Classical Mechanics by John R. Taylor Problems

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Contents

1 Newton's Laws of Motion

1.1

$$
\begin{aligned}
\mathbf{b} + \mathbf{c} &= 2\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}} \\
5\mathbf{b} + 2\mathbf{x} &= 7\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 2\hat{\mathbf{z}} \\
\mathbf{b} \cdot \mathbf{c} &= 1 \\
\mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \\
&= \hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}}
\end{aligned}
$$

1.5

$$
\begin{aligned}
\mathbf{v}_{\text{body}} &= \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}} \\
\mathbf{v}_{\text{face}} &= \hat{\mathbf{x}} + \hat{\mathbf{z}} \\
\mathbf{v}_{\text{body}} \cdot \mathbf{v}_{\text{face}} &= v_{\text{body}} v_{\text{face}} \cos\theta \\
2 &= \sqrt{6} \cos\theta \\
\cos\theta &= \frac{2}{\sqrt{6}} \\
\theta &= \arccos\frac{2}{\sqrt{6}} \\
&= 35.26^\circ
\end{aligned}
$$

1.11

The particle moves counterclockwise in an ellipse of width $2b$ and height $2c$. The angular speed is ω.

1.23

$$
\begin{aligned}
\mathbf{v} &= v\cos\theta\frac{\mathbf{b}}{b} - v\sin\theta\frac{\mathbf{b}\times\mathbf{c}}{bc} \\
&= \frac{\lambda}{b}\frac{\mathbf{b}}{b} - \frac{c}{b}\frac{\mathbf{b}\times\mathbf{c}}{bc} \\
&= \frac{\lambda\mathbf{b} - \mathbf{b}\times\mathbf{c}}{b^2}
\end{aligned}
$$

1.25

$$
\begin{aligned}
\frac{df}{dt} &= -3f \\
\frac{1}{f}\frac{df}{dt} &= -3 \\
\ln f &= -3t + c \\
f &= ce^{-3t}
\end{aligned}
$$

One constant.

1.35

$$
\begin{aligned}
F_x &= 0 \\
ma_x &= 0 \\
a_x &= 0 \\
v_x &= c_1 \\
&= v_o \cos\theta \\
r_x &= v_o \cos(\theta) t + c_2 \\
&= v_o \cos(\theta) t
\end{aligned}
$$

$$
\begin{aligned}
F_y &= 0 \\
ma_y &= 0 \\
a_y &= 0 \\
v_y &= c_3 \\
v_y &= 0 \\
r_y &= c_4 \\
r_y &= 0
\end{aligned}
$$

$$
\begin{aligned}
F_z &= -mg \\
ma_z &= -mg \\
a_z &= -g \\
v_z &= -gt + c_5 \\
&= v_o \sin\theta - gt \\
r_z &= v_o \sin(\theta) t - \frac{1}{2} g t^2 + c_6 \\
&= v_o \sin(\theta) t - \frac{1}{2} g t^2
\end{aligned}
$$

$$
\begin{aligned}
0 &= v_o \sin(\theta) t - \frac{1}{2} g t^2 \\
t &= \frac{2\sin(\theta) v_o}{g} \\
r_x &= v_o \cos(\theta) t \\
&= \frac{2\cos(\theta)\sin(\theta) v_o^2}{g} \\
&= \frac{\sin(2\theta) v_o^2}{g}
\end{aligned}
$$

1.37

(a)

$$\begin{aligned}
F &= -mg\sin\theta \\
ma &= -mg\sin\theta \\
a &= -g\sin\theta \\
v &= c_1 - gt\sin\theta \\
&= v_o - gt\sin\theta \\
x &= v_o t - \frac{1}{2}gt^2\sin\theta
\end{aligned}$$

(b)

$$t = \frac{2v_o}{g\sin\theta}$$

1.39

$$
\begin{aligned}
F_x &= -mg\sin\phi \\
ma_x &= -mg\sin\phi \\
a_x &= -g\sin\phi \\
v_x &= c_1 - gt\sin\phi \\
&= v_o\cos\theta - gt\sin\phi \\
r_x &= v_o t\cos\theta - \frac{1}{2}gt^2\sin\phi + c_2 \\
&= v_o t\cos\theta - \frac{1}{2}gt^2\sin\phi
\end{aligned}
$$

$$
\begin{aligned}
F_y &= -mg\cos\phi \\
ma_y &= -mg\cos\phi \\
a_y &= -g\cos\phi \\
v_y &= c_3 - gt\cos\phi \\
&= v_o\sin\theta - gt\cos\phi \\
r_y &= v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi + c_4 \\
&= v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi
\end{aligned}
$$

$$
\begin{aligned}
0 &= v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi \\
t &= \frac{2v_o\sin\theta}{g\cos\phi}
\end{aligned}
$$

$$
\begin{aligned}
r_x &= \frac{2v_o^2\cos\theta\sec\phi\sin\theta}{g} - \frac{2v_o^2\sec\phi\sin^2\theta\tan\phi}{g} \\
&= \frac{2v_o^2\sin\theta(\cos\theta\cos\phi - \sin\theta\sin\phi)}{g\cos^2\phi} \\
&= \frac{2v_o^2\sin\theta\cos(\theta+\phi)}{g\cos^2\phi}
\end{aligned}
$$

$$
\begin{aligned}
\frac{dr_x}{d\theta} &= \frac{2v_o^2}{g\cos^2\phi}[\cos\theta\cos(\theta+\phi) - \sin\theta\sin(\theta+\phi)] \\
&= \frac{2v_o^2\cos(2\theta+\phi)}{g\cos^2\phi} \\
0 &= \frac{2v_o^2\cos(2\theta+\phi)}{g\cos^2\phi} \\
&= \cos(2\theta+\phi) \\
2\theta+\phi &= \frac{\pi}{2} \\
\theta &= \frac{\pi}{4} - \frac{\phi}{2} \\
r_{x,\max} &= \frac{2v_o^2\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\phi}{2}+\phi\right)}{g\cos^2\phi} \\
&= \frac{v_o^2(1-\sin\phi)}{g\cos^2\phi} \\
&= \frac{v_o^2}{g(1+\sin\phi)}
\end{aligned}
$$

1.41

$$
\begin{aligned}
F &= ma \\
T &= m\frac{v^2}{R} \\
&= m\frac{(\omega R)^2}{R} \\
&= m\omega^2 R
\end{aligned}
$$

1.47

(a)

$$
\begin{aligned}
\rho &= \sqrt{x^2+y^2} \\
\phi &= \arctan\frac{y}{x} \\
z &= z
\end{aligned}
$$

ρ is the distance of P from the z-axis.

The use of r may be unfortunate because it suggests it's the distance of P from the origin.

(b) $\hat{\boldsymbol{\rho}}$ points away from the z-axis, $\hat{\boldsymbol{\phi}}$ points counter-clockwise around the z-axis, and $\hat{\mathbf{z}}$ points in the positive z direction.

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}} + \sqrt{x^2 + y^2}\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$$

(c)

$$\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{r}}{dt} \\
&= \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\frac{d\hat{\boldsymbol{\rho}}}{dt} + \dot{z}\hat{\mathbf{z}} + z\frac{d\hat{\mathbf{z}}}{dt} \\
&= \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}} \\
\mathbf{a} &= \frac{d\mathbf{v}}{dt} \\
&= \ddot{\rho}\hat{\boldsymbol{\rho}} + \dot{\rho}\frac{d\hat{\boldsymbol{\rho}}}{dt} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}} + \rho\ddot{\phi}\hat{\boldsymbol{\phi}} + \rho\dot{\phi}\frac{d\hat{\boldsymbol{\phi}}}{dt} + \ddot{z}\hat{\mathbf{z}} \\
&= \ddot{\rho}\hat{\boldsymbol{\rho}} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}} + \rho\ddot{\phi}\hat{\boldsymbol{\phi}} - \rho\dot{\phi}^2\hat{\boldsymbol{\rho}} + \ddot{z}\hat{\mathbf{z}} \\
&= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\boldsymbol{\rho}} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\mathbf{z}}
\end{aligned}$$

2 Projectiles and Charged Particles

2.1

$$\begin{aligned}
1 &= (1.6 \times 10^3)Dv \\
v &= \frac{1}{(1.6 \times 10^3)D} \\
&= 8.9\,\mathrm{mm/s}
\end{aligned}$$

When $v \gg 1\,\mathrm{cm/s}$ the drag force can be treated as purely quadratic. For a beach ball this becomes $v \gg 1\,\mathrm{mm/s}$.

