Introduction to Electrodynamics by David J. Griffiths Problems

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Contents

2 Electrostatics

2.1

(a) **0**

(b) The same as if only the opposite charge were present — all others are cancelled out.

2.2

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0} 2 \frac{q}{\mathscr{r}^2} \cos\theta \hat{\mathbf{x}} \\ &= \frac{1}{4\pi\epsilon_0} \frac{dq}{[(d/2)^2 + z^2]^{3/2}} \hat{\mathbf{x}}\end{aligned}$$

2.3

$$
\begin{aligned}
\mathbf{r} &= z\hat{\mathbf{z}} \\
\mathbf{r}' &= x\hat{\mathbf{x}} \\
\boldsymbol{\mathscr{r}} &= z\hat{\mathbf{z}} - x\hat{\mathbf{x}} \\
\mathscr{r} &= \sqrt{x^2 + z^2} \\
\hat{\boldsymbol{\mathscr{r}}} &= \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{\sqrt{x^2 + z^2}} \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_0^L \frac{\lambda}{x^2 + z^2} \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{\sqrt{x^2 + z^2}}\, dx \\
&= \frac{1}{4\pi\epsilon_0} \lambda \left(z\hat{\mathbf{z}} \int_0^L \frac{1}{(x^2 + z^2)^{3/2}}\, dx - \hat{\mathbf{x}} \int_0^L \frac{x}{(x^2 + z^2)}\, dx \right) \\
&= \frac{1}{4\pi\epsilon_0} \lambda \left[\frac{L}{z\sqrt{L^2 + z^2}} \hat{\mathbf{z}} - \left(\frac{1}{z} - \frac{1}{\sqrt{L^2 + z^2}} \right) \hat{\mathbf{x}} \right] \\
&= \frac{1}{4\pi\epsilon_0} \frac{\lambda}{z} \left[\left(-1 + \frac{z}{\sqrt{L^2 + z^2}} \right) \hat{\mathbf{x}} + \frac{L}{\sqrt{L^2 + z^2}} \hat{\mathbf{z}} \right]
\end{aligned}
$$

2.4

The electric field a distance z above the midpoint of a line segment of length $2L$ and uniform line charge λ is

$$
\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z\sqrt{z^2 + L^2}} \hat{\mathbf{z}}.
$$

Applying this to the four sides of the square, the horizontal components of opposite sides cancel leaving only the vertical component.

$$
\begin{aligned}
\cos\theta &= \frac{z}{\mathscr{r}} \\
&= \frac{z}{\sqrt{(a/2)^2 + z^2}} \\
\mathbf{E} &= 4 \left(\frac{1}{4\pi\epsilon_0} \frac{\lambda a}{\sqrt{(a/2)^2 + z^2}\sqrt{(a/2)^2 + (a/2)^2 + z^2}} \hat{\mathbf{z}} \right) \cos\theta \\
&= \frac{1}{4\pi\epsilon_0} \frac{4a\lambda z}{[(a/2)^2 + z^2]\sqrt{(a^2/2) + z^2}} \hat{\mathbf{z}}
\end{aligned}
$$

2.5

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\frac{\lambda r}{r^2+z^2}\cos\alpha\,d\theta\,\hat{\mathbf{z}}\\ &= \frac{1}{4\pi\epsilon_0}\frac{2\pi\lambda r z}{(r^2+z^2)^{3/2}}\hat{\mathbf{z}}\end{aligned}$$

2.6

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int\frac{dq}{\mathscr{r}^2}\cos\theta\hat{\mathbf{z}}\\ &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{R}\frac{\sigma}{r^2+z^2}\frac{z}{\sqrt{r^2+z^2}}r\,dr\,d\theta\hat{\mathbf{z}}\\ &= \frac{1}{4\pi\epsilon_0}2\pi\sigma z\int_0^{R}\frac{r}{(r^2+z^2)^{3/2}}\,dr\,\hat{\mathbf{z}}\\ &= \frac{1}{4\pi\epsilon_0}2\pi\sigma z\left(\frac{1}{z}-\frac{1}{\sqrt{R^2+z^2}}\right)\hat{\mathbf{z}}\end{aligned}$$

When $R\to\infty$

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}.$$

2.7

$$\mathbf{E} = \begin{cases}\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}} & z>R\\ \mathbf{0} & z<R\end{cases}$$

2.8

$$\mathbf{E} = \begin{cases}\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}} & z>R\\ \frac{1}{4\pi\epsilon_0}\frac{qz}{R^3}\hat{\mathbf{z}} & z<R\end{cases}$$

2.9

(a)

$$\begin{aligned}\rho &= \epsilon_0\nabla\cdot\mathbf{E}\\ &= \epsilon_0\frac{1}{r^2}\frac{\partial}{\partial r}(kr^5)\\ &= 5\epsilon_0 k r^2\end{aligned}$$

(b)

$$
\begin{aligned}
Q_{\text{enc}} &= \epsilon_0 \oint \mathbf{E} \cdot d\mathbf{a} \\
&= \epsilon_0 \int_0^{2\pi} \int_0^{\pi} kR^3 R\, d\theta R \sin\theta\, d\phi \\
&= 2\pi\epsilon_0 kR^5 [-\cos\theta]_0^{\pi} \\
&= 4\pi\epsilon_0 kR^5 \\
Q_{\text{enc}} &= \int_V \rho\, d\tau \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^{R} 5\epsilon_0 kr^2\, dr r\, d\theta r \sin\theta\, d\phi \\
&= 10\pi\epsilon_0 k \int_0^{\pi} \int_0^{R} r^4 \sin\theta\, dr\, d\theta \\
&= 2\pi\epsilon_0 kR^5 [-\cos\theta]_0^{\pi} \\
&= 4\pi\epsilon_0 kR^5
\end{aligned}
$$

2.10

If the charge was surrounded by 8 such cubes the total flux through all the cubes would be q/ϵ_0. There are 24 outside faces to the larger cube, so the total flux through the shaded face is $q/(24\epsilon_0)$.

2.11

$$
\begin{aligned}
\int \mathbf{E}_{\text{inside}} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
&= 0 \\
\mathbf{E}_{\text{inside}} &= \mathbf{0} \\
\int \mathbf{E}_{\text{outside}} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E_{\text{outside}} &= \frac{4\pi R^2 \sigma}{\epsilon_0} \\
\mathbf{E}_{\text{outside}} &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}
\end{aligned}
$$

2.12

$$\begin{aligned}\int \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\ 4\pi r^2 E &= \frac{\frac{4}{3}\pi r^3 \rho}{\epsilon_0} \\ \mathbf{E} &= \frac{r\rho}{3\epsilon_0}\hat{\mathbf{r}}\end{aligned}$$

2.13

$$\begin{aligned}\int \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\ 2\pi s l E &= \frac{l\lambda}{\epsilon_0} \\ \mathbf{E} &= \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{\mathbf{s}}\end{aligned}$$

2.14

$$\begin{aligned}Q_{\text{enc}} &= \int_V \rho \, d\tau \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{r} k r'^3 \sin\theta \, dr' \, d\theta \, d\phi \\ &= 2\pi k \int_0^{\pi} \left[\frac{1}{4} r'^4 \sin\theta\right]_0^r d\theta \\ &= \frac{1}{2}\pi k r^4 [-\cos\theta]_0^{\pi} \\ &= \pi k r^4 \\ \int \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\ 4\pi r^2 E &= \frac{\pi k r^4}{\epsilon_0} \\ \mathbf{E} &= \frac{k r^2}{4\epsilon_0}\hat{\mathbf{r}}\end{aligned}$$

2.15

(a) $\mathbf{E} = \mathbf{0}$

(b)

$$\begin{aligned}
Q_{\text{enc}} &= \int_0^{2\pi}\int_0^{\pi}\int_a^{r} k\sin\theta\, dr'\, d\theta\, d\phi \\
&= 4\pi k(r-a) \\
4\pi r^2 E &= \frac{4\pi k(r-a)}{\epsilon_0} \\
\mathbf{E} &= \frac{k(r-a)}{\epsilon_0 r^2}\hat{\mathbf{r}}
\end{aligned}$$

(c) $\mathbf{E} = \frac{k(b-a)}{\epsilon_0 r^2}\hat{\mathbf{r}}$

2.16

(a)

$$\begin{aligned}
Q_{\text{enc}} &= \pi s^2 l\rho \\
2\pi s l E &= \frac{\pi s^2 l \rho}{\epsilon_0} \\
\mathbf{E} &= \frac{s\rho}{2\epsilon_0}\hat{\mathbf{s}}
\end{aligned}$$

(b)

$$\mathbf{E} = \frac{a^2\rho}{2\epsilon_0 s}\hat{\mathbf{s}}$$

(c)

$$\mathbf{E} = \mathbf{0}$$

2.17

$$\begin{aligned}
2AE_{\text{inside}} &= \frac{2Ay\rho}{\epsilon_0} \\
\mathbf{E}_{\text{inside}} &= \frac{y\rho}{\epsilon_0} \\
\mathbf{E} &= \begin{cases} \frac{d\rho}{\epsilon_0} & d < y \\ \frac{y\rho}{\epsilon_0} & 0 < y < d \\ -\frac{y\rho}{\epsilon_0} & -d < y < 0 \\ -\frac{d\rho}{\epsilon_0} & y < -d \end{cases}
\end{aligned}$$

2.18

The electric field inside a uniformly charged solid sphere is

$$\mathbf{E} = \frac{r\rho}{3\epsilon_0}\hat{\mathbf{r}}.$$

$$\begin{aligned}
\mathbf{d} &= \mathbf{r}_1 - \mathbf{r}_2 \\
\mathbf{E} &= \frac{r_1\rho}{3\epsilon_0}\hat{\mathbf{r}}_1 - \frac{r_2\rho}{3\epsilon_0}\hat{\mathbf{r}}_2 \\
&= \frac{\rho}{3\epsilon_0}(\mathbf{r}_1 - \mathbf{r}_2) \\
&= \frac{\rho}{3\epsilon_0}\mathbf{d}
\end{aligned}$$

2.20

a is impossible because its curl is nonzero.

$$\begin{aligned}
V &= -\int_0^y 2kxy'\,dy' - \int_0^z 2kyz'\,dz \\
&= -2kx\left[\frac{1}{2}y'^2\right]_0^y - 2ky\left[\frac{1}{2}z'^2\right]_0^z \\
&= -k(xy^2 + yz^2) \\
-\nabla V &= k[y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}] \\
&= \mathbf{E}
\end{aligned}$$

2.21

$$
\begin{aligned}
\mathbf{E} &= \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} & r > R \\ \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3} & r < R \end{cases} \\
V_{\text{outside}}(r) &= -\int_\infty^r \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\,dr' \\
&= -\frac{1}{4\pi\epsilon_0}q\left[-\frac{1}{r'}\right]_\infty^r \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{r} \\
-\nabla V_{\text{outside}} &= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \\
&= \mathbf{E}_{\text{outside}} \\
V_{\text{inside}}(r) &= -\left(\int_\infty^R \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\,dr' + \int_R^r \frac{1}{4\pi\epsilon_0}\frac{qr'}{R^3}\,dr'\right) \\
&= -\left(-\frac{1}{4\pi\epsilon_0}\frac{q}{R} + \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\left[\frac{1}{2}r'^2\right]_R^r\right) \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left[3 - \left(\frac{r}{R}\right)^2\right] \\
-\nabla V_{\text{inside}} &= \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3}\hat{\mathbf{r}} \\
&= \mathbf{E}_{\text{inside}}
\end{aligned}
$$

2.22

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{\mathbf{s}} \\
V &= -\int_O^s \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s'}\,ds' \\
&= -\frac{1}{2\pi\epsilon_0}\lambda\ln\frac{s}{O} \\
-\nabla V &= \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{\mathbf{s}}
\end{aligned}
$$

