Advanced Engineering Mathematics Ordinary Differential Equations Notes

Chris Doble

February 2022

Contents

1 Introduction to Differential Equations

1.1 Definitions and Terminology

- An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation** (DE)

- An **ordinary DE** (ODE) is a DE that contains only ordinary (i.e. non-partial) derivatives of one or more functions with respect to a single independent variable

- A **partial DE** is a DE that contains only partial derivatives of one or more functions of two or more independent variables

- The **order** of a DE is the order of the highest derivative in the equation

- First order ODEs are sometimes written in the **differential form**

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

- n-th order ODEs in one dependent variable can be expressed by the **general form**

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

- It's possible to solve ODEs in the general form uniquely for the highest derivative $y^{(n)}$ in terms of the other $n + 1$ variables, allowing them to be expressed in the **normal form**

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

- An n-th order ODE is said be **linear** in the variable y if it can be expressed in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - g(x) = 0$$

  i.e. the dependent variable y and all of its derivatives aren't raised to a power or used in nonlinear functions like e^y or $\sin y$, and the coefficients a_0, a_1, …, a_n depend at most on the independent variable x

- A **nonlinear** ODE is one that is not linear

- A **solution** to an ODE is a function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, such that

$$F(x, \phi(x), \phi'(x), \ldots, \phi^n(x)) = 0 \text{ for all } x \text{ in } I.$$

- The **interval of definition**, **interval of validity**, or the **domain** of a solution is the interval over which the solution is valid

- A solution of a DE that is 0 on an interval I is said to be a **trivial solution**

- Because solutions to DEs must be differentiable over their interval of validity, discontinuities, etc. must be excluded from the interval

- An **explicit solution** to an ODE is one where the dependent variable is expressed solely in terms of the independent variable and constants

- An **implicit solution** to an ODE is a relation $G(x, y) = 0$ over an interval I provided there exists at least one function ϕ that satisfies the relation as well as the ODE on I

- When solving a first-order ODE we usually obtain a solution containing a single arbitrary constant or parameter c. A solution containing an arbitrary constant represents a set of solution called a **one-parameter family of solutions**

- When solving an n-th order DE we usually obtain an n**-parameter family of solutions**

- A solution of a DE that is free from arbitrary parameters is called a **particular solution**

- A **singular solution** is a solution to a DE that isn't a member of a family of solutions

- A **system of ODEs** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable. A solution of such a system is a differentiable function for each equation defined on a common interval I that satisfy each equation of the system on that interval

1.2 Initial Value Problems

- An **initial value problem** is the problem of solving a DE with some given **initial conditions**, e.g. solve

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)})$$

  subject to

$$y(x_0) = y_0,\ y'(x_0) = y_1,\ \dots,\ y^{(n-1)}(x_0) = y_{n-1}$$

- The domain of $y = f(x)$ differs depending on how it's considered:
  - As a function its domain is all real numbers for which it's defined
  - As a solution of a DE its domain is a single interval over which it's defined an differentiable
  - As a solution of an initial value problem its domain is a single interval over which it's defined, differentiable, and contains the initial conditions

- An initial value problem may not have any solutions. If it does it may have multiple.

- First-order initial value problems of the form

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

are guaranteed to have a unique solution over an interval I containing x_0 if $f(x, y)$ and $\partial f/\partial y$ are continuous

1.3 Differential Equations as Mathematical Models

- A **mathematical model** is a mathematical description of a system or phenomenon
- The **level of resolution** of a model determines how many variables are included in the model
- A simple model of the growth of a population P is

$$\frac{dP}{dt} = kP$$

where $k > 0$

- A simple model of radioactive decay of an amount of substance A is

$$\frac{dA}{dt} = kA$$

where $k < 0$

- Newton's empirical law of cooling/warming states that the rate of change of the temperature of a body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium

$$\frac{dT}{dt} = k(T - T_m)$$

2 First-Order Differential Equations

2.1 Solution Curves Without a Solution

- An ODE in which the independent variable doesn't appear is said to be **autonomous**, e.g.

