

# Classical Mechanics by John R. Taylor Notes

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# 1 Newton's Laws of Motion

## 1.2 Space and Time

- In cartesian coordinates the basis vectors don't depend on time so their derivatives are  $\mathbf{0}$ . This means that

$$\begin{aligned}\frac{d}{dt}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) &= \frac{dx}{dt}\hat{\mathbf{x}} + x\frac{d\hat{\mathbf{x}}}{dt} + \frac{dy}{dt}\hat{\mathbf{y}} + y\frac{d\hat{\mathbf{y}}}{dt} + \frac{dz}{dt}\hat{\mathbf{z}} + z\frac{d\hat{\mathbf{z}}}{dt} \\ &= \frac{dx}{dt}\hat{\mathbf{x}} + \frac{dy}{dt}\hat{\mathbf{y}} + \frac{dz}{dt}\hat{\mathbf{z}}\end{aligned}$$

as expected. However, in order coordinate systems (e.g. polar, spherical) the basis vectors may depend on time and their derivatives aren't  $\mathbf{0}$ .

## 1.4 Newton's First and Second Laws; Inertial Frames

- Newton's second law  $\mathbf{F} = m\mathbf{a}$  can be restated as  $\mathbf{F} = \dot{\mathbf{p}}$ .
- An inertial frame is one where Newton's first law holds. Typically this means the frame isn't accelerating or rotating.

## 1.5 The Third Law and Conservation of Momentum

- Forces that act along the line joining two objects are called **central forces**.
- The **principle of conservation of momentum** states that if the net external force  $\mathbf{F}_{\text{ext}}$  on an  $N$ -particle system is zero, the system's total momentum  $\mathbf{P}$  is constant.

## 1.7 Two-Dimensional Polar Coordinates

- In two-dimensional polar coordinates, the unit vectors  $\hat{\mathbf{r}}$  and  $\hat{\phi}$  depend on position and thus time. Their derivatives are

$$\begin{aligned}\frac{d\hat{\mathbf{r}}}{dt} &= \dot{\phi}\hat{\phi} \\ \frac{d\hat{\phi}}{dt} &= -\dot{\phi}\hat{\mathbf{r}}.\end{aligned}$$

Consequently, the derivatives of the position vector  $\mathbf{r} = r\hat{\mathbf{r}}$  are

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= \frac{d}{dt}(r\hat{\mathbf{r}}) \\ &= \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \\ &= \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\phi}\end{aligned}$$

and

$$\begin{aligned}
\frac{d^2 \mathbf{r}}{dt^2} &= \frac{d}{dt}(r\hat{\mathbf{r}} + r\dot{\phi}\hat{\phi}) \\
&= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\frac{d\hat{\phi}}{dt} \\
&= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\phi}\hat{\phi} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} - r\dot{\phi}^2\hat{\mathbf{r}} \\
&= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi}.
\end{aligned}$$

- In light of the above, Newton's second law in polar coordinates can be written

$$\begin{aligned}
F_r &= m(\ddot{r} - r\dot{\phi}^2) \\
F_\phi &= m(r\ddot{\phi} + 2\dot{r}\dot{\phi}).
\end{aligned}$$

## 2 Projectiles and Charged Particles

### 2.1 Air Resistance

- Air resistance depends on the speed  $v$  of the moving object. For many objects the direction of the air resistance force  $\mathbf{f}$  is opposite to  $\mathbf{v}$ , but not always. For example, the air resistance force on an airplane causes lift.
- An air resistance force can be described by the equation

$$\mathbf{f} = -f(v)\hat{\mathbf{v}}$$

where  $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$  gives the direction and  $f(v)$  gives the magnitude.

