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A WALRASIAN THEORY OF SOVEREIGN DEBT AUCTIONS WITH ASYMMETRIC
INFORMATION

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Working Paper 24890

<http://www.nber.org/papers/w24890>

NATIONAL BUREAU OF ECONOMIC RESEARCH

1050 Massachusetts Avenue

Cambridge, MA 02138

August 2018

We thank Fernando Alvarez, Markus Brunnermeier, Sergei Glebkin, Jakub Kastl, Felix Kubler, George Mailath, Aviv Nevo, Andy Postlewaite, Tommaso Porzio, Giacomo Rondina, Laura Veldkamp, Neil Wallace and seminar participants at Central Bank of Chile, EIEF, EUI, Penn State, Princeton, UBC, UCLA, UCSD, UPenn, San Francisco Fed, Wharton, Zurich, the 2017 Cowles Conference on General Equilibrium at Yale, the 2017 SED Meetings in Edinburgh, and the 2018 EIRS Conference in Barcelona for comments. The usual waiver of liability applies. Harold Cole received support from the NSF through grant #1726976. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

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[A Walrasian Theory of Sovereign Debt Auctions with Asymmetric Information](#)

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[NBER Working Paper No. 24890](#)

[August 2018](#)

[JEL No. D4,D44,D53,E4,E5,F34](#)

[ABSTRACT](#)

[How does investors' information about a country's fundamentals, and the fact that this information may be asymmetrically held, affect a country's financing cost? Motivated by this question, and by the observation that sovereign bonds are usually auctioned in large lots to a large number of potential investors, we develop a novel model of auctions with asymmetric information that relies on price-taking and rational expectations. We first characterize sovereign bond prices for different degrees of asymmetric information under two commonly-used protocols: discriminatory-price auctions and uniform-price auctions. We show that there is trade-off between these protocols if information is sufficiently asymmetric: expected bond yields are higher when pricing is discriminatory, but yield volatility is higher when pricing is uniform. We then study endogenous information acquisition and find that \(i\) discriminatory auctions may display multiple welfare-ranked informational equilibria, and \(ii\) investors are less likely to acquire information in uniform auctions.](#)

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1 Introduction

Governments finance their fiscal needs by selling bonds in sovereign debt auctions. In these auctions, a large number of bonds are usually sold at one time to a large number of investors. Investors can submit multiple bids and are often free to try and buy as many units of the bonds as they can afford.¹ Given the set of submitted bids, auctions constitute a series of rules that determine the price(s) of newly issued sovereign bonds and the corresponding revenues for the government (or, equivalently, the future debt burden implied by the auction outcome). The results of sovereign debt auctions thus play a critical role in determining governments' cost of financing deficits, their implementation of monetary policy, and even the extent to which they can successfully navigate internal and external macroeconomic shocks.

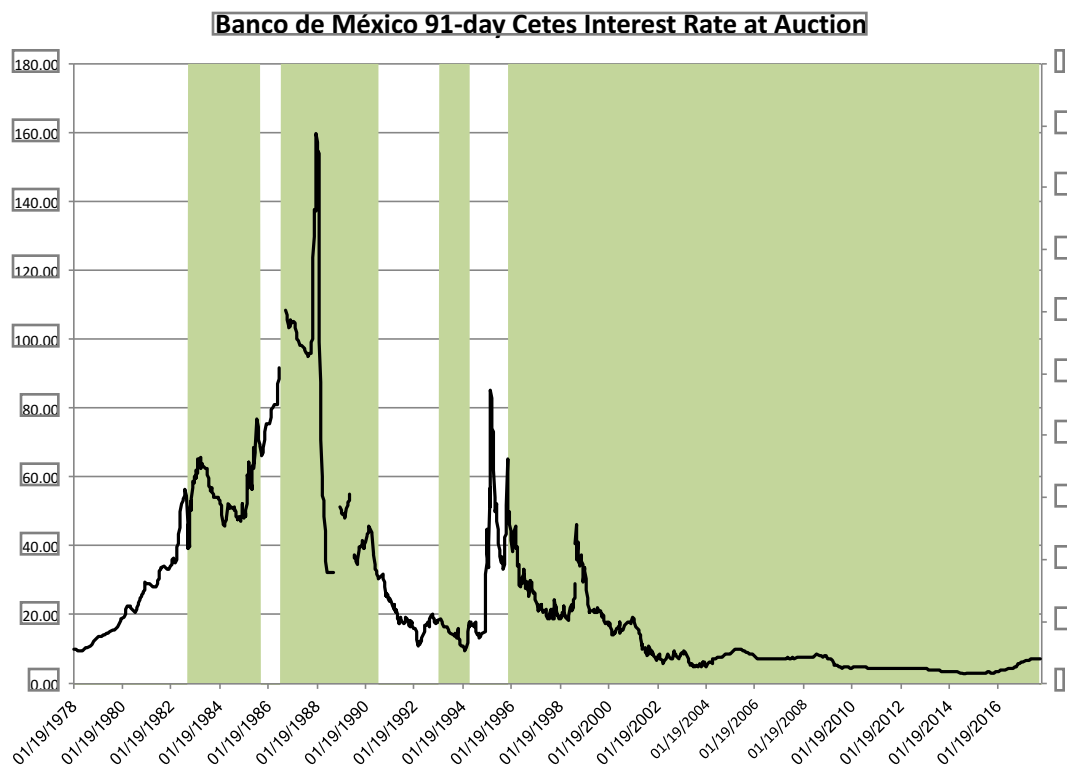
Understanding the evolution of primary market sovereign bond prices has proven to be challenging, however. Figure 1 illustrates this problem using interest rate data for 91-day CETES bonds issued by the Mexican government between 1978 and 2016. These bonds are domestically denominated, sold with small face values, and in large lots, to a wide variety of investors, using auctions that alternated between discriminating (shaded in the figure) and uniform price protocols.² During the long period displayed in the figure, annual bond yields went through periods of high turbulence (coincident to events such as the Latin American debt crisis of the 1980s and the "Tequila Crisis" of 1995) and periods of prolonged stability (the 2000s). Nevertheless, it has been difficult to attribute these price movements to shocks to particular set of "fundamentals," such as GDP growth or the government's debt service policy. Rather, certain shocks seem to be important drivers of bond yields during certain times but irrelevant in other periods. Deciphering the right mapping from shocks to prices is complicated by the fact that shocks will

¹Malvey, Archibald, and Flynn (1995) report that the U.S. Treasury typically receives 75-85 competitive bids or tenders, many of which come from the 37 primary deals. They also receive 850-900 noncompetitive tenders through the book-entry system and another 19,000 through TREASURY DIRECT.

²Cetes are zero-coupon bonds which investors can obtain directly online by using Cetesdirecto since 2010. Cetes remain among the most important public debt instruments in Mexico. In 2001, for example, Cetes alone represented 25% of all government securities, and were auctioned 180 times to 3,581 participating bidders. Mexico has switched the auction protocol in October 5, 2017 to uniform price auctions after more than two decades of using discriminatory-price auctions.

typically differ in the mechanism by which they affect bond prices: while shocks to GDP growth or debt service are likely impact investors' common value by altering the expected probability of default, liquidity or wealth shocks are more likely to affect private valuations. Furthermore, the information environment is likely to differ across shocks: while some fundamentals, such as GDP growth or inflation are often publicly observable (albeit perhaps with a lag), others are difficult or costly too learn and evaluate. In the Mexican context, a particularly pertinent example is knowledge of the inner workings of financial negotiations between Clinton and Congress over the 1995 bailout.³

Figure 1: Mexican Bond Auction Interest Rates



In line with these observations, Aguiar et al. (2016b) study emerging market interest rate spreads (in secondary markets against LIBOR) and show that they vary substantially across both countries and time. One standard explanation for this variation is uncer-

³On January 30, 1995, at exactly the moment when the Mexican government was informing the Clinton Administration that without an emergency injection of funds it would have to default, the Speaker of the House, Newt Gingrich, was informing the Clinton Administration that the bailout bill was stalled in the Congress. See Chun, John H. "Post-Modern Sovereign Debt Crisis: Did Mexico Need an International Bankruptcy Forum." *Fordham L. Rev.* 64 (1995): 2647.

tainty about the likelihood of default or renegotiation. While these spreads are partially and occasionally accounted for by country fundamentals, like debt-to-output ratios or the growth rate of output, the high (on average) spreads on emerging market debt relative to actual defaults suggest that the pricing of these bonds include a substantial premium. While some the fluctuations in this premium can be accounted for with aggregate risk-pricing factors (like the aggregate price-earning ratio, which is a measure of the market price of risk, or the VIX, which is a measure of the volatility premium), the bulk of it seemingly cannot.

In this paper we argue that (potentially endogenous) changes in investor information may be an important driver of the observed premium on sovereign debt. To make this point, we depart from the standard presumption that fundamentals are always in the information set of investors and explore how prices are determined if investors have to acquire at a cost information about non-public fundamentals (or pay a cost to process information about public fundamentals). This introduces the possibility that investors are asymmetric in their information sets, and that this asymmetry is reflected in bond prices.

To circumvent some of the challenges standard auction models face in determining equilibrium prices, we propose a novel auction model with three key characteristics: (i) the good being auctioned is perfectly divisible, (ii) the number of bidders is large, and (iii) there is both common uncertainty about the good quality and about the mass of investors who participate in the auction. Given these three characteristics, the price-quantity strategic aspects of standard auction theory become less relevant, and a price-taking, or Walrasian, analysis emerges as a good approximation.⁴ With price-taking, instead of studying strategic bidding and information processing, we can focus on a *rational expectations equilibrium* where the “market” is cleared using realistic auction rules. We refer to such auction approximated by price-taking as *Walrasian auctions* and show that they are particularly tractable, allowing for an analysis of the role of information in equilibrium price

⁴Recent auction literature shows that price-taking arises as the number of bidders get large. A recent example is Fudenberg, Mobius, and Szeidl (2007), who show that the equilibria of large double auctions with correlated private values are essentially fully revealing and approximate price-taking behavior when the number of bidders goes to infinity. Another is Reny and Perry (2006) who show a similar result when bidders have affiliated values and prices are on a fine grid.

determination and the role of auction rules in determining the volume and asymmetry of information endogenously acquired by investors. As a corollary, we show that using auction protocols may circumvent the Grossman and Stiglitz (1980) paradox: there are strict incentives to acquire information even when equilibrium prices are fully revealing ex-post.

As with Cetes in Mexico, sovereign debt auctions worldwide are generally conducted using one of two formats: uniform-price or discriminatory. Discriminatory-price auctions are slightly more prevalent, while the uniform-price auctions is the standard method used in the United States.⁵ Given the prevalence of both protocols, and the frequency at which some countries switch between protocols (as is clear for Mexico), we consider both types of auction protocols and examine their implication for both government and investor utility.

To maintain tractability while allowing for asymmetric information, we will assume that there are just two types of investors: the informed who know more about bond's default probability, and the uninformed who know less. We are then able to characterize the rational expectations equilibria of our auction model under both auction formats for different degrees of information asymmetry (as proxied by the fraction of informed investors). This success comes despite the fact that we deviate from the standard CARA-normal framework commonly used in portfolio choice models with asymmetric information thereby allowing for investor wealth effects.⁶ We view this as an attractive feature of a theory of government bond yields, in that it may help the theory speak to bond yields during both normal times and crises.

When the degree of asymmetric information among investors exogenous, we uncover

⁵The heterogeneity of treasury auction formats is well-documented. For example, the U.S. switched to a uniform-price format from a discriminating-price format in the 1970s, while Canada and Germany use the price-discriminating format. Bartolini and Cottarelli (2001) study a sample in which 39 out of 42 countries use discriminatory price auctions. Brenner, Galai, and Sade (2009) analyze a sample of 48 countries, out of which 24 use discriminatory-price auctions, 9 use uniform-price auctions and the rest use either both or an hybrid between the two.

⁶Portfolio choice models with asymmetric information in which prices do not perfectly reveal the quality of the bond ex-post are difficult to analyze because expectations, and hence excess demands, may not be continuous in the price of an asset. It is well known that proving the existence of an equilibrium when prices are not fully revealing is very difficult (see Allen and Jordan (1998)), and that this challenge is usually tackled assuming CARA-normal cases because this combination yields a tractable linear price function.

a level-volatility trade-off between the two protocols. While uniform-price auctions generate a higher sensitivity to demand shocks, the discriminatory-price auction generates a higher debt burden (lower average prices for the bond). Interestingly, this trade-off is only present when there is some degree of information asymmetry. When information (or the lack of information) is symmetric, both protocols are nearly identical in terms of average price and average exposure to demand shocks.

Our paper fills an important gap in the sovereign debt literature, which has typically focused on bond yields in secondary market, but has neglected the specifics of how a government sells its bonds and the role of investors' information in determining issue prices. This is surprising since issue prices, rather than secondary market prices, enter the government's budget constraint. To focus squarely on the determination of auction prices, we neglect some of the issues studied in the literature, but expand on others. First, most papers study sovereign default as the outcome of governments' strategic choice, but use a parsimonious model of investor optimization. We take the opposite route, and focus on the auction mechanics and investors choices while entirely neglecting strategic considerations on the part of the government.⁷ Second, most of the literature generates a fixed mapping between the bond quality and its price by assuming that investors are risk neutral and then requiring that the return, adjusted for the probability of default, equals the risk-free rate. We depart from these settings dramatically by assuming risk aversion and specific auction clearing rules. Finally, while there has been some attention to the impact of the timing of decisions and of debt maturity in sovereign markets (see Aguiar et al. (2016a)), the actual mechanics of how sovereign bonds are sold in reality through auctions and their impact on observed prices has been ignored. Our paper focuses on this neglected role of information acquisition in the context of an explicit auction model. Additionally, there has been a recent effort to empirically document the implications of different auction protocols and of the information sharing across dealers on the revenue of governments. For the former, see the survey by Hortaçsu (2011). For the latter, see Boyarchenko, Lucca, and Veldkamp (2017). Because our setting differs substantially from

⁷See for example Eaton and Gersovitz (1981), the review articles by Aguiar and Amador (2013) and Aguiar et al. (2016b), and the recent quantitative literature by Aguiar and Gopinath (2006), Arellano (2008), Chatterjee and Eyigungor (2012), Bocola and Dovis (2016).

these papers, we discuss in detail our paper's relationship with this literature once we have presented our model. Similarly, we also postpone a discussion of the broader relationship between our Walrasian auctions and the literature on general equilibrium theory and auctions.

2 Model with Exogenous Information Asymmetry

2.1 Environment

There are two periods, period 1 and period 2, and a single good (the numeraire). The economy is populated by a measure one of ex-ante identical risk-averse investors, and a government. The government is modeled mechanically: it needs to raise D units of the numeraire in period one by auctioning (multiple units of) a bond that promises repayment in period two.⁸ Without loss of generality, we study zero-coupon pure-discount bonds which promise a claim to one unit of the numeraire. Bonds are risky because the government only delivers the promised payment if it does not default. If the government defaults, then investors cannot recover any of their investment. If the amount of money raised at auction falls short of D , then government simply defaults on any bonds that it has already sold. (We can also take this to mean that the government defaults on the bonds coming due in period one).

The government's default probability, κ_θ is random and determined by the realization of an exogenous state of the world $\theta \in \{g, b\}$. We assume that $\kappa_g < \kappa_b$ and that the ex-ante probability of each state is given by $f(g)$ and $f(b)$ respectively, with $f(g) + f(b) = 1$. Since the default probability determines the expected repayment of the bond, we refer to the realization of κ as a *quality shock*, where the bond with κ_g is a *good quality bond* and the one with κ_b is a *bad quality bond*.⁹

⁸The supply of bonds being auctioned to raise D is therefore pinned down by the realized bids. This is meant to capture the impact of revenue needs on the government's cost of financing.

⁹It is straightforward to think of the possible κ_θ realizations being themselves governed by an aggregate public shock ν at the beginning of the period. We anticipate that such a public shock would be likely to play an important role in accounting for data like that in Figure 1. However, since the analysis that follows remains the same, we do not explicitly incorporate it.

Investors consume only in the second period. Accordingly, their objective is to maximize their expected utility over second period consumption given a strictly concave flow utility function U . Each investor has wealth W in period one and can either invest in a risk-free bond (storage) or the risky bond being auctioned by the government.

To allow for the possibility that prices are not fully revealing of θ ex-post, we introduce a *demand shock*. Specifically, we assume that a random share of investors η cannot participate in the auction and instead invests in the risk-free bond only. We assume that η is continuously distributed on the interval $\mathcal{H} = [0, \eta_M]$ according to a continuous density function $g(\eta)$ that is nonzero everywhere on the interior of the interval, with $\eta_M < 1$. We will refer to $s = (\theta, \eta)$ as the state of the world and to the set of states by $\mathcal{S} = \{g, b\} \times \mathcal{H}$.

A natural interpretation of η is that it represents the fraction of investors who suffer a liquidity or hedging shock. Another is that some investors may randomly have access to more favorable investment opportunities and thus do not invest in government bonds. In the context of the auction literature, the shock to demand coming through η can be therefore be thought of as a *correlated private value shock*, while the shock to the quality of the bond coming through θ is a *common value shock*. Note that the demand shock is isomorphic to a supply shock due to which the government needs to raise $D\psi$ at the auction, where $\psi = 1/(1 - \eta)$. We will later use this alternative interpretation in our numerical illustration.

There will be two types of investors at the auction: those who are *informed* (I) about θ and thus know its realization, and those who are *uninformed* (U) about θ and do not know its realization. We denote by $i \in \{I, U\}$ the type of investor and use $n \in [0, 1]$ to denote the share of informed investors. The remaining share $1 - n$ thus consists of uninformed investors. The fraction n thus determines the *degree of asymmetric information* in the sense that it measures the relative mass of investors with superior information about the quality of the bond. Consistent with our mechanical modeling of the government we assume that it observes neither θ nor η before the auction. This precludes signaling by the government. Because informed (uninformed) investors are otherwise identical, we can refer without loss of generality to a representative informed (uninformed) investor. Neither type of investor is informed about η , which means that all investors face some

uncertainty about the state of the world. We will show below that this implies that all investors face uncertainty about the minimum price at which they can buy the bond.

