Advanced Engineering Mathematics Ordinary Differential Equations Notes

Chris Doble

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Contents

1 Introduction to Differential Equations

1.1 Definitions and Terminology

- An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation** (DE)
- An **ordinary DE** (ODE) is a DE that contains only ordinary (i.e. non-partial) derivatives of one or more functions with respect to a single independent variable
- A **partial DE** is a DE that contains only partial derivatives of one or more functions of two or more independent variables
- The **order** of a DE is the order of the highest derivative in the equation
- First order ODEs are sometimes written in the **differential form**

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

- n-th order ODEs in one dependent variable can be expressed by the **general form**

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

- It's possible to solve ODEs in the general form uniquely for the highest derivative $y^{(n)}$ in terms of the other $n+1$ variables, allowing them to be expressed in the **normal form**

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

- An n-th order ODE is said be **linear** in the variable y if it can be expressed in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - g(x) = 0$$

i.e. the dependent variable y and all of its derivatives aren't raised to a power or used in nonlinear functions like e^y or $\sin y$, and the coefficients a_0, a_1, ..., a_n depend at most on the independent variable x

- A **nonlinear** ODE is one that is not linear
- A **solution** to an ODE is a function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, such that

$$F(x, \phi(x), \phi'(x), \ldots, \phi^n(x)) = 0 \text{ for all } x \text{ in } I.$$

- The **interval of definition**, **interval of validity**, or the **domain** of a solution is the interval over which the solution is valid
- A solution of a DE that is 0 on an interval I is said to be a **trivial solution**
- Because solutions to DEs must be differentiable over their interval of validity, discontinuities, etc. must be excluded from the interval
- An **explicit solution** to an ODE is one where the dependent variable is expressed solely in terms of the independent variable and constants
- An **implicit solution** to an ODE is a relation $G(x, y) = 0$ over an interval I provided there exists at least one function ϕ that satisfies the relation as well as the ODE on I
- When solving a first-order ODE we usually obtain a solution containing a single arbitrary constant or parameter c. A solution containing an arbitrary constant represents a set of solution called a **one-parameter family of solutions**
- When solving an n-th order DE we usually obtain an n**-parameter family of solutions**
- A solution of a DE that is free from arbitrary parameters is called a **particular solution**
- A **singular solution** is a solution to a DE that isn't a member of a family of solutions
- A **system of ODEs** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable. A solution of such a system is a differentiable function for each equation defined on a common interval I that satisfy each equation of the system on that interval

1.2 Initial Value Problems

- An **initial value problem** is the problem of solving a DE with some given **initial conditions**, e.g. solve

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)})$$

  subject to

$$y(x_0) = y_0,\ y'(x_0) = y_1,\ \dots,\ y^{(n-1)}(x_0) = y_{n-1}$$

- The domain of $y = f(x)$ differs depending on how it's considered:
  - As a function its domain is all real numbers for which it's defined
  - As a solution of a DE its domain is a single interval over which it's defined an differentiable
  - As a solution of an initial value problem its domain is a single interval over which it's defined, differentiable, and contains the initial conditions
- An initial value problem may not have any solutions. If it does it may have multiple.
- First-order initial value problems of the form

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

  are guaranteed to have a unique solution over an interval I containing x_0 if $f(x, y)$ and $\partial f/\partial y$ are continuous

1.3 Differential Equations as Mathematical Models

- A **mathematical model** is a mathematical description of a system or phenomenon
- The **level of resolution** of a model determines how many variables are included in the model
- A simple model of the growth of a population P is

$$\frac{dP}{dt} = kP$$

  where $k > 0$

- A simple model of radioactive decay of an amount of substance A is

$$\frac{dA}{dt} = kA$$

where $k < 0$

- Newton's empirical law of cooling/warming states that the rate of change of the temperature of a body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium

$$\frac{dT}{dt} = k(T - T_m)$$

2 First-Order Differential Equations

2.1 Solution Curves Without a Solution

- An ODE in which the independent variable doesn't appear is said to be **autonomous**, e.g.

$$\frac{dy}{dx} = f(y)$$

- A real number c is a **critical/equilibrium/stationary point** of an autonomous DE if it is a zero of f
- If c is a critial point of an autonomous DE, then $y(x) = c$ is a solution
- A solution of the form $y(x) = c$ is called an **equilibrium solution**
- We can draw several conclusions about the solutions of an autonomous DE with n critical points and $n + 1$ subregions bounded by the critical points:
  - If (x_0, y_0) is in a subregion, it remains in that subregion for all x
  - By continuity, $f(y) < 0$ or $f(y) > 0$ for all y in a subregion and thus $y(x)$ can't have maximum/minimum points or oscillate
  - If $y(x)$ is bounded above by a critical point c_1, it must approach $y(x) = c_1$ as $x \to -\infty$ or $x \to \infty$
  - If $y(x)$ is bounded above and below by critical points c_1 and c_2, it must approach $y(x) = c_1$ as $x \to -\infty$ and $y(x) = c_2$ as $x \to \infty$ or vice versa
  - If $y(x)$ is bounded below by a critical point c_1, it must approach $y(x) = c_1$ as $x \to -\infty$ or $x \to \infty$

