The Quantum Group $\mathrm{U}_q(\mathfrak{sl}_2)$

Talk 14 on Hopf Algebras and Tensor Categories

1 Recalling the Representation Theory of $\mathfrak{sl}_2$

1.1 Definition and Universal Enveloping Algebra

Let $\Bbbk$ be a field. The Lie algebra

$$\mathfrak{sl}_2 := \{A \in \mathrm{M}(2, \Bbbk) \mid \mathrm{tr}(A) = 0\}$$

admits the basis

$$e := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad f := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

and these basis elements satisfy the commutator relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Its universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ is therefore generated by the elements e, h, f subject to these relations, i.e.

$$\mathrm{U}(\mathfrak{sl}_2) \cong \Bbbk\langle e, h, f\rangle / (he - eh - 2e, hf - fh + 2f, ef - fe - h).$$

It follows from the theorem of Poincaré–Birkhoff–Witt that the algebra $\mathrm{U}(\mathfrak{sl}_2)$ admits the vector space basis

$$f^l h^m e^n \qquad \text{with } l, m, n \geq 0.$$

Let $\mathfrak{b}$ denote the Lie subalgebra of $\mathfrak{sl}_2$ consisting of (traceless) upper triangular matrices. It has the matrices e, h as a basis. Its universal enveloping algebra $\mathrm{U}(\mathfrak{b})$ has the PBW-basis $h^m e^n$ with $m, n \geq 0$, and it is a subalgebra of $\mathrm{U}(\mathfrak{sl}_2)$.

1.2 Representations of $\mathfrak{sl}_2$

Let V be any representation of $\mathfrak{sl}_2$ and let $\lambda \in \Bbbk$ be a scalar.

Definition 1.1. Let V be a representation of $\mathfrak{sl}_2$.

1. The *weight space* of V with respect to λ is $V_\lambda := \{v \in V \mid h.v = \lambda v\}$.
2. A nonzero weight vector v of V is *primitive* if $e.v = 0$.

3. The representation V is of *highest weight* λ if it is generated by a primitive weight vector of weight λ.

Proposition 1.2 (Shifting weight spaces). Let V be a representation of $\mathfrak{sl}_2$ and let $\lambda \in \Bbbk$. Then

$$e.V_\lambda \subseteq V_{\lambda+2}\,, \quad f.V_\lambda \subseteq V_{\lambda-2}\,.$$

Lemma 1.3. Let $\Bbbk$ be algebraically closed. Then every finite-dimensional irreducible representation of $\mathfrak{sl}_2$ is a highest weight representation.

There exists for every scalar $\lambda \in \Bbbk$ a universal representation of highest weight λ, the so-called Verma module:

Definition 1.4. For every scalar $\lambda \in \Bbbk$ let $\Bbbk_\lambda$ be the one-dimensional representation of $\mathfrak{b}$ whose underlying vector space is $\Bbbk$ and with action of $\mathfrak{b}$ given by

$$h.1 = \lambda\,, \quad e.1 = 0\,.$$

Lemma 1.5. The representation $\Bbbk_\lambda$ can be described as an $\mathrm{U}(\mathfrak{b})$-module as

$$\Bbbk_\lambda \cong \mathrm{U}(\mathfrak{b})/\langle e, h-\lambda\rangle\,.$$

Definition 1.6. The representation

$$\mathrm{M}(\lambda) \coloneqq \mathrm{U}(\mathfrak{sl}_2) \otimes_{\mathrm{U}(\mathfrak{b})} \Bbbk_\lambda$$

of $\mathfrak{sl}_2$ is the *Verma module* of highest weight λ.

Proposition 1.7. Let $\lambda \in \Bbbk$.

1. The Verma module $\mathrm{M}(\lambda)$ has the basis

$$v_i \coloneqq f^i \otimes 1 \qquad \text{with } i \geq 0,$$

   and the actions of e, h, f on this basis is given by

$$f.v_i = v_{i+1}\,, \quad h.v_i = (\lambda - 2i)v_i\,, \quad e.v_i = i(\lambda - i + 1)v_{i-1}\,.$$

   This action can be graphically described as in Figure 1.

Suppose that the field $\Bbbk$ is of characteristic zero.

2. The Verma module $\mathrm{M}(\lambda)$ is a representation of highest weight λ.
3. There exists for every representation V of $\mathfrak{sl}_2$ an isomorphism of vector spaces given by

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{sl}_2}(\mathrm{M}(\lambda), V) &\longrightarrow \{v \in V \mid v \text{ is of weight } \lambda \text{ with } e.v = 0\}\,,\\ \varphi &\longmapsto \varphi(1 \otimes 1)\,.\end{aligned}$$

   In particular

$$\mathrm{End}_{\mathfrak{sl}_2}(\mathrm{M}(\lambda)) = \Bbbk\,.$$

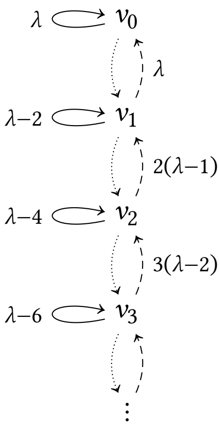

Figure 1: The Verma module $\mathrm{M}(\lambda)$.

4. The representation $\mathrm{M}(\lambda)$ is indecomposable.
5. a. If $\lambda \notin \mathbb{N}$ then the representation $\mathrm{M}(\lambda)$ is irreducible.

   b. If $\lambda = n \in \mathbb{N}$ then the representation $\mathrm{M}(\lambda)$ has a unique nonzero, proper subrepresentation, which is spanned by
$$v_i \qquad \text{with } i \geq n+1.$$
   This subrepresentation is isomorphic to $\mathrm{M}(-n-2)$.

Definition 1.8. Suppose that $\Bbbk$ is of characteristic zero and let $\lambda \in \Bbbk$.

1. For $\lambda \notin \mathbb{N}$ let $\mathrm{L}(\lambda) := \mathrm{M}(\lambda)$.
2. For $\lambda \in \mathbb{N}$ let $\mathrm{L}(\lambda) := \mathrm{M}(\lambda)/N$ where N is the unique nonzero, proper subrepresentation of $\mathrm{M}(\lambda)$.

Theorem 1.9. Let $\Bbbk$ be algebraically closed field of characteristic zero.

1. There is a one-to-one correspondence given by
$$\begin{aligned} \begin{Bmatrix} \text{irreducible highest weight} \\ \text{representations of } \mathfrak{sl}_2 \end{Bmatrix} &\longleftrightarrow \Bbbk\,, \\ \mathrm{L}(\lambda) &\longleftarrow\!\shortmid \lambda\,. \end{aligned}$$
2. The representation $\mathrm{L}(\lambda)$ is finite-dimensional if and only if $\lambda = n \in \mathbb{N}$, in which case
$$\dim(\mathrm{L}(n)) = n+1\,.$$
   The above correspondence does therefore restrict to a one-to-one correspondence
$$\begin{aligned} \begin{Bmatrix} \text{irreducible finite-dimensional} \\ \text{representations of } \mathfrak{sl}_2 \end{Bmatrix} &\longleftrightarrow \mathbb{N}\,, \\ L &\longmapsto \dim(L) - 1\,, \\ \mathrm{L}(n) &\longleftarrow\!\shortmid n\,. \end{aligned}$$

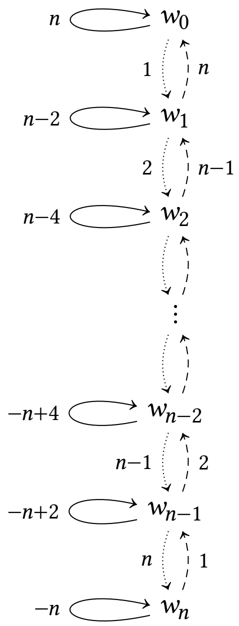

Figure 2: The irreducible representation $\mathrm{L}(n)$.

