Group Theory and MicroRobots

Domo arigato MicroRoboto!

If we assign each color in a MicroRobots board a unique integer $1, \dots, 6$, then a board becomes a 6×6 grid of pairs (i, j) where each possible pair in $\mathbb{Z}/6 \times \mathbb{Z}/6$ occurs exactly once. This way, we can view a MicroRobots board as a bijection

$$\mathbb{Z}/6 \times \mathbb{Z}/6 \xrightarrow{f} \mathbb{Z}/6 \times \mathbb{Z}/6 .$$

Here $f(j,k)$ = entry in j^{th} row, k^{th} column indexed from 0. Notation $f(j,k) = (f_1(j,k), f_2(j,k))$

The set $\mathcal{B}$ of these bijections forms a group which is isomorphic to the symmetric group S_{36}.

Def: We say a bijection $f \in \mathcal{B}$ is connected if the graph $\Gamma(f)$ defined by

$$\Gamma(f) \quad \begin{array}{ll} \text{vertices}: & \mathbb{Z}/6 \times \mathbb{Z}/6 \\ \text{edges}: & (a_1, a_2) \sim (b_1, b_2) \iff \begin{pmatrix} a_1 = a_2 \\ \text{OR} \\ b_1 = b_2 \end{pmatrix} \text{ and } \begin{pmatrix} f_1(a_1,a_2) = f_1(b_1,b_2) \\ \text{OR} \\ f_2(a_1,a_2) = f_2(b_1,b_2) \end{pmatrix} \end{array}$$

is also connected.

Equivalently, the MicroRobots board's graph is connected.

Key Group Theory Concept: important groups are defined by actions and invariance

Def: Define a subgroup of $\mathcal{B}$ by

$$G = \{ g \in \mathcal{B} \mid \Gamma(f \circ g) \cong \Gamma(f) \text{ for all } f \in \mathcal{B} \}$$

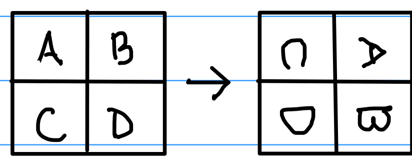
In other words, G is the group of automorphisms preserving graph structure!

Examples of things in G:

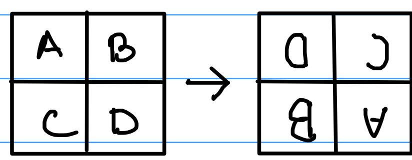
- row swaps $\quad g:(a,b) \mapsto (\sigma(a),b)$

  σ a permutation of $\{0,1,\dots,5\}$
- column swaps $\quad g:(a,b) \mapsto (a,\sigma(b))$

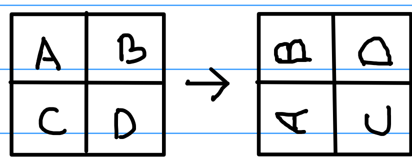
  σ a permutation of $\{0,1,\dots,5\}$
- global rotations



$(i,j) \mapsto (j, 5-i)$



$(i,j) \mapsto (5-i, 5-j)$



$(i,j) \mapsto (5-j, i)$

Since $\mathcal{B}$ is a group, it has inversion $f \mapsto f^{-1}$.

Def: We call $\Gamma(f^{-1})$ the dual of $\Gamma(f)$.

This is the "dual board" operation we noticed before!

Prop: $\Gamma(f)$ and $\Gamma(f^{-1})$ are isomorphic

Proof: The map taking (i,j) to $f(i,j)$ defines the graph isomorphism. $\square$

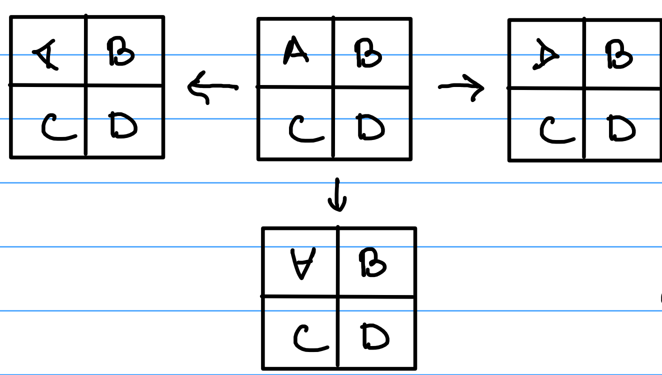
Prop: If $g, h \in G$ and $f \in B$ then $\Gamma(f) \cong \Gamma(h \circ f \circ g)$

Proof: $\Gamma(f) \cong \Gamma(f \circ g) \cong \Gamma(g^{-1} \circ f^{-1}) \cong \Gamma(g^{-1} \circ f^{-1} \circ h^{-1}) \cong \Gamma(h \circ f \circ g)$ $\square$

In MicroRobots, we are able to shuffle the board by rotating any of four distinguished 3x3 blocks or by swapping around these same blocks.