2.2 Auction Protocols

We now describe the auction protocol governing which investor bids the government accepts and at which price accepted bids are executed. A *bid* by an investor is defined to be a pair $\{P, B\}$ representing a commitment to purchase B units of the bond at a price no higher than P , should the government decide to accept the bid. Each investor is free to submit as many bids as desired at the beginning of the auction. The government treats each bid independently, sorts all bids from the highest to the lowest bid price, and accepts all bids in descending order until it raises D in revenue. We refer to the (highest possible) “lowest” accepted price as the *marginal price* \bar{P} . All bids at prices above the marginal price are accepted; all bids below are rejected. Hence we refer to these bids above the marginal price as *in the money*, and to bids below the marginal price as *out of the money*. (If there is excess demand at the marginal price, we assume that the government rations bids pro-rata. This does not occur in equilibrium, however.)

Investors lack commitment in two important dimensions. First, they cannot commit to honor any intertemporal contracts. We will take this to mean that they cannot borrow at the riskfree rate, nor can they short-sell the bond at the auction. Investors must therefore bid nonnegative quantities ($B \geq 0$) and can spend no more than their wealth W on bonds. Second, they cannot commit to credibly share their information about θ . Hence investors cannot directly acquire information from other investors. A unit of the bond is a claim to a real unit of the numeraire in period two. As this claim either pays 1 or 0, the range of possible prices is $P \in [0, 1]$. Since investors will typically find it optimal to submit multiple bids, we start by taking the investors’ strategy to be a bid function $B^I(P|\theta)$ for the informed and $B^U(P)$ for the uninformed.

The price at which an accepted bid is executed depends on the auction protocol. We consider two protocols that are widely used in a large multi-unit auctions of common-value goods, not just sovereign bonds. The first is the *discriminatory-price* (DP) auction in

which all accepted bids are executed at the bid price ("pay as you bid"). The second is the *uniform-price* (UP) auction in which all accepted bids are executed at the lowest accepted (or marginal) price. Government and investors take the auction protocol as given.

Denote the marginal price in state s by $\bar{P}(s)$. In a UP auction, the amount that the government raises in state s is

$$(1 - \eta) \int_{\bar{P}(s)}^1 [nB^I(P|\theta(s)) + (1 - n)B^U(P)] dP \bar{P}(s),$$

where $\theta(s)$ is the quality shock that corresponds to state s . The government's revenue is thus simply the marginal price $\bar{P}(s)$ multiplied by the accepted number of in the money bids at the marginal price. Revenue is increasing in the marginal price $\bar{P}(s)$, but the number of accepted bids is decreasing in the marginal price. A government that needs to raise more funds will thus have to accept a lower marginal price. As the auction clears when the demand equals D there may be multiple marginal prices at which the government raises D . As we are not interested in this particular source of multiplicity, we will focus on the equilibrium with the highest marginal price.

In a DP auction, the amount that the government raises in state s is

$$(1 - \eta) \int_{\bar{P}(s)}^1 [nB^I(P|\theta(s)) + (1 - n)B^U(P)] P dP$$

In contrast to the UP auction, revenue is always declining in the marginal price $\bar{P}(s)$. This is because executed prices are fixed at the bidding price while the number of accepted bids is decreasing in the marginal price. The marginal price is defined such that the government's revenue is equal to D . Hence it is immediately clear that a government that needs to raise more funds must accept a lower marginal price.

2.3 Bidding

Investors have rational expectations: the set of marginal prices, their probabilities, and the states associated with each marginal price are all common knowledge when submitting bids. After the auction is completed and the realization of the marginal price has been

revealed, informed and uninformed investors can make inferences with respect to the state. This is straightforward for informed investors since they know θ and can infer η by inverting the price schedule. Inference is harder for the uninformed. If the price $\bar{P}(s)$ is uniquely generated by a quality shock, then they too can infer the state perfectly. If there are two states (g, η_g) and (b, η_b) with a common price, then they will still be able to update their beliefs about the set of possible states and their probabilities from observing the price. However, they will not be able to uniquely identify the true state. Importantly, this ex-post information is of limited use because all of the investors must choose their bids *prior* to observing the realized marginal price. Inference thus helps investors form expectations regarding the states in which a given bid at price P is accepted, but the auction format precludes them from revising their bids given the information contained in prices ex-post. This ex-ante commitment to bids distinguishes our auction model from canonical noisy rational expectations models such as Grossman and Stiglitz (1980). The advantage of informed investors is that they know θ when submitting bids, and thus face uncertainty only about the values of η for which a given bid is accepted. Uninformed investors instead face uncertainty about both θ and η when forecasting the set of states for which a given bid is in the money.

In a DP auction, it is a strictly dominant strategy to bid only at possible marginal prices $\bar{P}(s)$. If the investor were to bid at a slightly higher price, the bid is accepted in the same states but the investor pays a higher price. In the UP auction it is a weakly dominating strategy to do so. We thus restrict investors to only bid at marginal prices. This means that we no longer need to think of our investors having a bidding strategy for each possible price but instead think of them as choosing how many bonds to bid for at the marginal price for each state. Since bids only happen at marginal prices, we drop the notation \bar{P} and just refer to P . We also switch to a starker specification for bids and prices.

Definition 1. For each state $s = (\theta, \eta) \in \mathcal{S}$, the marginal price is $P(\theta, \eta)$. The set of marginal prices is \mathcal{P} . An action for an uninformed investor is a function $B^U(\theta, \eta)$ denoting the number of bids at marginal price $P(\theta, \eta)$. An action for an informed investor is a function $B^I(\theta, \eta | \hat{\theta})$ denoting the number of bids at marginal price $P(\theta, \eta)$ when the realized quality shock is $\hat{\theta}$.

Remark 1. Bids in any two states (g, η_g) and (b, η_b) such that $P(g, \eta_g) = P(b, \eta_b)$, are perfect substitutes since they are accepted and rejected in the identical sets of states. Thus, the investor just chooses the total quantity $B(g, \eta_g) + B(b, \eta_b)$ to bid at the price $P = P(g, \eta_g) = P(b, \eta_b)$.

Remark 2. This stark specification allows us to directly compare our auction to a competitive equilibrium. It is also particularly helpful when the set of η 's is finite, so that the set of possible marginal prices is as well. In our original specification of actions as bids on the set of all potential prices $P \in [0, 1]$ this would mean that the bid function would be positive only at a finite set of points corresponding to those marginal prices. But even when η is continuous, the set of marginal prices is a strict subset of the set of potential prices.

3 Auction Equilibrium

3.1 Definition

We start defining the problem of the uninformed investor. If the government ends up defaulting in the second period, the uninformed investor simply consumes the unit payoff from his risk-free bonds, which we denote by $B_{RF}^U(s)$. If the government does not default, then the investor additionally consumes the unit payoff from his total holdings of the risky bond, which we denote by $B_R^U(s)$. Hence the expected payoff to an uninformed investor is the probability-weighted integral over the conditional payoffs in each state (θ, η) , i.e.

$$V^U \equiv \sum_{\theta \in \{g, b\}} \int_{\eta} \left\{ U(B_{RF}^U([\theta, \eta])) \kappa_{\theta} + U(B_{RF}^U([\theta, \eta]) + B_R^U([\theta, \eta])) (1 - \kappa_{\theta}) \right\} f(\theta) g(\eta) d\eta. \quad (1)$$

The total number of risky bonds purchased by an uninformed bidder in each state, $B_R^U(s)$, is the “sum” of in-the-money bids,¹⁰

$$B_R^U(s) \equiv \sum_{s': P(s') \geq P(s)} B^U(s'). \quad (2)$$

The total expenditure on risky bonds determines the investor’s holding of the risky bonds $B_{RF}^U(s)$. Since the price at which a bid is executed depends on the auction format and the state of the world s , so does the expenditure on risky bonds. Given that the price of risk-free bonds is normalized to one, we have

$$\begin{aligned} \text{UP auction : } B_{RF}^U(s) &= W - \sum_{s': P(s') \geq P(s)} B^U(s') P(s), \quad (3) \\ \text{DP auction : } B_{RF}^U(s) &= W - \sum_{s': P(s') \geq P(s)} B^U(s') P(s'). \quad (4) \end{aligned}$$

The short-sale and borrowing constraints for an uninformed investor are:

$$B^U(s) \geq 0 \text{ and } B_{RF}^U(s) \geq 0 \quad \forall s \in \mathcal{S}. \quad (5)$$

Definition 2 (Uninformed Investor Decision Problem). *The decision problem of an uninformed investor is to choose $B^U(s)$ for all $s \in \mathcal{S}$ to maximize (1) subject to (2), (3) and (5) for the UP auction, and subject to (2), (4) and (5) for the DP auction.*

Informed investors make bids conditional on the realized quality shock $\hat{\theta} \in \{g, b\}$. Given that it is optimal to bid only at prices that are marginal in some state s , informed investors thus choose bids $B^I(s, \hat{\theta})$ for all s and $\hat{\theta}$. We can further reduce the dimensionality of the problem by observing that it is a strictly dominant strategy to bid only at

¹⁰Formally, since the state s is composed of a discrete quality shock θ and a continuous demand shock η , we should define the total number of risky bonds purchased by an uninformed bidder in state $s = \{\theta, \bar{\eta}\}$ as $B_R^U(s) \equiv B_R^U(\theta, \bar{\eta}) \equiv \sum_{\theta} \int_{\eta: P(\theta, \eta) \geq P(\theta, \bar{\eta})} B(\theta, \eta) d\eta$. However, we find the abuse of notation in (2) useful for both expositional reasons, since it captures the insight that in-the-money-bids correspond to a sum over sets of prices above the marginal price while simplifying the equations, and because it allows for a transparent connection with the special case of our model in which η is assumed to be discrete, which we use in our discussion of DP auctions. We therefore use the convention that $\sum_{s': P(s') \geq P(s)} \equiv \sum_{\theta} \int_{\eta: P(\theta, \eta) \geq P(\theta, \bar{\eta})}$ throughout the paper.

prices that are marginal conditional on the realized θ . Hence we can restrict attention bid functions of the form $B^I([\theta, \eta], \theta)$. Define $B_R^I(s, \theta)$ and $B_{RF}^I(s, \theta)$ to be the total purchases of the risky bond and the risk-free bond conditional on (θ, η) and the auction protocol respectively. Then the expected payoff to an informed investor is

$$V^I \equiv \int_{\underline{\eta}}^{\bar{\eta}} \left\{ U(B_{RF}^I([\theta, \eta], \theta)) \kappa_{\theta} + U(B_R^I([\theta, \eta], \theta) + B_{RF}^I([\theta, \eta], \theta)) (1 - \kappa_{\theta}) \right\} g(\eta) d\eta \quad \forall \theta \in \{g, b\}. \quad (6)$$

The total purchases of risky bonds are

$$B_R^I(s, \theta) \equiv \sum_{s': P(s') \geq P(s)} B^I(s', \theta) \quad \forall \theta \in \{g, b\}, \quad (7)$$

and the holdings of the risk-free bond are

$$\text{UP auction: } B_{RF}^I(s, \theta) = W - \left[\sum_{s': P(s') \geq P(s)} B^I(s', \theta) \right] P(s) \quad (8)$$

$$\text{DP auction: } B_{RF}^I(s, \theta) = W - \left[\sum_{s': P(s') \geq P(s)} B^I(s', \theta) P(s') \right] \quad (9)$$

The short-sale and borrowing constraints are

$$B^I(s, \theta) \geq 0 \text{ and } B_{RF}^I(s, \theta) \geq 0 \quad \forall s \in \mathcal{S} \text{ and } \forall \theta \in \{g, b\}. \quad (10)$$

Definition 3 (Informed Investor Decision Problem). *The decision problem of an informed investor is to choose $B^I(s, \theta)$ for all $s \in \mathcal{S}$ and each realized $\theta \in \{g, b\}$ to maximize (6) subject to (7), (8) and (10) for the UP auction, and subject to (7), (9) and (10) for the DP auction.*

The auction-clearing constraint which ensures that the government raises the required revenue in every state s can then be stated compactly as

$$(1 - \eta) \left[W - B_{RF}^I(s, \hat{\theta}(s)) \right] + (1 - \eta) \left[W - B_{RF}^U(s) \right] = D \quad \forall s, \quad (11)$$

where $\hat{\theta}(s)$ denotes the quality shock associated with s .

The auction-clearing constraint must be satisfied state-by-state, even though bids are submitted prior to revelation of the state. This opens up the possibility that bids associ-

ated with the marginal price in state \bar{s} may be enough to clear the market in some other state s even though the marginal price associated with the two states were deemed to be distinct by investors. To ensure that this does not occur in equilibrium, we must impose a cross-state pricing restriction that we call the *bid overhang constraint*. This constraint requires that there *cannot* exist a state s such that $P(\bar{s}) > P(s)$, and at the marginal price $P(s)$, there is enough demand to generate the government's revenue needs in state s . As per our auction protocol, even though $P(s)$ may satisfy clearing in state s it cannot be an equilibrium price since the government would *also* be able to raise those funds at the higher price $P(\bar{s})$. The formal definition is as follows.

Definition 4 (Bid-Overhang Constraint). *For any $s \in \mathcal{S}$ and for all $\bar{s} \in \mathcal{S} : P(\bar{s}) > P(s)$, the bid overhang constraint for the UP and DP auction protocols is given by*

$$\begin{aligned}
 \text{UP : } (1 - \eta(s)) & \left\{ \begin{array}{l} (1 - n) \left[\sum_{s' : P(s') > P(\bar{s})} B^U(s') \right] \\ + n \left[\sum_{s' : P(s') > P(\bar{s})} B^I(s', \theta(s)) \right] \end{array} \right\} P(s) < D \quad (12) \\
 \text{DP : } (1 - \eta(s)) & \left\{ \begin{array}{l} (1 - n) \left[\sum_{s' : P(s') > P(\bar{s})} B^U(s') P(s') \right] \\ + n \left[\sum_{s' : P(s') > P(\bar{s})} B^I(s', \theta(s)) P(s') \right] \end{array} \right\} < D.
 \end{aligned}$$

Notice that $P(s)$ is obtained from the auction-clearing condition in state s when evaluating the demand at $P(s)$, and analogously for $P(\bar{s})$. If $P(\bar{s}) > P(s)$, the constraint implies that there cannot be excess demand in state s when demand is instead evaluated at $P(\bar{s})$.

One can see from inspection that the bid-overhang constraint cannot bind in the DP auction. This is because accepted bids are executed at the bid price, not the marginal price. Hence total revenue in state s when evaluated at $P(\bar{s})$ is always smaller than when evaluated at $P(s) < P(\bar{s})$ because fewer bids are executed. The bid-overhang constraint can and does bind in the UP auction, however. The reason is that there is a trade-off between the number of accepted bids and the price paid on all accepted bids in a UP auction. Hence total revenue may be higher when evaluated at $P(s)$ than when evaluated at $P(s) < P(\bar{s})$.

Definition 5 (Auction Equilibrium). *An auction equilibrium is a price schedule $P : \mathcal{S} \rightarrow [0, 1]$ and bidding functions $B^U : \mathcal{S} \rightarrow [0, \infty)$ and $B^I : \mathcal{S} \times \{g, b\} \rightarrow [0, \infty)$, such that*

1. each investor type's bid function solves their problem,
2. the auction-clearing condition (11) is satisfied for all $s \in \mathcal{S}$,
3. the bid-overhang constraint is satisfied at each $s \in \mathcal{S}$.

Remark 3. Formulating an equilibrium in this stark fashion, where bids are defined as functions directly of the state, is isomorphic to a more standard formulation where bids are defined as functions of prices. The price functions are the same, as $B^U(P(s)) = B^U(s)$ and 0 elsewhere, while $B^I(P(s), \theta) = B^I(s, \theta)$ and 0 elsewhere. The main difference is that the standard formulation defines the bid function over all potential prices rather than marginal prices only. Even though the two formulations are formally identical, the standard formulation is poorly behaved computationally as the bid function is discontinuous around marginal prices given that no investor bids at non-marginal prices.

Proposition 1. For both UP and DP auctions, the price function $P(\theta, \eta)$ is decreasing in η . Conditional on θ , the price function is continuous and differentiable almost everywhere.

Proof. For the DP auction, monotonicity in η follows directly from the auction-clearing condition, while for the UP auction it follows from the bid-overhang constraint. Monotonicity and boundedness of the price function then imply continuity and differentiability almost everywhere by Lebesgue's Theorem. \square

Corollary 2. A bid made at price $P(\theta, \eta)$ is in-the-money for all $\bar{\eta} \geq \eta$ given θ , and if there exists a $\bar{\eta}$ such that $P(\bar{\theta}, \bar{\eta}) = P(\theta, \eta)$ for $\bar{\theta} \neq \theta$, then it is in-the-money for all $\bar{\eta} \geq \eta$ given $\bar{\theta}$.

3.2 Comparison of Auction Equilibrium and Competitive Equilibrium

Now we have defined an auction equilibrium, we can compare its structure with that of a competitive equilibrium, defined as a tuple consisting of a equilibrium price and demand functions such that there is no excess supply of bonds at the equilibrium price.

It is evident that a DP auction equilibrium cannot be a competitive equilibrium: there is a single market-clearing price in the latter, while bonds trade at multiple distinct prices in the former. In UP auctions, however, bonds are always sold at a single price. In many

cases, a UP auction equilibrium will therefore turn out to be isomorphic to a standard competitive equilibrium with heterogeneous information. We establish this link here.

In UP auctions, uninformed investors choose $B^U(s)$ to maximize (1) subject to constraints (2), (3) and the short-sale constraint (5) (see Definition 2). The decision problem in a competitive equilibrium is identical except for the fact that the short-sale constraint applies to *total* purchases of bonds in each state s , rather than bid-by-bid. That is, in competitive equilibrium we replace (5) with a short-sale constraint of the form

$$B_R^U(s) \geq 0 \text{ and } B_{RF}^U(s) \geq 0 \forall s \in \mathcal{S}. \quad (13)$$

The same considerations hold for the informed investor. To check whether a UP auction equilibrium is isomorphic to a competitive equilibrium, it is thus sufficient to verify whether or not constraints (5) and (13) (and their analogues for the informed investor) imply each other. To go from constraint (5) to (13) we can construct the associated total risky bond purchases just by summing over the in-the-money bids. This implies that if (5) does not bind in any state s , then neither does (13).

