

The Quantum Group $U_q(\mathfrak{sl}_2)$

Talk 14 on Hopf Algebras and Tensor Categories

1. Recalling \mathfrak{sl}_2 -Theory

Let \mathbb{k} be a field. The Lie algebra

$$\mathfrak{sl}_2 := \{A \in M(2, \mathbb{k}) \mid \text{tr}(A) = 0\}$$

admits the basis

$$E := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

and these basis elements satisfy the commutator relations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H. \quad (1)$$

Its universal enveloping algebra

$$U(\mathfrak{sl}_2) := T(\mathfrak{sl}_2) / (XY - YX - [X, Y] \mid X, Y \in \mathfrak{sl}_2)$$

is generated by the elements E, H, F subject to the relations (1), i.e.

$$U(\mathfrak{sl}_2) \cong \mathbb{k}\langle E, H, F \rangle / ([H, E] - 2E, [H, F] + 2F, [E, F] - H).$$

The universal enveloping algebra $U(\mathfrak{sl}_2)$ is a Hopf algebra with comultiplication

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \varepsilon(X) = 0, \quad S(X) = 0 \quad \text{for every } X \in \mathfrak{sl}_2.$$

A representation of \mathfrak{sl}_2 is the same as an $U(\mathfrak{sl}_2)$ -module.

Theorem 1.1 (Poincaré–Birkhoff–Witt). The algebra $U(\mathfrak{sl}_2)$ admits the vector space basis

$$F^l H^m E^n \quad \text{with } l, m, n \in \mathbb{N}.$$

Theorem 1.2. Let \mathbb{k} be of characteristic zero.

1. Every finite-dimensional \mathfrak{sl}_2 -representation is semisimple.
2. The finite-dimensional irreducible \mathfrak{sl}_2 -representations are (up to isomorphism) given by certain representations $L(n)$ for $n \in \mathbb{N}$. This representation $L(n)$ has a basis w_0, \dots, w_n on which E, H, F act as depicted in Figure 1.

We refer to Appendix A.1 for more details on the representation theory of the Lie algebra \mathfrak{sl}_2 in characteristic zero.

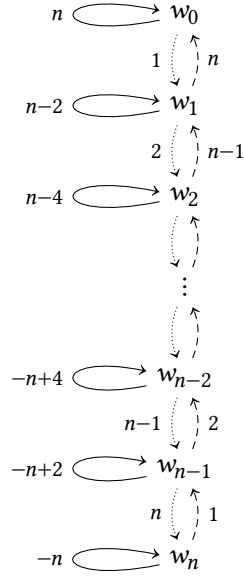


Figure 1: The irreducible representation $L(n)$ of $U(\mathfrak{sl}_2)$. Loops depict the action of H , dashed arrows the action of E and dotted arrows the action of F .

2. The Algebra $U_q(\mathfrak{sl}_2)$

Convention 2.1. In the following \mathbb{k} denotes a field of characteristic zero and q is an element of \mathbb{k} with $q \neq 0, 1, -1$.

Definition 2.2. The \mathbb{k} -algebra $U_q(\mathfrak{sl}_2)$ is given by the generators

$$E, \quad F, \quad K, \quad K^{-1}$$

subject to the relations

$$KK^{-1} = 1 = K^{-1}K, \quad KE = q^2EK, \quad KF = q^{-2}FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \quad (2)$$

Remark 2.3 (Choice of q). One often requires additional conditions on q , namely that

1. q is not a root of unity, or that
2. \mathbb{k} is the field $\mathbb{K}(q)$ over some other field \mathbb{K} , with q being the indeterminate.

Remark 2.4 (The case $q = 1$). The algebra $U_q(\mathfrak{sl})$ admits another useful presentations: One introduces the element

$$\tilde{H} := \frac{K - K^{-1}}{q - q^{-1}}$$

as an additional generator, and then adjust the relations (2). The resulting presentation does then make sense for any $q \in \mathbb{k}$, and one has

$$U_1(\mathfrak{sl}_2) \cong U(\mathfrak{sl}_2)[\sigma]/(\sigma^2 - 1)$$

given by

$$E \mapsto \sigma E, \quad \tilde{H} \mapsto \sigma H, \quad F \mapsto F, \quad K \mapsto \sigma. \quad (3)$$

We refer to Appendix A.2 for more details on this presentation.

Remark 2.5. One might think about E and F as the usual elements of \mathfrak{sl}_2 , but $U_q(\mathfrak{sl}_2)$ does not contain H . We will later see that the algebra $U_q(\mathfrak{sl})$ lives (up to some technical details) inside an $\mathbb{k}[[\hbar]]$ -algebra $U_{\hbar}(\mathfrak{sl}_2)$ that also contains H , and in which

$$q = e^{\hbar}, \quad K = e^{\hbar H}.$$

We may therefore think about the element K as

$$K = q^H.$$

Theorem 2.6 (PBW basis). The algebra $U_q(\mathfrak{sl}_2)$ has a vector space basis given by

$$F^l K^m E^n \quad \text{with } l, n \in \mathbb{N} \text{ and } m \in \mathbb{Z}$$

Proof. See Appendix A.4. □

We refer to Appendix A.5 for more remarks on the algebra structure of $U_q(\mathfrak{sl}_2)$.

3. Representation Theory of $U_q(\mathfrak{sl}_2)$

We will in this section focus on the finite-dimensional representation theory of $U_q(\mathfrak{sl}_2)$.

3.1. The Case $q = 1$

Every \mathfrak{sl}_2 -representation extends to a $U_1(\mathfrak{sl}_1)$ -module by letting σ act by either 1 or -1 . The resultings $U_1(\mathfrak{sl}_1)$ -modules are denoted by $L(\varepsilon, n)$ for $\varepsilon = \pm$ and $n \in \mathbb{N}$. One can conclude from Theorem 1.2 that every finite-dimensional $U_1(\mathfrak{sl}_2)$ -module is semisimple, and that the irreducible finite-dimensional $U_1(\mathfrak{sl}_2)$ -modules are given precisely given by $L(\pm, n)$. One can depict these irreducible modules as in Figure 2. We refer to Appendix A.3 for proofs of these claims.

We will keep the case of $U_1(\mathfrak{sl}_2)$ in the back of our minds while considering the following discussion.

3.2. Weight Space Decomposition

Convention 3.1. In the following q is an element of \mathbb{k} which is not a root of unity, unless otherwise specified.

Definition 3.2. Let M be an $U_q(\mathfrak{sl}_2)$ -module. For every scalar $\lambda \in \mathbb{k}^\times$ the associated *weight space* is given by

$$M_\lambda := \{m \in M \mid Km = \lambda m\}.$$

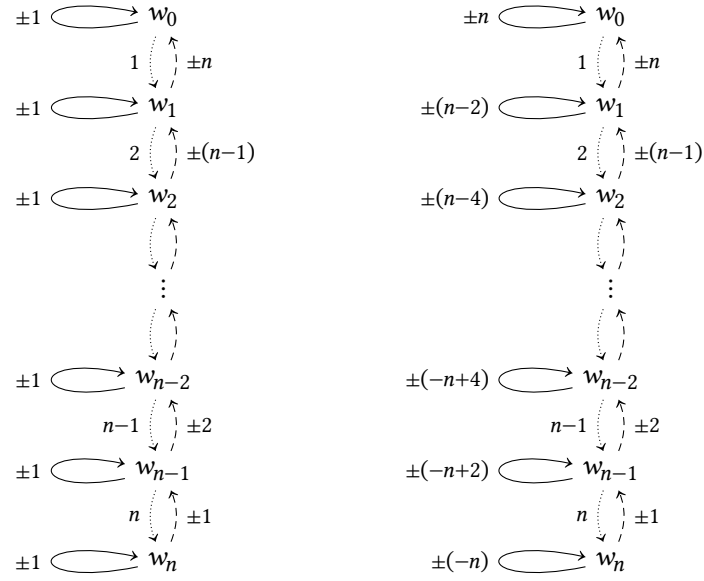


Figure 2: The irreducible representation $L(\pm, n)$ of $U_1(\mathfrak{sl}_2)$. On the left side loops depict the action of K , and on the right side they depict the action of \tilde{H} . On both sides dashed arrows depict the action of E and dotted arrows depict the action of F .

Theorem 3.3. Let M be an $U_q(\mathfrak{sl}_2)$ -module.

1. It holds for every scalar $\lambda \in \mathbb{k}^\times$ that

$$EM_\lambda \subseteq M_{q^2\lambda}, \quad FM_\lambda \subseteq M_{q^{-2}\lambda}.$$

2. If M is finite-dimensional then M decomposes into weight spaces, and all occurring weights are of the form $\pm q^n$ with $n \in \mathbb{Z}$.

Proof. See Appendix A.6. □

3.3. Verma Modules and Classifications

Definition 3.4. Let M be an $U_q(\mathfrak{sl}_2)$ -module.

1. A weight vector m is *primitive* if it is nonzero and $Em = 0$.
2. The module M is of *highest weight* λ if it is generated by a primitive weight vector of weight λ .

Proposition 3.5. Every irreducible, finite-dimensional $U_q(\mathfrak{sl}_2)$ -module is a highest weight module.

Proof. The assertion follows from Theorem 3.3. □

We will classify the irreducible highest-weight representations of $U_q(\mathfrak{sl}_2)$ and its irreducible finite-dimensional representations. We mirror the corresponding classifications of \mathfrak{sl}_2 -representations.

Definition 3.6. Let $U_q(\mathfrak{b})$ be the subalgebra of $U_q(\mathfrak{sl}_2)$ generated by E, K, K^{-1} .¹

Definition 3.7. Let $\lambda \in \mathbb{k}^\times$.

1. Let \mathbb{k}_λ be the one-dimensional $U_q(\mathfrak{b})$ -module whose underlying vector space is given by \mathbb{k} , together with the action

$$K \cdot 1 = \lambda, \quad E \cdot 1 = 0.$$

2. The *Verma module* associated to λ is the $U_q(\mathfrak{sl}_2)$ -module given by

$$M(\lambda) := U_q(\mathfrak{sl}_2) \otimes_{U_q(\mathfrak{b})} \mathbb{k}_\lambda.$$

Definition 3.8. For $q \in \mathbb{k}$ with $q \neq 0$ the n -th *quantum integer* is

$$[n]_q := q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1},$$

and thus for $q \neq 1, 0, -1$,

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The *quantum factorial* is

$$[n]_q! := [n]_q [n-1]_q \cdots [1]_q.$$

For every invertible element $x \in U_q(\mathfrak{sl}_2)$ and integer $n \in \mathbb{Z}$ let

$$[x, n]_q := \frac{q^n x - q^{-n} x^{-1}}{q - q^{-1}}.$$

Remark 3.9. For $q = 1$ we have $[n]_1 = n$ and $[n]_1! = n!$.

