List 2

Michał Balcerek
Computer Simulations of Stochastic Processes

03.2021

Problem 1. *Show that sum of two (or more, in general) normal variables need not to be normally distributed*

Note *Consider a one point distribution as a degenerate normal variable with variance* 0.

Problem 2 (Simulation of α-stable distribution). *Prepare a program which simulates a sample having symmetric α-stable distribution $\mathbf{S}(\alpha, \beta, \gamma, \delta, 0)$ and $\mathbf{S}(\alpha, \beta, \gamma, \delta, 1)$ which are given in definitions Def.* 1.7 *and Def.* 1.8 *in Chapter 1.*

Problem 3 (Characteristic function of symmetric α-stable distributions). *Compare empirical characteristic function calculated for samples of symmetric α-stable distribution ($\beta = 0$) with its exact value*

$$\varphi_X(t) = e^{-c|t|^\alpha}, \quad t \in \mathbb{R}$$

for some $c \geq 0$ (how is c connected with other parameters of α-stable distribution?). How could you simply explain that for symmetric α-stable distribution the characteristic function has no imaginary part?

Problem 4 (Comparing densities for known cases). *For cases:*

1. *normal distribution,*
2. *Cauchy distribution,*
3. *Lévy distribution,*

find appropriate α-stable distribution parameters. Afterwards, compare estimated densities (and, whenever possible, distribution functions) with their analytical counterparts presented in remarks $1 - 3$.

Remark 1 (Normal or Gaussian distribution).

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \quad x \in \mathbb{R}$$

for some $\mu \in \mathbb{R}, \sigma > 0$.

Remark 2 (Cauchy distribution)**.** *$X \sim$ Cauchy(γ, δ) if it has a density*

$$f(x) = \frac{1}{\pi} \frac{\gamma}{\gamma^2 + (x-\delta)^2}, \quad x \in \mathbb{R}$$

for some $\gamma > 0, \delta \in \mathbb{R}$.

Remark 3 (Lévy distribution)**.** *$X \sim$ Lévy(γ, δ) if it has a density*

$$f(x) = \sqrt{\frac{\gamma}{2\pi}} \frac{1}{(x-\delta)^{3/2}} \exp\left\{-\frac{\gamma}{2(x-\delta)}\right\}, \quad \delta < x < \infty$$

for some $\gamma > 0, \delta \in \mathbb{R}$.

Problem 5 (Tails of CDF and PDF)**.** *Using Monte Carlo simulations show that for $X \sim \mathbf{S}(\alpha, \beta, \gamma, \delta, 0)$ with $0 < \alpha < 2, -1 < \beta \leq 1$ the appropriate behavior follows as $t \to \infty$:*

- $\mathbb{P}(X > x) \sim \gamma^\alpha c_\alpha (1+\beta) t^{-\alpha}$,
- $f_X(t) \sim \alpha \gamma^\alpha c_\alpha (1+\beta) t^{-(\alpha+1)}$,

and for $-1 \leq \beta < 1$, as $-t \to -\infty$:

- $\mathbb{P}(X < -t) \sim \gamma^\alpha c_\alpha (1-\beta) t^{-\alpha}$,
- $f_X(-t) \sim \alpha \gamma^\alpha c_\alpha (1-\beta) t^{-(\alpha+1)}$.

In both cases, parameter $c_\alpha = \sin(\frac{\alpha\pi}{2})\Gamma(\alpha)/\pi$.