$$\frac{dy}{dx} = f(y)$$

- A real number c is a **critical/equilibrium/stationary point** of an autonomous DE if it is a zero of f
- If c is a critial point of an autonomous DE, then $y(x) = c$ is a solution

- A solution of the form $y(x) = c$ is called an **equilibrium solution**
- We can draw several conclusions about the solutions of an autonomous DE with n critical points and $n+1$ subregions bounded by the critical points:
  - If (x_0, y_0) is in a subregion, it remains in that subregion for all x
  - By continuity, $f(y) < 0$ or $f(y) > 0$ for all y in a subregion and thus $y(x)$ can't have maximum/minimum points or oscillate
  - If $y(x)$ is bounded above by a critical point c_1, it must approach $y(x) = c_1$ as $x \to -\infty$ or $x \to \infty$
  - If $y(x)$ is bounded above and below by critical points c_1 and c_2, it must approach $y(x) = c_1$ as $x \to -\infty$ and $y(x) = c_2$ as $x \to \infty$ or vice versa
  - If $y(x)$ is bounded below by a critical point c_1, it must approach $y(x) = c_1$ as $x \to -\infty$ or $x \to \infty$

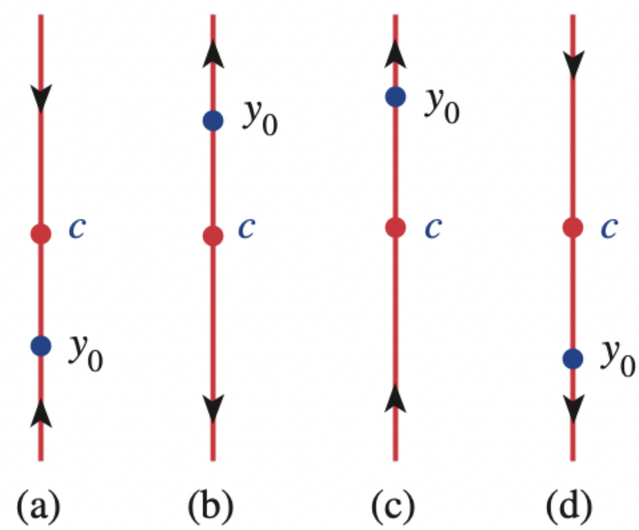

FIGURE 2.1.8 Critical point c is an attractor in (a), a repeller in (b), and semi-stable in (c) and (d)

- If $y(x)$ is a solution of an autonomous differential equation $dy/dx = f(y)$, then $y_1(x) = y(x-k)$, where k is a constant, is also a solution

2.2 Separable Equations

- A first-order ODE of the form
$$\frac{dy}{dx} = g(x)h(y)$$
is said to be **separable** or to have **separate variables**

- A separable first-order ODE can be solved by dividing both sides by $h(y)$ then integrating both sides with respect to x

$$\begin{aligned}
\frac{dy}{dx} &= g(x)h(y) \\
\frac{1}{h(y)}\frac{dy}{dx} &= g(x) \\
\int \frac{1}{h(y)}\frac{dy}{dx}\,dx &= \int g(x)\,dx \\
\int \frac{1}{h(y)}\,dy &= \int g(x)\,dx \\
H(y) &= G(x) + c
\end{aligned}$$

- Care should be taken when dividing by $h(y)$ as it removes constant solutions $y = r$ where $h(r) = 0$

2.3 Linear Equations

- A first-order DE of the form
$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$
or in standard form
$$\frac{dy}{dx} + P(x)y = f(x)$$
is said to be a **linear equation** in the dependent variable y

- When $g(x) = 0$ or $f(x) = 0$ the linear equation is said to be **homogeneous** and is solvable via separation of variables, otherwise it is **nonhomogeneous**

- The nonhomogeneous linear equation's solution is the sum of two solutions $y = y_c + y_p$ where y_c is a solution of the associated homogeneous equation
$$\frac{dy}{dx} + P(x)y = 0$$
and y_p is a particular solution of the nonhomogeneous equation

