Projective Geometry & Homography

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CIVE 497 – CIVE 700: Smart Structure Technology

UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING

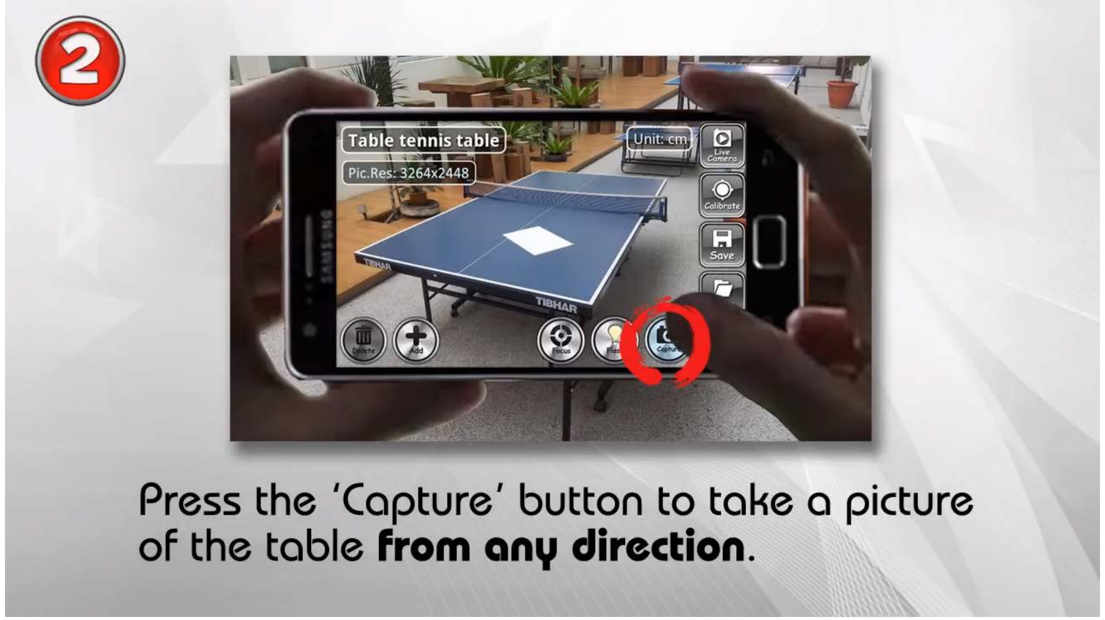
Last updated: 2018-01-31

ON 3D-CameraMeasure App

Android: https://play.google.com/store/apps/details?id=com.potatotree.on3dcamerameasure

Microsoft Lens

Measurement Demo



Measurement Demo (Continue)

Projective Geometry I and II

We will study this topic using

ECE 661: Computer Vision (by Avinash Kak)

- Lecture 2: World 2D: Representing and Manipulating Points, Lines And Conics Using Homogeneous Coordinates
- Lecture 3: World 2D: Projective Transformations and Transformation Groups
- Lecture 4: Characterization of Distortions Caused by Projective Imaging and the Principle of Point/Line Duality
- Lecture 5: Estimating a Plane-to-Plane Homography with Angle-to-Angle and Point-to-Point Correspondences

Course website: https://engineering.purdue.edu/kak/computervision/ECE661Folder/

Lecture 2: World 2D: Representing and Manipulating Points, Lines And Conics Using Homogeneous Coordinates

2-1: Point in the Homogeneous Coordinate

An arbitrary homogeneous vector representative of a point is of the form $\mathbf{x} = (x_1, x_2, x_3)^T$, representing the point $(x_1/x_3, x_2/x_3)^T$ in $\mathbb{R}^2$.

Example) $\mathbf{x_1} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$ $\mathbf{x_2} = \begin{pmatrix} 10 \\ 6 \\ 2 \end{pmatrix}$ $\mathbf{x_3} = \begin{pmatrix} 5k \\ 3k \\ k \end{pmatrix}, k \neq 0$ **up to a scale**

$\mathbf{x_1}, \mathbf{x_2}$, and $\mathbf{x_3}$ indicate the same point of (5, 3) in $\mathbb{R}^2$

$\mathbb{R}^n$: n-dimension real coordinate system

2-1: Line in the Homogeneous Coordinate (HC)

$$ax + by + c = 0$$

Line equation in $\mathbb{R}^2$

$$\mathrm{l} = (a, b, c)^\intercal$$

Line representation in HC

Example) $\mathrm{l}_{\mathbf{1}} = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}$ $\quad \mathrm{l}_{\mathbf{2}} = \begin{pmatrix} 9 \\ 12 \\ 9 \end{pmatrix}$ $\quad \mathrm{l}_{\mathbf{3}} = \begin{pmatrix} 3k \\ 4k \\ 3k \end{pmatrix}, k \neq 0$

$\mathbf{x_1}, \mathbf{x_2}$, and $\mathbf{x_3}$ indicate the same line of $3x + 4y + 3 = 0$ in $\mathbb{R}^2$

2-1: Points and Lines in the Homogeneous Coordinate (HC)

$$ax + by + c = 0 \qquad\qquad (x_1, x_2, x_3) \cdot (a, b, c)^\intercal = 0$$

The point $\mathbf{x}$ lies on the line $\mathbf{l}$ if and only if $\mathbf{x}\mathbf{l}^\intercal = \mathbf{l}\mathbf{x}^\intercal = 0$.

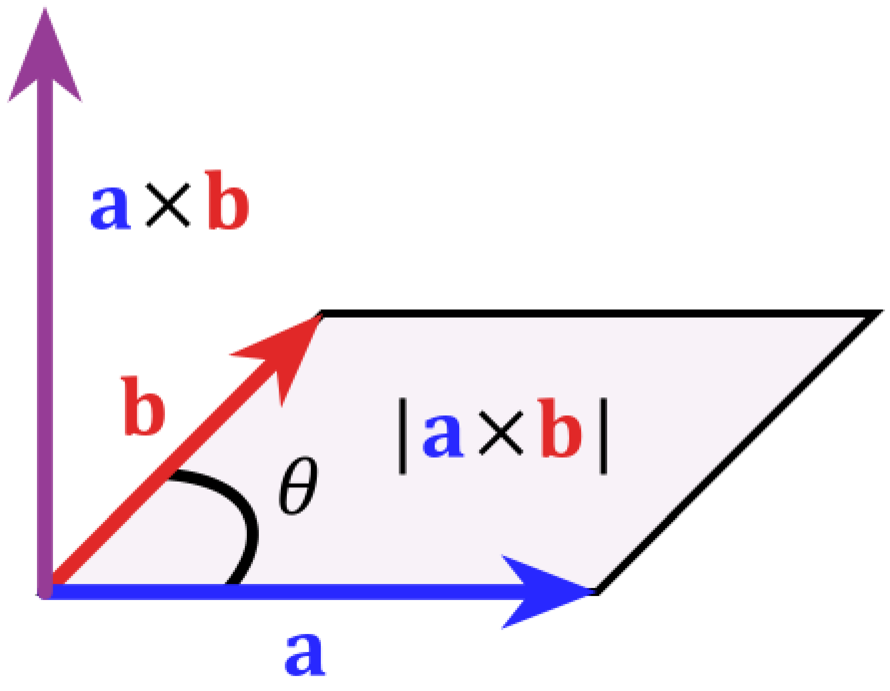
Given any two lines $\mathbf{l_1} = (a_1, b_1, c_1)$ and $\mathbf{l_1} = (a_2, b_2, c_2)$, the point ($\mathbf{x}$) of intersection of the two lines :

$$\mathbf{x} = \mathbf{l_1} \times \mathbf{l_2}$$

Given any two points $\mathbf{x_1} = (x_1, y_1, z_1)$ and $\mathbf{x_2} = (x_2, y_2, z_2)$, the line ($\mathbf{l}$) that passes through the two points :

$$\mathbf{l} = \mathbf{x_1} \times \mathbf{x_2}$$

Cross Product



$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

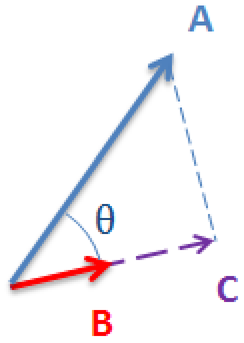
$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})\\ &= a_1b_1(\mathbf{i}\times\mathbf{i}) + a_1b_2(\mathbf{i}\times\mathbf{j}) + a_1b_3(\mathbf{i}\times\mathbf{k}) +\\ &\quad a_2b_1(\mathbf{j}\times\mathbf{i}) + a_2b_2(\mathbf{j}\times\mathbf{j}) + a_2b_3(\mathbf{j}\times\mathbf{k}) +\\ &\quad a_3b_1(\mathbf{k}\times\mathbf{i}) + a_3b_2(\mathbf{k}\times\mathbf{j}) + a_3b_3(\mathbf{k}\times\mathbf{k})\end{aligned}$$

$$\begin{aligned}\mathbf{a}\times\mathbf{b} = &\; -a_1b_1\mathbf{0} + a_1b_2\mathbf{k} - a_1b_3\mathbf{j}\\ &\; - a_2b_1\mathbf{k} - a_2b_2\mathbf{0} + a_2b_3\mathbf{i}\\ &\; + a_3b_1\mathbf{j} - a_3b_2\mathbf{i} - a_3b_3\mathbf{0}\\ = &\; (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\end{aligned}$$

https://en.wikipedia.org/wiki/Cross_product

2-2: Prove the Relationship using the Triple Scalar Identity

Dot product

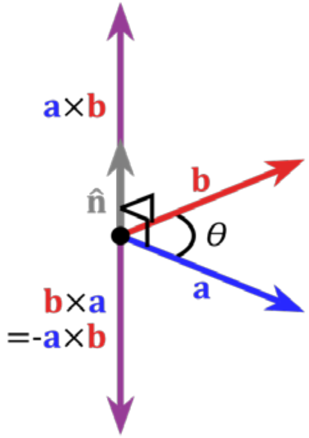


$$A \cdot B = |A||B|\cos(\theta)$$

if the magnitude of B is 1, then…

$$C = A \cdot B = |A|\cos(\theta)$$

Cross product



$$\mathbf{l}_1\mathbf{x} = \mathbf{l}_2\mathbf{x} = \mathbf{0}$$

$$\boldsymbol{a} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \mathbf{0}$$

$$\mathbf{l}_1 \cdot (\mathbf{l}_1 \times \mathbf{l}_2) = \mathbf{l}_2 \cdot (\mathbf{l}_1 \times \mathbf{l}_2) = \mathbf{0}$$

