The Quantum Group $\mathrm{U}_q(\mathfrak{sl}_2)$

Talk 14 on Hopf Algebras and Tensor Categories

1. Recalling $\mathfrak{sl}_2$-Theory

Let $\Bbbk$ be a field. The Lie algebra

$$\mathfrak{sl}_2 := \{A \in \mathrm{M}(2, \Bbbk) \mid \mathrm{tr}(A) = 0\}$$

admits the basis

$$E := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

and these basis elements satisfy the commutator relations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H. \tag{1}$$

Its universal enveloping algebra

$$\mathrm{U}(\mathfrak{sl}_2) := \mathrm{T}(\mathfrak{sl}_2)/(XY - YX - [X, Y] \mid X, Y \in \mathfrak{sl}_2)$$

is generated by the elements E, H, F subject to the relations (1), i.e.

$$\mathrm{U}(\mathfrak{sl}_2) \cong \Bbbk\langle E, H, F\rangle / \big([H, E] - 2E, [H, F] + 2F, [E, F] - H\big).$$

The universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ is a Hopf algebra with comultiplication

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \varepsilon(X) = 0, \quad S(X) = 0 \qquad \text{for every } X \in \mathfrak{sl}_2.$$

A representation of $\mathfrak{sl}_2$ is the same as an $\mathrm{U}(\mathfrak{sl}_2)$-module.

Theorem 1.1 (Poincaré–Birkhoff–Witt). The algebra $\mathrm{U}(\mathfrak{sl}_2)$ admits the vector space basis

$$F^l H^m E^n \qquad \text{with } l, m, n \in \mathbb{N}.$$

Theorem 1.2. Let $\Bbbk$ be of characteristic zero.

1. Every finite-dimensional $\mathfrak{sl}_2$-representation is semisimple.
2. The finite-dimensional irreducible $\mathfrak{sl}_2$-representation are (up to isomorphism) given by certain representations $\mathrm{L}(n)$ for $n \in \mathbb{N}$. This representation $\mathrm{L}(n)$ has a basis $w_0, \dots, w_n$ on which E, H, F act as depicted in Figure 1.

We refer to Appendix A.1 for more details on the representation theory of the Lie algebra $\mathfrak{sl}_2$ in characteristic zero.

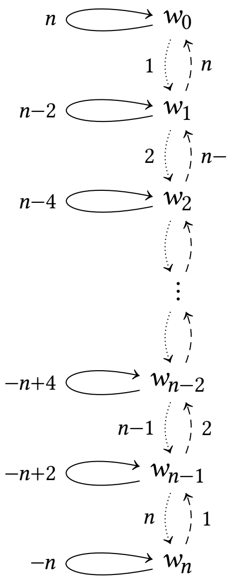

Figure 1: The irreducible representation $\mathrm{L}(n)$ of $\mathrm{U}(\mathfrak{sl}_2)$. Loops depict the action of H, dashed arrows the action of E and dotted arrows the action of F.

2. The Algebra $\mathrm{U}_q(\mathfrak{sl}_2)$

Convention 2.1. In the following $\Bbbk$ denotes a field of characteristic zero and q is an element of $\Bbbk$ with $q \neq 0, 1, -1$.

Definition 2.2. The $\Bbbk$-algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ is given by the generators

$$E\,, \quad K\,, \quad K^{-1}\,, \quad F$$

subject to the relations

$$KK^{-1} = 1 = K^{-1}K\,, \quad KE = q^2 EK\,, \quad KF = q^{-2}FK\,, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}\,. \tag{2}$$

Remark 2.3 (Choice of q). One often requires additional conditions on q, namely that

1. q is not a root of unity, or that
2. $\Bbbk$ is the field $\mathbb{K}(q)$ over some other field $\mathbb{K}$, with q being the indeterminate.

Remark 2.4 (The case $q = 1$). The algebra $\mathrm{U}_q(\mathfrak{sl})$ admits another useful presentation: One introduces the element

$$\widetilde{H} := \frac{K - K^{-1}}{q - q^{-1}}$$

as an additional generator, and then adjust the relations (2). This presentation of $\mathrm{U}_q(\mathfrak{sl}_2)$ does then make sense for any $q \in \Bbbk$, and for $q = 1$ one has

$$\mathrm{U}_1(\mathfrak{sl}_2) \cong \mathrm{U}(\mathfrak{sl}_2)[\sigma]/(\sigma^2 - 1)$$

given by

$$E \mapsto \sigma E\,, \quad \widetilde{H} \mapsto \sigma H\,, \quad F \mapsto F\,, \quad K \mapsto \sigma\,. \tag{3}$$

We refer to Appendix A.2 for more details on this presentation.

Remark 2.5. One might think about E and F as the usual elements of $\mathfrak{sl}_2$, but $\mathrm{U}_q(\mathfrak{sl}_2)$ does not contain the element H. We will later see that the algebra $\mathrm{U}_q(\mathfrak{sl})$ lives (up to some technical details) inside an $\Bbbk[\![\hbar]\!]$-algebra $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ that also contains H, and in which

$$q = \mathrm{e}^{\hbar}\,, \quad K = \mathrm{e}^{\hbar H}\,.$$

We may therefore think about the element K as

$$K = q^H\,.$$

Theorem 2.6 (PBW basis). The algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ has a vector space basis given by

$$F^l K^m E^n \qquad \text{with } l, n \in \mathbb{N} \text{ and } m \in \mathbb{Z}$$

Proof. See Appendix A.4. □

We refer to Appendix A.5 for more remarks on the algebra structure of $\mathrm{U}_q(\mathfrak{sl}_2)$.

3. Representation Theory of $\mathrm{U}_q(\mathfrak{sl}_2)$

We will in this section focus on the finite-dimensional representation theory of $\mathrm{U}_q(\mathfrak{sl}_2)$.

3.1. The Case $q = 1$

Every $\mathfrak{sl}_2$-representation extends to a $\mathrm{U}_1(\mathfrak{sl}_1)$-module by letting σ act by either 1 or -1. The resultings $\mathrm{U}_1(\mathfrak{sl}_1)$-modules are denoted by $\mathrm{L}(\varepsilon, n)$ for $\varepsilon = \pm$ and $n \in \mathbb{N}$. One can conclude from Theorem 1.2 that every finite-dimensional $\mathrm{U}_1(\mathfrak{sl}_2)$-module is semisimple, and that the irreducible finite-dimensional $\mathrm{U}_1(\mathfrak{sl}_2)$-modules are given precisely given by $\mathrm{L}(\pm, n)$. One can depict these irreducible modules as in Figure 2. We refer to Appendix A.3 for proofs of these claims.

We will keep the case of $\mathrm{U}_1(\mathfrak{sl}_2)$ in the back of our minds while considering the following discussion.

3.2. Weight Space Decomposition

Convention 3.1. In the following q is an element of $\Bbbk$ which is not a root of unity, unless otherwise specified.

Definition 3.2. Let M be an $\mathrm{U}_q(\mathfrak{sl}_2)$-module. For every scalar $\lambda \in \Bbbk^\times$ the associated *weight space* is given by

$$M_\lambda := \{m \in M \mid Km = \lambda m\}\,.$$

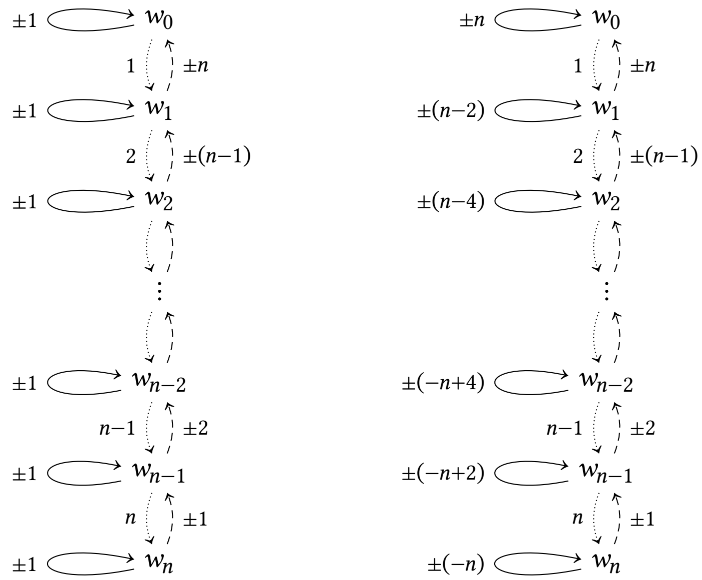

Figure 2: The irreducible representations $\mathrm{L}(\pm, n)$ of $\mathrm{U}_1(\mathfrak{sl}_2)$. On the left side loops depict the action of K, and on the right side they depict the action of $\widetilde{H}$. On both sides dashed arrows depict the action of E and dotted arrows depict the action of F.

Theorem 3.3. Let M be an $\mathrm{U}_q(\mathfrak{sl}_2)$-module.

1. It holds for every scalar $\lambda \in \Bbbk^\times$ that

$$EM_\lambda \subseteq M_{q^2\lambda}\,, \quad FM_\lambda \subseteq M_{q^{-2}\lambda}\,.$$

2. If M is finite-dimensional then M decomposes into weight spaces, and all occuring weights are of the form $\pm q^n$ with $n \in \mathbb{Z}$.

Proof. See Appendix A.6. $\square$

3.3. Verma Modules and Classifications

Definition 3.4. Let M be an $\mathrm{U}_q(\mathfrak{sl}_2)$-module.

1. A weight vector m is *primitive* if it is nonzero and $Em = 0$.
2. The module M is of *highest weight* λ if it is generated by a primitive weight vector of weight λ.

Proposition 3.5. Every irreducible, finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module is a highest weight module.

Proof. The assertion follows from Theorem 3.3. $\square$

We will classify the irreducible highest-weight representations of $\mathrm{U}_q(\mathfrak{sl}_2)$ and its irreducible finite-dimensional representations. We mirror the corresponding classifications of $\mathfrak{sl}_2$-representations.

Definition 3.6. Let $\mathrm{U}_q(\mathfrak{b})$ be the subalgebra of $\mathrm{U}_q(\mathfrak{sl}_2)$ generated by E, K, K^{-1}.[1]

Definition 3.7. Let $\lambda \in \Bbbk^\times$.

1. Let $\Bbbk_\lambda$ be the one-dimensional $\mathrm{U}_q(\mathfrak{b})$-module whose underlying vector space is given by $\Bbbk$, together with the action
$$K \cdot 1 = \lambda\,, \quad E \cdot 1 = 0\,.$$
2. The *Verma module* associated to λ is the $\mathrm{U}_q(\mathfrak{sl}_2)$-module given by
$$\mathrm{M}(\lambda) \coloneqq \mathrm{U}_q(\mathfrak{sl}_2) \otimes_{\mathrm{U}_q(\mathfrak{b})} \Bbbk_\lambda\,.$$

Definition 3.8. For $q \in \Bbbk$ with $q \neq 0$ the *n-th quantum integer* is
$$[n]_q \coloneqq q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}\,,$$
and thus for $q \neq 1, 0, -1$,
$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}\,.$$
The *quantum factorial* is
$$[n]_q! \coloneqq [n]_q[n-1]_q \cdots [1]_q\,.$$
For every invertible element $x \in \mathrm{U}_q(\mathfrak{sl}_2)$ and integer $n \in \mathbb{Z}$ let
$$[x, n]_q \coloneqq \frac{q^n x - q^{-n} x^{-1}}{q - q^{-1}}\,.$$

Remark 3.9. For $q = 1$ we have $[n]_1 = n$ and $[n]_1! = n!$.

