Linear

Algebra

$\boxed{\text{Def.}}$ If $V = (v_1, v_2, \cdots v_n)$ is a vector in $\mathbb{R}^n$, then the length of V, also called the norm of V or the magnitude of V, is denoted by $\|V\|$ and is denoted by the formula.

$$\|V\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots v_n^2}$$

$\boxed{\text{Def.}}$ If $U = (u_1, u_2, \cdots, u_n)$ and $V = (v_1, v_2 \cdots v_n)$ are vectors in $\mathbb{R}^n$, then the dot product of u and v also called the Euclidean inner product of u and v, is denoted by $U \cdot V$ and is defined by the formula

$$U \cdot V = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

$$= \|U\| \, \|V\| \cos\theta$$

where θ is the angle between U and V.

$\boxed{\text{Def.}}$ Two vectors u and v in $\mathbb{R}^n$ are said to be orthogonal if $u \cdot v = 0$, and a nonempty set of vectors in $\mathbb{R}^n$ is said to be an orthogonal set if each pair of distinct vectors in the set is orthogonal.

$\boxed{\text{Def.}}$ Two vectors u and v in $\mathbb{R}^n$ are said to be orthonormal if they are orthogonal and have length 1, and a set of vectors is said to be an orthonomal set if every vector in the set has length 1 and each pair of distinct vectors is orthogonal.

ex) $u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$u_1 \cdot u_2 = u_2 \cdot u_3 = u_1 \cdot u_3 = 0$ Orthogonal set!

$e_1 = \dfrac{u_1}{\|u_1\|}$, $e_2 = \dfrac{u_2}{\|u_2\|}$, $e_3 = \dfrac{u_3}{\|u_3\|}$ $\quad e_1 \cdot e_2 = e_1 \cdot e_3 = e_2 \cdot e_3 = 0$

Orthonomal set!

$\boxed{\text{Def.}}$ A nonempty set of vectors in $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if it is closed under scalar multiplication and addition.

$\boxed{\text{Theo.}}$ If $v_1, v_2, \cdots v_s$ are vectors in $\mathbb{R}^n$, then the set of all combinations

$$x = t_1 v_1 + t_2 v_2 + \cdots t_s v_s \quad \text{----} \quad (1)$$

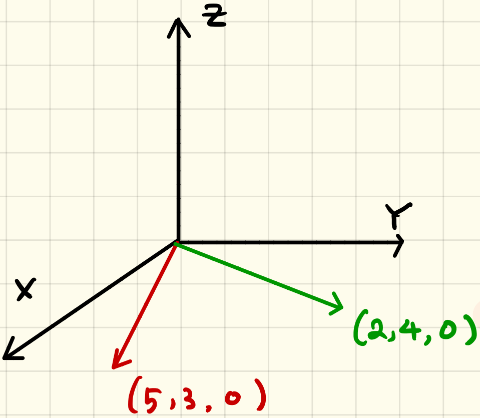
is a subspace of $\mathbb{R}^n$

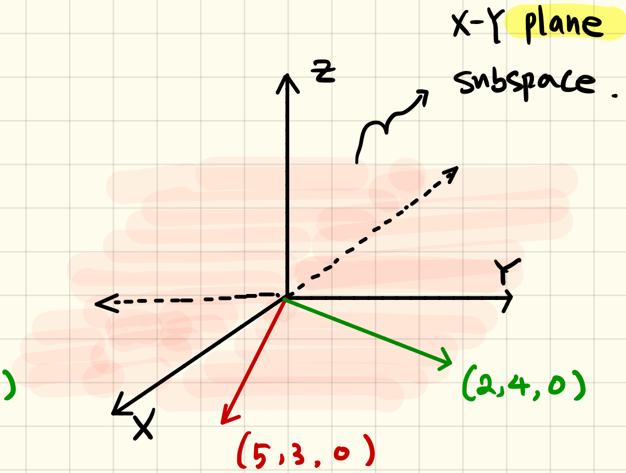
The subspace W of $\mathbb{R}^n$ whose vectors satisfy (1) is called the span of $v_1, v_2, \cdots v_s$ and is denoted by

$$\boxed{W = \text{span}\{v_1, v_2, \cdots v_s\}}$$

ex) $W_1 = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$, $W_2 = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right\}$