2.23

$$
\begin{aligned}
\mathbf{E} &= \begin{cases} \mathbf{0} & r < a \\ \frac{k(r-a)}{\epsilon_0 r^2}\hat{\mathbf{r}} & a < r < b \\ \frac{k(b-a)}{\epsilon_0 r^2}\hat{\mathbf{r}} & b < r \end{cases} \\
V(0) &= -\int_{\infty}^{0} E\, dr \\
&= -\left(\int_{\infty}^{b} \frac{k(b-a)}{\epsilon_0 r^2}\, dr + \int_{b}^{a} \frac{k(r-a)}{\epsilon_0 r^2}\, dr\right) \\
&= -\left(\frac{k(b-a)}{\epsilon_0}\left[-\frac{1}{r}\right]_{\infty}^{b} + \frac{k}{\epsilon_0}\left[\ln r + \frac{a}{r}\right]_{b}^{a}\right) \\
&= -\left[-\frac{k(b-a)}{\epsilon_0 b} + \frac{k}{\epsilon_0}\left(\ln a + 1 - \ln b - \frac{a}{b}\right)\right] \\
&= -\frac{k}{\epsilon_0}\left(-1 + \frac{a}{b} + \ln\frac{a}{b} + 1 - \frac{a}{b}\right) \\
&= \frac{k}{\epsilon_0}\ln\frac{b}{a}
\end{aligned}
$$

2.24

$$
\begin{aligned}
V(b) - V(0) &= -\int_{0}^{b} E\, dr \\
&= -\left(\int_{0}^{a} \frac{s\rho}{2\epsilon_0}\, ds + \int_{a}^{b} \frac{a^2\rho}{2\epsilon_0 s}\, ds\right) \\
&= -\left(\frac{\rho}{2\epsilon_0}\left[\frac{1}{2}s^2\right]_{0}^{a} + \frac{a^2\rho}{2\epsilon_0}\ln\frac{b}{a}\right) \\
&= -\left(\frac{a^2\rho}{4\epsilon_0} + \frac{a^2\rho}{2\epsilon_0}\ln\frac{b}{a}\right) \\
&= -\frac{a^2\rho}{4\epsilon_0}\left(1 + 2\ln\frac{a}{b}\right)
\end{aligned}
$$

2.25

(a)

$$
V = \frac{1}{4\pi\epsilon_0}\frac{2q}{\sqrt{(d/2)^2 + z^2}}
$$

(b)

$$
\begin{aligned}
V &= \frac{1}{4\pi\epsilon_0}\int_{-L}^{L}\frac{\lambda}{\sqrt{x^2+z^2}}\,dx \\
&= \frac{1}{4\pi\epsilon_0}\lambda\ln\left(1+\frac{2L(L+\sqrt{L^2+z^2})}{z^2}\right)
\end{aligned}
$$

(c)

$$
\begin{aligned}
V &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{R}\frac{\sigma}{\sqrt{r^2+z^2}}r\,dr\,d\theta \\
&= \frac{1}{4\pi\epsilon_0}2\pi\sigma(\sqrt{R^2+z^2}-z)
\end{aligned}
$$

2.26

$$
\begin{aligned}
V_{\text{bottom}} &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{h}\frac{\sqrt{2}\sigma z}{\sqrt{2}z}\,d\phi\,dz \\
&= \frac{\sigma h}{2\epsilon_0} \\
V_{\text{top}} &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{h}\frac{\sqrt{2}\sigma z}{\sqrt{z^2+(h-z)^2}}\,d\phi\,dz \\
&= \frac{\sqrt{2}\sigma}{2\epsilon_0}\int_0^{h}\frac{z}{\sqrt{z^2+(h-z)^2}}\,dz \\
&= \frac{\sigma h}{4\epsilon_0}\ln(3+2\sqrt{2}) \\
V_{\text{bottom}}-V_{\text{top}} &= \frac{\sigma h}{2\epsilon_0}\left[1-\frac{1}{2}\ln(3+2\sqrt{2})\right]
\end{aligned}
$$

2.28

$$
\begin{aligned}
V(r) &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{\rho r'^2\sin\theta}{\sqrt{r^2+r'^2-2rr'\cos\theta}}\,dr'\,d\theta\,d\phi \\
&= \frac{\rho}{2\epsilon_0}\int_0^{\pi}\int_0^{R}\frac{r'^2\sin\theta}{\sqrt{r^2+r'^2-2rr'\cos\theta}}\,dr'\,d\theta \\
&= \frac{\rho}{2\epsilon_0}\left(R^2-\frac{r^2}{3}\right) \\
&= \frac{q}{8\pi\epsilon_0 R}\left(3-\frac{r^2}{R^2}\right)
\end{aligned}
$$

2.31

(a)

$$W = \frac{q^2}{4\pi\epsilon_0 a}\left(\frac{1}{\sqrt{2}} - 2\right)$$

(b)

$$\begin{aligned}
W &= \frac{1}{4\pi\epsilon_0}\left(-\frac{q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a} - \frac{q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a}\right) \\
&= \frac{q^2}{2\pi\epsilon_0 a}\left(\frac{1}{\sqrt{2}} - 2\right)
\end{aligned}$$

2.32

$$\begin{aligned}
W &= \frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{a} \\
W &= K_1 + K_2 \\
\frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{a} &= \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 \\
\frac{1}{2\pi\epsilon_0}\frac{q_A q_B}{a} &= m_A v_A^2 + m_B v_B^2 \\
0 &= m_B v_B - m_A v_A \\
v_B &= \frac{m_A}{m_B} v_A \\
\frac{1}{2\pi\epsilon_0}\frac{q_A q_B}{a} &= m_A v_A^2 + m_B\left(\frac{m_A}{m_B} v_A\right)^2 \\
&= m_A v_A^2 + \frac{m_A^2}{m_B} v_A^2 \\
&= \frac{m_A(m_A + m_B)}{m_B} v_A^2 \\
v_A &= \sqrt{\frac{1}{2\pi\epsilon_0}\frac{q_A q_B}{(m_A + m_B)a}\frac{m_B}{m_A}} \\
v_B &= \sqrt{\frac{1}{2\pi\epsilon_0}\frac{q_A q_B}{(m_A + m_B)a}\frac{m_A}{m_B}}
\end{aligned}$$

2.33

$$\begin{aligned}
W &= \frac{1}{4\pi\epsilon_0}\left(-\frac{q^2}{a}+\frac{q^2}{2a}-\frac{q^2}{3a}+\dots\right) \\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{a}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} \\
&= -\frac{1}{4\pi\epsilon_0}\frac{q^2}{a}\ln 2
\end{aligned}$$

2.34

(a)

$$\begin{aligned}
V &= \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left[3-\left(\frac{r}{R}\right)^2\right] & r < R \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r} & r > R \end{cases} \\
W &= \frac{1}{2}\int \rho V\, d\tau \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\rho\frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left[3-\left(\frac{r}{R}\right)^2\right] r^2\sin\theta\, dr\, d\theta\, d\phi \\
&= \frac{q\rho}{8\epsilon_0 R}\int_0^{\pi}\int_0^{R}\left[3-\left(\frac{r}{R}\right)^2\right] r^2\sin\theta\, dr\, d\theta \\
&= \frac{q\rho R^2}{5\epsilon_0} \\
&= \frac{qR^2}{5\epsilon_0}\frac{q}{\frac{4}{3}\pi R^3} \\
&= \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}
\end{aligned}$$

(b)

$$\mathbf{E} = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} & r > R \\ \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3}\hat{\mathbf{r}} & r < R \end{cases}$$

$$E^2 = \begin{cases} \frac{1}{16\pi^2\epsilon_0^2}\frac{q^2}{r^4} & r > R \\ \frac{1}{16\pi^2\epsilon_0^2}\frac{q^2r^2}{R^6} & r < R \end{cases}$$

$$\begin{aligned}
W &= \frac{\epsilon_0}{2}\int E^2\,d\tau \\
&= \frac{\epsilon_0}{2}\left(\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{1}{16\pi^2\epsilon_0^2}\frac{q^2r^2}{R^6}r^2\sin\theta\,dr\,d\theta\,d\phi\right. \\
&\qquad \left.+\int_0^{2\pi}\int_0^{\pi}\int_R^{\infty}\frac{1}{16\pi^2\epsilon_0^2}\frac{q^2}{r^4}r^2\sin\theta\,dr\,d\theta\,d\phi\right) \\
&= \frac{\epsilon_0}{2}\frac{1}{16\pi^2\epsilon_0^2}2\pi q^2\left(\int_0^{\pi}\int_0^{R}\frac{r^4}{R^6}\sin\theta\,dr\,d\theta + \int_0^{\pi}\int_R^{\infty}\frac{1}{r^2}\sin\theta\,dr\,d\theta\right) \\
&= \frac{1}{16\pi\epsilon_0}q^2\left(\int_0^{\pi}\int_0^{R}\frac{r^4}{R^6}\sin\theta\,dr\,d\theta + \int_0^{\pi}\int_R^{\infty}\frac{1}{r^2}\sin\theta\,dr\,d\theta\right) \\
&= \frac{1}{16\pi\epsilon_0}q^2\left(\frac{2}{5R}+\frac{2}{R}\right) \\
&= \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}
\end{aligned}$$

(c)

$$\begin{aligned}
W &= \frac{\epsilon_0}{2}\left(\int_V E^2\,d\tau + \oint_S V\mathbf{E}\cdot d\mathbf{a}\right)\\
&= \frac{\epsilon_0}{2}\left(\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{1}{(4\pi\epsilon_0)^2}\frac{q^2r^2}{R^6}r^2\sin\theta\,dr\,d\theta\,d\phi\right.\\
&\qquad+\int_0^{2\pi}\int_0^{\pi}\int_R^{a}\frac{1}{(4\pi\epsilon_0)^2}\frac{q^2}{r^4}r^2\sin\theta\,dr\,d\theta\,d\phi\\
&\qquad\left.+\int_0^{2\pi}\int_0^{\pi}\frac{1}{4\pi\epsilon_0}\frac{q}{a}\frac{1}{4\pi\epsilon_0}\frac{q}{a^2}a^2\sin\theta\,d\theta\,d\phi\right)\\
&= \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}2\pi q^2\left(\int_0^{\pi}\int_0^{R}\frac{r^4}{R^6}\sin\theta\,dr\,d\theta\right.\\
&\qquad\left.+\int_0^{\pi}\int_R^{a}\frac{1}{r^2}\sin\theta\,dr\,d\theta+\int_0^{\pi}\frac{1}{a}\sin\theta\,d\theta\right)\\
&= \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}2\pi q^2\left[\frac{2}{5R}+2\left(\frac{1}{R}-\frac{1}{a}\right)+\frac{2}{a}\right]\\
&= \frac{1}{8\pi\epsilon_0}q^2\left[\frac{1}{5R}+\frac{1}{R}\right]\\
&= \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}
\end{aligned}$$

2.36

(a)

$$\mathbf{E} = \begin{cases}\mathbf{0} & r<a\\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} & a<r<b\\ \mathbf{0} & b<r\end{cases}$$

$$E^2 = \begin{cases}0 & r<a\\ \frac{1}{(4\pi\epsilon_0)^2}\frac{q^2}{r^4} & a<r<b\\ 0 & b<r\end{cases}$$

$$\begin{aligned}
W &= \frac{\epsilon_0}{2}\int E^2\,d\tau\\
&= \frac{\epsilon_0}{2}\int_0^{2\pi}\int_0^{\pi}\int_a^{b}\frac{1}{(4\pi\epsilon_0)^2}\frac{q^2}{r^4}r^2\sin\theta\,dr\,d\theta\,d\phi\\
&= \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}2\pi q^2\int_0^{\pi}\int_a^{b}\frac{\sin\theta}{r^2}\,dr\,d\theta\\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)
\end{aligned}$$

(b)

$$
\begin{aligned}
W_{\text{shell}} &= \frac{1}{8\pi\epsilon_0}\frac{q^2}{R} \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \\
\mathbf{E}_1 \cdot \mathbf{E}_2 &= -\frac{1}{(4\pi\epsilon_0)^2}\frac{q^2}{r^4} \\
W_{\text{total}} &= W_1 + W_2 + \epsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2 \, d\tau \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b}\right) - \epsilon_0 \int_0^{2\pi}\int_0^{\pi}\int_b^{\infty} \frac{1}{(4\pi\epsilon_0)^2}\frac{q^2}{r^4} r^2 \sin\theta \, dr \, d\theta \, d\phi \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b}\right) - \frac{1}{8\pi\epsilon_0} q^2 \int_0^{\pi}\int_b^{\infty} \frac{1}{r^2} \sin\theta \, dr \, d\theta \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b}\right) - \frac{1}{4\pi\epsilon_0} q^2 \int_b^{\infty} \frac{1}{r^2} \, dr \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b}\right) - \frac{1}{4\pi\epsilon_0}\frac{q^2}{b} \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{b}\right) \\
&= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)
\end{aligned}
$$

