

Advanced Engineering Mathematics Partial Differential Equations by Dennis G. Zill Problems

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12 Orthogonal Functions and Fourier Series

12.1 Orthogonal Functions

12.1.7

$$\begin{aligned}\int_0^{\pi/2} \sin mx \sin nx \, dx &= \frac{1}{2} \int_0^{\pi/2} [\cos(m-n)x - \cos(m+n)x] \, dx \\&= \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_0^{\pi/2} \\&= \frac{1}{2} \left(\frac{\sin(m-n)\pi/2}{m-n} - \frac{\sin(m+n)\pi/2}{m+n} \right) \\&= 0\end{aligned}$$

$$\begin{aligned}\|\sin nx\|^2 &= (\sin nx, \sin nx) \\&= \int_0^{\pi/2} \sin^2 nx \, dx \\&= \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2nx) \, dx \\&= \frac{1}{2} \left[x - \frac{1}{2n} \sin 2nx \right]_0^{\pi/2} \\&= \frac{\pi}{4} \\ \|\sin nx\| &= \frac{\sqrt{\pi}}{2}\end{aligned}$$

12.1.9

$$\begin{aligned}
\int_0^\pi \sin mx \sin nx \, dx &= \frac{1}{2} \int_0^\pi [\cos(m-n)x - \cos(m+n)x] \, dx \\
&= \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_0^\pi \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\|\sin nx\|^2 &= (\sin nx, \sin nx) \\
&= \int_0^\pi \sin^2 nx \, dx \\
&= \frac{1}{2} \int_0^\pi (1 - \cos 2nx) \, dx \\
&= \frac{1}{2} \left[x - \frac{1}{2n} \sin 2nx \right]_0^\pi \\
&= \frac{\pi}{2}
\end{aligned}$$

$$\|\sin nx\| = \sqrt{\frac{\pi}{2}}$$

12.1.21

$$T = 1$$

12.1.23

$$T = 2\pi$$

12.1.25

$$T = 2\pi$$

12.2 Fourier Series

12.2.1

$$\begin{aligned}p &= \pi \\a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\&= \frac{1}{\pi} \int_0^{\pi} dx \\&= 1 \\a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\&= \frac{1}{\pi} \int_0^{\pi} \cos nx dx \\&= \frac{1}{n\pi} [\sin nx]_0^{\pi} \\&= 0 \\b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\&= \frac{1}{\pi} \int_0^{\pi} \sin nx dx \\&= -\frac{1}{n\pi} [\cos nx]_0^{\pi} \\&= -\frac{1}{n\pi} [(-1)^n - 1] \\&= \frac{1 - (-1)^n}{n\pi} \\f(x) &= \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin nx\end{aligned}$$

The series converges to $\frac{1}{2}$ at the point of discontinuity.

12.2.3

$$\begin{aligned}p &= 1 \\a_0 &= \frac{3}{2} \\a_n &= \int_{-1}^0 \cos n\pi x \, dx + \int_0^1 x \cos n\pi x \, dx \\&= \frac{1}{n\pi} [\sin n\pi x]_{-1}^0 + \frac{1}{n\pi} \left[\frac{\cos n\pi x}{n\pi} + x \sin n\pi x \right]_0^1 \\&= \frac{(-1)^n - 1}{n^2 \pi^2} \\b_n &= \int_{-1}^0 \sin n\pi x \, dx + \int_0^1 x \sin n\pi x \, dx \\&= -\frac{1}{n\pi} [\cos n\pi x]_{-1}^0 + \frac{1}{n\pi} \left[\frac{\sin n\pi x}{n\pi} - x \cos n\pi x \right]_0^1 \\&= -\frac{1}{n\pi} \\f(x) &= \frac{3}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2 \pi^2} \cos n\pi x - \frac{1}{n\pi} \sin n\pi x \right]\end{aligned}$$

The series converges to $\frac{1}{2}$ at the point of discontinuity.

