# Advanced Engineering Mathematics Complex Analysis by Dennis G. Zill Notes

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## Contents



## 17 Functions of a Complex Variable

### 17.1 Complex Numbers

- A **complex number** is any number of the form

$$z = a + ib$$

where $a$ and $b$ are real numbers and $i$ is the imaginary unit such that $i^2 = -1$.

- The real number $a$ in the above complex number $z$ is called the **real part** of $z$ and the real number $b$ (not $ib$) is called the **imaginary part** of $z$.
- The real and imaginary parts of a complex number $z$ are denoted $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively.
- A real constant multiple of the imaginary unit, e.g. $6i$ is called a **pure imaginary number**.
- Two complex numbers are equal if their real and imaginary parts are equal.
- The addition and subtraction of complex numbers occur between the real and imaginary parts, e.g.

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

- The multiplication of complex numbers occurs elementwise as normal, e.g.

$$(a + bi)(c + di) = ac + adi + bci - bd.$$

- The **conjugate** of a complex number $z = a + ib$ is

$$\overline{z} = a - ib.$$

- The division of complex numbers occurs by multiplying the numerator and denominator by the conjugate of the denominator, e.g.

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{ac - adi + bci + bd}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}.\end{aligned}$$

- Conjugates have several interesting properties:

$$\begin{aligned}\overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2} \\ \overline{z_1 - z_2} &= \overline{z_1} - \overline{z_2} \\ \overline{z_1 z_2} &= \overline{z_1}\,\overline{z_2} \\ \overline{\frac{z_1}{z_2}} &= \frac{\overline{z_1}}{\overline{z_2}}.\end{aligned}$$

- The sum and product of a complex number $z = x + iy$ with its conjugate are real numbers

$$z + \overline{z} = 2x$$
$$z\overline{z} = x^2 + y^2$$

while the difference between a complex number and its conjugate is a purre imaginary number

$$z - \overline{z} = 2iy.$$

- The above properties let us define

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2} \text{ and } \mathrm{Im}(z) = \frac{z - \overline{z}}{2i}.$$

- The **complex plane** or **$z$-plane** is a coordinate system where the horizontal or $x$-axis is called the **real axis** and the vertical or $y$-axis is called the **imaginary axis**. Complex numbers can be plotted in this coordinate system by considering their real and imaginary parts an ordered pair corresponding their position.

- The **modulus** or **absolute value** of a complex number $z = x+iy$ denoted by $|z|$ is the real number

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\overline{z}}.$$

This is the distance between $z$ and the origin in the complex plane.

- If you consider two numbers in the complex plane as vectors, the length of their sum can't be longer than their individual lengths combined

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

This extends to any finite sum

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

and is known as the **triangle inequality**.

## 17.2 Powers and Roots

- A complex number can be expressed in **polar form**

$$z = (r\cos\theta) + i(r\sin\theta)$$

where $r = |z|$ is the nonnegative modulus of $z$ and $\theta = \arg z$ is the **argument** of $z$ — the angle between $z$ and the positive real axis measured in the counterclockwise direction.

- The argument of a complex number $z$ isn't unique as any multiply of $2\pi$ can be added to it. The **principle argument** of $z$ denoted $\operatorname{Arg} z$ is the argument of $z$ restricted to the intercal $-\pi \leq \operatorname{Arg} z \leq \pi$.

- Multiplication and division of complex numbers is simpler in polar form. For two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ we get

$$\begin{aligned} z_1 z_2 &= r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]. \end{aligned}$$

- The above formulas can be used to find integer powers of a complex number $z$

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

  where $n$ is an integer (including negative integers).

- **DeMoivre's formula** is a special case of the above where $r = 1$ so

$$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

- A number $w$ is said to be an ***n*th root** of a nonzero complex number $z$ if $w^n = z$. The $n$th roots of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ are

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$$

  where $k = 0, 1, 2, \ldots, n-1$.

- The root $w$ of a complex number $z$ obtained by using the principle argument of $z$ with $k = 0$ is called the **principle *n*th root** of $z$.

- Since the $n$th roots of a complex number have the same modulus they lie on a circle of radius $r^{1/n}$. The arguments of subsequent roots differ by $2\pi/n$ so they're also equally spaced around the circle.

## 17.3 Sets in the Complex Plane

- The points $z = x + iy$ that satisfy the equation

$$|z - z_0| = \rho$$

  for $\rho > 0$ lie on a circle of radius $\rho$ centred at the point $z_0$.

- The points $z$ satisfying the inequality $|z - z_0| < \rho$ for $\rho > 0$ lie within, but not on, a circle of radius $\rho$ centered at the point $z_0$. This set is called a **neighborhood** of $z_0$ or an **open disk**.

- A point $z_0$ is said to be an **interior point** of a set $S$ of the complex plane if there exists some neighborhood of $z_0$ that lies entirely within $S$.

