Introduction to Quantum Mechanics by David J. Griffiths Problems

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March 2023

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Part I
Theory

1   The Wave Function

1.1

(a)

$$\langle j^2 \rangle = \sum j^2 P(j)$$
$$= 14^2 \frac{1}{14} + 15^2 \frac{1}{14} + 16^2 \frac{3}{14} + 22^2 \frac{2}{14} + 24^2 \frac{2}{14} + 25^2 \frac{5}{14}$$
$$= \frac{3217}{7}$$
$$\approx 459.571$$
$$\langle j \rangle^2 = \left(\sum j P(j) \right)^2$$
$$= 441$$

4

(b)

$$\Delta j_{14} = -7$$
$$\Delta j_{15} = -6$$
$$\Delta j_{16} = -5$$
$$\Delta j_{22} = 1$$
$$\Delta j_{24} = 3$$
$$\Delta j_{25} = 4$$
$$\sigma^2 = \sum (\Delta j)^2 P(j)$$
$$= \frac{130}{7}$$
$$\approx 18.571$$

(c)

$$\sigma^2 = \sqrt{\langle j^2 \rangle - \langle j \rangle^2} = 18.571$$

1.2

(a)

$$\langle x^2 \rangle = \int_0^h x^2 \rho(x)\, dx$$
$$= \int_0^h \frac{x^{3/2}}{2\sqrt{h}}\, dx$$
$$= \frac{1}{2\sqrt{h}} \left[\frac{2}{5} x^{5/2} \right]_0^h$$
$$= \frac{h^2}{5}$$
$$\langle x \rangle^2 = \frac{h^2}{9}$$
$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$
$$= \sqrt{\frac{h^2}{5} - \frac{h^2}{9}}$$
$$= h\sqrt{\frac{4}{45}}$$
$$= \frac{2}{3\sqrt{5}} h$$

(b)

$$1 - \int_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} \rho(x)\, dx = 1 - \frac{1}{2\sqrt{h}}[2\sqrt{x}]_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma}$$

$$= 1 - \frac{1}{\sqrt{h}}\left(\sqrt{\frac{1}{3}h + \frac{2}{3\sqrt{5}}h} - \sqrt{\frac{1}{3}h - \frac{2}{3\sqrt{5}}h}\right)$$

$$= 1 - \left(\sqrt{\frac{1}{3} + \frac{2}{3\sqrt{5}}} - \sqrt{\frac{1}{3} - \frac{2}{3\sqrt{5}}}\right)$$

$$\approx 0.393$$

1.3

(a)

$$\rho(x) = Ae^{-\lambda(x-a)^2}$$

$$1 = \int_{-\infty}^{\infty} \rho(x)\, dx$$

$$= A\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2}\, dx$$

$$= A\sqrt{\frac{\pi}{\lambda}}$$

$$A = \sqrt{\frac{\lambda}{\pi}}$$

(b)

$$\langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} xe^{-\lambda(x-a)^2}\, dx$$

$$= a$$

$$\langle x^2 \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2}\, dx$$

$$= a^2 + \frac{1}{2\lambda}$$

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$= \sqrt{a^2 + \frac{1}{2\lambda} - a^2}$$

$$= \frac{1}{\sqrt{2\lambda}}$$

1.4

(a)

$$1 = \int_{-\infty}^{\infty} |\Psi(x,0)|^2 \, dx$$

$$= \left(\frac{A}{a}\right)^2 \int_0^a x^2 \, dx + \left(\frac{A}{b-a}\right)^2 \int_a^b (b-x)^2 \, dx$$

$$= \frac{1}{3}A^2 a + \left(\frac{A}{b-a}\right)^2 \left[-\frac{1}{3}(b-x)^3\right]_a^b$$

$$= \frac{1}{3}A^2 a + \frac{1}{3}A^2(b-a)$$

$$= \frac{1}{3}A^2 b$$

$$A = \sqrt{\frac{3}{b}}$$

(c) $x = a$

(d)

$$\int_0^a |\Psi(x,0)|^2 \, dx = \frac{3}{a^2 b}\left[\frac{1}{3}x^3\right]_0^a$$

$$= \frac{a}{b}$$

(e)

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\Psi(x,0)|^2 \, dx$$

$$= \frac{3}{a^2 b}\left[\frac{1}{4}x^4\right]_0^a + \frac{3}{b(b-a)^2}\int_a^b x(b-x)^2 \, dx$$

$$= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2}\int_a^b (b^2 x - 2bx^2 + x^3)\, dx$$

$$= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2}\left[\frac{1}{2}b^2 x^2 - \frac{2}{3}bx^3 + \frac{1}{4}x^4\right]_a^b$$

$$= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2}\left(\frac{1}{2}b^4 - \frac{2}{3}b^4 + \frac{1}{4}b^4 - \frac{1}{2}a^2 b^2 + \frac{2}{3}a^3 b - \frac{1}{4}a^4\right)$$

$$= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2}\frac{1}{12}(b-a)^3(3a+b)$$

$$= \frac{3a^2}{4b} + \frac{1}{4b}(3ab + b^2 - 3a^2 - ab)$$

$$= \frac{1}{2}a + \frac{1}{4}b$$

1.5

(a)

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t}$$

$$\Psi(x,0) = Ae^{-\lambda|x|}$$

$$1 = A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|}\, dx$$

$$= 2A^2 \int_{0}^{\infty} e^{-2\lambda x}\, dx$$

$$= 2A^2 \left[-\frac{1}{2\lambda} e^{-2\lambda x} \right]_{0}^{\infty}$$

$$= \frac{A^2}{\lambda}$$

$$A = \sqrt{\lambda}$$

(b)

$$\langle x \rangle = \int_{-\infty}^{\infty} x\lambda e^{-2\lambda|x|}\, dx$$

$$= \lambda \int_{-\infty}^{\infty} x e^{-2\lambda|x|}\, dx$$

$$= 0$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \lambda e^{-2\lambda|x|}\, dx$$

$$= 2\lambda \int_{0}^{\infty} x^2 e^{-2\lambda x}\, dx$$

$$= \frac{1}{2\lambda^2}$$

(c)

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$= \frac{1}{\sqrt{2}\lambda}$$

$$1 - \int_{-\sigma}^{\sigma} \lambda e^{-2\lambda|x|}\, dx = 1 - 2\lambda \int_{0}^{\sigma} e^{-2\lambda x}\, dx$$

$$= 1 - 2\lambda \left[-\frac{1}{2\lambda} e^{-2\lambda x} \right]_{0}^{\sigma}$$

$$= e^{-2\lambda\sigma}$$

$$= e^{-\sqrt{2}}$$

$$\approx 0.243$$

1.6

The chain rule requires that you apply it to both x and $|\Psi|^2$ which gives the same result

$$\frac{d\langle x \rangle}{dt} = \frac{d}{dt} \int x|\Psi|^2 \, dx$$

$$= \int \frac{d}{dt}(x|\Psi|^2) \, dx$$

$$= \int \left(0 \cdot |\Psi|^2 + x\frac{\partial |\Psi|^2}{\partial t} \right) dx$$

$$= \int x\frac{\partial |\Psi|^2}{\partial t} \, dx$$

1.8

$$i\hbar\frac{\partial}{\partial t}\left(e^{-iV_0 t/\hbar}\Psi \right) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(e^{-iV_0 t/\hbar}\Psi \right) + (V + V_0)\left(e^{-iV_0 t/\hbar}\Psi \right)$$

$$i\hbar\left(-\frac{iV_0}{\hbar}e^{-iV_0 t/\hbar}\Psi + e^{-iV_0 t/\hbar}\frac{\partial \Psi}{\partial t} \right) = -\frac{\hbar^2}{2m}e^{-iV_0 t/\hbar}\frac{\partial^2 \Psi}{\partial x^2} + Ve^{-iV_0 t/\hbar}\Psi + V_0 e^{-iV_0 t/\hbar}\Psi$$

$$V_0\Psi + i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi + V_0\Psi$$

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

$$\langle Q(x,p) \rangle = \int \left(e^{-iV_0 t/\hbar}\Psi \right)^* [Q(x, -i\hbar\partial/\partial x)] \, e^{-iV_0 t/\hbar}\Psi \, dx$$

$$= \int e^{iV_0 t/\hbar}\Psi^* [Q(x, -i\hbar\partial/\partial x)] \, e^{-iV_0 t/\hbar}\Psi \, dx$$

$$= \int \Psi^* [Q(x, -i\hbar\partial/\partial x)]\Psi \, dx$$

No effect on the expectation value.

1.9

(a)

$$\Psi(x,t) = Ae^{-a[(mx^2/\hbar)+it]}$$

$$1 = A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)}\, dx$$

$$= A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)}\, dx$$

$$= A^2 \sqrt{\frac{\pi\hbar}{2am}}$$

$$A^2 = \sqrt{\frac{2am}{\pi\hbar}}$$

$$A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}$$

(b)

$$\Psi = Ae^{-a[(mx^2/\hbar)+it]}$$

$$\frac{\partial\Psi}{\partial t} = -ia\Psi$$

$$\frac{\partial\Psi}{\partial x} = -\frac{2amx}{\hbar}\Psi$$

$$\frac{\partial^2\Psi}{\partial x^2} = -\frac{2am}{\hbar}\left(\Psi + x\frac{\partial\Psi}{\partial x}\right)$$

$$= -\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\Psi$$

$$V\Psi = i\hbar\frac{\partial\Psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}$$

$$= a\hbar\Psi - a\hbar\left(1 - \frac{2amx^2}{\hbar}\right)\Psi$$

$$V = a\hbar - a\hbar + 2a^2mx^2$$

$$= 2a^2mx^2$$

(c)

$$\langle x \rangle = A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)} x \, dx$$

$$= 0$$

$$\langle x^2 \rangle = A^2 \int_{-\infty}^{\infty} e^{-2a(mx^2/\hbar)} x^2 \, dx$$

$$= 2A^2 \int_{0}^{\infty} e^{-2a(mx^2/\hbar)} x^2 \, dx$$

$$= \frac{\hbar}{4am}$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \Psi^* \left[-i\hbar \frac{\partial}{\partial x} \right] \Psi \, dx$$

$$= -i\hbar \int_{-\infty}^{\infty} A e^{-a[(mx^2/\hbar)-it]} \left(-\frac{2amx}{\hbar} A e^{-a[(mx^2/\hbar)+it]} \right) dx$$

$$= 2iA^2 am \int_{-\infty}^{\infty} x e^{-2amx^2/\hbar} \, dx$$

$$= 0$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \Psi^* \left[-\hbar^2 \frac{\partial^2}{\partial x^2} \right] \Psi \, dx$$

$$= -\hbar^2 \int_{-\infty}^{\infty} A e^{-a[(mx^2/\hbar)-it]} \left[-\frac{2am}{\hbar} \left(1 - \frac{2amx^2}{\hbar} \right) A e^{-a[(mx^2/\hbar)+it]} \right] dx$$

$$= 2A^2 am\hbar \int_{-\infty}^{\infty} e^{-2amx^2/\hbar} \left(1 - \frac{2amx^2}{\hbar} \right) dx$$

$$= am\hbar$$

(d)

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$= \sqrt{\frac{\hbar}{4am}}$$

$$\sigma_p = \sqrt{am\hbar}$$

$$\sigma_x \sigma_p = \sqrt{\frac{1}{4}\hbar^2}$$

$$= \frac{1}{2}\hbar$$

$$\geq \frac{1}{2}\hbar$$

Yes, this is consistent with Heisenberg's uncertainty principle.

1.10

(a)

$$P(0) = 0$$

$$P(1) = \frac{2}{25}$$
$$= 0.08$$

$$P(2) = \frac{3}{25}$$
$$= 0.12$$

$$P(3) = \frac{1}{5}$$
$$= 0.2$$

$$P(4) = \frac{3}{25}$$
$$= 0.12$$

$$P(5) = \frac{3}{25}$$
$$= 0.2$$

$$P(6) = \frac{3}{25}$$
$$= 0.2$$

$$P(7) = \frac{1}{25}$$
$$= 0.04$$

$$P(8) = \frac{2}{25}$$
$$= 0.08$$

$$P(9) = \frac{3}{25}$$
$$= 0.12$$

(b) The most probable digit is 3, the median digit is 4, and the average value is $\frac{118}{25} = 4.72$.

(c) $\sigma = 2.474$

1.14

(a)

$$P_{ab}(t) = \int_a^b |\Psi(x,t)|^2 \, dx$$

$$\frac{dP_{ab}}{dt} = \frac{d}{dt} \int_a^b |\Psi(x,t)|^2 \, dx$$

$$= \int_a^b \frac{d}{dt} \left(|\Psi(x,t)|^2 \right) \, dx$$

$$= \int_a^b \frac{\partial}{\partial x} \left[\frac{i\hbar}{2m} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) \right] \, dx$$

$$= J(a,t) - J(b,t)$$

The units are s^{-1}.

(b)

$$\Psi(x,t) = A e^{-a[(mx^2/\hbar)+it]}$$

$$\frac{\partial \Psi}{\partial x} = -\frac{2amx}{\hbar} \Psi$$

$$\Psi^*(x,t) = A e^{-a[(mx^2/\hbar)-it]}$$

$$\frac{\partial \Psi^*}{\partial x} = -\frac{2amx}{\hbar} \Psi^*$$

$$J(x,t) = \frac{i\hbar}{2m} \left(\Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi}{\partial x} \right)$$

$$= \frac{i\hbar}{2m} \left[\Psi \left(-\frac{2amx}{\hbar} \Psi^* \right) - \Psi^* \left(-\frac{2amx}{\hbar} \Psi \right) \right]$$

$$= 0$$

1.15

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2\,dx = \int_{-\infty}^{\infty}\left(\frac{\partial\Psi_1^*}{\partial t}\Psi_2 + \Psi_1^*\frac{\partial\Psi_2}{\partial t}\right)\,dx$$

$$= \int_{-\infty}^{\infty}\left[\left(-i\frac{\hbar}{2m}\frac{\partial^2\Psi_1^*}{\partial x^2} + i\frac{V}{\hbar}\Psi_1^*\right)\Psi_2\right.$$

$$\left.+\Psi_1^*\left(i\frac{\hbar}{2m}\frac{\partial^2\Psi_2}{\partial x^2} - i\frac{V}{\hbar}\Psi_2\right)\right]\,dx$$

$$= i\frac{\hbar}{2m}\int_{-\infty}^{\infty}\left(\Psi_1^*\frac{\partial^2\Psi_2}{\partial x^2} - \frac{\partial^2\Psi_1^*}{\partial x^2}\Psi_2\right)\,dx$$

$$= i\frac{\hbar}{2m}\left[\Psi_1^*\frac{\partial\Psi_2}{\partial x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial}{\partial x}(\Psi_1^*\Psi_2)\,dx\right.$$

$$\left.\frac{\partial\Psi_1^*}{\partial x}\Psi_2\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial}{\partial x}(\Psi_1^*\Psi_2)\,dx\right]$$

$$= 0$$

1.16

(a)

$$1 = \int_{-a}^{a}A^2(a^2-x^2)^2\,dx$$

$$= A^2\int_0^a (a^2-x^2)^2\,dx$$

$$= \frac{16}{15}A^2a^5$$

$$A = \sqrt{\frac{15}{16a^5}}$$

(b)

$$\langle x\rangle = \int_{-a}^{a}xA(a^2-x^2)\,dx$$

$$= 0$$

(c)

$$\langle p\rangle = \int_{-a}^{a}\Psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi\,dx$$

$$= 2iA^2\hbar\int_{-a}^{a}x(a^2-x^2)\,dx$$

$$= 0$$

14

(d)

$$\langle x^2 \rangle = \int_{-a}^{a} \Psi^* x^2 \Psi \, dx$$

$$= A^2 \int_{-a}^{a} x^2 (a^2 - x^2)^2 \, dx$$

$$= A^2 \frac{16}{105} a^7$$

$$= \frac{a^2}{7}$$

(e)

$$\langle p^2 \rangle = \int_{-a}^{a} \Psi^* \left(-\hbar^2 \frac{\partial^2}{\partial x^2} \right) \Psi \, dx$$

$$= -\hbar^2 \int_{-a}^{a} A(a^2 - x^2)(-2A) \, dx$$

$$= 4A^2 \hbar^2 \int_{0}^{a} (a^2 - x^2) \, dx$$

$$= 4A^2 \hbar^2 \left[a^2 x - \frac{1}{3} x^3 \right]_{0}^{a}$$

$$= 4A^2 \hbar^2 \left(a^3 - \frac{1}{3} a^3 \right)$$

$$= \frac{8}{3} A^2 a^3 \hbar^2$$

$$= \frac{8}{3} \frac{15}{16a^5} a^3 \hbar^2$$

$$= \frac{5}{2} \frac{\hbar^2}{a^2}$$

(f)

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$= \sqrt{\frac{a^2}{7}}$$

$$= \frac{a}{\sqrt{7}}$$

(g)

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

$$= \sqrt{\frac{5}{2}} \frac{\hbar}{a}$$

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(h)

$$\sigma_x \sigma_p = \sqrt{\frac{5}{14}} \hbar$$

$$\geq \frac{1}{2} \hbar$$

1.18

(a)

$$\frac{h}{\sqrt{3mk_BT}} > d$$

$$\frac{\sqrt{3mk_BT}}{h} < \frac{1}{d}$$

$$T_{\text{electron}} < \frac{h^2}{3d^2 mk_B}$$

$$< 1.3 \times 10^5 \, \text{K}$$

$$T_{\text{nuclei}} < 2.5 \, \text{K}$$

(b)

$$PV = Nk_BT$$

$$\frac{V}{N} = \frac{k_BT}{P}$$

$$d = \left(\frac{k_BT}{P}\right)^{1/3}$$

$$\frac{h}{\sqrt{3mk_Bt}} > \left(\frac{k_BT}{P}\right)^{1/3}$$

$$T < \frac{1}{k_B} \left(\frac{h^2}{3m}\right)^{3/5} P^{2/5}$$

2   Time-Independent Schrödinger Equation

2.1

(a)

$$\int_{-\infty}^{\infty} |\Psi|^2 \, dx = \int_{-\infty}^{\infty} \Psi^* \Psi \, dx$$

$$= \int_{-\infty}^{\infty} \psi^* e^{i(E_0 - i\Gamma)t/\hbar} \psi e^{-i(E_0 + i\Gamma)t/\hbar} \, dx$$

$$= e^{2\Gamma t/\hbar} \int_{-\infty}^{\infty} |\psi|^2 \, dx$$

In order for this to equal 1 for all t, Γ must be 0.

(b) If $\psi(x)$ is a complex solution to the time-independent Schrödinger equation then so is $\psi^*(x)$ and $\psi(x) + \psi^*(x)$ which is real.

2.2

If ψ and its second derivative always have the same sign, ψ will increase or decrease without bound forever. This means there is no non-zero choice of constant A such that

$$\int_{-\infty}^{\infty} |A\Psi|^2 \, dx = 1$$

and thus the equation can't be normalised.

The classical analog of this is statements is that the potential energy of a system can't exceed its total energy.

2.3

The time-independent Schrödinger equation in an infinite square well is

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi.$$

If $E = 0$ then $\psi = Ax + B$ which isn't normalisable.
If $E < 0$ then $\psi = Ae^{kt} + Be^{-kt}$ where $k \in \mathbb{R}$ which also isn't normalisable.

2.4

$$\Psi_n(x,t) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-i(n^2\pi^2\hbar/2ma^2)t}$$

$$\langle x \rangle = \int_0^a \Psi_n^* x \Psi_n \, dx$$

$$= \frac{2}{a} \int_0^a x \sin^2\left(\frac{n\pi}{a}x\right) dx$$

$$= \frac{a}{2}$$

$$\langle x^2 \rangle = \int_0^a \Psi_n^* x^2 \Psi_n \, dx$$

$$= \frac{2}{a} \int_0^a x^2 \sin^2\left(\frac{n\pi}{a}x\right) dx$$

$$= a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right)$$

$$\langle p \rangle = \int_0^a \Psi_n^* \left(-i\hbar\frac{\partial}{\partial x}\right) \Psi_n \, dx$$

$$= -i\frac{2\hbar n\pi}{a^2} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \cos\left(\frac{n\pi}{a}x\right) dx$$

$$= 0$$

$$\langle p^2 \rangle = \int_0^a \Psi_n^* \left(-\hbar^2\frac{\partial^2}{\partial x^2}\right) \Psi_n \, dx$$

$$= \frac{2\hbar^2 n^2\pi^2}{a^3} \int_0^a \sin^2\left(\frac{n\pi}{a}x\right) dx$$

$$= \left(\frac{n\pi\hbar}{a}\right)^2$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$= \frac{a}{2}\sqrt{\frac{1}{3} - \frac{2}{n^2\pi^2}}$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

$$= \frac{n\pi\hbar}{a}$$

2.5

(a)

$$1 = \int_0^a A^2(\psi_1 + \psi_2)^2 \, dx$$

$$= A^2 \int_0^a (\psi_1^2 + 2\psi_1\psi_2 + \psi_2^2) \, dx$$

$$= \frac{2A^2}{a} \left[\int_0^a \sin^2\left(\frac{\pi}{a}x\right) dx + \int_0^a \sin^2\left(\frac{2\pi}{a}x\right) dx \right]$$

$$= 2A^2$$

$$A = \frac{1}{\sqrt{2}}$$

(b)

$$\Psi(x,t) = \frac{1}{\sqrt{2}} \left[\sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right) e^{-i\omega t} + \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi}{a}x\right) e^{-4i\omega t} \right]$$

$$|\Psi(x,t)|^2 = \Psi^*\Psi$$

$$= \frac{1}{a} \left[\sin\left(\frac{\pi}{a}x\right) e^{i\omega t} + \sin\left(\frac{2\pi}{a}x\right) e^{4i\omega t} \right]$$

$$\left[\sin\left(\frac{\pi}{a}x\right) e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right) e^{-4i\omega t} \right]$$

$$= \frac{1}{a} \left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) e^{-3i\omega t} \right.$$

$$\left. + \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) e^{3i\omega t} + \sin^2\left(\frac{2\pi}{a}x\right) \right]$$

$$= \frac{1}{a} \left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right) \right.$$

$$\left. + 2\sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) \cos(3\omega t) \right]$$

(c)

$$\langle x \rangle = \int_0^a \Psi^* x \Psi \, dx$$

$$= \int_0^a x|\Psi|^2 \, dx$$

$$= \frac{a}{2} \left[1 - \frac{32}{9\pi^2} \cos(3\omega t) \right]$$

(d)

$$\langle p \rangle = m \frac{d \langle x \rangle}{dt}$$

$$= \frac{16 a m \omega}{3 \pi^2} \sin(3\omega t)$$

$$= \frac{8\hbar}{3a} \sin(3\omega t)$$

(e) You can get E_1 or E_2 and the probability of getting each is $1/2$.
$H = \frac{1}{2}(E_1 + E_2)$ is the mean of the two possible energy values.

