psi \, dx \\
&= A^2 \int_{-a}^{a} x^2 (a^2 - x^2)^2 \, dx \\
&= A^2 \frac{16}{105} a^7 \\
&= \frac{a^2}{7}
\end{aligned}
$$

(e)

$$
\begin{aligned}
\left\langle p^2 \right\rangle &= \int_{-a}^{a} \Psi^* \left(-\hbar^2 \frac{\partial^2}{\partial x^2} \right) \Psi \, dx \\
&= -\hbar^2 \int_{-a}^{a} A(a^2 - x^2)(-2A) \, dx \\
&= 4A^2\hbar^2 \int_{0}^{a} (a^2 - x^2) \, dx \\
&= 4A^2\hbar^2 \left[a^2 x - \frac{1}{3} x^3 \right]_0^a \\
&= 4A^2\hbar^2 \left(a^3 - \frac{1}{3} a^3 \right) \\
&= \frac{8}{3} A^2 a^3 \hbar^2 \\
&= \frac{8}{3} \frac{15}{16a^5} a^3 \hbar^2 \\
&= \frac{5}{2} \frac{\hbar^2}{a^2}
\end{aligned}
$$

(f)

$$
\begin{aligned}
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{a^2}{7}} \\
&= \frac{a}{\sqrt{7}}
\end{aligned}
$$

(g)

$$
\begin{aligned}
\sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\
&= \sqrt{\frac{5}{2}} \frac{\hbar}{a}
\end{aligned}
$$

(h)

$$\begin{aligned}\sigma_x\sigma_p &= \sqrt{\frac{5}{14}}\hbar\\ &\geq \frac{1}{2}\hbar\end{aligned}$$

1.18

(a)

$$\begin{aligned}\frac{h}{\sqrt{3mk_BT}} &> d\\ \frac{\sqrt{3mk_BT}}{h} &< \frac{1}{d}\\ T_{\text{electron}} &< \frac{h^2}{3d^2mk_B}\\ &< 1.3\times 10^5\,\text{K}\\ T_{\text{nuclei}} &< 2.5\,\text{K}\end{aligned}$$

(b)

$$\begin{aligned}PV &= Nk_BT\\ \frac{V}{N} &= \frac{k_BT}{P}\\ d &= \left(\frac{k_BT}{P}\right)^{1/3}\\ \frac{h}{\sqrt{3mk_Bt}} &> \left(\frac{k_BT}{P}\right)^{1/3}\\ T &< \frac{1}{k_B}\left(\frac{h^2}{3m}\right)^{3/5}P^{2/5}\end{aligned}$$

2 Time-Independent Schrödinger Equation

2.1

(a)

$$\begin{aligned}\int_{-\infty}^{\infty}|\Psi|^2\,dx &= \int_{-\infty}^{\infty}\Psi^*\Psi\,dx\\ &= \int_{-\infty}^{\infty}\psi^*e^{i(E_0-i\Gamma)t/\hbar}\psi e^{-i(E_0+i\Gamma)t/\hbar}\,dx\\ &= e^{2\Gamma t/\hbar}\int_{-\infty}^{\infty}|\psi|^2\,dx\end{aligned}$$

In order for this to equal 1 for all t, Γ must be 0.

(b) If $\psi(x)$ is a complex solution to the time-independent Schrödinger equation then so is $\psi^*(x)$ and $\psi(x) + \psi^*(x)$ which is real.

2.2

If ψ and its second derivative always have the same sign, ψ will increase or decrease without bound forever. This means there is no non-zero choice of constant A such that

$$\int_{-\infty}^{\infty} |A\Psi|^2 \, dx = 1$$

and thus the equation can't be normalised.

The classical analog of this is statements is that the potential energy of a system can't exceed its total energy.

2.3

The time-independent Schrödinger equation in an infinite square well is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi.$$

If $E = 0$ then $\psi = Ax + B$ which isn't normalisable.

If $E < 0$ then $\psi = Ae^{kt} + Be^{-kt}$ where $k \in \mathbb{R}$ which also isn't normalisable.

2.4

$$
\begin{aligned}
\Psi_n(x,t) &= \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)e^{-i(n^2\pi^2\hbar/2ma^2)t} \\
\langle x\rangle &= \int_0^a \Psi_n^* x \Psi_n \, dx \\
&= \frac{2}{a}\int_0^a x\sin^2\left(\frac{n\pi}{a}x\right)\, dx \\
&= \frac{a}{2} \\
\left\langle x^2\right\rangle &= \int_0^a \Psi_n^* x^2 \Psi_n \, dx \\
&= \frac{2}{a}\int_0^a x^2\sin^2\left(\frac{n\pi}{a}x\right)\, dx \\
&= a^2\left(\frac{1}{3}-\frac{1}{2n^2\pi^2}\right) \\
\langle p\rangle &= \int_0^a \Psi_n^*\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi_n\, dx \\
&= -i\frac{2\hbar n\pi}{a^2}\int_0^a \sin\left(\frac{n\pi}{a}x\right)\cos\left(\frac{n\pi}{a}x\right)\, dx \\
&= 0 \\
\left\langle p^2\right\rangle &= \int_0^a \Psi_n^*\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)\Psi_n\, dx \\
&= \frac{2\hbar^2 n^2\pi^2}{a^3}\int_0^a \sin^2\left(\frac{n\pi}{a}x\right)\, dx \\
&= \left(\frac{n\pi\hbar}{a}\right)^2 \\
\sigma_x &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\
&= \frac{a}{2}\sqrt{\frac{1}{3}-\frac{2}{n^2\pi^2}} \\
\sigma_p &= \sqrt{\langle p^2\rangle - \langle p\rangle^2} \\
&= \frac{n\pi\hbar}{a}
\end{aligned}
$$

2.5

(a)

$$
\begin{aligned}
1 &= \int_0^a A^2(\psi_1+\psi_2)^2\,dx \\
&= A^2\int_0^a (\psi_1^2+2\psi_1\psi_2+\psi_2^2)\,dx \\
&= \frac{2A^2}{a}\left[\int_0^a \sin^2\left(\frac{\pi}{a}x\right)\,dx + \int_0^a \sin^2\left(\frac{2\pi}{a}x\right)\,dx\right] \\
&= 2A^2 \\
A &= \frac{1}{\sqrt{2}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi}{a}x\right)e^{-4i\omega t}\right] \\
|\Psi(x,t)|^2 &= \Psi^*\Psi \\
&= \frac{1}{a}\left[\sin\left(\frac{\pi}{a}x\right)e^{i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{4i\omega t}\right] \\
&\quad\ \left[\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{-4i\omega t}\right] \\
&= \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)e^{-3i\omega t}\right. \\
&\quad\ \left. + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)e^{3i\omega t} + \sin^2\left(\frac{2\pi}{a}x\right)\right] \\
&= \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right)\right. \\
&\quad\ \left. +2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right]
\end{aligned}
$$

(c)

$$
\begin{aligned}
\langle x\rangle &= \int_0^a \Psi^* x\Psi\,dx \\
&= \int_0^a x|\Psi|^2\,dx \\
&= \frac{a}{2}\left[1-\frac{32}{9\pi^2}\cos(3\omega t)\right]
\end{aligned}
$$

(d)

$$
\begin{aligned}
\langle p \rangle &= m\frac{d\langle x \rangle}{dt} \\
&= \frac{16am\omega}{3\pi^2}\sin(3\omega t) \\
&= \frac{8\hbar}{3a}\sin(3\omega t)
\end{aligned}
$$

(e) You can get E_1 or E_2 and the probability of getting each is 1/2. $H = \frac{1}{2}(E_1 + E_2)$ is the mean of the two possible energy values.

2.6

$$
\begin{aligned}
\Psi(x,0) &= A[\psi_1 + e^{i\phi}\psi_2] \\
1 &= \int_0^a |\Psi|^2 \, dx \\
&= \int_0^a \Psi^*\Psi \, dx \\
&= A^2 \int_0^a (\psi_1 + e^{-i\phi}\psi_2)(\psi_1 + e^{i\phi}\psi_2) \, dx \\
&= A^2 \int_0^a (\psi_1^2 + e^{i\phi}\psi_1\psi_2 + e^{-i\phi}\psi_1\psi_2 + \psi_2^2) \, dx \\
&= \frac{2A^2}{a}\int_0^a \left[\sin^2\left(\frac{\pi}{a}x\right) + e^{i\phi}\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\right. \\
&\qquad \left. e^{-i\phi}\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right)\right] \, dx \\
&= \frac{2A^2}{a}\int_0^a \left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos\phi\right. \\
&\qquad \left. + \sin^2\left(\frac{2\pi}{a}x\right)\right] \, dx \\
&= 2A^2 \\
A &= \frac{1}{\sqrt{2}} \\
\Psi(x,t) &= \frac{1}{\sqrt{a}}\left[\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{i(\phi - 4\omega t)}\right]
\end{aligned}
$$

$$
\begin{aligned}
|\Psi|^2 &= \Psi^*\Psi \\
&= \frac{1}{a}\left[\sin\left(\frac{\pi}{a}x\right)e^{i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{-i(\phi-4\omega t)}\right] \\
&\quad \left[\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{i(\phi-4\omega t)}\right] \\
&= \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)e^{i(\phi-3\omega t)}\right. \\
&\quad \left.\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)e^{-i(\phi-3\omega t)} + \sin^2\left(\frac{2\pi}{a}x\right)\right] \\
&= \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right)\right. \\
&\quad \left. +2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(\phi-3\omega t)\right] \\
\langle x\rangle &= \int_0^a \Psi^* x\Psi\,dx \\
&= \int_0^a x|\Psi|^2\,dx \\
&= \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t-\phi)\right]
\end{aligned}
$$

2.7

(a)

$$
\begin{aligned}
1 &= \int_0^a |\Psi|^2\,dx \\
&= A^2\left[\int_0^{a/2} x^2\,dx + \int_{a/2}^a (a-x)^2\,dx\right] \\
&= A^2\left\{\frac{1}{3}\left[\frac{a}{2}\right]^3 + \left[-\frac{1}{3}(a-x)^3\right]_{a/2}^a\right\} \\
&= A^2\left(\frac{a^3}{24} + \frac{a^3}{24}\right) \\
&= \frac{A^2a^3}{12} \\
A &= \frac{2\sqrt{3}}{\sqrt{a^3}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
c_n &= \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \Psi(x,0)\, dx \\
&= \sqrt{\frac{2}{a}} \left[\int_0^{a/2} \sin\left(\frac{n\pi}{a}x\right) Ax\, dx + \int_{a/2}^a \sin\left(\frac{n\pi}{a}x\right) A(a-x)\, dx\right] \\
&= \frac{2\sqrt{6}}{a^2} \left[\int_0^{a/2} x \sin\left(\frac{n\pi}{a}x\right)\, dx + \int_{a/2}^a (a-x) \sin\left(\frac{n\pi}{a}x\right)\, dx\right] \\
&= \frac{8\sqrt{6}}{n^2\pi^2} \sin^2\left(\frac{n\pi}{4}\right) \sin\left(\frac{n\pi}{2}\right) \\
&= \begin{cases} 0 & n \text{ even} \\ (-1)^{(n-1)/2} \frac{4\sqrt{6}}{n^2\pi^2} & n \text{ odd} \end{cases} \\
\Psi(x,t) &= \frac{4\sqrt{6}}{\pi^2} \sqrt{\frac{2}{a}} \sum_{n=1,3,5,\ldots}^{\infty} (-1)^{(n-1)/2} \frac{1}{n^2} \sin\left(\frac{n\pi}{a}x\right) e^{-i(n^2\pi^2\hbar/2ma^2)t}
\end{aligned}
$$

(c)

$$
\begin{aligned}
|c_1|^2 &= \left(\frac{4\sqrt{6}}{\pi^2}\right)^2 \\
&\approx 0.985
\end{aligned}
$$

(d)

$$
\begin{aligned}
E_n &= \frac{n^2\pi^2\hbar^2}{2ma^2} \\
\langle H \rangle &= \sum_{n=0}^{\infty} |c_{2n+1}|^2 E_{2n+1} \\
&= \sum_{n=0}^{\infty} \left(\frac{4\sqrt{6}}{(2n+1)^2\pi^2}\right)^2 \frac{(2n+1)^2\pi^2\hbar^2}{2ma^2} \\
&= \sum_{n=0}^{\infty} \frac{48\hbar^2}{(2n+1)^2 ma^2\pi^2} \\
&= \frac{48\hbar^2}{ma^2\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\
&= \frac{6\hbar^2}{ma^2}
\end{aligned}
$$

2.8

$$
\begin{aligned}
1 &= \int_0^{a/2} |\Psi|^2 \, dx \\
&= A^2 \int_0^{a/2} \, dx \\
&= \frac{aA^2}{2} \\
A &= \sqrt{\frac{2}{a}} \\
c_n &= \frac{2}{a} \int_0^{a/2} \sin\left(\frac{n\pi}{a} x\right) \, dx \\
|c_1|^2 &= \left(\frac{2}{\pi}\right)^2 \\
&\approx 0.405
\end{aligned}
$$

2.9

$$
\begin{aligned}
\Psi(x,0) &= Ax(a-x) \\
\langle H \rangle &= \int_0^a \Psi(x,0)^* \hat{H} \Psi(x,0) \, dx \\
&= \int_0^a \Psi(x,0)^* \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}\right) \Psi(x,0) \, dx \\
&= \frac{A^2 \hbar^2}{m} \int_0^a x(a-x) \, dx \\
&= \frac{30\hbar^2}{ma^5} \frac{a^3}{6} \\
&= \frac{5\hbar^2}{ma^2}
\end{aligned}
$$

2.10

(a)

$$
\begin{aligned}
\psi_2(x) &= \frac{1}{\sqrt{2!}}(\hat{a}_+)\psi_1 \\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}xe^{-\frac{m\omega}{2\hbar}x^2} \\
&= \frac{1}{\sqrt{2}\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(-\hbar\frac{d}{dx} + m\omega x\right)xe^{-\frac{m\omega}{2\hbar}x^2} \\
&= \frac{1}{\sqrt{2}\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-\hbar\left(e^{-\frac{m\omega}{2\hbar}x^2} - \frac{m\omega}{\hbar}x^2e^{-\frac{m\omega}{2\hbar}x^2}\right) + m\omega x^2e^{-\frac{m\omega}{2\hbar}x^2}\right] \\
&= \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)e^{-\frac{m\omega}{2\hbar}x^2}
\end{aligned}
$$

2.11

(a)

$$
\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} \psi_0^* x \psi_0 \, dx \\
&= \alpha^2 \int_{-\infty}^{\infty} x e^{-\frac{m\omega}{\hbar}x^2} \, dx \\
&= 0 \\
\langle p\rangle &= m\frac{d\langle x\rangle}{dt} \\
&= 0 \\
\langle x^2\rangle &= \int_{-\infty}^{\infty} \psi_0^* x^2 \psi_0 \, dx \\
&= \alpha^2 \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} \, dx \\
&= \frac{\hbar}{2m\omega} \\
\langle p^2\rangle &= \int_{-\infty}^{\infty} \psi_0^* \left(-\hbar^2 \frac{d^2}{dx^2}\right) \psi_0 \, dx \\
&= -\hbar^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\frac{m\omega}{2\hbar}x^2} \frac{d}{dx} \left(-\frac{m\omega}{\hbar} x e^{-\frac{m\omega}{2\hbar}x^2}\right) \, dx \\
&= \hbar^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{m\omega}{\hbar} \int_{-\infty}^{\infty} e^{-\frac{m\omega}{2\hbar}x^2} \left(e^{-\frac{m\omega}{2\hbar}x^2} - \frac{m\omega}{\hbar} x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right) \, dx \\
&= \hbar^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{m\omega}{\hbar} \int_{-\infty}^{\infty} \left(1 - \frac{m\omega}{\hbar}x^2\right) e^{-\frac{m\omega}{\hbar}x^2} \, dx \\
&= \hbar^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{m\omega}{\hbar} \frac{\hbar\sqrt{\pi}}{2\sqrt{\hbar m\omega}} \\
&= \frac{1}{2} m\hbar\omega
\end{aligned}
$$

$$
\begin{aligned}
\psi_1(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}xe^{-\frac{m\omega}{2\hbar}x^2} \\
\langle x\rangle &= 0 \\
\langle p\rangle &= m\frac{d\langle x\rangle}{dt} \\
&= 0 \\
\left\langle x^2\right\rangle &= \int_{-\infty}^{\infty}\psi_1^* x^2\psi_1\,dx \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{2m\omega}{\hbar}\int_{-\infty}^{\infty}x^4 e^{-\frac{m\omega}{\hbar}x^2}\,dx \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{2m\omega}{\hbar}\frac{3}{4}\sqrt{\pi}\left(\frac{\hbar}{m\omega}\right)^{5/2} \\
&= \frac{3}{2}\frac{\hbar}{m\omega} \\
\left\langle p^2\right\rangle &= \int_{-\infty}^{\infty}\psi_1^*\left(-\hbar^2\frac{d^2}{dx^2}\right)\psi_1\,dx \\
&= -\hbar^2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{2m\omega}{\hbar}\int_{-\infty}^{\infty}xe^{-\frac{m\omega}{2\hbar}x^2}\frac{d}{dx}\left(e^{-\frac{m\omega}{2\hbar}x^2}-\frac{m\omega}{\hbar}x^2e^{-\frac{m\omega}{2\hbar}x^2}\right)\,dx \\
&= -\hbar^2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{2m\omega}{\hbar}\int_{-\infty}^{\infty}xe^{-\frac{m\omega}{2\hbar}x^2}\left[-\frac{m\omega}{\hbar}xe^{-\frac{m\omega}{2\hbar}x^2}\right. \\
&\qquad \left.-\frac{2m\omega}{\hbar}xe^{-\frac{m\omega}{2\hbar}x^2}+\left(\frac{m\omega}{\hbar}\right)^2x^3e^{-\frac{m\omega}{2\hbar}x^2}\right]\,dx \\
&= 2\hbar^2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left(\frac{m\omega}{\hbar}\right)^2\int_{-\infty}^{\infty}x^2e^{-\frac{m\omega}{\hbar}x^2}\left(3-\frac{m\omega}{\hbar}x^2\right)\,dx \\
&= 2\hbar^2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left(\frac{m\omega}{\hbar}\right)^2\frac{3}{4}\sqrt{\pi}\left(\frac{h}{m\omega}\right)^{3/2} \\
&= \frac{3}{2}m\hbar\omega
\end{aligned}
$$

(b)

$$
\begin{aligned}
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{\hbar}{2m\omega}} \\
\sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\
&= \sqrt{\frac{m\hbar\omega}{2}} \\
\sigma_x \sigma_p &= \frac{\hbar}{2} \\
\sigma_x &= \sqrt{\frac{3\hbar}{2m\omega}} \\
\sigma_p &= \sqrt{\frac{3m\hbar\omega}{2}} \\
\sigma_x \sigma_p &= \frac{3}{2}\hbar
\end{aligned}
$$

(c)

$$
\begin{aligned}
\langle T \rangle &= \frac{\left\langle p^2 \right\rangle}{2m} \\
&= \frac{\hbar\omega}{4} \\
\langle V \rangle &= \frac{1}{2} m\omega^2 \left\langle x^2 \right\rangle \\
&= \frac{1}{4}\hbar\omega \\
\langle T \rangle &= \frac{\left\langle p^2 \right\rangle}{2m} \\
&= \frac{3}{4}\hbar\omega \\
\langle V \rangle &= \frac{1}{2} m\omega^2 \left\langle x^2 \right\rangle \\
&= \frac{3}{4}\hbar\omega
\end{aligned}
$$

2.12

$$
\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} \psi_n^* x \psi_n \, dx \\
&= \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}_+ + \hat{a}_-) \psi_n \, dx \\
&= \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{\infty} \psi_n^* (\sqrt{n+1}\psi_{n+1} + \sqrt{n}\psi_{n-1}) \, dx \\
&= 0 \\
\langle p \rangle &= \int_{-\infty}^{\infty} \psi_n^* p \psi_n \, dx \\
&= i\sqrt{\frac{\hbar m\omega}{2}} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}_+ - \hat{a}_-) \psi_n \, dx \\
&= i\sqrt{\frac{\hbar m\omega}{2}} \int_{-\infty}^{\infty} \psi_n^* (\sqrt{n+1}\psi_{n+1} - \sqrt{n}\psi_{n-1}) \, dx \\
&= 0 \\
\left\langle x^2 \right\rangle &= \int_{-\infty}^{\infty} \psi_n^* x^2 \psi_n \, dx \\
&= \frac{\hbar}{2m\omega} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}_+^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-^2) \psi_n \, dx \\
&= \frac{\hbar}{2m\omega} (2n+1) \int_{-\infty}^{\infty} |\psi_n|^2 \, dx \\
&= \frac{\hbar}{m\omega} \left(n + \frac{1}{2} \right) \\
\left\langle p^2 \right\rangle &= \int_{-\infty}^{\infty} \psi_n^* p^2 \psi_n \, dx \\
&= -\frac{\hbar m\omega}{2} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}_+^2 - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a} - +\hat{a}_-^2) \psi_n \, dx \\
&= \frac{\hbar m\omega}{2} (2n+1) \int_{-\infty}^{\infty} |\psi_n|^2 \, dx \\
&= \hbar m\omega \left(n + \frac{1}{2} \right) \\
\langle T \rangle &= \left\langle \frac{p^2}{2m} \right\rangle \\
&= \frac{1}{2}\hbar\omega \left(n + \frac{1}{2} \right)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_x &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\
&= \sqrt{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)} \\
\sigma_p &= \sqrt{\hbar m\omega\left(n+\frac{1}{2}\right)} \\
\sigma_x\sigma_p &= (2n+1)\frac{\hbar}{2} \\
&\geq \frac{\hbar}{2}
\end{aligned}
$$

2.13

(a)

$$
\begin{aligned}
\Psi(x,0) &= A[3\psi_0(x) + 4\psi_1(x)] \\
1 &= \int_{-\infty}^{\infty} |\Psi(x,0)|^2\,dx \\
&= A^2\int_{-\infty}^{\infty} [9\psi_0(x)^2 + 24\psi_0(x)\psi_1(x) + 16\psi_1(x)^2]\,dx \\
&= 25A^2 \\
A &= \frac{1}{5}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\Psi(x,t) &= \frac{1}{5}[3\psi_0(x)e^{-i\omega t/2} + 4\psi_1(x)e^{-3i\omega t/2}] \\
|\Psi(x,t)|^2 &= \Psi(x,t)^*\Psi(x,t) \\
&= \frac{1}{25}[3\psi_0(x)e^{i\omega t/2} + 4\psi_1(x)e^{3i\omega t/2}][3\psi_0(x)e^{-i\omega t/2} + 4\psi_1(x)e^{-3i\omega t/2}] \\
&= \frac{1}{25}[9\psi_0(x)^2 + 12\psi_0(x)\psi_1(x)e^{-i\omega t} + 12\psi_0(x)\psi_1(x)e^{i\omega t} + 16\psi_1(x)^2] \\
&= \frac{1}{25}[9\psi_0(x)^2 + 16\psi_1(x)^2 + 24\psi_0(x)\psi_1(x)\cos\omega t]
\end{aligned}
$$

(c)

$$
\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} \Psi^* x \Psi \, dx \\
&= \frac{1}{25} \int_{-\infty}^{\infty} x(9\psi_0^2 + 16\psi_1^2 + 24\psi_0\psi_1 \cos\omega t) \, dx \\
&= \frac{24}{25} \int_{-\infty}^{\infty} x\psi_0\psi_1 \cos(\omega t) \, dx \\
&= \frac{24}{25} \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \sqrt{\frac{2m\omega}{\hbar}} \cos(\omega t) \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} \, dx \\
&= \frac{24}{25} \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \sqrt{\frac{2m\omega}{\hbar}} \cos(\omega t) \frac{1}{2}\sqrt{\pi} \left(\frac{\hbar}{m\omega}\right)^{3/2} \\
&= \frac{24}{25} \sqrt{\frac{\hbar}{2m\omega}} \cos(\omega t) \\
\langle p \rangle &= m \frac{d \langle x \rangle}{dt} \\
&= -\frac{24}{25} \sqrt{\frac{\hbar m\omega}{2}} \sin(\omega t) \\
\frac{d \langle p \rangle}{dt} &= -\frac{24}{25} \sqrt{\frac{\hbar m\omega}{2}} \omega \cos(\omega t) \\
V &= \frac{1}{2} m\omega^2 x^2 \\
\frac{\partial V}{d\partial} &= m\omega^2 x \\
\left\langle -\frac{\partial V}{\partial x} \right\rangle &= -m\omega^2 \langle x \rangle \\
&= -\frac{24}{25} \sqrt{\frac{\hbar m\omega}{2}} \omega \cos(\omega t) \\
&= \frac{d \langle p \rangle}{dt}
\end{aligned}
$$

(d)

$$
\begin{aligned}
E_0 &= \frac{\hbar\omega}{2} \\
P(E_0) &= \frac{9}{25} \\
E_1 &= \frac{3\hbar\omega}{2} \\
P(E_1) &= \frac{16}{25}
\end{aligned}
$$

2.14

$$
\begin{aligned}
1-\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_{-\sqrt{\hbar/m\omega}}^{\sqrt{\hbar/m\omega}} e^{-m\omega x^2/\hbar}\,dx &= 1-\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\sqrt{\frac{\pi\hbar}{m\omega}}\operatorname{erf}1 \\
&= 0.157
\end{aligned}
$$

2.15

$$
\begin{aligned}
a_{j+2} &= \frac{-2(n-j)}{(j+1)(j+2)}a_j \\
a_3 &= -\frac{4}{3}a_1 \\
a_5 &= \frac{4}{15}a_1 \\
H_5(\xi) &= a_1\left(\xi-\frac{4}{3}\xi^3+\frac{4}{15}\xi^5\right) \\
&= \frac{1}{120}a_1(120\xi-160\xi^3+32\xi^5) \\
&= 32\xi^5-160\xi^3+120\xi \\
a_2 &= -6a_0 \\
a_4 &= \frac{-8}{12}a_2 \\
&= 4a_0 \\
a_6 &= \frac{-4}{30}a_4 \\
&= -\frac{8}{15}a_0 \\
H_6(\xi) &= a_0\left(1-6\xi^2+4\xi^4-\frac{8}{15}\xi^6\right) \\
&= \frac{1}{120}a_0(120-720\xi^2+480\xi^4-64\xi^6) \\
&= 64\xi^6-480\xi^4+720\xi^2-120
\end{aligned}
$$

2.17

$$\begin{aligned} Ae^{ikx} + Be^{-ikx} &= A[\cos(kx) + i\sin(kx)] + B[\cos(kx) - i\sin(kx)] \\ &= (A+B)\cos(kx) + i(A-B)\sin(kx) \\ C &= A + B \\ D &= i(A-B) \\ -iD &= A - B \\ A &= \frac{C - iD}{2} \\ B &= \frac{C + iD}{2} \end{aligned}$$

2.18

$$\begin{aligned} \Psi_k(x,t) &= Ae^{i\left(kx - \frac{\hbar k^2}{2m}t\right)} \\ J(x,t) &= \frac{i\hbar}{2m}\left(\Psi\frac{\partial \Psi^*}{\partial x} - \Psi^*\frac{\partial \Psi}{\partial x}\right) \\ &= \frac{\hbar k |A|^2}{m} \end{aligned}$$

The probability travels in the same direction as the wave.

2.20

(a)

$$\begin{aligned} \Psi(x,0) &= Ae^{-a|x|} \\ 1 &= \int_{-\infty}^{\infty} \Psi^*\Psi\, dx \\ &= |A|^2 \int_{-\infty}^{\infty} e^{-2a|x|}\, dx \\ &= 2|A|^2 \int_{0}^{\infty} e^{-2ax}\, dx \\ &= \frac{|A|^2}{a} \\ A &= \sqrt{a} \end{aligned}$$

(b)

$$
\begin{aligned}
\phi(k) &= \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|-ikx}\,dx \\
&= \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|}[\cos(kx) - i\sin(kx)]\,dx \\
&= \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|}\cos(kx)\,dx \\
&= \sqrt{\frac{a}{2\pi}} 2\int_{0}^{\infty} e^{-ax}\cos(kx)\,dx \\
&= \sqrt{\frac{a}{2\pi}} \frac{2a}{a^2+k^2}
\end{aligned}
$$

(c)

$$
\Psi(x,t) = \frac{a^{3/2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{a^2+k^2} e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)}\,dk
$$

2.21

(a)

$$
\begin{aligned}
\Psi(x,0) &= Ae^{-ax^2} \\
1 &= A^2 \int_{-\infty}^{\infty} e^{-2ax^2}\,dx \\
&= \sqrt{\frac{\pi}{2a}} A^2 \\
A &= \left(\frac{2a}{\pi}\right)^{1/4}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\phi(k) &= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-(ax^2+ikx)}\,dx \\
&= \frac{1}{(2\pi a)^{1/4}} e^{-k^2/4a} \\
\Psi(x,t) &= \frac{1}{\sqrt{2\pi}} \frac{1}{(2\pi a)^{1/4}} \int_{-\infty}^{\infty} e^{-\frac{k^2}{4a} + i\left(kx - \frac{\hbar k^2}{2m}t\right)}\,dk \\
\Psi(x,t) &= \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\gamma} e^{-ax^2/\gamma^2}
\end{aligned}
$$

(c)

$$
\begin{aligned}
|\Psi(x,t)|^2 &= \Psi^*\Psi \\
&= \left(\frac{2a}{\pi}\right)^{1/2} \frac{1}{\gamma^*} e^{-ax^2/(\gamma^*)^2} \frac{1}{\gamma} e^{-ax^2/\gamma^2} \\
&= \left(\frac{2a}{\pi}\right)^{1/2} \frac{1}{\sqrt{1-2i\hbar at/m}} e^{-ax^2/(1-2i\hbar at/m)} \\
&\quad \frac{1}{\sqrt{1+2i\hbar at/m}} e^{-ax^2/(1+2i\hbar at/m)} \\
&= \left(\frac{2a}{\pi}\right)^{1/2} \frac{1}{\sqrt{1+(2\hbar at/m)^2}} e^{-2ax^2/[1+(2a\hbar t/m)^2]} \\
&= \sqrt{\frac{2}{\pi}} w e^{-2w^2x^2}
\end{aligned}
$$

As t increases $|\Psi|^2$ flattens out and broadens.

(d)

$$
\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} \Psi^* x \Psi \, dx \\
&= 0 \\
\langle p \rangle &= m \frac{d\langle x \rangle}{dt} \\
&= 0 \\
\langle x^2 \rangle &= \int_{-\infty}^{\infty} \Psi^* x^2 \Psi \, dx \\
&= \sqrt{\frac{2}{\pi}} w \int_{-\infty}^{\infty} x^2 e^{-2w^2x^2} \, dx \\
&= \frac{1}{4w^2}
\end{aligned}
$$

2.22

(a) -25

(b) 1

(c) 0

2.26

$$
\begin{aligned}
F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-ikx} \, dx \\
&= \frac{1}{\sqrt{2\pi}} \\
f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, dk
\end{aligned}
$$

2.29

$$
\begin{aligned}
\psi(x) &= \begin{cases} Fe^{-\kappa x} & x > a \\ C\sin(lx) & 0 < x < a \\ -\psi(-x) & x < 0 \end{cases} \\
Fe^{-\kappa a} &= C\sin(la) \\
-\kappa Fe^{-\kappa a} &= lC\cos(la) \\
-\frac{1}{\kappa} &= \frac{1}{l}\tan(la) \\
\tan z &= -\frac{l}{\kappa} \\
&= -\frac{la}{\kappa a} \\
&= -\frac{z}{\sqrt{z_0^2 - z^2}}
\end{aligned}
$$

For large z_0 the intersections occur just below $z_n = n\pi$ so

$$
\begin{aligned}
z &= la \\
n\pi &\approx \frac{\sqrt{2m(E+V_0)}}{\hbar} a \\
E + V_0 &\approx \frac{n^2\pi^2\hbar^2}{2ma^2}.
\end{aligned}
$$

As z_0 decreases there are fewer and fewer bound states. When $z_0 < \pi/2$ there are no odd bound states.