Proposition 3.10. Let $\lambda \in \Bbbk^\times$.

1. The Verma module $\mathrm{M}(\lambda)$ has the basis
$$m_i \coloneqq F^i \otimes 1 \qquad \text{with } i \geq 0,$$
and the actions of E, K, F on this basis is given by
$$Fm_i = m_{i+1}\,, \quad Km_i = q^{-2i}\lambda m_i\,, \quad Em_i = [i]_q[\lambda, 1-i]_q m_{i-1}\,.$$
This action can be graphically described as in Figure 3.
2. The Verma module $\mathrm{M}(\lambda)$ is indecomposable.

[1] Here $\mathfrak{b}$ refers to the Lie subalgebra of $\mathfrak{sl}_2$ consisting of the traceless upper triangular matrices, see Appendix A.1.

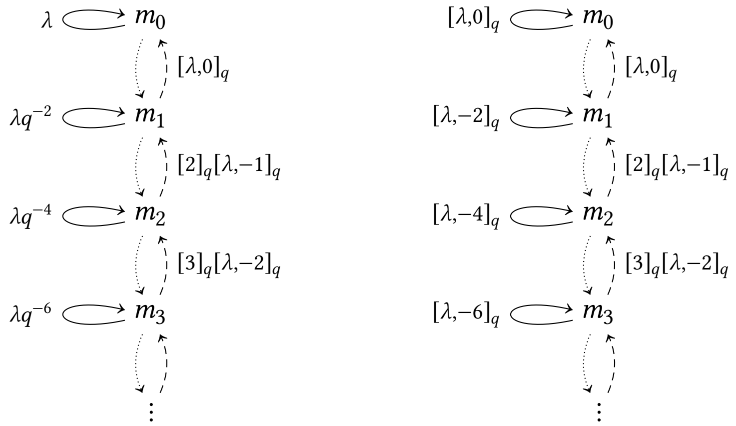

Figure 3: The Verma module $\mathrm{M}(\lambda)$ of $\mathrm{U}_q(\mathfrak{sl}_2)$. On the left side loops depict the action of K, an on the right side they depict the action of $\widetilde{H}$. On both sides the action of F is depicted by dotted arrows and the action of E by dashed arrows.

3. a. If $\lambda = \pm q^n$ for some $n \in \mathbb{N}$ then the Verma module $\mathrm{M}(\lambda)$ contains a unique nonzero, proper submodule N_λ, which is spanned by the elements

$$m_i \qquad \text{with } i \geq n+1.$$

This submodule is isomorphic to $\mathrm{M}(q^{-n-2}\lambda)$.

b. If $\lambda \neq \pm q^n$ for every $n \in \mathbb{N}$ then the Verma module $\mathrm{M}(\lambda)$ is irreducible.

Proof. See Appendix A.7. □

Definition 3.11. For every scalar $\lambda \in \Bbbk^\times$ let

$$\mathrm{L}(\lambda) := \begin{cases} \mathrm{M}(\lambda)/N_\lambda & \text{if } \lambda = \pm q^n \text{ for some } n \in \mathbb{N}, \\ \mathrm{M}(\lambda) & \text{otherwise.} \end{cases}$$

Theorem 3.12.

1. There is a one-to-one correspondence given by

$$\begin{aligned} \Bbbk^\times &\longmapsto \left\{ \begin{array}{c} \text{isomorphism clases of} \\ \text{highest-weight irreducible} \\ \mathrm{U}_q(\mathfrak{sl}_2)\text{-modules} \end{array} \right\}, \\ \lambda &\longmapsto \mathrm{L}(\lambda)\,. \end{aligned}$$

2. The module $\mathrm{L}(\lambda)$ is finite-dimensional if and only if $\lambda = \pm q^n$ for some $n \in \mathbb{N}$. The above

one-to-one correspondence does therefore restrict to a one-to-one correspondence given by

$$\begin{aligned} \{1,-1\}\times\mathbb{N} &\longmapsto \left\{\begin{matrix}\text{isomorphism clases of}\\ \text{finite-dimensional irreducible}\\ \mathrm{U}_q(\mathfrak{sl}_2)\text{-modules}\end{matrix}\right\}, \\ (\varepsilon,n) &\longmapsto \mathrm{L}(\varepsilon q^n)\,. \end{aligned}$$

We have for every $n\in\mathbb{N}$ that

$$\dim(\mathrm{L}(\pm q^n)) = n+1\,.$$

Remark 3.13.

1. For every $n\in\mathbb{Z}$ we have
$$[\pm q^n,-i+1]_q = \pm[n-i+1]_q\,.$$
On the rescalled basis $m_0,\dots,m_n$ of $\mathrm{L}(\pm q^n)$ given by
$$w_i := \frac{m_i}{[i]_q!}$$
the actions of E, K, F thus become
$$Ew_i = \pm[n-i+1]_q w_{i-1}\,,\quad Kw_i = \pm q^{n-2i}w_i\,,\quad Fw_i = [i+1]_q w_{i+1}\,.$$
The action of E, K, F on $\mathrm{L}(\pm q^n)$ can therefore be graphically be represented as in Figure 4.
2. We can consider again the element
$$\widetilde{H} := \frac{K-K^{-1}}{q-q^{-1}}$$
of $\mathrm{U}_q(\mathfrak{sl}_2)$. It acts on the weight space $M_{\lambda q^{-2i}}$ by the scalar $[\lambda,-2i]_q$. For $\lambda=\pm q^n$ this means
$$[\lambda,-2i]_q = [\pm q^n,-2i]_q = \pm[n-2i]_q\,.$$
The action of $\widetilde{H}$ on the Verma module $\mathrm{M}(\lambda)$ and irreducible modules $\mathrm{L}(\pm q^n)$ is therefore as depicted in Figure 3 and Figure 4.
3. We observe that for $q=1$ the descriptions of the irreducible $\mathrm{U}_q(\mathfrak{sl}_2)$-modules $\mathrm{L}(\pm q^n)$ from Figure 4 becomes the description of the irreducible $\mathrm{U}_1(\mathfrak{sl}_2)$-modules $\mathrm{L}(\pm,n)$ from Figure 2.

3.4. Semisimplicity of Finite-Dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-modules

Theorem 3.14. Every finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module is semisimple.

Proof. See Appendix A.8. □

Corollary 3.15. Let M, N be two finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-modules with $\dim M_\lambda = \dim N_\lambda$ for every $\lambda\in\Bbbk^\times$. Then $M\cong N$. □

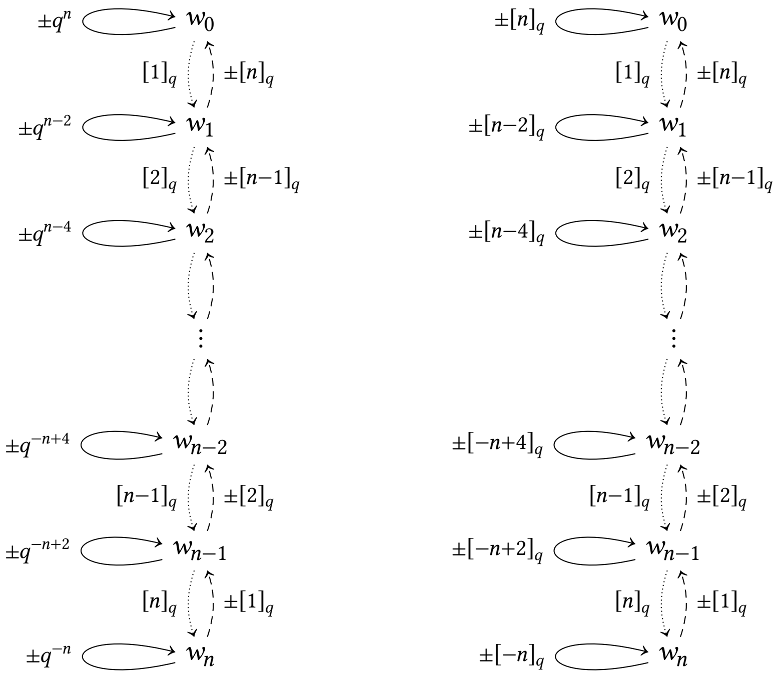

Figure 4: The irreducible representations $\mathrm{L}(\pm q^n)$ of $\mathrm{U}_q(\mathfrak{sl}_2)$. On the left side the loops depict the action of K, an on the right side they depict the action of $\widetilde{H}$. On both sides the action of F is depicted by dotted arrows and the action of E by dashed arrows.

4. Hopf Algebra Structure on $\mathrm{U}_q(\mathfrak{sl}_2)$

Proposition 4.1. The algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ becomes a Hopf algebra when endowed with the comultiplication Δ, the counit ε and the antipode S given by

$$\Delta(E) = E \otimes K + 1 \otimes E\,, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F\,, \quad \Delta(K) = K \otimes K\,,$$
$$\varepsilon(E) = 0\,, \quad \varepsilon(F) = 0\,, \quad \varepsilon(K) = 1$$
$$S(E) = -EK^{-1}\,, \quad S(F) = -KF\,, \quad S(K) = K^{-1}\,.$$

Proof. One checks that the proposed images of the algebra generators E, F, K, K^{-1} are compatible with the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$, and that the Hopf algebra diagram commute on these algebra generators. □

Definition 4.2. The Hopf algebra structure of $\mathrm{U}_q(\mathfrak{sl}_2)$ is given as in Proposition 4.1.

Remark 4.3. The Hopf algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ is neither commutative nor cocommutative. It is an example of a so-called *quantum group*.

Lemma 4.4. Let M, N be two finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-modules. Then

$$(M \otimes N)_\lambda = \bigoplus_{\mu\kappa=\lambda} M_\mu \otimes N_\kappa \,.$$

Proof. See Appendix A.9. □

Corollary 4.5. Let M, N be two finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-modules. Then

$$M \otimes N \cong N \otimes M \,.$$

Proof. This follows from Corollary 3.15 and Lemma 4.4. □

Warning 4.6. For two (finite-dimensional) $\mathrm{U}_q(\mathfrak{sl}_2)$-modules M, N the flip map

$$\tau \colon M \otimes N \to N \otimes M \,, \quad m \otimes n \mapsto n \otimes m$$

is in general not $\mathrm{U}_q(\mathfrak{sl}_2)$-linear.

Example 4.7. Indeed, let us consider $M = N = \mathrm{L}(q)$ with basis m_0, m_1. Then

$$F \cdot (m_0 \otimes m_1) = m_1 \otimes m_1 \neq q m_1 \otimes m_1 = F \cdot (m_1 \otimes m_0) \,.$$

There exists a quantum version of the Clebsch–Gordan formula, see Appendix A.10.

5. Outlook: The Deformation $\mathrm{U}_\hbar(\mathfrak{sl}_2)$

Definition 5.1. Let A be a Hopf algebra over $\Bbbk$. A *(formal) deformation* of a Hopf algebra A is a Hopf algebra over $\Bbbk[\![\hbar]\!]$ such that $A_\hbar = A[\![\hbar]\!]$ as $\Bbbk[\![\hbar]\!]$-modules and $A_\hbar / \hbar A_\hbar = A$ as Hopf algebras over $\Bbbk$.

