The Quantum Group $\mathrm{U}_q(\mathfrak{sl}_2)$

Talk 14 on Hopf Algebras and Tensor Categories

1. Recalling the Representation Theory of $\mathfrak{sl}_2$

1.1. Definition and Universal Enveloping Algebra

Let $\Bbbk$ be a field. The Lie algebra

$$\mathfrak{sl}_2 := \{A \in \mathrm{M}(2, \Bbbk) \mid \mathrm{tr}(A) = 0\}$$

admits the basis

$$E := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

and these basis elements satisfy the commutator relations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

Its universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ is therefore generated by the elements E, H, F subject to these relations, i.e.

$$\mathrm{U}(\mathfrak{sl}_2) \cong \Bbbk\langle E, H, F\rangle / \big([H, E] - 2E, [H, F] + 2F, [E, F] - H\big).$$

Theorem 1.1 (Poincaré–Birkhoff–Witt). The algebra $\mathrm{U}(\mathfrak{sl}_2)$ admits the vector space basis

$$F^l H^m E^n \qquad \text{with } l, m, n \in \mathbb{N}.$$

Theorem 1.2. Let $\Bbbk$ be of characteristic zero.

1. Every finite-dimensional $\mathfrak{sl}_2$-representation is semisimple.
2. The finite-dimensional irreducible $\mathfrak{sl}_2$-representation are (up to isomorphism) given by certain representations $\mathrm{L}(n)$ for $n \in \mathbb{N}$. This representation $\mathrm{L}(n)$ has a basis $w_0, \dots, w_n$ on which E, H, F act as depicted in Figure 1.

We refer to Appendix A.1 for more details on the representation theory of $\mathfrak{sl}_2$.

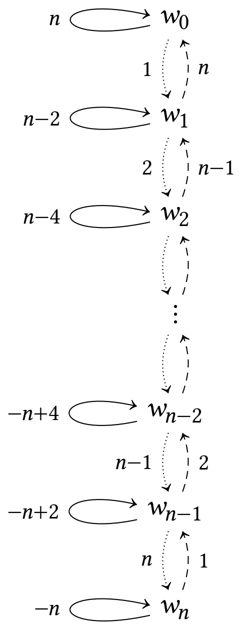

Figure 1: The irreducible representation $\mathrm{L}(n)$. Loops depict the action of H, dashed arrows the action of E and dotted arrows the action of F.

2. The Algebra $\mathrm{U}_q(\mathfrak{sl}_2)$

Convention 2.1. In the following $\Bbbk$ denotes a field of characteristic zero and q is an element of $\Bbbk$ with $q \neq 0, 1, -1$.

Definition 2.2. The $\Bbbk$-algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ is given by the generators

$$E\,, \quad F\,, \quad K\,, \quad K^{-1}$$

subject to the relations

$$KK^{-1} = 1 = K^{-1}K\,, \quad KE = q^2 EK\,, \quad KF = q^{-2} FK\,, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}\,. \tag{1}$$

Remark 2.3 (Choice of q). One often requires additional conditions on q, namely that

1. q is not a root of unity, or that
2. $\Bbbk$ is the field $\mathbb{K}(q)$ over some other field $\mathbb{K}$, with q being the indeterminate.

Remark 2.4. One might think about E and F as the usual elements of $\mathfrak{sl}_2$, but $\mathrm{U}_q(\mathfrak{sl}_2)$ does not contain H. We will later see that $\mathrm{U}_q(\mathfrak{sl})$ (at least morally speaking) lives inside an algebra $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ that also contains H, and in which

$$q = \mathrm{e}^{\hbar}\,, \quad K = \mathrm{e}^{\hbar H}\,.$$

We may therefore think about the element K as

$$K = q^H .$$

Remark 2.5 (The case $q = 1$). The algebra $\mathrm{U}_q(\mathfrak{sl})$ admits another useful presentations: One introduces the element

$$\widetilde{H} := \frac{K - K^{-1}}{q - q^{-1}}$$

as an additional generator, and then adjust the relations (1). The resulting presentation does then make sense for any $q \in \Bbbk$, and one has

$$\mathrm{U}_1(\mathfrak{sl}_2) \cong \mathrm{U}(\mathfrak{sl}_2)[\sigma]/(\sigma^2 - 1)$$

given by

$$E \mapsto \sigma E, \quad \widetilde{H} \mapsto \sigma H, \quad F \mapsto F, \quad K \mapsto \sigma .$$

We refer to Appendix A.2 for more details.

Every $\mathfrak{sl}_2$-representation extends to a $\mathrm{U}_1(\mathfrak{sl}_1)$-module by letting σ act by either 1 or -1. The resultings $\mathrm{U}_1(\mathfrak{sl}_1)$-modules are denoted by $\mathrm{L}(\varepsilon, n)$ for $\varepsilon = \pm$ and $n \in \mathbb{N}$. One can conclude from Theorem 1.2 that every finite-dimensional $\mathrm{U}_1(\mathfrak{sl}_2)$-module is semisimple, and that the irreducible finite-dimensional $\mathrm{U}_1(\mathfrak{sl}_2)$-modules are given precisely given by $\mathrm{L}(\pm, n)$. We refer to Appendix A.3 for more details.

Remark 2.6. There also exist other, more exotic presentations of $\mathrm{U}_q(\mathfrak{sl}_2)$. We refer to [ITW05] for an example.

Theorem 2.7 (PBW). The elements

$$F^l K^m E^n \qquad \text{with } l, n \in \mathbb{N} \text{ and } m \in \mathbb{Z}$$

are a basis of $\mathrm{U}_q(\mathfrak{sl}_2)$.

Proof. See Appendix A.4. □

We refer to Appendix A.5 for more remarks on the algebra structure of $\mathrm{U}_q(\mathfrak{sl}_2)$.

3. Representation Theory of $\mathrm{U}_q(\mathfrak{sl}_2)$

We will in this section focus on the finite-dimensional representation theory of $\mathrm{U}_q(\mathfrak{sl}_2)$, which generalizes that of $\mathrm{U}_1(\mathfrak{sl}_2)$ as discussed in Remark 2.5.

3.1. Weight Space Decomposition

Convention 3.1. In the following q is an element of $\Bbbk$ which is not a root of unity, unless otherwise specified.

Definition 3.2. Let M be an $\mathrm{U}_q(\mathfrak{sl}_2)$-module. For every scalar $\lambda \in \Bbbk^\times$ the associated *weight space* is given by

$$M_\lambda := \{m \in M \mid Km = \lambda m .\}$$

Proposition 3.3. Let M be an $\mathrm{U}_q(\mathfrak{sl}_2)$-module.

1. It holds for every scalar $\lambda \in \Bbbk^\times$ that

$$EM_\lambda \subseteq M_{q^2\lambda}, \quad FM_\lambda \subseteq M_{q^{-2}\lambda}.$$

Suppose now that M is finite-dimensional

2. If M is finite-dimensional then M decomposes into weight spaces, and all occuring weights are of the form $\pm q^n$ with $n \in \mathbb{Z}$.

Proof. See Appendix A.6. □

3.2. Verma Modules and Classifications

Definition 3.4. Let M be an $\mathrm{U}_q(\mathfrak{sl}_2)$-module.

1. A weight vector m is *primitive* if it is nonzero and $Em = 0$.
2. The module M is of *highest weight* λ if it is generated by a primitive weight vector of weight λ.

Proposition 3.5. Every irreducible, finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module is a highest weight module.

Proof. The assertion follows from Proposition 3.3. □

We will classify the irreducible highest-weight representations of $\mathrm{U}_q(\mathfrak{sl}_2)$ and its irreducible finite-dimensional representations. We mirror the corresponding classifications of $\mathfrak{sl}_2$-representations.

Definition 3.6. Let $\mathrm{U}_q(\mathfrak{b})$ be the subalgebra of $\mathrm{U}_q(\mathfrak{sl}_2)$ generated by E, K, K^{-1}.

Proposition 3.7.

1. The algebra $\mathrm{U}_q(\mathfrak{b})$ has the basis

$$K^n E^m \qquad \text{with } n \in \mathbb{Z} \text{ and } m \in \mathbb{N}$$

2. The algebra $\mathrm{U}_q(\mathfrak{b})$ is given with respect to its generators E, K, K^{-1} by the relations

$$KK^{-1} = 1 = K^{-1}K, \quad KE = q^2 EK.$$

Proof. See Appendix A.7. □

Definition 3.8. Let $\lambda \in \Bbbk^\times$.

1. Let $\Bbbk_\lambda$ be the one-dimensional $\mathrm{U}_q(\mathfrak{b})$-module whose underlying vector space is given by $\Bbbk$, together with the action
$$K \cdot 1 = \lambda\,, \quad E \cdot 1 = 0\,.$$
2. The *Verma module* associated to λ is the $\mathrm{U}_q(\mathfrak{sl}_2)$-module given by
$$\mathrm{M}(\lambda) := \mathrm{U}_q(\mathfrak{sl}_2) \otimes_{\mathrm{U}_q(\mathfrak{b})} \Bbbk_\lambda\,.$$

Definition 3.9. For $q \in \Bbbk$ with $q \neq 1$ the *n-th quantum integer* is
$$[n]_q := q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}\,,$$
and thus for $q \neq 1, 0, -1$,
$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}\,.$$
The *quantum factorial* is
$$[n]_q! := [n]_q [n-1]_q \cdots [1]_q\,.$$
For every invertible element $u \in \mathrm{U}_q(\mathfrak{sl}_2)$ and integer $n \in \mathbb{Z}$ let
$$[u, n]_q := \frac{q^n u - q^{-n} u^{-1}}{q - q^{-1}}\,.$$

Remark 3.10. For $q = 1$ we have $[n]_1 = n$ and $[n]_1! = n!$.