2.6

$$\Psi(x,0) = A[\psi_1 + e^{i\phi}\psi_2]$$

$$1 = \int_0^a |\Psi|^2 \, dx$$

$$= \int_0^a \Psi^* \Psi \, dx$$

$$= A^2 \int_0^a (\psi_1 + e^{-i\phi}\psi_2)(\psi_1 + e^{i\phi}\psi_2) \, dx$$

$$= A^2 \int_0^a (\psi_1^2 + e^{i\phi}\psi_1\psi_2 + e^{-i\phi}\psi_1\psi_2 + \psi_2^2) \, dx$$

$$= \frac{2A^2}{a} \int_0^a \left[\sin^2\left(\frac{\pi}{a}x\right) + e^{i\phi} \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) \right.$$

$$\left. e^{-i\phi} \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right) \right] \, dx$$

$$= \frac{2A^2}{a} \int_0^a \left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) \cos\phi \right.$$

$$\left. + \sin^2\left(\frac{2\pi}{a}x\right) \right] \, dx$$

$$= 2A^2$$

$$A = \frac{1}{\sqrt{2}}$$

$$\Psi(x,t) = \frac{1}{\sqrt{a}} \left[\sin\left(\frac{\pi}{a}x\right) e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right) e^{i(\phi - 4\omega t)} \right]$$

20

$$|\Psi|^2 = \Psi^*\Psi$$

$$= \frac{1}{a}\left[\sin\left(\frac{\pi}{a}x\right)e^{i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{-i(\phi-4\omega t)}\right]$$

$$\left[\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{i(\phi-4\omega t)}\right]$$

$$= \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)e^{i(\phi-3\omega t)}\right.$$

$$\left.\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)e^{-i(\phi-3\omega t)} + \sin^2\left(\frac{2\pi}{a}x\right)\right]$$

$$= \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right)\right.$$

$$\left. +2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(\phi-3\omega t)\right]$$

$$\langle x\rangle = \int_0^a \Psi^* x\Psi\, dx$$

$$= \int_0^a x|\Psi|^2\, dx$$

$$= \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t - \phi)\right]$$

2.7

(a)

$$1 = \int_0^a |\Psi|^2\, dx$$

$$= A^2\left[\int_0^{a/2} x^2\, dx + \int_{a/2}^a (a-x)^2\, dx\right]$$

$$= A^2\left\{\frac{1}{3}\left[\frac{a}{2}\right]^3 + \left[-\frac{1}{3}(a-x)^3\right]_{a/2}^a\right\}$$

$$= A^2\left(\frac{a^3}{24} + \frac{a^3}{24}\right)$$

$$= \frac{A^2 a^3}{12}$$

$$A = \frac{2\sqrt{3}}{\sqrt{a^3}}$$

(b)

$$c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \Psi(x,0)\, dx$$

$$= \sqrt{\frac{2}{a}} \left[\int_0^{a/2} \sin\left(\frac{n\pi}{a}x\right) Ax\, dx + \int_{a/2}^a \sin\left(\frac{n\pi}{a}x\right) A(a-x)\, dx \right]$$

$$= \frac{2\sqrt{6}}{a^2} \left[\int_0^{a/2} x \sin\left(\frac{n\pi}{a}x\right) dx + \int_{a/2}^a (a-x) \sin\left(\frac{n\pi}{a}x\right) dx \right]$$

$$= \frac{8\sqrt{6}}{n^2\pi^2} \sin^2\left(\frac{n\pi}{4}\right) \sin\left(\frac{n\pi}{2}\right)$$

$$= \begin{cases} 0 & n \text{ even} \\ (-1)^{(n-1)/2} \frac{4\sqrt{6}}{n^2\pi^2} & n \text{ odd} \end{cases}$$

$$\Psi(x,t) = \frac{4\sqrt{6}}{\pi^2} \sqrt{\frac{2}{a}} \sum_{n=1,3,5,\dots}^{\infty} (-1)^{(n-1)/2} \frac{1}{n^2} \sin\left(\frac{n\pi}{a}x\right) e^{-i(n^2\pi^2\hbar/2ma^2)t}$$

(c)

$$|c_1|^2 = \left(\frac{4\sqrt{6}}{\pi^2}\right)^2$$

$$\approx 0.985$$

(d)

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$$\langle H \rangle = \sum_{n=0}^{\infty} |c_{2n+1}|^2 E_{2n+1}$$

$$= \sum_{n=0}^{\infty} \left(\frac{4\sqrt{6}}{(2n+1)^2\pi^2}\right)^2 \frac{(2n+1)^2\pi^2\hbar^2}{2ma^2}$$

$$= \sum_{n=0}^{\infty} \frac{48\hbar^2}{(2n+1)^2 ma^2\pi^2}$$

$$= \frac{48\hbar^2}{ma^2\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

$$= \frac{6\hbar^2}{ma^2}$$

2.8

$$1 = \int_0^{a/2} |\Psi|^2 \, dx$$

$$= A^2 \int_0^{a/2} dx$$

$$= \frac{aA^2}{2}$$

$$A = \sqrt{\frac{2}{a}}$$

$$c_n = \frac{2}{a} \int_0^{a/2} \sin\left(\frac{n\pi}{a}x\right) dx$$

$$|c_1|^2 = \left(\frac{2}{\pi}\right)^2$$

$$\approx 0.405$$

2.9

$$\Psi(x,0) = Ax(a-x)$$

$$\langle H \rangle = \int_0^a \Psi(x,0)^* \hat{H} \Psi(x,0) \, dx$$

$$= \int_0^a \Psi(x,0)^* \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right) \Psi(x,0) \, dx$$

$$= \frac{A^2\hbar^2}{m} \int_0^a x(a-x) \, dx$$

$$= \frac{30\hbar^2}{ma^5}\frac{a^3}{6}$$

$$= \frac{5\hbar^2}{ma^2}$$

2.10

(a)

$$\psi_2(x) = \frac{1}{\sqrt{2!}}(\hat{a}_+)\psi_1$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}xe^{-\frac{m\omega}{2\hbar}x^2}$$

$$= \frac{1}{\sqrt{2}\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(-\hbar\frac{d}{dx} + m\omega x\right)xe^{-\frac{m\omega}{2\hbar}x^2}$$

$$= \frac{1}{\sqrt{2}\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-\hbar\left(e^{-\frac{m\omega}{2\hbar}x^2} - \frac{m\omega}{\hbar}x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right) + m\omega x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right]$$

$$= \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)e^{-\frac{m\omega}{2\hbar}x^2}$$

2.11

(a)

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi_0^* x \psi_0 \, dx$$

$$= \alpha^2 \int_{-\infty}^{\infty} x e^{-\frac{m\omega}{\hbar}x^2} \, dx$$

$$= 0$$

$$\langle p \rangle = m \frac{d \langle x \rangle}{dt}$$

$$= 0$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi_0^* x^2 \psi_0 \, dx$$

$$= \alpha^2 \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} \, dx$$

$$= \frac{\hbar}{2m\omega}$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \psi_0^* \left(-\hbar^2 \frac{d^2}{dx^2} \right) \psi_0 \, dx$$

$$= -\hbar^2 \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \int_{-\infty}^{\infty} e^{-\frac{m\omega}{2\hbar}x^2} \frac{d}{dx} \left(-\frac{m\omega}{\hbar} x e^{-\frac{m\omega}{2\hbar}x^2} \right) \, dx$$

$$= \hbar^2 \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \frac{m\omega}{\hbar} \int_{-\infty}^{\infty} e^{-\frac{m\omega}{2\hbar}x^2} \left(e^{-\frac{m\omega}{2\hbar}x^2} - \frac{m\omega}{\hbar} x^2 e^{-\frac{m\omega}{2\hbar}x^2} \right) \, dx$$

$$= \hbar^2 \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \frac{m\omega}{\hbar} \int_{-\infty}^{\infty} \left(1 - \frac{m\omega}{\hbar} x^2 \right) e^{-\frac{m\omega}{\hbar}x^2} \, dx$$

$$= \hbar^2 \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \frac{m\omega}{\hbar} \frac{\hbar\sqrt{\pi}}{2\sqrt{\hbar m\omega}}$$

$$= \frac{1}{2} m\hbar\omega$$

$$\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\,xe^{-\frac{m\omega}{2\hbar}x^2}$$

$$\langle x \rangle = 0$$

$$\langle p \rangle = m\frac{d\langle x \rangle}{dt}$$

$$= 0$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty}\psi_1^*x^2\psi_1\,dx$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{2m\omega}{\hbar}\int_{-\infty}^{\infty}x^4e^{-\frac{m\omega}{\hbar}x^2}\,dx$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{2m\omega}{\hbar}\frac{3}{4}\sqrt{\pi}\left(\frac{\hbar}{m\omega}\right)^{5/2}$$

$$= \frac{3}{2}\frac{\hbar}{m\omega}$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty}\psi_1^*\left(-\hbar^2\frac{d^2}{dx^2}\right)\psi_1\,dx$$

$$= -\hbar^2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{2m\omega}{\hbar}\int_{-\infty}^{\infty}xe^{-\frac{m\omega}{2\hbar}x^2}\frac{d}{dx}\left(e^{-\frac{m\omega}{2\hbar}x^2} - \frac{m\omega}{\hbar}x^2e^{-\frac{m\omega}{2\hbar}x^2}\right)\,dx$$

$$= -\hbar^2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{2m\omega}{\hbar}\int_{-\infty}^{\infty}xe^{-\frac{m\omega}{2\hbar}x^2}\left[-\frac{m\omega}{\hbar}xe^{-\frac{m\omega}{2\hbar}x^2}\right.$$

$$\left.-\frac{2m\omega}{\hbar}xe^{-\frac{m\omega}{2\hbar}x^2} + \left(\frac{m\omega}{\hbar}\right)^2x^3e^{-\frac{m\omega}{2\hbar}x^2}\right]\,dx$$

$$= 2\hbar^2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left(\frac{m\omega}{\hbar}\right)^2\int_{-\infty}^{\infty}x^2e^{-\frac{m\omega}{\hbar}x^2}\left(3 - \frac{m\omega}{\hbar}x^2\right)\,dx$$

$$= 2\hbar^2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left(\frac{m\omega}{\hbar}\right)^2\frac{3}{4}\sqrt{\pi}\left(\frac{h}{m\omega}\right)^{3/2}$$

$$= \frac{3}{2}m\hbar\omega$$

(b)

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

$$= \sqrt{\frac{m\hbar\omega}{2}}$$

$$\sigma_x\sigma_p = \frac{\hbar}{2}$$

$$\sigma_x = \sqrt{\frac{3\hbar}{2m\omega}}$$

$$\sigma_p = \sqrt{\frac{3m\hbar\omega}{2}}$$

$$\sigma_x\sigma_p = \frac{3}{2}\hbar$$

(c)

$$\langle T \rangle = \frac{\langle p^2 \rangle}{2m}$$

$$= \frac{\hbar\omega}{4}$$

$$\langle V \rangle = \frac{1}{2}m\omega^2 \langle x^2 \rangle$$

$$= \frac{1}{4}\hbar\omega$$

$$\langle T \rangle = \frac{\langle p^2 \rangle}{2m}$$

$$= \frac{3}{4}\hbar\omega$$

$$\langle V \rangle = \frac{1}{2}m\omega^2 \langle x^2 \rangle$$

$$= \frac{3}{4}\hbar\omega$$

2.12

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi_n^* x \psi_n \, dx$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}_+ + \hat{a}_-) \psi_n \, dx$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{\infty} \psi_n^* (\sqrt{n+1}\psi_{n+1} + \sqrt{n}\psi_{n-1}) \, dx$$

$$= 0$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi_n^* p \psi_n \, dx$$

$$= i\sqrt{\frac{\hbar m\omega}{2}} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}_+ - \hat{a}_-) \psi_n \, dx$$

$$= i\sqrt{\frac{\hbar m\omega}{2}} \int_{-\infty}^{\infty} \psi_n^* (\sqrt{n+1}\psi_{n+1} - \sqrt{n}\psi_{n-1}) \, dx$$

$$= 0$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi_n^* x^2 \psi_n \, dx$$

$$= \frac{\hbar}{2m\omega} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}_+^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-^2) \psi_n \, dx$$

$$= \frac{\hbar}{2m\omega} (2n+1) \int_{-\infty}^{\infty} |\psi_n|^2 \, dx$$

$$= \frac{\hbar}{m\omega} \left(n + \frac{1}{2} \right)$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \psi_n^* p^2 \psi_n \, dx$$

$$= -\frac{\hbar m\omega}{2} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}_+^2 - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a} - +\hat{a}_-^2) \psi_n \, dx$$

$$= \frac{\hbar m\omega}{2} (2n+1) \int_{-\infty}^{\infty} |\psi_n|^2 \, dx$$

$$= \hbar m\omega \left(n + \frac{1}{2} \right)$$

$$\langle T \rangle = \left\langle \frac{p^2}{2m} \right\rangle$$

$$= \frac{1}{2} \hbar \omega \left(n + \frac{1}{2} \right)$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$= \sqrt{\frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right)}$$

$$\sigma_p = \sqrt{\hbar m\omega\left(n + \frac{1}{2}\right)}$$

$$\sigma_x \sigma_p = (2n + 1)\frac{\hbar}{2}$$

$$\geq \frac{\hbar}{2}$$

2.13

(a)

$$\Psi(x, 0) = A[3\psi_0(x) + 4\psi_1(x)]$$

$$1 = \int_{-\infty}^{\infty} |\Psi(x, 0)|^2 \, dx$$

$$= A^2 \int_{-\infty}^{\infty} [9\psi_0(x)^2 + 24\psi_0(x)\psi_1(x) + 16\psi_1(x)^2] \, dx$$

$$= 25A^2$$

$$A = \frac{1}{5}$$

(b)

$$\Psi(x, t) = \frac{1}{5}[3\psi_0(x)e^{-i\omega t/2} + 4\psi_1(x)e^{-3i\omega t/2}]$$

$$|\Psi(x, t)|^2 = \Psi(x, t)^* \Psi(x, t)$$

$$= \frac{1}{25}[3\psi_0(x)e^{i\omega t/2} + 4\psi_1(x)e^{3i\omega t/2}][3\psi_0(x)e^{-i\omega t/2} + 4\psi_1(x)e^{-3i\omega t/2}]$$

$$= \frac{1}{25}[9\psi_0(x)^2 + 12\psi_0(x)\psi_1(x)e^{-i\omega t} + 12\psi_0(x)\psi_1(x)e^{i\omega t} + 16\psi_1(x)^2]$$

$$= \frac{1}{25}[9\psi_0(x)^2 + 16\psi_1(x)^2 + 24\psi_0(x)\psi_1(x)\cos\omega t]$$

(c)

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^* x \Psi \, dx$$

$$= \frac{1}{25} \int_{-\infty}^{\infty} x(9\psi_0^2 + 16\psi_1^2 + 24\psi_0\psi_1 \cos \omega t) \, dx$$

$$= \frac{24}{25} \int_{-\infty}^{\infty} x\psi_0\psi_1 \cos(\omega t) \, dx$$

$$= \frac{24}{25} \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \sqrt{\frac{2m\omega}{\hbar}} \cos(\omega t) \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} \, dx$$

$$= \frac{24}{25} \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} \sqrt{\frac{2m\omega}{\hbar}} \cos(\omega t) \frac{1}{2}\sqrt{\pi} \left(\frac{\hbar}{m\omega} \right)^{3/2}$$

$$= \frac{24}{25} \sqrt{\frac{\hbar}{2m\omega}} \cos(\omega t)$$

$$\langle p \rangle = m \frac{d\langle x \rangle}{dt}$$

$$= -\frac{24}{25} \sqrt{\frac{\hbar m\omega}{2}} \sin(\omega t)$$

$$\frac{d\langle p \rangle}{dt} = -\frac{24}{25} \sqrt{\frac{\hbar m\omega}{2}} \omega \cos(\omega t)$$

$$V = \frac{1}{2}m\omega^2 x^2$$

$$\frac{\partial V}{d\partial} = m\omega^2 x$$

$$\left\langle -\frac{\partial V}{\partial x} \right\rangle = -m\omega^2 \langle x \rangle$$

$$= -\frac{24}{25} \sqrt{\frac{\hbar m\omega}{2}} \omega \cos(\omega t)$$

$$= \frac{d\langle p \rangle}{dt}$$

(d)

$$E_0 = \frac{\hbar\omega}{2}$$

$$P(E_0) = \frac{9}{25}$$

$$E_1 = \frac{3\hbar\omega}{2}$$

$$P(E_1) = \frac{16}{25}$$

30

2.14

$$1 - \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\sqrt{\hbar/m\omega}}^{\sqrt{\hbar/m\omega}} e^{-m\omega x^2/\hbar} \, dx = 1 - \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \sqrt{\frac{\pi\hbar}{m\omega}} \operatorname{erf} 1$$

$$= 0.157$$

2.15

$$a_{j+2} = \frac{-2(n-j)}{(j+1)(j+2)} a_j$$

$$a_3 = -\frac{4}{3} a_1$$

$$a_5 = \frac{4}{15} a_1$$

$$H_5(\xi) = a_1 \left(\xi - \frac{4}{3}\xi^3 + \frac{4}{15}\xi^5\right)$$

$$= \frac{1}{120} a_1 (120\xi - 160\xi^3 + 32\xi^5)$$

$$= 32\xi^5 - 160\xi^3 + 120\xi$$

$$a_2 = -6a_0$$

$$a_4 = \frac{-8}{12} a_2$$

$$= 4a_0$$

$$a_6 = \frac{-4}{30} a_4$$

$$= -\frac{8}{15} a_0$$

$$H_6(\xi) = a_0 \left(1 - 6\xi^2 + 4\xi^4 - \frac{8}{15}\xi^6\right)$$

$$= \frac{1}{120} a_0 (120 - 720\xi^2 + 480\xi^4 - 64\xi^6)$$

$$= 64\xi^6 - 480\xi^4 + 720\xi^2 - 120$$

2.17

$$Ae^{ikx} + Be^{-ikx} = A[\cos(kx) + i\sin(kx)] + B[\cos(kx) - i\sin(kx)]$$
$$= (A+B)\cos(kx) + i(A-B)\sin(kx)$$
$$C = A + B$$
$$D = i(A - B)$$
$$-iD = A - B$$
$$A = \frac{C - iD}{2}$$
$$B = \frac{C + iD}{2}$$

2.18

$$\Psi_k(x,t) = Ae^{i\left(kx - \frac{\hbar k^2}{2m}t\right)}$$
$$J(x,t) = \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right)$$
$$= \frac{\hbar k|A|^2}{m}$$

The probability travels in the same direction as the wave.

2.20

(a)

$$\Psi(x,0) = Ae^{-a|x|}$$
$$1 = \int_{-\infty}^{\infty}\Psi^*\Psi\,dx$$
$$= |A|^2\int_{-\infty}^{\infty}e^{-2a|x|}\,dx$$
$$= 2|A|^2\int_{0}^{\infty}e^{-2ax}\,dx$$
$$= \frac{|A|^2}{a}$$
$$A = \sqrt{a}$$

(b)

$$\phi(k) = \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|-ikx}\, dx$$

$$= \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|}[\cos(kx) - i\sin(kx)]\, dx$$

$$= \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|}\cos(kx)\, dx$$

$$= \sqrt{\frac{a}{2\pi}}\, 2\int_{0}^{\infty} e^{-ax}\cos(kx)\, dx$$

$$= \sqrt{\frac{a}{2\pi}}\,\frac{2a}{a^2 + k^2}$$

(c)

$$\Psi(x,t) = \frac{a^{3/2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{a^2 + k^2} e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)}\, dk$$

2.21

(a)

$$\Psi(x,0) = Ae^{-ax^2}$$

$$1 = A^2 \int_{-\infty}^{\infty} e^{-2ax^2}\, dx$$

$$= \sqrt{\frac{\pi}{2a}}A^2$$

$$A = \left(\frac{2a}{\pi}\right)^{1/4}$$

(b)

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-(ax^2 + ikx)}\, dx$$

$$= \frac{1}{(2\pi a)^{1/4}} e^{-k^2/4a}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\frac{1}{(2\pi a)^{1/4}} \int_{-\infty}^{\infty} e^{-\frac{k^2}{4a} + i\left(kx - \frac{\hbar k^2}{2m}t\right)}\, dk$$

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\gamma} e^{-ax^2/\gamma^2}$$

33

(c)

$$|\Psi(x,t)|^2 = \Psi^*\Psi$$

$$= \left(\frac{2a}{\pi}\right)^{1/2} \frac{1}{\gamma^*} e^{-ax^2/(\gamma^*)^2} \frac{1}{\gamma} e^{-ax^2/\gamma^2}$$

$$= \left(\frac{2a}{\pi}\right)^{1/2} \frac{1}{\sqrt{1-2i\hbar at/m}} e^{-ax^2/(1-2i\hbar at/m)}$$

$$\frac{1}{\sqrt{1+2i\hbar at/m}} e^{-ax^2/(1+2i\hbar at/m)}$$

$$= \left(\frac{2a}{\pi}\right)^{1/2} \frac{1}{\sqrt{1+(2\hbar at/m)^2}} e^{-2ax^2/[1+(2a\hbar t/m)^2]}$$

$$= \sqrt{\frac{2}{\pi}} w e^{-2w^2x^2}$$

As t increases $|\Psi|^2$ flattens out and broadens.

(d)

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^* x \Psi \, dx$$

$$= 0$$

$$\langle p \rangle = m \frac{d\langle x \rangle}{dt}$$

$$= 0$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \Psi^* x^2 \Psi \, dx$$

$$= \sqrt{\frac{2}{\pi}} w \int_{-\infty}^{\infty} x^2 e^{-2w^2x^2} \, dx$$

$$= \frac{1}{4w^2}$$

2.22

(a) -25

(b) 1

(c) 0

2.26

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-ikx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, dk$$

2.29

$$\psi(x) = \begin{cases} F e^{-\kappa x} & x > a \\ C \sin(lx) & 0 < x < a \\ -\psi(-x) & x < 0 \end{cases}$$

$$F e^{-\kappa a} = C \sin(la)$$

$$-\kappa F e^{-\kappa a} = lC \cos(la)$$

$$-\frac{1}{\kappa} = \frac{1}{l} \tan(la)$$

$$\tan z = -\frac{l}{\kappa}$$

$$= -\frac{la}{\kappa a}$$

$$= -\frac{z}{\sqrt{z_0^2 - z^2}}$$

For large z_0 the intersections occur just below $z_n = n\pi$ so

$$z = la$$

$$n\pi \approx \frac{\sqrt{2m(E + V_0)}}{\hbar} a$$

$$E + V_0 \approx \frac{n^2 \pi^2 \hbar^2}{2ma^2}.$$

As z_0 decreases there are fewer and fewer bound states. When $z_0 < \pi/2$ there are no odd bound states.

2.30

$$\psi(x) = \begin{cases} Fe^{-\kappa x} & x > a \\ D\cos(lx) & 0 < x < a \\ \psi(-x) & x < 0 \end{cases}$$

$$1 = \int_{-\infty}^{\infty} |\psi|^2 \, dx$$

$$= 2\left(|D|^2 \int_0^a \cos^2(lx) \, dx + |F|^2 \int_a^\infty e^{-2\kappa x} \, dx \right)$$

$$= 2\left[|D|^2 \frac{2al + \sin(2al)}{4l} + \frac{|F|^2}{2\kappa} e^{-2\kappa a} \right]$$

$$= |D|^2 \left[a + \frac{\sin(2al)}{2l} + \frac{\cos^2(al)}{\kappa} \right]$$

$$= |D|^2 \left[a + \frac{2\sin(al)\cos(al)}{2l} + \frac{\cos^3(al)}{l\sin(al)} \right]$$

$$= |D|^2 \left\{ a + \frac{\cos(al)}{l\sin(al)} [\sin^2(al) + \cos^2(al)] \right\}$$

$$= |D|^2 \left[a + \frac{\cos(al)}{l\sin(al)} \right]$$

$$= |D|^2 \left[a + \frac{1}{l\tan(al)} \right]$$

$$= |D|^2 \left[a + \frac{1}{\kappa} \right]$$

$$D = \frac{1}{\sqrt{a + 1/\kappa}}$$

$$1 = \left\{ \frac{1}{a + 1/\kappa} \left[a + \frac{\sin(2al)}{2l} \right] + |F|^2 \frac{e^{-2\kappa a}}{\kappa} \right\}$$

$$\frac{(Fe^{-\kappa a})^2}{\kappa} = 1 - \frac{1}{a + 1/\kappa} \left[a + \frac{\sin(2al)}{2l} \right]$$

$$(Fe^{-\kappa a})^2 = \kappa - \frac{\kappa}{a + 1/\kappa} \left[a + \frac{\sin(2al)}{2l} \right]$$

$$= \frac{\kappa a + 1 - \kappa a - \kappa \sin(al)\cos(al)/l}{a + 1/\kappa}$$

$$= \frac{1 - \sin^2(al)}{a + 1/\kappa}$$

$$F = \frac{e^{\kappa a} \cos(al)}{\sqrt{a + 1/\kappa}}$$

2.31

$$1 = 2aV_0$$

$$V_0 = \frac{1}{2a}$$

$$z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$$

$$= \frac{a}{\hbar}\sqrt{\frac{m}{a}}$$

$$= \frac{\sqrt{am}}{\hbar}$$

$$\lim_{a\to 0} z_0 = 0$$

2.34

(a)

$$V(x) = \begin{cases} 0 & x \leq 0 \\ V_0 & x > 0 \end{cases}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

$$= -k^2\psi$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$$\psi = Ae^{ikx} + Be^{-ikx}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2m(E-V_0)}{\hbar^2}\psi$$

$$= \kappa^2\psi$$

$$\kappa = \frac{\sqrt{2m(V_0-E)}}{\hbar}$$

$$\psi = Fe^{-\kappa x}$$

$$A + B = F$$

$$ik(A-B) = -\kappa F$$

$$F = -i\frac{k}{\kappa}(A-B)$$

$$A + B = -i\frac{k}{\kappa}(A-B)$$

$$\left(1 - i\frac{k}{\kappa}\right)B = -\left(1 + i\frac{k}{\kappa}\right)A$$

$$B = -\frac{1+ik/\kappa}{1-ik/\kappa}A$$

$$R = \frac{|B|^2}{|A|^2}$$

$$= \left(-\frac{1+ik/\kappa}{1-ik/\kappa}\right)\left(-\frac{1-ik/\kappa}{1+ik/\kappa}\right)$$

$$= 1$$

(b)

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \\ Fe^{ilx} & x > 0 \end{cases}$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$$l = \frac{\sqrt{2m(E - V_0)}}{\hbar}$$

$$A + B = F$$

$$ik(A - B) = ilF$$

$$F = \frac{k}{l}(A - B)$$

$$A + B = \frac{k}{l}(A - B)$$

$$\left(\frac{k}{l} + 1\right)B = \left(\frac{k}{l} - 1\right)A$$

$$B = \frac{k/l - 1}{k/l + 1}A$$

$$R = \frac{|B|^2}{|A|^2}$$

$$= \left(\frac{k/l - 1}{k/l + 1}\right)^2$$

$$= \left(\frac{k - l}{k + l}\right)^2$$

$$= \frac{(k - l)^4}{(k^2 - l^2)^2}$$

$$k^2 - l^2 = \frac{2mE}{\hbar^2} - \frac{2m(E - V_0)}{\hbar^2}$$

$$= \frac{2m}{\hbar^2}V_0$$

$$k - l = \frac{\sqrt{2m}}{\hbar}(\sqrt{E} - \sqrt{E - V_0})$$

$$R = \frac{(\sqrt{E} - \sqrt{E - V_0})^4}{V_0^2}$$

(d)

$$B = F - A$$

$$F = \frac{k}{l}(A - F + A)$$

$$\left(1 + \frac{k}{l}\right)F = \frac{2k}{l}A$$

$$F = \frac{2k}{k+l}A$$

$$\frac{l}{k} = \sqrt{\frac{E - V_0}{E}}$$

$$T = \left|\frac{F}{A}\right|^2 \frac{l}{k}$$

$$= \left(\frac{2k}{k+l}\right)^2 \frac{l}{k}$$

$$= \frac{4kl}{(k+l)^2}$$

$$= \frac{4kl(k-l)^2}{(k^2 - l^2)^2}$$

$$= \frac{4\sqrt{E}\sqrt{E - V_0}(\sqrt{E} - \sqrt{E - V_0})^2}{V_0^2}$$

$$T + R = \frac{4kl}{(k+l)^2} + \frac{(k-l)^2}{(k+l)^2}$$

$$= \frac{4kl + k^2 - 2kl + l^2}{(k+l)^2}$$

$$= \frac{k^2 + 2kl + l^2}{(k+l)^2}$$

$$= \frac{(k+l)^2}{(k+l)^2}$$

$$= 1$$

2.35

(a)

$$V(x) = \begin{cases} 0 & x < 0 \\ -V_0 & x > 0 \end{cases}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

$$= -k^2\psi$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$$\psi = Ae^{ikx} + Be^{-ikx}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E + V_0)\psi$$

$$= -l^2\psi$$

$$l = \frac{\sqrt{2m(E + V_0)}}{\hbar}$$

$$\psi = Fe^{ilx}$$

$$A + B = F$$

$$ik(A - B) = ilF$$

$$k(A - B) = l(A + B)$$

$$(k + l)B = (k - l)A$$

$$B = \frac{k - l}{k + l}A$$

$$R = \left|\frac{B}{A}\right|^2$$

$$= \left(\frac{k - l}{k + l}\right)^2$$

$$= \left(\frac{\sqrt{E} - \sqrt{E + V_0}}{\sqrt{E} + \sqrt{E + V_0}}\right)^2$$

$$= \frac{1}{9}$$

(c)

$$T = 1 - R = \frac{8}{9}$$

2.36

$$V(x) = \begin{cases} 0 & |x| < a \\ \infty & |x| > a \end{cases}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

$$= -k^2\psi$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$$\psi = A\sin kx + B\cos kx$$

$$0 = -A\sin ka + B\cos ka$$

$$0 = A\sin ka + B\cos ka$$

$$B\cos ka = 0$$

$$k = \frac{n\pi}{2a},\ n = 1,3,5,\ldots$$

$$E = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}$$

$$\psi = B\cos\left(\frac{n\pi}{2a}x\right),\ n = 1,3,5,\ldots$$

$$1 = |B|^2\int_{-a}^{a}\cos^2\left(\frac{n\pi}{2a}x\right)\ dx$$

$$B = \frac{1}{\sqrt{a}}$$

$$\psi = \frac{1}{\sqrt{a}}\cos\left(\frac{n\pi}{2a}x\right),\ n = 1,3,5,\ldots$$

$$A\sin ka = 0$$

$$k = \frac{n\pi}{2a},\ n = 2,4,6,\ldots$$

$$E = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}$$

$$\psi = A\sin\left(\frac{n\pi}{2a}x\right),\ n = 2,4,6,\ldots$$

$$1 = |A|\int_{-a}^{a}\sin^2\left(\frac{n\pi}{2a}x\right)\ dx$$

$$A = \frac{1}{\sqrt{a}}$$

$$\psi = \frac{1}{\sqrt{a}}\sin\left(\frac{n\pi}{2a}x\right),\ n = 2,4,6,\ldots$$

