

# Projective Geometry & Homography

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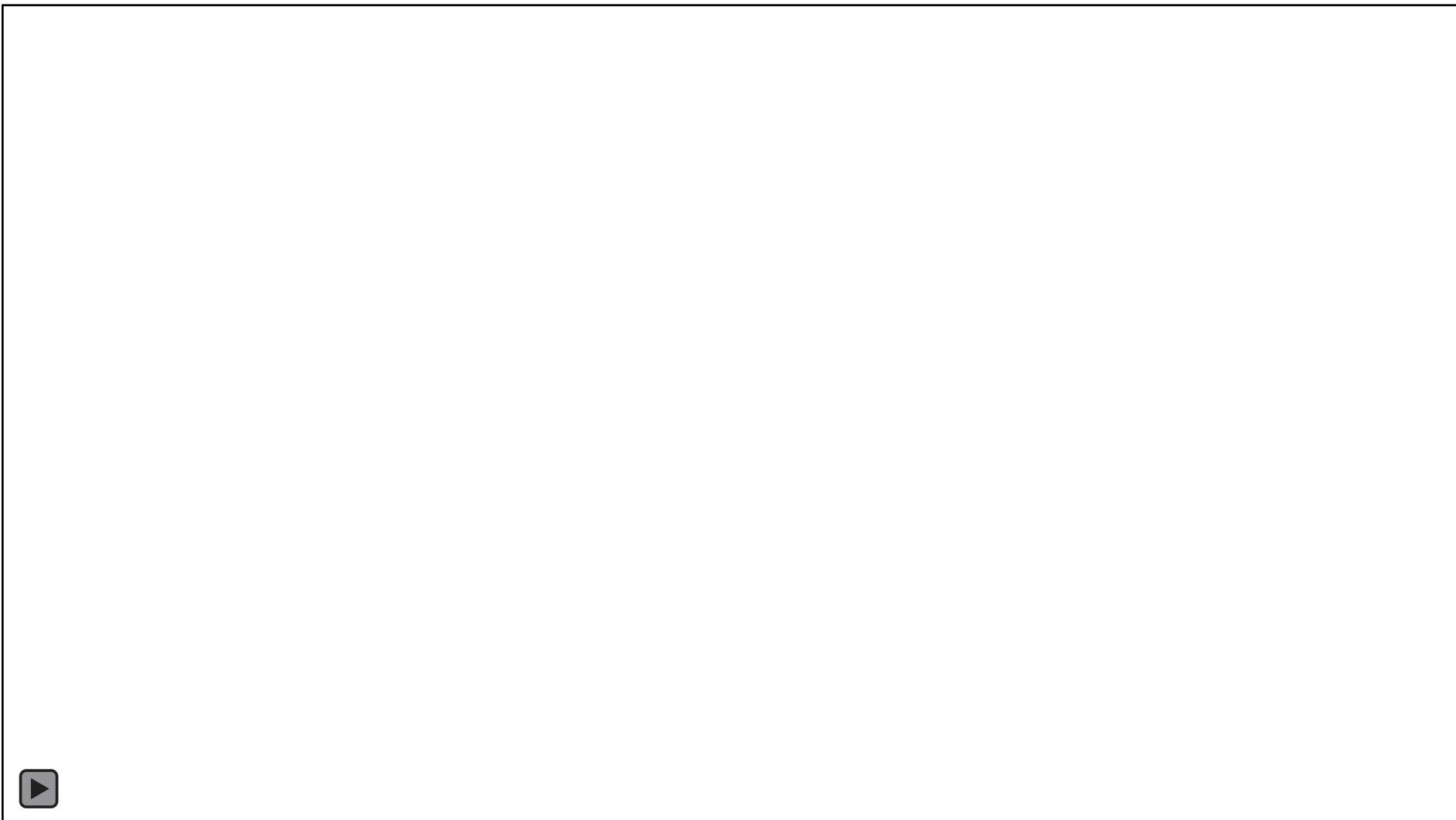
CIVE 497 – CIVE 700: Smart Structure Technology



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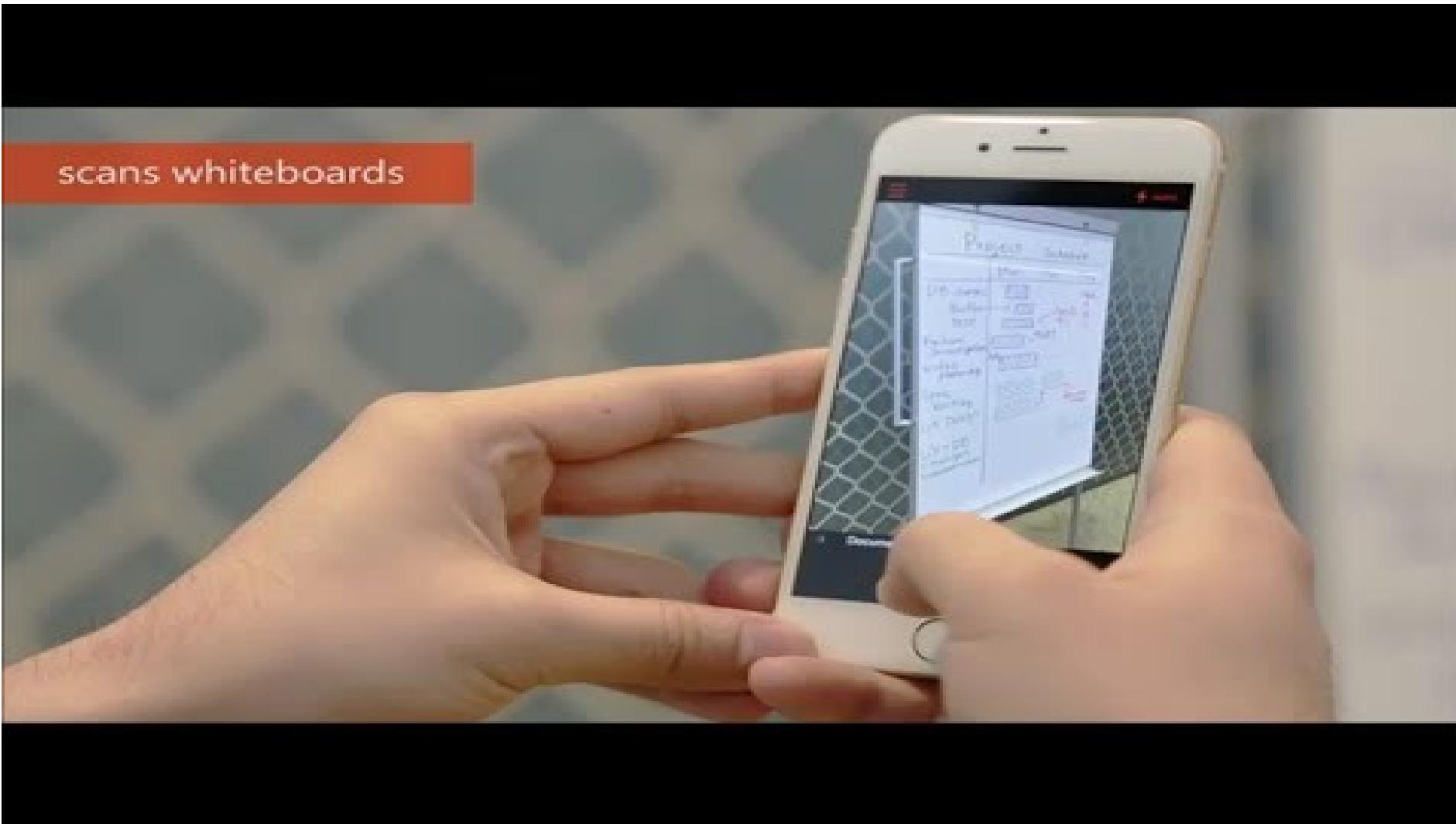
Last updated: 2018-01-31

# ON 3D-CameraMeasure App



Android: <https://play.google.com/store/apps/details?id=com.potatotree.on3dcamerameasure>

# Microsoft Lens



# Measurement Demo

2



Press the 'Capture' button to take a picture of the table **from any direction**.



# Measurement Demo (Continue)



# Projective Geometry I and II

We will study this topic using

ECE 661: Computer Vision (by Avinash Kak)

- [Lecture 2: World 2D: Representing and Manipulating Points, Lines And Conics Using Homogeneous Coordinates](#)
- [Lecture 3: World 2D: Projective Transformations and Transformation Groups](#)
- [Lecture 4: Characterization of Distortions Caused by Projective Imaging and the Principle of Point/Line Duality](#)
- [Lecture 5: Estimating a Plane-to-Plane Homography with Angle-to-Angle and Point-to-Point Correspondences](#)

Course website: <https://engineering.purdue.edu/kak/computervision/ECE661Folder/>

# Lecture 2: World 2D: Representing and Manipulating Points, Lines And Conics Using Homogeneous Coordinates

# 2-1: Point in the Homogeneous Coordinate

An arbitrary homogeneous vector representative of a point is of the form  $\mathbf{x} = (x_1, x_2, x_3)^T$ , representing the point  $(x_1/x_3, x_2/x_3)^T$  in  $\mathbb{R}^2$ .

**Example)**  $\mathbf{x}_1 = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$   $\mathbf{x}_2 = \begin{pmatrix} 10 \\ 6 \\ 2 \end{pmatrix}$   $\mathbf{x}_3 = \begin{pmatrix} 5k \\ 3k \\ k \end{pmatrix}, k \neq 0$  up to a scale

$\mathbf{x}_1$ ,  $\mathbf{x}_2$ , and  $\mathbf{x}_3$  indicate the same point of  $(5, 3)$  in  $\mathbb{R}^2$

$\mathbb{R}^n$ : n-dimension real coordinate system

## 2-1: Line in the Homogeneous Coordinate (HC)

$$ax + by + c = 0$$

$$\mathbf{l} = (a, b, c)^\top$$

Line equation in  $\mathbb{R}^2$

Line representation in HC

**Example)**  $\mathbf{l}_1 = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}$      $\mathbf{l}_2 = \begin{pmatrix} 9 \\ 12 \\ 9 \end{pmatrix}$      $\mathbf{l}_3 = \begin{pmatrix} 3k \\ 4k \\ 3k \end{pmatrix}, k \neq 0$

$\mathbf{x}_1, \mathbf{x}_2$ , and  $\mathbf{x}_3$  indicate the same line of  $3x + 4y + 3 = 0$  in  $\mathbb{R}^2$

## 2-1: Points and Lines in the Homogeneous Coordinate (HC)

$$ax + by + c = 0$$

$$(x_1, x_2, x_3) \cdot (a, b, c)^T = 0$$

The point  $\mathbf{x}$  lies on the line  $\mathbf{l}$  if and only if  $\mathbf{x}\mathbf{l}^T = \mathbf{l}\mathbf{x}^T = 0$ .

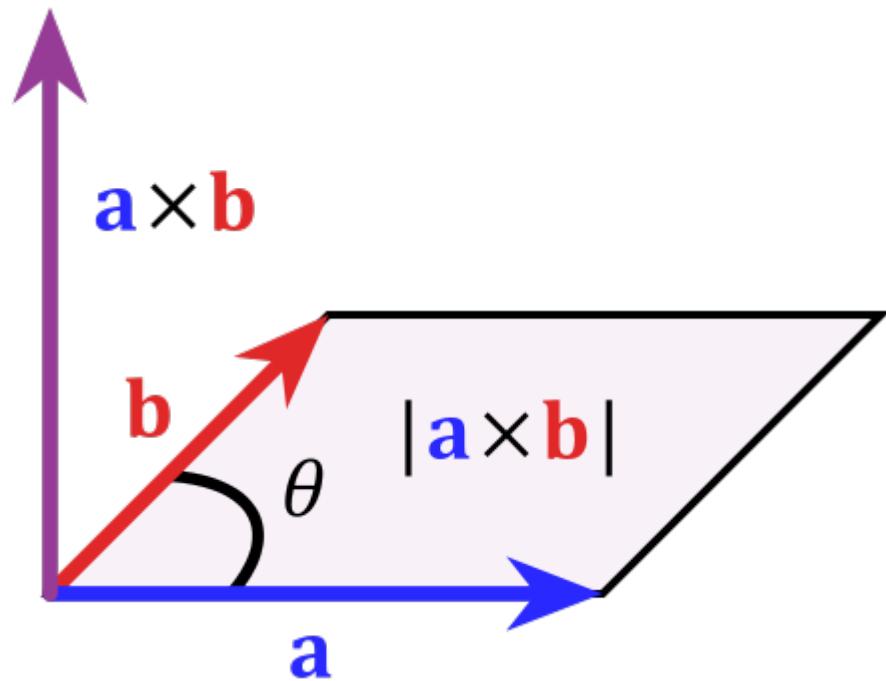
Given any two lines  $\mathbf{l}_1 = (a_1, b_1, c_1)$  and  $\mathbf{l}_2 = (a_2, b_2, c_2)$ , the point ( $\mathbf{x}$ ) of intersection of the two lines :

$$\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2$$

Given any two points  $\mathbf{x}_1 = (x_1, y_1, z_1)$  and  $\mathbf{x}_2 = (x_2, y_2, z_2)$ , the line ( $\mathbf{l}$ ) that passes through the two points :

$$\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$$

# Cross Product



$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

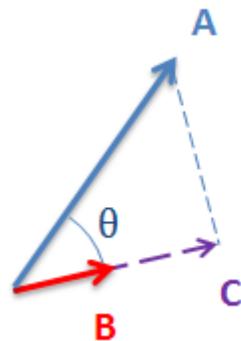
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= a_1b_1(\mathbf{i} \times \mathbf{i}) + a_1b_2(\mathbf{i} \times \mathbf{j}) + a_1b_3(\mathbf{i} \times \mathbf{k}) + \\ &\quad a_2b_1(\mathbf{j} \times \mathbf{i}) + a_2b_2(\mathbf{j} \times \mathbf{j}) + a_2b_3(\mathbf{j} \times \mathbf{k}) + \\ &\quad a_3b_1(\mathbf{k} \times \mathbf{i}) + a_3b_2(\mathbf{k} \times \mathbf{j}) + a_3b_3(\mathbf{k} \times \mathbf{k})\end{aligned}$$