2.30

$$\psi(x) = \begin{cases} Fe^{-\kappa x} & x > a \\ D\cos(lx) & 0 < x < a \\ \psi(-x) & x < 0 \end{cases}$$

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\psi|^2 \, dx \\
&= 2\left(|D|^2 \int_0^a \cos^2(lx)\, dx + |F|^2 \int_a^{\infty} e^{-2\kappa x}\, dx\right) \\
&= 2\left[|D|^2 \frac{2al + \sin(2al)}{4l} + \frac{|F|^2}{2\kappa} e^{-2\kappa a}\right] \\
&= |D|^2 \left[a + \frac{\sin(2al)}{2l} + \frac{\cos^2(al)}{\kappa}\right] \\
&= |D|^2 \left[a + \frac{2\sin(al)\cos(al)}{2l} + \frac{\cos^3(al)}{l\sin(al)}\right] \\
&= |D|^2 \left\{a + \frac{\cos(al)}{l\sin(al)}[\sin^2(al) + \cos^2(al)]\right\} \\
&= |D|^2 \left[a + \frac{\cos(al)}{l\sin(al)}\right] \\
&= |D|^2 \left[a + \frac{1}{l\tan(al)}\right] \\
&= |D|^2 \left[a + \frac{1}{\kappa}\right] \\
D &= \frac{1}{\sqrt{a + 1/\kappa}}
\end{aligned}$$

$$\begin{aligned}
1 &= \left\{\frac{1}{a + 1/\kappa}\left[a + \frac{\sin(2al)}{2l}\right] + |F|^2 \frac{e^{-2\kappa a}}{\kappa}\right\} \\
\frac{(Fe^{-\kappa a})^2}{\kappa} &= 1 - \frac{1}{a + 1/\kappa}\left[a + \frac{\sin(2al)}{2l}\right] \\
(Fe^{-\kappa a})^2 &= \kappa - \frac{\kappa}{a + 1/\kappa}\left[a + \frac{\sin(2al)}{2l}\right] \\
&= \frac{\kappa a + 1 - \kappa a - \kappa\sin(al)\cos(al)/l}{a + 1/\kappa} \\
&= \frac{1 - \sin^2(al)}{a + 1/\kappa} \\
F &= \frac{e^{\kappa a}\cos(al)}{\sqrt{a + 1/\kappa}}
\end{aligned}$$

2.31

$$
\begin{aligned}
1 &= 2aV_0 \\
V_0 &= \frac{1}{2a} \\
z_0 &= \frac{a}{\hbar}\sqrt{2mV_0} \\
&= \frac{a}{\hbar}\sqrt{\frac{m}{a}} \\
&= \frac{\sqrt{am}}{\hbar} \\
\lim_{a\to 0} z_0 &= 0
\end{aligned}
$$

2.34

(a)

$$\begin{aligned}
V(x) &= \begin{cases} 0 & x \leq 0 \\ V_0 & x > 0 \end{cases} \\
-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} &= E\psi \\
\frac{d^2\psi}{dx^2} &= -\frac{2mE}{\hbar^2}\psi \\
&= -k^2\psi \\
k &= \frac{\sqrt{2mE}}{\hbar} \\
\psi &= Ae^{ikx} + Be^{-ikx} \\
-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi &= E\psi \\
\frac{d^2\psi}{dx^2} &= -\frac{2m(E-V_0)}{\hbar^2}\psi \\
&= \kappa^2\psi \\
\kappa &= \frac{\sqrt{2m(V_0-E)}}{\hbar} \\
\psi &= Fe^{-\kappa x} \\
A + B &= F \\
ik(A-B) &= -\kappa F \\
F &= -i\frac{k}{\kappa}(A-B) \\
A + B &= -i\frac{k}{\kappa}(A-B) \\
\left(1 - i\frac{k}{\kappa}\right)B &= -\left(1 + i\frac{k}{\kappa}\right)A \\
B &= -\frac{1+ik/\kappa}{1-ik/\kappa}A \\
R &= \frac{|B|^2}{|A|^2} \\
&= \left(-\frac{1+ik/\kappa}{1-ik/\kappa}\right)\left(-\frac{1-ik/\kappa}{1+ik/\kappa}\right) \\
&= 1
\end{aligned}$$

(b)

$$
\begin{aligned}
\psi(x) &= \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \\ Fe^{ilx} & x > 0 \end{cases} \\
k &= \frac{\sqrt{2mE}}{\hbar} \\
l &= \frac{\sqrt{2m(E - V_0)}}{\hbar} \\
A + B &= F \\
ik(A - B) &= ilF \\
F &= \frac{k}{l}(A - B) \\
A + B &= \frac{k}{l}(A - B) \\
\left(\frac{k}{l} + 1\right) B &= \left(\frac{k}{l} - 1\right) A \\
B &= \frac{k/l - 1}{k/l + 1} A \\
R &= \frac{|B|^2}{|A|^2} \\
&= \left(\frac{k/l - 1}{k/l + 1}\right)^2 \\
&= \left(\frac{k - l}{k + l}\right)^2 \\
&= \frac{(k - l)^4}{(k^2 - l^2)^2} \\
k^2 - l^2 &= \frac{2mE}{\hbar^2} - \frac{2m(E - V_0)}{\hbar^2} \\
&= \frac{2m}{\hbar^2} V_0 \\
k - l &= \frac{\sqrt{2m}}{\hbar}(\sqrt{E} - \sqrt{E - V_0}) \\
R &= \frac{(\sqrt{E} - \sqrt{E - V_0})^4}{V_0^2}
\end{aligned}
$$

(d)

$$
\begin{aligned}
B &= F - A \\
F &= \frac{k}{l}(A - F + A) \\
\left(1 + \frac{k}{l}\right) F &= \frac{2k}{l} A \\
F &= \frac{2k}{k+l} A \\
\frac{l}{k} &= \sqrt{\frac{E - V_0}{E}} \\
T &= \left|\frac{F}{A}\right|^2 \frac{l}{k} \\
&= \left(\frac{2k}{k+l}\right)^2 \frac{l}{k} \\
&= \frac{4kl}{(k+l)^2} \\
&= \frac{4kl(k-l)^2}{(k^2 - l^2)^2} \\
&= \frac{4\sqrt{E}\sqrt{E - V_0}(\sqrt{E} - \sqrt{E - V_0})^2}{V_0^2} \\
T + R &= \frac{4kl}{(k+l)^2} + \frac{(k-l)^2}{(k+l)^2} \\
&= \frac{4kl + k^2 - 2kl + l^2}{(k+l)^2} \\
&= \frac{k^2 + 2kl + l^2}{(k+l)^2} \\
&= \frac{(k+l)^2}{(k+l)^2} \\
&= 1
\end{aligned}
$$

2.35

(a)

$$
\begin{aligned}
V(x) &= \begin{cases} 0 & x < 0 \\ -V_0 & x > 0 \end{cases} \\
-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} &= E\psi \\
\frac{d^2\psi}{dx^2} &= -\frac{2mE}{\hbar^2}\psi \\
&= -k^2\psi \\
k &= \frac{\sqrt{2mE}}{\hbar} \\
\psi &= Ae^{ikx} + Be^{-ikx} \\
-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi &= E\psi \\
\frac{d^2\psi}{dx^2} &= -\frac{2m}{\hbar^2}(E+V_0)\psi \\
&= -l^2\psi \\
l &= \frac{\sqrt{2m(E+V_0)}}{\hbar} \\
\psi &= Fe^{ilx} \\
A + B &= F \\
ik(A-B) &= ilF \\
k(A-B) &= l(A+B) \\
(k+l)B &= (k-l)A \\
B &= \frac{k-l}{k+l}A \\
R &= \left|\frac{B}{A}\right|^2 \\
&= \left(\frac{k-l}{k+l}\right)^2 \\
&= \left(\frac{\sqrt{E}-\sqrt{E+V_0}}{\sqrt{E}+\sqrt{E+V_0}}\right)^2 \\
&= \frac{1}{9}
\end{aligned}
$$

(c)

$$
T = 1 - R = \frac{8}{9}
$$

2.36

$$
\begin{aligned}
V(x) &= \begin{cases} 0 & |x| < a \\ \infty & |x| > a \end{cases} \\
-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} &= E\psi \\
\frac{d^2\psi}{dx^2} &= -\frac{2mE}{\hbar^2}\psi \\
&= -k^2\psi \\
k &= \frac{\sqrt{2mE}}{\hbar} \\
\psi &= A\sin kx + B\cos kx \\
0 &= -A\sin ka + B\cos ka \\
0 &= A\sin ka + B\cos ka \\
B\cos ka &= 0 \\
k &= \frac{n\pi}{2a},\ n = 1, 3, 5, \ldots \\
E &= \frac{n^2\pi^2\hbar^2}{2m(2a)^2} \\
\psi &= B\cos\left(\frac{n\pi}{2a}x\right),\ n = 1, 3, 5, \ldots \\
1 &= |B|^2\int_{-a}^{a}\cos^2\left(\frac{n\pi}{2a}x\right)\,dx \\
B &= \frac{1}{\sqrt{a}} \\
\psi &= \frac{1}{\sqrt{a}}\cos\left(\frac{n\pi}{2a}x\right),\ n = 1, 3, 5, \ldots \\
A\sin ka &= 0 \\
k &= \frac{n\pi}{2a},\ n = 2, 4, 6, \ldots \\
E &= \frac{n^2\pi^2\hbar^2}{2m(2a)^2} \\
\psi &= A\sin\left(\frac{n\pi}{2a}x\right),\ n = 2, 4, 6, \ldots \\
1 &= |A|\int_{-a}^{a}\sin^2\left(\frac{n\pi}{2a}x\right)\,dx \\
A &= \frac{1}{\sqrt{a}} \\
\psi &= \frac{1}{\sqrt{a}}\sin\left(\frac{n\pi}{2a}x\right),\ n = 2, 4, 6, \ldots
\end{aligned}
$$

2.37

$$\begin{aligned}
\Psi(x,0) &= A\sin^3\left(\frac{\pi}{a}x\right) \\
&= A\left[\frac{3}{4}\sin\left(\frac{\pi}{a}x\right) - \frac{1}{4}\sin\left(\frac{3\pi}{a}x\right)\right] \\
&= A\sqrt{\frac{a}{2}}\left[\frac{3}{4}\psi_1(x) - \frac{1}{4}\psi_3(x)\right] \\
1 &= |A|^2\frac{a}{2}\int_0^a \left[\frac{3}{4}\psi_1(x) - \frac{1}{4}\psi_3(x)\right]^2\,dx \\
&= |A|^2\frac{a}{2}\int_0^a \left[\frac{9}{16}\psi_1(x)^2 - \frac{3}{8}\psi_1(x)\psi_3(x) + \frac{1}{16}\psi_3(x)^2\right]dx \\
&= \frac{5}{16}a|A|^2 \\
A &= \frac{4}{\sqrt{5a}} \\
\Psi(x,0) &= \frac{1}{\sqrt{10}}[3\psi_1(x) - \psi_3(x)] \\
\Psi(x,t) &= \frac{1}{\sqrt{10}}[3\psi_1(x)e^{-iE_1t/\hbar} - \psi_3(x)e^{-iE_3t/\hbar}] \\
\langle x\rangle &= \int_0^a \Psi^* x\Psi\,dx \\
&= \frac{1}{10}\int_0^a x\left(9\psi_1^2 + \psi_3^2 - 3\psi_1\psi_3 e^{-i(E_3-E_1)t/\hbar} - 3\psi_1\psi_3 e^{-i(E_1-E_3)t/\hbar}\right)dx \\
&= \frac{1}{10}\int_0^a x\left[9\psi_1^2 + \psi_3^2 - 6\psi_1\psi_3\cos\left(\frac{E_3-E_1}{\hbar}t\right)\right]dx \\
&= \frac{1}{10}\left[9\langle x\rangle_1 + \langle x\rangle_3 - 6\cos\left(\frac{E_3-E_1}{\hbar}t\right)\int_0^a x\psi_1\psi_3\,dx\right] \\
&= \frac{1}{10}\left[\frac{9}{2}a + \frac{1}{2}a\right] \\
&= \frac{a}{2} \\
P(E_1) &= \frac{9}{10} \\
P(E_3) &= \frac{1}{10} \\
\langle E\rangle &= E_1P(E_1) + E_3P(E_3) \\
&= \frac{9\pi^2\hbar^2}{20ma^2} + \frac{9\pi^2\hbar^2}{20ma^2} \\
&= \frac{9\pi^2\hbar^2}{10ma^2}
\end{aligned}$$

2.38

(a)

$$\begin{aligned}
\Psi(x,t) &= \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar} \\
\Psi(x,0) &= \sum_{n=1}^{\infty} c_n \psi_n(x) \\
E_n T &= \frac{n^2\pi^2\hbar^2}{2ma^2}\frac{4ma^2}{\pi\hbar} \\
&= 2\pi n^2 \hbar \\
\Psi(x,T) &= \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n T/\hbar} \\
&= \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-2\pi i n^2} \\
&= \sum_{n=1}^{\infty} c_n \psi_n(x) \\
&= \Psi(x,0)
\end{aligned}$$

(b)

$$\begin{aligned}
E &= \frac{1}{2}mv^2 \\
v &= \sqrt{\frac{2E}{m}} \\
T &= \frac{2a}{v} \\
&= a\sqrt{\frac{2m}{E}}
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{4ma^2}{\pi\hbar} &= a\sqrt{\frac{2m}{E}} \\
\frac{16m^2a^2}{\pi^2\hbar^2} &= \frac{2m}{E} \\
E &= \frac{\pi^2\hbar^2}{8ma^2} \\
&= \frac{E_1}{4}
\end{aligned}$$

2.39

(a)

$$\Psi(x,0) = \begin{cases} \frac{2\sqrt{3}}{a\sqrt{a}}x & 0 \le x \le a/2 \\ \frac{2\sqrt{3}}{a\sqrt{a}}(a-x) & a/2 \le x \le a \end{cases}$$

$$\frac{d}{dx}\Psi(x,0) = \frac{2\sqrt{3}}{a\sqrt{a}}\left[1 - 2\theta\left(x - \frac{a}{2}\right)\right]$$

(b)

$$\frac{d^2}{dx^2}\Psi(x,0) = -\frac{4\sqrt{3}}{a\sqrt{a}}\delta\left(x - \frac{a}{2}\right)$$

(c)

$$\begin{aligned} \hat{H}\Psi(x,0) &= \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi(x,0) \\ &= \frac{\hbar^2}{2m}\frac{4\sqrt{3}}{a\sqrt{a}}\delta\left(x - \frac{a}{2}\right) + V(x)\Psi(x,0) \\ \langle H\rangle &= \int \Psi(x,0)^*\hat{H}\Psi(x,0)\,dx \\ &= \int_0^a \Psi(x,0)^*\left[\frac{\hbar^2}{2m}\frac{4\sqrt{3}}{a\sqrt{a}}\delta\left(x - \frac{a}{2}\right) + V(x)\Psi(x,0)\right]dx \\ &= \Psi\left(\frac{a}{2},0\right)^*\frac{\hbar^2}{2m}\frac{4\sqrt{3}}{a\sqrt{a}} + \int_0^a \Psi(x,0)^*V(x)\Psi(x,0)\,dx \\ &= \frac{6\hbar^2}{ma^2} \end{aligned}$$

2.40

(a)

$$\begin{aligned}
V(x) &= \frac{1}{2}m\omega^2 x^2 \\
\xi &= \sqrt{\frac{m\omega}{\hbar}}x \\
\psi_n(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2} \\
\Psi(x,0) &= A\left(1 - 2\sqrt{\frac{m\omega}{\hbar}}x\right)^2 e^{-\frac{m\omega}{2\hbar}x^2} \\
&= A\left(1 - 4\sqrt{\frac{m\omega}{\hbar}}x + \frac{4m\omega}{\hbar}x^2\right) e^{-\frac{m\omega}{2\hbar}x^2} \\
&= A\left(\frac{\pi\hbar}{m\omega}\right)^{1/4} \left[3\psi_0(x) - 2\sqrt{2}\psi_1(x) + 2\sqrt{2}\psi_2(x)\right] \\
1 &= A^2\sqrt{\frac{\pi\hbar}{m\omega}} \int_{-\infty}^{\infty} (3\psi_0 - 2\sqrt{2}\psi_1 + 2\sqrt{2}\psi_2)^2 \, dx \\
&= A^2\sqrt{\frac{\pi\hbar}{m\omega}} \int_{-\infty}^{\infty} (9\psi_0^2 - 12\sqrt{2}\psi_0\psi_1 + 12\sqrt{2}\psi_0\psi_2 + 8\psi_1^2 - 16\psi_1\psi_2 + 8\psi_2^2) \, dx \\
&= 25A^2\sqrt{\frac{\pi\hbar}{m\omega}} \\
A &= \frac{1}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \\
\Psi(x,0) &= \frac{3}{5}\psi_0(x) - \frac{2\sqrt{2}}{5}\psi_1(x) + \frac{2\sqrt{2}}{5}\psi_2(x)
\end{aligned}$$

(b)

$$
\begin{aligned}
E_0 &= \frac{\hbar\omega}{2} \\
P(E_0) &= \frac{9}{25} \\
E_1 &= \frac{3\hbar\omega}{2} \\
P(E_1) &= \frac{8}{25} \\
E_2 &= \frac{5\hbar\omega}{2} \\
P(E_2) &= \frac{8}{25} \\
\langle E\rangle &= \frac{\hbar\omega}{2}\frac{9}{25} + \frac{3\hbar\omega}{2}\frac{8}{25} + \frac{5\hbar\omega}{2}\frac{8}{25} \\
&= \frac{73}{50}\hbar\omega
\end{aligned}
$$

(c)

$$
\begin{aligned}
\xi &= \sqrt{\frac{m\omega}{\hbar}}x \\
\psi_n(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2} \\
\Psi(x,T) &= B\left(1 + 2\sqrt{\frac{m\omega}{\hbar}}x\right)^2 e^{-\frac{m\omega}{2\hbar}x^2} \\
&= B\left(1 + 4\sqrt{\frac{m\omega}{\hbar}}x + 4\frac{m\omega}{\hbar}x^2\right) e^{-\frac{m\omega}{2\hbar}x^2} \\
&= B\left(\frac{\pi\hbar}{m\omega}\right)^{1/4} \left[3\psi_0(x) + 2\sqrt{2}\psi_1(x) + 2\sqrt{2}\psi_2(x)\right] \\
1 &= |B|^2\sqrt{\frac{\pi\hbar}{m\omega}} \int_{-\infty}^{\infty} [3\psi_0 + 2\sqrt{2}\psi_1 + 2\sqrt{2}\psi_2]^2\, dx \\
&= |B|^2\sqrt{\frac{\pi\hbar}{m\omega}} \int_{-\infty}^{\infty} [9\psi_0^2 + 12\sqrt{2}\psi_0\psi_1 + 12\sqrt{2}\psi_0\psi_2 + 8\psi_1^2 + 16\psi_1\psi_2 + 8\psi_2^2]\, dx \\
&= 25|B|^2\sqrt{\frac{\pi\hbar}{m\omega}} \\
B &= \frac{1}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \\
\Psi(x,T) &= \frac{3}{5}\psi_0(x) + \frac{2\sqrt{2}}{5}\psi_1(x) + \frac{2\sqrt{2}}{5}\psi_2(x) \\
\Psi(x,t) &= \frac{3}{5}\psi_0(x)e^{-i\omega t/2} - \frac{2\sqrt{2}}{5}\psi_1(x)e^{-3i\omega t/2} + \frac{2\sqrt{2}}{5}\psi_2(x)e^{-5i\omega t/2} \\
&= e^{-i\omega t/2}\left[\frac{3}{5}\psi_0(x) - \frac{2\sqrt{2}}{5}\psi_1(x)e^{-i\omega t} + \frac{2\sqrt{2}}{5}e^{-2i\omega t}\right] \\
e^{-i\omega T} &= -1 \\
e^{-2i\omega T} &= 1 \\
T &= \frac{\pi}{\omega}
\end{aligned}
$$

2.41

The argument for calculating the allowed energies and wavefunctions is the same, except there is a boundary condition $\psi(0) = 0$. This leaves only $\psi_n(x)$ for odd n.

2.43

$$
\begin{aligned}
k &= \frac{\sqrt{2mE}}{\hbar} \\
\psi(x) &= \begin{cases} -A\sin kx + B\cos kx & -a < x \le 0 \\ A\sin kx + B\cos kx & 0 \le x < a \end{cases} \\
\Delta\left(\frac{d\psi}{dx}\right) &= 2Ak \\
\Delta\left(\frac{d\psi}{dx}\right) &= \frac{2m\alpha}{\hbar^2}\psi(0) \\
2Ak &= \frac{2m\alpha}{\hbar^2}B \\
B &= \frac{\hbar^2 k}{m\alpha}A \\
\psi(x) &= A\left(\sin kx + \frac{\hbar^2 k}{m\alpha}\cos kx\right) \\
0 &= A\sin ka + \frac{\hbar^2 k}{m\alpha}A\cos ka \\
\tan ka &= -\frac{\hbar^2 k}{m\alpha} \\
ka &\approx \frac{n\pi}{2},\ n = 1, 3, 5, \ldots \\
E &\approx \frac{n^2\pi^2\hbar^2}{2m(2a)^2},\ n = 1, 3, 5, \ldots \\
\psi(x) &= \begin{cases} A\sin kx - B\cos kx & -a < x \le 0 \\ A\sin kx + B\cos kx & 0 \le x < a \end{cases} \\
-B &= B \\
B &= 0 \\
\psi(x) &= A\sin kx \\
0 &= A\sin ka \\
ka &= \frac{n\pi}{2},\ n = 2, 4, 6, \ldots \\
\psi(x) &= A\sin\left(\frac{n\pi}{2a}x\right) \\
E &= \frac{n^2\pi^2\hbar^2}{2m(2a)^2},\ n = 2, 4, 6, \ldots
\end{aligned}
$$

2.44

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2}\psi_2 + V\psi_1\psi_2 &= E\psi_1\psi_2 \\
-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2}\psi_1 + V\psi_1\psi_2 &= E\psi_1\psi_2 \\
\psi_2\frac{d^2\psi_1}{dx^2} - \psi_1\frac{d^2\psi_2}{dx^2} &= 0 \\
\frac{d}{dx}\left(\psi_2\frac{d\psi_1}{dx} - \psi_1\frac{d\psi_2}{dx}\right) &= 0 \\
\psi_2\frac{d\psi_1}{dx} - \psi_1\frac{d\psi_2}{dx} &= c \\
c &= 0 \\
\psi_2\frac{d\psi_1}{dx} &= \psi_1\frac{d\psi_2}{dx} \\
\frac{1}{\psi_1}\frac{d\psi_1}{dx} &= \frac{1}{\psi_2}\frac{d\psi_2}{dx} \\
\ln\psi_1 &= \ln\psi_2 + c \\
\psi_1 &= A\psi_2
\end{aligned}
$$

2.45

(a)

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2}\psi_m + V(x)\psi_n\psi_m &= E_n\psi_n\psi_m \\
-\frac{\hbar^2}{2m}\frac{d^2\psi_m}{dx^2}\psi_n + V(x)\psi_n\psi_m &= E_m\psi_n\psi_m \\
\frac{d^2\psi_m}{dx^2}\psi_n - \frac{d^2\psi_n}{dx^2}\psi_m &= \frac{2m}{\hbar^2}(E_n - E_m)\psi_n\psi_m \\
\frac{d}{dx}\left(\frac{d\psi_m}{dx}\psi_n - \frac{d\psi_n}{dx}\psi_m\right) &= \frac{2m}{\hbar^2}(E_n - E_m)\psi_n\psi_m
\end{aligned}
$$

(b)

$$
\begin{aligned}
\int_{x_1}^{x_2}\frac{d}{dx}(\psi_m'\psi_n - \psi_n'\psi_m)\,dx &= \frac{2m}{\hbar^2}(E_n - E_m)\int_{x_1}^{x_2}\psi_n\psi_m\,dx \\
\psi_m'(x_2)\psi_n(x_2) - \psi_m'(x_1)\psi_n(x_1) &= \frac{2m}{\hbar^2}(E_n - E_m)\int_{x_1}^{x_2}\psi_n\psi_m\,dx
\end{aligned}
$$

2.53

(a)

$$
\frac{1}{1 - i\beta}\begin{pmatrix} i\beta & 1 \\ 1 & i\beta \end{pmatrix}
$$

(b)

$$\frac{e^{-2ika}}{\cos(2la) - i\frac{(k^2+l^2)}{2kl}\sin(2la)} \begin{pmatrix} i\frac{\sin(2la)}{2kl}(l^2-k^2) & 1 \\ 1 & i\frac{\sin(2la)}{2kl}(l^2-k^2) \end{pmatrix}$$

3 Formalism

3.1

(a)

$$\begin{aligned}\left|\int_a^b (f^*+g^*)(f+g)\,dx\right| &= \left|\int_a^b (f^*f + f^*g + g^*f + g^*g)\,dx\right| \\ &\leq \int_a^b |f|^2\,dx + \left|\int_a^b f^*g\,dx\right| + \left|\int_a^b g^*f\,dx\right| + \int_a^b |g|^2\,dx \\ &\leq \int_a^b |f|^2\,dx + 2\sqrt{\int_a^b |f|^2\,dx \int_a^b |g|^2\,dx} + \int_a^b |g|^2\,dx\end{aligned}$$

The set of all normalised functions isn't a vector space because e.g. multiplying a function by a constant also multiplies its integral by that constant meaning it's no longer a member of the vector space.

(b)

$$\begin{aligned}\langle\beta|\alpha\rangle &= \int_a^b \beta^*\alpha\,dx \\ &= \left(\int_a^b \alpha^*\beta\,dx\right)^* \\ &= \langle\alpha|\beta\rangle^* \\ \langle a|a\rangle &= \int_a^b |a|^2\,dx \\ &\geq 0\end{aligned}$$

If $\langle\alpha|\alpha\rangle = 0$ that implies $|\alpha|^2 = 0$ everywhere in the interval and thus $|\alpha\rangle = |0\rangle$.

$$\begin{aligned}\langle\alpha|\left(b\,|\beta\rangle + c\,|\gamma\rangle\right) &= \int_{x_1}^{x_2} \alpha^*(b\beta)\,dx + \int_{x_1}^{x_2} \alpha^*(c\gamma)\,dx \\ &= b\int_{x_1}^{x_2} \alpha^*\beta\,dx + c\int_{x_1}^{x_2} \alpha^*\gamma\,dx \\ &= b\,\langle\alpha|\beta\rangle + c\,\langle\alpha|\gamma\rangle\end{aligned}$$

3.2

(a)

$$\int_0^1 x^{2\nu}\,dx = \frac{1}{2\nu+1}\left[x^{2\nu+1}\right]_0^1$$

The integral is defined for $\nu > -1/2$. For the case $\nu = -1/2$

$$\int_0^1 x^{-1}\,dx = [\ln x]_0^1 = \ln 1 - \ln 0 = 0 - \infty.$$

So $f(x) = x^\nu$ is in Hilbert space for $\nu > -1/2$.

(b)

$$\begin{aligned}
\int_0^1 x\,dx &= \frac{1}{2} \\
\int_0^1 x^3\,dx &= \frac{1}{4} \\
\int_0^1 x^{-1}\,dx &= [\ln x]_0^1 \\
&= 0 - \infty
\end{aligned}$$

$f(x)$ and $xf(x)$ are in Hilbert space, but not $(d/dx)f(x)$.

3.4

(a)

$$\begin{aligned}
\langle f|(\hat{Q}+\hat{R})f\rangle &= \langle f|\hat{Q}f\rangle + \langle f|\hat{R}f\rangle \\
&= \langle \hat{Q}f|f\rangle + \langle \hat{R}f|f\rangle \\
&= \langle (\hat{Q}+\hat{R})f|f\rangle
\end{aligned}$$

(b)

$$\begin{aligned}
\langle f|\alpha\hat{Q}g\rangle &= \alpha\,\langle f|\hat{Q}g\rangle \\
&= \alpha\,\langle \hat{Q}f|g\rangle \\
\langle \alpha\hat{Q}f|g\rangle &= \alpha^*\,\langle \hat{Q}f|g\rangle \\
\alpha &= \alpha^*
\end{aligned}$$

α is real.

(c)

$$\begin{aligned}\langle f|\hat{Q}\hat{R}g\rangle &= \langle \hat{Q}f|\hat{R}g\rangle \\ &= \langle \hat{R}\hat{Q}f|g\rangle\end{aligned}$$

The product of the operators is hermitian when $\hat{Q}\hat{R} = \hat{R}\hat{Q}$ i.e. $[\hat{Q}, \hat{R}] = 0$.