2.37

$$
\begin{aligned}
r_1 &= r \\
E_1 &= \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1^2} \\
&= \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2} \\
r_2 &= \sqrt{a^2 + r^2 - 2ar\cos\theta} \\
E_2 &= \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_2^2} \\
&= \frac{1}{4\pi\epsilon_0}\frac{q_2}{a^2 + r^2 - 2ar\cos\theta} \\
\cos\alpha &= \frac{r - a\cos\theta}{\sqrt{a^2 + r^2 - 2ar\cos\theta}} \\
\mathbf{E}_1 \cdot \mathbf{E}_2 &= E_1 E_2 \cos\alpha \\
&= \frac{1}{(4\pi\epsilon_0)^2}\frac{q_1 q_2}{r^2(a^2 + r^2 - 2ar\cos\theta)}\frac{r - a\cos\theta}{\sqrt{a^2 + r^2 - 2ar\cos\theta}} \\
&= \frac{1}{(4\pi\epsilon_0)^2}\frac{q_1 q_2(r - a\cos\theta)}{r^2(a^2 + r^2 - 2ar\cos\theta)^{3/2}} \\
\epsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2 \, d\tau &= \epsilon_0 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{(4\pi\epsilon_0)^2}\frac{q_1 q_2(r - a\cos\theta)}{r^2(a^2 + r^2 - 2ar\cos\theta)^{3/2}} r^2 \sin\theta \, dr \, d\theta \, d\phi \\
&= \frac{q_1 q_2}{8\pi\epsilon_0}\int_0^{\pi}\int_0^{\infty} \frac{(r - a\cos\theta)\sin\theta}{(a^2 + r^2 - 2ar\cos\theta)^{3/2}} \, dr \, d\theta
\end{aligned}
$$

2.38

(a)

$$
\begin{aligned}
\sigma_R &= \frac{q}{4\pi R^2} \\
\sigma_a &= -\frac{q}{4\pi a^2} \\
\sigma_b &= \frac{q}{4\pi b^2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
V &= -\int_{\infty}^{b} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \, dr - \int_a^R \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \, dr \\
&= \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{b} + \frac{1}{R} - \frac{1}{a}\right)
\end{aligned}
$$

(c)

$$\sigma_b = 0$$
$$V = \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{R} - \frac{1}{a} \right)$$

2.39

(a)

$$\sigma_a = -\frac{q_a}{4\pi a^2}$$
$$\sigma_b = -\frac{q_b}{4\pi b^2}$$
$$\sigma_R = \frac{q_a + q_b}{4\pi R^2}$$

(b)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q_a + q_b}{r^2} \hat{\mathbf{r}}$$

(c)

$$\mathbf{E}_a = \frac{1}{4\pi\epsilon_0} \frac{q_a}{r^2} \hat{\mathbf{r}}$$
$$\mathbf{E}_b = \frac{1}{4\pi\epsilon_0} \frac{q_b}{r^2} \hat{\mathbf{r}}$$

(d)

$$\mathbf{0}$$

(e) a, b

2.40

(a) No. If it's close to the wall it will induce a surface charge and be attracted.

(b) No. If the conductor contains a cavity containing a like charge it will be repelled.

2.41

By Gauss's law, the electric field of each plate is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$
$$2A'E = \frac{A' \frac{Q}{A}}{\epsilon_0}$$
$$\mathbf{E} = \frac{Q}{2A\epsilon_0} \hat{\mathbf{n}}$$

so the field between the plates is zero and the field outside is $Q/A\epsilon_0\hat{\mathbf{n}}$, resulting in a pressure of

$$\begin{aligned} P &= \frac{\epsilon_0}{2}E^2 \\ &= \frac{\epsilon_0}{2}\frac{Q^2}{A^2\epsilon_0^2} \\ &= \frac{Q^2}{2A^2\epsilon_0} \end{aligned}$$

2.42

$$\begin{aligned} \mathbf{E}_{\text{above}} &= \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \\ \mathbf{f} &= \frac{1}{2}\sigma\mathbf{E}_{\text{above}} \\ &= \frac{1}{2}\frac{Q}{4\pi R^2}\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\hat{\mathbf{r}} \\ &= \frac{Q^2}{32\pi^2\epsilon_0 R^4}\hat{\mathbf{r}} \\ \mathbf{F} &= \int_0^{2\pi}\int_0^{\pi/2}\frac{Q^2}{32\pi^2\epsilon_0 R^4}\cos\theta R^2\sin\theta\, d\theta\, d\phi\hat{\mathbf{z}} \\ &= \frac{Q^2}{16\pi\epsilon_0 R^2}\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta\hat{\mathbf{z}} \\ &= \frac{Q^2}{32\pi\epsilon_0 R^2}\hat{\mathbf{z}} \end{aligned}$$

2.43

$$
\begin{aligned}
\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q}{\epsilon_0} \\
2\pi s L E &= \frac{Q}{\epsilon_0} \\
\mathbf{E} &= \frac{Q}{2\pi L \epsilon_0} \frac{1}{s} \hat{\mathbf{s}} \\
V &= -\int_b^a \frac{Q}{2\pi \epsilon_0 L} \frac{1}{s} \, dr \\
&= \frac{Q}{2\pi \epsilon_0 L} \ln \frac{b}{a} \\
C &= \frac{Q}{V} \\
&= \frac{2\pi \epsilon_0 L}{\ln b/a}
\end{aligned}
$$

So the capacitance per unit length is

$$
C = \frac{2\pi \epsilon_0}{\ln b/a}.
$$

2.44

(a)

$$
\begin{aligned}
P &= \frac{\epsilon_0}{2} E^2 \\
W &= F d \\
&= P A \epsilon \\
&= \frac{\epsilon_0}{2} E^2 A \epsilon
\end{aligned}
$$

(b)

$$
\frac{\epsilon_0}{2} E^2 A \epsilon
$$

2.46

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 3\frac{k}{r}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{k}{r}2\sin\theta\cos\theta\sin\phi\right) \\
&\quad + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{k}{r}\sin\theta\cos\phi\right) \\
&= \frac{3k}{r^2} + \frac{1}{r\sin\theta}\frac{2k}{r}\sin\phi(2\sin\theta\cos^2\theta - \sin^3\theta) - \frac{1}{r\sin\theta}\frac{k}{r}\sin\theta\sin\phi \\
&= \frac{3k}{r^2} + \frac{2k\sin\phi}{r^2}(2\cos^2\theta - \sin^2\theta) - \frac{k}{r^2}\sin\phi \\
&= \frac{k}{r^2}[3 + 2\sin\phi(2\cos^2\theta - \sin^2\theta) - \sin\phi] \\
&= \frac{k}{r^2}[3 + \sin\phi(4\cos^2\theta - 2\sin^2\theta - 1)] \\
&= \frac{k}{r^2}[3 + \sin\phi(6\cos^2\theta - 3)] \\
&= \frac{3k}{r^2}(1 + \cos 2\theta\sin\phi) \\
\rho &= \epsilon_0 \nabla\cdot\mathbf{E} \\
&= \frac{3k\epsilon_0}{r^2}(1 + \cos 2\theta\sin\phi)
\end{aligned}
$$

2.47

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}\hat{\mathbf{r}} \\
\rho &= \frac{Q}{\frac{4}{3}\pi R^3} \\
\rho\mathbf{E} &= \frac{3Q}{4\pi R^3}\frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}\hat{\mathbf{r}} \\
&= \frac{3r}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2\hat{\mathbf{r}} \\
F_z &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^R \frac{3r}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 \cos\theta r^2\sin\theta\, dr\, d\theta\, d\phi \\
&= \frac{3\pi}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 \int_0^{\pi/2}\int_0^R r^3\sin 2\theta\, dr\, d\theta \\
&= \frac{3\pi}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 \frac{R^4}{4} \\
&= \frac{3Q^2}{64\pi\epsilon_0 R^2}
\end{aligned}
$$

2.49

$$
\begin{aligned}
Q_{\text{enc}} &= \int_0^{2\pi}\int_0^{\pi}\int_0^{r} kr'^3 \sin\theta\, dr'\, d\theta\, d\phi \\
&= 2\pi k \int_0^{\pi}\int_0^{r} r'^3 \sin\theta\, dr'\, d\theta \\
&= \pi k r^4 \\
\oint \mathbf{E}\cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E &= \frac{\pi k r^4}{\epsilon_0} \\
\mathbf{E} &= \begin{cases} \frac{kr^2}{4\epsilon_0}\hat{\mathbf{r}} & r < R \\ \frac{kR^4}{4\epsilon_0 r^2}\hat{\mathbf{r}} & r > R \end{cases} \\
W &= \frac{\epsilon_0}{2}\left(\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{k^2 r^4}{16\epsilon_0^2} r^2 \sin\theta\, dr\, d\theta\, d\phi\right. \\
&\qquad \left.\int_0^{2\pi}\int_0^{\pi}\int_R^{\infty} \frac{k^2 R^8}{16\epsilon_0^2 r^4} r^2 \sin\theta\, dr\, d\theta\, d\phi\right) \\
&= \frac{\epsilon_0}{2} 2\pi \frac{k^2}{16\epsilon_0^2}\left(\int_0^{\pi}\int_0^{R} r^6 \sin\theta\, dr\, d\theta + \int_0^{\pi}\int_R^{\infty} \frac{R^8 \sin\theta}{r^2}\, dr\, d\theta\right) \\
&= \frac{\pi k^2}{16\epsilon_0}\left(\frac{2R^7}{7} + 2R^7\right) \\
&= \frac{\pi k^2 R^7}{7\epsilon_0}
\end{aligned}
$$

2.50

$$
\begin{aligned}
V(\mathbf{r}) &= A\frac{e^{-\lambda r}}{r} \\
\mathbf{E} &= -\nabla V \\
&= Ae^{-\lambda r}(1+\lambda r)\frac{\hat{\mathbf{r}}}{r^2} \\
\rho &= \epsilon_0 \nabla\cdot\mathbf{E} \\
&= \epsilon_0\left[Ae^{-\lambda r}(1+\lambda r)\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2} + \frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla\left(Ae^{-\lambda r}(1+\lambda r)\right)\right] \\
&= A\epsilon_0\left[4\pi\delta(\mathbf{r}) + \frac{\hat{\mathbf{r}}}{r^2}\cdot(-\lambda^2 e^{-\lambda r} r\hat{\mathbf{r}})\right] \\
&= A\epsilon_0\left(4\pi\delta(\mathbf{r}) - \frac{\lambda^2 e^{-\lambda r}}{r}\right)
\end{aligned}
$$

2.51

$$
\begin{aligned}
V &= \int \frac{1}{4\pi\epsilon_0}\frac{\sigma}{\mathscr{r}}\,dA \\
&= \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{R}\frac{r}{\sqrt{r^2+R^2-2rR\cos\theta}}\,dr\,d\theta \\
&= \frac{R\sigma}{4\pi\epsilon_0}\int_0^{2\pi}\left[\cos\theta\ln\left(1+\csc\frac{\theta}{2}\right)+2\sin\frac{\theta}{2}-1\right]\,d\theta \\
&= \frac{R\sigma}{\pi\epsilon_0}
\end{aligned}
$$

2.52

(a)

$$
\begin{aligned}
V_- &= \frac{1}{2\pi\epsilon_0}\lambda\ln\frac{s_-}{a} \\
&= \frac{1}{2\pi\epsilon_0}\lambda\ln\frac{\sqrt{(y+a)^2+z^2}}{a} \\
V_+ &= -\frac{1}{2\pi\epsilon_0}\lambda\ln\frac{s_+}{a} \\
&= -\frac{1}{2\pi\epsilon_0}\lambda\ln\frac{\sqrt{(y-a)^2+z^2}}{a} \\
V &= V_- + V_+ \\
&= \frac{1}{4\pi\epsilon_0}\lambda\ln\frac{(y+a)^2+z^2}{(y-a)^2+z^2}
\end{aligned}
$$

2.53

(a)

$$
\begin{aligned}
\nabla^2 V &= -\frac{\rho}{\epsilon_0} \\
\nabla\cdot\nabla V &= -\frac{\rho}{\epsilon_0} \\
\nabla\cdot\frac{dV}{dx}\hat{\mathbf{x}} &= -\frac{\rho}{\epsilon_0} \\
\frac{d^2V}{dx^2} &= -\frac{\rho}{\epsilon_0}
\end{aligned}
$$

(b)

$$qV = \frac{1}{2}mv^2$$
$$v = \sqrt{\frac{2qV}{m}}$$

(c)

$$I = A\rho v$$

(d)

$$\begin{aligned}\frac{d^2V}{dx^2} &= -\frac{I}{Av\epsilon_0}\\ &= -\frac{I}{A\epsilon_0}\sqrt{\frac{m}{2qV}}\\ &= \beta V^{-1/2}\end{aligned}$$

2.55

$$\begin{aligned}\rho &= \epsilon_0 \nabla \cdot \mathbf{E}\\ &= a\epsilon_0\end{aligned}$$

2.56

$$\begin{aligned}E &= \frac{3GM^2}{5R}\\ E_{\text{sun}} &= 2.3 \times 10^{41}\,\text{J}\\ t &= \frac{E_{\text{sun}}}{P}\\ &= 1.89 \times 10^{7}\,\text{years}\end{aligned}$$

3 Potentials

3.1

$$\begin{aligned}
V_{\text{ave}} &= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\left.\sqrt{z^2+R^2-2zR\cos\theta}\right|_0^\pi \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\left(\sqrt{z^2+R^2+2zR}-\sqrt{z^2+R^2-2zR}\right) \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\left(\sqrt{(z+R)^2}-\sqrt{(R-z)^2}\right) \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}(z+R-R+z) \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{R}
\end{aligned}$$

The average potential due to external charges is V_{center} and the average potential due to internal charges is

$$\frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{R}$$

so

$$V_{\text{ave}} = V_{\text{center}} + \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{R}.$$

3.2

A stable equilibrium is a minimum of potential energy. Laplace's equation doesn't allow for minimums, so they must be saddle points and the charge can escape.