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= -a_1b_1\mathbf{0} + a_1b_2\mathbf{k} - a_1b_3\mathbf{j} \\ &\quad - a_2b_1\mathbf{k} - a_2b_2\mathbf{0} + a_2b_3\mathbf{i} \\ &\quad + a_3b_1\mathbf{j} - a_3b_2\mathbf{i} - a_3b_3\mathbf{0} \\ &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\end{aligned}$$

## 2-2: Prove the Relationship using the Triple Scalar Identity

Dot product

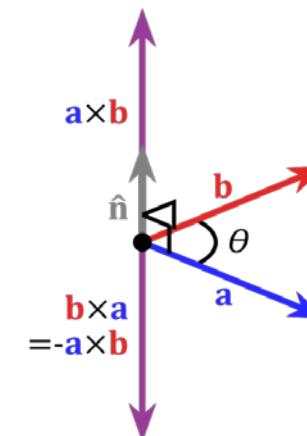


$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos(\theta)$$

if the magnitude of B is 1, then...

$$C = \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| \cos(\theta)$$

Cross product



$$\mathbf{l}_1 \mathbf{x} = \mathbf{l}_2 \mathbf{x} = \mathbf{0}$$

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$

$$\mathbf{l}_1 \cdot (\mathbf{l}_1 \times \mathbf{l}_2) = \mathbf{l}_2 \cdot (\mathbf{l}_1 \times \mathbf{l}_2) = \mathbf{0}$$

$$\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2$$

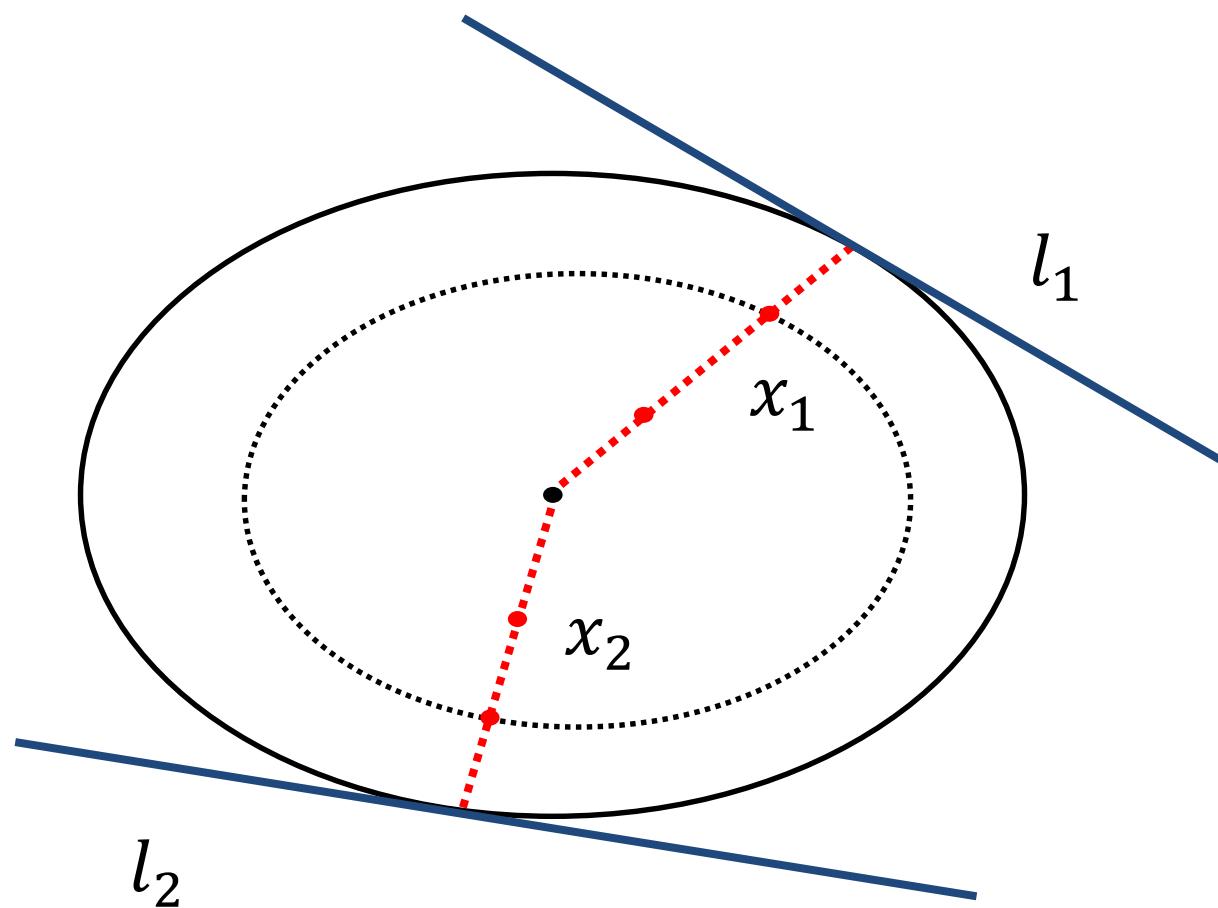
**Q0: Line passes through two (0,1) and (1,2)**

**Q1: Intersection point  $(p_x, p_y)$  of the two lines  $l_1$  and  $l_2$**

$l_1$  passes through two distinct points, (0,1) and (1,2).

$l_2$  passes through two distinct points, (4,0) and (5,3).

## 2-1 & 2-2: Visualization of Points and Lines in the Homogeneous Coordinate



## 2-2: Ideal Points

$$\mathbf{l}_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\mathbf{l}_2 = \begin{pmatrix} a \\ b \\ c' \end{pmatrix}$$

$$x = \mathbf{l}_1 \times \mathbf{l}_2 = \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}$$

### Ideal points

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$(u/w, v/w)$  in  $\mathbb{R}^2$

$(u/w, v/w)$  when  $w \rightarrow 0$

The point approach to infinity is along a specific direction and the direction being controlled by the value of  $u$  and  $v$ .

**Mathematically!!!**

**Valid the point and line relationship**

## 2-3: Vanishing Line

$$\mathbf{x}_1 = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \quad \mathbf{x}_2 = \begin{pmatrix} a' \\ b' \\ 0 \end{pmatrix}$$

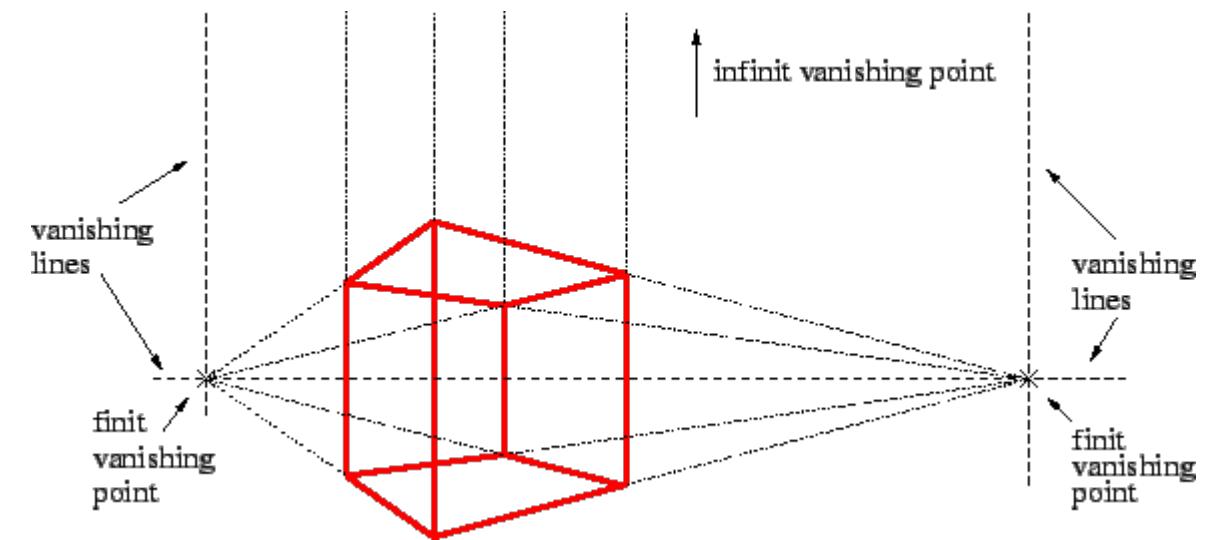
$$\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2 = \begin{pmatrix} 0 \\ 0 \\ ab' - a'b \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{l}_\infty = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

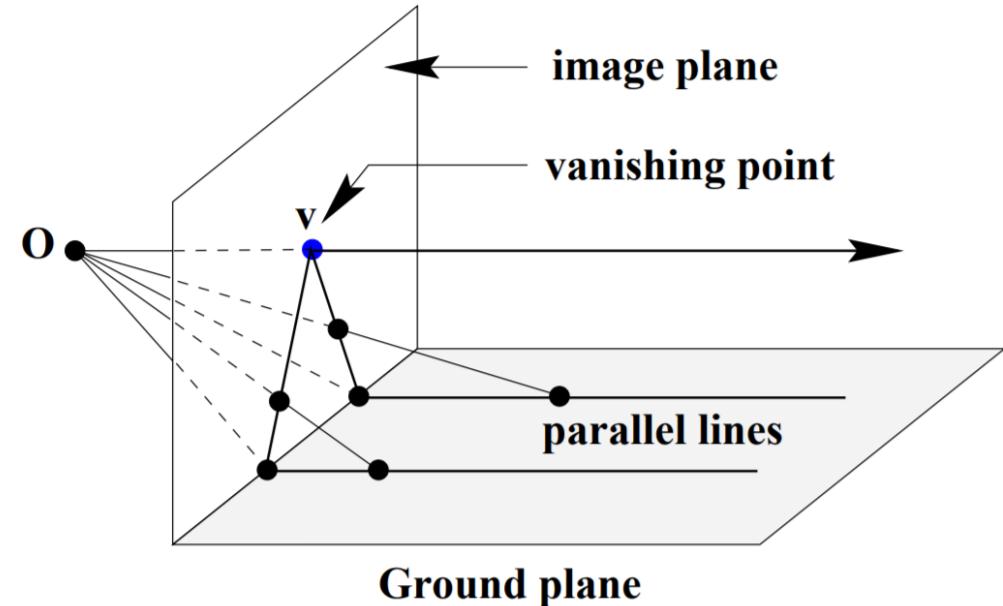
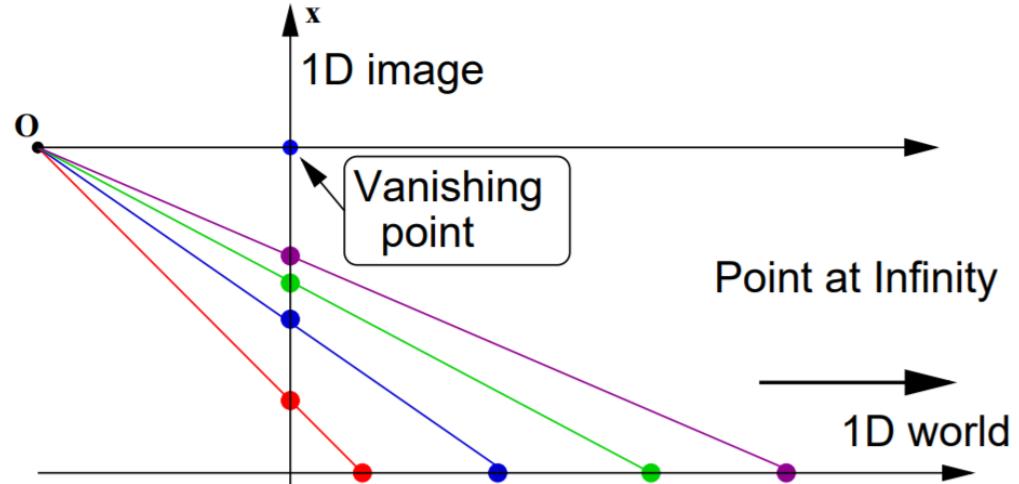
The camera image of the  $\mathbf{l}_\infty$  line is called the vanishing line.

Line at infinity

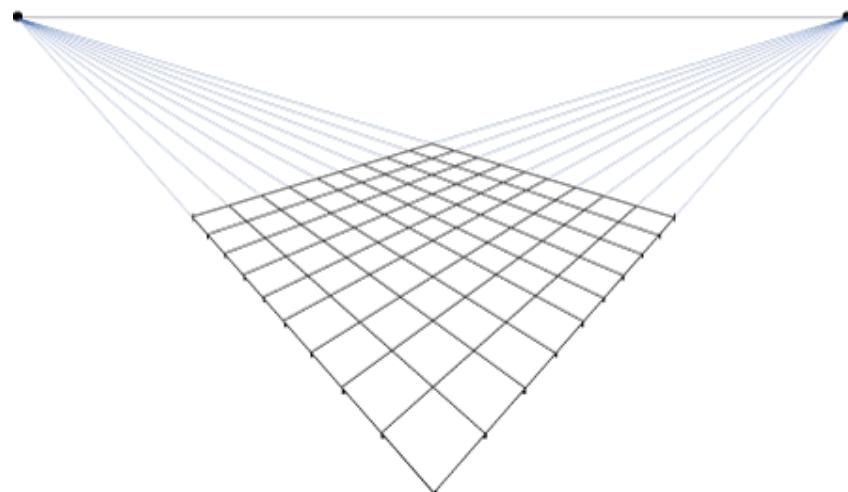
# Vanishing Point and Line



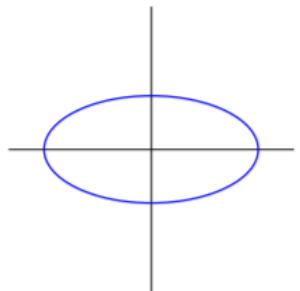
# Why All Parallel Lines Meet at One Point?



