

Introduction to Electrodynamics by David J. Griffiths Problems

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2 Electrostatics

2.1

- (a) **0**
- (b) The same as if only the opposite charge were present — all others are cancelled out.

2.2

$$\begin{aligned}
 \mathbf{E} &= \frac{1}{4\pi\epsilon_0} 2 \frac{q}{z^2} \cos\theta \hat{\mathbf{x}} \\
 &= \frac{1}{4\pi\epsilon_0} \frac{dq}{[(d/2)^2 + z^2]^{3/2}} \hat{\mathbf{x}}
 \end{aligned}$$

2.3

$$\begin{aligned}
\mathbf{r} &= z\hat{\mathbf{z}} \\
\mathbf{r}' &= x\hat{\mathbf{x}} \\
\mathbf{r} &= z\hat{\mathbf{z}} - x\hat{\mathbf{x}} \\
r &= \sqrt{x^2 + z^2} \\
\hat{\mathbf{r}} &= \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{\sqrt{x^2 + z^2}} \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_0^L \frac{\lambda}{x^2 + z^2} \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{\sqrt{x^2 + z^2}} dx \\
&= \frac{1}{4\pi\epsilon_0} \lambda \left(z\hat{\mathbf{z}} \int_0^L \frac{1}{(x^2 + z^2)^{3/2}} dx - \hat{\mathbf{x}} \int_0^L \frac{x}{(x^2 + z^2)} dx \right) \\
&= \frac{1}{4\pi\epsilon_0} \lambda \left[\frac{L}{z\sqrt{L^2 + z^2}} \hat{\mathbf{z}} - \left(\frac{1}{z} - \frac{1}{\sqrt{L^2 + z^2}} \right) \hat{\mathbf{x}} \right] \\
&= \frac{1}{4\pi\epsilon_0} \frac{\lambda}{z} \left[\left(-1 + \frac{z}{\sqrt{L^2 + z^2}} \right) \hat{\mathbf{x}} + \frac{L}{\sqrt{L^2 + z^2}} \hat{\mathbf{z}} \right]
\end{aligned}$$

2.4

The electric field a distance z above the midpoint of a line segment of length $2L$ and uniform line charge λ is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z\sqrt{z^2 + L^2}} \hat{\mathbf{z}}.$$

Applying this to the four sides of the square, the horizontal components of opposite sides cancel leaving only the vertical component.

$$\begin{aligned}
\cos \theta &= \frac{z}{r} \\
&= \frac{z}{\sqrt{(a/2)^2 + z^2}} \\
\mathbf{E} &= 4 \left(\frac{1}{4\pi\epsilon_0} \frac{\lambda a}{\sqrt{(a/2)^2 + z^2} \sqrt{(a/2)^2 + (a/2)^2 + z^2}} \hat{\mathbf{z}} \right) \cos \theta \\
&= \frac{1}{4\pi\epsilon_0} \frac{4a\lambda z}{[(a/2)^2 + z^2] \sqrt{(a/2)^2 + z^2}} \hat{\mathbf{z}}
\end{aligned}$$

2.5

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \frac{\lambda r}{r^2 + z^2} \cos \alpha \, d\theta \, \hat{\mathbf{z}} \\ &= \frac{1}{4\pi\epsilon_0} \frac{2\pi\lambda r z}{(r^2 + z^2)^{3/2}} \hat{\mathbf{z}}\end{aligned}$$

2.6

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int \frac{dq}{z^2} \cos \theta \hat{\mathbf{z}} \\ &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^R \frac{\sigma}{r^2 + z^2} \frac{z}{\sqrt{r^2 + z^2}} r \, dr \, d\theta \hat{\mathbf{z}} \\ &= \frac{1}{4\pi\epsilon_0} 2\pi\sigma z \int_0^R \frac{r}{(r^2 + z^2)^{3/2}} \, dr \hat{\mathbf{z}} \\ &= \frac{1}{4\pi\epsilon_0} 2\pi\sigma z \left(\frac{1}{z} - \frac{1}{\sqrt{R^2 + z^2}} \right) \hat{\mathbf{z}}\end{aligned}$$

When $R \rightarrow \infty$

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0} \hat{\mathbf{z}}.$$

2.7

$$\mathbf{E} = \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q}{z^2} \hat{\mathbf{z}} & z > R \\ \mathbf{0} & z < R \end{cases}$$

2.8

$$\mathbf{E} = \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q}{z^2} \hat{\mathbf{z}} & z > R \\ \frac{1}{4\pi\epsilon_0} \frac{qz}{R^3} \hat{\mathbf{z}} & z < R \end{cases}$$

2.9

(a)

$$\begin{aligned}\rho &= \epsilon_0 \nabla \cdot \mathbf{E} \\ &= \epsilon_0 \frac{1}{r^2} \frac{\partial}{\partial r} (kr^5) \\ &= 5\epsilon_0 kr^2\end{aligned}$$

(b)

$$\begin{aligned}
Q_{\text{enc}} &= \epsilon_0 \oint \mathbf{E} \cdot d\mathbf{a} \\
&= \epsilon_0 \int_0^{2\pi} \int_0^\pi kR^3 R d\theta R \sin \theta d\phi \\
&= 2\pi\epsilon_0 kR^5 [-\cos \theta]_0^\pi \\
&= 4\pi\epsilon_0 kR^5 \\
Q_{\text{enc}} &= \int_V \rho d\tau \\
&= \int_0^{2\pi} \int_0^\pi \int_0^R 5\epsilon_0 k r^2 dr r d\theta r \sin \theta d\phi \\
&= 10\pi\epsilon_0 k \int_0^\pi \int_0^R r^4 \sin \theta dr d\theta \\
&= 2\pi\epsilon_0 kR^5 [-\cos \theta]_0^\pi \\
&= 4\pi\epsilon_0 kR^5
\end{aligned}$$

2.10

If the charge was surrounded by 8 such cubes the total flux through all the cubes would be q/ϵ_0 . There are 24 outside faces to the larger cube, so the total flux through the shaded face is $q/(24\epsilon_0)$.

