Advanced Engineering Mathematics Ordinary Differential Equations Notes

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February 2022

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1 Introduction to Differential Equations

1.1 Definitions and Terminology

- An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation** (DE)
- An **ordinary DE** (ODE) is a DE that contains only ordinary (i.e. non-partial) derivatives of one or more functions with respect to a single independent variable
- A **partial DE** is a DE that contains only partial derivatives of one or more functions of two or more independent variables
- The **order** of a DE is the order of the highest derivative in the equation

- First order ODEs are sometimes written in the **differential form**

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

- n-th order ODEs in one dependent variable can be expressed by the **general form**

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

- It's possible to solve ODEs in the general form uniquely for the highest derivative $y^{(n)}$ in terms of the other $n+1$ variables, allowing them to be expressed in the **normal form**

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

- An n-th order ODE is said be **linear** in the variable y if it can be expressed in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - g(x) = 0$$

  i.e. the dependent variable y and all of its derivatives aren't raised to a power or used in nonlinear functions like e^y or $\sin y$, and the coefficients a_0, a_1, …, a_n depend at most on the independent variable x

- A **nonlinear** ODE is one that is not linear
- A **solution** to an ODE is a function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, such that

$$F(x, \phi(x), \phi'(x), \ldots, \phi^n(x)) = 0 \text{ for all } x \text{ in } I.$$

- The **interval of definition**, **interval of validity**, or the **domain** of a solution is the interval over which the solution is valid
- A solution of a DE that is 0 on an interval I is said to be a **trivial solution**
- Because solutions to DEs must be differentiable over their interval of validity, discontinuities, etc. must be excluded from the interval
- An **explicit solution** to an ODE is one where the dependent variable is expressed solely in terms of the independent variable and constants
- An **implicit solution** to an ODE is a relation $G(x, y) = 0$ over an interval I provided there exists at least one function ϕ that satisfies the relation as well as the ODE on I
- When solving a first-order ODE we usually obtain a solution containing a single arbitrary constant or parameter c. A solution containing an arbitrary constant represents a set of solution called a **one-parameter family of solutions**

- When solving an n-th order DE we usually obtain an **n-parameter family of solutions**
- A solution of a DE that is free from arbitrary parameters is called a **particular solution**
- A **singular solution** is a solution to a DE that isn't a member of a family of solutions
- A **system of ODEs** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable. A solution of such a system is a differentiable function for each equation defined on a common interval I that satisfy each equation of the system on that interval

1.2 Initial Value Problems

- An **initial value problem** is the problem of solving a DE with some given **initial conditions**, e.g. solve

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

subject to

$$y(x_0) = y_0,\, y'(x_0) = y_1,\, \ldots,\, y^{(n-1)}(x_0) = y_{n-1}$$

- The domain of $y = f(x)$ differs depending on how it's considered:
  - As a function its domain is all real numbers for which it's defined
  - As a solution of a DE its domain is a single interval over which it's defined an differentiable
  - As a solution of an initial value problem its domain is a single interval over which it's defined, differentiable, and contains the initial conditions
- An initial value problem may not have any solutions. If it does it may have multiple.
- First-order initial value problems of the form

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

are guaranteed to have a unique solution over an interval I containing x_0 if $f(x, y)$ and $\partial f / \partial y$ are continuous

1.3 Differential Equations as Mathematical Models

- A **mathematical model** is a mathematical description of a system or phenomenon
- The **level of resolution** of a model determines how many variables are included in the model
- A simple model of the growth of a population P is

$$\frac{dP}{dt} = kP$$

  where $k > 0$

- A simple model of radioactive decay of an amount of substance A is

$$\frac{dA}{dt} = kA$$

  where $k < 0$

- Newton's empirical law of cooling/warming states that the rate of change of the temperature of a body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium

$$\frac{dT}{dt} = k(T - T_m)$$

2 First-Order Differential Equations

2.1 Solution Curves Without a Solution

- An ODE in which the independent variable doesn't appear is said to be **autonomous**, e.g.

$$\frac{dy}{dx} = f(y)$$

- A real number c is a **critical/equilibrium/stationary point** of an autonomous DE if it is a zero of f
- If c is a critial point of an autonomous DE, then $y(x) = c$ is a solution
- A solution of the form $y(x) = c$ is called an **equilibrium solution**
- We can draw several conclusions about the solutions of an autonomous DE with n critical points and $n + 1$ subregions bounded by the critical points:
  - If (x_0, y_0) is in a subregion, it remains in that subregion for all x
  - By continuity, $f(y) < 0$ or $f(y) > 0$ for all y in a subregion and thus $y(x)$ can't have maximum/minimum points or oscillate

