Advanced Engineering Mathematics Ordinary Differential Equations Notes

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Contents

1 Introduction to Differential Equations

1.1 Definitions and Terminology

1.1.1 1

2, linear

1.1.2 3

4, linear

1.1.3 5

2, nonlinear

1.1.4 7

3, linear

1.1.5 9

no; yes

1.1.6 15

The domain of the function is $x \in [-2, \infty)$.

$$y' = 1 + \frac{2}{\sqrt{x+2}}$$

The largest interval of definition of the solution is $x \in (-2, \infty)$.

$$\begin{aligned}(y-x)y' &= y - x + 8\\(x + 4\sqrt{x+2} - x)(1 + \frac{2}{\sqrt{x+2}}) &= x + 4\sqrt{x+2} - x + 8\\4\sqrt{x+2} + 8 &= 4\sqrt{x+2} + 8\end{aligned}$$

1.1.7 17

The domain of the function is $x \in \mathbb{R}, x \neq \pm 2$.

$$y' = \frac{2x}{(4-x^2)^2}$$

The largest intervals of definition of the solution are $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

$$\begin{aligned}y' &= 2xy^2\\\frac{2x}{(4-x^2)^2} &= 2x\left(\frac{1}{4-x^2}\right)^2\\&= \frac{2x}{(4-x^2)^2}\end{aligned}$$

1.1.8 19

$$\begin{aligned}ln\frac{2X-1}{X-1} &= t\\2X - 1 &= (X-1)e^t\\(2-e^t)X &= 1 - e^t\\X &= \frac{e^t - 1}{e^t - 2}\end{aligned}$$

The solutions intervals of validity are $(\infty, \ln 2)$ and $(\ln 2, \infty)$.

$$
\begin{aligned}
\frac{dX}{dt} &= (X-1)(1-2X) \\
\frac{e^t}{e^t-2} - \frac{e^t(e^t-1)}{(e^t-2)^2} &= \left(\frac{e^t-1}{e^t-2} - 1\right)\left(1 - 2\frac{e^t-1}{e^t-2}\right) \\
\frac{e^t(e^t-2) - e^t(e^t-1)}{(e^t-2)^2} &= \left(\frac{e^t-1-e^t+2}{e^t-2}\right)\left(\frac{e^t-2-2e^t+2}{e^t-2}\right) \\
\frac{e^{2t}-2e^t-e^{2t}+e^t}{(e^t-2)^2} &= \left(\frac{1}{e^t-2}\right)\left(\frac{-e^t}{e^t-2}\right) \\
\frac{-e^t}{(e^t-2)^2} &= \frac{-e^t}{(e^t-2)^2}
\end{aligned}
$$

1.1.9 31

$m = -2$

1.1.10 33

$m = 2$ or 3

1.1.11 35

$m = -1$ or 0

1.1.12 37

$y = 2$

1.1.13 39

No constant solutions

1.2 Initial Value Problems

1.2.1 1

$$
\begin{aligned}
y(0) = -\frac{1}{3} &= \frac{1}{1+c_1e^{-(0)}} \\
-3 &= 1 + c_1 \\
c_1 &= -4
\end{aligned}
$$

$$
y = \frac{1}{1-4e^{-x}}
$$

1.2.2 3

$$\begin{aligned} y(2) = \frac{1}{3} &= \frac{1}{(2)^2 + c} \\ 3 &= 4 + c \\ c &= -1 \end{aligned}$$

$$y = \frac{1}{x^2 - 1}$$

$I = (1, \infty)$

1.2.3 5

$$\begin{aligned} y(0) = 1 &= \frac{1}{(0)^2 + c} \\ c &= 1 \end{aligned}$$

$$y = \frac{1}{x^2 + 1}$$

$I = (-\infty, \infty)$

1.2.4 7

$$\begin{aligned} x(0) = -1 &= c_1 \cos 0 + c_2 \sin 0 \\ c_1 &= -1 \end{aligned}$$

$$\begin{aligned} x'(0) = 8 &= -c_1 \sin 0 + c_2 \cos 0 \\ c_2 &= 8 \end{aligned}$$

$$x = -\cos t + 8 \sin t$$

1.2.5 9

$$\begin{aligned} x'\left(\frac{\pi}{6}\right) = 0 &= -c_1 \sin \frac{\pi}{6} + c_2 \cos \frac{\pi}{6} \\ &= -c_1 \frac{1}{2} + c_2 \frac{\sqrt{3}}{2} \\ c_1 &= \sqrt{3} c_2 \end{aligned}$$

$$\begin{aligned}
x\left(\frac{\pi}{6}\right) = \frac{1}{2} &= c_1 \cos\frac{\pi}{6} + c_2 \sin\frac{\pi}{6} \\
&= \frac{3}{2}c_2 + \frac{1}{2}c_2 \\
&= 2c_2 \\
c_2 &= \frac{1}{4}
\end{aligned}$$

$$y = \frac{\sqrt{3}}{4}\cos t + \frac{1}{4}\sin t$$

1.2.6 11

$$\begin{aligned}
y(0) = 1 &= c_1 e^{(0)} + c_2 e^{-(0)} \\
&= c_1 + c_2 \\
c_1 &= 1 - c_2
\end{aligned}$$

$$\begin{aligned}
y'(0) = 2 &= c_1 e^{(0)} - c_2 e^{-(0)} \\
&= 1 - c_2 - c_2 \\
c_2 &= -\frac{1}{2}
\end{aligned}$$

$$y = \frac{3}{2}e^{x} - \frac{1}{2}e^{-x}$$

1.2.7 13

$$\begin{aligned}
y(-1) = 5 &= c_1 e^{(-1)} + c_2 e^{-(-1)} \\
&= c_1 e^{-1} + c_2 e \\
c_1 &= 5e - c_2 e^2
\end{aligned}$$

$$\begin{aligned}
y'(-1) = -5 &= c_1 e^{(-1)} - c_2 e^{-(-1)} \\
&= 5e - c_2 e^2 - c_2 e \\
c_2 e(e+1) &= 5(e+1) \\
c_2 &= \frac{5}{e}
\end{aligned}$$

$$y = 5e^{-x-1}$$

1.2.8 15

$$y = 0$$

$$y = x^3$$

1.2.9 17

$$f(x, y) = y^{2/3}$$

$$\frac{\partial f}{\partial y} = \frac{2}{3y^{1/3}}$$

$y < 0$ or $y > 0$

1.2.10 19

$$f(x, y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$x < 0$ or $x > 0$

1.2.11 21

$$f(x, y) = \frac{x^2}{4 - y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2x^2 y}{(4 - y^2)^2}$$

$y < -2$, $-2 < y < 2$, or $y > 2$

1.2.12 23

$$f(x, y) = \frac{y^2}{x^2 + y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2} - \frac{2y^3}{(x^2 + y^2)^2}$$

$x \neq 0$ and $y \neq 0$

1.2.13 25

$$f(x, y) = \sqrt{y^2 - 9}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{y^2 - 9}}$$

Yes

1.2.14 27

No

1.2.15 29

(a) $y = cx$

(b)

$$f(x, y) = \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}$$

$x \neq 0$

(c) No, the function is not differentiable at $x = 0$

1.2.16 31

(a)

$$y' = \frac{1}{(x + c)^2} = y^2$$

(b)

$$y(0) = 1 = -\frac{1}{(0) + c} \Rightarrow c = -1 \Rightarrow y = \frac{1}{1 - x}$$

$I = (-\infty, 1)$

$$y(0) = -1 = -\frac{1}{(0) + c} \Rightarrow c = 1 \Rightarrow y = -\frac{1}{x + 1}$$

$I = (-1, \infty)$

1.2.17 39

$$\begin{aligned} y(0) = 0 &= c_1 \cos 3(0) + c_2 \sin 3(0) \\ c_1 &= 0 \end{aligned}$$

$$\begin{aligned} y\left(\frac{\pi}{6}\right) = -1 &= c_2 \sin 3\left(\frac{\pi}{6}\right) \\ c_2 &= -1 \end{aligned}$$

$$y = -\sin 3x$$

1.2.18 41

$$
\begin{aligned}
y'(0) = 0 &= -3c_1 \sin 3(0) + 3c_2 \cos 3(0) \\
c_2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
y'\left(\frac{\pi}{4}\right) = 0 &= -3c_1 \sin 3\left(\frac{\pi}{4}\right) \\
&= -\frac{3}{\sqrt{2}}c_1 \\
c_1 &= 0
\end{aligned}
$$

$$y = 0$$

1.2.19 43

$$
\begin{aligned}
y(0) = 0 &= c_1 \cos 3(0) + c_2 \sin 3(0) \\
c_1 &= 0
\end{aligned}
$$

$$
\begin{aligned}
y(\pi) = 4 &= c_2 \sin 3(\pi) \\
4 &= 0
\end{aligned}
$$

No solution

1.3 Differential Equations as Mathematical Models

1.3.1 1

$$\frac{dP}{dt} = kP + r$$

$$\frac{dP}{dt} = kP - r$$

1.3.2 3

$$\frac{dP}{dt} = k_b P - k_d P^2$$

1.3.3 7

$$\frac{dx}{dt} = kx(1000 - x)$$

1.3.4 9

$$\frac{dA}{dt} = -\frac{A}{100}$$

$$A(0) = 50\,\mathrm{lb}$$

1.3.5 11

$$\frac{dA}{dt} + \frac{7}{600-t}A = 6$$

1.3.6 13

$$\begin{aligned}
\frac{dV}{dt} &= -cA_h\sqrt{2gh} \\
A_w\frac{dh}{dt} &= -cA_h\sqrt{2gh} \\
\frac{dh}{dt} &= -\frac{cA_h\sqrt{2g}}{A_w}\sqrt{h} \\
&= -\frac{c\pi r_h^2\sqrt{2g}}{A_w}\sqrt{h} \\
&= -\frac{c\pi}{430}\sqrt{h}
\end{aligned}$$

1.3.7 15

$$L\frac{di}{dt} + Ri = E$$

1.3.8 17

$$m\frac{dv}{dt} = mg - kv^2$$

1.3.9 19

$$m\frac{d^2x}{dt^2} = -kx$$

1.3.10 21

$$\begin{aligned}
\frac{d}{dt}(mv) &= R - kv \\
\frac{dm}{dt}v + m\frac{dv}{dt} &= R - kv - mg
\end{aligned}$$

1.3.11 23

$$g = \frac{k}{R^2} \Rightarrow k = gR^2$$

$$\frac{d^2r}{dt^2} = -\frac{gR^2}{r^2}$$

1.3.12 25

$$\frac{dA}{dt} = k(M - A)$$

1.3.13 27

$$\frac{dx}{dt} = r - kx$$

1.3.14 29

$$\begin{aligned}
\frac{dy}{dx} &= \tan\theta \\
&= \tan\frac{\phi}{2} \\
&= \frac{1-\cos\phi}{\sin\phi} \\
&= \frac{1-x/r}{y/r} \\
&= \frac{r-x}{y} \\
&= \frac{\sqrt{x^2+y^2}-x}{y}
\end{aligned}$$

1.4 Chapter in Review

1.4.1 1

$$\frac{dy}{dx} = ky$$

1.4.2 3

$$y'' + k^2y = 0$$

1.4.3 5

$$y = c_1 e^x + c_2 x e^x$$

$$\begin{aligned} y' &= c_1 e^x + c_2 e^x + c_2 x e^x \\ &= y + c_2 e^x \end{aligned}$$

$$\begin{aligned} y'' &= c_1 e^x + c_2 e^x + c_2 e^x + c_2 x e^x \\ &= c_1 e^x + 2c_2 e^x + c_2 x e^x \\ &= y' + c_2 e^x \end{aligned}$$

$$y'' - 2y' + y = 0$$

1.4.4 7

a, d

1.4.5 9

b

1.4.6 11

b

1.4.7 13

$$y = ce^x$$

1.4.8 15

$$\frac{dy}{dx} = x^2 + y^2$$

1.4.9 17

(a) $(-\infty, \infty)$

(b) $(-\infty, 0)$ or $(0, \infty)$

1.4.10 19

$x_0 = -1$ and $I = (-\infty, 0)$ or $x_0 = 2$ and $I = (0, \infty)$

1.4.11 23

$$y = x\sin x + x\cos x$$

$$y' = \sin x + x\cos x + \cos x - x\sin x$$

$$\begin{aligned} y'' &= \cos x + \cos x - x\sin x - \sin x - \sin x - x\cos x \\ &= 2\cos x - 2\sin x - x\sin x - x\cos x \end{aligned}$$

$$\begin{aligned} y'' + y &= 2\cos x - 2\sin x - x\sin x - x\cos x + x\sin x + x\cos x \\ &= 2\cos x - 2\sin x \end{aligned}$$

$I = (-\infty, \infty)$

1.4.12 25

$$y = \sin(\ln x)$$

$$y' = \frac{1}{x}\cos(\ln x)$$

$$y'' = -\frac{1}{x^2}\cos(\ln x) - \frac{1}{x^2}\sin(\ln x)$$

$$\begin{aligned} x^2y'' + xy' + y &= -\cos(\ln x) - \sin(\ln x) + \cos(\ln x) + \sin(\ln x) \\ &= 0 \end{aligned}$$

$I = (0, \infty)$

1.4.13 35

$$\begin{aligned} y(0) = 0 &= c_1e^{-3(0)} + c_2e^{(0)} + 4(0) \\ &= c_1 + c_2 \\ c_1 &= -c_2 \end{aligned}$$

$$\begin{aligned} y'(0) = 0 &= -3c_1e^{-3(0)} + c_2e^{(0)} + 4 \\ &= -3c_1 + c_2 + 4 \\ c_2 &= 3c_1 - 4 \end{aligned}$$

$$c_1 = -(3c_1 - 4) \Rightarrow c_1 = 1 \Rightarrow c_2 = -1$$

$$y = e^{-3x} - e^x + 4x$$

1.4.14 37

$$\begin{aligned}
y(1) = -2 &= c_1 e^{-3(1)} + c_2 e^{(1)} + 4(1) \\
&= c_1 e^{-3} + c_2 e + 4 \\
c_1 &= -e^3 (c_2 e + 6)
\end{aligned}$$

$$\begin{aligned}
y'(1) = 4 &= -3c_1 e^{-3(1)} + c_2 e^{(1)} + 4 \\
&= -3c_1 e^{-3} + c_2 e + 4 \\
c_2 e &= 3c_1 e^{-3}
\end{aligned}$$

$$c_1 = -e^3(3c_1 e^{-3} + 6) = -3c_1 - 6e^3 \Rightarrow c_1 = -\frac{3}{2}e^3 \Rightarrow c_2 = -\frac{9}{2}e^{-1}$$

$$y = -\frac{3}{2}e^{3(1-x)} - \frac{9}{2}e^{x-1} + 4x$$

1.4.15 41

$y_0 = -3$, $y_1 = 0$

1.4.16 43

$$\begin{aligned}
\frac{d}{dt}(mv) &= F - mg \\
\frac{d}{dt}(\lambda x \frac{dx}{dt}) &= F - \lambda x g \\
x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + gx &= \frac{F}{\lambda} \\
x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 32x &= 5
\end{aligned}$$

2 First-Order Differential Equations

2.1 Solution Curves Without a Solution

2.1.1 21

0 is stable, 3 is unstable

2.1.2 23

2 is semi-stable

2.1.3 25

-2 is unstable, 0 is semi-stable, 2 is stable

2.1.4 27

-1 is stable, 0 is unstable

2.1.5 39

$P_0 < h/k$

2.1.6 41

$$g - \frac{k}{m}v^2 = 0 \Rightarrow v = \sqrt{\frac{gm}{k}}$$

2.2 Separable Equations

2.2.1 1

$$\begin{aligned}\frac{dy}{dx} &= \sin 5x \\ y &= -\frac{1}{5}\cos 5x + c\end{aligned}$$

2.2.2 3

$$\begin{aligned}dx + e^{3x}\,dy &= 0 \\ e^{-3x}\,dx + \,dy &= 0 \\ -\frac{1}{3}e^{-3x} + y &= c \\ y &= \frac{1}{3}e^{-3x} + c\end{aligned}$$

2.2.3 5

$$
\begin{aligned}
x\frac{dy}{dx} &= 4y \\
\frac{1}{4y}\,dy &= \frac{1}{x}\,dx \\
\frac{1}{4}\ln|4y| &= \ln|x| + c \\
\ln|4y| &= 4\ln|x| + c \\
4y &= e^{4\ln|x|+c} \\
&= c\left(e^{\ln|x|}\right)^4 \\
y &= cx^4
\end{aligned}
$$

2.2.4 7

$$
\begin{aligned}
\frac{dy}{dx} &= e^{3x+2y} \\
&= e^{3x}e^{2y} \\
e^{-2y}\,dy &= e^{3x}\,dx \\
-\frac{1}{2}e^{-2y} &= \frac{1}{3}e^{3x} + c \\
-3e^{-2y} &= 2e^{3x} + c
\end{aligned}
$$

2.2.5 9

$$
\begin{aligned}
y\ln x\frac{dx}{dy} &= \left(\frac{y+1}{x}\right)^2 \\
x^2\ln x\,dx &= \frac{(y+1)^2}{y}\,dy \\
x^3\left(\frac{\ln x}{3} - \frac{1}{9}\right) &= \frac{1}{2}y(y+4) + \ln y + c \\
\frac{1}{3}x^3\ln x - \frac{1}{9}x^3 &= \frac{1}{2}y^2 + 2y + \ln y + c
\end{aligned}
$$

2.2.6 11

$$\begin{aligned}
\csc y\,dx + \sec^2 x\,dy &= 0\\
\frac{1}{\sin y}\,dx + \frac{1}{\cos^2 x}\,dy &= 0\\
\cos^2 x\,dx + \sin y\,dy &= 0\\
\frac{1}{2}(1+\cos 2x)\,dx + \sin y\,dy &= 0\\
\frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) - \cos y + c &= 0\\
&4\cos y = 2x + \sin 2x + c
\end{aligned}$$

2.2.7 13

$$\begin{aligned}
(e^y+1)^2 e^{-y}\,dx + (e^x+1)^3 e^{-x}\,dy &= 0\\
\frac{e^x}{(e^x+1)^3}\,dx + \frac{e^y}{(e^y+1)^2} &= 0\\
-\frac{1}{2(e^x+1)^2} - \frac{1}{e^y+1} &= c\\
(e^x+1)^{-2} + 2(e^y+1)^{-1} &= c
\end{aligned}$$

2.2.8 15

$$\begin{aligned}
\frac{dS}{dr} &= kS\\
\frac{1}{S}\,dS &= k\,dr\\
\ln|S| &= kr + c\\
S &= ce^{kr}
\end{aligned}$$

2.2.9 17

$$\begin{aligned}
\frac{dP}{dt} &= P - P^2 \\
\frac{1}{P(1-P)}\,dP &= dt \\
\ln\frac{P}{1-P} &= t + c \\
\frac{P}{1-P} &= ce^t \\
P &= ce^t(1-P) \\
P &= \frac{ce^t}{1+ce^t}
\end{aligned}$$

2.2.10 19

$$\begin{aligned}
\frac{dy}{dx} &= \frac{xy+3x-y-3}{xy-2x+4y-8} \\
&= \frac{(x-1)(y+3)}{(x+4)(y-2)} \\
\frac{y-2}{y+3}\,dt &= \frac{x-1}{x+4}\,dx \\
y - 5\ln|y+3| &= x - 5\ln|x+4| + c \\
e^{y-5\ln|y+3|} &= e^{x-5\ln|x+4|+c} \\
\frac{e^y}{(y+3)^5} &= \frac{ce^x}{(x+4)^5} \\
c(x+4)^5e^y &= (y+3)^5e^x
\end{aligned}$$

2.2.11 21

$$\begin{aligned}
\frac{dy}{dx} &= x\sqrt{1-y^2} \\
(1-y^2)^{-1/2}\,dy &= x\,dx \\
\arcsin y &= \frac{1}{2}x^2 + c \\
y &= \sin\left(\frac{1}{2}x^2 + c\right)
\end{aligned}$$

2.2.12 23

$$\begin{aligned}
\frac{dx}{dt} &= 4(x^2+1) \\
\frac{1}{x^2+1}\,dx &= 4\,dt \\
\arctan x &= 4t + c \\
x &= \tan(4t+c)
\end{aligned}$$

$$\begin{aligned}
x\left(\frac{\pi}{4}\right) = 1 &= \tan\left(4\left(\frac{\pi}{4}\right)+c\right) \\
&= \tan(\pi + c) \\
c &= \arctan(1) - \pi \\
&= -\frac{3}{4}\pi
\end{aligned}$$

$$x = \tan\left(4t - \frac{3}{4}\pi\right)$$

2.2.13 25

$$\begin{aligned}
x^2\frac{dy}{dx} &= y - xy \\
&= y(1-x) \\
\frac{1}{y}\,dy &= \left(\frac{1}{x^2} - \frac{1}{x}\right)\,dx \\
\ln|y| &= -\frac{1}{x} - \ln|x| + c \\
y &= e^{-\frac{1}{x}-\ln|x|+c} \\
&= \frac{c}{xe^{1/x}}
\end{aligned}$$

$$\begin{aligned}
y(-1) = -1 &= \frac{c}{(-1)e^{1/(-1)}} \\
&= -ce \\
c &= e^{-1}
\end{aligned}$$

$$y = \frac{1}{xe^{1+1/x}}$$

2.2.14 29

$$
\begin{aligned}
\frac{dy}{dx} &= ye^{-x^2} \\
\frac{1}{y}\frac{dy}{dx} &= e^{-x^2} \\
\int_4^x \frac{1}{y}\frac{dy}{dx'}\,dx' &= \int_4^x e^{-x'^2}\,dx' \\
\ln|y|\big|_4^x &= \int_4^x e^{-x'^2}\,dx' \\
\ln|y(x)| - \ln|y(4)| &= \int_4^x e^{-x'^2}\,dx' \\
\ln|y(x)| &= \ln|y(4)| + \int_4^x e^{-x'^2}\,dx' \\
y(x) &= e^{\int_4^x e^{-x'^2}\,dx'}
\end{aligned}
$$

2.2.15 31

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{2x+1}{2y} \\
2y\,dy &= (2x+1)\,dx \\
y^2 &= x^2 + x + c \\
y &= \pm\sqrt{x^2+x+c}
\end{aligned}
$$

$$
\begin{aligned}
y(-2) = -1 &= -\sqrt{(-2)^2 + (-2) + c} \\
&= -\sqrt{2+c} \\
c &= -1
\end{aligned}
$$

$$y = -\sqrt{x^2+x-1}$$

$$I = \left(-\infty, -\frac{1-\sqrt{5}}{2}\right)$$

2.2.16 33

$$\begin{aligned}
e^y\,dx - e^{-x}\,dy &= 0 \\
e^x\,dx - e^{-y}\,dy &= 0 \\
e^x + e^{-y} &= c \\
\ln|e^{-y}| &= \ln|c - e^x| \\
y &= -\ln|c - e^x|
\end{aligned}$$

$$\begin{aligned}
y(0) = 0 &= -\ln|c - e^{(0)}| \\
1 &= c - 1 \\
c &= 2
\end{aligned}$$

$$y = -\ln|2 - e^x|$$

$$I = (-\infty, \ln 2)$$

2.3 Linear Equations

2.3.1 1

$$\begin{aligned}
\frac{dy}{dx} &= 5y \\
\ln|y| &= 5x + c \\
y &= ce^{5x}
\end{aligned}$$

$$I = (-\infty, \infty)$$

2.3.2 3

$$\begin{aligned}
\frac{dy}{dx} + y &= e^{3x} \\
e^x\frac{dy}{dx} + e^x y &= e^{4x} \\
\frac{d}{dx}(e^x y) &= e^{4x} \\
e^x y &= \frac{1}{4}e^{4x} + c \\
y &= \frac{1}{4}e^{3x} + ce^{-x}
\end{aligned}$$

$$I = (-\infty, \infty)$$

2.3.3 5

$$\begin{aligned}
y' + 3x^2 y &= x^2 \\
e^{x^3} y' + 3x^2 e^{x^3} y &= e^{x^3} x^2 \\
e^{x^3} y &= \frac{1}{3} e^{x^3} + c \\
y &= \frac{1}{3} + c e^{-x^3}
\end{aligned}$$

$$I = (-\infty, \infty)$$

2.3.4 7

$$\begin{aligned}
x^2 y' + xy &= 1 \\
y' + x^{-1} y &= x^{-2} \\
e^{\ln x} y' + x^{-1} e^{\ln x} y &= e^{\ln x} x^{-2} \\
\frac{d}{dx}(e^{\ln x} y) &= x^{-1} \\
\frac{d}{dx}(xy) &= x^{-1} \\
xy &= \ln x + c \\
y &= \frac{\ln x + c}{x}
\end{aligned}$$

$$I = (0, \infty)$$

2.3.5 9

$$\begin{aligned}
x \frac{dy}{dx} - y &= x^2 \sin x \\
\frac{dy}{dx} - x^{-1} y &= x \sin x \\
e^{-\ln x} \frac{dy}{dx} - x^{-1} e^{-\ln x} y &= e^{-\ln x} x \sin x \\
\frac{d}{dx}(e^{-\ln x} y) &= \sin x \\
x^{-1} y &= -\cos x + c \\
y &= cx - x \cos x
\end{aligned}$$

$$I = (0, \infty)$$

2.3.6 11

$$\begin{aligned}
x\frac{dy}{dx} + 4y &= x^3 - x \\
\frac{dy}{dx} + 4x^{-1}y &= x^2 - 1 \\
e^{4\ln x}\frac{dy}{dx} + 4x^{-1}e^{4\ln x}y &= e^{4\ln x}(x^2 - 1) \\
\frac{d}{dx}(e^{4\ln x}y) &= x^6 - x^4 \\
x^4y &= \frac{1}{7}x^7 - \frac{1}{5}x^5 + c \\
y &= \frac{1}{7}x^3 - \frac{1}{5}x^2 + cx^{-4}
\end{aligned}$$

$$I = (0, \infty)$$

2.3.7 13

$$\begin{aligned}
x^2y' + x(x+2)y &= e^x \\
y' + x^{-1}(x+2)y &= x^{-2}e^x \\
e^{x+2\ln x}y' + x^{-1}(x+2)e^{x+2\ln x}y &= e^{x+2\ln x}x^{-2}e^x \\
\frac{d}{dx}(e^x x^2 y) &= e^{2x} \\
e^x x^2 y &= \frac{1}{2}e^{2x} + c \\
y &= \frac{e^x}{2x^2} + \frac{c}{e^x x^2}
\end{aligned}$$

$$I = (0, \infty)$$

2.3.8 15

$$
\begin{aligned}
y\,dx - 4(x + y^6)\,dy &= 0 \\
y\frac{dx}{dy} - 4x - 4y^6 &= 0 \\
\frac{dx}{dy} - \frac{4}{y}x &= 4y^5 \\
e^{-4\ln y}\frac{dx}{dy} - \frac{4}{y}e^{-4\ln y}x &= 4e^{-4\ln y}y^5 \\
\frac{d}{dy}(e^{-4\ln y}x) &= 4y \\
y^{-4}x &= 2y^2 + c \\
x &= 2y^6 + cy^4
\end{aligned}
$$

$$I = (0, \infty)$$

2.3.9 17

$$
\begin{aligned}
\cos x\frac{dy}{dx} + (\sin x)y &= 1 \\
\frac{dy}{dx} + (\tan x)y &= \sec x \\
e^{\ln(\sec x)}\frac{dy}{dx} + (\tan x)e^{\ln(\sec x)}y &= e^{\ln(\sec x)}\sec x \\
\frac{d}{dx}(e^{\ln(\sec x)}y) &= \sec^2 x \\
y\sec x &= \tan x + c \\
y &= \sin x + c\cos x
\end{aligned}
$$

$$I = (-\pi/2, \pi/2)$$

2.3.10 19

$$
\begin{aligned}
(x+1)\frac{dy}{dx}+(x+2)y &= 2xe^{-x} \\
\frac{dy}{dx}+\frac{x+2}{x+1}y &= \frac{2xe^{-x}}{x+1} \\
e^{x+\ln|x+1|}\frac{dy}{dx}+\frac{x+2}{x+1}e^{x+\ln|x+1|}y &= e^{x+\ln|x+1|}\frac{2xe^{-x}}{x+1} \\
\frac{d}{dx}(e^{x+\ln|x+1|}y) &= 2x \\
e^{x}(x+1)y &= x^2+c \\
y &= \frac{x^2+c}{e^{x}(x+1)}
\end{aligned}
$$

$$I=(-1,\infty)$$

2.3.11 21

$$
\begin{aligned}
\frac{dr}{d\theta}+r\sec\theta &= \cos\theta \\
e^{\ln|\sec\theta+\tan\theta|}\frac{dr}{d\theta}+e^{\ln|\sec\theta+\tan\theta|}r\sec\theta &= e^{\ln|\sec\theta+\tan\theta|}\cos\theta \\
\frac{d}{d\theta}(e^{\ln|\sec\theta+\tan\theta|}r) &= 1+\sin\theta \\
(\sec\theta+\tan\theta)r &= \theta-\cos\theta+c \\
r &= \frac{\theta-\cos\theta+c}{\sec\theta+\tan\theta}
\end{aligned}
$$

$$I=(-\pi/2,\pi/2)$$

2.3.12 23

$$
\begin{aligned}
x\frac{dy}{dx}+(3x+1)y &= e^{-3x} \\
\frac{dy}{dx}+(3+x^{-1})y &= e^{-3x}x^{-1} \\
e^{3x+\ln|x|}\frac{dy}{dx}+(3+x^{-1})e^{3x+\ln|x|}y &= 1 \\
\frac{d}{dx}(e^{3x+\ln|x|}y) &= 1 \\
e^{3x}xy &= x+c \\
y &= \frac{x+c}{e^{3x}x}
\end{aligned}
$$

$$I=(0,\infty)$$

2.3.13 25

$$\begin{aligned} xy' + y &= e^x \\ y' + x^{-1}y &= e^x x^{-1} \\ e^{\ln|x|}y' + x^{-1}e^{\ln|x|}y &= e^x \\ \frac{d}{dx}(e^{\ln|x|}y) &= e^x \\ xy &= e^x + c \\ y &= \frac{e^x + c}{x} \end{aligned}$$

$$\begin{aligned} y(1) = 2 &= \frac{e^{(1)} + c}{(1)} \\ c &= 2 - e \end{aligned}$$

$$y = \frac{e^x + 2 - e}{x}$$

$$I = (0, \infty)$$

2.3.14 27

$$\begin{aligned} L\frac{di}{dt} + Ri &= E \\ \frac{di}{dt} + \frac{R}{L}i &= \frac{E}{L} \\ e^{Rt/L}\frac{di}{dt} + \frac{R}{L}e^{Rt/L}i &= \frac{E}{L}e^{Rt/L} \\ \frac{d}{dt}(e^{Rt/L}i) &= \frac{E}{L}e^{Rt/L} \\ e^{Rt/L}i &= \frac{E}{R}e^{Rt/L} + c \\ i &= \frac{E}{R} + ce^{-Rt/L} \end{aligned}$$

$$\begin{aligned} i(0) = i_0 &= \frac{E}{R} + ce^{-R(0)/L} \\ &= \frac{E}{R} + c \\ c &= i_0 - \frac{E}{R} \end{aligned}$$

$$i = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right) e^{-Rt/L}$$

$$I = (-\infty, \infty)$$

2.3.15 53

$$\begin{aligned} \frac{dE}{dt} &= -\frac{1}{RC}E \\ \frac{1}{E}\frac{dE}{dt} &= -\frac{1}{RC} \\ \ln|E| &= -\frac{1}{RC}t + c \\ E &= ce^{-t/RC} \end{aligned}$$

$$\begin{aligned} E(4) = E_0 &= ce^{-(4)/RC} \\ c &= E_0 e^{4/RC} \end{aligned}$$

$$E = E_0 e^{(4-t)/RC}$$

2.4 Exact Equations

2.4.1 1

$$f(x, y) = x^2 - x + g(y)$$

$$\frac{\partial f}{\partial y} = g'(y) = 3y + 7$$

$$g(y) = \frac{3}{2}y^2 + 7y$$

$$x^2 - x + \frac{3}{2}y^2 + 7y = c$$

2.4.2 3

$$f(x, y) = \frac{5}{2}x^2 + 4xy + g(y)$$

$$4x + g'(y) = 4x - 8y^3 \Rightarrow g'(y) = -8y^3$$

$$g(y) = -2y^4$$

$$\frac{5}{2}x^2 + 4xy - 2y^4 = c$$

2.4.3 5

$$f(x,y) = x^2y^2 - 3x + g(y)$$

$$2x^2y + g'(y) = 2x^2y + 4 \Rightarrow g'(y) = 4$$

$$g(y) = 4y$$

$$x^2y^2 - 3x + 4y = c$$

2.4.4 7

Not exact

2.4.5 9

$$f(x,y) = \frac{1}{2}x^2 - xy^3 - y^2\cos x + g(y)$$

$$-3xy^2 - 2y\cos x + g'(y) = -3xy^2 - 2y\cos x \Rightarrow g'(y) = 0$$

$$\frac{1}{2}x^2 - xy^3 - y^2\cos x = c$$

2.4.6 11

Not exact

2.4.7 13

$$f(x,y) = xy + g(x)$$

$$y + g'(x) = -2xe^x + y - 6x^2 \Rightarrow g'(x) = -2xe^x - 6x^2$$

$$g(x) = -2e^x(x-1) - 2x^3$$

$$xy - 2e^x(x-1) - 2x^3 = c$$

2.4.8 21

$$f(x,y) = \frac{1}{3}(x+y)^3 + g(y)$$

$$(x+y)^2 + g'(y) = 2xy + x^2 - 1 \Rightarrow g'(y) = -y^2 - 1$$

$$g(y) = -\frac{1}{3}y^3 - y$$

$$\frac{1}{3}(x+y)^3 - \frac{1}{3}y^3 - y = c$$

$$\frac{1}{3}(1+1)^3 - \frac{1}{3}1^3 - 1 = c \Rightarrow c = \frac{4}{3}$$

$$x^3 + 3x^2y + 3xy^2 - 3y = 4$$

2.4.9 23

$$f(x,y) = 4ty + t^2 - 5t + g(y)$$

$$4t + g'(y) = 6y + 4t - 1 \Rightarrow g'(y) = 6y - 1$$

$$g(y) = 3y^2 - y$$

$$4ty + t^2 - 5t + 3y^2 - y = c$$

$$4(-1)(2) + (-1)^2 - 5(-1) + 3(2)^2 - (2) = c \Rightarrow c = 8$$

$$4ty + t^2 - 5t + 3y^2 - y = 8$$

2.4.10 27

$$3y^2 + 4kxy^3 = 3y^2 + 40xy^3 \Rightarrow k = 10$$

2.4.11 31

$$M_y = 4y$$

$$N_x = 2y$$

$$\frac{M_y - N_x}{N} = \frac{4y - 2y}{2xy} = \frac{1}{x}$$

$$\mu(x) = e^{\ln x} = x$$

$$(2xy^2 + 3x^2)\, dx + 2x^2 y\, dy = 0$$

$$f(x,y) = x^2y^2 + x^3 + g(y)$$

$$2x^2y + g'(y) = 2x^2y \Rightarrow g'(y) = 0$$

$$x^2y^2 + x^3 = c$$

2.4.12 33

$$M_y = 6x$$

$$N_x = 18x$$

$$\frac{M_y - N_x}{N} = \frac{6x - 18x}{4y + 9x^2}$$

$$\frac{N_x - M_y}{M} = \frac{18x - 6x}{6xy} = \frac{2}{y}$$

$$\mu(y) = e^{2\ln y} = y^2$$

$$6xy^3\, dx + (4y^3 + 9x^2y^2)\, dy = 0$$

$$f(x,y) = 3x^2y^3 + g(y)$$

$$9x^2y^2 + g'(y) = 4y^3 + 9x^2y^2 \Rightarrow g'(y) = 4y^3$$

$$g(y) = y^4$$

$$3x^2y^3 + y^4 = c$$

2.4.13 37

$$M_y = 0$$

$$N_x = 2xy$$

$$\frac{N_x - M_y}{M} = \frac{2xy - 0}{x} = 2y$$

$$\mu(y) = e^{y^2}$$

$$e^{y^2} x\,dx + e^{y^2}(x^2y + 4y)\,dy = 0$$

$$f(x,y) = \frac{1}{2}e^{y^2}x^2 + g(y)$$

$$ye^{y^2}x^2 + g'(y) = e^{y^2}(x^2y + 4y) \Rightarrow g'(y) = 4e^{y^2}y$$

$$g(y) = 2e^{y^2}$$

$$\frac{1}{2}e^{y^2}x^2 + 2e^{y^2} = c$$

$$\frac{1}{2}e^{(0)^2}(4)^2 + 2e^{(0)^2} = c \Rightarrow c = 10$$

$$\frac{1}{2}e^{y^2}x^2 + 2e^{y^2} = 10$$

2.4.14 39

(c)

$$(0)^3 + 2(0)^2(-2) + (-2)^2 = c \Rightarrow c = 4$$

$$y^2 + 2x^2y + x^3 - 4 = 0$$

$$\begin{aligned} y &= \frac{-(2x^2) \pm \sqrt{(2x^2)^2 - 4(1)(x^3 - 4)}}{2(1)} \\ &= \frac{-2x^2 \pm \sqrt{4x^4 - 4(x^3 - 4)}}{2} \\ &= -x^2 \pm \sqrt{x^4 - x^3 + 4} \end{aligned}$$

2.4.15 45

(a)

$$xv\frac{dv}{dx} + v^2 = 32x \Rightarrow xv\,dv + (v^2 - 32x)\,dx = 0$$

$$M_x = v$$

$$N_v = 2v$$

$$\frac{M_x - N_v}{N} = \frac{v - 2v}{v^2 - 32x}$$

$$\frac{N_v - M_x}{M} = \frac{2v - v}{xv} = \frac{1}{x}$$

$$\mu(x) = e^{\ln x} = x$$

$$x^2v\,dv + (xv^2 - 32x^2)\,dx = 0$$

$$f(x, v) = \frac{1}{2}x^2v^2 + g(x)$$

$$xv^2 + g'(x) = xv^2 - 32x^2 \Rightarrow g'(x) = -32x^2$$

$$g(x) = -\frac{32}{3}x^3$$

$$\frac{1}{2}(3)^2(0)^2 - \frac{32}{3}(3)^3 = c \Rightarrow c = -288$$

$$\frac{1}{2}x^2v^2 - \frac{32}{3}x^3 = -288 \Rightarrow v = 8\sqrt{\frac{x}{3} - \frac{9}{x^2}}$$

(b) $v = 12.7\,\mathrm{ft/s}$

2.5 Solutions by Substitution

2.5.1 1

$$
\begin{aligned}
(x-y)\,dx + x\,dy &= 0 \\
(x-ux)\,dx + x(u\,dx + x\,du) &= 0 \\
x\,dx + x^2\,du &= 0 \\
x^{-1}\,dx + du &= 0 \\
\ln|x| + u &= c \\
\ln|x| + \frac{y}{x} &= c \\
y &= cx - x\ln|x|
\end{aligned}
$$

2.5.2 3

$$
\begin{aligned}
x\,dx + (y-2x)\,dy &= 0 \\
vy(v\,dy + y\,dv) + (y-2vy)\,dy &= 0 \\
(v^2y + y - 2vy)\,dy + vy^2\,dv &= 0 \\
y(v^2 - 2v + 1)\,dy + vy^2\,dv &= 0 \\
(v-1)^2\,dy + vy\,dv &= 0 \\
\frac{1}{y}\,dy + \frac{v}{(v-1)^2}\,dv &= 0 \\
\ln|y| + \frac{1}{1-v} + \ln|v-1| &= c \\
\ln|y| + \frac{1}{1-x/y} + \ln\left|\frac{x}{y} - 1\right| &= c \\
\ln|x-y| + \frac{y}{y-x} &= c \\
(y-x)\ln|x-y| + y &= c(y-x) \\
(x-y)\ln|x-y| &= y + c(x-y)
\end{aligned}
$$

2.5.3 5

$$
\begin{aligned}
(y^2 + yx)\,dx - x^2\,dy &= 0 \\
((ux)^2 + ux^2)\,dx - x^2(u\,dx + x\,du) &= 0 \\
u^2x^2\,dx - x^3\,du &= 0 \\
\frac{1}{x}\,dx - \frac{1}{u^2}\,du &= 0 \\
\ln|x| + \frac{1}{u} &= c \\
\ln|x| + \frac{x}{y} &= c \\
y &= \frac{x}{c - \ln|x|}
\end{aligned}
$$

2.5.4 7

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{y-x}{y+x} \\
(y+x)\,dy + (x-y)\,dx &= 0 \\
(ux+x)(u\,dx + x\,du) + (x - ux)\,dx &= 0 \\
(u^2x + x)\,dx + (ux^2 + x^2)\,du &= 0 \\
x(u^2+1)\,dx + x^2(u+1)\,du &= 0 \\
\frac{1}{x}\,dx + \frac{u+1}{u^2+1}\,du &= 0 \\
\ln|x| + \frac{1}{2}\ln|u^2+1| + \arctan u &= c \\
\ln|x^2+y^2| + 2\arctan\frac{y}{x} &= c
\end{aligned}
$$

