

Vibrations and Waves by George C. King Problems

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1 Simple Harmonic Motion

1.1

- (a) (i) $T = 4 \text{ s}$
(ii) $\omega = \frac{\pi}{2} \text{ rad/s}$
(iii) $\omega^2 = \frac{k}{m} \Rightarrow k = m\omega^2 = \frac{\pi^2}{8} \text{ N/m}$

1.2

- (a)

$$\begin{aligned}
x &= A \cos \omega t \\
&= A \cos 2\pi f t \\
v &= -2\pi f A \sin 2\pi f t \\
v_{\max} &= 2\pi f A \\
&= 1.38 \text{ m/s}
\end{aligned}$$

- (b)

$$\begin{aligned}
a &= -4\pi^2 f^2 A \cos 2\pi f t \\
a_{\max} &= 4\pi^2 f^2 A \\
&= 3.82 \times 10^3 \text{ m/s}^2
\end{aligned}$$

1.3

$$\begin{aligned}
a_{\max} &\leq g \\
4\pi^2 f^2 A &\leq g \\
f &\leq \sqrt{\frac{g}{4\pi^2 A}} \\
&\leq 1.11 \text{ Hz}
\end{aligned}$$

1.4

(a)

$$\frac{U}{E} = \frac{\frac{1}{2}k\left(\frac{1}{2}A\right)^2}{\frac{1}{2}kA^2} = \frac{1}{4} \Rightarrow \frac{K}{E} = \frac{3}{4}$$

(b) (i) The total energy will increase by a factor of 4

(ii) The maximum velocity will increase by a factor of 2

(iii) The maximum acceleration will increase by a factor of 2 and the period won't change

1.5

(a) $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = 0.41 \text{ J}$

(b)

$$\begin{aligned} E &= \frac{1}{2}kA^2 \\ A &= \sqrt{\frac{2E}{k}} \\ &= 4.5 \text{ cm} \\ \omega &= \sqrt{\frac{k}{m}} \\ &= \sqrt{\frac{1600}{3}} \\ &= \frac{40}{\sqrt{3}} \\ &= 23 \text{ rad/s} \\ x &= A \cos(\omega t + \phi) \\ \phi &= \arccos\left(\frac{x}{A}\right) - \omega t \\ &= 2.7 \text{ rad} \\ x &= 0.045 \cos(23t + 2.7) \text{ m} \end{aligned}$$

1.6

Using the angular frequency of system (b) ω_b as the baseline, the angular frequency of system (a) ω_a is

$$\begin{aligned}
F &= ma = -2kx \\
a &= -\frac{2k}{m}x \\
\omega_a &= \sqrt{\frac{2k}{m}} \\
&= \sqrt{2}\omega_b
\end{aligned}$$

and the angular frequency of system (c) ω_c is

$$\begin{aligned}
F &= ma = -\frac{k}{2}x \\
a &= -\frac{k}{2m}x \\
\omega_c &= \sqrt{\frac{k}{2m}} \\
&= \sqrt{\frac{1}{2}}\omega_b
\end{aligned}$$

1.7

- (a) The test tube experiences a bouyancy force of $F = Ag\rho x$ so its equation of motion is

$$\begin{aligned}
F &= ma = -Ag\rho x \\
a &= -\frac{Ag\rho}{m}x \\
\omega &= \sqrt{\frac{Ag\rho}{m}}
\end{aligned}$$

- (b) The work done by the bouyancy force when moving from equilibrium to x and thus the potential energy is

$$\begin{aligned}
U &= \int_0^x Ag\rho x' dx' \\
&= \frac{1}{2}Ag\rho x^2
\end{aligned}$$

so the total energy of the system is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}Ag\rho x^2$$

1.8

$$s \propto \text{kg}^\alpha \text{m}^\beta (\text{m/s}^2)^\gamma$$

so $\alpha = 0$, $\beta = 1/2$, and $\gamma = -1/2$ meaning

$$T \propto \sqrt{\frac{l}{g}}$$

1.9

(a)

$$x = A \cos \sqrt{\frac{g}{l}} t$$

$$v = -\sqrt{\frac{g}{l}} A \sin \sqrt{\frac{g}{l}} t$$

$$\begin{aligned} v_{\max} &= \sqrt{\frac{g}{l}} A \\ &= 0.018 \text{ m/s} \end{aligned}$$

(b) The pendulum reaches its maximum speed at the bottom of its swing which occurs after a quarter cycle

$$\frac{1}{4}T = \frac{1}{4} \frac{2\pi}{\omega} = \frac{\pi}{2\sqrt{g/l}} = 0.43 \text{ s}$$

1.10

$$I \frac{d^2\theta}{dt^2} = \tau$$

$$\frac{1}{3}ML^2 \frac{d^2\theta}{dt^2} = -kL \sin \theta L \cos \theta$$

$$\frac{1}{3}M \frac{d^2\theta}{dt^2} = -k\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{3k}{M}\theta$$

$$T = \frac{2\pi}{\omega}$$

$$= 2\pi \sqrt{\frac{M}{3k}}$$

1.11

(a)

$$\begin{aligned}
 F &= -\frac{dU}{dx} = -\left(\frac{6a}{x^7} - \frac{12b}{x^{13}}\right) \\
 0 &= \frac{12b}{x^{13}} - \frac{6a}{x^7} \\
 &= \frac{12b}{x^6} - 6a \\
 6a &= \frac{12b}{x^6} \\
 x^6 &= \frac{2b}{a} \\
 x &= \left(\frac{2b}{a}\right)^{1/6}
 \end{aligned}$$

1.12

(a)

$$\begin{aligned}
 K &= \frac{1}{2}Mv^2 + \int dK \\
 &= \frac{1}{2}Mv^2 + \int_0^L \frac{1}{2} \frac{m}{L} \left(\frac{l}{L}v\right)^2 dl \\
 &= \frac{1}{2}Mv^2 + \frac{1}{2} \frac{mv^2}{L^3} \int_0^L l^2 dl \\
 &= \frac{1}{2}Mv^2 + \frac{1}{2} \frac{mv^2}{L^3} \frac{1}{3}L^3 \\
 &= \frac{1}{2}Mv^2 + \frac{1}{6}mv^2 \\
 &= \frac{1}{2}(M + m/3)v^2 \\
 E &= K + U \\
 &= \frac{1}{2}(M + m/3)v^2 + \frac{1}{2}kx^2
 \end{aligned}$$

(b)

$$\omega = \sqrt{\frac{k}{M + m/3}}$$

1.13

(a)

$$\begin{aligned} K &= E - U \\ \frac{1}{2}mv^2 &= U(A) - U(x) \\ v &= \sqrt{2[U(A) - U(x)]/m} \end{aligned}$$

(b)

$$\begin{aligned} T &= 4 \int_0^A \frac{dx}{v} \\ &= 4 \int_0^A \sqrt{\frac{m}{2[U(A) - U(x)]}} dx \\ &= 4 \sqrt{\frac{m}{2U(A)}} \int_0^A \frac{dx}{\sqrt{1 - U(x)/U(A)}} \end{aligned}$$

(c)

$$\begin{aligned} T &= 4 \sqrt{\frac{m}{2\alpha A^n}} \int_0^A \frac{dx}{\sqrt{1 - (x/A)^n}} \\ &= 4 \sqrt{\frac{m}{2\alpha A^n}} \int_0^1 \frac{A d\xi}{\sqrt{1 - \xi^n}} \\ &= cA^{(n/2)-1} \end{aligned}$$

2 The Damped Harmonic Oscillator

2.1

$$\begin{aligned} \left(\frac{\gamma}{2}\right)^2 &= \omega_0^2 \\ \frac{b}{2m} &= \sqrt{\frac{k}{m}} \\ b &= 2m\sqrt{\frac{k}{m}} \\ &= 2m\sqrt{\frac{mg/x}{m}} \\ &= 2m\sqrt{\frac{g}{x}} \\ &= 64 \text{ kg/s} \end{aligned}$$

2.2

$$\begin{aligned}
 \frac{A_{n+1}}{A_n} &= 0.90 \\
 e^{-2.5\gamma/2} &= 0.90 \\
 e^{2.5\gamma/2} &= \frac{1}{0.90} \\
 \frac{2.5\gamma}{2} &= \ln \frac{1}{0.90} \\
 \gamma &= \frac{2}{2.5} \ln \frac{1}{0.90} \\
 &= 8.43 \times 10^{-2} \text{ s}^{-1} \\
 F &= -bv \\
 &= -(4.21 \times 10^{-2})v
 \end{aligned}$$

2.3

After 10 cycles the amplitude has decreased by a factor of 5/3. The energy of the system is proportional to the amplitude squared, so

$$\begin{aligned}
 E(300) &= E(0)e^{-300/\tau} \\
 e^{300/\tau} &= \frac{E(0)}{E(300)} \\
 \tau &= \frac{300}{\ln[E(0)/E(300)]} \\
 &= \frac{300}{\ln \frac{25}{9}} \\
 &= 294 \text{ s} \\
 Q &= \omega_0 \tau \\
 &= \frac{2\pi\tau}{T} \\
 &= 61.5
 \end{aligned}$$

2.4

$$\begin{aligned}
 \frac{E(10T)}{E_0} &= \frac{E_0 e^{-\gamma 10T}}{E_0} \\
 \frac{1}{2} &= e^{-\gamma 10T} \\
 \frac{E(50T)}{E_0} &= \frac{E_0 e^{-\gamma 50T}}{E_0} \\
 &= (e^{-\gamma 10T})^5 \\
 &= \left(\frac{1}{2}\right)^5 \\
 &= \frac{1}{32}
 \end{aligned}$$

2.5

(a)

$$\begin{aligned}
 Q_{0.01} &= 310 \\
 \omega_{0.01} &= 3.14 \text{ rad/s} \\
 Q_{0.30} &= 10.5 \\
 \omega_{0.30} &= 3.14 \text{ rad/s} \\
 Q_{1.00} &= 3.14 \\
 \omega_{1.00} &= 3.10 \text{ rad/s}
 \end{aligned}$$

(c)

$$\begin{aligned}
 \gamma^2/4 &= \pi^2 \\
 \gamma &= 2\pi \\
 x &= Ae^{-\pi t} + Bte^{-\pi t} \\
 A &= 10 \text{ mm} \\
 v &= -10\pi e^{-\pi t} + Be^{-\pi t} - \pi Bte^{-\pi t} \\
 0 &= -10\pi + B \\
 B &= 10\pi \\
 x &= 10e^{-\pi t} + 10\pi te^{-\pi t}
 \end{aligned}$$

