# Advanced Engineering Mathematics Partial Differential Equations by Dennis G. Zill Notes

Chris Doble

November 2023

## Contents



# 12 Orthogonal Functions and Fourier Series

## 12.1 Orthogonal Functions

- The **inner product** of two functions $f_1$ and $f_2$ on an interval $[a, b]$ is the number
$$(f_1, f_2) = \int_a^b f_1(x) f_2(x)\, dx.$$
- Two functions $f_1$ and $f_2$ are said to be orthogonal on an interval if $(f_1, f_2) = 0$.
- A set of real-valued functions $\{\phi_1(x), \phi_2(x), \ldots, \phi_n(x)\}$ is said to be **orthogonal** on an interval if
$$(\phi_i, \phi_j) = 0 \text{ for } i \neq j.$$
- The **square norm** of a function is
$$||\phi_n(x)||^2 = (\phi_n, \phi_n)$$
and thus its **norm** is
$$||\phi_n(x)|| = \sqrt{(\phi_n, \phi_n)}.$$
- An **orthonormal set** of functions is an orthogonal set of functions that all have a norm of 1.

- An orthogonal set can be made into an orthonormal set by dividing each member by its norm.

- If $\{\phi_n(x)\}$ is an infinite orthogonal set of functions on an interval $[a, b]$ and $f(x)$ is an arbitrary function, then it's possible to determine a set of coefficients $c_n, n = 0, 1, 2, \ldots$ such that

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) = c_0\phi_0(x) + c_1\phi_1(x) + \ldots + c_n\phi_n(x) + \ldots$$

  This is called an **orthogonal series expansion** of $f$ or a **generalized Fourier series** where the coefficients are given by

$$c_n = \frac{(f, \phi_n)}{||\phi_n||^2}.$$

- A set of real-valued functions $\{\phi_n(x)\}$ is said to be **orthogonal with respect to a weight function** $w(x)$ on the interval $[a, b]$ if

$$\int_a^b w(x)\phi_m(x)\phi_n(x)\, dx = 0, \;\; m \neq n.$$

## 12.2 Fourier Series

- The **Fourier series** of a function $f$ defined on the interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

  where

$$\begin{aligned} a_0 &= \frac{1}{p} \int_{-p}^{p} f(x)\, dx \\ a_n &= \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi}{p} x\, dx \\ b_n &= \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi}{p} x\, dx \end{aligned}$$

- At points of discontinuity in $f$, the Fourier series takes on the average of the values either side of it.

- The Fourier series of a function $f$ gives a **periodic extension** of the function outside the interval $(-p, p)$.

## 12.3 Fourier Cosine and Sine Series

- A function $f$ is said to be **even** if

$$f(-x) = f(x)$$

and **odd** if

$$f(-x) = -f(x).$$

- Even and odd functions have some interesting properties:
  - The product of two even functions is even.
  - The product of two odd functions is even.
  - The product of an even function and an odd function is odd.
  - If $f$ is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
  - If $f$ is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

- In light of this, if a function $f$ is even its Fourier coefficients are

$$\begin{aligned} a_0 &= \frac{2}{p}\int_0^p f(x)\,dx \\ a_n &= \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx \\ b_n &= 0. \end{aligned}$$

The series consists of cosine terms and is called the **Fourier cosine series**.

- Similarly, if $f$ is odd then

$$\begin{aligned} a_n &= 0,\ \ n = 0, 1, 2, \dots \\ b_n &= \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx. \end{aligned}$$

The series consists of sine terms and is called the **Fourier sine series**.

- Sometimes a Fourier series "overshoots" the original value of the function near discontinuities. This is called the **Gibbs phenomenon**.
- Taking the Fourier cosine series of a function $f$ over the interval $[0, L]$ effectively mirrors the function around the vertical axis.
- Taking the Fourier sine series of a function $f$ over the interval $[0, L]$ effectively rotates it 180° around the origin.
- A particular solution for a nonhomogeneous differential equation with a periodic driving force can be found by taking the Fourier transform of the driving force then using the method of undetermined coefficients to determine the coefficients.