

Introduction to Electrodynamics by David J.
Griffiths Problems

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2 Electrostatics

2.1

- (a) **0**
- (b) The same as if only the opposite charge were present — all others are cancelled out.

2.2

$$\begin{aligned}
 \mathbf{E} &= \frac{1}{4\pi\epsilon_0} 2 \frac{q}{z^2} \cos\theta \hat{\mathbf{x}} \\
 &= \frac{1}{4\pi\epsilon_0} \frac{dq}{[(d/2)^2 + z^2]^{3/2}} \hat{\mathbf{x}}
 \end{aligned}$$

2.3

$$\begin{aligned}
\mathbf{r} &= z\hat{\mathbf{z}} \\
\mathbf{r}' &= x\hat{\mathbf{x}} \\
\mathbf{r} &= z\hat{\mathbf{z}} - x\hat{\mathbf{x}} \\
r &= \sqrt{x^2 + z^2} \\
\hat{\mathbf{r}} &= \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{\sqrt{x^2 + z^2}} \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_0^L \frac{\lambda}{x^2 + z^2} \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{\sqrt{x^2 + z^2}} dx \\
&= \frac{1}{4\pi\epsilon_0} \lambda \left(z\hat{\mathbf{z}} \int_0^L \frac{1}{(x^2 + z^2)^{3/2}} dx - \hat{\mathbf{x}} \int_0^L \frac{x}{(x^2 + z^2)} dx \right) \\
&= \frac{1}{4\pi\epsilon_0} \lambda \left[\frac{L}{z\sqrt{L^2 + z^2}} \hat{\mathbf{z}} - \left(\frac{1}{z} - \frac{1}{\sqrt{L^2 + z^2}} \right) \hat{\mathbf{x}} \right] \\
&= \frac{1}{4\pi\epsilon_0} \frac{\lambda}{z} \left[\left(-1 + \frac{z}{\sqrt{L^2 + z^2}} \right) \hat{\mathbf{x}} + \frac{L}{\sqrt{L^2 + z^2}} \hat{\mathbf{z}} \right]
\end{aligned}$$

2.4

The electric field a distance z above the midpoint of a line segment of length $2L$ and uniform line charge λ is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z\sqrt{z^2 + L^2}} \hat{\mathbf{z}}.$$

Applying this to the four sides of the square, the horizontal components of opposite sides cancel leaving only the vertical component.

$$\begin{aligned}
\cos \theta &= \frac{z}{r} \\
&= \frac{z}{\sqrt{(a/2)^2 + z^2}} \\
\mathbf{E} &= 4 \left(\frac{1}{4\pi\epsilon_0} \frac{\lambda a}{\sqrt{(a/2)^2 + z^2} \sqrt{(a/2)^2 + (a/2)^2 + z^2}} \hat{\mathbf{z}} \right) \cos \theta \\
&= \frac{1}{4\pi\epsilon_0} \frac{4a\lambda z}{[(a/2)^2 + z^2] \sqrt{(a/2)^2 + z^2}} \hat{\mathbf{z}}
\end{aligned}$$

2.5

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \frac{\lambda r}{r^2 + z^2} \cos \alpha \, d\theta \, \hat{\mathbf{z}} \\ &= \frac{1}{4\pi\epsilon_0} \frac{2\pi\lambda r z}{(r^2 + z^2)^{3/2}} \hat{\mathbf{z}}\end{aligned}$$

2.6

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int \frac{dq}{z^2} \cos \theta \hat{\mathbf{z}} \\ &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^R \frac{\sigma}{r^2 + z^2} \frac{z}{\sqrt{r^2 + z^2}} r \, dr \, d\theta \hat{\mathbf{z}} \\ &= \frac{1}{4\pi\epsilon_0} 2\pi\sigma z \int_0^R \frac{r}{(r^2 + z^2)^{3/2}} \, dr \hat{\mathbf{z}} \\ &= \frac{1}{4\pi\epsilon_0} 2\pi\sigma z \left(\frac{1}{z} - \frac{1}{\sqrt{R^2 + z^2}} \right) \hat{\mathbf{z}}\end{aligned}$$

When $R \rightarrow \infty$

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0} \hat{\mathbf{z}}.$$

2.7

$$\mathbf{E} = \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q}{z^2} \hat{\mathbf{z}} & z > R \\ \mathbf{0} & z < R \end{cases}$$

2.8

$$\mathbf{E} = \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q}{z^2} \hat{\mathbf{z}} & z > R \\ \frac{1}{4\pi\epsilon_0} \frac{qz}{R^3} \hat{\mathbf{z}} & z < R \end{cases}$$

2.9

(a)

$$\begin{aligned}\rho &= \epsilon_0 \nabla \cdot \mathbf{E} \\ &= \epsilon_0 \frac{1}{r^2} \frac{\partial}{\partial r} (kr^5) \\ &= 5\epsilon_0 kr^2\end{aligned}$$

(b)

$$\begin{aligned}
Q_{\text{enc}} &= \epsilon_0 \oint \mathbf{E} \cdot d\mathbf{a} \\
&= \epsilon_0 \int_0^{2\pi} \int_0^\pi kR^3 R d\theta R \sin \theta d\phi \\
&= 2\pi\epsilon_0 kR^5 [-\cos \theta]_0^\pi \\
&= 4\pi\epsilon_0 kR^5 \\
Q_{\text{enc}} &= \int_V \rho d\tau \\
&= \int_0^{2\pi} \int_0^\pi \int_0^R 5\epsilon_0 k r^2 dr r d\theta r \sin \theta d\phi \\
&= 10\pi\epsilon_0 k \int_0^\pi \int_0^R r^4 \sin \theta dr d\theta \\
&= 2\pi\epsilon_0 kR^5 [-\cos \theta]_0^\pi \\
&= 4\pi\epsilon_0 kR^5
\end{aligned}$$

2.10

If the charge was surrounded by 8 such cubes the total flux through all the cubes would be q/ϵ_0 . There are 24 outside faces to the larger cube, so the total flux through the shaded face is $q/(24\epsilon_0)$.

2.11

$$\begin{aligned}
\int \mathbf{E}_{\text{inside}} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
&= 0 \\
\mathbf{E}_{\text{inside}} &= \mathbf{0} \\
\int \mathbf{E}_{\text{outside}} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E_{\text{outside}} &= \frac{4\pi R^2 \sigma}{\epsilon_0} \\
\mathbf{E}_{\text{outside}} &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}
\end{aligned}$$

2.12

$$\begin{aligned}\int \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\ 4\pi r^2 E &= \frac{\frac{4}{3}\pi r^3 \rho}{\epsilon_0} \\ \mathbf{E} &= \frac{r\rho}{3\epsilon_0} \hat{\mathbf{r}}\end{aligned}$$

2.13

$$\begin{aligned}\int \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\ 2\pi s l E &= \frac{l\lambda}{\epsilon_0} \\ \mathbf{E} &= \frac{1}{2\pi\epsilon_0} \frac{\lambda}{s} \hat{\mathbf{s}}\end{aligned}$$

2.14

$$\begin{aligned}Q_{\text{enc}} &= \int_V \rho \, d\tau \\ &= \int_0^{2\pi} \int_0^\pi \int_0^r k r'^3 \sin \theta \, dr' \, d\theta \, d\phi \\ &= 2\pi k \int_0^\pi \left[\frac{1}{4} r'^4 \sin \theta \right]_0^r d\theta \\ &= \frac{1}{2} \pi k r^4 [-\cos \theta]_0^\pi \\ &= \pi k r^4 \\ \int \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\ 4\pi r^2 E &= \frac{\pi k r^4}{\epsilon_0} \\ \mathbf{E} &= \frac{k r^2}{4\epsilon_0} \hat{\mathbf{r}}\end{aligned}$$

2.15

(a) $\mathbf{E} = \mathbf{0}$

(b)

$$\begin{aligned}
 Q_{\text{enc}} &= \int_0^{2\pi} \int_0^\pi \int_a^r k \sin \theta \, dr' \, d\theta \, d\phi \\
 &= 4\pi k(r-a) \\
 4\pi r^2 E &= \frac{4\pi k(r-a)}{\epsilon_0} \\
 \mathbf{E} &= \frac{k(r-a)}{\epsilon_0 r^2} \hat{\mathbf{r}}
 \end{aligned}$$

(c) $\mathbf{E} = \frac{k(b-a)}{\epsilon_0 r^2} \hat{\mathbf{r}}$

2.16

(a)

$$\begin{aligned}
 Q_{\text{enc}} &= \pi s^2 l \rho \\
 2\pi s l E &= \frac{\pi s^2 l \rho}{\epsilon_0} \\
 \mathbf{E} &= \frac{s\rho}{2\epsilon_0} \hat{\mathbf{s}}
 \end{aligned}$$

(b)

$$\mathbf{E} = \frac{a^2 \rho}{2\epsilon_0 s} \hat{\mathbf{s}}$$

(c)

$$\mathbf{E} = \mathbf{0}$$

2.17

$$\begin{aligned}
 2AE_{\text{inside}} &= \frac{2Ay\rho}{\epsilon_0} \\
 \mathbf{E}_{\text{inside}} &= \frac{y\rho}{\epsilon_0} \\
 \mathbf{E} &= \begin{cases} \frac{d\rho}{\epsilon_0} & d < y \\ \frac{y\rho}{\epsilon_0} & 0 < y < d \\ -\frac{y\rho}{\epsilon_0} & -d < y < 0 \\ -\frac{d\rho}{\epsilon_0} & y < -d \end{cases}
 \end{aligned}$$

2.18

The electric field inside a uniformly charged solid sphere is

$$\mathbf{E} = \frac{r\rho}{3\epsilon_0} \hat{\mathbf{r}}.$$

$$\begin{aligned}\mathbf{d} &= \mathbf{r}_1 - \mathbf{r}_2 \\ \mathbf{E} &= \frac{r_1\rho}{3\epsilon_0} \hat{\mathbf{r}}_1 - \frac{r_2\rho}{3\epsilon_0} \hat{\mathbf{r}}_2 \\ &= \frac{\rho}{3\epsilon_0} (\mathbf{r}_1 - \mathbf{r}_2) \\ &= \frac{\rho}{3\epsilon_0} \mathbf{d}\end{aligned}$$

2.20

a is impossible because its curl is nonzero.

$$\begin{aligned}V &= - \int_0^y 2kxy' dy' - \int_0^z 2kyz' dz \\ &= -2kx \left[\frac{1}{2}y'^2 \right]_0^y - 2ky \left[\frac{1}{2}z'^2 \right]_0^z \\ &= -k(xy^2 + yz^2) \\ -\nabla V &= k[y^2 \hat{\mathbf{x}} + (2xy + z^2) \hat{\mathbf{y}} + 2yz \hat{\mathbf{z}}] \\ &= \mathbf{E}\end{aligned}$$

2.21

$$\begin{aligned}
\mathbf{E} &= \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} & r > R \\ \frac{1}{4\pi\epsilon_0} \frac{qr}{R^3} & r < R \end{cases} \\
V_{\text{outside}}(r) &= - \int_{\infty}^r \frac{1}{4\pi\epsilon_0} \frac{q}{r'^2} dr' \\
&= - \frac{1}{4\pi\epsilon_0} q \left[-\frac{1}{r'} \right]_{\infty}^r \\
&= \frac{1}{4\pi\epsilon_0} \frac{q}{r} \\
-\nabla V_{\text{outside}} &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} \\
&= \mathbf{E}_{\text{outside}} \\
V_{\text{inside}}(r) &= - \left(\int_{\infty}^R \frac{1}{4\pi\epsilon_0} \frac{q}{r'^2} dr' + \int_R^r \frac{1}{4\pi\epsilon_0} \frac{qr'}{R^3} dr' \right) \\
&= - \left(-\frac{1}{4\pi\epsilon_0} \frac{q}{R} + \frac{1}{4\pi\epsilon_0} \frac{q}{R^3} \left[\frac{1}{2} r'^2 \right]_R^r \right) \\
&= \frac{1}{4\pi\epsilon_0} \frac{q}{2R} \left[3 - \left(\frac{r}{R} \right)^2 \right] \\
-\nabla V_{\text{inside}} &= \frac{1}{4\pi\epsilon_0} \frac{qr}{R^3} \hat{\mathbf{r}} \\
&= \mathbf{E}_{\text{inside}}
\end{aligned}$$

2.22

$$\begin{aligned}
\mathbf{E} &= \frac{1}{2\pi\epsilon_0} \frac{\lambda}{s} \hat{\mathbf{s}} \\
V &= - \int_O^s \frac{1}{2\pi\epsilon_0} \frac{\lambda}{s'} ds' \\
&= - \frac{1}{2\pi\epsilon_0} \lambda \ln \frac{s}{O} \\
-\nabla V &= \frac{1}{2\pi\epsilon_0} \frac{\lambda}{s} \hat{\mathbf{s}}
\end{aligned}$$

2.23

$$\begin{aligned}
\mathbf{E} &= \begin{cases} \mathbf{0} & r < a \\ \frac{k(r-a)}{\epsilon_0 r^2} \hat{\mathbf{r}} & a < r < b \\ \frac{k(b-a)}{\epsilon_0 r^2} \hat{\mathbf{r}} & b < r \end{cases} \\
V(0) &= - \int_{\infty}^0 E dr \\
&= - \left(\int_{\infty}^b \frac{k(b-a)}{\epsilon_0 r^2} dr + \int_b^a \frac{k(r-a)}{\epsilon_0 r^2} dr \right) \\
&= - \left(\frac{k(b-a)}{\epsilon_0} \left[-\frac{1}{r} \right]_{\infty}^b + \frac{k}{\epsilon_0} \left[\ln r + \frac{a}{r} \right]_b^a \right) \\
&= - \left[-\frac{k(b-a)}{\epsilon_0 b} + \frac{k}{\epsilon_0} \left(\ln a + 1 - \ln b - \frac{a}{b} \right) \right] \\
&= -\frac{k}{\epsilon_0} \left(-1 + \frac{a}{b} + \ln \frac{a}{b} + 1 - \frac{a}{b} \right) \\
&= \frac{k}{\epsilon_0} \ln \frac{b}{a}
\end{aligned}$$

2.24

$$\begin{aligned}
V(b) - V(0) &= - \int_0^b E dr \\
&= - \left(\int_0^a \frac{s\rho}{2\epsilon_0} ds + \int_a^b \frac{a^2\rho}{2\epsilon_0 s} ds \right) \\
&= - \left(\frac{\rho}{2\epsilon_0} \left[\frac{1}{2} s^2 \right]_0^a + \frac{a^2\rho}{2\epsilon_0} \ln \frac{b}{a} \right) \\
&= - \left(\frac{a^2\rho}{4\epsilon_0} + \frac{a^2\rho}{2\epsilon_0} \ln \frac{b}{a} \right) \\
&= -\frac{a^2\rho}{4\epsilon_0} \left(1 + 2 \ln \frac{a}{b} \right)
\end{aligned}$$

2.25

(a)

$$V = \frac{1}{4\pi\epsilon_0} \frac{2q}{\sqrt{(d/2)^2 + z^2}}$$

(b)

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0} \int_{-L}^L \frac{\lambda}{\sqrt{x^2 + z^2}} dx \\ &= \frac{1}{4\pi\epsilon_0} \lambda \ln \left(1 + \frac{2L(L + \sqrt{L^2 + z^2})}{z^2} \right) \end{aligned}$$

(c)

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^R \frac{\sigma}{\sqrt{r^2 + z^2}} r dr d\theta \\ &= \frac{1}{4\pi\epsilon_0} 2\pi\sigma(\sqrt{R^2 + z^2} - z) \end{aligned}$$

2.26

$$\begin{aligned} V_{\text{bottom}} &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^h \frac{\sqrt{2}\sigma z}{\sqrt{2}z} d\phi dz \\ &= \frac{\sigma h}{2\epsilon_0} \\ V_{\text{top}} &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^h \frac{\sqrt{2}\sigma z}{\sqrt{z^2 + (h-z)^2}} d\phi dz \\ &= \frac{\sqrt{2}\sigma}{2\epsilon_0} \int_0^h \frac{z}{\sqrt{z^2 + (h-z)^2}} dz \\ &= \frac{\sigma h}{4\epsilon_0} \ln(3 + 2\sqrt{2}) \\ V_{\text{bottom}} - V_{\text{top}} &= \frac{\sigma h}{2\epsilon_0} \left[1 - \frac{1}{2} \ln(3 + 2\sqrt{2}) \right] \end{aligned}$$

2.28

$$\begin{aligned} V(r) &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^\pi \int_0^R \frac{\rho r'^2 \sin \theta}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta}} dr' d\theta d\phi \\ &= \frac{\rho}{2\epsilon_0} \int_0^\pi \int_0^R \frac{r'^2 \sin \theta}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta}} dr' d\theta \\ &= \frac{\rho}{2\epsilon_0} \left(R^2 - \frac{r^2}{3} \right) \\ &= \frac{q}{8\pi\epsilon_0 R} \left(3 - \frac{r^2}{R^2} \right) \end{aligned}$$

2.31

(a)

$$W = \frac{q^2}{4\pi\epsilon_0 a} \left(\frac{1}{\sqrt{2}} - 2 \right)$$

(b)

$$\begin{aligned} W &= \frac{1}{4\pi\epsilon_0} \left(-\frac{q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a} - \frac{q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a} \right) \\ &= \frac{q^2}{2\pi\epsilon_0 a} \left(\frac{1}{\sqrt{2}} - 2 \right) \end{aligned}$$

2.32

$$W = \frac{1}{4\pi\epsilon_0} \frac{q_A q_B}{a}$$

$$W = K_1 + K_2$$

$$\frac{1}{4\pi\epsilon_0} \frac{q_A q_B}{a} = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2$$

$$\frac{1}{2\pi\epsilon_0} \frac{q_A q_B}{a} = m_A v_A^2 + m_B v_B^2$$

$$0 = m_B v_B - m_A v_A$$

$$v_B = \frac{m_A}{m_B} v_A$$

$$\frac{1}{2\pi\epsilon_0} \frac{q_A q_B}{a} = m_A v_A^2 + m_B \left(\frac{m_A}{m_B} v_A \right)^2$$

$$= m_A v_A^2 + \frac{m_A^2}{m_B} v_A^2$$

$$= \frac{m_A(m_A + m_B)}{m_B} v_A^2$$

$$v_A = \sqrt{\frac{1}{2\pi\epsilon_0} \frac{q_A q_B}{(m_A + m_B)a} \frac{m_B}{m_A}}$$

$$v_B = \sqrt{\frac{1}{2\pi\epsilon_0} \frac{q_A q_B}{(m_A + m_B)a} \frac{m_A}{m_B}}$$

2.33

$$\begin{aligned}
 W &= \frac{1}{4\pi\epsilon_0} \left(-\frac{q^2}{a} + \frac{q^2}{2a} - \frac{q^2}{3a} + \dots \right) \\
 &= \frac{1}{4\pi\epsilon_0} \frac{q^2}{a} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \\
 &= -\frac{1}{4\pi\epsilon_0} \frac{q^2}{a} \ln 2
 \end{aligned}$$

