Classical Mechanics by John R. Taylor Problems

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Contents

1 Newton's Laws of Motion

1.1

$$\mathbf{b} + \mathbf{c} = 2\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$$
$$5\mathbf{b} + 2\mathbf{x} = 7\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 2\hat{\mathbf{z}}$$
$$\mathbf{b} \cdot \mathbf{c} = 1$$
$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$
$$= \hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}}$$

1.5

$$\mathbf{v}_{\text{body}} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$$
$$\mathbf{v}_{\text{face}} = \hat{\mathbf{x}} + \hat{\mathbf{z}}$$
$$\mathbf{v}_{\text{body}} \cdot \mathbf{v}_{\text{face}} = v_{\text{body}} v_{\text{face}} \cos\theta$$
$$2 = \sqrt{6}\cos\theta$$
$$\cos\theta = \frac{2}{\sqrt{6}}$$
$$\theta = \arccos\frac{2}{\sqrt{6}}$$
$$= 35.26°$$

1.11

The particle moves counterclockwise in an ellipse of width $2b$ and height $2c$. The angular speed is ω.

1.23

$$\mathbf{v} = v\cos\theta\,\frac{\mathbf{b}}{b} - v\sin\theta\,\frac{\mathbf{b}\times\mathbf{c}}{bc}$$
$$= \frac{\lambda}{b}\frac{\mathbf{b}}{b} - \frac{c}{b}\frac{\mathbf{b}\times\mathbf{c}}{bc}$$
$$= \frac{\lambda\mathbf{b} - \mathbf{b}\times\mathbf{c}}{b^2}$$

1.25

$$\frac{df}{dt} = -3f$$
$$\frac{1}{f}\frac{df}{dt} = -3$$
$$\ln f = -3t + c$$
$$f = ce^{-3t}$$

One constant.

1.35

$$F_x = 0$$
$$ma_x = 0$$
$$a_x = 0$$
$$v_x = c_1$$
$$= v_o \cos \theta$$
$$r_x = v_o \cos(\theta)t + c_2$$
$$= v_o \cos(\theta)t$$

$$F_y = 0$$
$$ma_y = 0$$
$$a_y = 0$$
$$v_y = c_3$$
$$v_y = 0$$
$$r_y = c_4$$
$$r_y = 0$$

$$F_z = -mg$$
$$ma_z = -mg$$
$$a_z = -g$$
$$v_z = -gt + c_5$$
$$= v_o \sin \theta - gt$$
$$r_z = v_o \sin(\theta)t - \frac{1}{2}gt^2 + c_6$$
$$= v_o \sin(\theta)t - \frac{1}{2}gt^2$$

$$0 = v_o \sin(\theta)t - \frac{1}{2}gt^2$$
$$t = \frac{2\sin(\theta)v_o}{g}$$
$$r_x = v_o \cos(\theta)t$$
$$= \frac{2\cos(\theta)\sin(\theta)v_o^2}{g}$$
$$= \frac{\sin(2\theta)v_o^2}{g}$$

1.37

(a)

$$F = -mg\sin\theta$$
$$ma = -mg\sin\theta$$
$$a = -g\sin\theta$$
$$v = c_1 - gt\sin\theta$$
$$= v_o - gt\sin\theta$$
$$x = v_o t - \frac{1}{2}gt^2\sin\theta$$

(b)

$$t = \frac{2v_o}{g\sin\theta}$$

1.39

$$F_x = -mg\sin\phi$$
$$ma_x = -mg\sin\phi$$
$$a_x = -g\sin\phi$$
$$v_x = c_1 - gt\sin\phi$$
$$= v_o\cos\theta - gt\sin\phi$$
$$r_x = v_o t\cos\theta - \frac{1}{2}gt^2\sin\phi + c_2$$
$$= v_o t\cos\theta - \frac{1}{2}gt^2\sin\phi$$

$$F_y = -mg\cos\phi$$
$$ma_y = -mg\cos\phi$$
$$a_y = -g\cos\phi$$
$$v_y = c_3 - gt\cos\phi$$
$$= v_o\sin\theta - gt\cos\phi$$
$$r_y = v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi + c_4$$
$$= v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi$$

$$0 = v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi$$
$$t = \frac{2v_o\sin\theta}{g\cos\phi}$$

$$r_x = \frac{2v_o^2\cos\theta\sec\phi\sin\theta}{g} - \frac{2v_o^2\sec\phi\sin^2\theta\tan\phi}{g}$$
$$= \frac{2v_o^2\sin\theta(\cos\theta\cos\phi - \sin\theta\sin\phi)}{g\cos^2\phi}$$
$$= \frac{2v_o^2\sin\theta\cos(\theta + \phi)}{g\cos^2\phi}$$

$$\frac{dr_x}{d\theta} = \frac{2v_o^2}{g\cos^2\phi}[\cos\theta\cos(\theta+\phi) - \sin\theta\sin(\theta+\phi)]$$

