Advanced Engineering Mathematics Complex Analysis by Dennis G. Zill Problems

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Contents

17   Functions of a Complex Variable

17.1   Complex Numbers

17.1.1

$$3 + 3i$$

17.1.3

$$i^8 = (i^2)^4 = (-1)^4 = 1$$

17.1.5

$$7 - 13i$$

17.1.7

$$-7 + 5i$$

17.1.9

$$11 - 10i$$

17.1.11

$$-5 + 12i$$

17.1.13

$$-2i$$

17.1.15

$$\begin{aligned}
\frac{2 - 4i}{3 + 5i} &= \frac{(2 - 4i)(3 - 5i)}{34} \\
&= \frac{-14 - 22i}{34} \\
&= -\frac{7}{17} - \frac{11}{17}i
\end{aligned}$$

17.1.17

$$\begin{aligned}
\frac{(3 - i)(2 + 3i)}{1 + i} &= \frac{9 + 7i}{1 + i} \\
&= \frac{(9 + 7i)(1 - i)}{2} \\
&= \frac{16 - 2i}{2} \\
&= 8 - i
\end{aligned}$$

17.1.27

$$\begin{aligned}
\frac{1}{z} &= \frac{\overline{z}}{z\overline{z}} \\
&= \frac{x - iy}{x^2 + y^2} \\
\operatorname{Re}\left(\frac{1}{z}\right) &= \frac{x}{x^2 + y^2}
\end{aligned}$$

17.1.29

$$2z + 4\bar{z} - 4i = 2(x + iy) + 4(x - iy) - 4i$$
$$= 6x - 2(y + 2)i$$
$$\text{Im}(2z + 4\bar{z} - 4i) = -2y - 4$$

17.1.31

$$z - 1 - 3i = x + iy - 1 - 3i$$
$$= (x - 1) + (y - 3)i$$
$$|z| = \sqrt{(x - 1)^2 + (y - 3)^2}$$

17.1.33

$$2z = i(2 + 9i)$$
$$= -9 + 2i$$
$$z = -\frac{9}{2} + i$$

17.1.35

$$(x + iy)^2 = x^2 + 2xyi - y^2$$
$$= (x^2 - y^2) + 2xyi$$
$$x^2 = y^2$$
$$x = y$$
$$2xy = 1$$
$$x^2 = \frac{1}{2}$$
$$x = \frac{\sqrt{2}}{2}$$
$$z = \frac{\sqrt{2}}{2}(1 + i)$$

17.1.37

$$z + 2\overline{z} = x + iy + 2x - 2iy$$
$$= 3x - iy$$
$$\frac{2-i}{1+3i} = \frac{(2-i)(1-3i)}{10}$$
$$= \frac{-1-7i}{10}$$
$$3x - iy = \frac{-1-7i}{10}$$
$$x = -\frac{1}{30}$$
$$y = \frac{7}{10}$$
$$z = -\frac{1}{30} + \frac{7}{10}i$$

17.1.39

$$|10 + 8i| \approx 12.8$$
$$|11 - 6i| \approx 12.5$$

$11 - 6i$ is closer.

17.2   Powers and Roots

17.2.1

$$2(\cos 0 + i \sin 0)$$

17.2.3

$$-3[\cos(-\pi/2) + i \sin(-\pi/2)]$$

17.2.5

$$\sqrt{2}[\cos(\pi/4) + i \sin(\pi/4)]$$

17.2.7

$$2[\cos(5\pi/6) + i \sin(5\pi/6)]$$

17.2.9

$$\frac{3}{-1+i} = \frac{3(-1-i)}{2}$$
$$= \frac{-3-3i}{2}$$
$$= -\frac{3}{2} - \frac{3}{2}i$$
$$= \frac{3\sqrt{2}}{2}[\cos(5\pi/4) + i\sin(5\pi/4)]$$

17.2.11

$$-\frac{5\sqrt{3}}{2} - \frac{5}{2}i$$

17.2.13

$$5.54 + 2.30i$$

17.2.15

$$8[\cos(\pi/2) + i\sin(\pi/2)] = 8i$$
$$\frac{1}{2}[\cos(-\pi/4) + i\sin(-\pi/4)] = \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}i$$

17.2.21

$$(1 + \sqrt{3}i)^9 = \{2[\cos(\pi/3) + i\sin(\pi/3)]\}^9$$
$$= 512(\cos\pi + i\sin\pi)$$
$$= -512$$

17.2.23

$$\left(\frac{1}{2} + \frac{1}{2}i\right)^1 0 = \left\{\frac{\sqrt{2}}{2}[\cos(\pi/4) + i\sin(\pi/4)]\right\}^{10}$$
$$= \frac{1}{32}[\cos(\pi/2) + i\sin(\pi/2)]$$
$$= \frac{1}{32}i$$

17.2.27

$$w_k = 2[\cos(2\pi k/3) + i\sin(2\pi k/3)]$$
$$w_0 = 2$$
$$w_1 = -1 + \sqrt{3}i$$
$$w_2 = -1 - \sqrt{3}i$$

17.2.29

$$w_k = \cos(\pi/4 + k\pi) + i\sin(\pi/4 + k\pi)$$
$$w_0 = \frac{\sqrt{2}}{2}(1 + i)$$
$$w_1 = -\frac{\sqrt{2}}{2}(1 + i)$$

17.2.31

$$w_k = \sqrt{2}[\cos(\pi/3 + k\pi) + i\sin(\pi/3 + k\pi)]$$
$$w_0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i$$
$$w_1 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i$$

17.2.33

$$z^4 + 1 = 0$$
$$z^4 = -1$$
$$w_k = \cos(\pi/4 + k\pi/2) + \sin(\pi/4 + k\pi/2)$$
$$w_0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$
$$w_1 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$
$$w_2 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$
$$w_3 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

17.3   Sets in the Complex Plane

17.3.1

A vertical line at $\text{Re}(z) = 5$.

17.3.3

A horizontal line at $\mathrm{Im}(z) = -3$.

17.3.5

A circle of radius 2 centred at $3i$.

17.3.7

A circle of radius 5 centred at $4 - 3i$.

17.3.9

The region of the plane to the left of (but not including) $\mathrm{Re}(z) = -1$. It is a domain.

17.3.11

The region of the plane above (but not including) $\mathrm{Im}(z) = 3$. It is a domain.

17.3.13

The region of the plane between (but not including) $\mathrm{Re}(z) = 3$ and $\mathrm{Re}(z) = 5$. It is a domain.

17.3.15

$$
\begin{aligned}
z^2 &= (a + ib)^2 \\
&= a^2 - b^2 + 2iab \\
\mathrm{Re}(z^2) &= a^2 - b^2 \\
\mathrm{Re}(z^2) &> 0 \\
a^2 - b^2 &> 0 \\
a^2 &> b^2
\end{aligned}
$$

The region between $y = x$ and $y = -x$. Not a domain.

17.3.17

The region between $\theta = 0$ and $\theta = 2\pi/3$. Not a domain.

17.3.19

The region outside a circle of radius 1 centred at i. It is a domain.

17.3.21

The region between the circles of radius 2 and 3 centred at i. It is a domain.

17.3.23

$$y = -x$$

17.3.25

$$
\begin{aligned}
z^2 + \bar{z}^2 &= (a + ib)^2 + (a - ib)^2 \\
&= a^2 + 2iab - b^2 + a^2 - 2iab - b^2 \\
&= 2(a^2 - b^2) \\
2(a^2 - b^2) &= 2 \\
a^2 - b^2 &= 1 \\
a^2 &= b^2 + 1
\end{aligned}
$$

The hyperbola $x^2 - y^2 = 1$.

17.4 Functions of a Complex Variable

17.4.1

$$
\begin{aligned}
f(z) &= z^2 \\
&= (x + iy)^2 \\
&= x^2 - y^2 + 2ixy \\
u(x, y) &= x^2 - y^2 \\
&= x^2 - 4 \\
v(x, y) &= 2xy \\
&= 4x \\
x &= \frac{v}{4} \\
u &= \left(\frac{v}{4}\right)^2 - 4 \\
&= \frac{1}{16}v^2 - 4
\end{aligned}
$$

17.4.3

$$
\begin{aligned}
u &= -y^2 \\
v &= 0
\end{aligned}
$$

Line on the left half of the real axis.

17.4.5

$$u = 0$$
$$v = 2x^2$$

Line on the top half of the imaginary axis.

17.4.7
$$f(x) = (6x - 5) + i(6y + 9)$$

17.4.9
$$f(z) = (x^2 - y^2 - 3x) + i(2xy - 3y + 4)$$

17.4.11
$$f(z) = (x^3 - 3xy^2 - 4x) + i(3x^2y - y^3 - 4y)$$

17.4.13
$$f(z) = \left(x + \frac{x}{x^2 + y^2}\right) i \left(y - \frac{y}{x^2 + y^2}\right)$$

17.4.15

(a) $-4 + i$

(b) $3 - 9i$

(c) $1 + 86i$

17.4.17

(a) $14 - 20i$

(b) $-13 + 43i$

(c) $3 - 26i$

17.4.19
$$6 - 5i$$

17.4.21
$$-4i$$

17.4.27

$$f'(z) = 12z^2 - 2(3+i)z - 5$$

17.4.29

$$f'(z) = 2(z^2 - 4z + 8i) + (2z+1)(2z-4)$$
$$= 2z^2 - 8z + 16i + 4z^2 - 8z + 2z - 4$$
$$= 6z^2 - 14z - 4 + 16i$$

17.4.31

$$f'(z) = 6z(z^2 - 4i)^2$$

17.4.33

$$f'(z) = \frac{3(2z+i) - 2(3z - 4 + 8i)}{(2z+i)^2}$$
$$= \frac{6z + 3i - 6z + 8 - 16i}{(2z+i)^2}$$
$$= \frac{8 - 13i}{(2z+i)^2}$$

17.4.35

$3i$

17.4.37

$\pm 2i$

17.4.41

$$\frac{dx}{dt} = 2x$$
$$x = c_1 e^{2t}$$
$$\frac{dy}{dt} = 2y$$
$$y = c_2 e^{2t}$$

17.4.43

$$f(z) = \frac{1}{z}$$
$$= \frac{1}{x - iy}$$
$$= \frac{x + iy}{x^2 + y^2}$$
$$= \frac{x}{x^2 + y^2} + i\frac{y}{x^2 + y^2}$$
$$\frac{dx}{dt} = \frac{x}{x^2 + y^2}$$
$$\frac{dy}{dt} = \frac{y}{x^2 + y^2}$$
$$\frac{dy}{dx} = \frac{y}{x}$$
$$\frac{dy}{y} = \frac{dx}{x}$$
$$\ln y = \ln x + c_1$$
$$y = c_2 x$$

17.5   Cauchy-Riemann Equations

17.5.1

$$f(z) = z^3$$
$$= (x + iy)^3$$
$$= (x^2 + 2ixy - y^2)(x + iy)$$
$$= x^3 + ix^2 y + 2ix^2 y - 2xy^2 - xy^2 - iy^3$$
$$= (x^3 - 3xy^2) + i(3x^2 y - y^3)$$
$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$$
$$= \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -6xy$$
$$= -\frac{\partial v}{\partial x}$$

17.5.3

$$f(z) = \text{Re}(z)$$
$$= x$$
$$\frac{\partial u}{\partial x} = 1$$
$$\neq \frac{\partial v}{\partial y}$$

17.5.5

$$f(z) = 4z - 6\bar{z} + 3$$
$$= 4(x + iy) - 6(x - iy) + 3$$
$$= (-2x + 3) + 10iy$$
$$\frac{\partial u}{\partial x} = -2$$
$$\neq \frac{\partial v}{\partial y}$$

17.5.7

$$f(z) = x^2 + y^2$$
$$\frac{\partial u}{\partial x} = 2x$$
$$\neq \frac{\partial v}{\partial y}$$

17.5.9

$$f(z) = e^x \cos y + ie^x \sin y$$
$$u = e^x \cos y$$
$$\frac{\partial u}{\partial x} = e^x \cos y$$
$$\frac{\partial u}{\partial y} = -e^x \sin y$$
$$v = e^x \sin y$$
$$\frac{\partial v}{\partial x} = e^x \sin y$$
$$\frac{\partial v}{\partial y} = e^x \cos y$$

Analytic everywhere.