(d)

$$\begin{aligned}\langle \Psi|\hat{x}\Psi\rangle &= \int \Psi^* \hat{x}\Psi\, dx \\ &= \int \Psi^* \hat{x}^* \Psi\, dx \\ &= \int (\hat{x}\Psi)^* \Psi\, dx \\ &= \langle \hat{x}\Psi|\Psi\rangle \\ \langle \Psi|\hat{H}\Psi\rangle &= \int \Psi^* \hat{H}\Psi\, dx \\ &= \int \Psi^* \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right] \Psi\, dx \\ &= -\frac{\hbar^2}{2m}\int \Psi^* \frac{d^2\Psi}{dx^2}\, dx + \int \Psi^* V(x)\Psi\, dx \\ &= -\frac{\hbar^2}{2m}\left[\Psi^* \frac{d\Psi}{dx}\bigg|_{-\infty}^{\infty} - \int \frac{d\Psi^*}{dx}\frac{d\Psi}{dx}\, dx\right] + \langle V(x)\Psi|\Psi\rangle \\ &= \frac{\hbar^2}{2m}\left[\frac{d\Psi^*}{dx}\Psi\bigg|_{-\infty}^{\infty} - \int \frac{d^2\Psi^*}{dx}\Psi\, dx\right] + \langle V(x)\Psi|\Psi\rangle \\ &= -\frac{\hbar^2}{2m}\int \frac{d^2}{dx^2}\Psi^*\Psi\, dx + \langle V(x)\Psi|\Psi\rangle \\ &= \left\langle -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi \,\middle|\, \Psi \right\rangle + \langle V(x)\Psi|\Psi\rangle \\ &= \langle \hat{H}\Psi|\Psi\rangle\end{aligned}$$

3.5

(a)

$$\begin{aligned}x^\dagger &= x \\ i^\dagger &= -i \\ \left(\frac{d}{dx}\right)^\dagger &= -\frac{d}{dx}\end{aligned}$$

(b)

$$\begin{aligned}
\langle f|\hat{Q}\hat{R}g\rangle &= \int f^\dagger \hat{Q}\hat{R}g\,dx \\
&= \int (\hat{Q}^\dagger f)^\dagger \hat{R}g\,dx \\
&= \int (\hat{R}^\dagger \hat{Q}^\dagger f)^\dagger g\,dx \\
&= \langle \hat{R}^\dagger \hat{Q}^\dagger f|g\rangle
\end{aligned}$$

(c)

$$\begin{aligned}
\hat{a}_+ &= \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p}+m\omega x) \\
\langle f|\hat{a}g\rangle &= \left\langle f \,\middle|\, \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p}+m\omega x)g \right\rangle \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\,\langle f|(-i\hat{p}+m\omega x)g\rangle \\
&= \frac{1}{\sqrt{2\hbar m\omega}}(\langle f|-i\hat{p}g\rangle + \langle f|m\omega xg\rangle) \\
&= \frac{1}{\sqrt{2\hbar m\omega}}(\langle f|-i\hat{p}g\rangle + \langle m\omega xf|g\rangle) \\
&= \frac{1}{\sqrt{2\hbar m\omega}}(\langle i\hat{p}f|g\rangle + \langle m\omega xf|g\rangle) \\
&= \left\langle \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p}+m\omega x)f \,\middle|\, g \right\rangle \\
&= \langle \hat{a}_- f|g\rangle
\end{aligned}$$

3.6

$$\begin{aligned}
\langle f|\hat{Q}g\rangle &= \int_0^{2\pi} f^* \frac{d^2g}{d\phi^2}\,d\phi \\
&= f^*\frac{dg}{d\phi}\bigg|_0^{2\pi} - \int_0^{2\pi}\left(\frac{df}{d\phi}\right)^* \frac{dg}{d\phi}\,d\phi \\
&= -\left(\frac{df}{d\phi}\right)^* g\bigg|_0^{2\pi} + \int_0^{2\pi}\left(\frac{d^2f}{d\phi}\right)^* g\,d\phi \\
&= \langle \hat{Q}f|g\rangle
\end{aligned}$$

Yes, the operator is hermitian.

$$
\begin{aligned}
\hat{Q}f &= qf \\
\frac{d^2 f}{d\phi^2} &= qf \\
\frac{d^2 f}{d\phi^2} - qf &= 0 \\
f &= Ae^{\sqrt{q}\phi} + Be^{-\sqrt{q}\phi} \\
f(\phi + 2\pi) &= Ae^{\sqrt{q}(\phi+2\pi)} + Be^{\sqrt{q}(\phi+2\pi)} \\
&= Ae^{\sqrt{q}\phi}e^{2\pi\sqrt{q}} + Be^{\sqrt{q}\phi}e^{2\pi\sqrt{q}} \\
2\pi\sqrt{q} &= 1 \\
q &= -n^2,\ n = 0, 1, 2, \ldots
\end{aligned}
$$

The eigenfunctions are $f = Ae^{\pm\sqrt{q}\phi}$ and the eigenvalues are $q = 0, 1, 2, \ldots$ The spectrum is degenerate as there are two eigenfunctions associated with each eigenvalue $q > 0$.

3.7

(a)

$$
\begin{aligned}
h &= af + bg \\
\hat{Q}h &= \hat{Q}(af + bg) \\
&= \hat{Q}(af) + \hat{Q}(bg) \\
&= a\hat{Q}f + b\hat{Q}g \\
&= aqf + bqg \\
&= q(af + bg) \\
&= qh
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{d^2}{dx^2}e^x &= e^x \\
\frac{d^2}{dx^2}e^{-x} &= e^{-x} \\
f &= e^x + e^{-x} \\
g &= e^x - e^{-x}
\end{aligned}
$$

3.8

(a)

$$\begin{aligned}
\hat{Q} &= i\frac{d}{d\phi} \\
\hat{Q}f &= qf \\
i\frac{df}{d\phi} &= qf \\
\frac{df}{d\phi} + iqf &= 0 \\
f &= Ae^{-iq\phi} \\
e^{-2\pi iq} &= 1 \\
q &= 0, \pm 1, \pm 2, \dots \\
\int_0^{2\pi} Ae^{-iq\phi}Be^{-iq'\phi}\,d\phi &= AB\int_0^{2\pi} e^{-i(q+q')\phi}\,d\phi \\
&= 0
\end{aligned}$$

(b)

$$\begin{aligned}
\hat{Q} &= \frac{d^2}{d\phi^2} \\
\hat{Q}f &= qf \\
\frac{d^2 f}{d\phi^2} - qf &= 0 \\
f &= Ae^{\pm\sqrt{q}\phi} \\
q &= -n^2,\ n = 0, 1, 2, \dots \\
\int_0^{2\pi} Ae^{\pm\sqrt{q}\phi}Be^{\pm\sqrt{q'}\phi}\,d\phi &= AB\int_0^{2\pi} e^{\pm in\phi}e^{\pm in'\phi}\,d\phi \\
&= AB\int_0^{2\pi} e^{i(\pm n\pm n')\phi}\,d\phi \\
&= AB\left[\frac{1}{i(\pm n \pm n')}e^{i(\pm n\pm n')\phi}\right]_0^{2\pi} \\
&= AB\frac{1}{i(\pm n \pm n')}[e^{i(\pm n\pm n')2\pi} - 1] \\
&= 0
\end{aligned}$$

3.9

(a) Infinite square well

(b) Delta function barrier

(c) Delta function well

3.11

$$\begin{aligned}
\Psi_0(x,t) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} e^{-i\omega t/2} \\
\Phi_0(p,t) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} e^{-i\omega t/2}\, dx \\
&= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-i\omega t/2} \int_{-\infty}^{\infty} e^{-ipx/\hbar} e^{-m\omega x^2/2\hbar}\, dx \\
&= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-i\omega t/2} \sqrt{\frac{2\pi\hbar}{m\omega}} e^{-p^2/2\hbar m\omega} \\
&= \frac{1}{(\pi\hbar m\omega)^{1/4}} e^{-p^2/2\hbar m\omega} e^{-i\omega t/2} \\
\frac{p^2}{2m} &= \frac{\hbar\omega}{2} \\
p &= \pm\sqrt{\hbar m\omega} \\
1 - \int_{-\sqrt{\hbar m\omega}}^{\sqrt{\hbar m\omega}} |\Phi_0|^2\, dp &= 1 - \frac{1}{(\pi\hbar m\omega)^{1/4}} \int_{-\sqrt{\hbar m\omega}}^{\sqrt{\hbar m\omega}} e^{-p^2/\hbar m\omega}\, dp \\
&= 1 - \frac{1}{(\pi\hbar m\omega)^{1/2}} \sqrt{\pi\hbar m\omega}\, \mathrm{erf}\, 1 \\
&= 0.16
\end{aligned}$$

3.12

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)}\, dk \\
\Phi(p,t) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)}\, dk\right) dx \\
&= \frac{1}{2\pi\sqrt{\hbar}} \int_{-\infty}^{\infty} \phi(k) e^{-i\frac{\hbar k^2}{2m} t} \left(\int_{-\infty}^{\infty} e^{i(k-p/\hbar)x}\, dx\right) dk \\
&= \frac{1}{2\pi\sqrt{\hbar}} \int_{-\infty}^{\infty} \phi(k) e^{-i\frac{\hbar k^2}{2m} t} 2\pi\delta(k - p/\hbar)\, dk \\
&= \frac{1}{\sqrt{\hbar}} \int_{-\infty}^{\infty} \delta(k - p/\hbar)\phi(k) e^{-i\frac{\hbar k^2}{2m} t}\, dk \\
&= \frac{1}{\sqrt{\hbar}} \phi(p/\hbar) e^{-i\frac{p^2}{2\hbar m} t} \\
|\Phi(p,t)|^2 &= \frac{1}{\hbar} |\phi(p/\hbar)|^2
\end{aligned}$$

3.14

(a)

$$
\begin{aligned}
[\hat{A}+\hat{B},\hat{C}] &= (\hat{A}+\hat{B})\hat{C} - \hat{C}(\hat{A}+\hat{B}) \\
&= \hat{A}\hat{C} + \hat{B}\hat{C} - \hat{C}\hat{A} - \hat{C}\hat{B} \\
&= \hat{A}\hat{C} - \hat{C}\hat{A} + \hat{B}\hat{C} - \hat{C}\hat{B} \\
&= [\hat{A},\hat{C}] + [\hat{B},\hat{C}] \\
\hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B} &= \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} + \hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B} \\
&= \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} \\
&= [\hat{A}\hat{B},\hat{C}]
\end{aligned}
$$

(b)

$$
\begin{aligned}
[x^n,\hat{p}] &= \left[x^n, -i\hbar\frac{d}{dx}\right] \\
&= x^n\left(-i\hbar\frac{d}{dx}\right) - \left(-i\hbar\frac{d}{dx}\right)x^n \\
&= x^n\left(-i\hbar\frac{d}{dx}\right)(1) + i\hbar n x^{n-1} \\
&= i\hbar n x^{n-1}
\end{aligned}
$$

(c)

$$
\begin{aligned}
[f(x),\hat{p}]g(x) &= f(x)\left(-i\hbar\frac{d}{dx}\right)g(x) - \left(-i\hbar\frac{d}{dx}\right)[f(x)g(x)] \\
&= -i\hbar f(x)\frac{dg}{dx} + i\hbar\left[\frac{df}{dx}g(x) + f(x)\frac{dg}{dx}\right] \\
&= i\hbar\frac{df}{dx}g(x) \\
[f(x),\hat{p}] &= i\hbar\frac{df}{dx}
\end{aligned}
$$

(d)

$$\begin{aligned}
[\hat{H},\hat{a}_-]g &= \hat{H}\hat{a}_-g - \hat{a}_-\hat{H}g \\
&= \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)\hat{a}_-g - \hat{a}_-\hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)g \\
&= \hbar\omega\hat{a}_-\hat{a}_+\hat{a}_-g - \frac{1}{2}\hbar\omega\hat{a}_-g - \hbar\omega\hat{a}_-^2\hat{a}_+g + \frac{1}{2}\hbar\omega\hat{a}_-g \\
&= \hbar\omega\hat{a}_-\hat{a}_+\hat{a}_-g - \hbar\omega\hat{a}_-^2\hat{a}_+g \\
&= \hbar\omega\hat{a}_-(\hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+)g \\
&= -\hbar\omega\hat{a}_-g \\
[\hat{H},\hat{a}_-] &= -\hbar\omega\hat{a}_- \\
[\hat{H},\hat{a}_+]g &= \hat{H}\hat{a}_+g - \hat{a}_+\hat{H}g \\
&= \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)\hat{a}_+g - \hat{a}_+\hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)g \\
&= \hbar\omega\hat{a}_-\hat{a}_+^2g - \frac{1}{2}\hbar\omega\hat{a}_+g - \hbar\omega\hat{a}_+\hat{a}_-\hat{a}_+g + \frac{1}{2}\hbar\omega\hat{a}_+g \\
&= \hbar\omega\hat{a}_-\hat{a}_+^2g - \hbar\omega\hat{a}_+\hat{a}_-\hat{a}_+g \\
&= \hbar\omega(\hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_-)\hat{a}_+g \\
&= \hbar\omega\hat{a}_+g \\
[\hat{H},\hat{a}_+] &= \hbar\omega\hat{a}_+
\end{aligned}$$

3.15

$$\begin{aligned}
\left[x, \frac{p^2}{2m} + V\right] g &= x\left(\frac{p^2}{2m} + V\right) g - \left(\frac{p^2}{2m} + V\right) xg \\
&= x\frac{p^2}{2m} g + xVg - \frac{p^2}{2m} xg - Vxg \\
&= \frac{1}{2m}(xp^2 g - p^2 xg) \\
&= \frac{1}{2m}\left[-\hbar^2 x\frac{d^2 g}{dx^2} + \hbar^2 \frac{d}{dx}\left(g + x\frac{dg}{dx}\right)\right] \\
&= \frac{1}{2m}\left[-\hbar^2 x\frac{d^2 g}{dx^2} + \hbar^2 \left(\frac{dg}{dx} + \frac{dg}{dx} + x\frac{d^2 g}{dx^2}\right)\right] \\
&= \frac{\hbar^2}{m}\frac{dg}{dx} \\
\left[x, \frac{p^2}{2m} + V\right] &= \frac{\hbar^2}{m}\frac{d}{dx} \\
&= -\frac{\hbar}{im}\langle p\rangle \\
\sigma_x^2\sigma_H^2 &\geq \left(\frac{1}{2i}\left\langle\left[x, \frac{p^2}{2m} + V\right]\right\rangle\right)^2 \\
&= \frac{\hbar^2}{4m^2}|\langle p\rangle|^2 \\
\sigma_x\sigma_H &\geq \frac{\hbar}{2m}|\langle p\rangle|
\end{aligned}$$

This doesn't tell us much because for stationary states $\sigma_H = 0$ and $\langle p\rangle = 0$ so this says $0 \geq 0$.

3.17

$$\begin{aligned}
\left(-i\hbar\frac{d}{dx} - \langle p\rangle\right)\Psi &= ia(x - \langle x\rangle)\Psi \\
-i\hbar\frac{d\Psi}{dx} - \langle p\rangle\Psi &= ia(x - \langle x\rangle)\Psi \\
\frac{d\Psi}{dx} + \frac{\langle p\rangle + ia(x - \langle x\rangle)}{i\hbar}\Psi &= 0 \\
\frac{d\Psi}{dx} + \left[\frac{a}{\hbar}(x - \langle x\rangle) - i\frac{\langle p\rangle}{\hbar}\right]\Psi &= 0 \\
\frac{d\Psi}{dx} + \frac{a}{\hbar}x\Psi - \frac{a}{\hbar}\langle x\rangle\Psi - i\frac{\langle p\rangle}{\hbar}\Psi &= 0 \\
\Psi &= Ae^{-ax^2/2\hbar}e^{a\langle x\rangle x/\hbar}e^{i\langle p\rangle x/\hbar} \\
&= Be^{-a(x-\langle x\rangle)^2/2\hbar}e^{i\langle p\rangle x/\hbar}
\end{aligned}$$

3.18

(a)

$$\begin{aligned} Q &= 1 \\ \hat{Q} &= 1 \\ [\hat{H}, \hat{Q}] &= 0 \\ \frac{d}{dt}\langle Q \rangle &= 0 \end{aligned}$$

(b)

$$\begin{aligned} Q &= H \\ \hat{Q} &= \hat{H} \\ [\hat{H}, \hat{Q}] &= [\hat{H}, \hat{H}] \\ &= 0 \\ \frac{d}{dt}\langle H \rangle &= 0 \end{aligned}$$

(c)

$$\begin{aligned} Q &= x \\ \hat{Q} &= x \\ [\hat{H}, \hat{Q}] &= [\hat{H}, x] \\ &= -i\frac{\hbar}{m}\hat{p} \\ \frac{d}{dt}\langle x \rangle &= \frac{i}{\hbar}\left\langle -i\frac{\hbar}{m}p \right\rangle \\ &= \frac{\langle p \rangle}{m} \end{aligned}$$

(d)

$$\begin{aligned} Q &= p \\ \hat{Q} &= \hat{p} \\ [\hat{H}, \hat{Q}] &= [\hat{H}, \hat{p}] \\ &= i\hbar\frac{\partial V}{\partial x} \\ \frac{d}{dt}\langle p \rangle &= \frac{i}{\hbar}\left\langle i\hbar\frac{\partial V}{\partial x} \right\rangle \\ &= -\left\langle \frac{\partial V}{\partial x} \right\rangle \end{aligned}$$

3.19

(a)

$$\begin{aligned}
\frac{d^2}{dt^2}\langle x\rangle &= \frac{d}{dx}\left(\frac{\langle p\rangle}{m}\right)\\
&= -\frac{1}{m}\left\langle\frac{\partial V}{\partial x}\right\rangle\\
&= 0
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{d^2}{dt^2}\langle x\rangle &= \frac{d}{dx}\left(\frac{\langle p\rangle}{m}\right)\\
&= -\frac{1}{m}\left\langle\frac{\partial V}{\partial x}\right\rangle\\
&= -\omega^2\langle x\rangle\\
\frac{d^2}{dt^2}\langle x\rangle + \omega^2\langle x\rangle &= 0\\
\langle x\rangle &= A\sin\omega t + B\cos\omega t
\end{aligned}$$

3.20

$$\begin{aligned}
\Psi &= \frac{1}{\sqrt{2}}(\psi_1 e^{-iE_1t/\hbar} + \psi_2 e^{-iE_2t/\hbar})\\
\hat{H}\Psi &= \frac{1}{\sqrt{2}}(E_1\psi_1 e^{-iE_1t/\hbar} + E_2\psi_2 e^{-iE_2t/\hbar})\\
\hat{H}^2\Psi &= \frac{1}{\sqrt{2}}(E_1^2\psi_1 e^{-iE_1t/\hbar} + E_2^2\psi_2 e^{-iE_2t/\hbar})\\
\langle H^2\rangle &= \langle\Psi|\hat{H}^2\Psi\rangle\\
&= \frac{1}{2}\int_0^a (\psi_1^* e^{iE_1t/\hbar} + \psi_2^* e^{iE_2t/\hbar})(E_1^2\psi_1 e^{-iE_1t/\hbar} + E_2^2\psi_2 e^{-iE_2t/\hbar})\,dx\\
&= \frac{1}{2}\int_0^a (E_1^2|\psi_1|^2 + E_2^2\psi_1^*\psi_2^* e^{i(E_1-E_2)t/\hbar} + E_1^2\psi_2^*\psi_1 e^{i(E_2-E_1)t/\hbar}\\
&\qquad + E_2^2|\psi_2|^2)\,dx\\
&= \frac{1}{2}(E_1^2 + E_2^2)\\
\langle H\rangle &= \frac{1}{2}(E_1 + E_2)\\
\sigma_H^2 &= \langle H^2\rangle - \langle H\rangle^2\\
&= \frac{1}{2}(E_1^2 + E_2^2) - \frac{1}{4}(E_1 + E_2)^2\\
&= \frac{1}{4}(E_2 - E_1)^2
\end{aligned}$$

$$
\begin{aligned}
\omega &= \frac{\pi^2\hbar}{2ma^2} \\
\langle x \rangle &= \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t)\right] \\
\langle x^2 \rangle &= \frac{1}{2}\int_0^a (\psi_1^* e^{iE_1 t/\hbar} + \psi_2^* e^{iE_2 t/\hbar}) x^2 (\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar})\, dx \\
&= \frac{1}{2}\int_0^a x^2 (|\psi_1|^2 + \psi_1^*\psi_2 e^{i(E_1 - E_2)t/\hbar} + \psi_2^*\psi_1 e^{i(E_2 - E_1)t/\hbar} + |\psi_2|^2)\, dx \\
&= \frac{1}{a}\int_0^a x^2 \left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right)\right. \\
&\qquad \left. +2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right]\, dx \\
&= \frac{a^2}{144\pi^2}[-45 + 48\pi^2 - 256\cos(3\omega t)] \\
\sigma_x^2 &= \langle x^2 \rangle - \langle x \rangle^2 \\
&= \frac{a^2}{4}\left[\frac{1}{3} - \frac{5}{4\pi^2} - \left(\frac{32}{9\pi^2}\right)^2 \cos^2(3\omega t)\right] \\
\frac{d\langle x \rangle}{dt} &= \frac{16a\omega}{3\pi^2}\sin(3\omega t) \\
&= \frac{8\hbar}{3ma}\sin(3\omega t)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_H^2 \sigma_x^2 &\geq \left(\frac{\hbar}{2}\right)^2 \left|\frac{d\langle x \rangle}{dt}\right|^2 \\
\frac{1}{4}\left(\frac{3\pi^2\hbar^2}{2ma^2}\right)^2 \frac{a^2}{4}\left[\frac{1}{3} - \frac{5}{4\pi^2} - \left(\frac{32}{9\pi^2}\right)^2 \cos^2(3\omega t)\right] &\geq \left(\frac{\hbar}{2}\right)^2 \left[\frac{8\hbar}{3ma}\sin(3\omega t)\right]^2 \\
\left(\frac{3}{4}\right)^2 \left[\frac{1}{3} - \frac{5}{4\pi^2} - \left(\frac{32}{9\pi^2}\right)^2 \cos^2(3\omega t)\right] &\geq \left(\frac{8}{3\pi^2}\right)^2 \sin^2(3\omega t) \\
\frac{1}{3} - \frac{5}{4\pi^2} &\geq \left(\frac{32}{9\pi^2}\right)^2
\end{aligned}
$$

3.23

$$
\begin{aligned}
\hat{P}^2 |\beta\rangle &= \hat{P}(\hat{P} |\beta\rangle) \\
&= \hat{P}(\langle\alpha|\beta\rangle |\alpha\rangle) \\
&= \langle\alpha|\beta\rangle \hat{P} |\alpha\rangle \\
&= \langle\alpha|\beta\rangle (\langle\alpha|\alpha\rangle |\alpha\rangle) \\
&= \langle\alpha|\beta\rangle |\alpha\rangle \\
&= \hat{P} |\beta\rangle
\end{aligned}
$$

So, $\hat{P}^2 = \hat{P}$.

$$
\begin{aligned}
\hat{P} |\beta\rangle &= \lambda |\beta\rangle \\
\langle\alpha|\beta\rangle |\alpha\rangle &= \lambda |\beta\rangle
\end{aligned}
$$

If $|\beta\rangle$ is a constant multiple of $|\alpha\rangle$, then

$$
\begin{aligned}
\langle\alpha|c\alpha\rangle |\alpha\rangle &= \lambda c |\alpha\rangle \\
c \langle\alpha|\alpha\rangle |\alpha\rangle &= \lambda c |\alpha\rangle \\
c |\alpha\rangle &= \lambda c |\alpha\rangle
\end{aligned}
$$

thus $\lambda = 1$.
If $|\beta\rangle$ is orthogonal to $|\alpha\rangle$, then

$$
\begin{aligned}
\langle\alpha|\beta\rangle |\alpha\rangle &= \lambda |\beta\rangle \\
0 &= \lambda |\beta\rangle
\end{aligned}
$$

thus $\lambda = 0$.

3.24

$$
\begin{aligned}
\hat{Q} &= \hat{Q}^\dagger \\
Q_{mn} &= \langle e_m|\hat{Q}|e_n\rangle \\
Q_{nm} &= \langle e_n|\hat{Q}|e_m\rangle \\
Q_{nm}^* &= \langle e_n|\hat{Q}|e_m\rangle^* \\
&= \langle e_m|\hat{Q}^\dagger|e_n\rangle \\
&= \langle e_m|\hat{Q}|e_n\rangle \\
&= Q_{mn}
\end{aligned}
$$

3.25

$$
\begin{aligned}
\hat{H}\,|\alpha\rangle &= \lambda\,|\alpha\rangle \\
\epsilon[(\langle 1|\alpha\rangle + \langle 2|\alpha\rangle)\,|1\rangle + (\langle 1|\alpha\rangle - \langle 2|\alpha\rangle)\,|2\rangle] &= \lambda(\langle 1|\alpha\rangle\,|1\rangle + \langle 2|\alpha\rangle\,|2\rangle)
\end{aligned}
$$

From this we get two equations

$$
\begin{aligned}
\epsilon(\langle 1|\alpha\rangle + \langle 2|\alpha\rangle) &= \lambda\,\langle 1|\alpha\rangle \\
\epsilon(\langle 1|\alpha\rangle) - \langle 2|\alpha\rangle) &= \lambda\,\langle 2|\alpha\rangle
\end{aligned}
$$

If we let $|\alpha\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ then this becomes

$$
\begin{aligned}
\epsilon(a+b) &= \lambda a \\
\epsilon(a-b) &= \lambda b
\end{aligned}
$$

or in matrix form

$$
\epsilon \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}.
$$

The eigenvalues of this matrix are $\lambda = \pm\sqrt{2}\epsilon$ and the eigenvectors are $|1\rangle + (\sqrt{2} \pm 1)\,|2\rangle$.

3.26

(a)

$$
\begin{aligned}
\langle\alpha| &= -i\,\langle 1| - 2\,\langle 2| + i\,\langle 3| \\
\langle\beta| &= -i\,\langle 1| + 2\,\langle 3|
\end{aligned}
$$

(b)

$$
\begin{aligned}
\langle\alpha|\beta\rangle &= (-i\,\langle 1| - 2\,\langle 2| + i\,\langle 3|)(i\,|1\rangle + 2\,|3\rangle) \\
&= \langle 1|1\rangle - 2i\,\langle 1|3\rangle - 2i\,\langle 2|1\rangle - 4\,\langle 2|3\rangle - \langle 3|1\rangle + 2i\,\langle 3|3\rangle \\
&= 1 + 2i \\
\langle\beta|\alpha\rangle &= (-i\,\langle 1| + 2\,\langle 3|)(i\,|1\rangle - 2\,|2\rangle - i\,|3\rangle) \\
&= \langle 1|1\rangle + 2i\,\langle 1|2\rangle - \langle 1|3\rangle + 2i\,\langle 3|1\rangle - 4\,\langle 3|2\rangle - 2i\,\langle 3|3\rangle \\
&= 1 - 2i \\
&= \langle\alpha|\beta\rangle^*
\end{aligned}
$$

(c)

$$
\begin{aligned}
\hat{A} &= |\alpha\rangle \langle\beta| \\
\hat{A}\,|1\rangle &= |\alpha\rangle \langle\beta|1\rangle \\
&= |\alpha\rangle \langle 1|\beta\rangle^* \\
&= -i\,|\alpha\rangle \\
&= |1\rangle + 2i\,|2\rangle - |3\rangle \\
A_{11} &= \langle 1|\hat{A}|1\rangle \\
&= 1 \\
A_{21} &= 2i \\
A_{31} &= -1 \\
\hat{A}\,|2\rangle &= |\alpha\rangle \langle\beta|2\rangle \\
&= |\alpha\rangle \langle 2|\beta\rangle^* \\
&= 0 \\
A_{12} &= 0 \\
A_{22} &= 0 \\
A_{32} &= 0 \\
\hat{A}\,|3\rangle &= |\alpha\rangle \langle\beta|3\rangle \\
&= |\alpha\rangle \langle 3|\beta\rangle^* \\
&= 2\,|\alpha\rangle \\
&= 2i\,|1\rangle - 4\,|2\rangle - 2i\,|3\rangle \\
A_{13} &= 2i \\
A_{23} &= -4 \\
A_{33} &= -2i \\
A &= \begin{pmatrix} 1 & 0 & 2i \\ 2i & 0 & -4 \\ -1 & 0 & -2i \end{pmatrix}
\end{aligned}
$$

It's not hermitian.