3.3

$$
\begin{aligned}
0 &= \nabla^2 V \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) \\
&= \frac{1}{r^2}\left(2r\frac{\partial V}{\partial r} + r^2\frac{\partial^2 V}{\partial r^2}\right) \\
&= \frac{2}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} \\
V &= \frac{c_1}{r} + c_2 \\
0 &= \nabla^2 V \\
&= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) \\
&= \frac{1}{s}\left(\frac{\partial V}{\partial s} + s\frac{\partial^2 V}{\partial s^2}\right) \\
&= \frac{1}{s}\frac{\partial V}{\partial s} + \frac{\partial^2 V}{\partial s^2} \\
V &= c_1 + c_2 \ln s
\end{aligned}
$$

3.7

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} q^2 \left(-\frac{2}{(2d)^2} + \frac{2}{(4d)^2} - \frac{1}{(6d)^2}\right)\hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0}\frac{29q^2}{72d^2}\hat{\mathbf{z}}
\end{aligned}
$$

3.8

(a)

$$
\begin{aligned}
V(r,\theta) &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} + \frac{q'}{\sqrt{r^2+b^2-2rb\cos\theta}}\right] \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{Rq/a}{\sqrt{r^2+(R^2/a)^2-2r(R^2/a)\cos\theta}}\right] \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{q}{\sqrt{R^2+(ra/R)^2-2ra\cos\theta}}\right]
\end{aligned}
$$

(b)

$$\begin{aligned}
\sigma &= -\epsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=R} \\
&= \frac{q}{4\pi R}\frac{R^2-a^2}{(a^2+R^2-2aR\cos\theta)^{3/2}} \\
Q_{\text{induced}} &= \int_0^{2\pi}\int_0^{\pi} \sigma R^2 \sin\theta \, d\theta \, d\phi \\
&= \frac{qR(R^2-a^2)}{2}\int_0^{\pi} \frac{\sin\theta}{(a^2+R^2-2aR\cos\theta)^{3/2}} \, d\theta \\
&= \frac{qR(R^2-a^2)}{a(a^2-R^2)} \\
&= -\frac{R}{a}q \\
&= q'
\end{aligned}$$

(c)

$$\begin{aligned}
W &= \frac{1}{2}qV \\
&= \frac{1}{8\pi\epsilon_0}\frac{qq'}{a-b} \\
&= -\frac{1}{8\pi\epsilon_0}\frac{q^2R/a}{a-R^2/a} \\
&= -\frac{1}{8\pi\epsilon_0}\frac{q^2R}{a^2-R^2}
\end{aligned}$$

3.9

Place the second image charge at the centre of the sphere with charge

$$q'' = 4\pi\epsilon_0 R V_0.$$

$$
\begin{aligned}
F &= \frac{1}{4\pi\epsilon_0} q \left(\frac{q'}{(a-b)^2} + \frac{q''}{a^2} \right) \\
&= \frac{qq'}{4\pi\epsilon_0} \left(\frac{1}{(a-b)^2} - \frac{1}{a^2} \right) \\
&= \frac{qq'}{4\pi\epsilon_0} \frac{a^2 - (a-b)^2}{a^2(a-b)^2} \\
&= \frac{qq'}{4\pi\epsilon_0} \frac{b(2a-b)}{a^2(a-b)^2} \\
&= \frac{q(-Rq/a)}{4\pi\epsilon_0} \frac{(R^2/a)(2a - R^2/a)}{a^2(a - R^2/a)^2} \\
&= -\frac{q^2}{4\pi\epsilon_0} \left(\frac{R}{a} \right)^3 \frac{2a^2 - R^2}{(a^2 - R^2)^2}
\end{aligned}
$$

3.10

(a)

$$
V(x,y,z) = \frac{1}{4\pi\epsilon_0} \lambda \ln \frac{y^2 + (z+d)^2}{y^2 + (z-d)^2}
$$

(b)

$$
\begin{aligned}
\sigma &= -\epsilon_0 \left. \frac{\partial V}{\partial z} \right|_{z=0} \\
&= -\frac{d\lambda}{\pi(d^2 + y^2)}
\end{aligned}
$$

3.11

You need three charges: $-q$ at $(-a,b)$, $-q$ at $(a,-b)$, and q at $(-b,-a)$. The potential is

$$
V = \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{\sqrt{(x-a)^2 + (y-b)^2}} - \frac{1}{\sqrt{(x+a)^2 + (y-b)^2}} \right.
$$
$$
\left. - \frac{1}{\sqrt{(x-a)^2 + (y+b)^2}} + \frac{1}{\sqrt{(x+a)^2 + (y+b)^2}} \right).
$$

The force on q is

$$
\mathbf{F} = \frac{q^2}{16\pi\epsilon_0} \left[\left(\frac{a}{(a^2+b^2)^{3/2}} - \frac{1}{a^2} \right) \hat{\mathbf{x}} + \left(\frac{b}{(a^2+b^2)^{3/2}} - \frac{1}{b^2} \right) \hat{\mathbf{y}} \right].
$$

The work to bring q in from infinity is

$$W = \frac{q^2}{16\pi\epsilon_0}\left(\frac{1}{\sqrt{a^2+b^2}} - \frac{1}{a} - \frac{1}{b}\right).$$

3.12

Two infinitely long wires running parallel to the x-axis a distance $2a$ apart with charge densities λ and $-\lambda$ have cylindrical equipotential surfaces with centres at

$$y_0 = \pm a \coth \frac{2\pi\epsilon_0 V_0}{\lambda}$$

radii

$$R = a \operatorname{csch} \frac{2\pi\epsilon_0 V_0}{\lambda}.$$

We know the equipotential surfaces (the pipes) and want to find the wires so we can find the potential, so

$$\begin{aligned}
d &= a \coth \frac{2\pi\epsilon_0 V_0}{\lambda} \\
R &= a \operatorname{csch} \frac{2\pi\epsilon_0 V_0}{\lambda} \\
\frac{d}{R} &= \cosh \frac{2\pi\epsilon_0 V_0}{\lambda} \\
\operatorname{arcosh} \frac{d}{R} &= \frac{2\pi\epsilon_0 V_0}{\lambda} \\
\lambda &= \frac{2\pi\epsilon_0 V_0}{\operatorname{arcosh} d/R} \\
R &= a \operatorname{csch} \operatorname{arcosh} \frac{d}{R} \\
a &= \frac{R}{\operatorname{csch} \operatorname{arcosh} d/R} \\
&= (d+R)\sqrt{\frac{2d}{d+R} - 1} \\
&= \sqrt{d^2 - R^2}
\end{aligned}$$

thus the potential is

$$V = \frac{V_0}{2 \operatorname{arcosh} d/R} \ln \frac{(y + d^2 - R^2)^2 + z^2}{(y - d^2 + R^2)^2 + z^2}.$$

3.13

$$
\begin{aligned}
V_0(y) &= \begin{cases} V_0 & 0 \le y \le \frac{a}{2} \\ -V_0 & \frac{a}{2} \le y \le a \end{cases} \\
C_n &= \frac{2}{a}\left(\int_0^{a/2} V_0 \sin\frac{n\pi y}{a}\,dy - \int_{a/2}^{a} V_0 \sin\frac{n\pi y}{a}\,dy\right) \\
&= \frac{2V_0}{n\pi}\left(\cos\frac{n\pi y}{a}\bigg|_{a/2}^{a} - \cos\frac{n\pi y}{a}\bigg|_{0}^{a/2}\right) \\
&= \frac{2V_0}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2} - \cos\frac{n\pi}{2} + 1\right) \\
&= \frac{2V_0}{n\pi}\left(1 + (-1)^n - 2\cos\frac{n\pi}{2}\right) \\
&= \frac{2V_0}{n\pi}\begin{cases} 0 & n \text{ is odd or divisible by } 4 \\ 4 & \text{otherwise} \end{cases} \\
V &= \frac{8V_0}{\pi}\sum_{n=2,6,10,\ldots}^{\infty} \frac{1}{n} e^{-n\pi x/a} \sin\frac{n\pi y}{a}
\end{aligned}
$$

3.14

$$
\begin{aligned}
\sigma &= -\epsilon_0 \frac{\partial V}{\partial x} \\
&= \frac{4\epsilon_0 V_0 \sin\frac{\pi y}{a}}{a\left(1 - \cos\frac{2\pi y}{a}\right)} \\
&= \frac{2\epsilon_0 V_0}{a}\frac{1}{\sin \pi y/a}
\end{aligned}
$$

3.15

(a)

$$
\begin{aligned}
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} &= 0 \\
V(0,y) &= 0 \\
V(b,y) &= V_0(y) \\
V(x,0) &= 0 \\
V(x,a) &= 0 \\
V &= X(x)Y(y) \\
X''Y + XY'' &= 0 \\
\frac{X''}{X} + \frac{Y''}{Y} &= 0 \\
\frac{Y''}{Y} &= -\alpha^2 \\
Y'' + \alpha^2 Y &= 0 \\
Y &= c_1 \cos \alpha y + c_2 \sin \alpha y \\
Y &= c_2 \sin \alpha y \\
Y &= c_2 \sin \frac{n\pi y}{a}, n \in \mathbb{R} \\
\frac{X''}{X} &= \alpha^2 \\
X'' - \alpha^2 X &= 0 \\
X &= c_3 \cosh \alpha x + c_4 \sinh \alpha x \\
X &= c_4 \sinh \alpha x \\
&= c_4 \sinh \frac{n\pi x}{a}, n \in \mathbb{R} \\
V &= \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi x}{a} \sin \frac{n\pi y}{a} \\
V_0(y) &= V(b,y) \\
&= \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi y}{a} \\
C_n \sinh \frac{n\pi b}{a} &= \frac{2}{a} \int_0^a V_0(y) \sin \frac{n\pi y}{a}\, dy \\
C_n &= \frac{2}{a \sinh n\pi b/a} \int_0^a V_0(y) \sin \frac{n\pi y}{a}\, dy
\end{aligned}
$$

(b)

$$
\begin{aligned}
C_n &= \frac{2V_0}{a\sinh n\pi b/a}\int_0^a \sin\frac{n\pi y}{a}\,dy \\
&= \frac{2V_0}{a\sinh n\pi b/a}\,\frac{a[1-(-1)^n]}{n\pi} \\
&= \frac{2V_0[1-(-1)^n]}{n\pi\sinh n\pi b/a} \\
V &= \frac{2V_0}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n\sinh n\pi b/a}\sinh\frac{n\pi x}{a}\sin\frac{n\pi y}{a}
\end{aligned}
$$

