The Quantum Group $\mathrm{U}_q(\mathfrak{sl}_2)$

Talk 14 on Hopf Algebras and Tensor Categories

1. Recalling $\mathfrak{sl}_2$-Theory

Let $\Bbbk$ be a field. The Lie algebra

$$\mathfrak{sl}_2 := \{A \in \mathrm{M}(2, \Bbbk) \mid \mathrm{tr}(A) = 0\}$$

admits the basis

$$E := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

and these basis elements satisfy the commutator relations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H. \tag{1}$$

Its universal enveloping algebra

$$\mathrm{U}(\mathfrak{sl}_2) := \mathrm{T}(\mathfrak{sl}_2)/(XY - YX - [X, Y] \mid X, Y \in \mathfrak{sl}_2)$$

is generated by the elements E, H, F subject to the relations (1), i.e.

$$\mathrm{U}(\mathfrak{sl}_2) \cong \Bbbk\langle E, H, F\rangle / \big([H, E] - 2E, [H, F] + 2F, [E, F] - H\big).$$

The universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ is a Hopf algebra with comultiplication Δ, counit ε and antipode S given by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \varepsilon(X) = 0, \quad S(X) = -X \qquad \text{for every } X \in \mathfrak{sl}_2.$$

A representation of $\mathfrak{sl}_2$ is the same as an $\mathrm{U}(\mathfrak{sl}_2)$-module.

Theorem 1.1 (Poincaré–Birkhoff–Witt). The algebra $\mathrm{U}(\mathfrak{sl}_2)$ admits the vector space basis

$$F^l H^m E^n \qquad \text{with } l, m, n \in \mathbb{N}.$$

Theorem 1.2. Let $\Bbbk$ be of characteristic zero.

1. Every finite-dimensional $\mathfrak{sl}_2$-representation is semisimple.
2. The irreducible finite-dimensional $\mathfrak{sl}_2$-representation are (up to isomorphism) given by certain representations $\mathrm{L}(n)$ with $n \in \mathbb{N}$. This representation $\mathrm{L}(n)$ has a basis $w_0, \dots, w_n$ on which the elements E, H, F act as depicted in Figure 1.

We refer to Appendix A.1 for more details on the representation theory of the Lie algebra $\mathfrak{sl}_2$ in characteristic zero.

*Last changes on February 9, 2020. An up-to-date version is available online at https://gitlab.com/cionx/representation-theory-seminar-ws-19-20.

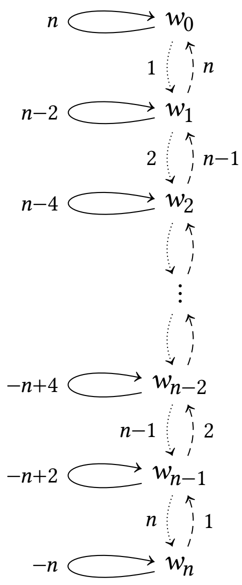

Figure 1: The irreducible representation $\mathrm{L}(n)$ of $\mathrm{U}(\mathfrak{sl}_2)$. Loops depict the action of H, dashed arrows the action of E and dotted arrows the action of F.

2. The Algebra $\mathrm{U}_q(\mathfrak{sl}_2)$

Convention 2.1. In the following $\Bbbk$ denotes a field of characteristic zero and q is an element of $\Bbbk$ with $q \neq 0, 1, -1$.

Definition 2.2. The $\Bbbk$-algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ is given by the generators

$$E\,, \quad K\,, \quad K^{-1}\,, \quad F$$

subject to the relations

$$KK^{-1} = 1 = K^{-1}K\,, \quad KE = q^2 EK\,, \quad KF = q^{-2}FK\,, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}\,. \tag{2}$$

Remark 2.3 (Choice of q). One often requires additional conditions on q, namely that

1. q is not a root of unity, or that
2. $\Bbbk$ is the function field $\mathbb{K}(q)$ over some other field $\mathbb{K}$ with indeterminate q.

Remark 2.4 (The case $q = 1$). The algebra $\mathrm{U}_q(\mathfrak{sl})$ admits another useful presentation: One introduces the element

$$\widetilde{H} := \frac{K - K^{-1}}{q - q^{-1}}$$

as an additional generator, and then adjust the relations (2). This presentation of $\mathrm{U}_q(\mathfrak{sl}_2)$ does then make sense for any parameter $q \in \Bbbk$, and for $q = 1$ one has an isomorphism

$$\mathrm{U}_1(\mathfrak{sl}_2) \cong \mathrm{U}(\mathfrak{sl}_2)[\sigma]/(\sigma^2 - 1)$$

given by

$$E \mapsto \sigma E\,, \quad \widetilde{H} \mapsto \sigma H\,, \quad F \mapsto F\,, \quad K \mapsto \sigma\,. \tag{3}$$

We refer to Appendix A.2 for more details on this presentation.

Remark 2.5. One might think about the algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ as containing the original elements E, F of $\mathfrak{sl}_2$, but not the element H. We will later see that the algebra $\mathrm{U}_q(\mathfrak{sl})$ lives (up to some technical details) inside a $\Bbbk[\![\hbar]\!]$-algebra $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ which then also contains H, and in which

$$q = \mathrm{e}^{\hbar}\,, \quad K = \mathrm{e}^{\hbar H}\,.$$

We may therefore think about the element K as

$$K = q^H\,.$$

Theorem 2.6 (PBW basis). The algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ has a vector space basis given by

$$F^l K^m E^n \qquad \text{with } l, n \in \mathbb{N} \text{ and } m \in \mathbb{Z}.$$

Proof. See Appendix A.4. □

We refer to Appendix A.5 for more information on the algebra structure of $\mathrm{U}_q(\mathfrak{sl}_2)$.

3. Representation Theory of $\mathrm{U}_q(\mathfrak{sl}_2)$

We will now focus on the finite-dimensional representation theory of $\mathrm{U}_q(\mathfrak{sl}_2)$.

3.1. The Case $q = 1$

For motivational purposes we consider first the special case $q = 1$.

By using the above isomorphism (3) we see that every $\mathfrak{sl}_2$-representation M extends to an $\mathrm{U}_1(\mathfrak{sl}_1)$-module by letting σ act on M by multiplication with either 1 or -1. Let us denote the $\mathrm{U}_1(\mathfrak{sl}_1)$-modules which result in this way from $M = \mathrm{L}(n)$ (where $n \in \mathbb{N}$) by $\mathrm{L}(\varepsilon, n)$ for $\varepsilon \in \{+, -\}$.

One can now conclude from Theorem 1.2 that every finite-dimensional $\mathrm{U}_1(\mathfrak{sl}_2)$-module is semisimple, and that the irreducible finite-dimensional $\mathrm{U}_1(\mathfrak{sl}_2)$-modules are (up to isomorphism) given by the modules $\mathrm{L}(\pm, n)$. One can depict these irreducible modules as in Figure 2. We refer to Appendix A.3 for proofs of these claims.

Throughout the following discussion of the general case of $\mathrm{U}_q(\mathfrak{sl}_2)$ we will keep the case of $\mathrm{U}_1(\mathfrak{sl}_2)$ in the back of our minds.

3.2. Weight Space Decomposition

One of the most useful tools in the study of the representation theory of $\mathfrak{sl}_2$ are weight space decompositions. By this we mean the decomposition of an $\mathfrak{sl}_2$-representation into eigenspaces with respect to the action of the element H of $\mathfrak{sl}_2$.

We would also like this tool to be available for studying the representation theory of $\mathrm{U}_q(\mathfrak{sl}_2)$. For this we replace the role of H by that of K.

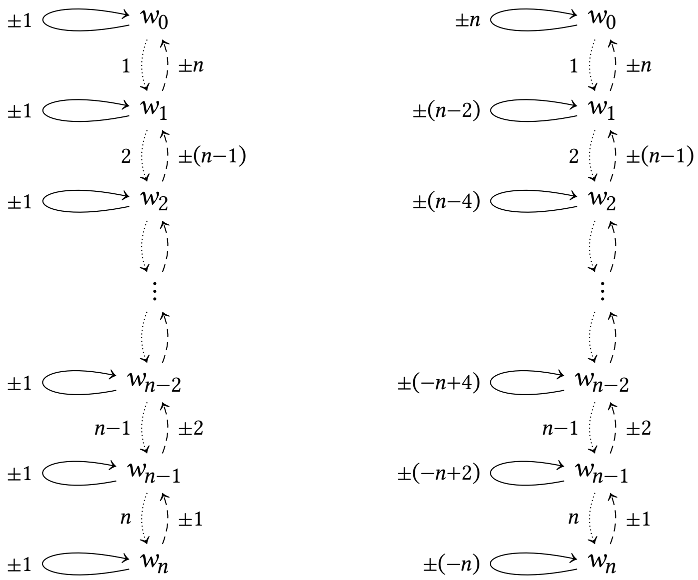

Figure 2: The irreducible representations $\mathrm{L}(\pm, n)$ of $\mathrm{U}_1(\mathfrak{sl}_2)$. On the left side loops depict the action of K, and on the right side they depict the action of $\widetilde{H}$. On both sides dashed arrows depict the action of E and dotted arrows depict the action of F.

Convention 3.1. In the following q is an element of $\Bbbk$ which is not a root of unity, unless otherwise specified.

Definition 3.2. Let M be an $\mathrm{U}_q(\mathfrak{sl}_2)$-module. For every scalar $\lambda \in \Bbbk^\times$ the associated *weight space* of M is given by

$$M_\lambda := \{m \in M \mid Km = \lambda m\}.$$

Theorem 3.3. Let M be an $\mathrm{U}_q(\mathfrak{sl}_2)$-module.

1. It holds for every scalar $\lambda \in \Bbbk^\times$ that

$$EM_\lambda \subseteq M_{q^2\lambda}, \quad FM_\lambda \subseteq M_{q^{-2}\lambda}.$$

2. If M is finite-dimensional then M decomposes into weight spaces, and all occuring weights are of the form $\pm q^n$ with $n \in \mathbb{Z}$.

Proof. See Appendix A.6. □

3.3. Verma Modules and Classifications

The weight space decomposition of finite-dimensional $\mathfrak{sl}_2$-representations leads to the classification of irreducible finite-dimensional $\mathfrak{sl}_2$-modules via the more general classification of irreducible highest-weight representations. Thanks to Theorem 3.3 we can mirror this approach for $\mathrm{U}_q(\mathfrak{sl}_2)$-modules.

Definition 3.4. Let M be an $\mathrm{U}_q(\mathfrak{sl}_2)$-module.

1. A weight vector m of M is *primitive* if it is nonzero and $Em = 0$.
2. The module M is of *highest weight* λ if it is generated by a primitive weight vector of weight λ.

Proposition 3.5. Every irreducible finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module is of highest weight.

Proof. The assertion follows from Theorem 3.3. □

We will now classify the irreducible highest-weight representations of $\mathrm{U}_q(\mathfrak{sl}_2)$ and determine which of these irreducible representations are finite-dimensional. Thanks to Proposition 3.5 this will give us a classification of the irreducible finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-modules.

Definition 3.6. Let $\mathrm{U}_q(\mathfrak{b})$ be the subalgebra of $\mathrm{U}_q(\mathfrak{sl}_2)$ generated by the elements E, K, K^{-1}.[1]

Definition 3.7. Let $\lambda \in \mathbb{k}^\times$.

1. Let $\mathbb{k}_\lambda$ be the one-dimensional $\mathrm{U}_q(\mathfrak{b})$-module whose underlying vector space is given by $\mathbb{k}$, together with the action of $\mathrm{U}_q(\mathfrak{b})$ given by

$$K \cdot 1 = \lambda\,, \quad E \cdot 1 = 0\,.$$

2. The *Verma module* associated to λ is the $\mathrm{U}_q(\mathfrak{sl}_2)$-module given by

$$\mathrm{M}(\lambda) := \mathrm{U}_q(\mathfrak{sl}_2) \otimes_{\mathrm{U}_q(\mathfrak{b})} \mathbb{k}_\lambda\,.$$

In the upcoming Proposition 3.10 we will describe the structure of the Verma module $\mathrm{M}(\lambda)$. This will require us to express the occuring coefficients – which will be rational expressions involving the highest weight λ and powers of q – in an efficient and insightful way. For this we will first introduce some standard notation.