2.34

(a)

$$\begin{aligned}
 V &= \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q}{2R} \left[3 - \left(\frac{r}{R} \right)^2 \right] & r < R \\ \frac{1}{4\pi\epsilon_0} \frac{q}{r} & r > R \end{cases} \\
 W &= \frac{1}{2} \int \rho V d\tau \\
 &= \frac{1}{2} \int_0^{2\pi} \int_0^\pi \int_0^R \rho \frac{1}{4\pi\epsilon_0} \frac{q}{2R} \left[3 - \left(\frac{r}{R} \right)^2 \right] r^2 \sin \theta dr d\theta d\phi \\
 &= \frac{q\rho}{8\epsilon_0 R} \int_0^\pi \int_0^R \left[3 - \left(\frac{r}{R} \right)^2 \right] r^2 \sin \theta dr d\theta \\
 &= \frac{q\rho R^2}{5\epsilon_0} \\
 &= \frac{qR^2}{5\epsilon_0} \frac{q}{\frac{4}{3}\pi R^3} \\
 &= \frac{1}{4\pi\epsilon_0} \frac{3q^2}{5R}
 \end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{E} &= \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} & r > R \\ \frac{1}{4\pi\epsilon_0} \frac{qr}{R^3} \hat{\mathbf{r}} & r < R \end{cases} \\
E^2 &= \begin{cases} \frac{1}{16\pi^2\epsilon_0^2} \frac{q^2}{r^4} & r > R \\ \frac{1}{16\pi^2\epsilon_0^2} \frac{q^2 r^2}{R^6} & r < R \end{cases} \\
W &= \frac{\epsilon_0}{2} \int E^2 d\tau \\
&= \frac{\epsilon_0}{2} \left(\int_0^{2\pi} \int_0^\pi \int_0^R \frac{1}{16\pi^2\epsilon_0^2} \frac{q^2 r^2}{R^6} r^2 \sin\theta \, dr \, d\theta \, d\phi \right. \\
&\quad \left. + \int_0^{2\pi} \int_0^\pi \int_R^\infty \frac{1}{16\pi^2\epsilon_0^2} \frac{q^2}{r^4} r^2 \sin\theta \, dr \, d\theta \, d\phi \right) \\
&= \frac{\epsilon_0}{2} \frac{1}{16\pi^2\epsilon_0^2} 2\pi q^2 \left(\int_0^\pi \int_0^R \frac{r^4}{R^6} \sin\theta \, dr \, d\theta + \int_0^\pi \int_R^\infty \frac{1}{r^2} \sin\theta \, dr \, d\theta \right) \\
&= \frac{1}{16\pi\epsilon_0} q^2 \left(\int_0^\pi \int_0^R \frac{r^4}{R^6} \sin\theta \, dr \, d\theta + \int_0^\pi \int_R^\infty \frac{1}{r^2} \sin\theta \, dr \, d\theta \right) \\
&= \frac{1}{16\pi\epsilon_0} q^2 \left(\frac{2}{5R} + \frac{2}{R} \right) \\
&= \frac{1}{4\pi\epsilon_0} \frac{3q^2}{5R}
\end{aligned}$$

(c)

$$\begin{aligned}
W &= \frac{\epsilon_0}{2} \left(\int_V E^2 d\tau + \oint_S V \mathbf{E} \cdot d\mathbf{a} \right) \\
&= \frac{\epsilon_0}{2} \left(\int_0^{2\pi} \int_0^\pi \int_0^R \frac{1}{(4\pi\epsilon_0)^2} \frac{q^2 r^2}{R^6} r^2 \sin \theta dr d\theta d\phi \right. \\
&\quad + \int_0^{2\pi} \int_0^\pi \int_R^a \frac{1}{(4\pi\epsilon_0)^2} \frac{q^2}{r^4} r^2 \sin \theta dr d\theta d\phi \\
&\quad \left. + \int_0^{2\pi} \int_0^\pi \frac{1}{4\pi\epsilon_0} \frac{q}{a} \frac{1}{4\pi\epsilon_0} \frac{q}{a^2} a^2 \sin \theta d\theta d\phi \right) \\
&= \frac{\epsilon_0}{2} \frac{1}{(4\pi\epsilon_0)^2} 2\pi q^2 \left(\int_0^\pi \int_0^R \frac{r^4}{R^6} \sin \theta dr d\theta \right. \\
&\quad \left. + \int_0^\pi \int_R^a \frac{1}{r^2} \sin \theta dr d\theta + \int_0^\pi \frac{1}{a} \sin \theta d\theta \right) \\
&= \frac{\epsilon_0}{2} \frac{1}{(4\pi\epsilon_0)^2} 2\pi q^2 \left[\frac{2}{5R} + 2 \left(\frac{1}{R} - \frac{1}{a} \right) + \frac{2}{a} \right] \\
&= \frac{1}{8\pi\epsilon_0} q^2 \left[\frac{1}{5R} + \frac{1}{R} \right] \\
&= \frac{1}{4\pi\epsilon_0} \frac{3q^2}{5R}
\end{aligned}$$

2.36

(a)

$$\begin{aligned}
\mathbf{E} &= \begin{cases} \mathbf{0} & r < a \\ \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} & a < r < b \\ \mathbf{0} & b < r \end{cases} \\
E^2 &= \begin{cases} 0 & r < a \\ \frac{1}{(4\pi\epsilon_0)^2} \frac{q^2}{r^4} & a < r < b \\ 0 & b < r \end{cases} \\
W &= \frac{\epsilon_0}{2} \int E^2 d\tau \\
&= \frac{\epsilon_0}{2} \int_0^{2\pi} \int_0^\pi \int_a^b \frac{1}{(4\pi\epsilon_0)^2} \frac{q^2}{r^4} r^2 \sin \theta dr d\theta d\phi \\
&= \frac{\epsilon_0}{2} \frac{1}{(4\pi\epsilon_0)^2} 2\pi q^2 \int_0^\pi \int_a^b \frac{\sin \theta}{r^2} dr d\theta \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)
\end{aligned}$$

(b)

$$\begin{aligned}
W_{\text{shell}} &= \frac{1}{8\pi\epsilon_0} \frac{q^2}{R} \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} \\
\mathbf{E}_1 \cdot \mathbf{E}_2 &= -\frac{1}{(4\pi\epsilon_0)^2} \frac{q^2}{r^4} \\
W_{\text{total}} &= W_1 + W_2 + \epsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2 d\tau \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} + \frac{1}{b} \right) - \epsilon_0 \int_0^{2\pi} \int_0^\pi \int_b^\infty \frac{1}{(4\pi\epsilon_0)^2} \frac{q^2}{r^4} r^2 \sin\theta dr d\theta d\phi \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} + \frac{1}{b} \right) - \frac{1}{8\pi\epsilon_0} q^2 \int_0^\pi \int_b^\infty \frac{1}{r^2} \sin\theta dr d\theta \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} + \frac{1}{b} \right) - \frac{1}{4\pi\epsilon_0} q^2 \int_b^\infty \frac{1}{r^2} dr \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} + \frac{1}{b} \right) - \frac{1}{4\pi\epsilon_0} \frac{q^2}{b} \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} + \frac{1}{b} - \frac{2}{b} \right) \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)
\end{aligned}$$

2.37

$$\begin{aligned}
r_1 &= r \\
E_1 &= \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_1^2} \\
&= \frac{1}{4\pi\epsilon_0} \frac{q_1}{r^2} \\
r_2 &= \sqrt{a^2 + r^2 - 2ar \cos \theta} \\
E_2 &= \frac{1}{4\pi\epsilon_0} \frac{q_2}{r_2^2} \\
&= \frac{1}{4\pi\epsilon_0} \frac{q_2}{a^2 + r^2 - 2ar \cos \theta} \\
\cos \alpha &= \frac{r - a \cos \theta}{\sqrt{a^2 + r^2 - 2ar \cos \theta}} \\
\mathbf{E}_1 \cdot \mathbf{E}_2 &= E_1 E_2 \cos \alpha \\
&= \frac{1}{(4\pi\epsilon_0)^2} \frac{q_1 q_2}{r^2 (a^2 + r^2 - 2ar \cos \theta)} \frac{r - a \cos \theta}{\sqrt{a^2 + r^2 - 2ar \cos \theta}} \\
&= \frac{1}{(4\pi\epsilon_0)^2} \frac{q_1 q_2 (r - a \cos \theta)}{r^2 (a^2 + r^2 - 2ar \cos \theta)^{3/2}} \\
\epsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2 d\tau &= \epsilon_0 \int_0^{2\pi} \int_0^\pi \int_0^\infty \frac{1}{(4\pi\epsilon_0)^2} \frac{q_1 q_2 (r - a \cos \theta)}{r^2 (a^2 + r^2 - 2ar \cos \theta)^{3/2}} r^2 \sin \theta dr d\theta d\phi \\
&= \frac{q_1 q_2}{8\pi\epsilon_0} \int_0^\pi \int_0^\infty \frac{(r - a \cos \theta) \sin \theta}{(a^2 + r^2 - 2ar \cos \theta)^{3/2}} dr d\theta
\end{aligned}$$

2.38

(a)

$$\begin{aligned}
\sigma_R &= \frac{q}{4\pi R^2} \\
\sigma_a &= -\frac{q}{4\pi a^2} \\
\sigma_b &= \frac{q}{4\pi b^2}
\end{aligned}$$

(b)

$$\begin{aligned}
V &= -\int_\infty^b \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} dr - \int_a^R \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} dr \\
&= \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{b} + \frac{1}{R} - \frac{1}{a} \right)
\end{aligned}$$

(c)

$$\sigma_b = 0$$

$$V = \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{R} - \frac{1}{a} \right)$$

2.39

(a)

$$\sigma_a = -\frac{q_a}{4\pi a^2}$$

$$\sigma_b = -\frac{q_b}{4\pi b^2}$$

$$\sigma_R = \frac{q_a + q_b}{4\pi R^2}$$

(b)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q_a + q_b}{r^2} \hat{\mathbf{r}}$$

(c)

$$\mathbf{E}_a = \frac{1}{4\pi\epsilon_0} \frac{q_a}{r^2} \hat{\mathbf{r}}$$

$$\mathbf{E}_b = \frac{1}{4\pi\epsilon_0} \frac{q_b}{r^2} \hat{\mathbf{r}}$$

(d)

$$\mathbf{0}$$

(e) a, b

2.40

(a) No. If it's close to the wall it will induce a surface charge and be attracted.

(b) No. If the conductor contains a cavity containing a like charge it will be repelled.

2.41

By Gauss's law, the electric field of each plate is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$2A'E = \frac{A' \frac{Q}{A}}{\epsilon_0}$$

$$\mathbf{E} = \frac{Q}{2A\epsilon_0} \hat{\mathbf{n}}$$

so the field between the plates is zero and the field outside is $Q/A\epsilon_0\hat{\mathbf{n}}$, resulting in a pressure of

$$\begin{aligned} P &= \frac{\epsilon_0}{2} E^2 \\ &= \frac{\epsilon_0}{2} \frac{Q^2}{A^2 \epsilon_0^2} \\ &= \frac{Q^2}{2A^2 \epsilon_0} \end{aligned}$$

2.42

$$\begin{aligned} \mathbf{E}_{\text{above}} &= \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}} \\ \mathbf{f} &= \frac{1}{2} \sigma \mathbf{E}_{\text{above}} \\ &= \frac{1}{2} \frac{Q}{4\pi R^2} \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \hat{\mathbf{r}} \\ &= \frac{Q^2}{32\pi^2 \epsilon_0 R^4} \hat{\mathbf{r}} \\ \mathbf{F} &= \int_0^{2\pi} \int_0^{\pi/2} \frac{Q^2}{32\pi^2 \epsilon_0 R^4} \cos \theta R^2 \sin \theta d\theta d\phi \hat{\mathbf{z}} \\ &= \frac{Q^2}{16\pi\epsilon_0 R^2} \int_0^{\pi/2} \cos \theta \sin \theta d\theta \hat{\mathbf{z}} \\ &= \frac{Q^2}{32\pi\epsilon_0 R^2} \hat{\mathbf{z}} \end{aligned}$$

2.43

$$\begin{aligned}
 \oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q}{\epsilon_0} \\
 2\pi s L E &= \frac{Q}{\epsilon_0} \\
 \mathbf{E} &= \frac{Q}{2\pi L \epsilon_0 s} \hat{\mathbf{s}} \\
 V &= - \int_b^a \frac{Q}{2\pi \epsilon_0 L s} \frac{1}{s} dr \\
 &= \frac{Q}{2\pi \epsilon_0 L} \ln \frac{b}{a} \\
 C &= \frac{Q}{V} \\
 &= \frac{2\pi \epsilon_0 L}{\ln b/a}
 \end{aligned}$$

So the capacitance per unit length is

$$C = \frac{2\pi \epsilon_0}{\ln b/a}.$$

2.44

(a)

$$\begin{aligned}
 P &= \frac{\epsilon_0}{2} E^2 \\
 W &= Fd \\
 &= PA\epsilon \\
 &= \frac{\epsilon_0}{2} E^2 A\epsilon
 \end{aligned}$$

(b)

$$\frac{\epsilon_0}{2} E^2 A\epsilon$$

2.46

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 3 \frac{k}{r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{k}{r} 2 \sin \theta \cos \theta \sin \phi \right) \\
&\quad + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left(\frac{k}{r} \sin \theta \cos \phi \right) \\
&= \frac{3k}{r^2} + \frac{1}{r \sin \theta} \frac{2k}{r} \sin \phi (2 \sin \theta \cos^2 \theta - \sin^3 \theta) - \frac{1}{r \sin \theta} \frac{k}{r} \sin \theta \sin \phi \\
&= \frac{3k}{r^2} + \frac{2k \sin \phi}{r^2} (2 \cos^2 \theta - \sin^2 \theta) - \frac{k}{r^2} \sin \phi \\
&= \frac{k}{r^2} [3 + 2 \sin \phi (2 \cos^2 \theta - \sin^2 \theta) - \sin \phi] \\
&= \frac{k}{r^2} [3 + \sin \phi (4 \cos^2 \theta - 2 \sin^2 \theta - 1)] \\
&= \frac{k}{r^2} [3 + \sin \phi (6 \cos^2 \theta - 3)] \\
&= \frac{3k}{r^2} (1 + \cos 2\theta \sin \phi) \\
\rho &= \epsilon_0 \nabla \cdot \mathbf{E} \\
&= \frac{3k\epsilon_0}{r^2} (1 + \cos 2\theta \sin \phi)
\end{aligned}$$

2.47

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \frac{Qr}{R^3} \hat{\mathbf{r}} \\
\rho &= \frac{Q}{\frac{4}{3}\pi R^3} \\
\rho \mathbf{E} &= \frac{3Q}{4\pi R^3} \frac{1}{4\pi\epsilon_0} \frac{Qr}{R^3} \hat{\mathbf{r}} \\
&= \frac{3r}{\epsilon_0} \left(\frac{Q}{4\pi R^3} \right)^2 \hat{\mathbf{r}} \\
F_z &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \frac{3r}{\epsilon_0} \left(\frac{Q}{4\pi R^3} \right)^2 \cos \theta r^2 \sin \theta \, dr \, d\theta \, d\phi \\
&= \frac{3\pi}{\epsilon_0} \left(\frac{Q}{4\pi R^3} \right)^2 \int_0^{\pi/2} \int_0^R r^3 \sin 2\theta \, dr \, d\theta \\
&= \frac{3\pi}{\epsilon_0} \left(\frac{Q}{4\pi R^3} \right)^2 \frac{R^4}{4} \\
&= \frac{3Q^2}{64\pi\epsilon_0 R^2}
\end{aligned}$$

2.49

$$\begin{aligned}
Q_{\text{enc}} &= \int_0^{2\pi} \int_0^\pi \int_0^r k r'^3 \sin \theta \, dr' \, d\theta \, d\phi \\
&= 2\pi k \int_0^\pi \int_0^r r'^3 \sin \theta \, dr' \, d\theta \\
&= \pi k r^4 \\
\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E &= \frac{\pi k r^4}{\epsilon_0} \\
\mathbf{E} &= \begin{cases} \frac{k r^2}{4\epsilon_0} \hat{\mathbf{r}} & r < R \\ \frac{k R^4}{4\epsilon_0 r^2} \hat{\mathbf{r}} & r > R \end{cases} \\
W &= \frac{\epsilon_0}{2} \left(\int_0^{2\pi} \int_0^\pi \int_0^R \frac{k^2 r^4}{16\epsilon_0^2} \sin \theta \, dr \, d\theta \, d\phi \right. \\
&\quad \left. \int_0^{2\pi} \int_0^\pi \int_R^\infty \frac{k^2 R^8}{16\epsilon_0^2 r^4} \sin \theta \, dr \, d\theta \, d\phi \right) \\
&= \frac{\epsilon_0}{2} 2\pi \frac{k^2}{16\epsilon_0^2} \left(\int_0^\pi \int_0^R r^6 \sin \theta \, dr \, d\theta + \int_0^\pi \int_R^\infty \frac{R^8 \sin \theta}{r^2} \, dr \, d\theta \right) \\
&= \frac{\pi k^2}{16\epsilon_0} \left(\frac{2R^7}{7} + 2R^7 \right) \\
&= \frac{\pi k^2 R^7}{7\epsilon_0}
\end{aligned}$$

2.50

$$\begin{aligned}
V(\mathbf{r}) &= A \frac{e^{-\lambda r}}{r} \\
\mathbf{E} &= -\nabla V \\
&= A e^{-\lambda r} (1 + \lambda r) \frac{\hat{\mathbf{r}}}{r^2} \\
\rho &= \epsilon_0 \nabla \cdot \mathbf{E} \\
&= \epsilon_0 \left[A e^{-\lambda r} (1 + \lambda r) \nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} + \frac{\hat{\mathbf{r}}}{r^2} \cdot \nabla (A e^{-\lambda r} (1 + \lambda r)) \right] \\
&= A \epsilon_0 \left[4\pi \delta(\mathbf{r}) + \frac{\hat{\mathbf{r}}}{r^2} \cdot (-\lambda^2 e^{-\lambda r} r \hat{\mathbf{r}}) \right] \\
&= A \epsilon_0 \left(4\pi \delta(\mathbf{r}) - \frac{\lambda^2 e^{-\lambda r}}{r} \right)
\end{aligned}$$

2.51

$$\begin{aligned}
V &= \int \frac{1}{4\pi\epsilon_0} \frac{\sigma}{z} dA \\
&= \frac{\sigma}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^R \frac{r}{\sqrt{r^2 + R^2 - 2rR\cos\theta}} dr d\theta \\
&= \frac{R\sigma}{4\pi\epsilon_0} \int_0^{2\pi} \left[\cos\theta \ln \left(1 + \csc \frac{\theta}{2} \right) + 2 \sin \frac{\theta}{2} - 1 \right] d\theta \\
&= \frac{R\sigma}{\pi\epsilon_0}
\end{aligned}$$

