

# The Quantum Group $U_q(\mathfrak{sl}_2)$

## Talk 14 on Hopf Algebras and Tensor Categories

### 1. Recalling $\mathfrak{sl}_2$ -Theory

Let  $\mathbb{k}$  be a field. The Lie algebra

$$\mathfrak{sl}_2 := \{A \in M(2, \mathbb{k}) \mid \text{tr}(A) = 0\}$$

admits the basis

$$E := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

and these basis elements satisfy the commutator relations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H. \quad (1)$$

Its universal enveloping algebra

$$U(\mathfrak{sl}_2) := T(\mathfrak{sl}_2) / (XY - YX - [X, Y] \mid X, Y \in \mathfrak{sl}_2)$$

is generated by the elements  $E, H, F$  subject to the relations (1), i.e.

$$U(\mathfrak{sl}_2) \cong \mathbb{k}\langle E, H, F \rangle / ([H, E] - 2E, [H, F] + 2F, [E, F] - H).$$

The universal enveloping algebra  $U(\mathfrak{sl}_2)$  is a Hopf algebra with comultiplication

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \varepsilon(X) = 0, \quad S(X) = 0 \quad \text{for every } X \in \mathfrak{sl}_2.$$

A representation of  $\mathfrak{sl}_2$  is the same as an  $U(\mathfrak{sl}_2)$ -module.

**Theorem 1.1** (Poincaré–Birkhoff–Witt). The algebra  $U(\mathfrak{sl}_2)$  admits the vector space basis

$$F^l H^m E^n \quad \text{with } l, m, n \in \mathbb{N}.$$

**Theorem 1.2.** Let  $\mathbb{k}$  be of characteristic zero.

1. Every finite-dimensional  $\mathfrak{sl}_2$ -representation is semisimple.
2. The finite-dimensional irreducible  $\mathfrak{sl}_2$ -representations are (up to isomorphism) given by certain representations  $L(n)$  for  $n \in \mathbb{N}$ . This representation  $L(n)$  has a basis  $w_0, \dots, w_n$  on which  $E, H, F$  act as depicted in Figure 1.

We refer to Appendix A.1 for more details on the representation theory of the Lie algebra  $\mathfrak{sl}_2$  in characteristic zero.

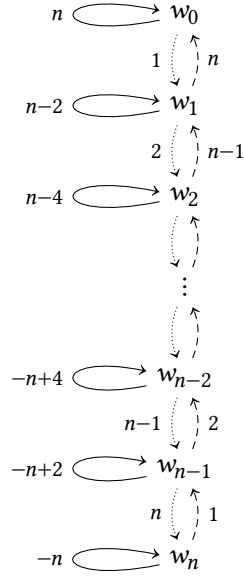


Figure 1: The irreducible representation  $L(n)$  of  $U(\mathfrak{sl}_2)$ . Loops depict the action of  $H$ , dashed arrows the action of  $E$  and dotted arrows the action of  $F$ .

## 2. The Algebra $U_q(\mathfrak{sl}_2)$

**Convention 2.1.** In the following  $\mathbb{k}$  denotes a field of characteristic zero and  $q$  is an element of  $\mathbb{k}$  with  $q \neq 0, 1, -1$ .

**Definition 2.2.** The  $\mathbb{k}$ -algebra  $U_q(\mathfrak{sl}_2)$  is given by the generators

$$E, \quad K, \quad K^{-1}, \quad F$$

subject to the relations

$$KK^{-1} = 1 = K^{-1}K, \quad KE = q^2EK, \quad KF = q^{-2}FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \quad (2)$$

**Remark 2.3** (Choice of  $q$ ). One often requires additional conditions on  $q$ , namely that

1.  $q$  is not a root of unity, or that
2.  $\mathbb{k}$  is the function field  $\mathbb{K}(q)$  over some other field  $\mathbb{K}$ , with  $q$  being the indeterminate.

**Remark 2.4** (The case  $q = 1$ ). The algebra  $U_q(\mathfrak{sl})$  admits another useful presentation: One introduces the element

$$\tilde{H} := \frac{K - K^{-1}}{q - q^{-1}}$$

as an additional generator, and then adjust the relations (2). This presentation of  $U_q(\mathfrak{sl}_2)$  does then make sense for any parameter  $q \in \mathbb{k}$ , and for  $q = 1$  one has an isomorphism

$$U_1(\mathfrak{sl}_2) \cong U(\mathfrak{sl}_2)[\sigma]/(\sigma^2 - 1)$$

given by

$$E \mapsto \sigma E, \quad \tilde{H} \mapsto \sigma H, \quad F \mapsto F, \quad K \mapsto \sigma. \quad (3)$$

We refer to Appendix A.2 for more details on this presentation.

**Remark 2.5.** One might think about the algebra  $U_q(\mathfrak{sl}_2)$  as containing the original elements  $E, F$  of  $\mathfrak{sl}_2$ , but not the element  $H$ . We will later see that the algebra  $U_q(\mathfrak{sl})$  lives (up to some technical details) inside an  $\mathbb{k}[[\hbar]]$ -algebra  $U_{\hbar}(\mathfrak{sl}_2)$  which then also contains  $H$ , and in which

$$q = e^{\hbar}, \quad K = e^{\hbar H}.$$

We may therefore think about the element  $K$  as

$$K = q^H.$$

**Theorem 2.6** (PBW basis). The algebra  $U_q(\mathfrak{sl}_2)$  has a vector space basis given by

$$F^l K^m E^n \quad \text{with } l, n \in \mathbb{N} \text{ and } m \in \mathbb{Z}$$

*Proof.* See Appendix A.4. □

We refer to Appendix A.5 for more information on the algebra structure of  $U_q(\mathfrak{sl}_2)$ .

### 3. Representation Theory of $U_q(\mathfrak{sl}_2)$

We will now focus on the finite-dimensional representation theory of  $U_q(\mathfrak{sl}_2)$ .

#### 3.1. The Case $q = 1$

For motivational purposes we consider first the special case  $q = 1$ .

By using the above isomorphism (3) we see that every  $\mathfrak{sl}_2$ -representation  $M$  extends to an  $U_1(\mathfrak{sl}_1)$ -module by letting  $\sigma$  act on  $M$  by either 1 or  $-1$ . Let us denote the  $U_1(\mathfrak{sl}_1)$ -modules which result from  $M = L(n)$  for  $n \in \mathbb{N}$  by  $L(\varepsilon, n)$  for  $\varepsilon \in \{+, -\}$ . One can conclude from Theorem 1.2 that every finite-dimensional  $U_1(\mathfrak{sl}_2)$ -module is semisimple, and that the irreducible finite-dimensional  $U_1(\mathfrak{sl}_2)$ -modules are given precisely by the modules  $L(\pm, n)$ . One can depict these irreducible modules as in Figure 2. We refer to Appendix A.3 for proofs of these claims.

We will keep the case of  $U_1(\mathfrak{sl}_2)$  in the back of our minds throughout the following discussion of the general case of  $U_q(\mathfrak{sl}_2)$ .

#### 3.2. Weight Space Decomposition

One of the useful tools in the representation theory of  $\mathfrak{sl}_2$  are weight space decompositions. By this we mean the decomposition of an  $\mathfrak{sl}_2$ -representation into eigenspaces with respect to the action of the element  $H$  of  $\mathfrak{sl}_2$ .

We also want this tool to be available for studying the representation theory of  $U_q(\mathfrak{sl}_2)$ . For this we replace the role of  $H$  by that of  $K$ .

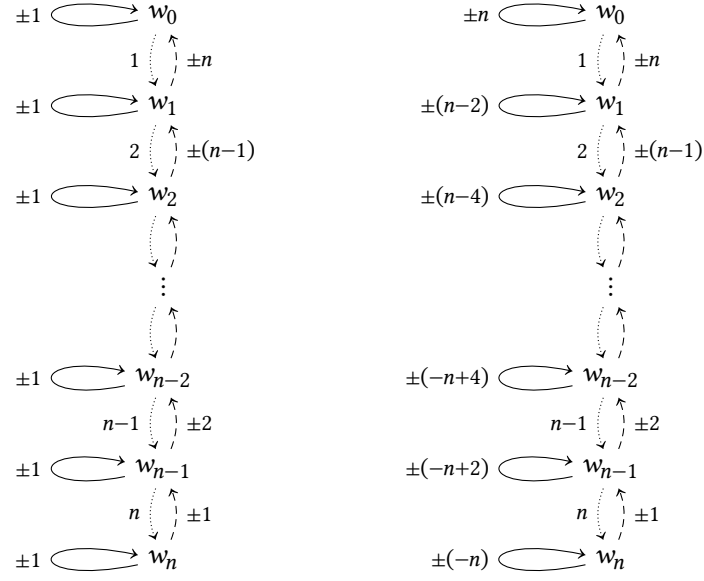


Figure 2: The irreducible representations  $L(\pm, n)$  of  $U_1(\mathfrak{sl}_2)$ . On the left side loops depict the action of  $K$ , and on the right side they depict the action of  $\tilde{H}$ . On both sides dashed arrows depict the action of  $E$  and dotted arrows depict the action of  $F$ .

**Convention 3.1.** In the following  $q$  is an element of  $\mathbb{k}$  which is not a root of unity, unless otherwise specified.

**Definition 3.2.** Let  $M$  be an  $U_q(\mathfrak{sl}_2)$ -module. For every scalar  $\lambda \in \mathbb{k}^\times$  the associated *weight space* is given by

$$M_\lambda := \{m \in M \mid Km = \lambda m\}.$$

**Theorem 3.3.** Let  $M$  be an  $U_q(\mathfrak{sl}_2)$ -module.

1. It holds for every scalar  $\lambda \in \mathbb{k}^\times$  that

$$EM_\lambda \subseteq M_{q^2\lambda}, \quad FM_\lambda \subseteq M_{q^{-2}\lambda}.$$

2. If  $M$  is finite-dimensional then  $M$  decomposes into weight spaces, and all occuring weights are of the form  $\pm q^n$  with  $n \in \mathbb{Z}$ .

*Proof.* See Appendix A.6. □

### 3.3. Verma Modules and Classifications

The weight space decompositions of finite-dimensional  $\mathfrak{sl}_2$ -representations leads to the classification of irreducible finite-dimensional  $\mathfrak{sl}_2$ -modules via the more general classification of irreducible highest-weight representations. Thanks to Theorem 3.3 we can mirror this approach for  $U_q(\mathfrak{sl}_2)$ -modules.

**Definition 3.4.** Let  $M$  be an  $U_q(\mathfrak{sl}_2)$ -module.

1. A weight vector  $m$  of  $M$  is *primitive* if it is nonzero and  $Em = 0$ .
2. The module  $M$  is of *highest weight*  $\lambda$  if it is generated by a primitive weight vector of weight  $\lambda$ .

**Proposition 3.5.** Every irreducible finite-dimensional  $U_q(\mathfrak{sl}_2)$ -module is a highest weight module.

*Proof.* The assertion follows from Theorem 3.3.  $\square$

We will now classify the irreducible highest-weight representations of  $U_q(\mathfrak{sl}_2)$  and determine which of these irreducibles are finite-dimensional. Thanks to Proposition 3.5 this will give us a classification of the irreducible finite-dimensional  $U_q(\mathfrak{sl}_2)$ -modules.

**Definition 3.6.** Let  $U_q(\mathfrak{b})$  be the subalgebra of  $U_q(\mathfrak{sl}_2)$  generated by the elements  $E, K, K^{-1}$ .<sup>1</sup>

**Definition 3.7.** Let  $\lambda \in \mathbb{k}^\times$ .

1. Let  $\mathbb{k}_\lambda$  be the one-dimensional  $U_q(\mathfrak{b})$ -module whose underlying vector space is given by  $\mathbb{k}$ , together with the action of  $U_q(\mathfrak{b})$  given by

$$K \cdot 1 = \lambda, \quad E \cdot 1 = 0.$$

2. The *Verma module* associated to  $\lambda$  is the  $U_q(\mathfrak{sl}_2)$ -module given by

$$M(\lambda) := U_q(\mathfrak{sl}_2) \otimes_{U_q(\mathfrak{b})} \mathbb{k}_\lambda.$$

In the upcoming Proposition 3.10 we will describe the structure of the Verma module  $M(\lambda)$ . This will require us to express the occurring coefficients – which will be rational expressions involving the highest weight  $\lambda$  and powers of  $q$  – in an efficient and insightful way. For this we will first introduce some standard notation.

**Definition 3.8.** For  $q \in \mathbb{k}$  with  $q \neq 0$  and  $n \in \mathbb{Z}$  the *n-th quantum integer* is

$$[n]_q := \begin{cases} q^{n-1} + q^{n-3} + \dots + q^{-n+3} + q^{-n+1} & \text{if } n \geq 0, \\ -(q^{n+1} + q^{n+3} + \dots + q^{-n-3} + q^{-n-1}) & \text{if } n \leq 0, \end{cases}$$

which can for  $q \neq 1, 0, -1$  be expressed as

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The *n-th quantum factorial* for  $n \in \mathbb{N}$  is given by

$$[n]_q! := [n]_q [n-1]_q \cdots [1]_q.$$

For every invertible element  $x \in U_q(\mathfrak{sl}_2)$  and integer  $n \in \mathbb{Z}$  let

$$[x, n]_q := \frac{q^n x - q^{-n} x^{-1}}{q - q^{-1}}.$$

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<sup>1</sup>Here  $\mathfrak{b}$  refers to the Lie subalgebra of  $\mathfrak{sl}_2$  consisting of the traceless upper triangular matrices, see Appendix A.1.

