Classical Mechanics by John R. Taylor Problems

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Contents

1 Newton's Laws of Motion

1.1

$$\mathbf{b} + \mathbf{c} = 2\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$$
$$5\mathbf{b} + 2\mathbf{x} = 7\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 2\hat{\mathbf{z}}$$
$$\mathbf{b} \cdot \mathbf{c} = 1$$
$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$
$$= \hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}}$$

1.5

$$\mathbf{v}_{\text{body}} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$$
$$\mathbf{v}_{\text{face}} = \hat{\mathbf{x}} + \hat{\mathbf{z}}$$
$$\mathbf{v}_{\text{body}} \cdot \mathbf{v}_{\text{face}} = v_{\text{body}} v_{\text{face}} \cos\theta$$
$$2 = \sqrt{6} \cos\theta$$
$$\cos\theta = \frac{2}{\sqrt{6}}$$
$$\theta = \arccos\frac{2}{\sqrt{6}}$$
$$= 35.26°$$

1.11

The particle moves counterclockwise in an ellipse of width $2b$ and height $2c$. The angular speed is ω.

1.23

$$\mathbf{v} = v\cos\theta\frac{\mathbf{b}}{b} - v\sin\theta\frac{\mathbf{b} \times \mathbf{c}}{bc}$$
$$= \frac{\lambda}{b}\frac{\mathbf{b}}{b} - \frac{c}{b}\frac{\mathbf{b} \times \mathbf{c}}{bc}$$
$$= \frac{\lambda\mathbf{b} - \mathbf{b} \times \mathbf{c}}{b^2}$$

1.25

$$\frac{df}{dt} = -3f$$
$$\frac{1}{f}\frac{df}{dt} = -3$$
$$\ln f = -3t + c$$
$$f = ce^{-3t}$$

One constant.

1.35

$$F_x = 0$$
$$ma_x = 0$$
$$a_x = 0$$
$$v_x = c_1$$
$$= v_o \cos \theta$$
$$r_x = v_o \cos(\theta)t + c_2$$
$$= v_o \cos(\theta)t$$

$$F_y = 0$$
$$ma_y = 0$$
$$a_y = 0$$
$$v_y = c_3$$
$$v_y = 0$$
$$r_y = c_4$$
$$r_y = 0$$

$$F_z = -mg$$
$$ma_z = -mg$$
$$a_z = -g$$
$$v_z = -gt + c_5$$
$$= v_o \sin \theta - gt$$
$$r_z = v_o \sin(\theta)t - \frac{1}{2}gt^2 + c_6$$
$$= v_o \sin(\theta)t - \frac{1}{2}gt^2$$

$$0 = v_o \sin(\theta)t - \frac{1}{2}gt^2$$
$$t = \frac{2\sin(\theta)v_o}{g}$$
$$r_x = v_o \cos(\theta)t$$
$$= \frac{2\cos(\theta)\sin(\theta)v_o^2}{g}$$
$$= \frac{\sin(2\theta)v_o^2}{g}$$

1.37

(a)

$$F = -mg\sin\theta$$
$$ma = -mg\sin\theta$$
$$a = -g\sin\theta$$
$$v = c_1 - gt\sin\theta$$
$$= v_o - gt\sin\theta$$
$$x = v_o t - \frac{1}{2}gt^2\sin\theta$$

(b)

$$t = \frac{2v_o}{g\sin\theta}$$

1.39

$$F_x = -mg\sin\phi$$

$$ma_x = -mg\sin\phi$$

$$a_x = -g\sin\phi$$

$$v_x = c_1 - gt\sin\phi$$

$$= v_o\cos\theta - gt\sin\phi$$

$$r_x = v_o t\cos\theta - \frac{1}{2}gt^2\sin\phi + c_2$$

$$= v_o t\cos\theta - \frac{1}{2}gt^2\sin\phi$$

$$F_y = -mg\cos\phi$$

$$ma_y = -mg\cos\phi$$

$$a_y = -g\cos\phi$$

$$v_y = c_3 - gt\cos\phi$$

$$= v_o\sin\theta - gt\cos\phi$$

$$r_y = v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi + c_4$$

$$= v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi$$

$$0 = v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi$$

$$t = \frac{2v_o\sin\theta}{g\cos\phi}$$

$$r_x = \frac{2v_o^2\cos\theta\sec\phi\sin\theta}{g} - \frac{2v_o^2\sec\phi\sin^2\theta\tan\phi}{g}$$

$$= \frac{2v_o^2\sin\theta(\cos\theta\cos\phi - \sin\theta\sin\phi)}{g\cos^2\phi}$$

$$= \frac{2v_o^2\sin\theta\cos(\theta+\phi)}{g\cos^2\phi}$$

$$\frac{dr_x}{d\theta} = \frac{2v_o^2}{g\cos^2\phi}[\cos\theta\cos(\theta+\phi) - \sin\theta\sin(\theta+\phi)]$$

$$= \frac{2v_o^2\cos(2\theta+\phi)}{g\cos^2\phi}$$

$$0 = \frac{2v_o^2\cos(2\theta+\phi)}{g\cos^2\phi}$$

$$= \cos(2\theta+\phi)$$

$$2\theta+\phi = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4} - \frac{\phi}{2}$$

$$r_{x,\max} = \frac{2v_o^2\sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right)\cos\left(\frac{\pi}{4} - \frac{\phi}{2} + \phi\right)}{g\cos^2\phi}$$

$$= \frac{v_o^2(1-\sin\phi)}{g\cos^2\phi}$$

$$= \frac{v_o^2}{g(1+\sin\phi)}$$

1.41

$$F = ma$$

$$T = m\frac{v^2}{R}$$

$$= m\frac{(\omega R)^2}{R}$$

$$= m\omega^2 R$$

1.47

(a)

$$\rho = \sqrt{x^2+y^2}$$

$$\phi = \arctan\frac{y}{x}$$

$$z = z$$

ρ is the distance of P from the z-axis.

The use of r may be unfortunate because it suggests it's the distance of P from the origin.

(b) $\hat{\boldsymbol{\rho}}$ points away from the z-axis, $\hat{\boldsymbol{\phi}}$ points counter-clockwise around the z-axis, and $\hat{\mathbf{z}}$ points in the positive z direction.

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}} + \sqrt{x^2 + y^2}\,\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$$

(c)

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$
$$= \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\frac{d\hat{\boldsymbol{\rho}}}{dt} + \dot{z}\hat{\mathbf{z}} + z\frac{d\hat{\mathbf{z}}}{dt}$$
$$= \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$
$$= \ddot{\rho}\hat{\boldsymbol{\rho}} + \dot{\rho}\frac{d\hat{\boldsymbol{\rho}}}{dt} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}} + \rho\ddot{\phi}\hat{\boldsymbol{\phi}} + \rho\dot{\phi}\frac{d\hat{\boldsymbol{\phi}}}{dt} + \ddot{z}\hat{\mathbf{z}}$$
$$= \ddot{\rho}\hat{\boldsymbol{\rho}} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}} + \rho\ddot{\phi}\hat{\boldsymbol{\phi}} - \rho\dot{\phi}^2\hat{\boldsymbol{\rho}} + \ddot{z}\hat{\mathbf{z}}$$
$$= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\boldsymbol{\rho}} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\mathbf{z}}$$

2   Projectiles and Charged Particles

2.1

$$1 = (1.6 \times 10^3)Dv$$
$$v = \frac{1}{(1.6 \times 10^3)D}$$
$$= 8.9\,\text{mm/s}$$

When $v \gg 1\,\text{cm/s}$ the drag force can be treated as purely quadratic. For a beach ball this becomes $v \gg 1\,\text{mm/s}$.

2.3

(a)

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} = \frac{(1/4)\rho A v^2}{3\pi\eta Dv}$$
$$= \frac{\rho\pi\left(\frac{D}{2}\right)^2 v}{12\pi\eta D}$$
$$= \frac{\rho D v}{48\eta}$$
$$= \frac{R}{48}$$

10

(b)

$$R = \frac{Dv\rho}{\eta} \approx 0.01$$

2.5

$$
\begin{aligned}
v_y(t) &= v_{\text{ter}} + (v_{\text{yo}} - v_{\text{ter}})e^{-t/\tau} \\
&= v_{\text{ter}} + (2v_{\text{ter}} - v_{\text{ter}})e^{-t/\tau} \\
&= v_{\text{ter}}(1 + e^{-t/\tau})
\end{aligned}
$$

The velocity starts at $2v_{\text{ter}}$ and asymptotically approaches v_{ter}.

2.7

$$
\begin{aligned}
F &= F(v) \\
m\dot{v} &= F(v) \\
m\frac{dv}{F(v)} &= dt \\
t &= \int_{v_{\text{o}}}^{v} m\frac{dv'}{F(v')}
\end{aligned}
$$

$$
\begin{aligned}
F &= F(v) \\
m\dot{v} &= F_{\text{o}} \\
v &= \frac{F_{\text{o}}}{m}t + c
\end{aligned}
$$

2.11

(a)

$$m\dot{v} = -mg - bv$$

$$\dot{v} = -g - kv$$

$$\frac{1}{-g - kv}\dot{v} = 1$$

$$-\frac{1}{k}\ln(-g - kv) = t + c$$

$$\ln(-g - kv) = c - \frac{t}{\tau}$$

$$-g - kv = Ae^{-t/\tau}$$

$$v = \tau(-g - Ae^{-t/\tau})$$

$$= -v_{\text{ter}} - \tau Ae^{-t/\tau}$$

$$v_{\text{o}} = -v_{\text{ter}} - \tau A$$

$$A = -k(v_{\text{o}} + v_{\text{ter}})$$

$$v = -v_{\text{ter}} + (v_{\text{o}} + v_{\text{ter}})e^{-t/\tau}$$

$$y = -v_{\text{ter}}t - \tau(v_{\text{o}} + v_{\text{ter}})e^{-t/\tau} + c$$

$$0 = -\tau(v_{\text{o}} + v_{\text{ter}}) + c$$

$$c = \tau(v_{\text{o}} + v_{\text{ter}})$$

$$y = -v_{\text{ter}}t + \tau(v_{\text{o}} + v_{\text{ter}})(1 - e^{-t/\tau})$$

(b)

$$0 = -v_{\text{ter}} + (v_{\text{o}} + v_{\text{ter}})e^{-t/\tau}$$

$$e^{-t/\tau} = \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}}$$

$$-\frac{t}{\tau} = \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}}$$

$$t = -\tau \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}}$$

$$y_{\max} = -v_{\text{ter}} \left(-\tau \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} \right) + \tau(v_{\text{o}} + v_{\text{ter}}) \left(1 - \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} \right)$$

$$= \tau v_{\text{ter}} \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} + \tau(v_{\text{o}} + v_{\text{ter}} - v_{\text{ter}})$$

$$= \tau \left(v_{\text{o}} + v_{\text{ter}} \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} \right)$$

$$= \tau \left[v_{\text{o}} - v_{\text{ter}} \ln \left(1 + \frac{v_{\text{o}}}{v_{\text{ter}}} \right) \right]$$

