Advanced Engineering Mathematics Complex Analysis by Dennis G. Zill Notes

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Contents

17 Functions of a Complex Variable

17.1 Complex Numbers

- A **complex number** is any number of the form

$$z = a + ib$$

  where a and b are real numbers and i is the imaginary unit such that $i^2 = -1$.

- The real number a in the above complex number z is called the **real part** of z and the real number b (not ib) is called the **imaginary part** of z.

- The real and imaginary parts of a complex number z are denoted $\text{Re}(z)$ and $\text{Im}(z)$, respectively.

- A real constant multiple of the imaginary unit, e.g. $6i$ is called a **pure imaginary number**.

- Two complex numbers are equal if their real and imaginary parts are equal.

- The addition and subtraction of complex numbers occur between the real and imaginary parts, e.g.

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

- The multiplication of complex numbers occurs elementwise as normal, e.g.

$$(a + bi)(c + di) = ac + adi + bci - bd.$$

- The **conjugate** of a complex number $z = a + ib$ is

$$\overline{z} = a - ib.$$

- The division of complex numbers occurs by multiplying the numerator and denominator by the conjugate of the denominator, e.g.

$$\begin{aligned}
\frac{a + bi}{c + di} &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\
&= \frac{ac - adi + bci + bd}{c^2 + d^2} \\
&= \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}.
\end{aligned}$$

- Conjugates have several interesting properties:

$$\begin{aligned}
\overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2} \\
\overline{z_1 - z_2} &= \overline{z_1} - \overline{z_2} \\
\overline{z_1 z_2} &= \overline{z_1}\,\overline{z_2} \\
\overline{\left(\frac{z_1}{z_2}\right)} &= \frac{\overline{z_1}}{\overline{z_2}}.
\end{aligned}$$

- The sum and product of a complex number $z = x + iy$ with its conjugate are real numbers

$$\begin{aligned}
z + \overline{z} &= 2x \\
z\overline{z} &= x^2 + y^2
\end{aligned}$$

while the difference between a complex number and its conjugate is a purre imaginary number

$$z - \overline{z} = 2iy.$$

- The above properties let us define

$$\text{Re}(z) = \frac{z + \overline{z}}{2} \text{ and } \text{Im}(z) = \frac{z - \overline{z}}{2i}.$$

- The **complex plane** or z-**plane** is a coordinate system where the horizontal or x-axis is called the **real axis** and the vertical or y-axis is called the **imaginary axis**. Complex numbers can be plotted in this coordinate system by considering their real and imaginary parts an ordered pair corresponding their position.

- The **modulus** or **absolute value** of a complex number $z = x + iy$ denoted by $|z|$ is the real number

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\overline{z}}.$$

This is the distance between z and the origin in the complex plane.

- If you consider two numbers in the complex plane as vectors, the length of their sum can't be longer than their individual lengths combined

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

This extends to any finite sum

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

and is known as the **triangle inequality**.

17.2   Powers and Roots

- A complex number can be expressed in **polar form**

$$z = (r \cos \theta) + i(r \sin \theta)$$

where $r = |z|$ is the nonnegative modulus of z and $\theta = \arg z$ is the **argument** of z — the angle between z and the positive real axis measured in the counterclockwise direction.

- The argument of a complex number z isn't unique as any multiply of 2π can be added to it. The **principle argument** of z denoted $\text{Arg}\, z$ is the argument of z restricted to the intercal $-\pi \leq \text{Arg}\, z \leq \pi$.

- Multiplication and division of complex numbers is simpler in polar form. For two complex numbers $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ we get

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)].$$

- The above formulas can be used to find integer powers of a complex number z
$$z^n = r^n(\cos n\theta + i \sin n\theta)$$
where n is an integer (including negative integers).

- **DeMoivre's formula** is a special case of the above where $r = 1$ so
$$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

- A number w is said to be an **nth root** of a nonzero complex number z if $w^n = z$. The nth roots of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ are
$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$
where $k = 0, 1, 2, \ldots, n-1$.

- The root w of a complex number z obtained by using the principle argument of z with $k = 0$ is called the **principle nth root** of z.

- Since the nth roots of a complex number have the same modulus they lie on a circle of radius $r^{1/n}$. The arguments of subsequent roots differ by $2\pi/n$ so they're also equally spaced around the circle.