- If every point $z$ of a set $S$ is an interior point, then $S$ is said to be an **open set**. An example of a set that isn't open is the set of points satisfying the inequality $\text{Re}(z) \geq 0$. This isn't open because it includes the line $\text{Re}(z) = 0$ and no points on that line are interior to the set because, no matter what $\rho$ you choose, some points in the neighborhood have $\text{Re}(z) < 0$.

- If every neighborhood of a point $z_0$ contains at least one point that is in a set $S$ and at least one point that is not in $S$, then $z_0$ is said to be a **boundary point** of $S$.

- The **boundary** of a set $S$ in the complex plane is the set of all boundary points of $S$.

- If any pair of points in a set $S$ can be connected by a polygonal line that lies entirely within the set, then $S$ is said to be **connected**.

- An open connected set is called a **domain**.

- A **region** is a set in the complex plane with all, some, or none of its boundary points. A region containing all of its boundary points is said to be **closed**.

## 17.4 Functions of a Complex Variable

- A **function** $f$ from a set $A$ to a set $B$ is a rule of correspondence that assigns to each element of $A$ one and only one element of $B$.

- If $b$ is the element of $B$ assigned to the element $a$ of $A$, $b$ is said to be the **image** of $a$ and is denoted $b = f(a)$.

- The set $A$ is called the **domain** of $f$.

- The set of all images in $B$ is called the **range** of $f$.

- If $A$ is a set of real numbers, $f$ is said to be a **function of a real variable** $\boldsymbol{x}$.

- If $A$ is a set of complex numbers, $f$ is said to be a **function of a complex varibale** $\boldsymbol{z}$ or a **complex function**.

- The image $w$ of a complex number $z$ is

$$w = f(z) = u(x, y) + iv(x, y)$$

  where $u$ and $v$ are the real and imaginary parts of $w$ and are real-valued functions.

- Although we cannot draw a graph of a complex function $w = f(z)$ (because it would require a four-dimensional coordinate system), it can be interpreted as a **mapping** or **transformation** from the $z$ plane to the $w$ plane.

- A complex function may be interpreted as a two-dimensional fluid flow by considering $w = f(z)$ as the fluid velocity vector at the point $z$. In that case, if $x(t) + iy(t)$ is a parametric representation of a particle's position over time then

$$\frac{dx}{dt} = u(x,y)$$
$$\frac{dy}{dt} = v(x,y)$$

and the family of solutions to this system of differential equations are called the **streamlines** of the flow associated with $f(z)$.

**Definition 17.4.1 Limit of a Function**

Suppose the function $f$ is defined in some neighborhood of $z_0$, except possibly at $z_0$ itself. Then $f$ is said to possess a **limit** at $z_0$, written

$$\lim_{z \to z_0} f(z) = L$$

if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

- For a function $f$ of a real variable $x$, the limit $\lim_{x \to x_0} f(x) = L$ means $f$ approaches $L$ as you approach from both the left and right. If however $f$ is a function of a complex variable it means $f$ approaches $L$ as you approach from any direction in the complex plane.

**Theorem 17.4.1 Limit of Sum, Product, Quotient**

Suppose $\lim_{z \to z_0} f(z) = L_1$ and $\lim_{z \to z_0} g(z) = L_2$. Then

(*i*) $\lim_{z \to z_0} [f(z) + g(z)] = L_1 + L_2$

(*ii*) $\lim_{z \to z_0} f(z)g(z) = L_1 L_2$

(*iii*) $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{L_1}{L_2}, \quad L_2 \neq 0.$

- A function $f$ is continuous at a point $z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

- A function $f$ defined by

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0, \;\; a_n \neq 0$$

where $n$ is a nonnegative integer and the coefficients $a_i$, $i = 0, 1, \ldots, n$, are complex constants is called a **polynomial** of degree $n$.

- Polynomials are continuous on the entire complex plane.
- A **rational function**

$$f(z) = \frac{g(z)}{h(z)}$$

  is continuous everywhere $h(z) \neq 0$.

**Definition 17.4.3 Derivative**

Suppose the complex function $f$ is defined in a neighborhood of a point $z_0$. The **derivative** of $f$ at $z_0$ is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{3}$$

provided this limit exists.

- In order for a complex function to be differentiable, the limit must approach the same value from every direction. This is a greater demand than in real variables. If you take an arbitrary complex function, there's a good chance it isn't differentiable.

**Definition 17.4.4 Analyticity at a Point**

A complex function $w = f(z)$ is said to be **analytic at a point** $z_0$ if $f$ is differentiable at $z_0$ and at every point in some neighborhood of $z_0$.

- Analyticity at a point is a neighborhood property. A function can be differentiable at a point but if the neighboring points aren't also differentiable, it's not analytic at that point.
- A function is analytic in a domain $D$ if it is analytic at every point in $D$.
- A function that is analytic everywhere is called an **entire function**.

## 17.5 Cauchy-Riemann Equations

**Theorem 17.5.1 Cauchy–Riemann Equations**

Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at $z$ the first-order partial derivatives of $u$ and $v$ exist and satisfy the **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1}$$

- If a complex function $f(z) = u(x, y) + iv(x, y)$ is analytic throughout a domain $D$, then the real functions $u$ and $v$ must satisfy the Cauchy-Riemann equations at every point in $D$.