Proposition 3.10. Let $\lambda \in \mathbb{k}^\times$.

1. The Verma module $M(\lambda)$ has the basis

$$m_i := F^i \otimes 1 \quad \text{with } i \geq 0,$$

and the actions of E, K, F on this basis is given by

$$Fm_i = m_{i+1}, \quad Km_i = q^{-2i} \lambda m_i, \quad Em_i = [i]_q [\lambda, 1 - i]_q m_{i-1}.$$

This action can be graphically described as in Figure 3.

2. The Verma module $M(\lambda)$ is indecomposable.

¹Here \mathfrak{b} refers to the Lie subalgebra of \mathfrak{sl}_2 consisting of the traceless upper triangular matrices, see Appendix A.1.

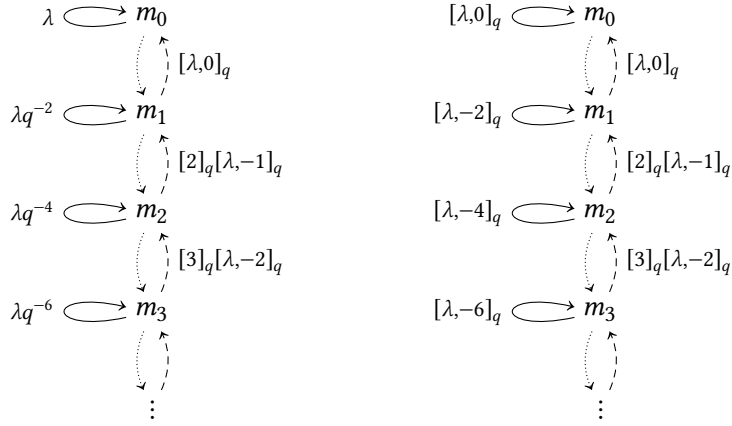


Figure 3: The Verma module $M(\lambda)$. On the left side the loops depict the action of K , and on the right side they depict the action of \tilde{H} . On both sides the action of F is depicted by dotted arrows and the action of E by dashed arrows.

3. a. If $\lambda = \pm q^n$ for some $n \in \mathbb{N}$ then the Verma module $M(\lambda)$ contains a unique nonzero, proper submodule N_λ , which is spanned by the elements

$$m_i \quad \text{with } i \geq n + 1.$$

This submodule is isomorphic to $M(q^{-n-2}\lambda)$.

- b. If $\lambda \neq \pm q^n$ for every $n \in \mathbb{N}$ then the Verma module $M(\lambda)$ is irreducible.

Proof. See Appendix A.7. □

Definition 3.11. For every scalar $\lambda \in \mathbb{k}^\times$ let

$$L(\lambda) := \begin{cases} M(\lambda)/N_\lambda & \text{if } \lambda = \pm q^n \text{ for some } n \in \mathbb{N}, \\ M(\lambda) & \text{otherwise.} \end{cases}$$

Theorem 3.12.

1. There is a one-to-one correspondence given by

$$\begin{aligned} \mathbb{k}^\times &\mapsto \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{highest-weight irreducible} \\ \text{U}_q(\mathfrak{sl}_2)\text{-modules} \end{array} \right\}, \\ \lambda &\mapsto L(\lambda). \end{aligned}$$

2. The module $L(\lambda)$ is finite-dimensional if and only if $\lambda = \pm q^n$ for some $n \in \mathbb{N}$. The above

one-to-one correspondence does therefore restrict to a one-to-one correspondence given by

$$\begin{aligned} \{1, -1\} \times \mathbb{N} &\mapsto \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{finite-dimensional irreducible} \\ \text{U}_q(\mathfrak{sl}_2)\text{-modules} \end{array} \right\}, \\ (\varepsilon, n) &\mapsto L(\varepsilon q^n). \end{aligned}$$

We have for every $n \in \mathbb{N}$ that

$$\dim(L(\pm q^n)) = n + 1.$$

Remark 3.13.

1. For every $n \in \mathbb{Z}$ we have

$$[\pm q^n, -i + 1]_q = \pm[n - i + 1]_q.$$

On the rescaled basis m_0, \dots, m_n of $L(\pm q^n)$ given by

$$w_i := \frac{m_i}{[i]_q!}$$

the actions of E, F thus become

$$E w_i = \pm[n - i + 1]_q w_{i-1}, \quad F w_i = [i + 1]_q w_{i+1}.$$

The action of E, H, K on $L(\pm q^n)$ can therefore be graphically be represented as in Figure 4

2. We can consider again the element

$$\tilde{H} := \frac{K - K^{-1}}{q - q^{-1}}$$

of $U_q(\mathfrak{sl}_2)$. It acts on the weight space $M_{\lambda q^{-2i}}$ by the scalar $[\lambda, -2i]_q$. For $\lambda = \pm q^n$ this means

$$[\lambda, -2i]_q = [\pm q^n, -2i]_q = \pm[n - 2i]_q.$$

The action of \tilde{H} on the Verma module $M(\lambda)$ and irreducible modules $L(\pm q^n)$ is therefore as depicted in Figure 3 and Figure 4.

3. We observe that for $q = 1$ the descriptions of the irreducible $U_q(\mathfrak{sl}_2)$ -modules $L(\pm q^n)$ from Figure 4 becomes the description of the irreducible $U_1(\mathfrak{sl}_2)$ -modules $L(\pm, n)$ from Figure 2.

3.4. Semisimplicity of Finite-Dimensional $U_q(\mathfrak{sl}_2)$ -modules

Theorem 3.14. Every finite-dimensional $U_q(\mathfrak{sl}_2)$ -module is semisimple.

Proof. See Appendix A.8. □

Corollary 3.15. Let M, N be two finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules with $\dim M_\lambda = \dim N_\lambda$ for every $\lambda \in \mathbb{k}^\times$. Then $M \cong N$. □

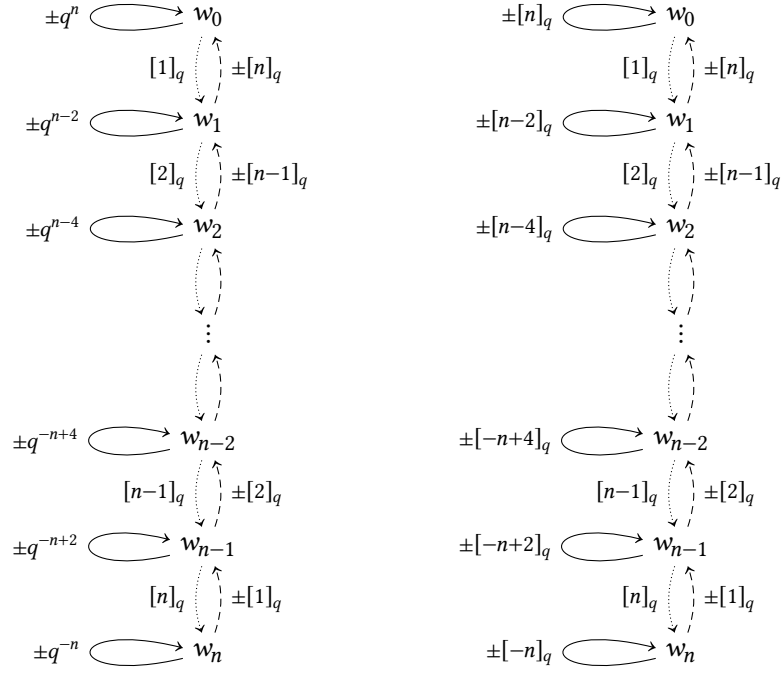


Figure 4: The irreducible representation $L(\pm q^n)$ of $U_q(\mathfrak{sl}_2)$. On the left side the loops depict the action of K , and on the right side they depict the action of \tilde{H} . On both sides the action of F is depicted by dotted arrows and the action of E by dashed arrows.

4. Hopf Algebra Structure on $U_q(\mathfrak{sl}_2)$

Proposition 4.1. The algebra $U_q(\mathfrak{sl}_2)$ becomes a Hopf algebra when endowed with the comultiplication

$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \quad \Delta(K) = K \otimes K,$$

the counit

$$\varepsilon(E) = 0, \quad \varepsilon(F) = 0, \quad \varepsilon(K) = 1$$

and the antipode

$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}.$$

Proof. One checks that the proposed images of the algebra generators E, F, K, K^{-1} are compatible with the defining relations of $U_q(\mathfrak{sl}_2)$, and that the Hopf algebra diagram commute on these algebra generators. \square

Definition 4.2. The Hopf algebra structure of $U_q(\mathfrak{sl}_2)$ is given as in Proposition 4.1.

Remark 4.3. The Hopf algebra $U_q(\mathfrak{sl}_2)$ is neither commutative nor cocommutative. It is an example of a so-called *quantum group*.

Lemma 4.4. Let M, N be two finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules. Then

$$(M \otimes N)_\lambda = \bigoplus_{\mu\kappa=\lambda} M_\mu \otimes N_\kappa.$$

Proof. See Appendix A.9. \square

Corollary 4.5. Let M, N be two finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules. Then

$$M \otimes N \cong N \otimes M.$$

Proof. This follows from Corollary 3.15 and Lemma 4.4. \square

Warning 4.6. For two (finite-dimensional) $U_q(\mathfrak{sl}_2)$ -modules M, N the flip map

$$\tau : M \otimes N \rightarrow N \otimes M, \quad m \otimes n \mapsto n \otimes m$$

is in general not $U_q(\mathfrak{sl}_2)$ -linear.

Example 4.7. Indeed, let us consider $M = N = L(q)$ with basis m_0, m_1 . Then

$$F \cdot (m_0 \otimes m_1) = m_1 \otimes m_1 \neq qm_1 \otimes m_1 = F \cdot (m_1 \otimes m_0).$$

There exists a quantum version of the Clebsch–Gordan formula, see Appendix A.10.