17.3   Sets in the Complex Plane

- The points $z = x + iy$ that satisfy the equation
$$|z - z_0| = \rho$$
for $\rho > 0$ lie on a circle of radius ρ centred at the point z_0.

- The points z satisfying the inequality $|z - z_0| < \rho$ for $\rho > 0$ lie within, but not on, a circle of radius ρ centered at the point z_0. This set is called a **neighborhood** of z_0 or an **open disk**.

- A point z_0 is said to be an **interior point** of a set S of the complex plane if there exists some neighborhood of z_0 that lies entirely within S.

- If every point z of a set S is an interior point, then S is said to be an **open set**. An example of a set that isn't open is the set of points satisfying the inequality $\text{Re}(z) \geq 0$. This isn't open because it includes the line $\text{Re}(z) = 0$ and no points on that line are interior to the set because, no matter what ρ you choose, some points in the neighborhood have $\text{Re}(z) < 0$.

- If every neighborhood of a point z_0 contains at least one point that is in a set S and at least one point that is not in S, then z_0 is said to be a **boundary point** of S.

- The **boundary** of a set S in the complex plane is the set of all boundary points of S.

- If any pair of points in a set S can be connected by a polygonal line that lies entirely within the set, then S is said to be **connected**.

- An open connected set is called a **domain**.

- A **region** is a set in the complex plane with all, some, or none of its boundary points. A region containing all of its boundary points is said to be **closed**.

17.4   Functions of a Complex Variable

- A **function** f from a set A to a set B is a rule of correspondence that assigns to each element of A one and only one element of B.

- If b is the element of B assigned to the element a of A, b is said to be the **image** of a and is denoted $b = f(a)$.

- The set A is called the **domain** of f.

- The set of all images in B is called the **range** of f.

- If A is a set of real numbers, f is said to be a **function of a real variable** x.

- If A is a set of complex numbers, f is said to be a **function of a complex varibale** z or a **complex function**.

- The image w of a complex number z is

$$w = f(z) = u(x, y) + iv(x, y)$$

where u and v are the real and imaginary parts of w and are real-valued functions.

- Although we cannot draw a graph of a complex function $w = f(z)$ (because it would require a four-dimensional coordinate system), it can be interpreted as a **mapping** or **transformation** from the z plane to the w plane.

- A complex function may be interpreted as a two-dimensional fluid flow by considering $w = f(z)$ as the fluid velocity vector at the point z. In that case, if $x(t) + iy(t)$ is a parametric representation of a particle's position over time then

$$\frac{dx}{dt} = u(x, y)$$
$$\frac{dy}{dt} = v(x, y)$$

and the family of solutions to this system of differential equations are called the **streamlines** of the flow associated with $f(z)$.

Definition 17.4.1   Limit of a Function

Suppose the function f is defined in some neighborhood of z_0, except possibly at z_0 itself. Then f is said to possess a **limit** at z_0, written

$$\lim_{z \to z_0} f(z) = L$$

if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

- For a function f of a real variable x, the limit $\lim_{x \to x_0} f(x) = L$ means f approaches L as you approach from both the left and right. If however f is a function of a complex variable it means f approaches L as you approach from any direction in the complex plane.

Theorem 17.4.1   Limit of Sum, Product, Quotient

Suppose $\lim_{z \to z_0} f(z) = L_1$ and $\lim_{z \to z_0} g(z) = L_2$. Then

(i) $\lim_{z \to z_0} [f(z) + g(z)] = L_1 + L_2$

(ii) $\lim_{z \to z_0} f(z)g(z) = L_1 L_2$

(iii) $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{L_1}{L_2}, \quad L_2 \neq 0.$

- A function f is continuous at a point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

- A function f defined by

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0, \ a_n \neq 0$$

where n is a nonnegative integer and the coefficients a_i, $i = 0, 1, \ldots, n$, are complex constants is called a **polynomial** of degree n.

- Polynomials are continuous on the entire complex plane.

- A **rational function**

$$f(z) = \frac{g(z)}{h(z)}$$

is continuous everywhere $h(z) \neq 0$.

Definition 17.4.3   Derivative

Suppose the complex function f is defined in a neighborhood of a point z_0. The **derivative** of f at z_0 is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{3}$$

provided this limit exists.