2.3

(a)

$$\begin{aligned}
\frac{f_{\mathrm{quad}}}{f_{\mathrm{lin}}} &= \frac{(1/4)\rho A v^2}{3\pi\eta D v} \\
&= \frac{\rho\pi\left(\frac{D}{2}\right)^2 v}{12\pi\eta D} \\
&= \frac{\rho D v}{48\eta} \\
&= \frac{R}{48}
\end{aligned}$$

(b)

$$R = \frac{Dv\rho}{\eta} \approx 0.01$$

2.5

$$\begin{aligned} v_y(t) &= v_{\text{ter}} + (v_{\text{yo}} - v_{\text{ter}})e^{-t/\tau} \\ &= v_{\text{ter}} + (2v_{\text{ter}} - v_{\text{ter}})e^{-t/\tau} \\ &= v_{\text{ter}}(1 + e^{-t/\tau}) \end{aligned}$$

The velocity starts at $2v_{\text{ter}}$ and asymptotically approaches v_{ter}.

2.7

$$\begin{aligned} F &= F(v) \\ m\dot{v} &= F(v) \\ m\frac{dv}{F(v)} &= dt \\ t &= \int_{v_\text{o}}^{v} m\frac{dv'}{F(v')} \end{aligned}$$

$$\begin{aligned} F &= F(v) \\ m\dot{v} &= F_\text{o} \\ v &= \frac{F_\text{o}}{m}t + c \end{aligned}$$

2.11

(a)

$$
\begin{aligned}
m\dot{v} &= -mg - bv \\
\dot{v} &= -g - kv \\
\frac{1}{-g-kv}\dot{v} &= 1 \\
-\frac{1}{k}\ln(-g-kv) &= t + c \\
\ln(-g-kv) &= c - \frac{t}{\tau} \\
-g - kv &= Ae^{-t/\tau} \\
v &= \tau(-g - Ae^{-t/\tau}) \\
&= -v_{\text{ter}} - \tau Ae^{-t/\tau}
\end{aligned}
$$

$$
\begin{aligned}
v_{\text{o}} &= -v_{\text{ter}} - \tau A \\
A &= -k(v_{\text{o}} + v_{\text{ter}})
\end{aligned}
$$

$$
v = -v_{\text{ter}} + (v_{\text{o}} + v_{\text{ter}})e^{-t/\tau}
$$

$$
y = -v_{\text{ter}}t - \tau(v_{\text{o}} + v_{\text{ter}})e^{-t/\tau} + c
$$

$$
\begin{aligned}
0 &= -\tau(v_{\text{o}} + v_{\text{ter}}) + c \\
c &= \tau(v_{\text{o}} + v_{\text{ter}})
\end{aligned}
$$

$$
y = -v_{\text{ter}}t + \tau(v_{\text{o}} + v_{\text{ter}})(1 - e^{-t/\tau})
$$

(b)

$$\begin{aligned}
0 &= -v_{\text{ter}} + (v_{\text{o}} + v_{\text{ter}})e^{-t/\tau} \\
e^{-t/\tau} &= \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} \\
-\frac{t}{\tau} &= \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} \\
t &= -\tau \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}}
\end{aligned}$$

$$\begin{aligned}
y_{\max} &= -v_{\text{ter}} \left(-\tau \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}}\right) + \tau(v_{\text{o}} + v_{\text{ter}}) \left(1 - \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}}\right) \\
&= \tau v_{\text{ter}} \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}} + \tau(v_{\text{o}} + v_{\text{ter}} - v_{\text{ter}}) \\
&= \tau \left(v_{\text{o}} + v_{\text{ter}} \ln \frac{v_{\text{ter}}}{v_{\text{o}} + v_{\text{ter}}}\right) \\
&= \tau \left[v_{\text{o}} - v_{\text{ter}} \ln \left(1 + \frac{v_{\text{o}}}{v_{\text{ter}}}\right)\right]
\end{aligned}$$

(c)

$$\begin{aligned}
y_{\max} &= \tau \left[v_{\text{o}} - v_{\text{ter}} \ln \left(1 + \frac{v_{\text{o}}}{v_{\text{ter}}}\right)\right] \\
&= \tau \left[v_{\text{o}} - g\tau \ln \left(1 + \frac{v_{\text{o}}}{g\tau}\right)\right] \\
&\approx \tau \left\{v_{\text{o}} - g\tau \left[\frac{v_{\text{o}}}{g\tau} - \frac{1}{2}\left(\frac{v_{\text{o}}}{g\tau}\right)^2\right]\right\} \\
&= \tau \left(v_{\text{o}} - v_{\text{o}} + \frac{1}{2}\frac{v_{\text{o}}^2}{g\tau}\right) \\
&= \frac{1}{2}\frac{v_{\text{o}}^2}{g}
\end{aligned}$$

2.13

$$
\begin{aligned}
v^2 &= \frac{2}{m}\int_{x_0}^{x} -kx'\,dx' \\
&= -\frac{2k}{m}\left(\frac{1}{2}x^2 - \frac{1}{2}x_0^2\right) \\
&= -\frac{k}{m}(x^2 - x_0^2) \\
v &= \sqrt{\frac{k}{m}(x_0^2 - x^2)} \\
&= \omega\sqrt{x_0^2 - x^2}
\end{aligned}
$$

$$
\begin{aligned}
\int_{x_0}^{x} \frac{1}{\sqrt{x_0^2 - x'^2}}\,dx' &= \int_0^t \omega\,dt \\
\arctan\frac{x}{\sqrt{x_0^2 - x^2}} - \arctan\frac{x_0}{\sqrt{x_0^2 - x_0^2}} &= \omega t \\
\arctan\frac{x}{\sqrt{x_0^2 - x^2}} &= \omega t + \frac{\pi}{2} \\
\frac{x}{\sqrt{x_0^2 - x^2}} &= \tan\left(\omega t + \frac{\pi}{2}\right) \\
&= -\cot\omega t \\
\frac{\sqrt{x_0^2 - x^2}}{x} &= -\tan\omega t \\
\sqrt{x_0^2 - x^2} &= -x\tan\omega t \\
x_0^2 - x^2 &= x^2\tan^2\omega t \\
x^2 &= \frac{x_0^2}{1 + \tan^2\omega t} \\
&= \frac{x_0^2\cos^2\omega t}{\cos^2\omega t + \sin^2\omega t} \\
&= x_0^2\cos^2\omega t \\
x &= x_0\cos\omega t
\end{aligned}
$$

2.15

$$\begin{aligned}
a_y &= -g \\
v_y &= v_{y0} - gt \\
y &= v_{y0}t - \frac{1}{2}gt^2 \\
0 &= v_{y0}t - \frac{1}{2}gt^2 \\
t &= \frac{2v_{y0}}{g} \\
x &= v_{x0}t \\
R &= \frac{2v_{x0}v_{y0}}{g}
\end{aligned}$$

2.19

(a)

$$\begin{aligned}
x &= v_{x0}t \\
y &= v_{y0}t - \frac{1}{2}gt^2 \\
&= \frac{v_{y0}}{v_{x0}}x - \frac{1}{2}g\left(\frac{x}{v_{x0}}\right)^2
\end{aligned}$$

(b)

$$\begin{aligned}
y &= \frac{v_{y0} + v_{\text{ter}}}{v_{x0}}x + v_{\text{ter}}\tau \ln\left(1 - \frac{x}{v_{x0}\tau}\right) \\
&\approx \frac{v_{y0}}{v_{x0}}x + \frac{g\tau}{v_{x0}}x - g\tau^2\left[\frac{x}{v_{x0}\tau} + \frac{1}{2}\left(\frac{x}{v_{x0}\tau}\right)^2\right] \\
&= \frac{v_{y0}}{v_{x0}}x - \frac{1}{2}g\left(\frac{x}{v_{x0}}\right)^2
\end{aligned}$$

