Dissipation and dispersion in finite difference solutions of hyperbolic PDEs

Numerical solutions of PDEs

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Summary

Hyperbolic problems

Usually transport wave-like phenomena with finite speed of propagation. Examples:

- 1d transport/advection eq. $u_t + au_x = 0$
- Conservation laws $u_t + (f(u))_x = 0$
- Wave eq. $u_{tt} + cu_{xx} = 0$

All of the prevoius can be grouped with the general tranport system of equations:

$$\boldsymbol{u}_t + A\boldsymbol{u}_x = 0$$

where A matrix which can depend on $t, x, \boldsymbol{u}$ and has a full set of real eigenvalues.
e.g. for the conservation law equation

$$u_t + (f(u))_x = 0 \Rightarrow u_t + A(u)u_x = 0$$

where $A(u) = \frac{\partial f}{\partial u}$.

- There is no dissipation: $\|u(t,\cdot)\|_{L_2} = \|u_0\|_{L_2}$
- Information propagates at finite speed
- Discontinuites in initial data are propagated $\Rightarrow$ discontinue solution
- CFL condition necessary for convergence of a finite difference scheme

Finite differences

If the PDE is defined in a domain $I \times \Omega$ where I is the time interval $[0, T_f]$ and Ω is the space domain of one variable $[a, b]$, we can discretize the PDE domain with $N_t \times N_x$ points. We can than define a general explicit difference scheme at time $t = n * \tau$ as

$$v_j^{n+1} = \sum_{i=-l}^{r} \beta_i v_{j+i}^n$$

where $j = l, ..., N_x - r - 1$, $n = 0, ..., N_t$ and v_j^n is a mesh function over the discretised domain.

If:

- PDE has constant coefficients
- Problem is defined on infinte mesh or has periodic boundary conditions

Can perform a Fourier analysis of how the difference scheme acts on the initial condition.
If $\hat{v}(t,\xi)$ is Fourier transform of the solution and $\iota = \sqrt{-1}$, we have

$$v(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\iota \xi x} \hat{v}(t,\xi) d\xi$$

The difference scheme gives

$$v(t+\tau,x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\iota\xi x}\hat{v}(t+\tau,\xi)d\xi=$$
$$=\frac{1}{\sqrt{2\pi}}\sum_{i=-l}^{r}\beta_i\int_{-\infty}^{\infty}e^{\iota\xi(x+ih)}\hat{v}(t,\xi)d\xi=$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\iota\xi x}\left(\sum_{i=-l}^{r}\beta_i e^{\iota\xi ih}\right)\hat{v}(t,\xi)d\xi$$

$$\sum_{i=-l}^{r}\beta_i e^{\iota\xi ih}=g(\zeta)\ \Rightarrow\hat{v}(t+\tau,\xi)=g(\zeta)\hat{v}(t,\xi)$$

$\zeta=\xi h,\ \tau=\frac{T_f}{N_t},\ h=\frac{b-a}{N_x}$

$g(\zeta)$ is the **amplification factor** and $\hat{v}(t, \xi) = g(\zeta)^n \hat{v}(0, \xi)$

- $\|g\| \Rightarrow$ analysis of dissipation of wave numbers
- $Arg(g) \Rightarrow$ analysis of dispersion of wave numbers
- If the PDE has s components than g is an $s \times s$ **amplification matrix** G. Dissipation and dispersion studied via the eigenvalues of G. [1]

Dissipation and dispersion in the finite difference scheme can occur even if the PDE has not these characteristics.

Advection equation

Consider the advection equation equipped with initial and boundary condition:

$$\begin{cases} u_t + au_x = 0, & x, t \in [a, b] \times [0, T_f] \\ u(0, x) = u_0, & x \in [a, b] \end{cases}$$

Boundary condition depends on the sign of a. e.g. if $a > 0$ the boundary condition reads $u(a, t) = f(t) \; \forall \; t \; \in [0, T_f]$

Upwind method

Considering a discretization of $[0, T_f] \times [a, b]$ and that U_j^n is mesh function approximating u solution of PDE we can discretize the operators as follows:

$$\begin{cases} \dfrac{U_j^{n+1} - U_j^n}{\tau} = \dfrac{U_j^n - U_{j-1}^n}{h}, & \text{if } a > 0 \\[2ex] \dfrac{U_j^{n+1} - U_j^n}{\tau} = \dfrac{U_{j+1}^n - U_j^n}{h}, & \text{if } a < 0 \end{cases}$$