17.5.11

$$f(z) = x + \sin x \cosh y + i(y + \cos x \sinh y)$$
$$u = x + \sin x \cosh y$$
$$\frac{\partial u}{\partial x} = 1 + \cos x \cosh y$$
$$\frac{\partial u}{\partial y} = \sin x \sinh y$$
$$v = y + \cos x \sinh y$$
$$\frac{\partial v}{\partial x} = -\sin x \sinh y$$
$$\frac{\partial v}{\partial y} = 1 + \cos x \cosh y$$

Analytic everywhere.

17.5.15

$$f(z) = 3x - y + 5 + i(ax + by - 3)$$
$$u = 3x - y + 5$$
$$\frac{\partial u}{\partial x} = 3$$
$$\frac{\partial u}{\partial y} = -1$$
$$v = ax + by - 3$$
$$\frac{\partial v}{\partial x} = a$$
$$\frac{\partial v}{\partial y} = b$$
$$a = 1$$
$$b = 3$$

17.5.17

$$f(z) = x^2 + y^2 + 2ixy$$
$$u = x^2 + y^2$$
$$\frac{\partial u}{\partial x} = 2x$$
$$\frac{\partial u}{\partial y} = 2y$$
$$v = 2xy$$
$$\frac{\partial v}{\partial x} = 2y$$
$$\frac{\partial v}{\partial y} = 2x$$

Only differentiable when $y = 0$.

17.5.19

$$f(z) = x^3 + 3xy^2 - x + i(y^3 + 3x^2y - y)$$
$$u = x^3 + 3xy^2 - x$$
$$\frac{\partial u}{\partial x} = 3x^2 + 3y^2 - 1$$
$$\frac{\partial u}{\partial y} = 6xy$$
$$v = y^3 + 3x^2y - y$$
$$\frac{\partial v}{\partial x} = 6xy$$
$$\frac{\partial v}{\partial y} = 3y^2 + 3x^2 - 1$$

Only differentiable when $x = 0$ or $y = 0$.

17.5.21

$$f(z) = e^x \cos y + ie^x \sin y$$
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= e^x \cos y + ie^x \sin y$$

17.5.23

$$u = x$$
$$\frac{\partial^2 u}{\partial x^2} = 0$$
$$\frac{\partial^2 u}{\partial y^2} = 0$$
$$\frac{\partial v}{\partial y} = 1$$
$$v = y + h(x)$$
$$h'(x) = 0$$
$$v = y + c$$
$$f(z) = x + i(y + c)$$

17.5.25

$$u = x^2 - y^2$$
$$\frac{\partial^2 u}{\partial x^2} = 2$$
$$\frac{\partial^2 u}{\partial y^2} = -2$$
$$\frac{\partial v}{\partial y} = 2x$$
$$v = 2xy + h(x)$$
$$2y = 2y + h'(x)$$
$$h'(x) = 0$$
$$h(x) = c$$
$$v = 2xy + c$$
$$f(z) = (x^2 - y^2) + i(2xy + c)$$

17.6   Exponential and Logarithmic Functions

17.6.1

$$\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

17.6.3

$$e^{-1}\frac{\sqrt{2}}{2}(1 + i)$$

17.6.5

$$-e^{\pi}$$

17.6.7

$$e^{1.5}(\cos 2 + i \sin 2) = -1.865 + 4.075i$$

17.6.9

$$\cos 5 + i \sin 5 = 0.2836 - 0.9589i$$

17.6.11

$$e^{1+5\pi i/4}e^{-1-\pi i/3} = e^{11\pi i/12}$$
$$= \cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}$$
$$= -0.9659 + 0.2588i$$

17.6.13

$$f(z) = e^{-iz}$$
$$= e^{-i(x+iy)}$$
$$= e^{y-ix}$$
$$= e^{y}(\cos x - i\sin x)$$

17.6.15

$$f(z) = e^{z^2}$$
$$= e^{x^2-y^2+2ixy}$$
$$= e^{x^2-y^2}[\cos(2xy) + i\sin(2xy)]$$

17.6.17

$$e^z = e^{x+iy}$$
$$= e^x(\cos y + i\sin y)$$
$$|e^z| = \sqrt{e^{2x}[\cos^2 y + \sin^2 y]}$$
$$= e^x$$

17.6.19

$$e^{z+\pi i} = e^{x+i(y+\pi)}$$
$$= e^x[\cos(y+\pi) + i\sin(y+\pi)]$$
$$= e^x[-\cos y - i\sin y]$$
$$= -e^x(\cos y + i\sin y)$$
$$e^{z-\pi i} = e^{x+i(y-\pi)}$$
$$= e^x[\cos(y-\pi) + i\sin(y-\pi)]$$
$$= e^x(-\cos y - i\sin y)$$
$$= -e^x(\cos y + i\sin y)$$

17.6.21

$$e^{\overline{z}} = e^{x-iy}$$
$$= e^x(\cos y - i\sin y)$$
$$u = e^x\cos y$$
$$v = -e^x\sin y$$
$$\frac{\partial u}{\partial x} = e^x\cos y$$
$$\neq \frac{\partial v}{\partial y}$$

17.6.23

$$\log_e 5 + i(\pi + 2n\pi) = 1.6094 + i(\pi + 2n\pi)$$

17.6.25

$$\log_e(2\sqrt{2}) + i\left(\frac{3}{4}\pi + 2n\pi\right) = 1.0397 + i\left(\frac{3}{4}\pi + 2n\pi\right)$$

17.6.27

$$\log_e(2\sqrt{2}) + i\left(\frac{1}{3}\pi + 2n\pi\right) = 1.0397 + i\left(\frac{1}{3}\pi + 2n\pi\right)$$

17.6.29

$$\log_e(6\sqrt{2}) - \frac{\pi}{4}i = 2.1383 - \frac{\pi}{4}i$$

17.6.31

$$\log_e 13 + 2.7468i = 2.5649 + 2.7468i$$

17.6.33

$$5 \left(\log_e 2 + \frac{\pi}{3} i \right) = 3.4657 - \frac{\pi}{3} i$$

17.6.35

$$z = \log_e 4 + i \left(\frac{\pi}{2} + 2n\pi \right) = 1.3863 + i \left(\frac{\pi}{2} + 2n\pi \right)$$

17.6.37

$$z - 1 = 2 + i \left(-\frac{\pi}{2} + 2n\pi \right)$$
$$z = 3 + i \left(-\frac{\pi}{2} + 2n\pi \right)$$

17.6.39

$$\ln(-i) = i \left(-\frac{\pi}{2} + 2n\pi \right)$$
$$(-i)^{4i} = e^{4i \ln(-i)}$$
$$= e^{4i \times i(-\pi/2 + 2n\pi)}$$
$$= e^{2\pi(1 - 4n)}$$

17.6.41

$$\ln(1 + i) = \log_e \sqrt{2} + i \left(\frac{\pi}{4} + 2n\pi \right)$$
$$(1 + i)^{(1+i)} = e^{(1+i) \ln(1+i)}$$
$$= e^{(1+i)[\log_e \sqrt{2} + i(\pi/4 + 2n\pi)]}$$
$$= e^{\log_e \sqrt{2} + i(\pi/4 + 2n\pi) + i \log_e \sqrt{2} - (\pi/4 + 2n\pi)}$$
$$= e^{(\log_e \sqrt{2} - \pi/4 - 2n\pi) + i(\log_e \sqrt{2} + \pi/4 + 2n\pi)}$$
$$= e^{-2n\pi} e^{(\log_e \sqrt{2} - \pi/4) + i(\log_e \sqrt{2} + \pi/4)}$$
$$= e^{-2n\pi} e^{\log_e \sqrt{2} - \pi/4} e^{i(\log_e \sqrt{2} + \pi/4)}$$
$$= e^{-2n\pi} (0.2739 + 0.5837i)$$

17.6.43

$$\mathrm{Ln}(-1) = \pi i$$
$$(-1)^{(-2i/\pi)} = e^{(-2i/\pi) \mathrm{Ln}(-1)}$$
$$= e^{(-2i/\pi)(\pi i)}$$
$$= e^2$$

17.6.47

(a)

$$(-1+i)^2 = -2i$$
$$\text{Ln}(-1+i)^2 = \text{Ln}(-2i)$$
$$= \log_e 2 - \frac{\pi}{2}i$$
$$2\,\text{Ln}(-1+i) = 2\log_e \sqrt{2} + \frac{3\pi}{2}i$$
$$\neq \text{Ln}(-1+i)^2$$

Not true

(b)

$$\text{Ln}\,i^3 = \text{Ln}(-i)$$
$$= -\frac{\pi}{2}i$$
$$3\,\text{Ln}\,i = \frac{3\pi}{2}i$$
$$\neq \text{Ln}\,i^3$$

Not true

(c)

$$\ln i^3 = i\left(-\frac{\pi}{2} + 2n\pi\right)$$
$$3\ln i = 3i\left(\frac{\pi}{2} + 2n\pi\right)$$
$$\neq \ln i^3$$

Not true

17.7 Trigonometric and Hyperbolic Functions

17.7.1

$$\cos(3i) = \cos 0 \cosh 3 - i\sin 0 \sinh 3$$
$$= \cosh 3$$
$$= 10.0677$$

17.7.3

$$\sin(\pi/4 + i) = \sin\frac{\pi}{4}\cosh 1 + i\cos\frac{\pi}{4}\sinh 1$$
$$= 1.0911 + 0.8309i$$

25

17.7.5

$$\begin{aligned}
\tan i &= \frac{\sin i}{\cos i} \\
&= \frac{\sin 0 \cosh 1 + i \cos 0 \sinh 1}{\cos 0 \cosh 1 + i \sin 0 \sinh 1} \\
&= \frac{i \sinh 1}{\cosh 1} \\
&= i \tanh 1 \\
&= 0.7615i
\end{aligned}$$

