Advanced Engineering Mathematics Partial Differential Equations by Dennis G. Zill Notes

Chris Doble

November 2023

Contents

12 Orthogonal Functions and Fourier Series

12.1 Orthogonal Functions

- The **inner product** of two functions f_1 and f_2 on an interval $[a, b]$ is the number
$$(f_1, f_2) = \int_a^b f_1(x) f_2(x)\, dx.$$
- Two functions f_1 and f_2 are said to be orthogonal on an interval if $(f_1, f_2) = 0$.

- A set of real-valued functions $\{\phi_1(x), \phi_2(x), \dots, \phi_n(x)\}$ is said to be **orthogonal** on an interval if

$$(\phi_i, \phi_j) = 0 \text{ for } i \neq j.$$

- The **square norm** of a function is

$$||\phi_n(x)||^2 = (\phi_n, \phi_n)$$

and thus its **norm** is

$$||\phi_n(x)|| = \sqrt{(\phi_n, \phi_n)}.$$

- An **orthonormal set** of functions is an orthogonal set of functions that all have a norm of 1.

- An orthogonal set can be made into an orthonormal set by dividing each member by its norm.

- If $\{\phi_n(x)\}$ is an infinite orthogonal set of functions on an interval $[a, b]$ and $f(x)$ is an arbitrary function, then it's possible to determine a set of coefficients $c_n, n = 0, 1, 2, \dots$ such that

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) = c_0 \phi_0(x) + c_1 \phi_1(x) + \dots + c_n \phi_n(x) + \dots$$

This is called an **orthogonal series expansion** of f or a **generalized Fourier series** where the coefficients are given by

$$c_n = \frac{(f, \phi_n)}{||\phi_n||^2}.$$

- A set of real-valued functions $\{\phi_n(x)\}$ is said to be **orthogonal with respect to a weight function** $w(x)$ on the interval $[a, b]$ if

$$\int_a^b w(x)\phi_m(x)\phi_n(x)\, dx = 0,\ m \neq n.$$

12.2 Fourier Series

- The **Fourier series** of a function f defined on the interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

where

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x)\, dx$$
$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi}{p} x\, dx$$
$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi}{p} x\, dx$$

- At points of discontinuity in f, the Fourier series takes on the average of the values either side of it.
- The Fourier series of a function f gives a **periodic extension** of the function outside the interval $(-p, p)$.

12.3 Fourier Cosine and Sine Series

- A function f is said to be **even** if

$$f(-x) = f(x)$$

and **odd** if

$$f(-x) = -f(x).$$

- Even and odd functions have some interesting properties:
  - The product of two even functions is even.
  - The product of two odd functions is even.
  - The product of an even function and an odd function is odd.
  - If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
  - If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.
- In light of this, if a function f is even its Fourier coefficients are

$$\begin{aligned} a_0 &= \frac{2}{p}\int_0^p f(x)\,dx \\ a_n &= \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx \\ b_n &= 0. \end{aligned}$$

The series consists of cosine terms and is called the **Fourier cosine series**.

- Similarly, if f is odd then

$$\begin{aligned} a_n &= 0, \;\; n = 0, 1, 2, \ldots \\ b_n &= \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx. \end{aligned}$$

The series consists of sine terms and is called the **Fourier sine series**.

- Sometimes a Fourier series "overshoots" the original value of the function near discontinuities. This is called the **Gibbs phenomenon**.
- Taking the Fourier cosine series of a function f over the interval $[0, L]$ effectively mirrors the function around the vertical axis.

- Taking the Fourier sine series of a function f over the interval $[0, L]$ effectively rotates it 180° around the origin.

- A particular solution for a nonhomogeneous differential equation with a periodic driving force can be found by taking the Fourier transform of the driving force then using the method of undetermined coefficients to determine the coefficients.

12.4 Complex Fourier Series

- The **complex Fourier series** of a function f defined on an interval $(-p, p)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p}$$

  where

$$c_n = \frac{1}{2p} \int_{-p}^{p} f(x) e^{-in\pi x/p} \, dx, \;\; n = 0, \pm 1, \pm 2, \dots$$

- The **fundamental period** of a Fourier series is $T = 2p$.

- The **fundamental angular frequency** of a Fourier series is $\omega = \frac{2\pi}{T}$.