2.11

$$\begin{aligned}
\int \mathbf{E}_{\text{inside}} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
&= 0 \\
\mathbf{E}_{\text{inside}} &= \mathbf{0} \\
\int \mathbf{E}_{\text{outside}} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E_{\text{outside}} &= \frac{4\pi R^2 \sigma}{\epsilon_0} \\
\mathbf{E}_{\text{outside}} &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}
\end{aligned}$$

2.12

$$\begin{aligned}\int \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\ 4\pi r^2 E &= \frac{\frac{4}{3}\pi r^3 \rho}{\epsilon_0} \\ \mathbf{E} &= \frac{r\rho}{3\epsilon_0} \hat{\mathbf{r}}\end{aligned}$$

2.13

$$\begin{aligned}\int \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\ 2\pi s l E &= \frac{l\lambda}{\epsilon_0} \\ \mathbf{E} &= \frac{1}{2\pi\epsilon_0} \frac{\lambda}{s} \hat{\mathbf{s}}\end{aligned}$$

2.14

$$\begin{aligned}Q_{\text{enc}} &= \int_V \rho \, d\tau \\ &= \int_0^{2\pi} \int_0^\pi \int_0^r k r'^3 \sin \theta \, dr' \, d\theta \, d\phi \\ &= 2\pi k \int_0^\pi \left[\frac{1}{4} r'^4 \sin \theta \right]_0^r d\theta \\ &= \frac{1}{2} \pi k r^4 [-\cos \theta]_0^\pi \\ &= \pi k r^4 \\ \int \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\ 4\pi r^2 E &= \frac{\pi k r^4}{\epsilon_0} \\ \mathbf{E} &= \frac{k r^2}{4\epsilon_0} \hat{\mathbf{r}}\end{aligned}$$

2.15

(a) $\mathbf{E} = \mathbf{0}$

(b)

$$\begin{aligned}
 Q_{\text{enc}} &= \int_0^{2\pi} \int_0^\pi \int_a^r k \sin \theta \, dr' \, d\theta \, d\phi \\
 &= 4\pi k(r-a) \\
 4\pi r^2 E &= \frac{4\pi k(r-a)}{\epsilon_0} \\
 \mathbf{E} &= \frac{k(r-a)}{\epsilon_0 r^2} \hat{\mathbf{r}}
 \end{aligned}$$

(c) $\mathbf{E} = \frac{k(b-a)}{\epsilon_0 r^2} \hat{\mathbf{r}}$

2.16

(a)

$$\begin{aligned}
 Q_{\text{enc}} &= \pi s^2 l \rho \\
 2\pi s l E &= \frac{\pi s^2 l \rho}{\epsilon_0} \\
 \mathbf{E} &= \frac{s\rho}{2\epsilon_0} \hat{\mathbf{s}}
 \end{aligned}$$

(b)

$$\mathbf{E} = \frac{a^2 \rho}{2\epsilon_0 s} \hat{\mathbf{s}}$$

(c)

$$\mathbf{E} = \mathbf{0}$$

2.17

$$\begin{aligned}
 2AE_{\text{inside}} &= \frac{2Ay\rho}{\epsilon_0} \\
 \mathbf{E}_{\text{inside}} &= \frac{y\rho}{\epsilon_0} \\
 \mathbf{E} &= \begin{cases} \frac{d\rho}{\epsilon_0} & d < y \\ \frac{y\rho}{\epsilon_0} & 0 < y < d \\ -\frac{y\rho}{\epsilon_0} & -d < y < 0 \\ -\frac{d\rho}{\epsilon_0} & y < -d \end{cases}
 \end{aligned}$$

2.18

The electric field inside a uniformly charged solid sphere is

$$\mathbf{E} = \frac{r\rho}{3\epsilon_0} \hat{\mathbf{r}}.$$

$$\begin{aligned}\mathbf{d} &= \mathbf{r}_1 - \mathbf{r}_2 \\ \mathbf{E} &= \frac{r_1\rho}{3\epsilon_0} \hat{\mathbf{r}}_1 - \frac{r_2\rho}{3\epsilon_0} \hat{\mathbf{r}}_2 \\ &= \frac{\rho}{3\epsilon_0} (\mathbf{r}_1 - \mathbf{r}_2) \\ &= \frac{\rho}{3\epsilon_0} \mathbf{d}\end{aligned}$$

2.20

a is impossible because its curl is nonzero.

$$\begin{aligned}V &= - \int_0^y 2kxy' dy' - \int_0^z 2kyz' dz \\ &= -2kx \left[\frac{1}{2}y'^2 \right]_0^y - 2ky \left[\frac{1}{2}z'^2 \right]_0^z \\ &= -k(xy^2 + yz^2) \\ -\nabla V &= k[y^2 \hat{\mathbf{x}} + (2xy + z^2) \hat{\mathbf{y}} + 2yz \hat{\mathbf{z}}] \\ &= \mathbf{E}\end{aligned}$$

