

Advanced Engineering Mathematics Vectors, Matrices, and Vector Calculus by Dennis G. Zill Problems

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1 Vectors

1.1 Vectors in 2-Space

1.1.1

(a) $3\mathbf{a} = 6\mathbf{i} + 12\mathbf{j}$

(b) $\mathbf{a} + \mathbf{b} = \mathbf{i} + 8\mathbf{j}$

(c) $\mathbf{a} - \mathbf{b} = 3\mathbf{i}$

(d) $\|\mathbf{a} + \mathbf{b}\| = \sqrt{1 + 8^2} = \sqrt{65}$

(e) $\|\mathbf{a} - \mathbf{b}\| = 3$

1.1.9

(a) $4\mathbf{a} - 2\mathbf{b} = \langle 6, -14 \rangle$

(b) $-3\mathbf{a} - 5\mathbf{b} = \langle 2, 4 \rangle$

1.1.15

$\overrightarrow{P_1P_2} = \langle 2, 5 \rangle$

1.1.19

$(1, 18)$

1.1.21

(a) Yes

(b) Yes

(c) Yes

(d) No

(e) Yes

(f) Yes

1.1.25

(a) $\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\langle 2, 2 \rangle}{\sqrt{2^2+2^2}} = \frac{1}{2\sqrt{2}} \langle 2, 2 \rangle = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

(b) $\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$

1.1.31

$2\frac{\mathbf{a}}{\|\mathbf{a}\|} = 2\frac{\langle 3, 7 \rangle}{\sqrt{3^2+7^2}} = \frac{2}{\sqrt{58}} \langle 3, 7 \rangle = \langle \frac{6}{\sqrt{58}}, \frac{14}{\sqrt{58}} \rangle$

1.1.37

$\mathbf{x} = -(\mathbf{a} + \mathbf{b})$

1.1.41

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$$

$$\mathbf{b} = \mathbf{i} + \mathbf{j}$$

$$\mathbf{c} = \mathbf{i} - \mathbf{j}$$

$$\mathbf{i} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$$

$$\mathbf{j} = \frac{1}{2}(\mathbf{b} - \mathbf{c})$$

$$\begin{aligned}\mathbf{a} &= 2\left(\frac{1}{2}(\mathbf{b} + \mathbf{c})\right) + 3\left(\frac{1}{2}(\mathbf{b} - \mathbf{c})\right) \\ &= \mathbf{b} + \mathbf{c} + \frac{3}{2}\mathbf{b} - \frac{3}{2}\mathbf{c} \\ &= \frac{5}{2}\mathbf{b} - \frac{1}{2}\mathbf{c}\end{aligned}$$

1.1.43

$$y = \frac{1}{4}x^2 + 1$$

$$y(2) = 2$$

$$y' = \frac{1}{2}x$$

$$y'(2) = 1$$

$$\mathbf{v} = \pm \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

1.1.45

(a)

$$\mathbf{F}_n = \mathbf{F} \cos \theta$$

$$\mathbf{F}_g = \mathbf{F} \sin \theta$$

$$||\mathbf{F}_f|| = \mu ||\mathbf{F}_n||$$

$$||-\mathbf{F}_g|| = \mu ||\mathbf{F}_n||$$

$$||-\mathbf{F} \sin \theta|| = \mu ||\mathbf{F} \cos \theta||$$

$$||\mathbf{F}|| \sin \theta = \mu ||\mathbf{F}|| \cos \theta$$

$$\tan \theta = \mu$$

(b) $\theta = \arctan \mu \approx 31^\circ$

1.1.47

$$\begin{aligned}
 F_x &= \frac{qQ}{4\pi\epsilon_0} \int_{-a}^a \frac{L dy}{2a(L^2 + y^2)^{3/2}} \\
 &= \frac{LqQ}{8\pi\epsilon_0} \int_{-a}^a (L^2 + y^2)^{-3/2} dy \\
 &= \frac{LqQ}{8\pi\epsilon_0} \frac{2a}{L^2\sqrt{a^2 + L^2}} \\
 &= \frac{aqQ}{4\pi\epsilon_0 L\sqrt{a^2 + L^2}} \\
 F_y &= -\frac{qQ}{4\pi\epsilon_0} \int_{-a}^a \frac{y dy}{2a(L^2 + y^2)^{3/2}} \\
 &= 0 \\
 \mathbf{F} &= \left\langle \frac{1}{4\pi\epsilon_0} \frac{qQ}{L\sqrt{a^2 + L^2}}, 0 \right\rangle
 \end{aligned}$$

1.1.49

Let the three sides of the triangle be vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} . The triangle is closed so it must be the case that

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0.$$

This gives

$$\mathbf{c} = -(\mathbf{a} + \mathbf{b}).$$

The vector from the midpoint of side \mathbf{a} to the midpoint of side \mathbf{b} is

$$\left(\mathbf{a} + \frac{1}{2}\mathbf{b}\right) - \frac{1}{2}\mathbf{a} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$$

which is parallel with \mathbf{c} and half its length.

1.2 Vectors in 3-Space

1.2.7

A plane at $z = 5$ parallel with the x - y plane.

1.2.9

A line parallel to the z axis at $x = 2$ and $y = 3$.

1.2.13

- (a) $(0, 5, 4)$, $(-2, 0, 4)$, $(-2, 5, 0)$
- (b) $(-2, 5, -2)$
- (c) $(3, 5, 4)$

1.2.15

The planes $x = 0$, $y = 0$, and $z = 0$.

1.2.17

$$(-1, 2, -3)$$

1.2.19

The planes $z = \pm 5$.

1.2.21

$$\sqrt{(6-3)^2 + (4+1)^2 + (8-2)^2} = \sqrt{9+25+36} = \sqrt{70}$$

1.2.31

$$\begin{aligned}\sqrt{(2-x)^2 + (1-2)^2 + (1-3)^2} &= \sqrt{21} \\ (2-x)^2 + 1 + 4 &= 21 \\ (2-x)^2 &= 16 \\ 2-x &= \pm 4 \\ x &= 2 \pm 4 \\ &= -2 \text{ or } 6\end{aligned}$$

1.2.33

$$(4, \frac{1}{2}, \frac{3}{2})$$

1.2.37

$$(-3, -6, 1)$$

1.3 Dot Product**1.3.1**

$$\mathbf{a} \cdot \mathbf{b} = 12$$

1.3.11

$$\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right) \mathbf{b} = \frac{12}{30} \mathbf{b} = \langle -\frac{2}{5}, \frac{4}{5}, 2 \rangle$$

1.3.13

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta = 25\sqrt{2}$$

1.3.17

$$\begin{aligned}\mathbf{a} \cdot \mathbf{v} &= 0 \\ 3x_1 + y_1 - 1 &= 0\end{aligned}$$

$$\begin{aligned}\mathbf{b} \cdot \mathbf{v} &= 0 \\ -3x_1 + 2y_2 + 2 &= 0\end{aligned}$$

$$\begin{aligned}3y_2 + 1 &= 0 \\ y_2 &= -\frac{1}{3}\end{aligned}$$

$$\begin{aligned}3x_1 - \frac{1}{3} - 1 &= 0 \\ x_1 &= \frac{4}{9}\end{aligned}$$

$$\mathbf{v} = \left\langle \frac{4}{9}, -\frac{1}{3}, 1 \right\rangle$$

1.3.19

$$\begin{aligned}\mathbf{a} \cdot \mathbf{c} &= \mathbf{a} \cdot \left(\mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \mathbf{a} \right) \\ &= \mathbf{a} \cdot \mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \mathbf{a} \cdot \mathbf{a} \\ &= 0\end{aligned}$$

1.3.21

$$\begin{aligned}\|\mathbf{a}\| &= \sqrt{3^2 + 1^2} \\ &= \sqrt{10}\end{aligned}$$

$$\begin{aligned}\|\mathbf{b}\| &= \sqrt{2^2 + 2^2} \\ &= \sqrt{8} \\ &= 2\sqrt{2}\end{aligned}$$

$$\mathbf{a} \cdot \mathbf{b} = 4$$

$$\begin{aligned}\theta &= \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \\ &= \arccos \frac{4}{(\sqrt{10})(2\sqrt{2})} \\ &= \arccos \frac{1}{\sqrt{5}} \\ &\approx 63^\circ\end{aligned}$$

1.3.25

$$\begin{aligned}\|\mathbf{a}\| &= \sqrt{1^2 + 2^2 + 3^2} \\ &= \sqrt{14}\end{aligned}$$

$$\begin{aligned}\cos \alpha &= \frac{1}{\sqrt{14}} \\ \alpha &\approx 75^\circ\end{aligned}$$

$$\begin{aligned}\cos \beta &= \frac{2}{\sqrt{14}} \\ \beta &\approx 58^\circ\end{aligned}$$

$$\begin{aligned}\cos \gamma &= \frac{3}{\sqrt{14}} \\ \gamma &\approx 37^\circ\end{aligned}$$

1.3.29

$$\begin{aligned}
 \overrightarrow{AD} &= \langle s, -s, s \rangle \\
 \|\overrightarrow{AD}\| &= s\sqrt{3} \\
 \overrightarrow{AB} &= \langle s, 0, 0 \rangle \\
 \|\overrightarrow{AB}\| &= s \\
 \theta &= \arccos \frac{\overrightarrow{AD} \cdot \overrightarrow{AB}}{\|\overrightarrow{AD}\| \|\overrightarrow{AB}\|} \\
 &= \arccos \frac{s^2}{s^2\sqrt{3}} \\
 &= \arccos \frac{1}{\sqrt{3}} \\
 &\approx 55^\circ
 \end{aligned}$$

1.3.33

$$\begin{aligned}
 \text{comp}_{\mathbf{b}} \mathbf{a} &= \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} \\
 &= \frac{5}{7}
 \end{aligned}$$

1.3.37

$$\begin{aligned}
 \text{comp}_{\overrightarrow{OP}} \mathbf{a} &= \frac{\mathbf{a} \cdot \overrightarrow{OP}}{\|\overrightarrow{OP}\|} \\
 &= \frac{72}{\sqrt{109}}
 \end{aligned}$$

1.3.39

$$\begin{aligned}
 \text{proj}_{\mathbf{b}} \mathbf{a} &= \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \right) \mathbf{b} \\
 &= \frac{35}{25} \mathbf{b} \\
 &= \left\langle -\frac{21}{5}, \frac{28}{5} \right\rangle
 \end{aligned}$$

1.3.43

$$\begin{aligned}
\mathbf{a} + \mathbf{b} &= \langle 3, 4 \rangle \\
\text{proj}_{\mathbf{a}+\mathbf{b}} \mathbf{a} &= \left(\frac{\mathbf{a} \cdot (\mathbf{a} + \mathbf{b})}{(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})} \right) (\mathbf{a} + \mathbf{b}) \\
&= \frac{24}{25} (\mathbf{a} + \mathbf{b}) \\
&= \left\langle \frac{72}{25}, \frac{96}{25} \right\rangle
\end{aligned}$$

1.3.45

$$W = \mathbf{F} \cdot \mathbf{d} = Fd \cos \theta = 1000$$

1.3.47

(a) $W = 0$

(b)

$$\begin{aligned}
\|\mathbf{d}\| &= \sqrt{4^2 + 3^2} \\
&= 5
\end{aligned}$$

$$\mathbf{F} = F \hat{\mathbf{d}}$$

$$= F \frac{\mathbf{d}}{\|\mathbf{d}\|}$$

$$= F \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$$

$$= \langle 24, 18 \rangle$$

$$W = \mathbf{F} \cdot \mathbf{d}$$

$$= 150 \text{ J}$$

1.4 Cross Product**1.4.1**

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 0 & 3 & 5 \end{vmatrix} \\
&= -5\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}
\end{aligned}$$

1.4.11

$$\begin{aligned}\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & -4 \\ -3 & 1 & 1 \end{vmatrix} \\ &= 6\mathbf{i} + 14\mathbf{j} + 4\mathbf{k}\end{aligned}$$

1.4.17

(a)

$$\begin{aligned}\mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} \\ &= \mathbf{j} - \mathbf{k} \\ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 0 & 1 & -1 \end{vmatrix} \\ &= -\mathbf{i} + \mathbf{j} + \mathbf{k}\end{aligned}$$

1.4.19

$2\mathbf{k}$

1.4.21

$$\begin{aligned}\mathbf{k} \times (2\mathbf{i} - \mathbf{j}) &= (\mathbf{k} \times 2\mathbf{i}) - (\mathbf{k} \times \mathbf{j}) \\ &= \mathbf{i} + 2\mathbf{j}\end{aligned}$$

1.4.23

$$\begin{aligned}[(2\mathbf{k}) \times (3\mathbf{j})] \times (4\mathbf{j}) &= (-6\mathbf{i}) \times (4\mathbf{j}) \\ &= -24\mathbf{k}\end{aligned}$$

1.4.37

$12\mathbf{i} - 9\mathbf{j} + 18\mathbf{k}$

1.4.53

$$\begin{aligned}
\mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 6 & -6 \\ \frac{5}{2} & 3 & \frac{1}{2} \end{vmatrix} \\
&= 21\mathbf{i} - 14\mathbf{j} - 21\mathbf{k} \\
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= 4 \times 21 + 6 \times (-14) \\
&= 0
\end{aligned}$$

They are coplanar.

1.5 Lines and Planes in 3-Space**1.5.1**

$$\mathbf{r} = \langle 1, 2, 1 \rangle + t\langle 2, 3, -3 \rangle$$

1.5.7

$$\begin{aligned}
x &= 2 + 4t \\
y &= 3 - 4t \\
z &= 5 + 3t
\end{aligned}$$

1.5.13

$$\begin{aligned}
x &= 1 + 9t \\
y &= 4 + 10t \\
z &= -9 + 7t \\
\frac{x-1}{9} &= \frac{y-4}{10} = \frac{z+9}{7}
\end{aligned}$$

1.5.19

$$\begin{aligned}
x &= 4 + 3t \\
y &= 6 + \frac{1}{2}t \\
z &= -7 - \frac{3}{2}t \\
\frac{x-4}{3} &= \frac{y-6}{1/2} = -\frac{z+7}{3/2}
\end{aligned}$$

1.5.23

$$\begin{aligned}x &= 6 + 2t \\y &= 4 - 3t \\z &= -2 + 6t\end{aligned}$$

1.5.25

$$\begin{aligned}x &= 2 + t \\y &= -2 \\z &= 15\end{aligned}$$

1.5.29

$$(0, 5, 15), (5, 0, \frac{15}{2}), (10, -5, 0)$$

1.5.31

$$\begin{aligned}4 + t_x &= 6 + 2t_x \\t_x &= -2\end{aligned}$$

$$\begin{aligned}5 + t_y &= 11 + 4t_y \\t_y &= -2\end{aligned}$$

$$\begin{aligned}-1 + 2t_z &= -3 + t_z \\t_z &= -2\end{aligned}$$

$$(2, 3, -5)$$

1.5.35

$$\begin{aligned}\mathbf{a} &= \langle -1, 2, -2 \rangle \\||\mathbf{a}|| &= 3 \\\mathbf{b} &= \langle 2, 3, -6 \rangle \\||\mathbf{b}|| &= 7 \\\theta &= \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}|| ||\mathbf{b}||} \\&\approx 40.37^\circ\end{aligned}$$

1.5.37

$$\begin{aligned}
 \mathbf{a} &= \langle 1, 1, 1 \rangle \\
 \mathbf{b} &= \langle -2, 1, -5 \rangle \\
 \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ -2 & 1 & -5 \end{vmatrix} \\
 &= \langle -6, 3, 3 \rangle \\
 x &= 4 - 6t \\
 y &= 1 + 3t \\
 z &= 6 + 3t
 \end{aligned}$$

1.5.39

$$\begin{aligned}
 \langle 2, -3, 4 \rangle \cdot (\mathbf{r} - \langle 5, 1, 3 \rangle) &= 0 \\
 2(x - 5) - 3(y - 1) + 4(z - 3) &= 0 \\
 2x - 3y + 4z - 19 &= 0
 \end{aligned}$$

1.5.45

$$\begin{aligned}
 \mathbf{a} &= \langle 3, 5, 2 \rangle \\
 \mathbf{b} &= \langle 2, 3, 1 \rangle \\
 \mathbf{c} &= \langle -1, -1, 4 \rangle \\
 \mathbf{a} - \mathbf{c} &= \langle 4, 6, -2 \rangle \\
 \mathbf{b} - \mathbf{c} &= \langle 3, 4, -3 \rangle \\
 (\mathbf{a} - \mathbf{c}) \times (\mathbf{b} - \mathbf{c}) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 4 & 6 & -2 \\ 3 & 4 & -3 \end{vmatrix} \\
 &= \langle -10, 6, -2 \rangle \\
 \mathbf{n} \cdot (\mathbf{r} - \mathbf{c}) &= 0 \\
 \langle -10, 6, -2 \rangle \cdot (\langle x, y, z \rangle - \langle -1, -1, 4 \rangle) &= 0 \\
 -10(x + 1) + 6(y + 1) - 2(z - 4) &= 0 \\
 -10x + 6y - 2z + 4 &= 0
 \end{aligned}$$

1.5.51

$$\begin{aligned}
 \langle 1, 1, -4 \rangle \cdot (\mathbf{r} - \langle 2, 3, -5 \rangle) &= 0 \\
 (x - 2) + (y - 3) - 4(z + 5) &= 0 \\
 x + y - 4z &= 25
 \end{aligned}$$

1.5.63

- (a) Not perpendicular
- (b) Not perpendicular
- (c) Perpendicular
- (d) Perpendicular

1.5.65

$$5x - 4y - 9t = 8$$

$$x + 4y + 3t = 4$$

$$6x - 6t = 12$$

$$x = 2 + t$$

$$y = \frac{1}{2} - t$$

$$z = t$$

1.5.69

$$2(1 + 2t) - 3(2 - t) + 2(-3t) = -7$$

$$t = -3$$

$$x = -5$$

$$y = 5$$

$$z = 9$$

1.5.73

$$x + y - 4t = 2$$

$$2x - y + t = 10$$

$$3x - 3t = 12$$

$$x = 4 + t$$

$$2(4 + t) - y + t = 10$$

$$8 + 2t - y + t = 10$$

$$y = -2 + 3t$$

$$z = t$$

$$x = 5 + t$$

$$y = 6 + 3t$$

$$z = -12 + t$$

1.5.75

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 5 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \langle -6, 2, 4 \rangle$$

$$\mathbf{n} \cdot (\mathbf{r} - \langle 4, 0, 1 \rangle) = 0$$

$$-6(x - 4) + 2y + 4(z - 1) = 0$$

$$-6x + 2y + 4z = -20$$

$$3x - y - 2z = 10$$

1.6 Vector Spaces**1.6.1**

Violates axiom 6

1.6.3

Violates axiom 10

1.6.5

Vector space

1.6.7

Violates axiom 2

1.6.9

Vector space

1.6.11

Subspace

1.6.13

Not a subspace

1.6.15

Subspace

1.6.17

Subspace

1.6.19

Not a subspace

1.6.23

(a)

$$k_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 + k_3 \mathbf{u}_3 = \mathbf{0}$$

$$k_1 \langle 1, 0, 0 \rangle + k_2 \langle 1, 1, 0 \rangle + k_3 \langle 1, 1, 1 \rangle = \mathbf{0}$$

$$k_3 = 0$$

$$k_2 + k_3 = 0$$

$$k_2 = 0$$

$$k_1 + k_2 + k_3 = 0$$

$$k_1 = 0$$

(b)

$$\mathbf{a} = 7\mathbf{u}_1 - 12\mathbf{u}_2 + 8\mathbf{u}_3$$

1.6.25

Dependent

1.6.27

Independent

1.6.29 $f(x)$ is undefined at $x = -3$ and $x = -1$.**1.6.31**

$$\begin{aligned}
||x|| &= \sqrt{(x, x)} \\
&= \sqrt{\int_0^{2\pi} x^2 dx} \\
&= \sqrt{\left[\frac{1}{3}x^3\right]_0^{2\pi}} \\
&= \sqrt{\frac{8}{3}\pi^3} \\
||\sin x|| &= \sqrt{(\sin x, \sin x)} \\
&= \sqrt{\int_0^{2\pi} \sin^2 x dx} \\
&= \sqrt{\left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_0^{2\pi}} \\
&= \sqrt{\pi}
\end{aligned}$$

1.7 Gram–Schmidt Orthogonalization Process

1.7.1

$$\begin{aligned}\left\langle \frac{12}{13}, \frac{5}{13} \right\rangle \cdot \left\langle \frac{5}{13}, -\frac{12}{13} \right\rangle &= 0 \\ \sqrt{\left(\frac{12}{13}\right)^2 + \left(\frac{5}{13}\right)^2} &= 1 \\ \mathbf{u} &= \left(\left\langle 4, 2 \right\rangle \cdot \left\langle \frac{12}{13}, \frac{5}{13} \right\rangle \right) \left\langle \frac{12}{13}, \frac{5}{13} \right\rangle \\ &\quad + \left(\left\langle 4, 2 \right\rangle \cdot \left\langle \frac{5}{13}, -\frac{12}{13} \right\rangle \right) \left\langle \frac{5}{13}, -\frac{12}{13} \right\rangle \\ &= \left(\frac{58}{13} \right) \left\langle \frac{12}{13}, \frac{5}{13} \right\rangle - \left(\frac{4}{13} \right) \left\langle \frac{5}{13}, -\frac{12}{13} \right\rangle\end{aligned}$$

1.7.3

$$\begin{aligned}\langle 1, 0, 1 \rangle \cdot \langle 0, 1, 0 \rangle &= 0 \\ \langle 1, 0, 1 \rangle \cdot \langle -1, 0, 1 \rangle &= 0 \\ \langle 0, 1, 0 \rangle \cdot \langle -1, 0, 1 \rangle &= 0 \\ B' &= \left\{ \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle, \langle 0, 1, 0 \rangle, \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle \right\} \\ \mathbf{u} &= -\frac{3}{\sqrt{2}} \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle + 7 \langle 0, 1, 0 \rangle - \frac{23}{\sqrt{2}} \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle\end{aligned}$$

1.7.5

(a)

$$B = \{\langle -3, 2 \rangle, \langle -1, -1 \rangle\}$$

$$\mathbf{v}_1 = \mathbf{u}_1$$

$$= \langle -3, 2 \rangle$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{v}_1} \mathbf{u}_2$$

$$= \langle -1, -1 \rangle - \left(\frac{\langle -1, -1 \rangle \cdot \langle -3, 2 \rangle}{\langle -3, 2 \rangle \cdot \langle -3, 2 \rangle} \right) \langle -3, 2 \rangle$$

$$= \langle -1, -1 \rangle - \frac{1}{13} \langle -3, 2 \rangle$$

$$= \left\langle -\frac{10}{13}, -\frac{15}{13} \right\rangle$$

$$\mathbf{w}_1 = \left\langle -\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right\rangle$$

$$\mathbf{w}_2 = \sqrt{\frac{169}{325}} \left\langle -\frac{10}{13}, -\frac{15}{13} \right\rangle$$

$$= \frac{\sqrt{13}}{5} \left\langle -\frac{10}{13}, -\frac{15}{13} \right\rangle$$

$$= \left\langle -\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}} \right\rangle$$

