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ABSTRACT

This paper describes the optimal level of government purchases in the presence of unemployment. The theoretical framework is a general-equilibrium matching model in which government purchases are valuable. When the unemployment gap is zero, the conventional Samuelson formula is valid. Otherwise, optimal government spending deviates from the Samuelson level to fill the unemployment gap partially. Hence, with a positive unemployment multiplier (so that increasing government purchases reduces unemployment), government spending should be higher than the Samuelson level when unemployment is inefficiently high and lower when unemployment is inefficiently low. We characterize the optimal level of stimulus spending. We find that stimulus spending is largest for a moderate unemployment multiplier. With larger multipliers, stimulus spending is smaller because less spending is required to fill the unemployment gap. With smaller multipliers, stimulus spending is smaller because there is more crowding out of private spending by public spending. We also find that stimulus spending increases with the elasticity of substitution between public and private consumption. With a zero elasticity (so that additional public workers dig and fill holes), stimulus spending is zero. With an infinite elasticity (so that private and public workers are perfect substitute), stimulus spending is large enough to fill the unemployment gap completely. Finally, the results hold whether taxes are nondistortionary or distortionary.

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1. Introduction

At the onset of the Great Recession governments responded to rising unemployment rates as they always do: by lowering interest rates. But the zero lower bound on the nominal interest rate rapidly became an issue, prompting governments to look for alternative stabilization policies. One policy that had been used before and was considered again, was adjusting government purchases. However, as most modern research on stabilization had focused on monetary policy, policymakers had to base their decisions on fragmentary theoretical results.

In this paper we revisit and expand existing results on the optimal use of government purchases for stabilization. The theoretical framework is a general-equilibrium matching model.[1] Government purchases are valuable and financed by taxes that keep the government budget balanced. We formulate the optimal policy with three sufficient statistics: (1) the unemployment gap, which is the difference between the current unemployment rate and the efficient unemployment rate; (2) the unemployment multiplier, which measures the reduction in unemployment rate caused by an increase in public consumption; and (3) the elasticity of substitution between public and private consumption, which describes the value of additional public consumption.[2]

Our starting point is the standard public-economics result that when productive efficiency holds, public goods should be provided to the point where the marginal rate of substitution between public and private consumption equals the marginal rate of transformation (Samuelson 1954).[3] The Samuelson formula remains valid in our model when the unemployment rate is efficient.

However, when unemployment is inefficiently high, productive efficiency fails and the Samuelson formula does not apply. Macroeconomics offers a basic insight for these situations: if the output multiplier (the effect on output of an increase in government purchases) is larger than one, government spending should increase. The logic is simple. With a multiplier above one, higher government spending leads not only to higher public consumption but also to higher private consumption, so it is desirable. During the Great Recession, this logic lead to a spurt of empirical

[1] For other papers using matching models to address optimal policy questions, see for instance Boone and Bovenberg (2002), Hungerbühler and Lehmann (2009), Lehmann, Parmentier and Van Der Linden (2011).

[2] For a description of the sufficient-statistics approach, see Chetty (2009). For other papers using the approach, see for instance Chetty (2006, 2008), Hendren (2015), Farhi and Gabaix (2015), Stantcheva (2015), Werquin (2016), and Jacquet and Lehmann (2016).

[3] For a survey of the literature on the optimal provision of public goods, see Kreiner and Verdelin (2012).

research to determine whether the output multiplier was above or below one. But the logic is incomplete. If more public spending yields more public consumption and more private consumption, public spending should increase forevermore, or until the multiplier falls below one.[4] Assuming that the multiplier becomes less than one at some point, a key question remains: now that there is a tradeoff between public and private consumption, what should the government do? This is also the question if the multiplier is below one to begin with.

To address this question, Mankiw and Weinzierl (2011) use a disequilibrium model. They find that government purchases should be above the Samuelson level to completely fill the output gap. However, the disequilibrium model is special. Its output multiplier is exactly one, so an increase in government activity does not affect private activity. There is no crowding out of private activity by government spending when government spending rises and no shift of resources back to private activity when government spending falls. This property has strong implications. For example, the social value of public spending is irrelevant for the optimal level of stimulus spending.

Using a matching model allows us to analyze optimal government spending when unemployment is inefficiently high and there is an actual tradeoff between public and private consumption. The tradeoff exists because public consumption crowds out private consumption. When the government increases its purchases, it increases the length of the queues to purchase goods and services and thus discourages household purchases, generating crowding out. The amount of crowding out may vary with the unemployment rate (Michaillat 2014).

We find that when the unemployment rate is inefficiently high or low, government purchases should deviate from the Samuelson level to partially fill the unemployment gap. Hence, when the unemployment multiplier is positive—such that an increase in government purchases reduces the unemployment rate—optimal public spending is above the Samuelson level whenever unemployment is inefficiently high. To understand the result, imagine that public consumption is at the Samuelson level, the unemployment multiplier is positive, and unemployment is inefficiently high. Keeping total consumption constant, increasing public consumption by 1 unit reduces pri-

[4] Robert Barro has made this point: "If the multiplier is greater than one, as is apparently assumed by Team Obama, the process is even more wonderful. In this case, real GDP rises by more than the increase in government purchases. Thus, in addition to the free airplane or bridge, we also have more goods and services left over to raise private consumption or investment. In this scenario, the added government spending is a good idea even if the bridge goes to nowhere, or if public employees are just filling useless holes. Of course, if this mechanism is genuine, one might ask why the government should stop with only $1 trillion of added purchases." See Robert J. Barro, "Government Spending Is No Free Lunch", Wall Street Journal, January 2009, available at http://www.wsj.com/articles/SB123258618204604599.

vate consumption by 1 unit. At the Samuelson level, the marginal utilities of public and private consumption are equalized. Hence, the increase in public consumption has no first-order effect on welfare so far. Since the unemployment multiplier is positive, increasing public consumption lowers unemployment. Since unemployment is inefficiently high, reducing unemployment raises total consumption. Now the increase in public consumption has a positive effect on welfare: it is therefore optimal to raise public spending.

Next, we study how much of the unemployment gap should be filled, and how much stimulus spending—the deviation of government spending from the Samuelson level—is required to achieve this desired outcome.

First, we examine how the optimal level of stimulus spending depends on the size of the unemployment multiplier. Nakamura and Steinsson (2014, pp.787–788) consider this question in a New Keynesian model. They obtain a formula that implicitly characterizes optimal government spending, and they infer that stimulus spending should be larger when the multiplier is larger. However, this bang-for-the-buck logic is incomplete because it omits the response of output to government spending in the implicit formula. To address this typical limitation of implicit formulas, we develop an explicit formula that describes optimal stimulus spending as a function of fixed quantities.

Our explicit formula shows that the optimal level of stimulus spending is increasing in the multiplier for small multipliers, is maximized for a moderate multiplier, and is decreasing in the multiplier for large multipliers. When the unemployment multiplier is small, the response of unemployment to public spending is small, so the bang-for-the-buck logic holds. However, when the multiplier becomes large, the response of unemployment to public spending becomes important and the logic breaks down. As the multiplier becomes large, crowding out of private consumption by public consumption becomes small, and optimal stimulus spending fills the unemployment gap nearly entirely. As less spending is required to fill the unemployment gap when the multiplier is larger, the optimal level of stimulus spending is decreasing in the multiplier.

Then, we examine how the optimal level of stimulus spending depends on the usefulness of additional public spending. We find that both the optimal level of stimulus spending and the fraction of the unemployment gap that should be filled are increasing in the elasticity of substitution between public and private consumption. There are two interesting limit cases. With a zero elasticity of substitution—new public workers dig and fill holes—public consumption beyond the Samuelson level is useless. Since public consumption crowds out private consumption, it is never optimal to provide public consumption beyond the Samuelson level. With an infinite elasticity of substitution, public and private consumption are interchangeable, so it is optimal to maximize total consumption, irrespective of its composition. Accordingly, it is optimal to fill the unemployment gap completely.

Finally, we examine how taxation affects the results. The results are initially derived with a fixed labor supply, and thus nondistortionary taxes. But a prevalent concern is that when government purchases are financed by distortionary taxes, the output multiplier could be negative as a result of the fall in labor supply caused by tax distortions, thus overturning the benefits of stimulus spending in slumps. In fact, empirical evidence for the United States suggests that the output multiplier is negative once tax distortions are taken into account (Barro and Redlick 2011). We therefore introduce endogenous labor supply and distortionary taxation. We find that all the results carry over. We also find that the unemployment multiplier equals the output multiplier net of the labor-supply response due to distortionary taxation. Since the labor-supply response is negative, the unemployment multiplier is larger than the output multiplier. With a strong negative labor-supply response, a negative output multiplier can coexist with a positive unemployment multiplier. Hence, we cannot use the sign of the output multiplier to infer the sign of the unemployment multiplier and determine whether stimulus spending is desirable in slumps. In fact, a reduction in labor supply leads to lower output but higher tightness and lower unemployment in the matching model (Michaillat and Saez 2015). Therefore, the unemployment multiplier is larger once tax distortions are taken into account. This implies that if stimulus spending is desirable in slumps under nondistortionary taxation, it is also desirable under distortionary taxation.

In this paper we limit ourselves to static considerations and skirt around dynamic considerations.[5] However, to gain a more complete understanding of how public spending can be used for stabilization, it would be useful to enrich our static analysis with dynamic elements. Several dynamic elements seem important: the intertemporal smoothing of distortionary taxes and the use of debt, as in Barro (1979); the distinction between temporary and permanent changes in government

[5] In the data the transitional dynamics of unemployment are extremely fast and inflation displays a large amount of inertia. Given these observations, we opt to abstract from the transitional dynamics of unemployment and from price dynamics. As a result, our dynamic model immediately converges to steady state from any initial condition. Thus, the analysis is essentially static even though the model is dynamic.

spending, as in Barro (1981); the dynamic response of inflation to public spending, especially in a liquidity trap, as in Woodford (2011) and Werning (2012); government investment on infrastructure projects, as in Baxter and King (1993); and the political process leading to the design of stimulus packages, as in Battaglini and Coate (2016).

2. A Matching Model for the Analysis of Government Purchases

This section presents the matching model used for the analysis of optimal government purchases. The model is dynamic and set in continuous time. There are two goods: private services, purchased by households, and public services, purchased by the government. Government purchases are valuable to households, as in Samuelson (1954). Government purchases contribute to macroeconomic stabilization by influencing the aggregate demand.

The supply structure of the model is borrowed from Michaillat and Saez (2015). Services are produced by self-employed households and traded on a matching market. Not all the available services are sold at any point in time, so there is always some unemployment. Whereas the supply structure is specific, the demand structure and price mechanism are generic. Following Chetty (2006, 2008), we will express the formula with sufficient statistics that summarize the relevant features of the demand structure and price mechanism.

2.1. The Supply Structure

The economy consists of a government and a measure 1 of identical households. Households are self-employed, producing and selling services on a matching market.[6] There are two types of services: private services and public services. Public and private services are produced using the same technology. The government purchases public services and households purchases private services. All services are sold on the same market at the same price.

Each household has a fixed productive capacity k; the productive capacity indicates the maximum amount of services that a household could sell at any point in time. Households sell $C(t)$ services to other households and $G(t)$ services to the government. The output of services is the

[6] We assume that households cannot consume their own labor services. For simplicity, we abstract from firms and assume that all production directly takes place within households. Michaillat and Saez (2015) show how the model can be modified to include firms hiring workers on a labor market and selling their production on a product market.

sum of household purchases and government purchases:

$$Y(t) = C(t) + G(t).$$

The matching process prevents households from selling their entire capacity so $Y(t) < k$.

The services are sold through long-term relationships. The relationships separate at rate $s > 0$. Since $Y(t)$ services are committed to existing relationships, households' idle capacity is $k - Y(t)$. The unemployment rate is the share of households' capacity that is idle: $u(t) = (k - Y(t))/k$.