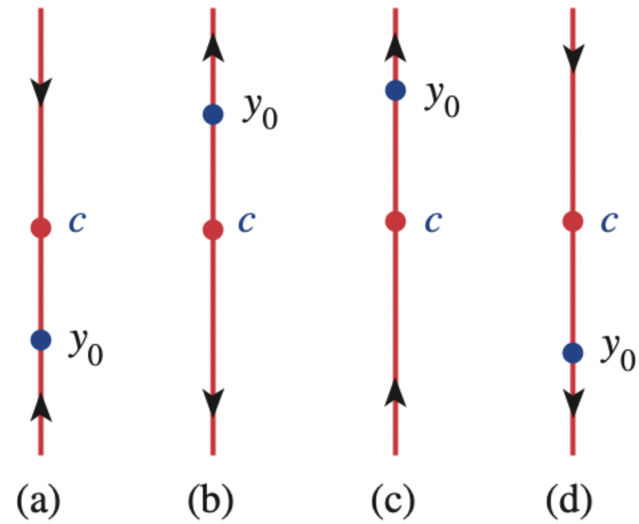

FIGURE 2.1.8 Critical point c is an attractor in (a), a repeller in (b), and semi-stable in (c) and (d)

- If $y(x)$ is a solution of an autonomous differential equation $dy/dx = f(y)$, then $y_1(x) = y(x-k)$, where k is a constant, is also a solution

2.2 Separable Equations

- A first-order ODE of the form
$$\frac{dy}{dx} = g(x)h(y)$$
is said to be **separable** or to have **separate variables**
- A separable first-order ODE can be solved by dividing both sides by $h(y)$ then integrating both sides with respect to x

$$
\begin{aligned}
\frac{dy}{dx} &= g(x)h(y) \\
\frac{1}{h(y)}\frac{dy}{dx} &= g(x) \\
\int \frac{1}{h(y)}\frac{dy}{dx}\,dx &= \int g(x)\,dx \\
\int \frac{1}{h(y)}\,dy &= \int g(x)\,dx \\
H(y) &= G(x) + c
\end{aligned}
$$

- Care should be taken when dividing by $h(y)$ as it removes constant solutions $y = r$ where $h(r) = 0$

2.3 Linear Equations

- A first-order DE of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

  or in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

  is said to be a **linear equation** in the dependent variable y

- When $g(x) = 0$ or $f(x) = 0$ the linear equation is said to be **homogeneous** and is solvable via separation of variables, otherwise it is **nonhomogeneous**

- The nonhomogeneous linear equation's solution is the sum of two solutions $y = y_c + y_p$ where y_c is a solution of the associated homogeneous equation

$$\frac{dy}{dx} + P(x)y = 0$$

  and y_p is a particular solution of the nonhomogeneous equation

- Nonhomogeneous linear equations can be solved via **variation of parameters**:

  1. Put it into standard form
  2. Determine the **integrating factor** $e^{\int P(x)\,dx}$
  3. Multiply by the integrating factor
  4. Recognise that the left hand side of the equation is the derivative of the product of the integrating factor and y

5. Integrate both sides of the equation
6. Solve for y

- The **general solution** of a DE is a family of solutions that contains all possible solutions (except singular solutions)
- A term $y = f(x)$ in a solution is called a **transient term** if $f(x) \to 0$ as $x \to \infty$
- When either $P(x)$ or $f(x)$ is a piecewise-defined function the equation is then referred to as a **piecewise-linear differential equation** that can be solved by solving each interval in isolation then choosing appropriate constants to ensure the overall solution is continuous
- The **error function** and **complementary error function** are defined

$$\operatorname{erf} x + \operatorname{erfc} x = 1$$
$$\left(\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt\right) + \left(\frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt\right) = 1$$

2.4 Exact Equations

- The **differential** of a function $z = f(x, y)$ is

$$dz = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy$$

- A differential expression $M(x, y)\, dx + N(x, y)\, dy$ is an **exact differential** in the region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$
- A first-order DE of the form

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

  is said to be an **exact equation** if the expression on the left side is an exact differential
- A necessary and sufficient condition that $M(x, y)\, dx + N(x, y)\, dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- Exact differentials can be solved by