3.27

(a)

$$\begin{aligned}
\hat{Q}\,|\alpha\rangle &= \hat{Q}\sum_{n=1}^{\infty}\langle e_n|\alpha\rangle\,|e_n\rangle \\
&= \hat{Q}\left(\sum_{n=1}^{\infty}|e_n\rangle\,\langle e_n|\right)|\alpha\rangle \\
&= \left(\sum_{n=1}^{\infty}\hat{Q}\,|e_n\rangle\,\langle e_n|\right)|\alpha\rangle \\
&= \left(\sum_{n=1}^{\infty}q_n\,|e_n\rangle\,\langle e_n|\right)|\alpha\rangle \\
\hat{Q} &= \sum_{n=1}^{\infty}q_n\,|e_n\rangle\,\langle e_n|
\end{aligned}$$

(b)

$$\begin{aligned}
\hat{Q} &= \sum_{n=1}^{\infty}q_n\,|e_n\rangle\,\langle e_n| \\
\hat{Q}^2 &= \left(\sum_{n=1}^{\infty}q_n\,|e_n\rangle\,\langle e_n|\right)\left(\sum_{l=1}^{\infty}q_l\,|e_l\rangle\,\langle e_l|\right) \\
&= \sum_{n=1}^{\infty}\sum_{l=0}^{\infty}q_n q_l\,|e_n\rangle\,\langle e_n|e_l\rangle\,\langle e_l| \\
&= \sum_{n=1}^{\infty}q_n^2\,|e_n\rangle\,\langle e_n| \\
e^{\hat{Q}} &= \sum_{n=1}^{\infty}e^{q_n}\,|e_n\rangle\,\langle e_n| \\
&= \sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}\frac{q_n^k}{k!}\right)|e_n\rangle\,\langle e_n| \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{n=1}^{\infty}q_n^k\,|e_n\rangle\,\langle e_n|\right) \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\hat{Q}^k \\
&= 1+\hat{Q}+\frac{1}{2}\hat{Q}^2+\frac{1}{3!}\hat{Q}^3+\cdots
\end{aligned}$$

3.28

(a)

$$
\begin{aligned}
\sin \hat{D} &= \hat{D} - \frac{D^3}{3!} + \frac{\hat{D}^5}{5!} - \frac{\hat{D}^7}{7!} + \cdots \\
(\sin \hat{D}) x^5 &= 5x^4 - 10x^2 + 1
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{1}{1 - \hat{D}/2} &= 1 + \frac{\hat{D}}{2} + \left(\frac{\hat{D}}{2}\right)^2 + \left(\frac{\hat{D}}{2}\right)^3 + \cdots \\
&= 1 + \frac{\hat{D}}{2} + \frac{\hat{D}^2}{4} + \frac{\hat{D}^3}{8} + \cdots \\
\frac{1}{1 - \hat{D}/2} \cos x &= \cos x - \frac{1}{2} \sin x - \frac{1}{4} \cos x + \frac{1}{8} \sin x + \cdots \\
&= \left(-\frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \cdots\right) \sin x + \left(1 - \frac{1}{4} + \frac{1}{16} - \cdots\right) \cos x \\
&= -\frac{2}{5} \sin x + \frac{4}{5} \cos x
\end{aligned}
$$

3.30

$$
\begin{aligned}
c_n(t) &= \langle n | S(t) \rangle \\
&= \left\langle n \,\middle|\, \int dx \, |x\rangle \langle x| \,\middle|\, S(t) \right\rangle \\
&= \int \langle n|x\rangle \langle x|S(t)\rangle \; dx \\
&= \int \langle x|n\rangle^* \, \Psi(x,t) \, dx \\
&= \int \psi_n(x)^* \Psi(x,t) \, dx
\end{aligned}
$$

3.31

$$
\begin{aligned}
|e_1\rangle &= 1 \\
\langle e_1|e_1\rangle &= \int_{-1}^{1} dx \\
&= 2 \\
|e_1'\rangle &= \frac{1}{\sqrt{2}} \\
|e_2\rangle &= x \\
\langle e_1'|e_2\rangle &= \int_{-1}^{1} \frac{1}{\sqrt{2}} x \, dx \\
&= \frac{1}{\sqrt{2}} \left[\frac{1}{2}x^2\right]_{-1}^{1} \\
&= 0 \\
\langle e_2|e_2\rangle &= \int_{-1}^{1} x^2 \, dx \\
&= \left[\frac{1}{3}x^3\right]_{-1}^{1} \\
&= \frac{2}{3} \\
|e_2'\rangle &= \sqrt{\frac{3}{2}} x \\
|e_3\rangle &= x^2 \\
\langle e_1'|e_3\rangle &= \int_{-1}^{1} \frac{1}{\sqrt{2}} x^2 \, dx \\
&= \frac{1}{\sqrt{2}} \left[\frac{1}{3}x^3\right]_{-1}^{1} \\
&= \frac{\sqrt{2}}{3} \\
\langle e_2'|e_3\rangle &= \int_{-1}^{1} \sqrt{\frac{3}{2}} x^3 \, dx \\
&= 0 \\
|e_3'\rangle &= x^2 - \frac{\sqrt{2}}{3} |e_1'\rangle \\
&= x^2 - \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
\langle e_3'|e_3'\rangle &= \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 \, dx \\
&= \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) \, dx \\
&= \left[\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x\right]_{-1}^{1} \\
&= \frac{2}{5} - \frac{4}{9} + \frac{2}{9} \\
&= \frac{18}{45} - \frac{20}{45} + \frac{10}{45} \\
&= \frac{8}{45} \\
|e_3''\rangle &= \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3}\right) \\
&= \sqrt{\frac{5}{2}} \left(\frac{3}{2}x^2 - \frac{1}{2}\right) \\
|e_4\rangle &= x^3 \\
\langle e_1'|e_4\rangle &= \int_{-1}^{1} \frac{1}{\sqrt{2}} x^3 \, dx \\
&= 0 \\
\langle e_2'|e_4\rangle &= \int_{-1}^{1} \sqrt{\frac{3}{2}} x^4 \, dx \\
&= \sqrt{\frac{3}{2}} \left[\frac{1}{5}x^5\right] \\
&= \frac{\sqrt{6}}{5} \\
\langle e_3''|e_4\rangle &= \int_{-1}^{1} \sqrt{\frac{5}{2}} \left(\frac{3}{2}x^2 - \frac{1}{2}\right) x^3 \, dx \\
&= \sqrt{\frac{5}{2}} \int_{-1}^{1} \left(\frac{3}{2}x^5 - \frac{1}{2}x^3\right) \, dx \\
&= 0 \\
|e_4'\rangle &= |e_4\rangle - \frac{\sqrt{6}}{5} |e_2'\rangle \\
&= x^3 - \frac{3}{5}x
\end{aligned}
$$

$$
\begin{aligned}
\langle e_4'|e_4'\rangle &= \int_{-1}^{1}\left(x^3-\frac{3}{5}x\right)^2\,dx \\
&= \int_{-1}^{1}\left(x^6-\frac{6}{5}x^4+\frac{9}{25}x^2\right)\,dx \\
&= \left[\frac{1}{7}x^7-\frac{6}{25}x^5+\frac{3}{25}x^3\right]_{-1}^{1} \\
&= \frac{2}{7}-\frac{12}{25}+\frac{6}{25} \\
&= \frac{50}{175}-\frac{84}{175}+\frac{42}{175} \\
&= \frac{8}{175} \\
|e_4''\rangle &= \sqrt{\frac{175}{8}}\left(x^3-\frac{3}{5}x\right) \\
&= \sqrt{\frac{7}{2}}\left(\frac{5}{2}x^3-\frac{3}{2}x\right)
\end{aligned}
$$

3.32

(a)

$$
\begin{aligned}
\langle\hat{Q}\rangle &= \langle\Psi|\hat{Q}|\Psi\rangle \\
&= \langle\Psi|\hat{Q}^\dagger|\Psi\rangle^* \\
&= -\langle\Psi|\hat{Q}|\Psi\rangle^* \\
&= -\langle\hat{Q}\rangle^*
\end{aligned}
$$

(b)

$$
\begin{aligned}
\hat{Q}\,|\psi\rangle &= \lambda\,|\psi\rangle \\
\langle\psi|\hat{Q}|\psi\rangle &= \langle\psi|\lambda|\psi\rangle \\
&= \lambda\,\langle\psi|\psi\rangle \\
\langle\psi|\hat{Q}|\psi\rangle^* &= \lambda^*\,\langle\psi|\psi\rangle^* \\
\langle\psi|\hat{Q}^\dagger|\psi\rangle &= \lambda^*\,\langle\psi|\psi\rangle \\
-\langle\psi|\hat{Q}|\psi\rangle &= \lambda^*\,\langle\psi|\psi\rangle \\
\langle\psi|\hat{Q}|\psi\rangle &= -\lambda^*\,\langle\psi|\psi\rangle \\
\hat{Q}\,|\psi\rangle &= -\lambda^*\,|\psi\rangle
\end{aligned}
$$

(c)

$$\begin{aligned}
\hat{Q}\,|f\rangle &= \lambda_1\,|f\rangle \\
\hat{Q}\,|g\rangle &= \lambda_2\,|g\rangle \\
\langle g|\hat{Q}|f\rangle &= \langle g|\lambda_1|f\rangle \\
&= \lambda_1\,\langle g|f\rangle \\
\langle f|\hat{Q}^\dagger|g\rangle &= \lambda_1^*\,\langle f|g\rangle \\
-\,\langle f|\hat{Q}|g\rangle &= \lambda_1^*\,\langle f|g\rangle \\
\langle f|\lambda_2|g\rangle &= -\lambda_1^*\,\langle f|g\rangle \\
\lambda_2\,\langle f|g\rangle &= -\lambda_1^*\,\langle f|g\rangle \\
\lambda_2 &= -\lambda_1^* \\
\lambda_2 &= \lambda_1 \\
\langle f|g\rangle &= 0
\end{aligned}$$

(d)

$$\begin{aligned}
[\hat{A},\hat{B}] &= \hat{A}\hat{B} - \hat{B}\hat{A} \\
[\hat{A},\hat{B}]^\dagger &= (\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger \\
&= \hat{A}^\dagger\hat{B}^\dagger - \hat{B}^\dagger\hat{A}^\dagger \\
&= -(\hat{A}\hat{B} - \hat{B}\hat{A}) \\
&= -[\hat{A},\hat{B}] \\
[\hat{A},\hat{B}]^\dagger &= \hat{A}^\dagger\hat{B}^\dagger - \hat{B}^\dagger\hat{A}^\dagger \\
&= -(\hat{A}\hat{B} - \hat{B}\hat{A}) \\
&= -[\hat{A},\hat{B}]
\end{aligned}$$

(e)

$$\begin{aligned}
\hat{Q}\,|q_n\rangle &= \lambda_n\,|q_n\rangle \\
&= (x+iy)\,|q_n\rangle \\
&= x\,|q_n\rangle + iy\,|q_n\rangle \\
&= \hat{X}\,|q_n\rangle + \hat{Y}\,|q_n\rangle \\
&= (\hat{X}+\hat{Y})\,|q_n\rangle
\end{aligned}$$

3.33

(a) ψ_1

(b) b_1 and b_2 with $P(b_1) = \frac{9}{25}$ and $P(b_2) = \frac{16}{25}$.

(c)

$$\begin{aligned}
\phi_1 &= \frac{3}{5}\psi_1 + \frac{4}{5}\psi_2 \\
\phi_2 &= \frac{4}{5}\psi_1 - \frac{3}{5}\psi_2 \\
P(a_1) &= P(b_1)\left(\frac{3}{5}\right)^2 + P(b_2)\left(\frac{4}{5}\right)^2 \\
&= \left(\frac{9}{25}\right)^2 + \left(\frac{16}{25}\right)^2 \\
&= \frac{337}{625} \\
&\approx 53.9\%
\end{aligned}$$

3.34

(a)

$$\begin{aligned}
\Phi_n(p,t) &= \frac{1}{\sqrt{2\pi\hbar}}\int_0^a e^{-ipx/\hbar}\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)e^{-iE_nt/\hbar}\,dx \\
&= \frac{1}{\sqrt{\pi\hbar a}}e^{-iE_nt/\hbar}\int_0^a e^{-ipx/\hbar}\sin\left(\frac{n\pi}{a}x\right)\,dx \\
&= \sqrt{\frac{a\pi}{\hbar}}\frac{ne^{-iE_nt/\hbar}}{(n\pi)^2-(ap/\hbar)^2}[1-(-1)^n e^{-ipa/\hbar}]
\end{aligned}$$

(b)

$$|\Phi_n(p,t)|^2 = \frac{a\pi}{\hbar}\frac{4n^2}{[(n\pi)^2-(ap/\hbar)^2]^2}\begin{cases}\cos^2\left(\frac{a}{2\hbar}p\right) & n \text{ odd} \\ \sin^2\left(\frac{a}{2\hbar}p\right) & n \text{ even}\end{cases}$$

3.35

$$
\begin{aligned}
\Psi(x,0) &= \begin{cases} \frac{1}{\sqrt{2n\lambda}} e^{i2\pi x/\lambda} & -n\lambda < x < n\lambda \\ 0 & \text{otherwise} \end{cases} \\
\Phi(p,0) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \Psi(x,0)\,dx \\
&= \frac{1}{2\sqrt{\pi\hbar n\lambda}} \int_{-n\lambda}^{n\lambda} e^{i(2\pi/\lambda - p/\hbar)x}\,dx \\
&= \frac{1}{2\sqrt{\pi\hbar n\lambda}} \frac{1}{i(2\pi/\lambda - p/\hbar)} \left[e^{i(2\pi/\lambda - p/\hbar)x} \right]_{-n\lambda}^{n\lambda} \\
&= \frac{1}{2\sqrt{\pi\hbar n\lambda}} \frac{1}{i(2\pi/\lambda - p/\hbar)} e^{i(2\pi/\lambda - p/\hbar)n\lambda} - e^{-i(2\pi/\lambda - p/\hbar)n\lambda} \\
&= \frac{1}{\sqrt{\pi\hbar n\lambda}} \frac{1}{2\pi/\lambda - p/\hbar} \sin\left[\left(\frac{2\pi}{\lambda} - \frac{p}{\hbar} \right) n\lambda \right] \\
&= \sqrt{\frac{\lambda\hbar}{n\pi}} \frac{1}{\lambda p - 2\pi\hbar} \sin\left(\frac{n\lambda}{\hbar} p \right) \\
w_x &= 2n\lambda \\
w_p &= \frac{2\pi\hbar}{n\lambda}
\end{aligned}
$$

As $n \to \infty$, $w_x \to \infty$ and $w_p \to 0$.

$$
\begin{aligned}
w_x w_p &= 2n\lambda \frac{2\pi\hbar}{n\lambda} \\
&= 4\pi\hbar \\
&\geq \frac{\hbar}{2}
\end{aligned}
$$

3.36

(a)

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} \left(\frac{A}{x^2 + a^2} \right)^2 dx \\
&= |A|^2 \int_{-\infty}^{\infty} \frac{1}{(x^2 + a^2)^2}\,dx \\
&= \frac{\pi |A|^2}{2a^3} \\
A &= a\sqrt{\frac{2a}{\pi}}
\end{aligned}
$$

(b)

$$\begin{aligned}
\langle x \rangle &= \langle \Psi | x | \Psi \rangle \\
&= \int_{-\infty}^{\infty} \Psi^* x \Psi \, dx \\
&= \frac{a^3}{\pi} \int_{-\infty}^{\infty} \frac{2x}{(x^2+a^2)^2} \, dx \\
u &= x^2 + a^2 \\
du &= 2x \, dx \\
\langle x \rangle &= \frac{a^3}{\pi} \int_{-\infty}^{\infty} \frac{1}{u^2} \\
&= \frac{a^3}{\pi} \left[-\frac{1}{u} \right]_{-\infty}^{\infty} \\
&= 0 \\
\langle x^2 \rangle &= \langle \Psi | x^2 \Psi \rangle \\
&= \int_{-\infty}^{\infty} \Psi^* x^2 \Psi \, dx \\
&= \frac{2a^3}{\pi} \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2} \, dx \\
&= a^2 \\
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= a
\end{aligned}$$

(c)

$$\begin{aligned}
\Phi(x,0) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} a \sqrt{\frac{2a}{\pi}} \frac{1}{x^2+a^2} \, dx \\
&= \frac{a\sqrt{a}}{\pi\sqrt{\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \frac{1}{x^2+a^2} \, dx \\
&= \sqrt{\frac{a}{\hbar}} e^{-|p|a/\hbar}
\end{aligned}$$

$$\frac{a}{\hbar} \int_{-\infty}^{\infty} e^{-2|p|a/\hbar} \, dp = 1$$

(d)

$$\begin{aligned}
\langle p \rangle &= \langle \Phi | p | \Phi \rangle \\
&= \int_{-\infty}^{\infty} \Phi^* p \Psi \, dp \\
&= \frac{a}{\hbar} \int_{-\infty}^{\infty} p e^{-2|p|a/\hbar} \, dp \\
&= 0 \\
\langle p^2 \rangle &= \frac{a}{\hbar} \int_{-\infty}^{\infty} p^2 e^{-2|p|a/\hbar} \, dp \\
&= \frac{\hbar^2}{2a^2} \\
\sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\
&= \frac{\hbar}{\sqrt{2} a}
\end{aligned}$$

(e)

$$\begin{aligned}
\sigma_x \sigma_p &= \frac{\hbar}{\sqrt{2}} \\
&\geq \frac{\hbar}{2}
\end{aligned}$$

3.37

$$\begin{aligned}
[\hat{H}, xp] &= x[\hat{H}, p] + [\hat{H}, x]p \\
&= i\hbar x \frac{dV}{dx} - \frac{i\hbar p^2}{m} \\
\frac{d}{dt} \langle xp \rangle &= \frac{i}{\hbar} \langle [\hat{H}, xp] \rangle \\
&= \frac{i}{\hbar} \left\langle i\hbar x \frac{dV}{dx} - \frac{i\hbar p^2}{m} \right\rangle \\
&= \left\langle \frac{p^2}{m} \right\rangle - \left\langle x \frac{dV}{dx} \right\rangle \\
&= 2 \langle T \rangle - \left\langle x \frac{dV}{dx} \right\rangle
\end{aligned}$$

The left hand side is 0 because expectation values are constant in stationary states.

$$
\begin{aligned}
V &= \frac{1}{2}m\omega^2x^2 \\
\frac{dV}{dx} &= m\omega^2 x \\
x\frac{dV}{dx} &= m\omega^2 x \\
&= 2V \\
2\left\langle T\right\rangle &= \left\langle x\frac{dV}{dx}\right\rangle \\
&= \left\langle 2V\right\rangle \\
&= 2\left\langle V\right\rangle \\
\left\langle T\right\rangle &= \left\langle V\right\rangle
\end{aligned}
$$

3.38

$$
\begin{aligned}
\Psi(x,0) &= \frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \\
\Psi(x,t) &= \frac{1}{\sqrt{2}}(\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar}) \\
0 &= \int_{-\infty}^{\infty} \Psi(x,0)\Psi(x,t)\,dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} (\psi_1 + \psi_2)(\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar})\,dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} (\psi_1^2 e^{-iE_1 t/\hbar} + \psi_1\psi_2 e^{-iE_2 t/\hbar} + \psi_1\psi_2 e^{-iE_1 t/\hbar} + \psi_2^2 e^{-iE_2 t/\hbar})\,dx \\
&= \frac{1}{2}(e^{-iE_1 t/\hbar} + e^{-iE_2 t/\hbar}) \\
&= e^{-iE_1 t/\hbar} + e^{-iE_2 t/\hbar} \\
e^{-iE_1 t/\hbar} &= -e^{-iE_2 t/\hbar} \\
&= e^{\pi i} e^{-iE_2 t/\hbar} \\
-\frac{iE_1 t}{\hbar} &= i\left(\pi - \frac{E_2 t}{\hbar}\right) \\
-\frac{E_1 t}{\hbar} &= \pi - \frac{E_2 t}{\hbar} \\
t &= \frac{\pi\hbar}{E_2 - E_1} \\
\Delta t &= \frac{t}{\pi} \\
&= \frac{\hbar}{E_2 - E_1} \\
\Delta E &= \frac{1}{2}(E_2 - E_1) \\
\Delta E \Delta t &= \frac{\hbar}{2}
\end{aligned}
$$

3.41

$$
\begin{aligned}
\Psi &= \frac{e^{i\theta_0}}{\sqrt{2}}\psi_0 e^{-i\omega t/2} + \frac{e^{i\theta_1}}{\sqrt{2}}\psi_1 e^{-3i\omega t/2} \\
\langle p \rangle &= \langle \Psi | p | \Psi \rangle \\
&= \frac{1}{2}\langle \psi_0 | p | \psi_0 \rangle + \frac{1}{2} e^{i(\theta_1-\theta_0)} e^{-i\omega t} \langle \psi_0 | p | \psi_1 \rangle \\
&\quad + \frac{1}{2} e^{i(\theta_0-\theta_1)} e^{i\omega t} \langle \psi_1 | p | \psi_0 \rangle + \frac{1}{2}\langle \psi_1 | p | \psi_1 \rangle \\
\langle \psi_0 | p | \psi_0 \rangle &= \frac{d}{dt}\langle \psi_0 | x | \psi_0 \rangle \\
&= 0 \\
\langle \psi_1 | p | \psi_1 \rangle &= \frac{d}{dt}\langle \psi_1 | x | \psi_1 \rangle \\
&= 0 \\
\langle p \rangle &= \frac{1}{2} e^{i(\theta_1-\theta_0-\omega t)} \langle \psi_0 | p | \psi_1 \rangle + \frac{1}{2} e^{-i(\theta_1-\theta_0-\omega t)} \langle \psi_1 | p | \psi_0 \rangle \\
\langle \psi_0 | p | \psi_1 \rangle &= -i\sqrt{\frac{\hbar m\omega}{2}} \\
\langle \psi_1 | p | \psi_0 \rangle &= i\sqrt{\frac{\hbar m\omega}{2}} \\
\langle p \rangle &= -\frac{1}{2} i e^{i(\theta_1-\theta_0-\omega t)} \sqrt{\frac{\hbar m\omega}{2}} + \frac{1}{2} i e^{-i(\theta_1-\theta_0-\omega t)} \sqrt{\frac{\hbar m\omega}{2}} \\
&= -\frac{1}{2} i \sqrt{\frac{\hbar m\omega}{2}} (e^{i(\theta_1-\theta_0-\omega t)} - e^{-i(\theta_1-\theta_0-\omega t)}) \\
&= \sqrt{\frac{\hbar m\omega}{2}} \sin(\theta_1-\theta_0-\omega t)
\end{aligned}
$$

The largest possible value of $\langle p \rangle$ is $\sqrt{\hbar m\omega/2}$. If it takes on this value at $t = 0$ then

$$\Psi(x,t) = \frac{1}{\sqrt{2}} e^{-i\omega t/2}(\psi_0 + i\psi_1 e^{-i\omega t}).$$

3.43

(a)

$$\begin{aligned}
|z|^2 &= \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 \\
&= \left(\frac{z+z^*}{2}\right)^2 + \left(\frac{z-z^*}{2i}\right)^2 \\
\sigma_A^2\sigma_B^2 &\geq \left(\frac{\langle f|g\rangle + \langle g|f\rangle}{2}\right)^2 + \left(\frac{\langle f|g\rangle - \langle g|f\rangle}{2i}\right)^2 \\
&= \left(\frac{\langle \hat{A}\hat{B}\rangle + \langle \hat{B}\hat{A}\rangle - 2\langle A\rangle\langle B\rangle}{2}\right)^2 + \left(\frac{\langle [\hat{A},\hat{B}]\rangle}{2i}\right)^2 \\
&= \frac{1}{4}[\langle -i[\hat{A},\hat{B}]\rangle^2 + (\langle \hat{A}\hat{B}\rangle + \langle \hat{B}\hat{A}\rangle - 2\langle A\rangle\langle B\rangle)^2] \\
&= \frac{1}{4}(\langle C\rangle^2 + \langle D\rangle^2)
\end{aligned}$$

(b)

$$\begin{aligned}
B &= A \\
\hat{C} &= -i[\hat{A},\hat{B}] \\
&= 0 \\
\hat{D} &= 2(\hat{A}^2 - \langle A\rangle^2) \\
\langle D\rangle &= \langle \Psi|D|\Psi\rangle \\
&= \langle \Psi|2(\hat{A}^2 - \langle A\rangle^2)|\Psi\rangle \\
&= 2(\langle \Psi|\hat{A}^2|\Psi\rangle - \langle A\rangle^2) \\
&= 2(\langle A^2\rangle - \langle A\rangle^2) \\
&= 2\sigma_A^2 \\
\sigma_A^2\sigma_B^2 &= \frac{1}{4}(\langle C\rangle^2 + \langle D\rangle^2) \\
\sigma_A^4 &\geq \sigma_A^4
\end{aligned}$$

3.44

(a) The eigenvalues of $\mathbf{H}$ are $a-b$, $a+b$, and c and the associated eigenvectors are $\begin{pmatrix}-1\\0\\1\end{pmatrix}$, $\begin{pmatrix}1\\0\\1\end{pmatrix}$, and $\begin{pmatrix}0\\1\\0\end{pmatrix}$, respectively.

$$|S(t)\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} e^{-ict/\hbar}$$

(b)

$$\begin{aligned}
\begin{pmatrix}1\\0\\0\end{pmatrix} &= \frac{1}{2}\left[\begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}-1\\0\\1\end{pmatrix}\right] \\
|S(t)\rangle &= \frac{1}{2}\left[\begin{pmatrix}1\\0\\1\end{pmatrix} e^{-i(a+b)t/\hbar} - \begin{pmatrix}-1\\0\\1\end{pmatrix} e^{-i(a-b)t/\hbar}\right] \\
&= \frac{1}{2}e^{-iat/\hbar}\begin{pmatrix}e^{-ibt/\hbar}+e^{ibt/\hbar}\\0\\e^{-ibt/\hbar}-e^{ibt/\hbar}\end{pmatrix} \\
&= e^{-iat/\hbar}\begin{pmatrix}\cos(bt/\hbar)\\0\\-i\sin(bt/\hbar)\end{pmatrix}
\end{aligned}$$

3.45

$$\begin{aligned}
\langle n|\hat{x}|S(t)\rangle &= \left\langle n \left| \hat{x}\sum_{n'=0}^{\infty} |n'\rangle\langle n'| \right| S(t)\right\rangle \\
&= \sum_{n'=0}^{\infty}\langle n|\hat{x}|n'\rangle\langle n'|S(t)\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}\sum_{n'=0}^{\infty}(\sqrt{n'}\delta_{n,n'-1}+\sqrt{n}\delta_{n',n-1})c_{n'}(t) \\
&= \sqrt{\frac{\hbar}{2m\omega}}[\sqrt{n+1}c_{n+1}(t)+\sqrt{n}c_{n-1}(t)]
\end{aligned}$$

3.46

(a)

$$h_1 = \hbar\omega$$
$$h_2 = 2\hbar\omega$$
$$h_3 = 2\hbar\omega$$
$$|h_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$|h_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$|h_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$a_1 = 2\lambda$$
$$a_2 = \lambda$$
$$a_3 = -\lambda$$
$$|a_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$|a_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
$$|a_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$
$$b_1 = 2\mu$$
$$b_2 = \mu$$
$$b_3 = -\mu$$
$$|b_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$|b_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$
$$|b_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

(b)

$$
\begin{aligned}
\langle H\rangle &= \langle S(0)|H|S(0)\rangle \\
&= \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\
&= \hbar\omega \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} c_1 \\ 2c_2 \\ 2c_3 \end{pmatrix} \\
&= \hbar\omega(|c_1|^2 + 2|c_2|^2 + 2|c_3|^2) \\
\langle A\rangle &= \lambda(c_1^* c_2 + c_2^* c_1 + 2|c_3|^2) \\
\langle B\rangle &= \mu(2|c_1|^2 + c_2^* c_3 + c_3^* c_2)
\end{aligned}
$$

(c)

$$|S(t)\rangle = c_1 \, |h_1\rangle \, e^{-i\omega t} + c_2 \, |h_2\rangle \, e^{-2i\omega t} + c_3 \, |h_3\rangle \, e^{-2i\omega t}$$

You could measure H as $\hbar\omega$ with probability $|c_1|^2$ or $2\hbar\omega$ with probability $|c_2|^2 + |c_3|^2$.

4 Quantum Mechanics in Three Dimensions

4.1

(a)

$$
\begin{aligned}
[r_i, r_j] &= 0 \\
[p_i, p_j] &= 0 \\
[x, p_x]f &= x\left(-i\hbar\frac{\partial}{\partial x}\right)f - \left(-i\hbar\frac{\partial}{\partial x}\right)(xf) \\
&= -i\hbar x\frac{\partial f}{\partial x} + i\hbar\left(f + x\frac{\partial f}{\partial x}\right) \\
&= i\hbar f \\
[p_x, x] &= \left(-i\hbar\frac{\partial}{\partial x}\right)(xf) - x\left(-i\hbar\frac{\partial}{\partial x}\right)f \\
&= -i\hbar\left(f + x\frac{\partial f}{\partial x}\right) + i\hbar x\frac{\partial f}{\partial x} \\
&= -i\hbar f \\
[x, p_y] &= x\left(-i\hbar\frac{\partial}{\partial y}\right)f - \left(-i\hbar\frac{\partial}{\partial y}\right)(xf) \\
&= -i\hbar x\frac{\partial f}{\partial y} + i\hbar x\frac{\partial f}{\partial y} \\
&= 0 \\
[r_i, p_j] &= i\hbar\delta_{ij} \\
[p_i, r_j] &= -i\hbar\delta_{ij}
\end{aligned}
$$

(b)

$$
\begin{aligned}
[\hat{H}, x]f &= \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)(xf) - x\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)f \\
&= -\frac{\hbar^2}{2m}\left[\frac{\partial}{\partial x}\left(f + x\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(x\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial x}\left(z\frac{\partial f}{\partial z}\right)\right] \\
&\quad + Vxf + \frac{\hbar^2}{2m}x\nabla^2 f - Vxf \\
&= -\frac{\hbar^2}{2m}\left(2\frac{\partial f}{\partial x} + x\nabla^2 f\right) + \frac{\hbar^2}{2m}x\nabla^2 f \\
&= -\frac{\hbar^2}{m}\frac{\partial f}{\partial x} \\
[\hat{H}, x] &= -\frac{\hbar^2}{m}\frac{\partial}{\partial x} \\
[\hat{H}, \mathbf{r}] &= -\frac{\hbar^2}{m}\nabla \\
\frac{d}{dt}\langle\mathbf{r}\rangle &= \frac{i}{\hbar}\langle[\hat{H}, \mathbf{r}]\rangle \\
&= \frac{1}{m}\langle -i\hbar\nabla\rangle \\
&= \frac{1}{m}\langle\mathbf{p}\rangle \\
[\hat{H}, \hat{p}_x]f &= \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\left(-i\hbar\frac{\partial}{\partial x}\right)f - \left(-i\hbar\frac{\partial}{\partial x}\right)\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)f \\
&= -i\hbar\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\frac{\partial f}{\partial x} + i\hbar\left(\frac{\partial}{\partial x}\right)\left(-\frac{\hbar^2}{2m}\nabla^2 f + Vf\right) \\
&= i\frac{\hbar^3}{2m}\nabla^2\frac{\partial f}{\partial x} - i\hbar V\frac{\partial f}{\partial x} - i\frac{\hbar^3}{2m}\nabla^2\frac{\partial f}{\partial x} + i\hbar\left(\frac{\partial V}{\partial x}f + V\frac{\partial f}{\partial x}\right) \\
&= i\hbar\frac{\partial V}{\partial x}f \\
[\hat{H}, \hat{p}_x] &= i\hbar\frac{\partial V}{\partial x} \\
[\hat{H}, \mathbf{p}] &= i\hbar\nabla V \\
\frac{d}{dt}\langle\mathbf{p}\rangle &= \frac{i}{\hbar}\langle[\hat{H}, \mathbf{p}]\rangle \\
&= \langle -\nabla V\rangle
\end{aligned}
$$

(c)

$$\begin{aligned}
\sigma_{r_i}\sigma_{p_j} &\geq \frac{1}{2i}\left\langle [r_i, p_j] \right\rangle \\
&= \frac{1}{2i} i\hbar\delta_{ij} \\
&= \frac{\hbar}{2}\delta_{ij}
\end{aligned}$$

4.2

(a)

$$\begin{aligned}
\psi(\mathbf{r}) &= X(x)Y(y)Z(z) \\
-\frac{\hbar^2}{2m}\nabla^2\psi &= E\psi \\
-\frac{\hbar^2}{2m}\left(\frac{\partial^2 X}{\partial x^2}YZ + X\frac{\partial^2 Y}{\partial y^2}Z + XY\frac{\partial^2 Z}{\partial z^2}\right) &= EXYZ \\
\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} &= -\frac{2m}{\hbar^2}E
\end{aligned}$$

The terms on the left hand side are each functions of a different variable, so they must be constant. Starting with the X term:

$$\begin{aligned}
\frac{X''}{X} &= -\alpha \\
X'' + \alpha X &= 0
\end{aligned}$$

If $\alpha < 0$

$$X = A_x e^{\sqrt{-\alpha} x} + B_x e^{-\sqrt{-\alpha} x}$$

If $\alpha = 0$

$$X = A_x x + B_x$$

If $\alpha > 0$

$$X = A_x \sin(\sqrt{\alpha} x) + B_x \cos(\sqrt{\alpha} x)$$

Boundary conditions require that $\alpha > 0$. $X(0) = 0$ so $B_x = 0$. $X(a) = 0$ so $\sqrt{\alpha} = n_x\pi/a \Rightarrow \alpha = n_x^2\pi^2/a^2$,

$$X = A_x \sin\left(\frac{n_x\pi}{a}x\right).$$

Repeating the above for Y and Z finds

$$\begin{aligned} Y &= A_y \sin\left(\frac{n_y \pi}{a} y\right) \\ Z &= A_z \sin\left(\frac{n_z \pi}{a} z\right) \end{aligned}$$

so

$$\psi(\mathbf{r}) = A_x A_y A_z \sin\left(\frac{n_x \pi}{a} x\right) \sin\left(\frac{n_y \pi}{a} y\right) \sin\left(\frac{n_z \pi}{a} z\right).$$

Assuming $A_x = A_y = A_z = A$ and normalising finds

$$\begin{aligned} 1 &= \int_0^a \int_0^a \int_0^a A^6 \sin^2\left(\frac{n_x \pi}{a} x\right) \sin^2\left(\frac{n_y \pi}{a} y\right) \sin^2\left(\frac{n_z \pi}{a} z\right)\, d^3\mathbf{r} \\ &= A^6 \frac{a^3}{8} \\ A^6 &= \frac{8}{a^3} \\ &= \left(\frac{2}{a}\right)^3 \\ A &= \sqrt{\frac{2}{a}} \end{aligned}$$

so

$$\psi(\mathbf{r}) = \left(\frac{2}{a}\right)^{3/2} \sin\left(\frac{n_x \pi}{a} x\right) \sin\left(\frac{n_y \pi}{a} y\right) \sin\left(\frac{n_z \pi}{a} z\right).$$

Finally

$$\begin{aligned} -\frac{2m}{\hbar^2} E &= -\alpha - \beta - \gamma \\ &= -\frac{\pi^2 n_x^2}{a^2} - \frac{\pi^2 n_y^2}{a^2} - \frac{\pi^2 n_z^2}{a^2} \\ E &= \frac{\pi^2 \hbar^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2),\ n_x, n_y, n_z = 1, 2, 3, \ldots \end{aligned}$$

