

# Advanced Engineering Mathematics Vectors, Matrices, and Vector Calculus by Dennis G. Zill

## Notes

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June 2023

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## 1 Vectors

### 1.1 Vectors in 2-Space

- The zero vector can be assigned any direction
- The vectors  $\mathbf{i}$  and  $\mathbf{j}$  are known as the **standard basis vectors** for  $\mathbb{R}^2$

### 1.2 Vectors in 3-Space

- In  $\mathbb{R}^3$  the octant in which all coordinates are positive is known as the **first octant**. There is no agreement for naming the other seven octants.

### 1.3 Dot Product

- The **dot product** is also known as the **inner product** or the **scalar product** and is denoted  $\mathbf{a} \cdot \mathbf{b}$
- Two non-zero vectors are orthogonal iff their dot product is 0
- The zero vector is considered orthogonal to all vectors
- The angles  $\alpha$ ,  $\beta$ , and  $\gamma$  between a vector and the unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$ , respectively are called the **direction angles** of the vector
- The cosines of a vectors direction angles (the **direction cosines**) can be calculated as

$$\begin{aligned}\cos \alpha &= \frac{\mathbf{a} \cdot \mathbf{i}}{\|\mathbf{a}\| \|\mathbf{i}\|} \\ &= \frac{a_1}{\|\mathbf{a}\|} \\ \cos \beta &= \frac{\mathbf{a} \cdot \mathbf{j}}{\|\mathbf{a}\| \|\mathbf{j}\|} \\ &= \frac{a_2}{\|\mathbf{a}\|} \\ \cos \gamma &= \frac{\mathbf{a} \cdot \mathbf{k}}{\|\mathbf{a}\| \|\mathbf{k}\|} \\ &= \frac{a_3}{\|\mathbf{a}\|}\end{aligned}$$

Equivalently, these can be calculated as the components of the unit vector  $\mathbf{a}/\|\mathbf{a}\|$ .

- To find the component of a vector  $\mathbf{a}$  in the direction of a vector  $\mathbf{b}$

$$\text{comp}_{\mathbf{b}} \mathbf{a} = \|\mathbf{a}\| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}$$

- To project a vector  $\mathbf{a}$  onto a vector  $\mathbf{b}$

$$\text{proj}_{\mathbf{b}} \mathbf{a} = (\text{comp}_{\mathbf{b}} \mathbf{a}) \frac{\mathbf{b}}{\|\mathbf{b}\|} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \right) \mathbf{b}$$

### 1.4 Cross Product

- The cross product is only defined in  $\mathbb{R}^3$
- The **scalar triple product** of vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  is defined as

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- The area of a parallelogram with sides  $\mathbf{a}$  and  $\mathbf{b}$  is  $\|\mathbf{a} \times \mathbf{b}\|$
- The area of a triangle with sides  $\mathbf{a}$  and  $\mathbf{b}$  is  $\frac{1}{2}\|\mathbf{a} \times \mathbf{b}\|$
- The volume of a parallelepiped with sides  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  is  $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$  iff  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are coplanar

## 1.5 Lines and Planes in 3-Space

- There is a unique line between any two points  $\mathbf{r}_1$  and  $\mathbf{r}_2$  in 3-space. The equation for that line is

$$\mathbf{r} = \mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1) = \mathbf{r}_1 + t\mathbf{a}$$

where  $t$  is called a **parameter**, the nonzero vector  $\mathbf{a}$  is called a **direction vector**, and its components are called **direction numbers**.

- Equating the components of the equation above we find

$$x = r_1 + ta_1$$

$$y = r_2 + ta_2$$

$$z = r_3 + ta_3.$$

These are the **parametric equations** for the line through  $\mathbf{r}_1$  and  $\mathbf{r}_2$ .

- By solving the parametric equations for  $t$  and equating the results we find the **symmetric equations** for the line

$$t = \frac{x - r_1}{a_1} = \frac{y - r_2}{a_2} = \frac{z - r_3}{a_3}.$$

- Given a point  $P_1$  and a vector  $\mathbf{n}$ , there exists only one plane containing  $P_1$  with  $\mathbf{n}$  normal. The vector from  $P_1$  to another point  $P$  on that plane will be perpendicular to  $\mathbf{n}$ , so the equation for the plane is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$

where  $\mathbf{r} = \overrightarrow{OP}$  and  $\mathbf{r}_1 = \overrightarrow{OP_1}$ . If

$$\mathbf{n} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$$

the cartesian form of this equation is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

and is called the **point-normal form**.

- The graph of any equation  $ax + by + cz + d = 0$ , where  $a$ ,  $b$ , and  $c$  are not all zero, is a plane with the normal vector  $\mathbf{n} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ .
- Given three noncollinear points, a normal vector can be found by forming two vectors from two pairs of points and take their cross product.
- A line and a plane that aren't parallel intersect at a single point.
- Two planes that aren't parallel must intersect in a line.