To purchase labor services, households and government advertise $v(t)$ vacancies. The rate at which new long-term relationships are formed is given by a Cobb-Douglas matching function

$$h(t) = \omega \cdot (k - Y(t))^{\eta} \cdot v(t)^{1-\eta},$$

where $\eta \in (0,1)$ is the matching elasticity, and $\omega > 0$ is the matching efficacy.

The market tightness $x(t)$ is the ratio of the two arguments in the matching function: $x(t) \equiv v(t)/(k - Y(t))$. With constant returns to scale in matching, the tightness determines the rates at which sellers and buyers enter into new long-term trading relationships. Each of the $k - Y(t)$ units of idle capacity is sold at rate $f(x(t)) = h(t)/(k - Y(t)) = \omega \cdot x(t)^{1-\eta}$ and each of the $v(t)$ vacancies is filled at rate $q(x(t)) = h(t)/v(t) = \omega \cdot x(t)^{-\eta}$. The selling rate $f(x)$ is increasing in x and the buying rate $q(x)$ is decreasing in x. Hence, when tightness is higher, it is easier to sell services but harder to buy them. A useful result is that $f(x) = q(x) \cdot x$.

Output is a state variable with law of motion $\dot{Y}(t) = f(x(t)) \cdot (k - Y(t)) - s \cdot Y(t)$. The term $f(x(t)) \cdot (k - Y(t))$ is the number of new relationships formed at time t. The term $s \cdot Y(t)$ is the number of existing relationships separated at time t. If $f(x)$ and s are constant over time, output converges to the steady-state level

$$Y(x,k) = \frac{f(x)}{f(x)+s} \cdot k. \tag{1}$$

In practice, output reaches this steady-state level quickly because market flows are large. Throughout the paper, we therefore simplify the analysis by modeling output as a jump variable equal to its steady-state value defined by (1). The function $Y(x,k)$ is increasing in x and k $[0,+\infty)$, with

$Y(0,k) = 0$ and $\lim_{x\to+\infty} Y(x,k) = k$. When tightness is higher, it is easier to sell services so output is higher. The elasticity of $Y(x,k)$ with respect to x is $(1-\eta)\cdot u(x)$.

The unemployment rate is directly related to output: $u = 1 - Y/k$. Hence, the simplification also implies that the unemployment rate is given by

$$u(x) = \frac{s}{s+f(x)}. \tag{2}$$

The function $u(x)$ is decreasing in x, with $u(0) = 1$ and $\lim_{x\to+\infty} u(x) = 0$. When tightness is higher, it is easier to sell services so the unemployment rate is lower. The elasticity of $u(x)$ with respect to x is $-(1-\eta)\cdot(1-u(x))$. In Appendix A, we show that in US data, the unemployment rate given by (2) and the actual employment rate are indistinguishable.[7]

Advertising vacancies is costly. Posting one vacancy costs $\rho > 0$ services per unit time. Hence, a total of $\rho \cdot v(t)$ services are spent at time t on filling vacancies. These services represent the resources devoted by households and government to matching with appropriate providers of services. These resources devoted to matching are not consumed by households in that they do not enter their utility function. Hence, households' consumption of private services, denoted by $c(t)$, equals household purchases net of the matching costs incurred by households, households' consumption of public services, denoted by $g(t)$, equals government purchases net of the matching costs incurred by the government, and households' total consumption of services, denoted by $y(t) = c(t) + g(t)$, equals output net of total matching costs. We refer to $c(t)$ as private consumption, $g(t)$ as public consumption, and $y(t)$ as total consumption.

Equation (1) and the definition of the market tightness imply that $s\cdot Y(t) = f(x(t))\cdot(k - Y(t)) = q(x(t))\cdot v(t)$. Hence, $y(t) = Y(t) - \rho\cdot v(t) = Y(t) - \rho\cdot s\cdot Y(t)/q(x(t))$. Consequently, $Y(t) = y(t)\cdot[1+\tau(x(t))]$, where

$$\tau(x) \equiv \frac{\rho\cdot s}{q(x) - \rho\cdot s}. \tag{3}$$

Hence, consuming one service requires to purchase $1+\tau$ services—one service for consumption plus τ services for matching. This logic implies that private consumption is related to household

[7] We follow the approach of Hall (2005) and Pissarides (2009), who simplify their analysis by ignoring the transitional dynamics of unemployment and assuming that unemployment is a function of tightness given by (2).

purchases by $c(t) = C(t)/[1+\tau(x(t))]$ and public consumption to government purchases by $g(t) = G(t)/[1+\tau(x(t))]$. The matching wedge $\tau(x)$ is positive and increasing on $[0, x^m)$, where $x^m \in (0, +\infty)$ is defined by $q(x^m) = \rho \cdot s$. In addition, $\lim_{x \to x^m} \tau(x) = +\infty$. When tightness is higher, it is more difficult to match with a seller so the matching wedge is higher. The elasticity of $\tau(x)$ with respect to x is $\eta \cdot (1+\tau(x))$.

We write total consumption as a function of tightness and capacity:

$$y(x,k) = \frac{1-u(x)}{1+\tau(x)} \cdot k. \tag{4}$$

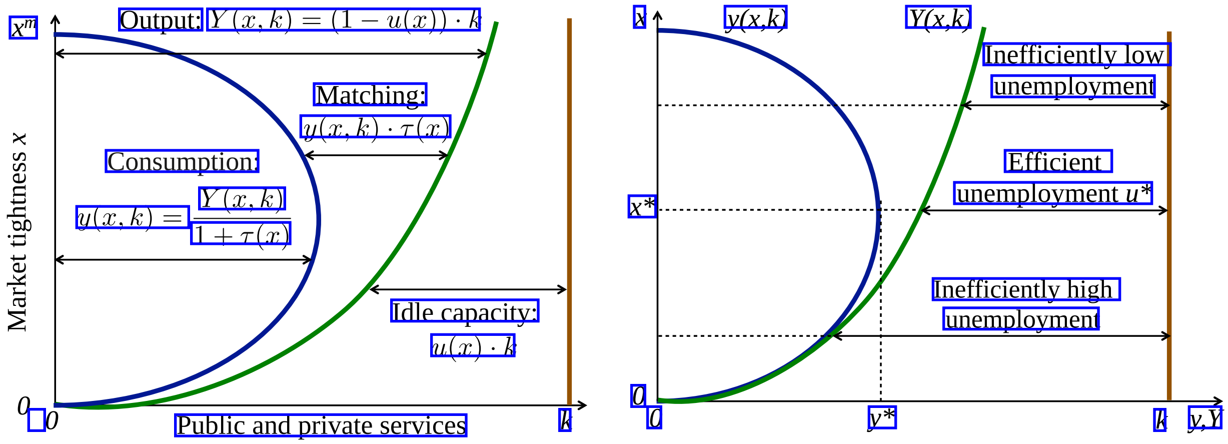
This function $y(x,k)$ plays a central role in the analysis because it gives the amount of services that can be allocated to consumption for a given tightness. The expression (4) shows that consumption is below the capacity k because some services are not sold ($u(x) > 0$) and some services are used for matching instead of consumption ($\tau(x) > 0$). The function $y(x,k)$ is defined for $x \in [0, x^m]$ and $k \geq 0$, positive, with $y(0,k) = 0$ and $y(x^m, k) = 0$. The elasticity of $y(x,k)$ with respect to x is $(1-\eta) \cdot u(x) - \eta \cdot \tau(x)$. Hence, the elasticity of $y(x,k)$ is $1-\eta$ at $x = 0$ and $-\infty$ at $x = x^m$, and it is strictly decreasing in x. Therefore, the function $y(x,k)$ is strictly increasing for $x \leq x^*$ and strictly decreasing for $x \geq x^*$, where the tightness x^* is implicitly defined by

$$\left.\frac{\partial y}{\partial x}\right|_k = 0 \quad \text{or} \quad (1-\eta) \cdot u(x^*) - \eta \cdot \tau(x^*) = 0. \tag{5}$$

As a consequence, the function $y(x,k)$ is maximized at $x = x^*$.

DEFINITION 1. *Tightness and unemployment rate are* efficient *if they maximize total consumption for a given productive capacity. The* efficient tightness*, denoted x^*, is defined by (5). The* efficient unemployment rate*, denoted u^*, is defined by $u^* = u(x^*)$. For an unemployment rate u, the* unemployment gap *is $u - u^*$.*

Figure 1 summarizes the supply side of the model. Panel A depicts how total consumption, output, and unemployment rate depend on tightness. Panel B depicts the efficient tightness, the efficient unemployment rate, situations in which tightness is inefficiently high and unemployment is inefficiently low, and situations in which tightness is inefficiently low and unemployment is inefficiently high. When unemployment is inefficiently high, too much of the economy's productive



A. The supply structure

B. Efficient and inefficient unemployment rates

Figure 1: The Matching Market

capacity is idle, and a marginal decrease in unemployment increases total consumption. When unemployment is inefficiently low, too many resources are devoted to purchasing labor services, and a further decrease in unemployment reduces total consumption. The efficient unemployment rate u^* is the unemployment rate underlying the efficiency condition of Hosios (1990).

2.2. The Demand Structure, Price Mechanism, and Equilibrium

In developing the demand structure of the model, we have two objectives. The first is to develop an aggregate demand rich enough to capture the different possible effects of government purchases, such as the stimulus of aggregate demand by government purchases or the crowding out of private consumption by public consumption.

These effects are not necessarily present in macroeconomic models. Typical matching models are an example. In these models, a free-entry condition determines equilibrium tightness. The free-entry condition is independent of government purchases so an increase in government purchases has no effect on tightness and total consumption. As discussed in Michaillat (2014), public consumption therefore crowds out private consumption one-for-one. To allow for less-than-full crowding out, we allow the aggregate demand to be less than perfectly tightness elastic. Typical disequilibrium models are another example: the demand for private consumption is unaffected by public consumption so there is no crowding out and the multiplier is exactly one (for example, Mankiw and Weinzierl 2011). The the multiplier also is exactly one in typical New Keynesian models when monetary policy is held constant (Woodford 2011). To allow for some crowding out, we allow the aggregate demand to be tightness elastic.

The advantage of a rich demand structure is to accommodate a wide range of multipliers. This facilitates the mapping between the model and the data. If the price of services does not respond to government purchases, utility is separable in private and public consumption, and taxation is lump sum, the output multiplier would be one in a disequilibrium model and zero in a matching model. In our model, the output multiplier can be anywhere between zero and one because the aggregate demand is neither perfectly tightness inelastic nor perfectly tightness elastic.[8]

The second objective is to develop an aggregate demand that is nondegenerate such that aggregate demand and supply intersect to give the equilibrium level of tightness. To avoid degeneracy, the aggregate demand curve needs to involve the tightness and price of services. This requirement on the aggregate demand is equivalent to preventing the demand and supply curves of a Walrasian market to be both price inelastic. Indeed, if both demand and supply are perfectly price inelastic (vertical in a standard price-quantity diagram), the equilibrium exists only if the two curves overlap and the intersection of the two curves does not determine a price. We want to avoid this problem.

To generate an aggregate demand satisfying the requirements, we introduce an asset in fixed supply and assume that households derive utility from holding this asset. Households spend part of their labor income on services and save part of it using the asset. Household hold the asset because they desire to smooth consumption over time and they derive utility from holding the asset.

An Example with Land. The asset that we introduce is land. Households derive utility from holding $l(t)$ units of land. This utility captures for instance the housing services provided by land. Land is traded on a perfectly competitive market. The supply of land is fixed at l_0. In equilibrium, the land market clears so $l(t) = l_0$. This demand structure is similar to the structure found in models that study the role of land and land prices in causing or propagating business cycles. For instance, models by Iacoviello (2005), Liu, Wang and Zha (2013), and Kocherlakota (2013) assume that land enters the utility function, is traded on a perfectly competitive market, and is in fixed supply.