12.3 Fourier Cosine and Sine Series

12.3.1

Odd

12.3.3

Neither

12.3.5

Even

12.3.7

Odd

12.3.9

Neither

12.3.11

$$\begin{aligned}
b_n &= -2\pi \int_0^1 \sin n\pi x \, dx \\
&= \frac{2}{n} [\cos n\pi x]_0^1 \\
&= \frac{2}{n} [(-1)^n - 1] \\
f &= \sum_{n=1}^{\infty} \frac{2}{n} [(-1)^n - 1] \sin n\pi x
\end{aligned}$$

12.3.13

$$\begin{aligned}
a_0 &= \frac{2}{\pi} \int_0^{\pi} x \, dx \\
&= \pi \\
a_n &= 2 \int_0^{\pi} x \cos nx \, dx \\
&= \frac{2}{n} \left[\frac{\cos nx}{n} + x \sin nx \right]_0^{\pi} \\
&= \frac{2[(-1)^n - 1]}{n^2} \\
f &= \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx
\end{aligned}$$

12.3.25

$$\begin{aligned}
 a_0 &= 2 \int_0^1 f(x) dx \\
 &= 1 \\
 a_n &= 2 \int_0^1 f(x) \cos n\pi x dx \\
 &= 2 \int_0^{1/2} \cos n\pi x dx \\
 &= \frac{2}{n\pi} [\sin n\pi x]_0^{1/2} \\
 &= \frac{2}{n\pi} \sin \frac{n\pi}{2} \\
 f &= \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n} \cos n\pi x
 \end{aligned}$$

$$\begin{aligned}
 b_n &= 2 \int_0^1 f(x) \sin n\pi x dx \\
 &= 2 \int_0^{1/2} \sin n\pi x dx \\
 &= -\frac{2}{n\pi} [\cos n\pi x]_0^{1/2} \\
 &= \frac{2}{n\pi} \left(1 - \cos \frac{n\pi}{2}\right) \\
 f &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - \cos \frac{n\pi}{2}}{n} \sin n\pi x
 \end{aligned}$$

12.3.27

$$\begin{aligned}
 a_0 &= \frac{4}{\pi} \int_0^{\pi/2} \cos x \, dx \\
 &= \frac{4}{\pi} [\sin x]_0^{\pi/2} \\
 &= \frac{4}{\pi} \\
 a_n &= \frac{4}{\pi} \int_0^{\pi/2} \cos x \cos 2nx \, dx \\
 &= \frac{2}{\pi} \int_0^{\pi/2} [\cos(1-2n)x + \cos(1+2n)x] \, dx \\
 &= \frac{2}{\pi} \left[\frac{\sin(1-2n)x}{1-2n} + \frac{\sin(1+2n)x}{1+2n} \right]_0^{\pi/2} \\
 &= \frac{2(-1)^n}{\pi} \left[\frac{1}{1-2n} + \frac{1}{1+2n} \right] \\
 &= \frac{2(-1)^n}{\pi} \frac{1+2n+1-2n}{(1-2n)(1+2n)} \\
 &= \frac{4(-1)^n}{\pi(1-2n)(1+2n)} \\
 f &= \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(1-2n)(1+2n)} \cos 2nx
 \end{aligned}$$

$$\begin{aligned}
 b_n &= \frac{4}{\pi} \int_0^{\pi/2} \cos x \sin 2nx \, dx \\
 &= \frac{2}{\pi} \int_0^{\pi/2} [\sin(2n+1)x + \sin(2n-1)x] \, dx \\
 &= -\frac{2}{\pi} \left[\frac{\cos(2n+1)x}{2n+1} + \frac{\cos(2n-1)x}{2n-1} \right]_0^{\pi/2} \\
 &= \frac{2}{\pi} \left(\frac{1}{2n+1} + \frac{1}{2n-1} \right) \\
 &= \frac{2}{\pi} \frac{4n}{4n^2-1} \\
 &= \frac{8n}{\pi(4n^2-1)} \\
 f &= \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2-1} \sin 2nx
 \end{aligned}$$