$$\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2$$

Quiz

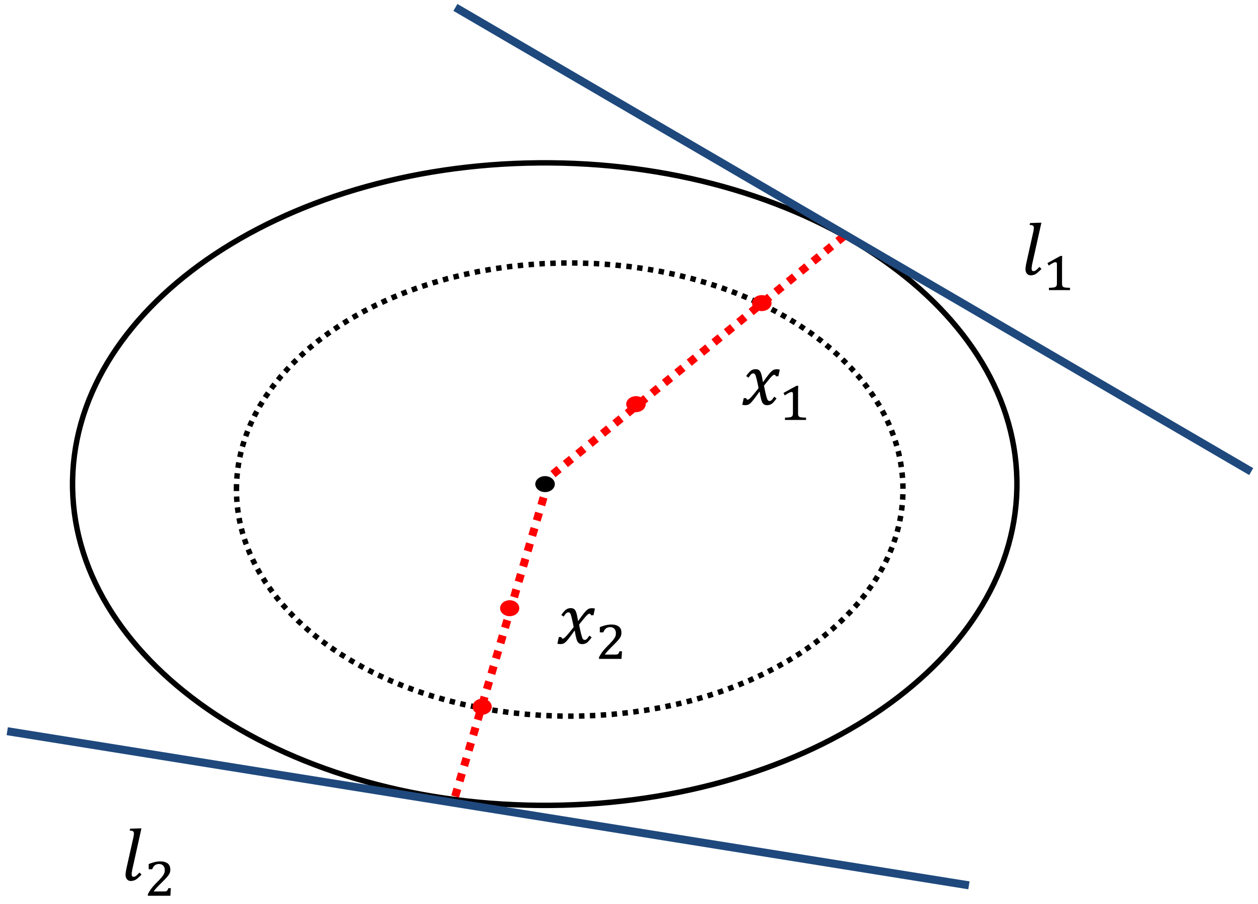
Q0: Line passes through two (0,1) and (1,2)

Q1: Intersection point (p_x, p_y) of the two lines l_1 and l_2

l_1 passes trhough two distinct points, (0,1) and (1,2).

l_2 passes trhough two distinct points, (4,0) and (5,3).

2-1 & 2-2: Visualization of Points and Lines in the Homogeneous Coordinate



2-2: Ideal Points

$$\mathbf{l}_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \mathbf{l}_2 = \begin{pmatrix} a \\ b \\ c' \end{pmatrix}$$

$$\boldsymbol{x} = \mathbf{l}_1 \times \mathbf{l}_2 = \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}$$

Ideal points

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \qquad (u/w, v/w) \text{ in } \mathbb{R}^2$$

$(u/w, v/w)$ when $w \to 0$

The point approach to infinity is along a specific direction and the direction being controlled by the value of u and v.

Mathematically!!!

Valid the point and line relationship

2-3: Vanishing Line

$$\mathbf{x_1} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \qquad \mathbf{x_2} = \begin{pmatrix} a' \\ b' \\ 0 \end{pmatrix}$$

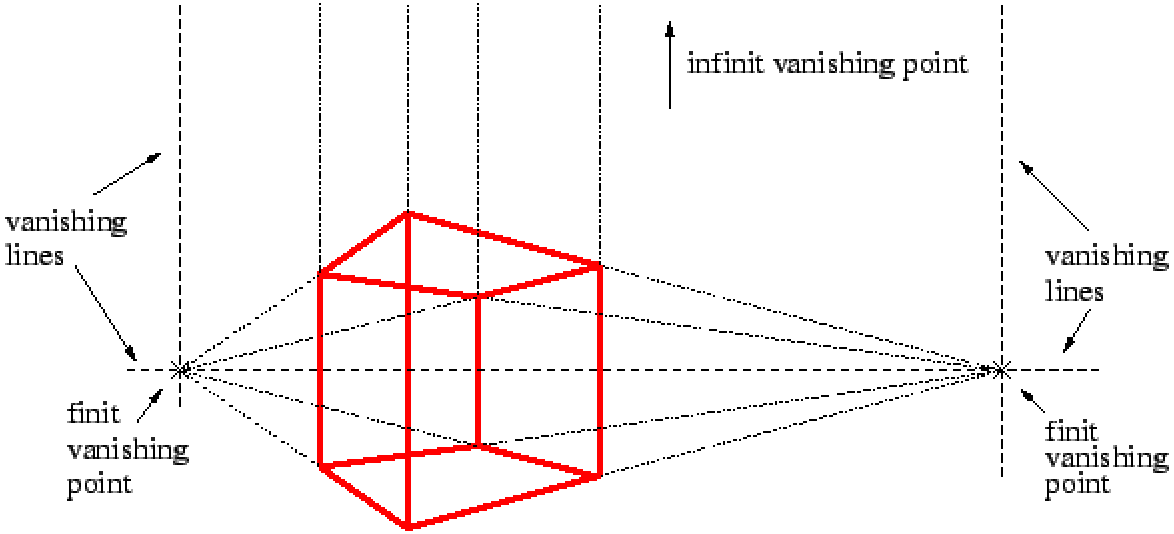
$$\mathbf{l} = \mathbf{x_1} \times \mathbf{x_2} = \begin{pmatrix} 0 \\ 0 \\ ab' - a'b \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{l}_\infty = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

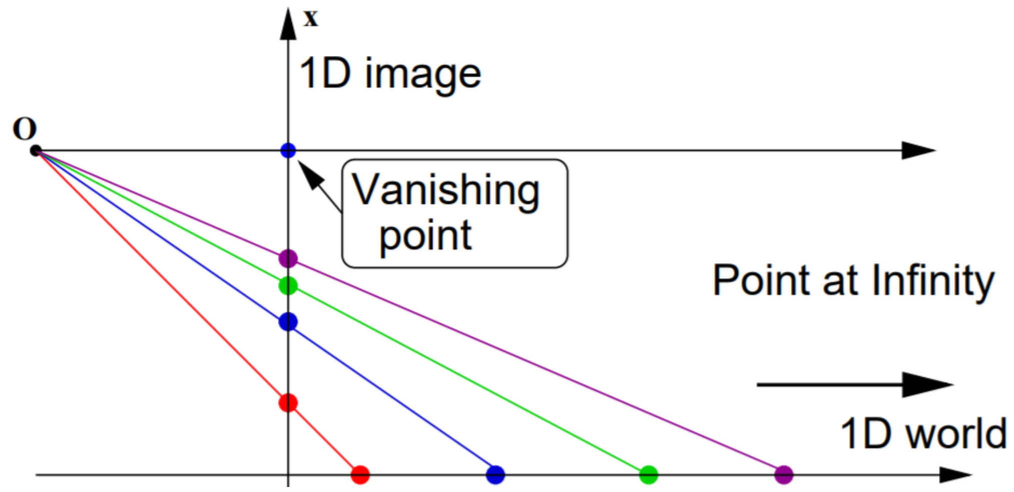
The camera image of the $\mathbf{l}_\infty$ line is called the vanishing line.

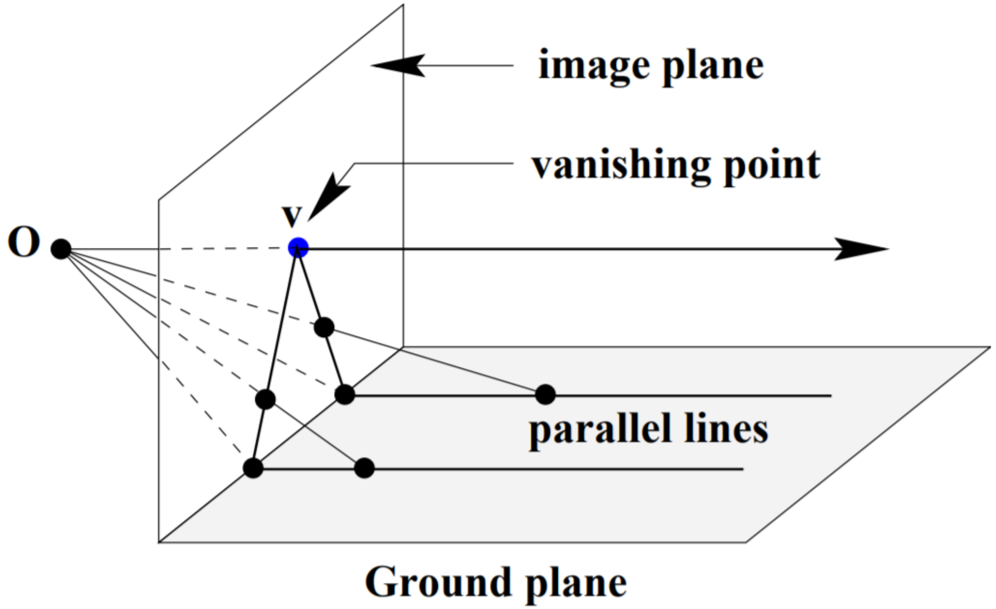
Line at infinity

Vanishing Point and Line

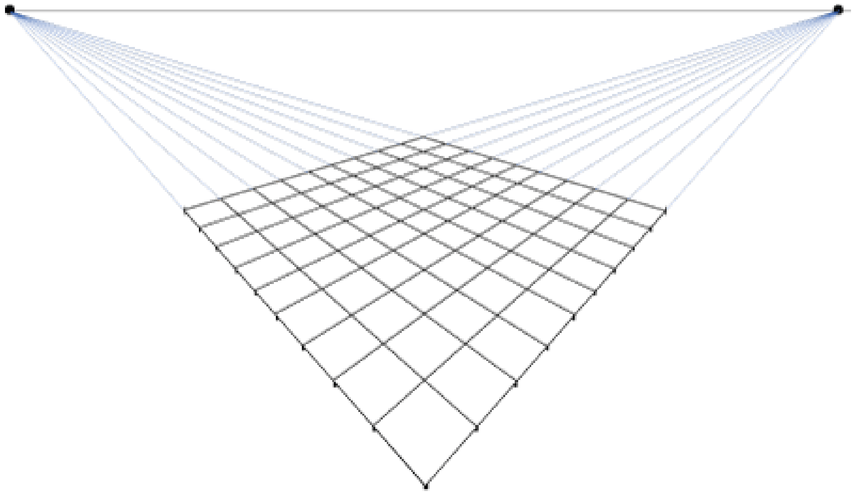


Why All Parallel Lines Meet at One Point?