- $f(v)$  can be approximated as

$$f(v) = f_{\text{lin}} + f_{\text{quad}} = bv + cv^2.$$

- The linear term  $f_{\text{lin}}$  arises from the viscous drag of the medium and is generally proportional to the projectile's linear size.
- The quadratic term  $f_{\text{quad}}$  arises from the fact that the projectile must accelerate the air with which it is continually colliding and it is proportional to the density of the medium and the cross-sectional area of the projectile.
- For a spherical projectile the coefficients  $b$  and  $c$  above have the form

$$b = \beta D \text{ and } c = \gamma D^2$$

where  $D$  is the diameter of the sphere and the coefficients  $\beta$  and  $\gamma$  depend on the nature of the medium. In air at STP they have approximate values

$$\beta = 1.6 \times 10^{-4} \text{ N s/m}^2$$

and

$$\gamma = 0.25 \text{ N s}^2/\text{m}^4.$$

- Depending on the natures of the medium and projectile it's often possible to neglect one of the terms in  $f(v)$ . To determine if this is the case we can calculate their ratio. For example, for a spherical projectile at STP

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} = \frac{cv^2}{bv} = \frac{\gamma D}{\beta} v = (1.6 \times 10^3 \text{ s/m}^2) Dv.$$

If the ratio is large  $f_{\text{lin}}$  can be ignored. If it's small  $f_{\text{quad}}$  can be ignored.

- The **Reynolds number** can be used to characterise the behaviour of an object in a fluid

$$R = \frac{\rho}{\mu} Dv$$

where  $\rho$  is the medium's density,  $\mu$  is its viscosity,  $D$  is the linear dimension of the projectile (diameter for spherical projectiles), and  $v$  is the projectile's speed. The quadratic force  $f_{\text{quad}}$  is dominant when the Reynolds number  $R$  is large and the linear force  $f_{\text{linear}}$  is dominant when it is small.

## 2.2 Linear Air Resistance

- When the quadratic drag force is negligible the equation of motion becomes

$$\begin{aligned}\mathbf{F} &= \mathbf{W} - \mathbf{f} \\ m\mathbf{a} &= m\mathbf{g} - b\mathbf{v} \\ m\dot{\mathbf{v}} &= m\mathbf{g} - b\mathbf{v}.\end{aligned}$$

This is a first-order differential equation for  $\mathbf{v}$  where the horizontal and vertical components can be separated to

$$\begin{aligned}m\dot{v}_x &= -bv_x \\ m\dot{v}_y &= mg - bv_y,\end{aligned}$$

each of which is easily solvable.

- The **terminal speed** of an object undergoing freefall and experiencing only linear drag is

$$v_{\text{ter}} = \frac{mg}{b}.$$

- The **characteristic time**

$$\tau = \frac{1}{k} = \frac{1}{b/m} = \frac{m}{b}$$

is a measure of the importance of air resistance.

- For horizontal motion with drag it's a measure of the time it takes for the projectile to reach  $1/e$  of its initial velocity.

- For freefall with drag it's a measure of the time it would take the projectile to reach its terminal velocity if it didn't experience drag

$$v_{\text{ter}} = g\tau.$$

- For freefall with drag it can also be used to gauge what percentage of its terminal velocity a projectile will reach after a certain time:

Time $t$	Percent of $v_{\text{ter}}$
0	0
$\tau$	63%
$2\tau$	86%
$3\tau$	95%

From this it can be seen that after  $t = 3\tau$  the projectile has effectively reached its terminal velocity.

## 2.4 Quadratic Air Resistance

- Equations of motion for quadratic air resistance can be solved analytically when the projectile moves in one dimension, but can only be solved numerically when it moves in multiple dimensions.
- When a projectile moves in one dimension and only experiences the force of air resistance (i.e. there are no other forces), the equation of motion is

$$m\dot{v} = -cv^2.$$

Using separation of variables the solution can be found to be

$$v(t) = \frac{v_0}{1 + t/\tau}$$

where

$$\tau = \frac{m}{cv_0}.$$

- As in the linear case,  $\tau$  is a measure of how long it takes for air resistance to slow down the projectile ( $v = v_0/2$  at  $t = \tau$ ).
- Integrating the equation for  $v(t)$  gives

$$x(t) = v_0\tau \ln \left( 1 + \frac{t}{\tau} \right).$$

- When a projectile moves in one dimension and experiences the forces of air resistance and weight, the equation of motion (with  $y$  down) is

$$m\dot{v} = mg - cv^2.$$

Using separation of variables the solution can be found to be

$$v(t) = v_{\text{ter}} \tanh \frac{gt}{v_{\text{ter}}}$$

where

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}}.$$

- Integrating the equation for  $v(t)$  gives

$$y = \frac{v_{\text{ter}}^2}{g} \ln \left( \cosh \frac{gt}{v_{\text{ter}}} \right).$$

