Advanced Engineering Mathematics Complex Analysis by Dennis G. Zill Problems

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Contents

17 Functions of a Complex Variable

17.1 Complex Numbers

17.1.1

$$3+3i$$

17.1.3

$$i^8=(i^2)^4=(-1)^4=1$$

17.1.5

$$7-13i$$

17.1.7

$$-7+5i$$

17.1.9

$$11-10i$$

17.1.11

$$-5+12i$$

17.1.13

$$-2i$$

17.1.15

$$\begin{aligned}\frac{2-4i}{3+5i}&=\frac{(2-4i)(3-5i)}{34}\\&=\frac{-14-22i}{34}\\&=-\frac{7}{17}-\frac{11}{17}i\end{aligned}$$

17.1.17

$$\begin{aligned}\frac{(3-i)(2+3i)}{1+i} &= \frac{9+7i}{1+i}\\ &= \frac{(9+7i)(1-i)}{2}\\ &= \frac{16-2i}{2}\\ &= 8-i\end{aligned}$$

17.1.27

$$\begin{aligned}\frac{1}{z} &= \frac{\overline{z}}{z\overline{z}}\\ &= \frac{x-iy}{x^2+y^2}\\ \mathrm{Re}\left(\frac{1}{z}\right) &= \frac{x}{x^2+y^2}\end{aligned}$$

17.1.29

$$\begin{aligned}2z+4\overline{z}-4i &= 2(x+iy)+4(x-iy)-4i\\ &= 6x-2(y+2)i\\ \mathrm{Im}(2z+4\overline{z}-4i) &= -2y-4\end{aligned}$$

17.1.31

$$\begin{aligned}z-1-3i &= x+iy-1-3i\\ &= (x-1)+(y-3)i\\ |z| &= \sqrt{(x-1)^2+(y-3)^2}\end{aligned}$$

17.1.33

$$\begin{aligned}2z &= i(2+9i)\\ &= -9+2i\\ z &= -\frac{9}{2}+i\end{aligned}$$

17.1.35

$$\begin{aligned}(x+iy)^2 &= x^2+2xyi-y^2\\ &= (x^2-y^2)+2xyi\\ x^2 &= y^2\\ x &= y\\ 2xy &= 1\\ x^2 &= \frac{1}{2}\\ x &= \frac{\sqrt{2}}{2}\\ z &= \frac{\sqrt{2}}{2}(1+i)\end{aligned}$$

17.1.37

$$\begin{aligned}z+2\overline{z} &= x+iy+2x-2iy\\ &= 3x-iy\\ \frac{2-i}{1+3i} &= \frac{(2-i)(1-3i)}{10}\\ &= \frac{-1-7i}{10}\\ 3x-iy &= \frac{-1-7i}{10}\\ x &= -\frac{1}{30}\\ y &= \frac{7}{10}\\ z &= -\frac{1}{30}+\frac{7}{10}i\end{aligned}$$

17.1.39

$$\begin{aligned}|10+8i| &\approx 12.8\\ |11-6i| &\approx 12.5\end{aligned}$$

$11-6i$ is closer.

17.2 Powers and Roots

17.2.1

$$2(\cos 0 + i\sin 0)$$

17.2.3

$$-3[\cos(-\pi/2) + i\sin(-\pi/2)]$$

17.2.5

$$\sqrt{2}[\cos(\pi/4) + i\sin(\pi/4)]$$

17.2.7

$$2[\cos(5\pi/6) + i\sin(5\pi/6)]$$

17.2.9

$$\begin{aligned}\frac{3}{-1+i} &= \frac{3(-1-i)}{2} \\ &= \frac{-3-3i}{2} \\ &= -\frac{3}{2} - \frac{3}{2}i \\ &= \frac{3\sqrt{2}}{2}[\cos(5\pi/4) + i\sin(5\pi/4)]\end{aligned}$$

17.2.11

$$-\frac{5\sqrt{3}}{2} - \frac{5}{2}i$$

17.2.13

$$5.54 + 2.30i$$

17.2.15

$$\begin{aligned}8[\cos(\pi/2) + i\sin(\pi/2)] &= 8i \\ \frac{1}{2}[\cos(-\pi/4) + i\sin(-\pi/4)] &= \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}i\end{aligned}$$

