

# Advanced Engineering Mathematics Ordinary Differential Equations Notes

Chris Doble

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# 1 Introduction to Differential Equations

## 1.1 Definitions and Terminology

- An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation** (DE)
- An **ordinary DE** (ODE) is a DE that contains only ordinary (i.e. non-partial) derivatives of one or more functions with respect to a single independent variable
- A **partial DE** is a DE that contains only partial derivatives of one or more functions of two or more independent variables
- The **order** of a DE is the order of the highest derivative in the equation
- First order ODEs are sometimes written in the **differential form**

$$M(x, y) dx + N(x, y) dy = 0$$

- $n$ -th order ODEs in one dependent variable can be expressed by the **general form**

$$F(x, y, y', \dots, y^{(n)}) = 0$$

- It's possible to solve ODEs in the general form uniquely for the highest derivative  $y^{(n)}$  in terms of the other  $n + 1$  variables, allowing them to be expressed in the **normal form**

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)})$$

- An  $n$ -th order ODE is said be **linear** in the variable  $y$  if it can be expressed in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y - g(x) = 0$$

i.e. the dependent variable  $y$  and all of its derivatives aren't raised to a power or used in nonlinear functions like  $e^y$  or  $\sin y$ , and the coefficients  $a_0, a_1, \dots, a_n$  depend at most on the independent variable  $x$

- A **nonlinear** ODE is one that is not linear
- A **solution** to an ODE is a function  $\phi$ , defined on an interval  $I$  and possessing at least  $n$  derivatives that are continuous on  $I$ , such that

$$F(x, \phi(x), \phi'(x), \dots, \phi^n(x)) = 0 \text{ for all } x \text{ in } I.$$

- The **interval of definition**, **interval of validity**, or the **domain** of a solution is the interval over which the solution is valid
- A solution of a DE that is 0 on an interval  $I$  is said to be a **trivial solution**
- Because solutions to DEs must be differentiable over their interval of validity, discontinuities, etc. must be excluded from the interval
- An **explicit solution** to an ODE is one where the dependent variable is expressed solely in terms of the independent variable and constants
- An **implicit solution** to an ODE is a relation  $G(x, y) = 0$  over an interval  $I$  provided there exists at least one function  $\phi$  that satisfies the relation as well as the ODE on  $I$
- When solving a first-order ODE we usually obtain a solution containing a single arbitrary constant or parameter  $c$ . A solution containing an arbitrary constant represents a set of solution called a **one-parameter family of solutions**
- When solving an  $n$ -th order DE we usually obtain an  **$n$ -parameter family of solutions**
- A solution of a DE that is free from arbitrary parameters is called a **particular solution**
- A **singular solution** is a solution to a DE that isn't a member of a family of solutions
- A **system of ODEs** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable. A solution of such a system is a differentiable function for each equation defined on a common interval  $I$  that satisfy each equation of the system on that interval

## 1.2 Initial Value Problems

- An **initial value problem** is the problem of solving a DE with some given **initial conditions**, e.g. solve

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)})$$

subject to

$$y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$$

- The domain of  $y = f(x)$  differs depending on how it's considered:
  - As a function its domain is all real numbers for which it's defined

- As a solution of a DE its domain is a single interval over which it's defined and differentiable
- As a solution of an initial value problem its domain is a single interval over which it's defined, differentiable, and contains the initial conditions
- An initial value problem may not have any solutions. If it does it may have multiple.
- First-order initial value problems of the form

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

are guaranteed to have a unique solution over an interval  $I$  containing  $x_0$  if  $f(x, y)$  and  $\partial f/\partial y$  are continuous