(c)

$$y_{\max} = \tau \left[v_{\text{o}} - v_{\text{ter}} \ln \left(1 + \frac{v_{\text{o}}}{v_{\text{ter}}} \right) \right]$$

$$= \tau \left[v_{\text{o}} - g\tau \ln \left(1 + \frac{v_{\text{o}}}{g\tau} \right) \right]$$

$$\approx \tau \left\{ v_{\text{o}} - g\tau \left[\frac{v_{\text{o}}}{g\tau} - \frac{1}{2} \left(\frac{v_{\text{o}}}{g\tau} \right)^2 \right] \right\}$$

$$= \tau \left(v_{\text{o}} - v_{\text{o}} + \frac{1}{2} \frac{v_{\text{o}}^2}{g\tau} \right)$$

$$= \frac{1}{2} \frac{v_{\text{o}}^2}{g}$$

2.13

$$v^2 = \frac{2}{m} \int_{x_0}^{x} -kx'\, dx'$$

$$= -\frac{2k}{m} \left(\frac{1}{2}x^2 - \frac{1}{2}x_0^2 \right)$$

$$= -\frac{k}{m}(x^2 - x_0^2)$$

$$v = \sqrt{\frac{k}{m}(x_0^2 - x^2)}$$

$$= \omega\sqrt{x_0^2 - x^2}$$

$$\int_{x_0}^{x} \frac{1}{\sqrt{x_0^2 - x'^2}}\, dx' = \int_{0}^{t} \omega\, dt$$

$$\arctan \frac{x}{\sqrt{x_0^2 - x^2}} - \arctan \frac{x_0}{\sqrt{x_0^2 - x_0^2}} = \omega t$$

$$\arctan \frac{x}{\sqrt{x_0^2 - x^2}} = \omega t + \frac{\pi}{2}$$

$$\frac{x}{\sqrt{x_0^2 - x^2}} = \tan\left(\omega t + \frac{\pi}{2}\right)$$

$$= -\cot \omega t$$

$$\frac{\sqrt{x_0^2 - x^2}}{x} = -\tan \omega t$$

$$\sqrt{x_0^2 - x^2} = -x \tan \omega t$$

$$x_0^2 - x^2 = x^2 \tan^2 \omega t$$

$$x^2 = \frac{x_0^2}{1 + \tan^2 \omega t}$$

$$= \frac{x_0^2 \cos^2 \omega t}{\cos^2 \omega t + \sin^2 \omega t}$$

$$= x_0^2 \cos^2 \omega t$$

$$x = x_0 \cos \omega t$$

14

2.15

$$a_y = -g$$

$$v_y = v_{y0} - gt$$

$$y = v_{y0}t - \frac{1}{2}gt^2$$

$$0 = v_{y0}t - \frac{1}{2}gt^2$$

$$t = \frac{2v_{y0}}{g}$$

$$x = v_{x0}t$$

$$R = \frac{2v_{x0}v_{y0}}{g}$$

2.19

(a)

$$x = v_{x0}t$$

$$y = v_{y0}t - \frac{1}{2}gt^2$$

$$= \frac{v_{y0}}{v_{x0}}x - \frac{1}{2}g\left(\frac{x}{v_{x0}}\right)^2$$

(b)

$$y = \frac{v_{y0} + v_{\text{ter}}}{v_{x0}}x + v_{\text{ter}}\tau \ln\left(1 - \frac{x}{v_{x0}\tau}\right)$$

$$\approx \frac{v_{y0}}{v_{x0}}x + \frac{g\tau}{v_{x0}}x - g\tau^2\left[\frac{x}{v_{x0}\tau} + \frac{1}{2}\left(\frac{x}{v_{x0}\tau}\right)^2\right]$$

$$= \frac{v_{y0}}{v_{x0}}x - \frac{1}{2}g\left(\frac{x}{v_{x0}}\right)^2$$

2.23

(a)

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}}$$

$$= \sqrt{\frac{mg}{\gamma D^2}}$$

$$= \sqrt{\frac{mg}{0.25D^2}}$$

$$= \sqrt{\frac{\rho\frac{4}{3}\pi\left(\frac{D}{2}\right)^3 g}{0.25D^2}}$$

$$= \sqrt{\frac{4\pi\rho Dg}{6}}$$

$$= 22\,\text{m/s}$$

(b)

$$m = \rho V$$

$$= \rho\frac{4}{3}\pi\left(\frac{D}{2}\right)^3$$

$$= \frac{\pi\rho D^3}{6}$$

$$D^2 = \left(\frac{6m}{\pi\rho}\right)^{2/3}$$

$$v_{\text{ter}} = \sqrt{\frac{mg}{0.25D^2}}$$

$$= \sqrt{\frac{mg}{0.25(6m/\pi\rho)^{2/3}}}$$

$$= 140\,\text{m/s}$$

(c)

$$v_{\text{ter}} = 107\,\text{m/s}$$

2.27

$$m\dot{v} = -mg\sin\theta - cv^2$$

$$-\frac{\sqrt{m}\arctan\frac{\sqrt{c}v}{\sqrt{gm\sin\theta}}}{\sqrt{cg\sin\theta}} = t + c_1$$

$$\arctan\frac{\sqrt{c}v}{\sqrt{gm\sin\theta}} = \sqrt{\frac{cg\sin\theta}{m}}(c_1 - t)$$

$$\frac{\sqrt{c}v}{\sqrt{gm\sin\theta}} = \tan\left[\sqrt{\frac{cg\sin\theta}{m}}(c_1 - t)\right]$$

$$v = \sqrt{\frac{gm\sin\theta}{c}}\tan\left[\sqrt{\frac{cg\sin\theta}{m}}(c_1 - t)\right]$$

$$v_0 = \sqrt{\frac{gm\sin\theta}{c}}\tan\left(\sqrt{\frac{cg\sin\theta}{m}}c_1\right)$$

$$c_1 = \sqrt{\frac{m}{cg\sin\theta}}\arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right)$$

$$v = \sqrt{\frac{gm\sin\theta}{c}}\tan\left[\arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right) - \sqrt{\frac{cg\sin\theta}{m}}t\right]$$

$$0 = \sqrt{\frac{gm\sin\theta}{c}}\tan\left[\arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right) - \sqrt{\frac{cg\sin\theta}{m}}t\right]$$

$$\sqrt{\frac{cg\sin\theta}{m}}t = \arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right)$$

$$t = \sqrt{\frac{m}{cg\sin\theta}}\arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right)$$

2.29

$$v(t) = v_{\text{ter}}\tanh\frac{gt}{v_{\text{ter}}}$$

$$v(1) = 9.6\,\text{m/s}$$

$$v(5) = 38\,\text{m/s}$$

$$v(10) = 48\,\text{m/s}$$

$$v(20) = 50\,\text{m/s}$$

$$v(30) = 50\,\text{m/s}$$

2.31

(a)

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}}$$
$$= \sqrt{\frac{mg}{0.25D^2}}$$
$$= 20.2\,\text{m/s}$$

(b)

$$y = -30 + \frac{v_{\text{ter}}^2}{g} \ln\left(\cosh\frac{gt}{v_{\text{ter}}}\right)$$
$$0 = -30 + \frac{v_{\text{ter}}^2}{g} \ln\left(\cosh\frac{gt}{v_{\text{ter}}}\right)$$
$$t = 2.78\,\text{s}$$
$$v(2.78) = 17.6\,\text{m/s}$$

2.33

(b)

$$\cosh z = \frac{e^z + e^{-z}}{2}$$
$$= \frac{1}{2}\left[\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots\right) + \left(1 - z + \frac{z^2}{2} - \frac{z^3}{6} + \cdots\right)\right]$$
$$= 1 + \frac{z^2}{2} + \frac{z^4}{24} + \frac{z^6}{720} + \cdots$$
$$\cos iz = 1 - \frac{(iz)^2}{2} + \frac{(iz)^4}{24} - \frac{(iz)^6}{720} + \cdots$$
$$= 1 + \frac{z^2}{2} + \frac{z^4}{24} + \frac{z^6}{720} + \cdots$$
$$= \cosh z$$
$$\sinh z = -i\sin iz$$

(c)

$$\frac{d}{dz}\cosh z = \frac{d}{dz}\left(\frac{e^z + e^{-z}}{2}\right)$$
$$= \frac{e^z - e^{-z}}{2}$$
$$= \sinh z$$
$$\frac{d}{dz}\sinh z = \frac{d}{dz}\left(\frac{e^z - e^{-z}}{2}\right)$$
$$= \frac{e^z + e^{-z}}{2}$$
$$= \cosh z$$

(d)

$$\cosh^2 z - \sinh^2 z = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2$$
$$= \frac{1}{4}(e^{2z} + 2 + e^{-2z} - e^{2z} + 2 - e^{-2z})$$
$$= 1$$

(e)

$$\int \frac{1}{\sqrt{1 + x^2}}\,dx = \int \frac{\cosh z}{\sqrt{1 + \sinh^2 z}}\,dz$$
$$= \int 1\,dz$$
$$= z$$
$$= \operatorname{arcsinh} x$$

2.35

(a)

$$m\dot{v} = mg - cv^2$$

$$\dot{v} = g\left(1 - \frac{v^2}{v_{\text{ter}}^2}\right)$$

$$\int_0^v \frac{1}{1 - v'^2/v_{\text{ter}}^2}\,dv' = \int_0^t g\,dt$$

$$v_{\text{ter}}\,\text{arctanh}\,\frac{v}{v_{\text{ter}}} = gt$$

$$v = v_{\text{ter}}\tanh\frac{gt}{v_{\text{ter}}}$$

$$y = \int_0^t v_{\text{ter}}\tanh\frac{gt'}{v_{\text{ter}}}\,dt'$$

$$= \frac{v_{\text{ter}}^2}{g}\ln\left[\cosh\left(\frac{gt}{v_{\text{ter}}}\right)\right]$$

(b)

$$v = g\tau\tanh\frac{t}{\tau}$$

$$y = g\tau^2\ln\left[\cosh\left(\frac{t}{\tau}\right)\right]$$

$$v(\tau) = g\tau\tanh 1$$

$$= 0.76v_{\text{ter}}$$

$$v(2\tau) = 0.96v_{\text{ter}}$$

$$v(3\tau) = 0.99v_{\text{ter}}$$

(c)

$$y = g\tau^2\ln\left[\cosh\left(\frac{t}{\tau}\right)\right]$$

$$= g\tau^2\ln\left(\frac{e^{t/\tau} + e^{-t/\tau}}{2}\right)$$

$$= g\tau^2\ln\left(\frac{e^{t/\tau}}{2}\right)$$

$$= g\tau^2(\ln e^{t/\tau} - \ln 2)$$

$$= g\tau t - g\tau^2\ln 2$$

$$= v_{\text{ter}}t - g\tau^2\ln 2$$

(d)

$$y = \frac{(v_{\text{ter}})^2}{g} \ln \left[\cosh \left(\frac{gt}{v_{\text{ter}}} \right) \right]$$

$$\approx \frac{(v_{\text{ter}})^2}{g} \ln \left[1 + \frac{1}{2} \left(\frac{gt}{v_{\text{ter}}} \right)^2 \right]$$

$$\approx \frac{(v_{\text{ter}})^2}{g} \frac{1}{2} \left(\frac{gt}{v_{\text{ter}}} \right)^2$$

$$= \frac{1}{2} g t^2$$

2.39

(a)

$$m\dot{v} = -cv^2 - 3$$

$$\int_{v_0}^{v} \frac{m}{-cv'^2 - 3} \, dv' = \int_0^t dt'$$

$$\frac{m}{\sqrt{3c}} \left[\arctan \left(\sqrt{\frac{c}{3}} v_0 \right) - \arctan \left(\sqrt{\frac{c}{3}} v \right) \right] = t$$