Proposition 3.11. Let $\lambda \in \Bbbk^\times$.

1. The Verma module $\mathrm{M}(\lambda)$ has the basis
$$m_i := F^i \otimes 1 \qquad \text{with } i \geq 0,$$
and the actions of E, K, F on this basis is given by
$$F m_i = m_{i+1}\,, \quad K m_i = q^{-2i} \lambda m_i\,, \quad E m_i = [i]_q [\lambda, 1 - i]_q m_{i-1}\,.$$
This action can be graphically described as in Figure 2.
2. The Verma module $\mathrm{M}(\lambda)$ is indecomposable.
3. a. If $\lambda = \pm q^n$ for some $n \in \mathbb{N}$ then the Verma module $\mathrm{M}(\lambda)$ contains a unique nonzero, proper submodule, which is spanned by the elements
$$m_i \qquad \text{with } i \geq n + 1.$$
This submodule is isomorphic to $\mathrm{M}(q^{-n-2}\lambda)$.
   b. If $\lambda \neq \pm q^n$ for every $n \in \mathbb{N}$ then the Verma module $\mathrm{M}(\lambda)$ is irreducible.

Proof. See Appendix A.8. ☐

Definition 3.12. Let $\lambda \in \Bbbk^\times$.

$$
\begin{tikzcd}
m_0 \arrow[loop left, "\lambda"] \arrow[d, dotted, bend right] \\
m_1 \arrow[loop left, "q^{-2}\lambda"] \arrow[u, dashed, bend right, "{[\lambda,0]_q}"'] \arrow[d, dotted, bend right] \\
m_2 \arrow[loop left, "q^{-4}\lambda"] \arrow[u, dashed, bend right, "{[2]_q[\lambda,-1]_q}"'] \arrow[d, dotted, bend right] \\
m_3 \arrow[loop left, "q^{-6}\lambda"] \arrow[u, dashed, bend right, "{[3]_q[\lambda,-2]_q}"'] \arrow[d, dotted, bend right] \\
\vdots \arrow[u, dashed, bend right]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
m_0 \arrow[loop left, "{[\lambda,0]_q}"] \arrow[d, dotted, bend right] \\
m_1 \arrow[loop left, "{[\lambda,-2]_q}"] \arrow[u, dashed, bend right, "{[\lambda,0]_q}"'] \arrow[d, dotted, bend right] \\
m_2 \arrow[loop left, "{[\lambda,-4]_q}"] \arrow[u, dashed, bend right, "{[2]_q[\lambda,-1]_q}"'] \arrow[d, dotted, bend right] \\
m_3 \arrow[loop left, "{[\lambda,-6]_q}"] \arrow[u, dashed, bend right, "{[3]_q[\lambda,-2]_q}"'] \arrow[d, dotted, bend right] \\
\vdots \arrow[u, dashed, bend right]
\end{tikzcd}
$$

Figure 2: The Verma module $\mathrm{M}(\lambda)$. On the left side the loopes depict the action of K, an on the right side they depict the action of $\widetilde{H}$. On both sides the action of F is depicted by dotted arrows and the action of E by dashed arrows.

1. If $\lambda \neq \pm q^n$ for every $n \in \mathbb{N}$ then $\mathrm{L}(\lambda) := \mathrm{M}(\lambda)$.
2. If $\lambda = \pm q^n$ for some $n \in \mathbb{N}$ then $\mathrm{L}(\lambda) := \mathrm{M}(\lambda)/N$ where λ is the unique nonzero, proper submodule of $\mathrm{M}(\lambda)$.

Theorem 3.13.

1. There is a one-to-one correspondence given by
$$
\begin{aligned}
\Bbbk^\times &\longmapsto \{\text{highest-weight irreducible } \mathrm{U}_q(\mathfrak{sl}_2)\text{-modules}\}\,, \\
\lambda &\longmapsto \mathrm{L}(\lambda)\,.
\end{aligned}
$$
2. The module $\mathrm{L}(\lambda)$ is finite-dimensional if and only if $\lambda = \pm q^n$ for some $n \in \mathbb{N}$, in which case
$$
\dim(\mathrm{L}(\lambda)) = n + 1\,.
$$
The above one-to-one correspondence does therefore restrict to a one-to-one correspondence given by
$$
\begin{aligned}
\{1, -1\} \times \mathbb{N} &\longmapsto \{\text{finite-dimensional irreducible } \mathrm{U}_q(\mathfrak{sl}_2)\text{-modules}\}\,, \\
(\varepsilon, n) &\longmapsto \mathrm{L}(\varepsilon q^n)\,.
\end{aligned}
$$

Remark 3.14.

1. For every $n \geq 0$ we have
$$
[\pm q^n, -i + 1]_q = \pm[n - i + 1]_q\,.
$$
On the rescalled basis $w_0, \dots, w_n$ of $\mathrm{L}(\pm q^n)$ given by
$$
w_i := \frac{v_i}{[i]_q!}
$$

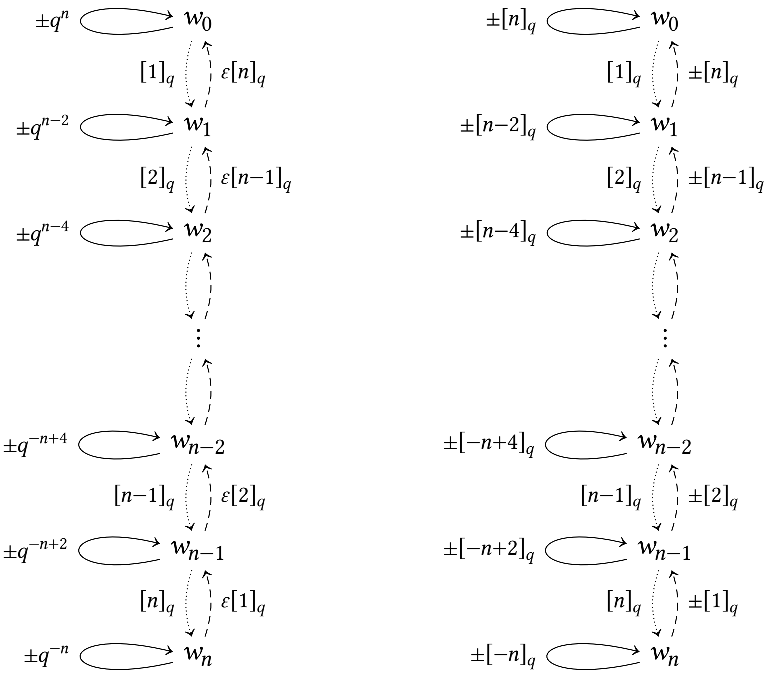

Figure 3: The irreducible representation $\mathrm{L}(\pm q^n)$. On the left side the loops depict the action of K, an on the right side they depict the action of $\widetilde{H}$. On both sides the action of F is depicted by dotted arrows and the action of E by dashed arrows.

the actions of E, F thus become

$$Ew_i = [n-i+1]_q w_{i-1}\,, \quad Fw_i = [i+1]_q w_{i+1}\,.$$

The action of E, H, K on $\mathrm{L}(\pm q^n)$ can therefore be graphically be represented as in Figure 3

2. We can consider again the element

$$\widetilde{H} := \frac{K - K^{-1}}{q - q^{-1}}$$

of $\mathrm{U}_q(\mathfrak{sl}_2)$. It acts on the weight space $M_{q^{-2i}\lambda}$ by the scalar $[\lambda, -2i]_q$. For $\lambda = \pm q^n$ this means

$$[\lambda, -2i]_q = [\pm q^n, -2i]_q = \pm[n-2i]_q\,.$$

The action of $\widetilde{H}$ on the Verma module $\mathrm{M}(\lambda)$ and irreducible modules $\mathrm{L}(\pm q^n)$ is therefore as depicted in Figure 2 and Figure 3.

3.3. Semisimplicity of Finite-Dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-modules

Theorem 3.15. Every finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module is semisimple.