- A **frequency spectrum** is a plot of the points $(n\omega, |c_n|)$ where ω is the fundamental angular frequency and c_n are the coefficients of the complex Fourier series. This can be useful to see how each harmonic contributes.

12.5 Sturm-Liouville Problem

- If a boundary value problem contains an arbitrary parameter λ, the values of λ for which the problem has nontrivial solutions are called the **eigenvalues** of the problem and the associated solutions are called the **eigenfunctions** of the problem.

- An orthogonal set of functions can be generated by solving a two-point boundary-value problem involing a linear second-order differential equation containing a parameter λ.

- A **regular Sturm-Liouville problem** is a boundary value problem

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

  subject to

$$A_1 y(a) + B_1 y'(a) = 0$$
$$A_2 y(b) + B_2 y'(b) = 0$$

  where p, q, r, and r' are real-valued functions continuous on an interval $[a, b]$, $r(x) > 0$ and $p(x) > 0$ for every x in that interval, the coefficients in the boundary conditions are real and independent of λ, A_1 and B_1 are not both zero, and A_2 and B_2 are not both zero.

- A boundary condition

$$A_1 y(a) + B_1 y'(a) = C$$

is said to be **homogeneous** if $C = 0$ and **nonhomogeneous** otherwise.

- A boundary-value problem consisting of a homogeneous differential equation and a homogeneous boundary condition is said to be homogeneous, otherwise it's nonhomogeneous.

- Multiple boundary conditions are said to be **separated** if each deals with values at a single point $x = a$ and **mixed** if each deals with values at multiple points $x = a, b, \ldots$

- If a boundary-value problem can be identified as a Sturm-Liouville problem we know it has several properties:

  - There exists an infinite number of real eigenvalues that can be arranged in increasing order $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$ such that $\lambda_n \to \infty$ as $n \to \infty$.
  - For each eigenvalue there is only one eigenfunction.
  - Eigenfunctions corresponding to different eigenvalues are linearly independent.
  - The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on the interval $[a, b]$.

- If a Sturm-Liouville problem has $r(a) = 0$ and boundary conditions are specified at $x = b$, or $r(b) = 0$ and boundary conditions are specified at $x = a$, then it is called a **singular boundary-value problem**.

- If a Sturm-Liouville problem has $r(a) = r(b)$ and boundary conditions $y(a) = y(b)$, $y'(a) = y'(b)$, then it is called a **periodic boundary-value problem**.

- If the solutions to a singular or periodic boundary-value problem are bounded on the interval $[a, b]$ then the orthogonality relation holds.

- Any second-order linear differential equation

$$a(x)y'' + b(x)y' + [c(x) + \lambda d(x)]y = 0$$

can be transformed into a Sturm-Liouville problem providing the coefficients are continuous and $a(x) \neq 0$ on the interval of interest. This can be done by:

1. dividing by a,
2. multiplying by the integrating factor $e^{\int (b/a)\,dx}$,
3. recognising that

$$e^{\int (b/a)\,dx} y'' + \frac{b}{a} e^{\int (b/a)\,dx} y' = \frac{d}{dx}\left[e^{\int (b/a)\,dx} y'\right],$$

4. and rewriting the equation as

$$\frac{d}{dx}\left[e^{\int (b/a)\,dx} y'\right] + \left(\frac{c}{a} e^{\int (b/a)\,dx} + \lambda \frac{d}{a} e^{\int (b/a)\,dx}\right)\lambda = 0$$

which is the desired form and lets us recognise

$$\begin{aligned} r(x) &= e^{\int (b/a)\,dx} \\ q(x) &= \frac{c}{a} e^{\int (b/a)\,dx} \\ p(x) &= \frac{d}{a} e^{\int (b/a)\,dx}. \end{aligned}$$

12.6 Bessel and Legendre Series

12.6.1 Fourier-Bessel Series

Definition 12.6.1 Fourier–Bessel Series

The **Fourier–Bessel series** of a function f defined on the interval $(0, b)$ is given by

(*i*)

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \tag{15}$$

$$c_i = \frac{2}{b^2 J_{n+1}^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x)\, dx, \tag{16}$$

where the α_i are defined by $J_n(\alpha b) = 0$.

(*ii*)

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \tag{17}$$

$$c_i = \frac{2\alpha_i^2}{(\alpha_i^2 b^2 - n^2 + h^2) J_n^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x)\, dx, \tag{18}$$

where the α_i are defined by $hJ_n(\alpha b) + \alpha b J_n'(\alpha b) = 0$.