Remark 5.2. The above definition is actually wrong. Instead of simply Hopf algebras over $\Bbbk[\![\hbar]\!]$ one needs to consider *topological Hopf algebras*. This means that for the comultiplication of $A_\hbar$ one has to replace the tensor product

$$A_\hbar \otimes_{\Bbbk[\![\hbar]\!]} A_\hbar$$

by its $\hbar$-adic completion

$$A_\hbar \mathbin{\widehat{\otimes}} A_\hbar \,.$$

In the given situation we have

$$A_\hbar \mathbin{\widehat{\otimes}} A_\hbar = A[\![\hbar]\!] \mathbin{\widehat{\otimes}} A[\![\hbar]\!] \cong (A \otimes A)[\![\hbar]\!]$$

as $\Bbbk[\![\hbar]\!]$-modules. This means that we must allow the comultiplication to take as values not only tensors, but actually power series of tensors.

Theorem 5.3. The universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ admits a Hopf algebra deformation

$$\mathrm{U}_\hbar(\mathfrak{sl}_2)$$

that is given by

$$[H,E] = 2E\,, \quad [H,F] = -2F\,, \quad [E,F] = \frac{\mathrm{e}^{\hbar H} - \mathrm{e}^{-\hbar H}}{\mathrm{e}^{\hbar} - \mathrm{e}^{-\hbar}}\,,$$
$$\Delta(E) = E \otimes \mathrm{e}^{\hbar H} + 1 \otimes E\,, \quad \Delta(H) = H \otimes 1 + 1 \otimes H\,, \quad \Delta(F) = F \otimes 1 + \mathrm{e}^{-\hbar H} \otimes F\,,$$
$$\varepsilon(E) = 0\,, \quad \varepsilon(H) = 0\,, \quad \varepsilon(F) = 0\,,$$
$$S(E) = -E\mathrm{e}^{-\hbar H}\,, \quad S(H) = -H\,, \quad S(F) = -\mathrm{e}^{\hbar H}F\,.$$

Remark 5.4.

1. In the algebra $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ we can consider the well-defined elements

   $$q := \mathrm{e}^{\hbar}\,, \quad K := \mathrm{e}^{\hbar H}\,.$$

   The elements q, E, K, K^{-1}, F satisfy the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$. We can thus (up to some technical details) regard $\mathrm{U}_q(\mathfrak{sl}_2)$ as a subalgebra of $\mathrm{U}_\hbar(\mathfrak{sl}_2)$.
2. In $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ we have both the element H and the element

   $$\widetilde{H} := [E,F] = \frac{K - K^{-1}}{q - q^{-1}} = \frac{\mathrm{e}^{\hbar H} - \mathrm{e}^{-\hbar H}}{\mathrm{e}^{\hbar} - \mathrm{e}^{-\hbar}}$$

   which is of the form

   $$\widetilde{H} = H + \text{terms of order } \hbar^2\,.$$

   We may think about $\widetilde{H}$ is a deformation of H (in an informal sense). We note that

   $$q \equiv 1\,, \quad K \equiv 1\,, \quad \widetilde{H} \equiv H \qquad (\mathrm{mod}\ \hbar).$$

Theorem 5.5 ([CP95, Proposition 6.4.10]). For every natural number $n \in \mathbb{N}$ let $L(n)$ be the free $\Bbbk[\![\hbar]\!]$-module of rank $n+1$ with basis $w_0, \dots, w_n$.

1. There exists a unique $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-module structure on $V(n)$ such that

   $$Hw_i := (n-2i)w_i\,, \quad Ew_i := [n-i+1]_q w_{i-1}\,, \quad Fw_i := [i+1]_q w_{i+1}\,.$$

   The actions of E, H, F can be graphically depicted as in Figure 5.
2. The $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-modules $V(n)$ is indecomposable.
3. The $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-module $V(n)$ reduces modulo $\hbar$ to the irreducible representations $\mathrm{L}(n)$ of $\mathrm{U}(\mathfrak{sl}_2)$.
4. The actions of K and $\widetilde{H}$ on $V(n)$ are given by

   $$Kw_i = q^{n-2i}w_i\,, \quad \widetilde{H}w_i = [n-2i]_q w_i\,.$$

   It follows that the module $V(n)$ becomes the irreducible representation $\mathrm{L}(q^n)$ of $\mathrm{U}_q(\mathfrak{sl}_2)$.

We refer to Appendix B for more a more detailed account about deformations of algebras and Hopf algebras.

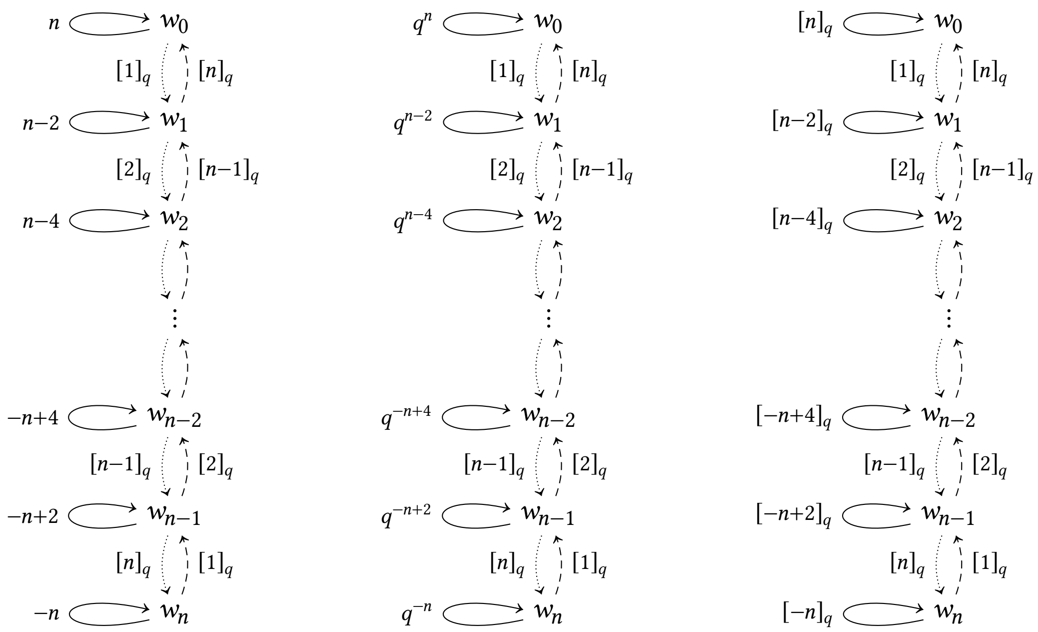

Figure 5: The indecomposable representation $V(n)$ of $\mathrm{U}(\mathfrak{sl}_2)$. On the left side loops depict the action of H, in the middle they depict the action of K, and on the right they depict the action of $\widetilde{H}$. Dashed arrows depict the action of E and dotted arrows the action of F.

A. Remarks and Proofs

A.1. Representation Theory of $\mathfrak{sl}_2$

Let $\mathfrak{b}$ denote the Lie subalgebra of $\mathfrak{sl}_2$ consisting of (traceless) upper triangular matrices. It has the matrices E, H as a basis. Its universal enveloping algebra $\mathrm{U}(\mathfrak{b})$ has the PBW-basis $H^m E^n$ with $m, n \geq 0$, and it is a subalgebra of $\mathrm{U}(\mathfrak{sl}_2)$.

A.1.1. Weight Spaces and Shifting of Weight Spaces

Definition A.1. Let V be a representation of $\mathfrak{sl}_2$.

1. The *weight space* of V with respect to λ is given by

$$V_\lambda := \{v \in V \mid H.v = \lambda v\}.$$

2. A nonzero weight vector v of V is *primitive* if $E.v = 0$.
3. The representation V is of *highest weight* λ if it is generated by a primitive weight vector of weight λ.

Proposition A.2 (Shifting weight spaces). Let V be a representation of $\mathfrak{sl}_2$ and let $\lambda \in \Bbbk$. Then

$$E.V_\lambda \subseteq V_{\lambda+2}, \quad F.V_\lambda \subseteq V_{\lambda-2}.$$

Proof. This follows from the commutator relations $[H, E] = 2E$ and $[H, F] = -2F$. □

Lemma A.3. Let $\Bbbk$ be algebraically closed. Then every finite-dimensional irreducible representation of $\mathfrak{sl}_2$ is a highest-weight representation.

A.1.2. Verma Modules

There exists for every scalar $\lambda \in \Bbbk$ a universal representation of highest weight λ, the so-called Verma module.

Definition A.4. For every scalar $\lambda \in \Bbbk$ let $\Bbbk_\lambda$ be the one-dimensional representation of $\mathfrak{b}$ which is given by $\Bbbk$ as its underlying vector space together with the action of $\mathfrak{b}$ on $\Bbbk$ given by

$$H.1 = \lambda, \quad E.1 = 0.$$

Lemma A.5. There is an isomorphism of $\mathrm{U}(\mathfrak{b})$-modules given by

$$\mathrm{U}(\mathfrak{b})/\langle E, H - \lambda\rangle \to \Bbbk_\lambda, \quad x \mapsto x.1.$$

Definition A.6. The *Verma module* of highest weight λ is the $\mathrm{U}(\mathfrak{sl}_2)$-module given by

$$\mathrm{M}(\lambda) := \mathrm{U}(\mathfrak{sl}_2) \otimes_{\mathrm{U}(\mathfrak{b})} \Bbbk_\lambda.$$

Convention A.7. From now on the field $\Bbbk$ is of characteristic zero.

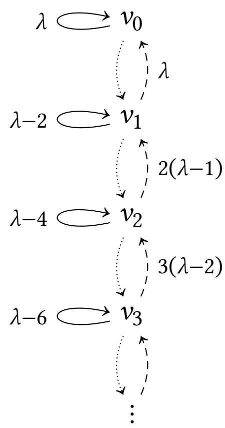

Figure 6: The Verma module $\mathrm{M}(\lambda)$ of $\mathrm{U}(\mathfrak{sl}_2)$. The action of H is depicted by loops, the action of F by dotted arrows and the action of E by dashed arrows.

Proposition A.8. Let $\lambda \in \Bbbk$.

1. The Verma module $\mathrm{M}(\lambda)$ has the vectors

$$v_i := F^i \otimes 1 \qquad \text{with } i \geq 0,$$

as a basis. The actions of E, H, F on this basis are given by

$$F.v_i = v_{i+1}\,, \quad H.v_i = (\lambda - 2i)v_i\,, \quad E.v_i = i(\lambda - i + 1)v_{i-1}\,.$$

This action can be graphically described as in Figure 6.

2. The Verma module $\mathrm{M}(\lambda)$ is a representation of highest weight λ.
3. There exists for every representation V of $\mathfrak{sl}_2$ an isomorphism of vector spaces given by

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{sl}_2}(\mathrm{M}(\lambda), V) &\longrightarrow \{v \in V \mid v \text{ is of weight } \lambda \text{ with } E.v = 0\}\,, \\ \varphi &\longmapsto \varphi(1 \otimes 1)\,. \end{aligned}$$

In particular

$$\mathrm{End}_{\mathfrak{sl}_2}(\mathrm{M}(\lambda)) = \Bbbk\,.$$

4. The representation $\mathrm{M}(\lambda)$ is indecomposable.
5. a. If $\lambda \notin \mathbb{N}$ then the representation $\mathrm{M}(\lambda)$ is irreducible.
   b. If $\lambda = n \in \mathbb{N}$ then the representation $\mathrm{M}(\lambda)$ has a unique nonzero, proper subrepresentation, which is spanned by the vectors

$$v_i \qquad \text{with } i \geq n + 1.$$

   This subrepresentation is isomorphic to $\mathrm{M}(-n-2)$.