2.5.5 9

$$
\begin{aligned}
-y\,dx + (x + \sqrt{xy})\,dy &= 0 \\
-ux\,dx + (x + \sqrt{ux^2})(u\,dx + x\,du) &= 0 \\
u\sqrt{ux^2}\,dx + (x^2 + x\sqrt{ux^2})\,du &= 0 \\
u^{3/2}x\,dx + x^2(1 + \sqrt{u})\,du &= 0 \\
\frac{1}{x}\,dx + \frac{1+\sqrt{u}}{u^{3/2}}\,du &= 0 \\
\frac{1}{x}\,dx + (u^{-3/2} + u^{-1})\,du &= 0 \\
\ln|x| - 2u^{-1/2} + \ln|u| &= c \\
\ln|x| - 2(y/x)^{-1/2} + \ln|y/x| &= c \\
\ln|y| - 2\sqrt{\frac{x}{y}} &= c \\
4\frac{x}{y} &= (\ln|y| - c)^2 \\
4x &= y(\ln|y| - c)^2
\end{aligned}
$$

2.5.6 11

$$
\begin{aligned}
xy^2\frac{dy}{dx} &= y^3 - x^3 \\
xy^2\,dy + (x^3 - y^3)\,dx &= 0 \\
x(ux)^2(u\,dx + x\,du) + (x^3 - (ux)^3)\,dx &= 0 \\
x^3\,dx + u^2x^4\,du &= 0 \\
x^{-1}\,dx + u^2\,du &= 0 \\
\ln|x| + \frac{1}{3}u^3 &= c \\
\ln|x| + \frac{1}{3}\left(\frac{y}{x}\right)^3 &= c
\end{aligned}
$$

$$
\ln|1| + \frac{1}{3}\left(\frac{2}{1}\right)^3 = c \Rightarrow c = \frac{8}{3} + \ln 1
$$

$$
\begin{aligned}
\ln|x| + \frac{1}{3}\left(\frac{y}{x}\right)^3 &= \frac{8}{3} \\
y^3 + 3x^3\ln|x| &= 8x^3
\end{aligned}
$$

2.5.7 13

$$
\begin{aligned}
(x + ye^{y/x})\,dx - xe^{y/x}\,dy &= 0 \\
(x + uxe^{u})\,dx - xe^{u}(u\,dx + x\,du) &= 0 \\
x\,dx - x^2 e^{u}\,du &= 0 \\
x^{-1}\,dx - e^{u}\,du &= 0 \\
\ln|x| - e^{u} &= c \\
\ln|x| - e^{y/x} &= c
\end{aligned}
$$

$$\ln|1| - e^{0/1} = c \Rightarrow c = -1$$

$$\ln|x| = e^{y/x} - 1$$

2.5.8 15

$$
\begin{aligned}
x\frac{dy}{dx} + y &= \frac{1}{y^2} \\
\frac{dy}{dx} + x^{-1}y &= x^{-1}y^{-2}
\end{aligned}
$$

$$u = y^{1-n} = y^3 \Rightarrow y = u^{1/3} \Rightarrow \frac{dy}{dx} = \frac{1}{3}u^{-2/3}\frac{du}{dx}$$

$$
\begin{aligned}
\frac{1}{3}u^{-2/3}\frac{du}{dx} + x^{-1}u^{1/3} &= x^{-1}u^{-2/3} \\
\frac{du}{dx} + 3x^{-1}u &= 3x^{-1} \\
e^{3\ln|x|}\frac{du}{dx} + 3x^{-1}e^{3\ln|x|}u &= 3x^2 \\
\frac{d}{dx}(x^3u) &= 3x^2 \\
x^3u &= x^3 + c \\
y^3 &= 1 + cx^{-3}
\end{aligned}
$$

2.5.9 17

$$
\begin{aligned}
\frac{dy}{dx} &= y(xy^3 - 1) \\
\frac{dy}{dx} + y &= xy^4
\end{aligned}
$$

$$u = y^{1-n} = y^{-3} \Rightarrow y = u^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}u^{-4/3}\frac{du}{dx}$$

$$\begin{aligned}
-\frac{1}{3}u^{-4/3}\frac{du}{dx} + u^{-1/3} &= xu^{-4/3} \\
\frac{du}{dx} - 3u &= -3x \\
e^{-3x}\frac{du}{dx} - 3e^{-3x}u &= -3e^{-3x}x \\
\frac{d}{dt}(e^{-3x}u) &= -3e^{-3x}x \\
e^{-3x}u &= e^{-3x}x + \frac{1}{3}e^{-3x} + c \\
u &= x + \frac{1}{3} + ce^{3x} \\
y^{-3} &= x + \frac{1}{3} + ce^{3x}
\end{aligned}$$

2.5.10 21

$$\begin{aligned}
x^2\frac{dy}{dx} - 2xy &= 3y^4 \\
\frac{dy}{dx} - 2x^{-1}y &= 3x^{-2}y^4
\end{aligned}$$

$$u = y^{1-n} = y^{-3} \Rightarrow y = u^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}u^{-4/3}\frac{du}{dx}$$

$$\begin{aligned}
-\frac{1}{3}u^{-4/3}\frac{du}{dx} - 2x^{-1}u^{-1/3} &= 3x^{-2}u^{-4/3} \\
\frac{du}{dx} + 6x^{-1}u &= -9x^{-2} \\
e^{6\ln|x|}\frac{du}{dx} + 6x^{-1}e^{6\ln|x|}u &= -9e^{6\ln|x|}x^{-2} \\
\frac{d}{dx}(x^6u) &= -9x^4 \\
x^6u &= -\frac{9}{5}x^5 + c \\
u &= -\frac{9}{5}x^{-1} + cx^{-6} \\
y^{-3} &= -\frac{9}{5}x^{-1} + cx^{-6}
\end{aligned}$$

$$\left(\frac{1}{2}\right)^{-3} = -\frac{9}{5}(1)^{-1} + c(1)^{-6} \Rightarrow c = \frac{49}{5}$$

$$y^{-3} = -\frac{9}{5}x^{-1} + \frac{49}{5}x^{-6}$$

2.5.11 23

Let $u = x + y + 1$ so $\frac{du}{dx} = 1 + \frac{dy}{dx}$ and

$$\begin{aligned}
\frac{du}{dx} - 1 &= u^2 \\
\frac{1}{u^2+1}\,du &= dx \\
\arctan u &= x + c \\
\arctan(x + y + 1) &= x + c \\
x + y + 1 &= \tan(x + c) \\
y &= -x - 1 + \tan(x + c)
\end{aligned}$$

2.5.12 25

Let $u = x + y$ so $\frac{du}{dx} = 1 + \frac{dy}{dx}$ and

$$\begin{aligned}
\frac{du}{dx} - 1 &= \tan^2 u \\
\frac{1}{1+\tan^2 u}\,du &= dx \\
\frac{1}{2}(u + \sin u \cos u) &= x + c \\
x + y + \sin(x + y)\cos(x + y) &= 2(x + c) \\
x + y + \frac{1}{2}\sin(2(x + y)) &= 2(x + c) \\
2x + 2y + \sin(2(x + y)) &= 4(x + c) \\
2y - 2x + \sin(2(x + y)) &= c
\end{aligned}$$

2.5.13 35

(a) Let $y = y_1 + u$ so $\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{du}{dx}$ but $\frac{dy_1}{dx} = P(x) + Q(x)y_1 + R(x)y_1^2$ so

$$\begin{aligned}
\frac{dy}{dx} &= P(x) + Q(x)y + R(x)y^2 \\
P(x) + Q(x)y_1 + R(x)y_1^2 + \frac{du}{dx} &= P(x) + Q(x)(y_1 + u) + R(x)(y_1 + u)^2 \\
\frac{du}{dx} &= Q(x)u + R(x)(2y_1 u + u^2) \\
\frac{du}{dx} - (Q(x) + 2R(x)y_1)u &= R(x)u^2
\end{aligned}$$

(b) Let $y = 2x^{-1} + u$ so $\frac{dy}{dx} = -2x^{-2} + \frac{du}{dx}$ and

$$\begin{aligned}
-\frac{2}{x^2} + \frac{du}{dx} &= -\frac{4}{x^2} - \frac{1}{x}\left(\frac{2}{x} + u\right) + \left(\frac{2}{x} + u\right)^2 \\
\frac{du}{dx} &= \frac{2}{x^2} - \frac{4}{x^2} - \frac{2}{x^2} - \frac{u}{x} + \frac{4}{x^2} + \frac{4u}{x} + u^2 \\
\frac{du}{dx} - \frac{3}{x}u &= u^2
\end{aligned}$$

Let $v = u^{1-n} = u^{-1}$ so $u = v^{-1}$ and $\frac{du}{dx} = -v^{-2}\frac{dv}{dx}$

$$\begin{aligned}
-v^{-2}\frac{dv}{dx} - \frac{3}{x}v^{-1} &= v^{-2} \\
\frac{dv}{dx} + \frac{3}{x}v &= -1 \\
e^{3\ln|x|}\frac{dv}{dx} + \frac{3}{x}e^{3\ln|x|}v &= -e^{3\ln|x|} \\
\frac{d}{dt}(x^3 v) &= -x^3 \\
x^3 v &= -\frac{1}{4}x^4 + c \\
\frac{1}{y - y_1} &= -\frac{1}{4}x + cx^{-3} \\
y &= y_1 + \left(-\frac{1}{4}x + cx^{-3}\right)^{-1} \\
&= \frac{2}{x} + \left(-\frac{1}{4}x + cx^{-3}\right)^{-1}
\end{aligned}$$

2.5.14 37

$$\begin{aligned}
\frac{dP}{dt} &= P(a - bP) \\
\frac{dP}{dt} - aP &= -bP^2
\end{aligned}$$

Let $u = P^{1-n} = P^{-1}$ so $P = u^{-1}$ and $\frac{dP}{dt} = -u^{-2}\frac{du}{dt}$

$$\begin{aligned}
-u^{-2}\frac{du}{dt} - au^{-1} &= -bu^{-2} \\
\frac{du}{dt} + au &= b \\
e^{at}\frac{du}{dt} + ae^{at}u &= be^{at} \\
\frac{d}{dt}(e^{at}u) &= be^{at} \\
e^{at}u &= \frac{b}{a}e^{at} + c \\
P^{-1} &= \frac{b}{a} + ce^{-at} \\
&= \frac{b + ce^{-at}}{a} \\
P &= \frac{a}{b + ce^{-at}}
\end{aligned}$$

2.6 A Numerical Method

2.6.1 1

$$\begin{aligned}
x_0 &= 1 & y_0 &= 5 \\
x_1 &= 1.1 & y_1 &= y_0 + hf(x_0, y_0) = 3.8000 \\
x_2 &= 1.2 & y_2 &= y_1 + hf(x_1, y_1) = 2.9800
\end{aligned}$$

$$\begin{aligned}
x_0 &= 1 & y_0 &= 5 \\
x_1 &= 1.05 & y_1 &= y_0 + hf(x_0, y_0) = 4.4000 \\
x_2 &= 1.1 & y_2 &= y_1 + hf(x_1, y_1) = 3.8950 \\
x_3 &= 1.15 & y_3 &= y_2 + hf(x_2, y_2) = 3.4708 \\
x_4 &= 1.2 & y_4 &= y_3 + hf(x_3, y_3) = 3.1152
\end{aligned}$$

2.7 Linear Models

2.7.1 1

$$P(t) = P_0e^{kt}$$

$$P(5) = 2P_0 = P_0e^{5k} \Rightarrow k = \frac{\ln 2}{5} = 0.139$$

$$P(t) = P_0 e^{0.139t}$$

$$3P_0 = P_0 e^{0.139t} \Rightarrow t = 7.9\,\text{years}$$

$$4P_0 = P_0 e^{0.139t} \Rightarrow t = 10\,\text{years}$$

2.7.2 5

$$A(t) = A_0 e^{kt}$$

$$A(3.3) = \frac{1}{2}A_0 = A_0 e^{3.3k} \Rightarrow k = -0.21$$

$$0.1A_0 = A_0 e^{-0.21t} \Rightarrow t = 11\,\text{hours}$$

2.7.3 9

$$\frac{dI}{dt} = kI \Rightarrow I(t) = ce^{kt}$$

$$I(3) = 0.25I_0 = I_0 e^{3k} \Rightarrow k = -0.462$$

$$I(15) = I_0 e^{-0.462(15)} = 0.001I_0$$

2.7.4 11

$$0.145A_0 = A_0 e^{-0.00012097t} \Rightarrow t = 15\,963\,\text{years}$$

2.7.5 13

$$\begin{aligned}
\frac{dT}{dt} &= k(T - T_m) \\
\frac{1}{T - T_m}\frac{dT}{dt} &= k \\
\ln(T - T_m) &= kt + c \\
T - T_m &= ce^{kt} \\
T &= T_m + ce^{kt} \\
&= 10 + 60e^{kt}
\end{aligned}$$

$$T(0.5) = 50 = 10 + 60e^{0.5k} \Rightarrow k = -0.811$$

$$T(1) = 36.7$$

$$15 = 10 + 60e^{-0.811t} \Rightarrow t = 3.06\,\text{min}$$

2.7.6 21

$$\begin{aligned}
\frac{dA}{dt} &= 4 - \frac{A}{50} \\
\frac{dA}{dt} + \frac{A}{50} &= 4 \\
\frac{d}{dt}(e^{t/50}A) &= 4e^{t/50} \\
e^{t/50}A &= 200e^{t/50} + c \\
A &= 200 + ce^{-t/50}
\end{aligned}$$

$$A(0) = 30 = 200 + ce^{-(0)/50} \Rightarrow c = -170$$

$$A(t) = 200 - 170e^{-t/50}$$

2.7.7 25

$$V(t) = 500 - 5t$$

$$\begin{aligned}
\frac{dA}{dt} &= 10 - \frac{10}{500-5t}A \\
\frac{dA}{dt} + \frac{10}{500-5t}A &= 10 \\
\frac{dA}{dt} - 2\frac{-5}{500-5t}A &= 10 \\
e^{-2\ln(500-5t)}\frac{dA}{dt} - 2\frac{-5}{500-5t}e^{-2\ln(500-5t)}A &= e^{-2\ln(500-5t)}10 \\
\frac{d}{dt}(A(500-5t)^{-2}) &= 10(500-5t)^{-2} \\
A(500-5t)^{-2} &= \frac{2}{500-5t} + c \\
A &= 2(500-5t) + c(500-5t)^2 \\
&= 1000 - 10t + c(500-5t)^2
\end{aligned}$$

$$A(0) = 0 \Rightarrow c = -0.004$$

$$A(t) = 1000 - 10t - 0.004(500-5t)^2 = 1000 - 10t - \frac{1}{10}(100-t)^2$$

The tank is empty at $t = 100$

2.7.8 29

$$i(t) = \frac{3}{5} - \frac{3}{5}e^{-500t}$$

$i \to \frac{3}{5}$ as $t \to \infty$

2.7.9 33

$$i(t) = \begin{cases} 60(1 - e^{-t/10}), & 0 \le t \le 20 \\ 383e^{-t/10}, & t > 20 \end{cases}$$

2.7.10 35

(a)

$$\begin{aligned} m\frac{dv}{dt} &= mg - kv \\ \frac{dv}{dt} + \frac{k}{m}v &= g \\ \frac{d}{dt}(e^{kt/m}v) &= e^{kt/m}g \\ e^{kt/m}v &= \frac{gm}{k}e^{kt/m} + c \\ v &= \frac{gm}{k} + ce^{-kt/m} \end{aligned}$$

$$v(0) = v_0 = \frac{gm}{k} + ce^{-k(0)/m} \Rightarrow c = v_0 - \frac{gm}{k}$$

$$v(t) = \frac{gm}{k} + \left(v_0 - \frac{gm}{k}\right)e^{-kt/m}$$

(b)

$$v_t = \frac{gm}{k}$$

(c)

$$s(t) = \frac{gm}{k}t - \frac{m}{k}\left(v_0 - \frac{gm}{k}\right)e^{-kt/m} + c$$

$$s(0) = 0 = -\frac{m}{k}\left(v_0 - \frac{gm}{k}\right) + c \Rightarrow c = \frac{m}{k}\left(v_0 - \frac{gm}{k}\right)$$

$$\begin{aligned} s(t) &= \frac{m}{k}\left(gt - \left(v_0 + \frac{gm}{k}\right)e^{-kt/m} + v_0 - \frac{gm}{k}\right) \\ &= \frac{m}{k}\left(gt + \left(v_0 - \frac{gm}{k}\right)\left(1 - e^{-kt/m}\right)\right) \end{aligned}$$

2.7.11 41

(a)

$$\begin{aligned}\frac{dP}{dt} &= k_1P - k_2P \\ &= (k_1 - k_2)P \\ P &= ce^{(k_1-k_2)t}\end{aligned}$$

2.7.12 43

(a) $x = r/k$

2.8 Nonlinear Models

2.8.1 1

(a) $N = 2000$

(b)

$$N = \frac{1}{0.0005 + (1 - 0.0005)e^{-t}}$$

$$N(10) = 1834$$

2.8.2 3

$P = 1.0 \times 10^6$

$$P = \frac{500}{0.0005 + (0.1 - 0.0005)e^{-0.1t}}$$

$$\begin{aligned}500000 &= \frac{500}{0.0005 + (0.1 - 0.0005)e^{-0.1t}} \\ 500000(0.0005 + (0.1 - 0.0005)e^{-0.1t}) &= 500 \\ e^{-0.1t} &= \frac{0.001 - 0.0005}{0.1 - 0.0005} \\ t &= 52.9\,\text{months}\end{aligned}$$

2.8.3 11

$29.3\,\mathrm{g}$; $60\,\mathrm{g}$; $0\,\mathrm{g}$; $30\,\mathrm{g}$

2.8.4 13

(a)

$$\begin{aligned}
\frac{dh}{dt} &= -\frac{A_h}{A_w}\sqrt{2gh} \\
\frac{1}{\sqrt{h}}\frac{dh}{dt} &= -\frac{A_h}{A_w}\sqrt{2g} \\
2\sqrt{h} &= -\frac{A_h}{A_w}\sqrt{2g}t + c \\
\sqrt{h} &= c - \frac{A_h}{A_w}\sqrt{\frac{g}{2}}t \\
h &= \left(c - \frac{A_h}{A_w}\sqrt{\frac{g}{2}}t\right)^2
\end{aligned}$$

$$h(0) = H = c^2 \Rightarrow c = \sqrt{H}$$

$$h = \left(\sqrt{H} - \frac{A_h}{A_w}\sqrt{\frac{g}{2}}t\right)^2 = \left(\sqrt{H} - 4\frac{A_h}{A_w}t\right)^2$$

Interval of definition is $[0, \frac{A_w\sqrt{H}}{4A_h}]$

(b) $1821\,\mathrm{s} = 30\,\mathrm{min}$

2.8.5 15

(a)

$$\begin{aligned}
\frac{dh}{dt} &= -\frac{5}{6h^{3/2}} \\
h^{3/2}\frac{dh}{dt} &= -\frac{5}{6} \\
\frac{2}{5}h^{5/2} &= -\frac{5}{6}t + c \\
h &= \left(c - \frac{25}{12}t\right)^{2/5}
\end{aligned}$$

$$h(0) = H = c^{2/5} \Rightarrow c = H^{5/2}$$

$$h = \left(H^{5/2} - \frac{25}{12}t\right)^{2/5}$$

$$0 = \left(H^{5/2} - \frac{25}{12}t\right)^{2/5} \Rightarrow t = \frac{12}{25}H^{5/2} = 858\,\mathrm{s}$$

(b)

$$\begin{aligned}
V(h) &= \pi r^2 \frac{h}{3} \\
&= \pi \left(h \tan\frac{\pi}{6}\right)^2 \frac{h}{3} \\
&= \pi \left(\frac{h}{\sqrt{3}}\right)^2 \frac{h}{3} \\
&= \frac{1}{9}\pi h^3
\end{aligned}$$

$$\begin{aligned}
\frac{dV}{dt} &= -cA_h\sqrt{2gh} \\
\frac{d}{dt}\left(\frac{1}{9}\pi h^3\right) &= -cA_h\sqrt{2gh} \\
\frac{1}{3}\pi h^2 \frac{dh}{dt} &= -cA_h\sqrt{2gh} \\
h^{3/2}\frac{dh}{dt} &= -\frac{24}{\pi}cA_h \\
\frac{2}{5}h^{5/2} &= c_1 - \frac{24}{\pi}cA_h t \\
h &= \left(c_1 - \frac{60}{\pi}cA_h t\right)^{2/5}
\end{aligned}$$

$$h(0) = H = c_1^{2/5} \Rightarrow c_1 = H^{5/2}$$

$$h = \left(H^{5/2} - \frac{60}{\pi}cA_h t\right)^{2/5}$$

$$\begin{aligned}
0 &= \left(H^{5/2} - \frac{60}{\pi}cA_h t\right)^{2/5} \\
t &= \frac{\pi H^{5/2}}{60cA_h} \\
&= 243\,\mathrm{s}
\end{aligned}$$

2.8.6 17

(a)

$$\begin{aligned}
m\frac{dv}{dt} &= mg - kv^2 \\
\frac{m}{mg-kv^2}\frac{dv}{dt} &= 1 \\
\sqrt{\frac{m}{gk}}\operatorname{arctanh}\left(\sqrt{\frac{k}{gm}}v\right) &= t + c_1 \\
v &= \sqrt{\frac{gm}{k}}\tanh\left(\sqrt{\frac{gk}{m}}(t+c_1)\right)
\end{aligned}$$

$$\begin{aligned}
v(0) = v_0 &= \sqrt{\frac{gm}{k}}\tanh c_1 \\
c_1 &= \operatorname{arctanh}\sqrt{\frac{k}{gm}}v_0
\end{aligned}$$

(b) $v_t = \sqrt{gm/k}$

(c)

$$\begin{aligned}
s &= \frac{m}{k}\ln\cosh\left(\sqrt{\frac{gk}{m}}t + c_1\right) + c_2 \\
c_2 &= -\frac{m}{k}\ln\cosh c_1
\end{aligned}$$

2.8.7 21

(a) $W = 0$, $W = 2$

(b)

$$\begin{aligned}
\frac{dW}{dt} &= W\sqrt{4-2W} \\
\frac{1}{W\sqrt{4-2W}}\frac{dW}{dt} &= 1 \\
-\operatorname{arctanh}\left(\frac{1}{2}\sqrt{4-2W}\right) &= t + c \\
\frac{1}{2}\sqrt{4-2W} &= \tanh(c-t) \\
W &= 2 - 2\tanh^2(c-t) \\
&= 2(1-\tanh^2(c-t)) \\
&= 2\operatorname{sech}^2(c-t)
\end{aligned}$$

2.9 Modeling with Systems of First-Order DEs

2.9.1 1

$$\begin{aligned}\frac{dx}{dt} &= -\lambda_1 x\\ \ln|x| &= -\lambda_1 t + c_1\\ x &= c_1 e^{-\lambda_1 t}\end{aligned}$$

$$x(0) = x_0 = c_1 e^{-\lambda_1(0)} \Rightarrow c_1 = x_0$$

$$x = x_0 e^{-\lambda_1 t}$$

$$\begin{aligned}\frac{dy}{dt} &= \lambda_1 x - \lambda_2 y\\ &= \lambda_1 x_0 e^{-\lambda_1 t} - \lambda_2 y\\ \frac{dy}{dt} + \lambda_2 y &= \lambda_1 x_0 e^{-\lambda_1 t}\\ \frac{d}{dt}(e^{\lambda_2 t} y) &= \lambda_1 x_0 e^{(\lambda_2 - \lambda_1)t}\\ e^{\lambda_2 t} y &= \frac{\lambda_1}{\lambda_2 - \lambda_1} x_0 e^{(\lambda_2 - \lambda_1)t} + c_2\\ y &= \frac{\lambda_1}{\lambda_2 - \lambda_1} x_0 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}\end{aligned}$$

$$y(0) = 0 = \frac{\lambda_1}{\lambda_2 - \lambda_1} x_0 e^{-\lambda_1(0)} + c_2 e^{-\lambda_2(0)} \Rightarrow c_2 = -\frac{\lambda_1}{\lambda_2 - \lambda_1} x_0$$

$$y = \frac{\lambda_1}{\lambda_2 - \lambda_1} x_0 (e^{-\lambda_1 t} - e^{-\lambda_2 t})$$

$$\begin{aligned}\frac{dz}{dt} &= \lambda_2 y\\ &= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} x_0 (e^{-\lambda_1 t} - e^{-\lambda_2 t})\\ z &= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} x_0 \left(-\frac{1}{\lambda_1} e^{-\lambda_1 t} + \frac{1}{\lambda_2} e^{-\lambda_2 t}\right) + c_3\\ &= \frac{\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t}}{\lambda_2 - \lambda_1} x_0 + c_3\end{aligned}$$

$$
\begin{aligned}
z(0) = 0 &= \frac{\lambda_1 e^{-\lambda_2(0)} - \lambda_2 e^{-\lambda_1(0)}}{\lambda_2 - \lambda_1} x_0 + c_3 \\
&= \frac{\lambda_1 - \lambda_2}{\lambda_2 - \lambda_1} x_0 + c_3 \\
c_3 &= x_0
\end{aligned}
$$

$$
z = \frac{\lambda_1(e^{-\lambda_2 t} - 1) + \lambda_2(1 - e^{-\lambda_1 t})}{\lambda_2 - \lambda_1} x_0
$$

2.9.2 3

$5\,\text{days},\ 20\,\text{days},\ 147\,\text{days}$

2.9.3 5

(a)

$$
\begin{aligned}
\frac{dP}{dt} &= -(\lambda_A + \lambda_C)P \\
P &= ce^{-(\lambda_A+\lambda_C)t}
\end{aligned}
$$

$$
P(0) = P_0 = ce^{-(\lambda_A+\lambda_C)(0)} \Rightarrow c = P_0
$$

$$
P = P_0 e^{-(\lambda_A+\lambda_C)t}
$$

(b)

$$
\frac{1}{2}P_0 = P_0 e^{-(\lambda_A+\lambda_C)t} \Rightarrow t = \frac{\ln 1/2}{-(\lambda_A + \lambda_C)} = 1.25 \times 10^9 \text{ years}
$$

(c)

$$
\begin{aligned}
\frac{dA}{dt} &= \lambda_A P \\
&= \lambda_A P_0 e^{-(\lambda_A+\lambda_C)t} \\
A &= -\frac{\lambda_A}{\lambda_A + \lambda_C} P_0 e^{-(\lambda_A+\lambda_C)t} + c
\end{aligned}
$$

$$
A(0) = 0 = -\frac{\lambda_A}{\lambda_A + \lambda_C} P_0 e^{-(\lambda_A+\lambda_C)(0)} + c \Rightarrow c = \frac{\lambda_A}{\lambda_A + \lambda_C} P_0
$$

$$
A = \frac{\lambda_A}{\lambda_A + \lambda_C} P_0 (1 - e^{-(\lambda_A+\lambda_C)t})
$$

$$\begin{aligned}\frac{dC}{dt} &= \lambda_C P \\ &= \lambda_C P_0 e^{-(\lambda_A+\lambda_C)t} \\ C &= -\frac{\lambda_C}{\lambda_A+\lambda_C} P_0 e^{-(\lambda_A+\lambda_C)t} + c\end{aligned}$$

$$C(0) = 0 = -\frac{\lambda_C}{\lambda_A+\lambda_C} P_0 e^{-(\lambda_A+\lambda_C)(0)} + c \Rightarrow c = \frac{\lambda_C}{\lambda_A+\lambda_C} P_0$$

$$C = \frac{\lambda_C}{\lambda_A+\lambda_C} P_0 (1 - e^{-(\lambda_A+\lambda_C)t})$$

(d)

$$\frac{\lambda_A}{\lambda_A+\lambda_C} = 10.5\%$$

$$\frac{\lambda_C}{\lambda_A+\lambda_C} = 89.5\%$$

2.9.4 7

$$\begin{aligned}\frac{dx_1}{dt} &= 6 - \frac{2}{25}x_1 + \frac{1}{50}x_2 \\ \frac{dx_2}{dt} &= \frac{2}{25}x_1 - \frac{2}{25}x_2\end{aligned}$$

2.9.5 9

(a)

$$V_1 = 100 + t$$

$$V_2 = 100 - t$$

$$\frac{dx_1}{dt} = \frac{3}{100-t}x_2 - \frac{2}{100+t}x_1$$

$$\frac{dx_2}{dt} = \frac{2}{100+t}x_1 - \frac{3}{100-t}x_2$$

(b)

$$\frac{dx_1}{dt} = -\frac{dx_2}{dt}$$

This makes sense because it's a closed system. Salt is moving from tank B to tank A.

$$x_1 = c - x_2$$

$$x_1(0) = c - x_2(0) \Rightarrow 100 = c - 50 \Rightarrow c = 150$$

$$\begin{aligned}
\frac{dx_2}{dt} &= \frac{2}{100+t}(150 - x_2) - \frac{3}{100-t}x_2 \\
&= \frac{300}{100+t} - \frac{2}{100+t}x_2 - \frac{3}{100-t}x_2 \\
\frac{dx_2}{dt} + \left(\frac{2}{100+t} + \frac{3}{100-t}\right)x_2 &= \frac{300}{100+t} \\
\frac{d}{dt}(e^{2\ln|100+t|-3ln|100-t|}x_2) &= \frac{300}{100+t}e^{2\ln|100+t|-3ln|100-t|} \\
\frac{d}{dt}\left(\frac{(100+t)^2}{(100-t)^3}x_2\right) &= \frac{300(100+t)}{(100-t)^3} \\
&= \frac{30000}{(100-t)^3} + \frac{300t}{(100-t)^3} \\
\frac{(100+t)^2}{(100-t)^3}x_2 &= \frac{15000}{(100-t)^2} + \frac{300(t-50)}{(100-t)^2} + c
\end{aligned}$$

$$\begin{aligned}
x_2 &= \frac{(100-t)^3}{(100+t)^2}\left(\frac{15000}{(100-t)^2} + \frac{300(t-50)}{(100-t)^2} + c\right) \\
x_2(0) = 50 &= \frac{100^3}{100^2}\left(\frac{15000}{100^2} + \frac{300(-50)}{100^2} + c\right) \\
&= 100c \\
c &= \frac{1}{2}
\end{aligned}$$

$$x_2(30) = 47.4\,\text{lb}$$

2.9.6 15

$$i_1 = i_2 + i_3$$

$$\begin{aligned} i_1 R_1 + \frac{di_2}{dt} L_1 + i_2 R_2 &= E(t) \\ (i_2 + i_3) R_1 + \frac{di_2}{dt} L_1 + i_2 R_2 &= E(t) \end{aligned}$$

$$\begin{aligned} i_1 R_1 + \frac{di_3}{dt} L_2 + i_3 R_3 &= E(t) \\ (i_2 + i_3) R_1 + \frac{di_3}{dt} L_2 + i_3 R_3 &= E(t) \end{aligned}$$

2.9.7 17

$i(0) = i_0$, $s(0) = n - i_0$, $r(0) = 0$; It's consistent because no one leaves the community

2.10 Chapter in Review

2.10.1 1

$y = -A/k$; repeller; attractor

2.10.2 3

$$\frac{dy}{dx} = (y-1)^2 (y-3)^2$$

2.10.3 5

$\frac{dy}{dx} = x^n$ is semi-stable for even n, unstable for odd n
$\frac{dy}{dx} = -x^n$ is semi-stable for even n, stable for odd n

2.10.4 9

$$\begin{aligned} (y^2+1)\,dx &= y \sec^2 x\, dy \\ \cos^2 x\, dx &= \frac{y}{y^2+1}\, dy \\ \frac{1}{2}(1+\cos 2x)\, dx &= \frac{1}{2} \frac{2y}{y^2+1}\, dy \\ x + \frac{1}{2} \sin 2x &= \ln|y^2+1| + c \\ 2x + \sin 2x &= 2 \ln|y^2+1| + c \end{aligned}$$

2.10.5 11

$$\begin{aligned}(6x+1)y^2\frac{dy}{dx}+3x^2+2y^3&=0\\(6x+1)y^2\,dy+(3x^2+2y^3)\,dx&=0\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial x}&=3x^2+2y^3\\f&=x^3+2xy^3+g(y)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y}=6xy^2+g'(y)&=6xy^2+y^2\\g'(y)&=y^2\\g(y)&=\frac{1}{3}y^3\end{aligned}$$

$$\begin{aligned}f(x,y)&=x^3+2xy^3+\frac{1}{3}y^3\\c&=x^3+2xy^3+\frac{1}{3}y^3\end{aligned}$$

2.10.6 13

$$\begin{aligned}t\frac{dQ}{dt}+Q&=t^4\ln t\\\frac{dQ}{dt}+\frac{1}{t}Q&=t^3\ln t\\\frac{d}{dt}(tQ)&=t^4\ln t\\tQ&=\frac{1}{25}t^5(5\ln t-1)+c\\Q&=\frac{1}{25}t^4(5\ln t-1)+ct^{-1}\end{aligned}$$

2.10.7 15

$$(8xy-2x)\,dx+(x^2+4)\,dy=0$$

$$M_y=8x$$

$$N_x=2x$$

$$\frac{M_y - N_x}{N} = \frac{6x}{x^2+4}$$

$$\mu(x) = e^{3\ln|x^2+4|} = (x^2+4)^3$$

$$(x^2+4)^3(8xy-2x)\,dx + (x^2+4)^4\,dy = 0$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= (x^2+4)^4 \\ f(x,y) &= (x^2+4)^4 y + g(x) \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial x} = 8x(x^2+4)^3 y + g'(x) &= (8xy-2x)(x^2+4)^3 \\ g'(x) &= -2x(x^2+4)^3 \\ g(x) &= -\frac{1}{4}(x^2+4)^4 \end{aligned}$$

$$\begin{aligned} c &= (y - \frac{1}{4})(x^2+4)^4 \\ y &= \frac{1}{4} + c(x^2+4)^{-4} \end{aligned}$$

2.10.8 17

$$\begin{aligned} 2\frac{dy}{dx} + (4\cos x)y &= x \\ \frac{dy}{dx} + (2\cos x)y &= \frac{1}{2}x \\ e^{2\sin x}\frac{dy}{dx} + (2\cos x)e^{2\sin x}y &= \frac{1}{2}xe^{2\sin x} \\ \frac{d}{dx}(e^{2\sin x}y) &= \frac{1}{2}xe^{2\sin x} \\ \int_0^x \frac{d}{dx}(e^{2\sin x'}y)\,dx' &= \int_0^x \frac{1}{2}x'e^{2\sin x'}\,dx' \\ e^{2\sin x}y - e^{2\sin 0} &= \int_0^x \frac{1}{2}x'e^{2\sin x'}\,dx' \\ y &= \frac{1}{e^{2\sin x}}\left(1 + \int_0^x \frac{1}{2}x'e^{2\sin x'}\,dx'\right) \end{aligned}$$

2.10.9 19

$$\begin{aligned} x\frac{dy}{dx} + 2y &= xe^{x^2} \\ \frac{dy}{dx} + \frac{2}{x}y &= e^{x^2} \\ \frac{d}{dt}(x^2 y) &= x^2 e^{x^2} \\ \int_1^x \frac{d}{dt}(x'^2 y)\,dx' &= \int_1^x x'^2 e^{x'^2}\,dx' \\ x^2 y - 3 &= \int_1^x x'^2 e^{x'^2}\,dx' \\ y &= \frac{3}{x^2} + \frac{1}{x^2}\int_1^x x'^2 e^{x'^2}\,dx' \end{aligned}$$

2.10.10 21

$$\begin{aligned} \frac{dy}{dx} + y &= e^{-x} \\ \frac{d}{dx}(e^x y) &= 1 \\ e^x y &= x + c_1 \\ y &= (x + c_1)e^{-x} \end{aligned}$$

$$y(0) = 5 = c_1$$

$$y = (x+5)e^{-x}$$

$$\begin{aligned} \frac{dy}{dx} + y &= 0 \\ \frac{d}{dt}(e^x y) &= 0 \\ e^x y &= c_2 \\ y &= c_2 e^{-x} \end{aligned}$$

$$(1+5)e^{-1} = c_2 e^{-1} \Rightarrow c_2 = 6$$

$$y = \begin{cases} (x+5)e^{-x} & 0 \le x < 1 \\ 6e^{-x} & x \ge 1 \end{cases}$$

2.10.11 23

$$\begin{aligned}
\sin x \frac{dy}{dx} + (\cos x)y &= 0 \\
\frac{dy}{dx} + (\cot x)y &= 0 \\
\frac{d}{dx}(y \sin x) &= 0 \\
y \sin x &= c \\
y &= c \csc x
\end{aligned}$$

$$y(7\pi/6) = -2 = c \csc \frac{7\pi}{6} \Rightarrow c = 1$$

$$y = \csc x$$

$$I = (\pi, 2\pi)$$

2.10.12 25

(a) Because $\sqrt{y}$ isn't defined for $y < 0$

(b)

$$\begin{aligned}
\frac{dy}{dx} &= \sqrt{y} \\
y^{-1/2}\frac{dy}{dx} &= 1 \\
2\sqrt{y} &= x + c \\
y &= \frac{1}{4}(x+c)^2
\end{aligned}$$

$$y(x_0) = y_0 = \frac{1}{4}(x_0 + c)^2 \Rightarrow c = \sqrt{4y_0} - x_0$$

$$y = \frac{1}{4}(x + \sqrt{4y_0} - x_0)^2$$

2.10.13 29

$$\begin{aligned}
\frac{dP}{dt} &= kP \\
P &= P_0 e^{kt} \\
P(45) &= 8.99 \times 10^9 \text{ people}
\end{aligned}$$

2.10.14 31

(a)

$$0.53A_0 = A_0 e^{-0.00012097t} \Rightarrow t = 5248 \text{ years ago}$$

(b) 3257 BC

2.10.15 35

(a)

$$\begin{aligned} k(T - T_m) &= 0 \\ T &= T_m \\ &= T_2 + B(T_1 - T) \\ &= \frac{BT_1 + T_2}{1 + B} \end{aligned}$$

T_m is the same

(b)

$$\begin{aligned} \frac{dT}{dt} &= k(T - T_m) \\ &= k(T - (T_2 + B(T_1 - T))) \\ &= k((1 + B)T - BT_1 - T_2) \\ \frac{dT}{dt} - k(1 + B)T &= -k(BT_1 + T_2) \\ \frac{d}{dt}(e^{-k(1+B)t}T) &= -k(BT_1 + T_2)e^{-k(1+B)t} \\ e^{-k(1+B)t}T &= \frac{BT_1 + T_2}{1 + B}e^{-k(1+B)t} + c \\ T &= \frac{BT_1 + T_2}{1 + B} + ce^{k(1+B)t} \end{aligned}$$

$$\begin{aligned} T(0) = T_1 &= \frac{BT_1 + T_2}{1 + B} + ce^{k(1+B)(0)} \\ c &= T_1 - \frac{BT_1 + T_2}{1 + B} \\ &= \frac{T_1(1 + B) - BT_1 - T_2}{1 + B} \\ &= \frac{T_1 - T_2}{1 + B} \end{aligned}$$

$$T = \frac{BT_1 + T_2 + (T_1 - T_2)e^{k(1+B)t}}{1 + B}$$

2.10.16 37

$$
\begin{aligned}
(k_1+k_2t)\frac{dq}{dt}+\frac{1}{C}q &= E_0\\
\frac{dq}{dt}+\frac{1}{C(k_1+k_2t)}q &= \frac{E_0}{k_1+k_2t}\\
\frac{d}{dt}(e^{\frac{\ln|C(k_1+k_2t)|}{Ck_2}}q) &= \frac{E_0}{k_1+k_2t}e^{\frac{\ln|C(k_1+k_2t)|}{Ck_2}}\\
\frac{d}{dt}((C(k_1+k_2t))^{1/Ck_2}q) &= \frac{E_0}{k_1+k_2t}(C(k_1+k_2t))^{1/Ck_2}\\
(C(k_1+k_2t))^{1/Ck_2}q &= E_0C(C(k_1+k_2t))^{1/Ck_2}+c\\
q &= E_0C+c(C(k_1+k_2t))^{-1/Ck_2}
\end{aligned}
$$

$$
\begin{aligned}
q(0)=q_0 &= E_0C+c(C(k_1+k_2(0)))^{-1/Ck_2}\\
q_0 &= E_0C+c(Ck_1)^{-1/Ck_2}\\
c &= (q_0-E_0C)(Ck_1)^{1/Ck_2}
\end{aligned}
$$

$$
\begin{aligned}
q &= E_0C+(q_0-E_0C)(Ck_1)^{1/Ck_2}(C(k_1+k_2t))^{-1/Ck_2}\\
&= E_0C+(q_0-E_0C)\left(\frac{k_1}{k_1+k_2t}\right)^{1/Ck_2}
\end{aligned}
$$

