Introduction to Electrodynamics by David J. Griffiths Notes

Chris Doble

December 2023

Contents

1 Vector Algebra

1.6 The Theory of Vector Fields

1.6.1 The Helmholtz Theorem

- The **Helmholtz theorem** states that a vector field $\mathbf{F}$ is uniquely determined if you're given its divergence $\nabla \cdot \mathbf{F}$, curl $\nabla \times \mathbf{F}$, and sufficient boundary conditions.

1.6.2 Potentials

- If the curl of a vector field vanishes everywhere, then it can be expressed as the gradient of a **scalar potential**

$$\nabla \times \mathbf{F} = \mathbf{0} \Leftrightarrow \mathbf{F} = -\nabla V.$$

- If the divergence of a vector field vanishes everywhere, then it can be expressed as the curl of a **vector potential**

$$\nabla \cdot \mathbf{F} = 0 \Leftrightarrow \mathbf{F} = \nabla \times \mathbf{A}.$$

2 Electrostatics

2.1 The Electric Field

2.1.2 Coulomb's Law

- **Couloumb's law** gives the force between two point charges q and Q

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathscr{r}} \hat{\boldsymbol{\mathscr{r}}}$$

where

$$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C}^2/(\mathrm{N\,m}^2)$$

is the **permittivity of free space** and $\boldsymbol{\mathscr{r}}$ is the separation vector between the two charges.

2.1.3 The Electric Field

- The **electric field E** is a vector field that varies from point to point and gives the force per unit charge that would be exerted on a test charge if placed at a particular point.

- For a collection of n source charges q_i at displacements $\boldsymbol{\mathscr{r}}_i$ from a test charge, the electric field is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{\mathscr{r}_i^2} \hat{\boldsymbol{\mathscr{r}}}.$$

2.1.4 Continuous Charge Distributions

- Couloumb's law for a continuous charge distribution is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{1}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}\, dq.$$

2.2 Divergence and Curl of Electrostatic Fields

2.2.1 Field Lines, Flux, and Gauss's Law

- **Gauss's law** states that the electric field flux through a closed surface is proportional to the amount of charge within that surface

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$$

or

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho.$$

2.2.4 The Curl of E

- The curl of an electric field is $\mathbf{0}$

$$\nabla \times \mathbf{E} = \mathbf{0}.$$

2.3 Electric Potential

2.3.1 Introduction to Potential

- The **electric potential** at a point $\mathbf{r}$ is defined as

$$V(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$$

where $\mathcal{O}$ is an agreed origin.

- The potential difference between two points $\mathbf{a}$ and $\mathbf{b}$ is

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}.$$

- The electric field and potential are also related by the equation

$$\mathbf{E} = -\nabla V.$$

2.3.2 Comments on Potential

- The choice of origin $\mathcal{O}$ in the definiton of vector potential only affects the absolute potential values, not potential differences. Typically it is chosen to be "at infinity" unless the charge distribution itself extends to infinity.
- Electric potential obeys the superposition principle.
- The units of electric potential is $\mathrm{N\,m/C} = \mathrm{J/C} = \mathrm{V}$.

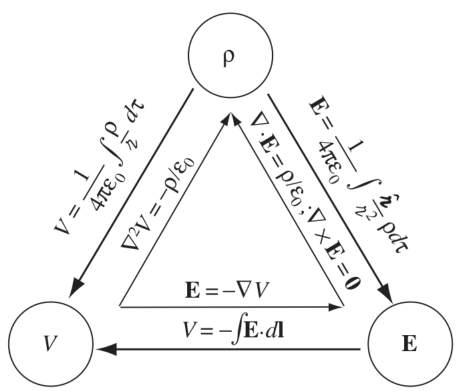
2.3.3 Poisson's Equation and Laplace's Equation

- If
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$
and
$$\mathbf{E} = -\nabla V$$
then
$$\begin{aligned} \nabla \cdot (-\nabla V) &= \frac{\rho}{\epsilon_0} \\ \nabla^2 V &= -\frac{\rho}{\epsilon_0}. \end{aligned}$$
This is known as **Poisson's equation**. In regions where $\rho = 0$ it reduces to **Laplace's equation**
$$\nabla^2 V = 0.$$

2.3.4 The Potential of a Localized Charge Distribution

- The potential of a continuous charge distribution is
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}} \, d\tau'$$
where the reference is point is set to infinity.

2.3.5 Boundary Conditions



- The normal component of the electric field is discontinuous by an amount σ/ϵ_0 at any boundary, i.e.
$$E_{\text{above}} - E_{\text{below}} = \frac{\sigma}{\epsilon_0}.$$

- The tangential component of the electric field is always continuous at any boundary.
- The electric potential is always continuous at any boundary, however because $\mathbf{E} = -\nabla V$, the gradient of the electric potential inherits the discontinuity at boundaries with surface charge, i.e.

  $$\nabla V_{\text{above}} - \nabla V_{\text{below}} = -\frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

  or

  $$\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{\sigma}{\epsilon_0}$$

  where

  $$\frac{\partial V}{\partial n} = \nabla V \cdot \hat{\mathbf{n}}.$$