2.37

$$\Psi(x,0) = A\sin^3\left(\frac{\pi}{a}x\right)$$

$$= A\left[\frac{3}{4}\sin\left(\frac{\pi}{a}x\right) - \frac{1}{4}\sin\left(\frac{3\pi}{a}x\right)\right]$$

$$= A\sqrt{\frac{a}{2}}\left[\frac{3}{4}\psi_1(x) - \frac{1}{4}\psi_3(x)\right]$$

$$1 = |A|^2\frac{a}{2}\int_0^a\left[\frac{3}{4}\psi_1(x) - \frac{1}{4}\psi_3(x)\right]^2 dx$$

$$= |A|^2\frac{a}{2}\int_0^a\left[\frac{9}{16}\psi_1(x)^2 - \frac{3}{8}\psi_1(x)\psi_3(x) + \frac{1}{16}\psi_3(x)^2\right] dx$$

$$= \frac{5}{16}a|A|^2$$

$$A = \frac{4}{\sqrt{5a}}$$

$$\Psi(x,0) = \frac{1}{\sqrt{10}}[3\psi_1(x) - \psi_3(x)]$$

$$\Psi(x,t) = \frac{1}{\sqrt{10}}[3\psi_1(x)e^{-iE_1 t/\hbar} - \psi_3(x)e^{-iE_3 t/\hbar}]$$

$$\langle x \rangle = \int_0^a \Psi^* x \Psi\, dx$$

$$= \frac{1}{10}\int_0^a x\left(9\psi_1^2 + \psi_3^2 - 3\psi_1\psi_3 e^{-i(E_3-E_1)t/\hbar} - 3\psi_1\psi_3 e^{-i(E_1-E_3)t/\hbar}\right) dx$$

$$= \frac{1}{10}\int_0^a x\left[9\psi_1^2 + \psi_3^2 - 6\psi_1\psi_3\cos\left(\frac{E_3-E_1}{\hbar}t\right)\right] dx$$

$$= \frac{1}{10}\left[9\langle x\rangle_1 + \langle x\rangle_3 - 6\cos\left(\frac{E_3-E_1}{\hbar}t\right)\int_0^a x\psi_1\psi_3\, dx\right]$$

$$= \frac{1}{10}\left[\frac{9}{2}a + \frac{1}{2}a\right]$$

$$= \frac{a}{2}$$

$$P(E_1) = \frac{9}{10}$$

$$P(E_3) = \frac{1}{10}$$

$$\langle E \rangle = E_1 P(E_1) + E_3 P(E_3)$$

$$= \frac{9\pi^2\hbar^2}{20ma^2} + \frac{9\pi^2\hbar^2}{20ma^2}$$

$$= \frac{9\pi^2\hbar^2}{10ma^2}$$

2.38

(a)

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}$$

$$\Psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

$$E_n T = \frac{n^2 \pi^2 \hbar^2}{2ma^2} \frac{4ma^2}{\pi \hbar}$$

$$= 2\pi n^2 \hbar$$

$$\Psi(x,T) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n T/\hbar}$$

$$= \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-2\pi i n^2}$$

$$= \sum_{n=1}^{\infty} c_n \psi_n(x)$$

$$= \Psi(x,0)$$

(b)

$$E = \frac{1}{2}mv^2$$

$$v = \sqrt{\frac{2E}{m}}$$

$$T = \frac{2a}{v}$$

$$= a\sqrt{\frac{2m}{E}}$$

(c)

$$\frac{4ma^2}{\pi \hbar} = a\sqrt{\frac{2m}{E}}$$

$$\frac{16m^2 a^2}{\pi^2 \hbar^2} = \frac{2m}{E}$$

$$E = \frac{\pi^2 \hbar^2}{8ma^2}$$

$$= \frac{E_1}{4}$$

2.39

(a)

$$\Psi(x,0) = \begin{cases} \frac{2\sqrt{3}}{a\sqrt{a}}x & 0 \le x \le a/2 \\ \frac{2\sqrt{3}}{a\sqrt{a}}(a-x) & a/2 \le x \le a \end{cases}$$

$$\frac{d}{dx}\Psi(x,0) = \frac{2\sqrt{3}}{a\sqrt{a}}\left[1 - 2\theta\left(x - \frac{a}{2}\right)\right]$$

(b)

$$\frac{d^2}{dx^2}\Psi(x,0) = -\frac{4\sqrt{3}}{a\sqrt{a}}\delta\left(x - \frac{a}{2}\right)$$

(c)

$$\hat{H}\Psi(x,0) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi(x,0)$$

$$= \frac{\hbar^2}{2m}\frac{4\sqrt{3}}{a\sqrt{a}}\delta\left(x - \frac{a}{2}\right) + V(x)\Psi(x,0)$$

$$\langle H \rangle = \int \Psi(x,0)^* \hat{H}\Psi(x,0)\,dx$$

$$= \int_0^a \Psi(x,0)^* \left[\frac{\hbar^2}{2m}\frac{4\sqrt{3}}{a\sqrt{a}}\delta\left(x - \frac{a}{2}\right) + V(x)\Psi(x,0)\right]dx$$

$$= \Psi\left(\frac{a}{2},0\right)^* \frac{\hbar^2}{2m}\frac{4\sqrt{3}}{a\sqrt{a}} + \int_0^a \Psi(x,0)^* V(x)\Psi(x,0)\,dx$$

$$= \frac{6\hbar^2}{ma^2}$$

2.40

(a)

$$V(x) = \frac{1}{2}m\omega^2 x^2$$

$$\xi = \sqrt{\frac{m\omega}{\hbar}} x$$

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$$

$$\Psi(x,0) = A\left(1 - 2\sqrt{\frac{m\omega}{\hbar}}x\right)^2 e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= A\left(1 - 4\sqrt{\frac{m\omega}{\hbar}}x + \frac{4m\omega}{\hbar}x^2\right) e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= A\left(\frac{\pi\hbar}{m\omega}\right)^{1/4} \left[3\psi_0(x) - 2\sqrt{2}\psi_1(x) + 2\sqrt{2}\psi_2(x)\right]$$

$$1 = A^2\sqrt{\frac{\pi\hbar}{m\omega}} \int_{-\infty}^{\infty} (3\psi_0 - 2\sqrt{2}\psi_1 + 2\sqrt{2}\psi_2)^2\, dx$$

$$= A^2\sqrt{\frac{\pi\hbar}{m\omega}} \int_{-\infty}^{\infty} (9\psi_0^2 - 12\sqrt{2}\psi_0\psi_1 + 12\sqrt{2}\psi_0\psi_2 + 8\psi_1^2 - 16\psi_1\psi_2 + 8\psi_2^2)\, dx$$

$$= 25A^2\sqrt{\frac{\pi\hbar}{m\omega}}$$

$$A = \frac{1}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\Psi(x,0) = \frac{3}{5}\psi_0(x) - \frac{2\sqrt{2}}{5}\psi_1(x) + \frac{2\sqrt{2}}{5}\psi_2(x)$$

(b)

$$E_0 = \frac{\hbar\omega}{2}$$

$$P(E_0) = \frac{9}{25}$$

$$E_1 = \frac{3\hbar\omega}{2}$$

$$P(E_1) = \frac{8}{25}$$

$$E_2 = \frac{5\hbar\omega}{2}$$

$$P(E_2) = \frac{8}{25}$$

$$\langle E \rangle = \frac{\hbar\omega}{2}\frac{9}{25} + \frac{3\hbar\omega}{2}\frac{8}{25} + \frac{5\hbar\omega}{2}\frac{8}{25}$$

$$= \frac{73}{50}\hbar\omega$$

(c)

$$\xi = \sqrt{\frac{m\omega}{\hbar}}x$$

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\xi^2/2}$$

$$\Psi(x,T) = B\left(1+2\sqrt{\frac{m\omega}{\hbar}}x\right)^2 e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= B\left(1+4\sqrt{\frac{m\omega}{\hbar}}x+4\frac{m\omega}{\hbar}x^2\right)e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= B\left(\frac{\pi\hbar}{m\omega}\right)^{1/4}\left[3\psi_0(x)+2\sqrt{2}\psi_1(x)+2\sqrt{2}\psi_2(x)\right]$$

$$1 = |B|^2\sqrt{\frac{\pi\hbar}{m\omega}}\int_{-\infty}^{\infty}[3\psi_0+2\sqrt{2}\psi_1+2\sqrt{2}\psi_2]^2\,dx$$

$$= |B|^2\sqrt{\frac{\pi\hbar}{m\omega}}\int_{-\infty}^{\infty}[9\psi_0^2+12\sqrt{2}\psi_0\psi_1+12\sqrt{2}\psi_0\psi_2+8\psi_1^2+16\psi_1\psi_2+8\psi_2^2]\,dx$$

$$= 25|B|^2\sqrt{\frac{\pi\hbar}{m\omega}}$$

$$B = \frac{1}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\Psi(x,T) = \frac{3}{5}\psi_0(x)+\frac{2\sqrt{2}}{5}\psi_1(x)+\frac{2\sqrt{2}}{5}\psi_2(x)$$

$$\Psi(x,t) = \frac{3}{5}\psi_0(x)e^{-i\omega t/2}-\frac{2\sqrt{2}}{5}\psi_1(x)e^{-3i\omega t/2}+\frac{2\sqrt{2}}{5}\psi_2(x)e^{-5i\omega t/2}$$

$$= e^{-i\omega t/2}\left[\frac{3}{5}\psi_0(x)-\frac{2\sqrt{2}}{5}\psi_1(x)e^{-i\omega t}+\frac{2\sqrt{2}}{5}e^{-2i\omega t}\right]$$

$$e^{-i\omega T} = -1$$

$$e^{-2i\omega T} = 1$$

$$T = \frac{\pi}{\omega}$$

2.41

The argument for calculating the allowed energies and wavefunctions is the same, except there is a boundary condition $\psi(0) = 0$. This leaves only $\psi_n(x)$ for odd n.

2.43

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$$\psi(x) = \begin{cases} -A\sin kx + B\cos kx & -a < x \le 0 \\ A\sin kx + B\cos kx & 0 \le x < a \end{cases}$$

$$\Delta\left(\frac{d\psi}{dx}\right) = 2Ak$$

$$\Delta\left(\frac{d\psi}{dx}\right) = \frac{2m\alpha}{\hbar^2}\psi(0)$$

$$2Ak = \frac{2m\alpha}{\hbar^2}B$$

$$B = \frac{\hbar^2 k}{m\alpha}A$$

$$\psi(x) = A\left(\sin kx + \frac{\hbar^2 k}{m\alpha}\cos kx\right)$$

$$0 = A\sin ka + \frac{\hbar^2 k}{m\alpha}A\cos ka$$

$$\tan ka = -\frac{\hbar^2 k}{m\alpha}$$

$$ka \approx \frac{n\pi}{2}, \, n = 1, 3, 5, \ldots$$

$$E \approx \frac{n^2\pi^2\hbar^2}{2m(2a)^2}, \, n = 1, 3, 5, \ldots$$

$$\psi(x) = \begin{cases} A\sin kx - B\cos kx & -a < x \le 0 \\ A\sin kx + B\cos kx & 0 \le x < a \end{cases}$$

$$-B = B$$

$$B = 0$$

$$\psi(x) = A\sin kx$$

$$0 = A\sin ka$$

$$ka = \frac{n\pi}{2}, \, n = 2, 4, 6, \ldots$$

$$\psi(x) = A\sin\left(\frac{n\pi}{2a}x\right)$$

$$E = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}, \, n = 2, 4, 6, \ldots$$

2.44

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2}\psi_2 + V\psi_1\psi_2 = E\psi_1\psi_2$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2}\psi_1 + V\psi_1\psi_2 = E\psi_1\psi_2$$

$$\psi_2\frac{d^2\psi_1}{dx^2} - \psi_1\frac{d^2\psi_2}{dx^2} = 0$$

$$\frac{d}{dx}\left(\psi_2\frac{d\psi_1}{dx} - \psi_1\frac{d\psi_2}{dx}\right) = 0$$

$$\psi_2\frac{d\psi_1}{dx} - \psi_1\frac{d\psi_2}{dx} = c$$

$$c = 0$$

$$\psi_2\frac{d\psi_1}{dx} = \psi_1\frac{d\psi_2}{dx}$$

$$\frac{1}{\psi_1}\frac{d\psi_1}{dx} = \frac{1}{\psi_2}\frac{d\psi_2}{dx}$$

$$\ln\psi_1 = \ln\psi_2 + c$$

$$\psi_1 = A\psi_2$$

2.45

(a)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2}\psi_m + V(x)\psi_n\psi_m = E_n\psi_n\psi_m$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_m}{dx^2}\psi_n + V(x)\psi_n\psi_m = E_m\psi_n\psi_m$$

$$\frac{d^2\psi_m}{dx^2}\psi_n - \frac{d^2\psi_n}{dx^2}\psi_m = \frac{2m}{\hbar^2}(E_n - E_m)\psi_n\psi_m$$

$$\frac{d}{dx}\left(\frac{d\psi_m}{dx}\psi_n - \frac{d\psi_n}{dx}\psi_m\right) = \frac{2m}{\hbar^2}(E_n - E_m)\psi_n\psi_m$$

(b)

$$\int_{x_1}^{x_2}\frac{d}{dx}(\psi_m'\psi_n - \psi_n'\psi_m)\,dx = \frac{2m}{\hbar^2}(E_n - E_m)\int_{x_1}^{x_2}\psi_n\psi_m\,dx$$

$$\psi_m'(x_2)\psi_n(x_2) - \psi_m'(x_1)\psi_n(x_1) = \frac{2m}{\hbar^2}(E_n - E_m)\int_{x_1}^{x_2}\psi_n\psi_m\,dx$$

2.53

(a)

$$\frac{1}{1 - i\beta}\begin{pmatrix} i\beta & 1 \\ 1 & i\beta \end{pmatrix}$$

(b)

$$\frac{e^{-2ika}}{\cos(2la) - i\frac{(k^2+l^2)}{2kl}\sin(2la)} \begin{pmatrix} i\frac{\sin(2la)}{2kl}(l^2 - k^2) & 1 \\ 1 & i\frac{\sin(2la)}{2kl}(l^2 - k^2) \end{pmatrix}$$

3   Formalism

3.1

(a)

$$\left| \int_a^b (f^* + g^*)(f + g)\, dx \right| = \left| \int_a^b (f^*f + f^*g + g^*f + g^*g)\, dx \right|$$

$$\leq \int_a^b |f|^2\, dx + \left| \int_a^b f^*g\, dx \right| + \left| \int_a^b g^*f\, dx \right| + \int_a^b |g|^2\, dx$$

$$\leq \int_a^b |f|^2\, dx + 2\sqrt{\int_a^b |f|^2\, dx \int_a^b |g|^2\, dx} + \int_a^b |g|^2\, dx$$

The set of all normalised functions isn't a vector space because e.g. multiplying a function by a constant also multiplies its integral by that constant meaning it's no longer a member of the vector space.

(b)

$$\langle \beta | \alpha \rangle = \int_a^b \beta^* \alpha\, dx$$

$$= \left(\int_a^b \alpha^* \beta\, dx \right)^*$$

$$= \langle \alpha | \beta \rangle^*$$

$$\langle a | a \rangle = \int_a^b |a|^2\, dx$$

$$\geq 0$$

If $\langle \alpha | \alpha \rangle = 0$ that implies $|\alpha|^2 = 0$ everywhere in the interval and thus $|\alpha\rangle = |0\rangle$.

$$\langle \alpha | (b\,|\beta\rangle + c\,|\gamma\rangle) = \int_{x_1}^{x_2} \alpha^*(b\beta)\, dx + \int_{x_1}^{x_2} \alpha^*(c\gamma)\, dx$$

$$= b \int_{x_1}^{x_2} \alpha^* \beta\, dx + c \int_{x_1}^{x_2} \alpha^* \gamma\, dx$$

$$= b\,\langle \alpha | \beta \rangle + c\,\langle \alpha | \gamma \rangle$$

51

3.2

(a)

$$\int_0^1 x^{2\nu} \, dx = \frac{1}{2\nu + 1} \left[x^{2\nu+1} \right]_0^1$$

The integral is defined for $\nu > -1/2$. For the case $\nu = -1/2$

$$\int_0^1 x^{-1} \, dx = [\ln x]_0^1 = \ln 1 - \ln 0 = 0 - \infty.$$

So $f(x) = x^\nu$ is in Hilbert space for $\nu > -1/2$.

(b)

$$\int_0^1 x \, dx = \frac{1}{2}$$

$$\int_0^1 x^3 \, dx = \frac{1}{4}$$

$$\int_0^1 x^{-1} \, dx = [\ln x]_0^1$$

$$= 0 - \infty$$

$f(x)$ and $xf(x)$ are in Hilbert space, but not $(d/dx)f(x)$.

3.4

(a)

$$\langle f|(\hat{Q} + \hat{R})f \rangle = \langle f|\hat{Q}f \rangle + \langle f|\hat{R}f \rangle$$
$$= \langle \hat{Q}f|f \rangle + \langle \hat{R}f|f \rangle$$
$$= \langle (\hat{Q} + \hat{R})f|f \rangle$$

(b)

$$\langle f|\alpha\hat{Q}g \rangle = \alpha \langle f|\hat{Q}g \rangle$$
$$= \alpha \langle \hat{Q}f|g \rangle$$
$$\langle \alpha\hat{Q}f|g \rangle = \alpha^* \langle \hat{Q}f|g \rangle$$
$$\alpha = \alpha^*$$

α is real.

(c)

$$\langle f|\hat{Q}\hat{R}g\rangle = \langle \hat{Q}f|\hat{R}g\rangle$$
$$= \langle \hat{R}\hat{Q}f|g\rangle$$

The product of the operators is hermitian when $\hat{Q}\hat{R} = \hat{R}\hat{Q}$ i.e. $[\hat{Q}, \hat{R}] = 0$.

(d)

$$\langle \Psi|\hat{x}\Psi\rangle = \int \Psi^* \hat{x}\Psi \, dx$$
$$= \int \Psi^* \hat{x}^*\Psi \, dx$$
$$= \int (\hat{x}\Psi)^*\Psi \, dx$$
$$= \langle \hat{x}\Psi|\Psi\rangle$$

$$\langle \Psi|\hat{H}\Psi\rangle = \int \Psi^* \hat{H}\Psi \, dx$$
$$= \int \Psi^* \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) \right] \Psi \, dx$$
$$= -\frac{\hbar^2}{2m} \int \Psi^* \frac{d^2\Psi}{dx^2} \, dx + \int \Psi^* V(x)\Psi \, dx$$
$$= -\frac{\hbar^2}{2m} \left[\Psi^* \frac{d\Psi}{dx} \Big|_{-\infty}^{\infty} - \int \frac{d\Psi^*}{dx}\frac{d\Psi}{dx} \, dx \right] + \langle V(x)\Psi|\Psi\rangle$$
$$= \frac{\hbar^2}{2m} \left[\frac{d\Psi^*}{dx} \Psi \Big|_{-\infty}^{\infty} - \int \frac{d^2\Psi^*}{dx} \Psi \, dx \right] + \langle V(x)\Psi|\Psi\rangle$$
$$= -\frac{\hbar^2}{2m} \int \frac{d^2}{dx^2} \Psi^* \Psi \, dx + \langle V(x)\Psi|\Psi\rangle$$
$$= \left\langle -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \Psi \,\Big|\, \Psi \right\rangle + \langle V(x)\Psi|\Psi\rangle$$
$$= \langle \hat{H}\Psi|\Psi\rangle$$

3.5

(a)

$$x^\dagger = x$$
$$i^\dagger = -i$$
$$\left(\frac{d}{dx} \right)^\dagger = -\frac{d}{dx}$$

53

(b)

$$\langle f|\hat{Q}\hat{R}g\rangle = \int f^\dagger \hat{Q}\hat{R}g\,dx$$
$$= \int (\hat{Q}^\dagger f)^\dagger \hat{R}g\,dx$$
$$= \int (\hat{R}^\dagger \hat{Q}^\dagger f)^\dagger g\,dx$$
$$= \langle \hat{R}^\dagger \hat{Q}^\dagger f|g\rangle$$

(c)

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega x)$$
$$\langle f|\hat{a}g\rangle = \left\langle f \left| \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega x)g \right.\right\rangle$$
$$= \frac{1}{\sqrt{2\hbar m\omega}}\,\langle f|(-i\hat{p} + m\omega x)g\rangle$$
$$= \frac{1}{\sqrt{2\hbar m\omega}}(\langle f| - i\hat{p}g\rangle + \langle f|m\omega xg\rangle)$$
$$= \frac{1}{\sqrt{2\hbar m\omega}}(\langle f| - i\hat{p}g\rangle + \langle m\omega xf|g\rangle)$$
$$= \frac{1}{\sqrt{2\hbar m\omega}}(\langle i\hat{p}f|g\rangle + \langle m\omega xf|g\rangle)$$
$$= \left\langle \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega x)f \left| g \right.\right\rangle$$
$$= \langle \hat{a}_- f|g\rangle$$

3.6

$$\langle f|\hat{Q}g\rangle = \int_0^{2\pi} f^* \frac{d^2 g}{d\phi^2}\,d\phi$$
$$= f^* \frac{dg}{d\phi}\bigg|_0^{2\pi} - \int_0^{2\pi} \left(\frac{df}{d\phi}\right)^* \frac{dg}{d\phi}\,d\phi$$
$$= -\left(\frac{df}{d\phi}\right)^* g\bigg|_0^{2\pi} + \int_0^{2\pi} \left(\frac{d^2 f}{d\phi}\right)^* g\,d\phi$$
$$= \langle \hat{Q}f|g\rangle$$

Yes, the operator is hermitian.

$$\hat{Q}f = qf$$

$$\frac{d^2 f}{d\phi^2} = qf$$

$$\frac{d^2 f}{d\phi^2} - qf = 0$$

$$f = Ae^{\sqrt{q}\phi} + Be^{-\sqrt{q}\phi}$$

$$f(\phi + 2\pi) = Ae^{\sqrt{q}(\phi+2\pi)} + Be^{\sqrt{q}(\phi+2\pi)}$$

$$= Ae^{\sqrt{q}\phi}e^{2\pi\sqrt{q}} + Be^{\sqrt{q}\phi}e^{2\pi\sqrt{q}}$$

$$2\pi\sqrt{q} = 1$$

$$q = -n^2, \; n = 0, 1, 2, \ldots$$

The eigenfunctions are $f = Ae^{\pm\sqrt{q}\phi}$ and the eigenvalues are $q = 0, 1, 2, \ldots$ The spectrum is degenerate as there are two eigenfunctions associated with each eigenvalue $q > 0$.

