Vibrations and Waves by George C. King Problems

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Contents

1 Simple Harmonic Motion

1.1

(a) (i) $T = 4\,\mathrm{s}$

(ii) $\omega = \frac{\pi}{2}$ rad/s

(iii) $\omega^2 = \frac{k}{m} \Rightarrow k = m\omega^2 = \frac{\pi^2}{8}$ N/m

1.2

(a)

$$\begin{aligned} x &= A\cos\omega t \\ &= A\cos 2\pi f t \\ v &= -2\pi f A \sin 2\pi f t \\ v_{\max} &= 2\pi f A \\ &= 1.38\,\mathrm{m/s} \end{aligned}$$

(b)

$$\begin{aligned} a &= -4\pi^2 f^2 A \cos 2\pi f t \\ a_{\max} &= 4\pi^2 f^2 A \\ &= 3.82 \times 10^3\,\mathrm{m/s^2} \end{aligned}$$

1.3

$$\begin{aligned} a_{\max} &\leq g \\ 4\pi^2 f^2 A &\leq g \\ f &\leq \sqrt{\frac{g}{4\pi^2 A}} \\ &\leq 1.11\,\mathrm{Hz} \end{aligned}$$

1.4

(a)

$$\frac{U}{E} = \frac{\frac{1}{2}k\left(\frac{1}{2}A\right)^2}{\frac{1}{2}kA^2} = \frac{1}{4} \Rightarrow \frac{K}{E} = \frac{3}{4}$$

(b) (i) The total energy will increase by a factor of 4

(ii) The maximum velocity will increase by a factor of 2

(iii) The maximum acceleration will increase by a factor of 2 and the period won't change

1.5

(a) $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = 0.41\,\mathrm{J}$

(b)

$$\begin{aligned}
E &= \frac{1}{2}kA^2 \\
A &= \sqrt{\frac{2E}{k}} \\
&= 4.5\,\mathrm{cm} \\
\omega &= \sqrt{\frac{k}{m}} \\
&= \sqrt{\frac{1600}{3}} \\
&= \frac{40}{\sqrt{3}} \\
&= 23\,\mathrm{rad/s} \\
x &= A\cos(\omega t + \phi) \\
\phi &= \arccos\left(\frac{x}{A}\right) - \omega t \\
&= 2.7\,\mathrm{rad} \\
x &= 0.045\cos(23t + 2.7)\,\mathrm{m}
\end{aligned}$$

1.6

Using the angular frequency of system (b) ω_b as the baseline, the angular frequency of system (a) ω_a is

$$\begin{aligned}
F = ma &= -2kx \\
a &= -\frac{2k}{m}x \\
\omega_a &= \sqrt{\frac{2k}{m}} \\
&= \sqrt{2}\omega_b
\end{aligned}$$

and the angular frequency of system (c) ω_c is

$$\begin{aligned}
F = ma &= -\frac{k}{2}x \\
a &= -\frac{k}{2m}x \\
\omega_c &= \sqrt{\frac{k}{2m}} \\
&= \sqrt{\frac{1}{2}}\omega_b
\end{aligned}$$

1.7

(a) The test tube experiences a bouyancy force of $F = Ag\rho x$ so its equation of motion is

$$\begin{aligned}
F = ma &= -Ag\rho x \\
a &= -\frac{Ag\rho}{m}x \\
\omega &= \sqrt{\frac{Ag\rho}{m}}
\end{aligned}$$

(b) The work done by the bouyancy force when moving from equilibrium to x and thus the potential energy is

$$\begin{aligned}
U &= \int_0^x Ag\rho x' \, dx' \\
&= \frac{1}{2}Ag\rho x^2
\end{aligned}$$

so the total energy of the system is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}Ag\rho x^2$$

1.8

$$\mathrm{s} \propto \mathrm{kg}^{\alpha}\,\mathrm{m}^{\beta}\,\left(\mathrm{m/s^2}\right)^{\gamma}$$

so $\alpha = 0$, $\beta = 1/2$, and $\gamma = -1/2$ meaning

$$T \propto \sqrt{\frac{l}{g}}$$

1.9

(a)

$$
\begin{aligned}
x &= A\cos\sqrt{\frac{g}{l}}t \\
v &= -\sqrt{\frac{g}{l}}A\sin\sqrt{\frac{g}{l}}t \\
v_{\max} &= \sqrt{\frac{g}{l}}A \\
&= 0.018\,\mathrm{m/s}
\end{aligned}
$$

(b) The pendulum reaches its maximum speed at the bottom of its swing which occurs after a quarter cycle

$$
\frac{1}{4}T = \frac{1}{4}\frac{2\pi}{\omega} = \frac{\pi}{2\sqrt{g/l}} = 0.43\,\mathrm{s}
$$