17.7.7

$$\begin{aligned}
\sec(\pi + i) &= \frac{1}{\cos(\pi + i)} \\
&= \frac{1}{\cos \pi \cosh 1 + \sin \pi \sinh 1} \\
&= -\frac{1}{\cosh 1} \\
&= -0.6480
\end{aligned}$$

17.7.9

$$\begin{aligned}
\cosh(\pi i) &= \cosh 0 \cos \pi + i \sinh 0 \sin \pi \\
&= -1
\end{aligned}$$

17.7.11

$$\begin{aligned}
\sinh(1 + \pi i/3) &= \sinh 1 \cos(\pi/3) + i \cosh 1 \sin(\pi/3) \\
&= 0.5876 + 1.3363i
\end{aligned}$$

17.7.15

$$\sin z = 2$$

$$\frac{e^{iz} - e^{-iz}}{2i} = 2$$

$$e^{iz} - e^{-iz} = 4i$$

$$e^{2iz} - 1 = 4ie^{iz}$$

$$e^{2iz} - 4ie^{iz} - 1 = 0$$

$$e^{iz} = \frac{4i \pm \sqrt{-16 + 4}}{2}$$

$$= (2 \pm \sqrt{3})i$$

$$iz = \log_e(2 \pm \sqrt{3}) + i(\pi/2 + 2n\pi)$$

$$z = (\pi/2 + 2n\pi) - i\log_e(2 \pm \sqrt{3})$$

17.7.17

$$\sinh z = -i$$

$$\frac{e^z - e^{-z}}{2} = -i$$

$$e^{2z} + 2ie^z - 1 = 0$$

$$e^z = \frac{-2i \pm \sqrt{-4 + 4}}{2}$$

$$= -i$$

$$z = \ln(-i)$$

$$= i\left(-\frac{\pi}{2} + 2n\pi\right)$$

17.7.19

$$\cos z = \sin z$$

$$\frac{e^{iz} + e^{-iz}}{2} = \frac{e^{iz} - e^{-iz}}{2i}$$

$$e^{iz} + e^{-iz} = \frac{e^{iz} - e^{-iz}}{i}$$

$$= -i(e^{iz} - e^{-iz})$$

$$e^{2iz} + 1 = -i(e^{2iz} - 1)$$

$$e^{2iz}(1 + i) = -1 + i$$

$$e^{2iz} = \frac{-1 + i}{1 + i}$$

$$= \frac{(-1 + i)(1 - i)}{(1 + i)(1 - i)}$$

$$= \frac{-1 + i + i + 1}{1 - i + i + 1}$$

$$= \frac{2i}{2}$$

$$= i$$

$$2iz = \ln i$$

$$= i\left(\frac{\pi}{2} + 2n\pi\right)$$

$$z = \frac{\pi}{4} + n\pi$$

17.7.21

$$\cos z = \cosh 2$$

$$\cos x \cosh y - i \sin x \sinh y = \cosh 2$$

$$y = \pm 2$$

$$x = 2n\pi$$

$$z = 2n\pi \pm 2i$$

17.8 Inverse Trigonometric and Hyperbolic Functions

17.8.1

$$\arcsin z = -i \ln[iz + (1 - z^2)^{1/2}]$$
$$\arcsin(-i) = -i \ln[i(-i) + (1 - (-i)^2)^{1/2}]$$
$$= -i \ln[1 \pm \sqrt{2}]$$
$$\ln(1 + \sqrt{2}) = \log_e(1 + \sqrt{2}) + 2n\pi i$$
$$\ln(1 - \sqrt{2}) = \ln\left(-\frac{1}{1 + \sqrt{2}}\right)$$
$$= -\ln[-(1 + \sqrt{2})]$$
$$= -[\log_e(1 + \sqrt{2}) + i(\pi + 2n\pi)]$$
$$= -\log_e(1 + \sqrt{2}) + i(\pi + 2n\pi)$$
$$\ln(1 \pm \sqrt{2}) = (-1)^n \log_e(1 + \sqrt{2}) + n\pi i$$
$$\arcsin(-i) = -i[(-1)^n \log_e(1 + \sqrt{2}) + n\pi i]$$
$$= n\pi - (-1)^n i \log_e(1 + \sqrt{2})$$
$$= n\pi + (-1)^{n+1} i \log_e(1 + \sqrt{2})$$

17.8.3

$$\arcsin 0 = -i \ln(\pm 1)$$
$$= -i(n\pi i)$$
$$= n\pi$$

17.8.5

$$\arccos 2 = -i \ln[2 + i(1 - 2^2)^{1/2}]$$
$$= -i \ln[2 \pm \sqrt{3}]$$
$$\ln(2 + \sqrt{3}) = \log_e(2 + \sqrt{3}) + 2n\pi i$$
$$\ln(2 - \sqrt{3}) = \log_e(2 - \sqrt{3}) + 2n\pi i$$
$$= -\log_e(2 + \sqrt{3}) + 2n\pi i$$
$$\ln(2 \pm \sqrt{3}) = \pm \log_e(2 + \sqrt{3}) + 2n\pi i$$
$$\arccos 2 = 2n\pi \pm i \log_e(2 + \sqrt{3})$$

17.8.7

$$\arccos \frac{1}{2} = -i \ln \left\{ \frac{1}{2} + i \left[1 - \left(\frac{1}{2} \right)^2 \right]^{1/2} \right\}$$

$$= -i \ln \left(\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \right)$$

$$\ln \left(\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \right) = i \left(\pm \frac{\pi}{3} + 2n\pi \right)$$

$$\arccos \frac{1}{2} = \pm \frac{\pi}{3} + 2n\pi$$

17.8.9

$$\arctan 1 = \frac{i}{2} \ln \frac{1+i}{-1+i}$$

$$\frac{1+i}{-1+i} = \frac{(1+i)(-1-i)}{(-1+i)(-1-i)}$$

$$= -i$$

$$\ln(-i) = i \left(-\frac{\pi}{2} + 2n\pi \right)$$

$$\arctan 1 = \frac{\pi}{4} + n\pi$$

17.8.11

$$\operatorname{arcsinh} \frac{4}{3} = \ln\left\{\frac{4}{3} + \left[\left(\frac{4}{3}\right)^2 + 1\right]^{1/2}\right\}$$

$$= \ln\left(\frac{4}{3} \pm \frac{5}{3}\right)$$

$$\ln\left(\frac{4}{3} + \frac{5}{3}\right) = \ln\frac{9}{3}$$

$$= \ln 3$$

$$= \log_e 3 + 2n\pi i$$

$$\ln\left(\frac{4}{3} - \frac{5}{3}\right) = \ln\left(-\frac{1}{3}\right)$$

$$= \log_e \frac{1}{3} + i(\pi + 2n\pi)$$

$$= -\log_e 3 + i(\pi + 2n\pi)$$

$$\operatorname{arcsinh} \frac{4}{3} = (-1)^n \log_e 3 + n\pi i$$

17.9   Chapter in Review

17.9.1

0, 32

17.9.3

$$\frac{3+4i}{3-4i} = \frac{(3+4i)^2}{(3-4i)(3+4i)}$$

$$= \frac{-7+24i}{25}$$

$$= -\frac{7}{25} + \frac{24}{25}i$$

$$\operatorname{Re}\left(\frac{z}{\bar{z}}\right) = -\frac{7}{25}$$

17.9.5

$$\frac{4i}{-3-4i} = \frac{(4i)(-3+4i)}{(-3-4i)(-3+4i)}$$
$$= \frac{-16-12i}{25}$$
$$= -\frac{16}{25} - \frac{12}{25}i$$
$$|z| = \sqrt{\left(\frac{16}{25}\right)^2 + \left(\frac{12}{25}\right)^2}$$
$$= \frac{4}{5}$$

17.9.7

False

17.9.9

$$e^z = 2i$$
$$z = \ln(2i)$$
$$= \log_e 2 + i\left(\frac{\pi}{2} + 2n\pi\right)$$

17.9.11

$$(1+i)^{(2+i)} = e^{(2+i)\ln(1+i)}$$
$$ln(1+i) = \log_e \sqrt{2} + \frac{\pi}{4}i$$
$$(2+i)\left(\log_e \sqrt{2} + \frac{\pi}{4}i\right) = 2\log_e \sqrt{2} + \frac{\pi}{2}i + i\log_e \sqrt{2} - \frac{\pi}{4}$$
$$= \left(2\log_e \sqrt{2} - \frac{\pi}{4}\right) + i\left(\log_e \sqrt{2} + \frac{\pi}{2}\right)$$
$$(1+i)^{(2+i)} = e^{2\log_e \sqrt{2} - \pi/4}\left[\cos\left(\log_e \sqrt{2} + \frac{\pi}{2}\right) + i\sin\left(\log_e \sqrt{2} + \frac{\pi}{2}\right)\right]$$
$$\approx -0.3097 + 0.8576i$$

17.9.13

False

17.9.15

$$\text{Ln}(-ie^3) = 3 - \frac{\pi}{2}i$$

17.9.21

$$z^2 = x^2 - y^2 + 2ixy$$
$$\text{Im}(z^2) \le 2$$
$$2xy \le 2$$

17.9.23

$$\frac{1}{\sqrt{x^2 + y^2}} \le 1$$

17.9.27

$$z^4 = 1 - i$$
$$z_k = 2^{1/8} e^{(-\pi/4 + 2k\pi)i/4}$$
$$= 2^{1/8} e^{i(k\pi/2 - \pi/16)}$$
$$z_0 = 1.0695 - 0.2127i$$
$$z_1 = 0.2127 + 1.0695i$$
$$z_2 = -1.0695 + 0.2127i$$
$$x_3 = -0.2127 - 1.0695i$$

18   Integration in the Complex Plane

18.1   Contour Integrals

18.1.1

$$z(t) = 2t + i(4t - 1)$$
$$z'(t) = 2 + 4i$$
$$f(z(t)) = (2t + 3) + i(4t - 1)$$
$$f(z(t))z'(t) = [(2t + 3) + i(4t - 1)](2 + 4i)$$
$$= (2t + 3)(2) + (2t + 3)(4i) + i(4t - 1)(2) + i(4t - 1)(4i)$$
$$= 4t + 6 + 8it + 12i + 8it - 2i - 16t + 4$$
$$= (-12t + 10) + i(16t + 10)$$
$$\int_C f(z)\,dz = \int_1^3 f(z(t))z'(t)\,dt$$
$$= \int_1^3 (-12t + 10)\,dt + i\int_1^3 (16t + 10)\,dt$$
$$= -28 + 84i$$