- Nonhomogeneous linear equations can be solved via **variation of parameters**:

  1. Put it into standard form
  2. Determine the **integrating factor** $e^{\int P(x)\,dx}$
  3. Multiply by the integrating factor
  4. Recognise that the left hand side of the equation is the derivative of the product of the integrating factor and y
  5. Integrate both sides of the equation
  6. Solve for y

- The **general solution** of a DE is a family of solutions that contains all possible solutions (except singular solutions)

- A term $y = f(x)$ in a solution is called a **transient term** if $f(x) \to 0$ as $x \to \infty$

- When either $P(x)$ or $f(x)$ is a piecewise-defined function the equation is then referred to as a **piecewise-linear differential equation** that can be solved by solving each interval in isolation then choosing appropriate constants to ensure the overall solution is continuous

- The **error function** and **complementary error function** are defined

$$\operatorname{erf} x + \operatorname{erfc} x = 1$$
$$\left(\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt\right) + \left(\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt\right) = 1$$

2.4 Exact Equations

- The **differential** of a function $z = f(x, y)$ is

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

- A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is an **exact differential** in the region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$

- A first-order DE of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

  is said to be an **exact equation** if the expression on the left side is an exact differential

- A necessary and sufficient condition that $M(x, y)\, dx + N(x, y)\, dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- Exact differentials can be solved by

  1. Integrating $M(x, y)$ with respect to x to find an expression for $f(x, y)$

$$\frac{\partial f}{\partial x} = M(x, y)$$
$$f(x, y) = \int M(x, y)\, dx + g(y)$$

  2. Differentiating $f(x, y)$ with respect to y and equating it to $N(x, y)$ to find $g'(y)$

$$\frac{\partial f}{\partial y} = N(x, y) = \frac{\partial}{\partial y} \int M(x, y)\, dx + g'(y)$$
$$g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, dx$$

  3. Integrating $g'(y)$ with respect to y to find $g(y)$ and substituting it into $f(x, y)$
  4. Equating $f(x, y)$ with an unknown constant c

- x and y can be swapped in the steps above (i.e. you can start by integrating $N(x, y)$ with respect to y, etc.)

- A nonexact DE $M(x, y)\, dx + N(x, y)\, dy = 0$ can sometimes be transformed into an exact DE by finding an appropriate integrating factor

  – If $(M_y - N_x)/N$ is a function of x alone, then an integrating factor is

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\, dx}$$

  – If $(N_x - M_y)/M$ is a function of y alone, then an integrating factor is

$$\mu(y) = e^{\int \frac{N_x - M_y}{M}\, dy}$$

2.5 Solutions by Substitution

- A function $f(x, y)$ is said to be a **homogeneous function** of degree α if

$$f(tx, ty) = t^{\alpha} f(x, y)$$

- A first-order DE of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be **homogeneous** if both M and N are homogeneous functions of the same degree

- To solve a homogeneous first-order DE:
  1. Rewrite it as
     $$M(x, y) = x^\alpha M(1, u) \text{ and } N(x, y) = x^\alpha N(1, u) \text{ where } u = y/x$$
     or
     $$M(x, y) = y^\alpha M(v, 1) \text{ and } N(x, y) = y^\alpha N(v, 1) \text{ where } v = x/y$$
  2. Substitute $y = ux$ and $dy = u\,dx + x\,du$ or $x = vy$ and $dx = v\,dy + y\,dv$ as appropriate
  3. Solve the resulting first-order separable DE
  4. Substitude $u = y/x$ or $v = x/y$ as appropriate

- The DE
$$\frac{dy}{dx} + P(x)y = f(x)y^n$$
where n is any real number is called **Bernoulli's equation**

- For $n = 0$ and $n = 1$ Bernoulli's equation is linear

- To solve Bernoulli's equation for $n \neq 0$ and $n \neq 1$:
  1. Substitude $y = u^{1/(1-n)}$ and $\frac{dy}{dx} = \frac{d}{dx}(u^{1/(1-n)})$
  2. Solve the resulting linear equation
  3. Substitude $u = y^{n-1}$