2.10.17 39

$$
\begin{aligned}
\frac{dh}{dt} &= -c\frac{\pi r_h^2}{\pi r_w^2}\sqrt{2gh}\\
\frac{1}{\sqrt{h}}\frac{dh}{dt} &= -8c(r_h/r_w)^2\\
2\sqrt{h} &= c_1-8c(r_h/r_w)^2t\\
h &= (c_1-4c(r_h/r_w)^2t)^2
\end{aligned}
$$

$$
h(0)=2=(c_1-4c(r_h/r_w)^2(0))^2 \Rightarrow c_1=\sqrt{2}
$$

$$
h=(\sqrt{2}-4c(r_h/r_w)^2t)^2=(\sqrt{2}-(1.63\times 10^{-5})t)^2
$$

2.10.18 43

$$\begin{aligned}
\frac{dx}{dt} &= k_1 x(\alpha - x) \\
\frac{1}{x(\alpha - x)}\frac{dx}{dt} &= k_1 \\
\left(\frac{1}{x} + \frac{1}{\alpha - x}\right)\frac{dx}{dt} &= \alpha k_1 \\
\ln|x| - \ln|\alpha - x| &= \alpha k_1 t + c_1 \\
\ln\left|\frac{x}{\alpha - x}\right| &= \alpha k_1 t + c_1 \\
\frac{x}{\alpha - x} &= c_1 e^{\alpha k_1 t} \\
x &= (\alpha - x) c_1 e^{\alpha k_1 t} \\
(c_1 e^{\alpha k_1 t} + 1)x &= \alpha c_1 e^{\alpha k_1 t} \\
x &= \frac{\alpha c_1 e^{\alpha k_1 t}}{c_1 e^{\alpha k_1 t} + 1}
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dt} &= k_2 x y \\
&= k_2 \frac{\alpha c_1 e^{\alpha k_1 t}}{c_1 e^{\alpha k_1 t} + 1} y \\
\frac{1}{y}\frac{dy}{dt} &= k_2 \frac{\alpha c_1 e^{\alpha k_1 t}}{c_1 e^{\alpha k_1 t} + 1} \\
\ln|y| &= \frac{k_2}{k_1} \ln|c_1 e^{\alpha k_1 t} + 1| + c_2 \\
y &= c_2 (c_1 e^{\alpha k_1 t} + 1)^{k_2/k_1}
\end{aligned}$$

2.10.19 45

$$\begin{aligned}
\frac{dP}{dt} &= kP \ln\frac{450}{P} \\
\frac{1}{P\ln(450/P)}\frac{dP}{dt} &= k \\
-\ln(\ln\frac{450}{P}) &= kt + c \\
\ln\frac{450}{P} &= ce^{-kt} \\
\frac{450}{P} &= e^{ce^{-kt}} \\
P &= \frac{450}{e^{ce^{-kt}}}
\end{aligned}$$

$$P(0) = 40 = \frac{450}{e^{ce^{-k(0)}}} \Rightarrow c = \ln\frac{450}{40} = 2.42$$

$$\begin{aligned}
P(15) = 95 &= \frac{450}{e^{2.42e^{-k(15)}}} \\
2.42e^{-15k} &= \ln\frac{450}{95} \\
k &= -\frac{\ln(\ln(450/95)/2.42)}{15} \\
&= 0.0295
\end{aligned}$$

$$P(30) = \frac{450}{e^{2.42e^{-0.0295(30)}}} = 166$$

2.10.20 47

$$\begin{aligned}
y &= c_1 x \\
\frac{dy}{dx} &= c_1
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{1}{c_1} \\
y &= -\frac{1}{c_1}x + c_2
\end{aligned}$$

2.10.21 49

$$\begin{aligned}
y &= -x - 1 + c_1 e^x \\
\frac{dy}{dx} &= c_1 e^x - 1
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{1}{c_1 e^x - 1} \\
y &= x - \ln(1 - c_1 e^x) + c_2
\end{aligned}$$

3 Higher-Order Differential Equations

3.1 Theory of Linear Equations

3.1.1 1

$$\begin{aligned}
y &= c_1 e^x + c_2 e^{-x} \\
0 &= c_1 e^{(0)} + c_2 e^{-(0)} \\
&= c_1 + c_2
\end{aligned}$$

$$\begin{aligned}
y' &= c_1 e^x - c_2 e^{-x} \\
1 &= c_1 e^{(0)} - c_2 e^{-(0)} \\
&= c_1 - c_2
\end{aligned}$$

$$c_2 = c_1 - 1 \Rightarrow 0 = c_1 + c_1 - 1 \Rightarrow c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}$$

$$y = \frac{1}{2}(e^x - x^{-x})$$

3.1.2 3

$$\begin{aligned}
y &= c_1 x + c_2 x \ln x \\
3 &= c_1(1) + c_2(1)\ln(1) \\
&= c_1
\end{aligned}$$

$$\begin{aligned}
y' &= 3 + c_2(1 + \ln x) \\
-1 &= 3 + c_2(1 + \ln(1)) \\
c_2 &= -4
\end{aligned}$$

$$y = 3x - 4x \ln x$$

3.1.3 9

$(-\infty, 2)$

3.1.4 11

(a)

$$\begin{aligned} y &= c_1 e^{x} + c_2 e^{-x} \\ 0 &= c_1 e^{(0)} + c_2 e^{-(0)} \\ &= c_1 + c_2 \end{aligned}$$

$$\begin{aligned} 1 &= c_1 e^{(1)} + c_2 e^{-(1)} \\ &= c_1 e + c_2 e^{-1} \\ &= c_1 e - c_1 e^{-1} \\ &= c_1 (e - e^{-1}) \\ c_1 &= \frac{1}{e - e^{-1}} \\ c_2 &= -\frac{1}{e - e^{-1}} \end{aligned}$$

$$y = \frac{e^{x} - e^{-x}}{e - e^{-1}}$$

(b)

$$\begin{aligned} y &= c_3 \cosh x + c_4 \sinh x \\ 0 &= c_3 \cosh 0 + c_4 \sinh 0 \\ &= c_3 \end{aligned}$$

$$\begin{aligned} y &= c_4 \sinh x \\ 1 &= c_4 \sinh 1 \\ c_4 &= \operatorname{csch} 1 \end{aligned}$$

$$y = (\operatorname{csch} 1) \sinh x$$

(c)

$$(\operatorname{csch} 1) \sinh x = \frac{2}{e - e^{-1}} \frac{e^{x} - e^{-x}}{2} = \frac{e^{x} - e^{-x}}{e - e^{-1}}$$

3.1.5 13

(a)

$$\begin{aligned}
y &= c_1 e^x \cos x + c_2 e^x \sin x \\
1 &= c_1 e^{(0)} \cos(0) + c_2 e^{(0)} \sin(0) \\
&= c_1
\end{aligned}$$

$$\begin{aligned}
y' &= e^x \cos x - e^x \sin x + c_2 e^x \sin x + c_2 e^x \cos x \\
0 &= e^{(\pi)} \cos(\pi) - e^{(\pi)} \sin(\pi) + c_2 e^{(\pi)} \sin(\pi) + c_2 e^{(\pi)} \cos(\pi) \\
&= -e^\pi - c_2 e^\pi \\
c_2 &= -1
\end{aligned}$$

$$y = e^x \cos x - e^x \sin x$$

(b)

$$\begin{aligned}
y &= c_1 e^x \cos x + c_2 e^x \sin x \\
1 &= c_1 e^{(0)} \cos(0) + c_2 e^{(0)} \sin(0) \\
&= c_1
\end{aligned}$$

$$\begin{aligned}
y &= c_1 e^x \cos x + c_2 e^x \sin x \\
-1 &= e^{(\pi)} \cos(\pi) + c_2 e^{(\pi)} \sin(\pi) \\
&= -e^\pi
\end{aligned}$$

No solution

(c)

$$c_1 = 1$$

$$\begin{aligned}
y &= c_1 e^x \cos x + c_2 e^x \sin x \\
1 &= e^{(\pi/2)} \cos\left(\frac{\pi}{2}\right) + c_2 e^{(\pi/2)} \sin\left(\frac{\pi}{2}\right) \\
&= c_2 e^{\pi/2} \\
c_2 &= e^{-\pi/2}
\end{aligned}$$

$$y = e^x \cos x + e^{x-\pi/2} \sin x$$

(d)

$$\begin{aligned} y &= c_1 e^x \cos x + c_2 e^x \sin x \\ 0 &= c_1 e^{(0)} \cos(0) + c_2 e^{(0)} \sin(0) \\ &= c_1 e \\ &= c_1 \end{aligned}$$

$$\begin{aligned} y &= c_1 e^x \cos x + c_2 e^x \sin x \\ 0 &= c_2 e^{(\pi)} \sin(\pi) \end{aligned}$$

$$y = c_2 e^x \sin x$$

3.1.6 15

Dependent

3.1.7 17

Dependent

3.1.8 19

Dependent

3.1.9 21

Independent

3.1.10 23

$$y'' - y' - 12y = 9e^{-3x} + 3e^{-3x} - 12e^{-3x} = 0$$

$$y'' - y' - 12y = 16e^{4x} - 4e^{4x} - 12e^{4x} = 0$$

Both functions are solutions of the differential equation and are linearly independent, so they form a fundamental set of solutions.

$$y = c_1 e^{-3x} + c_2 e^{4x}$$

3.1.11 25

$$\begin{aligned} y'' - 2y' + 5y &= e^x \cos 2x - 2e^x \sin 2x - 2e^x \sin 2x - 4e^x \cos 2x \\ &\quad - 2(e^x \cos 2x - 2e^x \sin 2x) + 5e^x \cos 2x \\ &= 0 \end{aligned}$$

$$\begin{aligned} y'' - 2y' + 5y &= e^x \sin 2x + 2e^x \cos 2x + 2e^x \cos 2x - 4e^x \sin 2x \\ &\quad - 2(e^x \sin 2x + 2e^x \cos 2x) + 5e^x \sin 2x \\ &= 0 \end{aligned}$$

$$\begin{aligned} W(e^x \cos 2x, e^x \sin 2x) &= \begin{vmatrix} e^x \cos 2x & e^x \sin 2x \\ e^x \cos 2x - 2e^x \sin 2x & e^x \sin 2x + 2e^x \cos 2x \end{vmatrix} \\ &= e^x \cos 2x(e^x \sin 2x + 2e^x \cos 2x) \\ &\quad - e^x \sin 2x(e^x \cos 2x - 2e^x \sin 2x) \\ &= e^{2x}(\sin 2x \cos 2x + 2\cos^2 2x - \sin 2x \cos 2x + 2\sin^2 2x) \\ &= 2e^{2x} \end{aligned}$$

Both functions are solutions to the differential equation and the Wronskian does not equal 0 for all x in the interval.

$$y = c_1 e^x \cos 2x + c_2 e^x \sin 2x$$

3.1.12 27

$$\begin{aligned} x^2 y'' - 6xy' + 12y &= x^2(6x) - 6x(3x^2) + 12(x^3) \\ &= 6x^3 - 18x^3 + 12x^3 \\ &= 0 \end{aligned}$$

$$\begin{aligned} x^2 y'' - 6xy' + 12y &= x^2(12x^2) - 6x(4x^3) + 12(x^4) \\ &= 12x^4 - 24x^4 + 12x^4 \\ &= 0 \end{aligned}$$

$$\begin{aligned} W(x^3, x^4) &= \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix} \\ &= (x^3)(4x^3) - (x^4)(3x^2) \\ &= 4x^6 - 3x^6 \\ &= x^6 \end{aligned}$$

Both functions are solutions to the differential equation and, because 0 isn't included in the interval, the Wronskian does not equal 0 for all x in the interval.

$$y = c_1 x^3 + c_2 x^4$$

3.1.13 35

(a)

$$y'' - 6y' + 5y = 12e^{2x} - 6(6e^{2x}) + 5(3e^{2x}) = -9e^{2x}$$

$$y'' - 6y' + 5y = 2 - 6(2x + 3) + 5(x^2 + 3x) = 5x^2 + 3x - 16$$

(b)

$$y = 3e^{2x} + x^2 + 3x$$

$$y = -\frac{1}{9}(3e^{2x}) - 2(x^2 + 3x) = -\frac{1}{3}e^{2x} - 2(x^2 + 3x)$$

3.2 Reduction of Order

3.2.1 1

$$\begin{aligned} y_2(x) &= u(x)e^{2x} \\ y_2'(x) &= u'(x)e^{2x} + 2u(x)e^{2x} \\ &= (u'(x) + 2u(x))e^{2x} \\ y_2''(x) &= u''(x)e^{2x} + 2u'(x)e^{2x} + 2u'(x)e^{2x} + 4u(x)e^{2x} \\ &= (u''(x) + 4u'(x) + 4u(x))e^{2x} \end{aligned}$$

$$\begin{aligned} y'' - 4y' + 4y &= 0 \\ (u''(x) + 4u'(x) + 4u(x))e^{2x} - 4(u'(x) + 2u(x))e^{2x} + 4u(x)e^{2x} &= 0 \\ u''(x) &= 0 \\ u'(x) &= c_1 \\ u(x) &= c_1 x + c_2 \end{aligned}$$

$$y_2(x) = xe^{2x}$$

3.2.2 3

$$
\begin{aligned}
y_2(x) &= u(x)y_1(x) \\
&= u(x)\cos 4x \\
y_2'(x) &= u'(x)\cos 4x - 4u(x)\sin 4x \\
y_2''(x) &= u''(x)\cos 4x - 4u'(x)\sin 4x - 4u'(x)\sin 4x - 16u(x)\cos 4x \\
&= u''(x)\cos 4x - 8u'(x)\sin 4x - 16u(x)\cos 4x
\end{aligned}
$$

$$
\begin{aligned}
y'' + 16y &= 0 \\
u''(x)\cos 4x - 8u'(x)\sin 4x - 16u(x)\cos 4x + 16u(x)\cos 4x &= 0 \\
u''(x)\cos 4x - 8u'(x)\sin 4x &= 0 \\
u''(x) - 8(\tan 4x)u'(x) &= 0
\end{aligned}
$$

$$
\begin{aligned}
e^{\int -8\tan 4x\,dx}u''(x) - 8(\tan 4x)e^{\int -8\tan 4x\,dx}u'(x) &= 0 \\
\frac{d}{dx}(u'(x)\cos^2 4x) &= 0 \\
u'(x)\cos^2 4x &= c_1 \\
u'(x) &= c_1\sec^2 4x \\
u(x) &= c_1\tan 4x + c_2
\end{aligned}
$$

$$y_2 = \sin 4x$$

3.2.3 5

$$
\begin{aligned}
y_2(x) &= u(x)y_1(x) \\
&= u(x)\cosh x \\
y_2'(x) &= u'(x)\cosh x + u(x)\sinh x \\
y_2''(x) &= u''(x)\cosh x + u'(x)\sinh x + u'(x)\sinh x + u(x)\cosh(x) \\
&= u''(x)\cosh x + 2u'(x)\sinh x + u(x)\cosh x
\end{aligned}
$$

$$
\begin{aligned}
y'' - y &= 0 \\
u''(x)\cosh x + 2u'(x)\sinh x + u(x)\cosh x - u(x)\cosh x &= 0 \\
u''(x)\cosh x + 2u'(x)\sinh x &= 0 \\
u''(x) + 2(\tanh x)u'(x) &= 0
\end{aligned}
$$

$$
\begin{aligned}
e^{\int 2\tanh x\,dx}u''(x) + 2(\tanh x)e^{\int 2\tanh x\,dx}u'(x) &= 0 \\
\frac{d}{dx}(u'(x)\cosh^2 x) &= 0 \\
u'(x)\cosh^2 x &= c_1 \\
u'(x) &= c_1 \operatorname{sech}^2 x \\
u(x) &= c_1 \tanh x + c_2
\end{aligned}
$$

$$
y_2 = \sinh x
$$

3.2.4 7

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \\
&= e^{2x/3} \int \frac{e^{4x/3}}{e^{4x/3}}\,dx \\
&= xe^{2x/3}
\end{aligned}
$$

3.2.5 9

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \\
&= x^4 \int \frac{x^7}{x^8}\,dx \\
&= x^4 \ln|x|
\end{aligned}
$$

3.2.6 11

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \\
&= \ln x \int \frac{1}{x \ln^2 x}\,dx \\
&= -1
\end{aligned}
$$

3.2.7 13

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \\
&= x \sin(\ln x) \int \frac{1}{x \sin^2(\ln x)}\,dx \\
&= x \cos(\ln x)
\end{aligned}
$$

3.2.8 15

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \\
&= (x+1) \int \frac{1-2x-x^2}{(x+1)^2}\,dx \\
&= (x+1)\left(-x - \frac{2}{x+1}\right) \\
&= x^2 + x + 2
\end{aligned}
$$

3.2.9 17

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \\
&= e^{-2x} \int \frac{e^{-\int 0\,dx}}{(e^{-2x})^2}\,dx \\
&= e^{-2x} \int e^{4x}\,dx \\
&= e^{2x}
\end{aligned}
$$

$$
y_p(x) = -\frac{1}{2}
$$

3.2.10 19

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \\
&= e^{x} \int \frac{e^{3x}}{e^{2x}}\,dx \\
&= e^{2x}
\end{aligned}
$$

$$
y_p(x) = \frac{5}{2}e^{3x}
$$

3.2.11 21

$$\begin{aligned}
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \\
&= x \int \frac{e^{-\int (1-x^{-1})\,dx}}{x^2}\,dx \\
&= x \int \frac{e^{\ln|x|-x}}{x^2}\,dx \\
&= x \int \frac{1}{xe^x}\,dx \\
&= x \int_{x_0}^{x} \frac{1}{te^t}\,dt
\end{aligned}$$

3.3 Homogeneous Linear Equations with Constant Coefficients

3.3.1 1

$$\begin{aligned}
4y'' + y' &= 0 \\
4m^2 + m &= 0 \\
m(4m+1) &= 0 \\
y &= c_1 + c_2 e^{-x/4}
\end{aligned}$$

3.3.2 3

$$\begin{aligned}
y'' - y' - 6y &= 0 \\
m^2 - m - 6 &= 0 \\
(m-3)(m+2) &= 0 \\
y &= c_1 e^{3x} + c_2 e^{-2x}
\end{aligned}$$

3.3.3 5

$$\begin{aligned}
y'' + 8y' + 16y &= 0 \\
m^2 + 8m + 16m &= 0 \\
(m+4)^2 &= 0 \\
y &= c_1 e^{-4x} + c_2 x e^{-4x}
\end{aligned}$$

3.3.4 7

$$\begin{aligned} 12y'' - 5y' - 2y &= 0 \\ 12m^2 - 5m - 2 &= 0 \\ \left(m - \frac{2}{3}\right)\left(m + \frac{1}{4}\right) &= 0 \\ y &= c_1 e^{2x/3} + c_2 e^{-x/4} \end{aligned}$$

3.3.5 9

$$\begin{aligned} y'' + 9y &= 0 \\ m^2 + 9 &= 0 \\ (m + 3i)(m - 3i) &= 0 \\ y &= c_1 \cos 3x + c_2 \sin 3x \end{aligned}$$

3.3.6 11

$$\begin{aligned} y'' - 4y' + 5y &= 0 \\ m^2 - 4m + 5 &= 0 \\ (m - (2 + i))(m - (2 - i)) &= 0 \\ y &= e^{2x}(c_1 \cos x + c_2 \sin x) \end{aligned}$$

3.3.7 13

$$\begin{aligned} 3y'' + 2y' + y &= 0 \\ 3m^2 + 2m + 1 &= 0 \\ \left(m - (-\frac{1}{3} + \frac{\sqrt{2}}{3}i)\right)\left(m - (-\frac{1}{3} - \frac{\sqrt{2}}{3}i)\right) &= 0 \\ y &= e^{-x/3}(c_1 \cos \frac{\sqrt{2}}{3}x + c_2 \sin \frac{\sqrt{2}}{3}x) \end{aligned}$$

3.3.8 15

$$\begin{aligned} y''' - 4y'' - 5y' &= 0 \\ m^3 - 4m^2 - 5m &= 0 \\ m(m^2 - 4m - 5) &= 0 \\ m(m - 5)(m + 1) &= 0 \\ y &= c_1 + c_2 e^{5x} + c_3 e^{-x} \end{aligned}$$

3.3.9 17

$$\begin{aligned}
y''' - 5y'' + 3y' + 9y &= 0\\
m^3 - 5m^2 + 3m + 9 &= 0\\
(m-3)^2(m+1) &= 0\\
y &= c_1 e^{3x} + c_2 x e^{3x} + c_3 e^{-x}
\end{aligned}$$

3.3.10 19

$$\begin{aligned}
u''' + u'' - 2u &= 0\\
m^3 + m^2 - 2 &= 0\\
(x-1)(x-(-1+i))(x-(-1-i)) &= 0\\
y &= c_1 e^{x} + e^{-x}(c_2 \cos x + \sin x)
\end{aligned}$$

3.3.11 21

$$\begin{aligned}
y''' + 3y'' + 3y' + y &= 0\\
m^3 + 3m^2 + 3m + 1 &= 0\\
(m+1)^3 &= 0\\
y &= c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}
\end{aligned}$$

3.3.12 23

$$\begin{aligned}
y^{(4)} + y''' + y'' &= 0\\
m^4 + m^3 + m^2 &= 0\\
m^2(m^2 + m + 1) &= 0\\
m^2\left(m - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(m - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right) &= 0\\
y = c_1 + c_2 x + e^{-x/2}\left(c_3 \cos \frac{\sqrt{3}}{2}x + c_4 \sin \frac{\sqrt{3}}{2}x\right)
\end{aligned}$$

3.3.13 25

$$\begin{aligned}
16y^{(4)} + 24y'' + 9y &= 0\\
16m^4 + 24m^2 + 9 &= 0\\
(4m^2 + 3)^2 &= 0\\
y &= c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x + c_3 x \cos \frac{\sqrt{3}}{2}x + c_4 x \sin \frac{\sqrt{3}}{2}x
\end{aligned}$$

3.3.14 27

$$\begin{aligned}
u^{(5)} + 5u^{(4)} - 2u^{(3)} - 10u'' + u' + 5u &= 0 \\
m^5 + 5m^4 - 2m^3 - 10m^2 + m + 5 &= 0 \\
(m+5)(m-1)^2(m+1)^2 &= 0 \\
u = c_1e^{-5r} + c_2e^r + c_3re^r + c_4e^{-r} + c_5re^{-r}
\end{aligned}$$

3.3.15 29

$$\begin{aligned}
y'' + 16y &= 0 \\
m^2 + 16 &= 0 \\
(m+4i)(m-4i) &= 0 \\
y &= c_1 \cos 4x + c_2 \sin 4x
\end{aligned}$$

$$y(0) = 2 = c_1 \cos 4(0) + c_2 \sin 4(0) \Rightarrow c_1 = 2$$

$$y' = -8 \sin 4x + 4c_2 \cos 4x$$

$$y'(0) = -2 = -8 \sin 4(0) + 4c_2 \cos 4(0) \Rightarrow c_2 = -\frac{1}{2}$$

$$y = 2 \cos 4x - \frac{1}{2} \sin 4x$$

3.3.16 31

$$\begin{aligned}
y'' - 4y' - 5y &= 0 \\
m^2 - 4m - 5 &= 0 \\
(m-5)(m+1) &= 0 \\
y &= c_1e^{-x} + c_2e^{5x}
\end{aligned}$$

$$y(1) = 0 = c_1e^{-1} + c_2e^5$$

$$y' = -c_1e^{-x} + 5c_2e^{5x}$$

$$y'(1) = 2 = -c_1e^{-1} + 5c_2e^5$$

$$2 = 6c_2e^5 \Rightarrow c_2 = \frac{1}{3e^5}$$

$$0 = c_1 e^{-1} + \frac{1}{3} \Rightarrow c_1 = -\frac{1}{3}e$$

$$y = -\frac{1}{3}e^{1-x} + \frac{1}{3}e^{5x-5}$$

3.3.17 33

$$\begin{aligned} y'' + y' + 2y &= 0 \\ m^2 + m + 2 &= 0 \\ \left(m - \left(-\frac{1}{2} + \frac{\sqrt{7}}{2}i\right)\right)\left(m - \left(-\frac{1}{2} - \frac{\sqrt{7}}{2}i\right)\right) &= 0 \\ y = e^{-x/2}\left(c_1 \cos\frac{\sqrt{7}}{2}x + c_2 \sin\frac{\sqrt{7}}{2}x\right) \end{aligned}$$

$$y(0) = 0 = e^{-(0)/2}\left(c_1 \cos\frac{\sqrt{7}}{2}(0) + c_2 \sin\frac{\sqrt{7}}{2}(0)\right) \Rightarrow c_1 = 0$$

$$y = c_2 e^{-x/2} \sin\frac{\sqrt{7}}{2}x$$

$$y' = -\frac{1}{2}c_2 e^{-x/2} \sin\frac{\sqrt{7}}{2}x + \frac{\sqrt{7}}{2}c_2 e^{-x/2} \cos\frac{\sqrt{7}}{2}x$$

$$y'(0) = 0 = \frac{\sqrt{7}}{2}c_2 \Rightarrow c_2 = 0$$

$$y = 0$$

3.3.18 37

$$\begin{aligned} y'' - 10y' + 25y &= 0 \\ m^2 - 10m + 25 &= 0 \\ (m-5)^2 &= 0 \\ y &= c_1 e^{5x} + c_2 x e^{5x} \end{aligned}$$

$$y(0) = 1 = c_1 e^{5(0)} + c_2(0)e^{5(0)} \Rightarrow c_1 = 1$$

$$y(1) = 0 = e^5 + c_2 e^5 \Rightarrow c_2 = -1$$

$$y = e^{5x} - xe^{5x}$$

3.3.19 39

$$
\begin{aligned}
y'' + y &= 0 \\
m^2 + 1 &= 0 \\
(m+i)(m-i) &= 0 \\
y = c_1 \cos x + c_2 \sin x &
\end{aligned}
$$

$$y' = -c_1 \sin x + c_2 \cos x$$

$$y'(0) = 0 = -c_1 \sin(0) + c_2 \cos(0) \Rightarrow c_2 = 0$$

$$y'(\pi/2) = 0 = -c_1 \sin(\pi/2) \Rightarrow c_1 = 0$$

$$y = 0$$

3.3.20 41

$$
\begin{aligned}
y'' - 3y = 0 & \\
y &= c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x}
\end{aligned}
$$

$$y(0) = 1 = c_1 + c_2$$

$$y' = \sqrt{3}c_1 e^{\sqrt{3}x} - \sqrt{3}c_2 e^{-\sqrt{3}x}$$

$$y'(0) = 5 = \sqrt{3}c_1 - \sqrt{3}c_2 \Rightarrow c_1 = \frac{5}{\sqrt{3}} + c_2$$

$$1 = \frac{5}{\sqrt{3}} + 2c_2 \Rightarrow c_2 = \frac{1}{2} - \frac{5}{2\sqrt{3}} \Rightarrow c_1 = \frac{1}{2} + \frac{5}{2\sqrt{3}}$$

$$y = \left(\frac{1}{2} + \frac{5}{2\sqrt{3}}\right) e^{\sqrt{3}x} + \left(\frac{1}{2} - \frac{5}{2\sqrt{3}}\right) e^{-\sqrt{3}x}$$

3.3.21 49

$$
\begin{aligned}
(m-1)(m-6) &= 0 \\
m^2 - 7m + 6 &= 0 \\
y'' - 7y' + 6y &= 0
\end{aligned}
$$

3.3.22 51

$$\begin{aligned} m(m-3) &= 0 \\ m^2 - 3m &= 0 \\ y'' - 3y' &= 0 \end{aligned}$$

3.3.23 53

$$\begin{aligned} (m-8i)(m+8i) &= 0 \\ m^2 + 64 &= 0 \\ y'' + 64y &= 0 \end{aligned}$$

3.3.24 55

$$\begin{aligned} (m-(1+i))(m-(1-i)) &= 0 \\ (m-1-i)(m-1+i) &= 0 \\ m^2 - m + im - m + 1 - i - im + i + 1 &= 0 \\ m^2 - 2m + 2 &= 0 \\ y'' - 2y' + 2y &= 0 \end{aligned}$$

3.3.25 57

$$\begin{aligned} m^2(m-7) &= 0 \\ m^3 - 7m^2 &= 0 \\ y''' - 7y'' &= 0 \end{aligned}$$

3.4 Undetermined Coefficients

3.4.1 1

$$\begin{aligned} y'' + 3y' + 2y &= 0 \\ m^2 + 3m + 2 &= 0 \\ (m+1)(m+2) &= 0 \end{aligned}$$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

$$y_p = A,\ y_p' = 0,\ y_p'' = 0$$

$$2A = 6 \Rightarrow A = 3$$

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + 3$$

3.4.2 3

$$\begin{aligned}y'' - 10y' + 25y &= 0\\ m^2 - 10m + 25 &= 0\\ (m-5)^2 &= 0\end{aligned}$$

$$y_c = c_1e^{5x} + c_2xe^{5x}$$

$$y_p = Ax + B,\ y_p' = A,\ y_p'' = 0$$

$$\begin{aligned}-10A + 25(Ax + B) &= 30x + 3\\ 25Ax - 10A + 25B &= 30x + 3\end{aligned}$$

$$25A = 30 \Rightarrow A = \frac{6}{5}$$

$$-10A + 25B = -10\left(\frac{6}{5}\right) + 25B = 3 \Rightarrow B = \frac{3}{5}$$

$$y = y_c + y_p = c_1e^{5x} + c_2xe^{5x} + \frac{6}{5}x + \frac{3}{5}$$

3.4.3 5

$$\begin{aligned}\frac{1}{4}y'' + y' + y &= 0\\ \frac{1}{4}m^2 + m + 1 &= 0\\ (m+2)^2 &= 0\end{aligned}$$

$$y_c = c_1e^{-2x} + c_2xe^{-2x}$$

$$y_p = Ax^2 + Bx + C,\ y_p' = 2Ax + B,\ y_p'' = 2A$$

$$\begin{aligned}\frac{1}{4}(2A) + 2Ax + B + Ax^2 + Bx + C &= x^2 - 2x\\ Ax^2 + (2A + B)x + (\frac{1}{2}A + B + C) &= x^2 - 2x\end{aligned}$$

$$A = 1,\ 2A + B = -2,\ \frac{1}{2}A + B + C = 0$$

$$2(1) + B = -2 \Rightarrow B = -4$$

$$\frac{1}{2}(1) + (-4) + C = 0 \Rightarrow \frac{7}{2}$$

$$y = y_c + y_p = c_1 e^{-2x} + c_2 x e^{-2x} + x^2 - 4x + \frac{7}{2}$$

3.4.4 7

$$\begin{aligned} y'' + 3y &= 0 \\ m^2 + 3 &= 0 \\ (m + i\sqrt{3})(m - i\sqrt{3}) &= 0 \end{aligned}$$

$$y_c = c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x$$

$$\begin{aligned} y_p &= (Ax^2 + Bx + C)e^{3x} \\ y_p' &= (2Ax + B)e^{3x} + 3(Ax^2 + Bx + C)e^{3x} \\ y_p'' &= 2Ae^{3x} + 3(2Ax + B)e^{3x} + 3(2Ax + B)e^{3x} + 9(Ax^2 + Bx + C)e^{3x} \\ &= 2Ae^{3x} + 6(2Ax + B)e^{3x} + 9(Ax^2 + Bx + C)e^{3x} \end{aligned}$$

$$\begin{aligned} 2A + 6(2Ax + B) + 9(Ax^2 + Bx + C) + 3(Ax^2 + Bx + C) &= -48x^2 \\ 12Ax^2 + 12(A + B)x + (2A + 6B + 12C) &= -48x^2 \end{aligned}$$

$$\begin{aligned} 12A = -48 &\Rightarrow A = -4 \\ 12(A + B) = 12(-4 + B) = 0 &\Rightarrow B = 4 \\ 2A + 6B + 12C = 2(-4) + 6(4) + 12C = 0 &\Rightarrow C = -\frac{4}{3} \end{aligned}$$

$$y = y_c + y_p = c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x + \left(-4x^2 + 4x - \frac{4}{3}\right) e^{3x}$$

3.4.5 9

$$\begin{aligned} y'' - y' &= 0 \\ m^2 - m &= 0 \\ m(m-1) &= 0 \end{aligned}$$

$$y_c = c_1 + c_2 e^x$$

$$y_p = Ax$$

$$-A = -3 \Rightarrow A = 3$$

$$y = y_c + y_p = c_1 + c_2 e^x + 3x$$

3.4.6 11

$$\begin{aligned} y'' - y' + \frac{1}{4}y &= 0 \\ m^2 - m + \frac{1}{4} &= 0 \\ \left(m - \frac{1}{2}\right)^2 &= 0 \end{aligned}$$

$$y_c = c_1 e^{x/2} + c_2 x e^{x/2}$$

$$\begin{aligned} y_p &= y_{p1} + y_{p2} \\ &= A + Bx^2 e^{x/2} \\ y_p' &= 2Bxe^{x/2} + \frac{1}{2}Bx^2 e^{x/2} \\ y_p'' &= 2Be^{x/2} + Bxe^{x/2} + Bxe^{x/2} + \frac{1}{4}Bx^2 e^{x/2} \\ &= 2Be^{x/2} + 2Bxe^{x/2} + \frac{1}{4}Bx^2 e^{x/2} \end{aligned}$$

$$\begin{aligned} 3 + e^{x/2} &= 2Be^{x/2} + 2Bxe^{x/2} + \frac{1}{4}Bx^2 e^{x/2} - 2Bxe^{x/2} - \frac{1}{2}Bx^2 e^{x/2} \\ &\quad + \frac{1}{4}(A + Bx^2 e^{x/2}) \\ &= \frac{1}{4}A + 2Be^{x/2} \end{aligned}$$

$$3 = \frac{1}{4}A \Rightarrow A = 12$$

$$1 = 2B \Rightarrow B = \frac{1}{2}$$

$$y = y_c + y_p = c_1 e^{x/2} + c_2 x e^{x/2} + 12 + \frac{1}{2}x^2 e^{x/2}$$

3.4.7 13

$$\begin{aligned} y'' + 4y &= 0 \\ m^2 + 4 &= 0 \\ (m+2i)(m-2i) &= 0 \end{aligned}$$

$$y_c = c_1 \cos 2x + c_2 \sin 2x$$

$$\begin{aligned} y_p &= Ax \cos 2x + Bx \sin 2x \\ y_p' &= A \cos 2x - 2Ax \sin 2x + B \sin 2x + 2Bx \cos 2x \\ &= A \cos 2x + 2Bx \cos 2x + B \sin 2x - 2Ax \sin 2x \\ y_p'' &= -2A \sin 2x + 2B \cos 2x - 4Bx \sin 2x + 2B \cos 2x - 2A \sin 2x - 4Ax \cos 2x \\ &= 4B \cos 2x - 4Ax \cos 2x - 4A \sin 2x - 4Bx \sin 2x \end{aligned}$$

$$\begin{aligned} 3 \sin 2x = &\ 4B \cos 2x - 4Ax \cos 2x - 4A \sin 2x - 4Bx \sin 2x \\ &+ 4(Ax \cos 2x + Bx \sin 2x) \\ = &\ 4B \cos 2x - 4A \sin 2x \end{aligned}$$

$$3 = -4A \Rightarrow A = -\frac{3}{4}$$

$$0 = 4B \Rightarrow B = 0$$

$$y = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x - \frac{3}{4}x \cos 2x$$

3.4.8 15

$$
\begin{aligned}
y'' + y &= 0 \\
m^2 + 1 &= 0 \\
(m+i)(m-i) &= 0
\end{aligned}
$$

$$y_c = c_1 \cos x + c_2 \sin x$$

$$
\begin{aligned}
y_p &= (Ax^2 + Bx)\cos x + (Cx^2 + Ex)\sin x \\
y_p' &= (2Ax + B)\cos x - (Ax^2 + Bx)\sin x + (2Cx + E)\sin x + (Cx^2 + Ex)\cos x \\
&= B\cos x + (2A + E)x\cos x + Cx^2\cos x + E\sin x + (2C - B)x\sin x \\
&\quad - Ax^2 \sin x \\
y_p'' &= -B\sin x + (2A + E)\cos x - (2A + E)x\sin x + 2Cx\cos x - Cx^2\sin x \\
&\quad + E\cos x + (2C - B)\sin x + (2C - B)x\cos x - 2Ax\sin x - Ax^2\cos x \\
&= 2(A + E)\cos x + (4C - B)x\cos x - Ax^2\cos x + 2(C - B)\sin x \\
&\quad - (4A + E)x\sin x - Cx^2\sin x
\end{aligned}
$$

$$
\begin{aligned}
2x\sin x &= 2(A + E)\cos x + (4C - B)x\cos x - Ax^2\cos x + 2(C - B)\sin x \\
&\quad - (4A + E)x\sin x - Cx^2\sin x + (Ax^2 + Bx)\cos x \\
&\quad + (Cx^2 + Ex)\sin x \\
&= 2(A + E)\cos x + 4Cx\cos x + 2(C - B)\sin x - 4Ax\sin x
\end{aligned}
$$

$$
\begin{aligned}
2 = -4A &\Rightarrow A = -\frac{1}{2} \\
0 = 2(A + E) &\Rightarrow E = \frac{1}{2} \\
0 = 4C &\Rightarrow C = 0 \\
0 = 2(C - B) &\Rightarrow B = 0
\end{aligned}
$$

$$y = y_c + y_p = c_1\cos x + c_2\sin x - \frac{1}{2}x^2\cos x + \frac{1}{2}x\sin x$$

3.4.9 17

$$
\begin{aligned}
y'' - 2y' + 5y &= 0 \\
m^2 - 2m + 5 &= 0 \\
(m - (1 + 2i))(m - (1 - 2i)) &= 0
\end{aligned}
$$

$$y_c = c_1 e^x \cos 2x + c_2 e^x \sin 2x$$

$$\begin{aligned}
y_p &= Axe^x \cos 2x + Bxe^x \sin 2x \\
y_p' &= Ae^x \cos 2x + Axe^x \cos 2x - 2Axe^x \sin 2x + Be^x \sin 2x + Bxe^x \sin 2x \\
&\quad + 2Bxe^x \cos 2x \\
&= Ae^x \cos 2x + (A+2B)xe^x \cos 2x + Be^x \sin 2x + (B-2A)xe^x \sin 2x \\
y_p'' &= Ae^x \cos 2x - 2Ae^x \sin 2x + (A+2B)e^x \cos 2x + (A+2B)xe^x \cos 2x \\
&\quad - 2(A+2B)xe^x \sin 2x + Be^x \sin 2x + 2Be^x \cos 2x + (B-2A)e^x \sin 2x \\
&\quad + (B-2A)xe^x \sin 2x + 2(B-2A)xe^x \cos 2x \\
&= (2A+4B)e^x \cos 2x + (4B-3A)xe^x \cos 2x + (2B-4A)e^x \sin 2x \\
&\quad - (4A+3B)xe^x \sin 2x
\end{aligned}$$

$$\begin{aligned}
e^x \cos 2x &= (2A+4B)e^x \cos 2x + (4B-3A)xe^x \cos 2x + (2B-4A)e^x \sin 2x \\
&\quad - (4A+3B)xe^x \sin 2x - 2(Ae^x \cos 2x + (A+2B)xe^x \cos 2x \\
&\quad + Be^x \sin 2x + (B-2A)xe^x \sin 2x) + 5(Axe^x \cos 2x \\
&\quad + Bxe^x \sin 2x) \\
&= 4Be^x \cos 2x - 4Ae^x \sin 2x
\end{aligned}$$

$$\begin{aligned}
1 = 4B &\Rightarrow B = \frac{1}{4} \\
0 = -4A &\Rightarrow A = 0
\end{aligned}$$

$$y = y_c + y_p = c_1 e^x \cos 2x + c_2 e^x \sin 2x + \frac{1}{4} xe^x \sin 2x$$

3.4.10 21

$$\begin{aligned}
y''' - 6y'' &= 0 \\
m^3 - 6m^2 &= 0 \\
m^2(m-6) &= 0
\end{aligned}$$

$$y_c = c_1 + c_2 x + c_3 e^{6x}$$

$$
\begin{aligned}
y_p &= Ax^2 + B\cos x + C\sin x \\
y_p' &= 2Ax - B\sin x + C\cos x \\
y_p'' &= 2A - B\cos x - C\sin x \\
y_p''' &= B\sin x - C\cos x
\end{aligned}
$$

$$
\begin{aligned}
3 - \cos x &= B\sin x - C\cos x - 6(2A - B\cos x - C\sin x) \\
&= -12A + (B + 6C)\sin x + (6B - C)\cos x
\end{aligned}
$$

$$
\begin{aligned}
3 = -12A \Rightarrow A &= -\frac{1}{4} \\
B + 6C + 6(6B - C) = 0 + 6(-1) \Rightarrow 37B = -6 \Rightarrow B &= -\frac{6}{37} \\
B + 6C = 0 \Rightarrow C &= \frac{1}{37}
\end{aligned}
$$

$$
y = y_c + y_p = c_1 + c_2 x + c_3 e^{6x} - \frac{1}{4}x^2 - \frac{6}{37}\cos x + \frac{1}{37}\sin x
$$