1. Integrating $M(x, y)$ with respect to x to find an expression for $f(x, y)$

$$\frac{\partial f}{\partial x} = M(x, y)$$
$$f(x, y) = \int M(x, y)\, dx + g(y)$$

2. Differentiating $f(x, y)$ with respect to y and equating it to $N(x, y)$ to find $g'(y)$

$$\frac{\partial f}{\partial y} = N(x, y) = \frac{\partial}{\partial y} \int M(x, y)\, dx + g'(y)$$
$$g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, dx$$

3. Integrating $g'(y)$ with respect to y to find $g(y)$ and substituting it into $f(x, y)$
4. Equating $f(x, y)$ with an unknown constant c

- x and y can be swapped in the steps above (i.e. you can start by integrating $N(x, y)$ with respect to y, etc.)
- A nonexact DE $M(x, y)\, dx + N(x, y)\, dy = 0$ can sometimes be transformed into an exact DE by finding an appropriate integrating factor
  - If $(M_y - N_x)/N$ is a function of x alone, then an integrating factor is

    $$\mu(x) = e^{\int \frac{M_y - N_x}{N}\, dx}$$

  - If $(N_x - M_y)/M$ is a function of y alone, then an integrating factor is

    $$\mu(y) = e^{\int \frac{N_x - M_y}{M}\, dy}$$

2.5 Solutions by Substitution

- A function $f(x, y)$ is said to be a **homogeneous function** of degree α if

  $$f(tx, ty) = t^{\alpha} f(x, y)$$

- A first-order DE of the form

  $$M(x, y)\, dx + N(x, y)\, dy = 0$$

  is said to be **homogeneous** if both M and N are homogeneous functions of the same degree

- To solve a homogeneous first-order DE:

  1. Rewrite it as

  $$M(x,y) = x^\alpha M(1,u) \text{ and } N(x,y) = x^\alpha N(1,u) \text{ where } u = y/x$$

  or

  $$M(x,y) = y^\alpha M(v,1) \text{ and } N(x,y) = y^\alpha N(v,1) \text{ where } v = x/y$$

  2. Substitute $y = ux$ and $dy = u\,dx + x\,du$ or $x = vy$ and $dx = v\,dy + y\,dv$ as appropriate
  3. Solve the resulting first-order separable DE
  4. Substitude $u = y/x$ or $v = x/y$ as appropriate

- The DE
  $$\frac{dy}{dx} + P(x)y = f(x)y^n$$
  where n is any real number is called **Bernoulli's equation**

- For $n = 0$ and $n = 1$ Bernoulli's equation is linear

- To solve Bernoulli's equation for $n \neq 0$ and $n \neq 1$:

  1. Substitude $y = u^{1/(1-n)}$ and $\frac{dy}{dx} = \frac{d}{dx}(u^{1/(1-n)})$
  2. Solve the resulting linear equation
  3. Substitude $u = y^{n-1}$

- A DE of the form
  $$\frac{dy}{dx} = f(Ax + By + C)$$
  can always be reduced to an equation with separable variables by means of the substitution
  $$u = Ax + By + C, B \neq 0$$

2.6 A Numerical Method

- Approximate values for points on a solution curve near an initial point can be calculated via a **linearization** of the solution curve — a straight line that has the same slope as the initial point and passes through it

- **Euler's method** approximates a solution curve by iteratively stepping along its linearizations

  $$y_{n+1} = y_n + hf(x_n, y_n)$$

  where h is the **step size**

2.9 Modeling with Systems of First-Order DEs

- In a system of DEs
$$\frac{dx}{dt} = g_1(t, x, y)$$
and
$$\frac{dy}{dt} = g_2(t, x, y)$$
if g_1 and g_2 are linear in x and y, i.e.
$$g_1(t, x, y) = c_1 x + c_2 y + f_1(t)$$
and
$$g_2(t, x, y) = c_3 x + c_4 y + f_2(t)$$
it is said to be a **linear system**

3 Higher-Order Differential Equations

3.1 Theory of Linear Equations

- An n**th-order initial-value problem (IVP)** is to solve
$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx_{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$
subject to
$$y(x_0) = y_0,\, y'(x_0) = y_1,\, \ldots,\, y^{(n-1)}(x_0) = y_{n-1}$$

- If $a_n(x)$, $a_{n-1}(x)$, …, $a_1(x)$, $a_0(x)$, and $g(x)$ are continuous on an interval I and $a_n(x) \neq 0$ for every x in the interval, then then a unique solution exists for the above IVP for every $x = x_0$ within the interval

- An **initial value problem** is when all of the constraints are located at the same point while a **boundary value problem** is when they're at different points