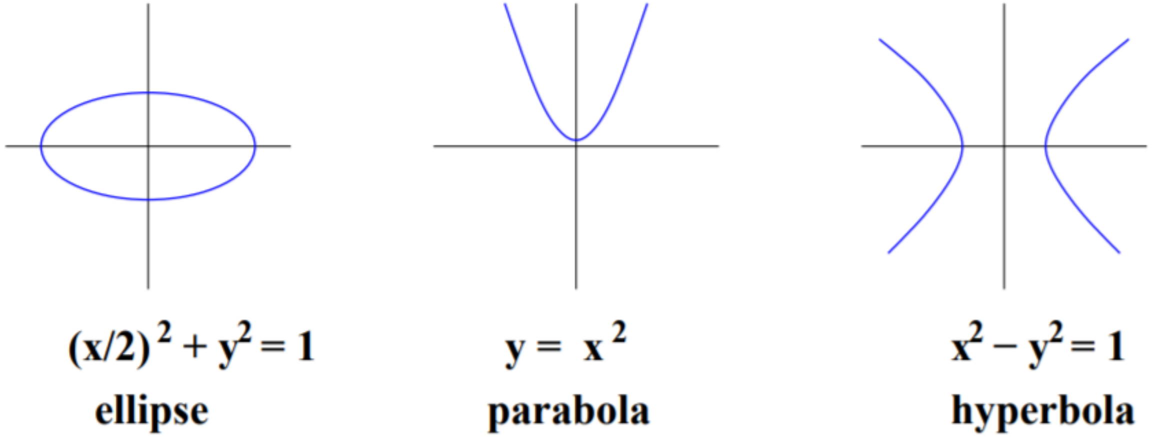


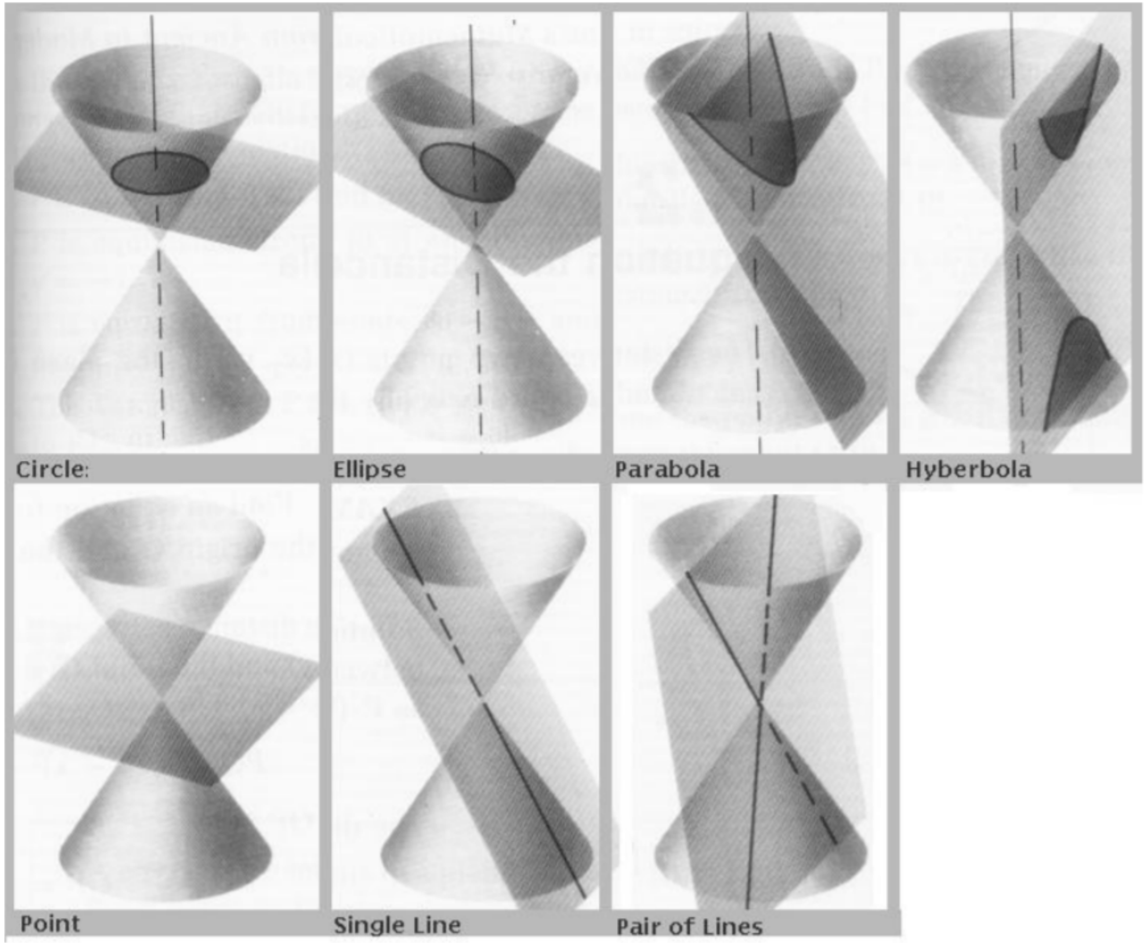


Vanishing line



2-3: Conic





$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

A conic has five degrees of freedom in general

<u>x and y can be represented by the conic equation</u>

2-3: Conics in HC

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

(x, y) in $\mathbb{R}^2$

$$a(\frac{x_1}{x_3})^2 + b\frac{x_1}{x_3}\frac{x_2}{x_3} + c(\frac{x_2}{x_3})^2 + d\left(\frac{x_1}{x_3}\right) + e\left(\frac{x_2}{x_3}\right) + f$$

(x_1, x_2, x_3) in $\mathbb{HC}$

$$= ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2$$

$$= 0$$

2-3: Conic in HC (Continue)

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

(x_1, x_2, x_3) in $\mathbb{HC}$

$$[x_1 \quad x_2 \quad x_3]\begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

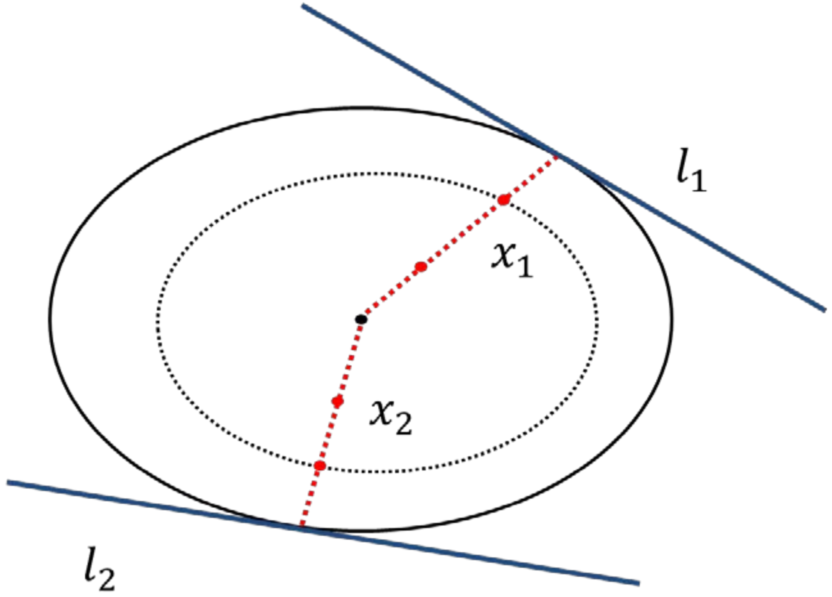
$$\mathbf{x}^{\mathsf{T}}\mathbf{C}\,\mathbf{x}=0$$

<u>**C** is the HC representation of a conic</u>

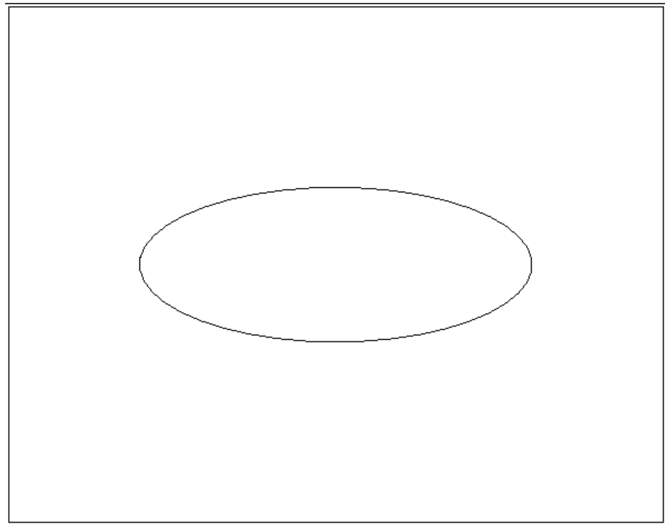
2-4: Conic Property

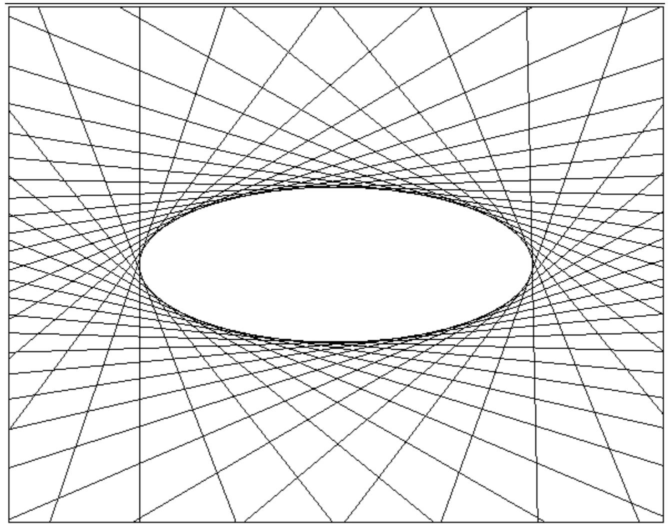
The HC representation of a conic gives us compact formulas for the tangent lines to a conic.

$$\mathbf{x}^{\mathsf{T}}\mathbf{l} = \mathbf{x}^{\mathsf{T}}\mathbf{C}\,\mathbf{x} = 0 \qquad\qquad \mathbf{l} = \mathbf{C}\,\mathbf{x}$$



2-4: Dual Conic



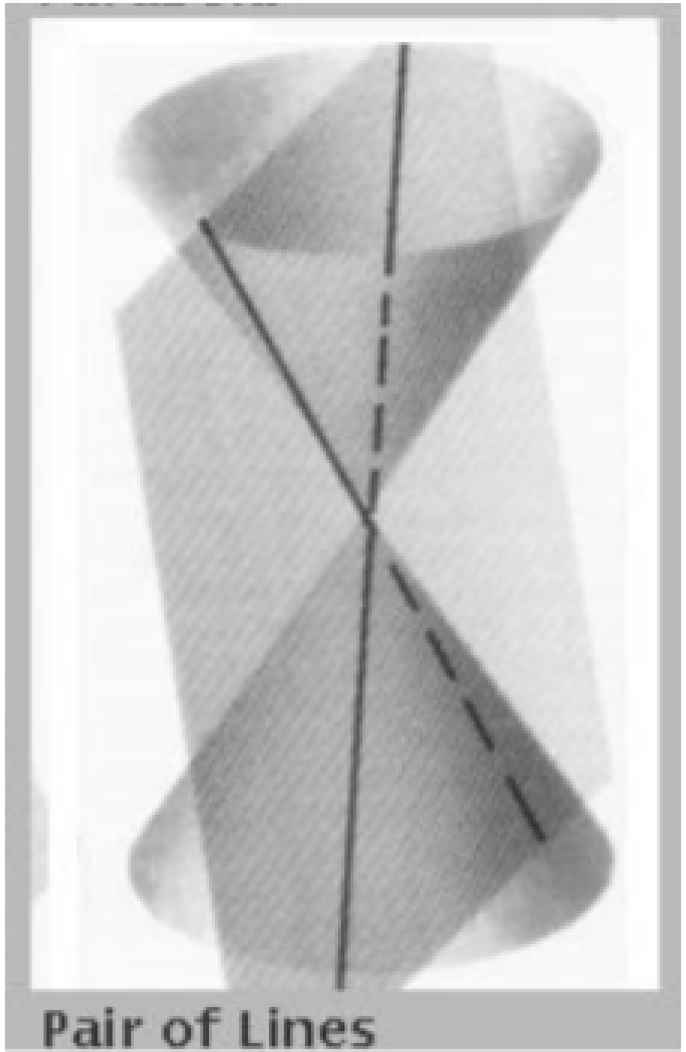


$$x^{\top} C\, x = 0$$

$x^{\top} l = 0 \qquad l = C\, x$

$$\begin{aligned} & x^{\top} C x \\ &= l^{T} C^{-T} C C^{-1} l \\ &= l^{T} C^{-T} l \\ &= l^{T}\, C^{-1} l = l^{T}\, C^{*} l \end{aligned}$$

2-4: Degenerate Conic



$$C = lm^{\top} + ml^{\top}$$

Two lines constitute a degenerate conic.