### 1.3 Differential Equations as Mathematical Models

- A **mathematical model** is a mathematical description of a system or phenomenon
- The **level of resolution** of a model determines how many variables are included in the model
- A simple model of the growth of a population  $P$  is

$$\frac{dP}{dt} = kP$$

where  $k > 0$

- A simple model of radioactive decay of an amount of substance  $A$  is

$$\frac{dA}{dt} = kA$$

where  $k < 0$

- Newton's empirical law of cooling/warming states that the rate of change of the temperature of a body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium

$$\frac{dT}{dt} = k(T - T_m)$$

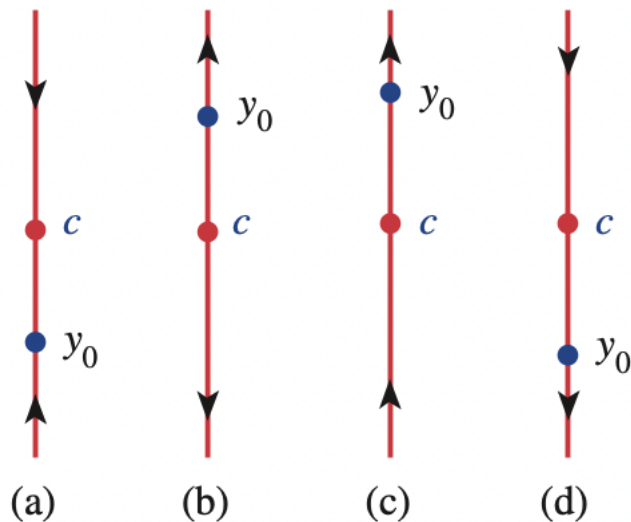
## 2 First-Order Differential Equations

### 2.1 Solution Curves Without a Solution

- An ODE in which the independent variable doesn't appear is said to be **autonomous**, e.g.

$$\frac{dy}{dx} = f(y)$$

- A real number  $c$  is a **critical/equilibrium/stationary point** of an autonomous DE if it is a zero of  $f$
- If  $c$  is a critical point of an autonomous DE, then  $y(x) = c$  is a solution
- A solution of the form  $y(x) = c$  is called an **equilibrium solution**
- We can draw several conclusions about the solutions of an autonomous DE with  $n$  critical points and  $n + 1$  subregions bounded by the critical points:
  - If  $(x_0, y_0)$  is in a subregion, it remains in that subregion for all  $x$
  - By continuity,  $f(y) < 0$  or  $f(y) > 0$  for all  $y$  in a subregion and thus  $y(x)$  can't have maximum/minimum points or oscillate
  - If  $y(x)$  is bounded above by a critical point  $c_1$ , it must approach  $y(x) = c_1$  as  $x \rightarrow -\infty$  or  $x \rightarrow \infty$
  - If  $y(x)$  is bounded above and below by critical points  $c_1$  and  $c_2$ , it must approach  $y(x) = c_1$  as  $x \rightarrow -\infty$  and  $y(x) = c_2$  as  $x \rightarrow \infty$  or vice versa
  - If  $y(x)$  is bounded below by a critical point  $c_1$ , it must approach  $y(x) = c_1$  as  $x \rightarrow -\infty$  or  $x \rightarrow \infty$



**FIGURE 2.1.8** Critical point  $c$  is an attractor in (a), a repeller in (b), and semi-stable in (c) and (d)

- If  $y(x)$  is a solution of an autonomous differential equation  $dy/dx = f(y)$ , then  $y_1(x) = y(x - k)$ , where  $k$  is a constant, is also a solution

## 2.2 Separable Equations

- A first-order ODE of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be **separable** or to have **separate variables**

- A separable first-order ODE can be solved by dividing both sides by  $h(y)$  then integrating both sides with respect to  $x$

$$\begin{aligned}
\frac{dy}{dx} &= g(x)h(y) \\
\frac{1}{h(y)} \frac{dy}{dx} &= g(x) \\
\int \frac{1}{h(y)} \frac{dy}{dx} dx &= \int g(x) dx \\
\int \frac{1}{h(y)} dy &= \int g(x) dx \\
H(y) &= G(x) + c
\end{aligned}$$

- Care should be taken when dividing by  $h(y)$  as it removes constant solutions  $y = r$  where  $h(r) = 0$