$$= \frac{2v_o^2\cos(2\theta+\phi)}{g\cos^2\phi}$$

$$0 = \frac{2v_o^2\cos(2\theta+\phi)}{g\cos^2\phi}$$

$$= \cos(2\theta+\phi)$$

$$2\theta + \phi = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4} - \frac{\phi}{2}$$

$$r_{x,\max} = \frac{2v_o^2\sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right)\cos\left(\frac{\pi}{4} - \frac{\phi}{2} + \phi\right)}{g\cos^2\phi}$$

$$= \frac{v_o^2(1 - \sin\phi)}{g\cos^2\phi}$$

$$= \frac{v_o^2}{g(1 + \sin\phi)}$$

1.41

$$F = ma$$

$$T = m\frac{v^2}{R}$$

$$= m\frac{(\omega R)^2}{R}$$

$$= m\omega^2 R$$

1.47

(a)

$$\rho = \sqrt{x^2 + y^2}$$

$$\phi = \arctan\frac{y}{x}$$

$$z = z$$

ρ is the distance of P from the z-axis.

The use of r may be unfortunate because it suggests it's the distance of P from the origin.

(b) $\hat{\boldsymbol{\rho}}$ points away from the z-axis, $\hat{\boldsymbol{\phi}}$ points counter-clockwise around the z-axis, and $\hat{\mathbf{z}}$ points in the positive z direction.

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}} + \sqrt{x^2 + y^2}\,\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$$

(c)

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

$$= \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\frac{d\hat{\boldsymbol{\rho}}}{dt} + \dot{z}\hat{\mathbf{z}} + z\frac{d\hat{\mathbf{z}}}{dt}$$

$$= \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

$$= \ddot{\rho}\hat{\boldsymbol{\rho}} + \dot{\rho}\frac{d\hat{\boldsymbol{\rho}}}{dt} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}} + \rho\ddot{\phi}\hat{\boldsymbol{\phi}} + \rho\dot{\phi}\frac{d\hat{\boldsymbol{\phi}}}{dt} + \ddot{z}\hat{\mathbf{z}}$$

$$= \ddot{\rho}\hat{\boldsymbol{\rho}} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}} + \rho\ddot{\phi}\hat{\boldsymbol{\phi}} - \rho\dot{\phi}^2\hat{\boldsymbol{\rho}} + \ddot{z}\hat{\mathbf{z}}$$

$$= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\boldsymbol{\rho}} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\mathbf{z}}$$

2 Projectiles and Charged Particles

2.1

$$1 = (1.6 \times 10^3)Dv$$

$$v = \frac{1}{(1.6 \times 10^3)D}$$

$$= 8.9\,\mathrm{mm/s}$$

When $v \gg 1\,\mathrm{cm/s}$ the drag force can be treated as purely quadratic. For a beach ball this becomes $v \gg 1\,\mathrm{mm/s}$.

2.3

(a)

$$\frac{f_{\mathrm{quad}}}{f_{\mathrm{lin}}} = \frac{(1/4)\rho A v^2}{3\pi\eta D v}$$

$$= \frac{\rho\pi\left(\frac{D}{2}\right)^2 v}{12\pi\eta D}$$

$$= \frac{\rho D v}{48\eta}$$

$$= \frac{R}{48}$$

8

(b)

$$R = \frac{Dv\rho}{\eta} \approx 0.01$$

2.5

$$
\begin{aligned}
v_y(t) &= v_{\text{ter}} + (v_{\text{yo}} - v_{\text{ter}})e^{-t/\tau} \\
&= v_{\text{ter}} + (2v_{\text{ter}} - v_{\text{ter}})e^{-t/\tau} \\
&= v_{\text{ter}}(1 + e^{-t/\tau})
\end{aligned}
$$

The velocity starts at $2v_{\text{ter}}$ and asymptotically approaches v_{ter}.

2.7

$$
\begin{aligned}
F &= F(v) \\
m\dot{v} &= F(v) \\
m\frac{dv}{F(v)} &= dt \\
t &= \int_{v_o}^{v} m\frac{dv'}{F(v')}
\end{aligned}
$$

$$
\begin{aligned}
F &= F(v) \\
m\dot{v} &= F_o \\
v &= \frac{F_o}{m}t + c
\end{aligned}
$$

