

Linear

Algebra

Def. If $V = (v_1, v_2, \dots, v_n)$ is a vector in \mathbb{R}^n ,

then the length of V , also called the norm of V or the magnitude of V , is denoted by $\|V\|$ and is denoted by the formula.

$$\|V\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \dots + v_n^2}$$

Def. If $U = (u_1, u_2, \dots, u_n)$ and $V = (v_1, v_2, \dots, v_n)$

are vectors in \mathbb{R}^n , then the dot product of U and V also called the Euclidean inner product of U and V , is denoted by $U \cdot V$ and is defined by the formula

$$U \cdot V = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$$= \|U\| \|V\| \cos \theta$$

where θ is the angle between U and V .

Def.

Two Vectors U and V in \mathbb{R}^n are said to be **orthogonal** if $U \cdot V = 0$, and a nonempty set of vectors in \mathbb{R}^n is said to be an **orthogonal set** if each pair of distinct vectors in the set is orthogonal.

Def.

Two Vectors U and V in \mathbb{R}^n are said to be **orthonormal** if they are orthogonal and have length 1, and a set of vectors is said to be an **orthonormal set** if every vector in the set has length 1 and each pair of distinct vectors is orthogonal.

Ex) $U_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $U_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $U_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$U_1 \cdot U_2 = U_2 \cdot U_3 = U_1 \cdot U_3 = 0 \quad \text{Orthogonal Set!}$$

$$e_1 = \frac{U_1}{\|U_1\|}, e_2 = \frac{U_2}{\|U_2\|}, e_3 = \frac{U_3}{\|U_3\|} \quad e_1 \cdot e_2 = e_1 \cdot e_3 = e_2 \cdot e_3 = 0$$

Orthonormal Set!

Def.

A nonempty set of vectors in \mathbb{R}^n is called a **subspace** of \mathbb{R}^n if it is closed under scalar multiplication and addition.

Theo.

If v_1, v_2, \dots, v_s are vectors in \mathbb{R}^n , then the set of all combinations

$$x = t_1 v_1 + t_2 v_2 + \dots + t_s v_s \quad \dots \quad (1)$$

is a subspace of \mathbb{R}^n

The subspace W of \mathbb{R}^n whose vectors satisfy (1) is called the span of v_1, v_2, \dots, v_s and is denoted by

$$W = \text{span}\{v_1, v_2, \dots, v_s\}$$

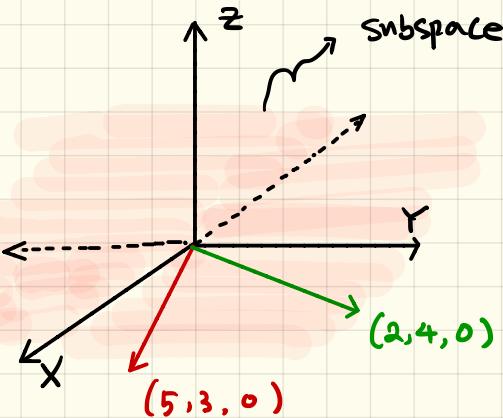
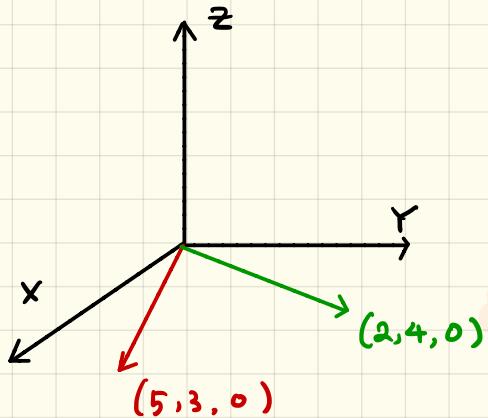
Ex)

$$W_1 = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}, \quad W_2 = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$$

W_1, W_2 are the same subspace.