Definition A.9. Let $\lambda \in \Bbbk$.

1. For $\lambda \notin \mathbb{N}$ let $\mathrm{L}(\lambda) := \mathrm{M}(\lambda)$.
2. For $\lambda \in \mathbb{N}$ let $\mathrm{L}(\lambda) := \mathrm{M}(\lambda)/N$ where N is the unique nonzero, proper subrepresentation of $\mathrm{M}(\lambda)$.

A.1.3. Classifications of Certain Irreducible Representations

Theorem A.10.

1. There is a one-to-one correspondence given by

$$\begin{aligned} \left\{ \begin{matrix} \text{irreducible highest weight} \\ \text{representations of } \mathfrak{sl}_2 \end{matrix} \right\} &\longleftrightarrow \Bbbk \,, \\ \mathrm{L}(\lambda) &\longleftarrow\!\!\dashv \lambda \,. \end{aligned}$$

2. The representation $\mathrm{L}(\lambda)$ is finite-dimensional if and only if $\lambda = n \in \mathbb{N}$, in which case

$$\dim(\mathrm{L}(n)) = n + 1 \,.$$

If $\Bbbk$ is algebraically closed (so that every irreducible finite-dimensional $\mathfrak{sl}_2$-representation is a highest-weight representation) then the above correspondence does therefore restrict to a one-to-one correspondence

$$\begin{aligned} \left\{ \begin{matrix} \text{irreducible finite-dimensional} \\ \text{representations of } \mathfrak{sl}_2 \end{matrix} \right\} &\longleftrightarrow \mathbb{N} \,, \\ L &\longmapsto \dim(L) - 1 \,, \\ \mathrm{L}(n) &\longleftarrow\!\!\dashv n \,. \end{aligned}$$

Remark A.11. Let $n \in \mathbb{N}$. The basis $v_0, \dots, v_n$ of $\mathrm{L}(n)$ can be rescaled to the basis

$$w_i := \frac{v_i}{i!} \,.$$

The actions of E and F then become

$$E.w_i = (n - i + 1)w_{i-1} \,, \quad F.w_i = (i + 1)w_{i+1} \,.$$

The actions of E, H, F on $\mathrm{L}(n)$ can now be graphically be represented as in Figure 1.

A.1.4. Semisimplicity of Finite-Dimensional Representations

Theorem A.12 (Weyl). Let $\Bbbk$ be algebraically closed. Every finite-dimensional representation of $\mathfrak{sl}_2$ is semisimple.

Proof. See [Hum72, Theorem 6.3]. □

Corollary A.13. Any finite-dimensional representation of $\mathfrak{sl}_2$ admits a weight space decomposition. All occuring weights are integral. □

The decomposition of a finite-dimensional representation of $\mathfrak{sl}_2$ into irreducible representations can be read off from its weight space decomposition. From this the following result can be shown.

Proposition A.14 (Clebsch–Gordan). Let n, m be natural numbers with $n \geq m$. Then

$$\mathrm{L}(n) \otimes \mathrm{L}(m) \cong \mathrm{L}(n+m) \oplus \mathrm{L}(n+m-2) \oplus \cdots \oplus \mathrm{L}(n-m)\,.$$

A.1.5. The General Case of Characteristic Zero

We have above used a few times the additional assumption that $\Bbbk$ is algebraically closed. We will now explain how to get rid of this assumption. For this we first recall some standard results about semisimplicity of algebras.

Lemma A.15. Let $\Bbbk$ be any field and let V be a finite-dimensional $\Bbbk$-vector spaces. Let A be a subalgebra of $\mathrm{End}_{\Bbbk}(V)$. Then A is semisimple if and only if V is semisimple as an A-module.

Proof. See [Lan02, XVII, §5, Proposition 4.7] and [MS16], or [Mil13, Proposition 5.13] □

Definition A.16. The *Jacobson radical* of a ring R is the intersection of all maximal left ideal of R. It is denoted by $\mathrm{J}(R)$.

Remark A.17. Let R be a ring. The irreducible R-modules are up to isomorphism precisely those R-modules of the form $R/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal left ideal in R. The Jacobson radical of R does therefore consists of precisely those elements of R which annihilate every irreducible R-module.

Proposition A.18. Let A be a finite-dimensional $\Bbbk$-algebra.

1. The Jacobson radical $\mathrm{J}(A)$ is a nilpotent, two-sided ideal in A.
2. Every nilpotent, two-sided ideal of A is contained in the Jacobson raidcal $\mathrm{J}(A)$. It is thus the unique maximal nilpotent, two-sided ideal.
3. The following conditions on A are equivalent:
   i. The algebra A is semisimple.
   ii. The Jacobson radical $J(A)$ vanishes.
   iii. The algebra A does not contain any nonzero nilpotent, two-sided ideal.

Proof. See [Lam01, §4]. □

Corollary A.19 ([Mil13, Proposition 5.11]). Let A be a finite-dimensional $\Bbbk$-algebra. Let $\mathbb{K}$ be a field extension of $\Bbbk$ and suppose that $\mathbb{K} \otimes_{\Bbbk} A$ is semisimple. Then A is semisimple.

Proof. We find for the Jacobson radical $\mathrm{J}(A)$ that $\mathbb{K} \otimes \mathrm{J}(A)$ is a nilpotent, two-sided ideal of $\mathbb{K} \otimes A$. It is thus contained in the Jacobson radical $\mathrm{J}(\mathbb{K} \otimes A)$. This Jacobson radical vanishes because $\mathbb{K} \otimes A$ is semisimple. It follows that $\mathbb{K} \otimes \mathrm{J}(A) = 0$ and thus $\mathrm{J}(A) = 0$. □

Corollary A.20. Let A be an $\Bbbk$-algebra and let M be a finite-dimensional A-module. Let $\mathbb{K}$ be a field extension of $\Bbbk$ and let

$$A_{\mathbb{K}} := \mathbb{K} \otimes_{\Bbbk} A\,, \quad M_{\mathbb{K}} := \mathbb{K} \otimes_{\Bbbk} M\,.$$

If $M_{\mathbb{K}}$ is semisimple as an $A_{\mathbb{K}}$-module then M is semisimple as an A-module.

Proof. We replace A by its image in $\operatorname{End}_{\Bbbk}(M)$ and then apply Lemma A.15 and Corollary A.19. □

Theorem A.21. Let $\Bbbk$ be a field of characteristic zero.

1. Every finite-dimensional $\mathfrak{sl}_2$-representation is semisimple.
2. Every finite-dimensional $\mathfrak{sl}_2$-representation decomposes into weight spaces, and all occuring weights are integral.
3. The irreducible finite-dimensional representations of $\mathfrak{sl}_2$ are given by $\mathrm{L}(n)$ for $n \in \mathbb{N}$.

Proof. Let $\mathbb{K}$ be an algebraic closure of $\Bbbk$.

1. We have previously seen that the assertion holds for $\mathfrak{sl}_2(\mathbb{K})$. We have

   $$\mathbb{K} \otimes \mathrm{U}(\mathfrak{sl}_2(\Bbbk)) \cong \mathrm{U}(\mathfrak{sl}_2(\mathbb{K}))$$

   whence the assertion follows for $\mathfrak{sl}_2(\Bbbk)$ from Corollary A.20.
2. The assertion holds for $\mathfrak{sl}_2(\mathbb{K})$, as previously seen. Let M be a finite-dimensional representaion of $\mathfrak{sl}_2(\Bbbk)$. Then $\mathbb{K} \otimes_{\Bbbk} M$ is a finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{K})$ and it follows that $\mathbb{K} \otimes_{\Bbbk} M$ decomposes into weight spaces as described. This means that $\mathbb{K} \otimes_{\Bbbk} M$ is annihilated by the element

   $$x := \prod_{j=-n}^{n} (H - j)$$

   of $\mathrm{U}(\mathfrak{sl}_2(\mathbb{K}))$ for some sufficiently large $n \geq 0$. It follows that x, regarded as an element of $\mathrm{U}(\mathfrak{sl}_2(\Bbbk))$, annihilates the original representation M. The assertion for $\mathfrak{sl}_2(\Bbbk)$ follows from this.
3. It follows from the previous assertion that every finite-dimensional irreducible $\mathfrak{sl}_2(\Bbbk)$-representation is a highest-weight representations. The classification of irreducible, finite-dimensional highest-weight representations of $\mathfrak{sl}_2$ works the same over every field of characteristic zero. We hence get for $\mathfrak{sl}_2(\Bbbk)$ the asserted classification. □

A.2. An Alternative Presentation for $\mathrm{U}_q(\mathfrak{sl}_2)$

Let $q \in \Bbbk$ and let U_q be the algebra given by the generators

$$E\,, \quad \widetilde{H}\,, \quad F\,, \quad K\,, \quad K^{-1}$$

and the relations

$$KK^{-1} = 1 = K^{-1}K\,, \quad KE = q^2 EK\,, \quad KF = q^{-2} FK\,,$$
$$[E, F] = \widetilde{H}\,, \quad (q - q^{-1})\widetilde{H} = K - K^{-1}\,,$$
$$[\widetilde{H}, E] = q(EK + K^{-1}E)\,, \quad [\widetilde{H}, F] = -q^{-1}(FK + K^{-1}F)\,.$$

Proposition A.22. There exists a unique homomorphism of algebras

$$\psi : U_q \to \mathrm{U}_q(\mathfrak{sl}_2)$$

that is given by

$$\psi(E) = E\,, \quad \psi(\widetilde{H}) = \frac{K - K^{-1}}{q - q^{-1}}\,, \quad \psi(F) = F\,, \quad \psi(K) = K\,,$$

and this homomorphism is an isomorphism.

Proof. See [Kas95, Proposition VI.2.1]. □

Proposition A.23. For $q = 1$ there exists a unique homomorphism of algebras

$$\varphi : U_1 \to \mathrm{U}(\mathfrak{sl}_2)[\sigma]/(\sigma^2 - 1)$$

that is given by

$$\varphi(E) = \sigma E\,, \quad \varphi(\widetilde{H}) = \sigma H\,, \quad \varphi(F) = F\,, \quad \varphi(K) = \sigma\,.$$

Proof. See [Kas95, Proof of Proposition VI.2.2]. □

Remark A.24. There also exist other, more exotic presentations of $\mathrm{U}_q(\mathfrak{sl}_2)$. We refer to [ITW05] for an example.

A.3. Representation Theory of $\mathrm{U}_1(\mathfrak{sl}_2)$

Let A denote the algebra $\mathrm{U}(\mathfrak{sl}_2)[\sigma]/(\sigma^2 - 1)$.

Let M be an $\mathfrak{sl}_2$-representation and let $\varepsilon = \pm 1$. The corresponding $\mathrm{U}(\mathfrak{sl}_2)$-module structure on M extends to an $\mathrm{U}(\mathfrak{sl}_2)[\sigma]$-module structure for which σ acts by multiplication with ε, because σ is central in $\mathrm{U}(\mathfrak{sl}_2)[\sigma]$. It follows from $\varepsilon^2 = 1$ that this induces a A-module structure on M as claimed in Remark 2.4.