12.3.35

$$\begin{aligned}
 a_0 &= \frac{1}{\pi} \int_0^{2\pi} x^2 dx \\
 &= \frac{8}{3} \pi^2 \\
 a_n &= \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx dx \\
 &= \frac{4}{n^2} \\
 b_n &= \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nx dx \\
 &= -\frac{4\pi}{n} \\
 f &= \frac{4}{3} \pi^2 + \sum_{n=1}^{\infty} \left(\frac{4}{n^2} \cos nx - \frac{4\pi}{n} \sin nx \right)
 \end{aligned}$$

12.3.43

$$\begin{aligned}
 b_n &= \frac{10}{\pi} \int_0^{\pi} \sin nt dt \\
 &= -\frac{10}{n\pi} [\cos nt]_0^{\pi} \\
 &= \frac{10}{n\pi} [1 - (-1)^n] \\
 f &= \frac{10}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} [1 - (-1)^n] \sin nt \\
 x_p(t) &= \sum_{n=1}^{\infty} B_n \sin nt \\
 m \frac{d^2 x}{dt^2} + kx &= f(t)
 \end{aligned}$$

$$\begin{aligned}
 -mn^2 B_n + kB_n &= \frac{10}{n\pi} [1 - (-1)^n] \\
 B_n &= \frac{10}{n\pi(k - mn^2)} [1 - (-1)^n] \\
 x_p(t) &= \frac{10}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n(k - mn^2)} \sin nt \\
 &= \frac{10}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n(10 - n^2)} \sin nt
 \end{aligned}$$

12.3.45

$$a_0 = \frac{2}{\pi} \int_0^\pi (2\pi t - t^2) dt$$

$$= \frac{2}{\pi} \left[\pi t^2 - \frac{1}{3} t^3 \right]_0^\pi$$

$$= \frac{2}{\pi} \left(\pi^3 - \frac{1}{3} \pi^3 \right)$$

$$= \frac{4}{3} \pi^2$$

$$a_n = \frac{2}{\pi} \int_0^\pi (2\pi t - t^2) \cos nt dt$$

$$= -\frac{4}{n^2}$$

$$f(t) = \frac{2}{3} \pi^2 - \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nt$$

$$x_p(t) = c + \sum_{n=1}^{\infty} B_n \cos nt$$

$$\frac{1}{4} \frac{d^2 x}{dt^2} + 12x = f(t)$$

$$-\frac{n^2}{4} \sum_{n=1}^{\infty} B_n \cos nt + 12c + 12 \sum_{n=1}^{\infty} B_n \cos nt = \frac{2}{3} \pi^2 - \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nt$$

$$12c = \frac{2}{3} \pi^2$$

$$c = \frac{\pi^2}{18}$$

$$-\frac{n^2}{4} B_n + 12B_n = -\frac{4}{n^2}$$

$$\frac{48 - n^2}{4} B_n = -\frac{4}{n^2}$$

$$B_n = -\frac{16}{n^2(48 - n^2)}$$

$$x_p(t) = \frac{\pi^2}{18} - \sum_{n=1}^{\infty} \frac{16}{n^2(48 - n^2)} \cos nt$$

$$= \frac{\pi^2}{18} + 16 \sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - 48)} \cos nt$$

12.4 Complex Fourier Series

12.4.1

$$\begin{aligned}T &= 4 \\p &= 2 \\c_n &= \frac{1}{4} \left(\int_0^2 e^{-in\pi x/2} dx - \int_{-2}^0 e^{-in\pi x/2} dx \right) \\&= \frac{1}{2in\pi} ([e^{-in\pi x/2}]_{-2}^0 - [e^{-in\pi x/2}]_0^2) \\&= \frac{2 - e^{in\pi} - e^{-in\pi}}{2in\pi} \\&= \frac{2 - \cos n\pi - i \sin n\pi - \cos n\pi + i \sin n\pi}{2in\pi} \\&= \frac{1 - \cos n\pi}{in\pi} \\&= \frac{1 - (-1)^n}{in\pi} \\f(x) &= \sum_{n=-\infty, n \neq 0}^{\infty} \frac{1 - (-1)^n}{in\pi} e^{in\pi x/2}\end{aligned}$$