2.52

(a)

$$\begin{aligned}
V_- &= \frac{1}{2\pi\epsilon_0} \lambda \ln \frac{s_-}{a} \\
&= \frac{1}{2\pi\epsilon_0} \lambda \ln \frac{\sqrt{(y+a)^2 + z^2}}{a} \\
V_+ &= -\frac{1}{2\pi\epsilon_0} \lambda \ln \frac{s_+}{a} \\
&= -\frac{1}{2\pi\epsilon_0} \lambda \ln \frac{\sqrt{(y-a)^2 + z^2}}{a} \\
V &= V_- + V_+ \\
&= \frac{1}{4\pi\epsilon_0} \lambda \ln \frac{(y+a)^2 + z^2}{(y-a)^2 + z^2}
\end{aligned}$$

2.53

(a)

$$\begin{aligned}
\nabla^2 V &= -\frac{\rho}{\epsilon_0} \\
\nabla \cdot \nabla V &= -\frac{\rho}{\epsilon_0} \\
\nabla \cdot \frac{dV}{dx} \hat{\mathbf{x}} &= -\frac{\rho}{\epsilon_0} \\
\frac{d^2 V}{dx^2} &= -\frac{\rho}{\epsilon_0}
\end{aligned}$$

(b)

$$qV = \frac{1}{2}mv^2$$
$$v = \sqrt{\frac{2qV}{m}}$$

(c)

$$I = A\rho v$$

(d)

$$\frac{d^2V}{dx^2} = -\frac{I}{Av\epsilon_0}$$
$$= -\frac{I}{A\epsilon_0}\sqrt{\frac{m}{2qV}}$$
$$= \beta V^{-1/2}$$

2.55

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E}$$
$$= a\epsilon_0$$

2.56

$$E = \frac{3GM^2}{5R}$$
$$E_{\text{sun}} = 2.3 \times 10^{41} \text{ J}$$
$$t = \frac{E_{\text{sun}}}{P}$$
$$= 1.89 \times 10^7 \text{ years}$$

3 Potentials

3.1

$$\begin{aligned} V_{\text{ave}} &= \frac{q}{4\pi\epsilon_0} \frac{1}{2zR} \left. \sqrt{z^2 + R^2 - 2zR \cos \theta} \right|_0^\pi \\ &= \frac{q}{4\pi\epsilon_0} \frac{1}{2zR} \left(\sqrt{z^2 + R^2 + 2zR} - \sqrt{z^2 + R^2 - 2zR} \right) \\ &= \frac{q}{4\pi\epsilon_0} \frac{1}{2zR} \left(\sqrt{(z+R)^2} - \sqrt{(R-z)^2} \right) \\ &= \frac{q}{4\pi\epsilon_0} \frac{1}{2zR} (z+R - R+z) \\ &= \frac{1}{4\pi\epsilon_0} \frac{q}{R} \end{aligned}$$

The average potential due to external charges is V_{center} and the average potential due to internal charges is

$$\frac{1}{4\pi\epsilon_0} \frac{Q_{\text{enc}}}{R}$$

so

$$V_{\text{ave}} = V_{\text{center}} + \frac{1}{4\pi\epsilon_0} \frac{Q_{\text{enc}}}{R}.$$

3.2

A stable equilibrium is a minimum of potential energy. Laplace's equation doesn't allow for minimums, so they must be saddle points and the charge can escape.

3.3

$$\begin{aligned}
0 &= \nabla^2 V \\
&= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) \\
&= \frac{1}{r^2} \left(2r \frac{\partial V}{\partial r} + r^2 \frac{\partial^2 V}{\partial r^2} \right) \\
&= \frac{2}{r} \frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} \\
V &= \frac{c_1}{r} + c_2 \\
0 &= \nabla^2 V \\
&= \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial V}{\partial s} \right) \\
&= \frac{1}{s} \left(\frac{\partial V}{\partial s} + s \frac{\partial^2 V}{\partial s^2} \right) \\
&= \frac{1}{s} \frac{\partial V}{\partial s} + \frac{\partial^2 V}{\partial s^2} \\
V &= c_1 + c_2 \ln s
\end{aligned}$$

3.7

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} q^2 \left(-\frac{2}{(2d)^2} + \frac{2}{(4d)^2} - \frac{1}{(6d)^2} \right) \hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0} \frac{29q^2}{72d^2} \hat{\mathbf{z}}
\end{aligned}$$

3.8

(a)

$$\begin{aligned}
V(r, \theta) &= \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{r^2 + a^2 - 2ra \cos \theta}} + \frac{q'}{\sqrt{r^2 + b^2 - 2rb \cos \theta}} \right] \\
&= \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{r^2 + a^2 - 2ra \cos \theta}} - \frac{Rq/a}{\sqrt{r^2 + (R^2/a)^2 - 2r(R^2/a) \cos \theta}} \right] \\
&= \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{r^2 + a^2 - 2ra \cos \theta}} - \frac{q}{\sqrt{R^2 + (ra/R)^2 - 2ra \cos \theta}} \right]
\end{aligned}$$

(b)

$$\begin{aligned}
\sigma &= -\epsilon_0 \left. \frac{\partial V}{\partial r} \right|_{r=R} \\
&= \frac{q}{4\pi R} \frac{R^2 - a^2}{(a^2 + R^2 - 2aR \cos \theta)^{3/2}} \\
Q_{\text{induced}} &= \int_0^{2\pi} \int_0^\pi \sigma R^2 \sin \theta \, d\theta \, d\phi \\
&= \frac{qR(R^2 - a^2)}{2} \int_0^\pi \frac{\sin \theta}{(a^2 + R^2 - 2aR \cos \theta)^{3/2}} \, d\theta \\
&= \frac{qR(R^2 - a^2)}{a(a^2 - R^2)} \\
&= -\frac{R}{a} q \\
&= q'
\end{aligned}$$

(c)

$$\begin{aligned}
W &= \frac{1}{2} qV \\
&= \frac{1}{8\pi\epsilon_0} \frac{qq'}{a - b} \\
&= -\frac{1}{8\pi\epsilon_0} \frac{q^2 R/a}{a - R^2/a} \\
&= -\frac{1}{8\pi\epsilon_0} \frac{q^2 R}{a^2 - R^2}
\end{aligned}$$

3.9

Place the second image charge at the centre of the sphere with charge

$$q'' = 4\pi\epsilon_0 R V_0.$$

$$\begin{aligned}
F &= \frac{1}{4\pi\epsilon_0} q \left(\frac{q'}{(a-b)^2} + \frac{q''}{a^2} \right) \\
&= \frac{qq'}{4\pi\epsilon_0} \left(\frac{1}{(a-b)^2} - \frac{1}{a^2} \right) \\
&= \frac{qq'}{4\pi\epsilon_0} \frac{a^2 - (a-b)^2}{a^2(a-b)^2} \\
&= \frac{qq'}{4\pi\epsilon_0} \frac{b(2a-b)}{a^2(a-b)^2} \\
&= \frac{q(-Rq/a)}{4\pi\epsilon_0} \frac{(R^2/a)(2a - R^2/a)}{a^2(a - R^2/a)^2} \\
&= -\frac{q^2}{4\pi\epsilon_0} \left(\frac{R}{a} \right)^3 \frac{2a^2 - R^2}{(a^2 - R^2)^2}
\end{aligned}$$

3.10

(a)

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \lambda \ln \frac{y^2 + (z+d)^2}{y^2 + (z-d)^2}$$

(b)

$$\begin{aligned}
\sigma &= -\epsilon_0 \left. \frac{\partial V}{\partial z} \right|_{z=0} \\
&= -\frac{d\lambda}{\pi(d^2 + y^2)}
\end{aligned}$$

3.11

You need three charges: $-q$ at $(-a, b)$, $-q$ at $(a, -b)$, and q at $(-b, -a)$. The potential is

$$\begin{aligned}
V &= \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{\sqrt{(x-a)^2 + (y-b)^2}} - \frac{1}{\sqrt{(x+a)^2 + (y-b)^2}} \right. \\
&\quad \left. - \frac{1}{\sqrt{(x-a)^2 + (y+b)^2}} + \frac{1}{\sqrt{(x+a)^2 + (y+b)^2}} \right).
\end{aligned}$$

The force on q is

$$\mathbf{F} = \frac{q^2}{16\pi\epsilon_0} \left[\left(\frac{a}{(a^2 + b^2)^{3/2}} - \frac{1}{a^2} \right) \hat{\mathbf{x}} + \left(\frac{b}{(a^2 + b^2)^{3/2}} - \frac{1}{b^2} \right) \hat{\mathbf{y}} \right].$$

The work to bring q in from infinity is

$$W = \frac{q^2}{16\pi\epsilon_0} \left(\frac{1}{\sqrt{a^2 + b^2}} - \frac{1}{a} - \frac{1}{b} \right).$$

3.12

Two infinitely long wires running parallel to the x -axis a distance $2a$ apart with charge densities λ and $-\lambda$ have cylindrical equipotential surfaces with centres at

$$y_0 = \pm a \coth \frac{2\pi\epsilon_0 V_0}{\lambda}$$

radii

$$R = a \operatorname{csch} \frac{2\pi\epsilon_0 V_0}{\lambda}.$$

We know the equipotential surfaces (the pipes) and want to find the wires so we can find the potential, so

$$\begin{aligned} d &= a \coth \frac{2\pi\epsilon_0 V_0}{\lambda} \\ R &= a \operatorname{csch} \frac{2\pi\epsilon_0 V_0}{\lambda} \\ \frac{d}{R} &= \cosh \frac{2\pi\epsilon_0 V_0}{\lambda} \\ \operatorname{arcosh} \frac{d}{R} &= \frac{2\pi\epsilon_0 V_0}{\lambda} \\ \lambda &= \frac{2\pi\epsilon_0 V_0}{\operatorname{arcosh} d/R} \\ R &= a \operatorname{csch} \operatorname{arcosh} \frac{d}{R} \\ a &= \frac{R}{\operatorname{csch} \operatorname{arcosh} d/R} \\ &= (d + R) \sqrt{\frac{2d}{d + R} - 1} \\ &= \sqrt{d^2 - R^2} \end{aligned}$$

thus the potential is

$$V = \frac{V_0}{2 \operatorname{arcosh} d/R} \ln \frac{(y + d^2 - R^2)^2 + z^2}{(y - d^2 + R^2)^2 + z^2}.$$

3.13

$$\begin{aligned}
V_0(y) &= \begin{cases} V_0 & 0 \leq y \leq \frac{a}{2} \\ -V_0 & \frac{a}{2} \leq y \leq a \end{cases} \\
C_n &= \frac{2}{a} \left(\int_0^{a/2} V_0 \sin \frac{n\pi y}{a} dy - \int_{a/2}^a V_0 \sin \frac{n\pi y}{a} dy \right) \\
&= \frac{2V_0}{n\pi} \left(\cos \frac{n\pi y}{a} \Big|_{a/2}^a - \cos \frac{n\pi y}{a} \Big|_0^{a/2} \right) \\
&= \frac{2V_0}{n\pi} \left(\cos n\pi - \cos \frac{n\pi}{2} - \cos \frac{n\pi}{2} + 1 \right) \\
&= \frac{2V_0}{n\pi} \left(1 + (-1)^n - 2 \cos \frac{n\pi}{2} \right) \\
&= \frac{2V_0}{n\pi} \begin{cases} 0 & n \text{ is odd or divisible by 4} \\ 4 & \text{otherwise} \end{cases} \\
V &= \frac{8V_0}{\pi} \sum_{n=2,6,10,\dots}^{\infty} \frac{1}{n} e^{-n\pi x/a} \sin \frac{n\pi y}{a}
\end{aligned}$$

3.14

$$\begin{aligned}
\sigma &= -\epsilon_0 \frac{\partial V}{\partial x} \\
&= \frac{4\epsilon_0 V_0 \sin \frac{\pi y}{a}}{a \left(1 - \cos \frac{2\pi y}{a} \right)} \\
&= \frac{2\epsilon_0 V_0}{a} \frac{1}{\sin \pi y/a}
\end{aligned}$$

3.15

(a)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$V(0, y) = 0$$

$$V(b, y) = V_0(y)$$

$$V(x, 0) = 0$$

$$V(x, a) = 0$$

$$V = X(x)Y(y)$$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\frac{Y''}{Y} = -\alpha^2$$

$$Y'' + \alpha^2 Y = 0$$

$$Y = c_1 \cos \alpha y + c_2 \sin \alpha y$$

$$Y = c_2 \sin \alpha y$$

$$Y = c_2 \sin \frac{n\pi y}{a}, n \in \mathbb{R}$$

$$\frac{X''}{X} = \alpha^2$$

$$X'' - \alpha^2 X = 0$$

$$X = c_3 \cosh \alpha x + c_4 \sinh \alpha x$$

$$X = c_4 \sinh \alpha x$$

$$= c_4 \sinh \frac{n\pi x}{a}, n \in \mathbb{R}$$

$$V = \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi x}{a} \sin \frac{n\pi y}{a}$$

$$V_0(y) = V(b, y)$$

$$= \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi y}{a}$$

$$C_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a V_0(y) \sin \frac{n\pi y}{a} dy$$

$$C_n = \frac{2}{a \sinh n\pi b/a} \int_0^a V_0(y) \sin \frac{n\pi y}{a} dy$$

(b)

$$\begin{aligned} C_n &= \frac{2V_0}{a \sinh n\pi b/a} \int_0^a \sin \frac{n\pi y}{a} dy \\ &= \frac{2V_0}{a \sinh n\pi b/a} \frac{a[1 - (-1)^n]}{n\pi} \\ &= \frac{2V_0[1 - (-1)^n]}{n\pi \sinh n\pi b/a} \\ V &= \frac{2V_0}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n \sinh n\pi b/a} \sinh \frac{n\pi x}{a} \sin \frac{n\pi y}{a} \end{aligned}$$

3.16

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V(0, y, z) = 0$$

$$V(a, y, z) = 0$$

$$V(x, 0, z) = 0$$

$$V(x, a, z) = 0$$

$$V(x, y, 0) = 0$$

$$V(x, y, a) = V_0$$

$$V = X(x)Y(y)Z(z)$$

$$X''YZ + XY''Z + XYZ'' = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

$$\frac{X''}{X} = -\alpha^2$$

$$\frac{Y''}{Y} = -\beta^2$$

$$\frac{Z''}{Z} = \alpha^2 + \beta^2$$

$$X'' + \alpha^2 X = 0$$

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$X = c_2 \sin \alpha x$$

$$X = c_2 \sin \frac{n\pi x}{a}, n \in \mathbb{R}$$

$$\frac{Y''}{Y} = -\beta^2$$

$$Y'' + \beta^2 Y = 0$$

$$Y = c_3 \cos \beta y + c_4 \sin \beta y$$

$$Y = c_4 \sin \beta y$$

$$Y = c_4 \sin \frac{m\pi y}{a}, m \in \mathbb{R}$$

$$\frac{Z''}{Z} = \alpha^2 + \beta^2$$

$$Z'' - (\alpha^2 + \beta^2)Z = 0$$

$$Z = c_5 \cosh \sqrt{\alpha^2 + \beta^2} z + c_6 \sinh \sqrt{\alpha^2 + \beta^2} z$$

$$= c_5 \cosh \pi \sqrt{(n/a)^2 + (m/a)^2} z$$

$$+ c_6 \sinh \pi \sqrt{(n/a)^2 + (m/a)^2} z$$

$$Z = c_6 \sinh \pi \sqrt{(n/a)^2 + (m/a)^2} z$$

$$\begin{aligned}
V &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{n,m} \sinh \left(\pi \sqrt{(n/a)^2 + (m/a)^2} z \right) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{a} \\
V_0 &= V(x, y, a) \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{n,m} \sinh \left(\pi \sqrt{n^2 + m^2} \right) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{a} \\
C_{n,m} &= \frac{4V_0}{a^2 \sinh \left(\pi \sqrt{n^2 + m^2} \right)} \int_0^a \int_0^a \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{a} dy dx \\
&= \frac{4V_0}{a^2 \sinh \left(\pi \sqrt{n^2 + m^2} \right)} \frac{a^2 [-1 + (-1)^m] [-1 + (-1)^n]}{nm\pi^2} \\
&= \frac{4V_0 [-1 + (-1)^n] [-1 + (-1)^m]}{nm\pi^2 \sinh \left(\pi \sqrt{n^2 + m^2} \right)} \\
&= \begin{cases} 0 & n \text{ even or } m \text{ even} \\ \frac{16V_0}{nm\pi^2 \sinh \left(\pi \sqrt{n^2 + m^2} \right)} & \text{otherwise} \end{cases} \\
V &= \frac{16V_0}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{nm} \frac{\sinh \left(\pi \sqrt{n^2 + m^2} z/a \right)}{\sinh \left(\pi \sqrt{n^2 + m^2} \right)} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{a}
\end{aligned}$$

3.17

$$\begin{aligned}
P_3(x) &= \frac{1}{2^3 3!} \left(\frac{d}{dx} \right)^3 (x^2 - 1)^3 \\
&= \frac{1}{48} \frac{d^3}{dx^3} [(x^2 - 1)^3] \\
&= \frac{1}{48} \frac{d^2}{dx^2} [6x(x^2 - 1)^2] \\
&= \frac{1}{48} \frac{d}{dx} [6(x^2 - 1)^2 + 24x^2(x^2 - 1)] \\
&= \frac{1}{48} [24x(x^2 - 1) + 48x(x^2 - 1) + 48x^3] \\
&= \frac{1}{48} (24x^3 - 24x + 48x^3 - 48x + 48x^3) \\
&= \frac{120}{48} x^3 - \frac{72}{48} x \\
&= \frac{5}{2} x^3 - \frac{3}{2} x \\
\frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) &= -12 \sin \theta \Theta \\
\Theta &= \frac{5}{2} \cos^3 \theta - \frac{3}{2} \cos \theta \\
\frac{d\Theta}{d\theta} &= -\frac{15}{2} \cos^2 \theta \sin \theta + \frac{3}{2} \sin \theta \\
\sin \theta \frac{d\Theta}{d\theta} &= -\frac{15}{2} \cos^2 \theta \sin^2 \theta + \frac{3}{2} \sin^2 \theta \\
\frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) &= \frac{3}{2} (1 - 5 \cos 2\theta) \sin 2\theta \\
\frac{3}{2} (1 - 5 \cos 2\theta) \sin 2\theta &= -12 \sin \theta \left(\frac{5}{2} \cos^3 \theta - \frac{3}{2} \cos \theta \right) \\
3(1 - 5 \cos 2\theta) \sin 2\theta &= 12 \sin \theta \cos \theta (3 - 5 \cos^2 \theta) \\
&= 6(3 - 5 \cos^2 \theta) \sin 2\theta \\
&= 6 \left(3 - 5 \frac{1 + \cos 2\theta}{2} \right) \sin 2\theta \\
&= 3(6 - 5 - 5 \cos 2\theta) \sin 2\theta \\
&= 3(1 - 5 \cos 2\theta) \sin 2\theta \\
\int_{-1}^1 P_1(x) P_3(x) dx &= \int_{-1}^1 x \left(\frac{5}{2} x^3 - \frac{3}{2} x \right) dx \\
&= \left[\frac{1}{2} x^5 - \frac{1}{2} x^3 \right]_{-1}^1 \\
&= \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \\
&= 0
\end{aligned}$$