2.23

(a)

$$
\begin{aligned}
v_{\text{ter}} &= \sqrt{\frac{mg}{c}} \\
&= \sqrt{\frac{mg}{\gamma D^2}} \\
&= \sqrt{\frac{mg}{0.25D^2}} \\
&= \sqrt{\frac{\rho \frac{4}{3}\pi \left(\frac{D}{2}\right)^3 g}{0.25D^2}} \\
&= \sqrt{\frac{4\pi\rho D g}{6}} \\
&= 22\,\mathrm{m/s}
\end{aligned}
$$

(b)

$$
\begin{aligned}
m &= \rho V \\
&= \rho \frac{4}{3}\pi \left(\frac{D}{2}\right)^3 \\
&= \frac{\pi\rho D^3}{6} \\
D^2 &= \left(\frac{6m}{\pi\rho}\right)^{2/3} \\
v_{\text{ter}} &= \sqrt{\frac{mg}{0.25D^2}} \\
&= \sqrt{\frac{mg}{0.25(6m/\pi\rho)^{2/3}}} \\
&= 140\,\mathrm{m/s}
\end{aligned}
$$

(c)

$$
v_{\text{ter}} = 107\,\mathrm{m/s}
$$

2.27

$$
\begin{aligned}
m\dot{v} &= -mg\sin\theta - cv^2 \\
-\frac{\sqrt{m}\arctan\frac{\sqrt{c}v}{\sqrt{gm\sin\theta}}}{\sqrt{cg\sin\theta}} &= t + c_1 \\
\arctan\frac{\sqrt{c}v}{\sqrt{gm\sin\theta}} &= \sqrt{\frac{cg\sin\theta}{m}}(c_1 - t) \\
\frac{\sqrt{c}v}{\sqrt{gm\sin\theta}} &= \tan\left[\sqrt{\frac{cg\sin\theta}{m}}(c_1 - t)\right] \\
v &= \sqrt{\frac{gm\sin\theta}{c}}\tan\left[\sqrt{\frac{cg\sin\theta}{m}}(c_1 - t)\right]
\end{aligned}
$$

$$
\begin{aligned}
v_0 &= \sqrt{\frac{gm\sin\theta}{c}}\tan\left(\sqrt{\frac{cg\sin\theta}{m}}c_1\right) \\
c_1 &= \sqrt{\frac{m}{cg\sin\theta}}\arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right)
\end{aligned}
$$

$$
v = \sqrt{\frac{gm\sin\theta}{c}}\tan\left[\arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right) - \sqrt{\frac{cg\sin\theta}{m}}t\right]
$$

$$
\begin{aligned}
0 &= \sqrt{\frac{gm\sin\theta}{c}}\tan\left[\arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right) - \sqrt{\frac{cg\sin\theta}{m}}t\right] \\
\sqrt{\frac{cg\sin\theta}{m}}t &= \arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right) \\
t &= \sqrt{\frac{m}{cg\sin\theta}}\arctan\left(\sqrt{\frac{c}{gm\sin\theta}}v_0\right)
\end{aligned}
$$

2.29

$$
\begin{aligned}
v(t) &= v_{\text{ter}}\tanh\frac{gt}{v_{\text{ter}}} \\
v(1) &= 9.6\,\text{m/s} \\
v(5) &= 38\,\text{m/s} \\
v(10) &= 48\,\text{m/s} \\
v(20) &= 50\,\text{m/s} \\
v(30) &= 50\,\text{m/s}
\end{aligned}
$$

2.31

(a)

$$\begin{aligned} v_{\text{ter}} &= \sqrt{\frac{mg}{c}} \\ &= \sqrt{\frac{mg}{0.25D^2}} \\ &= 20.2\,\text{m/s} \end{aligned}$$

(b)

$$\begin{aligned} y &= -30 + \frac{v_{\text{ter}}^2}{g} \ln\left(\cosh \frac{gt}{v_{\text{ter}}}\right) \\ 0 &= -30 + \frac{v_{\text{ter}}^2}{g} \ln\left(\cosh \frac{gt}{v_{\text{ter}}}\right) \\ t &= 2.78\,\text{s} \\ v(2.78) &= 17.6\,\text{m/s} \end{aligned}$$

2.33

(b)

$$\begin{aligned} \cosh z &= \frac{e^z + e^{-z}}{2} \\ &= \frac{1}{2}\left[\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots\right) + \left(1 - z + \frac{z^2}{2} - \frac{z^3}{6} + \cdots\right)\right] \\ &= 1 + \frac{z^2}{2} + \frac{z^4}{24} + \frac{z^6}{720} + \cdots \\ \cos iz &= 1 - \frac{(iz)^2}{2} + \frac{(iz)^4}{24} - \frac{(iz)^6}{720} + \cdots \\ &= 1 + \frac{z^2}{2} + \frac{z^4}{24} + \frac{z^6}{720} + \cdots \\ &= \cosh z \\ \sinh z &= -i \sin iz \end{aligned}$$

(c)

$$\begin{aligned}
\frac{d}{dz}\cosh z &= \frac{d}{dz}\left(\frac{e^z + e^{-z}}{2}\right) \\
&= \frac{e^z - e^{-z}}{2} \\
&= \sinh z \\
\frac{d}{dz}\sinh z &= \frac{d}{dz}\left(\frac{e^z - e^{-z}}{2}\right) \\
&= \frac{e^z + e^{-z}}{2} \\
&= \cosh z
\end{aligned}$$

(d)

$$\begin{aligned}
\cosh^2 z - \sinh^2 z &= \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2 \\
&= \frac{1}{4}(e^{2z} + 2 + e^{-2z} - e^{2z} + 2 - e^{-2z}) \\
&= 1
\end{aligned}$$

(e)

$$\begin{aligned}
\int \frac{1}{\sqrt{1+x^2}}\,dx &= \int \frac{\cosh z}{\sqrt{1+\sinh^2 z}}\,dz \\
&= \int 1\,dz \\
&= z \\
&= \operatorname{arcsinh} x
\end{aligned}$$

2.35

(a)

$$\begin{aligned}
m\dot{v} &= mg - cv^2 \\
\dot{v} &= g\left(1 - \frac{v^2}{v_{\text{ter}}^2}\right) \\
\int_0^v \frac{1}{1 - v'^2/v_{\text{ter}}^2}\, dv' &= \int_0^t g\, dt \\
v_{\text{ter}} \operatorname{arctanh} \frac{v}{v_{\text{ter}}} &= gt \\
v &= v_{\text{ter}} \tanh \frac{gt}{v_{\text{ter}}}
\end{aligned}$$

$$\begin{aligned}
y &= \int_0^t v_{\text{ter}} \tanh \frac{gt'}{v_{\text{ter}}} dt' \\
&= \frac{v_{\text{ter}}^2}{g} \ln\left[\cosh\left(\frac{gt}{v_{\text{ter}}}\right)\right]
\end{aligned}$$

(b)

$$\begin{aligned}
v &= g\tau \tanh \frac{t}{\tau} \\
y &= g\tau^2 \ln\left[\cosh\left(\frac{t}{\tau}\right)\right] \\
v(\tau) &= g\tau \tanh 1 \\
&= 0.76 v_{\text{ter}} \\
v(2\tau) &= 0.96 v_{\text{ter}} \\
v(3\tau) &= 0.99 v_{\text{ter}}
\end{aligned}$$

(c)

$$\begin{aligned}
y &= g\tau^2 \ln\left[\cosh\left(\frac{t}{\tau}\right)\right] \\
&= g\tau^2 \ln\left(\frac{e^{t/\tau} + e^{-t/\tau}}{2}\right) \\
&= g\tau^2 \ln\left(\frac{e^{t/\tau}}{2}\right) \\
&= g\tau^2 (\ln e^{t/\tau} - \ln 2) \\
&= g\tau t - g\tau^2 \ln 2 \\
&= v_{\text{ter}} t - g\tau^2 \ln 2
\end{aligned}$$

