

# Advanced Engineering Mathematics Ordinary Differential Equations Notes

Chris Doble

February 2022

## Contents

<b>1</b>	<b>Introduction to Differential Equations</b>	<b>8</b>
1.1	Definitions and Terminology . . . . .	8
1.1.1	1 . . . . .	8
1.1.2	3 . . . . .	8
1.1.3	5 . . . . .	8
1.1.4	7 . . . . .	8
1.1.5	9 . . . . .	9
1.1.6	15 . . . . .	9
1.1.7	17 . . . . .	9
1.1.8	19 . . . . .	9
1.1.9	31 . . . . .	10
1.1.10	33 . . . . .	10
1.1.11	35 . . . . .	10
1.1.12	37 . . . . .	10
1.1.13	39 . . . . .	10
1.2	Initial Value Problems . . . . .	10
1.2.1	1 . . . . .	10
1.2.2	3 . . . . .	11
1.2.3	5 . . . . .	11
1.2.4	7 . . . . .	11
1.2.5	9 . . . . .	11
1.2.6	11 . . . . .	12
1.2.7	13 . . . . .	12
1.2.8	15 . . . . .	13
1.2.9	17 . . . . .	13
1.2.10	19 . . . . .	13
1.2.11	21 . . . . .	13
1.2.12	23 . . . . .	13
1.2.13	25 . . . . .	13
1.2.14	27 . . . . .	14

1.2.15	29	14
1.2.16	31	14
1.2.17	39	14
1.2.18	41	15
1.2.19	43	15
1.3	Differential Equations as Mathematical Models	15
1.3.1	1	15
1.3.2	3	15
1.3.3	7	15
1.3.4	9	16
1.3.5	11	16
1.3.6	13	16
1.3.7	15	16
1.3.8	17	16
1.3.9	19	16
1.3.10	21	16
1.3.11	23	17
1.3.12	25	17
1.3.13	27	17
1.3.14	29	17
1.4	Chapter in Review	17
1.4.1	1	17
1.4.2	3	17
1.4.3	5	18
1.4.4	7	18
1.4.5	9	18
1.4.6	11	18
1.4.7	13	18
1.4.8	15	18
1.4.9	17	18
1.4.10	19	18
1.4.11	23	19
1.4.12	25	19
1.4.13	35	19
1.4.14	37	20
1.4.15	41	20
1.4.16	43	20
<b>2</b>	<b>First-Order Differential Equations</b>	<b>20</b>
2.1	Solution Curves Without a Solution	20
2.1.1	21	20
2.1.2	23	20
2.1.3	25	21
2.1.4	27	21
2.1.5	39	21
2.1.6	41	21

2.2	Separable Equations	21
2.2.1	1	21
2.2.2	3	21
2.2.3	5	22
2.2.4	7	22
2.2.5	9	22
2.2.6	11	23
2.2.7	13	23
2.2.8	15	23
2.2.9	17	24
2.2.10	19	24
2.2.11	21	24
2.2.12	23	25
2.2.13	25	25
2.2.14	29	26
2.2.15	31	26
2.2.16	33	27
2.3	Linear Equations	27
2.3.1	1	27
2.3.2	3	27
2.3.3	5	28
2.3.4	7	28
2.3.5	9	28
2.3.6	11	29
2.3.7	13	29
2.3.8	15	30
2.3.9	17	30
2.3.10	19	31
2.3.11	21	31
2.3.12	23	31
2.3.13	25	32
2.3.14	27	32
2.3.15	53	33
2.4	Exact Equations	33
2.4.1	1	33
2.4.2	3	33
2.4.3	5	34
2.4.4	7	34
2.4.5	9	34
2.4.6	11	34
2.4.7	13	34
2.4.8	21	35
2.4.9	23	35
2.4.10	27	35
2.4.11	31	36
2.4.12	33	36

	2.4.13	37	37
	2.4.14	39	37
	2.4.15	45	38
2.5	Solutions by Substitution		39
	2.5.1	1	39
	2.5.2	3	39
	2.5.3	5	40
	2.5.4	7	40
	2.5.5	9	41
	2.5.6	11	41
	2.5.7	13	42
	2.5.8	15	42
	2.5.9	17	42
	2.5.10	21	43
	2.5.11	23	44
	2.5.12	25	44
	2.5.13	35	44
	2.5.14	37	45
2.6	A Numerical Method		46
	2.6.1	1	46
2.7	Linear Models		46
	2.7.1	1	46
	2.7.2	5	47
	2.7.3	9	47
	2.7.4	11	47
	2.7.5	13	47
	2.7.6	21	48
	2.7.7	25	48
	2.7.8	29	49
	2.7.9	33	49
	2.7.10	35	49
	2.7.11	41	50
	2.7.12	43	50
2.8	Nonlinear Models		50
	2.8.1	1	50
	2.8.2	3	50
	2.8.3	11	50
	2.8.4	13	51
	2.8.5	15	51
	2.8.6	17	53
	2.8.7	21	53
2.9	Modeling with Systems of First-Order DEs		54
	2.9.1	1	54
	2.9.2	3	55
	2.9.3	5	55
	2.9.4	7	56

2.9.5	9	56
2.9.6	15	58
2.9.7	17	58
2.10	Chapter 2 in Review	58
2.10.1	1	58
2.10.2	3	58
2.10.3	5	58
2.10.4	9	58
2.10.5	11	59
2.10.6	13	59
2.10.7	15	59
2.10.8	17	60
2.10.9	19	61
2.10.10	21	61
2.10.11	23	62
2.10.12	25	62
2.10.13	29	62
2.10.14	31	63
2.10.15	35	63
2.10.16	37	64
2.10.17	39	64
2.10.18	43	65
2.10.19	45	65
2.10.20	47	66
2.10.21	49	66
<b>3</b>	<b>Higher-Order Differential Equations</b>	<b>67</b>
3.1	Theory of Linear Equations	67
3.1.1	1	67
3.1.2	3	67
3.1.3	9	67
3.1.4	11	68
3.1.5	13	69
3.1.6	15	70
3.1.7	17	70
3.1.8	19	70
3.1.9	21	70
3.1.10	23	70
3.1.11	25	71
3.1.12	27	71
3.1.13	35	72
3.2	Reduction of Order	72
3.2.1	1	72
3.2.2	3	73
3.2.3	5	73
3.2.4	7	74

	3.2.5	9	74
	3.2.6	11	74
	3.2.7	13	74
	3.2.8	15	75
	3.2.9	17	75
	3.2.10	19	75
	3.2.11	21	76
3.3	Homogeneous Linear Equations with Constant Coefficients . . .		76
	3.3.1	1	76
	3.3.2	3	76
	3.3.3	5	76
	3.3.4	7	77
	3.3.5	9	77
	3.3.6	11	77
	3.3.7	13	77
	3.3.8	15	77
	3.3.9	17	78
	3.3.10	19	78
	3.3.11	21	78
	3.3.12	23	78
	3.3.13	25	78
	3.3.14	27	79
	3.3.15	29	79
	3.3.16	31	79
	3.3.17	33	80
	3.3.18	37	80
	3.3.19	39	81
	3.3.20	41	81
	3.3.21	49	81
	3.3.22	51	82
	3.3.23	53	82
	3.3.24	55	82
	3.3.25	57	82
3.4	Undetermined Coefficients . . .		82
	3.4.1	1	82
	3.4.2	3	83
	3.4.3	5	83
	3.4.4	7	84
	3.4.5	9	85
	3.4.6	11	85
	3.4.7	13	86
	3.4.8	15	87
	3.4.9	17	87
	3.4.10	21	88
	3.4.11	27	89
	3.4.12	29	90

	3.4.13	31	91
	3.4.14	37	92
3.5	Variation of Parameters		93
	3.5.1	1	93
	3.5.2	3	94
	3.5.3	7	94
	3.5.4	9	95
	3.5.5	19	96
	3.5.6	21	98
	3.5.7	27	99
	3.5.8	29	100
3.6	Cauchy-Euler Equations		102
	3.6.1	1	102
	3.6.2	3	102
	3.6.3	5	102
	3.6.4	7	102
	3.6.5	9	103
	3.6.6	11	103
	3.6.7	13	103
	3.6.8	15	104
	3.6.9	17	104
	3.6.10	19	104
	3.6.11	21	105
	3.6.12	23	106
	3.6.13	25	107
	3.6.14	27	108
	3.6.15	29	108
	3.6.16	31	109
	3.6.17	33	110
	3.6.18	35	110
	3.6.19	37	110
	3.6.20	39	111
	3.6.21	41	111
	3.6.22	43	111
	3.6.23	51	112
3.7	Nonlinear Equations		112
	3.7.1	3	112
	3.7.2	5	113
	3.7.3	7	113
	3.7.4	9	114
	3.7.5	11	115
	3.7.6	13	116
	3.7.7	15	116
	3.7.8	19	117
3.8	Linear Models: Initial-Value Problems		117
	3.8.1	1	117

3.8.2	3	117
3.8.3	9	118
3.8.4	13	119
3.8.5	15	120
3.8.6	17	120
3.8.7	21	121
3.8.8	23	121
3.8.9	25	121
3.8.10	29	122
3.8.11	31	124
3.8.12	33	125
3.8.13	35	127
3.8.14	37	128
3.8.15	39	129
3.8.16	41	130
3.8.17	49	131
3.9	Linear Models: Boundary-Value Problems	132
3.9.1	1	132
3.9.2	3	133
3.9.3	11	134
3.9.4	13	135
3.9.5	15	135
3.9.6	17	136
3.9.7	19	136
3.9.8	23	137
3.9.9	27	137
3.9.10	29	137

# 1 Introduction to Differential Equations

## 1.1 Definitions and Terminology

### 1.1.1 1

2, linear

### 1.1.2 3

4, linear

### 1.1.3 5

2, nonlinear

### 1.1.4 7

3, linear



**1.1.5 9**

no; yes

**1.1.6 15**

The domain of the function is  $x \in [-2, \infty)$ .

$$y' = 1 + \frac{2}{\sqrt{x+2}}$$

The largest interval of definition of the solution is  $x \in (-2, \infty)$ .

$$\begin{aligned}(y-x)y' &= y-x+8 \\ (x+4\sqrt{x+2}-x)(1+\frac{2}{\sqrt{x+2}}) &= x+4\sqrt{x+2}-x+8 \\ 4\sqrt{x+2}+8 &= 4\sqrt{x+2}+8\end{aligned}$$

**1.1.7 17**

The domain of the function is  $x \in \mathbb{R}, x \neq \pm 2$ .

$$y' = \frac{2x}{(4-x^2)^2}$$

The largest intervals of definition of the solution are  $(-\infty, -2)$ ,  $(-2, 2)$ , and  $(2, \infty)$ .

$$\begin{aligned}y' &= 2xy^2 \\ \frac{2x}{(4-x^2)^2} &= 2x \left( \frac{1}{4-x^2} \right)^2 \\ &= \frac{2x}{(4-x^2)^2}\end{aligned}$$

**1.1.8 19**

$$\begin{aligned}\ln \frac{2X-1}{X-1} &= t \\ 2X-1 &= (X-1)e^t \\ (2-e^t)X &= 1-e^t \\ X &= \frac{e^t-1}{e^t-2}\end{aligned}$$

The solutions intervals of validity are  $(\infty, \ln 2)$  and  $(\ln 2, \infty)$ .

$$\begin{aligned}
\frac{dX}{dt} &= (X-1)(1-2X) \\
\frac{e^t}{e^t-2} - \frac{e^t(e^t-1)}{(e^t-2)^2} &= \left(\frac{e^t-1}{e^t-2} - 1\right) \left(1 - 2\frac{e^t-1}{e^t-2}\right) \\
\frac{e^t(e^t-2) - e^t(e^t-1)}{(e^t-2)^2} &= \left(\frac{e^t-1-e^t+2}{e^t-2}\right) \left(\frac{e^t-2-2e^t+2}{e^t-2}\right) \\
\frac{e^{2t}-2e^t-e^{2t}+e^t}{(e^t-2)^2} &= \left(\frac{1}{e^t-2}\right) \left(\frac{-e^t}{e^t-2}\right) \\
\frac{-e^t}{(e^t-2)^2} &= \frac{-e^t}{(e^t-2)^2}
\end{aligned}$$

### 1.1.9 31

$$m = -2$$

### 1.1.10 33

$$m = 2 \text{ or } 3$$

### 1.1.11 35

$$m = -1 \text{ or } 0$$

### 1.1.12 37

$$y = 2$$

### 1.1.13 39

No constant solutions

## 1.2 Initial Value Problems

### 1.2.1 1

$$\begin{aligned}
y(0) &= -\frac{1}{3} = \frac{1}{1+c_1e^{-(0)}} \\
-3 &= 1+c_1 \\
c_1 &= -4
\end{aligned}$$

$$y = \frac{1}{1-4e^{-x}}$$

**1.2.2 3**

$$y(2) = \frac{1}{3} = \frac{1}{(2)^2 + c}$$

$$3 = 4 + c$$

$$c = -1$$

$$y = \frac{1}{x^2 - 1}$$

$$I = (1, \infty)$$

**1.2.3 5**

$$y(0) = 1 = \frac{1}{(0)^2 + c}$$

$$c = 1$$

$$y = \frac{1}{x^2 + 1}$$

$$I = (-\infty, \infty)$$

**1.2.4 7**

$$x(0) = -1 = c_1 \cos 0 + c_2 \sin 0$$

$$c_1 = -1$$

$$x'(0) = 8 = -c_1 \sin 0 + c_2 \cos 0$$

$$c_2 = 8$$

$$x = -\cos t + 8 \sin t$$

**1.2.5 9**

$$x'\left(\frac{\pi}{6}\right) = 0 = -c_1 \sin \frac{\pi}{6} + c_2 \cos \frac{\pi}{6}$$

$$= -c_1 \frac{1}{2} + c_2 \frac{\sqrt{3}}{2}$$

$$c_1 = \sqrt{3}c_2$$

$$\begin{aligned}
x\left(\frac{\pi}{6}\right) &= \frac{1}{2} = c_1 \cos \frac{\pi}{6} + c_2 \sin \frac{\pi}{6} \\
&= \frac{3}{2}c_2 + \frac{1}{2}c_2 \\
&= 2c_2 \\
c_2 &= \frac{1}{4}
\end{aligned}$$

$$y = \frac{\sqrt{3}}{4} \cos t + \frac{1}{4} \sin t$$

### 1.2.6 11

$$\begin{aligned}
y(0) &= 1 = c_1 e^{(0)} + c_2 e^{-(0)} \\
&= c_1 + c_2 \\
c_1 &= 1 - c_2
\end{aligned}$$

$$\begin{aligned}
y'(0) &= 2 = c_1 e^{(0)} - c_2 e^{-(0)} \\
&= 1 - c_2 - c_2 \\
c_2 &= -\frac{1}{2}
\end{aligned}$$

$$y = \frac{3}{2}e^x - \frac{1}{2}e^{-x}$$

### 1.2.7 13

$$\begin{aligned}
y(-1) &= 5 = c_1 e^{(-1)} + c_2 e^{-(-1)} \\
&= c_1 e^{-1} + c_2 e \\
c_1 &= 5e - c_2 e^2
\end{aligned}$$

$$\begin{aligned}
y'(-1) &= -5 = c_1 e^{(-1)} - c_2 e^{-(-1)} \\
&= 5e - c_2 e^2 - c_2 e \\
c_2 e(e+1) &= 5(e+1) \\
c_2 &= \frac{5}{e}
\end{aligned}$$

$$y = 5e^{-x-1}$$

**1.2.8 15**

$$y = 0$$

$$y = x^3$$

**1.2.9 17**

$$f(x, y) = y^{2/3}$$

$$\frac{\partial f}{\partial y} = \frac{2}{3y^{1/3}}$$

$$y < 0 \text{ or } y > 0$$

**1.2.10 19**

$$f(x, y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$$x < 0 \text{ or } x > 0$$

**1.2.11 21**

$$f(x, y) = \frac{x^2}{4 - y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2x^2y}{(4 - y^2)^2}$$

$$y < -2, -2 < y < 2, \text{ or } y > 2$$

**1.2.12 23**

$$f(x, y) = \frac{y^2}{x^2 + y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2} - \frac{2y^3}{(x^2 + y^2)^2}$$

$$x \neq 0 \text{ and } y \neq 0$$

**1.2.13 25**

$$f(x, y) = \sqrt{y^2 - 9}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{y^2 - 9}}$$

Yes

**1.2.14 27**

No

**1.2.15 29**

(a)  $y = cx$

(b)

$$f(x, y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$$x \neq 0$$

(c) No, the function is not differentiable at  $x = 0$ **1.2.16 31**

(a)

$$y' = \frac{1}{(x+c)^2} = y^2$$

(b)

$$y(0) = 1 = -\frac{1}{(0)+c} \Rightarrow c = -1 \Rightarrow y = \frac{1}{1-x}$$

$$I = (-\infty, 1)$$

$$y(0) = -1 = -\frac{1}{(0)+c} \Rightarrow c = 1 \Rightarrow y = -\frac{1}{x+1}$$

$$I = (-1, \infty)$$

**1.2.17 39**

$$y(0) = 0 = c_1 \cos 3(0) + c_2 \sin 3(0)$$

$$c_1 = 0$$

$$y\left(\frac{\pi}{6}\right) = -1 = c_2 \sin 3\left(\frac{\pi}{6}\right)$$

$$c_2 = -1$$

$$y = -\sin 3x$$

**1.2.18 41**

$$y'(0) = 0 = -3c_1 \sin 3(0) + 3c_2 \cos 3(0)$$
$$c_2 = 0$$

$$y'\left(\frac{\pi}{4}\right) = 0 = -3c_1 \sin 3\left(\frac{\pi}{4}\right)$$
$$= -\frac{3}{\sqrt{2}}c_1$$
$$c_1 = 0$$

$$y = 0$$

**1.2.19 43**

$$y(0) = 0 = c_1 \cos 3(0) + c_2 \sin 3(0)$$
$$c_1 = 0$$

$$y(\pi) = 4 = c_2 \sin 3(\pi)$$
$$4 = 0$$

No solution

**1.3 Differential Equations as Mathematical Models**

**1.3.1 1**

$$\frac{dP}{dt} = kP + r$$

$$\frac{dP}{dt} = kP - r$$

**1.3.2 3**

$$\frac{dP}{dt} = k_b P - k_d P^2$$

**1.3.3 7**

$$\frac{dx}{dt} = kx(1000 - x)$$

**1.3.4 9**

$$\frac{dA}{dt} = -\frac{A}{100}$$

$$A(0) = 50 \text{ lb}$$

**1.3.5 11**

$$\frac{dA}{dt} + \frac{7}{600-t}A = 6$$

**1.3.6 13**

$$\begin{aligned}\frac{dV}{dt} &= -cA_h\sqrt{2gh} \\ A_w\frac{dh}{dt} &= -cA_h\sqrt{2gh} \\ \frac{dh}{dt} &= -\frac{cA_h\sqrt{2g}}{A_w}\sqrt{h} \\ &= -\frac{c\pi r_h^2\sqrt{2g}}{A_w}\sqrt{h} \\ &= -\frac{c\pi}{430}\sqrt{h}\end{aligned}$$

**1.3.7 15**

$$L\frac{di}{dt} + Ri = E$$

**1.3.8 17**

$$m\frac{dv}{dt} = mg - kv^2$$

**1.3.9 19**

$$m\frac{d^2x}{dt^2} = -kx$$

**1.3.10 21**

$$\begin{aligned}\frac{d}{dt}(mv) &= R - kv \\ \frac{dm}{dt}v + m\frac{dv}{dt} &= R - kv - mg\end{aligned}$$



**1.3.11 23**

$$g = \frac{k}{R^2} \Rightarrow k = gR^2$$

$$\frac{d^2r}{dt^2} = -\frac{gR^2}{r^2}$$

**1.3.12 25**

$$\frac{dA}{dt} = k(M - A)$$

**1.3.13 27**

$$\frac{dx}{dt} = r - kx$$

**1.3.14 29**

$$\begin{aligned}\frac{dy}{dx} &= \tan \theta \\ &= \tan \frac{\phi}{2} \\ &= \frac{1 - \cos \phi}{\sin \phi} \\ &= \frac{1 - x/r}{y/r} \\ &= \frac{r - x}{y} \\ &= \frac{\sqrt{x^2 + y^2} - x}{y}\end{aligned}$$

**1.4 Chapter in Review**

**1.4.1 1**

$$\frac{dy}{dx} = ky$$

**1.4.2 3**

$$y'' + k^2y = 0$$

**1.4.3 5**

$$y = c_1 e^x + c_2 x e^x$$

$$\begin{aligned} y' &= c_1 e^x + c_2 e^x + c_2 x e^x \\ &= y + c_2 e^x \end{aligned}$$

$$\begin{aligned} y'' &= c_1 e^x + c_2 e^x + c_2 e^x + c_2 x e^x \\ &= c_1 e^x + 2c_2 e^x + c_2 x e^x \\ &= y' + c_2 e^x \end{aligned}$$

$$y'' - 2y' + y = 0$$

**1.4.4 7**

a, d

**1.4.5 9**

b

**1.4.6 11**

b

**1.4.7 13**

$$y = c e^x$$

**1.4.8 15**

$$\frac{dy}{dx} = x^2 + y^2$$

**1.4.9 17**

(a)  $(-\infty, \infty)$

(b)  $(-\infty, 0)$  or  $(0, \infty)$

**1.4.10 19**

$x_0 = -1$  and  $I = (-\infty, 0)$  or  $x_0 = 2$  and  $I = (0, \infty)$

**1.4.11 23**

$$y = x \sin x + x \cos x$$

$$y' = \sin x + x \cos x + \cos x - x \sin x$$

$$\begin{aligned} y'' &= \cos x + \cos x - x \sin x - \sin x - \sin x - x \cos x \\ &= 2 \cos x - 2 \sin x - x \sin x - x \cos x \end{aligned}$$

$$\begin{aligned} y'' + y &= 2 \cos x - 2 \sin x - x \sin x - x \cos x + x \sin x + x \cos x \\ &= 2 \cos x - 2 \sin x \end{aligned}$$

$$I = (-\infty, \infty)$$

**1.4.12 25**

$$y = \sin(\ln x)$$

$$y' = \frac{1}{x} \cos(\ln x)$$

$$y'' = -\frac{1}{x^2} \cos(\ln x) - \frac{1}{x^2} \sin(\ln x)$$

$$\begin{aligned} x^2 y'' + x y' + y &= -\cos(\ln x) - \sin(\ln x) + \cos(\ln x) + \sin(\ln x) \\ &= 0 \end{aligned}$$

$$I = (0, \infty)$$

**1.4.13 35**

$$\begin{aligned} y(0) = 0 &= c_1 e^{-3(0)} + c_2 e^{(0)} + 4(0) \\ &= c_1 + c_2 \\ c_1 &= -c_2 \end{aligned}$$

$$\begin{aligned} y'(0) = 0 &= -3c_1 e^{-3(0)} + c_2 e^{(0)} + 4 \\ &= -3c_1 + c_2 + 4 \\ c_2 &= 3c_1 - 4 \end{aligned}$$

$$c_1 = -(3c_1 - 4) \Rightarrow c_1 = 1 \Rightarrow c_2 = -1$$

$$y = e^{-3x} - e^x + 4x$$

**1.4.14 37**

$$\begin{aligned}
y(1) &= -2 = c_1 e^{-3(1)} + c_2 e^{(1)} + 4(1) \\
&= c_1 e^{-3} + c_2 e + 4 \\
c_1 &= -e^3(c_2 e + 6)
\end{aligned}$$

$$\begin{aligned}
y'(1) &= 4 = -3c_1 e^{-3(1)} + c_2 e^{(1)} + 4 \\
&= -3c_1 e^{-3} + c_2 e + 4 \\
c_2 e &= 3c_1 e^{-3}
\end{aligned}$$

$$c_1 = -e^3(3c_1 e^{-3} + 6) = -3c_1 - 6e^3 \Rightarrow c_1 = -\frac{3}{2}e^3 \Rightarrow c_2 = -\frac{9}{2}e^{-1}$$

$$y = -\frac{3}{2}e^{3(1-x)} - \frac{9}{2}e^{x-1} + 4x$$

**1.4.15 41**

$$y_0 = -3, y_1 = 0$$

**1.4.16 43**

$$\begin{aligned}
\frac{d}{dt}(mv) &= F - mg \\
\frac{d}{dt}\left(\lambda x \frac{dx}{dt}\right) &= F - \lambda xg \\
x \frac{d^2 x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + gx &= \frac{F}{\lambda} \\
x \frac{d^2 x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 32x &= 5
\end{aligned}$$

**2 First-Order Differential Equations****2.1 Solution Curves Without a Solution****2.1.1 21**

0 is stable, 3 is unstable

**2.1.2 23**

2 is semi-stable

**2.1.3 25**

−2 is unstable, 0 is semi-stable, 2 is stable

**2.1.4 27**

−1 is stable, 0 is unstable

**2.1.5 39**

$$P_0 < h/k$$

**2.1.6 41**

$$g - \frac{k}{m}v^2 = 0 \Rightarrow v = \sqrt{\frac{gm}{k}}$$

**2.2 Separable Equations****2.2.1 1**

$$\begin{aligned}\frac{dy}{dx} &= \sin 5x \\ y &= -\frac{1}{5} \cos 5x + c\end{aligned}$$

**2.2.2 3**

$$\begin{aligned}dx + e^{3x} dy &= 0 \\ e^{-3x} dx + dy &= 0 \\ -\frac{1}{3}e^{-3x} + y &= c \\ y &= \frac{1}{3}e^{-3x} + c\end{aligned}$$

### 2.2.3 5

$$\begin{aligned}
 x \frac{dy}{dx} &= 4y \\
 \frac{1}{4y} dy &= \frac{1}{x} dx \\
 \frac{1}{4} \ln |4y| &= \ln |x| + c \\
 \ln |4y| &= 4 \ln |x| + c \\
 4y &= e^{4 \ln |x| + c} \\
 &= c \left( e^{\ln |x|} \right)^4 \\
 y &= cx^4
 \end{aligned}$$