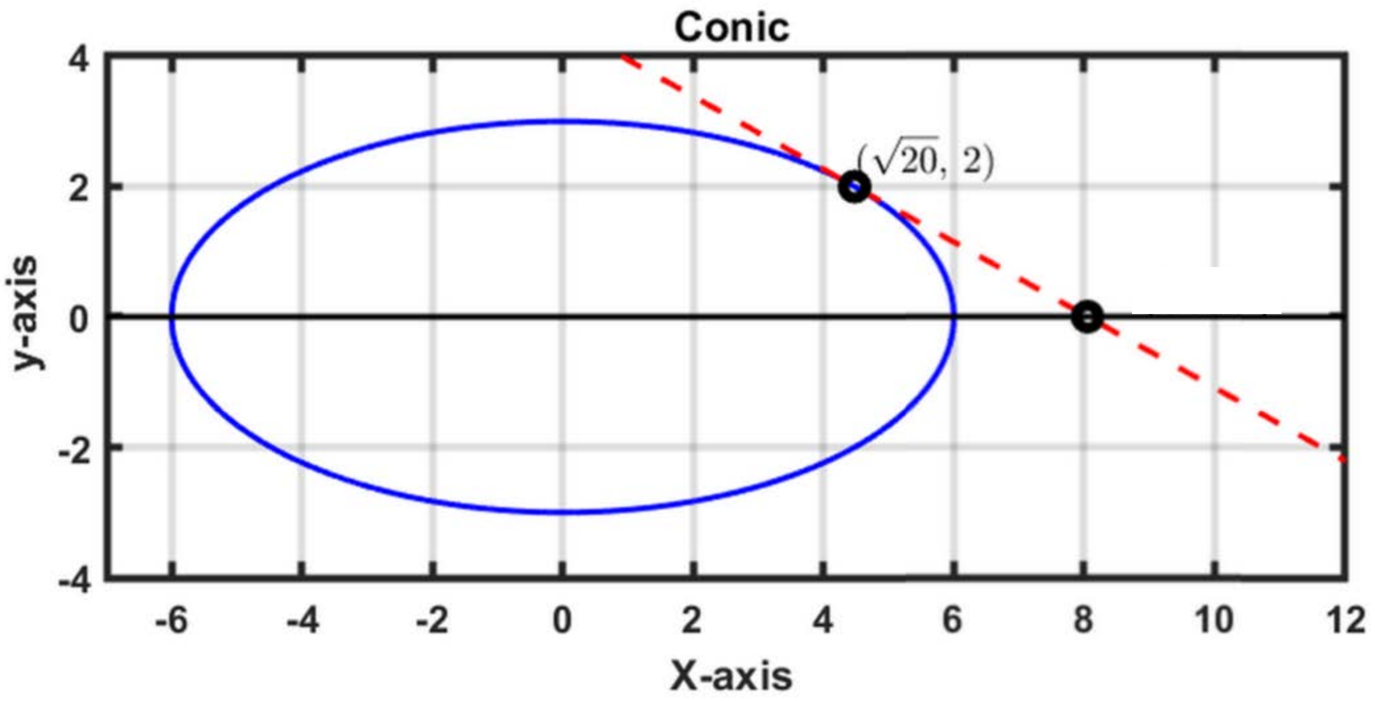
$$x^{\top}C\,x = x^{\top}(lm^{\top} + ml^{\top})x = 0$$

Two lines constitute a degenerate **<u>line conic</u>**.

$$l^{\top}C^{*}\,l = l^{\top}(xy^{\top} + xy^{\top})x = 0$$

Quiz

Q2: Given a conic $\frac{x^2}{6^2}+\frac{y^2}{3^2}-1=0$ (ellipse), compute a tagential line to this conic at ($\sqrt{20}\,,2)$ and its intersection point with x-axis, (p_x, p_y).



Summary

1. An arbitrary homogeneous vector representative of a point is of the form $\mathbf{x} = (x_1, x_2, x_3)^T$, representing the point $(x_1/x_3, x_2/x_3)^T$ in $\mathbb{R}^2$.

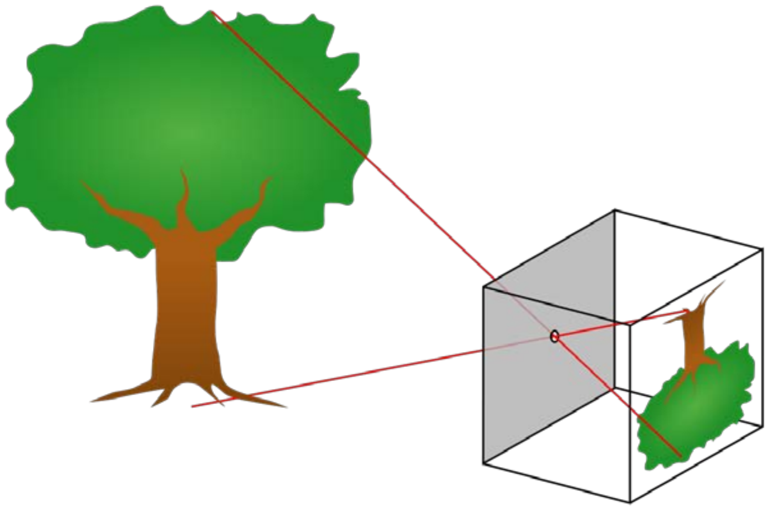
2. Line equation, $ax + by + c = 0$, in $\mathbb{R}^2$ is represented as $\mathrm{l} = (a, b, c)^\mathsf{T}$ in the homogeneous coordinate.

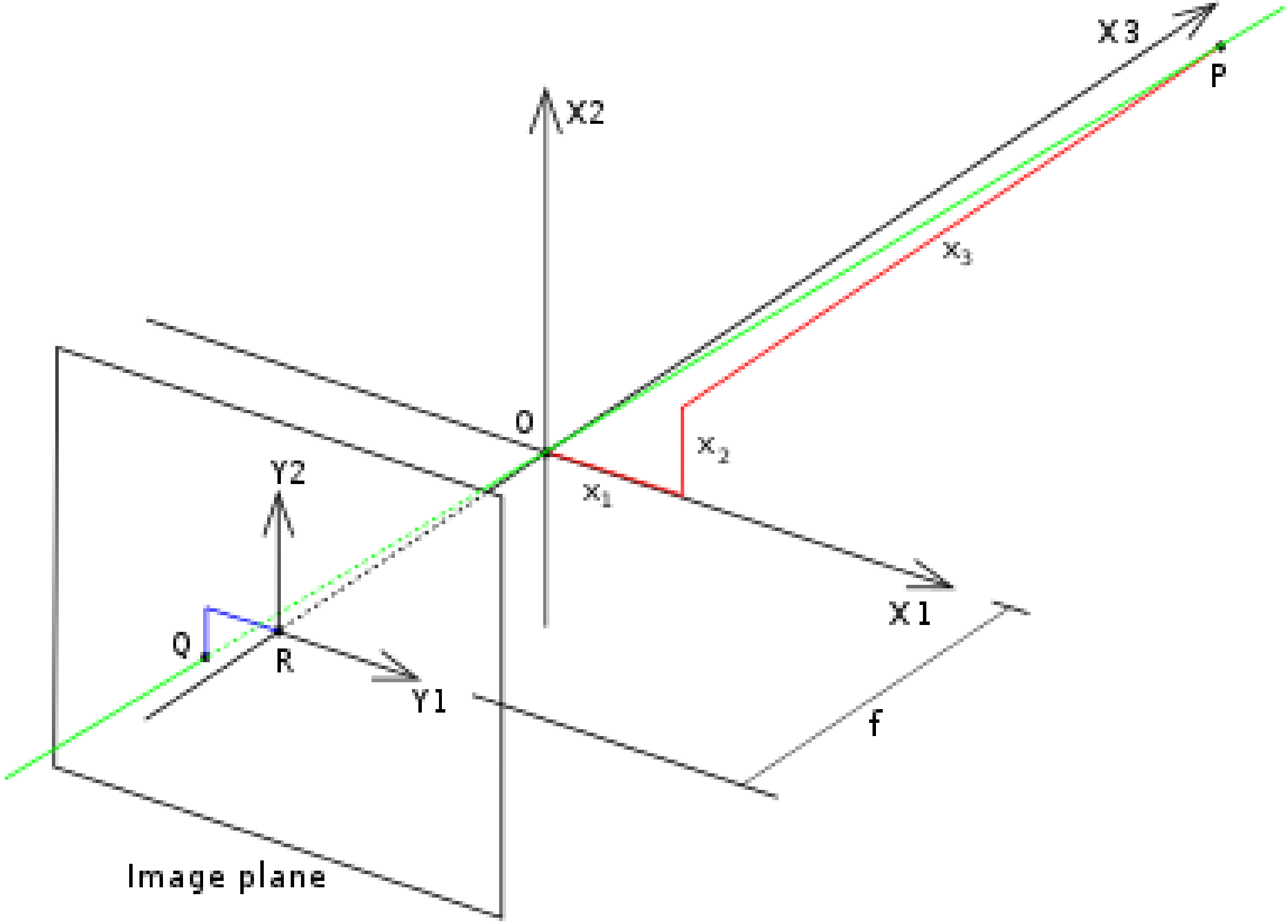
3. A conic, $ax^2 + bxy + cy^2 + dx + ey + f = 0$, in $\mathbb{R}^2$ become a 3x3 matrix, **C**, in the homogeneous coordinate,