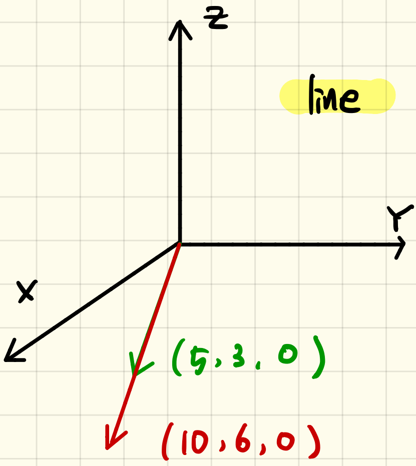
W_1, W_2 are the same subspace.





$$W_1 = \text{span}\left\{ \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \right\}, \quad W_2 = \text{span}\left\{ \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \right\}$$

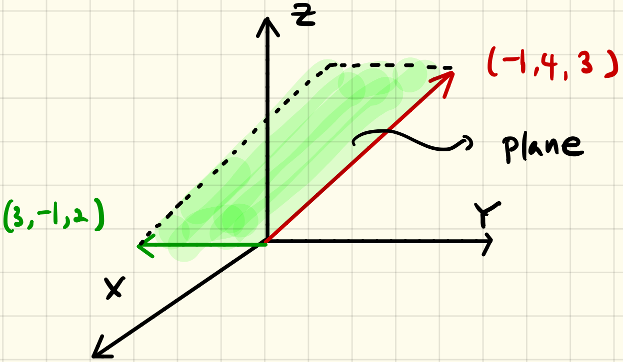
multiplication, addition.



$$W_1 = \text{span}\left\{ \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} \right\}$$

$$W_2 = \text{span}\left\{ \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ 6 \\ 0 \end{pmatrix} \right\}$$

$$W_3 = \text{span}\left\{ \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ 6 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -3 \\ 0 \end{pmatrix} \right\}$$



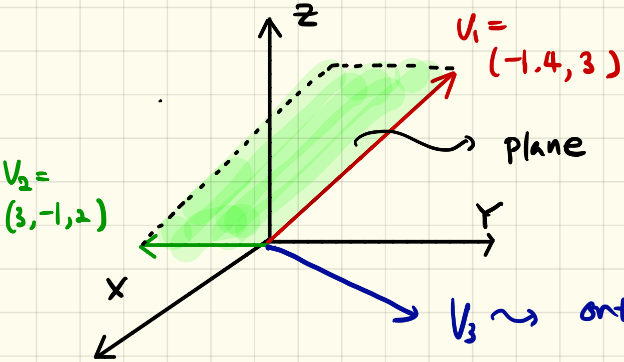
$W = \text{span}\left\{ \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \right\}$

If A is an $m \times n$ matrix,

- The row space of A, denoted by $\boxed{\text{row}(A)}$, is the subspace of $\mathbb{R}^n$ that is spaned by the $\boxed{\text{row vectors}}$ of A.

- The column space of A, denoted by $\boxed{\text{col}(A)}$, is the subspace of $\mathbb{R}^n$ that is spaned by the $\boxed{\text{column vectors}}$ of A

- The null space of A, denoted by $\boxed{\text{null}(A)}$, is the solution space of $\boxed{Ax = 0}$. This is a subspace of $\mathbb{R}^n$.



$$W = \text{span}\left\{ \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \right\}$$

$v_3 \rightsquigarrow$ orthogonal to v_1, v_2

$$A = \begin{bmatrix} -1 & 4 & 3 \\ 3 & -1 & 2 \end{bmatrix} \longrightarrow W = \text{row}(A).$$

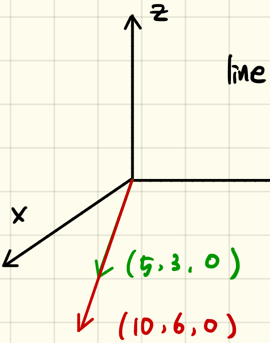
Let's $v_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$$v_1 \cdot v_3 = 0$$
$$v_2 \cdot v_3 = 0$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cdot v_3' = 0$$

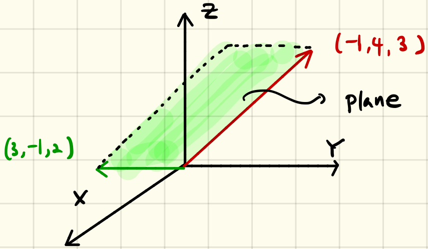
$$\text{null}(A) = \text{span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}_{v_3.} \right\}$$

$\boxed{\text{Def.}}$ The dimension of the row space of a matrix A is called the rank of A and is denoted by $\boxed{\text{rank}(A)}$; and the dimension of the null space of A is called the nullity of A and denoted by $\boxed{\text{nullity}(A)}$.