	Speed	Time
	15 m/s	6.34 s
(b)	10 m/s	18.4 s
	5 m/s	48.3 s
	0 m/s	142 s

2.41

$$m\dot{v} = -mg - cv^2$$

$$\dot{v} = -g\left[1 + \left(\frac{v}{v_{\text{ter}}}\right)^2\right]$$

$$v\frac{dv}{dy} = -g\left[1 + \left(\frac{v}{v_{\text{ter}}}\right)^2\right]$$

$$\int_{v_0}^{v} \frac{v'}{1 + (v'/v_{\text{ter}})^2}\, dv' = \int_0^y -g\, dy'$$

$$\frac{1}{2}v_{\text{ter}}^2[\ln(v_{\text{ter}}^2 + v^2) - \ln(v_{\text{ter}}^2 + v_0^2)] = -gy$$

$$\ln\frac{v_{\text{ter}}^2 + v^2}{v_{\text{ter}}^2 + v_0^2} = -\frac{2gy}{v_{\text{ter}}^2}$$

$$\frac{v_{\text{ter}}^2 + v^2}{v_{\text{ter}}^2 + v_0^2} = e^{-2gy/v_{\text{ter}}^2}$$

$$v = \sqrt{(v_{\text{ter}}^2 + v_0^2)e^{-2gy/v_{\text{ter}}^2} - v_{\text{ter}}^2}$$

$$0 = \sqrt{(v_{\text{ter}}^2 + v_0^2)e^{-2gy/v_{\text{ter}}^2} - v_{\text{ter}}^2}$$

$$v_{\text{ter}}^2 = (v_{\text{ter}}^2 + v_0^2)e^{-2gy/v_{\text{ter}}^2}$$

$$\frac{v_{\text{ter}}^2}{v_{\text{ter}}^2 + v_0^2} = e^{-2gy/v_{\text{ter}}^2}$$

$$-\frac{2gy}{v_{\text{ter}}^2} = \ln\frac{v_{\text{ter}}^2}{v_{\text{ter}}^2 + v_0^2}$$

$$y = -\frac{v_{\text{ter}}^2}{2g}\ln\frac{v_{\text{ter}}^2}{v_{\text{ter}}^2 + v_0^2}$$

$$= \frac{v_{\text{ter}}^2}{2g}\ln\frac{v_{\text{ter}}^2 + v_0^2}{v_{\text{ter}}^2}$$

$$y_{\max} = 17.6\,\text{m}$$

2.45

(a)

$$z = re^{i\theta}$$
$$= r(\cos\theta + i\sin\theta)$$
$$= \sqrt{x^2 + y^2}\left[\cos\left(\arctan\frac{y}{x}\right) + i\sin\left(\arctan\frac{y}{x}\right)\right]$$
$$= \sqrt{x^2 + y^2}\left(\frac{1}{\sqrt{1 + \frac{y^2}{x^2}}} + i\frac{y}{x\sqrt{1 + \frac{y^2}{x^2}}}\right)$$
$$= \sqrt{x^2 + y^2}\left(\frac{x}{\sqrt{x^2 + y^2}} + i\frac{y}{\sqrt{x^2 + y^2}}\right)$$
$$= x + iy$$

r is the distance between z and the origin, θ is the angle between the positive real axis and z.

(b)
$$z = \sqrt{3^2 + 4^2}e^{i\arctan 4/3} = 5e^{0.927i}$$

(c)
$$z = 2\cos-\frac{\pi}{3} + i2\sin-\frac{\pi}{3} = 1 - \sqrt{3}i$$

2.47

(a)

$$z + w = 9 + 4i$$
$$z - w = 3 + 12i$$
$$zw = (6 + 8i)(3 - 4i)$$
$$= 18 - 24i + 24i + 32$$
$$= 50$$
$$\frac{z}{w} = \frac{zw*}{ww*}$$
$$= \frac{(6 + 8i)(3 + 4i)}{(3 - 4i)(3 + 4i)}$$
$$= \frac{18 + 24i + 24i - 32}{9 + 12i - 12i + 16}$$
$$= \frac{-14 + 48i}{25}$$
$$= -\frac{14}{25} + \frac{48}{25}i$$

23

(b)

$$
\begin{aligned}
z + w &= \left(8\cos\frac{\pi}{3} + i8\sin\frac{\pi}{3}\right) + \left(4\cos\frac{\pi}{6} + i4\sin\frac{\pi}{6}\right) \\
&= (4 + 2\sqrt{3}) + i(4\sqrt{3} + 2) \\
z - w &= (4 - 2\sqrt{3}) + i(4\sqrt{3} - 2) \\
zw &= 32e^{i\pi/2} \\
&= 32i \\
\frac{z}{w} &= 2e^{i\pi/6} \\
&= \sqrt{3} + i
\end{aligned}
$$

2.49

(a)

$$
\begin{aligned}
z^2 &= (e^{i\theta})^2 \\
&= e^{i2\theta} \\
&= \cos 2\theta + i\sin 2\theta \\
z^2 &= (\cos\theta + i\sin\theta)^2 \\
&= \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta \\
&= \cos^2\theta + i\sin 2\theta - \sin^2\theta
\end{aligned}
$$

$$
\cos 2\theta = \cos^2\theta - \sin^2\theta
$$

$$
\begin{aligned}
\cos 2\theta + i\sin 2\theta &= \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta \\
i\sin 2\theta &= 2i\sin\theta\cos\theta \\
\sin 2\theta &= 2\sin\theta\cos\theta
\end{aligned}
$$

(b)

$$z^3 = (e^{i\theta})^3$$
$$= e^{i3\theta}$$
$$= \cos 3\theta + i \sin 3\theta$$
$$z^3 = (\cos \theta + i \sin \theta)^3$$
$$= \cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta$$

$$\cos 3\theta + i \sin 3\theta = \cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta$$
$$= \cos \theta (\cos^2 \theta - 3 \sin^2 \theta) + i(3 \cos^2 \theta \sin \theta - \sin^3 \theta)$$
$$= \cos \theta (\cos^2 \theta - 3 \sin^2 \theta) + i[3(1 - \sin^2 \theta) \sin \theta - \sin^3 \theta]$$
$$= \cos \theta (\cos^2 \theta - 3 \sin^2 \theta) + i(3 \sin \theta - 4 \sin^3 \theta)$$
$$\cos 3\theta = \cos \theta (\cos^2 \theta - 3 \sin^2 \theta)$$

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$$

2.53

$$\mathbf{B} = B_z\hat{\mathbf{z}}$$

$$\mathbf{E} = E_z\hat{\mathbf{z}}$$

$$\mathbf{v} \times \mathbf{B} = B_z v_y \hat{\mathbf{x}} - B_z v_x \hat{\mathbf{y}}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$= qB_z v_y \hat{\mathbf{x}} - qB_z v_x \hat{\mathbf{y}} + qE_z \hat{\mathbf{z}}$$

$$m\dot{v}_x = qB_z v_y$$

$$\dot{v}_x = \omega v_y$$

$$m\dot{v}_y = -qB_z v_x$$

$$\dot{v}_y = -\omega v_x$$

$$m\dot{v}_z = qE_z$$

$$\dot{v}_z = \frac{q}{m}E_z$$

$$\frac{d}{dt}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = c_1\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos\omega t - \begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin\omega t\right] + c_2\left[\begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos\omega t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin\omega t\right]$$

$$= \begin{pmatrix} c_1\sin\omega t - c_2\cos\omega t \\ c_1\cos\omega t + c_2\sin\omega t \end{pmatrix}$$

$$x = -\frac{c_1}{\omega}\cos\omega t - \frac{c_2}{\omega}\sin\omega t + x_0$$

$$y = \frac{c_1}{\omega}\sin\omega t - \frac{c_2}{\omega}\cos\omega t + y_0$$

$$v_z = \frac{q}{m}E_z t + v_{z0}$$

$$z = \frac{q}{2m}E_z t^2 + v_{z0}t + z_0$$

The particle moves in a helix oriented along the z-axis.

2.55

(a)

$$\mathbf{B} = B\hat{\mathbf{z}}$$
$$\mathbf{E} = E\hat{\mathbf{y}}$$
$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$
$$= Bqv_y\hat{\mathbf{x}} + q(E - Bv_x)\hat{\mathbf{y}}$$
$$\dot{v}_x = \frac{Bq}{m}v_y$$
$$= \omega v_y$$
$$\dot{v}_y = \frac{Eq}{m} - \frac{Bq}{m}v_x$$
$$= \frac{Eq}{m} - \omega v_x$$
$$\dot{v}_z = 0$$

The net force has no $\hat{\mathbf{z}}$ component, so the motion stays in the xy-plane.

(b)

$$0 = \frac{Eq}{m} - \omega v_x$$
$$v_x = \frac{Eq}{\omega m}$$
$$= \frac{E}{B}$$

(c)

$$\begin{pmatrix} \dot{v}_x \\ \dot{v}_y \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{Eq}{m} \end{pmatrix}$$

$$\mathbf{V}_c = \begin{pmatrix} c_1 \sin \omega t - c_2 \cos \omega t \\ c_1 \cos \omega t + c_2 \sin \omega t \end{pmatrix}$$

$$\mathbf{V}_p = \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{Eq}{m} \end{pmatrix}$$

$$= \begin{pmatrix} c_4 \omega \\ -c_3 \omega + \frac{Eq}{m} \end{pmatrix}$$

$$c_3 = \frac{Eq}{m\omega}$$

$$= \frac{E}{B}$$

$$= v_{\mathrm{dr}}$$

$$c_4 = 0$$

$$\mathbf{V}_p = \begin{pmatrix} v_{\mathrm{dr}} \\ 0 \end{pmatrix}$$

$$\mathbf{V} = \begin{pmatrix} c_1 \sin \omega t - c_2 \cos \omega t + v_{\mathrm{dr}} \\ c_1 \cos \omega t + c_2 \sin \omega t \end{pmatrix}$$

$$\begin{pmatrix} v_{x0} \\ 0 \end{pmatrix} = \begin{pmatrix} -c_2 + v_{\mathrm{dr}} \\ c_1 \end{pmatrix}$$

$$c_1 = 0$$

$$c_2 = v_{\mathrm{dr}} - v_{x0}$$

$$\mathbf{V} = \begin{pmatrix} v_{\mathrm{dr}} + (v_{x0} - v_{\mathrm{dr}}) \cos \omega t \\ -(v_{x0} - v_{\mathrm{dr}}) \sin \omega t \end{pmatrix}$$

(d)

$$x = v_{\mathrm{dr}} t + \frac{v_{x0} - v_{\mathrm{dr}}}{\omega} \sin \omega t + x_0$$

$$y = \frac{v_{x0} - v_{\mathrm{dr}}}{\omega} \cos \omega t + y_0$$

$$z = z_0$$

28

3   Momentum and Angular Momentum

3.3

$$mv_0 = \frac{m}{3}(v_1 + v_2\cos\theta + v_3\cos\theta)$$
$$3v_0 = v_0 + \sqrt{2}v_2$$
$$2v_0 = \sqrt{2}v_2$$
$$v_2 = \sqrt{2}v_0$$
$$\mathbf{v}_2 = \sqrt{2}v_0\left(\cos\frac{\pi}{4}\hat{\mathbf{x}} + \sin\frac{\pi}{4}\hat{\mathbf{y}}\right)$$
$$= v_0(\hat{\mathbf{x}} + \hat{\mathbf{y}})$$
$$\mathbf{v}_3 = v_0(\hat{\mathbf{x}} - \hat{\mathbf{y}})$$

3.7

$$v = v_{\text{ex}}\ln\frac{m_0}{m}$$
$$= 2079\,\text{m/s}$$
$$F = 25\,\text{MN}$$
$$W = 19.6\,\text{MN}$$

The thrust is 1.28 times the weight on Earth.