If M is irreducible then the resulting A-module is again irreducible since every A-submodule is in particular an $\mathfrak{sl}_2$-subrepresentation. It hence follows that the A-modules $\mathrm{L}(+, n)$ and $\mathrm{L}(-, n)$ that result from the irreducible $\mathfrak{sl}_2$-representation $\mathrm{L}(n)$ are again irreducible. These representations are pairwise non-isomorphic since the element $H\sigma$ of A (which corresponds to the element $\widetilde{H}$ of $\mathrm{U}_1(\mathfrak{sl}_2)$) acts on $\mathrm{L}(+, n)$ with highest weight n and on $\mathrm{L}(-, n)$ with highest weight $-n$.

Let now M be any finite-dimensional M-module. It follows from the relation $\sigma^2 = 1$ in A that the action of σ on A is diagonalizable with eigenvalues 1 and -1. We thus have

$$M = M_1 \oplus M_{-1}$$

with $M_\varepsilon := \{m \in M \mid \sigma m = \varepsilon m\}$ for $\varepsilon = \pm 1$. The action of σ on M is an A-module homomorphism because σ is central in A. The decomposition $M = M_1 \oplus M_{-1}$ is therefore one of A-modules.

We may regard both M_1 and M_{-1} as $\mathfrak{sl}_2$-representations by restriction. We then have decompositions into finite-dimensional irreducible $\mathfrak{sl}_2$-representations given by

$$M_1 \cong \mathrm{L}(n_1) \oplus \cdots \oplus \mathrm{L}(n_s)\,, \quad M_{-1} \cong \mathrm{L}(n'_1) \oplus \cdots \oplus \mathrm{L}(n'_t)\,.$$

We note that this is already a decomposition as A-modules since σ acts on M_1 and M_{-1} by multiplication with scalars. As A-modules we have

$$\mathrm{L}(n_i) = \mathrm{L}(+, n_i)\,, \quad \mathrm{L}(n'_i) = \mathrm{L}(-, n'_i)\,.$$

This shows that every finite-dimensional A-module decomposes into a direct sum of the irreducible A-modules $\mathrm{L}(\varepsilon, n)$.

A.4. PBW Basis for $\mathrm{U}_q(\mathfrak{sl}_2)$

We use in the following the notation introduced in Definition 3.8.

Lemma A.25. For every $r \geq 0$ we have

$$[E, F^r] = [r]_q F^{r-1} [K, 1-r]_q\,.$$

Proof. For $r = 0$ both sides vanish and for $r = 1$ this is one of the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$. For $r \geq 2$ the assertion follows by induction, see [Jan96, Appendix 1.3 (5)]. □

Corollary A.26. We have

$$\begin{aligned} F \cdot F^l K^m E^n &= F^{l+1} K^m E^n\,, \\ K^{\pm 1} \cdot F^l K^m E^n &= q^{\mp 2l} F^l K^{m \pm 1} E^n\,, \\ E \cdot F^l K^m E^n &= q^{-2m} F^l K^m E^{n+1} + \frac{[l]_q}{q - q^{-1}} (q^{1-l} F^{l-1} K^{m+1-l} E^n - q^{l-1} F^{l-1} K^{m+l-1} E^n)\,. \end{aligned}$$

Proof. This follows from Lemma A.25 and the two relations $KE = q^2 EK$ and $KF = q^{-2} FK$. □

Proof of Theorem 2.6. Let U be the linear subspace of $\mathrm{U}_q(\mathfrak{sl}_2)$ spanned by these given monomials. It follows from Corollary A.26 that $\mathrm{U}_q(\mathfrak{sl}_2)$ is a left ideal. It contains the elements $F^0 K^0 E^0 = 1$, whence $U = \mathrm{U}_q(\mathfrak{sl}_2)$. This shows that the given monomials are a vector space generating set.

The linear independence is shown in the usual representation-theoretic way: Let V be the free vector space with basis

$$X^l Y^n Z^m \qquad \text{with } l, n \in \mathbb{N} \text{ and } m \in \mathbb{Z}.$$

There exists an action of $\mathrm{U}_q(\mathfrak{sl}_2)$ on V by using the formulas from Corollary A.26, with $F^l K^m E^n$ replaced by $X^l Y^n Z^m$. (It has to be checked that this proposed action is compatible with the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$, see [Jan96, Appendix 1.5].) The elements

$$F^l K^m E^n \cdot X^0 Y^0 Z^0 = X^l Y^m Z^n$$

are linearly independent in V, whence the given monomials $F^l K^m E^n$ are linearly independent in $\mathrm{U}_q(\mathfrak{sl}_2)$. □

A.5. More on the Algebra Structure of $\mathrm{U}_q(\mathfrak{sl}_2)$

Remark A.27.

1. The universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ is noetherian and has no nonzero zero divisors. The same holds for $\mathrm{U}_q(\mathfrak{sl}_2)$, see [Kas95, Proposition VI.1.4] and [Jan96, Propositon 1.8].
2. The algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ admits a grading such that E, K, F are homogeneous with

   $$\deg(E) = 1\,, \quad \deg(F) = -1\,, \quad \deg(K) = 0\,.$$

   The degree d part of $\mathrm{U}_q(\mathfrak{sl}_2)$ has the basis

   $$F^l K^m E^n \qquad \text{with } n - l = d.$$

   This grading wan also be characterized in terms of the conjugation map

   $$\mathrm{U}_q(\mathfrak{sl}_2) \to \mathrm{U}_q(\mathfrak{sl}_2)\,, \quad x \mapsto KxK^{-1}\,.$$

   The degree d part of the grading is precisely the eigenspace with eigenvalue q^{2d}.

Proposition A.28.

1. There exists a unique algebra involution ω of $\mathrm{U}_q(\mathfrak{sl}_2)$ with

   $$\omega(E) = F\,, \qquad \omega(K) = K^{-1}\,, \quad \omega(F) = E\,.$$

2. There exists a unique algebra anti-involution τ of $\mathrm{U}_q(\mathfrak{sl}_2)$ with

   $$\tau(E) = E\,, \qquad \tau(K) = K^{-1}\,, \quad \tau(F) = F\,.$$

3. There exists a unique algebra isomorphism $\varphi_q \colon \mathrm{U}_q(\mathfrak{sl}_2) \to \mathrm{U}_{q^{-1}}(\mathfrak{sl}_2)$ with

   $$\varphi(E) = -F\,, \quad \varphi(K) = K^{-1} \qquad \varphi(F) = -E\,.$$

   The inverse of the isomorphism φ_q is given by $\varphi_{q^{-1}}$.
4. There exist unique algebra involutions σ_E and σ_F of $\mathrm{U}_q(\mathfrak{sl}_2)$ with

   $$\sigma_E(E) = -E\,, \quad \sigma_E(K) = -K\,, \quad \sigma_E(F) = F\,.$$

   and

   $$\sigma_F(E) = E\,, \quad \sigma_F(K) = -K\,, \quad \sigma_E(F) = -F\,.$$

Proof. One checks that the proposed images of E, F, $K^{\pm 1}$ are compatible with the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$. See also [Jan96, Lemma 1.2]. ☐

Remark A.29.

1. One can combine the above (anti-)isomorphisms to construct further (anti-)isomorphisms involving $\mathrm{U}_q(\mathfrak{sl}_2)$ and $\mathrm{U}_{q^{-1}}(\mathfrak{sl}_2)$.
2. It follows from the existence of these (anti-)isomorphisms that many formulas and propositions involving $\mathrm{U}_q(\mathfrak{sl}_2)$ have to satisfy certain symmetries.

A.6. Proof of Theorem 3.3

Lemma A.30. Let M be a finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module.

1. Both E and F act nilpotently on M.
2. For a sufficiently large power $r \geq 0$ (namely such that $F^r M = 0$) the module M is annihilated by
$$\prod_{j=-r}^{r} (K^2 - q^{2j}) \,.$$

Proof. See [Jan96, Proposition 2.1] and [Jan96, Proposition 2.3]. □

Proposition A.31. Every finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module decomposes into weight spaces. All occuring weights are of the form $\pm q^n$ for some $n \in \mathbb{Z}$.

Proof. Let M be a finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module and let k denote the action of K on M. It follows from Lemma A.30 that

$$0 = \prod_{n=-r}^{r} (k^2 - q^{2n}) = \prod_{n=-r}^{r} (k - q^n)(k + q^n) \,.$$

The roots $\pm q^n$ with $n = -r, \dots, r$ are pairwise distinct[2] whence it follows that k is diagonalizable with possible eigenvalues $\pm q^n$ for $n = -r, \dots, r$. □

A.7. Proof of Proposition 3.10

Proposition A.32.

1. The algebra $\mathrm{U}_q(\mathfrak{b})$ has the basis
$$K^n E^m \qquad \text{with } n \in \mathbb{Z} \text{ and } m \in \mathbb{N}$$
2. The algebra $\mathrm{U}_q(\mathfrak{b})$ is given with respect to its generators E, K, K^{-1} by the relations
$$KK^{-1} = 1 = K^{-1}K \,, \quad KE = q^2 EK \,.$$

Proof.

1. Let U be the linear subspace of $\mathrm{U}_q(\mathfrak{sl}_2)$ spanned by the monomials $K^n E^m$ with $n, m \in \mathbb{N}$. This linear subspace is contained in $\mathrm{U}_q(\mathfrak{b})$. It follows on the other hand from the relation $KE = q^2 EK$ that
$$K^n E^m \cdot K^{n'} E^{m'} = q^{2mn'} K^{n+n'} E^{m+m'}$$
for all $n, n', m, m' \in \mathbb{N}$, and we have $1 = K^0 E^0 \in U$. This shows that U is a subalgebra of $\mathrm{U}_q(\mathfrak{sl}_2)$ containing E, K, K^{-1}, and therefore containing $\mathrm{U}_q(\mathfrak{b})$. This shows together that $U = \mathrm{U}_q(\mathfrak{b})$.

[2] If $\pm q^n = \pm q^m$ then squaring both sides of this equation gives $q^{2n} = q^{2m}$ and thus $q^{2(n-m)} = 1$. It follows that $2(n-m) = 0$ because q is not a root of unity, and thus $n = m$.

2. Let U be the algebra given by generators E, K, K^{-1} and relations

$$KK^{-1} = 1 = K^{-1}K\,, \quad KE = q^2 EK\,.$$

There exists a unique algebra homomorphism $\varphi\colon U \to \mathrm{U}_q(\mathfrak{b})$ given by

$$\varphi(E) = E\,, \quad \varphi(K) = K\,.$$

In the same way as Theorem 2.6 one sees that U has a PBW-basis given by the monomials

$$K^n E^m \qquad \text{with } n \in \mathbb{Z} \text{ and } m \in \mathbb{N}.$$

It follows that the algebra homomorphism φ restricts to a bijection between the PBW-bases of U and $\mathrm{U}_q(\mathfrak{b})$ and is therefore an algebra isomorphism. □

We now show an extended version of Proposition 3.10

Proposition A.33. Let $\lambda \in \Bbbk^\times$.

1. We have $\Bbbk_\lambda \cong \mathrm{U}_q(\mathfrak{b})/\langle E, K - \lambda\rangle$ as an $\mathrm{U}_q(\mathfrak{b})$-module.
2. The Verma module $\mathrm{M}(\lambda)$ has the basis

$$m_i := F^i \otimes 1 \qquad \text{with } i \geq 0,$$

and the actions of E, K, F on this basis is given by

$$Fm_i = m_{i+1}\,, \quad Km_i = q^{-2i}\lambda m_i\,, \quad Em_i = [i]_q[\lambda, 1-i]_q m_{i-1}\,.$$

This action can be graphically described as in Figure 3.