2.6

$$\begin{aligned}
 \frac{\omega}{\omega_0} &= \frac{\omega_0 \sqrt{1 - 1/4Q^2}}{\omega_0} \\
 &= \sqrt{1 - 1/4Q^2} \\
 &= 1 - \frac{1/4Q^2}{2} + \dots \\
 &\approx 1 - \frac{Q^2}{8}
 \end{aligned}$$

2.7

The amplitude of each pendulum decreases over time by a factor of

$$\begin{aligned}
 \exp\left(-\frac{\gamma t}{2}\right) &= \exp\left(-\frac{bt}{2m}\right) \\
 &= \exp\left(-\frac{bt}{2 \cdot \frac{4}{3}\pi r^3 \rho}\right) \\
 &= \exp\left(-\frac{3bt}{8\pi r^3 \rho}\right) \\
 &= \exp\left(-\frac{3bt}{8\pi r^3}\right)^{1/\rho}.
 \end{aligned}$$

After 10 minutes the amplitude of oscillation of the aluminium pendulum has decreased to half of its initial value

$$\begin{aligned}
 \exp\left(-\frac{225b}{\pi r^3}\right)^{1/\rho_a} &= \frac{1}{2} \\
 \exp\left(-\frac{225b}{\pi r^3}\right) &= \left(\frac{1}{2}\right)^{\rho_a}
 \end{aligned}$$

so the brass pendulum's amplitude of oscillation has decreased by a factor of

$$\begin{aligned}
 \exp\left(-\frac{225b}{\pi r^3}\right)^{1/\rho_b} &= \left(\frac{1}{2}\right)^{\rho_a/\rho_b} \\
 &= 0.802
 \end{aligned}$$

2.8

(a)

$$\begin{aligned}
 x &= A \sin \omega t \\
 v &= \omega A \cos \omega t \\
 a &= -\omega^2 A \sin \omega t \\
 E &= \int_0^T \frac{K e^2 a^2}{c^3} dt \\
 &= \int_0^T \frac{K e^2 \omega^4 A^2 \sin^2 \omega t}{c^3} dt \\
 &= \frac{K e^2 \omega^4 A^2}{c^3} \int_0^{2\pi/\omega} \sin^2 \omega t dt \\
 &= \frac{K e^2 \omega^4 A^2}{c^3} \left[\frac{t}{2} - \frac{1}{4\omega} \sin 2\omega t \right]_0^{2\pi/\omega} \\
 &= \frac{K e^2 \omega^3 A^2 \pi}{c^3}
 \end{aligned}$$

(b)

$$\begin{aligned}
 Q &= \frac{\frac{1}{2} m \omega^2 A^2}{\frac{K e^2 \omega^3 A^2 \pi}{2\pi c^3}} \\
 &= \frac{c^3 m}{e^2 K \omega}
 \end{aligned}$$

(c)

$$\begin{aligned}
 \tau &= \frac{1}{\gamma} \\
 &= \frac{Q}{\omega} \\
 &= \frac{c^3 m}{e^2 K \omega^2} \\
 &= \frac{c^3 m}{e^2 K (2\pi(c/\lambda))^2} \\
 &= \frac{\lambda^2 c m}{4\pi^2 e^2 K} \\
 &\approx 1.13 \times 10^{-8} \text{ s}
 \end{aligned}$$

3 Forced Oscillations

3.1

$$\begin{aligned}
 A(2 \text{ rad/s}) &= 1.3 \times 10^{-2} \text{ m} \\
 \delta(2 \text{ rad/s}) &= 0.58^\circ \\
 A(20 \text{ rad/s}) &= 0.13 \text{ m} \\
 \delta(20 \text{ rad/s}) &= 90^\circ \\
 A(100 \text{ rad/s}) &= 5.2 \times 10^{-4} \text{ m} \\
 \delta(100 \text{ rad/s}) &= 179^\circ
 \end{aligned}$$

3.2

$$\begin{aligned}
 A(\omega) &= \frac{a\omega_0/\omega}{\sqrt{(\omega_0/\omega - \omega/\omega_0)^2 + 1/Q^2}} \\
 &= \frac{au}{\sqrt{(u - 1/u)^2 + 1/Q^2}} \\
 &= \frac{a}{\sqrt{(1 - 1/u^2)^2 + 1/u^2 Q^2}}
 \end{aligned}$$

$A(\omega)$ is maximised when the denominator is minimised which occurs when

$$\begin{aligned}
 \frac{d}{du}((1 - u^{-2})^2 + Q^{-2}u^{-2}) &= 0 \\
 2(1 - u^{-2})2u^{-3} - 2Q^{-2}u^{-3} &= 0 \\
 4(1 - u^{-2}) - 2Q^{-2} &= 0 \\
 \frac{4Q^2 - 2}{Q^2} &= \frac{4}{u^2} \\
 \frac{Q^2}{4Q^2 - 2} &= \frac{u^2}{4} \\
 \frac{4Q^2}{4Q^2 - 2} &= u^2 \\
 \frac{1}{1 - 1/2Q^2} &= u^2 \\
 \frac{1}{\sqrt{1 - 1/2Q^2}} &= \frac{\omega_0}{\omega_{\max}} \\
 \omega_{\max} &= \omega_0 \sqrt{1 - 1/2Q^2}
 \end{aligned}$$

at which point the amplitude will be

$$\begin{aligned}
A_{\max} &= \frac{a\omega_0/\omega_0\sqrt{1-1/2Q^2}}{\sqrt{\left(\frac{\omega_0}{\omega_0\sqrt{1-1/2Q^2}} - \frac{\omega_0\sqrt{1-1/2Q^2}}{\omega_0}\right)^2 + 1/Q^2}} \\
&= \frac{a/\sqrt{1-1/2Q^2}}{\sqrt{\left(\frac{1}{\sqrt{1-1/2Q^2}} - \sqrt{1-1/2Q^2}\right)^2 + 1/Q^2}} \\
&= \frac{a/\sqrt{1-1/2Q^2}}{\sqrt{\left(\frac{1-1+1/2Q^2}{\sqrt{1-1/2Q^2}}\right)^2 + 1/Q^2}} \\
&= \frac{a/\sqrt{1-1/2Q^2}}{\sqrt{\frac{1}{4Q^4(1-1/2Q^2)} + 1/Q^2}} \\
&= \frac{a/\sqrt{1-1/2Q^2}}{\sqrt{\frac{1+4Q^2(1-1/2Q^2)}{4Q^2(1-1/2Q^2)}}} \\
&= \frac{a}{\sqrt{1-1/2Q^2}} \sqrt{\frac{4Q^2(1-1/2Q^2)}{1+4Q^2(1-1/2Q^2)}} \\
&= a\sqrt{\frac{4Q^2}{1+4Q^2(1-1/2Q^2)}} \\
&= aQ\sqrt{\frac{4}{4(1-1/2Q^2)+1/Q^2}} \\
&= \frac{aQ}{\sqrt{1-1/2Q^2+1/4Q^2}} \\
&= \frac{aQ}{\sqrt{1-1/4Q^2}}
\end{aligned}$$

3.3

(a)

$$\begin{aligned}
\frac{\omega_0 - \omega_{\max}}{\omega_0} &= 1 - \sqrt{1-1/2Q^2} \\
&\approx 0.25 \%
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{A_{\max} - A_0}{A_0} &= \frac{1}{\sqrt{1-1/4Q^2}} - 1 \\
&\approx 0.13 \%
\end{aligned}$$

3.4

$$\begin{aligned}
 \overline{P}_{\max} &= \frac{F_0^2}{2m\gamma} \\
 &= 50 \text{ W} \\
 \overline{P}(\omega) &= \frac{F_0^2}{2m\omega_0 Q [4(\Delta\omega/\omega_0)^2 + 1/Q^2]} \\
 &= \frac{\overline{P}_{\max}}{Q^2 [4(\Delta\omega/\omega_0)^2 + 1/Q^2]} \\
 &= \frac{50}{625 [4(\Delta\omega/100)^2 + 1/625]} \\
 &= \frac{50}{\frac{1}{4}\Delta\omega^2 + 1}
 \end{aligned}$$

3.5

(a)

$$\frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \approx 398 \text{ Hz}$$

(b)

$$I = \frac{V_0}{R} = \frac{15}{75} = 0.2 \text{ A}$$

3.6

$$i^i = (e^{i\pi/2})^i = e^{-\pi/2} = 0.208$$

3.7

$$\begin{aligned}
 z &= Ae^{i(\omega t + \phi)} \\
 x &= \text{Re } z \\
 &= A \cos(\omega t + \phi) \\
 \frac{dz}{dt} &= i\omega Ae^{i(\omega t + \phi)} \\
 \frac{dx}{dt} &= \text{Re } \frac{dz}{dt} \\
 &= -\omega A \sin(\omega t + \phi) \\
 &= \omega A \cos(\omega t + \phi + \pi/2)
 \end{aligned}$$

$\frac{dx}{dt}$ is in advance of x by 90°

3.8

(a)

$$\begin{aligned}
m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + mg \frac{x - \xi}{l} &= 0 \\
m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + m \frac{g}{l} x &= m \frac{g}{l} a \cos \omega t \\
m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + m \omega_0^2 x &= m \omega_0^2 a \cos \omega t \\
\operatorname{Re} \left(m \frac{d^2 z}{dt^2} + b \frac{dz}{dt} + m \omega_0^2 z \right) &= \operatorname{Re}(m \omega_0^2 a e^{i \omega t})
\end{aligned}$$