2.11

(a)

$$m\dot{v} = -mg - bv$$

$$\dot{v} = -g - kv$$

$$\frac{1}{-g - kv}\dot{v} = 1$$

$$-\frac{1}{k}\ln(-g - kv) = t + c$$

$$\ln(-g - kv) = c - \frac{t}{\tau}$$

$$-g - kv = Ae^{-t/\tau}$$

$$v = \tau(-g - Ae^{-t/\tau})$$

$$= -v_{\text{ter}} - \tau Ae^{-t/\tau}$$

$$v_{\text{o}} = -v_{\text{ter}} - \tau A$$

$$A = -k(v_{\text{o}} + v_{\text{ter}})$$

$$v = -v_{\text{ter}} + (v_{\text{o}} + v_{\text{ter}})e^{-t/\tau}$$

$$y = -v_{\text{ter}}t - \tau(v_{\text{o}} + v_{\text{ter}})e^{-t/\tau} + c$$

$$0 = -\tau(v_{\text{o}} + v_{\text{ter}}) + c$$

$$c = \tau(v_{\text{o}} + v_{\text{ter}})$$

$$y = -v_{\text{ter}}t + \tau(v_{\text{o}} + v_{\text{ter}})(1 - e^{-t/\tau})$$

(b)

$$0 = -v_{\text{ter}} + (v_{\text{o}} + v_{\text{ter}})e^{-t/\tau}$$

$$e^{-t/\tau} = \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}}$$

$$-\frac{t}{\tau} = \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}}$$

$$t = -\tau \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}}$$

$$y_{\text{max}} = -v_{\text{ter}} \left(-\tau \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} \right) + \tau(v_{\text{o}} + v_{\text{ter}}) \left(1 - \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} \right)$$

$$= \tau v_{\text{ter}} \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} + \tau(v_{\text{o}} + v_{\text{ter}} - v_{\text{ter}})$$

$$= \tau \left(v_{\text{o}} + v_{\text{ter}} \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} \right)$$

$$= \tau \left[v_{\text{o}} - v_{\text{ter}} \ln \left(1 + \frac{v_{\text{o}}}{v_{\text{ter}}} \right) \right]$$

(c)

$$y_{\text{max}} = \tau \left[v_{\text{o}} - v_{\text{ter}} \ln \left(1 + \frac{v_{\text{o}}}{v_{\text{ter}}} \right) \right]$$

$$= \tau \left[v_{\text{o}} - g\tau \ln \left(1 + \frac{v_{\text{o}}}{g\tau} \right) \right]$$

$$\approx \tau \left\{ v_{\text{o}} - g\tau \left[\frac{v_{\text{o}}}{g\tau} - \frac{1}{2} \left(\frac{v_{\text{o}}}{g\tau} \right)^2 \right] \right\}$$

$$= \tau \left(v_{\text{o}} - v_{\text{o}} + \frac{1}{2} \frac{v_{\text{o}}^2}{g\tau} \right)$$

$$= \frac{1}{2} \frac{v_{\text{o}}^2}{g}$$

2.13

$$v^2 = \frac{2}{m} \int_{x_0}^{x} -kx' \, dx'$$

$$= -\frac{2k}{m} \left(\frac{1}{2}x^2 - \frac{1}{2}x_0^2 \right)$$

$$= -\frac{k}{m}(x^2 - x_0^2)$$

$$v = \sqrt{\frac{k}{m}(x_0^2 - x^2)}$$

$$= \omega \sqrt{x_0^2 - x^2}$$

$$\int_{x_0}^{x} \frac{1}{\sqrt{x_0^2 - x'^2}} \, dx' = \int_0^t \omega \, dt$$

$$\arctan \frac{x}{\sqrt{x_0^2 - x^2}} - \arctan \frac{x_0}{\sqrt{x_0^2 - x_0^2}} = \omega t$$

$$\arctan \frac{x}{\sqrt{x_0^2 - x^2}} = \omega t + \frac{\pi}{2}$$

$$\frac{x}{\sqrt{x_0^2 - x^2}} = \tan \left(\omega t + \frac{\pi}{2} \right)$$

$$= -\cot \omega t$$

$$\frac{\sqrt{x_0^2 - x^2}}{x} = -\tan \omega t$$

$$\sqrt{x_0^2 - x^2} = -x \tan \omega t$$

$$x_0^2 - x^2 = x^2 \tan^2 \omega t$$

$$x^2 = \frac{x_0^2}{1 + \tan^2 \omega t}$$

$$= \frac{x_0^2 \cos^2 \omega t}{\cos^2 \omega t + \sin^2 \omega t}$$

$$= x_0^2 \cos^2 \omega t$$

$$x = x_0 \cos \omega t$$

2.15

$$a_y = -g$$

$$v_y = v_{y0} - gt$$

$$y = v_{y0}t - \frac{1}{2}gt^2$$

$$0 = v_{y0}t - \frac{1}{2}gt^2$$

$$t = \frac{2v_{y0}}{g}$$

$$x = v_{x0}t$$

$$R = \frac{2v_{x0}v_{y0}}{g}$$

2.19

(a)

$$x = v_{x0}t$$

$$y = v_{y0}t - \frac{1}{2}gt^2$$

$$= \frac{v_{y0}}{v_{x0}}x - \frac{1}{2}g\left(\frac{x}{v_{x0}}\right)^2$$

(b)

