Advanced Engineering Mathematics Complex Analysis by Dennis G. Zill Notes

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Contents

17 Functions of a Complex Variable

17.1 Complex Numbers

- A **complex number** is any number of the form

$$z = a + ib$$

where a and b are real numbers and i is the imaginary unit such that $i^2 = -1$.

- The real number a in the above complex number z is called the **real part** of z and the real number b (not ib) is called the **imaginary part** of z.

- The real and imaginary parts of a complex number z are denoted $\text{Re}(z)$ and $\text{Im}(z)$, respectively.

- A real constant multiple of the imaginary unit, e.g. $6i$ is called a **pure imaginary number**.

- Two complex numbers are equal if their real and imaginary parts are equal.
- The addition and subtraction of complex numbers occur between the real and imaginary parts, e.g.

$$(a+bi)+(c+di)=(a+c)+(b+d)i.$$

- The multiplication of complex numbers occurs elementwise as normal, e.g.

$$(a+bi)(c+di)=ac+adi+bci-bd.$$

- The **conjugate** of a complex number $z=a+ib$ is

$$\overline{z}=a-ib.$$

- The division of complex numbers occurs by multiplying the numerator and denominator by the conjugate of the denominator, e.g.

$$\begin{aligned}\frac{a+bi}{c+di}&=\frac{(a+bi)(c-di)}{(c+di)(c-di)}\\&=\frac{ac-adi+bci+bd}{c^2+d^2}\\&=\frac{ac+bd}{c^2+d^2}+i\frac{bc-ad}{c^2+d^2}.\end{aligned}$$

- Conjugates have several interesting properties:

$$\begin{aligned}\overline{z_1+z_2}&=\overline{z_1}+\overline{z_2}\\\overline{z_1-z_2}&=\overline{z_1}-\overline{z_2}\\\overline{z_1z_2}&=\overline{z_1}\,\overline{z_2}\\\overline{\frac{z_1}{z_2}}&=\frac{\overline{z_1}}{\overline{z_2}}.\end{aligned}$$

- The sum and product of a complex number $z=x+iy$ with its conjugate are real numbers

$$\begin{aligned}z+\overline{z}&=2x\\z\overline{z}&=x^2+y^2\end{aligned}$$

  while the difference between a complex number and its conjugate is a purre imaginary number

$$z-\overline{z}=2iy.$$

- The above properties let us define

$$\mathrm{Re}(z)=\frac{z+\overline{z}}{2}\ \text{and}\ \mathrm{Im}(z)=\frac{z-\overline{z}}{2i}.$$

- The **complex plane** or **z-plane** is a coordinate system where the horizontal or x-axis is called the **real axis** and the vertical or y-axis is called the **imaginary axis**. Complex numbers can be plotted in this coordinate system by considering their real and imaginary parts an ordered pair corresponding their position.

- The **modulus** or **absolute value** of a complex number $z = x+iy$ denoted by $|z|$ is the real number

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$$

  This is the distance between z and the origin in the complex plane.

- If you consider two numbers in the complex plane as vectors, the length of their sum can't be longer than their individual lengths combined

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

  This extends to any finite sum

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

  and is known as the **triangle inequality**.

17.2 Powers and Roots

- A complex number can be expressed in **polar form**

$$z = (r\cos\theta) + i(r\sin\theta)$$

  where $r = |z|$ is the nonnegative modulus of z and $\theta = \arg z$ is the **argument** of z — the angle between z and the positive real axis measured in the counterclockwise direction.

- The argument of a complex number z isn't unique as any multiply of 2π can be added to it. The **principle argument** of z denoted $\operatorname{Arg} z$ is the argument of z restricted to the intercal $-\pi \leq \operatorname{Arg} z \leq \pi$.

- Multiplication and division of complex numbers is simpler in polar form. For two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ we get

$$\begin{aligned} z_1 z_2 &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]. \end{aligned}$$

- The above formulas can be used to find integer powers of a complex number z

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

  where n is an integer (including negative integers).