1.7.9

$$B = \{\langle 1, 1, 0 \rangle, \langle 1, 2, 2 \rangle, \langle 2, 2, 1 \rangle\}$$

$$\mathbf{v}_1 = \langle 1, 1, 0 \rangle$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{v}_1} \mathbf{u}_2$$

$$= \langle 1, 2, 2 \rangle - \left(\frac{\langle 1, 2, 2 \rangle \cdot \langle 1, 1, 0 \rangle}{\langle 1, 1, 0 \rangle \cdot \langle 1, 1, 0 \rangle} \right) \langle 1, 1, 0 \rangle$$

$$= \langle 1, 2, 2 \rangle - \frac{3}{2} \langle 1, 1, 0 \rangle$$

$$= \langle -\frac{1}{2}, \frac{1}{2}, 2 \rangle$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{\mathbf{v}_1} \mathbf{u}_3 - \text{proj}_{\mathbf{v}_2} \mathbf{u}_3$$

$$= \langle 2, 2, 1 \rangle - \left(\frac{\langle 2, 2, 1 \rangle \cdot \langle 1, 1, 0 \rangle}{\langle 1, 1, 0 \rangle \cdot \langle 1, 1, 0 \rangle} \right) \langle 1, 1, 0 \rangle$$

$$- \left(\frac{\langle 2, 2, 1 \rangle \cdot \langle -\frac{1}{2}, \frac{1}{2}, 2 \rangle}{\langle -\frac{1}{2}, \frac{1}{2}, 2 \rangle \cdot \langle -\frac{1}{2}, \frac{1}{2}, 2 \rangle} \right) \langle -\frac{1}{2}, \frac{1}{2}, 2 \rangle$$

$$= \langle 2, 2, 1 \rangle - 2 \langle 1, 1, 0 \rangle - \frac{4}{9} \langle -\frac{1}{2}, \frac{1}{2}, 2 \rangle$$

$$= \langle \frac{2}{9}, -\frac{2}{9}, \frac{1}{9} \rangle$$

$$\mathbf{w}_1 = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle$$

$$\mathbf{w}_2 = \langle -\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}} \rangle$$

$$\mathbf{w}_3 = 3 \langle \frac{2}{9}, -\frac{2}{9}, \frac{1}{9} \rangle$$

$$= \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle$$

1.7.17

$$B = \{1, x, x^2\}$$

$$\mathbf{v}_1 = \mathbf{u}_1$$

$$= 1$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{v}_1} \mathbf{u}_2$$

$$= \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1$$

$$= x - \frac{\int_{-1}^1 x \, dx}{\int_{-1}^1 dx}$$

$$= x - \frac{\left[\frac{1}{2} x^2 \right]_{-1}^1}{2}$$

$$= x$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{\mathbf{v}_1} \mathbf{u}_3 - \text{proj}_{\mathbf{v}_2} \mathbf{u}_3$$

$$= \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2$$

$$= x^2 - \frac{\int_{-1}^1 x^2 \, dx}{\int_{-1}^1 dx} - \frac{\int_{-1}^1 x^3 \, dx}{\int_{-1}^1 x^2 \, dx} x$$

$$= x^2 - \frac{\left[\frac{1}{3} x^3 \right]_{-1}^1}{2} - \frac{\left[\frac{1}{4} x^4 \right]_{-1}^1}{\left[\frac{1}{3} x^3 \right]_{-1}^1} x$$

$$= x^2 - \frac{1}{3}$$

1.7.19

$$\begin{aligned}
\|\mathbf{v}_1\|^2 &= \int_{-1}^1 dx \\
&= 2 \\
\mathbf{w}_1 &= \frac{1}{\sqrt{2}} \\
\|\mathbf{v}_2\|^2 &= \int_{-1}^1 x^2 dx \\
&= \left[\frac{1}{3} x^3 \right]_{-1}^1 \\
&= \frac{2}{3} \\
\mathbf{w}_2 &= \frac{\sqrt{3}}{\sqrt{6}} x \\
\|\mathbf{v}_3\|^2 &= \int_{-1}^1 \left(x^2 - \frac{1}{3} \right)^2 dx \\
&= \int_{-1}^1 \left(x^4 - \frac{2}{3} x^2 + \frac{1}{9} \right) dx \\
&= \left[\frac{1}{5} x^5 - \frac{2}{9} x^3 + \frac{1}{9} x \right]_{-1}^1 \\
&= \frac{1}{5} - \frac{2}{9} + \frac{1}{9} + \frac{1}{5} - \frac{2}{9} + \frac{1}{9} \\
&= \frac{2}{5} - \frac{2}{9} \\
&= \frac{8}{45} \\
\mathbf{w}_3 &= \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3} \right) \\
&= \frac{5}{2\sqrt{10}} (3x^2 - 1)
\end{aligned}$$

1.7.21

$$\begin{aligned}
(\mathbf{p}, \mathbf{w}_1) &= \int_{-1}^1 \frac{1}{\sqrt{2}} (9x^2 - 6x + 5) dx \\
&= \frac{1}{\sqrt{2}} [3x^3 - 3x^2 + 5x]_{-1}^1 \\
&= \frac{1}{\sqrt{2}} (3 - 3 + 5 + 3 + 3 + 5) \\
&= \frac{16}{\sqrt{2}} \\
(\mathbf{p}, \mathbf{w}_2) &= \int_{-1}^1 \frac{3}{\sqrt{6}} x (9x^2 - 6x + 5) dx \\
&= \frac{3}{\sqrt{6}} \left[\frac{9}{4} x^4 - 2x^3 + \frac{5}{2} x^2 \right]_{-1}^1 \\
&= \frac{3}{\sqrt{6}} \left(\frac{9}{4} - 2 + \frac{5}{2} - \frac{9}{4} - 2 - \frac{5}{2} \right) \\
&= \frac{3}{\sqrt{6}} \left(\frac{9}{4} - \frac{8}{4} + \frac{10}{4} - \frac{9}{4} - \frac{8}{4} - \frac{10}{4} \right) \\
&= -\frac{12}{\sqrt{6}} \\
(\mathbf{p}, \mathbf{w}_3) &= \int_{-1}^1 \frac{5}{2\sqrt{10}} (3x^2 - 1)(9x^2 - 6x + 5) dx \\
&= \frac{5}{2\sqrt{10}} \int_{-1}^1 (27x^4 - 18x^3 + 6x^2 + 6x - 5) dx \\
&= \frac{5}{2\sqrt{10}} \left[\frac{27}{5} x^5 - \frac{9}{2} x^4 + 2x^3 + 3x^2 - 5x \right]_{-1}^1 \\
&= \frac{5}{2\sqrt{10}} \left(\frac{27}{5} - \frac{9}{2} + 2 + 3 - 5 + \frac{27}{5} + \frac{9}{2} + 2 - 3 - 5 \right) \\
&= \frac{5}{2\sqrt{10}} \left(\frac{54}{10} - \frac{45}{10} + \frac{20}{10} + \frac{30}{10} - \frac{50}{10} + \frac{54}{10} + \frac{45}{10} + \frac{20}{10} - \frac{30}{10} - \frac{50}{10} \right) \\
&= \frac{5}{2\sqrt{10}} \frac{48}{10} \\
&= \frac{12}{\sqrt{10}} \\
\mathbf{p} &= \frac{16}{\sqrt{2}} \mathbf{w}_1 - \frac{12}{\sqrt{6}} \mathbf{w}_2 + \frac{12}{\sqrt{10}} \mathbf{w}_3
\end{aligned}$$

1.8 Chapter in Review

1.8.1

True

1.8.3

$$\mathbf{u} = \langle 5, -2, 1 \rangle$$

$$\mathbf{v} = \langle 2, 3, -4 \rangle$$

False

1.8.5

True

1.8.7

True

1.8.9

True

1.8.11

$$9\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

1.8.13

$$\begin{aligned} (-\mathbf{k}) \times (5\mathbf{j}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -1 \\ 0 & 5 & 0 \end{vmatrix} \\ &= 5\mathbf{i} \end{aligned}$$

1.8.15

$$\| -12\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} \| = \sqrt{12^2 + 4^2 + 6^2} = 14$$

1.8.17

$$\langle -6, 1, -7 \rangle$$

1.8.19

$$\begin{aligned}x &= 1 + t \\y &= -2 + 3t \\z &= -1 + 2t\end{aligned}$$

$$\begin{aligned}x + 2y - z &= 13 \\(1 + t) + 2(-2 + 3t) - (-1 + 2t) &= 13 \\1 + t - 4 + 6t + 1 - 2t &= 13 \\-2 + 5t &= 13 \\t &= 3\end{aligned}$$

$$\begin{aligned}x &= 4 \\y &= 7 \\z &= 5\end{aligned}$$

1.8.21

$$\begin{aligned}\overrightarrow{P_1P_2} &= \vec{P_2} - \vec{P_1} \\ \vec{P_2} &= \overrightarrow{P_1P_2} + \vec{P_1} \\ &= \langle 3, 5, -4 \rangle + \langle 2, 1, 7 \rangle \\ &= \langle 5, 6, 3 \rangle\end{aligned}$$

1.8.23

$$\mathbf{a} \cdot \mathbf{b} = -36\sqrt{2}$$

1.8.25

$$x = 12, y = -8, z = 6$$

1.8.27

$$\begin{aligned}\frac{1}{2}(\mathbf{a} \times \mathbf{b}) &= \frac{1}{2} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & -1 & 2 \end{vmatrix} \\ &= \frac{1}{2} \langle 5, -4, -7 \rangle \\ &= \left\langle \frac{5}{2}, -2, -\frac{7}{2} \right\rangle\end{aligned}$$

The area is $\sqrt{\left(\frac{5}{2}\right)^2 + (-2)^2 + \left(-\frac{7}{2}\right)^2} = \frac{3}{2}\sqrt{10}$

1.8.29

2

1.8.31

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -2 & 1 \end{vmatrix} \\ &= \langle 1, -1, -3 \rangle \\ \|\mathbf{a} \times \mathbf{b}\| &= \sqrt{11} \\ \text{norm}(\mathbf{a} \times \mathbf{b}) &= \left\langle \frac{1}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, -\frac{3}{\sqrt{11}} \right\rangle\end{aligned}$$

1.8.33

$$\text{comp}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} = \frac{10}{5} = 2$$

1.8.35

$$\begin{aligned}\mathbf{a} &= \langle 1, 2, -2 \rangle \\ \mathbf{b} &= \langle 4, 3, 0 \rangle \\ \mathbf{a} + \mathbf{b} &= \langle 5, 5, -2 \rangle \\ \text{proj}_{\mathbf{a}}(\mathbf{a} + \mathbf{b}) &= \left(\frac{(\mathbf{a} + \mathbf{b}) \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}} \right) \mathbf{a} \\ &= \frac{19}{9} \langle 1, 2, -2 \rangle \\ &= \left\langle \frac{19}{9}, \frac{38}{9}, -\frac{38}{9} \right\rangle\end{aligned}$$

1.8.37

(a)

(b) A plane with normal \mathbf{a}

1.8.39

$$\frac{x-7}{4} = \frac{y-3}{-2} = \frac{z+5}{6}$$

1.8.41

$$\begin{aligned}\langle -2, 3, 1 \rangle \cdot \langle 2, 1, 1 \rangle &= -4 + 3 + 1 \\ &= 0\end{aligned}$$

$$\begin{aligned}1 - 2t &= 1 + 2s \\ t &= -s\end{aligned}$$

$$3t = -4 + s$$

$$\begin{aligned}t &= -\frac{4}{3} + \frac{s}{3} \\ &= -\frac{4}{3} - \frac{t}{3}\end{aligned}$$

$$\begin{aligned}\frac{4}{3}t &= -\frac{4}{3} \\ t &= -1\end{aligned}$$

$$s = 1$$

$$\langle 3, -3, 0 \rangle$$

1.8.43

$$\mathbf{u} = \langle 1, 4, -2 \rangle$$

$$\mathbf{v} = \langle 1, 1, 3 \rangle$$

$$\mathbf{n} = \mathbf{u} \times \mathbf{v}$$

$$\begin{aligned}&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & -2 \\ 1 & 1 & 3 \end{vmatrix} \\ &= \langle 14, -5, -3 \rangle\end{aligned}$$

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{v}) = 0$$

$$\langle 14, -5, -3 \rangle \cdot (\langle x, y, z \rangle - \langle 1, 1, 3 \rangle) = 0$$

$$14(x - 1) - 5(y - 1) - 3(z - 3) = 0$$

$$14x - 5y - 3z = 0$$

1.8.45

$$\begin{aligned}\mathbf{F} &= \left\langle \frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}}, 0 \right\rangle \\ \mathbf{d} &= \langle 3, 3, 0 \rangle \\ \mathbf{F} \cdot \mathbf{d} &= 30\sqrt{2} \text{ J}\end{aligned}$$

1.8.47

$$\begin{aligned}\mathbf{F}_1 &= \langle 200, 0, 0 \rangle \\ \mathbf{F}_2 &= \left\langle \frac{200}{\sqrt{2}}, \frac{200}{\sqrt{2}}, 0 \right\rangle \\ \mathbf{F}_2 &= \mathbf{F}_1 + \mathbf{F}_3 \\ \mathbf{F}_3 &= \mathbf{F}_2 - \mathbf{F}_1 \\ &= \left\langle \frac{200}{\sqrt{2}}, \frac{200}{\sqrt{2}}, 0 \right\rangle - \langle 200, 0, 0 \rangle \\ &= \left\langle \frac{200}{\sqrt{2}} - 200, \frac{200}{\sqrt{2}}, 0 \right\rangle \\ \|\mathbf{F}_3\| &= \sqrt{\left(\frac{200}{\sqrt{2}} - 200\right)^2 + \left(\frac{200}{\sqrt{2}}\right)^2} \\ &= \sqrt{\frac{40000}{2} - \frac{80000}{\sqrt{2}} + 40000 + \frac{40000}{2}} \\ &= 200\sqrt{2\left(1 - \frac{1}{\sqrt{2}}\right)} \\ &\approx 153 \text{ lb}\end{aligned}$$

2 Matrices

2.1 Matrix Algebra

2.1.1

$$2 \times 4$$

2.1.3

$$3 \times 3$$

2.1.5

$$3 \times 4$$

2.1.7

No

2.1.9

No

2.1.11

$$\begin{aligned}x &= y - 2 \\ 3x - 2 &= y\end{aligned}$$

$$\begin{aligned}2x - 2 &= 2 \\ 2x &= 4 \\ x &= 2\end{aligned}$$

$$\begin{aligned}2 &= y - 2 \\ y &= 4\end{aligned}$$

2.1.13

$$\begin{aligned}c_{23} &= 9 \\ c_{12} &= 12\end{aligned}$$

2.1.15

$$(a) \begin{pmatrix} 2 & 11 \\ 2 & -1 \end{pmatrix}$$

$$(b) \begin{pmatrix} -6 & 1 \\ 14 & -19 \end{pmatrix}$$

$$(c) \begin{pmatrix} 2 & 28 \\ 12 & -12 \end{pmatrix}$$

2.1.17

$$(a) \begin{pmatrix} -11 & 6 \\ 17 & -22 \end{pmatrix}$$

$$(b) \begin{pmatrix} -32 & 27 \\ -4 & -1 \end{pmatrix}$$

$$(c) \begin{pmatrix} 19 & -18 \\ -30 & 31 \end{pmatrix}$$

$$(d) \begin{pmatrix} 19 & 6 \\ 3 & 22 \end{pmatrix}$$

2.1.21

$$(a) 180$$

$$(b) \begin{pmatrix} 4 & 8 & 10 \\ 8 & 16 & 20 \\ 10 & 20 & 25 \end{pmatrix}$$

$$(c) \begin{pmatrix} 6 \\ 12 \\ -5 \end{pmatrix}$$

2.1.23

$$(a) \begin{pmatrix} 7 & 38 \\ 10 & 75 \end{pmatrix}$$

$$(b) \begin{pmatrix} 7 & 38 \\ 10 & 75 \end{pmatrix}$$

2.1.25

$$\begin{pmatrix} -14 \\ 1 \end{pmatrix}$$

2.1.27

$$\begin{pmatrix} -38 \\ -2 \end{pmatrix}$$

2.1.29

$$4 \times 5$$

2.1.41

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

2.1.43

$$\begin{aligned} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} ax + \frac{1}{2}by \\ \frac{1}{2}bx + cy \end{pmatrix} \\ &= ax^2 + \frac{1}{2}bxy + \frac{1}{2}bxy + cy^2 \\ &= ax^2 + bxy + cy^2 \end{aligned}$$

2.1.45

$$\langle -1, 1 \rangle$$

2.1.47

$$\langle -2, 0 \rangle$$

2.1.49

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2.1.51

(b)

$$\begin{pmatrix} x_S \\ y_S \\ z_S \end{pmatrix} = \begin{pmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

2.2 Systems of Linear Algebraic Equations

2.2.1

$$\begin{pmatrix} 1 & -1 & | & 11 \\ 4 & 3 & | & -5 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -1 & | & 11 \\ 0 & 7 & | & -49 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -1 & | & 11 \\ 0 & 1 & | & -7 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & | & 4 \\ 0 & 1 & | & -7 \end{pmatrix}$$

2.2.5

$$\begin{pmatrix} 1 & -1 & -1 & \big| & -3 \\ 2 & 3 & 5 & \big| & 7 \\ 1 & -2 & 3 & \big| & -11 \end{pmatrix} \\
 \begin{pmatrix} 1 & -1 & -1 & \big| & -3 \\ 0 & 5 & 7 & \big| & 13 \\ 0 & -1 & 4 & \big| & -8 \end{pmatrix} \\
 \begin{pmatrix} 1 & -1 & -1 & \big| & -3 \\ 0 & 1 & \frac{7}{5} & \big| & \frac{13}{5} \\ 0 & -1 & 4 & \big| & -8 \end{pmatrix} \\
 \begin{pmatrix} 1 & -1 & -1 & \big| & -3 \\ 0 & 1 & \frac{7}{5} & \big| & \frac{13}{5} \\ 0 & 0 & \frac{27}{5} & \big| & -\frac{27}{5} \end{pmatrix} \\
 \begin{pmatrix} 1 & -1 & -1 & \big| & -3 \\ 0 & 1 & \frac{7}{5} & \big| & \frac{13}{5} \\ 0 & 0 & 1 & \big| & -1 \end{pmatrix} \\
 \begin{pmatrix} 1 & -1 & 0 & \big| & -4 \\ 0 & 1 & 0 & \big| & 4 \\ 0 & 0 & 1 & \big| & -1 \end{pmatrix} \\
 \begin{pmatrix} 1 & 0 & 0 & \big| & 0 \\ 0 & 1 & 0 & \big| & 4 \\ 0 & 0 & 1 & \big| & -1 \end{pmatrix}$$

2.3 Rank of a Matrix

2.3.1

$$\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \\
 \begin{pmatrix} 1 & -\frac{1}{3} \\ 1 & 3 \end{pmatrix} \\
 \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & \frac{10}{3} \end{pmatrix}$$

Rank 2

2.3.3

$$\begin{pmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ -1 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}$$
$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 6 & 3 & 9 \\ -1 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}$$
$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Rank 1

2.3.5

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 4 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 3 \\ 0 & 3 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 3 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 9 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

Rank 3

2.3.7

$$\begin{pmatrix} 1 & -2 \\ 3 & -6 \\ 7 & -1 \\ 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ 0 & 13 \\ 0 & 13 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ 0 & 13 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Rank 2

2.3.11

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & -3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

Linearly independent

2.3.15

5

2.3.17

$\text{rank}(\mathbf{A}) = 2$

2.4 Determinants

2.4.1

9

2.4.3

1

2.4.5

$$\begin{aligned}M_{33} &= \begin{vmatrix} 0 & 2 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} \\ &= -2 \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} \\ &= 2\end{aligned}$$

2.4.7

$$\begin{aligned}C_{34} &= (-1)^{3+4} \begin{vmatrix} 0 & 2 & 4 \\ 1 & 2 & -2 \\ 1 & 1 & 1 \end{vmatrix} \\ &= - \left(-2 \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} + 4 \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \right) \\ &= 10\end{aligned}$$

2.4.9

-7

2.4.11

17

2.4.13

$$\begin{aligned}(1 - \lambda)(2 - \lambda) - 6 &= 2 - \lambda - 2\lambda + \lambda^2 - 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda + 1)(\lambda - 4)\end{aligned}$$

2.4.15

-48

2.4.23

$$\begin{aligned}\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 2 & 3 & 4 \end{vmatrix} &= \begin{vmatrix} y & z \\ 3 & 4 \end{vmatrix} - \begin{vmatrix} x & z \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} x & y \\ 2 & 3 \end{vmatrix} \\ &= 4y - 3z - 4x + 2z + 3x - 2y \\ &= -x + 2y - z\end{aligned}$$

2.4.29

$$\begin{aligned}\begin{vmatrix} (-3 - \lambda) & 10 \\ 2 & (5 - \lambda) \end{vmatrix} &= 0 \\ (-3 - \lambda)(5 - \lambda) - 20 &= 0 \\ -15 + 3\lambda - 5\lambda + \lambda^2 - 20 &= 0 \\ \lambda^2 - 2\lambda - 35 &= 0 \\ (\lambda - 7)(\lambda + 5) &= 0 \\ \lambda &= -5 \text{ or } 7\end{aligned}$$

2.5 Properties of Determinants

2.5.1

8.5.4

2.5.3

8.5.7

2.5.5

8.5.5

2.5.7

8.5.3

2.5.9

8.5.1

2.5.11

-5

2.5.13

-5

2.5.15

5

2.5.17

80

2.5.19

-105

2.5.25

$$\begin{aligned}\mathbf{A}\mathbf{A} &= \mathbf{I} \\ \det \mathbf{A} \cdot \det \mathbf{A} &= \det \mathbf{I} \\ (\det \mathbf{A})^2 &= 1 \\ \det \mathbf{A} &= \pm 1\end{aligned}$$

2.5.27

$$\begin{aligned}\begin{vmatrix} a & a+1 & a+2 \\ b & b+1 & b+2 \\ c & c+1 & c+2 \end{vmatrix} &= \begin{vmatrix} a & 1 & 2 \\ b & 1 & 2 \\ c & 1 & 2 \end{vmatrix} \\ &= \begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} \\ &= 0\end{aligned}$$

2.5.29

$$\begin{aligned}\begin{vmatrix} 1 & 1 & 5 \\ 4 & 3 & 6 \\ 0 & -1 & 1 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 5 \\ 0 & -1 & -14 \\ 0 & -1 & 1 \end{vmatrix} \\ &= - \begin{vmatrix} 1 & 1 & 5 \\ 0 & 1 & 14 \\ 0 & 0 & 15 \end{vmatrix} \\ &= -15\end{aligned}$$

2.5.37

$$\begin{aligned}
 \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix} \\
 &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & c^2-a^2-(b+a)(c-a) \end{vmatrix} \\
 &= (b-a)(c^2-a^2-(b+a)(c-a)) \\
 &= (b-a)(c^2-a^2-bc+ab-ac+a^2) \\
 &= (b-a)(c^2+ab-ac-bc) \\
 &= (b-a)(c-a)(c-b)
 \end{aligned}$$

2.5.39

$$\begin{aligned}
 a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13} &= (-1)(4) + (2)(5) + (1)(-6) \\
 &= 0 \\
 a_{13}C_{12} + a_{23}C_{22} + a_{33}C_{32} &= (2)(5) + (1)(-7) + (1)(-3) \\
 &= 0
 \end{aligned}$$

2.5.41

$$\begin{aligned}
 \mathbf{A} + \mathbf{B} &= \begin{pmatrix} 10 & 0 \\ 0 & -3 \end{pmatrix} \\
 \det(\mathbf{A} + \mathbf{B}) &= -30 \\
 \det \mathbf{A} &= 10 \\
 \det \mathbf{B} &= -31 \\
 -30 &\neq 10 - 31
 \end{aligned}$$

2.6 Inverse of a Matrix

2.6.3

$$\begin{aligned}
 \det \mathbf{A} &= 9 \\
 \mathbf{A}^{-1} &= \frac{1}{9} \begin{pmatrix} 1 & 1 \\ -4 & 5 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{9} & \frac{1}{9} \\ -\frac{4}{9} & \frac{5}{9} \end{pmatrix}
 \end{aligned}$$

2.6.5

$$\det \mathbf{A} = 12$$

$$\begin{aligned}\mathbf{A}^{-1} &= \frac{1}{12} \begin{pmatrix} 2 & 0 \\ 3 & 6 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{6} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}\end{aligned}$$

2.6.7

$$\begin{aligned}\det \mathbf{A} &= (1) \begin{vmatrix} 4 & 4 \\ -1 & 1 \end{vmatrix} - (3) \begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} + (5) \begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} \\ &= 8 + 6 - 30 \\ &= -16\end{aligned}$$

$$\begin{aligned}\mathbf{A}^{-1} &= -\frac{1}{16} \begin{pmatrix} 8 & 2 & -6 \\ -8 & -4 & 4 \\ -8 & 6 & -2 \end{pmatrix}^T \\ &= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{8} & \frac{1}{4} & -\frac{3}{8} \\ \frac{3}{8} & -\frac{1}{4} & \frac{1}{8} \end{pmatrix}\end{aligned}$$

