Vibrations and Waves by A. P. French Problems

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Contents

1 Periodic motions

1.4

(a)

$$\begin{aligned} z &= Ae^{j\theta} \\ dz &= jAe^{j\theta}\,d\theta \\ &= jz\,d\theta \end{aligned}$$

The motion of the point is always perpendicular to its position.

(b)

$$\begin{aligned}
|2 + j\sqrt{3}| &= \sqrt{2^2 + \sqrt{3}^2} \\
&= \sqrt{7} \\
\arg(2 + j\sqrt{3}) &= \arctan \frac{\sqrt{3}}{2} \\
&= 41^\circ
\end{aligned}$$

$$\begin{aligned}
(2 - j\sqrt{3})^2 &= 4 - j4\sqrt{3} - 3 \\
&= 1 - j4\sqrt{3} \\
|1 - j4\sqrt{3}| &= \sqrt{1^2 + (4\sqrt{3})^2} \\
&= 7 \\
\arg(1 - j4\sqrt{3}) &= -\arctan 4\sqrt{3}
\end{aligned}$$

1.9

$$\begin{aligned}
\cos\theta + j\sin\theta &= e^{j\theta} \\
\cos\frac{\pi}{2} + j\sin\frac{\pi}{2} &= e^{j\frac{\pi}{2}} \\
j &= e^{j\frac{\pi}{2}} \\
j^j &= (e^{j\frac{\pi}{2}})^j \\
&= e^{-\frac{\pi}{2}} \\
&\approx 0.208
\end{aligned}$$

Yes, I would be willing to pay 20 cents because I could sell it to the mathematician and gain 0.8 cents.

1.10

$$
\begin{aligned}
y &= A\cos kx + B\sin kx \\
\frac{dy}{dx} &= -Ak\sin kx + Bk\cos kx \\
\frac{d^2y}{dx^2} &= -Ak^2\cos kx - Bk^2\sin kx \\
&= -k^2 y
\end{aligned}
$$

$$
\begin{aligned}
C &= \sqrt{A^2 + B^2} \\
\alpha &= \arctan\left(-\frac{B}{A}\right) \\
y &= C\cos(kx + \alpha) \\
&= C\,\mathrm{Re}[e^{j(kx+\alpha)}] \\
&= Re[(Ce^{j\alpha})e^{jkx}]
\end{aligned}
$$

1.11

(a)

$$
\begin{aligned}
x &= A\cos(\omega t + \alpha) \\
A &= 5\,\mathrm{cm} \\
f &= 1\,\mathrm{Hz} \\
\omega &= 2\pi f \\
&= 2\pi\,\mathrm{rad/s} \\
\alpha &= \pm\frac{\pi}{2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
x\left(\frac{8}{3}\right) &= 5\cos\left(2\pi\frac{8}{3} + \alpha\right) \\
&= \pm 4.33\,\mathrm{cm} \\
\frac{dx}{dt} &= -A\omega\sin(\omega t + \alpha) \\
\frac{dx}{dt}\left(\frac{8}{3}\right) &= \pm 15.7\,\mathrm{cm/s} \\
\frac{d^2x}{dt^2} &= -A\omega^2\cos(\omega t + \alpha) \\
\frac{d^2x}{dt^2}\left(\frac{8}{3}\right) &= \mp 171\,\mathrm{cm/s^2}
\end{aligned}
$$

1.12

(a)

$$\begin{aligned}
v &= 50\,\mathrm{cm/s} \\
T &= 6\,\mathrm{s} \\
\theta_0 &= 30^\circ \\
c &= 300\,\mathrm{cm} \\
A &= \frac{c}{2\pi} \\
&= \frac{150}{\pi}\,\mathrm{cm} \\
\omega &= \frac{2\pi}{T} \\
&= \frac{\pi}{3}\,\mathrm{rad/s} \\
\alpha &= \frac{\pi}{6}\,\mathrm{rad} \\
x &= \frac{150}{\pi}\cos\left(\frac{\pi}{3}t + \frac{\pi}{6}\right)
\end{aligned}$$

(b)

$$\begin{aligned}
x(2\,\mathrm{s}) &= -41.3\,\mathrm{cm} \\
\frac{dx}{dt} &= -50\sin\left(\frac{\pi}{3}t + \frac{\pi}{6}\right) \\
\frac{dx}{dt}(2\,\mathrm{s}) &= -25\,\mathrm{cm/s} \\
\frac{d^2x}{dt^2} &= -\frac{50\pi}{3}\cos\left(\frac{\pi}{3}t + \frac{\pi}{6}\right) \\
\frac{d^2x}{dt^2}(2\,\mathrm{s}) &= 45\,\mathrm{cm/s^2}
\end{aligned}$$