**Theorem 17.5.2 Criterion for Analyticity**

Suppose the real-valued functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a domain $D$. If $u$ and $v$ satisfy the Cauchy–Riemann equations at all points of $D$, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in $D$.

- The Cauchy-Riemann equations are derived assuming the function is differentiable at a particular point. That being the case, they can also be used as a formula for the derivative of the function

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

- Because analyticity implies differentiability, theorem 17.5.2 can also be used to determine if a function is differentiable at a point.

- A real-valued function $\phi(x, y)$ that has continuous second-order partial derivatives in a domain $D$ and satisfies Laplace's equation is said to be **harmonic** in $D$.

- If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain $D$ then the functions $u(x, y)$ and $v(x, y)$ are harmonic functions.

- If a given function $u(x, y)$ is harmonic in a domain $D$ it is sometimes possible to find another function $v(x, y)$ that is harmonic in $D$ such that $u(x, y) + iv(x, y)$ is analytic in $D$. The function $v$ is called the **harmonic conjugate function** of $u$.

- To find the harmonic conjugate function of a given function $u$:

  1. Take the first-order partial derivatives of $u$ with respect to $x$ and $y$.
  2. If $u(x, y) + iv(x, y)$ is analytic in a domain $D$ then $u$ and $v$ must satisfy the Cauchy-Riemann equations in $D$ from which we can find expressions for $\partial v/\partial x$ and $\partial v/\partial y$.
  3. Integrate $\partial v/\partial x$ with respect to $x$ to get an expression for $v$ with an unknown constant $h(y)$.
  4. Take the first-order partial derivative of $v$ with respect to $y$, equate it with the other expression for $\partial v/\partial y$, and solve for $h'(y)$.
  5. Integrate $h'(y)$ and substitute the result to find $v$.

## 17.6 Exponential and Logarithmic Functions

- The exponential function for complex numbers is defined as

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

- $e^z$ is analytic for all $z$, i.e. it's an entire function.
- Like its real-valued counterpart,

$$\frac{d}{dz}e^z = e^z,$$
$$e^{z_1}e^{z_2} = e^{z_1+z_2},$$

  and

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}.$$

- Since

$$e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1$$

  and

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z$$

  the complex function $f(z) = e^z$ is **periodic** with complex period $2\pi i$. Because of this complex periodicity an infinite horizontal strip of height $2\pi$ contains all possible values for the function. The strip $-\pi < y \leq \pi$ is called the **fundamental region**.

- For $z \neq 0$ and $\theta = \arg z$,

$$\ln z = \log_e |z| + i(\theta + 2n\pi),\ n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

  This means there are infinitely many values of the logarithm of a complex number $z$. This makes sense as the complex exponential is periodic.

- The **principal value** of $\ln z$ is the complex logarithm corresponding to $n = 0$ and $\theta = \operatorname{Arg} z$. It is denoted $\operatorname{Ln} z$.
- Some familiar properties of the real-valued logarithm hold for the complex-valued logarithm, e.g.

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$

  and

$$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2,$$

  however they don't necessarily hold for the principal value.

- $\operatorname{Ln} z$ is discontinuous and thus not analytic at $z = 0$ because $\ln z$ is undefined at $z = 0$ and on the negative real axis because $\operatorname{Arg} z$ is discontinuous there.

- The derivative of $\mathrm{Ln}\, z$ is
$$\frac{d}{dz}\mathrm{Ln}\, z = \frac{1}{z}.$$
- The complex power of a complex number is defined as
$$z^\alpha = e^{\alpha \ln z}, \ z \neq 0.$$
In general this is multiple-valued because $\ln z$ is multiple-valued — only if $\alpha = n, \ n = 0, \ \pm 1, \pm 2, \ldots$ is it single-valued. If $\ln z$ is replaced with $\mathrm{Ln}\, z$ then we get the **principle value** of $z^\alpha$.

## 17.7 Trigonometric and Hyperbolic Functions

**Definition 17.7.1 Trigonometric Sine and Cosine**

For any complex number $z = x + iy$,
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{2}$$

- The other trigonometric functions ($\tan z$, etc.) are defined as usual.
- Because $e^{iz}$ and $e^{-iz}$ are entire functions, $\sin z$ and $\cos z$ are also entire functions.
- $\sin z = 0$ for the real numbers $z = n\pi$, $n \in \mathbb{Z}$ and $\cos z = 0$ for the real numbers $z = (2n+1)\pi/2$, $n \in \mathbb{Z}$. This means that $\tan z$ and $\sec z$ are analytic except at the points where $\cos z = 0$ and $\cot z$ and $\sec z$ are analytic except at the points where $\sin z = 0$.
- The usual derivatives and trigonometric functions are still valid in the complex case.
- $\sin z$ can be expressed as
$$\sin z = \sin x \cosh y + i \cos x \sinh y$$
and $\cos z$ can be expressed as
$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$
- The only zeroes of $\sin z$ are the real numbers $z = n\pi$, $n \in \mathbb{Z}$ and the only zeroes of $\cos z$ are the real numbers $z = (2n+1)\pi/2$, $n \in \mathbb{Z}$.