## 2.3 Linear Equations

- A first-order DE of the form

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

or in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

is said to be a **linear equation** in the dependent variable  $y$

- When  $g(x) = 0$  or  $f(x) = 0$  the linear equation is said to be **homogeneous** and is solvable via separation of variables, otherwise it is **nonhomogeneous**
- The nonhomogeneous linear equation's solution is the sum of two solutions  $y = y_c + y_p$  where  $y_c$  is a solution of the associated homogeneous equation

$$\frac{dy}{dx} + P(x)y = 0$$

and  $y_p$  is a particular solution of the nonhomogeneous equation

- Nonhomogeneous linear equations can be solved via **variation of parameters**:
  1. Put it into standard form
  2. Determine the **integrating factor**  $e^{\int P(x) dx}$
  3. Multiply by the integrating factor
  4. Recognise that the left hand side of the equation is the derivative of the product of the integrating factor and  $y$

5. Integrate both sides of the equation

6. Solve for  $y$

- The **general solution** of a DE is a family of solutions that contains all possible solutions (except singular solutions)
- A term  $y = f(x)$  in a solution is called a **transient term** if  $f(x) \rightarrow 0$  as  $x \rightarrow \infty$
- When either  $P(x)$  or  $f(x)$  is a piecewise-defined function the equation is then referred to as a **piecewise-linear differential equation** that can be solved by solving each interval in isolation then choosing appropriate constants to ensure the overall solution is continuous
- The **error function** and **complementary error function** are defined

$$\operatorname{erf} x + \operatorname{erfc} x = 1$$
$$\left( \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \right) + \left( \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \right) = 1$$

## 2.4 Exact Equations

- The **differential** of a function  $z = f(x, y)$  is

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

- A differential expression  $M(x, y) dx + N(x, y) dy$  is an **exact differential** in the region  $R$  of the  $xy$ -plane if it corresponds to the differential of some function  $f(x, y)$
- A first-order DE of the form

$$M(x, y) dx + N(x, y) dy = 0$$

is said to be an **exact equation** if the expression on the left side is an exact differential

- A necessary and sufficient condition that  $M(x, y) dx + N(x, y) dy$  be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- Exact differentials can be solved by



1. Integrating  $M(x, y)$  with respect to  $x$  to find an expression for  $f(x, y)$

$$\frac{\partial f}{\partial x} = M(x, y)$$

$$f(x, y) = \int M(x, y) dx + g(y)$$

2. Differentiating  $f(x, y)$  with respect to  $y$  and equating it to  $N(x, y)$  to find  $g'(y)$

$$\frac{\partial f}{\partial y} = N(x, y) = \frac{\partial}{\partial y} \int M(x, y) dx + g'(y)$$

$$g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx$$

3. Integrating  $g'(y)$  with respect to  $y$  to find  $g(y)$  and substituting it into  $f(x, y)$
4. Equating  $f(x, y)$  with an unknown constant  $c$

- $x$  and  $y$  can be swapped in the steps above (i.e. you can start by integrating  $N(x, y)$  with respect to  $y$ , etc.)
- A nonexact DE  $M(x, y) dx + N(x, y) dy = 0$  can sometimes be transformed into an exact DE by finding an appropriate integrating factor

- If  $(M_y - N_x)/N$  is a function of  $x$  alone, then an integrating factor is

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

- If  $(N_x - M_y)/M$  is a function of  $y$  alone, then an integrating factor is

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

## 2.5 Solutions by Substitution

- A function  $f(x, y)$  is said to be a **homogeneous function** of degree  $\alpha$  if

$$f(tx, ty) = t^\alpha f(x, y)$$

- A first-order DE of the form

$$M(x, y) dx + N(x, y) dy = 0$$

is said to be **homogeneous** if both  $M$  and  $N$  are homogeneous functions of the same degree

- To solve a homogeneous first-order DE:

