List 4

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Computer Simulations of Stochastic Processes

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1 Multivariate stable distributions

In case when Γ is a discrete spectral measure with a finite number of point masses, i.e.,

$$\Gamma(\cdot) = \sum_{j=1}^{n} \gamma_j \delta_{\mathbf{s}_j}(\cdot), \tag{1}$$

where γ_j are the weights, and $\delta_{\mathbf{s}_j}$'s are point masses at the points $\mathbf{s}_j \in S_d, j = 1, 2 \ldots, n$. ($S_d = \{\mathbf{x} \in \mathbb{R}^d : ||x|| = 1\}$)
For such discrete spectral measure (1), the characteristic function of $\mathbf{X} \sim S_{\alpha,d}(\Gamma, \mu^{\mathbf{0}} = \mathbf{0})$

$$\mathbb{E} \exp\{i\langle \mathbf{X}, \mathbf{t}\rangle\} \tag{2}$$

takes the form

$$\phi^*(\mathbf{t}) = \exp\left(-\sum_{j=1}^{n} \psi_\alpha(\langle \mathbf{t}, \mathbf{s}_j\rangle)\gamma_j\right), \tag{3}$$

where ψ_α is given by

$$\psi_\alpha(u) = \begin{cases} |u|^\alpha(1 - i\mathrm{sign}(u)) \tan \frac{\pi\alpha}{2}, & \alpha \neq 1, \\ |u|(1 + i\frac{2}{\pi}\mathrm{sign}(u)) \log |u|, & \alpha = 1. \end{cases} \tag{4}$$

Following result from Modarres and Nolan, if $\mathbf{X}$ has a characteristic function (3), then:

$$\mathbf{X} \stackrel{D}{=} \begin{cases} \sum_{j=1}^{n} \gamma_j^{1/\alpha} Z_j \mathbf{s}_j, & \alpha \neq 1, \\ \sum_{j=1}^{n} \gamma_j^{1/\alpha} (Z_j + \frac{2}{\pi} \log \gamma_j)\mathbf{s}_j, & \alpha = 1, \end{cases}$$

where $Z_1, Z_2, \ldots, Z_n$ are iid totally skewed, standardized one dimensional α-stable random variables, i.e. $Z_i \sim S_\alpha(\beta = 1, \gamma = 1, \delta = 0)$. (When $\mu^{\mathbf{0}} \neq \mathbf{0}$, both cases above have a $+\mu^{\mathbf{0}}$ in them.)

Problem 1 (Multivariate stable distribution)**.** *Implement a method to simulate multivariate stable distribution with a discrete spectral measure.*

Problem 2. *Simulate a sample with*

- *Symmetric case* $\alpha = 0.9$ *and* $n = 6$ *point masses*
  - $\gamma_1 = 0.25$ *at* $\mathbf{s}_1 = (1, 0)$,
  - $\gamma_2 = 0.125$ *at* $\mathbf{s}_2 = (1/2, \sqrt{3}/2)$
  - $\gamma_3 = 0.25$ *at* $\mathbf{s}_3 = (-1/2, \sqrt{3}/2)$
  - $\gamma_4 = 0.25$ *at* $\mathbf{s}_4 = (-1, 0)$,
  - $\gamma_5 = 0.125$ *at* $\mathbf{s}_5 = (-1/2, -\sqrt{3}/2)$
  - $\gamma_6 = 0.25$ *at* $\mathbf{s}_6 = (1/2, -\sqrt{3}/2)$.
- *Non-symmetric case with* $\alpha = 1.6$ *with* $n = 5$ *point masses:*
  - $\gamma_1 = 0.1$ *at* $\mathbf{s}_1 = (1, 0)$,
  - $\gamma_2 = 0.3$ *at* $\mathbf{s}_2 = (\sqrt{3}/2, 1/2)$,
  - $\gamma_3 = 0.1$ *at* $\mathbf{s}_3 = (1/2, \sqrt{3}/2)$,
  - $\gamma_4 = 0.3$ *at* $\mathbf{s}_4 = (0, 1)$,
  - $\gamma_5 = 0.1$ *at* $\mathbf{s}_5 = (-1/2, \sqrt{3}/2)$.

Compare your results with the results presented in this paper in the figure 1-3.

Problem 3 (Independence of coordinates)**.** *When are the coordinates independent? (What should be the form of the spectral measure?) Try to simulate such cases.*

Problem 4 (Symmetry)**.** *When is the joint distribution symmetric? Try to simulate such cases.*

Useful functions/programs to use: scatterplot! Other, less reliable to estimate 2-d densities: histogram2, contour, ksdensity.