Proof. See Appendix A.9. $\square$

Corollary 3.16. Let M, N be two finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-modules with $\dim M_\lambda = \dim N_\lambda$ for every $\lambda \in \Bbbk^\times$. Then $M \cong N$. $\square$

4. Hopf Algebra Structure on $\mathrm{U}_q(\mathfrak{sl}_2)$

Proposition 4.1. The algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ becomes a Hopf algebra when endowed with the comultiplication

$$\Delta(E) = E \otimes 1 + K \otimes E\,, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F\,, \quad \Delta(K) = K \otimes K\,,$$

the counit

$$\varepsilon(E) = 0\,, \quad \varepsilon(F) = 0\,, \quad \varepsilon(K) = 1$$

and the antipode

$$S(E) = -K^{-1}E\,, \quad S(F) = -FK\,, \quad S(K) = K^{-1}\,.$$

Proof. One checks that the proposed images of the algebra generators E, F, K, K^{-1} are compatible with the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$, and that the Hopf algebra diagram commute on these algebra generators. $\square$

Convention 4.2. We will in the following regard $\mathrm{U}_q(\mathfrak{sl}_2)$ as a Hopf algebra as explained in Proposition 4.1.

Remark 4.3.

1. The Hopf algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ is neither commutative nor cocommutative. It is an example of a so-called *quantum group*.
2. In $\mathrm{U}_q(\mathfrak{sl}_2)$ we don't have $S^2 = \mathrm{id}$ but instead

   $$S^2(x) = K^{-1}xK$$

   for every $x \in \mathrm{U}_q(\mathfrak{sl}_2)$, as can be checked on E, K, F.

Lemma 4.4. Let M, N be two finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-modules. Then

$$(M \otimes N)_\lambda = \bigoplus_{\mu\kappa=\lambda} M_\mu \otimes N_\kappa\,.$$

Proof. See Appendix A.10. $\square$

Proposition 4.5.

1. For all $\delta, \varepsilon \in \{1, -1\}$ and $n, m \in \mathbb{N}$ with $n \geq m$ we have

   $$\mathrm{L}(\delta q^n) \otimes \mathrm{L}(\varepsilon q^m) \cong \mathrm{L}(\delta\varepsilon q^{n+m}) \oplus \mathrm{L}(\delta\varepsilon q^{n+m-2}) \oplus \cdots \oplus \mathrm{L}(\delta\varepsilon q^{n-m})\,.$$

2. Let M, N be two finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-modules. Then

$$M \otimes N \cong N \otimes M\,.$$

Proof. This follows from Corollary 3.16 and Lemma 4.4. □

Warning 4.6. For two (finite-dimensional) $\mathrm{U}_q(\mathfrak{sl}_2)$-modules M, N the flip map

$$\tau:\ M \otimes N \to N \otimes M\,,\quad m \otimes n \mapsto n \otimes m$$

is in general not $\mathrm{U}_q(\mathfrak{sl}_2)$-linear. Indeed, let us consider $M = N = \mathrm{L}(q)$ with basis m_0, m_1, so that

$$K^{-1}m_0 = q^{-1}m_0\,,\quad K^{-1}m_1 = qm_1\,,\quad Fm_0 = m_1\,,\quad Fm_1 = 0\,.$$

Then

$$F \cdot (m_0 \otimes m_1) = qm_1 \otimes m_1 \neq m_1 \otimes m_1 = F \cdot (m_1 \otimes m_0)\,.$$

5. Some Deformation Theory

We will in the following introduce a formal deformation $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ of the Hopf algebra $\mathrm{U}(\mathfrak{sl}_2)$ and gain a new understanding of $\mathrm{U}_q(\mathfrak{sl}_2)$.

5.1. Deformation of Algebras

The following is taken (at least in spirit) from [Bel18, §5.2] and [GS92].

Motivation 5.1. Deforming a $\Bbbk$-algebra A means – roughly speaking – that the multiplication on A is replaced by a perturbated multiplication $*$, in the sense that for all $a, b \in A$,

$$a * b = ab + \mu_1(a,b)\hbar + \mu_2(a,b)\hbar^2 + \cdots$$

for some bilinear terms $\mu_i(a,b)$. The limit $\hbar \to 0$ does then give back the original algebra A.

Definition 5.2. Let A be an $\Bbbk$-algebra. A *(formal) deformation* of A is an $\Bbbk[\![\hbar]\!]$-algebra $A_\hbar$ whose underlying $\Bbbk[\![\hbar]\!]$-module is $A[\![\hbar]\!]$ and for which $A_\hbar/\hbar A_\hbar = A$ as algebras.

Remark 5.3. Every $\Bbbk[\![\hbar]\!]$-bilinear multiplication

$$(-) * (-):\ A[\![\hbar]\!] \times A[\![\hbar]\!] \to A[\![\hbar]\!]\,.$$

satisfies the equality

$$\left(\sum_{i=0}^{\infty} a_i\hbar^i\right) * \left(\sum_{j=0}^{\infty} b_j\hbar^j\right) = \sum_{i,j=0}^{\infty} (a_i * b_j)\hbar^{i+j}\,.$$

The multiplication $*$ can therefore be characterized by the $\Bbbk$-bilinear maps $\mu_i:\ A\times A \to A$ such that

$$a * b = \mu_0(a,b) + \mu_1(a,b)\hbar + \mu_2(a,b)\hbar^2 + \cdots$$

The condition $A[\![\hbar]\!]/\hbar A[\![\hbar]\!] = A$ means that μ_0 is the original multiplication on A, whence

$$a * b = ab + \mu_1(a,b)\hbar + \mu_2(a,b)\hbar^2 + \cdots$$

That the multiplication $*$ is associative gives certain compatibility conditions on the μ_1, which we won't discuss here.

Example 5.4. Every $\Bbbk$-algebra A admits the *trivial deformation* $A[\![\hbar]\!]$ (i.e. the algebra of power series with its usual product). It corresponds to the choice $\mu_1, \mu_2, \ldots = 0$.

Theorem 5.5. The universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ admits a deformation with

$$[H,E] = 2E\,, \quad [H,F] = 2F\,, \quad [E,F] = \frac{\mathrm{e}^{\hbar H} - \mathrm{e}^{-\hbar H}}{\mathrm{e}^{\hbar} - \mathrm{e}^{-\hbar}} \tag{2}$$

Proof (sketch). Let P be the free algebra on the generators E, H, F. Let I be the two-sided ideal in $P[\![\hbar]\!]$ given by the relations (2). Let J be the closure of I in the $\hbar$-adic topology. Then J is again a two-sided ideal in $P[\![\hbar]\!]$. The described deformation can be realized as the quotient $P[\![\hbar]\!]/J$. We refer to [CP95, Definition-Proposition 6.4.3 ff.] for the specific details. □

Definition 5.6. The deformation of $\mathrm{U}(\mathfrak{sl}_2)$ from Theorem 5.5 is denoted by $\mathrm{U}_\hbar(\mathfrak{sl}_2)$.

Remark 5.7. In the algebra $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ we can consider the well-defined elements

$$q := \mathrm{e}^{\hbar}\,, \quad K := \mathrm{e}^{\hbar H}\,.$$

The elements E, F, K, K^{-1} satisfy the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$ and one should think about the algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ as somewhat of a subalgebra of $\mathrm{U}_\hbar(\mathfrak{sl}_2)$.

In $\mathrm{U}_\hbar$ we have both the element H and the element

$$\widetilde{H} = \frac{K - K^{-1}}{q - q^{-1}}\,,$$

which is of the form

$$\widetilde{H} = H + \text{terms of order } \hbar^2\,.$$

We may think about $\widetilde{H}$ is a deformation of H (in an informal sense). We note that

$$q \equiv 1\,, \quad K \equiv 1\,, \quad \widetilde{H} \equiv H \qquad (\mathrm{mod}\ \hbar).$$

We refer to Appendix A.11 for more information on deformations of algebras.

5.2. Recalling Completions

We also want define coalgebras (and bialgebras and Hopf algebras). For this we need to make sense of power series in tensor products $A[\![\hbar]\!] \otimes A[\![\hbar]\!]$, which does in general not make sense. This problem is solved by using the *completed tensor product.*

Definition 5.8. Let M be an $\Bbbk[\![\hbar]\!]$-module.

1. The *ℏ-adic completion* of M is the $\Bbbk[\![\hbar]\!]$-module

$$\widehat{M} := \lim_{n\geq 0}(M/\hbar^{n+1}M) = \left\{ (m_n)_{n\geq 0} \;\middle|\; \begin{array}{l} m_n \in M/\hbar^{n+1}M \text{ with} \\ m_{n+1} \equiv m_n \pmod{\hbar^{n+1}} \text{ for every } n \geq 0 \end{array} \right\}.$$

2. The *canonical homomorphism* $M \to \widehat{M}$ is given by $m \mapsto (\overline{m}, \overline{m}, \dots)$.
3. A $\Bbbk[\![\hbar]\!]$-module M is *complete* if the canonical homomorphism $M \to \widehat{M}$ is an isomorphism.

Remark 5.9.