- In order for a complex function to be differentiable, the limit must approach the same value from every direction. This is a greater demand than in real variables. If you take an arbitrary complex function, there's a good chance it isn't differentiable.

Definition 17.4.4   Analyticity at a Point

A complex function $w = f(z)$ is said to be **analytic at a point** z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

- Analyticity at a point is a neighborhood property. A function can be differentiable at a point but if the neighboring points aren't also differentiable, it's not analytic at that point.

- A function is analytic in a domain D if it is analytic at every point in D.

- A function that is analytic everywhere is called an **entire function**.

17.5   Cauchy-Riemann Equations

Theorem 17.5.1   Cauchy–Riemann Equations

Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivatives of u and v exist and satisfy the **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1}$$

- If a complex function $f(z) = u(x, y) + iv(x, y)$ is analytic throughout a domain D, then the real functions u and v must satisfy the Cauchy-Riemann equations at every point in D.

Theorem 17.5.2   Criterion for Analyticity

Suppose the real-valued functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a domain D. If u and v satisfy the Cauchy–Riemann equations at all points of D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

- The Cauchy-Riemann equations are derived assuming the function is differentiable at a particular point. That being the case, they can also be used as a formula for the derivative of the function

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

- Because analyticity implies differentiability, theorem 17.5.2 can also be used to determine if a function is differentiable at a point.

- A real-valued function $\phi(x, y)$ that has continuous second-order partial derivatives in a domain D and satisfies Laplace's equation is said to be **harmonic** in D.

- If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D then the functions $u(x, y)$ and $v(x, y)$ are harmonic functions.

- If a given function $u(x, y)$ is harmonic in a domain D it is sometimes possible to find another function $v(x, y)$ that is harmonic in D such that $u(x, y) + iv(x, y)$ is analytic in D. The function v is called the **harmonic conjugate function** of u.

- To find the harmonic conjugate function of a given function u:

  1. Take the first-order partial derivatives of u with respect to x and y.
  2. If $u(x, y) + iv(x, y)$ is analytic in a domain D then u and v must satisfy the Cauchy-Riemann equations in D from which we can find expressions for $\partial v/\partial x$ and $\partial v/\partial y$.
  3. Integrate $\partial v/\partial x$ with respect to x to get an expression for v with an unknown constant $h(y)$.
  4. Take the first-order partial derivative of v with respect to y, equate it with the other expression for $\partial v/\partial y$, and solve for $h'(y)$.
  5. Integrate $h'(y)$ and substitute the result to find v.

17.6   Exponential and Logarithmic Functions

- The exponential function for complex numbers is defined as

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

- e^z is analytic for all z, i.e. it's an entire function.

- Like its real-valued counterpart,

$$\frac{d}{dz}e^z = e^z,$$
$$e^{z_1}e^{z_2} = e^{z_1+z_2},$$

and

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}.$$

- Since
$$e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1$$
  and
$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z$$
  the complex function $f(z) = e^z$ is **periodic** with complex period $2\pi i$. Because of this complex periodicity an infinite horizontal strip of height 2π contains all possible values for the function. The strip $-\pi < y \le \pi$ is called the **fundamental region**.

- For $z \ne 0$ and $\theta = \arg z$,
$$\ln z = \log_e |z| + i(\theta + 2n\pi), \ \ n = 0, \pm 1, \pm 2, \ldots$$
  This means there are infinitely many values of the logarithm of a complex number z. This makes sense as the complex exponential is periodic.

- The **principal value** of $\ln z$ is the complex logarithm corresponding to $n = 0$ and $\theta = \operatorname{Arg} z$. It is denoted $\operatorname{Ln} z$.

- Some familiar properties of the real-valued logarithm hold for the complex-valued logarithm, e.g.
$$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$
  and
$$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2,$$
  however they don't necessarily hold for the principal value.

- $\operatorname{Ln} z$ is discontinuous and thus not analytic at $z = 0$ because $\ln z$ is undefined at $z = 0$ and on the negative real axis because $\operatorname{Arg} z$ is discontinuous there.