### 2.2.4 7

$$\begin{aligned}
 \frac{dy}{dx} &= e^{3x+2y} \\
 &= e^{3x} e^{2y} \\
 e^{-2y} dy &= e^{3x} dx \\
 -\frac{1}{2} e^{-2y} &= \frac{1}{3} e^{3x} + c \\
 -3e^{-2y} &= 2e^{3x} + c
 \end{aligned}$$

### 2.2.5 9

$$\begin{aligned}
 y \ln x \frac{dx}{dy} &= \left( \frac{y+1}{x} \right)^2 \\
 x^2 \ln x dx &= \frac{(y+1)^2}{y} dy \\
 x^3 \left( \frac{\ln x}{3} - \frac{1}{9} \right) &= \frac{1}{2} y(y+4) + \ln y + c \\
 \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 &= \frac{1}{2} y^2 + 2y + \ln y + c
 \end{aligned}$$

**2.2.6 11**

$$\begin{aligned}
 \csc y \, dx + \sec^2 x \, dy &= 0 \\
 \frac{1}{\sin y} \, dx + \frac{1}{\cos^2 x} \, dy &= 0 \\
 \cos^2 x \, dx + \sin y \, dy &= 0 \\
 \frac{1}{2}(1 + \cos 2x) \, dx + \sin y \, dy &= 0 \\
 \frac{1}{2} \left( x + \frac{1}{2} \sin 2x \right) - \cos y + c &= 0 \\
 4 \cos y &= 2x + \sin 2x + c
 \end{aligned}$$

**2.2.7 13**

$$\begin{aligned}
 (e^y + 1)^2 e^{-y} \, dx + (e^x + 1)^3 e^{-x} \, dy &= 0 \\
 \frac{e^x}{(e^x + 1)^3} \, dx + \frac{e^y}{(e^y + 1)^2} \, dy &= 0 \\
 -\frac{1}{2(e^x + 1)^2} - \frac{1}{e^y + 1} &= c \\
 (e^x + 1)^{-2} + 2(e^y + 1)^{-1} &= c
 \end{aligned}$$

**2.2.8 15**

$$\begin{aligned}
 \frac{dS}{dr} &= kS \\
 \frac{1}{S} \, dS &= k \, dr \\
 \ln |S| &= kr + c \\
 S &= ce^{kr}
 \end{aligned}$$

2.2.9 17

$$\begin{aligned}\frac{dP}{dt} &= P - P^2 \\ \frac{1}{P(1-P)} dP &= dt \\ \ln \frac{P}{1-P} &= t + c \\ \frac{P}{1-P} &= ce^t \\ P &= ce^t(1-P) \\ P &= \frac{ce^t}{1+ce^t}\end{aligned}$$

2.2.10 19

$$\begin{aligned}\frac{dy}{dx} &= \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8} \\ &= \frac{(x-1)(y+3)}{(x+4)(y-2)} \\ \frac{y-2}{y+3} dt &= \frac{x-1}{x+4} dx \\ y - 5 \ln |y+3| &= x - 5 \ln |x+4| + c \\ e^{y-5 \ln |y+3|} &= e^{x-5 \ln |x+4|+c} \\ \frac{e^y}{(y+3)^5} &= \frac{ce^x}{(x+4)^5} \\ c(x+4)^5 e^y &= (y+3)^5 e^x\end{aligned}$$

2.2.11 21

$$\begin{aligned}\frac{dy}{dx} &= x\sqrt{1-y^2} \\ (1-y^2)^{-1/2} dy &= x dx \\ \arcsin y &= \frac{1}{2}x^2 + c \\ y &= \sin \left( \frac{1}{2}x^2 + c \right)\end{aligned}$$



**2.2.12 23**

$$\begin{aligned}\frac{dx}{dt} &= 4(x^2 + 1) \\ \frac{1}{x^2 + 1} dx &= 4 dt \\ \arctan x &= 4t + c \\ x &= \tan(4t + c)\end{aligned}$$

$$\begin{aligned}x\left(\frac{\pi}{4}\right) &= 1 = \tan\left(4\left(\frac{\pi}{4}\right) + c\right) \\ &= \tan(\pi + c) \\ c &= \arctan(1) - \pi \\ &= -\frac{3}{4}\pi\end{aligned}$$

$$x = \tan\left(4t - \frac{3}{4}\pi\right)$$

**2.2.13 25**

$$\begin{aligned}x^2 \frac{dy}{dx} &= y - xy \\ &= y(1 - x) \\ \frac{1}{y} dy &= \left(\frac{1}{x^2} - \frac{1}{x}\right) dx \\ \ln |y| &= -\frac{1}{x} - \ln |x| + c \\ y &= e^{-\frac{1}{x} - \ln |x| + c} \\ &= \frac{c}{xe^{1/x}}\end{aligned}$$

$$\begin{aligned}y(-1) &= -1 = \frac{c}{(-1)e^{1/(-1)}} \\ &= -ce \\ c &= e^{-1}\end{aligned}$$

$$y = \frac{1}{xe^{1+1/x}}$$

**2.2.14 29**

$$\begin{aligned}\frac{dy}{dx} &= ye^{-x^2} \\ \frac{1}{y} \frac{dy}{dx} &= e^{-x^2} \\ \int_4^x \frac{1}{y} \frac{dy}{dx'} dx' &= \int_4^x e^{-x'^2} dx' \\ \ln |y|_4^x &= \int_4^x e^{-x'^2} dx' \\ \ln |y(x)| - \ln |y(4)| &= \int_4^x e^{-x'^2} dx' \\ \ln |y(x)| &= \ln |y(4)| + \int_4^x e^{-x'^2} dx' \\ y(x) &= e^{\int_4^x e^{-x'^2} dx'}\end{aligned}$$

**2.2.15 31**

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x+1}{2y} \\ 2y dy &= (2x+1) dx \\ y^2 &= x^2 + x + c \\ y &= \pm \sqrt{x^2 + x + c} \\ y(-2) &= -1 = -\sqrt{(-2)^2 + (-2) + c} \\ &= -\sqrt{2+c} \\ c &= -1 \\ y &= -\sqrt{x^2 + x - 1} \\ I &= \left( -\infty, -\frac{1-\sqrt{5}}{2} \right)\end{aligned}$$

### 2.2.16 33

$$e^y dx - e^{-x} dy = 0$$

$$e^x dx - e^{-y} dy = 0$$

$$e^x + e^{-y} = c$$

$$\ln |e^{-y}| = \ln |c - e^x|$$

$$y = -\ln |c - e^x|$$

$$y(0) = 0 = -\ln |c - e^{(0)}|$$

$$1 = c - 1$$

$$c = 2$$

$$y = -\ln |2 - e^x|$$

$$I = (-\infty, \ln 2)$$

## 2.3 Linear Equations

### 2.3.1 1

$$\frac{dy}{dx} = 5y$$

$$\ln |y| = 5x + c$$

$$y = ce^{5x}$$

$$I = (-\infty, \infty)$$

### 2.3.2 3

$$\frac{dy}{dx} + y = e^{3x}$$

$$e^x \frac{dy}{dx} + e^x y = e^{4x}$$

$$\frac{d}{dx}(e^x y) = e^{4x}$$

$$e^x y = \frac{1}{4} e^{4x} + c$$

$$y = \frac{1}{4} e^{3x} + ce^{-x}$$

$$I = (-\infty, \infty)$$

2.3.3 5

$$\begin{aligned}
 y' + 3x^2y &= x^2 \\
 e^{x^3}y' + 3x^2e^{x^3}y &= e^{x^3}x^2 \\
 e^{x^3}y &= \frac{1}{3}e^{x^3} + c \\
 y &= \frac{1}{3} + ce^{-x^3}
 \end{aligned}$$

$$I = (-\infty, \infty)$$

2.3.4 7

$$\begin{aligned}
 x^2y' + xy &= 1 \\
 y' + x^{-1}y &= x^{-2} \\
 e^{\ln x}y' + x^{-1}e^{\ln x}y &= e^{\ln x}x^{-2} \\
 \frac{d}{dx}(e^{\ln x}y) &= x^{-1} \\
 \frac{d}{dx}(xy) &= x^{-1} \\
 xy &= \ln x + c \\
 y &= \frac{\ln x + c}{x}
 \end{aligned}$$

$$I = (0, \infty)$$

2.3.5 9

$$\begin{aligned}
 x \frac{dy}{dx} - y &= x^2 \sin x \\
 \frac{dy}{dx} - x^{-1}y &= x \sin x \\
 e^{-\ln x} \frac{dy}{dx} - x^{-1}e^{-\ln x}y &= e^{-\ln x}x \sin x \\
 \frac{d}{dx}(e^{-\ln x}y) &= \sin x \\
 x^{-1}y &= -\cos x + c \\
 y &= cx - x \cos x
 \end{aligned}$$

$$I = (0, \infty)$$

**2.3.6 11**

$$\begin{aligned}
 x \frac{dy}{dx} + 4y &= x^3 - x \\
 \frac{dy}{dx} + 4x^{-1}y &= x^2 - 1 \\
 e^{4 \ln x} \frac{dy}{dx} + 4x^{-1} e^{4 \ln x} y &= e^{4 \ln x} (x^2 - 1) \\
 \frac{d}{dx} (e^{4 \ln x} y) &= x^6 - x^4 \\
 x^4 y &= \frac{1}{7} x^7 - \frac{1}{5} x^5 + c \\
 y &= \frac{1}{7} x^3 - \frac{1}{5} x^2 + c x^{-4}
 \end{aligned}$$

$$I = (0, \infty)$$

**2.3.7 13**

$$\begin{aligned}
 x^2 y' + x(x+2)y &= e^x \\
 y' + x^{-1}(x+2)y &= x^{-2} e^x \\
 e^{x+2 \ln x} y' + x^{-1}(x+2) e^{x+2 \ln x} y &= e^{x+2 \ln x} x^{-2} e^x \\
 \frac{d}{dx} (e^x x^2 y) &= e^{2x} \\
 e^x x^2 y &= \frac{1}{2} e^{2x} + c \\
 y &= \frac{e^x}{2x^2} + \frac{c}{e^x x^2}
 \end{aligned}$$

$$I = (0, \infty)$$

2.3.8 15

$$\begin{aligned}
 y \, dx - 4(x + y^6) \, dy &= 0 \\
 y \frac{dx}{dy} - 4x - 4y^6 &= 0 \\
 \frac{dx}{dy} - \frac{4}{y}x &= 4y^5 \\
 e^{-4 \ln y} \frac{dx}{dy} - \frac{4}{y} e^{-4 \ln y} x &= 4e^{-4 \ln y} y^5 \\
 \frac{d}{dy}(e^{-4 \ln y} x) &= 4y \\
 y^{-4} x &= 2y^2 + c \\
 x &= 2y^6 + cy^4
 \end{aligned}$$

$$I = (0, \infty)$$

2.3.9 17

$$\begin{aligned}
 \cos x \frac{dy}{dx} + (\sin x)y &= 1 \\
 \frac{dy}{dx} + (\tan x)y &= \sec x \\
 e^{\ln(\sec x)} \frac{dy}{dx} + (\tan x)e^{\ln(\sec x)}y &= e^{\ln(\sec x)} \sec x \\
 \frac{d}{dx}(e^{\ln(\sec x)}y) &= \sec^2 x \\
 y \sec x &= \tan x + c \\
 y &= \sin x + c \cos x
 \end{aligned}$$

$$I = (-\pi/2, \pi/2)$$

**2.3.10 19**

$$\begin{aligned}
 (x+1)\frac{dy}{dx} + (x+2)y &= 2xe^{-x} \\
 \frac{dy}{dx} + \frac{x+2}{x+1}y &= \frac{2xe^{-x}}{x+1} \\
 e^{x+\ln|x+1|}\frac{dy}{dx} + \frac{x+2}{x+1}e^{x+\ln|x+1|}y &= e^{x+\ln|x+1|}\frac{2xe^{-x}}{x+1} \\
 \frac{d}{dx}(e^{x+\ln|x+1|}y) &= 2x \\
 e^x(x+1)y &= x^2 + c \\
 y &= \frac{x^2 + c}{e^x(x+1)}
 \end{aligned}$$

$$I = (-1, \infty)$$

**2.3.11 21**

$$\begin{aligned}
 \frac{dr}{d\theta} + r \sec \theta &= \cos \theta \\
 e^{\ln|\sec \theta + \tan \theta|}\frac{dr}{d\theta} + e^{\ln|\sec \theta + \tan \theta|}r \sec \theta &= e^{\ln|\sec \theta + \tan \theta|}\cos \theta \\
 \frac{d}{d\theta}(e^{\ln|\sec \theta + \tan \theta|}r) &= 1 + \sin \theta \\
 (\sec \theta + \tan \theta)r &= \theta - \cos \theta + c \\
 r &= \frac{\theta - \cos \theta + c}{\sec \theta + \tan \theta}
 \end{aligned}$$

$$I = (-\pi/2, \pi/2)$$

**2.3.12 23**

$$\begin{aligned}
 x\frac{dy}{dx} + (3x+1)y &= e^{-3x} \\
 \frac{dy}{dx} + (3+x^{-1})y &= e^{-3x}x^{-1} \\
 e^{3x+\ln|x|}\frac{dy}{dx} + (3+x^{-1})e^{3x+\ln|x|}y &= 1 \\
 \frac{d}{dx}(e^{3x+\ln|x|}y) &= 1 \\
 e^{3x}xy &= x + c \\
 y &= \frac{x+c}{e^{3x}x}
 \end{aligned}$$

$$I = (0, \infty)$$

**2.3.13 25**

$$\begin{aligned}
 xy' + y &= e^x \\
 y' + x^{-1}y &= e^x x^{-1} \\
 e^{\ln|x|}y' + x^{-1}e^{\ln|x|}y &= e^x \\
 \frac{d}{dx}(e^{\ln|x|}y) &= e^x \\
 xy &= e^x + c \\
 y &= \frac{e^x + c}{x}
 \end{aligned}$$

$$\begin{aligned}
 y(1) &= 2 = \frac{e^{(1)} + c}{(1)} \\
 c &= 2 - e
 \end{aligned}$$

$$y = \frac{e^x + 2 - e}{x}$$

$$I = (0, \infty)$$

**2.3.14 27**

$$\begin{aligned}
 L \frac{di}{dt} + Ri &= E \\
 \frac{di}{dt} + \frac{R}{L}i &= \frac{E}{L} \\
 e^{Rt/L} \frac{di}{dt} + \frac{R}{L}e^{Rt/L}i &= \frac{E}{L}e^{Rt/L} \\
 \frac{d}{dt}(e^{Rt/L}i) &= \frac{E}{L}e^{Rt/L} \\
 e^{Rt/L}i &= \frac{E}{R}e^{Rt/L} + c \\
 i &= \frac{E}{R} + ce^{-Rt/L}
 \end{aligned}$$

$$\begin{aligned}
 i(0) = i_0 &= \frac{E}{R} + ce^{-R(0)/L} \\
 &= \frac{E}{R} + c \\
 c &= i_0 - \frac{E}{R}
 \end{aligned}$$



$$i = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right) e^{-Rt/L}$$

$$I = (-\infty, \infty)$$

**2.3.15 53**

$$\begin{aligned}\frac{dE}{dt} &= -\frac{1}{RC}E \\ \frac{1}{E} \frac{dE}{dt} &= -\frac{1}{RC} \\ \ln|E| &= -\frac{1}{RC}t + c \\ E &= ce^{-t/RC}\end{aligned}$$

$$\begin{aligned}E(4) &= E_0 = ce^{-(4)/RC} \\ c &= E_0 e^{4/RC}\end{aligned}$$

$$E = E_0 e^{(4-t)/RC}$$

## **2.4 Exact Equations**

**2.4.1 1**

$$f(x, y) = x^2 - x + g(y)$$

$$\frac{\partial f}{\partial y} = g'(y) = 3y + 7$$

$$g(y) = \frac{3}{2}y^2 + 7y$$

$$x^2 - x + \frac{3}{2}y^2 + 7y = c$$

**2.4.2 3**

$$f(x, y) = \frac{5}{2}x^2 + 4xy + g(y)$$

$$4x + g'(y) = 4x - 8y^3 \Rightarrow g'(y) = -8y^3$$

$$g(y) = -2y^4$$

$$\frac{5}{2}x^2 + 4xy - 2y^4 = c$$

**2.4.3 5**

$$f(x, y) = x^2y^2 - 3x + g(y)$$

$$2x^2y + g'(y) = 2x^2y + 4 \Rightarrow g'(y) = 4$$

$$g(y) = 4y$$

$$x^2y^2 - 3x + 4y = c$$

**2.4.4 7**

Not exact

**2.4.5 9**

$$f(x, y) = \frac{1}{2}x^2 - xy^3 - y^2 \cos x + g(y)$$

$$-3xy^2 - 2y \cos x + g'(y) = -3xy^2 - 2y \cos x \Rightarrow g'(y) = 0$$

$$\frac{1}{2}x^2 - xy^3 - y^2 \cos x = c$$

**2.4.6 11**

Not exact

**2.4.7 13**

$$f(x, y) = xy + g(x)$$

$$y + g'(x) = -2xe^x + y - 6x^2 \Rightarrow g'(x) = -2xe^x - 6x^2$$

$$g(x) = -2e^x(x - 1) - 2x^3$$

$$xy - 2e^x(x - 1) - 2x^3 = c$$

**2.4.8 21**

$$f(x, y) = \frac{1}{3}(x + y)^3 + g(y)$$

$$(x + y)^2 + g'(y) = 2xy + x^2 - 1 \Rightarrow g'(y) = -y^2 - 1$$

$$g(y) = -\frac{1}{3}y^3 - y$$

$$\frac{1}{3}(x + y)^3 - \frac{1}{3}y^3 - y = c$$

$$\frac{1}{3}(1 + 1)^3 - \frac{1}{3}1^3 - 1 = c \Rightarrow c = \frac{4}{3}$$

$$x^3 + 3x^2y + 3xy^2 - 3y = 4$$

**2.4.9 23**

$$f(x, y) = 4ty + t^2 - 5t + g(y)$$

$$4t + g'(y) = 6y + 4t - 1 \Rightarrow g'(y) = 6y - 1$$

$$g(y) = 3y^2 - y$$

$$4ty + t^2 - 5t + 3y^2 - y = c$$

$$4(-1)(2) + (-1)^2 - 5(-1) + 3(2)^2 - (2) = c \Rightarrow c = 8$$

$$4ty + t^2 - 5t + 3y^2 - y = 8$$

**2.4.10 27**

$$3y^2 + 4kxy^3 = 3y^2 + 40xy^3 \Rightarrow k = 10$$

**2.4.11 31**

$$M_y = 4y$$

$$N_x = 2y$$

$$\frac{M_y - N_x}{N} = \frac{4y - 2y}{2xy} = \frac{1}{x}$$

$$\mu(x) = e^{\ln x} = x$$

$$(2xy^2 + 3x^2) dx + 2x^2y dy = 0$$

$$f(x, y) = x^2y^2 + x^3 + g(y)$$

$$2x^2y + g'(y) = 2x^2y \Rightarrow g'(y) = 0$$

$$x^2y^2 + x^3 = c$$

**2.4.12 33**

$$M_y = 6x$$

$$N_x = 18x$$

$$\frac{M_y - N_x}{N} = \frac{6x - 18x}{4y + 9x^2}$$

$$\frac{N_x - M_y}{M} = \frac{18x - 6x}{6xy} = \frac{2}{y}$$

$$\mu(y) = e^{2 \ln y} = y^2$$

$$6xy^3 dx + (4y^3 + 9x^2y^2) dy = 0$$

$$f(x, y) = 3x^2y^3 + g(y)$$

$$9x^2y^2 + g'(y) = 4y^3 + 9x^2y^2 \Rightarrow g'(y) = 4y^3$$

$$g(y) = y^4$$

$$3x^2y^3 + y^4 = c$$

**2.4.13 37**

$$M_y = 0$$

$$N_x = 2xy$$

$$\frac{N_x - M_y}{M} = \frac{2xy - 0}{x} = 2y$$

$$\mu(y) = e^{y^2}$$

$$e^{y^2} x \, dx + e^{y^2} (x^2 y + 4y) \, dy = 0$$

$$f(x, y) = \frac{1}{2} e^{y^2} x^2 + g(y)$$

$$y e^{y^2} x^2 + g'(y) = e^{y^2} (x^2 y + 4y) \Rightarrow g'(y) = 4e^{y^2} y$$

$$g(y) = 2e^{y^2}$$

$$\frac{1}{2} e^{y^2} x^2 + 2e^{y^2} = c$$

$$\frac{1}{2} e^{(0)^2} (4)^2 + 2e^{(0)^2} = c \Rightarrow c = 10$$

$$\frac{1}{2} e^{y^2} x^2 + 2e^{y^2} = 10$$

**2.4.14 39**

(c)

$$(0)^3 + 2(0)^2(-2) + (-2)^2 = c \Rightarrow c = 4$$

$$y^2 + 2x^2 y + x^3 - 4 = 0$$

$$\begin{aligned} y &= \frac{-(2x^2) \pm \sqrt{(2x^2)^2 - 4(1)(x^3 - 4)}}{2(1)} \\ &= \frac{-2x^2 \pm \sqrt{4x^4 - 4(x^3 - 4)}}{2} \\ &= -x^2 \pm \sqrt{x^4 - x^3 + 4} \end{aligned}$$

**2.4.15 45**

(a)

$$xv \frac{dv}{dx} + v^2 = 32x \Rightarrow xv \, dv + (v^2 - 32x) \, dx = 0$$

$$M_x = v$$

$$N_v = 2v$$

$$\frac{M_x - N_v}{N} = \frac{v - 2v}{v^2 - 32x}$$

$$\frac{N_v - M_x}{M} = \frac{2v - v}{xv} = \frac{1}{x}$$

$$\mu(x) = e^{\ln x} = x$$

$$x^2 v \, dv + (xv^2 - 32x^2) \, dx = 0$$

$$f(x, v) = \frac{1}{2}x^2v^2 + g(x)$$

$$xv^2 + g'(x) = xv^2 - 32x^2 \Rightarrow g'(x) = -32x^2$$

$$g(x) = -\frac{32}{3}x^3$$

$$\frac{1}{2}(3)^2(0)^2 - \frac{32}{3}(3)^3 = c \Rightarrow c = -288$$

$$\frac{1}{2}x^2v^2 - \frac{32}{3}x^3 = -288 \Rightarrow v = 8\sqrt{\frac{x}{3} - \frac{9}{x^2}}$$

(b)  $v = 12.7 \text{ ft/s}$

## 2.5 Solutions by Substitution

### 2.5.1 1

$$\begin{aligned}(x - y) dx + x dy &= 0 \\(x - ux) dx + x(u dx + x du) &= 0 \\x dx + x^2 du &= 0 \\x^{-1} dx + du &= 0 \\\ln |x| + u &= c \\\ln |x| + \frac{y}{x} &= c \\y &= cx - x \ln |x|\end{aligned}$$

### 2.5.2 3

$$\begin{aligned}x dx + (y - 2x) dy &= 0 \\vy(v dy + y dv) + (y - 2vy) dy &= 0 \\(v^2 y + y - 2vy) dy + vy^2 dv &= 0 \\y(v^2 - 2v + 1) dy + vy^2 dv &= 0 \\(v - 1)^2 dy + vy dv &= 0 \\\frac{1}{y} dy + \frac{v}{(v - 1)^2} dv &= 0 \\\ln |y| + \frac{1}{1 - v} + \ln |v - 1| &= c \\\ln |y| + \frac{1}{1 - x/y} + \ln \left| \frac{x}{y} - 1 \right| &= c \\\ln |x - y| + \frac{y}{y - x} &= c \\(y - x) \ln |x - y| + y &= c(y - x) \\(x - y) \ln |x - y| &= y + c(x - y)\end{aligned}$$

**2.5.3 5**

$$\begin{aligned}
 (y^2 + yx) dx - x^2 dy &= 0 \\
 ((ux)^2 + ux^2) dx - x^2(u dx + x du) &= 0 \\
 u^2 x^2 dx - x^3 du &= 0 \\
 \frac{1}{x} dx - \frac{1}{u^2} du &= 0 \\
 \ln |x| + \frac{1}{u} &= c \\
 \ln |x| + \frac{x}{y} &= c \\
 y &= \frac{x}{c - \ln |x|}
 \end{aligned}$$

**2.5.4 7**

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{y - x}{y + x} \\
 (y + x) dy + (x - y) dx &= 0 \\
 (ux + x)(u dx + x du) + (x - ux) dx &= 0 \\
 (u^2 x + x) dx + (ux^2 + x^2) du &= 0 \\
 x(u^2 + 1) dx + x^2(u + 1) du &= 0 \\
 \frac{1}{x} dx + \frac{u + 1}{u^2 + 1} du &= 0 \\
 \ln |x| + \frac{1}{2} \ln |u^2 + 1| + \arctan u &= c \\
 \ln |x^2 + y^2| + 2 \arctan \frac{y}{x} &= c
 \end{aligned}$$



2.5.5 9

$$\begin{aligned}
 -y \, dx + (x + \sqrt{xy}) \, dy &= 0 \\
 -ux \, dx + (x + \sqrt{ux^2})(u \, dx + x \, du) &= 0 \\
 u\sqrt{ux^2} \, dx + (x^2 + x\sqrt{ux^2}) \, du &= 0 \\
 u^{3/2}x \, dx + x^2(1 + \sqrt{u}) \, du &= 0 \\
 \frac{1}{x} \, dx + \frac{1 + \sqrt{u}}{u^{3/2}} \, du &= 0 \\
 \frac{1}{x} \, dx + (u^{-3/2} + u^{-1}) \, du &= 0 \\
 \ln |x| - 2u^{-1/2} + \ln |u| &= c \\
 \ln |x| - 2(y/x)^{-1/2} + \ln |y/x| &= c \\
 \ln |y| - 2\sqrt{\frac{x}{y}} &= c \\
 4\frac{x}{y} &= (\ln |y| - c)^2 \\
 4x &= y(\ln |y| - c)^2
 \end{aligned}$$

