Dissipation and dispersion in finite difference solutions of hyperbolic PDEs

Numerical solutions of PDEs

Gabriele Cimador

Data Science and Scientific computing at Università di Trieste

February 3, 2023

Summary

Usually transport wave-like phenomena with finite speed of propagation. Examples:

- 1d transport/advection eq. $u_t + au_x = 0$
- Conservation laws $u_t + (f(u))_x = 0$
- Wave eq. $u_{tt} + cu_{xx} = 0$

All of the prevoius can be grouped with the general tranport system of equations:

$$\boldsymbol{u}_t + A\boldsymbol{u}_x = 0$$

where A matrix which can depend on $t, x, \boldsymbol{u}$ and has a full set of real eigenvalues.
e.g. for the conservation law equation

$$u_t + (f(u))_x = 0 \Rightarrow u_t + A(u)u_x = 0$$

where $A(u) = \frac{\partial f}{\partial u}$.

Some comments

- There is no dissipation: $\|u(t,\cdot)\|_{L_2} = \|u_0\|_{L_2}$
- Information propagates at finite speed
- Discontinuites in initial data is propagated $\Rightarrow$ discontinue solution
- CFL condition necessary for convergence of a finite difference scheme

Finite differences

If the PDE is defined in a domain $I \times \Omega$ where I is the time interval $[0, T_f]$ and Ω is the space domain of one variable $[a, b]$, we can discretize the PDE domain with $N_t \times N_x$ points. We can than define a general explicit difference scheme at time $t = t_n$ as

$$v_j^{n+1} = \sum_{i=-l}^{r} \beta_i v_{j+i}^{n}$$

where $j = l, ..., N_x - r - 1,\ n = 0, ..., N_t$ and v_j^n is a mesh function over the discretised domain.

If:

- PDE has constant coefficients
- Problem is defined on infinte mesh or has periodic boundary conditions

Can perform a Fourier analysis of how the difference scheme acts on the initial condition.
If $\hat{v}(t,\xi)$ is Fourier transform of the solution, we have
$v(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\iota \xi x} \hat{v}(t,\xi) d\xi$

The difference scheme gives

$$\int_{-\infty}^{\infty} e^{\iota\xi x}\hat{v}(t+\tau,\xi)d\xi = \sum_{i=-l}^{r}\beta_i \int_{-\infty}^{\infty} e^{\iota\xi(x+ih)}\hat{v}(t,\xi)d\xi$$

$$= \int_{-\infty}^{\infty} \sum_{i=-l}^{r}\beta_i e^{\iota\xi(x+ih)}\hat{v}(t,\xi)d\xi$$

$$\Rightarrow \hat{v}(t+\tau,\xi) = g(\zeta)\hat{v}(t,\xi),$$

$$\zeta = \xi h,\ \tau = \frac{T_f}{N_t},\ h = \frac{b-a}{N_x}$$

the amplification factor

Advection equation

We consider the advection equation equipped with initial and boundary condition:

$$\begin{cases} u_t + a u_x = 0, & x, t \in [a, b] \times [0, T_f] \\ u(0, x) = u_0, & x \in [a, b] \end{cases}$$

The boundary condition depends on the sign of a. e.g. if $a > 0$ the boundary condition reads $u(a, t) = f(t) \; \forall \; t \;\in [0, T_f]$

Upwind method

If U_j^n is a mesh function approximating u solution of PDE over a discretisation of $[a, b] \times [0, T_f]$ with space step h and time step τ, then:

$$U_j^{n+1} = \begin{cases} (1-\nu)U_j^n + \nu U_{j-1}^n & \text{if } a > 0 \\ (1+\nu)U_j^n - \nu U_{j+1}^n & \text{if } a < 0 \end{cases}$$

where $\nu = a\frac{\tau}{h}$, $n = 0, ..., N_t - 1$

tau