# Vibrations and Waves by A. P. French Problems

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## Contents

# 1 Periodic motions

## 1.4

(a)

$$\begin{aligned} z &= Ae^{j\theta} \\ dz &= jAe^{j\theta}\,d\theta \\ &= jz\,d\theta \end{aligned}$$

The motion of the point is always perpendicular to its position.

(b)

$$\begin{aligned} |2+j\sqrt{3}| &= \sqrt{2^2+\sqrt{3}^2} \\ &= \sqrt{7} \\ \arg(2+j\sqrt{3}) &= \arctan\frac{\sqrt{3}}{2} \\ &= 41^\circ \end{aligned}$$

$$\begin{aligned} (2-j\sqrt{3})^2 &= 4-j4\sqrt{3}-3 \\ &= 1-j4\sqrt{3} \\ |1-j4\sqrt{3}| &= \sqrt{1^2+(4\sqrt{3})^2} \\ &= 7 \\ \arg(1-j4\sqrt{3}) &= -\arctan 4\sqrt{3} \end{aligned}$$

## 1.9

$$\begin{aligned}
\cos\theta + j\sin\theta &= e^{j\theta} \\
\cos\frac{\pi}{2} + j\sin\frac{\pi}{2} &= e^{j\frac{\pi}{2}} \\
j &= e^{j\frac{\pi}{2}} \\
j^j &= (e^{j\frac{\pi}{2}})^j \\
&= e^{-\frac{\pi}{2}} \\
&\approx 0.208
\end{aligned}$$

Yes, I would be willing to pay 20 cents because I could sell it to the mathematician and gain 0.8 cents.

## 1.10

$$\begin{aligned}
y &= A\cos kx + B\sin kx \\
\frac{dy}{dx} &= -Ak\sin kx + Bk\cos kx \\
\frac{d^2y}{dx^2} &= -Ak^2\cos kx - Bk^2\sin kx \\
&= -k^2 y
\end{aligned}$$

$$\begin{aligned}
C &= \sqrt{A^2 + B^2} \\
\alpha &= \arctan\left(-\frac{B}{A}\right) \\
y &= C\cos(kx + \alpha) \\
&= C\,\mathrm{Re}[e^{j(kx+\alpha)}] \\
&= Re[(Ce^{j\alpha})e^{jkx}]
\end{aligned}$$

## 1.11

(a)

$$\begin{aligned}
x &= A\cos(\omega t + \alpha) \\
A &= 5\,\mathrm{cm} \\
f &= 1\,\mathrm{Hz} \\
\omega &= 2\pi f \\
&= 2\pi\,\mathrm{rad/s} \\
\alpha &= \pm\frac{\pi}{2}
\end{aligned}$$

(b)

$$\begin{aligned}
x\left(\frac{8}{3}\right) &= 5\cos\left(2\pi\frac{8}{3}+\alpha\right)\\
&= \pm 4.33\,\mathrm{cm}\\
\frac{dx}{dt} &= -A\omega\sin(\omega t+\alpha)\\
\frac{dx}{dt}\left(\frac{8}{3}\right) &= \pm 15.7\,\mathrm{cm/s}\\
\frac{d^2x}{dt^2} &= -A\omega^2\cos(\omega t+\alpha)\\
\frac{d^2x}{dt^2}\left(\frac{8}{3}\right) &= \mp 171\,\mathrm{cm/s^2}
\end{aligned}$$

## 1.12

(a)

$$\begin{aligned}
v &= 50\,\mathrm{cm/s}\\
T &= 6\,\mathrm{s}\\
\theta_0 &= 30^\circ\\
c &= 300\,\mathrm{cm}\\
A &= \frac{c}{2\pi}\\
&= \frac{150}{\pi}\,\mathrm{cm}\\
\omega &= \frac{2\pi}{T}\\
&= \frac{\pi}{3}\,\mathrm{rad/s}\\
\alpha &= \frac{\pi}{6}\,\mathrm{rad}\\
x &= \frac{150}{\pi}\cos\left(\frac{\pi}{3}t+\frac{\pi}{6}\right)
\end{aligned}$$

(b)

$$\begin{aligned}
x(2\,\mathrm{s}) &= -41.3\,\mathrm{cm}\\
\frac{dx}{dt} &= -50\sin\left(\frac{\pi}{3}t+\frac{\pi}{6}\right)\\
\frac{dx}{dt}(2\,\mathrm{s}) &= -25\,\mathrm{cm/s}\\
\frac{d^2x}{dt^2} &= -\frac{50\pi}{3}\cos\left(\frac{\pi}{3}t+\frac{\pi}{6}\right)\\
\frac{d^2x}{dt^2}(2\,\mathrm{s}) &= 45\,\mathrm{cm/s^2}
\end{aligned}$$