Definition 3.8. For $q \in \mathbb{k}$ with $q \neq 0$ and $n \in \mathbb{Z}$ the *n-th quantum integer* is

$$[n]_q := \begin{cases} q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1} & \text{if } n \geq 0, \\ -(q^{n+1} + q^{n+3} + \cdots + q^{-n-3} + q^{-n-1}) & \text{if } n \leq 0, \end{cases}$$

which can for $q \neq 1, 0, -1$ be expressed as

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}\,.$$

The *n-th quantum factorial* for $n \in \mathbb{N}$ is given by

$$[n]_q! := [n]_q [n-1]_q \cdots [1]_q\,.$$

For every invertible element $x \in \mathrm{U}_q(\mathfrak{sl}_2)$ and integer $n \in \mathbb{Z}$ let

$$[x, n]_q := \frac{q^n x - q^{-n} x^{-1}}{q - q^{-1}}\,.$$

[1] Here $\mathfrak{b}$ refers to the Lie subalgebra of $\mathfrak{sl}_2$ consisting of the traceless upper triangular matrices, see Appendix A.1.

$$
\begin{tikzcd}
m_0 \arrow[loop left, "\lambda"] \arrow[d, dotted, bend right] \\
m_1 \arrow[loop left, "\lambda q^{-2}"] \arrow[u, dashed, bend right, "{[\lambda,0]_q}"'] \arrow[d, dotted, bend right] \\
m_2 \arrow[loop left, "\lambda q^{-4}"] \arrow[u, dashed, bend right, "{[2]_q[\lambda,-1]_q}"'] \arrow[d, dotted, bend right] \\
m_3 \arrow[loop left, "\lambda q^{-6}"] \arrow[u, dashed, bend right, "{[3]_q[\lambda,-2]_q}"'] \arrow[d, dotted, bend right] \\
\vdots \arrow[u, dashed, bend right]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
m_0 \arrow[loop left, "{[\lambda,0]_q}"] \arrow[d, dotted, bend right] \\
m_1 \arrow[loop left, "{[\lambda,-2]_q}"] \arrow[u, dashed, bend right, "{[\lambda,0]_q}"'] \arrow[d, dotted, bend right] \\
m_2 \arrow[loop left, "{[\lambda,-4]_q}"] \arrow[u, dashed, bend right, "{[2]_q[\lambda,-1]_q}"'] \arrow[d, dotted, bend right] \\
m_3 \arrow[loop left, "{[\lambda,-6]_q}"] \arrow[u, dashed, bend right, "{[3]_q[\lambda,-2]_q}"'] \arrow[d, dotted, bend right] \\
\vdots \arrow[u, dashed, bend right]
\end{tikzcd}
$$

Figure 3: The Verma module $\mathrm{M}(\lambda)$ of $\mathrm{U}_q(\mathfrak{sl}_2)$. On the left side loops depict the action of K, and on the right side they depict the action of $\widetilde{H}$. On both sides the action of F is depicted by dotted arrows and the action of E by dashed arrows.

Remark 3.9. For $q = 1$ we have $[n]_1 = n$ for every $n \in \mathbb{Z}$ and $[n]_1! = n!$ for every $n \in \mathbb{N}$. We have $[0]_q = 0$, $[1]_q = 1$ and $[-n]_q = -[n]_q$.

Proposition 3.10. Let $\lambda \in \Bbbk^\times$.

1. The Verma module $\mathrm{M}(\lambda)$ has the basis
$$m_i := F^i \otimes 1 \qquad \text{with } i \geq 0,$$
and the actions of E, K, F on this basis are given by
$$Fm_i = m_{i+1}\,, \quad Km_i = \lambda q^{-2i} m_i\,, \quad Em_i = [i]_q[\lambda, 1-i]_q m_{i-1}\,.$$
This action can be graphically described as in Figure 3.
2. The Verma module $\mathrm{M}(\lambda)$ is indecomposable.
3. a. If $\lambda = \pm q^n$ for some $n \in \mathbb{N}$ then the Verma module $\mathrm{M}(\lambda)$ contains a unique nonzero, proper submodule N_λ, which is spanned by the elements
$$m_i \qquad \text{with } i \geq n+1.$$
This submodule is isomorphic to $\mathrm{M}(\pm q^{-n-2})$.
   b. If $\lambda \neq \pm q^n$ for every $n \in \mathbb{N}$ then the Verma module $\mathrm{M}(\lambda)$ is irreducible.

Proof. See Appendix A.7. □

Definition 3.11. For every scalar $\lambda \in \Bbbk^\times$ let
$$\mathrm{L}(\lambda) := \begin{cases} \mathrm{M}(\lambda)/N_\lambda & \text{if } \lambda = \pm q^n \text{ for some } n \in \mathbb{N}, \\ \mathrm{M}(\lambda) & \text{otherwise.} \end{cases}$$

Theorem 3.12.

1. There is a one-to-one correspondence given by

$$\begin{aligned} \Bbbk^\times &\longrightarrow \left\{ \begin{matrix} \text{isomorphism classes of irreducible} \\ \text{highest-weight } \mathrm{U}_q(\mathfrak{sl}_2)\text{-modules} \end{matrix} \right\}, \\ \lambda &\longmapsto \mathrm{L}(\lambda) . \end{aligned}$$

2. The module $\mathrm{L}(\lambda)$ is finite-dimensional if and only if $\lambda = \pm q^n$ for some $n \in \mathbb{N}$. The above one-to-one correspondence does therefore restrict to a one-to-one correspondence given by

$$\begin{aligned} \{1, -1\} \times \mathbb{N} &\longrightarrow \left\{ \begin{matrix} \text{isomorphism classes of irreducible} \\ \text{finite-dimensional } \mathrm{U}_q(\mathfrak{sl}_2)\text{-modules} \end{matrix} \right\}, \\ (\varepsilon, n) &\longmapsto \mathrm{L}(\varepsilon q^n) . \end{aligned}$$

   We have for every $n \in \mathbb{N}$ that

$$\dim \mathrm{L}(\pm q^n) = n + 1 .$$

Remark 3.13.

1. For every $n \in \mathbb{Z}$ we have

$$[\pm q^n, -i + 1]_q = \pm [n - i + 1]_q .$$

   On the rescaled basis $m_0, \dots, m_n$ of $\mathrm{L}(\pm q^n)$ given by

$$w_i := \frac{m_i}{[i]_q!}$$

   the actions of E, K, F thus become

$$E w_i = \pm [n - i + 1]_q w_{i-1} , \quad K w_i = \pm q^{n-2i} w_i , \quad F w_i = [i + 1]_q w_{i+1} .$$

   The action of E, K, F on $\mathrm{L}(\pm q^n)$ can therefore be graphically depicted as in Figure 4.

2. We can consider again the element $\widetilde{H}$ of $\mathrm{U}_q(\mathfrak{sl}_2)$ given by

$$\widetilde{H} := \frac{K - K^{-1}}{q - q^{-1}} .$$

   This element acts on the weight space $M_{\lambda q^{-2i}}$ by the scalar $[\lambda, -2i]_q$. For $\lambda = \pm q^n$ this scalar is given by

$$[\lambda, -2i]_q = [\pm q^n, -2i]_q = \pm [n - 2i]_q .$$

   The action of $\widetilde{H}$ on the Verma module $\mathrm{M}(\lambda)$ and the irreducible modules $\mathrm{L}(\pm q^n)$ is therefore as depicted in Figure 3 and Figure 4.

3. We observe that for $q = 1$ the descriptions of the irreducible $\mathrm{U}_q(\mathfrak{sl}_2)$-modules $\mathrm{L}(\pm q^n)$ from Figure 4 become the description of the irreducible $\mathrm{U}_1(\mathfrak{sl}_2)$-modules $\mathrm{L}(\pm, n)$ from Figure 2.

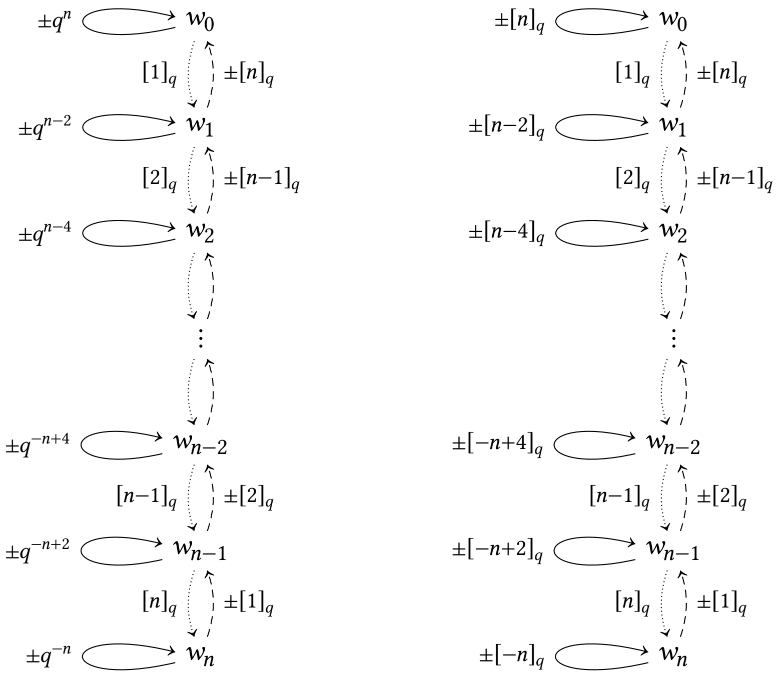

Figure 4: The irreducible representations $\mathrm{L}(\pm q^n)$ of $\mathrm{U}_q(\mathfrak{sl}_2)$. On the left side the loops depict the action of K, an on the right side they depict the action of $\widetilde{H}$. On both sides the action of F is depicted by dotted arrows and the action of E by dashed arrows.

3.4. Semisimplicity of Finite-Dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-modules

Theorem 3.14. Every finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module is semisimple.

Proof. See Appendix A.8. □

Corollary 3.15. Let M, N be two finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-modules with $\dim M_\lambda = \dim N_\lambda$ for every $\lambda \in \Bbbk^\times$. Then $M \cong N$. □

4. Hopf Algebra Structure on $\mathrm{U}_q(\mathfrak{sl}_2)$

Proposition 4.1. The algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ becomes a Hopf algebra when endowed with the comultiplication Δ, the counit ε and the antipode S given by

$$\Delta(E) = E \otimes K + 1 \otimes E\,, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F\,, \quad \Delta(K) = K \otimes K\,,$$
$$\varepsilon(E) = 0\,, \quad \varepsilon(F) = 0\,, \quad \varepsilon(K) = 1$$
$$S(E) = -EK^{-1}\,, \quad S(F) = -KF\,, \quad S(K) = K^{-1}\,.$$

Proof. One checks that the proposed images of the algebra generators E, K, K^{-1}, F are compatible with the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$, and thus giving rise to well-defined algebra homomorphisms

$$\Delta : \mathrm{U}_q(\mathfrak{sl}_2) \to \mathrm{U}_q(\mathfrak{sl}_2) \otimes \mathrm{U}_q(\mathfrak{sl}_2), \qquad \varepsilon : \mathrm{U}_q(\mathfrak{sl}_2) \to \Bbbk, \qquad S : \mathrm{U}_q(\mathfrak{sl}_2) \to \mathrm{U}_q(\mathfrak{sl}_2)^{\mathrm{op}}.$$

It can then be checked on the algebra generators E, K, K^{-1}, F that the required Hopf algebra diagrams commute. □

Definition 4.2. The Hopf algebra structure of $\mathrm{U}_q(\mathfrak{sl}_2)$ is given as described in Proposition 4.1.

Remark 4.3. The Hopf algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ is neither commutative nor cocommutative. It is an example of a so-called *quantum group*.

The newly-introduced Hopf algebra structure on $\mathrm{U}_q(\mathfrak{sl}_2)$ allows us to consider tensor products of $\mathrm{U}_q(\mathfrak{sl}_2)$-modules. For the original universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ its Hopf algebra structure is cocommutative, from which it follows that for any two $\mathfrak{sl}_2$-representations M, N the usual flip map $M \otimes N \to N \otimes M$ (given by $m \otimes n \mapsto n \otimes m$) is an isomorphism of representations. But $\mathrm{U}_q(\mathfrak{sl}_2)$ is not cocommutative anymore, so things change.

Lemma 4.4. Let M, N be two finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-modules. Then for every $\lambda \in \Bbbk^\times$,

$$(M \otimes N)_\lambda = \bigoplus_{\mu\kappa=\lambda} M_\mu \otimes N_\kappa .$$

Proof. See Appendix A.9. □

Corollary 4.5. Let M, N be two finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-modules. Then

$$M \otimes N \cong N \otimes M$$

as $\mathrm{U}_q(\mathfrak{sl}_2)$-modules.

Proof. This follows from Corollary 3.15 and Lemma 4.4. □

Warning 4.6. For two (finite-dimensional) $\mathrm{U}_q(\mathfrak{sl}_2)$-modules M, N the usual flip map

$$\tau : M \otimes N \to N \otimes M, \quad m \otimes n \mapsto n \otimes m$$

is in general not $\mathrm{U}_q(\mathfrak{sl}_2)$-linear, and thus not an isomorphism of $\mathrm{U}_q(\mathfrak{sl}_2)$-modules.

Example 4.7. Let us consider the U_q-modules M, N given by $\mathrm{L}(q)$, with basis m_0, m_1. Then on the one hand

$$F \cdot (m_0 \otimes m_1) = \underbrace{(Fm_0)}_{=m_1} \otimes m_1 + (K^{-1}m_0) \otimes \underbrace{(Fm_1)}_{=0} = m_1 \otimes m_1$$

while one the other hand

$$F \cdot (m_1 \otimes m_0) = \underbrace{(Fm_1)}_{=0} \otimes m_0 + \underbrace{(K^{-1}m_1)}_{=qm_1} \otimes \underbrace{(Fm_0)}_{=m_1} = qm_1 \otimes m_1 .$$

From the above assertions one can conclude a quantum version of the Clebsch–Gordan formula, which explains how a tensor product $\mathrm{L}(\delta q^n) \otimes \mathrm{L}(\varepsilon q^m)$ decomposes into irreducible representations. We refer to Appendix A.10 for more details.