(d)

$$\begin{aligned} y &= \frac{(v_{\text{ter}})^2}{g} \ln\left[\cosh\left(\frac{gt}{v_{\text{ter}}}\right)\right] \\ &\approx \frac{(v_{\text{ter}})^2}{g} \ln\left[1 + \frac{1}{2}\left(\frac{gt}{v_{\text{ter}}}\right)^2\right] \\ &\approx \frac{(v_{\text{ter}})^2}{g} \frac{1}{2}\left(\frac{gt}{v_{\text{ter}}}\right)^2 \\ &= \frac{1}{2} g t^2 \end{aligned}$$

2.39

(a)

$$\begin{aligned} m\dot{v} &= -cv^2 - 3 \\ \int_{v_0}^{v} \frac{m}{-cv'^2 - 3}\, dv' &= \int_0^t dt' \\ \frac{m}{\sqrt{3c}}\left[\arctan\left(\sqrt{\frac{c}{3}} v_0\right) - \arctan\left(\sqrt{\frac{c}{3}} v\right)\right] &= t \end{aligned}$$

(b)

Speed	Time
15 m/s	6.34 s
10 m/s	18.4 s
5 m/s	48.3 s
0 m/s	142 s

2.41

$$
\begin{aligned}
m\dot{v} &= -mg - cv^2 \\
\dot{v} &= -g\left[1 + \left(\frac{v}{v_{\text{ter}}}\right)^2\right] \\
v\frac{dv}{dy} &= -g\left[1 + \left(\frac{v}{v_{\text{ter}}}\right)^2\right] \\
\int_{v_0}^{v} \frac{v'}{1 + (v'/v_{\text{ter}})^2}\, dv' &= \int_0^y -g\, dy' \\
\frac{1}{2}v_{\text{ter}}^2[\ln(v_{\text{ter}}^2 + v^2) - \ln(v_{\text{ter}}^2 + v_0^2)] &= -gy \\
\ln\frac{v_{\text{ter}}^2 + v^2}{v_{\text{ter}}^2 + v_0^2} &= -\frac{2gy}{v_{\text{ter}}^2} \\
\frac{v_{\text{ter}}^2 + v^2}{v_{\text{ter}}^2 + v_0^2} &= e^{-2gy/v_{\text{ter}}^2} \\
v &= \sqrt{(v_{\text{ter}}^2 + v_0^2)e^{-2gy/v_{\text{ter}}^2} - v_{\text{ter}}^2}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \sqrt{(v_{\text{ter}}^2 + v_0^2)e^{-2gy/v_{\text{ter}}^2} - v_{\text{ter}}^2} \\
v_{\text{ter}}^2 &= (v_{\text{ter}}^2 + v_0^2)e^{-2gy/v_{\text{ter}}^2} \\
\frac{v_{\text{ter}}^2}{v_{\text{ter}}^2 + v_0^2} &= e^{-2gy/v_{\text{ter}}^2} \\
-\frac{2gy}{v_{\text{ter}}^2} &= \ln\frac{v_{\text{ter}}^2}{v_{\text{ter}}^2 + v_0^2} \\
y &= -\frac{v_{\text{ter}}^2}{2g}\ln\frac{v_{\text{ter}}^2}{v_{\text{ter}}^2 + v_0^2} \\
&= \frac{v_{\text{ter}}^2}{2g}\ln\frac{v_{\text{ter}}^2 + v_0^2}{v_{\text{ter}}^2}
\end{aligned}
$$

$$
y_{\max} = 17.6\,\text{m}
$$

2.45

(a)

$$\begin{aligned}
z &= re^{i\theta} \\
&= r(\cos\theta + i\sin\theta) \\
&= \sqrt{x^2+y^2}\left[\cos\left(\arctan\frac{y}{x}\right) + i\sin\left(\arctan\frac{y}{x}\right)\right] \\
&= \sqrt{x^2+y^2}\left(\frac{1}{\sqrt{1+\frac{y^2}{x^2}}} + i\frac{y}{x\sqrt{1+\frac{y^2}{x^2}}}\right) \\
&= \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} + i\frac{y}{\sqrt{x^2+y^2}}\right) \\
&= x + iy
\end{aligned}$$

r is the distance between z and the origin, θ is the angle between the positive real axis and z.

(b)

$$z = \sqrt{3^2+4^2}e^{i\arctan 4/3} = 5e^{0.927i}$$

(c)

$$z = 2\cos -\frac{\pi}{3} + i2\sin -\frac{\pi}{3} = 1 - \sqrt{3}i$$

2.47

(a)

$$\begin{aligned}
z + w &= 9 + 4i \\
z - w &= 3 + 12i \\
zw &= (6+8i)(3-4i) \\
&= 18 - 24i + 24i + 32 \\
&= 50 \\
\frac{z}{w} &= \frac{zw*}{ww*} \\
&= \frac{(6+8i)(3+4i)}{(3-4i)(3+4i)} \\
&= \frac{18+24i+24i-32}{9+12i-12i+16} \\
&= \frac{-14+48i}{25} \\
&= -\frac{14}{25} + \frac{48}{25}i
\end{aligned}$$

(b)

$$\begin{aligned}
z + w &= \left(8\cos\frac{\pi}{3} + i8\sin\frac{\pi}{3}\right) + \left(4\cos\frac{\pi}{6} + i4\sin\frac{\pi}{6}\right) \\
&= (4 + 2\sqrt{3}) + i(4\sqrt{3} + 2) \\
z - w &= (4 - 2\sqrt{3}) + i(4\sqrt{3} - 2) \\
zw &= 32e^{i\pi/2} \\
&= 32i \\
\frac{z}{w} &= 2e^{i\pi/6} \\
&= \sqrt{3} + i
\end{aligned}$$

2.49

(a)

$$\begin{aligned}
z^2 &= (e^{i\theta})^2 \\
&= e^{i2\theta} \\
&= \cos 2\theta + i\sin 2\theta \\
z^2 &= (\cos\theta + i\sin\theta)^2 \\
&= \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta \\
&= \cos^2\theta + i\sin 2\theta - \sin^2\theta
\end{aligned}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\begin{aligned}
\cos 2\theta + i\sin 2\theta &= \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta \\
i\sin 2\theta &= 2i\sin\theta\cos\theta \\
\sin 2\theta &= 2\sin\theta\cos\theta
\end{aligned}$$

(b)

$$
\begin{aligned}
z^3 &= (e^{i\theta})^3 \\
&= e^{i3\theta} \\
&= \cos 3\theta + i \sin 3\theta \\
z^3 &= (\cos\theta + i\sin\theta)^3 \\
&= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta
\end{aligned}
$$

$$
\begin{aligned}
\cos 3\theta + i\sin 3\theta &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\
&= \cos\theta(\cos^2\theta - 3\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta) \\
&= \cos\theta(\cos^2\theta - 3\sin^2\theta) + i[3(1-\sin^2\theta)\sin\theta - \sin^3\theta] \\
&= \cos\theta(\cos^2\theta - 3\sin^2\theta) + i(3\sin\theta - 4\sin^3\theta) \\
\cos 3\theta &= \cos\theta(\cos^2\theta - 3\sin^2\theta)
\end{aligned}
$$

$$
\sin 3\theta = 3\sin\theta - 4\sin^3\theta
$$

2.53

$$\begin{aligned}
\mathbf{B} &= B_z\hat{\mathbf{z}} \\
\mathbf{E} &= E_z\hat{\mathbf{z}} \\
\mathbf{v}\times\mathbf{B} &= B_z v_y\hat{\mathbf{x}} - B_z v_x\hat{\mathbf{y}} \\
\mathbf{F} &= q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \\
&= qB_z v_y\hat{\mathbf{x}} - qB_z v_x\hat{\mathbf{y}} + qE_z\hat{\mathbf{z}}
\end{aligned}$$