3.7

(a)

$$h = af + bg$$

$$\hat{Q}h = \hat{Q}(af + bg)$$

$$= \hat{Q}(af) + \hat{Q}(bg)$$

$$= a\hat{Q}f + b\hat{Q}g$$

$$= aqf + bqg$$

$$= q(af + bg)$$

$$= qh$$

(b)

$$\frac{d^2}{dx^2}e^x = e^x$$

$$\frac{d^2}{dx^2}e^{-x} = e^{-x}$$

$$f = e^x + e^{-x}$$

$$g = e^x - e^{-x}$$

3.8

(a)

$$\hat{Q} = i\frac{d}{d\phi}$$

$$\hat{Q}f = qf$$

$$i\frac{df}{d\phi} = qf$$

$$\frac{df}{d\phi} + iqf = 0$$

$$f = Ae^{-iq\phi}$$

$$e^{-2\pi iq} = 1$$

$$q = 0, \pm 1, \pm 2, \ldots$$

$$\int_0^{2\pi} Ae^{-iq\phi}Be^{-iq'\phi}\,d\phi = AB\int_0^{2\pi} e^{-i(q+q')\phi}\,d\phi$$

$$= 0$$

(b)

$$\hat{Q} = \frac{d^2}{d\phi^2}$$

$$\hat{Q}f = qf$$

$$\frac{d^2f}{d\phi^2} - qf = 0$$

$$f = Ae^{\pm\sqrt{q}\phi}$$

$$q = -n^2,\; n = 0, 1, 2, \ldots$$

$$\int_0^{2\pi} Ae^{\pm\sqrt{q}\phi}Be^{\pm\sqrt{q'}\phi}\,d\phi = AB\int_0^{2\pi} e^{\pm in\phi}e^{\pm in'\phi}\,d\phi$$

$$= AB\int_0^{2\pi} e^{i(\pm n\pm n')\phi}\,d\phi$$

$$= AB\left[\frac{1}{i(\pm n \pm n')}e^{i(\pm n\pm n')\phi}\right]_0^{2\pi}$$

$$= AB\frac{1}{i(\pm n \pm n')}[e^{i(\pm n\pm n')2\pi} - 1]$$

$$= 0$$

3.9

(a) Infinite square well

(b) Delta function barrier

(c) Delta function well

3.11

$$\Psi_0(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} e^{-i\omega t/2}$$

$$\Phi_0(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} e^{-i\omega t/2} \, dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-i\omega t/2} \int_{-\infty}^{\infty} e^{-ipx/\hbar} e^{-m\omega x^2/2\hbar} \, dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-i\omega t/2} \sqrt{\frac{2\pi\hbar}{m\omega}} e^{-p^2/2\hbar m\omega}$$

$$= \frac{1}{(\pi\hbar m\omega)^{1/4}} e^{-p^2/2\hbar m\omega} e^{-i\omega t/2}$$

$$\frac{p^2}{2m} = \frac{\hbar\omega}{2}$$

$$p = \pm\sqrt{\hbar m\omega}$$

$$1 - \int_{-\sqrt{\hbar m\omega}}^{\sqrt{\hbar m\omega}} |\Phi_0|^2 \, dp = 1 - \frac{1}{(\pi\hbar m\omega)^{1/4}} \int_{-\sqrt{\hbar m\omega}}^{\sqrt{\hbar m\omega}} e^{-p^2/\hbar m\omega} \, dp$$

$$= 1 - \frac{1}{(\pi\hbar m\omega)^{1/2}} \sqrt{\pi\hbar m\omega} \operatorname{erf} 1$$

$$= 0.16$$

3.12

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)} \, dk$$

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)} \, dk\right) dx$$

$$= \frac{1}{2\pi\sqrt{\hbar}} \int_{-\infty}^{\infty} \phi(k) e^{-i\frac{\hbar k^2}{2m} t} \left(\int_{-\infty}^{\infty} e^{i(k-p/\hbar)x} \, dx\right) dk$$

$$= \frac{1}{2\pi\sqrt{\hbar}} \int_{-\infty}^{\infty} \phi(k) e^{-i\frac{\hbar k^2}{2m} t} 2\pi\delta(k - p/\hbar) \, dk$$

$$= \frac{1}{\sqrt{\hbar}} \int_{-\infty}^{\infty} \delta(k - p/\hbar) \phi(k) e^{-i\frac{\hbar k^2}{2m} t} \, dk$$

$$= \frac{1}{\sqrt{\hbar}} \phi(p/\hbar) e^{-i\frac{p^2}{2\hbar m} t}$$

$$|\Phi(p,t)|^2 = \frac{1}{\hbar} |\phi(p/\hbar)|^2$$

3.14

(a)

$$\begin{aligned}
[\hat{A} + \hat{B}, \hat{C}] &= (\hat{A} + \hat{B})\hat{C} - \hat{C}(\hat{A} + \hat{B}) \\
&= \hat{A}\hat{C} + \hat{B}\hat{C} - \hat{C}\hat{A} - \hat{C}\hat{B} \\
&= \hat{A}\hat{C} - \hat{C}\hat{A} + \hat{B}\hat{C} - \hat{C}\hat{B} \\
&= [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}] \\
\hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B} &= \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} + \hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B} \\
&= \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} \\
&= [\hat{A}\hat{B}, \hat{C}]
\end{aligned}$$

(b)

$$\begin{aligned}
[x^n, \hat{p}] &= \left[x^n, -i\hbar\frac{d}{dx}\right] \\
&= x^n\left(-i\hbar\frac{d}{dx}\right) - \left(-i\hbar\frac{d}{dx}\right)x^n \\
&= x^n\left(-i\hbar\frac{d}{dx}\right)(1) + i\hbar n x^{n-1} \\
&= i\hbar n x^{n-1}
\end{aligned}$$

(c)

$$\begin{aligned}
[f(x), \hat{p}]g(x) &= f(x)\left(-i\hbar\frac{d}{dx}\right)g(x) - \left(-i\hbar\frac{d}{dx}\right)[f(x)g(x)] \\
&= -i\hbar f(x)\frac{dg}{dx} + i\hbar\left[\frac{df}{dx}g(x) + f(x)\frac{dg}{dx}\right] \\
&= i\hbar\frac{df}{dx}g(x) \\
[f(x), \hat{p}] &= i\hbar\frac{df}{dx}
\end{aligned}$$

(d)

$$[\hat{H}, \hat{a}_-]g = \hat{H}\hat{a}_-g - \hat{a}_-\hat{H}g$$

$$= \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)\hat{a}_-g - \hat{a}_-\hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)g$$

$$= \hbar\omega\hat{a}_-\hat{a}_+\hat{a}_-g - \frac{1}{2}\hbar\omega\hat{a}_-g - \hbar\omega\hat{a}_-^2\hat{a}_+g + \frac{1}{2}\hbar\omega\hat{a}_-g$$

$$= \hbar\omega\hat{a}_-\hat{a}_+\hat{a}_-g - \hbar\omega\hat{a}_-^2\hat{a}_+g$$

$$= \hbar\omega\hat{a}_-(\hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+)g$$

$$= -\hbar\omega\hat{a}_-g$$

$$[\hat{H}, \hat{a}_-] = -\hbar\omega\hat{a}_-$$

$$[\hat{H}, \hat{a}_+]g = \hat{H}\hat{a}_+g - \hat{a}_+\hat{H}g$$

$$= \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)\hat{a}_+g - \hat{a}_+\hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)g$$

$$= \hbar\omega\hat{a}_-\hat{a}_+^2g - \frac{1}{2}\hbar\omega\hat{a}_+g - \hbar\omega\hat{a}_+\hat{a}_-\hat{a}_+g + \frac{1}{2}\hbar\omega\hat{a}_+g$$

$$= \hbar\omega\hat{a}_-\hat{a}_+^2g - \hbar\omega\hat{a}_+\hat{a}_-\hat{a}_+g$$

$$= \hbar\omega(\hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_-)\hat{a}_+g$$

$$= \hbar\omega\hat{a}_+g$$

$$[\hat{H}, \hat{a}_+] = \hbar\omega\hat{a}_+$$

3.15

$$\left[x, \frac{p^2}{2m} + V\right]g = x\left(\frac{p^2}{2m} + V\right)g - \left(\frac{p^2}{2m} + V\right)xg$$

$$= x\frac{p^2}{2m}g + xVg - \frac{p^2}{2m}xg - Vxg$$

$$= \frac{1}{2m}(xp^2g - p^2xg)$$

$$= \frac{1}{2m}\left[-\hbar^2 x\frac{d^2g}{dx^2} + \hbar^2\frac{d}{dx}\left(g + x\frac{dg}{dx}\right)\right]$$

$$= \frac{1}{2m}\left[-\hbar^2 x\frac{d^2g}{dx^2} + \hbar^2\left(\frac{dg}{dx} + \frac{dg}{dx} + x\frac{d^2g}{dx^2}\right)\right]$$

$$= \frac{\hbar^2}{m}\frac{dg}{dx}$$

$$\left[x, \frac{p^2}{2m} + V\right] = \frac{\hbar^2}{m}\frac{d}{dx}$$

$$= -\frac{\hbar}{im}\langle p\rangle$$

$$\sigma_x^2\sigma_H^2 \geq \left(\frac{1}{2i}\left\langle\left[x, \frac{p^2}{2m} + V\right]\right\rangle\right)^2$$

$$= \frac{\hbar^2}{4m^2}|\langle p\rangle|^2$$

$$\sigma_x\sigma_H \geq \frac{\hbar}{2m}|\langle p\rangle|$$

This doesn't tell us much because for stationary states $\sigma_H = 0$ and $\langle p\rangle = 0$ so this says $0 \geq 0$.

3.17

$$\left(-i\hbar\frac{d}{dx} - \langle p\rangle\right)\Psi = ia(x - \langle x\rangle)\Psi$$

$$-i\hbar\frac{d\Psi}{dx} - \langle p\rangle\Psi = ia(x - \langle x\rangle)\Psi$$

$$\frac{d\Psi}{dx} + \frac{\langle p\rangle + ia(x - \langle x\rangle)}{i\hbar}\Psi = 0$$

$$\frac{d\Psi}{dx} + \left[\frac{a}{\hbar}(x - \langle x\rangle) - i\frac{\langle p\rangle}{\hbar}\right]\Psi = 0$$

$$\frac{d\Psi}{dx} + \frac{a}{\hbar}x\Psi - \frac{a}{\hbar}\langle x\rangle\Psi - i\frac{\langle p\rangle}{\hbar}\Psi = 0$$

$$\Psi = Ae^{-ax^2/2\hbar}e^{a\langle x\rangle x/\hbar}e^{i\langle p\rangle x/\hbar}$$

$$= Be^{-a(x - \langle x\rangle)^2/2\hbar}e^{i\langle p\rangle x/\hbar}$$

60

3.18

(a)

$$Q = 1$$
$$\hat{Q} = 1$$
$$[\hat{H}, \hat{Q}] = 0$$
$$\frac{d}{dt} \langle Q \rangle = 0$$

(b)

$$Q = H$$
$$\hat{Q} = \hat{H}$$
$$[\hat{H}, \hat{Q}] = [\hat{H}, \hat{H}]$$
$$= 0$$
$$\frac{d}{dt} \langle H \rangle = 0$$

(c)

$$Q = x$$
$$\hat{Q} = x$$
$$[\hat{H}, \hat{Q}] = [\hat{H}, x]$$
$$= -i\frac{\hbar}{m}\hat{p}$$
$$\frac{d}{dt} \langle x \rangle = \frac{i}{\hbar} \left\langle -i\frac{\hbar}{m}p \right\rangle$$
$$= \frac{\langle p \rangle}{m}$$

(d)

$$Q = p$$
$$\hat{Q} = \hat{p}$$
$$[\hat{H}, \hat{Q}] = [\hat{H}, \hat{p}]$$
$$= i\hbar\frac{\partial V}{\partial x}$$
$$\frac{d}{dt} \langle p \rangle = \frac{i}{\hbar} \left\langle i\hbar\frac{\partial V}{\partial x} \right\rangle$$
$$= -\left\langle \frac{\partial V}{\partial x} \right\rangle$$

3.19

(a)

$$\frac{d^2}{dt^2}\langle x \rangle = \frac{d}{dx}\left(\frac{\langle p \rangle}{m}\right)$$

$$= -\frac{1}{m}\left\langle \frac{\partial V}{\partial x}\right\rangle$$

$$= 0$$

(b)

$$\frac{d^2}{dt^2}\langle x \rangle = \frac{d}{dx}\left(\frac{\langle p \rangle}{m}\right)$$

$$= -\frac{1}{m}\left\langle \frac{\partial V}{\partial x}\right\rangle$$

$$= -\omega^2 \langle x \rangle$$

$$\frac{d^2}{dt^2}\langle x \rangle + \omega^2 \langle x \rangle = 0$$

$$\langle x \rangle = A\sin\omega t + B\cos\omega t$$

3.20

$$\Psi = \frac{1}{\sqrt{2}}(\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar})$$

$$\hat{H}\Psi = \frac{1}{\sqrt{2}}(E_1\psi_1 e^{-iE_1 t/\hbar} + E_2\psi_2 e^{-iE_2 t/\hbar})$$

$$\hat{H}^2\Psi = \frac{1}{\sqrt{2}}(E_1^2\psi_1 e^{-iE_1 t/\hbar} + E_2^2\psi_2 e^{-iE_2 t/\hbar})$$

$$\langle H^2 \rangle = \langle \Psi | \hat{H}^2 \Psi \rangle$$

$$= \frac{1}{2}\int_0^a (\psi_1^* e^{iE_1 t/\hbar} + \psi_2^* e^{iE_2 t/\hbar})(E_1^2\psi_1 e^{-iE_1 t/\hbar} + E_2^2\psi_2 e^{-iE_2 t/\hbar})\,dx$$

$$= \frac{1}{2}\int_0^a (E_1^2|\psi_1|^2 + E_2^2\psi_1^*\psi_2^* e^{i(E_1-E_2)t/\hbar} + E_1^2\psi_2^*\psi_1 e^{i(E_2-E_1)t/\hbar}$$

$$\qquad + E_2^2|\psi_2|^2)\,dx$$

$$= \frac{1}{2}(E_1^2 + E_2^2)$$

$$\langle H \rangle = \frac{1}{2}(E_1 + E_2)$$

$$\sigma_H^2 = \langle H^2 \rangle - \langle H \rangle^2$$

$$= \frac{1}{2}(E_1^2 + E_2^2) - \frac{1}{4}(E_1 + E_2)^2$$

$$= \frac{1}{4}(E_2 - E_1)^2$$

$$\omega = \frac{\pi^2 \hbar}{2ma^2}$$

$$\langle x \rangle = \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t)\right]$$

$$\langle x^2 \rangle = \frac{1}{2}\int_0^a (\psi_1^* e^{iE_1 t/\hbar} + \psi_2^* e^{iE_2 t/\hbar})x^2(\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar})\,dx$$

$$= \frac{1}{2}\int_0^a x^2(|\psi_1|^2 + \psi_1^*\psi_2 e^{i(E_1 - E_2)t/\hbar} + \psi_2^*\psi_1 e^{i(E_2 - E_1)t/\hbar} + |\psi_2|^2)\,dx$$

$$= \frac{1}{a}\int_0^a x^2\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right)\right.$$

$$\left. + 2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right]\,dx$$

$$= \frac{a^2}{144\pi^2}[-45 + 48\pi^2 - 256\cos(3\omega t)]$$

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$= \frac{a^2}{4}\left[\frac{1}{3} - \frac{5}{4\pi^2} - \left(\frac{32}{9\pi^2}\right)^2 \cos^2(3\omega t)\right]$$

$$\frac{d\langle x \rangle}{dt} = \frac{16a\omega}{3\pi^2}\sin(3\omega t)$$

$$= \frac{8\hbar}{3ma}\sin(3\omega t)$$

$$\sigma_H^2 \sigma_x^2 \geq \left(\frac{\hbar}{2}\right)^2 \left|\frac{d\langle x \rangle}{dt}\right|^2$$

$$\frac{1}{4}\left(\frac{3\pi^2\hbar^2}{2ma^2}\right)^2 \frac{a^2}{4}\left[\frac{1}{3} - \frac{5}{4\pi^2} - \left(\frac{32}{9\pi^2}\right)^2 \cos^2(3\omega t)\right] \geq \left(\frac{\hbar}{2}\right)^2\left[\frac{8\hbar}{3ma}\sin(3\omega t)\right]^2$$

$$\left(\frac{3}{4}\right)^2\left[\frac{1}{3} - \frac{5}{4\pi^2} - \left(\frac{32}{9\pi^2}\right)^2 \cos^2(3\omega t)\right] \geq \left(\frac{8}{3\pi^2}\right)^2 \sin^2(3\omega t)$$

$$\frac{1}{3} - \frac{5}{4\pi^2} \geq \left(\frac{32}{9\pi^2}\right)^2$$

3.23

$$\hat{P}^2 \ket{\beta} = \hat{P}(\hat{P}\ket{\beta})$$
$$= \hat{P}(\braket{\alpha|\beta}\ket{\alpha})$$
$$= \braket{\alpha|\beta}\hat{P}\ket{\alpha}$$
$$= \braket{\alpha|\beta}(\braket{\alpha|\alpha}\ket{\alpha})$$
$$= \braket{\alpha|\beta}\ket{\alpha}$$
$$= \hat{P}\ket{\beta}$$

So, $\hat{P}^2 = \hat{P}$.

$$\hat{P}\ket{\beta} = \lambda\ket{\beta}$$
$$\braket{\alpha|\beta}\ket{\alpha} = \lambda\ket{\beta}$$

If $\ket{\beta}$ is a constant multiple of $\ket{\alpha}$, then

$$\braket{\alpha|c\alpha}\ket{\alpha} = \lambda c\ket{\alpha}$$
$$c\braket{\alpha|\alpha}\ket{\alpha} = \lambda c\ket{\alpha}$$
$$c\ket{\alpha} = \lambda c\ket{\alpha}$$

thus $\lambda = 1$.
If $\ket{\beta}$ is orthogonal to $\ket{\alpha}$, then

$$\braket{\alpha|\beta}\ket{\alpha} = \lambda\ket{\beta}$$
$$0 = \lambda\ket{\beta}$$

thus $\lambda = 0$.

3.24

$$\hat{Q} = \hat{Q}^\dagger$$
$$Q_{mn} = \braket{e_m|\hat{Q}|e_n}$$
$$Q_{nm} = \braket{e_n|\hat{Q}|e_m}$$
$$Q_{nm}^* = \braket{e_n|\hat{Q}|e_m}^*$$
$$= \braket{e_m|\hat{Q}^\dagger|e_n}$$
$$= \braket{e_m|\hat{Q}|e_n}$$
$$= Q_{mn}$$

3.25

$$\hat{H}|\alpha\rangle = \lambda|\alpha\rangle$$
$$\epsilon[(\langle1|\alpha\rangle + \langle2|\alpha\rangle)|1\rangle + (\langle1|\alpha\rangle - \langle2|\alpha\rangle)|2\rangle] = \lambda(\langle1|\alpha\rangle|1\rangle + \langle2|\alpha\rangle|2\rangle)$$

From this we get two equations

$$\epsilon(\langle1|\alpha\rangle + \langle2|\alpha\rangle) = \lambda\langle1|\alpha\rangle$$
$$\epsilon(\langle1|\alpha\rangle) - \langle2|\alpha\rangle) = \lambda\langle2|\alpha\rangle$$

If we let $|\alpha\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ then this becomes

$$\epsilon(a + b) = \lambda a$$
$$\epsilon(a - b) = \lambda b$$

or in matrix form

$$\epsilon \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda = \pm\sqrt{2}\epsilon$ and the eigenvectors are $|1\rangle + (\sqrt{2} \pm 1)|2\rangle$.

3.26

(a)

$$\langle\alpha| = -i\langle1| - 2\langle2| + i\langle3|$$
$$\langle\beta| = -i\langle1| + 2\langle3|$$

(b)

$$\langle\alpha|\beta\rangle = (-i\langle1| - 2\langle2| + i\langle3|)(i|1\rangle + 2|3\rangle)$$
$$= \langle1|1\rangle - 2i\langle1|3\rangle - 2i\langle2|1\rangle - 4\langle2|3\rangle - \langle3|1\rangle + 2i\langle3|3\rangle$$
$$= 1 + 2i$$
$$\langle\beta|\alpha\rangle = (-i\langle1| + 2\langle3|)(i|1\rangle - 2|2\rangle - i|3\rangle)$$
$$= \langle1|1\rangle + 2i\langle1|2\rangle - \langle1|3\rangle + 2i\langle3|1\rangle - 4\langle3|2\rangle - 2i\langle3|3\rangle$$
$$= 1 - 2i$$
$$= \langle\alpha|\beta\rangle^*$$

(c)

$$\hat{A} = |\alpha\rangle \langle\beta|$$
$$\hat{A}\,|1\rangle = |\alpha\rangle \langle\beta|1\rangle$$
$$= |\alpha\rangle \langle1|\beta\rangle^*$$
$$= -i\,|\alpha\rangle$$
$$= |1\rangle + 2i\,|2\rangle - |3\rangle$$
$$A_{11} = \langle1|\hat{A}|1\rangle$$
$$= 1$$
$$A_{21} = 2i$$
$$A_{31} = -1$$
$$\hat{A}\,|2\rangle = |\alpha\rangle \langle\beta|2\rangle$$
$$= |\alpha\rangle \langle2|\beta\rangle^*$$
$$= 0$$
$$A_{12} = 0$$
$$A_{22} = 0$$
$$A_{32} = 0$$
$$\hat{A}\,|3\rangle = |\alpha\rangle \langle\beta|3\rangle$$
$$= |\alpha\rangle \langle3|\beta\rangle^*$$
$$= 2\,|\alpha\rangle$$
$$= 2i\,|1\rangle - 4\,|2\rangle - 2i\,|3\rangle$$
$$A_{13} = 2i$$
$$A_{23} = -4$$
$$A_{33} = -2i$$
$$A = \begin{pmatrix} 1 & 0 & 2i \\ 2i & 0 & -4 \\ -1 & 0 & -2i \end{pmatrix}$$

It's not hermitian.

3.27

(a)

$$\hat{Q}\,|\alpha\rangle = \hat{Q}\sum_{n=1}^{\infty} \langle e_n|\alpha\rangle\,|e_n\rangle$$

$$= \hat{Q}\left(\sum_{n=1}^{\infty} |e_n\rangle\,\langle e_n|\right)|\alpha\rangle$$

$$= \left(\sum_{n=1}^{\infty} \hat{Q}\,|e_n\rangle\,\langle e_n|\right)|\alpha\rangle$$

$$= \left(\sum_{n=1}^{\infty} q_n\,|e_n\rangle\,\langle e_n|\right)|\alpha\rangle$$

$$\hat{Q} = \sum_{n=1}^{\infty} q_n\,|e_n\rangle\,\langle e_n|$$

(b)

$$\hat{Q} = \sum_{n=1}^{\infty} q_n\,|e_n\rangle\,\langle e_n|$$

$$\hat{Q}^2 = \left(\sum_{n=1}^{\infty} q_n\,|e_n\rangle\,\langle e_n|\right)\left(\sum_{l=1}^{\infty} q_l\,|e_l\rangle\,\langle e_l|\right)$$

$$= \sum_{n=1}^{\infty}\sum_{l=0}^{\infty} q_n q_l\,|e_n\rangle\,\langle e_n|e_l\rangle\,\langle e_l|$$

$$= \sum_{n=1}^{\infty} q_n^2\,|e_n\rangle\,\langle e_n|$$

$$e^{\hat{Q}} = \sum_{n=1}^{\infty} e^{q_n}\,|e_n\rangle\,\langle e_n|$$

$$= \sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty} \frac{q_n^k}{k!}\right)|e_n\rangle\,\langle e_n|$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{n=1}^{\infty} q_n^k\,|e_n\rangle\,\langle e_n|\right)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\hat{Q}^k$$

$$= 1 + \hat{Q} + \frac{1}{2}\hat{Q}^2 + \frac{1}{3!}\hat{Q}^3 + \cdots$$

3.28

(a)

$$\sin \hat{D} = \hat{D} - \frac{\hat{D}^3}{3!} + \frac{\hat{D}^5}{5!} - \frac{\hat{D}^7}{7!} + \cdots$$

$$(\sin \hat{D})x^5 = 5x^4 - 10x^2 + 1$$

(b)

$$\frac{1}{1 - \hat{D}/2} = 1 + \frac{\hat{D}}{2} + \left(\frac{\hat{D}}{2}\right)^2 + \left(\frac{\hat{D}}{2}\right)^3 + \cdots$$

$$= 1 + \frac{\hat{D}}{2} + \frac{\hat{D}^2}{4} + \frac{\hat{D}^3}{8} + \cdots$$

$$\frac{1}{1 - \hat{D}/2} \cos x = \cos x - \frac{1}{2}\sin x - \frac{1}{4}\cos x + \frac{1}{8}\sin x + \cdots$$

$$= \left(-\frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \cdots\right)\sin x + \left(1 - \frac{1}{4} + \frac{1}{16} - \cdots\right)\cos x$$

$$= -\frac{2}{5}\sin x + \frac{4}{5}\cos x$$

3.30

$$c_n(t) = \langle n|S(t)\rangle$$

$$= \left\langle n \left| \int dx \, |x\rangle \langle x| \right| S(t) \right\rangle$$

$$= \int \langle n|x\rangle \langle x|S(t)\rangle \, dx$$

$$= \int \langle x|n\rangle^* \Psi(x,t) \, dx$$

$$= \int \psi_n(x)^* \Psi(x,t) \, dx$$

3.31

$$|e_1\rangle = 1$$

$$\langle e_1|e_1\rangle = \int_{-1}^{1} dx$$

$$= 2$$

$$|e_1'\rangle = \frac{1}{\sqrt{2}}$$

$$|e_2\rangle = x$$

$$\langle e_1'|e_2\rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} x \, dx$$

$$= \frac{1}{\sqrt{2}} \left[\frac{1}{2} x^2 \right]_{-1}^{1}$$

$$= 0$$

$$\langle e_2|e_2\rangle = \int_{-1}^{1} x^2 \, dx$$

$$= \left[\frac{1}{3} x^3 \right]_{-1}^{1}$$

$$= \frac{2}{3}$$

$$|e_2'\rangle = \sqrt{\frac{3}{2}} x$$

$$|e_3\rangle = x^2$$

$$\langle e_1'|e_3\rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} x^2 \, dx$$

$$= \frac{1}{\sqrt{2}} \left[\frac{1}{3} x^3 \right]_{-1}^{1}$$

$$= \frac{\sqrt{2}}{3}$$

$$\langle e_2'|e_3\rangle = \int_{-1}^{1} \sqrt{\frac{3}{2}} x^3 \, dx$$

$$= 0$$

$$|e_3'\rangle = x^2 - \frac{\sqrt{2}}{3} |e_1'\rangle$$

$$= x^2 - \frac{1}{3}$$

$$\langle e_3'|e_3'\rangle = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx$$