To go from constraint (13) to (5), we can construct the associated state-by-state bids given the total bond purchases in a competitive equilibrium, $\{B_R^I(s, \theta), B_R^U(s)\}$, using the difference between the risky bond purchases at s and those at the next highest price s' , i.e.

$$B^U(s) = B_R^U(s) - B_R^U(s'), \quad (14)$$

where $P(s') = \min \{P(s'') > P(s)\}$ for all $s'' \in \mathcal{S}$. The analogous object for the informed is constructed using their conditional total purchases $B_R^I(s, \theta(s))$.¹¹

We can then distinguish three scenarios. First, if the state-by-state bids are nonnegative in the UP auction equilibrium so that (5) does not bind, then the short-sale constraint cannot bind for total purchases and there is an associated competitive equilibrium. Second, if there is a state in which the short-sale constraint (5) binds in the UP auction equilibrium, there is an associated competitive equilibrium only if the short-sale constraint

¹¹As highlighted in Footnote 10, the mathematically accurate expression for state $s = \{\bar{\theta}, \bar{\eta}\}$ when demand shocks are continuously distributed is $B^U(s) \equiv B^U(\bar{\theta}, \bar{\eta}) \equiv \sum_{\theta} \frac{dB_R^U(\theta, \eta)}{d\eta}$

on total risky bond purchases also binds and total bids are zero. Third, there is no associated competitive equilibrium when there are states for which the nonnegativity constraint does not bind for total purchases in the competitive equilibrium, but do bind for individual bids in particular states in the UP auction. Hence UP auction equilibria are not necessarily isomorphic to competitive equilibria. The next proposition formalizes this discussion. Later on, we use this insight to guide our discussion of information acquisition incentives in auctions relative to canonical competitive models in the spirit of Grossman and Stiglitz (1980)

Proposition 3. *A UP auction equilibrium $\{P(s), B^I(s, \theta), B^U(s)\}$ in which the short-sale constraint on risky bond purchases only binds when total risky bond purchases are 0 for all $s \in S$ has an associated competitive equilibrium $\{P(s), B_{CE}^U(s), B_{CE}^I(s, \theta)\}$. Any competitive equilibrium $\{P(s), B_{CE}^I(s, \theta), B_{CE}^U(s)\}$ in which the associated bids $\{B^I(s, \theta), B^U(s)\}$ constructed using (14) are nonnegative and satisfy the bid-overhang constraint (12) is an auction equilibrium.*

Even if the differences in short-sale constraints in competitive equilibrium and the auction auction do not matter, so that the auction equilibrium has an associated competitive equilibrium, there is another condition that is unique to the auction protocol and may break the mapping between the two equilibria: the bid-overhang constraint. Hence the set of UP auction equilibria is a subset of the set of competitive equilibria. In the numerical examples, we will illustrate the mechanics underlying this observation.

3.3 Investor's Optimal Bidding

We now explicitly characterize investors' optimal bids in the two auction protocols.

3.3.1 Informed Investors

The informed investor's problem is relatively simply because they are informed about the realized quality shock θ^* . Hence they need not infer the bond's quality in states in which a given bid will be accepted, and the first-order condition for bid $B^I([\theta^*, \eta^*])$ for each state

$s = [\theta^*, \eta^*]$ at a UP auction is given by

$$\int_{\eta} \left(\begin{aligned} & -U'(B_{RF}^I([\theta^*, \eta])) \kappa_{\theta^*} P([\theta^*, \eta]) \\ & + U(B_{RF}^I([\theta^*, \eta])) \\ & + B_R^I([\theta^*, \eta]) \end{aligned} \right) (1 - \kappa_{\theta^*})(1 - P([\theta^*, \eta])) \mathcal{I}\{P([\theta^*, \eta^*]) \geq P([\theta^*, \eta])\} g(\eta) d\eta$$

$$- \chi^I([\theta^*, \eta^*]) = 0,$$

where $\chi^I(s)$ is the multiplier on the nonnegativity constraint, and $\mathcal{I}\{\cdot\}$ is an indicator function. The price $P([\theta^*, \eta])$ at which the investor buys the risky bond plays a dual role. On the one hand, it determines the set of states in which the bid is in-the-money, as captured by the indicator function. On the other hand, it also determines the price at which accepted bids are executed since the investor pays the marginal price $P([\theta^*, \eta])$ for all demand shocks given which the bid is in the money.

In the DP auction, the first-order condition is

$$\int_{\eta} \left(\begin{aligned} & -U'(B_{RF}^I([\theta^*, \eta])) \kappa_{\theta^*} P([\theta^*, \eta^*]) \\ & + U(B_{RF}^I([\theta^*, \eta])) \\ & + B_R^I([\theta^*, \eta]) \end{aligned} \right) (1 - \kappa_{\theta^*})(1 - P([\theta^*, \eta^*])) \mathcal{I}\{P([\theta^*, \eta^*]) \geq P([\theta^*, \eta])\} g(\eta) d\eta$$

$$- \chi^I([\theta^*, \eta^*]) = 0,$$

In contrast to the UP auction, the marginal price $P([\theta^*, \eta])$ now only determines the set of states the investor is in the money, but not the price at which each accepted bid is executed. Instead, all in-the-money bids are executed at the bid price.

3.3.2 Uninformed Investors

Now consider the uninformed investors' problem. Since these investors do not know the realized θ , they must form expectations regarding the quality of bond conditional on a given bid being executed. In order to rewrite the problem in terms of the expected probability of default conditional on the realized marginal price, we first explain how this inference problem is resolved.

Inference. Given that bids are executed depending on the realization of the marginal price and that the quality of a bond is fully pinned down by its default probability, the inference problem is equivalent to computing the expected default probability of a bond given the realization of a marginal price. We denote this conditional expected default probability by κ . For the informed, $\kappa(P(\theta, \eta)|\theta) = \kappa_\theta$ because they know the true θ . For the uninformed there are two cases:

1. For any (θ, η) such that $\nexists (\theta', \eta')$ with $P(\theta, \eta) = P(\theta', \eta')$, then $\kappa(P(\theta, \eta)) = \kappa_\theta$.
2. If there are two states (θ, η) and (θ', η') such that $P(\theta', \eta') = P(\theta, \eta)$ and $\theta' \neq \theta$ the solution to the uninformed investor's inference problem is as follows.

Given $P(\theta, \eta)$, define $\eta = \phi(P|\theta)$, where ϕ is the inverse function of the price with respect to η .¹² The probability h of an interval of prices $\mathbf{P} \subset \mathcal{P}$ conditional on θ is

$$h(\mathbf{P}|\theta) \equiv \int_{\{\eta: P(\theta, \eta) \in \mathbf{P}\}} g(\eta) d\eta \equiv \int_{\tilde{\mathbf{P}} \in \mathbf{P}} g(\phi(\tilde{P}|\theta)) \frac{\partial \phi(\tilde{P}|\theta)}{\partial \tilde{P}} d\tilde{P}$$

Note that the slope of the inverse function with respect to the price determines the size of the set of η 's that are associated with the prices in \mathbf{P} (given θ). The unconditional probability of the set of prices is then given by

$$h(\mathbf{P}) \equiv \sum_{\theta} f(\theta) h(\mathbf{P}|\theta)$$

and the probability of θ conditional on a price in \mathbf{P} is simply $f(\theta)h(\mathbf{P}|\theta)/h(\mathbf{P})$. We can define the probability density function of a particular price $P \in \mathbf{P}$, given θ , by shrinking the set $\mathbf{P} \rightarrow P$, and observing h in the limit, or

$$Pr(P|\theta) = \lim_{\mathbf{P} \rightarrow P} \frac{h(\mathbf{P}|\theta)}{\Delta(\mathbf{P})}$$

where $\Delta(\mathbf{P})$ is the length of the price interval. This then leads to the inferred default

¹²Note from proposition 1 that $P(\theta, \eta)$ is continuous almost everywhere and since rationing does not occur in equilibrium, strictly monotonic, thus it is invertible almost everywhere. See Rudin (1964, p. 90).

probability

$$\kappa(P) \equiv \frac{\sum_{\theta} f(\theta) Pr(P|\theta) \kappa_{\theta}}{\sum_{\theta} f(\theta) Pr(P|\theta)}$$

Optimal Bids. Given the inferred probability of default, and reordering states by marginal prices, the uninformed investor's payoff can be stated as

$$\sum_{\theta \in \{a,b\}} \int_{\eta} \left\{ U(B_{RF}^U([\theta, \eta]) \tilde{\kappa}(P([\theta, \eta]))) + U(B_{RF}^U([\theta, \eta]) + B_R^U([\theta, \eta])) (1 - \tilde{\kappa}(P([\theta, \eta]))) \right\} f(\theta) g(\eta) d\eta$$

The first-order conditions for $B^U([\theta^*, \eta^*])$ in a UP and DP auction thus are, respectively

$$\sum_{\theta \in \{a,b\}} \int_{\eta} \left\{ \begin{aligned} &= U'(B_{RF}^U([\theta, \eta]) \tilde{\kappa}(P([\theta, \eta]) P([\theta, \eta]))) \\ &+ U'(B_{RF}^U([\theta, \eta]) + B_R^U([\theta, \eta])) (1 - \tilde{\kappa}(P([\theta, \eta])) (1 - P([\theta, \eta]))) \end{aligned} \right\} \mathbb{I}\{P([\theta^*, \eta^*]) \geq P([\theta, \eta])\} f(\theta) g(\eta) d\eta - \chi^U([\theta^*, \eta^*]) = 0$$

and

$$\sum_{\theta \in \{a,b\}} \int_{\eta} \left\{ \begin{aligned} &= U'(B_{RF}^U([\theta, \eta]) \tilde{\kappa}(P([\theta, \eta]) P([\theta^*, \eta^*]))) \\ &+ U'(B_{RF}^U([\theta, \eta]) + B_R^U([\theta, \eta])) (1 - \tilde{\kappa}(P([\theta, \eta])) (1 - P([\theta^*, \eta^*]))) \end{aligned} \right\} \mathbb{I}\{P([\theta^*, \eta^*]) \geq P([\theta, \eta])\} f(\theta) g(\eta) d\eta - \chi^U([\theta^*, \eta^*]) = 0$$

Comparing this expression to the informed investors' first-order condition, it is clear that the uninformed face the same basic tradeoffs as the informed, but weighted by the expected default probability rather than the true default probability. As we show below, this leads to a form of adverse selection: when bidding at high marginal prices, uninformed investors also expect these bids to be accepted after bad quality shocks, leading to a downward revision of expected asset quality. Uninformed investors thus have weaker marginal incentives to bid at high prices than informed investors.

Proposition 4. Fixing the price schedule $P(s)$, both the informed and the uninformed bidders do strictly better under the UP protocol than the DP protocol so long as there exists a set of η 's of

positive measure at which they are making positive bids, fixing θ

Proof. The uninformed (informed) UP bidders simply have to replicate the bids of the uninformed (informed) DP bidders. Note that since $P(\theta, \eta_1) > P(\theta, \eta_2)$ if $\eta_1 < \eta_2$, they have lower expenditures on the risky bond in state (θ, η_2) under the UP protocol that they would under the DP protocol if they have positive bids in these two states. Hence they have higher purchases of the risk-free bond in (θ, η_2) . This raises their consumption in both the default and the repayment outcomes. \square

Since, fixing prices, the investors are always better off under the UP than the DP auctions, it follows that the government can only prefer the UP auction if it delivers higher prices in some appropriate average sense. Whether this will be the case is unclear since the substitution effect will encourage higher bidding under UP, but the income effect coming through the lower expenditures can have an ambiguous affect depending upon the curvature of the utility function U .

4 Characterization of UP Auction Equilibrium

We now characterize uniform-price auction equilibrium. (The characterization of the discriminatory-price auction equilibrium is in Section 5). In particular, we show how equilibrium prices depend on the fraction of informed investors n . We later use this to compute sovereign bond yields, the government's debt burden and investors' incentives to acquire information given n . Accordingly, we will use $P^{UP}(s; n)$ to denote the price function for each state and degree of asymmetric information in the UP auction. When there is no risk of confusion, we will simply write $P(s)$. In a similar fashion, we will use $B^{UP,U}(s; n)$ for the bids of the uninformed and $B^{UP,I}(s, \theta; n)$ for the bids of the informed, but use the simpler notation $B^U(s)$ and $B^I(s, \theta)$ when there is no risk of confusion.

4.1 Symmetric Benchmarks

We begin by considering the two symmetric benchmarks: the *symmetric ignorance equilibrium* in which no investor is informed ($n = 0$), and the *symmetric information equilibrium* in

which all investors are informed ($n = 1$).

4.1.1 Symmetric Ignorance

If there are no informed investors, bids and marginal prices cannot depend upon θ . Hence $P(g, \eta) = P(b, \eta)$ for all $\eta \in \mathcal{H}$ and we can simplify notation and write $P(\eta)$ for prices and $B(\eta)$ for bond purchases. Because $P(\eta)$ is declining in η , the set of demand of demand shocks for which bid $B(\eta)$ is in-the-money is $[\eta, \eta_M]$. Hence the auction-clearing condition in state η is

$$\left[\int_{\eta}^{\eta_M} B(\hat{\eta}) d\hat{\eta} \right] P(\eta) = \frac{D}{1-\eta} \quad (15)$$

This condition is satisfied only if $B(\eta) > 0$ for all η . Since all investors are symmetric, it follows that the short-sale constraint cannot bind for any η . Combining these observations with the budget constraint shows that holdings of risk-free bonds in state η are independent of $P(\eta)$:

$$B_{RF}(\eta) = W - \frac{D}{1-\eta}$$

As prices do not convey information about θ , the inference problem is trivial: the inferred default probability is the ex-ante default probability for all P , $\kappa(P) = \kappa^U \equiv f(g)\kappa_g + f(b)\kappa_b$. Since bid $B(\eta^*)$ in state η^* is in the money for all $\eta > \eta^*$, the system of first-order condition for $B(\eta^*)$ can be rewritten as

$$\int_{\eta^*}^{\eta_M} \left\{ -U'(B_{RF}(\eta)) \kappa^U P(\eta) \right\} \left\{ \pm U'(B_{RF}(\eta) \pm \int_{\eta}^{\eta_M} B(\hat{\eta}) d\hat{\eta}) (1 - \kappa^U)(1 - P(\eta)) \right\} g(\eta) d\eta = 0 \quad \text{for all } \eta^*$$

This system of equation is *block-recursive*: the term in brackets must be equal to zero for any interval $[\eta^*, \eta_M]$, and so it must also be zero for all η . Rewriting this terms as a function of only $P(\eta)$ for all η implies that $P(\eta)$ is the unique solution to

$$U\left(W - \frac{D}{1-\eta}\right) \kappa^U P(\eta) = U\left(W - \frac{D}{1-\eta} \pm \frac{D}{P(\eta)(1-\eta)}\right) (1 - \kappa^U)(1 - P(\eta)). \quad (16)$$

Remark 4 (Value of Information at $n = 0$). What is the value of information when noone else is informed? A single atomistic informed investor who is informed when noone else is faces the

same prices as the mass of uninformed investors. The difference is that the first-order condition is evaluated using the true state-contingent default probability κ_θ rather than κ^U . Since the left-hand side of (16) is increasing in κ and the right-hand side is decreasing, an atomistic informed investor earns a higher expected payoff by bidding more than the uninformed in the good state and less in the bad state.

4.1.2 Symmetric Information

If all investors are informed ($n = 1$), there is no inference problem with respect to the quality shock and all bids are contingent on θ . Hence we can compute the equilibrium conditional on the realized θ . The construction is analogous to the symmetric ignorance case, the only difference being that the first-order condition is evaluated using the true default probability κ_θ . The system of first-order conditions remains block-recursive and the marginal price $P(\theta, \eta)$ in state (θ, η) solves

$$U\left(W - \frac{D}{1-\eta}\right) \kappa_\theta P(\theta, \eta) = U\left(W - \frac{D}{1-\eta} + \frac{D}{P(\theta, \eta)(1-\eta)}\right) (1 - \kappa_\theta)(1 - P(\theta, \eta))$$

Remark 5 (Value of Information at $n = 1$). What is the cost of being uninformed when everyone else is informed? The answer may be zero. Specifically, UP auctions can generate a result similar to canonical rational expectations models with asymmetric information, namely that uninformed investors may be able to replicate the portfolio and payoffs of informed investors. The next section characterizes conditions such that this is the case.

4.1.3 Replication

In competitive equilibrium, there are no gains from acquiring information if prices are fully revealing (Grossman and Stiglitz 1980). A similar result holds in UP auction equilibrium, but under more stringent conditions. These additional restrictions arise because bids cannot be adjusted ex-post conditional on the information revealed in prices.

Proposition 5. In a UP auction, uninformed investors can replicate the total bids and ex-post payoffs of informed investors in every state (and hence informed investors' ex-ante payoff) if

1. Prices are fully revealing: each marginal price is associated with a unique state in \mathcal{S}

2. Ordering marginal prices and associated total bids of the informed from the highest to the lowest marginal price, the total bids of the informed are weakly ranked from lowest to highest.

Proof. Condition 1 ensures that the uninformed can accurately infer the state in which each bid is accepted. Condition 2 second ensures that the no short-sale constraint does not bind for marginal bids in any state. \square

Definition 6. Replication is *feasible* if the conditions in Proposition 5 are satisfied. Replication *fails* if the conditions are violated

Corollary 6. The conditions in Proposition 5 are satisfied in a UP auction equilibrium with n informed investors if $P(g, \eta_M; n) > P(b, 0; n)$ and $\mathcal{B}_R^I([g, \eta_M], g; n) < \mathcal{B}_R^I([b, 0], b; n)$

Proof. $P(\theta, \eta)$ is strictly decreasing in η by Proposition 1, and so informed investors' bids quantities are strictly increasing in η given θ . The first condition implies that $P(g, \eta) > P(b, \eta')$ for all η, η' , and so $P(s) \neq P(s')$ for any $s, s' \in \mathcal{S}$. Hence all uninformed bids on the high-price schedule are accepted in the bad state. The second condition ensures that the short-sale constraint does not bind for uninformed investors in the bad state when replicating the informed portfolio. \square

Even if uninformed investors can replicate the informed portfolio, they will generally not make the same marginal bids as the informed in every state. The reason is that, unlike the bids of the informed, all of the bids by uninformed investors on the $\theta = g$ schedule are also accepted when $\theta = b$. To replicate the informed portfolio, the uninformed must therefore make the same marginal bids as the informed at prices on the high-quality schedule, but bid less than the informed on the low-quality schedule. The cumulation of bids on both schedules is then such that the total bond holdings are identical to those of the informed. This observation also provides intuition as to when replication fails: if the second condition is violated, then the number of bonds accumulated through bids on the high-quality price schedule is so high that the uninformed would have to bid negative amounts on the low-quality schedule in order to obtain the same total portfolio. But this is ruled out by the short-sale constraint

4.2 Asymmetric Information: Special Case with Log Preferences

We now study the equilibrium with asymmetric information ($n \in (0, 1)$). To highlight how changes in n affect prices, we use a special case of our model in which investors have log preferences. This allows us to solve for bid functions in closed form.

