Classical Mechanics by John R. Taylor Notes

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1 Newton's Laws of Motion

1.2 Space and Time

- In cartesian coordinates the basis vectors don't depend on time so their derivatives are $\mathbf{0}$. This means that

$$\begin{aligned}\frac{d}{dt}(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}) &= \frac{dx}{dt}\hat{\mathbf{x}}+x\frac{d\hat{\mathbf{x}}}{dt}+\frac{dy}{dt}\hat{\mathbf{y}}+y\frac{d\hat{\mathbf{y}}}{dt}+\frac{dz}{dt}\hat{\mathbf{z}}+z\frac{d\hat{\mathbf{z}}}{dt}\\ &= \frac{dx}{dt}\hat{\mathbf{x}}+\frac{dy}{dt}\hat{\mathbf{y}}+\frac{dz}{dt}\hat{\mathbf{z}}\end{aligned}$$

  as expected. However, in order coordinate systems (e.g. polar, spherical) the basis vectors may depend on time and their derivatives aren't $\mathbf{0}$.

1.4 Newton's First and Second Laws; Inertial Frames

- Newton's second law $\mathbf{F} = m\mathbf{a}$ can be restated as $\mathbf{F} = \dot{\mathbf{p}}$.
- An inertial frame is one where Newton's first law holds. Typically this means the frame isn't accelerating or rotating.

1.5 The Third Law and Conservation of Momentum

- Forces that act along the line joining two objects are called **central forces**.
- The **principle of conservation of momentum** states that if the net external force $\mathbf{F}_{\text{ext}}$ on an N-particle system is zero, the system's total momentum $\mathbf{P}$ is constant.

1.7 Two-Dimensional Polar Coordinates

- In two-dimensional polar coordinates, the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ depend on position and thus time. Their derivatives are

$$\begin{aligned}\frac{d\hat{\mathbf{r}}}{dt} &= \dot{\phi}\hat{\boldsymbol{\phi}}\\ \frac{d\hat{\boldsymbol{\phi}}}{dt} &= -\dot{\phi}\hat{\mathbf{r}}.\end{aligned}$$

  Consequently, the derivatives of the position vector $\mathbf{r} = r\hat{\mathbf{r}}$ are

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= \frac{d}{dt}(r\hat{\mathbf{r}})\\ &= \dot{r}\hat{\mathbf{r}}+r\frac{d\hat{\mathbf{r}}}{dt}\\ &= \dot{r}\hat{\mathbf{r}}+r\dot{\phi}\hat{\boldsymbol{\phi}}\end{aligned}$$

and

$$\begin{aligned}
\frac{d^2\mathbf{r}}{dt^2} &= \frac{d}{dt}(\dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}) \\
&= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt} + \dot{r}\dot{\phi}\hat{\boldsymbol{\phi}} + r\ddot{\phi}\hat{\boldsymbol{\phi}} + r\dot{\phi}\frac{d\hat{\boldsymbol{\phi}}}{dt} \\
&= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{r}\dot{\phi}\hat{\boldsymbol{\phi}} + r\ddot{\phi}\hat{\boldsymbol{\phi}} - r\dot{\phi}^2\hat{\mathbf{r}} \\
&= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\boldsymbol{\phi}}.
\end{aligned}$$

- In light of the above, Newton's second law in polar coordinates can be written

$$\begin{aligned}
F_r &= m(\ddot{r} - r\dot{\phi}^2) \\
F_\phi &= m(r\ddot{\phi} + 2\dot{r}\dot{\phi}).
\end{aligned}$$