2.5.6 11

$$\begin{aligned}
 xy^2 \frac{dy}{dx} &= y^3 - x^3 \\
 xy^2 \, dy + (x^3 - y^3) \, dx &= 0 \\
 x(ux)^2(u \, dx + x \, du) + (x^3 - (ux)^3) \, dx &= 0 \\
 x^3 \, dx + u^2 x^4 \, du &= 0 \\
 x^{-1} \, dx + u^2 \, du &= 0 \\
 \ln |x| + \frac{1}{3} u^3 &= c \\
 \ln |x| + \frac{1}{3} \left(\frac{y}{x}\right)^3 &= c \\
 \ln |1| + \frac{1}{3} \left(\frac{2}{1}\right)^3 = c \Rightarrow c &= \frac{8}{3} + \ln 1 \\
 \ln |x| + \frac{1}{3} \left(\frac{y}{x}\right)^3 &= \frac{8}{3} \\
 y^3 + 3x^3 \ln |x| &= 8x^3
 \end{aligned}$$

2.5.7 13

$$\begin{aligned}
 (x + ye^{y/x}) dx - xe^{y/x} dy &= 0 \\
 (x + uxe^u) dx - xe^u(u dx + x du) &= 0 \\
 x dx - x^2 e^u du &= 0 \\
 x^{-1} dx - e^u du &= 0 \\
 \ln |x| - e^u &= c \\
 \ln |x| - e^{y/x} &= c
 \end{aligned}$$

$$\ln |1| - e^{0/1} = c \Rightarrow c = -1$$

$$\ln |x| = e^{y/x} - 1$$

2.5.8 15

$$\begin{aligned}
 x \frac{dy}{dx} + y &= \frac{1}{y^2} \\
 \frac{dy}{dx} + x^{-1}y &= x^{-1}y^{-2} \\
 u = y^{1-n} = y^3 \Rightarrow y &= u^{1/3} \Rightarrow \frac{dy}{dx} = \frac{1}{3}u^{-2/3} \frac{du}{dx}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{3}u^{-2/3} \frac{du}{dx} + x^{-1}u^{1/3} &= x^{-1}u^{-2/3} \\
 \frac{du}{dx} + 3x^{-1}u &= 3x^{-1} \\
 e^{3 \ln |x|} \frac{du}{dx} + 3x^{-1}e^{3 \ln |x|}u &= 3x^2 \\
 \frac{d}{dx}(x^3u) &= 3x^2 \\
 x^3u &= x^3 + c \\
 y^3 &= 1 + cx^{-3}
 \end{aligned}$$

2.5.9 17

$$\begin{aligned}
 \frac{dy}{dx} &= y(xy^3 - 1) \\
 \frac{dy}{dx} + y &= xy^4
 \end{aligned}$$

$$u = y^{1-n} = y^{-3} \Rightarrow y = u^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}u^{-4/3}\frac{du}{dx}$$

$$-\frac{1}{3}u^{-4/3}\frac{du}{dx} + u^{-1/3} = xu^{-4/3}$$

$$\frac{du}{dx} - 3u = -3x$$

$$e^{-3x}\frac{du}{dx} - 3e^{-3x}u = -3e^{-3x}x$$

$$\frac{d}{dt}(e^{-3x}u) = -3e^{-3x}x$$

$$e^{-3x}u = e^{-3x}x + \frac{1}{3}e^{-3x} + c$$

$$u = x + \frac{1}{3} + ce^{3x}$$

$$y^{-3} = x + \frac{1}{3} + ce^{3x}$$

**2.5.10 21**

$$x^2\frac{dy}{dx} - 2xy = 3y^4$$

$$\frac{dy}{dx} - 2x^{-1}y = 3x^{-2}y^4$$

$$u = y^{1-n} = y^{-3} \Rightarrow y = u^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}u^{-4/3}\frac{du}{dx}$$

$$-\frac{1}{3}u^{-4/3}\frac{du}{dx} - 2x^{-1}u^{-1/3} = 3x^{-2}u^{-4/3}$$

$$\frac{du}{dx} + 6x^{-1}u = -9x^{-2}$$

$$e^{6\ln|x|}\frac{du}{dx} + 6x^{-1}e^{6\ln|x|}u = -9e^{6\ln|x|}x^{-2}$$

$$\frac{d}{dx}(x^6u) = -9x^4$$

$$x^6u = -\frac{9}{5}x^5 + c$$

$$u = -\frac{9}{5}x^{-1} + cx^{-6}$$

$$y^{-3} = -\frac{9}{5}x^{-1} + cx^{-6}$$

$$\left(\frac{1}{2}\right)^{-3} = -\frac{9}{5}(1)^{-1} + c(1)^{-6} \Rightarrow c = \frac{49}{5}$$

$$y^{-3} = -\frac{9}{5}x^{-1} + \frac{49}{5}x^{-6}$$

### 2.5.11 23

Let  $u = x + y + 1$  so  $\frac{du}{dx} = 1 + \frac{dy}{dx}$  and

$$\begin{aligned}\frac{du}{dx} - 1 &= u^2 \\ \frac{1}{u^2 + 1} du &= dx \\ \arctan u &= x + c \\ \arctan(x + y + 1) &= x + c \\ x + y + 1 &= \tan(x + c) \\ y &= -x - 1 + \tan(x + c)\end{aligned}$$

### 2.5.12 25

Let  $u = x + y$  so  $\frac{du}{dx} = 1 + \frac{dy}{dx}$  and

$$\begin{aligned}\frac{du}{dx} - 1 &= \tan^2 u \\ \frac{1}{1 + \tan^2 u} du &= dx \\ \frac{1}{2}(u + \sin u \cos u) &= x + c \\ x + y + \sin(x + y) \cos(x + y) &= 2(x + c) \\ x + y + \frac{1}{2} \sin(2(x + y)) &= 2(x + c) \\ 2x + 2y + \sin(2(x + y)) &= 4(x + c) \\ 2y - 2x + \sin(2(x + y)) &= c\end{aligned}$$

### 2.5.13 35

(a) Let  $y = y_1 + u$  so  $\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{du}{dx}$  but  $\frac{dy_1}{dx} = P(x) + Q(x)y_1 + R(x)y_1^2$  so

$$\begin{aligned}
\frac{dy}{dx} &= P(x) + Q(x)y + R(x)y^2 \\
P(x) + Q(x)y_1 + R(x)y_1^2 + \frac{du}{dx} &= P(x) + Q(x)(y_1 + u) + R(x)(y_1 + u)^2 \\
\frac{du}{dx} &= Q(x)u + R(x)(2y_1u + u^2) \\
\frac{du}{dx} - (Q(x) + 2R(x)y_1)u &= R(x)u^2
\end{aligned}$$

(b) Let  $y = 2x^{-1} + u$  so  $\frac{dy}{dx} = -2x^{-2} + \frac{du}{dx}$  and

$$\begin{aligned}
-\frac{2}{x^2} + \frac{du}{dx} &= -\frac{4}{x^2} - \frac{1}{x} \left( \frac{2}{x} + u \right) + \left( \frac{2}{x} + u \right)^2 \\
\frac{du}{dx} &= \frac{2}{x^2} - \frac{4}{x^2} - \frac{2}{x^2} - \frac{u}{x} + \frac{4}{x^2} + \frac{4u}{x} + u^2 \\
\frac{du}{dx} - \frac{3}{x}u &= u^2
\end{aligned}$$

Let  $v = u^{1-n} = u^{-1}$  so  $u = v^{-1}$  and  $\frac{du}{dx} = -v^{-2} \frac{dv}{dx}$

$$\begin{aligned}
-v^{-2} \frac{dv}{dx} - \frac{3}{x}v^{-1} &= v^{-2} \\
\frac{dv}{dx} + \frac{3}{x}v &= -1 \\
e^{3 \ln |x|} \frac{dv}{dx} + \frac{3}{x}e^{3 \ln |x|}v &= -e^{3 \ln |x|} \\
\frac{d}{dt}(x^3v) &= -x^3 \\
x^3v &= -\frac{1}{4}x^4 + c \\
\frac{1}{y - y_1} &= -\frac{1}{4}x + cx^{-3} \\
y &= y_1 + \left( -\frac{1}{4}x + cx^{-3} \right)^{-1} \\
&= \frac{2}{x} + \left( -\frac{1}{4}x + cx^{-3} \right)^{-1}
\end{aligned}$$

**2.5.14 37**

$$\begin{aligned}
\frac{dP}{dt} &= P(a - bP) \\
\frac{dP}{dt} - aP &= -bP^2
\end{aligned}$$

Let  $u = P^{1-n} = P^{-1}$  so  $P = u^{-1}$  and  $\frac{dP}{dt} = -u^{-2} \frac{du}{dt}$

$$-u^{-2} \frac{du}{dt} - au^{-1} = -bu^{-2}$$

$$\frac{du}{dt} + au = b$$

$$e^{at} \frac{du}{dt} + ae^{at}u = be^{at}$$

$$\frac{d}{dt}(e^{at}u) = be^{at}$$

$$e^{at}u = \frac{b}{a}e^{at} + c$$

$$P^{-1} = \frac{b}{a} + ce^{-at}$$

$$= \frac{b + ce^{-at}}{a}$$

$$P = \frac{a}{b + ce^{-at}}$$

## 2.6 A Numerical Method

### 2.6.1 1

$x_0 = 1$	$y_0 = 5$
$x_1 = 1.1$	$y_1 = y_0 + hf(x_0, y_0) = 3.8000$
$x_2 = 1.2$	$y_2 = y_1 + hf(x_1, y_1) = 2.9800$

$x_0 = 1$	$y_0 = 5$
$x_1 = 1.05$	$y_1 = y_0 + hf(x_0, y_0) = 4.4000$
$x_2 = 1.1$	$y_2 = y_1 + hf(x_1, y_1) = 3.8950$
$x_3 = 1.15$	$y_3 = y_2 + hf(x_2, y_2) = 3.4708$
$x_4 = 1.2$	$y_4 = y_3 + hf(x_3, y_3) = 3.1152$

## 2.7 Linear Models

### 2.7.1 1

$$P(t) = P_0 e^{kt}$$

$$P(5) = 2P_0 = P_0 e^{5k} \Rightarrow k = \frac{\ln 2}{5} = 0.139$$

$$P(t) = P_0 e^{0.139t}$$

$$3P_0 = P_0 e^{0.139t} \Rightarrow t = 7.9 \text{ years}$$

$$4P_0 = P_0 e^{0.139t} \Rightarrow t = 10 \text{ years}$$

### 2.7.2 5

$$A(t) = A_0 e^{kt}$$

$$A(3.3) = \frac{1}{2}A_0 = A_0 e^{3.3k} \Rightarrow k = -0.21$$

$$0.1A_0 = A_0 e^{-0.21t} \Rightarrow t = 11 \text{ hours}$$

### 2.7.3 9

$$\frac{dI}{dt} = kI \Rightarrow I(t) = ce^{kt}$$

$$I(3) = 0.25I_0 = I_0 e^{3k} \Rightarrow k = -0.462$$

$$I(15) = I_0 e^{-0.462(15)} = 0.001I_0$$

### 2.7.4 11

$$0.145A_0 = A_0 e^{-0.00012097t} \Rightarrow t = 15\,963 \text{ years}$$

### 2.7.5 13

$$\begin{aligned}\frac{dT}{dt} &= k(T - T_m) \\ \frac{1}{T - T_m} \frac{dT}{dt} &= k \\ \ln(T - T_m) &= kt + c \\ T - T_m &= ce^{kt} \\ T &= T_m + ce^{kt} \\ &= 10 + 60e^{kt}\end{aligned}$$

$$T(0.5) = 50 = 10 + 60e^{0.5k} \Rightarrow k = -0.811$$

$$T(1) = 36.7$$

$$15 = 10 + 60e^{-0.811t} \Rightarrow t = 3.06 \text{ min}$$

**2.7.6 21**

$$\begin{aligned}\frac{dA}{dt} &= 4 - \frac{A}{50} \\ \frac{dA}{dt} + \frac{A}{50} &= 4 \\ \frac{d}{dt}(e^{t/50}A) &= 4e^{t/50} \\ e^{t/50}A &= 200e^{t/50} + c \\ A &= 200 + ce^{-t/50}\end{aligned}$$

$$A(0) = 30 = 200 + ce^{-(0)/50} \Rightarrow c = -170$$

$$A(t) = 200 - 170e^{-t/50}$$

**2.7.7 25**

$$V(t) = 500 - 5t$$

$$\begin{aligned}\frac{dA}{dt} &= 10 - \frac{10}{500 - 5t}A \\ \frac{dA}{dt} + \frac{10}{500 - 5t}A &= 10 \\ \frac{dA}{dt} - 2\frac{-5}{500 - 5t}A &= 10 \\ e^{-2 \ln(500-5t)} \frac{dA}{dt} - 2\frac{-5}{500 - 5t}e^{-2 \ln(500-5t)}A &= e^{-2 \ln(500-5t)}10 \\ \frac{d}{dt}(A(500 - 5t)^{-2}) &= 10(500 - 5t)^{-2} \\ A(500 - 5t)^{-2} &= \frac{2}{500 - 5t} + c \\ A &= 2(500 - 5t) + c(500 - 5t)^2 \\ &= 1000 - 10t + c(500 - 5t)^2\end{aligned}$$

$$A(0) = 0 \Rightarrow c = -0.004$$

$$A(t) = 1000 - 10t - 0.004(500 - 5t)^2 = 1000 - 10t - \frac{1}{10}(100 - t)^2$$

The tank is empty at  $t = 100$



**2.7.8 29**

$$i(t) = \frac{3}{5} - \frac{3}{5}e^{-500t}$$

$$i \rightarrow \frac{3}{5} \text{ as } t \rightarrow \infty$$

**2.7.9 33**

$$i(t) = \begin{cases} 60(1 - e^{-t/10}), & 0 \leq t \leq 20 \\ 383e^{-t/10}, & t > 20 \end{cases}$$

**2.7.10 35**

(a)

$$m \frac{dv}{dt} = mg - kv$$

$$\frac{dv}{dt} + \frac{k}{m}v = g$$

$$\frac{d}{dt}(e^{kt/m}v) = e^{kt/m}g$$

$$e^{kt/m}v = \frac{gm}{k}e^{kt/m} + c$$

$$v = \frac{gm}{k} + ce^{-kt/m}$$

$$v(0) = v_0 = \frac{gm}{k} + ce^{-k(0)/m} \Rightarrow c = v_0 - \frac{gm}{k}$$

$$v(t) = \frac{gm}{k} + \left(v_0 - \frac{gm}{k}\right)e^{-kt/m}$$

(b)

$$v_t = \frac{gm}{k}$$

(c)

$$s(t) = \frac{gm}{k}t - \frac{m}{k}\left(v_0 - \frac{gm}{k}\right)e^{-kt/m} + c$$

$$s(0) = 0 = -\frac{m}{k}\left(v_0 - \frac{gm}{k}\right) + c \Rightarrow c = \frac{m}{k}\left(v_0 - \frac{gm}{k}\right)$$

$$\begin{aligned} s(t) &= \frac{m}{k}\left(gt - \left(v_0 + \frac{gm}{k}\right)e^{-kt/m} + v_0 - \frac{gm}{k}\right) \\ &= \frac{m}{k}\left(gt + \left(v_0 - \frac{gm}{k}\right)(1 - e^{-kt/m})\right) \end{aligned}$$

**2.7.11 41**

(a)

$$\begin{aligned}\frac{dP}{dt} &= k_1P - k_2P \\ &= (k_1 - k_2)P \\ P &= ce^{(k_1 - k_2)t}\end{aligned}$$

**2.7.12 43**(a)  $x = r/k$ **2.8 Nonlinear Models****2.8.1 1**(a)  $N = 2000$ 

(b)

$$N = \frac{1}{0.0005 + (1 - 0.0005)e^{-t}}$$

$$N(10) = 1834$$

**2.8.2 3**

$$P = 1.0 \times 10^6$$

$$P = \frac{500}{0.0005 + (0.1 - 0.0005)e^{-0.1t}}$$

$$500000 = \frac{500}{0.0005 + (0.1 - 0.0005)e^{-0.1t}}$$

$$500000(0.0005 + (0.1 - 0.0005)e^{-0.1t}) = 500$$

$$e^{-0.1t} = \frac{0.001 - 0.0005}{0.1 - 0.0005}$$

$$t = 52.9 \text{ months}$$

**2.8.3 11**

29.3 g; 60 g; 0 g; 30 g

### 2.8.4 13

(a)

$$\begin{aligned}\frac{dh}{dt} &= -\frac{A_h}{A_w} \sqrt{2gh} \\ \frac{1}{\sqrt{h}} \frac{dh}{dt} &= -\frac{A_h}{A_w} \sqrt{2g} \\ 2\sqrt{h} &= -\frac{A_h}{A_w} \sqrt{2g}t + c \\ \sqrt{h} &= c - \frac{A_h}{A_w} \sqrt{\frac{g}{2}}t \\ h &= \left( c - \frac{A_h}{A_w} \sqrt{\frac{g}{2}}t \right)^2\end{aligned}$$

$$h(0) = H = c^2 \Rightarrow c = \sqrt{H}$$

$$h = \left( \sqrt{H} - \frac{A_h}{A_w} \sqrt{\frac{g}{2}}t \right)^2 = \left( \sqrt{H} - 4 \frac{A_h}{A_w}t \right)^2$$

Interval of definition is  $[0, \frac{A_w \sqrt{H}}{4A_h}]$

(b) 1821 s = 30 min

### 2.8.5 15

(a)

$$\begin{aligned}\frac{dh}{dt} &= -\frac{5}{6h^{3/2}} \\ h^{3/2} \frac{dh}{dt} &= -\frac{5}{6} \\ \frac{2}{5} h^{5/2} &= -\frac{5}{6}t + c \\ h &= \left( c - \frac{25}{12}t \right)^{2/5}\end{aligned}$$

$$h(0) = H = c^{2/5} \Rightarrow c = H^{5/2}$$

$$h = \left( H^{5/2} - \frac{25}{12}t \right)^{2/5}$$

$$0 = \left( H^{5/2} - \frac{25}{12}t \right)^{2/5} \Rightarrow t = \frac{12}{25} H^{5/2} = 858 \text{ s}$$

(b)

$$\begin{aligned}
 V(h) &= \pi r^2 \frac{h}{3} \\
 &= \pi \left( h \tan \frac{\pi}{6} \right)^2 \frac{h}{3} \\
 &= \pi \left( \frac{h}{\sqrt{3}} \right)^2 \frac{h}{3} \\
 &= \frac{1}{9} \pi h^3
 \end{aligned}$$

$$\begin{aligned}
 \frac{dV}{dt} &= -cA_h \sqrt{2gh} \\
 \frac{d}{dt} \left( \frac{1}{9} \pi h^3 \right) &= -cA_h \sqrt{2gh} \\
 \frac{1}{3} \pi h^2 \frac{dh}{dt} &= -cA_h \sqrt{2gh} \\
 h^{3/2} \frac{dh}{dt} &= -\frac{24}{\pi} cA_h \\
 \frac{2}{5} h^{5/2} &= c_1 - \frac{24}{\pi} cA_h t \\
 h &= \left( c_1 - \frac{60}{\pi} cA_h t \right)^{2/5}
 \end{aligned}$$

$$h(0) = H = c_1^{2/5} \Rightarrow c_1 = H^{5/2}$$

$$h = \left( H^{5/2} - \frac{60}{\pi} cA_h t \right)^{2/5}$$

$$0 = \left( H^{5/2} - \frac{60}{\pi} cA_h t \right)^{2/5}$$

$$\begin{aligned}
 t &= \frac{\pi H^{5/2}}{60 cA_h} \\
 &= 243 \text{ s}
 \end{aligned}$$

### 2.8.6 17

(a)

$$\begin{aligned}
 m \frac{dv}{dt} &= mg - kv^2 \\
 \frac{m}{mg - kv^2} \frac{dv}{dt} &= 1 \\
 \sqrt{\frac{m}{gk}} \operatorname{arctanh} \left( \sqrt{\frac{k}{gm}} v \right) &= t + c_1 \\
 v &= \sqrt{\frac{gm}{k}} \tanh \left( \sqrt{\frac{gk}{m}} (t + c_1) \right)
 \end{aligned}$$

$$\begin{aligned}
 v(0) = v_0 &= \sqrt{\frac{gm}{k}} \tanh c_1 \\
 c_1 &= \operatorname{arctanh} \sqrt{\frac{k}{gm}} v_0
 \end{aligned}$$

(b)  $v_t = \sqrt{gm/k}$

(c)

$$\begin{aligned}
 s &= \frac{m}{k} \ln \cosh \left( \sqrt{\frac{gk}{m}} t + c_1 \right) + c_2 \\
 c_2 &= -\frac{m}{k} \ln \cosh c_1
 \end{aligned}$$

### 2.8.7 21

(a)  $W = 0, W = 2$

(b)

$$\begin{aligned}
 \frac{dW}{dt} &= W\sqrt{4-2W} \\
 \frac{1}{W\sqrt{4-2W}} \frac{dW}{dt} &= 1 \\
 -\operatorname{arctanh} \left( \frac{1}{2} \sqrt{4-2W} \right) &= t + c \\
 \frac{1}{2} \sqrt{4-2W} &= \tanh(c-t) \\
 W &= 2 - 2 \tanh^2(c-t) \\
 &= 2(1 - \tanh^2(c-t)) \\
 &= 2 \operatorname{sech}^2(c-t)
 \end{aligned}$$

## 2.9 Modeling with Systems of First-Order DEs

### 2.9.1 1

$$\begin{aligned}\frac{dx}{dt} &= -\lambda_1 x \\ \ln |x| &= -\lambda_1 t + c_1 \\ x &= c_1 e^{-\lambda_1 t}\end{aligned}$$

$$x(0) = x_0 = c_1 e^{-\lambda_1(0)} \Rightarrow c_1 = x_0$$

$$x = x_0 e^{-\lambda_1 t}$$

$$\begin{aligned}\frac{dy}{dt} &= \lambda_1 x - \lambda_2 y \\ &= \lambda_1 x_0 e^{-\lambda_1 t} - \lambda_2 y \\ \frac{dy}{dt} + \lambda_2 y &= \lambda_1 x_0 e^{-\lambda_1 t} \\ \frac{d}{dt}(e^{\lambda_2 t} y) &= \lambda_1 x_0 e^{(\lambda_2 - \lambda_1)t} \\ e^{\lambda_2 t} y &= \frac{\lambda_1}{\lambda_2 - \lambda_1} x_0 e^{(\lambda_2 - \lambda_1)t} + c_2 \\ y &= \frac{\lambda_1}{\lambda_2 - \lambda_1} x_0 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}\end{aligned}$$

$$y(0) = 0 = \frac{\lambda_1}{\lambda_2 - \lambda_1} x_0 e^{-\lambda_1(0)} + c_2 e^{-\lambda_2(0)} \Rightarrow c_2 = -\frac{\lambda_1}{\lambda_2 - \lambda_1} x_0$$

$$y = \frac{\lambda_1}{\lambda_2 - \lambda_1} x_0 (e^{-\lambda_1 t} - e^{-\lambda_2 t})$$

$$\begin{aligned}\frac{dz}{dt} &= \lambda_2 y \\ &= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} x_0 (e^{-\lambda_1 t} - e^{-\lambda_2 t}) \\ z &= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} x_0 \left( -\frac{1}{\lambda_1} e^{-\lambda_1 t} + \frac{1}{\lambda_2} e^{-\lambda_2 t} \right) + c_3 \\ &= \frac{\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t}}{\lambda_2 - \lambda_1} x_0 + c_3\end{aligned}$$

$$\begin{aligned}
z(0) = 0 &= \frac{\lambda_1 e^{-\lambda_2(0)} - \lambda_2 e^{-\lambda_1(0)}}{\lambda_2 - \lambda_1} x_0 + c_3 \\
&= \frac{\lambda_1 - \lambda_2}{\lambda_2 - \lambda_1} x_0 + c_3 \\
c_3 &= x_0
\end{aligned}$$

$$z = \frac{\lambda_1(e^{-\lambda_2 t} - 1) + \lambda_2(1 - e^{-\lambda_1 t})}{\lambda_2 - \lambda_1} x_0$$

### 2.9.2 3

5 days, 20 days, 147 days

### 2.9.3 5

(a)

$$\begin{aligned}
\frac{dP}{dt} &= -(\lambda_A + \lambda_C)P \\
P &= ce^{-(\lambda_A + \lambda_C)t}
\end{aligned}$$

$$P(0) = P_0 = ce^{-(\lambda_A + \lambda_C)(0)} \Rightarrow c = P_0$$

$$P = P_0 e^{-(\lambda_A + \lambda_C)t}$$

(b)

$$\frac{1}{2}P_0 = P_0 e^{-(\lambda_A + \lambda_C)t} \Rightarrow t = \frac{\ln 1/2}{-(\lambda_A + \lambda_C)} = 1.25 \times 10^9 \text{ years}$$

(c)

$$\begin{aligned}
\frac{dA}{dt} &= \lambda_A P \\
&= \lambda_A P_0 e^{-(\lambda_A + \lambda_C)t} \\
A &= -\frac{\lambda_A}{\lambda_A + \lambda_C} P_0 e^{-(\lambda_A + \lambda_C)t} + c
\end{aligned}$$

$$A(0) = 0 = -\frac{\lambda_A}{\lambda_A + \lambda_C} P_0 e^{-(\lambda_A + \lambda_C)(0)} + c \Rightarrow c = \frac{\lambda_A}{\lambda_A + \lambda_C} P_0$$

$$A = \frac{\lambda_A}{\lambda_A + \lambda_C} P_0 (1 - e^{-(\lambda_A + \lambda_C)t})$$

$$\begin{aligned}
\frac{dC}{dt} &= \lambda_C P \\
&= \lambda_C P_0 e^{-(\lambda_A + \lambda_C)t} \\
C &= -\frac{\lambda_C}{\lambda_A + \lambda_C} P_0 e^{-(\lambda_A + \lambda_C)t} + c
\end{aligned}$$

$$C(0) = 0 = -\frac{\lambda_C}{\lambda_A + \lambda_C} P_0 e^{-(\lambda_A + \lambda_C)(0)} + c \Rightarrow c = \frac{\lambda_C}{\lambda_A + \lambda_C} P_0$$

$$C = \frac{\lambda_C}{\lambda_A + \lambda_C} P_0 (1 - e^{-(\lambda_A + \lambda_C)t})$$