$$\mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

Lecture 3: World 2D: Projective Transformations and Transformation Groups

Pinhole Camera Model

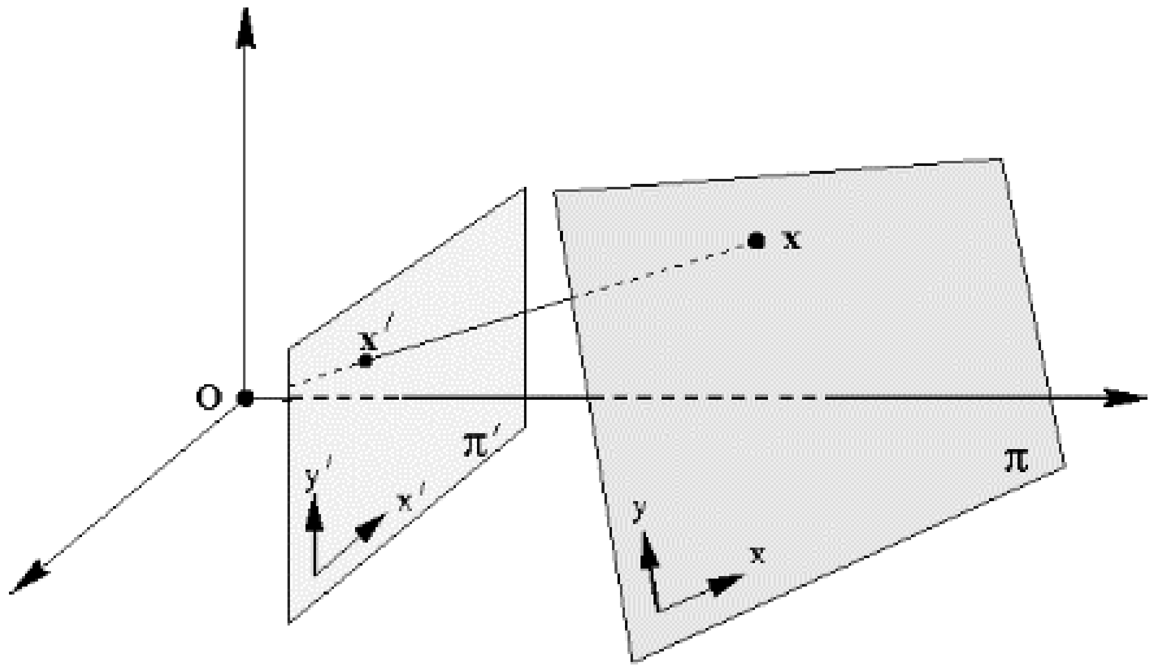




https://en.wikipedia.org/wiki/Pinhole_camera_model

3-1: Projective Transformation

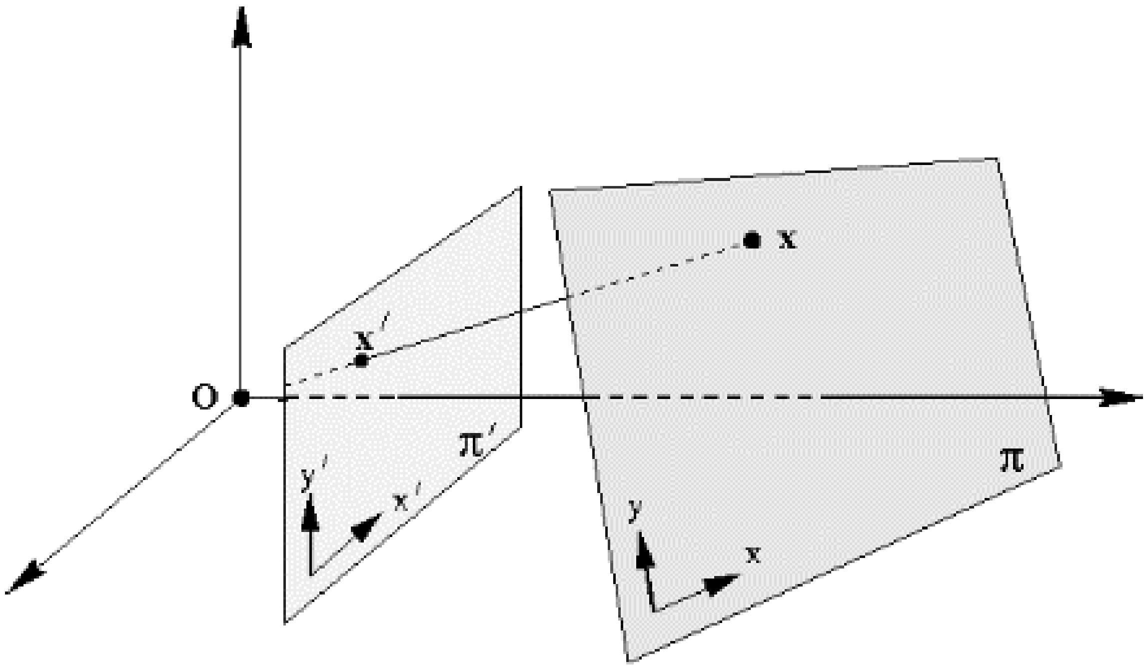
A planar projective transformation (homography) is a **linear** transformation on homogeneous 3-vectors, the transformation being represented by a **non-singular** 3x3 matrix H, as in



$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{H}\mathbf{x}$$

3-1: Projective Transformation (continue)



In planar perspective transformation, all rays that join a scene point **x** with its corresponding image point **x’** must pass through the same point that is referred to as the center of projection or the focal center. Obviously , an image formed with a planar perspective transformation will, in general, suffer from distortions including projective, affine, and similarity.

3-1: Property of a Homography

It always maps a straight line to a straight line.

$$\mathbf{x}' = \mathbf{H}\mathbf{x}$$

$$\mathbf{l}'^{\top}\mathbf{x}' = \mathbf{l}'^{\top}\mathbf{H}\mathbf{x} = (\mathbf{l}'^{\top}\mathbf{H})\mathbf{x} = \mathbf{l}\mathbf{x} = 0 \qquad \mathbf{l}'^{\top}\mathbf{H} = \mathbf{l} \qquad \mathbf{l}' = \mathbf{H}^{-\top}\mathbf{l}$$

Conics

$$\mathbf{x}^{\top}\mathbf{C}\,\mathbf{x} = 0$$

$$\begin{aligned}\mathbf{x}^{\top}\mathbf{C}\,\mathbf{x} &= (\mathbf{H}^{-1}\mathbf{x}')^{\top}\mathbf{C}\mathbf{H}^{-1}\mathbf{x}' \\ &= \mathbf{x}'^{\top}\boxed{\mathbf{H}^{-\top}\mathbf{C}\mathbf{H}^{-1}}\mathbf{x}' \\ &= \mathbf{x}'^{\top}\mathbf{C}'\mathbf{x}'\end{aligned}$$

3-2: Hierarchy of Transformation

Projective group

subgroup of

Affine group

subgroup of

Similarity group

subgroup of

Euclidean group

A similarity transform is an affine transform.

~~A projective transform is an affine transform.~~

An Euclidean transform is an affine transform.

An affine transform is an projective transform.

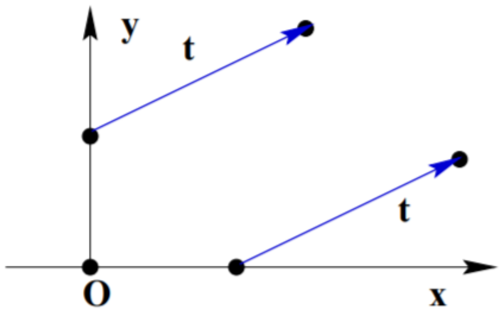
~~An similarity transform is an Euclidean transform.~~

Rigid body motions

1. *Translation* — 2 dof in 2D

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{x} + \mathbf{t}$$

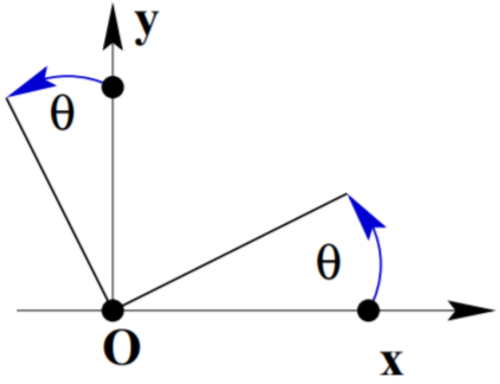


$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

2. *Rotation* — 1 dof in 2D

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{R}\mathbf{x}$$



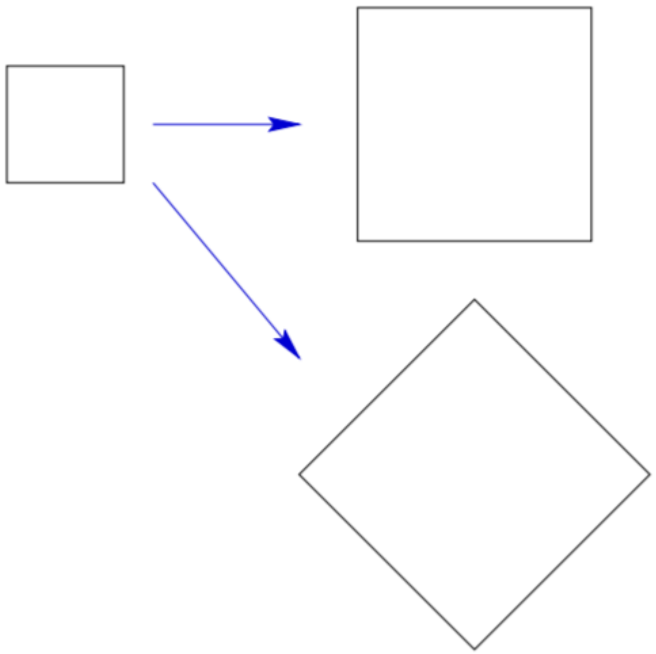
$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} cos\theta & -sin\theta & t_x \\ sin\theta & cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

3-4: Geometric Transformation (Similarity Transformation)

Preserve angles and ratios of lengths => Preserve "shape" (isotropic scale)



$$\mathbf{x}' = s\mathrm{R}\mathbf{x} + \mathbf{t}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} s & 0 & t_x \\ 0 & s & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

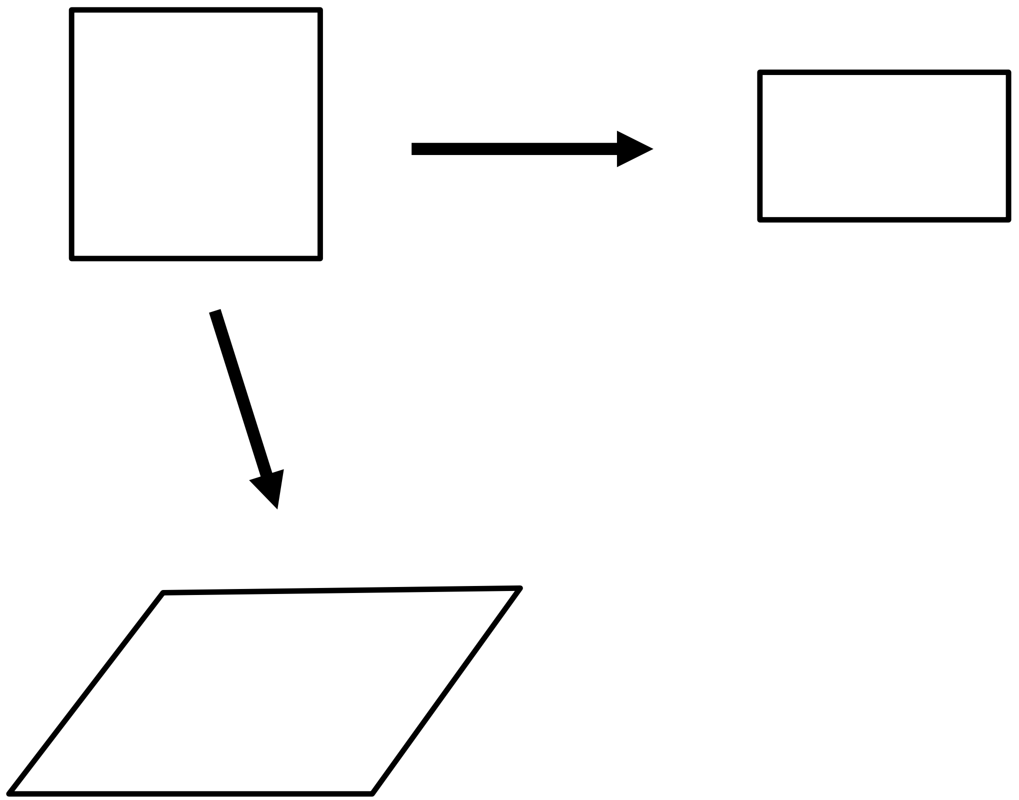
$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} s * cos\theta & -s * sin\theta & t_x \\ s * sin\theta & s * cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

3-3 & 3-4: Geometric Transformation (Affine Transformation)

Keep parallel lines parallel.

