Dissipation and dispersion in finite difference solutions of hyperbolic PDEs

Numerical solutions of PDEs

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Summary

1. Hyperbolic problems
   - What is an hyperbolic problem?
   - Some comments
2. Finite differences
3. Upwind method
4. Lax-Wendroff

Usually transport wave-like phenomena with finite speed of propagation. Examples:

- 1d transport/advection eq. $u_t + au_x = 0$
- Conservation laws $u_t + (f(u))_x = 0$
- Wave eq. $u_{tt} + cu_{xx} = 0$

All of the prevoius can be grouped with the general tranport system of equations:

$$\boldsymbol{u}_t + A\boldsymbol{u}_x = 0$$

where A matrix which can depend on $t, x, \boldsymbol{u}$ and has a full set of real eigenvalues.
e.g. for the conservation law equation

$$u_t + (f(u))_x = 0 \Rightarrow u_t + A(u)u_x = 0$$

where $A(u) = \frac{\partial f}{\partial u}$.

- There is no dissipation: $\|u(t,\cdot)\|_{L_2} = \|u_0\|_{L_2}$
- Information propagates at finite speed
- Discontinuites in initial data is propagated $\Rightarrow$ discontinue solution
- CFL condition necessary for convergence of a finite difference scheme

Finite differences

If the PDE is defined in a domain $I \times \Omega$ where I is the time interval $[0, T_f]$ and Ω is the space domain of one variable $[a, b]$, we can discretize the PDE domain with $N_t \times N_x$ points. We can than define a general explicit difference scheme at time $t = t_n$ as

$$v_j^{n+1} = \sum_{i=-l}^{r} \beta_i v_{j+i}^n$$

where $j = l, ..., N_x - r - 1$, $n = 0, ..., N_t$ and v_j^n is a mesh function over the discretised domain.

If:

- PDE has constant coefficients
- Problem is defined on infinte mesh or has periodic boundary conditions

Can perform a Fourier analysis of how the difference scheme acts on the initial condition.
If $\hat{v}(t,\xi)$ is Fourier transform of the solution, we have

$$v(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\iota \xi x} \hat{v}(t,\xi) d\xi$$

The difference scheme gives

$$\int_{-\infty}^{\infty} e^{\iota\xi x}\hat{v}(t+\tau,\xi)d\xi = \sum_{i=-l}^{r}\beta_i \int_{-\infty}^{\infty} e^{\iota\xi(x+ih)}\hat{v}(t,\xi)d\xi$$

$$= \int_{-\infty}^{\infty}\left(\sum_{i=-l}^{r}\beta_i e^{\iota\xi ih}\right) e^{\iota\xi x}\hat{v}(t,\xi)d\xi$$

$$\sum_{i=-l}^{r}\beta_i e^{\iota\xi ih} = g(\zeta) \;\Rightarrow \hat{v}(t+\tau,\xi) = g(\zeta)\hat{v}(t,\xi)$$

$\zeta = \xi h,\ \tau = \frac{T_f}{N_t},\ h = \frac{b-a}{N_x}$

$g(\zeta)$ is the **amplification factor** and $\hat{v}(t,\xi) = g(\zeta)^n \hat{v}(0,\xi)$

- $\|g\| \Rightarrow$ analysis of dissipation of wave numbers
- $Arg(g) \Rightarrow$ analysis of dispersion of wave numbers

Dissipation and dispersion in the finite difference scheme can occur even if the PDE has not these characteristics.

Advection equation

Consider the advection equation equipped with initial and boundary condition:

$$\begin{cases} u_t + au_x = 0, & x, t \in [a, b] \times [0, T_f] \\ u(0, x) = u_0, & x \in [a, b] \end{cases}$$

Boundary condition depends on the sign of a. e.g. if $a > 0$ the boundary condition reads $u(a, t) = f(t) \ \forall \ t \in [0, T_f]$

Upwind method

Considering a discretization of $[0, T_f] \times [a, b]$ and that U_j^n is mesh function approximating u solution of PDE we can discretize the operators as follows:

$$\begin{cases} \dfrac{U_j^{n+1} - U_j^n}{\tau} = \dfrac{U_j^n - U_{j-1}^n}{h}, & \text{if } a > 0 \\[2ex] \dfrac{U_j^{n+1} - U_j^n}{\tau} = \dfrac{U_{j+1}^n - U_j^n}{h}, & \text{if } a < 0 \end{cases}$$

$n = 0, \ldots, N_t - 1$

Upwind method

Can be rewritten as:

$$U_j^{n+1} = \begin{cases} (1-\nu)U_j^n + \nu U_{j-1}^n & \text{if } a > 0 \\ (1+\nu)U_j^n - \nu U_{j+1}^n & \text{if } a < 0 \end{cases}$$

where $\nu = a\dfrac{\tau}{h}, \; n = 0, \ldots, N_t - 1$

Amplification factor for $a > 0$ is $g(\zeta) = (1 - \nu) + \nu e^{-\iota \xi h}$:

- $\|g\| \; \leq \; 1 \; \forall \; \xi \iff 0 \leq |\nu| \leq 1$
- Dissipative scheme
- Monotone scheme: if $\nu \leq 1 \;\Rightarrow\; \max_j |v_j^{n+1}| \leq \max_j |v_j^n|$
- $Arg(g) = -\tan^{-1}\left(\dfrac{\nu \sin \xi h}{(1-\nu) + \nu \cos \xi h}\right)$

Dissipation can be explained also as follows:

$$
\begin{aligned}
0 &= \frac{U_j^{n+1} - U_j^n}{\tau} + a\frac{U_j^n - U_{j-1}^n}{h} = \\
&= \frac{U_j^{n+1} - U_j^n}{\tau} + a\frac{U_{j+1}^n - U_{j-1}^n}{2h} - \frac{a * h}{2}\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}
\end{aligned}
$$

i.e. it is a difference scheme for the parabolic PDE

$$u_t + au_x - \frac{ah}{2}u_{xx} = 0$$

So upwind scheme pollutes PDE with artificial diffusion.

Lax-Wendroff

It takes the form

$$U_j^{n+1} = \frac{1}{2}\nu(1+\nu)U_{j-1}^n + (1-\nu^2)U_j^n - \frac{1}{2}\nu(1-\nu)U_{j+1}^n$$

With amplification factor

$$g(\zeta) = 1 - 2\eta^2 \sin^2 \frac{1}{2}\zeta - \iota\eta \sin \zeta$$

- $|g|^2 = 1 - 4\nu^2(1-\nu^2)\sin^4 \tfrac{1}{2}\zeta \ \leq 1 \iff |\nu| \leq 1$
  Order of amplitude error ζ^4 when $\zeta = h\xi$ small
- $Arg(g) = -\tan^{-1} \dfrac{\nu \sin\zeta}{1 - 2\nu^2 \sin^2 \frac{1}{2}\zeta}$