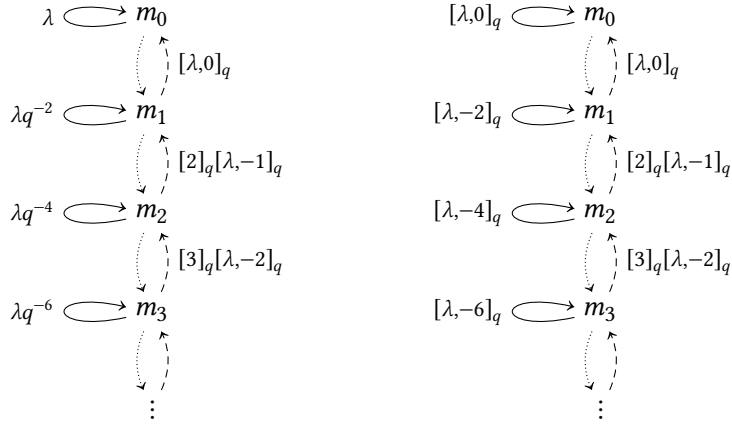


Figure 3: The Verma module  $M(\lambda)$  of  $U_q(\mathfrak{sl}_2)$ . On the left side loops depict the action of  $K$ , and on the right side they depict the action of  $\tilde{H}$ . On both sides the action of  $F$  is depicted by dotted arrows and the action of  $E$  by dashed arrows.

**Remark 3.9.** For  $q = 1$  we have  $[n]_1 = n$  for every  $n \in \mathbb{Z}$  and  $[n]_1! = n!$  for every  $n \in \mathbb{N}$ .

**Proposition 3.10.** Let  $\lambda \in \mathbb{k}^\times$ .

1. The Verma module  $M(\lambda)$  has the basis

$$m_i := F^i \otimes 1 \quad \text{with } i \geq 0,$$

and the actions of  $E, K, F$  on this basis is given by

$$Fm_i = m_{i+1}, \quad Km_i = \lambda q^{-2i} m_i, \quad Em_i = [i]_q [\lambda, 1 - i]_q m_{i-1}.$$

This action can be graphically described as in Figure 3.

2. The Verma module  $M(\lambda)$  is indecomposable.
3. a. If  $\lambda = \pm q^n$  for some  $n \in \mathbb{N}$  then the Verma module  $M(\lambda)$  contains a unique nonzero, proper submodule  $N_\lambda$ , which is spanned by the elements

$$m_i \quad \text{with } i \geq n + 1.$$

This submodule is isomorphic to  $M(\pm q^{-n-2})$ .

- b. If  $\lambda \neq \pm q^n$  for every  $n \in \mathbb{N}$  then the Verma module  $M(\lambda)$  is irreducible.

*Proof.* See Appendix A.7. □

**Definition 3.11.** For every scalar  $\lambda \in \mathbb{k}^\times$  let

$$L(\lambda) := \begin{cases} M(\lambda)/N_\lambda & \text{if } \lambda = \pm q^n \text{ for some } n \in \mathbb{N}, \\ M(\lambda) & \text{otherwise.} \end{cases}$$

**Theorem 3.12.**

1. There is a one-to-one correspondence given by

$$\begin{aligned} \mathbb{K}^\times &\longrightarrow \left\{ \begin{array}{l} \text{isomorphism classes of irreducible} \\ \text{highest-weight } U_q(\mathfrak{sl}_2)\text{-modules} \end{array} \right\}, \\ \lambda &\longmapsto L(\lambda). \end{aligned}$$

2. The module  $L(\lambda)$  is finite-dimensional if and only if  $\lambda = \pm q^n$  for some  $n \in \mathbb{N}$ . The above one-to-one correspondence does therefore restrict to a one-to-one correspondence given by

$$\begin{aligned} \{1, -1\} \times \mathbb{N} &\longrightarrow \left\{ \begin{array}{l} \text{isomorphism classes of irreducible} \\ \text{finite-dimensional } U_q(\mathfrak{sl}_2)\text{-modules} \end{array} \right\}, \\ (\varepsilon, n) &\longmapsto L(\varepsilon q^n). \end{aligned}$$

We have for every  $n \in \mathbb{N}$  that

$$\dim(L(\pm q^n)) = n + 1.$$

**Remark 3.13.**

1. For every  $n \in \mathbb{Z}$  we have

$$[\pm q^n, -i + 1]_q = \pm [n - i + 1]_q.$$

On the rescaled basis  $m_0, \dots, m_n$  of  $L(\pm q^n)$  given by

$$w_i := \frac{m_i}{[i]_q!}$$

the actions of  $E, K, F$  thus become

$$E w_i = \pm [n - i + 1]_q w_{i-1}, \quad K w_i = \pm q^{n-2i} w_i, \quad F w_i = [i + 1]_q w_{i+1}.$$

The action of  $E, K, F$  on  $L(\pm q^n)$  can therefore be graphically depicted as in Figure 4.

2. We can consider again the element

$$\tilde{H} := \frac{K - K^{-1}}{q - q^{-1}}$$

of  $U_q(\mathfrak{sl}_2)$ . This element acts on the weight space  $M_{\lambda q^{-2i}}$  by the scalar  $[\lambda, -2i]_q$ . For  $\lambda = \pm q^n$  this scalar is given by

$$[\lambda, -2i]_q = [\pm q^n, -2i]_q = \pm [n - 2i]_q.$$

The action of  $\tilde{H}$  on the Verma module  $M(\lambda)$  and the irreducible modules  $L(\pm q^n)$  is therefore as depicted in Figure 3 and Figure 4.

3. We observe that for  $q = 1$  the descriptions of the irreducible  $U_q(\mathfrak{sl}_2)$ -modules  $L(\pm q^n)$  from Figure 4 become the description of the irreducible  $U_1(\mathfrak{sl}_2)$ -modules  $L(\pm, n)$  from Figure 2.

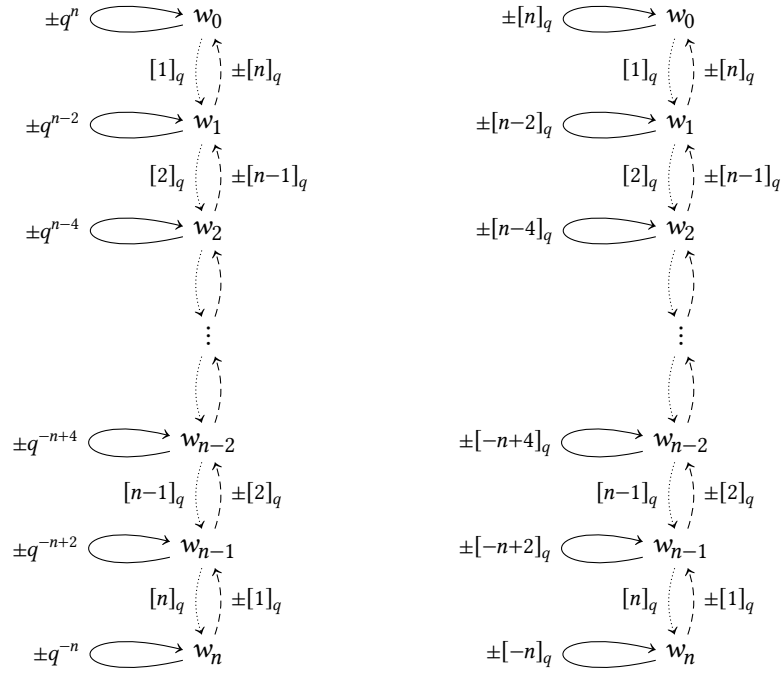


Figure 4: The irreducible representations  $L(\pm q^n)$  of  $U_q(\mathfrak{sl}_2)$ . On the left side the loops depict the action of  $K$ , an on the right side they depict the action of  $\tilde{H}$ . On both sides the action of  $F$  is depicted by dotted arrows and the action of  $E$  by dashed arrows.

### 3.4. Semisimplicity of Finite-Dimensional $U_q(\mathfrak{sl}_2)$ -modules

**Theorem 3.14.** Every finite-dimensional  $U_q(\mathfrak{sl}_2)$ -module is semisimple.

*Proof.* See Appendix A.8. □

**Corollary 3.15.** Let  $M, N$  be two finite-dimensional  $U_q(\mathfrak{sl}_2)$ -modules with  $\dim M_\lambda = \dim N_\lambda$  for every  $\lambda \in \mathbb{k}^\times$ . Then  $M \cong N$ . □

## 4. Hopf Algebra Structure on $U_q(\mathfrak{sl}_2)$

**Proposition 4.1.** The algebra  $U_q(\mathfrak{sl}_2)$  becomes a Hopf algebra when endowed with the comultiplication  $\Delta$ , the counit  $\varepsilon$  and the antipode  $S$  given by

$$\begin{aligned} \Delta(E) &= E \otimes K + 1 \otimes E, & \Delta(F) &= F \otimes 1 + K^{-1} \otimes F, & \Delta(K) &= K \otimes K, \\ \varepsilon(E) &= 0, & \varepsilon(F) &= 0, & \varepsilon(K) &= 1 \\ S(E) &= -EK^{-1}, & S(F) &= -KF, & S(K) &= K^{-1}. \end{aligned}$$



*Proof.* One checks that the proposed images of the algebra generators  $E, K, K^{-1}, F$  are compatible with the defining relations of  $U_q(\mathfrak{sl}_2)$ , giving rise to well-defined algebra homomorphisms

$$\Delta : U_q(\mathfrak{sl}_2) \rightarrow U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2), \quad \varepsilon : U_q(\mathfrak{sl}_2) \rightarrow \mathbb{k}, \quad S : U_q(\mathfrak{sl}_2) \rightarrow U_q(\mathfrak{sl}_2)^{\text{op}}.$$

It can then be checked on the algebra generators  $E, K, K^{-1}, F$  that the required Hopf algebra diagrams commute.  $\square$

**Definition 4.2.** The Hopf algebra structure of  $U_q(\mathfrak{sl}_2)$  is given as described in Proposition 4.1.

**Remark 4.3.** The Hopf algebra  $U_q(\mathfrak{sl}_2)$  is neither commutative nor cocommutative. It is an example of a so-called *quantum group*.

The newly-introduced Hopf algebra structure on  $U_q(\mathfrak{sl}_2)$  allows us to consider tensor products of  $U_q(\mathfrak{sl}_2)$ -modules. For the original universal enveloping algebra  $U(\mathfrak{sl}_2)$  its Hopf algebra structure is cocommutative, from which it follows that for any two  $\mathfrak{sl}_2$ -representations  $M, N$  the usual flip map  $M \otimes N \rightarrow N \otimes M$  (given by  $m \otimes n \mapsto n \otimes m$ ) is an isomorphism of representations. But  $U_q(\mathfrak{sl}_2)$  is not cocommutative anymore, so things change.

**Lemma 4.4.** Let  $M, N$  be two finite-dimensional  $U_q(\mathfrak{sl}_2)$ -modules. Then for every  $\lambda \in \mathbb{k}^\times$ ,

$$(M \otimes N)_\lambda = \bigoplus_{\mu\kappa=\lambda} M_\mu \otimes N_\kappa.$$

*Proof.* See Appendix A.9.  $\square$

**Corollary 4.5.** Let  $M, N$  be two finite-dimensional  $U_q(\mathfrak{sl}_2)$ -modules. Then

$$M \otimes N \cong N \otimes M$$

as  $U_q(\mathfrak{sl}_2)$ -modules.

*Proof.* This follows from Corollary 3.15 and Lemma 4.4.  $\square$

**Warning 4.6.** For two (finite-dimensional)  $U_q(\mathfrak{sl}_2)$ -modules  $M, N$  the usual flip map

$$\tau : M \otimes N \rightarrow N \otimes M, \quad m \otimes n \mapsto n \otimes m$$

is in general not  $U_q(\mathfrak{sl}_2)$ -linear, and thus not an isomorphism of  $U_q(\mathfrak{sl}_2)$ -modules.

**Example 4.7.** Let us consider the  $U_q$ -modules  $M, N$  given by  $L(q)$ , with basis  $m_0, m_1$ . Then on the one hand

$$F \cdot (m_0 \otimes m_1) = \underbrace{(Fm_0)}_{=m_1} \otimes m_1 + (K^{-1}m_0) \otimes \underbrace{(Fm_1)}_{=0} = m_1 \otimes m_1$$

while on the other hand

$$F \cdot (m_1 \otimes m_0) = \underbrace{(Fm_1)}_{=0} \otimes m_0 + \underbrace{(K^{-1}m_1)}_{=qm_1} \otimes \underbrace{(Fm_0)}_{=m_1} = qm_1 \otimes m_1.$$

From the above assertions one can conclude a quantum version of the Clebsch–Gordan formula, which explains how a tensor product  $L(\delta q^n) \otimes L(\varepsilon q^m)$  decomposes into irreducible representations. We refer to Appendix A.10 for more details.