Remark 1.10. Let $n \in \mathbb{N}$. The basis $v_0, \dots, v_n$ of $\mathrm{L}(n)$ can be rescaled to the basis

$$w_i := \frac{1}{i!} v_i \,.$$

The actions of e and f then become

$$e.w_i = (n - i + 1) w_{i-1} \,, \quad f.w_i = (i + 1) w_{i+1} \,.$$

The actions of e, h, f on $\mathrm{L}(n)$ can now be graphically be represented as in Figure 2.

Theorem 1.11 (Weyl). Let $\Bbbk$ be algebraically closed. Every finite-dimensional representation of $\mathfrak{sl}_2$ is semisimple.

Corollary 1.12. Any finite-dimensional representation of $\mathfrak{sl}_2$ admits a weight space decomposition. All occuring weights are integral.

The decomposition of a finite-dimensional representation of $\mathfrak{sl}_2$ into irreducible representations can be read off from its weight space decomposition. From this the following result can be shown:

Proposition 1.13 (Clebsch–Gordan). Let n, m be natural numbers with $n \geq m$. Then

$$\mathrm{L}(n) \otimes \mathrm{L}(m) \cong \mathrm{L}(n+m) \oplus \mathrm{L}(n+m-2) \oplus \cdots \oplus \mathrm{L}(n-m) \,.$$

2 Definition and Basic Properties of $\mathrm{U}_q(\mathfrak{sl}_2)$

In the following let $\Bbbk$ be a field of characteristic zero and let q be an element of $\Bbbk$ that is nonzero and not a root of unity.

Remark 2.1. One often takes the field $\Bbbk$ as $\mathbb{Q}(v)$ or $\mathbb{C}(v)$, and for q the indeterminate v.

Definition 2.2. The algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ is the $\Bbbk$-algebra generated by the elements

$$E\,, \quad F\,, \quad K\,, \quad K^{-1}$$

subject to the relations

$$KK^{-1} = 1 = K^{-1}K\,, \quad KE = q^2 EK\,, \quad KF = q^{-2} FK\,, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}\,.$$

Remark 2.3. We will later see that in the situation of Remark 2.1 the algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ lives inside a larger algebra $\mathrm{U}_\hbar(\mathfrak{sl}_2)$. This will be an $\Bbbk[\![\hbar]\!]$-algebra with

$$\mathrm{U}_\hbar(\mathfrak{sl}_2) \cong \mathrm{U}(\mathfrak{sl}_2)[\![\hbar]\!]$$

as $\Bbbk[\![\hbar]\!]$-modules. Then

$$q = \mathrm{e}^\hbar\,, \quad K = \mathrm{e}^{\hbar H}\,.$$

In an informal way, this means that

$$K = q^H\,.$$

Definition 2.4. The *n-th quantum integer* is

$$[n]_q := q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1} = \frac{q^n - q^{-n}}{q - q^{-1}}\,.$$

The *quantum factorial* is

$$[n]_q! := [n]_q [n-1]_q \cdots [1]_q\,.$$

For every invertible element $u \in \mathrm{U}_q(\mathfrak{sl}_2)$ and integer $n \in \mathbb{Z}$ let

$$[u, n]_q := \frac{q^n u - q^{-n} u^{-1}}{q - q^{-1}}\,.$$

Remark 2.5. We note that for all integers $n, m \in \mathbb{Z}$,

$$[\pm q^n, m]_q = \pm [n + m]_q\,.$$

Lemma 2.6. For every $r \geq 0$ we have

$$[E, F^r] = [r]_q F^{r-1} [K, 1 - r]_q\,.$$

Proof. By induction, see [Jan96, Appendix 1.3 (5)]. □

Corollary 2.7. We have

$$
\begin{aligned}
F \cdot F^l K^m E^n &= F^{l+1} K^m E^n \,, \\
K^{\pm 1} \cdot F^l K^m E^n &= q^{\mp 2l} F^l K^{m \pm 1} E^n \,, \\
E \cdot F^l K^m E^n &= q^{-2m} F^l K^m E^{n+1} + \frac{[l]_q}{q - q^{-1}} (q^{1-l} F^{l-1} K^{m+1-l} E^n - q^{l-1} F^{l-1} K^{m+l-1} E^n)
\end{aligned}
$$

Theorem 2.8 (PBW). The elements

$$
F^l K^m E^n \qquad \text{with } l, n \in \mathbb{N} \text{ and } m \in \mathbb{Z}
$$

are a basis of $\mathrm{U}_q(\mathfrak{sl}_2)$.

Proof. Let U be the linear subspace of $\mathrm{U}_q(\mathfrak{sl}_2)$ spanned by these monomials. It follows from Corollary 2.7 that $\mathrm{U}_q(\mathfrak{sl}_2)$ is a left ideal. It contains $F^0 K^0 E^0 = 1$, whence $U = \mathrm{U}_q(\mathfrak{sl}_2)$. This shows that the given monomials are a vector space generating set.

The linear independence is shown in the usual representation-theoretic way: Let V be the free vector space with basis

$$
X^l Y^n Z^m \qquad \text{with } l, n \in \mathbb{N} \text{ and } m \in \mathbb{Z}.
$$

There exists an action of $\mathrm{U}_q(\mathfrak{sl}_2)$ on V by using the formulas from Corollary 2.7, with $F^l K^m E^n$ replaced by $X^l Y^n Z^m$. (It has to be checked that this proposed action is compatible with the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$, see [Jan96, Appendix 1.5].) The elements

$$
F^l K^m E^n \cdot X^0 Y^0 Z^0 = X^l Y^m Z^n
$$

are linearly independent in V, whence the given monomials $F^l K^m E^n$ are linearly independent in $\mathrm{U}_q(\mathfrak{sl}_2)$. □

Remark 2.9.

1. The universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ is noetherian and has no nonzero zero divisors. The same holds for $\mathrm{U}_q(\mathfrak{sl}_2)$, see [Kas95, Proposition VI.1.4] and [Jan96, Propositon 1.8].
2. The algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ admits a grading such that E, K, F are homogeneous with

   $$
   \deg(E) = 1 \,, \quad \deg(F) = -1 \,, \quad \deg(K) = 0 \,.
   $$

   The degree d part of $\mathrm{U}_q(\mathfrak{sl}_2)$ has the basis

   $$
   F^l K^m E^n \qquad \text{with } n - l = d.
   $$

   The conjugation map

   $$
   \mathrm{U}_q(\mathfrak{sl}_2) \to \mathrm{U}_q(\mathfrak{sl}_2) \,, \quad x \mapsto K x K^{-1}
   $$

   has the degree d part as the eigenspace with eigenvalue q^{2d}.