5. Outlook: The Deformation $U_{\hbar}(\mathfrak{sl}_2)$

Definition 5.1. Let A be a Hopf algebra over \mathbb{k} . A *(formal) deformation* of a Hopf algebra A is a Hopf algebra over $\mathbb{k}[[\hbar]]$ such that $A_{\hbar} = A[[\hbar]]$ as $\mathbb{k}[[\hbar]]$ -modules and $A_{\hbar}/\hbar A_{\hbar} = A$ as Hopf algebras over \mathbb{k} .

Remark 5.2. The above definition is actually wrong. Instead of simply Hopf algebras over $\mathbb{k}[[\hbar]]$ one needs to consider *topological Hopf algebras*. This means that for the comultiplication of A_{\hbar} one has to replace the tensor product

$$A_{\hbar} \otimes_{\mathbb{k}[[\hbar]]} A_{\hbar}$$

by its \hbar -adic completion

$$A_{\hbar} \widehat{\otimes} A_{\hbar}.$$

In the given situation we have

$$A_{\hbar} \widehat{\otimes} A_{\hbar} = A[[\hbar]] \widehat{\otimes} A[[\hbar]] \cong (A \otimes A)[[\hbar]]$$

as $\mathbb{k}[[\hbar]]$ -modules. This means that we must allow the comultiplication to take as values not only tensors, but actually power series of tensors.

Theorem 5.3. The universal enveloping algebra $U(\mathfrak{sl}_2)$ admits a Hopf algebra deformation

$$U_{\hbar}(\mathfrak{sl}_2)$$

with

$$\begin{aligned} [H, E] &= 2E, \quad [H, F] = 2F, \quad [E, F] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}}, \\ \Delta(E) &= E \otimes e^{\hbar H} + 1 \otimes E, \quad \Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(F) = F \otimes 1 + e^{-\hbar H} \otimes F, \\ \varepsilon(E) &= 0, \quad \varepsilon(H) = 0, \quad \varepsilon(F) = 0, \\ S(E) &= -Ee^{-\hbar H}, \quad S(H) = -H, \quad S(F) = -e^{\hbar H}F. \end{aligned}$$

Remark 5.4.

1. In the algebra $U_{\hbar}(\mathfrak{sl}_2)$ we can consider the well-defined elements

$$q := e^{\hbar}, \quad K := e^{\hbar H}.$$

The elements E, F, K, K^{-1} satisfy the defining relations of $U_q(\mathfrak{sl}_2)$. We can thus (up to some technical details) regard $U_q(\mathfrak{sl}_2)$ as a subalgebra of $U_{\hbar}(\mathfrak{sl}_2)$. We refer to Appendix A.11.

2. In $U_{\hbar}(\mathfrak{sl}_2)$ we have both the element H and the element

$$\widetilde{H} := [E, F] = \frac{K - K^{-1}}{q - q^{-1}} = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}}$$

which is of the form

$$\widetilde{H} = H + \text{terms of order } \hbar^2.$$

We may think about \widetilde{H} is a deformation of H (in an informal sense). We note that

$$q \equiv 1, \quad K \equiv 1, \quad \widetilde{H} \equiv H \pmod{\hbar}.$$

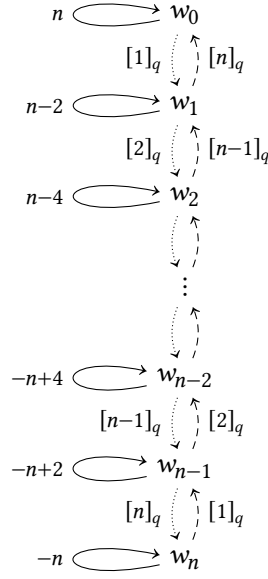


Figure 5: The irreducible representation $L(n)$ of $U(\mathfrak{sl}_2)$. Loops depict the action of H , dashed arrows the action of E and dotted arrows the action of F .

Theorem 5.5 ([CP95, Proposition 6.4.10]). For every natural number $n \in \mathbb{N}$ let $L(n)$ be the free $\mathbb{k}[[\hbar]]$ -module of rank $n + 1$ with basis w_0, \dots, w_n .

1. There exists a unique $U_{\hbar}(\mathfrak{sl}_2)$ -module structure on $V(n)$ such that

$$Hw_i := (n - 2i)w_i, \quad Ew_i := [n - i + 1]_q w_{i-1}, \quad Fw_i := [i + 1]_q w_{i+1}.$$

The actions of E, H, F can be graphically depicted as in Figure 5.

2. The $U_{\hbar}(\mathfrak{sl}_2)$ -modules $V(n)$ is indecomposable.
3. The $U_{\hbar}(\mathfrak{sl}_2)$ -module $V(n)$ reduces modulo \hbar to the irreducible representations $L(n)$ of $U(\mathfrak{sl}_2)$.
4. The actions of K and \tilde{H} on $V(n)$ is given by

$$Kv_i = q^{n-2i}v_i, \quad \tilde{H}v_i = [n - 2i]_q v_i.$$

It follows that the module $V(n)$ becomes the irreducible representation $L(q^n)$ of $U_q(\mathfrak{sl}_2)$, see Appendix A.12.

We refer to Appendix B for more a more detailed account about deformations of algebras and Hopf algebras.

A. Remarks and Proofs

A.1. Representation Theory of \mathfrak{sl}_2

Let \mathfrak{b} denote the Lie subalgebra of \mathfrak{sl}_2 consisting of (traceless) upper triangular matrices. It has the matrices E, H as a basis. Its universal enveloping algebra $U(\mathfrak{b})$ has the PBW-basis $H^m E^n$ with $m, n \geq 0$, and it is a subalgebra of $U(\mathfrak{sl}_2)$.

A.1.1. Weight Spaces and Shifting of Weight Spaces

Definition A.1. Let V be a representation of \mathfrak{sl}_2 .

1. The *weight space* of V with respect to λ is given by

$$V_\lambda := \{v \in V \mid H.v = \lambda v\}.$$

2. A nonzero weight vector v of V is *primitive* if $E.v = 0$.
3. The representation V is of *highest weight* λ if it is generated by a primitive weight vector of weight λ .

Proposition A.2 (Shifting weight spaces). Let V be a representation of \mathfrak{sl}_2 and let $\lambda \in \mathbb{k}$. Then

$$E.V_\lambda \subseteq V_{\lambda+2}, \quad F.V_\lambda \subseteq V_{\lambda-2}.$$

Proof. This follows from the commutator relations $[H, E] = 2E$ and $[H, F] = -2F$. \square

Lemma A.3. Let \mathbb{k} be algebraically closed. Then every finite-dimensional irreducible representation of \mathfrak{sl}_2 is a highest-weight representation.

A.1.2. Verma Modules

There exists for every scalar $\lambda \in \mathbb{k}$ a universal representation of highest weight λ , the so-called Verma module.

Definition A.4. For every scalar $\lambda \in \mathbb{k}$ let \mathbb{k}_λ be the one-dimensional representation of \mathfrak{b} which is given by \mathbb{k} as its underlying vector space together with the action of \mathfrak{b} on \mathbb{k} given by

$$H.1 = \lambda, \quad E.1 = 0.$$

Lemma A.5. There is an isomorphism of $U(\mathfrak{b})$ -modules given by

$$U(\mathfrak{b})/\langle E, H - \lambda \rangle \rightarrow \mathbb{k}_\lambda, \quad x \mapsto x.1.$$

Definition A.6. The *Verma module* of highest weight λ is the $U(\mathfrak{sl}_2)$ -module given by

$$M(\lambda) := U(\mathfrak{sl}_2) \otimes_{U(\mathfrak{b})} \mathbb{k}_\lambda.$$

Convention A.7. From now on the field \mathbb{k} is of characteristic zero.

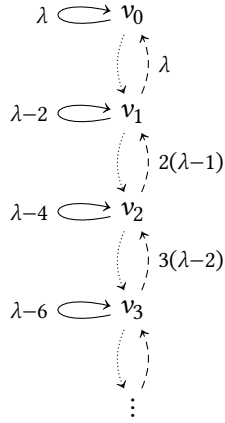


Figure 6: The Verma module $M(\lambda)$ of $U(\mathfrak{sl}_2)$. The action of H is depicted by loops, the action of F by dotted arrows and the action of E by dashed arrows.

Proposition A.8. Let $\lambda \in \mathbb{k}$.

1. The Verma module $M(\lambda)$ has the vectors

$$v_i := F^i \otimes 1 \quad \text{with } i \geq 0,$$

as a basis. The actions of E, H, F on this basis are given by

$$F.v_i = v_{i+1}, \quad H.v_i = (\lambda - 2i)v_i, \quad E.v_i = i(\lambda - i + 1)v_{i-1}.$$

This action can be graphically described as in Figure 6.

2. The Verma module $M(\lambda)$ is a representation of highest weight λ .
3. There exists for every representation V of \mathfrak{sl}_2 an isomorphism of vector spaces given by

$$\begin{aligned} \text{Hom}_{\mathfrak{sl}_2}(M(\lambda), V) &\longrightarrow \{v \in V \mid v \text{ is of weight } \lambda \text{ with } E.v = 0\}, \\ \varphi &\longmapsto \varphi(1 \otimes 1). \end{aligned}$$

In particular

$$\text{End}_{\mathfrak{sl}_2}(M(\lambda)) = \mathbb{k}.$$

4. The representation $M(\lambda)$ is indecomposable.
5. a. If $\lambda \notin \mathbb{N}$ then the representation $M(\lambda)$ is irreducible.
b. If $\lambda = n \in \mathbb{N}$ then the representation $M(\lambda)$ has a unique nonzero, proper subrepresentation, which is spanned by the vectors

$$v_i \quad \text{with } i \geq n + 1.$$

This subrepresentation is isomorphic to $M(-n - 2)$.

Definition A.9. Let $\lambda \in \mathbb{k}$.

1. For $\lambda \notin \mathbb{N}$ let $L(\lambda) := M(\lambda)$.
2. For $\lambda \in \mathbb{N}$ let $L(\lambda) := M(\lambda)/N$ where N is the unique nonzero, proper subrepresentation of $M(\lambda)$.

A.1.3. Classifications of Certain Irreducible Representations

Theorem A.10.