5. Outlook: The Deformation $\mathrm{U}_\hbar(\mathfrak{sl}_2)$

Definition 5.1. Let A be a Hopf algebra over $\Bbbk$. A *(formal) deformation* of A is a Hopf algebra over $\Bbbk[\![\hbar]\!]$ such that $A_\hbar = A[\![\hbar]\!]$ as $\Bbbk[\![\hbar]\!]$-modules and $A_\hbar/\hbar A_\hbar = A$ as Hopf algebras over $\Bbbk$.

Remark 5.2. The above definition is actually wrong. Instead of Hopf algebras over $\Bbbk[\![\hbar]\!]$ one needs to consider *topological Hopf algebras* (over $\Bbbk[\![\hbar]\!]$). This means that for the comultiplication $\Delta_\hbar$ of $A_\hbar$ one has to replace the tensor product

$$A_\hbar \otimes_{\Bbbk[\![\hbar]\!]} A_\hbar$$

by its $\hbar$-adic completion

$$A_\hbar \mathbin{\widehat{\otimes}} A_\hbar \,.$$

In the given situation we have

$$A_\hbar \mathbin{\widehat{\otimes}} A_\hbar = A[\![\hbar]\!] \mathbin{\widehat{\otimes}} A[\![\hbar]\!] \cong (A \otimes A)[\![\hbar]\!]$$

as $\Bbbk[\![\hbar]\!]$-modules. This means that we must allow $\Delta_\hbar$ to have as its values not only tensors, but actually power series of tensors.

Theorem 5.3. The universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ admits a Hopf algebra deformation

$$\mathrm{U}_\hbar(\mathfrak{sl}_2)$$

such that

$$\begin{gathered} [H,E] = 2E\,, \quad [H,F] = -2F\,, \quad [E,F] = \frac{\mathrm{e}^{\hbar H} - \mathrm{e}^{-\hbar H}}{\mathrm{e}^{\hbar} - \mathrm{e}^{-\hbar}}\,, \\ \Delta(E) = E \otimes \mathrm{e}^{\hbar H} + 1 \otimes E\,, \quad \Delta(H) = H \otimes 1 + 1 \otimes H\,, \quad \Delta(F) = F \otimes 1 + \mathrm{e}^{-\hbar H} \otimes F\,, \\ \varepsilon(E) = 0\,, \quad \varepsilon(H) = 0\,, \quad \varepsilon(F) = 0\,, \\ S(E) = -E\mathrm{e}^{-\hbar H}\,, \quad S(H) = -H\,, \quad S(F) = -\mathrm{e}^{\hbar H}F\,. \end{gathered}$$

Remark 5.4.

1. In the deformation $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ we can consider the well-defined elements

   $$q \coloneqq \mathrm{e}^{\hbar}\,, \quad K \coloneqq \mathrm{e}^{\hbar H}\,.$$

   The elements q, E, K, K^{-1}, F satisfy the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$. We can thus (up to some technical details) regard $\mathrm{U}_q(\mathfrak{sl}_2)$ as a subalgebra of $\mathrm{U}_\hbar(\mathfrak{sl}_2)$.

2. The deformation $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ contains both the element H of $\mathfrak{sl}_2$ and the element

   $$\widetilde{H} \coloneqq [E,F] = \frac{K - K^{-1}}{q - q^{-1}} = \frac{\mathrm{e}^{\hbar H} - \mathrm{e}^{-\hbar H}}{\mathrm{e}^{\hbar} - \mathrm{e}^{-\hbar}}\,,$$

   which is of the form

   $$\widetilde{H} = H + \text{terms of order at least } \hbar^2\,.$$

   We may think about the element $\widetilde{H}$ as a deformation of the element H (in an informal sense). We note that

   $$q \equiv 1\,, \quad K \equiv 1\,, \quad \widetilde{H} \equiv H \qquad (\mathrm{mod}\ \hbar).$$

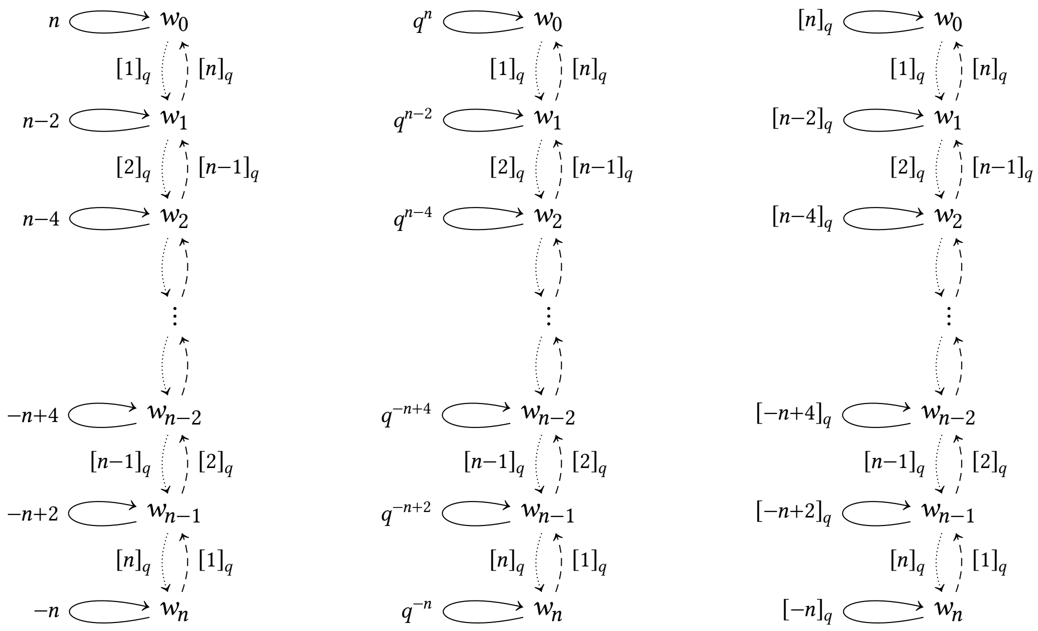

Figure 5: The indecomposable representation $V(n)$ of $\mathrm{U}_\hbar(\mathfrak{sl}_2)$. On the left side loops depict the action of H, in the middle they depict the action of K, and on the right they depict the action of $\widetilde{H}$. Dashed arrows depict the action of E and dotted arrows the action of F.

We finish this talk by explaining how the irreducible $\mathrm{U}_q(\mathfrak{sl}_2)$-modules $\mathrm{L}(q^n)$ relate to the irreducible $\mathfrak{sl}_2$-representations $\mathrm{L}(n)$ via the deformation $\mathrm{U}_\hbar(\mathfrak{sl}_2)$.

Theorem 5.5 ([CP95, Proposition 6.4.10]). For every natural number $n \in \mathbb{N}$ let $V(n)$ be the free $\Bbbk[\![\hbar]\!]$-module of rank $n+1$ with basis $w_0, \dots, w_n$.

1. The $\Bbbk[\![\hbar]\!]$-module structure of $V(n)$ extends uniquely to an $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-module structure such that
$$Hw_i := (n-2i)w_i\,, \quad Ew_i := [n-i+1]_q w_{i-1}\,, \quad Fw_i := [i+1]_q w_{i+1}\,.$$
The actions of E, H, F can be graphically depicted as in Figure 5.
2. The $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-modules $V(n)$ is indecomposable.
3. The $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-module $V(n)$ reduces modulo $\hbar$ to the irreducible representations $\mathrm{L}(n)$ of $\mathrm{U}(\mathfrak{sl}_2)$.
4. The actions of K and $\widetilde{H}$ on $V(n)$ are given by
$$Kw_i = q^{n-2i}w_i\,, \quad \widetilde{H}w_i = [n-2i]_q w_i\,.$$
It follows that the $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-module $V(n)$ becomes the irreducible $\mathrm{U}_q(\mathfrak{sl}_2)$-module $\mathrm{L}(q^n)$ by restriction (up to some technical details).

We refer to Appendix B for a more detailed account on deformations of algebras and Hopf algebras, and proper formulations of some of the above statements.

A. Remarks and Proofs

A.1. Representation Theory of $\mathfrak{sl}_2$

Let $\mathfrak{b}$ denote the Lie subalgebra of $\mathfrak{sl}_2$ consisting of (traceless) upper triangular matrices. It has the matrices E, H as a basis. Its universal enveloping algebra $\mathrm{U}(\mathfrak{b})$ has the PBW-basis

$$H^m E^n \qquad \text{with } m, n \in \mathbb{N},$$

and it is a subalgebra of $\mathrm{U}(\mathfrak{sl}_2)$.

A.1.1. Weight Spaces and Shifting of Weight Spaces

Definition A.1. Let V be a representation of $\mathfrak{sl}_2$.

1. The *weight space* of V with respect to λ is given by

$$V_\lambda := \{v \in V \mid H.v = \lambda v\}.$$

2. A nonzero weight vector v of V is *primitive* if $E.v = 0$.
3. The representation V is of *highest weight* λ if it is generated by a primitive weight vector of weight λ.

Proposition A.2 (Shifting weight spaces). Let V be a representation of $\mathfrak{sl}_2$ and let $\lambda \in \Bbbk$. Then

$$E.V_\lambda \subseteq V_{\lambda+2}, \quad F.V_\lambda \subseteq V_{\lambda-2}.$$

Proof. This follows from the commutator relations $[H, E] = 2E$ and $[H, F] = -2F$. □

Lemma A.3. Suppose that $\Bbbk$ is algebraically closed. Then every finite-dimensional irreducible representation of $\mathfrak{sl}_2$ is a highest-weight representation.

A.1.2. Verma Modules

There exists for every scalar $\lambda \in \Bbbk$ a universal representation of highest weight λ, the so-called Verma module.

Definition A.4. For every scalar $\lambda \in \Bbbk$ let $\Bbbk_\lambda$ be the one-dimensional representation of $\mathfrak{b}$ which is given by $\Bbbk$ as its underlying vector space, together with the action of $\mathfrak{b}$ on $\Bbbk$ given by

$$H.1 = \lambda, \quad E.1 = 0.$$

Lemma A.5. There is an isomorphism of $\mathrm{U}(\mathfrak{b})$-modules given by

$$\mathrm{U}(\mathfrak{b})/\langle E, H - \lambda \rangle \to \Bbbk_\lambda, \quad \overline{x} \mapsto x.1.$$

Definition A.6. The *Verma module* of highest weight λ is the $\mathrm{U}(\mathfrak{sl}_2)$-module given by

$$\mathrm{M}(\lambda) := \mathrm{U}(\mathfrak{sl}_2) \otimes_{\mathrm{U}(\mathfrak{b})} \Bbbk_\lambda.$$

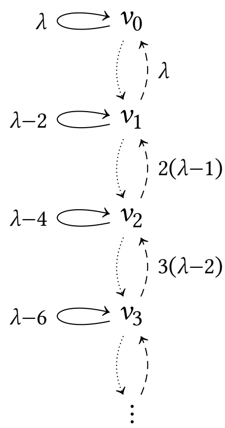

Figure 6: The Verma module $\mathrm{M}(\lambda)$ of $\mathrm{U}(\mathfrak{sl}_2)$. The action of H is depicted by loops, the action of F by dotted arrows and the action of E by dashed arrows.

Convention A.7. From now on the field $\Bbbk$ is of characteristic zero.

Proposition A.8. Let $\lambda \in \Bbbk$.

1. The Verma module $\mathrm{M}(\lambda)$ has the vectors

$$v_i := F^i \otimes 1 \qquad \text{with } i \geq 0$$

   as a basis. The actions of E, H, F on this basis are given by

$$F.v_i = v_{i+1}\,, \quad H.v_i = (\lambda - 2i)v_i\,, \quad E.v_i = i(\lambda - i + 1)v_{i-1}\,.$$

   These actions can be graphically depicted as in Figure 6.
2. The Verma module $\mathrm{M}(\lambda)$ is a representation of highest weight λ.
3. There exists for every representation V of $\mathfrak{sl}_2$ an isomorphism of vector spaces given by

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{sl}_2}(\mathrm{M}(\lambda), V) &\longrightarrow \{v \in V \mid v \text{ is of weight } \lambda \text{ with } E.v = 0\}\,, \\ \varphi &\longmapsto \varphi(1 \otimes 1)\,. \end{aligned}$$

   In particular

$$\mathrm{End}_{\mathfrak{sl}_2}(\mathrm{M}(\lambda)) = \Bbbk\,.$$

4. The representation $\mathrm{M}(\lambda)$ is indecomposable.
5. a. If $\lambda \notin \mathbb{N}$ then the representation $\mathrm{M}(\lambda)$ is irreducible.
   b. If $\lambda = n \in \mathbb{N}$ then the representation $\mathrm{M}(\lambda)$ has a unique nonzero, proper subrepresentation, which is spanned by the vectors

$$v_i \qquad \text{with } i \geq n + 1.$$

   This subrepresentation is isomorphic to $\mathrm{M}(-n-2)$.

Definition A.9. Let $\lambda \in \Bbbk$.

1. For $\lambda \notin \mathbb{N}$ let $\mathrm{L}(\lambda) := \mathrm{M}(\lambda)$.
2. For $\lambda \in \mathbb{N}$ let $\mathrm{L}(\lambda) := \mathrm{M}(\lambda)/N$ where N is the unique nonzero, proper subrepresentation of $\mathrm{M}(\lambda)$.

A.1.3. Classifications of Certain Irreducible Representations

Theorem A.10.