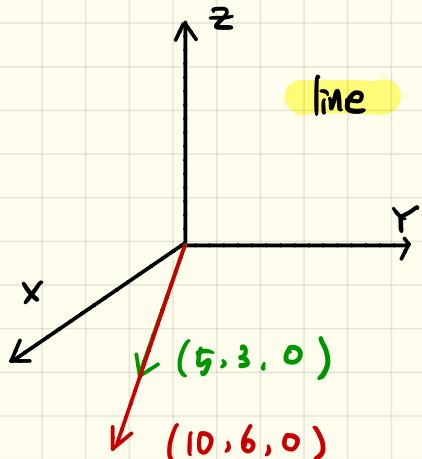
X-Y plane

Subspace.



$$W_1 = \text{span} \left\{ \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \right\}, \quad W_2 = \text{span} \left\{ \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \right\}$$

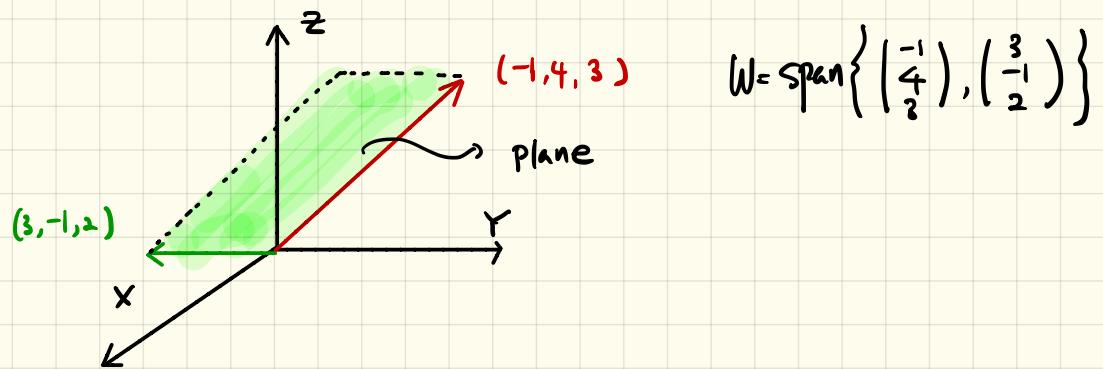
multiplication, addition.



$$W_1 = \text{span} \left\{ \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} \right\}$$

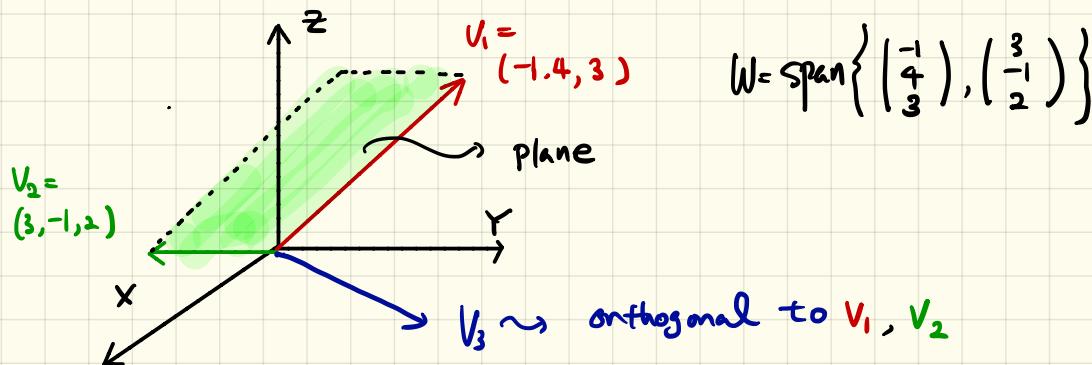
$$W_2 = \text{span} \left\{ \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ 6 \\ 0 \end{pmatrix} \right\}$$

$$W_3 = \text{span} \left\{ \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ 6 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -3 \\ 0 \end{pmatrix} \right\}$$



If A is an $m \times n$ matrix,

- The row space of A , denoted by $\text{row}(A)$, is the subspace of \mathbb{R}^m that is spanned by the row vectors of A .
- The column space of A , denoted by $\text{col}(A)$, is the subspace of \mathbb{R}^n that is spanned by the column vectors of A .
- The null space of A , denoted by $\text{null}(A)$, is the solution space of $\mathbf{Ax} = \mathbf{0}$. This is a subspace of \mathbb{R}^n .



$$A = \begin{bmatrix} -1 & 4 & 3 \\ 3 & -1 & 2 \end{bmatrix} \rightarrow W = \text{row}(A).$$

Let's $V_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$$V_1 \cdot V_3 = 0$$

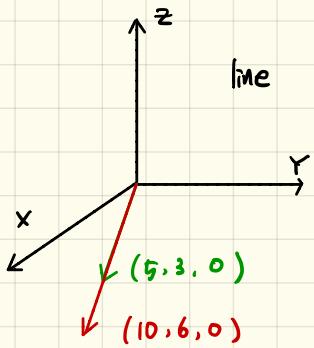
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \cdot V_3' = 0$$

$$V_2 \cdot V_3 = 0$$

$\sim V_3$.