1. More explicitely, an $\Bbbk[\![\hbar]\!]$-module M is complete if and only if there exists for every sequence $m_0, m_1, \dots$ of elements $m_n \in M$ with

$$m_{n+1} \equiv m_n \pmod{\hbar^{n+1}} \qquad \text{for every } n \geq 0$$

a unique element $m \in M$ with

$$m \equiv m_n \pmod{\hbar^{n+1}} \qquad \text{for every } n \geq 0.$$

2. Let M be a complete $\Bbbk[\![\hbar]\!]$-module Every sequence $(m_i)_{i\geq 0}$ of elements $m_i \in M$ defines a sequence $(s_n)_{n\geq 0}$ of partial sums

$$s_n := \sum_{i=0}^{n} \hbar^i m_i \,.$$

for every $n \geq 0$. By the completeness of M there exists a unique element $\sum_{i=0}^{\infty} \hbar^i m_i$ of M with

$$\sum_{i=0}^{\infty} \hbar^i m_i \equiv \sum_{i=0}^{n} \hbar^i m_i \pmod{\hbar^{n+1}} \qquad \text{for every } n \geq 0.$$

Example 5.10. For every $\Bbbk$-vector space the resulting $\Bbbk[\![\hbar]\!]$-module $V[\![\hbar]\!]$ is complete. For every sequence of elements $v_0, v_1, \dots \in V$ we have

$$\sum_{i=0}^{\infty} \hbar^i v_i = \sum_{i=0}^{\infty} v_i \hbar^i \,.$$

Definition 5.11. Let M, N be two $\Bbbk[\![\hbar]\!]$-modules. The *completed tensor product*

$$M \mathbin{\widehat{\otimes}} N$$

is the ℏ-adic completion of the tensor product $M \otimes_{\Bbbk[\![\hbar]\!]} N$.

Proposition 5.12. Let V, W be two $\Bbbk$-vector spaces. Then the $\Bbbk[\![\hbar]\!]$-linear map

$$V[\![\hbar]\!] \otimes_{\Bbbk[\![\hbar]\!]} W[\![\hbar]\!] \to (V \otimes W)[\![\hbar]\!] \,, \quad \left(\sum_{i=0}^{\infty} v_i \hbar^i \right) \otimes \left(\sum_{j=0}^{\infty} w_j \hbar^j \right) \mapsto \sum_{i,j=0}^{\infty} (v_i \otimes w_j) \hbar^{i+j}$$

extends along the canonical homomorphism

$$V \otimes W \to V \mathbin{\widehat{\otimes}} W$$

to an isomorphism of $\Bbbk[\![\hbar]\!]$-modules

$$V[\![\hbar]\!] \mathbin{\widehat{\otimes}} W[\![\hbar]\!] \to (V \otimes W)[\![\hbar]\!] \,.$$

5.3. Deformation of Hopf Algebras

The following is taken mostly from [CP95, Chapter 6].

Definition 5.13. A *topological Hopf algebra* consists of a complete $\Bbbk[\![\hbar]\!]$-module A together with $\Bbbk[\![\hbar]\!]$-linear maps

$$m \colon A \mathbin{\widehat{\otimes}} A \to A\,, \quad u \colon \Bbbk[\![\hbar]\!] \to A\,, \quad \Delta \colon A \to A \mathbin{\widehat{\otimes}} A\,, \quad \varepsilon \colon A \to \Bbbk[\![\hbar]\!]\,, \quad S \colon A \to A$$

such that the usual Hopf algebra diagrams commute.

Remark 5.14.

1. A topological Hopf algebra A is generally not an actual Hopf algebra, since the comultiplication
$$\Delta \colon A \to A \mathbin{\widehat{\otimes}} A$$
does in general not restrict to a map $A \to A \otimes A$.
2. If A is a topological Hopf algebra then $A/\hbar A$ becomes an Hopf algebra over $\Bbbk$. We note for this that
$$(A \mathbin{\widehat{\otimes}} A)/\hbar(A \mathbin{\widehat{\otimes}} A) \cong (A \otimes A)/\hbar(A \otimes A) \cong (A/\hbar A) \otimes (A/\hbar A)\,.$$

Definition 5.15. Let A be a Hopf algebra over $\Bbbk$. A *(formal) deformation* of A is a topological Hopf algebra $A_\hbar$ whose underlying $\Bbbk[\![\hbar]\!]$-module is $A[\![\hbar]\!]$ and for which $A_\hbar/\hbar A_\hbar = A$ as Hopf algebras.

Formal deformations of algebras, coalgebras and bialgebras are defined in the same way.

Remark 5.16. Let A be a Hopf algebra over $\Bbbk$ with deformation $A_\hbar$. By using the isomorphism

$$A[\![\hbar]\!] \mathbin{\widehat{\otimes}} A[\![\hbar]\!] \cong (A \otimes A)[\![\hbar]\!]$$

we can regard the structure maps of $A_\hbar$ as $\Bbbk[\![\hbar]\!]$-linear map

$$\begin{aligned} m_\hbar &\colon (A \otimes A)[\![\hbar]\!] \to A[\![\hbar]\!]\,, \\ u_\hbar &\colon \Bbbk[\![\hbar]\!] \to A[\![\hbar]\!]\,, \\ \Delta_\hbar &\colon A[\![\hbar]\!] \to (A \otimes A)[\![\hbar]\!]\,, \\ \varepsilon_\hbar &\colon A[\![\hbar]\!] \to \Bbbk[\![\hbar]\!]\,, \\ S_\hbar &\colon A[\![\hbar]\!] \to A[\![\hbar]\!] \end{aligned} \tag{3}$$

which are perturbations of the structure maps of A, i.e. they reduce modulo $\hbar$ to the structure maps of A.

Example 5.17.

1. Every Hopf algebra A admits the trivial deformation $A[\![\hbar]\!]$. In the form (3) the structure maps of this deformation are given by the $\Bbbk[\![\hbar]\!]$-linear extensions of the structure maps of A. A deformation of A is *trivial* if it is equivalent to the trivial deformation.

2. One an make the algebra deformation $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ of $\mathrm{U}(\mathfrak{sl}_2)$ into a Hopf algebra deformation via he comultiplication

$$\Delta_\hbar(H) = H \otimes 1 + 1 \otimes H\,, \quad \Delta_\hbar(E) = E \otimes K + 1 \otimes E\,, \quad \Delta_\hbar(F) = F \otimes 1 + K^{-1} \otimes F$$

the counit

$$\varepsilon_\hbar(H) = 0\,, \quad \varepsilon_\hbar(E) = 0\,, \quad \varepsilon_\hbar(F) = 0\,,$$

and the antipode

$$S_\hbar(H) = -H\,, \quad S_\hbar(E) = -K^{-1}E\,, \quad S_\hbar(F) = -FK\,.$$

We note that it follows from this formulas from $K = \mathrm{e}^{\hbar H}$ that

$$\Delta_\hbar(K) = K \otimes K\,, \quad \varepsilon_\hbar(K) = 1\,, \quad S_\hbar(K) = K^{-1}\,.$$

For the elements E, F, K, K^{-1} in $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ we hence regain the formulas for the Hopf algebra structure of $\mathrm{U}_q(\mathfrak{sl}_2)$.

5.4. Deformation of Representations

We lastly give an explanation of how the irreducible, finite-dimensional representations $\mathrm{L}(n)$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ can be used to construct the irreducible, finite-dimensional representations $\mathrm{L}(q^n)$ of $\mathrm{U}_q(\mathfrak{sl}_2)$, where $n \in \mathbb{N}$.

Theorem 5.18 ([CP95, Proposition 6.4.10]). For every natural number $n \in \mathbb{N}$ let $V(n)$ be the free $\Bbbk[\![\hbar]\!]$-module of rank $n+1$ with basis $v_0, \dots, v_n$.

1. There exists a unique $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-module structure on $V(n)$ such that

$$Hv_i := (n-2i)v_i\,, \quad Ev_i := [n-i+1]_q v_{i-1}\,, \quad Fv_i := [i+1]_q v_{i+1}\,.$$

2. The $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-modules $V(n)$ is indecomposable.
3. The $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-module $V(n)$ reduces modulo $\hbar$ to the irreducible representations $\mathrm{L}(n)$ of $\mathrm{U}(\mathfrak{sl}_2)$.
4. The action of K on $V(n)$ is given by

$$K \cdot v_i = q^{n-2i} v_i\,.$$

A. Remarks and Proofs

A.1. Representation Theory of $\mathfrak{sl}_2$

Let $\mathfrak{b}$ denote the Lie subalgebra of $\mathfrak{sl}_2$ consisting of (traceless) upper triangular matrices. It has the matrices e, h as a basis. Its universal enveloping algebra $\mathrm{U}(\mathfrak{b})$ has the PBW-basis $h^m e^n$ with $m, n \geq 0$, and it is a subalgebra of $\mathrm{U}(\mathfrak{sl}_2)$.

Definition A.1. Let V be a representation of $\mathfrak{sl}_2$.

1. The *weight space* of V with respect to λ is $V_\lambda := \{v \in V \mid h.v = \lambda v\}$.
2. A nonzero weight vector v of V is *primitive* if $e.v = 0$.
3. The representation V is of *highest weight* λ if it is generated by a primitive weight vector of weight λ.

Proposition A.2 (Shifting weight spaces). Let V be a representation of $\mathfrak{sl}_2$ and let $\lambda \in \Bbbk$. Then

$$e.V_\lambda \subseteq V_{\lambda+2}\,, \quad f.V_\lambda \subseteq V_{\lambda-2}\,.$$

Proof. This follows from the commutator relations $[H, E] = 2E$ and $[H, F] = -2F$. □

Lemma A.3. Let $\Bbbk$ be algebraically closed. Then every finite-dimensional irreducible representation of $\mathfrak{sl}_2$ is a highest weight representation.