- **DeMoivre's formula** is a special case of the above where $r = 1$ so

$$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

- A number w is said to be an ***n*th root** of a nonzero complex number z if $w^n = z$. The nth roots of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ are

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n-1$.

- The root w of a complex number z obtained by using the principle argument of z with $k = 0$ is called the **principle *n*th root** of z.

- Since the nth roots of a complex number have the same modulus they lie on a circle of radius $r^{1/n}$. The arguments of subsequent roots differ by $2\pi/n$ so they're also equally spaced around the circle.

17.3 Sets in the Complex Plane

- The points $z = x + iy$ that satisfy the equation

$$|z - z_0| = \rho$$

for $\rho > 0$ lie on a circle of radius ρ centred at the point z_0.

- The points z satisfying the inequality $|z - z_0| < \rho$ for $\rho > 0$ lie within, but not on, a circle of radius ρ centered at the point z_0. This set is called a **neighborhood** of z_0 or an **open disk**.

- A point z_0 is said to be an **interior point** of a set S of the complex plane if there exists some neighborhood of z_0 that lies entirely within S.

- If every point z of a set S is an interior point, then S is said to be an **open set**. An example of a set that isn't open is the set of points satisfying the inequality $\text{Re}(z) \geq 0$. This isn't open because it includes the line $\text{Re}(z) = 0$ and no points on that line are interior to the set because, no matter what ρ you choose, some points in the neighborhood have $\text{Re}(z) < 0$.

- If every neighborhood of a point z_0 contains at least one point that is in a set S and at least one point that is not in S, then z_0 is said to be a **boundary point** of S.

- The **boundary** of a set S in the complex plane is the set of all boundary points of S.

- If any pair of points in a set S can be connected by a polygonal line that lies entirely within the set, then S is said to be **connected**.

- An open connected set is called a **domain**.

- A **region** is a set in the complex plane with all, some, or none of its boundary points. A region containing all of its boundary points is said to be **closed**.

17.4 Functions of a Complex Variable

- A **function** f from a set A to a set B is a rule of correspondence that assigns to each element of A one and only one element of B.
- If b is the element of B assigned to the element a of A, b is said to be the **image** of a and is denoted $b = f(a)$.
- The set A is called the **domain** of f.
- The set of all images in B is called the **range** of f.
- If A is a set of real numbers, f is said to be a **function of a real variable** $\boldsymbol{x}$.
- If A is a set of complex numbers, f is said to be a **function of a complex varibale** $\boldsymbol{z}$ or a **complex function**.
- The image w of a complex number z is

  $$w = f(z) = u(x, y) + iv(x, y)$$

  where u and v are the real and imaginary parts of w and are real-valued functions.
- Although we cannot draw a graph of a complex function $w = f(z)$ (because it would require a four-dimensional coordinate system), it can be interpreted as a **mapping** or **transformation** from the z plane to the w plane.
- A complex function may be interpreted as a two-dimensional fluid flow by considering $w = f(z)$ as the fluid velocity vector at the point z. In that case, if $x(t) + iy(t)$ is a parametric representation of a particle's position over time then

  $$\frac{dx}{dt} = u(x, y)$$
  $$\frac{dy}{dt} = v(x, y)$$

  and the family of solutions to this system of differential equations are called the **streamlines** of the flow associated with $f(z)$.

Definition 17.4.1 Limit of a Function

Suppose the function f is defined in some neighborhood of z_0, except possibly at z_0 itself. Then f is said to possess a **limit** at z_0, written

$$\lim_{z \to z_0} f(z) = L$$

if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

- For a function f of a real variable x, the limit $\lim_{x \to x_0} f(x) = L$ means f approaches L as you approach from both the left and right. If however f is a function of a complex variable it means f approaches L as you approach from any direction in the complex plane.