If the short-sale constraint does not bind, the first-order condition (16) of investor $i \in \{I, U\}$ with log preferences for a bid at marginal price $P(s)$ is

$$\frac{\kappa_i(s)P(s)}{B_{RF}^i(s)} \equiv \frac{(1 - \kappa_i(s))(1 - P(s))}{B_{RF}^i(s) + B_R^i(s)}$$

where $\kappa_i(s)$ is investor i 's inferred default probability given that the bid was accepted at $P(s)$, and B_{RF}^i and $B_R^i(s) \equiv \int_{s': P(s') \geq P(s)} B^i(s')$ are total holdings of the risk-free and risky bond, respectively. The budget constraint implies that $B_{RF}^i(s) = W - P(s)B_R^i(s)$. Total expenditures on risky bonds for investor i are thus

$$P(s)B_R^i(s) \equiv \left[1 - \frac{\kappa_i(s)}{1 - P(s)} \right] W, \quad (17)$$

and are strictly decreasing in $P(s)$ and $\kappa_i(s)$, and strictly increasing in W .

To simplify notation, we use the following change of variables that expresses demand shocks η in terms of the supply of bonds per participating investors.

Definition 7. The supply of bonds per investor is $\psi \equiv \frac{1}{1-\eta}$, with $\psi \in [1, \psi_M]$ and $\psi_M \equiv \frac{1}{1-\eta_M}$.

Thus if only half of investors participate in the auction ($\eta = 0.5$), the per-capita supply of bonds doubles, $\psi = 2$. We can then equivalently express the state as $s = [\theta, \psi]$, and summarize investors' aggregated beliefs about the bond's default probability as follows.

Definition 8 (Average belief). The average belief about the risky bond's default probability in state (θ, ψ) given n is $\hat{\kappa}_\theta(\psi, n) \equiv n\kappa_\theta + (1-n)\kappa(s)$.

Holding fixed uninformed investors' beliefs, the average belief is increasing in n if $\theta = g$ and decreasing in n if $\theta = b$. The price spread across quality shocks will thus typically increase in n . Using this definition leads to an intuitive expression for bond prices.

Proposition 7. The marginal price in state $s = (\theta, \psi)$ given n is $P(s; n) \equiv 1 - \frac{\hat{\kappa}_\theta(\psi, n)}{1 - \frac{B}{W}\psi}$

Proof. Use (17) and impose auction clearing: $nP(s)B_n^I(s) + (1-n)P(s)B_n^U(s) = D\psi$. \square

Bond prices are thus declining in the average belief and the per-capita debt burden relative to wealth, and can be decomposed into a risk-neutral price and a risk premium:

$$P(s; n) = \underbrace{1 - \hat{\kappa}(\psi, n)}_{\text{Risk-neutral price}} = \underbrace{\frac{\hat{\kappa}_\theta(\psi, n)}{W}}_{\text{Risk premium}} \underbrace{\frac{1}{D\psi}}_{\text{Risk premium}}$$

Quality shocks that raise the expected default probability thus affect both the risk-neutral price and the risk-premium, while the demand shock operates solely through the risk premium by altering the per-capita risk exposure of participating investors. Prices in the benchmarks with symmetric ignorance and symmetric information follow immediately.

Corollary 8 (Prices in Symmetric Benchmarks). *The price schedules under symmetric information and symmetric ignorance are $P(\theta, \psi; n = 1) = 1 - \frac{\kappa_\theta}{1 - \frac{\kappa_\theta}{W}\psi}$ and $P(\psi; n = 0) = 1 - \frac{D}{1 - \frac{D}{W}\psi}$ respectively. Moreover, $P(g, \psi; n = 1) > P(\psi; n = 0) > P(b, \psi; n = 1)$ for all ψ*

Because prices are ranked by θ conditional on ψ , we will refer to the good (bad)-bond price schedule as high (low)-price schedule when there is no risk of confusion.

The next step is to show that equilibrium prices converge to the symmetric ignorance benchmark as n converges to zero, and that the equilibrium is equivalent to the symmetric information benchmark if $n > \eta_M$ and replication is feasible in the symmetric information benchmark.¹³ In proving this result, we show that the bid-overhang constraint binds if $n \leq \eta_M$ even if replication is feasible given $n = 1$. This implies that price schedules will overlap for sufficiently low n .

Proposition 9. *Fix $n \in (0, 1)$. Then the following is true in any UP auction equilibrium.*

- (i) *If replication is feasible for $n = 1$, then replication is feasible for all $n > \eta_M$ and fails if $n < \eta_M$. In this case, equilibrium prices are given by the symmetric information price*

¹³If replication is not feasible at $n = 1$, the system of uninformed first-order conditions can no longer be solved block recursively. The reason is that a bid in state s may determine total purchases in all states s' in which the no short-sale constraint binds, forcing total bids in those states to be the same. Hence the bid at s fully determining bond purchases in a set of states, and this will be reflected in the optimality condition associated with this choice. We discuss this in further detail in the Appendix and present a version of our numerical example in which this situation arises

schedule if $n > \eta_M$

(ii) The quality-contingent contingent price schedules converge to each other as $n \rightarrow 0$. That is, $\lim_{n \rightarrow 0} P([g, \eta]; n) = P([b, \eta]; n)$ for all $\eta \in (0, \eta_M)$.

Proof. First statement. Since replication is feasible at $n = 1$, the short-sale constraint does not bind in the informed portfolio. Thus, replication fails if and only if the bid-overhang constraint binds in at least one state. Since bids are smooth in ψ conditional on θ , the bid-overhang constraint (BOC) cannot bind in any two states (ψ, θ) and (ψ', θ) with $\psi \neq \psi'$. Since informed investors' bids are increasing in ψ and prices are ranked by θ given ψ , it follows that, if the BOC binds in some state, it first binds in the pair of states $(b, 1)$ and (g, ψ_M) . By replication, auction-clearing in (g, ψ_M) implies $P(g, \psi_M) B_R^U([g, \psi_M]) = D\psi_M$. Since $P(g, \psi_M) > P(b, 1)$, executing bids at $P(g, \psi_M)$ in state $(b, 1)$ implies that only uninformed bids are accepted. These bids are sufficient to clear the market in $(b, 1)$ if $(1 - n)P(g, \psi_M) B_R^U([g, \psi_M]) \geq D$. Hence the BOC is violated, and replication fails, if and only if $n \leq \eta_M$.

Second statement. Let $X^U([\theta, \eta]; n)$ denote uninformed investors' total expenditures on risky bonds. Then $\lim_{n \rightarrow 0} X^U([\theta, \eta]; n) \rightarrow D/(1 - \eta)$ for all θ by auction-clearing. Uninformed bids are made unconditionally on θ . Hence $\lim_{n \rightarrow 0} X^U([g, \eta]; n) \rightarrow X^U([b, \eta]; n)$ and thus $\lim_{n \rightarrow 0} P([g, \eta]; n) \rightarrow P([b, \eta]; n)$. By Proposition 1, $P(\theta, \eta; n)$ is strictly decreasing in η given θ and n . When $n \rightarrow 0$, prices must then be sorted by η . That is, there is always a ϵ small enough such that for $\eta' - \eta = \epsilon$, i.e. $P([\theta, \eta]; n) < P([\theta', \eta']; n) < P([\theta, \eta']; n)$. Since η is drawn from a closed interval, this implies that the price schedules must converge at every interior point in which the price schedules are continuous \square

What remains is a characterization of equilibrium prices for $n \leq \eta_M$. In this range, the bid-overhang constraint binds for some values of η , forcing an overlap in the high-quality and low-quality price schedules that renders replication infeasible. Prices are then determined as follows. Take any two states $s = [g, \psi_g]$ and $s' = [b, \psi_b]$ for which a binding constraint forces a common price, $P = P(s) = P(s')$. The respective auction-clearing conditions are

$$\eta \left(\frac{1 - \kappa_g - P}{1 - P} \right) + (1 - n) \left(\frac{1 - \kappa - P}{1 - P} \right) = \frac{D}{W} \psi_g \quad (18)$$

$$\text{and } n \max \left[\left(\frac{1 - \kappa_b - P}{1 - P} \right), 0 \right] + (1 - n) \left(\frac{1 - \kappa - P}{1 - P} \right) = \frac{D}{W} \psi_b. \quad (19)$$

The two endogenous variables determined by these equations are the common price P and uninformed investors' inferred default probability κ . Since $\kappa_b \leq \kappa \leq \kappa_g$, the informed demand weakly more than the uninformed if $\theta = g$ and weakly less if $\theta = b$. Hence the short-sale constraint may bind on the informed if $\theta = b$ (in particular, it binds if and only if $P \geq 1 - \kappa_b$) but not if $\theta = g$. By the same logic, it cannot bind on the uninformed. The combination of short-sale constraints and uninformed investors' endogenous belief leads to the possibility of endogenous jumps in demand and, as a consequence, multiple equilibria. Unfortunately, the endogeneity of beliefs means that the model is no longer analytically tractable. We thus use a numerical example to characterize the set of possible prices when the bid-overhang constraint binds

4.2.1 Numerical Example

The parameters we use in our numerical example are in Table 1, and are such that the two conditions for perfect replication in Proposition 5 are satisfied in the symmetric information benchmark. We will construct examples for different values of n on a grid from zero to one, for two possible values of the probability of state b , $f(b) \in \{0.25, 0.5\}$. (In the appendix we also consider parameter values such that replication fails in the symmetric information benchmark.) Comparative statics with respect to n and $f(b)$ are natural given our focus on asymmetric information, since there is little effective information asymmetry if n and $f(b)$ are close to zero or one.

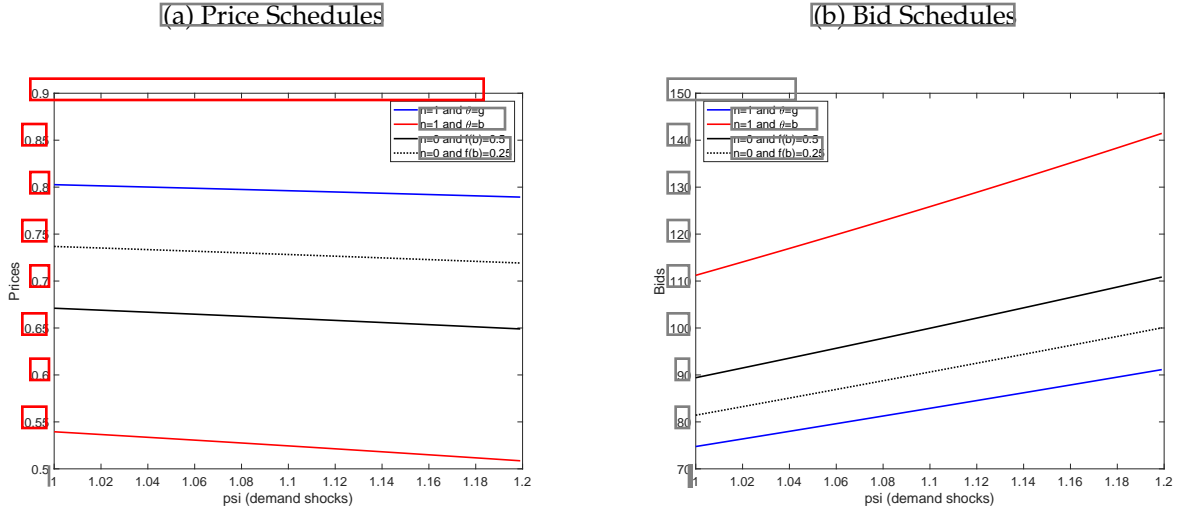
Table 1: Benchmark Parameterization

$\kappa_g = 0.15$	$\kappa_b = 0.35$
$W = 250$	$D = 60$
$\eta_M = 0.17$	$\Rightarrow \psi_M = 1.2$
$\psi \sim U[1, \Psi_M]$	

Panel (a) of Figure 2 shows the benchmark cases with symmetric information ($n = 1$) and symmetric ignorance ($n = 0$). Under symmetric information, equilibrium prices are

independent of $f(b)$ because all bids and prices are state-contingent. In the symmetric ignorance equilibrium, instead, prices do depend on $f(b)$ through its effect on the unconditional default probability κ^U . As $f(b)$ falls, the price schedules (in black) rise from the symmetric information low-quality price schedule (in red) to the symmetric information high-quality price schedule (in blue). The corresponding increasing bid schedules are in Panel (b).

Figure 2: *Symmetric Benchmarks for UP Auctions*



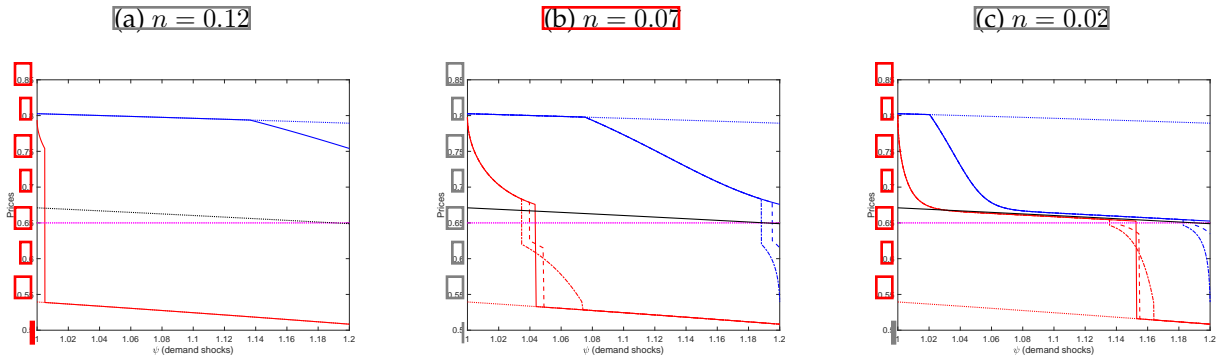
Proposition 9 shows that the equilibrium is equivalent to the symmetric information benchmark obtain for all $n > \eta_M$, and that the state-contingent price schedules will converge to each other as $n \rightarrow 0$. In the interval $n \in (0, \eta_M]$ instead, the bid-overhang constraint binds and price schedules overlap. To illustrate the resulting equilibrium, we plot equilibrium prices for $n \in \{0.12, 0.07, 0.02\}$ in Figure 3. Given that $\eta_M = 0.17$, the bid-overhang constraint binds in all three cases. Since per-capita bids are increasing in ψ conditional on θ , moreover, the state-contingent price schedules first overlap in the pair of states $(b, 1)$ and $(g, \hat{\psi}_g(n))$, where $\hat{\psi}_g(n)$ is defined such that uninformed investors' good-state bids evaluated at $P(g, \hat{\psi}_g)$ are sufficient to cover the supply of bonds in $(b, 1)$. (See the proof of Proposition 9 for an analogous construction.) Hence, the region of overlap is such that fewer investors participate conditional on a good than a bad shock.

Multiple equilibria with different prices can be sustained in our model due to the

following logic. Suppose first that the common price is low and the short-sale constraint does not bind, $P < 1 - \kappa_b$. (This threshold is plotted using pink-dotted horizontal lines.) Uninformed investors then infer from the low price that the bond is likely to be of low quality, and reduce their demand. But informed investors continue to buy, and fewer investors suffer a demand shock if $\theta = b$ than if $\theta = g$, allowing markets to clear and sustaining a low-price equilibrium.

Conversely, suppose that the common price is high and that the short-sale constraint binds on the informed. Observing a high price, uninformed investors now infer that the bond is likely to be of high quality, boosting their demand. Since informed investors do not participate, however, this is entirely consistent with market-clearing. Intuitively, short-sale constraints may thus prevent the information held by informed investors from being impounded into prices, allowing the model to sustain multiple equilibria.

Figure 3: Equilibrium Prices when the Bid-Overhang Constraint Binds

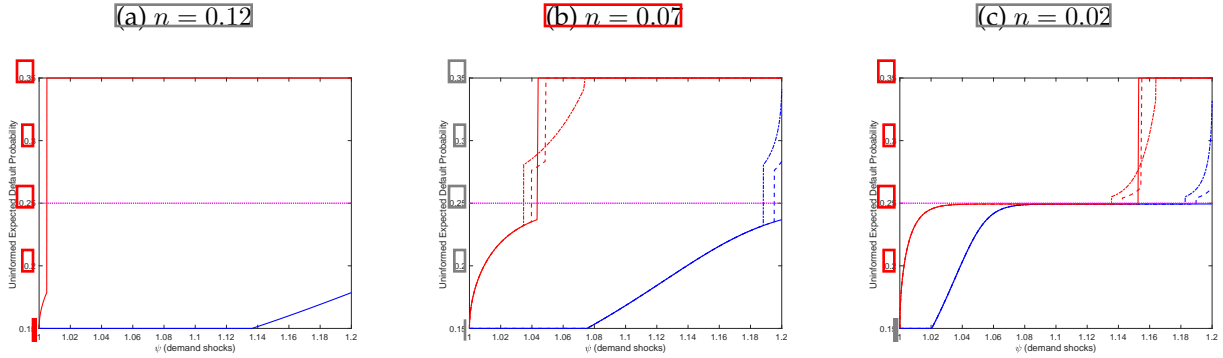


Accordingly, Figure 3 shows multiple potential equilibria using dashed and dotted lines. Panel (a) shows that there is a unique equilibrium when $n = 0.12$. The equilibrium is unique because the bid-overhang constraint binds for a relatively high ψ (recall $\hat{\eta} = n = 0.12$ and then $\hat{\psi} = 1/(1 - n) = 1.136$), leading to a relatively high common price when schedules first overlap. As a result, there is no level of pessimism by the uninformed that justifies another equilibrium with a lower price such that the informed are willing to bid in the bad state. Panel (b) shows three possible equilibria for $n = 0.07$, two with a discontinuous jump in the range of overlapping prices (in dashed and dotted lines). Finally, Panel (c) shows the outcome when $n = 0.02$. Here the number of informed is so

low that only the early jump to pessimism is an equilibrium. Finally, note that the bid-overhang constraint forces convergence towards the symmetric ignorance schedule (solid black in the figures) as n declines.¹⁴

To provide further insight into the construction of the various equilibria, Figure 4 graphs the inferred equilibrium beliefs of uninformed investors that sustain the equilibrium prices depicted in Figure 3. The vertical axes of the figure is bounded between $\kappa_g = 0.15$ and $\kappa_b = 0.35$. The pink-dotted line is the unconditional default probability $\kappa^U = 0.25$. The plots show the discontinuous changes in beliefs that sustain the jump in the range of overlapping prices. Jumps occur from a point below κ^U to a point above κ^U . As n declines and prices converge towards the symmetric ignorance schedules, the uninformed beliefs about the default probability also converge towards κ^U .