2.21

$$\begin{aligned}
\mathbf{E} &= \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} & r > R \\ \frac{1}{4\pi\epsilon_0} \frac{qr}{R^3} & r < R \end{cases} \\
V_{\text{outside}}(r) &= - \int_{\infty}^r \frac{1}{4\pi\epsilon_0} \frac{q}{r'^2} dr' \\
&= - \frac{1}{4\pi\epsilon_0} q \left[-\frac{1}{r'} \right]_{\infty}^r \\
&= \frac{1}{4\pi\epsilon_0} \frac{q}{r} \\
-\nabla V_{\text{outside}} &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} \\
&= \mathbf{E}_{\text{outside}} \\
V_{\text{inside}}(r) &= - \left(\int_{\infty}^R \frac{1}{4\pi\epsilon_0} \frac{q}{r'^2} dr' + \int_R^r \frac{1}{4\pi\epsilon_0} \frac{qr'}{R^3} dr' \right) \\
&= - \left(-\frac{1}{4\pi\epsilon_0} \frac{q}{R} + \frac{1}{4\pi\epsilon_0} \frac{q}{R^3} \left[\frac{1}{2} r'^2 \right]_R^r \right) \\
&= \frac{1}{4\pi\epsilon_0} \frac{q}{2R} \left[3 - \left(\frac{r}{R} \right)^2 \right] \\
-\nabla V_{\text{inside}} &= \frac{1}{4\pi\epsilon_0} \frac{qr}{R^3} \hat{\mathbf{r}} \\
&= \mathbf{E}_{\text{inside}}
\end{aligned}$$

2.22

$$\begin{aligned}
\mathbf{E} &= \frac{1}{2\pi\epsilon_0} \frac{\lambda}{s} \hat{\mathbf{s}} \\
V &= - \int_O^s \frac{1}{2\pi\epsilon_0} \frac{\lambda}{s'} ds' \\
&= - \frac{1}{2\pi\epsilon_0} \lambda \ln \frac{s}{O} \\
-\nabla V &= \frac{1}{2\pi\epsilon_0} \frac{\lambda}{s} \hat{\mathbf{s}}
\end{aligned}$$

2.23

$$\begin{aligned}
\mathbf{E} &= \begin{cases} \mathbf{0} & r < a \\ \frac{k(r-a)}{\epsilon_0 r^2} \hat{\mathbf{r}} & a < r < b \\ \frac{k(b-a)}{\epsilon_0 r^2} \hat{\mathbf{r}} & b < r \end{cases} \\
V(0) &= - \int_{\infty}^0 E dr \\
&= - \left(\int_{\infty}^b \frac{k(b-a)}{\epsilon_0 r^2} dr + \int_b^a \frac{k(r-a)}{\epsilon_0 r^2} dr \right) \\
&= - \left(\frac{k(b-a)}{\epsilon_0} \left[-\frac{1}{r} \right]_{\infty}^b + \frac{k}{\epsilon_0} \left[\ln r + \frac{a}{r} \right]_b^a \right) \\
&= - \left[-\frac{k(b-a)}{\epsilon_0 b} + \frac{k}{\epsilon_0} \left(\ln a + 1 - \ln b - \frac{a}{b} \right) \right] \\
&= -\frac{k}{\epsilon_0} \left(-1 + \frac{a}{b} + \ln \frac{a}{b} + 1 - \frac{a}{b} \right) \\
&= \frac{k}{\epsilon_0} \ln \frac{b}{a}
\end{aligned}$$

2.24

$$\begin{aligned}
V(b) - V(0) &= - \int_0^b E dr \\
&= - \left(\int_0^a \frac{s\rho}{2\epsilon_0} ds + \int_a^b \frac{a^2\rho}{2\epsilon_0 s} ds \right) \\
&= - \left(\frac{\rho}{2\epsilon_0} \left[\frac{1}{2} s^2 \right]_0^a + \frac{a^2\rho}{2\epsilon_0} \ln \frac{b}{a} \right) \\
&= - \left(\frac{a^2\rho}{4\epsilon_0} + \frac{a^2\rho}{2\epsilon_0} \ln \frac{b}{a} \right) \\
&= -\frac{a^2\rho}{4\epsilon_0} \left(1 + 2 \ln \frac{a}{b} \right)
\end{aligned}$$

2.25

(a)

$$V = \frac{1}{4\pi\epsilon_0} \frac{2q}{\sqrt{(d/2)^2 + z^2}}$$

(b)

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0} \int_{-L}^L \frac{\lambda}{\sqrt{x^2 + z^2}} dx \\ &= \frac{1}{4\pi\epsilon_0} \lambda \ln \left(1 + \frac{2L(L + \sqrt{L^2 + z^2})}{z^2} \right) \end{aligned}$$

(c)

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^R \frac{\sigma}{\sqrt{r^2 + z^2}} r dr d\theta \\ &= \frac{1}{4\pi\epsilon_0} 2\pi\sigma(\sqrt{R^2 + z^2} - z) \end{aligned}$$