3.18

(a)

$$\begin{aligned}
A_l &= \frac{V_0(2l+1)}{2R^l} \int_0^\pi P_l(\cos \theta) \sin \theta d\theta \\
&= \begin{cases} V_0 & l = 0 \\ 0 & l \neq 0 \end{cases} \\
V(r, \theta) &= \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) \\
&= V_0 \\
B_l &= \frac{V_0(2l+1)}{2} R^{l+1} \int_0^\pi P_l(\cos \theta) \sin \theta d\theta \\
&= \begin{cases} V_0 R & l = 0 \\ 0 & l \neq 0 \end{cases} \\
V(r, \theta) &= \frac{V_0 R}{r}
\end{aligned}$$

(b)

$$\begin{aligned}
A_l &= \frac{\sigma_0}{2\epsilon_0 R^{l-1}} \int_0^\pi P_l(\cos \theta) \sin \theta d\theta \\
&= \begin{cases} \frac{R\sigma_0}{\epsilon_0} & l = 0 \\ 0 & l \neq 0 \end{cases} \\
V(r, \theta) &= \frac{R\sigma_0}{\epsilon_0} \\
B_l &= A_l R^{2l+1} \\
&= \begin{cases} \frac{R^2\sigma_0}{\epsilon_0} & l = 0 \\ 0 & l \neq 0 \end{cases} \\
V(r, \theta) &= \frac{R^2\sigma_0}{\epsilon_0 r}
\end{aligned}$$

3.19

$$\begin{aligned}
V_0 &= k \cos 3\theta \\
A_l &= \frac{k(2l+1)}{2R^l} \int_0^\pi \cos 3\theta P_l(\cos \theta) \sin \theta d\theta \\
&= \begin{cases} -\frac{3k}{5R} & l=1 \\ \frac{8k}{5R^3} & l=3 \\ 0 & \text{otherwise} \end{cases} \\
V(r, \theta) &= -\frac{3k}{5R} r P_1(\cos \theta) + \frac{8k}{5R^3} r^3 P_3(\cos \theta) \\
&= \frac{kr}{5R} \left[-3P_1(\cos \theta) + \frac{8}{R^2} r^2 P_3(\cos \theta) \right] \\
B_l &= \frac{k(2l+1)}{2} R^{l+1} \int_0^\pi \cos 3\theta P_l(\cos \theta) \sin \theta d\theta \\
&= \begin{cases} -\frac{3kR^2}{5} & l=1 \\ \frac{8kR^4}{5} & l=3 \\ 0 & \text{otherwise} \end{cases} \\
V(r, \theta) &= -\frac{3kR^2}{5r^2} P_1(\cos \theta) + \frac{8kR^4}{5r^4} P_3(\cos \theta) \\
&= \frac{kR^2}{5r^2} \left[\frac{8R^2}{r^2} P_3(\cos \theta) - 3P_1(\cos \theta) \right] \\
\sigma(\theta) &= -\epsilon_0 \left(\frac{\partial V_{\text{above}}}{\partial r} - \frac{\partial V_{\text{below}}}{\partial r} \right) \\
&= \frac{\epsilon_0 k (12 \cos \theta + 35 \cos 3\theta)}{5R}
\end{aligned}$$

3.20

$$\begin{aligned}
V(r, \theta) &= \begin{cases} \sum_{l=0}^{\infty} \frac{2l+1}{2} \frac{r^l}{R^l} \left(\int_0^\pi V_0(\theta) P_l(\cos \theta) \sin \theta d\theta \right) P_l(\cos \theta) & r \leq R \\ \sum_{l=0}^{\infty} \frac{2l+1}{2} \frac{R^{l+1}}{r^{l+1}} \left(\int_0^\pi V_0(\theta) P_l(\cos \theta) \sin \theta d\theta \right) P_l(\cos \theta) & r \geq R \end{cases} \\
\sigma_0 &= -\epsilon_0 \left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r} \right) \Big|_{r=R} \\
&= \frac{\epsilon_0}{2R} \sum_{l=0}^{\infty} (2l+1)^2 C_l P_l(\cos \theta)
\end{aligned}$$

3.21

$$V(r, \theta) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} - E_0 \left(r - \frac{R^3}{r^2} \right) \cos \theta$$

3.22

(a)

$$\begin{aligned}
\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} &= \frac{\sigma}{2\epsilon_0} \left(\sqrt{r^2 + R^2} - r \right) \\
&= \frac{\sigma r}{2\epsilon_0} \left(\sqrt{1 + (R/r)^2} - 1 \right) \\
&= \frac{\sigma r}{2\epsilon_0} \left[\left(1 + \frac{(R/r)^2}{2} - \frac{(R/r)^4}{8} + \dots \right) - 1 \right] \\
&= \frac{\sigma}{2\epsilon_0} \left(\frac{R^2}{2r} - \frac{R^4}{8r^3} + \dots \right) \\
B_0 &= \frac{\sigma R^2}{4\epsilon_0} \\
B_1 &= 0 \\
B_2 &= -\frac{\sigma R^4}{16\epsilon_0}
\end{aligned}$$

(b)

$$\begin{aligned}
\sum_{l=0}^{\infty} A_l r^l &= \frac{\sigma}{2\epsilon_0} \left(\sqrt{r^2 + R^2} - r \right) \\
&= \frac{\sigma}{2\epsilon_0} \left(R \sqrt{1 + (r/R)^2} - r \right) \\
&= \frac{\sigma}{2\epsilon_0} \left[R \left(1 + \frac{(r/R)^2}{2} - \frac{(r/R)^4}{8} + \dots \right) - r \right] \\
&= \frac{\sigma}{2\epsilon_0} \left(R - r + \frac{r^2}{2R} - \frac{r^4}{8R^3} + \dots \right) \\
A_0 &= \frac{\sigma R}{2\epsilon_0} \\
A_1 &= -\frac{\sigma}{2\epsilon_0} \\
A_2 &= \frac{\sigma}{4\epsilon_0 R} \\
A'_0 &= A_0 \\
A'_1 &= -A_1 \\
A'_2 &= A_2
\end{aligned}$$

3.23

$$V(r, \theta) = \sum_{l=1}^{\infty} A_l r^l P_l(\cos \theta)$$

$$A_l = \frac{\sigma_0}{2\epsilon_0 R^{l-1}} \left(\int_0^{\pi/2} P_l(\cos \theta) \sin \theta d\theta - \int_{\pi/2}^{\pi} P_l(\cos \theta) \sin \theta d\theta \right)$$

$$A_0 = 0$$

$$A_1 = \frac{\sigma_0}{2\epsilon_0}$$

$$A_2 = 0$$

$$A_3 = -\frac{\sigma_0}{8\epsilon_0 R^2}$$

$$A_4 = 0$$

$$A_5 = \frac{\sigma_0}{16\epsilon_0 R^4}$$

$$A_6 = 0$$

$$V(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

$$B_l = A_l R^{2l+1}$$

$$B_0 = 0$$

$$B_1 = \frac{\sigma_0 R^3}{2\epsilon_0}$$

$$B_2 = 0$$

$$B_3 = -\frac{\sigma_0 R^5}{8\epsilon_0}$$

$$B_4 = 0$$

$$B_5 = \frac{\sigma_0 R^7}{16\epsilon_0}$$

$$B_6 = 0$$

3.24

$$\begin{aligned} 0 &= \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial V}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \\ &= \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial V}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 V}{\partial \phi^2} \end{aligned}$$

$$V(s, \phi) = S(s)\Phi(\phi)$$

$$\begin{aligned} 0 &= \frac{1}{s} \frac{\partial}{\partial s} (sS')\Phi + \frac{1}{s^2} S\Phi'' \\ &= \frac{1}{s} (S' + sS'')\Phi + \frac{1}{s^2} S\Phi'' \\ &= \frac{S'}{sS} + \frac{S''}{S} + \frac{\Phi''}{s^2\Phi} \\ &= \frac{s^2S'' + sS'}{S} + \frac{\Phi''}{\Phi} \end{aligned}$$

$$\frac{\Phi''}{\Phi} = 0$$

$$\Phi'' = 0$$

$$\Phi = c_1 + c_2\phi$$

$$\frac{\Phi''}{\Phi} = -n^2$$

$$\Phi'' + \alpha^2\Phi = 0$$

$$\Phi = c_3 \cos \alpha\phi + c_4 \sin \alpha\phi$$

$$\Phi(0) = \Phi(2\pi)$$

$$c_1 = c_3 \cos 2\pi\alpha + c_4 \sin 2\pi\alpha$$

$$\alpha = n, n \in \mathbb{R}$$

$$\Phi = c_3 \cos n\phi + c_4 \sin n\pi$$

$$\frac{s^2S'' + sS'}{S} = 0$$

$$s^2S'' + sS' = 0$$

$$sS'' + S' = 0$$

$$S = c_5 + c_6 \ln s$$

$$\begin{aligned}
\frac{s^2 S'' + s S'}{S} &= n^2 \\
s^2 S'' + s S' - n^2 S &= 0 \\
S &= s^m \\
S' &= m s^{m-1} \\
S'' &= m(m-1) s^{m-2} \\
m(m-1) s^m + m s^m - n^2 s^m &= 0 \\
m^2 - m + m - n^2 &= 0 \\
m^2 - n^2 &= 0 \\
(m+n)(m-n) &= 0 \\
S &= c_7 s^n + c_8 s^{-n} \\
V &= S(s) \Phi(\phi) \\
&= (c_1 + c_2 \phi)(c_5 + c_6 \ln s) \\
&\quad + \sum_{n=1}^{\infty} (c_7 s^n + c_8 s^{-n})(c_3 \cos n\phi + c_4 \sin n\phi) \\
&= c_1 + c_2 \ln s \\
&\quad + \sum_{n=1}^{\infty} [s^n (A_n \cos n\phi + B_n \sin n\phi) + s^{-n} (C_n \cos n\phi + D_n \sin n\phi)]
\end{aligned}$$

3.25

$$\begin{aligned}
V &= 0 \text{ at } s = R \\
V &\rightarrow -E_0 s \cos \phi \text{ as } s \rightarrow \infty \\
V &= \left(A_1 s + \frac{C_1}{s} \right) \cos \phi \\
0 &= A_1 R + \frac{C_1}{R} \\
C_2 &= -A_1 R^2 \\
A_1 &= -E_0 \\
V &= E_0 s \left(\frac{R^2}{s^2} - 1 \right) \cos \phi \\
\sigma &= -\epsilon_0 \left(\frac{\partial V_{\text{out}}}{\partial s} - \frac{\partial V_{\text{in}}}{\partial s} \right) \Big|_{s=R} \\
&= -\epsilon_0 \frac{\partial V_{\text{out}}}{\partial s} \Big|_{s=R} \\
&= 2\epsilon_0 E_0 \cos \phi
\end{aligned}$$

3.27

$$\begin{aligned}
V(z) &= \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{z^{(n+1)}} \int (r')^n P_n(\cos \alpha) \rho(\mathbf{r}') d\tau' \\
&= \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{z^{(n+1)}} \int_0^R \int_0^\pi \int_0^{2\pi} (r')^n P_n(\cos \theta) k \frac{R}{(r')^2} (R - 2r') \sin \theta (r')^2 \sin \theta dr' d\theta d\phi \\
&= \frac{kR}{2\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{z^{(n+1)}} \int_0^R \int_0^\pi (r')^n (R - 2r') \sin^2 \theta P_n(\cos \theta) dr' d\theta \\
&\approx \frac{1}{4\pi\epsilon_0} \frac{k\pi^2 R^5}{48z^3}
\end{aligned}$$

3.28

$$\begin{aligned}
V(r, \theta) &= \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int (r')^n P_n(\cos \alpha) \rho(\mathbf{r}') d\tau' \\
&= \frac{\lambda R}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int_0^{2\pi} R^n P_n(\sin \phi \sin \theta) d\phi \\
V_0 &= \frac{\lambda R}{4\pi\epsilon_0} \frac{2\pi}{r} \\
&= \frac{\lambda R}{2\epsilon_0 r} \\
V_1 &= 0 \\
V_2 &= -\frac{\lambda R}{4\pi\epsilon_0} \frac{1}{4r^3} \pi R^2 (1 + 3 \cos 2\theta) \\
&= -\frac{\lambda R^3}{8\epsilon_0 r^3} (3 \cos^2 \theta - 1)
\end{aligned}$$

3.29

$$\begin{aligned}
\mathbf{p} &= \sum_{i=1}^n q_i \mathbf{r}'_i \\
&= 2aq\hat{\mathbf{z}} \\
V_{\text{dip}}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \frac{2aq \cos \theta}{r^2}
\end{aligned}$$

3.30

(a)

$$\begin{aligned}
 \sigma &= k \cos \theta \\
 \mathbf{p} &= \int \mathbf{r}' \rho(\mathbf{r}') d\tau' \\
 &= \int_0^{2\pi} \int_0^\pi R(\sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}}) k \cos \theta R^2 \sin \theta d\theta d\phi \\
 &= \frac{1}{2} k R^3 \int_0^{2\pi} \int_0^\pi (\sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}}) \sin 2\theta d\theta d\phi \\
 &= \frac{4}{3} \pi R^3 k \hat{\mathbf{z}}
 \end{aligned}$$

(b)

$$\begin{aligned}
 V_{\text{dip}}(\mathbf{r}) &\approx \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} \\
 &= \frac{1}{4\pi\epsilon_0} \frac{4\pi R^3 k \cos \theta}{3r^2} \\
 &= \frac{kR^3}{3\epsilon_0} \frac{1}{r^2} \cos \theta \\
 V_{\text{dip}}(r, \theta) &= \frac{kR^3}{3\epsilon_0} \frac{1}{r^2} \cos \theta
 \end{aligned}$$

Higher multipoles are all 0.

3.32

(a)

$$\begin{aligned}
 Q &= 2q \\
 \mathbf{p} &= 3aq\hat{\mathbf{z}} \\
 V &\approx \frac{1}{4\pi\epsilon_0} \left(\frac{2q}{r} + \frac{3aq \cos \theta}{r^2} \right)
 \end{aligned}$$

(b)

$$\begin{aligned}
 Q &= 2q \\
 \mathbf{p} &= aq\hat{\mathbf{z}} \\
 V &\approx \frac{1}{4\pi\epsilon_0} \left(\frac{2q}{r} + \frac{aq \cos \theta}{r^2} \right)
 \end{aligned}$$

(c)

$$\begin{aligned}
 Q &= 2q \\
 \mathbf{p} &= 3aq\hat{\mathbf{y}} \\
 V &\approx \frac{1}{4\pi\epsilon_0} \left(\frac{2q}{r} + \frac{3aq \sin \theta \sin \phi}{r^2} \right)
 \end{aligned}$$

3.33

(a)

$$\begin{aligned}
 \mathbf{E} &= -\frac{1}{4\pi\epsilon_0} \frac{p}{a^3} \hat{\mathbf{z}} \\
 \mathbf{F} &= -\frac{1}{4\pi\epsilon_0} \frac{pq}{a^3} \hat{\mathbf{z}}
 \end{aligned}$$

(b)

$$\begin{aligned}
 \mathbf{E} &= \frac{1}{4\pi\epsilon_0} \frac{2p}{a^3} \hat{\mathbf{z}} \\
 \mathbf{F} &= \frac{1}{4\pi\epsilon_0} \frac{2pq}{a^3} \hat{\mathbf{z}}
 \end{aligned}$$

(c)

$$\begin{aligned}
 W &= \int \mathbf{F} \cdot d\mathbf{l} \\
 &= \int_0^{\pi/2} aq\mathbf{E} \cdot d\boldsymbol{\theta} \\
 &= \frac{1}{4\pi\epsilon_0} \frac{pq}{a^2} \int_0^{\pi/2} (2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) \cdot d\boldsymbol{\theta} \\
 &= \frac{1}{4\pi\epsilon_0} \frac{pq}{a^2} \int_0^{\pi/2} \sin\theta d\theta \\
 &= \frac{1}{4\pi\epsilon_0} \frac{pq}{a^2}
 \end{aligned}$$

3.34

$$\begin{aligned}
 Q &= -q \\
 \mathbf{p} &= qa\hat{\mathbf{z}} \\
 V &= \frac{1}{4\pi\epsilon_0} q \left(-\frac{1}{r} + \frac{a\cos\theta}{r^2} \right) \\
 \mathbf{E} &= -\nabla V \\
 &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} [(2a\cos\theta - r)\hat{\mathbf{r}} + a\sin\theta\hat{\boldsymbol{\theta}}]
 \end{aligned}$$

3.35

$$\begin{aligned}
\mathbf{p} &= \int \mathbf{r}' \rho(\mathbf{r}') d\tau' \\
&= \left(\int_0^{2\pi} \int_0^{\pi/2} \int_0^R r \cos \theta \rho_0 r^2 \sin \theta dr d\theta d\phi \right. \\
&\quad \left. - \int_0^{2\pi} \int_{\pi/2}^{\pi} \int_0^R r \cos \theta \rho_0 r^2 \sin \theta dr d\theta d\phi \right) \hat{\mathbf{z}} \\
&= \pi \rho_0 \left(\int_0^{\pi/2} \int_0^R r^3 \sin 2\theta dr d\theta - \int_{\pi/2}^{\pi} \int_0^R r^3 \sin 2\theta dr d\theta \right) \hat{\mathbf{z}} \\
&= \frac{1}{2} \pi \rho_0 R^4 \hat{\mathbf{z}} \\
\mathbf{E}_{\text{dip}}(r, \theta) &= \frac{1}{4\pi\epsilon_0} \frac{\pi \rho_0 R^4}{2r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}})
\end{aligned}$$

3.36

The factor of $1/4\pi\epsilon_0 r^3$ is the common, so the goal is to show that

$$\begin{aligned}
p(2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}}) &= 2(\mathbf{p} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} + p \sin \theta \hat{\boldsymbol{\theta}} \\
&= 2(\mathbf{p} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - (\mathbf{p} \cdot \hat{\boldsymbol{\theta}}) \hat{\boldsymbol{\theta}} \\
&= 3(\mathbf{p} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \hat{\mathbf{p}}.
\end{aligned}$$

3.37

$$\begin{aligned}
V_{\text{ave}} &= \frac{1}{4\pi R^2} \oint V da \\
\frac{dV_{\text{ave}}}{dR} &= \frac{1}{4\pi R^2} \oint \nabla V \cdot d\mathbf{a} \\
&= \frac{1}{4\pi R^2} \int \nabla^2 V d\tau \\
&= 0
\end{aligned}$$

3.38

$$\begin{aligned}
E_{qz} &= \frac{1}{4\pi\epsilon_0} \frac{q}{z^2} \cos \theta \\
&= \frac{1}{4\pi\epsilon_0} \frac{qd}{(x^2 + y^2 + d^2)^{3/2}} \\
0 &= E_{qz} + E_{\sigma z} \\
&= \frac{1}{4\pi\epsilon_0} \frac{qd}{(x^2 + y^2 + d^2)^{3/2}} - \frac{\sigma}{2\epsilon_0} \\
\sigma &= \frac{qd}{2\pi(x^2 + y^2 + d^2)^{3/2}}
\end{aligned}$$

3.39

$$E = \frac{q^2}{4\pi\epsilon_0} \left[\left(\sum_{n=1}^{\infty} \frac{1}{(2an - 2x)^2} \right) - \left(\sum_{n=0}^{\infty} \frac{1}{(2an + 2x)^2} \right) \right]$$

3.40

Set $V = 0$ at $x = 0$. The cylinder is a conductor and is thus an equipotential, so $V = 0$ at the surface. Place two infinite line charges within the cylinder at $x = \pm R^2/a$, giving

$$\begin{aligned}
V &= \frac{\lambda}{2\pi\epsilon_0} \left(\ln \frac{a}{\sqrt{s^2 + a^2 - 2sa \cos \phi}} - \ln \frac{a}{\sqrt{s^2 + a^2 + 2sa \cos \phi}} \right. \\
&\quad + \ln \frac{R^2/a}{\sqrt{s^2 + (R^2/a)^2 + 2s(R^2/a) \cos \phi}} \\
&\quad \left. - \ln \frac{R^2/a}{\sqrt{s^2 + (R^2/a)^2 - 2s(R^2/a) \cos \phi}} \right) \\
&= \frac{\lambda}{4\pi\epsilon_0} \left(\ln \frac{s^2 + a^2 + 2sa \cos \phi}{s^2 + a^2 - 2sa \cos \phi} + \ln \frac{(sa/R)^2 + R^2 - 2sa \cos \phi}{(sa/R)^2 + R^2 + 2sa \cos \phi} \right) \\
&= \frac{\lambda}{4\pi\epsilon_0} \ln \frac{(s^2 + a^2 + 2sa \cos \phi)[(sa/R)^2 + R^2 - 2sa \cos \phi]}{(s^2 + a^2 - 2sa \cos \phi)[(sa/R)^2 + R^2 + 2sa \cos \phi]}
\end{aligned}$$

3.41

(a) For a sphere of charge q , $q' + q'' = q \Rightarrow q'' = q - q'$ so

$$\begin{aligned} F &= \frac{q}{4\pi\epsilon_0} \left(\frac{q''}{a^2} + \frac{q'}{(a-b)^2} \right) \\ &= \frac{q}{4\pi\epsilon_0} \left(\frac{q}{a^2} - \frac{q'}{a^2} + \frac{q'}{(a-b)^2} \right) \\ &= \frac{q^2}{4\pi\epsilon_0 a^3} \left[a - R^3 \frac{2a^2 - R^2}{(a^2 - R^2)^2} \right] \end{aligned}$$

and solving for $F = 0$ gives $r = 5.663\,12\,\text{\AA}$.