18.1.3

$$z(t) = 3t + 2it$$
$$z'(t) = 3 + 2i$$

$$\int_C f(z)\,dz = \int_{-2}^{2} (3t + 2it)^2 (3 + 2i)\,dt$$

$$= \int_{-2}^{2} [(3 + 2i)t]^2 (3 + 2i)\,dt$$

$$= (3 + 2i)^3 \int_{-2}^{2} t^2\,dt$$

$$= (-9 + 46i)\frac{16}{3}$$

$$= -48 + \frac{736}{3}i$$

18.1.5

$$z(t) = e^{it}$$
$$z'(t) = ie^{it}$$

$$\int_C f(z)\,dz = \int_{-\pi/2}^{\pi/2} \frac{1 + e^{it}}{e^{it}} ie^{it}\,dt$$

$$= i \int_{-\pi/2}^{\pi/2} (1 + e^{it})\,dt$$

$$= i\left[t + \frac{1}{i}e^{it} \right]_{-\pi/2}^{\pi/2}$$

$$= i[t - ie^{it}]_{-\pi/2}^{\pi/2}$$

$$= i\left(\frac{\pi}{2} - ie^{\pi i/2} + \frac{\pi}{2} + ie^{-\pi i/2} \right)$$

$$= i(\pi + 2)$$

18.1.7

$$z(t) = \cos t + i\sin t$$
$$z'(t) = -\sin t + i\cos t$$

$$\int_C f(z)\,dz = \int_0^{2\pi} \cos t(-\sin t + i\cos t)\,dt$$

$$= \int_0^{2\pi} \left(-\frac{1}{2}\sin 2t + i\cos^2 t \right)\,dt$$

$$= \pi i$$

18.1.9

$$z(t) = (1 - t) + it$$
$$z'(t) = -1 + i$$
$$\int_C f(z)\, dz = \int_0^1 [(1 - t)^2 + it^3](-1 + i)\, dt$$
$$= \int_0^1 (1 - 2t + t^2 + it^3)(-1 + i)\, dt$$
$$= \int_0^1 (-1 + i + 2t - 2it - t^2 + it^2 - it^3 - t^3)\, dt$$
$$= \int_0^1 (-1 + 2t - t^2 - t^3)\, dt + i \int_0^1 (1 - 2t + t^2 - t^3)\, dt$$
$$= -\frac{7}{12} + \frac{1}{12}i$$

18.1.17

$$z(t) = 1 + it$$
$$z'(t) = i$$
$$\int_{C_1} f(z)\, dz = \int_0^1 i\, dt$$
$$= i$$
$$z(t) = (1 - t) + i(1 - t)$$
$$z'(t) = -(1 + i)$$
$$\int_{C_2} f(z)\, dz = -\int_0^1 (1 - t)(1 + i)\, dt$$
$$= -\int_0^1 (1 + i - t - it)\, dt$$
$$= -\frac{1}{2} - \frac{1}{2}i$$
$$z(t) = t$$
$$z'(t) = 1$$
$$\int_{C_3} f(z)\, dz = \int_0^1 t\, dt$$
$$= \frac{1}{2}$$
$$\int_C f(z)\, dz = \frac{1}{2}i$$

18.1.19

$$z(t) = 1 + it$$

$$z'(t) = i$$

$$\int_{C_1} f(z)\, dz = \int_0^1 (1+it)^2 i\, dt$$

$$= i \int_0^1 (1 + 2it - t^2)\, dt$$

$$= -1 + \frac{2}{3}i$$

$$z(t) = (1+i)(1-t)$$

$$z'(t) = -(1+i)$$

$$\int_{C_2} f(z)\, dz = -\int_0^1 [(1+i)(1-t)]^2 (1+i)\, dt$$

$$= -\int_0^1 (1 - t + i - it)^2 (1+i)\, dt$$

$$= \frac{2}{3} - \frac{2}{3}i$$

$$z(t) = t$$

$$z'(t) = 1$$

$$\int_{C_3} f(z)\, dz = \int_0^1 t^2\, dt$$

$$= \frac{1}{3}$$

$$\int_C f(z)\, dz = 0$$

18.1.23

$$z(t) = t + i(1 - t^2)$$

$$z'(t) = 1 - 2it$$

$$\int_C f(z)\, dz = \frac{4}{3} - \frac{5}{3}i$$

18.1.25

$$L = 10\pi$$

$$|z^2 + 1| \geq |z^2| - 1$$

$$\left| \frac{e^z}{z^2 + 1} \right| \leq \frac{|e^z|}{|z^2| - 1}$$

$$= \frac{e^5}{24}$$

$$= M$$

$$\left| \oint \frac{e^z}{z^2 + 1} \, dz \right| \leq ML$$

$$= \frac{5\pi e^5}{12}$$

18.1.27

$$z(t) = (1 + i)t, \; 0 \leq t \leq 1$$

$$L = \sqrt{2}$$

$$|z^2 + 4| = |2it^2 + 4|$$

$$\leq |2i + 4|$$

$$= \sqrt{20}$$

$$= 2\sqrt{5}$$

$$= M$$

$$\left| \oint (z^2 + 4) \, dz \right| \leq ML$$

$$= 2\sqrt{10}$$

18.1.33

$$z(t) = e^{it}$$

$$z'(t) = ie^{it}$$

$$\oint \overline{f(z)} \, dz = \int_0^{2\pi} 2e^{-it} ie^{it} \, dt$$

$$= 4\pi i$$

The circulation is 0 and the flux is 4π.

18.2 Cauchy-Goursat Theorem

18.2.1

$$z = e^{it}, \ 0 \le t \le 2\pi$$

$$z' = ie^{it}$$

$$\int (z^3 - 1 + 3i) \, dz = \int_0^{2\pi} [(e^{it})^3 - 1 + 3i]ie^{it} \, dt$$

$$= i \int_0^{2\pi} (e^{4it} - e^{it} + 3ie^{it}) \, dt$$

$$= \left[\frac{1}{4}e^{4it} - e^{it} + 3ie^{it} \right]_0^{2\pi}$$

$$= \frac{1}{4}e^{8\pi i} - e^{2\pi i} + 3ie^{2\pi i} - \frac{1}{4} + 1 - 3i$$

$$= \frac{1}{4} - 1 + 3i - \frac{1}{4} + 1 - 3i$$

$$= 0$$

18.2.9

$$\int_C \frac{1}{z} \, dz = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it} \, dt$$

$$= 2\pi i$$

18.2.11

$$\oint_C \left(z + \frac{1}{z} \right) dz = \oint_C \frac{1}{z^{-1}} \, dz + \oint_C \frac{1}{z} \, dz$$

$$= 0 + 2\pi i$$

$$= 2\pi i$$

18.2.13

$$z^2 - \pi^2 = (z + \pi)(z - \pi)$$

$$\frac{z}{z^2 - \pi^2} = \frac{1/2}{z + \pi} + \frac{1/2}{z - \pi}$$

$$\oint_C \frac{z}{z^2 - \pi^2} \, dz = \frac{1}{2} \oint \left(\frac{1}{z + \pi} + \frac{1}{z - \pi} \right) dz$$

$$= 0$$

18.2.15

(a)

$$\frac{2z+1}{z^2+z} = \frac{2z+1}{z(z+1)}$$

$$= \frac{1}{z} + \frac{1}{z+1}$$

$$\oint_C \frac{2z+1}{z^2+z}\,dz = \oint_C \frac{2z+1}{z(z+1)}\,dz$$

$$= \oint_C \left(\frac{1}{z} + \frac{1}{z+1}\right)\,dz$$

$$= 2\pi i$$

(b) $4\pi i$

(c) 0

18.2.17

(a)

$$\frac{-3z+2}{z^2-8z+12} = \frac{1}{z-1} - \frac{4}{z-6}$$

$$\oint_C \frac{-3z+2}{z^2-8z+12}\,dz = \oint \left(\frac{1}{z-1} - \frac{4}{z-6}\right)\,dz$$

$$= -8\pi i$$

(b) $-6\pi i$

18.2.19

$$\frac{z-1}{z(z-i)(z-3i)} = \frac{1}{3z} - \frac{1/2 - i/2}{z-i} + \frac{1/6 - i/2}{z-3i}$$

$$\oint_C \frac{z-1}{z(z-i)(z-3i)}\,dz = -\left(\frac{1}{2} - \frac{i}{2}\right)2\pi i$$

$$= -\pi(1+i)$$

18.2.21

$$\frac{8z - 3}{z^2 - z} = \frac{3}{z} + \frac{5}{z - 1}$$

$$\oint_C f(z)\, dz = \oint_{C_1} f(z)\, dz - \oint_{C_2} f(z)\, dz$$

$$= \oint_{C_1} \left(\frac{3}{z} + \frac{5}{z - 1} \right) dz - \oint_{C_2} \left(\frac{3}{z} + \frac{5}{z - 1} \right) dz$$

$$= 6\pi i - 10\pi i$$

$$= -4\pi i$$

18.2.23

$$\oint_C \left(\frac{e^z}{z + 3} - 3\bar{z} \right) dz = \oint_C \frac{e^z}{z + 3}\, dz - 3 \oint_C \bar{z}\, dz$$

$$= -3 \oint_0^{2\pi} e^{-it} i e^{it}\, dt$$

$$= -6\pi i$$

18.3   Independence of the Path

18.3.1

(a)

$$z(t) = i(t - 1), \ 0 \leq t \leq 2$$

$$z'(t) = i$$

$$\int_C (4z - 1)\, dz = \int_0^2 \{4[i(t - 1)] - 1\} i\, dt$$

$$= \int_0^2 [4(1 - t) - i]\, dt$$

$$= \left[4 \left(t - \frac{1}{2} t^2 \right) - it \right]_0^2$$

$$= -2i$$

(b)

$$F(z) = 2z^2 - z$$

$$\int_C (4z - 1)\, dz = F(i) - F(-i)$$

$$= [2(i)^2 - (i)] - [2(-i)^2 - (-i)]$$

$$= -2 - i + 2 - i$$

$$= -2i$$

18.3.3

$$z(-1) = -2 + 7i$$

$$z(1) = 2 - i$$

$$\int_C 2z \, dz = z^2 \big|_{-2+7i}^{2-i}$$

$$= (2-i)^2 - (-2+7i)^2$$

$$= 48 + 24i$$

18.3.5

$$\int_0^{3+i} z^2 \, dz = \frac{1}{3} z^3 \bigg|_0^{3+i}$$

$$= \frac{1}{3}(3+i)^3$$

$$= 6 + \frac{26}{3} i$$

18.3.7

$$\int_{1-i}^{1+i} z^3 \, dz = \frac{1}{4} z^4 \bigg|_{1-i}^{1+i}$$

$$= \frac{1}{4}[(1+i)^4 - (1-i)^4]$$

$$= 0$$

18.3.9

$$\int_{-i/2}^{1-i} (2z+1)^2 \, dz = \left. z + 2z^2 + \frac{4}{3} z^3 \right|_{-i/2}^{1-i}$$

$$= -\frac{7}{6} - \frac{22}{3} i$$

18.3.11

$$\int_{i/2}^{i} e^{\pi z} \, dz = \frac{1}{\pi} e^{\pi z} \bigg|_{i/2}^{i}$$

$$= -\frac{1}{\pi}(1+i)$$

18.3.13

$$\int_{\pi}^{\pi+2i} \sin\frac{z}{2}\, dz = -2\cos\frac{z}{2}\Big|_{\pi}^{\pi+2i}$$
$$= -2\cos\left(\frac{\pi}{2}+i\right)$$
$$= 2i\sinh 1$$
$$\approx 2.3504i$$