12.4.3

$$\begin{aligned}T &= 1 \\p &= \frac{1}{2} \\c_n &= \int_0^{1/4} e^{-2in\pi x} dx \\&= -\frac{1}{2in\pi} [e^{-2in\pi x}]_0^{1/4} \\&= \frac{1}{2in\pi} (1 - e^{-in\pi/2}) \\c_0 &= \frac{1}{4} \\f(x) &= \frac{1}{4} + \sum_{n=-\infty, n \neq 0}^{\infty} \frac{1 - e^{-in\pi/2}}{2in\pi} e^{2in\pi x}\end{aligned}$$

12.4.5

$$T = 2\pi$$

$$p = \pi$$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} dx$$

$$= \frac{i}{n}$$

$$c_0 = \frac{1}{2\pi} \int_0^{2\pi} x dx$$

$$= \pi$$

$$f(x) = \pi + \sum_{n=-\infty, n \neq 0}^{n=\infty} \frac{i}{n} e^{inx}$$

12.5 Sturm-Liouville Problem

12.5.1

$$y'' + \lambda y = 0$$

$$y'(0) = 0$$

$$y(1) + y'(1) = 0$$

$$\lambda = \alpha^2$$

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$y' = -\alpha c_1 \sin \alpha x + \alpha c_2 \cos \alpha x$$

$$c_2 = 0$$

$$y = c_1 \cos \alpha x$$

$$c_1 \cos \alpha - \alpha c_1 \sin \alpha = 0$$

$$c_1 \cos \alpha = \alpha c_1 \sin \alpha$$

$$\alpha \tan \alpha = 0$$

$$\alpha = \cot \alpha$$

$$\lambda_1 = 0.740174$$

$$y_1 = \cos 0.860334x$$

$$\lambda_2 = 11.734872$$

$$y_2 = \cos 3.42562x$$

$$\lambda_3 = 41.438831$$

$$y_3 = \cos 6.4373x$$

$$\lambda_4 = 90.808130$$

$$y_4 = \cos 9.52933x$$

12.5.5

$$\begin{aligned}(y_n, y_n) &= \int_0^1 \cos^2 \alpha_n x \, dx \\&= \frac{1}{2} \int_0^1 (1 + \cos 2\alpha_n x) \, dx \\&= \frac{1}{2} \left[x + \frac{1}{2\alpha_n} \sin 2\alpha_n x \right]_0^1 \\&= \frac{1}{2} \left(1 + \frac{1}{2\alpha_n} \sin 2\alpha_n \right) \\&= \frac{1}{2} \left(1 + \frac{1}{\alpha_n} \sin \alpha_n \cos \alpha_n \right) \\&= \frac{1}{2} (1 + \tan \alpha_n \sin \alpha_n \cos \alpha_n) \\&= \frac{1}{2} (1 + \sin^2 \alpha_n)\end{aligned}$$

12.5.7

(a)

$$x^2 y'' + xy' + \lambda y = 0$$

$$y(1) = 0$$

$$y(5) = 0$$

$$\lambda = \alpha^2$$

$$y = x^m$$

$$y' = mx^{m-1}$$

$$y'' = m(m-1)x^{m-2}$$

$$x^2 m(m-1)x^{m-2} + xmx^{m-1} + \alpha^2 x^m = 0$$

$$m(m-1) + m + \alpha^2 = 0$$

$$m^2 + \alpha^2 = 0$$

$$m = \pm i\alpha$$

$$y = c_1 \cos(\alpha \ln x) + c_2 \sin(\alpha \ln x)$$

$$0 = c_1$$

$$0 = c_2 \sin(\alpha \ln 5)$$

$$\alpha = \frac{n\pi}{\ln 5}$$

$$\lambda = \left(\frac{n\pi}{\ln 5} \right)^2$$

$$y_n = \sin \left(\frac{n\pi}{\ln 5} \ln x \right)$$

(b)