## 2 The superposition of periodic motions

### 2.1

(a)

$$\begin{aligned}
z &= \sin \omega t + \cos \omega t \\
&= \sqrt{2} \cos\left(\omega t - \frac{\pi}{4}\right) \\
&= \sqrt{2} e^{j\left(\omega t - \frac{\pi}{4}\right)}
\end{aligned}$$

(b)

$$\begin{aligned}
z &= \cos(\omega t - \pi/3) - \cos \omega t \\
&= \cos \omega t \cos \frac{\pi}{3} + \sin \omega t \sin \frac{\pi}{3} - \cos \omega t \\
&= -\frac{1}{2} \cos \omega t + \frac{\sqrt{3}}{2} \sin \omega t \\
&= \cos(\omega t + 2\pi/3) \\
&= e^{j(\omega t + 2\pi/3)}
\end{aligned}$$

(c)

$$\begin{aligned}
z &= 3 \cos \omega t + 2 \sin \omega t \\
&= \sqrt{13} \cos(\omega t + \arctan -2/3)
\end{aligned}$$

(d)

$$\begin{aligned}
z &= \sin \omega t - 2 \cos(\omega t - \pi/4) + \cos \omega t \\
&= \sin \omega t - 2(\cos \omega t \cos \pi/4 + \sin \omega t \sin \pi/4) + \cos \omega t \\
&= \sin \omega t - \sqrt{2} \cos \omega t - \sqrt{2} \sin \omega t + \cos \omega t \\
&= (1 - \sqrt{2}) \cos \omega t + (1 - \sqrt{2}) \sin \omega t \\
&= (1 - \sqrt{2}) \sqrt{2} \cos(\omega t - \pi/4) \\
&= (\sqrt{2} - 2) \cos(\omega t - \pi/4) \\
&= (2 - \sqrt{2}) \cos(\omega t + 3\pi/4)
\end{aligned}$$

### 2.2

$$
\begin{aligned}
x &= A_1 \cos\omega t + A_2 \cos(\omega t + \alpha_1) + A_3 \cos(\omega t + \alpha_1 + \alpha_2) \\
&= A_1 \cos\omega t + A_2(\cos\omega t \cos\alpha_1 - \sin\omega t \sin\alpha_1) \\
&\qquad + A_3(\cos\omega t \cos(\alpha_1 + \alpha_2) - \sin\omega t \sin(\alpha_1 + \alpha_2)) \\
&= (A_1 + A_2 \cos\alpha_1 + A_3 \cos(\alpha_1 + \alpha_2)) \cos\omega t \\
&\qquad - (A_2 \sin\alpha_1 + A_3 \sin(\alpha_1 + \alpha_2)) \sin\omega t \\
A &= \sqrt{(A_1 + A_2 \cos\alpha_1 + A_3 \cos(\alpha_1 + \alpha_2))^2 + (A_2 \sin\alpha_1 + A_3 \sin(\alpha_1 + \alpha_2))^2} \\
&\approx 0.52\,\mathrm{mm} \\
\alpha &= \arctan \frac{A_2 \sin\alpha_1 + A_3 \sin(\alpha_1 + \alpha_2)}{A_1 + A_2 \cos\alpha_1 + A_3 \cos(\alpha_1 + \alpha_2)} \\
&\approx 0.59\,\mathrm{rad} \\
&\approx 34^\circ
\end{aligned}
$$

### 2.3

The equation of motion is

$$x = 2A \cos\left(\frac{12\pi - 10\pi}{2} t\right) \cos\left(\frac{12\pi + 10\pi}{2} t\right)$$

with the variation in amplitude given by the term

$$2A \cos \pi t$$

so the beat period is $1\,\mathrm{s}$.

### 2.4

(a)

$$\omega = 2\pi, \mathrm{rad/s} \Rightarrow f = 1\,\mathrm{Hz}$$

(b)

$$\omega = \frac{25\pi}{2}\,\mathrm{rad/s} \Rightarrow f = \frac{25}{4}\,\mathrm{Hz}$$

(c)

$$\omega = \frac{3 + \pi}{2}\,\mathrm{rad/s} \Rightarrow f = \frac{3 + \pi}{4\pi}\,\mathrm{Hz}$$

# 3 The free vibrations of physical systems

## 3.1

$$
\begin{aligned}
F &= -kx \\
ma &= -kx \\
k &= -\frac{ma}{x} \\
&= 4.0 \times 10^{-5}\,\mathrm{N/m}
\end{aligned}
$$