## 5. Outlook: The Deformation $U_{\hbar}(\mathfrak{sl}_2)$

**Definition 5.1.** Let  $A$  be a Hopf algebra over  $\mathbb{k}$ . A (formal) deformation  $A$  is a Hopf algebra over  $\mathbb{k}[[\hbar]]$  such that  $A_{\hbar} = A[[\hbar]]$  as  $\mathbb{k}[[\hbar]]$ -modules and  $A_{\hbar}/\hbar A_{\hbar} = A$  as Hopf algebras over  $\mathbb{k}$ .

**Remark 5.2.** The above definition is actually wrong. Instead of Hopf algebras over  $\mathbb{k}[[\hbar]]$  one needs to consider *topological Hopf algebras* (over  $\mathbb{k}[[\hbar]]$ ). This means that for the comultiplication  $\Delta_{\hbar}$  of  $A_{\hbar}$  one has to replace the tensor product

$$A_{\hbar} \otimes_{\mathbb{k}[[\hbar]]} A_{\hbar}$$

by its  $\hbar$ -adic completion

$$A_{\hbar} \widehat{\otimes} A_{\hbar}.$$

In the given situation we have

$$A_{\hbar} \widehat{\otimes} A_{\hbar} = A[[\hbar]] \widehat{\otimes} A[[\hbar]] \cong (A \otimes A)[[\hbar]]$$

as  $\mathbb{k}[[\hbar]]$ -modules. This means that we must allow  $\Delta_{\hbar}$  to have as its values not only tensors, but actually power series of tensors.

**Theorem 5.3.** The universal enveloping algebra  $U(\mathfrak{sl}_2)$  admits a Hopf algebra deformation

$$U_{\hbar}(\mathfrak{sl}_2)$$

such that

$$\begin{aligned} [H, E] &= 2E, \quad [H, F] = -2F, \quad [E, F] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}}, \\ \Delta(E) &= E \otimes e^{\hbar H} + 1 \otimes E, \quad \Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(F) = F \otimes 1 + e^{-\hbar H} \otimes F, \\ \varepsilon(E) &= 0, \quad \varepsilon(H) = 0, \quad \varepsilon(F) = 0, \\ S(E) &= -Ee^{-\hbar H}, \quad S(H) = -H, \quad S(F) = -e^{\hbar H}F. \end{aligned}$$

**Remark 5.4.**

1. In the deformation  $U_{\hbar}(\mathfrak{sl}_2)$  we can consider the well-defined elements

$$q := e^{\hbar}, \quad K := e^{\hbar H}.$$

The elements  $q, E, K, K^{-1}, F$  satisfy the defining relations of  $U_q(\mathfrak{sl}_2)$ . We can thus (up to some technical details) regard  $U_{\hbar}(\mathfrak{sl}_2)$  as a subalgebra of  $U_q(\mathfrak{sl}_2)$ .

2. The deformation  $U_{\hbar}(\mathfrak{sl}_2)$  contains both the element  $H$  of  $\mathfrak{sl}_2$  and the element

$$\tilde{H} := [E, F] = \frac{K - K^{-1}}{q - q^{-1}} = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}},$$

which is of the form

$$\tilde{H} = H + \text{terms of order at least } \hbar^2.$$

We may think about the element  $\tilde{H}$  is a deformation of the element  $H$  (in an informal sense). We note that

$$q \equiv 1, \quad K \equiv 1, \quad \tilde{H} \equiv H \pmod{\hbar}.$$

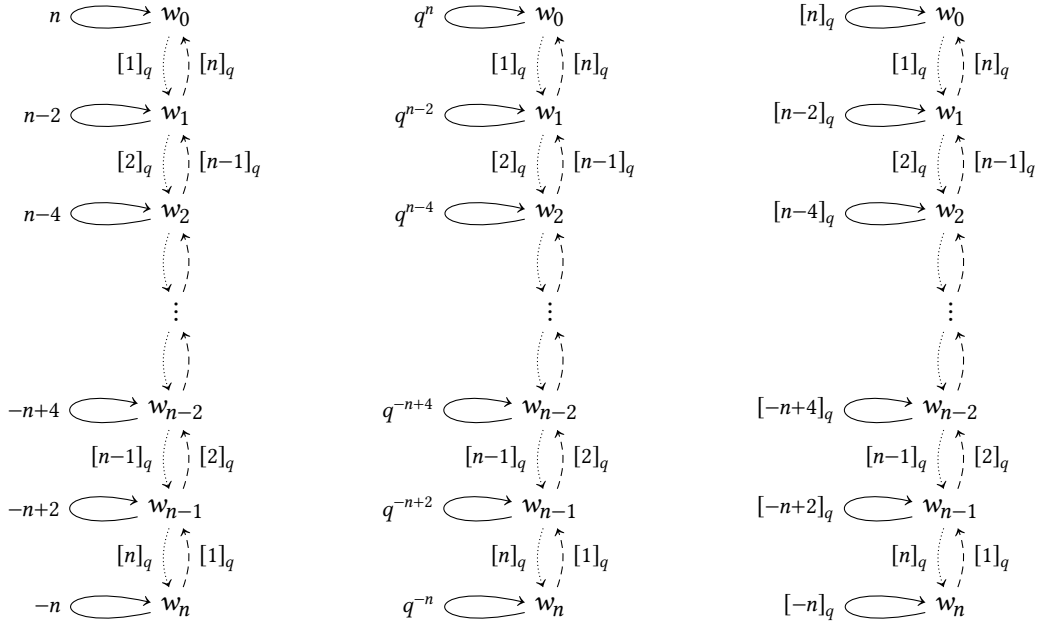


Figure 5: The indecomposable representation  $V(n)$  of  $U_{\hbar}(\mathfrak{sl}_2)$ . On the left side loops depict the action of  $H$ , in the middle they depict the action of  $K$ , and on the right they depict the action of  $\tilde{H}$ . Dashed arrows depict the action of  $E$  and dotted arrows the action of  $F$ .

**Theorem 5.5** ([CP95, Proposition 6.4.10]). For every natural number  $n \in \mathbb{N}$  let  $V(n)$  be the free  $\mathbb{k}[[\hbar]]$ -module of rank  $n + 1$  with basis  $w_0, \dots, w_n$ .

1. The  $\mathbb{k}[[\hbar]]$ -module structure of  $V(n)$  extends uniquely to an  $U_{\hbar}(\mathfrak{sl}_2)$ -module structure such that

$$Hw_i := (n - 2i)w_i, \quad Ew_i := [n - i + 1]_q w_{i-1}, \quad Fw_i := [i + 1]_q w_{i+1}.$$

The actions of  $E, H, F$  can be graphically depicted as in Figure 5.

2. The  $U_{\hbar}(\mathfrak{sl}_2)$ -modules  $V(n)$  is indecomposable.
3. The  $U_{\hbar}(\mathfrak{sl}_2)$ -module  $V(n)$  reduces modulo  $\hbar$  to the irreducible representations  $L(n)$  of  $U(\mathfrak{sl}_2)$ .
4. The actions of  $K$  and  $\tilde{H}$  on  $V(n)$  are given by

$$Kw_i = q^{n-2i}w_i, \quad \tilde{H}w_i = [n - 2i]_q w_i.$$

It follows that the  $U_{\hbar}(\mathfrak{sl}_2)$ -module  $V(n)$  becomes the irreducible  $U_q(\mathfrak{sl}_2)$ -module  $L(q^n)$  by restriction (up to some technical details).

We refer to Appendix B for a more detailed account on deformations of algebras and Hopf algebras, and proper formulations of some of the above statements.

## A. Remarks and Proofs

### A.1. Representation Theory of $\mathfrak{sl}_2$

Let  $\mathfrak{b}$  denote the Lie subalgebra of  $\mathfrak{sl}_2$  consisting of (traceless) upper triangular matrices. It has the matrices  $E, H$  as a basis. Its universal enveloping algebra  $U(\mathfrak{b})$  has the PBW-basis

$$H^m E^n \quad \text{with } m, n \geq \in \mathbb{N},$$

and it is a subalgebra of  $U(\mathfrak{sl}_2)$ .

#### A.1.1. Weight Spaces and Shifting of Weight Spaces

**Definition A.1.** Let  $V$  be a representation of  $\mathfrak{sl}_2$ .

1. The *weight space* of  $V$  with respect to  $\lambda$  is given by

$$V_\lambda := \{v \in V \mid H.v = \lambda v\}.$$

2. A nonzero weight vector  $v$  of  $V$  is *primitive* if  $E.v = 0$ .
3. The representation  $V$  is of *highest weight*  $\lambda$  if it is generated by a primitive weight vector of weight  $\lambda$ .

**Proposition A.2** (Shifting weight spaces). Let  $V$  be a representation of  $\mathfrak{sl}_2$  and let  $\lambda \in \mathbb{k}$ . Then

$$E.V_\lambda \subseteq V_{\lambda+2}, \quad F.V_\lambda \subseteq V_{\lambda-2}.$$

*Proof.* This follows from the commutator relations  $[H, E] = 2E$  and  $[H, F] = -2F$ . □

**Lemma A.3.** Suppose that  $\mathbb{k}$  be algebraically closed. Then every finite-dimensional irreducible representation of  $\mathfrak{sl}_2$  is a highest-weight representation.

#### A.1.2. Verma Modules

There exists for every scalar  $\lambda \in \mathbb{k}$  a universal representation of highest weight  $\lambda$ , the so-called Verma module.

**Definition A.4.** For every scalar  $\lambda \in \mathbb{k}$  let  $\mathbb{k}_\lambda$  be the one-dimensional representation of  $\mathfrak{b}$  which is given by  $\mathbb{k}$  as its underlying vector space together with the action of  $\mathfrak{b}$  on  $\mathbb{k}$  given by

$$H.1 = \lambda, \quad E.1 = 0.$$

**Lemma A.5.** There is an isomorphism of  $U(\mathfrak{b})$ -modules given by

$$U(\mathfrak{b})/\langle E, H - \lambda \rangle \rightarrow \mathbb{k}_\lambda, \quad \bar{x} \mapsto x.1.$$

**Definition A.6.** The *Verma module* of highest weight  $\lambda$  is the  $U(\mathfrak{sl}_2)$ -module given by

$$M(\lambda) := U(\mathfrak{sl}_2) \otimes_{U(\mathfrak{b})} \mathbb{k}_\lambda.$$

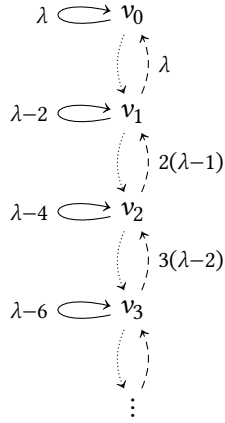


Figure 6: The Verma module  $M(\lambda)$  of  $U(\mathfrak{sl}_2)$ . The action of  $H$  is depicted by loops, the action of  $F$  by dotted arrows and the action of  $E$  by dashed arrows.

**Convention A.7.** From now on the field  $\mathbb{k}$  is of characteristic zero.

**Proposition A.8.** Let  $\lambda \in \mathbb{k}$ .

1. The Verma module  $M(\lambda)$  has the vectors

$$v_i := F^i \otimes 1 \quad \text{with } i \geq 0$$

as a basis. The actions of  $E, H, F$  on this basis are given by

$$F.v_i = v_{i+1}, \quad H.v_i = (\lambda - 2i)v_i, \quad E.v_i = i(\lambda - i + 1)v_{i-1}.$$

This action can be graphically depicted as in Figure 6.

2. The Verma module  $M(\lambda)$  is a representation of highest weight  $\lambda$ .
3. There exists for every representation  $V$  of  $\mathfrak{sl}_2$  an isomorphism of vector spaces given by

$$\begin{aligned} \text{Hom}_{\mathfrak{sl}_2}(M(\lambda), V) &\longrightarrow \{v \in V \mid v \text{ is of weight } \lambda \text{ with } E.v = 0\}, \\ \varphi &\longmapsto \varphi(1 \otimes 1). \end{aligned}$$

In particular

$$\text{End}_{\mathfrak{sl}_2}(M(\lambda)) = \mathbb{k}.$$

4. The representation  $M(\lambda)$  is indecomposable.
5. a. If  $\lambda \notin \mathbb{N}$  then the representation  $M(\lambda)$  is irreducible.  
b. If  $\lambda = n \in \mathbb{N}$  then the representation  $M(\lambda)$  has a unique nonzero, proper subrepresentation, which is spanned by the vectors

$$v_i \quad \text{with } i \geq n + 1.$$

This subrepresentation is isomorphic to  $M(-n - 2)$ .

