

# Dissipation and dispersion in finite difference solutions of hyperbolic PDEs

## Numerical solutions of PDEs

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# Summary

- 1 Hyperbolic problems
- 2 Finite differences
- 3 Upwind method
- 4 Lax-Wendroff
- 5 Leap-frog
- 6 Box method

# Hyperbolic problems

Usually transport wave-like phenomena with finite speed of propagation. Examples:

- 1d transport/advection eq.  $u_t + au_x = 0$
- Conservation laws  $u_t + (f(u))_x = 0$
- Wave eq.  $u_{tt} + cu_{xx} = 0$

All of the previous can be grouped with the general transport system of equations:

$$\mathbf{u}_t + A\mathbf{u}_x = 0$$

where  $A$  matrix which can depend on  $t, x, \mathbf{u}$  and has a full set of real eigenvalues.

e.g. for the conservation law equation

$$u_t + (f(u))_x = 0 \Rightarrow u_t + A(u)u_x = 0$$

where  $A(u) = \frac{\partial f}{\partial u}$ .

- There is no dissipation:  $\|u(t, \cdot)\|_{L_2} = \|u_0\|_{L_2}$
- Information propagates at finite speed
- Discontinuities in initial data are propagated  $\Rightarrow$  discontinuous solution
- CFL condition necessary for convergence of a finite difference scheme

# Finite differences

If the PDE is defined in a domain  $I \times \Omega$  where  $I$  is the time interval  $[0, T_f]$  and  $\Omega$  is the space domain of one variable  $[a, b]$ , we can discretize the PDE domain with  $N_t \times N_x$  points. We can then define a general explicit difference scheme at time  $t = t_n$  as

$$v_j^{n+1} = \sum_{i=-l}^r \beta_i v_{j+i}^n$$

where  $j = l, \dots, N_x - r - 1$ ,  $n = 0, \dots, N_t$  and  $v_j^n$  is a mesh function over the discretised domain.

If:

- PDE has constant coefficients
- Problem is defined on infinite mesh or has periodic boundary conditions

Can perform a Fourier analysis of how the difference scheme acts on the initial condition.

If  $\hat{v}(t, \xi)$  is Fourier transform of the solution and  $\iota = \sqrt{-1}$ , we have

$$v(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\iota \xi x} \hat{v}(t, \xi) d\xi$$

The difference scheme gives

$$\begin{aligned}\int_{-\infty}^{\infty} e^{\iota\xi x} \hat{v}(t + \tau, \xi) d\xi &= \sum_{i=-l}^r \beta_i \int_{-\infty}^{\infty} e^{\iota\xi(x+ih)} \hat{v}(t, \xi) d\xi \\ &= \int_{-\infty}^{\infty} \left( \sum_{i=-l}^r \beta_i e^{\iota\xi ih} \right) e^{\iota\xi x} \hat{v}(t, \xi) d\xi\end{aligned}$$

$$\sum_{i=-l}^r \beta_i e^{\iota\xi ih} = g(\zeta) \Rightarrow \hat{v}(t + \tau, \xi) = g(\zeta) \hat{v}(t, \xi)$$

$$\zeta = \xi h, \quad \tau = \frac{T_f}{N_t}, \quad h = \frac{b-a}{N_x}$$



$g(\zeta)$  is the **amplification factor** and  $\hat{v}(t, \xi) = g(\zeta)^n \hat{v}(0, \xi)$

- $\|g\| \Rightarrow$  analysis of dissipation of wave numbers
- $Arg(g) \Rightarrow$  analysis of dispersion of wave numbers
- If the PDE has  $s$  components then  $g$  is an  $s \times s$  **amplification matrix**  $G$ . Dissipation and dispersion studied via the eigenvalues of  $G$ .

Dissipation and dispersion in the finite difference scheme can occur even if the PDE has not these characteristics.

# Advection equation

Consider the advection equation equipped with initial and boundary condition:

$$\begin{cases} u_t + au_x = 0, & x, t \in [a, b] \times [0, T_f] \\ u(0, x) = u_0, & x \in [a, b] \end{cases}$$

Boundary condition depends on the sign of  $a$ . e.g. if  $a > 0$  the boundary condition reads  $u(a, t) = f(t) \forall t \in [0, T_f]$