(d)

$$\frac{\lambda_A}{\lambda_A + \lambda_C} = 10.5\%$$

$$\frac{\lambda_C}{\lambda_A + \lambda_C} = 89.5\%$$

**2.9.4    7**

$$\frac{dx_1}{dt} = 6 - \frac{2}{25}x_1 + \frac{1}{50}x_2$$

$$\frac{dx_2}{dt} = \frac{2}{25}x_1 - \frac{2}{25}x_2$$

**2.9.5    9**

(a)

$$V_1 = 100 + t$$

$$V_2 = 100 - t$$

$$\frac{dx_1}{dt} = \frac{3}{100 - t}x_2 - \frac{2}{100 + t}x_1$$

$$\frac{dx_2}{dt} = \frac{2}{100 + t}x_1 - \frac{3}{100 - t}x_2$$



(b)

$$\frac{dx_1}{dt} = -\frac{dx_2}{dt}$$

This makes sense because it's a closed system. Salt is moving from tank B to tank A.

$$x_1 = c - x_2$$

$$x_1(0) = c - x_2(0) \Rightarrow 100 = c - 50 \Rightarrow c = 150$$

$$\begin{aligned}\frac{dx_2}{dt} &= \frac{2}{100+t}(150 - x_2) - \frac{3}{100-t}x_2 \\ &= \frac{300}{100+t} - \frac{2}{100+t}x_2 - \frac{3}{100-t}x_2 \\ \frac{dx_2}{dt} + \left(\frac{2}{100+t} + \frac{3}{100-t}\right)x_2 &= \frac{300}{100+t} \\ \frac{d}{dt}(e^{2\ln|100+t|-3\ln|100-t|}x_2) &= \frac{300}{100+t}e^{2\ln|100+t|-3\ln|100-t|} \\ \frac{d}{dt}\left(\frac{(100+t)^2}{(100-t)^3}x_2\right) &= \frac{300(100+t)}{(100-t)^3} \\ &= \frac{30000}{(100-t)^3} + \frac{300t}{(100-t)^3} \\ \frac{(100+t)^2}{(100-t)^3}x_2 &= \frac{15000}{(100-t)^2} + \frac{300(t-50)}{(100-t)^2} + c\end{aligned}$$

$$\begin{aligned}x_2 &= \frac{(100-t)^3}{(100+t)^2} \left( \frac{15000}{(100-t)^2} + \frac{300(t-50)}{(100-t)^2} + c \right) \\ x_2(0) = 50 &= \frac{100^3}{100^2} \left( \frac{15000}{100^2} + \frac{300(-50)}{100^2} + c \right) \\ &= 100c \\ c &= \frac{1}{2}\end{aligned}$$

$$x_2(30) = 47.4 \text{ lb}$$

**2.9.6 15**

$$i_1 = i_2 + i_3$$

$$\begin{aligned} i_1 R_1 + \frac{di_2}{dt} L_1 + i_2 R_2 &= E(t) \\ (i_2 + i_3) R_1 + \frac{di_2}{dt} L_1 + i_2 R_2 &= E(t) \end{aligned}$$

$$\begin{aligned} i_1 R_1 + \frac{di_3}{dt} L_2 + i_3 R_3 &= E(t) \\ (i_2 + i_3) R_1 + \frac{di_3}{dt} L_2 + i_3 R_3 &= E(t) \end{aligned}$$

**2.9.7 17**

$i(0) = i_0$ ,  $s(0) = n - i_0$ ,  $r(0) = 0$ ; It's consistent because no one leaves the community

**2.10 Chapter 2 in Review****2.10.1 1**

$y = -A/k$ ; repeller; attractor

**2.10.2 3**

$$\frac{dy}{dx} = (y-1)^2(y-3)^2$$

**2.10.3 5**

$\frac{dy}{dx} = x^n$  is semi-stable for even  $n$ , unstable for odd  $n$   
 $\frac{dy}{dx} = -x^n$  is semi-stable for even  $n$ , stable for odd  $n$

**2.10.4 9**

$$\begin{aligned} (y^2 + 1) dx &= y \sec^2 x dy \\ \cos^2 x dx &= \frac{y}{y^2 + 1} dy \\ \frac{1}{2}(1 + \cos 2x) dx &= \frac{1}{2} \frac{2y}{y^2 + 1} dy \\ x + \frac{1}{2} \sin 2x &= \ln |y^2 + 1| + c \\ 2x + \sin 2x &= 2 \ln |y^2 + 1| + c \end{aligned}$$

**2.10.5 11**

$$(6x+1)y^2 \frac{dy}{dx} + 3x^2 + 2y^3 = 0$$

$$(6x+1)y^2 dy + (3x^2 + 2y^3) dx = 0$$

$$\frac{\partial f}{\partial x} = 3x^2 + 2y^3$$

$$f = x^3 + 2xy^3 + g(y)$$

$$\frac{\partial f}{\partial y} = 6xy^2 + g'(y) = 6xy^2 + y^2$$

$$g'(y) = y^2$$

$$g(y) = \frac{1}{3}y^3$$

$$f(x, y) = x^3 + 2xy^3 + \frac{1}{3}y^3$$

$$c = x^3 + 2xy^3 + \frac{1}{3}y^3$$

**2.10.6 13**

$$t \frac{dQ}{dt} + Q = t^4 \ln t$$

$$\frac{dQ}{dt} + \frac{1}{t}Q = t^3 \ln t$$

$$\frac{d}{dt}(tQ) = t^4 \ln t$$

$$tQ = \frac{1}{25}t^5(5 \ln t - 1) + c$$

$$Q = \frac{1}{25}t^4(5 \ln t - 1) + ct^{-1}$$

**2.10.7 15**

$$(8xy - 2x) dx + (x^2 + 4) dy = 0$$

$$M_y = 8x$$

$$N_x = 2x$$

$$\frac{M_y - N_x}{N} = \frac{6x}{x^2 + 4}$$

$$\mu(x) = e^{3 \ln |x^2+4|} = (x^2 + 4)^3$$

$$(x^2 + 4)^3(8xy - 2x) dx + (x^2 + 4)^4 dy = 0$$

$$\frac{\partial f}{\partial y} = (x^2 + 4)^4$$

$$f(x, y) = (x^2 + 4)^4 y + g(x)$$

$$\frac{\partial f}{\partial x} = 8x(x^2 + 4)^3 y + g'(x) = (8xy - 2x)(x^2 + 4)^3$$

$$g'(x) = -2x(x^2 + 4)^3$$

$$g(x) = -\frac{1}{4}(x^2 + 4)^4$$

$$c = (y - \frac{1}{4})(x^2 + 4)^4$$

$$y = \frac{1}{4} + c(x^2 + 4)^{-4}$$

### 2.10.8 17

$$2 \frac{dy}{dx} + (4 \cos x)y = x$$

$$\frac{dy}{dx} + (2 \cos x)y = \frac{1}{2}x$$

$$e^{2 \sin x} \frac{dy}{dx} + (2 \cos x)e^{2 \sin x} y = \frac{1}{2}x e^{2 \sin x}$$

$$\frac{d}{dx}(e^{2 \sin x} y) = \frac{1}{2}x e^{2 \sin x}$$

$$\int_0^x \frac{d}{dx}(e^{2 \sin x'} y) dx' = \int_0^x \frac{1}{2}x' e^{2 \sin x'} dx'$$

$$e^{2 \sin x} y - e^{2 \sin 0} = \int_0^x \frac{1}{2}x' e^{2 \sin x'} dx'$$

$$y = \frac{1}{e^{2 \sin x}} \left( 1 + \int_0^x \frac{1}{2}x' e^{2 \sin x'} dx' \right)$$

2.10.9 19

$$\begin{aligned}
 x \frac{dy}{dx} + 2y &= xe^{x^2} \\
 \frac{dy}{dx} + \frac{2}{x}y &= e^{x^2} \\
 \frac{d}{dt}(x^2y) &= x^2e^{x^2} \\
 \int_1^x \frac{d}{dt}(x'^2y) dx' &= \int_1^x x'^2e^{x'^2} dx' \\
 x^2y - 3 &= \int_1^x x'^2e^{x'^2} dx' \\
 y &= \frac{3}{x^2} + \frac{1}{x^2} \int_1^x x'^2e^{x'^2} dx'
 \end{aligned}$$

2.10.10 21

$$\begin{aligned}
 \frac{dy}{dx} + y &= e^{-x} \\
 \frac{d}{dx}(e^xy) &= 1 \\
 e^xy &= x + c_1 \\
 y &= (x + c_1)e^{-x}
 \end{aligned}$$

$$y(0) = 5 = c_1$$

$$y = (x + 5)e^{-x}$$

$$\begin{aligned}
 \frac{dy}{dx} + y &= 0 \\
 \frac{d}{dt}(e^xy) &= 0 \\
 e^xy &= c_2 \\
 y &= c_2e^{-x}
 \end{aligned}$$

$$(1 + 5)e^{-1} = c_2e^{-1} \Rightarrow c_2 = 6$$

$$y = \begin{cases} (x + 5)e^{-x} & 0 \leq x < 1 \\ 6e^{-x} & x \geq 1 \end{cases}$$

**2.10.11 23**

$$\sin x \frac{dy}{dx} + (\cos x)y = 0$$

$$\frac{dy}{dx} + (\cot x)y = 0$$

$$\frac{d}{dx}(y \sin x) = 0$$

$$y \sin x = c$$

$$y = c \csc x$$

$$y(7\pi/6) = -2 = c \csc \frac{7\pi}{6} \Rightarrow c = 1$$

$$y = \csc x$$

$$I = (\pi, 2\pi)$$

**2.10.12 25**

(a) Because  $\sqrt{y}$  isn't defined for  $y < 0$

(b)

$$\frac{dy}{dx} = \sqrt{y}$$

$$y^{-1/2} \frac{dy}{dx} = 1$$

$$2\sqrt{y} = x + c$$

$$y = \frac{1}{4}(x + c)^2$$

$$y(x_0) = y_0 = \frac{1}{4}(x_0 + c)^2 \Rightarrow c = \sqrt{4y_0} - x_0$$

$$y = \frac{1}{4}(x + \sqrt{4y_0} - x_0)^2$$

**2.10.13 29**

$$\frac{dP}{dt} = kP$$

$$P = P_0 e^{kt}$$

$$P(45) = 8.99 \times 10^9 \text{ people}$$

**2.10.14 31**

(a)

$$0.53A_0 = A_0 e^{-0.00012097t} \Rightarrow t = 5248 \text{ years ago}$$

(b) 3257 BC

**2.10.15 35**

(a)

$$\begin{aligned} k(T - T_m) &= 0 \\ T &= T_m \\ &= T_2 + B(T_1 - T) \\ &= \frac{BT_1 + T_2}{1 + B} \end{aligned}$$

$T_m$  is the same

(b)

$$\begin{aligned} \frac{dT}{dt} &= k(T - T_m) \\ &= k(T - (T_2 + B(T_1 - T))) \\ &= k((1 + B)T - BT_1 - T_2) \\ \frac{dT}{dt} - k(1 + B)T &= -k(BT_1 + T_2) \\ \frac{d}{dt}(e^{-k(1+B)t}T) &= -k(BT_1 + T_2)e^{-k(1+B)t} \\ e^{-k(1+B)t}T &= \frac{BT_1 + T_2}{1 + B}e^{-k(1+B)t} + c \\ T &= \frac{BT_1 + T_2}{1 + B} + ce^{k(1+B)t} \end{aligned}$$

$$\begin{aligned} T(0) = T_1 &= \frac{BT_1 + T_2}{1 + B} + ce^{k(1+B)(0)} \\ c &= T_1 - \frac{BT_1 + T_2}{1 + B} \\ &= \frac{T_1(1 + B) - BT_1 - T_2}{1 + B} \\ &= \frac{T_1 - T_2}{1 + B} \end{aligned}$$

$$T = \frac{BT_1 + T_2 + (T_1 - T_2)e^{k(1+B)t}}{1 + B}$$

**2.10.16 37**

$$\begin{aligned}
(k_1 + k_2 t) \frac{dq}{dt} + \frac{1}{C} q &= E_0 \\
\frac{dq}{dt} + \frac{1}{C(k_1 + k_2 t)} q &= \frac{E_0}{k_1 + k_2 t} \\
\frac{d}{dt} (e^{\frac{\ln |C(k_1 + k_2 t)|}{Ck_2}} q) &= \frac{E_0}{k_1 + k_2 t} e^{\frac{\ln |C(k_1 + k_2 t)|}{Ck_2}} \\
\frac{d}{dt} ((C(k_1 + k_2 t))^{1/Ck_2} q) &= \frac{E_0}{k_1 + k_2 t} (C(k_1 + k_2 t))^{1/Ck_2} \\
(C(k_1 + k_2 t))^{1/Ck_2} q &= E_0 C (C(k_1 + k_2 t))^{1/Ck_2} + c \\
q &= E_0 C + c (C(k_1 + k_2 t))^{-1/Ck_2}
\end{aligned}$$

$$\begin{aligned}
q(0) &= q_0 = E_0 C + c (C(k_1 + k_2(0)))^{-1/Ck_2} \\
q_0 &= E_0 C + c (Ck_1)^{-1/Ck_2} \\
c &= (q_0 - E_0 C) (Ck_1)^{1/Ck_2}
\end{aligned}$$

$$\begin{aligned}
q &= E_0 C + (q_0 - E_0 C) (Ck_1)^{1/Ck_2} (C(k_1 + k_2 t))^{-1/Ck_2} \\
&= E_0 C + (q_0 - E_0 C) \left( \frac{k_1}{k_1 + k_2 t} \right)^{1/Ck_2}
\end{aligned}$$

**2.10.17 39**

$$\begin{aligned}
\frac{dh}{dt} &= -c \frac{\pi r_h^2}{\pi r_w^2} \sqrt{2gh} \\
\frac{1}{\sqrt{h}} \frac{dh}{dt} &= -8c (r_h/r_w)^2 \\
2\sqrt{h} &= c_1 - 8c (r_h/r_w)^2 t \\
h &= (c_1 - 4c (r_h/r_w)^2 t)^2
\end{aligned}$$

$$h(0) = 2 = (c_1 - 4c (r_h/r_w)^2 (0))^2 \Rightarrow c_1 = \sqrt{2}$$

$$h = (\sqrt{2} - 4c (r_h/r_w)^2 t)^2 = (\sqrt{2} - (1.63 \times 10^{-5}) t)^2$$



2.10.18 43

$$\begin{aligned}
 \frac{dx}{dt} &= k_1 x(\alpha - x) \\
 \frac{1}{x(\alpha - x)} \frac{dx}{dt} &= k_1 \\
 \left( \frac{1}{x} + \frac{1}{\alpha - x} \right) \frac{dx}{dt} &= \alpha k_1 \\
 \ln |x| - \ln |\alpha - x| &= \alpha k_1 t + c_1 \\
 \ln \left| \frac{x}{\alpha - x} \right| &= \alpha k_1 t + c_1 \\
 \frac{x}{\alpha - x} &= c_1 e^{\alpha k_1 t} \\
 x &= (\alpha - x) c_1 e^{\alpha k_1 t} \\
 (c_1 e^{\alpha k_1 t} + 1)x &= \alpha c_1 e^{\alpha k_1 t} \\
 x &= \frac{\alpha c_1 e^{\alpha k_1 t}}{c_1 e^{\alpha k_1 t} + 1}
 \end{aligned}$$

$$\begin{aligned}
 \frac{dy}{dt} &= k_2 xy \\
 &= k_2 \frac{\alpha c_1 e^{\alpha k_1 t}}{c_1 e^{\alpha k_1 t} + 1} y \\
 \frac{1}{y} \frac{dy}{dt} &= k_2 \frac{\alpha c_1 e^{\alpha k_1 t}}{c_1 e^{\alpha k_1 t} + 1} \\
 \ln |y| &= \frac{k_2}{k_1} \ln |c_1 e^{\alpha k_1 t} + 1| + c_2 \\
 y &= c_2 (c_1 e^{\alpha k_1 t} + 1)^{k_2/k_1}
 \end{aligned}$$

2.10.19 45

$$\begin{aligned}
 \frac{dP}{dt} &= kP \ln \frac{450}{P} \\
 \frac{1}{P \ln(450/P)} \frac{dP}{dt} &= k \\
 -\ln \left( \ln \frac{450}{P} \right) &= kt + c \\
 \ln \frac{450}{P} &= ce^{-kt} \\
 \frac{450}{P} &= e^{ce^{-kt}} \\
 P &= \frac{450}{e^{ce^{-kt}}}
 \end{aligned}$$

$$P(0) = 40 = \frac{450}{e^{ce^{-k(0)}}} \Rightarrow c = \ln \frac{450}{40} = 2.42$$

$$\begin{aligned} P(15) = 95 &= \frac{450}{e^{2.42e^{-k(15)}}} \\ 2.42e^{-15k} &= \ln \frac{450}{95} \\ k &= -\frac{\ln(\ln(450/95)/2.42)}{15} \\ &= 0.0295 \end{aligned}$$

$$P(30) = \frac{450}{e^{2.42e^{-0.0295(30)}}} = 166$$

**2.10.20 47**

$$\begin{aligned} y &= c_1 x \\ \frac{dy}{dx} &= c_1 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{c_1} \\ y &= -\frac{1}{c_1}x + c_2 \end{aligned}$$

**2.10.21 49**

$$\begin{aligned} y &= -x - 1 + c_1 e^x \\ \frac{dy}{dx} &= c_1 e^x - 1 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{c_1 e^x - 1} \\ y &= x - \ln(1 - c_1 e^x) + c_2 \end{aligned}$$

### 3 Higher-Order Differential Equations

#### 3.1 Theory of Linear Equations

##### 3.1.1 1

$$\begin{aligned}y &= c_1 e^x + c_2 e^{-x} \\0 &= c_1 e^{(0)} + c_2 e^{-(0)} \\&= c_1 + c_2\end{aligned}$$

$$\begin{aligned}y' &= c_1 e^x - c_2 e^{-x} \\1 &= c_1 e^{(0)} - c_2 e^{-(0)} \\&= c_1 - c_2\end{aligned}$$

$$c_2 = c_1 - 1 \Rightarrow 0 = c_1 + c_1 - 1 \Rightarrow c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}$$

$$y = \frac{1}{2}(e^x - e^{-x})$$

##### 3.1.2 3

$$\begin{aligned}y &= c_1 x + c_2 x \ln x \\3 &= c_1(1) + c_2(1) \ln(1) \\&= c_1\end{aligned}$$

$$\begin{aligned}y' &= 3 + c_2(1 + \ln x) \\-1 &= 3 + c_2(1 + \ln(1)) \\c_2 &= -4\end{aligned}$$

$$y = 3x - 4x \ln x$$

##### 3.1.3 9

$$(-\infty, 2)$$

### 3.1.4 11

(a)

$$\begin{aligned}y &= c_1 e^x + c_2 e^{-x} \\0 &= c_1 e^{(0)} + c_2 e^{-(0)} \\&= c_1 + c_2\end{aligned}$$

$$\begin{aligned}1 &= c_1 e^{(1)} + c_2 e^{-(1)} \\&= c_1 e + c_2 e^{-1} \\&= c_1 e - c_1 e^{-1} \\&= c_1 (e - e^{-1}) \\c_1 &= \frac{1}{e - e^{-1}} \\c_2 &= -\frac{1}{e - e^{-1}}\end{aligned}$$

$$y = \frac{e^x - e^{-x}}{e - e^{-1}}$$

(b)

$$\begin{aligned}y &= c_3 \cosh x + c_4 \sinh x \\0 &= c_3 \cosh 0 + c_4 \sinh 0 \\&= c_3\end{aligned}$$

$$\begin{aligned}y &= c_4 \sinh x \\1 &= c_4 \sinh 1 \\c_4 &= \operatorname{csch} 1\end{aligned}$$

$$y = (\operatorname{csch} 1) \sinh x$$

(c)

$$(\operatorname{csch} 1) \sinh x = \frac{2}{e - e^{-1}} \frac{e^x - e^{-x}}{2} = \frac{e^x - e^{-x}}{e - e^{-1}}$$

### 3.1.5 13

(a)

$$\begin{aligned}y &= c_1 e^x \cos x + c_2 e^x \sin x \\1 &= c_1 e^{(0)} \cos(0) + c_2 e^{(0)} \sin(0) \\&= c_1\end{aligned}$$

$$\begin{aligned}y' &= e^x \cos x - e^x \sin x + c_2 e^x \sin x + c_2 e^x \cos x \\0 &= e^{(\pi)} \cos(\pi) - e^{(\pi)} \sin(\pi) + c_2 e^{(\pi)} \sin(\pi) + c_2 e^{(\pi)} \cos(\pi) \\&= -e^\pi - c_2 e^\pi \\c_2 &= -1\end{aligned}$$

$$y = e^x \cos x - e^x \sin x$$

(b)

$$\begin{aligned}y &= c_1 e^x \cos x + c_2 e^x \sin x \\1 &= c_1 e^{(0)} \cos(0) + c_2 e^{(0)} \sin(0) \\&= c_1\end{aligned}$$

$$\begin{aligned}y &= c_1 e^x \cos x + c_2 e^x \sin x \\-1 &= e^{(\pi)} \cos(\pi) + c_2 e^{(\pi)} \sin(\pi) \\&= -e^\pi\end{aligned}$$

No solution

(c)

$$c_1 = 1$$

$$\begin{aligned}y &= c_1 e^x \cos x + c_2 e^x \sin x \\1 &= e^{(\pi/2)} \cos\left(\frac{\pi}{2}\right) + c_2 e^{(\pi/2)} \sin\left(\frac{\pi}{2}\right) \\&= c_2 e^{\pi/2} \\c_2 &= e^{-\pi/2}\end{aligned}$$

$$y = e^x \cos x + e^{x-\pi/2} \sin x$$

(d)

$$\begin{aligned}y &= c_1 e^x \cos x + c_2 e^x \sin x \\0 &= c_1 e^{(0)} \cos(0) + c_2 e^{(0)} \sin(0) \\&= c_1 e \\&= c_1\end{aligned}$$

$$\begin{aligned}y &= c_1 e^x \cos x + c_2 e^x \sin x \\0 &= c_2 e^{(\pi)} \sin(\pi)\end{aligned}$$

$$y = c_2 e^x \sin x$$

### 3.1.6 15

Dependent

### 3.1.7 17

Dependent

### 3.1.8 19

Dependent

### 3.1.9 21

Independent

### 3.1.10 23

$$y'' - y' - 12y = 9e^{-3x} + 3e^{-3x} - 12e^{-3x} = 0$$

$$y'' - y' - 12y = 16e^{4x} - 4e^{4x} - 12e^{4x} = 0$$

Both functions are solutions of the differential equation and are linearly independent, so they form a fundamental set of solutions.

$$y = c_1 e^{-3x} + c_2 e^{4x}$$

### 3.1.11 25

$$\begin{aligned} y'' - 2y' + 5y &= e^x \cos 2x - 2e^x \sin 2x - 2e^x \sin 2x - 4e^x \cos 2x \\ &\quad - 2(e^x \cos 2x - 2e^x \sin 2x) + 5e^x \cos 2x \\ &= 0 \end{aligned}$$

$$\begin{aligned} y'' - 2y' + 5y &= e^x \sin 2x + 2e^x \cos 2x + 2e^x \cos 2x - 4e^x \sin 2x \\ &\quad - 2(e^x \sin 2x + 2e^x \cos 2x) + 5e^x \sin 2x \\ &= 0 \end{aligned}$$

$$\begin{aligned} W(e^x \cos 2x, e^x \sin 2x) &= \begin{vmatrix} e^x \cos 2x & e^x \sin 2x \\ e^x \cos 2x - 2e^x \sin 2x & e^x \sin 2x + 2e^x \cos 2x \end{vmatrix} \\ &= e^x \cos 2x(e^x \sin 2x + 2e^x \cos 2x) \\ &\quad - e^x \sin 2x(e^x \cos 2x - 2e^x \sin 2x) \\ &= e^{2x}(\sin 2x \cos 2x + 2 \cos^2 2x - \sin 2x \cos 2x + 2 \sin^2 2x) \\ &= 2e^{2x} \end{aligned}$$

Both functions are solutions to the differential equation and the Wronskian does not equal 0 for all  $x$  in the interval.

$$y = c_1 e^x \cos 2x + c_2 e^x \sin 2x$$

### 3.1.12 27

$$\begin{aligned} x^2 y'' - 6xy' + 12y &= x^2(6x) - 6x(3x^2) + 12(x^3) \\ &= 6x^3 - 18x^3 + 12x^3 \\ &= 0 \end{aligned}$$

$$\begin{aligned} x^2 y'' - 6xy' + 12y &= x^2(12x^2) - 6x(4x^3) + 12(x^4) \\ &= 12x^4 - 24x^4 + 12x^4 \\ &= 0 \end{aligned}$$

$$\begin{aligned} W(x^3, x^4) &= \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix} \\ &= (x^3)(4x^3) - (x^4)(3x^2) \\ &= 4x^6 - 3x^6 \\ &= x^6 \end{aligned}$$

Both functions are solutions to the differential equation and, because 0 isn't included in the interval, the Wronskian does not equal 0 for all  $x$  in the interval.