(b)

$$\begin{aligned}
m \frac{d^2}{dt^2} (A e^{i(\omega t - \delta)}) + b \frac{d}{dt} (A e^{i(\omega t - \delta)}) + m \omega_0^2 A e^{i(\omega t - \delta)} &= m \omega_0^2 a e^{i \omega t} \\
-m \omega^2 A e^{i(\omega t - \delta)} + i \omega b A e^{i(\omega t - \delta)} + m \omega_0^2 A e^{i(\omega t - \delta)} &= m \omega_0^2 a e^{i \omega t} \\
-m \omega^2 A + i \omega b A + m \omega_0^2 A &= m \omega_0^2 a e^{i \delta}
\end{aligned}$$

$$\begin{aligned}
-m \omega^2 A + m \omega_0^2 A &= m \omega_0^2 a \cos \delta \\
-\omega^2 A + \omega_0^2 A &= \omega_0^2 a \cos \delta \\
\frac{\omega_0^2 - \omega^2}{\omega_0^2 a} A &= \cos \delta
\end{aligned}$$

$$\begin{aligned}
\omega b A &= m \omega_0^2 a \sin \delta \\
\frac{\omega b}{m \omega_0^2 a} A &= \sin \delta
\end{aligned}$$

$$\begin{aligned}
\frac{\omega b}{m(\omega_0^2 - \omega^2)} &= \tan \delta \\
\frac{\omega \gamma}{\omega_0^2 - \omega^2} &= \tan \delta
\end{aligned}$$

$$\begin{aligned}
A &= \frac{\omega_0^2 a \cos \delta}{\omega_0^2 - \omega^2} \\
&= \frac{\omega_0^2 a}{\omega_0^2 - \omega^2} \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} \\
&= \frac{a \omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}}
\end{aligned}$$

3.9

(a)

$$\begin{aligned}
 A(t_{75}) &= A_0 e^{-\gamma t_{75}/2} \\
 \frac{A(t_{75})}{A_0} &= e^{-\gamma t_{75}/2} \\
 \ln \frac{A_0/e}{A_0} &= -\frac{\gamma t_{75}}{2} \\
 -1 &= -\frac{\gamma t_{75}}{2} \\
 \frac{2}{t_{75}} &= \gamma
 \end{aligned}$$

$$\begin{aligned}
 Q &= \frac{\omega_0}{\gamma} \\
 &= \frac{75 \cdot 2\pi}{t_{75}} \frac{t_{75}}{2} \\
 &= 75\pi
 \end{aligned}$$

(b)

$$\begin{aligned}
 A(\omega_0) &= a \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega_0^2)^2 + \omega_0^2 \gamma^2}} \\
 &= a \frac{\omega_0^2}{\sqrt{\omega_0^2 \gamma^2}} \\
 &= a \frac{\omega_0}{\gamma} \\
 &= aQ \\
 &= (0.5 \text{ mm}) 75\pi \\
 &= 0.12 \text{ m}
 \end{aligned}$$

(c)

$$\begin{aligned}
\frac{aQ}{2} &= a \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} \\
&\approx a \frac{\omega_0^2}{\sqrt{(2\omega_0(-\Delta\omega))^2 + \omega_0^2 \gamma^2}} \\
\frac{2\omega_0^2}{Q} &= \sqrt{4\omega_0^2 \Delta\omega^2 + \omega_0^2 \gamma^2} \\
\frac{4\omega_0^4}{Q^2} &= 4\omega_0^2 \Delta\omega^2 + \omega_0^2 \gamma^2 \\
4\omega_0^2 \Delta\omega^2 &= \frac{4\omega_0^4}{(\omega_0/\gamma)^2} - \omega_0^2 \gamma^2 \\
(2\Delta\omega)^2 &= 4\gamma^2 - \gamma^2 \\
2\Delta\omega &= \gamma\sqrt{3} \\
&= \frac{\omega_0}{Q} \sqrt{3} \\
&= \frac{\sqrt{g/l}}{Q} \sqrt{3} \\
&= 0.019 \text{ rad/s}
\end{aligned}$$

3.10

(a) (i)

$$\begin{aligned}
K &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}m(-\omega A \sin(\omega t - \delta))^2 \\
&= \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t - \delta)
\end{aligned}$$

(ii)

$$\begin{aligned}
U &= \frac{1}{2}kx^2 \\
&= \frac{1}{2}k(A \cos(\omega t - \delta))^2 \\
&= \frac{1}{2}kA^2 \cos^2(\omega t - \delta)
\end{aligned}$$

(iii)

$$\begin{aligned}
E &= \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t - \delta) + \frac{1}{2}kA^2 \cos^2(\omega t - \delta) \\
&= \frac{1}{2}mA^2[\omega^2 \sin^2(\omega t - \delta) + \omega_0^2 \cos^2(\omega t - \delta)]
\end{aligned}$$

(b)

$$\begin{aligned} 0 &= \frac{dE}{dt} \\ &= \frac{1}{2}mA^2[2\omega^3 \sin(\omega t - \delta) \cos(\omega t - \delta) - 2\omega_0^3 \cos(\omega t - \delta) \sin(\omega t - \delta)] \\ &= \frac{1}{2}mA^2 \sin(2(\omega t - \delta))(\omega^3 - \omega_0^3) \\ \omega &= \omega_0 \end{aligned}$$

$$\begin{aligned} E &= \frac{1}{2}mA^2[\omega_0^2 \sin^2(\omega_0 t - \delta) + \omega_0^2 \cos^2(\omega_0 t - \delta)] \\ &= \frac{1}{2}mA^2\omega_0^2 \end{aligned}$$

(c)

$$\begin{aligned} \overline{K} &= \frac{1}{T} \int_{t_0}^{t_0+T} \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t - \delta) dt \\ &= \frac{m\omega^2 A^2}{2T} \int_{t_0}^{t_0+T} \sin^2(\omega t - \delta) dt \\ &= \frac{m\omega^2 A^2}{4} \\ \overline{E} &= \frac{1}{T} \int_{t_0}^{t_0+T} \frac{1}{2}mA^2[\omega^2 \sin^2(\omega t - \delta) + \omega_0^2 \cos^2(\omega t - \delta)] dt \\ &= \frac{mA^2}{2T} \left(\int_{t_0}^{t_0+T} \omega^2 \sin^2(\omega t - \delta) dt + \int_{t_0}^{t_0+T} \omega_0^2 \cos^2(\omega t - \delta) dt \right) \\ &= \frac{mA^2}{4}(\omega^2 + \omega_0^2) \\ \frac{\overline{K}}{\overline{E}} &= \frac{m\omega^2 A^2}{4} \frac{4}{mA^2(\omega^2 + \omega_0^2)} \\ &= \frac{\omega^2}{\omega^2 + \omega_0^2} \\ &= \frac{1}{1 + (\omega_0/\omega)^2} \end{aligned}$$

(d)

$$\begin{aligned}
\overline{E} &= \overline{K} + \overline{U} \\
&= \frac{m\omega^2 A^2}{4} + \frac{kA^2}{4} \\
&= \frac{1}{4}mA^2(\omega^2 + \omega_0^2) \\
&= \frac{1}{4}m(\omega^2 + \omega_0^2) \left(\frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \right)^2 \\
&= \frac{1}{4}m(\omega^2 + \omega_0^2) \frac{F_0^2}{m^2[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2]} \\
&= \frac{F_0^2(\omega_0^2 + \omega^2)}{4m[(\omega_0^2 - \omega^2)^2 + \omega^2b^2/m^2]}
\end{aligned}$$

3.11

(a)

$$\begin{aligned}
x &= A \cos \omega t \\
v &= -\omega A \sin \omega t \\
W &= \int_0^T bv^2 dt \\
&= b \int_0^{2\pi/\omega} \omega^2 A^2 \sin^2 \omega t dt \\
&= \pi b \omega A^2
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{W}{E} &= \frac{\pi b \omega A^2}{\frac{1}{2}m\omega^2 A^2} \\
&= \frac{2\pi b}{m\omega}
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{W}{E} &= \frac{2\pi b}{m\omega_0} \\
&= \frac{2\pi\gamma}{\omega_0} \\
&= \frac{2\pi}{Q}
\end{aligned}$$

3.12

Let T' be the number of seconds in 8 days and $T = 2\pi\sqrt{l/g}$ be the period of the pendulum, then

$$\begin{aligned}
 Q &= 2\pi \frac{\text{stored energy}}{\text{energy dissipated/cycle}} \\
 &= 2\pi \frac{1}{2} m_1 \omega^2 A^2 \frac{\text{cycles in 8 days}}{\text{energy dissipated in 8 days}} \\
 &= \pi m_1 \frac{g}{l} A^2 \frac{T'/T}{m_2 g h} \\
 &= \frac{\pi m_1 A^2 T'}{m_2 l h T} \\
 &= \frac{m_1 A^2 \cdot 8 \cdot 24 \cdot 60 \cdot 60}{2 m_2 l h \sqrt{l/g}} \\
 &\approx 70
 \end{aligned}$$

4 Coupled Oscillators

4.1

(a)

$$\begin{aligned}
 \omega_1 &= \sqrt{\frac{g}{l}} \\
 &= 5.7 \text{ rad/s} \\
 \omega_2 &= \sqrt{\frac{g}{l} + \frac{2k}{m}} \\
 &= 6.0 \text{ rad/s}
 \end{aligned}$$

(b) The oscillation of the first pendulum is described by the equation

$$x_a = A \cos \frac{(\omega_2 - \omega_1)t}{2} \cos \frac{(\omega_2 + \omega_1)t}{2},$$

the amplitude of which temporarily becomes 0 at

$$\frac{(\omega_2 - \omega_1)t}{2} = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{\omega_2 - \omega_1} = 11.6 \text{ s.}$$