3.9

$$-\dot{m}v_{\text{ex}} = m_0 g$$
$$v_{\text{ex}} = -\frac{m_0 g}{\dot{m}}$$
$$= 2352\,\text{m/s}$$

3.11

(a)
$$m\dot{v} = -\dot{m}v_{\text{ex}} + F_{\text{ext}}$$

(b)

$$m\dot{v} = -\dot{m}v_{\mathrm{ex}} - mg$$

$$\dot{v} = -\frac{v_{\mathrm{ex}}}{m}\dot{m} - g$$

$$\int_0^t \dot{v}\,dt = \int_0^t \left(-\frac{v_{\mathrm{ex}}}{m}\dot{m} - g\right)\,dt$$

$$\int_0^v dv' = -v_{\mathrm{ex}}\int_{m_0}^m \frac{1}{m'}\,dm' - \int_0^t g\,dt'$$

$$v = -v_{\mathrm{ex}}\ln\frac{m}{m_0} - gt$$

$$= v_{\mathrm{ex}}\ln\frac{m_0}{m} - gt$$

(c)

$$v = 903\,\mathrm{m/s}$$

It would be $2079\,\mathrm{m/s}$ without gravity (2.3 times larger).

(d) The rocket wouldn't take off until it was light enough (from burning fuel) that its thrust was greater than its weight.

3.13

$$v = v_{\mathrm{ex}}\ln\frac{m_0}{m} - gt$$

$$= v_{\mathrm{ex}}\ln\frac{m_0}{m_0 - kt} - gt$$

$$y = v_{\mathrm{ex}}t - \frac{mv_{\mathrm{ex}}}{k}\ln\frac{m_0}{m} - \frac{1}{2}gt^2$$

$$y(2\,\mathrm{min}) = 40\,\mathrm{km}$$

3.15

$$M = m_1 + m_2 + m_3$$
$$= m_1 + m_1 + 10m_1$$
$$= 12m_1$$

$$\mathbf{R} = \frac{1}{12m_1}\left[m_1\begin{pmatrix}1\\1\\0\end{pmatrix} + m_2\begin{pmatrix}1\\-1\\0\end{pmatrix} + m_3\begin{pmatrix}0\\0\\0\end{pmatrix}\right]$$

$$= \frac{1}{12m_1}\begin{pmatrix}2m_1\\0\\0\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{6}\\0\\0\end{pmatrix}$$

3.17

If we let Earth be at the origin, then

$$R = \frac{dM_m}{M_e + M_m}$$
$$= 4630\,\mathrm{km}$$

The centre of mass is inside Earth.

3.19

(a) No external forces apply during the explosion so the path of the centre of mass would be unchanged.

(b) $100\,\mathrm{m}$ before the target.

(c) No.

3.21

$$\mathbf{R} = \frac{1}{M} \int \mathbf{r}\, dm$$

$$= \frac{2}{\sigma \pi R^2} \int \mathbf{r}\sigma\, dA$$

$$= \frac{2}{\pi R^2} \int_0^R \int_0^\pi \mathbf{r} r\, d\phi\, dr$$

$$= \frac{2}{\pi R^2} \int_0^R \int_0^\pi r(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}) r\, d\phi\, dr$$

$$= \frac{2}{\pi R^2} \int_0^R r^2 \int_0^\pi (\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}})\, d\phi\, dr$$

$$= \frac{2}{\pi R^2} \int_0^R r^2 [\sin\phi\hat{\mathbf{x}} - \cos\phi\hat{\mathbf{y}}]_0^\pi\, dr$$

$$= \frac{4}{\pi R^2} \int_0^R r^2\, dr\, \hat{\mathbf{y}}$$

$$= \frac{4R}{3\pi}\hat{\mathbf{y}}$$

3.25

$$L = L_0$$
$$I\omega = I_0\omega_0$$
$$mr^2\omega = mr_0^2\omega_0$$
$$\omega = \left(\frac{r_0}{r}\right)^2 \omega_0$$

3.29

$$I\omega = I_0\omega_0$$

$$\frac{2}{5}\left(\frac{4}{3}\pi R^3\rho\right) R^2\omega = \frac{2}{5}\left(\frac{4}{3}\pi R_0^3\rho\right) R_0^2\omega_0$$

$$\omega = \left(\frac{R_0}{R}\right)^5 \omega_0$$

If the radius doubles the angular velocity is $\omega_0/32$.

3.31

$$I = \int r^2 \, dm$$

$$= \int r^2 \sigma \, dA$$

$$= \frac{M}{\pi R^2} \int_0^R \int_0^{2\pi} r^3 \, d\phi \, dr$$

$$= \frac{1}{2} M R^2$$

3.33

$$I = \int r^2 \, dm$$

$$= \int r^2 \sigma \, dA$$

$$= \frac{M}{(2b)^2} \int_{-b}^b \int_{-b}^b (x^2 + y^2) \, dx \, dy$$

$$= \frac{M}{4b^2} \int_{-b}^b \left[\frac{1}{3} x^3 + x y^2 \right]_{-b}^b \, dy$$

$$= \frac{M}{4b} \int_{-b}^b \left(\frac{2}{3} b^2 + 2y^2 \right) dy$$

$$= \frac{M}{4b} \left[\frac{2}{3} b^2 y + \frac{2}{3} y^3 \right]_{-b}^b$$

$$= \frac{2}{3} M b^2$$

3.35

(b)

$$\Gamma_{\text{ext}} = RMg\sin\gamma$$

$$\dot{L} = \Gamma_{\text{ext}}$$
$$I\dot{\omega} = RMg\sin\gamma$$
$$\frac{3}{2}MR^2\dot{\omega} = RMg\sin\gamma$$
$$\dot{\omega} = \frac{2g\sin\gamma}{3R}$$

$$\dot{v} = R\dot{\omega}$$
$$= \frac{2}{3}g\sin\gamma$$

(c)

$$M\dot{v} = Mg\sin\gamma - f$$
$$f = M(g\sin\gamma - \dot{v})$$

$$\Gamma_{\text{ext}} = Rf$$
$$= RM(g\sin\gamma - \dot{v})$$

$$\dot{L} = \Gamma_{\text{ext}}$$
$$I\dot{\omega} = RM(g\sin\gamma - \dot{v})$$
$$\frac{1}{2}MR^2\dot{\omega} = RM(g\sin\gamma - \dot{v})$$
$$\dot{\omega} = \frac{2(g\sin\gamma - \dot{v})}{R}$$

$$\dot{v} = R\dot{\omega}$$
$$= 2(g\sin\gamma - \dot{v})$$
$$= \frac{2}{3}g\sin\gamma$$

3.37

(b)

$$\sum m_\alpha r'_\alpha = \sum m_\alpha (r_\alpha - R)$$
$$= \sum m_\alpha r_\alpha - \sum m_\alpha R$$
$$= MR - MR$$
$$= 0$$

4 Energy

4.3

(a)

$$\int_P^Q \mathbf{F} \cdot d\mathbf{r} = \int_P^O \mathbf{F} \cdot d\mathbf{r} + \int_O^Q \mathbf{F} \cdot d\mathbf{r}$$
$$= 0$$

(b)

$$x = 1 - t$$
$$y = t$$
$$\mathbf{r} = (1 - t)\hat{\mathbf{x}} + t\hat{\mathbf{y}}$$
$$d\mathbf{r} = (-\hat{\mathbf{x}} + \hat{\mathbf{y}})\, dt$$
$$\mathbf{F} \cdot d\mathbf{r} = (x + y)\, dt$$
$$= [(1 - t) + 1]\, dt$$
$$= dt$$
$$\int_P^Q \mathbf{F} \cdot d\mathbf{r} = \int_0^1 dt$$
$$= 1$$

(c)

$$\mathbf{r} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$$
$$d\mathbf{r} = (-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}})\, d\phi$$
$$\mathbf{F} \cdot d\mathbf{r} = (\sin^2\phi + \cos^2\phi)\, d\phi$$
$$= d\phi$$
$$\int_P^Q \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} d\phi$$
$$= \frac{\pi}{2}$$

4.7

(a)

$$\mathbf{F} = -m\gamma y^2 \hat{\mathbf{y}}$$

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r}$$

$$= -m\gamma \int_{y_1}^{y_2} y^2 \, dy$$

$$= -\frac{1}{3} m\gamma (y_2^3 - y_1^3)$$

$$U(\mathbf{r}) = \frac{1}{3} m\gamma y^3$$

(b) Assuming no friction

$$\frac{1}{2} m v^2 = \frac{1}{3} m\gamma h^3$$

$$v = \sqrt{\frac{2}{3} \gamma h^3}$$

4.9

(a)

$$U(x) = -\int_0^x -kx' \, dx'$$

$$= \frac{1}{2} k x^2$$

4.11

(a)

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = 2ay + 2bz$$

$$\frac{\partial f}{\partial z} = 2by + 2cz$$

(b)

$$\frac{\partial g}{\partial x} = -ay^2 z^3 \sin(axy^2 z^3)$$

$$\frac{\partial g}{\partial y} = -2axyz^3 \sin(axy^2 z^3)$$

$$\frac{\partial g}{\partial z} = -3axy^2 z^2 \sin(axy^2 z^3)$$

(c)

$$\frac{\partial h}{\partial x} = \frac{ax}{r}$$
$$\frac{\partial h}{\partial y} = \frac{ay}{r}$$
$$\frac{\partial h}{\partial z} = \frac{az}{r}$$

4.13

(a)

$$\nabla f = \frac{1}{r^2}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})$$

(b)

$$\nabla f = nr^{n-2}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})$$

(c)

$$\nabla f = \frac{g'(r)}{r}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})$$

4.15

$$df = \nabla f \cdot d\mathbf{r}$$
$$= (2, 4, 6) \cdot (0.01, 0.03, 0.05)$$
$$= 0.44$$
$$f(1.01, 1.03, 1.05) - f(1, 1, 1) = 0.4494$$

4.19

(a) An ellipse that is two times wider than it is tall.

