Dissipation and dispersion in finite difference solutions of hyperbolic PDEs

Numerical solutions of PDEs

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Summary

1. Hyperbolic problems
2. Finite differences
3. Upwind method
4. Lax-Wendroff
5. Leap-frog
6. Box method

Hyperbolic problems

Usually transport wave-like phenomena with finite speed of propagation. Examples:

- 1d transport/advection eq. $u_t + au_x = 0$
- Conservation laws $u_t + (f(u))_x = 0$
- Wave eq. $u_{tt} + cu_{xx} = 0$

[?]

All of the prevoius can be grouped with the general tranport system of equations:

$$\boldsymbol{u}_t + A\boldsymbol{u}_x = 0$$

where A matrix which can depend on $t, x, \boldsymbol{u}$ and has a full set of real eigenvalues.
e.g. for the conservation law equation

$$u_t + (f(u))_x = 0 \Rightarrow u_t + A(u)u_x = 0$$

where $A(u) = \frac{\partial f}{\partial u}$.

- There is no dissipation: $\|u(t,\cdot)\|_{L_2} = \|u_0\|_{L_2}$
- Information propagates at finite speed
- Discontinuites in initial data are propagated $\Rightarrow$ discontinue solution
- CFL condition necessary for convergence of a finite difference scheme

Finite differences

If the PDE is defined in a domain $I \times \Omega$ where I is the time interval $[0, T_f]$ and Ω is the space domain of one variable $[a, b]$, we can discretize the PDE domain with $N_t \times N_x$ points. We can than define a general explicit difference scheme at time $t = t_n$ as

$$v_j^{n+1} = \sum_{i=-l}^{r} \beta_i v_{j+i}^n$$

where $j = l, ..., N_x - r - 1$, $n = 0, ..., N_t$ and v_j^n is a mesh function over the discretised domain.

If:

- PDE has constant coefficients
- Problem is defined on infinte mesh or has periodic boundary conditions

Can perform a Fourier analysis of how the difference scheme acts on the initial condition.
If $\hat{v}(t,\xi)$ is Fourier transform of the solution and $\iota = \sqrt{-1}$, we have

$$v(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\iota \xi x} \hat{v}(t,\xi) d\xi$$

The difference scheme gives

$$\int_{-\infty}^{\infty} e^{\iota \xi x} \hat{v}(t+\tau, \xi) d\xi = \sum_{i=-l}^{r} \beta_i \int_{-\infty}^{\infty} e^{\iota \xi (x+ih)} \hat{v}(t, \xi) d\xi$$

$$= \int_{-\infty}^{\infty} \left(\sum_{i=-l}^{r} \beta_i e^{\iota \xi ih} \right) e^{\iota \xi x} \hat{v}(t, \xi) d\xi$$

$$\sum_{i=-l}^{r} \beta_i e^{\iota \xi ih} = g(\zeta) \ \Rightarrow \hat{v}(t+\tau, \xi) = g(\zeta) \hat{v}(t, \xi)$$

$\zeta = \xi h, \ \tau = \frac{T_f}{N_t}, \ h = \frac{b-a}{N_x}$

$g(\zeta)$ is the **amplification factor** and $\hat{v}(t,\xi) = g(\zeta)^n \hat{v}(0,\xi)$

- $\|g\| \Rightarrow$ analysis of dissipation of wave numbers
- $Arg(g) \Rightarrow$ analysis of dispersion of wave numbers
- If the PDE has s components than g is an $s \times s$ **amplification matrix** G. Dissipation and dispersion studied via the eigenvalues of G.

Dissipation and dispersion in the finite difference scheme can occur even if the PDE has not these characteristics.

Advection equation

Consider the advection equation equipped with initial and boundary condition:

$$\begin{cases} u_t + a u_x = 0, & x, t \in [a, b] \times [0, T_f] \\ u(0, x) = u_0, & x \in [a, b] \end{cases}$$

Boundary condition depends on the sign of a. e.g. if $a > 0$ the boundary condition reads $u(a, t) = f(t) \; \forall \; t \; \in [0, T_f]$

Upwind method

Considering a discretization of $[0, T_f] \times [a, b]$ and that U_j^n is mesh function approximating u solution of PDE we can discretize the operators as follows:

$$\begin{cases} \dfrac{U_j^{n+1} - U_j^n}{\tau} = \dfrac{U_j^n - U_{j-1}^n}{h}, & \text{if } a > 0 \\[2ex] \dfrac{U_j^{n+1} - U_j^n}{\tau} = \dfrac{U_{j+1}^n - U_j^n}{h}, & \text{if } a < 0 \end{cases}$$