2 The superposition of periodic motions

2.1

(a)

$$\begin{aligned}
z &= \sin\omega t + \cos\omega t \\
&= \sqrt{2}\cos\left(\omega t - \frac{\pi}{4}\right) \\
&= \sqrt{2}e^{j\left(\omega t - \frac{\pi}{4}\right)}
\end{aligned}$$

(b)

$$\begin{aligned}
z &= \cos(\omega t - \pi/3) - \cos \omega t \\
&= \cos \omega t \cos \frac{\pi}{3} + \sin \omega t \sin \frac{\pi}{3} - \cos \omega t \\
&= -\frac{1}{2} \cos \omega t + \frac{\sqrt{3}}{2} \sin \omega t \\
&= \cos(\omega t + 2\pi/3) \\
&= e^{j(\omega t + 2\pi/3)}
\end{aligned}$$

(c)

$$\begin{aligned}
z &= 3 \cos \omega t + 2 \sin \omega t \\
&= \sqrt{13} \cos(\omega t + \arctan -2/3)
\end{aligned}$$

(d)

$$\begin{aligned}
z &= \sin \omega t - 2\cos(\omega t - \pi/4) + \cos \omega t \\
&= \sin \omega t - 2(\cos \omega t \cos \pi/4 + \sin \omega t \sin \pi/4) + \cos \omega t \\
&= \sin \omega t - \sqrt{2} \cos \omega t - \sqrt{2} \sin \omega t + \cos \omega t \\
&= (1 - \sqrt{2}) \cos \omega t + (1 - \sqrt{2}) \sin \omega t \\
&= (1 - \sqrt{2})\sqrt{2} \cos(\omega t - \pi/4) \\
&= (\sqrt{2} - 2) \cos(\omega t - \pi/4) \\
&= (2 - \sqrt{2}) \cos(\omega t + 3\pi/4)
\end{aligned}$$

2.2

$$\begin{aligned}
x &= A_1 \cos \omega t + A_2 \cos(\omega t + \alpha_1) + A_3 \cos(\omega t + \alpha_1 + \alpha_2) \\
&= A_1 \cos \omega t + A_2(\cos \omega t \cos \alpha_1 - \sin \omega t \sin \alpha_1) \\
&\qquad + A_3(\cos \omega t \cos(\alpha_1 + \alpha_2) - \sin \omega t \sin(\alpha_1 + \alpha_2)) \\
&= (A_1 + A_2 \cos \alpha_1 + A_3 \cos(\alpha_1 + \alpha_2)) \cos \omega t \\
&\qquad - (A_2 \sin \alpha_1 + A_3 \sin(\alpha_1 + \alpha_2)) \sin \omega t \\
A &= \sqrt{(A_1 + A_2 \cos \alpha_1 + A_3 \cos(\alpha_1 + \alpha_2))^2 + (A_2 \sin \alpha_1 + A_3 \sin(\alpha_1 + \alpha_2))^2} \\
&\approx 0.52\,\mathrm{mm} \\
\alpha &= \arctan \frac{A_2 \sin \alpha_1 + A_3 \sin(\alpha_1 + \alpha_2)}{A_1 + A_2 \cos \alpha_1 + A_3 \cos(\alpha_1 + \alpha_2)} \\
&\approx 0.59\,\mathrm{rad} \\
&\approx 34^\circ
\end{aligned}$$

2.3

The equation of motion is

$$x = 2A\cos\left(\frac{12\pi - 10\pi}{2}t\right)\cos\left(\frac{12\pi + 10\pi}{2}t\right)$$

with the variation in amplitude given by the term

$$2A\cos\pi t$$

so the beat period is 1 s.

2.4

(a)

$$\omega = 2\pi, \text{rad/s} \Rightarrow f = 1\,\text{Hz}$$

(b)

$$\omega = \frac{25\pi}{2}\,\text{rad/s} \Rightarrow f = \frac{25}{4}\,\text{Hz}$$

(c)

$$\omega = \frac{3+\pi}{2}\,\text{rad/s} \Rightarrow f = \frac{3+\pi}{4\pi}\,\text{Hz}$$