3. The Verma module $\mathrm{M}(\lambda)$ is of highest weight λ, and every $\mathrm{U}_q(\mathfrak{sl})$-module of highest weight λ is a quotient of $\mathrm{M}(\lambda)$.
4. There exists for every $\mathrm{U}_q(\mathfrak{sl}_2)$-module M an isomorphism of vector spaces given by

$$\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{sl}_2)}(\mathrm{M}(\lambda), M) \cong \{m \in M \mid m \text{ is of weight } \lambda \text{ with } Em = 0\}\,.$$

It follows in particular that

$$\mathrm{End}_{\mathrm{U}_q(\mathfrak{sl}_2)}(\mathrm{M}(\lambda)) = \Bbbk\,.$$

5. The Verma module $\mathrm{M}(\lambda)$ is indecomposable.
6. a. If $\lambda = \pm q^n$ for some $n \in \mathbb{N}$ then the Verma module $\mathrm{M}(\lambda)$ contains a unique nonzero, proper submodule, which is spanned by the elements

$$m_i \qquad \text{with } i \geq n+1.$$

This submodule is isomorphic to $\mathrm{M}(q^{-n-2}\lambda)$.

b. If $\lambda \neq \pm q^n$ for every $n \in \mathbb{N}$ then the Verma module $\mathrm{M}(\lambda)$ is irreducible.

1. This follows from the PBW-basis of $\mathrm{U}_q(\mathfrak{b})$.

2. This follows from the PBW-basis of $\mathrm{U}_q(\mathfrak{sl}_2)$ and induction.
3. The Verma module $\mathrm{M}(\lambda)$ is generated by the primitive weight vector $1 \otimes 1$.
4. We have
$$\begin{aligned}\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{sl}_2)}(\mathrm{M}(\lambda), M) &\cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{b})}(\Bbbk_\lambda, M)\\ &\cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{b})}(\mathrm{U}_q(\mathfrak{b})/\langle K-\lambda, E\rangle, M)\\ &\cong \{m \in M \mid (K-\lambda)m = 0, Em = 0\}.\end{aligned}$$
5. The endomorphism algebra $\mathrm{End}_{\mathrm{U}_q(\mathfrak{sl}_2)}(\mathrm{M}(\lambda)) = \Bbbk$ does not contain any non-trivial idempotents.
6. This follows as for $\mathrm{U}(\mathfrak{sl}_2)$ since $[i]_q[\lambda, i-1]_q = 0$ if and only if $\lambda = \pm q^{i-1}$.

A.8. Proof of Theorem 3.14

Lemma A.34. If M is an highest-weight $\mathrm{U}_q(\mathfrak{sl}_2)$-module then
$$\mathrm{End}_{\mathrm{U}_q(\mathfrak{sl}_2)}(M) = \Bbbk\,.$$

Definition A.35. The *quantum Casimir element* is the element $C_q \in \mathrm{U}_q(\mathfrak{sl}_2)$ given by
$$C_q := EF + \frac{Kq^{-1} + K^{-1}q}{(q-q^{-1})^2}\,.$$

Lemma A.36.

1. The element C_q is central in $\mathrm{U}_q(\mathfrak{sl}_2)$.
2. The element C_q acts on every $\mathrm{U}_q(\mathfrak{sl}_2)$-module by module endomorphisms.
3. The element C_q acts for every scalar $\lambda \in \Bbbk^\times$ on the representation $\mathrm{L}(\lambda)$ by multiplication with the scalar
$$\frac{\lambda q + \lambda^{-1}q^{-1}}{(q-q^{-1})^2}\,.$$
4. The element C_q acts the same on $\mathrm{L}(\lambda)$ and $\mathrm{L}(\mu)$ if and only if $\lambda = \mu$ or $\lambda = \mu^{-1}q^{-2}$.

Proof.

1. It can be checked that C_q commutes with E, F, K by using the defining relations for $\mathrm{U}_q(\mathfrak{sl}_2)$.
2. This follows from the previous assertion.
3. It follows from the previous assertion and Lemma A.34 that C_q acts by a scalar. This scalar can be read off from the action on the primitive generator $1 \otimes 1$. It thus sufficies to show the assertion for $\mathrm{M}(\lambda)$, where it follows from Proposition 3.10.
4. This follows from the previous assertion. □

Corollary A.37. The quantum Casimir element C_q acts on every finite-dimensional, irreducible representation of $\mathrm{U}_q(\mathfrak{sl}_2)$ by a different scalar.

Proof. If $\lambda = \delta q^n$ and $\mu = \varepsilon q^m$ with $\delta, \varepsilon \in \{1, -1\}$ and $n, m \in \mathbb{N}$ then it cannot happen that $\lambda = \mu^{-1} q^{-2}$. The assertion thus follows from Lemma A.36. $\square$

Proof of Theorem 3.14 ([Jan96, Theorem 2.9]). Let M be any finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module and let c denote the action of C_q on M. We may assume that M is indecomposable. We can consider a composition series

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_r = M \tag{4}$$

with composition factors

$$M_i / M_{i-1} \cong \mathrm{L}(\varepsilon_i q^{n_i}) .$$

Letting c_i be the scalar by which C_q acts on $\mathrm{L}(\varepsilon_i q^{n_i})$, we have

$$(c - c_i) M_i \subseteq M_{i-1} .$$

It follows that $\prod_{i=1}^{r} (c - c_i)$ annihilates M and that c admits a generalized eigenspace decomposition with eigenvalues $c_1, \dots, c_r$. The resulting generalized eigenspaces are subrepresentations because c is a $\mathrm{U}_q(\mathfrak{sl}_2)$-module endomorphism. It follows that

$$c_1 = \cdots = c_r$$

because M is indecomposable, and thus

$$\varepsilon_1 q^{n_1} = \cdots = \varepsilon_r q^{n_r} =: \lambda$$

by Corollary A.37. It follows with the composition series (4) that

$$\dim(M_\mu) = r \dim(\mathrm{L}(\lambda)_\mu)$$

for every scalar $\mu \in \Bbbk^\times$. Thus M is of highest weight λ.

The short exact sequence

$$0 \to M_{r-1} \to M \to \mathrm{L}(\lambda) \to 0 \tag{5}$$

restricts to a short exact sequence

$$0 \to (M_{r-1})_\lambda \to M_\lambda \to \mathrm{L}(\lambda)_\lambda \to 0 .$$

It follows that the primitive generator v_0 of $\mathrm{L}(\lambda)$ has a preimage m_0 in M. The weight vector m_0 is primitive because M isof highest weight λ. It follows that there exists a homomorphism of $\mathrm{U}_q(\mathfrak{sl}_2)$-modules

$$\varphi : \ \mathrm{M}(\lambda) \to M , \quad 1 \otimes 1 \mapsto m_0 .$$

It follows from the finite-dimensionality of M that φ factors through a homomorphism

$$\psi : \ \mathrm{L}(\lambda) \to M , \quad \overline{1 \otimes 1} \mapsto m_0 .$$

This shows that the short exact sequence (5) splits, whence

$$M \cong M_{r-1} \oplus \mathrm{L}(\lambda) .$$

It follows by induction that $M_{r-1} \cong \mathrm{L}(\lambda)^{\oplus (r-1)}$ and thus altogether $M \cong \mathrm{L}(\lambda)^{\oplus r}$. $\square$

Remark A.38. The center of the universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ is a polynomial algebra, generated by the classical Casimir element $C = (ef + h^2 + fe)/4$. It can be shown that the center of $\mathrm{U}_q(\mathfrak{sl}_2)$ is again a polynomial algebra, now generated by the quantum Casimir element C_q. We refer to [Jan96, Proposition 2.18] for more details on this.

A.9. Proof of Lemma 4.4

We have

$$M_\mu \otimes N_\kappa \subseteq (M \otimes N)_{\mu\kappa}$$

for all $\mu, \kappa \in \Bbbk^\times$ since the element K is group-like in $\mathrm{U}_q(\mathfrak{sl}_2)$. Both M and N admits weight space decompositions

$$M = \bigoplus_\mu M_\mu\,, \quad N = \bigoplus_\kappa N_\kappa$$

and it follows that

$$M \otimes N = \left(\bigoplus_\mu M_\mu\right) \otimes \left(\bigoplus_\kappa N_\kappa\right) = \bigoplus_{\mu,\kappa} (M_\mu \otimes N_\kappa) \subseteq \bigoplus_\lambda M_\lambda \subseteq M \otimes N$$

It folllows with the inclusions $M_\mu \otimes N_\kappa \subseteq (M \otimes N)_{\mu\kappa}$ that already

$$(M \otimes N)_\lambda = \bigoplus_{\mu\kappa=\lambda} M_\mu \otimes N_\kappa$$

for every λ.

A.10. Clebsch--Gordan for $\mathrm{U}_q(\mathfrak{sl}_2)$

Proposition A.39. For all $\delta, \varepsilon \in \{1, -1\}$ and $n, m \in \mathbb{N}$ with $n \geq m$ we have

$$\mathrm{L}(\delta q^n) \otimes \mathrm{L}(\varepsilon q^m) \cong \mathrm{L}(\delta\varepsilon q^{n+m}) \oplus \mathrm{L}(\delta\varepsilon q^{n+m-2}) \oplus \cdots \oplus \mathrm{L}(\delta\varepsilon q^{n-m})\,.$$

Proof. This follows from Corollary 3.15 and Lemma 4.4. □

B. Deformation Theory

B.1. Deformations of Algebras

We will in the following introduce a formal deformation $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ of the Hopf algebra $\mathrm{U}(\mathfrak{sl}_2)$ and gain a new understanding of $\mathrm{U}_q(\mathfrak{sl}_2)$.

B.2. Deformation of Algebras

The following is taken (at least in spirit) from [Bel18, §5.2] and [GS92].

Motivation B.1. Deforming a $\Bbbk$-algebra A means – roughly speaking – that the multiplication on A is replaced by a perturbated multiplication $*$, in the sense that for all $a, b \in A$,

$$a * b = ab + \mu_1(a, b)\hbar + \mu_2(a, b)\hbar^2 + \cdots$$

for some bilinear terms $\mu_i(a, b)$. The limit $\hbar \to 0$ does then give back the original algebra A.

Definition B.2. Let A be an $\Bbbk$-algebra.

1. A *(formal) deformation* of A is an $\Bbbk[\![\hbar]\!]$-algebra $A_\hbar$ whose underlying $\Bbbk[\![\hbar]\!]$-module is $A[\![\hbar]\!]$ and for which $A_\hbar/\hbar A_\hbar = A$ as algebras.
2. Two deformations $A_\hbar$ and $A'_\hbar$ of the algebra A are *equivalent* if there exists an isomorphism of $\Bbbk[\![\hbar]\!]$-algebras
$$\varphi\colon\ A_\hbar \to A'_\hbar$$
such that the induced isomorphism of $\Bbbk$-algebras
$$A = A_\hbar/\hbar A_\hbar \to A'_\hbar/\hbar A'_\hbar = A$$
is the identity on A, i.e.
$$\varphi \equiv \mathrm{id}_A \pmod{\hbar}.$$
3. A deformation is *trivial* if it is equivalent to the trivial deformation (i.e. equivalent to the algebra of power series $A[\![\hbar]\!]$).