# Upwind method

Considering a discretization of  $[0, T_f] \times [a, b]$  and that  $U_j^n$  is mesh function approximating  $u$  solution of PDE we can discretize the operators as follows:

$$\begin{cases} \frac{U_j^{n+1} - U_j^n}{\tau} = \frac{U_j^n - U_{j-1}^n}{h}, & \text{if } a > 0 \\ \frac{U_j^{n+1} - U_j^n}{\tau} = \frac{U_{j+1}^n - U_j^n}{h}, & \text{if } a < 0 \end{cases}$$

$$n = 0, \dots, N_t - 1$$

# Upwind method

Can be rewritten as:

$$U_j^{n+1} = \begin{cases} (1 - \nu)U_j^n + \nu U_{j-1}^n & \text{if } a > 0 \\ (1 + \nu)U_j^n - \nu U_{j+1}^n & \text{if } a < 0 \end{cases}$$

where  $\nu = a \frac{\tau}{h}$ ,  $n = 0, \dots, N_t - 1$

Amplification factor for  $a > 0$  is  $g(\zeta) = (1 - \nu) + \nu e^{-i\xi h}$ :

- $\|g\| \leq 1 \quad \forall \xi \iff 0 \leq |\nu| \leq 1$
- Dissipative scheme
- Monotone scheme: if  $\nu \leq 1 \Rightarrow \max_j |v_j^{n+1}| \leq \max_j |v_j^n|$   
 $\Rightarrow$  stability even if  $a = a(x, t) \iff |a_j^n \frac{\tau}{h}| \leq 1$
- $Arg(g) = -\tan^{-1} \left( \frac{\nu \sin \xi h}{(1 - \nu) + \nu \cos \xi h} \right)$

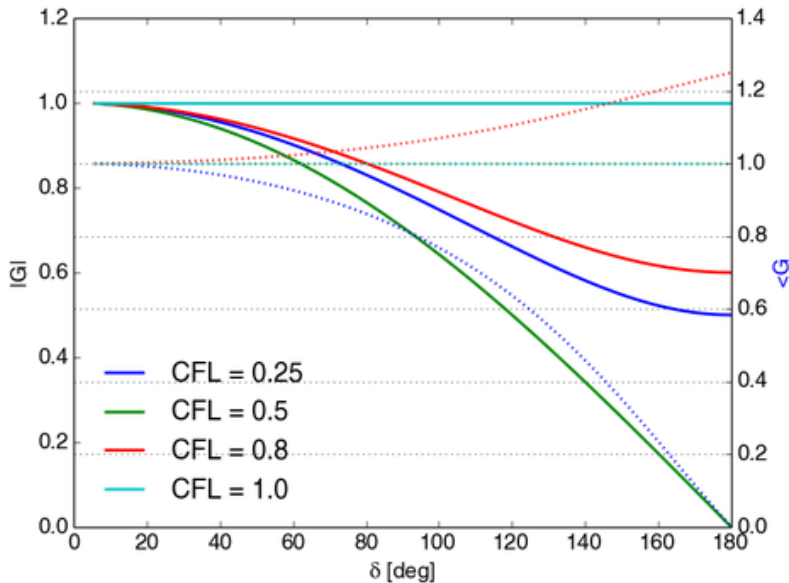
PDE admits plane-wave solutions of the form

$$u(t, x) = e^{i(\xi x + \omega t)}$$

where  $\xi$  = wave number,  $\omega$  = frequency.

PDE imposes the **dispersion relation**  $\omega = \omega(\xi)$  and **phase velocity**  $c = \frac{\omega}{\xi}$ . e.g. Advection eq. imposes  $\omega = -a\xi$  and  $c = -a$  independent on wave number.

To study dispersion, we can study the ratio  $\phi_e = \frac{\text{Arg}(g)}{\omega}$



Dissipation can be explained using the truncation error:

$$\frac{u(t + \tau, x) - u(t, x)}{\tau} = u_t + \frac{\tau}{2} u_{tt} + O(\tau^2)$$
$$\frac{u(t, x + h) - u(t, x)}{h} = u_x + \frac{h}{2} u_{xx} + O(h^2)$$

Thus using  $u_t = -au_x$  and  $u_{tt} = a^2 u_{xx}$  and writing the upwind scheme:

$$\begin{aligned} \frac{u(t + \tau, x) - u(t, x)}{\tau} + a \frac{u(t, x + h) - u(t, x)}{h} &= \\ &= \left( \frac{\tau a^2}{2} + \frac{ha}{2} \right) u_{xx} + O(\tau^2, h^2) \end{aligned}$$

So upwind is order 2 approximation of the dissipative PDE

$$u_t + au_x + \left( \frac{\tau a^2}{2} + \frac{ha}{2} \right) u_{xx} = 0$$



# Lax-Wendroff

It takes the form

$$U_j^{n+1} = \frac{1}{2}\nu(1+\nu)U_{j-1}^n + (1-\nu^2)U_j^n - \frac{1}{2}\nu(1-\nu)U_{j+1}^n$$

or

$$\frac{U_j^{n+1} - U_j^n}{\tau} + a \frac{U_{j+1}^n - U_{j-1}^n}{2h} - \frac{a^2\tau}{2} \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}$$