Figure 4: Impact of Information Acquisition on Uninformed Equilibrium Beliefs



Taken together, the figures show that multiplicity is partly sustained by both jumps in the price schedule and in the set of states for which prices overlap, so as to ensure market-clearing in all states. The logic underlying multiplicity (and also non-existence of equilibrium) in our model is thus similar to the one that pervades competitive equilibria with heterogeneous information more generally: prices are determined by local beliefs, but these can jump discontinuously while being consistent with optimality and updating.

¹⁴ An implicit assumption maintained here is that the symmetric ignorance price schedule lies everywhere above the threshold $1 - \kappa_b$ below which the short-sale constraint binds for informed investors given $\theta = b$. This implies that it is possible for a low-quality price schedule to exist that does not require jumps in the region of overlap in order to get informed investors to bid on low-quality bonds. Parameters may be such that this is not feasible for sufficiently low n , however. In this case, there is no equilibrium with a continuous low-quality price schedule in the range of overlap.

Appendix A provides further details on equilibrium multiplicity in UP auctions.

5 Characterization of DP Auction Equilibrium

We now characterize discriminatory-price auctions, and use $P^{DP}(s; n)$ to denote the price for each state and each n in the discriminatory protocol. Similarly, will use $B^{DP,U}(s; n)$ for the bids of the uninformed and $B^{DP,I}(s, \theta; n)$ for the bids of the informed. Whenever there is no risk of confusion, we use the simpler notation $P(s)$, $B^U(s)$, and $B^I(s, \theta)$. As before, we start with the symmetric benchmarks $n = 0$ and $n = 1$.

5.1 Symmetric Benchmarks

5.1.1 Symmetric Ignorance ($n = 0$)

When there are no informed investors, prices and bids cannot be contingent on θ . Hence we can simplify notation to $P(\eta)$ and $B(\eta)$, and the auction-clearing condition (15) to

$$(1 - \eta) \int_0^{\eta} B(\hat{\eta}) P(\hat{\eta}) d\hat{\eta} = D.$$

The first important difference between UP and DP auctions is that in-the-money bids are not executed at the marginal price $P(\eta)$ but rather at bid price $P(\hat{\eta})$. As a result, total expenditures must be monotonically increasing in η and bids must be strictly positive, $B(\eta) > 0$ for all $\eta \in \mathcal{H}$. This in turn implies that the short-sale constraint for the investors cannot bind for any η . The first-order condition for the uninformed investor at η^* thus is

$$\int_{\eta^*}^{\eta_M} \left[-U'(B_{RF}(\eta)) \kappa^U P(\eta^*) \right] \left[\pm U'(B_{RF}(\eta) \pm \int_0^{\eta} B(\hat{\eta}) d\hat{\eta}) \right] (1 - \kappa^U)(1 - P(\eta^*)) g(\eta) d\eta = 0. \quad (20)$$

This expression reveals the second important difference between DP and UP auctions: the cumulation of marginal utilities is determined by the holdings risk-free bonds, which are the residual of expenditures evaluated at the bid price rather than the marginal price. As a result, the system of first-order conditions is not block-recursive and must be solved

simultaneously. This introduces both computational and analytical complexity.

To discuss the main properties of the optimal bid function while maintaining tractability, we approximate the continuous distribution of the demand shock η with an (arbitrarily fine) discrete grid $\{\eta_0, \dots, \eta_J\}$ of length J . We index the demand shock by $j \in \{0, \dots, J\}$, with $\eta_0 = 0$ and $\eta_J = \eta_M$, and assume without loss of generality that η_j is strictly increasing in j . Define the following vectors of prices, returns, and bids

$$\begin{aligned} \vec{P} &= \begin{bmatrix} P(\eta_0) \\ \vdots \\ P(\eta_J) \end{bmatrix}, \quad (\mathbf{I} - \vec{P}) \equiv \begin{bmatrix} 1 - P(\eta_0) \\ \vdots \\ 1 - P(\eta_J) \end{bmatrix}, \quad \vec{B}^U \equiv \begin{bmatrix} B^U(\eta_0) \\ \vdots \\ B^U(\eta_J) \end{bmatrix} \end{aligned}$$

and the following triangular matrices of dimension $J \times J$

$$\mathbf{P} \equiv \begin{cases} P_{ij} = P(\eta_i) \text{ if } i \leq j \\ P_{ij} = 0 \text{ o.w.} \end{cases}, \quad \mathbf{I} - \mathbf{P} \equiv \begin{cases} 1 - P_{ij} = 1 - P(\eta_i) \text{ if } i \leq j \\ 1 - P_{ij} = 0 \text{ o.w.} \end{cases}$$

Price vector \mathbf{P} must then solve the stacked system of first-order conditions

$$U \left(W - \mathbf{P} \times \vec{B}^U \right) \cdot \vec{P} \cdot \kappa^U + U \left(W + [\mathbf{I} - \mathbf{P}] \times \vec{B}^U \right) \cdot [\mathbf{I} - \vec{P}] \cdot [1 - \kappa^U] = 0$$

As in the UP auction, the benefit of being informed when all other investors are uninformed is the ability to choose optimal right state-contingent quantities. Since prices are not state-contingent, an informed investor does not pay lower prices than uninformed investors when $n = 0$ even in a DP auction. This is no longer true if $n > 0$ and prices are contingent on the state, however

5.1.2 Symmetric Information

With symmetric information ($n = 1$), we can again separately construct the equilibrium for each θ . All equilibrium conditions are the same as in the symmetric ignorance case, except that investors use $\kappa_{\theta(\phi)}$ rather than κ^U to evaluate their first-order conditions. Hence

the system of first-order conditions is

$$= U^I \left(W - \mathbf{P} \times \vec{B}^I \right) \cdot \vec{P} \cdot \kappa_{\theta(\epsilon)} + U^U \left(W + [\mathbf{I} - \mathbf{P}] \times \vec{B}^I \right) \cdot [\mathbf{I} - \vec{P}] \cdot [\mathbf{I} - \kappa_{\theta(\epsilon)}] = 0.$$

5.1.3 No replication in DP auctions

In contrast to the UP auction, information is valuable in a DP auction even if *all* investors are informed and prices are fully revealing ex-post. The reason is that bids are executed at the bid price. Hence uninformed investors will typically pay higher prices than the informed even if they buy the same number of bonds in a given state. DP auctions thus offer a stark contrast to general equilibrium models with asymmetric information, in which the value of information is zero when prices are fully revealing ex-post. The following proposition formalizes this argument under mild regularity conditions. We then provide an example to clarify the mechanism.

Proposition 10. *Replication is not feasible in a DP auction if*

1. $\kappa_g \neq \kappa_b$ and $f(g)$ and $f(b)$ are both positive
2. the informed investor bids positive amounts for both $\theta = g$ and $\theta = b$ for some values of η .

Example 1. Consider an equilibrium in which all investors are informed. Assume that parameters are such that equilibrium prices differ across quality shocks at the smallest possible demand shock, $P(g, 0) > P(b, 0)$, and that informed investors bid positive quantities at both prices, $B^I(\theta, 0) > 0$ for all θ . (It is trivial to choose parameters such that these assumptions are satisfied). Suppose now that uninformed investors want to replicate the informed portfolio. Since an uninformed investor's bids at the high price are accepted in both states, he spends $P(g, 0)B^U([g, 0]) + P(b, 0)B^U([b, 0])$, while an informed investor spends $P(b, 0)B^I([b, 0])$. Hence even if $B^U([g, 0]) + B^U([b, 0]) = B^I([b, 0])$ and both investors buy the same quantity of risky bonds in state $(g, 0)$, the uninformed investor pays more and thus has fewer risk-free bonds in his portfolio.

5.2 Asymmetric Information – Special Case with Log Preferences

We now return to the special case with log preferences to study equilibrium prices under asymmetric information, $n \in (0, 1)$. Given the discrete grid for demand shock $\eta \in \mathcal{H}$,

there is a discrete set of states $\{s_0, \dots, s_M\}$ indexed by i , with $M = J \times 2$. Without loss of generality, order states in decreasing order of prices, $P(s_i) > P(s_{i+1})$, and let

$$\begin{aligned}
 & \begin{array}{c} \boxed{\bar{P}} \equiv \begin{bmatrix} P(s_0) \\ \vdots \\ P(s_J) \end{bmatrix} \quad \vec{B}^U \equiv \begin{bmatrix} B^U(s_0) \\ \vdots \\ B^U(s_J) \end{bmatrix} \quad (\mathbf{1} - \vec{P}) \equiv \begin{bmatrix} 1 - P(s_0) \\ \vdots \\ 1 - P(s_J) \end{bmatrix} \quad \kappa \equiv \begin{bmatrix} \kappa(\theta_0) \\ \vdots \\ \kappa(\theta_J) \end{bmatrix} \\
 \mathbf{P} \equiv \begin{cases} P_{ij} = P(s_i) \text{ if } i \leq j \\ P_{ij} = 0 \text{ o.w.} \end{cases} \quad \mathbf{1} - \mathbf{P} \equiv \begin{cases} 1 - P_{ij} = 1 - P(s_i) \text{ if } i \leq j \\ 1 - P_{ij} = 0 \text{ o.w.} \end{cases} \end{array}
 \end{aligned}$$

Given this notation, the system of first-order conditions that pins down the optimal bids of uninformed investors is

$$\begin{aligned}
 & \left(W - \mathbf{P} \times \vec{B}^U \right)^{-1} \cdot \vec{P} \cdot \kappa + \left(W + [\mathbf{1} - \mathbf{P}] \cdot \vec{B}^U \right)^{-1} \cdot [\mathbf{1} - \vec{P}] \cdot [1 - \kappa] \leq 0 \\
 & \text{and } B^U(s_i) = 0 \text{ if the inequality is slack}
 \end{aligned}$$

For informed investors, the analogous system of equations is

$$\left(W - \mathbf{P} \times \vec{B}^I \right)^{-1} \cdot \vec{P} \cdot \kappa_{\theta(s)} + \left(W + [\mathbf{1} - \mathbf{P}] \times \vec{B}^I \right)^{-1} \cdot [\mathbf{1} - \vec{P}] \cdot [1 - \kappa_{\theta(s)}] = 0$$

The joint solution to these systems of first-order necessary conditions converges arbitrarily closely to the solution under continuous demand shocks as \mathcal{H} becomes arbitrarily fine. (Of course, the linear algebra conditions will become infinite dimensional in the limit). They also highlight the fourth important difference between UP and DP auctions.

Remark 6. Consider an uninformed investor bidding \vec{B} at some marginal price \bar{P} . In a UP auction, the investor is concerned with inferring the default probability-weighted marginal utilities associated with state giving rise to marginal price \bar{P} . In a DP auction, the investor is instead concerned with inferring the default probability (and the cumulated marginal utilities) of the entire set of states in which (\vec{B}, \bar{P}) is in the money. That is, the investor must infer $\mathbb{E} \{ \kappa(P) | P \leq \bar{P} \}$ and the associated cumulation of marginal utilities, because the marginal utility from increasing a given bid is evaluated at the bid price rather than the marginal price.

Solving for equilibrium prices in DP auctions with asymmetric information is analytically intractable. Hence we use our numerical example from Table 1 to illustrate prices and comparative statics.

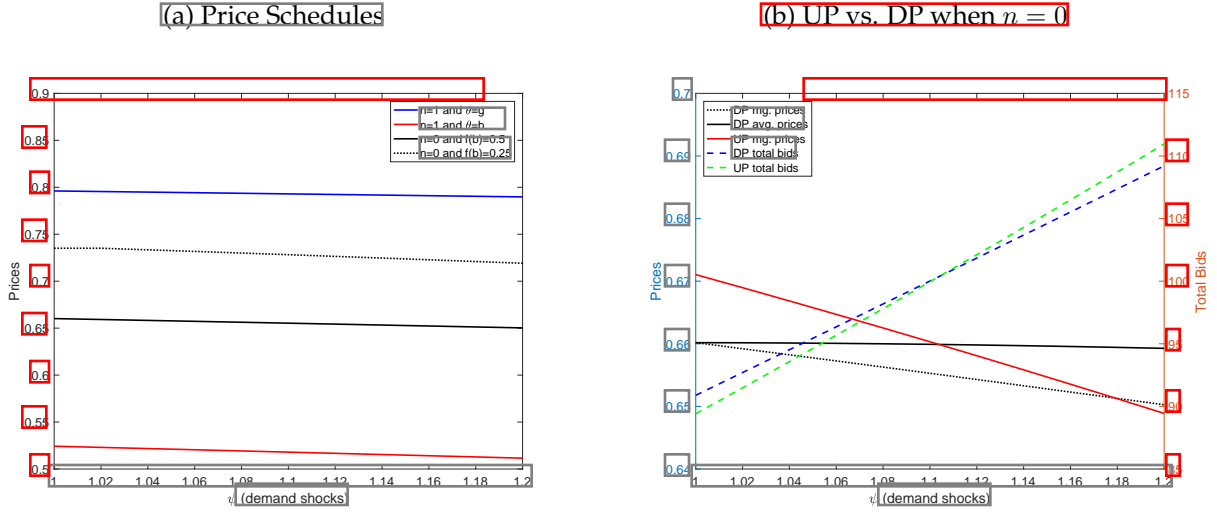
We first compute the equilibrium in the benchmarks with symmetric information ($n = 1$) and symmetric ignorance ($n = 0$). As before, we consider two values for the ex-ante probability of the bad state, $f(b) \in \{0.25, 0.5\}$. The price functions are in Panel (a) of Figure 5. As with the UP auction, the price schedules are independent of $f(b)$ under symmetric information, but they depend on $f(b)$ under symmetric ignorance. The graph shows that, as $f(b)$ decreases, the bad θ -state is less likely and the price schedule under symmetric ignorance (in black) gradually increases from the fully informed low-quality price schedule (in red) to the fully informed high-quality price schedule (in blue).

More generally, the benchmark price schedules look similar in terms of both shape and level to their UP analogs in Panel (a) of Figure 2. This is not a coincidence: in Appendix B we show formally that average prices are indeed the same in the symmetric benchmarks. To highlight the similarities UP and the DP auctions at the symmetric information benchmarks, Panel (b) of Figure 5 shows marginal prices and bid functions for $n = 0$. (Notice the change in scale relative to Panel (a)). Since there is price dispersion in the DP auction, we also plot the *average price* in the DP auction for each value of the supply shock ψ in solid black. The average price schedule is flatter than the marginal price schedule because the latter is strictly decreasing. In the UP auction, there is no price dispersion and so average and marginal prices coincide (plotted in red).¹⁵ By construction, if the average price in the DP auction is the same as the average price in the UP auction, then total bids per investor are also the same. At the same time, the average price schedule is flatter in DP auctions because a fraction of bids are locked in at high prices. Hence low marginal prices apply only to the residual bids needed to cover any additional demand shocks, while low marginal prices apply to *all* bids in UP auctions.

The marginal price in the UP auction when all investors participate (that is, $\psi = 1$) is higher than in the DP auction because bidders are willing to bid more aggressively

¹⁵The average and marginal prices in the DP auction also coincide at $\psi = 1$ since there is no price dispersion in this case. Hence there is price dispersion for all $\psi \neq 1$.

Figure 5: Symmetric Benchmarks for DP Auctions



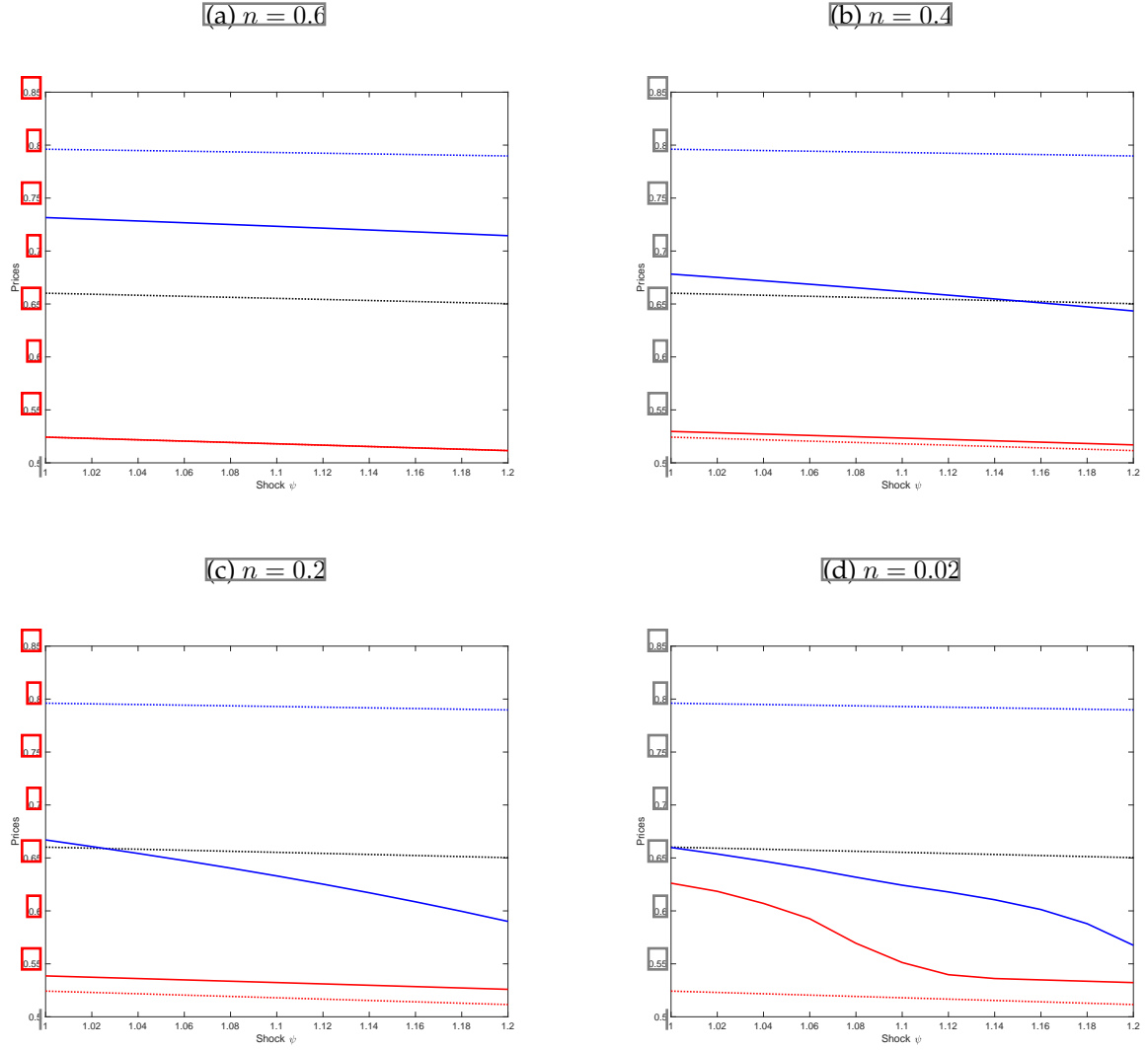
knowing that they do not have to pay this high price when fewer investors participate ($\psi > 1$). At the same time, the need to raise more additional revenue per capita in the UP auction as ψ increases causes the prices to fall faster than in DP, and marginal prices eventually cross.