3.16

$$
\begin{aligned}
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} &= 0 \\
V(0, y, z) &= 0 \\
V(a, y, z) &= 0 \\
V(x, 0, z) &= 0 \\
V(x, a, z) &= 0 \\
V(x, y, 0) &= 0 \\
V(x, y, a) &= V_0 \\
V &= X(x)Y(y)Z(z) \\
X''YZ + XY''Z + XYZ'' &= 0 \\
\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} &= 0 \\
\frac{X''}{X} &= -\alpha^2 \\
\frac{Y''}{Y} &= -\beta^2 \\
\frac{Z''}{Z} &= \alpha^2 + \beta^2 \\
X'' + \alpha^2 X &= 0 \\
X &= c_1 \cos \alpha x + c_2 \sin \alpha x \\
X &= c_2 \sin \alpha x \\
X &= c_2 \sin \frac{n\pi x}{a}, n \in \mathbb{R} \\
\frac{Y''}{Y} &= -\beta^2 \\
Y'' + \beta^2 Y &= 0 \\
Y &= c_3 \cos \beta y + c_4 \sin \beta y \\
Y &= c_4 \sin \beta y \\
Y &= c_4 \sin \frac{m\pi y}{a}, m \in \mathbb{R} \\
\frac{Z''}{Z} &= \alpha^2 + \beta^2 \\
Z'' - (\alpha^2 + \beta^2)Z &= 0 \\
Z &= c_5 \cosh \sqrt{\alpha^2 + \beta^2} z + c_6 \sinh \sqrt{\alpha^2 + \beta^2} z \\
&= c_5 \cosh \pi \sqrt{(n/a)^2 + (m/a)^2} z \\
&\quad + c_6 \sinh \pi \sqrt{(n/a)^2 + (m/a)^2} z \\
Z &= c_6 \sinh \pi \sqrt{(n/a)^2 + (m/a)^2} z
\end{aligned}
$$

$$
\begin{aligned}
V &= \sum_{n=1}^{\infty} \sum_{m=1^{\infty}} C_{n,m} \sinh\left(\pi\sqrt{(n/a)^2 + (m/a)^2} z\right) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a} \\
V_0 &= V(x, y, a) \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{n,m} \sinh\left(\pi\sqrt{n^2 + m^2}\right) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a} \\
C_{n,m} &= \frac{4V_0}{a^2 \sinh\left(\pi\sqrt{n^2+m^2}\right)} \int_0^a \int_0^a \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a}\, dy\, dx \\
&= \frac{4V_0}{a^2 \sinh\left(\pi\sqrt{n^2+m^2}\right)} \frac{a^2[-1+(-1)^m][-1+(-1)^n]}{nm\pi^2} \\
&= \frac{4V_0[-1+(-1)^n][-1+(-1)^m]}{nm\pi^2 \sinh\left(\pi\sqrt{n^2+m^2}\right)} \\
&= \begin{cases} 0 & n \text{ even or } m \text{ even} \\ \frac{16V_0}{nm\pi^2 \sinh\left(\pi\sqrt{n^2+m^2}\right)} & \text{otherwise} \end{cases} \\
V &= \frac{16V_0}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{nm} \frac{\sinh\left(\pi\sqrt{n^2+m^2} z/a\right)}{\sinh\left(\pi\sqrt{n^2+m^2}\right)} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a}
\end{aligned}
$$

3.17

$$
\begin{aligned}
P_3(x) &= \frac{1}{2^3 3!}\left(\frac{d}{dx}\right)^3 (x^2-1)^3 \\
&= \frac{1}{48}\frac{d^3}{dx^3}\left[(x^2-1)^3\right] \\
&= \frac{1}{48}\frac{d^2}{dx^2}\left[6x(x^2-1)^2\right] \\
&= \frac{1}{48}\frac{d}{dx}\left[6(x^2-1)^2 + 24x^2(x^2-1)\right] \\
&= \frac{1}{48}\left[24x(x^2-1) + 48x(x^2-1) + 48x^3\right] \\
&= \frac{1}{48}\left(24x^3 - 24x + 48x^3 - 48x + 48x^3\right) \\
&= \frac{120}{48}x^3 - \frac{72}{48}x \\
&= \frac{5}{2}x^3 - \frac{3}{2}x \\
\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) &= -12\sin\theta\Theta \\
\Theta &= \frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta \\
\frac{d\Theta}{d\theta} &= -\frac{15}{2}\cos^2\theta\sin\theta + \frac{3}{2}\sin\theta \\
\sin\theta\frac{d\Theta}{d\theta} &= -\frac{15}{2}\cos^2\theta\sin^2\theta + \frac{3}{2}\sin^2\theta \\
\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) &= \frac{3}{2}(1-5\cos 2\theta)\sin 2\theta \\
\frac{3}{2}(1-5\cos 2\theta)\sin 2\theta &= -12\sin\theta\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) \\
3(1-5\cos 2\theta)\sin 2\theta &= 12\sin\theta\cos\theta(3-5\cos^2\theta) \\
&= 6(3-5\cos^2\theta)\sin 2\theta \\
&= 6\left(3 - 5\frac{1+\cos 2\theta}{2}\right)\sin 2\theta \\
&= 3(6-5-5\cos 2\theta)\sin 2\theta \\
&= 3(1-5\cos 2\theta)\sin 2\theta \\
\int_{-1}^{1} P_1(x)P_3(x)\,dx &= \int_{-1}^{1} x\left(\frac{5}{2}x^3 - \frac{3}{2}x\right)\,dx \\
&= \left[\frac{1}{2}x^5 - \frac{1}{2}x^3\right]_{-1}^{1} \\
&= \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \\
&= 0
\end{aligned}
$$

3.18

(a)

$$
\begin{aligned}
A_l &= \frac{V_0(2l+1)}{2R^l}\int_0^\pi P_l(\cos\theta)\sin\theta\,d\theta \\
&= \begin{cases} V_0 & l=0 \\ 0 & l\neq 0 \end{cases} \\
V(r,\theta) &= \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \\
&= V_0 \\
B_l &= \frac{V_0(2l+1)}{2}R^{l+1}\int_0^\pi P_l(\cos\theta)\sin\theta\,d\theta \\
&= \begin{cases} V_0 R & l=0 \\ 0 & l\neq 0 \end{cases} \\
V(r,\theta) &= \frac{V_0 R}{r}
\end{aligned}
$$

(b)

$$
\begin{aligned}
A_l &= \frac{\sigma_0}{2\epsilon_0 R^{l-1}}\int_0^\pi P_l(\cos\theta)\sin\theta\,d\theta \\
&= \begin{cases} \frac{R\sigma_0}{\epsilon_0} & l=0 \\ 0 & l\neq 0 \end{cases} \\
V(r,\theta) &= \frac{R\sigma_0}{\epsilon_0} \\
B_l &= A_l R^{2l+1} \\
&= \begin{cases} \frac{R^2\sigma_0}{\epsilon_0} & l=0 \\ 0 & l\neq 0 \end{cases} \\
V(r,\theta) &= \frac{R^2\sigma_0}{\epsilon_0 r}
\end{aligned}
$$

3.19

$$\begin{aligned}
V_0 &= k\cos 3\theta \\
A_l &= \frac{k(2l+1)}{2R^l}\int_0^\pi \cos 3\theta P_l(\cos\theta)\sin\theta\,d\theta \\
&= \begin{cases} -\frac{3k}{5R} & l=1 \\ \frac{8k}{5R^3} & l=3 \\ 0 & \text{otherwise} \end{cases} \\
V(r,\theta) &= -\frac{3k}{5R} r P_1(\cos\theta) + \frac{8k}{5R^3} r^3 P_3(\cos\theta) \\
&= \frac{kr}{5R}\left[-3P_1(\cos\theta) + \frac{8}{R^2} r^2 P_3(\cos\theta)\right] \\
B_l &= \frac{k(2l+1)}{2} R^{l+1}\int_0^\pi \cos 3\theta P_l(\cos\theta)\sin\theta\,d\theta \\
&= \begin{cases} -\frac{3kR^2}{5} & l=1 \\ \frac{8kR^4}{5} & l=3 \\ 0 & \text{otherwise} \end{cases} \\
V(r,\theta) &= -\frac{3kR^2}{5r^2} P_1(\cos\theta) + \frac{8kR^4}{5r^4} P_3(\cos\theta) \\
&= \frac{kR^2}{5r^2}\left[\frac{8R^2}{r^2} P_3(\cos\theta) - 3P_1(\cos\theta)\right] \\
\sigma(\theta) &= -\epsilon_0\left(\frac{\partial V_{\text{above}}}{\partial r} - \frac{\partial V_{\text{below}}}{\partial r}\right) \\
&= \frac{\epsilon_0 k(12\cos\theta + 35\cos 3\theta)}{5R}
\end{aligned}$$

3.20

$$\begin{aligned}
V(r,\theta) &= \begin{cases} \sum_{l=0}^\infty \frac{2l+1}{2}\frac{r^l}{R^l}\left(\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta\right) P_l(\cos\theta) & r <= R \\ \sum_{l=0}^\infty \frac{2l+1}{2}\frac{R^{l+1}}{r^{l+1}}\left(\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta\right) P_l(\cos\theta) & r >= R \end{cases} \\
\sigma_0 &= -\epsilon_0\left.\left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\right)\right|_{r=R} \\
&= \frac{\epsilon_0}{2R}\sum_{l=0}^\infty (2l+1)^2 C_l P_l(\cos\theta)
\end{aligned}$$

3.21

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} - E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta$$

3.22

(a)

$$
\begin{aligned}
\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} &= \frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2}-r\right) \\
&= \frac{\sigma r}{2\epsilon_0}\left(\sqrt{1+(R/r)^2}-1\right) \\
&= \frac{\sigma r}{2\epsilon_0}\left[\left(1+\frac{(R/r)^2}{2}-\frac{(R/r)^4}{8}+\dots\right)-1\right] \\
&= \frac{\sigma}{2\epsilon_0}\left(\frac{R^2}{2r}-\frac{R^4}{8r^3}+\dots\right) \\
B_0 &= \frac{\sigma R^2}{4\epsilon_0} \\
B_1 &= 0 \\
B_2 &= -\frac{\sigma R^4}{16\epsilon_0}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\sum_{l=0}^{\infty} A_l r^l &= \frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2}-r\right) \\
&= \frac{\sigma}{2\epsilon_0}\left(R\sqrt{1+(r/R)^2}-r\right) \\
&= \frac{\sigma}{2\epsilon_0}\left[R\left(1+\frac{(r/R)^2}{2}-\frac{(r/R)^4}{8}+\dots\right)-r\right] \\
&= \frac{\sigma}{2\epsilon_0}\left(R-r+\frac{r^2}{2R}-\frac{r^4}{8R^3}+\dots\right) \\
A_0 &= \frac{\sigma R}{2\epsilon_0} \\
A_1 &= -\frac{\sigma}{2\epsilon_0} \\
A_2 &= \frac{\sigma}{4\epsilon_0 R} \\
A_0' &= A_0 \\
A_1' &= -A_1 \\
A_2' &= A_2
\end{aligned}
$$

3.23

$$
\begin{aligned}
V(r,\theta) &= \sum_{l=1}^{\infty} A_l r^l P_l(\cos\theta) \\
A_l &= \frac{\sigma_0}{2\epsilon_0 R^{l-1}} \left(\int_0^{\pi/2} P_l(\cos\theta)\sin\theta\, d\theta - \int_{\pi/2}^{\pi} P_l(\cos\theta)\sin\theta\, d\theta \right) \\
A_0 &= 0 \\
A_1 &= \frac{\sigma_0}{2\epsilon_0} \\
A_2 &= 0 \\
A_3 &= -\frac{\sigma_0}{8\epsilon_0 R^2} \\
A_4 &= 0 \\
A_5 &= \frac{\sigma_0}{16\epsilon_0 R^4} \\
A_6 &= 0 \\
V(r,\theta) &= \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \\
B_l &= A_l R^{2l+1} \\
B_0 &= 0 \\
B_1 &= \frac{\sigma_0 R^3}{2\epsilon_0} \\
B_2 &= 0 \\
B_3 &= -\frac{\sigma_0 R^5}{8\epsilon_0} \\
B_4 &= 0 \\
B_5 &= \frac{\sigma_0 R^7}{16\epsilon_0} \\
B_6 &= 0
\end{aligned}
$$