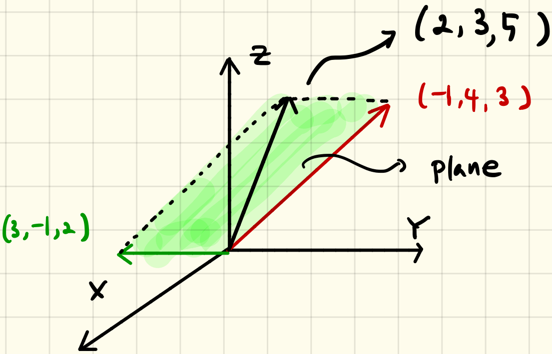
$$A = \begin{bmatrix} 5 & 3 & 0 \\ 10 & 6 & 0 \end{bmatrix}$$

$$\text{Rank}(A) = 1.$$



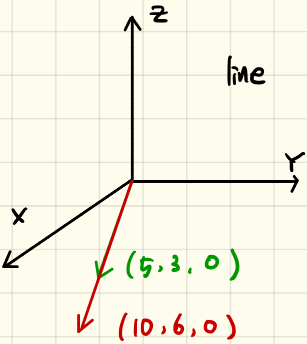
$$A = \begin{bmatrix} -1 & 4 & 3 \\ 3 & -1 & 2 \end{bmatrix}$$

$$\text{Rank}(A) = 2.$$



$$A = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

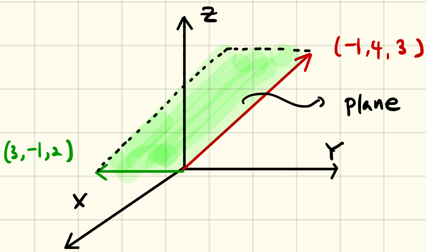
$$\text{Rank}(A) = 2.$$



$$A = \begin{bmatrix} 5 & 3 & 0 \\ 10 & 6 & 0 \end{bmatrix}$$

$\text{Rank}(A) = 1.$

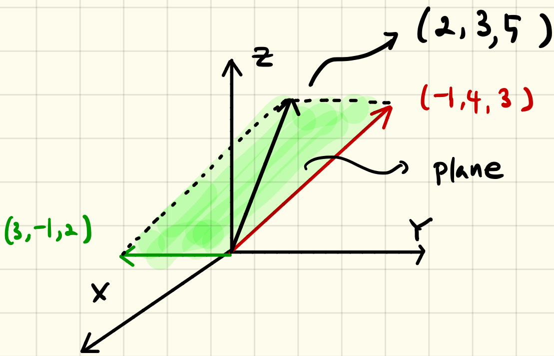
$\text{nullity}(A) = 2$



$$A = \begin{bmatrix} -1 & 4 & 3 \\ 3 & -1 & 2 \end{bmatrix}$$

$\text{Rank}(A) = 2.$

$\text{nullity}(A) = 1$



$$A = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

$\text{Rank}(A) = 2.$

$\text{nullity}(A) = 1$

If A is a $m \times n$ matrix, $n = \text{Rank}(A) + \text{nullity}(A).$

Ex).

$$x+y+z=6$$
$$2x+3y+z=9$$
$$3x+2y+3z=16$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 9 \\ 16 \end{bmatrix}$$

$$A \quad \mathbb{x} = b$$

$$A^{-1}A\mathbb{x} = A^{-1}b \qquad \mathbb{x} = A^{-1}b$$

Non-invertable case??

A is invertiable, $\text{Rank}(A) = 3$.

$$A^{-1} = \begin{bmatrix} -7 & 1 & 2 \\ 3 & 0 & -1 \\ 5 & -1 & -1 \end{bmatrix} \qquad \mathbb{x} = \begin{bmatrix} -7 & 1 & 2 \\ 3 & 0 & -1 \\ 5 & -1 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 9 \\ 16 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

$\text{nullity}(A) = 0$

Unique solution.