## 3.2

(a)

$$T_0 = 2\pi\sqrt{\frac{m}{k}}$$

(b) (i)

$$
\begin{aligned}
mx'' &= -2kx \\
x'' &= -\frac{2k}{m}x \\
T &= 2\pi\sqrt{\frac{m}{2k}} \\
&= \frac{T_0}{\sqrt{2}}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
mx'' &= -k\frac{x}{2} \\
x'' &= -\frac{k}{2m}x \\
T &= 2\pi\sqrt{\frac{2m}{k}} \\
&= \sqrt{2}T_0
\end{aligned}
$$

## 3.3

(a)

$$\begin{aligned}
y &= A\cos\omega t \\
y' &= -\omega A\sin\omega t \\
y'' &= -\omega^2 A\cos\omega t \\
g &= \omega^2 A\cos\omega t \\
\omega t &= \arccos\frac{g}{\omega^2 A} \\
t &= \frac{1}{\omega}\arccos\frac{g}{\omega^2 A} \\
y &= A\cos\arccos\frac{g}{\omega^2 A} \\
&= \frac{g}{\omega^2} \\
&= 2.5\,\mathrm{cm}
\end{aligned}$$

(b)

$$\begin{aligned}
v &= -\omega A\sin\omega t \\
&= -\omega A\sin\arccos\frac{g}{\omega^2 A} \\
&\approx 0.87\,\mathrm{m/s} \\
\frac{1}{2}mv^2 &= mgh \\
h &= \frac{v^2}{2g} \\
&\approx 3.8\,\mathrm{cm} \\
\Delta h &\approx 1.3\,\mathrm{cm}
\end{aligned}$$

## 3.4

(a)

$$\begin{aligned}
my'' &= -g\rho Ay \\
y'' &= -\frac{g\rho A}{m}y \\
\omega &= \sqrt{\frac{g\rho A}{m}} \\
&= \sqrt{\frac{g}{l}}
\end{aligned}$$

## 3.5

$$T = 2\pi\sqrt{\frac{2L}{3g}}$$

## 3.6

$$T = 2\pi\sqrt{\frac{d}{g}}$$

## 3.8

(a)

$$\begin{aligned}
mg &= \frac{AY}{l_0}x \\
x &= \frac{mgl_0}{AY} \\
&= \frac{mgl_0}{\pi(d/2)^2Y} \\
&= 0.25\,\mathrm{mm}
\end{aligned}$$

(b)

$$\begin{aligned}
F_u &= u\pi(d/2)^2 \\
&\approx 215.98\,\mathrm{N} \\
k &= \frac{AY}{L} \\
&= \frac{\pi(d/2)^2Y}{L} \\
&= \frac{\pi d^2Y}{4L} \\
&\approx 19\,634.95\,\mathrm{N/m} \\
F_u &= kx_u \\
x_u &= \frac{F_u}{k} \\
&\approx 1.1\,\mathrm{cm} \\
mgh &= \frac{1}{2}\frac{AY}{L}x_u^2 - mgx_u \\
h &= \frac{\pi(d/2)^2Yx_u^2}{2mgL} - x_u \\
&= \frac{\pi d^2Yx_u^2}{8mgL} - x_u \\
&= 0.23\,\mathrm{m}
\end{aligned}$$

### 3.9

(a)

$$
\begin{aligned}
\rho_{\text{steel}} &= 7850\,\mathrm{kg/m^3} \\
V_{\text{sphere}} &= \frac{4}{3}\pi r^3 \\
F_u &= Au \\
&= \pi r^2 u \\
&\approx 3455.75\,\mathrm{N} \\
mg &= F_u \\
m &= \frac{F_u}{g} \\
&\approx 352.3\,\mathrm{kg} \\
\rho V &= m \\
\rho \frac{4}{3}\pi r^3 &= m \\
r &= \sqrt[3]{\frac{3m}{4\pi\rho}} \\
&= 22\,\mathrm{cm}
\end{aligned}
$$

(b)

$$
\begin{aligned}
M &= -\frac{\pi n r^4}{2l}\theta \\
c &= \frac{\pi n r^4}{2l} \\
T &= 2\pi\sqrt{\frac{I}{c}} \\
&= 2\pi\sqrt{\frac{2MR^2/5}{\pi n r^4/2l}} \\
&= 2\pi\sqrt{\frac{4lMR^2}{5\pi n r^4}} \\
&= 66\,\mathrm{s}
\end{aligned}
$$