2.26

$$\begin{aligned} V_{\text{bottom}} &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^h \frac{\sqrt{2}\sigma z}{\sqrt{2}z} d\phi dz \\ &= \frac{\sigma h}{2\epsilon_0} \\ V_{\text{top}} &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^h \frac{\sqrt{2}\sigma z}{\sqrt{z^2 + (h-z)^2}} d\phi dz \\ &= \frac{\sqrt{2}\sigma}{2\epsilon_0} \int_0^h \frac{z}{\sqrt{z^2 + (h-z)^2}} dz \\ &= \frac{\sigma h}{4\epsilon_0} \ln(3 + 2\sqrt{2}) \\ V_{\text{bottom}} - V_{\text{top}} &= \frac{\sigma h}{2\epsilon_0} \left[1 - \frac{1}{2} \ln(3 + 2\sqrt{2}) \right] \end{aligned}$$

2.28

$$\begin{aligned} V(r) &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^\pi \int_0^R \frac{\rho r'^2 \sin \theta}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta}} dr' d\theta d\phi \\ &= \frac{\rho}{2\epsilon_0} \int_0^\pi \int_0^R \frac{r'^2 \sin \theta}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta}} dr' d\theta \\ &= \frac{\rho}{2\epsilon_0} \left(R^2 - \frac{r^2}{3} \right) \\ &= \frac{q}{8\pi\epsilon_0 R} \left(3 - \frac{r^2}{R^2} \right) \end{aligned}$$

2.31

(a)

$$W = \frac{q^2}{4\pi\epsilon_0 a} \left(\frac{1}{\sqrt{2}} - 2 \right)$$

(b)

$$\begin{aligned} W &= \frac{1}{4\pi\epsilon_0} \left(-\frac{q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a} - \frac{q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a} \right) \\ &= \frac{q^2}{2\pi\epsilon_0 a} \left(\frac{1}{\sqrt{2}} - 2 \right) \end{aligned}$$

2.32

$$W = \frac{1}{4\pi\epsilon_0} \frac{q_A q_B}{a}$$

$$W = K_1 + K_2$$

$$\frac{1}{4\pi\epsilon_0} \frac{q_A q_B}{a} = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2$$

$$\frac{1}{2\pi\epsilon_0} \frac{q_A q_B}{a} = m_A v_A^2 + m_B v_B^2$$

$$0 = m_B v_B - m_A v_A$$

$$v_B = \frac{m_A}{m_B} v_A$$

$$\frac{1}{2\pi\epsilon_0} \frac{q_A q_B}{a} = m_A v_A^2 + m_B \left(\frac{m_A}{m_B} v_A \right)^2$$

$$= m_A v_A^2 + \frac{m_A^2}{m_B} v_A^2$$

$$= \frac{m_A(m_A + m_B)}{m_B} v_A^2$$

$$v_A = \sqrt{\frac{1}{2\pi\epsilon_0} \frac{q_A q_B}{(m_A + m_B)a} \frac{m_B}{m_A}}$$

$$v_B = \sqrt{\frac{1}{2\pi\epsilon_0} \frac{q_A q_B}{(m_A + m_B)a} \frac{m_A}{m_B}}$$

2.33

$$\begin{aligned}
 W &= \frac{1}{4\pi\epsilon_0} \left(-\frac{q^2}{a} + \frac{q^2}{2a} - \frac{q^2}{3a} + \dots \right) \\
 &= \frac{1}{4\pi\epsilon_0} \frac{q^2}{a} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \\
 &= -\frac{1}{4\pi\epsilon_0} \frac{q^2}{a} \ln 2
 \end{aligned}$$

2.34

(a)

$$\begin{aligned}
 V &= \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q}{2R} \left[3 - \left(\frac{r}{R} \right)^2 \right] & r < R \\ \frac{1}{4\pi\epsilon_0} \frac{q}{r} & r > R \end{cases} \\
 W &= \frac{1}{2} \int \rho V d\tau \\
 &= \frac{1}{2} \int_0^{2\pi} \int_0^\pi \int_0^R \rho \frac{1}{4\pi\epsilon_0} \frac{q}{2R} \left[3 - \left(\frac{r}{R} \right)^2 \right] r^2 \sin \theta dr d\theta d\phi \\
 &= \frac{q\rho}{8\epsilon_0 R} \int_0^\pi \int_0^R \left[3 - \left(\frac{r}{R} \right)^2 \right] r^2 \sin \theta dr d\theta \\
 &= \frac{q\rho R^2}{5\epsilon_0} \\
 &= \frac{qR^2}{5\epsilon_0} \frac{q}{\frac{4}{3}\pi R^3} \\
 &= \frac{1}{4\pi\epsilon_0} \frac{3q^2}{5R}
 \end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{E} &= \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} & r > R \\ \frac{1}{4\pi\epsilon_0} \frac{qr}{R^3} \hat{\mathbf{r}} & r < R \end{cases} \\
E^2 &= \begin{cases} \frac{1}{16\pi^2\epsilon_0^2} \frac{q^2}{r^4} & r > R \\ \frac{1}{16\pi^2\epsilon_0^2} \frac{q^2 r^2}{R^6} & r < R \end{cases} \\
W &= \frac{\epsilon_0}{2} \int E^2 d\tau \\
&= \frac{\epsilon_0}{2} \left(\int_0^{2\pi} \int_0^\pi \int_0^R \frac{1}{16\pi^2\epsilon_0^2} \frac{q^2 r^2}{R^6} r^2 \sin\theta dr d\theta d\phi \right. \\
&\quad \left. + \int_0^{2\pi} \int_0^\pi \int_R^\infty \frac{1}{16\pi^2\epsilon_0^2} \frac{q^2}{r^4} r^2 \sin\theta dr d\theta d\phi \right) \\
&= \frac{\epsilon_0}{2} \frac{1}{16\pi^2\epsilon_0^2} 2\pi q^2 \left(\int_0^\pi \int_0^R \frac{r^4}{R^6} \sin\theta dr d\theta + \int_0^\pi \int_R^\infty \frac{1}{r^2} \sin\theta dr d\theta \right) \\
&= \frac{1}{16\pi\epsilon_0} q^2 \left(\int_0^\pi \int_0^R \frac{r^4}{R^6} \sin\theta dr d\theta + \int_0^\pi \int_R^\infty \frac{1}{r^2} \sin\theta dr d\theta \right) \\
&= \frac{1}{16\pi\epsilon_0} q^2 \left(\frac{2}{5R} + \frac{2}{R} \right) \\
&= \frac{1}{4\pi\epsilon_0} \frac{3q^2}{5R}
\end{aligned}$$