$$x^2 y'' + xy' + \lambda y = 0$$

$$y'' + \frac{1}{x} y' + \lambda \frac{1}{x^2} y = 0$$

$$e^{\ln x} y'' + \frac{1}{x} e^{\ln x} y' + \lambda e^{\ln x} \frac{1}{x^2} y = 0$$

$$\frac{d}{dx} (e^{\ln x} y') + \lambda e^{\ln x} \frac{1}{x^2} y = 0$$

$$\frac{d}{dx} (xy') + \lambda \frac{1}{x} y = 0$$

(c)

$$\int_1^5 \frac{1}{x} \sin \left(\frac{m\pi}{\ln 5} \ln x \right) \sin \left(\frac{n\pi}{\ln 5} \ln x \right) dx = 0, \quad m \neq n$$

12.5.9

$$\begin{aligned}
 xy'' + (1-x)y' + ny &= 0 \\
 y'' + \left(\frac{1}{x} - 1\right)y' + n\frac{1}{x}y &= 0 \\
 e^{\int\left(\frac{1}{x}-1\right)dx} &= e^{\ln(x)-x} \\
 &= xe^{-x} \\
 xe^{-x}y'' + \left(\frac{1}{x} - 1\right)xe^{-x}y' + n\frac{1}{x}xe^{-x}y &= 0 \\
 \frac{d}{dx}(xe^{-x}y') + ne^{-x}y &= 0 \\
 \int_0^\infty e^{-x}L_m(x)L_n(x)dx &= 0, \quad m \neq n
 \end{aligned}$$

12.6 Bessel and Legendre Series

12.6.1

$$\begin{aligned}
 J_1(3\alpha) &= 0 \\
 \alpha_1 &= 1.277 \\
 \alpha_2 &= 2.338 \\
 \alpha_3 &= 3.391 \\
 \alpha_4 &= 4.441
 \end{aligned}$$

12.6.3

$$\begin{aligned}
 J_0(2\alpha) &= 0 \\
 c_i &= \frac{1}{2J_1^2(2\alpha_i)} \int_0^2 xJ_0(\alpha_i x) dx \\
 &= \frac{1}{2J_1^2(2\alpha_i)} \int_0^2 \frac{d}{dx} \left[\frac{1}{\alpha_i} xJ_1(\alpha_i x) \right] dx \\
 &= \frac{1}{\alpha_i J_1(2\alpha_i)} \\
 f(x) &= \sum_{i=1}^{\infty} \frac{J_0(\alpha_i x)}{\alpha_i J_1(2\alpha_i)}
 \end{aligned}$$

12.6.5

$$J_0(2\alpha) + 2\alpha J_0'(2\alpha) = 0$$

$$h = 1$$

$$b = 2$$

$$\begin{aligned} c_i &= \frac{2\alpha_i^2}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)} \int_0^2 x J_0(\alpha_i x) dx \\ &= \frac{2\alpha_i^2}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)} \int_0^2 \frac{d}{dx} \left[\frac{1}{\alpha_i} x J_1(\alpha_i x) \right] dx \\ &= \frac{4\alpha_i J_1(2\alpha_i)}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)} \end{aligned}$$

$$f(x) = 4 \sum_{i=1}^{\infty} \frac{\alpha_i J_1(2\alpha_i)}{(4\alpha_i^2 + 1)J_0^2(2\alpha_i)} J_0(\alpha_i x)$$

12.6.7

$$f(x) = 5x, \quad 0 < x < 4$$

$$4J_1(4\alpha) + 4\alpha J_1'(4\alpha) = 0$$

$$h = 3$$

$$n = 1$$

$$b = 4$$

$$\begin{aligned} c_i &= \frac{2\alpha_i^2}{(16\alpha_i^2 + 8)J_1^2(4\alpha_i)} \int_0^4 5x^2 J_1(\alpha_i x) dx \\ &= \frac{10\alpha_i^2}{(16\alpha_i^2 + 8)J_1^2(4\alpha_i)} \int_0^4 \frac{d}{dx} \left[\frac{1}{\alpha_i} x^2 J_2(\alpha_i x) \right] dx \\ &= \frac{160\alpha_i J_2(4\alpha_i)}{(16\alpha_i^2 + 8)J_1^2(4\alpha_i)} \end{aligned}$$

$$f(x) = \sum_{i=1}^{\infty} \frac{20\alpha_i J_2(4\alpha_i)}{(2\alpha_i^2 + 1)J_1^2(4\alpha_i)} J_1(\alpha_i x)$$