17.2.21

$$\begin{aligned}(1+\sqrt{3}i)^9 &= \{2[\cos(\pi/3) + i\sin(\pi/3)]\}^9 \\ &= 512(\cos\pi + i\sin\pi) \\ &= -512\end{aligned}$$

17.2.23

$$
\begin{aligned}
\left(\frac{1}{2}+\frac{1}{2}i\right)^1 0 &= \left\{\frac{\sqrt{2}}{2}[\cos(\pi/4)+i\sin(\pi/4)]\right\}^{10} \\
&= \frac{1}{32}[\cos(\pi/2)+i\sin(\pi/2)] \\
&= \frac{1}{32}i
\end{aligned}
$$

17.2.27

$$
\begin{aligned}
w_k &= 2[\cos(2\pi k/3)+i\sin(2\pi k/3)] \\
w_0 &= 2 \\
w_1 &= -1+\sqrt{3}i \\
w_2 &= -1-\sqrt{3}i
\end{aligned}
$$

17.2.29

$$
\begin{aligned}
w_k &= \cos(\pi/4+k\pi)+i\sin(\pi/4+k\pi) \\
w_0 &= \frac{\sqrt{2}}{2}(1+i) \\
w_1 &= -\frac{\sqrt{2}}{2}(1+i)
\end{aligned}
$$

17.2.31

$$
\begin{aligned}
w_k &= \sqrt{2}[\cos(\pi/3+k\pi)+i\sin(\pi/3+k\pi)] \\
w_0 &= \frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}i \\
w_1 &= -\frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}i
\end{aligned}
$$

17.2.33

$$
\begin{aligned}
z^4 + 1 &= 0 \\
z^4 &= -1 \\
w_k &= \cos(\pi/4 + k\pi/2) + \sin(\pi/4 + k\pi/2) \\
w_0 &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \\
w_1 &= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \\
w_2 &= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \\
w_3 &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i
\end{aligned}
$$

17.3 Sets in the Complex Plane

17.3.1

A vertical line at $\mathrm{Re}(z) = 5$.

17.3.3

A horizontal line at $\mathrm{Im}(z) = -3$.

17.3.5

A circle of radius 2 centred at $3i$.

17.3.7

A circle of radius 5 centred at $4 - 3i$.

17.3.9

The region of the plane to the left of (but not including) $\mathrm{Re}(z) = -1$. It is a domain.

17.3.11

The region of the plane above (but not including) $\mathrm{Im}(z) = 3$. It is a domain.

17.3.13

The region of the plane between (but not including) $\mathrm{Re}(z) = 3$ and $\mathrm{Re}(z) = 5$. It is a domain.

17.3.15

$$\begin{aligned} z^2 &= (a+ib)^2 \\ &= a^2 - b^2 + 2iab \\ \mathrm{Re}(z^2) &= a^2 - b^2 \\ \mathrm{Re}(z^2) &> 0 \\ a^2 - b^2 &> 0 \\ a^2 &> b^2 \end{aligned}$$

The region between $y = x$ and $y = -x$. Not a domain.

17.3.17

The region between $\theta = 0$ and $\theta = 2\pi/3$. Not a domain.

17.3.19

The region outside a circle of radius 1 centred at i. It is a domain.

17.3.21

The region between the circles of radius 2 and 3 centred at i. It is a domain.

17.3.23

$$y = -x$$

17.3.25

$$\begin{aligned} z^2 + \overline{z}^2 &= (a+ib)^2 + (a-ib)^2 \\ &= a^2 + 2iab - b^2 + a^2 - 2iab - b^2 \\ &= 2(a^2 - b^2) \\ 2(a^2 - b^2) &= 2 \\ a^2 - b^2 &= 1 \\ a^2 &= b^2 + 1 \end{aligned}$$

The hyperbola $x^2 - y^2 = 1$.