1. There is a one-to-one correspondence given by

$$\begin{aligned} \Bbbk &\longrightarrow \{\text{irreducible highest-weight } \mathfrak{sl}_2\text{-representations}\}\,, \\ \lambda &\longmapsto \mathrm{L}(\lambda)\,. \end{aligned}$$

2. The representation $\mathrm{L}(\lambda)$ is finite-dimensional if and only if $\lambda = n \in \mathbb{N}$, in which case

$$\dim \mathrm{L}(n) = n + 1\,.$$

   If $\Bbbk$ is algebraically closed (so that every irreducible finite-dimensional $\mathfrak{sl}_2$-representation is a highest-weight representation) then the above correspondence does therefore restrict to a one-to-one correspondence given by

$$\begin{aligned} \mathbb{N} &\longrightarrow \{\text{irreducible finite-dimensional } \mathfrak{sl}_2\text{-representations}\}\,, \\ n &\longmapsto \mathrm{L}(n)\,. \end{aligned}$$

Remark A.11. Let $n \in \mathbb{N}$. The basis $v_0, \dots, v_n$ of $\mathrm{L}(n)$ can be rescaled to the basis

$$w_i := \frac{v_i}{i!}\,.$$

The actions of E, F on this basis are given by

$$E.w_i = (n - i + 1)w_{i-1}\,, \quad F.w_i = (i + 1)w_{i+1}\,.$$

The actions of E, H, F on $\mathrm{L}(n)$ can now be graphically presented as in Figure 1.

A.1.4. Semisimplicity of Finite-Dimensional Representations

Theorem A.12 (Weyl). Let $\Bbbk$ be algebraically closed. Every finite-dimensional representation of $\mathfrak{sl}_2$ is semisimple.

Proof. See [Hum72, Theorem 6.3]. ☐

Corollary A.13. Any finite-dimensional $\mathfrak{sl}_2$-representation admits a weight space decomposition. All occuring weights are integral. ☐

The decomposition of a finite-dimensional $\mathfrak{sl}_2$-representation into irreducible representations can be read off from its weight space decomposition. From this the following results can be shown.

Lemma A.14. Let M, N be two finite-dimensional $\mathfrak{sl}_2$-representations with $\dim(M_\lambda) = \dim(N_\lambda)$ for every $\lambda \in \Bbbk$. Then $M \cong N$.

Corollary A.15 (Clebsch–Gordan). Let n, m be natural numbers with $n \geq m$. Then

$$\mathrm{L}(n) \otimes \mathrm{L}(m) \cong \mathrm{L}(n+m) \oplus \mathrm{L}(n+m-2) \oplus \cdots \oplus \mathrm{L}(n-m)\,.$$

A.1.5. The General Case of Characteristic Zero

We have above used a few times the additional assumption that the field $\Bbbk$ is algebraically closed. We will now explain how to get rid of this assumption. For this we first recall some standard results about the semisimplicity of (finite-dimensional) algebras and representations.

Lemma A.16. Let $\Bbbk$ be any field and let V be a finite-dimensional $\Bbbk$-vector spaces. Let A be a subalgebra of $\mathrm{End}_\Bbbk(V)$. Then A is semisimple if and only if V is semisimple as an A-module.

Proof. See [Lan02, XVII, §5, Proposition 4.7] and [MS16], or [Mil13, Proposition 5.13] □

Definition A.17. The *Jacobson radical* of a ring R is the intersection of all maximal left ideal of R. It is denoted by $\mathrm{J}(R)$.

Remark A.18. Let R be a ring. The irreducible R-modules are up to isomorphism precisely those R-modules of the form $R/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal left ideal in R. The Jacobson radical of R does therefore consists of precisely those elements of R which annihilate every irreducible R-module.

Proposition A.19. Let A be a finite-dimensional $\Bbbk$-algebra.

1. The Jacobson radical $\mathrm{J}(A)$ is a nilpotent, two-sided ideal in A.
2. Every nilpotent, two-sided ideal of A is contained in the Jacobson radical $\mathrm{J}(A)$. The Jacobson radical $\mathrm{J}(A)$ is thus the unique maximal nilpotent, two-sided ideal in A.
3. The following conditions on A are equivalent:
   i. The algebra A is semisimple.
   ii. The Jacobson radical $\mathrm{J}(A)$ vanishes.
   iii. The algebra A does not contain any nonzero, nilpotent, two-sided ideal.

Proof. See [Lam01, §4]. □

Corollary A.20 ([Mil13, Proposition 5.11]). Let A be a finite-dimensional $\Bbbk$-algebra. Let $\mathbb{K}$ be a field extension of $\Bbbk$ and suppose that the $\mathbb{K}$-algebra $\mathbb{K} \otimes_\Bbbk A$ is semisimple. Then A is semisimple.

Proof. We find for the Jacobson radical $\mathrm{J}(A)$ that $\mathbb{K} \otimes_\Bbbk \mathrm{J}(A)$ is a nilpotent, two-sided ideal of $\mathbb{K} \otimes_\Bbbk A$. It follows that $\mathbb{K} \otimes_\Bbbk \mathrm{J}(A) = 0$ because $\mathbb{K} \otimes_\Bbbk A$ is semisimple. Thus $\mathrm{J}(A) = 0$. □

Corollary A.21. Let A be a $\Bbbk$-algebra and let M be a finite-dimensional A-module. Let $\mathbb{K}$ be a field extension of $\Bbbk$ and suppose that $\mathbb{K} \otimes_\Bbbk M$ is semisimple as an $(\mathbb{K} \otimes_\Bbbk A)$-module. Then M is semisimple as an A-module.

Proof. Let A' be the image of A in $\mathrm{End}_{\Bbbk}(M)$ and let $(\mathbb{K} \otimes_{\Bbbk} A)'$ be the image of $\mathbb{K} \otimes_{\Bbbk} A$ in $\mathrm{End}_{\mathbb{K}}(\mathbb{K} \otimes_{\Bbbk} M)$. Under the identification $\mathrm{End}_{\mathbb{K}}(\mathbb{K} \otimes_{\Bbbk} M) \cong \mathbb{K} \otimes \mathrm{End}_{\Bbbk}(M)$ we have

$$(\mathbb{K} \otimes_{\Bbbk} A)' \cong \mathbb{K} \otimes_{\Bbbk} A' .$$

That $\mathbb{K} \otimes_{\Bbbk} M$ is semisimple as an $(\mathbb{K} \otimes_{\Bbbk} A)$-module is equivalent to it being semisimple as an $(\mathbb{K} \otimes_{\Bbbk} A)'$-module. It thus follows from Lemma A.16 that $(\mathbb{K} \otimes_{\Bbbk} A)'$ is semisimple. Therefore A' is semisimple by Corollary A.21 and the above isomorphism. It follows that M is semisimple as an A'-module and thus an an A-module. □

Theorem A.22. Let $\Bbbk$ be a field of characteristic zero.

1. Every finite-dimensional $\mathfrak{sl}_2$-representation is semisimple.
2. Every finite-dimensional $\mathfrak{sl}_2$-representation decomposes into weight spaces, and all occuring weights are integral.
3. The irreducible finite-dimensional representations of $\mathfrak{sl}_2$ are given by $\mathrm{L}(n)$ with $n \in \mathbb{N}$.

Proof. Let $\mathbb{K}$ be an algebraic closure of $\Bbbk$.

1. The assertion holds for $\mathfrak{sl}_2(\mathbb{K})$ by Theorem A.12. We have

   $$\mathbb{K} \otimes_{\Bbbk} \mathrm{U}(\mathfrak{sl}_2(\Bbbk)) \cong \mathrm{U}(\mathfrak{sl}_2(\mathbb{K}))$$

   whence the assertion follows for $\mathfrak{sl}_2(\Bbbk)$ from Corollary A.21.
2. The assertion holds for $\mathfrak{sl}_2(\mathbb{K})$ by Corollary A.13. Let now M be a finite-dimensional representaion of $\mathfrak{sl}_2(\Bbbk)$. Then $\mathbb{K} \otimes_{\Bbbk} M$ is a finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{K})$ and it follows that $\mathbb{K} \otimes_{\Bbbk} M$ decomposes into weight spaces as described. This means that $\mathbb{K} \otimes_{\Bbbk} M$ is annihilated by the element

   $$x \coloneqq \prod_{j=-n}^{n} (H - j)$$

   of $\mathrm{U}(\mathfrak{sl}_2(\mathbb{K}))$ for some sufficiently large $n \geq 0$. It follows that x, regarded as an element of $\mathrm{U}(\mathfrak{sl}_2(\Bbbk))$, annihilates the original representation M. Thus, by linear algebra, the representation M decomposes into weight spaces with possible occuring weights $-n, \dots, n$.
3. It follows from the previous assertion that every finite-dimensional irreducible $\mathfrak{sl}_2(\Bbbk)$-representation is a highest-weight representations. But the classification of irreducible, finite-dimensional highest-weight representations of $\mathfrak{sl}_2$ works the same over every field of characteristic zero. □

A.2. An Alternative Presentation for $\mathrm{U}_q(\mathfrak{sl}_2)$

Let $q \in \Bbbk$ and let U_q be the $\Bbbk$-algebra given by the generators

$$E , \quad K , \quad K^{-1} , \quad \widetilde{H} , \quad F$$

and the relations

$$KK^{-1} = 1 = K^{-1}K\,, \quad KE = q^2 EK\,, \quad KF = q^{-2}FK\,,$$
$$[E, F] = \widetilde{H}\,, \quad (q - q^{-1})\widetilde{H} = K - K^{-1}\,,$$
$$[\widetilde{H}, E] = q(EK + K^{-1}E)\,, \quad [\widetilde{H}, F] = -q^{-1}(FK + K^{-1}F)\,.$$

Proposition A.23. There exists a unique homomorphism of algebras

$$\psi : U_q \to \mathrm{U}_q(\mathfrak{sl}_2)$$

that is given by

$$\psi(E) = E\,, \quad \psi(K) = K\,, \quad \psi(\widetilde{H}) = \frac{K - K^{-1}}{q - q^{-1}}\,, \quad \psi(F) = F\,,$$

and this homomorphism is an isomorphism.

Proof. See [Kas95, Proposition VI.2.1]. □

Proposition A.24. For $q = 1$ there exists a unique homomorphism of algebras

$$\varphi : U_1 \to \mathrm{U}(\mathfrak{sl}_2)[\sigma]/(\sigma^2 - 1)$$

that is given by

$$\varphi(E) = \sigma E\,, \quad \varphi(K) = \sigma\,. \quad \varphi(\widetilde{H}) = \sigma H\,, \quad \varphi(F) = F\,,$$

Proof. See [Kas95, Proof of Proposition VI.2.2]. □

Remark A.25. There also exist other, more exotic presentations of $\mathrm{U}_q(\mathfrak{sl}_2)$. We refer to [ITW05] for a specific example.

A.3. The Finite-Dimensional Representation Theory of $\mathrm{U}_1(\mathfrak{sl}_2)$

Let A denote the algebra $\mathrm{U}(\mathfrak{sl}_2)[\sigma]/(\sigma^2 - 1)$.

Let M be an $\mathfrak{sl}_2$-representation and let $\varepsilon \in \{1, -1\}$. The corresponding $\mathrm{U}(\mathfrak{sl}_2)$-module structure on M extends to an $\mathrm{U}(\mathfrak{sl}_2)[\sigma]$-module structure for which σ acts by multiplication with ε, because σ is central in $\mathrm{U}(\mathfrak{sl}_2)[\sigma]$. It follows from the identity $\varepsilon^2 = 1$ that this $\mathrm{U}(\mathfrak{sl}_2)[\sigma]$-module structure descends to an A-module structure on M as claimed in Remark 2.4.

If M is irreducible then the resulting A-module is again irreducible since every A-submodule is in particular an $\mathfrak{sl}_2$-subrepresentation. It hence follows that the A-modules $\mathrm{L}(+, n)$ and $\mathrm{L}(-, n)$ which result from the irreducible $\mathfrak{sl}_2$-representation $\mathrm{L}(n)$ are again irreducible. These representations are pairwise non-isomorphic since the element σH of A (which corresponds to the element $\widetilde{H}$ of $\mathrm{U}_1(\mathfrak{sl}_2)$) acts on $\mathrm{L}(+, n)$ and $\mathrm{L}(-, n)$ with different eigenvalues.

Let now M be any finite-dimensional M-module. It follows from the relation $\sigma^2 = 1$ in A that the action of σ on A is diagonalizable with eigenvalues 1 and -1. We thus have

$$M = M_1 \oplus M_{-1}$$

with $M_\varepsilon := \{m \in M \mid \sigma m = \varepsilon m\}$ for $\varepsilon \in \{1, -1\}$. The action of σ on M is an A-module homomorphism because σ is central in A. The decomposition $M = M_1 \oplus M_{-1}$ is therefore a decomposition of M into A-submodules.

We may regard both M_1 and M_{-1} as $\mathfrak{sl}_2$-representations by restriction. We then have decompositions of M_1, M_{-1} into finite-dimensional irreducible $\mathfrak{sl}_2$-representations given by

$$M_1 = N_1(n_1) \oplus \cdots \oplus N_1(n_s), \quad M_{-1} = N_{-1}(n'_1) \oplus \cdots \oplus N_{-1}(n'_t).$$

with

$$N_1(n_i) \cong \mathrm{L}(n_i), \quad N_{-1}(n'_i) \cong \mathrm{L}(n_i).$$

We note that this is already a decomposition of M as an A-module since σ acts on M_1 and M_{-1} by multiplication with scalars. As A-modules we have

$$N_1(n_i) \cong \mathrm{L}(+, n_i), \quad N_{-1}(n'_i) \cong \mathrm{L}(-, n'_i).$$

This shows that every finite-dimensional A-module decomposes into a direct sum of the irreducible A-modules $\mathrm{L}(\varepsilon, n)$.