4.2

(a)

$$5.0 \text{ mm} = \frac{1}{2}(C_1 + C_2)$$

$$5.0 \text{ mm} = \frac{1}{2}(C_1 - C_2)$$

$$10 \text{ mm} = C_1$$

$$0.0 \text{ mm} = C_2$$

(b) (i)

$$5.0 \text{ mm} = \frac{1}{2}(C_1 + C_2)$$

$$-5.0 \text{ mm} = \frac{1}{2}(C_1 - C_2)$$

$$0.0 \text{ mm} = C_1$$

$$10 \text{ mm} = C_2$$

(ii)

$$10 \text{ mm} = \frac{1}{2}(C_1 + C_2)$$

$$0 \text{ mm} = \frac{1}{2}(C_1 - C_2)$$

$$10 \text{ mm} = C_1$$

$$10 \text{ mm} = C_2$$

(iii)

$$10 \text{ mm} = \frac{1}{2}(C_1 + C_2)$$

$$5.0 \text{ mm} = \frac{1}{2}(C_1 - C_2)$$

$$15 \text{ mm} = C_1$$

$$5.0 \text{ mm} = C_2$$

4.3

$$m \frac{d^2 x_a}{dt^2} = kx_b - 2kx_a$$

$$m \frac{d^2 x_b}{dt^2} = kx_a - 2kx_b$$

$$m \frac{d^2 (x_a + x_b)}{dt^2} = -k(x_a + x_b)$$

$$m \frac{d^2 q_1}{dt^2} = -kq_1$$

$$\omega_1 = \sqrt{\frac{k}{m}}$$

$$m \frac{d^2 (x_a - x_b)}{dt^2} = -3k(x_a - x_b)$$

$$m \frac{d^2 q_2}{dt^2} = -3kq_2$$

$$\omega_2 = \sqrt{\frac{3k}{m}}$$

4.4

(a)

$$E_a = \frac{1}{2} m \omega^2 A^2$$

$$= \frac{1}{2} m A^2 \left(\frac{\omega_2 + \omega_1}{2} \right)^2 \cos^2 \frac{(\omega_2 - \omega_1)t}{2}$$

$$E_b = \frac{1}{2} m \omega^2 A^2$$

$$= \frac{1}{2} m \left(\frac{\omega_2 + \omega_1}{2} \right)^2 A^2 \sin^2 \frac{(\omega_2 - \omega_1)t}{2}$$

(b)

$$\frac{(\omega_2 - \omega_1)T}{2} = \pi$$

$$T = \frac{2\pi}{\omega_2 - \omega_1}$$

$$\omega = \frac{2\pi}{T}$$

$$= \omega_2 - \omega_1$$

4.5

(a)

$$m \frac{d^2 x_1}{dt^2} = -2mg \frac{x_1}{l} + mg \frac{x_2 - x_1}{l}$$

$$\frac{d^2 x_1}{dt^2} + \frac{3g}{l} x_1 - \frac{g}{l} x_2 = 0$$

$$m \frac{d^2 x_2}{dt^2} = -mg \frac{x_2 - x_1}{l}$$

$$\frac{d^2 x_2}{dt^2} - \frac{g}{l} x_1 + \frac{g}{l} x_2 = 0$$

(b)

$$-\omega^2 A \cos \omega t + \frac{3g}{l} A \cos \omega t - \frac{g}{l} B \cos \omega t = 0$$

$$A \left(\frac{3g}{l} - \omega^2 \right) = B \left(\frac{g}{l} \right)$$

$$-\omega^2 B \cos \omega t - \frac{g}{l} A \cos \omega t + \frac{g}{l} B \cos \omega t = 0$$

$$A \left(\frac{g}{l} \right) = B \left(\frac{g}{l} - \omega^2 \right)$$

$$\frac{3g/l - \omega^2}{g/l} = \frac{g/l}{g/l - \omega^2}$$

$$\left(\frac{3g}{l} - \omega^2 \right) \left(\frac{g}{l} - \omega^2 \right) = \left(\frac{g}{l} \right)^2$$

$$3 \left(\frac{g}{l} \right)^2 - \frac{3g}{l} \omega^2 - \frac{g}{l} \omega^2 + \omega^4 = \left(\frac{g}{l} \right)^2$$

$$(\omega^2)^2 - \frac{4g}{l} \omega^2 + 2 \left(\frac{g}{l} \right)^2 = 0$$

$$\omega^2 = \frac{\frac{4g}{l} \pm \sqrt{\frac{16g^2}{l^2} - 8 \left(\frac{g}{l} \right)^2}}{2}$$

$$= (2 \pm \sqrt{2}) \frac{g}{l}$$

$$\omega = \sqrt{(2 \pm \sqrt{2}) \frac{g}{l}}$$

$$\begin{aligned}
\frac{A}{B} &= \frac{g/l}{3g/l - \omega^2} \\
&= \frac{g}{3g - l\omega^2} \\
&= \frac{g}{3g - l(2 \pm \sqrt{2})\frac{g}{l}} \\
&= \frac{1}{3 - (2 \pm \sqrt{2})} \\
&= \frac{1}{1 \pm \sqrt{2}}
\end{aligned}$$

(c)

$$\begin{aligned}
T &= \frac{2\pi}{\omega} \\
&= \frac{2\pi}{\sqrt{(2 \pm \sqrt{2})\frac{g}{l}}} \\
&= 1.09 \text{ s or } 2.62 \text{ s}
\end{aligned}$$

4.6

(b)

$$m \frac{d^2 x_1}{dt^2} = k(x_2 - x_1)$$

$$\frac{d^2 x_1}{dt^2} + \omega_1^2 x_1 - \omega_1^2 x_2 = 0$$

$$\begin{aligned}
M \frac{d^2 x_2}{dt^2} &= -k(x_2 - x_1) + k(x_3 - x_2) \\
&= kx_1 - 2kx_2 + kx_3
\end{aligned}$$

$$\frac{d^2 x_2}{dt^2} - \omega_2^2 x_1 + 2\omega_2^2 x_2 - \omega_2^2 x_3 = 0$$

$$m \frac{d^2 x_3}{dt^2} = -k(x_3 - x_2)$$

$$\frac{d^2 x_3}{dt^2} - \omega_1^2 x_2 + \omega_1^2 x_3 = 0$$

(c)

$$x_1 = A \cos \omega t$$

$$x_2 = B \cos \omega t$$

$$x_3 = C \cos \omega t$$

$$-\omega^2 A \cos \omega t + \omega_1^2 A \cos \omega t - \omega_1^2 B \cos \omega t = 0$$

$$A(\omega_1^2 - \omega^2) = B(\omega_1^2)$$

$$-\omega^2 B \cos \omega t - \omega_2^2 A \cos \omega t + 2\omega_2^2 B \cos \omega t - \omega_2^2 C \cos \omega t = 0$$

$$(A + C)(\omega_2^2) = B(2\omega_2^2 - \omega^2)$$

$$-\omega^2 C \cos \omega t - \omega_1^2 B \cos \omega t + \omega_1^2 C \cos \omega t = 0$$

$$C(\omega_1^2 - \omega^2) = B(\omega_1^2)$$

$$(A + C)(\omega_1^2 - \omega^2) = B(2\omega_1^2)$$

$$\frac{\omega_2^2}{\omega_1^2 - \omega^2} = \frac{2\omega_2^2 - \omega^2}{2\omega_1^2}$$

$$\begin{aligned} 2\omega_1^2 \omega_2^2 &= (2\omega_2^2 - \omega^2)(\omega_1^2 - \omega^2) \\ &= 2\omega_1^2 \omega_2^2 - 2\omega_2^2 \omega^2 - \omega_1^2 \omega^2 + \omega^4 \end{aligned}$$

$$\begin{aligned} 0 &= (\omega^2)^2 - (\omega_1^2 + 2\omega_2^2)\omega^2 \\ &= \omega^2 - \omega_1^2 - 2\omega_2^2 \end{aligned}$$

$$\omega^2 = \omega_1^2 + 2\omega_2^2$$

$$= \frac{k}{m} + 2\frac{k}{M}$$

$$\omega = \sqrt{\frac{k(M+2m)}{Mm}}$$

(d)

$$\begin{aligned} \frac{\sqrt{\frac{k(M+2m)}{Mm}}}{\sqrt{\frac{k}{m}}} &= \sqrt{\frac{M+2m}{M}} \\ &= \sqrt{1+2m/M} \\ &= \sqrt{1+16/6} \\ &\approx 1.91 \end{aligned}$$

4.7

- (a) Let x_1 be the top mass's displacement from equilibrium and x_2 be the bottom mass's, then the equations of motion are

$$\begin{aligned} 3m \frac{d^2 x_1}{dt^2} &= -4kx_1 + k(x_2 - x_1) \\ &= -5kx_1 + kx_2 \\ \frac{d^2 x_1}{dt^2} + \frac{5k}{3m}x_1 - \frac{k}{3m}x_2 &= 0 \end{aligned}$$

$$\begin{aligned} m \frac{d^2 x_2}{dt^2} &= -k(x_2 - x_1) \\ &= kx_1 - kx_2 \\ \frac{d^2 x_2}{dt^2} - \frac{k}{m}x_1 + \frac{k}{m}x_2 &= 0 \end{aligned}$$

Assuming solutions of the form $x_1 = A \cos \omega t$ and $x_2 = B \cos \omega t$ and substituting into the above gives

$$\begin{aligned} -\omega^2 A \cos \omega t + \frac{5k}{3m} A \cos \omega t - \frac{k}{3m} B \cos \omega t &= 0 \\ A \left(\frac{5k}{3m} - \omega^2 \right) &= B \left(\frac{k}{3m} \right) \end{aligned}$$

$$\begin{aligned} -\omega^2 B \cos \omega t - \frac{k}{m} A \cos \omega t + \frac{k}{m} B \cos \omega t &= 0 \\ A \left(\frac{k}{m} \right) &= B \left(\frac{k}{m} - \omega^2 \right) \end{aligned}$$

$$\begin{aligned} \frac{\frac{5k}{3m} - \omega^2}{\frac{k}{m}} &= \frac{\frac{k}{3m}}{\frac{k}{m} - \omega^2} \\ \left(\frac{5k}{3m} - \omega^2 \right) \left(\frac{k}{m} - \omega^2 \right) &= \frac{1}{3} \left(\frac{k}{m} \right)^2 \\ \frac{5}{3} \left(\frac{k}{m} \right)^2 - \frac{5k}{3m} \omega^2 - \frac{k}{m} \omega^2 + \omega^4 &= \frac{1}{3} \left(\frac{k}{m} \right)^2 \\ (\omega^2)^2 - \frac{8}{3} \frac{k}{m} \omega^2 + \frac{4}{3} \left(\frac{k}{m} \right)^2 &= 0 \end{aligned}$$

$$\begin{aligned}
\omega^2 &= \frac{\frac{8}{3} \frac{k}{m} \pm \sqrt{\left(\frac{8}{3} \frac{k}{m}\right)^2 - \frac{16}{3} \left(\frac{k}{m}\right)^2}}{2} \\
&= \frac{\frac{8}{3} \frac{k}{m} \pm \sqrt{\frac{64}{9} \left(\frac{k}{m}\right)^2 - \frac{16}{3} \left(\frac{k}{m}\right)^2}}{2} \\
&= \frac{\frac{8}{3} \frac{k}{m} \pm \sqrt{\frac{16}{9} \left(\frac{k}{m}\right)^2}}{2} \\
&= \frac{\frac{8}{3} \frac{k}{m} \pm \frac{4}{3} \frac{k}{m}}{2} \\
&= \left(\frac{4}{3} \pm \frac{2}{3}\right) \frac{k}{m} \\
\omega &= \sqrt{\frac{2k}{3m}} \text{ or } \sqrt{\frac{2k}{m}}
\end{aligned}$$

(b)

$$A \left(\frac{k}{m} \right) = B \left(\frac{k}{m} - \omega^2 \right)$$

$$A \left(\frac{k}{m} \right) = B \left(\frac{k}{m} - \frac{2k}{3m} \right)$$

$$= B \left(\frac{k}{3m} \right)$$

$$A = \frac{1}{3} B$$

$$A \left(\frac{k}{m} \right) = B \left(\frac{k}{m} - \frac{2k}{m} \right)$$

$$= B \left(-\frac{k}{m} \right)$$

$$A = -B$$

So the first normal mode is

$$x_1 = A \cos \sqrt{\frac{2k}{3m}} t$$

$$x_2 = 3A \cos \sqrt{\frac{2k}{3m}} t$$

where the masses oscillate in phase and the lower mass has an amplitude 3 times greater than the upper mass.