$$y = \frac{v_{y0} + v_{\text{ter}}}{v_{x0}}x + v_{\text{ter}}\tau \ln\left(1 - \frac{x}{v_{x0}\tau}\right)$$

$$\approx \frac{v_{y0}}{v_{x0}}x + \frac{g\tau}{v_{x0}}x - g\tau^2\left[\frac{x}{v_{x0}\tau} + \frac{1}{2}\left(\frac{x}{v_{x0}\tau}\right)^2\right]$$

$$= \frac{v_{y0}}{v_{x0}}x - \frac{1}{2}g\left(\frac{x}{v_{x0}}\right)^2$$

2.23

(a)

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}}$$

$$= \sqrt{\frac{mg}{\gamma D^2}}$$

$$= \sqrt{\frac{mg}{0.25 D^2}}$$

$$= \sqrt{\frac{\rho \frac{4}{3}\pi \left(\frac{D}{2}\right)^3 g}{0.25 D^2}}$$

$$= \sqrt{\frac{4\pi \rho D g}{6}}$$

$$= 22\,\text{m/s}$$

(b)

$$m = \rho V$$

$$= \rho \frac{4}{3}\pi \left(\frac{D}{2}\right)^3$$

$$= \frac{\pi \rho D^3}{6}$$

$$D^2 = \left(\frac{6m}{\pi \rho}\right)^{2/3}$$

$$v_{\text{ter}} = \sqrt{\frac{mg}{0.25 D^2}}$$

$$= \sqrt{\frac{mg}{0.25 (6m/\pi \rho)^{2/3}}}$$

$$= 140\,\text{m/s}$$

(c)

$$v_{\text{ter}} = 107\,\text{m/s}$$

2.27

$$m\dot{v} = -mg\sin\theta - cv^2$$

$$-\frac{\sqrt{m}\arctan\frac{\sqrt{c}v}{\sqrt{gm\sin\theta}}}{\sqrt{cg\sin\theta}} = t + c_1$$

$$\arctan\frac{\sqrt{c}v}{\sqrt{gm\sin\theta}} = \sqrt{\frac{cg\sin\theta}{m}}(c_1 - t)$$

$$\frac{\sqrt{c}v}{\sqrt{gm\sin\theta}} = \tan\left[\sqrt{\frac{cg\sin\theta}{m}}(c_1 - t)\right]$$

$$v = \sqrt{\frac{gm\sin\theta}{c}}\tan\left[\sqrt{\frac{cg\sin\theta}{m}}(c_1 - t)\right]$$

$$v_0 = \sqrt{\frac{gm\sin\theta}{c}}\tan\left(\sqrt{\frac{cg\sin\theta}{m}}c_1\right)$$

$$c_1 = \sqrt{\frac{m}{cg\sin\theta}}\arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right)$$

$$v = \sqrt{\frac{gm\sin\theta}{c}}\tan\left[\arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right) - \sqrt{\frac{cg\sin\theta}{m}}t\right]$$

$$0 = \sqrt{\frac{gm\sin\theta}{c}}\tan\left[\arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right) - \sqrt{\frac{cg\sin\theta}{m}}t\right]$$

$$\sqrt{\frac{cg\sin\theta}{m}}t = \arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right)$$

$$t = \sqrt{\frac{m}{cg\sin\theta}}\arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right)$$

2.29

$$v(t) = v_{\text{ter}}\tanh\frac{gt}{v_{\text{ter}}}$$

$$v(1) = 9.6\,\text{m/s}$$

$$v(5) = 38\,\text{m/s}$$

$$v(10) = 48\,\text{m/s}$$

$$v(20) = 50\,\text{m/s}$$

$$v(30) = 50\,\text{m/s}$$

2.31

(a)

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}}$$

$$= \sqrt{\frac{mg}{0.25D^2}}$$

$$= 20.2 \, \text{m/s}$$

(b)

$$y = -30 + \frac{v_{\text{ter}}^2}{g} \ln \left(\cosh \frac{gt}{v_{\text{ter}}} \right)$$

$$0 = -30 + \frac{v_{\text{ter}}^2}{g} \ln \left(\cosh \frac{gt}{v_{\text{ter}}} \right)$$

$$t = 2.78 \, \text{s}$$

$$v(2.78) = 17.6 \, \text{m/s}$$

2.33

(b)