There exists for every scalar $\lambda \in \Bbbk$ a universal representation of highest weight λ, the so-called Verma module:

Definition A.4. For every scalar $\lambda \in \Bbbk$ let $\Bbbk_\lambda$ be the one-dimensional representation of $\mathfrak{b}$ whose underlying vector space is $\Bbbk$ and with action of $\mathfrak{b}$ given by

$$h.1 = \lambda\,, \quad e.1 = 0\,.$$

Lemma A.5. The representation $\Bbbk_\lambda$ can be described as an $\mathrm{U}(\mathfrak{b})$-module as

$$\Bbbk_\lambda \cong \mathrm{U}(\mathfrak{b})/\langle e, h - \lambda\rangle\,.$$

Definition A.6. The representation

$$\mathrm{M}(\lambda) := \mathrm{U}(\mathfrak{sl}_2) \otimes_{\mathrm{U}(\mathfrak{b})} \Bbbk_\lambda$$

of $\mathfrak{sl}_2$ is the *Verma module* of highest weight λ.

Proposition A.7. Let $\lambda \in \Bbbk$.

1. The Verma module $\mathrm{M}(\lambda)$ has the basis

   $$v_i := f^i \otimes 1 \qquad \text{with } i \geq 0,$$

   and the actions of e, h, f on this basis is given by

   $$f.v_i = v_{i+1}\,, \quad h.v_i = (\lambda - 2i)v_i\,, \quad e.v_i = i(\lambda - i + 1)v_{i-1}\,.$$

   This action can be graphically described as in Figure 4.

Suppose that the field $\Bbbk$ is of characteristic zero.

2. The Verma module $\mathrm{M}(\lambda)$ is a representation of highest weight λ.

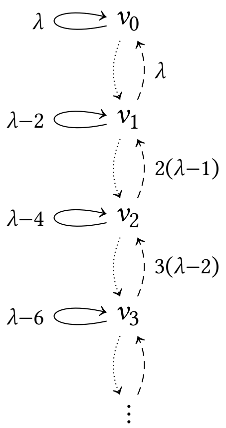

Figure 4: The Verma module $\mathrm{M}(\lambda)$.

3. There exists for every representation V of $\mathfrak{sl}_2$ an isomorphism of vector spaces given by

$$\begin{aligned}\operatorname{Hom}_{\mathfrak{sl}_2}(\mathrm{M}(\lambda), V) &\longrightarrow \{v \in V \mid v \text{ is of weight } \lambda \text{ with } e.v = 0\}\,,\\ \varphi &\longmapsto \varphi(1 \otimes 1)\,.\end{aligned}$$

   In particular

$$\operatorname{End}_{\mathfrak{sl}_2}(\mathrm{M}(\lambda)) = \Bbbk\,.$$

4. The representation $\mathrm{M}(\lambda)$ is indecomposable.
5. a. If $\lambda \notin \mathbb{N}$ then the representation $\mathrm{M}(\lambda)$ is irreducible.
   b. If $\lambda = n \in \mathbb{N}$ then the representation $\mathrm{M}(\lambda)$ has a unique nonzero, proper subrepresentation, which is spanned by

$$v_i \qquad \text{with } i \geq n+1.$$

   This subrepresentation is isomorphic to $\mathrm{M}(-n-2)$.

Definition A.8. Suppose that $\Bbbk$ is of characteristic zero and let $\lambda \in \Bbbk$.

1. For $\lambda \notin \mathbb{N}$ let $\mathrm{L}(\lambda) := \mathrm{M}(\lambda)$.
2. For $\lambda \in \mathbb{N}$ let $\mathrm{L}(\lambda) := \mathrm{M}(\lambda)/N$ where N is the unique nonzero, proper subrepresentation of $\mathrm{M}(\lambda)$.

Theorem A.9. Let $\Bbbk$ be algebraically closed field of characteristic zero.

1. There is a one-to-one correspondence given by

$$\begin{aligned}\left\{\begin{matrix}\text{irreducible highest weight}\\ \text{representations of } \mathfrak{sl}_2\end{matrix}\right\} &\longleftrightarrow \Bbbk\,,\\ \mathrm{L}(\lambda) &\longleftarrow\!\shortmid\; \lambda\,.\end{aligned}$$

2. The representation $\mathrm{L}(\lambda)$ is finite-dimensional if and only if $\lambda = n \in \mathbb{N}$, in which case

$$\dim(\mathrm{L}(n)) = n + 1\,.$$

The above correspondence does therefore restrict to a one-to-one correspondence

$$\begin{aligned} \left\{\begin{matrix}\text{irreducible finite-dimensional} \\ \text{representations of } \mathfrak{sl}_2\end{matrix}\right\} &\longleftrightarrow \mathbb{N}\,, \\ L &\longmapsto \dim(L) - 1\,, \\ \mathrm{L}(n) &\longleftarrow\!\shortmid n\,. \end{aligned}$$

Remark A.10. Let $n \in \mathbb{N}$. The basis $v_0, \dots, v_n$ of $\mathrm{L}(n)$ can be rescaled to the basis

$$w_i := \frac{1}{i!} v_i\,.$$

The actions of e and f then become

$$e.w_i = (n - i + 1)w_{i-1}\,, \quad f.w_i = (i + 1)w_{i+1}\,.$$

The actions of e, h, f on $\mathrm{L}(n)$ can now be graphically be represented as in Figure 1.

Theorem A.11 (Weyl). Let $\Bbbk$ be algebraically closed. Every finite-dimensional representation of $\mathfrak{sl}_2$ is semisimple.

Corollary A.12. Any finite-dimensional representation of $\mathfrak{sl}_2$ admits a weight space decomposition. All occuring weights are integral.

The decomposition of a finite-dimensional representation of $\mathfrak{sl}_2$ into irreducible representations can be read off from its weight space decomposition. From this the following result can be shown:

Proposition A.13 (Clebsch–Gordan). Let n, m be natural numbers with $n \geq m$. Then

$$\mathrm{L}(n) \otimes \mathrm{L}(m) \cong \mathrm{L}(n + m) \oplus \mathrm{L}(n + m - 2) \oplus \cdots \oplus \mathrm{L}(n - m)\,.$$

A.2. An alternative presentation for $\mathrm{U}_q(\mathfrak{sl}_2)$

Let $q \in \Bbbk$ and let U_q be the algebra given by the generators

$$E\,, \quad \widetilde{H}\,, \quad F\,, \quad K\,, \quad K^{-1}$$

and the relations

$$\begin{gathered} KK^{-1} = 1 = K^{-1}K\,, \quad KE = q^2 EK\,, \quad KF = q^{-2} FK\,, \\ [E, F] = \widetilde{H}\,, \quad (q - q^{-1})\widetilde{H} = K - K^{-1}\,, \\ [\widetilde{H}, E] = q(EK + K^{-1}E)\,, \quad [\widetilde{H}, F] = -q^{-1}(FK + K^{-1}F)\,. \end{gathered}$$

Proposition A.14. There exists a unique homomorphism of algebras

$$\psi : U_q \to \mathrm{U}_q(\mathfrak{sl}_2)$$

that is given by

$$\psi(E) = E\,, \quad \psi(\widetilde{H}) = \frac{K - K^{-1}}{q - q^{-1}}\,, \quad \psi(F) = F\,, \quad \psi(K) = K\,,$$

and this homomorphism is an isomorphism.

Proof. See [Kas95, Proposition VI.2.1]. □

Proposition A.15. For $q = 1$ there exists a unique homomorphism of algebras

$$\varphi : U_1 \to \mathrm{U}(\mathfrak{sl}_2)[\sigma]/(\sigma^2 - 1)$$

that is given by

$$\varphi(E) = \sigma E\,, \quad \varphi(\widetilde{H}) = \sigma H\,, \quad \varphi(F) = F\,, \quad \varphi(K) = \sigma\,.$$

Proof. See [Kas95, Proof of Proposition VI.2.2]. □

A.3. Representation Theory of $\mathrm{U}_1(\mathfrak{sl}_2)$

Let A denote the algebra $\mathrm{U}(\mathfrak{sl}_2)[\sigma]/(\sigma^2 - 1)$.

Let M be an $\mathfrak{sl}_2$-representation and let $\varepsilon = \pm 1$. The corresponding $\mathrm{U}(\mathfrak{sl}_2)$-module structure on M extends to an $\mathrm{U}(\mathfrak{sl}_2)[\sigma]$-module structure for which σ acts by multiplication with ε, because σ is central in $\mathrm{U}(\mathfrak{sl}_2)[\sigma]$. It follows from $\varepsilon^2 = 1$ that this induces a A-module structure on M as claimed in Remark 2.5.

If M is irreducible then the resulting A-module is again irreducible since every A-submodule is in particular an $\mathfrak{sl}_2$-subrepresentation. It hence follows that the A-modules $\mathrm{L}(+, n)$ and $\mathrm{L}(-, n)$ that result from the irreducible $\mathfrak{sl}_2$-representation $\mathrm{L}(n)$ are again irreducible. These representations are pairwise non-isomorphic since the element $H\sigma$ of A (which corresponds to the element $\widetilde{H}$ of $\mathrm{U}_1(\mathfrak{sl}_2)$) acts on $\mathrm{L}(+, n)$ with highest weight n and on $\mathrm{L}(-, n)$ with highest weight $-n$.