1.10

$$
\begin{aligned}
I\frac{d^2\theta}{dt^2} &= \tau \\
\frac{1}{3}ML^2\frac{d^2\theta}{dt^2} &= -kL\sin\theta L\cos\theta \\
\frac{1}{3}M\frac{d^2\theta}{dt^2} &= -k\theta \\
\frac{d^2\theta}{dt^2} &= -\frac{3k}{M}\theta \\
T &= \frac{2\pi}{\omega} \\
&= 2\pi\sqrt{\frac{M}{3k}}
\end{aligned}
$$

1.11

(a)

$$
\begin{aligned}
F = -\frac{dU}{dx} &= -\left(\frac{6a}{x^7} - \frac{12b}{x^{13}}\right) \\
0 &= \frac{12b}{x^{13}} - \frac{6a}{x^7} \\
&= \frac{12b}{x^6} - 6a \\
6a &= \frac{12b}{x^6} \\
x^6 &= \frac{2b}{a} \\
x &= \left(\frac{2b}{a}\right)^{1/6}
\end{aligned}
$$

1.12

(a)

$$
\begin{aligned}
K &= \frac{1}{2}Mv^2 + \int dK \\
&= \frac{1}{2}Mv^2 + \int_0^L \frac{1}{2}\frac{m}{L}\left(\frac{l}{L}v\right)^2 dl \\
&= \frac{1}{2}Mv^2 + \frac{1}{2}\frac{mv^2}{L^3}\int_0^L l^2\, dl \\
&= \frac{1}{2}Mv^2 + \frac{1}{2}\frac{mv^2}{L^3}\frac{1}{3}L^3 \\
&= \frac{1}{2}Mv^2 + \frac{1}{6}mv^2 \\
&= \frac{1}{2}(M + m/3)v^2 \\
E &= K + U \\
&= \frac{1}{2}(M + m/3)v^2 + \frac{1}{2}kx^2
\end{aligned}
$$

(b)

$$
\omega = \sqrt{\frac{k}{M + m/3}}
$$

1.13

(a)

$$
\begin{aligned}
K &= E - U \\
\frac{1}{2}mv^2 &= U(A) - U(x) \\
v &= \sqrt{2[U(A) - U(x)]/m}
\end{aligned}
$$

(b)

$$
\begin{aligned}
T &= 4\int_0^A \frac{dx}{v} \\
&= 4\int_0^A \sqrt{\frac{m}{2[U(A) - U(x)]}}\,dx \\
&= 4\sqrt{\frac{m}{2U(A)}}\int_0^A \frac{dx}{\sqrt{1 - U(x)/U(A)}}
\end{aligned}
$$

(c)

$$
\begin{aligned}
T &= 4\sqrt{\frac{m}{2\alpha A^n}}\int_0^A \frac{dx}{\sqrt{1 - (x/A)^n}} \\
&= 4\sqrt{\frac{m}{2\alpha A^n}}\int_0^1 \frac{A\,d\xi}{\sqrt{1 - \xi^n}} \\
&= cA^{(n/2)-1}
\end{aligned}
$$

2 The Damped Harmonic Oscillator

2.1

$$
\begin{aligned}
\left(\frac{\gamma}{2}\right)^2 &= \omega_0^2 \\
\frac{b}{2m} &= \sqrt{\frac{k}{m}} \\
b &= 2m\sqrt{\frac{k}{m}} \\
&= 2m\sqrt{\frac{mg/x}{m}} \\
&= 2m\sqrt{\frac{g}{x}} \\
&= 64\,\mathrm{kg/s}
\end{aligned}
$$

2.2

$$
\begin{aligned}
\frac{A_{n+1}}{A_n} &= 0.90 \\
e^{-2.5\gamma/2} &= 0.90 \\
e^{2.5\gamma/2} &= \frac{1}{0.90} \\
\frac{2.5\gamma}{2} &= \ln\frac{1}{0.90} \\
\gamma &= \frac{2}{2.5}\ln\frac{1}{0.90} \\
&= 8.43 \times 10^{-2}\,\mathrm{s}^{-1} \\
F &= -bv \\
&= -(4.21 \times 10^{-2})v
\end{aligned}
$$

2.3

After 10 cycles the amplitude has decreased by a factor of 5/3. The energy of the system is proportional to the amplitude squared, so

$$
\begin{aligned}
E(300) &= E(0)e^{-300/\tau} \\
e^{300/\tau} &= \frac{E(0)}{E(300)} \\
\tau &= \frac{300}{\ln[E(0)/E(300)]} \\
&= \frac{300}{\ln\frac{25}{9}} \\
&= 294\,\mathrm{s} \\
Q &= \omega_0\tau \\
&= \frac{2\pi\tau}{T} \\
&= 61.5
\end{aligned}
$$