Vanishing line

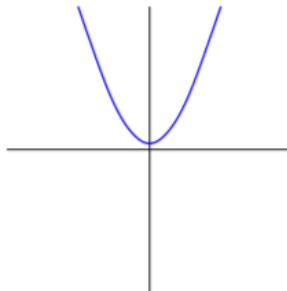


## 2-3: Conic



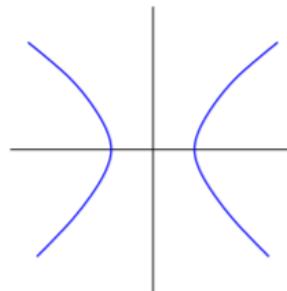
$$(x/2)^2 + y^2 = 1$$

ellipse



$$y = x^2$$

parabola

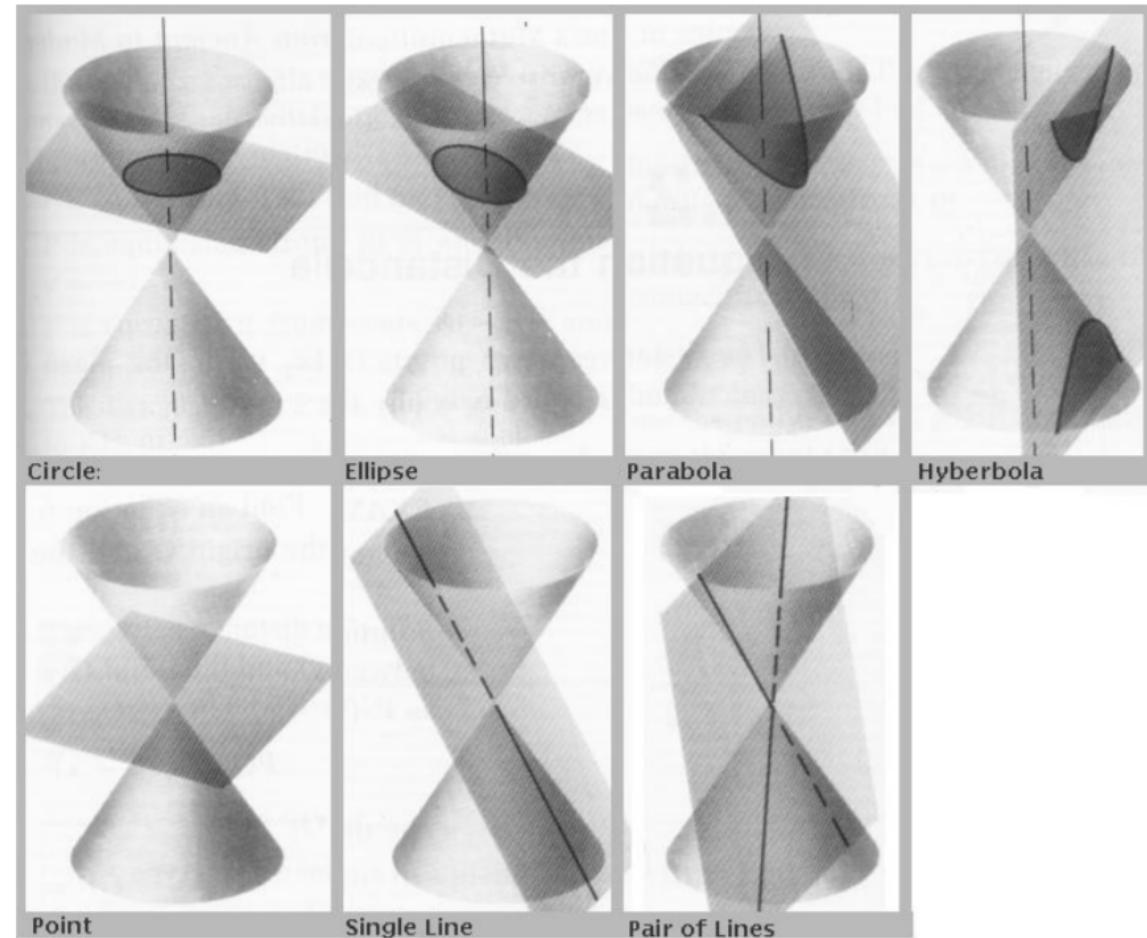


$$x^2 - y^2 = 1$$

hyperbola

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

A conic has five degrees of freedom in general



x and y can be represented by the conic equation

## 2-3: Conics in HC

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \quad (x, y) \text{ in } \mathbb{R}^2$$

$$a\left(\frac{x_1}{x_3}\right)^2 + b\frac{x_1}{x_3}\frac{x_2}{x_3} + c\left(\frac{x_2}{x_3}\right)^2 + d\left(\frac{x_1}{x_3}\right) + e\left(\frac{x_2}{x_3}\right) + f \\ (x_1, x_2, x_3) \text{ in } \mathbb{HC}$$

$$= ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2$$

$$= 0$$

## 2-3: Conic in HC (Continue)

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

$(x_1, x_2, x_3)$  in  $\mathbb{HC}$

$$[x_1 \ x_2 \ x_3] \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

$$\mathbf{x}^\top \mathbf{C} \mathbf{x} = 0$$

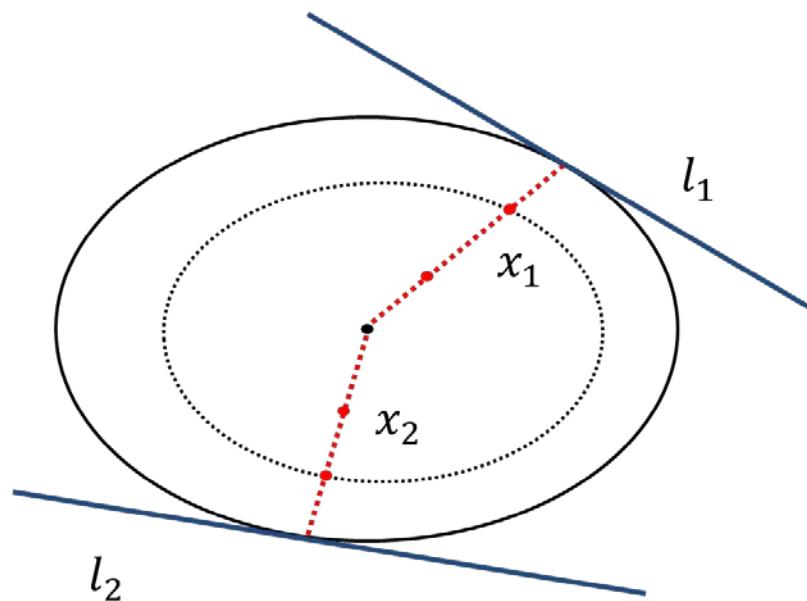
$\mathbf{C}$  is the HC representation of a conic

## 2-4: Conic Property

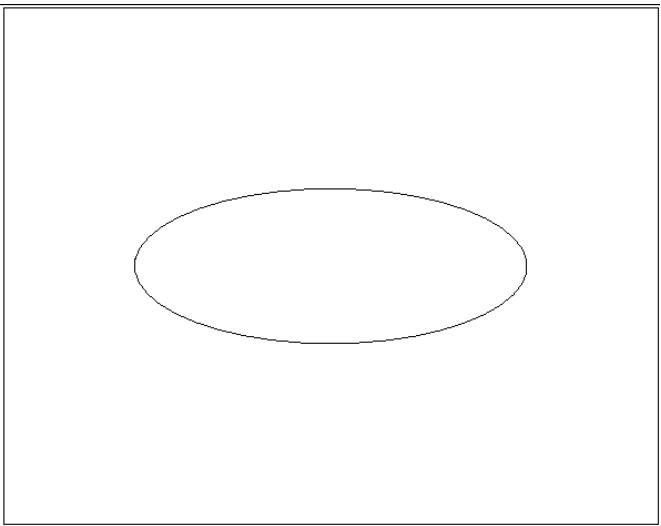
The HC representation of a conic gives us compact formulas for the **tangent lines** to a conic.

$$\mathbf{x}^T \mathbf{l} = \mathbf{x}^T \mathbf{C} \mathbf{x} = 0$$

$$\mathbf{l} = \mathbf{C} \mathbf{x}$$

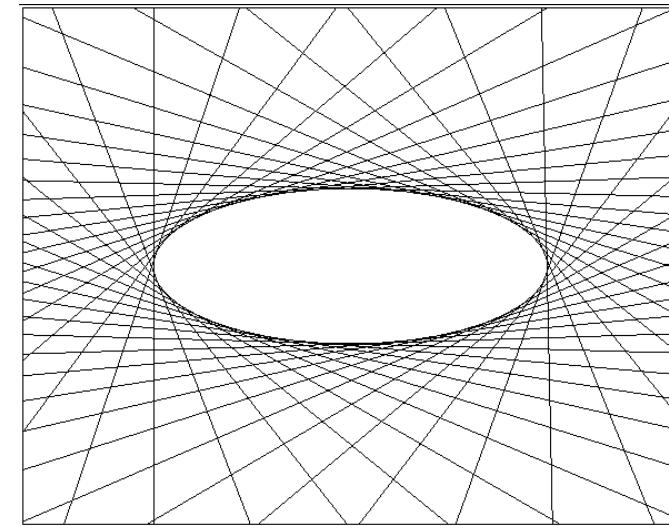


## 2-4: Dual Conic



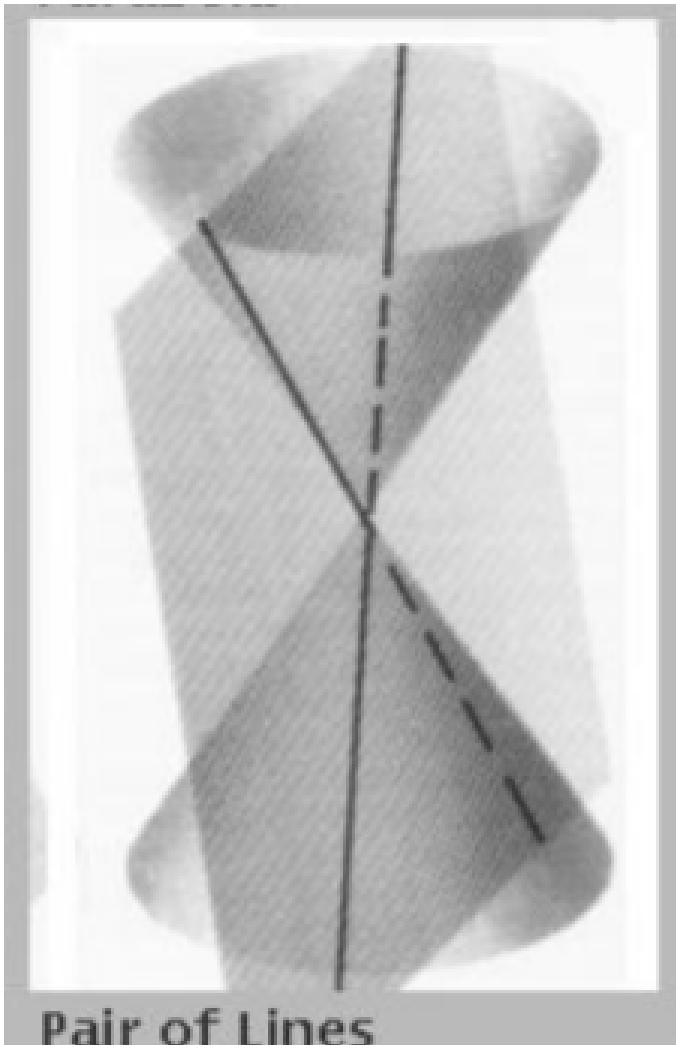
$$\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$$

$$\mathbf{x}^T \mathbf{l} = 0 \quad \mathbf{l} = \mathbf{C} \mathbf{x}$$



$$\begin{aligned}\mathbf{x}^T \mathbf{C} \mathbf{x} \\ &= \mathbf{l}^T \mathbf{C}^{-T} \mathbf{C} \mathbf{C}^{-1} \mathbf{l} \\ &= \mathbf{l}^T \mathbf{C}^{-T} \mathbf{l} \\ &= \mathbf{l}^T \mathbf{C}^{-1} \mathbf{l} = \mathbf{l}^T \mathbf{C}^* \mathbf{l}\end{aligned}$$

## 2-4: Degenerate Conic



Pair of Lines

$$C = lm^T + ml^T$$

Two lines constitute a degenerate conic.

$$x^T C x = x^T (lm^T + ml^T) x = 0$$

Two lines constitute a degenerate line conic.

$$l^T C^* l = l^T (xy^T + yx^T) l = 0$$

# Previous Lecture

Dual Conic

$$X^T C X = 0$$

$$\underline{X^T l = 0} \quad \underline{l = CX}$$

Dual Conic !!

$$\begin{aligned} X^T C X &= l^T C^{-T} C C^{-1} l \\ &= l^T C^{-1} l \\ &= l^T C^* l \end{aligned}$$

