Advanced Engineering Mathematics Vectors, Matrices, and Vector Calculus by Dennis G. Zill Notes

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Contents

1 Vectors

1.1 Vectors in 2-Space

- The zero vector can be assigned any direction
- The vectors $\mathbf{i}$ and $\mathbf{j}$ are known as the **standard basis vectors** for $\mathbb{R}^2$

1.2 Vectors in 3-Space

- In $\mathbb{R}^3$ the octant in which all coordinates are positive is known as the **first octant**. There is no agreement for naming the other seven octants.

1.3 Dot Product

- The **dot product** is also known as the **inner product** or the **scalar product** and is denoted $\mathbf{a} \cdot \mathbf{b}$

- Two non-zero vectors are orthogonal iff their dot product is 0
- The zero vector is considered orthogonal to all vectors
- The angles α, β, and γ between a vector and the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively are called the **direction angles** of the vector
- The cosines of a vectors direction angles (the **direction cosines**) can be calculated as

$$\begin{aligned}
\cos\alpha &= \frac{\mathbf{a}\cdot\mathbf{i}}{||\mathbf{a}||||\mathbf{i}||} \\
&= \frac{a_1}{||\mathbf{a}||} \\
\cos\beta &= \frac{\mathbf{a}\cdot\mathbf{j}}{||\mathbf{a}||||\mathbf{j}||} \\
&= \frac{a_2}{||\mathbf{a}||} \\
\cos\gamma &= \frac{\mathbf{a}\cdot\mathbf{k}}{||\mathbf{a}||||\mathbf{k}||} \\
&= \frac{a_3}{||\mathbf{a}||}
\end{aligned}$$

  Equivalently, these can be calculated as the components of the unit vector $\mathbf{a}/||\mathbf{a}||$.

- To find the component of a vector $\mathbf{a}$ in the direction of a vector $\mathbf{b}$

$$\mathrm{comp}_{\mathbf{b}}\mathbf{a} = ||\mathbf{a}||\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{b}||}$$

- To project a vector $\mathbf{a}$ onto a vector $\mathbf{b}$

$$\mathrm{proj}_{\mathbf{b}}\mathbf{a} = (\mathrm{comp}_{\mathbf{b}}\mathbf{a})\frac{\mathbf{b}}{||\mathbf{b}||} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b}$$

1.4 Cross Product

- The cross product is only defined in $\mathbb{R}^3$
- The **scalar triple product** of vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is defined as

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- The area of a parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$ is $||\mathbf{a}\times\mathbf{b}||$

- The area of a triangle with sides $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{2}||\mathbf{a} \times \mathbf{b}||$
- The volume of a paralleleipied with sides $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ iff $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar

1.5 Lines and Planes in 3-Space

- There is a unique line between any two points $\mathbf{r_1}$ and $\mathbf{r_2}$ in 3-space. The equation for that line is

$$\mathbf{r} = \mathbf{r_1} + t(\mathbf{r_2} - \mathbf{r_1}) = \mathbf{r_1} + t\mathbf{a}$$

  where t is called a **parameter**, the nonzero vector $\mathbf{a}$ is called a **direction vector**, and its components are called **direction numbers**.

- Equating the components of the equation above we find

$$\begin{aligned} x &= r_1 + ta_1 \\ y &= r_2 + ta_2 \\ z &= r_3 + ta_3. \end{aligned}$$

  These are the **parametric equations** for the line through $\mathbf{r_1}$ and $\mathbf{r_2}$.

- By solving the parametric equations for t and equating the results we find the **symmetric equations** for the line

$$t = \frac{x - r_1}{a_1} = \frac{y - r_2}{a_2} = \frac{z - r_3}{a_3}.$$

- Given a point P_1 and a vector $\mathbf{n}$, there exists only one plane containing P_1 with $\mathbf{n}$ normal. The vector from P_1 to another point P on that plane will be perpendicular to $\mathbf{n}$, so the equation for the plane is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$

  where $\mathbf{r} = \overrightarrow{OP}$ and $\mathbf{r_1} = \overrightarrow{OP_1}$. If

$$\mathbf{n} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$$

  the cartesian form of this equation is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

  and is called the **point-normal form**.