**Definition 17.7.2 Hyperbolic Sine and Cosine**

For any complex number $z = x + iy$,
$$\sinh z = \frac{e^{z} - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^{z} + e^{-z}}{2}. \tag{10}$$

- The complex trigonometric functions can be expressed in terms of the complex hyperbolic functions and vice versa

$$\sin z = -i \sinh(iz), \qquad \cos z = \cosh(iz)$$
$$\sinh z = -i \sin(iz), \qquad \cosh z = \cos(iz).$$

- $\sinh z$ can be expressed as

$$\sinh z = \sinh x \cos y + i \cosh x \sin y$$

and $\cosh z$ can be expressed as

$$\cosh z = \cosh x \cos y + i \sinh x \sin y.$$

- The zeroes of $\sinh z$ are $z = n\pi i$, $n \in \mathbb{Z}$ and the zeroes of $\cosh z$ are $z = (2n+1)\pi i/2$, $n \in \mathbb{Z}$.

- $\sin z$ and $\cos z$ are $2\pi$ periodic while $\sinh z$ and $\cosh z$ are $2\pi i$ periodic.

## 17.8 Inverse Trigonometric and Hyperbolic Functions

- Because the complex trigonometric functions are multi-valued, their inverse functions are also multi-valued.

- The definitions of those inverse functions are

$$\begin{aligned}
\arcsin z &= -i \ln[iz + (1-z^2)^{1/2}], \\
\arccos z &= -i \ln[z + i(1-z^2)^{1/2}], \text{ and} \\
\arctan z &= \frac{i}{2} \ln \frac{i+z}{i-z}.
\end{aligned}$$

- The derivatives of the inverse trigonometric functions are

$$\begin{aligned}
\frac{d}{dz} \arcsin z &= \frac{1}{(1-z^2)^{1/2}}, \\
\frac{d}{dz} \arccos z &= \frac{-1}{(1-z^2)^{1/2}}, \text{ and} \\
\frac{d}{dz} \arctan z &= \frac{1}{1+z^2}.
\end{aligned}$$

- The definitions of the hyperbolic inverse functions and their derivatives are

$$\sinh^{-1} z = \ln\left[z + (z^2+1)^{1/2}\right]$$

$$\cosh^{-1} z = \ln\left[z + (z^2-1)^{1/2}\right]$$

$$\tanh^{-1} z = \frac{1}{2}\ln\frac{1+z}{1-z}$$

$$\frac{d}{dz}\sinh^{-1} z = \frac{1}{(z^2+1)^{1/2}}$$

$$\frac{d}{dz}\cosh^{-1} z = \frac{1}{(z^2-1)^{1/2}}$$

$$\frac{d}{dz}\tanh^{-1} z = \frac{1}{1-z^2}.$$

# 18 Integration in the Complex Plane

## 18.1 Contour Integrals

- In complex variables, a piecewise smooth curve $C$ is called a **contour** or **path**. An integral of a complex function $f(z)$ on $C$ is denoted $\int_C f(z)\,dz$ or $\oint_C f(z)\,dz$ if $C$ is closed — this is called a **contour integral** or a **complex integral**.

1. Let $f(z) = u(x, y) + iv(x, y)$ be defined at all points on a smooth curve $C$ defined by $x = x(t)$, $y = y(t)$, $a \le t \le b$.
2. Divide $C$ into $n$ subarcs according to the partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b]$. The corresponding points on the curve $C$ are $z_0 = x_0 + iy_0 = x(t_0) + iy(t_0)$, $z_1 = x_1 + iy_1 = x(t_1) + iy(t_1), \ldots, z_n = x_n + iy_n = x(t_n) + iy(t_n)$. Let $\Delta z_k = z_k - z_{k-1}$, $k = 1, 2, \ldots, n$.
3. Let $\|P\|$ be the **norm** of the partition, that is, the maximum value of $|\Delta z_k|$.
4. Choose a sample point $z_k^* = x_k^* + iy_k^*$ on each subarc. See **FIGURE 18.1.1**.
5. Form the sum $\sum_{k=1}^{n} f(z_k^*)\,\Delta z_k$.

**Definition 18.1.1 Contour Integral**

Let $f$ be defined at points of a smooth curve $C$ defined by $x = x(t)$, $y = y(t)$, $a \le t \le b$. The **contour integral** of $f$ along $C$ is

$$\int_C f(z)\,dz = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(z_k^*)\,\Delta z_k. \tag{1}$$

**Theorem 18.1.1 Evaluation of a Contour Integral**

If $f$ is continuous on a smooth curve $C$ given by $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, then

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt. \tag{3}$$

- If a complex function $f$ is continuous on a smooth curve $C$ and if $|f(z)| \leq M$ for all $z$ on $C$, then

$$\left|\int_C f(z)\,dz\right| \leq ML,$$

where

$$L = \int_a^b |z'(t)|\,dt$$

is the length of $C$. This is sometimes called the **ML-intequality**.