Degenerate Conic

If the matrix  $C$  is not of full rank

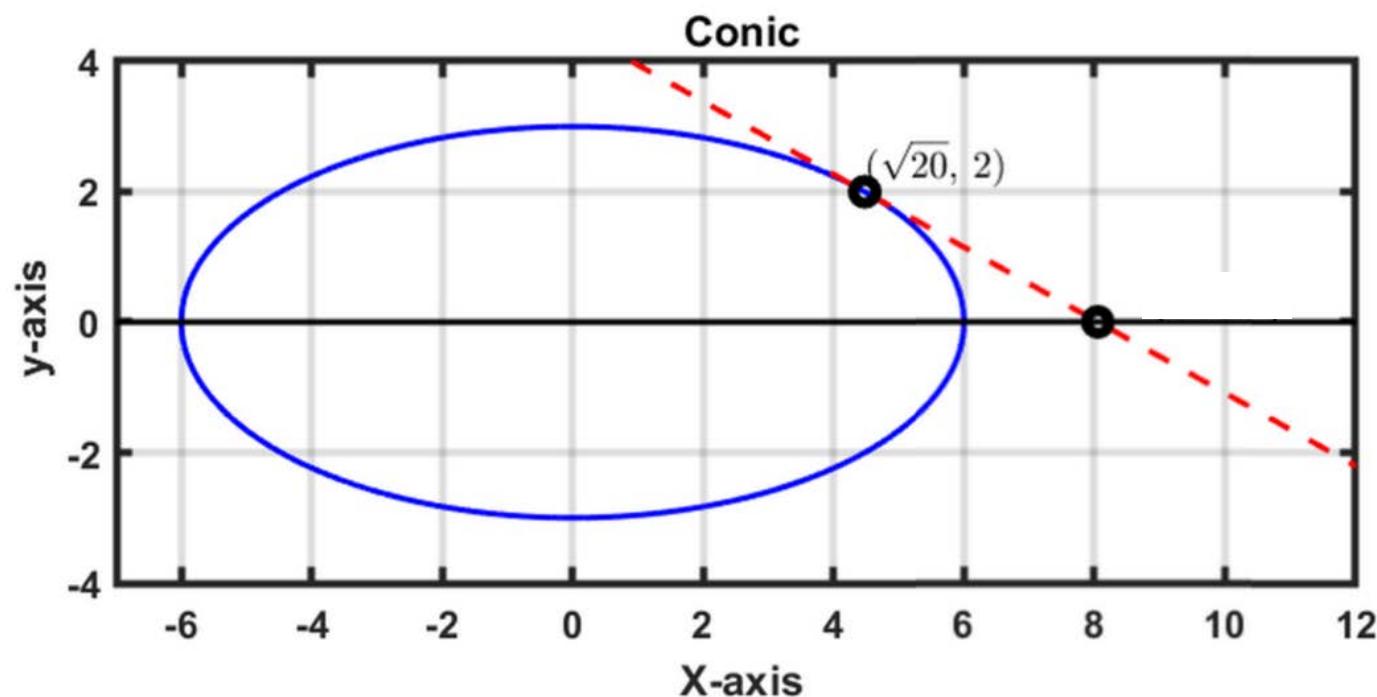
two lines (rank 2) one line (rank 1).

Degenerate Conic  $X^T C X = X^T (l m^T + m l^T) X = 0$

Degenerate line conic  $l^T C^* l = l^T (x y^T + y x^T) l = 0$

# Quiz

Q2: Given a conic  $\frac{x^2}{6^2} + \frac{y^2}{3^2} - 1 = 0$  (ellipse), compute a tangential line to this conic at  $(\sqrt{20}, 2)$  and its intersection point with x-axis,  $(p_x, p_y)$ .



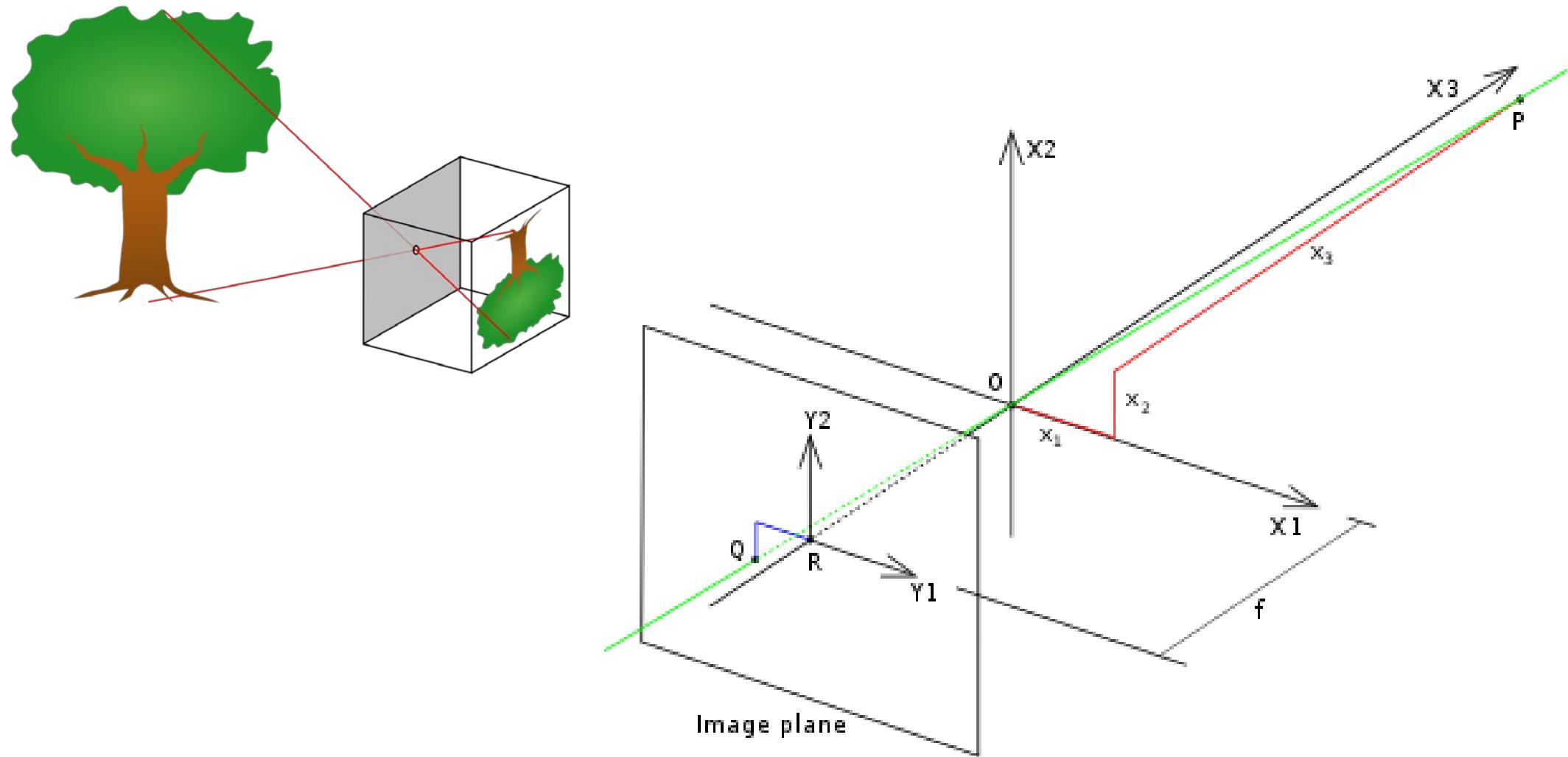
## Summary

1. An arbitrary homogeneous vector representative of a point is of the form  $\mathbf{x} = (x_1, x_2, x_3)^T$ , representing the point  $(x_1/x_3, x_2/x_3)^T$  in  $\mathbb{R}^2$ .
2. Line equation,  $ax + by + c = 0$ , in  $\mathbb{R}^2$  is represented as  $\mathbf{l} = (a, b, c)^\top$  in the homogeneous coordinate.
3. A conic,  $ax^2 + bxy + cy^2 + dx + ey + f = 0$ , in  $\mathbb{R}^2$  become a 3x3 matrix,  $\mathbf{C}$ , in the homogeneous coordinate,

$$\mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

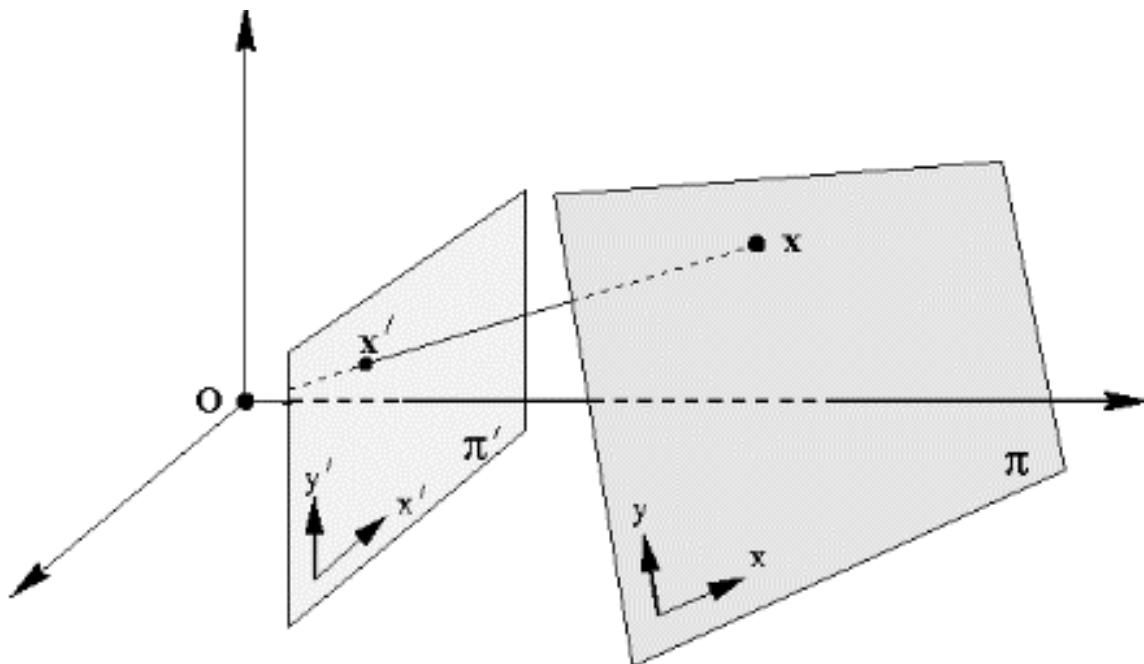
# Lecture 3: World 2D: Projective Transformations and Transformation Groups

# Pinhole Camera Model



### 3-1: Projective Transformation

A planar projective transformation (homography) is a linear transformation on homogeneous 3-vectors, the transformation being represented by a non-singular 3x3 matrix  $H$ , as in



$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{H}\mathbf{x}$$

## 3-1: Projective Transformation (continue)

In planar perspective transformation, all rays that join a scene point  $x$  with its corresponding image point  $x'$  must pass through the same point that is referred to as the center of projection or the focal center. Obviously, an image formed with a planar perspective transformation will, in general, suffer from distortions including projective, affine, and similarity.

In planar perspective transformation, all rays that join a scene point  $x$  with its corresponding image point  $x'$  must pass through the same point that is referred to as the center of projection or the focal center. Obviously, an image formed with a planar perspective transformation will, in general, suffer from distortions including projective, affine, and similarity.

### 3-1: Property of a Homography

It always maps a straight line to a straight line.

$$\mathbf{x}' = \mathbf{H}\mathbf{x}$$

$$\mathbf{l}'^\top \mathbf{x}' = \mathbf{l}'^\top \mathbf{H}\mathbf{x} = (\mathbf{l}'^\top \mathbf{H})\mathbf{x} = \mathbf{l}\mathbf{x} = 0$$

$$\mathbf{l}'^\top \mathbf{H} = \mathbf{l}$$

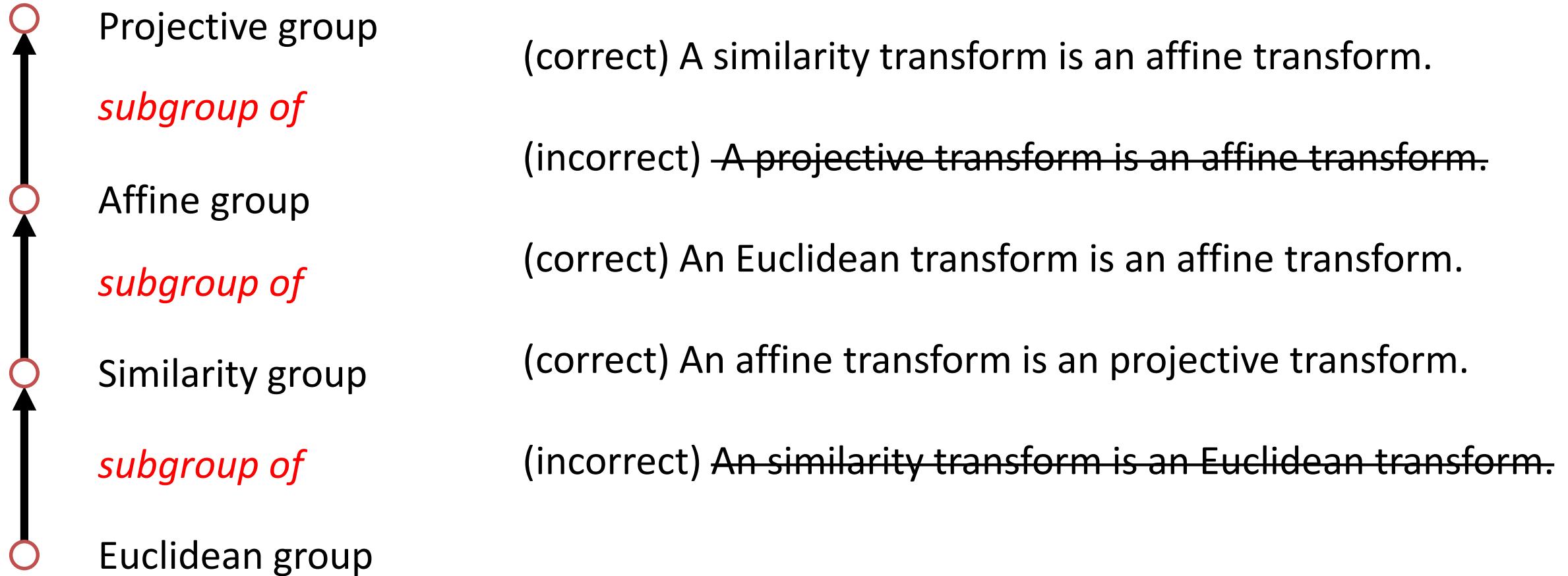
$$\mathbf{l}' = \mathbf{H}^{-\top} \mathbf{l}$$