3 Representation Theory of $\mathrm{U}_q(\mathfrak{sl}_2)$

3.1 Decomposition into Weight Spaces

Definition 3.1. Let M be an $\mathrm{U}_q(\mathfrak{sl}_2)$-module.

1. For every scalar $\lambda \in \Bbbk^\times$ the associated *weight space* is given by
$$M_\lambda := \{m \in M \mid Km = \lambda m .\}$$
2. A weight vector m is *primitive* if it is nonzero and $Em = 0$.
3. The module M is *of highest weight* λ if it is generated by a primitive weight vector of weight λ.

Proposition 3.2 (Shifting weight spaces). Let M be an $\mathrm{U}_q(\mathfrak{sl}_2)$-module. For every scalar $\lambda \in \Bbbk^\times$,
$$EM_\lambda \subseteq M_{q^2\lambda}\,, \quad FM_\lambda \subseteq M_{q^{-2}\lambda}\,.$$

Lemma 3.3. Let M be a finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module.

1. Both E and F act nilpotently on M.
2. For a sufficiently large power $r \geq 0$ (namely such that $F^r M = 0$) the module M is annihilated by
$$\prod_{j=-r}^{r} (K^2 - q^{2j})\,.$$

Proof. See [Jan96, Proposition 2.1] and [Jan96, Proposition 2.3]. □

Proposition 3.4. Every finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module decomposes into weight spaces. All occuring weights are of the form $\pm q^n$ for some $n \in \mathbb{Z}$.

Proof. Let M be a finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module and let k denote the action of K on M. It follows from Lemma 3.3 that
$$0 = \prod_{n=-r}^{r} (k^2 - q^{2n}) = \prod_{n=-r}^{r} (k - q^n)(k + q^n)\,.$$
The roots $\pm q^n$ with $n = -r, \dots, r$ are pairwise distinct[1] whence it follows that k is diagonalizable with possible eigenvalues $\pm q^n$ for $n = -r, \dots, r$. □

Corollary 3.5. Every irreducible, finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module is a highest weight module.

[1]If $\pm q^n = \pm q^m$ then squaring both sides of this equation gives $q^{2n} = q^{2m}$ and thus $q^{2(n-m)} = 1$. It follows that $2(n-m) = 0$ because q is not a root of unity, and thus $n = m$.

3.2 Verma Modules and Classifications

Definition 3.6. Let $\mathrm{U}_q(\mathfrak{b})$ be the algebra given by generators K, K^{-1}, E and relations

$$KK^{-1} = 1 = K^{-1}K\,, \quad KE = q^2 EK\,.$$

Proposition 3.7. The algebra $\mathrm{U}_q(\mathfrak{b})$ has the basis

$$K^n E^m \qquad \text{with } n \in \mathbb{Z} \text{ and } m \in \mathbb{N}$$

Proof. This can be shown in the same way as Theorem 2.8. □

We wan therefore regard $\mathrm{U}(\mathfrak{b})$ as the subalgebra of $\mathrm{U}_q(\mathfrak{sl}_2)$ given generated by K, K^{-1}, E. By using $\mathrm{U}_q(\mathfrak{b})$ we can again define Verma modules, and classify both irreducible highest weight representations and irreducible, finite-dimensional representations.

Definition 3.8. Let $\lambda \in \Bbbk^\times$.

1. Let $\Bbbk_\lambda$ be the one-dimensional $\mathrm{U}_q(\mathfrak{b})$-module whose underlying vector space is given by $\Bbbk$ with
$$K \cdot 1 = \lambda\,, \quad E \cdot 1 = 0\,.$$
2. The *Verma module* associated to λ is the $\mathrm{U}_q(\mathfrak{sl}_2)$-module given by
$$\mathrm{M}(\lambda) \coloneqq \mathrm{U}_q(\mathfrak{sl}_2) \otimes_{\mathrm{U}_q(\mathfrak{b})} \Bbbk_\lambda\,.$$

Proposition 3.9. Let $\lambda \in \Bbbk^\times$.

1. We have $\Bbbk_\lambda \cong \mathrm{U}_q(\mathfrak{b})/\langle E, K-\lambda\rangle$ as an $\mathrm{U}_q(\mathfrak{b})$-module.
2. The Verma module $\mathrm{M}(\lambda)$ has the basis
$$m_i \coloneqq F^i \otimes 1 \qquad \text{with } i \geq 0,$$
and the actions of E, K, F on this basis is given by
$$Fm_i = m_{i+1}\,, \quad Km_i = q^{-2i}\lambda m_i\,, \quad Em_i = [i]_q[\lambda, 1-i]_q m_{i-1}\,.$$
This action can be graphically described as in Figure 3.
3. The Verma module $\mathrm{M}(\lambda)$ is of highest weight λ.
4. There exists for every $\mathrm{U}_q(\mathfrak{sl}_2)$-module M an isomorphism of vector spaces given by
$$\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{sl}_2)}(\mathrm{M}(\lambda), M) \cong \{m \in M \mid m \text{ is of weight } \lambda \text{ with } Em = 0\}\,.$$
It follows in particular that
$$\mathrm{End}_{\mathrm{U}_q(\mathfrak{sl}_2)}(\mathrm{M}(\lambda)) = \Bbbk\,.$$
5. The Verma module $\mathrm{M}(\lambda)$ is indecomposable.
6. a. If $\lambda \neq \pm q^n$ for every $n \in \mathbb{N}$ then the Verma module $\mathrm{M}(\lambda)$ is irreducible.

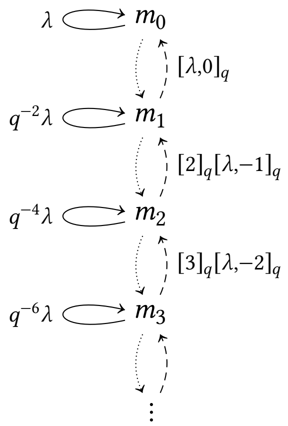

Figure 3: The Verma module $\mathrm{M}(\lambda)$.

b. If $\lambda = \pm q^n$ for some $n \in \mathbb{N}$ then the Verma module $\mathrm{M}(\lambda)$ contains a unique nonzero, proper submodule, which is spanned by

$$m_i \qquad \text{with } i \geq n+1.$$

This submodule is isomorphic to $\mathrm{M}(q^{-n-2)}\lambda)$.

Proof.

1. This follows from the PBW-basis of $\mathrm{U}_q(\mathfrak{b})$.
2. This follows from the PBW-basis of $\mathrm{U}_q(\mathfrak{sl}_2)$ and induction.
3. The Verma module $\mathrm{M}(\lambda)$ is generated by the primitive weight vector $1 \otimes 1$.
4. We have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{sl}_2)}(\mathrm{M}(\lambda), M) &\cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{b})}(\Bbbk_\lambda, M)\\ &\cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{b})}(\mathrm{U}_q(\mathfrak{b})/\langle K-\lambda, E\rangle, M)\\ &\cong \{m \in M \mid (K-\lambda)m = 0, Em = 0\}\,.\end{aligned}$$

5. The endomorphism algebra $\mathrm{End}_{\mathrm{U}_q(\mathfrak{sl}_2)}(\mathrm{M}(\lambda)) = \Bbbk$ does not contain any non-trivial idempotents.
6. This follows as for $\mathrm{U}(\mathfrak{sl}_2)$ since $[i]_q[\lambda, i-1]_q = 0$ if and only if $\lambda = \pm q^{i-1}$. □

Definition 3.10. Let $\lambda \in \Bbbk^\times$.