$n = 0, \ldots, N_t - 1$

Upwind method

Can be rewritten as:

$$U_j^{n+1} = \begin{cases} (1-\nu)U_j^n + \nu U_{j-1}^n & \text{if } a > 0 \\ (1+\nu)U_j^n - \nu U_{j+1}^n & \text{if } a < 0 \end{cases}$$

where $\nu = a\dfrac{\tau}{h}$, $n = 0, \ldots, N_t - 1$

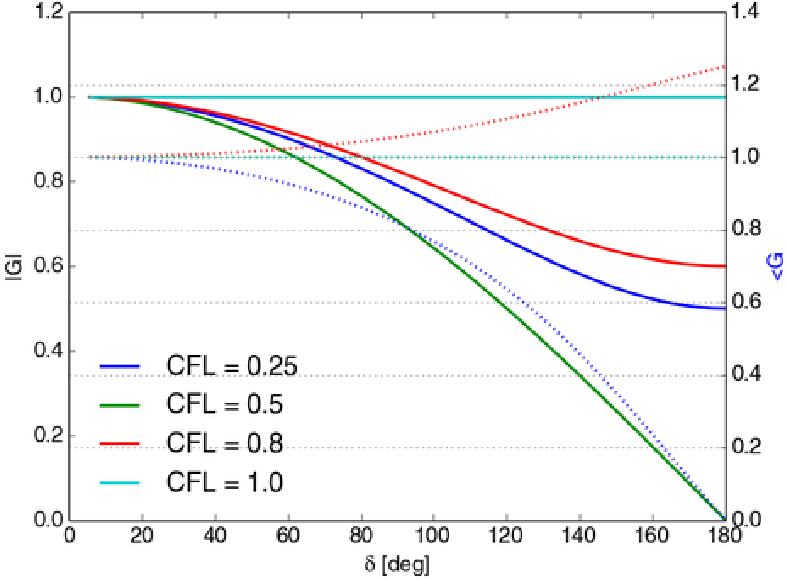
Amplification factor for $a > 0$ is $g(\zeta) = (1-\nu) + \nu e^{-\iota \xi h}$:

- $\|g\| \leq 1 \; \forall \, \xi \iff 0 \leq |\nu| \leq 1$
- Dissipative scheme
- Monotone scheme: if $\nu \leq 1 \Rightarrow \max_j |v_j^{n+1}| \leq \max_j |v_j^n|$
  $\Rightarrow$ stability even if $a = a(x,t) \iff |a_j^n \frac{\tau}{h}| \leq 1$
- $Arg(g) = -\tan^{-1}\left(\dfrac{\nu \sin \xi h}{(1-\nu) + \nu \cos \xi h}\right)$

PDE admits plane-wave solutions of the form

$$u(t,x) = e^{\iota(\xi x + \omega t)}$$

where $\xi =$ wave number, $\omega =$ frequency.
PDE imposes the **dispersion relation** $\omega = \omega(\xi)$ and **phase velocity** $c = \frac{\omega}{\xi}$. e.g. Advection eq. imposes $\omega = -a\xi$ and $c = -a$ independent on wave number.
To study dispersion, we can study the ratio $\phi_e = \frac{Arg(g)}{\omega}$



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δ [deg]
CFL = 0.25
CFL = 0.5
CFL = 0.8
CFL = 1.0

Dissipation can be explained using the truncation error:

$$\frac{u(t+\tau,x)-u(t,x)}{\tau} = u_t + \frac{\tau}{2}u_{tt} + O(\tau^2)$$
$$\frac{u(t,x+h)-u(t,x)}{h} = u_x + \frac{h}{2}u_{xx} + O(h^2)$$

Thus using $u_t = -au_x$ and $u_{tt} = a^2 u_{xx}$ and writing the upwind scheme:

$$\frac{u(t+\tau,x)-u(t,x)}{\tau} + a\frac{u(t,x+h)-u(t,x)}{h} =$$
$$= \left(\frac{\tau a^2}{2} + \frac{ha}{2}\right) u_{xx} + O(\tau^2, h^2)$$

So upwind is order 2 approximation of the dissipative PDE $u_t + au_x + \left(\frac{\tau a^2}{2} + \frac{ha}{2}\right) u_{xx} = 0$

Lax-Wendroff

It takes the form

$$U_j^{n+1} = \frac{1}{2}\nu(1+\nu)U_{j-1}^n + (1-\nu^2)U_j^n - \frac{1}{2}\nu(1-\nu)U_{j+1}^n$$