Conics

$$\mathbf{x}^\top \mathbf{C} \mathbf{x} = 0$$

$$\begin{aligned}\mathbf{x}^\top \mathbf{C} \mathbf{x} &= (\mathbf{H}^{-1} \mathbf{x}')^\top \mathbf{C} \mathbf{H}^{-1} \mathbf{x}' \\ &= \mathbf{x}'^\top \boxed{\mathbf{H}^{-\top} \mathbf{C} \mathbf{H}^{-1}} \mathbf{x}' \\ &= \mathbf{x}'^\top \mathbf{C}' \mathbf{x}'\end{aligned}$$

## 3-2: Hierarchy of Transformation



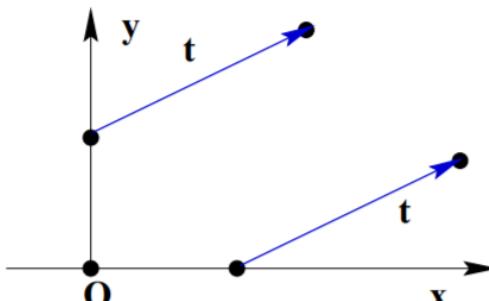
# 3-4: Geometric Transformation (Euclidean Transformation)

## Rigid body motions

1. Translation — 2 dof in 2D

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

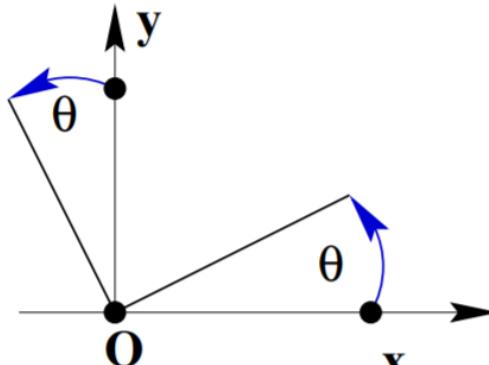
$$\mathbf{x}' = \mathbf{x} + \mathbf{t}$$



2. Rotation — 1 dof in 2D

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{R}\mathbf{x}$$



$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t}$$

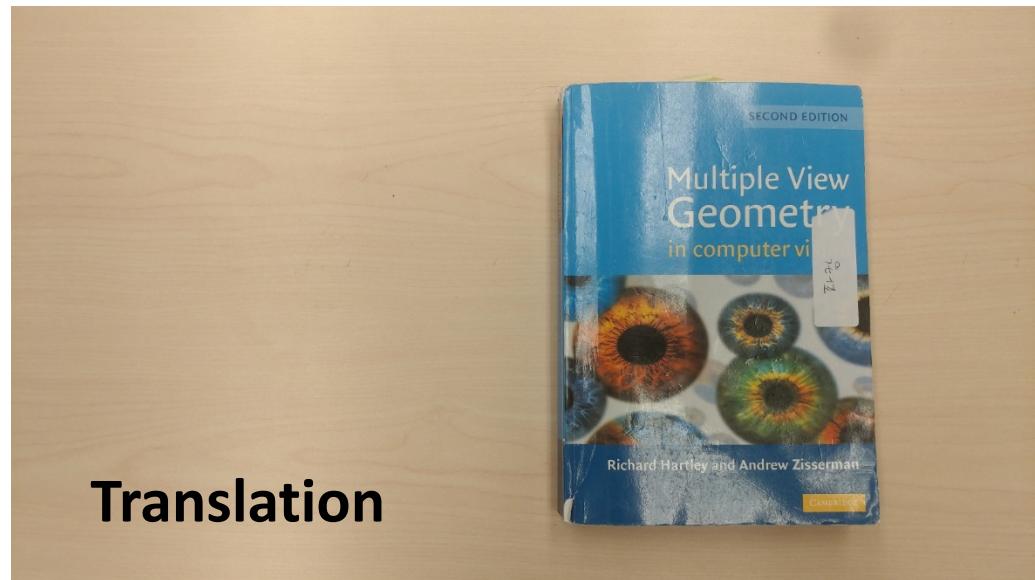
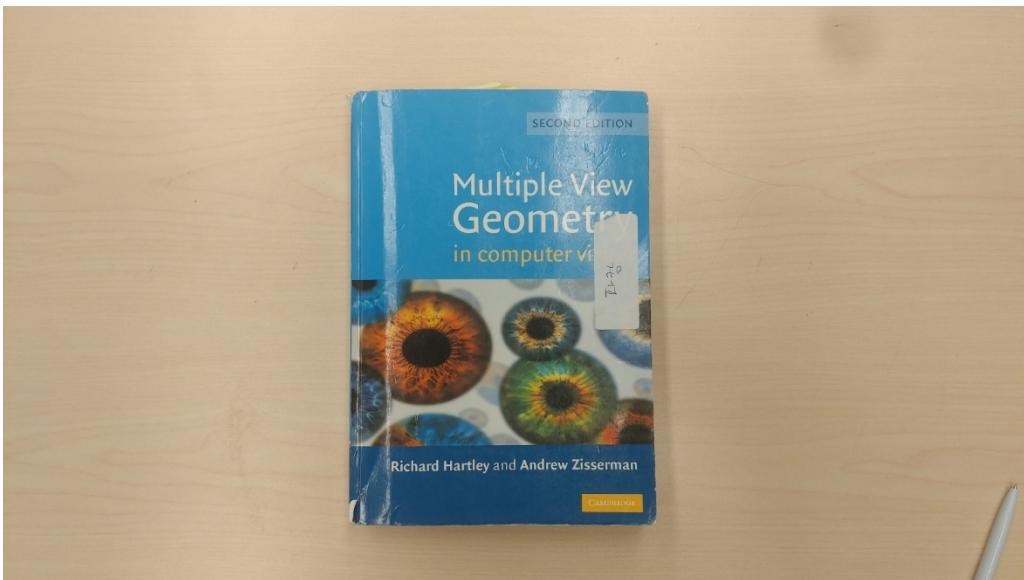
$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & t_x \\ \sin \theta & \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

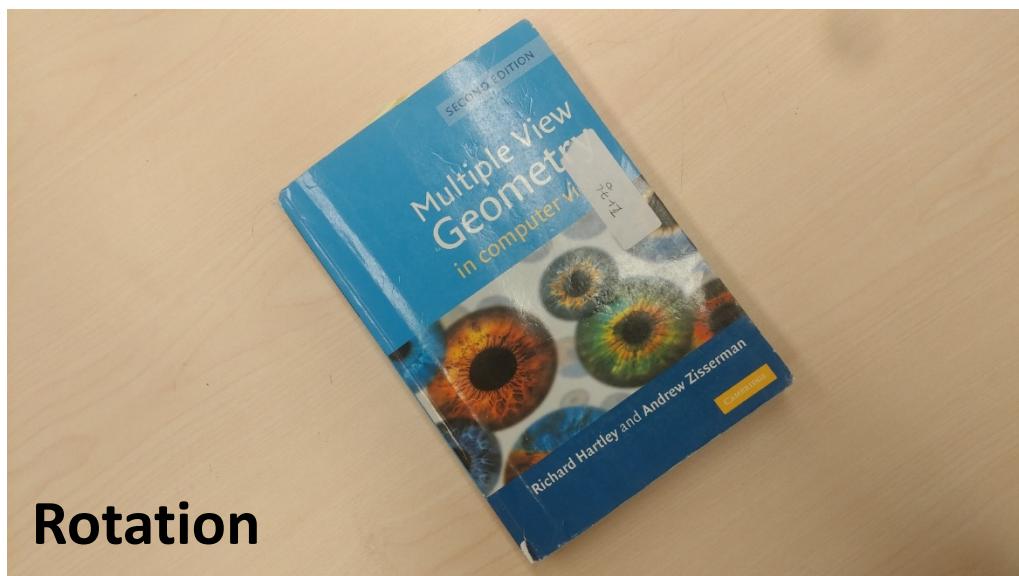
$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

See tutorials

# Example



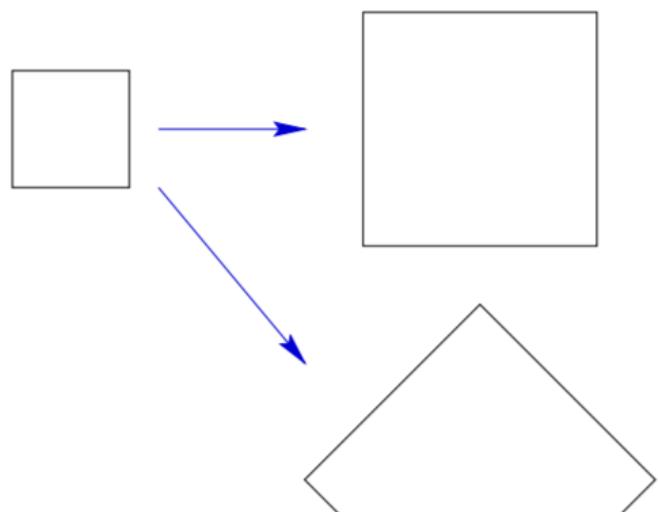
Translation



Rotation

## 3-4: Geometric Transformation (Similarity Transformation)

Preserve angles and ratios of lengths => Preserve “shape” (isotropic scale)



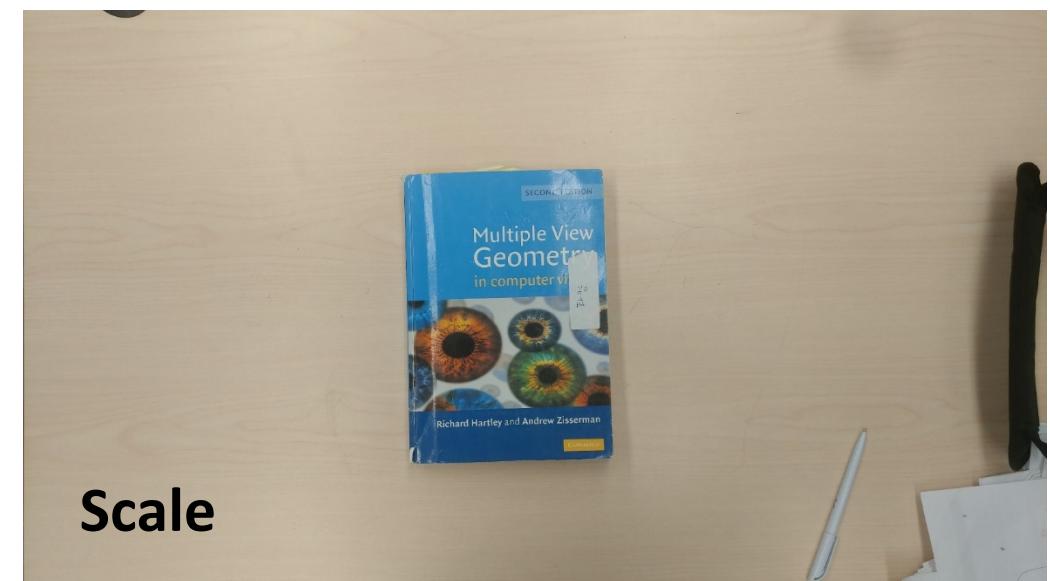
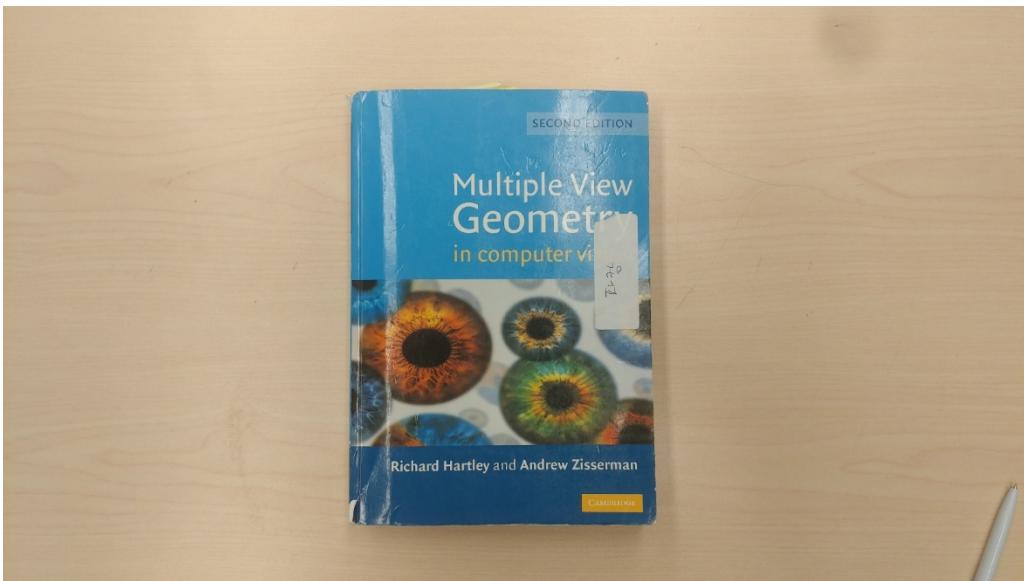
$$\mathbf{x}' = sR\mathbf{x} + \mathbf{t}$$