This comparison between UP and DP protocols under symmetric ignorance is revealing of a more general property of DP auctions: the average price at which the government raises funds is less sensitive to demand shocks. That is, average prices are less volatile in DP auctions even though (or precisely because) there is more price dispersion. The next section shows that the same observation obtains for $n \in (0, 1)$.

5.2.1 Comparative statics with respect to n

Figure 6 shows equilibrium prices as we shrink n from one to zero in a DP auction. Despite the mathematical similarity, the economic forces underpinning equilibrium prices under asymmetric information are quite different across auction protocols (see Figure 3 for a comparison). Two of the critical mechanisms in UP auctions are absent in DP auctions: perfect replication is always infeasible, and the bid-overhang constraint never binds. Instead, adverse selection of uninformed bidders and the concentration of default risk on informed investors play a more prominent role. Each example in Figure 6 is cho-

Figure 6: DP auction equilibrium as n falls.



sen to highlight one of these forces.

Observe first that the spread between the high-quality and low-quality price schedules is increasing in n . When n is large, uninformed investors thus face a substantial adverse selection problem: if they bid on the high-quality schedule, they will overpay in the bad state since the government is sure to accept their high-state bids and executes them at the bid price. For n sufficiently large ($n = 0.6$ in our example), the uninformed may therefore refrain from placing any bids on the high-price schedule. This has two effects. First, the an uninformed investors who bids only on the low-price schedule knows

that, conditional on a bid being accepted, the state must be bad. Hence they choose the same portfolio as informed investors when $\theta = b$, and the low-quality schedule is locally independent of n . Second, precisely because the uninformed do not participate at high prices, informed investors have to buy more bonds per capita in the good state, and are therefore disproportionately exposed to the government's default risk. Since there are fewer participants as n decreases (above and beyond the lack of participation generated by the demand shock), the high-price schedule must fall

When n is sufficiently small ($n = 0.4$ in our example), prices on the high-quality schedule are low enough such that the uninformed investors are less worried about adverse selection and begin to bid on both price schedules. Since bids on the high-price schedule are also executed in the bad state, there is less residual demand that needs to be met by marginal bids on the low-price schedule. Hence the low-price schedule rises

The adverse selection effect continues to operate as n decreases further (to $n = 0.1$ in our example). In particular, because the per-capita bids of the uninformed remain below those of the informed on the high-price schedule, reductions in n continue to further concentrate default risk in informed portfolios. This forces a large fraction of the high-quality schedule to drop below the uninformed price schedule. That is, the adverse selection effect may be severe enough that prices are lower than in the uninformed equilibrium conditional on bad news and good news. Finally, when n is very small ($n = 0.02$ in the figure), price schedules start overlapping. Uninformed investors are now willing to participate fully on both schedules, and, as the next result shows, prices converge to the uninformed price schedule as $n \rightarrow 0$.

Corollary 11. $\lim_{n \rightarrow 0} P([g, \eta]; n) = P([b, \eta]; n)$ for all $\eta \in (0, \eta_M)$ in a DP auction equilibrium.

Proof. The proof of the analogous statement in Proposition 9 applies. □

Remark 7. Despite the fact that bid overhang never binds in DP auction, it is instructive to revisit its role in our auction model here. Specifically, once the uninformed start bidding on the high-quality schedule, their overhanging bids impact the equilibrium prices at the low-quality schedule in a manner reminiscent of the way the bid-overhang constraint affected the UP auction: they drag up the prices on the low-quality schedule by forcing the execution of some high-price bids

originally intended for a different state. Nevertheless, the mechanism is quite different, because the uninformed investors' accumulated bids reduce the available supply of bonds for the informed and thereby raise prices. Another key difference is while the bid-overhang constraint restricted the set of equilibria in the UP auction, in DP auctions the overhanging bids act on the nature of the equilibrium. The bid-overhang does not induce a pooling of prices as the inference is not at each price but instead in-the-money set. The bid-overhang operates as soon as uninformed investors start bidding at prices in the high-quality schedule (around 0.4 in the example) and not when they become the marginal investor (at $n = \eta_M$ as in the UP auction example). Even though the bid-overhang constraint operates differently in both protocols, it remains the force that guarantees convergence of price schedules in both as $n \rightarrow 0$.

5.2.2 Comparative statics with respect to $f(b)$

What if a government's quality improves and the probability that a country is in the bad state, $f(b)$, declines? Interestingly, there is not much effect when a large number of investors are informed. In the UP auction, there is perfect replication when $n > \eta_M$ (the bid-overhang not binding) and the two schedules are independent on $f(b)$ conditional on θ . In the DP auction, conditional on the uninformed not participating on the high-quality schedule, prices are also independent of $f(b)$. The probability of a bad bond $f(b)$, however, changes the ex-ante expected probability of default κ^U , which is critical in determining the symmetric ignorance schedule towards which prices converge as $n \rightarrow 0$. Indeed, once the bid-overhang constraint operates, at relatively low n , the parameter $f(b)$ does affect prices. In the UP auction, the point at which the bid-overhang constraint binds is just η_M , independent of $f(b)$. Conditional on binding, $f(b)$ affects the inference problem and thus the shape of the overlapping price schedules. In the DP auction, a lower $f(b)$ also reduces the likelihood that the uninformed buys an overpriced bad bond, speeding up and increasing the participation of the uninformed on the high-quality schedule. This in turn implies that the low price schedule begins to rise for higher values of n . Contrary to the UP auction, the point at which the bid-overhang constraint operates thus does depend on $f(b)$. This is summarized in the next proposition.

6 Comparison Between UP and DP Protocols

6.1 Level and Volatility of Sovereign Yields

We now examine the implications of the choice auction protocol from the perspective of the government for different values of n . Rather than explicitly specifying a government payoff function, we simply assume that the government (i) prefers to sell bonds at high prices (or equivalently, low yields), and (ii) dislikes price volatility stemming from demand shocks, since these are outside its control.

The yield on a government bond sold at price P is equal to the promised return,

$$Yield = \frac{1 - P}{P}$$

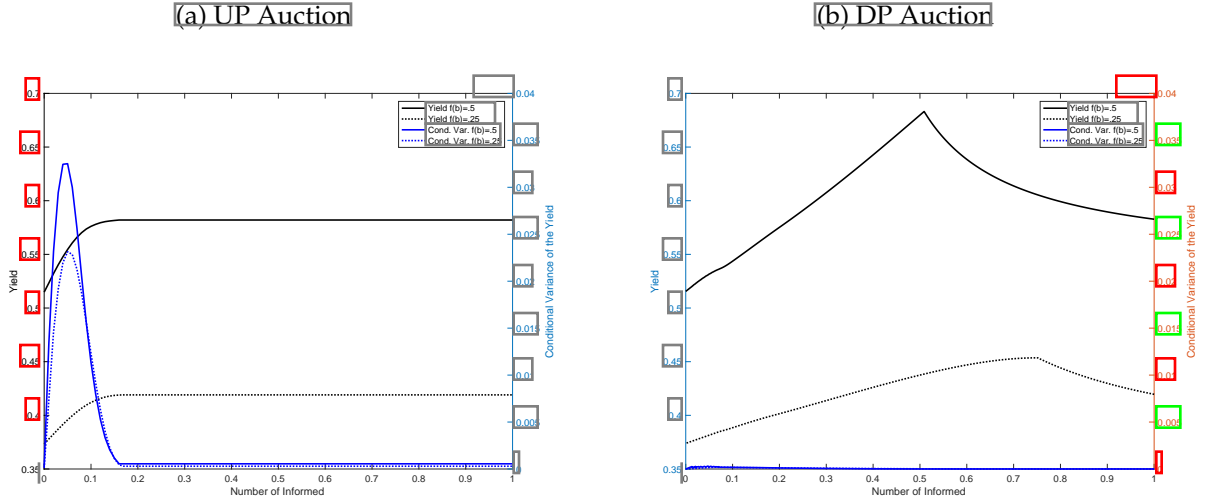
In UP auctions, all bonds are sold at the same price, and we can simply compute the equilibrium yield using the unique marginal price. In DP auctions, we compute a quantity-weighted average yield using the individual yields on all sold bonds. This average yield captures the risk-neutral component of the government's payoff. Note that the government's debt burden can be defined as $D/P = D(1 + Yield)$. Hence the government faces a higher average debt burden if bonds trade at a higher average yield.

The risky component of the government's payoff is given by the variation in the average yield conditional on the quality of the bond, and captures the government's exposure to demand shocks.¹⁶ Figure 7 plots average yield (the risk-neutral component) and its average conditional variance (the risky component) for both types of auctions.

Compare first the benchmarks of symmetric information ($n = 1$) and ignorance ($n = 0$). Given that average prices and total bids are the same across auction protocols under symmetry, so are average yields. The average conditional variance of the yield, while small in both cases, is substantially lower in the DP auction, however. As we discussed

¹⁶There is also an ex-ante variance across quality bonds. Because we have made the default probabilities so different in order to allow for perfect replication in the UP auction, the differences across quality shocks swamp the conditional variance in our numerical example. However, this would not be true when these differences were smaller. For example, if $f(g) \rightarrow 0$, almost all bond yield volatility is due to demand shocks rather than quality shocks. For this reason, we have chosen to focus on the behavior of the average conditional variance.

Figure 7: Bond Yields and Conditional Variances



when describing Figure 5, this is because the lion's share of bids in a DP auction is executed at the highest state-contingent price. Specifically, yield volatility is determined by the slope of price schedules conditional on a θ -state, and these are flatter in DP auctions. The average yield is lower under symmetric ignorance than under symmetric information.

Next turn to asymmetric information, $n \in (0, 1)$. The behavior of yields and conditional variances now differs qualitatively across protocols. In UP auctions, the average yield rises monotonically until the point of perfect replication, and is constant thereafter as the equilibrium becomes invariant with respect to n . In contrast, the average conditional variance of the yield shows a strong non-monotonicity in the region in which the bid-overhang constraint binds. The reason is that the bid-overhang constraint forces price pooling, thereby steepening the slope of overlapping price schedules as n grows. Steep price schedules imply that equilibrium prices are highly sensitive to demand shocks.

In DP auctions, the average yield is hump-shaped, increasing for low levels of n and declining for high levels of n . If n is low, uninformed investors bid on both schedules, but face adverse selection because bid on the high-quality schedule are also accepted in the bad state. This depresses prices on the high-quality schedule to an extent that is not fully compensated by an increase of the low-quality schedule. Indeed, the adverse selection effect grows in importance as n rises, up to the point where uninformed investor stop

bidding on the high-quality schedule. When only informed investors participate, further increases in n lead to lower per-capita risk exposure. As a result, informed investors' information rents are gradually competed away as n grows. (Bad-state prices are unaffected because the uninformed choose the same portfolio as the informed conditional on the bad state.) This *cannibalization effect* raises prices in the good state and reduces the yield as $n \rightarrow 1$. In contrast to UP auctions, the conditional variance is small for all n . The reason again is that the lion's share of expenditures is made at the highest bid price. Large demand shocks thus necessitate only a marginal price adjustment. Furthermore, the absence of the bid-overhang constraint means that price schedules need not be as steep as in a UP auction, reducing the sensitivity to demand shocks.

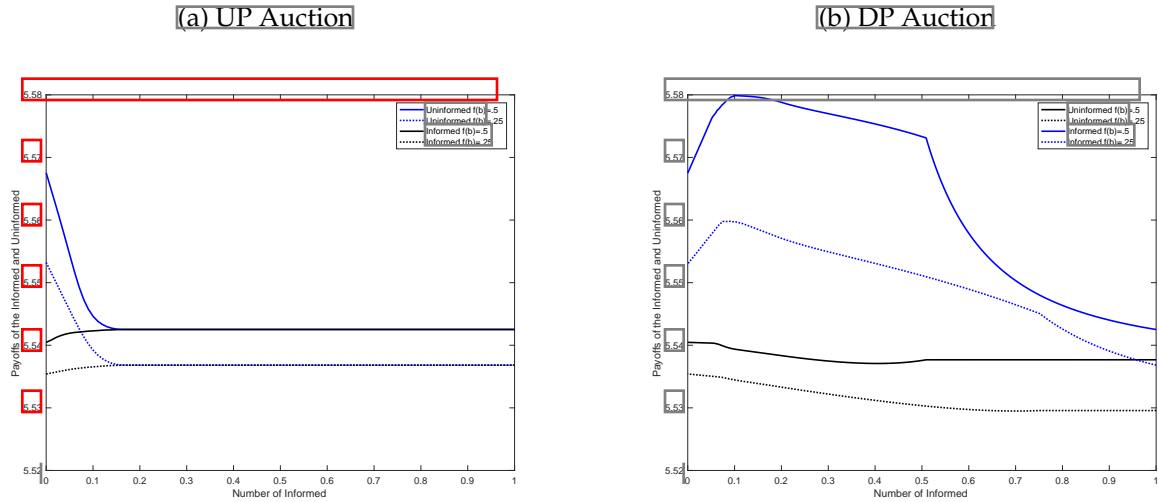
The choice of auction protocol thus leads to a *risk-return tradeoff* between average yields and volatility for the government. When information is symmetric, both protocols deliver the same average prices, but DP auctions offer lower exposure to demand shocks. When n is relatively low, average yields are similar (and increasing in n), but UP protocols feature more exposure to demand shocks. When n is relatively large (in particular, when n is such that the UP auction allows for perfect replication), the UP auction still displays a larger conditional variance, but a smaller average yield. The resolution of this tradeoff depends on the government's preferences for lower average yields versus volatility.

Note that Figure 7 shows average yields and conditional variances for two different values of $f(b)$. When the ex-ante probability that the bond is of low quality is smaller ($f(b) = 0.25$ instead of $f(b) = 0.5$), average yields and volatility falls for all n . This is quite natural as the bond is less likely to default. More interestingly, the peak in average yields in DP auctions occurs for higher n when $f(b)$ is lower. This is because uninformed investors are less afraid of bidding on the high-quality schedule because the risk of overpaying in the bad state is lower. Hence they continue participating in both states for relatively high values of n . In the figure, the peak in average yields occurs at around $n = 0.5$ when $f(b) = 0.5$ and around $n = 0.75$ when $f(b) = 0.25$.

6.2 Investors' Payoffs and Information Acquisition

We now show the implications of the two auction protocols for investor welfare. Figure 8 plots investors' expected utility for different values of n and for two levels of $f(b)$. Informed investor utility is in black, while uninformed investor utility is in blue.

Figure 8: Payoffs to Informed and Uninformed



In the benchmark symmetric ignorance case, the utility levels obtained by informed and uninformed investors, respectively, are the same in both protocols. This is because prices are not contingent on θ at $n = 0$, and so there are no price differences that can be differentially exploited across protocols by an atomistic informed investors. This is not the case in the symmetric information benchmark ($n = 1$), however. While informed investors obtain the same payoffs in both protocols, the uninformed payoff is substantially different. In the UP auction, informed investors can perfectly replicate the informed portfolio, and they thus obtain the same same payoff as well. In the DP auction, perfect replication is not attainable because the uninformed would pay higher prices in the bad state. Hence they earn achieve lower utility than uninformed investors in a UP auction.

When information is asymmetric, the comparative statics of both informed and uninformed investor utility with respect to n are qualitatively different across auction protocols. First, consider uninformed investors. In a UP auction, uninformed utility is increasing in n because (partial) portfolio replication allows the uninformed to free ride

on the information impounded into prices by informed investors, of which there is more when there are more informed investors. In a DP auction, instead, the degree of adverse selection is increasing in n , and hence so is uninformed utility.

Next, consider informed investors. In a UP auction, all investors pay the same price. Since informed investors bid more aggressively than uninformed investors on average, an increase in n raises prices and lowers rents. Hence informed investor utility is decreasing in n until $n = \eta_M$. (Beyond this point, uninformed investors can perfectly replicate and payoffs are invariant to n .) In a DP auction, bids are executed at the bid price, and uninformed investors face adverse selection. If n is small to begin with, increases in n increase the severity of adverse selection, and uninformed investors participate less and less. This reduced participation lowers prices on average, leading to an increase in infra-marginal rents and utility earned by informed investors. Once we reach the point where the uninformed no longer participate on the high-state schedule, the comparative static reverses. Now, further increases in n just serve to increase competition for bonds in the high-state, leading to price increases that erode information rents. (The utility level obtained in the low state is constant, since both informed and uninformed investor choose the same portfolio.) Importantly, the value of information is strictly positive even in the limit $n \rightarrow 1$ where prices are fully revealing ex-post.

The figure also shows the comparison between two different levels of $f(b)$. A lower probability of the bad state increases marginal prices, leading to lower infra-marginal rents on all bond purchases. Hence investors are worse off if $f(b) = 0.25$. While the effect of $f(b)$ just scales down prices for each n in the UP auction (as we discussed $f(b)$ does not change the point at which the bid-overhang constraint binds in UP auctions), in DP auctions a lower $f(b)$ implies that uninformed participate earlier in the high-quality schedule, generating a different non-monotonicity in informed utility. Lastly, note that the cost of not participating in the high-state for the uninformed in a DP auction are higher when the probability of being in the good state is high, even in the benchmark with $n = 1$. (That is, the payoffs decline more for the uninformed than for the informed in the symmetric information benchmark).