12.6.15

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ x & 0 < x < 1 \end{cases}$$

$$c_n = \frac{2n+1}{2} \int_0^1 x P_n(x) dx$$

$$c_0 = \frac{1}{4}$$

$$c_1 = \frac{1}{2}$$

$$c_2 = \frac{5}{16}$$

$$c_3 = 0$$

$$c_4 = -\frac{3}{32}$$

$$c_5 = 0$$

$$c_6 = \frac{13}{256}$$

12.6.21

$$c_0 = \frac{1}{2}$$

$$c_1 = \frac{5}{8}$$

$$c_2 = -\frac{3}{16}$$

$$c_3 = \frac{13}{128}$$

12.7 Chapter in Review**12.7.1**

$$(x^2 - 1, x^5) = \int_{-\pi}^{\pi} (x^2 - 1)x^5 dx$$

$$= \int_{-\pi}^{\pi} (x^7 - x^5) dx$$

$$= \left[\frac{1}{8}x^8 - \frac{1}{6}x^6 \right]_{-\pi}^{\pi}$$

$$= \frac{1}{8}\pi^8 - \frac{1}{6}\pi^6 - \frac{1}{8}\pi^8 + \frac{1}{6}\pi^6$$

$$= 0$$

True

12.7.3

Fourier cosine

12.7.5

False

12.7.7

5.5, 1, 0

12.7.9

True

12.7.13

$$f(x) = |x| - x, \quad -1 < x < 1$$

$$L = 2$$

$$p = 1$$

$$a_0 = \int_{-1}^1 (|x| - x) dx$$

$$= \int_{-1}^0 -2x dx$$

$$= -[x^2]_{-1}^0$$

$$= -(0 - 1)$$

$$= 1$$

$$a_n = \int_{-1}^1 (|x| - x) \cos n\pi x dx$$

$$= -2 \int_{-1}^0 x \cos n\pi x dx$$

$$= \frac{2[(-1)^n - 1]}{n^2 \pi^2}$$

$$b_n = \int_{-1}^1 (|x| - x) \sin n\pi x dx$$

$$= -2 \int_{-1}^0 x \sin n\pi x dx$$

$$= \frac{2(-1)^n}{n\pi}$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{n^2 \pi^2} \cos n\pi x + \frac{2(-1)^n}{n\pi} \sin n\pi x$$

12.7.17

$$x^2 y'' + xy' + 9\lambda y = 0$$

$$y'(1) = 0$$

$$y(e) = 0$$

$$y = x^m$$

$$y' = mx^{m-1}$$

$$y'' = m(m-1)x^{m-2}$$

$$x^2 m(m-1)x^{m-2} + mx^{m-1} + 9\lambda x^m = 0$$

$$m(m-1) + m + 9\lambda = 0$$

$$m^2 + 9\lambda = 0$$

$$m = \pm 3\sqrt{\lambda}i$$

$$y = c_1 \cos(3\sqrt{\lambda} \ln x) + c_2 \sin(3\sqrt{\lambda} \ln x)$$

$$y' = \frac{3\sqrt{\lambda}}{x} [c_2 \cos(3\sqrt{\lambda} \ln x) - c_1 \sin(3\sqrt{\lambda} \ln x)]$$