1. There is a one-to-one correspondence given by

$$\begin{aligned} \left\{ \begin{array}{l} \text{irreducible highest weight} \\ \text{representations of } \mathfrak{sl}_2 \end{array} \right\} &\longleftrightarrow \mathbb{k}, \\ L(\lambda) &\longleftrightarrow \lambda. \end{aligned}$$

2. The representation $L(\lambda)$ is finite-dimensional if and only if $\lambda = n \in \mathbb{N}$, in which case

$$\dim(L(n)) = n + 1.$$

If \mathbb{k} is algebraically closed (so that every irreducible finite-dimensional \mathfrak{sl}_2 -representation is a highest-weight representation) then the above correspondence does therefore restrict to a one-to-one correspondence

$$\begin{aligned} \left\{ \begin{array}{l} \text{irreducible finite-dimensional} \\ \text{representations of } \mathfrak{sl}_2 \end{array} \right\} &\longleftrightarrow \mathbb{N}, \\ L &\longmapsto \dim(L) - 1, \\ L(n) &\longleftrightarrow n. \end{aligned}$$

Remark A.11. Let $n \in \mathbb{N}$. The basis v_0, \dots, v_n of $L(n)$ can be rescaled to the basis

$$w_i := \frac{1}{i!} v_i.$$

The actions of E and F then become

$$E.w_i = (n - i + 1)w_{i-1}, \quad F.w_i = (i + 1)w_{i+1}.$$

The actions of E, H, F on $L(n)$ can now be graphically be represented as in Figure 1.

A.1.4. Semisimplicity of Finite-Dimensional Representations

Theorem A.12 (Weyl). Let \mathbb{k} be algebraically closed. Every finite-dimensional representation of \mathfrak{sl}_2 is semisimple.

Corollary A.13. Any finite-dimensional representation of \mathfrak{sl}_2 admits a weight space decomposition. All occurring weights are integral.

The decomposition of a finite-dimensional representation of \mathfrak{sl}_2 into irreducible representations can be read off from its weight space decomposition. From this the following result can be shown.

Proposition A.14 (Clebsch–Gordan). Let n, m be natural numbers with $n \geq m$. Then

$$L(n) \otimes L(m) \cong L(n+m) \oplus L(n+m-2) \oplus \cdots \oplus L(n-m).$$

A.1.5. The General Case of Characteristic Zero

We have above used a few times the additional assumption that \mathbb{k} is algebraically closed. We will now explain how to get rid of this assumption. For this we first recall some standard results about semisimplicity of algebras.

Lemma A.15. Let \mathbb{k} be any field and let V be a finite-dimensional \mathbb{k} -vector spaces. Let A be a subalgebra of $\text{End}_{\mathbb{k}}(V)$. Then A is semisimple if and only if V is semisimple as an A -module.

Proof. See [Lan02, XVII, §5, Proposition 4.7] and [MS16]. \square

Definition A.16. The *Jacobson radical* of a ring R is the intersection of all maximal left ideal of R . It is denoted by $J(R)$.

Remark A.17. Let R be a ring. The irreducible R -modules are up to isomorphism precisely those R -modules of the form R/\mathfrak{m} , where \mathfrak{m} is a maximal left ideal in R . The Jacobson radical of R does therefore consists of precisely those elements of R which annihilate every irreducible R -module.

Proposition A.18. Let A be a finite-dimensional \mathbb{k} -algebra.

1. The Jacobson radical $J(A)$ is a nilpotent, two-sided ideal in A .
2. Every nilpotent, two-sided ideal of A is contained in the Jacobson radical $J(A)$. It is thus the unique maximal nilpotent, two-sided ideal.
3. The following conditions on A are equivalent:
 - i. The algebra A is semisimple.
 - ii. The Jacobson radical $J(A)$ vanishes.
 - iii. The algebra A does not contain any nonzero nilpotent, two-sided ideal.

Proof. See [Lam01, §4]. \square

Corollary A.19. Let A be a finite-dimensional \mathbb{k} -algebra. Let K be a field extension of \mathbb{k} and suppose that $K \otimes_{\mathbb{k}} A$ is semisimple. Then A is semisimple.

Corollary A.20. Let A be an \mathbb{k} -algebra and let M be a finite-dimensional A -module. Let K be a field extension of \mathbb{k} and let

$$A_K := K \otimes_{\mathbb{k}} A, \quad M_K := K \otimes_{\mathbb{k}} M.$$

If M_K is irreducible as an A_K -module then M is irreducible as an A -module.

Remark A.21. Corollary A.20 asserts that extension of scalars reflects irreducibility of finite-dimensional modules. It is generally not true that it preserves irreducibility: it may happen that M is irreducible as an A -module but $M_{\mathbb{K}}$ is not irreducible as an $A_{\mathbb{K}}$ -module.

Theorem A.22. Let \mathbb{k} be a field of characteristic zero.

1. Every finite-dimensional \mathfrak{sl}_2 -representation is semisimple.
2. Every finite-dimensional \mathfrak{sl}_2 -representation decomposes into weight spaces, and all occurring weights are integral.
3. The irreducible finite-dimensional representations of \mathfrak{sl}_2 are given by $L(n)$ for $n \in \mathbb{N}$.

A.2. An Alternative Presentation for $U_q(\mathfrak{sl}_2)$

Let $q \in \mathbb{k}$ and let U_q be the algebra given by the generators

$$E, \quad \tilde{H}, \quad F, \quad K, \quad K^{-1}$$

and the relations

$$\begin{aligned} KK^{-1} &= 1 = K^{-1}K, & KE &= q^2EK, & KF &= q^{-2}FK, \\ [E, F] &= \tilde{H}, & (q - q^{-1})\tilde{H} &= K - K^{-1}, \\ [\tilde{H}, E] &= q(EK + K^{-1}E), & [\tilde{H}, F] &= -q^{-1}(FK + K^{-1}F). \end{aligned}$$

Proposition A.23. There exists a unique homomorphism of algebras

$$\psi : U_q \rightarrow U_q(\mathfrak{sl}_2)$$

that is given by

$$\psi(E) = E, \quad \psi(\tilde{H}) = \frac{K - K^{-1}}{q - q^{-1}}, \quad \psi(F) = F, \quad \psi(K) = K,$$

and this homomorphism is an isomorphism.

Proof. See [Kas95, Proposition VI.2.1]. □

Proposition A.24. For $q = 1$ there exists a unique homomorphism of algebras

$$\varphi : U_1 \rightarrow U(\mathfrak{sl}_2)[\sigma]/(\sigma^2 - 1)$$

that is given by

$$\varphi(E) = \sigma E, \quad \varphi(\tilde{H}) = \sigma H, \quad \varphi(F) = F, \quad \varphi(K) = \sigma.$$

Proof. See [Kas95, Proof of Proposition VI.2.2]. □

Remark A.25. There also exist other, more exotic presentations of $U_q(\mathfrak{sl}_2)$. We refer to [ITW05] for an example.

A.3. Representation Theory of $U_1(\mathfrak{sl}_2)$

Let A denote the algebra $U(\mathfrak{sl}_2)[\sigma]/(\sigma^2 - 1)$.

Let M be an \mathfrak{sl}_2 -representation and let $\varepsilon = \pm 1$. The corresponding $U(\mathfrak{sl}_2)$ -module structure on M extends to an $U(\mathfrak{sl}_2)[\sigma]$ -module structure for which σ acts by multiplication with ε , because σ is central in $U(\mathfrak{sl}_2)[\sigma]$. It follows from $\varepsilon^2 = 1$ that this induces a A -module structure on M as claimed in Remark 2.4.

If M is irreducible then the resulting A -module is again irreducible since every A -submodule is in particular an \mathfrak{sl}_2 -subrepresentation. It hence follows that the A -modules $L(+, n)$ and $L(-, n)$ that result from the irreducible \mathfrak{sl}_2 -representation $L(n)$ are again irreducible. These representations are pairwise non-isomorphic since the element $H\sigma$ of A (which corresponds to the element \tilde{H} of $U_1(\mathfrak{sl}_2)$) acts on $L(+, n)$ with highest weight n and on $L(-, n)$ with highest weight $-n$.

Let now M be any finite-dimensional M -module. It follows from the relation $\sigma^2 = 1$ in A that the action of σ on A is diagonalizable with eigenvalues 1 and -1 . We thus have

$$M = M_1 \oplus M_{-1}$$

with $M_\varepsilon := \{m \in M \mid \sigma m = \varepsilon m\}$ for $\varepsilon = \pm 1$. The action of σ on M is an A -module homomorphism because σ is central in A . The decomposition $M = M_1 \oplus M_{-1}$ is therefore one of A -modules.

We may regard both M_1 and M_{-1} as \mathfrak{sl}_2 -representations by restriction. We then have decompositions into finite-dimensional irreducible \mathfrak{sl}_2 -representations given by

$$M_1 \cong L(n_1) \oplus \cdots \oplus L(n_s), \quad M_{-1} \cong L(n'_1) \oplus \cdots \oplus L(n'_t).$$

We note that this is already a decomposition as A -modules since σ acts on M_1 and M_{-1} by multiplication with scalars. As A -modules we have

$$L(n_i) = L(+, n_i), \quad L(n'_i) = L(-, n'_i).$$

This shows that every finite-dimensional A -module decomposes into a direct sum of the irreducible A -modules $L(\varepsilon, n)$.

A.4. PBW Basis for $U_q(\mathfrak{sl}_2)$

We use in the following the notation introduced in Definition 3.8.

Lemma A.26. For every $r \geq 0$ we have

$$[E, F^r] = [r]_q F^{r-1} [K, 1 - r]_q.$$

Proof. For $r = 0$ both sides vanish and for $r = 1$ this is one of the defining relations of $U_q(\mathfrak{sl}_2)$. For $r \geq 2$ the assertion follows by induction, see [Jan96, Appendix 1.3 (5)]. \square

Corollary A.27. We have

$$\begin{aligned} F \cdot F^l K^m E^n &= F^{l+1} K^m E^n, \\ K^{\pm 1} \cdot F^l K^m E^n &= q^{\mp 2l} F^l K^{m \pm 1} E^n, \\ E \cdot F^l K^m E^n &= q^{-2m} F^l K^m E^{n+1} + \frac{[l]_q}{q - q^{-1}} (q^{1-l} F^{l-1} K^{m+1-l} E^n - q^{l-1} F^{l-1} K^{m+l-1} E^n). \end{aligned}$$

Proof. This follows from Lemma A.26 and the two relations $KE = q^2EK$ and $KF = q^{-2}FK$. \square

Proof of Theorem 2.6. Let U be the linear subspace of $U_q(\mathfrak{sl}_2)$ spanned by these given monomials. It follows from Corollary A.27 that $U_q(\mathfrak{sl}_2)$ is a left ideal. It contains the elements $F^0K^0E^0 = 1$, whence $U = U_q(\mathfrak{sl}_2)$. This shows that the given monomials are a vector space generating set.

