Advanced Engineering Mathematics Partial Differential Equations by Dennis G. Zill Problems

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Contents

12 Orthogonal Functions and Fourier Series

12.1 Orthogonal Functions

12.1.7

$$
\begin{aligned}
\int_0^{\pi/2} \sin mx \sin nx \, dx &= \frac{1}{2}\int_0^{\pi/2} [\cos(m-n)x - \cos(m+n)x] \, dx \\
&= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_0^{\pi/2} \\
&= \frac{1}{2}\left(\frac{\sin(m-n)\pi/2}{m-n} - \frac{\sin(m+n)\pi/2}{m+n}\right) \\
&= 0 \\
||\sin nx||^2 &= (\sin nx, \sin nx) \\
&= \int_0^{\pi/2} \sin^2 nx \, dx \\
&= \frac{1}{2}\int_0^{\pi/2} (1 - \cos 2nx) \, dx \\
&= \frac{1}{2}\left[x - \frac{1}{2n}\sin 2nx\right]_0^{\pi/2} \\
&= \frac{\pi}{4} \\
||\sin nx|| &= \frac{\sqrt{\pi}}{2}
\end{aligned}
$$

12.1.9

$$
\begin{aligned}
\int_0^{\pi} \sin mx \sin nx \, dx &= \frac{1}{2} \int_0^{\pi} [\cos(m-n)x - \cos(m+n)x] \, dx \\
&= \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_0^{\pi} \\
&= 0 \\
|| \sin nx ||^2 &= (\sin nx, \sin nx) \\
&= \int_0^{\pi} \sin^2 nx \, dx \\
&= \frac{1}{2} \int_0^{\pi} (1 - \cos 2nx) \, dx \\
&= \frac{1}{2} \left[x - \frac{1}{2n} \sin 2nx \right]_0^{\pi} \\
&= \frac{\pi}{2} \\
|| \sin nx || &= \sqrt{\frac{\pi}{2}}
\end{aligned}
$$

12.1.21

$$T = 1$$

12.1.23

$$T = 2\pi$$

12.1.25

$$T = 2\pi$$

12.2 Fourier Series

12.2.1

$$
\begin{aligned}
p &= \pi \\
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} dx \\
&= 1 \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \cos nx\,dx \\
&= \frac{1}{n\pi}[\sin nx]_0^{\pi} \\
&= 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin nx\,dx \\
&= -\frac{1}{n\pi}[\cos nx]_0^{\pi} \\
&= -\frac{1}{n\pi}[(-1)^n - 1] \\
&= \frac{1-(-1)^n}{n\pi} \\
f(x) &= \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx
\end{aligned}
$$

The series converges to $\frac{1}{2}$ at the point of discontinuity.

12.2.3

$$
\begin{aligned}
p &= 1 \\
a_0 &= \frac{3}{2} \\
a_n &= \int_{-1}^{0} \cos n\pi x \, dx + \int_0^1 x \cos n\pi x \, dx \\
&= \frac{1}{n\pi} [\sin n\pi x]_{-1}^{0} + \frac{1}{n\pi} \left[\frac{\cos n\pi x}{n\pi} + x \sin n\pi x \right]_0^1 \\
&= \frac{(-1)^n - 1}{n^2\pi^2} \\
b_n &= \int_{-1}^{0} \sin n\pi x \, dx + \int_0^1 x \sin n\pi x \, dx \\
&= -\frac{1}{n\pi} [\cos n\pi x]_{-1}^{0} + \frac{1}{n\pi} \left[\frac{\sin n\pi x}{n\pi} - x \cos n\pi x \right]_0^1 \\
&= -\frac{1}{n\pi} \\
f(x) &= \frac{3}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2\pi^2} \cos n\pi x - \frac{1}{n\pi} \sin n\pi x \right]
\end{aligned}
$$

The series converges to $\frac{1}{2}$ at the point of discontinuity.