## 3.10

(a)

$$
\begin{aligned}
Y &= \frac{\text{stress}}{\text{strain}} \\
&= \frac{F/A}{\Delta l/l_0} \\
&= \frac{mg/A}{\Delta l/l_0} \\
&= \frac{mgl_0}{\Delta l A} \\
&= 5.9 \times 10^{11}\,\mathrm{N/m^2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
y &= \frac{4L^3}{Yab^3}F \\
F &= \frac{Yab^3}{4L^3}y \\
k &= \frac{Yab^3}{4L^3} \\
\omega_y &= \sqrt{\frac{k}{m}} \\
&= \sqrt{\frac{Yab^3}{4L^3m}} \\
\omega_x &= \sqrt{\frac{Ya^3b}{4L^3m}} \\
\frac{\omega_y}{\omega_x} &= \sqrt{\frac{ab^3}{a^3b}} \\
&= \frac{b}{a}
\end{aligned}
$$

(c) 3/2

## 3.11

(a)

$$\omega = \sqrt{\frac{A\gamma p}{lm}}$$

## 3.14

(a)

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

(b)

$$\begin{aligned}
\omega &= \frac{\sqrt{3}}{2}\omega_0 \\
\omega^2 &= \frac{3}{4}\omega_0^2 \\
\omega_0^2 - \frac{\gamma^2}{4} &= \frac{3}{4}\omega_0^2 \\
\frac{1}{4}\omega_0^2 &= \frac{\gamma^2}{4} \\
\omega_0^2 &= \gamma^2 \\
\omega_0 &= \gamma \\
&= \frac{b}{m} \\
b &= m\omega_0 \\
&= m\sqrt{\frac{k}{m}} \\
&= 4\,\mathrm{N/(m/s)}
\end{aligned}$$

## 3.15

(a)

$$\begin{aligned}
\overline{E}_0 e^{-\gamma} &= \frac{1}{2}\overline{E}_0 \\
e^{-\gamma} &= \frac{1}{2} \\
-\gamma &= \ln\frac{1}{2} \\
\gamma &= \ln 2 \\
Q_0 &= \frac{\omega_0}{\gamma} \\
&= \frac{2\pi f}{\gamma} \\
&= \frac{512\pi}{\ln 2} \\
&\approx 2321
\end{aligned}$$

(b)

$$Q = 2Q_0$$

(c)

$$
\begin{aligned}
\gamma &= \frac{1}{4} \\
Q &= \frac{\omega_0}{\gamma} \\
&= 4\sqrt{\frac{k}{m}} \\
&= 12 \\
\gamma &= \frac{b}{m} \\
b &= \gamma m \\
&= 0.025\,\mathrm{N/(m/s)}
\end{aligned}
$$

## 3.16

(a)

$$
\begin{aligned}
x &= A \sin \omega t \\
v &= \omega A \cos \omega t \\
a &= -\omega^2 A \sin \omega t \\
E &= \int_0^{1/f} \frac{Ke^2}{c^3} (-\omega^2 A \sin \omega t)^2 \, dt \\
&= \frac{Ke^2 \omega^4 A^2}{c^3} \int_0^{1/f} \sin^2 \omega t \, dt \\
&= \frac{Ke^2 \omega^4 A^2}{c^3} \left[ \frac{t}{2} - \frac{1}{4\omega} \sin 2\omega t \right]_0^{1/f} \\
&= \frac{Ke^2 \omega^4 A^2}{c^3} \left( \frac{1}{2f} - \frac{1}{4\omega} \sin 2\omega \frac{1}{f} \right) \\
&= \frac{Ke^2 (2\pi f)^4 A^2}{2fc^3} \\
&= \frac{8\pi^4 Ke^2 f^3 A^2}{c^3}
\end{aligned}
$$

(b)

$$\begin{aligned}
E_0 &= \frac{1}{2}mv^2 \\
&= \frac{m(\omega A)^2}{2} \\
&= 2\pi^2 A^2 f^2 m \\
\frac{Q}{\pi}E &= E_0\left(1-\frac{1}{e}\right) \\
\frac{Q}{\pi}\frac{8\pi^4 Kq^2 f^3 A^2}{c^3} &= 2\pi^2 A^2 f^2 m\left(1-\frac{1}{e}\right) \\
Q\frac{4\pi Kq^2 f}{c^3} &= m\left(1-\frac{1}{e}\right) \\
Q &= \frac{c^3 m}{4\pi f Kq^2}\left(1-\frac{1}{e}\right)
\end{aligned}$$