Remark B.3. Every $\Bbbk[\![\hbar]\!]$-bilinear multiplication
$$(-) * (-)\colon\ A[\![\hbar]\!] \times A[\![\hbar]\!] \to A[\![\hbar]\!].$$
satisfies the equality
$$\left(\sum_{i=0}^{\infty} a_i \hbar^i\right) * \left(\sum_{j=0}^{\infty} b_j \hbar^j\right) = \sum_{i,j=0}^{\infty} (a_i * b_j)\hbar^{i+j}.$$
The multiplication $*$ can therefore be characterized by the $\Bbbk$-bilinear maps $\mu_i\colon\ A\times A \to A$ such that
$$a * b = \mu_0(a,b) + \mu_1(a,b)\hbar + \mu_2(a,b)\hbar^2 + \cdots$$
The condition $A[\![\hbar]\!]/\hbar A[\![\hbar]\!] = A$ means that μ_0 is the original multiplication on A, whence
$$a * b = ab + \mu_1(a,b)\hbar + \mu_2(a,b)\hbar^2 + \cdots$$
That the multiplication $*$ is associative gives certain compatibility conditions on the μ_1, which we won't discuss here.

Example B.4. Every $\Bbbk$-algebra A admits the *trivial deformation* $A[\![\hbar]\!]$ (i.e. the algebra of power series with its usual product). It corresponds to the choice $\mu_1, \mu_2, \ldots = 0$.

Theorem B.5. The universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ admits a deformation with
$$[H,E] = 2E\,, \quad [H,F] = -2F\,, \quad [E,F] = \frac{\mathrm{e}^{\hbar H} - \mathrm{e}^{-\hbar H}}{\mathrm{e}^{\hbar} - \mathrm{e}^{-\hbar}} \tag{6}$$

Proof (sketch). Let P be the free algebra on the generators E, H, F. Let I be the two-sided ideal in $P[\![\hbar]\!]$ given by the relations (6). Let J be the closure of I in the $\hbar$-adic topology. Then J is again a two-sided ideal in $P[\![\hbar]\!]$. The described deformation can be realized as the quotient $P[\![\hbar]\!]/J$. We refer to [CP95, Definition-Proposition 6.4.3 ff.] for the specific details. □

Definition B.6. The deformation of $\mathrm{U}(\mathfrak{sl}_2)$ from Theorem B.5 is denoted by $\mathrm{U}_\hbar(\mathfrak{sl}_2)$.

Remark B.7.

In the algebra $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ we can consider the well-defined elements

$$q := \mathrm{e}^{\hbar}\,, \quad K := \mathrm{e}^{\hbar H}\,.$$

The elements q, E, F, K, K^{-1} satisfy the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$.

Let us consider the field of Laurent polynomials $\Bbbk(\!(\hbar)\!)$ and the extension of scalars

$$\Bbbk(\!(\hbar)\!) \otimes_{\Bbbk[\![\hbar]\!]} \mathrm{U}_\hbar(\mathfrak{sl}_2)\,,$$

which is given as an $\Bbbk(\!(\hbar)\!)$-module by

$$\Bbbk(\!(\hbar)\!) \otimes_{\Bbbk[\![\hbar]\!]} \mathrm{U}_\hbar(\mathfrak{sl}_2) = \Bbbk[\![\hbar]\!][\hbar^{-1}] \otimes_{\Bbbk[\![\hbar]\!]} \mathrm{U}(\mathfrak{sl}_2)[\![\hbar]\!] \cong \mathrm{U}(\mathfrak{sl}_2)[\![\hbar]\!][\hbar^{-1}] \cong \mathrm{U}(\mathfrak{sl}_2)(\!(\hbar)\!)\,.$$

The field $\Bbbk(\!(\hbar)\!)$ contains the subfield $\Bbbk(q)$, and we get from the above observation an homomorphism of $\Bbbk(q)$-algebras

$$\mathrm{U}_q(\mathfrak{sl}_2) \to \Bbbk(\!(\hbar)\!) \otimes_{\Bbbk[\![\hbar]\!]} \mathrm{U}(\mathfrak{sl}_2)$$

where $\mathrm{U}_q(\mathfrak{sl}_2)$ is defined over $\Bbbk(q)$.

In $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ we have both the element H and the element

$$\widetilde{H} = \frac{K - K^{-1}}{q - q^{-1}}\,,$$

which is of the form

$$\widetilde{H} = H + \text{terms of order } \hbar^2\,.$$

We may think about $\widetilde{H}$ is a deformation of H (in an informal sense). We note that

$$q \equiv 1\,, \quad K \equiv 1\,, \quad \widetilde{H} \equiv H \qquad (\mathrm{mod}\ \hbar).$$

Definition B.8. The deformation of $\mathrm{U}(\mathfrak{sl}_2)$ from Theorem B.5 is denoted by $\mathrm{U}_\hbar(\mathfrak{sl}_2)$.

Remark B.9. In the algebra $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ we can consider the well-defined elements

$$q := \mathrm{e}^{\hbar}\,, \quad K := \mathrm{e}^{\hbar H}\,.$$

The elements E, F, K, K^{-1} satisfy the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$ and one should think about the algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ as somewhat of a subalgebra of $\mathrm{U}_\hbar(\mathfrak{sl}_2)$.

In $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ we have both the element H and the element

$$\widetilde{H} = \frac{K - K^{-1}}{q - q^{-1}}\,,$$

which is of the form

$$\widetilde{H} = H + \text{terms of order } \hbar^2\,.$$

We may think about $\widetilde{H}$ is a deformation of H (in an informal sense). We note that

$$q \equiv 1\,, \quad K \equiv 1\,, \quad \widetilde{H} \equiv H \qquad (\mathrm{mod}\ \hbar).$$

Remark B.10. One can study the deformation theory of an $\Bbbk$-algebra via homological algebra: The *Hochschild cochain complex* of A is given by

$$\mathrm{C}^n_{\mathrm{Hoch}}(A) \coloneqq \mathrm{Hom}_{\Bbbk}(A^{\otimes n}, A)$$

together with certain differentials. The cohomology of this chain complex is the *Hochschild cohomology* of A, which is denoted by

$$\mathrm{HH}^n(A) \coloneqq \mathrm{H}^n\left(\mathrm{C}^\bullet_{\mathrm{Hoch}}\right).$$

One of the connections between deformation theory and Hochschild cohomology is that in the case of

$$\mathrm{HH}^2(A) = 0$$

every deformation of A is trivial.

Warning B.11. Let $A_\hbar$ be a deformation of an $\Bbbk$-algebra A with $\mathrm{HH}^2(A) = 0$. The above criterion shows that $A_\hbar$ is equivalent to $A[\![\hbar]\!]$, but it does not provide an explicit isomorphism.

Example B.12. Let $\mathfrak{g}$ be a semisimple Lie algebra. It can be shown that

$$\mathrm{HH}^2(\mathrm{U}(\mathfrak{g})) = 0,$$

see [GS92, Theorem 2] or [Sch16, Exercise 2.8.1, Bonus]. Therefore all deformations of $\mathrm{U}(\mathfrak{g})$ (as an algebra) are trivial.

It follows in particular that the every algebra deformation of $\mathrm{U}(\mathfrak{sl}_2)$ is trivial. An explicit equivalence between $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ and $\mathrm{U}(\mathfrak{sl}_2)[\![\hbar]\!]$ is constructed in [CP95, Proposition 4.6.4].

B.3. Background on Completions

We also want define coalgebras (and bialgebras and Hopf algebras). For this we need to make sense of power series in tensor products $A[\![\hbar]\!] \otimes A[\![\hbar]\!]$, which does in general not make sense. This problem is solved by using the *completed tensor product.*

Definition B.13. Let M be an $\Bbbk[\![\hbar]\!]$-module.

1. The *$\hbar$-adic completion* of M is the $\Bbbk[\![\hbar]\!]$-module

$$\widehat{M} \coloneqq \varprojlim_{n \geq 0} (M/\hbar^{n+1}M) = \left\{ (m_n)_{n\geq 0} \,\middle|\, \begin{array}{l} m_n \in M/\hbar^{n+1}M \text{ with} \\ m_{n+1} \equiv m_n \pmod{\hbar^{n+1}} \text{ for every } n \geq 0 \end{array} \right\}.$$

2. The *canonical homomorphism* $M \to \widehat{M}$ is given by $m \mapsto (\overline{m}, \overline{m}, \dots)$.
3. A $\Bbbk[\![\hbar]\!]$-module M is *complete* if the canonical homomorphism $M \to \widehat{M}$ is an isomorphism.

Remark B.14.

1. More explicitely, an $\Bbbk[\![\hbar]\!]$-module M is complete if and only if there exists for every sequence $m_0, m_1, \dots$ of elements $m_n \in M$ with
$$m_{n+1} \equiv m_n \pmod{\hbar^{n+1}} \qquad \text{for every } n \geq 0$$
a unique element $m \in M$ with
$$m \equiv m_n \pmod{\hbar^{n+1}} \qquad \text{for every } n \geq 0.$$
2. Let M be a complete $\Bbbk[\![\hbar]\!]$-module Every sequence $(m_i)_{i\geq 0}$ of elements $m_i \in M$ defines a sequence $(s_n)_{n\geq 0}$ of partial sums
$$s_n := \sum_{i=0}^{n} \hbar^i m_i \,.$$
for every $n \geq 0$. By the completeness of M there exists a unique element $\sum_{i=0}^{\infty} \hbar^i m_i$ of M with
$$\sum_{i=0}^{\infty} \hbar^i m_i \equiv \sum_{i=0}^{n} \hbar^i m_i \pmod{\hbar^{n+1}} \qquad \text{for every } n \geq 0.$$

Example B.15.

1. Every finite-dimensional $\Bbbk[\![\hbar]\!]$-module M is complete since $\hbar^n M = 0$ for some sufficiently large power n.
2. For every $\Bbbk$-vector space the resulting $\Bbbk[\![\hbar]\!]$-module $V[\![\hbar]\!]$ is complete. For every sequence of elements $v_0, v_1, \dots \in V$ we have
$$\sum_{i=0}^{\infty} \hbar^i v_i = \sum_{i=0}^{\infty} v_i \hbar^i \,.$$

Proposition B.16. Let M, N be two $\Bbbk[\![\hbar]\!]$-modules.