2.6.15

$$\begin{aligned}&\left(\begin{array}{cc|cc} 6 & -2 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{array} \right) \\ &\left(\begin{array}{cc|cc} 1 & -\frac{1}{3} & \frac{1}{6} & 0 \\ 0 & 1 & 0 & \frac{1}{4} \end{array} \right) \\ &\left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{6} & \frac{1}{12} \\ 0 & 1 & 0 & \frac{1}{4} \end{array} \right)\end{aligned}$$

2.6.17

$$\begin{pmatrix} 1 & 3 & | & 1 & 0 \\ 5 & 3 & | & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & | & 1 & 0 \\ 0 & -12 & | & -5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & | & 1 & 0 \\ 0 & 1 & | & \frac{5}{12} & -\frac{1}{12} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & | & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & | & \frac{5}{12} & -\frac{1}{12} \end{pmatrix}$$

2.6.27

$$\begin{aligned} (\mathbf{AB})^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} \\ &= \begin{pmatrix} \frac{2}{3} & \frac{4}{3} \\ -\frac{1}{3} & \frac{5}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{5}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ -\frac{17}{12} & \frac{55}{12} \end{pmatrix} \end{aligned}$$

2.6.29

$$\begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$$

2.6.31

$$\begin{aligned} \begin{pmatrix} 4 & -3 \\ x & -4 \end{pmatrix} &= \frac{1}{3x-16} \begin{pmatrix} -4 & -x \\ 3 & 4 \end{pmatrix}^T \\ &= \frac{1}{3x-16} \begin{pmatrix} -4 & 3 \\ -x & 4 \end{pmatrix} \\ -1 &= \frac{1}{3x-16} \\ 16-3x &= 1 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

2.6.45

$$\begin{aligned}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 4 \\ 14 \end{pmatrix} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= -\frac{1}{3} \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 14 \end{pmatrix} \\ &= -\frac{1}{3} \begin{pmatrix} -18 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ -2 \end{pmatrix}\end{aligned}$$

2.6.49

$$\begin{aligned}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix} \\ \det \mathbf{A} &= \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} \\ &= 1 - 6 \\ &= -5 \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= -\frac{1}{5} \begin{pmatrix} 1 & 5 & -6 \\ -1 & -5 & 1 \\ -1 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix} \\ &= -\frac{1}{5} \begin{pmatrix} 1 & -1 & -1 \\ 5 & -5 & 0 \\ -6 & 1 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix} \\ &= -\frac{1}{5} \begin{pmatrix} -10 \\ -20 \\ 30 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix}\end{aligned}$$

2.6.55

$$\begin{aligned}\det \begin{pmatrix} 1 & 2 & -1 \\ 4 & -1 & 1 \\ 5 & 1 & -2 \end{pmatrix} &= (1) \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} - (2) \begin{vmatrix} 4 & 1 \\ 5 & -2 \end{vmatrix} + (-1) \begin{vmatrix} 4 & -1 \\ 5 & 1 \end{vmatrix} \\ &= 1 + 26 - 9 \\ &= 18\end{aligned}$$

Only trivial solution

2.7 Cramer's Rule

2.7.1

$$\mathbf{A} = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$$

$$\det \mathbf{A} = 10$$

$$\det \mathbf{A}_1 = -6$$

$$\det \mathbf{A}_2 = 12$$

$$x_1 = -\frac{3}{5}$$

$$x_2 = \frac{6}{5}$$

2.7.11

$$\mathbf{A} = \begin{pmatrix} 2-k & k \\ k & 3-k \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\begin{aligned} \det \mathbf{A} &= (2-k)(3-k) - k^2 \\ &= 6 - 5k \end{aligned}$$

$$\begin{aligned} \det \mathbf{A}_1 &= 4(3-k) - 3k \\ &= 12 - 7k \end{aligned}$$

$$\begin{aligned} \det \mathbf{A}_2 &= 3(2-k) - 4k \\ &= 6 - 7k \end{aligned}$$

$$x_1 = \frac{12 - 7k}{6 - 5k}$$

$$x_2 = \frac{6 - 7k}{6 - 5k}$$

The system is inconsistent when $k = \frac{6}{5}$

2.7.13

$$\mathbf{A} = \begin{pmatrix} \cos 25^\circ & -\cos 15^\circ \\ \sin 25^\circ & \sin 15^\circ \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 300 \end{pmatrix}$$

$$\begin{aligned} \det \mathbf{A} &= \cos 25^\circ \sin 15^\circ + \cos 15^\circ \sin 25^\circ \\ &= \sin 40^\circ \end{aligned}$$

$$\det \mathbf{A}_1 = 300 \cos 15^\circ$$

$$\det \mathbf{A}_2 = 300 \cos 25^\circ$$

$$\begin{aligned} T_1 &= \frac{300 \cos 15^\circ}{\sin 40^\circ} \\ &\approx 451 \text{ lb} \end{aligned}$$

$$\begin{aligned} T_2 &= \frac{300 \cos 25^\circ}{\sin 40^\circ} \\ &\approx 423 \text{ lb} \end{aligned}$$

2.8 The Eigenvalue Problem

2.8.1

\mathbf{K}_3 with $\lambda = -1$

2.8.3

\mathbf{K}_3 with $\lambda = 0$

2.8.5

\mathbf{K}_2 with $\lambda = 3$

\mathbf{K}_3 with $\lambda = 1$

2.8.7

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} -1 - \lambda & 2 \\ -7 & 8 - \lambda \end{vmatrix} \\ &= (-1 - \lambda)(8 - \lambda) + 14 \\ &= -8 + \lambda - 8\lambda + \lambda^2 + 14 \\ &= \lambda^2 - 7\lambda + 6 \\ &= (\lambda - 1)(\lambda - 6) \\ \lambda_1 &= 1 \\ \lambda_2 &= 6 \end{aligned}$$

$$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ -7 & 7 & 0 \end{array} \right)$$

$$x_1 = x_2$$

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -7 & 2 & 0 \\ -7 & 2 & 0 \end{array} \right)$$

$$x_1 = \frac{2}{7}x_2$$

$$\mathbf{X}_2 = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

Nonsingular

2.8.9

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} -8 - \lambda & -1 \\ 16 & -\lambda \end{vmatrix} \\ &= -\lambda(-8 - \lambda) + 16 \\ &= 8\lambda + \lambda^2 + 16 \\ &= (\lambda + 4)^2 \end{aligned}$$

$$\lambda_1 = \lambda_2 = -4$$

$$\left(\begin{array}{cc|c} -4 & -1 & 0 \\ 16 & 4 & 0 \end{array} \right)$$

$$x_1 = -\frac{1}{4}x_2$$

$$\mathbf{X}_1 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

Nonsingular

2.8.11

$$\begin{aligned}
\det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} -1 - \lambda & 2 \\ -5 & 1 - \lambda \end{vmatrix} \\
&= (-1 - \lambda)(1 - \lambda) + 10 \\
&= -1 + \lambda - \lambda + \lambda^2 + 10 \\
&= \lambda^2 + 9 \\
&= (\lambda - 3i)(\lambda + 3i) \\
\lambda_1 &= 3i \\
\lambda_2 &= -3i
\end{aligned}$$

$$\left(\begin{array}{cc|c} -1 - 3i & 2 & 0 \\ -5 & 1 - 3i & 0 \end{array} \right)$$

$$\mathbf{X}_1 = \begin{pmatrix} 1 - 3i \\ 5 \end{pmatrix}$$

$$\mathbf{X}_2 = \begin{pmatrix} 1 + 3i \\ 5 \end{pmatrix}$$

Nonsingular

2.8.23

$$\begin{aligned}
\det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} 5 - \lambda & 1 \\ 1 & 5 - \lambda \end{vmatrix} \\
&= (5 - \lambda)^2 - 1 \\
&= 25 - 10\lambda + \lambda^2 - 1 \\
&= \lambda^2 - 10\lambda + 24 \\
&= (\lambda - 4)(\lambda - 6) \\
\lambda_1 &= 4 \\
\lambda_2 &= 6
\end{aligned}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda'_1 = \frac{1}{4}$$

$$\lambda'_2 = \frac{1}{6}$$

$$\mathbf{X}'_1 = \mathbf{X}_1$$

$$\mathbf{X}'_2 = \mathbf{X}_2$$

2.9 Powers of Matrices

2.9.1

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= (1 - \lambda)(5 - \lambda) + 8 \\ &= 5 - \lambda - 5\lambda + \lambda^2 + 8 \\ &= \lambda^2 - 6\lambda + 13 \end{aligned}$$

$$\mathbf{A}^2 = 6\mathbf{A} - 13\mathbf{I}$$

$$\begin{aligned} \begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix} &= 6 \begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix} - 13 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} -7 & -12 \\ 24 & 17 \end{pmatrix} &= \begin{pmatrix} -7 & -12 \\ 24 & 17 \end{pmatrix} \end{aligned}$$

2.9.3

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix} \\ \det(\mathbf{A} - \lambda \mathbf{I}) &= (-1 - \lambda)(4 - \lambda) - 6 \\ &= -4 + \lambda - 4\lambda + \lambda^2 - 6 \\ &= \lambda^2 - 3\lambda - 10 \\ &= (\lambda - 5)(\lambda + 2)\end{aligned}$$

$$\begin{aligned}\lambda^m &= c_0 + c_1 \lambda \\ (-2)^m &= c_0 - 2c_1 \\ (5)^m &= c_0 + 5c_1\end{aligned}$$

$$\begin{aligned}(-2)^m + \frac{2}{5}(5)^m &= \frac{7}{5}c_0 \\ \frac{5}{7}(-2)^m + \frac{2}{7}(5)^m &= c_0\end{aligned}$$

$$\begin{aligned}(5)^m - (-2)^m &= 7c_1 \\ \frac{1}{7}(5)^m - \frac{1}{7}(-2)^m &= c_1\end{aligned}$$

$$\begin{aligned}\mathbf{A}^m &= \frac{1}{7} \begin{pmatrix} 5^m + 6(-2)^m & 3(5)^m - 3(-2)^m \\ 2(5)^m - 2(-2)^m & 6(5)^m + (-2)^m \end{pmatrix} \\ \mathbf{A}^3 &= \frac{1}{7} \begin{pmatrix} 5^3 + 6(-2)^3 & 3(5)^3 - 3(-2)^3 \\ 2(5)^3 - 2(-2)^3 & 6(5)^3 + (-2)^3 \end{pmatrix} \\ &= \begin{pmatrix} 11 & 57 \\ 38 & 106 \end{pmatrix}\end{aligned}$$

2.9.5

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} 8 & 5 \\ 4 & 0 \end{pmatrix} \\ \det(\mathbf{A} - \lambda \mathbf{I}) &= -\lambda(8 - \lambda) - 20 \\ &= -8\lambda + \lambda^2 - 20 \\ &= \lambda^2 - 8\lambda - 20 \\ &= (\lambda - 10)(\lambda + 2)\end{aligned}$$

$$\begin{aligned}\lambda^m &= c_0 + c_1\lambda \\ (-2)^m &= c_0 - 2c_1 \\ 10^m &= c_0 + 10c_1\end{aligned}$$

$$\begin{aligned}10^m + 5(-2)^m &= 6c_0 \\ \frac{1}{6}(10^m + 5(-2)^m) &= c_0 \\ \frac{1}{12}(2 \cdot 10^m + 10(-2)^m) &= c_0\end{aligned}$$

$$\begin{aligned}10^m - (-2)^m &= 12c_1 \\ \frac{1}{12}(10^m - (-2)^m) &= c_1\end{aligned}$$

$$\begin{aligned}\mathbf{A}^m &= \frac{1}{12} \begin{pmatrix} 10 \cdot 10^m + 2(-2)^m & 5 \cdot 10^m - 5(-2)^m \\ 4 \cdot 10^m - 4(-2)^m & 2 \cdot 10^m + 10(-2)^m \end{pmatrix} \\ \mathbf{A}^5 &= \begin{pmatrix} 83328 & 41680 \\ 33344 & 16640 \end{pmatrix}\end{aligned}$$

2.9.11

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 7 & 3 \\ -3 & 1 \end{pmatrix} \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= (7 - \lambda)(1 - \lambda) + 9 \\
&= 7 - 7\lambda - \lambda + \lambda^2 + 9 \\
&= \lambda^2 - 8\lambda + 16 \\
&= (\lambda - 4)^2
\end{aligned}$$

$$\begin{aligned}
\lambda^m &= c_0 + c_1 \lambda \\
m\lambda^{m-1} &= c_1
\end{aligned}$$

$$4^{m-1}m = c_1$$

$$\begin{aligned}
4^m &= c_0 + 4^m m \\
4^m(1 - m) &= c_0
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}^m &= \begin{pmatrix} 4^m(1 - m) + 7 \cdot 4^{m-1}m & 3 \cdot 4^{m-1}m \\ -3 \cdot 4^{m-1}m & 4^m(1 - m) + 4^{m-1}m \end{pmatrix} \\
&= 4^m \begin{pmatrix} 1 - m + \frac{7}{4}m & \frac{3}{4}m \\ -\frac{3}{4}m & 1 - m + \frac{1}{4}m \end{pmatrix} \\
&= 4^m \begin{pmatrix} 1 + \frac{3}{4}m & \frac{3}{4}m \\ -\frac{3}{4}m & 1 - \frac{3}{4}m \end{pmatrix} \\
\mathbf{A}^6 &= 4^6 \begin{pmatrix} 1 + \frac{3}{4}6 & \frac{3}{4}6 \\ -\frac{3}{4}6 & 1 - \frac{3}{4}6 \end{pmatrix} \\
&= 4^5 \begin{pmatrix} 4 + 18 & 18 \\ -18 & 4 - 18 \end{pmatrix} \\
&= 4^5 \begin{pmatrix} 22 & 18 \\ -18 & -14 \end{pmatrix} \\
&= \begin{pmatrix} 22528 & 18432 \\ -18432 & -14336 \end{pmatrix}
\end{aligned}$$

2.9.13

(a)

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$$

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= (1 - \lambda)(3 - \lambda) - 3 \\ &= 3 - \lambda - 3\lambda + \lambda^2 - 3 \\ &= \lambda^2 - 4\lambda \\ &= \lambda(\lambda - 4) \end{aligned}$$

$$\begin{aligned} \lambda^m &= c_1 \lambda \\ 4^m &= 4c_1 \\ c_1 &= 4^{m-1} \end{aligned}$$

$$\mathbf{A}^m = 4^{m-1} \mathbf{A}$$

2.9.15

$$\mathbf{A} = \begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}$$

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= (2 - \lambda)(3 - \lambda) + 4 \\ &= 6 - 2\lambda - 3\lambda + \lambda^2 + 4 \\ &= \lambda^2 - 5\lambda + 10 \end{aligned}$$

$$10\mathbf{I} = 5\mathbf{A} - \mathbf{A}^2$$

$$\mathbf{I} = \frac{1}{2}\mathbf{A} - \frac{1}{10}\mathbf{A}^2$$

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{2}\mathbf{I} - \frac{1}{10}\mathbf{A} \\ &= \begin{pmatrix} \frac{3}{10} & \frac{2}{5} \\ -\frac{1}{10} & \frac{1}{5} \end{pmatrix} \end{aligned}$$

2.10 Orthogonal Matrices**2.10.5**

Orthogonal

2.10.7

Orthogonal

2.10.9

Not orthogonal

2.10.11

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 9 \\ 9 & 1 \end{pmatrix} \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= (1 - \lambda)^2 - 81 \\
&= \lambda^2 - 2\lambda + 1 - 81 \\
&= \lambda^2 - 2\lambda - 80 \\
&= (\lambda - 10)(\lambda + 8)
\end{aligned}$$

$$\begin{aligned}
&\begin{pmatrix} 9 & 9 \\ 9 & 9 \end{pmatrix} \\
\mathbf{X}_1 &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&\begin{pmatrix} -9 & 9 \\ 9 & -9 \end{pmatrix} \\
\mathbf{X}_2 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}
\end{aligned}$$

2.10.13

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= (1 - \lambda)(9 - \lambda) - 9 \\
&= 9 - \lambda - 9\lambda + \lambda^2 - 9 \\
&= \lambda^2 - 10\lambda \\
&= \lambda(\lambda - 10)
\end{aligned}$$

$$\begin{aligned}
&\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \\
\mathbf{X}_1 &= \begin{pmatrix} 3 \\ -1 \end{pmatrix}
\end{aligned}$$

$$\begin{pmatrix} -9 & 3 \\ 3 & -1 \end{pmatrix}$$

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}$$

2.10.19

$$\frac{3}{5}a + \frac{4}{5}b = 0$$

$$3a = -4b$$

$$a = -\frac{4}{3}b$$

$$a = -\frac{4}{5}$$

$$b = \frac{3}{5}$$

2.10.21

$$(b) \ \lambda_1 = -2, \lambda_2 = -2, \lambda_3 = 4$$

(c)

$$\mathbf{W}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\begin{aligned} \mathbf{V}_2 &= \mathbf{K}_2 - (\mathbf{K}_2 \cdot \mathbf{W}_1) \mathbf{W}_1 \\ &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} |\mathbf{V}_2| &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (-1)^2} \\ &= \sqrt{\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} \mathbf{W}_2 &= \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

2.11 Approximation of Eigenvalues

2.11.1

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \\ \mathbf{X}_0 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \mathbf{A}^5 \mathbf{X}_0 &= 32 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \mathbf{K}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \lambda_1 &= \frac{\mathbf{A} \mathbf{X}_5 \cdot \mathbf{X}_5}{\mathbf{X}_5 \cdot \mathbf{X}_5} \\ &= 2\end{aligned}$$

2.11.3

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} 2 & 4 \\ 3 & 13 \end{pmatrix} \\ \mathbf{X}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \mathbf{X}_2 &= \begin{pmatrix} \frac{3}{8} \\ 1 \end{pmatrix} \\ \mathbf{X}_3 &= \begin{pmatrix} 0.3363 \\ 1 \end{pmatrix} \\ \mathbf{X}_4 &= \begin{pmatrix} 0.3335 \\ 1 \end{pmatrix} \\ \mathbf{X}_5 &= \begin{pmatrix} 0.3333 \\ 1 \end{pmatrix} \\ \mathbf{K}_1 &= \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \\ \lambda_1 &= \frac{\mathbf{A} \mathbf{K}_1 \cdot \mathbf{K}_1}{\mathbf{K}_1 \cdot \mathbf{K}_1} \\ &= 14\end{aligned}$$

2.11.7

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} \\
\mathbf{X}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathbf{A}^5 \mathbf{X}_1 &= \begin{pmatrix} 0.5008 \\ 1 \end{pmatrix} \\
\lambda_1 &= 7 \\
\mathbf{K}_1 &= \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \\
\mathbf{B} &= \mathbf{A} - \lambda_1 \mathbf{K}_1 \mathbf{K}_1^T \\
&= \begin{pmatrix} \frac{8}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{2}{5} \end{pmatrix} \\
\mathbf{B}^5 \mathbf{X}_1 &= \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \\
\lambda_2 &= 2
\end{aligned}$$

2.11.11

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} \\
\det \mathbf{A} &= 1 \\
\mathbf{A}^{-1} &= \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix} \\
\mathbf{X}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
(\mathbf{A}^{-1})^5 \mathbf{X}_1 &= \begin{pmatrix} 1 \\ -0.7913 \end{pmatrix} \\
\lambda'_1 &\approx 4.78 \\
\lambda_1 &\approx 0.21
\end{aligned}$$

2.11.13

(a)

$$\begin{aligned}
EI \frac{d^2 y}{dx^2} + Py &= 0 \\
EI \left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right) + Py_i &= 0 \\
EI(y_{i+1} - 2y_i + y_{i-1}) + Ph^2 y_i &= 0
\end{aligned}$$

(b)

$$\begin{aligned}EI(y_2 - 2y_1) + Ph^2y_1 &= 0 \\EI(2y_1 - y_2) &= Ph^2y_1\end{aligned}$$

$$\begin{aligned}EI(y_3 - 2y_2 + y_1) + Ph^2y_2 &= 0 \\EI(-y_1 + 2y_2 - y_3) &= Ph^2y_2\end{aligned}$$

$$\begin{aligned}EI(-2y_3 + y_2) + Ph^2y_3 &= 0 \\EI(-y_2 + 2y_3) &= Ph^2y_3\end{aligned}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \frac{PL^2}{16EI} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

(c)

$$\begin{aligned}\det \mathbf{A} &= 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} \\ &= 4\end{aligned}$$

$$\begin{aligned}\mathbf{A}^{-1} &= \frac{1}{\det \mathbf{A}} \text{adj } \mathbf{A} \\ &= \frac{1}{4} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}^T \\ &= \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}\end{aligned}$$

(d)

$$\begin{aligned}\mathbf{X}_1 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ (\mathbf{A}^{-1})^6 \mathbf{X}_1 &= \begin{pmatrix} 0.7071 \\ 1 \\ 0.7071 \end{pmatrix} \\ \lambda'_1 &\approx 1.7071 \\ \lambda_1 &\approx 0.59\end{aligned}$$

(e)

$$\begin{aligned}\lambda_1 &= \frac{PL^2}{16EI} \\ P &= \frac{16EI\lambda_1}{L^2} \\ &\approx \frac{9.44EI}{L^2}\end{aligned}$$

2.12 Diagonalization

2.12.1

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \\ \det(\mathbf{A} - \lambda\mathbf{I}) &= (2 - \lambda)(4 - \lambda) - 3 \\ &= 8 - 2\lambda - 4\lambda + \lambda^2 - 3 \\ &= \lambda^2 - 6\lambda + 5 \\ &= (\lambda - 5)(\lambda - 1) \\ \lambda_1 &= 1 \\ \lambda_2 &= 5\end{aligned}$$

$$\begin{aligned}&\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \\ \mathbf{X}_1 &= \begin{pmatrix} -3 \\ 1 \end{pmatrix} \\ &\begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix} \\ \mathbf{X}_2 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \\ \mathbf{P} &= \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \\ \mathbf{P}^{-1} &= \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \\ \mathbf{D} &= \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}\end{aligned}$$

2.12.3

Not diagonalisable

2.12.5

$$\mathbf{P} = \begin{pmatrix} 13 & 1 \\ 2 & 1 \end{pmatrix}$$
$$\mathbf{D} = \begin{pmatrix} -7 & 0 \\ 0 & 4 \end{pmatrix}$$

2.12.7

$$\mathbf{P} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$\mathbf{D} = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

2.12.21

$$\mathbf{P} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

2.12.23

$$\mathbf{P} = \begin{pmatrix} \sqrt{\frac{2}{5}} & -\sqrt{\frac{5}{2}} \\ 1 & 1 \end{pmatrix}$$
$$\mathbf{D} = \begin{pmatrix} 10 & 0 \\ 0 & 3 \end{pmatrix}$$

2.12.35

$$\mathbf{A} = \mathbf{PDP}^{-1}$$
$$= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$$

2.12.39

$$\begin{aligned}\mathbf{D} &= \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \\ \mathbf{P} &= \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \\ \mathbf{A}^5 &= \mathbf{PD}^5\mathbf{P}^{-1} \\ &= \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 32 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} 21 & 11 \\ 22 & 10 \end{pmatrix}\end{aligned}$$

2.13 LU-Factorisation

2.13.1

$$\begin{aligned}\mathbf{A} &= \mathbf{LU} \\ \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} \\ &= \begin{pmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \end{pmatrix} \\ u_{11} &= 2 \\ u_{12} &= -2 \\ l_{21} &= \frac{1}{2} \\ u_{22} &= 3 \\ \mathbf{L} &= \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \\ \mathbf{U} &= \begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix}\end{aligned}$$

2.13.3

$$\begin{pmatrix} -1 & 4 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \end{pmatrix}$$

$$u_{11} = -1$$

$$u_{12} = 4$$

$$l_{21} = -2$$

$$u_{22} = 10$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} -1 & 4 \\ 0 & 10 \end{pmatrix}$$

2.13.11

$$\begin{pmatrix} 3 & 9 \\ 1 & 11 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & 9 \\ 0 & 8 \end{pmatrix}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} 3 & 9 \\ 0 & 8 \end{pmatrix}$$

2.13.13

$$\begin{pmatrix} -4 & -2 \\ -1 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} -4 & -2 \\ 0 & -\frac{5}{2} \end{pmatrix}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{1}{4} & 1 \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} -4 & -2 \\ 0 & -\frac{5}{2} \end{pmatrix}$$