Theorem 17.4.1 Limit of Sum, Product, Quotient

Suppose $\lim_{z \to z_0} f(z) = L_1$ and $\lim_{z \to z_0} g(z) = L_2$. Then

(*i*) $\lim_{z \to z_0} [f(z) + g(z)] = L_1 + L_2$

(*ii*) $\lim_{z \to z_0} f(z)g(z) = L_1 L_2$

(*iii*) $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{L_1}{L_2}, \quad L_2 \neq 0.$

- A function f is continuous at a point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

- A function f defined by

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0, \; a_n \neq 0$$

where n is a nonnegative integer and the coefficients a_i, $i = 0, 1, \ldots, n$, are complex constants is called a **polynomial** of degree n.

- Polynomials are continuous on the entire complex plane.
- A **rational function**

$$f(z) = \frac{g(z)}{h(z)}$$

is continuous everywhere $h(z) \neq 0$.

Definition 17.4.3 Derivative

Suppose the complex function f is defined in a neighborhood of a point z_0. The **derivative** of f at z_0 is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{3}$$

provided this limit exists.

- In order for a complex function to be differentiable, the limit must approach the same value from every direction. This is a greater demand than in real variables. If you take an arbitrary complex function, there's a good chance it isn't differentiable.

Definition 17.4.4 Analyticity at a Point

A complex function $w = f(z)$ is said to be **analytic at a point** z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

- Analyticity at a point is a neighborhood property. A function can be differentiable at a point but if the neighboring points aren't also differentiable, it's not analytic at that point.
- A function is analytic in a domain D if it is analytic at every point in D.
- A function that is analytic everywhere is called an **entire function**.

17.5 Cauchy-Riemann Equations

Theorem 17.5.1 Cauchy–Riemann Equations

Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivatives of u and v exist and satisfy the **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1}$$

- If a complex function $f(z) = u(x, y) + iv(x, y)$ is analytic throughout a domain D, then the real functions u and v must satisfy the Cauchy-Riemann equations at every point in D.

Theorem 17.5.2 Criterion for Analyticity

Suppose the real-valued functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a domain D. If u and v satisfy the Cauchy–Riemann equations at all points of D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

- The Cauchy-Riemann equations are derived assuming the function is differentiable at a particular point. That being the case, they can also be used as a formula for the derivative of the function

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

- Because analyticity implies differentiability, theorem 17.5.2 can also be used to determine if a function is differentiable at a point.
- A real-valued function $\phi(x, y)$ that has continuous second-order partial derivatives in a domain D and satisfies Laplace's equation is said to be **harmonic** in D.
- If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D then the functions $u(x, y)$ and $v(x, y)$ are harmonic functions.
- If a given function $u(x, y)$ is harmonic in a domain D it is sometimes possible to find another function $v(x, y)$ that is harmonic in D such that $u(x, y) + iv(x, y)$ is analytic in D. The function v is called the **harmonic conjugate function** of u.
- To find the harmonic conjugate function of a given function u:
    1. Take the first-order partial derivatives of u with respect to x and y.
    2. If $u(x, y) + iv(x, y)$ is analytic in a domain D then u and v must satisfy the Cauchy-Riemann equations in D from which we can find expressions for $\partial v/\partial x$ and $\partial v/\partial y$.
    3. Integrate $\partial v/\partial x$ with respect to x to get an expression for v with an unknown constant $h(y)$.
    4. Take the first-order partial derivative of v with respect to y, equate it with the other expression for $\partial v/\partial y$, and solve for $h'(y)$.
    5. Integrate $h'(y)$ and substitute the result to find v.

17.6 Exponential and Logarithmic Functions

- The exponential function for complex numbers is defined as

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

- e^z is analytic for all z, i.e. it's an entire function.
- Like its real-valued counterpart,

$$\frac{d}{dz}e^z = e^z,$$
$$e^{z_1}e^{z_2} = e^{z_1+z_2},$$

  and

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}.$$

- Since
$$e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1$$
and
$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z$$
the complex function $f(z) = e^z$ is **periodic** with complex period $2\pi i$. Because of this complex periodicity an infinite horizontal strip of height 2π contains all possible values for the function. The strip $-\pi < y \leq \pi$ is called the **fundamental region**.