(b)

$$
\begin{aligned}
E_1 &= 3\frac{\pi^2\hbar^2}{2ma^2},\, n = 1 \\
E_2 &= 6\frac{\pi^2\hbar^2}{2ma^2},\, n = 3 \\
E_3 &= 9\frac{\pi^2\hbar^2}{2ma^2},\, n = 3 \\
E_4 &= 11\frac{\pi^2\hbar^2}{2ma^2},\, n = 3 \\
E_5 &= 12\frac{\pi^2\hbar^2}{2ma^2},\, n = 1 \\
E_6 &= 14\frac{\pi^2\hbar^2}{2ma^2},\, n = 6
\end{aligned}
$$

(c)

$$
\begin{aligned}
E_{14} &= 27\frac{\pi^2\hbar^2}{2ma^2} \\
d &= 4
\end{aligned}
$$

4.3

(a)

$$\begin{aligned}
\phi(r,\theta,\phi) &= Ae^{-r/a} \\
\frac{\partial\psi}{\partial r} &= -\frac{A}{a}e^{-r/a} \\
\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) &= \frac{\partial}{\partial r}\left(-\frac{A}{a}r^2e^{-r/a}\right) \\
&= -\frac{A}{a}\left(2re^{-r/a}-\frac{1}{a}r^2e^{-r/a}\right) \\
&= \frac{A}{a^2}r^2\left(1-\frac{2a}{r}\right)e^{-r/a} \\
\frac{\partial\psi}{\partial\theta} &= 0 \\
\frac{\partial^2\psi}{\partial\phi^2} &= 0 \\
-\frac{\hbar^2}{2m}\frac{A}{a^2}\left(1-\frac{2a}{r}\right)e^{-r/a}+V(r)Ae^{-r/a} &= EAe^{-r/a} \\
-\frac{\hbar^2}{2ma^2}\left(1-\frac{2a}{r}\right)+V(r) &= E \\
-\frac{\hbar^2}{2ma^2} &= E \\
V(r) &= -\frac{\hbar^2}{mar}
\end{aligned}$$

(b)

$$\begin{aligned}
\phi(r,\theta,\phi) &= Ae^{-r^2/a^2} \\
\frac{\partial\psi}{\partial r} &= -\frac{2}{a^2}rAe^{-r^2/a^2} \\
\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) &= \frac{\partial}{\partial r}\left(-\frac{2}{a^2}r^3Ae^{-r^2/a^2}\right) \\
&= -\frac{2A}{a^2}\left(3r^2e^{-r^2/a^2}-\frac{2}{a^2}r^4e^{-r^2/a^2}\right) \\
&= \frac{2A}{a^2}r^2\left(\frac{2}{a^2}r^2-3\right)e^{-r^2/a^2}
\end{aligned}$$

$$-\frac{\hbar^2}{2m}\frac{2A}{a^2}\left(\frac{2}{a^2}r^2-3\right)e^{-r^2/a^2}+V(r)Ae^{-r^2/a^2}=EAe^{-r^2/a^2}$$
$$-\frac{\hbar^2}{2m}\frac{2}{a^2}\left(\frac{2}{a^2}r^2-3\right)+V(r)=E$$
$$\frac{\hbar^2}{2m}\frac{2}{a^2}3=E$$
$$\frac{3\hbar^2}{ma^2}=E$$
$$\frac{3\hbar^2}{ma^2}+\frac{\hbar^2}{2m}\frac{2}{a^2}\left(\frac{2}{a^2}r^2-3\right)=V(r)$$
$$\frac{2\hbar^2}{ma^4}r^2=V(r)$$

4.4

$$\begin{aligned}
Y_0^0 &= \frac{1}{2\sqrt{\pi}}\\
Y_2^1 &= \frac{1}{2}\sqrt{\frac{5}{6\pi}}e^{i\phi}P_2^1(\cos\theta)\\
&= \frac{1}{2}\sqrt{\frac{5}{6\pi}}e^{i\phi}(-1)^1(1-\cos^2\theta)^{1/2}\left(\frac{d}{dx}\right)P_2(\cos\theta)\\
&= -\frac{1}{2}\sqrt{\frac{5}{6\pi}}e^{i\phi}\sin\theta\left(\frac{d\theta}{dx}\frac{d}{d\theta}\right)\frac{1}{2}(3\cos^2\theta-1)\\
&= -\frac{1}{4}\sqrt{\frac{5}{6\pi}}e^{i\phi}\sin\theta\left(-\frac{1}{\sin\theta}\right)(-6\cos\theta\sin\theta)\\
&= -\frac{1}{2}\sqrt{\frac{15}{2\pi}}e^{i\phi}\sin\theta\cos\theta\\
\int|Y_0^0|^2\,d\Omega &= \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\sin\theta\,d\theta\,d\phi\\
&= 1\\
\int|Y_2^1|^2\,d\Omega &= \frac{15}{8\pi}\int_0^{\pi}\int_0^{2\pi}\sin^3\theta\cos^2\theta\,d\theta\,d\phi\\
&= 1\\
\int(Y_0^0)^*Y_2^1\,d\Omega &= -\frac{1}{2\sqrt{\pi}}\frac{1}{2}\sqrt{\frac{15}{2\pi}}\int_0^{\pi}\int_0^{2\pi}e^{i\phi}\sin^2\theta\cos\theta\,d\theta\,d\phi\\
&= 0
\end{aligned}$$

4.5

$$
\begin{aligned}
\Theta(\theta) &= A \ln\left(\tan\frac{\theta}{2}\right) \\
0 &= \sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) \\
&= \frac{1}{2} A \sin\theta \frac{d}{d\theta}\left(\sin\theta \csc\frac{\theta}{2}\sec\frac{\theta}{2}\right) \\
&= \frac{1}{2} A \sin\theta \left(\cos\theta \csc\frac{\theta}{2}\sec\frac{\theta}{2} - \frac{1}{2}\csc^2\frac{\theta}{2}\sin\theta + \frac{1}{2}\sec^2\frac{\theta}{2}\sin\theta\right) \\
&= 0
\end{aligned}
$$

It's not a valid physical solution because it blows up at $\theta = 0$ and $\theta = \pi$.

4.6

$$
\begin{aligned}
Y_\ell^{-m} &= \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell+m)!}{(\ell-m)!}} e^{-im\phi} P_\ell^{-m}(\cos\theta) \\
&= \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell+m)!}{(\ell-m)!}} e^{-im\phi} (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^{m}(\cos\theta) \\
&= (-1)^m \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} e^{-im\phi} P_\ell^{m}(\cos\theta) \\
&= (-1)^m (Y_\ell^m)^*
\end{aligned}
$$

4.7

$$
\begin{aligned}
Y_3^2(\theta,\phi) &= \sqrt{\frac{7}{480\pi}}e^{2i\phi}P_3^2(\cos\theta) \\
&= \frac{1}{4}\sqrt{\frac{105}{2\pi}}e^{2i\phi}\sin^2\theta\cos\theta \\
Y_\ell^\ell(\theta,\phi) &= \sqrt{\frac{2\ell+1}{4\pi(2\ell)!}}e^{i\ell\phi}P_\ell^\ell(\cos\theta) \\
P_\ell^\ell(x) &= (-1)^\ell(1-x^2)^{\ell/2}\left(\frac{d}{dx}\right)^\ell P_\ell(x) \\
&= (-1)^\ell(1-x^2)^{\ell/2}\left(\frac{d}{dx}\right)^\ell\left[\frac{1}{2^\ell\ell!}\left(\frac{d}{dx}\right)^\ell(x^2-1)^\ell\right] \\
&= (-1)^\ell\frac{1}{2^\ell\ell!}(1-x^2)^{\ell/2}\left(\frac{d}{dx}\right)^{2\ell}(x^2-1)^\ell \\
&= (-1)^\ell\frac{(2\ell)!}{2^\ell\ell!}(1-x^2)^{\ell/2} \\
Y_\ell^\ell(\theta,\phi) &= \sqrt{\frac{2\ell+1}{4\pi(2\ell)!}}e^{i\ell\phi}(-1)^\ell\frac{(2\ell)!}{2^\ell\ell!}(1-\cos^2\theta)^{\ell/2} \\
&= \frac{1}{\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\left(-\frac{1}{2}e^{i\phi}\sin\theta\right)^\ell
\end{aligned}
$$

4.9

(a)

$$
\begin{aligned}
n_1(x) &= -(-x)\left(\frac{1}{x}\frac{d}{dx}\right)\frac{\cos x}{x} \\
&= -\frac{\sin x}{x} - \frac{\cos x}{x^2} \\
n_2(x) &= -(-x)^2\left(\frac{1}{x}\frac{d}{dx}\right)^2\frac{\cos x}{x} \\
&= -x^2\left(\frac{1}{x}\frac{d}{dx}\right)\left[\frac{1}{x}\frac{d}{dx}\left(\frac{\cos x}{x}\right)\right] \\
&= -x^2\left(\frac{1}{x}\frac{d}{dx}\right)\left[\frac{1}{x}\left(-\frac{\sin x}{x} - \frac{\cos x}{x^2}\right)\right] \\
&= x\frac{d}{dx}\left(\frac{\sin x}{x^2} + \frac{\cos x}{x^3}\right) \\
&= x\left(\frac{\cos x}{x^2} - \frac{2\sin x}{x^3} - \frac{\sin x}{x^3} - \frac{3\cos x}{x^4}\right) \\
&= \left(\frac{1}{x} - \frac{3}{x^3}\right)\cos x - \frac{3\sin x}{x^2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
n_1(x) &\approx -\frac{x}{x} - \frac{1}{x^2} \\
&= -1 - \frac{1}{x^2} \\
n_2(x) &\approx \left(\frac{1}{x} - \frac{3}{x^3}\right) - \frac{3}{x^2}x \\
&= -\frac{2}{x} - \frac{3}{x^3}
\end{aligned}
$$

4.10

(a)

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+\frac{\hbar^2}{2m}\frac{2}{r^2}u &= Eu\\
\frac{d^2u}{dr^2}-\frac{2}{r^2}u &= -\frac{2mE}{\hbar^2}u\\
\frac{d^2u}{dr^2} &= \left(\frac{2}{r^2}-k^2\right)u\\
j_1(kr) &= \frac{\sin(kr)}{(kr)^2}-\frac{\cos(kr)}{kr}\\
u &= Arj_1(kr)\\
&= Ar\left[\frac{\sin(kr)}{(kr)^2}-\frac{\cos(kr)}{kr}\right]
\end{aligned}$$

$$\begin{aligned}
\frac{du}{dr} &= A\left[\frac{\sin(kr)}{(kr)^2}-\frac{\cos(kr)}{kr}\right]\\
&\quad + Ar\left[\frac{k\cos(kr)}{(kr)^2}-\frac{2\sin(kr)}{k^2r^3}+\frac{k\sin(kr)}{kr}+\frac{\cos(kr)}{kr^2}\right]\\
&= \frac{A}{kr}\left[\frac{\sin(kr)}{kr}-\cos(kr)\right]\\
&\quad + \frac{A}{k}\left[\frac{2\cos(kr)}{r}-\frac{2\sin(kr)}{kr^2}+k\sin(kr)\right]\\
\frac{d^2u}{dr^2} &= \frac{A(-2+k^2r^2)(kr\cos(kr)-\sin(kr))}{k^2r^3}\\
&= \frac{2-k^2r^2}{r^2}Ar\left[\frac{\sin(kr)}{(kr)^2}-\frac{\cos(kr)}{kr}\right]\\
&= \left(\frac{2}{r^2}-k^2\right)u
\end{aligned}$$

(b)

$$
\begin{aligned}
0 &= \frac{\sin x}{x^2} - \frac{\cos x}{x} \\
&= \frac{\sin x}{x} - \cos x \\
\cos x &= \frac{\sin x}{x} \\
x &= \tan x \\
x &\approx \left(N + \frac{1}{2}\right)\pi,\ n \in \mathbb{Z} \\
ka &= x \\
\frac{\sqrt{2mE_{N1}}}{\hbar} a &\approx \left(N + \frac{1}{2}\right)\pi \\
E_{N1} &= \frac{\hbar^2\pi^2}{2ma^2}\left(N + \frac{1}{2}\right)^2,\ N \in \mathbb{Z}
\end{aligned}
$$

4.12

$$
\begin{aligned}
R_{n\ell}(r) &= \frac{1}{r}\left(\frac{r}{an}\right)^{\ell+1} e^{-r/an} v(r/an) \\
R_{30} &= e^{-r/3a}\frac{c_0}{3a}\left[1 - 2\left(\frac{r}{3a}\right) - \frac{1}{3}\left(\frac{r}{3a}\right)^2\right] \\
R_{31} &= \frac{1}{r}\left(\frac{r}{3a}\right)^2 c_0\left[1 - \frac{1}{2}\left(\frac{r}{3a}\right)\right] \\
R_{32} &= \frac{1}{r}\left(\frac{r}{3a}\right)^3 c_0
\end{aligned}
$$

4.13

(a)

$$
\begin{aligned}
R_{20}(r) &= \frac{c_0}{2a}\left(1 - \frac{r}{2a}\right) e^{-r/2a} \\
1 &= \int_0^\infty |R_{20}|^2 r^2\, dr \\
&= \frac{c_0^2}{4a^2}\int_0^\infty \left(1 - \frac{r}{2a}\right)^2 e^{-r/a} r^2\, dr \\
&= \frac{ac_0^2}{2} \\
c_0 &= \sqrt{\frac{2}{a}} \\
R_{20}(r) &= \frac{1}{\sqrt{2}} a^{-3/2}\left(1 - \frac{r}{2a}\right) e^{-r/2a} \\
\psi_{200} &= R_{20} Y_0^0 \\
&= \frac{1}{\sqrt{2\pi a}} \frac{1}{2a}\left(1 - \frac{r}{2a}\right) e^{-r/2a}
\end{aligned}
$$

(b)

$$
\begin{aligned}
R_{21}(r) &= \frac{c_0}{4a^2} r e^{-r/2a} \\
1 &= \frac{c_0^2}{16a^4}\int_0^\infty r^4 e^{-r/a}\, dr \\
&= \frac{24}{16} a c_0^2 \\
c_0 &= \sqrt{\frac{2}{3a}} \\
R_{21}(r) &= \sqrt{\frac{2}{3a}} \frac{1}{4a^2} r e^{-r/2a} \\
\psi_{21\pm1} &= R_{21} Y_1^1 \\
&= \mp \frac{1}{\sqrt{\pi a}} \frac{1}{8a^2} r e^{-r/2a} \sin\theta e^{\pm i\phi} \\
\psi_{210} &= \frac{1}{\sqrt{2\pi a}} \frac{1}{4a^2} r e^{-r/2a} \cos\theta
\end{aligned}
$$

4.14

(a)

$$
\begin{aligned}
L_q &= \frac{e^x}{q!}\left(\frac{d}{dx}\right)^q (e^{-x}x^q) \\
L_0 &= 1 \\
L_1 &= e^x \frac{d}{dx}(xe^{-x}) \\
&= e^x(e^{-x} - xe^{-x}) \\
&= 1 - x \\
L_2 &= \frac{1}{2}e^x \left(\frac{d}{dx}\right)^2 (x^2e^{-x}) \\
&= \frac{1}{2}e^x \frac{d}{dx}(2xe^{-x} - x^2e^{-x}) \\
&= \frac{1}{2}e^x(2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x}) \\
&= 1 - 2x + \frac{1}{2}x^2 \\
L_3 &= \frac{1}{6}e^x \left(\frac{d}{dx}\right)^3 (x^3e^{-x}) \\
&= \frac{1}{6}e^x \left(\frac{d}{dx}\right)^2 (3x^2e^{-x} - x^3e^{-x}) \\
&= \frac{1}{6}e^x \frac{d}{dx}(6xe^{-x} - 6x^2e^{-x} + x^3e^{-x}) \\
&= \frac{1}{6}e^x(6e^{-x} - 18xe^{-x} + 9x^2e^{-x} - x^3e^{-x}) \\
&= 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3
\end{aligned}
$$

4.15

(a)

$$\begin{aligned}
\psi_{100} &= \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \\
\langle r \rangle &= \int \psi_{100}^* r \psi_{100} \\
&= \frac{1}{\pi a^3} \int_0^\infty \int_0^\pi \int_0^{2\pi} r^3 e^{-2r/a} \sin\theta \, dr \, d\theta \, d\phi \\
&= \frac{4}{a^3} \int_0^\infty r^3 e^{-2r/a} \, dr \\
&= \frac{3}{2} a \\
\langle r^2 \rangle &= \frac{4}{a^3} \int_0^\infty r^4 e^{-2r/a} \, dr \\
&= 3a^2
\end{aligned}$$

(b)

$$\begin{aligned}
\langle x \rangle &= 0 \\
\langle x^2 \rangle &= a^2
\end{aligned}$$

(c)

$$\begin{aligned}
\psi_{211} &= R_{21} Y_1^1 \\
&= -\frac{1}{\sqrt{\pi a}} \frac{1}{8a^2} r e^{-r/2a} \sin\theta e^{i\phi} \\
\langle x^2 \rangle &= \int \psi_{211}^* x^2 \psi_{211} \\
&= \frac{1}{64\pi a^5} \int_0^\infty \int_0^\pi \int_0^{2\pi} r^6 e^{-r/a} \sin^5\theta \cos^2\phi \, dr \, d\theta \, d\phi \\
&= 12a^2
\end{aligned}$$

4.16

$$\begin{aligned}
P(r) &= \int |\psi_{100}|^2 r^2 \sin\theta \, d\theta \, d\phi \\
&= \frac{4}{a^3} r^2 e^{-2r/a} \\
P'(r) &= \frac{4}{a^3} \left(2r e^{-2r/a} - \frac{2}{a} r^2 e^{-2r/a} \right) \\
&= \frac{8}{a^3} r \left(1 - \frac{r}{a} \right) e^{-2r/a} \\
r_{\max} &= a
\end{aligned}$$

4.18

(a)

$$\begin{aligned}
\Psi(\mathbf{r},t) &= \frac{1}{\sqrt{2}}(\psi_{211}+\psi_{21-1})e^{-iE_2t/\hbar} \\
&= \frac{1}{\sqrt{2}}R_{21}(Y_1^1+Y_1^{-1})e^{-iE_1t/4\hbar} \\
&= \frac{1}{4a^2}\frac{1}{\sqrt{3a}}re^{-r/2a}\left(-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}+\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}\right)e^{-iE_1t/4\hbar} \\
&= -i\frac{1}{4a^2}\frac{1}{\sqrt{2\pi a}}re^{-r/2a}\sin\theta\sin\phi\, e^{-iE_1t/4\hbar}
\end{aligned}$$

(b)

$$\begin{aligned}
\langle V\rangle &= \int\Psi^*\left(-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}\right)\Psi \\
&= -\frac{e^2}{4\pi\epsilon_0}\frac{1}{16a^4}\frac{1}{2\pi a}\int_0^\infty\int_0^\pi\int_0^{2\pi}r^2e^{-r/a}\sin^2\theta\sin^2\phi\frac{1}{r}r^2\sin\theta\,dr\,d\theta\,d\phi \\
&= -\frac{e^2}{128\pi^2\epsilon_0a^5}\int_0^\infty\int_0^\pi\int_0^{2\pi}r^3e^{-r/a}\sin^3\theta\sin^2\phi\,dr\,d\theta\,d\phi \\
&= -\frac{e^2}{4\pi\epsilon_0}\frac{1}{4a} \\
&= -6.8\,\mathrm{eV}
\end{aligned}$$

4.19

$$
\begin{aligned}
E_1(Z) &= -\left[\frac{m_e}{2\hbar^2}\left(\frac{Ze^2}{4\pi\epsilon_0}\right)^2\right] \\
&= Z^2 E_1(1) \\
&= (-13.6\,\mathrm{eV})Z^2 \\
E_n(Z) &= \frac{E_1(Z)}{n^2} \\
&= \frac{(-13.6\,\mathrm{eV})Z^2}{n^2} \\
&= Z^2 E_n(1) \\
a(Z) &= \frac{a}{Z} \\
\mathcal{R}(Z) &= Z^2 R(1) \\
\frac{1}{\lambda} &= Z^2 R \\
\lambda &= \frac{1}{Z^2 R} \\
\lambda_{Z=2} &= \frac{1}{4R} \\
&\approx 2.28 \times 10^{-8}\,\mathrm{m} \\
&= 22.8\,\mathrm{nm}
\end{aligned}
$$

4.20

(a)

$$V = -\frac{GMm}{r}$$

(b)

$$a_g = \frac{\hbar^2}{GMm^2}$$

(c)

$$
\begin{aligned}
E_c &= \frac{1}{2}mv^2 - \frac{GMm}{r_0} \\
\frac{GMm}{r_0^2} &= \frac{mv^2}{r_0} \\
\frac{1}{2}mv^2 &= \frac{GMm}{2r_0} \\
E_c &= -\frac{GMm}{2r_0} \\
E_n &= -\left[\frac{m}{2\hbar^2}(GMm)^2\right]\frac{1}{n^2} \\
&= -\frac{G^2M^2m^3}{2\hbar^2}\frac{1}{n^2} \\
-\frac{GMm}{2r_0} &= -\frac{G^2M^2m^3}{2\hbar^2}\frac{1}{n^2} \\
\frac{\hbar^2}{GMm^2r_0} &= \frac{1}{n^2} \\
\frac{a_g}{r_0} &= \frac{1}{n^2} \\
n &= \sqrt{\frac{r_0}{a_g}}
\end{aligned}
$$

4.21

$$
\begin{aligned}
\langle f_\ell^m | L_+L_- f_\ell^m \rangle &= \langle f_\ell^m | (L^2 - L_z^2 + \hbar L_z) f_\ell^m \rangle \\
&= \langle f_\ell^m | L^2 f_\ell^m \rangle - \langle f_\ell^m | L_z^2 f_\ell^m \rangle + \langle f_\ell^m | \hbar L_z f_\ell^m \rangle \\
&= \langle f_\ell^m | \hbar^2 \ell(\ell+1) f_\ell^m \rangle - \langle f_\ell^m | \hbar^2 m^2 f_\ell^m \rangle + \langle f_\ell^m | \hbar^2 m f_\ell m \rangle \\
&= \hbar^2 \ell(\ell+1) \langle f_\ell^m | f_\ell^m \rangle - \hbar^2 m^2 \langle f_\ell^m | f_\ell^m \rangle + \hbar^2 m \langle f_\ell^m | f_\ell^m \rangle \\
&= \hbar^2 \ell(\ell+1) - \hbar^2 m^2 + \hbar^2 m \\
&= \hbar^2 [\ell(\ell+1) - m(m-1)] \\
\langle f_\ell^m | L_+L_- f_\ell^m \rangle &= \langle L_- f_\ell^m | L_- f_\ell^m \rangle \\
&= \langle B_\ell^m f_\ell^{m-1} | B_\ell^m f_\ell^{m-1} \rangle \\
&= |B_\ell^m|^2 \\
|B_\ell^m|^2 &= \hbar^2 [\ell(\ell+1) - m(m-1)] \\
B_\ell^m &= \hbar\sqrt{\ell(\ell+1) - m(m-1)}
\end{aligned}
$$

The same argument for $\langle f_\ell^m | L_-L_+ f_\ell^m \rangle$ finds A_ℓ^m.

4.22

(a)

$$\begin{aligned}
[r_i, p_j] &= i\hbar\delta_{ij} \\
[r_i, r_j] &= 0 \\
[p_i, p_j] &= 0 \\
[L_z, x] &= [xp_y - yp_x, x] \\
&= [xp_y, x] - [yp_x, x] \\
&= x[p_y, x] + [x, x]p_y - y[p_x, x] - [y, x]p_x \\
&= i\hbar y \\
[L_z, y] &= [xp_y - yp_x, y] \\
&= [xp_y, y] - [yp_x, y] \\
&= x[p_y, y] + [x, y]p_y - y[p_x, y] - [y, y]p_x \\
&= -i\hbar x \\
[L_z, z] &= [xp_y - yp_x, z] \\
&= [xp_y, z] - [yp_x, z] \\
&= x[p_y, z] + [x, z]p_y - y[p_x, z] - [p_x, z]y \\
&= 0 \\
[L_z, p_x] &= [xp_y - yp_x, p_x] \\
&= [xp_y, p_x] - [yp_x, p_x] \\
&= x[p_y, p_x] + [x, p_x]p_y - y[p_x, p_x] - [y, p_x]p_x \\
&= i\hbar p_y \\
[L_z, p_y] &= [xp_y - yp_x, p_y] \\
&= [xp_y, p_y] - [yp_x, p_y] \\
&= x[p_y, p_y] + [x, p_y]p_y - y[p_x, p_y] - [y, p_y]p_x \\
&= -i\hbar p_x \\
[L_z, p_z] &= [xp_y - yp_x, p_z] \\
&= [xp_y, p_z] - [yp_x, p_z] \\
&= x[p_y, p_z] + [x, p_z]p_y - y[p_x, p_z] - [y, p_z]p_x \\
&= 0
\end{aligned}$$

(b)

$$\begin{aligned}
[L_z, L_x] &= [L_z, yp_z - zp_y] \\
&= [L_z, yp_z] - [L_z, zp_y] \\
&= y[L_z, p_z] + [L_z, y]p_z - z[L_z, p_y] - [L_z, z]p_y \\
&= -i\hbar xp_z + i\hbar zp_x \\
&= i\hbar L_y
\end{aligned}$$

(c)

$$
\begin{aligned}
[L_z, r^2] &= [L_z, x^2 + y^2 + z^2] \\
&= [L_z, x^2] + [L_z, y^2] + [L_z, z^2] \\
&= x[L_z, x] + [L_z, x]x + y[L_z, y] + [L_z, y]y + z[L_z, z] + [L_z, z]z \\
&= 2i\hbar xy - 2i\hbar xy \\
&= 0 \\
[L_z, p^2] &= [L_z, p_x^2 + p_y^2 + p_z^2] \\
&= [L_z, p_x^2] + [L_z, p_y^2] + [L_z, p_z^2] \\
&= p_x[L_z, p_x] + [L_z, p_x]p_x + p_y[L_z, p_y] + [L_z, p_y]p_y + p_z[L_z, p_z] + [L_z, p_z]p_z \\
&= 2i\hbar p_x p_y - 2i\hbar p_x p_y \\
&= 0
\end{aligned}
$$

4.24

(a)

$$
\begin{aligned}
L_+ L_- f &= \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left[-\hbar e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right)\right] f \\
&= -\hbar^2 e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left[e^{-i\phi}\left(\frac{\partial f}{\partial\theta} - i\cot\theta\frac{\partial f}{\partial\phi}\right)\right] \\
&= -\hbar^2 e^{i\phi}\left[e^{-i\phi}\left(\frac{\partial^2 f}{\partial\theta^2} + i\csc^2\theta\frac{\partial f}{\partial\phi} - i\cot\theta\frac{\partial^2 f}{\partial\phi\partial\theta}\right)\right. \\
&\qquad \left. + \cot\theta e^{-i\phi}\left(\frac{\partial f}{\partial\theta} - i\cot\theta\frac{\partial f}{\partial\phi}\right) + i\cot\theta e^{-i\phi}\left(\frac{\partial^2 f}{\partial\phi\partial\theta} - i\cot\theta\frac{\partial^2 f}{\partial\phi^2}\right)\right] \\
&= -\hbar^2\left(\frac{\partial^2 f}{\partial\theta^2} + i\csc^2\theta\frac{\partial f}{\partial\phi} + \cot\theta\frac{\partial f}{\partial\theta} - i\cot^2\theta\frac{\partial f}{\partial\phi} + \cot^2\theta\frac{\partial^2 f}{\partial\phi^2}\right) \\
&= -\hbar^2\left(\frac{\partial^2 f}{\partial\theta^2} + i\frac{\partial f}{\partial\phi} + \cot\theta\frac{\partial f}{\partial\theta} + \cot^2\theta\frac{\partial^2 f}{\partial\phi^2}\right) \\
L_+ L_- &= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right)
\end{aligned}
$$

(b)

$$
\begin{aligned}
L^2 &= L_+L_- + L_z^2 - \hbar L_z \\
&= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right) - \hbar^2\frac{\partial^2}{\partial\phi^2} + i\hbar^2\frac{\partial}{\partial\phi} \\
&= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi} + \frac{\partial^2}{\partial\phi^2} - i\frac{\partial}{\partial\phi}\right) \\
&= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\phi^2}\right] \\
&= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]
\end{aligned}
$$

4.25

(a)

$$L_+Y_\ell^\ell = 0$$

(b)

$$\begin{aligned}
L_z Y_\ell^\ell &= \hbar \ell Y_\ell^\ell \\
-i\hbar \frac{\partial Y_\ell^\ell}{\partial \phi} &= \hbar \ell Y_\ell^\ell \\
\frac{\partial Y_\ell^\ell}{\partial \phi} &= i\ell Y_\ell^\ell \\
Y_\ell^\ell &= e^{i\ell\phi}\Theta(\theta) \\
0 &= L_+ Y_\ell^\ell \\
&= \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi}\right) e^{i\ell\phi}\Theta(\theta) \\
&= \hbar e^{i\phi}\left[e^{i\ell\phi}\frac{\partial\Theta}{\partial\theta} - \ell\cot\theta e^{i\ell\phi}\Theta(\theta)\right] \\
&= \frac{\partial\Theta}{\partial\theta} - \ell\cot\theta\Theta(\theta) \\
\frac{1}{\Theta(\theta)}\frac{\partial\Theta}{\partial\theta} &= \ell\cot\theta \\
\int \frac{1}{\Theta(\theta)}\, d\Theta &= \ell \int \frac{\cos\theta}{\sin\theta}\, d\theta \\
\ln\Theta &= \ell\ln\sin\theta + c_1 \\
&= \ln(\sin^\ell\theta) + c_1 \\
\ln\frac{\Theta}{\sin^\ell\theta} &= c_1 \\
\frac{\Theta}{\sin^\ell\theta} &= c_2 \\
\Theta &= c_2 \sin^\ell\theta \\
Y_\ell^\ell &= c e^{i\ell\phi}\sin^\ell\theta
\end{aligned}$$

4.26

$$
\begin{aligned}
Y_2^1(\theta,\phi) &= -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi} \\
&= -\frac{1}{2}\sqrt{\frac{15}{8\pi}}\sin(2\theta)e^{i\phi} \\
L_+Y_2^1 &= \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)\left[-\frac{1}{2}\sqrt{\frac{15}{8\pi}}\sin(2\theta)e^{i\phi}\right] \\
&= -\frac{1}{2}\sqrt{\frac{15}{8\pi}}\hbar e^{i\phi}[2\cos(2\theta)e^{i\phi}-\cot\theta\sin(2\theta)e^{i\phi}] \\
&= \sqrt{\frac{15}{8\pi}}\hbar(e^{i\phi}\sin\theta)^2 \\
&= \hbar\sqrt{(2-1)(2+1+1)}Y_2^1 \\
&= 2\hbar Y_2^1 \\
Y_2^1 &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}(e^{i\phi}\sin\theta)^2
\end{aligned}
$$

4.28

$$
\begin{aligned}
r &= \frac{e^2}{4\pi\epsilon_0 mc^2} \\
&= 0.0796\,\mathrm{m} \\
\frac{1}{2}\hbar &= I\omega \\
&= \frac{2}{5}mr^2\frac{v}{r} \\
v &= \frac{5\hbar}{4mr} \\
&= 5.14\times10^{10}\,\mathrm{m/s}
\end{aligned}
$$

This is 100 times the speed of light, so no it doesn't make sense.