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

$$= \frac{1}{2} \left[\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots \right) + \left(1 - z + \frac{z^2}{2} - \frac{z^3}{6} + \cdots \right) \right]$$

$$= 1 + \frac{z^2}{2} + \frac{z^4}{24} + \frac{z^6}{720} + \cdots$$

$$\cos iz = 1 - \frac{(iz)^2}{2} + \frac{(iz)^4}{24} - \frac{(iz)^6}{720} + \cdots$$

$$= 1 + \frac{z^2}{2} + \frac{z^4}{24} + \frac{z^6}{720} + \cdots$$

$$= \cosh z$$

$$\sinh z = -i \sin iz$$

(c)

$$\frac{d}{dz}\cosh z = \frac{d}{dz}\left(\frac{e^z + e^{-z}}{2}\right)$$
$$= \frac{e^z - e^{-z}}{2}$$
$$= \sinh z$$
$$\frac{d}{dz}\sinh z = \frac{d}{dz}\left(\frac{e^z - e^{-z}}{2}\right)$$
$$= \frac{e^z + e^{-z}}{2}$$
$$= \cosh z$$

(d)

$$\cosh^2 z - \sinh^2 z = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2$$
$$= \frac{1}{4}(e^{2z} + 2 + e^{-2z} - e^{2z} + 2 - e^{-2z})$$
$$= 1$$

(e)

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \int \frac{\cosh z}{\sqrt{1+\sinh^2 z}}\,dz$$
$$= \int 1\,dz$$
$$= z$$
$$= \operatorname{arcsinh} x$$

17

2.35

(a)

$$m\dot{v} = mg - cv^2$$

$$\dot{v} = g\left(1 - \frac{v^2}{v_{\text{ter}}^2}\right)$$

$$\int_0^v \frac{1}{1 - v'^2/v_{\text{ter}}^2}\, dv' = \int_0^t g\, dt$$

$$v_{\text{ter}} \operatorname{arctanh} \frac{v}{v_{\text{ter}}} = gt$$

$$v = v_{\text{ter}} \tanh \frac{gt}{v_{\text{ter}}}$$

$$y = \int_0^t v_{\text{ter}} \tanh \frac{gt'}{v_{\text{ter}}} dt'$$

$$= \frac{v_{\text{ter}}^2}{g} \ln\left[\cosh\left(\frac{gt}{v_{\text{ter}}}\right)\right]$$

(b)

$$v = g\tau \tanh \frac{t}{\tau}$$

$$y = g\tau^2 \ln\left[\cosh\left(\frac{t}{\tau}\right)\right]$$

$$v(\tau) = g\tau \tanh 1$$

$$= 0.76 v_{\text{ter}}$$

$$v(2\tau) = 0.96 v_{\text{ter}}$$

$$v(3\tau) = 0.99 v_{\text{ter}}$$

(c)

$$y = g\tau^2 \ln\left[\cosh\left(\frac{t}{\tau}\right)\right]$$

$$= g\tau^2 \ln\left(\frac{e^{t/\tau} + e^{-t/\tau}}{2}\right)$$

$$= g\tau^2 \ln\left(\frac{e^{t/\tau}}{2}\right)$$

$$= g\tau^2 (\ln e^{t/\tau} - \ln 2)$$

$$= g\tau t - g\tau^2 \ln 2$$

$$= v_{\text{ter}} t - g\tau^2 \ln 2$$

(d)

$$y = \frac{(v_{\text{ter}})^2}{g} \ln \left[\cosh \left(\frac{gt}{v_{\text{ter}}} \right) \right]$$

$$\approx \frac{(v_{\text{ter}})^2}{g} \ln \left[1 + \frac{1}{2} \left(\frac{gt}{v_{\text{ter}}} \right)^2 \right]$$

$$\approx \frac{(v_{\text{ter}})^2}{g} \frac{1}{2} \left(\frac{gt}{v_{\text{ter}}} \right)^2$$

$$= \frac{1}{2} g t^2$$

2.39

(a)

$$m\dot{v} = -cv^2 - 3$$

$$\int_{v_0}^{v} \frac{m}{-cv'^2 - 3} \, dv' = \int_0^t dt'$$

$$\frac{m}{\sqrt{3c}} \left[\arctan \left(\sqrt{\frac{c}{3}} v_0 \right) - \arctan \left(\sqrt{\frac{c}{3}} v \right) \right] = t$$

	Speed	Time
	15 m/s	6.34 s
(b)	10 m/s	18.4 s
	5 m/s	48.3 s
	0 m/s	142 s

2.41

$$m\dot{v} = -mg - cv^2$$

$$\dot{v} = -g\left[1 + \left(\frac{v}{v_{\text{ter}}}\right)^2\right]$$

$$v\frac{dv}{dy} = -g\left[1 + \left(\frac{v}{v_{\text{ter}}}\right)^2\right]$$

$$\int_{v_0}^{v} \frac{v'}{1 + (v'/v_{\text{ter}})^2}\, dv' = \int_0^y -g\, dy'$$

$$\frac{1}{2}v_{\text{ter}}^2[\ln(v_{\text{ter}}^2 + v^2) - \ln(v_{\text{ter}}^2 + v_0^2)] = -gy$$