(b)

$$\nabla f = (2x, 8y, 0)$$
$$\nabla f|_{(1,1,1)} = (2, 8, 0)$$
$$\mathbf{n} = (1, 4, 0)/\sqrt{17}$$

4.21

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{GMm}{r^3}x & -\frac{GMm}{r^3}y & -\frac{GMm}{r^3}z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(-\frac{GMm}{r^3}z \right) - \frac{\partial}{\partial z}\left(-\frac{GMm}{r^3}y \right) \right]\hat{\mathbf{x}}$$

$$- \left[\frac{\partial}{\partial x}\left(-\frac{GMm}{r^3}z \right) - \frac{\partial}{\partial z}\left(-\frac{GMm}{r^3}x \right) \right]\hat{\mathbf{y}}$$

$$+ \left[\frac{\partial}{\partial x}\left(-\frac{GMm}{r^3}y \right) - \frac{\partial}{\partial y}\left(-\frac{GMm}{r^3}x \right) \right]\hat{\mathbf{z}}$$

$$= -GMm\left[\left(-\frac{3yz}{r^5} + \frac{3yz}{r^5} \right)\hat{\mathbf{x}} + \left(-\frac{3xz}{r^5} + \frac{3xz}{r^5} \right)\hat{\mathbf{y}} \right.$$

$$\left. \left(-\frac{3xy}{r^5} + \frac{3xy}{r^4} \right)\hat{\mathbf{z}} \right]$$

$$= \mathbf{0}$$

$$U(\mathbf{r}) = -\int_{\infty}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}'$$

$$= -\int_{\infty}^{r} -\frac{GMm}{r'^2}\, dr'$$

$$= GMm\left[-\frac{1}{r'} \right]_{\infty}^{r}$$

$$= GMm\left(-\frac{1}{r} + \frac{1}{\infty} \right)$$

$$= -\frac{GMm}{r}$$

4.23

(a)

$$\nabla \times \mathbf{F} = \mathbf{0}$$

$$U = -\int_{\mathbf{0}}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}$$

$$= -\left(\int_0^x kx\, dx + \int_0^y 2ky\, dy + \int_0^z 3kz\, dz \right)$$

$$= -\frac{1}{2}k(x^2 + 2y^2 + 3z^2)$$

(b)

$$\nabla \times \mathbf{F} = \mathbf{0}$$

$$U = -\int_{\mathbf{0}}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}$$

$$= -\left(\int_0^x ky\,dx + \int_0^y kx\,dy\right)$$

$$= -kxy$$

(c)

$$\nabla \times \mathbf{F} = 2k\hat{\mathbf{z}}$$

Not conservative

4.29

(a) The mass will oscillate around $x = 0$.

(b)

$$t = \sqrt{\frac{m}{2}} \int_0^A \frac{dx}{\sqrt{kA^4 - kx^4}}$$

$$= \sqrt{\frac{m}{2k}} \int_0^A \frac{dx}{\sqrt{A^4 - x^4}}$$

$$\tau = 4t$$

(d) $\tau \approx 3.71\,\mathrm{s}$

4.31

(a)

$$E = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_2 - m_1)gx$$

(b)

$$(m_1 + m_2)\ddot{x} = (m_1 - m_2)g$$

$$c = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_2 - m_1)gx$$

$$0 = (m_1 + m_2)\dot{x}\ddot{x} + (m_2 - m_1)g\dot{x}$$

$$(m_1 + m_2)\ddot{x} = (m_1 - m_2)g$$

4.35

(a)

$$E = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}^2 + (m_2 - m_1)gx$$

(b)

$$0 = \left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}\ddot{x} + (m_2 - m_1)g\dot{x}$$

$$\left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} = (m_1 - m_2)g$$

$$m_1\ddot{x} = m_1 g - T_2$$
$$T_2 = m_1 g - m_1\ddot{x}$$

$$m_2\ddot{x} = T_1 - m_2 g$$
$$T_1 = m_2\ddot{x} + m_2 g$$

$$\omega = -\frac{\dot{x}}{R}$$
$$\dot{\omega} = -\frac{\ddot{x}}{R}$$

$$I\dot{\omega} = (T_1 - T_2)R$$
$$-I\frac{\ddot{x}}{R} = (m_2\ddot{x} + m_2 g - m_1 g - m_1\ddot{x})R$$
$$\left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} = (m_1 - m_2)g$$

4.37

(a)

$$U(\phi) = MgR(1 - \cos\phi) - mgR\phi$$

(b)

$$\frac{dU(\phi)}{d\phi} = MgR\sin\phi - mgR$$
$$= gR(M\sin\phi - m)$$
$$0 = gR(M\sin\phi - m)$$
$$m = M\sin\phi$$

$$\frac{d^2U(\phi)}{d\phi^2} = MgR\cos\phi$$

There is a position of equilibrium at $\phi = \arcsin\frac{m}{M}$. It is stable if $\phi < \frac{\pi}{2}$, i.e. $m < M$.

4.51

$$U(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = U_{\text{int}} + U_{\text{ext}}$$
$$= [U_{12}(\mathbf{r}_1 - \mathbf{r}_2) + U_{13}(\mathbf{r}_1 - \mathbf{r}_3) + U_{14}(\mathbf{r}_1 - \mathbf{r}_4) + U_{23}(\mathbf{r}_2 - \mathbf{r}_3)$$
$$+ U_{24}(\mathbf{r}_2 - \mathbf{r}_4) + U_{34}(\mathbf{r}_3 - \mathbf{r}_4)] + [U_1(\mathbf{r}_1) + U_2(\mathbf{r}_2) + U_3(\mathbf{r}_3)$$
$$+ U_4(\mathbf{r}_4)]$$
$$\mathbf{F}_3 = \mathbf{F}_{3,\text{int}} + \mathbf{F}_{3,\text{ext}}$$
$$= [\mathbf{F}_{13} + \mathbf{F}_{23} + \mathbf{F}_{34}] + \mathbf{F}_{3,\text{ext}}$$
$$= -\nabla_3 U_{13} - \nabla_3 U_{23} - \nabla_3 U_{34} - \nabla_3 U_{3,\text{ext}}$$
$$= -\nabla_3 U$$

4.53

(a)

$$F = ma$$
$$\frac{ke^2}{r^2} = m\frac{v^2}{r}$$
$$v^2 = \frac{ke^2}{mr}$$
$$K = \frac{1}{2}mv^2$$
$$= \frac{ke^2}{2r}$$
$$U = -\frac{ke^2}{r}$$
$$K = -\frac{1}{2}U$$

(b)

$$E = K_1 + K_2 + U_{12} + U_{1p} + U_{2p}$$
$$= \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 - ke^2 \left(\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_{12}} \right)$$

(c)

$$E_{\text{before}} = K_1 + K_2 + U_1 + U_2 + U_{12}$$
$$= T_2 + \frac{ke^2}{2r} - \frac{ke^2}{r}$$
$$= T_2 - \frac{ke^2}{2r}$$

$$E_{\text{after}} = K_1' + K_2' + U_1 + U_2 + U_{12}$$
$$= T_1' + \frac{ke^2}{2r'} - \frac{ke^2}{r'}$$
$$= T_1' - \frac{ke^2}{2r'}$$
$$T_2 - \frac{ke^2}{2r} = T_1' - \frac{ke^2}{2r'}$$
$$T_1' = T_2 + \frac{ke^2}{2} \left(\frac{1}{r'} - \frac{1}{r} \right)$$

5 Oscillations

5.3

$$U(\phi) = mgl(1 - \cos \phi)$$
$$\approx mgl \left(1 - 1 + \frac{1}{2}\phi^2 \right)$$
$$= \frac{1}{2}mgl\phi^2$$
$$k = mgl$$

5.5

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$
$$= C_1(\cos\omega t + i\sin\omega t) + C_2(\cos\omega t - i\sin\omega t)$$
$$= (C_1 + C_2)\cos\omega t + i(C_1 - C_2)\sin\omega t$$
$$= B_1\cos\omega t + B_2\sin\omega t$$
$$B_1 = C_1 + C_2$$
$$B_2 = i(C_1 - C_2)$$

$$x(t) = A\cos(\omega t - \delta)$$
$$A = \sqrt{B_1^2 + B_2^2}$$
$$\delta = \arctan\frac{B_2}{B_1}$$

$$x(t) = C\,\mathrm{Re}\,e^{\omega t}$$
$$C = Ae^{-i\delta}$$

5.7

(a) $B_1 = x_0$, $B_2 = \frac{v_0}{\omega}$

(b)

$$\omega = \sqrt{\frac{k}{m}}$$
$$= 10\,\mathrm{rad/s}$$
$$B_1 = 3.0\,\mathrm{m}$$
$$B_2 = 5.0\,\mathrm{m}$$

(c) $x = 0\,\mathrm{m}$ at $t = 0.26\,\mathrm{s}$, $\dot{x} = 0\,\mathrm{m/s}$ at $t = 0.10\,\mathrm{s}$.

5.9

$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2$$

$$\frac{k}{m} = \left(\frac{v}{A}\right)^2$$

$$\tau = \frac{2\pi}{\omega}$$

$$= \frac{2\pi}{\sqrt{k/m}}$$

$$= \frac{2\pi}{v/A}$$

$$= 1.05\,\text{s}$$

5.11

$$\frac{1}{2}kx_1^2 + \frac{1}{2}mv_1^2 = \frac{1}{2}kx_2^2 + \frac{1}{2}mv_2^2$$

$$kx_1^2 + mv_1^2 = kx_2^2 + mv_2^2$$

$$\frac{k}{m}x_1^2 + v_1^2 = \frac{k}{m}x_2^2 + v_2^2$$

$$\omega^2(x_1^2 - x_2^2) = v_2^2 - v_1^2$$

$$\omega = \sqrt{\frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}}$$

$$\frac{1}{2}kA^2 = \frac{1}{2}kx_1^2 + \frac{1}{2}mv_1^2$$

$$A^2 = x_1^2 + \frac{m}{k}v_1^2$$

$$A = \sqrt{x_1^2 + \frac{v_1^2}{\omega^2}}$$

$$= \sqrt{x_1^2 + v_1^2\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$$

$$= \sqrt{\frac{x_1^2(v_2^2 - v_1^2) + v_1^2(x_1^2 - x_2^2)}{v_2^2 - v_1^2}}$$

$$= \sqrt{\frac{x_2^2v_1^2 - x_1^2v_2^2}{v_1^2 - v_2^2}}$$

5.13

$$U(r) = U_0 \left(\frac{r}{R} + \lambda^2 \frac{R}{r} \right)$$

$$\frac{dU(r)}{dr} = U_0 \left(\frac{1}{R} - \lambda^2 \frac{R}{r^2} \right)$$

$$0 = \frac{dU(r_0)}{dr}$$

$$= U_0 \left(\frac{1}{R} - \lambda^2 \frac{R}{r_0^2} \right)$$

$$\frac{1}{R} = \lambda^2 \frac{R}{r_0^2}$$

$$r_0 = \lambda R$$

$$U(r_0 + x) - U(r_0) = U_0 \left(\frac{r_0 + x}{R} + \lambda^2 \frac{R}{r_0 + x} \right) - U_0 \left(\frac{r_0}{R} + \lambda^2 \frac{R}{r_0} \right)$$

$$= U_0 \left[\frac{1}{R} x + \lambda^2 R \left(\frac{1}{r_0 + x} - \frac{1}{r_0} \right) \right]$$

$$\approx U_0 \left[\frac{1}{R} x + \lambda^2 R \left(\frac{1}{r_0} - \frac{x}{r_0^2} + \frac{x^2}{r_0^3} - \frac{1}{r_0} \right) \right]$$

$$= U_0 \left[\frac{1}{R} x + \lambda^2 R \left(\frac{x^2}{r_0^3} - \frac{x}{r_0^2} \right) \right]$$

$$= U_0 \left[\frac{1}{R} x + \lambda^2 R \left(\frac{x^2}{(\lambda R)^3} - \frac{x}{(\lambda R)^2} \right) \right]$$

$$= \frac{U_0 x^2}{\lambda R^2}$$

$$= \frac{1}{2} \left(\frac{2U_0}{\lambda R^2} \right) x^2$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{2U_0}{\lambda m R^2}}$$