- Boundary value problems may have many, one, or no solutions

- When $g(x) = 0$ the DE is said to be **homogeneous**, otherwise it's **non-homogeneous**

- The symbol D is called a **differential operator** because it transforms a differentiable function into another function
$$Dy = \frac{dy}{dx}$$

- Higher-order derivatives can be expressed as

$$D^n = \frac{d^n y}{dx^n}$$

- An n**th-order differential operator** is defined to be

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)$$

- As a consequence of the properties of differentiation

$$D(cf(x)) = cDf(x)$$

  and

$$D\{f(x) + g(x)\} = Df(x) + Dg(x)$$

- The superposition principle for homogeneous linear nth-order differential equation states that if y_1, y_2, …, y_k are solutions of the equation on an interval I then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

  where c_i are arbitrary constants is also a solution on the interval

- A set of functions $f_1(x)$, $f_2(x)$, …, $f_n(x)$ is said to be **linearly dependent** on an interval I if there exists constants c_1, c_2, …, c_n, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

  for every x in the interval. Otherwise it is said to tbe **linearly independent**

- The **Wronskian** of a set of n functions that are $n-1$ times differentiable is defined as

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

- If y_1, y_2, …, y_n are n solutions to a homogeneous linear nth-order differential equation on an interval I then the set of solutions is **linearly independent** on I iff $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in the interval

- Any set of n linearly independent solutions of a homogeneous linear nth-order differential equation on an interval I is said to be a **fundamental set of solutions** on the interval

- If y_1, y_2, ..., y_n are a fundamental set of solutions of a homogeneous linear nth-order DE on an interval I then the **general solution** of the equation on the interval is

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

  where c_i are arbitrary constants

- Another way of saying the above is that any solution on the interval can be expressed as a linear combination of the fundamental set of solutions

- A linear combination of a fundamental set of solutions of a homogenous linear nth-order DE

$$y_c(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

  is called the **complementary function** of associated nonhomogenous DEs

- If y_p is any particular solution to a nonhomogeneous linear nth-order DE on an interval I and y_1, y_2, ..., y_n are a fundamental set of solutions of the associated homogeneous DE on I, then the **general solution** of the equation on the interval is

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x)$$

  where c_i are arbitrary constants

- Another way of saying the above is that any solution on the interval can be expressed as $y = y_c + y_p$

- The superposition for nonhomogeneous linear nth-order differential equations states that if y_{p_1}, y_{p_2}, ..., y_{p_k} are k particular solutions of a nonhomogeneous lienar nth-order differential equation on an interval I corresponding, in turn, to k distinct functions g_1, g_2, ..., g_k, then

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x)$$

  is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x)$$

3.2 Reduction of Order

- The **reduction of order** method requires knowledge of one non-trivial solution and comprises the following steps:

  1. Recognise that the ratio of two linearly independent functions isn't constant, i.e.

$$u(x) = \frac{y_1(x)}{y_2(x)} \text{ or } y_2(x) = u(x)y_1(x)$$

2. Substitute $y_2(x) = u(x)y_1(x)$ into the DE — this will result in a DE involving only u'' and u' which can be treated as a linear first-order DE in $u' = w$
3. Solve for w
4. Substitute $w = u'$
5. Integrate to find u
6. Multiply by y_1 to find y_2

- A formula for the above on a DE in standard form

$$y'' + P(x)y' + Q(x)y = 0$$

is

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx$$

3.3 Homogeneous Linear Equations with Constant Coefficients

- All solutions to homogenous linear DEs

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where a_i are real constants and $a_n \neq 0$ are either exponential functions or constructed from exponential functions

- Substituting a solution $y = e^{mx}$ we find

$$e^{mx}(a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0) = 0$$

where the term in brackets is called the **auxiliary equation** of the DE

- Thus, the solution $y = e^{mx}$ is valid if m is a root of the auxiliary equation
- Real roots correspond to solutions of the form

$$y = ce^{mx}$$

- Complex roots $\alpha \pm i\beta$ correspond to solutions of the form

$$y_1 = c_1 e^{\alpha x} \cos \beta x \text{ and } y_2 = c_2 e^{\alpha x} \sin \beta x$$

- A root m of multiplicity k corresponds to the solutions

$$e^{mx},\ xe^{mx},\ x^2 e^{mx},\ \ldots,\ x^{k-1} e^{mx}$$

3.4 Undetermined Coefficients

- The **method of undetermined coefficients** may be used to find a particular solution to nonhomogenous linear differential equations where the input function is comprised of constants, polynomials, exponentials $e^{\alpha x}$, sines, and cosines
- To apply the method you:
  1. Solve the associated homogeneous equation
  2. Assume the particular solution has the same form as the input function
  3. If a term in the proposed solution is present in the complementary function, multiply it by x^n where n is the smallest positive integer that removes the duplication
  4. Substitute the proposed solution into the DE
  5. Solve for the unknown constants