3.4.11 27

$$
\begin{aligned}
y'' + 4y &= 0 \\
m^2 + 4 &= 0 \\
(m + 2i)(m - 2i) &= 0
\end{aligned}
$$

$$
y_c = c_1 \cos 2x + c_2 \sin 2x
$$

$$
\begin{aligned}
y_p &= A \\
y_p' &= 0 \\
y_p'' &= 0
\end{aligned}
$$

$$
\begin{aligned}
0 + 4A &= -2 \\
A &= -\frac{1}{2}
\end{aligned}
$$

$$
y = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{2}
$$

$$
\begin{aligned}
y\left(\frac{\pi}{8}\right) = \frac{1}{2} &= c_1 \cos\frac{\pi}{4} + c_2 \sin\frac{\pi}{4} - \frac{1}{2} \\
&= \frac{\sqrt{2}c_1}{2} + \frac{\sqrt{2}c_2}{2} - \frac{1}{2} \\
2 &= \sqrt{2}c_1 + \sqrt{2}c_2
\end{aligned}
$$

$$
\begin{aligned}
y'\left(\frac{\pi}{8}\right) = 2 &= -2c_1 \sin\frac{\pi}{4} + 2c_2 \cos\frac{\pi}{4} \\
&= -\sqrt{2}c_1 + \sqrt{2}c_2
\end{aligned}
$$

$$
\begin{aligned}
4 &= 2\sqrt{2}c_2 \\
c_2 &= \frac{4}{2\sqrt{2}} \\
&= \sqrt{2}
\end{aligned}
$$

$$
\begin{aligned}
2 &= \sqrt{2}c_1 + 2 \\
c_1 &= 0
\end{aligned}
$$

$$
y = \sqrt{2}\sin 2x - \frac{1}{2}
$$

3.4.12 29

$$
\begin{aligned}
5y'' + y' &= 0 \\
5m^2 + m &= 0 \\
m(5m+1) &= 0
\end{aligned}
$$

$$
y_c = c_1 + c_2 e^{-x/5}
$$

$$
\begin{aligned}
y_p &= Ax^2 + Bx \\
y_p' &= 2Ax + B \\
y_p'' &= 2A
\end{aligned}
$$

$$-6x = 10A + 2Ax + B$$

$$\begin{aligned} -6 &= 2A \Rightarrow A = -3 \\ 0 = 10A + B = 10(-3) + B &\Rightarrow B = 30 \end{aligned}$$

$$y = y_c + y_p = c_1 + c_2 e^{-x/5} - 3x^2 + 30x$$

$$\begin{aligned} y(0) = 0 &= c_1 + c_2 e^{-(0)/5} - 3(0)^2 + 30(0) \\ &= c_1 + c_2 \end{aligned}$$

$$\begin{aligned} y'(0) = -10 &= -\frac{1}{5} c_2 e^{-(0)/5} - 6(0) + 30 \\ &= -\frac{1}{5} c_2 + 30 \\ c_2 &= 200 \end{aligned}$$

$$c_1 = -200$$

$$y = -200 + 200e^{-x/5} - 3x^2 + 30x$$

3.4.13 31

$$\begin{aligned} y'' + 4y' + 5y &= 0 \\ m^2 + 4m + 5 &= 0 \\ (m - (-2 + i))(m - (-2 - i)) &= 0 \end{aligned}$$

$$y_c = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x$$

$$\begin{aligned} y_p &= Ae^{-4x} \\ y_p' &= -4Ae^{-4x} \\ y_p'' &= 16Ae^{-4x} \end{aligned}$$

$$\begin{aligned}
35e^{-4x} &= 16Ae^{-4x} + 4(-4Ae^{-4x}) + 5(Ae^{-4x}) \\
&= 5Ae^{-4x} \\
A &= 7
\end{aligned}$$

$$y = y_c + y_p = c_1e^{-2x}\cos x + c_2e^{-2x}\sin x + 7e^{-4x}$$

$$\begin{aligned}
y(0) = -3 &= c_1e^{-2(0)}\cos(0) + c_2e^{-2(0)}\sin(0) + 7e^{-4(0)} \\
&= c_1 + 7 \\
c_1 &= -10
\end{aligned}$$

$$\begin{aligned}
y'(0) = 1 &= -2c_1e^{-2(0)}\cos(0) - c_1e^{-2(0)}\sin(0) - 2c_2e^{-2(0)}\sin(0) \\
&\quad - 28e^{-4(0)} + c_2e^{-2(0)}\cos(0) \\
&= -2(-10) - 28 + c_2 \\
c_2 &= 9
\end{aligned}$$

$$y = -10e^{-2x}\cos x + 9e^{-2x}\sin x + 7e^{-4x}$$

3.4.14 37

$$\begin{aligned}
y'' + y &= 0 \\
m^2 + 1 &= 0 \\
(m+i)(m-i) &= 0
\end{aligned}$$

$$y_c = c_1\cos x + c_2\sin x$$

$$\begin{aligned}
y_p &= Ax^2 + Bx + C \\
y_p' &= 2Ax + B \\
y_p'' &= 2A
\end{aligned}$$

$$x^2 + 1 = 2A + Ax^2 + Bx + C$$

$$\begin{aligned}
A &= 1 \\
B &= 0 \\
1 = 2A + C = 2(1) + C \Rightarrow C &= -1
\end{aligned}$$

$$y = y_c + y_p = c_1 \cos x + c_2 \sin x + x^2 - 1$$

$$\begin{aligned} y(0) = 5 &= c_1 \cos(0) + c_2 \sin(0) + (0)^2 - 1 \\ &= c_1 - 1 \\ c_1 &= 6 \end{aligned}$$

$$\begin{aligned} y(1) = 0 &= 6\cos(1) + c_2 \sin(1) + (1)^2 - 1 \\ c_2 &= -6 \cot 1 \end{aligned}$$

$$y = 6\cos x - 6(\cot 1)\sin x + x^2 - 1$$

3.5 Variation of Parameters

3.5.1 1

$$\begin{aligned} y'' + y &= 0 \\ m^2 + 1 &= 0 \\ (m+i)(m-i) &= 0 \end{aligned}$$

$$\begin{aligned} W &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \\ &= \cos^2 x + \sin^2 x \\ &= 1 \\ u_1' &= \frac{W_1}{W} \\ &= \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} \\ &= \tan x \\ u_1 &= \ln|\cos x| \\ u_2' &= \frac{W_2}{W} \\ &= \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} \\ &= 1 \\ u_2 &= x \end{aligned}$$

$$y = y_c + y_p = c_1 \cos x + c_2 \sin x + \ln|\cos x|\cos x + x \sin x$$

3.5.2 3

$$
\begin{aligned}
y'' + y &= 0 \\
m^2 + 1 &= 0 \\
(m+i)(m-i) &= 0
\end{aligned}
$$

$$
y_c = c_1 \cos x + c_2 \sin x
$$

$$
\begin{aligned}
W &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \\
&= cos^2 x + \sin^2 x \\
&= 1 \\
u_1' &= \frac{W_1}{W} \\
&= \begin{vmatrix} 0 & \sin x \\ \sin x & \cos x \end{vmatrix} \\
&= -\sin^2 x \\
u_1 &= \frac{1}{4}\sin 2x - \frac{x}{2} \\
u_2' &= \frac{W_2}{W} \\
&= \begin{vmatrix} \cos x & 0 \\ -\sin x & \sin x \end{vmatrix} \\
&= (\cos x)\sin x \\
&= \frac{1}{2}\sin 2x \\
u_2 &= -\frac{1}{4}\cos 2x
\end{aligned}
$$

$$
\begin{aligned}
y &= y_c + y_p \\
&= c_1 \cos x + c_2 \sin x + \left(\frac{1}{4}\sin 2x - \frac{x}{2}\right)\cos x - \frac{1}{4}(\cos 2x)\sin x \\
&= c_1 \cos x + c_2 \sin x - \frac{1}{2} x \cos x
\end{aligned}
$$

3.5.3 7

$$
\begin{aligned}
y'' - y &= 0 \\
m^2 - 1 &= 0 \\
(m+1)(m-1) &= 0
\end{aligned}
$$

$$y_c = c_1 e^x + c_2 e^{-x}$$

$$\begin{aligned}
W &= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} \\
&= -2 \\
u_1' &= \frac{W_1}{W} \\
&= -\frac{1}{2} \begin{vmatrix} 0 & e^{-x} \\ \cosh x & -e^{-x} \end{vmatrix} \\
&= \frac{1}{2} e^{-x} \cosh x \\
&= \frac{1 + e^{-2x}}{4} \\
u_1 &= \frac{1}{4}x - \frac{1}{8}e^{-2x} \\
u_2' &= \frac{W_2}{W} \\
&= -\frac{1}{2} \begin{vmatrix} e^x & 0 \\ e^x & \cosh x \end{vmatrix} \\
&= -\frac{1}{2} e^x \cosh x \\
&= -\frac{e^{2x} + 1}{4} \\
u_2 &= -\frac{1}{8}e^{2x} - \frac{1}{4}x
\end{aligned}$$

$$\begin{aligned}
y &= y_c + y_p \\
&= c_1 e^x + c_2 e^{-x} + \left(\frac{1}{4}x - \frac{1}{8}e^{-2x}\right) e^x - \left(\frac{1}{8}e^{2x} + \frac{1}{4}x\right) e^{-x} \\
&= c_1 e^x + c_2 e^{-x} + \frac{1}{4}xe^x - \frac{1}{4}xe^{-x} \\
&= c_1 e^x + c_2 e^{-x} + \frac{1}{2}x \sinh x
\end{aligned}$$

3.5.4 9

$$\begin{aligned}
y'' - 9y &= 0 \\
m^2 - 9 &= 0 \\
(m+3)(m-3) &= 0
\end{aligned}$$

$$y_c = c_1 e^{-3x} + c_2 e^{3x}$$

$$\begin{aligned} W &= \begin{vmatrix} e^{-3x} & e^{3x} \\ -3e^{-3x} & 3e^{3x} \end{vmatrix} \\ &= 6 \\ u_1' &= \frac{W_1}{W} \\ &= \frac{1}{6} \begin{vmatrix} 0 & e^{3x} \\ \frac{9x}{e^{3x}} & 3e^{3x} \end{vmatrix} \\ &= -\frac{3}{2}x \\ u_1 &= -\frac{3}{4}x^2 \\ u_2' &= \frac{W_2}{W} \\ &= \frac{1}{6} \begin{vmatrix} e^{-3x} & 0 \\ -3e^{-3x} & \frac{9x}{e^{3x}} \end{vmatrix} \\ &= \frac{3}{2}xe^{-6x} \\ u_2 &= -\frac{1}{24}e^{-6x} - \frac{1}{4}xe^{-6x} \end{aligned}$$

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 e^{-3x} + c_2 e^{3x} - \frac{3}{4}x^2 e^{-3x} - \frac{1}{24}e^{-6x}(1+6x)e^{3x} \\ &= c_1 e^{-3x} + c_2 e^{3x} - \frac{1}{4}xe^{-3x} - \frac{3}{4}x^2 e^{-3x} \end{aligned}$$

3.5.5 19

$$\begin{aligned} 4y'' - y &= 0 \\ 4m^2 - 1 &= 0 \\ (2m+1)(2m-1) &= 0 \end{aligned}$$

$$y_c = c_1 e^{-x/2} + c_2 e^{x/2}$$

$$
\begin{aligned}
W &= \begin{vmatrix} e^{-x/2} & e^{x/2} \\ -\frac{1}{2}e^{-x/2} & \frac{1}{2}e^{x/2} \end{vmatrix} \\
&= 1 \\
u_1' &= \frac{W_1}{W} \\
&= \begin{vmatrix} 0 & e^{x/2} \\ \frac{1}{4}xe^{x/2} & \frac{1}{2}e^{x/2} \end{vmatrix} \\
&= -\frac{1}{4}xe^{x} \\
u_1 &= \frac{1}{4}e^{x} - \frac{1}{4}xe^{x} \\
u_2' &= \frac{W_2}{W} \\
&= \begin{vmatrix} e^{-x/2} & 0 \\ -\frac{1}{2}e^{-x/2} & \frac{1}{4}xe^{x/2} \end{vmatrix} \\
&= \frac{1}{4}x \\
u_2 &= \frac{1}{8}x^2
\end{aligned}
$$

$$
\begin{aligned}
y &= y_c + y_p \\
&= c_1e^{-x/2} + c_2e^{x/2} + \left(\frac{1}{4}e^{x} - \frac{1}{4}xe^{x}\right)e^{-x/2} + \frac{1}{8}x^2e^{x/2} \\
&= c_1e^{-x/2} + c_2e^{x/2} - \frac{1}{4}xe^{x/2} + \frac{1}{8}x^2e^{x/2}
\end{aligned}
$$

$$
\begin{aligned}
y(0) = 1 &= c_1e^{-(0)/2} + c_2e^{(0)/2} - \frac{1}{4}(0)e^{(0)/2} + \frac{1}{8}(0)^2e^{(0)/2} \\
&= c_1 + c_2
\end{aligned}
$$

$$
\begin{aligned}
y'(0) = 0 &= -\frac{1}{2}c_1e^{-(0)/2} + \frac{1}{2}c_2e^{(0)/2} - \frac{1}{4}e^{(0)/2} - \frac{1}{8}(0)e^{(0)/2} + \frac{1}{4}(0)e^{(0)/2} \\
&\quad + \frac{1}{16}(0)^2e^{(0)/2} \\
&= -\frac{1}{2}c_1 + \frac{1}{2}c_2 - \frac{1}{4} \\
\frac{1}{2} &= -c_1 + c_2
\end{aligned}
$$

$$\frac{3}{2} = 2c_2 \Rightarrow c_2 = \frac{3}{4} \Rightarrow c_1 = \frac{1}{4}$$

$$y = \frac{1}{4}e^{-x/2} + \frac{3}{4}e^{x/2} - \frac{1}{4}xe^{x/2} + \frac{1}{8}x^2e^{x/2}$$

3.5.6 21

$$\begin{aligned} y'' + 2y' - 8y &= 0 \\ m^2 + 2m - 8 &= 0 \\ (m+4)(m-2) &= 0 \end{aligned}$$

$$y_c = c_1e^{-4x} + c_2e^{2x}$$

$$\begin{aligned} W &= \begin{vmatrix} e^{-4x} & e^{2x} \\ -4e^{-4x} & 2e^{2x} \end{vmatrix} \\ &= 6e^{-2x} \\ u_1' &= \frac{W_1}{W} \\ &= \frac{\begin{vmatrix} 0 & e^{2x} \\ 2e^{-2x} - e^{-x} & 2e^{2x} \end{vmatrix}}{6e^{-2x}} \\ &= \frac{e^x - 2}{6e^{-2x}} \\ &= \frac{1}{6}e^{3x} - \frac{1}{3}e^{2x} \\ u_1 &= \frac{1}{18}e^{3x} - \frac{1}{6}e^{2x} \\ u_2' &= \frac{W_2}{W} \\ &= \frac{\begin{vmatrix} e^{-4x} & 0 \\ -4e^{-4x} & 2e^{-2x} - e^{-x} \end{vmatrix}}{6e^{-2x}} \\ &= \frac{2e^{-6x} - e^{-5x}}{6e^{-2x}} \\ &= \frac{1}{3}e^{-4x} - \frac{1}{6}e^{-3x} \\ u_2 &= -\frac{1}{12}e^{-4x} + \frac{1}{18}e^{-3x} \end{aligned}$$

$$\begin{aligned}
y &= y_c + y_p \\
&= c_1 e^{-4x} + c_2 e^{2x} + \left(\frac{1}{18}e^{3x} - \frac{1}{6}e^{2x}\right)e^{-4x} + \left(\frac{1}{18}e^{-3x} - \frac{1}{12}e^{-4x}\right)e^{2x} \\
&= c_1 e^{-4x} + c_2 e^{2x} + \frac{1}{9}e^{-x} - \frac{1}{4}e^{-2x}
\end{aligned}$$

$$\begin{aligned}
y(0) = 1 &= c_1 e^{-4(0)} + c_2 e^{2(0)} + \frac{1}{9}e^{-(0)} - \frac{1}{4}e^{-2(0)} \\
&= c_1 + c_2 + \frac{1}{9} - \frac{1}{4} \\
\frac{41}{36} &= c_1 + c_2
\end{aligned}$$

$$\begin{aligned}
y'(0) = 0 &= -4c_1 e^{-4(0)} + 2c_2 e^{2(0)} - \frac{1}{9}e^{-(0)} + \frac{1}{2}e^{-2(0)} \\
&= -4c_1 + 2c_2 - \frac{1}{9} + \frac{1}{2} \\
-\frac{7}{18} &= -4c_1 + 2c_2
\end{aligned}$$

$$\frac{41}{9} - \frac{7}{18} = 6c_2 \Rightarrow c_2 = \frac{25}{36} \Rightarrow c_1 = \frac{16}{36} = \frac{4}{9}$$

$$y = \frac{4}{9}e^{-4x} + \frac{25}{36}e^{2x} + \frac{1}{9}e^{-x} - \frac{1}{4}e^{-2x}$$

3.5.7 27

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = \frac{1}{\sqrt{x}}$$

$$
\begin{aligned}
W &= \begin{vmatrix} x^{-1/2}\cos x & x^{-1/2}\sin x \\ -\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x & -\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x \end{vmatrix} \\
&= x^{-1/2}\cos x\left(-\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x\right) \\
&\quad - x^{-1/2}\sin x\left(-\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x\right) \\
&= -\frac{1}{2}x^{-2}(\cos x)\sin x + x^{-1}\cos^2 x + \frac{1}{2}x^{-2}(\cos x)\sin x + x^{-1}\sin^2 x \\
&= x^{-1} \\
u_1' &= \frac{W_1}{W} \\
&= x\begin{vmatrix} 0 & x^{-1/2}\sin x \\ x^{-1/2} & -\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x \end{vmatrix} \\
&= -\sin x \\
u_1 &= \cos x \\
u_2' &= x\begin{vmatrix} x^{-1/2}\cos x & 0 \\ -\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x & x^{-1/2} \end{vmatrix} \\
&= \cos x \\
u_2 &= \sin x
\end{aligned}
$$

$$
\begin{aligned}
y &= c_1x^{-1/2}\cos x + c_2x^{-1/2}\sin x + x^{-1/2}\cos^2 x + x^{-1/2}\sin^2 x \\
&= c_1x^{-1/2}\cos x + c_2x^{-1/2}\sin x + x^{-1/2}
\end{aligned}
$$

3.5.8 29

$$
\begin{aligned}
y''' + y' &= 0 \\
m^3 + m &= 0 \\
m(m^2+1) &= 0 \\
m(m+i)(m-i) &= 0
\end{aligned}
$$

$$
y_c = c_1 + c_2\cos x + c_3\sin x
$$

$$
\begin{aligned}
W &= \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} \\
&= \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} - (\cos x) \begin{vmatrix} 0 & \cos x \\ 0 & -\sin x \end{vmatrix} + (\sin x) \begin{vmatrix} 0 & -\sin x \\ 0 & -\cos x \end{vmatrix} \\
&= \sin^2 x + \cos^2 x \\
&= 1 \\
u_1' &= \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \tan x & -\cos x & -\sin x \end{vmatrix} \\
&= -(\cos x) \begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix} + (\sin x) \begin{vmatrix} 0 & -\sin x \\ \tan x & -\cos x \end{vmatrix} \\
&= (\cos^2 x) \tan x + (\sin^2 x) \tan x \\
&= \tan x \\
u_1 &= -\ln|\cos x| \\
u_2' &= \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \tan x & -\sin x \end{vmatrix} \\
&= \begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix} \\
&= -\sin x \\
u_2 &= \cos x \\
u_3' &= \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \tan x \end{vmatrix} \\
&= \begin{vmatrix} -\sin x & 0 \\ -\cos x & \tan x \end{vmatrix} \\
&= -(\sin x) \tan x \\
&= -(\sin^2 x) \sec x \\
&= -(1 - \cos^2) \sec x \\
&= \cos x - \sec x \\
u_3 &= \sin x - \ln|\sec x + \tan x|
\end{aligned}
$$

$$
\begin{aligned}
y &= c_1 + c_2 \cos x + c_3 \sin x - \ln|\cos x| + \cos^2 x + \sin^2 x \\
&\quad - (\sin x) \ln|\sec x + \tan x| \\
&= c_1 + c_2 \cos x + c_3 \sin x - \ln|\cos x| - (\sin x) \ln|\sec x + \tan x|
\end{aligned}
$$

3.6 Cauchy-Euler Equations

3.6.1 1

$$\begin{aligned} x^2y'' - 2y &= 0 \\ m(m-1) - 2 &= 0 \\ m^2 - m - 2 &= 0 \\ (m-2)(m+1) &= 0 \end{aligned}$$

$$y = c_1x^2 + c_2x^{-1}$$

3.6.2 3

$$\begin{aligned} xy'' + y' &= 0 \\ x^2y'' + xy' &= 0 \\ m(m-1) + m &= 0 \\ m^2 - m + m &= 0 \\ m^2 &= 0 \end{aligned}$$

$$y = c_1 + c_2 \ln x$$

3.6.3 5

$$\begin{aligned} x^2y'' + xy' + 4y &= 0 \\ m(m-1) + m + 4 &= 0 \\ m^2 - m + m + 4 &= 0 \\ m^2 + 4 &= 0 \\ (m+2i)(m-2i) &= 0 \end{aligned}$$

$$y = c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)$$

3.6.4 7

$$\begin{aligned} x^2y'' - 3xy' - 2y &= 0 \\ m(m-1) - 3m - 2 &= 0 \\ m^2 - m - 3m - 2 &= 0 \\ m^2 - 4m - 2 &= 0 \\ (m - (2+\sqrt{6}))(m - (2-\sqrt{6})) &= 0 \end{aligned}$$

$$y = c_1 x^{2+\sqrt{6}} + c_2 x^{2-\sqrt{6}}$$

3.6.5 9

$$\begin{aligned}
25x^2y'' + 25xy' + y &= 0 \\
25m(m-1) + 25m + 1 &= 0 \\
25m^2 - 25m + 25m + 1 &= 0 \\
25m^2 + 1 &= 0 \\
(m + \frac{1}{5}i)(m - \frac{1}{5}i) &= 0
\end{aligned}$$

$$y = c_1 \cos(\frac{1}{5}\ln x) + c_2 \sin(\frac{1}{5}\ln x)$$

3.6.6 11

$$\begin{aligned}
x^2y'' + 5xy' + 4y &= 0 \\
m(m-1) + 5m + 4 &= 0 \\
m^2 - m + 5m + 4 &= 0 \\
m^2 + 4m + 2 &= 0 \\
(m+2)^2 &= 0
\end{aligned}$$

$$y = c_1 x^{-2} + c_2 x^{-2} \ln x$$

3.6.7 13

$$\begin{aligned}
3x^2y'' + 6xy' + y &= 0 \\
3m(m-1) + 6m + 1 &= 0 \\
3m^2 - 3m + 6m + 1 &= 0 \\
3m^2 + 3m + 1 &= 0 \\
\left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{6}\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{6}\right)\right) &= 0
\end{aligned}$$

$$y = c_1 x^{-1/2} \cos\left(\frac{\sqrt{3}}{6}\ln x\right) + c_2 x^{-1/2} \sin\left(\frac{\sqrt{3}}{6}\ln x\right)$$

3.6.8 15

$$\begin{aligned}
x^3y''' - 6y &= 0\\
m(m-1)(m-2) - 6 &= 0\\
(m^2-m)(m-2) - 6 &= 0\\
m^3 - 2m^2 - m^2 + 2m - 6 &= 0\\
m^3 - 3m^2 + 2m - 6 &= 0\\
(m-3)(m-i\sqrt{2})(m+i\sqrt{2}) &= 0
\end{aligned}$$

$$y = c_1x^3 + c_2\cos(\sqrt{2}\ln x) + c_3\sin(\sqrt{2}\ln x)$$

3.6.9 17

$$\begin{aligned}
xy^{(4)} + 6y''' &= 0\\
m(m-1)(m-2)(m-3) + 6m(m-1)(m-2) &= 0\\
(m^2-m)(m^2-5m+6) + (6m^2-6m)(m-2) &= 0\\
m^4 - 5m^3 + 6m^2 - m^3 + 5m^2 - 6m + 6m^3 - 12m^2 - 6m^2 + 12m &= 0\\
m^4 - 7m^2 + 6m &= 0\\
m(m^3 - 7m + 6) &= 0\\
m(m+3)(m-1)(m-2) &= 0
\end{aligned}$$

$$y = c_1 + c_2x^{-3} + c_3x + c_4x^2$$

3.6.10 19

$$\begin{aligned}
xy'' - 4y' &= 0\\
m(m-1) - 4m &= 0\\
m^2 - m - 4m &= 0\\
m^2 - 5m &= 0\\
m(m-5) &= 0
\end{aligned}$$

$$y_c = c_1 + c_2x^5$$

$$
\begin{aligned}
W &= \begin{vmatrix} 1 & x^5 \\ 0 & 5x^4 \end{vmatrix} \\
&= 5x^4 \\
u_1' &= \frac{\begin{vmatrix} 0 & x^5 \\ x^3 & 5x^4 \end{vmatrix}}{5x^4} \\
&= -\frac{x^8}{5x^4} \\
&= -\frac{1}{5}x^4 \\
u_1 &= -\frac{1}{25}x^5 \\
u_2' &= \frac{\begin{vmatrix} 1 & 0 \\ 0 & x^3 \end{vmatrix}}{5x^4} \\
&= \frac{x^3}{5x^4} \\
&= \frac{1}{5x} \\
u_2 &= \frac{1}{5}\ln x
\end{aligned}
$$

$$
\begin{aligned}
y &= y_c + y_p \\
&= c_1 + c_2x^5 - \frac{1}{25}x^5 + \frac{1}{5}x^5 \ln x \\
&= c_1 + c_2x^5 + \frac{1}{5}x^5 \ln x
\end{aligned}
$$

3.6.11 21

$$
\begin{aligned}
x^2y'' - xy' + y &= 0 \\
m(m-1) - m + 1 &= 0 \\
m^2 - m - m + 1 &= 0 \\
m^2 - 2m + 1 &= 0 \\
(m-1)^2 &= 0
\end{aligned}
$$

$$
y_c = c_1x + c_2x \ln x
$$

$$
\begin{aligned}
W &= \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} \\
&= x \\
u_1' &= x^{-1} \begin{vmatrix} 0 & x\ln x \\ 2x^{-1} & 1+\ln x \end{vmatrix} \\
&= -2x^{-1}\ln x \\
u_1 &= -\ln^2 x \\
u_2' &= x^{-1} \begin{vmatrix} x & 0 \\ 1 & 2x^{-1} \end{vmatrix} \\
&= 2x^{-1} \\
u_2 &= 2\ln x \\
y &= y_c + y_p \\
&= c_1 x + c_2 x\ln x + x\ln^2 x
\end{aligned}
$$

3.6.12 23

$$
\begin{aligned}
x^2y'' + xy' - y = 0 \\
m(m-1) + m - 1 = 0 \\
m^2 - m + m - 1 = 0 \\
m^2 - 1 = 0 \\
(m+1)(m-1) = 0
\end{aligned}
$$

$$
y_c = c_1 x^{-1} + c_2 x
$$

$$
\begin{aligned}
W &= \begin{vmatrix} x^{-1} & x \\ -x^{-2} & 1 \end{vmatrix} \\
&= x^{-1} + x^{-1} \\
&= 2x^{-1} \\
u_1' &= \frac{1}{2}x \begin{vmatrix} 0 & x \\ x^{-2}\ln x & 1 \end{vmatrix} \\
&= -\frac{1}{2}\ln x \\
u_1 &= -\frac{1}{2}x((\ln x) - 1) \\
&= \frac{1}{2}x - \frac{1}{2}x\ln x \\
u_2' &= \frac{1}{2}x \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & x^{-2}\ln x \end{vmatrix} \\
&= \frac{1}{2}x^{-2}\ln x \\
u_2 &= -\frac{1+\ln x}{2x} \\
y &= y_c + y_p \\
&= c_1x^{-1} + c_2x + \frac{1}{2} - \frac{1}{2}\ln x - \frac{1}{2} - \frac{1}{2}\ln x \\
&= c_1x^{-1} + c_2x - \ln x
\end{aligned}
$$

3.6.13 25

$$
\begin{aligned}
x^2y'' + 3xy' &= 0 \\
m(m-1) + 3m &= 0 \\
m^2 - m + 3m &= 0 \\
m^2 + 2m &= 0 \\
m(m+2) &= 0
\end{aligned}
$$

$$
y = c_1 + c_2x^{-2}
$$

$$
\begin{aligned}
y(1) = 0 &= c_1 + c_2(1)^{-2} \\
&= c_1 + c_2 \\
y'(1) = 4 &= -2c_2(1)^{-3} \\
&= -2c_2 \\
c_2 &= -2 \\
c_1 &= 2 \\
y &= 2 - 2x^{-2}
\end{aligned}
$$

3.6.14 27

$$
\begin{aligned}
x^2y'' + xy' + y &= 0 \\
m(m-1) + m + 1 &= 0 \\
m^2 - m + m + 1 &= 0 \\
m^2 + 1 &= 0 \\
(m+i)(m-i) &= 0
\end{aligned}
$$

$$
y = c_1 \cos(\ln x) + c_2 \sin(\ln x)
$$

$$
\begin{aligned}
y(1) = 1 &= c_1 \cos(\ln 1) + c_2 \sin(\ln 1) \\
&= c_1 \cos 0 + c_2 \sin 0 \\
&= c_1 \\
y' &= -x^{-1} \sin(\ln x) + c_2 x^{-1} \cos(\ln x) \\
y'(1) = 2 &= -1^{-1} \sin(\ln 1) + c_2 1^{-1} \cos(\ln 1) \\
&= c_2 \\
y &= y_c + y_p \\
&= \cos(\ln x) + 2\sin(\ln x)
\end{aligned}
$$

3.6.15 29

$$
\begin{aligned}
xy'' + y' &= 0 \\
m(m-1) + m &= 0 \\
m^2 - m + m &= 0 \\
m^2 &= 0
\end{aligned}
$$

$$
y_c = c_1 + c_2 \ln x
$$

$$
\begin{aligned}
W &= \begin{vmatrix} 1 & \ln x \\ 0 & x^{-1} \end{vmatrix} \\
&= x^{-1} \\
u_1' &= x \begin{vmatrix} 0 & \ln x \\ 1 & x^{-1} \end{vmatrix} \\
&= -x \ln x \\
u_1 &= \frac{1}{4}x^2(1 - 2\ln x) \\
u_2' &= x \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\
&= x \\
u_2 &= \frac{1}{2}x^2 \\
y &= y_c + y_p \\
&= c_1 + c_2 \ln x + \frac{1}{4}x^2(1 - 2\ln x) + \frac{1}{2}x^2 \ln x \\
&= c_1 + c_2 \ln x + \frac{1}{4}x^2 \\
y(1) = 1 &= c_1 + c_2 \ln(1) + \frac{1}{4}(1)^2 \\
c_1 &= \frac{3}{4} \\
y' &= c_2 x^{-2} + \frac{1}{2}x \\
y'(1) = -\frac{1}{2} &= c_2(1)^{-2} + \frac{1}{2}(1) \\
c_2 &= -1 \\
y &= \frac{3}{4} - \ln x + \frac{1}{4}x^2
\end{aligned}
$$

3.6.16 31

$$
\begin{aligned}
xy'' - 7xy' + 12y &= 0 \\
m(m-1) - 7m + 12 &= 0 \\
m^2 - m - 7m + 12 &= 0 \\
m^2 - 8m + 12 &= 0 \\
(m-2)(m-6) &= 0
\end{aligned}
$$

$$
\begin{aligned}
y &= c_1x^2 + c_2x^6 \\
y(0) = 0 &= c_1(0)^2 + c_2(0)^6 \\
y(1) = 0 &= c_1(1)^2 + c_2(1)^6 \\
&= c_1 + c_2 \\
c_2 &= -c_1 \\
y &= c_1x^2 - c_1x^6 \\
&= c_1(x^2 - x^6)
\end{aligned}
$$

3.6.17 33

$$y = c_1x^4 + c_2x^{-2}$$

$$
\begin{aligned}
(m-4)(m+2) &= 0 \\
m(m-1) - m - 8 &= 0 \\
x^2y'' - xy' - 8 &= 0
\end{aligned}
$$

3.6.18 35

$$y = c_1x^{-3} + c_2x^{-3}\ln x$$

$$
\begin{aligned}
(m+3)^2 &= 0 \\
m^2 + 6m + 9 &= 0 \\
m(m-1) + 7m + 9 &= 0 \\
x^2y'' + 7xy' + 9y &= 0
\end{aligned}
$$

3.6.19 37

$$y = c_1\cos(\ln x) + c_2\sin(\ln x)$$

$$
\begin{aligned}
(m+i)(m-i) &= 0 \\
m^2 + 1 &= 0 \\
m(m-1) + m + 1 &= 0 \\
x^2y'' + xy' + y &= 0
\end{aligned}
$$

3.6.20 39

$$\begin{aligned}(x+3)^2y'' - 8(x+3)y' + 14y &= 0\\ m(m-1) - 8m + 14 &= 0\\ m^2 - m - 8m + 14 &= 0\\ (m-2)(m-7) &= 0\end{aligned}$$

$$y = c_1(x+3)^2 + c_2(x+3)^7$$

3.6.21 41

$$\begin{aligned}(x+2)^2y'' + (x+2)y' + y &= 0\\ m(m-1) + m + 1 &= 0\\ m^2 - m + m + 1 &= 0\\ (m+i)(m-i) &= 0\end{aligned}$$

$$y = c_1\cos(\ln(x+2)) + c_2\sin(\ln(x+2))$$

3.6.22 43

$$\begin{aligned}z(t) &= y(e^t)\\ z'(t) &= e^t y'(e^t)\\ y'(e^t) &= e^{-t}z'(t)\\ &= x^{-1}z'(t)\\ z''(t) &= e^t y'(e^t) + e^{2t}y''(e^t)\\ &= z'(t) + e^{2t}y''(e^t)\\ y''(e^t) &= x^{-2}z'(t) + x^{-2}z''(t)\end{aligned}$$

$$\begin{aligned}x^2y'' + 9xy' - 20y &= 0\\ z' + z'' + 9z' - 20z &= 0\\ z'' + 8z' - 20z &= 0\\ m^2 + 8m - 20 &= 0\\ (m+10)(m-2) &= 0\end{aligned}$$

$$\begin{aligned}z &= c_1e^{-10t} + c_2e^{2t}\\ y &= c_1x^{-10} + c_2x^2\end{aligned}$$

3.6.23 51

$$\begin{aligned}
T'' + r^{-1}T' - r^{-2}T &= 0 \\
r^2T'' + rT' - T &= 0 \\
m(m-1) + m - 1 &= 0 \\
m^2 - m + m - 1 &= 0 \\
(m+1)(m-1) &= 0
\end{aligned}$$

$$\begin{aligned}
T &= c_1 r^{-1} + c_2 r \\
T(2) = T_0 &= c_1(2)^{-1} + c_2(2) \\
&= \frac{1}{2}c_1 + 2c_2 \\
T' &= -c_1 r^{-2} + c_2 \\
T'(1) = 0 &= -c_1(1)^{-2} + c_2 \\
&= c_2 - c_1 \\
c_1 &= c_2 \\
T_0 &= \frac{1}{2}c_1 + 2c_2 \\
c_1 = c_2 &= \frac{2}{5}T_0
\end{aligned}$$

$$T = \frac{2}{5}T_0(r + r^{-1})$$

3.7 Nonlinear Equations

3.7.1 3

$$\begin{aligned}
y'' + (y')^2 + 1 &= 0 \\
u' + u^2 + 1 &= 0 \\
u' &= -(u^2 + 1) \\
\frac{1}{u^2+1}u' &= -1 \\
\arctan u &= c - x \\
u &= \tan(c - x) \\
y' &= \tan(c - x) \\
y &= \ln(\cos(c_1 - x)) + c_2
\end{aligned}$$

3.7.2 5

$$\begin{aligned}
x^2y'' + (y')^2 &= 0\\
x^2u' + u^2 &= 0\\
u^{-2}u' &= -x^{-2}\\
-u^{-1} &= x^{-1} + c\\
u &= -\frac{x}{cx+1}\\
y &= \frac{1}{c_1^2}\ln(c_1x+1) - \frac{1}{c_1}x + c_2
\end{aligned}$$

3.7.3 7

$$\begin{aligned}
yy'' + (y')^2 + 1 &= 0\\
yuu' + u^2 + 1 &= 0\\
\frac{u}{u^2+1}u' &= -y^{-1}\\
\frac{1}{2}\ln|u^2+1| &= -\ln|y| + c_1\\
\sqrt{u^2+1} &= c_1y^{-1}\\
u^2+1 &= c_1y^{-2}\\
y' &= \sqrt{c_1y^{-2}-1}\\
\frac{1}{\sqrt{c_1y^{-2}-1}}y' &= 1\\
-y\sqrt{\frac{c_1}{y^2}-1} &= x + c_2\\
y^2\left(\frac{c_1}{y^2}-1\right) &= (x+c_2)^2\\
c_1 - y^2 &= (x+c_2)^2\\
y^2 &= c_1 - (x+c_2)^2
\end{aligned}$$

3.7.4 9

$$\begin{aligned}
y'' + 2y(y')^3 &= 0 \\
uu' + 2yu^3 &= 0 \\
u^{-2}u' &= -2y \\
-u^{-1} &= -y^2 + c_1 \\
u &= \frac{1}{y^2 + c_1} \\
(y^2 + c_1)y' &= 1 \\
\frac{1}{3}y^3 + c_1 y &= x + c_2
\end{aligned}$$

3.7.5 11

$$\begin{aligned}
y'' + yy' &= 0 \\
uu' + yu &= 0 \\
u' &= -y \\
u &= -\frac{1}{2}y^2 + c_1 \\
\frac{-2}{y^2 + c_1}y' &= 1 \\
-\frac{2\arctan(y/c_1)}{c_1} &= x + c_2 \\
\arctan(y/c_1) &= -\frac{c_1}{2}(x + c_2) \\
&= c_2 - \frac{c_1}{2}x \\
y &= c_1 \tan\left(c_2 - \frac{c_1}{2}x\right) \\
y(0) = 1 &= c_1 \tan\left(c_2 - \frac{c_1}{2}(0)\right) \\
&= c_1 \tan c_2 \\
y'(0) = -1 &= -\frac{1}{2}c_1^2 \sec^2\left(c_2 - \frac{c_1(0)}{2}\right) \\
&= -\frac{1}{2}c_1^2 \sec^2 c_2 \\
-1 &= -\frac{1}{2}\left(\frac{1}{\tan c_2}\right)^2 \sec^2 c_2 \\
&= -\frac{1}{2}\frac{\cos^2 c_2}{\sin^2 c_2}\frac{1}{\cos^2 c_2} \\
2 &= \frac{1}{\sin^2 c_2} \\
c_2 &= \arcsin\frac{1}{\sqrt{2}} \\
&= \frac{\pi}{4} \\
c_1 &= 1 \\
y &= \tan\left(\frac{\pi}{4} - \frac{1}{2}x\right)
\end{aligned}$$

3.7.6 13

$$\begin{aligned}
xy'' &= y' + (y')^3 \\
xu' &= u + u^3 \\
u' - x^{-1}u &= x^{-1}u^3
\end{aligned}$$

Let $u = v^{-1/2}$ and $u' = -\frac{1}{2}v^{-3/2}v'$

$$\begin{aligned}
-\frac{1}{2}v^{-3/2}v' - x^{-1}v^{-1/2} &= x^{-1}v^{-3/2} \\
v' + 2x^{-1}v &= -2x^{-1} \\
e^{2\ln|x|}v' + 2x^{-1}e^{2\ln|x|}v &= -2x^{-1}e^{2\ln|x|} \\
\frac{d}{dx}(x^2v) &= -2x \\
x^2v &= -x^2 + c_1 \\
u^{-2} &= c_1x^{-2} - 1 \\
y' &= \frac{1}{\sqrt{c_1x^{-2} - 1}} \\
y &= c_2 - x\sqrt{c_1x^{-2} - 1} \\
&= c_2 - \sqrt{c_1 - x^2} \\
&= c_2 - \frac{1}{c_1}\sqrt{1 - c_1^2x^2}
\end{aligned}$$