12.3 Fourier Cosine and Sine Series

12.3.1

Odd

12.3.3

Neither

12.3.5

Even

12.3.7

Odd

12.3.9

Neither

12.3.11

$$
\begin{aligned}
b_n &= -2\pi \int_0^1 \sin n\pi x \, dx \\
&= \frac{2}{n}[\cos n\pi x]_0^1 \\
&= \frac{2}{n}[(-1)^n - 1] \\
f &= \sum_{n=1}^{\infty} \frac{2}{n}[(-1)^n - 1] \sin n\pi x
\end{aligned}
$$

12.3.13

$$
\begin{aligned}
a_0 &= \frac{2}{\pi} \int_0^\pi x \, dx \\
&= \pi \\
a_n &= 2 \int_0^\pi x \cos nx \, dx \\
&= \frac{2}{n} \left[\frac{\cos nx}{n} + x \sin nx \right]_0^\pi \\
&= \frac{2[(-1)^n - 1]}{n^2} \\
f &= \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx
\end{aligned}
$$

12.3.25

$$\begin{aligned}
a_0 &= 2\int_0^1 f(x)\,dx \\
&= 1 \\
a_n &= 2\int_0^1 f(x)\cos n\pi x\,dx \\
&= 2\int_0^{1/2} \cos n\pi x\,dx \\
&= \frac{2}{n\pi}[\sin n\pi x]_0^{1/2} \\
&= \frac{2}{n\pi}\sin\frac{n\pi}{2} \\
f &= \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n}\cos n\pi x
\end{aligned}$$

$$\begin{aligned}
b_n &= 2\int_0^1 f(x)\sin n\pi x\,dx \\
&= 2\int_0^{1/2} \sin n\pi x\,dx \\
&= -\frac{2}{n\pi}[\cos n\pi x]_0^{1/2} \\
&= \frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right) \\
f &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos\frac{n\pi}{2}}{n}\sin n\pi x
\end{aligned}$$

12.3.27

$$
\begin{aligned}
a_0 &= \frac{4}{\pi}\int_0^{\pi/2} \cos x \, dx \\
&= \frac{4}{\pi}[\sin x]_0^{\pi/2} \\
&= \frac{4}{\pi} \\
a_n &= \frac{4}{\pi}\int_0^{\pi/2} \cos x \cos 2nx \, dx \\
&= \frac{2}{\pi}\int_0^{\pi/2} [\cos(1-2n)x + \cos(1+2n)x] \, dx \\
&= \frac{2}{\pi}\left[\frac{\sin(1-2n)x}{1-2n} + \frac{\sin(1+2n)x}{1+2n}\right]_0^{\pi/2} \\
&= \frac{2(-1)^n}{\pi}\left[\frac{1}{1-2n} + \frac{1}{1+2n}\right] \\
&= \frac{2(-1)^n}{\pi}\frac{1+2n+1-2n}{(1-2n)(1+2n)} \\
&= \frac{4(-1)^n}{\pi(1-2n)(1+2n)} \\
f &= \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(1-2n)(1+2n)}\cos 2nx
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{4}{\pi}\int_0^{\pi/2} \cos x \sin 2nx \, dx \\
&= \frac{2}{\pi}\int_0^{\pi/2} [\sin(2n+1)x + \sin(2n-1)x] \, dx \\
&= -\frac{2}{\pi}\left[\frac{\cos(2n+1)x}{2n+1} + \frac{\cos(2n-1)x}{2n-1}\right]_0^{\pi/2} \\
&= \frac{2}{\pi}\left(\frac{1}{2n+1} + \frac{1}{2n-1}\right) \\
&= \frac{2}{\pi}\frac{4n}{4n^2-1} \\
&= \frac{8n}{\pi(4n^2-1)} \\
f &= \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^1-1}\sin 2nx
\end{aligned}
$$

12.3.35

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} x^2\,dx \\
&= \frac{8}{3}\pi^2 \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx \\
&= \frac{4}{n^2} \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\,dx \\
&= -\frac{4\pi}{n} \\
f &= \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)
\end{aligned}$$

12.3.43

$$\begin{aligned}
b_n &= \frac{10}{\pi}\int_0^{\pi}\sin nt\,dt \\
&= -\frac{10}{n\pi}[\cos nt]_0^{\pi} \\
&= \frac{10}{n\pi}[1-(-1)^n] \\
f &= \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[1-(-1)^n]\sin nt \\
x_p(t) &= \sum_{n=1}^{\infty} B_n\sin nt \\
m\frac{d^2x}{dt^2} + kx &= f(t)
\end{aligned}$$

$$\begin{aligned}
-mn^2B_n + kB_n &= \frac{10}{n\pi}[1-(-1)^n] \\
B_n &= \frac{10}{n\pi(k-mn^2)}[1-(-1)^n] \\
x_p(t) &= \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n(k-mn^2)}\sin nt \\
&= \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n(10-n^2)}\sin nt
\end{aligned}$$