## 1.6 Vector Spaces

- The length of a vector is called its **norm**
- The process of multiplying a vector by the reciprocal of its norm is called **normalizing** the vector
- Two nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$  in  $\mathbb{R}^n$  are said to be orthogonal if  $\mathbf{a} \cdot \mathbf{b} = 0$

### Definition 7.6.1 Vector Space

Let  $V$  be a set of elements on which two operations called **vector addition** and **scalar multiplication** are defined. Then  $V$  is said to be a **vector space** if the following 10 properties are satisfied.

#### Axioms for Vector Addition:

- (i) If  $\mathbf{x}$  and  $\mathbf{y}$  are in  $V$ , then  $\mathbf{x} + \mathbf{y}$  is in  $V$ .
- (ii) For all  $\mathbf{x}, \mathbf{y}$  in  $V$ ,  $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ . ← commutative law
- (iii) For all  $\mathbf{x}, \mathbf{y}, \mathbf{z}$  in  $V$ ,  $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$ . ← associative law
- (iv) There is a unique vector  $\mathbf{0}$  in  $V$  such that  
 $\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x}$ . ← zero vector
- (v) For each  $\mathbf{x}$  in  $V$ , there exists a vector  $-\mathbf{x}$  such that  
 $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$ . ← negative of a vector

#### Axioms for Scalar Multiplication:

- (vi) If  $k$  is any scalar and  $\mathbf{x}$  is in  $V$ , then  $k\mathbf{x}$  is in  $V$ .
- (vii)  $k(\mathbf{x} + \mathbf{y}) = k\mathbf{x} + k\mathbf{y}$  ← distributive law
- (viii)  $(k_1 + k_2)\mathbf{x} = k_1\mathbf{x} + k_2\mathbf{x}$  ← distributive law
- (ix)  $k_1(k_2\mathbf{x}) = (k_1k_2)\mathbf{x}$
- (x)  $1\mathbf{x} = \mathbf{x}$

- If a subset  $W$  of a vector space  $V$  is itself a vector space under the operations of vector addition and scalar multiplication defined on  $V$ , then  $W$  is called a **subspace** of  $V$
- Every vector space has at least two subspaces: itself and the zero subspace  $\{\mathbf{0}\}$
- A set of vectors  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$  is said to be **linearly independent** if the only constants satisfying the equation

$$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \dots + k_n\mathbf{x}_n = \mathbf{0}$$

are  $k_1 = k_2 = \dots = k_n = 0$ . If the set of vectors is not linearly independent it is said to be **linearly dependent**.

- If a set of vectors  $B = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$  in a vector space  $V$  is linearly independent and every vector in  $V$  can be expressed as a linear combination of vectors in  $B$  then  $B$  is said to be a **basis** for  $V$ .
- The number of vectors in a basis  $B$  for a vector space  $V$  is said to be the **dimension** of the space.

- If the basis of a vector space contains a finite number of vectors, then the space is **finite dimensional**; otherwise it is **infinite dimensional**.
- If  $S$  denotes any set of vectors  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$  in a vector space  $V$ , then the set of all linear combinations of the vectors in  $S$

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots + c_n\mathbf{x}_n$$

is called the **span** of the vectors and is denoted  $\text{Span}(S)$ .

- $\text{Span}(S)$  is a subspace of  $V$  and is said to be a subspace spanned by its vectors  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ .
- If  $V = \text{Span}(S)$  then  $S$  is said to be a **spanning set** for the vector space  $V$  or that  $S$  **spans**  $V$ .

## 1.7 Gram–Schmidt Orthogonalization Process

- An **orthonormal basis** is a basis whose vectors are mutually orthogonal and are unit vectors.
- If  $B = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$  is an orthonormal basis for  $\mathbb{R}^n$  then an arbitrary vector  $\mathbf{u}$  can be expressed as

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{u} \cdot \mathbf{w}_2)\mathbf{w}_2 + \dots + (\mathbf{u} \cdot \mathbf{w}_n)\mathbf{w}_n$$

- The **Gram-Schmidt Orthogonalization Process** is a process for converting any basis of a vector space into an orthonormal basis. First the basis vectors are made orthogonal to each other, then they are normalized. More specifically, to convert a basis  $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  into an orthogonal basis  $B' = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$

1. Let  $\mathbf{v}_1 = \mathbf{u}_1$
2. Let  $\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{v}_1} \mathbf{u}_2$
3. ...
4. Let  $\mathbf{v}_n = \mathbf{u}_n - \text{proj}_{\mathbf{v}_1} \mathbf{u}_n - \text{proj}_{\mathbf{v}_2} \mathbf{u}_n - \dots - \text{proj}_{\mathbf{v}_{n-1}} \mathbf{u}_n$

and to convert  $B'$  into an orthonormal basis  $B'' = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ , normalize each  $\mathbf{v}_i$ ,  $i = 1, 2, \dots, n$ .