5.17

(a)

$$x(t) = A_x \cos \omega_x t$$
$$y(t) = A_y \cos(\omega_y t - \delta)$$
$$\frac{\omega_x}{\omega_y} = \frac{p}{q}$$
$$\omega_x \tau = 2\pi p$$
$$\omega_y \tau = 2\pi q$$
$$(\omega_x + \omega_y)\tau = 2\pi(p + q)$$
$$\tau = \frac{2\pi(p + q)}{\omega_x + \omega_y}$$
$$x(\tau) = A_x \cos\left(\omega_x \frac{2\pi(p + q)}{\omega_x + \omega_y}\right)$$
$$= A_x \cos\left(\frac{2\pi(p + q)}{1 + \omega_y/\omega_x}\right)$$
$$= A_x \cos\left(\frac{2\pi(p + q)}{1 + q/p}\right)$$
$$= A_x \cos\left(2\pi \frac{p(p + q)}{p + q}\right)$$
$$y(\tau) = A_y \cos\left(\omega_y \frac{2\pi(p + q)}{\omega_x + \omega_y} - \delta\right)$$
$$= A_y \cos\left(2\pi \frac{p + q}{1 + \omega_x/\omega_y} - \delta\right)$$
$$= A_y \cos\left(2\pi \frac{q(p + q)}{p + q} - \delta\right)$$

5.23

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x}$$
$$= \dot{x}(m\ddot{x} + kx)$$
$$= -b\dot{x}^2$$

5.25

(a)

$$\tau = \frac{2\pi}{\omega_1}$$

(b)

$$0 = Ae^{-\beta t}\cos\omega_1 t$$
$$= \cos\omega_1 t$$
$$t = \frac{\frac{\pi}{2} + n\pi}{\omega_1}, n \in \mathbb{Z}$$

The time between successive zeroes is π/ω_1. The period τ is twice this.

(c)

$$e^{-\beta\tau} = e^{-(\omega_0/2)(2\pi/\omega_1)}$$
$$= e^{-\pi\omega_0/\omega_1}$$
$$= e^{-\pi\omega_0/\sqrt{\omega_0^2 - (\omega_0/2)^2}}$$
$$= e^{-\pi\omega_0/\sqrt{3\omega_0^2/4}}$$
$$= e^{-\pi\sqrt{4/3}}$$
$$\approx 0.027$$

5.29

$$\tau_0 = 1\,\text{s}$$
$$\omega_0 = 2\pi f$$
$$= \frac{2\pi}{\tau}$$
$$= 2\pi\,\text{rad/s}$$
$$\frac{1}{2} = e^{-\beta \tau_1}$$
$$= e^{-2\pi\beta/\omega_1}$$
$$= e^{-2\pi\beta/\sqrt{\omega_0^2 - \beta^2}}$$
$$\ln\frac{1}{2} = -\frac{2\pi\beta}{\sqrt{\omega_0^2 - \beta^2}}$$
$$\sqrt{\omega_0^2 - \beta^2}\,\ln\frac{1}{2} = -2\pi\beta$$
$$(\omega_0^2 - \beta^2)\ln^2\frac{1}{2} = 4\pi^2\beta^2$$
$$\omega_0^2\ln^2\frac{1}{2} = \left(4\pi^2 + \ln^2\frac{1}{2}\right)\beta^2$$
$$\beta = \pm\frac{\ln\frac{1}{2}}{\sqrt{4\pi^2 + \ln^2\frac{1}{2}}}\omega_0$$
$$\approx 0.11\omega_0$$
$$\tau_1 = \frac{2\pi}{\omega_1}$$
$$= \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}}$$
$$= \frac{2\pi}{\sqrt{\omega_0^2 - 0.0121\omega_0^2}}$$
$$\approx 1.006\,\text{s}$$

5.43

(a)

$$4mg = 4kx$$
$$k = \frac{mg}{x}$$
$$\approx 4 \times 10^4\,\text{N/m}$$

(b)

$$\omega_0 = \sqrt{\frac{2k}{m}} = 40\,\text{rad/s} \approx 6\,\text{Hz}$$

(c)

$$5\,\mathrm{m/s} \approx 18\,\mathrm{km/h}$$

6 Calculus of Variations

6.5

$$|AP| = \frac{|AB|}{2}\operatorname{crd}\left(\frac{\pi}{2} - \theta\right)$$
$$= \frac{|AB|}{2}2\sin\left(\frac{\frac{\pi}{2} - \theta}{2}\right)$$
$$= |AB|\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$|PB| = \frac{|AB|}{2}\operatorname{crd}\left(\frac{\pi}{2} + \theta\right)$$
$$= \frac{|AB|}{2}2\sin\left(\frac{\frac{\pi}{2} + \theta}{2}\right)$$
$$= |AB|\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

$$S = |AP| + |PB|$$
$$= |AB|\left[\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + \sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right]$$

$$\frac{dS}{d\theta} = |AB|\left[-\frac{1}{2}\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + \frac{1}{2}\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right]$$
$$= \frac{1}{2}|AB|\left[\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right) - \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]$$

$$\frac{dS}{d\theta} = 0$$
$$\frac{\pi}{4} + \frac{\theta}{2} = \frac{\pi}{4} - \frac{\theta}{2}$$
$$\theta = 0$$

$$\frac{d^2S}{d\theta^2} = -\frac{1}{4}|AB|\left[\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + \sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right]$$
$$\left.\frac{d^2S}{d\theta^2}\right|_{\theta=0} = -\frac{\sqrt{2}}{4}|AB|$$

$\frac{dS}{d\theta} = 0$ and $\frac{d^2S}{d\theta^2} < 0$ at $\theta = 0$ so it is a maximum.

6.7

$$S = \int_P^Q dS$$

$$= \int_P^Q \sqrt{(R\,d\phi)^2 + (dz)^2}$$

$$= \int_{z_1}^{z_2} \sqrt{1 + \left(R\frac{d\phi}{dz}\right)^2}\, dz$$

$$f\left(z, \phi, \frac{d\phi}{dz}\right) = \sqrt{1 + \left(R\frac{d\phi}{dz}\right)^2}$$

$$\frac{\partial f}{\partial \phi} = 0$$

$$\frac{\partial f}{\partial \phi'} = \frac{R^2 \frac{d\phi}{dz}}{\sqrt{1 + \left(R\frac{d\phi}{dz}\right)^2}}$$

$$\frac{\partial f}{\partial \phi} - \frac{d}{dz}\frac{\partial f}{\partial \phi'} = 0$$

$$\frac{R^2 \frac{d\phi}{dz}}{\sqrt{1 + \left(R\frac{d\phi}{dz}\right)^2}} = c_1$$

$$R^4 \left(\frac{d\phi}{dz}\right)^2 = c_1 \left[1 + \left(R\frac{d\phi}{dz}\right)^2\right]$$

$$(R^4 - c_1 R^2) \left(\frac{d\phi}{dz}\right)^2 = c_1$$

$$\frac{d\phi}{dz} = \sqrt{\frac{c_1}{R^4 - c_1 R^2}}$$

$$= c_2$$

$$\phi = c_2 z + c_3$$

6.9

$$f(x, y, y') = y'^2 + yy' + y^2$$

$$\frac{\partial f}{\partial y} = y' + 2y$$

$$\frac{\partial f}{\partial y'} = 2y' + y$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 2y'' + y'$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$y' + 2y - 2y'' - y' = 0$$

$$y'' = y$$

$$y = c_1 e^x + c_2 e^{-x}$$

$$y(0) = 0$$

$$= c_1 + c_2$$

$$y(1) = 1$$

$$= c_1 e + c_2 e^{-1}$$

$$1 = -c_2 e + c_2 e^{-1}$$

$$= c_2(e^{-1} - e)$$

$$c_2 = \frac{1}{e^{-1} - e}$$

$$c_1 = \frac{1}{e - e^{-1}}$$

$$y = \frac{e^x}{e - e^{-1}} + \frac{e^{-x}}{e^{-1} - e}$$

$$= \frac{e^x - e^{-x}}{e - e^{-1}}$$

$$= \frac{\sinh x}{\sinh 1}$$

6.11

$$f(x, y, y') = \sqrt{x(1 + y'^2)}$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = \frac{\sqrt{x} y'}{\sqrt{1 + y'^2}}$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0$$

$$\frac{\sqrt{x} y'}{\sqrt{1 + y'^2}} = c_1$$

$$xy'^2 = c_1(1 + y'^2)$$

$$(x - c_1)y'^2 = c_1$$

$$y' = \sqrt{\frac{c_1}{x - c_1}}$$

$$y = 2c_1 \sqrt{\frac{x - c_1}{c_1}} + c_2$$

7 Lagrange's Equations

7.1

$$K = \frac{1}{2}m\mathbf{v}^2$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$U = mgz$$

$$\mathcal{L} = K - U$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

$$F_x = \frac{\partial \mathcal{L}}{\partial x}$$

$$= 0$$

$$F_y = \frac{\partial \mathcal{L}}{\partial y}$$

$$= 0$$

$$F_z = \frac{\partial \mathcal{L}}{\partial z}$$

$$= -mg$$

7.3

$$\mathcal{L} = K - U$$
$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2)$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$
$$m\ddot{x} = -kx$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}$$
$$m\ddot{y} = -ky$$

The mass is a harmonic oscillator in each dimension.

7.5

$$df = \frac{\partial f}{\partial r}\,dr + \frac{\partial f}{\partial \theta}\,d\theta$$
$$d\mathbf{r} = dr + r\,d\theta$$
$$df = \nabla f \cdot d\mathbf{r}$$
$$= (\nabla f)_r\,dr + (\nabla f)_\theta r\,d\theta$$
$$(\nabla f)_r = \frac{\partial f}{\partial r}$$
$$(\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}$$
$$\nabla f = \frac{\partial f}{\partial r}\,dr + \frac{1}{r}\frac{\partial f}{\partial \theta}\,d\theta$$

7.7

(a)
$$m_i\ddot{\mathbf{r}}_i = -\nabla U_i$$

(b)
$$\mathcal{L}(\mathbf{r}_1,\ldots,\mathbf{r}_n,\dot{\mathbf{r}}_1,\ldots,\dot{\mathbf{r}}_n) = \frac{1}{2}m_1\dot{r}_1^2 + \ldots + \frac{1}{2}m_n\dot{r}_n^2 - U(\mathbf{r}_1,\ldots,\mathbf{r}_n)$$

7.9

$$x = R\cos\phi$$
$$y = R\sin\phi$$
$$\phi = \arctan\frac{y}{x}$$

54

7.11

$$x = A\cos\omega t + l\sin\phi$$

$$y = l\cos\phi$$

$$\phi = \arctan\frac{x - A\cos\omega t}{y}$$

7.15

$$\mathcal{L} = K - U$$

$$= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_2 gx$$

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}}$$

$$(m_1 + m_2)\ddot{x} = m_2 g$$

$$\ddot{x} = \frac{m_2}{m_1 + m_2}g$$

7.17

$$\mathcal{L} = K - U$$

$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}m_2\dot{x}^2 - (m_2 gx - m_1 gx)$$

$$= \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}^2 + (m_1 - m_2)gx$$

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}}$$

$$(m_1 - m_2)g = \left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x}$$

$$\ddot{x} = \frac{(m_1 - m_2)g}{m_1 + m_2 + \frac{I}{R^2}}$$

7.21

$$\mathcal{L} = K - U$$

$$= \frac{1}{2}mv^2$$

$$= \frac{1}{2}m[\dot{r}^2 + (r\dot{\phi})^2]$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\omega^2)$$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}$$

$$m\ddot{r} = mr\omega^2$$

$$\ddot{r} = r\omega^2$$

$$r = c_1 e^{\omega t} + c_2 e^{-\omega t}$$

If the bead is initially at rest at the origin

$$0 = c_1 + c_2$$

$$0 = c_1\omega - c_2\omega$$

$$= c_1 - c_2$$

$$c_1 = 0$$

$$c_2 = 0$$

thus the bead stays at the origin.
   If it is released from a point $r_0 > 0$

$$r_0 = c_1 + c_2$$

$$0 = c_1 - c_2$$

$$c_1 = c_2$$

$$= \frac{r_0}{2}$$

$$r = \frac{r_0}{2}(e^{\omega t} + e^{-\omega t})$$

r eventually grows as $\frac{r_0}{2}e^{\omega t}$.