The price of services in terms of land is $p(t)$. If the market for services was perfectly compet-

[8] If the price of services responds to government purchases, utility is not separable, and taxation is distortionary, the output multiplier can be negative or larger than one.

itive, the price would be determined such that the supply and demand of services are equal. On a matching market things are different; we specify a general price mechanism that determines the price of services: $p(t) = p(g(t))$. Given the price of services, tightness adjusts such that supply equals demand on the market for services.

The government balances its budget at all times using a lump-sum tax $T(t) = G(t)$ levied on households.

The household derives utility from consuming $c(t)$ private services and $g(t)$ public services, and from owning $l(t)$ units of land. Its instantaneous utility function is separable: $\mathscr{U}(c(t), g(t)) + \mathscr{V}(l(t))$. The function $\mathscr{U}$ is strictly increasing in its two arguments and concave. The function $\mathscr{V}$ is strictly increasing and concave. The household's utility at time 0 is

$$\int_0^{+\infty} e^{-\delta \cdot t} \cdot [\mathscr{U}(c(t), g(t)) + \mathscr{V}(l(t))]\, dt \tag{6}$$

where $\delta > 0$ is the subjective discount rate. The law of motion of the representative household's holding of land is

$$\dot{l}(t) = p(t) \cdot (1 - u(x(t))) \cdot k - p(t) \cdot (1 + \tau(x(t))) \cdot c(t) - T(t). \tag{7}$$

The household takes $l(0) = 0$ and the paths of $x(t)$, $g(t)$, $p(t)$, and $T(t)$ as given. It chooses the paths of $c(t)$ and $l(t)$ to maximize (6) subject to (7) and a standard no-Ponzi condition. To solve the household's problem, we set up the current-value Hamiltonian:

$$\mathscr{H}(t, c(t), l(t)) = \mathscr{U}(c(t), g(t)) + \mathscr{V}(l(t)) + \lambda(t)\,[p(t)\,(1 - u(x(t)))\,k - p(t)\,(1 + \tau(x(t)))\,c(t) - T(t)]$$

with control variable $c(t)$, state variable $l(t)$, and current-value costate variable $\lambda(t)$. The first-order conditions for an interior solution to the maximization problem are $\partial\mathscr{H}/\partial c = 0$, $\partial\mathscr{H}/\partial l = \delta \cdot \lambda(t) - \dot{\lambda}(t)$, and the appropriate transversality condition. These conditions imply

$$\frac{\partial \mathscr{U}}{\partial c}(c(t), g(t)) = \lambda(t) \cdot p(t) \cdot (1 + \tau(x(t))), \tag{8}$$

$$\mathscr{V}'(l(t)) = \delta \cdot \lambda(t) - \dot{\lambda}(t). \tag{9}$$

Since $\mathscr{U}$ and $\mathscr{V}$ are concave, these conditions are necessary and sufficient.

We now describe the equilibrium for a given public consumption, g. An equilibrium consists of paths for $[x(t), c(t), y(t), l(t), p(t), \lambda(t)]_{t=0}^{\infty}$ that satisfy $c(t)+g=y(t)$, $y(t)=y(x(t))$, $l(t)=l_0$, $p(t)=p(g)$, and equations (8) and (9). The first condition is the resource constraint, the second is the equality of supply and demand on the market for services, the third is the equality of supply and demand on the market for land, the fourth is the price mechanism on the market for services, and the fifth and sixth are the first-order conditions of the household's utility-maximization problem.

The equilibrium can be represented as a dynamical system of dimension 1. The only variable in the system is the costate variable $\lambda(t)$. All the other variables can be recovered from $\lambda(t)$. The variable $\lambda(t)$ satisfies the differential equation $\dot{\lambda}(t) = \delta \cdot \lambda(t) - \mathscr{V}'(l_0)$. The steady-state value of the costate variable is $\lambda = \mathscr{V}'(l_0)/\delta > 0$. Since $\delta > 0$, we infer that the dynamical system is a source. As there is no state variable, the system immediately jumps to the steady state from any initial condition.

For a given g, the equilibrium is always in steady state. The welfare associated with the equilibrium therefore is the instantaneous welfare, $\mathscr{U}(c,g)+\mathscr{V}(l_0)$. For the policy analysis, we need to relate c to g; as $c = y(x,k)-g$, we need to relate x to g. We can do that using the property that supply equals demand on the market for services.

In steady state, the desired amount of private consumption $c(x,p,g)$ is implicitly defined by

(10)
$$\frac{\partial \mathscr{U}}{\partial c}(c,g) = \frac{p\cdot(1+\tau(x))\cdot \mathscr{V}'(l_0)}{\delta}.$$

One unit of land held forever yields utility $\mathscr{V}'(l_0)/\delta$); with one unit of land the household can purchase $1/[p\cdot(1+\tau(x))]$ services yielding utility $(\partial\mathscr{U}/\partial c)/[p\cdot(1+\tau(x))]$. The equation therefore implies that at the margin the household is indifferent between purchasing one unit of land and spending the same amount on private services. The equation determines the aggregate demand, $c(x,p,g)+g$. Aggregate demand shocks are shocks to the marginal utility of land, $\mathscr{V}'(l_0)$, or to the time discount rate, δ; with higher marginal utility of land or lower time discount rate, households desire to save more and consume less, which depresses aggregate demand.

The constraint that aggregate supply $y(x,k)$ equals aggregate demand $c(x,p,g)+g$ imposes

$$y(x,k) = c(x, p(g), g) + g. \tag{11}$$

This equation implicitly defines equilibrium tightness as a function $x(g)$ of public consumption.

The Generic Case. In Appendix B we provide three other examples of demand structure and price mechanism: one with an unproduced good, one with money, and one with bonds. These three examples yield the same equilibrium structure as the land example.

In all the examples, the equilibrium immediately converges to steady state. In the steady-state equilibrium, tightness is a function of the provision of public services: $x = x(g)$. The tightness function $x(g)$ is determined by households' utility function, the price mechanism, and the supply structure. It summarizes everything that the government needs to know about the demand structure in the economy. With knowledge of the tightness function $x(g)$ and the supply structure of the economy, the government will determine the optimal provision of public services. The optimal g maximizes $\mathscr{U}(c,g) = \mathscr{U}(y(x(g),k) - g, g)$. The utility over the asset does not enter welfare because the asset is in fixed supply.

In general on the matching market the price mechanism does not guarantee efficiency. Since search is random, prices are determined in a situation of bilateral monopoly, and there are no market forces ensuring that the price of services relative to the asset maintains the unemployment rate at its efficient level (Pissarides 2000, Chapter 8). Since prices do not ensure efficiency, policies are useful to stabilize the economy—that is, bring unemployment closer to its efficient level.

In our theory, we take as given all other potential policies that could affect economic activity. These other policies could fully stabilize the economy. Our analysis would trivially apply in that case. But these policies may not be able to keep unemployment at its efficient level.[9] Our analysis concentrates on these cases. In our example with bonds, monetary policy could adjust the nominal interest rate to bring the economy to efficiency, but constraints such as the zero lower bound prevent monetary policy to achieve this goal perfectly. In the other examples taxes could affect relative

[9] We take the same approach as Eggertsson and Woodford (2006) and Farhi and Werning (2016). Eggertsson and Woodford study optimal monetary policy taking fiscal policy as given and assuming that fiscal policy fails to eliminate the decline in interest rate that led to a liquidity trap. Farhi and Werning study macroprudential policies assuming that monetary policy and tax instruments are unable to undo price rigidities and achieve the first-best allocation.

prices and bring the economy closer to efficiency, but the taxes may be difficult to implement or have costs making it suboptimal to bring unemployment to its efficient level.

3. Optimal Government Purchases Formulas

This section describes the optimal level of government purchases in the model of Section 2. We obtain formulas that relate the optimal level of government purchases to the Samuelson level, the unemployment gap, the unemployment multiplier, and the elasticity of substitution between public and private consumption.

The rest of the paper will complement this section in several ways. Section 4 shows that the formulas obtained here under exogenous labor supply remain the same under endogenous labor supply and distortionary taxation. Section 5 explains the relationship between the unemployment multiplier used in our formulas and the output multiplier commonly used in academic and policy discussions. Section 6 offers a numerical application to the Great Recession in the United States. The application shows how to calibrate the formulas and provides orders of magnitude for the optimal policy response at the onset of Great Recession. Last, Section 7 uses simulations to check the accuracy of our formulas, which are derived using first-order approximations.

3.1. An Implicit Formula

Households derive instantaneous utility $\mathscr{U}(c,g)$ from private consumption c and public consumption g. As discussed above, the utility function $\mathscr{U}$ is strictly increasing in c and g and concave. In addition, we assume that $\mathscr{U}$ is such that the marginal rate of substitution between public and private consumption

$$MRS_{gc} \equiv \frac{\partial \mathscr{U}/\partial g}{\partial \mathscr{U}/\partial c}$$

is a decreasing function of $g/c = G/C$. For instance, $\mathscr{U}$ could be a constant-elasticity-of-substitution utility function.[10] For convenience, we also assume that $MRS_{gc}(0) \geq 1$.

The welfare of an equilibrium is $\mathscr{U}(c,g)$. Since $c = y(x,k) - g$, the welfare is $\mathscr{U}(y(x,k) - g, g)$. Hence, given a tightness function $x(g)$, the government chooses g to maximize $\mathscr{U}(y(x(g),k) -$

[10] More generally, $\mathscr{U}$ could be a homothetic utility function of the form $\mathscr{U}(c,g) = \mathscr{N}(n(c,g)))$ where the function $\mathscr{N}$ is increasing and the function n is increasing in c and g, concave, and homogeneous of degree 1. In that case, as n

$g,g)$. We assume that $g \mapsto \mathscr{U}(y(x(g),k)-g,g)$ is well behaved: it admits a unique extremum and the extremum is a maximum.[11] Under this assumption, first-order conditions are necessary and sufficient to describe the optimum of the government's problem.

The first-order condition of the government's problem is

$$0 = \frac{\partial \mathscr{U}}{\partial g} - \frac{\partial \mathscr{U}}{\partial c} + \frac{\partial \mathscr{U}}{\partial c} \cdot \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}.$$

This equation shows that an increase in government purchases affects welfare through three channels: (1) it raises public consumption (first term in the right-hand side); (2) for a given level of total consumption, it reduces private consumption one-for-one (second term in the right-hand side); and (3) it affects the level of total consumption and thus the level of private consumption (third term in the right-hand side). Dividing the first-order condition by $\partial \mathscr{U}/\partial c$ yields the following proposition:

PROPOSITION 1. *In the model of Section 2, optimal government purchases satisfy*

$$1 = \underbrace{MRS_{gc}}_{\text{Samuelson formula}} + \underbrace{\frac{\partial y}{\partial x} \cdot \frac{dx}{dg}}_{\text{correction}}. \tag{12}$$

In a matching model the optimal government purchases formula adds the correction $(\partial y/\partial x) \cdot (dx/dg)$ to the formula of Samuelson (1954) for the provision of public goods. The correction term measures the value derived from government spending through stabilization. Indeed, when productive efficiency holds, total consumption is maximized ($\partial y/\partial x = 0$), the economy does not need stabilization, and the correction term is necessarily zero. But when productive efficiency fails, total consumption is below its maximum, so the economy would benefit from a stabilization policy that brings total consumption closer to its maximum. The correction term is the product of the effect

is homogeneous of degree 1, its partial derivatives $\partial n/\partial c$ and $\partial n/\partial g$ are homogeneous of degree 0. This implies

$$MRS_{gc} = \frac{\frac{\partial n}{\partial g}(c,g) \cdot \mathscr{N}'(n(c,g))}{\frac{\partial n}{\partial c}(c,g) \cdot \mathscr{N}'(n(c,g))} = \frac{\frac{\partial n}{\partial g}\left(1, \frac{g}{c}\right)}{\frac{\partial n}{\partial c}\left(\frac{c}{g}, 1\right)}.$$

We see that MRS_{gc} is a function of g/c. Moreover, as n is increasing in c and g, $\partial n/\partial g > 0$ and $\partial n/\partial c > 0$; and as n is concave, $\partial n/\partial g$ is decreasing in its second argument while $\partial n/\partial c$ is decreasing in its first argument. We conclude that MRS_{gc} is decreasing in $g/c = G/C$.

