Dissipation and dispersion in finite difference solutions of hyperbolic PDEs

Numerical solutions of PDEs

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Summary

Usually transport wave-like phenomena with finite speed of propagation. Examples:

- 1d transport/advection eq. $u_t + au_x = 0$
- Conservation laws $u_t + (f(u))_x = 0$
- Wave eq. $u_{tt} + cu_{xx} = 0$

All of the prevoius can be grouped with the general tranport system of equations:

$$\boldsymbol{u}_t + A\boldsymbol{u}_x = 0$$

where A matrix which can depend on $t, x, \boldsymbol{u}$ and has a full set of real eigenvalues.
e.g. for the conservation law equation

$$u_t + (f(u))_x = 0 \Rightarrow u_t + A(u)u_x = 0$$

where $A(u) = \frac{\partial f}{\partial u}$.

- There is no dissipation: $\|u(t,\cdot)\|_{L_2} = \|u_0\|_{L_2}$
- Information propagates at finite speed
- Discontinuites in initial data is propagated $\Rightarrow$ discontinue solution
- CFL condition necessary for convergence of a finite difference scheme

Advection equation

We consider the advection equation equipped with initial and boundary condition:

$$\begin{cases} u_t + a u_x = 0, & x, t \in [a, b] \times [0, T_f] \\ u(0, x) = u_0, & x \in [a, b] \end{cases}$$

The boundary condition depends on the sign of a. e.g. if $a > 0$ the boundary condition reads $u(a, t) = f(t) \; \forall \; t \;\; \in [0, T_f]$

Upwind method

If U_j^n is a mesh function approximating u solution of PDE over a discretisation of $[a, b] \times [0, T_f]$ with space step h and time step τ, then:

$$U_j^{n+1} = \begin{cases} (1-\nu)U_j^n + \nu U_{j-1}^n & \text{if } a > 0 \\ (1+\nu)U_j^n - \nu U_{j+1}^n & \text{if } a < 0 \end{cases}$$

where $\nu = a\frac{\tau}{h}$, $n = 0, ..., N_t - 1$