4.29

(a)

$$
\begin{aligned}
[S_x,S_y] &= S_xS_y-S_yS_x \\
&= \begin{pmatrix} i\hbar^2/2 & 0 \\ 0 & -i\hbar^2/2 \end{pmatrix} \\
&= i\hbar S_z \\
[S_y,S_z] &= i\hbar S_x \\
[S_z,S_x] &= i\hbar S_y
\end{aligned}
$$

(b)

$$
\begin{aligned}
\sigma_x\sigma_x &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \delta_{xx} \\
\delta_x\delta_y &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
&= i\epsilon_{xyz}\sigma_z \\
\delta_x\delta_z &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= i\epsilon_{xzy}\sigma_y \\
\delta_y\delta_x &= \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix} \\
&= i\epsilon_{yxz}\sigma_z \\
\delta_y\delta_y &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \delta_{yy} \\
\delta_y\delta_z &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\
&= i\epsilon_{yzx}\sigma_x \\
\delta_z\delta_x &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= i\epsilon_{zxy}\sigma_y \\
\delta_z\delta_y &= \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \\
&= i\epsilon_{zyx}\sigma_x \\
\delta_z\delta_z &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \delta_{zz}
\end{aligned}
$$

4.30

(a)

$$
\begin{aligned}
1 &= \chi^\dagger\chi \\
&= |A|^2 \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} \\
&= 25|A|^2 \\
A &= \pm\frac{1}{5}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\chi &= \frac{1}{5}\begin{pmatrix}3i\\4\end{pmatrix}\\
P\left(\frac{\hbar}{2}\right) &= \frac{1}{2}\left|\frac{1}{5}(3i+4)\right|^2\\
&= \frac{1}{50}(-3i+4)(3i+4)\\
&= \frac{1}{2}\\
\langle S_x\rangle &= 0\\
\langle S_y\rangle &= \chi^\dagger S_y\chi\\
&= \frac{\hbar}{50}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}3i\\4\end{pmatrix}\\
&= \frac{\hbar}{50}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}0 & -4i\\ -3 & 0\end{pmatrix}\\
&= -\frac{12}{25}\hbar\\
\langle S_z\rangle &= \frac{\hbar}{50}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}3i\\4\end{pmatrix}\\
&= \frac{\hbar}{50}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}3i\\-4\end{pmatrix}\\
&= -\frac{7}{50}\hbar
\end{aligned}
$$

(c)

$$
\begin{aligned}
\langle S_x^2\rangle &= \frac{\hbar^2}{100}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}3i\\4\end{pmatrix}\\
&= \frac{\hbar^2}{4}\\
\sigma_{S_x} &= \sqrt{\langle S_x^2\rangle - \langle S_x\rangle^2}\\
&= \frac{\hbar}{2}\\
\langle S_y^2\rangle &= \frac{\hbar^2}{4}\\
\sigma_{S_y} &= \sqrt{\frac{1}{4}\hbar^2 - \frac{144}{625}\hbar^2}\\
&= \frac{7}{50}\hbar\\
\sigma_{S_z} &= \frac{12}{25}\hbar
\end{aligned}
$$

(d)

$$
\begin{aligned}
\sigma_{S_x}\sigma_{S_y} &= \frac{7}{100}\hbar^2 \\
&= \frac{\hbar}{2}|\left\langle S_z \right\rangle| \\
&\geq \frac{\hbar}{2}|\left\langle S_z \right\rangle| \\
\sigma_{S_y}\sigma_{S_z} &= \frac{42}{625}\hbar^2 \\
&\geq \frac{\hbar}{2}|\left\langle S_x \right\rangle| \\
&= 0 \\
\sigma_{S_z}\sigma_{S_x} &= \frac{6}{25}\hbar^2 \\
&= \frac{\hbar}{2}|\left\langle S_y \right\rangle| \\
&\geq \frac{\hbar}{2}|\left\langle S_z \right\rangle|
\end{aligned}
$$

4.31

$$
\begin{aligned}
\chi &= \begin{pmatrix} a \\ b \end{pmatrix} \\
\langle S_x \rangle &= \frac{\hbar}{2}(a^*b + ab^*) \\
\langle S_y \rangle &= i\frac{\hbar}{2}(ab^* - a^*b) \\
\langle S_z \rangle &= \frac{\hbar}{2}(|a|^2 - |b|^2) \\
\langle S_x^2 \rangle &= \frac{\hbar^2}{4}(|a|^2 + |b|^2) \\
\langle S_y^2 \rangle &= \frac{\hbar^2}{4}(|a|^2 + |b|^2) \\
\langle S_z^2 \rangle &= \frac{\hbar^2}{4}(|a|^2 + |b|^2) \\
\langle S^2 \rangle &= 3\frac{\hbar^2}{4}(|a|^2 + |b|^2) \\
\langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle &= 3\frac{\hbar^2}{4}(|a|^2 + |b|^2) \\
&= \langle S^2 \rangle
\end{aligned}
$$

4.32

(a)

$$
\begin{aligned}
0 &= \begin{vmatrix} -\lambda & -i\hbar/2 \\ i\hbar/2 & -\lambda \end{vmatrix} \\
&= \lambda^2 - \frac{\hbar^2}{4} \\
\lambda &= \pm\frac{\hbar}{2} \\
\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \pm\frac{\hbar}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
\begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix} &= \pm\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
\beta &= \pm i\alpha \\
\chi_+^{(y)} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \\
\chi_-^{(y)} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\chi &= \begin{pmatrix} a \\ b \end{pmatrix} \\
&= \left(\frac{a-ib}{\sqrt{2}}\right)\chi_+^{(x)} + \left(\frac{a+ib}{\sqrt{2}}\right)\chi_-^{(y)}
\end{aligned}
$$

You could get $\frac{\hbar}{2}$ with probability $\frac{|a-ib|^2}{2}$ or $-\frac{\hbar}{2}$ with probability $\frac{|a+ib|^2}{2}$.

$$
\begin{aligned}
\frac{1}{2}|a-ib|^2 + \frac{1}{2}|a+ib|^2 &= \frac{1}{2}(a+ib)(a-ib) + \frac{1}{2}(a-ib)(a+ib) \\
&= \frac{1}{2}(a^2+b^2) + \frac{1}{2}(a^2+b^2) \\
&= 1
\end{aligned}
$$

(c) $\hbar^2/2$ with probability 1.

4.33

$$\begin{aligned}
S_r &= S \cdot \hat{\mathbf{r}} \\
&= S_x \sin\theta\cos\phi + S_y \sin\theta\sin\phi + S_z\cos\theta \\
&= \frac{\hbar}{2}\left[\begin{pmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\sin\theta\sin\phi \\ i\sin\theta\sin\phi & 0 \end{pmatrix}\right. \\
&\qquad \left. + \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix}\right] \\
&= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\begin{vmatrix} \frac{\hbar}{2}\cos\theta - \lambda & \frac{\hbar}{2}e^{-i\phi}\sin\theta \\ \frac{\hbar}{2}e^{i\phi}\sin\theta & -\frac{\hbar}{2}\cos\theta - \lambda \end{vmatrix} &= \left(\frac{\hbar}{2}\cos\theta - \lambda\right)\left(-\frac{\hbar}{2}\cos\theta - \lambda\right) \\
&\quad - \left(\frac{\hbar}{2}e^{-i\phi}\sin\theta\right)\left(\frac{\hbar}{2}e^{i\phi}\sin\theta\right) \\
&= -\frac{\hbar^2}{4}\cos^2\theta - \lambda\frac{\hbar}{2}\cos\theta + \lambda\frac{\hbar}{2}\cos\theta + \lambda^2 \\
&\quad - \frac{\hbar^2}{4}\sin^2\theta \\
&= \lambda^2 - \frac{\hbar^2}{4} \\
\lambda &= \pm\frac{\hbar}{2}
\end{aligned}$$

4.35

(a)

$$\begin{aligned}
\chi(t) &= \begin{pmatrix} \cos(\alpha/2)e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)e^{-i\gamma B_0 t/2} \end{pmatrix} \\
\frac{1}{2}|a+b|^2 &= \frac{1}{2}[\cos(\alpha/2)e^{-i\gamma B_0 t/2} + \sin(\alpha/2)e^{i\gamma B_0 t/2}] \\
&\qquad [\cos(\alpha/2)e^{i\gamma B_0 t/2} + \sin(\alpha/2)e^{-i\gamma B_0 t/2}] \\
&= \frac{1}{2}[\cos^2(\alpha/2) + \cos(\alpha/2)\sin(\alpha/2)e^{-i\gamma B_0 t} \\
&\qquad + \cos(\alpha/2)\sin(\alpha/2)e^{i\gamma B_0 t} + \sin^2(\alpha/2)] \\
&= \frac{1}{2}[1 + \sin\alpha\cos(\gamma B_0 t)]
\end{aligned}$$

(b)

$$\begin{aligned}\frac{1}{2}|a-ib|^2 &= \frac{1}{2}[\cos(\alpha/2)e^{-i\gamma B_0 t/2} + i\sin(\alpha/2)e^{i\gamma B_0 t/2}] \\ &\quad [\cos(\alpha/2)e^{i\gamma B_0 t/2} - i\sin(\alpha/2)e^{-i\gamma B_0 t/2}] \\ &= \frac{1}{2}[\cos^2(\alpha/2) - i\cos(\alpha/2)\sin(\alpha/2)e^{-i\gamma B_0 t} \\ &\quad + i\cos(\alpha/2)\sin(\alpha/2)e^{i\gamma B_0 t} + \sin^2(\alpha/2)] \\ &= \frac{1}{2}[1-\sin\alpha\sin(\gamma B_0 t)]\end{aligned}$$

(c)

$$|a|^2 = \cos^2\frac{\alpha}{2}$$

4.36

(a)

$$\begin{aligned}\mathbf{B} &= B_0\cos(\omega t)\hat{k} \\ H &= -\gamma\mathbf{B}\cdot\mathbf{S} \\ &= -\gamma B_0\cos(\omega t)S_z \\ &= -\frac{\gamma B_0\hbar}{2}\cos(\omega t)\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\end{aligned}$$

(b)

$$\begin{aligned}
i\hbar\frac{\partial\chi}{\partial t} &= H\chi \\
i\hbar\begin{pmatrix} a'(t) \\ b'(t) \end{pmatrix} &= -\frac{\gamma B_0\hbar}{2}\cos(\omega t)\begin{pmatrix} a(t) \\ -b(t) \end{pmatrix} \\
i\hbar a'(t) &= -\frac{\gamma B_0\hbar}{2}\cos(\omega t)a(t) \\
\frac{1}{a(t)}a'(t) &= i\frac{\gamma B_0}{2}\cos(\omega t) \\
\ln[a(t)] &= i\frac{\gamma B_0}{2\omega}\sin(\omega t) + c_1 \\
a(t) &= Ae^{i\gamma B_0\sin(\omega t)/2\omega} \\
i\hbar b'(t) &= \frac{\gamma B_0\hbar}{2}\cos(\omega t)b(t) \\
\frac{1}{b(t)}b'(t) &= -i\frac{\gamma B_0}{2}\cos(\omega t) \\
\ln[b(t)] &= -i\frac{\gamma B_0}{2\omega}\sin(\omega t) + c_2 \\
b(t) &= Be^{-i\gamma B_0\sin(\omega t)/2\omega} \\
\chi &= \begin{pmatrix} Ae^{i\gamma B_0\sin(\omega t)/2\omega} \\ Be^{-i\gamma B_0\sin(\omega t)/2\omega} \end{pmatrix} \\
1 &= \frac{1}{2}|A+B|^2 \\
&= \frac{1}{2}(A^* + B^*)(A+B) \\
&= \frac{1}{2}(|A|^2 + A^*B + B^*A + |B|^2) \\
0 &= \frac{1}{2}|A-B|^2 \\
&= \frac{1}{2}(A^* - B^*)(A-B) \\
&= \frac{1}{2}(|A|^2 - A^*B - B^*A + |B|^2) \\
\chi &= \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\gamma B_0\sin(\omega t)/2\omega} \\ e^{-i\gamma B_0\sin(\omega t)/2\omega} \end{pmatrix}
\end{aligned}$$

(c)

$$\begin{aligned}\frac{A-B}{\sqrt{2}} &= \frac{1}{2}[e^{i\gamma B_0\sin(\omega t)/2\omega} - e^{-i\gamma B_0\sin(\omega t)/2\omega}] \\ &= i\sin\left[\frac{\gamma B_0}{2\omega}\sin(\omega t)\right] \\ \frac{1}{2}|A-B|^2 &= \sin^2\left[\frac{\gamma B_0}{2\omega}\sin(\omega t)\right]\end{aligned}$$

(d)

$$|B_0| = \frac{\omega\pi}{\gamma}$$

4.37

(a)

$$\begin{aligned}S_-\left|10\right\rangle &= S_-\left[\frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle)\right] \\ &= \frac{1}{\sqrt{2}}[(S_-^{(1)}\left|\uparrow\right\rangle)\left|\downarrow\right\rangle + \left|\uparrow\right\rangle(S_-^{(2)}\left|\downarrow\right\rangle) + (S_-^{(1)}\left|\downarrow\right\rangle)\left|\uparrow\right\rangle + \left|\downarrow\right\rangle(S_-^{(2)}\left|\uparrow\right\rangle)] \\ &= \frac{1}{\sqrt{2}}[\hbar\left|\downarrow\downarrow\right\rangle + \hbar\left|\downarrow\downarrow\right\rangle] \\ &= \sqrt{2}\hbar\left|1\,{-1}\right\rangle\end{aligned}$$

(b)

$$\begin{aligned}S_+\left|00\right\rangle &= S_+\left[\frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle)\right] \\ &= \frac{1}{\sqrt{2}}[(S_+^{(1)}\left|\uparrow\right\rangle)\left|\downarrow\right\rangle + \left|\uparrow\right\rangle(S_+^{(2)}\left|\downarrow\right\rangle) - (S_+^{(1)}\left|\downarrow\right\rangle)\left|\uparrow\right\rangle - \left|\downarrow\right\rangle(S_+^{(2)}\left|\uparrow\right\rangle)] \\ &= \frac{1}{\sqrt{2}}(\hbar\left|\uparrow\uparrow\right\rangle - \hbar\left|\uparrow\uparrow\right\rangle) \\ &= 0 \\ S_-\left|00\right\rangle &= S_-\left[\frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle)\right] \\ &= \frac{1}{\sqrt{2}}[(S_-^{(1)}\left|\uparrow\right\rangle)\left|\downarrow\right\rangle + \left|\uparrow\right\rangle(S_-^{(2)}\left|\downarrow\right\rangle) - (S_-^{(1)}\left|\downarrow\right\rangle)\left|\uparrow\right\rangle - \left|\downarrow\right\rangle(S_-^{(2)}\left|\uparrow\right\rangle)] \\ &= \frac{1}{\sqrt{2}}(\hbar\left|\downarrow\downarrow\right\rangle - \hbar\left|\downarrow\downarrow\right\rangle) \\ &= 0\end{aligned}$$

(c)

$$\begin{aligned}
S^{(1)} \cdot S^{(2)} |\uparrow\uparrow\rangle &= (S_x^{(1)} |\uparrow\rangle)(S_x^{(2)} |\uparrow\rangle) + (S_y^{(1)} |\uparrow\rangle)(S_y^{(2)} |\uparrow\rangle) \\
&\quad + (S_z^{(1)} |\uparrow\rangle)(S_z^{(2)} |\uparrow\rangle) \\
&= \left(\frac{\hbar}{2} |\downarrow\rangle\right)\left(\frac{\hbar}{2} |\downarrow\rangle\right) + \left(\frac{i\hbar}{2} |\downarrow\rangle\right)\left(\frac{i\hbar}{2} |\downarrow\rangle\right) \\
&\quad + \left(\frac{\hbar}{2} |\uparrow\rangle\right)\left(\frac{\hbar}{2} |\uparrow\rangle\right) \\
&= \frac{\hbar^2}{4} |\downarrow\downarrow\rangle - \frac{\hbar^2}{4} |\downarrow\downarrow\rangle + \frac{\hbar^2}{4} |\uparrow\uparrow\rangle \\
S^2 |\uparrow\uparrow\rangle &= \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\frac{\hbar^2}{4}\right) |\uparrow\uparrow\rangle \\
&= 2\hbar^2 |\uparrow\uparrow\rangle \\
S^{(1)} \cdot S^{(2)} |\downarrow\downarrow\rangle &= (S_x^{(1)} |\downarrow\rangle)(S_x^{(2)} |\downarrow\rangle) + (S_y^{(1)} |\downarrow\rangle)(S_y^{(2)} |\downarrow\rangle) \\
&\quad + (S_z^{(1)} |\downarrow\rangle)(S_z^{(2)} |\downarrow\rangle) \\
&= \left(\frac{\hbar}{2} |\uparrow\rangle\right)\left(\frac{\hbar}{2} |\uparrow\rangle\right) + \left(-\frac{i\hbar}{2} |\uparrow\rangle\right)\left(-\frac{i\hbar}{2} |\uparrow\rangle\right) \\
&\quad \left(-\frac{\hbar}{2} |\downarrow\rangle\right)\left(-\frac{\hbar}{2} |\downarrow\rangle\right) \\
&= \frac{\hbar^2}{4} |\uparrow\uparrow\rangle - \frac{\hbar^2}{4} |\uparrow\uparrow\rangle + \frac{\hbar^2}{4} |\downarrow\downarrow\rangle \\
S^2 |\downarrow\downarrow\rangle &= \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\frac{\hbar^2}{4}\right) |\downarrow\downarrow\rangle \\
&= 2\hbar^2 |\downarrow\downarrow\rangle
\end{aligned}$$

4.38

(a) $3/2, 1/2$

(b) $0, 1$

4.40

(a)

$$|31\rangle = \frac{1}{\sqrt{15}} |22\rangle |1 - 1\rangle + \sqrt{\frac{8}{15}} |21\rangle |10\rangle + \sqrt{\frac{6}{15}} |20\rangle |11\rangle$$

$2\hbar$ with probability $1/15$, $\hbar$ with probability $8/15$, and 0 with probability $6/15$.

4.46

(a)

$$\begin{aligned}
V(r) &= \frac{1}{2}m\omega^2 r^2 \\
&= \frac{1}{2}m\omega^2(x^2+y^2+z^2) \\
\psi &= XYZ \\
-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi &= E\psi \\
-\frac{\hbar^2}{2m}\left(\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}\right)+\frac{1}{2}m\omega^2(x^2+y^2+z^2) &= E \\
-\frac{\hbar^2}{2m}\frac{X''}{X}+\frac{1}{2}m\omega^2x^2 &= c_1 \\
-\frac{\hbar^2}{2m}X''+\frac{1}{2}m\omega^2x^2X &= c_1X \\
c_1 &= E_{x_n} \\
&= \left(n_x+\frac{1}{2}\right)\hbar\omega \\
\frac{\hbar^2}{2m}\left(\frac{Y''}{Y}+\frac{Z''}{Z}\right)-\frac{1}{2}m\omega^2(y^2+z^2)+E &= c_1 \\
\frac{\hbar^2}{2m}\frac{Y''}{Y}-\frac{1}{2}m\omega^2y^2+E &= c_2 \\
-\frac{\hbar^2}{2m}Y''+\frac{1}{2}m\omega^2y^2Y &= (E-c_2)Y \\
E-c_2 &= \left(n_y+\frac{1}{2}\right)\hbar\omega \\
c_1-\frac{\hbar^2}{2m}\frac{Z''}{Z}+\frac{1}{2}m\omega^2z^2 &= c_2 \\
-\frac{\hbar^2}{2m}Z''+\frac{1}{2}m\omega^2z^2Z &= (c_2-c_1)Z \\
c_2-c_1 &= \left(n_z+\frac{1}{2}\right)\hbar\omega \\
c_2 &= (n_x+n_z+1)\hbar\omega \\
E &= \left(n_x+n_y+n_z+\frac{3}{2}\right)\hbar\omega \\
&= \left(n+\frac{3}{2}\right)\hbar\omega
\end{aligned}$$

4.48

(b)

$$
\begin{aligned}
V &= -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r} \\
\nabla V &= \frac{\partial V}{\partial r}\hat{\mathbf{r}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2}\hat{\mathbf{r}} \\
2\langle T\rangle &= \langle \mathbf{r}\cdot\nabla V\rangle \\
&= \left\langle \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}\right\rangle \\
&= -\langle V\rangle \\
E_n &= \langle T\rangle + \langle V\rangle \\
&= \langle T\rangle - 2\langle T\rangle \\
&= -\langle T\rangle \\
\langle T\rangle &= -E_n \\
\langle V\rangle &= 2E_n
\end{aligned}
$$

(c)

$$
\begin{aligned}
V &= \frac{1}{2}m\omega^2 r^2 \\
\nabla V &= m\omega^2 r\,\hat{\mathbf{r}} \\
2\langle T\rangle &= \langle m\omega^2 r^2\rangle \\
&= 2\langle V\rangle \\
\langle T\rangle &= \langle V\rangle \\
E_n &= \langle T\rangle + \langle V\rangle \\
&= 2\langle T\rangle \\
\langle T\rangle &= \frac{E_n}{2} \\
\langle V\rangle &= \frac{E_n}{2}
\end{aligned}
$$

4.52

(a)

$$
\begin{aligned}
\psi_{321} &= \frac{1}{9a}\frac{1}{\sqrt{30a}}e^{-r/3a}\frac{4r^2}{9a^2}L_0^5\left(\frac{2r}{3a}\right)Y_2^1(\theta,\phi) \\
&= -\frac{1}{\sqrt{\pi a}}\frac{r^2}{81a^3}e^{-r/3a}\cos\theta\sin\theta e^{i\phi}
\end{aligned}
$$

(b)

$$\begin{aligned}
\int |\psi_{321}|^2 &= \frac{1}{81^2\pi a^7}\int_0^\infty\int_0^\pi\int_0^{2\pi} r^6 e^{-2r/3a}\cos^2\theta\sin^3\theta\,dr\,d\theta\,d\phi \\
&= \frac{2}{81^2 a^7}\int_0^\infty\int_0^\pi r^6 e^{-2r/3a}\cos^2\theta\sin^3\theta\,dr\,d\theta \\
&= \frac{8}{15\cdot 81^2 a^7}\int_0^\infty r^6 e^{-2r/3a}\,dr \\
&= 1
\end{aligned}$$

(c)

$$\begin{aligned}
\langle r^s\rangle &= \int \psi_{321}^* r^s \psi_{321} \\
&= \frac{8}{98415 a^7}\int_0^\infty r^s r^6 e^{-2r/3a}\,dr \\
&= \frac{8}{98415 a^7}(s+6)!\left(\frac{3a}{2}\right)^{s+7} \\
&= \frac{1}{720}(s+6!)\left(\frac{3a}{2}\right)^{s} \\
&= \frac{(s+6)!}{6!}\left(\frac{3a}{2}\right)^{s}
\end{aligned}$$

Finite for $s > -7$

4.53

(a)

$$\psi_{433} = -\frac{1}{\sqrt{\pi a}}\frac{1}{6144 a^4}e^{-r/4a}r^3\sin^3\theta e^{3i\phi}$$

(b)

$$\begin{aligned}
\langle r\rangle &= \frac{1}{6144^2\pi a^9}\int_0^\infty\int_0^\pi\int_0^{2\pi} r^9 e^{-r/2a}\sin^7\theta\,dr\,d\theta\,d\phi \\
&= \frac{2}{6144^2 a^9}\int_0^\infty\int_0^\pi r^9 e^{-r/2a}\sin^7\theta\,dr\,d\theta \\
&= \frac{64}{35\cdot 6144^2 a^9}\int_0^\infty r^9 e^{-r/2a}\,dr \\
&= \frac{64}{35\cdot 6144^2 a^9}371589120 a^1 0 \\
&= 18a
\end{aligned}$$

(c)

$$
\begin{aligned}
(L_x^2 + L_y^2)\psi_{433} &= (L^2 - L_z^2)\psi_{433} \\
&= [\hbar^2 3(3+1) - \hbar^2 3^2]\psi_{433} \\
&= 3\hbar^2\psi_{433}
\end{aligned}
$$

Probability 1.

4.54

(a)

$$
\begin{aligned}
\psi_{100} &= \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \\
P_{\text{inside}} &= \frac{1}{\pi a^3} \int_0^b \int_0^\pi \int_0^{2\pi} e^{-2r/a} r^2 \sin\theta \, dr \, d\theta \, d\phi \\
&= \frac{2}{a^3} \int_0^b \int_0^\pi e^{-2r/a} r^2 \sin\theta \, dr \, d\theta \\
&= \frac{4}{a^3} \int_0^b e^{-2r/a} r^2 \, dr \\
&= \frac{4}{a^3} \frac{1}{4} a \left[a^2 - \left(a^2 + 2ab + 2b^2\right) e^{-2b/a} \right] \\
&= 1 - \left[1 + 2\frac{b}{a} + 2\left(\frac{b}{a}\right)^2 \right] e^{-2b/a}
\end{aligned}
$$

(b)

$$
\begin{aligned}
P_{\text{inside}} &= 1 - \left(1 + \epsilon + \frac{1}{2}\epsilon^2\right) e^{-\epsilon} \\
&= 1 - \left(1 + \epsilon + \frac{1}{2}\epsilon^2\right)\left(1 - \epsilon + \frac{\epsilon^2}{2} - \frac{\epsilon^3}{6} + \cdots\right) \\
&= 1 - \left(1 - \epsilon + \frac{\epsilon^2}{2} - \frac{\epsilon^3}{6} + \cdots\right) - \left(\epsilon - \epsilon^2 + \frac{\epsilon^3}{2} - \cdots\right) \\
&\quad - \left(\frac{1}{2}\epsilon^2 - \frac{1}{2}\epsilon^3 + \cdots\right) \\
&= \frac{\epsilon^3}{6} + \cdots \\
&= \frac{4}{3}\left(\frac{b}{a}\right)^3 + \cdots
\end{aligned}
$$

(c)

$$\begin{aligned}
P_{\text{inside}} &\approx \frac{4\pi b^3}{3}|\psi(0)|^2 \\
&= \frac{4\pi b^3}{3}\frac{1}{\pi a^3} \\
&= \frac{4}{3}\left(\frac{b}{a}\right)^3
\end{aligned}$$

(d)

$$P_{\text{inside}} \approx 1.07 \times 10^{-14}$$

4.55

(a)

$$\begin{aligned}
R_{n(n-1)}(r) &= c_0 \frac{1}{r}\rho^{\ell+1}e^{-\rho} \\
&= c_0 \frac{1}{r}\left(\frac{r}{an}\right)^n e^{-r/an} \\
&= N_n r^{n-1} e^{-r/an} \\
1 &= N_n^2 \int_0^\infty r^{2n} e^{-2r/an}\,dr \\
u &= \frac{2r}{an} \\
r &= \frac{anu}{2} \\
dr &= \frac{an}{2}\,du \\
1 &= N_n^2 \int_0^\infty \left(\frac{anu}{2}\right)^{2n} e^{-u}\frac{an}{2}\,du \\
&= N_n^2 \left(\frac{an}{2}\right)^{2n+1} \int_0^\infty u^{2n}e^{-u}\,du \\
&= N_n^2 \left(\frac{an}{2}\right)^{2n+1} \Gamma(2n+1) \\
&= N_n^2 \left(\frac{an}{2}\right)^{2n+1} (2n)! \\
N_n &= \left(\frac{2}{an}\right)^n \sqrt{\frac{2}{an(2n)!}}
\end{aligned}$$

(b)

$$
\begin{aligned}
\langle r \rangle &= \int R^* r R \\
&= N_n^2 \int_0^\infty r^{2n+1} e^{-2r/an}\, dr \\
u &= \frac{2r}{an} \\
r &= \frac{anu}{2} \\
dr &= \frac{an}{2}\, du \\
\langle r \rangle &= N_n^2 \int_0^\infty \left(\frac{anu}{2}\right)^{2n+1} e^{-u} \frac{an}{2}\, du \\
&= \left(n + \frac{1}{2}\right) na \\
\langle r^2 \rangle &= \int R^* r^2 R \\
&= \left(\frac{2}{an}\right)^{2n} \frac{2}{an(2n)!} \int_0^\infty r^{2n+2} e^{-2r/an}\, dr \\
&= \left(n + \frac{1}{2}\right)(n+1)(na)^2
\end{aligned}
$$

(c)

$$
\begin{aligned}
\sigma_r &= \sqrt{\langle r^2 \rangle - \langle r \rangle^2} \\
&= \sqrt{\left(n + \frac{1}{2}\right)(n+1)(na)^2 - \left(n + \frac{1}{2}\right)^2 (na)^2} \\
&= \left(n + \frac{1}{2}\right) na \sqrt{\frac{n+1}{n+\frac{1}{2}} - 1} \\
&= \langle r \rangle \sqrt{\frac{n+1-n-\frac{1}{2}}{n+\frac{1}{2}}} \\
&= \frac{\langle r \rangle}{\sqrt{2n+1}}
\end{aligned}
$$

4.57

(a)

$$\begin{aligned}
A &= x^2 \\
B &= L_z \\
[A, B] &= [x^2, L_z] \\
&= x[x, L_z] + [x, L_z]x \\
&= x[x, xp_y - yp_x] + [x, xp_y - yp_x]x \\
&= x[x, xp_y] - x[x, yp_x] + [x, xp_y]x - [x, yp_x]x \\
&= x^2[x, p_y] + x[x, x]p_y - xy[x, p_x] - x[x, y]p_x \\
&\qquad + x[x, p_y]x + [x, x]p_y x - y[x, p_x]x - [x, y]p_x x \\
&= -i\hbar xy - i\hbar xy \\
&= -2i\hbar xy \\
\sigma_A^2 \sigma_B^2 &\geq \left(\frac{1}{2i}\left\langle [A, B] \right\rangle\right)^2 \\
&= \left(\frac{1}{2i}\left\langle -2i\hbar xy \right\rangle\right)^2 \\
&= \hbar^2 \left\langle xy \right\rangle^2 \\
\sigma_A \sigma_B &= \hbar |\langle xy \rangle|
\end{aligned}$$

(b)

$$\begin{aligned}
L_z \psi_{n\ell m} &= m\hbar \psi_{n\ell m} \\
L_z^2 \psi_{n\ell m} &= m^2 \hbar^2 \psi_{n\ell m} \\
\sigma_B &= \sqrt{\langle L_z^2 \rangle - \langle L_z \rangle^2} \\
&= \sqrt{m^2\hbar^2 - m^2\hbar^2} \\
&= 0
\end{aligned}$$

(c)

$$\langle xy \rangle = 0$$

4.58

(a)

$$\begin{aligned}
1 &= \chi^\dagger \chi \\
&= |A|^2 \begin{pmatrix} 1+2i & 2 \end{pmatrix} \begin{pmatrix} 1-2i \\ 2 \end{pmatrix} \\
&= |A|^2 [(1+2i)(1-2i)+4] \\
&= 9|A|^2 \\
A &= \frac{1}{3}
\end{aligned}$$

(b)

$$\chi = \frac{1}{3} \begin{pmatrix} 1-2i \\ 2 \end{pmatrix}$$

You could get $\frac{\hbar}{2}$ with probability $\frac{1}{9}(1+2i)(1-2i) = \frac{5}{9}$ or $-\frac{\hbar}{2}$ with probability $\frac{4}{9}$.

$$\langle S_z \rangle = \left(\frac{5}{9} - \frac{4}{9} \right) \frac{\hbar}{2} = \frac{\hbar}{18}$$

(c) You could get $\frac{\hbar}{2}$ with probability

$$\frac{1}{9} \frac{(3+2i)(3-2i)}{2} = \frac{13}{18}$$

or $-\frac{\hbar}{2}$ with probability

$$\frac{1}{9} \frac{(-1+2i)(-1-2i)}{2} = \frac{5}{18}.$$

$$\langle S_x \rangle = \left(\frac{13}{18} - \frac{5}{18} \right) \frac{\hbar}{2} = \frac{2}{9}\hbar$$

(d) You could get $\frac{\hbar}{2}$ with probability

$$\frac{1}{9} \frac{(1+4i)(1-4i)}{2} = \frac{17}{18}$$

or $-\frac{\hbar}{2}$ with probability

$$\frac{1}{18}.$$

$$\langle S_y \rangle = \left(\frac{17}{18} - \frac{1}{18} \right) \frac{\hbar}{2} = \frac{4}{9}\hbar$$

4.61

The basis states are $|\frac{3}{2}\rangle$, $|\frac{1}{2}\rangle$, $|-\frac{1}{2}\rangle$, and $|-\frac{3}{2}\rangle$. The matrix corresponding to the raising operator is

$$S_+ = \hbar \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the matrix corresponding to the lowering operator is

$$S_- = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}.$$

$S_\pm = S_x \pm S_y$ so

$$\begin{aligned} S_x &= \frac{S_+ + S_-}{2} \\ &= \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}. \end{aligned}$$

The eigenvalues of S_x are $\frac{3}{2}\hbar$, $\frac{1}{2}\hbar$, $-\frac{1}{2}\hbar$, and $-\frac{3}{2}\hbar$, as expected.