- The graph of any equation $ax + by + cz + d = 0$, where a, b, and c are not all zero, is a plane with the normal vector $\mathbf{n} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$.
- Given three noncollinear points, a normal vector can be found by forming two vectors from two pairs of points and take their cross product.
- A line and a plane that aren't parellel intersect at a single point.
- Two planes that aren't parallel must intersect in a line.

1.6 Vector Spaces

- The length of a vector is called its **norm**
- The process of multipying a vector by the reciprocal of its norm is called **normalizing** the vector
- Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ are said to be orthogonal if $\mathbf{a} \cdot \mathbf{b} = 0$

Definition 7.6.1 Vector Space

Let V be a set of elements on which two operations called **vector addition** and **scalar multiplication** are defined. Then V is said to be a **vector space** if the following 10 properties are satisfied.

Axioms for Vector Addition:

(*i*) If $\mathbf{x}$ and $\mathbf{y}$ are in V, then $\mathbf{x} + \mathbf{y}$ is in V.

(*ii*) For all $\mathbf{x}, \mathbf{y}$ in V, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$. ← commutative law

(*iii*) For all $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in V, $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$. ← associative law

(*iv*) There is a unique vector $\mathbf{0}$ in V such that

$$\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x}.$$ ← zero vector

(*v*) For each $\mathbf{x}$ in V, there exists a vector $-\mathbf{x}$ such that

$$\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}.$$ ← negative of a vector

Axioms for Scalar Multiplication:

(*vi*) If k is any scalar and $\mathbf{x}$ is in V, then $k\mathbf{x}$ is in V.

(*vii*) $k(\mathbf{x} + \mathbf{y}) = k\mathbf{x} + k\mathbf{y}$ ← distributive law

(*viii*) $(k_1 + k_2)\mathbf{x} = k_1\mathbf{x} + k_2\mathbf{x}$ ← distributive law

(*ix*) $k_1(k_2\mathbf{x}) = (k_1k_2)\mathbf{x}$

(*x*) $1\mathbf{x} = \mathbf{x}$

- If a subset W of a vector space V is itself a vector space under the operations of vector addition and scalar multiplication defined on V, then W is called a **subspace** of V
- Every vector space has at least two subspaces: itself and the zero subspace $\{\mathbf{0}\}$
- A set of vectors $\{\mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_n}\}$ is said to be **linearly independent** if the only constants satisfying the equation

$$k_1\mathbf{x_1} + k_2\mathbf{x_2} + \cdots + k_n\mathbf{x_n} = \mathbf{0}$$

  are $k_1 = k_2 = \cdots = k_n = 0$. If the set of vectors is not linearly independent it is said to be **linearly dependent**.
- If a set of vectors $B = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ in a vector space V is linearly independent and every vector in V can be expressed as a linear combination of vectors in B then B is said to be a **basis** for V.
- The number of vectors in a basis B for a vector space V is said to be the **dimension** of the space.

- If the basis of a vector space contains a finite number of vectors, then the space is **finite dimensional**; otherwise it is **infinite dimensional**.

- If S denotes any set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ in a vector space V, then the set of all linear combinations of the vectors in S

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

  is called the **span** of the vectors and is denoted $\mathrm{Span}(S)$.

- $\mathrm{Span}(S)$ is a subspace of V and is said to be a subspace spanned by its vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$.

- If $V = \mathrm{Span}(S)$ then S is said to be a **spanning set** for the vector space V or that S **spans** V.

1.7 Gram–Schmidt Orthogonalization Process

- An **orthonormal basis** is a basis whose vectors are mutually orthogonal and are unit vectors.