The linear independence is shown in the usual representation-theoretic way: Let V be the free vector space with basis

$$X^l Y^n Z^m \quad \text{with } l, n \in \mathbb{N} \text{ and } m \in \mathbb{Z}.$$

There exists an action of $U_q(\mathfrak{sl}_2)$ on V by using the formulas from Corollary A.27, with $F^l K^m E^n$ replaced by $X^l Y^n Z^m$. (It has to be checked that this proposed action is compatible with the defining relations of $U_q(\mathfrak{sl}_2)$, see [Jan96, Appendix 1.5].) The elements

$$F^l K^m E^n \cdot X^0 Y^0 Z^0 = X^l Y^m Z^n$$

are linearly independent in V , whence the given monomials $F^l K^m E^n$ are linearly independent in $U_q(\mathfrak{sl}_2)$. \square

A.5. More on the Algebra Structure of $U_q(\mathfrak{sl}_2)$

Remark A.28.

1. The universal enveloping algebra $U(\mathfrak{sl}_2)$ is noetherian and has no nonzero zero divisors. The same holds for $U_q(\mathfrak{sl}_2)$, see [Kas95, Proposition VI.1.4] and [Jan96, Proposition 1.8].
2. The algebra $U_q(\mathfrak{sl}_2)$ admits a grading such that E, K, F are homogeneous with

$$\deg(E) = 1, \quad \deg(F) = -1, \quad \deg(K) = 0.$$

The degree d part of $U_q(\mathfrak{sl}_2)$ has the basis

$$F^l K^m E^n \quad \text{with } n - l = d.$$

This grading can also be characterized in terms of the conjugation map

$$U_q(\mathfrak{sl}_2) \rightarrow U_q(\mathfrak{sl}_2), \quad x \mapsto KxK^{-1}.$$

The degree d part of the grading is precisely the eigenspace with eigenvalue q^{2d} .

Proposition A.29.

1. There exists a unique algebra involution ω of $U_q(\mathfrak{sl}_2)$ with

$$\omega(E) = F, \quad \omega(K) = K^{-1}, \quad \omega(F) = E.$$

2. There exists a unique algebra anti-involution τ of $U_q(\mathfrak{sl}_2)$ with

$$\tau(E) = E, \quad \tau(K) = K^{-1}, \quad \tau(F) = F.$$

3. There exists a unique algebra isomorphism $\varphi_q : U_q(\mathfrak{sl}_2) \rightarrow U_{q^{-1}}(\mathfrak{sl}_2)$ with

$$\varphi(E) = -F, \quad \varphi(K) = K^{-1}, \quad \varphi(F) = -E.$$

The inverse of the isomorphism φ_q is given by $\varphi_{q^{-1}}$.

4. There exist unique algebra involutions σ_E and σ_F of $U_q(\mathfrak{sl}_2)$ with

$$\sigma_E(E) = -E, \quad \sigma_E(K) = -K, \quad \sigma_E(F) = F.$$

and

$$\sigma_F(E) = E, \quad \sigma_F(K) = -K, \quad \sigma_F(F) = -F.$$

Proof. One checks that the proposed images of $E, F, K^{\pm 1}$ are compatible with the defining relations of $U_q(\mathfrak{sl}_2)$. See also [Jan96, Lemma 1.2]. \square

Remark A.30.

1. One can combine the above (anti-)isomorphisms to construct further (anti-)isomorphisms involving $U_q(\mathfrak{sl}_2)$ and $U_{q^{-1}}(\mathfrak{sl}_2)$.
2. It follows from the existence of these (anti-)isomorphisms that many formulas and propositions involving $U_q(\mathfrak{sl}_2)$ have to satisfy certain symmetries.

A.6. Proof of Theorem 3.3

Lemma A.31. Let M be a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module.

1. Both E and F act nilpotently on M .
2. For a sufficiently large power $r \geq 0$ (namely such that $F^r M = 0$) the module M is annihilated by

$$\prod_{j=-r}^r (K^2 - q^{2j}).$$

Proof. See [Jan96, Proposition 2.1] and [Jan96, Proposition 2.3]. \square

Proposition A.32. Every finite-dimensional $U_q(\mathfrak{sl}_2)$ -module decomposes into weight spaces. All occurring weights are of the form $\pm q^n$ for some $n \in \mathbb{Z}$.

Proof. Let M be a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module and let k denote the action of K on M . It follows from Lemma A.31 that

$$0 = \prod_{n=-r}^r (k^2 - q^{2n}) = \prod_{n=-r}^r (k - q^n)(k + q^n).$$

The roots $\pm q^n$ with $n = -r, \dots, r$ are pairwise distinct² whence it follows that k is diagonalizable with possible eigenvalues $\pm q^n$ for $n = -r, \dots, r$. \square

²If $\pm q^n = \pm q^m$ then squaring both sides of this equation gives $q^{2n} = q^{2m}$ and thus $q^{2(n-m)} = 1$. It follows that $2(n-m) = 0$ because q is not a root of unity, and thus $n = m$.

A.7. Proof of Proposition 3.10

Proposition A.33.

1. The algebra $U_q(\mathfrak{b})$ has the basis

$$K^n E^m \quad \text{with } n \in \mathbb{Z} \text{ and } m \in \mathbb{N}$$

2. The algebra $U_q(\mathfrak{b})$ is given with respect to its generators E, K, K^{-1} by the relations

$$KK^{-1} = 1 = K^{-1}K, \quad KE = q^2 EK.$$

Proof.

1. Let U be the linear subspace of $U_q(\mathfrak{sl}_2)$ spanned by the monomials $K^n E^m$ with $n, m \in \mathbb{N}$. This linear subspace is contained in $U_q(\mathfrak{b})$. It follows on the other hand from the relation $KE = q^2 EK$ that

$$K^n E^m \cdot K^{n'} E^{m'} = q^{2mn'} K^{n+n'} E^{m+m'}$$

for all $n, n', m, m' \in \mathbb{N}$, and we have $1 = K^0 E^0 \in U$. This shows that U is a subalgebra of $U_q(\mathfrak{sl}_2)$ containing E, K, K^{-1} , and therefore containing $U_q(\mathfrak{b})$. This shows together that $U = U_q(\mathfrak{b})$.

2. Let U be the algebra given by generators E, K, K^{-1} and relations

$$KK^{-1} = 1 = K^{-1}K, \quad KE = q^2 EK.$$

There exists a unique algebra homomorphism $\varphi : U \rightarrow U_q(\mathfrak{b})$ given by

$$\varphi(E) = E, \quad \varphi(K) = K.$$

In the same way as Theorem 2.6 one sees that U has a PBW-basis given by the monomials

$$K^n E^m \quad \text{with } n \in \mathbb{Z} \text{ and } m \in \mathbb{N}.$$

It follows that the algebra homomorphism φ restricts to a bijection between the PBW-bases of U and $U_q(\mathfrak{b})$ and is therefore an algebra isomorphism. \square

We now show an extended version of Proposition 3.10

Proposition A.34. Let $\lambda \in \mathbb{k}^\times$.

1. We have $\mathbb{k}_\lambda \cong U_q(\mathfrak{b})/\langle E, K - \lambda \rangle$ as an $U_q(\mathfrak{b})$ -module.
2. The Verma module $M(\lambda)$ has the basis

$$m_i := F^i \otimes 1 \quad \text{with } i \geq 0,$$

and the actions of E, K, F on this basis is given by

$$Fm_i = m_{i+1}, \quad Km_i = q^{-2i}\lambda m_i, \quad Em_i = [i]_q[\lambda, 1 - i]_q m_{i-1}.$$

This action can be graphically described as in Figure 3.

3. The Verma module $M(\lambda)$ is of highest weight λ , and every $U_q(\mathfrak{sl})$ -module of highest weight λ is a quotient of $M(\lambda)$.
4. There exists for every $U_q(\mathfrak{sl}_2)$ -module M an isomorphism of vector spaces given by

$$\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(M(\lambda), M) \cong \{m \in M \mid m \text{ is of weight } \lambda \text{ with } Em = 0\}.$$

It follows in particular that

$$\mathrm{End}_{U_q(\mathfrak{sl}_2)}(M(\lambda)) = \mathbb{k}.$$

5. The Verma module $M(\lambda)$ is indecomposable.
6. a. If $\lambda = \pm q^n$ for some $n \in \mathbb{N}$ then the Verma module $M(\lambda)$ contains a unique nonzero, proper submodule, which is spanned by the elements

$$m_i \quad \text{with } i \geq n + 1.$$

This submodule is isomorphic to $M(q^{-n-2}\lambda)$.

- b. If $\lambda \neq \pm q^n$ for every $n \in \mathbb{N}$ then the Verma module $M(\lambda)$ is irreducible.

1. This follows from the PBW-basis of $U_q(\mathfrak{b})$.
2. This follows from the PBW-basis of $U_q(\mathfrak{sl}_2)$ and induction.
3. The Verma module $M(\lambda)$ is generated by the primitive weight vector $1 \otimes 1$.
4. We have

$$\begin{aligned} \mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(M(\lambda), M) &\cong \mathrm{Hom}_{U_q(\mathfrak{b})}(\mathbb{k}_\lambda, M) \\ &\cong \mathrm{Hom}_{U_q(\mathfrak{b})}(U_q(\mathfrak{b})/\langle K - \lambda, E \rangle, M) \\ &\cong \{m \in M \mid (K - \lambda)m = 0, Em = 0\}. \end{aligned}$$

5. The endomorphism algebra $\mathrm{End}_{U_q(\mathfrak{sl}_2)}(M(\lambda)) = \mathbb{k}$ does not contain any non-trivial idempotents.
6. This follows as for $U(\mathfrak{sl}_2)$ since $[i]_q[\lambda, i - 1]_q = 0$ if and only if $\lambda = \pm q^{i-1}$.