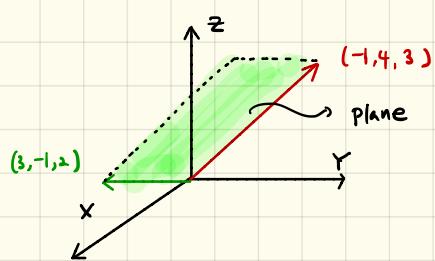
$$\text{null}(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

Def. The dimension of the row space of a matrix A is called the rank of A and is denoted by $\text{rank}(A)$; and the dimension of the null space of A is called the nullity of A and denoted by $\text{nullity}(A)$.



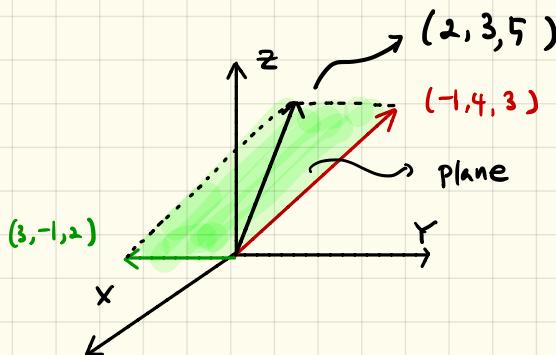
$$A = \begin{bmatrix} 5 & 3 & 0 \\ 10 & 6 & 0 \end{bmatrix}$$

$$\text{Rank}(A) = 1.$$



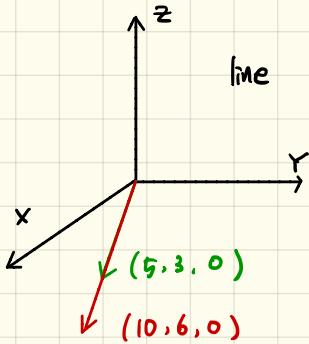
$$A = \begin{bmatrix} -1 & 4 & 3 \\ 3 & -1 & 2 \end{bmatrix}$$

$$\text{Rank}(A) = 2.$$



$$A = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

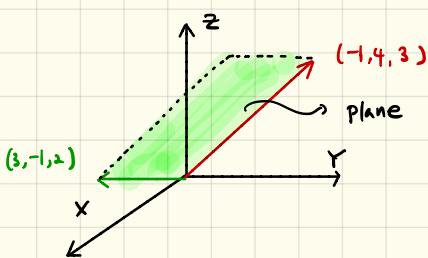
$$\text{Rank}(A) = 2.$$



$$A = \begin{bmatrix} 5 & 3 & 0 \\ 10 & 6 & 0 \end{bmatrix}$$

$\text{Rank}(A) = 1$.

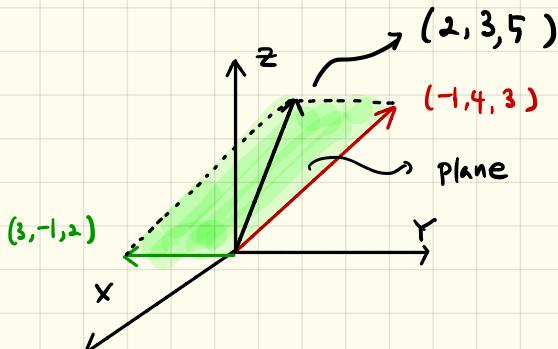
$\text{nullity}(A) = 2$



$$A = \begin{bmatrix} -1 & 4 & 3 \\ 3 & -1 & 2 \end{bmatrix}$$

$\text{Rank}(A) = 2$.

$\text{nullity}(A) = 1$



$$A = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

$\text{Rank}(A) = 2$.

$\text{nullity}(A) = 1$

If A is a $m \times n$ matrix, $n = \text{Rank}(A) + \text{nullity}(A)$.

Ex).

$$x + y + z = 6$$

$$2x + 3y + z = 9$$

$$3x + 2y + 3z = 16$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 16 \end{bmatrix}$$

$$A \quad \mathbf{x} = b$$

$$A^T A \mathbf{x} = A^{-1} b \quad \mathbf{x} = A^{-1} b$$

Non-invertible Case??