$$\begin{aligned}
m\dot{v}_x &= qB_z v_y \\
\dot{v}_x &= \omega v_y
\end{aligned}$$

$$\begin{aligned}
m\dot{v}_y &= -qB_z v_x \\
\dot{v}_y &= -\omega v_x
\end{aligned}$$

$$\begin{aligned}
m\dot{v}_z &= qE_z \\
\dot{v}_z &= \frac{q}{m}E_z
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\begin{pmatrix} v_x \\ v_y \end{pmatrix} &= \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} \\
\begin{pmatrix} v_x \\ v_y \end{pmatrix} &= c_1\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos\omega t - \begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin\omega t\right] + c_2\left[\begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos\omega t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin\omega t\right] \\
&= \begin{pmatrix} c_1\sin\omega t - c_2\cos\omega t \\ c_1\cos\omega t + c_2\sin\omega t \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
x &= -\frac{c_1}{\omega}\cos\omega t - \frac{c_2}{\omega}\sin\omega t + x_0 \\
y &= \frac{c_1}{\omega}\sin\omega t - \frac{c_2}{\omega}\cos\omega t + y_0
\end{aligned}$$

$$\begin{aligned}
v_z &= \frac{q}{m}E_z t + v_{z0} \\
z &= \frac{q}{2m}E_z t^2 + v_{z0}t + z_0
\end{aligned}$$

The particle moves in a helix oriented along the z-axis.

2.55

(a)

$$
\begin{aligned}
\mathbf{B} &= B\hat{\mathbf{z}} \\
\mathbf{E} &= E\hat{\mathbf{y}} \\
\mathbf{F} &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
&= Bqv_y\hat{\mathbf{x}} + q(E - Bv_x)\hat{\mathbf{y}} \\
\dot{v}_x &= \frac{Bq}{m}v_y \\
&= \omega v_y \\
\dot{v}_y &= \frac{Eq}{m} - \frac{Bq}{m}v_x \\
&= \frac{Eq}{m} - \omega v_x \\
\dot{v}_z &= 0
\end{aligned}
$$

The net force has no $\hat{\mathbf{z}}$ component, so the motion stays in the xy-plane.

(b)

$$
\begin{aligned}
0 &= \frac{Eq}{m} - \omega v_x \\
v_x &= \frac{Eq}{\omega m} \\
&= \frac{E}{B}
\end{aligned}
$$

(c)

$$\begin{aligned}
\begin{pmatrix}\dot{v}_x\\ \dot{v}_y\end{pmatrix} &= \begin{pmatrix}0 & \omega\\ -\omega & 0\end{pmatrix}\begin{pmatrix}v_x\\ v_y\end{pmatrix} + \begin{pmatrix}0\\ \frac{Eq}{m}\end{pmatrix}\\
\mathbf{V}_c &= \begin{pmatrix}c_1\sin\omega t - c_2\cos\omega t\\ c_1\cos\omega t + c_2\sin\omega t\end{pmatrix}\\
\mathbf{V}_p &= \begin{pmatrix}c_3\\ c_4\end{pmatrix}\\
\begin{pmatrix}0\\ 0\end{pmatrix} &= \begin{pmatrix}0 & \omega\\ -\omega & 0\end{pmatrix}\begin{pmatrix}c_3\\ c_4\end{pmatrix} + \begin{pmatrix}0\\ \frac{Eq}{m}\end{pmatrix}\\
&= \begin{pmatrix}c_4\omega\\ -c_3\omega + \frac{Eq}{m}\end{pmatrix}\\
c_3 &= \frac{Eq}{m\omega}\\
&= \frac{E}{B}\\
&= v_{\mathrm{dr}}\\
c_4 &= 0\\
\mathbf{V}_p &= \begin{pmatrix}v_{\mathrm{dr}}\\ 0\end{pmatrix}\\
\mathbf{V} &= \begin{pmatrix}c_1\sin\omega t - c_2\cos\omega t + v_{\mathrm{dr}}\\ c_1\cos\omega t + c_2\sin\omega t\end{pmatrix}\\
\begin{pmatrix}v_{x0}\\ 0\end{pmatrix} &= \begin{pmatrix}-c_2 + v_{\mathrm{dr}}\\ c_1\end{pmatrix}\\
c_1 &= 0\\
c_2 &= v_{\mathrm{dr}} - v_{x0}\\
\mathbf{V} &= \begin{pmatrix}v_{\mathrm{dr}} + (v_{x0} - v_{\mathrm{dr}})\cos\omega t\\ -(v_{x0} - v_{\mathrm{dr}})\sin\omega t\end{pmatrix}
\end{aligned}$$

(d)

$$\begin{aligned}
x &= v_{\mathrm{dr}}t + \frac{v_{x0} - v_{\mathrm{dr}}}{\omega}\sin\omega t + x_0\\
y &= \frac{v_{x0} - v_{\mathrm{dr}}}{\omega}\cos\omega t + y_0\\
z &= z_0
\end{aligned}$$

3 Momentum and Angular Momentum

3.3

$$\begin{aligned}
mv_0 &= \frac{m}{3}(v_1 + v_2\cos\theta + v_3\cos\theta) \\
3v_0 &= v_0 + \sqrt{2}v_2 \\
2v_0 &= \sqrt{2}v_2 \\
v_2 &= \sqrt{2}v_0 \\
\mathbf{v}_2 &= \sqrt{2}v_0\left(\cos\frac{\pi}{4}\hat{\mathbf{x}} + \sin\frac{\pi}{4}\hat{\mathbf{y}}\right) \\
&= v_0(\hat{\mathbf{x}} + \hat{\mathbf{y}}) \\
\mathbf{v}_3 &= v_0(\hat{\mathbf{x}} - \hat{\mathbf{y}})
\end{aligned}$$

3.7

$$\begin{aligned}
v &= v_{\text{ex}} \ln\frac{m_0}{m} \\
&= 2079\,\text{m/s} \\
F &= 25\,\text{MN} \\
W &= 19.6\,\text{MN}
\end{aligned}$$

The thrust is 1.28 times the weight on Earth.

3.9

$$\begin{aligned}
-\dot{m}v_{\text{ex}} &= m_0 g \\
v_{\text{ex}} &= -\frac{m_0 g}{\dot{m}} \\
&= 2352\,\text{m/s}
\end{aligned}$$

3.11

(a)

$$m\dot{v} = -\dot{m}v_{\text{ex}} + F_{\text{ext}}$$

(b)

$$
\begin{aligned}
m\dot{v} &= -\dot{m}v_{\text{ex}} - mg \\
\dot{v} &= -\frac{v_{\text{ex}}}{m}\dot{m} - g \\
\int_0^t \dot{v}\,dt &= \int_0^t \left(-\frac{v_{\text{ex}}}{m}\dot{m} - g\right)\,dt \\
\int_0^v dv' &= -v_{\text{ex}} \int_{m_0}^{m} \frac{1}{m'}\,dm' - \int_0^t g\,dt' \\
v &= -v_{\text{ex}} \ln\frac{m}{m_0} - gt \\
&= v_{\text{ex}} \ln\frac{m_0}{m} - gt
\end{aligned}
$$

(c)

$$v = 903\,\text{m/s}$$

It would be $2079\,\text{m/s}$ without gravity (2.3 times larger).

(d) The rocket wouldn't take off until it was light enough (from burning fuel) that its thrust was greater than its weight.