7.23

$$K = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m(\dot{x} + \dot{X})^2$$

$$= \frac{1}{2}m(\dot{x} - A\omega \sin \omega t)^2$$

$$= \frac{1}{2}m(\dot{x}^2 - 2\dot{x}A\omega \sin \omega t + A^2\omega^2 \sin^2 \omega t)$$

$$U = \frac{1}{2}kx^2$$

$$\mathcal{L} = K - U$$

$$= \frac{1}{2}m(\dot{x}^2 - 2\dot{x}A\omega \sin \omega t + A^2\omega^2 \sin^2 \omega t) - \frac{1}{2}kx^2$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$-kx = \frac{d}{dt}\left[\frac{1}{2}m\left(2\dot{x} - 2A\omega \sin \omega t\right)\right]$$

$$= m\ddot{x} - Am\omega^2 \cos \omega t$$

$$\ddot{x} + \frac{k}{m}x = A\omega^2 \cos \omega t$$

$$\ddot{x} + \omega_0^2 x = B \cos \omega t$$

7.27

$$K = \frac{1}{2}(4m)\dot{x}_1^2 + \frac{1}{2}(3m)(\dot{x}_1 - \dot{x}_2)^2 + \frac{1}{2}m(\dot{x}_1 + \dot{x}_2)^2$$

$$= 2m\dot{x}_1^2 + \frac{3}{2}m(\dot{x}_1^2 - 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2) + \frac{1}{2}m(\dot{x}_1^2 + 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2)$$

$$= \frac{1}{2}m(4\dot{x}_1^2 + 3\dot{x}_1^2 - 6\dot{x}_1\dot{x}_2 + 3\dot{x}_2^2 + \dot{x}_1^2 + 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2)$$

$$= 2m(2\dot{x}_1^2 - \dot{x}_1\dot{x}_2 + \dot{x}_2^2)$$

$$U = 4mgx_1 + 3mg(x_2 - x_1) - mg(x_1 + x_2)$$

$$= mg(4x_1 + 3x_2 - 3x_1 - x_1 - x_2)$$

$$= 2mgx_2$$

$$\mathcal{L} = K - U$$

$$= 2m(2\dot{x}_1^2 - \dot{x}_1\dot{x}_2 + \dot{x}_2^2) - 2mgx_2$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1}$$

$$0 = \frac{d}{dt}(8m\dot{x}_1 - 2m\dot{x}_2)$$

$$= 8m\ddot{x}_1 - 2m\ddot{x}_2$$

$$\ddot{x}_1 = \frac{1}{4}\ddot{x}_2$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_2}$$

$$-2mg = \frac{d}{dt}(-2m\dot{x}_1 + 4m\dot{x}_2)$$

$$= -2m\ddot{x}_1 + 4m\ddot{x}_2$$

$$\ddot{x}_2 = \frac{1}{2}\ddot{x}_1 - \frac{1}{2}g$$

$$\ddot{x}_1 = \frac{1}{4}\left(\frac{1}{2}\ddot{x}_1 - \frac{1}{2}g\right)$$

$$= \frac{1}{8}\ddot{x}_1 - \frac{1}{8}g$$

$$\frac{7}{8}\ddot{x}_1 = -\frac{1}{8}g$$

$$\ddot{x}_1 = -\frac{1}{7}g$$

The acceleration of the mass $4m$ is $\frac{1}{7}g$ downwards.

7.29

$$x = R\cos\omega t + l\sin\phi$$

$$\dot{x} = -R\omega\sin\omega t + l\dot{\phi}\cos\phi$$

$$y = R\sin\omega t - l\cos\phi$$

$$\dot{y} = R\omega\cos\omega t + l\dot{\phi}\sin\phi$$

$$K = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m[R^2\omega^2 + l^2\dot{\phi}^2 + 2R\omega l\dot{\phi}\sin(\phi - \omega t)]$$

$$U = mgy$$

$$= mg(R\sin\omega t - l\cos\phi)$$

$$\mathcal{L} = K - U$$

$$= \frac{1}{2}m[R^2\omega^2 + l^2\dot{\phi}^2 + 2R\omega l\dot{\phi}\sin(\phi - \omega t)] - mg(R\sin\omega t - l\cos\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}$$

$$mR\omega l\dot{\phi}\cos(\phi - \omega t) - mgl\sin\phi = \frac{d}{dt}[ml^2\dot{\phi} + mR\omega l\sin(\phi - \omega t)]$$

$$= ml^2\ddot{\phi} + mR\omega l\cos(\phi - \omega t)(\dot{\phi} - \omega)$$

$$l\ddot{\phi} = R\omega^2\cos(\phi - \omega t) - g\sin\phi$$

7.31

(a)

$$X = x + L\sin\phi$$
$$\dot{X} = \dot{x} + L\dot{\phi}\cos\phi$$
$$y = -L\cos\phi$$
$$\dot{y} = L\dot{\phi}\sin\phi$$

$$K = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Mv^2$$
$$= \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}LM\dot{\phi}(2\dot{x}\cos\phi + L\dot{\phi})$$

$$U = \frac{1}{2}kx^2 - MgL\cos\phi$$

$$\mathcal{L} = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}LM\dot{\phi}(2\dot{x}\cos\phi + L\dot{\phi}) - \frac{1}{2}kx^2 + MgL\cos\phi$$

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}}$$
$$-kx = (m+M)\ddot{x} + LM(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}$$
$$M(L\ddot{\phi} + \ddot{x}\cos\phi) = -Mg\sin\phi$$

(b)

$$-kx = (m+M)\ddot{x} + LM(\ddot{\phi} - \phi\dot{\phi}^2)$$
$$\approx (m+M)\ddot{x} + LM\ddot{\phi}$$

$$M(L\ddot{\phi} + \ddot{x}) = -Mg\phi$$

7.33

$$X = -x\cos\omega t$$
$$\dot{X} = -\dot{x}\cos\omega t + \omega x\sin\omega t$$
$$y = x\sin\omega t$$
$$\dot{y} = \dot{x}\sin\omega t + \omega x\cos\omega t$$
$$v^2 = \dot{X}^2 + \dot{y}^2$$
$$= (-\dot{x}\cos\omega t + \omega x\sin\omega t)^2 + (\dot{x}\sin\omega t + \omega x\cos\omega t)^2$$
$$= \dot{x}^2 + \omega^2 x^2$$
$$K = \frac{1}{2}mv^2$$
$$= \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2)$$
$$U = mgy$$
$$= mgx\sin\omega t$$
$$\mathcal{L} = K - U$$
$$= \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2) - mgx\sin\omega t$$
$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$
$$\ddot{x} - \omega^2 x = -g\sin\omega t$$
$$x = x_0\cosh\omega t + \frac{g}{2\omega^2}(\sin\omega - \sinh\omega t)$$

7.35

$$x = R\cos\omega t + R\cos(\omega t + \phi)$$
$$\dot{x} = -R[\omega\sin\omega t + (\omega + \dot{\phi})\sin(\omega t + \phi)]$$
$$y = R[\sin\omega t + \sin(\omega t + \phi)]$$
$$\dot{y} = R[\omega\cos\omega t + (\omega + \dot{\phi})\cos(\omega t + \phi)]$$
$$v^2 = R^2[\omega^2 + (\omega + \dot{\phi})^2 + 2\omega(\omega + \dot{\phi})\cos\phi]$$
$$\mathcal{L} = K - U$$
$$= \frac{1}{2}mv^2$$
$$= \frac{1}{2}mR^2[\omega^2 + (\omega + \dot{\phi})^2 + 2\omega(\omega + \dot{\phi})\cos\phi]$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$-m\omega(\omega + \dot{\phi})R^2 \sin\phi = mR^2(\ddot{\phi} - \omega\dot{\phi}\sin\phi)$$

$$-\omega^2 \sin\phi = \ddot{\phi}$$

This is the equation of motion for a simple pendulum. For small ϕ the angular frequency is ω.

7.37

(a)

$$K = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}m\dot{r}^2$$

$$= m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2$$

$$U = mgr$$

$$\mathcal{L} = K - U$$

$$= m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - mgr$$

(b)

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}$$

$$mr\dot{\phi}^2 - mg = \frac{d}{dt}(2m\dot{r})$$

$$= 2m\ddot{r}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$0 = \frac{d}{dt}(mr^2\dot{\phi})$$

$$\ell = mr^2\dot{\phi}$$

(c)

$$\dot{\phi} = \frac{\ell}{mr^2}$$

$$mr\left(\frac{\ell}{mr^2}\right)^2 - mg = 2m\ddot{r}$$

$$\frac{\ell^2}{mr^3} - mg = 2m\ddot{r}$$

$$m\ddot{r} = \frac{\ell^2}{2mr^3} - \frac{mg}{2}$$

$$\frac{\ell^2}{2mr_0^3} - \frac{mg}{2} = 0$$

$$\frac{2mr_0^3}{\ell^2} = \frac{2}{mg}$$

$$r_0 = \sqrt[3]{\frac{\ell^2}{m^2 g}}$$

7.39

(a)

$$K = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}I_\theta \dot{\theta}^2 + \frac{1}{2}I_\phi \dot{\phi}^2$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}m(r\sin\theta)^2\dot{\phi}^2$$

$$= \frac{1}{2}m[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2]$$

$$\mathcal{L} = K - U$$

$$= \frac{1}{2}m[\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2] - U(r)$$

(b)

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}$$

$$mr\dot{\theta}^2 + mr\sin^2\theta\dot{\phi}^2 - \frac{dU(r)}{dr} = m\ddot{r}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}}$$

$$mr^2\sin\theta\cos\theta\dot{\phi}^2 = \frac{d}{dt}(mr^2\dot{\theta})$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$0 = \frac{d}{dt}(mr^2\sin^2\theta\dot{\phi})$$

(c) The motion remains in the equatorial plane.