TABLE 3.4.1 Trial Particular Solutions

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A\cos 4x + B\sin 4x$
6. $\cos 4x$	$A\cos 4x + B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

3.5 Variation of Parameters

- The **method of variation of parameters** can be used to find a particular solution of a nonhomogeneous linear nth-order DE
- To apply the method you:
  1. Solve the homogeneous equation to find the complementary function
  2. Assume the solution has the form

     $$y_p = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x)$$

     where n is the order of the equation and y_i are the fundamental set of solutions from the complementary equation
  3. Convert to standard form by dividing by the leading coefficient

     $$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

4. Solve the system of linear equations

$$\begin{aligned} y_1 u_1' + \cdots + y_n u_n' &= 0 \\ y_1' u_1' + \cdots + y_n' u_n' &= 0 \\ &\vdots \\ y_1^{(n-1)} u_1' + \cdots + y_n^{(n-1)} y_n' &= 0 \\ y_1^{(n)} u_1' + \cdots + y_n^{(n)} u_n' &= f(x) \end{aligned}$$

via Cramer's method:

(a) Compute the Wronskian of y_i

$$W = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \ddots & \vdots \\ y_1^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

(b) Compute u_i' for $i = 1, \ldots, n$ where

$$u_i' = \frac{W_i}{W}$$

and W_i is the determinant of the matrix formed by replacing the ith column of the Wronskian matrix with the column vector

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(x) \end{bmatrix}$$

5. Integrate each u_i' to find u_i

3.6 Cauchy-Euler Equations

- A **Cauchy-Euler equation** is a linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

- To solve a homogeneous Cauchy-Euler equation you:

1. Assume the equation has a solution of the form $y = x^m$, giving

$$\begin{aligned} a_n x^n \frac{d^n y}{dx^n} &= a_n x^n m(m-1)(m-2) \cdots (m-n+1) x^{m-n} \\ &= a_n m(m-1)(m-2) \cdots (m-n+1) x^m \end{aligned}$$

and the equation then becomes

$$f(m)x^m = 0$$

where $f(m)$ is a polynomial in m known as the auxiliary or characteristic equation, the roots of which form the general solution

2. Solve the auxiliary equation where
   - A real root m corresponds to a solution

     $$y = cx^m$$

   - Complex roots $\alpha \pm i\beta$ correspond to solutions

     $$x^\alpha(c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x))$$

   - A root m of multiplicity k corresponds to solutions

     $$x^m,\ x^m\ln x,\ x^m(\ln x)^2\ \ \ldots,\ x^m(\ln x)^{k-1}$$

- To solve a nonhomogeneous Cauchy-Euler euqation you:

  1. Solve the associated homogeneous equation
  2. Find a particular solution via variation of parameters

3.7 Nonlinear Equations

- The superposition principle does not hold for nonlinear equations
- Nonlinear second order DEs of the form $F(x, y', y'') = 0$ where y is missing can sometimes be solved by:

  1. Substitute $u = y'$ (and thus $u' = y''$)
  2. Solve the resulting DE for u
  3. Integrate to find y

- Nonlinear second order DEs of the form $F(y, y', y'') = 0$ where x is missing can sometimes by solved by:

  1. Substitute $u = y'$ and

     $$y'' = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy}$$

  2. Solve the resulting DE for u
  3. Integrate to find y

- Nonlinear initial-value problems can sometimes be solved by substituting the initial conditions into a Taylor series centred at x_0. The initial conditions can also be substituted into subsequent derivatives to add further terms to the series

3.10 Green's Functions

- Green's functions are useful because they allow you to express the solution of a DE in terms of the input function $g(x)$, making it easy to see how different input functions change the solution

3.10.1 Initial-Value Problems

- The solution of a second-order IVP

$$y'' + P(x)y' + Q(x)y = f(x),\ y(x_0) = y_0,\ y'(x_0) = y_1$$

can be expressed as

$$y = y_h + y_p$$

where y_h is the solution to the associated homogeneous equation with nonhomogeneous initial conditions

$$y'' + P(x)y' + Q(x)y' = 0,\ y(x_0) = y_0,\ y'(x_0) = y_1$$

and y_p is the solution to the nonhomogeneous equation with homogeneous initial conditions

$$y'' + P(x)y' + y = f(x),\ y(x_0) = 0,\ y'(x_0) = 0$$

- If $P(x)$ and $Q(x)$ are constant y_h can be found via the auxiliary / characteristic equation