## 2.5 Motion of a Charge in a Uniform Magnetic Field

- When a particle of charge  $q$  moves in a magnetic field  $\mathbf{B} = (0, 0, B_z)$  with velocity  $\mathbf{v} = (v_x, v_y, v_z)$  it experiences a force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = q(v_y B, -v_x B, 0).$$

This gives the coupled equations of motion

$$\begin{aligned} m\dot{v}_x &= qBv_y \\ m\dot{v}_y &= -qBv_x \\ m\dot{v}_z &= 0 \end{aligned}$$

or

$$\begin{aligned} \dot{v}_x &= \omega v_y \\ \dot{v}_y &= -\omega v_x \\ \dot{v}_z &= 0 \end{aligned}$$

where  $\omega = qB/m$  is called the **cyclotron frequency**.

- If we define a complex value

$$\eta = v_x + iv_y,$$

its derivative is

$$\begin{aligned} \dot{\eta} &= \dot{v}_x + i\dot{v}_y \\ &= \omega v_y - i\omega v_x \\ &= -i\omega\eta \end{aligned}$$

which has the solution

$$\eta = Ae^{-i\omega t}.$$

## 3 Momentum and Angular Momentum

### 3.1 Conservation of Momentum

- The **principle of conservation of momentum** states that if the net external force  $\mathbf{F}_{\text{ext}}$  on an  $N$ -particle system is zero, the system's total mechanical momentum  $\mathbf{P} = \sum m_{\alpha} \mathbf{v}_{\alpha}$  is constant.

### 3.2 Rockets

- Newton's second law for a rocket is

$$m\dot{v} = -\dot{m}v_{\text{ex}}$$

where  $\dot{m}$  is the (negative) rate of change of the mass of the rocket and  $v_{\text{ex}}$  is the velocity of the exhaust. The quantity on the right hand side of the equation is called the **thrust**.

- The equation above can be solved by separation of variables giving

$$v - v_0 = v_{\text{ex}} \ln \frac{m_0}{m}$$

which is often called the **rocket equation**.

### 3.3 The Center of Mass

- The **centre of mass** of a system is defined to be

$$\mathbf{R} = \frac{1}{M} \sum_{\alpha=1}^N m_{\alpha} \mathbf{r}_{\alpha}$$

where  $M$  is the total mass of all particles in the system,  $m_{\alpha}$  is the mass of particle  $\alpha$ , and  $\mathbf{r}_{\alpha}$  is the vector from the origin to particle  $\alpha$ .

- The total momentum of a system can be written in terms of its centre of mass

$$\mathbf{P} = \sum_{\alpha} \mathbf{p}_{\alpha} = \sum_{\alpha} m_{\alpha} \dot{\mathbf{r}}_{\alpha} = M \dot{\mathbf{R}}$$

i.e. the total momentum of  $N$  particles is equivalent to that of a single particle of mass  $M$  with velocity equal to that of the centre of mass.

- Differentiating the above we find

$$\begin{aligned} \frac{d}{dt} \mathbf{P} &= \frac{d}{dt} (M \dot{\mathbf{R}}) \\ \mathbf{F}_{\text{ext}} &= M \ddot{\mathbf{R}} \end{aligned}$$

i.e. the centre of mass moves as if it was a single particle of mass  $M$  subject to the net external force on the system.

- When a body is continuous the expression for its centre of mass becomes an integral

$$\mathbf{R} = \frac{1}{M} \int \mathbf{r} dm = \frac{1}{M} \int \rho \mathbf{r} dV.$$

### 3.4 Angular Momentum for a Single Particle

- The **angular momentum** of a particle relative to an origin  $O$  is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

where  $\mathbf{r}$  is measured relative to  $O$ .