[11] We know that $x \mapsto y(x,k)$ has a unique extremum and this extremum is a maximum. Furthermore, $\mathscr{U}$ is concave. Therefore, $g \mapsto x(g)$ needs to be well behaved for $g \mapsto \mathscr{U}(y(x(g),k)-g,g)$ to satisfy the assumption.

of government purchases on tightness, dx/dg, times the effect of tightness on total consumption, $\partial y/\partial x$, so it measures the effect of government spending on total consumption. Therefore, the correction term is positive whenever government spending contributes to stabilization by increasing total consumption.[12]

Formula 12 implies that the Samuelson formula remains valid in the presence of unemployment as long as productive efficiency is satisfied. The Samuelson formula was originally derived in a neoclassical model, which is very different from our matching model, so this result was not completely obvious. This result suggests that the Samuelson formula is robust: it applies to a broad range of models as long as productive efficiency holds.

An implication of the formula is that the provision of public services is sometimes justified even if public services are not as valuable as private services. Imagine that one public service is never as valuable as one private service: $\partial \mathscr{U}/\partial g < \partial \mathscr{U}/\partial c$ or $MRS_{gc} < 1$ for any (c, g). In that case the Samuelson formula indicates that no public services should be provided: $g = 0$. But in a matching model, it may be optimal to provide some public services for stabilization. This happens when the correction term evaluated at $g = 0$ is greater than $1 - MRS_{gc}(0)$.

Formula (12) remains abstract, so we rework it to express it with sufficient statistics. We start by introducing three sufficient statistics.

DEFINITION 2. *The* Samuelson level *of G/C, denoted $(G/C)^*$, is defined by*

$$MRS_{gc}((G/C)^*) = 1. \tag{13}$$

The elasticity of substitution between public and private consumption, *denoted ε, is defined by*

$$\frac{1}{\varepsilon} = -\frac{d\ln(MRS_{gc})}{d\ln(G/C)}. \tag{14}$$

Throughout we refer to the elasticity evaluated at $(G/C)^*$. The unemployment multiplier, *denoted*

[12] Our formula is similar to formula (45) in Woodford (2011), formula (27) in Nakamura and Steinsson (2014), and formula (27) in Farhi and Werning (2012): all the formulas correct the Samuelson formula with a term arising from stabilization motives. Our formula is also closely related to the optimal unemployment insurance formula (23) in Landais, Michaillat and Saez (2016): the two formulas show that in matching models standard public-economics formulas needs to be corrected with a term that is positive whenever the policy improves welfare through tightness.

m, *is defined by*

$$m = -\frac{y}{1-u} \cdot \frac{du}{dg} \tag{15}$$

The Samuelson level is the level of G/C that satisfies the Samuelson formula. When G/C is at the Samuelson level, the marginal values from public and private consumption are equal. Since $MRS_{gc}(0) \geq 1$ and MRS_{gc} is decreasing in G/C, the Samuelson level is always well defined. The Samuelson level is determined by households' preferences, so it does not depend on economic conditions.

The elasticity of substitution between public and private consumption measures how the marginal rate of substitution between public and private consumption varies with G/C. Since MRS_{gc} is decreasing in G/C, the elasticity of substitution is positive. A lower elasticity of substitution implies that the marginal value of public consumption, measured relative to the marginal value of private consumption, decreases faster with G/C. The elasticity of substitution has two interesting limits. When $\varepsilon \to 0$, public and private consumption are perfect complement.[13] This means that a certain number of public services are needed for a given economy, but beyond that number, additional public services have zero value—additional public workers dig and fill holes. In that case, the marginal rate of substitution falls to zero as soon the required number of public services is provided. When $\varepsilon \to +\infty$, the public and private consumption are perfect substitute.[14] This means that households are equally happy to consume a unit of private or public services. In that case, the marginal rate of substitution is constant at 1.

The unemployment multiplier measures the percent increase in employment, $1-u$, when public consumption increases by 1 percent of total consumption. In practice, $1-u \approx 1$ so $m \approx -du/(dg/y)$. Hence, the unemployment multiplier approximately gives the decrease of the unemployment rate, measured in percentage points, when public consumption increases by 1 percent of total consumption.

The Samuelson level and the elasticity of substitution reflect the value that society places on public services. We will not estimate them, but treat them as an input for policy design. On the other hand, we will obtain estimates of the unemployment multiplier from the empirical literature.

[13] The Leontief utility function $\mathscr{U}(c,g) = \min\{(1-\gamma)\cdot c, \gamma\cdot g\}$ has $\varepsilon = 0$.
[14] The linear utility function $\mathscr{U}(c,g) = c + g$ has $\varepsilon \to +\infty$.

Next, we express the components of formula (12) with the sufficient statistics.

LEMMA 1. *The term $1 - MRS_{gc}$ can be approximated as follows:*

$$1 - MRS_{gc} \approx \frac{1}{\varepsilon} \cdot \frac{G/C - (G/C)^*}{(G/C)^*}. \tag{16}$$

The approximation is valid up to a remainder that is $O([G/C - (G/C)^*]^2)$. The term $\partial y/\partial x$ can be approximated as follows:

$$\frac{x}{y} \cdot \frac{\partial y}{\partial x} \approx \frac{u - u^*}{1 - u^*}. \tag{17}$$

The approximation is valid up to a remainder that is $O([u - u^*]^2)$. Last, the term dx/dg satisfies

$$\frac{y}{x} \cdot \frac{dx}{dg} = \frac{1}{(1-\eta) \cdot u} \cdot m. \tag{18}$$

Proof. Since MRS_{gc} is a function of G/C, we obtain (16) from a first-order Taylor approximation of $MRS_{gc}(G/C)$ at $(G/C)^*$, using the facts that $MRS_{gc}((G/C)^*) = 1$ and that the derivative of MRS_{gc} with respect to G/C at $(G/C)^*$ is $-1/(\varepsilon \cdot (G/C)^*)$.

Next, we write $\partial \ln(y)/\partial \ln(x)$ as a function of u:

$$\frac{\partial \ln(y)}{\partial \ln(x)} = (1-\eta) \cdot u - \eta \cdot \tau(u)$$

The function $\tau(u)$ is defined by $\tau(u) = \tau(x(u))$, where $\tau(x)$ is given by (3) and $x(u) = u^{-1}(u)$ is the inverse of the function $u(x)$ given by (2). We have

$$\tau'(u) = \tau'(x) \cdot x'(u) = \frac{\tau'(x)}{u'(x(u))} = \frac{\eta \cdot (1+\tau) \cdot \tau/x}{-(1-\eta) \cdot (1-u) \cdot u/x} = -\frac{\eta \cdot (1+\tau) \cdot \tau}{(1-\eta) \cdot (1-u) \cdot u}.$$

Since $(1-\eta) \cdot u^* = \eta \cdot \tau(u^*)$, we have $\tau'(u^*) = -(1+\tau(u^*))/(1-u^*)$ and

$$-\eta \cdot \tau'(u^*) = \frac{\eta + \eta \cdot \tau(u^*)}{1-u^*} = \frac{\eta + (1-\eta) \cdot u^*}{1-u^*} = \eta + \frac{u^*}{1-u^*}.$$

Hence, the derivative of $\partial \ln(y)/\partial \ln(x)$ with respect to u at u^* is $(1-\eta) - \eta \cdot \tau'(u^*) = 1/(1-u^*)$. Furthermore, $\partial \ln(y)/\partial \ln(x) = 0$ at u^*. Thus, a first-order Taylor expansion of $\partial \ln(y)/\partial \ln(x)$ at

u^* yields (17).

Finally, since the elasticity of $1-u(x)$ with respect to x is $(1-\eta)\cdot u$, we find that

$$m = -\frac{y}{g}\cdot\frac{g}{1-u}\cdot\frac{du}{dg} = \frac{y}{g}\cdot\frac{d\ln(1-u)}{d\ln(g)} = \frac{y}{g}\cdot(1-\eta)\cdot u\cdot\frac{d\ln(x)}{d\ln(g)} = \frac{y}{x}\cdot(1-\eta)\cdot u\cdot\frac{dx}{dg}.$$

We obtain (18) by rearranging this equation. □

Equation (16) shows that $1-MRS_{gc}$ depends on the deviation of G/C from the Samuelson level and on the elasticity of substitution between public and private consumption. This equation directly follows from the definition of the elasticity of substitution.

Equation (17) shows that the elasticity of total consumption with respect to tightness is approximately equal to the unemployment gap (as $1-u^* \approx 1$). Hence, the unemployment gap is a precise measure of productive inefficiency. Imagine tightness increases by 20 percent. If the unemployment gap is 10 percentage points total consumption increases by 2 percent; but if the unemployment gap is only 5 percentage points, total consumption only increases by 1 percent.

Last, equation (18) relates the unemployment multiplier to the effect of public consumption on tightness (dx/dg). This equation directly follows from the relationship between unemployment and tightness, given by (2).

Using the results from Lemma 1, we can rewrite formula (12) with sufficient statistics:

PROPOSITION 2. *In the model of Section 2, optimal government purchases satisfy*

(19)
$$\frac{G/C-(G/C)^*}{(G/C)^*} \approx \frac{\varepsilon}{1-\eta}\cdot m\cdot\frac{u-u^*}{u^*}.$$

The approximation is valid up to a remainder that is $O([u-u^*]^2+[G/C-(G/C)^*]^2)$.

Proof. We obtain (19) by taking (12), approximating $1-MRS_{gc}$ with (16), approximating $\partial y/\partial x$ with (17), and rewriting dx/dg with (18). □

Formula (19) relates the optimal level of government purchases (G/C) to several sufficient statistics: the matching elasticity (η), the unemployment multiplier (m), the Samuelson level ($(G/C)^*$), the elasticity of substitution between public and private consumption (ε), and the unemployment gap ($u-u^*$). The formula allows us to derive several qualitative properties of optimal government purchases.

The formula shows that the welfare-maximizing level of government purchases depends on a government-spending multiplier, confirming an intuition that macroeconomists have had for a long time. Academic and policy discussions about government spending, however, usually revolve around the output multiplier, not the unemployment multiplier. The output and unemployment multipliers are closely related, as we will show in Section 5. But the unemployment multiplier has an advantage over the output multiplier: it remains a valid sufficient statistics when taxation is distortionary, which is not the case of the output multiplier.

However, the multiplier is not sufficient to measure the effect of government purchases on welfare because an increase in government purchases also affects the composition of households' consumption. This is why of substitution between public and private consumption also enters the formula. The elasticity matters because it determines how quickly the marginal value of public services fades when government purchases increase.

Formula (19) allows us to determine under which conditions public spending should be higher than the Samuelson level. We find that when the unemployment multiplier is positive—such that an increase in government purchases reduces the unemployment rate—optimal public spending is above the Samuelson level whenever unemployment is inefficiently high. On the other hand, when the unemployment multiplier is negative—such that an increase in government purchases raises the unemployment rate—optimal public spending is above the Samuelson level whenever unemployment is inefficiently low. It is only if the unemployment multiplier is zero that government purchases should always be at the Samuelson level. These results are summarized in Table 1.