$$= \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx$$

$$= \left[\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x\right]_{-1}^{1}$$

$$= \frac{2}{5} - \frac{4}{9} + \frac{2}{9}$$

$$= \frac{18}{45} - \frac{20}{45} + \frac{10}{45}$$

$$= \frac{8}{45}$$

$$|e_3''\rangle = \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3}\right)$$

$$= \sqrt{\frac{5}{2}} \left(\frac{3}{2}x^2 - \frac{1}{2}\right)$$

$$|e_4\rangle = x^3$$

$$\langle e_1'|e_4\rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}}x^3 \, dx$$

$$= 0$$

$$\langle e_2'|e_4\rangle = \int_{-1}^{1} \sqrt{\frac{3}{2}}x^4 \, dx$$

$$= \sqrt{\frac{3}{2}} \left[\frac{1}{5}x^5\right]$$

$$= \frac{\sqrt{6}}{5}$$

$$\langle e_3''|e_4\rangle = \int_{-1}^{1} \sqrt{\frac{5}{2}} \left(\frac{3}{2}x^2 - \frac{1}{2}\right) x^3 \, dx$$

$$= \sqrt{\frac{5}{2}} \int_{-1}^{1} \left(\frac{3}{2}x^5 - \frac{1}{2}x^3\right) dx$$

$$= 0$$

$$|e_4'\rangle = |e_4\rangle - \frac{\sqrt{6}}{5}|e_2'\rangle$$

$$= x^3 - \frac{3}{5}x$$

$$\langle e_4'|e_4'\rangle = \int_{-1}^{1}\left(x^3 - \frac{3}{5}x\right)^2 dx$$

$$= \int_{-1}^{1}\left(x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2\right) dx$$

$$= \left[\frac{1}{7}x^7 - \frac{6}{25}x^5 + \frac{3}{25}x^3\right]_{-1}^{1}$$

$$= \frac{2}{7} - \frac{12}{25} + \frac{6}{25}$$

$$= \frac{50}{175} - \frac{84}{175} + \frac{42}{175}$$

$$= \frac{8}{175}$$

$$|e_4''\rangle = \sqrt{\frac{175}{8}}\left(x^3 - \frac{3}{5}x\right)$$

$$= \sqrt{\frac{7}{2}}\left(\frac{5}{2}x^3 - \frac{3}{2}x\right)$$

3.32

(a)

$$\langle\hat{Q}\rangle = \langle\Psi|\hat{Q}|\Psi\rangle$$

$$= \langle\Psi|\hat{Q}^\dagger|\Psi\rangle^*$$

$$= -\langle\Psi|\hat{Q}|\Psi\rangle^*$$

$$= -\langle\hat{Q}\rangle^*$$

(b)

$$\hat{Q}|\psi\rangle = \lambda|\psi\rangle$$

$$\langle\psi|\hat{Q}|\psi\rangle = \langle\psi|\lambda|\psi\rangle$$

$$= \lambda\langle\psi|\psi\rangle$$

$$\langle\psi|\hat{Q}|\psi\rangle^* = \lambda^*\langle\psi|\psi\rangle^*$$

$$\langle\psi|\hat{Q}^\dagger|\psi\rangle = \lambda^*\langle\psi|\psi\rangle$$

$$-\langle\psi|\hat{Q}|\psi\rangle = \lambda^*\langle\psi|\psi\rangle$$

$$\langle\psi|\hat{Q}|\psi\rangle = -\lambda^*\langle\psi|\psi\rangle$$

$$\hat{Q}|\psi\rangle = -\lambda^*|\psi\rangle$$

(c)

$$\hat{Q}\,|f\rangle = \lambda_1\,|f\rangle$$
$$\hat{Q}\,|g\rangle = \lambda_2\,|g\rangle$$
$$\langle g|\hat{Q}|f\rangle = \langle g|\lambda_1|f\rangle$$
$$= \lambda_1\,\langle g|f\rangle$$
$$\langle f|\hat{Q}^\dagger|g\rangle = \lambda_1^*\,\langle f|g\rangle$$
$$-\langle f|\hat{Q}|g\rangle = \lambda_1^*\,\langle f|g\rangle$$
$$\langle f|\lambda_2|g\rangle = -\lambda_1^*\,\langle f|g\rangle$$
$$\lambda_2\,\langle f|g\rangle = -\lambda_1^*\,\langle f|g\rangle$$
$$\lambda_2 = -\lambda_1^*$$
$$\lambda_2 = \lambda_1$$
$$\langle f|g\rangle = 0$$

(d)

$$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$
$$[\hat{A},\hat{B}]^\dagger = (\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger$$
$$= \hat{A}^\dagger\hat{B}^\dagger - \hat{B}^\dagger\hat{A}^\dagger$$
$$= -(\hat{A}\hat{B} - \hat{B}\hat{A})$$
$$= -[\hat{A},\hat{B}]$$
$$[\hat{A},\hat{B}]^\dagger = \hat{A}^\dagger\hat{B}^\dagger - \hat{B}^\dagger\hat{A}^\dagger$$
$$= -(\hat{A}\hat{B} - \hat{B}\hat{A})$$
$$= -[\hat{A},\hat{B}]$$

(e)

$$\hat{Q}\,|q_n\rangle = \lambda_n\,|q_n\rangle$$
$$= (x + iy)\,|q_n\rangle$$
$$= x\,|q_n\rangle + iy\,|q_n\rangle$$
$$= \hat{X}\,|q_n\rangle + \hat{Y}\,|q_n\rangle$$
$$= (\hat{X} + \hat{Y})\,|q_n\rangle$$

3.33

(a) ψ_1

(b) b_1 and b_2 with $P(b_1) = \frac{9}{25}$ and $P(b_2) = \frac{16}{25}$.

(c)

$$\phi_1 = \frac{3}{5}\psi_1 + \frac{4}{5}\psi_2$$

$$\phi_2 = \frac{4}{5}\psi_1 - \frac{3}{5}\psi_2$$

$$P(a_1) = P(b_1)\left(\frac{3}{5}\right)^2 + P(b_2)\left(\frac{4}{5}\right)^2$$

$$= \left(\frac{9}{25}\right)^2 + \left(\frac{16}{25}\right)^2$$

$$= \frac{337}{625}$$

$$\approx 53.9\%$$

3.34

(a)

$$\Phi_n(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_0^a e^{-ipx/\hbar} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-iE_nt/\hbar} \, dx$$

$$= \frac{1}{\sqrt{\pi\hbar a}} e^{-iE_nt/\hbar} \int_0^a e^{-ipx/\hbar} \sin\left(\frac{n\pi}{a}x\right) \, dx$$

$$= \sqrt{\frac{a\pi}{\hbar}} \frac{ne^{-iE_nt/\hbar}}{(n\pi)^2 - (ap/\hbar)^2}[1 - (-1)^n e^{-ipa/\hbar}]$$

(b)

$$|\Phi_n(p,t)|^2 = \frac{a\pi}{\hbar} \frac{4n^2}{[(n\pi)^2 - (ap/\hbar)^2]^2} \begin{cases} \cos^2\left(\frac{a}{2\hbar}p\right) & n \text{ odd} \\ \sin^2\left(\frac{a}{2\hbar}p\right) & n \text{ even} \end{cases}$$

3.35

$$\Psi(x,0) = \begin{cases} \frac{1}{\sqrt{2n\lambda}} e^{i2\pi x/\lambda} & -n\lambda < x < n\lambda \\ 0 & \text{otherwise} \end{cases}$$

$$\Phi(p,0) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \Psi(x,0)\, dx$$

$$= \frac{1}{2\sqrt{\pi\hbar n\lambda}} \int_{-n\lambda}^{n\lambda} e^{i(2\pi/\lambda - p/\hbar)x}\, dx$$

$$= \frac{1}{2\sqrt{\pi\hbar n\lambda}} \frac{1}{i(2\pi/\lambda - p/\hbar)} \left[e^{i(2\pi/\lambda - p/\hbar)x} \right]_{-n\lambda}^{n\lambda}$$

$$= \frac{1}{2\sqrt{\pi\hbar n\lambda}} \frac{1}{i(2\pi/\lambda - p/\hbar)} e^{i(2\pi/\lambda - p/\hbar)n\lambda} - e^{-i(2\pi/\lambda - p/\hbar)n\lambda}$$

$$= \frac{1}{\sqrt{\pi\hbar n\lambda}} \frac{1}{2\pi/\lambda - p/\hbar} \sin\left[\left(\frac{2\pi}{\lambda} - \frac{p}{\hbar}\right)n\lambda\right]$$

$$= \sqrt{\frac{\lambda\hbar}{n\pi}} \frac{1}{\lambda p - 2\pi\hbar} \sin\left(\frac{n\lambda}{\hbar}p\right)$$

$$w_x = 2n\lambda$$

$$w_p = \frac{2\pi\hbar}{n\lambda}$$

As $n \to \infty$, $w_x \to \infty$ and $w_p \to 0$.

$$w_x w_p = 2n\lambda \frac{2\pi\hbar}{n\lambda}$$

$$= 4\pi\hbar$$

$$\geq \frac{\hbar}{2}$$

3.36

(a)

$$1 = \int_{-\infty}^{\infty} \left(\frac{A}{x^2 + a^2}\right)^2 dx$$

$$= |A|^2 \int_{-\infty}^{\infty} \frac{1}{(x^2 + a^2)^2}\, dx$$

$$= \frac{\pi|A|^2}{2a^3}$$

$$A = a\sqrt{\frac{2a}{\pi}}$$

74

(b)

$$\langle x \rangle = \langle \Psi | x | \Psi \rangle$$

$$= \int_{-\infty}^{\infty} \Psi^* x \Psi \, dx$$

$$= \frac{a^3}{\pi} \int_{-\infty}^{\infty} \frac{2x}{(x^2 + a^2)^2} \, dx$$

$$u = x^2 + a^2$$

$$du = 2x \, dx$$

$$\langle x \rangle = \frac{a^3}{\pi} \int_{-\infty}^{\infty} \frac{1}{u^2}$$

$$= \frac{a^3}{\pi} \left[-\frac{1}{u} \right]_{-\infty}^{\infty}$$

$$= 0$$

$$\langle x^2 \rangle = \langle \Psi | x^2 \Psi \rangle$$

$$= \int_{-\infty}^{\infty} \Psi^* x^2 \Psi \, dx$$

$$= \frac{2a^3}{\pi} \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)^2} \, dx$$

$$= a^2$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$= a$$

(c)

$$\Phi(x, 0) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} a \sqrt{\frac{2a}{\pi}} \frac{1}{x^2 + a^2} \, dx$$

$$= \frac{a\sqrt{a}}{\pi\sqrt{\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \frac{1}{x^2 + a^2} \, dx$$

$$= \sqrt{\frac{a}{\hbar}} e^{-|p|a/\hbar}$$

$$\frac{a}{\hbar} \int_{-\infty}^{\infty} e^{-2|p|a/\hbar} \, dp = 1$$

(d)

$$\langle p \rangle = \langle \Phi | p | \Phi \rangle$$

$$= \int_{-\infty}^{\infty} \Phi^* p \Psi \, dp$$

$$= \frac{a}{\hbar} \int_{-\infty}^{\infty} p e^{-2|p|a/\hbar} \, dp$$

$$= 0$$

$$\langle p^2 \rangle = \frac{a}{\hbar} \int_{-\infty}^{\infty} p^2 e^{-2|p|a/\hbar} \, dp$$

$$= \frac{\hbar^2}{2a^2}$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

$$= \frac{\hbar}{\sqrt{2}a}$$

(e)

$$\sigma_x \sigma_p = \frac{\hbar}{\sqrt{2}}$$

$$\geq \frac{\hbar}{2}$$

3.37

$$[\hat{H}, xp] = x[\hat{H}, p] + [\hat{H}, x]p$$

$$= i\hbar x \frac{dV}{dx} - \frac{i\hbar p^2}{m}$$

$$\frac{d}{dt} \langle xp \rangle = \frac{i}{\hbar} \langle [\hat{H}, xp] \rangle$$

$$= \frac{i}{\hbar} \left\langle i\hbar x \frac{dV}{dx} - \frac{i\hbar p^2}{m} \right\rangle$$

$$= \left\langle \frac{p^2}{m} \right\rangle - \left\langle x \frac{dV}{dx} \right\rangle$$

$$= 2 \langle T \rangle - \left\langle x \frac{dV}{dx} \right\rangle$$

The left hand side is 0 because expectation values are constant in stationary states.

$$V = \frac{1}{2}m\omega^2 x^2$$

$$\frac{dV}{dx} = m\omega^2 x$$

$$x\frac{dV}{dx} = m\omega^2 x$$

$$= 2V$$

$$2\langle T \rangle = \left\langle x\frac{dV}{dx} \right\rangle$$

$$= \langle 2V \rangle$$

$$= 2\langle V \rangle$$

$$\langle T \rangle = \langle V \rangle$$

3.38

$$\Psi(x,0) = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$$

$$\Psi(x,t) = \frac{1}{\sqrt{2}}(\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar})$$

$$0 = \int_{-\infty}^{\infty} \Psi(x,0)\Psi(x,t)\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} (\psi_1 + \psi_2)(\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar})\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} (\psi_1^2 e^{-iE_1 t/\hbar} + \psi_1\psi_2 e^{-iE_2 t/\hbar} + \psi_1\psi_2 e^{-iE_1 t/\hbar} + \psi_2^2 e^{-iE_2 t/\hbar})\,dx$$

$$= \frac{1}{2}(e^{-iE_1 t/\hbar} + e^{-iE_2 t/\hbar})$$

$$= e^{-iE_1 t/\hbar} + e^{-iE_2 t/\hbar}$$

$$e^{-iE_1 t/\hbar} = -e^{-iE_2 t/\hbar}$$

$$= e^{\pi i} e^{-iE_2 t/\hbar}$$

$$-\frac{iE_1 t}{\hbar} = i\left(\pi - \frac{E_2 t}{\hbar}\right)$$

$$-\frac{E_1 t}{\hbar} = \pi - \frac{E_2 t}{\hbar}$$

$$t = \frac{\pi\hbar}{E_2 - E_1}$$

$$\Delta t = \frac{t}{\pi}$$

$$= \frac{\hbar}{E_2 - E_1}$$

$$\Delta E = \frac{1}{2}(E_2 - E_1)$$

$$\Delta E \Delta t = \frac{\hbar}{2}$$

3.41

$$\Psi = \frac{e^{i\theta_0}}{\sqrt{2}}\psi_0 e^{-i\omega t/2} + \frac{e^{i\theta_1}}{\sqrt{2}}\psi_1 e^{-3i\omega t/2}$$

$$\langle p \rangle = \langle \Psi | p | \Psi \rangle$$

$$= \frac{1}{2}\langle \psi_0 | p | \psi_0 \rangle + \frac{1}{2}e^{i(\theta_1 - \theta_0)}e^{-i\omega t}\langle \psi_0 | p | \psi_1 \rangle$$

$$\quad + \frac{1}{2}e^{i(\theta_0 - \theta_1)}e^{i\omega t}\langle \psi_1 | p | \psi_0 \rangle + \frac{1}{2}\langle \psi_1 | p | \psi_1 \rangle$$

$$\langle \psi_0 | p | \psi_0 \rangle = \frac{d}{dt}\langle \psi_0 | x | \psi_0 \rangle$$

$$= 0$$

$$\langle \psi_1 | p | \psi_1 \rangle = \frac{d}{dt}\langle \psi_1 | x | \psi_1 \rangle$$

$$= 0$$

$$\langle p \rangle = \frac{1}{2}e^{i(\theta_1 - \theta_0 - \omega t)}\langle \psi_0 | p | \psi_1 \rangle + \frac{1}{2}e^{-i(\theta_1 - \theta_0 - \omega t)}\langle \psi_1 | p | \psi_0 \rangle$$

$$\langle \psi_0 | p | \psi_1 \rangle = -i\sqrt{\frac{\hbar m \omega}{2}}$$

$$\langle \psi_1 | p | \psi_0 \rangle = i\sqrt{\frac{\hbar m \omega}{2}}$$

$$\langle p \rangle = -\frac{1}{2}ie^{i(\theta_1 - \theta_0 - \omega t)}\sqrt{\frac{\hbar m \omega}{2}} + \frac{1}{2}ie^{-i(\theta_1 - \theta_0 - \omega t)}\sqrt{\frac{\hbar m \omega}{2}}$$

$$= -\frac{1}{2}i\sqrt{\frac{\hbar m \omega}{2}}(e^{i(\theta_1 - \theta_0 - \omega t)} - e^{-i(\theta_1 - \theta_0 - \omega t)})$$

$$= \sqrt{\frac{\hbar m \omega}{2}}\sin(\theta_1 - \theta_0 - \omega t)$$

The largest possible value of $\langle p \rangle$ is $\sqrt{\hbar m \omega / 2}$. If it takes on this value at $t = 0$ then

$$\Psi(x,t) = \frac{1}{\sqrt{2}}e^{-i\omega t/2}(\psi_0 + i\psi_1 e^{-i\omega t}).$$

3.43

(a)

$$|z|^2 = \text{Re}(z)^2 + \text{Im}(z)^2$$

$$= \left(\frac{z + z^*}{2}\right)^2 + \left(\frac{z - z^*}{2i}\right)^2$$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{\langle f|g\rangle + \langle g|f\rangle}{2}\right)^2 + \left(\frac{\langle f|g\rangle - \langle g|f\rangle}{2i}\right)^2$$

$$= \left(\frac{\langle \hat{A}\hat{B}\rangle + \langle \hat{B}\hat{A}\rangle - 2\langle A\rangle\langle B\rangle}{2}\right)^2 + \left(\frac{\langle [\hat{A}, \hat{B}]\rangle}{2i}\right)^2$$

$$= \frac{1}{4}[\langle -i[\hat{A}, \hat{B}]\rangle^2 + (\langle \hat{A}\hat{B}\rangle + \langle \hat{B}\hat{A}\rangle - 2\langle A\rangle\langle B\rangle)^2]$$

$$= \frac{1}{4}(\langle C\rangle^2 + \langle D\rangle^2)$$

(b)

$$B = A$$

$$\hat{C} = -i[\hat{A}, \hat{B}]$$

$$= 0$$

$$\hat{D} = 2(\hat{A}^2 - \langle A\rangle^2)$$

$$\langle D\rangle = \langle \Psi|D|\Psi\rangle$$

$$= \langle \Psi|2(\hat{A}^2 - \langle A\rangle^2)|\Psi\rangle$$

$$= 2(\langle \Psi|\hat{A}^2|\Psi\rangle - \langle A\rangle^2)$$

$$= 2(\langle A^2\rangle - \langle A\rangle^2)$$

$$= 2\sigma_A^2$$

$$\sigma_A^2 \sigma_B^2 = \frac{1}{4}(\langle C\rangle^2 + \langle D\rangle^2)$$

$$\sigma_A^4 \geq \sigma_A^4$$

3.44

(a) The eigenvalues of **H** are $a - b$, $a + b$, and c and the associated eigenvectors

are $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, respectively.

$$|S(t)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-ict/\hbar}$$

(b)

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right]$$

$$|S(t)\rangle = \frac{1}{2} \left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-i(a+b)t/\hbar} - \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-i(a-b)t/\hbar} \right]$$

$$= \frac{1}{2} e^{-iat/\hbar} \begin{pmatrix} e^{-ibt/\hbar} + e^{ibt/\hbar} \\ 0 \\ e^{-ibt/\hbar} - e^{ibt/\hbar} \end{pmatrix}$$

$$= e^{-iat/\hbar} \begin{pmatrix} \cos(bt/\hbar) \\ 0 \\ -i\sin(bt/\hbar) \end{pmatrix}$$

3.45

$$\langle n|\hat{x}|S(t)\rangle = \left\langle n \left| \hat{x} \sum_{n'=0}^{\infty} |n'\rangle \langle n'| \right| S(t) \right\rangle$$

$$= \sum_{n'=0}^{\infty} \langle n|\hat{x}|n'\rangle \langle n'|S(t)\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \sum_{n'=0}^{\infty} (\sqrt{n'}\delta_{n,n'-1} + \sqrt{n}\delta_{n',n-1})c_{n'}(t)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} [\sqrt{n+1}c_{n+1}(t) + \sqrt{n}c_{n-1}(t)]$$

3.46

(a)

$$h_1 = \hbar\omega$$
$$h_2 = 2\hbar\omega$$
$$h_3 = 2\hbar\omega$$
$$|h_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$|h_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$|h_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$a_1 = 2\lambda$$
$$a_2 = \lambda$$
$$a_3 = -\lambda$$
$$|a_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$|a_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
$$|a_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$
$$b_1 = 2\mu$$
$$b_2 = \mu$$
$$b_3 = -\mu$$
$$|b_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$|b_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$
$$|b_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

(b)

$$\langle H \rangle = \langle S(0)|H|S(0) \rangle$$

$$= \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= \hbar\omega \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} c_1 \\ 2c_2 \\ 2c_3 \end{pmatrix}$$

$$= \hbar\omega(|c_1|^2 + 2|c_2|^2 + 2|c_3|^2)$$

$$\langle A \rangle = \lambda(c_1^* c_2 + c_2^* c_1 + 2|c_3|^2)$$

$$\langle B \rangle = \mu(2|c_1|^2 + c_2^* c_3 + c_3^* c_2)$$

(c)

$$|S(t)\rangle = c_1 |h_1\rangle e^{-i\omega t} + c_2 |h_2\rangle e^{-2i\omega t} + c_3 |h_3\rangle e^{-2i\omega t}$$

You could measure H as $\hbar\omega$ with probability $|c_1|^2$ or $2\hbar\omega$ with probability $|c_2|^2 + |c_3|^2$.

4 Quantum Mechanics in Three Dimensions

4.1

(a)

$$[r_i, r_j] = 0$$
$$[p_i, p_j] = 0$$
$$[x, p_x]f = x\left(-i\hbar\frac{\partial}{\partial x}\right)f - \left(-i\hbar\frac{\partial}{\partial x}\right)(xf)$$
$$= -i\hbar x\frac{\partial f}{\partial x} + i\hbar\left(f + x\frac{\partial f}{\partial x}\right)$$
$$= i\hbar f$$
$$[p_x, x] = \left(-i\hbar\frac{\partial}{\partial x}\right)(xf) - x\left(-i\hbar\frac{\partial}{\partial x}\right)f$$
$$= -i\hbar\left(f + x\frac{\partial f}{\partial x}\right) + i\hbar x\frac{\partial f}{\partial x}$$
$$= -i\hbar f$$
$$[x, p_y] = x\left(-i\hbar\frac{\partial}{\partial y}\right)f - \left(-i\hbar\frac{\partial}{\partial y}\right)(xf)$$
$$= -i\hbar x\frac{\partial f}{\partial y} + i\hbar x\frac{\partial f}{\partial y}$$
$$= 0$$
$$[r_i, p_j] = i\hbar\delta_{ij}$$
$$[p_i, r_j] = -i\hbar\delta_{ij}$$

(b)

$$[\hat{H}, x]f = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)(xf) - x\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)f$$

$$= -\frac{\hbar^2}{2m}\left[\frac{\partial}{\partial x}\left(f + x\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(x\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial x}\left(z\frac{\partial f}{\partial z}\right)\right]$$

$$\qquad + Vxf + \frac{\hbar^2}{2m}x\nabla^2 f - Vxf$$

$$= -\frac{\hbar^2}{2m}\left(2\frac{\partial f}{\partial x} + x\nabla^2 f\right) + \frac{\hbar^2}{2m}x\nabla^2 f$$

$$= -\frac{\hbar^2}{m}\frac{\partial f}{\partial x}$$

$$[\hat{H}, x] = -\frac{\hbar^2}{m}\frac{\partial}{\partial x}$$

$$[\hat{H}, \mathbf{r}] = -\frac{\hbar^2}{m}\nabla$$

$$\frac{d}{dt}\langle\mathbf{r}\rangle = \frac{i}{\hbar}\langle[\hat{H}, \mathbf{r}]\rangle$$

$$= \frac{1}{m}\langle-i\hbar\nabla\rangle$$

$$= \frac{1}{m}\langle\mathbf{p}\rangle$$

$$[\hat{H}, \hat{p}_x]f = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\left(-i\hbar\frac{\partial}{\partial x}\right)f - \left(-i\hbar\frac{\partial}{\partial x}\right)\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)f$$

$$= -i\hbar\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\frac{\partial f}{\partial x} + i\hbar\left(\frac{\partial}{\partial x}\right)\left(-\frac{\hbar^2}{2m}\nabla^2 f + Vf\right)$$

$$= i\frac{\hbar^3}{2m}\nabla^2\frac{\partial f}{\partial x} - i\hbar V\frac{\partial f}{\partial x} - i\frac{\hbar^3}{2m}\nabla^2\frac{\partial f}{\partial x} + i\hbar\left(\frac{\partial V}{\partial x}f + V\frac{\partial f}{\partial x}\right)$$

$$= i\hbar\frac{\partial V}{\partial x}f$$

$$[\hat{H}, \hat{p}_x] = i\hbar\frac{\partial V}{\partial x}$$

$$[\hat{H}, \mathbf{p}] = i\hbar\nabla V$$

$$\frac{d}{dt}\langle\mathbf{p}\rangle = \frac{i}{\hbar}\langle[\hat{H}, \mathbf{p}]\rangle$$

$$= \langle-\nabla V\rangle$$

(c)

$$\sigma_{r_i}\sigma_{p_j} \geq \frac{1}{2i}\langle[r_i,p_j]\rangle$$
$$= \frac{1}{2i}i\hbar\delta_{ij}$$
$$= \frac{\hbar}{2}\delta_{ij}$$

4.2

(a)

$$\psi(\mathbf{r}) = X(x)Y(y)Z(z)$$
$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$$
$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 X}{\partial x^2}YZ + X\frac{\partial^2 Y}{\partial y^2}Z + XY\frac{\partial^2 Z}{\partial z^2}\right) = EXYZ$$
$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -\frac{2m}{\hbar^2}E$$