3.7.7 15

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 + \cdots$$

$$y'' = x + y^2$$

$$\begin{aligned}
y(0) &= 1 \\
y'(0) &= 1 \\
y''(0) &= (0) + (1)^2 = 1 \\
y''' &= 1 + 2yy' \\
y'''(0) &= 1 + 2(1)(1) = 3 \\
y^{(4)} &= 2y'^2 + 2yy'' \\
y^{(4)}(0) &= 2(1)^2 + 2(1)(1) = 4 \\
y^{(5)} &= 6y'y'' + 2yy''' \\
y^{(5)}(0) &= 6(1)(1) + 2(1)(3) = 12
\end{aligned}$$

$$y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{10}x^5$$

3.7.8 19

$$\begin{aligned}
\frac{y''}{[1 + (y')^2]^{3/2}} &= 1 \\
\frac{y'}{\sqrt{(y')^2 + 1}} &= x \\
y' &= x\sqrt{(y')^2 + 1} \\
(y')^2 &= x^2((y')^2 + 1) \\
(1 - x^2)(y')^2 &= x^2 \\
(y')^2 &= \frac{x^2}{1 - x^2} \\
y' &= \frac{x}{\sqrt{1 - x^2}} \\
y &= -\sqrt{1 - x^2}
\end{aligned}$$

3.8 Linear Models: Initial-Value Problems

3.8.1 1

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = \frac{\sqrt{2}\pi}{8}$$

3.8.2 3

$$W = mg \Rightarrow m = W/g = \frac{3}{4}$$

$$W = kx \Rightarrow k = W/x = 72$$

$$\begin{aligned}
m\frac{d^2x}{dt^2} &= -kx \\
\frac{d^2x}{dt^2} + \frac{k}{m}x &= 0 \\
m^2 + \frac{k}{m} &= 0 \\
m^2 + \omega^2 &= 0 \\
(m + i\omega)(m - i\omega) &= 0
\end{aligned}$$

$$x = c_1 \cos \omega t + c_2 \sin \omega t$$

$$
\begin{aligned}
x(0) = -\frac{1}{4} &= c_1 \cos\omega(0) + c_2 \sin\omega(0) \\
&= c_1 \\
x'(0) = 0 &= 3\omega \sin\omega(0) + c_2\omega \cos\omega(0) \\
&= c_2
\end{aligned}
$$

$$
x = -\frac{1}{4}\cos 4\sqrt{6}t
$$

3.8.3 9

(a)

$$
\begin{aligned}
x(0) = \frac{1}{2} &= c_1 \cos\omega(0) + c_2 \sin\omega(0) \\
&= c_1 \\
x'(0) = \frac{3}{2} &= -\frac{1}{2}\omega \sin\omega(0) + c_2\omega \cos\omega(0) \\
&= c_2\omega \\
c_2 &= \frac{3}{2}\sqrt{\frac{m}{k}} \\
&= \frac{3}{4}
\end{aligned}
$$

$$
x = \frac{1}{2}\cos 2t + \frac{3}{4}\sin 2t
$$

(b)

$$
\begin{aligned}
A &= \sqrt{c_1^2 + c_2^2} \\
&= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2} \\
&= \sqrt{\frac{1}{4} + \frac{9}{16}} \\
&= \sqrt{\frac{4}{16} + \frac{9}{16}} \\
&= \sqrt{\frac{13}{16}} \\
&= \frac{\sqrt{13}}{4} \\
\tan\phi &= \frac{c_1}{c_2} \\
\phi &= \arctan \frac{1}{2}\frac{4}{3} \\
&= \arctan \frac{2}{3} \\
&= 0.558
\end{aligned}
$$

$$x = \frac{\sqrt{13}}{4}\sin(2t + 0.558)$$

(c)

$$
\begin{aligned}
\tan\phi &= \frac{c_2}{c_1} \\
&= \frac{3}{4}\frac{2}{1} \\
&= \frac{3}{2} \\
\phi &= \arctan \frac{3}{2} \\
&= 0.983
\end{aligned}
$$

$$x = \frac{\sqrt{13}}{4}\cos(2t - 0.983)$$

3.8.4 13

$$W = k_1 x_1 \Rightarrow k_1 = \frac{W}{x_1} = 40$$

$$W = k_2 x_2 \Rightarrow k_2 = \frac{W}{x_2} = 120$$

$$k_{\text{eff}} = k_1 + k_2 = 160$$

$$\begin{aligned}
x(0) = 0 &= c_1 \cos\omega(0) + c_2 \sin\omega(0) \\
&= c_2 \\
x'(0) = 2 &= c_2\omega\cos\omega(0) \\
&= c_2\omega \\
c_2 &= 2\sqrt{\frac{m}{k_{\text{eff}}}} \\
&= 2\sqrt{\frac{5}{1280}} \\
&= \frac{1}{8}
\end{aligned}$$

$$x = \frac{1}{8}\sin 16t$$

3.8.5 15

$$k_{\text{eff}} = \frac{k_1 k_2}{k_1 + k_2} = 30$$

$$\begin{aligned}
x'(0) = 2 &= c_2\omega\cos\omega(0) \\
&= c_2\omega \\
c_2 &= 2\sqrt{\frac{m}{k_{\text{eff}}}} \\
&= \frac{1}{2\sqrt{3}}
\end{aligned}$$

$$x = \frac{\sqrt{3}}{6}\sin 4\sqrt{3}t$$

3.8.6 17

$$k_{\text{eff}} = 2k$$

The spring is twice as rigid

3.8.7 21

(a) Above

(b) Upward

3.8.8 23

(a) Below

(b) Upward

3.8.9 25

$$
\begin{aligned}
mx'' &= -kx - x' \\
x'' + \frac{1}{m}x' + \frac{k}{m}x &= 0 \\
x'' + 2\lambda x' + \omega^2 x &= 0 \\
r^2 + 2\lambda r + \omega^2 &= 0 \\
(r - (-\lambda + \sqrt{\lambda^2 - \omega^2}))(r - (-\lambda - \sqrt{\lambda^2 - \omega^2})) &= 0
\end{aligned}
$$

$$
\begin{aligned}
2\lambda &= \frac{1}{m} \\
\lambda &= \frac{1}{2m} \\
&= \frac{g}{2W} \\
&= 4 \\
\omega^2 &= \frac{k}{m} \\
&= \frac{gk}{W} \\
&= 16 \\
\sqrt{\lambda^2 - \omega^2} &= \sqrt{16 - 16} \\
&= 0
\end{aligned}
$$

$$
x = c_1 e^{-4t} + c_2 t e^{-4t}
$$

$$
\begin{aligned}
x(0) = -1 &= c_1 e^{-4(0)} + c_2(0)e^{-4(0)} \\
&= c_1 \\
x'(0) = 8 &= 4e^{-4(0)} + c_2(e^{-4(0)} - 4(0)e^{-4(0)}) \\
&= 4 + c_2 \\
c_2 &= 4
\end{aligned}
$$

$$x = -e^{-4t} + 4te^{-4t}$$

$$
\begin{aligned}
0 &= -e^{-4t} + 4te^{-4t} \\
&= -1 + 4t \\
t &= \frac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
x' &= 0 \\
4e^{-4t} + 4e^{-4x} - 16te^{-4t} &= 0 \\
4 + 4 - 16t &= 0 \\
t &= \frac{1}{2}
\end{aligned}
$$

$$x\left(\frac{1}{2}\right) = -e^{-4(1/2)} + 4\left(\frac{1}{2}\right)e^{-4(1/2)} = e^{-2}$$

3.8.10 29

(a)

$$
\begin{aligned}
k &= W/x = 2 \\
m &= W/g = 0.1 \\
\lambda &= \frac{\beta}{2m} = 2 \\
\omega^2 &= \frac{k}{m} = 20 \\
\sqrt{\lambda^2 - \omega^2} &= \sqrt{4 - 20} = 4i
\end{aligned}
$$

$$x = c_1 e^{-2t}\cos 4t + c_2 e^{-2t}\sin 4t$$

$$
\begin{aligned}
x(0) = -1 &= c_1 e^{-2(0)} \cos 4(0) + c_2 e^{-2(0)} \sin 4(0) \\
&= c_1 \\
x'(0) = 0 &= 2e^{-2(0)} \cos 4(0) + 4e^{-2(0)} \sin 4(0) - 2c_2 e^{-2(0)} \sin 4(0) \\
&\quad + 4c_2 e^{-2(0)} \cos 4(0) \\
&= 2 + 4c_2 \\
c_2 &= -\frac{1}{2}
\end{aligned}
$$

$$
x = -e^{-2t} \cos 4t - \frac{1}{2} e^{-2t} \sin 4t
$$

(b)

$$
\begin{aligned}
A &= \sqrt{c_1^2 + c_2^2} \\
&= \frac{\sqrt{5}}{2} \\
\tan \phi &= \frac{c_1}{c_2} \\
&= 2 \\
\phi &= \arctan 2 \\
&= 4.25 \\
x &= \frac{\sqrt{5}}{2} e^{-2t} \sin(4t + 4.25)
\end{aligned}
$$

(c)

$$
\begin{aligned}
\frac{\sqrt{5}}{2} e^{-2t} \sin(4t + 4.25) &= 0 \\
4t + 4.25 &= 3\pi \\
t &= 1.29
\end{aligned}
$$

3.8.11 31

$$\begin{aligned} k &= W/x \\ &= 5 \\ m &= W/g \\ &= \frac{5}{16} \\ \lambda &= \frac{\beta}{2m} \\ &= \frac{8}{5}\beta \\ \omega^2 &= \frac{k}{m} \\ &= 16 \\ \lambda^2 - \omega^2 &= \left(\frac{8}{5}\beta\right)^2 - 16 \\ &= \frac{64}{25}\beta^2 - 16 \end{aligned}$$

(a)

$$\begin{aligned} \lambda^2 - \omega^2 &> 0 \\ \frac{64}{25}\beta^2 - 16 &> 0 \\ \beta &> \frac{5}{2} \end{aligned}$$

(b)

$$\beta = \frac{5}{2}$$

(c)

$$\beta < \frac{5}{2}$$

3.8.12 33

$$
\begin{aligned}
\beta &= \frac{1}{2} \\
k &= W/x = 6 \\
m &= W/g = \frac{1}{2} \\
\lambda &= \frac{\beta}{2m} = \frac{1}{2} \\
\omega^2 &= \frac{k}{m} = 12 \\
\sqrt{\lambda^2 - \omega^2} &= \sqrt{\frac{1}{4} - 12} \\
&= \sqrt{-\frac{47}{4}} \\
&= \frac{\sqrt{47}}{2} i
\end{aligned}
$$

$$
\begin{aligned}
mx'' &= -kx - \beta x' + f(t) \\
mx'' + \beta x' + kx &= f(t) \\
x'' + 2\lambda x' + \omega^2 x &= \frac{1}{m} f(t) \\
r^2 + 2\lambda r + \omega^2 &= 0 \\
(r - (-\lambda + \sqrt{\lambda^2 - \omega^2}))(r - (-\lambda - \sqrt{\lambda^2 - \omega^2})) &= 0
\end{aligned}
$$

$$
x_c = c_1 e^{-t/2} \cos \frac{\sqrt{47}}{2} t + c_2 e^{-t/2} \sin \frac{\sqrt{47}}{2} t
$$

$$
x'' + x' + 12x = 20 \cos 3t
$$

$$
\begin{aligned}
x_p &= A \cos 3t + B \sin 3t \\
x_p' &= -3A \sin 3t + 3B \cos 3t \\
x_p'' &= -9A \cos 3t - 9B \sin 3t
\end{aligned}
$$

$$
\begin{aligned}
20 \cos 3t &= -9A \cos 3t - 9B \sin 3t - 3A \sin 3t + 3B \cos 3t \\
&\quad + 12(A \cos 3t + B \sin 3t) \\
&= 3(A + B) \cos 3t + 3(B - A) \sin 3t
\end{aligned}
$$

$$
\begin{aligned}
20 &= 3(A+B) \\
0 &= 3(B-A) \\
20 &= 6B \\
B &= \frac{10}{3} \\
A &= \frac{10}{3}
\end{aligned}
$$

$$
\begin{aligned}
x &= x_c + x_p \\
&= c_1 e^{-t/2} \cos \frac{\sqrt{47}}{2} t + c_2 e^{-t/2} \sin \frac{\sqrt{47}}{2} t + \frac{10}{3} \cos 3t + \frac{10}{3} \sin 3t
\end{aligned}
$$

$$
\begin{aligned}
x(0) = 2 &= c_1 e^{-(0)/2} \cos \frac{\sqrt{47}}{2}(0) + c_2 e^{-(0)/2} \sin \frac{47}{2}(0) + \frac{10}{3} \cos 3(0) \\
&\quad + \frac{10}{3} \sin 3(0) \\
&= c_1 + \frac{10}{3} \\
c_1 &= -\frac{4}{3} \\
x'(0) = 0 &= \frac{2}{3} e^{-(0)/2} \cos \frac{\sqrt{47}}{2}(0) + \frac{2\sqrt{47}}{3} e^{-(0)/2} \sin \frac{\sqrt{47}}{2}(0) \\
&\quad - \frac{1}{2} c_2 e^{-(0)/2} \sin \frac{\sqrt{47}}{2}(0) + \frac{\sqrt{47}}{2} c_2 e^{-(0)/2} \cos \frac{\sqrt{47}}{2}(0) \\
&\quad - 10 \sin 3(0) + 10 \cos 3(0) \\
&= \frac{2}{3} + \frac{\sqrt{47}}{2} c_2 + 10 \\
c_2 &= -\frac{64}{3\sqrt{47}}
\end{aligned}
$$

$$
x = e^{-t/2} \left(-\frac{4}{3} \cos \frac{\sqrt{47}}{2} t - \frac{64}{3\sqrt{47}} \sin \frac{\sqrt{47}}{2} t \right) + \frac{10}{3} (\cos 3t + \sin 3t)
$$

3.8.13 35

$$\begin{aligned}
m &= 1 \\
k &= W/x = mg/x = 16 \\
\beta &= 8 \\
\lambda &= \beta/2m = 4 \\
\omega^2 &= k/m = 16 \\
\sqrt{\lambda^2 - \omega^2} &= 0
\end{aligned}$$

$$x'' + 8x' + 16x = 8\sin 4t$$

$$x_c = c_1 e^{-4t} + c_2 t e^{-4t}$$

$$\begin{aligned}
x_p &= A\cos 4t + B\sin 4t \\
x_p' &= -4A\sin 4t + 4B\cos 4t \\
x_p'' &= -16A\cos 4t - 16B\sin 4t \\
8\sin 4t &= -16A\cos 4t - 16B\sin 4t + 8(-4A\sin 4t + 4B\cos 4t) \\
&\quad + 16(A\cos 4t + B\sin 4t) \\
&= 32B\cos 4t - 32A\sin 4t \\
A &= -\frac{1}{4} \\
B &= 0
\end{aligned}$$

$$x = c_1 e^{-4t} + c_2 t e^{-4t} - \frac{1}{4}\cos 4t$$

$$\begin{aligned}
x(0) = 0 &= c_1 e^{-4(0)} + c_2(0)e^{-4(0)} - \frac{1}{4}\cos 4(0) \\
&= c_1 - \frac{1}{4} \\
c_1 &= \frac{1}{4} \\
x'(0) = 0 &= -e^{-4(0)} + c_2 e^{-4(0)} - 4c_2(0)e^{-4(0)} + \sin 4(0) \\
&= -1 + c_2 \\
c_2 &= 1
\end{aligned}$$

$$x = \frac{1}{4}e^{-4t} + te^{-4t} - \frac{1}{4}\cos 4t$$

3.8.14 37

$$
\begin{aligned}
m &= 2 \\
k &= 32
\end{aligned}
$$

$$
\begin{aligned}
2x'' + 32x &= 68e^{-2t}\cos 4t \\
x'' + 16x &= 34e^{-2t}\cos 4t
\end{aligned}
$$

$$
\begin{aligned}
x'' + 16x &= 0 \\
(r+4i)(r-4i) &= 0
\end{aligned}
$$

$$x_c = c_1 \cos 4t + c_2 \sin 4t$$

$$
\begin{aligned}
x_p &= Ae^{-2t}\cos 4t + Be^{-2t}\sin 4t \\
x_p' &= -2Ae^{-2t}\cos 4t - 4Ae^{-2t}\sin 4t - 2Be^{-2t}\sin 4t + 4Be^{-2t}\cos 4t \\
x_p'' &= 4Ae^{-2t}\cos 4t + 8Ae^{-2t}\sin 4t + 8Ae^{-2t}\sin 4t - 16Ae^{-2t}\cos 4t \\
&\quad + 4Be^{-2t}\sin 4t - 8Be^{-2t}\cos 4t - 8Be^{-2t}\cos 4t - 16Be^{-2t}\sin 4t \\
&= (-12A - 16B)e^{-2t}\cos 4t + (16A - 12B)e^{-2t}\sin 4t
\end{aligned}
$$

$$
\begin{aligned}
34e^{-2t}\cos 4t &= (-12A - 16B)e^{-2t}\cos 4t + (16A - 12B)e^{-2t}\sin 4t \\
&\quad + 16(Ae^{-2t}\cos 4t + Be^{-2t}\sin 4t) \\
&= (4A - 16B)e^{-2t}\cos 4t + (16A + 4B)e^{-2t}\sin 4t
\end{aligned}
$$

$$
\begin{aligned}
34 &= 4A - 16B \\
0 &= 16A + 4B \\
34 &= 68A \\
A &= \frac{1}{2} \\
B &= -2
\end{aligned}
$$

$$x = c_1 \cos 4t + c_2 \sin 4t + e^{-2t}\left(\frac{1}{2}\cos 4t - 2\sin 4t\right)$$

$$
\begin{aligned}
x(0) = 0 &= c_1 \cos 4(0) + c_2 \sin 4(0) + e^{-2(0)} \left(\frac{1}{2} \cos 4(0) - 2 \sin 4(0) \right) \\
&= c_1 + \frac{1}{2} \\
c_1 &= -\frac{1}{2} \\
x'(0) = 0 &= 2 \sin 4(0) + 4c_2 \cos 4(0) - e^{-2(0)} \cos 4(0) - 2e^{-2(0)} \sin 4(0) \\
&\quad + 4e^{-2(0)} \sin 4(0) - 8e^{-2(0)} \cos 4(0) \\
&= 4c_2 - 1 - 8 \\
c_2 &= \frac{9}{4}
\end{aligned}
$$

$$
x = -\frac{1}{2} \cos 4t + \frac{9}{4} \sin 4t + e^{-2t} \left(\frac{1}{2} \cos 4t - 2 \sin 4t \right)
$$

3.8.15 39

(a)

$$
\begin{aligned}
mx'' + \beta x' + kx &= kh \\
x'' + \frac{\beta}{m} x' + \frac{k}{m} x &= \frac{k}{m} h \\
x'' + 2\lambda x' + \omega^2 x &= \omega^2 h
\end{aligned}
$$

(b)

$$
x'' + 4x' + 8x = 40 \cos t
$$

$$
\begin{aligned}
r^2 + 4r + 8 &= 0 \\
(r - (-2 + 2i))(r - (-2 - 2i)) &= 0
\end{aligned}
$$

$$
x_c = c_1 e^{-2t} \cos 2t + c_2 e^{-2t} \sin 2t
$$

$$
\begin{aligned}
x_p &= A \cos t + B \sin t \\
x_p' &= -A \sin t + B \cos t \\
x_p'' &= -A \cos t - B \sin t
\end{aligned}
$$

$$
\begin{aligned}
40\cos t &= -A\cos t - B\sin t + 4(-A\sin t + B\cos t) + 8(A\cos t + B\sin t)\\
&= (7A+4B)\cos t + (7B-4A)\sin t
\end{aligned}
$$

$$
\begin{aligned}
40 &= 7A + 4B\\
0 &= -4A + 7B\\
160 &= 65B\\
B &= \frac{32}{13}\\
A &= \frac{56}{13}
\end{aligned}
$$

$$
x = c_1 e^{-2t}\cos 2t + c_2 e^{-2t}\sin 2t + \frac{56}{13}\cos t + \frac{32}{13}\sin t
$$

$$
\begin{aligned}
x(0) = 0 &= c_1 e^{-2(0)}\cos 2(0) + c_2 e^{-2(0)}\sin 2(0) + \frac{56}{13}\cos(0) + \frac{32}{13}\sin(0)\\
&= c_1 + \frac{56}{13}\\
c_1 &= -\frac{56}{13}\\
x'(0) = 0 &= \frac{112}{13}e^{-2(0)}\cos 2(0) + \frac{112}{13}e^{-2(0)}\sin 2(0) - 2c_2 e^{-2(0)}\sin 2(0)\\
&\quad + 2c_2 e^{-2(0)}\cos 2(0) - \frac{56}{13}\sin(0) + \frac{32}{13}\cos(0)\\
&= \frac{112}{13} + 2c_2 + \frac{32}{13}\\
c_2 &= -\frac{72}{13}
\end{aligned}
$$

$$
x = e^{-2t}\left(-\frac{56}{13}\cos 2t - \frac{72}{13}\sin 2t\right) + \frac{56}{13}\cos t + \frac{32}{13}\sin t
$$

3.8.16 41

$$
x'' + 4x = -5\sin 2t + 3\cos 2t
$$

$$
\begin{aligned}
x'' + 4x &= 0\\
(r+2i)(r-2i) &= 0\\
x_c &= c_1\cos 2t + c_2\sin 2t
\end{aligned}
$$

$$
\begin{aligned}
x_p &= At\cos 2t + Bt\sin 2t \\
x_p' &= A\cos 2t - 2At\sin 2t + B\sin 2t + 2Bt\cos 2t \\
x_p'' &= -2A\sin 2t - 2A\sin 2t - 4At\cos 2t + 2B\cos 2t + 2B\cos 2t \\
&\quad - 4Bt\sin 2t \\
&= -4A\sin 2t - 4At\cos 2t + 4B\cos 2t - 4Bt\sin 2t \\
-5\sin 2t + 3\cos 2t &= -4A\sin 2t - 4At\cos 2t + 4B\cos 2t - 4Bt\sin 2t \\
&\quad + 4(At\cos 2t + Bt\sin 2t) \\
&= -4A\sin 2t + 4B\cos 2t
\end{aligned}
$$

$$
\begin{aligned}
-5 = -4A \Rightarrow A = \frac{5}{4} \\
3 = 4B \Rightarrow B = \frac{3}{4}
\end{aligned}
$$

$$
x = c_1\cos 2t + c_2\sin 2t + \frac{5}{4}t\cos 2t + \frac{3}{4}t\sin 2t
$$

$$
\begin{aligned}
x(0) = -1 &= c_1\cos 2(0) + c_2\sin 2(0) + \frac{5}{4}(0)\cos 2(0) + \frac{3}{4}(0)\sin 2(0) \\
&= c_1 \\
x'(0) = 1 &= 2\sin 2(0) + 2c_2\cos 2(0) + \frac{5}{4}\cos 2(0) - \frac{5}{2}(0)\sin 2(0) + \frac{3}{4}\sin 2(0) \\
&\quad + \frac{3}{2}(0)\cos 2(0) \\
&= 2c_2 + \frac{5}{4} \\
c_2 &= -\frac{1}{8}
\end{aligned}
$$

$$
x = -\cos 2t - \frac{1}{8}\sin 2t + \frac{5}{4}t\cos 2t + \frac{3}{4}t\sin 2t
$$

3.8.17 49

$$
\begin{aligned}
0.05q'' + 2q' + 100q &= 0 \\
q'' + 40q' + 2000q &= 0 \\
r^2 + 40r + 2000 &= 0 \\
(r - (-20 + 40i))(r - (-20 - 40i)) &= 0
\end{aligned}
$$

$$q = c_1 e^{-20t} \cos 40t + c_2 e^{-20t} \sin 40t$$

$$\begin{aligned}
q(0) = 5 &= c_1 e^{-20(0)} \cos 40(0) + c_2 e^{-20(0)} \sin 40(0) \\
&= c_1 \\
q'(0) = 0 &= -100 e^{-20(0)} \cos 40(0) - 200 e^{-20(0)} \sin 40(0) - 20 c_2 e^{-20(0)} \sin 40(0) \\
&\quad + 40 c_2 e^{-20(0)} \cos 40(0) \\
&= -100 + 40 c_2 \\
c_2 &= \frac{5}{2}
\end{aligned}$$

$$q = e^{-20t}(5 \cos 40t + \frac{5}{2} \sin 40t)$$

$$q(0.01) = 4.57\,\mathrm{C}$$

$$\begin{aligned}
e^{-20t}(5 \cos 40t + \frac{5}{2} \sin 40t) &= 0 \\
5 \cos 40t + \frac{5}{2} \sin 40t &= 0 \\
2 \cos 40t + \sin 40t &= 0 \\
\sqrt{5} \cos(40t + \arctan(-1/2)) &= 0 \\
40t + \arctan(-1/2) &= \arccos 0 \\
40t &= \arccos 0 - \arctan(-1/2) \\
t &= \frac{1}{40}(\arccos 0 - \arctan(-1/2)) \\
&= 0.0509\,\mathrm{s}
\end{aligned}$$

3.9 Linear Models: Boundary-Value Problems

3.9.1 1

(a)

$$EIy^{(4)} = w_0$$

$$y(0) = 0,\ y'(0) = 0,\ y''(L) = 0,\ y'''(L) = 0$$

$$
\begin{aligned}
y^{(4)} &= \frac{w_0}{EI} \\
y''' &= \frac{w_0}{EI}x + c_1 \\
y'' &= \frac{w_0}{2EI}x^2 + c_1x + c_2 \\
y' &= \frac{w_0}{6EI}x^3 + \frac{1}{2}c_1x^2 + c_2x + c_3 \\
y &= \frac{w_0}{24EI}x^4 + \frac{1}{6}c_1x^3 + \frac{1}{2}c_2x^2 + c_3x + c_4
\end{aligned}
$$

$$
\begin{aligned}
y(0) = 0 &= c_4 \\
y'(0) = 0 &= c_3 \\
y'''(L) = 0 &= \frac{w_0}{EI}(L) + c_1 \\
c_1 &= -\frac{Lw_0}{EI} \\
y''(L) = 0 &= \frac{w_0}{2EI}(L)^2 - \frac{Lw_0}{EI}(L) + c_2 \\
c_2 &= \frac{L^2w_0}{2EI}
\end{aligned}
$$

$$
y = \frac{w_0}{24EI}\left(x^4 - 4Lx^3 + 6L^2x^2\right)
$$

(b)

$$
y = x^4 - 4Lx^3 + 6L^2x^2
$$

3.9.2 3

(a)

$$
\begin{aligned}
y^{(4)} &= \frac{w_0}{EI} \\
y''' &= \frac{w_0}{EI}x + c_1 \\
y'' &= \frac{w_0}{2EI}x^2 + c_1x + c_2 \\
y' &= \frac{w_0}{6EI}x^3 + \frac{1}{2}c_1x^2 + c_2x + c_3 \\
y &= \frac{w_0}{24EI}x^4 + \frac{1}{6}c_1x^3 + \frac{1}{2}c_2x^2 + c_3x + c_4
\end{aligned}
$$

$$
\begin{aligned}
y(0) &= 0 = c_4 \\
y'(0) &= 0 = c_3 \\
y(L) &= 0 = \frac{w_0}{24EI}(L)^4 + \frac{1}{6}c_1(L)^3 + \frac{1}{2}c_2(L)^2 \\
&= \frac{w_0}{12EI}L^2 + \frac{1}{3}c_1 L + c_2 \\
y''(L) &= 0 = \frac{w_0}{2EI}(L)^2 + c_1(L) + c_2 \\
0 &= \frac{5L^2 w_0}{12EI} + \frac{2}{3}c_1 L \\
c_1 &= -\frac{5Lw_0}{8EI} \\
0 &= \frac{L^2 w_0}{2EI} - \frac{5L^2 w_0}{8EI} + c_2 \\
c_2 &= \frac{L^2 w_0}{8EI}
\end{aligned}
$$

$$
\begin{aligned}
y &= \frac{w_0}{24EI}x^4 - \frac{5L^2 w_0}{48EI}x^3 + \frac{L^2 w_0}{16EI}x^2 \\
&= \frac{w_0}{48EI}(2x^4 - 5Lx^3 + 3L^2x^2)
\end{aligned}
$$

(b)

$$
y = 2x^4 - 5Lx^3 + 3L^2x^2
$$

3.9.3 11

$$
\begin{aligned}
y'' + \lambda y &= 0 \\
m^2 + \lambda &= 0 \\
(m + \sqrt{\lambda}i)(m - \sqrt{\lambda}i) &= 0
\end{aligned}
$$

$$
y = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x
$$

$$
\begin{aligned}
y(0) &= 0 = c_1 \cos\sqrt{\lambda}(0) + c_2 \sin\sqrt{\lambda}(0) \\
&= c_1 \\
y(\pi) &= 0 = c_2 \sin\sqrt{\lambda}(\pi) \\
\sqrt{\lambda}\pi &= n\pi \\
\lambda &= n^2
\end{aligned}
$$

$$
y = \sin nx \text{ for } n = 1, 2, 3, \ldots
$$

3.9.4 13

$$y'' + \lambda y = 0$$

$$y = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$$

$$y' = -\sqrt{\lambda} c_1 \sin \sqrt{\lambda} x + \sqrt{\lambda} c_2 \cos \sqrt{\lambda} x$$

$$\begin{aligned}
y'(0) = 0 &= -\sqrt{\lambda} c_1 \sin \sqrt{\lambda}(0) + \sqrt{\lambda} c_2 \cos \sqrt{\lambda}(0) \\
&= \sqrt{\lambda} c_2 \\
&= c_2 \\
y(L) = 0 &= c_1 \cos \sqrt{\lambda}(L) \\
\sqrt{\lambda} L &= n\pi - \frac{\pi}{2} \text{ for } n = 1, 2, 3, \ldots \\
&= \frac{2n-1}{2}\pi \\
\lambda &= \left(\frac{(2n-1)\pi}{2L} \right)^2
\end{aligned}$$

$$y = \cos \frac{(2n-1)\pi}{2L} x$$

3.9.5 15

$$\begin{aligned}
y'' + \lambda y &= 0 \\
m^2 + \lambda &= 0 \\
(m + \sqrt{\lambda} i)(m - \sqrt{\lambda} i) &= 0
\end{aligned}$$

$$y = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$$

$$y' = -\sqrt{\lambda} c_1 \sin \sqrt{\lambda} x + \sqrt{\lambda} c_2 \cos \sqrt{\lambda} x$$

$$\begin{aligned}
y'(0) = 0 &= -\sqrt{\lambda} c_1 \sin \sqrt{\lambda}(0) + \sqrt{\lambda} c_2 \cos \sqrt{\lambda}(0) \\
&= \sqrt{\lambda} c_2 \\
&= c_2 \\
y(\pi) = 0 &= c_1 \cos \sqrt{\lambda}(\pi) \\
\sqrt{\lambda} \pi &= \frac{\pi}{2} + n\pi \\
\lambda &= \left(\frac{1}{2} + n \right)^2
\end{aligned}$$

$$y = \cos\left(\frac{1}{2} + n\right)x$$

3.9.6 17

$$\begin{aligned} y'' + 2y' + (\lambda + 1)y &= 0 \\ m^2 + 2m + \lambda + 1 &= 0 \\ (m - (-1 + \sqrt{\lambda}i))(m - (-1 - \sqrt{\lambda}i)) &= 0 \end{aligned}$$

$$y = c_1 e^{-x} \cos\sqrt{\lambda}x + c_2 e^{-x} \sin\sqrt{\lambda}x$$

$$\begin{aligned} y(0) = 0 &= c_1 e^{-(0)} \cos\sqrt{\lambda}(0) + c_2 e^{-(0)} \sin\sqrt{\lambda}(0) \\ &= c_1 \\ y(5) = 0 &= c_2 e^{-(5)} \sin\sqrt{\lambda}(5) \\ 5\sqrt{\lambda} &= n\pi \\ \lambda &= \left(\frac{n\pi}{5}\right)^2 \end{aligned}$$

$$y = e^{-x} \sin\frac{n\pi}{5}x$$

3.9.7 19

$$\begin{aligned} x^2y'' + xy' + \lambda y &= 0 \\ m(m-1) + m + \lambda &= 0 \\ m^2 - m + m + \lambda &= 0 \\ (m + \sqrt{\lambda}i)(m - \sqrt{\lambda}i) &= 0 \end{aligned}$$

$$y = c_1 \cos(\sqrt{\lambda}\ln x) + c_2 \sin(\sqrt{\lambda}\ln x)$$

$$\begin{aligned} y(1) = 0 &= c_1 \cos(\sqrt{\lambda}\ln(1)) + c_2 \sin(\sqrt{\lambda}\ln(1)) \\ &= c_1 \\ y(e^\pi) = 0 &= c_2 \sin(\sqrt{\lambda}\ln(e^\pi)) \\ \pi\sqrt{\lambda} &= \pi n \\ \lambda &= n^2 \text{ for } n = 0, 1, 2, \ldots \end{aligned}$$

$$y = \sin(n \ln x)$$

3.9.8 23

$L/4$, $2L/4$, and $3L/4$

3.9.9 27

$$\begin{aligned} Ty'' + \rho\omega^2 y &= 0 \\ y'' + \frac{\rho\omega^2}{T} y &= 0 \\ m^2 + \frac{\rho\omega^2}{T} &= 0 \\ \left(m + \sqrt{\frac{\rho\omega^2}{T}} i\right)\left(m - \sqrt{\frac{\rho\omega^2}{T}} i\right) &= 0 \end{aligned}$$

$$y = c_1 \cos\sqrt{\frac{\rho\omega^2}{T}} x + c_2 \sin\sqrt{\frac{\rho\omega^2}{T}} x$$

$$\begin{aligned} &y(0) = 0 = c_1 \\ &y(L) = 0 = c_2 \sin\sqrt{\frac{\rho\omega^2}{T}} (L) \\ &\sqrt{\frac{\rho\omega^2}{T}} L = n\pi \end{aligned}$$

$$\omega = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}} \text{ for } n = 0, 1, 2, \ldots$$

$$y = \sin\frac{n\pi}{L} x$$

3.9.10 29

$$\begin{aligned} ru'' + 2u' &= 0 \\ r^2 u'' + 2ru' &= 0 \\ m(m-1) + 2m &= 0 \\ m^2 - m + 2m &= 0 \\ m^2 + m &= 0 \\ m(m+1) &= 0 \end{aligned}$$

$$
\begin{aligned}
u &= c_1 + c_2 r^{-1} \\
u(a) = u_0 &= c_1 + c_2(a)^{-1} \\
u(b) = u_1 &= c_1 + c_2(b)^{-1} \\
u_0 - u_1 &= \frac{c_2}{a} - \frac{c_2}{b} \\
&= c_2 \frac{b-a}{ab} \\
c_2 &= \frac{ab(u_0 - u_1)}{b-a} \\
u_0 &= c_1 + \frac{b(u_0 - u_1)}{b-a} \\
c_1 &= u_0 - \frac{b(u_0 - u_1)}{b-a} \\
&= \frac{u_0(b-a) - b(u_0 - u_1)}{b-a} \\
&= \frac{bu_1 - au_0}{b-a}
\end{aligned}
$$

$$
u = \frac{u_0 - u_1}{b-a} \frac{ab}{r} + \frac{bu_1 - au_0}{b-a}
$$

3.10 Green's Functions

3.10.1 1

$$
\begin{aligned}
y'' - 16y &= 0 \\
m^2 - 16 &= 0 \\
(m+4)(m-4) &= 0 \\
y_1 &= e^{-4x} \\
y_2 &= e^{4x} \\
W(t) &= y_1(t)y_2'(t) - y_1'(t)y_2(t) \\
&= 8 \\
G(x,t) &= \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} \\
&= \frac{e^{4(x-t)} - e^{-4(x-t)}}{8} \\
&= \frac{1}{4}\sinh 4(x-t) \\
y_p(x) &= \int_{x_0}^{x} G(x,t)f(t)\,dt \\
&= \int_{x_0}^{x} \frac{1}{4}\sinh 4(x-t)f(t)\,dt
\end{aligned}
$$

3.10.2 3

$$\begin{aligned}
y'' + 2y' + y &= 0 \\
m^2 + 2m + 1 &= 0 \\
(m+1)^2 &= 0 \\
y_1 &= e^{-x} \\
y_2 &= xe^{-x} \\
W(t) &= e^{-t}(e^{-t} - te^{-t}) + te^{-2t} \\
&= e^{-2t} \\
G(x,t) &= \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} \\
&= \frac{e^{-t}xe^{-x} - e^{-x}te^{-t}}{e^{-2t}} \\
&= xe^{t-x} - te^{t-x} \\
&= (x-t)e^{t-x} \\
y_p(x) &= \int_{x_0}^{x} G(x,t)f(t)\,dt \\
&= \int_{x_0}^{x} (x-t)e^{t-x}f(t)\,dt
\end{aligned}$$

3.10.3 5

$$\begin{aligned}
y'' + 9y &= 0 \\
m^2 + 9 &= 0 \\
(m+3i)(m-3i) &= 0 \\
y_1 &= \cos 3x \\
y_2 &= \sin 3x \\
W(t) &= 3\cos^2 3x + 3\sin^2 3x \\
&= 3 \\
G(x,t) &= \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} \\
&= \frac{(\cos 3t)\sin 3x - (\cos 3x)\sin 3t}{3} \\
&= \frac{1}{3}\sin 3(x-t) \\
y_p(x) &= \int_{x_0}^{x} G(x,t)f(t)\,dt \\
&= \int_{x_0}^{x} \frac{1}{3}(\sin 3(x-t))f(t)\,dt
\end{aligned}$$

3.10.4 7

$$y = c_1 e^{-4x} + c_2 e^{4x} + \int_{x_0}^{x} \frac{1}{4} \sinh 4(x-t) t e^{-2t} \, dt$$

3.10.5 9

$$y = c_1 e^{-x} + c_2 x e^{-x} + \int_{x_0}^{x} (x-t) e^{t-x} e^{-t} \, dt$$

3.10.6 11

$$y = c_1 \cos 3x + c_2 \sin 3x + \int_{x_0}^{x} \frac{1}{3} (\sin 3(x-t))(t + \sin t) \, dt$$

3.10.7 13

$$
\begin{aligned}
y'' - 4y &= 0\\
m^2 - 4 &= 0\\
(m+2)(m-2) &= 0\\
y_1 &= e^{-2x}\\
y_2 &= e^{2x}\\
y_h &= c_1 e^{-2x} + c_2 e^{2x}\\
0 &= c_1 e^{-2(0)} + c_2 e^{2(0)}\\
&= c_1 + c_2\\
0 &= -2c_1 e^{-2(0)} + 2c_2 e^{2(0)}\\
&= -2c_1 + 2c_2\\
c_1 &= 0\\
c_2 &= 0\\
W(t) &= 4\\
G(x,t) &= \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}\\
&= \frac{1}{2}\sinh 2(x-t)\\
y_p &= \int_{x_0}^{x} G(x,t)f(t)\,dt\\
&= \int_0^x \frac{1}{2}(\sinh 2(x-t))e^{2t}\,dt\\
&= \frac{1}{2}\int_0^x \frac{e^{2x} - e^{4t-2x}}{2}\,dt\\
&= \frac{1}{4}xe^{2x} - \frac{1}{8}\sinh 2x\\
y &= y_h + y_p\\
&= \frac{1}{4}xe^{2x} - \frac{1}{8}\sinh 2x
\end{aligned}
$$

3.10.8 15

$$
\begin{aligned}
y'' - 10y' + 25y &= 0 \\
m^2 - 10m + 25 &= 0 \\
(m-5)^2 &= 0 \\
y_1 &= e^{5x} \\
y_2 &= xe^{5x} \\
y_h &= c_1 e^{5x} + c_2 x e^{5x} \\
0 &= c_1 e^{5(0)} + c_2 (0) e^{5(0)} \\
&= c_1 \\
0 &= c_2 e^{5(0)} + 5c_2 (0) e^{5(0)} \\
&= c_2 \\
y_h &= 0 \\
W(t) &= e^{10x} \\
G(x,t) &= \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} \\
&= (x-t)e^{5(x-t)} \\
y_p &= \int_0^x G(x,t) f(t)\, dt \\
&= \int_0^x (x-t)e^{5(x-t)} e^{5t}\, dt \\
&= \int_0^x (x-t)e^{5x}\, dt \\
&= \frac{1}{2} x^2 e^{5x} \\
y &= y_h + y_p \\
&= y_p
\end{aligned}
$$

3.10.9 19

$$
\begin{aligned}
y'' - 4y &= 0 \\
y_h &= c_1 e^{-2x} + c_2 e^{2x} \\
1 &= c_1 e^{-2(0)} + c_2 e^{2(0)} \\
&= c_1 + c_2 \\
-4 &= -2c_1 e^{-2(0)} + 2c_2 e^{2(0)} \\
&= -2c_1 + 2c_2 \\
&= -2c_1 + 2(1 - c_1) \\
&= -4c_1 + 2 \\
c_1 &= \frac{3}{2} \\
c_2 &= -\frac{1}{2} \\
y_h &= \frac{3}{2}e^{-2x} - \frac{1}{2}e^{2x} \\
y &= y_h + y_p \\
&= \frac{3}{2}e^{-2x} - \frac{1}{2}e^{2x} + \frac{1}{4}xe^{2x} - \frac{e^{2x} - e^{-2x}}{16} \\
&= \frac{25}{16}e^{-2x} - \frac{9}{16}e^{2x} + \frac{1}{4}xe^{2x}
\end{aligned}
$$