3.24

$$
\begin{aligned}
0 &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \\
&= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} \\
V(s,\phi) &= S(s)\Phi(\phi) \\
0 &= \frac{1}{s}\frac{\partial}{\partial s}(sS')\Phi + \frac{1}{s^2}S\Phi'' \\
&= \frac{1}{s}(S' + sS'')\Phi + \frac{1}{s^2}S\Phi'' \\
&= \frac{S'}{sS} + \frac{S''}{S} + \frac{\Phi''}{s^2\Phi} \\
&= \frac{s^2S'' + sS'}{S} + \frac{\Phi''}{\Phi} \\
\frac{\Phi''}{\Phi} &= 0 \\
\Phi'' &= 0 \\
\Phi &= c_1 + c_2\phi \\
\frac{\Phi''}{\Phi} &= -n^2 \\
\Phi'' + \alpha^2\Phi &= 0 \\
\Phi &= c_3\cos\alpha\phi + c_4\sin\alpha\phi \\
\Phi(0) &= \Phi(2\pi) \\
c_1 &= c_3\cos 2\pi\alpha + c_4\sin 2\pi\alpha \\
\alpha &= n, n \in \mathbb{R} \\
\Phi &= c_3\cos n\phi + c_4\sin n\pi \\
\frac{s^2S'' + sS'}{S} &= 0 \\
s^2S'' + sS' &= 0 \\
sS'' + S' &= 0 \\
S &= c_5 + c_6\ln s
\end{aligned}
$$

$$
\begin{aligned}
\frac{s^2S'' + sS'}{S} &= n^2 \\
s^2S'' + sS' - n^2S &= 0 \\
S &= s^m \\
S' &= ms^{m-1} \\
S'' &= m(m-1)s^{m-2} \\
m(m-1)s^m + ms^m - n^2s^m &= 0 \\
m^2 - m + m - n^2 &= 0 \\
m^2 - n^2 &= 0 \\
(m+n)(m-n) &= 0 \\
S &= c_7s^n + c_8s^{-n} \\
V &= S(s)\Phi(\phi) \\
&= (c_1 + c_2\phi)(c_5 + c_6\ln s) \\
&\qquad + \sum_{n=1}^{\infty}(c_7s^n + c_8s^{-n})(c_3\cos n\phi + c_4\sin n\phi) \\
&= c_1 + c_2\ln s \\
&\qquad + \sum_{n=1}^{\infty}[s^n(A_n\cos n\phi + B_n\sin n\phi) + s^{-n}(C_n\cos n\phi + D_n\sin n\phi)]
\end{aligned}
$$

3.25

$$
\begin{aligned}
&V = 0 \text{ at } s = R \\
&V \to -E_0 s\cos\phi \text{ as } s \to \infty \\
&V = \left(A_1 s + \frac{C_1}{s}\right)\cos\phi \\
&0 = A_1R + \frac{C_1}{R} \\
&C_2 = -A_1R^2 \\
&A_1 = -E_0 \\
&V = E_0 s\left(\frac{R^2}{s^2} - 1\right)\cos\phi \\
&\sigma = -\epsilon_0\left(\frac{\partial V_{\text{out}}}{\partial s} - \frac{\partial V_{\text{in}}}{\partial s}\right)\bigg|_{s=R} \\
&\quad = -\epsilon_0\left.\frac{\partial V_{\text{out}}}{\partial s}\right|_{s=R} \\
&\quad = 2\epsilon_0 E_0\cos\phi
\end{aligned}
$$

3.27

$$
\begin{aligned}
V(z) &= \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{z^{(n+1)}} \int (r')^n P_n(\cos\alpha) \rho(\mathbf{r}') \, d\tau' \\
&= \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{z^{(n+1)}} \int_0^R \int_0^\pi \int_0^{2\pi} (r')^n P_n(\cos\theta) k \frac{R}{(r')^2} (R - 2r') \sin\theta (r')^2 \sin\theta \, dr' \, d\theta \, d\phi \\
&= \frac{kR}{2\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{z^{(n+1)}} \int_0^R \int_0^\pi (r')^n (R - 2r') \sin^2\theta P_n(\cos\theta) \, dr' \, d\theta \\
&\approx \frac{1}{4\pi\epsilon_0} \frac{k\pi^2 R^5}{48 z^3}
\end{aligned}
$$

3.28

$$
\begin{aligned}
V(r,\theta) &= \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int (r')^n P_n(\cos\alpha) \rho(\mathbf{r}') \, d\tau' \\
&= \frac{\lambda R}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int_0^{2\pi} R^n P_n(\sin\phi \sin\theta) \, d\phi \\
V_0 &= \frac{\lambda R}{4\pi\epsilon_0} \frac{2\pi}{r} \\
&= \frac{\lambda R}{2\epsilon_0 r} \\
V_1 &= 0 \\
V_2 &= -\frac{\lambda R}{4\pi\epsilon_0} \frac{1}{4r^3} \pi R^2 (1 + 3\cos 2\theta) \\
&= -\frac{\lambda R^3}{8\epsilon_0 r^3} (3\cos^2\theta - 1)
\end{aligned}
$$

3.29

$$
\begin{aligned}
\mathbf{p} &= \sum^{n} q_i \mathbf{r}'_i \\
&= 2aq\hat{\mathbf{z}} \\
V_{\text{dip}}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \frac{2aq\cos\theta}{r^2}
\end{aligned}
$$

3.30

(a)

$$\begin{aligned}
\sigma &= k\cos\theta \\
\mathbf{p} &= \int \mathbf{r}'\,\rho(\mathbf{r}')\,d\tau' \\
&= \int_0^{2\pi}\int_0^{\pi} R(\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}})k\cos\theta R^2\sin\theta\,d\theta\,d\phi \\
&= \frac{1}{2}kR^3\int_0^{2\pi}\int_0^{\pi}(\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}})\sin 2\theta\,d\theta\,d\phi \\
&= \frac{4}{3}\pi R^3 k\hat{\mathbf{z}}
\end{aligned}$$

(b)

$$\begin{aligned}
V_{\text{dip}}(\mathbf{r}) &\approx \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \\
&= \frac{1}{4\pi\epsilon_0}\frac{4\pi R^3 k\cos\theta}{3r^2} \\
&= \frac{kR^3}{3\epsilon_0}\frac{1}{r^2}\cos\theta \\
V_{\text{dip}}(r,\theta) &= \frac{kR^3}{3\epsilon_0}\frac{1}{r^2}\cos\theta
\end{aligned}$$

Higher multipoles are all 0.

3.32

(a)

$$\begin{aligned}
Q &= 2q \\
\mathbf{p} &= 3aq\hat{\mathbf{z}} \\
V &\approx \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{r} + \frac{3aq\cos\theta}{r^2}\right)
\end{aligned}$$

(b)

$$\begin{aligned}
Q &= 2q \\
\mathbf{p} &= aq\hat{\mathbf{z}} \\
V &\approx \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{r} + \frac{aq\cos\theta}{r^2}\right)
\end{aligned}$$

(c)

$$\begin{aligned}
Q &= 2q \\
\mathbf{p} &= 3aq\hat{\mathbf{y}} \\
V &\approx \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{r} + \frac{3aq\sin\theta\sin\phi}{r^2}\right)
\end{aligned}$$

3.33

(a)

$$\begin{aligned}
\mathbf{E} &= -\frac{1}{4\pi\epsilon_0}\frac{p}{a^3}\hat{\mathbf{z}} \\
\mathbf{F} &= -\frac{1}{4\pi\epsilon_0}\frac{pq}{a^3}\hat{\mathbf{z}}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{2p}{a^3}\hat{\mathbf{z}} \\
\mathbf{F} &= \frac{1}{4\pi\epsilon_0}\frac{2pq}{a^3}\hat{\mathbf{z}}
\end{aligned}$$

(c)

$$\begin{aligned}
W &= \int \mathbf{F}\cdot d\mathbf{l} \\
&= \int_0^{\pi/2} aq\mathbf{E}\cdot d\boldsymbol{\theta} \\
&= \frac{1}{4\pi\epsilon_0}\frac{pq}{a^2}\int_0^{\pi/2}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})\cdot d\boldsymbol{\theta} \\
&= \frac{1}{4\pi\epsilon_0}\frac{pq}{a^2}\int_0^{\pi/2}\sin\theta\, d\theta \\
&= \frac{1}{4\pi\epsilon_0}\frac{pq}{a^2}
\end{aligned}$$

3.34

$$\begin{aligned}
Q &= -q \\
\mathbf{p} &= qa\hat{\mathbf{z}} \\
V &= \frac{1}{4\pi\epsilon_0}q\left(-\frac{1}{r} + \frac{a\cos\theta}{r^2}\right) \\
\mathbf{E} &= -\nabla V \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}[(2a\cos\theta - r)\hat{\mathbf{r}} + a\sin\theta\hat{\boldsymbol{\theta}}]
\end{aligned}$$

3.35

$$\begin{aligned}
\mathbf{p} &= \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau' \\
&= \left(\int_0^{2\pi}\int_0^{\pi/2}\int_0^R r\cos\theta\rho_0 r^2\sin\theta\,dr\,d\theta\,d\phi \right.\\
&\qquad \left. - \int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^R r\cos\theta\rho_0 r^2\sin\theta\,dr\,d\theta\,d\phi\right)\hat{\mathbf{z}} \\
&= \pi\rho_0\left(\int_0^{\pi/2}\int_0^R r^3\sin 2\theta\,dr\,d\theta - \int_{\pi/2}^{\pi}\int_0^R r^3\sin 2\theta\,dr\,d\theta\right)\hat{\mathbf{z}} \\
&= \frac{1}{2}\pi\rho_0 R^4\hat{\mathbf{z}} \\
\mathbf{E}_{\text{dip}}(r,\theta) &= \frac{1}{4\pi\epsilon_0}\frac{\pi\rho_0 R^4}{2r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})
\end{aligned}$$

3.36

The factor of $1/4\pi\epsilon_0 r^3$ is the common, so the goal is to show that

$$\begin{aligned}
p(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) &= 2(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} + p\sin\theta\hat{\boldsymbol{\theta}} \\
&= 2(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - (\mathbf{p}\cdot\hat{\boldsymbol{\theta}})\hat{\boldsymbol{\theta}} \\
&= 3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{p}}.
\end{aligned}$$

3.37

$$\begin{aligned}
V_{\text{ave}} &= \frac{1}{4\pi R^2}\oint V\,da \\
\frac{dV_{\text{ave}}}{dR} &= \frac{1}{4\pi R^2}\oint \nabla V\cdot d\mathbf{a} \\
&= \frac{1}{4\pi R^2}\int \nabla^2 V\,d\tau \\
&= 0
\end{aligned}$$

3.38

$$
\begin{aligned}
E_{qz} &= \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\cos\theta \\
&= \frac{1}{4\pi\epsilon_0}\frac{qd}{(x^2+y^2+d^2)^{3/2}} \\
0 &= E_{qz} + E_{\sigma z} \\
&= \frac{1}{4\pi\epsilon_0}\frac{qd}{(x^2+y^2+d^2)^{3/2}} - \frac{\sigma}{2\epsilon_0} \\
\sigma &= \frac{qd}{2\pi(x^2+y^2+d^2)^{3/2}}
\end{aligned}
$$

3.39

$$
E = \frac{q^2}{4\pi\epsilon_0}\left[\left(\sum_{n=1}^{\infty}\frac{1}{(2an-2x)^2}\right) - \left(\sum_{n=0}^{\infty}\frac{1}{(2an+2x)^2}\right)\right]
$$

3.40

Set $V = 0$ at $x = 0$. The cylinder is a conductor and is thus an equipotential, so $V = 0$ at the surface. Place two infinite line charges within the cylinder at $x = \pm R^2/a$, giving

$$
\begin{aligned}
V = \frac{\lambda}{2\pi\epsilon_0}\Bigg(&\ln\frac{a}{\sqrt{s^2+a^2-2sa\cos\phi}} - \ln\frac{a}{\sqrt{s^2+a^2+2sa\cos\phi}} \\
&+ \ln\frac{R^2/a}{\sqrt{s^2+(R^2/a)^2+2s(R^2/a)\cos\phi}} \\
&- \ln\frac{R^2/a}{\sqrt{s^2+(R^2/a)^2-2s(R^2/a)\cos\phi}}\Bigg) \\
= \frac{\lambda}{4\pi\epsilon_0}&\left(\ln\frac{s^2+a^2+2sa\cos\phi}{s^2+a^2-2sa\cos\phi} + \ln\frac{(sa/R)^2+R^2-2sa\cos\phi}{(sa/R)^2+R^2+2sa\cos\phi}\right) \\
= \frac{\lambda}{4\pi\epsilon_0}&\ln\frac{(s^2+a^2+2sa\cos\phi)[(sa/R)^2+R^2-2sa\cos\phi]}{(s^2+a^2-2sa\cos\phi)[(sa/R)^2+R^2+2sa\cos\phi]}
\end{aligned}
$$

3.41

(a) For a sphere of charge q, $q' + q'' = q \Rightarrow q'' = q - q'$ so

$$\begin{aligned} F &= \frac{q}{4\pi\epsilon_0}\left(\frac{q''}{a^2} + \frac{q'}{(a-b)^2}\right) \\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{q}{a^2} - \frac{q'}{a^2} + \frac{q'}{(a-b)^2}\right) \\ &= \frac{q^2}{4\pi\epsilon_0 a^3}\left[a - R^3\,\frac{2a^2 - R^2}{(a^2 - R^2)^2}\right] \end{aligned}$$

and solving for $F = 0$ gives $r = 5.663\,12\,\text{Å}$.