1. If $\lambda \neq \pm q^n$ for every $n \in \mathbb{N}$ then $\mathrm{L}(\lambda) \coloneqq \mathrm{M}(\lambda)$.
2. If $\lambda = \pm q^n$ for some $n \in \mathbb{N}$ then $\mathrm{L}(\lambda) \coloneqq \mathrm{M}(\lambda)/N$ where λ is the unique nonzero, proper submodule of $\mathrm{M}(\lambda)$.

Theorem 3.11.

1. There is a one-to-one correspondence given by

$$\begin{aligned} \Bbbk^\times &\longmapsto \{\text{highest weight irreducible } \mathrm{U}_q(\mathfrak{sl}_2)\text{-modules}\} \\ \lambda &\longmapsto \mathrm{L}(\lambda)\,. \end{aligned}$$

2. The module $\mathrm{L}(\lambda)$ is finite-dimensional if and only if $\lambda = \pm q^n$ for some $n \in \mathbb{N}$, in which case

$$\dim(\mathrm{L}(\lambda)) = n + 1\,.$$

   The above one-to-one correspondence does therefore restrict to a one-to-one correspondence given by

$$\begin{aligned} \{1, -1\} \times \mathbb{N} &\longmapsto \{\text{finite-dimensional irreducible } \mathrm{U}_q(\mathfrak{sl}_2)\text{-modules}\} \\ (\varepsilon, n) &\longmapsto \mathrm{L}(\varepsilon q^n)\,. \end{aligned}$$

Remark 3.12. For every $n \geq 0$ we have

$$[\pm q^n, -i + 1] = \pm[n - i + 1]_q\,.$$

On the rescalled basis $w_0, \dots, w_n$ of $\mathrm{L}(\pm q^n)$ given by

$$w_i := \frac{1}{[i]_q!} v_i$$

the actions of E and F thus become

$$Ew_i = \pm[n - i + 1]_q w_{i-1}\,, \quad Fw_i = [i + 1]_q w_{i+1}\,.$$

The action of E, H, K on $\mathrm{L}(\pm q^n)$ can therefore be graphically be represented as in Figure 4

3.3 Semisimplicity of Finite-Dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-modules

Lemma 3.13. If M is an highest-weight $\mathrm{U}_q(\mathfrak{sl}_2)$-module then

$$\mathrm{End}_{\mathrm{U}_q(\mathfrak{sl}_2)}(M) = \Bbbk\,.$$

Definition 3.14. The *quantum Casimir element* is the element $C_q \in \mathrm{U}_q(\mathfrak{sl}_2)$ given by

$$C_q := EF + \frac{Kq^{-1} + K^{-1}q}{(q - q^{-1})^2}\,.$$

Lemma 3.15.

1. The element C_q is central in $\mathrm{U}_q(\mathfrak{sl}_2)$.
2. The element C_q acts on every $\mathrm{U}_q(\mathfrak{sl}_2)$-module by module endomorphisms.
3. The element C_q acts for every scalar $\lambda \in \Bbbk^\times$ on the representation $\mathrm{L}(\lambda)$ by multiplication with the scalar

$$\frac{\lambda q + \lambda^{-1} q^{-1}}{(q - q^{-1})^2}\,.$$

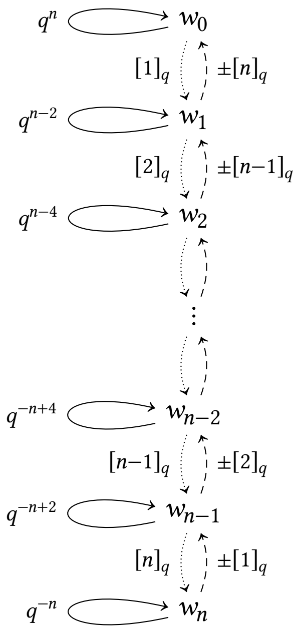

Figure 4: The irreducible representation $\mathrm{L}(\pm q^n)$.

4. The element C_q acts the same on $\mathrm{L}(\lambda)$ and $\mathrm{L}(\mu)$ if and only if $\lambda = \mu$ or $\lambda = \mu^{-1}q^{-2}$.

Proof.

1. It can be checked that C_q commutes with E, F, K by using the defining relations for $\mathrm{U}_q(\mathfrak{sl}_2)$.
2. This follows from the previous assertion.
3. It follows from the previous assertion and Lemma 3.13 that C_q acts by a scalar. This scalar can be read off from the action on the primitive generator $1 \otimes 1$. It thus sufficies to show the assertion for $\mathrm{M}(\lambda)$, where it follows from Proposition 3.9.
4. This follows from the previous assertion. $\square$

Corollary 3.16. The quantum Casimir element C_q acts on every finite-dimensional, irreducible representation of $\mathrm{U}_q(\mathfrak{sl}_2)$ by a different scalar.

Proof. If $\lambda = \delta q^n$ and $\mu = \varepsilon q^m$ with $\delta, \varepsilon \in \{1, -1\}$ and $n, m \in \mathbb{N}$ then it cannot happen that $\lambda = \mu^{-1}q^{-2}$. The assertion thus follows from Lemma 3.15. $\square$

Theorem 3.17. Every finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module is semisimple.

Proof ([Jan96, Theorem 2.9]). Let M be a finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module and let c denote the action of C_q on M. We may assume that M is indecomposable. We can consider a composition series

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_r = M \tag{1}$$

with composition factors

$$M_i/M_{i-1} \cong \mathrm{L}(\varepsilon_i q^{n_i})\,.$$

Letting c_i be the scalar by which C_q acts on $\mathrm{L}(\varepsilon_i q^{n_i})$, we have

$$(c - c_i)M_i \subseteq M_{i-1}\,.$$

It follows that $\prod_{i=1}^{r}(c - c_i)$ annihilates M and that c admits a generalized eigenspace decomposition with eigenvalues $c_1, \dots, c_r$. The resulting generalized eigenspaces are subrepresentations because c is a $\mathrm{U}_q(\mathfrak{sl}_2)$-module endomorphism. It follows that

$$c_1 = \cdots = c_r$$

because M is indecomposable, and thus

$$\varepsilon_1 q^{n_1} = \cdots = \varepsilon_r q^{n_r} =: \lambda$$

by Corollary 3.16. It follows with the composition series (1) that

$$\dim(M_\mu) = r\dim(\mathrm{L}(\lambda)_\mu)$$

for every scalar $\mu \in \Bbbk^\times$. Thus M is of highest weight λ.