3.10.10 21

$$
\begin{aligned}
y_h &= c_1 e^{5x} + c_2 x e^{5x} \\
-1 &= c_1 e^{5(0)} + c_2 (0) e^{5(0)} \\
&= c_1 \\
1 &= -5e^{5(0)} + c_2 e^{5(0)} + 5c_2(0)e^{5(0)} \\
&= -5 + c_2 \\
c_2 &= 6 \\
y_h &= -e^{5x} + 6xe^{5x} \\
y &= y_h + y_p \\
&= -e^{5x} + 6xe^{5x} + \frac{1}{2}x^2 e^{5x}
\end{aligned}
$$

3.10.11 31

$$
\begin{aligned}
y'' - y &= 0 \\
m^2 - 1 &= 0 \\
(m+1)(m-1) &= 0 \\
y_1 &= e^{-x} \\
y_2 &= e^{x} \\
y_h &= c_1 e^{-x} + c_2 e^{x} \\
8 &= c_1 e^{-(0)} + c_2 e^{(0)} \\
&= c_1 + c_2 \\
2 &= -c_1 e^{-(0)} + c_2 e^{(0)} \\
&= -c_1 + c_2 \\
&= -c_1 + (8 - c_1) \\
c_1 &= 3 \\
c_2 &= 5 \\
y_h &= 3e^{-x} + 5e^{x} \\
y_p &= \begin{cases} \int_0^x -\sinh(x-t)\,dt & \text{for } x < 0 \\ \int_0^x \sinh(x-t)\,dt & \text{for } x \geq 0 \end{cases} \\
&= \begin{cases} 1 - \cosh x & \text{for } x < 0 \\ (\cosh x) - 1 & \text{for } x \geq 0 \end{cases} \\
y &= \begin{cases} \frac{5}{2}e^{-x} + \frac{9}{2}e^{x} + 1 & \text{for } x < 0 \\ \frac{7}{2}e^{-x} + \frac{11}{2}e^{x} - 1 & \text{for } x \geq 0 \end{cases}
\end{aligned}
$$

3.10.12 35

$$
\begin{aligned}
y'' &= 0 \\
y_c &= c_1 + c_2 x \\
y_1 &= x \\
y_2 &= x - 1 \\
W(t) &= \begin{vmatrix} x & x-1 \\ 1 & 1 \end{vmatrix} \\
&= x - (x-1) \\
&= 1 \\
G(x,t) &= \begin{cases} t(x-1) & 0 \le t \le x \\ x(t-1) & x \le t \le 1 \end{cases} \\
y_p(x) &= \int_a^b G(x,t) f(t)\, dt \\
&= \int_0^x t(x-1) f(t)\, dt + \int_x^1 x(t-1) f(t)\, dt \\
&= (x-1) \int_0^x t f(t)\, dt + x \int_x^1 (t-1) f(t)\, dt
\end{aligned}
$$

3.10.13 37

$$
\begin{aligned}
y &= (x-1) \int_0^x t\, dt + x \int_x^1 (t-1)\, dt \\
&= \frac{1}{2}(x-1)x^2 + x \left(\frac{1}{2} - 1 - \frac{1}{2}x^2 + x \right) \\
&= \frac{1}{2}x^2 - \frac{1}{2}x^2 + \frac{1}{2}x - x - \frac{1}{2}x^2 + x^2 \\
&= \frac{1}{2}x(x-1)
\end{aligned}
$$

3.11 Nonlinear Models

3.11.1 7

$$
\begin{aligned}
x'' + xe^{0.01x} &= 0 \\
x'' + x &= 0
\end{aligned}
$$

3.11.2 15

(a)

$$\begin{aligned} xx'' + (x')^2 + 32x &= 160 \\ 32k &= 160 \\ k &= 5 \end{aligned}$$

(b) Because there would be 5 lb of rope in the air which equals the upwards force

(c) $v = \sqrt{160} = 4\sqrt{10} = 12.65\,\text{ft/s}$

3.11.3 17

(a)

$$\begin{aligned} \theta_1'' + \frac{g}{l}\theta_1 &= 0 \\ \theta_1'' + \omega_1^2\theta_1 &= 0 \\ (m - \omega_1 i)(m + \omega_1 i) &= 0 \\ \theta_{1c} &= c_1 \cos\omega_1 t + c_2 \sin\omega_1 t \\ \theta_0 &= c_1 \cos\omega_1(0) + c_2 \sin\omega_1(0) \\ &= c_1 \\ 0 &= -\theta_0 \sin\omega_1(0) + c_2 \cos\omega_1(0) \\ &= c_2 \\ \theta_1 &= \theta_0 \cos\omega_1 t \end{aligned}$$

(b)

$$\begin{aligned} \theta_0 \cos\omega_1 T &= 0 \\ \omega_1 T &= \arccos 0 \\ T &= \frac{\arccos 0}{\omega_1} \\ &= \frac{\pi}{2}\sqrt{\frac{l}{g}} \end{aligned}$$

(c)

$$
\begin{aligned}
\theta_2 &= c_3 \cos \omega_2 t + c_4 \sin \omega_2 t \\
0 &= c_3 \cos \sqrt{\frac{4g}{l}} \frac{\pi}{2} \sqrt{\frac{l}{g}} + c_4 \sin \sqrt{\frac{4g}{l}} \frac{\pi}{2} \sqrt{\frac{l}{g}} \\
&= c_3 \cos \pi + c_4 \sin \pi \\
&= -c_3 \\
\theta_1' &= -\omega_1 \theta_0 \sin \omega_1 t \\
l\theta_1'(T) &= \frac{l}{4} \theta_2'(T) \\
-l\omega_1 \theta_0 \sin \omega_1 T &= \frac{l}{4} c_4 \omega_2 \cos \omega_2 T \\
-l\sqrt{\frac{g}{l}} \theta_0 \sin \sqrt{\frac{g}{l}} \frac{\pi}{2} \sqrt{\frac{l}{g}} &= \frac{l}{4} c_4 \sqrt{\frac{4g}{l}} \cos \sqrt{\frac{4g}{l}} \frac{\pi}{2} \sqrt{\frac{l}{g}} \\
-\theta_0 \sin \frac{\pi}{2} &= \frac{1}{2} c_4 \cos \pi \\
-\theta_0 &= -\frac{1}{2} c_4 \\
c_4 &= 2\theta_0 \\
\theta_2 &= 2\theta_0 \sin 2\sqrt{\frac{g}{l}} t
\end{aligned}
$$

3.12 Solving Systems of Linear Equations

3.12.1 1

$$
\begin{aligned}
(D-2)x + y &= 0 \\
Dy - x &= 0 \\
D(D-2)x + Dy - Dy + x &= 0 \\
(D^2 - 2D + 1)x &= 0 \\
(D-1)^2 x &= 0 \\
x &= c_1 e^t + c_2 t e^t \\
(D-2)x + y + D(D-2)y - (D-2)x &= 0 \\
(D^2 - 2D + 1)y &= 0 \\
(D-1)^2 y &= 0 \\
y &= c_3 e^t + c_4 t e^t \\
c_3 e^t + c_4 e^t + c_4 t e^t - c_1 e^t - c_2 t e^t &= 0 \\
(c_4 - c_2) t e^t + (c_3 + c_4 - c_1) e^t &= 0 \\
c_4 &= c_2 \\
c_3 &= c_1 - c_2 \\
x &= c_1 e^t + c_2 t e^t \\
y &= (c_1 - c_2) e^t + c_2 t e^t
\end{aligned}
$$

3.12.2 3

$$
\begin{aligned}
Dx + y &= t \\
Dy - x &= -t \\
D^2x + Dy - Dy + x &= 1 + t \\
(D^2+1)x &= t + 1 \\
x_c &= c_1 \cos t + c_2 \sin t \\
x_p &= At + B \\
x_p' &= A \\
x_p'' &= 0 \\
0 + At + B &= t + 1 \\
A &= 1 \\
B &= 1 \\
x_p &= t + 1 \\
x &= c_1 \cos t + c_2 \sin t + t + 1 \\
Dx + y + D^2y - Dx &= t - 1 \\
(D^2+1)y &= t - 1 \\
y_c &= c_3 \cos t + c_4 \sin t \\
y_p &= At + B \\
y_p' &= A \\
y_p'' &= 0 \\
0 + At + B &= t - 1 \\
A &= 1 \\
B &= -1 \\
y_p &= t - 1 \\
y &= c_3 \cos t + c_4 \sin t + t - 1 \\
-c_1 \sin t + c_2 \cos t + 1 + c_3 \cos t + c_4 \sin t + t - 1 &= t \\
(c_2 + c_3) \cos t + (c_4 - c_1) \sin t + t &= t \\
c_3 &= -c_2 \\
c_4 &= c_1 \\
x &= c_1 \cos t + c_2 \sin t + t + 1 \\
y &= -c_2 \cos t + c_1 \sin t + t - 1
\end{aligned}
$$

3.12.3 5

$$\begin{aligned}
(D^2+5)x-2y&=0\\
-2x+(D^2+2)y&=0\\
(D^2+5)(D^2+2)x-2(D^2+2)y-4x+2(D^2+2)y&=0\\
(D^4+7D^2+6)x&=0\\
(D^2+1)(D^2+6)x&=0
\end{aligned}$$

$$x=c_1\cos t+c_2\sin t+c_3\cos\sqrt{6}t+c_4\sin\sqrt{6}t$$

$$\begin{aligned}
2(D^2+5)x-4y-2(D^2+5)x+(D^2+2)(D^2+5)y&=0\\
(D^4+7D^2+6)y&=0\\
(D^2+1)(D^2+6)&=0
\end{aligned}$$

$$y=c_5\cos t+c_6\sin t+c_7\cos\sqrt{6}t+c_8\sin\sqrt{6}t$$

$$\begin{aligned}
0&=-c_1\cos t-c_2\sin t-6c_3\cos\sqrt{6}t-6c_4\sin\sqrt{6}t+5c_1\cos t+5c_2\sin t\\
&\quad+5c_3\cos\sqrt{6}t+5c_4\sin\sqrt{6}t-2c_5\cos t-2c_6\sin t-2c_7\cos\sqrt{6}t\\
&\quad-2c_8\sin\sqrt{6}t\\
&=(-c_1+5c_1-2c_5)\cos t+(-c_2+5c_2-2c_6)\sin t\\
&\quad+(-6c_3+5c_3-2c_7)\cos\sqrt{6}t+(-6c_4+5c_4-2c_8)\sin\sqrt{6}t\\
&=(4c_1-2c_5)\cos t+(4c_2-2c_6)\sin t-(c_3+2c_7)\cos\sqrt{6}t\\
&\quad-(c_4+2c_8)\sin\sqrt{6}t\\
c_5&=2c_1\\
c_6&=2c_2\\
c_7&=-\frac{1}{2}c_3\\
c_8&=-\frac{1}{2}c_4\\
x&=c_1\cos t+c_2\sin t+c_3\cos\sqrt{6}t+c_4\sin\sqrt{6}t\\
y&=2c_1\cos t+2c_2\sin t-\frac{1}{2}c_3\cos\sqrt{6}t-\frac{1}{2}c_4\sin\sqrt{6}t
\end{aligned}$$

3.12.4 7

$$
\begin{aligned}
D^2x - 4y &= e^t \\
-4x + D^2y &= -e^t \\
D^4x - 4D^2y - 16x + 4D^2y &= e^t - 4e^t \\
(D^4 - 16)x &= -3e^t \\
(D^2+4)(D^2-4)x &= -3e^t
\end{aligned}
$$

$$x_c = c_1 \cos 2t + c_2 \sin 2t + c_3 e^{-2t} + c_4 e^{2t}$$

$$
\begin{aligned}
x_p^{(n)} &= Ae^t \\
Ae^t - 16Ae^t &= -3e^t \\
-15Ae^t &= -3e^t \\
A &= \frac{1}{5} \\
x_p &= \frac{1}{5}e^t \\
x &= c_1 \cos 2t + c_2 \sin 2t + c_3 e^{-2t} + c_4 e^{2t} + \frac{1}{5}e^t
\end{aligned}
$$

$$
\begin{aligned}
4D^2x - 16y - 4D^2x + D^4y &= 4e^t - e^t \\
(D^4 - 16)y &= 3e^t \\
(D^2+4)(D^2-4)y &= 3e^t
\end{aligned}
$$

$$y_c = c_5 \cos 2t + c_6 \sin 2t + c_7 e^{-2t} + c_8 e^{2t}$$

$$
\begin{aligned}
y_p^{(n)} &= Ae^t \\
Ae^t - 16Ae^t &= 3e^t \\
-15Ae^t &= 3e^t \\
A &= -\frac{1}{5} \\
y_p &= -\frac{1}{5}e^t \\
y &= c_5 \cos 2t + c_6 \sin 2t + c_7 e^{-2t} + c_8 e^{2t} - \frac{1}{5}e^t
\end{aligned}
$$

$$
\begin{aligned}
e^t &= -4c_1\cos 2t - 4c_2\sin 2t + 4c_3e^{-2t} + 4c_4e^{2t} + \frac{1}{5}e^t - 4c_5\cos 2t - 4c_6\sin 2t \\
&\quad - 4c_7e^{-2t} - 4c_8e^{2t} + \frac{4}{5}e^t \\
&= -4(c_1+c_5)\cos 2t - 4(c_2+c_6)\sin 2t + 4(c_3-c_7)e^{-2t} + 4(c_4-c_8)e^{2t} \\
&\quad + e^t \\
c_5 &= -c_1 \\
c_6 &= -c_2 \\
c_7 &= c_3 \\
c_8 &= c_4 \\
x &= c_1\cos 2t + c_2\sin 2t + c_3e^{-2t} + c_4e^{2t} + \frac{1}{5}e^t \\
y &= -c_1\cos 2t - c_2\sin 2t + c_3e^{-2t} + c_4e^{2t} - \frac{1}{5}e^t
\end{aligned}
$$

3.12.5 21

$$
\begin{aligned}
(D+5)x + y &= 0 \\
-4x + (D+1)y &= 0 \\
(D+1)(D+5)x + (D+1)y + 4x - (D+1)y &= 0 \\
(D^2+6D+9)x &= 0 \\
(D+3)^2x &= 0 \\
x &= c_1e^{-3t} + c_2te^{-3t} \\
4(D+5)x + 4y - 4(D+5)x + (D+1)(D+5)y &= 0 \\
(D^2+6D+9)y &= 0 \\
(D+3)^2y &= 0 \\
y &= c_3e^{-3t} + c_4te^{-3t}
\end{aligned}
$$

$$
\begin{aligned}
0 &= -3c_1e^{-3t} + c_2e^{-3t} - 3c_2te^{-3t} + 5c_1e^{-3t} + 5c_2te^{-3t} + c_3e^{-3t} + c_4te^{-3t} \\
&= (2c_1 + c_2 + c_3)e^{-3t} + (2c_2 + c_4)te^{-3t} \\
c_4 &= -2c_2 \\
c_3 &= -2c_1 - c_2 \\
x &= c_1e^{-3t} + c_2te^{-3t} \\
y &= -(2c_1 + c_2)e^{-3t} - 2c_2te^{-3t}
\end{aligned}
$$

$$
\begin{aligned}
0 &= c_1 e^{-3(1)} + c_2(1)e^{-3(1)} \\
&= c_1 e^{-3} + c_2 e^{-3} \\
&= c_1 + c_2 \\
1 &= -(2c_1 + c_2)e^{-3(1)} - 2c_2(1)e^{-3(1)} \\
e^3 &= -2c_1 - 3c_2 \\
c_2 &= -e^3 \\
c_1 &= e^3 \\
x &= e^{3(1-t)} - te^{3(1-t)} \\
y &= -e^{3(1-t)} + 2te^{3(1-t)}
\end{aligned}
$$

3.12.6 23

$$
\begin{aligned}
mx'' &= 0 \\
my'' &= -mg \\
x &= c_1 t + c_2 \\
y &= -\frac{1}{2}gt^2 + c_3 t + c_4
\end{aligned}
$$

3.13 Chapter in Review

3.13.1 1

$y = 0$

3.13.2 3

False

3.13.3 5

8

3.13.4 7

$$
\begin{aligned}
x'' + 16x &= 0 \\
m^2 + 16 &= 0 \\
(m+4i)(m-4i) &= 0 \\
x &= c_1 \cos 4t + c_2 \sin 4t \\
1 &= c_1 \cos 4(0) + c_2 \sin 4(0) \\
&= c_1 \\
-3 &= -4 \sin 4(0) + 4c_2 \cos 4(0) \\
&= 4c_2 \\
c_2 &= -\frac{3}{4} \\
x &= \cos 4t - \frac{3}{4} \sin 4t
\end{aligned}
$$

The amplitude is $\sqrt{1^2 + \left(-\frac{3}{4}\right)^2} = 1.25\,\mathrm{m}$

3.13.5 9

$(-\infty, \infty)$
$(0, \infty)$

3.13.6 11

(a) $y = c_1 e^{3x} + c_2 e^{-5x} + c_3 x e^{-5x} + c_4 e^x + c_5 x e^5 + c_6 x^2 e^x$

(b) $y = c_1 x^3 + c_2 x^{-5} + c_3 (\ln x) x^{-5} + c_4 x + c_5 (\ln x) x + c_6 (\ln x)^2 x^2$

3.13.7 13

$$
\begin{aligned}
y'' - 2y' - 2y &= 0 \\
m^2 - 2m - 2 &= 0 \\
(m - (1+\sqrt{3}))(m - (1-\sqrt{3})) &= 0 \\
y &= c_1 e^{(1+\sqrt{3})x} + c_2 e^{(1-\sqrt{3})x}
\end{aligned}
$$

3.13.8 15

$$
\begin{aligned}
y''' + 10y'' + 25y' &= 0 \\
m^3 + 10m^2 + 25m &= 0 \\
m(m^2 + 10m + 25) &= 0 \\
m(m+5)^2 &= 0 \\
y &= c_1 + c_2 e^{-5x} + c_3 x e^{-5x}
\end{aligned}
$$

3.13.9 17

$$
\begin{aligned}
3y''' + 10y'' + 15y' + 4y &= 0 \\
3m^3 + 10m^2 + 15m + 4 &= 0 \\
\left(m + \frac{1}{3}\right)(m^2 + 3m + 4) &= 0 \\
\left(m + \frac{1}{3}\right)\left(m - \left(-\frac{3}{2} + \frac{\sqrt{7}}{2}i\right)\right)\left(m - \left(-\frac{3}{2} - \frac{\sqrt{7}}{2}i\right)\right) &= 0
\end{aligned}
$$

$$
y = c_1 e^{-x/3} + e^{-3x/2}\left(c_2 \cos\frac{\sqrt{7}}{2}x + c_3 \sin\frac{\sqrt{7}}{2}x\right)
$$

3.13.10 19

$$
\begin{aligned}
y'' - 3y' + 5y &= 4x^3 - 2x \\
m^2 - 3m + 5 &= 0 \\
\left(m - \left(\frac{3}{2} + \frac{\sqrt{11}}{2}i\right)\right)\left(m - \left(\frac{3}{2} - \frac{\sqrt{11}}{2}i\right)\right) &= 0
\end{aligned}
$$

$$
y_c = e^{3x/2}\left(c_1 \cos\frac{\sqrt{11}}{2}x + c_2 \sin\frac{\sqrt{11}}{2}x\right)
$$

$$
\begin{aligned}
y_p &= Ax^3 + Bx^2 + Cx + D \\
y_p' &= 3Ax^2 + 2Bx + C \\
y_p'' &= 6Ax + 2B \\
4x^3 - 2x &= 6Ax + 2B - 3(3Ax^2 + 2Bx + C) + 5(Ax^3 + Bx^2 + Cx + D) \\
&= 5Ax^3 + (-9A + 5B)x^2 + (6A - 6B + 5C)x + 2B - 3C + 5D \\
A &= \frac{4}{5} \\
B &= \frac{36}{25} \\
C &= \frac{46}{125} \\
D &= -\frac{222}{625} \\
y_p &= \frac{4}{5}x^3 + \frac{36}{25}x^2 + \frac{46}{125}x - \frac{222}{625} \\
y &= e^{3x/2}\left(c_1 \cos\frac{\sqrt{11}}{2}x + c_2 \sin\frac{\sqrt{11}}{2}x\right) + \frac{4}{5}x^3 + \frac{36}{25}x^2 + \frac{46}{125}x - \frac{222}{625}
\end{aligned}
$$

3.13.11 21

$$\begin{aligned}
y''' - 5y'' + 6y' &= 8 + 2\sin x \\
m^3 - 5m^2 + 6m &= 0 \\
m(m^2 - 5m + 6) &= 0 \\
m(m-2)(m-3) &= 0 \\
y_c &= c_1 + c_2e^{2x} + c_3e^{3x} \\
y_p &= Ax + B\cos x + C\sin x \\
y_p' &= A - B\sin x + C\cos x \\
y_p'' &= -B\cos x - C\sin x \\
y_p''' &= B\sin x - C\cos x \\
8 + 2\sin x &= B\sin x - C\cos x - 5(-B\cos x - C\sin x) \\
&\quad + 6(A - B\sin x + C\cos x) \\
&= 6A + 5(B+C)\cos x + 5(C-B)\sin x \\
A &= \frac{4}{3} \\
0 &= B + C \\
2 &= 5C - 5B \\
&= -10B \\
B &= -\frac{1}{5} \\
C &= \frac{1}{5} \\
y_p &= \frac{4}{3}x - \frac{1}{5}\cos x + \frac{1}{5}\sin x \\
y &= c_1 + c_2e^{2x} + c_3e^{3x} + \frac{4}{3}x - \frac{1}{5}\cos x + \frac{1}{5}\sin x
\end{aligned}$$

3.13.12 23

$$
\begin{aligned}
y'' - 2y' + 2y &= e^x \tan x \\
m^2 - 2m + 2 &= 0 \\
(m - (1+i))(m - (1-i)) &= 0 \\
y_c &= c_1 e^x \cos x + c_2 e^x \sin x \\
y_1 &= e^x \cos x \\
y_2 &= e^x \sin x \\
W &= \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \sin x + e^x \cos x \end{vmatrix} \\
&= e^x \cos x (e^x \sin x + e^x \cos x) \\
&\quad - e^x \sin x (e^x \cos x - e^x \sin x) \\
&= e^{2x} (\cos x) \sin x + e^{2x} \cos^2 x - e^{2x} (\cos x) \sin x \\
&\quad + e^{2x} \sin^2 x \\
&= e^{2x}
\end{aligned}
$$

$$
\begin{aligned}
u_1' &= e^{-2x} \begin{vmatrix} 0 & e^x \sin x \\ e^x \tan x & e^x \sin x + e^x \cos x \end{vmatrix} \\
&= -(\sin x) \tan x \\
u_1 &= \sin x - \ln(\sec x + \tan x) \\
u_2' &= e^{-2x} \begin{vmatrix} e^x \cos x & 0 \\ e^x \cos x - e^x \sin x & e^x \tan x \end{vmatrix} \\
&= \sin x \\
u_2 &= -\cos x \\
y_p &= u_1 y_1 + u_2 y_2 \\
&= e^x \cos x (\sin x - \ln(\sec x + \tan x)) - e^x (\cos x) \sin x \\
&= -e^x (\cos x) \ln(\sec x + \tan x) \\
y &= c_1 e^x \cos x + c_2 e^x \sin x - e^x (\cos x) \ln(\sec x + \tan x)
\end{aligned}
$$

3.13.13 25

$$
\begin{aligned}
6x^2 y'' + 5xy' - y &= 0 \\
6m(m-1) + 5m - 1 &= 0 \\
6m^2 - m - 1 &= 0 \\
\left(m + \frac{1}{3}\right)\left(m - \frac{1}{2}\right) &= 0 \\
y &= c_1 x^{-1/3} + c_2 x^{1/2}
\end{aligned}
$$

3.13.14 27

$$\begin{aligned}
x^2y'' - 4xy' + 6y &= 2x^4 + x^2 \\
m(m-1) - 4m + 6 &= 0 \\
m^2 - 5m + 6 &= 0 \\
(m-3)(m-2) &= 0 \\
y_c &= c_1x^2 + c_2x^3 \\
y'' - 4x^{-1}y' + 6x^{-2}y &= 2x^2 + 1 \\
y_1 &= x^2 \\
y_2 &= x^3 \\
W &= \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} \\
&= x^4 \\
u_1' &= x^{-4}\begin{vmatrix} 0 & x^3 \\ 2x^2+1 & 3x^2 \end{vmatrix} \\
&= -2x - x^{-1} \\
u_1 &= -x^2 - \ln x \\
u_2' &= x^{-4}\begin{vmatrix} x^2 & 0 \\ 2x & 2x^2+1 \end{vmatrix} \\
&= 2 + x^{-2} \\
u_2 &= 2x - x^{-1} \\
y_p &= u_1y_1 + u_2y_2 \\
&= (-x^2 - \ln x)x^2 + (2x - x^{-1})x^3 \\
&= x^4 - x^2\ln x - x^2 \\
y &= c_1x^2 + c_2x^3 + x^4 - x^2\ln x
\end{aligned}$$

3.13.15 29

(a)

$$y'' + \omega^2 y = \sin\alpha x$$

(a) $\omega \neq \alpha$

$$y = c_1\cos\omega x + c_2\sin\omega x + A\cos\alpha x + B\sin\alpha x$$

(b) $\omega = \alpha$

$$y = c_1\cos\omega x + c_2\sin\omega x + Ax\cos\alpha x + Bx\sin\alpha x$$

(b)

$$y'' - \omega^2 y = e^{\alpha x}$$

(a) $\omega \neq \alpha$

$$y = c_1 e^{-\omega x} + c_2 e^{\omega x} + A e^{\alpha x}$$

(b) $\omega = \alpha$

$$y = c_1 e^{-\omega x} + c_2 e^{\omega x} + A x e^{\alpha x}$$

3.13.16 33

$$\begin{aligned}
y'' - 2y' + 2y &= 0 \\
m^2 - 2m + 2 &= 0 \\
(m - (1 - i))(m - (1 + i)) &= 0 \\
y_c &= c_1 e^x \cos x + c_2 e^x \sin x \\
0 &= c_1 e^{\pi/2} \cos \frac{\pi}{2} + c_2 e^{\pi/2} \sin \frac{\pi}{2} \\
&= c_2 e^{pi/2} \\
&= c_2 \\
-1 &= c_1 e^{\pi} \cos \pi \\
c_1 &= e^{-\pi} \\
y &= e^{x-\pi} \cos x
\end{aligned}$$

3.13.17 35

$$
\begin{aligned}
y'' - y &= x + \sin x \\
m^2 - 1 &= 0 \\
(m+1)(m-1) &= 0 \\
y_c &= c_1 e^{-x} + c_2 e^{x} \\
y_p &= Ax + B + C\cos x + D\sin x \\
y_p' &= A - C\sin x + D\cos x \\
y_p'' &= -C\cos x - D\sin x \\
x + \sin x &= -C\cos x - D\sin x - Ax - B - C\cos x - D\sin x \\
&= -Ax - B - 2C\cos x - 2D\sin x \\
A &= -1 \\
B &= 0 \\
C &= 0 \\
D &= -\frac{1}{2} \\
y_p &= -x - \frac{1}{2}\sin x \\
y &= c_1 e^{-x} + c_2 e^{x} - x - \frac{1}{2}\sin x \\
2 &= c_1 e^{-(0)} + c_2 e^{(0)} - (0) - \frac{1}{2}\sin(0) \\
&= c_1 + c_2 \\
3 &= -c_1 e^{-(0)} + c_2 e^{(0)} - 1 - \frac{1}{2}\cos(0) \\
&= -c_1 + c_2 - 1 - \frac{1}{2} \\
\frac{9}{2} &= -c_1 + c_2 \\
&= -c_1 + (2 - c_1) \\
&= 2 - 2c_1 \\
c_1 &= -\frac{5}{4} \\
c_2 &= \frac{13}{4} \\
y &= -\frac{5}{4}e^{-x} + \frac{13}{4}e^{x} - x - \frac{1}{2}\sin x
\end{aligned}
$$

3.13.18 37

$$
\begin{aligned}
y'y'' &= 4x \\
uu' &= 4x \\
\frac{1}{2}u^2 &= 2x^2 + c_1 \\
\frac{1}{2}(y')^2 &= 2x^2 + c_1 \\
y' &= \sqrt{4x^2 + c_1} \\
2 &= \sqrt{4(1)^2 + c_1} \\
c_1 &= 0 \\
y' &= 2x \\
y &= x^2 + c_2 \\
5 &= (1)^2 + c_2 \\
c_2 &= 4 \\
y &= x^2 + 4
\end{aligned}
$$

3.13.19 41

$$
\begin{aligned}
(D-2)x + (D-2)y &= 1 \\
Dx + (2D-1)y &= 3 \\
(2D-1)1 - (D-2)3 &= (2D-1)(D-2)x + (2D-1)(D-2)y \\
&\quad - D(D-2)x - (2D-1)(D-2)y \\
5 &= (D^2 - 3D + 2)x \\
x_c &= c_1 e^t + c_2 e^{2t} \\
x_p &= \frac{5}{2} \\
x &= c_1 e^t + c_2 e^{2t} + \frac{5}{2} \\
D1 - (D-2)3 &= D(D-2)x + D(D-2)y - D(D-2)x \\
&\quad - (D-2)(2D-1)y \\
-6 &= (D^2 - 3D + 2)y \\
y_c &= c_3 e^t + c_4 e^{2t} \\
y_p &= -3 \\
y &= c_3 e^t + c_4 e^{2t} - 3 \\
1 &= c_1 e^t + 2c_2 e^{2t} - 2c_1 e^t - 2c_2 e^{2t} - 5 + c_3 e^t \\
&\quad + 2c_4 e^{2t} - 2c_3 e^t - 2c_4 e^{2t} + 6 \\
&= -(c_1 + c_3)e^t + 1 \\
c_3 &= -c_1 \\
3 &= c_1 e^t + 2c_2 e^{2t} + 2c_3 e^t + 4c_4 e^{2t} - c_3 e^t - c_4 e^{2t} + 3 \\
&= (c_1 + c_3)e^t + (2c_2 + 3c_4)e^{2t} + 3 \\
c_4 &= -\frac{2}{3} c_2 \\
x &= c_1 e^t + c_2 e^{2t} + \frac{5}{2} \\
y &= -c_1 e^t - \frac{2}{3} c_2 e^{2t} - 3
\end{aligned}
$$

3.13.20 43

$$
\begin{aligned}
(D-2)x - y &= -e^t \\
-3x + (D-4)y &= -7e^t
\end{aligned}
$$

$$
\begin{aligned}
(D-4)(D-2)x-(D-4)y-3x+(D-4)y &= (D-4)-e^t-7e^t \\
(D^2-2D-4D+8)x-3x &= -e^t+4e^t-7e^t \\
(D^2-6D+5)x &= -4e^t \\
(D-1)(D-5)x &= -4e^t \\
x_c &= c_1e^t+c_2e^{5t} \\
x_p &= Ate^t \\
x_p' &= Ae^t+Ate^t \\
x_p'' &= 2Ae^t+Ate^t \\
-4e^t &= 2Ae^t+Ate^t-6(Ae^t+Ate^t) \\
&\quad +5Ate^t \\
&= -4Ae^t \\
A &= 1 \\
x_p &= te^t \\
x &= c_1e^t+c_2e^{5t}+te^t
\end{aligned}
$$

$$
\begin{aligned}
3(D-2)x-3y-3(D-2)x+(D-2)(D-4)y &= -3e^t-7(D-2)e^t \\
(D^2-6D+5)y &= 4e^t \\
(D-1)(D-5)y &= 4e^t \\
y_c &= c_3e^t+c_4e^{5t} \\
y_p &= Ate^t \\
y_p' &= Ae^t+Ate^t \\
y_p'' &= 2Ae^t+Ate^t \\
4e^t &= 2Ae^t+Ate^t \\
&\quad -6(Ae^t+Ate^t)+5Ate^t \\
&= -4Ae^t \\
A &= -1 \\
y_p &= -te^t \\
y &= c_3e^t+c_4e^{5t}-te^t
\end{aligned}
$$

$$
\begin{aligned}
-e^t &= c_1e^t + 5c_2e^{5t} + e^t + te^t - 2c_1e^t - 2c_2e^{5t} - 2te^t - c_3e^t - c_4e^{5t} + te^t \\
&= (1 - c_1 - c_3)e^t + (3c_2 - c_4)e^{5t} \\
-1 &= 1 - c_1 - c_3 \\
c_3 &= 2 - c_1 \\
c_4 &= 3c_2 \\
x &= c_1e^t + c_2e^{5t} + te^t \\
y &= (2 - c_1)e^t + 3c_2e^{5t} - te^t
\end{aligned}
$$

3.13.21 45

$$
\begin{aligned}
T &= \frac{2\pi}{\omega} \\
&= 2\pi\sqrt{\frac{m}{k}} \\
3 &= 2\pi\sqrt{\frac{m}{k}} \\
\left(\frac{3}{2\pi}\right)^2 &= \frac{m}{k} \\
k &= \frac{4\pi^2 m}{9} \\
2 &= 2\pi\sqrt{\frac{m-8}{k}} \\
&= 2\pi\sqrt{\frac{m-8}{\frac{4\pi^2 m}{9}}} \\
1 &= \pi\sqrt{\frac{9(m-8)}{4\pi^2 m}} \\
\frac{1}{\pi^2} &= \frac{9(m-8)}{4\pi^2 m} \\
4m &= 9m - 72 \\
5m &= 72 \\
m &= \frac{72}{5}
\end{aligned}
$$

3.13.22 47

$$
\begin{aligned}
x'' + \frac{\beta}{m}x' + \frac{k}{m}x = 0 \\
x'' + 2\lambda x' + \omega^2 x = 0
\end{aligned}
$$

We want

$$
\begin{aligned}
\lambda^2 - \omega^2 &\geq 0 \\
\left(\frac{\beta}{2m}\right)^2 - \frac{k}{m} &\geq 0 \\
\frac{\beta^2}{4m} - k &\geq 0 \\
\frac{\beta^2}{4m} &\geq k \\
\beta^2 &\geq 4km \\
\frac{\beta^2}{4k} &\geq m \\
\frac{4^2}{8} &\geq m \\
m &\leq 2
\end{aligned}
$$

3.13.23 49

(a)

$$\begin{aligned}
q'' + 10000q &= 100\sin 50t \\
m^2 + 10000 &= 0 \\
(m+100i)(m-100i) &= 0 \\
q_c &= c_1\cos 100t + c_2 \sin 100t \\
q_p &= A\cos 50t + B\sin 50t \\
q_p' &= -50A\sin 50t + 50B\cos 50t \\
q_p'' &= -2500A\cos 50t - 2500B\sin 50t \\
100\sin 50t &= -2500A\cos 50t - 2500B\sin 50t \\
&\quad + 10000(A\cos 50t + B\sin 50t) \\
&= 7500A\cos 50t + 7500B\sin 50t \\
A &= 0 \\
B &= \frac{1}{75} \\
q_p &= \frac{1}{75}\sin 50t \\
q &= c_1\cos 100t + c_2\sin 100t + \frac{1}{75}\sin 50t \\
0 &= c_1\cos 100(0) + c_2\sin 100(0) + \frac{1}{75}\sin 50(0) \\
&= c_1 \\
0 &= 100c_2\cos 100(0) + \frac{2}{3}\cos 50(0) \\
&= 100c_2 + \frac{2}{3} \\
c_2 &= -\frac{1}{150} \\
q &= -\frac{1}{150}\sin 100t + \frac{1}{75}\sin 50t
\end{aligned}$$

(b)

$$i = -\frac{2}{3}\cos 100t + \frac{2}{3}\cos 50t$$

3.13.24 53

$$x'' + \frac{k}{m}x = 0$$

3.13.25 55

$$
\begin{aligned}
y'' + y &= \tan x \\
m^2 + 1 &= 0 \\
(m+i)(m-i) &= 0 \\
y_h &= c_1 \cos x + c_2 \sin x \\
2 &= c_1 \cos(0) + c_2 \sin(0) \\
&= c_1 \\
-5 &= -2\sin(0) + c_2 \cos(0) \\
&= c_2 \\
y_h &= 2\cos x - 5 \sin x \\
y_1 &= \cos x \\
y_2 &= \sin x \\
W(t) &= \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} \\
&= \cos^2 t + \sin^2 t \\
&= 1 \\
G(x,t) &= (\cos t)\sin x - (\cos x)\sin t \\
y_p &= \int_0^x ((\cos t)\sin x - (\cos x)\sin t)\tan t \, dt \\
&= \int_0^x ((\sin x)\sin t - (\cos x)(\sin t)\tan t) \, dt \\
&= \sin x \int_0^x \sin t \, dt - \cos x \int_0^x (\sin t)\tan t \, dt \\
&= \sin x[-\cos t]_0^x - \cos x[\ln|\sec t + \tan t| - \sin t]_0^x \\
&= (\sin x)(1 - \cos x) - (\cos x)(\ln|\sec x + \tan x| - \sin x \\
&\qquad - \ln|\sec 0 + \tan 0| + \sin 0) \\
&= \sin x - (\cos x)\sin x - (\cos x)\ln|\sec x + \tan x| + (\cos x)\sin x \\
&= \sin x - (\cos x)\ln|\sec x + \tan x| \\
y &= 2\cos x - 5\sin x + \sin x - (\cos x)\ln|\sec x + \tan x|
\end{aligned}
$$

3.13.26 57

(a)

$$
\begin{aligned}
0 &= \theta'' + \frac{g}{l}\theta \\
0 &= (m + \sqrt{g/l}i)(m - \sqrt{g/l}i) \\
\theta_c &= c_1 \cos\sqrt{\frac{g}{l}}t + c_2 \sin\sqrt{\frac{g}{l}}t \\
0 &= c_1 \\
c_2 &= \omega_0\sqrt{\frac{l}{g}} \\
\theta_c &= \omega_0\sqrt{\frac{l}{g}}\sin\sqrt{\frac{g}{l}}t
\end{aligned}
$$

(b)

$$
\begin{aligned}
\theta' &= 0 \\
\omega_0 \cos\sqrt{\frac{g}{l}}t_{\max} &= 0 \\
\sqrt{\frac{g}{l}}t_{\max} &= \frac{\pi}{2} + n\pi \\
t_{\max} &= \sqrt{\frac{l}{g}}\left(\frac{\pi}{2} + n\pi\right) \\
\theta_{\max} &= \omega_0\sqrt{\frac{l}{g}}\sin\sqrt{\frac{g}{l}}t_{\max} \\
&= \omega_0\sqrt{\frac{l}{g}} \\
\omega_0 &= \theta_{\max}\sqrt{\frac{g}{l}} \\
V &= l\omega_0 \\
&= \theta_{\max}\sqrt{gl} \\
v_b &= \left(\frac{m_w + m_b}{m_b}\right)\theta_{\max}\sqrt{gl}
\end{aligned}
$$

(c)

$$\begin{aligned}
\cos\theta_{\max} &= \frac{l-h}{l} \\
1-\frac{\theta_{\max}^2}{2} &= \frac{l-h}{l} \\
\theta_{\max} &= \sqrt{2\left(1-\frac{l-h}{l}\right)} \\
v_b &= \left(\frac{m_w+m_b}{m_w}\right)\sqrt{lg}\sqrt{2\left(1-\frac{l-h}{l}\right)} \\
&= \left(\frac{m_w+m_b}{m_w}\right)\sqrt{2gh}
\end{aligned}$$

3.13.27 59

$$\begin{aligned}
\theta''(0) &= -\frac{g}{l}\sin\theta(0) \\
&= -\frac{g}{l}\sin\frac{\pi}{6} \\
&= -\frac{g}{2l} \\
\theta'''(0) &= \frac{d}{dt}\left(-\frac{g}{l}\sin\theta\right) \\
&= -\frac{g}{l}(\cos\theta)\theta' \\
&= -\frac{g}{l}(\cos\frac{\pi}{6})0 \\
&= 0 \\
\theta^{(4)}(0) &= \frac{d}{dt}\left(-\frac{g}{l}(\cos\theta)\theta'\right) \\
&= \frac{g}{l}(\sin\theta)\theta'^2 - \frac{g}{l}(\cos\theta)\theta'' \\
&= \frac{\sqrt{3}g^2}{4l^2} \\
\theta(t) &= \theta(0) + \frac{\theta'(0)}{1!}t + \frac{\theta''(0)}{2!}t^2 + \frac{\theta'''(0)}{3!}t^3 + \frac{\theta^{(4)}(0)}{4!}t^4 + \cdots \\
&= \frac{\pi}{6} - \frac{g}{4l}t^2 + \frac{\sqrt{3}g^2}{96l^2}t^4 + \cdots
\end{aligned}$$