(c)

$$\begin{aligned}
W &= \frac{\epsilon_0}{2} \left(\int_V E^2 d\tau + \oint_S V \mathbf{E} \cdot d\mathbf{a} \right) \\
&= \frac{\epsilon_0}{2} \left(\int_0^{2\pi} \int_0^\pi \int_0^R \frac{1}{(4\pi\epsilon_0)^2} \frac{q^2 r^2}{R^6} r^2 \sin \theta dr d\theta d\phi \right. \\
&\quad + \int_0^{2\pi} \int_0^\pi \int_R^a \frac{1}{(4\pi\epsilon_0)^2} \frac{q^2}{r^4} r^2 \sin \theta dr d\theta d\phi \\
&\quad \left. + \int_0^{2\pi} \int_0^\pi \frac{1}{4\pi\epsilon_0} \frac{q}{a} \frac{1}{4\pi\epsilon_0} \frac{q}{a^2} a^2 \sin \theta d\theta d\phi \right) \\
&= \frac{\epsilon_0}{2} \frac{1}{(4\pi\epsilon_0)^2} 2\pi q^2 \left(\int_0^\pi \int_0^R \frac{r^4}{R^6} \sin \theta dr d\theta \right. \\
&\quad \left. + \int_0^\pi \int_R^a \frac{1}{r^2} \sin \theta dr d\theta + \int_0^\pi \frac{1}{a} \sin \theta d\theta \right) \\
&= \frac{\epsilon_0}{2} \frac{1}{(4\pi\epsilon_0)^2} 2\pi q^2 \left[\frac{2}{5R} + 2 \left(\frac{1}{R} - \frac{1}{a} \right) + \frac{2}{a} \right] \\
&= \frac{1}{8\pi\epsilon_0} q^2 \left[\frac{1}{5R} + \frac{1}{R} \right] \\
&= \frac{1}{4\pi\epsilon_0} \frac{3q^2}{5R}
\end{aligned}$$

2.36

(a)

$$\begin{aligned}
\mathbf{E} &= \begin{cases} \mathbf{0} & r < a \\ \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} & a < r < b \\ \mathbf{0} & b < r \end{cases} \\
E^2 &= \begin{cases} 0 & r < a \\ \frac{1}{(4\pi\epsilon_0)^2} \frac{q^2}{r^4} & a < r < b \\ 0 & b < r \end{cases} \\
W &= \frac{\epsilon_0}{2} \int E^2 d\tau \\
&= \frac{\epsilon_0}{2} \int_0^{2\pi} \int_0^\pi \int_a^b \frac{1}{(4\pi\epsilon_0)^2} \frac{q^2}{r^4} r^2 \sin \theta dr d\theta d\phi \\
&= \frac{\epsilon_0}{2} \frac{1}{(4\pi\epsilon_0)^2} 2\pi q^2 \int_0^\pi \int_a^b \frac{\sin \theta}{r^2} dr d\theta \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)
\end{aligned}$$

(b)

$$\begin{aligned}
W_{\text{shell}} &= \frac{1}{8\pi\epsilon_0} \frac{q^2}{R} \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} \\
\mathbf{E}_1 \cdot \mathbf{E}_2 &= -\frac{1}{(4\pi\epsilon_0)^2} \frac{q^2}{r^4} \\
W_{\text{total}} &= W_1 + W_2 + \epsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2 d\tau \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} + \frac{1}{b} \right) - \epsilon_0 \int_0^{2\pi} \int_0^\pi \int_b^\infty \frac{1}{(4\pi\epsilon_0)^2} \frac{q^2}{r^4} r^2 \sin\theta dr d\theta d\phi \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} + \frac{1}{b} \right) - \frac{1}{8\pi\epsilon_0} q^2 \int_0^\pi \int_b^\infty \frac{1}{r^2} \sin\theta dr d\theta \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} + \frac{1}{b} \right) - \frac{1}{4\pi\epsilon_0} q^2 \int_b^\infty \frac{1}{r^2} dr \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} + \frac{1}{b} \right) - \frac{1}{4\pi\epsilon_0} \frac{q^2}{b} \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} + \frac{1}{b} - \frac{2}{b} \right) \\
&= \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)
\end{aligned}$$