The terms on the left hand side are each functions of a different variable, so they must be constant. Starting with the X term:

$$\frac{X''}{X} = -\alpha$$
$$X'' + \alpha X = 0$$

If $\alpha < 0$

$$X = A_x e^{\sqrt{-\alpha}x} + B_x e^{-\sqrt{-\alpha}x}$$

If $\alpha = 0$

$$X = A_x x + B_x$$

If $\alpha > 0$

$$X = A_x\sin(\sqrt{\alpha}x) + B_x\cos(\sqrt{\alpha}x)$$

Boundary conditions require that $\alpha > 0$. $X(0) = 0$ so $B_x = 0$. $X(a) = 0$ so $\sqrt{\alpha} = n_x\pi/a \Rightarrow \alpha = n_x^2\pi^2/a^2$,

$$X = A_x\sin\left(\frac{n_x\pi}{a}x\right).$$

Repeating the above for Y and Z finds

$$Y = A_y \sin\left(\frac{n_y \pi}{a} y\right)$$

$$Z = A_z \sin\left(\frac{n_z \pi}{a} z\right)$$

so

$$\psi(\mathbf{r}) = A_x A_y A_z \sin\left(\frac{n_x \pi}{a} x\right) \sin\left(\frac{n_y \pi}{a} y\right) \sin\left(\frac{n_z \pi}{a} z\right).$$

Assuming $A_x = A_y = A_z = A$ and normalising finds

$$1 = \int_0^a \int_0^a \int_0^a A^6 \sin^2\left(\frac{n_x \pi}{a} x\right) \sin^2\left(\frac{n_y \pi}{a} y\right) \sin^2\left(\frac{n_z \pi}{a} z\right) d^3\mathbf{r}$$

$$= A^6 \frac{a^3}{8}$$

$$A^6 = \frac{8}{a^3}$$

$$= \left(\frac{2}{a}\right)^3$$

$$A = \sqrt{\frac{2}{a}}$$

so

$$\psi(\mathbf{r}) = \left(\frac{2}{a}\right)^{3/2} \sin\left(\frac{n_x \pi}{a} x\right) \sin\left(\frac{n_y \pi}{a} y\right) \sin\left(\frac{n_z \pi}{a} z\right).$$

Finally

$$-\frac{2m}{\hbar^2} E = -\alpha - \beta - \gamma$$

$$= -\frac{\pi^2 n_x^2}{a^2} - \frac{\pi^2 n_y^2}{a^2} - \frac{\pi^2 n_z^2}{a^2}$$

$$E = \frac{\pi^2 \hbar^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2), \; n_x, n_y, n_z = 1, 2, 3, \ldots$$

(b)

$$E_1 = 3\frac{\pi^2\hbar^2}{2ma^2}, \, n = 1$$

$$E_2 = 6\frac{\pi^2\hbar^2}{2ma^2}, \, n = 3$$

$$E_3 = 9\frac{\pi^2\hbar^2}{2ma^2}, \, n = 3$$

$$E_4 = 11\frac{\pi^2\hbar^2}{2ma^2}, \, n = 3$$

$$E_5 = 12\frac{\pi^2\hbar^2}{2ma^2}, \, n = 1$$

$$E_6 = 14\frac{\pi^2\hbar^2}{2ma^2}, \, n = 6$$

(c)

$$E_{14} = 27\frac{\pi^2\hbar^2}{2ma^2}$$

$$d = 4$$

4.3

(a)

$$\phi(r, \theta, \phi) = Ae^{-r/a}$$

$$\frac{\partial \psi}{\partial r} = -\frac{A}{a}e^{-r/a}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) = \frac{\partial}{\partial r}\left(-\frac{A}{a}r^2 e^{-r/a}\right)$$

$$= -\frac{A}{a}\left(2re^{-r/a} - \frac{1}{a}r^2 e^{-r/a}\right)$$

$$= \frac{A}{a^2}r^2\left(1 - \frac{2a}{r}\right)e^{-r/a}$$

$$\frac{\partial \psi}{\partial \theta} = 0$$

$$\frac{\partial^2 \psi}{\partial \phi^2} = 0$$

$$-\frac{\hbar^2}{2m}\frac{A}{a^2}\left(1 - \frac{2a}{r}\right)e^{-r/a} + V(r)Ae^{-r/a} = EAe^{-r/a}$$

$$-\frac{\hbar^2}{2ma^2}\left(1 - \frac{2a}{r}\right) + V(r) = E$$

$$-\frac{\hbar^2}{2ma^2} = E$$

$$V(r) = -\frac{\hbar^2}{mar}$$

(b)

$$\phi(r, \theta, \phi) = Ae^{-r^2/a^2}$$

$$\frac{\partial \psi}{\partial r} = -\frac{2}{a^2}rAe^{-r^2/a^2}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) = \frac{\partial}{\partial r}\left(-\frac{2}{a^2}r^3 Ae^{-r^2/a^2}\right)$$

$$= -\frac{2A}{a^2}\left(3r^2 e^{-r^2/a^2} - \frac{2}{a^2}r^4 e^{-r^2/a^2}\right)$$

$$= \frac{2A}{a^2}r^2\left(\frac{2}{a^2}r^2 - 3\right)e^{-r^2/a^2}$$

$$-\frac{\hbar^2}{2m}\frac{2A}{a^2}\left(\frac{2}{a^2}r^2 - 3\right)e^{-r^2/a^2} + V(r)Ae^{-r^2/a^2} = EAe^{-r^2/a^2}$$

$$-\frac{\hbar^2}{2m}\frac{2}{a^2}\left(\frac{2}{a^2}r^2 - 3\right) + V(r) = E$$

$$\frac{\hbar^2}{2m}\frac{2}{a^2}3 = E$$

$$\frac{3\hbar^2}{ma^2} = E$$

$$\frac{3\hbar^2}{ma^2} + \frac{\hbar^2}{2m}\frac{2}{a^2}\left(\frac{2}{a^2}r^2 - 3\right) = V(r)$$

$$\frac{2\hbar^2}{ma^4}r^2 = V(r)$$

4.4

$$Y_0^0 = \frac{1}{2\sqrt{\pi}}$$

$$Y_2^1 = \frac{1}{2}\sqrt{\frac{5}{6\pi}}e^{i\phi}P_2^1(\cos\theta)$$

$$= \frac{1}{2}\sqrt{\frac{5}{6\pi}}e^{i\phi}(-1)^1(1-\cos^2\theta)^{1/2}\left(\frac{d}{dx}\right)P_2(\cos\theta)$$

$$= -\frac{1}{2}\sqrt{\frac{5}{6\pi}}e^{i\phi}\sin\theta\left(\frac{d\theta}{dx}\frac{d}{d\theta}\right)\frac{1}{2}(3\cos^2\theta - 1)$$

$$= -\frac{1}{4}\sqrt{\frac{5}{6\pi}}e^{i\phi}\sin\theta\left(-\frac{1}{\sin\theta}\right)(-6\cos\theta\sin\theta)$$

$$= -\frac{1}{2}\sqrt{\frac{15}{2\pi}}e^{i\phi}\sin\theta\cos\theta$$

$$\int |Y_0^0|^2\, d\Omega = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\sin\theta\, d\theta\, d\phi$$

$$= 1$$

$$\int |Y_2^1|^2\, d\Omega = \frac{15}{8\pi}\int_0^\pi\int_0^{2\pi}\sin^3\theta\cos^2\theta\, d\theta\, d\phi$$

$$= 1$$

$$\int (Y_0^0)^* Y_2^1\, d\Omega = -\frac{1}{2\sqrt{\pi}}\frac{1}{2}\sqrt{\frac{15}{2\pi}}\int_0^\pi\int_0^{2\pi}e^{i\phi}\sin^2\theta\cos\theta\, d\theta\, d\phi$$

$$= 0$$

4.5

$$\Theta(\theta) = A \ln\left(\tan\frac{\theta}{2}\right)$$

$$0 = \sin\theta \frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$

$$= \frac{1}{2}A\sin\theta\frac{d}{d\theta}\left(\sin\theta\csc\frac{\theta}{2}\sec\frac{\theta}{2}\right)$$

$$= \frac{1}{2}A\sin\theta\left(\cos\theta\csc\frac{\theta}{2}\sec\frac{\theta}{2} - \frac{1}{2}\csc^2\frac{\theta}{2}\sin\theta + \frac{1}{2}\sec^2\frac{\theta}{2}\sin\theta\right)$$

$$= 0$$

It's not a valid physical solution because it blows up at $\theta = 0$ and $\theta = \pi$.

4.6

$$Y_\ell^{-m} = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell+m)!}{(\ell-m)!}}e^{-im\phi}P_\ell^{-m}(\cos\theta)$$

$$= \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell+m)!}{(\ell-m)!}}e^{-im\phi}(-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(\cos\theta)$$

$$= (-1)^m\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}e^{-im\phi}P_\ell^m(\cos\theta)$$

$$= (-1)^m(Y_\ell^m)^*$$

4.7

$$Y_3^2(\theta, \phi) = \sqrt{\frac{7}{480\pi}} e^{2i\phi} P_3^2(\cos\theta)$$

$$= \frac{1}{4}\sqrt{\frac{105}{2\pi}} e^{2i\phi} \sin^2\theta \cos\theta$$

$$Y_\ell^\ell(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi(2\ell)!}} e^{i\ell\phi} P_\ell^\ell(\cos\theta)$$

$$P_\ell^\ell(x) = (-1)^\ell(1-x^2)^{\ell/2}\left(\frac{d}{dx}\right)^\ell P_\ell(x)$$

$$= (-1)^\ell(1-x^2)^{\ell/2}\left(\frac{d}{dx}\right)^\ell\left[\frac{1}{2^\ell\ell!}\left(\frac{d}{dx}\right)^\ell(x^2-1)^\ell\right]$$

$$= (-1)^\ell\frac{1}{2^\ell\ell!}(1-x^2)^{\ell/2}\left(\frac{d}{dx}\right)^{2\ell}(x^2-1)^\ell$$

$$= (-1)^\ell\frac{(2\ell)!}{2^\ell\ell!}(1-x^2)^{\ell/2}$$

$$Y_\ell^\ell(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi(2\ell)!}} e^{i\ell\phi}(-1)^\ell\frac{(2\ell)!}{2^\ell\ell!}(1-\cos^2\theta)^{\ell/2}$$

$$= \frac{1}{\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\left(-\frac{1}{2}e^{i\phi}\sin\theta\right)^\ell$$

4.9

(a)

$$n_1(x) = -(-x)\left(\frac{1}{x}\frac{d}{dx}\right)\frac{\cos x}{x}$$

$$= -\frac{\sin x}{x} - \frac{\cos x}{x^2}$$

$$n_2(x) = -(-x)^2\left(\frac{1}{x}\frac{d}{dx}\right)^2\frac{\cos x}{x}$$

$$= -x^2\left(\frac{1}{x}\frac{d}{dx}\right)\left[\frac{1}{x}\frac{d}{dx}\left(\frac{\cos x}{x}\right)\right]$$

$$= -x^2\left(\frac{1}{x}\frac{d}{dx}\right)\left[\frac{1}{x}\left(-\frac{\sin x}{x} - \frac{\cos x}{x^2}\right)\right]$$

$$= x\frac{d}{dx}\left(\frac{\sin x}{x^2} + \frac{\cos x}{x^3}\right)$$

$$= x\left(\frac{\cos x}{x^2} - \frac{2\sin x}{x^3} - \frac{\sin x}{x^3} - \frac{3\cos x}{x^4}\right)$$

$$= \left(\frac{1}{x} - \frac{3}{x^3}\right)\cos x - \frac{3\sin x}{x^2}$$

(b)

$$n_1(x) \approx -\frac{x}{x} - \frac{1}{x^2}$$

$$= -1 - \frac{1}{x^2}$$

$$n_2(x) \approx \left(\frac{1}{x} - \frac{3}{x^3}\right) - \frac{3}{x^2}x$$

$$= -\frac{2}{x} - \frac{3}{x^3}$$

4.10

(a)

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \frac{\hbar^2}{2m}\frac{2}{r^2}u = Eu$$

$$\frac{d^2u}{dr^2} - \frac{2}{r^2}u = -\frac{2mE}{\hbar^2}u$$

$$\frac{d^2u}{dr^2} = \left(\frac{2}{r^2} - k^2\right)u$$

$$j_1(kr) = \frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr}$$

$$u = Arj_1(kr)$$

$$= Ar\left[\frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr}\right]$$

$$\frac{du}{dr} = A\left[\frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr}\right]$$

$$+ Ar\left[\frac{k\cos(kr)}{(kr)^2} - \frac{2\sin(kr)}{k^2r^3} + \frac{k\sin(kr)}{kr} + \frac{\cos(kr)}{kr^2}\right]$$

$$= \frac{A}{kr}\left[\frac{\sin(kr)}{kr} - \cos(kr)\right]$$

$$+ \frac{A}{k}\left[\frac{2\cos(kr)}{r} - \frac{2\sin(kr)}{kr^2} + k\sin(kr)\right]$$

$$\frac{d^2u}{dr^2} = \frac{A(-2+k^2r^2)(kr\cos(kr) - \sin(kr))}{k^2r^3}$$

$$= \frac{2-k^2r^2}{r^2}Ar\left[\frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr}\right]$$

$$= \left(\frac{2}{r^2} - k^2\right)u$$

(b)

$$0 = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$
$$= \frac{\sin x}{x} - \cos x$$
$$\cos x = \frac{\sin x}{x}$$
$$x = \tan x$$
$$x \approx \left(N + \frac{1}{2}\right)\pi,\ n \in \mathbb{Z}$$
$$ka = x$$
$$\frac{\sqrt{2mE_{N1}}}{\hbar}a \approx \left(N + \frac{1}{2}\right)\pi$$
$$E_{N1} = \frac{\hbar^2\pi^2}{2ma^2}\left(N + \frac{1}{2}\right)^2,\ N \in \mathbb{Z}$$

4.12

$$R_{n\ell}(r) = \frac{1}{r}\left(\frac{r}{an}\right)^{\ell+1}e^{-r/an}v(r/an)$$
$$R_{30} = e^{-r/3a}\frac{c_0}{3a}\left[1 - 2\left(\frac{r}{3a}\right) - \frac{1}{3}\left(\frac{r}{3a}\right)^2\right]$$
$$R_{31} = \frac{1}{r}\left(\frac{r}{3a}\right)^2 c_0\left[1 - \frac{1}{2}\left(\frac{r}{3a}\right)\right]$$
$$R_{32} = \frac{1}{r}\left(\frac{r}{3a}\right)^3 c_0$$

4.13

(a)

$$R_{20}(r) = \frac{c_0}{2a}\left(1 - \frac{r}{2a}\right)e^{-r/2a}$$

$$1 = \int_0^\infty |R_{20}|^2 r^2\, dr$$

$$= \frac{c_0^2}{4a^2}\int_0^\infty \left(1 - \frac{r}{2a}\right)^2 e^{-r/a} r^2\, dr$$

$$= \frac{ac_0^2}{2}$$

$$c_0 = \sqrt{\frac{2}{a}}$$

$$R_{20}(r) = \frac{1}{\sqrt{2}}a^{-3/2}\left(1 - \frac{r}{2a}\right)e^{-r/2a}$$

$$\psi_{200} = R_{20}Y_0^0$$

$$= \frac{1}{\sqrt{2\pi a}}\frac{1}{2a}\left(1 - \frac{r}{2a}\right)e^{-r/2a}$$

(b)

$$R_{21}(r) = \frac{c_0}{4a^2}re^{-r/2a}$$

$$1 = \frac{c_0^2}{16a^4}\int_0^\infty r^4 e^{-r/a}\, dr$$

$$= \frac{24}{16}ac_0^2$$

$$c_0 = \sqrt{\frac{2}{3a}}$$

$$R_{21}(r) = \sqrt{\frac{2}{3a}}\frac{1}{4a^2}re^{-r/2a}$$

$$\psi_{21\pm1} = R_{21}Y_1^1$$

$$= \mp\frac{1}{\sqrt{\pi a}}\frac{1}{8a^2}re^{-r/2a}\sin\theta e^{\pm i\phi}$$

$$\psi_{210} = \frac{1}{\sqrt{2\pi a}}\frac{1}{4a^2}re^{-r/2a}\cos\theta$$

4.14

(a)

$$L_q = \frac{e^x}{q!} \left(\frac{d}{dx} \right)^q (e^{-x} x^q)$$

$$L_0 = 1$$

$$L_1 = e^x \frac{d}{dx} (x e^{-x})$$
$$= e^x (e^{-x} - x e^{-x})$$
$$= 1 - x$$

$$L_2 = \frac{1}{2} e^x \left(\frac{d}{dx} \right)^2 (x^2 e^{-x})$$
$$= \frac{1}{2} e^x \frac{d}{dx} (2x e^{-x} - x^2 e^{-x})$$
$$= \frac{1}{2} e^x (2 e^{-x} - 2x e^{-x} - 2x e^{-x} + x^2 e^{-x})$$
$$= 1 - 2x + \frac{1}{2} x^2$$

$$L_3 = \frac{1}{6} e^x \left(\frac{d}{dx} \right)^3 (x^3 e^{-x})$$
$$= \frac{1}{6} e^x \left(\frac{d}{dx} \right)^2 (3x^2 e^{-x} - x^3 e^{-x})$$
$$= \frac{1}{6} e^x \frac{d}{dx} (6x e^{-x} - 6x^2 e^{-x} + x^3 e^{-x})$$
$$= \frac{1}{6} e^x (6 e^{-x} - 18x e^{-x} + 9x^2 e^{-x} - x^3 e^{-x})$$
$$= 1 - 3x + \frac{3}{2} x^2 - \frac{1}{6} x^3$$

4.15

(a)

$$\psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

$$\langle r \rangle = \int \psi_{100}^* r \psi_{100}$$

$$= \frac{1}{\pi a^3} \int_0^\infty \int_0^\pi \int_0^{2\pi} r^3 e^{-2r/a} \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{4}{a^3} \int_0^\infty r^3 e^{-2r/a} \, dr$$

$$= \frac{3}{2} a$$

$$\langle r^2 \rangle = \frac{4}{a^3} \int_0^\infty r^4 e^{-2r/a} \, dr$$

$$= 3a^2$$

(b)

$$\langle x \rangle = 0$$

$$\langle x^2 \rangle = a^2$$

(c)

$$\psi_{211} = R_{21} Y_1^1$$

$$= -\frac{1}{\sqrt{\pi a}} \frac{1}{8a^2} r e^{-r/2a} \sin\theta e^{i\phi}$$

$$\langle x^2 \rangle = \int \psi_{211}^* x^2 \psi_{211}$$

$$= \frac{1}{64\pi a^5} \int_0^\infty \int_0^\pi \int_0^{2\pi} r^6 e^{-r/a} \sin^5\theta \cos^2\phi \, dr \, d\theta \, d\phi$$

$$= 12a^2$$

4.16

$$P(r) = \int |\psi_{100}|^2 r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{4}{a^3} r^2 e^{-2r/a}$$

$$P'(r) = \frac{4}{a^3} \left(2r e^{-2r/a} - \frac{2}{a} r^2 e^{-2r/a} \right)$$

$$= \frac{8}{a^3} r \left(1 - \frac{r}{a} \right) e^{-2r/a}$$

$$r_{\max} = a$$

4.18

(a)

$$\Psi(\mathbf{r}, t) = \frac{1}{\sqrt{2}}(\psi_{211} + \psi_{21-1})e^{-iE_2 t/\hbar}$$

$$= \frac{1}{\sqrt{2}}R_{21}(Y_1^1 + Y_1^{-1})e^{-iE_1 t/4\hbar}$$

$$= \frac{1}{4a^2}\frac{1}{\sqrt{3a}}re^{-r/2a}\left(-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} + \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}\right)e^{-iE_1 t/4\hbar}$$

$$= -i\frac{1}{4a^2}\frac{1}{\sqrt{2\pi a}}re^{-r/2a}\sin\theta\sin\phi\, e^{-iE_1 t/4\hbar}$$

(b)

$$\langle V \rangle = \int \Psi^*\left(-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}\right)\Psi$$

$$= -\frac{e^2}{4\pi\epsilon_0}\frac{1}{16a^4}\frac{1}{2\pi a}\int_0^\infty\int_0^\pi\int_0^{2\pi} r^2 e^{-r/a}\sin^2\theta\sin^2\phi\frac{1}{r}r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= -\frac{e^2}{128\pi^2\epsilon_0 a^5}\int_0^\infty\int_0^\pi\int_0^{2\pi} r^3 e^{-r/a}\sin^3\theta\sin^2\phi\,dr\,d\theta\,d\phi$$

$$= -\frac{e^2}{4\pi\epsilon_0}\frac{1}{4a}$$

$$= -6.8\,\text{eV}$$

4.19

$$E_1(Z) = -\left[\frac{m_e}{2\hbar^2}\left(\frac{Ze^2}{4\pi\epsilon_0}\right)^2\right]$$

$$= Z^2 E_1(1)$$

$$= (-13.6\,\text{eV})Z^2$$

$$E_n(Z) = \frac{E_1(Z)}{n^2}$$

$$= \frac{(-13.6\,\text{eV})Z^2}{n^2}$$

$$= Z^2 E_n(1)$$

$$a(Z) = \frac{a}{Z}$$

$$\mathcal{R}(Z) = Z^2 R(1)$$

$$\frac{1}{\lambda} = Z^2 R$$

$$\lambda = \frac{1}{Z^2 R}$$

$$\lambda_{Z=2} = \frac{1}{4R}$$

$$\approx 2.28 \times 10^{-8}\,\text{m}$$

$$= 22.8\,\text{nm}$$

4.20

(a)

$$V = -\frac{GMm}{r}$$

(b)

$$a_g = \frac{\hbar^2}{GMm^2}$$

(c)

$$E_c = \frac{1}{2}mv^2 - \frac{GMm}{r_0}$$

$$\frac{GMm}{r_0^2} = \frac{mv^2}{r_0}$$

$$\frac{1}{2}mv^2 = \frac{GMm}{2r_0}$$

$$E_c = -\frac{GMm}{2r_0}$$

$$E_n = -\left[\frac{m}{2\hbar^2}(GMm)^2\right]\frac{1}{n^2}$$

$$= -\frac{G^2M^2m^3}{2\hbar^2}\frac{1}{n^2}$$

$$-\frac{GMm}{2r_0} = -\frac{G^2M^2m^3}{2\hbar^2}\frac{1}{n^2}$$

$$\frac{\hbar^2}{GMm^2r_0} = \frac{1}{n^2}$$

$$\frac{a_g}{r_0} = \frac{1}{n^2}$$

$$n = \sqrt{\frac{r_0}{a_g}}$$

4.21

$$\langle f_\ell^m | L_+ L_- f_\ell^m \rangle = \langle f_\ell^m | (L^2 - L_z^2 + \hbar L_z) f_\ell^m \rangle$$

$$= \langle f_\ell^m | L^2 f_\ell^m \rangle - \langle f_\ell^m | L_z^2 f_\ell^m \rangle + \langle f_\ell^m | \hbar L_z f_\ell^m \rangle$$

$$= \langle f_\ell^m | \hbar^2 \ell(\ell+1) f_\ell^m \rangle - \langle f_\ell^m | \hbar^2 m^2 f_\ell^m \rangle + \langle f_\ell^m | \hbar^2 m f_\ell m \rangle$$

$$= \hbar^2 \ell(\ell+1) \langle f_\ell^m | f_\ell^m \rangle - \hbar^2 m^2 \langle f_\ell^m | f_\ell^m \rangle + \hbar^2 m \langle f_\ell^m | f_\ell^m \rangle$$

$$= \hbar^2 \ell(\ell+1) - \hbar^2 m^2 + \hbar^2 m$$

$$= \hbar^2 [\ell(\ell+1) - m(m-1)]$$

$$\langle f_\ell^m | L_+ L_- f_\ell^m \rangle = \langle L_- f_\ell^m | L_- f_\ell^m \rangle$$

$$= \langle B_\ell^m f_\ell^{m-1} | B_\ell^m f_\ell^{m-1} \rangle$$

$$= |B_\ell^m|^2$$

$$|B_\ell^m|^2 = \hbar^2 [\ell(\ell+1) - m(m-1)]$$

$$B_\ell^m = \hbar \sqrt{\ell(\ell+1) - m(m-1)}$$

The same argument for $\langle f_\ell^m | L_- L_+ f_\ell^m \rangle$ finds A_ℓ^m.