**Definition A.9.** Let  $\lambda \in \mathbb{k}$ .

1. For  $\lambda \notin \mathbb{N}$  let  $L(\lambda) := M(\lambda)$ .
2. For  $\lambda \in \mathbb{N}$  let  $L(\lambda) := M(\lambda)/N$  where  $N$  is the unique nonzero, proper subrepresentation of  $M(\lambda)$ .

### A.1.3. Classifications of Certain Irreducible Representations

**Theorem A.10.**

1. There is a one-to-one correspondence given by

$$\begin{aligned} \mathbb{k} &\longrightarrow \{\text{irreducible highest-weight } \mathfrak{sl}_2\text{-representations}\}, \\ \lambda &\longmapsto L(\lambda). \end{aligned}$$

2. The representation  $L(\lambda)$  is finite-dimensional if and only if  $\lambda = n \in \mathbb{N}$ , in which case

$$\dim(L(n)) = n + 1.$$

If  $\mathbb{k}$  is algebraically closed (so that every irreducible finite-dimensional  $\mathfrak{sl}_2$ -representation is a highest-weight representation) then the above correspondence does therefore restrict to a one-to-one correspondence

$$\begin{aligned} \mathbb{N} &\longrightarrow \{\text{irreducible finite-dimensional } \mathfrak{sl}_2\text{-representations}\}, \\ n &\longmapsto L(n). \end{aligned}$$

**Remark A.11.** Let  $n \in \mathbb{N}$ . The basis  $v_0, \dots, v_n$  of  $L(n)$  can be rescaled to the basis

$$w_i := \frac{v_i}{i!}.$$

The actions of  $E, F$  on this basis are given by

$$E.w_i = (n - i + 1)w_{i-1}, \quad F.w_i = (i + 1)w_{i+1}.$$

The actions of  $E, H, F$  on  $L(n)$  can now be graphically presented as in Figure 1.

### A.1.4. Semisimplicity of Finite-Dimensional Representations

**Theorem A.12 (Weyl).** Let  $\mathbb{k}$  be algebraically closed. Every finite-dimensional representation of  $\mathfrak{sl}_2$  is semisimple.

*Proof.* See [Hum72, Theorem 6.3]. □

**Corollary A.13.** Any finite-dimensional  $\mathfrak{sl}_2$ -representation admits a weight space decomposition. All occurring weights are integral. □

The decomposition of a finite-dimensional  $\mathfrak{sl}_2$ -representation into irreducible representations can be read off from its weight space decomposition. From this the following results can be shown.

**Lemma A.14.** Let  $M, N$  be two finite-dimensional  $\mathfrak{sl}_2$ -representations with  $\dim(M_\lambda) = \dim(N_\lambda)$  for every  $\lambda \in \mathbb{k}$ . Then  $M \cong N$ .

**Corollary A.15** (Clebsch–Gordan). Let  $n, m$  be natural numbers with  $n \geq m$ . Then

$$L(n) \otimes L(m) \cong L(n+m) \oplus L(n+m-2) \oplus \cdots \oplus L(n-m).$$

### A.1.5. The General Case of Characteristic Zero

We have above used a few times the additional assumption that  $\mathbb{k}$  is algebraically closed. We will now explain how to get rid of this assumption. For this we first recall some standard results about the semisimplicity of (finite-dimensional) algebras and representations.

**Lemma A.16.** Let  $\mathbb{k}$  be any field and let  $V$  be a finite-dimensional  $\mathbb{k}$ -vector spaces. Let  $A$  be a subalgebra of  $\text{End}_{\mathbb{k}}(V)$ . Then  $A$  is semisimple if and only if  $V$  is semisimple as an  $A$ -module.

*Proof.* See [Lan02, XVII, §5, Proposition 4.7] and [MS16], or [Mil13, Proposition 5.13] □

**Definition A.17.** The *Jacobson radical* of a ring  $R$  is the intersection of all maximal left ideal of  $R$ . It is denoted by  $J(R)$ .

**Remark A.18.** Let  $R$  be a ring. The irreducible  $R$ -modules are up to isomorphism precisely those  $R$ -modules of the form  $R/\mathfrak{m}$ , where  $\mathfrak{m}$  is a maximal left ideal in  $R$ . The Jacobson radical of  $R$  does therefore consists of precisely those elements of  $R$  which annihilate every irreducible  $R$ -module.

**Proposition A.19.** Let  $A$  be a finite-dimensional  $\mathbb{k}$ -algebra.

1. The Jacobson radical  $J(A)$  is a nilpotent, two-sided ideal in  $A$ .
2. Every nilpotent, two-sided ideal of  $A$  is contained in its Jacobson radical  $J(A)$ . The Jacobson radical  $J(A)$  is thus the unique maximal nilpotent, two-sided ideal in  $A$ .
3. The following conditions on  $A$  are equivalent:
  - i. The algebra  $A$  is semisimple.
  - ii. The Jacobson radical  $J(A)$  vanishes.
  - iii. The algebra  $A$  does not contain any nonzero, nilpotent, two-sided ideal.

*Proof.* See [Lam01, §4]. □

**Corollary A.20** ([Mil13, Proposition 5.11]). Let  $A$  be a finite-dimensional  $\mathbb{k}$ -algebra. Let  $\mathbb{K}$  be a field extension of  $\mathbb{k}$  and suppose that the  $\mathbb{K}$ -algebra  $\mathbb{K} \otimes_{\mathbb{k}} A$  is semisimple. Then  $A$  is semisimple.

*Proof.* We find for the Jacobson radical  $J(A)$  that  $\mathbb{K} \otimes_{\mathbb{k}} J(A)$  is a nilpotent, two-sided ideal of  $\mathbb{K} \otimes_{\mathbb{k}} A$ . It is thus contained in the Jacobson radical  $J(\mathbb{K} \otimes_{\mathbb{k}} A)$ . This Jacobson radical vanishes because  $\mathbb{K} \otimes_{\mathbb{k}} A$  is semisimple. It follows that  $\mathbb{K} \otimes_{\mathbb{k}} J(A) = 0$  and thus  $J(A) = 0$ . □

**Corollary A.21.** Let  $A$  be a  $\mathbb{k}$ -algebra and let  $M$  be a finite-dimensional  $A$ -module. Let  $\mathbb{K}$  be a field extension of  $\mathbb{k}$  and suppose that  $\mathbb{K} \otimes_{\mathbb{k}} M$  is semisimple as an  $(\mathbb{K} \otimes_{\mathbb{k}} A)$ -module. Then  $M$  is semisimple as an  $A$ -module.

*Proof.* Let  $A'$  be the image of  $A$  in  $\text{End}_{\mathbb{k}}(M)$  and let  $(\mathbb{K} \otimes_{\mathbb{k}} A)'$  be the image of  $\mathbb{K} \otimes_{\mathbb{k}} A$  in  $\text{End}_{\mathbb{K}}(\mathbb{K} \otimes_{\mathbb{k}} M)$ . Under the identification  $\text{End}_{\mathbb{K}}(\mathbb{K} \otimes_{\mathbb{k}} M) \cong \mathbb{K} \otimes \text{End}_{\mathbb{k}}(M)$  we have

$$(\mathbb{K} \otimes_{\mathbb{k}} A)' \cong \mathbb{K} \otimes_{\mathbb{k}} A'.$$

That  $\mathbb{K} \otimes_{\mathbb{k}} M$  is semisimple as an  $(\mathbb{K} \otimes_{\mathbb{k}} A)$ -module is equivalent to it being semisimple as an  $(\mathbb{K} \otimes_{\mathbb{k}} A)'$ -module. It follows from Lemma A.16 that  $(\mathbb{K} \otimes_{\mathbb{k}} A)'$  is semisimple. Thus  $A'$  is semisimple by Corollary A.21 and the above isomorphism. It follows that  $M$  is semisimple as an  $A'$ -module and thus an  $A$ -module.  $\square$

**Theorem A.22.** Let  $\mathbb{k}$  be a field of characteristic zero.

1. Every finite-dimensional  $\mathfrak{sl}_2$ -representation is semisimple.
2. Every finite-dimensional  $\mathfrak{sl}_2$ -representation decomposes into weight spaces, and all occurring weights are integral.
3. The irreducible finite-dimensional representations of  $\mathfrak{sl}_2$  are given by  $L(n)$  for  $n \in \mathbb{N}$ .

*Proof.* Let  $\mathbb{K}$  be an algebraic closure of  $\mathbb{k}$ .

1. We have previously seen that the assertion holds for  $\mathfrak{sl}_2(\mathbb{K})$ . We have

$$\mathbb{K} \otimes U(\mathfrak{sl}_2(\mathbb{k})) \cong U(\mathfrak{sl}_2(\mathbb{K}))$$

whence the assertion follows for  $\mathfrak{sl}_2(\mathbb{k})$  from Corollary A.21.

2. The assertion holds for  $\mathfrak{sl}_2(\mathbb{K})$ , as previously seen. Let  $M$  be a finite-dimensional representation of  $\mathfrak{sl}_2(\mathbb{k})$ . Then  $\mathbb{K} \otimes_{\mathbb{k}} M$  is a finite-dimensional representation of  $\mathfrak{sl}_2(\mathbb{K})$  and it follows that  $\mathbb{K} \otimes_{\mathbb{k}} M$  decomposes into weight spaces as described. This means that  $\mathbb{K} \otimes_{\mathbb{k}} M$  is annihilated by the element

$$x := \prod_{j=-n}^n (H - j)$$

of  $U(\mathfrak{sl}_2(\mathbb{K}))$  for some sufficiently large  $n \geq 0$ . It follows that  $x$ , regarded as an element of  $U(\mathfrak{sl}_2(\mathbb{k}))$ , annihilates the original representation  $M$ . The assertion for  $\mathfrak{sl}_2(\mathbb{k})$  follows from this.

3. It follows from the previous assertion that every finite-dimensional irreducible  $\mathfrak{sl}_2(\mathbb{k})$ -representation is a highest-weight representations. The classification of irreducible, finite-dimensional highest-weight representations of  $\mathfrak{sl}_2$  works the same over every field of characteristic zero. From this we then get the assertion for  $\mathfrak{sl}_2(\mathbb{k})$ .  $\square$



### A.2. An Alternative Presentation for $U_q(\mathfrak{sl}_2)$

Let  $q \in \mathbb{k}$  and let  $U_q$  be the algebra given by the generators

$$E, \quad K, \quad K^{-1}, \quad \tilde{H}, \quad F$$

and the relations

$$\begin{aligned} KK^{-1} &= 1 = K^{-1}K, \quad KE = q^2EK, \quad KF = q^{-2}FK, \\ [E, F] &= \tilde{H}, \quad (q - q^{-1})\tilde{H} = K - K^{-1}, \\ [\tilde{H}, E] &= q(EK + K^{-1}E), \quad [\tilde{H}, F] = -q^{-1}(FK + K^{-1}F). \end{aligned}$$

**Proposition A.23.** There exists a unique homomorphism of algebras

$$\psi : U_q \rightarrow U_q(\mathfrak{sl}_2)$$

that is given by

$$\psi(E) = E, \quad \psi(K) = K, \quad \psi(\tilde{H}) = \frac{K - K^{-1}}{q - q^{-1}}, \quad \psi(F) = F,$$

and this homomorphism is an isomorphism.

*Proof.* See [Kas95, Proposition VI.2.1]. □

**Proposition A.24.** For  $q = 1$  there exists a unique homomorphism of algebras

$$\varphi : U_1 \rightarrow U(\mathfrak{sl}_2)[\sigma]/(\sigma^2 - 1)$$

that is given by

$$\varphi(E) = \sigma E, \quad \varphi(K) = \sigma, \quad \varphi(\tilde{H}) = \sigma H, \quad \varphi(F) = F,$$

*Proof.* See [Kas95, Proof of Proposition VI.2.2]. □

**Remark A.25.** There also exist other, more exotic presentations of  $U_q(\mathfrak{sl}_2)$ . We refer to [ITW05] for a specific example.

### A.3. Representation Theory of $U_1(\mathfrak{sl}_2)$

Let  $A$  denote the algebra  $U(\mathfrak{sl}_2)[\sigma]/(\sigma^2 - 1)$ .

Let  $M$  be an  $\mathfrak{sl}_2$ -representation and let  $\varepsilon \in \{1, -1\}$ . The corresponding  $U(\mathfrak{sl}_2)$ -module structure on  $M$  extends to an  $U(\mathfrak{sl}_2)[\sigma]$ -module structure for which  $\sigma$  acts by multiplication with  $\varepsilon$ , because  $\sigma$  is central in  $U(\mathfrak{sl}_2)[\sigma]$ . It follows from the identity  $\varepsilon^2 = 1$  that this  $U(\mathfrak{sl}_2)[\sigma]$ -module structure descends to an  $A$ -module structure on  $M$  as claimed in Remark 2.4.