- The derivative of $\operatorname{Ln} z$ is
$$\frac{d}{dz} \operatorname{Ln} z = \frac{1}{z}.$$

- The complex power of a complex number is defined as
$$z^\alpha = e^{\alpha \ln z}, \ z \ne 0.$$
  In general this is multiple-valued because $\ln z$ is multiple-valued — only if $\alpha = n, \ n = 0, \pm 1, \pm 2, \ldots$ is it single-valued. If $\ln z$ is replaced with $\operatorname{Ln} z$ then we get the **principle value** of z^α.

17.7 Trigonometric and Hyperbolic Functions

Definition 17.7.1   Trigonometric Sine and Cosine

For any complex number $z = x + iy$,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \text{and} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{2}$$

- The other trigonometric functions ($\tan z$, etc.) are defined as usual.

- Because e^{iz} and e^{-iz} are entire functions, $\sin z$ and $\cos z$ are also entire functions.

- $\sin z = 0$ for the real numbers $z = n\pi$, $n \in \mathbb{Z}$ and $\cos z = 0$ for the real numbers $z = (2n + 1)\pi/2$, $n \in \mathbb{Z}$. This means that $\tan z$ and $\sec z$ are analytic except at the points where $\cos z = 0$ and $\cot z$ and $\sec z$ are analytic except at the points where $\sin z = 0$.

- The usual derivatives and trigonometric functions are still valid in the complex case.

- $\sin z$ can be expressed as

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

and $\cos z$ can be expressed as

$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$

- The only zeroes of $\sin z$ are the real numbers $z = n\pi$, $n \in \mathbb{Z}$ and the only zeroes of $\cos z$ are the real numbers $z = (2n + 1)\pi/2$, $n \in \mathbb{Z}$.

Definition 17.7.2   Hyperbolic Sine and Cosine

For any complex number $z = x + iy$,

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \qquad \text{and} \qquad \cosh z = \frac{e^{z} + e^{-z}}{2}. \tag{10}$$

- The complex trigonometric functions can be expressed in terms of the complex hyperbolic functions and vice versa

$$\sin z = -i \sinh(iz), \qquad\qquad \cos z = \cosh(iz)$$
$$\sinh z = -i \sin(iz), \qquad\qquad \cosh z = \cos(iz).$$

10

- $\sinh z$ can be expressed as

$$\sinh z = \sinh x \cos y + i \cosh x \sin y$$

and $\cosh z$ can be expressed as

$$\cosh z = \cosh x \cos y + i \sinh x \sin y.$$

- The zeroes of $\sinh z$ are $z = n\pi i$, $n \in \mathbb{Z}$ and the zeroes of $\cosh z$ are $z = (2n+1)\pi i/2$, $n \in \mathbb{Z}$.

- $\sin z$ and $\cos z$ are 2π periodic while $\sinh z$ and $\cosh z$ are $2\pi i$ periodic.

17.8   Inverse Trigonometric and Hyperbolic Functions

- Because the complex trigonometric functions are multi-valued, their inverse functions are also multi-valued.

- The definitions of those inverse functions are

$$\arcsin z = -i \ln[iz + (1 - z^2)^{1/2}],$$
$$\arccos z = -i \ln[z + i(1 - z^2)^{1/2}], \text{ and}$$
$$\arctan z = \frac{i}{2} \ln \frac{i+z}{i-z}.$$

- The derivatives of the inverse trigonometric functions are

$$\frac{d}{dz} \arcsin z = \frac{1}{(1 - z^2)^{1/2}},$$
$$\frac{d}{dz} \arccos z = \frac{-1}{(1 - z^2)^{1/2}}, \text{ and}$$
$$\frac{d}{dz} \arctan z = \frac{1}{1 + z^2}.$$

- The definitions of the hyperbolic inverse functions and their derivatives are

$$\sinh^{-1}z = \ln\left[z + (z^2 + 1)^{1/2}\right]$$

$$\cosh^{-1}z = \ln\left[z + (z^2 - 1)^{1/2}\right]$$

$$\tanh^{-1}z = \frac{1}{2}\ln\frac{1 + z}{1 - z}$$

$$\frac{d}{dz}\sinh^{-1}z = \frac{1}{(z^2 + 1)^{1/2}}$$

$$\frac{d}{dz}\cosh^{-1}z = \frac{1}{(z^2 - 1)^{1/2}}$$

$$\frac{d}{dz}\tanh^{-1}z = \frac{1}{1 - z^2}.$$

18 Integration in the Complex Plane

18.1 Contour Integrals

- In complex variables, a piecewise smooth curve C is called a **contour** or **path**. An integral of a complex function $f(z)$ on C is denoted $\int_C f(z)\,dz$ or $\oint_C f(z)\,dz$ if C is closed — this is called a **contour integral** or a **complex integral**.