2.4

$$\begin{aligned}
\frac{E(10T)}{E_0} &= \frac{E_0 e^{-\gamma 10T}}{E_0} \\
\frac{1}{2} &= e^{-\gamma 10T} \\
\frac{E(50T)}{E_0} &= \frac{E_0 e^{-\gamma 50T}}{E_0} \\
&= (e^{-\gamma 10T})^5 \\
&= \left(\frac{1}{2}\right)^5 \\
&= \frac{1}{32}
\end{aligned}$$

2.5

(a)

$$\begin{aligned}
Q_{0.01} &= 310 \\
\omega_{0.01} &= 3.14\,\mathrm{rad/s} \\
Q_{0.30} &= 10.5 \\
\omega_{0.30} &= 3.14\,\mathrm{rad/s} \\
Q_{1.00} &= 3.14 \\
\omega_{1.00} &= 3.10\,\mathrm{rad/s}
\end{aligned}$$

(c)

$$\begin{aligned}
\gamma^2/4 &= \pi^2 \\
\gamma &= 2\pi \\
x &= Ae^{-\pi t} + Bte^{-\pi t} \\
A &= 10\,\mathrm{mm} \\
v &= -10\pi e^{-\pi t} + Be^{-\pi t} - \pi Bte^{-\pi t} \\
0 &= -10\pi + B \\
B &= 10\pi \\
x &= 10e^{-\pi t} + 10\pi te^{-\pi t}
\end{aligned}$$

2.6

$$\begin{aligned}\frac{\omega}{\omega_0} &= \frac{\omega_0\sqrt{1-1/4Q^2}}{\omega_0} \\ &= \sqrt{1-1/4Q^2} \\ &= 1 - \frac{1/4Q^2}{2} + \cdots \\ &\approx 1 - \frac{Q^2}{8}\end{aligned}$$

2.7

The amplitude of each pendulum decreases over time by a factor of

$$\begin{aligned}\exp\left(-\frac{\gamma t}{2}\right) &= \exp\left(-\frac{bt}{2m}\right) \\ &= \exp\left(-\frac{bt}{2\cdot\frac{4}{3}\pi r^3\rho}\right) \\ &= \exp\left(-\frac{3bt}{8\pi r^3\rho}\right) \\ &= \exp\left(-\frac{3bt}{8\pi r^3}\right)^{1/\rho}.\end{aligned}$$

After 10 minutes the amplitude of oscillation of the aluminium pendulum has decreased to half of its initial value

$$\begin{aligned}\exp\left(-\frac{225b}{\pi r^3}\right)^{1/\rho_a} &= \frac{1}{2} \\ \exp\left(-\frac{225b}{\pi r^3}\right) &= \left(\frac{1}{2}\right)^{\rho_a}\end{aligned}$$

so the brass pendulum's amplitude of oscillation has decreased by a factor of

$$\begin{aligned}\exp\left(-\frac{225b}{\pi r^3}\right)^{1/\rho_b} &= \left(\frac{1}{2}\right)^{\rho_a/\rho_b} \\ &= 0.802\end{aligned}$$

2.8

(a)

$$
\begin{aligned}
x &= A \sin \omega t \\
v &= \omega A \cos \omega t \\
a &= -\omega^2 A \sin \omega t \\
E &= \int_0^T \frac{K e^2 a^2}{c^3}\, dt \\
&= \int_0^T \frac{K e^2 \omega^4 A^2 \sin^2 \omega t}{c^3}\, dt \\
&= \frac{K e^2 \omega^4 A^2}{c^3} \int_0^{2\pi/\omega} \sin^2 \omega t\, dt \\
&= \frac{K e^2 \omega^4 A^2}{c^3} \left[\frac{t}{2} - \frac{1}{4\omega} \sin 2\omega t \right]_0^{2\pi/\omega} \\
&= \frac{K e^2 \omega^3 A^2 \pi}{c^3}
\end{aligned}
$$

(b)

$$
\begin{aligned}
Q &= \frac{\frac{1}{2} m \omega^2 A^2}{\frac{K e^2 \omega^3 A^2 \pi}{2\pi c^3}} \\
&= \frac{c^3 m}{e^2 K \omega}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\tau &= \frac{1}{\gamma} \\
&= \frac{Q}{\omega} \\
&= \frac{c^3 m}{e^2 K \omega^2} \\
&= \frac{c^3 m}{e^2 K \left(2\pi (c/\lambda)\right)^2} \\
&= \frac{\lambda^2 c m}{4\pi^2 e^2 K} \\
&\approx 1.13 \times 10^{-8}\,\mathrm{s}
\end{aligned}
$$