$$y = c_1x^3 + c_2x^4$$

### 3.1.13 35

(a)

$$y'' - 6y' + 5y = 12e^{2x} - 6(6e^{2x}) + 5(3e^{2x}) = -9e^{2x}$$

$$y'' - 6y' + 5y = 2 - 6(2x + 3) + 5(x^2 + 3x) = 5x^2 + 3x - 16$$

(b)

$$y = 3e^{2x} + x^2 + 3x$$

$$y = -\frac{1}{9}(3e^{2x}) - 2(x^2 + 3x) = -\frac{1}{3}e^{2x} - 2(x^2 + 3x)$$

## 3.2 Reduction of Order

### 3.2.1 1

$$\begin{aligned} y_2(x) &= u(x)e^{2x} \\ y_2'(x) &= u'(x)e^{2x} + 2u(x)e^{2x} \\ &= (u'(x) + 2u(x))e^{2x} \\ y_2''(x) &= u''(x)e^{2x} + 2u'(x)e^{2x} + 2u'(x)e^{2x} + 4u(x)e^{2x} \\ &= (u''(x) + 4u'(x) + 4u(x))e^{2x} \end{aligned}$$

$$\begin{aligned} y'' - 4y' + 4y &= 0 \\ (u''(x) + 4u'(x) + 4u(x))e^{2x} - 4(u'(x) + 2u(x))e^{2x} + 4u(x)e^{2x} &= 0 \\ u''(x) &= 0 \\ u'(x) &= c_1 \\ u(x) &= c_1x + c_2 \end{aligned}$$

$$y_2(x) = xe^{2x}$$



### 3.2.2 3

$$\begin{aligned}
y_2(x) &= u(x)y_1(x) \\
&= u(x) \cos 4x \\
y_2'(x) &= u'(x) \cos 4x - 4u(x) \sin 4x \\
y_2''(x) &= u''(x) \cos 4x - 4u'(x) \sin 4x - 4u'(x) \sin 4x - 16u(x) \cos 4x \\
&= u''(x) \cos 4x - 8u'(x) \sin 4x - 16u(x) \cos 4x
\end{aligned}$$

$$\begin{aligned}
y'' + 16y &= 0 \\
u''(x) \cos 4x - 8u'(x) \sin 4x - 16u(x) \cos 4x + 16u(x) \cos 4x &= 0 \\
u''(x) \cos 4x - 8u'(x) \sin 4x &= 0 \\
u''(x) - 8(\tan 4x)u'(x) &= 0
\end{aligned}$$

$$\begin{aligned}
e^{\int -8 \tan 4x \, dx} u''(x) - 8(\tan 4x) e^{\int -8 \tan 4x \, dx} u'(x) &= 0 \\
\frac{d}{dx} (u'(x) \cos^2 4x) &= 0 \\
u'(x) \cos^2 4x &= c_1 \\
u'(x) &= c_1 \sec^2 4x \\
u(x) &= c_1 \tan 4x + c_2
\end{aligned}$$

$$y_2 = \sin 4x$$

### 3.2.3 5

$$\begin{aligned}
y_2(x) &= u(x)y_1(x) \\
&= u(x) \cosh x \\
y_2'(x) &= u'(x) \cosh x + u(x) \sinh x \\
y_2''(x) &= u''(x) \cosh x + u'(x) \sinh x + u'(x) \sinh x + u(x) \cosh(x) \\
&= u''(x) \cosh x + 2u'(x) \sinh x + u(x) \cosh x
\end{aligned}$$

$$\begin{aligned}
y'' - y &= 0 \\
u''(x) \cosh x + 2u'(x) \sinh x + u(x) \cosh x - u(x) \cosh x &= 0 \\
u''(x) \cosh x + 2u'(x) \sinh x &= 0 \\
u''(x) + 2(\tanh x)u'(x) &= 0
\end{aligned}$$

$$\begin{aligned}
e^{\int 2 \tanh x \, dx} u''(x) + 2(\tanh x) e^{\int 2 \tanh x \, dx} u'(x) &= 0 \\
\frac{d}{dx}(u'(x) \cosh^2 x) &= 0 \\
u'(x) \cosh^2 x &= c_1 \\
u'(x) &= c_1 \operatorname{sech}^2 x \\
u(x) &= c_1 \tanh x + c_2
\end{aligned}$$

$$y_2 = \sinh x$$

### 3.2.4 7

$$\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x) \, dx}}{y_1^2(x)} \, dx \\
&= e^{2x/3} \int \frac{e^{4x/3}}{e^{4x/3}} \, dx \\
&= x e^{2x/3}
\end{aligned}$$

### 3.2.5 9

$$\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x) \, dx}}{y_1^2(x)} \, dx \\
&= x^4 \int \frac{x^7}{x^8} \, dx \\
&= x^4 \ln |x|
\end{aligned}$$

### 3.2.6 11

$$\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x) \, dx}}{y_1^2(x)} \, dx \\
&= \ln x \int \frac{1}{x \ln^2 x} \, dx \\
&= -1
\end{aligned}$$

### 3.2.7 13

$$\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x) \, dx}}{y_1^2(x)} \, dx \\
&= x \sin(\ln x) \int \frac{1}{x \sin^2(\ln x)} \, dx \\
&= x \cos(\ln x)
\end{aligned}$$

**3.2.8 15**

$$\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx \\
&= (x+1) \int \frac{1-2x-x^2}{(x+1)^2} dx \\
&= (x+1) \left( -x - \frac{2}{x+1} \right) \\
&= x^2 + x + 2
\end{aligned}$$

**3.2.9 17**

$$\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx \\
&= e^{-2x} \int \frac{e^{-\int 0 dx}}{(e^{-2x})^2} dx \\
&= e^{-2x} \int e^{4x} dx \\
&= e^{2x}
\end{aligned}$$

$$y_p(x) = -\frac{1}{2}$$

**3.2.10 19**

$$\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx \\
&= e^x \int \frac{e^{3x}}{e^{2x}} dx \\
&= e^{2x}
\end{aligned}$$

$$y_p(x) = \frac{5}{2}e^{3x}$$

**3.2.11 21**

$$\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx \\
&= x \int \frac{e^{-\int (1-x^{-1}) dx}}{x^2} dx \\
&= x \int \frac{e^{\ln|x|-x}}{x^2} dx \\
&= x \int \frac{1}{xe^x} dx \\
&= x \int_{x_0}^x \frac{1}{te^t} dt
\end{aligned}$$

**3.3 Homogeneous Linear Equations with Constant Coefficients****3.3.1 1**

$$\begin{aligned}
4y'' + y' &= 0 \\
4m^2 + m &= 0 \\
m(4m + 1) &= 0 \\
y &= c_1 + c_2e^{-x/4}
\end{aligned}$$

**3.3.2 3**

$$\begin{aligned}
y'' - y' - 6y &= 0 \\
m^2 - m - 6 &= 0 \\
(m - 3)(m + 2) &= 0 \\
y &= c_1e^{3x} + c_2e^{-2x}
\end{aligned}$$

**3.3.3 5**

$$\begin{aligned}
y'' + 8y' + 16y &= 0 \\
m^2 + 8m + 16 &= 0 \\
(m + 4)^2 &= 0 \\
y &= c_1e^{-4x} + c_2xe^{-4x}
\end{aligned}$$

**3.3.4 7**

$$\begin{aligned}
12y'' - 5y' - 2y &= 0 \\
12m^2 - 5m - 2 &= 0 \\
\left(m - \frac{2}{3}\right)\left(m + \frac{1}{4}\right) &= 0 \\
y &= c_1 e^{2x/3} + c_2 e^{-x/4}
\end{aligned}$$

**3.3.5 9**

$$\begin{aligned}
y'' + 9y &= 0 \\
m^2 + 9 &= 0 \\
(m + 3i)(m - 3i) &= 0 \\
y &= c_1 \cos 3x + c_2 \sin 3x
\end{aligned}$$

**3.3.6 11**

$$\begin{aligned}
y'' - 4y' + 5y &= 0 \\
m^2 - 4m + 5 &= 0 \\
(m - (2 + i))(m - (2 - i)) &= 0 \\
y &= e^{2x}(c_1 \cos x + c_2 \sin x)
\end{aligned}$$

**3.3.7 13**

$$\begin{aligned}
3y'' + 2y' + y &= 0 \\
3m^2 + 2m + 1 &= 0 \\
\left(m - \left(-\frac{1}{3} + \frac{\sqrt{2}}{3}i\right)\right)\left(m - \left(-\frac{1}{3} - \frac{\sqrt{2}}{3}i\right)\right) &= 0 \\
y &= e^{-x/3}\left(c_1 \cos \frac{\sqrt{2}}{3}x + c_2 \sin \frac{\sqrt{2}}{3}x\right)
\end{aligned}$$

**3.3.8 15**

$$\begin{aligned}
y''' - 4y'' - 5y' &= 0 \\
m^3 - 4m^2 - 5m &= 0 \\
m(m^2 - 4m - 5) &= 0 \\
m(m - 5)(m + 1) &= 0 \\
y &= c_1 + c_2 e^{5x} + c_3 e^{-x}
\end{aligned}$$

**3.3.9 17**

$$\begin{aligned}
 y''' - 5y'' + 3y' + 9y &= 0 \\
 m^3 - 5m^2 + 3m + 9 &= 0 \\
 (m - 3)^2(m + 1) &= 0 \\
 y &= c_1 e^{3x} + c_2 x e^{3x} + c_3 e^{-x}
 \end{aligned}$$

**3.3.10 19**

$$\begin{aligned}
 u''' + u'' - 2u &= 0 \\
 m^3 + m^2 - 2 &= 0 \\
 (x - 1)(x - (-1 + i))(x - (-1 - i)) &= 0 \\
 y &= c_1 e^x + e^{-x}(c_2 \cos x + \sin x)
 \end{aligned}$$

**3.3.11 21**

$$\begin{aligned}
 y''' + 3y'' + 3y' + y &= 0 \\
 m^3 + 3m^2 + 3m + 1 &= 0 \\
 (m + 1)^3 &= 0 \\
 y &= c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}
 \end{aligned}$$

**3.3.12 23**

$$\begin{aligned}
 y^{(4)} + y''' + y'' &= 0 \\
 m^4 + m^3 + m^2 &= 0 \\
 m^2(m^2 + m + 1) &= 0 \\
 m^2 \left( m - \left( -\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \right) \left( m - \left( -\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) \right) &= 0 \\
 y = c_1 + c_2 x + e^{-x/2} \left( c_3 \cos \frac{\sqrt{3}}{2}x + c_4 \sin \frac{\sqrt{3}}{2}x \right)
 \end{aligned}$$

**3.3.13 25**

$$\begin{aligned}
 16y^{(4)} + 24y'' + 9y &= 0 \\
 16m^4 + 24m^2 + 9 &= 0 \\
 (4m^2 + 3)^2 &= 0 \\
 y &= c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x + c_3 x \cos \frac{\sqrt{3}}{2}x + c_4 x \sin \frac{\sqrt{3}}{2}x
 \end{aligned}$$

**3.3.14 27**

$$\begin{aligned}
 u^{(5)} + 5u^{(4)} - 2u^{(3)} - 10u'' + u' + 5u &= 0 \\
 m^5 + 5m^4 - 2m^3 - 10m^2 + m + 5 &= 0 \\
 (m + 5)(m - 1)^2(m + 1)^2 &= 0 \\
 u = c_1 e^{-5r} + c_2 e^r + c_3 r e^r + c_4 e^{-r} + c_5 r e^{-r}
 \end{aligned}$$

**3.3.15 29**

$$\begin{aligned}
 y'' + 16y &= 0 \\
 m^2 + 16 &= 0 \\
 (m + 4i)(m - 4i) &= 0 \\
 y = c_1 \cos 4x + c_2 \sin 4x
 \end{aligned}$$

$$y(0) = 2 = c_1 \cos 4(0) + c_2 \sin 4(0) \Rightarrow c_1 = 2$$

$$y' = -8 \sin 4x + 4c_2 \cos 4x$$

$$y'(0) = -2 = -8 \sin 4(0) + 4c_2 \cos 4(0) \Rightarrow c_2 = -\frac{1}{2}$$

$$y = 2 \cos 4x - \frac{1}{2} \sin 4x$$

**3.3.16 31**

$$\begin{aligned}
 y'' - 4y' - 5y &= 0 \\
 m^2 - 4m - 5 &= 0 \\
 (m - 5)(m + 1) &= 0 \\
 y = c_1 e^{-x} + c_2 e^{5x}
 \end{aligned}$$

$$y(1) = 0 = c_1 e^{-1} + c_2 e^5$$

$$y' = -c_1 e^{-x} + 5c_2 e^{5x}$$

$$y'(1) = 2 = -c_1 e^{-1} + 5c_2 e^5$$

$$2 = 6c_2 e^5 \Rightarrow c_2 = \frac{1}{3e^5}$$

$$0 = c_1 e^{-1} + \frac{1}{3} \Rightarrow c_1 = -\frac{1}{3}e$$

$$y = -\frac{1}{3}e^{1-x} + \frac{1}{3}e^{5x-5}$$

### 3.3.17 33

$$y'' + y' + 2y = 0$$

$$m^2 + m + 2 = 0$$

$$\left(m - \left(-\frac{1}{2} + \frac{\sqrt{7}}{2}i\right)\right) \left(m - \left(-\frac{1}{2} - \frac{\sqrt{7}}{2}i\right)\right) = 0$$

$$y = e^{-x/2} \left( c_1 \cos \frac{\sqrt{7}}{2}x + c_2 \sin \frac{\sqrt{7}}{2}x \right)$$

$$y(0) = 0 = e^{-(0)/2} \left( c_1 \cos \frac{\sqrt{7}}{2}(0) + c_2 \sin \frac{\sqrt{7}}{2}(0) \right) \Rightarrow c_1 = 0$$

$$y = c_2 e^{-x/2} \sin \frac{\sqrt{7}}{2}x$$

$$y' = -\frac{1}{2}c_2 e^{-x/2} \sin \frac{\sqrt{7}}{2}x + \frac{\sqrt{7}}{2}c_2 e^{-x/2} \cos \frac{\sqrt{7}}{2}x$$

$$y'(0) = 0 = \frac{\sqrt{7}}{2}c_2 \Rightarrow c_2 = 0$$

$$y = 0$$

### 3.3.18 37

$$y'' - 10y' + 25y = 0$$

$$m^2 - 10m + 25 = 0$$

$$(m - 5)^2 = 0$$

$$y = c_1 e^{5x} + c_2 x e^{5x}$$

$$y(0) = 1 = c_1 e^{5(0)} + c_2(0)e^{5(0)} \Rightarrow c_1 = 1$$

$$y(1) = 0 = e^5 + c_2 e^5 \Rightarrow c_2 = -1$$

$$y = e^{5x} - x e^{5x}$$



**3.3.19 39**

$$\begin{aligned}y'' + y &= 0 \\m^2 + 1 &= 0 \\(m + i)(m - i) &= 0 \\y &= c_1 \cos x + c_2 \sin x\end{aligned}$$

$$y' = -c_1 \sin x + c_2 \cos x$$

$$y'(0) = 0 = -c_1 \sin(0) + c_2 \cos(0) \Rightarrow c_2 = 0$$

$$y'(\pi/2) = 0 = -c_1 \sin(\pi/2) \Rightarrow c_1 = 0$$

$$y = 0$$

**3.3.20 41**

$$\begin{aligned}y'' - 3y &= 0 \\y &= c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x}\end{aligned}$$

$$y(0) = 1 = c_1 + c_2$$

$$y' = \sqrt{3}c_1 e^{\sqrt{3}x} - \sqrt{3}c_2 e^{-\sqrt{3}x}$$

$$y'(0) = 5 = \sqrt{3}c_1 - \sqrt{3}c_2 \Rightarrow c_1 = \frac{5}{\sqrt{3}} + c_2$$

$$1 = \frac{5}{\sqrt{3}} + 2c_2 \Rightarrow c_2 = \frac{1}{2} - \frac{5}{2\sqrt{3}} \Rightarrow c_1 = \frac{1}{2} + \frac{5}{2\sqrt{3}}$$

$$y = \left(\frac{1}{2} + \frac{5}{2\sqrt{3}}\right) e^{\sqrt{3}x} + \left(\frac{1}{2} - \frac{5}{2\sqrt{3}}\right) e^{-\sqrt{3}x}$$

**3.3.21 49**

$$\begin{aligned}(m - 1)(m - 6) &= 0 \\m^2 - 7m + 6 &= 0 \\y'' - 7y' + 6y &= 0\end{aligned}$$

**3.3.22 51**

$$m(m-3) = 0$$

$$m^2 - 3m = 0$$

$$y'' - 3y' = 0$$

**3.3.23 53**

$$(m-8i)(m+8i) = 0$$

$$m^2 + 64 = 0$$

$$y'' + 64y = 0$$

**3.3.24 55**

$$(m-(1+i))(m-(1-i)) = 0$$

$$(m-1-i)(m-1+i) = 0$$

$$m^2 - m + im - m + 1 - i - im + i + 1 = 0$$

$$m^2 - 2m + 2 = 0$$

$$y'' - 2y' + 2y = 0$$

**3.3.25 57**

$$m^2(m-7) = 0$$

$$m^3 - 7m^2 = 0$$

$$y''' - 7y'' = 0$$

**3.4 Undetermined Coefficients****3.4.1 1**

$$y'' + 3y' + 2y = 0$$

$$m^2 + 3m + 2 = 0$$

$$(m+1)(m+2) = 0$$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

$$y_p = A, y'_p = 0, y''_p = 0$$

$$2A = 6 \Rightarrow A = 3$$

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + 3$$

### 3.4.2 3

$$y'' - 10y' + 25y = 0$$

$$m^2 - 10m + 25 = 0$$

$$(m - 5)^2 = 0$$

$$y_c = c_1 e^{5x} + c_2 x e^{5x}$$

$$y_p = Ax + B, y'_p = A, y''_p = 0$$

$$-10A + 25(Ax + B) = 30x + 3$$

$$25Ax - 10A + 25B = 30x + 3$$

$$25A = 30 \Rightarrow A = \frac{6}{5}$$

$$-10A + 25B = -10\left(\frac{6}{5}\right) + 25B = 3 \Rightarrow B = \frac{3}{5}$$

$$y = y_c + y_p = c_1 e^{5x} + c_2 x e^{5x} + \frac{6}{5}x + \frac{3}{5}$$

### 3.4.3 5

$$\frac{1}{4}y'' + y' + y = 0$$

$$\frac{1}{4}m^2 + m + 1 = 0$$

$$(m + 2)^2 = 0$$

$$y_c = c_1 e^{-2x} + c_2 x e^{-2x}$$

$$y_p = Ax^2 + Bx + C, y'_p = 2Ax + B, y''_p = 2A$$

$$\frac{1}{4}(2A) + 2Ax + B + Ax^2 + Bx + C = x^2 - 2x$$

$$Ax^2 + (2A + B)x + \left(\frac{1}{2}A + B + C\right) = x^2 - 2x$$

$$A = 1, 2A + B = -2, \frac{1}{2}A + B + C = 0$$

$$2(1) + B = -2 \Rightarrow B = -4$$

$$\frac{1}{2}(1) + (-4) + C = 0 \Rightarrow \frac{7}{2}$$

$$y = y_c + y_p = c_1 e^{-2x} + c_2 x e^{-2x} + x^2 - 4x + \frac{7}{2}$$

### 3.4.4 7

$$y'' + 3y = 0$$

$$m^2 + 3 = 0$$

$$(m + i\sqrt{3})(m - i\sqrt{3}) = 0$$

$$y_c = c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x$$

$$y_p = (Ax^2 + Bx + C)e^{3x}$$

$$y'_p = (2Ax + B)e^{3x} + 3(Ax^2 + Bx + C)e^{3x}$$

$$\begin{aligned} y''_p &= 2Ae^{3x} + 3(2Ax + B)e^{3x} + 3(2Ax + B)e^{3x} + 9(Ax^2 + Bx + C)e^{3x} \\ &= 2Ae^{3x} + 6(2Ax + B)e^{3x} + 9(Ax^2 + Bx + C)e^{3x} \end{aligned}$$

$$2A + 6(2Ax + B) + 9(Ax^2 + Bx + C) + 3(Ax^2 + Bx + C) = -48x^2$$

$$12Ax^2 + 12(A + B)x + (2A + 6B + 12C) = -48x^2$$

$$12A = -48 \Rightarrow A = -4$$

$$12(A + B) = 12(-4 + B) = 0 \Rightarrow B = 4$$

$$2A + 6B + 12C = 2(-4) + 6(4) + 12C = 0 \Rightarrow C = -\frac{4}{3}$$

$$y = y_c + y_p = c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x + \left(-4x^2 + 4x - \frac{4}{3}\right)e^{3x}$$

### 3.4.5 9

$$\begin{aligned}y'' - y' &= 0 \\m^2 - m &= 0 \\m(m - 1) &= 0\end{aligned}$$

$$y_c = c_1 + c_2e^x$$

$$y_p = Ax$$

$$-A = -3 \Rightarrow A = 3$$

$$y = y_c + y_p = c_1 + c_2e^x + 3x$$

### 3.4.6 11

$$\begin{aligned}y'' - y' + \frac{1}{4}y &= 0 \\m^2 - m + \frac{1}{4} &= 0 \\ \left(m - \frac{1}{2}\right)^2 &= 0\end{aligned}$$

$$y_c = c_1e^{x/2} + c_2xe^{x/2}$$

$$\begin{aligned}y_p &= y_{p1} + y_{p2} \\&= A + Bx^2e^{x/2}\end{aligned}$$

$$y'_p = 2Bxe^{x/2} + \frac{1}{2}Bx^2e^{x/2}$$

$$\begin{aligned}y''_p &= 2Be^{x/2} + Bxe^{x/2} + Bxe^{x/2} + \frac{1}{4}Bx^2e^{x/2} \\&= 2Be^{x/2} + 2Bxe^{x/2} + \frac{1}{4}Bx^2e^{x/2}\end{aligned}$$

$$\begin{aligned}3 + e^{x/2} &= 2Be^{x/2} + 2Bxe^{x/2} + \frac{1}{4}Bx^2e^{x/2} - 2Bxe^{x/2} - \frac{1}{2}Bx^2e^{x/2} \\&\quad + \frac{1}{4}(A + Bx^2e^{x/2}) \\&= \frac{1}{4}A + 2Be^{x/2}\end{aligned}$$

$$3 = \frac{1}{4}A \Rightarrow A = 12$$

$$1 = 2B \Rightarrow B = \frac{1}{2}$$

$$y = y_c + y_p = c_1 e^{x/2} + c_2 x e^{x/2} + 12 + \frac{1}{2} x^2 e^{x/2}$$

### 3.4.7 13

$$y'' + 4y = 0$$

$$m^2 + 4 = 0$$

$$(m + 2i)(m - 2i) = 0$$

$$y_c = c_1 \cos 2x + c_2 \sin 2x$$

$$y_p = Ax \cos 2x + Bx \sin 2x$$

$$y'_p = A \cos 2x - 2Ax \sin 2x + B \sin 2x + 2Bx \cos 2x$$

$$= A \cos 2x + 2Bx \cos 2x + B \sin 2x - 2Ax \sin 2x$$

$$y''_p = -2A \sin 2x + 2B \cos 2x - 4Bx \sin 2x + 2B \cos 2x - 2A \sin 2x - 4Ax \cos 2x$$

$$= 4B \cos 2x - 4Ax \cos 2x - 4A \sin 2x - 4Bx \sin 2x$$

$$\begin{aligned} 3 \sin 2x &= 4B \cos 2x - 4Ax \cos 2x - 4A \sin 2x - 4Bx \sin 2x \\ &\quad + 4(Ax \cos 2x + Bx \sin 2x) \\ &= 4B \cos 2x - 4A \sin 2x \end{aligned}$$

$$3 = -4A \Rightarrow A = -\frac{3}{4}$$

$$0 = 4B \Rightarrow B = 0$$

$$y = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x - \frac{3}{4} x \cos 2x$$

### 3.4.8 15

$$\begin{aligned}y'' + y &= 0 \\m^2 + 1 &= 0 \\(m + i)(m - i) &= 0\end{aligned}$$

$$y_c = c_1 \cos x + c_2 \sin x$$

$$\begin{aligned}y_p &= (Ax^2 + Bx) \cos x + (Cx^2 + Ex) \sin x \\y'_p &= (2Ax + B) \cos x - (Ax^2 + Bx) \sin x + (2Cx + E) \sin x + (Cx^2 + Ex) \cos x \\&= B \cos x + (2A + E)x \cos x + Cx^2 \cos x + E \sin x + (2C - B)x \sin x \\&\quad - Ax^2 \sin x \\y''_p &= -B \sin x + (2A + E) \cos x - (2A + E)x \sin x + 2Cx \cos x - Cx^2 \sin x \\&\quad + E \cos x + (2C - B) \sin x + (2C - B)x \cos x - 2Ax \sin x - Ax^2 \cos x \\&= 2(A + E) \cos x + (4C - B)x \cos x - Ax^2 \cos x + 2(C - B) \sin x \\&\quad - (4A + E)x \sin x - Cx^2 \sin x\end{aligned}$$

$$\begin{aligned}2x \sin x &= 2(A + E) \cos x + (4C - B)x \cos x - Ax^2 \cos x + 2(C - B) \sin x \\&\quad - (4A + E)x \sin x - Cx^2 \sin x + (Ax^2 + Bx) \cos x \\&\quad + (Cx^2 + Ex) \sin x \\&= 2(A + E) \cos x + 4Cx \cos x + 2(C - B) \sin x - 4Ax \sin x\end{aligned}$$

$$\begin{aligned}2 &= -4A \Rightarrow A = -\frac{1}{2} \\0 &= 2(A + E) \Rightarrow E = \frac{1}{2} \\0 &= 4C \Rightarrow C = 0 \\0 &= 2(C - B) \Rightarrow B = 0\end{aligned}$$

$$y = y_c + y_p = c_1 \cos x + c_2 \sin x - \frac{1}{2}x^2 \cos x + \frac{1}{2}x \sin x$$

