List 5

Michał Balcerek
Computer Simulations of Stochastic Processes

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Problem 1 (Estimating spectral measure)**.** *For a set $A \subset S_d$, define the cone generated by A to be*

$$\mathrm{Cone}(A) = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{X}| > 0, \mathbf{x}/|\mathbf{x}| \in A\} = \{r\mathbf{a} : r > 0, \mathbf{a} \in A\}.$$

Theorem (Corollary 6.20, Araujo and Gine, 1980) states that:

$$\lim_{r\to\infty} \frac{\mathbb{P}(\mathbf{X} \in \mathrm{Cone}(A), |\mathbf{X}| > r)}{\mathbb{P}(|\mathbf{X}| > r)} = \frac{\Gamma(A)}{\Gamma(S_d)}.$$

In other words, the mass that Γ assigns to A determines the tail behavior of $\mathbf{X}$ in the "direction" of A.
We can use this theorem to estimate the specral measure. It is usueally called the Rachev-Xin-Chen method.

$$\widehat{\Gamma}(A) = \mathrm{const.} \frac{\#\{\mathbf{X_i} : \mathbf{X_i} \in \mathrm{Cone}(A), |\mathbf{X_i}| > r\}}{\#\{\mathbf{X_i} : |\mathbf{X}| > r\}}.$$

Use different values of $r > 0$ and see which give you proper results.

Problem 2 (Sub-Gaussian vectors, uniform spectral measure)**.** *Choose a random variable $A \sim S_{\alpha/2}(\gamma = \left(\cos\frac{\pi\alpha}{4}\right)^{2/\alpha}, \beta = 1, \delta = 0)$ with $\alpha < 2$. Let $\mathbf{G} = [G_1, G_2, \ldots, G_d]$ be a zero mean Gaussian vector in $\mathbb{R}^d$ independent of A. Then the random vector*

$$\mathbf{X} = [A^{1/2}G_1, A^{1/2}G_2, \ldots, A^{1/2}G_d] \tag{1}$$

has symmetric α-stable distribution in $\mathbb{R}^d$ and is called a sub-Gaussian $S\alpha S$ random vector in $\mathbb{R}^d$ with underlying Gaussian vector $\mathbf{G}$.
Your goal is to check if following statements, assuming the underlying Gaussian vector $\mathbf{G}$ has i.i.d. components with variance σ^2.

- *The spectral measure of such vector is uniform.*
- *The characteristic function of $\mathbf{X}$ is of the form:*

$$\mathbb{E}\exp\left\{i\sum_{k=1}^{d}\theta_k X_k\right\} = \exp\left\{-2^{-\alpha/2}\sigma^\alpha|\theta|^\alpha\right\}$$

How does this properties change, if we assume covariance matrix Σ of vector $\mathbf{G}$?