$n = 0, \ldots, N_t - 1$

Upwind method

Can be rewritten as:

$$U_j^{n+1} = \begin{cases} (1-\nu)U_j^n + \nu U_{j-1}^n & \text{if } a > 0 \\ (1+\nu)U_j^n - \nu U_{j+1}^n & \text{if } a < 0 \end{cases}$$

where $\nu = a\dfrac{\tau}{h}, \;\; n = 0, ..., N_t - 1$

Amplification factor for $a > 0$ is $g(\zeta) = (1-\nu) + \nu e^{-\iota\xi h}$:

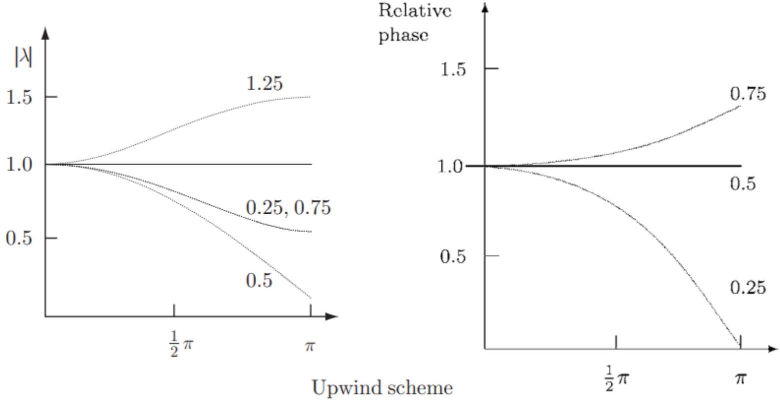
- $\|g\| \ \leq \ 1 \ \forall\ \xi \iff 0 \leq |\nu| \leq 1$
- Dissipative scheme
- Monotone scheme: if $\nu \leq 1 \ \Rightarrow\ \max_j |v_j^{n+1}| \leq \max_j |v_j^n|$
  $\Rightarrow$ stability even if $a = a(x,t) \iff |a_j^n \frac{\tau}{h}| \leq 1$
- $Arg(g) = -\tan^{-1}\left(\dfrac{\nu \sin \xi h}{(1-\nu) + \nu \cos \xi h}\right)$

PDE admits plane-wave solutions of the form

$$u(t,x) = e^{\iota(\xi x + \omega t)}$$

where $\xi =$ wave number, $\omega =$ frequency.
PDE imposes the **dispersion relation** $\omega = \omega(\xi)$ and **phase velocity** $c = \frac{\omega}{\xi}$. e.g. Advection eq. imposes $\omega = -a\xi$ and $c = -a$ independent on wave number.
To study dispersion, we can study the ratio $\phi_e = \frac{Arg(g)}{\omega}$

[2]



Upwind scheme

Dissipation can be explained using the truncation error:

$$\frac{u(t+\tau,x)-u(t,x)}{\tau} = u_t + \frac{\tau}{2}u_{tt} + O(\tau^2)$$
$$\frac{u(t,x+h)-u(t,x)}{h} = u_x + \frac{h}{2}u_{xx} + O(h^2)$$

Thus using $u_t = -au_x$ and $u_{tt} = a^2 u_{xx}$ and writing the upwind scheme:

$$\frac{u(t+\tau,x)-u(t,x)}{\tau} + a\frac{u(t,x+h)-u(t,x)}{h} =$$
$$= \left(\frac{\tau a^2}{2} + \frac{ha}{2}\right) u_{xx} + O(\tau^2, h^2)$$

So upwind is order 2 approximation of the dissipative PDE $u_t + au_x + \left(\frac{\tau a^2}{2} + \frac{ha}{2}\right) u_{xx} = 0$