### 3.4.9 17

$$\begin{aligned}y'' - 2y' + 5y &= 0 \\m^2 - 2m + 5 &= 0 \\(m - (1 + 2i))(m - (1 - 2i)) &= 0\end{aligned}$$

$$y_c = c_1 e^x \cos 2x + c_2 e^x \sin 2x$$

$$\begin{aligned}
y_p &= A x e^x \cos 2x + B x e^x \sin 2x \\
y'_p &= A e^x \cos 2x + A x e^x \cos 2x - 2A x e^x \sin 2x + B e^x \sin 2x + B x e^x \sin 2x \\
&\quad + 2B x e^x \cos 2x \\
&= A e^x \cos 2x + (A + 2B) x e^x \cos 2x + B e^x \sin 2x + (B - 2A) x e^x \sin 2x \\
y''_p &= A e^x \cos 2x - 2A e^x \sin 2x + (A + 2B) e^x \cos 2x + (A + 2B) x e^x \cos 2x \\
&\quad - 2(A + 2B) x e^x \sin 2x + B e^x \sin 2x + 2B e^x \cos 2x + (B - 2A) e^x \sin 2x \\
&\quad + (B - 2A) x e^x \sin 2x + 2(B - 2A) x e^x \cos 2x \\
&= (2A + 4B) e^x \cos 2x + (4B - 3A) x e^x \cos 2x + (2B - 4A) e^x \sin 2x \\
&\quad - (4A + 3B) x e^x \sin 2x
\end{aligned}$$

$$\begin{aligned}
e^x \cos 2x &= (2A + 4B) e^x \cos 2x + (4B - 3A) x e^x \cos 2x + (2B - 4A) e^x \sin 2x \\
&\quad - (4A + 3B) x e^x \sin 2x - 2(A e^x \cos 2x + (A + 2B) x e^x \cos 2x \\
&\quad + B e^x \sin 2x + (B - 2A) x e^x \sin 2x) + 5(A x e^x \cos 2x \\
&\quad + B x e^x \sin 2x) \\
&= 4B e^x \cos 2x - 4A e^x \sin 2x
\end{aligned}$$

$$\begin{aligned}
1 &= 4B \Rightarrow B = \frac{1}{4} \\
0 &= -4A \Rightarrow A = 0
\end{aligned}$$

$$y = y_c + y_p = c_1 e^x \cos 2x + c_2 e^x \sin 2x + \frac{1}{4} x e^x \sin 2x$$

### 3.4.10 21

$$\begin{aligned}
y''' - 6y'' &= 0 \\
m^3 - 6m^2 &= 0 \\
m^2(m - 6) &= 0
\end{aligned}$$

$$y_c = c_1 + c_2 x + c_3 e^{6x}$$



$$\begin{aligned}
y_p &= Ax^2 + B \cos x + C \sin x \\
y'_p &= 2Ax - B \sin x + C \cos x \\
y''_p &= 2A - B \cos x - C \sin x \\
y'''_p &= B \sin x - C \cos x
\end{aligned}$$

$$\begin{aligned}
3 - \cos x &= B \sin x - C \cos x - 6(2A - B \cos x - C \sin x) \\
&= -12A + (B + 6C) \sin x + (6B - C) \cos x
\end{aligned}$$

$$3 = -12A \Rightarrow A = -\frac{1}{4}$$

$$B + 6C + 6(6B - C) = 0 + 6(-1) \Rightarrow 37B = -6 \Rightarrow B = -\frac{6}{37}$$

$$B + 6C = 0 \Rightarrow C = \frac{1}{37}$$

$$y = y_c + y_p = c_1 + c_2 x + c_3 e^{6x} - \frac{1}{4}x^2 - \frac{6}{37} \cos x + \frac{1}{37} \sin x$$

### 3.4.11 27

$$y'' + 4y = 0$$

$$m^2 + 4 = 0$$

$$(m + 2i)(m - 2i) = 0$$

$$y_c = c_1 \cos 2x + c_2 \sin 2x$$

$$y_p = A$$

$$y'_p = 0$$

$$y''_p = 0$$

$$0 + 4A = -2$$

$$A = -\frac{1}{2}$$

$$y = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{2}$$

$$\begin{aligned}
y\left(\frac{\pi}{8}\right) &= \frac{1}{2} = c_1 \cos \frac{\pi}{4} + c_2 \sin \frac{\pi}{4} - \frac{1}{2} \\
&= \frac{\sqrt{2}c_1}{2} + \frac{\sqrt{2}c_2}{2} - \frac{1}{2} \\
2 &= \sqrt{2}c_1 + \sqrt{2}c_2
\end{aligned}$$

$$\begin{aligned}
y'\left(\frac{\pi}{8}\right) &= 2 = -2c_1 \sin \frac{\pi}{4} + 2c_2 \cos \frac{\pi}{4} \\
&= -\sqrt{2}c_1 + \sqrt{2}c_2
\end{aligned}$$

$$\begin{aligned}
4 &= 2\sqrt{2}c_2 \\
c_2 &= \frac{4}{2\sqrt{2}} \\
&= \sqrt{2}
\end{aligned}$$

$$\begin{aligned}
2 &= \sqrt{2}c_1 + 2 \\
c_1 &= 0
\end{aligned}$$

$$y = \sqrt{2} \sin 2x - \frac{1}{2}$$

### 3.4.12 29

$$\begin{aligned}
5y'' + y' &= 0 \\
5m^2 + m &= 0 \\
m(5m + 1) &= 0
\end{aligned}$$

$$y_c = c_1 + c_2 e^{-x/5}$$

$$\begin{aligned}
y_p &= Ax^2 + Bx \\
y'_p &= 2Ax + B \\
y''_p &= 2A
\end{aligned}$$

$$-6x = 10A + 2Ax + B$$

$$-6 = 2A \Rightarrow A = -3$$

$$0 = 10A + B = 10(-3) + B \Rightarrow B = 30$$

$$y = y_c + y_p = c_1 + c_2 e^{-x/5} - 3x^2 + 30x$$

$$\begin{aligned} y(0) = 0 &= c_1 + c_2 e^{-(0)/5} - 3(0)^2 + 30(0) \\ &= c_1 + c_2 \end{aligned}$$

$$\begin{aligned} y'(0) = -10 &= -\frac{1}{5}c_2 e^{-(0)/5} - 6(0) + 30 \\ &= -\frac{1}{5}c_2 + 30 \\ c_2 &= 200 \end{aligned}$$

$$c_1 = -200$$

$$y = -200 + 200e^{-x/5} - 3x^2 + 30x$$

### 3.4.13 31

$$y'' + 4y' + 5y = 0$$

$$m^2 + 4m + 5 = 0$$

$$(m - (-2 + i))(m - (-2 - i)) = 0$$

$$y_c = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x$$

$$y_p = Ae^{-4x}$$

$$y'_p = -4Ae^{-4x}$$

$$y''_p = 16Ae^{-4x}$$

$$\begin{aligned}
35e^{-4x} &= 16Ae^{-4x} + 4(-4Ae^{-4x}) + 5(Ae^{-4x}) \\
&= 5Ae^{-4x} \\
A &= 7
\end{aligned}$$

$$y = y_c + y_p = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x + 7e^{-4x}$$

$$\begin{aligned}
y(0) = -3 &= c_1 e^{-2(0)} \cos(0) + c_2 e^{-2(0)} \sin(0) + 7e^{-4(0)} \\
&= c_1 + 7 \\
c_1 &= -10
\end{aligned}$$

$$\begin{aligned}
y'(0) = 1 &= -2c_1 e^{-2(0)} \cos(0) - c_1 e^{-2(0)} \sin(0) - 2c_2 e^{-2(0)} \sin(0) \\
&\quad - 28e^{-4(0)} + c_2 e^{-2(0)} \cos(0) \\
&= -2(-10) - 28 + c_2 \\
c_2 &= 9
\end{aligned}$$

$$y = -10e^{-2x} \cos x + 9e^{-2x} \sin x + 7e^{-4x}$$

### 3.4.14 37

$$\begin{aligned}
y'' + y &= 0 \\
m^2 + 1 &= 0 \\
(m+i)(m-i) &= 0
\end{aligned}$$

$$y_c = c_1 \cos x + c_2 \sin x$$

$$\begin{aligned}
y_p &= Ax^2 + Bx + C \\
y'_p &= 2Ax + B \\
y''_p &= 2A
\end{aligned}$$

$$x^2 + 1 = 2A + Ax^2 + Bx + C$$

$$\begin{aligned}
A &= 1 \\
B &= 0 \\
1 = 2A + C = 2(1) + C &\Rightarrow C = -1
\end{aligned}$$

$$y = y_c + y_p = c_1 \cos x + c_2 \sin x + x^2 - 1$$

$$\begin{aligned} y(0) = 5 &= c_1 \cos(0) + c_2 \sin(0) + (0)^2 - 1 \\ &= c_1 - 1 \\ c_1 &= 6 \end{aligned}$$

$$\begin{aligned} y(1) = 0 &= 6 \cos(1) + c_2 \sin(1) + (1)^2 - 1 \\ c_2 &= -6 \cot 1 \end{aligned}$$

$$y = 6 \cos x - 6(\cot 1) \sin x + x^2 - 1$$

### 3.5 Variation of Parameters

#### 3.5.1 1

$$\begin{aligned} y'' + y &= 0 \\ m^2 + 1 &= 0 \\ (m + i)(m - i) &= 0 \end{aligned}$$

$$\begin{aligned} W &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \\ &= \cos^2 x + \sin^2 x \\ &= 1 \\ u'_1 &= \frac{W_1}{W} \\ &= \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} \\ &= \tan x \\ u_1 &= \ln |\cos x| \\ u'_2 &= \frac{W_2}{W} \\ &= \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} \\ &= 1 \\ u_2 &= x \end{aligned}$$

$$y = y_c + y_p = c_1 \cos x + c_2 \sin x + \ln |\cos x| \cos x + x \sin x$$

### 3.5.2 3

$$\begin{aligned}y'' + y &= 0 \\m^2 + 1 &= 0 \\(m + i)(m - i) &= 0\end{aligned}$$

$$y_c = c_1 \cos x + c_2 \sin x$$

$$\begin{aligned}W &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \\&= \cos^2 x + \sin^2 x \\&= 1 \\u'_1 &= \frac{W_1}{W} \\&= \begin{vmatrix} 0 & \sin x \\ \sin x & \cos x \end{vmatrix} \\&= -\sin^2 x \\u_1 &= \frac{1}{4} \sin 2x - \frac{x}{2} \\u'_2 &= \frac{W_2}{W} \\&= \begin{vmatrix} \cos x & 0 \\ -\sin x & \sin x \end{vmatrix} \\&= (\cos x) \sin x \\&= \frac{1}{2} \sin 2x \\u_2 &= -\frac{1}{4} \cos 2x\end{aligned}$$

$$\begin{aligned}y &= y_c + y_p \\&= c_1 \cos x + c_2 \sin x + \left( \frac{1}{4} \sin 2x - \frac{x}{2} \right) \cos x - \frac{1}{4} (\cos 2x) \sin x \\&= c_1 \cos x + c_2 \sin x - \frac{1}{2} x \cos x\end{aligned}$$

### 3.5.3 7

$$\begin{aligned}y'' - y &= 0 \\m^2 - 1 &= 0 \\(m + 1)(m - 1) &= 0\end{aligned}$$

$$y_c = c_1 e^x + c_2 e^{-x}$$

$$\begin{aligned} W &= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} \\ &= -2 \\ u_1' &= \frac{W_1}{W} \\ &= -\frac{1}{2} \begin{vmatrix} 0 & e^{-x} \\ \cosh x & -e^{-x} \end{vmatrix} \\ &= \frac{1}{2} e^{-x} \cosh x \\ &= \frac{1 + e^{-2x}}{4} \\ u_1 &= \frac{1}{4} x - \frac{1}{8} e^{-2x} \\ u_2' &= \frac{W_2}{W} \\ &= -\frac{1}{2} \begin{vmatrix} e^x & 0 \\ e^x & \cosh x \end{vmatrix} \\ &= -\frac{1}{2} e^x \cosh x \\ &= -\frac{e^{2x} + 1}{4} \\ u_2 &= -\frac{1}{8} e^{2x} - \frac{1}{4} x \end{aligned}$$

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 e^x + c_2 e^{-x} + \left( \frac{1}{4} x - \frac{1}{8} e^{-2x} \right) e^x - \left( \frac{1}{8} e^{2x} + \frac{1}{4} x \right) e^{-x} \\ &= c_1 e^x + c_2 e^{-x} + \frac{1}{4} x e^x - \frac{1}{4} x e^{-x} \\ &= c_1 e^x + c_2 e^{-x} + \frac{1}{2} x \sinh x \end{aligned}$$

### 3.5.4 9

$$\begin{aligned} y'' - 9y &= 0 \\ m^2 - 9 &= 0 \\ (m+3)(m-3) &= 0 \end{aligned}$$

$$y_c = c_1 e^{-3x} + c_2 e^{3x}$$

$$W = \begin{vmatrix} e^{-3x} & e^{3x} \\ -3e^{-3x} & 3e^{3x} \end{vmatrix}$$

$$= 6$$

$$u_1' = \frac{W_1}{W}$$

$$= \frac{1}{6} \begin{vmatrix} 0 & e^{3x} \\ \frac{9x}{e^{3x}} & 3e^{3x} \end{vmatrix}$$

$$= -\frac{3}{2}x$$

$$u_1 = -\frac{3}{4}x^2$$

$$u_2' = \frac{W_2}{W}$$

$$= \frac{1}{6} \begin{vmatrix} e^{-3x} & 0 \\ -3e^{-3x} & \frac{9x}{e^{3x}} \end{vmatrix}$$

$$= \frac{3}{2}x e^{-6x}$$

$$u_2 = -\frac{1}{24}e^{-6x} - \frac{1}{4}x e^{-6x}$$

$$y = y_c + y_p$$

$$= c_1 e^{-3x} + c_2 e^{3x} - \frac{3}{4}x^2 e^{-3x} - \frac{1}{24}e^{-6x}(1 + 6x)e^{3x}$$

$$= c_1 e^{-3x} + c_2 e^{3x} - \frac{1}{4}x e^{-3x} - \frac{3}{4}x^2 e^{-3x}$$

### 3.5.5 19

$$4y'' - y = 0$$

$$4m^2 - 1 = 0$$

$$(2m + 1)(2m - 1) = 0$$

$$y_c = c_1 e^{-x/2} + c_2 e^{x/2}$$



$$\begin{aligned}
W &= \begin{vmatrix} e^{-x/2} & e^{x/2} \\ -\frac{1}{2}e^{-x/2} & \frac{1}{2}e^{x/2} \end{vmatrix} \\
&= 1 \\
u_1' &= \frac{W_1}{W} \\
&= \begin{vmatrix} 0 & e^{x/2} \\ \frac{1}{4}xe^{x/2} & \frac{1}{2}e^{x/2} \end{vmatrix} \\
&= -\frac{1}{4}xe^x \\
u_1 &= \frac{1}{4}e^x - \frac{1}{4}xe^x \\
u_2' &= \frac{W_2}{W} \\
&= \begin{vmatrix} e^{-x/2} & 0 \\ -\frac{1}{2}e^{-x/2} & \frac{1}{4}xe^{x/2} \end{vmatrix} \\
&= \frac{1}{4}x \\
u_2 &= \frac{1}{8}x^2
\end{aligned}$$

$$\begin{aligned}
y &= y_c + y_p \\
&= c_1e^{-x/2} + c_2e^{x/2} + \left(\frac{1}{4}e^x - \frac{1}{4}xe^x\right)e^{-x/2} + \frac{1}{8}x^2e^{x/2} \\
&= c_1e^{-x/2} + c_2e^{x/2} - \frac{1}{4}xe^{x/2} + \frac{1}{8}x^2e^{x/2}
\end{aligned}$$

$$\begin{aligned}
y(0) = 1 &= c_1e^{-(0)/2} + c_2e^{(0)/2} - \frac{1}{4}(0)e^{(0)/2} + \frac{1}{8}(0)^2e^{(0)/2} \\
&= c_1 + c_2
\end{aligned}$$

$$\begin{aligned}
y'(0) = 0 &= -\frac{1}{2}c_1e^{-(0)/2} + \frac{1}{2}c_2e^{(0)/2} - \frac{1}{4}e^{(0)/2} - \frac{1}{8}(0)e^{(0)/2} + \frac{1}{4}(0)e^{(0)/2} \\
&\quad + \frac{1}{16}(0)^2e^{(0)/2} \\
&= -\frac{1}{2}c_1 + \frac{1}{2}c_2 - \frac{1}{4} \\
\frac{1}{2} &= -c_1 + c_2
\end{aligned}$$

$$\frac{3}{2} = 2c_2 \Rightarrow c_2 = \frac{3}{4} \Rightarrow c_1 = \frac{1}{4}$$

$$y = \frac{1}{4}e^{-x/2} + \frac{3}{4}e^{x/2} - \frac{1}{4}xe^{x/2} + \frac{1}{8}x^2e^{x/2}$$

### 3.5.6 21

$$y'' + 2y' - 8y = 0$$

$$m^2 + 2m - 8 = 0$$

$$(m + 4)(m - 2) = 0$$

$$y_c = c_1e^{-4x} + c_2e^{2x}$$

$$W = \begin{vmatrix} e^{-4x} & e^{2x} \\ -4e^{-4x} & 2e^{2x} \end{vmatrix}$$

$$= 6e^{-2x}$$

$$u'_1 = \frac{W_1}{W}$$

$$= \frac{\begin{vmatrix} 0 & e^{2x} \\ 2e^{-2x} - e^{-x} & 2e^{2x} \end{vmatrix}}{6e^{-2x}}$$

$$= \frac{e^x - 2}{6e^{-2x}}$$

$$= \frac{1}{6}e^{3x} - \frac{1}{3}e^{2x}$$

$$u_1 = \frac{1}{18}e^{3x} - \frac{1}{6}e^{2x}$$

$$u'_2 = \frac{W_2}{W}$$

$$= \frac{\begin{vmatrix} e^{-4x} & 0 \\ -4e^{-4x} & 2e^{-2x} - e^{-x} \end{vmatrix}}{6e^{-2x}}$$

$$= \frac{2e^{-6x} - e^{-5x}}{6e^{-2x}}$$

$$= \frac{1}{3}e^{-4x} - \frac{1}{6}e^{-3x}$$

$$u_2 = -\frac{1}{12}e^{-4x} + \frac{1}{18}e^{-3x}$$

$$\begin{aligned}
y &= y_c + y_p \\
&= c_1 e^{-4x} + c_2 e^{2x} + \left( \frac{1}{18} e^{3x} - \frac{1}{6} e^{2x} \right) e^{-4x} + \left( \frac{1}{18} e^{-3x} - \frac{1}{12} e^{-4x} \right) e^{2x} \\
&= c_1 e^{-4x} + c_2 e^{2x} + \frac{1}{9} e^{-x} - \frac{1}{4} e^{-2x}
\end{aligned}$$

$$\begin{aligned}
y(0) = 1 &= c_1 e^{-4(0)} + c_2 e^{2(0)} + \frac{1}{9} e^{-(0)} - \frac{1}{4} e^{-2(0)} \\
&= c_1 + c_2 + \frac{1}{9} - \frac{1}{4} \\
\frac{41}{36} &= c_1 + c_2
\end{aligned}$$

$$\begin{aligned}
y'(0) = 0 &= -4c_1 e^{-4(0)} + 2c_2 e^{2(0)} - \frac{1}{9} e^{-(0)} + \frac{1}{2} e^{-2(0)} \\
&= -4c_1 + 2c_2 - \frac{1}{9} + \frac{1}{2} \\
-\frac{7}{18} &= -4c_1 + 2c_2
\end{aligned}$$

$$\frac{41}{9} - \frac{7}{18} = 6c_2 \Rightarrow c_2 = \frac{25}{36} \Rightarrow c_1 = \frac{16}{36} = \frac{4}{9}$$

$$y = \frac{4}{9} e^{-4x} + \frac{25}{36} e^{2x} + \frac{1}{9} e^{-x} - \frac{1}{4} e^{-2x}$$

### 3.5.7 27

$$y'' + \frac{1}{x} y' + \left( 1 - \frac{1}{4x^2} \right) y = \frac{1}{\sqrt{x}}$$

$$\begin{aligned}
W &= \begin{vmatrix} x^{-1/2} \cos x & x^{-1/2} \sin x \\ -\frac{1}{2}x^{-3/2} \cos x - x^{-1/2} \sin x & -\frac{1}{2}x^{-3/2} \sin x + x^{-1/2} \cos x \end{vmatrix} \\
&= x^{-1/2} \cos x \left( -\frac{1}{2}x^{-3/2} \sin x + x^{-1/2} \cos x \right) \\
&\quad - x^{-1/2} \sin x \left( -\frac{1}{2}x^{-3/2} \cos x - x^{-1/2} \sin x \right) \\
&= -\frac{1}{2}x^{-2}(\cos x) \sin x + x^{-1} \cos^2 x + \frac{1}{2}x^{-2}(\cos x) \sin x + x^{-1} \sin^2 x \\
&= x^{-1} \\
u_1' &= \frac{W_1}{W} \\
&= x \begin{vmatrix} 0 & x^{-1/2} \sin x \\ x^{-1/2} & -\frac{1}{2}x^{-3/2} \sin x + x^{-1/2} \cos x \end{vmatrix} \\
&= -\sin x \\
u_1 &= \cos x \\
u_2' &= x \begin{vmatrix} x^{-1/2} \cos x & 0 \\ -\frac{1}{2}x^{-3/2} \cos x - x^{-1/2} \sin x & x^{-1/2} \end{vmatrix} \\
&= \cos x \\
u_2 &= \sin x
\end{aligned}$$

$$\begin{aligned}
y &= c_1 x^{-1/2} \cos x + c_2 x^{-1/2} \sin x + x^{-1/2} \cos^2 x + x^{-1/2} \sin^2 x \\
&= c_1 x^{-1/2} \cos x + c_2 x^{-1/2} \sin x + x^{-1/2}
\end{aligned}$$

### 3.5.8 29

$$\begin{aligned}
y''' + y' &= 0 \\
m^3 + m &= 0 \\
m(m^2 + 1) &= 0 \\
m(m+i)(m-i) &= 0
\end{aligned}$$

$$y_c = c_1 + c_2 \cos x + c_3 \sin x$$

$$\begin{aligned}
W &= \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} \\
&= \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} - (\cos x) \begin{vmatrix} 0 & \cos x \\ 0 & -\sin x \end{vmatrix} + (\sin x) \begin{vmatrix} 0 & -\sin x \\ 0 & -\cos x \end{vmatrix} \\
&= \sin^2 x + \cos^2 x \\
&= 1
\end{aligned}$$

$$\begin{aligned}
u_1' &= \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \tan x & -\cos x & -\sin x \end{vmatrix} \\
&= -(\cos x) \begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix} + (\sin x) \begin{vmatrix} 0 & -\sin x \\ \tan x & -\cos x \end{vmatrix} \\
&= (\cos^2 x) \tan x + (\sin^2 x) \tan x \\
&= \tan x
\end{aligned}$$

$$u_1 = -\ln |\cos x|$$

$$\begin{aligned}
u_2' &= \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \tan x & -\sin x \end{vmatrix} \\
&= \begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix} \\
&= -\sin x
\end{aligned}$$

$$u_2 = \cos x$$

$$\begin{aligned}
u_3' &= \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \tan x \end{vmatrix} \\
&= \begin{vmatrix} -\sin x & 0 \\ -\cos x & \tan x \end{vmatrix} \\
&= -(\sin x) \tan x \\
&= -(\sin^2 x) \sec x \\
&= -(1 - \cos^2) \sec x \\
&= \cos x - \sec x
\end{aligned}$$

$$u_3 = \sin x - \ln |\sec x + \tan x|$$

$$\begin{aligned}
y &= c_1 + c_2 \cos x + c_3 \sin x - \ln |\cos x| + \cos^2 x + \sin^2 x \\
&\quad - (\sin x) \ln |\sec x + \tan x| \\
&= c_1 + c_2 \cos x + c_3 \sin x - \ln |\cos x| - (\sin x) \ln |\sec x + \tan x|
\end{aligned}$$

## 3.6 Cauchy-Euler Equations

### 3.6.1 1

$$\begin{aligned}x^2 y'' - 2y &= 0 \\m(m-1) - 2 &= 0 \\m^2 - m - 2 &= 0 \\(m-2)(m+1) &= 0\end{aligned}$$

$$y = c_1 x^2 + c_2 x^{-1}$$

### 3.6.2 3

$$\begin{aligned}xy'' + y' &= 0 \\x^2 y'' + xy' &= 0 \\m(m-1) + m &= 0 \\m^2 - m + m &= 0 \\m^2 &= 0\end{aligned}$$

$$y = c_1 + c_2 \ln x$$

### 3.6.3 5

$$\begin{aligned}x^2 y'' + xy' + 4y &= 0 \\m(m-1) + m + 4 &= 0 \\m^2 - m + m + 4 &= 0 \\m^2 + 4 &= 0 \\(m+2i)(m-2i) &= 0\end{aligned}$$

$$y = c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)$$

### 3.6.4 7

$$\begin{aligned}x^2 y'' - 3xy' - 2y &= 0 \\m(m-1) - 3m - 2 &= 0 \\m^2 - m - 3m - 2 &= 0 \\m^2 - 4m - 2 &= 0 \\(m - (2 + \sqrt{6}))(m - (2 - \sqrt{6})) &= 0\end{aligned}$$

$$y = c_1 x^{2+\sqrt{6}} + c_2 x^{2-\sqrt{6}}$$

**3.6.5 9**

$$\begin{aligned} 25x^2y'' + 25xy' + y &= 0 \\ 25m(m-1) + 25m + 1 &= 0 \\ 25m^2 - 25m + 25m + 1 &= 0 \\ 25m^2 + 1 &= 0 \\ (m + \frac{1}{5}i)(m - \frac{1}{5}i) &= 0 \end{aligned}$$

$$y = c_1 \cos(\frac{1}{5} \ln x) + c_2 \sin(\frac{1}{5} \ln x)$$

**3.6.6 11**

$$\begin{aligned} x^2y'' + 5xy' + 4y &= 0 \\ m(m-1) + 5m + 4 &= 0 \\ m^2 - m + 5m + 4 &= 0 \\ m^2 + 4m + 2 &= 0 \\ (m+2)^2 &= 0 \end{aligned}$$

$$y = c_1 x^{-2} + c_2 x^{-2} \ln x$$

**3.6.7 13**

$$\begin{aligned} 3x^2y'' + 6xy' + y &= 0 \\ 3m(m-1) + 6m + 1 &= 0 \\ 3m^2 - 3m + 6m + 1 &= 0 \\ 3m^2 + 3m + 1 &= 0 \\ \left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{6}\right)\right) \left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{6}\right)\right) &= 0 \\ y = c_1 x^{-1/2} \cos\left(\frac{\sqrt{3}}{6} \ln x\right) + c_2 x^{-1/2} \sin\left(\frac{\sqrt{3}}{6} \ln x\right) \end{aligned}$$