4.22

(a)

$$[r_i, p_j] = i\hbar\delta_{ij}$$
$$[r_i, r_j] = 0$$
$$[p_i, p_j] = 0$$
$$[L_z, x] = [xp_y - yp_x, x]$$
$$= [xp_y, x] - [yp_x, x]$$
$$= x[p_y, x] + [x, x]p_y - y[p_x, x] - [y, x]p_x$$
$$= i\hbar y$$
$$[L_z, y] = [xp_y - yp_x, y]$$
$$= [xp_y, y] - [yp_x, y]$$
$$= x[p_y, y] + [x, y]p_y - y[p_x, y] - [y, y]p_x$$
$$= -i\hbar x$$
$$[L_z, z] = [xp_y - yp_x, z]$$
$$= [xp_y, z] - [yp_x, z]$$
$$= x[p_y, z] + [x, z]p_y - y[p_x, z] - [p_x, z]y$$
$$= 0$$
$$[L_z, p_x] = [xp_y - yp_x, p_x]$$
$$= [xp_y, p_x] - [yp_x, p_x]$$
$$= x[p_y, p_x] + [x, p_x]p_y - y[p_x, p_x] - [y, p_x]p_x$$
$$= i\hbar p_y$$
$$[L_z, p_y] = [xp_y - yp_x, p_y]$$
$$= [xp_y, p_y] - [yp_x, p_y]$$
$$= x[p_y, p_y] + [x, p_y]p_y - y[p_x, p_y] - [y, p_y]p_x$$
$$= -i\hbar p_x$$
$$[L_z, p_z] = [xp_y - yp_x, p_z]$$
$$= [xp_y, p_z] - [yp_x, p_z]$$
$$= x[p_y, p_z] + [x, p_z]p_y - y[p_x, p_z] - [y, p_z]p_x$$
$$= 0$$

(b)

$$[L_z, L_x] = [L_z, yp_z - zp_y]$$
$$= [L_z, yp_z] - [L_z, zp_y]$$
$$= y[L_z, p_z] + [L_z, y]p_z - z[L_z, p_y] - [L_z, z]p_y$$
$$= -i\hbar xp_z + i\hbar zp_x$$
$$= i\hbar L_y$$

(c)

$$
\begin{aligned}
[L_z, r^2] &= [L_z, x^2 + y^2 + z^2] \\
&= [L_z, x^2] + [L_z, y^2] + [L_z, z^2] \\
&= x[L_z, x] + [L_z, x]x + y[L_z, y] + [L_z, y]y + z[L_z, z] + [L_z, z]z \\
&= 2i\hbar xy - 2i\hbar xy \\
&= 0 \\
[L_z, p^2] &= [L_z, p_x^2 + p_y^2 + p_z^2] \\
&= [L_z, p_x^2] + [L_z, p_y^2] + [L_z, p_z^2] \\
&= p_x[L_z, p_x] + [L_z, p_x]p_x + p_y[L_z, p_y] + [L_z, p_y]p_y + p_z[L_z, p_z] + [L_z, p_z]p_z \\
&= 2i\hbar p_x p_y - 2i\hbar p_x p_y \\
&= 0
\end{aligned}
$$

4.24

(a)

$$
\begin{aligned}
L_+ L_- f &= \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left[-\hbar e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right)\right]f \\
&= -\hbar^2 e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left[e^{-i\phi}\left(\frac{\partial f}{\partial\theta} - i\cot\theta\frac{\partial f}{\partial\phi}\right)\right] \\
&= -\hbar^2 e^{i\phi}\left[e^{-i\phi}\left(\frac{\partial^2 f}{\partial\theta^2} + i\csc^2\theta\frac{\partial f}{\partial\phi} - i\cot\theta\frac{\partial^2 f}{\partial\phi\partial\theta}\right)\right. \\
&\qquad \left. + \cot\theta e^{-i\phi}\left(\frac{\partial f}{\partial\theta} - i\cot\theta\frac{\partial f}{\partial\phi}\right) + i\cot\theta e^{-i\phi}\left(\frac{\partial^2 f}{\partial\phi\partial\theta} - i\cot\theta\frac{\partial^2 f}{\partial\phi^2}\right)\right] \\
&= -\hbar^2\left(\frac{\partial^2 f}{\partial\theta^2} + i\csc^2\theta\frac{\partial f}{\partial\phi} + \cot\theta\frac{\partial f}{\partial\theta} - i\cot^2\theta\frac{\partial f}{\partial\phi} + \cot^2\theta\frac{\partial^2 f}{\partial\phi^2}\right) \\
&= -\hbar^2\left(\frac{\partial^2 f}{\partial\theta^2} + i\frac{\partial f}{\partial\phi} + \cot\theta\frac{\partial f}{\partial\theta} + \cot^2\theta\frac{\partial^2 f}{\partial\phi^2}\right) \\
L_+ L_- &= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right)
\end{aligned}
$$

(b)

$$L^2 = L_+L_- + L_z^2 - \hbar L_z$$

$$= -\hbar^2 \left(\frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \cot^2 \theta \frac{\partial^2}{\partial \phi^2} + i \frac{\partial}{\partial \phi} \right) - \hbar^2 \frac{\partial^2}{\partial \phi^2} + i\hbar^2 \frac{\partial}{\partial \phi}$$

$$= -\hbar^2 \left(\frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \cot^2 \theta \frac{\partial^2}{\partial \phi^2} + i \frac{\partial}{\partial \phi} + \frac{\partial^2}{\partial \phi^2} - i \frac{\partial}{\partial \phi} \right)$$

$$= -\hbar^2 \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \cot^2 \theta \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial \phi^2} \right]$$

$$= -\hbar^2 \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]$$

4.25

(a)

$$L_+ Y_\ell^\ell = 0$$

(b)

$$L_z Y_\ell^\ell = \hbar \ell Y_\ell^\ell$$

$$-i\hbar \frac{\partial Y_\ell^\ell}{\partial \phi} = \hbar \ell Y_\ell^\ell$$

$$\frac{\partial Y_\ell^\ell}{\partial \phi} = i\ell Y_\ell^\ell$$

$$Y_\ell^\ell = e^{i\ell\phi} \Theta(\theta)$$

$$0 = L_+ Y_\ell^\ell$$

$$= \hbar e^{i\phi} \left(\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right) e^{i\ell\phi} \Theta(\theta)$$

$$= \hbar e^{i\phi} \left[e^{i\ell\phi} \frac{\partial \Theta}{\partial \theta} - \ell \cot \theta e^{i\ell\phi} \Theta(\theta) \right]$$

$$= \frac{\partial \Theta}{\partial \theta} - \ell \cot \theta \Theta(\theta)$$

$$\frac{1}{\Theta(\theta)} \frac{\partial \Theta}{\partial \theta} = \ell \cot \theta$$

$$\int \frac{1}{\Theta(\theta)} d\Theta = \ell \int \frac{\cos \theta}{\sin \theta} d\theta$$

$$\ln \Theta = \ell \ln \sin \theta + c_1$$

$$= \ln(\sin^\ell \theta) + c_1$$

$$\ln \frac{\Theta}{\sin^\ell \theta} = c_1$$

$$\frac{\Theta}{\sin^\ell \theta} = c_2$$

$$\Theta = c_2 \sin^\ell \theta$$

$$Y_\ell^\ell = c e^{i\ell\phi} \sin^\ell \theta$$

4.26

$$Y_2^1(\theta, \phi) = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\phi}$$

$$= -\frac{1}{2}\sqrt{\frac{15}{8\pi}} \sin(2\theta)e^{i\phi}$$

$$L_+ Y_2^1 = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left[-\frac{1}{2}\sqrt{\frac{15}{8\pi}} \sin(2\theta)e^{i\phi}\right]$$

$$= -\frac{1}{2}\sqrt{\frac{15}{8\pi}}\hbar e^{i\phi}[2\cos(2\theta)e^{i\phi} - \cot\theta\sin(2\theta)e^{i\phi}]$$

$$= \sqrt{\frac{15}{8\pi}}\hbar(e^{i\phi}\sin\theta)^2$$

$$= \hbar\sqrt{(2-1)(2+1+1)}Y_2^1$$

$$= 2\hbar Y_2^1$$

$$Y_2^1 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}(e^{i\phi}\sin\theta)^2$$

4.28

$$r = \frac{e^2}{4\pi\epsilon_0 mc^2}$$

$$= 0.0796\,\text{m}$$

$$\frac{1}{2}\hbar = I\omega$$

$$= \frac{2}{5}mr^2\frac{v}{r}$$

$$v = \frac{5\hbar}{4mr}$$

$$= 5.14\times 10^{10}\,\text{m/s}$$

This is 100 times the speed of light, so no it doesn't make sense.

4.29

(a)

$$[S_x, S_y] = S_x S_y - S_y S_x$$

$$= \begin{pmatrix} i\hbar^2/2 & 0 \\ 0 & -i\hbar^2/2 \end{pmatrix}$$

$$= i\hbar S_z$$

$$[S_y, S_z] = i\hbar S_x$$

$$[S_z, S_x] = i\hbar S_y$$

(b)

$$\sigma_x\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \delta_{xx}$$

$$\delta_x\delta_y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$= i\epsilon_{xyz}\sigma_z$$

$$\delta_x\delta_z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= i\epsilon_{xzy}\sigma_y$$

$$\delta_y\delta_x = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}$$

$$= i\epsilon_{yxz}\sigma_z$$

$$\delta_y\delta_y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \delta_{yy}$$

$$\delta_y\delta_z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$= i\epsilon_{yzx}\sigma_x$$

$$\delta_z\delta_x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= i\epsilon_{zxy}\sigma_y$$

$$\delta_z\delta_y = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$= i\epsilon_{zyx}\sigma_x$$

$$\delta_z\delta_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \delta_{zz}$$

4.30

(a)

$$1 = \chi^\dagger\chi$$

$$= |A|^2 \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

$$= 25|A|^2$$

$$A = \pm\frac{1}{5}$$

(b)

$$\chi = \frac{1}{5}\begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

$$P\left(\frac{\hbar}{2}\right) = \frac{1}{2}\left|\frac{1}{5}(3i+4)\right|^2$$

$$= \frac{1}{50}(-3i+4)(3i+4)$$

$$= \frac{1}{2}$$

$$\langle S_x \rangle = 0$$

$$\langle S_y \rangle = \chi^\dagger S_y \chi$$

$$= \frac{\hbar}{50}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

$$= \frac{\hbar}{50}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 0 & -4i \\ -3 & 0 \end{pmatrix}$$

$$= -\frac{12}{25}\hbar$$

$$\langle S_z \rangle = \frac{\hbar}{50}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

$$= \frac{\hbar}{50}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 3i \\ -4 \end{pmatrix}$$

$$= -\frac{7}{50}\hbar$$

(c)

$$\langle S_x^2 \rangle = \frac{\hbar^2}{100}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}$$

$$\sigma_{S_x} = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2}$$

$$= \frac{\hbar}{2}$$

$$\langle S_y^2 \rangle = \frac{\hbar^2}{4}$$

$$\sigma_{S_y} = \sqrt{\frac{1}{4}\hbar^2 - \frac{144}{625}\hbar^2}$$

$$= \frac{7}{50}\hbar$$

$$\sigma_{S_z} = \frac{12}{25}\hbar$$

(d)

$$\sigma_{S_x}\sigma_{S_y} = \frac{7}{100}\hbar^2$$
$$= \frac{\hbar}{2}|\langle S_z \rangle|$$
$$\geq \frac{\hbar}{2}|\langle S_z \rangle|$$
$$\sigma_{S_y}\sigma_{S_z} = \frac{42}{625}\hbar^2$$
$$\geq \frac{\hbar}{2}|\langle S_x \rangle|$$
$$= 0$$
$$\sigma_{S_z}\sigma_{S_x} = \frac{6}{25}\hbar^2$$
$$= \frac{\hbar}{2}|\langle S_y \rangle|$$
$$\geq \frac{\hbar}{2}|\langle S_z \rangle|$$

4.31

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}$$
$$\langle S_x \rangle = \frac{\hbar}{2}(a^*b + ab^*)$$
$$\langle S_y \rangle = i\frac{\hbar}{2}(ab^* - a^*b)$$
$$\langle S_z \rangle = \frac{\hbar}{2}(|a|^2 - |b|^2)$$
$$\langle S_x^2 \rangle = \frac{\hbar^2}{4}(|a|^2 + |b|^2)$$
$$\langle S_y^2 \rangle = \frac{\hbar^2}{4}(|a|^2 + |b|^2)$$
$$\langle S_z^2 \rangle = \frac{\hbar^2}{4}(|a|^2 + |b|^2)$$
$$\langle S^2 \rangle = 3\frac{\hbar^2}{4}(|a|^2 + |b|^2)$$
$$\langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle = 3\frac{\hbar^2}{4}(|a|^2 + |b|^2)$$
$$= \langle S^2 \rangle$$

4.32

(a)

$$0 = \begin{vmatrix} -\lambda & -i\hbar/2 \\ i\hbar/2 & -\lambda \end{vmatrix}$$

$$= \lambda^2 - \frac{\hbar^2}{4}$$

$$\lambda = \pm\frac{\hbar}{2}$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix} = \pm\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\beta = \pm i\alpha$$

$$\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

(b)

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \left(\frac{a-ib}{\sqrt{2}}\right)\chi_+^{(x)} + \left(\frac{a+ib}{\sqrt{2}}\right)\chi_-^{(y)}$$

You could get $\frac{\hbar}{2}$ with probability $\frac{|a-ib|^2}{2}$ or $-\frac{\hbar}{2}$ with probability $\frac{|a+ib|^2}{2}$.

$$\frac{1}{2}|a-ib|^2 + \frac{1}{2}|a+ib|^2 = \frac{1}{2}(a+ib)(a-ib) + \frac{1}{2}(a-ib)(a+ib)$$

$$= \frac{1}{2}(a^2+b^2) + \frac{1}{2}(a^2+b^2)$$

$$= 1$$

(c) $\hbar^2/2$ with probability 1.

4.33

$$S_r = S \cdot \hat{\mathbf{r}}$$

$$= S_x \sin\theta\cos\phi + S_y \sin\theta\sin\phi + S_z \cos\theta$$

$$= \frac{\hbar}{2}\left[\begin{pmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\sin\theta\sin\phi \\ i\sin\theta\sin\phi & 0 \end{pmatrix}\right.$$

$$\left. + \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix}\right]$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}$$

$$\begin{vmatrix} \frac{\hbar}{2}\cos\theta - \lambda & \frac{\hbar}{2}e^{-i\phi}\sin\theta \\ \frac{\hbar}{2}e^{i\phi}\sin\theta & -\frac{\hbar}{2}\cos\theta - \lambda \end{vmatrix} = \left(\frac{\hbar}{2}\cos\theta - \lambda\right)\left(-\frac{\hbar}{2}\cos\theta - \lambda\right)$$

$$- \left(\frac{\hbar}{2}e^{-i\phi}\sin\theta\right)\left(\frac{\hbar}{2}e^{i\phi}\sin\theta\right)$$

$$= -\frac{\hbar^2}{4}\cos^2\theta - \lambda\frac{\hbar}{2}\cos\theta + \lambda\frac{\hbar}{2}\cos\theta + \lambda^2$$

$$- \frac{\hbar^2}{4}\sin^2\theta$$

$$= \lambda^2 - \frac{\hbar^2}{4}$$

$$\lambda = \pm\frac{\hbar}{2}$$

4.35

(a)

$$\chi(t) = \begin{pmatrix} \cos(\alpha/2)e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)e^{-i\gamma B_0 t/2} \end{pmatrix}$$

$$\frac{1}{2}|a + b|^2 = \frac{1}{2}[\cos(\alpha/2)e^{-i\gamma B_0 t/2} + \sin(\alpha/2)e^{i\gamma B_0 t/2}]$$

$$[\cos(\alpha/2)e^{i\gamma B_0 t/2} + \sin(\alpha/2)e^{-i\gamma B_0 t/2}]$$

$$= \frac{1}{2}[\cos^2(\alpha/2) + \cos(\alpha/2)\sin(\alpha/2)e^{-i\gamma B_0 t}$$

$$+ \cos(\alpha/2)\sin(\alpha/2)e^{i\gamma B_0 t} + \sin^2(\alpha/2)]$$

$$= \frac{1}{2}[1 + \sin\alpha\cos(\gamma B_0 t)]$$

(b)

$$\frac{1}{2}|a - ib|^2 = \frac{1}{2}[\cos(\alpha/2)e^{-i\gamma B_0 t/2} + i\sin(\alpha/2)e^{i\gamma B_0 t/2}]$$
$$[\cos(\alpha/2)e^{i\gamma B_0 t/2} - i\sin(\alpha/2)e^{-i\gamma B_0 t/2}]$$
$$= \frac{1}{2}[\cos^2(\alpha/2) - i\cos(\alpha/2)\sin(\alpha/2)e^{-i\gamma B_0 t}$$
$$+ i\cos(\alpha/2)\sin(\alpha/2)e^{i\gamma B_0 t} + \sin^2(\alpha/2)]$$
$$= \frac{1}{2}[1 - \sin\alpha\sin(\gamma B_0 t)]$$

(c)

$$|a|^2 = \cos^2\frac{\alpha}{2}$$

4.36

(a)

$$\mathbf{B} = B_0\cos(\omega t)\hat{k}$$
$$H = -\gamma\mathbf{B}\cdot\mathbf{S}$$
$$= -\gamma B_0\cos(\omega t)S_z$$
$$= -\frac{\gamma B_0\hbar}{2}\cos(\omega t)\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$$

(b)

$$i\hbar\frac{\partial\chi}{\partial t} = H\chi$$

$$i\hbar\begin{pmatrix}a'(t)\\b'(t)\end{pmatrix} = -\frac{\gamma B_0\hbar}{2}\cos(\omega t)\begin{pmatrix}a(t)\\-b(t)\end{pmatrix}$$

$$i\hbar a'(t) = -\frac{\gamma B_0\hbar}{2}\cos(\omega t)a(t)$$

$$\frac{1}{a(t)}a'(t) = i\frac{\gamma B_0}{2}\cos(\omega t)$$

$$\ln[a(t)] = i\frac{\gamma B_0}{2\omega}\sin(\omega t) + c_1$$

$$a(t) = Ae^{i\gamma B_0\sin(\omega t)/2\omega}$$

$$i\hbar b'(t) = \frac{\gamma B_0\hbar}{2}\cos(\omega t)b(t)$$

$$\frac{1}{b(t)}b'(t) = -i\frac{\gamma B_0}{2}\cos(\omega t)$$

$$\ln[b(t)] = -i\frac{\gamma B_0}{2\omega}\sin(\omega t) + c_2$$

$$b(t) = Be^{-i\gamma B_0\sin(\omega t)/2\omega}$$

$$\chi = \begin{pmatrix}Ae^{i\gamma B_0\sin(\omega t)/2\omega}\\Be^{-i\gamma B_0\sin(\omega t)/2\omega}\end{pmatrix}$$

$$1 = \frac{1}{2}|A+B|^2$$

$$= \frac{1}{2}(A^*+B^*)(A+B)$$

$$= \frac{1}{2}(|A|^2 + A^*B + B^*A + |B|^2)$$

$$0 = \frac{1}{2}|A-B|^2$$

$$= \frac{1}{2}(A^*-B^*)(A-B)$$

$$= \frac{1}{2}(|A|^2 - A^*B - B^*A + |B|^2)$$

$$\chi = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\gamma B_0\sin(\omega t)/2\omega}\\e^{-i\gamma B_0\sin(\omega t)/2\omega}\end{pmatrix}$$

113

(c)

$$\frac{A-B}{\sqrt{2}} = \frac{1}{2}[e^{i\gamma B_0 \sin(\omega t)/2\omega} - e^{-i\gamma B_0 \sin(\omega t)/2\omega}]$$

$$= i\sin\left[\frac{\gamma B_0}{2\omega}\sin(\omega t)\right]$$

$$\frac{1}{2}|A-B|^2 = \sin^2\left[\frac{\gamma B_0}{2\omega}\sin(\omega t)\right]$$

(d)

$$|B_0| = \frac{\omega\pi}{\gamma}$$

4.37

(a)

$$S_- |10\rangle = S_-\left[\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\right]$$

$$= \frac{1}{\sqrt{2}}[(S_-^{(1)}|\uparrow\rangle)|\downarrow\rangle + |\uparrow\rangle(S_-^{(2)}|\downarrow\rangle) + (S_-^{(1)}|\downarrow\rangle)|\uparrow\rangle + |\downarrow\rangle(S_-^{(2)}|\uparrow\rangle)]$$

$$= \frac{1}{\sqrt{2}}[\hbar|\downarrow\downarrow\rangle + \hbar|\downarrow\downarrow\rangle]$$

$$= \sqrt{2}\hbar|1-1\rangle$$

(b)

$$S_+ |00\rangle = S_+\left[\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\right]$$

$$= \frac{1}{\sqrt{2}}[(S_+^{(1)}|\uparrow\rangle)|\downarrow\rangle + |\uparrow\rangle(S_+^{(2)}|\downarrow\rangle) - (S_+^{(1)}|\downarrow\rangle)|\uparrow\rangle - |\downarrow\rangle(S_+^{(2)}|\uparrow\rangle)]$$

$$= \frac{1}{\sqrt{2}}(\hbar|\uparrow\uparrow\rangle - \hbar|\uparrow\uparrow\rangle)$$

$$= 0$$

$$S_- |00\rangle = S_-\left[\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\right]$$

$$= \frac{1}{\sqrt{2}}[(S_-^{(1)}|\uparrow\rangle)|\downarrow\rangle + |\uparrow\rangle(S_-^{(2)}|\downarrow\rangle) - (S_-^{(1)}|\downarrow\rangle)|\uparrow\rangle - |\downarrow\rangle(S_-^{(2)}|\uparrow\rangle)]$$

$$= \frac{1}{\sqrt{2}}(\hbar|\downarrow\downarrow\rangle - \hbar|\downarrow\downarrow\rangle)$$

$$= 0$$

(c)

$$S^{(1)} \cdot S^{(2)} \ket{\uparrow\uparrow} = (S_x^{(1)} \ket{\uparrow})(S_x^{(2)} \ket{\uparrow}) + (S_y^{(1)} \ket{\uparrow})(S_y^{(2)} \ket{\uparrow})$$
$$+ (S_z^{(1)} \ket{\uparrow})(S_z^{(2)} \ket{\uparrow})$$
$$= \left(\frac{\hbar}{2} \ket{\downarrow}\right)\left(\frac{\hbar}{2} \ket{\downarrow}\right) + \left(\frac{i\hbar}{2} \ket{\downarrow}\right)\left(\frac{i\hbar}{2} \ket{\downarrow}\right)$$
$$+ \left(\frac{\hbar}{2} \ket{\uparrow}\right)\left(\frac{\hbar}{2} \ket{\uparrow}\right)$$
$$= \frac{\hbar^2}{4} \ket{\downarrow\downarrow} - \frac{\hbar^2}{4} \ket{\downarrow\downarrow} + \frac{\hbar^2}{4} \ket{\uparrow\uparrow}$$
$$S^2 \ket{\uparrow\uparrow} = \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\frac{\hbar^2}{4}\right) \ket{\uparrow\uparrow}$$
$$= 2\hbar^2 \ket{\uparrow\uparrow}$$
$$S^{(1)} \cdot S^{(2)} \ket{\downarrow\downarrow} = (S_x^{(1)} \ket{\downarrow})(S_x^{(2)} \ket{\downarrow}) + (S_y^{(1)} \ket{\downarrow})(S_y^{(2)} \ket{\downarrow})$$
$$+ (S_z^{(1)} \ket{\downarrow})(S_z^{(2)} \ket{\downarrow})$$
$$= \left(\frac{\hbar}{2} \ket{\uparrow}\right)\left(\frac{\hbar}{2} \ket{\uparrow}\right) + \left(-\frac{i\hbar}{2} \ket{\uparrow}\right)\left(-\frac{i\hbar}{2} \ket{\uparrow}\right)$$
$$\left(-\frac{\hbar}{2} \ket{\downarrow}\right)\left(-\frac{\hbar}{2} \ket{\downarrow}\right)$$
$$= \frac{\hbar^2}{4} \ket{\uparrow\uparrow} - \frac{\hbar^2}{4} \ket{\uparrow\uparrow} + \frac{\hbar^2}{4} \ket{\downarrow\downarrow}$$
$$S^2 \ket{\downarrow\downarrow} = \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\frac{\hbar^2}{4}\right) \ket{\downarrow\downarrow}$$
$$= 2\hbar^2 \ket{\downarrow\downarrow}$$

4.38

(a) $3/2, 1/2$

(b) $0, 1$

4.40

(a)

$$\ket{31} = \frac{1}{\sqrt{15}} \ket{22}\ket{1-1} + \sqrt{\frac{8}{15}} \ket{21}\ket{10} + \sqrt{\frac{6}{15}} \ket{20}\ket{11}$$

$2\hbar$ with probability $1/15$, $\hbar$ with probability $8/15$, and 0 with probability $6/15$.