2 Projectiles and Charged Particles

2.1 Air Resistance

- Air resistance depends on the speed v of the moving object. For many objects the direction of the air resistance force $\mathbf{f}$ is opposite to $\mathbf{v}$, but not always. For example, the air resistance force on an airplane causes lift.
- An air resistance force can be described by the equation

$$\mathbf{f} = -f(v)\hat{\mathbf{v}}$$

  where $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$ gives the direction and $f(v)$ gives the magnitude.
- $f(v)$ can be approximated as

$$f(v) = f_{\text{lin}} + f_{\text{quad}} = bv + cv^2.$$

- The linear term f_{lin} arises from the viscous drag of the medium and is generally proportional to the projectile's linear size.
- The quadratic term f_{quad} arises from the fact that the projectile must accelerate the air with which it is continually colliding and it is proportional to the density of the medium and the cross-sectional area of the projectile.
- For a spherical projectile the coefficients b and c above have the form

$$b = \beta D \text{ and } c = \gamma D^2$$

  where D is the diameter of the sphere and the coefficients β and γ depend on the nature of the medium. In air at STP they have approximate values

$$\beta = 1.6 \times 10^{-4}\,\mathrm{N\,s/m^2}$$

  and

$$\gamma = 0.25\,\mathrm{N\,s^2/m^4}.$$

- Depending on the natures of the medium and projectile it's often possible to neglect one of the terms in $f(v)$. To determine if this is the case we can calculate their ratio. For example, for a spherical projectile at STP

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} = \frac{cv^2}{bv} = \frac{\gamma D}{\beta} v = (1.6 \times 10^3 \,\text{s/m}^2) Dv.$$

  If the ratio is large f_{lin} can be ignored. If it's small f_{quad} can be ignored.

- The **Reynolds number** can be used to characterise the behaviour of an object in a fluid

$$R = \frac{\rho}{\mu} Dv$$

  where ρ is the medium's density, μ is its viscosity, D is the linear dimension of the projectile (diameter for spherical projectiles), and v is the projectile's speed. The quadratic force f_{quad} is dominant when the Reynolds number R is large and the linear force f_{linear} is dominant when it is small.

2.2 Linear Air Resistance

- When the quadratic drag force is negligible the equation of motion becomes

$$\begin{aligned}
\mathbf{F} &= \mathbf{W} - \mathbf{f} \\
m\mathbf{a} &= m\mathbf{g} - b\mathbf{v} \\
m\dot{\mathbf{v}} &= m\mathbf{g} - b\mathbf{v}.
\end{aligned}$$

  This is a first-order differential equation for $\mathbf{v}$ where the horizontal and vertical components can be separated to

$$\begin{aligned}
m\dot{v}_x &= -bv_x \\
m\dot{v}_y &= mg - bv_y,
\end{aligned}$$

  each of which is easily solvable.

- The **terminal speed** of an object undergoing freefall and experiencing only linear drag is

$$v_{\text{ter}} = \frac{mg}{b}.$$

- The **characteristic time**

$$\tau = \frac{1}{k} = \frac{1}{b/m} = \frac{m}{b}$$

  is a measure of the importance of air resistance.

  - For horizontal motion with drag it's a measure of the time it takes for the projectile to reach $1/e$ of its initial velocity.

- For freefall with drag it's a measure of the time it would take the projectile to reach its terminal velocity if it didn't experience drag

$$v_{\text{ter}} = g\tau.$$

- For freefall with drag it can also be used to gauge what percentage of its terminal velocity a projectile will reach after a certain time:

Time t	Percent of v_{ter}
0	0
τ	63%
2τ	86%
3τ	95%

From this it can be seen that after $t = 3\tau$ the projectile has effectively reached its terminal velocity.

2.4 Quadratic Air Resistance

- Equations of motion for quadratic air resistance can be solved analytically when the projectile moves in one dimension, but can only be solved numerically when it moves in multiple dimensions.

- When a projectile moves in one dimension and only experiences the force of air resistance (i.e. there are no other forces), the equation of motion is

$$m\dot{v} = -cv^2.$$

Using separation of variables the solution can be found to be

$$v(t) = \frac{v_0}{1 + t/\tau}$$

where

$$\tau = \frac{m}{cv_0}.$$

- As in the linear case, τ is a measure of how long it takes for air resistance to slow down the projectile ($v = v_0/2$ at $t = \tau$).