## Amplification factor

$$g(\zeta) = 1 - 2\nu^2 \sin^2 \frac{1}{2}\zeta - \nu \sin \zeta$$

- $|g|^2 = 1 - 4\nu^2(1 - \nu^2) \sin^4 \frac{1}{2}\zeta \leq 1 \iff |\nu| \leq 1$   
Order of amplitude error  $\zeta^4$  when  $\zeta = h\xi$  small
- $Arg(g) = -\tan^{-1} \frac{\nu \sin \zeta}{1 - 2\nu^2 \sin^2 \frac{1}{2}\zeta}$

Hyperbolic problems  
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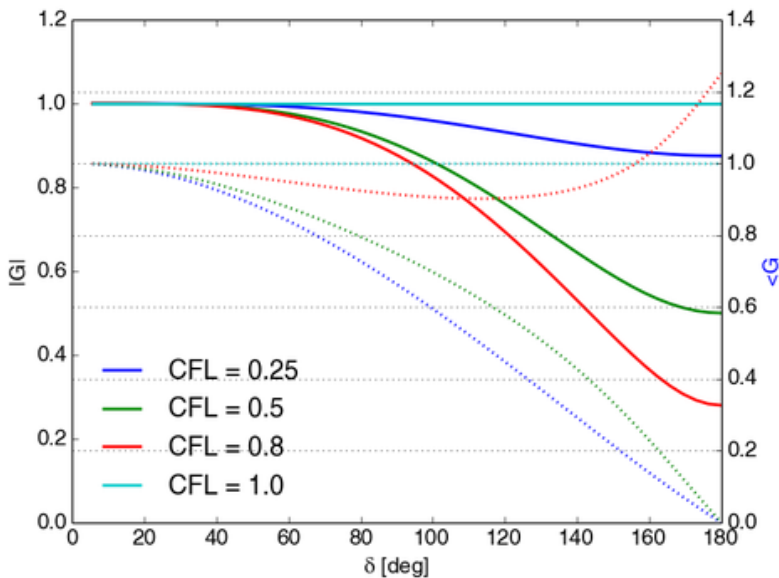
Finite differences  
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Upwind method  
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Lax-Wendroff  
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Leap-frog  
oo

Box method  
ooooooo



Modified PDE analysis:

$$\frac{u(t + \tau, x) - u(t, x)}{\tau} = u_t + \frac{\tau}{2} u_{tt} + \frac{\tau^2}{6} u_{ttt} + O(\tau^3)$$

$$\frac{u(t, x + h) - u(t, x - h)}{2h} = u_x + \frac{h^2}{6} u_{xxx} + O(h^4)$$

$$\frac{u(t, x + h) - 2u(t, x) + u(t, x - h))}{h^2} = u_{xx} + O(h^2)$$

Thus the truncation error of the Lax-Wendroff scheme might be read as:

$$\begin{aligned} & \frac{U_j^{n+1} - U_j^n}{\tau} + a \frac{U_{j+1}^n - U_{j-1}^n}{2h} - \frac{a^2 \tau}{2} \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} = \\ & = u_t + \frac{\tau}{2} u_{tt} + \frac{\tau^2}{6} u_{ttt} + a u_x + a \frac{h^2}{6} u_{xxx} - \frac{a^2 \tau}{2} u_{xx} + O(\tau^3, h^2) \end{aligned}$$

thus using  $u_t = -au_x$ ,  $u_{tt} = a^2 u_{xx}$  and  $u_{ttt} = -a^3 u_{xxx}$  we obtain:

$$\begin{aligned}
 &= \cancel{-au_x} + \cancel{\frac{a^2\tau}{2}u_{xx}} - \frac{\tau^2 a^3}{6}u_{xxx} + \cancel{au_x} + a\frac{h^2}{6}u_{xxx} - \cancel{\frac{a^2\tau}{2}u_{xx}} + O(\tau^3, h^2) = \\
 &= \left(\frac{ah^2}{6} - \frac{\tau^2 a^3}{6}\right)u_{xxx} + O(\tau^3, h^2) = \\
 &= -\frac{ah^2}{6}(1 - \mu^2 a^2)u_{xxx} + O(\tau^3, h^2), \quad \mu = \frac{\tau}{h}
 \end{aligned}$$

So Lax-Wendroff approximates with an higher order the PDE

$$u_t + au_x + \frac{ah^2}{6}(1 - \mu^2 a^2)u_{xxx}$$

Which is **dispersive**.

# Lax-Wendroff with variable coefficient

Now  $a = a(x, t) \Rightarrow$  Fourier analysis not possible  $\Rightarrow$  can freeze coefficients in constant values in the range of the actual coefficient functions  $\Rightarrow$  check stability of freezed coefficients.