1. Rewrite it as

$$M(x, y) = x^\alpha M(1, u) \text{ and } N(x, y) = x^\alpha N(1, u) \text{ where } u = y/x$$

or

$$M(x, y) = y^\alpha M(v, 1) \text{ and } N(x, y) = y^\alpha N(v, 1) \text{ where } v = x/y$$

2. Substitute  $y = ux$  and  $dy = u dx + x du$  or  $x = vy$  and  $dx = v dy + y dv$  as appropriate
3. Solve the resulting first-order separable DE
4. Substitute  $u = y/x$  or  $v = x/y$  as appropriate

- The DE

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

where  $n$  is any real number is called **Bernoulli's equation**

- For  $n = 0$  and  $n = 1$  Bernoulli's equation is linear
- To solve Bernoulli's equation for  $n \neq 0$  and  $n \neq 1$ :

1. Substitute  $y = u^{1/(1-n)}$  and  $\frac{dy}{dx} = \frac{d}{dx}(u^{1/(1-n)})$
2. Solve the resulting linear equation
3. Substitute  $u = y^{n-1}$

- A DE of the form

$$\frac{dy}{dx} = f(Ax + By + C)$$

can always be reduced to an equation with separable variables by means of the substitution

$$u = Ax + By + C, B \neq 0$$

## 2.6 A Numerical Method

- Approximate values for points on a solution curve near an initial point can be calculated via a **linearization** of the solution curve — a straight line that has the same slope as the initial point and passes through it
- **Euler's method** approximates a solution curve by iteratively stepping along its linearizations

$$y_{n+1} = y_n + hf(x_n, y_n)$$

where  $h$  is the **step size**

## 2.9 Modeling with Systems of First-Order DEs

- In a system of DEs

$$\frac{dx}{dt} = g_1(t, x, y)$$

and

$$\frac{dy}{dt} = g_2(t, x, y)$$

if  $g_1$  and  $g_2$  are linear in  $x$  and  $y$ , i.e.

$$g_1(t, x, y) = c_1x + c_2y + f_1(t)$$

and

$$g_2(t, x, y) = c_3x + c_4y + f_2(t)$$

it is said to be a **linear system**

## 3 Higher-Order Differential Equations

### 3.1 Theory of Linear Equations

- An  **$n$ th-order initial-value problem (IVP)** is to solve

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

subject to

$$y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$$

- If  $a_n(x)$ ,  $a_{n-1}(x)$ ,  $\dots$ ,  $a_1(x)$ ,  $a_0(x)$ , and  $g(x)$  are continuous on an interval  $I$  and  $a_n(x) \neq 0$  for every  $x$  in the interval, then a unique solution exists for the above IVP for every  $x = x_0$  within the interval
- An **initial value problem** is when all of the constraints are located at the same point while a **boundary value problem** is when they're at different points
- Boundary value problems may have many, one, or no solutions
- When  $g(x) = 0$  the DE is said to be **homogeneous**, otherwise it's **non-homogeneous**
- The symbol  $D$  is called a **differential operator** because it transforms a differentiable function into another function

$$Dy = \frac{dy}{dx}$$

- Higher-order derivatives can be expressed as

$$D^n = \frac{d^n y}{dx^n}$$

- An  **$n$ th-order differential operator** is defined to be

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)$$

- As a consequence of the properties of differentiation

$$D(cf(x)) = cDf(x)$$

and

$$D\{f(x) + g(x)\} = Df(x) + Dg(x)$$

- The superposition principle for homogeneous linear  $n$ th-order differential equation states that if  $y_1, y_2, \dots, y_k$  are solutions of the equation on an interval  $I$  then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

where  $c_i$  are arbitrary constants is also a solution on the interval

- A set of functions  $f_1(x), f_2(x), \dots, f_n(x)$  is said to be **linearly dependent** on an interval  $I$  if there exists constants  $c_1, c_2, \dots, c_n$ , not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every  $x$  in the interval. Otherwise it is said to be **linearly independent**