A.4. PBW-Basis for $\mathrm{U}_q(\mathfrak{sl}_2)$

We use in the following the notations introduced in Definition 3.8.

Lemma A.26. For every $r \geq 0$ we have

$$[E, F^r] = [r]_q F^{r-1} [K, 1-r]_q.$$

Proof. For $r = 0$ both sides vanish and for $r = 1$ this is one of the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$. For $r \geq 2$ the assertion follows by induction, see [Jan96, Appendix 1.3 (5)]. □

Corollary A.27. We have

$$\begin{aligned} F \cdot F^l K^m E^n &= F^{l+1} K^m E^n, \\ K^{\pm 1} \cdot F^l K^m E^n &= q^{\mp 2l} F^l K^{m \pm 1} E^n, \\ E \cdot F^l K^m E^n &= q^{-2m} F^l K^m E^{n+1} + \frac{[l]_q}{q - q^{-1}} (q^{1-l} F^{l-1} K^{m+1-l} E^n - q^{l-1} F^{l-1} K^{m+l-1} E^n). \end{aligned}$$

Proof. This follows from Lemma A.26 and the two relations $KE = q^2 EK$ and $KF = q^{-2} FK$. □

Proof of Theorem 2.6. Let U be the linear subspace of $\mathrm{U}_q(\mathfrak{sl}_2)$ spanned by these given monomials. It follows from Corollary A.27 that U is a left ideal in $\mathrm{U}_q(\mathfrak{sl}_2)$. It contains the elements $F^0 K^0 E^0 = 1$, whence U equals $\mathrm{U}_q(\mathfrak{sl}_2)$. This shows that the given monomials are a vector space generating set of $\mathrm{U}_q(\mathfrak{sl}_2)$.

The linear independence of the monomials $X^l Y^m Z^n$ is shown in the usual representation-theoretic way. Let V be the free vector space with basis

$$X^l Y^m Z^n \qquad \text{with } l, n \in \mathbb{N} \text{ and } m \in \mathbb{Z}.$$

There exists an action of $\mathrm{U}_q(\mathfrak{sl}_2)$ on V by using the formulas from Corollary A.27, with $F^l K^m E^n$ replaced by $X^l Y^m Z^n$. (It has to be checked that this proposed action is compatible with the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$, for which we refer to [Jan96, Appendix 1.5].) The elements

$$F^l K^m E^n \cdot X^0 Y^0 Z^0 = X^l Y^m Z^n$$

are linearly independent in V, whence the given monomials $F^l K^m E^n$ are linearly independent in $\mathrm{U}_q(\mathfrak{sl}_2)$. □

A.5. More on the Algebra Structure of $\mathrm{U}_q(\mathfrak{sl}_2)$

Remark A.28.

1. The universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ is noetherian and has no nonzero zero divisors. The same holds for $\mathrm{U}_q(\mathfrak{sl}_2)$, see [Kas95, Proposition VI.1.4] and [Jan96, Propositon 1.8].
2. The algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ admits a grading such that E, K, F are homogeneous with

   $$\deg(E) = 1\,, \quad \deg(K) = 0\,, \quad \deg(F) = -1\,.$$

   The degree d part of $\mathrm{U}_q(\mathfrak{sl}_2)$ has the vector space basis

   $$F^l K^m E^n \qquad \text{with } n - l = d.$$

   This grading can also be characterized in terms of the conjugation map

   $$\mathrm{U}_q(\mathfrak{sl}_2) \to \mathrm{U}_q(\mathfrak{sl}_2)\,, \quad x \mapsto KxK^{-1}\,.$$

   The degree d part of the grading is precisely the eigenspace of this conjugation map for the eigenvalue q^{2d}.

Proposition A.29.

1. There exists a unique algebra involution ω of $\mathrm{U}_q(\mathfrak{sl}_2)$ with

   $$\omega(E) = F\,, \qquad \omega(K) = K^{-1}\,, \quad \omega(F) = E\,.$$

2. There exists a unique algebra anti-involution τ of $\mathrm{U}_q(\mathfrak{sl}_2)$ with

   $$\tau(E) = E\,, \qquad \tau(K) = K^{-1}\,, \quad \tau(F) = F\,.$$

3. There exists a unique algebra isomorphism $\varphi_q \colon\ \mathrm{U}_q(\mathfrak{sl}_2) \to \mathrm{U}_{q^{-1}}(\mathfrak{sl}_2)$ with

   $$\varphi_q(E) = -F\,, \quad \varphi_q(K) = K^{-1} \quad \varphi_q(F) = -E\,.$$

   The inverse of the isomorphism φ_q is given by $\varphi_{q^{-1}}$.
4. There exist unique algebra involutions σ_E and σ_F of $\mathrm{U}_q(\mathfrak{sl}_2)$ with

   $$\sigma_E(E) = -E\,, \quad \sigma_E(K) = -K\,, \quad \sigma_E(F) = F\,.$$

   and

   $$\sigma_F(E) = E\,, \quad \sigma_F(K) = -K\,, \quad \sigma_E(F) = -F\,.$$

Proof. One checks that the proposed images of E, K, K^{-1}, F are compatible with the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$. See also [Jan96, Lemma 1.2]. □

Remark A.30.

1. One can combine the above (anti-)isomorphisms to construct further (anti-)isomorphisms involving $\mathrm{U}_q(\mathfrak{sl}_2)$ and $\mathrm{U}_{q^{-1}}(\mathfrak{sl}_2)$.
2. It follows from the existence of these (anti-)isomorphisms that many formulas and propositions involving $\mathrm{U}_q(\mathfrak{sl}_2)$ have to satisfy certain symmetries. Let us give an example:

   The algebra involution σ_E induces a (set-theoretic) involution on the class of $\mathrm{U}_q(\mathfrak{sl}_2)$-modules. For every module M the corresponding module M' has the same underlying vector space as M, the element F acts the same on both M and M', and the actions of E and K on M and M' differ only by a sign.

   It follows in particular that when the module M is irreducible, finite-dimensional and of highest weight λ, then the module M' is again irreducible and finite-dimensional, but now of highest weight $-\lambda$. This explains why in the classification of irreducible finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-modules (as given in Theorem 3.12) not only the modules $\mathrm{L}(q^n)$ occur, but also the modules $\mathrm{L}(-q^n)$. Indeed, for $M = \mathrm{L}(q^n)$ we have $M' = \mathrm{L}(-q^n)$.

A.6. Proof of Theorem 3.3

Lemma A.31. Let M be a finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module.

1. Both E and F act nilpotently on M.
2. For a sufficiently large power $r \geq 0$ (namely such that $F^r M = 0$) the module M is annihilated by the element
$$\prod_{j=-r}^{r} (K^2 - q^{2j}) .$$

Proof. See [Jan96, Proposition 2.1] and [Jan96, Proposition 2.3]. □

Proposition A.32. Every finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module decomposes into weight spaces. All occuring weights are of the form $\pm q^n$ for some $n \in \mathbb{Z}$.

Proof. Let M be a finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module and let k denote the action of K on M. It follows from Lemma A.31 that

$$0 = \prod_{n=-r}^{r} (k^2 - q^{2n}) = \prod_{n=-r}^{r} (k - q^n)(k + q^n) .$$

The roots $\pm q^n$ for $n = -r, \dots, r$ are pairwise distinct[2] whence it follows that k is diagonalizable with possible eigenvalues $\pm q^n$ for $n = -r, \dots, r$. □

[2]If $\pm q^n = \pm q^m$ then squaring both sides of this equation gives $q^{2n} = q^{2m}$ and thus $q^{2(n-m)} = 1$. It follows that $2(n - m) = 0$ because q is not a root of unity, and thus $n = m$.

A.7. Proof of Proposition 3.10

Proposition A.33.

1. The algebra $\mathrm{U}_q(\mathfrak{b})$ has the basis

$$K^n E^m \qquad \text{with } n \in \mathbb{Z} \text{ and } m \in \mathbb{N}.$$

2. The algebra $\mathrm{U}_q(\mathfrak{b})$ is given with respect to its generators E, K, K^{-1} by the relations

$$KK^{-1} = 1 = K^{-1}K\,, \quad KE = q^2 EK\,. \tag{4}$$

Proof.

1. Let U be the linear subspace of $\mathrm{U}_q(\mathfrak{sl}_2)$ spanned by the monomials $K^n E^m$ with $n, m \in \mathbb{N}$. This linear subspace is contained in $\mathrm{U}_q(\mathfrak{b})$. It follows on the other hand from the relation $KE = q^2 EK$ that

$$K^n E^m \cdot K^{n'} E^{m'} = q^{2mn'} K^{n+n'} E^{m+m'}$$

for all $n, n', m, m' \in \mathbb{N}$, and we have $1 = K^0 E^0 \in U$. This shows that U is a subalgebra of $\mathrm{U}_q(\mathfrak{sl}_2)$ containing E, K, K^{-1}, and therefore containing $\mathrm{U}_q(\mathfrak{b})$. Together this shows that $U = \mathrm{U}_q(\mathfrak{b})$.

2. Let U be the algebra given by generators E, K, K^{-1} and relations (4). There exists a unique algebra homomorphism $\varphi \colon U \to \mathrm{U}_q(\mathfrak{b})$ given by

$$\varphi(E) = E\,, \quad \varphi(K) = K\,.$$

In the same way as for Theorem 2.6 one can show that the algebra U has a PBW-basis given by the monomials

$$K^n E^m \qquad \text{with } n \in \mathbb{Z} \text{ and } m \in \mathbb{N}.$$

It follows that the algebra homomorphism φ restricts to a bijection between the PBW-bases of U and $\mathrm{U}_q(\mathfrak{b})$ and is therefore an algebra isomorphism. □

We now show an extended version of Proposition 3.10.

Proposition A.34. Let $\lambda \in \mathbb{k}^\times$.

1. We have $\mathbb{k}_\lambda \cong \mathrm{U}_q(\mathfrak{b})/\langle E, K-\lambda\rangle$ as $\mathrm{U}_q(\mathfrak{b})$-modules. Such an isomorphism is given by

$$\mathrm{U}_q(\mathfrak{b})/\langle E, K-\lambda\rangle \to \mathbb{k}_\lambda\,, \quad \overline{x} \mapsto x.1\,.$$

2. The Verma module $\mathrm{M}(\lambda)$ has the basis

$$m_i \coloneqq F^i \otimes 1 \qquad \text{with } i \geq 0,$$

and the actions of E, K, F on this basis are given by

$$Fm_i = m_{i+1}\,, \quad Km_i = \lambda q^{-2i} m_i\,, \quad Em_i = [i]_q[\lambda, 1-i]_q m_{i-1}\,.$$

This action can be graphically depicted as in Figure 3.

3. The Verma module $\mathrm{M}(\lambda)$ is of highest weight λ.
4. There exists for every $\mathrm{U}_q(\mathfrak{sl}_2)$-module M an isomorphism of vector spaces given by

$$\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{sl}_2)}(\mathrm{M}(\lambda), M) \cong \{m \in M \mid m \text{ is of weight } \lambda \text{ with } Em = 0\}.$$

   It follows in particular that

$$\mathrm{End}_{\mathrm{U}_q(\mathfrak{sl}_2)}(\mathrm{M}(\lambda)) = \Bbbk .$$

5. The Verma module $\mathrm{M}(\lambda)$ is indecomposable.
6. a. If $\lambda = \pm q^n$ for some $n \in \mathbb{N}$ then the Verma module $\mathrm{M}(\lambda)$ contains a unique nonzero, proper submodule, which is spanned by the elements

$$m_i \qquad \text{with } i \geq n+1.$$

   This submodule is isomorphic to $\mathrm{M}(\pm q^{-n-2})$.

   b. If $\lambda \neq \pm q^n$ for every $n \in \mathbb{N}$ then the Verma module $\mathrm{M}(\lambda)$ is irreducible.

Proof.

1. This follows from the PBW-basis of $\mathrm{U}_q(\mathfrak{b})$.
2. The first assertion follows from the PBW-bases of $\mathrm{U}_q(\mathfrak{sl}_2)$ and $\mathrm{U}_q(\mathfrak{b})$, and the second assertion by induction.
3. The Verma module $\mathrm{M}(\lambda)$ is generated by the primitive weight vector $1 \otimes 1$.
4. We have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{sl}_2)}(\mathrm{M}(\lambda), M) &= \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{sl}_2)}(\mathrm{U}_q(\mathfrak{sl}_2) \otimes_{\mathrm{U}_q(\mathfrak{b})} \Bbbk_\lambda, M) \\
&\cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{b})}(\Bbbk_\lambda, M) \\
&\cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{b})}(\mathrm{U}_q(\mathfrak{b})/\langle K-\lambda, E\rangle, M) \\
&\cong \{m \in M \mid (K-\lambda)m = 0, Em = 0\} \\
&= \{m \in M \mid Km = \lambda m, Em = 0\}.
\end{aligned}$$

5. The endomorphism ring $\mathrm{End}_{\mathrm{U}_q(\mathfrak{sl}_2)}(\mathrm{M}(\lambda)) = \Bbbk$ does not admit any non-trivial idempotents.
6. This follows as for $\mathrm{U}(\mathfrak{sl}_2)$ since $[i]_q[\lambda, i-1]_q = 0$ if and only if $\lambda = \pm q^{i-1}$. □

A.8. Proof of Theorem 3.14

Lemma A.35. If M is an highest-weight $\mathrm{U}_q(\mathfrak{sl}_2)$-module then

$$\mathrm{End}_{\mathrm{U}_q(\mathfrak{sl}_2)}(M) = \Bbbk .$$

Definition A.36. The *quantum Casimir element* is the element $C \in \mathrm{U}_q(\mathfrak{sl}_2)$ given by

$$C := EF + \frac{Kq^{-1} + K^{-1}q}{(q-q^{-1})^2} .$$

Lemma A.37.