1. For every homomorphism of $\Bbbk[\![\hbar]\!]$-module $f \colon M \to N$ there exists a unique module homomorphism $\widehat{f} \colon \widehat{M} \to \widehat{N}$ that makes the following square diagram commute:
$$\begin{tikzcd} \widehat{M} \arrow[r, "\widehat{f}"] & \widehat{N} \\ M \arrow[u] \arrow[r, "f"] & N \arrow[u] \end{tikzcd}$$
The homomorphism $\widehat{f}$ is given by
$$\widehat{f}\Big((\overline{m_0}, \overline{m_1}, \dots)\Big) = \Big(\overline{f(m_0)}, \overline{f(m_1)}, \dots\Big) \,.$$
2. The assignment $\widehat{(-)}$ defines a functor
$$\widehat{(-)} \colon \Bbbk[\![\hbar]\!]\textbf{-Mod} \to \Bbbk[\![\hbar]\!]\textbf{-Mod} \,.$$

3. If M, N are complete then
$$f\left(\sum_{i=0}^{\infty} \hbar^i m_i\right) = \sum_{i=0}^{\infty} \hbar^i f(m_i)$$
for every sequence of elements $m_0, m_1, \dots, \in M$.
4. If N is complete then every homomorphism $M \to N$ extends uniquely to a homomorphism $\widehat{M} \to N$.
$$\begin{tikzcd} \widehat{M} \arrow[r, dashed, "\exists!"] & N \\ M \arrow[u] \arrow[ur] & \end{tikzcd}$$
5. If V is any $\Bbbk$-vector space and N is complete then every $\Bbbk$-linear map $f : V \to N$ extends uniquely to a $\Bbbk[\![\hbar]\!]$-linear linear map $f' : V[\![\hbar]\!] \to N$.
$$\begin{tikzcd} V[\![\hbar]\!] \arrow[r, dashed, "\exists!"] & N \\ V \arrow[u] \arrow[ur] & \end{tikzcd}$$
The homomorphism f' is given by
$$f'\left(\sum_{i=0}^{\infty} \hbar^i v_i\right) = \sum_{i=0}^{\infty} \hbar^i f(v_i)\,.$$
6. The canonical homomorphism $M \to \widehat{M}$ induces an isomorphism of $\Bbbk$-vector spaces
$$M/\hbar M \longrightarrow \widehat{M}/\hbar\widehat{M}\,.$$

Remark B.17. Let M be a $\Bbbk[\![\hbar]\!]$-module. There exists a unique topology on M for which a basis is given by the sets
$$m + \hbar^{n+1} M$$
with $m \in M$ and $n \geq 0$. This topology is the *$\hbar$-adic topology* on M. It makes $\Bbbk[\![\hbar]\!]$ into a topological ring and every $\Bbbk[\![\hbar]\!]$-module into a topological $\Bbbk[\![\hbar]\!]$-module. The completion $\widehat{M}$ is then the usual topological completion of M.

Definition B.18. Let M, N be two $\Bbbk[\![\hbar]\!]$-modules. The *completed tensor product*
$$M \mathbin{\widehat{\otimes}} N$$
is the $\hbar$-adic completion of the tensor product $M \otimes_{\Bbbk[\![\hbar]\!]} N$.

Proposition B.19. Let V, W be two $\Bbbk$-vector spaces. Then the $\Bbbk[\![\hbar]\!]$-linear map
$$V[\![\hbar]\!] \otimes_{\Bbbk[\![\hbar]\!]} W[\![\hbar]\!] \to (V \otimes W)[\![\hbar]\!]\,, \quad \left(\sum_{i=0}^{\infty} v_i \hbar^i\right) \otimes \left(\sum_{j=0}^{\infty} w_j \hbar^j\right) \mapsto \sum_{i,j=0}^{\infty} (v_i \otimes w_j)\hbar^{i+j}$$

extends along the canonical homomorphism

$$V \otimes W \to V \mathbin{\widehat{\otimes}} W$$

to an isomorphism of $\Bbbk[\![\hbar]\!]$-modules

$$V[\![\hbar]\!] \mathbin{\widehat{\otimes}} W[\![\hbar]\!] \to (V \otimes W)[\![\hbar]\!] \,.$$

B.4. Deformation of Hopf Algebras

The following is taken mostly from [CP95, Chapter 6].

Definition B.20.

1. A *topological Hopf algebra* consists of a complete $\Bbbk[\![\hbar]\!]$-module A together with $\Bbbk[\![\hbar]\!]$-linear maps

$$m \colon A \mathbin{\widehat{\otimes}} A \to A \,, \quad u \colon \Bbbk[\![\hbar]\!] \to A \,, \quad \Delta \colon A \to A \mathbin{\widehat{\otimes}} A \,, \quad \varepsilon \colon A \to \Bbbk[\![\hbar]\!] \,, \quad S \colon A \to A$$

   such that the usual Hopf algebra diagrams commute.

2. The terms *topological algebra*, *topological coalgebra* and *topological bialgebra* are defined analogous to topological Hopf algebras.

Remark B.21.

1. A topological Hopf algebra A is generally not an actual Hopf algebra, since the comultiplication

$$\Delta \colon A \to A \mathbin{\widehat{\otimes}} A$$

   does in general not restrict to a map $A \to A \otimes A$.

2. If A is a topological Hopf algebra then $A/\hbar A$ becomes an Hopf algebra over $\Bbbk$. We note for this that

$$(A \mathbin{\widehat{\otimes}} A)/\hbar(A \mathbin{\widehat{\otimes}} A) \cong (A \otimes A)/\hbar(A \otimes A) \cong (A/\hbar A) \otimes (A/\hbar A) \,.$$

Remark B.22. A topological algebra in the sense of Definition B.20 is precisely the same as an $\Bbbk[\![\hbar]\!]$-algebra which is complete as an $\Bbbk[\![\hbar]\!]$-module.

Indeed, suppose first that (A, m, u) is a topological algebra. Then the multiplication

$$m \colon A \mathbin{\widehat{\otimes}} A \to A$$

restricts via the composition with the canonical homomorphism

$$A \otimes A \to A \mathbin{\widehat{\otimes}} A$$

to a multiplication

$$m' \colon A \otimes A \to A \,.$$

Then (A, m', u) is an $\Bbbk[\![\hbar]\!]$-algebra (and A is by definition complete).

Suppose on the other hand that (A, m', u) is an $\Bbbk[\![\hbar]\!]$-algebra where A is complete. Then the multiplication map

$$m' : A \otimes A \to A$$

extends by the completeness of A uniquely to a $\Bbbk[\![\hbar]\!]$-linear map

$$m : A \mathbin{\widehat{\otimes}} A \to A \,.$$

Then (A, m, u) is a topological algebra (by the denseness of $A \otimes A$ in $A \mathbin{\widehat{\otimes}} A$, etc.).

Definition B.23. Let A be a Hopf algebra.

1. A *(formal) deformation* of A is a topological Hopf algebra $A_\hbar$ whose underlying $\Bbbk[\![\hbar]\!]$-module is $A[\![\hbar]\!]$ and for which $A_\hbar/\hbar A_\hbar = A$ as Hopf algebras.
2. (Formal) deformations of coalgebras and bialgebras are defined in the way as for algebras and Hopf algebras.
3. Two Hopf algebra deformations $A_\hbar$ and $A'_\hbar$ of A are *equivalent* if there exists an isomorphism of Hopf algebras
$$\varphi : A_\hbar \to A'_\hbar$$
such that the induced isomorphism of Hopf algebras
$$A = A_\hbar/\hbar A_\hbar \to A'_\hbar/\hbar A'_\hbar = A$$
is the identity, i.e. φ is the identity modulo $\hbar$.
Equivalence of deformations of coalgebras and bialgebras is defined in the same way.
4. A deformation is *trivial* if it is equivalent to the trivial deformation (i.e. $A[\![\hbar]\!]$).
5. A Hopf algebra deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a *quantum universal enveloping algebra*.

Remark B.24. Let A be a Hopf algebra over $\Bbbk$ with deformation $A_\hbar$. By using the isomorphism

$$A[\![\hbar]\!] \mathbin{\widehat{\otimes}} A[\![\hbar]\!] \cong (A \otimes A)[\![\hbar]\!]$$

we can regard the structure maps of $A_\hbar$ as $\Bbbk[\![\hbar]\!]$-linear map

$$\begin{aligned} m_\hbar &: (A \otimes A)[\![\hbar]\!] \to A[\![\hbar]\!] \,, \\ u_\hbar &: \Bbbk[\![\hbar]\!] \to A[\![\hbar]\!] \,, \\ \Delta_\hbar &: A[\![\hbar]\!] \to (A \otimes A)[\![\hbar]\!] \,, \\ \varepsilon_\hbar &: A[\![\hbar]\!] \to \Bbbk[\![\hbar]\!] \,, \\ S_\hbar &: A[\![\hbar]\!] \to A[\![\hbar]\!] \end{aligned} \tag{7}$$

which are perturbations of the structure maps of A, i.e. they reduce modulo $\hbar$ to the structure maps of A.

We can for example characterize the comultiplication $\Delta_\hbar$ of $A_\hbar$ by a sequence of bilinear map

$$\Delta_i : A \to A \otimes A$$

such that

$$\Delta_\hbar(a) = \Delta_0(a) + \Delta_1(a)\hbar + \Delta_2(a)\hbar^2 + \cdots$$

for every $a \in A$. Here Δ_0 needs to be the original comultiplication from A.

Example B.25.

1. Every Hopf algebra A admits the trivial deformation $A[\![\hbar]\!]$. In the form (7) the structure maps of this deformation are given by the $\Bbbk[\![\hbar]\!]$-linear extensions of the structure maps of A.
2. One an make the algebra deformation $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ of $\mathrm{U}(\mathfrak{sl}_2)$ into a Hopf algebra deformation via the comultiplication

$$\Delta_\hbar(H) = H \otimes 1 + 1 \otimes H\,, \quad \Delta_\hbar(E) = E \otimes K + 1 \otimes E\,, \quad \Delta_\hbar(F) = F \otimes 1 + K^{-1} \otimes F$$

the counit

$$\varepsilon_\hbar(H) = 0\,, \quad \varepsilon_\hbar(E) = 0\,, \quad \varepsilon_\hbar(F) = 0\,,$$

and the antipode

$$S_\hbar(H) = -H\,, \quad S_\hbar(E) = -K^{-1}E\,, \quad S_\hbar(F) = -FK\,.$$

We note that it follows from these formulas for the element $K = \mathrm{e}^{\hbar H}$ that

$$\Delta_\hbar(K) = K \otimes K\,, \quad \varepsilon_\hbar(K) = 1\,, \quad S_\hbar(K) = K^{-1}\,.$$

For the elements E, F, K, K^{-1} in $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ we hence regain the formulas for the Hopf algebra structure of $\mathrm{U}_q(\mathfrak{sl}_2)$.

We lastly give an explanation of how the irreducible, finite-dimensional representations $\mathrm{L}(n)$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ can be used to construct the irreducible, finite-dimensional representations $\mathrm{L}(q^n)$ of $\mathrm{U}_q(\mathfrak{sl}_2)$, where $n \in \mathbb{N}$.

Theorem B.26 ([CP95, Proposition 6.4.10]). For every natural number $n \in \mathbb{N}$ let $V(n)$ be the free $\Bbbk[\![\hbar]\!]$-module of rank $n+1$ with basis $v_0, \dots, v_n$.

1. There exists a unique $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-module structure on $V(n)$ such that

$$Hv_i := (n-2i)v_i\,, \quad Ev_i := [n-i+1]_q v_{i-1}\,, \quad Fv_i := [i+1]_q v_{i+1}\,.$$

2. The $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-modules $V(n)$ is indecomposable.
3. The $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-module $V(n)$ reduces modulo $\hbar$ to the irreducible representations $\mathrm{L}(n)$ of $\mathrm{U}(\mathfrak{sl}_2)$.
4. The actions of K and $\widetilde{H}$ on $V(n)$ is given by

$$Kv_i = q^{n-2i}v_i\,, \quad \widetilde{H}v_i = [n-2i]_q v_i\,.$$

It follows that

$$\mathrm{L}(q^n) \cong \langle 1 \otimes v_0, \dots, 1 \otimes v_n \rangle_{\Bbbk(q)} \subseteq \Bbbk(\!(\hbar)\!) \otimes_{\Bbbk[\![\hbar]\!]} V(n)$$

as $\mathrm{U}_q(\mathfrak{sl}_2)$-modules.

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