3.43

(a)

$$\lim_{r \rightarrow \infty} V_{\text{above}}(r, \theta) = 0$$

$$V_{\text{below}}(a, \theta) = V_0$$

$$V_{\text{above}}(b, \theta) = V_{\text{below}}(b, \theta)$$

$$\left. \frac{\partial V_{\text{above}}}{\partial r} \right|_{r=b} - \left. \frac{\partial V_{\text{below}}}{\partial r} \right|_{r=b} = -\frac{k \cos \theta}{\epsilon_0}$$

$$V_{\text{above}}(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

$$\left. \frac{\partial V_{\text{above}}}{\partial r} \right|_{r=b} = \sum_{l=0}^{\infty} -(l+1) \frac{B_l}{b^{l+2}} P_l(\cos \theta)$$

$$V_{\text{below}}(r, \theta) = \sum_{l=0}^{\infty} \left(C_l r^l + \frac{D_l}{r^{l+1}} \right) P_l(\cos \theta)$$

$$V_0 = V_{\text{below}}(a, \theta)$$

$$= \sum_{l=0}^{\infty} \left(C_l a^l + \frac{D_l}{a^{l+1}} \right) P_l(\cos \theta)$$

$$V_0 = C_0 + \frac{D_0}{a}$$

$$0 = C_l a^l + \frac{D_l}{a^{l+1}}, l \neq 0$$

$$\left. \frac{\partial V_{\text{below}}}{\partial r} \right|_{r=b} = \sum_{l=0}^{\infty} \left(C_l l b^{l-1} - (l+1) \frac{D_l}{b^{l+2}} \right) P_l(\cos \theta)$$

$$V_{\text{above}}(b, \theta) = V_{\text{below}}(b, \theta)$$

$$\sum_{l=0}^{\infty} \frac{B_l}{b^{l+1}} P_l(\cos \theta) = \sum_{l=0}^{\infty} \left(C_l b^l + \frac{D_l}{b^{l+1}} \right) P_l(\cos \theta)$$

$$\frac{B_l}{b^{l+1}} = C_l b^l + \frac{D_l}{b^{l+1}}$$

$$-\frac{k \cos \theta}{\epsilon_0} = \sum_{l=0}^{\infty} \left[-(l+1) \frac{B_l}{b^{l+2}} - C_l l b^{l-1} + (l+1) \frac{D_l}{b^{l+2}} \right] P_l(\cos \theta)$$

$$-\frac{k}{\epsilon_0} = -2 \frac{B_1}{b^3} - C_1 + 2 \frac{D_1}{b^3}$$

$$0 = -(l+1) \frac{B_l}{b^{l+2}} - C_l l b^{l-1} + (l+1) \frac{D_l}{b^{l+2}}, l \neq 1$$

$$B_0 = a V_0$$

$$C_0 = 0$$

$$D_0 = a V_0$$

$$\begin{aligned}
B_1 &= \frac{(b^3 - a^3)k}{3\epsilon_0} \\
C_1 &= \frac{k}{3\epsilon_0} \\
D_1 &= -\frac{a^3 k}{3\epsilon_0} \\
B_l &= 0 \\
C_l &= 0 \\
D_l &= 0 \\
V &= \begin{cases} \frac{aV_0}{r} + \frac{(r^3 - a^3)k \cos \theta}{3\epsilon_0 r^2} & a \leq r \leq b \\ \frac{aV_0}{r} + \frac{(b^3 - a^3)k \cos \theta}{3\epsilon_0 r^2} & r \geq b \end{cases}
\end{aligned}$$

(b)

$$\begin{aligned}
\sigma &= -\epsilon_0 \left. \frac{\partial V_{\text{below}}}{\partial r} \right|_{r=a} \\
&= \frac{\epsilon_0 V_0}{a} - k \cos \theta
\end{aligned}$$

(c)

$$\begin{aligned}
Q &= \oint \sigma_i dA \\
&= \int_0^{2\pi} \int_0^\pi \left(\frac{\epsilon_0 V_0}{a} - k \cos \theta \right) a^2 \sin \theta d\theta d\phi \\
&= 2\pi \int_0^\pi \left(a\epsilon_0 V_0 \sin \theta - \frac{1}{2} a^2 k \sin 2\theta \right) d\theta \\
&= 4\pi\epsilon_0 a V_0 \\
V &\approx \frac{aV_0}{r} \\
\frac{1}{4\pi\epsilon_0} \frac{Q}{r} &= \frac{aV_0}{r} \\
Q &= 4\pi\epsilon_0 a V_0
\end{aligned}$$

3.44

$$\begin{aligned}
V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int (r')^n P_n(\cos \alpha) \rho(\mathbf{r}') d\tau' \\
&= \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int_{-a}^a z^n P_n(\cos \theta) \frac{Q}{2a} dz \\
&= \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{Q}{2ar^{(n+1)}} P_n(\cos \theta) \left[\frac{1}{n+1} z^{n+1} \right]_{-a}^a \\
&= \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{Q}{2a(n+1)r^{(n+1)}} P_n(\cos \theta) [a^{n+1} - (-1)^{n+1} a^{n+1}] \\
&= \frac{Q}{4\pi\epsilon_0} \frac{1}{r} \left[1 + \frac{1}{3} \left(\frac{a}{r} \right)^2 P_2(\cos \theta) + \frac{1}{5} \left(\frac{a}{r} \right)^4 P_4(\cos \theta) + \dots \right]
\end{aligned}$$

3.45

$$V = a_0 + b_0 \ln s + \sum_{k=1}^{\infty} [s^k (a_k \cos k\phi + b_k \sin k\phi) + s^{-k} (c_k \cos k\phi + d_k \sin k\phi)]$$

$$\begin{aligned}
\lim_{s \rightarrow \infty} V_{\text{above}}(s, \phi) &= 0 \\
V_{\text{above}}(R, \phi) &= V_{\text{below}}(R, \phi) \\
&= \begin{cases} -\sigma_0/\epsilon_0 & 0 \leq \phi \leq \pi \\ \sigma_0/\epsilon_0 & \pi \leq \phi \leq 2\pi \end{cases} \\
\left. \frac{\partial V_{\text{above}}}{\partial s} \right|_{s=R} - \left. \frac{\partial V}{\partial s} \right|_{s=R} &= -\frac{1}{\epsilon_0} \sigma \\
V_{\text{above}}(s, \phi) &= \sum_{k=1}^{\infty} s^{-k} (c_k \cos k\phi + d_k \sin k\phi) \\
\left. \frac{\partial V_{\text{above}}}{\partial s} \right|_{s=R} &= \sum_{k=1}^{\infty} -k R^{-(k+1)} (c_k \cos k\phi + d_k \sin k\phi) \\
V_{\text{below}}(s, \phi) &= e_0 + \sum_{k=1}^{\infty} s^k (e_k \cos k\phi + f_k \sin k\phi) \\
\left. \frac{\partial V_{\text{below}}}{\partial s} \right|_{s=R} &= \sum_{k=1}^{\infty} k R^{k-1} (e_k \cos k\phi + f_k \sin k\phi) \\
\sum_{k=1}^{\infty} R^{-k} (c_k \cos k\phi + d_k \sin k\phi) &= e_0 + \sum_{k=1}^{\infty} R^k (e_k \cos k\phi + f_k \sin k\phi) \\
e_0 &= 0 \\
R^{-k} c_k &= R^k e_k \\
R^{-k} d_k &= R^k f_k \\
-\frac{\sigma}{\epsilon_0} &= \sum_{k=1}^{\infty} \left[-k R^{-(k+1)} (c_k \cos k\phi + d_k \sin k\phi) - k R^{k-1} (e_k \cos k\phi + f_k \sin k\phi) \right] \\
&= \sum_{k=1}^{\infty} -k \left[\left(R^{-(k+1)} c_k + R^{k-1} e_k \right) \cos k\phi + \left(R^{-(k+1)} d_k + R^{k-1} f_k \right) \sin k\phi \right] \\
\frac{1}{\pi} \int_0^{2\pi} -\frac{\sigma}{\epsilon_0} \cos k\phi d\phi &= -k (R^{-(k+1)} c_k + R^{k-1} e_k) \\
\frac{\sigma_0 (\sin 2k\pi - 2 \sin k\pi)}{k\pi\epsilon_0} &= -k (R^{-(k+1)} c_k + R^{k-1} e_k) \\
\frac{1}{\pi} \int_0^{2\pi} -\frac{\sigma}{\epsilon_0} \sin k\phi d\phi &= -k (R^{-(k+1)} d_k + R^{k-1} f_k) \\
\frac{4\sigma_0 \cos k\pi \sin^2 k\pi/2}{k\pi\epsilon_0} &= -k (R^{-(k+1)} d_k + R^{k-1} f_k)
\end{aligned}$$

$$\begin{aligned}
c_k &= 0 \\
d_k &= \frac{2R^{k+1}\sigma_0 \cos k\pi \sin^2 k\pi/2}{k^2\pi\epsilon_0} \\
e_k &= 0 \\
f_k &= -\frac{2R^{-(k-1)}\sigma_0 \cos k\pi \sin^2 k\pi/2}{k^2\pi\epsilon_0} \\
V_{\text{above}} &= \frac{2\sigma_0}{\pi\epsilon_0} \sum_{k=1}^{\infty} s^{-k} \frac{R^{k+1} \cos k\pi \sin^2 k\pi/2}{k^2} \sin k\phi \\
&= -\frac{2\sigma_0}{\pi\epsilon_0} \sum_{k=1,3,5,\dots}^{\infty} \frac{1}{k^2} R^{k+1} s^{-k} \sin k\phi \\
V_{\text{below}} &= -\frac{2\sigma_0}{\pi\epsilon_0} \sum_{k=1}^{\infty} s^k \frac{R^{-(k-1)} \cos k\pi \sin^2 k\pi/2}{k^2} \sin k\phi \\
&= \frac{2\sigma_0}{\pi\epsilon_0} \sum_{k=1,3,5,\dots}^{\infty} \frac{1}{k^2} R^{-(k-1)} s^k \sin k\phi \\
V &= \frac{2R\sigma_0}{\pi\epsilon_0} \begin{cases} \sum_{k=1,3,5,\dots}^{\infty} \frac{1}{k^2} (s/R)^k \sin k\phi & s \leq R \\ -\sum_{k=1,3,5,\dots}^{\infty} \frac{1}{k^2} (R/s)^k \sin k\phi & s \geq R \end{cases}
\end{aligned}$$

3.46

(a)

$$\frac{1}{4\pi\epsilon_0} \frac{1}{r} \int_{-a}^a k \cos \frac{\pi z}{2a} dz = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \frac{4ak}{\pi}$$

(b)

$$\frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \int_{-a}^a z \cos \theta k \sin \frac{\pi z}{a} dz = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \frac{2a^2 k \cos \theta}{\pi}$$

(c)

$$\frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int_{-a}^a z^2 P_2(\cos \theta) k \cos \frac{\pi z}{a} dz = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \left(-\frac{4a^3 k}{\pi^2} \right) P_2(\cos \theta)$$

3.47

(a)

$$\begin{aligned}
 \mathbf{E}_{\text{ave}} &= \frac{1}{\frac{4}{3}\pi R^3} \int \mathbf{E} d\tau \\
 &= \frac{1}{4\pi\epsilon_0} \frac{1}{\frac{4}{3}\pi R^3} \int \frac{q}{r^2} \hat{\mathbf{r}} d\tau' \\
 \mathbf{E}_{\text{ave}} &= \int \frac{1}{4\pi\epsilon_0} \frac{\rho}{r^2} \hat{\mathbf{r}} d\tau' \\
 &= \frac{1}{4\pi\epsilon_0} \frac{1}{\frac{4}{3}\pi R^3} \int \frac{q}{r^2} \hat{\mathbf{r}} d\tau'
 \end{aligned}$$

(b)

$$\begin{aligned}
 \mathbf{p} &= q\mathbf{r} \\
 \oint \mathbf{E} \cdot d\mathbf{A} &= \frac{Q_{\text{encl}}}{\epsilon_0} \\
 4\pi r^2 E &= \frac{\frac{4}{3}\pi r^3 \rho}{\epsilon_0} \\
 \mathbf{E} &= \frac{r\rho}{3\epsilon_0} \hat{\mathbf{r}} \\
 &= -\frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}}{R^3}
 \end{aligned}$$

(c)

$$\begin{aligned}
 \mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 + \dots \\
 &= -\frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}_1}{R^3} - \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}_2}{R^3} + \dots \\
 &= -\frac{1}{4\pi\epsilon_0} \frac{1}{R^3} (\mathbf{p}_1 + \mathbf{p}_2 + \dots) \\
 &= -\frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}}{R^3}
 \end{aligned}$$

(d)

$$\begin{aligned}
\mathbf{E}_{\text{ave}} &= \frac{1}{\frac{4}{3}\pi R^3} \int \mathbf{E} d\tau' \\
&= \frac{1}{4\pi\epsilon_0} \frac{1}{\frac{4}{3}\pi R^3} \int \frac{q}{r^2} \hat{\mathbf{r}} d\tau' \\
\mathbf{E}_r &= -\frac{1}{4\pi\epsilon_0} \int \frac{\rho}{r^2} \hat{\mathbf{r}} d\tau' \\
&= \frac{1}{4\pi\epsilon_0} \frac{1}{\frac{4}{3}\pi R^3} \int \frac{q}{r^2} \hat{\mathbf{r}} d\tau' \\
\rho &= -\frac{q}{\frac{4}{3}\pi R^3} \\
Q &= \frac{4}{3}\pi R^3 \rho \\
&= -q \\
\mathbf{E}_r &= \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{z}}
\end{aligned}$$

This is the electric field at the origin.