$$\ln\frac{v_{\text{ter}}^2 + v^2}{v_{\text{ter}}^2 + v_0^2} = -\frac{2gy}{v_{\text{ter}}^2}$$

$$\frac{v_{\text{ter}}^2 + v^2}{v_{\text{ter}}^2 + v_0^2} = e^{-2gy/v_{\text{ter}}^2}$$

$$v = \sqrt{(v_{\text{ter}}^2 + v_0^2)e^{-2gy/v_{\text{ter}}^2} - v_{\text{ter}}^2}$$

$$0 = \sqrt{(v_{\text{ter}}^2 + v_0^2)e^{-2gy/v_{\text{ter}}^2} - v_{\text{ter}}^2}$$

$$v_{\text{ter}}^2 = (v_{\text{ter}}^2 + v_0^2)e^{-2gy/v_{\text{ter}}^2}$$

$$\frac{v_{\text{ter}}^2}{v_{\text{ter}}^2 + v_0^2} = e^{-2gy/v_{\text{ter}}^2}$$

$$-\frac{2gy}{v_{\text{ter}}^2} = \ln\frac{v_{\text{ter}}^2}{v_{\text{ter}}^2 + v_0^2}$$

$$y = -\frac{v_{\text{ter}}^2}{2g}\ln\frac{v_{\text{ter}}^2}{v_{\text{ter}}^2 + v_0^2}$$

$$= \frac{v_{\text{ter}}^2}{2g}\ln\frac{v_{\text{ter}}^2 + v_0^2}{v_{\text{ter}}^2}$$

$$y_{\max} = 17.6\,\text{m}$$

2.45

(a)

$$z = re^{i\theta}$$
$$= r(\cos\theta + i\sin\theta)$$
$$= \sqrt{x^2 + y^2}\left[\cos\left(\arctan\frac{y}{x}\right) + i\sin\left(\arctan\frac{y}{x}\right)\right]$$
$$= \sqrt{x^2 + y^2}\left(\frac{1}{\sqrt{1 + \frac{y^2}{x^2}}} + i\frac{y}{x\sqrt{1 + \frac{y^2}{x^2}}}\right)$$
$$= \sqrt{x^2 + y^2}\left(\frac{x}{\sqrt{x^2 + y^2}} + i\frac{y}{\sqrt{x^2 + y^2}}\right)$$
$$= x + iy$$

r is the distance between z and the origin, θ is the angle between the positive real axis and z.

(b)

$$z = \sqrt{3^2 + 4^2}e^{i\arctan 4/3} = 5e^{0.927i}$$

(c)

$$z = 2\cos-\frac{\pi}{3} + i2\sin-\frac{\pi}{3} = 1 - \sqrt{3}i$$

2.47

(a)

$$z + w = 9 + 4i$$
$$z - w = 3 + 12i$$
$$zw = (6 + 8i)(3 - 4i)$$
$$= 18 - 24i + 24i + 32$$
$$= 50$$
$$\frac{z}{w} = \frac{zw*}{ww*}$$
$$= \frac{(6 + 8i)(3 + 4i)}{(3 - 4i)(3 + 4i)}$$
$$= \frac{18 + 24i + 24i - 32}{9 + 12i - 12i + 16}$$
$$= \frac{-14 + 48i}{25}$$
$$= -\frac{14}{25} + \frac{48}{25}i$$

21

(b)

$$z + w = \left(8\cos\frac{\pi}{3} + i8\sin\frac{\pi}{3}\right) + \left(4\cos\frac{\pi}{6} + i4\sin\frac{\pi}{6}\right)$$
$$= (4 + 2\sqrt{3}) + i(4\sqrt{3} + 2)$$
$$z - w = (4 - 2\sqrt{3}) + i(4\sqrt{3} - 2)$$
$$zw = 32e^{i\pi/2}$$
$$= 32i$$
$$\frac{z}{w} = 2e^{i\pi/6}$$
$$= \sqrt{3} + i$$

2.49

(a)

$$z^2 = (e^{i\theta})^2$$
$$= e^{i2\theta}$$
$$= \cos 2\theta + i\sin 2\theta$$
$$z^2 = (\cos\theta + i\sin\theta)^2$$
$$= \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta$$
$$= \cos^2\theta + i\sin 2\theta - \sin^2\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta + i\sin 2\theta = \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta$$
$$i\sin 2\theta = 2i\sin\theta\cos\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

(b)

$$z^3 = (e^{i\theta})^3$$
$$= e^{i3\theta}$$
$$= \cos 3\theta + i \sin 3\theta$$
$$z^3 = (\cos\theta + i\sin\theta)^3$$
$$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$$\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$
$$= \cos\theta(\cos^2\theta - 3\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$
$$= \cos\theta(\cos^2\theta - 3\sin^2\theta) + i[3(1 - \sin^2\theta)\sin\theta - \sin^3\theta]$$
$$= \cos\theta(\cos^2\theta - 3\sin^2\theta) + i(3\sin\theta - 4\sin^3\theta)$$
$$\cos 3\theta = \cos\theta(\cos^2\theta - 3\sin^2\theta)$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