- If $\mathbf{T}$ is the unit tangent vector to a positively oriented simple closed curve $C$ then

$$\oint_C f \cdot \mathbf{T}\,ds = \mathrm{Re}\left(\oint \overline{f(z)}\,dz\right)$$

is called the **circulation** around $C$ and measures the tendency of the flow to rotate the curve $C$.

- If $\mathbf{N}$ is the normal vector to a positive oriented simple closed curve $C$ then

$$\oint_C f \cdot \mathbf{N}\,ds = \mathrm{Im}\left(\oint \overline{f(z)}\,dz\right)$$

is called the **net flux** across $C$ and measures the difference between the rates at which fluid enters and exits the region bounded by $C$.

## 18.2 Cauchy-Goursat Theorem

- A domain $D$ is said to be **simply connected** if every simple closed contour $C$ lying entirely in $D$ can be shrunk to a point without leaving $D$, i.e. the domain has no holes in it.

- A domain that is not simply connected is called a **multiply connected domain**. A domain with one hole is called **doubly connected**, a domain with two holes **triply connected**, etc.

**Theorem 18.2.1 Cauchy–Goursat Theorem**

Suppose a function $f$ is analytic in a simply connected domain $D$. Then for every simple closed contour $C$ in $D$, $\oint_C f(z)\,dz = 0$.

- An alternative way of stating the Cauchy-Goursat Theorem is: if $f$ is analytic at all points on and within a simple closed contour $C$, then $\oint_C f(z)\,dz = 0$.

- If $D$ is a double connected domain and $C$ and $C_1$ are simple closed contours such that $C_1$ surrounds the "hole" in the domain and is interior to $C$, then

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz.$$

  This is called the principle of **deformation of contours** since $C_1$ can be considered a continuous deformation of the contour $C$ (or vice versa) under which the value of the integal doesn't change.

- If $z_0$ is a constant complex number interior to a simple closed contour $C$, then

$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & n = 1 \\ 0 & n \text{ an integer} \neq 1 \end{cases}.$$

**Theorem 18.2.2 Cauchy–Goursat Theorem for Multiply Connected Domains**

Suppose $C, C_1, \ldots, C_n$ are simple closed curves with a positive orientation such that $C_1, C_2, \ldots, C_n$ are interior to $C$ but the regions interior to each $C_k$, $k = 1, 2, \ldots, n$, have no points in common. If $f$ is analytic on each contour and at each point interior to $C$ but exterior to all the $C_k$, $k = 1, 2, \ldots, n$, then

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz. \tag{6}$$

## 18.3 Independence of the Path

- Let $z_0$ and $z_1$ be points in a domain $D$. A contour integral $\int_C f(z)\,dz$ is said to be **independent of the path** if its value is the same for all contours $C$ in $D$ with an initial point $z_0$ and a terminal point $z_1$.

- If $f$ is an analytic function in a simply connected domain $D$, then $\int_C f(z)\,dz$ is independent of path $C$.

- Suppose $f$ is continuous in a domain $D$. If there exists a function $F$ such that $F'(z) = f(z)$ for each $z$ in $D$, then $F$ is called the **antiderivative** of $f$.

- The general antiderivative of a complex function includes a complex integration constant.

- Suppose $f$ is continuous in a domain $D$ and $F$ is an antiderivative of $f$ in $D$. Then for any contour $C$ in $D$ with initial point $z_0$ and terminal point $z_1$,

$$\int_C f(z)\,dz = F(z_1) - F(z_0).$$

- A consequence of the above is that if $C$ is closed, then

$$\oint_C f(z)\, dz = 0.$$

- If $f$ is analytic in a simply connected domain $D$, then $f$ has an antiderivative in $D$; this, there exists a function $F$ such that $F'(z) = f(z)$ for all $z$ in $D$.
- Suppose $f$ and $g$ are analytic in a simply connected domain $D$ that contains the contour $C$. If $z_0$ and $z_1$ are the initial and terminal points of $C$, then the **integration by parts** formula is valid in $D$:

$$\int_{z_0}^{z_1} f(z)g'(z)\, dz = f(z)g(z)|_{z_0}^{z_1} - \int_{z_0}^{z_1} f'(z)g(z)\, dz.$$

## 18.4 Cauchy's Integral Formulas

**Theorem 18.4.1 Cauchy's Integral Formula**

Let $f$ be analytic in a simply connected domain $D$, and let $C$ be a simple closed contour lying entirely within $D$. If $z_0$ is any point within $C$, then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz. \tag{1}$$

- Cauchy's integral formula is useful when a contour integral has the form

$$\oint \frac{f(z)}{z - z_0}\, dz$$

in which case you know its value is $2\pi i f(z_0)$.