3.13

$$
\begin{aligned}
v &= v_{\text{ex}} \ln\frac{m_0}{m} - gt \\
&= v_{\text{ex}} \ln\frac{m_0}{m_0 - kt} - gt \\
y &= v_{\text{ex}}t - \frac{mv_{\text{ex}}}{k}\ln\frac{m_0}{m} - \frac{1}{2}gt^2 \\
y(2\,\text{min}) &= 40\,\text{km}
\end{aligned}
$$

3.15

$$
\begin{aligned}
M &= m_1 + m_2 + m_3 \\
&= m_1 + m_1 + 10m_1 \\
&= 12m_1 \\
\mathbf{R} &= \frac{1}{12m_1}\left[m_1\begin{pmatrix}1\\1\\0\end{pmatrix} + m_2\begin{pmatrix}1\\-1\\0\end{pmatrix} + m_3\begin{pmatrix}0\\0\\0\end{pmatrix}\right] \\
&= \frac{1}{12m_1}\begin{pmatrix}2m_1\\0\\0\end{pmatrix} \\
&= \begin{pmatrix}\frac{1}{6}\\0\\0\end{pmatrix}
\end{aligned}
$$

3.17

If we let Earth be at the origin, then

$$
\begin{aligned}
R &= \frac{dM_m}{M_e + M_m} \\
&= 4630\,\mathrm{km}
\end{aligned}
$$

The centre of mass is inside Earth.

3.19

(a) No external forces apply during the explosion so the path of the centre of mass would be unchanged.

(b) 100 m before the target.

(c) No.

3.21

$$
\begin{aligned}
\mathbf{R} &= \frac{1}{M}\int \mathbf{r}\,dm \\
&= \frac{2}{\sigma\pi R^2}\int \mathbf{r}\sigma\,dA \\
&= \frac{2}{\pi R^2}\int_0^R\int_0^\pi \mathbf{r}r\,d\phi\,dr \\
&= \frac{2}{\pi R^2}\int_0^R\int_0^\pi r(\cos\phi\hat{\mathbf{x}}+\sin\phi\hat{\mathbf{y}})r\,d\phi\,dr \\
&= \frac{2}{\pi R^2}\int_0^R r^2\int_0^\pi (\cos\phi\hat{\mathbf{x}}+\sin\phi\hat{\mathbf{y}})\,d\phi\,dr \\
&= \frac{2}{\pi R^2}\int_0^R r^2[\sin\phi\hat{\mathbf{x}}-\cos\phi\hat{\mathbf{y}}]_0^\pi\,dr \\
&= \frac{4}{\pi R^2}\int_0^R r^2\,dr\,\hat{\mathbf{y}} \\
&= \frac{4R}{3\pi}\hat{\mathbf{y}}
\end{aligned}
$$

3.25

$$
\begin{aligned}
L &= L_0 \\
I\omega &= I_0\omega_0 \\
mr^2\omega &= mr_0^2\omega_0 \\
\omega &= \left(\frac{r_0}{r}\right)^2\omega_0
\end{aligned}
$$

3.29

$$
\begin{aligned}
I\omega &= I_0\omega_0 \\
\frac{2}{5}\left(\frac{4}{3}\pi R^3\rho\right)R^2\omega &= \frac{2}{5}\left(\frac{4}{3}\pi R_0^3\rho\right)R_0^2\omega_0 \\
\omega &= \left(\frac{R_0}{R}\right)^5\omega_0
\end{aligned}
$$

If the radius doubles the angular velocity is $\omega_0/32$.

3.31

$$
\begin{aligned}
I &= \int r^2 \, dm \\
&= \int r^2 \sigma \, dA \\
&= \frac{M}{\pi R^2} \int_0^R \int_0^{2\pi} r^3 \, d\phi \, dr \\
&= \frac{1}{2} M R^2
\end{aligned}
$$

3.33

$$
\begin{aligned}
I &= \int r^2 \, dm \\
&= \int r^2 \sigma \, dA \\
&= \frac{M}{(2b)^2} \int_{-b}^{b} \int_{-b}^{b} (x^2 + y^2) \, dx \, dy \\
&= \frac{M}{4b^2} \int_{-b}^{b} \left[\frac{1}{3} x^3 + x y^2 \right]_{-b}^{b} dy \\
&= \frac{M}{4b} \int_{-b}^{b} \left(\frac{2}{3} b^2 + 2 y^2 \right) dy \\
&= \frac{M}{4b} \left[\frac{2}{3} b^2 y + \frac{2}{3} y^3 \right]_{-b}^{b} \\
&= \frac{2}{3} M b^2
\end{aligned}
$$

3.35

(b)

$$\Gamma_{\text{ext}} = RMg\sin\gamma$$

$$\begin{aligned}\dot{L} &= \Gamma_{\text{ext}} \\ I\dot{\omega} &= RMg\sin\gamma \\ \frac{3}{2}MR^2\dot{\omega} &= RMg\sin\gamma \\ \dot{\omega} &= \frac{2g\sin\gamma}{3R}\end{aligned}$$

$$\begin{aligned}\dot{v} &= R\dot{\omega} \\ &= \frac{2}{3}g\sin\gamma\end{aligned}$$

(c)

$$\begin{aligned}M\dot{v} &= Mg\sin\gamma - f \\ f &= M(g\sin\gamma - \dot{v})\end{aligned}$$

$$\begin{aligned}\Gamma_{\text{ext}} &= Rf \\ &= RM(g\sin\gamma - \dot{v})\end{aligned}$$

$$\begin{aligned}\dot{L} &= \Gamma_{\text{ext}} \\ I\dot{\omega} &= RM(g\sin\gamma - \dot{v}) \\ \frac{1}{2}MR^2\dot{\omega} &= RM(g\sin\gamma - \dot{v}) \\ \dot{\omega} &= \frac{2(g\sin\gamma - \dot{v})}{R}\end{aligned}$$

$$\begin{aligned}\dot{v} &= R\dot{\omega} \\ &= 2(g\sin\gamma - \dot{v}) \\ &= \frac{2}{3}g\sin\gamma\end{aligned}$$

3.37

(b)

$$\begin{aligned}\sum m_\alpha r'_\alpha &= \sum m_\alpha (r_\alpha - R)\\ &= \sum m_\alpha r_\alpha - \sum m_\alpha R\\ &= MR - MR\\ &= 0\end{aligned}$$

4 Energy

4.3

(a)

$$\begin{aligned}\int_P^Q \mathbf{F}\cdot d\mathbf{r} &= \int_P^O \mathbf{F}\cdot d\mathbf{r} + \int_O^Q \mathbf{F}\cdot d\mathbf{r}\\ &= 0\end{aligned}$$

(b)

$$\begin{aligned}x &= 1-t\\ y &= t\\ \mathbf{r} &= (1-t)\hat{\mathbf{x}} + t\hat{\mathbf{y}}\\ d\mathbf{r} &= (-\hat{\mathbf{x}} + \hat{\mathbf{y}})\,dt\\ \mathbf{F}\cdot d\mathbf{r} &= (x+y)\,dt\\ &= [(1-t)+1]\,dt\\ &= dt\\ \int_P^Q \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \,dt\\ &= 1\end{aligned}$$

(c)

$$\begin{aligned}\mathbf{r} &= \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}\\ d\mathbf{r} &= (-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}})\,d\phi\\ \mathbf{F}\cdot d\mathbf{r} &= (\sin^2\phi + \cos^2\phi)\,d\phi\\ &= d\phi\\ \int_P^Q \mathbf{F}\cdot d\mathbf{r} &= \int_0^{\pi/2} d\phi\\ &= \frac{\pi}{2}\end{aligned}$$

4.7

(a)

$$\begin{aligned}
\mathbf{F} &= -m\gamma y^2\hat{\mathbf{y}} \\
W &= \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r} \\
&= -m\gamma \int_{y_1}^{y_2} y^2\, dy \\
&= -\frac{1}{3}m\gamma(y_2^3 - y_1^3) \\
U(\mathbf{r}) &= \frac{1}{3}m\gamma y^3
\end{aligned}$$

(b) Assuming no friction

$$\begin{aligned}
\frac{1}{2}mv^2 &= \frac{1}{3}m\gamma h^3 \\
v &= \sqrt{\frac{2}{3}\gamma h^3}
\end{aligned}$$

4.9

(a)

$$\begin{aligned}
U(x) &= -\int_0^x -kx'\, dx' \\
&= \frac{1}{2}kx^2
\end{aligned}$$

4.11

(a)

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 0 \\
\frac{\partial f}{\partial y} &= 2ay + 2bz \\
\frac{\partial f}{\partial z} &= 2by + 2cz
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\partial g}{\partial x} &= -ay^2z^3\sin(axy^2z^3) \\
\frac{\partial g}{\partial y} &= -2axyz^3\sin(axy^2z^3) \\
\frac{\partial g}{\partial z} &= -3axy^2z^2\sin(axy^2z^3)
\end{aligned}$$

(c)

$$\begin{aligned}\frac{\partial h}{\partial x} &= \frac{ax}{r}\\ \frac{\partial h}{\partial y} &= \frac{ay}{r}\\ \frac{\partial h}{\partial z} &= \frac{az}{r}\end{aligned}$$

4.13

(a)

$$\nabla f = \frac{1}{r^2}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})$$

(b)

$$\nabla f = nr^{n-2}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})$$

(c)

$$\nabla f = \frac{g'(r)}{r}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})$$

4.15

$$\begin{aligned}df &= \nabla f \cdot d\mathbf{r}\\ &= (2, 4, 6) \cdot (0.01, 0.03, 0.05)\\ &= 0.44\\ f(1.01, 1.03, 1.05) - f(1, 1, 1) &= 0.4494\end{aligned}$$

4.19

(a) An ellipse that is two times wider than it is tall.