A.8. Proof of Theorem 3.14

Lemma A.35. If M is an highest-weight $U_q(\mathfrak{sl}_2)$ -module then

$$\mathrm{End}_{U_q(\mathfrak{sl}_2)}(M) = \mathbb{k}.$$

Definition A.36. The *quantum Casimir element* is the element $C_q \in U_q(\mathfrak{sl}_2)$ given by

$$C_q := EF + \frac{Kq^{-1} + K^{-1}q}{(q - q^{-1})^2}.$$

Lemma A.37.

1. The element C_q is central in $U_q(\mathfrak{sl}_2)$.

2. The element C_q acts on every $U_q(\mathfrak{sl}_2)$ -module by module endomorphisms.
3. The element C_q acts for every scalar $\lambda \in \mathbb{k}^\times$ on the representation $L(\lambda)$ by multiplication with the scalar

$$\frac{\lambda q + \lambda^{-1} q^{-1}}{(q - q^{-1})^2}.$$

4. The element C_q acts the same on $L(\lambda)$ and $L(\mu)$ if and only if $\lambda = \mu$ or $\lambda = \mu^{-1} q^{-2}$.

Proof.

1. It can be checked that C_q commutes with E, F, K by using the defining relations for $U_q(\mathfrak{sl}_2)$.
2. This follows from the previous assertion.
3. It follows from the previous assertion and Lemma A.35 that C_q acts by a scalar. This scalar can be read off from the action on the primitive generator $1 \otimes 1$. It thus suffices to show the assertion for $M(\lambda)$, where it follows from Proposition 3.10.
4. This follows from the previous assertion. □

Corollary A.38. The quantum Casimir element C_q acts on every finite-dimensional, irreducible representation of $U_q(\mathfrak{sl}_2)$ by a different scalar.

Proof. If $\lambda = \delta q^n$ and $\mu = \varepsilon q^m$ with $\delta, \varepsilon \in \{1, -1\}$ and $n, m \in \mathbb{N}$ then it cannot happen that $\lambda = \mu^{-1} q^{-2}$. The assertion thus follows from Lemma A.37. □

Proof of Theorem 3.14 ([Jan96, Theorem 2.9]). Let M be any finite-dimensional $U_q(\mathfrak{sl}_2)$ -module and let c denote the action of C_q on M . We may assume that M is indecomposable. We can consider a composition series

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \dots \subsetneq M_r = M \tag{4}$$

with composition factors

$$M_i / M_{i-1} \cong L(\varepsilon_i q^{n_i}).$$

Letting c_i be the scalar by which C_q acts on $L(\varepsilon_i q^{n_i})$, we have

$$(c - c_i)M_i \subseteq M_{i-1}.$$

It follows that $\prod_{i=1}^r (c - c_i)$ annihilates M and that c admits a generalized eigenspace decomposition with eigenvalues c_1, \dots, c_r . The resulting generalized eigenspaces are subrepresentations because c is a $U_q(\mathfrak{sl}_2)$ -module endomorphism. It follows that

$$c_1 = \dots = c_r$$

because M is indecomposable, and thus

$$\varepsilon_1 q^{n_1} = \dots = \varepsilon_r q^{n_r} =: \lambda$$

by Corollary A.38. It follows with the composition series (4) that

$$\dim(M_\mu) = r \dim(L(\lambda)_\mu)$$

for every scalar $\mu \in \mathbb{k}^\times$. Thus M is of highest weight λ .

The short exact sequence

$$0 \rightarrow M_{r-1} \rightarrow M \rightarrow L(\lambda) \rightarrow 0 \quad (5)$$

restricts to a short exact sequence

$$0 \rightarrow (M_{r-1})_\lambda \rightarrow M_\lambda \rightarrow L(\lambda)_\lambda \rightarrow 0.$$

It follows that the primitive generator v_0 of $L(\lambda)$ has a preimage m_0 in M . The weight vector m_0 is primitive because M is of highest weight λ . It follows that there exists a homomorphism of $U_q(\mathfrak{sl}_2)$ -modules

$$\varphi : L(\lambda) \rightarrow M, \quad 1 \otimes 1 \mapsto m_0.$$

It follows from the finite-dimensionality of M that φ factors through a homomorphism

$$\psi : L(\lambda) \rightarrow M, \quad \overline{1 \otimes 1} \mapsto m_0.$$

This shows that the short exact sequence (5) splits, whence

$$M \cong M_{r-1} \oplus L(\lambda).$$

It follows by induction that $M_{r-1} \cong L(\lambda)^{\oplus(r-1)}$ and thus altogether $M \cong L(\lambda)^{\oplus r}$. \square

Remark A.39. The center of the universal enveloping algebra $U(\mathfrak{sl}_2)$ is a polynomial algebra, generated by the classical Casimir element $C = (ef + h^2 + fe)/4$. It can be shown that the center of $U_q(\mathfrak{sl}_2)$ is again a polynomial algebra, now generated by the quantum Casimir element C_q . We refer to [Jan96, Proposition 2.18] for more details on this.

A.9. Proof of Lemma 4.4

We have

$$M_\mu \otimes N_\kappa \subseteq (M \otimes N)_{\mu\kappa}$$

for all $\mu, \kappa \in \mathbb{k}^\times$ since the element K is group-like in $U_q(\mathfrak{sl}_2)$. Both M and N admits weight space decompositions

$$M = \bigoplus_{\mu} M_{\mu}, \quad N = \bigoplus_{\kappa} N_{\kappa}$$

and it follows that

$$M \otimes N = \left(\bigoplus_{\mu} M_{\mu} \right) \otimes \left(\bigoplus_{\kappa} N_{\kappa} \right) = \bigoplus_{\mu, \kappa} (M_{\mu} \otimes N_{\kappa}) \subseteq \bigoplus_{\lambda} M_{\lambda} \subseteq M \otimes N$$

It follows with the inclusions $M_{\mu} \otimes N_{\kappa} \subseteq (M \otimes N)_{\mu\kappa}$ that already

$$(M \otimes N)_{\lambda} = \bigoplus_{\mu\kappa=\lambda} M_{\mu} \otimes N_{\kappa}$$

for every λ .

A.10. Clebsch--Gordan for $U_q(\mathfrak{sl}_2)$

Proposition A.40. For all $\delta, \varepsilon \in \{1, -1\}$ and $n, m \in \mathbb{N}$ with $n \geq m$ we have

$$L(\delta q^n) \otimes L(\varepsilon q^m) \cong L(\delta \varepsilon q^{n+m}) \oplus L(\delta \varepsilon q^{n+m-2}) \oplus \cdots \oplus L(\delta \varepsilon q^{n-m}).$$

Proof. This follows from Corollary 3.15 and Lemma 4.4. □

A.11. $U_q(\mathfrak{sl}_2)$ inside of $U_{\hbar}(\mathfrak{sl}_2)$

We consider the field of Laurent polynomials $\mathbb{k}((\hbar))$ and the extension of scalars

$$\mathbb{k}((\hbar)) \otimes_{\mathbb{k}[[\hbar]]} U_{\hbar}(\mathfrak{sl}_2),$$

which is given as an $\mathbb{k}((\hbar))$ -module by

$$\mathbb{k}((\hbar)) \otimes_{\mathbb{k}[[\hbar]]} U_{\hbar}(\mathfrak{sl}_2) = \mathbb{k}[[\hbar]][\hbar^{-1}] \otimes_{\mathbb{k}[[\hbar]]} U(\mathfrak{sl}_2)[[\hbar]] \cong U(\mathfrak{sl}_2)[[\hbar]][\hbar^{-1}] \cong U(\mathfrak{sl}_2)((\hbar)).$$

The field $\mathbb{k}((\hbar))$ contains the subfield $\mathbb{k}(q)$, and we get an algebra homomorphism

$$U_q(\mathfrak{sl}_2) \rightarrow \mathbb{k}((\hbar)) \otimes_{\mathbb{k}[[\hbar]]} U(\mathfrak{sl}_2)$$

where $U_q(\mathfrak{sl}_2)$ is defined over $\mathbb{k}(q)$.

A.12. From $V(n)$ to $L(q^n)$

It follows that

$$L(q^n) \cong \langle 1 \otimes v_0, \dots, 1 \otimes v_n \rangle_{\mathbb{k}(q)} \subseteq \mathbb{k}((\hbar)) \otimes_{\mathbb{k}[[\hbar]]} V(n)$$

as $U_q(\mathfrak{sl}_2)$ -modules.

B. Deformation Theory

B.1. Deformations of Algebras

We will in the following introduce a formal deformation $U_{\hbar}(\mathfrak{sl}_2)$ of the Hopf algebra $U(\mathfrak{sl}_2)$ and gain a new understanding of $U_q(\mathfrak{sl}_2)$.

B.2. Deformation of Algebras

The following is taken (at least in spirit) from [Bel18, §5.2] and [GS92].

Motivation B.1. Deforming a \mathbb{k} -algebra A means – roughly speaking – that the multiplication on A is replaced by a perturbed multiplication $*$, in the sense that for all $a, b \in A$,

$$a * b = ab + \mu_1(a, b)\hbar + \mu_2(a, b)\hbar^2 + \cdots$$

for some bilinear terms $\mu_i(a, b)$. The limit $\hbar \rightarrow 0$ does then give back the original algebra A .

Definition B.2. Let A be an \mathbb{k} -algebra.

1. A *(formal) deformation* of A is an $\mathbb{k}[[\hbar]]$ -algebra A_\hbar whose underlying $\mathbb{k}[[\hbar]]$ -module is $A[[\hbar]]$ and for which $A_\hbar/\hbar A_\hbar = A$ as algebras.
2. Two deformations A_\hbar and A'_\hbar of the algebra A are *equivalent* if there exists an isomorphism of $\mathbb{k}[[\hbar]]$ -algebras

$$\varphi : A_\hbar \rightarrow A'_\hbar$$

such that the induced isomorphism of \mathbb{k} -algebras

$$A = A_\hbar/\hbar A_\hbar \rightarrow A'_\hbar/\hbar A'_\hbar = A$$

is the identity, i.e. φ is the identity modulo \hbar .