3.48

(a)

$$\begin{aligned}
\mathbf{E}_{\text{dip}}(r, \theta) &= \frac{p}{4\pi\epsilon_0 r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}}) \\
&= \frac{p}{4\pi\epsilon_0 r^3} [2 \cos \theta (\sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}}) \\
&\quad + \sin \theta (\cos \theta \cos \phi \hat{\mathbf{x}} + \cos \theta \sin \phi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}})] \\
&= \frac{p}{4\pi\epsilon_0 r^3} [3 \cos \theta \sin \theta \cos \phi \hat{\mathbf{x}} + 3 \cos \theta \sin \theta \sin \phi \hat{\mathbf{y}} \\
&\quad + (2 \cos^2 \theta - \sin^2 \theta) \hat{\mathbf{z}}] \\
\mathbf{E}_{\text{ave}} &= \frac{1}{\frac{4}{3}\pi R^3} \int \mathbf{E}_{\text{dip}} d\tau' \\
&= \frac{3p}{16\pi^2\epsilon_0 R^3} \int_0^{2\pi} \int_0^\pi \int_0^R \frac{1}{r^3} [3 \cos \theta \sin \theta \cos \phi \hat{\mathbf{x}} \\
&\quad + 3 \cos \theta \sin \theta \sin \phi \hat{\mathbf{y}} + (2 \cos^2 \theta - \sin^2 \theta) \hat{\mathbf{z}}] r^2 \sin \theta dr d\theta d\phi \\
&= \frac{3p}{16\pi^2\epsilon_0 R^3} \int_0^{2\pi} \int_0^\pi \int_0^R \frac{1}{r} [3 \cos \theta \sin^2 \theta \cos \phi \hat{\mathbf{x}} \\
&\quad + 3 \cos \theta \sin^2 \theta \sin \phi \hat{\mathbf{y}} + (2 \cos^2 \theta - \sin^2 \theta) \sin \theta \hat{\mathbf{z}}] dr d\theta d\phi \\
&= \mathbf{0}
\end{aligned}$$

(b)

$$-\frac{1}{4\pi\epsilon_0} \frac{\mathbf{P}}{R^3} = \frac{1}{\frac{4}{3}\pi R^3} \int \mathbf{E} d\tau'$$
$$\mathbf{E} = -\frac{\mathbf{P}}{3\epsilon_0} \delta^3(\mathbf{r})$$

3.50

(a)

$$\begin{aligned} \int \mathbf{E}_1 \cdot \mathbf{E}_2 d\tau &= \int (-\nabla V_1) \cdot \mathbf{E}_2 d\tau \\ &= \int [V_1(\nabla \cdot \mathbf{E}_2) - \nabla \cdot (V_1 \mathbf{E}_2)] d\tau \\ &= \int \frac{\rho_2 V_1}{\epsilon_0} d\tau - \int \nabla \cdot (V_1 \mathbf{E}_2) d\tau \\ &= \int \frac{\rho_2 V_1}{\epsilon_0} d\tau - \oint V_1 \mathbf{E}_2 \cdot d\mathbf{a} \\ &= \int \frac{\rho_2 V_1}{\epsilon_0} d\tau \\ \int \mathbf{E}_1 \cdot \mathbf{E}_2 d\tau &= \int \mathbf{E}_1 \cdot (-\nabla V_2) d\tau \\ &= \int [V_2(\nabla \cdot \mathbf{E}_1) - \nabla \cdot (V_2 \mathbf{E}_1)] d\tau \\ &= \int \frac{\rho_1 V_2}{\epsilon_0} d\tau - \int \nabla \cdot (V_2 \mathbf{E}_1) d\tau \\ &= \int \frac{\rho_1 V_2}{\epsilon_0} d\tau - \oint V_2 \mathbf{E}_1 \cdot d\mathbf{a} \\ &= \int \frac{\rho_1 V_2}{\epsilon_0} d\tau \\ \int \frac{\rho_1 V_2}{\epsilon_0} d\tau &= \int \frac{\rho_2 V_1}{\epsilon_0} d\tau \\ \int \rho_1 V_2 d\tau &= \int \rho_2 V_1 d\tau \end{aligned}$$

(b)

$$\begin{aligned}Q_a &= \int_a \rho_1 d\tau \\&= Q\end{aligned}$$

$$\begin{aligned}Q_b &= \int_b \rho_1 d\tau \\&= 0\end{aligned}$$

$$V_{1b} = V_{ab}$$

$$\begin{aligned}Q_a &= \int_a \rho_2 d\tau \\&= 0\end{aligned}$$

$$\begin{aligned}Q_b &= \int_b \rho_2 d\tau \\&= Q\end{aligned}$$

$$V_{2a} = V_{ba}$$

$$\begin{aligned}\int \rho_1 V_2 d\tau &= \int_a \rho_1 V_2 d\tau + \int_b \rho_1 V_2 d\tau \\&= V_{2a} \int_a \rho_1 d\tau + V_{2b} \int \rho_1 d\tau \\&= V_{ba} Q\end{aligned}$$

$$\begin{aligned}\int \rho_2 V_1 d\tau &= \int_a \rho_2 V_1 d\tau + \int_b \rho_2 V_1 d\tau \\&= V_{1a} \int_a \rho_2 d\tau + V_{1b} \int \rho_2 d\tau \\&= V_{ab} Q\end{aligned}$$

$$V_{ba} Q = V_{ab} Q$$

$$V_{ba} = V_{ab}$$

3.51

(a)

$$\begin{aligned}\int \rho_2 V_1 d\tau &= Q_{l2} V_{l1} + Q_{x2} V_{x1} + Q_{r2} V_{r1} \\ &= 0\end{aligned}$$

$$\begin{aligned}\int \rho_1 V_2 d\tau &= Q_{l1} V_{l2} + Q_{x1} V_{x2} + Q_{r1} V_{r2} \\ &= q \frac{x}{d} V_0 + Q_2 V_0 \\ Q_2 &= -\frac{qx}{d}\end{aligned}$$

$$\begin{aligned}\int \rho_2 V_1 d\tau &= Q_{l2} V_{l1} + Q_{x2} V_{x1} + Q_{r2} V_{r1} \\ &= 0\end{aligned}$$

$$\begin{aligned}\int \rho_1 V_2 d\tau &= Q_{l1} V_{l2} + Q_{x1} V_{x2} + Q_{r1} V_{r2} \\ &= Q_1 V_0 + q \left(1 - \frac{x}{d}\right) V_0 \\ Q_1 &= q \left(\frac{x}{d} - 1\right)\end{aligned}$$

(b)

$$\begin{aligned}
\int \rho_2 V_1 d\tau &= Q_{a2} V_{a1} + Q_{r2} V_{r1} + Q_{b2} V_{b1} \\
&= 0 \\
V(a, \theta) &= 0 \\
V(b, \theta) &= V_0 \\
V(r, \theta) &= \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta) \\
0 &= \sum_{l=0}^{\infty} \left(A_l a^l + \frac{B_l}{a^{l+1}} \right) P_l(\cos \theta) \\
0 &= A_l a^l + \frac{B_l}{a^{l+1}} \\
B_l &= -A_l a^{2l+1} \\
V(r, \theta) &= \sum_{l=0}^{\infty} A_l \left(r^l - \frac{a^{2l+1}}{r^{l+1}} \right) P_l(\cos \theta) \\
V_0 &= \sum_{l=0}^{\infty} A_l \left(b^l - \frac{a^{2l+1}}{b^{l+1}} \right) P_l(\cos \theta) \\
&= A_0 \left(1 - \frac{a}{b} \right) \\
A_0 &= \frac{b}{b-a} V_0 \\
A_n &= 0, n \neq 0 \\
V(r, \theta) &= V_0 \frac{b}{b-a} \left(1 - \frac{a}{r} \right) \\
\int \rho_1 V_2 d\tau &= Q_{r1} V_{r2} + Q_{b1} V_{b2} \\
&= q V_0 \frac{b}{b-a} \left(1 - \frac{a}{r} \right) + Q_2 V_0 \\
Q_2 &= -\frac{qb}{b-a} \left(1 - \frac{a}{r} \right)
\end{aligned}$$

$$\begin{aligned}
\int \rho_2 V_1 d\tau &= Q_{a2} V_{a1} + Q_{r2} V_{r1} + Q_{b2} V_{b1} \\
&= 0 \\
V(a, \theta) &= V_0 \\
V(b, \theta) &= 0 \\
0 &= \sum_{l=0}^{\infty} \left(A_l b^l + \frac{B_l}{b^{l+1}} \right) P_l(\cos \theta) \\
0 &= A_l b^l \frac{B_l}{b^{l+1}} \\
B_l &= -A_l b^{2l+1} \\
V(r, \theta) &= \sum_{l=0}^{\infty} A_l \left(r^l - \frac{b^{2l+1}}{r^{l+1}} \right) P_l(\cos \theta) \\
V_0 &= \sum_{l=0}^{\infty} A_l \left(a^l - \frac{b^{2l+1}}{a^{l+1}} \right) P_l(\cos \theta) \\
V_0 &= A_0 \left(1 - \frac{b}{a} \right) \\
A_0 &= V_0 \frac{a}{a-b} \\
V(r, \theta) &= V_0 \frac{a}{a-b} \left(1 - \frac{b}{r} \right) \\
\int \rho_1 V_2 d\tau &= Q_{a1} V_{a2} + Q_{r1} V_{r2} + Q_{b1} V_{b2} \\
&= Q_1 V_0 + q V_0 \frac{a}{a-b} \left(1 - \frac{b}{r} \right) \\
Q_1 &= -\frac{qa}{a-b} \left(1 - \frac{b}{r} \right)
\end{aligned}$$

3.52

(a)

$$\begin{aligned}
V_{\text{quad}}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int (r')^2 P_2(\cos \alpha) \rho(\mathbf{r}') d\tau' \\
\int (r')^2 P_2(\cos \alpha) \rho(\mathbf{r}') d\tau' &= \int (r')^2 \left[\frac{1}{2} (3 \cos^2 \alpha - 1) \right] \rho(\mathbf{r}') d\tau' \\
&= \frac{1}{2} \int (r')^2 [3(\hat{\mathbf{r}}' \cdot \hat{\mathbf{r}})^2 - 1] \rho(\mathbf{r}') d\tau' \\
&= \frac{1}{2} \int [3(\mathbf{r}' \cdot \hat{\mathbf{r}})^2 - (r')^2] \rho(\mathbf{r}') d\tau' \\
&= \frac{1}{2} \int [3(\mathbf{r}' \cdot \hat{\mathbf{r}})^2 - (r')^2 (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}})] \rho(\mathbf{r}') d\tau' \\
&= \frac{1}{2} \int \left[3 \sum_{i,j=1}^3 r'_i r'_j \hat{r}_i \hat{r}_j - (r')^2 \sum_{i,j=1}^3 \hat{r}_i \hat{r}_j \delta_{ij} \right] \rho(\mathbf{r}') d\tau' \\
&= \sum_{i,j=1}^3 \hat{r}_i \hat{r}_j \frac{1}{2} \int [3r'_i r'_j - (r')^2 \delta_{ij}] \rho(\mathbf{r}') d\tau' \\
&= \sum_{i,j=1}^3 \hat{r}_i \hat{r}_j Q_{ij}
\end{aligned}$$

(b)

$$\begin{aligned}
Q_{11} &= 0 \\
Q_{12} &= \frac{3a^2 q}{2} \\
Q_{13} &= 0 \\
Q_{21} &= \frac{3a^2 q}{2} \\
Q_{22} &= 0 \\
Q_{23} &= 0 \\
Q_{31} &= 0 \\
Q_{32} &= 0 \\
Q_{33} &= 0
\end{aligned}$$

4 Electric Fields in Matter

4.1

$$V(x) = 500 \frac{x}{d} = 500\,000x$$

$$\mathbf{E} = -\nabla V = -500\,000 \text{ N/C}$$

$$\alpha = 4\pi\epsilon_0(0.667 \times 10^{-30})$$

$$= 7.42 \times 10^{-41} \text{ C}^2 \text{ m/N}$$

$$\alpha E = qd$$

$$d = \frac{\alpha E}{q}$$

$$= 2.32 \times 10^{-16} \text{ m}$$

$$\frac{d}{R} = 4.6 \times 10^{-6}$$

$$V = 1.88 \times 10^8 \text{ V}$$

4.2

$$\begin{aligned}
\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E &= \frac{1}{\epsilon_0} \int_0^{2\pi} \int_0^\pi \int_0^r \rho(r') r'^2 \sin \theta \, dr' \, d\theta \, d\phi \\
&= \frac{4\pi}{\epsilon_0} \int_0^r \frac{q}{\pi a^3} e^{-2r'/a} r'^2 \, dr' \\
&= \frac{4q}{\epsilon_0 a^3} \int_0^r e^{-2r'/a} r'^2 \, dr' \\
&= \frac{4q}{\epsilon_0 a^3} \frac{1}{4} a \left[a^2 - e^{-2r/a} (a^2 + 2ar + 2r^2) \right] \\
&= \frac{q}{\epsilon_0 a^2} \left[a^2 - e^{-2r/a} (a^2 + 2ar + 2r^2) \right] \\
\mathbf{E}_e(r) &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \left[1 - e^{-2r/a} \left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2} \right) \right] \hat{\mathbf{r}} \\
E &= \frac{1}{4\pi\epsilon_0} \frac{q}{d^2} \left(\frac{4}{3} \frac{d^3}{a^3} \right) \\
&= \frac{1}{4\pi\epsilon_0} \frac{4}{3a^3} (qd) \\
&= \frac{1}{4\pi\epsilon_0} \frac{4}{3a^3} p \\
&= \frac{1}{4\pi\epsilon_0} \frac{4}{3a^3} \alpha E \\
\alpha &= 3\pi\epsilon_0 a^3
\end{aligned}$$

4.3

$$\begin{aligned}
\rho(r) &= kr \\
Q_{\text{enc}} &= \int_0^{2\pi} \int_0^\pi \int_0^r kr'^3 \sin \theta \, dr' \, d\theta \, d\phi \\
&= \pi kr^4 \\
\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E &= \frac{\pi kr^4}{\epsilon_0} \\
\mathbf{E} &= \frac{kr^2}{4\epsilon_0} \\
E &= \frac{kd^2}{4\epsilon_0} \\
d &= \sqrt{\frac{4\epsilon_0 E}{k}} \\
p &= ed \\
&= 2e \sqrt{\frac{\epsilon_0}{k}} \sqrt{E}
\end{aligned}$$

p is proportional to \sqrt{E} .

4.4

$$\begin{aligned}
E_q &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \\
p &= \alpha E_q \\
&= \frac{1}{4\pi\epsilon_0} \frac{\alpha q}{r^2} \\
\mathbf{E}_{\text{dip}} &= \frac{1}{4\pi\epsilon_0} \frac{p}{r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}}) \\
E_{\text{dip}} &= \frac{1}{4\pi\epsilon_0} \frac{2p}{r^3} \\
&= \frac{1}{4\pi\epsilon_0} \frac{2}{r^3} \left(\frac{1}{4\pi\epsilon_0} \frac{\alpha q}{r^2} \right) \\
&= \left(\frac{1}{4\pi\epsilon_0} \right)^2 \frac{2\alpha q}{r^5} \\
F &= \left(\frac{1}{4\pi\epsilon_0} \right)^2 \frac{2\alpha q^2}{r^5}
\end{aligned}$$

4.5

$$\begin{aligned}
\mathbf{E}_{p_1} &= -\frac{1}{4\pi\epsilon_0} \frac{p_1}{r^3} \hat{\mathbf{z}} \\
\boldsymbol{\tau}_{p_2} &= \mathbf{p}_2 \times \mathbf{E}_{p_1} \\
&= p_2 \hat{\mathbf{x}} \times -\frac{1}{4\pi\epsilon_0} \frac{p_1}{r^3} \hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{p_1 p_2}{r^3} \hat{\mathbf{y}} \\
\mathbf{E}_{p_2} &= \frac{1}{4\pi\epsilon_0} \frac{2p_2}{r^3} \hat{\mathbf{x}} \\
\boldsymbol{\tau}_{p_1} &= \mathbf{p}_1 \times \mathbf{E}_{p_2} \\
&= p_1 \hat{\mathbf{z}} \times -\frac{1}{4\pi\epsilon_0} \frac{2p_2}{r^3} \hat{\mathbf{x}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{2p_1 p_2}{r^3} \hat{\mathbf{y}}
\end{aligned}$$

4.6

$$\begin{aligned}
\mathbf{E}_i &= \frac{1}{4\pi\epsilon_0} \frac{p}{8z^3} (\sin \theta \hat{\mathbf{x}} + 2 \cos \theta \hat{\mathbf{y}}) \\
\boldsymbol{\tau} &= \mathbf{p} \times \mathbf{E}_i \\
&= p \begin{pmatrix} \sin \theta \\ \cos \theta \\ 0 \end{pmatrix} \times \frac{1}{4\pi\epsilon_0} \frac{p}{8z^3} \begin{pmatrix} \sin \theta \\ 2 \cos \theta \\ 0 \end{pmatrix} \\
&= \frac{1}{4\pi\epsilon_0} \frac{p^2 \sin 2\theta}{16z^3} \hat{\mathbf{z}}
\end{aligned}$$

The dipole will come to rest at $\theta = n\pi, n \in \mathbb{Z}$.

4.8

$$\begin{aligned}
U &= -\mathbf{p}_1 \cdot \mathbf{E} \\
&= -\mathbf{p}_1 \cdot \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} [3(\mathbf{p}_2 \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{p}_2] \\
&= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} [\mathbf{p}_1 \cdot \mathbf{p}_2 - 3(\mathbf{p}_1 \cdot \hat{\mathbf{r}})(\mathbf{p}_2 \cdot \hat{\mathbf{r}})]
\end{aligned}$$

4.9

(a)

$$\begin{aligned}
\mathbf{E} &= \frac{q}{4\pi\epsilon_0} \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{(x^2 + y^2 + z^2)^{3/2}} \\
F_x &= (\mathbf{p} \cdot \nabla) \mathbf{E} \\
&= \left(p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} + p_z \frac{\partial}{\partial z} \right) \frac{q}{4\pi\epsilon_0} \frac{x}{(x^2 + y^2 + z^2)^{3/2}} \\
&= \frac{q}{4\pi\epsilon_0} \left[p_x \left(\frac{1}{(x^2 + y^2 + z^2)^{3/2}} - 3x \frac{x}{(x^2 + y^2 + z^2)^{5/2}} \right) \right. \\
&\quad \left. + p_y \left(-3x \frac{y}{(x^2 + y^2 + z^2)^{5/2}} \right) + p_z \left(-3x \frac{z}{(x^2 + y^2 + z^2)^{5/2}} \right) \right] \\
&= \frac{q}{4\pi\epsilon_0} \left(\frac{p_x}{r^3} - 3x \frac{p_x x + p_y y + p_z z}{r^5} \right) \\
&= \frac{q}{4\pi\epsilon_0} \left[\frac{\mathbf{p}}{r^3} - \frac{3\mathbf{r}(\mathbf{p} \cdot \mathbf{r})}{r^5} \right]_x \\
\mathbf{F} &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} [\mathbf{p} - 3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}]
\end{aligned}$$

(b)

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} [3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]$$

4.10

(a)

$$\begin{aligned}
\mathbf{P} &= k\mathbf{r} \\
\sigma_b &= \mathbf{P} \cdot \hat{\mathbf{n}} \\
&= kR \\
\rho_b &= -\nabla \cdot \mathbf{P} \\
&= -\frac{1}{r^2} \frac{\partial}{\partial r} (kr^3) \\
&= -3k
\end{aligned}$$

(b)

$$\begin{aligned}
\oint \mathbf{E}_{\text{inside}} \cdot d\mathbf{a} &= \frac{Q_{\text{encl}}}{\epsilon_0} \\
4\pi r^2 E_{\text{inside}} &= \frac{1}{\epsilon_0} \int_0^{2\pi} \int_0^\pi \int_0^r -3kr'^2 \sin \theta \, dr' \, d\theta \, d\phi \\
&= -\frac{12\pi k}{\epsilon_0} \int_0^r r'^2 \, dr' \\
&= -\frac{4\pi kr^3}{\epsilon_0} \\
\mathbf{E}_{\text{inside}} &= -\frac{kr}{\epsilon_0} \hat{\mathbf{r}} \\
&= -\frac{k}{\epsilon_0} \mathbf{r} \\
\oint \mathbf{E}_{\text{outside}} \cdot d\mathbf{a} &= \frac{Q_{\text{encl}}}{\epsilon_0} \\
4\pi r^2 E_{\text{outside}} &= \frac{1}{\epsilon_0} \left(4\pi kR^3 - \int_0^{2\pi} \int_0^\pi \int_0^R 3kr^2 \sin \theta \, dr \, d\theta \, d\phi \right) \\
&= \frac{1}{\epsilon_0} (4\pi kR^3 - 4\pi kR^3) \\
\mathbf{E}_{\text{outside}} &= \mathbf{0}
\end{aligned}$$

4.11

$$\begin{aligned}
\sigma_b &= \mathbf{P} \cdot \hat{\mathbf{n}} \\
&= \begin{cases} 0 & \text{on the side} \\ P & \text{on one end} \\ -P & \text{on the other end} \end{cases} \\
\rho_b &= -\nabla \cdot \mathbf{P} \\
&= 0
\end{aligned}$$