## 3.17

(a)

$$\begin{aligned}
V &= \pi r^2 y_{\text{left}} \\
V &= \pi (2r)^2 y_{\text{right}} \\
\pi r^2 y_{\text{left}} &= \pi (2r)^2 y_{\text{right}} \\
y_{\text{right}} &= \frac{1}{4} y_{\text{left}} \\
\frac{y_{\text{left}}}{2} + \frac{y_{\text{right}}}{2} &= \frac{y_{\text{left}}}{2} + \frac{y_{\text{left}}}{8} \\
&= \frac{5}{8} y_{\text{left}} \\
U &= mg\frac{5}{8}y \\
&= \frac{5}{8}\rho\pi r^2 ygy \\
&= \frac{5}{8}g\rho\pi r^2 y^2
\end{aligned}$$

(b)

$$
\begin{aligned}
r(x) &= r + \frac{x}{l}r \\
&= r\left(1 + \frac{x}{l}\right) \\
\frac{dy}{dt}\pi r^2 &= v\pi r(x)^2 \\
&= v\pi\left[r\left(1 + \frac{x}{l}\right)\right]^2 \\
v &= \frac{dy}{dt}\frac{1}{\left(1 + \frac{x}{l}\right)^2} \\
m &= \rho\pi r(x)^2\,dx \\
&= \rho\pi\left[r\left(1 + \frac{x}{l}\right)\right]^2\,dx \\
&= \rho\pi r^2\left(1 + \frac{x}{l}\right)^2\,dx \\
dK &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}\rho\pi r^2\left(1 + \frac{x}{l}\right)^2\,dx\left(\frac{dy}{dt}\frac{1}{\left(1 + \frac{x}{l}\right)^2}\right)^2 \\
&= \frac{1}{2}\rho\frac{\pi r^2\,dx}{(1 + x/l)^2}\left(\frac{dy}{dt}\right)^2
\end{aligned}
$$

(c)

$$
\begin{aligned}
K &= \frac{1}{2}\rho\pi r^2 h\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}\rho\pi(2r)^2 h\left(\frac{dy}{dt}\right)^2 + \int_0^l dK \\
&= \frac{5}{2}\rho\pi r^2 h\left(\frac{dy}{dt}\right)^2 + \int_0^l \frac{1}{2}\rho\frac{\pi r^2\,dx}{(1 + x/l)^2}\left(\frac{dy}{dt}\right)^2 \\
&= \frac{5}{2}\rho\pi r^2 h\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}\rho\pi r^2\int_0^l \frac{1}{(1 + x/l)^2}\,dx\left(\frac{dy}{dt}\right)^2 \\
&= \frac{1}{4}\rho\pi r^2\left(l + \frac{5}{2}h\right)\left(\frac{dy}{dt}\right)^2
\end{aligned}
$$

(d)

$$
\begin{aligned}
K + U &= E \\
\frac{1}{4}\rho\pi r^2\left(l + \frac{5}{2}h\right)\left(\frac{dy}{dt}\right)^2 + \frac{5}{8}g\rho\pi r^2 y^2 &= E \\
m &= \frac{1}{2}\rho\pi r^2\left(l + \frac{5}{2}h\right) \\
k &= \frac{5}{4}g\rho\pi r^2 \\
T &= 2\pi\sqrt{\frac{m}{k}} \\
&= 2\pi\sqrt{\frac{\frac{1}{2}\rho\pi r^2\left(l + \frac{5}{2}h\right)}{\frac{5}{4}g\rho\pi r^2}} \\
&= 2\pi\sqrt{\frac{2h}{g}}
\end{aligned}
$$

## 3.19

(a)

$$
m\frac{d^2x}{dt^2} + 2k(x + l - l_0) = 0
$$

(b)

$$
\begin{aligned}
T &= k(l' - l_0) \\
&= k(\sqrt{l^2 + y^2} - l_0) \\
F &= 2T\sin\theta \\
&= 2k(\sqrt{l^2 + y^2} - l_0)\frac{y}{\sqrt{l^2 + y^2}} \\
&= 2k\left(1 - \frac{l_0}{\sqrt{l^2 + y^2}}\right)y \\
&\approx 2k\left(1 - \frac{l_0}{l}\right)y
\end{aligned}
$$

$$
m\frac{d^2y}{dt^2} + 2k\left(1 - \frac{l_0}{l}\right)y = 0
$$

(c)

$$
\begin{aligned}
T_x &= 2\pi\sqrt{\frac{m}{2k}} \\
T_y &= 2\pi\sqrt{\frac{m}{2k\left(1-\frac{l_0}{l}\right)}} \\
\frac{T_x}{T_y} &= \frac{2\pi\sqrt{m/2k}}{2\pi\sqrt{\frac{m}{2k(1-l/l_0)}}} \\
&= \sqrt{\frac{m}{2k}\frac{2k(1-l/l_0)}{m}} \\
&= \sqrt{1-l/l_0}
\end{aligned}
$$