4 The Laplace Transform

4.1 Definition of the Laplace Transform

4.1.1 1

$$
\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^1 e^{-st}(-1)\,dt + \int_1^\infty e^{-st}\,dt \\
&= \left[\frac{e^{-st}}{s}\right]_0^1 - \left[\frac{e^{-st}}{s}\right]_1^\infty \\
&= \frac{e^{-s}-1}{s} + \frac{e^{-s}}{s} \\
&= \frac{2e^{-s}-1}{s}
\end{aligned}
$$

4.1.2 3

$$
\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^1 e^{-st}t\,dt + \int_1^\infty e^{-st}\,dt \\
&= \left[-\frac{e^{-st}}{s}t\right]_0^1 + \int_0^1 \frac{e^{-st}}{s}\,dt + \frac{e^{-s}}{s} \\
&= -\frac{e^{-s}}{s} + \frac{1}{s}\left[-\frac{e^{-st}}{s}\right]_0^1 + \frac{e^{-s}}{s} \\
&= -\frac{e^{-s}}{s} + \frac{1-e^{-s}}{s^2} + \frac{e^{-s}}{s} \\
&= \frac{1-e^{-s}}{s^2}
\end{aligned}
$$

4.1.3 5

$$
\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\pi e^{-st}\sin t\,dt \\
&= \left[-\frac{e^{-st}}{s}\sin t\right]_0^\pi + \int_0^\pi \frac{e^{-st}}{s}\cos t\,dt \\
&= \frac{1}{s}\left(\left[-\frac{e^{-st}}{s}\cos t\right]_0^\pi - \int_0^\pi \frac{e^{-st}}{s}\sin t\,dt\right) \\
&= \frac{1}{s}\left(-\frac{1}{s}\left[-e^{-\pi s}-1\right] - \frac{1}{s}\mathcal{L}\{f(t)\}\right) \\
&= \frac{1}{s^2}(e^{-\pi s}+1) - \frac{1}{s^2}\mathcal{L}\{f(t)\} \\
\left(1+\frac{1}{s^2}\right)\mathcal{L}\{f(t)\} &= \frac{e^{-\pi s}+1}{s^2} \\
\mathcal{L}\{f(t)\} &= \frac{e^{-\pi s}+1}{s^2+1}
\end{aligned}
$$

4.1.4 11

$$
\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st}e^{t+7}\,dt \\
&= e^7\int_0^\infty e^{(1-s)t}\,dt \\
&= e^7\left[\frac{e^{(1-s)t}}{1-s}\right]_0^\infty \\
&= \frac{e^7}{s-1} \text{ for } s>1
\end{aligned}
$$

4.1.5 13

$$
\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st}te^{4t}\,dt \\
&= \int_0^\infty te^{(4-s)t}\,dt \\
&= \left[\frac{e^{(4-s)t}}{4-s}t\right]_0^\infty + \frac{1}{s-4}\int_0^\infty e^{(4-s)t}\,dt \\
&= \frac{1}{s-4}\left[\frac{e^{(4-s)t}}{4-s}\right]_0^\infty \\
&= \frac{1}{(s-4)^2} \text{ for } s>4
\end{aligned}
$$

4.1.6 15

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st}e^{-t}\sin t\,dt \\
&= \int_0^\infty e^{-(1+s)t}\sin t\,dt \\
&= \left[-\frac{e^{-(1+s)t}}{1+s}\sin t\right]_0^\infty + \frac{1}{1+s}\int_0^\infty e^{-(1+s)t}\cos t\,dt \\
&= \frac{1}{1+s}\left(\left[-\frac{e^{-(1+s)t}}{1+s}\cos t\right]_0^\infty - \frac{1}{1+s}\int_0^\infty e^{-(1+s)t}\sin t\,dt\right) \\
&= \frac{1}{1+s}\left(\frac{1}{1+s} - \frac{1}{1+s}\mathcal{L}\{f(t)\}\right) \\
&= \frac{1}{(1+s)^2} - \frac{1}{(1+s)^2}\mathcal{L}\{f(t)\} \\
\mathcal{L}\{f(t)\} &= \frac{1}{s^2+2s+2}
\end{aligned}$$

4.1.7 19

$$\mathcal{L}\{2t^4\} = 2\mathcal{L}\{t^4\} = \frac{48}{s^5}$$

4.1.8 21

$$\mathcal{L}\{4t-10\} = 4\mathcal{L}\{t\} - 10\mathcal{L}\{1\} = \frac{4}{s^2} - \frac{10}{s}$$

4.1.9 23

$$\mathcal{L}\{t^2+6t-3\} = \mathcal{L}\{t^2\} + 6\mathcal{L}\{t\} - 3\mathcal{L}\{1\} = \frac{2}{s^3} + \frac{6}{s^2} - \frac{3}{s}$$

4.1.10 25

$$\begin{aligned}
\mathcal{L}\{(t+1)^3\} &= \mathcal{L}\{t^3+3t^2+3t+1\} \\
&= \mathcal{L}\{t^3\} + 3\mathcal{L}\{t^2\} + 3\mathcal{L}\{t\} + \mathcal{L}\{1\} \\
&= \frac{6}{s^4} + \frac{6}{s^3} + \frac{3}{s^2} + \frac{1}{s}
\end{aligned}$$

4.1.11 27

$$\mathcal{L}\{1+e^{4t}\} = \mathcal{L}\{1\} + \mathcal{L}\{e^{4t}\} = \frac{1}{s} + \frac{1}{s-4}$$

4.1.12 29

$$\begin{aligned}\mathcal{L}\{(1+e^{2t})^2\} &= \mathcal{L}\{1+2e^{2t}+e^{4t}\}\\ &= \mathcal{L}\{1\}+2\mathcal{L}\{e^{2t}\}+\mathcal{L}\{e^{4t}\}\\ &= \frac{1}{s}+\frac{2}{s-2}+\frac{1}{s-4}\end{aligned}$$

4.1.13 33

$$\begin{aligned}\mathcal{L}\{\sinh kt\} &= \mathcal{L}\left\{\frac{1}{2}(e^{kt}-e^{-kt})\right\}\\ &= \frac{1}{2(s-k)}-\frac{1}{2(s+k)}\\ &= \frac{(s+k)-(s-k)}{2(s-k)(s+k)}\\ &= \frac{2k}{2(s^2-k^2)}\\ &= \frac{k}{s^2-k^2}\end{aligned}$$

4.1.14 35

$$\begin{aligned}\mathcal{L}\{e^t\sinh t\} &= \mathcal{L}\left\{\frac{1}{2}(e^{2t}-1)\right\}\\ &= \frac{1}{2(s-2)}-\frac{1}{2s}\end{aligned}$$

4.1.15 37

$$\begin{aligned}\mathcal{L}\{\sin 2t\cos 2t\} &= \mathcal{L}\left\{\frac{1}{2}\sin 4t\right\}\\ &= \frac{2}{s^2+16}\end{aligned}$$

4.1.16 39

$$\begin{aligned}\mathcal{L}\{\sin(4t+5)\} &= \mathcal{L}\{\sin 4t\cos 5+\cos 4t\sin 5\}\\ &= \frac{4\cos 5}{s^2+16}+\frac{s\sin 5}{s^2+16}\end{aligned}$$

4.1.17 43

$$\mathcal{L}\{t^{-1/2}\} = \frac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\frac{\pi}{s}}$$

4.1.18 45

$$\mathcal{L}\{t^{3/2}\} = \frac{\Gamma(5/2)}{s^{5/2}} = \frac{\frac{3}{2}\Gamma(3/2)}{s^{5/2}} = \frac{\frac{3}{2}\frac{1}{2}\Gamma(1/2)}{s^{5/2}} = \frac{3\sqrt{\pi}}{4s^{5/2}}$$

4.2 The Inverse Transform and Transforms of Derivatives

4.2.1 1

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2}$$

4.2.2 3

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{48}{s^5}\right\} = t - 2t^4$$

4.2.3 5

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{(s+1)^3}{s^4}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{3}{s^2} + \frac{3}{s^3} + \frac{1}{s^4}\right\} \\ &= 1 + 3t + \frac{3t^2}{2} + \frac{t^3}{6}\end{aligned}$$

4.2.4 7

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-2}\right\} = t - 1 + e^{2t}$$

4.2.5 9

$$\mathcal{L}^{-1}\left\{\frac{1}{4s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s+1/4}\right\} = \frac{1}{4}e^{-t/4}$$

4.2.6 11

$$\mathcal{L}^{-1}\left\{\frac{5}{s^2+49}\right\} = \frac{5}{7}\sin 7t$$

4.2.7 13

$$\mathcal{L}^{-1}\left\{\frac{4s}{4s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1/4}\right\} = \cos\frac{1}{2}t$$

4.2.8 15

$$\mathcal{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\} = 2\cos 3t - 2\sin 3t$$

4.2.9 17

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+3)}\right\} = \frac{1}{3} - \frac{1}{3}e^{-3t}$$

4.2.10 19

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+3)(s-1)}\right\} = \frac{3}{4}e^{-3t} + \frac{1}{4}e^{t}$$

4.2.11 21

$$\mathcal{L}^{-1}\left\{\frac{0.9s}{(s-0.1)(s+0.2)}\right\} = 0.3e^{0.1t} + 0.6e^{-0.2t}$$

4.2.12 23

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-2)(s-3)(s-6)}\right\} = \frac{1}{2}e^{2t} - e^{3t} + \frac{1}{2}e^{6t}$$

4.2.13 25

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3+5s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+5)}\right\} = \frac{1}{5} - \frac{1}{5}\cos\sqrt{5}t$$

4.2.14 27

$$\mathcal{L}^{-1}\left\{\frac{2s-4}{(s^2+s)(s^2+1)}\right\} = -4 + 3e^{-t} + \cos t + 3\sin t$$

4.2.15 29

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\} = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t$$

4.2.16 31

$$\begin{aligned}
\mathcal{L}\{a\sin bt - b\sin at\} &= a\frac{b}{s^2+b^2} - b\frac{a}{s^2+a^2} \\
&= ab\frac{s^2+a^2-s^2-b^2}{(s^2+a^2)(s^2+b^2)} \\
&= ab\frac{a^2-b^2}{(s^2+a^2)(s^2+b^2)} \\
\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} &= \frac{a\sin bt - b\sin at}{ab(a^2-b^2)}
\end{aligned}$$

4.2.17 33

$$\begin{aligned}
y' - y &= 1 \\
sY(s) - y(0) - Y(s) &= \frac{1}{s} \\
(s-1)Y(s) &= \frac{1}{s} \\
Y(s) &= \frac{1}{s(s-1)} \\
&= \frac{1}{s-1} - \frac{1}{s} \\
y(t) &= e^t - 1
\end{aligned}$$

4.2.18 35

$$\begin{aligned}
y' + 6y &= e^{4t} \\
sY(s) - y(0) + 6Y(s) &= \frac{1}{s-4} \\
(s+6)Y(s) &= \frac{1}{s-4} + 2 \\
Y(s) &= \frac{1}{(s-4)(s+6)} + \frac{2}{s+6} \\
&= \frac{1}{10(s-4)} - \frac{1}{10(s+6)} + \frac{2}{s+6} \\
y(t) &= \frac{1}{10}e^{4t} + \frac{19}{10}e^{-6t}
\end{aligned}$$

4.2.19 37

$$\begin{aligned}
y'' + 5y' + 4y &= 0 \\
s^2Y(s) - sy(0) - y'(0) + 5(sY(s) - y(0)) + 4Y(s) &= 0 \\
s^2Y(s) - s + 5sY(s) - 5 + 4Y(s) &= 0 \\
(s^2 + 5s + 4)Y(s) &= s + 5 \\
Y(s) &= \frac{s+5}{(s+1)(s+4)} \\
&= \frac{4}{3(s+1)} - \frac{1}{3(s+4)} \\
y(t) &= \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}
\end{aligned}$$

4.2.20 39

$$
\begin{aligned}
y'' + y &= \sqrt{2}\sin\sqrt{2}t \\
s^2Y(s) - sy(0) - y'(0) + Y(s) &= \frac{2}{s^2+2} \\
(s^2+1)Y(s) &= \frac{2}{s^2+2} + 10s \\
Y(s) &= \frac{2}{(s^2+1)(s^2+2)} + \frac{10s}{s^2+1} \\
&= \frac{2}{s^2+1} - \frac{2}{s^2+2} + \frac{10s}{s^2+1} \\
y(s) &= 2\sin t - \sqrt{2}\sin\sqrt{2}t + 10\cos t
\end{aligned}
$$

4.3 Translation Theorems

4.3.1 1

$$\frac{1}{(s-10)^2}$$

4.3.2 3

$$\frac{6}{(s+2)^4}$$

4.3.3 5

$$\mathcal{L}\{t(e^t + e^{2t})^2\} = \mathcal{L}\{t(e^{2t} + 2e^{3t} + e^{4t})\} = \frac{1}{(s-2)^2} + \frac{2}{(s-3)^2} + \frac{1}{(s-4)^2}$$

4.3.4 7

$$\frac{3}{(s-1)^2+9}$$

4.3.5 9

$$
\begin{aligned}
\mathcal{L}\{(1 - e^t + 3e^{-4t})\cos 5t\} &= \mathcal{L}\{\cos 5t - e^t\cos 5t + 3e^{-4t}\cos 5t\} \\
&= \frac{s}{s^2+25} - \frac{s-1}{(s-1)^2+25} + \frac{3(s+4)}{(s+4)^2+25}
\end{aligned}
$$

4.3.6 11

$$\frac{1}{2}t^2e^{-2t}$$

4.3.7 13

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2-6s+10}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2+1}\right\}\\ &= e^{3t}\sin t\end{aligned}$$

4.3.8 15

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\} &= \mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2+1}\right\}\\ &= \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}-\frac{2}{(s+2)^2+1}\right\}\\ &= e^{-2t}\cos t-2e^{-2t}\sin t\end{aligned}$$

4.3.9 17

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2}-\frac{1}{(s+1)^2}\right\}\\ &= e^{-t}-te^{-t}\end{aligned}$$

4.3.10 21

$$\begin{aligned}y'+4y &= e^{-4t}\\ sY(s)-y(0)+4Y(s) &= \frac{1}{s+4}\\ (s+4)Y(s) &= \frac{1}{s+4}+2\\ Y(s) &= \frac{1}{(s+4)^2}+\frac{2}{s+4}\\ y(t) &= te^{-4t}+2e^{-4t}\end{aligned}$$

4.3.11 23

$$\begin{aligned}y''+2y'+y &= 0\\ s^2Y(s)-sy(0)-y'(0)+2[sY(s)-y(0)]+Y(s) &= 0\\ (s^2+2s+1)Y(0) &= s+3\\ Y(s) &= \frac{s+3}{(s+1)^2}\\ &= \frac{s+1}{(s+1)^2}+\frac{2}{(s+1)^2}\\ y(t) &= e^{-t}+2te^{-t}\end{aligned}$$

4.3.12 25

$$\begin{aligned}
y'' - 6y' + 9y &= t \\
s^2Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 9Y(s) &= \frac{1}{s^2} \\
(s^2 - 6 + 9)Y(s) &= 1 + \frac{1}{s^2} \\
Y(s) &= \frac{1}{(s-3)^2} + \frac{1}{s^2(s-3)^2} \\
&= \frac{1}{(s-3)^2} + \frac{1}{9s^2} + \frac{2}{27s} \\
&\quad - \frac{2}{27(s-3)} + \frac{1}{9(s-3)^2} \\
y(t) &= te^{3t} + \frac{1}{9}t + \frac{2}{27} - \frac{2}{27}e^{3t} \\
&\quad + \frac{1}{9}te^{3t}
\end{aligned}$$

4.3.13 27

$$\begin{aligned}
y'' - 6y' + 13y &= 0 \\
s^2Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 13Y(s) &= 0 \\
(s^2 - 6s + 13)Y(s) &= -3 \\
Y(s) &= -\frac{3}{s^2 - 6s + 13} \\
&= -\frac{3}{(s-3)^2 + 4} \\
&= -\frac{3}{2}\frac{2}{(s-3)^2 + 4} \\
y(t) &= -\frac{3}{2}e^{3t}\sin 2t
\end{aligned}$$

4.3.14 29

$$
\begin{aligned}
y'' - y' &= e^t \cos t \\
s^2Y(s) - sy(0) - y'(0) - sY(s) + y(0) &= \frac{s-1}{(s-1)^2+1} \\
(s^2 - s)Y(s) &= \frac{s-1}{(s-1)^2+1} \\
s(s-1)Y(s) &= \frac{s-1}{(s-1)^2+1} \\
Y(s) &= \frac{1}{s((s-1)^2+1)} \\
&= \frac{1}{s^2-2s+2} - \frac{s}{2(s^2-2s+2)} + \frac{1}{2s} \\
&= \frac{1}{(s-1)^2+1} - \frac{1}{2}\frac{s-1}{(s-1)^2+1} \\
&\quad - \frac{1}{2}\frac{1}{(s-1)^2+1} + \frac{1}{2s} \\
&= \frac{1}{2}e^t \sin t - \frac{1}{2}e^t \cos t + \frac{1}{2}
\end{aligned}
$$

4.3.15 31

$$
\begin{aligned}
y'' + 2y' + y &= 0 \\
s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + Y(s) &= 0 \\
(s^2 + 2s + 1)Y(s) &= c_1 s + 2c_1 + 2 \\
(s+1)^2 Y(s) &= c_1(s+1) + c_1 + 2 \\
Y(s) &= \frac{c_1(s+1)}{(s+1)^2} + \frac{c_1}{(s+1)^2} \\
&\quad + \frac{2}{(s+1)^2} \\
y(t) &= c_1 e^{-t} + (c_1+2)te^{-t} \\
2 &= c_1 e^{-(1)} + (c_1+2)(1)e^{-(1)} \\
&= 2e^{-1}(c_1+1) \\
c_1 &= e - 1 \\
y(t) &= (e-1)e^{-t} + (e+1)te^{-t}
\end{aligned}
$$

4.3.16 33

$$\begin{aligned}
mx'' &= -\beta x' - kx \\
\frac{1}{8}x'' &= -\frac{7}{8}x' - 2x
\end{aligned}$$

$$\begin{aligned}
x'' + 7x' + 16x &= 0 \\
s^2X(s) - sx(0) - x'(0) + 7[sX(s) - x(0)] & \\
+16X(s) &= 0 \\
(s^2 + 7s + 16)X(s) + \frac{3}{2}s + \frac{21}{2} &= 0 \\
\left(\left(s + \frac{7}{2}\right)^2 + \frac{15}{4}\right)X(s) &= -\frac{3}{2}s - \frac{21}{2}
\end{aligned}$$

$$\begin{aligned}
X(s) &= -\frac{3}{2}\frac{s + \frac{7}{2}}{\left(s + \frac{7}{2}\right)^2 + \frac{15}{4}} \\
&\quad - \frac{7\sqrt{15}}{10}\frac{\frac{\sqrt{15}}{2}}{\left(s + \frac{7}{2}\right)^2 + \frac{15}{4}} \\
x(t) &= -\frac{3}{2}e^{-7t/2}\cos\frac{\sqrt{15}}{2}t \\
&\quad - \frac{7\sqrt{15}}{10}e^{-7t/2}\sin\frac{\sqrt{15}}{2}t
\end{aligned}$$

4.3.17 37

$$\frac{e^{-s}}{s^2}$$

4.3.18 39

$$\frac{e^{-2s}}{s^2} + 2\frac{e^{-2s}}{s}$$

4.3.19 41

$$\begin{aligned}
\mathcal{L}\{\cos 2t\mathcal{U}(t - \pi)\} &= e^{-\pi s}\mathcal{L}\{\cos 2(t + \pi)\} \\
&= e^{-\pi s}\mathcal{L}\{\cos 2t\} \\
&= e^{-\pi s}\frac{s}{s^2 + 4}
\end{aligned}$$

4.3.20 43

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2e^{-2s}}{s^3}\right\} \\ &= \frac{1}{2}(t-2)^2\mathcal{U}(t-2)\end{aligned}$$

4.3.21 45

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} &= \sin(t-\pi)\mathcal{U}(t-\pi) \\ &= -(\sin t)\mathcal{U}(t-\pi)\end{aligned}$$

4.3.22 47

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s} - \frac{e^{-s}}{s+1}\right\} \\ &= \left(1 - e^{-(t-1)}\right)\mathcal{U}(t-1)\end{aligned}$$

4.3.23 49

c

4.3.24 51

f

4.3.25 53

a

4.3.26 55

$$\begin{aligned}f(t) &= 2 - 4\mathcal{U}(t-3) \\ \mathcal{L}\{f(t)\} &= \frac{2}{s} - \frac{4e^{-3s}}{s}\end{aligned}$$

4.3.27 57

$$\begin{aligned}f(t) &= t^2\mathcal{U}(t-1) \\ \mathcal{L}\{f(t)\} &= e^{-s}\mathcal{L}\{(t+1)^2\} \\ &= e^{-s}\mathcal{L}\{t^2+2t+1\} \\ &= e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)\end{aligned}$$

4.3.28 59

$$\begin{aligned} f(t) &= t - t\mathcal{U}(t-2) \\ \mathcal{L}\{f(t)\} &= \frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) \end{aligned}$$

4.3.29 61

$$\begin{aligned} f(t) &= \mathcal{U}(t-a) - \mathcal{U}(t-b) \\ \mathcal{L}\{f(t)\} &= \frac{e^{-as}}{s} - \frac{e^{-bs}}{s} \end{aligned}$$

4.3.30 63

$$\begin{aligned} y' + y &= 5\mathcal{U}(t-1) \\ sY(s) + y(0) + Y(s) &= \frac{5e^{-s}}{s} \\ Y(s) &= \frac{5e^{-s}}{s(s+1)} \\ &= \frac{5e^{-s}}{s} - \frac{5e^{-s}}{s+1} \\ y(t) &= (5 - 5e^{-(t-1)})\mathcal{U}(t-1) \end{aligned}$$

4.3.31 65

$$\begin{aligned} y' + 2y &= t - t\mathcal{U}(t-1) \\ sY(s) - y(0) + 2Y(s) &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} \\ Y(s) &= \frac{1}{s^2(s+2)} - \frac{e^{-s}}{s^2(s+2)} - \frac{e^{-s}}{s(s+2)} \\ &= \frac{1}{2s^2} + \frac{1}{4(s+2)} - \frac{1}{4s} - \frac{e^{-s}}{2s^2} - \frac{e^{-s}}{4(s+2)} + \frac{e^{-s}}{4s} - \frac{e^{-s}}{2s} \\ &\quad + \frac{e^{-s}}{2(s+2)} \\ y(t) &= \frac{1}{2}t + \frac{1}{4}e^{-2t} - \frac{1}{4} - \frac{1}{2}(t-1)\mathcal{U}(t-1) \\ &\quad + \frac{1}{4}e^{-2(t-1)}\mathcal{U}(t-1) - \frac{1}{4}\mathcal{U}(t-1) \end{aligned}$$

4.3.32 67

$$
\begin{aligned}
y'' + 4y &= \sin t\,\mathcal{U}(t-2\pi) \\
s^2Y(s) - sy(0) - y'(0) + 4Y(s) &= \frac{e^{-2\pi s}}{s^2+1} \\
(s^2+4)Y(s) &= \frac{e^{-2\pi s}}{s^2+1} + s \\
Y(s) &= \frac{e^{-2\pi s}}{(s^2+1)(s^2+4)} + \frac{s}{s^2+4} \\
&= \frac{e^{-2\pi s}}{3(s^2+1)} - \frac{e^{-2\pi s}}{3(s^2+4)} + \frac{s}{s^2+4} \\
y(t) &= \frac{1}{3}\sin(t-2\pi)\mathcal{U}(t-2\pi) \\
&\quad - \frac{1}{6}\sin 2(t-2\pi)\mathcal{U}(t-2\pi) + \cos 2t
\end{aligned}
$$

4.3.33 69

$$
\begin{aligned}
y'' + y &= \mathcal{U}(t-\pi) - \mathcal{U}(t-2\pi) \\
s^2Y(s) - sy(0) - y'(0) + Y(s) &= \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s} \\
(s^2+1)Y(s) &= \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s} + 1 \\
Y(s) &= \frac{e^{-\pi s}}{s(s^2+1)} - \frac{e^{-2\pi s}}{s(s^2+1)} + \frac{1}{s^2+1} \\
&= \frac{e^{-\pi s}}{s} - \frac{se^{-\pi s}}{s^2+1} - \frac{e^{-2\pi s}}{s} + \frac{se^{-2\pi s}}{s^2+1} + \frac{1}{s^2+1} \\
y(t) &= \sin t + (1-\cos(t-\pi))\mathcal{U}(t-\pi) \\
&\quad - (1-\cos(t-2\pi))\mathcal{U}(t-2\pi)
\end{aligned}
$$

4.3.34 71

$$
\begin{aligned}
x'' + 16x &= 20t - 20t\mathcal{U}(t-5) \\
s^2X(s) + sx(0) + x'(0) + 16X(s) &= \frac{20}{s^2} - \frac{20e^{-5s}}{s^2} - \frac{100e^{-5s}}{s} \\
X(s) &= \frac{20}{s^2(s^2+16)} - \frac{20e^{-5s}}{s^2(s^2+16)} - \frac{100e^{-5s}}{s(s^2+16)} \\
&= \frac{5}{4s^2} - \frac{5}{4(s^2+16)} - \frac{5e^{-5s}}{4s^2} \\
&\quad + \frac{5e^{-5s}}{4(s^2+16)} - \frac{25e^{-5s}}{4s} + \frac{25se^{-5s}}{4(s^2+16)} \\
x(t) &= \frac{5}{4}t - \frac{5}{16}\sin 4t - \frac{5}{4}(t-5)\mathcal{U}(t-5) \\
&\quad + \frac{5}{16}\sin(4(t-5))\mathcal{U}(t-5) - \frac{25}{4}\mathcal{U}(t-5) \\
&\quad + \frac{25}{4}\cos(4(t-5))\mathcal{U}(t-5)
\end{aligned}
$$

4.4 Additional Operational Properties

4.4.1 1

$$\frac{1}{(s+10)^2}$$

4.4.2 3

$$\frac{s^2-4}{(s^2+4)^2}$$

4.4.3 5

$$\frac{2s}{(s^2-1)^2}$$

4.4.4 7

$$\frac{12(s-2)}{((s-2)^2+36)^2}$$

4.4.5 9

$$
\begin{aligned}
y' + y &= t \sin t \\
sY(s) - y(0) + Y(s) &= \frac{2s}{(s^2+1)^2} \\
Y(s) &= \frac{2s}{(s+1)(s^2+1)^2} \\
&= \frac{1}{s+1} \frac{2s}{(s^2+1)^2} \\
y(t) &= \int_0^t \tau e^{-(t-\tau)} \sin \tau \, d\tau \\
&= -\frac{1}{2} e^{-t} + \frac{1}{2} t \sin t - \frac{1}{2} t \cos t + \frac{1}{2} \cos t
\end{aligned}
$$

4.4.6 11

$$
\begin{aligned}
y'' + 9y &= \cos 3t \\
s^2 Y(s) - sy(0) - y'(0) + 9Y(s) &= \frac{s}{s^2+9} \\
(s^2+9)Y(s) &= 5 + 2s + \frac{s}{s^2+9} \\
Y(s) &= \frac{5}{s^2+9} + \frac{2s}{s^2+9} + \frac{s}{(s^2+9)^2} \\
y(t) &= \frac{5}{3} \sin 3t + 2 \cos 3t \\
&\quad + \int_0^t \frac{1}{3} \sin 3\tau \cos 3(t-\tau) \, d\tau \\
&= \frac{5}{3} \sin 3t + 2 \cos 3t + \frac{1}{6} t \sin 3t
\end{aligned}
$$

4.4.7 13

$$
\begin{aligned}
y'' + 16y &= \cos 4t - \cos 4t \mathcal{U}(t-\pi) \\
s^2 Y(s) - sy(0) - y'(0) + 16Y(s) &= \frac{s}{s^2+16} - e^{-\pi s} \mathcal{L}\{\cos(4(t+\pi))\} \\
(s^2+16)Y(s) &= 1 + \frac{s}{s^2+16} - \frac{e^{-\pi s} s}{s^2+16} \\
Y(s) &= \frac{1}{s^2+16} + \frac{s}{(s^2+16)^2} - \frac{e^{-\pi s} s}{(s^2+16)^2} \\
y(t) &= \frac{1}{4} \sin 4t + \frac{1}{8} t \sin 4t \\
&\quad - \frac{1}{8} (t-\pi) \sin 4(t-\pi) \mathcal{U}(t-\pi)
\end{aligned}
$$

4.4.8 17

$$\begin{aligned}
ty'' - y' &= 2t^2 \\
-\frac{d}{ds}(s^2Y(s) - sy(0) - y'(0)) - sY(s) + y(0) &= \frac{4}{s^3} \\
-(2sY(s) + s^2Y'(s)) - sY(s) &= \frac{4}{s^3} \\
-sY'(s) - 3Y(s) &= \frac{4}{s^4} \\
-m - 3 &= 0 \\
m &= -3 \\
Y_c &= cs^{-3} \\
Y_p &= \frac{4}{s^4} \\
Y(s) &= \frac{c}{s^3} + \frac{4}{s^4} \\
y(t) &= ct^2 + \frac{2}{3}t^3
\end{aligned}$$

4.4.9 19

$$\begin{aligned}
f * g &= \int_0^t 12(t - \tau)\tau^2 \, d\tau \\
&= 12 \int_0^t (t\tau^2 - \tau^3) \, d\tau \\
&= 12 \left[\frac{1}{3}t\tau^3 - \frac{1}{4}\tau^4 \right]_0^t \\
&= 4t^4 - 3t^4 \\
&= t^4 \\
\mathcal{L}\{f * g\} &= \frac{24}{s^5}
\end{aligned}$$

4.4.10 21

$$
\begin{aligned}
f * g &= \int_0^t e^{-\tau} e^{t-\tau}\, d\tau \\
&= \int_0^t e^{t-2\tau}\, d\tau \\
&= -\frac{1}{2}[e^{t-2\tau}]_0^t \\
&= -\frac{1}{2}(e^{-t} - e^t) \\
&= \sinh t \\
\mathcal{L}\{f * g\} &= \frac{1}{s^2 - 1}
\end{aligned}
$$

4.4.11 23

$$\frac{6}{s^5}$$

4.4.12 25

$$\frac{1}{s+1} \frac{s-1}{(s-1)^2+1}$$

4.4.13 27

$$\frac{1}{s(s-1)}$$

4.4.14 29

$$\frac{1}{s} \frac{s+1}{(s+1)^2+1}$$

4.4.15 31

$$\frac{1}{s^2} \frac{1}{s-1}$$

4.4.16 33

$$\begin{aligned}\mathcal{L}\left\{t\int_0^t \sin\tau\,d\tau\right\} &= -\frac{d}{ds}\frac{\mathcal{L}\{\sin t\}}{s}\\ &= -\frac{d}{ds}\frac{1}{s^3+s}\\ &= \frac{3s^2+1}{(s^3+s)^2}\\ &= \frac{3s^2+1}{s^2(s^2+1)^2}\end{aligned}$$

4.4.17 35

$$\int_0^t e^\tau\,d\tau = e^t - 1$$

4.4.18 36

$$\int_0^t (e^\tau - 1)\,d\tau = e^t - t - 1$$

4.4.19 37

$$\int_0^t (e^\tau - \tau - 1)\,d\tau = e^t - \frac{1}{2}t^2 - t - 1$$

4.4.20 41

$$\begin{aligned}f(t) + \int_0^t (t-\tau)f(\tau)\,d\tau &= t\\ F(s) + \frac{1}{s^2}F(s) &= \frac{1}{s^2}\\ \left(\frac{s^2+1}{s^2}\right)F(s) &= \frac{1}{s^2}\\ F(s) &= \frac{1}{s^2+1}\\ f(t) &= \sin t\end{aligned}$$

4.4.21 43

$$
\begin{aligned}
f(t) &= te^t + \int_0^t \tau f(t-\tau)\,d\tau \\
F(s) &= \frac{1}{(s-1)^2} + \frac{F(s)}{s^2} \\
\frac{s^2-1}{s^2}F(s) &= \frac{1}{(s-1)^2} \\
F(s) &= \frac{s^2}{(s^2-1)(s-1)^2} \\
&= -\frac{1}{8(s+1)} + \frac{1}{8(s-1)} + \frac{3}{4(s-1)^2} + \frac{1}{2(s-1)^3} \\
&= -\frac{1}{8(s+1)} + \frac{1}{8(s-1)} - \frac{3}{4}\frac{d}{ds}\left(\frac{1}{s-1}\right) + \frac{1}{4}\frac{d^2}{ds^2}\left(\frac{1}{s-1}\right) \\
f(t) &= -\frac{1}{8}e^{-t} + \frac{1}{8}e^t + \frac{3}{4}te^t + \frac{1}{4}t^2e^t
\end{aligned}
$$

4.4.22 45

$$
\begin{aligned}
f(t) + \int_0^t f(\tau)\,d\tau &= 1 \\
F(s) + \frac{F(s)}{s} &= \frac{1}{s} \\
\frac{s+1}{s}F(s) &= \frac{1}{s} \\
F(s) &= \frac{1}{s+1} \\
f(t) &= e^{-t}
\end{aligned}
$$

4.4.23 47

$$
\begin{aligned}
f(t) &= 1 + t - \frac{8}{3}\int_0^t (\tau - t)^3 f(\tau)\, d\tau \\
F(s) &= \frac{1}{s} + \frac{1}{s^2} + \frac{16}{s^4} F(s) \\
\frac{s^4 - 16}{s^4} F(s) &= \frac{1}{s} + \frac{1}{s^2} \\
F(s) &= \frac{s^3}{s^4 - 16} + \frac{s^2}{s^4 - 16} \\
&= \frac{s}{2(s^2+4)} + \frac{1}{4(s-2)} + \frac{1}{4(s+2)} + \frac{1}{2(s^2+4)} - \frac{1}{8(s+2)} \\
&\quad + \frac{8}{(s-2)} \\
f(t) &= \frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t + \frac{3}{8}e^{2t} + \frac{1}{8}e^{-2t}
\end{aligned}
$$

4.4.24 49

$$
\begin{aligned}
y'(t) &= 1 - \sin t - \int_0^t y(\tau)\, d\tau \\
sY(s) - y(0) &= \frac{1}{s} - \frac{1}{s^2+1} - \frac{Y(s)}{s} \\
\frac{s^2+1}{s} Y(s) &= \frac{1}{s} - \frac{1}{s^2+1} \\
Y(s) &= \frac{1}{s^2+1} - \frac{s}{(s^2+1)^2} \\
y(t) &= \sin t - \frac{1}{2} t \sin t
\end{aligned}
$$

4.4.25 55

$$
\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{1}{1 - e^{-2as}} \left(\int_0^a e^{-st}\, dt - \int_a^{2a} e^{-st}\, dt \right) \\
&= \frac{1}{1 - e^{-2as}} \left(\left[\frac{1}{s} e^{-st} \right]_a^{2a} - \left[\frac{1}{s} e^{-st} \right]_0^a \right) \\
&= \frac{1}{s(1 - e^{-2as})} (e^{-2as} - 2e^{-as} + 1) \\
&= \frac{(1 - e^{-as})^2}{s(1 + e^{-as})(1 - e^{-as})} \\
&= \frac{1 - e^{-as}}{s(1 + e^{-as})}
\end{aligned}
$$

4.4.26 57

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-bs}}\int_0^b e^{-st}\frac{a}{b}t\,dt \\
&= \frac{a}{b(1-e^{-bs})}\frac{1-e^{-bs}(bs+1)}{s^2} \\
&= \frac{a}{b}\frac{1}{1-e^{-bs}}\left(\frac{1-e^{-bs}}{s^2}-\frac{bse^{-bs}}{s^2}\right) \\
&= \frac{a}{s}\left(\frac{1}{bs}-\frac{1}{e^{bs}-1}\right)
\end{aligned}$$

4.4.27 59

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-\pi s}}\int_0^\pi e^{-st}\sin t\,dt \\
&= \frac{1}{1-e^{-\pi s}}\frac{e^{-\pi s}+1}{s^2+1} \\
&= \frac{e^{\pi s}+1}{e^{\pi s}-1}\frac{1}{s^2+1} \\
&= \frac{\coth \pi s/2}{s^2+1}
\end{aligned}$$

4.5 The Dirac Delta Function

4.5.1 1

$$\begin{aligned}
y'-3y &= \delta(t-2) \\
sY(s)-3Y(s) &= e^{-2s} \\
Y(s) &= \frac{e^{-2s}}{s-3} \\
y(t) &= e^{3(t-2)}\mathcal{U}(t-2)
\end{aligned}$$

4.5.2 3

$$\begin{aligned}
y''+y &= \delta(t-2\pi) \\
s^2Y(s)-sy(0)-y'(0)+Y(S) &= e^{-2\pi s} \\
(s^2+1)Y(s) &= 1+e^{-2\pi s} \\
Y(s) &= \frac{1}{s^2+1}+\frac{e^{-2\pi s}}{s^2+1} \\
y(t) &= \sin t+\sin t\mathcal{U}(t-2\pi)
\end{aligned}$$

4.5.3 5

$$\begin{aligned}
y'' + y &= \delta(t - \pi/2) + \delta(t - 3\pi/2) \\
s^2Y(s) - sy(0) - y'(0) + Y(s) &= e^{-\pi s/2} + e^{-3\pi s/2} \\
Y(s) &= \frac{e^{-\pi s/2}}{s^2+1} + \frac{e^{-3\pi s/2}}{s^2+1} \\
&= \sin t\mathcal{U}(t - \pi/2) + \sin t\mathcal{U}(t - 3\pi/2) \\
&= -\cos t\mathcal{U}(t - \pi/2) + \cos t\mathcal{U}(t - 3\pi/2)
\end{aligned}$$

4.5.4 7

$$\begin{aligned}
y'' + 2y' &= \delta(t - 1) \\
s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] &= e^{-s} \\
(s^2 + 2s)Y(s) &= 1 + e^{-s} \\
Y(s) &= \frac{1}{s(s+2)} + \frac{e^{-s}}{s(s+2)} \\
&= \frac{1}{2s} - \frac{1}{2(s+2)} + \frac{e^{-s}}{2s} - \frac{e^{-s}}{2(s+2)} \\
y(t) &= \frac{1}{2} - \frac{1}{2}e^{-2t} + \frac{1}{2}\mathcal{U}(t-1) \\
&\quad - \frac{1}{2}e^{-2(t-1)}\mathcal{U}(t-1)
\end{aligned}$$

4.5.5 9

$$\begin{aligned}
y'' + 4y' + 5y &= \delta(t - 2\pi) \\
s^2Y(s) + 4sY(s) + 5Y(s) &= e^{-2\pi s} \\
Y(s) &= \frac{e^{-2\pi s}}{s^2 + 4s + 5} \\
&= \frac{e^{-2\pi s}}{(s+2)^2 + 1} \\
&= e^{-2(t-2\pi)} \sin t\mathcal{U}(t - 2\pi)
\end{aligned}$$

4.5.6 11

$$\begin{aligned}
y'' + 4y' + 13y &= \delta(t-\pi) + \delta(t-3\pi) \\
s^2Y(s) - s + 4[sY(s) - 1] + 13Y(s) &= e^{-\pi s} + e^{-3\pi s} \\
((s+2)^2 + 9)Y(s) &= 4 + s + e^{-\pi s} + e^{-3\pi s} \\
Y(s) &= \frac{4}{(s+2)^2+9} + \frac{s}{(s+2)^2+9} \\
&\quad + \frac{e^{-\pi s}}{(s+2)^2+9} + \frac{e^{-3\pi s}}{(s+2)^2+9} \\
y(t) &= \frac{2}{3}e^{-2t}\sin 3t + e^{-2t}\cos 3t \\
&\quad + \frac{1}{3}e^{-2(t-\pi)}\sin 3(t-\pi)\mathcal{U}(t-\pi) \\
&\quad + \frac{1}{3}e^{-2(t-3\pi)}\sin 3(t-3\pi)\mathcal{U}(t-3\pi)
\end{aligned}$$

4.5.7 13

$$\begin{aligned}
y'' + y &= \sum_{k=1}^{\infty}\delta(t-k\pi) \\
s^2Y(s) - sy(0) - y'(0) + Y(s) &= \sum_{k=1}^{\infty} e^{-k\pi s} \\
(s^2+1)Y(s) &= 1 + \sum_{k=1}^{\infty} e^{-k\pi s} \\
Y(s) &= \frac{1}{s^2+1} + \sum_{k=1}^{\infty}\frac{e^{-k\pi s}}{s^2+1} \\
y(t) &= \sin t + \sin t\sum_{k=1}^{\infty}(-1)^k\mathcal{U}(t-k\pi)
\end{aligned}$$