- For $z \neq 0$ and $\theta = \arg z$,
$$\ln z = \log_e |z| + i(\theta + 2n\pi), \;\; n = 0, \, \pm 1, \, \pm 2, \, \ldots$$
This means there are infinitely many values of the logarithm of a complex number z. This makes sense as the complex exponential is periodic.

- The **principal value** of $\ln z$ is the complex logarithm corresponding to $n = 0$ and $\theta = \operatorname{Arg} z$. It is denoted $\operatorname{Ln} z$.

- Some familiar properties of the real-valued logarithm hold for the complex-valued logarithm, e.g.
$$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$
and
$$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2,$$
however they don't necessarily hold for the principal value.

- $\operatorname{Ln} z$ is discontinuous and thus not analytic at $z = 0$ because $\ln z$ is undefined at $z = 0$ and on the negative real axis because $\operatorname{Arg} z$ is discontinuous there.

- The derivative of $\operatorname{Ln} z$ is
$$\frac{d}{dz} \operatorname{Ln} z = \frac{1}{z}.$$

- The complex power of a complex number is defined as
$$z^\alpha = e^{\alpha \ln z}, \;\; z \neq 0.$$
In general this is multiple-valued because $\ln z$ is multiple-valued — only if $\alpha = n, \;\; n = 0, \;\pm 1, \, \pm 2, \, \ldots$ is it single-valued. If $\ln z$ is replaced with $\operatorname{Ln} z$ then we get the **principle value** of z^α.

17.7 Trigonometric and Hyperbolic Functions

Definition 17.7.1 Trigonometric Sine and Cosine

For any complex number $z = x + iy$,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{2}$$

- The other trigonometric functions ($\tan z$, etc.) are defined as usual.
- Because e^{iz} and e^{-iz} are entire functions, $\sin z$ and $\cos z$ are also entire functions.
- $\sin z = 0$ for the real numbers $z = n\pi$, $n \in \mathbb{Z}$ and $\cos z = 0$ for the real numbers $z = (2n+1)\pi/2$, $n \in \mathbb{Z}$. This means that $\tan z$ and $\sec z$ are analytic except at the points where $\cos z = 0$ and $\cot z$ and $\sec z$ are analytic except at the points where $\sin z = 0$.
- The usual derivatives and trigonometric functions are still valid in the complex case.
- $\sin z$ can be expressed as

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

  and $\cos z$ can be expressed as

$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$

- The only zeroes of $\sin z$ are the real numbers $z = n\pi$, $n \in \mathbb{Z}$ and the only zeroes of $\cos z$ are the real numbers $z = (2n+1)\pi/2$, $n \in \mathbb{Z}$.

Definition 17.7.2 Hyperbolic Sine and Cosine

For any complex number $z = x + iy$,

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^{z} + e^{-z}}{2}. \tag{10}$$

- The complex trigonometric functions can be expressed in terms of the complex hyperbolic functions and vice versa

$$\begin{aligned} \sin z &= -i \sinh(iz), & \cos z &= \cosh(iz) \\ \sinh z &= -i \sin(iz), & \cosh z &= \cos(iz). \end{aligned}$$

- $\sinh z$ can be expressed as

$$\sinh z = \sinh x \cos y + i \cosh x \sin y$$

and $\cosh z$ can be expressed as

$$\cosh z = \cosh x \cos y + i \sinh x \sin y.$$

- The zeroes of $\sinh z$ are $z = n\pi i$, $n \in \mathbb{Z}$ and the zeroes of $\cosh z$ are $z = (2n+1)\pi i/2$, $n \in \mathbb{Z}$.
- $\sin z$ and $\cos z$ are 2π periodic while $\sinh z$ and $\cosh z$ are $2\pi i$ periodic.