18.3.15

$$\int_{\pi i}^{2\pi i} \cosh z\, dz = \sinh(2\pi i) - \sinh(\pi i)$$
$$= 0$$

18.3.17

$$\int_{C} \frac{1}{z}\, dz = \operatorname{Ln} 4e^{\pi i/2} - \operatorname{Ln} 4e^{-\pi i/2}$$
$$= \log_e 4 + \frac{\pi}{2}i - \log_e 4 + \frac{\pi}{2}i$$
$$= \pi i$$

18.3.19

$$\int_{-4i}^{4i} \frac{1}{z^2}\, dz = -\frac{1}{z}\Big|_{-4i}^{4i}$$
$$= -\frac{1}{4i} - \frac{1}{4i}$$
$$= -\frac{1}{2i}$$
$$= \frac{i}{2}$$

18.3.21

$$\int_{\pi}^{i} e^z \cos z\, dz = \frac{1}{2}\int_{\pi}^{i} \left[e^{z(1+i)} + e^{z(1-i)}\right] dz$$
$$= \frac{1}{2}\left(\frac{e^{z(1+i)}}{1+i} + \frac{e^{z(1-i)}}{1-i}\Big|_{\pi}^{i}\right)$$
$$\approx 11.4928 + 0.9667i$$

18.3.23

$$\int_i^{1+i} z e^z \, dz = z e^z \big|_i^{1+i} - \int_i^{1+i} e^z \, dz$$
$$= (1+i)e^{1+i} - ie^i - [e^z]_i^{1+i}$$
$$= (1+i)e^{1+i} - ie^i - e^{1+i} + e^i$$
$$\approx -0.9055 + 1.7698i$$

18.4 Cauchy's Integral Formulas

18.4.1

$8\pi i$

18.4.3

$2\pi i e^{\pi i} = -2\pi i$

18.4.5

$2\pi i[(-2i)^2 - 3(-2i) + 4i] = 2\pi i(-4 + 10i) = -20\pi - 8\pi i$

18.4.7

(a)

$$\oint_C \frac{z^2}{z^2 + 4} \, dz = \oint_C \frac{\frac{z^2}{z+2i}}{z - 2i} \, dz$$
$$= 2\pi i \frac{(2i)^2}{(2i) + 2i}$$
$$= -2\pi$$

(b)

$$\oint_C \frac{z^2}{z^2 + 4} \, dz = \oint_C \frac{\frac{z^2}{z-2i}}{z + 2i} \, dz$$
$$= 2\pi i \frac{(-2i)^2}{(-2i) - 2i}$$
$$= 2\pi i \frac{-4}{-4i}$$
$$= 2\pi$$

18.4.9

$$\oint_C \frac{z^2+4}{z^2-5iz-4}\,dz = \oint_C \frac{z^2+4}{(z-i)(z-4i)}\,dz$$

$$= \oint_C \frac{\frac{z^2+4}{z-i}}{z-4i}\,dz$$

$$= -8\pi$$

18.4.11

$$\frac{2\pi i}{2!}\left.\frac{d^2}{dz^2}(e^{z^2})\right|_{z=i} = \pi i \left.\frac{d}{dz}(2ze^{z^2})\right|_{z=i}$$

$$= \pi i \left.\left(2e^{z^2}+4z^2e^{z^2}\right)\right|_{z=i}$$

$$= \pi i(2e^{-1}-4e^{-1})$$

$$= -\frac{2\pi}{e}i$$

18.4.13

$$\oint_C \frac{\cos 2z}{z^5}\,dz = \frac{2\pi i}{4!}\left.\frac{d^4}{dz^4}(\cos 2z)\right|_{z=0}$$

$$= \frac{\pi}{12}i(16\cos 2z)|_{z=0}$$

$$= \frac{4}{3}\pi i$$

18.4.19

$$\oint_C \left(\frac{e^{2iz}}{z^4}-\frac{z^4}{(z-i)^3}\right)\,dz = \frac{2\pi i}{3!}\left.\frac{d^3}{dz^3}(e^{2iz})\right|_{z=0} - \frac{2\pi i}{2!}\left.\frac{d^2}{dz^2}(z^4)\right|_{z=i}$$

$$= \frac{\pi}{3}i(-8ie^{2iz})|_{z=0} - \pi i(12z^2)|_{z=i}$$

$$= \frac{8}{3}\pi + 12\pi i$$

18.4.21

$$\oint_C \frac{1}{z^3(z-1)^2}\, dz = \oint_{C_1} \frac{\frac{1}{(z-1)^2}}{z^3}\, dz + \oint_{C_2} \frac{\frac{1}{z^3}}{(z-1)^2}\, dz$$

$$= \frac{2\pi i}{2!}\, \frac{d^2}{dz^2}\left(\frac{1}{(z-1)^2}\right)\Bigg|_{z=0}$$

$$+ 2\pi i\, \frac{d}{dz}\left(\frac{1}{z^3}\right)\Bigg|_{z=1}$$

$$= 6\pi i - 6\pi i$$

$$= 0$$

18.4.23

$$\oint_C \frac{3z+1}{z(z-2)^2}\, dz = \oint_{C_1} \frac{\frac{3z+1}{z}}{(z-2)^2}\, dz - \oint_{C_2} \frac{\frac{3z+1}{(z-2)^2}}{z}\, dz$$

$$= 2\pi i\, \frac{d}{dz}\left(\frac{3z+1}{z}\right)\Bigg|_{z=2} - 2\pi i\, \frac{3z+1}{(z-2)^2}\Bigg|_{z=0}$$

$$= -\frac{1}{2}pii - \frac{1}{2}\pi i$$

$$= -\pi i$$

18.5 Chapter in Review

18.5.1

True

18.5.3

True

18.5.5

0

18.5.7

$$\oint_C \frac{z^3 + e^z}{(z + \pi i)^3}\, dz = \frac{2\pi i}{2!}\, \frac{d^2}{dz^2}(z^3 + e^z)\Bigg|_{z=-\pi i}$$

$$= \pi i(6z + e^z)|_{z=-\pi i}$$

$$= \pi i(-6\pi i - 1)$$

$$= 6\pi^2 - \pi i$$

18.5.9

$$\oint_C \frac{1}{(z - z_0)(z - z_1)} \, dz = \oint_{C_1} \frac{\frac{1}{z - z_0}}{z - z_1} \, dz + \oint_{C_2} \frac{\frac{1}{z - z_1}}{z - z_0} \, dz$$

$$= 2\pi i \frac{1}{z_1 - z_0} + 2\pi i \frac{1}{z_0 - z_1}$$

$$= 0$$

True

18.5.11

$2\pi i$ if $n = -1$, 0 otherwise.

18.5.13

$$z_1 = -4 + iy, \; 0 \le y \le 2$$
$$z_1' = i$$
$$z_2 = x + 2i, \; -4 \le x \le 3$$
$$z_2' = 1$$
$$z_3 = 3 + i(2 - y), \; 0 \le y \le 2$$
$$z_3' = -i$$

$$\int_C (x + iy) \, dz = \int_0^2 (-4 + iy)i \, dy + \int_{-4}^3 (x + 2i) \, dx + \int_0^2 [3 + i(2 - y)](-i) \, dy$$

$$= \int_0^2 (-y - 4i) \, dy + \int_{-4}^3 (x + 2i) \, dx - \int_0^2 [(y - 2) + 3i] \, dy$$

$$= \left[-\frac{1}{2}y^2 - 4iy \right]_0^2 + \left[\frac{1}{2}x^2 + 2ix \right]_{-4}^3 - \left[\frac{1}{2}y^2 - 2y + 3iy \right]_0^2$$

$$= -2 - 8i + \frac{9}{2} + 6i - 8 + 8i - 2 + 4 - 6i$$

$$= -\frac{7}{2}$$

18.5.15

$$\int_C |z^2| \, dz = \int_0^2 |(t + it^2)^2|(1 + 2it) \, dt$$

$$= \int_0^2 (t^2 + t^4)(1 + 2it) \, dt$$

$$= \frac{136}{15} + \frac{88}{3}i$$

18.5.17

0

18.5.19

$$z(-1) = 1$$
$$z(1) = 1 + 4i$$
$$\int_C \sin z \, dz = -\cos z \big|_1^{1+4i}$$
$$= \cos 1 - \cos(1 + 4i)$$
$$\approx -14.2144 + 22.9637i$$

18.5.21

$2\pi i$

18.5.23

$$\oint_C \frac{e^{-2z}}{z^4} \, dz = \frac{2\pi i}{3!} \frac{d^3}{dz^3} (e^{-2z}) \bigg|_{z=0}$$
$$= -\frac{8}{3}\pi i$$

18.5.25

$$\oint_C \frac{1}{2z^2 + 7z + 3} \, dz = \oint_C \frac{\frac{1}{z+3}}{2z + 1} \, dz$$
$$= \oint_C \frac{\frac{1}{2(z+3)}}{z + \frac{1}{2}} \, dz$$
$$= 2\pi i \frac{1}{2\left(-\frac{1}{2} + 3\right)}$$
$$= \frac{2}{5}\pi i$$

18.5.27

2π

19 Series and Residues

19.1 Sequences and Series

19.1.1

$5i, -5, -5i, 5, 5i$

19.1.3

$0, 2, 0, 2, 0$

19.1.5

$$\lim_{n\to\infty} \frac{3ni + 2}{n + ni} = \frac{3i}{1 + i}$$

Converges

19.1.7

$$
\begin{aligned}
\lim_{n\to\infty} \frac{(ni + 2)^2}{n^2 i} &= \lim_{n\to\infty} \frac{-n^2 + 4ni + 4}{n^2 i} \\
&= \lim_{n\to\infty} \frac{-1 + 4i/n + 4/n^2}{i} \\
&= -\frac{1}{i}
\end{aligned}
$$