The second normal mode is

$$\begin{aligned}x_1 &= A \cos \sqrt{\frac{2k}{m}}t \\x_2 &= -A \cos \sqrt{\frac{2k}{m}}t\end{aligned}$$

where the masses oscillate 180° out of phase with equal amplitude.

4.8

There are 5 normal modes in the transverse direction.

4.9

(a)

$$\begin{aligned}M \frac{d^2 x_1}{dt^2} + (k_1 + k_2)x_1 - k_2 x_2 &= F_0 \cos \omega t \\m \frac{d^2 x_2}{dt^2} - k_2 x_1 + k_2 x_2 &= 0\end{aligned}$$

(b)

$$\begin{aligned}-M\omega^2 A \cos \omega t + (k_1 + k_2)A \cos \omega t - k_2 B \cos \omega t &= F_0 \cos \omega t \\A(k_1 + k_2 - M\omega^2) &= Bk_2 + F_0\end{aligned}$$

$$\begin{aligned}-m\omega^2 B \cos \omega t - k_2 A \cos \omega t + k_2 B \cos \omega t &= 0 \\Ak_2 &= B(k_2 - m\omega^2)\end{aligned}$$

$$\begin{aligned}B &= A \frac{k_2}{k_2 - m\omega^2} \\A(k_1 + k_2 - M\omega^2) &= A \frac{k_2^2}{k_2 - m\omega^2} + F_0 \\A \left(k_1 + k_2 - M\omega^2 - \frac{k_2^2}{k_2 - m\omega^2} \right) &= F_0 \\A \left(\frac{(k_1 + k_2 - M\omega^2)(k_2 - m\omega^2) - k_2^2}{k_2 - m\omega^2} \right) &= F_0 \\ \frac{F_0(k_2 - m\omega^2)}{(k_1 + k_2 - M\omega^2)(k_2 - m\omega^2) - k_2^2} &= A\end{aligned}$$

$$\begin{aligned}
A &= \frac{Bk_2 + F_0}{k_1 + k_2 - M\omega^2} \\
B &= \frac{Bk_2 + F_0}{k_1 + k_2 - M\omega^2} \frac{k_2}{k_2 - m\omega^2} \\
&= \frac{Bk_2^2 + F_0k_2}{(k_1 + k_2 - M\omega^2)(k_2 - m\omega^2)} \\
B \left(1 - \frac{k_2^2}{(k_1 + k_2 - M\omega^2)(k_2 - m\omega^2)} \right) &= \frac{F_0k_2}{(k_1 + k_2 - M\omega^2)(k_2 - m\omega^2)} \\
B[(k_1 + k_2 - M\omega^2)(k_2 - m\omega^2) - k_2^2] &= F_0k_2 \\
\frac{F_0k_2}{(k_1 + k_2 - M\omega^2)(k_2 - m\omega^2) - k_2^2} &= B
\end{aligned}$$

(c)

$$\begin{aligned}
A &= \frac{F_0 \left(k_2 - m \frac{k_1}{M} \right)}{\left(k_1 + k_2 - M \frac{k_1}{M} \right) \left(k_2 - m \frac{k_1}{M} \right) - k_2^2} \\
&= \frac{F_0 \left(k_2 - \frac{k_2}{k_1} k_1 \right)}{\left(k_1 + k_2 - k_1 \right) \left(k_2 - \frac{k_2}{k_1} k_1 \right) - k_2^2} \\
&= 0
\end{aligned}$$

4.10

(a)

$$\begin{aligned}
m \frac{d^2 x_1}{dt^2} &= -kx_1 + k(x_2 - x_1) \\
m \frac{d^2 x_1}{dt^2} + 2kx_1 - kx_2 &= 0 \\
m \frac{d^2 x_2}{dt^2} &= -k(x_2 - x_1) + k(x_3 - x_2) \\
m \frac{d^2 x_2}{dt^2} - kx_1 + 2kx_2 - kx_3 &= 0 \\
m \frac{d^2 x_3}{dt^2} &= -k(x_3 - x_2) - kx_3 \\
m \frac{d^2 x_3}{dt^2} - kx_2 + 2kx_3 &= 0
\end{aligned}$$

$$\begin{aligned}
-m\omega^2 A \cos \omega t + 2kA \cos \omega t - kB \cos \omega t &= 0 \\
A(2k - m\omega^2) &= B(k)
\end{aligned}$$

$$\begin{aligned}
-m\omega^2 B \cos \omega t - kA \cos \omega t + 2kB \cos \omega t - kC \cos \omega t &= 0 \\
(A + C)(k) &= B(2k - m\omega^2)
\end{aligned}$$

$$\begin{aligned}
-m\omega^2 C \cos \omega t - kB \cos \omega t + 2kC \cos \omega t &= 0 \\
C(2k - m\omega^2) &= B(k) \\
(A + C)(2k - m\omega^2) &= B(2k)
\end{aligned}$$

$$\begin{aligned}
\frac{k}{2k - m\omega^2} &= \frac{2k - m\omega^2}{2k} \\
(2k - m\omega^2)^2 &= 2k^2 \\
4k^2 - 4km\omega^2 + m^2\omega^4 &= 2k^2 \\
(\omega^2)^2 - 4\frac{k}{m}\omega^2 + 2\left(\frac{k}{m}\right)^2 &= 0 \\
\omega^2 &= \frac{4\frac{k}{m} \pm \sqrt{(4\frac{k}{m})^2 - 8\left(\frac{k}{m}\right)^2}}{2} \\
&= \frac{4\frac{k}{m} \pm \sqrt{16\left(\frac{k}{m}\right)^2 - 8\left(\frac{k}{m}\right)^2}}{2} \\
\omega &= \sqrt{(2 \pm \sqrt{2})\frac{k}{m}}
\end{aligned}$$

(b) (i) For $\omega = \sqrt{\frac{2k}{m}}$

$$\begin{aligned}
A\left(2k - m\frac{2k}{m}\right) &= B(k) \\
0 &= B(k)
\end{aligned}$$

so $B = 0$ and

$$\begin{aligned}
(A + C)(k) &= 0 \\
A &= -C
\end{aligned}$$

(ii) For $\omega = \sqrt{(2 + \sqrt{2})\frac{k}{m}}$

$$\begin{aligned} A\left(2k - m(2 + \sqrt{2})\frac{k}{m}\right) &= B(k) \\ A(2k - 2k - \sqrt{2}k) &= B(k) \\ A &= -\frac{1}{\sqrt{2}}B \end{aligned}$$

and

$$\begin{aligned} C\left(2k - m(2 + \sqrt{2})\frac{k}{m}\right) &= B(k) \\ C(2k - 2k - \sqrt{2}k) &= B(k) \\ C &= -\frac{1}{\sqrt{2}}B \\ &= A \end{aligned}$$

(iii) For $\omega = \sqrt{(2 - \sqrt{2})\frac{k}{m}}$

$$\begin{aligned} A\left(2k - m(2 - \sqrt{2})\frac{k}{m}\right) &= B(k) \\ A &= \frac{1}{\sqrt{2}}B \end{aligned}$$

and

$$\begin{aligned} C\left(2k - m(2 - \sqrt{2})\frac{k}{m}\right) &= B(k) \\ C &= \frac{1}{\sqrt{2}}B \\ &= A \end{aligned}$$

5 Travelling Waves

5.1

$$y = A \cos \left(\frac{2\pi}{\lambda} x + 2\pi f t \right)$$

$$= 15 \cos(0.25x + 75t)$$

$$A = 15 \text{ mm}$$

$$\lambda = 25 \text{ mm}$$

$$f = 12 \text{ Hz}$$

$$v = 30 \text{ cm/s}$$

The wave is travelling in the negative x direction

5.2

$$A = 0.15 \text{ m}$$

$$f = 10 \text{ Hz}$$

$$v = 50 \text{ m/s}$$

$$y = A \cos \left(\frac{2\pi}{v/f} x - 2\pi f t \right)$$

$$= 0.15 \cos(1.3x - 63t)$$

5.3

(a) (i)

$$y = A \sin 2\pi f(t - x/v)$$

$$\frac{\partial y}{\partial t} = 2\pi f A \cos 2\pi f(t - x/v)$$

$$\frac{\partial^2 y}{\partial t^2} = -(2\pi f)^2 A \sin 2\pi f(t - x/v)$$

$$\frac{\partial y}{\partial x} = -\frac{2\pi f}{v} A \cos 2\pi f(t - x/v)$$

$$\frac{\partial^2 y}{\partial x^2} = -\left(\frac{2\pi f}{v}\right)^2 A \sin 2\pi f(t - x/v)$$

$$-(2\pi f)^2 A \sin 2\pi f(t - x/v) = v^2 \left(-\left(\frac{2\pi f}{v}\right)^2 A \sin 2\pi f(t - x/v) \right)$$