## 2 Matrices

### 2.1 Matrix Algebra

- Vectors can be written as horizontal or vertical arrays of numbers

- A **matrix** is any rectangular array of numbers or functions

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- The numbers or functions in the array are called the **elements** or **entries** of the matrix
- If a matrix has  $m$  rows and  $n$  columns we say that its **size** is  $m$  by  $n$  or  $m \times n$
- An  $n \times n$  matrix is called a **square** matrix of **order**  $n$
- The entry in the  $i$ th row and the  $j$ th column of an  $m \times n$  matrix **A** is written  $a_{ij}$
- An  $m \times 1$  matrix

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

is called a **column vector**

- A  $1 \times n$  matrix

$$(a_1 \quad a_2 \quad \cdots \quad a_n)$$

is called a **row vector**

#### Definition 8.1.6 Matrix Multiplication

Let **A** be a matrix having  $m$  rows and  $p$  columns, and let **B** be a matrix having  $p$  rows and  $n$  columns. The **product** **AB** is the  $m \times n$  matrix

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1p}b_{p1} & \cdots & a_{11}b_{1n} + a_{12}b_{2n} + \cdots + a_{1p}b_{pn} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2p}b_{p1} & \cdots & a_{21}b_{1n} + a_{22}b_{2n} + \cdots + a_{2p}b_{pn} \\ \vdots & & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mp}b_{p1} & \cdots & a_{m1}b_{1n} + a_{m2}b_{2n} + \cdots + a_{mp}b_{pn} \end{pmatrix} \\ &= \left( \sum_{k=1}^p a_{ik}b_{kj} \right)_{m \times n}. \end{aligned}$$

- Matrix multiplication is associative, i.e.  $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
- Matrix multiplication is distributive, i.e.  $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$  and  $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$
- The **transpose** of an  $m \times n$  matrix  $\mathbf{A}$  is an  $n \times m$  matrix  $\mathbf{A}^T$

$$\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

i.e. the matrix is flipped along the main diagonal

#### Theorem 8.1.2 Properties of Transpose

Suppose  $\mathbf{A}$  and  $\mathbf{B}$  are matrices and  $k$  a scalar. Then

- |  |                                  |
|--|----------------------------------|
| (i) $(\mathbf{A}^T)^T = \mathbf{A}$                              | ← transpose of a transpose       |
| (ii) $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$ | ← transpose of a sum             |
| (iii) $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$              | ← transpose of a product         |
| (iv) $(k\mathbf{A})^T = k\mathbf{A}^T$                           | ← transpose of a scalar multiple |

- A matrix that consists of all zero entries is called a **zero matrix**
- A square matrix is said to be a **triangular matrix** if all of its entries above or below the main diagonal are zeroes. More specifically they are called **lower triangular** and **upper triangular** matrices, respectively.
- A square matrix is called a **diagonal matrix** if all entries not on the main diagonal are 0.
- A square matrix whose entries on the main diagonal are all equal is called a **scalar matrix**
- A square matrix that has the property  $\mathbf{A} = \mathbf{A}^T$  is called a **symmetric matrix**

## 2.2 Systems of Linear Algebraic Equations

- In a linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned}$$

the values  $a_{ij}$  are called the **coefficients** and the values  $b_n$  are called the **constants**

- If all the constants are zero the system is said to be **homogeneous**, otherwise it is **nonhomogeneous**
- A linear system is said to be **consistent** if it has at least one solution, otherwise it's **inconsistent**
- A linear system can be transformed into an equivalent system (i.e. one that has the same solutions) via three elementary operations:
  1. Multiply an equation by a nonzero constant
  2. Interchange the positions of equations in the system
  3. Add a multiple of one equation to any other equation

- A linear system can be represented by an **augmented matrix**, e.g.

$$\left( \begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array} \right)$$

- We say that two matrices are **row equivalent** if one can be obtained from the other via a series of elementary row operations
- **Gaussian elimination** is the process of applying elementary row operations to a matrix to put it into **row-echelon form** where:
  1. The first nonzero entry in a row is a 1
  2. In subsequent rows, the first 1 entry appears to the right of the 1 entry in earlier rows
  3. Rows consisting of all zeroes are at the bottom of the matrix
- **Gauss-Jordan elimination** is the same as Gaussian elimination with an additional constraint that puts the matrix into **reduced row-echelon form** where a column containing a first entry 1 has zeroes everywhere else
- A homogeneous linear system always has a trivial solution where all variables are equal to zero and will have an infinite number of nontrivial solutions if the number of equations  $m$  is less than the number of variables  $n$ , i.e.  $m < n$
- If  $\mathbf{X}_1$  is a solution to  $\mathbf{AX} = \mathbf{0}$ , then so is  $c\mathbf{X}_1$  for any constant  $c$
- If  $\mathbf{X}_1$  and  $\mathbf{X}_2$  are solutions of  $\mathbf{AX} = \mathbf{0}$ , then so is  $\mathbf{X}_1 + \mathbf{X}_2$
- If a linear system contains more equations than variables it is said to be **overdetermined**; if it contains fewer equations than variables it is said to be **underdetermined**