2.37

$$\begin{aligned}
r_1 &= r \\
E_1 &= \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_1^2} \\
&= \frac{1}{4\pi\epsilon_0} \frac{q_1}{r^2} \\
r_2 &= \sqrt{a^2 + r^2 - 2ar \cos \theta} \\
E_2 &= \frac{1}{4\pi\epsilon_0} \frac{q_2}{r_2^2} \\
&= \frac{1}{4\pi\epsilon_0} \frac{q_2}{a^2 + r^2 - 2ar \cos \theta} \\
\cos \alpha &= \frac{r - a \cos \theta}{\sqrt{a^2 + r^2 - 2ar \cos \theta}} \\
\mathbf{E}_1 \cdot \mathbf{E}_2 &= E_1 E_2 \cos \alpha \\
&= \frac{1}{(4\pi\epsilon_0)^2} \frac{q_1 q_2}{r^2 (a^2 + r^2 - 2ar \cos \theta)} \frac{r - a \cos \theta}{\sqrt{a^2 + r^2 - 2ar \cos \theta}} \\
&= \frac{1}{(4\pi\epsilon_0)^2} \frac{q_1 q_2 (r - a \cos \theta)}{r^2 (a^2 + r^2 - 2ar \cos \theta)^{3/2}} \\
\epsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2 d\tau &= \epsilon_0 \int_0^{2\pi} \int_0^\pi \int_0^\infty \frac{1}{(4\pi\epsilon_0)^2} \frac{q_1 q_2 (r - a \cos \theta)}{r^2 (a^2 + r^2 - 2ar \cos \theta)^{3/2}} r^2 \sin \theta dr d\theta d\phi \\
&= \frac{q_1 q_2}{8\pi\epsilon_0} \int_0^\pi \int_0^\infty \frac{(r - a \cos \theta) \sin \theta}{(a^2 + r^2 - 2ar \cos \theta)^{3/2}} dr d\theta
\end{aligned}$$

2.38

(a)

$$\begin{aligned}
\sigma_R &= \frac{q}{4\pi R^2} \\
\sigma_a &= -\frac{q}{4\pi a^2} \\
\sigma_b &= \frac{q}{4\pi b^2}
\end{aligned}$$

(b)

$$\begin{aligned}
V &= -\int_\infty^b \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} dr - \int_a^R \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} dr \\
&= \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{b} + \frac{1}{R} - \frac{1}{a} \right)
\end{aligned}$$

(c)

$$\sigma_b = 0$$

$$V = \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{R} - \frac{1}{a} \right)$$

2.39

(a)

$$\sigma_a = -\frac{q_a}{4\pi a^2}$$

$$\sigma_b = -\frac{q_b}{4\pi b^2}$$

$$\sigma_R = \frac{q_a + q_b}{4\pi R^2}$$

(b)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q_a + q_b}{r^2} \hat{\mathbf{r}}$$

(c)

$$\mathbf{E}_a = \frac{1}{4\pi\epsilon_0} \frac{q_a}{r^2} \hat{\mathbf{r}}$$

$$\mathbf{E}_b = \frac{1}{4\pi\epsilon_0} \frac{q_b}{r^2} \hat{\mathbf{r}}$$

(d)

$$\mathbf{0}$$

(e) a, b

2.40

(a) No. If it's close to the wall it will induce a surface charge and be attracted.

(b) No. If the conductor contains a cavity containing a like charge it will be repelled.

2.41

By Gauss's law, the electric field of each plate is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$2A'E = \frac{A' \frac{Q}{A}}{\epsilon_0}$$

$$\mathbf{E} = \frac{Q}{2A\epsilon_0} \hat{\mathbf{n}}$$

so the field between the plates is zero and the field outside is $Q/A\epsilon_0\hat{\mathbf{n}}$, resulting in a pressure of

$$\begin{aligned} P &= \frac{\epsilon_0}{2} E^2 \\ &= \frac{\epsilon_0}{2} \frac{Q^2}{A^2 \epsilon_0^2} \\ &= \frac{Q^2}{2A^2 \epsilon_0} \end{aligned}$$

2.42

$$\begin{aligned} \mathbf{E}_{\text{above}} &= \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}} \\ \mathbf{f} &= \frac{1}{2} \sigma \mathbf{E}_{\text{above}} \\ &= \frac{1}{2} \frac{Q}{4\pi R^2} \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \hat{\mathbf{r}} \\ &= \frac{Q^2}{32\pi^2 \epsilon_0 R^4} \hat{\mathbf{r}} \\ \mathbf{F} &= \int_0^{2\pi} \int_0^{\pi/2} \frac{Q^2}{32\pi^2 \epsilon_0 R^4} \cos \theta R^2 \sin \theta d\theta d\phi \hat{\mathbf{z}} \\ &= \frac{Q^2}{16\pi\epsilon_0 R^2} \int_0^{\pi/2} \cos \theta \sin \theta d\theta \hat{\mathbf{z}} \\ &= \frac{Q^2}{32\pi\epsilon_0 R^2} \hat{\mathbf{z}} \end{aligned}$$

2.43

$$\begin{aligned}
 \oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q}{\epsilon_0} \\
 2\pi s L E &= \frac{Q}{\epsilon_0} \\
 \mathbf{E} &= \frac{Q}{2\pi L \epsilon_0} \frac{1}{s} \hat{\mathbf{s}} \\
 V &= - \int_b^a \frac{Q}{2\pi \epsilon_0 L} \frac{1}{s} dr \\
 &= \frac{Q}{2\pi \epsilon_0 L} \ln \frac{b}{a} \\
 C &= \frac{Q}{V} \\
 &= \frac{2\pi \epsilon_0 L}{\ln b/a}
 \end{aligned}$$