**Theorem 18.4.2 Cauchy's Integral Formula for Derivatives**

Let $f$ be analytic in a simply connected domain $D$, and let $C$ be a simple closed contour lying entirely within $D$. If $z_0$ is any point interior to $C$, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz. \tag{6}$$

- Cauchy's integral formula for derivatives is useful when a contour integral has the form

$$\oint \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

in which case you know its value is $\frac{2\pi i}{n!} f^{(n)}(z_0)$.

- **Liouville's theorem** states that the only bounded entire functions are constants.

# 19 Series and Residues

## 19.1 Sequences and Series

- A **sequence** is a function whose domain is the set of positive integers, i.e. for each integer $n = 1, 2, 3, \ldots$ we assign a complex number $z_n$.
- If $\lim_{n\to\infty} z_n = L$ we say the sequence $\{z_n\}$ is **convergent**. In otherwords, $\{z_n\}$ converges to the number $L$ if, for every positive number $\varepsilon$, an $N$ can be found such that $|z_n - L| < \varepsilon$ whenever $n > N$.
- A sequence $\{z_n\}$ converges to a complex number $L$ if and only if $\mathrm{Re}(z_n)$ converges to $\mathrm{Re}(L)$ and $\mathrm{Im}(z_n)$ converges to $\mathrm{Im}(L)$.
- An **infinite series** of complex numbers

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + \cdots$$

  is **convergent** if the sequence of partial sums $\{S_n\}$, where

$$S_n = z_1 + z_2 + \cdots + z_n$$

  converges. If $S_n \to L$ as $n \to \infty$, we say that the **sum** of the series is $L$.
- The sum of the geometric series

$$\sum_{k=1}^{\infty} az^{k-1} = a + az + az^2 + \cdots$$

  converges to

$$\frac{a}{1-z}$$

  when $|z| < 1$ and diverges otherwise.
- If $\sum_{k=1}^{\infty} z_k$ converges, then $\lim_{n\to\infty} z_n = 0$.
- If $\lim_{n\to\infty} z_n \neq 0$ then the series $\sum_{k=1}^{\infty} z_k$ diverges.
- An infinite series $\sum_{k=1}^{\infty} z_k$ is said to be **absolutely convergent** if $\sum_{k=1}^{\infty} |z_k|$ converges. Absolute convergence implies convergence.

**Theorem 19.1.4 Ratio Test**

Suppose $\sum_{k=1}^{\infty} z_k$ is a series of nonzero complex terms such that

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = L. \tag{9}$$

(*i*) If $L < 1$, then the series converges absolutely.
(*ii*) If $L > 1$ or $L = \infty$, then the series diverges.
(*iii*) If $L = 1$, the test is inconclusive.

**Theorem 19.1.5 Root Test**

Suppose $\sum_{k=1}^{\infty} z_k$ is a series of complex terms such that

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} = L. \tag{10}$$

(*i*) If $L < 1$, then the series converges absolutely.
(*ii*) If $L > 1$ or $L = \infty$, then the series diverges.
(*iii*) If $L = 1$, the test is inconclusive.

- An infinite series of the form

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

where the coefficients $a_k$ are complex constants is called a **power series** in $z - z_0$. The power series is said to be **centred at $z_0$**, and the complex point $z_0$ is referred to as the **centre** of the series.

- Every complex power series has a **radius of convergence $R$** where $R$ is a real number. The power series converges for all $z$ within the **circle of convergence** $|z - z_0| < R$ and diverges for $|z - z_0| > R$. The series may converge at some, all, or none of the points on the actual circle of convergence.

- For a power series

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

the ratio test depends only on the coefficients $a_k$. If

1. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \neq 0$, the radius of convergence is $R = 1/L$;
2. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$, the radius of convergence is $\infty$;
3. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, the radius of convergence is $R = 0$.

## 19.2 Taylor Series

- A power series $\sum_{k=1}^{\infty} a_k(z-z_0)^k$ has a radius of convergence $R$. For each complex number $z$ within the circle of convergence, when substituted into the power series it converges to a unique value $L$. This defines a function $f$ that maps each $z$ to its corresponding $L$.

**Theorem 19.2.1 Continuity**

A power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ represents a continuous function $f$ within its circle of convergence $|z - z_0| = R$, $R \neq 0$.

**Theorem 19.2.2 Term-by-Term Integration**

A power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ can be integrated term by term within its circle of convergence $|z - z_0| = R$, $R \neq 0$, for every contour $C$ lying entirely within the circle of convergence.

**Theorem 19.2.3 Term-by-Term Differentiation**

A power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ can be differentiated term by term within its circle of convergence $|z - z_0| = R$, $R \neq 0$.

**Theorem 19.2.4 Taylor's Theorem**

Let $f$ be analytic within a domain $D$ and let $z_0$ be a point in $D$. Then $f$ has the series representation

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \tag{8}$$

valid for the largest circle $C$ with center at $z_0$ and radius $R$ that lies entirely within $D$.