12.3.45

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^\pi (2\pi t - t^2)\,dt \\
&= \frac{2}{\pi}\left[\pi t^2 - \frac{1}{3}t^3\right]_0^\pi \\
&= \frac{2}{\pi}\left(\pi^3 - \frac{1}{3}\pi^3\right) \\
&= \frac{4}{3}\pi^2 \\
a_n &= \frac{2}{\pi}\int_0^\pi (2\pi t - t^2)\cos nt\,dt \\
&= -\frac{4}{n^2} \\
f(t) &= \frac{2}{3}\pi^2 - \sum_{n=1}^\infty \frac{4}{n^2}\cos nt \\
x_p(t) &= c + \sum_{n=1}^\infty B_n \cos nt \\
\frac{1}{4}\frac{d^2x}{dt^2} + 12x &= f(t)
\end{aligned}$$

$$\begin{aligned}
-\frac{n^2}{4}\sum_{n=1}^\infty B_n\cos nt + 12c + 12\sum_{n=1}^\infty B_n\cos nt &= \frac{2}{3}\pi^2 - \sum_{n=1}^\infty \frac{4}{n^2}\cos nt \\
12c &= \frac{2}{3}\pi^2 \\
c &= \frac{\pi^2}{18} \\
-\frac{n^2}{4}B_n + 12B_n &= -\frac{4}{n^2} \\
\frac{48-n^2}{4}B_n &= -\frac{4}{n^2} \\
B_n &= -\frac{16}{n^2(48-n^2)} \\
x_p(t) &= \frac{\pi^2}{18} - \sum_{n=1}^\infty \frac{16}{n^2(48-n^2)}\cos nt \\
&= \frac{\pi^2}{18} + 16\sum_{n=1}^\infty \frac{1}{n^2(n^2-48)}\cos nt
\end{aligned}$$

12.4 Complex Fourier Series

12.4.1

$$
\begin{aligned}
T &= 4 \\
p &= 2 \\
c_n &= \frac{1}{4}\left(\int_0^2 e^{-in\pi x/2}\,dx - \int_{-2}^0 e^{-in\pi x/2}\,dx\right) \\
&= \frac{1}{2in\pi}([e^{-in\pi x/2}]_{-2}^0 - [e^{-in\pi x/2}]_0^2) \\
&= \frac{2 - e^{in\pi} - e^{-in\pi}}{2in\pi} \\
&= \frac{2 - \cos n\pi - i\sin n\pi - \cos n\pi + i\sin n\pi}{2in\pi} \\
&= \frac{1 - \cos n\pi}{in\pi} \\
&= \frac{1 - (-1)^n}{in\pi} \\
f(x) &= \sum_{n=-\infty,\ n\neq 0}^{\infty} \frac{1-(-1)^n}{in\pi} e^{in\pi x/2}
\end{aligned}
$$

12.4.3

$$
\begin{aligned}
T &= 1 \\
p &= \frac{1}{2} \\
c_n &= \int_0^{1/4} e^{-2in\pi x}\,dx \\
&= -\frac{1}{2in\pi}[e^{-2in\pi x}]_0^{1/4} \\
&= \frac{1}{2in\pi}(1 - e^{-in\pi/2}) \\
c_0 &= \frac{1}{4} \\
f(x) &= \frac{1}{4} + \sum_{n=-\infty,\ n\neq 0}^{\infty} \frac{1 - e^{-in\pi/2}}{2in\pi} e^{2in\pi x}
\end{aligned}
$$

12.4.5

$$
\begin{aligned}
T &= 2\pi \\
p &= \pi \\
c_n &= \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} \, dx \\
&= \frac{i}{n} \\
c_0 &= \frac{1}{2\pi} \int_0^{2\pi} x \, dx \\
&= \pi \\
f(x) &= \pi + \sum_{n=-\infty,\ n\neq 0}^{n=\infty} \frac{i}{n} e^{inx}
\end{aligned}
$$