(ii)

$$\begin{aligned}y &= A \sin \frac{2\pi}{\lambda}(x + vt) \\ \frac{\partial y}{\partial t} &= \frac{2\pi v}{\lambda} A \cos \frac{2\pi}{\lambda}(x + vt) \\ \frac{\partial^2 y}{\partial t^2} &= - \left(\frac{2\pi v}{\lambda} \right)^2 A \sin \frac{2\pi}{\lambda}(x + vt) \\ \frac{\partial y}{\partial x} &= \frac{2\pi}{\lambda} A \cos \frac{2\pi}{\lambda}(x + vt) \\ \frac{\partial^2 y}{\partial x^2} &= - \left(\frac{2\pi}{\lambda} \right)^2 A \sin \frac{2\pi}{\lambda}(x + vt) \\ - \left(\frac{2\pi v}{\lambda} \right)^2 A \sin \frac{2\pi}{\lambda}(x + vt) &= v^2 \left(- \left(\frac{2\pi}{\lambda} \right)^2 A \sin \frac{2\pi}{\lambda}(x + vt) \right)\end{aligned}$$

(iii)

$$\begin{aligned}y &= A \sin 2\pi(x/\lambda - t/T) \\ \frac{\partial y}{\partial t} &= - \frac{2\pi}{T} A \cos 2\pi(x/\lambda - t/T) \\ \frac{\partial^2 y}{\partial t^2} &= - \left(\frac{2\pi}{T} \right)^2 A \sin 2\pi(x/\lambda - t/T) \\ \frac{\partial y}{\partial x} &= \frac{2\pi}{\lambda} A \cos 2\pi(x/\lambda - t/T) \\ \frac{\partial^2 y}{\partial x^2} &= - \left(\frac{2\pi}{\lambda} \right)^2 A \sin 2\pi(x/\lambda - t/T) \\ - \left(\frac{2\pi}{T} \right)^2 A \sin 2\pi \left(\frac{x}{\lambda} - \frac{t}{T} \right) &= v^2 \left(- \left(\frac{2\pi}{\lambda} \right)^2 A \sin 2\pi \left(\frac{x}{\lambda} - \frac{t}{T} \right) \right) \\ \frac{1}{T^2} &= \frac{v^2}{\lambda^2} \\ f^2 &= f^2\end{aligned}$$

(iv)

$$\begin{aligned}y &= Ae^{i(\omega t + kx)} \\ \frac{\partial y}{\partial t} &= i\omega Ae^{i(\omega t + kx)} \\ \frac{\partial^2 y}{\partial t^2} &= -\omega^2 Ae^{i(\omega t + kx)} \\ \frac{\partial y}{\partial x} &= ikAe^{i(\omega t + kx)} \\ \frac{\partial^2 y}{\partial x^2} &= -k^2 Ae^{i(\omega t + kx)} \\ -\omega^2 Ae^{i(\omega t + kx)} &= v^2 \left(-k^2 Ae^{i(\omega t + kx)} \right) \\ \omega^2 &= v^2 k^2 \\ (2\pi f)^2 &= v^2 \left(\frac{2\pi}{\lambda} \right)^2 \\ f^2 &= \left(\frac{v}{\lambda} \right)^2 \\ &= f^2\end{aligned}$$

(v)

$$\begin{aligned}y &= A \cos(\omega_1 t - k_1 x) + B \cos(\omega_2 t - k_2 x) \\ \frac{\partial y}{\partial t} &= -\omega_1 A \sin(\omega_1 t - k_1 x) - \omega_2 B \sin(\omega_2 t - k_2 x) \\ \frac{\partial^2 y}{\partial t^2} &= -\omega_1^2 A \cos(\omega_1 t - k_1 x) - \omega_2^2 B \cos(\omega_2 t - k_2 x) \\ \frac{\partial y}{\partial x} &= k_1 A \sin(\omega_1 t - k_1 x) + k_2 B \sin(\omega_2 t - k_2 x) \\ \frac{\partial^2 y}{\partial x^2} &= -k_1^2 A \cos(\omega_1 t - k_1 x) - k_2^2 B \cos(\omega_2 t - k_2 x) \\ \omega_1^2 A + \omega_2^2 B &= v^2 (k_1^2 A + k_2^2 B) \\ \omega_1^2 A + \omega_2^2 B &= \omega_1^2 A + \omega_2^2 B\end{aligned}$$

(b)

$$\begin{aligned}
 \psi &= A \sin(k_1 x + k_2 y + k_3 z - \omega t) \\
 \frac{\partial^2 \psi}{\partial t^2} &= -\omega^2 A \sin(k_1 x + k_2 y + k_3 z - \omega t) \\
 \frac{\partial^2 \psi}{\partial x^2} &= -k_1^2 A \sin(k_1 x + k_2 y + k_3 z - \omega t) \\
 \frac{\partial^2 \psi}{\partial y^2} &= -k_2^2 A \sin(k_1 x + k_2 y + k_3 z - \omega t) \\
 \frac{\partial^2 \psi}{\partial z^2} &= -k_3^2 A \sin(k_1 x + k_2 y + k_3 z - \omega t) \\
 -\frac{\omega^2}{v^2} &= -k_1^2 - k_2^2 - k_3^2 \\
 v &= \frac{\omega}{\sqrt{k_1^2 + k_2^2 + k_3^2}}
 \end{aligned}$$

5.4

- (a) No difference
- (b) 180° out of phase

5.5

$$\begin{aligned}
 y(x, t) &= A \exp -\frac{(x^2 - vt)^2}{a^2} \\
 y(x + \delta x, t + \delta t) &= A \exp -\frac{(x + \delta x - v(t + \delta t))^2}{a^2} \\
 &= A \exp -\frac{(x - vt)^2}{a^2}
 \end{aligned}$$

5.6

- (a) (i) $f = c/\lambda = 200 \text{ kHz}$
- (ii) $f = 6 \times 10^{14} \text{ Hz}$
- (iii) $f = 3 \times 10^{18} \text{ Hz}$
- (iv) $f = c/\lambda = c/(2\pi/k) = ck/(2\pi) = 1 \times 10^8 \text{ Hz}$
- (v) $f = 69 \text{ kHz}$
- (b)

$$\begin{aligned}
 \lambda_{20} &= 17 \text{ m} \\
 \lambda_{15} &= 2.3 \text{ cm} \\
 \lambda_{440} &= 78 \text{ cm}
 \end{aligned}$$

5.7

(a)

$$\begin{aligned}
 \text{m/s} &= \sqrt{\text{N m/kg}} \\
 &= \sqrt{\text{kg} \cdot (\text{m/s}^2) \cdot \text{m/kg}} \\
 &= \sqrt{(\text{m/s})^2} \\
 &= \text{m/s}
 \end{aligned}$$

(b) The string with the smallest mass / the thinnest string, i.e. the high E string

5.8

(a) (i)

$$\begin{aligned}
 L &= 25 \text{ m} \\
 m &= 100 \text{ g} \\
 T &= 10 \text{ N} \\
 f &= 25 \text{ Hz} \\
 A &= 15 \text{ mm} \\
 v &= \sqrt{\frac{T}{\mu}} \\
 &= 50 \text{ m/s}
 \end{aligned}$$

(ii)

$$\begin{aligned}
 y &= A \sin \left(\frac{2\pi}{f} x - 2\pi f t \right) \\
 y' &= -2\pi f A \cos \left(\frac{2\pi}{f} x - 2\pi f t \right) \\
 y'_{\text{max}} &= 2\pi f A \\
 &= 2.4 \text{ m/s}
 \end{aligned}$$

(b)

$$L = 0.75 \text{ m}$$

$$m = 125 \text{ g}$$

$$T = 2.5 \text{ N}$$

$$\begin{aligned} v &= \sqrt{\frac{S}{\sigma}} \\ &= \sqrt{\frac{T/L}{m/L^2}} \\ &= \sqrt{\frac{LT}{m}} \\ &= 3.9 \text{ m/s} \end{aligned}$$

5.9

(a)

$$\begin{aligned} T(y) &= \mu y g \\ v &= \sqrt{\frac{T}{\mu}} \\ &= \sqrt{\frac{\mu y g}{\mu}} \\ &= \sqrt{y g} \end{aligned}$$

(b)

$$\begin{aligned} t &= 2 \int_0^L \frac{1}{v} dy \\ &= 2 \int_0^L \frac{1}{\sqrt{gy}} dy \\ &= \frac{2}{\sqrt{g}} [2\sqrt{y}]_0^L \\ &= 4\sqrt{\frac{L}{g}} \\ &= 2.0 \text{ s} \end{aligned}$$

5.10

(a)

$$\mu = 30 \text{ g/m}$$

$$T = 12 \text{ N}$$

$$f = 150 \text{ Hz}$$

$$A = 1.5 \text{ cm}$$

$$P = \frac{1}{2} \mu \omega^2 A^2 v$$

$$= \frac{1}{2} \mu (2\pi f)^2 A^2 \sqrt{\frac{T}{\mu}}$$

$$= 60 \text{ W}$$

(b) (i) $P = 240 \text{ W}$

(ii) $P = 15 \text{ W}$

5.11

(a)

$$I_1 = \frac{k}{r_1^2}$$

$$I_2 = \frac{k}{r_2^2}$$

$$= I_1 \left(\frac{r_1}{r_2} \right)^2$$

(b)

$$I_1 = 25 \text{ W/m}^2$$

$$r_1 = 1.0 \text{ m}$$

$$I_2 = 1.0 \text{ W/m}^2$$

$$r_2 = \sqrt{\frac{I_1}{I_2}} r_1$$

$$= 5.0 \text{ m}$$

5.12

$$P_{\text{earth}} \approx 1.5 \text{ kW}$$

$$P_{\text{jupiter}} \approx 57 \text{ W}$$

5.13

(a)

$$\begin{aligned}
 \mu_1 &= 1.0 \text{ g/cm} \\
 \mu_2 &= 4.0 \text{ g/cm} \\
 A_1 &= 3.0 \text{ cm} \\
 \lambda_1 &= 25 \text{ cm} \\
 v_1 &= \sqrt{\frac{T}{\mu_1}} \\
 f\lambda_1 &= \sqrt{\frac{T}{\mu_1}} \\
 \lambda_1\sqrt{\mu_1} &= \frac{\sqrt{T}}{f} \\
 \lambda_1\sqrt{\mu_1} &= \lambda_2\sqrt{\mu_2} \\
 \lambda_2 &= \lambda_1\sqrt{\frac{\mu_1}{\mu_2}} \\
 &= 12.5 \text{ cm} \\
 A_2 &= A_1 \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}} \\
 &= 2.0 \text{ cm}
 \end{aligned}$$

(b)

$$\begin{aligned}
 B_1 &= A_1 \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}} \\
 &= -1 \text{ cm}
 \end{aligned}$$

1/9 of the power is reflected

5.14

(a) The refractive index and wave number are directly proportional, so

$$\begin{aligned}
 B_1 &= A_1 \frac{k_1 - nk_1}{k_1 + nk_1} \\
 &= A_1 \frac{1 - n}{1 + n} \\
 &= -0.2A_1
 \end{aligned}$$

Intensity is proportional to amplitude squared, so 0.04 of the incident light intensity is reflected.