- A DE of the form
$$\frac{dy}{dx} = f(Ax + By + C)$$
can always be reduced to an equation with separable variables by means of the substitution
$$u = Ax + By + C, B \neq 0$$

2.6 A Numerical Method

- Approximate values for points on a solution curve near an initial point can be calculated via a **linearization** of the solution curve — a straight line that has the same slope as the initial point and passes through it

- **Euler's method** approximates a solution curve by iteratively stepping along its linearizations
$$y_{n+1} = y_n + hf(x_n, y_n)$$
where h is the **step size**

2.9 Modeling with Systems of First-Order DEs

- In a system of DEs
$$\frac{dx}{dt} = g_1(t, x, y)$$
and
$$\frac{dy}{dt} = g_2(t, x, y)$$
if g_1 and g_2 are linear in x and y, i.e.
$$g_1(t, x, y) = c_1x + c_2y + f_1(t)$$
and
$$g_2(t, x, y) = c_3x + c_4y + f_2(t)$$
it is said to be a **linear system**

3 Higher-Order Differential Equations

3.1 Theory of Linear Equations

- An n**th-order initial-value problem (IVP)** is to solve
$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx_{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$
subject to
$$y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1}$$

- If $a_n(x)$, $a_{n-1}(x)$, …, $a_1(x)$, $a_0(x)$, and $g(x)$ are continuous on an interval I and $a_n(x) \neq 0$ for every x in the interval, then then a unique solution exists for the above IVP for every $x = x_0$ within the interval

- An **initial value problem** is when all of the constraints are located at the same point while a **boundary value problem** is when they're at different points

- Boundary value problems may have many, one, or no solutions

- When $g(x) = 0$ the DE is said to be **homogeneous**, otherwise it's **non-homogeneous**

- The symbol D is called a **differential operator** because it transforms a differentiable function into another function
$$Dy = \frac{dy}{dx}$$

- Higher-order derivatives can be expressed as

$$D^n = \frac{d^n y}{dx^n}$$

- An **nth-order differential operator** is defined to be

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)$$

- As a consequence of the properties of differentiation

$$D(cf(x)) = cDf(x)$$

  and

$$D\{f(x) + g(x)\} = Df(x) + Dg(x)$$

- The superposition principle for homogeneous linear nth-order differential equation states that if y_1, y_2, …, y_k are solutions of the equation on an interval I then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

  where c_i are arbitrary constants is also a solution on the interval

- A set of functions $f_1(x)$, $f_2(x)$, …, $f_n(x)$ is said to be **linearly dependent** on an interval I if there exists constants c_1, c_2, …, c_n, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

  for every x in the interval. Otherwise it is said to tbe **linearly independent**

- The **Wronskian** of a set of n functions that are $n-1$ times differentiable is defined as

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

- If y_1, y_2, …, y_n are n solutions to a homogeneous linear nth-order differential equation on an interval I then the set of solutions is **linearly independent** on I iff $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in the interval

- Any set of n linearly independent solutions of a homogeneous linear nth-order differential equation on an interval I is said to be a **fundamental set of solutions** on the interval

- If y_1, y_2, ..., y_n are a fundamental set of solutions of a homogeneous linear nth-order DE on an interval I then the **general solution** of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

  where c_i are arbitrary constants

- Another way of saying the above is that any solution on the interval can be expressed as a linear combination of the fundamental set of solutions

- A linear combination of a fundamental set of solutions of a homogenous linear nth-order DE

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

  is called the **complementary function** of associated nonhomogenous DEs

- If y_p is any particular solution to a nonhomogeneous linear nth-order DE on an interval I and y_1, y_2, ..., y_n are a fundamental set of solutions of the associated homogeneous DE on I, then the **general solution** of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

  where c_i are arbitrary constants

- Another way of saying the above is that any solution on the interval can be expressed as $y = y_c + y_p$