To understand these results, imagine that public consumption is at the Samuelson level, the unemployment multiplier is positive, and unemployment is inefficiently high. Keeping total consumption constant, increasing public consumption by 1 unit reduces private consumption by 1 unit. At the Samuelson level, the marginal utilities of public and private consumption are equalized. Hence, the increase in public consumption has no first-order effect on welfare so far. Since the unemployment multiplier is positive, increasing public consumption lowers unemployment. Since unemployment is inefficiently high, reducing unemployment raises total consumption. Once the effect on government purchases on unemployment is accounted for, the increase in public consumption has a positive effect on welfare. Thus, it is optimal to raise public spending above the

Table 1: The Optimal Level of Government Purchases Compared to the Samuelson Level

	Unemployment multiplier		
Unemployment gap	$m < 0$	$m = 0$	$m > 0$
$u - u^* > 0$	lower	same	higher
$u - u^* = 0$	same	same	same
$u - u^* < 0$	higher	same	lower

Notes: This table compares the optimal level of G/C to the Samuelson level of G/C, given by (13). The unemployment multiplier m is given by (15). The efficient unemployment rate u^* is given by (5). The table is obtained from formula (19).

Samuelson level.

In fact, the sign of the unemployment multiplier determines the cyclicality of optimal government purchases. Michaillat and Saez (2016) show that the unemployment gap is countercyclical in US data. This empirical result, combined with our theoretical analysis, suggests that government purchases should be countercyclical if the unemployment multiplier is positive, acyclical if the unemployment multiplier is zero, and procyclical if the unemployment multiplier is negative.

Formula (19) implies that the unemployment rate is inefficiently high or low, optimal government purchases deviate from the Samuelson level to reduce the unemployment gap. However, government purchases should fill the unemployment gap only partially, not completely. Indeed, if the gap was completely filled by government spending, we would have $u = u^*$ but $G/C \neq (G/C)^*$, which would violate the formula. Overall, the formula shows that optimal government purchases depart from the Samuelson level to push unemployment toward its efficient level.

3.2. An Explicit Formula

Formula (19) characterizes the optimal level of government purchases implicitly. Since the right-hand-side of the formula is endogenous to the policy, we cannot read the optimal level of G/C off the formula. This means that we cannot use the formula to answer quantitative questions such as "How much should government purchases increase if the unemployment rate rises to 9%?" and "How does the amplitude of the increase depend on the multiplier or the value of new public services?". This is a typical limitation of the sufficient-statistics approach, for which it has often been criticized (Chetty 2009). Here we develop an explicit formula that addresses this limitation and provides quantitative insights about optimal government purchases.

We assume that the unemployment rate is initially at an inefficient level $u_0 \neq u^*$. As government purchases change, unemployment endogenously responds. By describing this endogenous response, we obtain the following explicit formula:

PROPOSITION 3. *Consider the model of Section 2, and assume that the economy initially is at an equilibrium $[(G/C)^*, u_0]$. Then optimal government purchases satisfy*

$$\frac{G/C-(G/C)^*}{(G/C)^*} \approx \frac{\varepsilon}{1-\eta} \cdot \frac{m_0}{1+z\cdot\frac{\varepsilon}{1-\eta}\cdot m_0^2} \cdot \frac{u_0-u^*}{u^*}, \tag{20}$$

where m_0 is the value of the unemployment multiplier m at $[(G/C)^*, u_0]$ and $z = (G/Y)^* \cdot [1-(G/Y)^*] \cdot (1-u_0)/u^*$. Once optimal government purchases are in place, the unemployment rate is

$$u \approx u^* + \frac{1}{1+z\cdot\frac{\varepsilon}{1-\eta}\cdot m_0^2} \cdot (u_0-u^*). \tag{21}$$

The approximations (20) and (21) are valid up to a remainder that is $O([u_0-u^*]^2+[G/C-(G/C)^*]^2)$.

Proof. The economy starts at an equilibrium $[(G/C)^*, u_0]$, where the unemployment rate u_0 is inefficient. Since $u_0 \neq u^*$, the optimal G/C departs from $(G/C)^*$. In (19), the multiplier m and unemployment rate u are functions of G/C, so they respond as G/C moves away from $(G/C)^*$, and we cannot read the optimal G/C off the formula. In this proof, we derive a formula giving the optimal G/C as a function of fixed quantities.

First, we express the equilibrium values of all variables as functions of $[u, G/C]$. Lemma 1 showed that x and τ can be written as functions of u. Since $y = (1-u)\cdot k/(1+\tau)$, we can also write y as a function of u. Since $g = y\cdot(G/C)/[1+G/C]$, g can be written as a function of u and G/C. As $c = y-g$, c can also be written as a function of u and G/C. Last, since $C = c\cdot(1+\tau)$, $C = c\cdot(1+\tau)$, and $C = c\cdot(1+\tau)$, we can write C, G, and Y as functions of u and G/C.

Of course not all pairs $[u, G/C]$ describe an equilibrium: the only pairs describing an equilibrium are those consistent with the equilibrium condition $u = u(x(g))$, where g is the function of u and G/C described above and $x(g)$ is the function implicitly defined by (11). This equilibrium condition defines the unemployment rate as an implicit function $u(G/C)$ of G/C. The pairs describing an equilibrium therefore are the pairs $[u(G/C), G/C]$ for any $G/C > 0$. These pairs describe equilibria with different government policies.

We start by linking u to u_0 and G/C. We write a first-order Taylor expansion of $u(G/C)$ around $u((G/C)^*) = u_0$, and we subtract u^* from both sides and then divide both sides by u^*:

$$\frac{u-u^*}{u^*} = \frac{u_0-u^*}{u^*} + \frac{1}{u^*} \cdot \frac{du}{d\ln(G/C)}((G/C)^*) \cdot \frac{G/C-(G/C)^*}{(G/C)^*} + O([G/C-(G/C)^*]^2). \tag{22}$$

We turn to the term $m \cdot (u-u^*)/u^*$. The unemployment multiplier m is a function of the variables of the model so it can be written as a function of $[u, G/C]$, and as u is a function of G/C in an equilibrium, m can be written as a function of G/C. A first-order Taylor expansion of $m(G/C)$ around $m((G/C)^*) = m_0$ yields

$$m = m_0 + \frac{dm}{dG/C}((G/C)^*) \cdot [G/C-(G/C)^*] + O([G/C-(G/C)^*]^2). \tag{23}$$

Multiplying this expression for m with the expression (22) for $(u-u^*)/u^*$, we obtain

$$\begin{aligned} m \cdot \frac{u-u^*}{u^*} = m_0 \cdot \frac{u_0-u^*}{u^*} + m_0 \cdot \frac{1}{u^*} \cdot \frac{du}{d\ln(G/C)}((G/C)^*) \cdot \frac{G/C-(G/C)^*}{(G/C)^*} \\ + O([G/C-(G/C)^*]^2 + [u_0-u^*]^2). \end{aligned} \tag{24}$$

We now compute $du/d\ln(G/C)((G/C)^*)$. First, we decompose the derivative:

$$\frac{du}{d\ln(G/C)}((G/C)^*) = \frac{du}{d\ln(g)}((G/C)^*) \cdot \frac{d\ln(g)}{d\ln(G/C)}((G/C)^*). \tag{25}$$

Equation (15) implies that $du/d\ln(g)((G/C)^*) = m_0 \cdot (1-u_0) \cdot (G/Y)^*$. Next, we compute $\Omega \equiv d\ln(g)/d\ln(G/C)$ at $G/C = (G/C)^*$ and $u = u_0$. The term Ω is a function of the variables of the model so it can be written as a function of $[u, G/C]$. A first-order Taylor expansion around $[u^*, (G/C)^*]$ gives $\Omega(u_0, (G/C)^*) = \Omega(u^*, (G/C)^*) + (\partial\Omega/\partial u) \cdot [u_0-u^*] + O([u_0-u^*]^2)$. Once they are plugged into (25) and then (24), all the terms except $\Omega(u^*, (G/C)^*)$ would be part of the second-order term $O([G/C-(G/C)^*]^2 + [u_0-u^*]^2)$. Thus, we only need to calculate $\Omega(u^*, (G/C)^*)$. We have $\ln(G/C) = \ln(g) - \ln(y(x(G/C), k) - g)$; differentiating with respect to $\ln(G/C)$ yields

$$1 = \Omega - \frac{Y}{C} \cdot \frac{\partial\ln(y)}{\partial\ln(x)} \frac{d\ln(x)}{d\ln(G/C)} + \frac{G}{C} \cdot \Omega. \tag{26}$$

At $[u^*, (G/C)^*]$, $\partial \ln(y)/\partial \ln(x) = 0$, so the equation implies that $\Omega(u^*, (G/C)^*) = (C/Y)^*$.

Combining all the results, we find that

$$m \cdot \frac{u-u^*}{u^*} = m_0 \cdot \frac{u_0-u^*}{u^*} + m_0^2 \cdot z \cdot \frac{G/C-(G/C)^*}{(G/C)^*} + O([G/C-(G/C)^*]^2 + [u_0-u^*]^2).$$

with $z = (G/Y)^* \cdot (C/Y)^* \cdot (1-u_0)/u^*$. Plugging this equation into formula (19) yields (20), up to a remainder. Equation (19) includes a remainder that is $O([u-u^*]^2 + [G/C-(G/C)^*]^2)$. Equation implies that $(u-u^*)^2$ is $O([u_0-u^*]^2 + [G/C-(G/C)^*]^2)$. We conclude that the remainder in formula (19) is $O([u_0-u^*]^2 + [G/C-(G/C)^*]^2)$. The sum of the remainders in the last equation and in formula (19) therefore is $O([u_0-u^*]^2 + [G/C-(G/C)^*]^2)$.

To finish the proof, we derive (21). As we have just discussed, the remainder in formula (19) is $O([u_0-u^*]^2 + [G/C-(G/C)^*]^2)$. Using equations (22) and (23), we infer that we can rewrite formula (19) as follows:

$$\frac{G/C-(G/C)^*}{(G/C)^*} = \frac{\varepsilon}{1-\eta} \cdot m_0 \cdot \frac{u-u^*}{u^*} + O([u_0-u^*]^2 + [G/C-(G/C)^*]^2).$$

We can replace the m in formula (19) by m_0 because all the terms after m_0 in the Taylor expansion (23) are absorbed by the second-order term $O([u_0-u^*]^2 + [G/C-(G/C)^*]^2)$ once they are multiplier by $u-u^*$. Replacing the left-hand side of the last equation by the right-hand side in (20), and dividing everything by $\varepsilon \cdot m_0/(1-\eta)$, we obtain (21). □

Formula (20) links the departure of the optimal government purchases from the Samuelson level to the initial unemployment gap (u_0-u^*), measured before government purchases have been adjusted from the Samuelson level, the elasticity of substitution between public and private consumption (ε), the matching elasticity (η), and the unemployment multiplier at u_0 (m_0). The formula is explicit because the right-hand side is not endogenous to the policy. Formula (21) links the unemployment rate once optimal government purchases are in place to the same statistics.

The multiplier in the formula (m_0) is the multiplier when the unemployment rate is u_0. If the unemployment multiplier does not depend on unemployment, then m_0 can be measured by the average unemployment multiplier over the business cycle. However, a growing body of empirical evidence suggests that the unemployment multiplier could be higher when the unemployment rate is higher and output lower.[15] Hence, to obtain accurate measures of m_0, the empirical estimation should probably allow the multiplier to depend on the underlying unemployment rate.

Formulas (20) and (21) confirm that optimal government purchases deviate from the Samuelson level to partially fill the initial unemployment gap. The formulas also show that the optimal government purchases and the unemployment gap under the optimal policy crucially depend on the elasticity of substitution between public and private consumption and the unemployment multiplier. The following proposition fleshes out this result.