6.2.1 Endogenizing Information Acquisition

We now endogenize the share of informed investors n by allowing for information acquisition. All investors are initially uninformed. After learning whether they will make it to the auction (that is, after learning whether they were hit by the demand shock), investors can learn the true value of θ by paying a *utility cost* of K . An investor chooses to become informed if the value of information exceeds its cost, and will remain uninformed otherwise. Denote by $V^U(n)$ the expected utility of an uninformed investor given n (see Equation 1), and by $V^I(n)$ the expected utility of an informed investor given n (see Equation 6). The dependence on n reflects the fact that equilibrium prices will depend on the share of informed investors, as shown in Figure 8. Then the following optimality conditions determine the equilibrium level of n

$$V^I(n) - K \geq V^U(n) \text{ if } n > 0 \quad (21)$$

$$V^I(n) - K \leq V^U(n) \text{ if } n < 1. \quad (22)$$

In an interior equilibrium with $n \in (0, 1)$, both conditions must simultaneously hold with equality. An equilibrium with information acquisition is defined as follows.

Definition 9. *For both the UP and DP auctions, an equilibrium of the model with endogenous information acquisition consists of the measure of informed traders n^* , a price schedule $P(s)$, a bid schedule for the uninformed $B^U(s)$, and a bid schedule for the informed $B^I(s, \theta)$. The bid schedules must be solutions to the investors' problems given P and satisfy the short-sale constraints. The bids and price schedules must satisfy the debt-overhang constraints and auction clearing for all (θ, η) , and n^* must satisfy the information acquisition criterion in (21) and (22).*

Given that investors are ex-ante identical, the value of information is simply the difference in expected utility obtained by informed and uninformed investors, $V^I(n) - V^U(n)$. We show this *payoff gap* in Figure 9. There are large differences across auction protocols, which has important consequences for equilibrium information acquisition. Specifically, the equilibrium share of informed investors n^* is determined by equalizing the payoff gap and the information acquisition cost K .

Figure 9: Payoff Differentials Between Informed and Uninformed

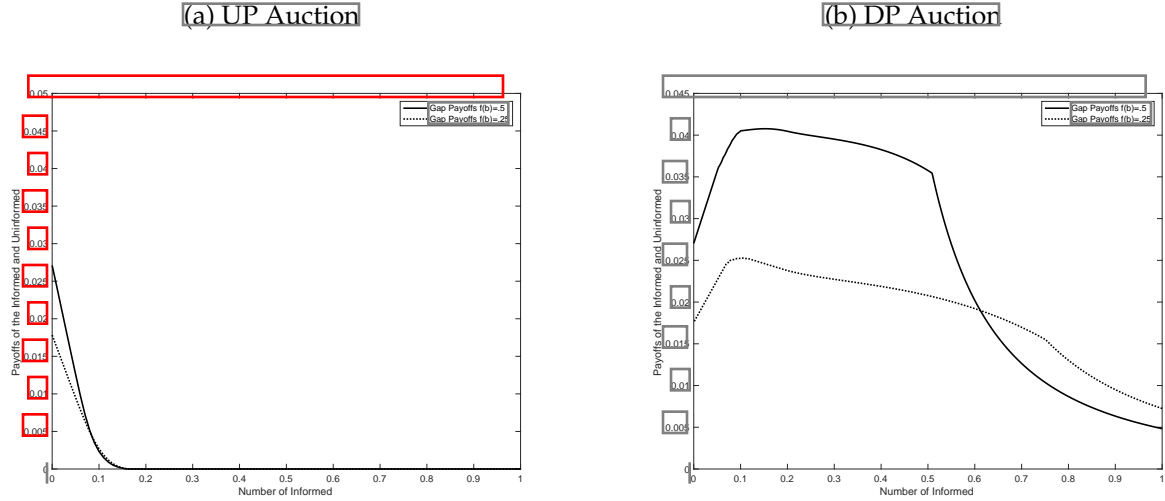


Figure 9 has certain features which numerical exploration suggests are common to this class of models. If we let n_{pr}^* denote the equilibrium share of informed investors in auction protocol $pr \in \{UP, DP\}$. Then there are thresholds $\bar{K} > \hat{K} > \underline{K} > 0$ such that:

1. if $K > \bar{K}$, then there is a unique equilibrium in each protocol, with $n_{DP}^* = n_{UP}^* = 0$.
2. if $\bar{K} > K > \hat{K}$, then there is a unique equilibrium with $n_{UP}^* = 0$ in the UP auction. In a DP auction, there are two stable equilibria in which $n_{DP}^* \in \{0, n(K)\}$. $n(K)$ is strictly decreasing in K . In addition, there exists an unstable equilibrium in which $n_{DP}^* \in (0, n(K))$.
3. if $\hat{K} > K > \underline{K}$, then $0 < n_{UP}^* < n_{DP}^* < 1$.
4. if $K < \underline{K}$, there is a unique equilibrium in each protocol, with $n_{UP}^* \in (0, 1)$ and $n_{DP}^* = 1$.
5. $n_{UP}^* < n_{DP}^*$. If replication is feasible given symmetric information, then $n_{UP}^* < 1$ $\forall K > 0$.

The fact that there are fewer informed with UP auctions does not mean that there is less information in prices. In our baseline example, when $f(b) = 0.5$, then $\bar{K} = 0.041$ (which is the value of the maximum gap in a DP auction), $\hat{K} = 0.018$ (which is the value of the

gap in both auction protocols at $n = 0$), and $\underline{K} = 0.005$ (which is the value of the gap for DP auctions at $n = 1$).

6.2.2 Relationship with Grossman and Stiglitz (1980)

A classical question in general equilibrium theory pertains to where the information contained in prices comes from. Grossman and Stiglitz (1980) argued that price-taking investors have no incentive to look at or acquire their private information if this information is already encapsulated in the price. But if their demands do not reflect their private information, then how do prices reflect this information in the first place? If acquiring information is costly, this question manifests itself as a nonexistence problem known as the Grossman-Stiglitz paradox: when information is costly and prices are fully revealing, no individual wants to acquire information. We now discuss in detail how our auction model circumvents this difficulty.

We can replicate the nonexistence problem in our UP auction if parameters are such that prices are fully revealing ex-post given $n = 1$ and *we simply assume that the bid-overhang constraint does not exist*. In the absence of the bid-overhang constraint, replication at $n = 1$ implies that replication is feasible for all $n > 0$. Hence the value of information is zero for all $n > 0$, but positive for $n = 0$. It follows that there is a non-existence problem if the cost of information is strictly between zero and the value of information at $n = 0$.

Taking into account the bid-overhang constraint, which follows endogenously from the auction protocol, eliminates this non-existence problem. Specifically, the bid-overhang constraint hinders perfect replication for low levels of n , eliminating the discontinuity of information gains at $n = 0$, and thus inducing equilibrium existence. Grossman and Stiglitz (1980) proposed a solution by adding a second source of noise to prevent the price system being invertible.¹⁷ In contrast we do not need to impose sufficient noise to prevent prices from being fully revealing. Instead, the auction protocol forces pooling when the fraction of informed investors is low enough, endogenously ensuring that prices are not

¹⁷The existence of equilibria when the shocks are continuous and hence states can have the same price is well known to be problematic, see Allen and Jordan (1998) for a survey of the existence literature. The combination of CARA preferences and jointly normal shocks was key to the construction of an equilibrium in Grossman and Stiglitz (1980), though recently Breon-Drish (2015) has developed a characterization that drops joint normality.

fully revealing for any fixed level of noise. Hence an equilibrium with costly information acquisition exists and lies endogenously in this range.

The contrast is even starker in a DP auction. In these auctions, the fact that bids are executed at the bid price provides incentives to acquire information even when prices are fully revealing ex-post. The fact that informed investors pay the correct marginal price in every state thus allows for informational efficiency even with costly information acquisition.

7 Relationship to Literature

This paper is motivated by the complex dynamics of sovereign debt yields in primary market auctions, particularly in emerging economies. We have already discussed the contribution of our model to the sovereign debt literature. In this section we discuss how our work relates three additional strands of literature. The first concerns the foundations of general equilibrium theory (GE) and the question of where market-clearing prices come from. The second concerns the question of how the private information held by multiple investors is aggregated into prices. The third consists of auction-theoretic models with a large number of bidders and perfectly divisible goods with uncertain common value, and the empirical application of such models to sovereign bonds.

With respect to the question of where prices come from, a key characteristic of GE theory is that the price vector is an endogenous object that is not chosen by anyone, yet is determined by the accumulated actions of individuals who think of themselves as unable to affect prices. To get around this conundrum, Walras made up his fictional “auctioneer” that matches total supply and total demand in perfectly competitive market with perfect information and no transaction costs. But this has long been considered a thought experiment that did not adequately address the basic question of prices (see Hahn (1989) for a discussion).

One approach to endogenizing the choice of prices is the market games literature. This literature introduces a fuller description of the environment in which all endogenous objects are selected by the agents (including prices) based upon noncooperative game the-

ory.¹⁸ Examples of this market game approach include Rubinstein and Wolinsky (1985)'s sequential bargaining model in which buyers and sellers are paired up under complete information each period.¹⁹

The price problem, however, becomes more severe once prices have to simultaneously clear markets and aggregate information, as in Lucas (1972), Radner (1979) and Grossman and Stiglitz (1980). Because agents "need to know" both the price function and the realized price in order to make inferences and determine their individual demands, this leads to the complementary question of where the information in prices comes from.²⁰ The relevance this question is perhaps best exemplified by the seminal work of Grossman and Stiglitz (1980), who discuss how fully revealing information prices are logically impossible.

But even if there exists a fully revealing equilibrium, there is an implementability problem because it may not be possible to find a trading mechanism that induces it. Kyle (1989) proposes a resolution using non-competitive rational expectations model in which agents submit limit orders (or demand schedules, as in our case) taking into account their effect on the equilibrium price. In a similar vein, Jackson (1991) proposes an environment in which the number of agents is finite and all agents internalize that the extent of information in prices depends upon the demand schedule they submit, while Golosov, Lorenzoni, and Tsyvinski (2014) propose a decentralized approach which features a sequence of bilateral meetings with take-it-or-leave-it offers.²¹ Finally, Vives (2014) and Gaballo and Ordonez (2017) propose settings with large centralized markets in which the valuation of each trader has both common and private value components, and the costly signal bundles information about these two components, such that prices can be fully revealing and yet there are incentives to acquire information.

¹⁸See Gale (2000) for a survey of this literature and for a discussion of the alternative cooperative approach.

¹⁹Gale (1987) shows that these sequential bargaining models converge to a common price equilibrium as the number of agents gets large.

²⁰Dubey, Geanakoplos, and Shubik (1987) consider the Nash equilibrium of a sequential trading game with incomplete information where traders make quantity offers to buy and sell and the price is determined by the ratio of the total buy versus sell offers. Here information revelation occurs largely in one-step through the vector of different prices for the different goods.

²¹A related contribution is Albagh, Hellwig, and Tsyvinski (2014) who develop a dynamic REE with dispersed information in which information enters nonlinearly into prices.

Our paper speaks to both of these problematic aspects of GE by using the structure of an auction to provide answer to the question of where prices come from, and under what conditions there are incentives to acquire and impound information into prices even when these are fully revealing ex-post. By specifying the auction protocol, our model features a specific order of moves. First, investors submit their bids (where each bid is a price-quantity pair). Second, a specific protocol is used to select the bids which are accepted and the prices at which they are executed. Information revelation occurs after the marginal price at the auction is revealed. The information that is revealed may be complete, as in REE model, but still there are incentives to acquire information because the information is revealed only once bids can no longer be changed. A related paper which takes an auction-based approach to micro found REE is Milgrom (1981). He considers, however, an auction in which bidders are restricted in the units they can buy and where the price is not clouded by a demand (or supply) shocks. Our paper relaxes both of these aspects.

Our paper also contributes to the auction literature on multi-unit auctions more generally. In contrast to the core of the auction literature, which is based on selling single object to bidders with either independent private values²² or correlated values²³ with a focus on strategic bidding, an adequate treatment of goods such as treasury bonds requires extending these models multi-unit auctions. The challenge, however, is solving an equilibrium that involves bidders with a double dimensional strategic problem: choosing both bid quantities and bid prices.²⁴ By assuming the limiting case of price taking, our setting allows to change the focus from bidding strategic considerations to how the auction protocol determine prices by aggregating bids and motivating information acquisition. In single-unit auctions, Eso and White (2004) consider the interaction of ex-post risk and auction protocols in determining bid prices.

Recent work tackles these questions of how multiunit auctions determine prices in equilibrium from an empirical perspective. Hortaçsu and McAdams (2010) develop a

²²See Vickrey (1962), Harris and Raviv (1981), Myerson (1979) and Maskin and Riley (1985).

²³See Milgrom and Weber (1982) and McAfee and McMillan (1987).

²⁴See Wilson (1979), Engelbrecht-Wiggans and Kahn (1998), Perry and Reny (1999), Kagel and Levin (2001) and McAdams (2006).

model based on Wilson (1979) of a multi-unit discriminatory-price auction with a finite set of potential risk-neutral bidders with symmetric and independent private values, using data from Turkish Treasury auctions to estimate those bidders' private values. In their model the symmetric equilibrium bidding functions depend on how an individual bid changes the probability distribution of the market clearing price, a complicated object to characterize analytically. Given this theoretical difficulty, they construct a non-parametric consistent estimator of the distribution exploiting a resampling technique.²⁵

In contrast to this empirical approach, our work is based on the presumption that bidders' valuations of the auctioned treasury bonds are perfectly correlated (common value) instead of independent (private value). Unfortunately, sorting out which assumption is more applicable for treasury bonds is challenging. As argued by Laffont and Vuong (1996), "common value" and "independent value" auctions are (nearly) observationally equivalent, unless there are exogenous variations that allow for identification. As a response, Hortaçsu and Kastl (2012) use Canadian treasury auctions to test whether common values or private values are a better representation of the motivations to buy treasury bonds. Even though they conclude that there is no evidence that dealers, who observe the bids of costumers, learn about fundamentals from those costumers, this is not *prima facie* evidence that dealers follow private values, but instead that they may have superior information than costumers about the common value.²⁶

A second relevant difference between our approach and this empirical literature is that we assume investors are risk averse, not risk neutral. This departure is not only relevant for the interpretation of the shading factor (as argued by Wilson (1979)) but also critical

²⁵Kastl (2011) extended Wilson model, which is based on continuous and differentiable functions, to more realistic discrete-step functions, showing that in such case only upper and lower bounds on private valuations can be identified, which he does by using the previously methodology on Czech bills auctions.

²⁶In Canada, some bidders (dealers) are allowed to observe the bonds of another set of bidders (costumers) when preparing their own bids. In a private value auction, observing the bid of a costumers only gives information about the competition the dealer faces (and then the probability of winning the auction) but not about the fundamental value of the bond. In this case the dealer would not revise the bid if this was higher than the observed competing bid. In a common value price auction, however, observing a costumers bid induce learning not only of competition but also of the fundamental, leading to a revision of the intended bid also when the bid was higher than the observed competing bid. Since Canadian auctions are discriminatory, this testable implication is not as straightforward, but they propose a correction. With a similar methodology applied to uniform-price auctions of U.S. Treasury bills, Hortaçsu, Kastl, and Zhang (2017) estimate that the informational advantage of primary dealers leads them to higher yield bids as a response to a larger ability to bid-shade their bids.

for thinking about the reaction of bond prices to shocks in volatile times (as highlighted by Gupta and Lamba (2017)). The third difference is our modeling of several correlated shocks, departing heavily from the assumptions of independent realizations across bidders. The quality shock, the demand shocks and the signal that all informed investors receive are perfectly correlated, which implies that bid shading only happens for uninformed investors in response to the possibility of adverse selection, but not because of competitive forces.

Closer to our setting, Boyarchenko, Lucca, and Veldkamp (2017) study an auction environment with risk averse investors that are asymmetrically informed about the common value of a bond. They assume that some investors have both superior information and market power, calibrate the model to U.S. Treasury auctions and show that information sharing across investors increase government revenues, as investors are willing to invest more as they become better informed. By focusing on the assumption that investors are price-takers we are able to study the effect of asymmetric information on prices instead of on the effect of strategic considerations on prices for both uniform and discriminatory price auctions.

8 Conclusion

We study the determination of sovereign bond yields in primary market auctions. We view this as an important departure from the literature on sovereign debt that focuses on secondary market prices, because primary market prices are an input to the government's budget constraint. We start from the perspective that accounting for the evolution of bond prices at issuance necessitates a wide range of shocks, some public, some private, some learnable, some not. Some of these shocks may affect common factors, like the probability of default, while others may affect the private valuation of the government's bonds, like liquidity shocks. This leads us to develop a rich model of sovereign debt auctions featuring shocks to both the default probability of the bond (about which investors can be heterogeneously informed), and a correlated private shock that determines auction participation (about which no investor is informed). This setting provides new insights into

the impact of different auction protocols on the role and extent of information asymmetry on bond prices in the presence of these shocks.

We find that these two protocols behave the same in terms of payoffs and yields when information (or lack of information) is symmetric, albeit with discriminatory-price auctions offering a lower variation in the yields in response to demand shocks. These symmetric cases can be thought of as occurring during tranquil periods when there is no information to be obtained about the government's future likelihood of default, or during eventful periods in which this information is publicly and freely available. Our model also implies that when there is valuable information that can be learned at a cost, this induces adverse selection that can lead to a wide dispersion in realized auction prices and high average yields in discriminatory-price auctions, as uninformed investors are reluctant to participate. While we find that the gains from acquiring information and the adverse selection effects are smaller in uniform price auctions, the conditional variance of prices is higher under this protocol. Finally, we show that the fraction of informed investors is never higher in uniform price auctions than in discriminatory auctions, with these last ones also displaying multiple equilibria (one with no investor informed and another with asymmetric information).

These results contribute to the wide discussion, dating back at least to Friedman (1960), of whether sovereign debt auctions should be conducted with a uniform-price or a price-discriminating protocol.²⁷ Our results suggest that the answer to this question is far from straightforward and depends on the nature of shocks, the relative preference of the government for low average yields or low volatility, the cost of information that determines the extent to which investors acquire information and the impact of asymmetric information on the bidding behavior of the uninformed.