(d)

$$
\begin{aligned}
x &= A_x\cos\left(\sqrt{\frac{2k}{m}}t+\phi_x\right) \\
A_0 &= A_x\cos\phi_x \\
0 &= -\sqrt{\frac{2k}{m}}A_x\sin\phi_x \\
\tan\phi_x &= 0 \\
\phi_x &= 0 \\
A_x &= A_0 \\
x &= A_0\cos\sqrt{\frac{2k}{m}}t
\end{aligned}
$$

$$
\begin{aligned}
y &= A_y\cos\left(\sqrt{\frac{2k(1-l_0/l)}{m}}t+\phi_y\right) \\
A_0 &= A_y\cos\phi_y \\
0 &= -\sqrt{\frac{2k(1-l_0/l)}{m}}A_y\sin\phi_y \\
\tan\phi_y &= 0 \\
\phi_y &= 0 \\
A_y &= A_0 \\
y &= A_0\cos\sqrt{\frac{2k(1-l_0/l)}{m}}t
\end{aligned}
$$

# 4 Forced vibrations and resonance

## 4.3

(a)

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

$$\begin{aligned}
\omega &= \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \\
&= \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \\
&= \sqrt{\frac{80}{0.2} - \frac{4^2}{4 \cdot 0.2^2}} \\
&= \sqrt{300} \\
&= 10\sqrt{3} \\
T &= \frac{2\pi}{\omega} \\
&= \frac{2\pi}{10\sqrt{3}} \\
&= \frac{\pi}{5\sqrt{3}}\,\mathrm{s}
\end{aligned}$$

(b)

$$\begin{aligned}
A &= \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \\
&= \frac{F_0}{m\sqrt{\left(\frac{k}{m} - \omega^2\right)^2 + \left(\frac{b}{m}\omega\right)^2}} \\
&\approx 1.3\,\mathrm{cm}
\end{aligned}$$

## 4.4

$$\begin{aligned}
mg &= kh \\
k &= \frac{mg}{h}
\end{aligned}$$

$$\begin{aligned}
mg &= bu \\
b &= \frac{mg}{u}
\end{aligned}$$

(a)

$$m\frac{d^2x}{dt^2} + \frac{mg}{u}\frac{dx}{dt} + \frac{mg}{h}x = 0$$

(b)

$$
\begin{aligned}
\omega &= \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \\
&= \sqrt{\frac{g}{h} - \frac{g^2}{4u^2}} \\
&= \sqrt{\frac{g}{h} - \frac{g^2}{36gh}} \\
&= \sqrt{\frac{g}{h} - \frac{g}{36h}} \\
&= \sqrt{\frac{35g}{36h}}
\end{aligned}
$$

(c)

$$\frac{1}{\gamma} = \frac{u}{g} = \frac{3\sqrt{gh}}{g} = 3\sqrt{\frac{h}{g}}\,\mathrm{s}$$

(d)

$$Q = \frac{\omega_0}{\gamma} = \sqrt{\frac{g}{h}} \cdot 3\sqrt{\frac{h}{g}} = 3$$

(e)

$$\delta = \frac{\pi}{2}$$

(f)

$$
\begin{aligned}
A &= \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}} \\
&= \frac{F_0}{m\sqrt{\left(\frac{g}{h}-\omega^2\right)+\left(\frac{g}{u}\omega\right)^2}} \\
&= \frac{F_0}{m\sqrt{\left(\frac{g}{h}-\frac{2g}{h}\right)^2+\left(\frac{g}{3\sqrt{gh}}\sqrt{\frac{2g}{h}}\right)^2}} \\
&= \frac{F_0}{m\sqrt{\left(\frac{g}{h}\right)^2+\left(\frac{\sqrt{2}}{3}\frac{g}{h}\right)^2}} \\
&= \frac{F_0}{m\sqrt{\left(\frac{g}{h}\right)^2+\frac{2}{9}\left(\frac{g}{h}\right)^2}} \\
&= \frac{F_0}{m\sqrt{\frac{11}{9}\left(\frac{g}{h}\right)^2}} \\
&= \sqrt{\frac{9}{11}}\frac{F_0 h}{gm} \\
&= \sqrt{\frac{9}{11}}h \\
&\approx 0.9h
\end{aligned}
$$

## 4.6

(a)

$$
\begin{aligned}
m\left(\frac{d^2y}{dt^2}+\frac{d^2\eta}{dt^2}\right) &= -kx-b\frac{dy}{dt} \\
\frac{d^2y}{dt^2}+\gamma\frac{dy}{dt}+\omega_0^2y &= -\frac{d^2\eta}{dt^2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{d^2y}{dt^2}+\gamma\frac{dy}{dt}+\omega_0^2y &= C\omega^2\cos\omega t \\
y &= A(\omega)\cos(\omega t+\delta(\omega)) \\
A(\omega) &= \frac{C\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+(\omega\omega_0/Q)^2}} \\
\tan\delta(\omega) &= \frac{\omega\omega_0/Q}{\omega_0^2-\omega^2}
\end{aligned}
$$