1. The element C is central in $\mathrm{U}_q(\mathfrak{sl}_2)$.
2. The element C acts on every $\mathrm{U}_q(\mathfrak{sl}_2)$-module by module endomorphisms.
3. Let $\lambda \in \mathbb{k}^\times$ and let M be an $\mathrm{U}_q(\mathfrak{sl}_2)$-module of highest weight λ. The element C acts on M by multiplication with the scalar
$$\frac{\lambda q + \lambda^{-1} q^{-1}}{(q - q^{-1})^2}\,.$$
This holds in particular for $M = \mathrm{L}(\lambda)$.
4. The element C acts the same on $\mathrm{L}(\lambda)$ and $\mathrm{L}(\mu)$ if and only if $\lambda = \mu$ or $\lambda = \mu^{-1} q^{-2}$.

Proof.

1. It suffices to check that the element C commutes with the elements E, K, F. This follows from the defining relations for $\mathrm{U}_q(\mathfrak{sl}_2)$.
2. This follows from the previous assertion.
3. It suffices to show the assertion for the Verma module $\mathrm{M}(\lambda)$ because M is a quotient of $\mathrm{M}(\lambda)$. For $\mathrm{M}(\lambda)$ the assertion follows from Proposition 3.10.
4. This follows from the previous assertion. □

Corollary A.38. The quantum Casimir element C acts on every finite-dimensional irreducible representation of $\mathrm{U}_q(\mathfrak{sl}_2)$ by a different scalar.

Proof. If $\lambda = \delta q^n$ and $\mu = \varepsilon q^m$ with $\delta, \varepsilon \in \{1, -1\}$ and $n, m \in \mathbb{N}$ then it cannot happen that $\lambda = \mu^{-1} q^{-2}$. The assertion thus follows from Lemma A.37. □

Proof of Theorem 3.14 ([Jan96, Theorem 2.9]). Let M be any finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module and let c denote the action of C on M. We may assume that M is indecomposable. We consider a composition series
$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_r = M \tag{5}$$
with composition factors
$$M_i / M_{i-1} \cong \mathrm{L}(\varepsilon_i q^{n_i})\,.$$
Letting c_i be the scalar by which C acts on $\mathrm{L}(\varepsilon_i q^{n_i})$, we have
$$(c - c_i) M_i \subseteq M_{i-1}\,.$$
It follows that $\prod_{i=1}^r (c - c_i)$ annihilates M. Therefore c admits a generalized eigenspace decomposition with eigenvalues $c_1, \dots, c_r$. The resulting generalized eigenspaces are subrepresentations because c is a $\mathrm{U}_q(\mathfrak{sl}_2)$-module endomorphism. It follows that
$$c_1 = \cdots = c_r$$
because M is indecomposable, and thus
$$\varepsilon_1 q^{n_1} = \cdots = \varepsilon_r q^{n_r} =: \lambda$$

by Corollary A.38. It therefore follows from the composition series (5) that

$$\dim(M_\mu) = r \dim(\mathrm{L}(\lambda)_\mu)$$

for every scalar $\mu \in \Bbbk^\times$. Thus M is of highest weight λ.

The short exact sequence of $\mathrm{U}_q(\mathfrak{sl}_2)$-modules

$$0 \to M_{r-1} \to M \to \mathrm{L}(\lambda) \to 0 \tag{6}$$

restricts to a short exact sequence of vector spaces

$$0 \to (M_{r-1})_\lambda \to M_\lambda \to \mathrm{L}(\lambda)_\lambda \to 0\,.$$

It follows that the primitive generator $\overline{1 \otimes 1}$ of $\mathrm{L}(\lambda)$ has a preimage m_0 in M. The weight vector m_0 is primitive because M is of highest weight λ. It follows that there exists a homomorphism of $\mathrm{U}_q(\mathfrak{sl}_2)$-modules

$$\varphi\colon\ \mathrm{M}(\lambda) \to M\,, \quad 1 \otimes 1 \mapsto m_0\,.$$

It further follows from the finite-dimensionality of M that φ factors through a homomorphism

$$\psi\colon\ \mathrm{L}(\lambda) \to M\,, \quad \overline{1 \otimes 1} \mapsto m_0\,.$$

This shows that the short exact sequence (6) splits, whence

$$M \cong M_{r-1} \oplus \mathrm{L}(\lambda)\,.$$

It follows by induction that $M_{r-1} \cong \mathrm{L}(\lambda)^{\oplus(r-1)}$ and thus altogether $M \cong \mathrm{L}(\lambda)^{\oplus r}$. □

Remark A.39. The center of the universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ is a polynomial algebra, generated by the classical Casimir element $C = (ef + h^2 + fe)/4$. It can be shown that the center of $\mathrm{U}_q(\mathfrak{sl}_2)$ is again a polynomial algebra, now generated by the quantum Casimir element C. We refer to [Jan96, Proposition 2.18] for more details on this.

A.9. Proof of Lemma 4.4

We have

$$M_\mu \otimes N_\kappa \subseteq (M \otimes N)_{\mu\kappa}$$

for all $\mu, \kappa \in \Bbbk^\times$ since the element K is group-like in $\mathrm{U}_q(\mathfrak{sl}_2)$. Both M and N admits weight space decompositions

$$M = \bigoplus_\mu M_\mu\,, \quad N = \bigoplus_\kappa N_\kappa$$

and it follows that

$$M \otimes N = \left(\bigoplus_\mu M_\mu\right) \otimes \left(\bigoplus_\kappa N_\kappa\right) = \bigoplus_{\mu,\kappa} (M_\mu \otimes N_\kappa) \subseteq \bigoplus_\lambda M_\lambda \subseteq M \otimes N$$

It follows with the inclusions $M_\mu \otimes N_\kappa \subseteq (M \otimes N)_{\mu\kappa}$ that for every $\lambda \in \Bbbk^\times$ we already have the equality

$$(M \otimes N)_\lambda = \bigoplus_{\mu\kappa = \lambda} M_\mu \otimes N_\kappa\,.$$

A.10. Clebsch–Gordan for $\mathrm{U}_q(\mathfrak{sl}_2)$

Proposition A.40. For all $\delta, \varepsilon \in \{1, -1\}$ and $n, m \in \mathbb{N}$ with $n \geq m$ we have

$$\mathrm{L}(\delta q^n) \otimes \mathrm{L}(\varepsilon q^m) \cong \mathrm{L}(\delta\varepsilon q^{n+m}) \oplus \mathrm{L}(\delta\varepsilon q^{n+m-2}) \oplus \cdots \oplus \mathrm{L}(\delta\varepsilon q^{n-m})\,.$$

Proof. This follows from Corollary 3.15 and Lemma 4.4. □

B. A Glimpse of Deformation Theory

B.1. Background on Completions

Definition B.1. An $\Bbbk[\![\hbar]\!]$-module M is *complete* if for every sequence of elements $m_0, m_1, \ldots \in M$ with

$$m_{n+1} \equiv m_n \pmod{\hbar^{n+1}} \qquad \text{for every } n \geq 0$$

there exists a unique element m of M with

$$m \equiv m_n \pmod{\hbar^{n+1}} \qquad \text{for every } n \geq 0.$$

Definition B.2. Let M be an $\Bbbk[\![\hbar]\!]$-module.

1. The *$\hbar$-adic completion* of M is the $\Bbbk[\![\hbar]\!]$-module

$$\widehat{M} := \lim_{n\geq 0}(M/\hbar^{n+1}M) = \left\{ (m_n)_{n\geq 0} \,\middle|\, \begin{array}{l} m_n \in M/\hbar^{n+1}M \text{ with} \\ m_{n+1} \equiv m_n \pmod{\hbar^{n+1}} \text{ for every } n \geq 0 \end{array} \right\}.$$

2. The *canonical homomorphism* $M \to \widehat{M}$ is given by $m \mapsto (\overline{m}, \overline{m}, \ldots)$.

Proposition B.3. Let M be an $\Bbbk[\![\hbar]\!]$-module.

1. The completion $\widehat{M}$ is complete.
2. The module M is complete if and only if the canonical homomorphism $M \to \widehat{M}$ is an isomorphism.

Remark B.4. Let M be a complete $\Bbbk[\![\hbar]\!]$-module. Every sequence $(m_i)_{i\geq 0}$ of elements $m_i \in M$ defines a sequence $(s_n)_{n\geq 0}$ of partial sums

$$s_n := \sum_{i=0}^{n} \hbar^i m_i\,.$$

for every $n \geq 0$. These partial sums satisfy the relation

$$s_{n+1} \equiv s_n \pmod{\hbar^{n+1}} \qquad \text{for every } n \geq 0\,.$$

There hence exists by the completeness of M a unique element $\sum_{i=0}^{\infty} \hbar^i m_i$ of M with

$$\sum_{i=0}^{\infty} \hbar^i m_i \equiv \sum_{i=0}^{n} \hbar^i m_i \pmod{\hbar^{n+1}} \qquad \text{for every } n \geq 0.$$

Example B.5.

1. Every finite-dimensional $\Bbbk[\![\hbar]\!]$-module M is complete since $\hbar^{n+1}M = 0$ for some sufficiently large power n.[3]
2. For every $\Bbbk$-vector space V the resulting $\Bbbk[\![\hbar]\!]$-module $V[\![\hbar]\!]$ is complete. For every sequence $v_0, v_1, \dots$ of elements $v_i \in V$ we have

$$\sum_{i=0}^{\infty} \hbar^i v_i = \sum_{i=0}^{\infty} v_i \hbar^i \,.$$

Proposition B.6. Let M, N be two $\Bbbk[\![\hbar]\!]$-modules.

1. For every homomorphism of $\Bbbk[\![\hbar]\!]$-modules $f : M \to N$ there exists a unique module homomorphism $\widehat{f} : \widehat{M} \to \widehat{N}$ that makes the following square diagram commute.

$$\begin{tikzcd} \widehat{M} \arrow[r, "\widehat{f}"] & \widehat{N} \\ M \arrow[u] \arrow[r, "f"] & N \arrow[u] \end{tikzcd}$$

   The homomorphism $\widehat{f}$ is given by

$$\widehat{f}\Big((\overline{m_0}, \overline{m_1}, \dots)\Big) = \Big(\overline{f(m_0)}, \overline{f(m_1)}, \dots\Big) \,.$$

2. The assignment $\widehat{(-)}$ defines a functor

$$\widehat{(-)} : \Bbbk[\![\hbar]\!]\text{-}\mathbf{Mod} \to \Bbbk[\![\hbar]\!]\text{-}\mathbf{Mod} \,.$$

3. If M, N are complete then

$$f\left(\sum_{i=0}^{\infty} \hbar^i m_i\right) = \sum_{i=0}^{\infty} \hbar^i f(m_i)$$

   for every sequence of elements $m_0, m_1, \dots \in M$.
4. If N is complete then every homomorphism $M \to N$ extends uniquely to a homomorphism $\widehat{M} \to N$.

$$\begin{tikzcd} \widehat{M} \arrow[r, dashed, "\exists!"] & N \\ M \arrow[u] \arrow[ur] & \end{tikzcd}$$

[3] Indeed, it follows from the finite-dimensionality of M that the annihilator of M is a nonzero ideal in $\Bbbk[\![\hbar]\!]$. It is of the form $(\hbar^n)$ for some $n \geq 0$, because every ideal in $\Bbbk[\![\hbar]\!]$ is of the form. Thus $\hbar^n M = 0$.

5. If V is any $\Bbbk$-vector space and N is complete then every $\Bbbk$-linear map $f : V \to N$ extends uniquely to a $\Bbbk[\![\hbar]\!]$-linear linear map $V[\![\hbar]\!] \to N$.

$$\begin{tikzcd} V[\![\hbar]\!] \arrow[r, dashed, "\exists!"] & N \\ V \arrow[u] \arrow[ur] & \end{tikzcd}$$

   The homomorphism f' is given by

$$f'\left(\sum_{i=0}^{\infty} v_i \hbar^i\right) = \sum_{i=0}^{\infty} \hbar^i f(v_i)\,.$$

6. The canonical homomorphism $M \to \widehat{M}$ induces an isomorphism of $\Bbbk$-vector spaces

$$M/\hbar M \longrightarrow \widehat{M}/\hbar\widehat{M}\,.$$

Remark B.7 (The $\hbar$-adic topology). Let M be a $\Bbbk[\![\hbar]\!]$-module.