17.4 Functions of a Complex Variable

17.4.1

$$\begin{aligned}
f(z) &= z^2 \\
&= (x+iy)^2 \\
&= x^2 - y^2 + 2ixy \\
u(x,y) &= x^2 - y^2 \\
&= x^2 - 4 \\
v(x,y) &= 2xy \\
&= 4x \\
x &= \frac{v}{4} \\
u &= \left(\frac{v}{4}\right)^2 - 4 \\
&= \frac{1}{16}v^2 - 4
\end{aligned}$$

17.4.3

$$\begin{aligned}
u &= -y^2 \\
v &= 0
\end{aligned}$$

Line on the left half of the real axis.

17.4.5

$$\begin{aligned}
u &= 0 \\
v &= 2x^2
\end{aligned}$$

Line on the top half of the imaginary axis.

17.4.7

$$f(x) = (6x-5) + i(6y+9)$$

17.4.9

$$f(z) = (x^2 - y^2 - 3x) + i(2xy - 3y + 4)$$

17.4.11

$$f(z) = (x^3 - 3xy^2 - 4x) + i(3x^2y - y^3 - 4y)$$

17.4.13

$$f(z) = \left(x + \frac{x}{x^2+y^2}\right) i \left(y - \frac{y}{x^2+y^2}\right)$$

17.4.15

(a) $-4+i$

(b) $3-9i$

(c) $1+86i$

17.4.17

(a) $14-20i$

(b) $-13+43i$

(c) $3-26i$

17.4.19

$$6-5i$$

17.4.21

$$-4i$$

17.4.27

$$f'(z) = 12z^2 - 2(3+i)z - 5$$

17.4.29

$$\begin{aligned} f'(z) &= 2(z^2-4z+8i) + (2z+1)(2z-4) \\ &= 2z^2 - 8z + 16i + 4z^2 - 8z + 2z - 4 \\ &= 6z^2 - 14z - 4 + 16i \end{aligned}$$

17.4.31

$$f'(z) = 6z(z^2-4i)^2$$

17.4.33

$$\begin{aligned} f'(z) &= \frac{3(2z+i) - 2(3z-4+8i)}{(2z+i)^2} \\ &= \frac{6z+3i-6z+8-16i}{(2z+i)^2} \\ &= \frac{8-13i}{(2z+i)^2} \end{aligned}$$

17.4.35

$3i$

17.4.37

$\pm 2i$

17.4.41

$$\begin{aligned} \frac{dx}{dt} &= 2x \\ x &= c_1 e^{2t} \\ \frac{dy}{dt} &= 2y \\ y &= c_2 e^{2t} \end{aligned}$$

17.4.43

$$
\begin{aligned}
f(z) &= \frac{1}{\bar{z}} \\
&= \frac{1}{x - iy} \\
&= \frac{x + iy}{x^2 + y^2} \\
&= \frac{x}{x^2 + y^2} + i\frac{y}{x^2 + y^2} \\
\frac{dx}{dt} &= \frac{x}{x^2 + y^2} \\
\frac{dy}{dt} &= \frac{y}{x^2 + y^2} \\
\frac{dy}{dx} &= \frac{y}{x} \\
\frac{dy}{y} &= \frac{dx}{x} \\
\ln y &= \ln x + c_1 \\
y &= c_2 x
\end{aligned}
$$

17.5 Cauchy-Riemann Equations

17.5.1

$$
\begin{aligned}
f(z) &= z^3 \\
&= (x + iy)^3 \\
&= (x^2 + 2ixy - y^2)(x + iy) \\
&= x^3 + ix^2y + 2ix^2y - 2xy^2 - xy^2 - iy^3 \\
&= (x^3 - 3xy^2) + i(3x^2y - y^3) \\
\frac{\partial u}{\partial x} &= 3x^2 - 3y^2 \\
&= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -6xy \\
&= -\frac{\partial v}{\partial x}
\end{aligned}
$$

17.5.3

$$\begin{aligned}
f(z) &= \mathrm{Re}(z)\\
&= x\\
\frac{\partial u}{\partial x} &= 1\\
&\neq \frac{\partial v}{\partial y}
\end{aligned}$$

17.5.5

$$\begin{aligned}
f(z) &= 4z - 6\overline{z} + 3\\
&= 4(x+iy) - 6(x-iy) + 3\\
&= (-2x+3) + 10iy\\
\frac{\partial u}{\partial x} &= -2\\
&\neq \frac{\partial v}{\partial y}
\end{aligned}$$