(d)

$$\begin{aligned}
T_0 &= \frac{2\pi}{\omega_0} \\
\omega_0 &= \frac{2\pi}{T_0} \\
&= \frac{\pi}{15}\,\mathrm{s}
\end{aligned}$$

$$Q = 2$$

$$\begin{aligned}
T &= \frac{2\pi}{\omega} \\
\omega &= \frac{2\pi}{T} \\
&= \frac{\pi}{600}\,\mathrm{s}
\end{aligned}$$

$$\begin{aligned}
A &= C\omega^2 \\
C &= \frac{A}{\omega^2} \\
&= A\left(\frac{600}{\pi}\right)^2 \\
&\approx 3.65 \times 10^{-5}
\end{aligned}$$

$$A(\omega) \approx 2.28 \times 10^{-8}\,\mathrm{m}$$

## 4.8

(a)

$$\begin{aligned}
m\frac{d^2x}{dt^2} &= -b\frac{dx}{dt} \\
m\frac{dv}{dt} &= -bv \\
\frac{1}{v}\frac{dv}{dt} &= -\frac{b}{m} \\
\ln v &= -\frac{b}{m}t + c \\
v &= e^{(-bt/m)+c} \\
&= v_0 e^{-\gamma t} \\
x &= C - \frac{v_0}{\gamma}e^{-\gamma t}
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} &= \frac{F_0}{m}\cos\omega t \\
x &= A\cos(\omega t - \delta) \\
\frac{dx}{dt} &= -A\omega\sin(\omega t - \delta) \\
\frac{d^2x}{dt^2} &= -A\omega^2\cos(\omega t - \delta) \\
-A\omega^2\cos(\omega t - \delta) - \gamma A\omega\sin(\omega t - \delta) &= \frac{F_0}{m}\cos\omega t \\
-A\omega^2(\cos\omega t\cos\delta + \sin\omega t\sin\delta) & \\
-\gamma A\omega(\sin\omega t\cos\delta - \cos\omega t\sin\delta) &= \frac{F_0}{m}\cos\omega t \\
A\omega(\gamma\sin\delta - \omega\cos\delta)\cos\omega t & \\
-A\omega(\gamma\cos\delta + \omega\sin\delta)\sin\omega t &= \frac{F_0}{m}\cos\omega t
\end{aligned}$$

$$\begin{aligned}
A\omega(\gamma\cos\delta + \omega\sin\delta) &= 0 \\
\gamma\cos\delta + \omega\sin\delta &= 0 \\
\tan\delta &
\end{aligned}$$

$$\begin{aligned}
A\omega(\gamma\sin\delta - \omega\cos\delta) &= \frac{F_0}{m} \\
\gamma\sin\delta - \omega\cos\delta &= -\sqrt{\gamma^2+\omega^2}\cos\left(\delta + \arctan\frac{\gamma}{\omega}\right) \\
&= -\sqrt{\gamma^2+\omega^2}\cos(\delta + \arctan(-\tan\delta)) \\
&= -\sqrt{\gamma^2+\omega^2}\cos(\delta - \arctan(\tan\delta)) \\
&= -\sqrt{\gamma^2+\omega^2} \\
-A\omega\sqrt{\gamma^2+\omega^2} &= \frac{F_0}{m} \\
|A| &= \frac{F_0}{m\omega\sqrt{\gamma^2+\omega^2}}
\end{aligned}$$

## 4.9

(a)

$$
\begin{aligned}
x &= A \sin \omega t \\
\frac{dx}{dt} &= A\omega \cos \omega t \\
W &= \int dW \\
&= \int F\, dx \\
&= \int b \frac{dx}{dt}\, dx \\
&= b \int \left( \frac{dx}{dt} \right)^2\, dt \\
&= b \int_0^T (A\omega \cos \omega t)^2\, dt \\
&= A^2 b \omega^2 \int_0^{2\pi/\omega} \cos^2 \omega t\, dt \\
&= A^2 b \omega^2 \left[ \frac{t}{2} + \frac{1}{4\omega} \sin 2\omega t \right]_0^{2\pi/\omega} \\
&= A^2 b \omega \pi
\end{aligned}
$$