DEFINITION 3. *For a level of government spending G/C, the level of* stimulus spending *is $G/C - (G/C)^*$.*

PROPOSITION 4. *Assume that the unemployment multiplier m_0 is positive. If the initial unemployment rate is inefficiently high ($u_0 > u^*$), optimal stimulus spending is positive ($G/C > (G/C)^*$) but the unemployment rate under the optimal policy remains inefficiently high ($u > u^*$).*

The optimal level of stimulus spending is 0 when $m_0 = 0$, increasing in m_0 for $m_0 \in [0, m^\dagger]$, maximized at $m_0 = m^\dagger$, decreasing in m_0 for $m_0 \in [m^\dagger, +\infty]$, and 0 for $m_0 \to +\infty$. The maximizing multiplier is

$$m^\dagger = \sqrt{\frac{1-\eta}{z \cdot \epsilon}} \tag{27}$$

and the maximum level of stimulus spending is

$$\frac{(G/C)^\dagger - (G/C)^*}{(G/C)^*} = \frac{1}{2} \cdot \sqrt{\frac{\epsilon}{(1-\eta) \cdot z}} \cdot \frac{u_0 - u^*}{u^*}. \tag{28}$$

The unemployment rate under the optimal policy is u_0 when $m_0 = 0$, decreasing in m_0 for $m_0 \in (0, +\infty)$, and u^* for $m_0 \to +\infty$.

The optimal level of stimulus spending is 0 when $\epsilon = 0$, increasing in ϵ for $\epsilon \in (0, +\infty)$, and converges to

$$\frac{G/C - (G/C)^*}{(G/C)^*} = \frac{1}{z \cdot m_0} \cdot \frac{u_0 - u^*}{u^*}$$

[15] For US evidence, see for instance Shoag (2011), Auerbach and Gorodnichenko (2012), and Nakamura and Steinsson (2014). For international evidence, see for instance Auerbach and Gorodnichenko (2013), Holden and Sparrman (2016), and Jorda and Taylor (2016).

when $\varepsilon \to +\infty$. The unemployment rate under the optimal policy is u_0 when $\varepsilon = 0$, decreasing in ε for $\varepsilon \in (0, +\infty)$, and u^* for $\varepsilon \to +\infty$.

Proof. The results directly follow from (20) and (21). □

For concreteness the proposition considers positive unemployment multipliers and unemployment gaps, so optimal stimulus spending is positive.[16]

The first part of the proposition studies how the optimal level of stimulus spending depends on the size of the unemployment multiplier. There is a presumption that stimulus spending should be larger when the unemployment multiplier is larger. At first glance, formula (19) seems to justify this bang-for-the-buck logic: for a given $u - u^*$, a larger m seems to indicate that the optimal $G/C - (G/C)^*$ is larger. In fact, Nakamura and Steinsson (2014, pp.787–788) derive a formula similar to (19) in a new Keynesian model and conclude that "the extent of extra desirable spending will depend on the size of the multiplier, with a larger multiplier implying that more spending is desirable." However, this bang-for-the-buck logic is incomplete because it omits the response of unemployment to government spending in formula (19). The multiplier also determines how strongly unemployment responds to government purchases, and the link between the size of the multiplier and the optimal level of stimulus spending is more complicated than suggested by the bang-for-the-buck logic.

The proposition shows that a higher unemployment multiplier does not necessarily entail a stronger response of government purchases to fluctuations in unemployment. Instead, the optimal level of stimulus spending for a given unemployment gap is a hump-shaped function of the unemployment multiplier: the optimal level of stimulus spending is increasing in the multiplier until a threshold, and decreasing in the multiplier after that. The multiplier that calls for the highest level of stimulus spending is given by (27). The highest level of stimulus spending a given unemployment gap is given by (28). We infer that the bang-for-the-buck logic applies when the multiplier is below (27), but not when the multiplier is above (27).

There is a simple intuition behind this result. First, consider a small multiplier. We can neglect the feedback of government spending on unemployment because the multiplier is small. So in (19) we can set $u = u_0$, and it becomes clear that G/C increases with the multiplier. Intuitively, for small

[16] We consider a positive unemployment multiplier because this case is the most empirically relevant, as discussed in Section 6. We could derive similar results with a negative unemployment multiplier.

multipliers, the optimal level of government purchases is determined by the level of crowding out of private consumption by public consumption. A higher multiplier means that there is less crowding out and thus that government purchases are more desirable.

Next, consider a large multiplier. Government purchases is such a potent policy that it can close the unemployment gap without distorting much the allocation of output between public and private consumption. Hence, the optimal policy broadly is to fill the unemployment gap $u_0 - u^*$. In fact, unemployment under the optimal policy converges to efficient as the multiplier becomes large, indicating that the unemployment gap is converging to zero. As the unemployment multiplier rises, fewer government purchases are required to fill this unemployment gap, so the optimal level of stimulus spending decreases with the multiplier.

The proposition also shows that a higher elasticity of substitution between public and private consumption entails a stronger response of government purchases to fluctuations in unemployment. Both the optimal level of stimulus spending and the share of the unemployment gap filled under the optimal policy are increasing in the elasticity of substitution.

There are two interesting special cases. The first special case is $\varepsilon \to 0$. In this situation, additional public services have zero value, so additional public workers dig and fill holes. In that case, the optimal stimulus spending is zero, irrespective of the unemployment rate and multiplier. Intuitively, public consumption beyond the Samuelson level is useless. Since public consumption always crowds out private consumption, it is never optimal to provide more public consumption than in the Samuelson formula.

The second special case is $\varepsilon \to +\infty$. In this situation, the public services provided by the government perfectly substitute for the private services purchased by households. In that case, optimal stimulus spending completely fills the unemployment gap such that $u = u^*$. This result holds even if the multiplier is very small and government purchases severely crowd out household purchases. Intuitively, public and private consumption are interchangeable, so it is optimal to maximize total consumption, irrespective of its composition. Accordingly, it is optimal to completely fill the unemployment gap.

The result that government purchases should be used to fill the unemployment gap when $\varepsilon \to \infty$ is reminiscent of the Keynesian result that government purchases should be used to fill the output gap. Mankiw and Weinzierl (2011) formally derive the Keynesian result with a disequilibrium model. In their model, there is no crowding out of household purchases by government purchases. If government purchases have some value, government purchases should be increased until the output gap is entirely filled. But the logics behind the Keynesian result and our result when $\varepsilon \to \infty$ are completely different. The Keynesian result arises because there is no crowding out of private consumption by public consumption. In our model there is crowding out, but the composition of total consumption does not matter when $\varepsilon \to \infty$, so the crowding out has no welfare cost.

In reality, it is likely that public services have some value at the margin without being perfect substitutes for private services; that is, $\varepsilon > 0$ but $\varepsilon < +\infty$. In that case the optimal level of stimulus spending is positive, but not sufficient to completely fill the unemployment gap. Hence, even with optimal government purchases, the unemployment rate remains inefficient. This result arises because government purchases are not a first-best policy.

4. Government Purchases Financed by a Distortionary Income Tax

So far we have abstracted from tax distortions. In this section we extend the model by introducing an endogenous labor supply and a distortionary income tax. In this extended model, an increase in government purchases affects the aggregate demand, as before, and the aggregate supply, because it triggers an increase in taxes that affects households' labor supply.

We consider two types of taxes: a linear income tax, and a nonlinear income tax implemented following the benefit principle. Using a linear income tax is the traditional approach in public economics and the standard approach in macroeconomics. Using the benefit principle is the modern approach in public economics. In both cases, the results from Section 3 remain valid.

4.1. The Traditional Approach to Taxation

The representative household supplies a productive capacity $k(t)$ at some utility cost. The choice of $k(t)$ is akin to a labor-supply decision. The instantaneous utility function becomes $\mathscr{U}(c(t), g(t)) - \mathscr{W}(k(t)) + \mathscr{V}(b(t))$, where the function $\mathscr{W}$ is strictly increasing in k and convex. Since timing is not ambiguous, we simplify notation by dropping the time index t.

In this subsection, the government uses a linear income tax at rate τ^L to finance government purchases. The household's labor income becomes $(1-\tau^L) \cdot Y(x,k) = (1-\tau^L) \cdot (1-u(x)) \cdot k$. To finance public consumption g, the tax rate must be $\tau^L = G/Y = g/y$.

The household chooses k to maximize utility. For all the demand structures considered in Section 2 and Appendix B, the first-order condition with respect to k is

$$\mathscr{W}'(k) = \lambda \cdot (1-\tau^L) \cdot (1-u(x)),$$

where λ is the costate variable associated with real wealth in the household's Hamiltonian. We combine this equation with the first-order condition with respect to c, given by (8). We obtain

$$MRS_{kc} = (1-\tau^L) \cdot \frac{1-u(x)}{1+\tau(x)}, \tag{29}$$

where $MRS_{kc} \equiv (\partial \mathscr{W}/\partial k)/(\partial \mathscr{U}/\partial c)$ is the marginal rate of substitution between labor and private consumption. This equation describes households' optimal labor supply. It says that the marginal rate of substitution between labor and consumption has to be equal to the post-tax real wage. Indeed, one unit of labor is only sold with probability $1-u$. When it is sold, it only yields $1/(1+\tau)$ units of consumption. Hence, the effective real wage is $(1-u)/(1+\tau)$ and the post-tax real wage is $(1-\tau^L)\cdot(1-u)/(1+\tau)$.

The supply decision is distorted by the income tax: a higher τ^L implies a lower k. In fact, equation (29) implicitly defines a function $k(g)$ that describes how the aggregate productive capacity responds to a change in government purchases and the associated change in the income tax rate. Since the income tax is distortionary, the function $k(g)$ is decreasing in g.

The welfare of an equilibrium is $\mathscr{U}(c,g) - \mathscr{W}(k)$. Given a tightness function $x(g)$ and a capacity function $k(g)$, the government chooses g to maximize $\mathscr{U}(y(x(g),k(g)) - g, g) - \mathscr{W}(k(g))$. The first-order condition of the government's problem is

$$0 = \frac{\partial \mathscr{U}}{\partial g} - \frac{\partial \mathscr{U}}{\partial c} - \mathscr{W}'(k) \cdot \frac{dk}{dg} + \frac{\partial \mathscr{U}}{\partial c} \cdot \frac{\partial y}{\partial k} \cdot \frac{dk}{dg} + \frac{\partial \mathscr{U}}{\partial c} \cdot \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}.$$

Dividing the condition by $\partial \mathscr{U}/\partial c$, we obtain

$$1 = MRS_{gc} - \left(MRS_{kc} - \frac{\partial y}{\partial k}\right) \cdot \frac{dk}{dg} + \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}.$$

Households' optimal labor supply, given by (29), implies that $MRS_{kc} = (1-\tau^L)\cdot(\partial y/\partial k)$. The government's budget constraint implies that $\tau^L = g/y$. Last, by definition, $\partial y/\partial k = y/k$. Hence, with the linear income tax, formula (12) is altered as follows:

PROPOSITION 5. *In the model of Section 4.1, optimal government purchases satisfy*

$$\underbrace{1-\frac{d\ln(k)}{d\ln(g)} = MRS_{gc}}_{\text{modified Samuelson formula}} + \underbrace{\frac{\partial y}{\partial x}\cdot\frac{dx}{dg}}_{\text{correction}}. \tag{30}$$