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$



3-4: Algebraic Understanding on Affine Transformation

Singular value decomposition (SVD)

$$A=UDV^{T}=(UV^{T})(VDV^{T})=R(\theta)R(-\phi)DR(\phi)$$

U, V: Orthonormal matrix
D: Diagonal matrix

Product of two orthonormal matrices become an orthonormal matrix

$$Q^TQ = I$$

$$R^TR = I,$$

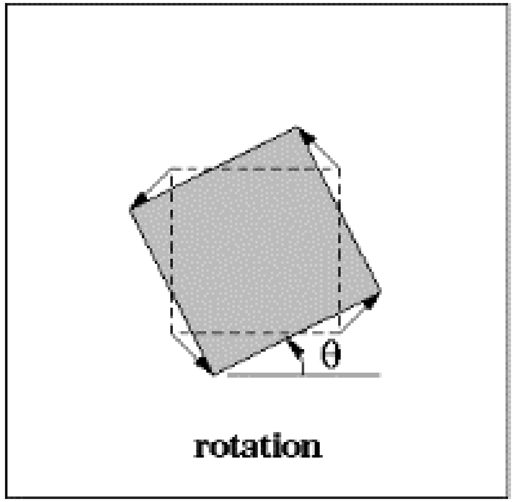
$$(QR)^T(QR) = R^T(Q^TQ)R = R^TR = I.$$

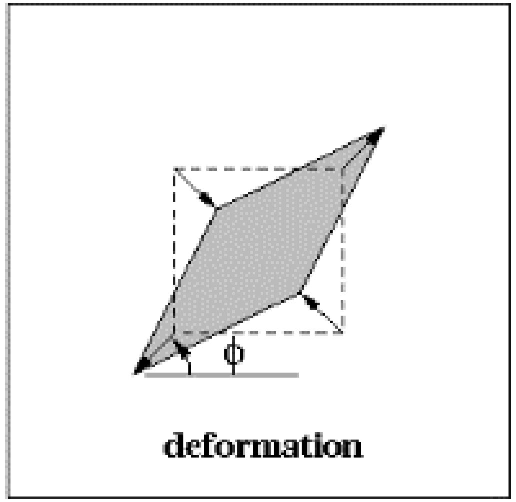
Every orthonormal matrix having determinant 1 acts as a rotation.

3-4: Algebraic Understanding on Affine Transformation

$$A = UDV^{T} = (UV^{T})(VDV^{T}) = R(\theta)R(\phi)\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}R(\phi)$$

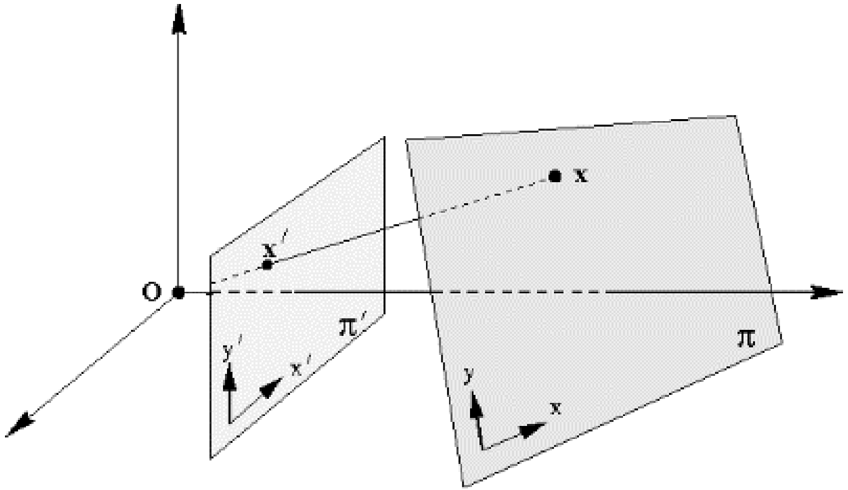
The affine matrix A is seen to be the concatenation of a rotation (by ϕ); a scaling by λ_1 and λ_2 respectively in the (rotated) x and y directions ; a rotation back (by $-\phi$); and finally another rotation (by θ). The only “new” geometry, compared to a similarity, is the non-isotropic scaling.

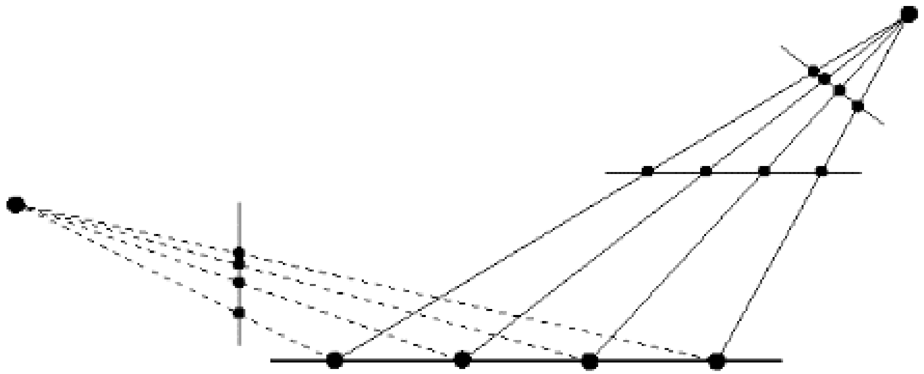




3-4: Geometric Transformation (Projective Transformation)

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$





$Cross\ ratio$

3-4: Decomposition of a Project Transformation

Similarity

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} s * cos\theta & -s * sin\theta & t_x \\ s * sin\theta & s * cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} \mathrm{sR} & \mathrm{t} \\ 0^\intercal & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Affine

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\mathrm{H}=\mathrm{H_S H_A H_P} = \begin{bmatrix} \mathrm{sR} & \mathrm{t} \\ 0^\intercal & 1 \end{bmatrix} \begin{bmatrix} \mathrm{K} & 0 \\ 0^\intercal & 1 \end{bmatrix} \begin{bmatrix} \mathrm{I} & 0 \\ \mathrm{v}^\intercal & v \end{bmatrix}$$

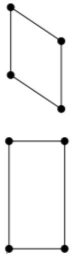
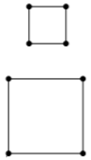
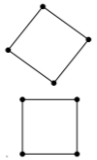
3-4: Decomposition of a Project Transformation (Continue)

$$H = H_S H_A H_P = \begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} K & 0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ v^T & v \end{bmatrix}$$

$$H^{-1} = {H_P}^{-1} {H_A}^{-1} {H_S}^{-1}$$

$$H = H_P H_A H_S = \begin{bmatrix} I & 0 \\ v^T & v \end{bmatrix} \begin{bmatrix} K & 0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$$

Summary

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

Lecture 4: Characterization of Distortions Caused by Projective Imaging and the Principle of Point/Line Duality

4-1: Projective Distortion

No distortion

Projective distortion

Lens distortion

4-1: Affine Transformation

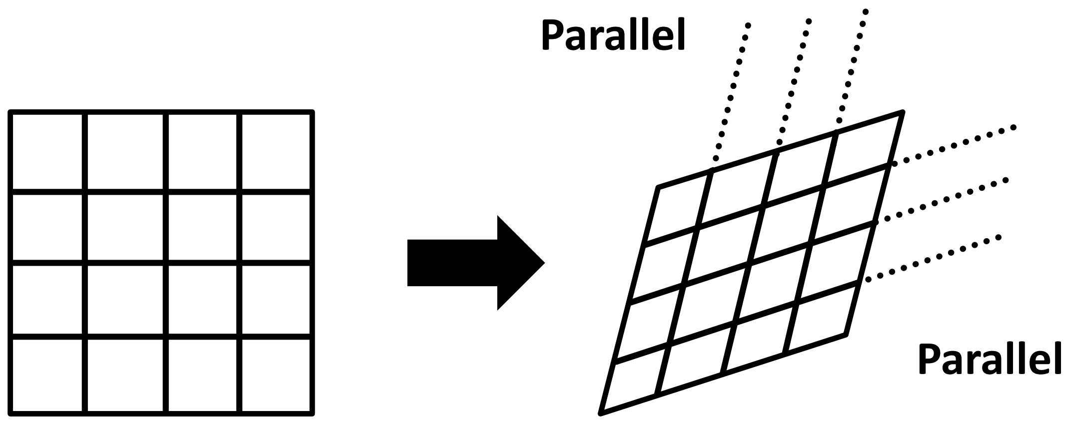
<u>A projective transformation H is affine if and only if l_∞ is mapped to l_∞.</u>

$$\mathbf{l}_\infty = \mathbf{H}^{-\mathsf{T}} \mathbf{l}_\infty$$

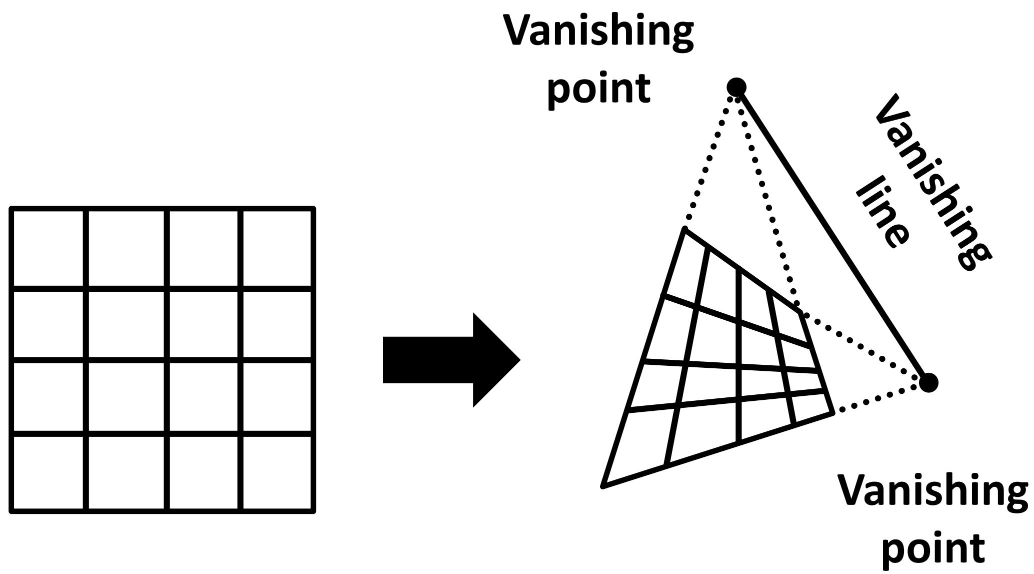
$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} \mathrm{A} & \mathrm{t} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \mathrm{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} \quad \mathrm{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\mathrm{H} = \begin{bmatrix} \mathrm{A} & \mathrm{t} \\ 0 & 1 \end{bmatrix} \qquad \mathbf{H}^{-\mathsf{T}} = \begin{bmatrix} \mathbf{A}^{-\mathsf{T}} & 0 \\ -\mathrm{t}\mathbf{A}^{-\mathsf{T}} & 1 \end{bmatrix}$$