The short exact sequence

$$0 \to M_{r-1} \to M \to \mathrm{L}(\lambda) \to 0 \tag{2}$$

restricts to a short exact sequence

$$0 \to (M_{r-1})_\lambda \to M_\lambda \to \mathrm{L}(\lambda)_\lambda \to 0\,.$$

It follows that the primitive generator v_0 of $\mathrm{L}(\lambda)$ has a preimage m_0 in M. The weight vector m_0 is primitive because M isof highest weight λ. It follows that there exists a homomorphism of $\mathrm{U}_q(\mathfrak{sl}_2)$-modules

$$\varphi\colon\ \mathrm{M}(\lambda) \to M\,, \quad 1 \otimes 1 \mapsto m_0\,.$$

It follows from the finite-dimensionality of M that φ factors through a homomorphism

$$\psi\colon\ \mathrm{L}(\lambda) \to M\,, \quad \overline{1 \otimes 1} \mapsto m_0\,.$$

This shows that the short exact sequence (2) splits, whence

$$M \cong M_{r-1} \oplus \mathrm{L}(\lambda)\,.$$

It follows by induction that $M_{r-1} \cong \mathrm{L}(\lambda)^{\oplus(r-1)}$ and thus altogether $M \cong \mathrm{L}(\lambda)^{\oplus r}$. ☐

Remark 3.18. The center of the universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ is a polynomial algebra, generated by the classical Casimir element $C = (ef + h^2 + fe)/4$. It can be shown that the center of $\mathrm{U}_q(\mathfrak{sl}_2)$ is again a polynomial algebra, now generated by the quantum Casimir element C_q. We refer to [Jan96, Proposition 2.18] for more details on this.

Corollary 3.19. Let M, N be two finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-modules with $\dim M_\lambda = \dim N_\lambda$ for every $\lambda \in \Bbbk^\times$. Then $M \cong N$.

Proof. One can read off the decomposition of a finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-module into irreducible represenations from the dimensions of its weight spaces. ☐

4 Hopf Algebra Structure on $\mathrm{U}_q(\mathfrak{sl}_2)$

Proposition 4.1. The algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ becomes a Hopf algebra when endowed with the comultiplication

$$\Delta(E) = E \otimes 1 + K \otimes E\,, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F\,, \quad \Delta(K) = K \otimes K\,,$$

the counit

$$\varepsilon(E) = 0\,, \quad \varepsilon(F) = 0\,, \quad \varepsilon(K) = 1$$

and the antipode

$$S(E) = -K^{-1}E\,, \quad S(F) = -FK\,, \quad S(K) = K^{-1}\,.$$

Proof. One checks that the proposed images of the algebra generators E, F, K, K^{-1} are compatible with the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$, and that the Hopf algebra diagram commute on these algebra generators. □

Convention 4.2. We will in the following regard $\mathrm{U}_q(\mathfrak{sl}_2)$ as a Hopf algebra as explained in Proposition 4.1.

Remark 4.3.

1. The Hopf algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ is neither commutative nor cocommutative. It is an example of a so-called *quantum group*.
2. In $\mathrm{U}_q(\mathfrak{sl}_2)$ we have $S^2 \neq \mathrm{id}$ but instead

$$S^2(x) = K^{-1}xK$$

for every $x \in \mathrm{U}_q(\mathfrak{sl}_2)$, as can be checked on the algebra generators E, $K^{\pm 1}$, F.

Lemma 4.4. Let M, N be two finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-modules. Then

$$(M \otimes N)_\lambda = \bigoplus_{\mu\kappa=\lambda} M_\mu \otimes N_\kappa\,.$$

Proof. We have

$$M_\mu \otimes N_\kappa \subseteq (M \otimes N)_{\mu\kappa}$$

for all $\mu, \kappa \in \Bbbk^\times$ since the element K is group-like in $\mathrm{U}_q(\mathfrak{sl}_2)$. The assertion follows since both M and N decompose into weight spaces. □

Proposition 4.5 (Clebsch–Gordan). For all $\delta, \varepsilon \in \{1, -1\}$ and $n, m \in \mathbb{N}$ with $n \geq m$ we have

$$\mathrm{L}(\delta q^n) \otimes \mathrm{L}(\varepsilon q^m) \cong \mathrm{L}(\delta\varepsilon q^{n+m}) \oplus \mathrm{L}(\delta\varepsilon q^{n+m-2}) \oplus \cdots \oplus \mathrm{L}(\delta\varepsilon q^{n-m})\,.$$

Proof. This follows from Lemma 4.4 and Corollary 3.19. □

Corollary 4.6. Let M, N be two finite-dimensional $\mathrm{U}_q(\mathfrak{sl}_2)$-modules. Then

$$M \otimes N \cong N \otimes M\,.$$

Proof. The assertion holds by Proposition 4.5 when M, N are irreducible. It follows for arbitrary finite-dimensional modules by Theorem 3.17. $\square$

Warning 4.7. For two (finite-dimensional) $\mathrm{U}_q(\mathfrak{sl}_2)$-modules M, N the flip map

$$\tau : M \otimes N \to N \otimes M , \quad m \otimes n \mapsto n \otimes m$$

is in general not $\mathrm{U}_q(\mathfrak{sl}_2)$-linear. Indeed, let us consider $M = N = \mathrm{L}(q)$ with basis m_0, m_1, so that

$$K^{-1}m_0 = q^{-1}m_0 , \quad K^{-1}m_1 = qm_1 , \quad Fm_0 = m_1 , \quad Fm_1 = 0 .$$

Then on the one hand

$$F \cdot (m_0 \otimes m_1) = \underbrace{(Fm_0)}_{=m_1} \otimes \underbrace{(K^{-1}m_1)}_{qm_1} + m_0 \otimes \underbrace{(Fm_1)}_{=0} = qm_1 \otimes m_1$$

while on the other hand

$$F \cdot (m_1 \otimes m_0) = \underbrace{(Fm_1)}_{=0} \otimes \underbrace{(K^{-1}m_0)}_{=q^{-1}m_0} + m_1 \otimes \underbrace{(Fm_0)}_{=m_1} = m_1 \otimes m_1 .$$

5 Recalling Completions

We want in the following consider $\Bbbk[\![\hbar]\!]$-modules in which infinite sums

$$m_0 + \hbar m_1 + \hbar^2 m_2 + \cdots = \sum_{i=0}^{\infty} \hbar^i m_i$$

make sense. For this we think about such an infinite sum as a sequence of finite sums

$$s_n = \sum_{i=0}^{n} \hbar^i m_i \quad \text{such that} \quad s_{n+1} \equiv s_n \pmod{\hbar^{n+1}}$$

for every $n \geq 0$.

Definition 5.1. Let M be an $\Bbbk[\![\hbar]\!]$-module. The *$\hbar$-adic completion* of M is the $\Bbbk[\![\hbar]\!]$-module

$$\widehat{M} := \lim_{n \geq 0}(M/\hbar^{n+1}M) = \left\{ (m_n)_{n\geq 0} \,\middle|\, \begin{array}{l} m_n \in M/\hbar^{n+1}M \text{ with} \\ m_{n+1} \equiv m_n \pmod{\hbar^{n+1}} \text{ for every } n \geq 0 \end{array} \right\} .$$

The *canonical homomorphism* $M \to \widehat{M}$ is given by $m \mapsto (\overline{m}, \overline{m}, \dots)$.

Definition 5.2. A $\Bbbk[\![\hbar]\!]$-module M is *complete* if the canonical homomorphism $M \to \widehat{M}$ is an isomorphism. We denote by $\Bbbk[\![\hbar]\!]\text{-}\mathbf{Mod}^{\mathrm{comp}}$ the full subcategory of $\Bbbk[\![\hbar]\!]\text{-}\mathbf{Mod}$ whose objects are the complete $\Bbbk[\![\hbar]\!]$-modules.