- If $y(x)$ is bounded above by a critical point c_1, it must approach $y(x) = c_1$ as $x \to -\infty$ or $x \to \infty$
- If $y(x)$ is bounded above and below by critical points c_1 and c_2, it must approach $y(x) = c_1$ as $x \to -\infty$ and $y(x) = c_2$ as $x \to \infty$ or vice versa
- If $y(x)$ is bounded below by a critical point c_1, it must approach $y(x) = c_1$ as $x \to -\infty$ or $x \to \infty$

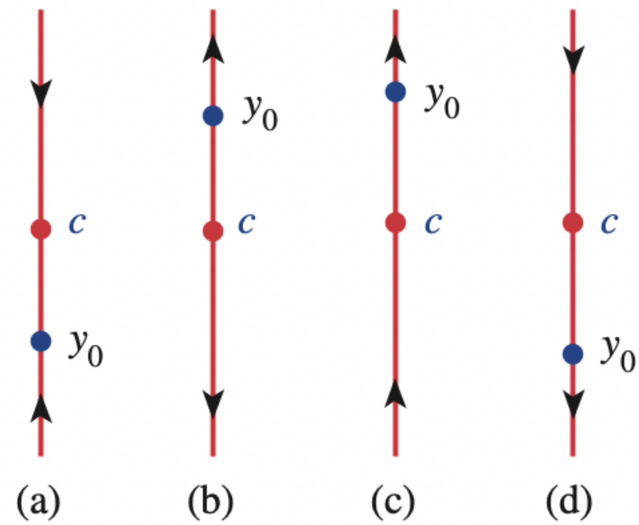

FIGURE 2.1.8 Critical point c is an attractor in (a), a repeller in (b), and semi-stable in (c) and (d)

- If $y(x)$ is a solution of an autonomous differential equation $dy/dx = f(y)$, then $y_1(x) = y(x - k)$, where k is a constant, is also a solution

2.2 Separable Equations

- A first-order ODE of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be **separable** or to have **separate variables**

- A separable first-order ODE can be solved by dividing both sides by $h(y)$ then integrating both sides with respect to x

$$
\begin{aligned}
\frac{dy}{dx} &= g(x)h(y) \\
\frac{1}{h(y)}\frac{dy}{dx} &= g(x) \\
\int \frac{1}{h(y)}\frac{dy}{dx}\,dx &= \int g(x)\,dx \\
\int \frac{1}{h(y)}\,dy &= \int g(x)\,dx \\
H(y) &= G(x) + c
\end{aligned}
$$

- Care should be taken when dividing by $h(y)$ as it removes constant solutions $y = r$ where $h(r) = 0$

2.3 Linear Equations

- A first-order DE of the form

  $$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

  or in standard form

  $$\frac{dy}{dx} + P(x)y = f(x)$$

  is said to be a **linear equation** in the dependent variable y

- When $g(x) = 0$ or $f(x) = 0$ the linear equation is said to be **homogeneous** and is solvable via separation of variables, otherwise it is **nonhomogeneous**

- The nonhomogeneous linear equation's solution is the sum of two solutions $y = y_c + y_p$ where y_c is a solution of the associated homogeneous equation

  $$\frac{dy}{dx} + P(x)y = 0$$

  and y_p is a particular solution of the nonhomogeneous equation

- Nonhomogeneous linear equations can be solved via **variation of parameters**:

  1. Put it into standard form
  2. Determine the **integrating factor** $e^{\int P(x)\,dx}$
  3. Multiply by the integrating factor
  4. Recognise that the left hand side of the equation is the derivative of the product of the integrating factor and y

5. Integrate both sides of the equation
6. Solve for y

- The **general solution** of a DE is a family of solutions that contains all possible solutions (except singular solutions)

- A term $y = f(x)$ in a solution is called a **transient term** if $f(x) \to 0$ as $x \to \infty$

- When either $P(x)$ or $f(x)$ is a piecewise-defined function the equation is then referred to as a **piecewise-linear differential equation** that can be solved by solving each interval in isolation then choosing appropriate constants to ensure the overall solution is continuous