- If y_1 and y_2 form a fundamental set of solutions to the associated homogeneous equation, then y_p is given by

$$y_p(x) = \int_{x_0}^{x} G(x,t)f(t)\,dt$$

where $G(x,y)$ is the Green's function for the differential equation

$$G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}$$

and $W(t)$ is the Wronskian

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

3.10.2 Boundary Value Problems

- If y_1 and y_2 are linearly independent solutions of

$$y'' + P(x)y' + Q(x) = 0$$

on $[a, b]$ and satisfy the boundary conditions

$$A_1 y_1(a) + B_1 y_1(a) = 0$$

and

$$A_2 y_2(b) + B_2 y_2(b) = 0$$

then the BVP

$$y'' + P(x)y' + Q(x)y = f(x)$$

subject to the same boundary conditions has a particular solution

$$y_p(x) = \int_a^b G(x,t) f(t)\, dt$$

where $G(x, t)$ is the Green's function for the differential equation

$$G(x,y) = \begin{cases} \frac{y_1(t)y_2(x)}{W(t)} & a \le t \le x \\ \frac{y_1(x)y_2(t)}{W(t)} & x \le t \le b \end{cases}$$

and $W(t)$ is the Wronskian

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

3.12 Solving Systems of Linear Equations

- Systems of linear differential equations can be solved in a similar manner to systems of equations, namely by adding and subtracting multiples of different equations to eliminate particular variables
- We can also apply the differential operator D as part of the elimination process
- Once you have an equation for each dependent variable it's important to substitute them back into the original differential equation to determine the constraints on the parameters — not all of them can be chosen arbitrarily

4 The Laplace Transform

4.1 Definition of the Laplace Transform

- If a function $f(t)$ is defined for $t \ge 0$ and the limit

$$\int_0^\infty K(s,t) f(t)\, dt = \lim_{b \to \infty} \int_0^b K(s,t) f(t)\, dt$$

exist, the integral is said to **exist** or be **convergent**, otherwise it does not exist or is **divergent**

- If a function $f(t)$ is defined for $t \geq 0$ then the limit

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

is called the **Laplace transform** of f providing the integral converges

- $\mathcal{L}$ is a linear transform, i.e.

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$$

- A function is said to be **piecewise continuous** on $[0, \infty)$ if, in any interval defined by $0 \leq a \leq t \leq b$, there are at most a finite number of points t_k, $k = 1, 2, \ldots, n$ $(t_{k-1} < t_k)$, at which f has finite discontinuities and is continuous on each open interval defined by $t_{k-1} < t < t_k$

- A function is said to be of **exponential order** if there exists constants c, $M > 0$, and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$

- If $f(t)$ is piecewise continuous on the interval $[0, \infty)$ and of exponential order, then $\mathcal{L}\{f(t)\}$ exists for $s > c$

4.2 The Inverse Transform and Transforms of Derivatives

- $\mathcal{L}^{-1}$ is a linear transform, i.e.

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathcal{L}^{-1}\{F(s)\} + \beta \mathcal{L}^{-1}\{G(s)\}$$

- If f, f', ..., $f^{(n-1)}$ are continuous on $[0, \infty)$, are of exponential order, and $f^{(n)}$ is piecewise continuous on $[0, \infty)$, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

where $F(s) = \mathcal{L}\{f(t)\}$

- The Laplace transform can be used to solve linear IVPs:

  1. Take the Laplace transform of the DE, resulting in an algebraic equation in $F(s) = \mathcal{L}\{f(s)\}$ where $f(s)$ is the goal
  2. Solve the equation for $F(s)$
  3. Apply the inverse Laplace transform to find $f(s)$

4.3 Translation Theorems

- The **first translation theorem** states that if

$$\mathcal{L}\{f(t)\} = F(s)$$

then

$$\mathcal{L}\{e^{at} f(t)\} = F(s - a)$$

and

$$\mathcal{L}^{-1}\{F(s - a)\} = \mathcal{L}^{-1}\{F(s)|_{s \to s-a}\} = e^{at} f(t)$$

- The **unit step function** or **Heaviside function** is definde to be

$$\mathcal{U}(t-a) = \begin{cases} 0 & 0 \le t < a \\ 1 & t \ge a \end{cases}$$

- The **second translation theorem** states that if $a > 0$ and

$$\mathcal{L}\{f(s)\} = F(s)$$

then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

and

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$$

- If f and $\mathcal{U}$ aren't shifted by the same amount when applying the second translation theorem, an alternate form can be applied

$$\mathcal{L}\{f(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$$

4.4 Additional Operational Properties

- If $F(s) = \mathcal{L}\{f(t)\}$ and $n = 1, 2, 3, \ldots$, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