2.13.21

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix}$$

$$\mathbf{LY} = \mathbf{B}$$

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$y_1 = 1$$

$$y_2 = -\frac{5}{2}$$

$$\mathbf{UX} = \mathbf{Y}$$

$$\begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{5}{2} \end{pmatrix}$$

$$x_2 = -\frac{5}{6}$$

$$2x_1 - 2\left(-\frac{5}{6}\right) = 1$$

$$2x_1 = -\frac{2}{3}$$

$$x_1 = -\frac{1}{3}$$

$$\mathbf{X} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{5}{6} \end{pmatrix}$$

2.13.23

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} -1 & 4 \\ 0 & 10 \end{pmatrix}$$

$$\mathbf{LY} = \mathbf{B}$$

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix}$$

$$y_1 = 15$$

$$y_2 = 35$$

$$\mathbf{UX} = \mathbf{Y}$$

$$\begin{pmatrix} -1 & 4 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 35 \end{pmatrix}$$

$$x_2 = \frac{7}{2}$$

$$-x_1 + 4\frac{7}{2} = 15$$

$$x_1 = -1$$

$$\mathbf{X} = \begin{pmatrix} -1 \\ \frac{7}{2} \end{pmatrix}$$

2.13.31

$$\mathbf{LY} = \mathbf{B}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

$$y_1 = 2$$

$$y_2 = 2$$

$$y_3 = -3$$

$$\mathbf{UX} = \mathbf{Y}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

$$x_3 = -3$$

$$x_2 = 5$$

$$x_1 = 0$$

$$\mathbf{X} = \begin{pmatrix} 0 \\ 5 \\ -3 \end{pmatrix}$$

2.14 Cryptography

2.14.1

$$\mathbf{B} = \mathbf{AM}$$

$$= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 19 & 5 & 14 & 4 & 0 \\ 8 & 5 & 12 & 16 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 35 & 15 & 38 & 36 & 0 \\ 27 & 10 & 26 & 20 & 0 \end{pmatrix}$$

2.14.7

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & -3 \\ -5 & 8 \end{pmatrix}$$

$$\mathbf{M} = \mathbf{A}^{-1}\mathbf{B}$$

$$= \begin{pmatrix} 19 & 20 & 21 & 4 & 25 \\ 0 & 8 & 1 & 18 & 4 \end{pmatrix}$$

$$= \text{STUDY HARD}$$

2.14.11

$$\mathbf{A}^{-1}\mathbf{B} = \mathbf{M}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 17 & 16 \\ -30 & -31 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ m_{21} & m_{22} \end{pmatrix}$$

$$17a_{11} - 30a_{12} = 4$$

$$16a_{11} - 31a_{12} = 1$$

$$17a_{11} - \frac{30}{31}16a_{11} = 4 - \frac{30}{31}$$

$$527a_{11} - 480a_{11} = 124 - 30$$

$$47a_{11} = 94$$

$$a_{11} = 2$$

$$-30a_{12} + \frac{17}{16}31a_{12} = 4 - \frac{17}{16}$$

$$-480a_{12} + 527a_{12} = 64 - 17$$

$$47a_{12} = 47$$

$$a_{12} = 1$$

$$\begin{pmatrix} 2 & 1 \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 5 & 25 \\ -6 & -50 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ 1 & 25 \end{pmatrix}$$

$$5a_{21} - 6a_{22} = 1$$

$$25a_{21} - 50a_{22} = 25$$

$$5a_{21} - \frac{6}{50}25a_{21} = 1 - \frac{6}{50}25$$

$$250a_{21} - 150a_{21} = 50 - 150$$

$$100a_{21} = -100$$

$$a_{21} = -1$$

$$4a_{22} = -4$$

$$a_{22} = -1$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{B} = \text{DAD I NEED MONEY TODAY}$$

2.15 Chapter in Review

2.15.1

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{pmatrix}$$

2.15.3

$$\mathbf{AB} = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$$
$$\mathbf{BA} = (11)$$

2.15.5

False

2.15.7

$$\det\left(\frac{1}{2}\mathbf{A}\right) = \frac{5}{8}$$
$$\det -\mathbf{A}^T = -5$$

2.15.9

0

2.15.11

False

2.15.13

True

2.15.15

False

2.15.17

True

2.15.19

False

2.15.23

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}^2$$

2.15.25

$$\begin{pmatrix} 5 & -1 & 1 & | & -9 \\ 2 & 4 & 0 & | & 27 \\ 1 & 1 & 5 & | & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{1}{5} & \frac{1}{5} & | & -\frac{9}{5} \\ 1 & 2 & 0 & | & \frac{27}{2} \\ 1 & 1 & 5 & | & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{1}{5} & \frac{1}{5} & | & -\frac{9}{5} \\ 0 & \frac{11}{5} & -\frac{1}{5} & | & \frac{153}{5} \\ 0 & \frac{6}{5} & \frac{24}{5} & | & \frac{10}{5} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{1}{5} & \frac{1}{5} & | & -\frac{9}{5} \\ 0 & 1 & -\frac{1}{11} & | & \frac{153}{22} \\ 0 & 1 & 4 & | & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{1}{5} & \frac{1}{5} & | & -\frac{9}{5} \\ 0 & 1 & -\frac{1}{11} & | & \frac{153}{22} \\ 0 & 0 & \frac{45}{11} & | & \frac{45}{22} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{1}{5} & \frac{1}{5} & | & -\frac{9}{5} \\ 0 & 1 & -\frac{1}{11} & | & \frac{153}{22} \\ 0 & 0 & 1 & | & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{1}{5} & 0 & | & -\frac{19}{10} \\ 0 & 1 & 0 & | & 7 \\ 0 & 0 & 1 & | & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & | & -\frac{1}{2} \\ 0 & 1 & 0 & | & 7 \\ 0 & 0 & 1 & | & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} -\frac{1}{2} \\ 7 \\ \frac{1}{2} \end{pmatrix}$$

2.15.29

240

2.15.31

$$\begin{vmatrix} 1 & -1 & 1 \\ 5 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = (1) \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + (1) \begin{vmatrix} 5 & -1 \\ 1 & 1 \end{vmatrix} + (1) \begin{vmatrix} 5 & 1 \\ 1 & 2 \end{vmatrix} \\
= 18$$

The matrix is nonsingular so the system only has the trivial solution.

2.15.35

$$\begin{aligned}
\det \mathbf{A} &= \begin{vmatrix} 1 & 2 & -3 \\ 2 & -4 & 3 \\ 0 & 4 & 6 \end{vmatrix} \\
&= (1) \begin{vmatrix} -4 & 3 \\ 4 & 6 \end{vmatrix} - (2) \begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} + (-3) \begin{vmatrix} 2 & -4 \\ 0 & 4 \end{vmatrix} \\
&= -36 - 24 - 24 \\
&= -84 \\
x_1 &= \frac{\det \mathbf{A}_1}{\det \mathbf{A}} \\
&= -\frac{1}{2} \\
x_2 &= \frac{\det \mathbf{A}_2}{\det \mathbf{A}} \\
&= \frac{1}{4} \\
x_3 &= \frac{\det \mathbf{A}_3}{\det \mathbf{A}} \\
&= \frac{2}{3}
\end{aligned}$$

2.15.37

$$\begin{aligned}
\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} X \\ Y \end{pmatrix} \\
x &= \frac{\begin{vmatrix} X & \sin \theta \\ Y & \cos \theta \end{vmatrix}}{\begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix}} \\
&= X \cos \theta - Y \sin \theta \\
y &= \frac{\begin{vmatrix} \cos \theta & X \\ -\sin \theta & Y \end{vmatrix}}{\begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix}} \\
&= X \sin \theta + Y \cos \theta
\end{aligned}$$

2.15.39

$$\begin{aligned}
\begin{pmatrix} 2 & 3 & -1 \\ 1 & -2 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 6 \\ -3 \\ 9 \end{pmatrix} \\
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \frac{1}{3} \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 1 \\ 4 & 6 & 7 \end{pmatrix} \begin{pmatrix} 6 \\ -3 \\ 9 \end{pmatrix} \\
&= \begin{pmatrix} 7 \\ 5 \\ 23 \end{pmatrix}
\end{aligned}$$

2.15.41

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \\ \det(\mathbf{A} - \lambda \mathbf{I}) &= (1 - \lambda)(3 - \lambda) - 8 \\ &= 3 - \lambda - 3\lambda + \lambda^2 - 8 \\ &= \lambda^2 - 4\lambda - 5 \\ &= (\lambda - 5)(\lambda + 1) \\ \lambda_1 &= -1 \\ \lambda_2 &= 5\end{aligned}$$

$$\begin{aligned}&\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \\ \mathbf{K}_1 &= \begin{pmatrix} 1 & -1 \end{pmatrix} \\ &\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \\ \mathbf{K}_2 &= \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}\end{aligned}$$

2.15.43

$$\begin{aligned}
 \mathbf{A} &= \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \\
 \det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} \\
 &= (3-\lambda) \begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 4 & 3-\lambda \end{vmatrix} + 4 \begin{vmatrix} 2 & -\lambda \\ 4 & 2 \end{vmatrix} \\
 &= (3-\lambda)(-\lambda(3-\lambda) - 4) - 2(2(3-\lambda) - 8) + 4(4 + 4\lambda) \\
 &= (3-\lambda)(\lambda^2 - 3\lambda - 4) - 2(-2 - 2\lambda) + 16 + 16\lambda \\
 &= 3\lambda^2 - 9\lambda - 12 - \lambda^3 + 3\lambda^2 + 4\lambda + 4 + 4\lambda + 16 + 16\lambda \\
 &= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 \\
 &= -(\lambda - 8)(\lambda + 1)^2
 \end{aligned}$$

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{2}{5} & -\frac{4}{5} \\ 1 & -4 & 1 \\ 1 & \frac{1}{2} & -\frac{5}{4} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & -\frac{11}{5} & \frac{9}{5} \\ 0 & \frac{9}{10} & -\frac{9}{20} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{X}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{X}_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

2.15.47

$$a^2 + b^2 + c^2 = 1$$

$$-a \frac{1}{\sqrt{2}} + c \frac{1}{\sqrt{2}} = 0$$

$$a = c$$

$$a \frac{1}{\sqrt{3}} + b \frac{1}{\sqrt{3}} + c \frac{1}{\sqrt{3}} = 0$$

$$a + b + c = 0$$

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

2.15.57

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 2 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$$

$$\mathbf{A} = \mathbf{LU}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 2 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$$= \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{pmatrix}$$

$$u_{11} = 1$$

$$u_{12} = 1$$

$$u_{13} = 1$$

$$l_{21} = 1$$

$$u_{22} = -3$$

$$u_{23} = 2$$

$$l_{31} = 2$$

$$l_{32} = \frac{2}{3}$$

$$u_{33} = -\frac{19}{3}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & \frac{2}{3} & 1 \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & -\frac{19}{3} \end{pmatrix}$$

$$\mathbf{AX} = \mathbf{B}$$

$$\mathbf{LUX} = \mathbf{B}$$

$$\mathbf{LY} = \mathbf{B}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & \frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ -\frac{19}{3} \end{pmatrix}$$

$$\mathbf{UX} = \mathbf{Y}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & -\frac{19}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ -\frac{19}{3} \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

3 Vector Calculus

3.1 Vector Functions

3.1.11

$$x = t$$

$$y = t$$

$$z = 2t^2$$

$$\mathbf{r} = \langle t, t, 2t^2 \rangle$$

3.1.13

$$x = 3 \cos t$$

$$z = 9 - 9 \cos^2 t$$

$$= 9(1 - \cos^2 t)$$

$$= 9 \sin^2 t$$

$$9 \cos^2 t + y^2 = 9$$

$$y^2 = 9 \sin^2 t$$

$$y = 3 \sin t$$

$$\mathbf{r} = \langle 3 \cos t, 3 \sin t, 9 \sin^2 t \rangle$$

3.1.15

$$\mathbf{r}(t) = \left\langle \frac{\sin 2t}{t}, (t-2)^5, t \ln t \right\rangle$$
$$\lim_{t \rightarrow 0^+} \mathbf{r}(t) = \langle 2, -32, 0 \rangle$$

3.1.17

$$\mathbf{r}(t) = \langle \ln t, 1, 0 \rangle$$
$$\mathbf{r}'(t) = \left\langle \frac{1}{t}, 0, 0 \right\rangle$$
$$\mathbf{r}''(t) = \left\langle -\frac{1}{t^2}, 0, 0 \right\rangle$$

3.1.19

$$\mathbf{r}(t) = \langle te^{2t}, t^3, 4t^2 - t \rangle$$
$$\mathbf{r}'(t) = \langle e^{2t} + 2te^{2t}, 3t^2, 8t - 1 \rangle$$
$$\mathbf{r}''(t) = \langle 4e^{2t} + 4te^{2t}, 6t, 8 \rangle$$

3.1.25

$$x = 2 + t$$
$$y = 2 + 2t$$
$$z = \frac{8}{3} + 4t$$

3.1.27

$$\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$$

3.1.29

$$\begin{aligned} \frac{d}{dt}[\mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t))] &= \mathbf{r}(t) \cdot \frac{d}{dt}(\mathbf{r}'(t) \times \mathbf{r}''(t)) \\ &= \mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}'''(t)) \end{aligned}$$

3.1.31

$$\frac{d}{dt} \left[\mathbf{r}_1(2t) + \mathbf{r}_2 \left(\frac{1}{t} \right) \right] = 2\mathbf{r}'_1(2t) - \frac{1}{t^2} \mathbf{r}'_2 \left(\frac{1}{t} \right)$$

3.1.33

$$\int_{-1}^2 \langle t, 3t^2, 4t^3 \rangle dt = \langle \frac{3}{2}, 9, 15 \rangle$$

3.1.35

$$\int \langle te^t, -e^{-2t}, te^{t^2} \rangle dt = \langle e^t(t-1), \frac{1}{2}e^{-2t}, \frac{1}{2}e^{t^2} \rangle + \mathbf{c}$$

3.1.37

$$\begin{aligned}\mathbf{r}(t) &= \langle 6t, 3t^2, t^3 \rangle + \mathbf{c} \\ \mathbf{r}(0) &= \mathbf{r}_0 \\ \mathbf{c} &= \mathbf{r}_0 \\ \mathbf{r}(t) &= \langle 6t+1, 3t^2-2, t^3+1 \rangle\end{aligned}$$

3.1.39

$$\begin{aligned}\mathbf{r}'(t) &= \langle 6(t^2-1), 7-6\sqrt{t}, 2(t-1) \rangle \\ \mathbf{r}(t) &= \langle 6\left(\frac{1}{3}t^3-t+1\right), 7t-4t^{3/2}-3, 2\left(\frac{1}{2}t^2-t\right) \rangle\end{aligned}$$

3.1.41

$$\begin{aligned}\mathbf{r}'(t) &= \langle -a \sin t, a \cos t, c \rangle \\ s &= \int_0^{2\pi} \|\mathbf{r}'(t)\| dt \\ &= \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + c^2} dt \\ &= \int_0^{2\pi} \sqrt{a^2 + c^2} dt \\ &= 2\pi \sqrt{a^2 + c^2}\end{aligned}$$

3.1.43

$$\begin{aligned}
\mathbf{r}'(t) &= \langle e^t(\cos 2t - 2 \sin 2t), e^t(\sin 2t + 2 \cos 2t), e^t \rangle \\
s &= \int_0^{3\pi} \|\mathbf{r}'(t)\| dt \\
&= \int_0^{3\pi} \sqrt{(e^t(\cos 2t - 2 \sin 2t))^2 + (e^t(\sin 2t + 2 \cos 2t))^2 + (e^t)^2} dt \\
&= \int_0^{3\pi} e^t \sqrt{(\cos 2t - 2 \sin 2t)^2 + (\sin 2t + 2 \cos 2t)^2 + 1} dt \\
&= \int_0^{3\pi} e^t \sqrt{5 \cos^2 2t + 5 \sin^2 2t + 1} dt \\
&= \sqrt{6} \int_0^{3\pi} e^t dt \\
&= \sqrt{6}(e^{3\pi} - 1)
\end{aligned}$$

3.1.45

$$\begin{aligned}
\mathbf{r}(t) &= \langle a \cos t, a \sin t \rangle \\
\mathbf{r}'(t) &= \langle -a \sin t, a \cos t \rangle \\
\|\mathbf{r}'(t)\| &= \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} \\
&= a \\
s &= \int_0^t a du \\
&= at \\
t &= \frac{s}{a} \\
\mathbf{r}(s) &= \langle a \cos \frac{s}{a}, a \sin \frac{s}{a} \rangle \\
\mathbf{r}'(s) &= \langle -\sin \frac{s}{a}, \cos \frac{s}{a} \rangle \\
\|\mathbf{r}'(s)\| &= \sqrt{\sin^2 \frac{s}{a} + \cos^2 \frac{s}{a}} \\
&= 1
\end{aligned}$$

3.1.47

$$\begin{aligned}
||\mathbf{r}(t)|| &= c \\
\sqrt{f(t)^2 + g(t)^2 + h(t)^2} &= c \\
\frac{1}{\sqrt{f(t)^2 + g(t)^2 + h(t)^2}}(2f(t)f'(t) + 2g(t)g'(t) + 2h(t)h'(t)) &= 0 \\
\frac{2(f(t)f'(t) + g(t)g'(t) + h(t)h'(t))}{||\mathbf{r}(t)||} &= 0 \\
\mathbf{r}(t) \cdot \mathbf{r}'(t) &= 0
\end{aligned}$$

Either $||\mathbf{r}'(t)|| = 0$ or they're perpendicular.

3.2 Motion on a Curve

3.2.1

$$\begin{aligned}
\mathbf{r}(t) &= \langle t^2, \frac{1}{4}t^4, 0 \rangle \\
\mathbf{r}'(t) &= \langle 2t, t^3, 0 \rangle \\
||\mathbf{r}'(1)|| &= ||\langle 2, 1, 0 \rangle|| \\
&= \sqrt{2^2 + 1^2 + 0^2} \\
&= \sqrt{5}
\end{aligned}$$

3.2.9

$$\begin{aligned}
\mathbf{r}(t) &= \langle t^2, t^3 - 2t, t^2 - 5t \rangle \\
z &= 0 \\
t^2 - 5t &= 0 \\
t(t - 5) &= 0 \\
t_1 &= 0 \\
t_2 &= 5 \\
\mathbf{r}(0) &= \langle 0, 0, 0 \rangle \\
\mathbf{r}(5) &= \langle 25, 115, 0 \rangle \\
\mathbf{r}'(t) &= \langle 2t, 3t^2 - 2, 2t - 5 \rangle \\
\mathbf{r}'(0) &= \langle 0, -2, -5 \rangle \\
\mathbf{r}'(5) &= \langle 10, 73, 5 \rangle \\
\mathbf{r}''(t) &= \langle 2, 6t, 2 \rangle \\
\mathbf{r}''(0) &= \langle 2, 0, 2 \rangle \\
\mathbf{r}''(5) &= \langle 2, 30, 2 \rangle
\end{aligned}$$

3.2.11

(a)

$$\mathbf{r}''(t) = \langle 0, -g \rangle$$

$$\mathbf{r}'(t) = \langle 240\sqrt{3}, 240 - gt \rangle$$

$$\mathbf{r}(t) = \langle 240\sqrt{3}t, 240t - \frac{1}{2}gt^2 \rangle$$

$$x(t) = 240\sqrt{3}t$$

$$\begin{aligned} y(t) &= 240t - \frac{1}{2}gt^2 \\ &= 240t - 16t^2 \end{aligned}$$

(b)

$$240 - 32t = 0$$

$$t = \frac{15}{2}$$

$$\begin{aligned} y\left(\frac{15}{2}\right) &= 240\frac{15}{2} - 16\left(\frac{15}{2}\right)^2 \\ &= 1800 - 900 \\ &= 900 \text{ ft} \end{aligned}$$

(c)

$$y(t) = 0$$

$$240t - 16t^2 = 0$$

$$t(240 - 16t) = 0$$

$$t_1 = 0$$

$$t_2 = 15$$

$$\begin{aligned} x(15) &= 240\sqrt{3}(15) \\ &= 3600\sqrt{3} \\ &\approx 6235 \text{ ft} \end{aligned}$$

(d)

$$\begin{aligned} \|\mathbf{v}(15)\| &= \sqrt{(240\sqrt{3})^2 + (240 - 32(15))^2} \\ &= 480 \text{ ft/s} \end{aligned}$$

3.2.23

$$\begin{aligned}
 \mathbf{r}(t) &= \langle r_0 \cos \omega t, r_0 \sin \omega t \rangle \\
 \mathbf{v}(t) &= \langle -\omega r_0 \sin \omega t, \omega r_0 \cos \omega t \rangle \\
 v &= \|\mathbf{v}(t)\| \\
 &= \sqrt{(-\omega r_0 \sin \omega t)^2 + (\omega r_0 \cos \omega t)^2} \\
 &= \omega r_0 \\
 \mathbf{a}(t) &= \langle -\omega^2 r_0 \cos \omega t, -\omega^2 r_0 \sin \omega t \rangle \\
 &= -\omega^2 \mathbf{r}(t) \\
 a &= \|\mathbf{a}(t)\| \\
 &= \|\omega^2 \mathbf{r}(t)\| \\
 &= \omega^2 r_0 \\
 &= \frac{v^2}{r_0}
 \end{aligned}$$

3.2.25

$$\begin{aligned}
 m'g &= mg - ma \\
 m' &= m \left(1 - \frac{v^2}{gr} \right) \\
 &\approx 191.3 \text{ lb}
 \end{aligned}$$

3.2.27

$$\mathbf{v}(t) = \langle 6t^2x, -4ty^2, 2t(z+1) \rangle$$

$$\begin{aligned}\frac{dx}{dt} &= 6t^2x \\ \frac{1}{x} \frac{dx}{dt} &= 6t^2 \\ \ln x &= 2t^3 + c_1 \\ x &= c_1 e^{2t^3}\end{aligned}$$

$$\begin{aligned}\frac{dy}{dt} &= -4ty^2 \\ \frac{1}{y^2} \frac{dy}{dt} &= -4t \\ -\frac{1}{y} &= -2t^2 + c_2 \\ y &= \frac{1}{2t^2 + c_2}\end{aligned}$$

$$\begin{aligned}\frac{dz}{dt} &= 2t(z+1) \\ \frac{1}{z+1} \frac{dz}{dt} &= 2t \\ \ln(z+1) &= t^2 + c_3 \\ z+1 &= c_3 e^{t^2} \\ z &= c_3 e^{t^2} - 1\end{aligned}$$

$$\mathbf{r}(t) = \langle c_1 e^{2t^3}, \frac{1}{2t^2 + c_2}, c_3 e^{t^2} - 1 \rangle$$

3.2.29

(a)

$$\begin{aligned}\boldsymbol{\tau} &= \mathbf{r} \times \mathbf{F} \\ &= \mathbf{r} \times \left(-k \frac{Mm}{r^2} \frac{\mathbf{r}}{r} \right) \\ &= -\frac{kMm}{r^3} (\mathbf{r} \times \mathbf{r}) \\ &= \mathbf{0}\end{aligned}$$

- (b) Torque is the derivative of angular momentum with respect to time. If there's no torque angular momentum doesn't change.