(d) The motion remains on that line of longitude.

7.41

$$z = k\rho^2$$

$$\dot{z} = 2k\rho\dot{\rho}$$

$$\phi = \omega t$$

$$\dot{\phi} = \omega$$

$$K = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}I_\phi\omega^2 + \frac{1}{2}m\dot{z}^2$$

$$= \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\omega^2 + \frac{1}{2}m(2k\rho\dot{\rho})^2$$

$$= \frac{1}{2}m[\dot{\rho}^2 + (\rho\omega)^2 + (2k\rho\dot{\rho})^2]$$

$$U = mgz$$

$$= mgk\rho^2$$

$$\mathcal{L} = K - U$$

$$= \frac{1}{2}m[\dot{\rho}^2 + (\rho\omega)^2 + (2k\rho\dot{\rho})^2] - mgk\rho^2$$

$$\frac{\partial \mathcal{L}}{\partial \rho} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\rho}}$$

$$m\rho\omega^2 + 4mk^2\rho\dot{\rho}^2 - 2mgk\rho = \frac{d}{dt}(m\dot{\rho} + 4mk^2\rho^2\dot{\rho})$$

$$= m\ddot{\rho} + 8mk^2\rho\dot{\rho}^2 + 4mk^2\rho^2\ddot{\rho}$$

$$(1 + 4k^2\rho^2)\ddot{\rho} + 4k^2\rho\dot{\rho}^2 = (\omega^2 - 2gk)\rho$$

8 Two-Body Central Force Problems

8.1

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r}$$

$$= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M} + \frac{m_2}{M}(\mathbf{r}_1 - \mathbf{r}_2)$$

$$= \frac{(m_1 + m_2)\mathbf{r}_1}{M}$$

$$= \mathbf{r}_1$$

$$\mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}$$

$$= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M} - \frac{m_1}{M}(\mathbf{r}_1 - \mathbf{r}_2)$$

$$= \frac{(m_1 + m_2)\mathbf{r}_2}{M}$$

$$= \mathbf{r}_2$$

$$K = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2$$

$$= \frac{1}{2}M\left(\frac{m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2}{M}\right)^2 + \frac{1}{2}\frac{m_1 m_2}{M}(\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2)^2$$

$$= \frac{1}{2}\frac{m_1^2\dot{\mathbf{r}}_1^2 + m_1 m_2\dot{\mathbf{r}}_1\dot{\mathbf{r}}_2 + m_2^2\dot{\mathbf{r}}_2^2}{M} + \frac{1}{2}\frac{m_1 m_2}{M}(\dot{\mathbf{r}}_1^2 - 2\dot{\mathbf{r}}_1\dot{\mathbf{r}}_2 + \dot{\mathbf{r}}_2^2)$$

$$= \frac{1}{2}\left(\frac{m_1^2 + m_1 m_2}{M}\dot{\mathbf{r}}_1^2 + \frac{m_2^2 + m_1 m_2}{M}\dot{\mathbf{r}}_2^2\right)$$

$$= \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2$$

8.2

(a)

$$\mathcal{L} = K - U$$

$$= \frac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{r}}_2^2 - m_1 g z_1 - m_2 g z_2 - U(r)$$

$$= \frac{1}{2} m_1 \left(\dot{\mathbf{R}} + \frac{m_2}{M} \dot{\mathbf{r}} \right)^2 + \frac{1}{2} m_2 \left(\dot{\mathbf{R}} - \frac{m_1}{M} \dot{\mathbf{r}} \right)^2 - m_1 g \left(Z + \frac{m_2}{M} z \right)$$

$$\quad - m_2 g \left(Z - \frac{m_1}{M} z \right) - U(r)$$

$$= \frac{1}{2} m_1 \left(\dot{\mathbf{R}}^2 + 2 \frac{m_2}{M} \dot{\mathbf{r}} \dot{\mathbf{R}} + \frac{m_2^2}{M^2} \dot{\mathbf{r}}^2 \right) + \frac{1}{2} m_2 \left(\dot{\mathbf{R}}^2 - 2 \frac{m_1}{M} \dot{\mathbf{r}} \dot{\mathbf{R}} + \frac{m_1^2}{M^2} \dot{\mathbf{r}}^2 \right)$$

$$\quad - m_1 g \left(Z + \frac{m_2}{M} z \right) - m_2 g \left(Z - \frac{m_1}{M} z \right) - U(r)$$

$$= \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \mu \dot{\mathbf{r}}^2 - M g Z - U(r)$$

$$= \left(\frac{1}{2} M \dot{\mathbf{R}}^2 - M g Z \right) + \left(\frac{1}{2} \mu \dot{\mathbf{r}}^2 - U(r) \right)$$

$$= \mathcal{L}_{\text{cm}} + \mathcal{L}_{\text{rel}}$$

8.3

$$\dot{\mathbf{R}} = \frac{m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2}{M}$$

$$\mathcal{L}_{\text{cm}} = \frac{1}{2}M\dot{\mathbf{R}}^2 - MgZ$$

$$= \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - MgZ$$

$$\frac{\partial\mathcal{L}_{\text{cm}}}{\partial Z} = \frac{d}{dt}\frac{\partial\mathcal{L}_{\text{cm}}}{\partial\dot{Z}}$$

$$-Mg = M\ddot{Z}$$

$$Z = \dot{R}_0 t - \frac{1}{2}gt^2$$

$$= \frac{m_1}{M}v_0 t - \frac{1}{2}gt^2$$

$$\mathcal{L}_{\text{rel}} = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - \frac{1}{2}k(r-L)^2$$

$$\frac{\partial\mathcal{L}_{\text{rel}}}{\partial r} = \frac{d}{dt}\frac{\partial\mathcal{L}_{\text{rel}}}{\partial\dot{r}}$$

$$-k(r-L) = \mu\ddot{r}$$

$$\omega = \sqrt{\frac{k}{\mu}}$$

$$r = \frac{v_0}{\omega}\sin\omega t + L$$

$$z_1 = Z + \frac{m_2}{M}r$$

$$= \frac{m_1}{M}v_0 t - \frac{1}{2}gt^2 + \frac{m_2}{M}\left(\frac{v_0}{\omega}\sin\omega t + L\right)$$

$$z_2 = Z - \frac{m_1}{M}r$$

$$= \frac{m_1}{M}v_0 t - \frac{1}{2}gt^2 - \frac{m_1}{M}\left(\frac{v_0}{\omega}\sin\omega t + L\right)$$

8.7

(a)

$$m_1 \frac{v^2}{r} = \frac{Gm_1m_2}{r^2}$$

$$v = \sqrt{\frac{Gm_2}{r}}$$

$$T = \frac{2\pi r}{v}$$

$$= 2\pi r \sqrt{\frac{r}{Gm_2}}$$

(b)

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{GM\mu}{r}$$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}$$

$$\mu r \dot{\phi}^2 - \frac{GM\mu}{r^2} = \mu \ddot{r}$$

$$r\left(\frac{v}{r}\right)^2 = \frac{GM}{r}$$

$$v = \sqrt{\frac{GM}{r}}$$

$$T = \frac{2\pi r}{v}$$

$$= \frac{2\pi r^{3/2}}{\sqrt{GM}}$$

This is equal to the period found in part (a) when $m_2 \to \infty$.

(c) $T = 0.703\,\text{years}$

8.9

(a)

$$\mathcal{L} = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - \frac{1}{2}k(|\mathbf{r}_1 - \mathbf{r}_2| - L)^2$$

$$= \frac{1}{2}m_1\left(\dot{\mathbf{R}} + \frac{m_2}{M}\dot{\mathbf{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\mathbf{R}} - \frac{m_1}{M}\dot{\mathbf{r}}\right)^2 - \frac{1}{2}k(r - L)^2$$

$$= \frac{1}{2}m_1\left(\dot{\mathbf{R}}^2 + 2\frac{m_2}{M}\dot{\mathbf{r}}\dot{\mathbf{R}} + \frac{m_2^2}{M^2}\dot{\mathbf{r}}^2\right) + \frac{1}{2}m_2\left(\dot{\mathbf{R}}^2 - 2\frac{m_1}{M}\dot{\mathbf{r}}\dot{\mathbf{R}} + \frac{m_1^2}{M^2}\dot{\mathbf{r}}^2\right)$$

$$\quad - \frac{1}{2}k(r - L)^2$$

$$= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - \frac{1}{2}k(r - L)^2$$

$$= \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{2}k(r - L)^2$$

(b)

$$\frac{\partial \mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}}$$

$$0 = M\ddot{X}$$

$$X = \dot{X}_0 t + X_0$$

$$\frac{\partial \mathcal{L}}{\partial Y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Y}}$$

$$0 = M\ddot{Y}$$

$$Y = \dot{Y}_0 t + Y_0$$

$$\frac{\partial \mathcal{L}}{\partial Z} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Z}}$$

$$0 = M\ddot{Z}$$

$$Z = \dot{Z}_0 t + Z_0$$

(c)

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}$$

$$\mu r \dot{\phi}^2 - k(r - L) = \mu \ddot{r}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$0 = \frac{d}{dt}(\mu r^2 \dot{\phi})$$

$$\mu r^2 \dot{\phi} = \ell$$

r remains constant:

$$\mu r \dot{\phi}^2 - k(r - L) = 0$$

$$\dot{\phi} = \sqrt{\frac{k(r - L)}{\mu r}}$$

$$r = \frac{\mu r \dot{\phi}^2}{k} + L$$

ϕ remains constant:

$$\mu \ddot{r} = -k(r - L)$$

$$\ddot{r} = -\frac{k}{\mu}(r - L)$$

$$= -\frac{2k}{m_1}(r - L)$$

$$= -\omega^2(r - L)$$

$$\omega = \sqrt{\frac{2k}{m_1}}$$

$$r = L + A\cos(\omega t - \delta)$$

71

(b)

$$U_{\text{eff}}(r) = U(r) + U_{\text{cf}}(r)$$

$$= \frac{1}{2}kr^2 + \frac{\ell^2}{2\mu r^2}$$

$$\frac{dU_{\text{eff}}(r)}{dr} = kr - \frac{\ell^2}{\mu r^3}$$

$$0 = kr - \frac{\ell^2}{\mu r^3}$$

$$r = \sqrt[4]{\frac{\ell^2}{\mu k}}$$

(c) $\omega = \sqrt{\frac{4k}{\mu}}$

8.15

$$\tau^2 = \frac{4\pi^2 \mu}{\gamma}a^3$$

$$= \frac{4\pi^2 \mu}{G\mu M}a^3$$

$$= \frac{4\pi^2}{G(M_s + m)}a^3$$

The constant would vary around 0.101%.

8.19

$$\frac{r_{\text{min}}}{r_{\text{max}}} = \frac{1-\epsilon}{1+\epsilon}$$

$$0.712 = \frac{1-\epsilon}{1+\epsilon}$$

$$0.712(1+\epsilon) = 1 - \epsilon$$

$$1.712\epsilon = 0.288$$

$$\epsilon = 0.168$$

$$c = a(1 - \epsilon^2)$$

$$= \frac{r_{\text{min}} + r_{\text{max}}}{2}(1 - \epsilon^2)$$

$$= 7795\,\text{km}$$

$$= 1424\,\text{km above the Earth's surface}$$