- The **Wronskian** of a set of  $n$  functions that are  $n - 1$  times differentiable is defined as

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

- If  $y_1, y_2, \dots, y_n$  are  $n$  solutions to a homogeneous linear  $n$ th-order differential equation on an interval  $I$  then the set of solutions is **linearly independent** on  $I$  iff  $W(y_1, y_2, \dots, y_n) \neq 0$  for every  $x$  in the interval
- Any set of  $n$  linearly independent solutions of a homogeneous linear  $n$ th-order differential equation on an interval  $I$  is said to be a **fundamental set of solutions** on the interval

- If  $y_1, y_2, \dots, y_n$  are a fundamental set of solutions of a homogeneous linear  $n$ th-order DE on an interval  $I$  then the **general solution** of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x)$$

where  $c_i$  are arbitrary constants

- Another way of saying the above is that any solution on the interval can be expressed as a linear combination of the fundamental set of solutions
- A linear combination of a fundamental set of solutions of a homogenous linear  $n$ th-order DE

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x)$$

is called the **complementary function** of associated nonhomogenous DEs

- If  $y_p$  is any particular solution to a nonhomogeneous linear  $n$ th-order DE on an interval  $I$  and  $y_1, y_2, \dots, y_n$  are a fundamental set of solutions of the associated homogeneous DE on  $I$ , then the **general solution** of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x) + y_p(x)$$

where  $c_i$  are arbitrary constants

- Another way of saying the above is that any solution on the interval can be expressed as  $y = y_c + y_p$
- The superposition for nonhomogeneous linear  $n$ th-order differential equations states that if  $y_{p_1}, y_{p_2}, \dots, y_{p_k}$  are  $k$  particular solutions of a nonhomogeneous linear  $n$ th-order differential equation on an interval  $I$  corresponding, in turn, to  $k$  distinct functions  $g_1, g_2, \dots, g_k$ , then

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x) + \dots + y_{p_k}(x)$$

is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_0(x)y = g_1(x) + g_2(x) + \dots + g_k(x)$$

## 3.2 Reduction of Order

- The **reduction of order** method requires knowledge of one non-trivial solution and comprises the following steps:

1. Recognise that the ratio of two linearly independent functions isn't constant, i.e.

$$u(x) = \frac{y_1(x)}{y_2(x)} \text{ or } y_2(x) = u(x)y_1(x)$$

2. Substitute  $y_2(x) = u(x)y_1(x)$  into the DE — this will result in a DE involving only  $u''$  and  $u'$  which can be treated as a linear first-order DE in  $u' = w$
  3. Solve for  $w$
  4. Substitute  $w = u'$
  5. Integrate to find  $u$
  6. Multiply by  $y_1$  to find  $y_2$
- A formula for the above on a DE in standard form

$$y'' + P(x)y' + Q(x)y = 0$$

is

$$y_2 = y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx$$

### 3.3 Homogeneous Linear Equations with Constant Coefficients

- All solutions to homogenous linear DEs

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where  $a_i$  are real constants and  $a_n \neq 0$  are either exponential functions or constructed from exponential functions

- Substituting a solution  $y = e^{mx}$  we find

$$e^{mx}(a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0) = 0$$

where the term in brackets is called the **auxiliary equation** of the DE

- Thus, the solution  $y = e^{mx}$  is valid if  $m$  is a root of the auxiliary equation
- Real roots correspond to solutions of the form

$$y = ce^{mx}$$

- Complex roots  $\alpha \pm i\beta$  correspond to solutions of the form

$$y_1 = c_1 e^{\alpha x} \cos \beta x \text{ and } y_2 = c_2 e^{\alpha x} \sin \beta x$$