3.3 Curvature and Components of Acceleration

3.3.1

$$\begin{aligned}
 \mathbf{r}(t) &= \langle t \cos t - \sin t, t \sin t + \cos t, t^2 \rangle \\
 \mathbf{r}'(t) &= \langle -t \sin t, t \cos t, 2t \rangle \\
 \|\mathbf{r}'(t)\| &= \sqrt{(-t \sin t)^2 + (t \cos t)^2 + (2t)^2} \\
 &= t \sqrt{\sin^2 t + \cos^2 t + 4} \\
 &= \sqrt{5}t \\
 \mathbf{T} &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \\
 &= \frac{1}{\sqrt{5}} \langle -\sin t, \cos t, 2 \rangle
 \end{aligned}$$

3.3.3

$$\begin{aligned}
\mathbf{r}(t) &= \langle a \cos t, a \sin t, ct \rangle \\
\mathbf{r}'(t) &= \langle -a \sin t, a \cos t, c \rangle \\
\|\mathbf{r}'(t)\| &= \sqrt{(-a \sin t)^2 + (a \cos t)^2 + (c)^2} \\
&= \sqrt{a^2 + c^2} \\
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \\
&= \frac{1}{\sqrt{a^2 + c^2}} \langle -a \sin t, a \cos t, c \rangle \\
\frac{d\mathbf{T}}{dt} &= \frac{1}{\sqrt{a^2 + c^2}} \langle -a \cos t, -a \sin t, 0 \rangle \\
\left\| \frac{d\mathbf{T}}{dt} \right\| &= \frac{a}{\sqrt{a^2 + c^2}} \\
\mathbf{N} &= \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|} \\
&= \langle -\cos t, -\sin t, 0 \rangle \\
\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\
&= -\frac{1}{\sqrt{a^2 + c^2}} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -a \sin t & a \cos t & c \\ \cos t & \sin t & 0 \end{vmatrix} \\
&= \frac{1}{\sqrt{a^2 + c^2}} \langle c \sin t, -c \cos t, a \rangle \\
\kappa &= \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} \\
&= \frac{a}{a^2 + c^2}
\end{aligned}$$

3.3.5

$$\begin{aligned}
\mathbf{r}(t) &= \langle 2 \cos t, 2 \sin t, 3t \rangle \\
\mathbf{r}(\pi/4) &= \langle \sqrt{2}, \sqrt{2}, \frac{3\pi}{4} \rangle \\
\mathbf{B}(t) &= \frac{1}{\sqrt{13}} \langle 3 \sin t, -3 \cos t, 2 \rangle \\
\mathbf{B}(\pi/4) &= \frac{1}{\sqrt{13}} \langle \frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 2 \rangle \\
0 &= \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) \\
&= \mathbf{B}(\pi/4) \cdot (\mathbf{r} - \mathbf{r}(\pi/4)) \\
&= \frac{1}{\sqrt{13}} \langle \frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 2 \rangle \cdot \left(\langle x, y, z \rangle - \langle \sqrt{2}, \sqrt{2}, \frac{3\pi}{4} \rangle \right) \\
&= \frac{1}{\sqrt{13}} \langle \frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 2 \rangle \cdot \langle x - \sqrt{2}, y - \sqrt{2}, z - \frac{3\pi}{4} \rangle \\
&= \frac{3}{\sqrt{26}}(x - \sqrt{2}) - \frac{3}{\sqrt{26}}(y - \sqrt{2}) + \frac{2}{\sqrt{13}} \left(z - \frac{3\pi}{4} \right) \\
&= \frac{3}{\sqrt{2}}x - 3 - \frac{3}{\sqrt{2}}y + 3 + 2z - \frac{3\pi}{2} \\
\frac{3\pi}{2} &= \frac{3}{\sqrt{2}}x - \frac{3}{\sqrt{2}}y + 2z \\
3\pi &= 3\sqrt{2}x - 3\sqrt{2}y + 4z
\end{aligned}$$

3.3.7

$$\begin{aligned}
\mathbf{r}(t) &= \langle 1, t, t^2 \rangle \\
\mathbf{r}'(t) &= \langle 0, 1, 2t \rangle \\
\|\mathbf{r}'(t)\| &= \sqrt{1 + 4t^2} \\
\mathbf{r}''(t) &= \langle 0, 0, 2 \rangle \\
a_T &= \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} \\
&= \frac{4t}{\sqrt{1 + 4t^2}} \\
\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} \\
&= \langle 2, 0, 0 \rangle \\
a_N &= \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} \\
&= \frac{2}{\sqrt{1 + 4t^2}}
\end{aligned}$$

3.3.17

$$\begin{aligned}
\mathbf{r}(t) &= \langle a \cos t, b \sin t, ct \rangle \\
\mathbf{r}'(t) &= \langle -a \sin t, b \cos t, c \rangle \\
\|\mathbf{r}'(t)\| &= \sqrt{(-a \sin t)^2 + (b \cos t)^2 + c^2} \\
&= \sqrt{a^2 \sin^2 t + b^2 \cos^2 t + c^2} \\
\mathbf{r}''(t) &= \langle -a \cos t, -b \sin t, 0 \rangle \\
\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -a \sin t & b \cos t & c \\ -a \cos t & -b \sin t & 0 \end{vmatrix} \\
&= \langle bc \sin t, -ac \cos t, ab \rangle \\
\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= \sqrt{(bc \sin t)^2 + (-ac \cos t)^2 + (ab)^2} \\
&= \sqrt{b^2 c^2 \sin^2 t + a^2 c^2 \cos^2 t + a^2 b^2} \\
\kappa &= \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \\
&= \frac{\sqrt{b^2 c^2 \sin^2 t + a^2 c^2 \cos^2 t + a^2 b^2}}{(a^2 \sin^2 t + b^2 \cos^2 t + c^2)^{3/2}}
\end{aligned}$$

3.3.23

$$\begin{aligned}y &= x^2 \\ \kappa &= \frac{2}{(1+4x^2)^{3/2}} \\ \rho &= \frac{1}{\kappa} \\ &= \frac{(1+4x^2)^{3/2}}{2} \\ \kappa(0) &= 2 \\ \kappa(1) &= \frac{2}{5\sqrt{5}}\end{aligned}$$

The curve is sharper at $(0, 0)$.

3.4 Partial Derivatives

3.4.13

$$\begin{aligned}z &= x^2 - xy^2 + 4y^5 \\ \frac{\partial z}{\partial x} &= 2x - y^2 \\ \frac{\partial z}{\partial y} &= -2xy + 20y^4\end{aligned}$$

3.4.15

$$\begin{aligned}z &= 5x^4y^3 - x^2y^6 + 6x^5 - 4y \\ \frac{\partial z}{\partial x} &= 20x^3y^3 - 2xy^6 + 30x^4 \\ \frac{\partial z}{\partial y} &= 15x^4y^2 - 6x^2y^5 - 4\end{aligned}$$

3.4.33

$$\begin{aligned}
z &= \ln(x^2 + y^2) \\
\frac{\partial z}{\partial x} &= \frac{2x}{x^2 + y^2} \\
\frac{\partial^2 z}{\partial x^2} &= \frac{2}{x^2 + y^2} - \frac{4x^2}{(x^2 + y^2)^2} \\
\frac{\partial z}{\partial y} &= \frac{2y}{x^2 + y^2} \\
\frac{\partial^2 z}{\partial y^2} &= \frac{2}{x^2 + y^2} - \frac{4y^2}{(x^2 + y^2)^2} \\
\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} &= \frac{2}{x^2 + y^2} - \frac{4x^2}{(x^2 + y^2)^2} + \frac{2}{x^2 + y^2} - \frac{4y^2}{(x^2 + y^2)^2} \\
&= \frac{4}{x^2 + y^2} - \frac{4(x^2 + y^2)}{(x^2 + y^2)^2} \\
&= 0
\end{aligned}$$

3.4.39

$$\begin{aligned}
z &= e^{uv^2} \\
u &= x^3 \\
v &= x - y^2 \\
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \\
&= (v^2 e^{uv^2})(3x^2) + (2uv e^{uv^2})(1) \\
&= 3e^{uv^2} v^2 x^2 + 2e^{uv^2} uv \\
\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} \\
&= (v^2 e^{uv^2})(0) + (2uv e^{uv^2})(-2y) \\
&= -4uvye^{uv^2}
\end{aligned}$$

3.4.49

$$\begin{aligned}
 z &= \ln(u^2 + v^2) \\
 u &= t^2 \\
 v &= t^{-2} \\
 \frac{dz}{dt} &= \frac{\partial z}{\partial u} \frac{du}{dt} + \frac{\partial z}{\partial v} \frac{dv}{dt} \\
 &= \frac{2u}{u^2 + v^2} 2t + \frac{2v}{u^2 + v^2} - 2t^{-3} \\
 &= \frac{4(ut - vt^{-3})}{u^2 + v^2}
 \end{aligned}$$

3.4.57

$$\begin{aligned}
 A(x, y, \theta) &= \frac{1}{2}xy \sin \theta \\
 \frac{dA}{dt} &= \frac{\partial A}{\partial x} \frac{dx}{dt} + \frac{\partial A}{\partial y} \frac{dy}{dt} + \frac{\partial A}{\partial \theta} \frac{d\theta}{dt} \\
 &= \frac{1}{2}y \sin \theta \frac{dx}{dt} + \frac{1}{2}x \sin \theta \frac{dy}{dt} + \frac{1}{2}xy \cos \theta \frac{d\theta}{dt} \\
 &\approx 5.31 \text{ cm}^2/\text{s}
 \end{aligned}$$

3.5 Directional Derivative

3.5.1

$$\nabla f = \langle 2x - 3x^2y^2, -2x^3y + 4y^3 \rangle$$

3.5.3

$$\nabla F = \left\langle \frac{y^2}{z^3}, \frac{2xy}{z^3}, -\frac{3xy^2}{z^4} \right\rangle$$

3.5.5

$$\begin{aligned}
 \nabla f &= \langle 2x, -8y \rangle \\
 \nabla f(2, 4) &= \langle 4, -32 \rangle
 \end{aligned}$$

3.5.7

$$\begin{aligned}
 \nabla F &= \langle 2xz^2 \sin 4y, 4x^2z^2 \cos 4y, 2x^2z \sin 4y \rangle \\
 \nabla F(-2, \pi/3, 1) &= \langle 2\sqrt{3}, -8, -4\sqrt{3} \rangle
 \end{aligned}$$

3.5.9

$$\begin{aligned} D_{\mathbf{u}}f &= \langle 2x, 2y \rangle \cdot \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle \\ &= \sqrt{3}x + y \end{aligned}$$

3.5.11

$$\begin{aligned} D_{\mathbf{u}}f &= \langle 15x^2y^6, 30x^3y^5 \rangle \cdot \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle \\ &= \frac{15\sqrt{3}}{2}x^2y^6 + 15x^3y^5 \\ D_{\mathbf{u}}f(-1, 1) &= \frac{15\sqrt{3}}{2} - 15 \\ &= 15 \left(\frac{\sqrt{3}}{2} - 1 \right) \\ &= \frac{15}{2}(\sqrt{3} - 2) \end{aligned}$$

3.5.21

$$\begin{aligned} \mathbf{u} &= \frac{\langle -4, -1 \rangle}{\|\langle -4, -1 \rangle\|} \\ &= \left\langle -\frac{4}{\sqrt{17}}, -\frac{1}{\sqrt{17}} \right\rangle \\ D_{\mathbf{u}}f &= \langle 2(x-y), -2(x-y) \rangle \cdot \left\langle -\frac{4}{\sqrt{17}}, -\frac{1}{\sqrt{17}} \right\rangle \\ &= \frac{2}{\sqrt{17}}(-4(x-y) + (x-y)) \\ &= \frac{6}{\sqrt{17}}(y-x) \\ D_{\mathbf{u}}f(4, 2, 4) &= \frac{6}{\sqrt{17}}(2-4) \\ &= -\frac{12}{\sqrt{17}} \end{aligned}$$

3.5.23

$$\begin{aligned} \nabla f &= \langle 2e^{2x} \sin y, e^{2x} \cos y \rangle \\ \nabla f(0, \pi/4) &= \left\langle \sqrt{2}, \frac{1}{\sqrt{2}} \right\rangle \\ \sqrt{2 + \frac{1}{2}} &= \sqrt{\frac{5}{2}} \end{aligned}$$

3.5.27

$$\begin{aligned}
\nabla f &= \langle 2x \sec^2(x^2 + y^2), 2y \sec^2(x^2 + y^2) \rangle \\
-\nabla f(\sqrt{\pi/6}, \sqrt{\pi/6}) &= -\langle 2\sqrt{\frac{\pi}{6}} \sec^2\left(\frac{\pi}{6} + \frac{\pi}{6}\right), 2\sqrt{\frac{\pi}{6}} \sec^2\left(\frac{\pi}{6} + \frac{\pi}{6}\right) \rangle \\
&= -8\sqrt{\frac{\pi}{6}} \langle 1, 1 \rangle
\end{aligned}$$

3.5.33

(a)

$$\mathbf{u}_0 = \frac{\langle 3, -4 \rangle}{\|\langle 3, -4 \rangle\|} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$$

(b)

$$\mathbf{u}_{\max} = \frac{\langle 4, 3 \rangle}{\|\langle 4, 3 \rangle\|} = \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$$

(c)

$$\mathbf{u}_{\min} = \left\langle -\frac{4}{5}, -\frac{3}{5} \right\rangle$$

3.5.37

$$\nabla f = \langle 3x^2 - 12, 2y - 10 \rangle$$

$$3x^2 - 12 = 0$$

$$x^2 = 4$$

$$x = \pm 2$$

$$2y - 10 = 0$$

$$y = 5$$

3.5.43

$$\nabla f = \langle 3x^2 + y^3 + ye^{xy}, -2y^2 + 3xy^2 + xe^{xy} \rangle$$

$$f = x^3 + xy^3 + e^{xy} - \frac{2}{3}y^3$$

3.6 Tangent Planes and Normal Lines

3.6.13

$$\begin{aligned}F(x, y, z) &= z - x^2 - y^2 \\ \nabla F &= \langle -2x, -2y, 1 \rangle \\ \langle -4, -1, 17 \rangle\end{aligned}$$

3.6.15

$$\begin{aligned}x^2 + y^2 + z^2 &= 9 \\ F(x, y, z) &= x^2 + y^2 + z^2 \\ \nabla F &= \langle 2x, 2y, 2z \rangle \\ \nabla F(-2, 2, 1) &= \langle -4, 4, 2 \rangle \\ \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) &= 0 \\ \langle -4, 4, 2 \rangle \cdot (\langle x, y, z \rangle - \langle -2, 2, 1 \rangle) &= 0 \\ -4(x + 2) + 4(y - 2) + 2(z - 1) &= 0 \\ -4x - 8 + 4y - 8 + 2z - 2 &= 0 \\ -2x + 2y + z &= 9\end{aligned}$$

3.6.17

$$\begin{aligned}x^2 - y^2 - 3z^2 &= 5 \\ F(x, y, z) &= x^2 - y^2 - 3z^2 \\ \nabla F &= \langle 2x, -2y, -6z \rangle \\ \nabla F(6, 2, 3) &= \langle 12, -4, -18 \rangle \\ 12(x - 6) - 4(y - 2) - 18(z - 3) &= 0 \\ 12x - 72 - 4y + 8 - 18z + 54 &= 0 \\ 6x - 2y - 9z &= 5\end{aligned}$$

3.6.21

$$\begin{aligned}
z &= \cos(2x + y) \\
F(x, y, z) &= \cos(2x + y) - z \\
\nabla F &= \langle -2\sin(2x + y), -\sin(2x + y), -1 \rangle \\
\nabla F(\pi/2, \pi/4, -1/\sqrt{2}) &= \langle \sqrt{2}, 1/\sqrt{2}, -1 \rangle \\
0 &= \sqrt{2} \left(x - \frac{\pi}{2} \right) + \frac{1}{\sqrt{2}} \left(y - \frac{\pi}{4} \right) - \left(z + \frac{1}{\sqrt{2}} \right) \\
&= \sqrt{2}x - \frac{1}{\sqrt{2}}\pi + \frac{1}{\sqrt{2}}y - \frac{1}{\sqrt{2}}\frac{\pi}{4} - z - \frac{1}{\sqrt{2}} \\
&= 2x - \pi + y - \frac{\pi}{4} - \sqrt{2}z - 1 \\
1 + \frac{5\pi}{4} &= 2x + y - \sqrt{2}z
\end{aligned}$$

3.6.25

$$\begin{aligned}
x^2 + y^2 + z^2 &= 7 \\
F(x, y, z) &= x^2 + y^2 + z^2 \\
\nabla F &= \langle 2x, 2y, 2z \rangle \\
\langle 2x, 2y, 2z \rangle &= k\langle 2, 4, 6 \rangle \\
x^2 + (2x)^2 + (3x)^2 &= 7 \\
x^2 + 4x^2 + 9x^2 &= 7 \\
14x^2 &= 7 \\
x^2 &= \frac{1}{2} \\
x &= \pm \frac{1}{\sqrt{2}} \\
\left\langle \frac{1}{\sqrt{2}}, \sqrt{2}, \frac{3}{\sqrt{2}} \right\rangle \\
\left\langle -\frac{1}{\sqrt{2}}, -\sqrt{2}, -\frac{3}{\sqrt{2}} \right\rangle
\end{aligned}$$

3.6.27

$$x^2 + 4x + y^2 + z^2 - 2z = 11$$

$$F(x, y, z) = x^2 + 4x + y^2 + z^2 - 2z$$

$$\nabla F = \langle 2x + 4, 2y, 2z - 2 \rangle$$

$$2x + 4 = 0$$

$$x = -2$$

$$2y = 0$$

$$y = 0$$

$$2z - 2 \neq 0$$

$$z \neq 1$$

$$(-2)^2 + 4(-2) + z^2 - 2z = 11$$

$$4 - 8 + z^2 - 2z = 11$$

$$z^2 - 2z - 15 = 0$$

$$(z - 5)(z + 3) = 0$$

$$(-2, 0, -3)$$

$$(-2, 0, 5)$$

3.6.33

$$x^2 + 2y^2 + z^2 = 4$$

$$F(x, y, z) = x^2 + 2y^2 + z^2$$

$$\nabla F = \langle 2x, 4y, 2z \rangle$$

$$\nabla F(1, -1, 1) = \langle 2, -4, 2 \rangle$$

$$\mathbf{n}(t) = \langle 1 + 2t, -1 - 4t, 1 + 2t \rangle$$

3.6.35

$$\begin{aligned}z &= 4x^2 + 9y^2 + 1 \\F(x, y, z) &= 4x^2 + 9y^2 - z + 1 \\ \nabla F &= \langle 8x, 18y, -1 \rangle \\ \nabla F \left(\frac{1}{2}, \frac{1}{3}, 3 \right) &= \langle 4, 6, -1 \rangle \\ \mathbf{n}(t) &= \left\langle \frac{1}{2} + 4t, \frac{1}{3} + 6t, 3 - t \right\rangle \\ \frac{x - \frac{1}{2}}{4} &= \frac{y - \frac{1}{3}}{6} = 3 - z\end{aligned}$$

3.7 Curl and Divergence

3.7.7

$$\begin{aligned}\mathbf{F} &= \langle xz, yz, xy \rangle \\ \nabla \times \mathbf{F} &= \langle x - y, x - y, 0 \rangle \\ \nabla \cdot \mathbf{F} &= 2z\end{aligned}$$

3.7.35

$$\begin{aligned}\mathbf{F} &= \langle xy, 4yz^2, 2xz \rangle \\ \nabla \times \mathbf{F} &= \langle -8yz, -2z, -x \rangle \\ \nabla \times (\nabla \times \mathbf{F}) &= \langle 2, 1 - 8y, 8z \rangle\end{aligned}$$

3.7.45

$$\begin{aligned}\nabla \cdot \mathbf{F} &= 2xyz - 2xyz + 1 \\ &= 1\end{aligned}$$

If \mathbf{F} were the curl of another vector field then $\nabla \cdot \mathbf{F}$ would be 0 but it's not.

3.8 Line Integrals

3.8.1

$$\begin{aligned}G(x, y) &= 2xy \\x &= 5 \cos t \\y &= 5 \sin t \\\int_C G(x, y) dx &= \int_0^{\pi/4} 2(5 \cos t)(5 \sin t)(-5 \sin t) dt \\&= -250 \int_0^{\pi/4} \cos t \sin^2 t dt \\&= -125 \int_0^{\pi/4} (\cos t - \cos t \cos 2t) dt \\&= -125 \int_0^{\pi/4} \left[\cos t - \frac{1}{2}(\cos t + \cos 3t) \right] dt \\&= -125 \left[\sin t - \frac{1}{2} \left(\sin t + \frac{1}{3} \sin 3t \right) \right]_0^{\pi/4} \\&= -125 \left[\frac{1}{\sqrt{2}} - \frac{1}{2} \left(\frac{1}{\sqrt{2}} + \frac{1}{3} \frac{1}{\sqrt{2}} \right) \right] \\&= -125 \left(\frac{1}{\sqrt{2}} - \frac{4}{6\sqrt{2}} \right) \\&= -\frac{125}{3\sqrt{2}}\end{aligned}$$

$$\begin{aligned}
\int_C G(x, y) dy &= \int_0^{\pi/4} 2(5 \cos t)(5 \sin t)(5 \cos t) dt \\
&= 250 \int_0^{\pi/4} \cos^2 t \sin t dt \\
&= 125 \int_0^{\pi/4} (\sin t + \cos 2t \sin t) dt \\
&= 125 \int_0^{\pi/4} \left[\sin t + \frac{1}{2}(\sin 3t - \sin t) \right] dt \\
&= 125 \left[-\cos t + \frac{1}{2} \left(-\frac{1}{3} \cos 3t + \cos t \right) \right]_0^{\pi/4} \\
&= 125 \left[-\frac{1}{\sqrt{2}} + \frac{1}{2} \left(\frac{1}{3} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) + 1 - \frac{1}{2} \left(-\frac{1}{3} + 1 \right) \right] \\
&= 125 \left[-\frac{1}{\sqrt{2}} + \frac{4}{6\sqrt{2}} + 1 + \frac{1}{6} - \frac{1}{2} \right] \\
&= 125 \left[\frac{2}{3} - \frac{2}{6\sqrt{2}} \right] \\
&= 125 \left[\frac{4}{6} - \frac{\sqrt{2}}{6} \right] \\
&= \frac{125}{6}(4 - \sqrt{2})
\end{aligned}$$

$$\begin{aligned}
\int_C G(x, y) ds &= \int_0^{\pi/4} 2(5 \cos t)(5 \sin t) \sqrt{[-5 \sin t]^2 + [5 \cos t]^2} dt \\
&= 50 \int_0^{\pi/4} \cos t \sin t \sqrt{25 \sin^2 t + 25 \cos^2 t} dt \\
&= 250 \int_0^{\pi/4} \cos t \sin t dt \\
&= 125 \int_0^{\pi/4} \sin 2t dt \\
&= 125 \left[-\frac{1}{2} \cos 2t \right]_0^{\pi/4} \\
&= \frac{125}{2}
\end{aligned}$$

3.8.5

$$G(x, y, z) = z$$

$$x = \cos t$$

$$y = \sin t$$

$$z = t$$

$$\begin{aligned}\int_C G(x, y, z) dx &= \int_0^{\pi/2} (t)(-\sin t) dt \\ &= [t \cos t - \sin t]_0^{\pi/2} \\ &= -1\end{aligned}$$

$$\begin{aligned}\int_C G(x, y, z) dy &= \int_0^{\pi/2} (t)(\cos t) dt \\ &= [t \sin t + \cos t]_0^{\pi/2} \\ &= \frac{\pi}{2} - 1\end{aligned}$$

$$\begin{aligned}\int_C G(x, y, z) dz &= \int_0^{\pi/2} t dt \\ &= \left[\frac{1}{2} t^2 \right]_0^{\pi/2} \\ &= \frac{\pi^2}{8}\end{aligned}$$

$$\begin{aligned}\int_C G(x, y, z) ds &= \int_0^{\pi/2} t \sqrt{\sin^2 t + \cos^2 t + 1} dt \\ &= \sqrt{2} \left[\frac{1}{2} t^2 \right]_0^{\pi/2} \\ &= \frac{\sqrt{2} \pi^2}{8}\end{aligned}$$

3.8.7

$$\begin{aligned}
\int_C (2x + y) dx + xy dy &= \int_{-1}^2 (2x + x + 3) dx + x(x + 3) dx \\
&= \int_{-1}^2 (x^2 + 6x + 3) dx \\
&= \left[\frac{1}{3}x^3 + 3x^2 + 3x \right]_{-1}^2 \\
&= \left(\frac{8}{3} + 12 + 6 \right) - \left(-\frac{1}{3} + 3 - 3 \right) \\
&= 21
\end{aligned}$$

3.8.9

$$\begin{aligned}
\int_C (2x + y) dx + xy dy &= \int_{-1}^2 (2x + 2) dx + \int_2^5 2y dy \\
&= [x^2 + 2x]_{-1}^2 + [y^2]_2^5 \\
&= (4 + 4) - (1 - 2) + (25) - (4) \\
&= 30
\end{aligned}$$