### 3.6.8 15

$$\begin{aligned}
 x^3 y''' - 6y &= 0 \\
 m(m-1)(m-2) - 6 &= 0 \\
 (m^2 - m)(m-2) - 6 &= 0 \\
 m^3 - 2m^2 - m^2 + 2m - 6 &= 0 \\
 m^3 - 3m^2 + 2m - 6 &= 0 \\
 (m-3)(m-i\sqrt{2})(m+i\sqrt{2}) &= 0
 \end{aligned}$$

$$y = c_1 x^3 + c_2 \cos(\sqrt{2} \ln x) + c_3 \sin(\sqrt{2} \ln x)$$

### 3.6.9 17

$$\begin{aligned}
 xy^{(4)} + 6y''' &= 0 \\
 m(m-1)(m-2)(m-3) + 6m(m-1)(m-2) &= 0 \\
 (m^2 - m)(m^2 - 5m + 6) + (6m^2 - 6m)(m-2) &= 0 \\
 m^4 - 5m^3 + 6m^2 - m^3 + 5m^2 - 6m + 6m^3 - 12m^2 - 6m^2 + 12m &= 0 \\
 m^4 - 7m^2 + 6m &= 0 \\
 m(m^3 - 7m + 6) &= 0 \\
 m(m+3)(m-1)(m-2) &= 0
 \end{aligned}$$

$$y = c_1 + c_2 x^{-3} + c_3 x + c_4 x^2$$

### 3.6.10 19

$$\begin{aligned}
 xy'' - 4y' &= 0 \\
 m(m-1) - 4m &= 0 \\
 m^2 - m - 4m &= 0 \\
 m^2 - 5m &= 0 \\
 m(m-5) &= 0
 \end{aligned}$$

$$y_c = c_1 + c_2 x^5$$



$$\begin{aligned}
W &= \begin{vmatrix} 1 & x^5 \\ 0 & 5x^4 \end{vmatrix} \\
&= 5x^4 \\
u_1' &= \frac{\begin{vmatrix} 0 & x^5 \\ x^3 & 5x^4 \end{vmatrix}}{5x^4} \\
&= -\frac{x^8}{5x^4} \\
&= -\frac{1}{5}x^4 \\
u_1 &= -\frac{1}{25}x^5 \\
u_2' &= \frac{\begin{vmatrix} 1 & 0 \\ 0 & x^3 \end{vmatrix}}{5x^4} \\
&= \frac{x^3}{5x^4} \\
&= \frac{1}{5x} \\
u_2 &= \frac{1}{5} \ln x
\end{aligned}$$

$$\begin{aligned}
y &= y_c + y_p \\
&= c_1 + c_2 x^5 - \frac{1}{25}x^5 + \frac{1}{5}x^5 \ln x \\
&= c_1 + c_2 x^5 + \frac{1}{5}x^5 \ln x
\end{aligned}$$

### 3.6.11 21

$$\begin{aligned}
x^2 y'' - xy' + y &= 0 \\
m(m-1) - m + 1 &= 0 \\
m^2 - m - m + 1 &= 0 \\
m^2 - 2m + 1 &= 0 \\
(m-1)^2 &= 0
\end{aligned}$$

$$y_c = c_1 x + c_2 x \ln x$$

$$\begin{aligned}
W &= \begin{vmatrix} x & x \ln x \\ 1 & 1 + \ln x \end{vmatrix} \\
&= x \\
u_1' &= x^{-1} \begin{vmatrix} 0 & x \ln x \\ 2x^{-1} & 1 + \ln x \end{vmatrix} \\
&= -2x^{-1} \ln x \\
u_1 &= -\ln^2 x \\
u_2' &= x^{-1} \begin{vmatrix} x & 0 \\ 1 & 2x^{-1} \end{vmatrix} \\
&= 2x^{-1} \\
u_2 &= 2 \ln x \\
y &= y_c + y_p \\
&= c_1 x + c_2 x \ln x + x \ln^2 x
\end{aligned}$$

**3.6.12 23**

$$\begin{aligned}
x^2 y'' + xy' - y &= 0 \\
m(m-1) + m - 1 &= 0 \\
m^2 - m + m - 1 &= 0 \\
m^2 - 1 &= 0 \\
(m+1)(m-1) &= 0
\end{aligned}$$

$$y_c = c_1 x^{-1} + c_2 x$$

$$\begin{aligned}
W &= \begin{vmatrix} x^{-1} & x \\ -x^{-2} & 1 \end{vmatrix} \\
&= x^{-1} + x^{-1} \\
&= 2x^{-1} \\
u_1' &= \frac{1}{2}x \begin{vmatrix} 0 & x \\ x^{-2} \ln x & 1 \end{vmatrix} \\
&= -\frac{1}{2} \ln x \\
u_1 &= -\frac{1}{2}x((\ln x) - 1) \\
&= \frac{1}{2}x - \frac{1}{2}x \ln x \\
u_2' &= \frac{1}{2}x \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & x^{-2} \ln x \end{vmatrix} \\
&= \frac{1}{2}x^{-2} \ln x \\
u_2 &= -\frac{1 + \ln x}{2x} \\
y &= y_c + y_p \\
&= c_1x^{-1} + c_2x + \frac{1}{2} - \frac{1}{2} \ln x - \frac{1}{2} - \frac{1}{2} \ln x \\
&= c_1x^{-1} + c_2x - \ln x
\end{aligned}$$

### 3.6.13 25

$$\begin{aligned}
x^2y'' + 3xy' &= 0 \\
m(m-1) + 3m &= 0 \\
m^2 - m + 3m &= 0 \\
m^2 + 2m &= 0 \\
m(m+2) &= 0
\end{aligned}$$

$$y = c_1 + c_2x^{-2}$$

$$\begin{aligned}
y(1) &= 0 = c_1 + c_2(1)^{-2} \\
&= c_1 + c_2 \\
y'(1) &= 4 = -2c_2(1)^{-3} \\
&= -2c_2 \\
c_2 &= -2 \\
c_1 &= 2 \\
y &= 2 - 2x^{-2}
\end{aligned}$$

### 3.6.14 27

$$\begin{aligned}
x^2 y'' + xy' + y &= 0 \\
m(m-1) + m + 1 &= 0 \\
m^2 - m + m + 1 &= 0 \\
m^2 + 1 &= 0 \\
(m+i)(m-i) &= 0
\end{aligned}$$

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$$

$$\begin{aligned}
y(1) &= 1 = c_1 \cos(\ln 1) + c_2 \sin(\ln 1) \\
&= c_1 \cos 0 + c_2 \sin 0 \\
&= c_1 \\
y' &= -x^{-1} \sin(\ln x) + c_2 x^{-1} \cos(\ln x) \\
y'(1) &= 2 = -1^{-1} \sin(\ln 1) + c_2 1^{-1} \cos(\ln 1) \\
&= c_2 \\
y &= y_c + y_p \\
&= \cos(\ln x) + 2 \sin(\ln x)
\end{aligned}$$

### 3.6.15 29

$$\begin{aligned}
xy'' + y' &= 0 \\
m(m-1) + m &= 0 \\
m^2 - m + m &= 0 \\
m^2 &= 0
\end{aligned}$$

$$y_c = c_1 + c_2 \ln x$$

$$\begin{aligned}
W &= \begin{vmatrix} 1 & \ln x \\ 0 & x^{-1} \end{vmatrix} \\
&= x^{-1} \\
u_1' &= x \begin{vmatrix} 0 & \ln x \\ 1 & x^{-1} \end{vmatrix} \\
&= -x \ln x \\
u_1 &= \frac{1}{4}x^2(1 - 2 \ln x) \\
u_2' &= x \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\
&= x \\
u_2 &= \frac{1}{2}x^2 \\
y &= y_c + y_p \\
&= c_1 + c_2 \ln x + \frac{1}{4}x^2(1 - 2 \ln x) + \frac{1}{2}x^2 \ln x \\
&= c_1 + c_2 \ln x + \frac{1}{4}x^2 \\
y(1) = 1 &= c_1 + c_2 \ln(1) + \frac{1}{4}(1)^2 \\
c_1 &= \frac{3}{4} \\
y' &= c_2 x^{-2} + \frac{1}{2}x \\
y'(1) = -\frac{1}{2} &= c_2(1)^{-2} + \frac{1}{2}(1) \\
c_2 &= -1 \\
y &= \frac{3}{4} - \ln x + \frac{1}{4}x^2
\end{aligned}$$

### 3.6.16 31

$$\begin{aligned}
xy'' - 7xy' + 12y &= 0 \\
m(m-1) - 7m + 12 &= 0 \\
m^2 - m - 7m + 12 &= 0 \\
m^2 - 8m + 12 &= 0 \\
(m-2)(m-6) &= 0
\end{aligned}$$

$$\begin{aligned}
y &= c_1x^2 + c_2x^6 \\
y(0) = 0 &= c_1(0)^2 + c_2(0)^6 \\
y(1) = 0 &= c_1(1)^2 + c_2(1)^6 \\
&= c_1 + c_2 \\
c_2 &= -c_1 \\
y &= c_1x^2 - c_1x^6 \\
&= c_1(x^2 - x^6)
\end{aligned}$$

**3.6.17 33**

$$y = c_1x^4 + c_2x^{-2}$$

$$\begin{aligned}
(m-4)(m+2) &= 0 \\
m(m-1) - m - 8 &= 0 \\
x^2y'' - xy' - 8 &= 0
\end{aligned}$$

**3.6.18 35**

$$y = c_1x^{-3} + c_2x^{-3} \ln x$$

$$\begin{aligned}
(m+3)^2 &= 0 \\
m^2 + 6m + 9 &= 0 \\
m(m-1) + 7m + 9 &= 0 \\
x^2y'' + 7xy' + 9y &= 0
\end{aligned}$$

**3.6.19 37**

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$$

$$\begin{aligned}
(m+i)(m-i) &= 0 \\
m^2 + 1 &= 0 \\
m(m-1) + m + 1 &= 0 \\
x^2y'' + xy' + y &= 0
\end{aligned}$$

**3.6.20 39**

$$(x+3)^2 y'' - 8(x+3)y' + 14y = 0$$

$$m(m-1) - 8m + 14 = 0$$

$$m^2 - m - 8m + 14 = 0$$

$$(m-2)(m-7) = 0$$

$$y = c_1(x+3)^2 + c_2(x+3)^7$$

**3.6.21 41**

$$(x+2)^2 y'' + (x+2)y' + y = 0$$

$$m(m-1) + m + 1 = 0$$

$$m^2 - m + m + 1 = 0$$

$$(m+i)(m-i) = 0$$

$$y = c_1 \cos(\ln(x+2)) + c_2 \sin(\ln(x+2))$$

**3.6.22 43**

$$z(t) = y(e^t)$$

$$z'(t) = e^t y'(e^t)$$

$$y'(e^t) = e^{-t} z'(t)$$

$$= x^{-1} z'(t)$$

$$z''(t) = e^t y'(e^t) + e^{2t} y''(e^t)$$

$$= z'(t) + e^{2t} y''(e^t)$$

$$y''(e^t) = x^{-2} z'(t) + x^{-2} z''(t)$$

$$x^2 y'' + 9xy' - 20y = 0$$

$$z' + z'' + 9z' - 20z = 0$$

$$z'' + 8z' - 20z = 0$$

$$m^2 + 8m - 20 = 0$$

$$(m+10)(m-2) = 0$$

$$z = c_1 e^{-10t} + c_2 e^{2t}$$

$$y = c_1 x^{-10} + c_2 x^2$$

### 3.6.23 51

$$T'' + r^{-1}T' - r^{-2}T = 0$$

$$r^2T'' + rT' - T = 0$$

$$m(m-1) + m - 1 = 0$$

$$m^2 - m + m - 1 = 0$$

$$(m+1)(m-1) = 0$$

$$T = c_1r^{-1} + c_2r$$

$$T(2) = T_0 = c_1(2)^{-1} + c_2(2)$$

$$= \frac{1}{2}c_1 + 2c_2$$

$$T' = -c_1r^{-2} + c_2$$

$$T'(1) = 0 = -c_1(1)^{-2} + c_2$$

$$= c_2 - c_1$$

$$c_1 = c_2$$

$$T_0 = \frac{1}{2}c_1 + 2c_2$$

$$c_1 = c_2 = \frac{2}{5}T_0$$

$$T = \frac{2}{5}T_0(r + r^{-1})$$

## 3.7 Nonlinear Equations

### 3.7.1 3

$$y'' + (y')^2 + 1 = 0$$

$$u' + u^2 + 1 = 0$$

$$u' = -(u^2 + 1)$$

$$\frac{1}{u^2 + 1}u' = -1$$

$$\arctan u = c - x$$

$$u = \tan(c - x)$$

$$y' = \tan(c - x)$$

$$y = \ln(\cos(c_1 - x)) + c_2$$



### 3.7.2 5

$$x^2 y'' + (y')^2 = 0$$

$$x^2 u' + u^2 = 0$$

$$u^{-2} u' = -x^{-2}$$

$$-u^{-1} = x^{-1} + c$$

$$u = -\frac{x}{cx + 1}$$

$$y = \frac{1}{c_1^2} \ln(c_1 x + 1) - \frac{1}{c_1} x + c_2$$

### 3.7.3 7

$$yy'' + (y')^2 + 1 = 0$$

$$yuu' + u^2 + 1 = 0$$

$$\frac{u}{u^2 + 1} u' = -y^{-1}$$

$$\frac{1}{2} \ln |u^2 + 1| = -\ln |y| + c_1$$

$$\sqrt{u^2 + 1} = c_1 y^{-1}$$

$$u^2 + 1 = c_1 y^{-2}$$

$$y' = \sqrt{c_1 y^{-2} - 1}$$

$$\frac{1}{\sqrt{c_1 y^{-2} - 1}} y' = 1$$

$$-y \sqrt{\frac{c_1}{y^2} - 1} = x + c_2$$

$$y^2 \left( \frac{c_1}{y^2} - 1 \right) = (x + c_2)^2$$

$$c_1 - y^2 = (x + c_2)^2$$

$$y^2 = c_1 - (x + c_2)^2$$

**3.7.4 9**

$$y'' + 2y(y')^3 = 0$$

$$uu' + 2yu^3 = 0$$

$$u^{-2}u' = -2y$$

$$-u^{-1} = -y^2 + c_1$$

$$u = \frac{1}{y^2 + c_1}$$

$$(y^2 + c_1)y' = 1$$

$$\frac{1}{3}y^3 + c_1y = x + c_2$$

3.7.5 11

$$y'' + yy' = 0$$

$$uu' + yu = 0$$

$$u' = -y$$

$$u = -\frac{1}{2}y^2 + c_1$$

$$\frac{-2}{y^2 + c_1}y' = 1$$

$$-\frac{2 \arctan(y/c_1)}{c_1} = x + c_2$$

$$\arctan(y/c_1) = -\frac{c_1}{2}(x + c_2)$$

$$= c_2 - \frac{c_1}{2}x$$

$$y = c_1 \tan\left(c_2 - \frac{c_1}{2}x\right)$$

$$y(0) = 1 = c_1 \tan\left(c_2 - \frac{c_1}{2}(0)\right)$$

$$= c_1 \tan c_2$$

$$y'(0) = -1 = -\frac{1}{2}c_1^2 \sec^2\left(c_2 - \frac{c_1(0)}{2}\right)$$

$$= -\frac{1}{2}c_1^2 \sec^2 c_2$$

$$-1 = -\frac{1}{2}\left(\frac{1}{\tan c_2}\right)^2 \sec^2 c_2$$

$$= -\frac{1 \cos^2 c_2}{2 \sin^2 c_2} \frac{1}{\cos^2 c_2}$$

$$2 = \frac{1}{\sin^2 c_2}$$

$$c_2 = \arcsin \frac{1}{\sqrt{2}}$$

$$= \frac{\pi}{4}$$

$$c_1 = 1$$

$$y = \tan\left(\frac{\pi}{4} - \frac{1}{2}x\right)$$

### 3.7.6 13

$$\begin{aligned}xyy'' &= y' + (y')^3 \\xu' &= u + u^3 \\u' - x^{-1}u &= x^{-1}u^3\end{aligned}$$

Let  $u = v^{-1/2}$  and  $u' = -\frac{1}{2}v^{-3/2}v'$

$$\begin{aligned}-\frac{1}{2}v^{-3/2}v' - x^{-1}v^{-1/2} &= x^{-1}v^{-3/2} \\v' + 2x^{-1}v &= -2x^{-1} \\e^{2\ln|x|}v' + 2x^{-1}e^{2\ln|x|}v &= -2x^{-1}e^{2\ln|x|} \\\frac{d}{dx}(x^2v) &= -2x \\x^2v &= -x^2 + c_1 \\u^{-2} &= c_1x^{-2} - 1 \\y' &= \frac{1}{\sqrt{c_1x^{-2} - 1}} \\y &= c_2 - x\sqrt{c_1x^{-2} - 1} \\&= c_2 - \sqrt{c_1 - x^2} \\&= c_2 - \frac{1}{c_1}\sqrt{1 - c_1^2x^2}\end{aligned}$$

### 3.7.7 15

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 + \dots$$

$$y'' = x + y^2$$

$$\begin{aligned}y(0) &= 1 \\y'(0) &= 1 \\y''(0) &= (0) + (1)^2 = 1 \\y''' &= 1 + 2yy' \\y'''(0) &= 1 + 2(1)(1) = 3 \\y^{(4)} &= 2y'^2 + 2yy'' \\y^{(4)}(0) &= 2(1)^2 + 2(1)(1) = 4 \\y^{(5)} &= 6y'y'' + 2yy''' \\y^{(5)}(0) &= 6(1)(1) + 2(1)(3) = 12\end{aligned}$$

$$y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{10}x^5$$

**3.7.8** 19

$$\begin{aligned}\frac{y''}{[1 + (y')^2]^{3/2}} &= 1 \\ \frac{y'}{\sqrt{(y')^2 + 1}} &= x \\ y' &= x\sqrt{(y')^2 + 1} \\ (y')^2 &= x^2((y')^2 + 1) \\ (1 - x^2)(y')^2 &= x^2 \\ (y')^2 &= \frac{x^2}{1 - x^2} \\ y' &= \frac{x}{\sqrt{1 - x^2}} \\ y &= -\sqrt{1 - x^2}\end{aligned}$$

### 3.8 Linear Models: Initial-Value Problems

**3.8.1** 1

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = \frac{\sqrt{2}\pi}{8}$$

**3.8.2** 3

$$W = mg \Rightarrow m = W/g = \frac{3}{4}$$

$$W = kx \Rightarrow k = W/x = 72$$

$$\begin{aligned}m \frac{d^2x}{dt^2} &= -kx \\ \frac{d^2x}{dt^2} + \frac{k}{m}x &= 0 \\ m^2 + \frac{k}{m} &= 0 \\ m^2 + \omega^2 &= 0 \\ (m + i\omega)(m - i\omega) &= 0\end{aligned}$$

$$x = c_1 \cos \omega t + c_2 \sin \omega t$$

$$\begin{aligned}
x(0) &= -\frac{1}{4} = c_1 \cos \omega(0) + c_2 \sin \omega(0) \\
&= c_1 \\
x'(0) &= 0 = 3\omega \sin \omega(0) + c_2 \omega \cos \omega(0) \\
&= c_2
\end{aligned}$$

$$x = -\frac{1}{4} \cos 4\sqrt{6}t$$

### 3.8.3 9

(a)

$$\begin{aligned}
x(0) &= \frac{1}{2} = c_1 \cos \omega(0) + c_2 \sin \omega(0) \\
&= c_1 \\
x'(0) &= \frac{3}{2} = -\frac{1}{2} \omega \sin \omega(0) + c_2 \omega \cos \omega(0) \\
&= c_2 \omega \\
c_2 &= \frac{3}{2} \sqrt{\frac{m}{k}} \\
&= \frac{3}{4}
\end{aligned}$$

$$x = \frac{1}{2} \cos 2t + \frac{3}{4} \sin 2t$$

(b)

$$\begin{aligned}
 A &= \sqrt{c_1^2 + c_2^2} \\
 &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2} \\
 &= \sqrt{\frac{1}{4} + \frac{9}{16}} \\
 &= \sqrt{\frac{4}{16} + \frac{9}{16}} \\
 &= \sqrt{\frac{13}{16}} \\
 &= \frac{\sqrt{13}}{4} \\
 \tan \phi &= \frac{c_1}{c_2} \\
 \phi &= \arctan \frac{1}{2} \frac{4}{3} \\
 &= \arctan \frac{2}{3} \\
 &= 0.558
 \end{aligned}$$

$$x = \frac{\sqrt{13}}{4} \sin(2t + 0.558)$$

(c)

$$\begin{aligned}
 \tan \phi &= \frac{c_2}{c_1} \\
 &= \frac{3}{4} \frac{2}{1} \\
 &= \frac{3}{2} \\
 \phi &= \arctan \frac{3}{2} \\
 &= 0.983
 \end{aligned}$$

$$x = \frac{\sqrt{13}}{4} \cos(2t - 0.983)$$

### 3.8.4 13

$$W = k_1 x_1 \Rightarrow k_1 = \frac{W}{x_1} = 40$$

$$W = k_2 x_2 \Rightarrow k_2 = \frac{W}{x_2} = 120$$

$$k_{\text{eff}} = k_1 + k_2 = 160$$

$$\begin{aligned} x(0) = 0 &= c_1 \cos \omega(0) + c_2 \sin \omega(0) \\ &= c_2 \end{aligned}$$

$$\begin{aligned} x'(0) = 2 &= c_2 \omega \cos \omega(0) \\ &= c_2 \omega \end{aligned}$$

$$\begin{aligned} c_2 &= 2 \sqrt{\frac{m}{k_{\text{eff}}}} \\ &= 2 \sqrt{\frac{5}{1280}} \\ &= \frac{1}{8} \end{aligned}$$

$$x = \frac{1}{8} \sin 16t$$

### 3.8.5 15

$$k_{\text{eff}} = \frac{k_1 k_2}{k_1 + k_2} = 30$$

$$\begin{aligned} x'(0) = 2 &= c_2 \omega \cos \omega(0) \\ &= c_2 \omega \end{aligned}$$

$$\begin{aligned} c_2 &= 2 \sqrt{\frac{m}{k_{\text{eff}}}} \\ &= \frac{1}{2\sqrt{3}} \end{aligned}$$

$$x = \frac{\sqrt{3}}{6} \sin 4\sqrt{3}t$$

### 3.8.6 17

$$k_{\text{eff}} = 2k$$

The spring is twice as rigid



**3.8.7 21**

- (a) Above
- (b) Upward

**3.8.8 23**

- (a) Below
- (b) Upward

**3.8.9 25**

$$mx'' = -kx - x'$$

$$x'' + \frac{1}{m}x' + \frac{k}{m}x = 0$$

$$x'' + 2\lambda x' + \omega^2 x = 0$$

$$r^2 + 2\lambda r + \omega^2 = 0$$

$$(r - (-\lambda + \sqrt{\lambda^2 - \omega^2}))(r - (-\lambda - \sqrt{\lambda^2 - \omega^2})) = 0$$

$$2\lambda = \frac{1}{m}$$

$$\lambda = \frac{1}{2m}$$

$$= \frac{g}{2W}$$

$$= 4$$

$$\omega^2 = \frac{k}{m}$$

$$= \frac{gk}{W}$$

$$= 16$$

$$\sqrt{\lambda^2 - \omega^2} = \sqrt{16 - 16}$$

$$= 0$$

$$x = c_1 e^{-4t} + c_2 t e^{-4t}$$

$$\begin{aligned}
x(0) &= -1 = c_1 e^{-4(0)} + c_2(0) e^{-4(0)} \\
&= c_1 \\
x'(0) &= 8 = 4e^{-4(0)} + c_2(e^{-4(0)} - 4(0)e^{-4(0)}) \\
&= 4 + c_2 \\
c_2 &= 4
\end{aligned}$$

$$x = -e^{-4t} + 4te^{-4t}$$

$$\begin{aligned}
0 &= -e^{-4t} + 4te^{-4t} \\
&= -1 + 4t \\
t &= \frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
x' &= 0 \\
4e^{-4t} + 4e^{-4x} - 16te^{-4t} &= 0 \\
4 + 4 - 16t &= 0 \\
t &= \frac{1}{2}
\end{aligned}$$

$$x\left(\frac{1}{2}\right) = -e^{-4(1/2)} + 4\left(\frac{1}{2}\right)e^{-4(1/2)} = e^{-2}$$

### 3.8.10 29

(a)

$$\begin{aligned}
k &= W/x = 2 \\
m &= W/g = 0.1 \\
\lambda &= \frac{\beta}{2m} = 2 \\
\omega^2 &= \frac{k}{m} = 20 \\
\sqrt{\lambda^2 - \omega^2} &= \sqrt{4 - 20} = 4i
\end{aligned}$$

$$x = c_1 e^{-2t} \cos 4t + c_2 e^{-2t} \sin 4t$$

$$\begin{aligned}
x(0) &= -1 = c_1 e^{-2(0)} \cos 4(0) + c_2 e^{-2(0)} \sin 4(0) \\
&= c_1 \\
x'(0) &= 0 = 2e^{-2(0)} \cos 4(0) + 4e^{-2(0)} \sin 4(0) - 2c_2 e^{-2(0)} \sin 4(0) \\
&\quad + 4c_2 e^{-2(0)} \cos 4(0) \\
&= 2 + 4c_2 \\
c_2 &= -\frac{1}{2}
\end{aligned}$$

$$x = -e^{-2t} \cos 4t - \frac{1}{2} e^{-2t} \sin 4t$$

(b)

$$\begin{aligned}
A &= \sqrt{c_1^2 + c_2^2} \\
&= \frac{\sqrt{5}}{2} \\
\tan \phi &= \frac{c_1}{c_2} \\
&= 2 \\
\phi &= \arctan 2 \\
&= 4.25 \\
x &= \frac{\sqrt{5}}{2} e^{-2t} \sin(4t + 4.25)
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{\sqrt{5}}{2} e^{-2t} \sin(4t + 4.25) &= 0 \\
4t + 4.25 &= 3\pi \\
t &= 1.29
\end{aligned}$$