- A root  $m$  of multiplicity  $k$  corresponds to the solutions

$$e^{mx}, xe^{mx}, x^2 e^{mx}, \dots, x^{k-1} e^{mx}$$

### 3.4 Undetermined Coefficients

- The **method of undetermined coefficients** may be used to find a particular solution to nonhomogenous linear differential equations where the input function is comprised of constants, polynomials, exponentials  $e^{\alpha x}$ , sines, and cosines
- To apply the method you:
  1. Solve the associated homogeneous equation
  2. Assume the particular solution has the same form as the input function
  3. If a term in the proposed solution is present in the complementary function, multiply it by  $x^n$  where  $n$  is the smallest positive integer that removes the duplication
  4. Substitute the proposed solution into the DE
  5. Solve for the unknown constants

**TABLE 3.4.1** Trial Particular Solutions

$g(x)$	Form of $y_p$
1. 1 (any constant)	$A$
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A \cos 4x + B \sin 4x$
6. $\cos 4x$	$A \cos 4x + B \sin 4x$
7. $e^{5x}$	$Ae^{5x}$
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. $x^2 e^{5x}$	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$
11. $5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
12. $xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$

### 3.5 Variation of Parameters

- The **method of variation of parameters** can be used to find a particular solution of a nonhomogeneous linear  $n$ th-order DE
- To apply the method you:
  1. Solve the homogeneous equation to find the complementary function
  2. Assume the solution has the form

$$y_p = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x)$$

where  $n$  is the order of the equation and  $y_i$  are the fundamental set of solutions from the complementary equation

3. Convert to standard form by dividing by the leading coefficient

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

4. Solve the system of linear equations

$$\begin{aligned}
 y_1 u'_1 + \cdots + y_n u'_n &= 0 \\
 y'_1 u'_1 + \cdots + y'_n u'_n &= 0 \\
 &\vdots \\
 y_1^{(n-1)} u'_1 + \cdots + y_n^{(n-1)} u'_n &= 0 \\
 y_1^{(n)} u'_1 + \cdots + y_n^{(n)} u'_n &= f(x)
 \end{aligned}$$

via Cramer's method:

(a) Compute the Wronskian of  $y_i$

$$W = \begin{vmatrix} y_1 & \cdots & y_n \\ y'_1 & \cdots & y'_n \\ \vdots & \ddots & \vdots \\ y_1^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

(b) Compute  $u'_i$  for  $i = 1, \dots, n$  where

$$u'_i = \frac{W_i}{W}$$

and  $W_i$  is the determinant of the matrix formed by replacing the  $i$ th column of the Wronskian matrix with the column vector

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(x) \end{bmatrix}$$

5. Integrate each  $u'_i$  to find  $u_i$

### 3.6 Cauchy-Euler Equations

- A **Cauchy-Euler equation** is a linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

- To solve a homogeneous Cauchy-Euler equation you:

1. Assume the equation has a solution of the form  $y = x^m$ , giving

$$\begin{aligned}
 a_n x^n \frac{d^n y}{dx^n} &= a_n x^n m(m-1)(m-2) \cdots (m-n+1) x^{m-n} \\
 &= a_n m(m-1)(m-2) \cdots (m-n+1) x^m
 \end{aligned}$$



and the equation then becomes

$$f(m)x^m = 0$$

where  $f(m)$  is a polynomial in  $m$  known as the auxiliary or characteristic equation, the roots of which form the general solution

2. Solve the auxiliary equation where

– A real root  $m$  corresponds to a solution

$$y = cx^m$$

– Complex roots  $\alpha \pm i\beta$  correspond to solutions

$$x^\alpha (c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x))$$

– A root  $m$  of multiplicity  $k$  corresponds to solutions

$$x^m, x^m \ln x, x^m (\ln x)^2 \dots, x^m (\ln x)^{k-1}$$

- To solve a nonhomogeneous Cauchy-Euler equation you:
  1. Solve the associated homogeneous equation
  2. Find a particular solution via variation of parameters