- Taking the derivative of angular momentum gives

$$\begin{aligned} \frac{d}{dt} \mathbf{L} &= \frac{d}{dt} (\mathbf{r} \times \mathbf{p}) \\ \dot{\mathbf{L}} &= \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} \\ &= \mathbf{r} \times \mathbf{F} \\ &= \boldsymbol{\tau}. \end{aligned}$$

In other words, the rate of change in angular momentum about an origin  $O$  is equal to the net torque about that origin.

- We can simplify some one-particle problems by choosing the origin such that the net torque is 0 and thus angular momentum is constant.

### 3.5 Angular Momentum for Several Particles

- The **total angular momentum** of a system is

$$\mathbf{L} = \sum_{\alpha=1}^N \mathbf{L}_{\alpha} = \sum_{\alpha=1}^N \mathbf{r}_{\alpha} \times \mathbf{p}_{\alpha}.$$

- Differentiating the above

$$\dot{\mathbf{L}} = \sum_{\alpha} \dot{\mathbf{L}}_{\alpha} = \sum_{\alpha} \mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha} = \boldsymbol{\tau}_{\text{ext}}$$

we find that the rate of change of the total angular momentum of the system is equal to the net torque on the system.

- The **principle of conservation of angular momentum** states that if the net external torque on a system is 0, the system's total angular momentum is constant. This assumes that all internal forces are central and obey Newton's third law.
- The principle of conservation of momentum and the result  $\dot{\mathbf{L}} = \boldsymbol{\tau}_{\text{ext}}$  also hold if  $\mathbf{L}$  and  $\boldsymbol{\tau}_{\text{ext}}$  are measured about the centre of mass, even if the centre of mass is being accelerated and is thus not an inertial frame.



## 4 Energy

### 4.1 Kinetic Energy and Work

- The **work-kinetic-energy theorem** states that the change in a particle's kinetic energy between two points is equal to the work done by the net force on the particle between those two points

$$\Delta K = \int_1^2 \mathbf{F} \cdot d\mathbf{r}.$$

### 4.2 Potential Energy and Conservative Forces

- A force  $\mathbf{F}$  acting on a particle is considered **conservative** if:
  - $\mathbf{F}$  depends only on the particle's position  $\mathbf{r}$  (and not on its velocity  $\mathbf{v}$ , time  $t$ , or any other variable), and
  - for any two points 1 and 2, the work done by  $\mathbf{F}$  is the same for all paths between 1 and 2.
- Only conservative forces have associated **potential energy** functions.
- The potential energy function  $U(\mathbf{r})$  of a conservative force  $\mathbf{F}$  is defined as

$$U(\mathbf{r}) = - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'$$

where  $\mathbf{r}_0$  is an arbitrary point at which  $U(\mathbf{r}_0)$  is defined to be 0.

- The **principle of conservation of energy** states that if all the forces acting on a particle are conservative, each with its corresponding potential energy function  $U_i(\mathbf{r})$ , the **total mechanical energy**

$$E = K + U = K + U_1(\mathbf{r}) + \cdots + U_n(\mathbf{r}),$$

is constant in time.

- If nonconservative forces do work then the total energy of the system changes by that amount

$$\Delta E = W_{\text{nc}}.$$

### 4.3 Force as the Gradient of Potential Energy

- A conservative force  $\mathbf{F}$  can be expressed as the negative gradient of its potential energy function  $U$

$$\mathbf{F} = -\nabla U.$$

#### 4.4 The Second Condition that $\mathbf{F}$ be Conservative

- A force  $\mathbf{F}$  is conservative if  $\nabla \times \mathbf{F} = \mathbf{0}$ .

#### 4.5 Time-Dependent Potential Energy

- If a time-dependent force  $\mathbf{F}(t)$  has the property  $\nabla \times \mathbf{F}(t) = \mathbf{0}$  it's still possible to define an associated potential energy function  $U(\mathbf{r}, t)$  where  $\mathbf{F}(t) = -\nabla U(t)$  but it's no longer guaranteed that total mechanical energy is conserved over time.