(b)

$$\begin{aligned}\nabla f &= (2x, 8y, 0)\\ \nabla f|_{(1,1,1)} &= (2, 8, 0)\\ \mathbf{n} &= (1, 4, 0)/\sqrt{17}\end{aligned}$$

4.21

$$
\begin{aligned}
\mathbf{F} &= -\frac{GMm}{r^2}\hat{\mathbf{r}} \\
\nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{GMm}{r^3}x & -\frac{GMm}{r^3}y & -\frac{GMm}{r^3}z \end{vmatrix} \\
&= \left[\frac{\partial}{\partial y}\left(-\frac{GMm}{r^3}z\right) - \frac{\partial}{\partial z}\left(-\frac{GMm}{r^3}y\right)\right]\hat{\mathbf{x}} \\
&\quad - \left[\frac{\partial}{\partial x}\left(-\frac{GMm}{r^3}z\right) - \frac{\partial}{\partial z}\left(-\frac{GMm}{r^3}x\right)\right]\hat{\mathbf{y}} \\
&\quad + \left[\frac{\partial}{\partial x}\left(-\frac{GMm}{r^3}y\right) - \frac{\partial}{\partial y}\left(-\frac{GMm}{r^3}x\right)\right]\hat{\mathbf{z}} \\
&= -GMm\left[\left(-\frac{3yz}{r^5} + \frac{3yz}{r^5}\right)\hat{\mathbf{x}} + \left(-\frac{3xz}{r^5} + \frac{3xz}{r^5}\right)\hat{\mathbf{y}}\right. \\
&\quad \left.\left(-\frac{3xy}{r^5} + \frac{3xy}{r^4}\right)\hat{\mathbf{z}}\right] \\
&= \mathbf{0} \\
U(\mathbf{r}) &= -\int_{\boldsymbol{\infty}}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}' \\
&= -\int_{\infty}^{r} -\frac{GMm}{r'^2}\,dr' \\
&= GMm\left[-\frac{1}{r'}\right]_{\infty}^{r} \\
&= GMm\left(-\frac{1}{r} + \frac{1}{\infty}\right) \\
&= -\frac{GMm}{r}
\end{aligned}
$$

4.23

(a)

$$
\begin{aligned}
\nabla \times \mathbf{F} &= \mathbf{0} \\
U &= -\int_{\mathbf{0}}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} \\
&= -\left(\int_0^x kx\,dx + \int_0^y 2ky\,dy + \int_0^z 3kz\,dz\right) \\
&= -\frac{1}{2}k(x^2 + 2y^2 + 3z^2)
\end{aligned}
$$

(b)

$$
\begin{aligned}
\nabla \times \mathbf{F} &= \mathbf{0} \\
U &= -\int_{\mathbf{0}}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} \\
&= -\left(\int_0^x ky\, dx + \int_0^y kx\, dy \right) \\
&= -kxy
\end{aligned}
$$

(c)

$$\nabla \times \mathbf{F} = 2k\hat{\mathbf{z}}$$

Not conservative

4.29

(a) The mass will oscillate around $x = 0$.

(b)

$$
\begin{aligned}
t &= \sqrt{\frac{m}{2}} \int_0^A \frac{dx}{\sqrt{kA^4 - kx^4}} \\
&= \sqrt{\frac{m}{2k}} \int_0^A \frac{dx}{\sqrt{A^4 - x^4}} \\
\tau &= 4t
\end{aligned}
$$

(d) $\tau \approx 3.71\,\mathrm{s}$

4.31

(a)

$$E = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_2 - m_1)gx$$

(b)

$$(m_1 + m_2)\ddot{x} = (m_1 - m_2)g$$

$$
\begin{aligned}
c &= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_2 - m_1)gx \\
0 &= (m_1 + m_2)\dot{x}\ddot{x} + (m_2 - m_1)g\dot{x} \\
(m_1 + m_2)\ddot{x} &= (m_1 - m_2)g
\end{aligned}
$$

4.35

(a)

$$E = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}^2 + (m_2 - m_1)gx$$

(b)

$$0 = \left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}\ddot{x} + (m_2 - m_1)g\dot{x}$$

$$\left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} = (m_1 - m_2)g$$

$$\begin{aligned} m_1\ddot{x} &= m_1 g - T_2 \\ T_2 &= m_1 g - m_1\ddot{x} \end{aligned}$$

$$\begin{aligned} m_2\ddot{x} &= T_1 - m_2 g \\ T_1 &= m_2\ddot{x} + m_2 g \end{aligned}$$

$$\begin{aligned} \omega &= -\frac{\dot{x}}{R} \\ \dot{\omega} &= -\frac{\ddot{x}}{R} \end{aligned}$$

$$\begin{aligned} I\dot{\omega} &= (T_1 - T_2)R \\ -I\frac{\ddot{x}}{R} &= (m_2\ddot{x} + m_2 g - m_1 g - m_1\ddot{x})\,R \\ \left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} &= (m_1 - m_2)g \end{aligned}$$

4.37

(a)

$$U(\phi) = MgR(1 - \cos\phi) - mgR\phi$$

(b)

$$\begin{aligned}
\frac{dU(\phi)}{d\phi} &= MgR\sin\phi - mgR \\
&= gR(M\sin\phi - m) \\
0 &= gR(M\sin\phi - m) \\
m &= M\sin\phi
\end{aligned}$$

$$\frac{d^2U(\phi)}{d\phi^2} = MgR\cos\phi$$

There is a position of equilibrium at $\phi = \arcsin\frac{m}{M}$. It is stable if $\phi < \frac{\pi}{2}$, i.e. $m < M$.

4.51

$$\begin{aligned}
U(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) &= U_{\text{int}} + U_{\text{ext}} \\
&= [U_{12}(\mathbf{r}_1 - \mathbf{r}_2) + U_{13}(\mathbf{r}_1 - \mathbf{r}_3) + U_{14}(\mathbf{r}_1 - \mathbf{r}_4) + U_{23}(\mathbf{r}_2 - \mathbf{r}_3) \\
&\quad + U_{24}(\mathbf{r}_2 - \mathbf{r}_4) + U_{34}(\mathbf{r}_3 - \mathbf{r}_4)] + [U_1(\mathbf{r}_1) + U_2(\mathbf{r}_2) + U_3(\mathbf{r}_3) \\
&\quad + U_4(\mathbf{r}_4)] \\
\mathbf{F}_3 &= \mathbf{F}_{3,\text{int}} + \mathbf{F}_{3,\text{ext}} \\
&= [\mathbf{F}_{13} + \mathbf{F}_{23} + \mathbf{F}_{34}] + \mathbf{F}_{3,\text{ext}} \\
&= -\nabla_3 U_{13} - \nabla_3 U_{23} - \nabla_3 U_{34} - \nabla_3 U_{3,\text{ext}} \\
&= -\nabla_3 U
\end{aligned}$$