If  $M$  is irreducible then the resulting  $A$ -module is again irreducible since every  $A$ -submodule is in particular an  $\mathfrak{sl}_2$ -subrepresentation. It hence follows that the  $A$ -modules  $L(+, n)$  and  $L(-, n)$

that result from the irreducible  $\mathfrak{sl}_2$ -representation  $L(n)$  are again irreducible. These representations are pairwise non-isomorphic since the element  $H\sigma$  of  $A$  (which corresponds to the element  $\tilde{H}$  of  $U_1(\mathfrak{sl}_2)$ ) acts on  $L(+, n)$  and  $L(-, n)$  with different eigenvalues.

Let now  $M$  be any finite-dimensional  $M$ -module. It follows from the relation  $\sigma^2 = 1$  in  $A$  that the action of  $\sigma$  on  $A$  is diagonalizable with eigenvalues 1 and  $-1$ . We thus have

$$M = M_1 \oplus M_{-1}$$

with  $M_\varepsilon := \{m \in M \mid \sigma m = \varepsilon m\}$  for  $\varepsilon \in \{1, -1\}$ . The action of  $\sigma$  on  $M$  is an  $A$ -module homomorphism because  $\sigma$  is central in  $A$ . The decomposition  $M = M_1 \oplus M_{-1}$  is therefore a decomposition of  $M$  into  $A$ -submodules.

We may regard both  $M_1$  and  $M_{-1}$  as  $\mathfrak{sl}_2$ -representations by restriction. We then have decompositions of  $M_1, M_{-1}$  into finite-dimensional irreducible  $\mathfrak{sl}_2$ -representations given by

$$M_1 \cong L(n_1) \oplus \cdots \oplus L(n_s), \quad M_{-1} \cong L(n'_1) \oplus \cdots \oplus L(n'_t).$$

We note that this is already a decomposition as  $A$ -modules since  $\sigma$  acts on  $M_1$  and  $M_{-1}$  by multiplication with scalars. As  $A$ -modules we have

$$L(n_i) = L(+, n_i), \quad L(n'_i) = L(-, n'_i).$$

This shows that every finite-dimensional  $A$ -module decomposes into a direct sum of the irreducible  $A$ -modules  $L(\varepsilon, n)$ .

#### A.4. PBW-Basis for $U_q(\mathfrak{sl}_2)$

We use in the following the notations introduced in Definition 3.8.

**Lemma A.26.** For every  $r \geq 0$  we have

$$[E, F^r] = [r]_q F^{r-1} [K, 1 - r]_q.$$

*Proof.* For  $r = 0$  both sides vanish and for  $r = 1$  this is one of the defining relations of  $U_q(\mathfrak{sl}_2)$ . For  $r \geq 2$  the assertion follows by induction, see [Jan96, Appendix 1.3 (5)].  $\square$

**Corollary A.27.** We have

$$\begin{aligned} F \cdot F^l K^m E^n &= F^{l+1} K^m E^n, \\ K^{\pm 1} \cdot F^l K^m E^n &= q^{\mp 2l} F^l K^{m \pm 1} E^n, \\ E \cdot F^l K^m E^n &= q^{-2m} F^l K^m E^{n+1} + \frac{[l]_q}{q - q^{-1}} (q^{1-l} F^{l-1} K^{m+1-l} E^n - q^{l-1} F^{l-1} K^{m+l-1} E^n). \end{aligned}$$

*Proof.* This follows from Lemma A.26 and the two relations  $KE = q^2 EK$  and  $KF = q^{-2} FK$ .  $\square$

*Proof of Theorem 2.6.* Let  $U$  be the linear subspace of  $U_q(\mathfrak{sl}_2)$  spanned by these given monomials. It follows from Corollary A.27 that  $U$  is a left ideal in  $U_q(\mathfrak{sl}_2)$ . It contains the elements  $F^0 K^0 E^0 = 1$ , whence  $U$  equals  $U_q(\mathfrak{sl}_2)$ . This shows that the given monomials are a vector space generating set of  $U_q(\mathfrak{sl}_2)$ .

The linear independence of the monomials  $X^l Y^m Z^n$  is shown in the usual representation-theoretic way. Let  $V$  be the free vector space with basis

$$X^l Y^m Z^n \quad \text{with } l, n \in \mathbb{N} \text{ and } m \in \mathbb{Z}.$$

There exists an action of  $U_q(\mathfrak{sl}_2)$  on  $V$  by using the formulas from Corollary A.27, with  $F^l K^m E^n$  replaced by  $X^l Y^m Z^n$ . (It has to be checked that this proposed action is compatible with the defining relations of  $U_q(\mathfrak{sl}_2)$ , see [Jan96, Appendix 1.5].) The elements

$$F^l K^m E^n \cdot X^0 Y^0 Z^0 = X^l Y^m Z^n$$

are linearly independent in  $V$ , whence the given monomials  $F^l K^m E^n$  are linearly independent in  $U_q(\mathfrak{sl}_2)$ .  $\square$

## A.5. More on the Algebra Structure of $U_q(\mathfrak{sl}_2)$

**Remark A.28.**

1. The universal enveloping algebra  $U(\mathfrak{sl}_2)$  is noetherian and has no nonzero zero divisors. The same holds for  $U_q(\mathfrak{sl}_2)$ , see [Kas95, Proposition VI.1.4] and [Jan96, Proposition 1.8].
2. The algebra  $U_q(\mathfrak{sl}_2)$  admits a grading such that  $E, K, F$  are homogeneous with

$$\deg(E) = 1, \quad \deg(F) = -1, \quad \deg(K) = 0.$$

The degree  $d$  part of  $U_q(\mathfrak{sl}_2)$  has the basis

$$F^l K^m E^n \quad \text{with } n - l = d.$$

This grading can also be characterized in terms of the conjugation map

$$U_q(\mathfrak{sl}_2) \rightarrow U_q(\mathfrak{sl}_2), \quad x \mapsto KxK^{-1}.$$

The degree  $d$  part of the grading is precisely the eigenspace of this conjugation map for the eigenvalue  $q^{2d}$ .

**Proposition A.29.**

1. There exists a unique algebra involution  $\omega$  of  $U_q(\mathfrak{sl}_2)$  with

$$\omega(E) = F, \quad \omega(K) = K^{-1}, \quad \omega(F) = E.$$

2. There exists a unique algebra anti-involution  $\tau$  of  $U_q(\mathfrak{sl}_2)$  with

$$\tau(E) = E, \quad \tau(K) = K^{-1}, \quad \tau(F) = F.$$

3. There exists a unique algebra isomorphism  $\varphi_q : U_q(\mathfrak{sl}_2) \rightarrow U_{q^{-1}}(\mathfrak{sl}_2)$  with

$$\varphi(E) = -F, \quad \varphi(K) = K^{-1}, \quad \varphi(F) = -E.$$

The inverse of the isomorphism  $\varphi_q$  is given by  $\varphi_{q^{-1}}$ .

4. There exist unique algebra involutions  $\sigma_E$  and  $\sigma_F$  of  $U_q(\mathfrak{sl}_2)$  with

$$\sigma_E(E) = -E, \quad \sigma_E(K) = -K, \quad \sigma_E(F) = F.$$

and

$$\sigma_F(E) = E, \quad \sigma_F(K) = -K, \quad \sigma_F(F) = -F.$$

*Proof.* One checks that the proposed images of  $E, K, K^{-1}, F$  are compatible with the defining relations of  $U_q(\mathfrak{sl}_2)$ . See also [Jan96, Lemma 1.2].  $\square$

**Remark A.30.**

1. One can combine the above (anti-)isomorphisms to construct further (anti-)isomorphisms involving  $U_q(\mathfrak{sl}_2)$  and  $U_{q^{-1}}(\mathfrak{sl}_2)$ .
2. It follows from the existence of these (anti-)isomorphisms that many formulas and propositions involving  $U_q(\mathfrak{sl}_2)$  have to satisfy certain symmetries.

## A.6. Proof of Theorem 3.3

**Lemma A.31.** Let  $M$  be a finite-dimensional  $U_q(\mathfrak{sl}_2)$ -module.

1. Both  $E$  and  $F$  act nilpotently on  $M$ .
2. For a sufficiently large power  $r \geq 0$  (namely such that  $F^r M = 0$ ) the module  $M$  is annihilated by the element

$$\prod_{j=-r}^r (K^2 - q^{2j}).$$

*Proof.* See [Jan96, Proposition 2.1] and [Jan96, Proposition 2.3].  $\square$

**Proposition A.32.** Every finite-dimensional  $U_q(\mathfrak{sl}_2)$ -module decomposes into weight spaces. All occurring weights are of the form  $\pm q^n$  for some  $n \in \mathbb{Z}$ .

*Proof.* Let  $M$  be a finite-dimensional  $U_q(\mathfrak{sl}_2)$ -module and let  $k$  denote the action of  $K$  on  $M$ . It follows from Lemma A.31 that

$$0 = \prod_{n=-r}^r (k^2 - q^{2n}) = \prod_{n=-r}^r (k - q^n)(k + q^n).$$

The roots  $\pm q^n$  for  $n = -r, \dots, r$  are pairwise distinct<sup>2</sup> whence it follows that  $k$  is diagonalizable with possible eigenvalues  $\pm q^n$  with  $n = -r, \dots, r$ .  $\square$

<sup>2</sup>If  $\pm q^n = \pm q^m$  then squaring both sides of this equation gives  $q^{2n} = q^{2m}$  and thus  $q^{2(n-m)} = 1$ . It follows that  $2(n-m) = 0$  because  $q$  is not a root of unity, and thus  $n = m$ .

### A.7. Proof of Proposition 3.10

**Proposition A.33.**

1. The algebra  $U_q(\mathfrak{b})$  has the basis

$$K^n E^m \quad \text{with } n \in \mathbb{Z} \text{ and } m \in \mathbb{N}.$$

2. The algebra  $U_q(\mathfrak{b})$  is given with respect to its generators  $E, K, K^{-1}$  by the relations

$$KK^{-1} = 1 = K^{-1}K, \quad KE = q^2 EK.$$

*Proof.*

1. Let  $U$  be the linear subspace of  $U_q(\mathfrak{sl}_2)$  spanned by the monomials  $K^n E^m$  with  $n, m \in \mathbb{N}$ . This linear subspace is contained in  $U_q(\mathfrak{b})$ . It follows on the other hand from the relation  $KE = q^2 EK$  that

$$K^n E^m \cdot K^{n'} E^{m'} = q^{2mn'} K^{n+n'} E^{m+m'}$$

for all  $n, n', m, m' \in \mathbb{N}$ , and we have  $1 = K^0 E^0 \in U$ . This shows that  $U$  is a subalgebra of  $U_q(\mathfrak{sl}_2)$  containing  $E, K, K^{-1}$ , and therefore containing  $U_q(\mathfrak{b})$ . Together this shows that  $U = U_q(\mathfrak{b})$ .

2. Let  $U$  be the algebra given by generators  $E, K, K^{-1}$  and relations

$$KK^{-1} = 1 = K^{-1}K, \quad KE = q^2 EK.$$

There exists a unique algebra homomorphism  $\varphi : U \rightarrow U_q(\mathfrak{b})$  given by

$$\varphi(E) = E, \quad \varphi(K) = K.$$

In the same way as for Theorem 2.6 one can show that the algebra  $U$  has a PBW-basis given by the monomials

$$K^n E^m \quad \text{with } n \in \mathbb{Z} \text{ and } m \in \mathbb{N}.$$

It follows that the algebra homomorphism  $\varphi$  restricts to a bijection between the PBW-bases of  $U$  and  $U_q(\mathfrak{b})$  and is therefore an algebra isomorphism.  $\square$

We now show an extended version of Proposition 3.10.

**Proposition A.34.** Let  $\lambda \in \mathbb{k}^\times$ .

1. We have  $\mathbb{k}_\lambda \cong U_q(\mathfrak{b})/\langle E, K - \lambda \rangle$  as  $U_q(\mathfrak{b})$ -modules. Such an isomorphism is given by

$$U_q(\mathfrak{b})/\langle E, K - \lambda \rangle \rightarrow \mathbb{k}_\lambda, \quad \bar{x} \mapsto x.1.$$

2. The Verma module  $M(\lambda)$  has the basis

$$m_i := F^i \otimes 1 \quad \text{with } i \geq 0,$$

and the actions of  $E, K, F$  on this basis are given by

$$Fm_i = m_{i+1}, \quad Km_i = \lambda q^{-2i} m_i, \quad Em_i = [i]_q [\lambda, 1 - i]_q m_{i-1}.$$

This action can be graphically depicted as in Figure 3.

3. The Verma module  $M(\lambda)$  is of highest weight  $\lambda$ .
4. There exists for every  $U_q(\mathfrak{sl}_2)$ -module  $M$  an isomorphism of vector spaces given by

$$\text{Hom}_{U_q(\mathfrak{sl}_2)}(M(\lambda), M) \cong \{m \in M \mid m \text{ is of weight } \lambda \text{ with } Em = 0\}.$$

It follows in particular that

$$\text{End}_{U_q(\mathfrak{sl}_2)}(M(\lambda)) = \mathbb{k}.$$

5. The Verma module  $M(\lambda)$  is indecomposable.
6. a. If  $\lambda = \pm q^n$  for some  $n \in \mathbb{N}$  then the Verma module  $M(\lambda)$  contains a unique nonzero, proper submodule, which is spanned by the elements

$$m_i \quad \text{with } i \geq n + 1.$$

This submodule is isomorphic to  $M(\pm q^{-n-2})$ .