- If functions f and g are piecewise continuous on the interval $[0, \infty)$ then the **convolution** of f and g, denoted $f * g$, is a function defined by the integral

$$f * g = \int_0^t f(\tau)g(t-\tau)\, d\tau$$

- The **convolution theorem** states that if f and g are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$

and

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g$$

- Under the convolution theorem if $g(t) = 1$ then $\mathcal{L}\{g(t)\} = G(s) = 1/s$,

$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{F(s)}{s},$$

and

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\, d\tau$$

- **Volterra integral equations** have the form

$$f(t) = g(t) + \int_0^t f(\tau)g(t-\tau)\,d\tau$$

and can be solved by using the convolution theorem while taking the Laplace transform

- An **integro-differential equation** is an equation that involves both integrals and derivatives of a function

- If $f(t)$ is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period T, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

4.5 The Dirac Delta Function

- A **unit impulse** function is defined as

$$\delta_a(t-t_0) = \begin{cases} 0 & 0 \le t < t_0 - a \\ \frac{1}{2a} & t_0 - a \le t < t_0 + a \\ 0 & t_0 + a \le t \end{cases}$$

and it possesses the property

$$\int_0^\infty \delta_a(t-t_0)\,dt = 1$$

- The **Dirac delta function** is defined as

$$\delta(t-t_0) = \lim_{a\to 0} \delta_a(t-t_0)$$

and has the properties

$$\delta(t-t_0) = \begin{cases} \infty & t = t_0 \\ 0 & t \ne t_0 \end{cases}$$

and

$$\int_0^\infty \delta(t-t_0)\,dt = 1$$

- For $t_0 > 0$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$

5 Series Solutions of Linear Differential Equations

5.1 Solutions about Ordinary Points

- A **power series** is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

- A point x_0 is said to be an **ordinary point** of the differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

  if both $P(x)$ and $Q(x)$ are analytic at x_0. A point that is not an ordinary point is said to be a **singular point** of the equation.

- If $x = x_0$ is an ordinary point of the differential equation above, we can always find two linearly independent solutions in the form of a power series centred at x_0. Such a solution is said to be a **solution about the ordinary point** x_0

- A series solution converges at least on some interval $|x - x_0| < R$ where R is the distance from x_0 to the closest singular point

- A series solution can be found for a homogeneous linear second-order differential equation by

  1. Assume the solution is of the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

     and thus

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

     and

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$$

  2. Substitute the assumed solution into the DE
  3. Group the summations
  4. Find a recurrence relation for the coefficients which will result in all coefficients being expressed in terms of c_0 or c_1
  5. Group terms by c_0 and c_1, giving

$$y(x) = c_0 y_1(x) + c_1 y_2(x)$$

     where $y_1(x)$ and $y_2(x)$ are the two linearly independent solutions

5.2 Solutions about Singular Points

- A singular point x_0 is said to be a **regular singular point** of the differential equation
$$y'' + P(x)y' + Q(x)y = 0$$
if the functions $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are both analytic at x_0. A singular point that is not regular is said to be an **irregular singular point** of the equation.

- This, if $x - x_0$ appears at most to the first power in the denominator of $P(x)$ and at most to the second power of the denominator of $Q(x)$, then $x = x_0$ is a regular singular point

- **Frobenius' theorem** states that if $x = x_0$ is a regular singular point of the differential equation
$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$
then there exists at least one nonzero solution of the form
$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$
where r is a constant to be determined

- When applying Frobenius' theorem, r can be determined by equating the total coefficient of the lowest power of x to 0 and solving for r. This coefficient is called the **indicial equation** and its solutions the **indicial roots** or **exponents**

- Frobenius' theorem can be applied like so:

  1. Assume the solution is of the form
  $$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$
  where $x = x_0$ is a regular singular point and thus
  $$y' = \sum_{n=0}^{\infty} (n + r)c_n (x - x_0)^{n+r-1}$$
  and
  $$y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)c_n (x - x_0)^{n+r-2}$$
  2. Substitute the assumed solution into the DE
  3. Group the summations

4. Solve the indicial equation to determine the value(s) of r
5. Solve the recurrence relation(s) given by the value(s) of r to determine constants
6. Use the constants to determine the solution(s)

- Assuming the indicial roots are real and $r_1 > r_2$, there are three cases to consider:

  1. If r_1 and r_2 are distinct and don't differ by an integer, then there exist two linearly independent solutions of the form

  $$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}$$

     and

  $$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

  2. If $r_1 - r_2 = N$ where N is a positive integer, then there exist two linearly independent solutions of the form

  $$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \; c_0 \neq 0$$

     and

  $$y_2(x) = Cy_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, \; b_0 \neq 0$$

     where C is a constant that may be zero

  3. If $r_1 = r_2$, then there exist two linearly independent solutions of the form

  $$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \; c_0 \neq 0$$

     and

  $$y_2(x) = y_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

- In cases 2 and 3 above it may not be possible to find a second solution. Instead a second solution can be found using the first solution and reduction of order