3.8.11

$$\begin{aligned}
\int_0^1 x^2 dx + x(2x) dx &= \int_0^1 3x^2 dx \\
&= [x^3]_0^1 \\
&= 1
\end{aligned}$$

3.8.13

$$\int_0^1 dx = 1$$

3.8.19

$$\begin{aligned}
 \int_{-2}^2 x^2 dx &= \left[\frac{1}{3} x^3 \right]_{-2}^2 \\
 &= \frac{16}{3} \\
 \int_2^{-2} (x^2 + 4 - x^2) dx - 2x \sqrt{4 - x^2} \frac{-x}{\sqrt{4 - x^2}} dx &= \int_2^{-2} (2x^2 + 4) dx \\
 &= \left[\frac{2}{3} x^3 + 4x \right]_2^{-2} \\
 &= \left(-\frac{16}{3} - 8 \right) - \left(\frac{16}{3} + 8 \right) \\
 &= -\frac{32}{3} - 16 \\
 &= -\frac{80}{3}
 \end{aligned}$$

$$\oint (x^2 + y^2) dx - 2xy dy = -\frac{64}{3}$$

3.8.21

$$\begin{aligned}\int_{-1}^1 -x^2 dx &= \left[-\frac{1}{3}x^3\right]_{-1}^1 \\ &= -\frac{2}{3}\end{aligned}$$

$$\begin{aligned}\int_{-1}^1 -y^2 dy &= \left[-\frac{1}{3}y^3\right]_{-1}^1 \\ &= -\frac{2}{3}\end{aligned}$$

$$\begin{aligned}\int_1^{-1} x^2 dx &= \left[\frac{1}{3}x^3\right]_1^{-1} \\ &= -\frac{2}{3}\end{aligned}$$

$$\begin{aligned}\int_1^{-1} y^2 dy &= \left[\frac{1}{3}y^3\right]_1^{-1} \\ &= -\frac{2}{3}\end{aligned}$$

$$\oint_C x^2 y^3 dx - xy^2 dy = -\frac{8}{3}$$

3.8.29

$$\begin{aligned}\mathbf{r}'(t) &= \langle -2e^{-2t}, e^t \rangle \\ \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\ln 2} \langle e^{3t}, -e^{-3t} \rangle \cdot \langle -2e^{-2t}, e^t \rangle dt \\ &= \int_0^{\ln 2} (-2e^t - e^{-2t}) dt \\ &= \left[-2e^t + \frac{1}{2}e^{-2t}\right]_0^{\ln 2} \\ &= -4 + \frac{1}{8} + 2 - \frac{1}{2} \\ &= -\frac{19}{8}\end{aligned}$$

3.8.31

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_1^e \langle \ln x, x \rangle \cdot \langle 1, \frac{1}{x} \rangle dx \\ &= \int_1^e (\ln x + 1) dx \\ &= [x(\ln x - 1) + x]_1^e \\ &= e\end{aligned}$$

3.8.35

$$\begin{aligned}\mathbf{r}(t) &= \langle 3 \cos \theta, 3 \sin \theta \rangle \\ \mathbf{r}'(t) &= \langle -3 \sin \theta, 3 \cos \theta \rangle \\ \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (-3a \sin \theta + 3b \cos \theta) d\theta \\ &= [3a \cos \theta + 3b \sin \theta]_0^{2\pi} \\ &= 0\end{aligned}$$

3.8.40

$$\begin{aligned}\rho(x, y) &= kx \\ m &= \int_C \rho(x, y) ds \\ &= \int_0^\pi kx \sqrt{[-\sin t]^2 + [\cos t]^2} dt \\ &= k \int_0^\pi (1 + \cos t) dt \\ &= k[t + \sin t]_0^\pi \\ &= k\pi\end{aligned}$$

3.8.41

$$\begin{aligned}
 M_y &= \int_C x \rho(x, y) \, ds \\
 &= \int_0^\pi k(1 + \cos t)^2 \sqrt{[-\sin t]^2 + [\cos t]^2} \, dt \\
 &= k \int_0^\pi (1 + \cos t)^2 \, dt \\
 &= k \left[\frac{1}{4}(6t + 8 \sin t + \sin 2t) \right]_0^\pi \\
 &= \frac{3}{2} k \pi
 \end{aligned}$$

$$\begin{aligned}
 \bar{x} &= \frac{M_y}{m} \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 M_x &= \int_C y \rho(x, y) \, ds \\
 &= \int_0^\pi (\sin t) k(1 + \cos t) \sqrt{[-\sin t]^2 + [\cos t]^2} \, dt \\
 &= k \int_0^\pi (\sin t + \cos t \sin t) \, dt \\
 &= k \int_0^\pi \left(\sin t + \frac{1}{2} \sin 2t \right) \, dt \\
 &= k \left[-\cos t - \frac{1}{4} \cos 2t \right]_0^\pi \\
 &= k \left(1 - \frac{1}{4} + 1 + \frac{1}{4} \right) \\
 &= 2k
 \end{aligned}$$

$$\begin{aligned}
 \bar{y} &= \frac{M_x}{m} \\
 &= \frac{2}{\pi}
 \end{aligned}$$

3.9 Independence of the Path

3.9.1

$$\begin{aligned}\phi &= \frac{1}{3}x^3 + g(y) \\ g'(y) &= y^2 \\ \phi &= \frac{1}{3}x^3 + \frac{1}{3}y^3 \\ \phi(2, 2) - \phi(0, 0) &= \frac{16}{3}\end{aligned}$$

3.9.3

$$\begin{aligned}\phi &= \frac{1}{2}x^2 + 2xy + g(y) \\ 2x + g'(y) &= 2x - y \\ g'(y) &= -y \\ g(y) &= -\frac{1}{2}y^2 \\ \phi &= \frac{1}{2}x^2 + 2xy - \frac{1}{2}y^2 \\ \phi(3, 2) - \phi(1, 0) &= \left(\frac{1}{2}9 + 12 - \frac{1}{2}4\right) - \left(\frac{1}{2}\right) \\ &= 14\end{aligned}$$

3.9.11

$$\begin{aligned}\frac{\partial P}{\partial y} &= 12x^3y^2 \\ \frac{\partial Q}{\partial x} &= 12x^3y^2 \\ \phi &= x^4y^3 + 3x + g(y) \\ 3x^4y^2 + g'(y) &= 3x^4y^2 + 1 \\ g'(y) &= 1 \\ g(y) &= y \\ \phi &= x^4y^3 + 3x + y\end{aligned}$$

3.9.13

$$\begin{aligned}\frac{\partial P}{\partial y} &= 2y \cos xy^2 - 2xy^3 \sin xy^2 \\ \frac{\partial Q}{\partial x} &= -2y \sin xy^2 - 2xy^3 \cos xy^2\end{aligned}$$

Not conservative

3.9.15

$$\begin{aligned}\frac{\partial P}{\partial y} &= 1 \\ \frac{\partial Q}{\partial x} &= 1 \\ \phi &= \frac{1}{4}x^4 + xy + g(y) \\ x + g'(y) &= x + y^3 \\ g'(y) &= y^3 \\ g(y) &= \frac{1}{4}y^4 \\ \phi &= \frac{1}{4}x^4 + xy + \frac{1}{4}y^4\end{aligned}$$

3.9.17

$$\begin{aligned}\phi &= x^2 + e^{-y}x + g(y) \\ -e^{-y}x + g'(y) &= 4y - xe^{-y} \\ g'(y) &= 4y \\ g(y) &= 2y^2 \\ \phi &= x^2 + e^{-y}x + 2y^2 \\ \phi(1, 1) - \phi(0, 0) &= (1 + e^{-1} + 2) - (0) \\ &= 3 + e^{-1}\end{aligned}$$

3.9.19

$$\begin{aligned}\phi &= xyz + g(y, z) \\ xz + g'(y, z) &= xz \\ \phi &= xyz \\ \phi(2, 4, 8) - \phi(1, 1, 1) &= 63\end{aligned}$$

3.9.21

$$\begin{aligned}
\phi &= x^2 \sin y + xe^{3z} + g(y, z) \\
x^2 \cos y + g'(y, z) &= x^2 \cos y \\
g'(y, z) &= 0 \\
\phi &= x^2 \sin y + xe^{3z} + g(z) \\
3xe^{3z} + g'(z) &= 3xe^{3z} + 5 \\
g'(z) &= 5 \\
g(z) &= 5z \\
\phi &= x^2 \sin y + xe^{3z} + 5z \\
\phi(2, \pi/2, 1) - \phi(1, 0, 0) &= \left((2)^2 \sin \frac{\pi}{2} + (2)e^{3(1)} + 5(1) \right) - (1) \\
&= 8 + 2e^3
\end{aligned}$$

3.9.25

$$\begin{aligned}
\phi &= xy + yz \cos x + g(y, z) \\
x + z \cos x + g'(y, z) &= x + z \cos x \\
g'(y, z) &= 0 \\
g'(y, z) &= h(z) \\
y \cos x + h'(z) &= y \cos z \\
h'(z) &= 0 \\
\phi &= xy + yz \cos x \\
\phi(\pi, 1, 4) - \phi(0, 4, 0) &= \pi - 4
\end{aligned}$$

3.9.27

$$\begin{aligned}
P &= -Gm_1m_2 \frac{x}{(x^2 + y^2 + z^3)^{3/2}} \\
Q &= -Gm_1m_2 \frac{y}{(x^2 + y^2 + z^3)^{3/2}} \\
R &= -Gm_1m_2 \frac{z}{(x^2 + y^2 + z^3)^{3/2}} \\
\phi &= \frac{Gm_1m_2}{\sqrt{x^2 + y^2 + z^2}} \\
&= \frac{Gm_1m_2}{||\mathbf{r}||}
\end{aligned}$$

3.10 Double Integrals

3.10.1

$$\begin{aligned}\int_{-1}^3 (6xy - 5e^y) dx &= [3x^2y - 5xe^y]_{-1}^3 \\ &= (3(3)^2y - 5(3)e^y) - (3(-1)^2y - 5(-1)e^y) \\ &= 27y - 15e^y - 3y - 5e^y \\ &= 24y - 20e^y\end{aligned}$$

3.10.3

$$\begin{aligned}\int_1^{3x} x^3 e^{xy} dy &= [x^2 e^{xy}]_1^{3x} \\ &= x^2 e^{3x^2} - x^2 e^x\end{aligned}$$

3.10.13

$$\begin{aligned}\int_0^1 \int_0^x x^3 y^2 dy dx &= \int_0^1 \left[\frac{1}{3} x^3 y^3 \right]_0^x dx \\ &= \int_0^1 \frac{1}{3} x^6 dx \\ &= \frac{1}{21}\end{aligned}$$

3.10.15

$$\begin{aligned}\int_0^1 \int_{x^3}^{x^2} (2x + 4y + 1) dy dx &= \int_0^1 [2xy + 2y^2 + y]_{x^3}^{x^2} dx \\ &= \int_0^1 (2x^3 + 2x^4 + x^2 - 2x^4 - 2x^6 - x^3) dx \\ &= \int_0^1 (-2x^6 + x^3 + x^2) dx \\ &= \left[-\frac{2}{7}x^7 + \frac{1}{4}x^4 + \frac{1}{3}x^3 \right]_0^1 \\ &= -\frac{2}{7} + \frac{1}{4} + \frac{1}{3} \\ &= -\frac{24}{84} + \frac{21}{84} + \frac{28}{84} \\ &= \frac{25}{84}\end{aligned}$$

3.10.23

$$\begin{aligned}
2 \int_{-2}^2 \int_0^{\sqrt{4-y^2}} (4-y) \, dx \, dy &= 2 \int_{-2}^2 [4x - xy]_0^{\sqrt{4-y^2}} \, dy \\
&= 2 \int_{-2}^2 (4-y) \sqrt{4-y^2} \, dy \\
&= 16\pi
\end{aligned}$$

3.10.25

$$\begin{aligned}
z &= 6 - 2x - y \\
0 &= 6 - 2x - y \\
y &= 6 - 2x \\
\int_0^3 \int_0^{6-2x} (6 - 2x - y) \, dy \, dx &= \int_0^3 \left[(6 - 2x)y - \frac{1}{2}y^2 \right]_0^{6-2x} \, dx \\
&= \int_0^3 \frac{1}{2}(6 - 2x)^2 \, dx \\
&= \frac{1}{2} \left[-\frac{1}{6}(6 - 2x)^3 \right]_0^3 \\
&= -\frac{1}{12}(0 - 216) \\
&= 18
\end{aligned}$$

3.10.27

$$\begin{aligned}
\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{1}{2}(4 - x + y) \, dy \, dx &= \frac{1}{2} \int_0^2 \left[(4-x)y + \frac{1}{2}y^2 \right]_0^{\sqrt{4-x^2}} \, dx \\
&= \frac{1}{2} \int_0^2 \left[(4-x)\sqrt{4-x^2} + \frac{1}{2}(4-x^2) \right] \, dx \\
&= 2\pi
\end{aligned}$$

3.10.35

$$\begin{aligned}\int_0^1 \int_x^1 x^2 \sqrt{1+y^4} dy dx &= \int_0^1 \int_0^y x^2 \sqrt{1+y^4} dx dy \\&= \int_0^1 \sqrt{1+y^4} \left[\frac{1}{3} x^3 \right]_0^y dy \\&= \frac{1}{3} \int_0^1 y^3 \sqrt{1+y^4} dy \\&= \frac{1}{18} [(1+y^4)^{3/2}]_0^1 \\&= \frac{1}{18} (2\sqrt{2} - 1)\end{aligned}$$

3.11 Double Integrals in Polar Coordinates

3.11.1

$$\begin{aligned}\int_0^{2\pi} \int_0^{3+3\sin\theta} r dr d\theta &= \int_0^{2\pi} \left[\frac{1}{2} r^2 \right]_0^{3+3\sin\theta} d\theta \\&= \frac{1}{2} \int_0^{2\pi} (3+3\sin\theta)^2 d\theta \\&= \frac{1}{2} \int_0^{2\pi} (9+18\sin\theta+9\sin^2\theta) d\theta \\&= \frac{1}{2} \int_0^{2\pi} \left[9+18\sin\theta+\frac{9}{2}(1-\cos 2\theta) \right] d\theta \\&= \frac{1}{2} \left[9\theta-18\cos\theta+\frac{9}{2}\left(\theta-\frac{1}{2}\sin 2\theta\right) \right]_0^{2\pi} \\&= \frac{1}{2} [(18\pi-18+9\pi)-(-18)] \\&= \frac{1}{2} 27\pi\end{aligned}$$

3.11.5

$$\begin{aligned}
4 \int_{-\pi/6}^{\pi/6} \int_0^{5 \cos 3\theta} r \, dr \, d\theta &= 2 \int_{-\pi/6}^{\pi/6} [r^2]_0^{5 \cos 3\theta} d\theta \\
&= 50 \int_{-\pi/6}^{\pi/6} \cos^2 3\theta \, d\theta \\
&= 25 \int_{-\pi/6}^{\pi/6} (1 + \cos 6\theta) \, d\theta \\
&= 25 \left[\theta + \frac{1}{6} \sin 6\theta \right]_{-\pi/6}^{\pi/6} \\
&= 25 \left[\left(\frac{\pi}{6} \right) - \left(-\frac{\pi}{6} \right) \right] \\
&= \frac{25\pi}{3}
\end{aligned}$$

3.11.25

$$\begin{aligned}
\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2 + y^2} \, dy \, dx &= \int_0^\pi \int_0^3 r^2 \, dr \, d\theta \\
&= \int_0^\pi \left[\frac{1}{3} r^3 \right]_0^3 d\theta \\
&= 9 \int_0^\pi d\theta \\
&= 9\pi
\end{aligned}$$

3.12 Green's Theorem

3.12.1

$$\begin{aligned}\int_0^1 x \, dx &= \left[\frac{1}{2} x^2 \right]_0^1 \\ &= \frac{1}{2} \\ \int_0^3 y \, dy &= \left[\frac{1}{2} y^2 \right]_0^3 \\ &= \frac{9}{2} \\ \int_1^0 (x - 3x) \, dx + x(3x)3 \, dx &= \int_1^0 (9x^2 - 2x) \, dx \\ &= [3x^3 - x^2]_1^0 \\ &= -2 \\ \oint_C (x - y) \, dx + xy \, dy &= 3\end{aligned}$$

$$\begin{aligned}\iint_R (y + 1) \, dA &= \int_0^1 \int_0^{3x} (y + 1) \, dy \, dx \\ &= \int_0^1 \left[\frac{1}{2} y^2 + y \right]_0^{3x} \, dx \\ &= \int_0^1 \left(\frac{9}{2} x^2 + 3x \right) \, dx \\ &= \left[\frac{3}{2} x^3 + \frac{3}{2} x^2 \right]_0^1 \\ &= 3\end{aligned}$$

3.12.5

$$\begin{aligned}\oint_C 2y \, dx + 5x \, dy &= \iint_R 3 \, dA \\ &= 75\pi\end{aligned}$$

3.12.7

$$\begin{aligned}
 \oint_C (x^4 - 2y^3) dx + (2x^3 - y^4) dy &= \iint_R (6x^2 + 6y^2) dA \\
 &= \int_0^{2\pi} \int_0^2 6r^3 dr d\theta \\
 &= \frac{3}{2} \int_0^{2\pi} [r^4]_0^2 d\theta \\
 &= 24 \int_0^{2\pi} d\theta \\
 &= 48\pi
 \end{aligned}$$

3.12.9

$$\begin{aligned}
 \oint_C 2xy dx + 3xy^2 dy &= \iint_R (3y^2 - 2x) dA \\
 &= \int_1^2 \int_2^{2x} (3y^2 - 2x) dy dx \\
 &= \int_1^2 [y^3 - 2xy]_2^{2x} dx \\
 &= \int_1^2 (8x^3 - 4x^2 + 4x - 8) dx \\
 &= \left[2x^4 - \frac{4}{3}x^3 + 2x^2 - 8x \right]_1^2 \\
 &= \left(32 - \frac{32}{3} + 8 - 16 \right) - \left(2 - \frac{4}{3} + 2 - 8 \right) \\
 &= \frac{96}{3} - \frac{32}{3} + \frac{24}{3} - \frac{48}{3} - \frac{6}{3} + \frac{4}{3} - \frac{6}{3} + \frac{24}{3} \\
 &= \frac{56}{3}
 \end{aligned}$$

3.12.15

$$\begin{aligned}
 \oint_C ay dx + bx dy &= \iint_R (b - a) dA \\
 &= (b - a)A
 \end{aligned}$$

3.12.19

$$\begin{aligned}
A &= \oint_C x \, dy \\
&= \int_0^{2\pi} (a \cos^3 t)(3a \sin^2 t \cos t) \, dt \\
&= 3a^2 \int_0^{2\pi} \cos^4 t \sin^2 t \, dt \\
&= \frac{3a^2\pi}{8}
\end{aligned}$$

3.12.23

$$\begin{aligned}
\oint_C (4x^2 - y^3) \, dx + (x^3 + y^2) \, dy &= \iint_R (3x^2 + 3y^2) \, dA \\
&= \iint_R 3r^2 \, dA \\
&= 3 \int_0^{2\pi} \int_1^2 r^3 \, dr \, d\theta \\
&= 3 \int_0^{2\pi} \left[\frac{1}{4} r^4 \right]_1^2 \, d\theta \\
&= \frac{45}{4} \int_0^{2\pi} d\theta \\
&= \frac{45}{2} \pi
\end{aligned}$$

3.12.25

$$\begin{aligned}
 \frac{\partial P}{\partial y} &= \frac{-3y^2}{(x^2 + y^2)^2} + \frac{4y^4}{(x^2 + y^2)^3} \\
 &= \frac{-3y^2(x^2 + y^2) + 4y^4}{(x^2 + y^2)^3} \\
 &= \frac{y^4 - 3x^2y^2}{(x^2 + y^2)^3} \\
 \frac{\partial Q}{\partial x} &= \frac{y^2}{(x^2 + y^2)^2} - \frac{4x^2y^2}{(x^2 + y^2)^3} \\
 &= \frac{y^2(x^2 + y^2) - 4x^2y^2}{(x^2 + y^2)^3} \\
 &= \frac{y^4 - 3x^2y^2}{(x^2 + y^2)^3} \\
 \oint_C \frac{-y^3}{(x^2 + y^2)^2} dx + \frac{xy^2}{(x^2 + y^2)^2} dy &= \int_0^{2\pi} (\sin^4 \theta + \cos^2 \theta \sin^2 \theta) dt \\
 &= \pi
 \end{aligned}$$

3.12.27

$$\begin{aligned}
 \iint_R x^2 dA &= \oint_C \frac{1}{3} x^3 dy \\
 &= \frac{1}{3} \int_0^{2\pi} (3 \cos \theta)^3 (2 \cos \theta) d\theta \\
 &= 18 \int_0^{2\pi} \cos^4 \theta d\theta \\
 &= \frac{27\pi}{2}
 \end{aligned}$$

3.12.29

$$\begin{aligned}
 \oint_C (x - y) dx + (x + y) dy &= \iint_R 2 dA \\
 &= \frac{3}{2} \pi
 \end{aligned}$$

3.12.33

$$\begin{aligned}\mathbf{F} &= \langle -y, x \rangle \\ &= \langle -r \sin \theta, r \cos \theta \rangle \\ \oint -y \, dx + x \, dy &= \int_0^{2\pi} [(-r \sin \theta)(-r \sin \theta) + (r \cos \theta)(r \cos \theta)] \, d\theta \\ &= \int_0^{2\pi} [r^2 \sin^2 \theta + r^2 \cos^2 \theta] \, d\theta \\ &= \int_0^{2\pi} r^2 \, d\theta \\ &= \int_0^{2\pi} (1 + \cos \theta)^2 \, d\theta \\ &= 3\pi\end{aligned}$$

3.13 Surface Integrals

3.13.1

$$\begin{aligned}2x + 3y + 4z &= 12 \\ z &= 3 - \frac{1}{2}x - \frac{3}{4}y \\ y &= 4 - \frac{2}{3}x \\ A &= \iint_S dS \\ &= \int_0^6 \int_0^{4-\frac{2}{3}x} \sqrt{1 + \left[-\frac{1}{2}\right]^2 + \left[-\frac{3}{4}\right]^2} \, dy \, dx \\ &= \frac{\sqrt{29}}{4} \int_0^6 \left(4 - \frac{2}{3}x\right) \, dx \\ &= \frac{\sqrt{29}}{4} \left[4x - \frac{1}{3}x^2\right]_0^6 \\ &= 3\sqrt{29}\end{aligned}$$

3.13.3

$$\begin{aligned}
x^2 + z^2 &= 16 \\
z &= \sqrt{16 - x^2} \\
A &= \int_0^2 \int_0^5 \sqrt{1 + \left[-\frac{x}{\sqrt{16 - x^2}} \right]^2} dy dx \\
&= 5 \int_0^2 \sqrt{1 + \frac{x^2}{16 - x^2}} dx \\
&= 5 \int_0^2 \sqrt{\frac{16}{16 - x^2}} dx \\
&= 20 \int_0^2 \frac{1}{\sqrt{16 - x^2}} dx \\
&= 20 \left[\arcsin \frac{x}{4} \right]_0^2 \\
&= \frac{10}{3} \pi
\end{aligned}$$

3.13.15

$$\begin{aligned}
\iint_S G(x, y, z) dS &= \int_0^{\sqrt{2}} \int_0^4 x \sqrt{1 + [-2x]^2} dy dx \\
&= \int_0^{\sqrt{2}} \int_0^4 x \sqrt{1 + 4x^2} dy dx \\
&= 4 \int_0^{\sqrt{2}} x \sqrt{1 + 4x^2} dx \\
&= 4 \left[\frac{1}{12} (4x^2 + 1)^{3/2} \right]_0^{\sqrt{2}} \\
&= \frac{26}{3}
\end{aligned}$$

3.13.17

$$\begin{aligned}
 \iint_S G(x, y, z) dS &= 2 \int_0^{\sqrt{1-x^2}} \int_{-1}^1 xz^3 \sqrt{1 + \left[\frac{x}{\sqrt{x^2+y^2}} \right]^2 + \left[\frac{y}{\sqrt{x^2+y^2}} \right]^2} dA \\
 &= 2\sqrt{2} \int_0^{\sqrt{1-x^2}} \int_{-1}^1 x(x^2+y^2)^{3/2} dx dy \\
 &= 2\sqrt{2} \int_0^{\sqrt{1-x^2}} \left[\frac{1}{5} (x^2+y^2)^{5/2} \right]_{-1}^1 dy \\
 &= 0
 \end{aligned}$$