$$y'(1) = 0$$

$$0 = 3\sqrt{\lambda}c_2$$

$$c_2 = 0$$

$$y(e) = 0$$

$$0 = c_1 \cos 3\sqrt{\lambda}$$

$$= \cos 3\sqrt{\lambda}$$

$$3\sqrt{\lambda} = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}$$

$$= \frac{2n+1}{2}\pi$$

$$\lambda_n = \left(\frac{2n+1}{6}\pi \right)^2$$

$$y_n = \cos \left(\frac{2n+1}{2}\pi \ln x \right)$$

13 Boundary-Value Problems in Rectangular Coordinates

13.1 Separable Partial Differential Equations

13.1.1

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$$

$$u = X(x)Y(y)$$

$$X'Y = XY'$$

$$\frac{X'}{X} = \frac{Y'}{Y}$$

$$\frac{X'}{X} = \lambda$$

$$\ln X = \lambda x + c_1$$

$$X = c_1 e^{\lambda x}$$

$$\frac{Y'}{Y} = \lambda$$

$$\ln Y = \lambda y + c_2$$

$$Y = c_2 e^{\lambda y}$$

$$u = XY$$

$$= c_1 c_2 e^{\lambda(x+y)}$$

$$= c_3 e^{\lambda(x+y)}$$

13.1.3

$$u_x + u_y = u$$
$$X'Y + XY' = XY$$

$$\frac{X'}{X}Y + Y' = Y$$

$$\frac{X'}{X} + \frac{Y'}{Y} = 1$$

$$\frac{X'}{X} = 1 - \frac{Y'}{Y}$$

$$\frac{X'}{X} = \lambda$$

$$X = c_1 e^{\lambda x}$$

$$1 - \frac{Y'}{Y} = \lambda$$

$$Y' + (\lambda - 1)Y = 0$$

$$Y = c_2 e^{-(\lambda-1)y}$$

$$u = c_3 e^{\lambda x - (\lambda-1)y}$$

13.1.5

$$\begin{aligned}
x \frac{\partial u}{\partial x} &= y \frac{\partial u}{\partial y} \\
xX'Y &= yXY' \\
x \frac{X'}{X} &= y \frac{Y'}{Y}
\end{aligned}$$

$$\begin{aligned}
x \frac{X'}{X} &= \lambda \\
\frac{X'}{X} &= \frac{\lambda}{x} \\
\ln X &= \lambda \ln x + c_1 \\
X &= c_1 x^\lambda
\end{aligned}$$

$$Y = c_2 y^\lambda$$

$$\begin{aligned}
u &= XY \\
&= c_3 (xy)^\lambda
\end{aligned}$$

13.1.7

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} &= 0 \\
X''Y + X'Y' + XY'' &= 0
\end{aligned}$$

Not separable.

13.1.9

$$k \frac{\partial^2 u}{\partial x^2} - u = \frac{\partial u}{\partial t}, \quad k > 0$$

$$kTX'' - TX = T'X$$

$$k \frac{X''}{X} - 1 = \frac{T'}{T}$$

$$\frac{T'}{T} = \lambda$$

$$T' - \lambda T = 0$$

$$T = c_1 e^{\lambda t}$$

$$k \frac{X''}{X} - 1 = \lambda$$

$$X'' - \frac{\lambda+1}{k} X = 0$$

$$X = \begin{cases} c_1 \cos \sqrt{\frac{\lambda+1}{k}} x + c_2 \sin \sqrt{\frac{\lambda+1}{k}} x & \lambda < -1 \\ c_1 x + c_2 & \lambda = -1 \\ c_1 \cosh \sqrt{\frac{\lambda+1}{k}} x + c_2 \sinh \sqrt{\frac{\lambda+1}{k}} x & \lambda > -1 \end{cases}$$

$$u = TX$$

$$= \begin{cases} e^{\lambda t} \left(c_1 \cos \sqrt{\frac{\lambda+1}{k}} x + c_2 \sin \sqrt{\frac{\lambda+1}{k}} x \right) & \lambda < -1 \\ e^{\lambda t} (c_1 x + c_2) & \lambda = -1 \\ e^{\lambda t} \left(c_1 \cosh \sqrt{\frac{\lambda+1}{k}} x + c_2 \sinh \sqrt{\frac{\lambda+1}{k}} x \right) & \lambda > -1 \end{cases}$$