1. There exists a unique topology on M for which a basis is given by the sets

$$m + \hbar^{n+1}M \qquad \text{with } m \in M \text{ and } n \geq 0.$$

   This topology is the *$\hbar$-adic topology* on M. It makes $\Bbbk[\![\hbar]\!]$ into a topological ring and every $\Bbbk[\![\hbar]\!]$-module into a topological $\Bbbk[\![\hbar]\!]$-module.

2. The $\hbar$-adic topology on M is generated by a semimetric $d_\hbar$ that is given by

$$d_\hbar(m, m') := \inf\left\{\frac{1}{n} \;\middle|\; n \geq 1, m \equiv m' \pmod{\hbar^{n+1}}\right\}.$$

   The semimetric $d_\hbar$ is already an ultra-semimetric. More explicitely, it satisfies for any three elements $m, m', m'' \in M$ the inequality

$$d_\hbar(m, m'') \leq \max(d_\hbar(m, m'), d_\hbar(m', m''))\,.$$

   This is a strengthening of the usual triangle inequality.

3. The module M is complete if and only if it is complete with respect to the semimetric $d_\hbar$, i.e. if and only if every Cauchy-sequence in M has a unique limit in M. This entails that the semimetric $d_\hbar$ is actually a metric (by the uniqueness of these limits).

   The completion $\widehat{M}$ together with the canonical homomorphism $M \to \widehat{M}$ is a completion of M in the usual topological sense.

4. The $\hbar$-adic topology on M is Hausdorff if and only if

$$\bigcap_{n \geq 0} \hbar^{n+1} M = \{0\}\,.$$

   This is in particular the case if M is complete. This can be seen topologically, since $d_\hbar$ is then a metric, and algebraically, by considering the explicit description of the completion $\widehat{M} \cong M$.

Definition B.8. Let M, N be two $\Bbbk[\![\hbar]\!]$-modules. The *completed tensor product*

$$M \mathbin{\widehat{\otimes}} N$$

is the $\hbar$-adic completion of the tensor product $M \otimes_{\Bbbk[\![\hbar]\!]} N$.

Proposition B.9. Let V, W be two $\Bbbk$-vector spaces. Then the $\Bbbk[\![\hbar]\!]$-linear map

$$V[\![\hbar]\!] \otimes_{\Bbbk[\![\hbar]\!]} W[\![\hbar]\!] \to (V \otimes W)[\![\hbar]\!] , \quad \left(\sum_{i=0}^{\infty} v_i \hbar^i\right) \otimes \left(\sum_{j=0}^{\infty} w_j \hbar^j\right) \mapsto \sum_{i,j=0}^{\infty} (v_i \otimes w_j) \hbar^{i+j}$$

extends along the canonical homomorphism

$$V[\![\hbar]\!] \otimes_{\Bbbk[\![\hbar]\!]} W[\![\hbar]\!] \to V[\![\hbar]\!] \mathbin{\widehat{\otimes}} W[\![\hbar]\!]$$

to an isomorphism of $\Bbbk[\![\hbar]\!]$-modules

$$V[\![\hbar]\!] \mathbin{\widehat{\otimes}} W[\![\hbar]\!] \to (V \otimes W)[\![\hbar]\!] .$$

B.2. Deformations of Algebras

The following is taken (at least in spirit) from [Bel18, §5.2] and [GS92].

Motivation B.10. Deforming a $\Bbbk$-algebra A means – roughly speaking – that the multiplication on A is replaced by a perturbated multiplication $*$, in the sense that for all $a, b \in A$,

$$a * b = ab + \mu_1(a,b)\hbar + \mu_2(a,b)\hbar^2 + \cdots$$

for some bilinear terms $\mu_i(a,b)$. Here $\hbar$ may be thought about as a "sufficiently small number". The limit $\hbar \to 0$ does then give back the original algebra A.

Definition B.11. Let A be an $\Bbbk$-algebra.

1. A *(formal) deformation* of A is an $\Bbbk[\![\hbar]\!]$-algebra $A_\hbar$ whose underlying $\Bbbk[\![\hbar]\!]$-module is $A[\![\hbar]\!]$ and for which $A_\hbar / \hbar A_\hbar = A$ as algebras.
2. Two deformations $A_\hbar$, $A'_\hbar$ of A are *equivalent* if there exists an isomorphism of $\Bbbk[\![\hbar]\!]$-algebras

$$\varphi : A_\hbar \to A'_\hbar$$

   such that the induced isomorphism of $\Bbbk$-algebras

$$A = A_\hbar / \hbar A_\hbar \to A'_\hbar / \hbar A'_\hbar = A$$

   is the identity of A. In other words, the isomorphism φ is the identity of A modulo $\hbar$.

Remark B.12. Every $\Bbbk[\![\hbar]\!]$-bilinear multiplication

$$(-) * (-) \colon\ A[\![\hbar]\!] \times A[\![\hbar]\!] \to A[\![\hbar]\!] .$$

satisfies the equality

$$\left(\sum_{i=0}^{\infty} a_i \hbar^i\right) * \left(\sum_{j=0}^{\infty} b_j \hbar^j\right) = \sum_{i,j=0}^{\infty} (a_i * b_j) \hbar^{i+j}$$

by the completeness of $A[\![\hbar]\!]$. The multiplication $*$ can therefore be characterized by the sequence of $\Bbbk$-bilinear maps $\mu_i \colon\ A \times A \to A$ such that

$$a * b = \mu_0(a, b) + \mu_1(a, b)\hbar + \mu_2(a, b)\hbar^2 + \cdots$$

The condition $A[\![\hbar]\!]/\hbar A[\![\hbar]\!] = A$ means that μ_0 is the original multiplication on A, whence

$$a * b = ab + \mu_1(a, b)\hbar + \mu_2(a, b)\hbar^2 + \cdots$$

That the multiplication $*$ is associative gives certain compatibility conditions on the μ_i, which we won't discuss here.

Example B.13. Every $\Bbbk$-algebra A admits the *trivial deformation* $A[\![\hbar]\!]$ (i.e. the algebra of power series with its usual product). It corresponds to the choice $\mu_1, \mu_2, \ldots = 0$.

Definition B.14. A deformation of an $\Bbbk[\![\hbar]\!]$-algebra A is *trivial* if it is equivalent to the trivial deformation (i.e. equivalent to the algebra of power series $A[\![\hbar]\!]$).

Theorem B.15. The universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ admits a deformation such that

$$[H, E] = 2E\,, \quad [H, F] = -2F\,, \quad [E, F] = \frac{\mathrm{e}^{\hbar H} - \mathrm{e}^{-\hbar H}}{\mathrm{e}^{\hbar} - \mathrm{e}^{-\hbar}}\,. \tag{7}$$

Proof (sketch). Let P be the free $\Bbbk$-algebra on the generators E, H, F. Let I be the two-sided ideal in $P[\![\hbar]\!]$ given by the relations (7). Let J be the closure of I in the $\hbar$-adic topology of $P[\![\hbar]\!]$. Then J is again a two-sided ideal in $P[\![\hbar]\!]$ and the described deformation can be realized as the quotient $P[\![\hbar]\!]/J$. We refer to [CP95, Definition-Proposition 6.4.3 ff.] for the specific details. □

Definition B.16. The deformation of $\mathrm{U}(\mathfrak{sl}_2)$ from Theorem B.15 is denoted by $\mathrm{U}_\hbar(\mathfrak{sl}_2)$.

Remark B.17.

1. In the algebra $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ we can consider the well-defined elements

$$q \coloneqq \mathrm{e}^{\hbar}\,, \quad K \coloneqq \mathrm{e}^{\hbar H}\,.$$

   The elements q, E, K, K^{-1}, F satisfy the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$, and we want to regard $\mathrm{U}_q(\mathfrak{sl}_2)$ as a subalgebra of $\mathrm{U}_\hbar(\mathfrak{sl}_2)$. But we run into a small technical problem. The algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ needs to be defined over a field which contains the element q. But the algebra $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ is defined over $\Bbbk[\![\hbar]\!]$, and this ring does not contain any field containing q.

To fix this problem we extend $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ to an $\Bbbk(\!(\hbar)\!)$-algebra and then define $\mathrm{U}_q(\mathfrak{sl}_2)$ over the subfield $\Bbbk(q)$ of $\Bbbk(\!(\hbar)\!)$.[4] We thus consider the extension of scalars

$$\Bbbk(\!(\hbar)\!) \otimes_{\Bbbk[\![\hbar]\!]} \mathrm{U}_\hbar(\mathfrak{sl}_2)\,,$$

which is given as an $\Bbbk(\!(\hbar)\!)$-module by

$$\Bbbk(\!(\hbar)\!) \otimes_{\Bbbk[\![\hbar]\!]} \mathrm{U}_\hbar(\mathfrak{sl}_2) = \Bbbk[\![\hbar]\!][\hbar^{-1}] \otimes_{\Bbbk[\![\hbar]\!]} \mathrm{U}(\mathfrak{sl}_2)[\![\hbar]\!] \cong \mathrm{U}(\mathfrak{sl}_2)[\![\hbar]\!][\hbar^{-1}] \cong \mathrm{U}(\mathfrak{sl}_2)(\!(\hbar)\!)\,.$$

The field $\Bbbk(\!(\hbar)\!)$ contains the subfield $\Bbbk(q)$, and we get from the above observation an homomorphism of $\Bbbk(q)$-algebras

$$\mathrm{U}_q(\mathfrak{sl}_2) \to \Bbbk(\!(\hbar)\!) \otimes_{\Bbbk[\![\hbar]\!]} \mathrm{U}(\mathfrak{sl}_2)$$

where $\mathrm{U}_q(\mathfrak{sl}_2)$ is defined over $\Bbbk(q)$.

2. In $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ we have both the original element H of $\mathfrak{sl}_2$ as well as the new element

$$\widetilde{H} = \frac{K - K^{-1}}{q - q^{-1}}\,,$$

which is of the form

$$\widetilde{H} = H + \text{terms of order at least } \hbar^2\,.$$

We may think about the element $\widetilde{H}$ is a deformation of the original element H (in an informal sense). We note that

$$q \equiv 1\,, \quad K \equiv 1\,, \quad \widetilde{H} \equiv H \pmod{\hbar}.$$

Remark B.18. One can study the deformation theory of a $\Bbbk$-algebra A through (co)homological methods. The *Hochschild cochain complex* of A is given by the vector spaces

$$\mathrm{C}^n_{\mathrm{Hoch}}(A) := \mathrm{Hom}_{\Bbbk}(A^{\otimes n}, A)$$

together with certain differentials. The cohomology of this chain complex is the *Hochschild cohomology* of A, which is denoted by

$$\mathrm{HH}^n(A) := \mathrm{H}^n(\mathrm{C}^\bullet_{\mathrm{Hoch}})\,.$$

One of the relations between deformation theory and Hochschild cohomology is that in the case of

$$\mathrm{HH}^2(A) = 0$$

every deformation of A is trivial, see [GS92, Theorem 2].

Warning B.19. Let A be a $\Bbbk$-algebra with $\mathrm{HH}^2(A) = 0$, and let $A_\hbar$ be a deformation of A. The above criterion shows that $A_\hbar$ is equivalent to $A[\![\hbar]\!]$, but it does not provide an explicit isomorphism that realizes this equivalence.

[4] Here $\Bbbk(\!(\hbar)\!)$ denotes the field of Laurent series with coefficients in $\Bbbk$. Its elements are series of the form $\sum_{i=n}^{\infty} a_i \hbar^i$ with $n \in \mathbb{Z}$ and $a_i \in \Bbbk$ for every $i \geq n$. The field of Laurent series $\Bbbk(\!(\hbar)\!)$ is the field of fractions of the ring of power series $\Bbbk[\![\hbar]\!]$, and to pass from $\Bbbk[\![\hbar]\!]$ to $\Bbbk(\!(\hbar)\!)$ it sufficies to localize at $\hbar$.

Example B.20.

1. Let $\mathfrak{g}$ be a semisimple Lie algebra over a field $\Bbbk$ of characteristic zero. It can be shown that

$$\mathrm{HH}^2(\mathrm{U}(\mathfrak{g})) = 0\,,$$

see [GS92, Theorem 2] or [Sch16, Exercise 2.8.1, Bonus]. Therefore all deformations of $\mathrm{U}(\mathfrak{g})$ (as an algebra) are trivial.

2. It follows in particular that every algebra deformation of $\mathrm{U}(\mathfrak{sl}_2)$ is trivial. An explicit equivalence between $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ and $\mathrm{U}(\mathfrak{sl}_2)[\![\hbar]\!]$ is constructed in [CP95, Proposition 4.6.4].

B.3. Deformation of Hopf Algebras

The following is taken mostly from [CP95, Chapter 6].

Definition B.21.

1. A *topological Hopf algebra* consists of a complete $\Bbbk[\![\hbar]\!]$-module A together with $\Bbbk[\![\hbar]\!]$-linear maps

$$m\colon\; A \mathbin{\widehat{\otimes}} A \to A\,, \quad u\colon\; \Bbbk[\![\hbar]\!] \to A\,, \quad \Delta\colon\; A \to A \mathbin{\widehat{\otimes}} A\,, \quad \varepsilon\colon\; A \to \Bbbk[\![\hbar]\!]\,, \quad S\colon\; A \to A$$

such that the usual Hopf algebra diagrams commute.

2. The terms *topological algebra*, *topological coalgebra* and *topological bialgebra* are defined in the same way.

Remark B.22.

1. A topological algebra in the sense of Definition B.21 is precisely the same as an $\Bbbk[\![\hbar]\!]$-algebra that is complete as an $\Bbbk[\![\hbar]\!]$-module.