**3.8.11 31**

$$k = W/x$$

$$= 5$$

$$m = W/g$$

$$= \frac{5}{16}$$

$$\lambda = \frac{\beta}{2m}$$

$$= \frac{8}{5}\beta$$

$$\omega^2 = \frac{k}{m}$$

$$= 16$$

$$\lambda^2 - \omega^2 = \left(\frac{8}{5}\beta\right)^2 - 16$$

$$= \frac{64}{25}\beta^2 - 16$$

(a)

$$\lambda^2 - \omega^2 > 0$$

$$\frac{64}{25}\beta^2 - 16 > 0$$

$$\beta > \frac{5}{2}$$

(b)

$$\beta = \frac{5}{2}$$

(c)

$$\beta < \frac{5}{2}$$

**3.8.12 33**

$$\beta = \frac{1}{2}$$

$$k = W/x = 6$$

$$m = W/g = \frac{1}{2}$$

$$\lambda = \frac{\beta}{2m} = \frac{1}{2}$$

$$\omega^2 = \frac{k}{m} = 12$$

$$\begin{aligned}\sqrt{\lambda^2 - \omega^2} &= \sqrt{\frac{1}{4} - 12} \\ &= \sqrt{-\frac{47}{4}} \\ &= \frac{\sqrt{47}}{2}i\end{aligned}$$

$$mx'' = -kx - \beta x' + f(t)$$

$$mx'' + \beta x' + kx = f(t)$$

$$x'' + 2\lambda x' + \omega^2 x = \frac{1}{m}f(t)$$

$$r^2 + 2\lambda r + \omega^2 = 0$$

$$(r - (-\lambda + \sqrt{\lambda^2 - \omega^2}))(r - (-\lambda - \sqrt{\lambda^2 - \omega^2})) = 0$$

$$x_c = c_1 e^{-t/2} \cos \frac{\sqrt{47}}{2}t + c_2 e^{-t/2} \sin \frac{\sqrt{47}}{2}t$$

$$x'' + x' + 12x = 20 \cos 3t$$

$$x_p = A \cos 3t + B \sin 3t$$

$$x'_p = -3A \sin 3t + 3B \cos 3t$$

$$x''_p = -9A \cos 3t - 9B \sin 3t$$

$$\begin{aligned}20 \cos 3t &= -9A \cos 3t - 9B \sin 3t - 3A \sin 3t + 3B \cos 3t \\ &\quad + 12(A \cos 3t + B \sin 3t) \\ &= 3(A + B) \cos 3t + 3(B - A) \sin 3t\end{aligned}$$

$$20 = 3(A + B)$$

$$0 = 3(B - A)$$

$$20 = 6B$$

$$B = \frac{10}{3}$$

$$A = \frac{10}{3}$$

$$x = x_c + x_p$$

$$= c_1 e^{-t/2} \cos \frac{\sqrt{47}}{2} t + c_2 e^{-t/2} \sin \frac{\sqrt{47}}{2} t + \frac{10}{3} \cos 3t + \frac{10}{3} \sin 3t$$

$$x(0) = 2 = c_1 e^{-(0)/2} \cos \frac{\sqrt{47}}{2} (0) + c_2 e^{-(0)/2} \sin \frac{\sqrt{47}}{2} (0) + \frac{10}{3} \cos 3(0)$$

$$+ \frac{10}{3} \sin 3(0)$$

$$= c_1 + \frac{10}{3}$$

$$c_1 = -\frac{4}{3}$$

$$x'(0) = 0 = \frac{2}{3} e^{-(0)/2} \cos \frac{\sqrt{47}}{2} (0) + \frac{2\sqrt{47}}{3} e^{-(0)/2} \sin \frac{\sqrt{47}}{2} (0)$$

$$- \frac{1}{2} c_2 e^{-(0)/2} \sin \frac{\sqrt{47}}{2} (0) + \frac{\sqrt{47}}{2} c_2 e^{-(0)/2} \cos \frac{\sqrt{47}}{2} (0)$$

$$- 10 \sin 3(0) + 10 \cos 3(0)$$

$$= \frac{2}{3} + \frac{\sqrt{47}}{2} c_2 + 10$$

$$c_2 = -\frac{64}{3\sqrt{47}}$$

$$x = e^{-t/2} \left( -\frac{4}{3} \cos \frac{\sqrt{47}}{2} t - \frac{64}{3\sqrt{47}} \sin \frac{\sqrt{47}}{2} t \right) + \frac{10}{3} (\cos 3t + \sin 3t)$$

3.8.13 35

$$m = 1$$

$$k = W/x = mg/x = 16$$

$$\beta = 8$$

$$\lambda = \beta/2m = 4$$

$$\omega^2 = k/m = 16$$

$$\sqrt{\lambda^2 - \omega^2} = 0$$

$$x'' + 8x' + 16x = 8 \sin 4t$$

$$x_c = c_1 e^{-4t} + c_2 t e^{-4t}$$

$$x_p = A \cos 4t + B \sin 4t$$

$$x'_p = -4A \sin 4t + 4B \cos 4t$$

$$x''_p = -16A \cos 4t - 16B \sin 4t$$

$$\begin{aligned} 8 \sin 4t &= -16A \cos 4t - 16B \sin 4t + 8(-4A \sin 4t + 4B \cos 4t) \\ &\quad + 16(A \cos 4t + B \sin 4t) \\ &= 32B \cos 4t - 32A \sin 4t \end{aligned}$$

$$A = -\frac{1}{4}$$

$$B = 0$$

$$x = c_1 e^{-4t} + c_2 t e^{-4t} - \frac{1}{4} \cos 4t$$

$$x(0) = 0 = c_1 e^{-4(0)} + c_2(0) e^{-4(0)} - \frac{1}{4} \cos 4(0)$$

$$= c_1 - \frac{1}{4}$$

$$c_1 = \frac{1}{4}$$

$$x'(0) = 0 = -e^{-4(0)} + c_2 e^{-4(0)} - 4c_2(0) e^{-4(0)} + \sin 4(0)$$

$$= -1 + c_2$$

$$c_2 = 1$$

$$x = \frac{1}{4} e^{-4t} + t e^{-4t} - \frac{1}{4} \cos 4t$$

3.8.14 37

$$m = 2$$

$$k = 32$$

$$2x'' + 32x = 68e^{-2t} \cos 4t$$

$$x'' + 16x = 34e^{-2t} \cos 4t$$

$$x'' + 16x = 0$$

$$(r + 4i)(r - 4i) = 0$$

$$x_c = c_1 \cos 4t + c_2 \sin 4t$$

$$x_p = Ae^{-2t} \cos 4t + Be^{-2t} \sin 4t$$

$$x'_p = -2Ae^{-2t} \cos 4t - 4Ae^{-2t} \sin 4t - 2Be^{-2t} \sin 4t + 4Be^{-2t} \cos 4t$$

$$\begin{aligned} x''_p &= 4Ae^{-2t} \cos 4t + 8Ae^{-2t} \sin 4t + 8Ae^{-2t} \sin 4t - 16Ae^{-2t} \cos 4t \\ &\quad + 4Be^{-2t} \sin 4t - 8Be^{-2t} \cos 4t - 8Be^{-2t} \cos 4t - 16Be^{-2t} \sin 4t \\ &= (-12A - 16B)e^{-2t} \cos 4t + (16A - 12B)e^{-2t} \sin 4t \end{aligned}$$

$$\begin{aligned} 34e^{-2t} \cos 4t &= (-12A - 16B)e^{-2t} \cos 4t + (16A - 12B)e^{-2t} \sin 4t \\ &\quad + 16(Ae^{-2t} \cos 4t + Be^{-2t} \sin 4t) \\ &= (4A - 16B)e^{-2t} \cos 4t + (16A + 4B)e^{-2t} \sin 4t \end{aligned}$$

$$34 = 4A - 16B$$

$$0 = 16A + 4B$$

$$34 = 68A$$

$$A = \frac{1}{2}$$

$$B = -2$$

$$x = c_1 \cos 4t + c_2 \sin 4t + e^{-2t} \left( \frac{1}{2} \cos 4t - 2 \sin 4t \right)$$



$$\begin{aligned}
x(0) = 0 &= c_1 \cos 4(0) + c_2 \sin 4(0) + e^{-2(0)} \left( \frac{1}{2} \cos 4(0) - 2 \sin 4(0) \right) \\
&= c_1 + \frac{1}{2} \\
c_1 &= -\frac{1}{2} \\
x'(0) = 0 &= 2 \sin 4(0) + 4c_2 \cos 4(0) - e^{-2(0)} \cos 4(0) - 2e^{-2(0)} \sin 4(0) \\
&\quad + 4e^{-2(0)} \sin 4(0) - 8e^{-2(0)} \cos 4(0) \\
&= 4c_2 - 1 - 8 \\
c_2 &= \frac{9}{4} \\
x &= -\frac{1}{2} \cos 4t + \frac{9}{4} \sin 4t + e^{-2t} \left( \frac{1}{2} \cos 4t - 2 \sin 4t \right)
\end{aligned}$$

### 3.8.15 39

(a)

$$\begin{aligned}
mx'' + \beta x' + kx &= kh \\
x'' + \frac{\beta}{m}x' + \frac{k}{m}x &= \frac{k}{m}h \\
x'' + 2\lambda x' + \omega^2 x &= \omega^2 h
\end{aligned}$$

(b)

$$x'' + 4x' + 8x = 40 \cos t$$

$$\begin{aligned}
r^2 + 4r + 8 &= 0 \\
(r - (-2 + 2i))(r - (-2 - 2i)) &= 0
\end{aligned}$$

$$x_c = c_1 e^{-2t} \cos 2t + c_2 e^{-2t} \sin 2t$$

$$\begin{aligned}
x_p &= A \cos t + B \sin t \\
x'_p &= -A \sin t + B \cos t \\
x''_p &= -A \cos t - B \sin t
\end{aligned}$$

$$\begin{aligned}
40 \cos t &= -A \cos t - B \sin t + 4(-A \sin t + B \cos t) + 8(A \cos t + B \sin t) \\
&= (7A + 4B) \cos t + (7B - 4A) \sin t
\end{aligned}$$

$$\begin{aligned}
40 &= 7A + 4B \\
0 &= -4A + 7B \\
160 &= 65B \\
B &= \frac{32}{13} \\
A &= \frac{56}{13}
\end{aligned}$$

$$x = c_1 e^{-2t} \cos 2t + c_2 e^{-2t} \sin 2t + \frac{56}{13} \cos t + \frac{32}{13} \sin t$$

$$\begin{aligned}
x(0) = 0 &= c_1 e^{-2(0)} \cos 2(0) + c_2 e^{-2(0)} \sin 2(0) + \frac{56}{13} \cos(0) + \frac{32}{13} \sin(0) \\
&= c_1 + \frac{56}{13} \\
c_1 &= -\frac{56}{13} \\
x'(0) = 0 &= \frac{112}{13} e^{-2(0)} \cos 2(0) + \frac{112}{13} e^{-2(0)} \sin 2(0) - 2c_2 e^{-2(0)} \sin 2(0) \\
&\quad + 2c_2 e^{-2(0)} \cos 2(0) - \frac{56}{13} \sin(0) + \frac{32}{13} \cos(0) \\
&= \frac{112}{13} + 2c_2 + \frac{32}{13} \\
c_2 &= -\frac{72}{13}
\end{aligned}$$

$$x = e^{-2t} \left( -\frac{56}{13} \cos 2t - \frac{72}{13} \sin 2t \right) + \frac{56}{13} \cos t + \frac{32}{13} \sin t$$

### 3.8.16 41

$$x'' + 4x = -5 \sin 2t + 3 \cos 2t$$

$$\begin{aligned}
x'' + 4x &= 0 \\
(r + 2i)(r - 2i) &= 0 \\
x_c &= c_1 \cos 2t + c_2 \sin 2t
\end{aligned}$$

$$\begin{aligned}
x_p &= At \cos 2t + Bt \sin 2t \\
x'_p &= A \cos 2t - 2At \sin 2t + B \sin 2t + 2Bt \cos 2t \\
x''_p &= -2A \sin 2t - 2A \sin 2t - 4At \cos 2t + 2B \cos 2t + 2B \cos 2t \\
&\quad - 4Bt \sin 2t \\
&= -4A \sin 2t - 4At \cos 2t + 4B \cos 2t - 4Bt \sin 2t \\
-5 \sin 2t + 3 \cos 2t &= -4A \sin 2t - 4At \cos 2t + 4B \cos 2t - 4Bt \sin 2t \\
&\quad + 4(At \cos 2t + Bt \sin 2t) \\
&= -4A \sin 2t + 4B \cos 2t
\end{aligned}$$

$$\begin{aligned}
-5 &= -4A \Rightarrow A = \frac{5}{4} \\
3 &= 4B \Rightarrow B = \frac{3}{4}
\end{aligned}$$

$$x = c_1 \cos 2t + c_2 \sin 2t + \frac{5}{4}t \cos 2t + \frac{3}{4}t \sin 2t$$

$$\begin{aligned}
x(0) &= -1 = c_1 \cos 2(0) + c_2 \sin 2(0) + \frac{5}{4}(0) \cos 2(0) + \frac{3}{4}(0) \sin 2(0) \\
&= c_1 \\
x'(0) &= 1 = 2 \sin 2(0) + 2c_2 \cos 2(0) + \frac{5}{4} \cos 2(0) - \frac{5}{2}(0) \sin 2(0) + \frac{3}{4} \sin 2(0) \\
&\quad + \frac{3}{2}(0) \cos 2(0) \\
&= 2c_2 + \frac{5}{4} \\
c_2 &= -\frac{1}{8}
\end{aligned}$$

$$x = -\cos 2t - \frac{1}{8} \sin 2t + \frac{5}{4}t \cos 2t + \frac{3}{4}t \sin 2t$$

**3.8.17 49**

$$\begin{aligned}
0.05q'' + 2q' + 100q &= 0 \\
q'' + 40q' + 2000q &= 0 \\
r^2 + 40r + 2000 &= 0 \\
(r - (-20 + 40i))(r - (-20 - 40i)) &= 0
\end{aligned}$$

$$q = c_1 e^{-20t} \cos 40t + c_2 e^{-20t} \sin 40t$$

$$\begin{aligned} q(0) = 5 &= c_1 e^{-20(0)} \cos 40(0) + c_2 e^{-20(0)} \sin 40(0) \\ &= c_1 \end{aligned}$$

$$\begin{aligned} q'(0) = 0 &= -100e^{-20(0)} \cos 40(0) - 200e^{-20(0)} \sin 40(0) - 20c_2 e^{-20(0)} \sin 40(0) \\ &\quad + 40c_2 e^{-20(0)} \cos 40(0) \\ &= -100 + 40c_2 \\ c_2 &= \frac{5}{2} \end{aligned}$$

$$q = e^{-20t} \left( 5 \cos 40t + \frac{5}{2} \sin 40t \right)$$

$$q(0.01) = 4.57 \text{ C}$$

$$\begin{aligned} e^{-20t} \left( 5 \cos 40t + \frac{5}{2} \sin 40t \right) &= 0 \\ 5 \cos 40t + \frac{5}{2} \sin 40t &= 0 \\ 2 \cos 40t + \sin 40t &= 0 \\ \sqrt{5} \cos(40t + \arctan(-1/2)) &= 0 \\ 40t + \arctan(-1/2) &= \arccos 0 \\ 40t &= \arccos 0 - \arctan(-1/2) \\ t &= \frac{1}{40} (\arccos 0 - \arctan(-1/2)) \\ &= 0.0509 \text{ s} \end{aligned}$$

### 3.9 Linear Models: Boundary-Value Problems

#### 3.9.1 1

(a)

$$EIy^{(4)} = w_0$$

$$y(0) = 0, y'(0) = 0, y''(L) = 0, y'''(L) = 0$$

$$\begin{aligned}
y^{(4)} &= \frac{w_0}{EI} \\
y''' &= \frac{w_0}{EI}x + c_1 \\
y'' &= \frac{w_0}{2EI}x^2 + c_1x + c_2 \\
y' &= \frac{w_0}{6EI}x^3 + \frac{1}{2}c_1x^2 + c_2x + c_3 \\
y &= \frac{w_0}{24EI}x^4 + \frac{1}{6}c_1x^3 + \frac{1}{2}c_2x^2 + c_3x + c_4
\end{aligned}$$

$$\begin{aligned}
y(0) &= 0 = c_4 \\
y'(0) &= 0 = c_3 \\
y'''(L) &= 0 = \frac{w_0}{EI}(L) + c_1 \\
c_1 &= -\frac{Lw_0}{EI} \\
y''(L) &= 0 = \frac{w_0}{2EI}(L)^2 - \frac{Lw_0}{EI}(L) + c_2 \\
c_2 &= \frac{L^2w_0}{2EI}
\end{aligned}$$

$$y = \frac{w_0}{24EI} (x^4 - 4Lx^3 + 6L^2x^2)$$

(b)

$$y = x^4 - 4Lx^3 + 6L^2x^2$$

### 3.9.2 3

(a)

$$\begin{aligned}
y^{(4)} &= \frac{w_0}{EI} \\
y''' &= \frac{w_0}{EI}x + c_1 \\
y'' &= \frac{w_0}{2EI}x^2 + c_1x + c_2 \\
y' &= \frac{w_0}{6EI}x^3 + \frac{1}{2}c_1x^2 + c_2x + c_3 \\
y &= \frac{w_0}{24EI}x^4 + \frac{1}{6}c_1x^3 + \frac{1}{2}c_2x^2 + c_3x + c_4
\end{aligned}$$

$$\begin{aligned}
y(0) &= 0 = c_4 \\
y'(0) &= 0 = c_3 \\
y(L) &= 0 = \frac{w_0}{24EI}(L)^4 + \frac{1}{6}c_1(L)^3 + \frac{1}{2}c_2(L)^2 \\
&= \frac{w_0}{12EI}L^2 + \frac{1}{3}c_1L + c_2 \\
y''(L) &= 0 = \frac{w_0}{2EI}(L)^2 + c_1(L) + c_2 \\
0 &= \frac{5L^2w_0}{12EI} + \frac{2}{3}c_1L \\
c_1 &= -\frac{5Lw_0}{8EI} \\
0 &= \frac{L^2w_0}{2EI} - \frac{5L^2w_0}{8EI} + c_2 \\
c_2 &= \frac{L^2w_0}{8EI}
\end{aligned}$$

$$\begin{aligned}
y &= \frac{w_0}{24EI}x^4 - \frac{5L^2w_0}{48EI}x^3 + \frac{L^2w_0}{16EI}x^2 \\
&= \frac{w_0}{48EI}(2x^4 - 5Lx^3 + 3L^2x^2)
\end{aligned}$$

(b)

$$y = 2x^4 - 5Lx^3 + 3L^2x^2$$

### 3.9.3 11

$$\begin{aligned}
y'' + \lambda y &= 0 \\
m^2 + \lambda &= 0 \\
(m + \sqrt{\lambda}i)(m - \sqrt{\lambda}i) &= 0 \\
y &= c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x \\
y(0) &= 0 = c_1 \cos \sqrt{\lambda}(0) + c_2 \sin \sqrt{\lambda}(0) \\
&= c_1 \\
y(\pi) &= 0 = c_2 \sin \sqrt{\lambda}(\pi) \\
\sqrt{\lambda}\pi &= n\pi \\
\lambda &= n^2 \\
y &= \sin nx \text{ for } n = 1, 2, 3, \dots
\end{aligned}$$

### 3.9.4 13

$$y'' + \lambda y = 0$$

$$y = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x$$

$$y' = -\sqrt{\lambda}c_1 \sin \sqrt{\lambda}x + \sqrt{\lambda}c_2 \cos \sqrt{\lambda}x$$

$$\begin{aligned} y'(0) = 0 &= -\sqrt{\lambda}c_1 \sin \sqrt{\lambda}(0) + \sqrt{\lambda}c_2 \cos \sqrt{\lambda}(0) \\ &= \sqrt{\lambda}c_2 \\ &= c_2 \end{aligned}$$

$$y(L) = 0 = c_1 \cos \sqrt{\lambda}(L)$$

$$\begin{aligned} \sqrt{\lambda}L &= n\pi - \frac{\pi}{2} \text{ for } n = 1, 2, 3, \dots \\ &= \frac{2n-1}{2}\pi \end{aligned}$$

$$\lambda = \left( \frac{(2n-1)\pi}{2L} \right)^2$$

$$y = \cos \frac{(2n-1)\pi}{2L}x$$

### 3.9.5 15

$$y'' + \lambda y = 0$$

$$m^2 + \lambda = 0$$

$$(m + \sqrt{\lambda}i)(m - \sqrt{\lambda}i) = 0$$

$$y = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x$$

$$y' = -\sqrt{\lambda}c_1 \sin \sqrt{\lambda}x + \sqrt{\lambda}c_2 \cos \sqrt{\lambda}x$$

$$\begin{aligned} y'(0) = 0 &= -\sqrt{\lambda}c_1 \sin \sqrt{\lambda}(0) + \sqrt{\lambda}c_2 \cos \sqrt{\lambda}(0) \\ &= \sqrt{\lambda}c_2 \\ &= c_2 \end{aligned}$$

$$y(\pi) = 0 = c_1 \cos \sqrt{\lambda}(\pi)$$

$$\begin{aligned} \sqrt{\lambda}\pi &= \frac{\pi}{2} + n\pi \\ \lambda &= \left( \frac{1}{2} + n \right)^2 \end{aligned}$$

$$y = \cos\left(\frac{1}{2} + n\right)x$$

### 3.9.6 17

$$y'' + 2y' + (\lambda + 1)y = 0$$

$$m^2 + 2m + \lambda + 1 = 0$$

$$(m - (-1 + \sqrt{\lambda}i))(m - (-1 - \sqrt{\lambda}i)) = 0$$

$$y = c_1 e^{-x} \cos \sqrt{\lambda}x + c_2 e^{-x} \sin \sqrt{\lambda}x$$

$$\begin{aligned} y(0) = 0 &= c_1 e^{-(0)} \cos \sqrt{\lambda}(0) + c_2 e^{-(0)} \sin \sqrt{\lambda}(0) \\ &= c_1 \end{aligned}$$

$$y(5) = 0 = c_2 e^{-(5)} \sin \sqrt{\lambda}(5)$$

$$5\sqrt{\lambda} = n\pi$$

$$\lambda = \left(\frac{n\pi}{5}\right)^2$$

$$y = e^{-x} \sin \frac{n\pi}{5}x$$

### 3.9.7 19

$$x^2 y'' + xy' + \lambda y = 0$$

$$m(m-1) + m + \lambda = 0$$

$$m^2 - m + m + \lambda = 0$$

$$(m + \sqrt{\lambda}i)(m - \sqrt{\lambda}i) = 0$$

$$y = c_1 \cos(\sqrt{\lambda} \ln x) + c_2 \sin(\sqrt{\lambda} \ln x)$$

$$\begin{aligned} y(1) = 0 &= c_1 \cos(\sqrt{\lambda} \ln(1)) + c_2 \sin(\sqrt{\lambda} \ln(1)) \\ &= c_1 \end{aligned}$$

$$y(e^\pi) = 0 = c_2 \sin(\sqrt{\lambda} \ln(e^\pi))$$

$$\pi\sqrt{\lambda} = \pi n$$

$$\lambda = n^2 \text{ for } n = 0, 1, 2, \dots$$

$$y = \sin(n \ln x)$$



### 3.9.8 23

$L/4$ ,  $2L/4$ , and  $3L/4$

### 3.9.9 27

$$Ty'' + \rho\omega^2 y = 0$$

$$y'' + \frac{\rho\omega^2}{T} y = 0$$

$$m^2 + \frac{\rho\omega^2}{T} = 0$$

$$\left(m + \sqrt{\frac{\rho\omega^2}{T}}i\right)\left(m - \sqrt{\frac{\rho\omega^2}{T}}i\right) = 0$$

$$y = c_1 \cos \sqrt{\frac{\rho\omega^2}{T}}x + c_2 \sin \sqrt{\frac{\rho\omega^2}{T}}x$$

$$y(0) = 0 = c_1$$

$$y(L) = 0 = c_2 \sin \sqrt{\frac{\rho\omega^2}{T}}(L)$$

$$\sqrt{\frac{\rho\omega^2}{T}}L = n\pi$$

$$\omega = \frac{n\pi}{L} \sqrt{\frac{T}{\rho}} \text{ for } n = 0, 1, 2, \dots$$

$$y = \sin \frac{n\pi}{L}x$$

### 3.9.10 29

$$ru'' + 2u' = 0$$

$$r^2u'' + 2ru' = 0$$

$$m(m-1) + 2m = 0$$

$$m^2 - m + 2m = 0$$

$$m^2 + m = 0$$

$$m(m+1) = 0$$

$$\begin{aligned}
u &= c_1 + c_2 r^{-1} \\
u(a) = u_0 &= c_1 + c_2(a)^{-1} \\
u(b) = u_1 &= c_1 + c_2(b)^{-1} \\
u_0 - u_1 &= \frac{c_2}{a} - \frac{c_2}{b} \\
&= c_2 \frac{b-a}{ab} \\
c_2 &= \frac{ab(u_0 - u_1)}{b-a} \\
u_0 &= c_1 + \frac{b(u_0 - u_1)}{b-a} \\
c_1 &= u_0 - \frac{b(u_0 - u_1)}{b-a} \\
&= \frac{u_0(b-a) - b(u_0 - u_1)}{b-a} \\
&= \frac{bu_1 - au_0}{b-a} \\
u &= \frac{u_0 - u_1}{b-a} \frac{ab}{r} + \frac{bu_1 - au_0}{b-a}
\end{aligned}$$