3.43

(a)

$$\begin{aligned}
\lim_{r\to\infty} V_{\text{above}}(r,\theta) &= 0 \\
V_{\text{below}}(a,\theta) &= V_0 \\
V_{\text{above}}(b,\theta) &= V_{\text{below}}(b,\theta) \\
\left.\frac{\partial V_{\text{above}}}{\partial r}\right|_{r=b} - \left.\frac{\partial V_{\text{below}}}{\partial r}\right|_{r=b} &= -\frac{k\cos\theta}{\epsilon_0} \\
V_{\text{above}}(r,\theta) &= \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \\
\left.\frac{\partial V_{\text{above}}}{\partial r}\right|_{r=b} &= \sum_{l=0}^{\infty} -(l+1)\frac{B_l}{b^{l+2}} P_l(\cos\theta) \\
V_{\text{below}}(r,\theta) &= \sum_{l=0}^{\infty} \left(C_l r^l + \frac{D_l}{r^{l+1}}\right) P_l(\cos\theta) \\
V_0 &= V_{\text{below}}(a,\theta) \\
&= \sum_{l=0}^{\infty} \left(C_l a^l + \frac{D_l}{a^{l+1}}\right) P_l(\cos\theta) \\
V_0 &= C_0 + \frac{D_0}{a} \\
0 &= C_l a^l + \frac{D_l}{a^{l+1}}, l \neq 0 \\
\left.\frac{\partial V_{\text{below}}}{\partial r}\right|_{r=b} &= \sum_{l=0}^{\infty} \left(C_l l b^{l-1} - (l+1)\frac{D_l}{b^{l+2}}\right) P_l(\cos\theta) \\
V_{\text{above}}(b,\theta) &= V_{\text{below}}(b,\theta) \\
\sum_{l=0}^{\infty} \frac{B_l}{b^{l+1}} P_l(\cos\theta) &= \sum_{l=0}^{\infty} \left(C_l b^l + \frac{D_l}{b^{l+1}}\right) P_l(\cos\theta) \\
\frac{B_l}{b^{l+1}} &= C_l b^l + \frac{D_l}{b^{l+1}} \\
-\frac{k\cos\theta}{\epsilon_0} &= \sum_{l=0}^{\infty} \left[-(l+1)\frac{B_l}{b^{l+2}} - C_l l b^{l-1} + (l+1)\frac{D_l}{b^{l+2}}\right] P_l(\cos\theta) \\
-\frac{k}{\epsilon_0} &= -2\frac{B_1}{b^3} - C_1 + 2\frac{D_1}{b^3} \\
0 &= -(l+1)\frac{B_l}{b^{l+2}} - C_l l b^{l-1} + (l+1)\frac{D_l}{b^{l+2}}, l \neq 1 \\
B_0 &= aV_0 \\
C_0 &= 0 \\
D_0 &= aV_0
\end{aligned}$$

$$
\begin{aligned}
B_1 &= \frac{(b^3 - a^3)k}{3\epsilon_0} \\
C_1 &= \frac{k}{3\epsilon_0} \\
D_1 &= -\frac{a^3 k}{3\epsilon_0} \\
B_l &= 0 \\
C_l &= 0 \\
D_l &= 0 \\
V &= \begin{cases} \frac{aV_0}{r} + \frac{(r^3 - a^3)k\cos\theta}{3\epsilon_0 r^2} & a \le r \le b \\ \frac{aV_0}{r} + \frac{(b^3 - a^3)k\cos\theta}{3\epsilon_0 r^2} & r \ge b \end{cases}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\sigma &= -\epsilon_0 \left.\frac{\partial V_{\text{below}}}{\partial r}\right|_{r=a} \\
&= \frac{\epsilon_0 V_0}{a} - k\cos\theta
\end{aligned}
$$

(c)

$$
\begin{aligned}
Q &= \oint \sigma_i \, dA \\
&= \int_0^{2\pi} \int_0^{\pi} \left(\frac{\epsilon_0 V_0}{a} - k\cos\theta \right) a^2 \sin\theta \, d\theta \, d\phi \\
&= 2\pi \int_0^{\pi} \left(a\epsilon_0 V_0 \sin\theta - \frac{1}{2} a^2 k \sin 2\theta \right) \, d\theta \\
&= 4\pi\epsilon_0 a V_0 \\
V &\approx \frac{aV_0}{r} \\
\frac{1}{4\pi\epsilon_0} \frac{Q}{r} &= \frac{aV_0}{r} \\
Q &= 4\pi\epsilon_0 a V_0
\end{aligned}
$$

3.44

$$\begin{aligned}
V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')\,d\tau' \\
&= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int_{-a}^{a} z^n P_n(\cos\theta)\frac{Q}{2a}\,dz \\
&= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{Q}{2ar^{(n+1)}}P_n(\cos\theta)\left[\frac{1}{n+1}z^{n+1}\right]_{-a}^{a} \\
&= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{Q}{2a(n+1)r^{(n+1)}}P_n(\cos\theta)[a^{n+1}-(-1)^{n+1}a^{n+1}] \\
&= \frac{Q}{4\pi\epsilon_0}\frac{1}{r}\left[1+\frac{1}{3}\left(\frac{a}{r}\right)^2 P_2(\cos\theta)+\frac{1}{5}\left(\frac{a}{r}\right)^4 P_4(\cos\theta)+\dots\right]
\end{aligned}$$

3.45

$$V = a_0 + b_0\ln s + \sum_{k=1}^{\infty}[s^k(a_k\cos k\phi + b_k\sin k\phi) + s^{-k}(c_k\cos k\phi + d_k\sin k\phi)]$$

$$
\begin{aligned}
\lim_{s\to\infty} V_{\text{above}}(s,\phi) &= 0 \\
V_{\text{above}}(R,\phi) &= V_{\text{below}}(R,\phi) \\
&= \begin{cases} -\sigma_0/\epsilon_0 & 0 \le \phi \le \pi \\ \sigma_0/\epsilon_0 & \pi \le \phi \le 2\pi \end{cases} \\
\left.\frac{\partial V_{\text{above}}}{\partial s}\right|_{s=R} - \left.\frac{\partial V}{\partial s}\right|_{s=R} &= -\frac{1}{\epsilon_0}\sigma \\
V_{\text{above}}(s,\phi) &= \sum_{k=1}^{\infty} s^{-k}(c_k \cos k\phi + d_k \sin k\phi) \\
\left.\frac{\partial V_{\text{above}}}{\partial s}\right|_{s=R} &= \sum_{k=1}^{\infty} -kR^{-(k+1)}(c_k \cos k\phi + d_k \sin k\phi) \\
V_{\text{below}}(s,\phi) &= e_0 + \sum_{k=1}^{\infty} s^{k}(e_k \cos k\phi + f_k \sin k\phi) \\
\left.\frac{\partial V_{\text{below}}}{\partial s}\right|_{s=R} &= \sum_{k=1}^{\infty} kR^{k-1}(e_k \cos k\phi + f_k \sin k\phi) \\
\sum_{k=1}^{\infty} R^{-k}(c_k \cos k\phi + d_k \sin k\phi) &= e_0 + \sum_{k=1}^{\infty} R^{k}(e_k \cos k\phi + f_k \sin k\phi) \\
e_0 &= 0 \\
R^{-k}c_k &= R^k e_k \\
R^{-k}d_k &= R^k f_k
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\sigma}{\epsilon_0} &= \sum_{k=1}^{\infty}\left[-kR^{-(k+1)}(c_k \cos k\phi + d_k \sin k\phi) - kR^{k-1}(e_k \cos k\phi + f_k \sin k\phi)\right] \\
&= \sum_{k=1}^{\infty} -k\left[\left(R^{-(k+1)}c_k + R^{k-1}e_k\right)\cos k\phi + \left(R^{-(k+1)}d_k + R^{k-1}f_k\right)\sin k\phi\right] \\
\frac{1}{\pi}\int_0^{2\pi} -\frac{\sigma}{\epsilon_0}\cos k\phi \, d\phi &= -k(R^{-(k+1)}c_k + R^{k-1}e_k) \\
\frac{\sigma_0(\sin 2k\pi - 2\sin k\pi)}{k\pi\epsilon_0} &= -k(R^{-(k+1)}c_k + R^{k-1}e_k) \\
\frac{1}{\pi}\int_0^{2\pi} -\frac{\sigma}{\epsilon_0}\sin k\phi \, d\phi &= -k(R^{-(k+1)}d_k + R^{k-1}f_k) \\
\frac{4\sigma_0 \cos k\pi \sin^2 k\pi/2}{k\pi\epsilon_0} &= -k(R^{-(k+1)}d_k + R^{k-1}f_k)
\end{aligned}
$$

$$
\begin{aligned}
c_k &= 0 \\
d_k &= \frac{2R^{k+1}\sigma_0 \cos k\pi \sin^2 k\pi/2}{k^2\pi\epsilon_0} \\
e_k &= 0 \\
f_k &= -\frac{2R^{-(k-1)}\sigma_0 \cos k\pi \sin^2 k\pi/2}{k^2\pi\epsilon_0} \\
V_{\text{above}} &= \frac{2\sigma_0}{\pi\epsilon_0}\sum_{k=1}^{\infty} s^{-k}\frac{R^{k+1}\cos k\pi \sin^2 k\pi/2}{k^2}\sin k\phi \\
&= -\frac{2\sigma_0}{\pi\epsilon_0}\sum_{k=1,3,5,\ldots}^{\infty}\frac{1}{k^2}R^{k+1}s^{-k}\sin k\phi \\
V_{\text{below}} &= -\frac{2\sigma_0}{\pi\epsilon_0}\sum_{k=1}^{\infty} s^{k}\frac{R^{-(k-1)}\cos k\pi \sin^2 k\pi/2}{k^2}\sin k\phi \\
&= \frac{2\sigma_0}{\pi\epsilon_0}\sum_{k=1,3,5,\ldots}^{\infty}\frac{1}{k^2}R^{-(k-1)}s^{k}\sin k\phi \\
V &= \frac{2R\sigma_0}{\pi\epsilon_0}\begin{cases}\sum_{k=1,3,5,\ldots}^{\infty}\frac{1}{k^2}(s/R)^k\sin k\phi & s \le R \\ -\sum_{k=1,3,5,\ldots}^{\infty}\frac{1}{k^2}(R/s)^k\sin k\phi & s \ge R\end{cases}
\end{aligned}
$$

3.46

(a)

$$\frac{1}{4\pi\epsilon_0}\frac{1}{r}\int_{-a}^{a} k\cos\frac{\pi z}{2a}\,dz = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\frac{4ak}{\pi}$$

(b)

$$\frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int_{-a}^{a} z\cos\theta k\sin\frac{\pi z}{a}\,dz = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\frac{2a^2k\cos\theta}{\pi}$$

(c)

$$\frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int_{-a}^{a} z^2P_2(\cos\theta)k\cos\frac{\pi z}{a}\,dz = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left(-\frac{4a^3k}{\pi^2}\right)P_2(\cos\theta)$$

3.47

(a)

$$
\begin{aligned}
\mathbf{E}_{\text{ave}} &= \frac{1}{\frac{4}{3}\pi R^3}\int \mathbf{E}\,d\tau \\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{\frac{4}{3}\pi R^3}\int \frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau' \\
\mathbf{E}_{\text{ave}} &= \int \frac{1}{4\pi\epsilon_0}\frac{\rho}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau' \\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{\frac{4}{3}\pi R^3}\int \frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau'
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathbf{p} &= q\mathbf{r} \\
\oint \mathbf{E}\cdot d\mathbf{A} &= \frac{Q_{\text{encl}}}{\epsilon_0} \\
4\pi r^2 E &= \frac{\frac{4}{3}\pi r^3\rho}{\epsilon_0} \\
\mathbf{E} &= \frac{r\rho}{3\epsilon_0}\hat{\mathbf{r}} \\
&= -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 + \ldots \\
&= -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}_1}{R^3} - \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}_2}{R^3} + \ldots \\
&= -\frac{1}{4\pi\epsilon_0}\frac{1}{R^3}(\mathbf{p}_1 + \mathbf{p}_2 + \ldots) \\
&= -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}
\end{aligned}
$$

(d)

$$
\begin{aligned}
\mathbf{E}_{\text{ave}} &= \frac{1}{\frac{4}{3}\pi R^3}\int \mathbf{E}\,d\tau' \\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{\frac{4}{3}\pi R^3}\int \frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau' \\
\mathbf{E}_r &= -\frac{1}{4\pi\epsilon_0}\int \frac{\rho}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau' \\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{\frac{4}{3}\pi R^3}\int \frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau' \\
\rho &= -\frac{q}{\frac{4}{3}\pi R^3} \\
Q &= \frac{4}{3}\pi R^3\rho \\
&= -q \\
\mathbf{E}_r &= \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{z}}
\end{aligned}
$$

This is the electric field at the origin.