- If $B = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is an orthonormal basis for $\mathbb{R}^n$ then an arbitrary vector $\mathbf{u}$ can be expressed as

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{u} \cdot \mathbf{w}_2)\mathbf{w}_2 + \cdots + (\mathbf{u} \cdot \mathbf{w}_n)\mathbf{w}_n$$

- The **Gram-Schmidt Orthogonalization Process** is a process for converting any basis of a vector space into an orthonormal basis. First the basis vectors are made orthogonal to each other, then they are normalized. More specifically, to convert a basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ into an orthogonal basis $B' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

  1. Let $\mathbf{v}_1 = \mathbf{u}_1$
  2. Let $\mathbf{v}_2 = \mathbf{u}_2 - \mathrm{proj}_{\mathbf{v}_1}\mathbf{u}_2$
  3. $\ldots$
  4. Let $\mathbf{v}_n = \mathbf{u}_n - \mathrm{proj}_{\mathbf{v}_1}\mathbf{u}_n - \mathrm{proj}_{\mathbf{v}_2}\mathbf{u}_n - \cdots - \mathrm{proj}_{\mathbf{v}_{n-1}}\mathbf{u}_n$

  and to convert B' into an orthonormal basis $B'' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$, normalize each $\mathbf{v}_i$, $i = 1, 2, \ldots, n$.

2 Matrices

2.1 Matrix Algebra

- Vectors can be written as horizontal or vertical arrays of numbers

- A **matrix** is any rectangular array of numbers or functions

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- The numbers or functions in the array are called the **elements** or **entries** of the matrix
- If a matrix has m rows and n columns we say that its **size** is m by n or $m \times n$
- An $n \times n$ matrix is called a **square** matrix of **order** n
- The entry in the ith row and the jth column of an $m \times n$ matrix **A** is written a_{ij}
- An $m \times 1$ matrix

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

is called a **column vector**

- A $1 \times n$ matrix

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

is called a **row vector**

Definition 8.1.6 Matrix Multiplication

Let **A** be a matrix having m rows and p columns, and let **B** be a matrix having p rows and n columns. The **product AB** is the $m \times n$ matrix

$$\mathbf{AB} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1p}b_{p1} & \cdots & a_{11}b_{1n} + a_{12}b_{2n} + \cdots + a_{1p}b_{pn} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2p}b_{p1} & \cdots & a_{21}b_{1n} + a_{22}b_{2n} + \cdots + a_{2p}b_{pn} \\ \vdots & & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mp}b_{p1} & \cdots & a_{m1}b_{1n} + a_{m2}b_{2n} + \cdots + a_{mp}b_{pn} \end{pmatrix}$$

$$= \left(\sum_{k=1}^{p} a_{ik}b_{kj} \right)_{m \times n}.$$

- Matrix multiplication is associative, i.e. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
- Matrix multiplication is distributive, i.e. $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$ and $(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{BA}+\mathbf{CA}$
- The **transpose** of an $m \times n$ matrix $\mathbf{A}$ is an $n \times m$ matrix $\mathbf{A}^T$

$$\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

i.e. the matrix is flipped along the main diagonal

Theorem 8.1.2 Properties of Transpose

Suppose $\mathbf{A}$ and $\mathbf{B}$ are matrices and k a scalar. Then

(*i*) $(\mathbf{A}^T)^T = \mathbf{A}$ ← transpose of a transpose
(*ii*) $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$ ← transpose of a sum
(*iii*) $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ ← transpose of a product
(*iv*) $(k\mathbf{A})^T = k\mathbf{A}^T$ ← transpose of a scalar multiple

- A matrix that consists of all zero entries is called a **zero matrix**
- A square matrix is said to be a **triangular matrix** if all of its entries above or below the main diagonal are zeroes. More specifically they are called **lower triangular** and **upper triangular** matrices, respectively.
- A square matrix is called a **diagonal matrix** if all entries not on the main diagonal are 0.
- A square matrix whose entries on the main diagonal are all equal is called a **scalar matrix**
- A square matrix that has the property $\mathbf{A} = \mathbf{A}^T$ is called a **symmetric matrix**