   Indeed, suppose first that (A, m, u) is a topological algebra. Then the multiplication

$$m\colon\; A \mathbin{\widehat{\otimes}} A \to A$$

restricts via the composition with the canonical homomorphism

$$A \otimes A \to A \mathbin{\widehat{\otimes}} A$$

to a multiplication

$$m'\colon\; A \otimes A \to A\,.$$

Then (A, m', u) is an $\Bbbk[\![\hbar]\!]$-algebra (and A is by definition complete).

   Suppose on the other hand that (A, m', u) is an $\Bbbk[\![\hbar]\!]$-algebra where A is complete. Then the multiplication map

$$m'\colon\; A \otimes A \to A$$

extends by Proposition B.6 and the completeness of A uniquely to a $\Bbbk[\![\hbar]\!]$-linear map

$$m\colon\; A \mathbin{\widehat{\otimes}} A \to A\,.$$

Then (A, m, u) is a topological algebra.[5]

2. A topological coalgebra C on the other hand is in general not an actual coalgebra, since the comultiplication
$$\Delta \colon C \to C \mathbin{\widehat{\otimes}} C$$
does not necessarily restrict to a map $C \to C \otimes C$. This problem is inherited by topological bialgebras and topological Hopf algebras.

3. If A is a topological Hopf algebra then $A/\hbar A$ becomes an Hopf algebra over $\Bbbk$ (in the usual, non-topological sense) because
$$(A \mathbin{\widehat{\otimes}} A)/\hbar(A \mathbin{\widehat{\otimes}} A) \cong (A \otimes_{\Bbbk[\![\hbar]\!]} A)/\hbar(A \otimes_{\Bbbk[\![\hbar]\!]} A) \cong (A/\hbar A) \otimes_{\Bbbk} (A/\hbar A)\,.$$ [6]

Definition B.23. Let A be a Hopf algebra.

1. A *(formal) deformation* of A is a topological Hopf algebra $A_\hbar$ whose underlying $\Bbbk[\![\hbar]\!]$-module is $A[\![\hbar]\!]$ and for which $A_\hbar/\hbar A_\hbar = A$ as Hopf algebras.
2. Two Hopf algebra deformations $A_\hbar$ and $A'_\hbar$ of A are *equivalent* if there exists an isomorphism of Hopf algebras
$$\varphi \colon A_\hbar \to A'_\hbar$$
such that the induced isomorphism of Hopf algebras
$$A = A_\hbar/\hbar A_\hbar \to A'_\hbar/\hbar A'_\hbar = A$$
is the identity of A, i.e. φ is the identity of A modulo $\hbar$.
3. (Formal) deformations of coalgebras and bialgebras are defined in the same way as for algebras and Hopf algebras. Equivalence of deformations of coalgebras and bialgebras is also defined in the same way as for algebras and Hopf algebras.
4. A Hopf algebra deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a *quantum universal enveloping algebra.*

Remark B.24. Let A be a Hopf algebra over $\Bbbk$ with and let $A_\hbar$ be a deformation of A. By using the isomorphism
$$A[\![\hbar]\!] \mathbin{\widehat{\otimes}} A[\![\hbar]\!] \cong (A \otimes A)[\![\hbar]\!]$$
we can regard the structure maps of $A_\hbar$ as $\Bbbk[\![\hbar]\!]$-linear map
$$\begin{aligned} m_\hbar &\colon (A \otimes A)[\![\hbar]\!] \to A[\![\hbar]\!]\,, \\ u_\hbar &\colon \Bbbk[\![\hbar]\!] \to A[\![\hbar]\!]\,, \\ \Delta_\hbar &\colon A[\![\hbar]\!] \to (A \otimes A)[\![\hbar]\!]\,, \\ \varepsilon_\hbar &\colon A[\![\hbar]\!] \to \Bbbk[\![\hbar]\!]\,, \\ S_\hbar &\colon A[\![\hbar]\!] \to A[\![\hbar]\!] \end{aligned} \tag{8}$$

[5] For the commutativity of the associativity diagram one needs to check that two maps $A \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}} A \to A$ coincide. It follows from the completeness of A and the denseness of $A \otimes A \otimes A$ in $A \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}} A$ that is sufficies to check this equality on elements of the non-completed tensor product $A \otimes A \otimes A$. There is holds by assumption.

[6] Recall from commutative algebra that if R is a commutative ring, S is a commutative R-algebra and M, N are two R-modules then $S \otimes_R (M \otimes_R N) \cong (S \otimes_R M) \otimes_S (S \otimes_R N)$. In other words, extension of scalars preserves tensor products. For $R = \Bbbk[\![\hbar]\!]$ and $S = \Bbbk[\![\hbar]\!]/(\hbar) = \Bbbk$ this gives the second isomorphism.

which are perturbations of the structure maps of A, i.e. they reduce modulo $\hbar$ to the original structure maps of A.

We can for example characterize the comultiplication of $A_\hbar$ by a sequence of $\Bbbk$-bilinear maps

$$\Delta_i : A \to A \otimes A$$

such that

$$\Delta_\hbar(a) = \Delta_0(a) + \Delta_1(a)\hbar + \Delta_2(a)\hbar^2 + \cdots$$

for every $a \in A$. Here Δ_0 needs to be the original comultiplication from A, and the coassociativity of $\Delta_\hbar$ gives certain conditions on the bilinear maps Δ_i.

Example B.25.

1. Every Hopf algebra A admits the trivial deformation $A[\![\hbar]\!]$. In the form (8) the structure maps of this deformation are given by the $\Bbbk[\![\hbar]\!]$-linear extensions of the structure maps of A. A deformation is *trivial* if it is equivalent to the trivial deformation (i.e. $A[\![\hbar]\!]$).
2. One can make the algebra deformation $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ of $\mathrm{U}(\mathfrak{sl}_2)$ into a Hopf algebra deformation via the comultiplication

$$\Delta_\hbar(H) = H \otimes 1 + 1 \otimes H\,, \quad \Delta_\hbar(E) = E \otimes K + 1 \otimes E\,, \quad \Delta_\hbar(F) = F \otimes 1 + K^{-1} \otimes F\,,$$

the counit

$$\varepsilon_\hbar(H) = 0\,, \quad \varepsilon_\hbar(E) = 0\,, \quad \varepsilon_\hbar(F) = 0\,,$$

and the antipode

$$S_\hbar(H) = -H\,, \quad S_\hbar(E) = -K^{-1}E\,, \quad S_\hbar(F) = -FK\,.$$

We note that it follows from these formulas for the element $K = \mathrm{e}^{\hbar H}$ that

$$\Delta_\hbar(K) = K \otimes K\,, \quad \varepsilon_\hbar(K) = 1\,, \quad S_\hbar(K) = K^{-1}\,.$$

For the elements q, E, K, K^{-1}, F in $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ we thus regain the formulas for the Hopf algebra structure of $\mathrm{U}_q(\mathfrak{sl}_2)$.

We finish this overview by explaining how the irreducible, finite-dimensional representations $\mathrm{L}(n)$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ can be used to construct the irreducible, finite-dimensional representations $\mathrm{L}(q^n)$ of $\mathrm{U}_q(\mathfrak{sl}_2)$, where $n \in \mathbb{N}$.

Theorem B.26 ([CP95, Proposition 6.4.10]). For every natural number $n \in \mathbb{N}$ let $V(n)$ be the free $\Bbbk[\![\hbar]\!]$-module of rank $n+1$ with basis $v_0, \dots, v_n$.

1. There exists a unique $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-module structure on $V(n)$ such that

$$Hv_i := (n-2i)v_i\,, \quad Ev_i := [n-i+1]_q v_{i-1}\,, \quad Fv_i := [i+1]_q v_{i+1}\,.$$

2. The $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-modules $V(n)$ is indecomposable.
3. The $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-module $V(n)$ reduces modulo $\hbar$ to the irreducible representations $\mathrm{L}(n)$ of $\mathrm{U}(\mathfrak{sl}_2)$.

4. The actions of K and $\widetilde{H}$ on $V(n)$ are given by

$$K v_i = q^{n-2i} v_i \,, \quad \widetilde{H} v_i = [n-2i]_q v_i \,.$$

It follows that the $\mathrm{U}_q(\mathfrak{sl}_2)$-module $\mathrm{L}(q^n)$ can (up to isomorphism of $\mathrm{U}_q(\mathfrak{sl}_2)$-modules) be realized as the $\mathrm{U}_q(\mathfrak{sl}_2)$-submodule of $\Bbbk(\!(\hbar)\!) \otimes_{\Bbbk[\![\hbar]\!]} V(n)$ given by $\langle 1 \otimes v_0, \dots, 1 \otimes v_n \rangle_{\Bbbk(q)}$.

Remark B.27. We have here only considered a specific deformation $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ of $\mathrm{U}(\mathfrak{sl}_2)$. But this choice of a deformation of $\mathrm{U}(\mathfrak{sl}_2)$ is not canonical, and there are also other deformations one can consider. One such deformation, as explained in [CP95, §6.4, F], is given by

$$\begin{gathered} [H,E] = 4(1 - \mathrm{e}^{-2\hbar E})/\hbar \,, \quad [H,F] = -2F - \hbar H^2 \,, \quad [E,F] = H \,, \\ \Delta(E) = E \otimes 1 + 1 \otimes E \,, \quad \Delta(H) = H \otimes 1 + \mathrm{e}^{-2\hbar E} \otimes H \,, \quad \Delta(F) = F \otimes 1 + \mathrm{e}^{-2\hbar E} \otimes F \,, \\ \varepsilon(E) = 0 \,, \quad \varepsilon(H) = 0 \,, \quad \varepsilon(F) = 0 \,, \\ S(E) = -E \,, \quad S(H) = -\mathrm{e}^{2\hbar E} H \,, \quad S(F) = -\mathrm{e}^{2\hbar E} F \,. \end{gathered}$$

References

[Bel18] Pieter Belmans. *Hochschild (co)homology, and the Hochschild–Kostant–Rosenberg decomposition*. Advanced topics in algebra (V5A5). July 9, 2018. 97 pp. URL: https://pbelmans.ncag.info/teaching/hh-2018 (visited on January 19, 2020). unpublished.

[CP95] Vyjayanthi Chari and Andrew N. Pressley. *A Guide to Quantum Groups*. First paperback edition (with corrections). Cambridge University Press, July 1995. xvi+651 pp. ISBN: 978-0-521-55884-6.

[GS92] Murray Gerstenhaber and Samuel D. Schack. “Algebras, Bialgebras, Quantum Groups and Algebraic Deformations”. In: *Deformation Theory and Quantum Groups with Applications to Mathematical Physics*. AMS-IMS-SIAM Joint Summer Research Conference on Deformation Theory of Algebras and Quantization with Applications to Physics (University of Massachusetts, July 14–20, 1990). Ed. by Murray Gerstenhaber and Jim Stasheff. Contemporary Mathematics 134. American Mathematical Society, 1992, pp. 51–93. ISBN: 978-0-8218-5141-8. DOI: 10.1090/conm/134.

[Hum72] James Humphreys. *Introduction to Lie Algebras and Representation Theory*. Graduate Texts in Mathematics 9. New York: Springer Verlag, 1972. xiii+173 pp. ISBN: 978-0-387-90053-7. DOI: 10.1007/978-1-4612-6398-2.

[ITW05] Tatsuro Ito, Paul Terwilliger, and Chih-wen Weng. *The quantum algebra $U_q(\mathfrak{sl}_2)$ and its equitable presentation*. July 22, 2005. arXiv: math/0507477v1 [math.QA].

[Jan96] Jens Carsten Jantzen. *Lectures on Quantum Groups*. Graduate Studies in Mathematics 6. American Mathematical Society, 1996. viii+266 pp. ISBN: 978-0-8218-0478-0. DOI: 10.1090/gsm/006.

[Kas95] Christian Kassel. *Quantum Groups*. Graduate Texts in Mathematics 155. New York: Springer Verlag, 1995. xii+534 pp. ISBN: 978-0-387-94370-1. DOI: 10.1007/978-1-4612-0783-2.

[Lam01] Tsit-Yuen Lam. *A First Course in Noncommutative Rings*. 2nd ed. Graduate Texts in Mathematics 131. New York: Springer Verlag, 2001. xix+388 pp. ISBN: 978-0-387-95183-6. DOI: 10.1007/978-1-4419-8616-0.

[Lan02] Serge Lang. *Algebra*. 3rd ed. Graduate Texts in Mathematics 211. New York: Springer Verlag, 2002. xv+914 pp. ISBN: 978-0-387-95385-4. DOI: 10.1007/978-1-4613-0041-0.

[Mil13] James Stuart Milne. *Lie Algebras, Algebraic Groups, and Lie Groups*. 2013. 186 pp. URL: https://www.jmilne.org/math/CourseNotes/ala.html (visited on January 24, 2020). unpublished.

[MS16] Jendrik Stelzner. *Proposition 4.7 on Lang's semisimple rings part*. December 14, 2016. URL: https://math.stackexchange.com/q/2058567 (visited on January 24, 2020).

[Sch16] Travis Schedler. "Deformations of algebras in noncommutative geometry". In: *Noncommutative algebraic geometry*. Mathematical Sciences Research Institute Publications 64. Cambridge: Cambridge University Press, 2016, pp. 71–165. ISBN: 978-1-107-57003-0.