3.48

(a)

$$
\begin{aligned}
\mathbf{E}_{\text{dip}}(r,\theta) &= \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) \\
&= \frac{p}{4\pi\epsilon_0 r^3}[2\cos\theta(\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}) \\
&\quad + \sin\theta(\cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}})] \\
&= \frac{p}{4\pi\epsilon_0 r^3}[3\cos\theta\sin\theta\cos\phi\,\hat{\mathbf{x}} + 3\cos\theta\sin\theta\sin\phi\,\hat{\mathbf{y}} \\
&\quad + (2\cos^2\theta - \sin^2\theta)\hat{\mathbf{z}}] \\
\mathbf{E}_{\text{ave}} &= \frac{1}{\frac{4}{3}\pi R^3}\int \mathbf{E}_{\text{dip}}\,d\tau' \\
&= \frac{3p}{16\pi^2\epsilon_0 R^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{1}{r^3}[3\cos\theta\sin\theta\cos\phi\,\hat{\mathbf{x}} \\
&\quad + 3\cos\theta\sin\theta\sin\phi\,\hat{\mathbf{y}} + (2\cos^2\theta - \sin^2\theta)\hat{\mathbf{z}}]r^2\sin\theta\,dr\,d\theta\,d\phi \\
&= \frac{3p}{16\pi^2\epsilon_0 R^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{1}{r}[3\cos\theta\sin^2\theta\cos\phi\,\hat{\mathbf{x}} \\
&\quad + 3\cos\theta\sin^2\theta\sin\phi\,\hat{\mathbf{y}} + (2\cos^2\theta - \sin^2\theta)\sin\theta\,\hat{\mathbf{z}}]\,dr\,d\theta\,d\phi \\
&= \mathbf{0}
\end{aligned}
$$

(b)

$$-\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3} = \frac{1}{\frac{4}{3}\pi R^3}\int \mathbf{E}\,d\tau'$$
$$\mathbf{E} = -\frac{\mathbf{p}}{3\epsilon_0}\delta^3(\mathbf{r})$$

3.50

(a)

$$\begin{aligned}
\int \mathbf{E}_1 \cdot \mathbf{E}_2\,d\tau &= \int (-\nabla V_1) \cdot \mathbf{E}_2\,d\tau \\
&= \int [V_1(\nabla \cdot \mathbf{E}_2) - \nabla \cdot (V_1\mathbf{E}_2)]\,d\tau \\
&= \int \frac{\rho_2 V_1}{\epsilon_0}\,d\tau - \int \nabla \cdot (V_1\mathbf{E}_2)\,d\tau \\
&= \int \frac{\rho_2 V_1}{\epsilon_0}\,d\tau - \oint V_1\mathbf{E}_2 \cdot d\mathbf{a} \\
&= \int \frac{\rho_2 V_1}{\epsilon_0}\,d\tau \\
\int \mathbf{E}_1 \cdot \mathbf{E}_2\,d\tau &= \int \mathbf{E}_1 \cdot (-\nabla V_2)\,d\tau \\
&= \int [V_2(\nabla \cdot \mathbf{E}_1) - \nabla \cdot (V_2\mathbf{E}_1)]\,d\tau \\
&= \int \frac{\rho_1 V_2}{\epsilon_0}\,d\tau - \int \nabla \cdot (V_2\mathbf{E}_1)\,d\tau \\
&= \int \frac{\rho_1 V_2}{\epsilon_0}\,d\tau - \oint V_2\mathbf{E}_1 \cdot\, d\mathbf{a} \\
&= \int \frac{\rho_1 V_2}{\epsilon_0}\,d\tau \\
\int \frac{\rho_1 V_2}{\epsilon_0}\,d\tau &= \int \frac{\rho_2 V_1}{\epsilon_0}\,d\tau \\
\int \rho_1 V_2\,d\tau &= \int \rho_2 V_1\,d\tau
\end{aligned}$$

(b)

$$
\begin{aligned}
Q_a &= \int_a \rho_1 \, d\tau \\
&= Q \\
Q_b &= \int_b \rho_1 \, d\tau \\
&= 0 \\
V_{1b} &= V_{ab} \\
Q_a &= \int_a \rho_2 \, d\tau \\
&= 0 \\
Q_b &= \int_b \rho_2 \, d\tau \\
&= Q \\
V_{2a} &= V_{ba} \\
\int \rho_1 V_2 \, d\tau &= \int_a \rho_1 V_2 \, d\tau + \int_b \rho_1 V_2 \, d\tau \\
&= V_{2a} \int_a \rho_1 \, d\tau + V_{2b} \int \rho_1 \, d\tau \\
&= V_{ba} Q \\
\int \rho_2 V_1 \, d\tau &= \int_a \rho_2 V_1 \, d\tau + \int_b \rho_2 V_1 \, d\tau \\
&= V_{1a} \int_a \rho_2 \, d\tau + V_{1b} \int \rho_2 \, d\tau \\
&= V_{ab} Q \\
V_{ba} Q &= V_{ab} Q \\
V_{ba} &= V_{ab}
\end{aligned}
$$

3.51

(a)

$$\begin{aligned}
\int \rho_2 V_1 \, d\tau &= Q_{l2}V_{l1} + Q_{x2}V_{x1} + Q_{r2}V_{r1} \\
&= 0 \\
\int \rho_1 V_2 \, d\tau &= Q_{l1}V_{l2} + Q_{x1}V_{x2} + Q_{r1}V_{r2} \\
&= q\frac{x}{d}V_0 + Q_2 V_0 \\
Q_2 &= -\frac{qx}{d}
\end{aligned}$$

$$\begin{aligned}
\int \rho_2 V_1 \, d\tau &= Q_{l2}V_{l1} + Q_{x2}V_{x1} + Q_{r2}V_{r1} \\
&= 0 \\
\int \rho_1 V_2 \, d\tau &= Q_{l1}V_{l2} + Q_{x1}V_{x2} + Q_{r1}V_{r2} \\
&= Q_1 V_0 + q\left(1 - \frac{x}{d}\right)V_0 \\
Q_1 &= q\left(\frac{x}{d} - 1\right)
\end{aligned}$$

(b)

$$
\begin{aligned}
\int \rho_2 V_1 \, d\tau &= Q_{a2}V_{a1} + Q_{r2}V_{r1} + Q_{b2}V_{b1} \\
&= 0 \\
V(a,\theta) &= 0 \\
V(b,\theta) &= V_0 \\
V(r,\theta) &= \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \\
0 &= \sum_{l=0}^{\infty} \left(A_l a^l + \frac{B_l}{a^{l+1}} \right) P_l(\cos\theta) \\
0 &= A_l a^l + \frac{B_l}{a^{l+1}} \\
B_l &= -A_l a^{2l+1} \\
V(r,\theta) &= \sum_{l=0}^{\infty} A_l \left(r^l - \frac{a^{2l+1}}{r^{l+1}} \right) P_l(\cos\theta) \\
V_0 &= \sum_{l=0}^{\infty} A_l \left(b^l - \frac{a^{2l+1}}{b^{l+1}} \right) P_l(\cos\theta) \\
&= A_0 \left(1 - \frac{a}{b} \right) \\
A_0 &= \frac{b}{b-a} V_0 \\
A_n &= 0, \; n \neq 0 \\
V(r,\theta) &= V_0 \frac{b}{b-a} \left(1 - \frac{a}{r} \right) \\
\int \rho_1 V_2 \, d\tau &= Q_{r1}V_{r2} + Q_{b1}V_{b2} \\
&= qV_0 \frac{b}{b-a} \left(1 - \frac{a}{r} \right) + Q_2 V_0 \\
Q_2 &= -\frac{qb}{b-a} \left(1 - \frac{a}{r} \right)
\end{aligned}
$$

$$
\begin{aligned}
\int \rho_2 V_1 \, d\tau &= Q_{a2} V_{a1} + Q_{r2} V_{r1} + Q_{b2} V_{b1} \\
&= 0 \\
V(a,\theta) &= V_0 \\
V(b,\theta) &= 0 \\
0 &= \sum_{l=0}^{\infty} \left(A_l b^l + \frac{B_l}{b^{l+1}} \right) P_l(\cos\theta) \\
0 &= A_l b^l \frac{B_l}{b^{l+1}} \\
B_l &= -A_l b^{2l+1} \\
V(r,\theta) &= \sum_{l=0}^{\infty} A_l \left(r^l - \frac{b^{2l+1}}{r^{l+1}} \right) P_l(\cos\theta) \\
V_0 &= \sum_{l=0}^{\infty} A_l \left(a^l - \frac{b^{2l+1}}{a^{l+1}} \right) P_l(\cos\theta) \\
V_0 &= A_0 \left(1 - \frac{b}{a} \right) \\
A_0 &= V_0 \frac{a}{a-b} \\
V(r,\theta) &= V_0 \frac{a}{a-b} \left(1 - \frac{b}{r} \right) \\
\int \rho_1 V_2 \, d\tau &= Q_{a1} V_{a2} + Q_{r1} V_{r2} + Q_{b1} V_{b2} \\
&= Q_1 V_0 + q V_0 \frac{a}{a-b} \left(1 - \frac{b}{r} \right) \\
Q_1 &= -\frac{qa}{a-b} \left(1 - \frac{b}{r} \right)
\end{aligned}
$$

3.52

(a)

$$
\begin{aligned}
V_{\text{quad}}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int (r')^2 P_2(\cos\alpha)\rho(\mathbf{r}')\,d\tau' \\
\int (r')^2 P_2(\cos\alpha)\rho(\mathbf{r}')\,d\tau' &= \int (r')^2\left[\frac{1}{2}(3\cos^2\alpha - 1)\right]\rho(\mathbf{r}')\,d\tau' \\
&= \frac{1}{2}\int (r')^2[3(\hat{\mathbf{r}}'\cdot\hat{\mathbf{r}})^2 - 1]\rho(\mathbf{r}')\,d\tau' \\
&= \frac{1}{2}\int [3(\mathbf{r}'\cdot\hat{\mathbf{r}})^2 - (r')^2]\rho(\mathbf{r}')\,d\tau' \\
&= \frac{1}{2}\int [3(\mathbf{r}'\cdot\hat{\mathbf{r}})^2 - (r')^2(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})]\rho(\mathbf{r}')\,d\tau' \\
&= \frac{1}{2}\int \left[3\sum_{i,j=1}^{3} r'_i r'_j \hat{r}_i\hat{r}_j - (r')^2\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j\delta_{ij}\right]\rho(\mathbf{r}')\,d\tau' \\
&= \sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j\frac{1}{2}\int [3r'_i r'_j - (r')^2\delta_{ij}]\rho(\mathbf{r}')\,d\tau' \\
&= \sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j Q_{ij}
\end{aligned}
$$

(b)

$$
\begin{aligned}
Q_{11} &= 0 \\
Q_{12} &= \frac{3a^2q}{2} \\
Q_{13} &= 0 \\
Q_{21} &= \frac{3a^2q}{2} \\
Q_{22} &= 0 \\
Q_{23} &= 0 \\
Q_{31} &= 0 \\
Q_{32} &= 0 \\
Q_{33} &= 0
\end{aligned}
$$