

# Advanced Engineering Mathematics Partial Differential Equations by Dennis G. Zill Notes

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## 12 Orthogonal Functions and Fourier Series

### 12.1 Orthogonal Functions

- The **inner product** of two functions  $f_1$  and  $f_2$  on an interval  $[a, b]$  is the number

$$(f_1, f_2) = \int_a^b f_1(x)f_2(x) dx.$$

- Two functions  $f_1$  and  $f_2$  are said to be orthogonal on an interval if  $(f_1, f_2) = 0$ .
- A set of real-valued functions  $\{\phi_1(x), \phi_2(x), \dots, \phi_n(x)\}$  is said to be **orthogonal** on an interval if

$$(\phi_i, \phi_j) = 0 \text{ for } i \neq j.$$

- The **square norm** of a function is

$$||\phi_n(x)||^2 = (\phi_n, \phi_n)$$

and thus its **norm** is

$$||\phi_n(x)|| = \sqrt{(\phi_n, \phi_n)}.$$

- An **orthonormal set** of functions is an orthogonal set of functions that all have a norm of 1.

- An orthogonal set can be made into an orthonormal set by dividing each member by its norm.
- If  $\{\phi_n(x)\}$  is an infinite orthogonal set of functions on an interval  $[a, b]$  and  $f(x)$  is an arbitrary function, then it's possible to determine a set of coefficients  $c_n, n = 0, 1, 2, \dots$  such that

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) = c_0 \phi_0(x) + c_1 \phi_1(x) + \dots + c_n \phi_n(x) + \dots$$

This is called an **orthogonal series expansion** of  $f$  or a **generalized Fourier series** where the coefficients are given by

$$c_n = \frac{(f, \phi_n)}{\|\phi_n\|^2}.$$

- A set of real-valued functions  $\{\phi_n(x)\}$  is said to be **orthogonal with respect to a weight function**  $w(x)$  on the interval  $[a, b]$  if

$$\int_a^b w(x) \phi_m(x) \phi_n(x) dx = 0, \quad m \neq n.$$

## 12.2 Fourier Series

- The **Fourier series** of a function  $f$  defined on the interval  $(-p, p)$  is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{p} \int_{-p}^p f(x) dx \\ a_n &= \frac{1}{p} \int_{-p}^p f(x) \cos \frac{n\pi}{p} x dx \\ b_n &= \frac{1}{p} \int_{-p}^p f(x) \sin \frac{n\pi}{p} x dx \end{aligned}$$

- At points of discontinuity in  $f$ , the Fourier series takes on the average of the values either side of it.
- The Fourier series of a function  $f$  gives a **periodic extension** of the function outside the interval  $(-p, p)$ .

## 12.3 Fourier Cosine and Sine Series

- A function  $f$  is said to be **even** if

$$f(-x) = f(x)$$

and **odd** if

$$f(-x) = -f(x).$$

- Even and odd functions have some interesting properties:
  - The product of two even functions is even.
  - The product of two odd functions is even.
  - The product of an even function and an odd function is odd.
  - If  $f$  is even, then  $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$ .
  - If  $f$  is odd, then  $\int_{-a}^a f(x) dx = 0$ .
- In light of this, if a function  $f$  is even its Fourier coefficients are

$$\begin{aligned}a_0 &= \frac{2}{p} \int_0^p f(x) dx \\a_n &= \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi}{p} x dx \\b_n &= 0.\end{aligned}$$

The series consists of cosine terms and is called the **Fourier cosine series**.

- Similarly, if  $f$  is odd then

$$\begin{aligned}a_n &= 0, \quad n = 0, 1, 2, \dots \\b_n &= \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x dx.\end{aligned}$$

The series consists of sine terms and is called the **Fourier sine series**.

- Sometimes a Fourier series “overshoots” the original value of the function near discontinuities. This is called the **Gibbs phenomenon**.
- Taking the Fourier cosine series of a function  $f$  over the interval  $[0, L]$  effectively mirrors the function around the vertical axis.
- Taking the Fourier sine series of a function  $f$  over the interval  $[0, L]$  effectively rotates it  $180^\circ$  around the origin.
- A particular solution for a nonhomogeneous differential equation with a periodic driving force can be found by taking the Fourier transform of the driving force then using the method of undetermined coefficients to determine the coefficients.

## 12.4 Complex Fourier Series

- The **complex Fourier series** of a function  $f$  defined on an interval  $(-p, p)$  is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p}$$

where

$$c_n = \frac{1}{2p} \int_{-p}^p f(x) e^{-in\pi x/p} dx, \quad n = 0, \pm 1, \pm 2, \dots$$

- The **fundamental period** of a Fourier series is  $T = 2p$ .
- The **fundamental angular frequency** of a Fourier series is  $\omega = \frac{2\pi}{T}$ .
- A **frequency spectrum** is a plot of the points  $(n\omega, |c_n|)$  where  $\omega$  is the fundamental angular frequency and  $c_n$  are the coefficients of the complex Fourier series. This can be useful to see how each harmonic contributes.