List 3

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Computer Simulations of Stochastic Processes

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Remark 1 (Series representation, Samorodnitsky, Taqqu). *Let $\{\epsilon_i\}_{i\in\mathbb{N}}, \{W_i\}_{i\in\mathbb{N}}, \{\Gamma_i\}_{i\in\mathbb{N}}$ be three independent sequences of random variables. Here ϵ_i is i.i.d. sequence of coin toss results:*

$$\mathbb{P}(\epsilon_i = \pm 1) = \frac{1}{2},$$

W_i is an i.i.d. sequence of random variables (not necessarily positive) with finite absolute α-th moment ($\mathbb{E}|W_i|^\alpha < \infty$). Finally, let $\Gamma_1, \Gamma_2, \ldots$ be a sequence of arrival times of a Poisson process with unit arrival rate, i.e. $\Gamma_i = \sum_{k=1}^{i} e_k$, where the e_k's are i.i.d. exponential random variable with $\mathbb{E}(e_k) = 1$. The $\Gamma_1, \Gamma_2, \ldots$ are, therefore, dependent and they are not identically distributed.
Suppose $0 < \alpha < 2$. Then the sum

$$\sum_{i=1}^{\infty} \epsilon_i \Gamma_i^{-1/\alpha} W_i$$

converges a.s. to a random variable X whose distribution is stable $S_\alpha(\sigma = (c_\alpha^{-1}\mathbb{E}|W_1|^\alpha)^{1/\alpha}, \beta = 0, \mu = 0)$. $c_\alpha = \frac{2\sin\frac{\pi\alpha}{2}\Gamma(\alpha)}{\pi}$.

Question *How to modify the distribution of ϵ_k's to obtain non-symmetric stable distribution?*

Problem 1 (Series representation). *Implement program simulating α-stable distribution using series representation described in Remark 1. Choose different distributions of W_i's and discuss the influence where we cut the presented series.*

Problem 2 (Series representation). *Answer the question stated in Remark 1.*

Problem 3 (Series represenation). *Use already implemented methods (such as looking at tails of empirical distribution function or comparing characteristic function) to check if your implementation of series representation is correct.*

Problem 4. *Using properties of tail behaviour of CDF, propose a estimator of parameter α.*

Problem 5. *Using properties of characteristic function, propose estimators of parameters α and scale γ.*

Note *You can only consider estimating parameters α, γ, δ: first, consider symmetric stable distribution. Then consdier a case when shift $\delta \neq 0$.*

Note 2 *For full step-wise solution, see **Problem 6**.*

Problem 6 (Estimating all parameters – Borak, Härdle, Weron, p. 15)**. *Advanced method.*** *The regression method is based on the following observations concerning the characteristic function $\varphi(u)$ (in Nolan's $\mathbf{S}(\alpha, \beta, \gamma, \delta; \mathbf{1}$ characterization). First, we can derive:*

$$\ln(-\ln|\varphi(u)|^2) = \ln(2\gamma^\alpha) + \alpha \ln|u|. \tag{1}$$

Also, considering the real and imaginary part of the characteristic function we get

$$\arctan\left(\frac{\Im(\varphi(u))}{\Re(\varphi(u))}\right) = \begin{cases} \gamma^\alpha |u|^\alpha \beta \tan\frac{\pi\alpha}{2} \mathrm{sign} u + \delta u, & \alpha \neq 1, \\ -\gamma|u|\beta\frac{2}{\pi}\mathrm{sign} u + \delta u, & \alpha \neq 1. \end{cases} \tag{2}$$

Equation (1) depends only on α and γ and suggest that we estimate these parameters by regressing $y = \ln(-\ln|\varphi(u)|^2)$ on $w = \ln|u|$ in the model

$$y_k = m + \alpha w_k + \varepsilon_k, \quad k = 1, \ldots, K$$

where u_k is an appropriate set of real numbers, $m = \ln(2\gamma^\alpha)$, and ε_k denotes an error term. Koutrouvelis (1980) proposed to use $u_k = \frac{\pi k}{25}, k = 1, 2, \ldots, K$ with K ranging between 9 and 134 for different estimates of α and sample sizes.
Once $\hat{\alpha}$ and $\hat{\sigma}$ have been obtained and α and σ have been fixed at these values, estimates of β and δ can be obtained using (2). Next, the regressions are repeated with $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{\delta}$ as the initial parameters. The iterations continue until a prespecified convergence criterion is satisfied.