Remark 5.3.

1. More explicitely, an $\Bbbk[\![\hbar]\!]$-module M is complete if and only if there exists for every sequence $m_0, m_1, \dots$ of elements $m_n \in M$ with

$$m_{n+1} \equiv m_n \pmod{\hbar^{n+1}} \qquad \text{for every } n \geq 0$$

a unique element $m \in M$ with

$$m \equiv m_n \pmod{\hbar^{n+1}} \qquad \text{for every } n \geq 0.$$

2. Let M be a complete $\Bbbk[\![\hbar]\!]$-module Every sequence $(m_i)_{i\geq 0}$ of elements $m_i \in M$ defines a sequence $(s_n)_{n\geq 0}$ of partial sums

$$s_n := \sum_{i=0}^{n} \hbar^i m_i \quad \text{with} \quad s_{n+1} \equiv s_n \pmod{h^{n+1}}$$

for every $n \geq 0$. By the completeness of M there exists a unique element of M, which will be denoted by $\sum_{i=0}^{\infty} \hbar^i m_i$, such that

$$\sum_{i=0}^{\infty} \hbar^i m_i \equiv \sum_{i=0}^{n} \hbar^i m_i \pmod{\hbar^{n+1}} \qquad \text{for every } n \geq 0.$$

Example 5.4.

1. Every finite-dimensional $\Bbbk[\![\hbar]\!]$-module M is complete since $\hbar^n M = 0$ for some sufficiently large power n.
2. For every $\Bbbk$-vector space the resulting $\Bbbk[\![\hbar]\!]$-module $V[\![\hbar]\!]$ is complete. For every sequence of elements $v_0, v_1, \dots \in V$ we have

$$\sum_{i=0}^{\infty} \hbar^i v_i = \sum_{i=0}^{\infty} v_i \hbar^i \,.$$

Proposition 5.5. Let M, N be two $\Bbbk[\![\hbar]\!]$-modules.

1. For every homomorphism of $\Bbbk[\![\hbar]\!]$-module $f : M \to N$ there exists a unique module homomorphism $\widehat{f} : \widehat{M} \to \widehat{N}$ that makes the following square diagram commute:

$$\begin{tikzcd} \widehat{M} \arrow[r, "\widehat{f}"] & \widehat{N} \\ M \arrow[u] \arrow[r, "f"] & N \arrow[u] \end{tikzcd}$$

The homomorphism $\widehat{f}$ is given by

$$\widehat{f}\left((\overline{m_0}, \overline{m_1}, \dots)\right) = \left(\overline{f(m_0)}, \overline{f(m_1)}, \dots\right).$$

2. The assignment $\widehat{(-)}$ defines a functor

$$\widehat{(-)} : \Bbbk[\![\hbar]\!]\text{-}\mathbf{Mod} \to \Bbbk[\![\hbar]\!]\text{-}\mathbf{Mod}\,.$$

3. If M, N are complete then

$$f\left(\sum_{i=0}^{\infty} \hbar^i m_i\right) = \sum_{i=0}^{\infty} \hbar^i f(m_i)$$

for every sequence of elements $m_0, m_1, \dots, \in M$.

4. If N is complete then every homomorphism $M \to N$ extends uniquely to a homomorphism $\widehat{M} \to N$.

$$\begin{tikzcd} \widehat{M} \arrow[r, dashed, "\exists!"] & N \\ M \arrow[u] \arrow[ur] & \end{tikzcd}$$

5. If V is any $\Bbbk$-vector space and N is complete then every $\Bbbk$-linear map $f : V \to N$ extends uniquely to a $\Bbbk[\![\hbar]\!]$-linear linear map $f' : V[\![\hbar]\!] \to N$.

$$\begin{tikzcd} V[\![\hbar]\!] \arrow[r, dashed, "\exists!"] & N \\ V \arrow[u] \arrow[ur] & \end{tikzcd}$$

The homomorphism f' is given by

$$f'\left(\sum_{i=0}^{\infty} \hbar^i v_i\right) = \sum_{i=0}^{\infty} \hbar^i f(v_i)\,.$$

6. The canonical homomorphism $M \to \widehat{M}$ induces an isomorphism of $\Bbbk$-vector spaces

$$M/\hbar M \longrightarrow \widehat{M}/\hbar\widehat{M}\,.$$

Definition 5.6. Let M, N be two $\Bbbk[\![\hbar]\!]$-modules. The *completed tensor product*

$$M \mathbin{\widehat{\otimes}} N$$

is the $\hbar$-adic completion of the tensor product $M \otimes_{\Bbbk[\![\hbar]\!]} N$.

Proposition 5.7. Let V, W be two $\Bbbk$-vector spaces. Then the $\Bbbk[\![\hbar]\!]$-linear map

$$V[\![\hbar]\!] \otimes_{\Bbbk[\![\hbar]\!]} W[\![\hbar]\!] \to (V \otimes W)[\![\hbar]\!]\,, \quad \left(\sum_{i=0}^{\infty} v_i \hbar^i\right) \otimes \left(\sum_{j=0}^{\infty} w_j \hbar^j\right) \mapsto \sum_{i,j=0}^{\infty} (v_i \otimes w_j)\hbar^{i+j}$$

extends to an isomorphism

$$V[\![\hbar]\!] \mathbin{\widehat{\otimes}} W[\![\hbar]\!] \to (V \otimes W)[\![\hbar]\!]\,.$$

Remark 5.8. Let M be a $\Bbbk[\![\hbar]\!]$-module. There exists a unique topology on M for which a basis is given by the sets

$$m + \hbar^{n+1} M$$

with $m \in M$ and $n \geq 0$. This topology is the *$\hbar$-adic topology* on M. It makes $\Bbbk[\![\hbar]\!]$ into a topological ring and every $\Bbbk[\![\hbar]\!]$-module into a topological $\Bbbk[\![\hbar]\!]$-module. The completion $\widehat{M}$ is then the usual topological completion of M.

6 Deformation Theory

We now want to study in which way the Hopf algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ is a deformation of the usual enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$.

6.1 Deformation of Algebras

The following is taken (in spirit) from [Bel18, §5.2] and [GS92].

Motivation 6.1. Deforming a $\Bbbk$-algebra A means – roughly speaking – that the multiplication on A is replaced by a perturbated multiplication $*$, in the sense that for all $a, b \in A$,

$$a * b = ab + \mu_1(a, b)\hbar + \mu_2(a, b)\hbar^2 + \cdots$$

for some bilinear terms $\mu_i(a, b)$. The limit $\hbar \to 0$ does then give back the original algebra A.

We present in the following one possible formalization of this intuition.

Definition 6.2.