- The **error function** and **complementary error function** are defined

$$\operatorname{erf} x + \operatorname{erfc} x = 1$$
$$\left(\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt\right) + \left(\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt\right) = 1$$

2.4 Exact Equations

- The **differential** of a function $z = f(x, y)$ is

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

- A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is an **exact differential** in the region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$

- A first-order DE of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

  is said to be an **exact equation** if the expression on the left side is an exact differential

- A necessary and sufficient condition that $M(x, y)\,dx + N(x, y)\,dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- Exact differentials can be solved by

1. Integrating $M(x, y)$ with respect to x to find an expression for $f(x, y)$

$$\frac{\partial f}{\partial x} = M(x, y)$$
$$f(x, y) = \int M(x, y)\, dx + g(y)$$

2. Differentiating $f(x, y)$ with respect to y and equating it to $N(x, y)$ to find $g'(y)$

$$\frac{\partial f}{\partial y} = N(x, y) = \frac{\partial}{\partial y} \int M(x, y)\, dx + g'(y)$$
$$g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, dx$$

3. Integrating $g'(y)$ with respect to y to find $g(y)$ and substituting it into $f(x, y)$
4. Equating $f(x, y)$ with an unknown constant c

- x and y can be swapped in the steps above (i.e. you can start by integrating $N(x, y)$ with respect to y, etc.)
- A nonexact DE $M(x, y)\, dx + N(x, y)\, dy = 0$ can sometimes be transformed into an exact DE by finding an appropriate integrating factor
  - If $(M_y - N_x)/N$ is a function of x alone, then an integrating factor is
  $$\mu(x) = e^{\int \frac{M_y - N_x}{N}\, dx}$$
  - If $(N_x - M_y)/M$ is a function of y alone, then an integrating factor is
  $$\mu(y) = e^{\int \frac{N_x - M_y}{M}\, dy}$$

2.5 Solutions by Substitution

- A function $f(x, y)$ is said to be a **homogeneous function** of degree α if
$$f(tx, ty) = t^{\alpha} f(x, y)$$
- A first-order DE of the form
$$M(x, y)\, dx + N(x, y)\, dy = 0$$
is said to be **homogeneous** if both M and N are homogeneous functions of the same degree

- To solve a homogeneous first-order DE:

  1. Rewrite it as

     $$M(x,y) = x^\alpha M(1,u) \text{ and } N(x,y) = x^\alpha N(1,u) \text{ where } u = y/x$$

     or

     $$M(x,y) = y^\alpha M(v,1) \text{ and } N(x,y) = y^\alpha N(v,1) \text{ where } v = x/y$$

  2. Substitute $y = ux$ and $dy = u\,dx + x\,du$ or $x = vy$ and $dx = v\,dy + y\,dv$ as appropriate
  3. Solve the resulting first-order separable DE
  4. Substitude $u = y/x$ or $v = x/y$ as appropriate

- The DE
  $$\frac{dy}{dx} + P(x)y = f(x)y^n$$
  where n is any real number is called **Bernoulli's equation**

- For $n = 0$ and $n = 1$ Bernoulli's equation is linear

- To solve Bernoulli's equation for $n \neq 0$ and $n \neq 1$:

  1. Substitude $y = u^{1/(1-n)}$ and $\frac{dy}{dx} = \frac{d}{dx}(u^{1/(1-n)})$
  2. Solve the resulting linear equation
  3. Substitude $u = y^{n-1}$

- A DE of the form
  $$\frac{dy}{dx} = f(Ax + By + C)$$
  can always be reduced to an equation with separable variables by means of the substitution
  $$u = Ax + By + C, B \neq 0$$

2.6 A Numerical Method

- Approximate values for points on a solution curve near an initial point can be calculated via a **linearization** of the solution curve — a straight line that has the same slope as the initial point and passes through it

- **Euler's method** approximates a solution curve by iteratively stepping along its linearizations

  $$y_{n+1} = y_n + hf(x_n, y_n)$$

  where h is the **step size**

2.9 Modeling with Systems of First-Order DEs

- In a system of DEs
$$\frac{dx}{dt} = g_1(t, x, y)$$
and
$$\frac{dy}{dt} = g_2(t, x, y)$$
if g_1 and g_2 are linear in x and y, i.e.
$$g_1(t, x, y) = c_1 x + c_2 y + f_1(t)$$
and
$$g_2(t, x, y) = c_3 x + c_4 y + f_2(t)$$
it is said to be a **linear system**