## 4.11

(a)

$$
\begin{aligned}
A &= \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \\
&= \frac{2/0.2}{\sqrt{\left(\frac{80}{0.2} - 30^2\right)^2 + \left(\frac{4}{0.2} 30\right)^2}} \\
&\approx 1.3\,\mathrm{cm}
\end{aligned}
$$

$$
\begin{aligned}
\tan \delta &= \frac{\gamma\omega}{\omega_0^2 - \omega^2} \\
\delta &= \arctan \frac{\frac{b}{m}\omega}{\frac{k}{m} - \omega^2} \\
&\approx 130^\circ
\end{aligned}
$$

(b)

$$
\begin{aligned}
x &= A\cos(\omega t - \delta) \\
\frac{dx}{dt} &= -A\omega\sin(\omega t - \delta) \\
W &= \int dW \\
&= \int F\,dx \\
&= \int b\frac{dx}{dt}\,dx \\
&= b\int_0^T \left(\frac{dx}{dt}\right)^2\,dt \\
&= b\int_0^T (-A\omega\sin(\omega t - \delta))^2\,dt \\
&= A^2 b\omega^2 \int_0^{2\pi/\omega} \sin^2(\omega t - \delta)\,dt \\
&= A^2 b\omega\pi \\
&\approx 6.4 \times 10^{-2}\,\mathrm{J}
\end{aligned}
$$

(c)

$$
P = \frac{W}{t} = 0.31\,\mathrm{W}
$$

## 4.12

(a)

$$
\begin{aligned}
mg &= kx \\
k &= \frac{mg}{x} \\
&\approx 785\,\mathrm{N/m}
\end{aligned}
$$

$$
\begin{aligned}
\omega_0 &= \sqrt{\frac{k}{m}} \\
&\approx 19.8\,\mathrm{rad/s}
\end{aligned}
$$

(b)

$$
\begin{aligned}
A &= BQ \\
&= 1.5\,\mathrm{cm}
\end{aligned}
$$

(c)

$$\begin{aligned}\overline{P}(\omega) &= \frac{1.082B^2 m\omega_0^5}{2Q}\frac{1}{0.0016\omega_0^2 + \omega_0^2/Q^2}\\ &\approx 0.093\,\mathrm{W}\end{aligned}$$

## 4.13

(a)

$$\omega_0 = 40\,\mathrm{rad/s}$$

$$\begin{aligned}2 &= \frac{\omega_0}{Q}\\ Q &= \frac{\omega_0}{2}\\ &= 20\end{aligned}$$

(b)

$$\begin{aligned}E_0 e^{-2\pi n/Q} &= E_0 e^{-5}\\ -\frac{2\pi n}{Q} &= -5\\ n &= \frac{5Q}{2\pi}\\ &\approx 16\end{aligned}$$

## 4.14

(a)

$$\begin{aligned}1.02\omega_0 - 0.98\omega_0 &= \frac{\omega_0}{Q}\\ 0.04\omega_0 &= \frac{\omega_0}{Q}\\ Q &= 25\end{aligned}$$

(b)

$$\gamma = 0.04\omega_0$$

(c)

$$\begin{aligned}\frac{E_0 - E(2\pi/\omega_0)}{E_0} &= 1 - \frac{E_0 e^{-0.04\omega_0(2\pi/\omega_0)}}{E_0}\\ &= 1 - e^{-2\pi\cdot 0.04}\\ &\approx 22\%\end{aligned}$$

(d)

$$\omega_0' = \sqrt{2}\omega_0$$

(e)

$$Q' = \sqrt{2}Q$$

(f)

$$\overline{P}_m' = \overline{P}_m$$

(g)

$$E_0' = E_0$$

## 4.15

(b)

$$\frac{\omega_1}{Q} = \frac{\omega_1}{5}$$
$$Q = 5$$

(c)

$$\begin{aligned} Q &= \frac{\omega_0}{\gamma} \\ &= \sqrt{\frac{k}{m}}\frac{m}{b} \\ b &= \frac{1}{5}\sqrt{km} \end{aligned}$$

## 4.17

(a)

$$W = PT = 10 \cdot \frac{2\pi}{10^6} = \frac{2\pi}{10^5} = 6.28 \times 10^{-5}\,\mathrm{J}$$

(b)

$$\begin{aligned} W &= E_0(1 - e^{-\gamma(2\pi/\omega)}) \\ E_0 &= \frac{W}{1 - e^{-\gamma(2\pi/\omega)}} \\ &\approx 1.03 \times 10^{-3}\,\mathrm{J} \end{aligned}$$

(c)

$$\frac{1}{\gamma} = \frac{1}{(1.005 - 0.995) \times 10^6} \approx 1 \times 10^{-4}\,\mathrm{s}$$