- Integrating the equation for $v(t)$ gives

$$x(t) = v_0\tau \ln\left(1 + \frac{t}{\tau}\right).$$

- When a projectile moves in one dimension and experiences the forces of air resistance and weight, the equation of motion (with y down) is

$$m\dot{v} = mg - cv^2.$$

Using separation of variables the solution can be found to be

$$v(t) = v_{\text{ter}} \tanh \frac{gt}{v_{\text{ter}}}$$

where

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}}.$$

- Integrating the equation for $v(t)$ gives

$$y = \frac{v_{\text{ter}}^2}{g} \ln \left(\cosh \frac{gt}{v_{\text{ter}}} \right).$$

2.5 Motion of a Charge in a Uniform Magnetic Field

- When a particle of charge q moves in a magnetic field $\mathbf{B} = (0, 0, B_z)$ with velocity $\mathbf{v} = (v_x, v_y, v_z)$ it experiences a force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = q(v_y B, -b_x B, 0).$$

This gives the coupled equations of motion

$$\begin{aligned} m\dot{v}_x &= qBv_y \\ m\dot{v}_y &= -qBv_x \\ m\dot{v}_z &= 0 \end{aligned}$$

or

$$\begin{aligned} \dot{v}_x &= \omega v_y \\ \dot{v}_y &= -\omega v_x \\ \dot{v}_z &= 0 \end{aligned}$$

where $\omega = qB/m$ is called the **cyclotron frequency**.

- If we define a complex value

$$\eta = v_x + iv_y,$$

its derivative is

$$\begin{aligned} \dot{\eta} &= \dot{v}_x + i\dot{v}_y \\ &= \omega v_y - i\omega v_x \\ &= -i\omega\eta \end{aligned}$$

which has the solution

$$\eta = Ae^{-i\omega t}.$$

3 Momentum and Angular Momentum

3.1 Conservation of Momentum

- The **principle of conservation of momentum** states that if the net external force $\mathbf{F}_{\text{ext}}$ on an N-particle system is zero, the system's total mechanical momentum $\mathbf{P} = \sum m_\alpha v_\alpha$ is constant.

3.2 Rockets

- Newton's second law for a rocket is

$$m\dot{v} = -\dot{m}v_{\text{ex}}$$

where $\dot{m}$ is the (negative) rate of change of the mass of the rocket and v_{ex} is the velocity of the exhaust. The quantity on the right hand side of the equation is called the **thrust**.

- The equation above can be solved by separation of variables giving

$$v - v_0 = v_{\text{ex}} \ln \frac{m_0}{m}$$

which is often called the **rocket equation**.

3.3 The Center of Mass

- The **centre of mass** of a system is defined to be

$$\mathbf{R} = \frac{1}{M} \sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha$$

where M is the total mass of all particles in the system, m_α is the mass of particle α, and $\mathbf{r}_\alpha$ is the vector from the origin to particle α.

- The total momentum of a system can be written in terms of its centre of mass

$$\mathbf{P} = \sum_\alpha \mathbf{p}_\alpha = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha = M\dot{\mathbf{R}}$$

i.e. the total momentum of N particles is equivalent to that of a single particle of mass M with velocity equal to that of the centre of mass.

- Differentiating the above we find

$$\begin{aligned} \frac{d}{dt}\mathbf{P} &= \frac{d}{dt}(M\dot{\mathbf{R}}) \\ \mathbf{F}_{\text{ext}} &= M\ddot{\mathbf{R}} \end{aligned}$$

i.e. the centre of mass moves as if it was a single particle of mass M subject to the net external force on the system.