Def: We define the shuffle group to be the subgroup Sh of B generated by

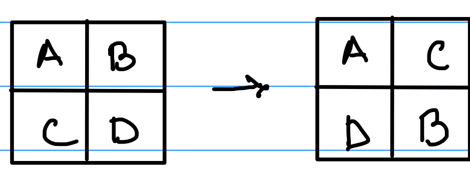
- rotations



$\cong (\mathbb{Z}/4)^4$

etc.

- card permutations



etc. $\cong S_4$

As a group, Sh is isomorphic to a semidirect product $(\mathbb{Z}/4)^4 \rtimes S_4$ corresponding to rotation and permutation because the rotation subgroup is normal.

Prop: Sh is isomorphic to the semidirect product $(\mathbb{Z}/4)^4 \rtimes S_4$ with the group operation

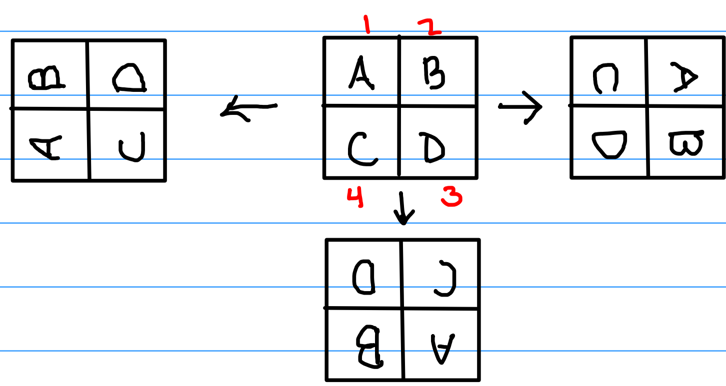
$$(a_1, a_2, a_3, a_4, \sigma)\cdot(b_1, b_2, b_3, b_4, \omega) = (a_{\omega(1)}+b_1, a_{\omega(2)}+b_2, a_{\omega(3)}+b_3, a_{\omega(4)}+b_4, \sigma\circ\omega)$$

Proof: The action of Sh on card layouts is fully faithful. If we let $(a_1, a_2, a_3, a_4, \sigma)$ represent a board layout so that card i has orientation a_i and position $\sigma(i)$, then we can identify card layouts with elements of $(\mathbb{Z}/4)^4 \rtimes S_4$. Furthermore the group product is exactly the action of Sh. $\square$

From here on out we will simply use Sh and $(\mathbb{Z}/4)^4 \rtimes S_4$ interchangeably.

Many elements of Sh are also elements of G. For example this is true of the global rotations:

$(3,3,3,3,(4321))$



$(1,1,1,1,(1234))$

$(2,2,2,2,(14)(23))$

Prop: $G \cap Sh$ is the subgroup of Sh generated by

$$(1,1,1,1,(1234)),\quad (2,2,2,2,e),\quad (0,0,0,0,(12)(34))$$

It has order 2^4.

Proof: This is just by inspection. We can probably find an even better way. $\square$

Now remember our goal is to create a MicroRobots board which exhibits some property of graphs, even after the action of Sh, or even better preserves iso. class.

Now we do something clever. Notice for $f, g \in B$ conjugate

$$g \circ f = f \circ g^f \quad \text{where } g^f := f^{-1} \circ g \circ f$$

Thus if $g \in G$, then $\Gamma(f) \cong \Gamma(f \circ g^f)$.

This leads to the following theorem.

Theorem: Let $f \in B$ and consider the set of double-cosets

$$Sh_f := G^f \cap Sh \backslash Sh / G \cap Sh \quad \text{where } G^f = \{g^f : g \in G\}$$

If P is a property of the graph $\Gamma(f)$ and for at least one representative g of every equivalence class in Sh_f $\Gamma(f \circ g)$ satisfies P then $\Gamma(f \circ g)$ has property P for all $g \in Sh$.

Proof:
Let $[g_1], [g_2], \dots, [g_r]$ be the distinct equivalence classes in Sh_f. Then for $g \in Sh$, there exists $h_1, h_2 \in G$ and $1 \le k \le r$ satisfying $g = h_1^f g_k h_2$.
Therefore

$$\Gamma(f \circ g) \cong \Gamma(h_1 \circ f \circ g_k \circ h_2) \cong \Gamma(f \circ g_k) \text{ which has property } P \quad \square$$

The previous theorem is really useful because it cuts down tremendously on what we need to check, from $|Sh| = 2^{11}\cdot 3$ different graphs down to just $|Sh_f|$ which in practice can be really small.

An important special case is when $f \in G$ since then

$$Sh_f = G \cap Sh \backslash Sh / G \cap Sh$$

Another interesting case is when $f \in Sh$ since then

$$Sh_f = (G \cap Sh)^f \backslash Sh / G \cap Sh$$

If we are specifically interested in boards f whose graphs $\Gamma(f)$ have the property that $\Gamma(f \circ h)$ is connected, for all $h \in Sh$, the following gives many examples.