(b)

$$\begin{aligned}n_1 &= 1.39 \\ \lambda_1 &= 550 \text{ nm} \\ f &= \frac{c}{\lambda} \\ &= 5.5 \times 10^{14} \text{ Hz} \\ n_2 &= 1.5 \\ \lambda_2 &= \frac{v}{f} \\ &= \frac{c}{fn} \\ &= 392 \text{ nm} \\ L &= \frac{\lambda_2}{4} \\ &= 98 \text{ nm}\end{aligned}$$

(c)

$$2\pi \times \frac{2L}{\lambda_2} = 2\pi \Rightarrow L = \frac{\lambda_2}{2} = 198 \text{ nm}$$

5.15

(a)

$$\begin{aligned}m \frac{\partial^2 y_r}{\partial t^2} &= -T \sin \theta_1 - T \sin \theta_2 \\ \frac{\partial^2 y_r}{\partial t^2} &= -\frac{T}{m} \left[\frac{y_r - y_{r-1}}{a} + \frac{y_r - y_{r+1}}{a} \right] \\ &= \frac{T}{m} \left[\frac{y_{r+1} - y_r}{a} - \frac{y_r - y_{r-1}}{a} \right]\end{aligned}$$

(b)

$$\begin{aligned}\frac{\partial^2 y_r}{\partial t^2} &= \frac{T}{\delta x m} [y(x + \delta x) - 2y(x) + y(x - \delta x)] \\ &= \frac{T}{\delta x m} \left[y(x) + \delta x \frac{\partial y}{\partial x} + \frac{1}{2}(\delta x)^2 \frac{\partial^2 y}{\partial x^2} - 2y(x) + y(x) - \delta x \frac{\partial y}{\partial x} \right. \\ &\quad \left. + \frac{1}{2}(\delta x)^2 \frac{\partial^2 y}{\partial x^2} \right] \\ &= \frac{T}{\delta x m} \left[(\delta x)^2 \frac{\partial^2 y}{\partial x^2} \right] \\ &= \frac{T}{m/\delta x} \frac{\partial^2 y}{\partial x^2} \\ &= \frac{T}{\mu} \frac{\partial^2 y}{\partial x^2}\end{aligned}$$

6 Standing Waves

6.1

(a)

$$\begin{aligned}T &= 98.1 \text{ N} \\ L &= 0.50 \text{ m} \\ \mu &= 0.05 \text{ kg/m} \\ v &= \sqrt{\frac{T}{\mu}} \\ &= 44.3 \text{ m/s} \\ \lambda_1 &= \frac{2L}{n} \\ &= 1.0 \text{ m} \\ f &= \frac{v}{\lambda} \\ &= 44.3 \text{ Hz}\end{aligned}$$

(b)

$$y(x, t) = A \sin\left(\frac{\pi}{L}x\right) \cos(2\pi ft)$$

$$\frac{\partial y}{\partial t} = -2\pi f A \sin\left(\frac{\pi}{L}x\right) \sin(2\pi ft)$$

$$\begin{aligned}\frac{\partial y}{\partial t}_{\max} &= 2\pi f A \\ &= 8.35 \text{ m/s}\end{aligned}$$

$$\frac{\partial^2 y}{\partial t^2} = -(2\pi f)^2 A \sin\left(\frac{\pi}{L}x\right) \cos(2\pi ft)$$

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2}_{\max} &= (2\pi f)^2 A \\ &= 2324 \text{ m/s}^2\end{aligned}$$

6.2

(a)

$$f = 262 \text{ Hz}$$

$$\mu = 0.04 \text{ kg/m}$$

$$T = 200 \text{ N}$$

$$\begin{aligned}v &= \sqrt{\frac{T}{\mu}} \\ &= 70.7 \text{ m/s}\end{aligned}$$

$$\begin{aligned}\lambda &= \frac{v}{f} \\ &= 0.270 \text{ m}\end{aligned}$$

(b)

$$L = \frac{\lambda}{2} = 0.135 \text{ m}$$

(c)

$$f = 262 \text{ Hz}$$

$$\begin{aligned}\lambda &= \frac{v}{f} \\ &= 1.31 \text{ m}\end{aligned}$$

6.3

(a)

$$\begin{aligned}
 \lambda_n &= \frac{2L}{n} \\
 &= 0.55 \text{ m} \\
 \lambda_{n+1} &= \frac{2L}{n+1} \\
 &= 0.44 \text{ m} \\
 \lambda_n n &= \lambda_{n+1} (n+1) \\
 n(\lambda_n - \lambda_{n+1}) &= \lambda_{n+1} \\
 n &= \frac{\lambda_{n+1}}{\lambda_n - \lambda_{n+1}} \\
 &= 4 \\
 L &= \frac{\lambda_n n}{2} \\
 &= 1.1 \text{ m}
 \end{aligned}$$

(b)

$$\begin{aligned}
 \frac{\lambda}{2} &= 0.5 \text{ cm} \\
 \lambda &= 1.0 \text{ cm} \\
 f &= \frac{c}{\lambda} \\
 &= 3.0 \times 10^{10} \text{ Hz}
 \end{aligned}$$

6.4

$$\begin{aligned}
 y_1 + y_2 &= A \cos(\omega t - kx) + RA \cos(\omega t + kx) \\
 &= A \cos \omega t \cos kx + A \sin \omega t \sin kx + RA \cos \omega t \cos kx \\
 &\quad - RA \sin \omega t \sin kx \\
 &= A \cos \omega t \cos kx + A \sin \omega t \sin kx + RA \cos \omega t \cos kx \\
 &\quad - RA \sin \omega t \sin kx + RA \cos \omega t \cos kx - RA \cos \omega t \cos kx \\
 &= 2RA \cos \omega t \cos kx + (1 - R)A(\cos \omega t \cos kx + \sin \omega t \sin kx) \\
 &= 2RA \cos \omega t \cos kx + (1 - R)A \cos(\omega t - kx)
 \end{aligned}$$

The maximum amplitude occurs when both terms are maximised at which point the amplitude is

$$2RA + (1 - R)A = (2R + 1 - R)A = (R + 1)A$$

while the minimum amplitude occurs when the first term is 0 at which point the amplitude is

$$(1 - R)A$$

meaning the ratio of the maximum and minimum amplitudes is

$$\frac{(R + 1)A}{(1 - R)A} = \frac{1 + R}{1 - R}$$

6.5

(a)

$$\begin{aligned} f_2 &= \frac{v}{\lambda_1/2} \\ &= 2 \frac{v}{\lambda_1} \\ &= 2f_1 \\ &= 880 \text{ Hz} \\ f_3 &= 1320 \text{ Hz} \end{aligned}$$

The wave velocity doesn't change

(b)

$$\begin{aligned} f_n &= nf_1 \\ n &= \frac{f_n}{f_1} \\ &\approx 34 \end{aligned}$$

(c)

$$\begin{aligned} f_1 &= \frac{v}{2L} \\ v &= 2f_1L \\ f &= \frac{v}{2L'} \\ L' &= \frac{v}{2f} \\ &= \frac{2f_1L}{2f} \\ &= \frac{f_1}{f}L \\ &= 26.9 \text{ cm} \end{aligned}$$

The violinist should place their finger 5.1 cm from the end of the string

6.6

(a)

$$\begin{aligned}
 f_{\min} &= 20 \text{ Hz} \\
 f_{\max} &= 15 \text{ kHz} \\
 f_{\min} \cdot 2^n &= f_{\max} \\
 n &= \frac{\ln \frac{f_{\max}}{f_{\min}}}{\ln 2} \\
 &= 9
 \end{aligned}$$

(b)

$$\begin{aligned}
 f_{\min} &= 200 \text{ kHz} \\
 f_{\max} &= 2.4 \times 10^{20} \text{ Hz} \\
 n &= \frac{\ln \frac{f_{\max}}{f_{\min}}}{\ln 2} \\
 &= 50
 \end{aligned}$$

6.7

$$\begin{aligned}
 \frac{\delta f}{f} &= \frac{f' - f}{f} \\
 &= \frac{f'}{f} - 1 \\
 &= \frac{v'/\lambda}{v/\lambda} - 1 \\
 &= \frac{v'}{v} - 1 \\
 &= \frac{\sqrt{(T + \delta T)/\mu}}{\sqrt{T/\mu}} - 1 \\
 &= \sqrt{\frac{T + \delta T}{T}} - 1 \\
 &= \sqrt{1 + \frac{\delta T}{T}} - 1 \\
 &\approx 1 + \frac{1}{2} \frac{\delta T}{T} - 1 \\
 &= \frac{1}{2} \frac{\delta T}{T}
 \end{aligned}$$

6.8

(a)

$$\begin{aligned}
 f_{E2} &= 4f_{E1} \\
 \frac{v_{E2}}{\lambda} &= 4 \frac{v_{E1}}{\lambda} \\
 v_{E2} &= 4v_{E1} \\
 \sqrt{\frac{T}{\mu_{E2}}} &= 4 \sqrt{\frac{T}{\mu_{E1}}} \\
 \mu_{E2} &= \frac{1}{16} \mu_{E1} \\
 \pi \left(\frac{d_{E2}}{2} \right)^2 \rho &= \frac{1}{16} \pi \left(\frac{d_{E1}}{2} \right)^2 \rho \\
 d_{E2} &= \frac{1}{4} d_{E1} \\
 d_{E1} &= 4d_{E2} \\
 &= 1.2 \text{ mm}
 \end{aligned}$$