So the capacitance per unit length is

$$C = \frac{2\pi \epsilon_0}{\ln b/a}.$$

2.44

(a)

$$\begin{aligned}
 P &= \frac{\epsilon_0}{2} E^2 \\
 W &= Fd \\
 &= PA\epsilon \\
 &= \frac{\epsilon_0}{2} E^2 A\epsilon
 \end{aligned}$$

(b)

$$\frac{\epsilon_0}{2} E^2 A\epsilon$$

2.46

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 3 \frac{k}{r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{k}{r} 2 \sin \theta \cos \theta \sin \phi \right) \\
&\quad + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left(\frac{k}{r} \sin \theta \cos \phi \right) \\
&= \frac{3k}{r^2} + \frac{1}{r \sin \theta} \frac{2k}{r} \sin \phi (2 \sin \theta \cos^2 \theta - \sin^3 \theta) - \frac{1}{r \sin \theta} \frac{k}{r} \sin \theta \sin \phi \\
&= \frac{3k}{r^2} + \frac{2k \sin \phi}{r^2} (2 \cos^2 \theta - \sin^2 \theta) - \frac{k}{r^2} \sin \phi \\
&= \frac{k}{r^2} [3 + 2 \sin \phi (2 \cos^2 \theta - \sin^2 \theta) - \sin \phi] \\
&= \frac{k}{r^2} [3 + \sin \phi (4 \cos^2 \theta - 2 \sin^2 \theta - 1)] \\
&= \frac{k}{r^2} [3 + \sin \phi (6 \cos^2 \theta - 3)] \\
&= \frac{3k}{r^2} (1 + \cos 2\theta \sin \phi) \\
\rho &= \epsilon_0 \nabla \cdot \mathbf{E} \\
&= \frac{3k\epsilon_0}{r^2} (1 + \cos 2\theta \sin \phi)
\end{aligned}$$

2.47

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \frac{Qr}{R^3} \hat{\mathbf{r}} \\
\rho &= \frac{Q}{\frac{4}{3}\pi R^3} \\
\rho \mathbf{E} &= \frac{3Q}{4\pi R^3} \frac{1}{4\pi\epsilon_0} \frac{Qr}{R^3} \hat{\mathbf{r}} \\
&= \frac{3r}{\epsilon_0} \left(\frac{Q}{4\pi R^3} \right)^2 \hat{\mathbf{r}} \\
F_z &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \frac{3r}{\epsilon_0} \left(\frac{Q}{4\pi R^3} \right)^2 \cos \theta r^2 \sin \theta \, dr \, d\theta \, d\phi \\
&= \frac{3\pi}{\epsilon_0} \left(\frac{Q}{4\pi R^3} \right)^2 \int_0^{\pi/2} \int_0^R r^3 \sin 2\theta \, dr \, d\theta \\
&= \frac{3\pi}{\epsilon_0} \left(\frac{Q}{4\pi R^3} \right)^2 \frac{R^4}{4} \\
&= \frac{3Q^2}{64\pi\epsilon_0 R^2}
\end{aligned}$$

2.49

$$\begin{aligned}
Q_{\text{enc}} &= \int_0^{2\pi} \int_0^\pi \int_0^r k r'^3 \sin \theta \, dr' \, d\theta \, d\phi \\
&= 2\pi k \int_0^\pi \int_0^r r'^3 \sin \theta \, dr' \, d\theta \\
&= \pi k r^4 \\
\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\
4\pi r^2 E &= \frac{\pi k r^4}{\epsilon_0} \\
\mathbf{E} &= \begin{cases} \frac{k r^2}{4\epsilon_0} \hat{\mathbf{r}} & r < R \\ \frac{k R^4}{4\epsilon_0 r^2} \hat{\mathbf{r}} & r > R \end{cases} \\
W &= \frac{\epsilon_0}{2} \left(\int_0^{2\pi} \int_0^\pi \int_0^R \frac{k^2 r^4}{16\epsilon_0^2} \sin \theta \, dr \, d\theta \, d\phi \right. \\
&\quad \left. \int_0^{2\pi} \int_0^\pi \int_R^\infty \frac{k^2 R^8}{16\epsilon_0^2 r^4} \sin \theta \, dr \, d\theta \, d\phi \right) \\
&= \frac{\epsilon_0}{2} 2\pi \frac{k^2}{16\epsilon_0^2} \left(\int_0^\pi \int_0^R r^6 \sin \theta \, dr \, d\theta + \int_0^\pi \int_R^\infty \frac{R^8 \sin \theta}{r^2} \, dr \, d\theta \right) \\
&= \frac{\pi k^2}{16\epsilon_0} \left(\frac{2R^7}{7} + 2R^7 \right) \\
&= \frac{\pi k^2 R^7}{7\epsilon_0}
\end{aligned}$$

2.50

$$\begin{aligned}
V(\mathbf{r}) &= A \frac{e^{-\lambda r}}{r} \\
\mathbf{E} &= -\nabla V \\
&= A e^{-\lambda r} (1 + \lambda r) \frac{\hat{\mathbf{r}}}{r^2} \\
\rho &= \epsilon_0 \nabla \cdot \mathbf{E} \\
&= \epsilon_0 \left[A e^{-\lambda r} (1 + \lambda r) \nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} + \frac{\hat{\mathbf{r}}}{r^2} \cdot \nabla (A e^{-\lambda r} (1 + \lambda r)) \right] \\
&= A \epsilon_0 \left[4\pi \delta(\mathbf{r}) + \frac{\hat{\mathbf{r}}}{r^2} \cdot (-\lambda^2 e^{-\lambda r} r \hat{\mathbf{r}}) \right] \\
&= A \epsilon_0 \left(4\pi \delta(\mathbf{r}) - \frac{\lambda^2 e^{-\lambda r}}{r} \right)
\end{aligned}$$