A is invertible, Rank(A) = 3.

$$A^{-1} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & 0 & -1 \\ 5 & -1 & -1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} -1 & 1 & 2 \\ 3 & 0 & -1 \\ 5 & -1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \\ 16 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \end{aligned}$$

$$\text{Nullity}(A) = 0$$

Unique solution.

Ex).

$$x + y + z = 6$$

$$2x + 3y + z = 9$$

$$3x + 2y + 3z = 16$$

$$x + y + z - 6 = 0$$

$$2x + 3y + z - 9 = 0$$

$$3x + 2y + 3z - 16 = 0$$

$$c \neq 0$$

$$\begin{bmatrix} 1 & 1 & 1 & -6 \\ 2 & 3 & 1 & -9 \\ 3 & 2 & 3 & -16 \end{bmatrix} \begin{bmatrix} cx \\ cy \\ cz \\ c \end{bmatrix} = 0$$

↓

$$A \quad *' = 0$$

$$\text{Rank}(A) = 3$$

$$\text{Nullity}(A) = 1$$

$$\text{Null}(A) = \begin{bmatrix} 0.1796 \\ -0.3592 \\ -0.8980 \\ -0.1796 \end{bmatrix}$$

$$c=1$$

$$\therefore x = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \begin{array}{l} \xrightarrow{x} \\ \xrightarrow{y} \\ \xrightarrow{z} \end{array}$$

Singular Value Decomposition (SVD)

The Singular Value Decomposition of an $m \times n$ matrix M is a factorization of the form $U \Sigma V^T$ where

U is an $m \times m$ unitary matrix

Σ is an $m \times n$ rectangular diagonal matrix

V is an $n \times n$ unitary matrix

There real of unitary matrix is
an orthogonal matrix.

$$U U^T = I.$$

$$V V^T = I.$$

$m < n$

$\sigma_1 > \sigma_2 > \sigma_3 \dots > \sigma_m$

$$m \begin{bmatrix} U_1 & U_2 & \dots & U_m \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_m \end{bmatrix} \begin{bmatrix} | & | & | \\ 0 & \ddots & \\ | & & | \end{bmatrix} \begin{bmatrix} V_1 & \dots & V_n \end{bmatrix} = M.$$

$U \qquad \qquad \qquad \Sigma \qquad \qquad \qquad V$

$$m \begin{bmatrix} u_1, u_2, \dots, u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = M.$$

Ex.

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U_1, U_2, U_3 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \\ & -\sigma_3 \end{bmatrix}$$

$$M = [\sigma_1 U_1, \sigma_2 U_2, \dots, \sigma_m U_m, 0] \begin{bmatrix} U_1 \\ \vdots \\ U_n \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1 U_{11} & \sigma_2 U_{12} \\ \sigma_1 U_{21} & \sigma_2 U_{22} \\ \sigma_1 U_{31} & \sigma_2 U_{32} \end{bmatrix} = \begin{bmatrix} \sigma_1 U_1 & \sigma_2 U_2 \end{bmatrix}$$

Here, $U_i : m \times 1$

$$= \sigma_1 U_1 V_1 + \sigma_2 U_2 V_2 + \dots + \sigma_m U_m V_m$$

$V_i : 1 \times n$

$m < n$.

Let's pick one vector from V_{m+1}, \dots, V_n

$$MV_i = \sigma_1 U_1 V_1 + \sigma_2 U_2 V_2 + \dots + \sigma_m U_m V_m$$

= 0

because there
(are orthogonal)
basis.

$\therefore V_{m+1}, \dots, V_n$ is null vectors of M

$$\begin{bmatrix} 1 & 1 & 1 & -6 \\ 2 & 3 & 1 & -9 \\ 3 & 2 & 3 & -16 \end{bmatrix} \begin{bmatrix} cx \\ cy \\ cz \\ c \end{bmatrix} = 0$$

$A \quad \mathbf{x}' = 0$

```
A = [ 1 1 1 -6; 2 3 1 -9; 3 2 3 -16];
null(A)'
```

```
ans = 1x4
0.1796 -0.3592 -0.8980 -0.1796
```

```
[U,S,V] = svd(A);
```

```
V(:,end)'
```

```
ans = 1x4
0.1796 -0.3592 -0.8980 -0.1796
```

$A \in 3 \times 4$

$$A = U \sum V^T$$

$\downarrow \quad / \quad |$

$3 \times 3 \quad 3 \times 4 \quad 4 \times 4$

`transpose(A*V(:,end))`

```
ans = 1x3
10^-15 x
0.2220 0.2220 0.8882
```

* for printing the vector in a row.