17.5.7

$$\begin{aligned}
f(z) &= x^2 + y^2\\
\frac{\partial u}{\partial x} &= 2x\\
&\neq \frac{\partial v}{\partial y}
\end{aligned}$$

17.5.9

$$\begin{aligned}
f(z) &= e^x \cos y + ie^x \sin y\\
u &= e^x \cos y\\
\frac{\partial u}{\partial x} &= e^x \cos y\\
\frac{\partial u}{\partial y} &= -e^x \sin y\\
v &= e^x \sin y\\
\frac{\partial v}{\partial x} &= e^x \sin y\\
\frac{\partial v}{\partial y} &= e^x \cos y
\end{aligned}$$

Analytic everywhere.

17.5.11

$$
\begin{aligned}
f(z) &= x + \sin x \cosh y + i(y + \cos x \sinh y) \\
u &= x + \sin x \cosh y \\
\frac{\partial u}{\partial x} &= 1 + \cos x \cosh y \\
\frac{\partial u}{\partial y} &= \sin x \sinh y \\
v &= y + \cos x \sinh y \\
\frac{\partial v}{\partial x} &= -\sin x \sinh y \\
\frac{\partial v}{\partial y} &= 1 + \cos x \cosh y
\end{aligned}
$$

Analytic everywhere.

17.5.15

$$
\begin{aligned}
f(z) &= 3x - y + 5 + i(ax + by - 3) \\
u &= 3x - y + 5 \\
\frac{\partial u}{\partial x} &= 3 \\
\frac{\partial u}{\partial y} &= -1 \\
v &= ax + by - 3 \\
\frac{\partial v}{\partial x} &= a \\
\frac{\partial v}{\partial y} &= b \\
a &= 1 \\
b &= 3
\end{aligned}
$$

17.5.17

$$
\begin{aligned}
f(z) &= x^2 + y^2 + 2ixy \\
u &= x^2 + y^2 \\
\frac{\partial u}{\partial x} &= 2x \\
\frac{\partial u}{\partial y} &= 2y \\
v &= 2xy \\
\frac{\partial v}{\partial x} &= 2y \\
\frac{\partial v}{\partial y} &= 2x
\end{aligned}
$$

Only differentiable when $y = 0$.

17.5.19

$$
\begin{aligned}
f(z) &= x^3 + 3xy^2 - x + i(y^3 + 3x^2y - y) \\
u &= x^3 + 3xy^2 - x \\
\frac{\partial u}{\partial x} &= 3x^2 + 3y^2 - 1 \\
\frac{\partial u}{\partial y} &= 6xy \\
v &= y^3 + 3x^2y - y \\
\frac{\partial v}{\partial x} &= 6xy \\
\frac{\partial v}{\partial y} &= 3y^2 + 3x^2 - 1
\end{aligned}
$$

Only differentiable when $x = 0$ or $y = 0$.

17.5.21

$$
\begin{aligned}
f(z) &= e^x \cos y + ie^x \sin y \\
f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\
&= e^x \cos y + ie^x \sin y
\end{aligned}
$$

17.5.23

$$
\begin{aligned}
u &= x \\
\frac{\partial^2 u}{\partial x^2} &= 0 \\
\frac{\partial^2 u}{\partial y^2} &= 0 \\
\frac{\partial v}{\partial y} &= 1 \\
v &= y + h(x) \\
h'(x) &= 0 \\
v &= y + c \\
f(z) &= x + i(y + c)
\end{aligned}
$$

17.5.25

$$
\begin{aligned}
u &= x^2 - y^2 \\
\frac{\partial^2 u}{\partial x^2} &= 2 \\
\frac{\partial^2 u}{\partial y^2} &= -2 \\
\frac{\partial v}{\partial y} &= 2x \\
v &= 2xy + h(x) \\
2y &= 2y + h'(x) \\
h'(x) &= 0 \\
h(x) &= c \\
v &= 2xy + c \\
f(z) &= (x^2 - y^2) + i(2xy + c)
\end{aligned}
$$