Our model can potentially speak to the kind of data we see in sovereign debt auction, such as the case of Mexican bonds in figure 1. During crisis periods, when the range of potential default probabilities increase, the model predicts a sharp rise and highly volatile

²⁷Friedman proposed (pp 64-65) that the U.S. Treasury abandons its previous price-discriminating practice and make all awards at the stopout price instead of at differing prices down through that price. The U.S. Treasury finally adopted this uniform-price protocol for all auctions of 2-year and 5-year notes on September 3, 1992. An excellent summary of this discussion is Chari and Weber (1992). Earlier discussions about Friedman's proposal include Goldstein (1962), Friedman (1963), Rieber (1964) and Friedman (1964).

interest rates, as those we see during Mexican "Tequila Crisis" of 1995. When the level of a country's indebtedness increase there is a substantial pressure for information acquisition and information asymmetry (specially under discriminatory-price auctions), decreasing participation of uninformed investors under the threat of adverse selection and increasing interest rates even when the quality shock is good. This reduction in participation is also reminiscent of the decline in bids and failed auctions that we commonly observe during crises. At the same time, under either symmetric information or symmetric ignorance when there is little heterogeneity, the additional risk premia associated with demand shocks is small. This illustrates how the model can also accommodate the relatively low volatility of the Mexican Cetes interest rates in recent years.

In a follow-up paper (Cole, Neuhann, and Ordóñez (2016)) we examine the implications of discriminatory-price auctions within a two-country setting. We use the insights developed here to discuss spillovers of information across countries and the role of secondary markets. We show that the sources of complementarities on information acquisition inherent to discriminatory-price auctions extend from cross-states to cross-bonds and generate spillovers in sovereign spreads even in the absence of other linkages.

Our model is also applicable to a number of other important cases, including auctions of liquidity infusion by central banks, electricity, emission permits, gas, oil, and mineral rights. The key requirement is that the auction involves a sufficiently "thick" market for a homogenous divisible good of uncertain quality so as to make the price-taking assumption a close approximation to reality. Our model also provides a potential mechanism to micro-found competitive equilibria for the case of the uniform-price protocol and to break the circularity inherent in having prices and quantities determined simultaneously.

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A More Details on UP Auctions

A.1 Non-existence and Multiplicity

Here we discuss more formally multiplicity and discuss how nonexistence may arise. As these two features are related we discuss them in the same section.

First, recall that the auction clearing condition when the quality of the bond is $\theta = g$ (equation (18)) is,

$$n \left(\frac{1 - \kappa_g - P}{1 - P} \right) + (1 - n) \left(\frac{1 - \kappa - P}{1 - P} \right) \equiv \frac{D}{W} \psi_g.$$

When the bid-overhang constraint does not bind, then there is perfect replication given our set of parameters. When the bid-overhang constraint binds, the price P that solves for auction clearing when $\theta = g$ is the same as the one that solves auction clearing when $\theta = b$. There may be, however, two versions of auction clearing when $\theta = b$, as in clear from equation (equation (19))

1. The short-sale binds on the informed in the bad state ($P > 1 - \kappa_b$)

$$(1 - n) \left(\frac{1 - \kappa - P}{1 - P} \right) \equiv \frac{D}{W} \psi_b. \quad (23)$$

2. The short-sale constraint does not bind on the informed in the bad state ($P < 1 - \kappa_b$)

$$n \left(\frac{1 - \kappa_b - P}{1 - P} \right) + (1 - n) \left(\frac{1 - \kappa - P}{1 - P} \right) \equiv \frac{D}{W} \psi_b. \quad (24)$$

As the price P is the same in the good and the bad θ -state when the bid-overhang constraint binds, we can subtract the auction clearing of the two θ -states in both states cases, which gives us

1. The short-sale binds on the informed in the bad state ($P > 1 - \kappa_b$)

$$n \left(\frac{1 - \kappa_g}{1 - P} \right) \equiv \frac{D}{W} (\psi_g - \psi_b)$$

2. The short-sale constraint does not bind on the informed in the bad state ($P < 1 - \kappa_b$)

$$n \left(\frac{\kappa_b - \kappa_g}{1 - P} \right) \equiv \frac{D}{W} (\psi_g - \psi_b)$$

In this construction there is a monotonic relation between prices P and the supply gap $\psi_g - \psi_b$. Further, as the demand by the uninformed investors in both states is the same, the gap has to be covered solely by the informed investors. This is the reason why the beliefs of the uninformed about the likelihood of each state, κ , disappears from these equations and the reason why n multiplies this extra demand.

We can now obtain analytical expressions for the prices in both cases

1. The short-sale binds on the informed in the bad state ($P > 1 - \kappa_b$)

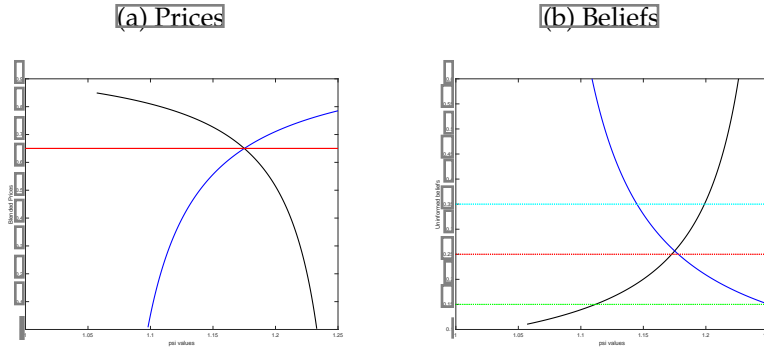
$$P = 1 - \frac{\kappa_g}{1 - \frac{D}{nW}(\psi_g - \psi_b)} \quad (25)$$

2. The short-sale constraint does not bind on the informed in the bad state ($P < 1 - \kappa_b$)

$$P = 1 - \frac{\kappa_b - \kappa_g}{\frac{D}{nW}(\psi_g - \psi_b)} \quad (26)$$

Notice that in the first case (when the short sale constraint binds) the price is decreasing in the gap $\psi_g - \psi_b$, while in the second case (when the short sale constraint does not bind) the price is increasing in the gap $\psi_g - \psi_b$. This is illustrated by the black and blue lines respectively in the first panel of Figure A.1, which is constructed by fixing ψ_b and assuming $n = 0.2$.

Figure A.1: Prices and beliefs with and without SS constraints



An important property to notice is that there is a unique gap level for which the price is the same, with and without the short-selling constraint binding. This is when

$$(\psi_g - \psi_b)^* = \frac{\kappa_b - \kappa_g}{\kappa_b - \frac{D}{nW}}$$

At this gap, when the two price functions of the Panel (a) of Figure 3 cross, $P = 1 - \kappa_b$, represented by the red horizontal line. As the decreasing (black) function is constructed under the presumption that the short sale constraint binds ($P > 1 - \kappa_b$), this is consistent only when the gap is smaller than $(\psi_g - \psi_b)^*$. Similarly, the increasing (blue) function is constructed under the presumption that the short sale constraint does not bind ($P < 1 - \kappa_b$), which is also consistent only when the gap is smaller than $(\psi_g - \psi_b)^*$. This is, no gap above $(\psi_g - \psi_b)^*$ can be rationalized by any price: either the price is too high if we assume the short-sale constraint does not bind (in which case it would), or too low if we assume that the short-sale constraint binds (in which case it would not).

We can now use one of the auction clearing conditions (either for the good or bad θ -state) to back out the inference parameter κ for the uninformed that rationalizes the price corresponding to a given supply gap. Regardless of the short-sale constraint binding or not, there is a negative relation between the price and the inference κ . This implies that, when the short selling constraint binds, as the supply gap increases the price declines, only consistent with an increasing κ (more pessimism about the state (the black function in Panel (b) of Figure 3). Similarly, when the short selling constraint does not bind, as the supply gap increases the price also increases, only consistent with a decreasing κ (more optimism about the state (the blue function in Panel (b) of Figure 3). As this inference is bounded between κ_b and κ_g , as shown in the figure, there is a constrained range of inferences that is consistent with an equilibrium

With these relations that arise from auction clearing we can now discuss the forces behind the construction of the equilibria displayed in the text. Since the bid-overhang constraint always binds first in the high price schedule when $\eta = 1 - \frac{1}{\psi} = n$, denote as $\psi_n = \frac{1}{1-n}$ the value of ψ for which the bid-overhang constraint binds first such that $\psi_b = 1$ (or $\eta = 0$). The price P that is common to the two states $[g, \phi_n]$ and $[b, 1]$ and covers the gap $\psi_n - 1$ is always consistent with the short sale constraint binding in state b . To see this note that P is determined in such situation by

$$P = 1 - \frac{\kappa_g}{1 - \frac{D}{nW}(\psi_n - 1)} = 1 - \frac{\kappa_g}{1 - \frac{D}{(1-n)W}}$$

as $\psi_n - 1 = \frac{1}{1-n} - 1 = \frac{n}{1-n}$. Notice that this price simultaneously solves auction clearing in states b and g when $\kappa = \kappa_g$, regardless of n .²⁸

Since the short sale constraint for the informed in the bad state (case 1) is what determines prices at the point in which the bid-overhang constraint and prices are continuous, prices walk down the decreasing (black) function in Panel (a) of the figure, determining κ and then the consistent supply gap $\psi_g - \psi_b$, according to the equation for bayesian updating in the text.

Notice, however, that as long as the blue and black functions in Panel (b) of the figure are inside the range of plausible κ , for each supply gap to the left of $(\psi_g - \psi_b)^*$ there are two consistent prices, and two consistent κ , one in which the short-sale constraint for the informed in the bad state binds and another in which it does not. As explained in the text, one price is high and consistent with pessimism by the uninformed and the other is low and consistent with optimism by the uninformed

The fact that for the same supply gap there may be two consistent prices imply that the prices may show a discontinuous jump from one function to another. In other words, the price would run downwards following the black function of Panel (a), jump at some point

²⁸If n is sufficiently large, the common price that is consistent with the bid-overhang constraint binding for the first time may be consistent with the short-sale constraint not binding, displaying an immediate discontinuous jump down. To see this, replace a common price P in states $(b, 1)$ and (g, ψ_n) from equation (26) in the auction clearing consistent with no short selling constraint from equation (24), the uninformed investors' inference that make those two conditions consistent is $\tilde{\kappa} = \frac{W-D}{D}(\kappa_b - \kappa_g) - \frac{n}{1-n}\kappa_b$. The expected probability of default can never be higher than the maximum one feasible, $\tilde{\kappa} < \kappa_b$, which only happens if $n > 1 - \frac{\kappa_b - D}{\kappa_b - \kappa_g} \frac{D}{W-D}$.

to the blue function and keep going down on that function. Consistently, this implies that initially the uninformed become more pessimistic (going up the black function in Panel (b)), then jumping to the blue function and keep becoming more pessimistic.

It is trivial to see that there is a continuous range in which this jump may occur, so there is an indeterminate number of equilibria characterized by different point at which prices discontinuously change. All these equilibria are characterized by a single discontinuous decline in prices, as prices cannot increase in equilibrium.

The possible nonexistence arises from the constraint that beliefs are consistent with Bayesian updating. A given κ that is consistent with an equilibrium price for a gap $\psi_g - \psi_b$ determines the slope of the price function that is consistent with such inference. For instance, if κ is low, this implies that the price applies for a relatively large mass of ψ points in the good schedule compared to the bad schedule, as discussed in the main text. Then, κ also determines the speed at which the gap $\psi_g - \psi_b$ changes along the schedules. When the gap is forced to the right of the point in which the two lines cross in Panel (a), there is no price that clears the auction.

In short, as long as prices for all states are consistent with a gap smaller than $(\psi_g - \psi_b)^*$ there is multiplicity. When for some states prices are forced to be consistent with a gap larger than $(\psi_g - \psi_b)^*$ there is non-existence.

A.2 Prices when the Short-Sale Constraint Binds on the Uninformed

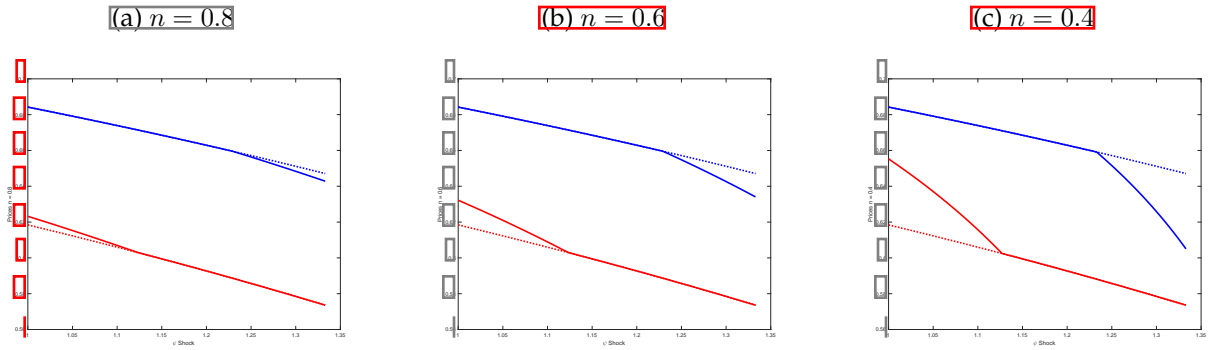
To illustrate how prices change in a situation in which the short-sale constraint binds for the uninformed investors in the prices that characterizes the symmetric information situation, we use default probabilities that are closer to each other, so that those price schedules are not very far apart. More precisely, in the example below we use $\kappa_g = 0.24$ and $\kappa_b = 0.29$. This change still guarantees that prices in the symmetric information case do not overlap, but now the uninformed wants to bid more at the price corresponding to the state $[g, \psi_M]$ than what they want to bid at the price corresponding to $[b, 1]$, which is the sufficient condition that breaks condition ii) for perfect replication in proposition 5.

When there is no overlap in the price schedule, the binding region is particularly simple. Binding will occur over intervals of the form $[\psi_g, \psi_M]$ on the $\theta = g$ schedule and $[1, \psi_b]$ on the $\theta = b$ schedule. In this case, the total risky bond purchases will be equal in this range, and the f.o.c. ignoring the short-sale constraint hold at the endpoints; i.e. $B^U(g, \theta_g) = B^U(b, \theta_b)$. In contrast to the previous analysis, the f.o.c. does not hold state by state, but the integral of the f.o.c. over the ranges will equal 0, as $s = (g, \psi_g)$ is the point at which the short-sale constraint starts binding. It is easy to see that starting from any $s = (g, \psi_g + \epsilon)$ the integral will turn negative, and extending the integral beyond (b, η_b) will turn it positive. Note that, as discussed in section 3.2, when this occurs, an auction equilibrium of the UP model no longer has an associated competitive equilibrium, as this is a case of nonnegativity constraints binding for bids in a particular range but not binding for total purchases on that range.

In Figure A.2 we plot the price functions for $n = 0.8, 0.6$ and 0.4 . As is clear from the figures, the lack of perfect replication has an effect on the price schedules. Because the bids of the uninformed do not adjust to the price over the range that the short-sale binds, it follows that all of the adjustment to match the change in per capita supply must be

done by the informed. As they shrink in number, this requires a larger change in the price to induce them change their risk exposure and cover those extra bonds. In the high price schedule the uninformed bid a fixed-amount starting from the point $(g, \psi g)$ in which the short-sale constraint binds. The need for extra bonds has to be compensated by informed investors, which depresses prices in equilibrium in the (blue) region $[\psi g, \psi_M]$. In contrast, in the low price schedule, uninformed are bidding more than informed until the point $(b, \psi b)$ at which the short-sale constraint stop binding. The extra demand inflates the prices in the (red) region $[1, \psi b]$.

Figure A.2: Prices with Binding Short-Sales Constraint on the Uninformed



Eventually, when n becomes small enough the price at the bottom of the high-price schedule will fall below the binding region for the short-sale constraint of the uninformed, and this price will now be the same as a top point on the low-price schedule where outcomes must be determined point-by-point along with the inference parameter κ of the uninformed, just as in the discussion above.

Finally, notice that in all these cases, the number of informed investors is above the threshold for the bid-overhang constraint to start binding. Indeed, the threshold at which the bid-overhang constraint starts binding when the short-sale constraint binds on the uninformed is not η_M anymore, but smaller. Intuitively, total bids of uninformed investors at the bottom of the high-price schedule are depressed relative to the informed bidders, and then it is more difficult for the uninformed investors alone to cover the revenue needs of the government in the state $[b, 1]$.

B Revenue Equivalence under Symmetry

Here we show that under the two auctions, with symmetric ignorance and symmetric information, the average yields and investors' payoffs are the same. Take a fine grid on the η 's indexed by $j = 1, \dots, J$. Define the f.o.c. kernel for a bond B_i in state j (where $j \geq i$) under the the UP auction as

$$X_{ij}^{UP} \equiv \frac{(1-\kappa)[1-P_j]}{W - \frac{B}{1-\eta_j} + \frac{B-1}{1-\eta_j} P_j} = \frac{\kappa^U P_j}{W - \frac{B}{1-\eta_j}} \quad \square$$

which does not depend on i , and then $X_{ij}^{UP} = X_j^{UP}$. Since η is distributed uninformally, the f.o.c. for a bond B_i can be expressed as

$$\sum_{j=i}^J X_j^{UP} = 0 \text{ for all } i,$$

from where it follows that

$$X_j^{UP} = 0 \text{ for all } j$$

Similarly, define the f.o.c. kernel for a bond B_i in state j (where $j \geq i$) under the the DP auction as

$$X_{ij}^{DP} = \frac{(1-\kappa)[1-P_i]}{W - \frac{D}{1-\eta_j} + \frac{D}{1-\eta_j \tilde{P}_j}} = \frac{\kappa^U P_i}{W - \frac{D}{1-\eta_j}}$$

where \tilde{P}_j is the average price that satisfies the condition

$$\tilde{P}_j * \mathcal{B}_{R,j} = \frac{D}{1-\eta_j}$$

where $\mathcal{B}_{R,j}$ is the number of risky bonds sold in state η_j . This average price must be equal to the marginal price when $j = 1$, will decline more slowly than the marginal price as j increases. The first-order condition for bond B_i can then be expressed as

$$\sum_{j=i}^J X_{ij}^{DP} = 0 \text{ for all } i$$

Note that if P_j does not decline very much so P_1 is close to P_J , then $\tilde{P}_j \simeq P_j \simeq P_{j'}$. In this case, the condition becomes very close to that in the UP auction, and

$$X_{ij}^{DP} \simeq 0$$

In figure 2 we compare prices in the symmetric ignorance case with the "average price" in the DP auction. What we see is that while the marginal prices are fairly flat, with the DP price schedule being flatter than the UP schedule, the "average price" paid schedule in a DP auction is even flatter. Given this, it is unsurprising in light of the above discussion that the average yield is also close, and that the conditional variance of the yield is much lower under the DP protocol.