4.6 Systems of Linear Differential Equations

4.6.1 1

$$
\begin{aligned}
x' + x - y &= 0 \\
sX(s) + X(s) - Y(s) &= 0 \\
(s+1)X(s) - Y(s) &= 0 \\
-2x + y' &= 0 \\
-2X(s) + sY(s) - 1 &= 0 \\
(s^2 + s - 2)Y(s) &= s + 1 \\
Y(s) &= \frac{s}{(s-1)(s+2)} + \frac{1}{(s-1)(s+2)} \\
Y(s) &= \frac{1}{3(s+2)} + \frac{2}{3(s-1)} \\
y(t) &= \frac{1}{3}e^{-2t} + \frac{2}{3}e^{t} \\
(s+1)X(s) - \frac{1}{3(s+2)} - \frac{2}{3(s-1)} &= 0 \\
X(s) &= \frac{1}{3(s+1)(s+2)} + \frac{2}{3(s-1)(s+1)} \\
&= \frac{1}{3(s-1)} - \frac{1}{3(s+2)} \\
x(t) &= \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t}
\end{aligned}
$$

4.6.2 3

$$
\begin{aligned}
x' - x + 2y &= 0 \\
sX(x) + 1 - X(s) + 2Y(s) &= 0 \\
(s-1)X(s) + 2Y(s) &= -1 \\
-5x + y' + y &= 0 \\
-5X(s) + sY(s) - 2 + Y(s) &= 0 \\
-5X(s) + (s+1)Y(s) &= 2 \\
(s^2+9)Y(s) &= -7 + 2s \\
Y(s) &= -\frac{7}{s^2+9} + \frac{2s}{s^2+9} \\
y(t) &= -\frac{7}{3}\sin 3t + 2\cos 3t \\
(s-1)X(s) &= -1 + \frac{14}{s^2+9} - \frac{4s}{s^2+9} \\
X(s) &= -\frac{1}{s-1} + \frac{14}{(s-1)(s^2+9)} - \frac{4s}{(s-1)(s^2+9)} \\
&= -\frac{s}{s^2+9} - \frac{5}{s^2+9} \\
x(t) &= -\cos 3t - \frac{5}{3}\sin 3t
\end{aligned}
$$

4.6.3 5

$$
\begin{aligned}
2x' - 2x + y' &= 1 \\
2sX(s) - 2X(s) + sY(s) &= \frac{1}{s} \\
2(s-1)X(s) + sY(s) &= \frac{1}{s} \\
x' - 3x + y' - 3y &= 2 \\
sX(s) - 3X(s) + sY(s) - 3Y(s) &= \frac{2}{s} \\
(s-3)X(s) + (s-3)Y(s) &= \frac{2}{s} \\
(s-2)(s-3)X(s) &= -1 - \frac{3}{s} \\
X(s) &= -\frac{1}{(s-2)(s-3)} - \frac{3}{s(s-2)(s-3)} \\
&= -\frac{2}{s-3} + \frac{5}{2(s-2)} - \frac{1}{2s} \\
x(t) &= \frac{5}{2}e^{2t} - 2e^{3t} - \frac{1}{2} \\
Y(s) &= \frac{2}{s(s-3)} + \frac{1}{(s-2)(s-3)} + \frac{3}{s(s-2)(s-3)} \\
&= \frac{8}{3(s-3)} - \frac{1}{6s} - \frac{5}{2(s-2)} \\
y(t) &= \frac{8}{3}e^{3t} - \frac{1}{6} - \frac{5}{2}e^{2t}
\end{aligned}
$$

4.6.4 7

$$\begin{aligned} x'' + x - y &= 0 \\ s^2X(s) - sx(0) - x'(0) + X(s) - Y(s) &= 0 \\ (s^2+1)X(s) - Y(s) &= -2 \end{aligned}$$

$$\begin{aligned} -x + y'' + y &= 0 \\ -X(s) + s^2Y(s) - sy(0) - y'(0) + Y(s) &= 0 \\ -X(s) + (s^2+1)Y(s) &= 1 \end{aligned}$$

$$\begin{aligned} (s^2+1)^2X(s) - (s^2+1)Y(s) - X(s) + (s^2+1)Y(s) &= -2(s^2+1) + 1 \\ (s^4+2s^2)X(s) &= -2s^2 - 1 \\ X(s) &= -\frac{2}{s^2+2} - \frac{1}{s^2(s^2+2)} \\ &= -\frac{3}{2(s^2+2)} - \frac{1}{2s^2} \\ x(t) &= -\frac{3}{4}\sqrt{2}\sin\sqrt{2}t - \frac{1}{2}t \end{aligned}$$

$$\begin{aligned} \frac{3}{2(s^2+2)} + \frac{1}{2s^2} + (s^2+1)Y(s) &= 1 \\ Y(s) &= \frac{1}{s^2+1} \\ &\quad - \frac{3}{2(s^2+1)(s^2+2)} \\ &\quad - \frac{1}{2s^2(s^2+1)} \\ &= \frac{3}{2(s^2+2)} - \frac{1}{2s^2} \\ y(t) &= \frac{3}{4}\sqrt{2}\sin\sqrt{2}t - \frac{1}{2}t \end{aligned}$$

4.6.5 9

$$\begin{aligned}
x'' + y'' &= t^2 \\
X(s) + Y(s) &= \frac{8}{s} + \frac{2}{s^5} \\
x'' - y'' &= 4t \\
X(s) - Y(s) &= \frac{8}{s} + \frac{4}{s^4} \\
X(s) &= \frac{8}{s} + \frac{2}{s^4} + \frac{1}{s^5} \\
x(t) &= 8 + \frac{1}{3}t^3 + \frac{1}{24}t^4 \\
Y(s) &= -\frac{2}{s^4} + \frac{1}{s^5} \\
y(t) &= -\frac{1}{3}t^3 + \frac{1}{24}t^4
\end{aligned}$$

4.6.6 11

$$\begin{aligned}
x'' + 3y' + 3y &= 0 \\
s^2X(s) + 3(s+1)Y(s) &= 2
\end{aligned}$$

$$\begin{aligned}
x'' + 3y &= te^{-t} \\
s^2X(s) + 3Y(s) &= 2 + \frac{1}{(s+1)^2} \\
3sY(s) &= -\frac{1}{(s+1)^2} \\
Y(s) &= \frac{1}{3(s+1)} + \frac{1}{3(s+1)^2} - \frac{1}{3s} \\
y(t) &= \frac{1}{3}e^{-t} + \frac{1}{3}te^{-t} - \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
s^2X(s) &= 2 - \frac{1}{s+1} + \frac{1}{s} \\
X(s) &= \frac{2}{s^2} - \frac{1}{s^2(s+1)} + \frac{1}{s^3} \\
&= \frac{1}{s^2} + \frac{1}{s} - \frac{1}{s+1} + \frac{1}{s^3} \\
x(t) &= t + 1 - e^{-t} + \frac{1}{2}t^2
\end{aligned}$$

4.6.7 13

$$\begin{aligned}
m_1x_1'' &= -k_1x_1 + k_2(x_2 - x_1)\\
x_1'' &= -3x_1 + 2(x_2 - x_1)\\
x_1'' + 5x_1 - 2x_2 &= 0\\
s^2X_1(s) - sx_1(0) - x_1'(0) + 5X_1(s) - 2X_2(s) &= 0\\
(s^2+5)X_1(s) - 2X_2(s) &= 1
\end{aligned}$$

$$\begin{aligned}
m_2x_2'' &= -k_2(x_2 - x_1)\\
x_2'' &= -2(x_2 - x_1)\\
-2x_1 + x_2'' + 2x_2 &= 0\\
-2X_1(s) + s^2X_2(s) - sx_2(0) - x_2'(0) + 2X_2(s) &= 0\\
-2X_1(s) + (s^2+2)X_2(s) &= s
\end{aligned}$$

$$\begin{aligned}
&(s^2+2)(s^2+5)X_1(s) - 2(s^2+2)X_2(s) - 4X_1(s)\\
&\qquad\qquad +2(s^2+2)X_2(s) = s^2 + 2s + 2\\
(s^2+2)(s^2+5)X_1(s) - 4X_1(s) &= (s+1)^2 + 1\\
(s^2+1)(s^2+6)X_1(s) &= (s+1)^2 + 1\\
X_1(s) &= \frac{(s+1)^2}{(s^2+1)(s^2+6)}\\
&\quad + \frac{1}{(s^2+1)(s^2+6)}\\
&= \frac{2s+1}{5\,(s^2+1)} + \frac{4-2s}{5\,(s^2+6)}\\
x_1(t) &= \frac{2}{5}\cos t + \frac{1}{5}\sin t\\
&\quad + \frac{2\sqrt{6}}{15}\sin\sqrt{6}t\\
&\quad - \frac{2}{5}\cos\sqrt{6}t
\end{aligned}$$

$$\begin{aligned}
X_2(s) &= s + \frac{4s+2}{5(s^2+2)(s^2+1)}\\
&\quad + \frac{8-4s}{5(s^2+2)(s^2+6)}\\
&= \frac{s-2}{5(s^2+6)} + \frac{2(2s+1)}{5(s^2+1)}\\
x_2(t) &= \frac{1}{5}\cos\sqrt{6}t\\
&\quad - \frac{\sqrt{6}}{15}\sin\sqrt{6}t\\
&\quad + \frac{4}{5}\cos t + \frac{2}{5}\sin t
\end{aligned}$$

4.6.8 21

(a)

$$\begin{aligned}
mx'' &= 0\\
m[s^2X(s) - sx(0) - x'(0)] &= 0\\
ms^2X(s) &= mvcos\theta\\
X(s) &= \frac{v\cos\theta}{s^2}\\
x(t) &= v(\cos\theta)t
\end{aligned}$$

$$\begin{aligned}
my'' &= -mg\\
s^2Y(s) - sy(0) - y'(0) &= -\frac{g}{s}\\
s^2Y(s) &= v\sin\theta - \frac{g}{s}\\
Y(s) &= \frac{v\sin\theta}{s^2} - \frac{g}{s^3}\\
y(t) &= v(\sin\theta)t - \frac{1}{2}gt^2
\end{aligned}$$

(b)

$$\begin{aligned}
x(t) &= v(\cos\theta)t\\
t &= \frac{x(t)}{v\cos\theta}\\
y(x) &= v(\sin\theta)\frac{x}{v\cos\theta} - \frac{1}{2}g\left(\frac{x}{v\cos\theta}\right)^2\\
&= -\frac{g}{2v^2\cos^2\theta}x^2 + (\tan\theta)x
\end{aligned}$$

(c) The solutions of $y(x)$ are $x = 0$ and $x = \frac{v^2}{g}\sin 2\theta$

$$\begin{aligned}
\frac{v_0^2}{g}\sin 2\left(\frac{\pi}{2} - \theta\right) &= \frac{v_0^2}{g}\sin(\pi - 2\theta)\\
&= \frac{v_0^2}{g}\sin 2\theta
\end{aligned}$$

(d) The projectile reaches its maximum height when

$$
\begin{aligned}
y'(x) = 0 &= -\frac{g}{v^2\cos^2\theta}x + \tan\theta \\
x &= \frac{v^2\cos^2\theta}{g}\tan\theta \\
&= \frac{v^2\sin 2\theta}{2g} \\
y(x) &= -\frac{g}{2v^2\cos^2\theta}\left(\frac{v^2\sin 2\theta}{2g}\right)^2 + (\tan\theta)\frac{v^2\sin 2\theta}{2g} \\
&= -\frac{g}{2v^2\cos^2\theta}\frac{v^4\sin^2\theta\cos^2\theta}{g^2} + \frac{v^2\sin 2\theta\tan\theta}{2g} \\
&= -\frac{v^2\sin^2\theta}{2g} + \frac{v^2\sin 2\theta\tan\theta}{2g} \\
&= \frac{v^2}{2g}(\sin 2\theta\tan\theta - \sin^2\theta) \\
&= \frac{v^2}{2g}\sin^2\theta
\end{aligned}
$$

(e) $R_{38} = 2729\,\text{ft},\ H_{38} = 533\,\text{ft},\ R_{52} = 2729\,\text{ft},\ H_{52} = 873\,\text{ft}$

(f) $y(t) = 0$ at $t = 2v(\sin\theta)/g$, $y'(t) = 0$ at $t = v(\sin\theta)/g$
$t_{38} = 11.54\,\text{s},\ t'_{38} = 5.77\,\text{s},\ t_{52} = 14.78\,\text{s},\ t'_{52} = 7.39\,\text{s}$

4.7 Chapter in Review

4.7.1 1

$$
\begin{aligned}
f(t) &= t - 2(t-1)\mathcal{U}(t-1) \\
\mathcal{L}\{f(t)\} &= \frac{1}{s^2} - \frac{2e^{-s}}{s^2}
\end{aligned}
$$

4.7.2 3

False

4.7.3 5

True

4.7.4 7

$$
\mathcal{L}\{e^{-7t}\} = \frac{1}{s+7}
$$

4.7.5 9

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$$

4.7.6 11

$$\mathcal{L}\{t \sin 2t\} = -\frac{d}{ds}\frac{2}{s^2+4} = \frac{4s}{(s^2+4)^2}$$

4.7.7 13

$$\mathcal{L}^{-1}\left\{\frac{20}{s^6}\right\} = \frac{1}{6}t^5$$

4.7.8 15

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-5)^3}\right\} = \frac{1}{2}e^{5t}t^2$$

4.7.9 17

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{s^2-10s+29}\right\} &= \mathcal{L}^{-1}\left\{\frac{s-5}{(s-5)^2+4} + \frac{5}{(s-5)^2+4}\right\} \\ &= e^{5t}\cos 2t + \frac{5}{2}e^{5t}\sin 2t\end{aligned}$$

4.7.10 19

$$\mathcal{L}^{-1}\left\{\frac{s+\pi}{s^2+\pi^2}e^{-s}\right\} = (\cos\pi(t-1) + \sin\pi(t-1))\mathcal{U}(t-1)$$

4.7.11 21

-5

4.7.12 23

$$e^{-k(s-a)}F(s-a)$$

4.7.13 25

$$y = f(t)\mathcal{U}(t-t_0)$$

4.7.14 27

$$y = f(t-t_0)\mathcal{U}(t-t_0)$$

4.7.15 29

$$f = t + (1-t)\mathcal{U}(t-1) - \mathcal{U}(t-4)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-4s}$$

$$\mathcal{L}\{e^t f(t)\} = \frac{1}{(s-1)^2} - \frac{1}{(s-1)^2}e^{-(s-1)} - \frac{1}{s-1}e^{-4(s-1)}$$

4.7.16 31

$$f(t) = 2 + (t-2)\mathcal{U}(t-2)$$

$$\mathcal{L}\{f(t)\} = \frac{2}{s} + e^{-2s}\frac{1}{s^2}$$

$$\mathcal{L}\{e^t f(t)\} = \frac{2}{s-1} + e^{-2(s-1)}\frac{1}{(s-1)^2}$$

4.7.17 35

$$\begin{aligned} y'' - 2y' + y &= e^t \\ s^2Y(s) - sy(0) - y'(0) - 2sY(s) + 2y(0) + Y(s) &= \frac{1}{s-1} \\ (s^2 - 2s + 1)Y(s) &= 5 + \frac{1}{s-1} \\ Y(s) &= \frac{5}{(s-1)^2} + \frac{1}{(s-1)^3} \\ y(t) &= 5e^t t + \frac{1}{2}e^t t^2 \end{aligned}$$

4.7.18 37

$$\begin{aligned} y'' + 6y' + 5y &= t - t\mathcal{U}(t-2) \\ s^2Y(s) - sy(0) + 6[sY(s) - y(0)] + 5Y(s) &= \frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) \\ (s^2 + 6s + 5)Y(s) &= 6 + s + \frac{1}{s^2} - e^{-2s}\frac{1}{s^2} - e^{-2s}\frac{2}{s} \end{aligned}$$

$$
\begin{aligned}
Y(s) &= \frac{6}{(s+1)(s+5)} + \frac{s}{(s+1)(s+5)} + \frac{1}{s^2(s+1)(s+5)} \\
&\quad - e^{-2s}\frac{1}{s^2(s+1)(s+5)} - e^{-2s}\frac{2}{s(s+1)(s+5)} \\
y(t) &= \frac{1}{5}t - \frac{6}{25} + \frac{3}{2}e^{-t} - \frac{13}{50}e^{-5t} \\
&\quad - \left(\frac{1}{5}(t-2) + \frac{4}{25} - \frac{1}{4}e^{-(t-2)} + \frac{9}{100}e^{-5(t-2)}\right)\mathcal{U}(t-2)
\end{aligned}
$$

4.7.19 39

$$
\begin{aligned}
y'(t) &= \cos t + \int_0^t y(\tau)\cos(t-\tau)\,d\tau \\
sY(s) - 1 &= \frac{s}{s^2+1} + Y(s)\frac{s}{s^2+1} \\
\left(s - \frac{s}{s^2+1}\right)Y(s) &= 1 + \frac{s}{s^2+1} \\
\left(\frac{s(s^2+1)-s}{s^2+1}\right)Y(s) &= \frac{s^2+1+s}{s^2+1} \\
\left(\frac{s^3+s-s}{s^2+1}\right)Y(s) &= \frac{s^2+s+1}{s^2+1} \\
Y(s) &= \frac{s^2+s+1}{s^3} \\
&= \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} \\
y(t) &= 1 + t + \frac{1}{2}t^2
\end{aligned}
$$

4.7.20 41

$$
\begin{aligned}
x' + y &= t \\
sX(s) + Y(s) &= 1 + \frac{1}{s^2}
\end{aligned}
$$

$$
\begin{aligned}
4x + y' &= 0 \\
4X(s) + sY(s) &= 2
\end{aligned}
$$

$$
\begin{aligned}
s^2X(s) + sY(s) - 4X(s) - sY(s) &= s + \frac{1}{s} - 2 \\
(s^2 - 4)X(s) &= s + \frac{1}{s} - 2 \\
X(s) &= \frac{s}{(s-2)(s+2)} + \frac{1}{s(s-2)(s+2)} \\
&\quad - \frac{2}{(s-2)(s+2)} \\
&= \frac{1}{8(s-2)} - \frac{1}{4s} + \frac{9}{8(s+2)} \\
x(t) &= \frac{1}{8}e^{2t} - \frac{1}{4} + \frac{9}{8}e^{-2t}
\end{aligned}
$$

$$
\begin{aligned}
Y(s) &= 1 + \frac{1}{s^2} - \frac{s}{8(s-2)} + \frac{1}{4} - \frac{9s}{8(s+2)} \\
&= \frac{1}{s^2} + \frac{9}{4(s+2)} - \frac{1}{4(s-2)} \\
y(t) &= t + \frac{9}{4}e^{-2t} - \frac{1}{4}e^{2t}
\end{aligned}
$$

4.7.21 43

$$\begin{aligned}
10i + 2\int_0^t i(\tau)\,d\tau &= 2(t^2+t) \\
10I(s) + 2\frac{I(s)}{s} &= \frac{4}{s^3} + \frac{2}{s^2} \\
\left(10 + \frac{2}{s}\right) I(s) &= \frac{4}{s^3} + \frac{2}{s^2} \\
\left(\frac{10s+2}{s}\right) I(s) &= \frac{4}{s^3} + \frac{2}{s^2} \\
I(s) &= \frac{4}{s^2(10s+2)} + \frac{2}{s(10s+2)} \\
&= \frac{2}{s^2(5s+1)} + \frac{1}{s(5s+1)} \\
&= \frac{2}{s^2} - \frac{9}{s} + \frac{45}{5s+1} \\
i(t) &= 2t - 9 + \frac{45}{5}e^{-t/5}
\end{aligned}$$

5 Series Solutions of Linear Differential Equations

5.1 Solutions about Ordinary Points

5.1.1 1

$$\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{n+1}x^{n+1}}{\frac{2^n}{n}x^n}\right| = 2|x|\lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

The series converges for $|x| < 1/2$, so the radius of convergence is $R = 1/2$. When $x < 0$,

$$\lim_{n\to\infty}\frac{2^n}{n}x^n = \lim_{n\to\infty}\frac{2^n}{n}(-1)^n(-x)^n$$

which converges when $x = -1/2$ so the interval of convergence is $[-1/2, 1/2)$

5.1.2 3

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{10^{n+1}}(x-5)^{n+1}}{\frac{(-1)^n}{10^n}(x-5)^n}\right| = \frac{|x-5|}{10}$$

The series converges for $|x-5| < 10$, so the radius of convergence is $R = 10$. When $x = 15$,

$$\lim_{n\to\infty}\frac{(-1)^k}{10^k}10^k = \lim_{n\to\infty}(-1)^k$$

which doesn't converge. When $x = -5$,

$$\lim_{n\to\infty} \frac{(-1)^k}{10^k}(-10)^k = \lim_{n\to\infty} \frac{(-1)^k}{10^k}(-1)^k(10)^k = \lim_{n\to\infty}(-1)^2k = 1$$

which also doesn't converge, so the interval of convergence is $(-5, 15)$.

5.1.3 5

$$x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315}$$

5.1.4 7

$$1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720}$$

$(-\pi/2, \pi/2)$

5.1.5 9

$$\sum_{n=1}^{\infty} nc_n x^{n+2} = \sum_{k=3}^{\infty}(k-2)c_{k-2}x^k$$

5.1.6 11

$$\begin{aligned}\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6c_n x^{n+1} &= 2c_1 + \sum_{n=2}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6c_n x^{n+1} \\ &= 2c_1 + \sum_{k=1}^{\infty} 2(k+1)c_{k+1}x^k + \sum_{k=1}^{\infty} 6c_{k-1}x^k \\ &= 2c_1 \sum_{k=1}^{\infty}[2(k+1)c_{k+1} + 6c_{k-1}]x^k\end{aligned}$$

5.1.7 15

$$\begin{aligned}(x^2-25)y'' + 2xy' + y = 0 \\ y'' + \frac{2x}{(x-5)(x+5)}y' + \frac{1}{(x-5)(x+5)}y = 0\end{aligned}$$

$R_0 = 5$, $R_1 = 4$

5.1.8 17

$$
\begin{aligned}
y'' - 3xy &= 0 \\
\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - 3x\sum_{n=0}^{\infty} c_n x^n &= 0 \\
\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} 3c_n x^{n+1} &= 0 \\
2c_2 + \sum_{n=3}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} 3c_n x^{n+1} &= 0 \\
2c_2 + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} 3c_{k-1}x^k &= 0 \\
2c_2 + \sum_{k=1}^{\infty} [c_{k+2}(k+2)(k+1) - 3c_{k-1}]x^k &= 0
\end{aligned}
$$

$c_2 = 0$

$$
\begin{aligned}
3\cdot 2\cdot c_3 - 3c_0 &= \frac{3!}{1!}c_3 - 3c_0 = 0 \Rightarrow c_3 = \frac{1!}{3!}3c_0 \\
4\cdot 3\cdot c_4 - 3c_1 &= \frac{4!}{2!}c_4 - 3c_1 = 0 \Rightarrow c_4 = \frac{2!}{4!}3c_1 \\
5\cdot 4\cdot c_5 - 3c_2 &= \frac{5!}{3!}c_5 = 0 \Rightarrow c_5 = 0 \\
6\cdot 5\cdot c_6 - 3c_3 &= \frac{6!}{4!}c_6 - 3c_3 \Rightarrow c_6 = \frac{4!}{6!}\frac{1!}{3!}3^2 c_0 \\
7\cdot 6\cdot c_7 - 3c_4 &= \frac{7!}{5!}c_7 - 3c_4 \Rightarrow c_7 = \frac{5!}{7!}\frac{2!}{4!}3^2 c_1 \\
8\cdot 7\cdot c_8 - 3c_5 &= \frac{8!}{6!}c_8 = 0 \Rightarrow c_8 = 0 \\
9\cdot 8\cdot c_9 - 3c_6 &= \frac{9!}{8!}c_9 - 3c_6 \Rightarrow c_9 = \frac{8!}{9!}\frac{4!}{6!}\frac{1!}{3!}3^3 c_0 \\
10\cdot 9\cdot c_10 - 3c_7 &= \frac{10!}{9!}c_10 - 3c_7 \Rightarrow c_10 = \frac{9!}{10!}\frac{5!}{7!}\frac{2!}{4!}3^3 c_1
\end{aligned}
$$

$$
\begin{aligned}
y &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots \\
&= c_0 + c_1 x + \frac{1}{3\cdot 2} 3^1 c_0 x^3 + \frac{1}{4\cdot 3} 3^1 c_1 x^4 + \frac{1}{6\cdot 5\cdot 3\cdot 2} 3^2 c_0 x^6 \\
&\quad + \frac{1}{7\cdot 6\cdot 4\cdot 3} 3^2 c_1 x^7 + \cdots \\
&= c_0 \left(1 + \frac{1}{3\cdot 2} 3^1 x^3 + \frac{1}{6\cdot 5\cdot 3\cdot 2} 3^2 x^6 + \cdots\right) \\
&\quad + c_1 \left(x + \frac{1}{4\cdot 3} 3^1 x^4 + \frac{1}{7\cdot 6\cdot 4\cdot 3} 3^2 x^7 + \cdots\right)
\end{aligned}
$$

5.1.9 19

$$
\begin{aligned}
y'' - 2xy' + y &= 0 \\
\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - 2x \sum_{n=1}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n &= 0 \\
\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} 2c_n n x^n + \sum_{n=0}^{\infty} c_n x^n &= 0 \\
2c_2 + \sum_{n=3}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} 2c_n n x^n + c_0 + \sum_{n=1}^{\infty} c_n x^n &= 0 \\
c_0 + 2c_2 + \sum_{n=1}^{\infty} c_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} 2c_n n x^n + \sum_{n=1}^{\infty} c_n x^n &= 0 \\
c_0 + 2c_2 + \sum_{n=1}^{\infty} [c_{n+2}(n+2)(n+1) + (1-2n)c_n]x^n &= 0
\end{aligned}
$$

$$
c_0 + 2c_2 = 0 \Rightarrow c_2 = -\frac{1}{2} c_0
$$

$$
\begin{aligned}
3\cdot 2\cdot c_3 - c_1 = 0 &\Rightarrow c_3 = \frac{1}{3\cdot 2} c_1 \\
4\cdot 3\cdot c_4 - 3c_2 = 0 &\Rightarrow c_4 = -\frac{3}{4\cdot 3\cdot 2} c_0 \\
5\cdot 4\cdot c_5 - 5c_3 = 0 &\Rightarrow c_5 = \frac{5}{5\cdot 4\cdot 3\cdot 2} c_1 \\
6\cdot 5\cdot c_6 - 7c_4 = 0 &\Rightarrow c_6 = -\frac{7\cdot 3}{6\cdot 5\cdot 4\cdot 3\cdot 2} c_0 \\
7\cdot 6\cdot c_7 - 9c_5 = 0 &\Rightarrow c_7 = \frac{9\cdot 5}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} c_1
\end{aligned}
$$

$$
\begin{aligned}
y &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + x_7 x^7 + \cdots \\
&= c_0 + c_1 x - \frac{1}{2} c_0 x^2 + \frac{1}{3!} c_1 x^3 - \frac{3}{4!} c_0 x^4 + \frac{5}{5!} c_1 x^5 - \frac{21}{6!} c_0 x^6 + \frac{45}{7!} c_1 x^7 + \cdots \\
&= c_0 \left(1 - \frac{1}{2!} x^2 - \frac{3}{4!} x^4 - \frac{21}{6!} x^6 + \cdots \right) \\
&\quad + c_1 \left(x + \frac{1}{3!} x^3 + \frac{5}{5!} x^5 + \frac{45}{7!} x^7 + \cdots \right)
\end{aligned}
$$

5.1.10 21

$$
\begin{aligned}
y'' + x^2 y' + xy &= 0 \\
\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + \sum_{n=1}^{\infty} c_n n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+1} &= 0 \\
2c_2 + 6c_3 x + \sum_{n=4}^{\infty} c_n n(n-1) x^{n-2} + \sum_{n=1}^{\infty} c_n n x^{n+1} + c_0 x + \sum_{n=1}^{\infty} c_n x^{n+1} &= 0 \\
2c_2 + 6c_3 x + c_0 x + \sum_{n=2}^{\infty} [c_{n+2}(n+2)(n+1) + c_{n-1} n] x^n &= 0
\end{aligned}
$$

$$c_2 = 0$$

$$6c_3 + c_0 = 0 \Rightarrow c_3 = -\frac{1}{3!} c_0$$

$$
\begin{aligned}
4 \cdot 3 \cdot c_4 + 2c_1 = 0 &\Rightarrow c_4 = -\frac{2^2}{4!} c_1 \\
5 \cdot 4 \cdot c_5 + 3c_2 = 0 &\Rightarrow c_5 = 0 \\
6 \cdot 5 \cdot c_6 + 4c_3 = 0 &\Rightarrow c_6 = \frac{4^2}{6!} c_0 \\
7 \cdot 6 \cdot c_7 + 5c_4 = 0 &\Rightarrow c_7 = \frac{5^2 \cdot 2^2}{7!} c_1 \\
8 \cdot 7 \cdot c_8 + 6c_5 = 0 &\Rightarrow c_8 = 0 \\
9 \cdot 8 \cdot c_9 + 7c_6 = 0 &\Rightarrow c_9 = -\frac{7^2 \cdot 4^2}{9!} c_0 \\
10 \cdot 9 \cdot c_{10} + 8c_7 = 0 &\Rightarrow c_{10} = -\frac{8^2 \cdot 5^2 \cdot 2^2}{10!} c_1
\end{aligned}
$$

$$
\begin{aligned}
y &= c_0 + c_1 x + c_2 x^2 + c_3 + \cdots \\
&= c_0 + c_1 x - \frac{1}{3!} c_0 x^3 - \frac{2^2}{4!} c_1 x^4 + \frac{4^2}{6!} c_0 x^6 + \frac{5^2 \cdot 2^2}{7!} c_1 x^7 + \cdots \\
&= c_0 \left(1 - \frac{1}{3!} x^3 + \frac{4^2}{6!} x^6 + \cdots \right) \\
&\quad + c_1 \left(x - \frac{2^2}{4!} x^4 + \frac{5^2 \cdot 2^2}{7!} x^7 + \cdots \right)
\end{aligned}
$$

5.2 Solutions about Singular Points

5.2.1 1

$$x^3 y'' + 4x^2 y' + 3y = 0 \Rightarrow y'' + \frac{4}{x} y' + \frac{3}{x^3} y = 0$$

$x = 0$ is an irregular singular point

5.2.2 3

$$(x^2 - 9)^2 y'' + (x+3)y' + 2y = 0 \Rightarrow y'' \frac{1}{(x+3)(x-3)^2} y' + \frac{2}{(x+3)^2(x-3)^2} y = 0$$

$x = -3$ is a regular singular point, $x = 3$ is an irregular singular point

5.2.3 5

$$(x^3 + 4x)y'' - 2xy' + 6y = 0 \Rightarrow y'' - \frac{2}{(x+2i)(x-2i)} + \frac{6}{x(x+2i)(x-2i)} y = 0$$

$x = 0$, $x = -2i$, $x = 2i$ regular

5.2.4 7

$$(x^2 + x - 6)y'' + (x+3)y' + (x-2)y = 0 \Rightarrow y'' + \frac{1}{x-2} + \frac{1}{x+3} y = 0$$

$x = -3$, $x = 2$ regular

5.2.5 9

$$y'' + \frac{3}{x^2(x-2)(x-5)(x+5)} y' + \frac{7}{x^3(x-5)(x-2)^2} y = 0$$

$x = -5$, $x = 2$, $x = 5$ regular
$x = 0$ irregular

5.2.6 11

$$
\begin{aligned}
(x^2-1)y''+5(x+1)y'+(x^2-x)y &= 0\\
(x+1)(x-1)y''+5(x+1)y'+x(x-1)y &= 0\\
y''+\frac{5}{x-1}y'+\frac{x}{x+1}y &= 0
\end{aligned}
$$

$$
\begin{aligned}
(x-1)^2y''+(x-1)5y'+(x-1)^2\frac{x}{x+1}y &= 0\\
p(x) &= 5\\
q(x) &= (x-1)^2\frac{x}{x+1}
\end{aligned}
$$

$$
\begin{aligned}
(x+1)^2y''+(x+1)^2\frac{5}{x-1}y'+(x+1)xy &= 0\\
p(x) &= (x+1)\frac{5}{x-1}\\
q(x) &= (x+1)x
\end{aligned}
$$

5.2.7 13

$$
\begin{aligned}
x^2y''+\left(\frac{5}{3}x+x^2\right)y'-\frac{1}{3}y &= 0\\
y''+\left(\frac{5}{3}x^{-1}+1\right)y'-\frac{1}{3}x^{-2}y &= 0\\
p(x) = xP(x) &= \frac{5}{3}+x\\
q(x) = x^2Q(x) &= -\frac{1}{3}\\
a_0 &= \frac{5}{3}\\
b_0 &= -\frac{1}{3}\\
r(r-1)+a_0r+b_0 &= 0\\
r^2-r+\frac{5}{3}r-\frac{1}{3} &= 0\\
r^2+\frac{2}{3}r-\frac{1}{3} &= 0\\
(r+1)(r-1/3) &= 0\\
r_1 &= \frac{1}{3}\\
r_2 &= -1
\end{aligned}
$$

5.2.8 15

$$
\begin{aligned}
2xy'' - y' + 2y &= 0 \\
y'' - \frac{1}{2}x^{-1}y' + x^{-1}y &= 0 \\
p(x) &= xP(x) = -\frac{1}{2} \\
q(x) &= x^2Q(x) = x \\
a_0 &= -\frac{1}{2} \\
b_0 &= 0 \\
r(r-1) + a_0 r + b_0 &= 0 \\
r^2 - r - \frac{1}{2}r &= 0 \\
r^2 - \frac{3}{2}r &= 0 \\
r\left(r - \frac{3}{2}\right) &= 0 \\
r_1 &= \frac{3}{2} \\
r_2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} - \sum_{n=0}^{\infty}\frac{1}{2}(n+r)c_n x^{n+r-2} + \sum_{n=0}^{\infty}c_n x^{n+r-1} &= 0 \\
x^r\Bigg[r(r-1)c_0x^{-2} - \frac{1}{2}rc_0x^{-2} + \sum_{n=1}^{\infty}(n+r)(n+r-1)c_n x^{n-2} \\
- \sum_{n=1}^{\infty}\frac{1}{2}(n+r)c_n x^{n-2} + \sum_{n=0}^{\infty}c_n x^{n-1}\Bigg] &= 0 \\
x^r\left[r\left(r - \frac{3}{2}\right)c_0x^{-2} + \sum_{n=-1}^{\infty}\left((n+r+2)\left(n+r+\frac{1}{2}\right)c_{n+2} + c_{n+1}\right)x^n\right] &= 0
\end{aligned}
$$

$$\left(n+\frac{7}{2}\right)(n+2)c_{n+2}+c_{n+1}=0$$

$$\begin{aligned}
c_{n+2}&=-\frac{c_{n+1}}{\left(n+\frac{7}{2}\right)(n+2)}\\
c_1&=-\frac{c_0}{\frac{5}{2}\frac{2}{2}}=-\frac{2c_0}{5}\\
c_2&=-\frac{c_1}{\frac{7}{2}\frac{4}{2}}=\frac{2^2c_0}{7\cdot 5\cdot 2}\\
c_3&=-\frac{c_2}{\frac{9}{2}\frac{6}{2}}=-\frac{2^3c_0}{9\cdot 7\cdot 5\cdot 3!}
\end{aligned}$$

$$(n+2)\left(n+\frac{1}{2}\right)c_{n+2}+c_{n+1}=0$$

$$\begin{aligned}
c_{n+2}&=-\frac{c_{n+1}}{(n+2)\left(n+\frac{1}{2}\right)}\\
c_1&=-\frac{c_0}{1\cdot -\frac{1}{2}}=2c_0\\
c_2&=-\frac{c_1}{2\cdot\frac{1}{2}}=-2c_0\\
c_3&=-\frac{c_2}{3\cdot\frac{3}{2}}=\frac{2^3c_0}{3\cdot 3!}
\end{aligned}$$

$$\begin{aligned}
y(x)=C_1x^{3/2}&\left(1-\frac{2}{5}x+\frac{2^2}{7\cdot 5\cdot 2}x^2-\frac{2^3}{9\cdot 7\cdot 5\cdot 3!}x^3+\cdots\right)\\
&+C_2\left(1+2x-2x^2+\frac{2^3}{3\cdot 3!}x^3+\cdots\right)
\end{aligned}$$

5.2.9 17

$$
\begin{aligned}
4xy'' + \frac{1}{2}y' + y &= 0 \\
\sum_{n=0}^{\infty} 4(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty} \frac{1}{2}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \\
x^r \left[4r(r-1)c_0 x^{-1} + \sum_{n=1}^{\infty} 4(n+r)(n+r-1)c_n x^{n-1} + \frac{1}{2} r c_0 x^{-1} \right. & \\
\left. + \sum_{n=1}^{\infty} \frac{1}{2}(n+r)c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n \right] &= 0 \\
x^r \left[r\left(4r - \frac{7}{2}\right) c_0 x^{-1} + \sum_{n=0}^{\infty} \left((n+r+1)\left(4n + 4r + \frac{1}{2}\right) c_{n+1} + c_n \right) x^n \right] &= 0
\end{aligned}
$$

$$
\begin{aligned}
r\left(4r - \frac{7}{2}\right) &= 0 \\
r_1 &= \frac{7}{8} \\
r_2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
\left(n + \frac{7}{8} + 1\right)\left(4n + \frac{7}{2} + \frac{1}{2}\right) c_{n+1} + c_n &= 0 \\
\left(n + \frac{15}{8}\right)(4n+4)\, c_{n+1} + c_n &= 0 \\
c_{n+1} &= -\frac{c_n}{4\left(n + \frac{15}{8}\right)(n+1)} \\
c_1 &= -\frac{c_0}{4 \cdot \frac{15}{8} \cdot 1} = -\frac{2c_0}{15} \\
c_2 &= -\frac{c_1}{4 \cdot \frac{23}{8} \cdot 2} = -\frac{c_1}{23} = \frac{2^2 c_0}{23 \cdot 15 \cdot 2} \\
c_3 &= -\frac{c_2}{4 \cdot \frac{31}{8} \cdot 3} \\
&= -\frac{2c_2}{93} \\
&= -\frac{2^3 c_0}{31 \cdot 23 \cdot 15 \cdot 3!}
\end{aligned}
$$

$$(n+1)\left(4n+\frac{1}{2}\right)c_{n+1}+c_n=0$$

$$c_{n+1}=-\frac{c_n}{(n+1)\left(4n+\frac{1}{2}\right)}$$

$$c_1=-2c_0$$

$$c_2=-\frac{c_1}{2\cdot\frac{9}{2}}=\frac{2^2c_0}{9\cdot 2}$$

$$c_3=-\frac{c_2}{3\cdot\frac{17}{2}}=-\frac{2^3c_0}{17\cdot 9\cdot 3!}$$

$$y=C_1x^{7/8}\left(1-\frac{2}{15}x+\frac{2^2}{23\cdot 15\cdot 2}x^2-\frac{2^3}{31\cdot 23\cdot 15\cdot 3!}x^3+\cdots\right)$$
$$+C_2\left(1-2x+\frac{2^2}{9\cdot 2}x^2-\frac{2^3}{17\cdot 9\cdot 3!}x^3+\cdots\right)$$

5.2.10 25

$$xy''+2y'-xy=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-1}+\sum_{n=0}^{\infty}2(n+r)c_nx^{n+r-1}-\sum_{n=0}^{\infty}c_nx^{n+r+1}=0$$

$$x^r\left[r(r-1)c_0x^{-1}+r(r+1)c_1+\sum_{n=2}^{\infty}(n+r)(n+r-1)c_nx^{n-1}\right.$$
$$\left.+2rc_0x^{-1}+2(r+1)c_1+\sum_{n=2}^{\infty}2(n+r)c_nx^{n-1}-\sum_{n=0}^{\infty}c_nx^{n+1}\right]=0$$

$$x^r\left[r(r+1)c_0x^{-1}+(r+1)(r+2)c_1\right.$$
$$\left.+\sum_{n=1}^{\infty}\left((n+r+1)(n+r+2)c_{n+1}-c_{n-1}\right)x^n\right]=0$$

$$r(r+1)=0$$

$$r_1=0$$

$$r_2=-1$$

$$(r+1)(r+2)c_1 = 0 \Rightarrow 2c_1 = 0 \Rightarrow c_1 = 0$$

$$(n+1)(n+2)c_{n+1} - c_{n-1} = 0$$

$$\begin{aligned}
c_{n+1} &= \frac{c_{n-1}}{(n+1)(n+2)} \\
c_2 &= \frac{c_0}{3!} \\
c_3 &= \frac{c_1}{3 \cdot 4} = 0 \\
c_4 &= \frac{c_2}{4 \cdot 5} = \frac{c_0}{5!} \\
c_5 &= \frac{c_3}{5 \cdot 6} = 0 \\
c_6 &= \frac{c_4}{6 \cdot 7} = \frac{c_0}{7!}
\end{aligned}$$

$$\begin{aligned}
y_1(x) &= C_1 \left[1 + \frac{1}{3!}x^2 + \frac{1}{5!}x^4 + \cdots\right] \\
&= C_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n} \\
&= C_1 \frac{\sinh x}{x} \\
y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \\
&= -\frac{\cosh x}{x} \\
y &= \frac{1}{x}(C_1 \sinh x + C_2 \cosh x)
\end{aligned}$$