### 3.7 Nonlinear Equations

- The superposition principle does not hold for nonlinear equations
- Nonlinear second order DEs of the form  $F(x, y', y'') = 0$  where  $y$  is missing can sometimes be solved by:
  1. Substitute  $u = y'$  (and thus  $u' = y''$ )
  2. Solve the resulting DE for  $u$
  3. Integrate to find  $y$
- Nonlinear second order DEs of the form  $F(y, y', y'') = 0$  where  $x$  is missing can sometimes be solved by:

1. Substitute  $u = y'$  and

$$y'' = \frac{du}{dy} \frac{dy}{dx} = u \frac{du}{dy}$$

2. Solve the resulting DE for  $u$

3. Integrate to find  $y$

- Nonlinear initial-value problems can sometimes be solved by substituting the initial conditions into a Taylor series centred at  $x_0$ . The initial conditions can also be substituted into subsequent derivatives to add further terms to the series

### 3.10 Green's Functions

- Green's functions are useful because they allow you to express the solution of a DE in terms of the input function  $g(x)$ , making it easy to see how different input functions change the solution

#### 3.10.1 Initial-Value Problems

- The solution of a second-order IVP

$$y'' + P(x)y' + Q(x)y = f(x), y(x_0) = y_0, y'(x_0) = y_1$$

can be expressed as

$$y = y_h + y_p$$

where  $y_h$  is the solution to the associated homogeneous equation with nonhomogeneous initial conditions

$$y'' + P(x)y' + Q(x)y = 0, y(x_0) = y_0, y'(x_0) = y_1$$

and  $y_p$  is the solution to the nonhomogeneous equation with homogeneous initial conditions

$$y'' + P(x)y' + Q(x)y = f(x), y(x_0) = 0, y'(x_0) = 0$$

- If  $P(x)$  and  $Q(x)$  are constant  $y_h$  can be found via the auxiliary / characteristic equation
- If  $y_1$  and  $y_2$  form a fundamental set of solutions to the associated homogeneous equation, then  $y_p$  is given by

$$y_p(x) = \int_{x_0}^x G(x, t) f(t) dt$$

where  $G(x, y)$  is the Green's function for the differential equation

$$G(x, t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}$$

and  $W(t)$  is the Wronskian

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

#### 3.10.2 Boundary Value Problems

- If  $y_1$  and  $y_2$  are linearly independent solutions of

$$y'' + P(x)y' + Q(x)y = 0$$

on  $[a, b]$  and satisfy the boundary conditions

$$A_1 y_1(a) + B_1 y_1(b) = 0$$

and

$$A_2 y_2(a) + B_2 y_2(b) = 0$$

then the BVP

$$y'' + P(x)y' + Q(x)y = f(x)$$

subject to the same boundary conditions has a particular solution

$$y_p(x) = \int_a^b G(x, t) f(t) dt$$

where  $G(x, t)$  is the Green's function for the differential equation

$$G(x, y) = \begin{cases} \frac{y_1(t)y_2(x)}{W(t)} & a \leq t \leq x \\ \frac{y_1(x)y_2(t)}{W(t)} & x \leq t \leq b \end{cases}$$

and  $W(t)$  is the Wronskian

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

### 3.12 Solving Systems of Linear Equations

- Systems of linear differential equations can be solved in a similar manner to systems of equations, namely by adding and subtracting multiples of different equations to eliminate particular variables
- We can also apply the differential operator  $D$  as part of the elimination process
- Once you have an equation for each dependent variable it's important to substitute them back into the original differential equation to determine the constraints on the parameters — not all of them can be chosen arbitrarily

## 4 The Laplace Transform

### 4.1 Definition of the Laplace Transform

- If a function  $f(t)$  is defined for  $t \geq 0$  and the limit

$$\int_0^\infty K(s, t) f(t) dt = \lim_{b \rightarrow \infty} \int_0^b K(s, t) f(t) dt$$

exist, the integral is said to **exist** or be **convergent**, otherwise it does not exist or is **divergent**

- If a function  $f(t)$  is defined for  $t \geq 0$  then the limit

$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t) dt$$

is called the **Laplace transform** of  $f$  providing the integral converges

- $\mathcal{L}$  is a linear transform, i.e.