Converges

19.1.9

Diverges

19.1.11

$$
\begin{aligned}
\frac{4n + 3in}{2n + i} &= \frac{(4n + 3in)(2n - i)}{(2n + i)(2n - i)} \\
&= \frac{8n^2 - 4in + 6in^2 + 3n}{4n^2 + 1} \\
&= \frac{8n^2 + 3n}{4n^2 + 1} + i\frac{6n^2 - 4n}{4n^2 + 1}
\end{aligned}
$$

$$\lim_{n\to\infty} \frac{8n^2 + 3n}{4n^2 + 1} = 2$$

$$\lim_{n\to\infty} \frac{6n^2 - 4n}{4n^2 + 1} = \frac{3}{2}$$

19.1.13

$$S_1 = \frac{1}{1+2i} - \frac{1}{2+2i}$$

$$S_2 = \frac{1}{1+2i} - \frac{1}{2+2i} + \frac{1}{2+2i} - \frac{1}{3+2i}$$

$$S_n = \frac{1}{1+2i} - \frac{1}{n+1+2i}$$

$$\lim_{n\to\infty} S_n = \frac{1}{1+2i}$$

19.1.15

$$|1 - i| = \sqrt{2}$$

Divergent

19.1.17

$$\left|\frac{i}{2}\right| = \frac{1}{2}$$

$$\frac{i/2}{1 - i/2} = \frac{i}{2 - i}$$

$$= \frac{i(2+i)}{(2-i)(2+i)}$$

$$= \frac{-1 + 2i}{5}$$

19.1.19

$$\frac{2}{1+2i} = \frac{2(1-2i)}{(1+2i)(1-2i)}$$

$$= \frac{2-4i}{5}$$

$$\left|\frac{2}{1+2i}\right| = \sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$

$$= \sqrt{\frac{4}{25} + \frac{16}{25}}$$

$$= \sqrt{\frac{20}{25}}$$

$$= \sqrt{\frac{4}{5}}$$

$$= \frac{2}{\sqrt{5}}$$

$$= \frac{2\sqrt{5}}{5}$$

$$\frac{3}{1-\frac{2}{1+2i}} = \frac{3(1+2i)}{1+2i-2}$$

$$= \frac{3+6i}{-1+2i}$$

$$= \frac{(3+6i)(-1-2i)}{(-1+2i)(-1-2i)}$$

$$= \frac{-3-6i-6i+12}{5}$$

$$= \frac{9-12i}{5}$$

19.1.21

$$a_k = \frac{1}{(1-2i)^{k+1}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(1-2i)^{n+1}}{(1-2i)^{n+2}}\right|$$

$$= \left|\frac{1}{1-2i}\right|$$

$$= \left|\frac{1+2i}{(1-2i)(1+2i)}\right|$$

$$= \left|\frac{1+2i}{5}\right|$$

$$= \frac{\sqrt{5}}{5}$$

$$R = \sqrt{5}$$

The circle of convergence is $|z - 2i| = \sqrt{5}$.

19.1.23

$$a_k = \frac{(-1)^k}{k2^k}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}/[(n+1)2^{(n+1)}]}{(-1)^n/(n2^n)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n2^n}{(n+1)2^{(n+1)}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n}{2(n+1)}\right|$$

$$= \frac{1}{2}$$

$$R = 2$$

The circle of convergence is $|z - 1 - i| = 2$.

19.1.25

$$a_k = (1+3i)^k$$

$$\lim_{n\to\infty}\left|\frac{(1+3i)^{n+1}}{(1+3i)^n}\right| = \lim_{n\to\infty}|1+3i|$$

$$= \sqrt{10}$$

$$R = \frac{1}{\sqrt{10}}$$

The circle of convergence is $|z - i| = \frac{1}{\sqrt{10}}$.

19.1.27

$$a_k = \frac{1}{5^{2k}}$$

$$\lim_{n \to \infty} \left| \frac{5^{2n}}{5^{2(n+1)}} \right| = \frac{1}{25}$$

$$R = 25$$

The circle of convergence is $|z - 4 - 3i| = 25$.

19.2   Taylor Series

19.2.1

$$f(z) = \frac{z}{1+z}$$
$$= z(1 - z + z^2 - z^3 + \cdots)$$
$$= z - z^2 + z^3 - z^4 + \cdots$$
$$= \sum_{k=1}^{\infty} (-1)^{k+1} z^k$$
$$R = 1$$

19.2.3

$$f(z) = \frac{1}{(1+2z)^2}$$
$$\frac{1}{1+2z} = 1 - 2z + 4z^2 - 8z^3 + 16z^4 - \cdots$$
$$= \sum_{k=0}^{\infty} (-1)^k (2z)^k$$
$$\frac{d}{dz} \frac{1}{1+2z} = -\frac{2}{(1+2z)^2}$$
$$= \sum_{k=1}^{\infty} (-1)^k 2k (2z)^{k-1}$$
$$f(z) = \sum_{k=1}^{\infty} (-1)^{k-1} k (2z)^{k-1}$$
$$R = \frac{1}{2}$$

19.2.5

$$f(z) = e^{-2z}$$

$$= \sum_{k=0}^{\infty} \frac{(-2z)^k}{k!}$$

$$R = \infty$$

19.2.7

$$f(z) = \sinh z$$

$$= \frac{e^z - e^{-z}}{2}$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \frac{z^k - (-z)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$$

$$R = \infty$$

19.2.9

$$f(z) = \cos \frac{z}{2}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(z/2)^{2k}}{(2k)!}$$

$$R = \infty$$

19.2.11

$$f(z) = \sin z^2$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{z^{4k+2}}{(2k+1)!}$$

$$R = \infty$$

19.2.13

$$f(z) = \frac{1}{z}$$

$$f'(z) = -\frac{1}{z^2}$$

$$f''(z) = \frac{2}{z^3}$$

$$f'''(z) = -\frac{6}{z^4}$$

$$f^{(n)}(z) = (-1)^n \frac{n!}{z^{n+1}}$$

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(z-1)^k$$

$$= \sum_{k=0}^{\infty} (-1)^k (z-1)^k$$

$$R = 1$$

19.2.15

$$f(z) = \frac{1}{3-z}$$

$$f'(z) = \frac{1}{(3-z)^2}$$

$$f''(z) = \frac{2}{(3-z)^3}$$

$$f^{(n)}(z) = \frac{n!}{(3-z)^{n+1}}$$

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(2i)}{k!}(z-2i)^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{(3-2i)^{k+1}}(z-2i)^k$$

$$R = \sqrt{13}$$

19.2.17

$$f(z) = \frac{z-1}{3-z}$$

$$f^{(n)}(z) = (-1)^{n+1}\frac{2n!}{(z-3)^{n+1}}$$

$$f(z) = \sum_{k=1}^{\infty}\frac{f^{(k)}(1)}{k!}(z-1)^k$$

$$= \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2}{(-2)^{k+1}}(z-1)^k$$

$$= \sum_{k=1}^{\infty}\frac{(z-1)^k}{2^k}$$

$$R = 2$$

19.2.19

$$f(z) = \cos z$$

$$f'(z) = -\sin z$$

$$f''(z) = -\cos z$$

$$f'''(z) = \sin z$$

$$f^{(4)}(z) = \cos z$$

$$f(z) = \sum_{k=0}^{\infty}\frac{f^{(k)}(\pi/4)}{k!}(z-\pi/4)^k$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2(1!)}(z-\pi/4) - \frac{\sqrt{2}}{2(2!)}(z-\pi/4)^2 + \frac{\sqrt{2}}{2(3!)}(z-\pi/4)^3 + \cdots$$

$$R = \infty$$

19.2.21

$$f(z) = e^z$$

$$= \sum_{k=0}^{\infty}\frac{f^{(k)}(3i)}{k!}(z-3i)^k$$

$$= e^{3i}\sum_{k=0}^{\infty}\frac{(z-3i)^k}{k!}$$

$$R = \infty$$

19.2.23

$$f(z) = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots$$

19.2.25

$$f(z) = \frac{i}{(z-i)(z-2i)}$$

$$= \frac{1}{z-2i} - \frac{1}{z-i}$$

$$f^{(n)}(z) = (-1)^n n! \left(\frac{1}{(z-2i)^{n+1}} - \frac{1}{(z-i)^{n+1}} \right)$$

$$f^{(n)}(0) = (-1)^n n! \left(\frac{1}{(-2i)^{n+1}} - \frac{1}{(-i)^{n+1}} \right)$$

$$= n! \left(\frac{1}{i^{n+1}} - \frac{1}{(2i)^{n+1}} \right)$$

$$= \frac{n!}{i^{n+1}} \left(1 - \frac{1}{2^{n+1}} \right)$$

$$f(z) = \sum_{k=0}^{\infty} \frac{1 - \frac{1}{2^{k+1}}}{i^{k+1}} z^k$$

$$R = 1$$

19.2.27

$R = \sqrt{20} = 2\sqrt{5}$

19.2.29

$$f(z) = \frac{1}{z+2}$$

$$f^{(n)}(z) = (-1)^n \frac{n!}{(z+2)^{n+1}}$$

$$f_1(z) = \sum_{k=0}^{\infty} (-1)^k (z+1)^k$$

$$R_1 = 1$$

$$f_2(z) = \sum_{k=0}^{\infty} (-1)^k \frac{(z-i)^k}{(2+i)^{k+1}}$$

$$R_2 = \sqrt{5}$$

19.3   Laurent Series

19.3.1

$$f(z) = \frac{\cos z}{z}$$
$$= \frac{1}{z}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right)$$
$$= \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} + \cdots$$

19.3.3

$$f(z) = e^{-1/z^2}$$
$$= 1 - \frac{1}{1!z^2} + \frac{1}{2!z^4} - \frac{1}{3!z^6} + \cdots$$

19.3.5

$$f(z) = \frac{e^z}{z-1}$$
$$= \frac{e^{1+z-1}}{z-1}$$
$$= \frac{e}{z-1}\left(1 + \frac{z-1}{1!} + \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} + \cdots\right)$$
$$= \frac{e}{z-1} + e + \frac{e(z-1)}{2!} + \frac{e(z-1)^2}{3!} + \cdots$$

19.3.7

$$f(z) = \frac{1}{z(z-3)}$$
$$= -\frac{1}{3z}\frac{1}{1-\frac{z}{3}}$$
$$= -\frac{1}{3z}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \cdots\right)$$
$$= -\frac{1}{3z} - \frac{1}{9} - \frac{z}{27} - \frac{z^2}{81} - \cdots$$

19.3.9

$$f(z) = \frac{1}{z(z-3)}$$

$$= \frac{1}{z-3} \frac{1}{3+z-3}$$

$$= \frac{1}{3(z-3)} \frac{1}{1+\frac{z-3}{3}}$$

$$= \frac{1}{3(z-3)} \left[1 - \frac{z-3}{3} + \left(\frac{z-3}{3}\right)^2 - \left(\frac{z-3}{3}\right)^3 + \cdots \right]$$

$$= \frac{1}{3(z-3)} - \frac{1}{3^2} + \frac{z-3}{3^3} - \frac{(z-3)^2}{3^4} + \cdots$$

19.3.11

$$f(z) = \frac{1}{z(z-3)}$$

$$= \frac{1}{3(z-3)} - \frac{1}{3z}$$

$$= \frac{1}{3(1+z-4)} - \frac{1}{3(4+z-4)}$$

$$= \frac{1}{3(z-4)} \frac{1}{1+\frac{1}{z-4}} - \frac{1}{12} \frac{1}{1+\frac{z-4}{4}}$$

$$= \frac{1}{3(z-4)} \left(1 - \frac{1}{z-4} + \frac{1}{(z-4)^2} - \frac{1}{(z-4)^3} + \cdots \right)$$

$$\qquad - \frac{1}{12} \left[1 - \frac{z-4}{4} + \left(\frac{z-4}{4}\right)^2 - \left(\frac{z-4}{4}\right)^3 + \cdots \right]$$

$$= \cdots - \frac{1}{3(z-4)^2} + \frac{1}{3(z-4)} - \frac{1}{12} + \frac{z-4}{3 \cdot 4^2} - \frac{(z-4)^2}{3 \cdot 4^3} + \cdots$$