- b. If  $\lambda \neq \pm q^n$  for every  $n \in \mathbb{N}$  then the Verma module  $M(\lambda)$  is irreducible.

1. This follows from the PBW-basis of  $U_q(\mathfrak{b})$ .
2. This follows from the PBW-bases of  $U_q(\mathfrak{sl}_2)$  and  $U_q(\mathfrak{b})$  and induction.
3. The Verma module  $M(\lambda)$  is generated by the primitive weight vector  $1 \otimes 1$ .
4. We have

$$\begin{aligned} \text{Hom}_{U_q(\mathfrak{sl}_2)}(M(\lambda), M) &= \text{Hom}_{U_q(\mathfrak{sl}_2)}(U_q(\mathfrak{sl}_2) \otimes_{U_q(\mathfrak{b})} \mathbb{k}_\lambda, M) \\ &\cong \text{Hom}_{U_q(\mathfrak{b})}(\mathbb{k}_\lambda, M) \\ &\cong \text{Hom}_{U_q(\mathfrak{b})}(U_q(\mathfrak{b}) / \langle K - \lambda, E \rangle, M) \\ &\cong \{m \in M \mid (K - \lambda)m = 0, Em = 0\}. \end{aligned}$$

5. The endomorphism algebra  $\text{End}_{U_q(\mathfrak{sl}_2)}(M(\lambda)) = \mathbb{k}$  does not contain any non-trivial idempotents.
6. This follows as for  $U(\mathfrak{sl}_2)$  since  $[i]_q[\lambda, i - 1]_q = 0$  if and only if  $\lambda = \pm q^{i-1}$ .

## A.8. Proof of Theorem 3.14

**Lemma A.35.** If  $M$  is an highest-weight  $U_q(\mathfrak{sl}_2)$ -module then

$$\text{End}_{U_q(\mathfrak{sl}_2)}(M) = \mathbb{k}.$$

**Definition A.36.** The *quantum Casimir element* is the element  $C_q \in U_q(\mathfrak{sl}_2)$  given by

$$C_q := EF + \frac{Kq^{-1} + K^{-1}q}{(q - q^{-1})^2}.$$

**Lemma A.37.**

1. The element  $C_q$  is central in  $U_q(\mathfrak{sl}_2)$ .

2. The element  $C_q$  acts on every  $U_q(\mathfrak{sl}_2)$ -module by module endomorphisms.
3. The element  $C_q$  acts for every scalar  $\lambda \in \mathbb{k}^\times$  on the representation  $L(\lambda)$  by multiplication with the scalar

$$\frac{\lambda q + \lambda^{-1} q^{-1}}{(q - q^{-1})^2}.$$

4. The element  $C_q$  acts the same on  $L(\lambda)$  and  $L(\mu)$  if and only if  $\lambda = \mu$  or  $\lambda = \mu^{-1} q^{-2}$ .

*Proof.*

1. It suffices to check that the element  $C_q$  commutes with the elements  $E, F, K$ . This follows from the defining relations for  $U_q(\mathfrak{sl}_2)$ .
2. This follows from the previous assertion.
3. It follows from the previous assertion and Lemma A.35 that  $C_q$  acts by a scalar. This scalar can be read off from the action on the primitive generator  $\overline{1 \otimes 1}$ . It thus suffices to show the assertion for  $M(\lambda)$ , where it follows from Proposition 3.10.
4. This follows from the previous assertion. □

**Corollary A.38.** The quantum Casimir element  $C_q$  acts on every finite-dimensional, irreducible representation of  $U_q(\mathfrak{sl}_2)$  by a different scalar.

*Proof.* If  $\lambda = \delta q^n$  and  $\mu = \varepsilon q^m$  with  $\delta, \varepsilon \in \{1, -1\}$  and  $n, m \in \mathbb{N}$  then it cannot happen that  $\lambda = \mu^{-1} q^{-2}$ . The assertion thus follows from Lemma A.37. □

*Proof of Theorem 3.14 ([Jan96, Theorem 2.9]).* Let  $M$  be any finite-dimensional  $U_q(\mathfrak{sl}_2)$ -module and let  $c$  denote the action of  $C_q$  on  $M$ . We may assume that  $M$  is indecomposable. We consider a composition series

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \dots \subsetneq M_r = M \quad (4)$$

with composition factors

$$M_i / M_{i-1} \cong L(\varepsilon_i q^{n_i}).$$

Letting  $c_i$  be the scalar by which  $C_q$  acts on  $L(\varepsilon_i q^{n_i})$ , we have

$$(c - c_i)M_i \subseteq M_{i-1}.$$

It follows that  $\prod_{i=1}^r (c - c_i)$  annihilates  $M$ . Therefore  $c$  admits a generalized eigenspace decomposition with eigenvalues  $c_1, \dots, c_r$ . The resulting generalized eigenspaces are subrepresentations because  $c$  is a  $U_q(\mathfrak{sl}_2)$ -module endomorphism. It follows that

$$c_1 = \dots = c_r$$

because  $M$  is indecomposable, and thus

$$\varepsilon_1 q^{n_1} = \dots = \varepsilon_r q^{n_r} =: \lambda$$

by Corollary A.38. It therefore follows from the composition series (4) that

$$\dim(M_\mu) = r \dim(L(\lambda)_\mu)$$

for every scalar  $\mu \in \mathbb{k}^\times$ . Thus  $M$  is of highest weight  $\lambda$ .

The short exact sequence of  $U_q(\mathfrak{sl}_2)$ -modules

$$0 \rightarrow M_{r-1} \rightarrow M \rightarrow L(\lambda) \rightarrow 0 \quad (5)$$

restricts to a short exact sequence of vector spaces

$$0 \rightarrow (M_{r-1})_\lambda \rightarrow M_\lambda \rightarrow L(\lambda)_\lambda \rightarrow 0.$$

It follows that the primitive generator  $\overline{1 \otimes 1}$  of  $L(\lambda)$  has a preimage  $m_0$  in  $M$ . The weight vector  $m_0$  is primitive because  $M$  is of highest weight  $\lambda$ . It follows that there exists a homomorphism of  $U_q(\mathfrak{sl}_2)$ -modules

$$\varphi : M(\lambda) \rightarrow M, \quad 1 \otimes 1 \mapsto m_0.$$

It further follows from the finite-dimensionality of  $M$  that  $\varphi$  factors through a homomorphism

$$\psi : L(\lambda) \rightarrow M, \quad \overline{1 \otimes 1} \mapsto m_0.$$

This shows that the short exact sequence (5) splits, whence

$$M \cong M_{r-1} \oplus L(\lambda).$$

It follows by induction that  $M_{r-1} \cong L(\lambda)^{\oplus(r-1)}$  and thus altogether  $M \cong L(\lambda)^{\oplus r}$ .  $\square$

**Remark A.39.** The center of the universal enveloping algebra  $U(\mathfrak{sl}_2)$  is a polynomial algebra, generated by the classical Casimir element  $C = (ef + h^2 + fe)/4$ . It can be shown that the center of  $U_q(\mathfrak{sl}_2)$  is again a polynomial algebra, now generated by the quantum Casimir element  $C_q$ . We refer to [Jan96, Proposition 2.18] for more details on this.

#### A.9. Proof of Lemma 4.4

We have

$$M_\mu \otimes N_\kappa \subseteq (M \otimes N)_{\mu\kappa}$$

for all  $\mu, \kappa \in \mathbb{k}^\times$  since the element  $K$  is group-like in  $U_q(\mathfrak{sl}_2)$ . Both  $M$  and  $N$  admits weight space decompositions

$$M = \bigoplus_{\mu} M_{\mu}, \quad N = \bigoplus_{\kappa} N_{\kappa}$$

and it follows that

$$M \otimes N = \left( \bigoplus_{\mu} M_{\mu} \right) \otimes \left( \bigoplus_{\kappa} N_{\kappa} \right) = \bigoplus_{\mu, \kappa} (M_{\mu} \otimes N_{\kappa}) \subseteq \bigoplus_{\lambda} M_{\lambda} \subseteq M \otimes N$$

It follows with the inclusions  $M_{\mu} \otimes N_{\kappa} \subseteq (M \otimes N)_{\mu\kappa}$  that for any  $\lambda \in \mathbb{k}^\times$  already

$$(M \otimes N)_{\lambda} = \bigoplus_{\mu\kappa=\lambda} M_{\mu} \otimes N_{\kappa}.$$



### A.10. Clebsch–Gordan for $U_q(\mathfrak{sl}_2)$

**Proposition A.40.** For all  $\delta, \varepsilon \in \{1, -1\}$  and  $n, m \in \mathbb{N}$  with  $n \geq m$  we have

$$L(\delta q^n) \otimes L(\varepsilon q^m) \cong L(\delta \varepsilon q^{n+m}) \oplus L(\delta \varepsilon q^{n+m-2}) \oplus \dots \oplus L(\delta \varepsilon q^{n-m}).$$

*Proof.* This follows from Corollary 3.15 and Lemma 4.4. □

## B. Deformation Theory

### B.1. Deformations of Algebras

We will in the following introduce a formal deformation  $U_{\hbar}(\mathfrak{sl}_2)$  of the Hopf algebra  $U(\mathfrak{sl}_2)$  and gain a new way of understanding  $U_q(\mathfrak{sl}_2)$ .

### B.2. Background on Completions

**Definition B.1.** An  $\mathbb{k}[[\hbar]]$ -module  $M$  is *complete* if for every sequence  $m_0, m_1, \dots$  of elements  $m_n$  of  $M$  with

$$m_{n+1} \equiv m_n \pmod{\hbar^{n+1}} \quad \text{for every } n \geq 0$$

there exists a unique element  $m$  of  $M$  with

$$m \equiv m_n \pmod{\hbar^{n+1}} \quad \text{for every } n \geq 0.$$

**Definition B.2.** Let  $M$  be an  $\mathbb{k}[[\hbar]]$ -module.

1. The  $\hbar$ -adic completion of  $M$  is the  $\mathbb{k}[[\hbar]]$ -module

$$\widehat{M} := \lim_{n \geq 0} (M / \hbar^{n+1} M) = \left\{ (m_n)_{n \geq 0} \left| \begin{array}{l} m_n \in M / \hbar^{n+1} M \text{ with} \\ m_{n+1} \equiv m_n \pmod{\hbar^{n+1}} \text{ for every } n \geq 0 \end{array} \right. \right\}.$$

2. The canonical homomorphism  $M \rightarrow \widehat{M}$  is given by  $m \mapsto (\overline{m}, \overline{m}, \dots)$ .

**Proposition B.3.** Let  $M$  be an  $\mathbb{k}[[\hbar]]$ -module.

1. The completion  $\widehat{M}$  is complete.
2. The module  $M$  is complete if and only if the canonical homomorphism  $M \rightarrow \widehat{M}$  is an isomorphism.

**Remark B.4.** Let  $M$  be a complete  $\mathbb{k}[[\hbar]]$ -module. Every sequence  $(m_i)_{i \geq 0}$  of elements  $m_i \in M$  defines a sequence  $(s_n)_{n \geq 0}$  of partial sums

$$s_n := \sum_{i=0}^n \hbar^i m_i.$$

for every  $n \geq 0$ . These partial sums satisfy the relation

$$s_{n+1} \equiv s_n \pmod{\hbar^{n+1}} \quad \text{for every } n \geq 0.$$

There hence exists by the completeness of  $M$  a unique element  $\sum_{i=0}^{\infty} \hbar^i m_i$  of  $M$  with

$$\sum_{i=0}^{\infty} \hbar^i m_i \equiv \sum_{i=0}^n \hbar^i m_i \pmod{\hbar^{n+1}} \quad \text{for every } n \geq 0.$$

**Example B.5.**

1. Every finite-dimensional  $\mathbb{k}[[\hbar]]$ -module  $M$  is complete since  $\hbar^{n+1}M = 0$  for some sufficiently large power  $n$ .<sup>3</sup>
2. For every  $\mathbb{k}$ -vector space  $V$  the resulting  $\mathbb{k}[[\hbar]]$ -module  $V[[\hbar]]$  is complete. For every sequence  $v_0, v_1, \dots$  of elements  $v_i \in V$  we have

$$\sum_{i=0}^{\infty} \hbar^i v_i = \sum_{i=0}^{\infty} v_i \hbar^i.$$

**Proposition B.6.** Let  $M, N$  be two  $\mathbb{k}[[\hbar]]$ -modules.

1. For every homomorphism of  $\mathbb{k}[[\hbar]]$ -modules  $f : M \rightarrow N$  there exists a unique module homomorphism  $\hat{f} : \hat{M} \rightarrow \hat{N}$  that makes the following square diagram commute.