12.5 Sturm-Liouville Problem

12.5.1

$$
\begin{aligned}
y'' + \lambda y &= 0 \\
y'(0) &= 0 \\
y(1) + y'(1) &= 0 \\
\lambda &= \alpha^2 \\
y &= c_1 \cos \alpha x + c_2 \sin \alpha x \\
y' &= -\alpha c_1 \sin \alpha x + \alpha c_2 \cos \alpha x \\
c_2 &= 0 \\
y &= c_1 \cos \alpha x
\end{aligned}
$$

$$
\begin{aligned}
c_1 \cos \alpha - \alpha c_1 \sin \alpha &= 0 \\
c_1 \cos \alpha &= \alpha c_1 \sin \alpha \\
\alpha \tan \alpha &= 0 \\
\alpha &= \cot \alpha \\
\lambda_1 &= 0.740174 \\
y_1 &= \cos 0.860334x \\
\lambda_2 &= 11.734872 \\
y_2 &= \cos 3.42562x \\
\lambda_3 &= 41.438831 \\
y_3 &= \cos 6.4373x \\
\lambda_4 &= 90.808130 \\
y_4 &= \cos 9.52933x
\end{aligned}
$$

12.5.5

$$
\begin{aligned}
(y_n, y_n) &= \int_0^1 \cos^2 \alpha_n x \, dx \\
&= \frac{1}{2} \int_0^1 (1 + \cos 2\alpha_n x) \, dx \\
&= \frac{1}{2} \left[x + \frac{1}{2\alpha_n} \sin 2\alpha_n x \right]_0^1 \\
&= \frac{1}{2} \left(1 + \frac{1}{2\alpha_n} \sin 2\alpha_n \right) \\
&= \frac{1}{2} \left(1 + \frac{1}{\alpha_n} \sin \alpha_n \cos \alpha_n \right) \\
&= \frac{1}{2} \left(1 + \tan \alpha_n \sin \alpha_n \cos \alpha_n \right) \\
&= \frac{1}{2} (1 + \sin^2 \alpha_n)
\end{aligned}
$$

12.5.7

(a)

$$\begin{aligned} x^2y'' + xy' + \lambda y &= 0 \\ y(1) &= 0 \\ y(5) &= 0 \end{aligned}$$

$$\begin{aligned} \lambda &= \alpha^2 \\ y &= x^m \\ y' &= mx^{m-1} \\ y'' &= m(m-1)x^{m-2} \\ x^2m(m-1)x^{m-2} + xmx^{m-1} + \alpha^2x^m &= 0 \\ m(m-1) + m + \alpha^2 &= 0 \\ m^2 + \alpha^2 &= 0 \\ m &= \pm i\alpha \\ y &= c_1\cos(\alpha \ln x) + c_2 \sin(\alpha \ln x) \end{aligned}$$

$$\begin{aligned} 0 &= c_1 \\ 0 &= c_2 \sin(\alpha \ln 5) \\ \alpha &= \frac{n\pi}{\ln 5} \\ \lambda &= \left(\frac{n\pi}{\ln 5}\right)^2 \end{aligned}$$

$$y_n = \sin\left(\frac{n\pi}{\ln 5}\ln x\right)$$

(b)

$$\begin{aligned} x^2y'' + xy' + \lambda y &= 0 \\ y'' + \frac{1}{x}y' + \lambda\frac{1}{x^2}y &= 0 \\ e^{\ln x}y'' + \frac{1}{x}e^{\ln x}y' + \lambda e^{\ln x}\frac{1}{x^2}y &= 0 \\ \frac{d}{dx}(e^{\ln x}y') + \lambda e^{\ln x}\frac{1}{x^2}y &= 0 \\ \frac{d}{dx}(xy') + \lambda\frac{1}{x}y &= 0 \end{aligned}$$

(c)

$$\int_1^5 \frac{1}{x}\sin\left(\frac{m\pi}{\ln 5}\ln x\right)\sin\left(\frac{n\pi}{\ln 5}\ln x\right) = 0,\ m \neq n$$

12.5.9

$$\begin{aligned}
xy'' + (1-x)y' + ny &= 0 \\
y'' + \left(\frac{1}{x} - 1\right) y' + n\frac{1}{x}y &= 0 \\
e^{\int \left(\frac{1}{x}-1\right) dx} &= e^{\ln(x)-x} \\
&= xe^{-x} \\
xe^{-x}y'' + \left(\frac{1}{x} - 1\right) xe^{-x}y' + n\frac{1}{x}xe^{-x}y &= 0 \\
\frac{d}{dx}(xe^{-x}y') + ne^{-x}y &= 0 \\
\int_0^\infty e^{-x}L_m(x)L_n(x)\,dx &= 0,\ m \neq n
\end{aligned}$$