4.53

(a)

$$\begin{aligned}
F &= ma \\
\frac{ke^2}{r^2} &= m\frac{v^2}{r} \\
v^2 &= \frac{ke^2}{mr} \\
K &= \frac{1}{2}mv^2 \\
&= \frac{ke^2}{2r} \\
U &= -\frac{ke^2}{r} \\
K &= -\frac{1}{2}U
\end{aligned}$$

(b)

$$
\begin{aligned}
E &= K_1 + K_2 + U_{12} + U_{1p} + U_{2p} \\
&= \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 - ke^2\left(\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_{12}}\right)
\end{aligned}
$$

(c)

$$
\begin{aligned}
E_{\text{before}} &= K_1 + K_2 + U_1 + U_2 + U_{12} \\
&= T_2 + \frac{ke^2}{2r} - \frac{ke^2}{r} \\
&= T_2 - \frac{ke^2}{2r}
\end{aligned}
$$

$$
\begin{aligned}
E_{\text{after}} &= K_1' + K_2' + U_1 + U_2 + U_{12} \\
&= T_1' + \frac{ke^2}{2r'} - \frac{ke^2}{r'} \\
&= T_1' - \frac{ke^2}{2r'} \\
T_2 - \frac{ke^2}{2r} &= T_1' - \frac{ke^2}{2r'} \\
T_1' &= T_2 + \frac{ke^2}{2}\left(\frac{1}{r'} - \frac{1}{r}\right)
\end{aligned}
$$

5 Oscillations

5.3

$$
\begin{aligned}
U(\phi) &= mgl(1 - \cos\phi) \\
&\approx mgl\left(1 - 1 + \frac{1}{2}\phi^2\right) \\
&= \frac{1}{2}mgl\phi^2 \\
k &= mgl
\end{aligned}
$$

5.5

$$
\begin{aligned}
x(t) &= C_1 e^{i\omega t} + C_2 e^{-i\omega t} \\
&= C_1(\cos\omega t + i\sin\omega t) + C_2(\cos\omega t - i\sin\omega t) \\
&= (C_1 + C_2)\cos\omega t + i(C_1 - C_2)\sin\omega t \\
&= B_1\cos\omega t + B_2\sin\omega t \\
B_1 &= C_1 + C_2 \\
B_2 &= i(C_1 - C_2)
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= A\cos(\omega t - \delta) \\
A &= \sqrt{B_1^2 + B_2^2} \\
\delta &= \arctan\frac{B_2}{B_1}
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= C\,\mathrm{Re}\,e^{\omega t} \\
C &= Ae^{-i\delta}
\end{aligned}
$$

5.7

(a) $B_1 = x_0$, $B_2 = \frac{v_0}{\omega}$

(b)

$$
\begin{aligned}
\omega &= \sqrt{\frac{k}{m}} \\
&= 10\,\mathrm{rad/s} \\
B_1 &= 3.0\,\mathrm{m} \\
B_2 &= 5.0\,\mathrm{m}
\end{aligned}
$$

(c) $x = 0\,\mathrm{m}$ at $t = 0.26\,\mathrm{s}$, $\dot{x} = 0\,\mathrm{m/s}$ at $t = 0.10\,\mathrm{s}$.

5.9

$$
\begin{aligned}
\frac{1}{2}kA^2 &= \frac{1}{2}mv^2 \\
\frac{k}{m} &= \left(\frac{v}{A}\right)^2 \\
\tau &= \frac{2\pi}{\omega} \\
&= \frac{2\pi}{\sqrt{k/m}} \\
&= \frac{2\pi}{v/A} \\
&= 1.05\,\mathrm{s}
\end{aligned}
$$

5.11

$$
\begin{aligned}
\frac{1}{2}kx_1^2 + \frac{1}{2}mv_1^2 &= \frac{1}{2}kx_2^2 + \frac{1}{2}mv_2^2 \\
kx_1^2 + mv_1^2 &= kx_2^2 + mv_2^2 \\
\frac{k}{m}x_1^2 + v_1^2 &= \frac{k}{m}x_2^2 + v_2^2 \\
\omega^2(x_1^2 - x_2^2) &= v_2^2 - v_1^2 \\
\omega &= \sqrt{\frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}kA^2 &= \frac{1}{2}kx_1^2 + \frac{1}{2}mv_1^2 \\
A^2 &= x_1^2 + \frac{m}{k}v_1^2 \\
A &= \sqrt{x_1^2 + \frac{v_1^2}{\omega^2}} \\
&= \sqrt{x_1^2 + v_1^2\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}} \\
&= \sqrt{\frac{x_1^2(v_2^2 - v_1^2) + v_1^2(x_1^2 - x_2^2)}{v_2^2 - v_1^2}} \\
&= \sqrt{\frac{x_2^2 v_1^2 - x_1^2 v_2^2}{v_1^2 - v_2^2}}
\end{aligned}
$$

5.13

$$\begin{aligned}
U(r) &= U_0\left(\frac{r}{R}+\lambda^2\frac{R}{r}\right)\\
\frac{dU(r)}{dr} &= U_0\left(\frac{1}{R}-\lambda^2\frac{R}{r^2}\right)\\
0 &= \frac{dU(r_0)}{dr}\\
&= U_0\left(\frac{1}{R}-\lambda^2\frac{R}{r_0^2}\right)\\
\frac{1}{R} &= \lambda^2\frac{R}{r_0^2}\\
r_0 &= \lambda R
\end{aligned}$$

$$\begin{aligned}
U(r_0+x)-U(r_0) &= U_0\left(\frac{r_0+x}{R}+\lambda^2\frac{R}{r_0+x}\right)-U_0\left(\frac{r_0}{R}+\lambda^2\frac{R}{r_0}\right)\\
&= U_0\left[\frac{1}{R}x+\lambda^2 R\left(\frac{1}{r_0+x}-\frac{1}{r_0}\right)\right]\\
&\approx U_0\left[\frac{1}{R}x+\lambda^2 R\left(\frac{1}{r_0}-\frac{x}{r_0^2}+\frac{x^2}{r_0^3}-\frac{1}{r_0}\right)\right]\\
&= U_0\left[\frac{1}{R}x+\lambda^2 R\left(\frac{x^2}{r_0^3}-\frac{x}{r_0^2}\right)\right]\\
&= U_0\left[\frac{1}{R}x+\lambda^2 R\left(\frac{x^2}{(\lambda R)^3}-\frac{x}{(\lambda R)^2}\right)\right]\\
&= \frac{U_0x^2}{\lambda R^2}\\
&= \frac{1}{2}\left(\frac{2U_0}{\lambda R^2}\right)x^2
\end{aligned}$$

$$\begin{aligned}
\omega &= \sqrt{\frac{k}{m}}\\
&= \sqrt{\frac{2U_0}{\lambda m R^2}}
\end{aligned}$$

5.17

(a)

$$
\begin{aligned}
x(t) &= A_x \cos \omega_x t \\
y(t) &= A_y \cos(\omega_y t - \delta) \\
\frac{\omega_x}{\omega_y} &= \frac{p}{q} \\
\omega_x \tau &= 2\pi p \\
\omega_y \tau &= 2\pi q \\
(\omega_x + \omega_y)\tau &= 2\pi(p+q) \\
\tau &= \frac{2\pi(p+q)}{\omega_x + \omega_y} \\
x(\tau) &= A_x \cos\left(\omega_x \frac{2\pi(p+q)}{\omega_x + \omega_y}\right) \\
&= A_x \cos\left(\frac{2\pi(p+q)}{1 + \omega_y/\omega_x}\right) \\
&= A_x \cos\left(\frac{2\pi(p+q)}{1 + q/p}\right) \\
&= A_x \cos\left(2\pi \frac{p(p+q)}{p+q}\right) \\
y(\tau) &= A_y \cos\left(\omega_y \frac{2\pi(p+q)}{\omega_x + \omega_y} - \delta\right) \\
&= A_y \cos\left(2\pi \frac{p+q}{1 + \omega_x/\omega_y} - \delta\right) \\
&= A_y \cos\left(2\pi \frac{q(p+q)}{p+q} - \delta\right)
\end{aligned}
$$