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\hat{f}} & \hat{N} \\ \uparrow & & \uparrow \\ M & \xrightarrow{f} & N \end{array}$$

The homomorphism  $\hat{f}$  is given by

$$\hat{f}(\overline{(m_0, m_1, \dots)}) = \overline{(f(m_0), f(m_1), \dots)}.$$

2. The assignment  $(-)^{\widehat{\phantom{x}}}$  defines a functor

$$(-)^{\widehat{\phantom{x}}} : \mathbb{k}[[\hbar]]\text{-}\mathbf{Mod} \rightarrow \mathbb{k}[[\hbar]]\text{-}\mathbf{Mod}.$$

3. If  $M, N$  are complete then

$$f\left(\sum_{i=0}^{\infty} \hbar^i m_i\right) = \sum_{i=0}^{\infty} \hbar^i f(m_i)$$

for every sequence  $m_0, m_1, \dots$  of elements  $m_n \in M$ .

4. If  $N$  is complete then every homomorphism  $M \rightarrow N$  extends uniquely to a homomorphism  $\hat{M} \rightarrow N$ .

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\exists!} & N \\ \uparrow & \nearrow & \\ M & & \end{array}$$

---

<sup>3</sup>Indeed, it follows from the finite-dimensionality of  $M$  that the annihilator of  $M$  is a nonzero ideal in  $\mathbb{k}[[\hbar]]$ . It is of the form  $(\hbar^n)$  for some  $n \geq 0$ , because every ideal in  $\mathbb{k}[[\hbar]]$  is of the form  $(\hbar^n)$ . Thus  $\hbar^n M = 0$ .

5. If  $V$  is any  $\mathbb{k}$ -vector space and  $N$  is complete then every  $\mathbb{k}$ -linear map  $f : V \rightarrow N$  extends uniquely to a  $\mathbb{k}[[\hbar]]$ -linear linear map  $V[[\hbar]] \rightarrow N$ .

$$\begin{array}{ccc} V[[\hbar]] & \xrightarrow{\exists!} & N \\ \uparrow & \nearrow & \\ V & & \end{array}$$

The homomorphism  $f'$  is given by

$$f' \left( \sum_{i=0}^{\infty} \hbar^i v_i \right) = \sum_{i=0}^{\infty} \hbar^i f(v_i).$$

6. The canonical homomorphism  $M \rightarrow \widehat{M}$  induces an isomorphism of  $\mathbb{k}$ -vector spaces

$$M/\hbar M \longrightarrow \widehat{M}/\hbar \widehat{M}.$$

**Remark B.7** (The  $\hbar$ -adic topology). Let  $M$  be a  $\mathbb{k}[[\hbar]]$ -module.

1. There exists a unique topology on  $M$  for which a basis is given by the sets

$$m + \hbar^{n+1}M \quad \text{with } m \in M \text{ and } n \geq 0.$$

This topology is the  $\hbar$ -adic topology on  $M$ . It makes  $\mathbb{k}[[\hbar]]$  into a topological ring and every  $\mathbb{k}[[\hbar]]$ -module into a topological  $\mathbb{k}[[\hbar]]$ -module.

2. The  $\hbar$ -adic topology on  $M$  is generated by a semimetric  $d_{\hbar}$  that is given by

$$d_{\hbar}(m, m') := \inf \left\{ \frac{1}{n} \mid n \geq 1, m \equiv m' \pmod{\hbar^{n+1}} \right\}.$$

The semimetric  $d_{\hbar}$  is already an ultra-semimetric. More explicitly, it satisfies for any three elements  $m, m', m'' \in M$  the inequality

$$d_{\hbar}(m, m'') \leq \max(d_{\hbar}(m, m'), d_{\hbar}(m', m'')).$$

This is a strengthening of the usual triangle inequality.

3. The module  $M$  is complete if and only if it is complete with respect to the semimetric  $d_{\hbar}$ , i.e. if and only if for every Cauchy-sequence in  $M$  has a unique limit. This entails that the semimetric  $d_{\hbar}$  is actually a metric (by the uniqueness of these limits).

The completion  $\widehat{M}$  together with the canonical homomorphism  $M \rightarrow \widehat{M}$  is a completion of  $M$  in the usual topological sense.

4. The  $\hbar$ -adic topology on  $M$  is Hausdorff if and only if

$$\bigcap_{n \geq 0} \hbar^{n+1}M = \{0\}.$$

This is in particular the case if  $M$  is complete. This can be seen topologically, since  $d_{\hbar}$  is then a metric, and algebraically, by considering the explicit description of the completion  $\widehat{M} \cong M$ .

**Definition B.8.** Let  $M, N$  be two  $\mathbb{k}[[\hbar]]$ -modules. The *completed tensor product*

$$M \widehat{\otimes} N$$

is the  $\hbar$ -adic completion of the tensor product  $M \otimes_{\mathbb{k}[[\hbar]]} N$ .

**Proposition B.9.** Let  $V, W$  be two  $\mathbb{k}$ -vector spaces. Then the  $\mathbb{k}[[\hbar]]$ -linear map

$$V[[\hbar]] \otimes_{\mathbb{k}[[\hbar]]} W[[\hbar]] \rightarrow (V \otimes W)[[\hbar]], \quad \left( \sum_{i=0}^{\infty} v_i \hbar^i \right) \otimes \left( \sum_{j=0}^{\infty} w_j \hbar^j \right) \mapsto \sum_{i,j=0}^{\infty} (v_i \otimes w_j) \hbar^{i+j}$$

extends along the canonical homomorphism

$$V \otimes W \rightarrow V \widehat{\otimes} W$$

to an isomorphism of  $\mathbb{k}[[\hbar]]$ -modules

$$V[[\hbar]] \widehat{\otimes} W[[\hbar]] \rightarrow (V \otimes W)[[\hbar]].$$

### B.3. Deformation of Algebras

The following is taken (at least in spirit) from [Bel18, §5.2] and [GS92].

**Motivation B.10.** Deforming a  $\mathbb{k}$ -algebra  $A$  means – roughly speaking – that the multiplication on  $A$  is replaced by a perturbed multiplication  $*$ , in the sense that for all  $a, b \in A$ ,

$$a * b = ab + \mu_1(a, b)\hbar + \mu_2(a, b)\hbar^2 + \dots$$

for some bilinear terms  $\mu_i(a, b)$ . The limit  $\hbar \rightarrow 0$  does then give back the original algebra  $A$ .

**Definition B.11.** Let  $A$  be an  $\mathbb{k}$ -algebra.

1. A *(formal) deformation* of  $A$  is an  $\mathbb{k}[[\hbar]]$ -algebra  $A_{\hbar}$  whose underlying  $\mathbb{k}[[\hbar]]$ -module is  $A[[\hbar]]$  and for which  $A_{\hbar}/\hbar A_{\hbar} = A$  as algebras.
2. Two deformations  $A_{\hbar}, A'_{\hbar}$  of  $A$  are *equivalent* if there exists an isomorphism of  $\mathbb{k}[[\hbar]]$ -algebras

$$\varphi : A_{\hbar} \rightarrow A'_{\hbar}$$

such that the induced isomorphism of  $\mathbb{k}$ -algebras

$$A = A_{\hbar}/\hbar A_{\hbar} \rightarrow A'_{\hbar}/\hbar A'_{\hbar} = A$$

is the identity of  $A$ . In other words, the isomorphism  $\varphi$  is the identity of  $A$  modulo  $\hbar$ .

**Remark B.12.** Every  $\mathbb{k}[[\hbar]]$ -bilinear multiplication

$$(-) * (-) : A[[\hbar]] \times A[[\hbar]] \rightarrow A[[\hbar]].$$

satisfies the equality

$$\left( \sum_{i=0}^{\infty} a_i \hbar^i \right) * \left( \sum_{j=0}^{\infty} b_j \hbar^j \right) = \sum_{i,j=0}^{\infty} (a_i * b_j) \hbar^{i+j}.$$

The multiplication  $*$  can therefore be characterized by the  $\mathbb{k}$ -bilinear maps  $\mu_i : A \times A \rightarrow A$  such that

$$a * b = \mu_0(a, b) + \mu_1(a, b) \hbar + \mu_2(a, b) \hbar^2 + \dots$$

The condition  $A[[\hbar]]/\hbar A[[\hbar]] = A$  means that  $\mu_0$  is the original multiplication on  $A$ , whence

$$a * b = ab + \mu_1(a, b) \hbar + \mu_2(a, b) \hbar^2 + \dots$$

That the multiplication  $*$  is associative gives certain compatibility conditions on the  $\mu_i$ , which we won't discuss here.

**Example B.13.** Every  $\mathbb{k}$ -algebra  $A$  admits the *trivial deformation*  $A[[\hbar]]$  (i.e. the algebra of power series with its usual product). It corresponds to the choice  $\mu_1, \mu_2, \dots = 0$ .

A deformation of  $A$  *trivial* if it is equivalent to the trivial deformation (i.e. equivalent to the algebra of power series  $A[[\hbar]]$ ).

**Theorem B.14.** The universal enveloping algebra  $U(\mathfrak{sl}_2)$  admits a deformation such that

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}}. \quad (6)$$

*Proof (sketch).* Let  $P$  be the free  $\mathbb{k}$ -algebra on the generators  $E, H, F$ . Let  $I$  be the two-sided ideal in  $P[[\hbar]]$  given by the relations (6). Let  $J$  be the closure of  $I$  in the  $\hbar$ -adic topology of  $P[[\hbar]]$ . Then  $J$  is again a two-sided ideal in  $P[[\hbar]]$ . The described deformation can be realized as the quotient  $P[[\hbar]]/J$ . We refer to [CP95, Definition-Proposition 6.4.3 ff.] for the specific details.  $\square$

**Definition B.15.** The deformation of  $U(\mathfrak{sl}_2)$  from Theorem B.14 is denoted by  $U_{\hbar}(\mathfrak{sl}_2)$ .

**Remark B.16.** 1. In the algebra  $U_{\hbar}(\mathfrak{sl}_2)$  we can consider the well-defined elements

$$q := e^{\hbar}, \quad K := e^{\hbar H}.$$

The elements  $q, E, K, K^{-1}, F$  satisfy the defining relations of  $U_q(\mathfrak{sl}_2)$ , and we want to regard  $U_q(\mathfrak{sl}_2)$  as a subalgebra of  $U_{\hbar}(\mathfrak{sl}_2)$ .

We run into a small technical problem. The algebra  $U_q(\mathfrak{sl}_2)$  needs to be defined over a field which contains the element  $q$ . But the algebra  $U_{\hbar}(\mathfrak{sl}_2)$  is defined over  $\mathbb{k}[[\hbar]]$ , and this ring does not contain any field containing  $q$ . To fix this problem we extend  $U_{\hbar}(\mathfrak{sl}_2)$  to an  $\mathbb{k}((\hbar))$ -algebra and then define  $U_q(\mathfrak{sl}_2)$  over the subfield  $\mathbb{k}(q)$  of  $\mathbb{k}((\hbar))$ .

So let us consider the field of Laurent polynomials  $\mathbb{k}((\hbar))$  and the extension of scalars

$$\mathbb{k}((\hbar)) \otimes_{\mathbb{k}[[\hbar]]} U_{\hbar}(\mathfrak{sl}_2),$$

which is given as an  $\mathbb{k}((\hbar))$ -module by

$$\mathbb{k}((\hbar)) \otimes_{\mathbb{k}[[\hbar]]} U_{\hbar}(\mathfrak{sl}_2) = \mathbb{k}[[\hbar]][\hbar^{-1}] \otimes_{\mathbb{k}[[\hbar]]} U(\mathfrak{sl}_2)[[\hbar]] \cong U(\mathfrak{sl}_2)[[\hbar]][\hbar^{-1}] \cong U(\mathfrak{sl}_2)((\hbar)).$$

The field  $\mathbb{k}(\hbar)$  contains the subfield  $\mathbb{k}(q)$ , and we get from the above observation an homomorphism of  $\mathbb{k}(q)$ -algebras

$$U_q(\mathfrak{sl}_2) \rightarrow \mathbb{k}(\hbar) \otimes_{\mathbb{k}[[\hbar]]} U(\mathfrak{sl}_2)$$

where  $U_q(\mathfrak{sl}_2)$  is defined over  $\mathbb{k}(q)$ .

2. In  $U_{\hbar}(\mathfrak{sl}_2)$  we have both the element  $H$  and the element

$$\tilde{H} = \frac{K - K^{-1}}{q - q^{-1}},$$

which is of the form

$$\tilde{H} = H + \text{terms of order at least } \hbar^2.$$

We may think about the element  $\tilde{H}$  is a deformation of the original element  $H$  (in an informal sense). We note that

$$q \equiv 1, \quad K \equiv 1, \quad \tilde{H} \equiv H \pmod{\hbar}.$$

**Definition B.17.** The deformation of  $U(\mathfrak{sl}_2)$  from Theorem B.14 is denoted by  $U_{\hbar}(\mathfrak{sl}_2)$ .