4.13

$$\begin{aligned}
\oint \mathbf{E}_{\text{inside}} \cdot d\mathbf{a} &= \frac{Q_{\text{encl}}}{\epsilon_0} \\
2\pi s L E_{\text{inside}} &= \frac{\pi s^2 L \rho}{\epsilon_0} \\
\mathbf{E}_{\text{inside}} &= \frac{\rho s}{2\epsilon_0} \hat{\mathbf{s}} \\
\mathbf{E}_{\text{inside}} &= \frac{\rho}{2\epsilon_0} (\mathbf{s}_+ - \mathbf{s}_-) \\
&= -\frac{\rho d}{2\epsilon_0} \\
&= -\frac{\mathbf{P}}{2\epsilon_0} \\
\oint \mathbf{E}_{\text{outside}} \cdot d\mathbf{a} &= \frac{Q_{\text{encl}}}{\epsilon_0} \\
2\pi s L E_{\text{outside}} &= \frac{\pi a^2 L \rho}{\epsilon_0} \\
\mathbf{E}_{\text{outside}} &= \frac{a^2 \rho}{2\epsilon_0 s} \hat{\mathbf{s}} \\
\mathbf{E}_{\text{outside}} &= \frac{a^2 \rho}{2\epsilon_0} \left(\frac{\hat{\mathbf{s}}_+}{s_+} - \frac{\hat{\mathbf{s}}_-}{s_-} \right)
\end{aligned}$$

4.14

$$\begin{aligned}
\oint_S \mathbf{P} \cdot \hat{\mathbf{n}} - \int_V \nabla \cdot \mathbf{P} d\tau' &= \oint_S \mathbf{P} \cdot \hat{\mathbf{n}} - \oint_S \mathbf{P} \cdot \hat{\mathbf{n}} \\
&= 0
\end{aligned}$$

4.15

(a)

$$\sigma_{b,a} = \mathbf{P} \cdot \hat{\mathbf{n}}$$

$$= -\frac{k}{a}$$

$$\sigma_{b,b} = \frac{k}{b}$$

$$\rho_b = -\nabla \cdot \mathbf{P}$$

$$= -\frac{1}{r^2} \frac{\partial}{\partial r}(kr)$$

$$= -\frac{k}{r^2}$$

$$\mathbf{E} = \mathbf{0}, r < a$$

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$4\pi r^2 E = \frac{1}{\epsilon_0} \left(-4\pi a^2 \frac{k}{a} - \int_0^{2\pi} \int_0^\pi \int_a^r \frac{k}{r'^2} r'^2 \sin \theta \, dr' \, d\theta \, d\phi \right)$$

$$= \frac{1}{\epsilon_0} [-4\pi a k - 4\pi k(r - a)]$$

$$\mathbf{E} = -\frac{k}{\epsilon_0 r} \hat{\mathbf{r}}, a < r < b$$

$$\mathbf{E} = \mathbf{0}$$

(b)

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$

$$\mathbf{D} = \mathbf{0}$$

$$\epsilon_0 \mathbf{E} + \mathbf{P} = \mathbf{0}$$

$$\mathbf{E} = \mathbf{0}$$

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$

$$\mathbf{D} = \mathbf{0}$$

$$\epsilon_0 \mathbf{E} + \mathbf{P} = \mathbf{0}$$

$$\begin{aligned} \mathbf{E} &= -\frac{\mathbf{P}}{\epsilon_0} \\ &= -\frac{k}{\epsilon_0 r} \hat{\mathbf{r}} \end{aligned}$$

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$

$$\mathbf{D} = \mathbf{0}$$

$$\epsilon_0 \mathbf{E} + \mathbf{P} = \mathbf{0}$$

$$\mathbf{E} = \mathbf{0}$$

4.16

(a)

$$\mathbf{E} = \mathbf{E}_0 + \frac{1}{3\epsilon_0} \mathbf{P}$$

$$\mathbf{D} = \mathbf{D}_0 - \frac{2}{3} \mathbf{P}$$

4.18

(a)

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$2AE = \frac{A\sigma}{\epsilon_0}$$

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0} \hat{\mathbf{n}}$$

$$\mathbf{E}_{\text{vac}} = -\frac{\sigma}{\epsilon_0} \hat{\mathbf{y}}$$

$$\begin{aligned} \mathbf{D} &= \epsilon_0 \mathbf{E}_{\text{vac}} \\ &= -\sigma \hat{\mathbf{y}} \end{aligned}$$

(b)

$$\begin{aligned}\mathbf{E}_1 &= \frac{1}{\epsilon_1} \mathbf{D} \\ &= -\frac{\sigma}{2\epsilon_0} \hat{\mathbf{y}} \\ \mathbf{E}_2 &= \frac{1}{\epsilon_2} \mathbf{D} \\ &= -\frac{\sigma}{1.5\epsilon_0} \hat{\mathbf{y}}\end{aligned}$$

(c)

$$\begin{aligned}\mathbf{P}_1 &= \epsilon_0 \chi_{e1} \mathbf{E}_1 \\ &= -\frac{\sigma}{2} \hat{\mathbf{y}} \\ \mathbf{P}_2 &= \epsilon_0 \chi_{e2} \mathbf{E}_2 \\ &= -\frac{\sigma}{3} \hat{\mathbf{y}}\end{aligned}$$

(d)

$$\begin{aligned}V &= a \frac{\sigma}{2\epsilon_0} + a \frac{2\sigma}{3\epsilon_0} \\ &= \frac{a\sigma}{\epsilon_0} \left(\frac{3}{6} + \frac{4}{6} \right) \\ &= \frac{7a\sigma}{6\epsilon_0}\end{aligned}$$

(e)

$$\begin{aligned}\sigma_{b1,\text{top}} &= \mathbf{P}_1 \cdot \hat{\mathbf{y}} \\ &= -\frac{\sigma}{2} \\ \sigma_{b1,\text{bottom}} &= \mathbf{P}_1 \cdot -\hat{\mathbf{y}} \\ &= \frac{\sigma}{2} \\ \sigma_{b2,\text{top}} &= \mathbf{P}_2 \cdot \hat{\mathbf{y}} \\ &= -\frac{\sigma}{3} \\ \sigma_{b2,\text{bottom}} &= \mathbf{P}_2 \cdot -\hat{\mathbf{y}} \\ &= \frac{\sigma}{3}\end{aligned}$$

(f)

$$\begin{aligned}\mathbf{E}_1 &= -\frac{\sigma}{\epsilon_0}\hat{\mathbf{y}} + \frac{\sigma}{2\epsilon_0}\hat{\mathbf{y}} \\ &= -\frac{\sigma}{2\epsilon_0}\hat{\mathbf{y}} \\ \mathbf{E}_2 &= -\frac{\sigma}{\epsilon_0}\hat{\mathbf{y}} + \frac{\sigma}{3\epsilon_0}\hat{\mathbf{y}} \\ &= -\frac{2\sigma}{3\epsilon_0}\hat{\mathbf{y}}\end{aligned}$$

4.19

$$\begin{aligned}Q &= A\sigma \\ E_{\text{vac}} &= \frac{\sigma}{\epsilon_0} \\ V_{\text{vac}} &= \frac{d\sigma}{\epsilon_0} \\ C_{\text{vac}} &= \frac{Q}{V_{\text{vac}}} \\ &= \frac{A\epsilon_0}{d} \\ E' &= \frac{1}{\epsilon_r}E_{\text{vac}} \\ &= \frac{\sigma}{\epsilon_0\epsilon_r} \\ V' &= \frac{d\sigma}{2\epsilon_0} + \frac{d\sigma}{2\epsilon_0\epsilon_r} \\ &= \frac{d\sigma}{2\epsilon_0} \left(1 + \frac{1}{\epsilon_r}\right) \\ &= \frac{d\sigma(1 + \epsilon_r)}{2\epsilon_0\epsilon_r} \\ C' &= \frac{Q}{V'} \\ &= \frac{2A\epsilon_0\epsilon_r}{d(1 + \epsilon_r)} \\ \frac{C'}{C_{\text{vac}}} &= \frac{2\epsilon_r}{1 + \epsilon_r}\end{aligned}$$

$$\begin{aligned}
E &= \frac{\sigma}{\epsilon_0} \\
\sigma &= \epsilon_0 E \\
&= \epsilon_0 \frac{V}{d} \\
P &= \epsilon_0 \chi_e E \\
&= \epsilon_0 \chi_e \frac{V}{d} \\
\sigma_b &= -\epsilon_0 \chi_e \frac{V}{d} \\
\sigma_{\text{total}} &= \sigma_f + \sigma_b \\
\epsilon_0 \frac{V}{d} &= \sigma_f - \epsilon_0 \chi_e \frac{V}{d} \\
\sigma_f &= \epsilon_0 \frac{V}{d} + \epsilon_0 \chi_e \frac{V}{d} \\
&= \epsilon_0 \frac{V}{d} (1 + \chi_e) \\
&= \frac{\epsilon_0 \epsilon_r V}{d} \\
C' &= \frac{Q'}{V} \\
&= \frac{1}{V} \left(\frac{A\sigma}{2} + \frac{A\sigma_f}{2} \right) \\
&= \frac{A}{2V} \left(\frac{\epsilon_0 V}{d} + \frac{\epsilon_0 \epsilon_r V}{d} \right) \\
&= \frac{A\epsilon_0(1 + \epsilon_r)}{2d} \\
\frac{C'}{C} &= \frac{1 + \epsilon_r}{2}
\end{aligned}$$

4.20

$$\begin{aligned}
\oint \mathbf{D} \cdot d\mathbf{a} &= Q_{\text{f}} \\
4\pi r^2 D &= \frac{4}{3}\pi r^3 \rho \\
\mathbf{D} &= \frac{r\rho}{3} \hat{\mathbf{r}} \\
\mathbf{D} &= \epsilon \mathbf{E} \\
\mathbf{E} &= \frac{1}{\epsilon} \mathbf{D} \\
&= \frac{r\rho}{3\epsilon_0 \epsilon_r} \hat{\mathbf{r}}, \quad r < R \\
\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E &= \frac{1}{\epsilon_0} \frac{4}{3}\pi R^3 \rho \\
\mathbf{E} &= \frac{R^3 \rho}{3\epsilon_0 r^2} \hat{\mathbf{r}}, \quad r > R \\
V &= - \int_{\infty}^0 \mathbf{E} \cdot d\mathbf{l} \\
&= - \left(\int_{\infty}^R \frac{R^3 \rho}{3\epsilon_0 r^2} dr + \int_R^0 \frac{r\rho}{3\epsilon_0 \epsilon_r} dr \right) \\
&= - \left(\frac{R^3 \rho}{3\epsilon_0} \left[-\frac{1}{r} \right]_{\infty}^R + \frac{\rho}{3\epsilon_0 \epsilon_r} \left[\frac{1}{2} r^2 \right]_R^0 \right) \\
&= - \left(-\frac{R^2 \rho}{3\epsilon_0} - \frac{R^2 \rho}{6\epsilon_0 \epsilon_r} \right) \\
&= \frac{R^2 \rho}{3\epsilon_0} \left(1 + \frac{1}{2\epsilon_r} \right)
\end{aligned}$$

4.21

$$\begin{aligned}
\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
2\pi s L E &= \frac{\pi a^2 L \rho}{\epsilon_0} \\
\mathbf{E} &= \frac{a^2 \rho}{2\epsilon_0 s} \hat{\mathbf{s}}, \quad a < s < b \\
\mathbf{E} &= \frac{\mathbf{E}_{\text{vac}}}{\epsilon_r} \\
&= \frac{a^2 \rho}{2\epsilon_0 \epsilon_r s} \hat{\mathbf{s}}, \quad b < s < c \\
V &= - \int_c^a \mathbf{E} \cdot d\mathbf{l} \\
&= - \left(\int_c^b \frac{a^2 \rho}{2\epsilon_0 \epsilon_r s} ds + \int_b^a \frac{a^2 \rho}{2\epsilon_0 s} ds \right) \\
&= - \left(\frac{a^2 \rho}{2\epsilon_0 \epsilon_r} \ln \frac{b}{c} + \frac{a^2 \rho}{2\epsilon_0} \ln \frac{a}{b} \right) \\
&= \frac{a^2 \rho}{2\epsilon_0} \left(\ln \frac{b}{a} + \frac{1}{\epsilon_r} \ln \frac{c}{b} \right) \\
C &= \frac{Q}{V} \\
&= \frac{\pi a^2 \rho}{\frac{a^2 \rho}{2\epsilon_0} \left(\ln \frac{b}{a} + \frac{1}{\epsilon_r} \ln \frac{c}{b} \right)} \\
&= \frac{2\pi \epsilon_0}{\ln \frac{b}{a} + \frac{1}{\epsilon_r} \ln \frac{c}{b}}
\end{aligned}$$

4.22

$$\begin{aligned}
V_{\text{in}}(a, \phi) &= V_{\text{out}}(a, \phi) \\
\epsilon \left. \frac{\partial V_{\text{in}}}{\partial s} \right|_{s=a} &= \epsilon_0 \left. \frac{\partial V_{\text{out}}}{\partial s} \right|_{s=a} \\
V_{\text{out}} &\rightarrow -E_0 s \cos \phi, \text{ for } s \gg a \\
V_{\text{in}} &= \sum_{k=1}^{\infty} s^k (a_k \cos k\phi + b_k \sin k\phi) \\
V_{\text{out}} &= -E_0 s \cos \phi + \sum_{k=1}^{\infty} s^{-k} (c_k \cos k\phi + d_k \sin k\phi) \\
\sum_{k=1}^{\infty} a^k (a_k \cos k\phi + b_k \sin k\phi) &= -E_0 a \cos \phi + \sum_{k=1}^{\infty} a^{-k} (c_k \cos k\phi + d_k \sin k\phi) \\
aa_1 &= \frac{c_1}{a} - E_0 a \\
a_1 &= \frac{c_1}{a^2} - E_0 \\
a^k a_k &= a^{-k} c_k \\
a_k &= a^{-2k} c_k, \text{ for } k \neq 1 \\
a^k b_k &= a^{-k} d_k \\
b_k &= a^{-2k} d_k \\
\epsilon_0 (1 + \chi_e) \sum_{k=1}^{\infty} k a^{k-1} (a_k \cos k\phi + b_k \sin k\phi) &= \epsilon_0 \left(-E_0 \cos \phi - \sum_{k=1}^{\infty} k a^{-(k+1)} (c_k \cos k\phi + d_k \sin k\phi) \right) \\
(1 + \chi_e) a_1 &= - \left(E_0 + \frac{c_1}{a^2} \right) \\
a_1 &= - \frac{1}{1 + \chi_e} \left(E_0 + \frac{c_1}{a^2} \right) \\
\frac{c_1}{a^2} - E_0 &= - \frac{1}{1 + \chi_e} \left(E_0 + \frac{c_1}{a^2} \right) \\
c_1 - a^2 E_0 &= - \frac{1}{1 + \chi_e} (a^2 E_0 + c_1) \\
c_1 \frac{2 + \chi_e}{1 + \chi_e} &= a^2 E_0 \frac{\chi_e}{1 + \chi_e} \\
c_1 &= a^2 E_0 \frac{\chi_e}{2 + \chi_e} \\
a_1 &= E_0 \left(\frac{\chi_e}{2 + \chi_e} - 1 \right) \\
&= - \frac{E_0}{1 + \chi_e/2}
\end{aligned}$$

$$\begin{aligned}
(1 + \chi_e) a^{k-1} a_k &= -a^{-(k+1)} c_k \\
a_k &= -\frac{1}{1 + \chi_e} a^{-2k} c_k \\
a^{-2k} c_k &= -\frac{1}{1 + \chi_e} a^{-2k} c_k \\
c_k &= 0 \\
a_k &= 0 \\
(1 + \chi_e) a^{k-1} b_k &= -a^{-(k+1)} d_k \\
b_k &= -\frac{1}{1 + \chi_e} a^{-2k} d_k \\
a^{-2k} d_k &= -\frac{1}{1 + \chi_e} a^{-2k} d_k \\
d_k &= 0 \\
b_k &= 0 \\
V_{\text{in}} &= -\frac{E_0}{1 + \chi_e/2} s \cos \phi \\
&= -\frac{E_0}{1 + \chi_e/2} x \\
\mathbf{E}_{\text{in}} &= -\frac{\partial V_{\text{in}}}{\partial x} \hat{\mathbf{x}} \\
&= \frac{\mathbf{E}_0}{1 + \chi_e/2}
\end{aligned}$$

4.23

$$\begin{aligned}
\mathbf{P}_0 &= \epsilon_0 \chi_e \mathbf{E}_0 \\
\mathbf{E}_1 &= -\frac{1}{3\epsilon_0} \mathbf{P}_0 \\
&= -\frac{\chi_e}{3} \mathbf{E}_0 \\
\mathbf{P}_1 &= \epsilon_0 \chi_e \mathbf{E}_1 \\
&= -\frac{\epsilon_0 \chi_e^2}{3} \mathbf{E}_0 \\
\mathbf{E}_2 &= -\frac{1}{3\epsilon_0} \mathbf{P}_1 \\
&= \frac{\chi_e^2}{9} \mathbf{E}_0 \\
\mathbf{E}_n &= \left(-\frac{\chi_e}{3}\right)^n \mathbf{E}_0 \\
\mathbf{E} &= \sum_{n=0}^{\infty} \mathbf{E}_n \\
&= \mathbf{E}_0 \sum_{n=0}^{\infty} \left(-\frac{\chi_e}{3}\right)^n \\
&= \frac{3}{3 + \chi_e} \mathbf{E}_0 \\
&= \frac{3}{2 + \epsilon_r} \mathbf{E}_0
\end{aligned}$$

4.26

$$\begin{aligned}
 \mathbf{D} &= \begin{cases} \mathbf{0} & r < a \\ \frac{Q}{4\pi r^2} \hat{\mathbf{r}} & a < r < b \\ \frac{Q}{4\pi r^2} \hat{\mathbf{r}} & b < r \end{cases} \\
 \mathbf{E} &= \begin{cases} \mathbf{0} & r < a \\ \frac{1}{4\pi\epsilon_0(1+\chi_e)} \frac{Q}{r^2} \hat{\mathbf{r}} & a < r < b \\ \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}} & b < r \end{cases} \\
 \mathbf{D} \cdot \mathbf{E} &= \begin{cases} 0 & r < a \\ \frac{Q^2}{16\pi^2\epsilon_0(1+\chi_e)r^4} & a < r < b \\ \frac{Q^2}{16\pi^2\epsilon_0 r^4} & b < r \end{cases} \\
 W &= \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} d\tau \\
 &= \frac{1}{2} \int_0^{2\pi} \int_0^\pi \int_0^\infty \mathbf{D} \cdot \mathbf{E} r^2 \sin\theta dr d\theta d\phi \\
 &= \frac{Q^2}{8\pi\epsilon_0} \left(\frac{1}{1+\chi_e} \int_a^b \frac{1}{r^2} dr + \int_b^\infty \frac{1}{r^2} dr \right) \\
 &= \frac{Q^2}{8\pi\epsilon_0} \left[\frac{1}{1+\chi_e} \left(\frac{1}{a} - \frac{1}{b} \right) + \frac{1}{b} \right] \\
 &= \frac{Q^2}{8\pi\epsilon_0(1+\chi_e)} \left(\frac{1}{a} + \frac{\chi_e}{b} \right)
 \end{aligned}$$