1. Let A be an $\Bbbk$-algebra. A *formal deformation* of A is an $\Bbbk[\![\hbar]\!]$-algebra $A_\hbar$ together with an isomorphism of $\Bbbk[\![\hbar]\!]$-modules

   $$\varphi \colon A_\hbar \to A[\![\hbar]\!]$$

   such that the induced isomorphism of vector spaces

   $$\overline{\varphi} \colon A_\hbar / \hbar A_\hbar \to A[\![\hbar]\!] / \hbar A[\![\hbar]\!] = A$$

   is an isomorphism of $\Bbbk$-algebras.
2. Two formal deformations $A_\hbar$ and $A'_\hbar$ of A are *equivalent* if there exists an in isomorphism of $\Bbbk[\![\hbar]\!]$-algebras

   $$\psi \colon A_\hbar \to A'_\hbar$$

   such that the following square diagram commutes:

   $$\begin{tikzcd} A_\hbar/\hbar A_\hbar \arrow[r, "\overline{\psi}"] \arrow[d, "\overline{\varphi}"'] & A'_\hbar/\hbar A'_\hbar \arrow[d, "\overline{\varphi'}"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

Remark 6.3.

1. A deformation of an $\Bbbk$-algebra is (up to equivalence) given by an $\Bbbk[\![\hbar]\!]$-algebra structure

$$(-) * (-) : \ A[\![\hbar]\!] \times A[\![\hbar]\!] \to A[\![\hbar]\!] .$$

such that $A[\![\hbar]\!]/\hbar A[\![\hbar]\!] = A$ as $\Bbbk$-algebras. The $\Bbbk[\![\hbar]\!]$-bilinearity of the mulitplication $*$ ensures that it satisfies the equality

$$\left(\sum_{i=0}^{\infty} a_i \hbar^i\right) * \left(\sum_{j=0}^{\infty} b_j \hbar^j\right) = \sum_{i,j=0}^{\infty} (a_i * b_j) \hbar^{i+j} .$$

The multiplication $*$ can therefore be characterized by the $\Bbbk$-bilinear maps $\mu_i : \ A \times A \to A$ such that

$$a * b = \mu_0(a, b) + \mu_1(a, b)\hbar + \mu_2(a, b)\hbar^2 + \cdots$$

The condition $A[\![\hbar]\!]/\hbar A[\![\hbar]\!] = A$ means that μ_0 is the original multiplication on A, whence

$$a * b = ab + \mu_1(a, b)\hbar + \mu_2(a, b)\hbar^2 + \cdots$$

That the multiplication $*$ is associative gives certain compatibility conditions on the μ_1, which we won't discuss here.

2. Every $\Bbbk[\![\hbar]\!]$-linear linear map $\varphi : \ A[\![\hbar]\!] \to A[\![\hbar]\!]$ is uniquely given by a sequence of $\Bbbk$-linear maps $\varphi_i : \ A \to A$ such that for every $a \in A$,

$$\varphi(a) = \varphi_0(a) + \varphi_1(a)\hbar + \varphi_2(a)\hbar^2 + \cdots$$

Two deformations $*$ and $*'$ are equivalent if and only if there exists an isomorphism of $\Bbbk[\![\hbar]\!]$-algebras $\varphi : \ A[\![\hbar]\!] \to A[\![\hbar]\!]$ for which $\varphi_0 = \mathrm{id}$. In other words,

$$\varphi \equiv \mathrm{id} \pmod{\hbar} .$$

Example 6.4. Every $\Bbbk$-algebra A admits the *trivial deformation* $A[\![\hbar]\!]$ (i.e. the algebra of power series with its usual product). It corresponds to the choice $\mu_1, \mu_2, \ldots = 0$. A deformation is *trivial* if it is equivalent to $A[\![\hbar]\!]$.

Theorem 6.5. The universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ admits a deformation, whose underlying $\Bbbk[\![\hbar]\!]$-module is given by $\mathrm{U}(\mathfrak{sl}_2)[\![\hbar]\!]$, such that

$$[H, E] = 2E , \quad [H, F] = 2F , \quad [E, F] = \frac{\mathrm{e}^{\hbar H} - \mathrm{e}^{-\hbar H}}{\mathrm{e}^{\hbar} - \mathrm{e}^{-\hbar}} = H + O(\hbar^2) . \tag{3}$$

Proof (sketch). Let P be the free algebra on the generators E, H, F. Let I be the two-sided ideal in $P[\![\hbar]\!]$ given by the relations (3). Let J be the closure of I in the $\hbar$-adic topology. Then J is again a two-sided ideal in $P[\![\hbar]\!]$. The described deformation can be realized as the quotient $P[\![\hbar]\!]/J$. We refer to [CP95, Definition-Proposition 6.4.3 ff.] for the specific details. □

Definition 6.6. The deformation of $\mathrm{U}(\mathfrak{sl}_2)$ from Theorem 6.5 is denoted by $\mathrm{U}_\hbar(\mathfrak{sl}_2)$.

Remark 6.7.

1. In the deformation $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ one can consider the (well-defined!) elements

$$q := \mathrm{e}^{\hbar}\,, \quad K := \mathrm{e}^{\hbar H}\,.$$

   The elements E, F, K, K^{-1} satisfy the defining relations of $\mathrm{U}_q(\mathfrak{sl}_2)$.

Remark 6.8. One can study the deformation theory of an $\Bbbk$-algebra via homological algebra: The *Hochschild cochain complex* of A is given by

$$\mathrm{C}^n_{\mathrm{Hoch}}(A) := \mathrm{Hom}_{\Bbbk}(A^{\otimes n}, A)$$

together with certain differentials. Its cohomology is the *Hochschild cohomology* of A, which is denoted by

$$\mathrm{HH}^n(A) := \mathrm{H}^n\big(\mathrm{C}^\bullet_{\mathrm{Hoch}}\big)\,.$$

One of the connections between deformation theory and Hochschild cohomology is that in the case of

$$\mathrm{HH}^2(A) = 0$$

every deformation of A is trivial. It can be shown that this happen for $A = \mathrm{U}(\mathfrak{g})$ when $\mathfrak{g}$ is a semisimple Lie algeba. (See [GS92, Theorem 2].)

It follows in particular that the every deformation of $\mathrm{U}(\mathfrak{sl}_2)$ as an algebra is trivial. An explicit equivalence between $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ and $\mathrm{U}(\mathfrak{sl}_2)[\![\hbar]\!]$ is constructed in [CP95, Proposition 4.6.4].

6.2 Deformation of Hopf Algebras

The following is taken mostly from [CP95, Chapter 6].

Definition 6.9.

1. A *topological Hopf algebra* is a complete $\Bbbk[\![\hbar]\!]$-module A together with $\Bbbk[\![\hbar]\!]$-linear maps

$$m \colon A \mathbin{\widehat{\otimes}} A \to A\,, \quad u \colon \Bbbk[\![\hbar]\!] \to A\,, \quad \Delta \colon A \to A \mathbin{\widehat{\otimes}} A\,, \quad \varepsilon \colon A \to \Bbbk[\![\hbar]\!]\,, \quad S \colon A \to A$$

   such that the usual Hopf algebra diagrams commute.

2. The terms *topological algebra*, *topological coalgebra* and *topological bialgebra* are defined analogous.

Remark 6.10.

1. A topological algebra in the sense of Definition 6.9 is the same as an $\Bbbk[\![\hbar]\!]$-algebra which is complete as an $\Bbbk[\![\hbar]\!]$-module.

   Indeed, if (A, m, u) is a topological algebra then the multiplication

$$m \colon A \mathbin{\widehat{\otimes}} A \to A$$

restricts via the composition with the canonical homomorphism

$$A \otimes A \to A \mathbin{\widehat{\otimes}} A$$

to a multiplication

$$m' : A \otimes A \to A .$$

Then (A, m', u) is an $\Bbbk[\![\hbar]\!]$-algebra (and A is by definition complete).