Formula (30) is not the same as formula (12). However, the two formulas have the same structure once the Samuelson formula is modified to account for distortionary taxation.[17] Our optimal government purchases formula can be written as the modified Samuelson formula plus a correction equal to $(\partial y/\partial x)\cdot(dx/dg)$. The statistic $1-d\ln(k)/d\ln(g) > 1$ in the modified Samuelson formula is the marginal cost of funds. It is more than one because the linear income tax distorts the labor supply so it costs more to produce one public service than one private service. Because the marginal cost of funds is greater than one, the modified Samuelson formula recommends a lower level of government purchases than the regular formula. Although the Samuelson level of government purchases is lower with a linear income tax, the correction to the Samuelson formula is the same, so the deviation of optimal government purchases from the Samuelson level remains the same. In fact, the implicit formula (19) remains valid with a linear income tax:

DEFINITION 4. *The* Samuelson level *of G/C, denoted $(G/C)^*$, is defined by $MRS_{gc}((G/C)^*) = 1-d\ln(k)/d\ln(g)$, where the elasticity $d\ln(k)/d\ln(g) < 0$ is evaluated when government purchases are set optimally.*

PROPOSITION 6. *In the model of Section 4.1, optimal government purchases satisfy* (19) *once the Samuelson level $(G/C)^*$ is defined as in Definition 4.*

Proof. The Samuelson level (G/C^*) satisfies $MRS_{gc} = 1-d\ln(k)/d\ln(g)$, and the optimal G/C satisfies (30). Hence, $MRS_{gc}((G/C)^*) - MRS_{gc}(G/C) = (\partial y/\partial x)\cdot(dx/dg)$. As in Lemma 1,

[17] The modified Samuelson formula was developed by Stiglitz and Dasgupta (1971) and Atkinson and Stern (1974) to describe the optimal provision of public goods with a linear income tax. A large literature has built on these two papers. See Ballard and Fullerton (1992) and Kreiner and Verdelin (2012) for surveys and Slemrod and Yitzhaki (2001) and Kleven and Kreiner (2006) for recent contributions.

we have $MRS_{gc}((G/C)^*) - MRS_{gc}(G/C) = (1/\varepsilon) \cdot [(G/C - (G/C)^*)/(G/C)^*]$. Moreover, (17) and (18) remain valid. Combining these results, we obtain (19). □

The unemployment multiplier in formula (19) is a policy elasticity, in the sense of Hendren (2015). It measures the change in unemployment for a change in government purchases accompanied by the change in taxes maintaining a balanced government budget. In Section 3 taxes are not distortionary, so the unemployment multiplier should be measured using a policy reform in which taxes are nondistortionary. Here taxes are distortionary, so the unemployment multiplier should be measured using a policy reform in which the tax change distorts the labor supply. Moreover, Michaillat and Saez (2015) show that tightness rises and the unemployment rate falls in response to a reduction in labor supply. Therefore, we expect unemployment multipliers to be larger under distortionary taxation than under nondistortionary taxation.

The explicit formula (20) also remains valid with a linear income tax:

PROPOSITION 7. *Consider the model of Section 4.1, and assume that the economy is at an equilibrium $[(G/C)^*, u_0]$. Then the optimal stimulus spending satisfies (20) and the unemployment rate once optimal government purchases are in place satisfies (21), where the statistic z is generalized to allow for supply-side responses:*

$$z = (G/Y)^* \cdot [1 - (G/Y)^*] \cdot \frac{1 - u_0}{u^*} \cdot \frac{1}{1 - d\ln(k)/d\ln(g)}.$$

The elasticity $d\ln(k)/d\ln(g)$ is evaluated at $[(G/C)^*, u^*]$.

Proof. Since formula (19) remains valid in the model of Section 4.1, the proof can follow the same steps as the proof of Proposition 3. The only difference occurs once we reach equation (26). With a supply-side response to taxation, the equation becomes

$$1 = \Omega - \frac{Y}{C} \cdot \frac{\partial \ln(y)}{\partial \ln(x)} \frac{d\ln(x)}{d\ln(G/C)} - \frac{Y}{C} \cdot \frac{\partial \ln(y)}{\partial \ln(k)} \cdot \frac{d\ln(k)}{d\ln(g)} \cdot \Omega + \frac{G}{C} \cdot \Omega.$$

At $[u^*, (G/C)^*]$, $\partial \ln(y)/\partial \ln(x) = 0$. Furthermore, $\partial \ln(y)/\partial \ln(k) = 1$. So the equation implies that $\Omega(u^*, (G/C)^*) = (C/Y)^*/(1 - d\ln(k)/d\ln(g))$. Using the new expression for $\Omega(u^*, (G/C)^*)$, we conclude this proof as the proof of Proposition 3. □

4.2. The Modern Approach to Taxation

With homogenous households and endogenous labor supply, the government should raise revenue with a lump-sum tax to avoid distortions. In fact, with endogenous labor supply and a lump-sum tax, all the results obtained with exogenous labor supply in Section 3 would remain valid.

With heterogenous households, on the other hand, a distortionary tax system can be justified. It is justified if the government values redistribution (Diamond and Mirrlees 1971). It is also justified if low-income workers are unable to pay a uniform lump-sum tax. These two reasons explain why real tax systems are distortionary.

To improve the realism of the analysis of Section 3, we now assume that labor supply is endogenous and government purchases are financed by a distortionary income tax. However, because households are homogenous in the model, our welfare analysis cannot capture the benefits of the distortionary income tax, such as higher redistribution. Our welfare analysis can only includes the cost of the distortionary income tax: a deadweight loss proportional to the labor-supply response. Because it considers the cost but not the benefits of the distortionary income tax, our welfare analysis is bound to be biased.

To overcome this bias, we follow the modern approach to taxation developed in public economics. This approach consists in using a nonlinear income tax schedule $T(k)$ and implementing tax reforms following the benefit principle. With the income tax, the household's disposable income becomes $(1-u(x))\cdot(k-T(k))$. Because the tax distortions exist for reasons unrelated to the provision of public goods, the benefit principle argues that an increase in government purchases should be financed by a tax increase that does not alter labor-supply distortions.[18] More precisely, a change in government purchases is financed by a tax change designed to leave individual utility unchanged for any level of labor supply. As a consequence, the labor-supply decision is unaffected by the policy reform. Then, a Pareto improvement is possible if the reform generates a government budget surplus or deficit. A surplus can be redistributed to households, creating a Pareto improvement. With a deficit, the opposite of the reform generates a surplus, making a Pareto improvement possible. Hence, the formula for optimal government purchases ensures that any reform leaves the

[18] The benefit principle was introduced by Hylland and Zeckhauser (1979) and generalized by Kaplow (1996, 1998). For surveys of the modern approach to taxation, see Kaplow (2004) and Kreiner and Verdelin (2012). The modern approach is closely related to the approach of Christiansen (1981), Sandmo (1998), Coate (2000), Jacobs (2016), and others, who study the optimal provision of public goods when taxes optimally satisfy redistributive objectives.

government budget balanced. We find that formula (12) does just that:

PROPOSITION 8. *In the model of Section 4.2, optimal government purchases satisfy* (12).

Proof. We start from an equilibrium $[c,g,x,k]$. To ease notation, we introduce $\phi(x) \equiv (1-u(x))/(1+\tau(x))$. In steady state, households' expenses equal their disposable income: $(1+\tau(x))\cdot c = (1-u(x))\cdot(k-T(k))$ so $c = \phi(x)\cdot(k-T(k))$.

We implement a small change dg. This change triggers a small change dx in tightness. We follow the benefit principle. The change dg is funded by a tax change $dT(k)$ designed to keep the household's utility constant for any choice of k. For all k, $dT(k)$ satisfies

$$\mathscr{U}(\phi(x)\cdot[k-T(k)],g) = \mathscr{U}(\phi(x+dx)\cdot[k-T(k)-dT(k)],g+dg). \tag{31}$$

The left-hand side and right-hand side of this equation define two functions of k. Since the functions are equal for all k, they have the same derivative with respect to k. For all k,

$$\frac{\partial\mathscr{U}}{\partial c}(\phi(x)\cdot[k-T(k)],g)\cdot\phi(x)\cdot\left[1-T'(k)\right]$$
$$=\frac{\partial\mathscr{U}}{\partial c}(\phi(x+dx)\cdot[k-T(k)-dT(k)],g+dg)\cdot\phi(x+dx)\cdot\left[1-T'(k)-dT'(k)\right].$$

The choice of k is given by an equation similar to (29). Hence, before the reform the household chooses k such that the left-hand side equals $\mathscr{W}'(k)$, and after the reform the household chooses k such that the right-hand side equals $\mathscr{W}'(k)$. Since the left-hand side and right-hand side are equal for all k, the household does not change his choice of k after the reform so $dk=0$. In sum, with the change $dT(k)$ in the tax system, the labor supply is unaffected by the change dg.

Taking a first-order expansion of the right-hand side of (31) and subtracting the left-hand side from the right-hand side, we obtain

$$\frac{\partial\mathscr{U}}{\partial c}\cdot\left[\phi'(x)\cdot(k-T(k))\cdot dx-\phi(x)\cdot dT(k)\right]+\frac{\partial\mathscr{U}}{\partial g}\cdot dg=0.$$

Dividing by $\partial\mathscr{U}/\partial c$ and re-arranging, we obtain

$$T(k)\cdot\phi'(x)\cdot dx+\phi(x)\cdot dT(k) = MRS_{gc}\cdot dg+\phi'(x)\cdot k\cdot dx$$

Accordingly, the effect of the reform on the government budget balance $R = \phi(x) \cdot T(k) - g$ is

$$dR = T(k) \cdot \phi'(x) \cdot dx + \phi(x) \cdot dT(k) - dg = [MRS_{gc} - 1] \cdot dg + \phi'(x) \cdot k \cdot dx = [MRS_{gc} - 1] \cdot dg + \frac{\partial y}{\partial x} \cdot dx.$$

(We used $dk = 0$ and $\phi'(x) \cdot k = \partial y / \partial x$.) At the optimum, $dR = 0$, so (12) holds. □

Since formula (12) remains valid with endogenous labor supply under the benefit principle, and since capacity k is not affected by changes in government purchases under the benefit principle, the other results from Section 3 also hold:

PROPOSITION 9. *In the model of Section 4.2, optimal government purchases satisfy formulas (19) and (20). The unemployment rate under the optimal policy satisfies formula (21). The statistics in the formulas are defined as in Section 2.*

5. The Link Between Unemployment Multiplier and Output Multiplier

The results in Sections 3 and 4 are based on the unemployment multiplier. But academic and policy discussions about government spending usually involve the output multiplier. In this section, we explain how the unemployment and output multipliers are related, and we recast the results in terms of the output multiplier. We find that the unemployment multiplier equals the output multiplier net of supply-side responses. Hence, when taxation is nondistortionary, unemployment and output multipliers are the same, but when taxation is distortionary, they are substantially different and can even have the opposite sign.

First, we introduce the empirical unemployment multiplier. This multiplier will be a bridge between the theoretical unemployment multiplier that enters our formulas and the output multiplier.