3.13.19

$$\begin{aligned}
 x^2 + y^2 + z^2 &= 36 \\
 z &= \sqrt{36 - x^2 - y^2}
 \end{aligned}$$

$$\begin{aligned}
 \iint_R (x^2 + y^2)z \sqrt{1 + \left[-\frac{x}{\sqrt{36-x^2-y^2}} \right]^2 + \left[-\frac{y}{\sqrt{36-x^2-y^2}} \right]^2} dA \\
 \int_0^6 \int_0^{\pi/2} r^2 \sqrt{36-r^2} \sqrt{1 + \frac{r^2}{36-r^2}} r d\theta dr \\
 6 \int_0^6 \int_0^{\pi/2} r^3 d\theta dr \\
 3\pi \int_0^6 r^3 dr \\
 3\pi \left[\frac{1}{4} r^4 \right]_0^6 \\
 972\pi
 \end{aligned}$$

3.13.25

$$x + 2y + 3z = 6$$

$$y = 3 - \frac{1}{2}x - \frac{3}{2}z$$

$$0 = 3 - \frac{1}{2}x - \frac{3}{2}z$$

$$\frac{3}{2}z = 3 - \frac{1}{2}x$$

$$z = 2 - \frac{1}{3}x$$

$$0 = 2 - \frac{1}{3}x$$

$$x = 6$$

$$\begin{aligned} \iint_S (3z^2 + 4yz) \, dS &= \int_0^6 \int_0^{2-\frac{1}{3}x} (3z^2 + 4yz) \sqrt{1 + \left[-\frac{1}{2}\right]^2 + \left[-\frac{3}{2}\right]^2} \, dz \, dx \\ &= \frac{\sqrt{14}}{2} \int_0^6 \int_0^{2-\frac{1}{3}x} \left[3z^2 + 4 \left(3 - \frac{1}{2}x - \frac{3}{2}z \right) z \right] \, dz \, dx \\ &= \frac{\sqrt{14}}{2} \int_0^6 \int_0^{2-\frac{1}{3}x} (-3z^2 - 2xz + 12z) \, dz \, dx \\ &= \frac{\sqrt{14}}{2} \int_0^6 [-z^3 - xz^2 + 6z^2]_0^{2-\frac{1}{3}x} \, dx \\ &= \frac{\sqrt{14}}{2} \int_0^6 \left[-\left(2 - \frac{1}{3}x \right)^3 + (6-x) \left(2 - \frac{1}{3}x \right)^2 \right] \, dx \\ &= \frac{\sqrt{14}}{2} \int_0^6 \left(-\frac{2}{27}x^3 + \frac{4}{3}x^2 - 8x + 16 \right) \, dx \\ &= \frac{\sqrt{14}}{2} \left[-\frac{1}{54}x^4 + \frac{4}{9}x^3 - 4x^2 + 16x \right]_0^6 \\ &= 12\sqrt{14} \end{aligned}$$

3.13.27

$$x + y + z = 1$$

$$z = 1 - x - y$$

$$0 = 1 - x - y$$

$$y = 1 - x$$

$$\begin{aligned}\iint_S kx^2 dS &= \int_0^1 \int_0^{1-x} kx^2 \sqrt{1 + [-1]^2 + [-1]^2} dy dx \\&= \sqrt{3}k \int_0^1 (x^2 - x^3) dx \\&= \sqrt{3}k \left[\frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^1 \\&= \frac{\sqrt{3}}{12}k\end{aligned}$$

3.13.29

$$\begin{aligned}
 \mathbf{F} &= \langle x, 2z, y \rangle \\
 g(x, y, z) &= y^2 + z^2 - 4 \\
 \nabla g &= \langle 0, 2y, 2z \rangle \\
 \|\nabla g\| &= \sqrt{4y^2 + 4z^2} \\
 &= 2\sqrt{y^2 + z^2} \\
 &= 4 \\
 \mathbf{n} &= \frac{\nabla g}{\|\nabla g\|} \\
 &= \left\langle 0, \frac{y}{2}, \frac{z}{2} \right\rangle \\
 z &= \sqrt{4 - y^2} \\
 \iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS &= \int_0^3 \int_0^2 \left(yz + \frac{1}{2}yz \right) \sqrt{1 + \left[-\frac{y}{\sqrt{4 - y^2}} \right]^2} \, dy \, dx \\
 &= \int_0^3 \int_0^2 \frac{3}{2}yz \sqrt{1 + \frac{y^2}{4 - y^2}} \, dy \, dx \\
 &= \int_0^3 \int_0^2 \frac{3}{2}yz \sqrt{\frac{4}{4 - y^2}} \, dy \, dx \\
 &= 3 \int_0^3 \int_0^2 y \, dy \, dx \\
 &= 3 \int_0^3 \left[\frac{1}{2}y^2 \right]_0^2 \, dx \\
 &= 6 \int_0^3 dx \\
 &= 18
 \end{aligned}$$

3.13.31

$$\begin{aligned}
\mathbf{F} &= \langle x, y, z \rangle \\
g(x, y, z) &= 5 - x^2 - y^2 - z \\
\nabla g &= \langle -2x, -2y, -1 \rangle \\
\|\nabla g\| &= \sqrt{4x^2 + 4y^2 + 1} \\
\mathbf{n} &= \langle -2x, -2y, -1 \rangle / \sqrt{4x^2 + 4y^2 + 1} \\
z &= 5 - x^2 - y^2 \\
0 &= 5 - x^2 - y^2 \\
y^2 &= 5 - x^2 \\
\mathbf{F} \cdot \mathbf{n} &= \frac{-2x^2 - 2y^2 - z}{\sqrt{4x^2 + 4y^2 + 1}} \\
y &= \sqrt{5 - x^2} \\
\iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS &= 2 \int_{-2}^2 \int_0^{\sqrt{4-x^2}} \frac{-2x^2 - 2y^2 - z}{\sqrt{4x^2 + 4y^2 + 1}} \sqrt{1 + [-2x]^2 + [-2y]^2} \, dy \, dx \\
&= 2 \int_{-2}^2 \int_0^{\sqrt{4-x^2}} [-2x^2 - 2y^2 - (5 - x^2 - y^2)] \, dy \, dx \\
&= 2 \int_{-2}^2 \int_0^{\sqrt{4-x^2}} (-x^2 - y^2 - 5) \, dy \, dx \\
&= 2 \int_{-2}^2 \left[-\frac{1}{3}y^3 - (x^2 + 5)y \right]_0^{\sqrt{4-x^2}} dx \\
&= 2 \int_{-2}^2 \left(-\frac{1}{3}(4-x^2)^{3/2} - (x^2 + 5)\sqrt{4-x^2} \right) dx \\
&= -28\pi
\end{aligned}$$

3.13.35

$$\begin{aligned}
\mathbf{F} &= \langle y^2, x^2, 5z \rangle \\
\iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, dS &= 5\pi \\
z &= x^2 + y^2 \\
g(x, y, z) &= x^2 + y^2 - z \\
\nabla g &= \langle 2x, 2y, -1 \rangle \\
\|\nabla g\| &= \sqrt{4x^2 + 4y^2 + 1} \\
\mathbf{n} &= \frac{\langle 2x, 2y, -1 \rangle}{\sqrt{4x^2 + 4y^2 + 1}} \\
\iint_{S_2} \mathbf{F} \cdot \mathbf{n} \, dS &= \iint_R \frac{2xy^2 + 2x^2y - 5z}{\sqrt{4x^2 + 4y^2 + 1}} \sqrt{1 + [2x]^2 + [2y]^2} \, dA \\
&= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [2xy^2 + 2x^2y - 5(x^2 + y^2)] \, dy \, dx \\
&= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2xy^2 + 2x^2y - 5x^2 - 5y^2) \, dy \, dx \\
&= \int_{-1}^1 \left[\frac{2}{3}xy^3 + x^2y^2 - 5x^2y - \frac{5}{3}y^3 \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \, dx \\
&= \int_{-1}^1 \left[2 \left(\frac{2}{3}x - \frac{5}{3} \right) (1-x^2)^{3/2} - 10x^2\sqrt{1-x^2} \right] \, dx \\
&= -\frac{5}{2}\pi \\
\iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= 5\pi - \frac{5}{2}\pi \\
&= \frac{5}{2}\pi
\end{aligned}$$

3.13.37

$$\begin{aligned}
 T(x, y, z) &= x^2 + y^2 + z^2 \\
 &= r^2 \\
 \mathbf{F} &= -\nabla T \\
 &= \langle -2r, 0, 0 \rangle \\
 \iint_R \mathbf{F} \cdot \mathbf{n} \, dA &= \int_0^{2\pi} \int_0^\pi -2a^3 \sin \theta \, d\theta \, d\phi \\
 &= -4\pi a^3 \int_0^\pi \sin \theta \, d\theta \\
 &= -4\pi a^3 [-\cos \theta]_0^\pi \\
 &= -8\pi a^3
 \end{aligned}$$

3.13.39

$$\begin{aligned}
 \mathbf{E} &= \frac{kq}{r^2} \hat{\mathbf{r}} \\
 \iint_S \mathbf{E} \cdot \mathbf{n} \, dA &= \int_0^{2\pi} \int_0^\pi \frac{kq}{a^2} a^2 \sin \theta \, d\theta \, d\phi \\
 &= 2\pi kq [-\cos \theta]_0^\pi \\
 &= 4\pi kq
 \end{aligned}$$

3.14 Stokes' Theorem

3.14.1

$$\begin{aligned}
 \mathbf{F} &= \langle 5y, -5x, 3 \rangle \\
 \mathbf{r} &= \langle 2 \cos t, 2 \sin t, 1 \rangle \\
 d\mathbf{r} &= \langle -2 \sin t, 2 \cos t, 0 \rangle \, dt \\
 \oint \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (-20 \sin^2 t - 20 \cos^2 t) \, dt \\
 &= -40\pi \\
 \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 5y & -5x & 3 \end{vmatrix} \\
 &= \langle 0, 0, -10 \rangle \\
 \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS &= -40\pi
 \end{aligned}$$

3.14.5

$$\begin{aligned}
\mathbf{F} &= \langle 2z + x, y - z, x + y \rangle \\
\nabla \times \mathbf{F} &= \langle 2, 1, 0 \rangle \\
x + y + z &= 1 \\
z &= 1 - x - y \\
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \\
&= 3 \int_0^1 \int_0^{1-x} dy \, dx \\
&= 3 \int_0^1 (1 - x) \, dx \\
&= 3 \left[x - \frac{1}{2}x^2 \right]_0^1 \\
&= \frac{3}{2}
\end{aligned}$$

3.14.7

$$\begin{aligned}
\mathbf{F} &= \langle xy, 2yz, xz \rangle \\
\nabla \times \mathbf{F} &= \langle -2y, -z, -x \rangle \\
z &= 1 - y \\
\mathbf{n} &= \langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \\
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \\
&= \int_0^2 \int_0^1 \left(-\frac{z}{\sqrt{2}} - \frac{x}{\sqrt{2}} \right) \sqrt{2} \, dy \, dx \\
&= \int_0^2 \int_0^1 (y - x - 1) \, dy \, dx \\
&= \int_0^2 \left[\frac{1}{2}y^2 - (x + 1)y \right]_0^1 \, dx \\
&= \int_0^2 \left(-\frac{1}{2} - x \right) \, dx \\
&= \left[-\frac{1}{2}x - \frac{1}{2}x^2 \right]_0^2 \\
&= -3
\end{aligned}$$

3.14.9

$$\begin{aligned}
\mathbf{F} &= \langle y^3, -x^3, z^3 \rangle \\
\nabla \times \mathbf{F} &= \langle 0, 0, -3x^2 - 3y^2 \rangle \\
\mathbf{n} &= \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle \\
\oint \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \\
&= \frac{1}{\sqrt{3}} \iint_S (-3x^2 - 3y^2) dS \\
&= \iint_S (-3x^2 - 3y^2) dA \\
&= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (-3x^2 - 3y^2) dy dx \\
&= \int_{-1}^1 [-3x^2 y - y^3]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \\
&= \int_{-1}^1 [-6x^2 \sqrt{1-x^2} - 2(1-x^2)^{3/2}] dx \\
&= -\frac{3\pi}{2}
\end{aligned}$$

3.14.11

$$\mathbf{F} = \langle x, x^3 y^2, z \rangle$$

$$\nabla \times \mathbf{F} = \langle 0, 0, 3x^2 y^2 \rangle$$

$$z = \sqrt{4 - 4x^2 - y^2}$$

$$z^2 = 4 - 4x^2 - y^2$$

$$0 = 4 - 4x^2 - y^2$$

$$y = 2\sqrt{1 - x^2}$$

$$g(x, y, z) = 4x^2 + y^2 + z^2 - 4$$

$$\nabla g = \langle 8x, 2y, 2z \rangle$$

$$||\nabla g|| = \sqrt{64x^2 + 4y^2 + 4z^2}$$

$$= 2\sqrt{16x^2 + y^2 + z^2}$$

$$\mathbf{n} = \frac{\nabla g}{||\nabla g||}$$

$$= \frac{\langle 4x, y, z \rangle}{\sqrt{16x^2 + y^2 + z^2}}$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

$$= \iint_S \frac{3x^2 y^2 z}{\sqrt{16x^2 + y^2 + z^2}} dS$$

$$= \iint_S \frac{3x^2 y^2 \sqrt{4 - 4x^2 - y^2}}{\sqrt{16x^2 + y^2 + 4 - 4x^2 - y^2}} dS$$

$$= \iint_S 3x^2 y^2 dA$$

$$= \int_{-1}^1 \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} 3x^2 y^2 dy dx$$

$$= \int_{-1}^1 [x^2 y^3]_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} dx$$

$$= 16 \int_{-1}^1 x^2 (1 - x^2)^{3/2} dx$$

$$= \pi$$

3.14.13

$$\mathbf{F} = \langle 6yz, 5x, yze^{x^2} \rangle$$

$$z = \frac{1}{4}x^2 + y^2$$

$$4 = \frac{1}{4}x^2 + y^2$$

$$y = \sqrt{4 - \frac{1}{4}x^2}$$

$$\mathbf{r}(t) = \langle 4 \cos t, 2 \sin t, 4 \rangle$$

$$d\mathbf{r} = \langle -4 \sin t, 2 \cos t, 0 \rangle dt$$

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^{2\pi} [6(2 \sin t)(4)(-4 \sin t) + 5(4 \cos t)(2 \cos t)] dt \\ &= \int_0^{2\pi} (-192 \sin^2 t + 40 \cos^2 t) dt \\ &= 8 \int_0^{2\pi} \left[-12(1 - \cos 2t) + \frac{5}{2}(1 + \cos 2t) \right] dt \\ &= 8 \left[-12t + 6 \sin 2t + \frac{5}{2}t + \frac{5}{4} \sin 2t \right] dt_0^{2\pi} \\ &= -152\pi \end{aligned}$$

3.14.17

$$\mathbf{F} = \langle z^2 e^{x^2}, xy^2, \arctan y \rangle$$

$$\nabla \times \mathbf{F} = \left\langle \frac{1}{y^2 + 1}, 2ze^{x^2}, y^2 \right\rangle$$

$$z = 0$$

$$g(x, y, z) = z$$

$$\nabla g = \langle 0, 0, 1 \rangle$$

$$\|\nabla g\| = 1$$

$$\mathbf{n} = \langle 0, 0, 1 \rangle$$

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \\ &= \iint_S y^2 dS \\ &= \iint_S y^2 dA \\ &= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} y^2 dy dx \\ &= \int_{-3}^3 \left[\frac{1}{3} y^3 \right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx \\ &= \frac{2}{3} \int_{-3}^3 (9 - x^2)^{3/2} dx \\ &= \frac{81}{4} \pi \end{aligned}$$

3.15 Triple Integrals

3.15.1

$$\begin{aligned}\int_2^4 \int_{-2}^2 \int_{-1}^1 (x + y + z) \, dx \, dy \, dz &= \int_2^4 \int_{-2}^2 \left[\frac{1}{2}x^2 + (y + z)x \right]_{-1}^1 \\&= \int_2^4 \int_{-2}^2 (2y + 2z) \, dy \, dz \\&= \int_2^4 [y^2 + 2zy]_{-2}^2 \, dz \\&= \int_2^4 8z \, dz \\&= [4z^2]_2^4 \\&= 48\end{aligned}$$

3.15.9

$$\begin{aligned}\iiint_D z \, dV &= \int_0^5 \int_1^3 \int_y^{y+2} z \, dx \, dy \, dz \\&= \int_0^5 \int_1^3 2z \, dy \, dz \\&= \int_0^5 4z \, dz \\&= [2z^2]_0^5 \\&= 50\end{aligned}$$

3.15.11

$$\begin{aligned}&\int_0^2 \int_0^{4-2y} \int_{x+2y}^4 F(x, y, z) \, dz \, dx \, dy \\&\int_0^4 \int_0^{z/2} \int_0^{z-2y} F(x, y, z) \, dx \, dy \, dz \\&\int_0^2 \int_{2y}^4 \int_0^{z-2y} F(x, y, z) \, dx \, dz \, dy \\&\int_0^4 \int_0^z \int_0^{(z-x)/2} F(x, y, z) \, dy \, dx \, dz \\&\int_0^4 \int_x^4 \int_0^{(z-x)/2} F(x, y, z) \, dy \, dz \, dx \\&\int_0^4 \int_0^{2-x/2} \int_{x+2y}^4 F(x, y, z) \, dz \, dy \, dx\end{aligned}$$

3.15.13

(a)

$$\int_0^2 \int_{x^3}^8 \int_0^4 dz \, dy \, dx$$

(b)

$$\int_0^8 \int_0^4 \int_0^{\sqrt[3]{y}} dx \, dz \, dy$$

(c)

$$\int_0^4 \int_0^2 \int_{x^3}^8 dy \, dx \, dz$$

3.15.21

$$\begin{aligned} \int_0^3 \int_{-\sqrt{2}}^{\sqrt{2}} \int_{y^2}^{4-y^2} dx \, dy \, dz &= \int_0^3 \int_{-\sqrt{2}}^{\sqrt{2}} (4 - 2y^2) \, dy \, dz \\ &= \int_0^3 \left[4y - \frac{2}{3}y^3 \right]_{-\sqrt{2}}^{\sqrt{2}} dz \\ &= \int_0^3 \left[\left(4\sqrt{2} - \frac{2}{3}2^{3/2} \right) - \left(-4\sqrt{2} + \frac{2}{3}2^{3/2} \right) \right] dz \\ &= \int_0^3 \left(8\sqrt{2} - \frac{4}{3}2^{3/2} \right) dz \\ &= \frac{16}{3}\sqrt{2} \int_0^3 dz \\ &= 16\sqrt{2} \end{aligned}$$

3.15.25

$$\begin{aligned}
 m &= \int_0^4 \int_0^2 \int_{x^3}^8 kz \, dy \, dx \, dz \\
 &= \int_0^4 kz \int_0^2 (8 - x^3) \, dx \, dz \\
 &= \int_0^4 kz \left[8x - \frac{1}{4}x^4 \right]_0^2 \, dz \\
 &= 12k \int_0^4 z \, dz \\
 &= 12k \left[\frac{1}{2}z^2 \right]_0^4 \\
 &= 96k \\
 M_{xy} &= \int_0^4 \int_0^2 \int_{x^3}^8 kz^2 \, dy \, dx \, dz \\
 &= k \int_0^4 z^2 \int_0^2 (8 - x^3) \, dx \, dz \\
 &= k \int_0^4 z^2 \left[8x - \frac{1}{4}x^4 \right]_0^2 \, dz \\
 &= 12k \int_0^4 z^2 \, dz \\
 &= 12k \left[\frac{1}{3}z^3 \right]_0^4 \\
 &= 256k \\
 M_{xz} &= \int_0^4 \int_0^2 \int_{x^3}^8 kyz \, dy \, dx \, dz \\
 &= k \int_0^4 z \int_0^2 \left[\frac{1}{2}y^2 \right]_{x^3}^8 \, dx \, dz \\
 &= \frac{k}{2} \int_0^4 z \int_0^2 (64 - x^6) \, dx \, dz \\
 &= \frac{k}{2} \int_0^4 z \left[64x - \frac{1}{7}x^7 \right]_0^2 \, dz \\
 &= \frac{384}{7}k \int_0^4 z \, dz \\
 &= \frac{384}{7}k \left[\frac{1}{2}z^2 \right]_0^4 \\
 &= \frac{3072}{7}k
 \end{aligned}$$

$$\begin{aligned}
M_{yz} &= \int_0^4 \int_0^2 \int_{x^3}^8 kxz \, dy \, dx \, dz \\
&= k \int_0^4 z \int_0^2 x(8 - x^3) \, dx \, dz \\
&= k \int_0^4 z \left[4x^2 - \frac{1}{5}x^5 \right]_0^2 \, dz \\
&= \frac{48}{5}k \int_0^4 z \, dz \\
&= \frac{48}{5}k \left[\frac{1}{2}z^2 \right]_0^4 \\
&= \frac{384}{5}k \\
\bar{x} &= \frac{M_{yz}}{m} \\
&= \frac{384k/5}{96k} \\
&= \frac{4}{5} \\
\bar{y} &= \frac{M_{xz}}{m} \\
&= \frac{3072k/7}{96k} \\
&= \frac{32}{7} \\
\bar{z} &= \frac{M_{xy}}{m} \\
&= \frac{256k}{96k} \\
&= \frac{8}{3}
\end{aligned}$$

3.15.29

$$\begin{aligned}
m &= \iiint \rho \, dV \\
&= \int_{-1}^1 \int_{2y+2}^{8-y} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x + y + 4) \, dx \, dz \, dy
\end{aligned}$$

3.15.35

$$\left(10 \cos \frac{3\pi}{4}, 10 \sin \frac{3\pi}{4}, 5 \right) = \left(-5\sqrt{2}, 5\sqrt{2}, 5 \right)$$

3.15.39

$$\left(\sqrt{2}, -\frac{\pi}{4}, -9\right)$$

3.15.43

$$r^2 + z^2 = 25$$

3.15.45

$$r^2 - z^2 = 1$$

3.15.47

$$z = x^2 + y^2$$

3.15.49

$$x = 5$$

3.15.51

$$r^2 = 4$$

$$r^2 + z^2 = 16$$

$$z = 0$$

$$\begin{aligned}\iiint dV &= \int_0^{2\pi} \int_0^2 \int_0^{\sqrt{16-r^2}} r dz dr d\theta \\ &= 2\pi \int_0^2 r \sqrt{16-r^2} dr\end{aligned}$$

Let $u = 16 - r^2$ so $du = -2r dr$

$$\begin{aligned}-\pi \int_{16}^{12} \sqrt{u} du &= -\pi \left[\frac{2}{3} u^{3/2} \right]_{16}^{12} \\ &= -\frac{2}{3} \pi (12^{3/2} - 16^{3/2}) \\ &= \frac{2}{3} \pi (64 - 8 \cdot 3^{3/2})\end{aligned}$$

3.15.53

$$\begin{aligned}
 z &= r^2 \\
 r^2 &= 25 \\
 z &= 0 \\
 \iiint dV &= \int_0^{2\pi} \int_0^5 \int_0^{r^2} r \, dz \, dr \, d\theta \\
 &= 2\pi \int_0^5 r^3 \, dr \\
 &= 2\pi \left[\frac{1}{4} r^4 \right]_0^5 \\
 &= \frac{625}{2} \pi
 \end{aligned}$$

3.15.59

(a)

$$\begin{aligned}
 &(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \\
 &\left(\frac{2}{3} \sin \frac{\pi}{2} \cos \frac{\pi}{6}, \frac{2}{3} \sin \frac{\pi}{2} \sin \frac{\pi}{6}, \frac{2}{3} \cos \frac{\pi}{2} \right) \\
 &\quad \left(\frac{\sqrt{3}}{3}, \frac{1}{3}, 0 \right)
 \end{aligned}$$