(*iii*)

$$f(x) = c_1 + \sum_{i=2}^{\infty} c_i J_0(\alpha_i x) \tag{19}$$

$$c_1 = \frac{2}{b^2} \int_0^b x f(x)\, dx, \quad c_i = \frac{2}{b^2 J_0^2(\alpha_i b)} \int_0^b x J_0(\alpha_i x) f(x)\, dx, \tag{20}$$

where the α_i are defined by $J_0'(\alpha b) = 0$.

- The Fourier-Bessel series converges to f where it is continuous and

$$\frac{f(x+) + f(x-)}{2}$$

where it is discontinuous.

12.6.2 Fourier-Legendre Series

Definition 12.6.2 Fourier–Legendre Series

The **Fourier–Legendre series** of a function f defined on the interval $(-1, 1)$ is given by

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \tag{21}$$

where

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx. \tag{22}$$

- The Fourier-Legendre series converges to f where it is continuous and

$$\frac{f(x+) + f(x-)}{2}$$

where it is discontinuous.

13 Boundary-Value Problems in Rectangular Coordinates

13.1 Separable Partial Differential Equations

- Like ordinary differential equations (ODEs), partial differential equations (PDEs) can be linear or nonlinear. If the dependent variable and its partial derivatives only appear to the first power, it's a linear PDE.

- The general form of a **linear second-order partial differential equation** is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G.$$

  When $G(x, y) = 0$ the equation is said to be **homogeneous**, otherwise it's **nonhomogeneous**.

- Under the method of **separation of variables** we assume that the solution of a PDE is a product of functions of each independent variable, e.g. if we seek a solution with independent variables x and y we assume it has the form $u = X(x)Y(y)$. With this assumption it's sometimes possible to reduce the PDE into multiple independent ODEs.

- A key step during the process of applying the method of separation of variables is when the equation has been reduced to a form like

$$F(X, X', X'') = G(Y, Y', Y'').$$

  Remembering that X and Y are functions of a single variable, this means that varying x independently of y or vice versa affects one side of the equation but not the other. In order for them to remain equal this means both sides must be constant. This lets us equate each side with a **separation constant**, giving us an ODE that can be solved on its own.

- The **superposition principle** states that if $u_1, u_2, \ldots, u_n$ are solutions of a homogeneous linear partial differential equation, then the linear combination

$$u = c_1u_1 + c_2u_2 + \ldots + c_nu_k$$

  is also a solution.

- A linear second-order partial differential equation in two independent variables with constant coefficients

$$A\frac{\partial^2 u}{\partial^2 x} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 y}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

  can be classified as one of three types:

  - **hyperbolic** if $B^2 - 4AC > 0$,
  - **parabolic** if $B^2 - 4AC = 0$, and
  - **elliptic** if $B^2 - 4AC < 0$.

13.2 Classical PDEs and Boundary-Value Problems

- The **one-dimensional heat equation**

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},\ k > 0$$

  describes the diffusion of thermal energy through a one-dimensional rod where

$$k = \frac{K}{\gamma\rho}$$

  is called the **thermal diffusivity** of the rod, K is its thermal conductivity, γ is its specific heat, and ρ is its density.

- The **one-dimensional wave equation**

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

describes the motion of a wave through a taut string where

$$a^2 = \frac{T}{\rho},$$

T is the tension in the string, and ρ is its density per unit length.

- If a PDE depends on time t, the state of the system at $t = 0$ can be used as **initial conditions** to help determine the solution.
- **Boundary conditions** state the value of the solution at particular points, e.g. if the ends of a string are fixed in place, and can also help determine the solution.
- There are three types of boundary conditions:
  - **Dirichlet conditions** specify the value of u at a particular location, e.g. a particular point on a string is fixed in place.
  - **Neumann conditions** specify the value of $\partial u/\partial n$ at a particular location, e.g. a particular point on a string always has 0 velocity.
  - **Robin conditions** specify the value of $\partial u/\partial n + hu$ where h is a constant, e.g. thermal energy is lost at a constant rate at the end of a rod.

13.3 Heat Equation

- Sometimes a solution of a PDE may not satisfy a given boundary condition or initial value. However, if we know that the solution is a member of an infinite orthogonal set of solutions (e.g. via the Sturm-Liouville theorem) then we may be able to use the superposition principle to select constants c_i that do satisfy the boundary condition or initial value.