But how can be sure that scheme is stable?

*Def.:* Difference scheme is **dissipative** of order  $2r$  if  $\exists \delta > 0$  such that  $\forall$  freezed coefficient schemes  $G(\zeta)$  satisfy:

$$\rho(G(\zeta)) \leq e^{\tilde{\alpha}\tau} (1 - \delta|\zeta|^{2r}) \quad \forall |\zeta| \leq \pi$$

For hyperbolic PDEs  $\tilde{\alpha} = 0$

**Theorem:** If the linear PDE is well-posed and the difference method consistent, dissipative of order  $2r$  and has coefficients that are Lipschitz continuous in  $x$  and  $t$ , then the scheme is stable if:

- PDE is strictly hyperbolic (i.e.  $u_t + Au_x = 0$  where  $A$  has real distinct eigenvalues)
- PDE is symmetric hyperbolic and difference scheme is symmetric and accurate of order at least  $2r - 2$

If  $\frac{\tau}{h} \max(a_l) = \mu \max(a_l) < 1$ , amplification matrix of LW  $G$  gives:

$$|\lambda_l(\zeta)| \leq 1 - \delta_l |\zeta|^4$$

$$\delta_l = \frac{1}{4} \mu^2 |a_l|^2 (1 - \mu^2 |a_l|^2)$$

Thus for the theorem LW is strictly dissipative and hence stable.

# Leap-frog scheme

$$\frac{U_j^{n+1} - U_j^{n-1}}{2\tau} + a \frac{U_{j+1}^n - U_{j-1}^n}{2h} = 0$$

Oss.: need special procedure to obtain  $U_1$ , e.g. with Lax-Wendroff.



Fourier analysis gives

$$g(\zeta) = -\iota\nu \sin \zeta \pm \sqrt{1 - \nu^2 \sin^2 \zeta}$$

- $|g|^2 = 1 \iff \nu \leq 1 \Rightarrow$  no dissipation
- Presence of two solutions  $\Rightarrow$  spurious solution mode

# A finite volume scheme - Box method

Suppose conservation law  $u_t + f(u)_x = 0$  and integrate it over a region  $\Omega = [t_n, t_{n+1}] \times [x_j, x_{j+1}]$ . Gauss divergence theorem gives

$$\begin{aligned} \iint_{\Omega} (u_t + f_x) dx dt &\equiv \iint_{\Omega} \operatorname{div}(f, u) dx dt \\ &= \oint_{\partial\Omega} [f dt - u dx] \end{aligned}$$

Approximating the integral with the trapezoidal rule gives the box method:

$$\frac{U_{j+1}^{n+1} + U_j^{n+1} - U_{j+1}^n - U_j^n}{2\tau} + \frac{F_{j+1}^{n+1} + F_{j+1}^n - F_j^{n+1} - F_j^n}{2h} = 0$$

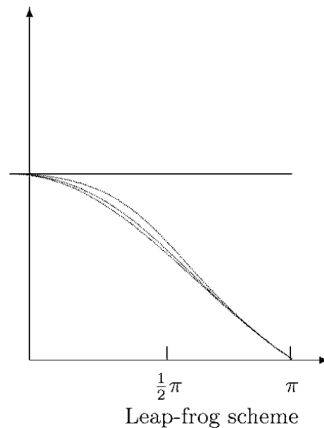
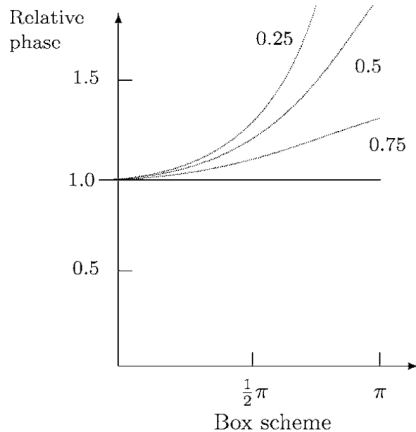
For the advection equation  $f = au$  and the method becomes:

$$U_{j+1}^{n+1} = U_j^n + (1 + \nu)^{-1}(1 - \nu)(U_{j+1}^n - U_j^{n+1})$$

where  $\nu = a\frac{\tau}{h}$

Amplification factor  $g(\zeta) = \frac{\cos \frac{1}{2}\zeta - \iota\nu \sin \frac{1}{2}\zeta}{\cos \frac{1}{2}\zeta + \iota\nu \sin \frac{1}{2}\zeta}$ :

- $|g| = 1 \ \forall \ \nu \Rightarrow$  unconditionally stable, no dissipation
- $Arg(g) = -2 \tan^{-1} \nu \tan \frac{1}{2}\zeta$



cons law - burgers

bibliography