- The superposition for nonhomogeneous linear nth-order differential equations states that if y_{p_1}, y_{p_2}, ..., y_{p_k} are k particular solutions of a nonhomogeneous lienar nth-order differential equation on an interval I corresponding, in turn, to k distinct functions g_1, g_2, ..., g_k, then

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x)$$

  is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x)$$

3.2 Reduction of Order

- The **reduction of order** method requires knowledge of one non-trivial solution and comprises the following steps:

  1. Recognise that the ratio of two linearly independent functions isn't constant, i.e.

$$u(x) = \frac{y_1(x)}{y_2(x)} \text{ or } y_2(x) = u(x)y_1(x)$$

2. Substitute $y_2(x) = u(x)y_1(x)$ into the DE — this will result in a DE involving only u'' and u' which can be treated as a linear first-order DE in $u' = w$
3. Solve for w
4. Substitute $w = u'$
5. Integrate to find u
6. Multiply by y_1 to find y_2

- A formula for the above on a DE in standard form

$$y'' + P(x)y' + Q(x)y = 0$$

is

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx$$

3.3 Homogeneous Linear Equations with Constant Coefficients

- All solutions to homogenous linear DEs

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where a_i are real constants and $a_n \neq 0$ are either exponential functions or constructed from exponential functions

- Substituting a solution $y = e^{mx}$ we find

$$e^{mx}(a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0) = 0$$

where the term in brackets is called the **auxiliary equation** of the DE

- Thus, the solution $y = e^{mx}$ is valid if m is a root of the auxiliary equation
- Real roots correspond to solutions of the form

$$y = ce^{mx}$$

- Complex roots $\alpha \pm i\beta$ correspond to solutions of the form

$$y_1 = c_1 e^{\alpha x} \cos \beta x \text{ and } y_2 = c_2 e^{\alpha x} \sin \beta x$$

- A root m of multiplicity k corresponds to the solutions

$$e^{mx},\ xe^{mx},\ x^2 e^{mx},\ \ldots,\ x^{k-1} e^{mx}$$

3.4 Undetermined Coefficients

- The **method of undetermined coefficients** may be used to find a particular solution to nonhomogenous linear differential equations where the input function is comprised of constants, polynomials, exponentials $e^{\alpha x}$, sines, and cosines
- To apply the method you:
  1. Solve the associated homogeneous equation
  2. Assume the particular solution has the same form as the input function
  3. If a term in the proposed solution is present in the complementary function, multiply it by x^n where n is the smallest positive integer that removes the duplication
  4. Substitute the proposed solution into the DE
  5. Solve for the unknown constants

TABLE 3.4.1 Trial Particular Solutions

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A \cos 4x + B \sin 4x$
6. $\cos 4x$	$A \cos 4x + B \sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$
11. $5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
12. $xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$

3.5 Variation of Parameters

- The **method of variation of parameters** can be used to find a particular solution of a nonhomogeneous linear nth-order DE
- To apply the method you:
  1. Solve the homogeneous equation to find the complementary function
  2. Assume the solution has the form
  $$y_p = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x)$$
  where n is the order of the equation and y_i are the fundamental set of solutions from the complementary equation
  3. Convert to standard form by dividing by the leading coefficient
  $$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

4. Solve the system of linear equations

$$
\begin{aligned}
y_1 u_1' + \cdots + y_n u_n' &= 0 \\
y_1' u_1' + \cdots + y_n' u_n' &= 0 \\
&\vdots \\
y_1^{(n-1)} u_1' + \cdots + y_n^{(n-1)} y_n' &= 0 \\
y_1^{(n)} u_1' + \cdots + y_n^{(n)} u_n' &= f(x)
\end{aligned}
$$

via Cramer's method:

(a) Compute the Wronskian of y_i

$$
W = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \ddots & \vdots \\ y_1^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}
$$

(b) Compute u_i' for $i = 1, \ldots, n$ where

$$
u_i' = \frac{W_i}{W}
$$

and W_i is the determinant of the matrix formed by replacing the ith column of the Wronskian matrix with the column vector

$$
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(x) \end{bmatrix}
$$

5. Integrate each u_i' to find u_i