12.6 Bessel and Legendre Series

12.6.1

$$\begin{aligned}
J_1(3\alpha) &= 0 \\
\alpha_1 &= 1.277 \\
\alpha_2 &= 2.338 \\
\alpha_3 &= 3.391 \\
\alpha_4 &= 4.441
\end{aligned}$$

12.6.3

$$\begin{aligned}
J_0(2\alpha) &= 0 \\
c_i &= \frac{1}{2J_1^2(2\alpha_i)} \int_0^2 xJ_0(\alpha_i x)\,dx \\
&= \frac{1}{2J_1^2(2\alpha_i)} \int_0^2 \frac{d}{dx}\left[\frac{1}{\alpha_i}xJ_1(\alpha_i x)\right]\,dx \\
&= \frac{1}{\alpha_i J_1(2\alpha_i)} \\
f(x) &= \sum_{i=1}^{\infty} \frac{J_0(\alpha_i x)}{\alpha_i J_1(2\alpha_i)}
\end{aligned}$$

12.6.5

$$
\begin{aligned}
J_0(2\alpha) + 2\alpha J_0'(2\alpha) &= 0 \\
h &= 1 \\
b &= 2 \\
c_i &= \frac{2\alpha_i^2}{(4\alpha_i^2+1)J_0^2(2\alpha_i)} \int_0^2 xJ_0(\alpha_i x)\,dx \\
&= \frac{2\alpha_i^2}{(4\alpha_i^2+1)J_0^2(2\alpha_i)} \int_0^2 \frac{d}{dx}\left[\frac{1}{\alpha_i} xJ_1(\alpha_i x)\right] dx \\
&= \frac{4\alpha_i J_1(2\alpha_i)}{(4\alpha_i^2+1)J_0^2(2\alpha_i)} \\
f(x) &= 4\sum_{i=1}^{\infty} \frac{\alpha_i J_1(2\alpha_i)}{(4\alpha_i^2+1)J_0^2(2\alpha_i)} J_0(\alpha_i x)
\end{aligned}
$$

12.6.7

$$
\begin{aligned}
f(x) &= 5x,\ 0 < x < 4 \\
4J_1(4\alpha) + 4\alpha J_1'(4\alpha) &= 0 \\
h &= 3 \\
n &= 1 \\
b &= 4 \\
c_i &= \frac{2\alpha_i^2}{(16\alpha_i^2+8)J_1^2(4\alpha_i)} \int_0^4 5x^2 J_1(\alpha_i x)\,dx \\
&= \frac{10\alpha_i^2}{(16\alpha_i^2+8)J_1^2(4\alpha_i)} \int_0^4 \frac{d}{dx}\left[\frac{1}{\alpha_i} x^2 J_2(\alpha_i x)\right] dx \\
&= \frac{160\alpha_i J_2(4\alpha_i)}{(16\alpha_i^2+8)J_1^2(4\alpha_i)} \\
f(x) &= \sum_{i=1}^{\infty} \frac{20\alpha_i J_2(4\alpha_i)}{(2\alpha_i^2+1)J_1^2(4\alpha_i)} J_1(\alpha_i x)
\end{aligned}
$$

12.6.15

$$
\begin{aligned}
f(x) &= \begin{cases} 0 & -1 < x < 0 \\ x & 0 < x < 1 \end{cases} \\
c_n &= \frac{2n+1}{2} \int_0^1 x P_n(x)\, dx \\
c_0 &= \frac{1}{4} \\
c_1 &= \frac{1}{2} \\
c_2 &= \frac{5}{16} \\
c_3 &= 0 \\
c_4 &= -\frac{3}{32} \\
c_5 &= 0 \\
c_6 &= \frac{13}{256}
\end{aligned}
$$

12.6.21

$$
\begin{aligned}
c_0 &= \frac{1}{2} \\
c_1 &= \frac{5}{8} \\
c_2 &= -\frac{3}{16} \\
c_3 &= \frac{13}{128}
\end{aligned}
$$