- The radius of convergence of a Taylor series is the distance from the centre $z_0$ to the nearest isolated singularity: a point at which the series fails to be analytic but is analytic at all points in some neighborhood of the point.

## 19.3 Laurent Series

- If a complex function $f$ fails to be analytic at a point $z = z_0$, then this point is said to be a **singularity** or a **singular point** of the function.
- Suppose $z = z_0$ is a singularity of a complex function $f$. It is said to be an **isolated singularity** if there exists some **deleted neighborhood**, or **punctured open disk**, $0 < |z - z_0| < R$ of $z_0$ in which $f$ is analytic.
- A singular point $z = z_0$ of a complex function $f$ is said to be **nonisolated** if every neighborhood of $z_0$ contains at least one singularity of $f$ other than $z_0$.

**Theorem 19.3.1 Laurent's Theorem**

Let $f$ be analytic within the annular domain $D$ defined by $r < |z - z_0| < R$. Then $f$ has the series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \tag{3}$$

valid for $r < |z - z_0| < R$. The coefficients $a_k$ are given by

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}}\, ds, \quad k = 0, \pm 1, \pm 2, \ldots, \tag{4}$$

where $C$ is a simple closed curve that lies entirely within $D$ and has $z_0$ in its interior (see **FIGURE 19.3.1**).

- Under Laurent's theorem, the part of $f(z)$ with negative powers of $z - z_0$ is called the **principle part** and the part with positive powers is called the **analytic part**.
- The coefficient formula of theorem 19.3.1 isn't used often. Generally $f$ is decomposed into functions for which the series are known (e.g. $\cos z$, $e^z$, etc.), and those series are combined to find the Laurent series.

## 19.4 Zeroes and Poles

- An isolated singularity $z = z_0$ can be categorised based on the number of terms contained in the principal part of its Laurent expansion (the part with negative powers).
  - If the principal part is zero, i.e. the Laurent expansion consists only of parts with nonnegative powers, then $z = z_0$ is called a **removable singularity**.
  - If the principal part contains a finite number of nonzero terms, then $z = z_0$ is called a **pole**. If the last nonzero coefficient of the principal part is $a_{-n}, n \geq 1$ then we say that $z = z_0$ is a **pole of order $n$**. A pole of order 1 is called a **simple pole**.
  - If the principal part contains infinitely many nonzero terms, then $z = z_0$ is called an **essential singularity**.

| $z = z_0$ | Laurent Series |
|---|---|
| Removable singularity | $a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$ |
| Pole of order $n$ | $\dfrac{a_{-n}}{(z - z_0)^n} + \dfrac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \dfrac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$ |
| Simple pole | $\dfrac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$ |
| Essential singularity | $\cdots + \dfrac{a_{-2}}{(z - z_0)^2} + \dfrac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$ |

- A point $z_0$ is said to be a **zero** of a function $f$ if $f(z_0) = 0$.
- A point $z_0$ is said to be a **zero of order $n$** of a function $f$ if $f(z_0) = 0,\ f'(z_0) = 0,\ \ldots,\ f^{(n-1)}(z_0) = 0$ but $f^{(n)}(z_0) \neq 0$.

**Theorem 19.4.1 Pole of Order $n$**

If the functions $f$ and $g$ are analytic at $z = z_0$ and $f$ has a zero of order $n$ at $z = z_0$ and $g(z_0) \neq 0$, then the function $F(z) = g(z)/f(z)$ has a pole of order $n$ at $z = z_0$.

- Theorem 19.4.1 can sometimes be used to determine the poles of a function by inspection, e.g. in
$$F(z) = \frac{2z + 5}{z - 1}$$
the denominator has a zero of order 1 at $z = 1$ and the numerator is nonzero at that point so $F$ has a simple pole at $z = 1$.

## 19.5 Residues and Residue Theorem

- If a complex function $f$ has an isolated singularity at a point $z_0$ then it has a Laurent series representation
$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k.$$
The coefficient $a_{-1}$ of $1/(z - z_0)$ is called the **residue** of $f$ at $z_0$ and is denoted
$$a_{-1} = \operatorname{Res}(f(z), z_0).$$

**Theorem 19.5.1 Residue at a Simple Pole**

If $f$ has a simple pole at $z = z_0$, then

$$\operatorname{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z). \tag{1}$$

**Theorem 19.5.2 Residue at a Pole of Order $n$**

If $f$ has a pole of order $n$ at $z = z_0$, then

$$\operatorname{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z). \tag{2}$$

- Suppose a complex function $f$ can be written as a quotient $f(z) = g(z)/h(z)$ where $g$ and $h$ are analytic at $z = z_0$. If $g(z_0) \neq 0$ and $h$ has a zero of order 1 at $z_0$, then $f$ has a simple pole at $z = z_0$ and

$$\operatorname{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)}.$$

**Theorem 19.5.3 Cauchy's Residue Theorem**

Let $D$ be a simply connected domain and $C$ a simple closed contour lying entirely within $D$. If a function $f$ is analytic on and within $C$, except at a finite number of singular points $z_1, z_2, \ldots, z_n$ within $C$, then

$$\oint_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k). \tag{5}$$

- L'Hôpital's rule is valid for complex analysis.