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx$$

5.3 Special Functions

- The equation
$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$
is called **Bessel's equation of order $\boldsymbol{\nu}$** where $nu \geq 0$

- The equation
$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$
is called **Legendre's equation of order $\boldsymbol{n}$** where n is a nonnegative integer

5.3.1 Bessel Functions

- The indicial roots are $r_1 = \nu$ and $r_2 = -\nu$

- $\Gamma(x)$ is the gamma function and it has the property that
$$\Gamma(1 + \alpha) = \alpha\Gamma(\alpha)$$

- The first solution is
$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1 + \nu + n)} \left(\frac{x}{2}\right)^{2n+\nu}$$
and it converges on $[0, \infty)$ if $\nu \geq 0$

- The second solution is
$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1 - \nu + n)} \left(\frac{x}{2}\right)^{2n-\nu}$$
and, depending on the value of ν, may contain negative powers of x and thus it converges on the interval $(0, \infty)$

- These solutions are known as **Bessel functions of the first kind** of order ν and $-\nu$

- The general solution to a Bessel equation of order ν is
$$y = c_1 J_\nu(x) + c_2 J_{-\nu}(x),\ v \neq \text{integer}$$

- The function
$$Y_\nu(x) = \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$$
is called the **Bessel function of the second kind** of order ν

- A general solution to a Bessel function of order ν is
$$y = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

- Sometimes it's possible to transform a DE into a Bessel function via a change of variable, e.g. by substituting $t = \alpha x$ in the **parametric Bessel function of order ν**

$$x^2 y'' + xy' + (\alpha^2 x^2 - \nu^2) y = 0$$

  it can be transformed into

$$t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + (t^2 - \nu^2) y = 0$$

  which has the general solution

$$y = c_1 J_\nu(t) + c_2 Y_\nu(t)$$

  or

$$y = c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x)$$

6 Numerical Solutions of Ordinary Differential Equations

6.1 Euler Methods and Error Analysis

- **Round-off error** occurs when calculators or computers round off numbers to fit within the limits of what they can represent
- If we assume that y_n is accurate, then the difference between the computed and actual values of y_{n+1} is called the **local truncation error**, **formula error**, or **discretization error**
- The upper bound on the absolute error of the local truncation error for Euler's formula is

$$M \frac{h^2}{2!}$$

  where

$$M = \max_{x_n < x < x_{n+1}} |y''(x)|$$

- The local truncation error for Euler's method is $O(h^2)$
- If $e(h)$ denotes the error in a numerical calculation depending on h, then $e(h)$ is said to be $O(h^n)$ if there is a constant C and a positive integer n such that $|e(h)| \leq Ch^n$
- If y_n isn't necessarily accurate, i.e. it contains its own local truncation error, the difference between the computer and actual values of y_{n+1} is called the **global truncation error** (this may be greater than the local truncation error as its affected by the truncation errors of previous values)

- The global truncation error for Euler's method is $O(h)$
- If a method for the numerical solution of a differential equation has local truncation error $O(h^{\alpha+1})$ then the global truncation error is $O(h^{\alpha})$
- The **improved Euler's method** uses an average of the gradients at the original point and the point predicted by Euler's method

$$y_{n+1} = y_n + h\frac{f(x_n, y_n) + f(x_n, y^*_{n+1})}{2}$$

  where

$$y^*_{n+1} = y_n + hf(x_n, y_n)$$

- The local truncation error for the improved Euler's method is $O(h^3)$ and the global truncation error is $O(h^2)$

6.2 Runge-Kutta Methods

- Runge-Kutta methods are methods for obtaining approximate solutions to first-order initial value problems
- There are Runge-Kutta methods of different orders
- Each Runge-Kutta method is a weighted average of slopes over the interval $x_n < x < x_{n+1}$

$$y_{n+1} = y_n + h(w_1k_1 + w_2k_2 + \cdots + w_mk_m)$$

  where m is the order of the method
- Euler's method is said to be a first-order Runge-Kutta method
- The improved Euler's method is said to be a second-order Runge-Kutta method
- The local truncation error for RK4 is $y^{(5)}(c)/5!$ or $O(h^5)$ and the global truncation error is $O(h^4)$
- Numerical methods that use a variable step size are called **adaptive methods**
- One of the more popular adaptive methods is the **Runge-Kutta-Fehlberg method** or the RKF45 method