19.3.13

$$f(z) = \frac{1}{(z-1)(z-2)}$$

$$= \frac{1}{1-z} - \frac{1}{2-z}$$

$$= -\frac{1}{z}\frac{1}{1-\frac{1}{z}} - \frac{1}{2}\frac{1}{1-\frac{z}{2}}$$

$$= -\frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right)$$

$$\quad - \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots\right)$$

$$= \cdots - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{z}{2^2} - \frac{z^2}{2^3} - \cdots$$

19.3.15

$$f(z) = \frac{1}{(z-1)(z-2)}$$

$$= -\frac{1}{z-1} + \frac{1}{z-2}$$

$$= -\frac{1}{z-1} + \frac{1}{-1+z-1}$$

$$= -\frac{1}{z-1} - \frac{1}{1-(z-1)}$$

$$= -\frac{1}{z-1} - 1 - (z-1) - (z-1)^2 - \cdots$$

19.3.17

$$f(z) = \frac{z}{(z+1)(z-2)}$$

$$= \frac{1}{3(z+1)} + \frac{2}{3(z-2)}$$

$$= \frac{1}{3(z+1)} + \frac{2}{3(-3+z+1)}$$

$$= \frac{1}{3(z+1)} - \frac{2}{9\left(1 - \frac{z+1}{3}\right)}$$

$$= \frac{1}{3(z+1)} - \frac{2}{9}\left[1 + \frac{z+1}{3} + \left(\frac{z+1}{3}\right)^2 + \left(\frac{z+1}{3}\right)^3 + \cdots\right]$$

$$= \frac{1}{3(z+1)} - \frac{2}{3^2} - \frac{2(z+1)}{3^3} - \frac{2(z+1)^2}{3^4} - \cdots$$

19.3.19

$$f(z) = \frac{z}{(z+1)(z-2)}, \ 1 < |z| < 2$$

$$= \frac{2}{3(z-2)} + \frac{1}{3(z+1)}$$

$$= -\frac{1}{3\left(1 - \frac{z}{2}\right)} + \frac{1}{3z\left(1 + \frac{1}{z}\right)}$$

$$= -\frac{1}{3}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right]$$

$$\quad + \frac{1}{3z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \cdots\right)$$

$$= \cdots - \frac{1}{3z^2} + \frac{1}{3z} - \frac{1}{3} - \frac{z}{3 \cdot 2} - \frac{z^2}{3 \cdot 2^2} - \cdots$$

19.4   Zeroes and Poles

19.4.1

$$f(z) = \frac{e^{2z} - 1}{z}$$

$$= -\frac{1}{z} + \frac{1}{z}\left(1 + \frac{2z}{1!} + \frac{4z^2}{2!} + \cdots\right)$$

$$= \frac{2}{1!} + \frac{4z}{2!} + \frac{8z^2}{3!} + \cdots$$

$$f(0) = 2$$

19.4.3

$z = -2 + i$, order 2.

19.4.5

$$f(z) = z^4 + z^2$$

$$= z^2(z^2 + 1)$$

$$= z^2(z + i)(z - i)$$

Zeroes are: $z = 0$ order 2, $z = \pm i$ order 1.

19.4.7

$$f(z) = e^{2z} - e^z$$

Zeroes are $2\pi n i$, $n \in \mathbb{R}$ order 1.

19.4.9

$$f(z) = z(1 - \cos z^2)$$
$$= z\left[1 - \left(1 - \frac{z^4}{2!} + \frac{z^8}{4!} - \cdots\right)\right]$$
$$= z\left(\frac{z^4}{2!} - \frac{z^8}{4!} + \frac{z^{12}}{6!} - \cdots\right)$$
$$= z^5\left(\frac{1}{2!} - \frac{z^4}{4!} + \frac{z^8}{6!} - \cdots\right)$$

Order 5.

19.4.11

$$f(z) = 1 - e^{z-1}$$
$$= 1 - \left[1 + \frac{z-1}{1!} + \frac{(z-1)^2}{2!} + \cdots\right]$$
$$= -\frac{z-1}{1!} - \frac{(z-1)^2}{2!} - \cdots$$
$$= -(z-1)\left[1 + \frac{z-1}{2!} + \frac{(z-1)^2}{3!} + \cdots\right]$$

Order 1.

19.4.13

$$f(z) = \frac{3z - 1}{z^2 + 2z + 5}$$
$$= \frac{3z - 1}{(z + 1 + 2i)(z + 1 - 2i)}$$

Poles are $z = -1 \pm 2i$ both of order 1.

19.4.15

$$f(z) = \frac{1 + 4i}{(z + 2)(z + i)^4}$$

Poles are $z = -2$ order 1, and $z = -i$ order 4.

19.4.17

$$f(z) = \tan z$$
$$= \frac{\sin z}{\cos z}$$

Poles are $z = \frac{\pi}{2} + n\pi$, $n \in \mathbb{R}$ order 1.

19.5   Residues and Residue Theorem

19.5.1

$$f(z) = \frac{2}{(z-1)(z+4)}$$
$$= \frac{2}{z-1}\frac{1}{5+z-1}$$
$$= \frac{2}{5(z-1)}\frac{1}{1+\frac{z-1}{5}}$$
$$= \frac{2}{5(z-1)}\left[1 - \frac{z-1}{5} + \left(\frac{z-1}{5}\right)^2 - \cdots\right]$$
$$= \frac{2}{5(z-1)} - \frac{2}{5^2} + \frac{2(z-1)}{5^3} - \cdots$$
$$\mathrm{Res}(f(z),1) = \frac{2}{5}$$

19.5.3

$$f(z) = \frac{4z-6}{z(2-z)}$$
$$= \frac{4z-6}{2z}\frac{1}{1-\frac{z}{2}}$$
$$= \left(2 - \frac{3}{z}\right)\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right]$$
$$\mathrm{Res}(f(z),0) = -3$$

19.5.5

$$f(z) = e^{-2/z^2}$$
$$= 1 - \frac{2}{z^2} + \frac{4}{2! \cdot z^4} + \cdots$$
$$\mathrm{Res}(f(z),0) = 0$$

19.5.7

$$f(z) = \frac{z}{z^2 + 16}$$

$$= \frac{z}{(z + 4i)(z - 4i)}$$

$$\mathrm{Res}(f(z), -4i) = \lim_{z \to -4i}(z + 4i)\frac{z}{(z + 4i)(z - 4i)}$$

$$= \frac{-4i}{-8i}$$

$$= \frac{1}{2}$$

$$\mathrm{Res}(f(z), 4i) = \frac{1}{2}$$

19.5.9

$$f(z) = \frac{1}{z^4 + z^3 - 2z^2}$$

$$= \frac{1}{z^2(z^2 + z - 2)}$$

$$= \frac{1}{z^2(z - 1)(z + 2)}$$

$$\mathrm{Res}(f(z), 1) = \frac{1}{3}$$

$$\mathrm{Res}(f(z), -2) = -\frac{1}{12}$$

$$\mathrm{Res}(f(z), 0) = \lim_{z \to 0}\frac{d}{dz}\frac{1}{(z - 1)(z + 2)}$$

$$= -\frac{1}{4}$$

19.5.11

$$f(z) = \frac{5z^2 - 4z + 3}{(z + 1)(z + 2)(z + 3)}$$

$$\mathrm{Res}(f(z), -1) = 6$$

$$\mathrm{Res}(f(z), -2) = -31$$

$$\mathrm{Res}(f(z), -3) = 30$$

19.5.13

$$f(z) = \frac{\cos z}{z^2(z-\pi)^3}$$

$$\mathrm{Res}(f(z), 0) = -\frac{3}{\pi^4}$$

$$\mathrm{Res}(f(z), \pi) = \frac{1}{2}\left(\frac{1}{\pi^2} - \frac{6}{\pi^4}\right)$$

19.5.15

$$f(z) = \sec z$$

$$= \frac{1}{\cos z}$$

$$\mathrm{Res}\left(f(z), \frac{\pi}{2} + n\pi\right) = \frac{1}{-\sin\left(\frac{\pi}{2} + n\pi\right)}$$

$$= (-1)^{n+1}$$

19.5.17

(a) 0

(b)

$$\mathrm{Res}(f(z), 1) = \frac{1}{9}$$

$$\oint_C \frac{1}{(z-1)(z+2)^2}\, dz = \frac{2}{9}\pi i$$

(c)

$$\mathrm{Res}(f(z), -2) = -\frac{1}{9}$$

$$\oint_C \frac{1}{(z-1)(z+2)^2}\, dz = 0$$

19.5.19

(a)

$$f(z) = z^3 e^{-1/z^2}$$

$$= z^3 \left(1 - \frac{1}{1! \cdot z^2} + \frac{1}{2! \cdot z^4} - \frac{1}{3! \cdot z^6} + \cdots \right)$$

$$= z^3 - \frac{z}{1!} + \frac{1}{2! \cdot z} - \frac{1}{3! \cdot z^3} + \cdots$$

$$\text{Res}(f(z), 0) = \frac{1}{2}$$

$$\oint_C f(z)\, dz = \pi i$$

(b) πi

(c) 0

19.5.21

$$f(z) = \frac{1}{z^2 + 4z + 13}$$

$$= \frac{1}{(z + 2 + 3i)(z + 2 - 3i)}$$

$$\text{Res}(f(z), -2 + 3i) = \lim_{z \to -2 + 3i} (z + 2 - 3i) f(z)$$

$$= \frac{1}{6i}$$

$$= -\frac{1}{6} i$$

$$\oint_C f(z)\, dz = \frac{\pi}{3}$$

19.5.23

$$f(z) = \frac{z}{z^4 - 1}$$

$$= \frac{z}{(z^2 + 1)(z^2 - 1)}$$

$$= \frac{z}{(z + i)(z - i)(z + 1)(z - 1)}$$

$$\oint_C f(z)\, dz = 2\pi i \left(-\frac{1}{4} - \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \right)$$

$$= 0$$

19.5.25

$$f(z) = \frac{ze^z}{z^2 - 1}$$

$$= \frac{ze^z}{(z+1)(z-1)}$$

$$\oint_C f(z)\, dz = 2\pi i \left(\frac{1}{2e} + \frac{e}{2} \right)$$

$$= 2\pi i \frac{e + e^{-1}}{2}$$

$$= 2\pi i \cosh 1$$

19.6   Evaluation of Real Integrals

19.6.1

$$\int_0^{2\pi} \frac{1}{1 + 0.5\sin\theta}\, d\theta = \oint_C \frac{1}{1 + \frac{1}{2}\frac{1}{2i}(z - z^{-1})} \frac{dz}{iz}$$

$$= \oint_C \frac{1}{iz + \frac{1}{4}z^2 - \frac{1}{4}}\, dz$$

$$= 4\oint_C \frac{1}{z^2 + 4iz - 1}\, dz$$

$$= 4\oint_C \frac{1}{[z + (2 + \sqrt{3})i][z + (2 - \sqrt{3})i]}\, dz$$

$$= 4 \cdot 2\pi i\, \mathrm{Res}(f(z), (-2 + \sqrt{3})i)$$

$$= \frac{4\pi}{\sqrt{3}}$$

19.6.3

$$\int_0^{2\pi} \frac{\cos\theta}{3 + \sin\theta}\, d\theta = \oint_C \frac{\frac{1}{2}(z + z^{-1})}{3 + \frac{1}{2i}(z - z^{-1})} \frac{dz}{iz}$$