(b)

$$\begin{aligned}
 f_{E2} &= 330 \text{ Hz} \\
 L &= 65 \text{ cm} \\
 \rho &= 7.7 \times 10^3 \text{ kg/m}^3 \\
 f_{E2} &= \frac{v}{\lambda} \\
 &= \frac{\sqrt{\frac{T}{\mu}}}{\lambda} \\
 &= \frac{\sqrt{\frac{T}{\pi \left(\frac{d}{2} \right)^2 \rho}}}{\lambda} \\
 T &= (f_{E2} \lambda)^2 \pi \left(\frac{d}{2} \right)^2 \rho \\
 &= 100 \text{ N} \\
 T_{\text{total}} &= 6T \\
 &= 600 \text{ N}
 \end{aligned}$$

(c)

$$\begin{aligned}
 f_{E2} &= \frac{\sqrt{\frac{T}{\pi \left(\frac{d}{2}\right)^2 \frac{\rho}{6}}}}{\lambda} \\
 (f_{E2}\lambda)^2 &= \frac{T}{\pi \left(\frac{d}{2}\right)^2 \frac{\rho}{6}} \\
 \left(\frac{d}{2}\right)^2 &= \frac{T}{\pi (f_{E2}\lambda)^2 \frac{\rho}{6}} \\
 d &= 2\sqrt{\frac{T}{\pi (f_{E2}\lambda)^2 \frac{\rho}{6}}} \\
 &= 0.73 \text{ mm}
 \end{aligned}$$

6.9

(a)

$$\begin{aligned}
 \frac{M}{3} \frac{d^2 x_1}{dt^2} &= -\frac{4T}{L}x_1 + \frac{4T}{L}(x_2 - x_1) \\
 &= \frac{4T}{L}(x_2 - 2x_1) \\
 \frac{M}{3} \frac{d^2 x_2}{dt^2} &= -\frac{4T}{L}(x_2 - x_1) + \frac{4T}{L}(x_3 - x_2) \\
 &= \frac{4T}{L}(x_3 - 2x_2 + x_1) \\
 \frac{M}{3} \frac{d^2 x_3}{dt^2} &= -\frac{4T}{L}(x_3 - x_2) - \frac{4T}{L}x_3 \\
 &= -\frac{4T}{L}(2x_3 - x_2)
 \end{aligned}$$

Let $x_1 = Ae^{i\omega t}$, $x_2 = Be^{i\omega t}$, and $x_3 = Ce^{i\omega t}$, then

$$\begin{aligned}
-\frac{M}{3}\omega^2 Ae^{i\omega t} &= \frac{4T}{L}(Be^{i\omega t} - 2Ae^{i\omega t}) \\
-\frac{M}{3}\omega^2 A &= \frac{4T}{L}(B - 2A) \\
(\omega^2 - 2\alpha)A + \alpha B &= 0
\end{aligned}$$

$$\begin{aligned}
-\frac{M}{3}\omega^2 Be^{i\omega t} &= \frac{4T}{L}(Ce^{i\omega t} - 2Be^{i\omega t} + Ae^{i\omega t}) \\
-\frac{M}{3}\omega^2 B &= \frac{4T}{L}(C - 2B + A) \\
\alpha A + (\omega^2 - 2\alpha)B + \alpha C &= 0
\end{aligned}$$

$$\begin{aligned}
-\frac{M}{3}\omega^2 Ce^{i\omega t} &= -\frac{4T}{L}(2Ce^{i\omega t} - Be^{i\omega t}) \\
-\frac{M}{3}\omega^2 C &= -\frac{4T}{L}(2C - B) \\
\alpha B + (\omega^2 - 2\alpha)C &= 0
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} \omega^2 - 2\alpha & \alpha & 0 \\ \alpha & \omega^2 - 2\alpha & \alpha \\ 0 & \alpha & \omega^2 - 2\alpha \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} &= 0 \\
(\omega^2 - 2\alpha)[(\omega^2 - 2\alpha)^2 - \alpha^2] - \alpha[\alpha(\omega^2 - 2\alpha)] &= 0 \\
(\omega^2 - 2\alpha)(\omega^4 - 4\alpha\omega^2 + 2\alpha^2) &= 0
\end{aligned}$$

$$\begin{aligned}
\omega_{1,3}^2 &= \frac{4\alpha \pm \sqrt{16\alpha^2 - 8\alpha^2}}{2} \\
&= \frac{4\alpha \pm 2\alpha\sqrt{2}}{2} \\
&= (2 \pm \sqrt{2})\alpha \\
\omega_2^2 &= 2\alpha
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\omega_1}{2\pi} &= \frac{\sqrt{(2 - \sqrt{2})\alpha}}{2\pi} \\
&= \frac{\sqrt{2 - \sqrt{2}}}{2\pi} \sqrt{\frac{12T}{LM}} \\
&= \frac{\sqrt{3(2 - \sqrt{2})}}{\pi} \sqrt{\frac{T}{LM}} \\
&\approx 0.442 \sqrt{\frac{T}{LM}}
\end{aligned}$$

$$\begin{aligned}
\frac{\omega_2}{2\pi} &= \frac{\sqrt{2\alpha}}{2\pi} \\
&= \frac{\sqrt{2}}{2\pi} \sqrt{\frac{12T}{LM}} \\
&= \frac{\sqrt{6}}{\pi} \sqrt{\frac{T}{LM}} \\
&\approx 0.780 \sqrt{\frac{T}{LM}}
\end{aligned}$$

$$\begin{aligned}
\frac{\omega_3}{2\pi} &= \frac{\sqrt{(2 + \sqrt{2})\alpha}}{2\pi} \\
&= \frac{\sqrt{2 + \sqrt{2}}}{2\pi} \sqrt{\frac{12T}{LM}} \\
&= \frac{\sqrt{3(2 + \sqrt{2})}}{\pi} \sqrt{\frac{T}{LM}} \\
&\approx 1.02
\end{aligned}$$

$$\begin{aligned}
f_n &= \frac{\omega_n}{2\pi} \\
&= \frac{1}{2\pi} \frac{n\pi v}{L} \\
&= \frac{nv}{2L} \\
&= \frac{n}{2L} \sqrt{\frac{T}{\mu}} \\
&= \frac{n}{2L} \sqrt{\frac{T}{M/L}} \\
&= \frac{n}{2L} \sqrt{\frac{LT}{M}} \\
&= \frac{n}{2} \sqrt{\frac{T}{LM}} \\
f_1 &= \frac{1}{2} \sqrt{\frac{T}{LM}} \\
f_2 &= \sqrt{\frac{T}{LM}} \\
f_3 &= \frac{3}{2} \sqrt{\frac{T}{LM}}
\end{aligned}$$

6.10

(a)

$$\begin{aligned}
 f_n &= \frac{nc}{2L} \\
 n &= \frac{2f_n L}{c} \\
 \frac{2L}{c} \left[f_0 + \frac{\Delta f}{2} - \left(f_0 - \frac{\Delta f}{2} \right) \right] &= \frac{2L}{c} \Delta f \\
 &= 30 \text{ normal frequencies}
 \end{aligned}$$

(b)

$$\begin{aligned}
 1 &= \frac{2L}{c} \Delta f \\
 L &= \frac{c}{2\Delta f} \\
 &= 3.3 \text{ cm}
 \end{aligned}$$

6.11

The 3rd, 6th, and 9th normal modes

6.12

$$\begin{aligned}
 A_n &= \frac{2}{L} \int_0^L \sin\left(\frac{n\pi}{L}x\right) \alpha x \, dx \\
 &= \frac{2\alpha}{L} \left(\left[-\frac{L}{n\pi} \cos\left(\frac{n\pi}{L}x\right) x \right]_0^L - \int_0^L -\frac{L}{n\pi} \cos\left(\frac{n\pi}{L}x\right) \, dx \right) \\
 &= \frac{2\alpha}{L} \left((-1)^{n+1} \frac{L^2}{n\pi} + \frac{L}{n\pi} \left[\frac{L}{n\pi} \sin\left(\frac{n\pi}{L}x\right) \right]_0^L \right) \\
 &= (-1)^{n+1} \frac{2\alpha L}{n\pi} \\
 f(x) &= \frac{2\alpha L}{\pi} \left[\sin\left(\frac{\pi x}{L}\right) - \frac{1}{2} \sin\left(\frac{2\pi x}{L}\right) + \frac{1}{3} \sin\left(\frac{3\pi x}{L}\right) - \dots \right]
 \end{aligned}$$

6.13

(a)

$$\begin{aligned}
 A_n &= \frac{8d}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) \\
 \omega_n &= \frac{n\pi v}{L} \\
 &= \frac{n\pi}{L} \sqrt{\frac{T}{\mu}} \\
 E &= \frac{1}{4} \mu L \sum_n A_n^2 \omega_n^2 \\
 &= \frac{1}{4} \mu L \sum_n \frac{64d^2}{n^4 \pi^4} \sin^2\left(\frac{n\pi}{2}\right) \frac{n^2 \pi^2}{L^2} \frac{T}{\mu} \\
 &= \frac{16d^2 T}{\pi^2 L} \sum_n \frac{1}{n^2} \sin^2\left(\frac{n\pi}{2}\right) \\
 &= \frac{16d^2 T}{\pi^2 L} \sum_n \frac{1}{(2n-1)^2} \\
 &= \frac{16d^2 T}{\pi^2 L} \frac{\pi^2}{8} \\
 &= \frac{2d^2 T}{L}
 \end{aligned}$$

(b)

$$\begin{aligned}
 E_n &= \frac{1}{4} \mu L \frac{64d^2}{n^4 \pi^4} \sin^2\left(\frac{n\pi}{2}\right) \frac{n^2 \pi^2}{L^2} \frac{T}{\mu} \\
 &= \frac{16d^2 T}{n^2 \pi^2 L} \\
 E_{1,3,5} &= \frac{16d^2 T}{\pi^2 L} \left(1 + \frac{1}{9} + \frac{1}{25}\right) \\
 &\approx 1.87 \frac{d^2 T}{L} \\
 &\approx 93.5\% \text{ of } E
 \end{aligned}$$

6.14

It's a square wave

6.15

(a)

$$\begin{aligned} E_n &= \frac{1}{2}mv^2 \\ &= \frac{p^2}{2m} \\ &= \frac{(h/\lambda)^2}{2m} \\ &= \frac{h^2}{2\lambda^2m} \\ &= \frac{h^2}{2(2L/n)^2m} \\ &= \frac{h^2n^2}{8L^2m} \end{aligned}$$

(b)

$$E_1 = \frac{h^2}{8L^2m} = 1.52 \times 10^{-18} \text{ J} \approx 10 \text{ eV}$$