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} s & 0 & t_x \\ 0 & s & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} s * \cos\theta & -s * \sin\theta & t_x \\ s * \sin\theta & s * \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

See tutorials

# Example



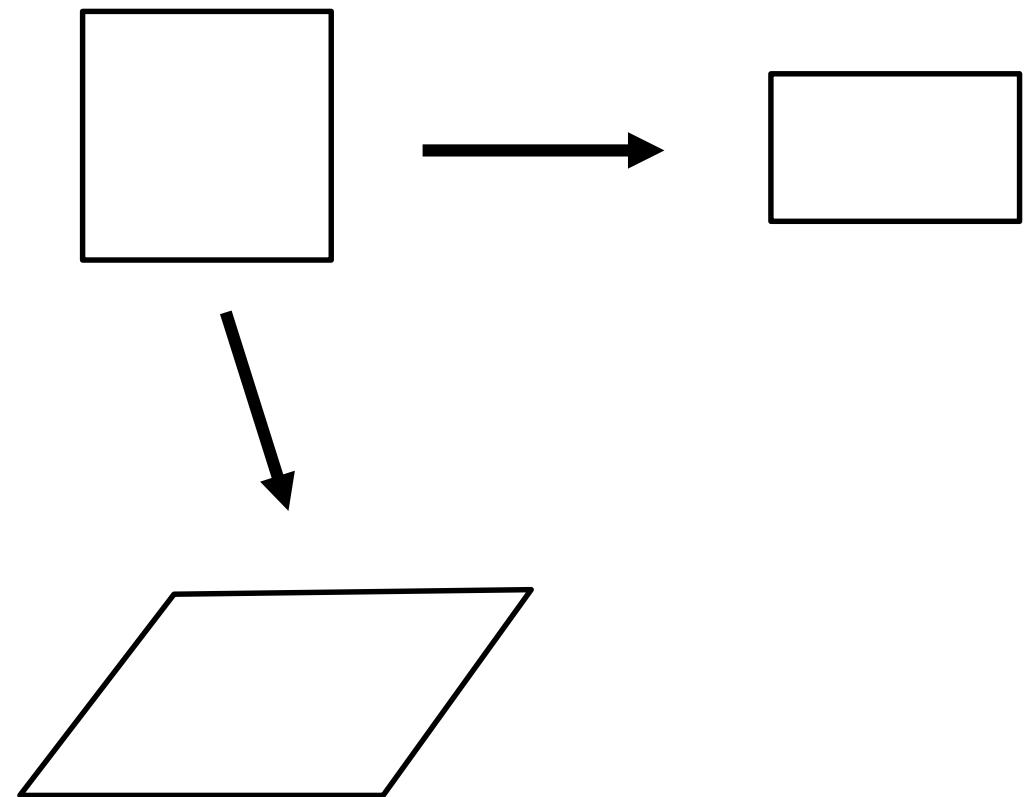
Scale

## 3-3 & 3-4: Geometric Transformation (Affine Transformation)

Keep parallel lines parallel.

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$



## 3-4: Algebraic Understanding on Affine Transformation

### Singular value decomposition (SVD)

$$A = UDV^T = (UV^T)(VDV^T) = R(\theta)R(-\phi)DR(\phi)$$

**U, V: Orthonormal matrix**

**D: Diagonal matrix**

Product of two orthonormal matrices become an orthonormal matrix

Every orthonormal matrix having determinant 1 acts as a rotation.

$$Q^T Q = I$$

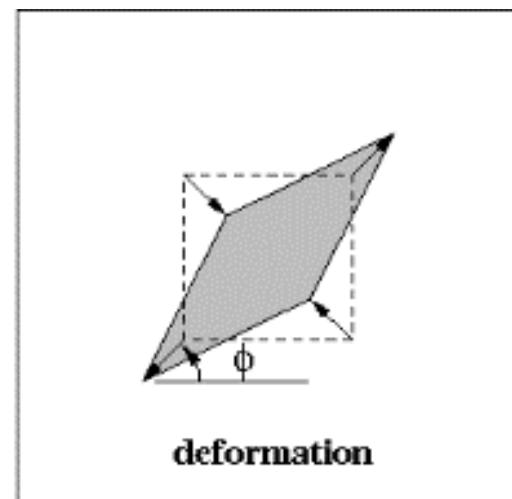
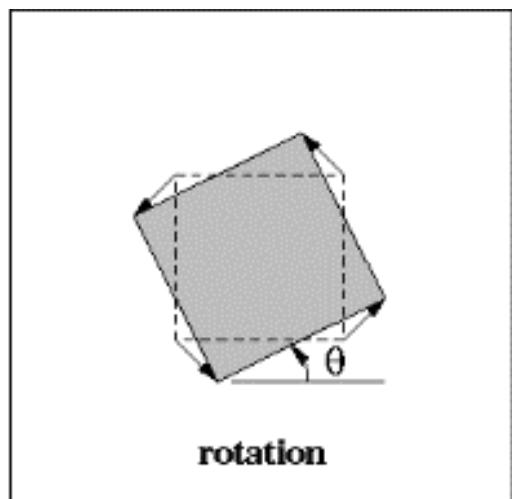
$$R^T R = I,$$

$$(QR)^T (QR) = R^T (Q^T Q) R = R^T R = I.$$

## 3-4: Algebraic Understanding on Affine Transformation

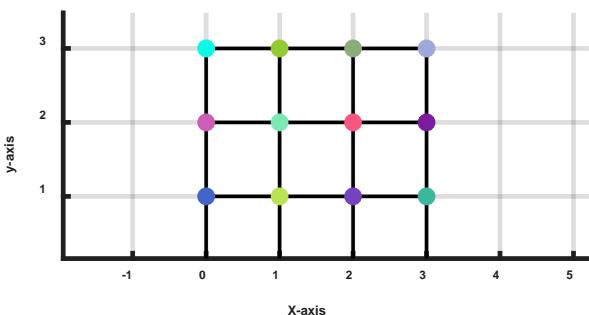
$$A = UDV^T = (UV^T)(VDV^T) = R(\theta)R(\phi) \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} R(\phi)$$

The affine matrix A is seen to be the concatenation of a rotation (by  $\phi$ ); a scaling by  $\lambda_1$  and  $\lambda_2$  respectively in the (rotated) x and y directions ; a rotation back (by  $-\phi$ ); and finally another rotation (by  $\theta$ ). The only “new” geometry, compared to a similarity, is the non-isotropic scaling.

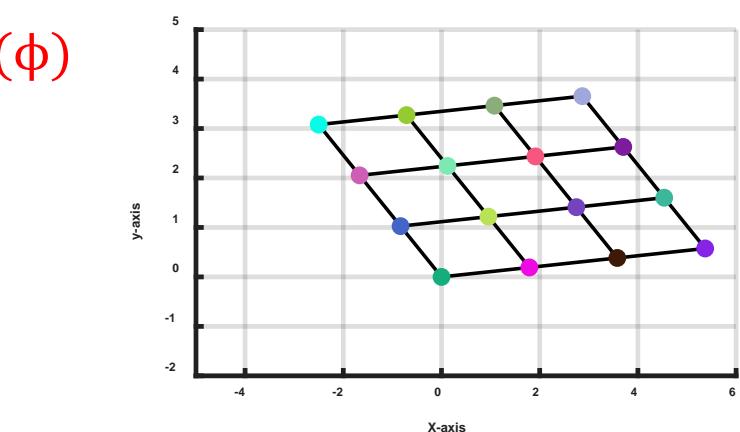
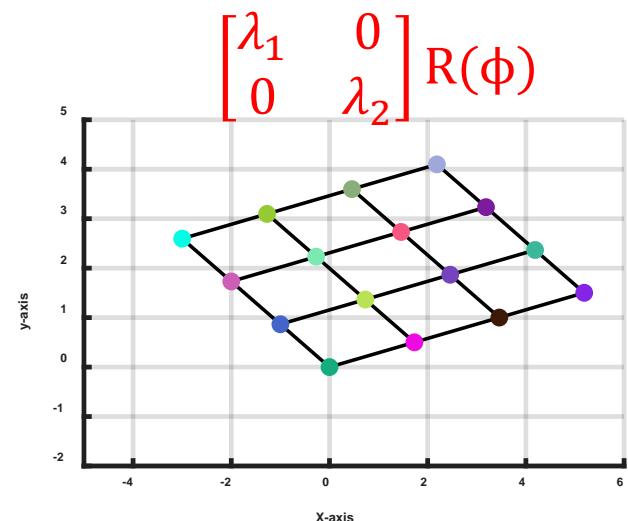
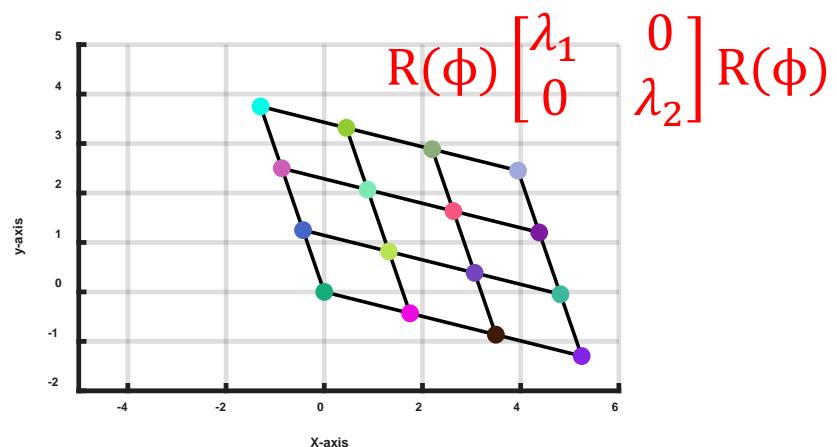
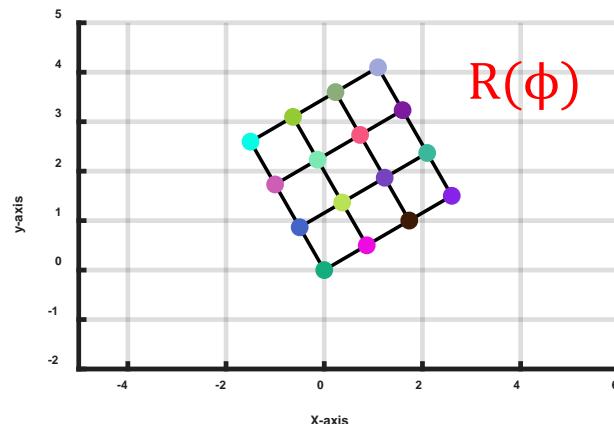


**See tutorials**

# Example: Decomposition of Affine Transformation



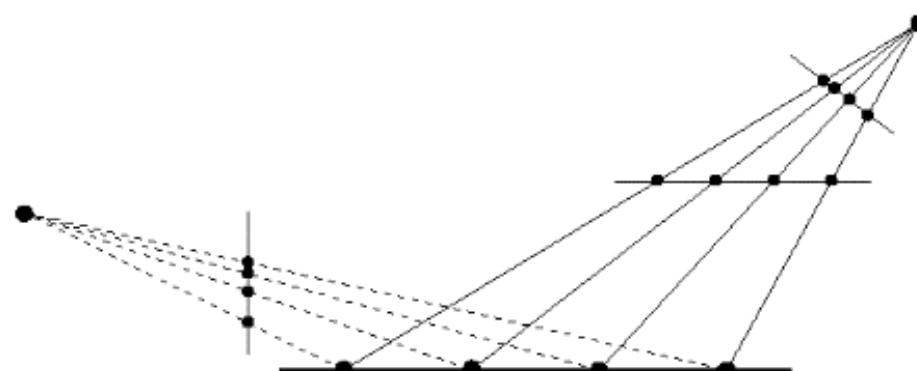
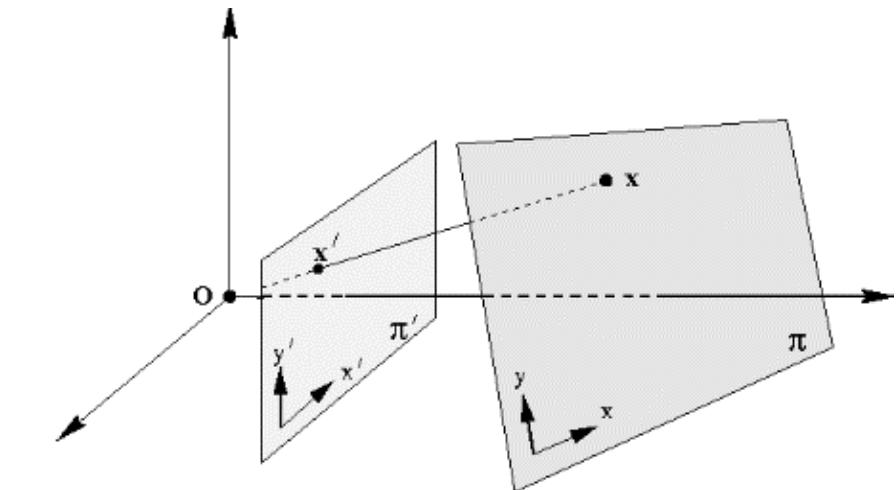
$$A = R(\theta)R(\phi) \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} R(\phi)$$



$R(\theta)R(\phi) \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} R(\phi)$

## 3-4: Geometric Transformation (Projective Transformation)

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$



Cross ratio

### 3-4: Decomposition of a Project Transformation

**Similarity**

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} s * \cos\theta & -s * \sin\theta & t_x \\ s * \sin\theta & s * \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = [sR \quad t] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

**Affine**

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$H = H_S H_A H_P = [sR \quad t] [K \quad 0] [I \quad 0] \\ [0^T \quad 1] [0^T \quad 1] [v^T \quad v]$$