4.46

(a)

$$V(r) = \frac{1}{2}m\omega^2 r^2$$

$$= \frac{1}{2}m\omega^2(x^2 + y^2 + z^2)$$

$$\psi = XYZ$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\left(\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}\right) + \frac{1}{2}m\omega^2(x^2 + y^2 + z^2) = E$$

$$-\frac{\hbar^2}{2m}\frac{X''}{X} + \frac{1}{2}m\omega^2 x^2 = c_1$$

$$-\frac{\hbar^2}{2m}X'' + \frac{1}{2}m\omega^2 x^2 X = c_1 X$$

$$c_1 = E_{x_n}$$

$$= \left(n_x + \frac{1}{2}\right)\hbar\omega$$

$$\frac{\hbar^2}{2m}\left(\frac{Y''}{Y} + \frac{Z''}{Z}\right) - \frac{1}{2}m\omega^2(y^2 + z^2) + E = c_1$$

$$\frac{\hbar^2}{2m}\frac{Y''}{Y} - \frac{1}{2}m\omega^2 y^2 + E = c_2$$

$$-\frac{\hbar^2}{2m}Y'' + \frac{1}{2}m\omega^2 y^2 Y = (E - c_2)Y$$

$$E - c_2 = \left(n_y + \frac{1}{2}\right)\hbar\omega$$

$$c_1 - \frac{\hbar^2}{2m}\frac{Z''}{Z} + \frac{1}{2}m\omega^2 z^2 = c_2$$

$$-\frac{\hbar^2}{2m}Z'' + \frac{1}{2}m\omega^2 z^2 Z = (c_2 - c_1)Z$$

$$c_2 - c_1 = \left(n_z + \frac{1}{2}\right)\hbar\omega$$

$$c_2 = (n_x + n_z + 1)\hbar\omega$$

$$E = \left(n_x + n_y + n_z + \frac{3}{2}\right)\hbar\omega$$

$$= \left(n + \frac{3}{2}\right)\hbar\omega$$

4.48

(b)

$$V = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

$$\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{r}}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2}\hat{\mathbf{r}}$$

$$2\langle T\rangle = \langle \mathbf{r}\cdot\nabla V\rangle$$

$$= \left\langle \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}\right\rangle$$

$$= -\langle V\rangle$$

$$E_n = \langle T\rangle + \langle V\rangle$$

$$= \langle T\rangle - 2\langle T\rangle$$

$$= -\langle T\rangle$$

$$\langle T\rangle = -E_n$$

$$\langle V\rangle = 2E_n$$

(c)

$$V = \frac{1}{2}m\omega^2 r^2$$

$$\nabla V = m\omega^2 r\,\hat{\mathbf{r}}$$

$$2\langle T\rangle = \langle m\omega^2 r^2\rangle$$

$$= 2\langle V\rangle$$

$$\langle T\rangle = \langle V\rangle$$

$$E_n = \langle T\rangle + \langle V\rangle$$

$$= 2\langle T\rangle$$

$$\langle T\rangle = \frac{E_n}{2}$$

$$\langle V\rangle = \frac{E_n}{2}$$

4.52

(a)

$$\psi_{321} = \frac{1}{9a}\frac{1}{\sqrt{30a}}e^{-r/3a}\frac{4r^2}{9a^2}L_0^5\left(\frac{2r}{3a}\right)Y_2^1(\theta,\phi)$$

$$= -\frac{1}{\sqrt{\pi a}}\frac{r^2}{81a^3}e^{-r/3a}\cos\theta\sin\theta e^{i\phi}$$

(b)

$$\int |\psi_{321}|^2 = \frac{1}{81^2 \pi a^7} \int_0^\infty \int_0^\pi \int_0^{2\pi} r^6 e^{-2r/3a} \cos^2\theta \sin^3\theta \, dr \, d\theta \, d\phi$$

$$= \frac{2}{81^2 a^7} \int_0^\infty \int_0^\pi r^6 e^{-2r/3a} \cos^2\theta \sin^3\theta \, dr \, d\theta$$

$$= \frac{8}{15 \cdot 81^2 a^7} \int_0^\infty r^6 e^{-2r/3a} \, dr$$

$$= 1$$

(c)

$$\langle r^s \rangle = \int \psi_{321}^* r^s \psi_{321}$$

$$= \frac{8}{98415 a^7} \int_0^\infty r^s r^6 e^{-2r/3a} \, dr$$

$$= \frac{8}{98415 a^7} (s+6)! \left(\frac{3a}{2}\right)^{s+7}$$

$$= \frac{1}{720} (s+6!) \left(\frac{3a}{2}\right)^s$$

$$= \frac{(s+6)!}{6!} \left(\frac{3a}{2}\right)^s$$

Finite for $s > -7$

4.53

(a)

$$\psi_{433} = -\frac{1}{\sqrt{\pi a}} \frac{1}{6144 a^4} e^{-r/4a} r^3 \sin^3\theta e^{3i\phi}$$

(b)

$$\langle r \rangle = \frac{1}{6144^2 \pi a^9} \int_0^\infty \int_0^\pi \int_0^{2\pi} r^9 e^{-r/2a} \sin^7\theta \, dr \, d\theta \, d\phi$$

$$= \frac{2}{6144^2 a^9} \int_0^\infty \int_0^\pi r^9 e^{-r/2a} \sin^7\theta \, dr \, d\theta$$

$$= \frac{64}{35 \cdot 6144^2 a^9} \int_0^\infty r^9 e^{-r/2a} \, dr$$

$$= \frac{64}{35 \cdot 6144^2 a^9} 371589120 a^{1}0$$

$$= 18a$$

(c)

$$(L_x^2 + L_y^2)\psi_{433} = (L^2 - L_z^2)\psi_{433}$$
$$= [\hbar^2 3(3+1) - \hbar^2 3^2]\psi_{433}$$
$$= 3\hbar^2 \psi_{433}$$

Probability 1.

4.54

(a)

$$\psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

$$P_{\text{inside}} = \frac{1}{\pi a^3} \int_0^b \int_0^\pi \int_0^{2\pi} e^{-2r/a} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{2}{a^3} \int_0^b \int_0^\pi e^{-2r/a} r^2 \sin\theta \, dr \, d\theta$$

$$= \frac{4}{a^3} \int_0^b e^{-2r/a} r^2 \, dr$$

$$= \frac{4}{a^3} \frac{1}{4} a \left[a^2 - \left(a^2 + 2ab + 2b^2 \right) e^{-2b/a} \right]$$

$$= 1 - \left[1 + 2\frac{b}{a} + 2\left(\frac{b}{a} \right)^2 \right] e^{-2b/a}$$

(b)

$$P_{\text{inside}} = 1 - \left(1 + \epsilon + \frac{1}{2}\epsilon^2 \right) e^{-\epsilon}$$

$$= 1 - \left(1 + \epsilon + \frac{1}{2}\epsilon^2 \right) \left(1 - \epsilon + \frac{\epsilon^2}{2} - \frac{\epsilon^3}{6} + \cdots \right)$$

$$= 1 - \left(1 - \epsilon + \frac{\epsilon^2}{2} - \frac{\epsilon^3}{6} + \cdots \right) - \left(\epsilon - \epsilon^2 + \frac{\epsilon^3}{2} - \cdots \right)$$

$$\quad - \left(\frac{1}{2}\epsilon^2 - \frac{1}{2}\epsilon^3 + \cdots \right)$$

$$= \frac{\epsilon^3}{6} + \cdots$$

$$= \frac{4}{3} \left(\frac{b}{a} \right)^3 + \cdots$$

(c)

$$P_{\text{inside}} \approx \frac{4\pi b^3}{3} |\psi(0)|^2$$
$$= \frac{4\pi b^3}{3} \frac{1}{\pi a^3}$$
$$= \frac{4}{3} \left(\frac{b}{a} \right)^3$$

(d)

$$P_{\text{inside}} \approx 1.07 \times 10^{-14}$$

4.55

(a)

$$R_{n(n-1)}(r) = c_0 \frac{1}{r} \rho^{\ell+1} e^{-\rho}$$
$$= c_0 \frac{1}{r} \left(\frac{r}{an} \right)^n e^{-r/an}$$
$$= N_n r^{n-1} e^{-r/an}$$
$$1 = N_n^2 \int_0^\infty r^{2n} e^{-2r/an} \, dr$$
$$u = \frac{2r}{an}$$
$$r = \frac{anu}{2}$$
$$dr = \frac{an}{2} \, du$$
$$1 = N_n^2 \int_0^\infty \left(\frac{anu}{2} \right)^{2n} e^{-u} \frac{an}{2} \, du$$
$$= N_n^2 \left(\frac{an}{2} \right)^{2n+1} \int_0^\infty u^{2n} e^{-u} \, du$$
$$= N_n^2 \left(\frac{an}{2} \right)^{2n+1} \Gamma(2n+1)$$
$$= N_n^2 \left(\frac{an}{2} \right)^{2n+1} (2n)!$$
$$N_n = \left(\frac{2}{an} \right)^n \sqrt{\frac{2}{an(2n)!}}$$

(b)

$$\langle r \rangle = \int R^* r R$$

$$= N_n^2 \int_0^\infty r^{2n+1} e^{-2r/an} \, dr$$

$$u = \frac{2r}{an}$$

$$r = \frac{anu}{2}$$

$$dr = \frac{an}{2} \, du$$

$$\langle r \rangle = N_n^2 \int_0^\infty \left(\frac{anu}{2}\right)^{2n+1} e^{-u} \frac{an}{2} \, du$$

$$= \left(n + \frac{1}{2}\right) na$$

$$\langle r^2 \rangle = \int R^* r^2 R$$

$$= \left(\frac{2}{an}\right)^{2n} \frac{2}{an(2n)!} \int_0^\infty r^{2n+2} e^{-2r/an} \, dr$$

$$= \left(n + \frac{1}{2}\right)(n + 1)(na)^2$$

(c)

$$\sigma_r = \sqrt{\langle r^2 \rangle - \langle r \rangle^2}$$

$$= \sqrt{\left(n + \frac{1}{2}\right)(n + 1)(na)^2 - \left(n + \frac{1}{2}\right)^2 (na)^2}$$

$$= \left(n + \frac{1}{2}\right) na \sqrt{\frac{n + 1}{n + \frac{1}{2}} - 1}$$

$$= \langle r \rangle \sqrt{\frac{n + 1 - n - \frac{1}{2}}{n + \frac{1}{2}}}$$

$$= \frac{\langle r \rangle}{\sqrt{2n + 1}}$$

121

4.57

(a)

$$A = x^2$$
$$B = L_z$$
$$[A, B] = [x^2, L_z]$$
$$= x[x, L_z] + [x, L_z]x$$
$$= x[x, xp_y - yp_x] + [x, xp_y - yp_x]x$$
$$= x[x, xp_y] - x[x, yp_x] + [x, xp_y]x - [x, yp_x]x$$
$$= x^2[x, p_y] + x[x, x]p_y - xy[x, p_x] - x[x, y]p_x$$
$$\quad + x[x, p_y]x + [x, x]p_y x - y[x, p_x]x - [x, y]p_x x$$
$$= -i\hbar xy - i\hbar xy$$
$$= -2i\hbar xy$$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [A, B] \rangle \right)^2$$
$$= \left(\frac{1}{2i} \langle -2i\hbar xy \rangle \right)^2$$
$$= \hbar^2 \langle xy \rangle^2$$
$$\sigma_A \sigma_B = \hbar |\langle xy \rangle|$$

(b)

$$L_z \psi_{n\ell m} = m\hbar \psi_{n\ell m}$$
$$L_z^2 \psi_{n\ell m} = m^2 \hbar^2 \psi_{n\ell m}$$
$$\sigma_B = \sqrt{\langle L_z^2 \rangle - \langle L_z \rangle^2}$$
$$= \sqrt{m^2\hbar^2 - m^2\hbar^2}$$
$$= 0$$

(c)

$$\langle xy \rangle = 0$$

4.58

(a)

$$1 = \chi^\dagger \chi$$

$$= |A|^2 \begin{pmatrix} 1 + 2i & 2 \end{pmatrix} \begin{pmatrix} 1 - 2i \\ 2 \end{pmatrix}$$

$$= |A|^2 [(1 + 2i)(1 - 2i) + 4]$$

$$= 9|A|^2$$

$$A = \frac{1}{3}$$

(b)

$$\chi = \frac{1}{3} \begin{pmatrix} 1 - 2i \\ 2 \end{pmatrix}$$

You could get $\frac{\hbar}{2}$ with probability $\frac{1}{9}(1 + 2i)(1 - 2i) = \frac{5}{9}$ or $-\frac{\hbar}{2}$ with probability $\frac{4}{9}$.

$$\langle S_z \rangle = \left(\frac{5}{9} - \frac{4}{9} \right) \frac{\hbar}{2} = \frac{\hbar}{18}$$

(c) You could get $\frac{\hbar}{2}$ with probability

$$\frac{1}{9} \frac{(3 + 2i)(3 - 2i)}{2} = \frac{13}{18}$$

or $-\frac{\hbar}{2}$ with probability

$$\frac{1}{9} \frac{(-1 + 2i)(-1 - 2i)}{2} = \frac{5}{18}.$$

$$\langle S_x \rangle = \left(\frac{13}{18} - \frac{5}{18} \right) \frac{\hbar}{2} = \frac{2}{9} \hbar$$

(d) You could get $\frac{\hbar}{2}$ with probability

$$\frac{1}{9} \frac{(1 + 4i)(1 - 4i)}{2} = \frac{17}{18}$$

or $-\frac{\hbar}{2}$ with probability

$$\frac{1}{18}.$$

$$\langle S_y \rangle = \left(\frac{17}{18} - \frac{1}{18} \right) \frac{\hbar}{2} = \frac{4}{9} \hbar$$

123

4.61

The basis states are $|\frac{3}{2}\rangle$, $|\frac{1}{2}\rangle$, $|-\frac{1}{2}\rangle$, and $|-\frac{3}{2}\rangle$. The matrix corresponding to the raising operator is

$$S_+ = \hbar \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the matrix corresponding to the lowering operator is

$$S_- = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}.$$

$S_\pm = S_x \pm S_y$ so

$$S_x = \frac{S_+ + S_-}{2}$$

$$= \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}.$$

The eigenvalues of S_x are $\frac{3}{2}\hbar$, $\frac{1}{2}\hbar$, $-\frac{1}{2}\hbar$, and $-\frac{3}{2}\hbar$, as expected.

4.64

(a) You would get $2\hbar^2$ with probability 1.

(b) You could get 0 with probability $\frac{1}{3}$ or $3\hbar$ with probability $\frac{2}{3}$.

(c) You would get $\frac{3}{4}\hbar^2$ with probability 1.

(d) You could get $\frac{\hbar}{2}$ with probability $\frac{1}{3}$ or $-\frac{\hbar}{2}$ with probability $\frac{2}{3}$.

(e)

$$\sqrt{\frac{1}{3}}\left|\frac{1}{2}\frac{1}{2}\right\rangle|10\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2}-\frac{1}{2}\right\rangle|11\rangle = \frac{1}{\sqrt{3}}\left(\sqrt{\frac{2}{3}}\left|\frac{3}{2}\frac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|\frac{1}{2}\frac{1}{2}\right\rangle\right)$$

$$+ \sqrt{\frac{2}{3}}\left(\frac{1}{\sqrt{3}}\left|\frac{3}{2}\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2}\frac{1}{2}\right\rangle\right)$$

$$= \sqrt{\frac{8}{9}}\left|\frac{3}{2}\frac{1}{2}\right\rangle + \sqrt{\frac{1}{9}}\left|\frac{1}{2}\frac{1}{2}\right\rangle$$

You could measure $\frac{15}{4}\hbar^2$ with probability $\frac{8}{9}$ or $\frac{3}{4}\hbar^2$ with probability $\frac{1}{9}$.

(f) You would measure $\frac{\hbar}{2}$ with probability 1.

(g)

$$R_{21}^* \left(\sqrt{\frac{1}{3}} (Y_1^0)^* \chi_+^\dagger + \sqrt{\frac{2}{3}} (Y_1^1)^* \chi_-^\dagger \right) R_{21} \left(\sqrt{\frac{1}{3}} Y_1^0 \chi_+ + \sqrt{\frac{2}{3}} Y_1^1 \chi_- \right)$$

$$= \frac{1}{3} |R_{21}|^2 (|Y_1^0|^2 + 2|Y_1^1|^2)$$

$$= \frac{1}{3} \frac{1}{24a^5} r^2 e^{-r/a} \left(\frac{3}{4\pi} \cos^2 \theta + \frac{3}{4\pi} \sin^2 \theta \right)$$

$$= \frac{1}{96\pi a^5} r^2 e^{-r/a}$$

(h)

$$\frac{1}{3} |R_{21}|^2 \int_0^\pi \int_0^{2\pi} |Y_1^0|^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{3} \frac{1}{24a^5} r^2 e^{-r/a} \int_0^\pi \int_0^{2\pi} \frac{3}{4\pi} \cos^2 \theta \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{48a^5} r^2 e^{-r/a} \int_0^\pi \cos^2 \theta \sin\theta \, d\theta$$

$$= \frac{1}{72a^5} r^2 e^{-r/a}$$

4.70

(a) At $t = 0$ all particles are in the spin-up state with respect to the z axis $|\uparrow\rangle$. At $t = t_1$ the spin of a fraction f of the particles is measured with respect to the z axis (the z group) and the remaining fraction $1 - f$ is measured with respect to the x axis (the x group). There is no change in the z group so the fraction of particles in the state $|\uparrow\rangle$ is f. The x group is evenly split, so the fraction of particles in either the $|\leftarrow\rangle$ or $|\rightarrow\rangle$ states is $\frac{1}{2}(1 - f)$. At $t = t_2$ the spin of all particles is measured with respect to the x axis. The z group is evenly split and the x group is unchanged so the fraction of particles in either the $|\leftarrow\rangle$ or $|\rightarrow\rangle$ states is $\frac{1}{2}f + \frac{1}{2}(1 - f) = \frac{1}{2}$. The subensemble of particles in the $|\rightarrow\rangle$ state is comprised of particles from both the z and x groups. Of those from the z group, all of them were in the spin-up state at $t = t_1$, and of those from the x group, they also were all in the spin-up state at $t = t_1$. In conclusion: all of the particles in the subensemble had spin-up in the first measurement.

(b)

$$P_+ = \cos^2(\theta_{ab}/2)$$
$$P_- = \sin^2(\theta_{ab}/2)$$
$$P_{++} = \cos^2(\theta_{ab}/2)\cos^2(\theta_{bc}/2)$$
$$P_{+-} = \cos^2(\theta_{ab}/2)\sin^2(\theta_{bc}/2)$$
$$P_{-+} = \sin^2(\theta_{ab}/2)\sin^2(\theta_{bc}/2)$$
$$P_{--} = \sin^2(\theta_{ab}/2)\cos^2(\theta_{bc}/2)$$
$$\frac{P_{++}}{P_{++}+P_{-+}} = \frac{1}{1+P_{-+}/P_{++}}$$
$$= [1+\tan^2(\theta_{ab}/2)\tan^2(\theta_{bc}/2)]^{-1}$$

4.71

(a)

$$\psi_{21m} = \frac{1}{\sqrt{24a^5}} re^{-r/2a} Y_1^m(\theta, \phi)$$

$$\psi_{21-1} = \frac{1}{\sqrt{24a^5}} re^{-r/2a} \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi}$$

$$= \frac{1}{\sqrt{64\pi a^5}} re^{-r/2a} \sin\theta e^{-i\phi}$$

$$\psi_{210} = \frac{1}{\sqrt{24a^5}} re^{-r/2a} \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$= \frac{1}{\sqrt{32\pi a^5}} re^{-r/2a} \cos\theta$$

$$\psi_{211} = -\frac{1}{\sqrt{24a^5}} re^{-r/2a} \sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}$$

$$= -\frac{1}{\sqrt{64\pi a^5}} re^{-r/2a} \sin\theta e^{i\phi}$$

$$\psi_{2p_x} = \frac{1}{\sqrt{32\pi a^3}} \frac{x}{a} e^{-r/2a}$$

$$= \frac{1}{\sqrt{32\pi a^5}} re^{-r/2a} \sin\theta \cos\phi$$

$$= \frac{1}{\sqrt{2}} (\psi_{21-1} - \psi_{211})$$

$$\psi_{2p_y} = \frac{1}{\sqrt{32\pi a^3}} \frac{y}{a} e^{-r/2a}$$

$$= \frac{1}{\sqrt{32\pi a^5}} re^{-r/2a} \sin\theta \sin\phi$$

$$= \frac{i}{\sqrt{2}} (\psi_{21-1} + \psi_{211})$$

$$\psi_{2p_z} = \frac{1}{\sqrt{32\pi a^3}} \frac{z}{a} e^{-r/2a}$$

$$= \frac{1}{\sqrt{32\pi a^5}} re^{-r/2a} \cos\theta$$

$$= \psi_{210}$$

5 Identical Particles

5.1

(a)

$$\mathbf{R} + \frac{\mu}{m_1}\mathbf{r} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} + \frac{m_2}{m_1 + m_2}(\mathbf{r}_1 - \mathbf{r}_2)$$

$$= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_2\mathbf{r}_1 - m_2\mathbf{r}_2}{m_1 + m_2}$$

$$= \mathbf{r}_1$$

$$\mathbf{R} - \frac{\mu}{m_2}\mathbf{r} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} - \frac{m_1}{m_1 + m_2}(\mathbf{r}_1 - \mathbf{r}_2)$$

$$= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 - m_1\mathbf{r}_1 + m_1\mathbf{r}_2}{m_1 + m_2}$$

$$= \mathbf{r}_2$$

Let $\mathbf{r} = (x, y, z)$ and $\mathbf{R} = (X, Y, Z)$, then

$$(\nabla_1)_x = \frac{\partial}{\partial x_1}$$

$$= \frac{\partial X}{\partial x_1}\frac{\partial}{\partial X} + \frac{\partial x}{\partial x_1}\frac{\partial}{\partial x}$$

$$= \frac{\mu}{m_2}(\nabla_R)_x + (\nabla_r)_x$$

$$\nabla_1 = \frac{\mu}{m_2}\nabla_R + \nabla_r$$

$$(\nabla_2)_x = \frac{\partial}{\partial x_2}$$

$$= \frac{\partial X}{\partial x_2}\frac{\partial}{\partial X} + \frac{\partial x}{\partial x_2}\frac{\partial}{\partial x}$$

$$= \frac{\mu}{m_1}(\nabla_R)_x - (\nabla_r)_x$$

$$\nabla_2 = \frac{\mu}{m_1}\nabla_R - \nabla_r$$

(b)

$$\nabla_1^2 \psi = \nabla_1 \cdot \nabla_1 \psi$$

$$= \nabla_1 \cdot \left(\frac{\mu}{m_2} \nabla_R \psi + \nabla_r \psi \right)$$

$$= \frac{\mu}{m_2} \nabla_R \cdot \left(\frac{\mu}{m_2} \nabla_R \psi + \nabla_r \psi \right) + \nabla_r \cdot \left(\frac{\mu}{m_2} \nabla_R \psi + \nabla_r \psi \right)$$

$$= \left(\frac{\mu}{m_2} \right)^2 \nabla_R^2 \psi + 2 \frac{\mu}{m_2} (\nabla_r \cdot \nabla_R) \psi + \nabla_r^2 \psi$$

$$\nabla_2^2 \psi = \nabla_2 \cdot \nabla_2 \psi$$

$$= \nabla_2 \cdot \left(\frac{\mu}{m_1} \nabla_R \psi - \nabla_r \psi \right)$$

$$= \frac{\mu}{m_1} \nabla_R \cdot \left(\frac{\mu}{m_1} \nabla_R \psi - \nabla_r \psi \right) - \nabla_r \left(\frac{\mu}{m_1} \nabla_R \psi - \nabla_r \psi \right)$$

$$= \left(\frac{\mu}{m_1} \right)^2 \nabla_R^2 \psi - 2 \frac{\mu}{m_1} (\nabla_r \cdot \nabla_R) \psi + \nabla_r^2 \psi$$

$$-\frac{\hbar^2}{2m_1} \nabla_1^2 \psi - \frac{\hbar^2}{2m_2} \nabla_2^2 \psi + V\psi = E\psi$$

$$-\frac{\hbar^2}{2m_1} \left[\left(\frac{\mu}{m_2} \right)^2 \nabla_R^2 \psi + 2 \frac{\mu}{m_2} (\nabla_r \cdot \nabla_R) \psi + \nabla_r^2 \psi \right]$$

$$-\frac{\hbar^2}{2m_2} \left[\left(\frac{\mu}{m_1} \right)^2 \nabla_R^2 \psi - 2 \frac{\mu}{m_1} (\nabla_r \cdot \nabla_R) \psi + \nabla_r^2 \psi \right] + V\psi = E\psi$$

$$-\frac{\hbar^2 \mu^2}{2} \left(\frac{1}{m_1 m_2^2} + \frac{1}{m_1^2 m_2} \right) \nabla_R^2 \psi - \frac{\hbar^2}{2} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \nabla_r^2 \psi + V\psi = E\psi$$

$$-\frac{\hbar^2}{2(m_1 + m_2)} \nabla_R^2 \psi - \frac{\hbar^2}{2\mu} \nabla_r^2 \psi + V\psi = E\psi$$

(c)

$$\psi(\mathbf{R}, \mathbf{r}) = \psi_R(\mathbf{R}) \psi_r(\mathbf{r})$$

$$-\frac{\hbar^2}{2(m_1 + m_2)} \nabla_R^2 \psi_R \psi_r - \frac{\hbar^2}{2\mu} \nabla_r^2 \psi_r \psi_R + V(\mathbf{r}) \psi_R \psi_r = E \psi_R \psi_r$$

$$-\frac{\hbar^2}{2(m_1 + m_2)} \frac{\nabla_R^2 \psi_R}{\psi_R} - \frac{\hbar^2}{2\mu} \frac{\nabla_r^2 \psi_r}{\psi_r} + V(\mathbf{r}) = E$$

$$\left(-\frac{\hbar^2}{2(m_1 + m_2)} \frac{\nabla_R^2 \psi_R}{\psi_R} - E_R \right) + \left(-\frac{\hbar^2}{2\mu} \frac{\nabla_r^2 \psi_r}{\psi_r} + V(\mathbf{r}) - E_r \right) = 0$$

5.2

(a)

$$\mu = \frac{m_e m_p}{m_e + m_p}$$

$$E_1 = -\frac{\mu}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2$$

$$\approx -13.5982\,\text{eV}$$

(b)

$$\Delta E_\text{H} = E_{3_H} - E_{2_H}$$

$$= \frac{E_{1_H}}{9} - \frac{E_{1_H}}{4}$$

$$= \left(\frac{1}{9} - \frac{1}{4}\right)E_{1_H}$$

$$\approx 1.888\,\text{eV}$$

$$\lambda_H = 6.569 \times 10^{-7}\,\text{m}$$

$$E_{1_D} = \frac{m_e(m_n + m_p)}{m_e + m_n + m_p}\frac{1}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2$$

$$\Delta E_\text{D} = E_{3_D} - E_{2_D}$$

$$= \left(\frac{1}{9} - \frac{1}{4}\right)E_{1_D}$$

$$\approx 1.889\,\text{eV}$$

$$\lambda_D = 6.567 \times 10^{-7}\,\text{m}$$

$$\Delta\lambda = 1.79 \times 10^{-10}\,\text{m}$$

(c)

$$\mu = \frac{m_e^2}{m_e + m_2}$$

$$= \frac{1}{2}m_e$$

$$E_1' = \frac{1}{2}E_1$$

$$= -6.8\,\text{eV}$$

(d)

$$\mu_p = \frac{m_e m_p}{m_e + m_p}$$

$$\mu_\mu = \frac{m_\mu m_p}{m_\mu + m_p}$$

$$\frac{\mu_\mu}{\mu_p} = \frac{m_\mu m_p}{m_\mu + m_p} \cdot \frac{m_e + m_p}{m_e m_p}$$

$$= \frac{m_\mu(m_e + m_p)}{m_e(m_\mu + m_p)}$$

$$= 206.77 \frac{m_e + m_p}{206.77 m_e + m_p}$$

$$\approx 185.943$$

$$\frac{1}{\lambda} = R'\left(1 - \frac{1}{4}\right)$$

$$= \frac{3}{4}R'$$

$$\lambda = \frac{4}{3R'}$$

$$\approx 6.54 \times 10^{-10}\,\text{m}$$