Lax-Wendroff

It takes the form

$$U_j^{n+1} = \frac{1}{2}\nu(1+\nu)U_{j-1}^n + (1-\nu^2)U_j^n - \frac{1}{2}\nu(1-\nu)U_{j+1}^n$$

or

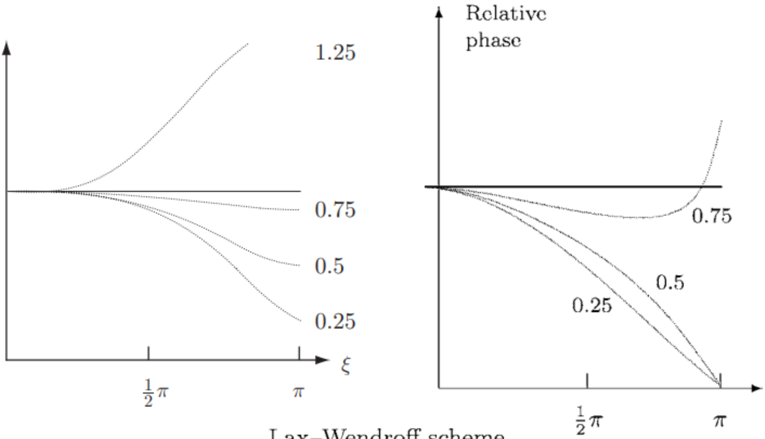
$$\frac{U_j^{n+1} - U_j^n}{\tau} + a\frac{U_{j+1}^n - U_{j-1}^n}{2h} - \frac{a^2\tau}{2}\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}$$

Amplification factor

$$g(\zeta) = 1 - 2\nu^2 \sin^2 \frac{1}{2}\zeta - \iota\nu \sin \zeta$$

- $|g|^2 = 1 - 4\nu^2(1-\nu^2)\sin^4 \frac{1}{2}\zeta \ \leq 1 \iff |\nu| \leq 1$
  Order of amplitude error ζ^4 when $\zeta = h\xi$ small
- $Arg(g) = -\tan^{-1} \dfrac{\nu \sin \zeta}{1 - 2\nu^2 \sin^2 \frac{1}{2}\zeta}$

[2]



Lax–Wendroff scheme

Modified PDE analysis:

$$\frac{u(t+\tau,x)-u(t,x)}{\tau} = u_t + \frac{\tau}{2}u_{tt} + \frac{\tau^2}{6}u_{ttt} + O(\tau^3)$$
$$\frac{u(t,x+h)-u(t,x-h)}{2h} = u_x + \frac{h^2}{6}u_{xxx} + O(h^4)$$
$$\frac{u(t,x+h)-2u(t,x)+u(t,x-h)}{h^2} = u_{xx} + O(h^2)$$

Thus the truncation error of the Lax-Wendroff scheme might be read as:

$$\frac{U_j^{n+1}-U_j^n}{\tau} + a\frac{U_{j+1}^n - U_{j-1}^n}{2h} - \frac{a^2\tau}{2}\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} =$$
$$= u_t + \frac{\tau}{2}u_{tt} + \frac{\tau^2}{6}u_{ttt} + au_x + a\frac{h^2}{6}u_{xxx} - \frac{a^2\tau}{2}u_{xx} + O(\tau^3, h^2)$$

thus using $u_t = -au_x$, $u_{tt} = a^2 u_{xx}$ and $u_{ttt} = -a^3 u_{xxx}$ we obtain:

$$= \cancel{-au_x} + \cancel{\frac{a^2\tau}{2}u_{xx}} - \frac{\tau^2 a^3}{6}u_{xxx} + \cancel{au_x} + a\frac{h^2}{6}u_{xxx} - \cancel{\frac{a^2\tau}{2}u_{xx}} + O(\tau^3, h^2) =$$

$$= (\frac{ah^2}{6} - \frac{\tau^2 a^3}{6})u_{xxx} + O(\tau^3, h^2) =$$

$$= -\frac{ah^2}{6}(1 - \mu^2 a^2)u_{xxx} + O(\tau^3, h^2),\ \mu = \frac{\tau}{h}$$

So Lax-Wendroff approximates with an higher order the PDE

$$u_t + au_x + \frac{ah^2}{6}(1 - \mu^2 a^2)u_{xxx}[1]$$

Which is **dispersive**.