## 3-4: Decomposition of a Project Transformation (Continue)

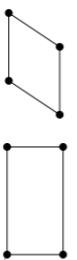
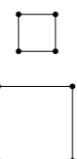
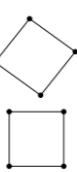
$$H = H_S H_A H_P = \begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} K & 0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ v^T & v \end{bmatrix}$$

$$H^{-1} = H_P^{-1} H_A^{-1} H_S^{-1}$$

Matrix inverse  
See tutorials

$$H = H_P H_A H_S = \begin{bmatrix} I & 0 \\ v^T & v \end{bmatrix} \begin{bmatrix} K & 0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$$

# Summary

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, <b>order of contact</b> : intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $l_\infty$ .
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, <b>I</b> , <b>J</b> (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

# Lecture 4: Characterization of Distortions Caused by Projective Imaging and the Principle of Point/Line Duality



# 4-1: Projective Distortion



No distortion



Projective distortion



Lens distortion

## 4-1: Affine Transformation

**A projective transformation  $H$  is affine if and only if  $l_\infty$  is mapped to  $l_\infty$ .**

$$l_\infty = H^{-T} l_\infty$$

It always maps a straight line to a straight line.

$$x' = Hx$$

$$l'^T x' = l'^T Hx = (l'^T H)x = l x = 0$$

$$l'^T H = l$$

$$l' = H^{-T} l$$

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

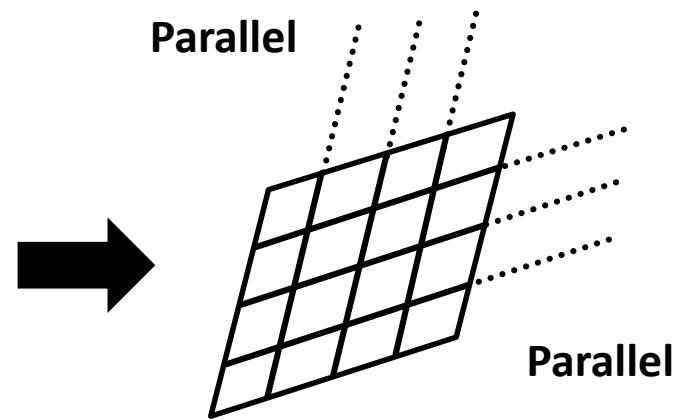
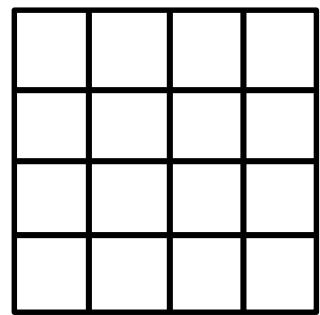
$$t = \begin{bmatrix} t_x \\ t_y \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$l_\infty = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

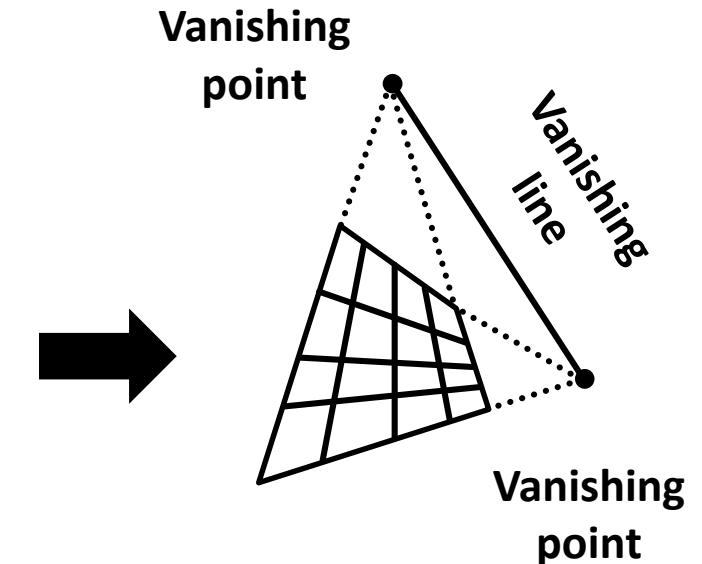
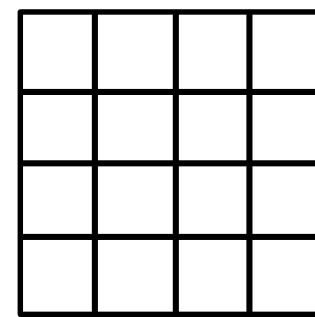
$$H = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix}$$

$$H^{-T} = \begin{bmatrix} A^{-T} & 0 \\ -tA^{-T} & 1 \end{bmatrix}$$

## 4-2: Projective Transformation



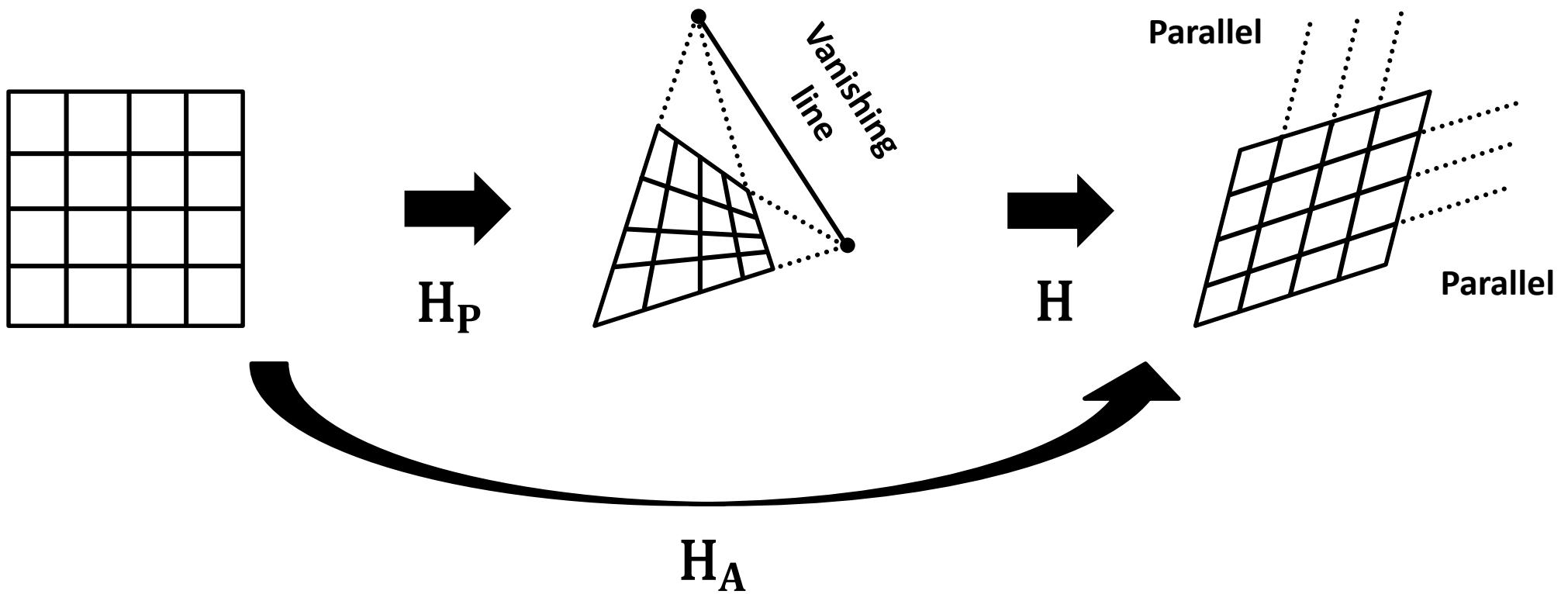
Affine Transformation



Projective Transformation

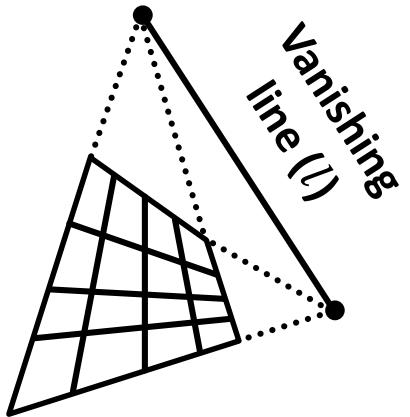
Whereas every affine transformation maps  $l_\infty$  to  $l_\infty$ , a general projective transform maps  $l_\infty$  to a physical line that we call the vanishing line.

## 4-2: Affine Rectification

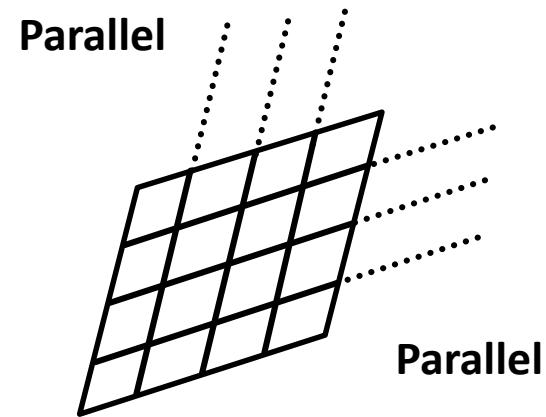


## 4-2: Affine Rectification (Continue)

$$l = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$$



$$\xrightarrow{H}$$



$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$

Invariant

See tutorials

See lecture note

# Lecture 5: Characterization of Distortions Caused by Projective Imaging and the Principle of Point/Line Duality

# Previous Lecture

Dual Conic

$$X^T C X = 0$$

$$X^T l = 0 \quad l = CX$$

Dual Conic !!

$$\begin{aligned} X^T C X &= l^T C^{-T} C C^{-1} l \\ &= l^T C^{-1} l \\ &= l^T C^* l \end{aligned}$$

Degenerate Conic

If the matrix  $C$  is not of full rank

two lines (rank 2) one line (rank 1).

Degenerate Conic  $X^T C X = X^T (l m^T + m l^T) X = 0$

Degenerate line conic  $l^T C^* l = l^T (X Y^T + Y X^T) l = 0$

## 4-3 & 4-4: Circular Points

A projective transformation  $H$  is affine if and only if  $l_\infty$  is mapped to  $l_\infty$ .

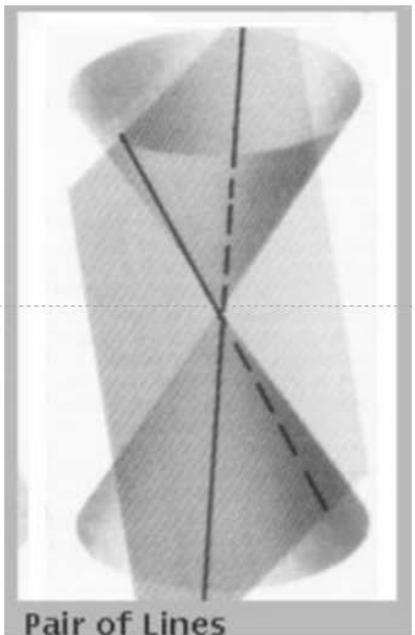
A projective transformation  $H$  is similarity if and only if circular points are mapped to circular points.

$$I = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad J = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

*i: complex number*

$$\begin{aligned} & \begin{bmatrix} s * \cos\theta & -s * \sin\theta & t_x \\ s * \sin\theta & s * \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} s * \cos\theta - s * i * \sin\theta \\ s * \sin\theta + s * i * \cos\theta \\ 0 \end{pmatrix} \\ &\cong \begin{pmatrix} \cos\theta - i * \sin\theta \\ \sin\theta + i * \cos\theta \\ 0 \end{pmatrix} \cong (\cos\theta - i * \sin\theta) \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \cong \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \end{aligned}$$

## 4-4: Dual Degenerate Conic



Pair of Lines

$$C = lm^\top + ml^\top$$

Two lines constitute a degenerate conic.