Let now M be any finite-dimensional M-module. It follows from the relation $\sigma^2 = 1$ in A that the action of σ on A is diagonalizable with eigenvalues 1 and -1. We thus have

$$M = M_1 \oplus M_{-1}$$

with $M_\varepsilon \coloneqq \{m \in M \mid \sigma m = \varepsilon m\}$ for $\varepsilon = \pm 1$. The action of σ on M is an A-module homomorphism because σ is central in A. The decomposition $M = M_1 \oplus M_{-1}$ is therefore one of A-modules.

We may regard both M_1 and M_{-1} as $\mathfrak{sl}_2$-representations by restriction. We then have decompositions into finite-dimensional irreducible $\mathfrak{sl}_2$-representations given by

$$M_1 \cong \mathrm{L}(n_1) \oplus \cdots \oplus \mathrm{L}(n_s)\,, \quad M_{-1} \cong \mathrm{L}(n'_1) \oplus \cdots \oplus \mathrm{L}(n'_t)\,.$$

We note that this is already a decomposition as A-modules since σ acts on M_1 and M_{-1} by multiplication with scalars. As A-modules we have

$$\mathrm{L}(n_i) = \mathrm{L}(+, n_i)\,, \quad \mathrm{L}(n'_i) = \mathrm{L}(-, n'_i)\,.$$

This shows that every finite-dimensional A-module decomposes into a direct sum of the irreducible A-modules $\mathrm{L}(\varepsilon, n)$.

A.4. PBW Basis for $\mathrm{U}_q(\mathfrak{sl}_2)$

We use in the following the notation introduced in Definition 3.9.

Lemma A.16. For every $r \geq 0$ we have

$$[E, F^r] = [r]_q F^{r-1} [K, 1-r]_q \,.$$

Proof. For $r = 0$ both sides vanish and for $r = 1$ this is one of the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$. For $r \geq 2$ the assertion follows by induction, see [Jan96, Appendix 1.3 (5)]. ☐

Corollary A.17. We have

$$\begin{aligned} F \cdot F^l K^m E^n &= F^{l+1} K^m E^n \,, \\ K^{\pm 1} \cdot F^l K^m E^n &= q^{\mp 2l} F^l K^{m \pm 1} E^n \,, \\ E \cdot F^l K^m E^n &= q^{-2m} F^l K^m E^{n+1} + \frac{[l]_q}{q - q^{-1}} (q^{1-l} F^{l-1} K^{m+1-l} E^n - q^{l-1} F^{l-1} K^{m+l-1} E^n) \,. \end{aligned}$$

Proof. This follows from Lemma A.16 and the two relations $KE = q^2 EK$ and $KF = q^{-2} FK$. ☐

Proof of Theorem 2.7. Let U be the linear subspace of $\mathrm{U}_q(\mathfrak{sl}_2)$ spanned by these given monomials. It follows from Corollary A.17 that $\mathrm{U}_q(\mathfrak{sl}_2)$ is a left ideal. It contains the elements $F^0 K^0 E^0 = 1$, whence $U = \mathrm{U}_q(\mathfrak{sl}_2)$. This shows that the given monomials are a vector space generating set.

The linear independence is shown in the usual representation-theoretic way: Let V be the free vector space with basis

$$X^l Y^n Z^m \qquad \text{with } l, n \in \mathbb{N} \text{ and } m \in \mathbb{Z}.$$

There exists an action of $\mathrm{U}_q(\mathfrak{sl}_2)$ on V by using the formulas from Corollary A.17, with $F^l K^m E^n$ replaced by $X^l Y^n Z^m$. (It has to be checked that this proposed action is compatible with the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$, see [Jan96, Appendix 1.5].) The elements

$$F^l K^m E^n \cdot X^0 Y^0 Z^0 = X^l Y^m Z^n$$

are linearly independent in V, whence the given monomials $F^l K^m E^n$ are linearly independent in $\mathrm{U}_q(\mathfrak{sl}_2)$. ☐

A.5. More on the Algebra Structure of $\mathrm{U}_q(\mathfrak{sl}_2)$

Remark A.18.

1. The universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ is noetherian and has no nonzero zero divisors. The same holds for $\mathrm{U}_q(\mathfrak{sl}_2)$, see [Kas95, Proposition VI.1.4] and [Jan96, Propositon 1.8].
2. The algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ admits a grading such that E, K, F are homogeneous with

$$\deg(E) = 1 \,, \quad \deg(F) = -1 \,, \quad \deg(K) = 0 \,.$$

The degree d part of $\mathrm{U}_q(\mathfrak{sl}_2)$ has the basis

$$F^l K^m E^n \qquad \text{with } n - l = d.$$

This grading wan also be characterized in terms of the conjugation map

$$\mathrm{U}_q(\mathfrak{sl}_2) \to \mathrm{U}_q(\mathfrak{sl}_2)\,, \quad x \mapsto KxK^{-1}\,.$$

The degree d part of the grading is precisely the eigenspace with eigenvalue q^{2d}.

Proposition A.19.

1. There exists a unique algebra involution ω of $\mathrm{U}_q(\mathfrak{sl}_2)$ with

$$\omega(E) = F\,, \qquad \omega(K) = K^{-1}\,, \quad \omega(F) = E\,.$$

2. There exists a unique algebra anti-involution τ of $\mathrm{U}_q(\mathfrak{sl}_2)$ with

$$\tau(E) = E\,, \qquad \tau(K) = K^{-1}\,, \quad \tau(F) = F\,.$$

3. There exists a unique algebra isomorphism $\varphi_q : \mathrm{U}_q(\mathfrak{sl}_2) \to \mathrm{U}_{q^{-1}}(\mathfrak{sl}_2)$ with

$$\varphi(E) = -F\,, \quad \varphi(K) = K^{-1} \qquad \varphi(F) = -E\,.$$

   The inverse of the isomorphism φ_q is given by $\varphi_{q^{-1}}$.
4. There exist unique algebra involutions σ_E and σ_F of $\mathrm{U}_q(\mathfrak{sl}_2)$ with

$$\sigma_E(E) = -E\,, \quad \sigma_E(K) = -K\,, \quad \sigma_E(F) = F\,.$$

   and

$$\sigma_F(E) = E\,, \quad \sigma_F(K) = -K\,, \quad \sigma_E(F) = -F\,.$$

Proof. One checks that the proposed images of E, F, $K^{\pm 1}$ are compatible with the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$. See also [Jan96, Lemma 1.2]. □

Remark A.20.

1. One can combine the above (anti-)isomorphisms to construct further (anti-)isomorphisms involving $\mathrm{U}_q(\mathfrak{sl}_2)$ and $\mathrm{U}_{q^{-1}}(\mathfrak{sl}_2)$.
2. It follows from the existence of these (anti-)isomorphisms that many formulas and propositions involving $\mathrm{U}_q(\mathfrak{sl}_2)$ have to satisfy certain symmetries.

A.6. Proof of Proposition 3.3

Lemma A.21. Let M be a finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module.

1. Both E and F act nilpotently on M.
2. For a sufficiently large power $r \geq 0$ (namely such that $F^r M = 0$) the module M is annihilated by
$$\prod_{j=-r}^{r} (K^2 - q^{2j}) \,.$$

Proof. See [Jan96, Proposition 2.1] and [Jan96, Proposition 2.3]. □

Proposition A.22. Every finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module decomposes into weight spaces. All occuring weights are of the form $\pm q^n$ for some $n \in \mathbb{Z}$.

Proof. Let M be a finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module and let k denote the action of K on M. It follows from Lemma A.21 that
$$0 = \prod_{n=-r}^{r} (k^2 - q^{2n}) = \prod_{n=-r}^{r} (k - q^n)(k + q^n) \,.$$
The roots $\pm q^n$ with $n = -r, \dots, r$ are pairwise distinct[1] whence it follows that k is diagonalizable with possible eigenvalues $\pm q^n$ for $n = -r, \dots, r$. □

A.7. Proof of Proposition 3.7

1. Let U be the linear subspace of $\mathrm{U}_q(\mathfrak{sl}_2)$ spanned by the monomials $K^n E^m$ with $n, m \in \mathbb{N}$. This linear subspace is contained in $\mathrm{U}_q(\mathfrak{b})$. It follows on the other hand from the relation $KE = q^2 EK$ that
$$K^n E^m \cdot K^{n'} E^{m'} = q^{2mn'} K^{n+n'} E^{m+m'}$$
for all $n, n', m, m' \in \mathbb{N}$, and we have $1 = K^0 E^0 \in U$. This shows that U is a subalgebra of $\mathrm{U}_q(\mathfrak{sl}_2)$ containing E, K, K^{-1}, and therefore containing $\mathrm{U}_q(\mathfrak{b})$. This shows together that $U = \mathrm{U}_q(\mathfrak{b})$.
2. Let U be the algebra given by generators E, K, K^{-1} and relations
$$KK^{-1} = 1 = K^{-1}K \,, \quad KE = q^2 EK \,.$$
There exists a unique algebra homomorphism $\varphi \colon U \to \mathrm{U}_q(\mathfrak{b})$ given by
$$\varphi(E) = E \,, \quad \varphi(K) = K \,.$$
In the same way as Theorem 2.7 one sees that U has a PBW-basis given by the monomials
$$K^n E^m \qquad \text{with } n \in \mathbb{Z} \text{ and } m \in \mathbb{N}.$$
It follows that the algebra homomorphism φ restricts to a bijection between the PBW-bases of U and $\mathrm{U}_q(\mathfrak{b})$ and is therefore an algebra isomorphism.