4-2: Projective Transformation



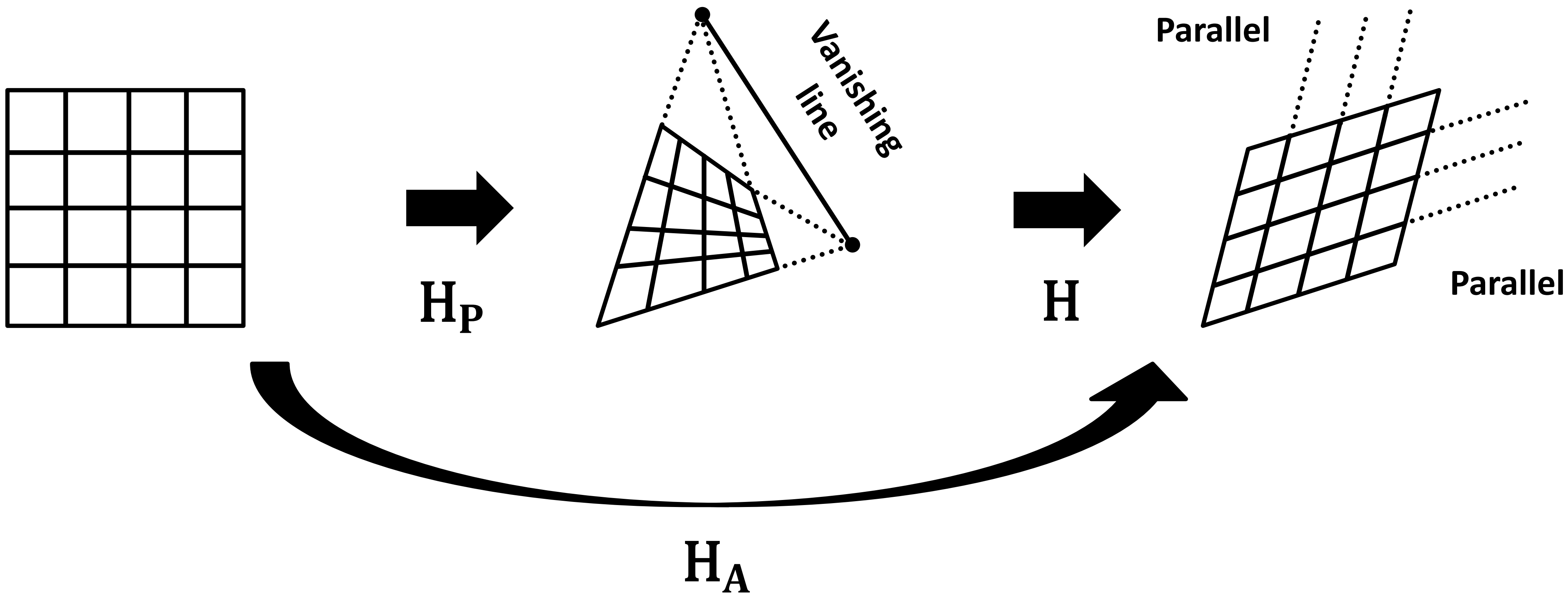
Affine Transformation



Projective Transformation

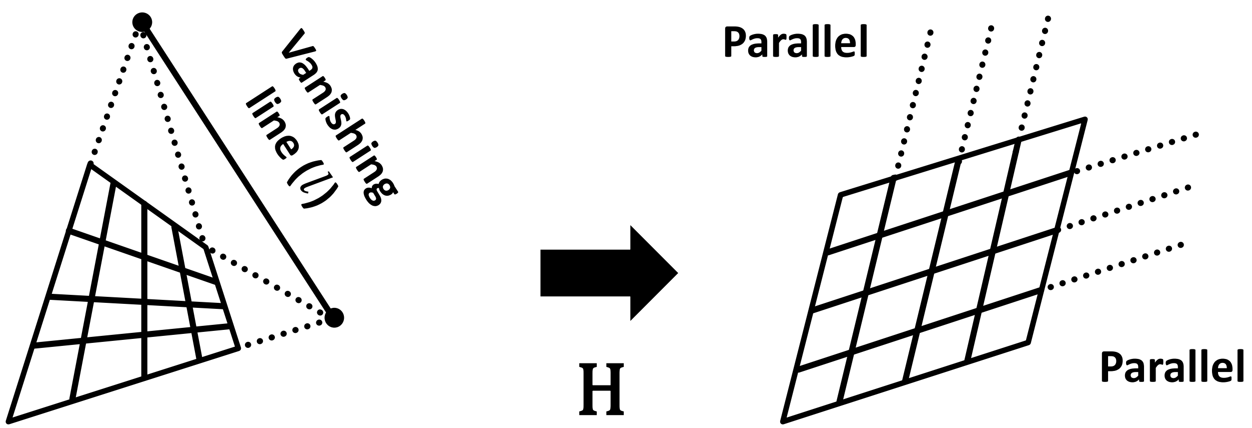
Whereas every affine transformation maps $\mathbf{l}_\infty$ to $\mathbf{l}_\infty$, a general projective transform maps $\mathbf{l}_\infty$ to a physical line that we call the vanishing line.

4-2: Affine Rectification



4-2: Affine Rectification (Continue)

$$l = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$$



$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$

Lecture 5: Characterization of Distortions Caused by Projective Imaging and the Principle of Point/Line Duality

4-3 & 4-4: Circular Points

A projective transformation H is affine if and only if l_∞ is mapped to l_∞.

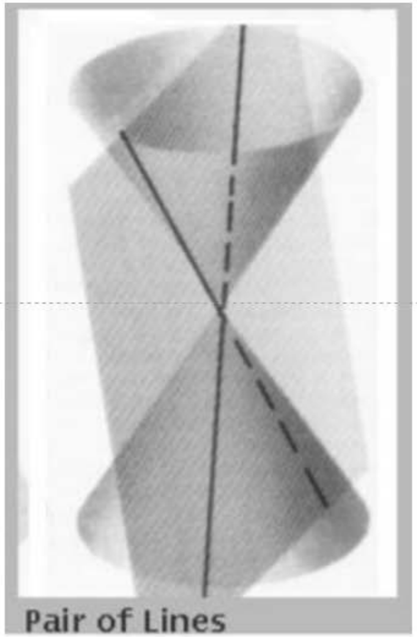
<u>A projective transformation H is similarity if and only if circular points are mapped to circular points.</u>

$$\mathbf{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad \mathbf{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$i: complex\ number$$

$$\begin{bmatrix} s * cos\theta & -s * sin\theta & t_x \\ s * sin\theta & s * cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} s * cos\theta - s * i * sin\theta \\ s * sin\theta + s * i * cos\theta \\ 0 \end{pmatrix}$$
$$\cong \begin{pmatrix} cos\theta - i * sin\theta \\ sin\theta + i * cos\theta \\ 0 \end{pmatrix} \cong (cos\theta - i * sin\theta) \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \cong \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$

4-4: Dual Degenerate Conic



$$\mathrm{C} = \mathrm{l}\mathrm{m}^\top + \mathrm{m}\mathrm{l}^\top$$

Two lines constitute a degenerate conic.

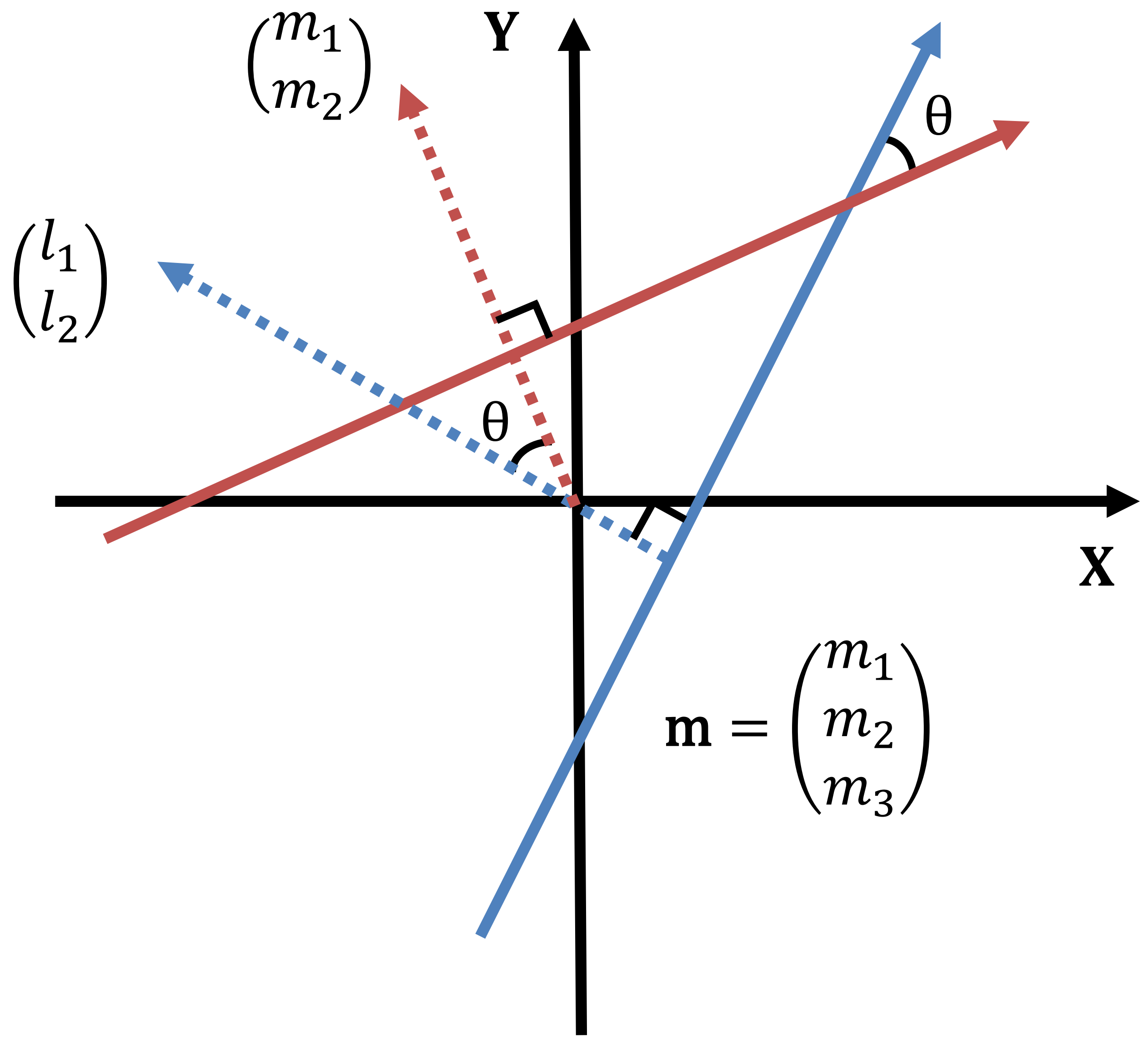
$$\mathrm{x}^\top \mathrm{C}\,\mathrm{x} = \mathrm{x}^\top(\mathrm{l}\mathrm{m}^\top + \mathrm{m}\mathrm{l}^\top)\mathrm{x} = 0$$

Two lines constitute a degenerate **line conic**.

$$\mathrm{l}^\top \mathrm{C}^*\,\mathrm{l} = \mathrm{l}^\top(\mathrm{x}\mathrm{y}^\top + \mathrm{x}\mathrm{y}^\top)\mathrm{x} = 0$$

$$\mathbf{C}_\infty^* = \begin{pmatrix}1\\ i\\ 0\end{pmatrix}\begin{pmatrix}1 & -i & 0\end{pmatrix} + \begin{pmatrix}1\\ -i\\ 0\end{pmatrix}\begin{pmatrix}1 & i & 0\end{pmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

5-1: Angles between Two Lines

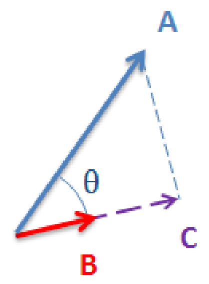


$$\mathbf{l} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$$

$$\mathbf{l}: l_1 x + l_2 y + l_3 = 0$$

$$\mathbf{m}: m_1 x + m_2 y + m_3 = 0$$

Dot product



$$A \cdot B = |A||B|\cos(\theta)$$

if the magnitude of B is 1, then...

$$C = A \cdot B = |A|\cos(\theta)$$

$$\boldsymbol{cos\theta} = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

5-1: Representation of an angle using Dual Degenerate Conic

$\mathbf{l}: l_1 x + l_2 y + l_3 = 0$

$\mathbf{m}: m_1 x + m_2 y + m_3 = 0$

$$\mathbf{l} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \quad \mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

$$cos\theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

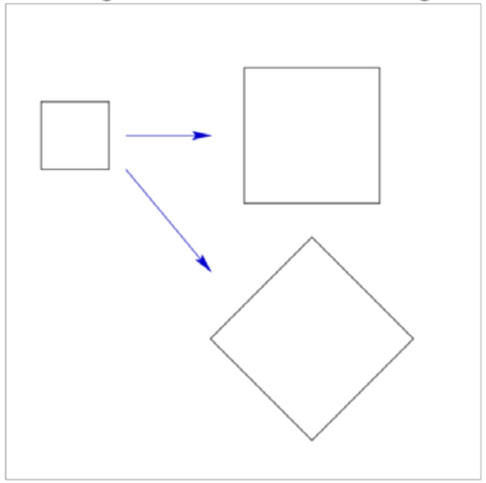
$$cos\theta = \frac{l^T \mathrm{C}_\infty^* m}{\sqrt{(l^T \mathrm{C}_\infty^* l)(m^T \mathrm{C}_\infty^* m)}}$$

$$\mathbf{C}_\infty^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5-2: Similarity Rectification

<u>A projective transformation H is similarity if and only if circular points are mapped to circular points.</u>

Preserve angles and ratios of lengths => Preserve "shape" (isotropic scale)



$$x' = sRx + t$$

Similarity transformation

$$cos\theta = \frac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* l)(m^T C_\infty^* m)}}$$

$$l^T C_\infty^* m = 0$$

when lines l and m are orthogonal.