3. A deformation is *trivial* if it is equivalent to the trivial deformation (i.e. $A[[\hbar]]$).

Remark B.3. Every $\mathbb{k}[[\hbar]]$ -bilinear multiplication

$$(-) * (-) : A[[\hbar]] \times A[[\hbar]] \rightarrow A[[\hbar]].$$

satisfies the equality

$$\left(\sum_{i=0}^{\infty} a_i \hbar^i \right) * \left(\sum_{j=0}^{\infty} b_j \hbar^j \right) = \sum_{i,j=0}^{\infty} (a_i * b_j) \hbar^{i+j}.$$

The multiplication $*$ can therefore be characterized by the \mathbb{k} -bilinear maps $\mu_i : A \times A \rightarrow A$ such that

$$a * b = \mu_0(a, b) + \mu_1(a, b)\hbar + \mu_2(a, b)\hbar^2 + \dots$$

The condition $A[[\hbar]]/\hbar A[[\hbar]] = A$ means that μ_0 is the original multiplication on A , whence

$$a * b = ab + \mu_1(a, b)\hbar + \mu_2(a, b)\hbar^2 + \dots$$

That the multiplication $*$ is associative gives certain compatibility conditions on the μ_i , which we won't discuss here.

Example B.4. Every \mathbb{k} -algebra A admits the *trivial deformation* $A[[\hbar]]$ (i.e. the algebra of power series with its usual product). It corresponds to the choice $\mu_1, \mu_2, \dots = 0$.

Theorem B.5. The universal enveloping algebra $U(\mathfrak{sl}_2)$ admits a deformation with

$$[H, E] = 2E, \quad [H, F] = 2F, \quad [E, F] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}} \quad (6)$$

Proof (sketch). Let P be the free algebra on the generators E, H, F . Let I be the two-sided ideal in $P[[\hbar]]$ given by the relations (6). Let J be the closure of I in the \hbar -adic topology. Then J is again a two-sided ideal in $P[[\hbar]]$. The described deformation can be realized as the quotient $P[[\hbar]]/J$. We refer to [CP95, Definition-Proposition 6.4.3 ff.] for the specific details. \square

Definition B.6. The deformation of $U(\mathfrak{sl}_2)$ from Theorem B.5 is denoted by $U_\hbar(\mathfrak{sl}_2)$.

Remark B.7. In the algebra $U_{\hbar}(\mathfrak{sl}_2)$ we can consider the well-defined elements

$$q := e^{\hbar}, \quad K := e^{\hbar H}.$$

The elements E, F, K, K^{-1} satisfy the defining relations of $U_q(\mathfrak{sl}_2)$ and one should think about the algebra $U_q(\mathfrak{sl}_2)$ as somewhat of a subalgebra of $U_{\hbar}(\mathfrak{sl}_2)$.

In $U_{\hbar}(\mathfrak{sl}_2)$ we have both the element H and the element

$$\tilde{H} = \frac{K - K^{-1}}{q - q^{-1}},$$

which is of the form

$$\tilde{H} = H + \text{terms of order } \hbar^2.$$

We may think about \tilde{H} is a deformation of H (in an informal sense). We note that

$$q \equiv 1, \quad K \equiv 1, \quad \tilde{H} \equiv H \pmod{\hbar}.$$

Definition B.8. The deformation of $U(\mathfrak{sl}_2)$ from Theorem B.5 is denoted by $U_{\hbar}(\mathfrak{sl}_2)$.

Remark B.9. In the algebra $U_{\hbar}(\mathfrak{sl}_2)$ we can consider the well-defined elements

$$q := e^{\hbar}, \quad K := e^{\hbar H}.$$

The elements E, F, K, K^{-1} satisfy the defining relations of $U_q(\mathfrak{sl}_2)$ and one should think about the algebra $U_q(\mathfrak{sl}_2)$ as somewhat of a subalgebra of $U_{\hbar}(\mathfrak{sl}_2)$.

In $U_{\hbar}(\mathfrak{sl}_2)$ we have both the element H and the element

$$\tilde{H} = \frac{K - K^{-1}}{q - q^{-1}},$$

which is of the form

$$\tilde{H} = H + \text{terms of order } \hbar^2.$$

We may think about \tilde{H} is a deformation of H (in an informal sense). We note that

$$q \equiv 1, \quad K \equiv 1, \quad \tilde{H} \equiv H \pmod{\hbar}.$$

Remark B.10. One can study the deformation theory of an \mathbb{k} -algebra via homological algebra: The *Hochschild cochain complex* of A is given by

$$C_{\text{Hoch}}^n(A) := \text{Hom}_{\mathbb{k}}(A^{\otimes n}, A)$$

together with certain differentials. The cohomology of this chain complex is the *Hochschild cohomology* of A , which is denoted by

$$\text{HH}^n(A) := H^n(C_{\text{Hoch}}^{\bullet}).$$

One of the connections between deformation theory and Hochschild cohomology is that in the case of

$$\text{HH}^2(A) = 0$$

every deformation of A is trivial.

Warning B.11. Let A_{\hbar} be a deformation of an \mathbb{k} -algebra A with $\mathrm{HH}^2(A) = 0$. The above criterion shows that A_{\hbar} is equivalent to $A[[\hbar]]$, but it does not provide an explicit isomorphism.

Example B.12. Let \mathfrak{g} be a semisimple Lie algebra. It can be shown that

$$\mathrm{HH}^2(\mathrm{U}(\mathfrak{g})) = 0$$

whence all deformations of $\mathrm{U}(\mathfrak{g})$ are trivial. (See [GS92, Theorem 2].)

It follows in particular that the every algebra deformation of $\mathrm{U}(\mathfrak{sl}_2)$ is trivial. An explicit equivalence between $\mathrm{U}_{\hbar}(\mathfrak{sl}_2)$ and $\mathrm{U}(\mathfrak{sl}_2)[[\hbar]]$ is constructed in [CP95, Proposition 4.6.4].

B.3. More on Completions

We also want define coalgebras (and bialgebras and Hopf algebras). For this we need to make sense of power series in tensor products $A[[\hbar]] \otimes A[[\hbar]]$, which does in general not make sense. This problem is solved by using the *completed tensor product*.

Definition B.13. Let M be an $\mathbb{k}[[\hbar]]$ -module.

1. The \hbar -adic completion of M is the $\mathbb{k}[[\hbar]]$ -module

$$\widehat{M} := \lim_{n \geq 0} (M / \hbar^{n+1} M) = \left\{ (m_n)_{n \geq 0} \mid \begin{array}{l} m_n \in M / \hbar^{n+1} M \text{ with} \\ m_{n+1} \equiv m_n \pmod{\hbar^{n+1}} \text{ for every } n \geq 0 \end{array} \right\}.$$

2. The canonical homomorphism $M \rightarrow \widehat{M}$ is given by $m \mapsto (\overline{m}, \overline{m}, \dots)$.
3. A $\mathbb{k}[[\hbar]]$ -module M is *complete* if the canonical homomorphism $M \rightarrow \widehat{M}$ is an isomorphism.

Remark B.14.

1. More explicitly, an $\mathbb{k}[[\hbar]]$ -module M is complete if and only if there exists for every sequence m_0, m_1, \dots of elements $m_n \in M$ with

$$m_{n+1} \equiv m_n \pmod{\hbar^{n+1}} \quad \text{for every } n \geq 0$$

a unique element $m \in M$ with

$$m \equiv m_n \pmod{\hbar^{n+1}} \quad \text{for every } n \geq 0.$$

2. Let M be a complete $\mathbb{k}[[\hbar]]$ -module. Every sequence $(m_i)_{i \geq 0}$ of elements $m_i \in M$ defines a sequence $(s_n)_{n \geq 0}$ of partial sums

$$s_n := \sum_{i=0}^n \hbar^i m_i.$$

for every $n \geq 0$. By the completeness of M there exists a unique element $\sum_{i=0}^{\infty} \hbar^i m_i$ of M with

$$\sum_{i=0}^{\infty} \hbar^i m_i \equiv \sum_{i=0}^n \hbar^i m_i \pmod{\hbar^{n+1}} \quad \text{for every } n \geq 0.$$

Example B.15.

1. Every finite-dimensional $\mathbb{k}[[\hbar]]$ -module M is complete since $\hbar^n M = 0$ for some sufficiently large power n .
2. For every \mathbb{k} -vector space the resulting $\mathbb{k}[[\hbar]]$ -module $V[[\hbar]]$ is complete. For every sequence of elements $v_0, v_1, \dots \in V$ we have

$$\sum_{i=0}^{\infty} \hbar^i v_i = \sum_{i=0}^{\infty} v_i \hbar^i.$$

Proposition B.16. Let M, N be two $\mathbb{k}[[\hbar]]$ -modules.

1. For every homomorphism of $\mathbb{k}[[\hbar]]$ -module $f : M \rightarrow N$ there exists a unique module homomorphism $\widehat{f} : \widehat{M} \rightarrow \widehat{N}$ that makes the following square diagram commute:

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\widehat{f}} & \widehat{N} \\ \uparrow & & \uparrow \\ M & \xrightarrow{f} & N \end{array}$$

The homomorphism \widehat{f} is given by

$$\widehat{f}(\overline{m_0}, \overline{m_1}, \dots) = (\overline{f(m_0)}, \overline{f(m_1)}, \dots).$$

2. The assignment $(-)^{\widehat{}}$ defines a functor

$$(-)^{\widehat{}} : \mathbb{k}[[\hbar]]\text{-Mod} \rightarrow \mathbb{k}[[\hbar]]\text{-Mod}.$$

3. If M, N are complete then

$$f\left(\sum_{i=0}^{\infty} \hbar^i m_i\right) = \sum_{i=0}^{\infty} \hbar^i f(m_i)$$

for every sequence of elements $m_0, m_1, \dots \in M$.

4. If N is complete then every homomorphism $M \rightarrow N$ extends uniquely to a homomorphism $\widehat{M} \rightarrow N$.