2.53

$$\mathbf{B} = B_z\hat{\mathbf{z}}$$
$$\mathbf{E} = E_z\hat{\mathbf{z}}$$
$$\mathbf{v}\times\mathbf{B} = B_z v_y\hat{\mathbf{x}} - B_z v_x\hat{\mathbf{y}}$$
$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$
$$= qB_z v_y\hat{\mathbf{x}} - qB_z v_x\hat{\mathbf{y}} + qE_z\hat{\mathbf{z}}$$

$$m\dot{v}_x = qB_z v_y$$
$$\dot{v}_x = \omega v_y$$

$$m\dot{v}_y = -qB_z v_x$$
$$\dot{v}_y = -\omega v_x$$

$$m\dot{v}_z = qE_z$$
$$\dot{v}_z = \frac{q}{m}E_z$$

$$\frac{d}{dt}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix}$$
$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = c_1\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos\omega t - \begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin\omega t\right] + c_2\left[\begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos\omega t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin\omega t\right]$$
$$= \begin{pmatrix} c_1\sin\omega t - c_2\cos\omega t \\ c_1\cos\omega t + c_2\sin\omega t \end{pmatrix}$$

$$x = -\frac{c_1}{\omega}\cos\omega t - \frac{c_2}{\omega}\sin\omega t + x_0$$
$$y = \frac{c_1}{\omega}\sin\omega t - \frac{c_2}{\omega}\cos\omega t + y_0$$

$$v_z = \frac{q}{m}E_z t + v_{z0}$$
$$z = \frac{q}{2m}E_z t^2 + v_{z0}t + z_0$$

The particle moves in a helix oriented along the z-axis.

2.55

(a)

$$\mathbf{B} = B\hat{\mathbf{z}}$$
$$\mathbf{E} = E\hat{\mathbf{y}}$$
$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$
$$= Bqv_y\hat{\mathbf{x}} + q(E - Bv_x)\hat{\mathbf{y}}$$
$$\dot{v}_x = \frac{Bq}{m}v_y$$
$$= \omega v_y$$
$$\dot{v}_y = \frac{Eq}{m} - \frac{Bq}{m}v_x$$
$$= \frac{Eq}{m} - \omega v_x$$
$$\dot{v}_z = 0$$

The net force has no $\hat{\mathbf{z}}$ component, so the motion stays in the xy-plane.

(b)

$$0 = \frac{Eq}{m} - \omega v_x$$
$$v_x = \frac{Eq}{\omega m}$$
$$= \frac{E}{B}$$

(c)

$$\begin{pmatrix} \dot{v}_x \\ \dot{v}_y \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{Eq}{m} \end{pmatrix}$$

$$\mathbf{V}_c = \begin{pmatrix} c_1 \sin \omega t - c_2 \cos \omega t \\ c_1 \cos \omega t + c_2 \sin \omega t \end{pmatrix}$$

$$\mathbf{V}_p = \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{Eq}{m} \end{pmatrix}$$

$$= \begin{pmatrix} c_4 \omega \\ -c_3 \omega + \frac{Eq}{m} \end{pmatrix}$$

$$c_3 = \frac{Eq}{m\omega}$$

$$= \frac{E}{B}$$

$$= v_{\mathrm{dr}}$$

$$c_4 = 0$$

$$\mathbf{V}_p = \begin{pmatrix} v_{\mathrm{dr}} \\ 0 \end{pmatrix}$$

$$\mathbf{V} = \begin{pmatrix} c_1 \sin \omega t - c_2 \cos \omega t + v_{\mathrm{dr}} \\ c_1 \cos \omega t + c_2 \sin \omega t \end{pmatrix}$$

$$\begin{pmatrix} v_{x0} \\ 0 \end{pmatrix} = \begin{pmatrix} -c_2 + v_{\mathrm{dr}} \\ c_1 \end{pmatrix}$$

$$c_1 = 0$$

$$c_2 = v_{\mathrm{dr}} - v_{x0}$$

$$\mathbf{V} = \begin{pmatrix} v_{\mathrm{dr}} + (v_{x0} - v_{\mathrm{dr}}) \cos \omega t \\ -(v_{x0} - v_{\mathrm{dr}}) \sin \omega t \end{pmatrix}$$