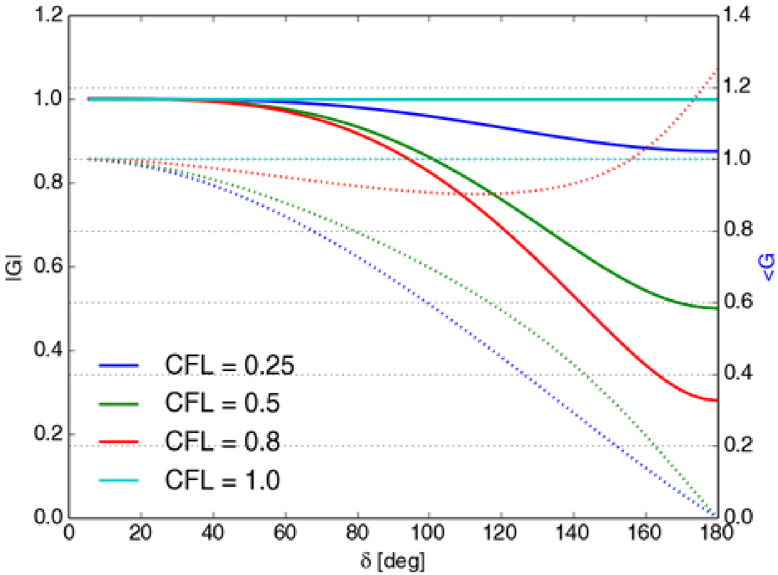
or

$$\frac{U_j^{n+1} - U_j^n}{\tau} + a\frac{U_{j+1}^n - U_{j-1}^n}{2h} - \frac{a^2\tau}{2}\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}$$

Amplification factor

$$g(\zeta) = 1 - 2\eta^2 \sin^2 \frac{1}{2}\zeta - \iota\eta \sin\zeta$$

- $|g|^2 = 1 - 4\nu^2(1-\nu^2)\sin^4 \frac{1}{2}\zeta \;\; \leq 1 \iff |\nu| \leq 1$
  Order of amplitude error ζ^4 when $\zeta = h\xi$ small
- $Arg(g) = -\tan^{-1} \dfrac{\nu \sin\zeta}{1 - 2\nu^2 \sin^2 \frac{1}{2}\zeta}$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20
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60
80
100
120
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160
180
δ [deg]
CFL = 0.25
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CFL = 0.8
CFL = 1.0

Modified PDE analysis:

$$\frac{u(t+\tau,x)-u(t,x)}{\tau} = u_t + \frac{\tau}{2}u_{tt} + \frac{\tau^2}{6}u_{ttt} + O(\tau^3)$$

$$\frac{u(t,x+h)-u(t,x-h)}{2h} = u_x + \frac{h^2}{6}u_{xxx} + O(h^4)$$

$$\frac{u(t,x+h)-2u(t,x)+u(t,x-h)}{h^2} = u_{xx} + O(h^2)$$

Thus the truncation error of the Lax-Wendroff scheme might be read as:

$$\frac{U_j^{n+1}-U_j^n}{\tau} + a\frac{U_{j+1}^n - U_{j-1}^n}{2h} - \frac{a^2\tau}{2}\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} =$$
$$= u_t + \frac{\tau}{2}u_{tt} + \frac{\tau^2}{6}u_{ttt} + au_x + a\frac{h^2}{6}u_{xxx} - \frac{a^2\tau}{2}u_{xx} + O(\tau^3, h^2)$$

thus using $u_t = -au_x$, $u_{tt} = a^2 u_{xx}$ and $u_{ttt} = -a^3 u_{xxx}$ we obtain:

$$= \cancel{-au_x} + \cancel{\frac{a^2\tau}{2}} u_{xx} - \frac{\tau^2 a^3}{6} u_{xxx} + \cancel{au_x} + a\frac{h^2}{6} u_{xxx} - \cancel{\frac{a^2\tau}{2}} u_{xx} + O(\tau^3, h^2) =$$

$$= (\frac{ah^2}{6} - \frac{\tau^2 a^3}{6}) u_{xxx} + O(\tau^3, h^2) =$$

$$= -\frac{ah^2}{6}(1 - \mu^2 a^2) u_{xxx} + O(\tau^3, h^2), \ \ \mu = \frac{\tau}{h}$$

So Lax-Wendroff approximates with an higher order the PDE

$$u_t + au_x + \frac{ah^2}{6}(1 - \mu^2 a^2) u_{xxx}$$

Which is **dispersive**.