1. Let $f(z) = u(x, y) + iv(x, y)$ be defined at all points on a smooth curve C defined by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$.
2. Divide C into n subarcs according to the partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b]$. The corresponding points on the curve C are $z_0 = x_0 + iy_0 = x(t_0) + iy(t_0)$, $z_1 = x_1 + iy_1 = x(t_1) + iy(t_1), \ldots, z_n = x_n + iy_n = x(t_n) + iy(t_n)$. Let $\Delta z_k = z_k - z_{k-1}$, $k = 1, 2, \ldots, n$.
3. Let $\|P\|$ be the **norm** of the partition, that is, the maximum value of $|\Delta z_k|$.
4. Choose a sample point $z_k^* = x_k^* + iy_k^*$ on each subarc. See **FIGURE 18.1.1**.
5. Form the sum $\sum_{k=1}^{n} f(z_k^*)\,\Delta z_k$.

Definition 18.1.1　Contour Integral

Let f be defined at points of a smooth curve C defined by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$. The **contour integral** of f along C is

$$\int_C f(z)\,dz = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(z_k^*)\,\Delta z_k. \tag{1}$$

If f is continuous on a smooth curve C given by $z(t) = x(t) + iy(t)$, $a \le t \le b$, then

$$\int_C f(z)\, dz \;=\; \int_a^b f(z(t))z'(t)\, dt. \tag{3}$$

- If a complex function f is continuous on a smooth curve C and if $|f(z)| \le M$ for all z on C, then

$$\left| \int_C f(z)\, dz \right| \le ML,$$

  where

$$L = \int_a^b |z'(t)|\, dt$$

  is the length of C. This is sometimes called the **ML-intequality**.

- If $\mathbf{T}$ is the unit tangent vector to a positively oriented simple closed curve C then

$$\oint_C f \cdot \mathbf{T}\, ds = \mathrm{Re}\left(\oint \overline{f(z)}\, dz \right)$$

  is called the **circulation** around C and measures the tendency of the flow to rotate the curve C.

- If $\mathbf{N}$ is the normal vector to a positive oriented simple closed curve C then

$$\oint_C f \cdot \mathbf{N}\, ds = \mathrm{Im}\left(\oint \overline{f(z)}\, dz \right)$$

  is called the **net flux** across C and measures the difference between the rates at which fluid enters and exits the region bounded by C.

18.2   Cauchy-Goursat Theorem

- A domain D is said to be **simply connected** if every simple closed contour C lying entirely in D can be shrunk to a point without leaving D, i.e. the domain has no holes in it.

- A domain that is not simply connected is called a **multiply connected domain**. A domain with one hole is called **doubly connected**, a domain with two holes **triply connected**, etc.

Suppose a function f is analytic in a simply connected domain D. Then for every simple closed contour C in D, $\oint_C f(z)\, dz = 0$.

- An alternative way of stating the Cauchy-Goursat Theorem is: if f is analytic at all points on and within a simple closed contour C, then $\oint_C f(z)\,dz = 0$.

- If D is a double connected domain and C and C_1 are simple closed contours such that C_1 surrounds the "hole" in the domain and is interior to C, then

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz.$$

This is called the principle of **deformation of contours** since C_1 can be considered a continuous deformation of the contour C (or vice versa) under which the value of the integal doesn't change.

- If z_0 is a constant complex number interior to a simple closed contour C, then
$$\oint_C \frac{dz}{(z - z_0)^n} = \begin{cases} 2\pi i & n = 1 \\ 0 & n \text{ an integer} \neq 1 \end{cases}.$$

Theorem 18.2.2   Cauchy–Goursat Theorem for Multiply Connected Domains

Suppose $C, C_1, \ldots, C_n$ are simple closed curves with a positive orientation such that $C_1, C_2, \ldots, C_n$ are interior to C but the regions interior to each C_k, $k = 1, 2, \ldots, n$, have no points in common. If f is analytic on each contour and at each point interior to C but exterior to all the C_k, $k = 1, 2, \ldots, n$, then

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz. \tag{6}$$