List 9

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Computer Simulations of Stochastic Processes

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Definition 1 (Fractional Brownian motion, FBM)**.** *[Samorodnitsky, Taqqu 1994] Let* $H \in (0,1)$*. Then standard fractional Brownian motion* $\{B_H(t), t \in \mathbb{R}\}$ *has the integral representation*

$$B_H(t) = \frac{1}{C_H} \int_{-\infty}^{\infty} \left((t-x)_+^{H-1/2} - (-x)_+^{H-1/2} \right) Z_2(\mathrm{d}x), \quad t \in \mathbb{R},$$

where

$$C_H = \left\{ \int_0^{\infty} \left((1+x)^{H-1/2} - x^{H-1/2} \right)^2 \mathrm{d}x + \frac{1}{2H} \right\}^{1/2}.$$

When $H = 1/2, C_{1/2} = 1$ *and such representation is to be interpreted* $\int_0^t Z_2(\mathrm{d}x)$ *if* $t \geq 0$ *or* $-\int_t^0 Z_2(\mathrm{d}x)$ *if* $t < 0$*, that is as a integral representation of Brownian motion.*

Definition 2 (Fractional α-stable motion, FSM)**.** *Let* $H \in (0,1), \alpha \in (0,2]$*. Then fractional* α*-stable motion* $\{L_H(t), t \in \mathbb{R}\}$ *has the integral representation*

$$L_H(t) = \int_{-\infty}^{\infty} \left((t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha} \right) Z_\alpha(\mathrm{d}x), \quad t \in \mathbb{R}.$$

Note that for $\alpha = 2$ *you obtain fractional Brownian motion (up to a constant).*

Definition 3 (Ornstein-Uhlenbeck process)**.** *Process* $\{X(t), t \in \mathbb{R}\}$ *is called* α*-stable Ornstein-Uhlenbeck is it has the following representation*

$$X(t) = \int_{-\infty}^{t} e^{-\lambda(t-x)} \mathrm{d}Z_\alpha(x), \text{ or equivalently}$$

$$X(t) - X(s)e^{-\lambda(t-s)} = \int_s^t e^{-\lambda(t-x)} \mathrm{d}Z_\alpha(x)$$

Your task is to simulate $\{X(t)\}$ *for* $t \geq 0$ *assuming some* $X(0) = X_0 \in \mathbb{R}$*.*

Problem 1 (Integral representation of stable process)**.** *Using integral representation simulate (approximate) the following processes:*

- *fractional Brownian motion;*
- *fractional α-stable motion;*
- *Ornstein-Uhlenbeck process.*

Problem 2. *Apply previously considered estimators:*

- *quantile lines,*
- *characteristic function,*

and consider estimation of autocorrelation function (or autocodifference). Apply them to either process or its increments (whichever is more appropriate).