(b)

$$\begin{aligned}
 &(\rho \sin \phi, \theta, \rho \cos \phi) \\
 &\left(\frac{2}{3} \sin \frac{\pi}{2}, \frac{\pi}{6}, \frac{2}{3} \cos \frac{\pi}{2} \right) \\
 &\quad \left(\frac{2}{3}, \frac{\pi}{6}, 0 \right)
 \end{aligned}$$

3.15.63

$$\begin{aligned}
 &\left(\sqrt{x^2 + y^2 + z^2}, \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \arctan \frac{y}{x} \right) \\
 &\quad \left(\sqrt{50}, \frac{\pi}{2}, -\frac{3\pi}{4} \right)
 \end{aligned}$$

3.15.67

$$\rho = 8$$

3.15.69

$$\begin{aligned}(\rho \cos \phi)^2 &= 3(\rho \sin \phi \cos \theta)^2 + 3(\rho \sin \phi \sin \theta)^2 \\ \rho^2 \cos^2 \phi &= 3\rho^2 \sin^2 \phi \cos^2 \theta + 3\rho^2 \sin^2 \phi \sin^2 \theta \\ \cos^2 \phi &= 3\sin^2 \phi (\cos^2 \theta + \sin^2 \theta) \\ \cos^2 \phi &= 3\sin^2 \phi \\ \tan \phi &= \frac{1}{\sqrt{3}} \\ \phi &= \pm \frac{\pi}{6}\end{aligned}$$

3.15.71

$$x^2 + y^2 + z^2 = 100$$

3.15.73

$$z = 2$$

3.15.75

$$\begin{aligned}\rho \cos \phi &= \sqrt{(\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2} \\ \cos \phi &= \sin \phi \\ \phi &= \frac{\pi}{4} \\ \rho &= 3\end{aligned}$$

$$\begin{aligned}\iiint dV &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^3 r^2 \sin \phi \, dr \, d\phi \, d\theta \\ &= 2\pi \int_0^{\pi/4} \sin \phi \left[\frac{1}{3} r^3 \right]_0^3 d\phi \\ &= 18\pi \int_0^{\pi/4} \sin \phi \, d\phi \\ &= 18\pi [-\cos \phi]_0^{\pi/4} \\ &= 18\pi \left(1 - \frac{1}{\sqrt{2}} \right) \\ &= 9\pi(2 - \sqrt{2})\end{aligned}$$

3.15.77

$$\begin{aligned}
 \phi &= \frac{\pi}{6} \\
 \rho \cos \phi &= 2 \\
 \iiint dV &= \int_0^{\pi/2} \int_0^{\pi/6} \int_0^{2 \sec \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
 &= \frac{\pi}{2} \int_0^{\pi/6} \sin \phi \left[\frac{1}{3} \rho^3 \right]_0^{2 \sec \phi} d\phi \\
 &= \frac{4}{3} \pi \int_0^{\pi/6} \sec^2 \phi \tan \phi \, d\phi
 \end{aligned}$$

Let $u = \tan \phi$ so $du = \sec^2 \phi \, d\phi$

$$\begin{aligned}
 \frac{4}{3} \pi \int_0^{1/\sqrt{3}} u \, du &= \frac{4}{3} \pi \left[\frac{1}{2} u^2 \right]_0^{1/\sqrt{3}} \\
 &= \frac{2}{9} \pi
 \end{aligned}$$

3.15.81

$$\rho \cos \phi = 4$$

$$\rho = 4 \sec \phi$$

$$\rho = 5$$

$$5 \cos \phi = 4$$

$$\phi = \arccos \frac{4}{5}$$

$$\begin{aligned} \iiint dV &= \int_0^{2\pi} \int_0^{\arccos 4/5} \int_{4 \sec \phi}^5 k \rho \sin \phi \, d\rho \, d\phi \, d\theta \\ &= 2\pi k \int_0^{\arccos 4/5} \sin \phi \left[\frac{1}{2} \rho^2 \right]_{4 \sec \phi}^5 d\phi \\ &= \pi k \int_0^{\arccos 4/5} \sin \phi (25 - 16 \sec^2 \phi) \, d\phi \\ &= \pi k \int_0^{\arccos 4/5} (25 \sin \phi - 16 \sec \phi \tan \phi) \, d\phi \\ &= \pi k [-25 \cos \phi - 16 \sec \phi]_0^{\arccos 4/5} \\ &= \pi k \left[\left(-25 \frac{4}{5} - 16 \frac{5}{4} \right) - (-25 - 16) \right] \\ &= \pi k (-20 - 20 + 25 + 16) \\ &= \pi k \end{aligned}$$

3.16 Divergence Theorem

3.16.1

$$\mathbf{F} = \langle xy, yz, xz \rangle$$

$$\int_0^1 \int_0^1 x(1) \, dx \, dy = \frac{1}{2}$$

$$\int_0^1 \int_0^1 -(0)z \, dx \, dz = 0$$

$$\int_0^1 \int_0^1 -(1)z \, dx \, dz = \frac{1}{2}$$

$$\int_0^1 \int_0^1 (1)y \, dy \, dz = \frac{1}{2}$$

$$\int_0^1 \int_0^1 -(0)y \, dy \, dz = 0$$

$$\int_0^1 \int_0^1 -x(0) \, dx \, dy = 0$$

$$\iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS = \frac{3}{2}$$

$$\nabla \cdot \mathbf{F} = x + y + z$$

$$\begin{aligned} \iiint_D (\nabla \cdot \mathbf{F}) \, dV &= \int_0^1 \int_0^1 \int_0^1 (x + y + z) \, dx \, dy \, dz \\ &= \int_0^1 \int_0^1 \left[\frac{1}{2}x^2 + (y + z)x \right]_0^1 \, dy \, dz \\ &= \int_0^1 \int_0^1 \left(\frac{1}{2} + y + z \right) \, dy \, dz \\ &= \int_0^1 \left[\frac{1}{2}y^2 + \left(\frac{1}{2} + z \right)y \right]_0^1 \, dz \\ &= \int_0^1 (1 + z) \, dz \\ &= \left[z + \frac{1}{2}z^2 \right]_0^1 \\ &= \frac{3}{2} \end{aligned}$$

3.16.3

$$\begin{aligned}
\mathbf{F} &= \langle x^3, y^3, z^3 \rangle \\
\nabla \cdot \mathbf{F} &= 3x^2 + 3y^2 + 3z^2 \\
&= 3r^2 \\
\iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS &= \iiint_D (\nabla \cdot \mathbf{F}) \, dV \\
&= \int_0^{2\pi} \int_0^\pi \int_0^a 3r^4 \sin \phi \, dr \, d\phi \, d\theta \\
&= 6\pi \int_0^\pi \sin \phi \left[\frac{1}{5} r^5 \right]_0^a \, d\phi \\
&= \frac{6}{5} a^5 \pi \int_0^\pi \sin \phi \, d\phi \\
&= \frac{6}{5} a^5 \pi [-\cos \phi]_0^\pi \\
&= \frac{12}{5} a^5 \pi
\end{aligned}$$

3.16.5

$$\begin{aligned}
\mathbf{F} &= \langle y^2, xz^3, (z-1)^2 \rangle \\
\nabla \cdot \mathbf{F} &= 2z - 2 \\
\iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS &= \iiint_D (\nabla \cdot \mathbf{F}) \, dV \\
&= \int_1^5 \int_0^{2\pi} \int_0^4 (2z - 2)r \, dr \, d\theta \, dz \\
&= 2\pi \int_1^5 (2z - 2) \left[\frac{1}{2} r^2 \right]_0^4 \, dz \\
&= 16\pi [z^2 - 2z]_1^5 \\
&= 16\pi [(25 - 10) - (1 - 2)] \\
&= 256\pi
\end{aligned}$$

3.16.9

$$\begin{aligned}
 \mathbf{F} &= \frac{\langle x, y, z \rangle}{x^2 + y^2 + z^2} \\
 \nabla \cdot \mathbf{F} &= \frac{1}{x^2 + y^2 + z^2} \\
 &= \frac{1}{r^2} \\
 \iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS &= \iiint_D (\nabla \cdot \mathbf{F}) \, dV \\
 &= \int_0^{2\pi} \int_0^\pi \int_a^b \sin \phi \, dr \, d\phi \, d\theta \\
 &= 2\pi(b-a)[- \cos \phi]_0^\pi \\
 &= 4\pi(b-a)
 \end{aligned}$$

3.17 Change of Variables in Multiple Integrals

3.17.1

$$\begin{aligned}
 (0, 0) \\
 (-2, 8) \\
 (16, 20) \\
 (14, 28)
 \end{aligned}$$

3.17.7

$$\begin{aligned}
 \frac{\partial(x, y)}{\partial(u, v)} &= \begin{vmatrix} -ve^{-u} & e^{-u} \\ ve^u & e^u \end{vmatrix} \\
 &= -2v
 \end{aligned}$$

3.17.9

$$\begin{aligned}
 \frac{\partial(u, v)}{\partial(x, y)} &= \begin{vmatrix} -\frac{2y}{x^3} & \frac{1}{x^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} \\
 &= -\frac{3y^2}{x^4} \\
 \frac{\partial(x, y)}{\partial(u, v)} &= \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}} \\
 &= -\frac{x^4}{3y^2} \\
 &= -\frac{1}{3u^2}
 \end{aligned}$$

3.17.13

$$\left(-\frac{8}{3}, \frac{5}{3}\right) \rightarrow (-6, -1)$$

$$(0, 3) \rightarrow (-6, 3)$$

$$(4, -1) \rightarrow (6, 3)$$

$$\left(\frac{4}{3}, -\frac{7}{3}\right) \rightarrow (6, -1)$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}$$

$$= 3$$

$$\iint_R (x + y) \, dA = \int_{-1}^3 \int_{-6}^6 \frac{1}{3} v \, du \, dv$$

$$= 4 \left[\frac{1}{2} v^2 \right]_{-1}^3$$

$$= 16$$

3.17.15

$$y = \left(\frac{1}{2} y^2 \right)$$

$$= \frac{1}{4} y^4$$

$$y = \sqrt[3]{4}$$

$$x = \frac{1}{2} 4^{2/3}$$

$$\left(\frac{1}{2} 4^{2/3}, \sqrt[3]{4} \right) \rightarrow (1, 2)$$

$$(2, 2) \rightarrow (2, 2)$$

$$\left(\sqrt[3]{4}, \frac{1}{2} 4^{2/3} \right) \rightarrow (2, 1)$$

$$(1, 1) \rightarrow (1, 1)$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix}$$

$$= 3$$

$$\iint_R \frac{y^2}{x} dA = \int_1^2 \int_1^2 \frac{1}{3} v du dv$$

$$= \frac{1}{3} \left[\frac{1}{2} v^2 \right]_1^2$$

$$= \frac{1}{2}$$

3.17.23

$$\begin{aligned}
 u &= y - x \\
 v &= y + x \\
 (0, 0) &\rightarrow (0, 0) \\
 (0, 1) &\rightarrow (1, 1) \\
 (1, 0) &\rightarrow (-1, 1) \\
 \frac{\partial(u, v)}{\partial(x, y)} &= \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} \\
 &= -2 \\
 \int_0^1 \int_0^{1-x} e^{(y-x)/(y+x)} dy dx &= \int_0^1 \int_{-v}^v -\frac{1}{2} e^{u/v} du dv \\
 &= -\frac{1}{2} \int_0^1 \left[v e^{u/v} \right]_{-v}^v dv \\
 &= -\frac{1}{2} \int_0^1 v(e - e^{-1}) dv \\
 &= -\frac{1}{2} (e - e^{-1}) \left[\frac{1}{2} v^2 \right]_0^1 \\
 &= -\frac{1}{4} (e - e^{-1})
 \end{aligned}$$

3.17.27

$$\begin{aligned}
 u &= xy \\
 v &= xy^{1.4} \\
 \frac{\partial(u, v)}{\partial(x, y)} &= \begin{vmatrix} y & x \\ y^{1.4} & 1.4xy^{0.4} \end{vmatrix} \\
 &= 1.4xy^{1.4} - xy^{1.4} \\
 &= \frac{1}{0.4} xy^{1.4} \\
 \iint_R dA &= \int_a^b \int_c^d \frac{1}{0.4v} dv du \\
 &= \frac{1}{0.4} \int_a^b [\ln v]_c^d du \\
 &= \frac{1}{0.4} \ln \frac{d}{c} \int_a^b du \\
 &= \frac{1}{0.4} (b - a) \ln \frac{d}{c}
 \end{aligned}$$

3.17.29

$$\begin{aligned}
 u &= \frac{x}{5} \\
 v &= \frac{y}{3} \\
 u^2 + v^2 &= 1 \\
 \frac{\partial(u, v)}{\partial(x, y)} &= \begin{vmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{3} \end{vmatrix} \\
 &= \frac{1}{15} \\
 \iint_R \left(\frac{x^2}{25} + \frac{y^2}{9} \right) dA &= \iint_{R'} (u^2 + v^2) 15 dA' \\
 &= 15 \iint_0^{2\pi} \int_0^1 r^3 dr d\theta \\
 &= 30\pi \left[\frac{1}{4} r^4 \right] \\
 &= \frac{15}{2} \pi
 \end{aligned}$$

3.18 Chapter in Review

3.18.1

$$\|\mathbf{r}(t)\| = \sqrt{\cos^2 t + \cos^2 t + 2 \sin^2 t} = \sqrt{2}$$

True

3.18.5

False

3.18.7

$$\frac{\partial P}{\partial y} = 2y = \frac{\partial Q}{\partial x}$$

The vector field is conservative so line integrals are independent of path.
True.

3.18.9

False

3.18.11

False

3.18.13

True

3.18.15

True

3.18.17

True

3.18.19

$$\mathbf{F} = \nabla\phi = -\frac{\langle x, y \rangle}{(x^2 + y^2)^{3/2}}$$

3.18.21

$$-6t + t + t^2 = -4$$

$$t^2 - 5t + 4 = 0$$

$$(t - 1)(t - 4) = 0$$

$$t = 1, 4$$

$$\mathbf{r}'(t) = \langle 6, 1, 2t \rangle$$

$$\mathbf{r}'(1) = \langle 6, 1, 2 \rangle$$

$$\mathbf{r}'(4) = \langle 6, 1, 8 \rangle$$

$$\mathbf{r}''(t) = \langle 0, 0, 2 \rangle$$

$$\mathbf{r}''(1) = \langle 0, 0, 2 \rangle$$

$$\mathbf{r}''(4) = \langle 0, 0, 2 \rangle$$

3.18.23

$$\begin{aligned}
\mathbf{a}(t) &= \langle \sqrt{2} \sin t, \sqrt{2} \cos t, 0 \rangle \\
\mathbf{v}(t) &= \langle -\sqrt{2} \cos t, \sqrt{2} \sin t, 0 \rangle + \mathbf{c}_1 \\
\langle -1, 1, 1 \rangle &= \langle -\sqrt{2} \cos \frac{\pi}{4}, \sqrt{2} \sin \frac{\pi}{4}, 0 \rangle + \mathbf{c}_1 \\
&= \langle -1, 1, 0 \rangle + \mathbf{c}_1 \\
\mathbf{c}_1 &= 1 \\
\mathbf{v}(t) &= \langle -\sqrt{2} \cos t, \sqrt{2} \sin t, 1 \rangle \\
\mathbf{r}(t) &= \langle -\sqrt{2} \sin t, -\sqrt{2} \cos t, t \rangle + \mathbf{c}_2 \\
\langle 1, 2, \frac{\pi}{4} \rangle &= \langle -\sqrt{2} \sin \frac{\pi}{4}, -\sqrt{2} \cos \frac{\pi}{4}, \frac{\pi}{4} \rangle + \mathbf{c}_2 \\
&= \langle -1, -1, \frac{\pi}{4} \rangle + \mathbf{c}_2 \\
\mathbf{c}_2 &= \langle 2, 3, 0 \rangle \\
\mathbf{r}(t) &= \langle 2 - \sqrt{2} \sin t, 3 - \sqrt{2} \cos t, t \rangle \\
\mathbf{r}\left(\frac{3\pi}{4}\right) &= \langle 2 - \sqrt{2} \sin \frac{3\pi}{4}, 3 - \sqrt{2} \cos \frac{3\pi}{4}, \frac{3\pi}{4} \rangle \\
&= \langle 1, 4, \frac{3\pi}{4} \rangle
\end{aligned}$$

3.18.27

$$\begin{aligned}
\nabla f &= \langle 2xy - y^2, x^2 - 2xy \rangle \\
D_{\mathbf{u}}f &= \nabla f \cdot \frac{\mathbf{u}}{\|\mathbf{u}\|} \\
&= \langle 2xy - y^2, x^2 - 2xy \rangle \cdot \frac{\langle 2, 6 \rangle}{\sqrt{40}} \\
&= \frac{6x^2 - 8xy - 2y^2}{\sqrt{40}}
\end{aligned}$$

3.18.29

(a) 2

(b)

$$\begin{aligned}\mathbf{u} &= \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle \\ D_{\mathbf{u}}f &= \nabla f \cdot \mathbf{u} \\ &= \langle 2xy^4, 4x^2y^3 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle \\ &= \sqrt{2}xy^4 - 2\sqrt{2}x^2y^3 \\ (D_{\mathbf{u}}f)(1, 1) &= \sqrt{2}(1)(1)^4 - 2\sqrt{2}(1)^2(1)^3 \\ &= -\sqrt{2}\end{aligned}$$

(c) 4

3.18.31

$$\begin{aligned}g(x, y, z) &= z - \sin xy \\ \nabla g(x, y, z) &= \langle -y \cos xy, -x \cos xy, 1 \rangle \\ \nabla g\left(\frac{1}{2}, \frac{2}{3}\pi, \frac{1}{2}\sqrt{3}\right) &= \left\langle -\frac{\pi}{3}, -\frac{1}{4}, 1 \right\rangle \\ 0 &= \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) \\ &= \left\langle -\frac{\pi}{3}, -\frac{1}{4}, 1 \right\rangle \cdot \left(\langle x, y, z \rangle - \left\langle \frac{1}{2}, \frac{2}{3}\pi, \frac{1}{2}\sqrt{3} \right\rangle \right) \\ &= -\frac{\pi}{3} \left(x - \frac{1}{2} \right) - \frac{1}{4} \left(y - \frac{2}{3}\pi \right) + \left(z - \frac{1}{2}\sqrt{3} \right) \\ &= -\frac{\pi}{3}x + \frac{\pi}{6} - \frac{1}{4}y + \frac{\pi}{6} + z - \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} - \frac{\pi}{3} &= -\frac{\pi}{3}x - \frac{1}{4}y + z \\ 6\sqrt{3} - 4\pi &= -4\pi x - 3y + 12z \\ 4\pi - 6\sqrt{3} &= 4\pi x + 3y - 12z\end{aligned}$$

3.18.33

(a)

$$\begin{aligned}\int_0^1 \int_x^{2x} \sqrt{1-x^2} \, dy \, dx &= \int_0^1 x \sqrt{1-x^2} \, dx \\ &= \left[-\frac{1}{3}(1-x^2)^{3/2} \right]_0^1 \\ &= \frac{1}{3}\end{aligned}$$

(b)

$$\int_0^2 \int_{y/2}^y \sqrt{1-x^2} \, dx \, dy$$

3.18.37

$$\begin{aligned} V &= \int_0^3 \int_0^{2\pi} \int_{z/3}^z r \, dr \, d\theta \, dz \\ &= 2\pi \int_0^3 \left[\frac{1}{2} r^2 \right]_{z/3}^z dz \\ &= \frac{8}{9} \pi \int_0^3 z^2 \, dz \\ &= \frac{8}{9} \pi \left[\frac{1}{3} z^3 \right]_0^3 \\ &= 8\pi \end{aligned}$$

3.18.39

$$\nabla \cdot \mathbf{F} = 2xy + 2xy + 2xy = 6xy$$

3.18.41

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

3.18.43

$$\begin{aligned} \int_C \frac{z^2}{x^2 + y^2} \, ds &= \int_{\pi}^{2\pi} \frac{4t^2}{\cos^2 2t + \sin^2 2t} \sqrt{4 \sin^2 t + 4 \cos^2 t + 4} \, dt \\ &= 8\sqrt{2} \int_{\pi}^{2\pi} t^2 \, dt \\ &= 8\sqrt{2} \left[\frac{1}{3} t^3 \right]_{\pi}^{2\pi} \\ &= \frac{56}{3} \sqrt{2} \pi^3 \end{aligned}$$

3.18.45

The vector field is conservative so the line integral is independent of path. We can first integrate from $(0, 0)$ to $(1, 0)$, then from $(1, 0)$ to $(1, -2)$.

$$\begin{aligned}
\int_C 3x^2y^2 dx + (2x^3y - 3y^2) dy &= \int_0^1 3x^2(0)^2 dx + \int_{-2}^0 (2(1)^3y - 3y^2) dy \\
&= \int_{-2}^0 (2y - 3y^2) dy \\
&= [y^2 - y^3]_{-2}^0 \\
&= 12
\end{aligned}$$

3.18.47

$$\begin{aligned}
\int_C y \sin \pi z dx + x^2 e^y dy + 3xyz dz &= \int_0^1 (t^2 \sin \pi t^3 + 2t^3 e^{t^2} + 6t^8) dt \\
&= 2 + \frac{2}{3\pi}
\end{aligned}$$

3.18.49

$$\begin{aligned}
\mathbf{F} &= \langle x \sin y, y \sin x \rangle \\
\int_0^{\pi/2} x \sin(0) dx &= 0 \\
\int_0^{\pi} y dy &= \frac{\pi^2}{2}
\end{aligned}$$

3.18.51

$$\begin{aligned}
\iint_S \frac{z}{xy} dS &= \int_1^4 \int_1^3 \frac{\sqrt{z}}{y} \sqrt{1 + \left[\frac{1}{2\sqrt{z}} \right]^2} dy dz \\
&= \int_1^4 \int_1^3 \frac{\sqrt{z}}{y} \sqrt{1 + \frac{1}{4z}} dy dz \\
&= \int_1^4 \sqrt{z + \frac{1}{4}} \int_1^3 \frac{1}{y} dy dz \\
&= \frac{2}{3} \ln 3 \left[\left(z + \frac{1}{4} \right)^{3/2} \right]_1^4 \\
&= \frac{2}{3} \ln 3 \left[\left(\frac{17}{4} \right)^{3/2} - \left(\frac{5}{4} \right)^{3/2} \right] \\
&= \frac{1}{12} (17^{3/2} - 5^{3/2}) \ln 3
\end{aligned}$$

3.18.53

$$\begin{aligned}
 r &= \sqrt{x^2 + y^2 + z^2} \\
 \nabla \left(\frac{1}{r} \right) &= -\frac{1}{r^3} \mathbf{r} \\
 \mathbf{F} &= c \nabla \left(\frac{1}{r} \right) \\
 &= -\frac{c}{r^3} \mathbf{r} \\
 &= -\frac{c}{r^2} \hat{\mathbf{r}} \\
 \mathbf{n} &= \hat{\mathbf{r}} \\
 \iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS &= \int_0^{2\pi} \int_0^\pi -c \sin \phi \, d\phi \, d\theta \\
 &= -2\pi c [-\cos \phi]_0^\pi \\
 &= -4\pi c
 \end{aligned}$$

3.18.55

$$\begin{aligned}
 \nabla \cdot \mathbf{F} &= \nabla \cdot \left(-\frac{c}{r^2} \hat{\mathbf{r}} \right) \\
 &= \frac{1}{r^2} \frac{\partial [r^2 (-c/r^2)]}{\partial r} \\
 &= 0 \\
 \iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS &= \iiint_V (\nabla \cdot \mathbf{F}) \, dV \\
 &= 0
 \end{aligned}$$

3.18.57

$$\begin{aligned}
 \mathbf{F} &= \langle -2y, 3x, 10z \rangle \\
 \nabla \times \mathbf{F} &= \langle 0, 0, 5 \rangle \\
 \oint \mathbf{F} \cdot d\mathbf{r} &= \iint_R (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \\
 &= \iint_R 5 \, dS \\
 &= 125\pi
 \end{aligned}$$

3.18.59

$$\begin{aligned}\mathbf{F} &= \langle x, y, z \rangle \\ \nabla \cdot \mathbf{F} &= 3 \\ \iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS &= \iiint_V (\nabla \cdot \mathbf{F}) \, dV \\ &= \iiint_C 3 \, dV \\ &= 3\pi\end{aligned}$$