[1] If $\pm q^n = \pm q^m$ then squaring both sides of this equation gives $q^{2n} = q^{2m}$ and thus $q^{2(n-m)} = 1$. It follows that $2(n-m) = 0$ because q is not a root of unity, and thus $n = m$.

A.8. Proof of Proposition 3.11

We shown an extended version of Proposition 3.11

Proposition A.23. Let $\lambda \in \Bbbk^\times$.

1. We have $\Bbbk_\lambda \cong \mathrm{U}_q(\mathfrak{b})/\langle E, K - \lambda\rangle$ as an $\mathrm{U}_q(\mathfrak{b})$-module.
2. The Verma module $\mathrm{M}(\lambda)$ has the basis

$$m_i := F^i \otimes 1 \qquad \text{with } i \geq 0,$$

   and the actions of E, K, F on this basis is given by

$$Fm_i = m_{i+1}\,, \quad Km_i = q^{-2i}\lambda m_i\,, \quad Em_i = [i]_q[\lambda, 1-i]_q m_{i-1}\,.$$

   This action can be graphically described as in Figure 2.
3. The Verma module $\mathrm{M}(\lambda)$ is of highest weight λ, and every $\mathrm{U}_q(\mathfrak{sl})$-module of highest weight λ is a quotient of $\mathrm{M}(\lambda)$.
4. There exists for every $\mathrm{U}_q(\mathfrak{sl}_2)$-module M an isomorphism of vector spaces given by

$$\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{sl}_2)}(\mathrm{M}(\lambda), M) \cong \{m \in M \mid m \text{ is of weight } \lambda \text{ with } Em = 0\}\,.$$

   It follows in particular that

$$\mathrm{End}_{\mathrm{U}_q(\mathfrak{sl}_2)}(\mathrm{M}(\lambda)) = \Bbbk\,.$$

5. The Verma module $\mathrm{M}(\lambda)$ is indecomposable.
6. a. If $\lambda = \pm q^n$ for some $n \in \mathbb{N}$ then the Verma module $\mathrm{M}(\lambda)$ contains a unique nonzero, proper submodule, which is spanned by the elements

$$m_i \qquad \text{with } i \geq n+1.$$

   This submodule is isomorphic to $\mathrm{M}(q^{-n-2}\lambda)$.

   b. If $\lambda \neq \pm q^n$ for every $n \in \mathbb{N}$ then the Verma module $\mathrm{M}(\lambda)$ is irreducible.

1. This follows from the PBW-basis of $\mathrm{U}_q(\mathfrak{b})$.
2. This follows from the PBW-basis of $\mathrm{U}_q(\mathfrak{sl}_2)$ and induction.
3. The Verma module $\mathrm{M}(\lambda)$ is generated by the primitive weight vector $1 \otimes 1$.
4. We have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{sl}_2)}(\mathrm{M}(\lambda), M) &\cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{b})}(\Bbbk_\lambda, M)\\ &\cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{b})}(\mathrm{U}_q(\mathfrak{b})/\langle K-\lambda, E\rangle, M)\\ &\cong \{m \in M \mid (K-\lambda)m = 0, Em = 0\}\,.\end{aligned}$$

5. The endomorphism algebra $\mathrm{End}_{\mathrm{U}_q(\mathfrak{sl}_2)}(\mathrm{M}(\lambda)) = \Bbbk$ does not contain any non-trivial idempotents.
6. This follows as for $\mathrm{U}(\mathfrak{sl}_2)$ since $[i]_q[\lambda, i-1]_q = 0$ if and only if $\lambda = \pm q^{i-1}$.

A.9. Proof of Theorem 3.15

Lemma A.24. If M is an highest-weight $\mathrm{U}_q(\mathfrak{sl}_2)$-module then

$$\mathrm{End}_{\mathrm{U}_q(\mathfrak{sl}_2)}(M) = \Bbbk\,.$$

Definition A.25. The *quantum Casimir element* is the element $C_q \in \mathrm{U}_q(\mathfrak{sl}_2)$ given by

$$C_q := EF + \frac{Kq^{-1} + K^{-1}q}{(q-q^{-1})^2}\,.$$

Lemma A.26.

1. The element C_q is central in $\mathrm{U}_q(\mathfrak{sl}_2)$.
2. The element C_q acts on every $\mathrm{U}_q(\mathfrak{sl}_2)$-module by module endomorphisms.
3. The element C_q acts for every scalar $\lambda \in \Bbbk^\times$ on the representation $\mathrm{L}(\lambda)$ by multiplication with the scalar
$$\frac{\lambda q + \lambda^{-1}q^{-1}}{(q-q^{-1})^2}\,.$$
4. The element C_q acts the same on $\mathrm{L}(\lambda)$ and $\mathrm{L}(\mu)$ if and only if $\lambda = \mu$ or $\lambda = \mu^{-1}q^{-2}$.

Proof.

1. It can be checked that C_q commutes with E, F, K by using the defining relations for $\mathrm{U}_q(\mathfrak{sl}_2)$.
2. This follows from the previous assertion.
3. It follows from the previous assertion and Lemma A.24 that C_q acts by a scalar. This scalar can be read off from the action on the primitive generator $1 \otimes 1$. It thus sufficies to show the assertion for $\mathrm{M}(\lambda)$, where it follows from Proposition 3.11.
4. This follows from the previous assertion. □

Corollary A.27. The quantum Casimir element C_q acts on every finite-dimensional, irreducible representation of $\mathrm{U}_q(\mathfrak{sl}_2)$ by a different scalar.

Proof. If $\lambda = \delta q^n$ and $\mu = \varepsilon q^m$ with $\delta, \varepsilon \in \{1, -1\}$ and $n, m \in \mathbb{N}$ then it cannot happen that $\lambda = \mu^{-1}q^{-2}$. The assertion thus follows from Lemma A.26. □

Proof of Theorem 3.15 ([Jan96, Theorem 2.9]). Let M be any finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module and let c denote the action of C_q on M. We may assume that M is indecomposable. We can consider a composition series

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_r = M \tag{4}$$

with composition factors

$$M_i / M_{i-1} \cong \mathrm{L}(\varepsilon_i q^{n_i})\,.$$

Letting c_i be the scalar by which C_q acts on $\mathrm{L}(\varepsilon_i q^{n_i})$, we have

$$(c - c_i)M_i \subseteq M_{i-1}\,.$$

It follows that $\prod_{i=1}^{r}(c - c_i)$ annihilates M and that c admits a generalized eigenspace decomposition with eigenvalues $c_1, \dots, c_r$. The resulting generalized eigenspaces are subrepresentations because c is a $\mathrm{U}_q(\mathfrak{sl}_2)$-module endomorphism. It follows that

$$c_1 = \cdots = c_r$$

because M is indecomposable, and thus

$$\varepsilon_1 q^{n_1} = \cdots = \varepsilon_r q^{n_r} =: \lambda$$

by Corollary A.27. It follows with the composition series (4) that

$$\dim(M_\mu) = r \dim(\mathrm{L}(\lambda)_\mu)$$

for every scalar $\mu \in \Bbbk^\times$. Thus M is of highest weight λ.

The short exact sequence

$$0 \to M_{r-1} \to M \to \mathrm{L}(\lambda) \to 0 \tag{5}$$

restricts to a short exact sequence

$$0 \to (M_{r-1})_\lambda \to M_\lambda \to \mathrm{L}(\lambda)_\lambda \to 0\,.$$

It follows that the primitive generator v_0 of $\mathrm{L}(\lambda)$ has a preimage m_0 in M. The weight vector m_0 is primitive because M isof highest weight λ. It follows that there exists a homomorphism of $\mathrm{U}_q(\mathfrak{sl}_2)$-modules

$$\varphi\colon\ \mathrm{M}(\lambda) \to M\,, \quad 1 \otimes 1 \mapsto m_0\,.$$

It follows from the finite-dimensionality of M that φ factors through a homomorphism

$$\psi\colon\ \mathrm{L}(\lambda) \to M\,, \quad \overline{1 \otimes 1} \mapsto m_0\,.$$

This shows that the short exact sequence (5) splits, whence

$$M \cong M_{r-1} \oplus \mathrm{L}(\lambda)\,.$$

It follows by induction that $M_{r-1} \cong \mathrm{L}(\lambda)^{\oplus(r-1)}$ and thus altogether $M \cong \mathrm{L}(\lambda)^{\oplus r}$. □

Remark A.28. The center of the universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ is a polynomial algebra, generated by the classical Casimir element $C = (ef + h^2 + fe)/4$. It can be shown that the center of $\mathrm{U}_q(\mathfrak{sl}_2)$ is again a polynomial algebra, now generated by the quantum Casimir element C_q. We refer to [Jan96, Proposition 2.18] for more details on this.