Ex).

$$x+y+z = 6$$
$$2x+3y+z = 9$$
$$3x+2y+3z = 16$$

$$x+y+z-6 = 0$$
$$2x+3y+z-9 = 0$$
$$3x+2y+3z-16 = 0$$

$$C \neq 0$$

$$Cx+Cy+Cz-6C = 0$$
$$2Cx+3Cy+Cz-9C = 0$$
$$3Cx+2Cy+3Cz-16C = 0$$

$$\begin{bmatrix} 1 & 1 & 1 & -6 \\ 2 & 3 & 1 & -9 \\ 3 & 2 & 3 & -16 \end{bmatrix} \begin{bmatrix} Cx \\ Cy \\ Cz \\ C \end{bmatrix} = 0$$

$$A \qquad x' \qquad = 0$$

$$\mathrm{Rank}(A) = 3 \qquad \mathrm{nullity}(A) = 1$$

$$\mathrm{Null}(A) = \begin{bmatrix} 0.1796 \\ -0.3592 \\ -0.8980 \\ -0.1796 \end{bmatrix}$$

$$C=1$$

$$\therefore x = \begin{array}{cc} \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} & \begin{array}{c} x \\ y \\ z \end{array} \end{array}$$

Singular value Decomposition (SVD)

the singular value decomposition of an $m \times n$ matrix M is a factorization of the form $U\Sigma V$ where

$$\begin{cases} U \text{ is an } m \times m \text{ unitary matrix} \\ \Sigma \text{ is an } m \times n \text{ rectangular diagonal matrix} \\ V \text{ is an } n \times n \text{ unitary matrix} \end{cases}$$

There real of unitary matrix is an orthogonal matrix.

$UU^T = I$.

$VU^T = I$.

$m < n$

$\sigma_1 > \sigma_2 > \sigma_3 \cdots > \sigma_m$

$$\begin{array}{c} \\ m \end{array} \underbrace{\overset{m}{\begin{bmatrix} u_1 & u_2 & \cdots & u_m \\ & & & \end{bmatrix}}}_{U} \overset{n}{\left[\begin{array}{cccc|c} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & 0 \\ & & & \sigma_m & \end{array}\right]} \underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{V} = M.$$

$$\begin{array}{c} \\ m \end{array}\underbrace{\begin{bmatrix} u_1 & u_2 & \cdots & u_m \\ & & & \end{bmatrix}}_{U}^{m \times m} \overset{m \times n}{\begin{bmatrix} \sigma_1 & & & | \\ & \sigma_2 & & 0 \\ & & \ddots & \\ & & \sigma_m & | \end{bmatrix}} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}}_{V}^{n \times n} = M.$$

Ex.

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U_1 U_2 U_3 \\ \\ \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \\ - 0 & - \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1 U_{11} & \sigma_2 U_{12} \\ \sigma_1 U_{21} & \sigma_2 U_{22} \\ \sigma_1 U_{31} & \sigma_2 U_{32} \end{bmatrix} = \begin{bmatrix} \sigma_1 U_1 & \sigma_2 U_2 \\ & \end{bmatrix}$$

$$M = \begin{bmatrix} \sigma_1 u_1 & \sigma_2 u_2 & \cdots & \sigma_m u_m, & -0- \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix}$$

Here, $u_i : m \times 1$

$V_i : 1 \times n$

$m < n$.

$$= \sigma_1 u_1 V_1 + \sigma_2 u_2 V_2 \cdots + \sigma_m u_m V_m$$

Let's pick one vector from $V_{m+1} \cdots V_n$

V_i

$$MV_i = \sigma_1 u_1 \underbrace{V_1 \cdot V_i}_{0} + \sigma_2 u_2 \underbrace{V_2 \cdot V_i}_{0} + \cdots + \sigma_m u_m \underbrace{V_m \cdot V_i}_{0}$$

(because there are orthongonal basis.)

$$= 0$$

$\therefore V_{m+1} \cdots V_n$ is null vectors of M

$$\begin{bmatrix} 1 & 1 & 1 & -6 \\ 2 & 3 & 1 & -9 \\ 3 & 2 & 3 & -16 \end{bmatrix} \begin{bmatrix} c_x \\ c_y \\ c_z \\ c \end{bmatrix} = 0$$

$$A \quad x' = 0$$

$A : 3 \times 4$

$$A = \underset{3\times 3}{U}\ \underset{3\times 4}{\Sigma}\ \underset{4\times 4}{V}$$

```
A = [ 1 1 1 -6; 2 3 1 -9;3 2 3 -16];
null(A)'
```

```
ans = 1×4
      0.1796   -0.3592   -0.8980   -0.1796
```

```
[U,S,V] = svd(A);

V(:,end)'
```

```
ans = 1×4
      0.1796   -0.3592   -0.8980   -0.1796
```

```
transpose(A*V(:,end))
```

```
ans = 1×3
10^-15 ×
      0.2220    0.2220    0.8882
```

* for printing the vector in a row.