(d)

$$x = v_{\mathrm{dr}} t + \frac{v_{x0} - v_{\mathrm{dr}}}{\omega} \sin \omega t + x_0$$

$$y = \frac{v_{x0} - v_{\mathrm{dr}}}{\omega} \cos \omega t + y_0$$

$$z = z_0$$

3 Momentum and Angular Momentum

3.3

$$mv_0 = \frac{m}{3}(v_1 + v_2\cos\theta + v_3\cos\theta)$$
$$3v_0 = v_0 + \sqrt{2}v_2$$
$$2v_0 = \sqrt{2}v_2$$
$$v_2 = \sqrt{2}v_0$$
$$\mathbf{v}_2 = \sqrt{2}v_0\left(\cos\frac{\pi}{4}\hat{\mathbf{x}} + \sin\frac{\pi}{4}\hat{\mathbf{y}}\right)$$
$$= v_0(\hat{\mathbf{x}} + \hat{\mathbf{y}})$$
$$\mathbf{v}_3 = v_0(\hat{\mathbf{x}} - \hat{\mathbf{y}})$$

3.7

$$v = v_{\text{ex}}\ln\frac{m_0}{m}$$
$$= 2079\,\text{m/s}$$
$$F = 25\,\text{MN}$$
$$W = 19.6\,\text{MN}$$

The thrust is 1.28 times the weight on Earth.

3.9

$$-\dot{m}v_{\text{ex}} = m_0 g$$
$$v_{\text{ex}} = -\frac{m_0 g}{\dot{m}}$$
$$= 2352\,\text{m/s}$$

3.11

(a)
$$m\dot{v} = -\dot{m}v_{\text{ex}} + F_{\text{ext}}$$

(b)

$$m\dot{v} = -\dot{m}v_{\text{ex}} - mg$$

$$\dot{v} = -\frac{v_{\text{ex}}}{m}\dot{m} - g$$

$$\int_0^t \dot{v}\,dt = \int_0^t \left(-\frac{v_{\text{ex}}}{m}\dot{m} - g\right) dt$$

$$\int_0^v dv' = -v_{\text{ex}}\int_{m_0}^m \frac{1}{m'}\,dm' - \int_0^t g\,dt'$$

$$v = -v_{\text{ex}}\ln\frac{m}{m_0} - gt$$

$$= v_{\text{ex}}\ln\frac{m_0}{m} - gt$$

(c)
$$v = 903\,\text{m/s}$$

It would be $2079\,\text{m/s}$ without gravity (2.3 times larger).

(d) The rocket wouldn't take off until it was light enough (from burning fuel) that its thrust was greater than its weight.

3.13

$$v = v_{\text{ex}}\ln\frac{m_0}{m} - gt$$

$$= v_{\text{ex}}\ln\frac{m_0}{m_0 - kt} - gt$$

$$y = v_{\text{ex}}t - \frac{mv_{\text{ex}}}{k}\ln\frac{m_0}{m} - \frac{1}{2}gt^2$$

$$y(2\,\text{min}) = 40\,\text{km}$$

3.15

$$M = m_1 + m_2 + m_3$$
$$= m_1 + m_1 + 10m_1$$
$$= 12m_1$$

$$\mathbf{R} = \frac{1}{12m_1} \left[m_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + m_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + m_3 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right]$$

$$= \frac{1}{12m_1} \begin{pmatrix} 2m_1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{6} \\ 0 \\ 0 \end{pmatrix}$$

3.17

If we let Earth be at the origin, then

$$R = \frac{dM_m}{M_e + M_m}$$
$$= 4630 \, \text{km}$$

The centre of mass is inside Earth.

3.19

(a) No external forces apply during the explosion so the path of the centre of mass would be unchanged.

(b) 100 m before the target.

(c) No.

3.21

$$\mathbf{R} = \frac{1}{M} \int \mathbf{r} \, dm$$

$$= \frac{2}{\sigma \pi R^2} \int \mathbf{r} \sigma \, dA$$

$$= \frac{2}{\pi R^2} \int_0^R \int_0^\pi \mathbf{r} r \, d\phi \, dr$$

$$= \frac{2}{\pi R^2} \int_0^R \int_0^\pi r(\cos\phi \hat{\mathbf{x}} + \sin\phi \hat{\mathbf{y}}) r \, d\phi \, dr$$

$$= \frac{2}{\pi R^2} \int_0^R r^2 \int_0^\pi (\cos\phi \hat{\mathbf{x}} + \sin\phi \hat{\mathbf{y}}) \, d\phi \, dr$$

$$= \frac{2}{\pi R^2} \int_0^R r^2 [\sin\phi \hat{\mathbf{x}} - \cos\phi \hat{\mathbf{y}}]_0^\pi \, dr$$

$$= \frac{4}{\pi R^2} \int_0^R r^2 \, dr \, \hat{\mathbf{y}}$$

$$= \frac{4R}{3\pi} \hat{\mathbf{y}}$$

3.25

$$L = L_0$$

$$I\omega = I_0\omega_0$$

$$mr^2\omega = mr_0^2\omega_0$$

$$\omega = \left(\frac{r_0}{r}\right)^2 \omega_0$$