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\exists!} & N \\ \uparrow & \nearrow & \\ M & & \end{array}$$

5. If V is any \mathbb{k} -vector space and N is complete then every \mathbb{k} -linear map $f : V \rightarrow N$ extends uniquely to a $\mathbb{k}[[\hbar]]$ -linear map $f' : V[[\hbar]] \rightarrow N$.

$$\begin{array}{ccc} V[[\hbar]] & \xrightarrow{\exists!} & N \\ \uparrow & \nearrow & \\ V & & \end{array}$$

The homomorphism f' is given by

$$f' \left(\sum_{i=0}^{\infty} \hbar^i v_i \right) = \sum_{i=0}^{\infty} \hbar^i f(v_i).$$

6. The canonical homomorphism $M \rightarrow \widehat{M}$ induces an isomorphism of \mathbb{k} -vector spaces

$$M/\hbar M \longrightarrow \widehat{M}/\hbar \widehat{M}.$$

Remark B.17. Let M be a $\mathbb{k}[[\hbar]]$ -module. There exists a unique topology on M for which a basis is given by the sets

$$m + \hbar^{n+1} M$$

with $m \in M$ and $n \geq 0$. This topology is the \hbar -adic topology on M . It makes $\mathbb{k}[[\hbar]]$ into a topological ring and every $\mathbb{k}[[\hbar]]$ -module into a topological $\mathbb{k}[[\hbar]]$ -module. The completion \widehat{M} is then the usual topological completion of M .

Definition B.18. Let M, N be two $\mathbb{k}[[\hbar]]$ -modules. The *completed tensor product*

$$M \widehat{\otimes} N$$

is the \hbar -adic completion of the tensor product $M \otimes_{\mathbb{k}[[\hbar]]} N$.

Proposition B.19. Let V, W be two \mathbb{k} -vector spaces. Then the $\mathbb{k}[[\hbar]]$ -linear map

$$V[[\hbar]] \otimes_{\mathbb{k}[[\hbar]]} W[[\hbar]] \rightarrow (V \otimes W)[[\hbar]], \quad \left(\sum_{i=0}^{\infty} v_i \hbar^i \right) \otimes \left(\sum_{j=0}^{\infty} w_j \hbar^j \right) \mapsto \sum_{i,j=0}^{\infty} (v_i \otimes w_j) \hbar^{i+j}$$

extends along the canonical homomorphism

$$V \otimes W \rightarrow V \widehat{\otimes} W$$

to an isomorphism of $\mathbb{k}[[\hbar]]$ -modules

$$V[[\hbar]] \widehat{\otimes} W[[\hbar]] \rightarrow (V \otimes W)[[\hbar]].$$

B.4. Deformation of Hopf Algebras

The following is taken mostly from [CP95, Chapter 6].

Definition B.20. 1. A *topological Hopf algebra* consists of a complete $\mathbb{k}[[\hbar]]$ -module A together with $\mathbb{k}[[\hbar]]$ -linear maps

$$m : A \widehat{\otimes} A \rightarrow A, \quad u : \mathbb{k}[[\hbar]] \rightarrow A, \quad \Delta : A \rightarrow A \widehat{\otimes} A, \quad \varepsilon : A \rightarrow \mathbb{k}[[\hbar]], \quad S : A \rightarrow A$$

such that the usual Hopf algebra diagrams commute.

2. The terms *topological algebra*, *topological coalgebra* and *topological bialgebra* are defined analogous to topological Hopf algebras.

Remark B.21.

1. A topological Hopf algebra A is generally not an actual Hopf algebra, since the comultiplication

$$\Delta : A \rightarrow A \widehat{\otimes} A$$

does in general not restrict to a map $A \rightarrow A \otimes A$.

2. If A is a topological Hopf algebra then $A/\hbar A$ becomes an Hopf algebra over \mathbb{k} . We note for this that

$$(A \widehat{\otimes} A)/\hbar(A \widehat{\otimes} A) \cong (A \otimes A)/\hbar(A \otimes A) \cong (A/\hbar A) \otimes (A/\hbar A).$$

Remark B.22. A topological algebra in the sense of Definition B.20 is precisely the same as an $\mathbb{k}[[\hbar]]$ -algebra which is complete as an $\mathbb{k}[[\hbar]]$ -module.

Indeed, suppose first that (A, m, u) is a topological algebra. Then the multiplication

$$m : A \widehat{\otimes} A \rightarrow A$$

restricts via the composition with the canonical homomorphism

$$A \otimes A \rightarrow A \widehat{\otimes} A$$

to a multiplication

$$m' : A \otimes A \rightarrow A.$$

Then (A, m', u) is an $\mathbb{k}[[\hbar]]$ -algebra (and A is by definition complete).

Suppose on the other hand that (A, m', u) is an $\mathbb{k}[[\hbar]]$ -algebra where A is complete. Then the multiplication map

$$m' : A \otimes A \rightarrow A$$

extends by the completeness of A uniquely to a $\mathbb{k}[[\hbar]]$ -linear map

$$m : A \widehat{\otimes} A \rightarrow A.$$

Then (A, m, u) is a topological algebra (by the denseness of $A \otimes A$ in $A \widehat{\otimes} A$, etc.).

Definition B.23. Let A be a Hopf algebra.

1. A *(formal) deformation* of A is a topological Hopf algebra A_\hbar whose underlying $\mathbb{k}[[\hbar]]$ -module is $A[[\hbar]]$ and for which $A_\hbar/\hbar A_\hbar = A$ as Hopf algebras.
2. (Formal) deformations of coalgebras and bialgebras are defined in the way as for algebras and Hopf algebras.
3. Two Hopf algebra deformations A_\hbar and A'_\hbar of A are *equivalent* if there exists an isomorphism of Hopf algebras

$$\varphi : A_\hbar \rightarrow A'_\hbar$$

such that the induced isomorphism of Hopf algebras

$$A = A_\hbar/\hbar A_\hbar \rightarrow A'_\hbar/\hbar A'_\hbar = A$$

is the identity, i.e. φ is the identity modulo \hbar .

Equivalence of deformations of coalgebras and bialgebras is defined in the same way.

4. A deformation is *trivial* if it is equivalent to the trivial deformation (i.e. $A[[\hbar]]$).
5. A Hopf algebra deformation of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra \mathfrak{g} is a *quantum universal enveloping algebra*.

Remark B.24. Let A be a Hopf algebra over \mathbb{k} with deformation A_{\hbar} . By using the isomorphism

$$A[[\hbar]] \hat{\otimes} A[[\hbar]] \cong (A \otimes A)[[\hbar]]$$

we can regard the structure maps of A_{\hbar} as $\mathbb{k}[[\hbar]]$ -linear map

$$\begin{aligned} m_{\hbar} &: (A \otimes A)[[\hbar]] \rightarrow A[[\hbar]], \\ u_{\hbar} &: \mathbb{k}[[\hbar]] \rightarrow A[[\hbar]], \\ \Delta_{\hbar} &: A[[\hbar]] \rightarrow (A \otimes A)[[\hbar]], \\ \varepsilon_{\hbar} &: A[[\hbar]] \rightarrow \mathbb{k}[[\hbar]], \\ S_{\hbar} &: A[[\hbar]] \rightarrow A[[\hbar]] \end{aligned} \tag{7}$$

which are perturbations of the structure maps of A , i.e. they reduce modulo \hbar to the structure maps of A .

We can for example characterize the comultiplication Δ_{\hbar} of A_{\hbar} by a sequence of bilinear map

$$\Delta_i : A \rightarrow A \otimes A$$

such that

$$\Delta_{\hbar}(a) = \Delta_0(a) + \Delta_1(a)\hbar + \Delta_2(a)\hbar^2 + \dots$$

for every $a \in A$. Here Δ_0 needs to be the original comultiplication from A .

Example B.25.

1. Every Hopf algebra A admits the trivial deformation $A[[\hbar]]$. In the form (7) the structure maps of this deformation are given by the $\mathbb{k}[[\hbar]]$ -linear extensions of the structure maps of A .
2. One can make the algebra deformation $U_{\hbar}(\mathfrak{sl}_2)$ of $U(\mathfrak{sl}_2)$ into a Hopf algebra deformation via the comultiplication

$$\Delta_{\hbar}(H) = H \otimes 1 + 1 \otimes H, \quad \Delta_{\hbar}(E) = E \otimes K + 1 \otimes E, \quad \Delta_{\hbar}(F) = F \otimes 1 + K^{-1} \otimes F$$

the counit

$$\varepsilon_{\hbar}(H) = 0, \quad \varepsilon_{\hbar}(E) = 0, \quad \varepsilon_{\hbar}(F) = 0,$$

and the antipode

$$S_{\hbar}(H) = -H, \quad S_{\hbar}(E) = -K^{-1}E, \quad S_{\hbar}(F) = -FK.$$

We note that it follows from these formulas for the element $K = e^{\hbar H}$ that

$$\Delta_{\hbar}(K) = K \otimes K, \quad \varepsilon_{\hbar}(K) = 1, \quad S_{\hbar}(K) = K^{-1}.$$

For the elements E, F, K, K^{-1} in $U_{\hbar}(\mathfrak{sl}_2)$ we hence regain the formulas for the Hopf algebra structure of $U_q(\mathfrak{sl}_2)$.

We lastly give an explanation of how the irreducible, finite-dimensional representations $L(n)$ of the universal enveloping algebra $U(\mathfrak{sl}_2)$ can be used to construct the irreducible, finite-dimensional representations $L(q^n)$ of $U_q(\mathfrak{sl}_2)$, where $n \in \mathbb{N}$.

Theorem B.26 ([CP95, Proposition 6.4.10]). For every natural number $n \in \mathbb{N}$ let $V(n)$ be the free $\mathbb{k}[[\hbar]]$ -module of rank $n + 1$ with basis v_0, \dots, v_n .

1. There exists a unique $U_{\hbar}(\mathfrak{sl}_2)$ -module structure on $V(n)$ such that

$$Hv_i := (n - 2i)v_i, \quad Ev_i := [n - i + 1]_q v_{i-1}, \quad Fv_i := [i + 1]_q v_{i+1}.$$

2. The $U_{\hbar}(\mathfrak{sl}_2)$ -modules $V(n)$ is indecomposable.
3. The $U_{\hbar}(\mathfrak{sl}_2)$ -module $V(n)$ reduces modulo \hbar to the irreducible representations $L(n)$ of $U(\mathfrak{sl}_2)$.
4. The actions of K and \tilde{H} on $V(n)$ is given by

$$Kv_i = q^{n-2i}v_i, \quad \tilde{H}v_i = [n - 2i]_q v_i.$$

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