**Remark B.18.** In the algebra  $U_{\hbar}(\mathfrak{sl}_2)$  we can consider the well-defined elements

$$q := e^{\hbar}, \quad K := e^{\hbar H}.$$

The elements  $E, F, K, K^{-1}$  satisfy the defining relations of  $U_q(\mathfrak{sl}_2)$  and one should think about the algebra  $U_q(\mathfrak{sl}_2)$  as somewhat of a subalgebra of  $U_{\hbar}(\mathfrak{sl}_2)$ .

In  $U_{\hbar}(\mathfrak{sl}_2)$  we have both the element  $H$  and the element

$$\tilde{H} = \frac{K - K^{-1}}{q - q^{-1}},$$

which is of the form

$$\tilde{H} = H + \text{terms of order } \hbar^2.$$

We may think about  $\tilde{H}$  is a deformation of  $H$  (in an informal sense). We note that

$$q \equiv 1, \quad K \equiv 1, \quad \tilde{H} \equiv H \pmod{\hbar}.$$

**Remark B.19.** One can study the deformation theory of an  $\mathbb{k}$ -algebra via homological algebra. The *Hochschild cochain complex* of  $A$  is given by

$$C_{\text{Hoch}}^n(A) := \text{Hom}_{\mathbb{k}}(A^{\otimes n}, A)$$

together with certain differentials. The cohomology of this chain complex is the *Hochschild cohomology* of  $A$ , which is denoted by

$$\text{HH}^n(A) := H^n(C_{\text{Hoch}}^{\bullet}).$$

One of the connections between deformation theory and Hochschild cohomology is that in the case of

$$\text{HH}^2(A) = 0$$

every deformation of  $A$  is trivial.

**Warning B.20.** Let  $A$  be a  $\mathbb{k}$ -algebra with  $\mathrm{HH}^2(A) = 0$ , and let  $A_{\hbar}$  be a deformation of  $A$ . The above criterion shows that  $A_{\hbar}$  is equivalent to  $A[[\hbar]]$ , but it does not provide an explicit isomorphism.

**Example B.21.**

1. Let  $\mathfrak{g}$  be a semisimple Lie algebra over a field  $\mathbb{k}$  of characteristic zero. It can be shown that

$$\mathrm{HH}^2(U(\mathfrak{g})) = 0,$$

see [GS92, Theorem 2] or [Sch16, Exercise 2.8.1, Bonus]. Therefore all deformations of  $U(\mathfrak{g})$  (as an algebra) are trivial.

2. It follows in particular that every algebra deformation of  $U(\mathfrak{sl}_2)$  is trivial. An explicit equivalence between  $U_{\hbar}(\mathfrak{sl}_2)$  and  $U(\mathfrak{sl}_2)[[\hbar]]$  is constructed in [CP95, Proposition 4.6.4].

## B.4. Deformation of Hopf Algebras

The following is taken mostly from [CP95, Chapter 6].

**Definition B.22.**

1. A *topological Hopf algebra* consists of a complete  $\mathbb{k}[[\hbar]]$ -module  $A$  together with  $\mathbb{k}[[\hbar]]$ -linear maps

$$m : A \hat{\otimes} A \rightarrow A, \quad u : \mathbb{k}[[\hbar]] \rightarrow A, \quad \Delta : A \rightarrow A \hat{\otimes} A, \quad \varepsilon : A \rightarrow \mathbb{k}[[\hbar]], \quad S : A \rightarrow A$$

such that the usual Hopf algebra diagrams commute.

2. The terms *topological algebra*, *topological coalgebra* and *topological bialgebra* are defined in the same way.

**Remark B.23.**

1. A topological coalgebra  $C$  is generally not an actual Hopf algebra, since the comultiplication

$$\Delta : C \rightarrow C \hat{\otimes} C$$

does in general not restrict to a map  $C \rightarrow C \otimes C$ . This problem is inherited by topological bialgebras and topological Hopf algebras.

2. If  $A$  is a topological Hopf algebra then  $A/\hbar A$  becomes an Hopf algebra over  $\mathbb{k}$  because

$$(A \hat{\otimes} A)/\hbar(A \hat{\otimes} A) \cong (A \otimes A)/\hbar(A \otimes A) \cong (A/\hbar A) \otimes (A/\hbar A).$$

**Remark B.24.** A topological algebra in the sense of Definition B.22 is precisely the same as an  $\mathbb{k}[[\hbar]]$ -algebra that is complete as an  $\mathbb{k}[[\hbar]]$ -module.

Indeed, suppose first that  $(A, m, u)$  is a topological algebra. Then the multiplication

$$m : A \hat{\otimes} A \rightarrow A$$

restricts via the composition with the canonical homomorphism

$$A \otimes A \rightarrow A \hat{\otimes} A$$

to a multiplication

$$m' : A \otimes A \rightarrow A.$$

Then  $(A, m', u)$  is an  $\mathbb{k}[[\hbar]]$ -algebra (and  $A$  is by definition complete).

Suppose on the other hand that  $(A, m', u)$  is an  $\mathbb{k}[[\hbar]]$ -algebra where  $A$  is complete. Then the multiplication map

$$m' : A \otimes A \rightarrow A$$

extends by Proposition B.6 and the completeness of  $A$  uniquely to a  $\mathbb{k}[[\hbar]]$ -linear map

$$m : A \hat{\otimes} A \rightarrow A.$$

Then  $(A, m, u)$  is a topological algebra.<sup>4</sup>

**Definition B.25.** Let  $A$  be a Hopf algebra.

1. A *(formal) deformation* of  $A$  is a topological Hopf algebra  $A_{\hbar}$  whose underlying  $\mathbb{k}[[\hbar]]$ -module is  $A[[\hbar]]$  and for which  $A_{\hbar}/\hbar A_{\hbar} = A$  as Hopf algebras.
2. Two Hopf algebra deformations  $A_{\hbar}$  and  $A'_{\hbar}$  of  $A$  are *equivalent* if there exists an isomorphism of Hopf algebras

$$\varphi : A_{\hbar} \rightarrow A'_{\hbar}$$

such that the induced isomorphism of Hopf algebras

$$A = A_{\hbar}/\hbar A_{\hbar} \rightarrow A'_{\hbar}/\hbar A'_{\hbar} = A$$

is the identity of  $A$ , i.e.  $\varphi$  is the identity of  $A$  modulo  $\hbar$ .

3. (Formal) deformations of coalgebras and bialgebras are defined in the way as for algebras and Hopf algebras. Equivalence of deformations of coalgebras and bialgebras is defined in the same way.
4. A Hopf algebra deformation of the universal enveloping algebra  $U(\mathfrak{g})$  of a Lie algebra  $\mathfrak{g}$  is a *quantum universal enveloping algebra*.

**Remark B.26.** Let  $A$  be a Hopf algebra over  $\mathbb{k}$  with deformation  $A_{\hbar}$ . By using the isomorphism

$$A[[\hbar]] \hat{\otimes} A[[\hbar]] \cong (A \otimes A)[[\hbar]]$$

we can regard the structure maps of  $A_{\hbar}$  as  $\mathbb{k}[[\hbar]]$ -linear map

$$\begin{aligned} m_{\hbar} &: (A \otimes A)[[\hbar]] \rightarrow A[[\hbar]], \\ u_{\hbar} &: \mathbb{k}[[\hbar]] \rightarrow A[[\hbar]], \\ \Delta_{\hbar} &: A[[\hbar]] \rightarrow (A \otimes A)[[\hbar]], \\ \varepsilon_{\hbar} &: A[[\hbar]] \rightarrow \mathbb{k}[[\hbar]], \\ S_{\hbar} &: A[[\hbar]] \rightarrow A[[\hbar]] \end{aligned} \tag{7}$$

<sup>4</sup>For the commutativity of the coassociativity diagram one needs to check that two maps  $A \hat{\otimes} A \hat{\otimes} A \rightarrow A$  coincide. It follows from the completeness of  $A$  and the denseness of  $A \otimes A \otimes A$  in  $A \hat{\otimes} A \hat{\otimes} A$  that it suffices to check this equality on elements of the non-completed tensor product  $A \otimes A \otimes A$ . This holds by assumption.



which are perturbations of the structure maps of  $A$ , i.e. they reduce modulo  $\hbar$  to the original structure maps of  $A$ .

We can for example characterize the comultiplication  $\Delta_\hbar$  of  $A_\hbar$  by a sequence of  $\mathbb{k}$ -bilinear maps

$$\Delta_i : A \rightarrow A \otimes A$$

such that

$$\Delta_\hbar(a) = \Delta_0(a) + \Delta_1(a)\hbar + \Delta_2(a)\hbar^2 + \dots$$

for every  $a \in A$ . Here  $\Delta_0$  needs to be the original comultiplication from  $A$ , and the coassociativity of  $\Delta_\hbar$  gives certain conditions on the bilinear maps  $\Delta_i$ .

**Example B.27.**

1. Every Hopf algebra  $A$  admits the trivial deformation  $A[[\hbar]]$ . In the form (7) the structure maps of this deformation are given by the  $\mathbb{k}[[\hbar]]$ -linear extensions of the structure maps of  $A$ .

A deformation is *trivial* if it is equivalent to the trivial deformation (i.e.  $A[[\hbar]]$ ).

2. One can make the algebra deformation  $U_\hbar(\mathfrak{sl}_2)$  of  $U(\mathfrak{sl}_2)$  into a Hopf algebra deformation via the comultiplication

$$\Delta_\hbar(H) = H \otimes 1 + 1 \otimes H, \quad \Delta_\hbar(E) = E \otimes K + 1 \otimes E, \quad \Delta_\hbar(F) = F \otimes 1 + K^{-1} \otimes F,$$

the counit

$$\varepsilon_\hbar(H) = 0, \quad \varepsilon_\hbar(E) = 0, \quad \varepsilon_\hbar(F) = 0,$$

and the antipode

$$S_\hbar(H) = -H, \quad S_\hbar(E) = -K^{-1}E, \quad S_\hbar(F) = -FK.$$

We note that it follows from these formulas for the element  $K = e^{\hbar H}$  that

$$\Delta_\hbar(K) = K \otimes K, \quad \varepsilon_\hbar(K) = 1, \quad S_\hbar(K) = K^{-1}.$$

For the elements  $E, K, K^{-1}, F$  in  $U_\hbar(\mathfrak{sl}_2)$  we hence regain the formulas for the Hopf algebra structure of  $U_q(\mathfrak{sl}_2)$ .

We lastly give an explanation of how the irreducible, finite-dimensional representations  $L(n)$  of the universal enveloping algebra  $U(\mathfrak{sl}_2)$  can be used to construct the irreducible, finite-dimensional representations  $L(q^n)$  of  $U_q(\mathfrak{sl}_2)$ , where  $n \in \mathbb{N}$ .

**Theorem B.28** ([CP95, Proposition 6.4.10]). For every natural number  $n \in \mathbb{N}$  let  $V(n)$  be the free  $\mathbb{k}[[\hbar]]$ -module of rank  $n + 1$  with basis  $v_0, \dots, v_n$ .

1. There exists a unique  $U_\hbar(\mathfrak{sl}_2)$ -module structure on  $V(n)$  such that

$$Hv_i := (n - 2i)v_i, \quad Ev_i := [n - i + 1]_q v_{i-1}, \quad Fv_i := [i + 1]_q v_{i+1}.$$

2. The  $U_\hbar(\mathfrak{sl}_2)$ -modules  $V(n)$  is indecomposable.

3. The  $U_{\hbar}(\mathfrak{sl}_2)$ -module  $V(n)$  reduces modulo  $\hbar$  to the irreducible representations  $L(n)$  of  $U(\mathfrak{sl}_2)$ .
4. The actions of  $K$  and  $\tilde{H}$  on  $V(n)$  are given by

$$Kv_i = q^{n-2i}v_i, \quad \tilde{H}v_i = [n-2i]_q v_i.$$

It follows that

$$L(q^n) \cong \langle 1 \otimes v_0, \dots, 1 \otimes v_n \rangle_{\mathbb{k}(q)} \subseteq \mathbb{k}((\hbar)) \otimes_{\mathbb{k}[[\hbar]]} V(n)$$

as  $U_q(\mathfrak{sl}_2)$ -modules.

**Remark B.29.** We have so far consider a specific deformation  $U_{\hbar}(\mathfrak{sl}_2)$  of  $U(\mathfrak{sl}_2)$ . But the choice of a deformation of  $U(\mathfrak{sl}_2)$  is not canonical, and there are also other deformations one can consider. One such deformation, as explained in [CP95, §6.4, F], is given by

$$\begin{aligned} [H, E] &= 4(1 - e^{-2\hbar E})/\hbar, & [H, F] &= -2F - \hbar H^2, & [E, F] &= H, \\ \Delta(E) &= E \otimes 1 + 1 \otimes E, & \Delta(H) &= H \otimes 1 + e^{-2\hbar E} \otimes H, & \Delta(F) &= F \otimes 1 + e^{-2\hbar E} \otimes F, \\ \varepsilon(E) &= 0, & \varepsilon(H) &= 0, & \varepsilon(F) &= 0, \\ S(E) &= -E, & S(H) &= -e^{2\hbar E} H, & S(F) &= -e^{2\hbar E} F. \end{aligned}$$

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