4.27

$$\begin{aligned}
\mathbf{E} &= \begin{cases} -\frac{\mathbf{P}}{3\epsilon_0} & r < R \\ \frac{R^3 P}{3\epsilon_0 r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}}) & r > R \end{cases} \\
E^2 &= \begin{cases} \frac{P^2}{9\epsilon_0^2} & r < R \\ \frac{R^6 P^2}{9\epsilon_0^2 r^6} (1 + 3 \cos^2 \theta) & r > R \end{cases} \\
W &= \frac{\epsilon_0}{2} \int E^2 d\tau \\
&= \frac{\epsilon_0}{2} \int_0^{2\pi} \int_0^\pi \int_0^\infty E^2 r^2 \sin \theta dr d\theta d\phi \\
&= \epsilon_0 \pi \left(\frac{P^2}{9\epsilon_0^2} \int_0^\pi \int_0^R r^2 \sin \theta dr d\theta + \frac{R^6 P^2}{9\epsilon_0^2} \int_0^\pi \int_R^\infty \frac{1 + 3 \cos^2 \theta}{r^4} \sin \theta dr d\theta \right) \\
&= \epsilon_0 \pi \left(\frac{2P^2 R^3}{27\epsilon_0^2} + \frac{4P^2 R^3}{27\epsilon_0^2} \right) \\
&= \frac{2\pi P^2 R^3}{9\epsilon_0} \\
W &= \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} d\tau
\end{aligned}$$

4.28

$$\begin{aligned}
\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{encl}}}{\epsilon_0} \\
2\pi sLE &= \frac{\lambda L}{\epsilon_0} \\
\mathbf{E} &= \frac{\lambda}{2\pi\epsilon_0 s} \hat{\mathbf{s}} \\
V &= \int_a^b \mathbf{E} \cdot d\mathbf{l} \\
&= \frac{\lambda}{2\pi\epsilon_0} \ln \frac{b}{a} \\
\oint \mathbf{D} \cdot d\mathbf{a} &= Q_{\text{encl},f} \\
2\pi sLD &= \lambda' L \\
\mathbf{D} &= \frac{\lambda'}{2\pi s} \hat{\mathbf{s}} \\
\mathbf{E} &= \frac{\lambda'}{2\pi\epsilon s} \hat{\mathbf{s}} \\
V &= \int_a^b \mathbf{E} \cdot d\mathbf{l} \\
&= \frac{\lambda'}{2\pi\epsilon} \ln \frac{b}{a} \\
\frac{\lambda'}{2\pi\epsilon} \ln \frac{b}{a} &= \frac{\lambda}{2\pi\epsilon_0} \ln \frac{b}{a} \\
\lambda' &= \epsilon_r \lambda \\
Q &= \lambda' h + \lambda(\ell - h) \\
&= (\epsilon_r h + \ell - h)\lambda \\
&= (\chi_e h + \ell)\lambda \\
C &= \frac{Q}{V} \\
&= (\chi_e h + \ell)\lambda \frac{2\pi\epsilon_0}{\lambda \ln(b/a)} \\
&= 2\pi\epsilon_0 \frac{\chi_e h + \ell}{\ln(b/a)} \\
\frac{dC}{dh} &= \frac{2\pi\epsilon_0 \chi_e}{\ln(b/a)}
\end{aligned}$$

$$\begin{aligned}
F_{\text{up}} &= \frac{1}{2} V^2 \frac{dC}{dh} \\
&= \frac{\pi \epsilon_0 \chi_e V^2}{\ln(b/a)} \\
F_{\text{down}} &= mg \\
&= \pi(b^2 - a^2) h \rho g \\
F_{\text{down}} &= F_{\text{up}} \\
\pi(b^2 - a^2) h \rho g &= \frac{\pi \epsilon_0 \chi_e V^2}{\ln(b/a)} \\
h &= \frac{\epsilon_0 \chi_e V^2}{\rho g(b^2 - a^2) \ln(b/a)}
\end{aligned}$$

4.29

(a)

$$\begin{aligned}
\mathbf{E}_1 &= -\frac{1}{4\pi\epsilon_0} \frac{p_1}{r^3} \hat{\mathbf{z}} \\
\mathbf{F}_2 &= \frac{1}{4\pi\epsilon_0} \frac{3p_1 p_2}{r^4} \hat{\mathbf{z}} \\
\mathbf{E}_2 &= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} [3(\mathbf{p}_2 \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{p}_2] \\
&= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \left[\frac{3(\mathbf{p}_2 \cdot \mathbf{r}) \mathbf{r}}{r^2} - \mathbf{p}_2 \right] \\
&= \frac{p_2}{4\pi\epsilon_0} \frac{3y(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) - (x^2 + y^2 + z^2)\hat{\mathbf{y}}}{(x^2 + y^2 + z^2)^{5/2}} \\
&= \frac{p_2}{4\pi\epsilon_0} \frac{3xy\hat{\mathbf{x}} + (-x^2 + 2y^2 - z^2)\hat{\mathbf{y}} + 3yz\hat{\mathbf{z}}}{(x^2 + y^2 + z^2)^{5/2}} \\
E_{2z} &= \frac{p_2}{4\pi\epsilon_0} \frac{3yz}{(x^2 + y^2 + z^2)^{5/2}} \\
\nabla E_{2z}|_{x=0, y=-r, z=0} &= -\frac{3p_2}{4\pi\epsilon_0 r^4} \hat{\mathbf{z}} \\
\mathbf{F}_1 &= -\frac{1}{4\pi\epsilon_0} \frac{3p_1 p_2}{r^4} \hat{\mathbf{z}}
\end{aligned}$$

Newton's third law is obeyed.

(b)

$$\begin{aligned}
\boldsymbol{\tau}_2 &= \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{p}_1 \times \mathbf{E}_1 \\
&= r\hat{\mathbf{y}} \times \frac{1}{4\pi\epsilon_0} \frac{3p_1p_2}{r^4} \hat{\mathbf{z}} - \frac{1}{4\pi\epsilon_0} \frac{p_1p_2}{r^3} \hat{\mathbf{x}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{2p_1p_2}{r^3} \hat{\mathbf{x}}
\end{aligned}$$

This is opposite to the torque experienced by \mathbf{p}_1 .

4.30

The electric field is perpendicular to each plate and “curves” towards the other. At $y = 0$ the electric field is purely vertical. \mathbf{p} only has a y component so $(\mathbf{p} \cdot \nabla)\mathbf{E} = p_y \nabla E_y$. The potential difference between the two plates is constant and as x increases the distance between them also increases, meaning the magnitude of the electric field decreases. E_y is negative at $y = 0$ so this means it increases with increasing x and thus that the x component of ∇E_y is positive. E_y is constant for small changes in y around $y = 0$ so the y component of ∇E_y is 0. We assume the plates are very long in the z direction so the z component of ∇E_y is also 0. This means the dipole experiences a force in the x direction.

4.31

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{Q(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})}{(x^2 + y^2 + z^2)^{3/2}} \\
E_y &= \frac{1}{4\pi\epsilon_0} \frac{Qy}{(x^2 + y^2 + z^2)^{3/2}} \\
\nabla E_y|_{x=R, y=z=0} &= \frac{1}{4\pi\epsilon_0} \frac{Q}{R^3} \hat{\mathbf{y}} \\
\mathbf{F} &= (\mathbf{p} \cdot \nabla)\mathbf{E} \\
&= p_y \nabla E_y \\
&= \frac{Q}{4\pi\epsilon_0} \frac{\mathbf{p}}{R^3}
\end{aligned}$$

4.32

(a)

$$\begin{aligned}
 \mathbf{p} &= \alpha \mathbf{E} \\
 \mathbf{F} &= (\mathbf{p} \cdot \nabla) \mathbf{E} \\
 &= (\alpha \mathbf{E} \cdot \nabla) \mathbf{E} \\
 &= \alpha (\mathbf{E} \cdot \nabla) \mathbf{E} \\
 \nabla(E^2) &= 2\mathbf{E} \times (\nabla \times \mathbf{E}) + 2(\mathbf{E} \cdot \nabla) \mathbf{E} \\
 (\mathbf{E} \cdot \nabla) \mathbf{E} &= \frac{1}{2} \nabla(E^2) \\
 \mathbf{F} &= \frac{1}{2} \alpha \nabla(E^2)
 \end{aligned}$$

4.33

$$\begin{aligned}
 \mathbf{P} &= k\mathbf{r} \\
 &= k(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \\
 \sigma_b &= \mathbf{P} \cdot \hat{\mathbf{n}} \\
 Q_{\text{face}} &= \int \sigma_b da \\
 &= \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} k \left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + \frac{a}{2}\hat{\mathbf{z}} \right) \cdot \hat{\mathbf{z}} dx dy \\
 &= \frac{ak}{2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} dx dy \\
 &= \frac{a^3 k}{2} \\
 Q_{\text{surface}} &= 3a^3 k \\
 \rho_b &= -\nabla \cdot \mathbf{P} \\
 &= -3k \\
 Q_{\text{volume}} &= \rho V \\
 &= -3a^3 k \\
 Q_{\text{total}} &= 0
 \end{aligned}$$

4.34

$$\begin{aligned}
\epsilon &= \epsilon_0 \left(1 + \frac{x}{d}\right) \\
\oint \mathbf{D} \cdot d\mathbf{a} &= Q_{\text{enc},f} \\
AD &= A\sigma_f \\
\mathbf{D} &= \sigma_f \hat{\mathbf{x}} \\
\mathbf{E} &= \frac{\mathbf{D}}{\epsilon} \\
&= \frac{\sigma_f}{\epsilon_0(1 + x/d)} \hat{\mathbf{x}} \\
V &= \int_0^d \mathbf{E} \cdot d\mathbf{l} \\
&= \frac{\sigma_f}{\epsilon_0} \int_0^d \frac{1}{1 + x/d} dx \\
&= \frac{\sigma_f}{\epsilon_0} d \ln 2 \\
\sigma_f &= \frac{\epsilon_0 V}{d \ln 2} \\
\mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} \\
\mathbf{P} &= \mathbf{D} - \epsilon_0 \mathbf{E} \\
&= \sigma_f \hat{\mathbf{x}} - \frac{\sigma_f}{1 + x/d} \hat{\mathbf{x}} \\
&= \sigma_f \left(1 - \frac{1}{1 + x/d}\right) \hat{\mathbf{x}} \\
&= \frac{\epsilon_0 V x}{d(d + x) \ln 2} \hat{\mathbf{x}} \\
\sigma_b|_{x=0} &= (\mathbf{P} \cdot -\hat{\mathbf{x}})|_{x=0} \\
&= 0 \\
\sigma_b|_{x=d} &= (\mathbf{P} \cdot \hat{\mathbf{x}})|_{x=d} \\
&= \frac{\epsilon_0 V}{2d \ln 2} \\
Q_{\text{surface}} &= A \frac{\epsilon_0 V}{2d \ln 2} \\
\rho_b &= -\nabla \cdot \mathbf{P} \\
&= -\frac{\epsilon_0 V}{(d + x)^2 \ln 2}
\end{aligned}$$

$$\begin{aligned}
Q_{\text{volume}} &= \int_0^d A \rho_b dx \\
&= -A \frac{\epsilon_0 V}{\ln 2} \int_0^d \frac{1}{(d+x)^2} dx \\
&= -A \frac{\epsilon_0 V}{2d \ln 2}
\end{aligned}$$

4.35

$$\begin{aligned}
\oint \mathbf{D} \cdot d\mathbf{a} &= Q_{\text{encl},f} \\
4\pi r^2 D &= q \\
\mathbf{D} &= \frac{q}{4\pi r^2} \hat{\mathbf{r}} \\
\mathbf{E} &= \frac{\mathbf{D}}{\epsilon} \\
&= \frac{1}{4\pi\epsilon_0} \frac{q}{(1+\chi_e)r^2} \hat{\mathbf{r}} \\
\mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} \\
\mathbf{P} &= \mathbf{D} - \epsilon_0 \mathbf{E} \\
&= \frac{q}{4\pi r^2} \hat{\mathbf{r}} - \frac{q}{4\pi(1+\chi_e)r^2} \hat{\mathbf{r}} \\
&= \frac{q}{4\pi r^2} \left(1 - \frac{1}{1+\chi_e}\right) \hat{\mathbf{r}} \\
&= \frac{q\chi_e}{4\pi(1+\chi_e)r^2} \hat{\mathbf{r}} \\
\rho_b &= -\nabla \cdot \mathbf{P} \\
&= -\frac{q\chi_e}{4\pi(1+\chi_e)} \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) \\
&= -\frac{q\chi_e}{1+\chi_e} \delta^3(\mathbf{r}) \\
\sigma_b &= \mathbf{P} \cdot \hat{\mathbf{n}} \\
&= \frac{q\chi_e}{4\pi(1+\chi_e)R^2} \\
Q_{\text{surface}} &= A\sigma_b \\
&= \frac{q\chi_e}{1+\chi_e}
\end{aligned}$$

4.36

$$\begin{aligned}
D_1^\perp - D_2^\perp &= \sigma_f \\
\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp &= 0 \\
\epsilon_1 E_1^\perp &= \epsilon_2 E_2^\perp \\
E_1^\parallel - E_2^\parallel &= 0 \\
E_1^\parallel &= E_2^\parallel \\
\tan \theta_2 &= 1 \\
\tan \theta_2 &= \frac{1}{\epsilon_2/\epsilon_1} \\
&= \frac{\epsilon_1}{\epsilon_2} \\
\frac{\tan \theta_2}{\tan \theta_1} &= \frac{\epsilon_2}{\epsilon_1}
\end{aligned}$$

If an electric field moved from air to a convex dielectric lens, $\epsilon_2/\epsilon_0 > 1$ meaning it would bend away from the normal. The lens would defocus the field.

4.39

(a)

$$\begin{aligned}
V &= \begin{cases} V_0 & r \leq R \\ V_0 \frac{R}{r} & r \geq R \end{cases} \\
\mathbf{E} &= -\nabla V \\
&= \begin{cases} \mathbf{0} & r \leq R \\ V_0 \frac{R}{r^2} \hat{\mathbf{r}} & r \geq R \end{cases} \\
\mathbf{P} &= \epsilon_0 \chi_e V_0 \frac{R}{r^2} \hat{\mathbf{r}}, \quad z < 0 \\
\sigma_b &= \mathbf{P} \cdot \hat{\mathbf{n}} \\
&= -\frac{\epsilon_0 \chi_e V_0}{R}, \text{ on the surface that touches the sphere} \\
\oint \sigma_f da &= Q_{\text{total}} \\
4\pi R^2 \sigma_f &= 4\pi \epsilon_0 V_0 R \\
\sigma_f &= \begin{cases} \epsilon_0 V_0 / R & \text{northern hemisphere} \\ \epsilon_0 V_0 (1 + \chi_e) / R & \text{southern hemisphere} \end{cases}
\end{aligned}$$

(b)

$$\begin{aligned}\sigma &= \sigma_b + \sigma_f \\ &= \frac{\epsilon_0 V_0}{R} \\ Q_{\text{total}} &= 4\pi\epsilon_0 V_0 R \\ V &= V_0 \frac{R}{r}\end{aligned}$$

5 Magnetostatics

5.1

The charge is positive.

$$\begin{aligned}a^2 + (R - d)^2 &= R^2 \\ a^2 + R^2 - 2dR + d^2 &= R^2 \\ 2dR &= a^2 + d^2 \\ R &= \frac{a^2 + d^2}{2d} \\ p &= qBR \\ &= \frac{qB(a^2 + d^2)}{2d}\end{aligned}$$

5.2

$$\begin{aligned}y(t) &= C_1 \cos \omega t + C_2 \sin \omega t + \frac{E}{B}t + C_3 \\ y'(t) &= -C_1 \omega \sin \omega t + C_2 \omega \cos \omega t + \frac{E}{B} \\ z(t) &= C_2 \cos \omega t - C_1 \sin \omega t + C_4 \\ z'(t) &= -C_2 \omega \sin \omega t - C_1 \omega \cos \omega t\end{aligned}$$

(a)

$$\begin{aligned}0 &= C_1 + C_3 \\ \frac{E}{B} &= C_2\omega + \frac{E}{B} \\ 0 &= C_2 + C_4 \\ 0 &= -C_1\omega \\ C_1 &= 0 \\ C_2 &= 0 \\ C_3 &= 0 \\ C_4 &= 0 \\ y(t) &= \frac{E}{B}t \\ z(t) &= 0\end{aligned}$$

(b)

$$\begin{aligned}0 &= C_1 + C_3 \\ \frac{E}{2B} &= C_2\omega + \frac{E}{B} \\ 0 &= C_2 + C_4 \\ 0 &= -C_1\omega \\ C_1 &= 0 \\ C_2 &= -\frac{E}{2B\omega} \\ C_3 &= 0 \\ C_4 &= \frac{E}{2B\omega} \\ y(t) &= -\frac{E}{2B\omega} \sin \omega t + \frac{E}{B}t \\ z(t) &= -\frac{E}{2B\omega} \cos \omega t + \frac{E}{2B\omega}\end{aligned}$$

(c)

$$\begin{aligned}
0 &= C_1 + C_3 \\
\frac{E}{B} &= C_2\omega + \frac{E}{B} \\
0 &= C_2 + C_4 \\
\frac{E}{B} &= -C_1\omega \\
C_1 &= -\frac{E}{B\omega} \\
C_2 &= 0 \\
C_3 &= \frac{E}{B\omega} \\
C_4 &= 0 \\
y(t) &= -\frac{E}{B\omega} \cos \omega t + \frac{E}{B}t + \frac{E}{B\omega} \\
z(t) &= \frac{E}{B\omega} \sin \omega t
\end{aligned}$$

5.3

(a)

$$\begin{aligned}
\mathbf{0} &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
0 &= E + vB \\
v &= \frac{E}{B}
\end{aligned}$$

(b)

$$\begin{aligned}
qBR &= p \\
&= mv \\
&= m\frac{E}{B} \\
\frac{q}{m} &= \frac{E}{B^2R}
\end{aligned}$$

5.4

The forces on the sides at $y = \pm a/2$ cancel, leaving the sides at $z = \pm a/2$ which both experience an upwards force of $F = IB = a^2Ik/2$ meaning the total force is $F_{\text{total}} = a^2Ik$ in the positive z direction.

5.5

(a)

$$K = \frac{I}{2\pi a}$$

(b)

$$\begin{aligned}
 J(s) &= \frac{k}{s} \\
 I &= \int J(s) \, da \\
 &= \int_0^a \int_0^{2\pi} \frac{k}{s} \, d\phi \, ds \\
 &= 2\pi a k \\
 k &= \frac{I}{2\pi a} \\
 J(s) &= \frac{I}{2\pi a s}
 \end{aligned}$$

5.6

(a)

$$K = \sigma \omega r$$

(b)

$$\mathbf{J} = \rho \omega r \sin \theta \hat{\phi} = \frac{3Q\omega r \sin \theta}{4\pi R^3} \hat{\phi}$$