13.5 Laplace's Equation

- A **Dirichlet problem** is the problem of finding a function which solves a PDE in a particular region given the values the function should take on the boundaries of that region.
- The **maximum principle** states that a solution of Laplace's equation takes on its maximum and minimum values on the boundary of the region in which the solution is defined. Also, the solution can have no relative maxima or minima in the region.
- A Dirichlet problem for a rectangle can be solved via separation of variables when homogeneous boundary conditions are specified on two parallel boundaries, but not when the boundary conditions of all four sides are nonhomogeneous. This can be solved by breaking the problem in two: one that has homogeneous boundary conditions on the x axis and another that has them on the y axis. By the superposition principle, adding the solutions to these problems will be a solution to the original problem.

13.6 Nonhomogeneous Boundary-Value Problems

- A BVP involving a time-independent nonhomogeneous equation and time-independent boundary conditions like

$$\begin{aligned} k\frac{\partial^2 u}{\partial x^2} + F(x) &= \frac{\partial u}{\partial t} \\ u(0,t) &= u_0 \\ u(L,t) &= u_1 \\ u(x,0) &= f(x) \end{aligned}$$

  can be solved by substituting

$$u(x,t) = v(x,t) + \psi(x).$$

  This decomposes the problem into two: an ODE in $\psi(x)$ with nonhomogeneous boundary conditions

$$\begin{aligned} k\psi'' + F(x) &= 0 \\ \psi(0) &= u_1 \\ \psi(1) &= u_2, \end{aligned}$$

  and a PDE in $v(x,t)$ with homogeneous boundary conditions

$$\begin{aligned} k\frac{\partial^2 v}{\partial x^2} &= \frac{\partial v}{\partial t} \\ v(0,t) &= 0 \\ v(L,t) &= 0 \\ v(x,0) &= f(x) - \psi(x). \end{aligned}$$

  We can then solve the ODE for $\psi(x)$, substitute the result into the PDE, and solve for $v(x,t)$, giving us $u(x,t)$.

- A BVP involving a time-dependent nonhomogeneous equation and time-dependeny boundary conditions like

$$\begin{aligned} k\frac{\partial^2 u}{\partial x^2} + F(x,t) &= \frac{\partial u}{\partial t} \\ u(0,t) &= u_0(t) \\ u(L,t) &= u_1(t) \\ u(x,0) &= f(x) \end{aligned}$$

  can be solved via the following steps:

  1. Substitute

$$\begin{aligned} u(x,t) &= v(x,t) + \psi(x,t) \\ &= v(x,t) + u_0(t) + \frac{x}{L}[u_1(t) - u_0(t)] \end{aligned}$$

which changes the BVP to

$$\begin{aligned} k\frac{\partial^2 v}{\partial x^2} + G(x,t) &= \frac{\partial v}{\partial t} \\ v(0,t) &= 0 \\ v(L,t) &= 0 \\ v(x,0) &= f(x) - \psi(x,0) \end{aligned}$$

where $G(x,t) = F(x,t) - \frac{\partial \psi}{\partial t}$.

2. Assuming $G(x,t)$ can be expressed as a Fourier sine series, find the coefficients of that series $G_n(t)$.
3. Assuming $v(x,t)$ can also be expressed as a Fourier sine series, express the BVP in $v(x,t)$ using such a series and the series for $G(x,t)$.
4. Equate the coefficients of the two series to get an ODE for the $v(x,t)$ series coefficients $v_n(t)$.
5. Solve the ODE to find $v_n(t)$ and thus express $v(x,t)$ as a Fourier sine series with coefficients C_n.
6. Use the boundary conditions of the BVP in $v(x,t)$ to solve for C_n.
7. Subsitute C_n to find $v(x,t)$.
8. Substitute $v(x,t)$ to find $u(x,t)$.

- If a BVP includes a time-dependent term $F(x,t)$ in the PDE but has homogeneous boundary conditions, $\psi(x,t)$ will be 0 and there's no need to substitute $u(x,t) = v(x,t) + \psi(x,t)$. The BVP can be solved by proceeding from step 2 above, replacing $G(x,t)$ with $F(x,t)$ and $v(x,t)$ with $u(x,t)$.