Suppose on the other hand that (A, m', u) is an $\Bbbk[\![\hbar]\!]$-algebra where A is complete. Then the multiplication map

$$m' : A \otimes A \to A$$

extends by the completeness of A uniquely to a $\Bbbk[\![\hbar]\!]$-linear map

$$m : A \mathbin{\widehat{\otimes}} A \to A .$$

Then (A, m, u) is a topological algebra (by the denseness of $A \otimes A$ in $A \mathbin{\widehat{\otimes}} A$, etc.).

Thus every topological algebra in the sense of Definition 6.9 is a $\Bbbk[\![\hbar]\!]$-algebra.

2. A topological coalgebra (C, Δ, ε) on the other hand is generally not a $\Bbbk[\![\hbar]\!]$-coalgebra algebra, since the comultiplication

$$\Delta : C \to C \mathbin{\widehat{\otimes}} C$$

does in general not restrict to a map $C \to C \otimes C$. Topological bialgebras and topological Hopf algebras inherit this problem.

3. If A is a topological Hopf algebra then $A/\hbar A$ becomes an Hopf algebra over $\Bbbk$. We note for this that

$$(A \mathbin{\widehat{\otimes}} A)/\hbar(A \mathbin{\widehat{\otimes}} A) \cong (A \otimes A)/\hbar(A \otimes A) \cong (A/\hbar A) \otimes (A/\hbar A) .$$

The analogous assertion holds for topological algebras, topological coalgebra and topological bialgebras.

Definition 6.11. Let A be a Hopf algebra over $\Bbbk$.

1. A *formal deformation* of A is a topological Hopf algebra $A_\hbar$ together with an isomorphism of $\Bbbk[\![\hbar]\!]$-modules

$$\varphi : A_\hbar \to A[\![\hbar]\!]$$

such that the induced isomorphism of vector spacse

$$\overline{\varphi} : A_\hbar/\hbar A_\hbar \to A_\hbar$$

is an isomorphism of Hopf algebras. Formal deformations of algebras, coalgebras and bialgebras are defined in the same way.

2. Two formal deformations $A_\hbar$ and $A'_\hbar$ are equivalent if there exists an isomorphism of topological Hopf algebras

$$\psi : A_\hbar \to A'_\hbar$$

that makes the following square diagram commute:

$$\begin{tikzcd} A_\hbar/\hbar A_\hbar \arrow[r, "\overline{\psi}"] \arrow[d, "\overline{\varphi}"'] & A'_\hbar/\hbar A'_\hbar \arrow[d, "\overline{\varphi'}"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

3. A Hopf algebra deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a *quantum universal enveloping algebra (QUE)*.

Remark 6.12.

1. Let A be a Hopf algebra over $\Bbbk$ with deformation $A_\hbar$. We may assume (up to equivalence) that $A_\hbar = A[\![\hbar]\!]$ as $\Bbbk[\![\hbar]\!]$-modules. By using the isomorphism

$$A[\![\hbar]\!] \mathbin{\widehat{\otimes}} A[\![\hbar]\!] \cong (A \otimes A)[\![\hbar]\!]$$

we can regard the structure maps of $A_\hbar$ as $\Bbbk[\![\hbar]\!]$-linear map

$$\begin{aligned} m_\hbar &: (A \otimes A)[\![\hbar]\!] \to A[\![\hbar]\!], \\ u_\hbar &: \Bbbk[\![\hbar]\!] \to A[\![\hbar]\!], \\ \Delta_\hbar &: A[\![\hbar]\!] \to (A \otimes A)[\![\hbar]\!], \\ \varepsilon_\hbar &: A[\![\hbar]\!] \to \Bbbk[\![\hbar]\!], \\ S_\hbar &: A[\![\hbar]\!] \to A[\![\hbar]\!] \end{aligned} \tag{4}$$

which are perturbations of the structure maps of A, i.e. they reduce module $\hbar$ to the structure maps of A. We can for example characterize the comultiplication $\Delta_\hbar$ by a sequence of bilinear map $\Delta_i : A \to A \otimes A$ such that

$$\Delta_\hbar(a) = \Delta_0(a) + \Delta_1(a)\hbar + \Delta_2(a)\hbar^2 + \cdots$$

for every $a \in A$. Here Δ_0 needs to be the original comultiplication from A.

2. Definition 6.11 agrees with Definition 6.2 for algebras.

Example 6.13.

1. Every Hopf algebra A admits the trivial deformation $A[\![\hbar]\!]$. In the form Equation (4) the structure maps of this deformation are given by the $\Bbbk[\![\hbar]\!]$-linear extensions of the structure maps of A. A deformation of A is *trivial* if it is equivalent to the trivial deformation.
2. One an make the algebra deformation $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ of $\mathrm{U}(\mathfrak{sl}_2)$ into a Hopf algebra deformation via he comultiplication

$$\Delta_\hbar(H) = H \otimes 1 + 1 \otimes H, \quad \Delta_\hbar(E) = E \otimes K + 1 \otimes E, \quad \Delta_\hbar(F) = F \otimes 1 + K^{-1} \otimes F$$

the counit

$$\varepsilon_\hbar(H) = 0, \quad \varepsilon_\hbar(E) = 0, \quad \varepsilon_\hbar(F) = 0,$$

and the antipode

$$S_\hbar(H) = -H\,, \quad S_\hbar(E) = -K^{-1}E\,, \quad S_\hbar(F) = -FK\,.$$

We note that it follows from this formulas from $K = e^{\hbar H}$ that

$$\Delta_\hbar(K) = K \otimes K\,, \quad \varepsilon_\hbar(K) = 1\,, \quad S_\hbar(K) = K^{-1}\,.$$

For the elements E, F, K, K^{-1} in $\mathrm{U}_\hbar(\mathfrak{sl}_2)$ we hence regain the formulas for the Hopf algebra structure of $\mathrm{U}_q(\mathfrak{sl}_2)$.

6.3 Deformation of Representations

We lastly give an explanation of how the irreducible, finite-dimensional representations $\mathrm{L}(n)$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ can be used to construct the irreducible, finite-dimensional representations $\mathrm{L}(q^n)$ of $\mathrm{U}_q(\mathfrak{sl}_2)$, where $n \in \mathbb{N}$.

Theorem 6.14 ([CP95, Proposition 6.4.10]). For every natural number $n \in \mathbb{N}$ let $V(n)$ be the free $\Bbbk[\![\hbar]\!]$-module of rank $n+1$ with basis $v_0, \dots, v_n$.

1. There exists a unique $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-module structure on $V(n)$ such that

$$Hv_i := (n-2i)v_i\,, \quad Ev_i := [n-i+1]_q v_{i-1}\,, \quad Fv_i := [i+1]_q v_{i+1}\,.$$

2. The $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-modules $V(n)$ is indecomposable.
3. The $\mathrm{U}_\hbar(\mathfrak{sl}_2)$-module $V(n)$ reduces modulo $\hbar$ to the irreducible representations $\mathrm{L}(n)$ of $\mathrm{U}(\mathfrak{sl}_2)$.
4. The action of K on $V(n)$ is given by

$$K \cdot v_i = q^{n-2i} v_i\,.$$

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