2.51

$$\begin{aligned}
V &= \int \frac{1}{4\pi\epsilon_0} \frac{\sigma}{z} dA \\
&= \frac{\sigma}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^R \frac{r}{\sqrt{r^2 + R^2 - 2rR\cos\theta}} dr d\theta \\
&= \frac{R\sigma}{4\pi\epsilon_0} \int_0^{2\pi} \left[\cos\theta \ln \left(1 + \csc \frac{\theta}{2} \right) + 2 \sin \frac{\theta}{2} - 1 \right] d\theta \\
&= \frac{R\sigma}{\pi\epsilon_0}
\end{aligned}$$

2.52

(a)

$$\begin{aligned}
V_- &= \frac{1}{2\pi\epsilon_0} \lambda \ln \frac{s_-}{a} \\
&= \frac{1}{2\pi\epsilon_0} \lambda \ln \frac{\sqrt{(y+a)^2 + z^2}}{a} \\
V_+ &= -\frac{1}{2\pi\epsilon_0} \lambda \ln \frac{s_+}{a} \\
&= -\frac{1}{2\pi\epsilon_0} \lambda \ln \frac{\sqrt{(y-a)^2 + z^2}}{a} \\
V &= V_- + V_+ \\
&= \frac{1}{4\pi\epsilon_0} \lambda \ln \frac{(y+a)^2 + z^2}{(y-a)^2 + z^2}
\end{aligned}$$

2.53

(a)

$$\begin{aligned}
\nabla^2 V &= -\frac{\rho}{\epsilon_0} \\
\nabla \cdot \nabla V &= -\frac{\rho}{\epsilon_0} \\
\nabla \cdot \frac{dV}{dx} \hat{\mathbf{x}} &= -\frac{\rho}{\epsilon_0} \\
\frac{d^2 V}{dx^2} &= -\frac{\rho}{\epsilon_0}
\end{aligned}$$

(b)

$$qV = \frac{1}{2}mv^2$$
$$v = \sqrt{\frac{2qV}{m}}$$

(c)

$$I = A\rho v$$

(d)

$$\begin{aligned}\frac{d^2V}{dx^2} &= -\frac{I}{Av\epsilon_0} \\ &= -\frac{I}{A\epsilon_0}\sqrt{\frac{m}{2qV}} \\ &= \beta V^{-1/2}\end{aligned}$$

2.55

$$\begin{aligned}\rho &= \epsilon_0 \nabla \cdot \mathbf{E} \\ &= a\epsilon_0\end{aligned}$$

2.56

$$\begin{aligned}E &= \frac{3GM^2}{5R} \\ E_{\text{sun}} &= 2.3 \times 10^{41} \text{ J} \\ t &= \frac{E_{\text{sun}}}{P} \\ &= 1.89 \times 10^7 \text{ years}\end{aligned}$$

3 Potentials

3.1

$$\begin{aligned} V_{\text{ave}} &= \frac{q}{4\pi\epsilon_0} \frac{1}{2zR} \left. \sqrt{z^2 + R^2 - 2zR \cos \theta} \right|_0^\pi \\ &= \frac{q}{4\pi\epsilon_0} \frac{1}{2zR} \left(\sqrt{z^2 + R^2 + 2zR} - \sqrt{z^2 + R^2 - 2zR} \right) \\ &= \frac{q}{4\pi\epsilon_0} \frac{1}{2zR} \left(\sqrt{(z+R)^2} - \sqrt{(R-z)^2} \right) \\ &= \frac{q}{4\pi\epsilon_0} \frac{1}{2zR} (z+R - R+z) \\ &= \frac{1}{4\pi\epsilon_0} \frac{q}{R} \end{aligned}$$

The average potential due to external charges is V_{center} and the average potential due to internal charges is

$$\frac{1}{4\pi\epsilon_0} \frac{Q_{\text{enc}}}{R}$$

so

$$V_{\text{ave}} = V_{\text{center}} + \frac{1}{4\pi\epsilon_0} \frac{Q_{\text{enc}}}{R}.$$

3.2

A stable equilibrium is a minimum of potential energy. Laplace's equation doesn't allow for minimums, so they must be saddle points and the charge can escape.

3.3

$$\begin{aligned}
0 &= \nabla^2 V \\
&= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) \\
&= \frac{1}{r^2} \left(2r \frac{\partial V}{\partial r} + r^2 \frac{\partial^2 V}{\partial r^2} \right) \\
&= \frac{2}{r} \frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2}
\end{aligned}$$

$$V = \frac{c_1}{r} + c_2$$

$$\begin{aligned}
0 &= \nabla^2 V \\
&= \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial V}{\partial s} \right) \\
&= \frac{1}{s} \left(\frac{\partial V}{\partial s} + s \frac{\partial^2 V}{\partial s^2} \right) \\
&= \frac{1}{s} \frac{\partial V}{\partial s} + \frac{\partial^2 V}{\partial s^2}
\end{aligned}$$

$$V = c_1 + c_2 \ln s$$