Lax-Wendroff with variable coefficient

Now $a = a(x, t) \Rightarrow$ Fourier analysis not possible $\Rightarrow$ can freeze coefficients in constant values in the range of the actual coefficient functions $\Rightarrow$ check stability of freezed coefficients.
But how can be sure that scheme is stable?
Def.: Difference scheme is **dissipative** of order $2r$ if $\exists\ \delta > 0$ such that $\forall$ freezed coefficient schemes $G(\zeta)$ satisfy:

$$\rho(G(\zeta)) \leq e^{\tilde{\alpha}\tau}(1 - \delta|\zeta|^{2r})\ \forall\ |\zeta| \leq \pi$$

For hyperbolic PDEs $\tilde{\alpha} = 0$

Theorem: If the linear PDE is well-posed and the difference method consistent, dissipative of order $2r$ and has coefficients that are Lipschitz continuous in x and t, then the scheme is stable if:

- PDE is strictly hyperbolic (i.e. $u_t + Au_x = 0$ where A has real dinstinct eigenvalues)

- PDE is symmetric hyperbolic and difference scheme is symmetric and accurate of order at least $2r - 2$

If $\frac{\tau}{h} max(a_l) = \mu max(a_l) < 1$, amplification matrix of LW G gives:

$$|\lambda_l(\zeta)| \leq 1 - \delta_l |\zeta|^4$$
$$\delta_l = \frac{1}{4}\mu^2 |a_l|^2 (1 - \mu^2 |a_l|^2)$$

Thus for the theorem LW is strictly dissipative and hence stable. [1]

Leap-frog scheme

$$\frac{U_j^{n+1} - U_j^{n-1}}{2\tau} + a\frac{U_{j+1}^n - U_{j-1}^n}{2h} = 0$$

Oss.: need special procedure to obtain U_1, e.g. with Lax-Wendroff.

Fourier analysis gives

$$g(\zeta) = -\iota\nu \sin\zeta \pm \sqrt{1 - \nu^2 \sin^2 \zeta}$$

- $|g|^2 = 1 \iff \nu \leq 1 \Rightarrow$ no dissipation
- Presence of two solutions $\Rightarrow$ spurious solution mode

A finite volume scheme - Box method

Suppose conservation law $u_t + f(u)_x = 0$ and integrate it over a region $\Omega = [t_n, t_{n+1}] \times [x_j, x_{j+1}]$. Gauss divergence theorem gives

$$\begin{aligned} 0 = \iint_\Omega (u_t + f_x) dxdt &\equiv \iint_\Omega div(f, u) dxdt \\ &= \oint_{\partial\Omega} [fdt - udx] \end{aligned}$$

Approximating the integral with the trapezoidal rule gives the box method:

$$\frac{U_{j+1}^{n+1} + U_j^{n+1} - U_{j+1}^n - U_j^n}{2\tau} + \frac{F_{j+1}^{n+1} + F_{j+1}^n - F_j^{n+1} - F_j^n}{2h} = 0$$

For the advection equation $f = au$ and the method becomes:

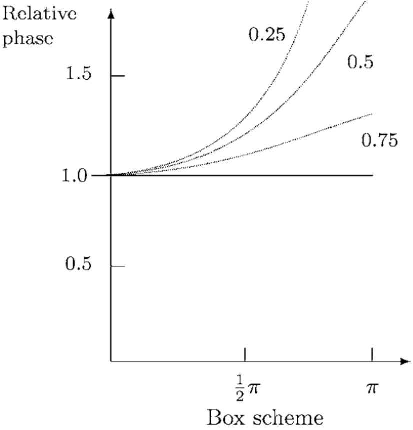
$$U_{j+1}^{n+1} = U_j^n + (1+\nu)^{-1}(1-\nu)(U_{j+1}^n - U_j^{n+1})$$

where $\nu = a\frac{\tau}{h}$

Amplification factor $g(\zeta) = \dfrac{\cos\frac{1}{2}\zeta - \iota\nu\sin\frac{1}{2}\zeta}{\cos\frac{1}{2}\zeta + \iota\nu\sin\frac{1}{2}\zeta}$:

- $|g| = 1 \; \forall \; \nu \;\; \Rightarrow$ unconditionally stable, no dissipation
- $Arg(g) = -2\tan^{-1}\nu\tan\frac{1}{2}\zeta$

[2]


Relative phase
1.5
1.0
0.5
0.25
0.5
0.75
$\frac{1}{2}\pi$
π
Box scheme
$\frac{1}{2}\pi$
π
Leap-frog scheme

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