<u>Theorem</u>: If $f = g \circ \tilde{g}$ for some $g \in G$ and $\tilde{g} \in Sh$ then

$$\Gamma(f \circ h) \text{ is connected for all } h \in Sh.$$

<u>Proof</u>: Consider the identity function id. The associated board is

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Each 3x3 card is connected within itself. Also, no matter what arrangement they are in, the four different centers will be connected to each other.

Thus, $\Gamma(h) = \Gamma(id \circ h)$ is connected for all $h \in Sh$.
Therefore since $\tilde{h} \circ h \in Sh$ for $h \in Sh$

$$\Gamma(g \circ \tilde{g} \circ h) \cong \Gamma(\tilde{g} \circ h) \text{ is connected for all } h \in Sh.$$

$\square$

Example: The board corresponding to $(1,1,1,1,e)$ is

1,3	2,3	3,3	1,6	2,6	3,6
1,2	2,2	3,2	1,5	2,5	3,5
1,1	2,1	3,1	1,4	2,4	3,4
4,3	5,3	6,3	4,6	5,6	6,6
4,2	5,2	6,2	4,5	5,5	6,5
4,1	5,1	6,1	4,4	5,4	6,4

and no matter how we mix up the cards, the graph of the board will remain connected.

Here's a more nontrivial example where $f \notin G$ and the subgroup $G^f \cap Sh$ is nontrivial.

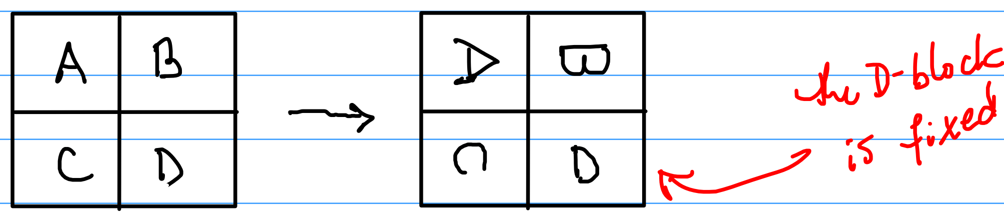
Ex: Consider the function f corresponding to the board

1,1	4,1	1,2	5,1	2,1	5,2
4,4	6,6	4,2	2,4	5,5	2,2
1,4	4,3	1,3	5,4	2,3	5,3
6,1	3,1	6,2	6,5	4,5	3,6
3,4	5,6	3,2	2,5	4,6	3,5
6,4	3,3	6,3	1,6	1,5	2,6

This function satisfies the property that for $g \in G$ defined by the column 4-cycle

$$g(i,j) = \begin{cases} (i, j+1) & , \; j = 1,2,3 \\ (i, 1) & , \; j = 4 \\ (i, j) & , \; j = 5,6 \end{cases}$$

we find $gf = f^{-1} \circ g \circ f$ is the triple rotation



which is represented in $(\mathbb{Z}/4)^4 \rtimes S_4$ by $(1,1,1,0,e)$.

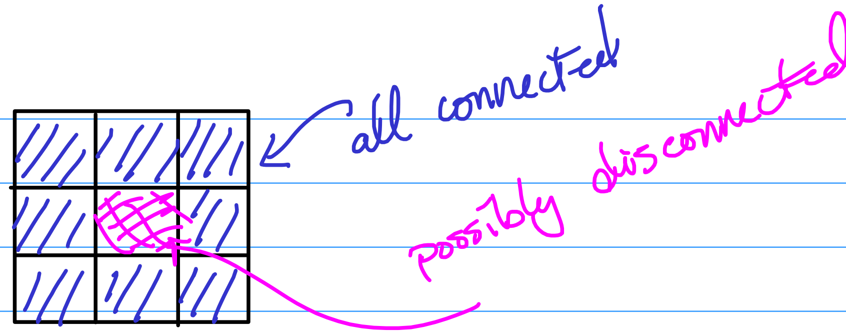
Therefore we need only check members of

$$\langle (1,1,1,0,e) \rangle \backslash (\mathbb{Z}/4)^4 \rtimes S_4 / (G \cap Sh)$$

which cuts down the usual conditions by a factor of four. We can check these to show that the graph $\Gamma(f \circ g)$ is connected for all $g \in Sh$.

$\square$

It is interesting to observe the symmetry induced by the condition $gf = (1,1,1,0,e)$. It forces all four cards to have connected rims as depicted below:



This makes the orientation of a particular card completely inconsequential.

<u>Question</u>: Does $G^f \cap Sh$ being nontrivial guarantee some symmetry which forces the associated graphs to always be connected, even after shuffling?

<u>Question</u>: What does the actual MicroRobots game board look like as a function f? How does its conjugation $g \mapsto g^f$ behave?