Lax-Wendroff with variable coefficient

Now $a = a(x,t) \Rightarrow$ Fourier analysis not possible $\Rightarrow$ can freeze coefficients in constant values in the range of the actual coefficient functions $\Rightarrow$ check stability of freezed coefficients.
But how can be sure that scheme is stable?
Def.: Difference scheme is **dissipative** of order $2r$ if $\exists\ \delta > 0$ such that $\forall$ freezed coefficient schemes $G(\zeta)$ satisfy:

$$\rho(G(\zeta)) \leq e^{\tilde{\alpha}\tau}(1 - \delta|\zeta|^{2r})\ \forall\ |\zeta| \leq \pi$$

For hyperbolic PDEs $\tilde{\alpha} = 0$

Theorem: If the linear PDE is well-posed and the difference method consistent, dissipative of order $2r$ and has coefficients that are Lipschitz continuous in x and t, then the scheme is stable if:

- PDE is strictly hyperbolic (i.e. $u_t + Au_x = 0$ where A has real dinstinct eigenvalues)
- PDE is symmetric hyperbolic and difference scheme is symmetric and accurate of order at least $2r - 2$

If $\frac{\tau}{h} max(a_l) = \mu max(a_l) < 1$, amplification matrix of LW G gives:

$$|\lambda_l(\zeta)| \leq 1 - \delta_l|\zeta|^4$$
$$\delta_l = \frac{1}{4}\mu^2|a_l|^2(1 - \mu^2|a_l|^2)$$

Thus for the theorem LW is strictly dissipative and hence stable.

Leap-frog scheme

$$\frac{U_j^{n+1} - U_j^{n-1}}{2\tau} + a\frac{U_{j+1}^n - U_{j-1}^n}{2h} = 0$$

Oss.: need special procedure to obtain U_1, e.g. with Lax-Wendroff.

Fourier analysis gives

$$g(\zeta) = -\iota\nu \sin\zeta \pm \sqrt{1 - \nu^2 \sin^2\zeta}$$

- $|g|^2 = 1 \iff \nu \leq 1 \Rightarrow$ no dissipation
- Presence of two solutions $\Rightarrow$ spurious solution mode

A finite volume scheme - Box method

Suppose conservation law $u_t + f(u)_x = 0$ and integrate it over a region $\Omega = [t_n, t_{n+1}] \times [x_j, x_{j+1}]$. Gauss divercence theorem gives

$$\begin{aligned} \iint_\Omega (u_t + f_x) dxdt &\equiv \iint_\Omega div(f, u) dxdt \\ &= \oint_{\partial\Omega} [fdt - udx] \end{aligned}$$

Approximating the integral with the trapezoidal rule gives the box method:

$$\frac{U_{j+1}^{n+1} + U_j^{n+1} - U_{j+1}^n - U_j^n}{2\tau} + \frac{F_{j+1}^{n+1} + F_{j+1}^n - F_j^{n+1} - F_j^n}{2h} = 0$$

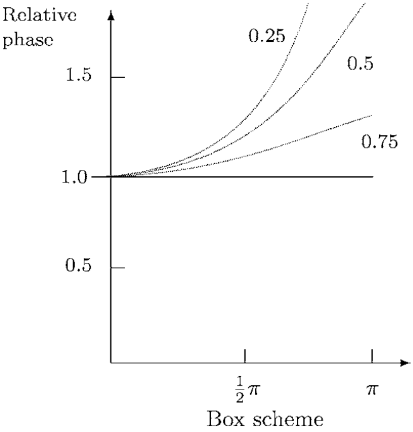
For the advection equation $f = au$ and the method becomes:

$$U_{j+1}^{n+1} = U_j^n + (1+\nu)^{-1}(1-\nu)(U_{j+1}^n - U_j^{n+1})$$

where $\nu = a\frac{\tau}{h}$

Amplification factor $g(\zeta) = \dfrac{\cos\frac{1}{2}\zeta - \iota\nu\sin\frac{1}{2}\zeta}{\cos\frac{1}{2}\zeta + \iota\nu\sin\frac{1}{2}\zeta}$:

- $|g| = 1 \; \forall \; \nu \;\Rightarrow$ unconditionally stable, no dissipation
- $Arg(g) = -2\tan^{-1}\nu\tan\frac{1}{2}\zeta$


Relative phase
1.5
1.0
0.5
0.25
0.5
0.75
$\frac{1}{2}\pi$
π
Box scheme
$\frac{1}{2}\pi$
π
Leap-frog scheme

cons law - burgers

bibliography