A.10. Proof of Lemma 4.4

We have

$$M_\mu \otimes N_\kappa \subseteq (M \otimes N)_{\mu\kappa}$$

for all $\mu, \kappa \in \Bbbk^\times$ since the element K is group-like in $\mathrm{U}_q(\mathfrak{sl}_2)$. Both M and N admits weight space decompositions

$$M = \bigoplus_\mu M_\mu\,, \quad N = \bigoplus_\kappa N_\kappa$$

and it follows that

$$M \otimes N = \left(\bigoplus_\mu M_\mu\right) \otimes \left(\bigoplus_\kappa N_\kappa\right) = \bigoplus_{\mu,\kappa} (M_\mu \otimes N_\kappa) \subseteq \bigoplus_\lambda M_\lambda \subseteq M \otimes N$$

It folllows with the inclusions $M_\mu \otimes N_\kappa \subseteq (M \otimes N)_{\mu\kappa}$ that already

$$(M \otimes N)_\lambda = \bigoplus_{\mu\kappa=\lambda} M_\mu \otimes N_\kappa$$

for every λ.

A.11. More on Deformation of Algebras

Definition A.29. Let A be an $\Bbbk$-algebra.

1. Two deformations $A_\hbar$ and $A'_\hbar$ of the algebra A are *equivalent* if there exists an isomorphism of $\Bbbk[\![\hbar]\!]$-algebras

   $$\varphi : A_\hbar \to A'_\hbar$$

   such that the induced isomorphism of $\Bbbk$-algebras

   $$A = A_\hbar / \hbar A_\hbar \to A'_\hbar / \hbar A'_\hbar = A$$

   is the identity, i.e. φ is the identity modulo $\hbar$.
2. A deformation is *trivial* if it is equivalent to the trivial deformation (i.e. $A[\![\hbar]\!]$).

Remark A.30. One can study the deformation theory of an $\Bbbk$-algebra via homological algebra: The *Hochschild cochain complex* of A is given by

$$\mathrm{C}^n_{\mathrm{Hoch}}(A) \coloneqq \mathrm{Hom}_{\Bbbk}(A^{\otimes n}, A)$$

together with certain differentials. The cohomology of this chain complex is the *Hochschild cohomology* of A, which is denoted by

$$\mathrm{HH}^n(A) \coloneqq \mathrm{H}^n\big(\mathrm{C}^\bullet_{\mathrm{Hoch}}\big)\,.$$

One of the connections between deformation theory and Hochschild cohomology is that in the case of

$$\mathrm{HH}^2(A) = 0$$

every deformation of A is trivial.

Warning A.31. Let $A_\hbar$ be a deformation of an $\Bbbk$-algebra A with $\mathrm{HH}^2(A) = 0$. The above criterion shows that $A_\hbar$ is equivalent to $A[\![\hbar]\!]$, but it does not provide an explicit isomorphism.

Example A.32. Let $\mathfrak{g}$ be a semisimple Lie algebra. It can be shown that

$$\mathrm{HH}^2(\mathrm{U}(\mathfrak{g})) = 0$$

whence all deformations of $\mathrm{U}(\mathfrak{g})$ are trivial. (See [GS92, Theorem 2].)

It follows in particular that the every algebra deformation of $\mathrm{U}(\mathfrak{sl}_2)$ is trivial. An explicit equivalence between $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ and $\mathrm{U}(\mathfrak{sl}_2)[\![\hbar]\!]$ is constructed in [CP95, Proposition 4.6.4].

A.12. More on Completions

Example A.33. Every finite-dimensional $\Bbbk[\![\hbar]\!]$-module M is complete since $\hbar^n M = 0$ for some sufficiently large power n.

Proposition A.34. Let M, N be two $\Bbbk[\![\hbar]\!]$-modules.

1. For every homomorphism of $\Bbbk[\![\hbar]\!]$-module $f\colon M \to N$ there exists a unique module homomorphism $\widehat{f}\colon \widehat{M} \to \widehat{N}$ that makes the following square diagram commute:

$$\begin{tikzcd} \widehat{M} \arrow[r, "\widehat{f}"] & \widehat{N} \\ M \arrow[u] \arrow[r, "f"] & N \arrow[u] \end{tikzcd}$$

The homomorphism $\widehat{f}$ is given by

$$\widehat{f}\Big((\overline{m_0}, \overline{m_1}, \dots)\Big) = \Big(\overline{f(m_0)}, \overline{f(m_1)}, \dots\Big).$$

2. The assignment $\widehat{(-)}$ defines a functor

$$\widehat{(-)}\colon \Bbbk[\![\hbar]\!]\text{-}\mathbf{Mod} \to \Bbbk[\![\hbar]\!]\text{-}\mathbf{Mod}.$$

3. If M, N are complete then

$$f\left(\sum_{i=0}^{\infty} \hbar^i m_i\right) = \sum_{i=0}^{\infty} \hbar^i f(m_i)$$

for every sequence of elements $m_0, m_1, \dots, \in M$.

4. If N is complete then every homomorphism $M \to N$ extends uniquely to a homomorphism $\widehat{M} \to N$.

$$\begin{tikzcd} \widehat{M} \arrow[r, dashed, "\exists!"] & N \\ M \arrow[u] \arrow[ur] & \end{tikzcd}$$

5. If V is any $\Bbbk$-vector space and N is complete then every $\Bbbk$-linear map $f : V \to N$ extends uniquely to a $\Bbbk[\![\hbar]\!]$-linear linear map $f' : V[\![\hbar]\!] \to N$.

$$\begin{tikzcd} V[\![\hbar]\!] \arrow[r, dashed, "\exists!"] & N \\ V \arrow[u] \arrow[ur] & \end{tikzcd}$$

The homomorphism f' is given by

$$f'\left(\sum_{i=0}^{\infty} \hbar^i v_i\right) = \sum_{i=0}^{\infty} \hbar^i f(v_i) .$$

6. The canonical homomorphism $M \to \widehat{M}$ induces an isomorphism of $\Bbbk$-vector spaces

$$M/\hbar M \longrightarrow \widehat{M}/\hbar\widehat{M} .$$

Remark A.35. Let M be a $\Bbbk[\![\hbar]\!]$-module. There exists a unique topology on M for which a basis is given by the sets

$$m + \hbar^{n+1} M$$

with $m \in M$ and $n \geq 0$. This topology is the *$\hbar$-adic topology* on M. It makes $\Bbbk[\![\hbar]\!]$ into a topological ring and every $\Bbbk[\![\hbar]\!]$-module into a topological $\Bbbk[\![\hbar]\!]$-module. The completion $\widehat{M}$ is then the usual topological completion of M.

A.13. More on Deformation of Hopf Algebras

Definition A.36. The terms *topological algebra*, *topological coalgebra* and *topological bialgebra* are defined analogous to topological Hopf algebras.

Remark A.37. A topological algebra in the sense of Definition A.36 is the same as an $\Bbbk[\![\hbar]\!]$-algebra which is complete as an $\Bbbk[\![\hbar]\!]$-module.

Indeed, suppose first that (A, m, u) is a topological algebra. Then the multiplication

$$m : A \mathbin{\widehat{\otimes}} A \to A$$

restricts via the composition with the canonical homomorphism

$$A \otimes A \to A \mathbin{\widehat{\otimes}} A$$

to a multiplication

$$m' : A \otimes A \to A .$$

Then (A, m', u) is an $\Bbbk[\![\hbar]\!]$-algebra (and A is by definition complete).

Suppose on the other hand that (A, m', u) is an $\Bbbk[\![\hbar]\!]$-algebra where A is complete. Then the multiplication map

$$m' : A \otimes A \to A$$

extends by the completeness of A uniquely to a $\Bbbk[\![\hbar]\!]$-linear map

$$m : A \mathbin{\widehat{\otimes}} A \to A .$$

Then (A, m, u) is a topological algebra (by the denseness of $A \otimes A$ in $A \mathbin{\widehat{\otimes}} A$, etc.).

Definition A.38. Let A be a Hopf algebra.

1. (Formal) deformations of coalgebras and bialgebras are defined in the way as for algebras and Hopf algebras.
2. Two Hopf algebra deformations $A_\hbar$ and $A'_\hbar$ of A are *equivalent* if there exists an isomorphism of Hopf algebras
$$\varphi\colon\ A_\hbar \to A'_\hbar$$
such that the induced isomorphism of Hopf algebras
$$A = A_\hbar/\hbar A_\hbar \to A'_\hbar/\hbar A'_\hbar = A$$
is the identity, i.e. φ is the identity modulo $\hbar$.
   Equivalence of deformations of coalgebras and bialgebras is defined in the same way.
3. A deformation is *trivial* if it is equivalent to the trivial deformation (i.e. $A[\![\hbar]\!]$).
4. A Hopf algebra deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a *quantum universal enveloping algebra (QUE).*

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