4.64

(a) You would get $2\hbar^2$ with probability 1.

(b) You could get 0 with probability $\frac{1}{3}$ or $3\hbar$ with probability $\frac{2}{3}$.

(c) You would get $\frac{3}{4}\hbar^2$ with probability 1.

(d) You could get $\frac{\hbar}{2}$ with probability $\frac{1}{3}$ or $-\frac{\hbar}{2}$ with probability $\frac{2}{3}$.

(e)

$$\begin{aligned} \sqrt{\frac{1}{3}}\left|\frac{1}{2}\frac{1}{2}\right\rangle|10\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2}{-\frac{1}{2}}\right\rangle|11\rangle &= \frac{1}{\sqrt{3}}\left(\sqrt{\frac{2}{3}}\left|\frac{3}{2}\frac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|\frac{1}{2}\frac{1}{2}\right\rangle\right) \\ &\quad + \sqrt{\frac{2}{3}}\left(\frac{1}{\sqrt{3}}\left|\frac{3}{2}\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2}\frac{1}{2}\right\rangle\right) \\ &= \sqrt{\frac{8}{9}}\left|\frac{3}{2}\frac{1}{2}\right\rangle + \sqrt{\frac{1}{9}}\left|\frac{1}{2}\frac{1}{2}\right\rangle \end{aligned}$$

You could measure $\frac{15}{4}\hbar^2$ with probability $\frac{8}{9}$ or $\frac{3}{4}\hbar^2$ with probability $\frac{1}{9}$.

(f) You would measure $\frac{\hbar}{2}$ with probability 1.

(g)

$$
\begin{aligned}
&R_{21}^* \left(\sqrt{\frac{1}{3}} (Y_1^0)^* \chi_+^\dagger + \sqrt{\frac{2}{3}} (Y_1^1)^* \chi_-^\dagger \right) R_{21} \left(\sqrt{\frac{1}{3}} Y_1^0 \chi_+ + \sqrt{\frac{2}{3}} Y_1^1 \chi_- \right) \\
&= \frac{1}{3} |R_{21}|^2 (|Y_1^0|^2 + 2|Y_1^1|^2) \\
&= \frac{1}{3} \frac{1}{24a^5} r^2 e^{-r/a} \left(\frac{3}{4\pi} \cos^2\theta + \frac{3}{4\pi} \sin^2\theta \right) \\
&= \frac{1}{96\pi a^5} r^2 e^{-r/a}
\end{aligned}
$$

(h)

$$
\begin{aligned}
&\frac{1}{3} |R_{21}|^2 \int_0^\pi \int_0^{2\pi} |Y_1^0|^2 \sin\theta \, d\theta \, d\phi \\
&= \frac{1}{3} \frac{1}{24a^5} r^2 e^{-r/a} \int_0^\pi \int_0^{2\pi} \frac{3}{4\pi} \cos^2\theta \sin\theta \, d\theta \, d\phi \\
&= \frac{1}{48a^5} r^2 e^{-r/a} \int_0^\pi \cos^2\theta \sin\theta \, d\theta \\
&= \frac{1}{72a^5} r^2 e^{-r/a}
\end{aligned}
$$

4.70

(a) At $t = 0$ all particles are in the spin-up state with respect to the z axis $|\uparrow\rangle$. At $t = t_1$ the spin of a fraction f of the particles is measured with respect to the z axis (the z group) and the remaining fraction $1 - f$ is measured with respect to the x axis (the x group). There is no change in the z group so the fraction of particles in the state $|\uparrow\rangle$ is f. The x group is evenly split, so the fraction of particles in either the $|\leftarrow\rangle$ or $|\rightarrow\rangle$ states is $\frac{1}{2}(1 - f)$. At $t = t_2$ the spin of all particles is measured with respect to the x axis. The z group is evenly split and the x group is unchanged so the fraction of particles in either the $|\leftarrow\rangle$ or $|\rightarrow\rangle$ states is $\frac{1}{2}f + \frac{1}{2}(1 - f) = \frac{1}{2}$. The subensemble of particles in the $|\rightarrow\rangle$ state is comprised of particles from both the z and x groups. Of those from the z group, all of them were in the spin-up state at $t = t_1$, and of those from the x group, they also were all in the spin-up state at $t = t_1$. In conclusion: all of the particles in the subensemble had spin-up in the first measurement.

(b)

$$\begin{aligned}
P_+ &= \cos^2(\theta_{ab}/2) \\
P_- &= \sin^2(\theta_{ab}/2) \\
P_{++} &= \cos^2(\theta_{ab}/2)\cos^2(\theta_{bc}/2) \\
P_{+-} &= \cos^2(\theta_{ab}/2)\sin^2(\theta_{bc}/2) \\
P_{-+} &= \sin^2(\theta_{ab}/2)\sin^2(\theta_{bc}/2) \\
P_{--} &= \sin^2(\theta_{ab}/2)\cos^2(\theta_{bc}/2) \\
\frac{P_{++}}{P_{++}+P_{-+}} &= \frac{1}{1+P_{-+}/P_{++}} \\
&= [1+\tan^2(\theta_{ab}/2)\tan^2(\theta_{bc}/2)]^{-1}
\end{aligned}$$

4.71

(a)

$$
\begin{aligned}
\psi_{21m} &= \frac{1}{\sqrt{24a^5}} r e^{-r/2a} Y_1^m(\theta,\phi) \\
\psi_{21-1} &= \frac{1}{\sqrt{24a^5}} r e^{-r/2a} \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi} \\
&= \frac{1}{\sqrt{64\pi a^5}} r e^{-r/2a} \sin\theta e^{-i\phi} \\
\psi_{210} &= \frac{1}{\sqrt{24a^5}} r e^{-r/2a} \sqrt{\frac{3}{4\pi}} \cos\theta \\
&= \frac{1}{\sqrt{32\pi a^5}} r e^{-r/2a} \cos\theta \\
\psi_{211} &= -\frac{1}{\sqrt{24a^5}} r e^{-r/2a} \sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \\
&= -\frac{1}{\sqrt{64\pi a^5}} r e^{-r/2a} \sin\theta e^{i\phi} \\
\psi_{2p_x} &= \frac{1}{\sqrt{32\pi a^3}} \frac{x}{a} e^{-r/2a} \\
&= \frac{1}{\sqrt{32\pi a^5}} r e^{-r/2a} \sin\theta\cos\phi \\
&= \frac{1}{\sqrt{2}} (\psi_{21-1} - \psi_{211}) \\
\psi_{2p_y} &= \frac{1}{\sqrt{32\pi a^3}} \frac{y}{a} e^{-r/2a} \\
&= \frac{1}{\sqrt{32\pi a^5}} r e^{-r/2a} \sin\theta\sin\phi \\
&= \frac{i}{\sqrt{2}} (\psi_{21-1} + \psi_{211}) \\
\psi_{2p_z} &= \frac{1}{\sqrt{32\pi a^3}} \frac{z}{a} e^{-r/2a} \\
&= \frac{1}{\sqrt{32\pi a^5}} r e^{-r/2a} \cos\theta \\
&= \psi_{210}
\end{aligned}
$$

5 Identical Particles

5.1

(a)

$$\begin{aligned}
\mathbf{R} + \frac{\mu}{m_1}\mathbf{r} &= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} + \frac{m_2}{m_1 + m_2}(\mathbf{r}_1 - \mathbf{r}_2) \\
&= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_2\mathbf{r}_1 - m_2\mathbf{r}_2}{m_1 + m_2} \\
&= \mathbf{r}_1 \\
\mathbf{R} - \frac{\mu}{m_2}\mathbf{r} &= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} - \frac{m_1}{m_1 + m_2}(\mathbf{r}_1 - \mathbf{r}_2) \\
&= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 - m_1\mathbf{r}_1 + m_1\mathbf{r}_2}{m_1 + m_2} \\
&= \mathbf{r}_2
\end{aligned}$$

Let $\mathbf{r} = (x, y, z)$ and $\mathbf{R} = (X, Y, Z)$, then

$$\begin{aligned}
(\nabla_1)_x &= \frac{\partial}{\partial x_1} \\
&= \frac{\partial X}{\partial x_1}\frac{\partial}{\partial X} + \frac{\partial x}{\partial x_1}\frac{\partial}{\partial x} \\
&= \frac{\mu}{m_2}(\nabla_R)_x + (\nabla_r)_x \\
\nabla_1 &= \frac{\mu}{m_2}\nabla_R + \nabla_r \\
(\nabla_2)_x &= \frac{\partial}{\partial x_2} \\
&= \frac{\partial X}{\partial x_2}\frac{\partial}{\partial X} + \frac{\partial x}{\partial x_2}\frac{\partial}{\partial x} \\
&= \frac{\mu}{m_1}(\nabla_R)_x - (\nabla_r)_x \\
\nabla_2 &= \frac{\mu}{m_1}\nabla_R - \nabla_r
\end{aligned}$$

(b)

$$
\begin{aligned}
\nabla_1^2\psi &= \nabla_1 \cdot \nabla_1\psi \\
&= \nabla_1 \cdot \left(\frac{\mu}{m_2}\nabla_R\psi + \nabla_r\psi\right) \\
&= \frac{\mu}{m_2}\nabla_R \cdot \left(\frac{\mu}{m_2}\nabla_R\psi + \nabla_r\psi\right) + \nabla_r \cdot \left(\frac{\mu}{m_2}\nabla_R\psi + \nabla_r\psi\right) \\
&= \left(\frac{\mu}{m_2}\right)^2 \nabla_R^2\psi + 2\frac{\mu}{m_2}(\nabla_r \cdot \nabla_R)\psi + \nabla_r^2\psi \\
\nabla_2^2\psi &= \nabla_2 \cdot \nabla_2\psi \\
&= \nabla_2 \cdot \left(\frac{\mu}{m_1}\nabla_R\psi - \nabla_r\psi\right) \\
&= \frac{\mu}{m_1}\nabla_R \cdot \left(\frac{\mu}{m_1}\nabla_R\psi - \nabla_r\psi\right) - \nabla_r \left(\frac{\mu}{m_1}\nabla_R\psi - \nabla_r\psi\right) \\
&= \left(\frac{\mu}{m_1}\right)^2 \nabla_R^2\psi - 2\frac{\mu}{m_1}(\nabla_r \cdot \nabla_R)\psi + \nabla_r^2\psi
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\hbar^2}{2m_1}\nabla_1^2\psi - \frac{\hbar^2}{2m_2}\nabla_2^2\psi + V\psi &= E\psi \\
-\frac{\hbar^2}{2m_1}\left[\left(\frac{\mu}{m_2}\right)^2 \nabla_R^2\psi + 2\frac{\mu}{m_2}(\nabla_r \cdot \nabla_R)\psi + \nabla_r^2\psi\right] & \\
-\frac{\hbar^2}{2m_2}\left[\left(\frac{\mu}{m_1}\right)^2 \nabla_R^2\psi - 2\frac{\mu}{m_1}(\nabla_r \cdot \nabla_R)\psi + \nabla_r^2\psi\right] + V\psi &= E\psi \\
-\frac{\hbar^2\mu^2}{2}\left(\frac{1}{m_1m_2^2} + \frac{1}{m_1^2m_2}\right)\nabla_R^2\psi - \frac{\hbar^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\nabla_r^2\psi + V\psi &= E\psi \\
-\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V\psi &= E\psi
\end{aligned}
$$

(c)

$$
\begin{aligned}
\psi(\mathbf{R},\mathbf{r}) &= \psi_R(\mathbf{R})\psi_r(\mathbf{r}) \\
-\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi_R\psi_r - \frac{\hbar^2}{2\mu}\nabla_r^2\psi_r\psi_R + V(\mathbf{r})\psi_R\psi_r &= E\psi_R\psi_r \\
-\frac{\hbar^2}{2(m_1+m_2)}\frac{\nabla_R^2\psi_R}{\psi_R} - \frac{\hbar^2}{2\mu}\frac{\nabla_r^2\psi_r}{\psi_r} + V(\mathbf{r}) &= E \\
\left(-\frac{\hbar^2}{2(m_1+m_2)}\frac{\nabla_R^2\psi_R}{\psi_R} - E_R\right) + \left(-\frac{\hbar^2}{2\mu}\frac{\nabla_r^2\psi_r}{\psi_r} + V(\mathbf{r}) - E_r\right) &= 0
\end{aligned}
$$

5.2

(a)

$$\mu = \frac{m_e m_p}{m_e + m_p}$$
$$E_1 = -\frac{\mu}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2$$
$$\approx -13.5982\,\mathrm{eV}$$

(b)

$$\begin{aligned}
\Delta E_{\mathrm{H}} &= E_{3_H} - E_{2_H} \\
&= \frac{E_{1_H}}{9} - \frac{E_{1_H}}{4} \\
&= \left(\frac{1}{9} - \frac{1}{4}\right) E_{1_H} \\
&\approx 1.888\,\mathrm{eV} \\
\lambda_H &= 6.569 \times 10^{-7}\,\mathrm{m} \\
E_{1_D} &= \frac{m_e(m_n + m_p)}{m_e + m_n + m_p}\frac{1}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \\
\Delta E_{\mathrm{D}} &= E_{3_D} - E_{2_D} \\
&= \left(\frac{1}{9} - \frac{1}{4}\right) E_{1_D} \\
&\approx 1.889\,\mathrm{eV} \\
\lambda_D &= 6.567 \times 10^{-7}\,\mathrm{m} \\
\Delta\lambda &= 1.79 \times 10^{-10}\,\mathrm{m}
\end{aligned}$$

(c)

$$\begin{aligned}
\mu &= \frac{m_e^2}{m_e + m_2} \\
&= \frac{1}{2} m_e \\
E_1' &= \frac{1}{2} E_1 \\
&= -6.8\,\mathrm{eV}
\end{aligned}$$

(d)

$$
\begin{aligned}
\mu_p &= \frac{m_e m_p}{m_e + m_p} \\
\mu_\mu &= \frac{m_\mu m_p}{m_\mu + m_p} \\
\frac{\mu_\mu}{\mu_p} &= \frac{m_\mu m_p}{m_\mu + m_p} \cdot \frac{m_e + m_p}{m_e m_p} \\
&= \frac{m_\mu (m_e + m_p)}{m_e (m_\mu + m_p)} \\
&= 206.77 \frac{m_e + m_p}{206.77 m_e + m_p} \\
&\approx 185.943 \\
\frac{1}{\lambda} &= R' \left(1 - \frac{1}{4}\right) \\
&= \frac{3}{4} R' \\
\lambda &= \frac{4}{3R'} \\
&\approx 6.54 \times 10^{-10}\,\mathrm{m}
\end{aligned}
$$

5.4

(a)

$$
\begin{aligned}
1 &= \int |\psi|^2 \\
&= |A|^2 \int [\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) \pm \psi_b(\mathbf{r}_1)\psi_a(\mathbf{r}_2)]^* [\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) \pm \psi_b(\mathbf{r}_1)\psi_a(\mathbf{r}_2)] \\
&= |A|^2 \int |\psi_a(\mathbf{r}_1)|^2 |\psi_b(\mathbf{r}_2)|^2 \pm \psi_a(\mathbf{r}_1)^* \psi_b(\mathbf{r}_1) \psi_a(\mathbf{r}_2) \psi_b(\mathbf{r}_2)^* \\
&\qquad \pm \psi_a(\mathbf{r}_1) \psi_b(\mathbf{r}_1)^* \psi_a(\mathbf{r}_2)^* \psi_b(\mathbf{r}_2) + |\psi_a(\mathbf{r}_2)|^2 |\psi_b(\mathbf{r}_1)|^2 \\
&= 2|A|^2 \\
A &= \frac{1}{\sqrt{2}}
\end{aligned}
$$

(b)

$$\begin{aligned}
1 &= \int |\psi|^2 \\
&= |A|^2 \int [2\psi_a(\mathbf{r}_1)\psi_a(\mathbf{r}_2)]^*[2\psi_a(\mathbf{r}_1)\psi_a(\mathbf{r}_2)] \\
&= 4|A|^2 \int \psi_a^*(\mathbf{r}_1)\psi_a(\mathbf{r}_1)\psi_a^*(\mathbf{r}_2)\psi_a(\mathbf{r}_2) \\
A &= \frac{1}{2}
\end{aligned}$$

5.5

(a)

$$\begin{aligned}
\hat{H} &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} \\
\hat{H}\psi_{12} &= -\frac{\hbar^2}{2m}\frac{\sqrt{2}}{a}\left[-\left(\frac{\pi}{a}\right)^2 \sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right)\right. \\
&\quad + \left(\frac{2\pi}{a}\right)^2 \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right) \\
&\quad - \left(\frac{2\pi}{a}\right)^2 \sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) \\
&\quad \left. + \left(\frac{\pi}{a}\right)^2 \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right] \\
&= -\frac{\hbar^2}{2m}\frac{\sqrt{2}}{a}\left[-5\left(\frac{\pi}{a}\right)^2 \sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right)\right. \\
&\quad \left. +5\left(\frac{\pi}{a}\right)^2 \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right] \\
&= 5\left(\frac{\pi}{a}\right)^2 \frac{\hbar^2}{2m}\psi \\
E &= \frac{5\pi^2\hbar^2}{2ma^2} \\
&= 5K
\end{aligned}$$

5.6

(a)

$$
\begin{aligned}
\langle x \rangle_l &= \frac{2}{a}\int_0^a x \sin^2\left(\frac{l\pi}{a}x\right) dx \\
&= \frac{a}{2} \\
\langle x^2 \rangle_l &= \frac{2}{a}\int_0^a x^2 \sin^2\left(\frac{l\pi}{a}x\right) dx \\
&= a^2\left(\frac{1}{3} - \frac{1}{2(l\pi)^2}\right) \\
\langle (x_1 - x_2)^2 \rangle &= \langle x^2 \rangle_l + \langle x^2 \rangle_n - 2\langle x \rangle_l \langle x \rangle_n \\
&= a^2\left(\frac{1}{3} - \frac{1}{2(l\pi)^2}\right) + a^2\left(\frac{1}{3} - \frac{1}{2(n\pi)^2}\right) - \frac{a^2}{2} \\
&= a^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{l^2} + \frac{1}{n^2}\right)\right]
\end{aligned}
$$

(b)

$$
\begin{aligned}
\langle x \rangle_{ln} &= \frac{2}{a}\int x \sin\left(\frac{l\pi}{a}x\right)\sin\left(\frac{n\pi}{a}\right) dx \\
&= \frac{4aln[-1 + (-1)^{l+n}]}{\pi^2(l^2 - n^2)^2} \\
&= \begin{cases} \frac{a(-8ln)}{\pi^2(l^2-n^2)^2} & \text{if } l \text{ and } n \text{ have opposite parity} \\ 0 & \text{if } l \text{ and } n \text{ have same parity} \end{cases} \\
\langle (x_1 - x_2)^2 \rangle &= a^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{l^2} + \frac{1}{n^2}\right)\right] - \frac{128a^2l^2n^2}{\pi^4(l^2 - n^2)^4}
\end{aligned}
$$

(c)

$$
\langle (x_1 - x_2)^2 \rangle = a^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{l^2} + \frac{1}{n^2}\right)\right] + \frac{128a^2l^2n^2}{\pi^4(l^2 - n^2)^4}
$$

5.7

(a) (i)

$$\psi(x_1, x_2) = \psi_0(x_1)\psi_1(x_2)$$

(ii)

$$\psi(x_1, x_2) = \frac{1}{2}[\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)]$$

(iii)

$$\psi(x_1, x_2) = \frac{1}{2}[\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2)]$$

(b) (i)

$$\begin{aligned}\left\langle (x_1 - x_2)^2 \right\rangle &= \left\langle x^2 \right\rangle_0 + \left\langle x^2 \right\rangle_1 - 2\left\langle x \right\rangle_0 \left\langle x \right\rangle_1 \\ &= \frac{4\hbar}{2m\omega}\end{aligned}$$

5.8

(a)

$$\psi(x_1, x_2, x_3) = \psi_a(x_1)\psi_b(x_2)\psi_c(x_3)$$

(b)

$$\begin{aligned}\psi(x_1, x_2, x_3) = \frac{1}{\sqrt{6}}[&\psi_a(x_1)\psi_b(x_2)\psi_c(x_3) + \psi_a(x_1)\psi_c(x_2)\psi_b(x_3) \\ &+ \psi_b(x_1)\psi_a(x_2)\psi_c(x_3) + \psi_b(x_1)\psi_c(x_2)\psi_a(x_3) \\ &+ \psi_c(x_1)\psi_a(x_2)\psi_b(x_3) + \psi_c(x_1)\psi_b(x_2)\psi_a(x_3)]\end{aligned}$$

(c)

$$\begin{aligned}\begin{vmatrix} \psi_a(x_1) & \psi_b(x_1) & \psi_c(x_1) \\ \psi_a(x_2) & \psi_b(x_2) & \psi_c(x_2) \\ \psi_a(x_3) & \psi_b(x_3) & \psi_c(x_3) \end{vmatrix} &= \psi_a(x_1)[\psi_b(x_2)\psi_c(x_3) - \psi_c(x_2)\psi_b(x_3)] \\ &\quad - \psi_b(x_1)[\psi_a(x_2)\psi_c(x_3) - \psi_c(x_2)\psi_a(x_3)] \\ &\quad + \psi_c(x_1)[\psi_a(x_2)\psi_b(x_3) - \psi_b(x_2)\psi_a(x_3)] \\ \psi(x_1, x_2, x_3) &= \frac{1}{\sqrt{6}}[\psi_a(x_1)\psi_b(x_2)\psi_c(x_3) - \psi_a(x_1)\psi_c(x_2)\psi_b(x_3) \\ &\quad - \psi_b(x_1)\psi_a(x_2)\psi_c(x_3) + \psi_b(x_1)\psi_c(x_2)\psi_a(x_3) \\ &\quad + \psi_c(x_1)\psi_a(x_2)\psi_b(x_3) - \psi_c(x_1)\psi_b(x_2)\psi_a(x_3)]\end{aligned}$$

5.13

(a)

$$\begin{aligned}E_{\text{before}} &= 4\left(\frac{E_1}{4} + \frac{E_1}{4}\right) \\ &= 2E_1 \\ E_{\text{after}} &= 4\left(\frac{E_1}{4} + E_1\right) \\ &= 5E_1 \\ E_{\text{electron}} &= 4\frac{E_1}{4} + \Delta E \\ &= -2E_1 \\ &= 27.2\,\text{eV}\end{aligned}$$

(b)

$$
\begin{aligned}
\mathcal{R}' &= \frac{m_e}{4\pi c\hbar^3}\left(\frac{2e^2}{4\pi\epsilon_0}\right)^2 \\
&= 4\mathcal{R} \\
\frac{1}{\lambda} &= 4\mathcal{R}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)
\end{aligned}
$$

5.14

(a) If the electrons were identical bosons the total wavefunction would be symmetric — a symmetric spatial wavefunction would be paired with a triplet spin state and an antisymmetric spatial wavefunction would be paired with a singlet spin state.

(b) If the electrons were distinguishable particles there would be no (anti)symmetry requirement on the total wavefunction, so every energy level could be in the singlet or triplet state.

5.15

(a)

$$
\begin{aligned}
\left\langle \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} \right\rangle &= \int \psi_0^* \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} \psi_0 \\
&= \int \frac{1}{|r_1 - r_2|} |\psi_0|^2 \, d^3\mathbf{r}_2 \, d^3\mathbf{r}_1 \\
&= \frac{64}{\pi^2 a^6} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta_2}} \\
&\quad e^{-4(r_1+r_2)/a} r_2^2 \sin\theta_2 \, dr_2 \, d\theta_2 \, d\phi_2 \, d^3\mathbf{r}_1 \\
&= \frac{128}{\pi a^6} \int_0^\infty \frac{r_1 + r_2 - |r_1 - r_2|}{r_1 r_2} e^{-4(r_1+r_2)/a} r_2^2 \, dr_2 \, d^3\mathbf{r}_1 \\
&= \frac{256}{\pi a^6} \left(\int_0^{r_1} \frac{r_2^2}{r_1} e^{-4(r_1+r_2)/a} \, dr_2 \, d^3\mathbf{r}_1 \right. \\
&\quad \left. + \int_{r_1}^\infty r_2 e^{-4(r_1+r_2)/a} \, dr_2 \, d^3\mathbf{r}_1 \right) \\
&= \frac{256}{\pi a^6} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{a^2 e^{-8r_1/a} \left[a \left(e^{4r_1/a} - 1 \right) - 2r_1 \right]}{32 r_1} \\
&\quad r_1^2 \sin\theta_1 \, dr_1 \, d\theta_1 \, d\phi_1 \\
&= \frac{8}{\pi a^4} \int_0^\infty \int_0^\pi \int_0^{2\pi} r_1 e^{-8r_1/a} \left[a \left(e^{4r_1/a} - 1 \right) - 2r_1 \right] \\
&\quad \sin\theta_1 \, dr_1 \, d\theta_1 \, d\phi_1 \\
&= \frac{32}{a^4} \int_0^\infty r_1 e^{-8r_1/a} \left[a \left(e^{4r_1/a} - 1 \right) - 2r_1 \right] \, dr_1 \\
&= \frac{5}{4a}
\end{aligned}
$$

(b)

$$
\begin{aligned}
E_{\text{interaction}} &= \frac{e^2}{4\pi\epsilon_0} \left\langle \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} \right\rangle \\
&\approx 34 \, \text{eV} \\
E_0' &= E_0 + E_{\text{interaction}} \\
&\approx -75 \, \text{eV}
\end{aligned}
$$

5.16

$$
\begin{aligned}
\psi_0(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) &= \psi_{100}(\mathbf{r}_1) \psi_{100}(\mathbf{r}_2) \psi_{200}(\mathbf{r}_3) \\
E_1' &= 9(E_1 + E_1 + E_2) \\
&\approx -275.4 \, \text{eV}
\end{aligned}
$$

5.17

(a)
- Hydrogen $(1s)^1$
- Helium $(1s)^2$
- Lithium $(1s)^2(2s)^1$
- Beryllium $(1s)^2(2s)^2$
- Boron $(1s)^2(2s)^2(2p)^1$
- Carbon $(1s)^2(2s)^2(2p)^2$
- Nitrogen $(1s)^2(2s)^2(2p)^3$
- Oxygen $(1s)^2(2s)^2(2p)^4$
- Fluorine $(1s)^2(2s)^2(2p)^5$
- Neon $(1s)^2(2s)^2(2p)^6$

(b)
- Hydrogen $^2S_{1/2}$
- Helium $^1S_0$
- Lithium $^2S_{1/2}$
- Beryllium $^1S_0$
- Boron $^2P_{1/2}$, $^2P_{3/2}$
- Carbon $^1S_0$, $^3S_1$, $^1P_1$, $^3P_0$, $^3P_1$, $^3P_2$, $^1D_2$, $^3D_1$, $^3D_2$, $^3D_3$
- Nitrogen $^2S_{1/2}$, $^4S_{3/2}$, $^2P_{1/2}$, $^2P_{3/2}$, $^4P_{1/2}$, $^4P_{3/2}$, $^4P_{5/2}$, $^2D_{3/2}$, $^2D_{5/2}$, $^4D_{1/2}$, $^4D_{3/2}$, $^4D_{5/2}$, $^4D_{7/2}$, $^2F_{5/2}$, $^2F_{7/2}$, $^4F_{3/2}$, $^4F_{5/2}$, $^4F_{7/2}$, $^4F_{9/2}$

5.18

(a) Orthohelium should have lower energy than parahelium.

(b) Hund's first rule says that the ground state of carbon has $S = 1$. This is a triplet state and is thus symmetric. That means the spatial wavefunction needs to be antisymmetric. Hund's second rule suggests we should choose $L = 2$. If we go to the "top of the ladder" for $L = 2$ we find $|2, 2\rangle = |1, 1\rangle_1 |1, 1\rangle_2$ which is symmetric and thus combining it with the triplet state would violate the Pauli exclusion principle. We must choose $L = 1$.

(c) Boron can be $^2P_{1/2}$ or $^2P_{3/2}$. The $2p$ subshell can contain up to 6 electrons so for boron it is less than half filled and thus the lowest energy level has $J = |L - S| = |1 - \frac{1}{2}| = \frac{1}{2}$ or $^2P_{1/2}$.

(d) For carbon the highest total spin is $S = 1$ which is a triplet state and is thus symmetric. Top of the ladder for $L = 0$ is $|0, 0\rangle = |0, 0\rangle_1 |0, 0\rangle_2$ which is symmetric, for $L = 1$ is $|1, 1\rangle = |1, 1\rangle |0, 0\rangle$ which is antisymmetric, and $L = 2$ is $|1, 1\rangle_1 |1, 1\rangle_2$ which is symmetric. Thus $L = 1$. The $2p$ subshell is less than half filled and thus the lowest energy level has $J = |L - S| = |1 - 1| = 0$ or $^3P_0$.

5.19

$S = 2,\ L = 6,\ J = 8$, the $n = 1, 2,$ and 3 shells are filled, the $n = 4$ and 5 shells are partially filled.