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$$

- A function is said to be **piecewise continuous** on  $[0, \infty)$  if, in any interval defined by  $0 \leq a \leq t \leq b$ , there are at most a finite number of points  $t_k$ ,  $k = 1, 2, \dots, n$  ( $t_{k-1} < t_k$ ), at which  $f$  has finite discontinuities and is continuous on each open interval defined by  $t_{k-1} < t < t_k$
- A function is said to be of **exponential order** if there exists constants  $c$ ,  $M > 0$ , and  $T > 0$  such that  $|f(t)| \leq Me^{ct}$  for all  $t > T$
- If  $f(t)$  is piecewise continuous on the interval  $[0, \infty)$  and of exponential order, then  $\mathcal{L}\{f(t)\}$  exists for  $s > c$

## 4.2 The Inverse Transform and Transforms of Derivatives

- $\mathcal{L}^{-1}$  is a linear transform, i.e.

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathcal{L}^{-1}\{F(s)\} + \beta \mathcal{L}^{-1}\{G(s)\}$$

- If  $f, f', \dots, f^{(n-1)}$  are continuous on  $[0, \infty)$ , are of exponential order, and  $f^{(n)}$  is piecewise continuous on  $[0, \infty)$ , then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$

where  $F(s) = \mathcal{L}\{f(t)\}$

- The Laplace transform can be used to solve linear IVPs:
  1. Take the Laplace transform of the DE, resulting in an algebraic equation in  $F(s) = \mathcal{L}\{f(s)\}$  where  $f(s)$  is the goal
  2. Solve the equation for  $F(s)$
  3. Apply the inverse Laplace transform to find  $f(s)$

## 4.3 Translation Theorems

- The **first translation theorem** states that if

$$\mathcal{L}\{f(t)\} = F(s)$$

then

$$\mathcal{L}\{e^{at} f(t)\} = F(s - a)$$

and

$$\mathcal{L}^{-1}\{F(s - a)\} = \mathcal{L}^{-1}\{F(s)|_{s \rightarrow s-a}\} = e^{at} f(t)$$

- The **unit step function** or **Heaviside function** is defined to be

$$\mathcal{U}(t-a) = \begin{cases} 0 & 0 \leq t < a \\ 1 & t \geq a \end{cases}$$

- The **second translation theorem** states that if  $a > 0$  and

$$\mathcal{L}\{f(s)\} = F(s)$$

then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

and

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$$

- If  $f$  and  $\mathcal{U}$  aren't shifted by the same amount when applying the second translation theorem, an alternate form can be applied

$$\mathcal{L}\{f(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$$

#### 4.4 Additional Operational Properties

- If  $F(s) = \mathcal{L}\{f(t)\}$  and  $n = 1, 2, 3, \dots$ , then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

- If functions  $f$  and  $g$  are piecewise continuous on the interval  $[0, \infty)$  then the **convolution** of  $f$  and  $g$ , denoted  $f * g$ , is a function defined by the integral

$$f * g = \int_0^t f(\tau)g(t-\tau) d\tau$$

- The **convolution theorem** states that if  $f$  and  $g$  are piecewise continuous on  $[0, \infty)$  and of exponential order, then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$

and

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g$$

- Under the convolution theorem if  $g(t) = 1$  then  $\mathcal{L}\{g(t)\} = G(s) = 1/s$ ,

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{F(s)}{s},$$

and

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau) d\tau$$

- **Volterra integral equations** have the form

$$f(t) = g(t) + \int_0^t f(\tau)g(t - \tau) d\tau$$

and can be solved by using the convolution theorem while taking the Laplace transform

- An **integro-differential equation** is an equation that involves both integrals and derivatives of a function
- If  $f(t)$  is piecewise continuous on  $[0, \infty)$ , of exponential order, and periodic with period  $T$ , then

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-st}} \int_0^T e^{-st} f(t) dt$$