## 19.6 Evaluation of Real Integrals

- An integral of the form

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

where $F$ is a rational function can be evaluated by converting it to a complex integral where the contour is the unit circle centred at the origin

$$\oint_C F\left(\frac{1}{2}(z + z^{-1}), \frac{1}{2i}(z - z^{-1})\right) \frac{dz}{iz}$$

where $C$ is $|z| = 1$.

- An improper integral of the form $\int_{-\infty}^{\infty} f(x)\, dx$ is defined in terms of two limits

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{r \to \infty} \int_{-r}^{0} f(x)\, dx + \lim_{R \to \infty} \int_0^R f(x)\, dx.$$

If both limits exist, the integral is said to be **convergent**. If one or both of the limits fail to exist the integral is said to be **divergent**.

- If we know a priori that an improper integral converges we can evaluate it with a single limit
$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx.$$
However, this limit may exist even if the improper integral is divergent in which case it is called the **Cauchy principal value** and is denoted
$$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx.$$

- An integral of the form
$$\int_{-\infty}^{\infty} f(x)\,dx$$
where $f(x) = P(x)/Q(x)$ is continuous on $(-\infty,\infty)$ can be evaluated by replacing $x$ with the complex variable $z$ and integrating over a closed contour $C$ consisting of the interval $[-R,R]$ on the real axis and a semicircle $C_R$ of radius large enough to enclose all the poles of $f(z) = P(z)/Q(z)$ in the upper half-plane $\text{Re}(z) > 0$. By Cauchy's residue theorem we have
$$\oint_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i\sum_{k=1}^{n}\text{Res}(f(z),z_k)$$
and if we assume $\int_{C_R} f(z)\,dz \to 0$ as $R\to\infty$ we get
$$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\sum_{k=1}^{n}\text{Res}(f(z),z_k).$$

**Theorem 19.6.1 Behavior of Integral as $R\to\infty$**

Suppose $f(z) = P(z)/Q(z)$, where the degree of $P(z)$ is $n$ and the degree of $Q(z)$ is $m \geq n+2$. If $C_R$ is a semicircular contour $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, then $\int_{C_R} f(z)\,dz \to 0$ as $R\to\infty$.

- Integrals of the form $\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx$ and $\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx$, $\alpha > 0$ are referred to as **Fourier integrals**. They appear as the real an imaginary parts in the improper integal $\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx$.

- When $f(x) = P(x)/Q(x)$ is continuous on $(-\infty,\infty)$ we can evaluate both forms of Fourier integrals at the same time by considering the integral $\int_C f(z)e^{i\alpha z}\,dz$ where $\alpha > 0$ and and the contour $C$ consists of the interval $[-R,R]$ on the real axis and a semicircular contour $C_R$ with radius large enough to enclose the poles of $f(z)$ in the upper half-plane.

**Theorem 19.6.2 Behavior of Integral as $R \to \infty$**

Suppose $f(z) = P(z)/Q(z)$, where the degree of $P(z)$ is $n$ and the degree of $Q(z)$ is $m \geq n + 1$. If $C_R$ is a semicircular contour $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, and $\alpha > 0$, then $\int_{C_R} (P(z)/Q(z))e^{i\alpha z}\,dz \to 0$ as $R \to \infty$.

- The above approaches to evaluating integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$, $\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx$, and $\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx$ all assume $f(x)$ is continuous on $(-\infty, \infty)$. If that's not the case and $f(x)$ has a pole at $z = c$ we instead use an **indented contour** where a semicircular contour centred at $z = c$ is included to bypass the pole.

**Theorem 19.6.3 Behavior of Integral as $r \to 0$**

Suppose $f$ has a simple pole $z = c$ on the real axis. If $C_r$ is the contour defined by $z = c + re^{i\theta}$, $0 \leq \theta \leq \pi$, then

$$\lim_{r\to 0} \int_{C_r} f(z)\,dz = \pi i \operatorname{Res}(f(z), c).$$

- Using the above theorem we can evaluate an integral where $f(x)$ has a pole on the real axis at $z = c$ by replacing $x$ with the complex variable $z$ and integrating over a closed contour $C$ consisting of the interval $[-R, c-r]$, a positively-oriented semicircle $C_r$ of radius $r$ centred at $z = c$, the interval $[c+r, R]$, and a semicircle $C_R$ of radius $R$ centred at $z = 0$. By Cauchy's residue theorem we have

$$\oint_C = \int_{-R}^{c-r} + \int_{-C_r} + \int_{c+r}^{R} + \int_{C_R} = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k)$$

and by theorem 19.6.3 as we take the limit $R \to \infty$ and $r \to 0$ we get

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = \pi i \operatorname{Res}(f(z), c) + 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k).$$