- When a body is continuous the expression for its centre of mass becomes an integral
$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\,dm = \frac{1}{M}\int \rho\mathbf{r}\,dV.$$

3.4 Angular Momentum for a Single Particle

- The **angular momentum** of a particle relative to an origin O is
$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$
where $\mathbf{r}$ is measured relative to O.
- Taking the derivative of angular momentum gives
$$\begin{aligned} \frac{d}{dt}\mathbf{L} &= \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) \\ \dot{\mathbf{L}} &= \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} \\ &= \mathbf{r} \times \mathbf{F} \\ &= \boldsymbol{\tau}. \end{aligned}$$
In other words, the rate of change in angular momentum about an origin O is equal to the net torque about that origin.
- We can simplify some one-particle problems by choosing the origin such that the net torque is 0 and thus angular momentum is constant.

3.5 Angular Momentum for Several Particles

- The **total angular momentum** of a system is
$$\mathbf{L} = \sum_{\alpha=1}^{N} \mathbf{L}_\alpha = \sum_{\alpha=1}^{N} \mathbf{r}_\alpha \times \mathbf{p}_\alpha.$$
- Differentiating the above
$$\dot{\mathbf{L}} = \sum_\alpha \dot{\mathbf{L}}_\alpha = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_\alpha = \boldsymbol{\tau}_{\text{ext}}$$
we find that the rate of change of the total angular momentum of the system is equal to the net torque on the system.
- The **principle of conservation of angular momentum** states that if the net external torque on a system is 0, the system's total angular momentum is constant. This assumes that all internal forces are central and obey Newton's third law.
- The principle of conservation of momentum and the result $\dot{\mathbf{L}} = \boldsymbol{\tau}_{\text{ext}}$ also hold if $\mathbf{L}$ and $\boldsymbol{\tau}_{\text{ext}}$ are measured about the centre of mass, even if the centre of mass is being accelerated and is thus not an interial frame.

4 Energy

4.1 Kinetic Energy and Work

- The **work-kinetic-energy theorem** states that the change in a particle's kinetic energy between two points is equal to the work done by the net force on the particle between those two points

$$\Delta K = \int_1^2 \mathbf{F} \cdot d\mathbf{r}.$$

4.2 Potential Energy and Conservative Forces

- A force $\mathbf{F}$ acting on a particle is considered **conservative** if:
  - $\mathbf{F}$ depends only on the particle's position $\mathbf{r}$ (and not on its velocity $\mathbf{v}$, time t, or any other variable), and
  - for any two points 1 and 2, the work done by $\mathbf{F}$ is the same for all paths between 1 and 2.
- Only conservative forces have associated **potential energy** functions.
- The potential energy function $U(\mathbf{r})$ of a conservative force $\mathbf{F}$ is defined as

$$U(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}')\, d\mathbf{r}'$$

  where $\mathbf{r}_0$ is an arbitrary point at which $U(\mathbf{r}_0)$ is defined to be 0.
- The **principle of conservation of energy** states that if all the forces acting on a particle are conservative, each with its corresponding potential energy function $U_i(\mathbf{r})$, the **total mechanical energy**

$$E = K + U = K + U_1(\mathbf{r}) + \cdots + U_n(\mathbf{r}),$$

  is constant in time.
- If nonconservative forces do work then the total energy of the system changes by that amount

$$\Delta E = W_{\mathrm{nc}}.$$

4.3 Force as the Gradient of Potential Energy

- A conservative force $\mathbf{F}$ can be expressed as the negative gradient of its potential energy function U

$$\mathbf{F} = -\nabla U.$$

4.4 The Second Condition that F be Conservative

- A force $\mathbf{F}$ is conservative if $\nabla \times \mathbf{F} = \mathbf{0}$.

4.5 Time-Dependent Potential Energy

- If a time-dependent force $\mathbf{F}(t)$ has the property $\nabla \times \mathbf{F}(t) = \mathbf{0}$ it's still possible to define an associated potential energy function $U(\mathbf{r}, t)$ where $\mathbf{F}(t) = -\nabla U(t)$ but it's no longer guaranteed that total mechanical energy is conserved over time.

4.8 Central Forces

- A central force is conservative if and only if it's spherically symmetric.