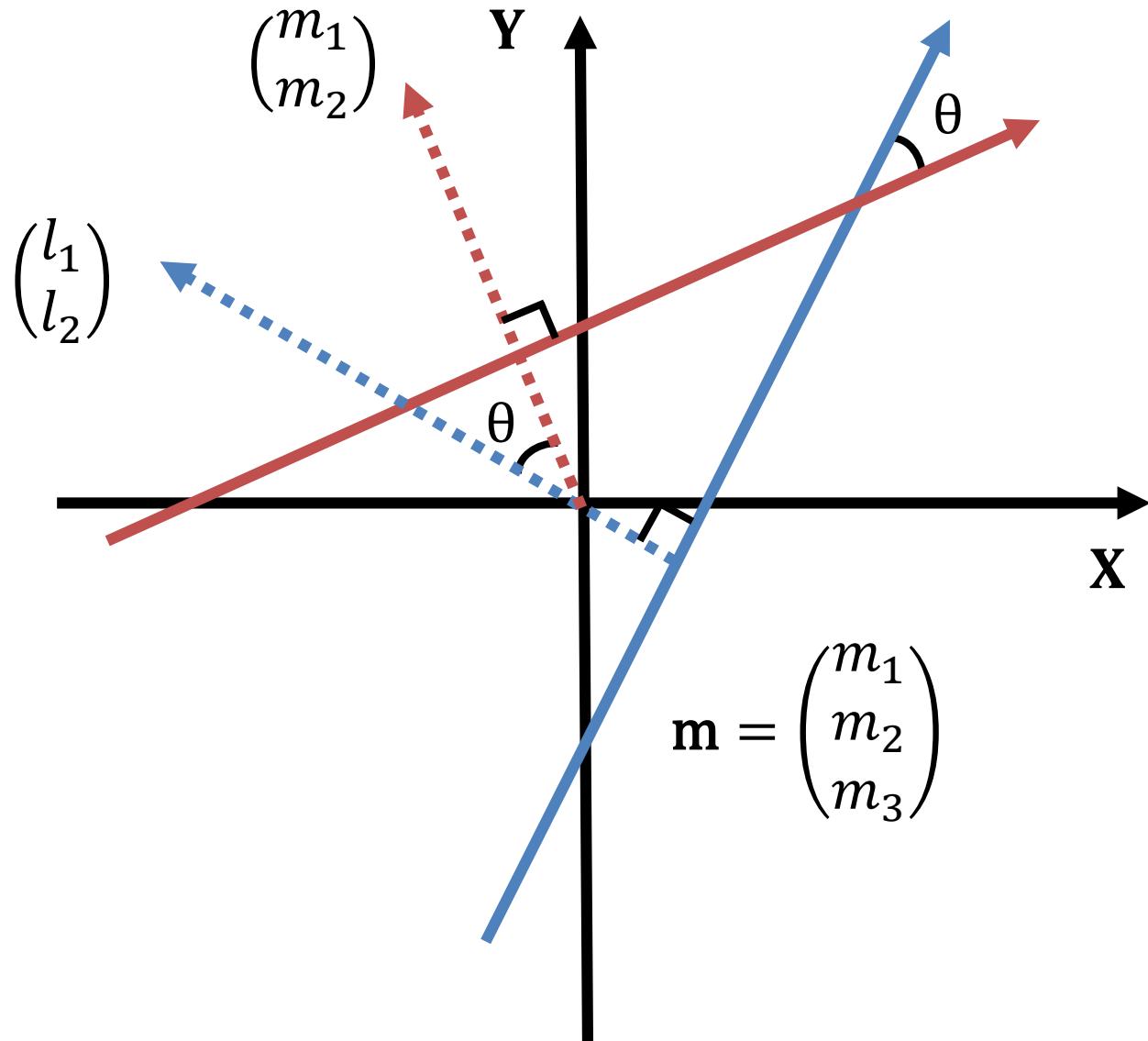
$$x^\top C x = x^\top (lm^\top + ml^\top) x = 0$$

Two lines constitute a degenerate line conic.

$$l^\top C^* l = l^\top (xy^\top + xy^\top) x = 0$$

$$C_\infty^* = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} (1 \quad -i \quad 0) + \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} (1 \quad i \quad 0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## 5-1: Angles between Two Lines

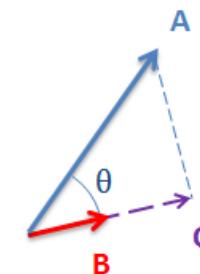


$$l = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$$

$$l: l_1x + l_2y + l_3 = 0$$

$$m: m_1x + m_2y + m_3 = 0$$

Dot product



$$A \cdot B = |A||B| \cos(\theta)$$

if the magnitude of  $B$  is 1, then...

$$C = A \cdot B = |A| \cos(\theta)$$

$$\cos\theta = \frac{l_1m_1 + l_2m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

## 5-1: Representation of an angle using Dual Degenerate Conic

$$\mathbf{l}: l_1x + l_2y + l_3 = 0$$

$$\mathbf{m}: m_1x + m_2y + m_3 = 0$$

$$\mathbf{l} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \quad \mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

$$\cos\theta = \frac{l_1m_1 + l_2m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

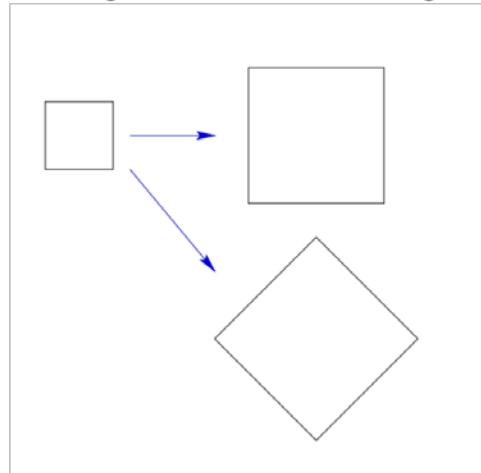
$$\cos\theta = \frac{\mathbf{l}^T \mathbf{C}_\infty^* \mathbf{m}}{\sqrt{(\mathbf{l}^T \mathbf{C}_\infty^* \mathbf{l})(\mathbf{m}^T \mathbf{C}_\infty^* \mathbf{m})}}$$

$$\mathbf{C}_\infty^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## 5-2: Similarity Rectification

A projective transformation  $H$  is similarity if and only if circular points are mapped to circular points.

Preserve angles and ratios of lengths => Preserve “shape” (isotropic scale)



Similarity  
transformation

$$x' = sRx + t$$

$$\cos\theta = \frac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* l)(m^T C_\infty^* m)}}$$

$l^T C_\infty^* m = 0$   
when lines  $l$  and  $m$  are orthogonal.

## 5-2: Metric Rectification using $C_\infty^*$

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{H}\mathbf{x}$$

$$\mathbf{l}' = \mathbf{H}^{-\top} \mathbf{l}$$

$$\mathbf{H} = \mathbf{H}_P \mathbf{H}_A \mathbf{H}_S = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^\top & v \end{bmatrix} \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{sR} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

$$\mathbf{l}^\top \mathbf{C}_\infty^* \mathbf{m}$$

$$= \mathbf{l}^\top \mathbf{H}^{-1} \mathbf{H} \mathbf{C}_\infty^* \mathbf{H}^\top \mathbf{H}^{-\top} \mathbf{m}$$

$$= (\mathbf{H}^{-\top} \mathbf{l})^\top \mathbf{H} \mathbf{C}_\infty^* \mathbf{H}^\top (\mathbf{H}^{-\top} \mathbf{m})$$

$$= \mathbf{l}'^\top \mathbf{C}_\infty^* \mathbf{m}'$$

$$\mathbf{C}_\infty^* \mathbf{l}' = \mathbf{H} \mathbf{C}_\infty^* \mathbf{H}^\top$$

$$= (\mathbf{H}_P \mathbf{H}_A \mathbf{H}_S) \mathbf{C}_\infty^* (\mathbf{H}_P \mathbf{H}_A \mathbf{H}_S)^\top$$

$$= (\mathbf{H}_P \mathbf{H}_A \mathbf{H}_S) \mathbf{C}_\infty^* \mathbf{H}_S^\top \mathbf{H}_A^\top \mathbf{H}_P^\top$$

$$= \mathbf{H}_P \mathbf{H}_A (\mathbf{H}_S \mathbf{C}_\infty^* \mathbf{H}_S^\top) \mathbf{H}_A^\top \mathbf{H}_P^\top$$

$$= \mathbf{H}_P \mathbf{H}_A \mathbf{C}_\infty^* \mathbf{H}_A^\top \mathbf{H}_P^\top$$

$$\mathbf{H}_S \mathbf{C}_{\infty}^* \mathbf{H}_S^T = \mathbf{C}_{\infty}^*$$

$$\mathbf{C}_{\infty}^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad H_S = \begin{bmatrix} s * \cos\theta & -s * \sin\theta & t_x \\ s * \sin\theta & s * \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} s * \cos\theta & -s * \sin\theta & t_x \\ s * \sin\theta & s * \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s * \cos\theta & s * \sin\theta & 0 \\ -s * \sin\theta & s * \cos\theta & 0 \\ t_x & t_y & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## 5-2: Metric Rectification using $C_{\infty}^*$

$$C_{\infty}^* = H C_{\infty}^* H^T$$

$$= (H_P H_A H_S) C_{\infty}^* (H_P H_A H_S)^T$$

$$= (H_P H_A H_S) C_{\infty}^* H_S^T H_A^T H_P^T$$

$$= H_P H_A (H_S C_{\infty}^* H_S^T) H_A^T H_P^T$$

$$= H_P H_A C_{\infty}^* H_A^T H_P^T$$

$$= \begin{bmatrix} I & 0 \\ v^T & \nu \end{bmatrix} \begin{bmatrix} K & 0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} K^T & 0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} I & v^T \\ 0^T & \nu \end{bmatrix}$$

$$= \begin{bmatrix} KK^T & KK^T v \\ v^T KK^T & v^T KK^T v \end{bmatrix}$$

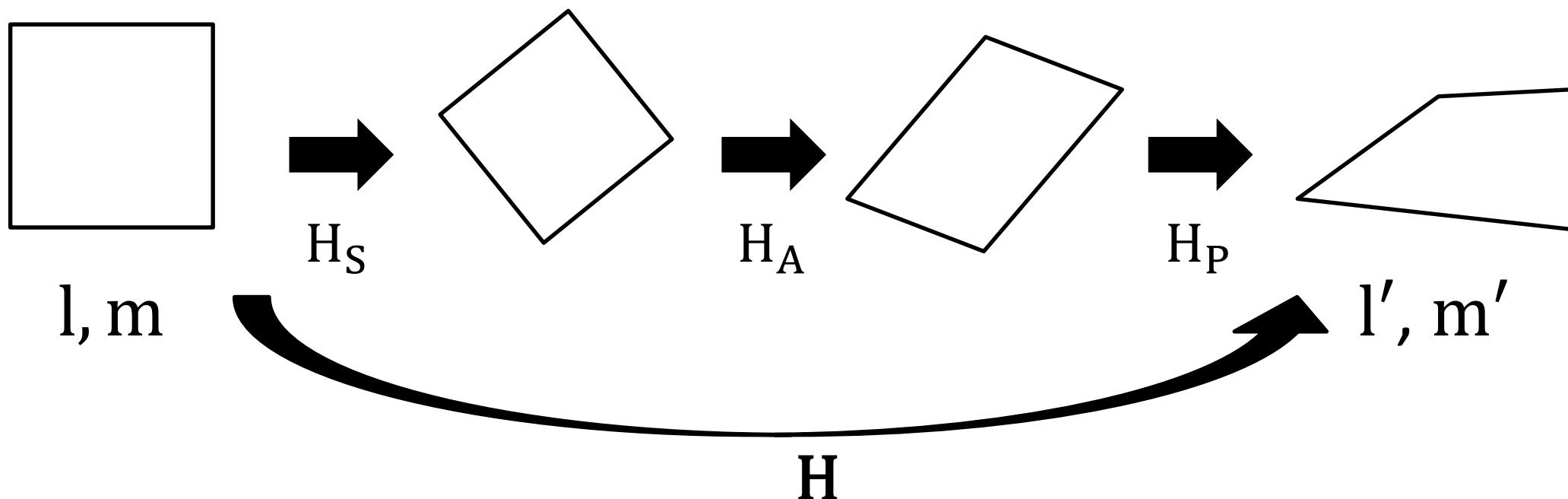
$$H = H_P H_A H_S = \begin{bmatrix} I & 0 \\ v^T & \nu \end{bmatrix} \begin{bmatrix} K & 0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$$

- 6-dof problem (5-dof up to scale)
- K: 4-dof
- v: 2-dof

## 5-2: Metric Rectification using $C_{\infty}^*$

$$l^T C_{\infty}^* m = l'^T C_{\infty}^* m' = l^T \begin{bmatrix} K K^T & K K^T v \\ v^T K K^T & v^T K K^T v \end{bmatrix} m' = 0$$

$$H = H_P H_A H_S = \begin{bmatrix} I & 0 \\ v^T & v \end{bmatrix} \begin{bmatrix} K & 0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$$



## Metric Rectification I: Two-step Method (Example 2.26)

$$\mathbf{l}^T \mathbf{C}_\infty^* \mathbf{m} = \mathbf{l}'^T \mathbf{C}_\infty^* \mathbf{m}' = \mathbf{l}'^T \begin{bmatrix} \mathbf{K} \mathbf{K}^T & 0 \\ 0 & 0 \end{bmatrix} \mathbf{m}' = 0$$

$$\mathbf{H} = \mathbf{H}_A \mathbf{H}_S = \begin{bmatrix} \mathbf{K} & 0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{sR} & \mathbf{t} \\ 0^T & 1 \end{bmatrix}$$

- **3-dof problem (2-dof up to scale)**

$$\mathbf{l}'^T \begin{bmatrix} \mathbf{K} \mathbf{K}^T & 0 \\ 0 & 0 \end{bmatrix} \mathbf{m}' = [l_1 \quad l_2 \quad l_3] \begin{bmatrix} s_1 & s_2 & 0 \\ s_2 & s_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = 0$$

$$[l_1 m_1 \quad l_1 m_2 + l_2 m_1 \quad l_2 m_2] \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = 0$$

## Metric Rectification II: One-step Method (Example 2.27)

$$l^T C_{\infty}^* m = l'^T C_{\infty}^* m' = l'^T \begin{bmatrix} K K^T & K K^T v \\ v^T K K^T & v^T K K^T v \end{bmatrix} m' = 0$$

- 6-dof problem (5-dof up to scale)

$$C = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

$$l'^T \begin{bmatrix} K K^T & K K^T v \\ v^T K K^T & v^T K K^T v \end{bmatrix} m' = [l_1 \quad l_2 \quad l_3] \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = 0$$

$$[l_1 m_1 \quad l_1 m_2 + l_2 m_1 \quad l_1 m_3 + l_3 m_1 \quad l_2 m_2 \quad l_2 m_3 + l_3 m_2 \quad l_3 m_3] \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = 0$$

## 5-3: Point Correspondences for Estimating a Homography

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} \cong \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} \cong \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} \cong \begin{pmatrix} xh_{11} + yh_{12} + h_{13} \\ xh_{21} + yh_{22} + h_{23} \\ xh_{31} + yh_{32} + h_{33} \end{pmatrix}$$

$$x' = \frac{xh_{11} + yh_{12} + h_{13}}{xh_{31} + yh_{32} + h_{33}}$$

$$y = \frac{xh_{21} + yh_{22} + h_{23}}{xh_{31} + yh_{32} + h_{33}}$$

$$\begin{bmatrix} x & y & 1 & 0 & 0 & 0 & -xx' & -x'y & -x' \\ 0 & 0 & 0 & x & y & 1 & -xy' & -y'y & -y' \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = 0$$