13.1.11

$$\begin{aligned}
a^2 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t^2} \\
a^2 T X'' &= T'' X \\
a^2 \frac{X''}{X} &= \frac{T''}{T}
\end{aligned}$$

$$\begin{aligned}
\frac{T''}{T} &= \lambda \\
T'' - \lambda T &= 0
\end{aligned}$$

$$T = \begin{cases} c_1 \cos \sqrt{\lambda} t + c_2 \sin \sqrt{\lambda} t & \lambda < 0 \\ c_1 t + c_2 & \lambda = 0 \\ c_1 \cosh \sqrt{\lambda} t + c_2 \sinh \sqrt{\lambda} t & \lambda > 0 \end{cases}$$

$$\begin{aligned}
a^2 \frac{X''}{X} &= \lambda \\
X'' - \frac{\lambda}{a^2} X &= 0
\end{aligned}$$

$$X = \begin{cases} c_1 \cos \frac{\sqrt{\lambda}}{a} x + c_2 \sin \frac{\sqrt{\lambda}}{a} x & \lambda < 0 \\ c_1 x + c_2 & \lambda = 0 \\ c_1 \cosh \frac{\sqrt{\lambda}}{a} x + c_2 \sinh \frac{\sqrt{\lambda}}{a} x & \lambda > 0 \end{cases}$$

$$\begin{aligned}
u &= T X \\
&= \begin{cases} (c_1 \cos \sqrt{\lambda} t + c_2 \sin \sqrt{\lambda} t)(c_3 \cos \frac{\sqrt{\lambda}}{a} x + c_4 \sin \frac{\sqrt{\lambda}}{a} x) & \lambda < 0 \\ (c_1 t + c_2)(c_3 x + c_4) & \lambda = 0 \\ (c_1 \cosh \sqrt{\lambda} t + c_2 \sinh \sqrt{\lambda} t)(c_3 \cosh \frac{\sqrt{\lambda}}{a} x + c_4 \sinh \frac{\sqrt{\lambda}}{a} x) & \lambda > 0 \end{cases}
\end{aligned}$$

13.1.17

Elliptic

13.1.19

Parabolic

13.1.21

Hyperbolic

13.1.23

Parabolic

13.1.25

Hyperbolic

13.2 Classical PDEs and Boundary-Value Problems**13.2.1**

$$\begin{aligned}
 k^2 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t} \\
 u(0, t) &= 0 \\
 \left. \frac{\partial u}{\partial x} \right|_{x=L} &= 0 \\
 u(x, 0) &= f(x)
 \end{aligned}$$

13.2.3

$$\begin{aligned}
 k^2 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t} \\
 u(0, t) &= 100 \\
 \left. \frac{\partial u}{\partial x} \right|_{x=L} &= -hu(L, t) \\
 u(x, 0) &= f(x)
 \end{aligned}$$

13.2.5

$$\begin{aligned}
 k^2 \frac{\partial^2 u}{\partial x^2} - hu &= \frac{\partial u}{\partial t} \\
 u(0, t) &= \sin \frac{\pi}{L} t \\
 u(L, t) &= 0 \\
 u(x, 0) &= f(x)
 \end{aligned}$$

13.2.7

$$\begin{aligned}
a^2 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t^2} \\
u(0, t) &= 0 \\
u(L, t) &= 0 \\
u(x, 0) &= x(L - x) \\
\left. \frac{\partial u}{\partial t} \right|_{t=0} &= 0
\end{aligned}$$

13.2.9

$$\begin{aligned}
a^2 \frac{\partial^2 u}{\partial x^2} - c \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial t^2} \\
u(0, t) &= 0 \\
u(L, t) &= \sin \pi t \\
u(x, 0) &= f(x) \\
\left. \frac{\partial u}{\partial t} \right|_{t=0} &= 0
\end{aligned}$$

13.2.11

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\
\left. \frac{\partial u}{\partial x} \right|_{x=0} &= 0 \\
\left. \frac{\partial u}{\partial y} \right|_{y=0} &= 0 \\
u(x, 2) &= 0 \\
u(4, y) &= f(y)
\end{aligned}$$