Vibrations and Waves by George C. King Notes

Chris Doble

April 2022

Contents

1 Simple Harmonic Motion

- The equation of motion for a simple harmonic oscillator is

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

  where

$$\omega^2 = \frac{k}{m}$$

- The general solution of the equation of motion for a simple harmonic oscillator is

$$x = A\cos(\omega t + \phi)$$

  or equivalently

$$x = a\cos\omega t + b\sin\omega t$$

- The angular frequency ω is determined entirely by properties of the oscillator, e.g. its mass and spring coefficient

- The total energy of a harmonic oscillator is

$$E = \frac{1}{2}kA^2$$

- Nearly all potential wells have a shape that is parabolic when sufficiently close to the equilibrium position, so most oscillating systems will oscillate with SHM when the amplitude of oscillation is small

- The vibrations of nuclei in a molecule can be modeled by SHM, but only a discrete set of vibrational energies is possible, namely

$$\frac{1}{2}\hbar\omega,\ \frac{3}{2}\hbar\omega,\ \frac{5}{2}\hbar\omega,\ \ldots$$

  where $\hbar$ is Planck's constant divided by 2π

- The total energy of a system undergoing SHM is always given by an expression of the form

$$E = \frac{1}{2}\alpha v^2 + \frac{1}{2}\beta x^2$$

  where α and β are physical constants — if we obtain this equation during the analysis of a system we know we have SHM

- The equation of motion for a system described by the energy equation above is

$$\frac{d^2x}{dt^2} = -\frac{\beta}{\alpha}x$$

2 The Damped Harmonic Oscillator

- The equation of motion of a damped harmonic oscillator is

$$\begin{aligned} F = ma &= -kx - bv \\ m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx &= 0 \\ \frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x &= 0 \end{aligned}$$

  where $\gamma = b/m$ and $\omega_0^2 = k/m$

- ω_0 is known as the **natural frequency of oscillation**, i.e. the oscillation frequency if there were no damping

- **Light damping / underdamped**

  - The motion is still oscillatory but the amplitude decreases expontentially
  - This occurs when $\gamma^2/4 < \omega_0^2$
  - The general solution is

$$x = A_0 e^{-\gamma t/2}\cos(\omega t + \phi)$$

    where A_0 is the initial amplitude

- Successive maxima decrease by the same fractional amount

$$\frac{A_n}{A_{n+1}} = e^{\gamma T/2}$$

- The natural logarithm of A_n/A_{n+1} is called the **logarithmic decrement**

$$\ln\left(\frac{A_n}{A_{n+1}}\right) = \frac{\gamma T}{2}$$

- **Heavy damping / overdamped**
  - The motion is not oscillatory and returns sluggishly to the equilibrium position
  - This occurs when $\gamma^2/4 > \omega_0^2$
  - The general solution is

$$\begin{aligned} x &= e^{-\gamma t/2}[Ae^{\alpha t} + Be^{-\alpha t}] \\ &= Ae^{(\alpha-\gamma/2)t} + Be^{-(\alpha+\gamma/2)t} \end{aligned}$$

  where $\alpha = \sqrt{\gamma^2/4 - \omega_0^2}$

- **Critical damping**
  - The motion is not oscillatory and returns as quickly as possible to the equilibrium position
  - This occurs when $\gamma^2/4 = \omega_0^2$
  - The general solution is

$$x = Ae^{-\gamma t/2} + Bte^{-\gamma t/2}$$

- The total energy of an underdamped system decreases over time

$$E = E_0 e^{-\gamma t}$$

  where E_0 is the initial energy of the system

- The **decay time** or **time constant** of the system $\tau = 1/\gamma$ is the time it takes for its energy to decrease by a factor of e

- The **quality factor** of a harmonic oscillator is a dimensionless value that gives a measure of the degree of damping

$$Q = \frac{\omega_0}{\gamma}$$

  where large values indicate little damping and small values indicate more damping

- The quality factor can also be used as a measure of the energy lost per cycle $2\pi/Q$ or the energy lost per radian $1/Q$