5-2: Metric Rectification using C^*_∞

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{H}\mathbf{x}$$

$$\mathbf{l}' = \mathbf{H}^{-\top}\mathbf{l}$$

$$\mathrm{H} = \mathrm{H_P H_A H_S} = \begin{bmatrix} \mathrm{I} & 0 \\ \mathrm{v}^\top & v \end{bmatrix} \begin{bmatrix} \mathrm{K} & 0 \\ 0^\top & 1 \end{bmatrix} \begin{bmatrix} \mathrm{sR} & \mathrm{t} \\ 0^\top & 1 \end{bmatrix}$$

$$\mathrm{l}^\top \mathrm{C}^*_\infty \mathrm{m}$$
$$= \mathrm{l}^\top \mathrm{H}^{-1} \mathrm{H} \mathrm{C}^*_\infty \mathrm{H}^\top \mathrm{H}^{-\top} \mathrm{m}$$
$$= (\mathrm{H}^{-\top}\mathrm{l})^\top \mathrm{H} \mathrm{C}^*_\infty \mathrm{H}^\top (\mathrm{H}^{-\top}\mathrm{m})$$
$$= \mathrm{l}'^\top \mathrm{C}^*_\infty{}' \mathrm{m}'$$

$$\mathrm{C}^*_\infty{}' = \mathrm{H} \mathrm{C}^*_\infty \mathrm{H}^\top$$
$$= (\mathrm{H}_P \mathrm{H}_A \mathrm{H}_S)\, \mathrm{C}^*_\infty\, (\mathrm{H}_P \mathrm{H}_A \mathrm{H}_S)^\top$$
$$= (\mathrm{H}_P \mathrm{H}_A \mathrm{H}_S)\, \mathrm{C}^*_\infty \mathrm{H_S}^\top \mathrm{H_A}^\top \mathrm{H_P}^\top$$
$$= \mathrm{H}_P \mathrm{H}_A (\mathrm{H}_S\, \mathrm{C}^*_\infty \mathrm{H_S}^\top) \mathrm{H_A}^\top \mathrm{H_P}^\top$$
$$= \mathrm{H}_P \mathrm{H}_A \mathrm{C}^*_\infty \mathrm{H_A}^\top \mathrm{H_P}^\top$$

5-2: Metric Rectification using C_∞^*

$C_\infty^{*}{}' = H C_\infty^* H^\top$

$= (H_P H_A H_S)\, C_\infty^*\, (H_P H_A H_S)^\top$

$= (H_P H_A H_S)\, C_\infty^* H_S{}^\top H_A{}^\top H_P{}^\top$

$= H_P H_A (H_S\, C_\infty^* H_S{}^\top) H_A{}^\top H_P{}^\top$

$= H_P H_A C_\infty^* H_A{}^\top H_P{}^\top$

$$H = H_P H_A H_S = \begin{bmatrix} I & 0 \\ v^\top & v \end{bmatrix} \begin{bmatrix} K & 0 \\ 0^\top & 1 \end{bmatrix} \begin{bmatrix} sR & t \\ 0^\top & 1 \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ v^\top & v \end{bmatrix} \begin{bmatrix} K & 0 \\ 0^\top & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} K^\top & 0 \\ 0^\top & 1 \end{bmatrix} \begin{bmatrix} I & v^\top \\ 0^\top & v \end{bmatrix}$$

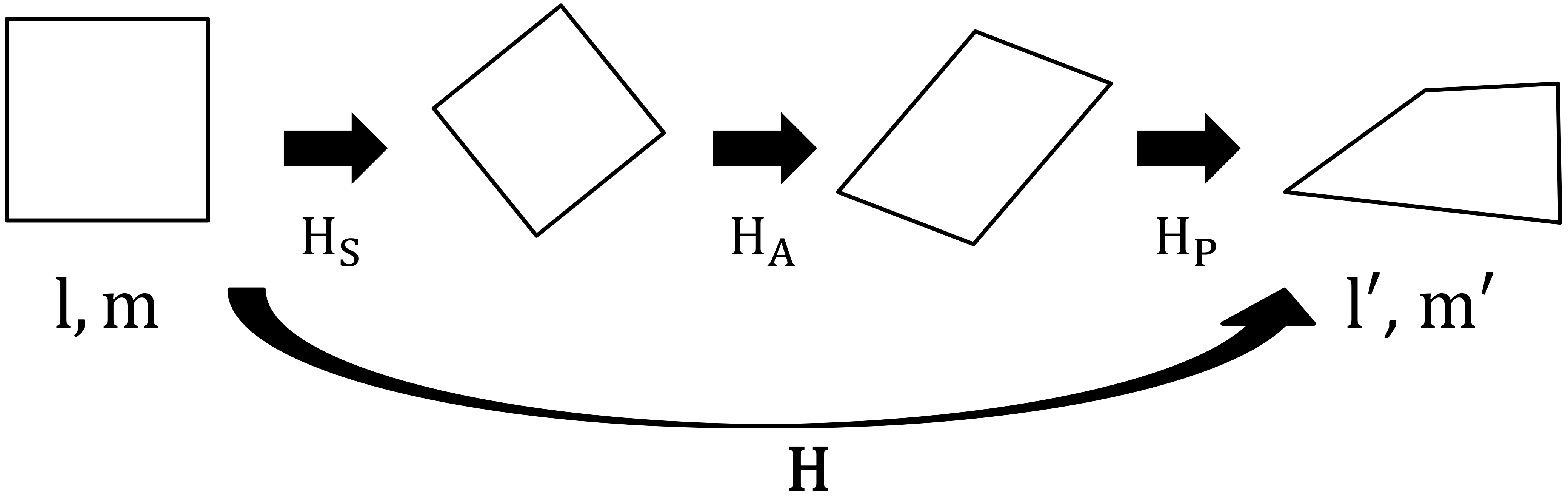
- **6-dof problem (5-dof up to scale)**
- **K: 4-dof**
- **V: 2-dof**

$$= \begin{bmatrix} KK^\top & KK^\top v \\ v^\top KK^\top & v^\top KK^\top v \end{bmatrix}$$

5-2: Metric Rectification using C^*_∞

$$l^\top C^*_\infty m = l'^\top C^*_\infty{}' m' = l'^\top \begin{bmatrix} KK^\top & KK^\top v \\ v^\top KK^\top & v^\top KK^\top v \end{bmatrix} m' = 0$$

$$H = H_P H_A H_S = \begin{bmatrix} I & 0 \\ v^\top & v \end{bmatrix} \begin{bmatrix} K & 0 \\ 0^\top & 1 \end{bmatrix} \begin{bmatrix} sR & t \\ 0^\top & 1 \end{bmatrix}$$



Metric Rectification I: Two-step Method (Example 2.26)

$$\mathrm{l}^{\intercal}\mathrm{C}_{\infty}^{*}\mathrm{m} = \mathrm{l}'^{\intercal}\mathrm{C}_{\infty}^{*}{}'\mathrm{m}' = \mathrm{l}'^{\intercal}\begin{bmatrix}\mathrm{KK}^{\intercal} & 0 \\ 0 & 0\end{bmatrix}\mathrm{m}' = 0$$

$$\mathrm{H} = \mathrm{H}_{\mathrm{A}}\mathrm{H}_{\mathrm{S}} = \begin{bmatrix}\mathrm{K} & 0 \\ 0^{\intercal} & 1\end{bmatrix}\begin{bmatrix}\mathrm{sR} & \mathrm{t} \\ 0^{\intercal} & 1\end{bmatrix}$$

- **3-dof problem (2-dof up to scale)**

$$\mathrm{l}'^{\intercal}\begin{bmatrix}\mathrm{KK}^{\intercal} & 0 \\ 0 & 0\end{bmatrix}\mathrm{m}' = \begin{bmatrix}l_1 & l_2 & l_3\end{bmatrix}\begin{bmatrix}s_1 & s_2 & 0 \\ s_2 & s_3 & 0 \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}m_1 \\ m_2 \\ m_3\end{bmatrix} = 0$$

$$\begin{bmatrix}l_1 m_1 & l_1 m_2 + l_2 m_1 & l_2 m_2\end{bmatrix}\begin{bmatrix}s_1 \\ s_2 \\ s_3\end{bmatrix} = 0$$

Metric Rectification II: One-step Method (Example 2.27)

$$\mathrm{l}^{\mathsf{T}} C_{\infty}^{*} \mathrm{m} = \mathrm{l}'^{\mathsf{T}} C_{\infty}^{*}{}' \mathrm{m}' = \mathrm{l}'^{\mathsf{T}} \begin{bmatrix} \mathrm{KK}^{\mathsf{T}} & \mathrm{KK}^{\mathsf{T}}\mathrm{v} \\ \mathrm{v}^{\mathsf{T}}\mathrm{KK}^{\mathsf{T}} & \mathrm{v}^{\mathsf{T}}\mathrm{KK}^{\mathsf{T}}\mathrm{v} \end{bmatrix} \mathrm{m}' = 0$$

- **6-dof problem (5-dof up to scale)**

$$\mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

$$\mathrm{l}'^{\mathsf{T}} \begin{bmatrix} \mathrm{KK}^{\mathsf{T}} & \mathrm{KK}^{\mathsf{T}}\mathrm{v} \\ \mathrm{v}^{\mathsf{T}}\mathrm{KK}^{\mathsf{T}} & \mathrm{v}^{\mathsf{T}}\mathrm{KK}^{\mathsf{T}}\mathrm{v} \end{bmatrix} \mathrm{m}' = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix} \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} l_1 m_1 & l_1 m_2 + l_2 m_1 & l_1 m_3 + l_3 m_1 & l_2 m_2 & l_2 m_3 + l_3 m_2 & l_3 m_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = 0$$

5-3: Point Correspondences for Estimating a Homography

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} \cong \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} \cong \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} \cong \begin{pmatrix} xh_{11} + yh_{12} + h_{13} \\ xh_{21} + yh_{22} + h_{23} \\ xh_{31} + yh_{32} + h_{33} \end{pmatrix}$$

$$\begin{bmatrix} x & y & 1 & 0 & 0 & 0 & -xx' & -x'y & -x' \\ 0 & 0 & 0 & x & y & 1 & -xy' & -y'y & -y' \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = 0$$

$$x' = \frac{xh_{11} + yh_{12} + h_{13}}{xh_{31} + yh_{32} + h_{33}}$$

$$y = \frac{xh_{21} + yh_{22} + h_{23}}{xh_{31} + yh_{32} + h_{33}}$$