5.20

$$
\begin{aligned}
E_{\text{tot}} &= \frac{\hbar^2 (3\pi^2 N d)^{5/3}}{10\pi^2 m} V^{-2/3} \\
&= \frac{3Nd}{5} E_F \\
\frac{E_{\text{tot}}}{Nd} &= \frac{3}{5} E_F
\end{aligned}
$$

5.21

(a)

$$
\begin{aligned}
\rho_{\text{electron}} &= \frac{N_A \rho_{\text{mass}}}{m_a V} \\
&= 8.50 \times 10^{28}\,\text{m}^{-3} \\
E_F &= \frac{\hbar^2}{2m} (3\rho\pi^2)^{2/3} \\
&= 7.05\,\text{eV}
\end{aligned}
$$

(b)

$$
\begin{aligned}
E_F &= \frac{1}{2} m v^2 \\
v &= \sqrt{\frac{2E_F}{m}} \\
&= 1.57 \times 10^6\,\text{m/s}
\end{aligned}
$$

It is safe to assume the electrons in copper are nonrelativistic.

(c)

$$
\begin{aligned}
k_B T &= E_F \\
T &= \frac{E_F}{k_B} \\
&= 8.18 \times 10^4\,\text{K}
\end{aligned}
$$

(d)

$$
\begin{aligned}
P &= \frac{(3\pi^2)^{2/3} \hbar^2}{5m} \rho^{5/3} \\
&= 3.84 \times 10^{10}\,\text{Pa}
\end{aligned}
$$

5.22

$$\begin{aligned}
\rho_{\text{electron}} &= \frac{\rho_{\text{mass}}}{m_a} \\
&= 1.64 \times 10^{28}\,\text{m}^{-3} \\
E_F &= \frac{\hbar^2}{2m}(3\rho_{\text{electron}}\pi^2)^{2/3} \\
&= 4.28 \times 10^{-4}\,\text{eV} \\
k_B T_F &= E_F \\
T_F &= \frac{E_F}{k_B} \\
&= 4.96\,\text{K}
\end{aligned}$$

5.23

$$\begin{aligned}
P &= \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3} \\
&= \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\left(\frac{Nd}{V}\right)^{5/3} \\
\frac{dP}{dV} &= -\frac{5}{3}\frac{(3\pi^2)^{2/3}\hbar^2}{5m}\frac{(Nd)^{5/3}}{V^{8/3}} \\
&= -\frac{5}{3}\frac{P}{V} \\
B &= -V\frac{dP}{dV} \\
&= \frac{5}{3}P \\
B_{\text{copper}} &= 6.4 \times 10^{10}\,\text{Pa}
\end{aligned}$$

5.24

(a)

$$\begin{aligned}
\psi(x) &= A\sin(kx) + B\cos(kx) \\
A\sin(ka) &= [e^{iqa} - \cos(ka)]B \\
\psi(x) &= \frac{[e^{iqa} - \cos(ka)]B\sin(kx)}{\sin(ka)} + B\cos(kx) \\
&= \frac{Be^{iqa}}{\sin(ka)}\left\{\sin(kx) + e^{-iqa}\sin[k(a-x)]\right\}
\end{aligned}$$

5.25

$$\begin{aligned}
10 &= \beta \\
&= \frac{m\alpha a}{\hbar^2} \\
a^2 &= \frac{10\hbar^2}{m}\frac{a}{\alpha} \\
a &\approx 8.73 \times 10^{-10}\,\mathrm{m} \\
-1 &= \cos z + \beta\frac{\sin z}{z} \\
z &= 2.62768 \\
ka &= 2.62768 \\
\frac{\sqrt{2mE}}{\hbar} &= \frac{2.62768}{a} \\
E &= \frac{2.62768^2\hbar^2}{2ma^2} \\
&\approx 0.345\,\mathrm{eV}
\end{aligned}$$

5.29

(a) Each particle can be in one of three states, so $3^3 = 27$.

(b)

$$\begin{gathered}
aaa \\
aab \\
abb \\
aac \\
acc \\
bbb \\
bbc \\
bcc \\
ccc \\
abc
\end{gathered}$$

10

(c) Fermions can't be in the same state, so 1.

5.30

$$
\begin{aligned}
E_{n_x,n_y} &= \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{l_x^2}+\frac{n_y^2}{l_y^2}\right)\\
&= \frac{\hbar^2k^2}{2m}\\
\frac{\pi^2}{l_xl_y} &= \frac{\pi^2}{A}\\
\frac{1}{4}(\pi k_F^2) &= \frac{Nd}{2}\left(\frac{\pi^2}{A}\right)\\
k_F &= \sqrt{2\sigma\pi}\\
E_F &= \frac{\hbar^2}{2m}k_F^2\\
&= \frac{\pi\hbar^2\sigma}{m}
\end{aligned}
$$

5.32

(a)

$$
\begin{aligned}
\frac{1}{8}\left(\frac{4}{3}\pi k_\pm^3\right) &= N_\pm\left(\frac{\pi^3}{V}\right)\\
k_\pm &= \left(\frac{6\pi^2N_\pm}{V}\right)^{1/3}\\
dE &= \frac{\hbar^2k^2}{2m}\frac{V}{2\pi^2}k^2\,dk\\
&= \frac{\hbar^2V}{4\pi^2m}k^4\,dk\\
E_{\text{total}} &= \frac{\hbar^2V}{4\pi^2m}\left(\int_0^{k_+}k^4\,dk+\int_0^{k_-}k^4\,dk\right)\\
&= \frac{\hbar^2V}{20\pi^2m}\left[\left(\frac{6\pi^2N_+}{V}\right)^{5/3}+\left(\frac{6\pi^2N_-}{V}\right)^{5/3}\right]\\
&= \frac{\hbar^2}{20\pi^2m}[(6\pi^2N_+)^{5/3}+(6\pi^2N_-)^{5/3}]V^{-2/3}
\end{aligned}
$$

If $N_+ = N_- = N$ then

$$
E_{\text{total}} = \frac{\hbar^2(6\pi^2N)^{5/3}}{10\pi^2m}V^{-2/3}
$$

6 Symmetries & Conservation Laws

6.1

(a) $\psi(x, y, z)$ mirrored across the xy plane becomes $\psi(x, y, -z)$, then rotated 180° around the z axis becomes $\psi(-x, -y, -z)$.

(b) In spherical coordinates, mirorring across the xy plan turns θ into $\pi - \theta$ and rotating 180° around the z axis ϕ becomes $\phi + \pi$ so $\psi(r, \theta, \phi)$ becomes $\psi(r, \pi - \theta, \phi + \pi)$.

(c)

$$\begin{aligned}
\hat{\Pi}\psi_{n\ell m}(r, \theta, \phi) &= \hat{\Pi}[R_{n\ell}(r)Y_\ell^m(\theta, \phi)] \\
&= R_{n\ell}(r)Y_\ell^m(\pi - \theta, \phi + \pi) \\
&= (-1)^\ell R_{n\ell}(r)Y_\ell^m(\theta, \phi) \\
&= (-1)^\ell \psi_{n\ell m}(r, \theta, \phi)
\end{aligned}$$

6.2

$$\begin{aligned}
\hat{U}^\dagger &= \exp(i\hat{Q})^\dagger \\
&= \left(1 + i\hat{Q} - \frac{1}{2!}\hat{Q}^2 - \frac{1}{3!}i\hat{Q}^3 + \frac{1}{4!}\hat{Q}^4 + \cdots\right)^\dagger \\
&= 1 - i\hat{Q}^\dagger - \frac{1}{2!}(\hat{Q}^\dagger)^2 + \frac{1}{3!}i(\hat{Q}^\dagger)^3 + \frac{1}{4!}(\hat{Q}^\dagger)^4 + \cdots \\
&= 1 - i\hat{Q} - \frac{1}{2!}\hat{Q}^2 + \frac{1}{3!}i\hat{Q}^3 + \frac{1}{4!}\hat{Q}^4 + \cdots \\
&= \exp(-i\hat{Q}) \\
\hat{U}^\dagger\hat{U} &= \exp(-i\hat{Q})\exp(i\hat{Q}) \\
&= 1
\end{aligned}$$

6.3

$$\begin{aligned}
\hat{p}'f(x) &= \hat{T}(a)^\dagger\hat{p}\hat{T}(a)f(x) \\
&= \hat{T}(-a)\left(-i\hbar\frac{d}{dx}\right)f(x - a) \\
&= \hat{T}(-a)[-i\hbar f'(x - a)] \\
&= -i\hbar f'(x) \\
&= \hat{p}
\end{aligned}$$

6.4

$$\begin{aligned}\hat{Q}'(\hat{x},\hat{p})f(x) &= \hat{T}(a)^\dagger \hat{Q}(\hat{x},\hat{p})\hat{T}(a)f(x)\\
&= \hat{T}(-a)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}\hat{x}^m\hat{p}^m\right) f(x-a)\\
&= \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}(\hat{x}+a)^m\hat{p}^m\right) f(x)\\
&= \hat{Q}(\hat{x}+a,\hat{p})f(x)\end{aligned}$$

6.5

$$\begin{aligned}\psi(x) &= e^{iqx}u(x)\\
\psi(x-a) &= e^{-iqa}\psi(x)\\
e^{iq(x-a)}u(x-a) &= e^{-iqa}e^{iqx}u(x)\\
&= e^{iq(x-a)}u(x)\\
u(x-a) &= u(x)\end{aligned}$$

6.7

(a)

$$\begin{aligned}\hat{T}(a)\psi(x_1,x_2) &= \psi(x_1-a, x_2-a)\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}(-a)^n\frac{d^n}{dx_1^n}\psi(x_1,x_2-a)\\
&= \exp\left(-\frac{ia}{\hbar}\hat{p}_1\right)\psi(x_1,x_2-a)\\
&= \exp\left(-\frac{ia}{\hbar}\hat{p}_1\right)\sum_{n=0}^{\infty}\frac{1}{n!}(-a)^n\frac{d^n}{dx_2^n}\psi(x_1,x_2)\\
&= \exp\left[-\frac{ia}{\hbar}(\hat{p}_1+\hat{p}_2)\right]\psi(x_1,x_2)\\
&= \exp\left(-\frac{ia}{\hbar}\hat{P}\right)\psi(x_1,x_2)\end{aligned}$$

6.8

(a)

$$\begin{aligned}
\langle \phi | \hat{\Pi} \psi \rangle &= \int_{-\infty}^{\infty} \phi^*(x) \hat{\Pi} \psi(x)\, dx \\
&= \int_{-\infty}^{\infty} \phi^*(x) \psi(-x)\, dx \\
u &= -x \\
du &= -dx \\
\langle \phi | \hat{\Pi} \psi \rangle &= \int_{\infty}^{-\infty} \phi^*(-u) \psi(u) (-du) \\
&= \int_{-\infty}^{\infty} \phi^*(-u) \psi(u)\, du \\
&= \int_{-\infty}^{\infty} \hat{\Pi} \phi^*(u) \psi(u)\, du \\
&= \langle \hat{\Pi} \phi | \psi \rangle \\
\hat{\Pi} &= \hat{\Pi}^\dagger
\end{aligned}$$

(b)

$$\begin{aligned}
\hat{\Pi} f(x) &= \lambda f(x) \\
f(-x) &= \lambda f(x) \\
\hat{\Pi} f(-x) &= \lambda f(-x) \\
f(x) &= \lambda f(-x) \\
f(x) &= \lambda(\lambda f(x)) \\
&= \lambda^2 f(x) \\
\lambda &= \pm 1
\end{aligned}$$

6.9

(a)

$$\begin{aligned}
\hat{\Pi}^\dagger \hat{f} \hat{\Pi} &= \hat{f} \\
\hat{f} \hat{\Pi} &= \hat{\Pi} \hat{f} \\
[\hat{\Pi}, \hat{f}] &= \hat{\Pi} \hat{f} - \hat{f} \hat{\Pi} \\
&= 0 \\
\hat{\Pi}^\dagger \hat{g} \hat{\Pi} &= -\hat{g} \\
\hat{g} \hat{\Pi} &= -\hat{\Pi} \hat{g} \\
\{\hat{\Pi}, \hat{g}\} &= \hat{\Pi} \hat{g} + \hat{g} \hat{\Pi} \\
&= 0
\end{aligned}$$

(b)

$$
\begin{aligned}
\hat{\Pi}^\dagger \hat{\mathbf{V}} \hat{\Pi} &= -\hat{\mathbf{V}} \\
\hat{\mathbf{V}} \hat{\Pi} &= -\hat{\Pi} \hat{\mathbf{V}} \\
\{\hat{\Pi}, \hat{\mathbf{V}}\} &= \hat{\Pi} \hat{\mathbf{V}} + \hat{\mathbf{V}} \hat{\Pi} \\
&= 0 \\
\hat{\Pi}^\dagger \hat{\mathbf{W}} \hat{\Pi} &= \hat{\mathbf{W}} \\
\hat{\mathbf{W}} \hat{\Pi} &= \hat{\Pi} \hat{\mathbf{W}} \\
[\hat{\Pi}, \hat{\mathbf{W}}] &= \hat{\Pi} \hat{\mathbf{W}} - \hat{\mathbf{W}} \hat{\Pi} \\
&= 0
\end{aligned}
$$

6.10

$$
\begin{aligned}
\hat{x}' |x\rangle &= \hat{\Pi}^\dagger \hat{x} \hat{\Pi} |x\rangle \\
&= \hat{\Pi}^\dagger \hat{x} |-x\rangle \\
&= -x \hat{\Pi}^\dagger |-x\rangle \\
&= -x \hat{\Pi} |-x\rangle \\
&= -x |x\rangle \\
&= -\hat{x} |x\rangle \\
\hat{x}' &= -\hat{x} \\
\hat{p}' |p\rangle &= \hat{\Pi}^\dagger \hat{p} \hat{\Pi} |p\rangle \\
&= \hat{\Pi}^\dagger \hat{p} |-p\rangle \\
&= -p \hat{\Pi}^\dagger |-p\rangle \\
&= -p \hat{\Pi} |-p\rangle \\
&= -p |p\rangle \\
&= -\hat{p} |p\rangle \\
\hat{p}' &= -\hat{p}
\end{aligned}
$$

6.11

$$
\begin{aligned}
\langle n'\ell' m' | \hat{\mathbf{L}} | n\ell m\rangle &= \langle n'\ell' m' | \hat{\Pi}^\dagger \hat{\mathbf{L}} \hat{\Pi} | n\ell m\rangle \\
&= \langle n'\ell' m' | (-1)^{\ell'} \hat{\mathbf{L}} (-1)^{\ell} | n\ell m\rangle \\
&= (-1)^{\ell+\ell'} \langle n'\ell' m' | \hat{\mathbf{L}} | n\ell m\rangle
\end{aligned}
$$

So $\langle n'\ell' m | \hat{\mathbf{L}} | n\ell m\rangle = \mathbf{0}$ if $\ell + \ell'$ is odd.

6.12

$$
\begin{aligned}
[\hat{\Pi}, \hat{S}_x] &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \frac{\hbar}{2} \left[\begin{pmatrix} b & a \\ d & c \end{pmatrix} - \begin{pmatrix} c & d \\ a & b \end{pmatrix} \right] \\
&= \frac{\hbar}{2} \begin{pmatrix} (b-c) & (a-d) \\ (d-a) & (c-b) \end{pmatrix} \\
&= \mathbf{0} \\
a &= d \\
b &= c \\
[\hat{\Pi}, \hat{S}_y] &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{i\hbar}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{i\hbar}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \frac{i\hbar}{2} \left[\begin{pmatrix} b & -a \\ d & -c \end{pmatrix} - \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \right] \\
&= \frac{i\hbar}{2} \begin{pmatrix} (b+c) & (d-a) \\ (d-a) & -(b+c) \end{pmatrix} \\
&= \mathbf{0} \\
a &= d \\
b + c &= 0 \\
b &= 0 \\
c &= 0 \\
\hat{\Pi} &= \pi \mathbf{I}
\end{aligned}
$$

6.13

(a)

$$\langle \mathbf{p}_e \rangle = \langle \psi_{100} | \hat{\mathbf{p}}_e | \psi_{100} \rangle$$

$\ell + \ell' = 0 + 0 = 0$ is even so by Laporte's rule $\langle \mathbf{p}_e \rangle = \mathbf{0}$.

(b)

$$
\begin{aligned}
\psi &= \frac{1}{\sqrt{2}}(\psi_{200} + \psi_{210}) \\
\langle \hat{\mathbf{p}}_e \rangle &= \langle \psi | \hat{\mathbf{p}}_e | \psi \rangle \\
&= \left\langle \frac{1}{\sqrt{2}}(\psi_{200} + \psi_{210}) \middle| \hat{\mathbf{p}}_e \middle| \frac{1}{\sqrt{2}}(\psi_{200} + \psi_{210}) \right\rangle \\
&= \frac{1}{2}(\langle \psi_{200} | \hat{\mathbf{p}}_e | \psi_{200} \rangle + \langle \psi_{200} | \hat{\mathbf{p}}_e | \psi_{210} \rangle + \langle \psi_{210} | \hat{\mathbf{p}}_e | \psi_{200} \rangle + \langle \psi_{210} | \hat{\mathbf{p}}_e | \psi_{210} \rangle) \\
&= \frac{1}{2}(\langle \psi_{200} | \hat{\mathbf{p}}_e | \psi_{210} \rangle + \langle \psi_{210} | \hat{\mathbf{p}}_e | \psi_{200} \rangle) \\
&= -\frac{e}{2}(\langle \psi_{200} | \hat{\mathbf{r}} | \psi_{210} \rangle + \langle \psi_{210} | \hat{\mathbf{r}} | \psi_{200} \rangle)
\end{aligned}
$$

6.15

$$
\begin{aligned}
\hat{f}' &= \hat{R}^\dagger \hat{f} \hat{R} \\
&= \exp\left(\frac{i\varphi}{\hbar}\hat{\mathbf{n}} \cdot \hat{\mathbf{L}}\right) \hat{f} \exp\left(-\frac{i\varphi}{\hbar}\hat{\mathbf{n}} \cdot \hat{\mathbf{L}}\right) \\
&\approx \left(1 + \frac{i\delta}{\hbar}\hat{\mathbf{n}} \cdot \hat{\mathbf{L}}\right) \hat{f} \left(1 - \frac{i\delta}{\hbar}\hat{\mathbf{n}} \cdot \hat{\mathbf{L}}\right) \\
&= \hat{f} + \frac{i\delta}{\hbar}\hat{\mathbf{n}} \cdot [\hat{\mathbf{L}}, \hat{f}] \\
&= \hat{f}
\end{aligned}
$$

6.16

$$
\begin{aligned}
[\hat{L}_i, \hat{V}_j] &= i\hbar\epsilon_{ijk}\hat{V}_k \\
\epsilon_{ijk} &= \begin{cases} 1 & ijk = 123,\ 231,\ 312 \\ -1 & ijk = 132,\ 213,\ 321 \\ 0 & \text{otherwise} \end{cases} \\
\hat{R}_{\hat{\mathbf{y}}}(\delta) &= \exp\left(-\frac{i\delta}{\hbar}\hat{\mathbf{y}}\cdot\hat{\mathbf{L}}\right) \\
&= 1 - \frac{i\delta}{\hbar}\hat{L}_y \\
\hat{\mathbf{V}}' &= \hat{R}_{\hat{\mathbf{y}}}(\delta)^\dagger\hat{\mathbf{V}}\hat{R}_{\hat{\mathbf{y}}}(\delta) \\
&= \left(1 + \frac{i\delta}{\hbar}\hat{L}_y\right)\hat{\mathbf{V}}\left(1 - \frac{i\delta}{\hbar}\hat{L}_y\right) \\
&\approx \hat{\mathbf{V}} + \frac{i\delta}{\hbar}[\hat{L}_y, \hat{\mathbf{V}}] \\
&= \begin{pmatrix} \hat{V}_x + \frac{i\delta}{\hbar}[\hat{L}_y, \hat{V}_x] \\ \hat{V}_y + \frac{i\delta}{\hbar}[\hat{L}_y, \hat{V}_y] \\ \hat{V}_z + \frac{i\delta}{\hbar}[\hat{L}_y, \hat{V}_z] \end{pmatrix} \\
&= \begin{pmatrix} \hat{V}_x + \delta\hat{V}_z \\ \hat{V}_y \\ \hat{V}_z - \delta V_x \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & \delta \\ 0 & 1 & 0 \\ -\delta & 0 & 1 \end{pmatrix}\hat{\mathbf{V}} \\
D &= \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix}
\end{aligned}
$$

6.18

(a)

$$
\begin{aligned}
[\hat{\Pi}, \hat{T}(a)]f(x) &= \hat{\Pi}\hat{T}(a)f(x) - \hat{T}(a)\hat{\Pi}f(x) \\
&= \hat{\Pi}f(x-a) - \hat{T}(a)f(-x) \\
&= f(-x+a) - f(-x-a)
\end{aligned}
$$

(b)

$$
\begin{aligned}
f_p(x) &= \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \\
\hat{\Pi} f_p(x) &= \frac{1}{\sqrt{2\pi\hbar}} e^{ip(-x)/\hbar} \\
&= f_{-p}(x)
\end{aligned}
$$

(c)

$$
\begin{aligned}
\hat{T}(a)\left[\frac{1}{\sqrt{\pi\hbar}}\cos\left(\frac{px}{\hbar}\right)\right] &= \frac{1}{\sqrt{\pi\hbar}}\cos\left(\frac{p(x-a)}{\hbar}\right) \\
&= \frac{1}{\sqrt{\pi\hbar}}\cos\left(\frac{px}{\hbar}-\frac{pa}{\hbar}\right) \\
-\frac{pa}{\hbar} &= \arctan\left(-\frac{b}{a}\right) \\
\tan\left(-\frac{pa}{\hbar}\right) &= -\frac{b}{a} \\
b &= -a\tan\left(-\frac{pa}{\hbar}\right) \\
\frac{1}{\sqrt{\pi\hbar}} &= \sqrt{a^2+b^2} \\
\frac{1}{\pi\hbar} &= a^2+b^2 \\
&= a^2 + a^2\tan^2\left(-\frac{pa}{\hbar}\right) \\
&= a^2\left[1+\tan^2\left(-\frac{pa}{\hbar}\right)\right] \\
&= a^2\sec^2\left(-\frac{pa}{\hbar}\right) \\
a &= \frac{1}{\sqrt{\pi\hbar}}\cos\left(-\frac{pa}{\hbar}\right) \\
\frac{1}{\pi\hbar} &= \frac{1}{\pi\hbar}\cos^2\left(-\frac{pa}{\hbar}\right) + b^2 \\
b &= \frac{1}{\sqrt{\pi\hbar}}\sin\left(-\frac{pa}{\hbar}\right) \\
\hat{T}(a)\left[\frac{1}{\sqrt{\pi\hbar}}\cos\left(\frac{px}{\hbar}\right)\right] &= \frac{1}{\sqrt{\pi\hbar}}\cos\left(-\frac{pa}{\hbar}\right)\cos\left(\frac{px}{\hbar}\right) \\
&\quad + \frac{1}{\sqrt{\pi\hbar}}\sin\left(-\frac{pa}{\hbar}\right)\sin\left(\frac{px}{\hbar}\right)
\end{aligned}
$$

6.19

(a)

$$
\begin{aligned}
[\hat{L}_i, \hat{V}_j] &= i\hbar\epsilon_{ijk}\hat{V}_k \\
\epsilon_{ijk} &= \begin{cases} 1 & ijk = 123,\ 231,\ 312 \\ -1 & ijk = 132,\ 213,\ 321 \\ 0 & \text{otherwise} \end{cases} \\
\hat{V}_\pm &= \hat{V}_x \pm i\hat{V}_y \\
[\hat{L}_z, \hat{V}_\pm] &= [\hat{L}_z, \hat{V}_x \pm i\hat{V}_y] \\
&= [\hat{L}_z, \hat{V}_x] \pm i[\hat{L}_z, \hat{V}_y] \\
&= i\hbar\hat{V}_y \pm i(-i\hbar\hat{V}_x) \\
&= \hbar(\pm\hat{V}_x + i\hat{V}_y) \\
[\hat{L}_z, \hat{V}_+] &= \hbar\hat{V}_+ \\
[\hat{L}_z, \hat{V}_-] &= -\hbar\hat{V}_-
\end{aligned}
$$

6.20

$$
\begin{aligned}
0 &= \langle n'\ell'm'|[\hat{L}_-, \hat{f}]|n\ell m\rangle \\
&= \langle n'\ell'm'|\hat{L}_-\hat{f}|n\ell m\rangle - \langle n'\ell'm'|\hat{f}\hat{L}_-|n\ell m\rangle \\
&= A_{\ell'}^{m'}\,\langle n'\ell'(m'+1)|\hat{f}|n\ell m\rangle - B_\ell^m\,\langle n'\ell'm'|\hat{f}|n\ell(m-1)\rangle \\
\ell' &= \ell \\
m' &= m-1 \\
A_{\ell'}^{m'} &= \hbar\sqrt{\ell'(\ell'+1) - m'(m'+1)} \\
&= \hbar\sqrt{\ell(\ell+1) - m(m-1)} \\
B_\ell^m &= \hbar\sqrt{\ell(\ell+1) - m(m-1)} \\
\langle n'\ell m|\hat{f}|n\ell m\rangle &= \langle n'\ell(m-1)|\hat{f}|n\ell(m-1)\rangle \\
\langle n'\ell m|\hat{f}|n\ell m\rangle &= \langle n'\ell(m+1)|\hat{f}|n\ell(m+1)\rangle
\end{aligned}
$$

6.21

$$
\begin{aligned}
\psi &= \frac{1}{\sqrt{2}}(\psi_{211} + \psi_{21-1}) \\
\langle\psi|r|\psi\rangle &= \frac{1}{2}(\langle\psi_{211}| + \langle\psi_{21-1}|)r(|\psi_{211} + |\psi_{21-1}\rangle\rangle) \\
&= \frac{1}{2}(\langle\psi_{211}|r|\psi_{211}\rangle + \langle\psi_{211}|r|\psi_{21-1}\rangle + \langle\psi_{21-1}|r|\psi_{211}\rangle + \langle\psi_{21-1}|r|\psi_{21-1}\rangle) \\
&= \frac{1}{2}(\langle\psi_{211}|r|\psi_{211}\rangle + \langle\psi_{21-1}|r|\psi_{21-1}\rangle) \\
&= \langle 21||r||21\rangle \\
&= 30a^2
\end{aligned}
$$

6.22

(a)

$$
\begin{aligned}
[\hat{L}_i, \hat{V}_j] &= i\hbar\epsilon_{ijk}\hat{V}_k \\
\epsilon_{ijk} &= \begin{cases} 1 & ijk = 123,\ 231,\ 312 \\ -1 & ijk = 132,\ 213,\ 321 \\ 0 & \text{otherwise} \end{cases} \\
[\hat{L}_z, \hat{V}_z] &= i\hbar\epsilon_{11k}\hat{V}_k \\
&= 0 \\
[\hat{L}_z, \hat{V}_\pm] &= [\hat{L}_z, \hat{V}_x \pm i\hat{V}_y] \\
&= [\hat{L}_z, \hat{V}_x] \pm i[\hat{L}_z, \hat{V}_y] \\
&= i\hbar\hat{V}_y \pm i(-i\hbar\hat{V}_x) \\
&= \hbar(\pm\hat{V}_x + i\hat{V}_y) \\
&= \pm\hbar\hat{V}_\pm \\
[\hat{L}_\pm, \hat{V}_\pm] &= [\hat{L}_x \pm i\hat{L}_y, \hat{V}_x \pm i\hat{V}_y] \\
&= [\hat{L}_x \pm i\hat{L}_y, \hat{V}_x] \pm i[\hat{L}_x \pm i\hat{L}_y, \hat{V}_y] \\
&= [\hat{L}_x, \hat{V}_x] \pm i[\hat{L}_y, \hat{V}_x] \pm i[\hat{L}_x, \hat{V}_y] - [\hat{L}_y, \hat{V}_y] \\
&= \pm i(-i\hbar\hat{V}_z + i\hbar\hat{V}_z) \\
&= 0 \\
[\hat{L}_\pm, \hat{V}_z] &= [\hat{L}_x \pm i\hat{L}_y, \hat{V}_z] \\
&= [\hat{L}_x, \hat{V}_z] \pm i[\hat{L}_y, \hat{V}_z] \\
&= -i\hbar\hat{V}_y \pm i(i\hbar\hat{V}_x) \\
&= \hbar(\mp\hat{V}_x - i\hat{V}_y) \\
&= \mp\hbar\hat{V}_\pm \\
[\hat{L}_\pm, \hat{V}_\mp] &= [\hat{L}_x \pm i\hat{L}_y, \hat{V}_x \mp i\hat{V}_y] \\
&= [\hat{L}_x \pm i\hat{L}_y, \hat{V}_x] \mp i[\hat{L}_x \pm i\hat{L}_y, \hat{V}_y] \\
&= [\hat{L}_x, \hat{V}_x] \pm i[\hat{L}_y, \hat{V}_x] \mp i[\hat{L}_x, \hat{V}_y] + [\hat{L}_y, \hat{V}_y] \\
&= \pm i([\hat{L}_y, \hat{V}_x] - [\hat{L}_x, \hat{V}_y]) \\
&= \pm i(-i\hbar\hat{V}_z - i\hbar\hat{V}_z) \\
&= \pm 2\hbar\hat{V}_z
\end{aligned}
$$

(b)

$$
\begin{aligned}
0 &= \langle n'\ell'm'|[\hat{L}_z, \hat{V}_z]|n\ell m\rangle \\
&= \langle n'\ell'm'|\hat{L}_z\hat{V}_z|n\ell m\rangle - \langle n'\ell'm'|\hat{V}_z\hat{L}_z|n\ell m\rangle \\
&= \hbar m' \langle n'\ell'm'|\hat{V}_z|n\ell m\rangle - \hbar m \langle n'\ell'm'|\hat{V}_z|n\ell m\rangle \\
&= (m' - m) \langle n'\ell'm'|\hat{V}_z|n\ell m\rangle
\end{aligned}
$$

so $\langle n'\ell'm'|\hat{V}_z|n\ell m\rangle = 0$ unless $m' = m$.

6.26

$$
\begin{aligned}
\hat{p}_H(t)\psi_n(x) &= \hat{U}^\dagger(t)\hat{p}\hat{U}(t)\psi_n(x) \\
&= e^{i\hat{H}t/\hbar} i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)e^{-i\hat{H}t/\hbar}\psi_n(x) \\
&= \sqrt{\frac{\hbar m\omega}{2}} e^{-iE_n t/\hbar} e^{i\hat{H}t/\hbar}(\hat{a}_+ - \hat{a}_-)\psi_n(x) \\
&= \sqrt{\frac{\hbar m\omega}{2}} e^{-iE_n t/\hbar} e^{i\hat{H}t/\hbar}[\sqrt{n+1}\psi_{n+1}(x) - \sqrt{n}\psi_{n-1}(x)] \\
&= \sqrt{\frac{\hbar m\omega}{2}} e^{-iE_n t/\hbar}[\sqrt{n+1}e^{iE_{n+1}t/\hbar}\psi_{n+1}(x) \\
&\quad - \sqrt{n}e^{iE_{n-1}t/\hbar}\psi_{n-1}(x)] \\
&= \sqrt{\frac{\hbar m\omega}{2}}[\sqrt{n+1}e^{i\omega t}\psi_{n+1}(x) - \sqrt{n}e^{-i\omega t}\psi_{n-1}(x)] \\
\hat{p}_H &= \sqrt{\frac{\hbar m\omega}{2}}(e^{i\omega t}\hat{a}_+ - e^{-i\omega t}\hat{a}_-)
\end{aligned}
$$