$$= \oint_C \frac{z + z^{-1}}{z^2 + 6iz - 1}\, dz$$

$$= \oint_C \frac{z^2 + 1}{z(z^2 + 6iz - 1)}\, dz$$

$$= \oint_C \frac{z^2 + 1}{z[z + (3 + 2\sqrt{2})i][z + (3 - 2\sqrt{2})i]}\, dz$$

$$= 2\pi i[\mathrm{Res}(f(z), 0) + \mathrm{Res}(f(z), (-3 + 2\sqrt{2})i)]$$

$$= 2\pi i(-1 + 1)$$

$$= 0$$

19.6.5

$$\int_0^\pi \frac{1}{2-\cos\theta}\,d\theta = \frac{1}{2}\int_{-\pi}^{\pi}\frac{1}{2-\cos\theta}\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\frac{1}{2+\cos t}\,dt$$

$$= \frac{1}{2}\oint_C \frac{1}{2+\frac{1}{2}(z+z^{-1})}\frac{dz}{iz}$$

$$= \frac{1}{i}\oint_C \frac{1}{z^2+4z+1}\,dz$$

$$= \frac{1}{i}\oint_C \frac{1}{(z+2+\sqrt{3})(z+2-\sqrt{3})}\,dz$$

$$= \frac{1}{i}2\pi i\,\mathrm{Res}(f(z),-2+\sqrt{3})$$

$$= \frac{\pi}{\sqrt{3}}$$

19.6.11

$$\int_{-\infty}^{\infty}\frac{1}{x^2-2x+2}\,dx = \oint_C \frac{1}{(z-1-i)(z-1+i)}\,dz$$

$$= 2\pi i\,\mathrm{Res}(f(z),1+i)$$

$$= \pi$$

19.6.13

$$\int_{-\infty}^{\infty}\frac{1}{(x^2+4)^2}\,dx = \oint_C \frac{1}{(z+2i)^2(z-2i)^2}\,dz$$

$$= 2\pi i\,\mathrm{Res}(f(z),2i)$$

$$= 2\pi i\,\lim_{z\to 2i}\frac{d}{dz}\frac{1}{(z+2i)^2}$$

$$= \frac{\pi}{16}$$

19.6.15

$$\int_{-\infty}^{\infty}\frac{1}{(x^2+1)^3}\,dx = \oint_C \frac{1}{(z+i)^3(z-i)^3}\,dz$$

$$= 2\pi i\,\mathrm{Res}(f(z),i)$$

$$= 2\pi i\frac{1}{2}\lim_{z\to i}\frac{d^2}{dz^2}\frac{1}{(z+i)^3}$$

$$= \frac{3\pi}{8}$$

19.6.21

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = \text{Re}\left(\oint_C \frac{1}{z^2+1}e^{iz}\,dz\right)$$

$$= \text{Re}\left(\oint_C \frac{1}{(z+i)(z-i)}e^{iz}\,dz\right)$$

$$= \text{Re}[2\pi i\,\text{Res}(f(z),i)]$$

$$= \frac{\pi}{e}$$

19.6.23

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+1}\,dx = \text{Im}\left(\oint_C \frac{z}{z^2+1}e^{iz}\,dz\right)$$

$$= \text{Im}\left(\frac{z}{(z+i)(z-i)}e^{iz}\,dz\right)$$

$$= \text{Im}[2\pi i\,\text{Res}(f(z),i)]$$

$$= \frac{\pi}{e}$$

19.6.31

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \text{Im}\left(\oint_C \frac{e^{iz}}{z}\,dz\right)$$

$$= \text{Im}[\pi i\,\text{Res}(f(z),0)]$$

$$= \pi$$

19.7   Chapter in Review

19.7.1

True

19.7.3

$$f(z) = \frac{1}{z-3}$$

$$= -\frac{1}{3}\frac{1}{1-\frac{z}{3}}$$

$$= -\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\cdots\right)$$

$$= -\frac{1}{3}-\frac{z}{9}-\frac{z^2}{27}-\frac{z^3}{81}-\cdots$$

False

19.7.5

$$f(z) = e^{1/(z-1)}$$

$$= 1 + \frac{1}{1! \cdot (z-1)} + \frac{1}{2! \cdot (z-1)^2} + \frac{1}{3! \cdot (z-1)^3} + \cdots$$

True

19.7.7

True

19.7.9

$$f(z) = \cot \pi z$$
$$= \frac{\cos \pi z}{\sin \pi z}$$
$$\operatorname{Res}(f(z), 0) = \frac{\cos 0}{\pi \cos 0}$$
$$= \frac{1}{\pi}$$

19.7.11

$$\lim_{n \to \infty} \left| \frac{\frac{(z-i)^{n+1}}{(2+i)^{n+2}}}{\frac{(z-i)^n}{(2+i)^{n+1}}} \right| = \left| \frac{z-i}{2+i} \right|$$
$$\left| \frac{z-i}{2+i} \right| = 1$$
$$\frac{|z-i|}{\sqrt{5}} = 1$$
$$|z-i| = \sqrt{5}$$

19.7.13

$$f(z) = e^z \cos z$$

$$= e^z \frac{e^{iz} + e^{-iz}}{2}$$

$$= \frac{e^{z(1+i)} + e^{z(1-i)}}{2}$$

$$f^{(n)}(z) = \frac{(1+i)^n e^{z(1+i)} + (1-i)^n e^{z(1-i)}}{2}$$

$$f^{(n)}(0) = \frac{(1+i)^n + (1-i)^n}{2}$$

$$= (\sqrt{2})^n \cos \frac{n\pi}{4}$$

$$f(z) = 1 + \sum_{k=1}^{\infty} \frac{(\sqrt{2})^k \cos(k\pi/4)}{k!} z^k$$

19.7.15

$$f(z) = \frac{1 - e^{iz}}{z^4}$$

$$= \frac{1}{z^4} - \frac{1}{z^4} \left(1 + \frac{iz}{1!} - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \cdots \right)$$

$$= -\frac{i}{1! \cdot z^3} + \frac{1}{2! \cdot z^2} + \frac{i}{3! \cdot z} - \frac{1}{4!} - \frac{iz}{5!} + \cdots$$

19.7.17

$$f(z) = (z - i)^2 \sin \frac{1}{z - i}$$

$$= (z - i)^2 \left[\frac{1}{z - i} - \frac{1}{3! \cdot (z - i)^3} + \frac{1}{5! \cdot (z - i)^5} - \cdots \right]$$

$$= (z - i) - \frac{1}{3! \cdot (z - i)} + \frac{1}{5! \cdot (z - i)^3} - \cdots$$

19.7.19

(a)

$$f(z) = \frac{2}{z^2 - 4z + 3}$$

$$= \frac{1}{1-z} - \frac{1}{3}\frac{1}{1-\frac{z}{3}}$$

$$= (1 + z + z^2 + \cdots) - \frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \cdots\right)$$

$$= \frac{2}{3} + \frac{8}{9}z + \frac{26}{27}z^2 + \cdots$$

(b)

$$f(z) = \frac{2}{z^2 - 4z + 3}$$

$$= -\frac{1}{3}\frac{1}{1-\frac{z}{3}} - \frac{1}{z}\frac{1}{1-\frac{1}{z}}$$

$$= -\frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \cdots\right) - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)$$

$$= \cdots - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{3} - \frac{z}{3^2} - \frac{z^2}{3^3} - \cdots$$

(c)

$$f(z) = \frac{2}{z^2 - 4z + 3}$$

$$= \frac{1}{z-3} - \frac{1}{z-1}$$

$$= \frac{1}{z}\frac{1}{1-\frac{3}{z}} - \frac{1}{z}\frac{1}{1-\frac{1}{z}}$$

$$= \frac{1}{z}\left(1 + \frac{3}{z} + \frac{3^2}{z^2} + \cdots\right) - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)$$

$$= \frac{1}{z}\left(\frac{2}{z} + \frac{8}{z^2} + \frac{26}{z^3} + \cdots\right)$$

$$= \frac{2}{z^2} + \frac{8}{z^3} + \frac{26}{z^4} + \cdots$$

(d)

$$f(z) = \frac{2}{z^2 - 4z + 3}$$

$$= \frac{2}{(z-1)(z-3)}$$

$$= \frac{2}{(z-1)(-2+z-1)}$$

$$= -\frac{1}{z-1}\frac{1}{1-\frac{z-1}{2}}$$

$$= -\frac{1}{z-1}\left[1 + \frac{z-1}{2} + \frac{(z-1)^2}{2^2} + \cdots\right]$$

$$= -\frac{1}{z-1} - \frac{1}{2} - \frac{z-1}{2^2} - \cdots$$

19.7.21

$$\oint_C \frac{2z+5}{z(z+2)(z-1)^4}\,dz = 2\pi i[\text{Res}(f(z), 0) + \text{Res}(f(z), -2)]$$

$$= 2\pi i\left(\frac{5}{2} - \frac{1}{162}\right)$$

$$= \frac{404}{81}\pi i$$

19.7.23

$$2\sin z - 1 = 0$$

$$\sin z = \frac{1}{2}$$

$$z = \frac{\pi}{6}$$

$$\oint_C \frac{1}{2\sin z - 1}\,dz = 2\pi i\,\text{Res}\left(f(z), \frac{\pi}{6}\right)$$

$$= 2\pi i\frac{1}{2\cos\frac{\pi}{6}}$$

$$= \frac{2}{\sqrt{3}}\pi i$$

19.7.25

$$\oint_C \frac{e^{2z}}{z^4 + 2z^3 + 2z^2}\, dz = \oint_C \frac{e^{2z}}{z^2(z^2 + 2z + 2)}\, dz$$

$$= \oint_C \frac{e^{2z}}{z^2(z + 1 + i)(z + 1 - i)}\, dz$$

$$= 2\pi i[\text{Res}(f(z), 0) + \text{Res}(f(z), -1 - i) + \text{Res}(f(z), -1 + i)]$$

$$= 2\pi i \left(\frac{1}{2} + \frac{1}{4}e^{-2(1+i)} + \frac{1}{4}e^{-2(1-i)} \right)$$

$$= (\pi + \pi e^{-2}\cos 2)i$$

19.7.31

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 2x + 2)(x^2 + 1)^2}\, dx = \oint_C \frac{z^2}{(z + 1 + i)(z + 1 - i)(z + i)^2(z - i)^2}\, dz$$

$$= 2\pi i[\text{Res}(f(z), -1 + i) + \text{Res}(f(z), i)]$$

$$= 2\pi i \left(\frac{3}{25} - \frac{4}{25}i - \frac{3}{25} + \frac{9}{100}i \right)$$

$$= \frac{7}{50}\pi$$