17.8 Inverse Trigonometric and Hyperbolic Functions

- Because the complex trigonometric functions are multi-valued, their inverse functions are also multi-valued.
- The definitions of those inverse functions are

$$\begin{aligned} \arcsin z &= -i \ln[iz + (1-z^2)^{1/2}], \\ \arccos z &= -i \ln[z + i(1-z^2)^{1/2}], \text{ and} \\ \arctan z &= \frac{i}{2} \ln \frac{i+z}{i-z}. \end{aligned}$$

- The derivatives of the inverse trigonometric functions are

$$\begin{aligned} \frac{d}{dz} \arcsin z &= \frac{1}{(1-z^2)^{1/2}}, \\ \frac{d}{dz} \arccos z &= \frac{-1}{(1-z^2)^{1/2}}, \text{ and} \\ \frac{d}{dz} \arctan z &= \frac{1}{1+z^2}. \end{aligned}$$

- The definitions of the hyperbolic inverse functions and their derivatives are

$$\sinh^{-1} z = \ln\left[z + (z^2 + 1)^{1/2}\right]$$

$$\cosh^{-1} z = \ln\left[z + (z^2 - 1)^{1/2}\right]$$

$$\tanh^{-1} z = \frac{1}{2}\ln\frac{1+z}{1-z}$$

$$\frac{d}{dz}\sinh^{-1} z = \frac{1}{(z^2+1)^{1/2}}$$

$$\frac{d}{dz}\cosh^{-1} z = \frac{1}{(z^2-1)^{1/2}}$$

$$\frac{d}{dz}\tanh^{-1} z = \frac{1}{1-z^2}.$$

18 Integration in the Complex Plane

18.1 Contour Integrals

- In complex variables, a piecewise smooth curve C is called a **contour** or **path**. An integral of a complex function $f(z)$ on C is denoted $\int_C f(z)\,dz$ or $\oint_C f(z)\,dz$ if C is closed — this is called a **contour integral** or a **complex integral**.

1. Let $f(z) = u(x, y) + iv(x, y)$ be defined at all points on a smooth curve C defined by $x = x(t)$, $y = y(t)$, $a \le t \le b$.
2. Divide C into n subarcs according to the partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b]$. The corresponding points on the curve C are $z_0 = x_0 + iy_0 = x(t_0) + iy(t_0)$, $z_1 = x_1 + iy_1 = x(t_1) + iy(t_1), \ldots, z_n = x_n + iy_n = x(t_n) + iy(t_n)$. Let $\Delta z_k = z_k - z_{k-1}$, $k = 1, 2, \ldots, n$.
3. Let $\|P\|$ be the **norm** of the partition, that is, the maximum value of $|\Delta z_k|$.
4. Choose a sample point $z_k^* = x_k^* + iy_k^*$ on each subarc. See **FIGURE 18.1.1**.
5. Form the sum $\sum_{k=1}^{n} f(z_k^*)\,\Delta z_k$.

Definition 18.1.1 Contour Integral

Let f be defined at points of a smooth curve C defined by $x = x(t)$, $y = y(t)$, $a \le t \le b$. The **contour integral** of f along C is

$$\int_C f(z)\,dz = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(z_k^*)\,\Delta z_k. \qquad (1)$$

Theorem 18.1.1 Evaluation of a Contour Integral

If f is continuous on a smooth curve C given by $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, then

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt. \tag{3}$$

- If a complex function f is continuous on a smooth curve C and if $|f(z)| \leq M$ for all z on C, then

  $$\left|\int_C f(z)\,dz\right| \leq ML,$$

  where

  $$L = \int_a^b |z'(t)|\,dt$$

  is the length of C. This is sometimes called the **ML-intequality**.

- If **T** is the unit tangent vector to a positively oriented simple closed curve C then

  $$\oint_C f \cdot \mathbf{T}\,ds = \mathrm{Re}\left(\oint \overline{f(z)}\,dz\right)$$

  is called the **circulation** around C and measures the tendency of the flow to rotate the curve C.

- If **N** is the normal vector to a positive oriented simple closed curve C then

  $$\oint_C f \cdot \mathbf{N}\,ds = \mathrm{Im}\left(\oint \overline{f(z)}\,dz\right)$$

  is called the **net flux** across C and measures the difference between the rates at which fluid enters and exits the region bounded by C.