DEFINITION 5. *The* empirical unemployment multiplier, *denoted M, is defined by*

$$M = -\frac{Y}{1-u} \cdot \frac{du}{dG}. \tag{32}$$

The empirical unemployment multiplier measures the percent increase of the employment rate, $1-u$, when government purchases increase by 1 percent of GDP. In practice, $1-u \approx 1$ so $M \approx -du/(dG/Y)$. This multiplier approximately measures the decrease of the unemployment rate when government purchases increase by 1 percent of GDP. The empirical unemployment multiplier is a close cousin of the theoretical unemployment multiplier:

LEMMA 2. *The theoretical and empirical unemployment multipliers are related by*

$$m = \frac{M}{1 - \frac{G}{Y} \cdot \frac{\eta}{1-\eta} \cdot \frac{\tau}{u} \cdot M}. \tag{33}$$

Proof. As $G = (1 + \tau(x(g))) \cdot g$ and the elasticity of $1 + \tau(x)$ with respect to x is $\eta \cdot \tau$, we find that

$$\frac{d\ln(G)}{d\ln(g)} = 1 + \eta \cdot \tau \cdot \frac{d\ln(x)}{d\ln(g)} = 1 + \frac{g}{y} \cdot \frac{\eta}{1-\eta} \cdot \frac{\tau}{u} \cdot m, \tag{34}$$

where the last equality is obtained using (18). Furthermore, the definitions of m and M imply that

$$m = -\frac{y}{1-u} \cdot \frac{du}{dg} = -\frac{Y/(1+\tau(x))}{1-u} \cdot \frac{du}{dG} \cdot \frac{dG}{dg} = \frac{g}{G} \cdot M \cdot \frac{dG}{dg} = M \cdot \frac{d\ln(G)}{d\ln(g)}. \tag{35}$$

We plug into (35) the expression for $d\ln(G)/d\ln(g)$ obtained in (34):

$$m = M + \frac{g}{y} \cdot \frac{\eta}{1-\eta} \cdot \frac{\tau}{u} \cdot M \cdot m.$$

We obtain (33) by rearranging this equation. □

The theoretical unemployment multiplier, m, and the empirical unemployment multiplier, M, always have the same sign. Moreover, m is increasing in M. While m is always larger than M in absolute value, they are not very different in practice.

The empirical unemployment multiplier and output multiplier are closely related. We first consider the case in which government purchases and taxes do not distort capacity.

PROPOSITION 10. *When a change in government purchases and the associated change in taxes do not distort the capacity supplied by households, as in the models of Section 2 and Section 4.2, the empirical unemployment multiplier and the output multiplier are equal: $M = dY/dG$.*

Proof. Consider a change in government purchases dG that does not distort the capacity k supplied by households. This change leads to a change du in unemployment and, since $Y = (1-u) \cdot k$, to a change $dY = -du \cdot k$ in output. Hence, $dY/dG = k \cdot (-du/dG) = [Y/(1-u)] \cdot (-du/dG) = M$. □

With exogenous labor supply or under the benefit principle, all our formulas can be directly expressed with the output multiplier by replacing m by the function of M given by (33) and then substituting dY/dG for M, since the two are identical. After this manipulation, it is clear that the output multiplier matters for optimal government purchases exactly in the same way as the unemployment multiplier m.

When government purchases and the associated taxes distort capacity, things are different:

PROPOSITION 11. *When a change in government purchases and the associated change in taxes distort the capacity supplied by households, as in the model of Section 4.1, the empirical unemployment multiplier equals the output multiplier net of the supply-side response:*

$$M = \frac{dY}{dG} - \frac{Y}{k} \cdot \frac{dk}{dG} > \frac{dY}{dG}.$$

Proof. Output is $Y = (1-u) \cdot k$ so

$$\frac{dY}{dG} = -k \cdot \frac{du}{dG} + (1-u) \cdot \frac{dk}{dG} = -\frac{Y}{1-u} \cdot \frac{du}{dG} + \frac{Y}{k} \cdot \frac{dk}{dG} = M + \frac{Y}{k} \cdot \frac{dk}{dG}.$$

Since taxes are distortionary, $dk/dG < 0$ and $M > dY/dG$. □

The proposition establishes that the empirical unemployment multiplier is the output multiplier net of the supply-side response $(Y/k) \cdot (dk/dG)$. The supply-side response measures the percentage change in labor supply when government purchases increase by 1 percent of GDP. As the linear income tax is distortionary, the supply-side response is negative and the unemployment multiplier is larger than the output multiplier.

The unemployment multiplier is the correct sufficient statistic whether labor supply is exogenous or endogenous, and whether taxation is distortionary or not. With exogenous labor supply or under the benefit principle, output and unemployment multipliers are the same so the output multiplier matters for optimal government purchases exactly in the same way as the unemployment multiplier. With distortionary taxation, things are different. Optimal government purchases depend on the unemployment multiplier, and the unemployment multiplier is equal to the output multiplier net of supply-side responses, so the output multiplier cannot be used directly to design optimal government purchases.

Empirical evidence hints that unemployment multiplier and output multiplier may not have the same sign in the United States. Barro and Redlick (2011) find that the output multiplier is negative (around -0.6) once tax distortions are taken into account. On the other hand, Monacelli, Perotti and Trigari (2010) and Ramey (2013) finds that the empirical unemployment multiplier is positive (around 0.5). One potential issue is that the unemployment multipliers are estimated using deficit-financed increases in government purchases, not balanced-budget increases (Barro and Redlick 2011). This issue is not very important here because unemployment multipliers are larger once the negative response of labor supply to distortionary taxes is taken into account. This is because a reduction in labor supply leads to higher tightness and lower unemployment in the matching model (Michaillat and Saez 2015). Hence, 0.5 is a lower bound for the balanced-budget empirical unemployment multiplier.

Theory also shows that unemployment multiplier and output multiplier may have the opposite sign. In the matching model, the unemployment multiplier is necessarily positive, whereas the output multiplier can be negative if labor-supply distortions are strong enough.

When the output multiplier is negative and the unemployment multiplier is positive, government purchases and the taxes used to finance them should be above the Samuelson level in slumps. This is surprising. How can lowering output through higher taxes be optimal when output is already inefficiently low? To understand this result, imagine that government purchases are the Samuelson level and that unemployment is inefficiently high. A small increase in government purchases reduces unemployment, reduces labor supply, and increase public consumption, which are all good for welfare; but it reduces output and thus private consumption, which is bad for welfare. At the Samuelson level, the cost of lower private consumption offsets the benefit of higher public consumption and lower labor supply; the only remaining effect on welfare is the positive effect from lower unemployment. (When the unemployment rate is efficient, variations in unemployment have no first-order effects on welfare so the Samuelson level is optimal.) Overall, the small increase in government purchases raises welfare. Hence, it is optimal to raise government purchases above the Samuelson level. At the same time, the increase in government spending and taxes lowers output because the reduction in output due to lower labor supply dominates the increase in output due to lower unemployment. But, because the reduction in output due to lower labor supply is already internalized in the modified Samuelson formula, it is the increase in output due to lower unemployment that determines the deviation from the modified Samuelson formula.

6. A Numerical Application: The Great Recession in the United States

In this section we use our explicit formula, given by (20), to compute the optimal response of government purchases to the increase in unemployment observed at the onset of the Great Recession in the United States. We examine how the optimal policy response varies with the unemployment multiplier and the elasticity of substitution between public and private consumption. The results are displayed in Figure 2.

The starting point of the numerical application is 2008:Q3 in the United States. Then, the unemployment rate was $u = 6\%$ and government purchases were $G/C = 19.7\%$.[19] To simplify the analysis, we consider the unemployment rate in 2008:Q3 as efficient: $u^* = 6\%$. This choice is reasonable as the unemployment rate in 2008:Q3 is close to its 25-year average, and there is a presumption, going back at least to Okun (1963), that the economy is efficient on average. If unemployment is efficient in 2008:Q3, it is optimal for the government to follow the Samuelson formula. To simplify the analysis further, we therefore set the Samuelson level to the government purchases in 2008:Q3: $(G/C)^* = 19.7\%$. This choice seems reasonable as G/C in 2008:Q3 is equal its 25-year average.[20]

In 2008, a shock hit the US economy and unemployment started rising toward an inefficient level $u_0 > u^*$. In our model unemployment immediately reaches the higher level u_0, but in reality unemployment slowly rises to u_0. The challenge for policymakers is to forecast u_0 before that level is reached. In the winter 2008–2009, when the US government was designing their stimulus package, they forecasted $u_0 = 9\%$ (Romer and Bernstein 2009, Figure 1). We therefore use $u_0 = 9\%$ for the numerical application.

To apply formula (20), we need estimates of three statistics: the elasticity of substitution between public and private consumption (ϵ), the matching elasticity (η), and the unemployment multiplier at u_0 (m_0). First, the elasticity ϵ reflect the value that society places on public services.

[19] We measure u as the quarterly average of the seasonally adjusted monthly unemployment rate constructed by the Bureau of Labor Statistics (BLS) from the Current Population survey (CPS). To construct G/C, we measure G as employment in the government industry and C as employment in the private industry. Both G and C are quarterly averages of seasonally adjusted monthly series constructed by the BLS from the Current Employment Statistics survey.

[20] Over the 1990–2014 period, the average unemployment rate is 6.1% and the average of G/C is 19.7%.

It is an input into the design of the optimal policy, so we consider several values that span the range of possibilities: $\varepsilon = 0.5$, $\varepsilon = 1$, and $\varepsilon = 2$.[21] Second, Landais, Michaillat and Saez (2015) review the literature estimating matching functions on labor market data, and pick $\eta = 0.6$ as their preferred estimate. We follow them and set $\eta = 0.6$.

Last, we determine a plausible range for the multiplier m_0. Since the theoretical multiplier m_0 is difficult to observe, we report estimates of the empirical multiplier M_0.[22] We then translate M_0 into m_0 using (33). In fact, as above, we can set $(G/Y)^* = 0.197/(1+0.197) = 16.5\%$, $\eta = 0.6$, and $u_0 = 9\%$. The series on resources devoted to matching constructed by Landais, Michaillat and Saez (2015) indicate that when the unemployment rate is around 9%, as in 2009:Q2, then $\tau = 1.7\%$. So we set $\tau_0 = 1.7\%$. Under this calibration,

$$m_0 = \frac{M_0}{1 - 0.165 \times 2 \times (1.7/9) \times M_0} = \frac{M_0}{1 + 0.062 \times M_0}. \tag{36}$$

The unemployment multiplier M_0 is estimated by measuring the response of the unemployment rate (in percentage points) when government purchases increase by 1% of GDP. Monacelli, Perotti and Trigari (2010) estimate a structural vector autoregression (SVAR) on US data for 1954–2006 and find unemployment multipliers between 0.2 and 0.6, depending on how the multiplier is defined. Ramey (2013) estimates SVARs on US data for 1939–2008. All the identification schemes and sample periods considered give unemployment multipliers between 0.2 and 0.5, except one specification giving an unemployment multiplier of 1.[23] As discussed by Barro and Redlick (2011), one potential issue is that these multipliers are estimated using deficit-financed increases in government purchases, not balanced-budget increases. This issue is not very important here because unemployment multipliers are larger once the negative response of labor supply to distortionary taxes is taken into account. This is because a reduction in labor supply leads to higher tightness and lower unemployment in the matching model (Michaillat and Saez 2015). Hence, these estimates are a lower bound for the balanced-budget unemployment multiplier.

[21] The case $\varepsilon = 1$ corresponds to a Cobb-Douglas utility function: $\mathscr{U}(c,g) = c^{1-\gamma} \cdot g^{\gamma}$.

[22] The theoretical multiplier m_0 is difficult to directly estimate because it measures the response to a change in government consumption g, which is government purchases G minus the resources devoted by the government for matching and therefore difficult to observe. The empirical multiplier M_0 is much easier to estimate because it measures the response to a change in government purchases G, which are reported in national accounts.

[23] See Monacelli, Perotti and Trigari (2010, pp.533–536) and Ramey (2013, pp.40–42).

We can also measure the unemployment multiplier M_0 from estimates of the output multiplier obtained under deficit financing. Indeed, under the standard assumption that households are Ricardian, deficit financing is equivalent to lump-sum taxation. Then, under the typical assumption that the utility function $\mathscr{U}(c,g)$ is separable, changes in government purchases financed with lump-sum taxation have no effect on the labor supply. Therefore, unemployment multiplier and output multipliers are the same (Proposition 10).[24] Ramey (2011, Table 1) reports that in aggregate analyses on postwar US data, deficit-financed output multipliers are between 0.6 and 1.6. This evidence indicates that when the tax changes used to finance changes in government purchases are nondistortionary, the unemployment multiplier could be in the 0.6–1.6 range.

Overall, the average unemployment multiplier is likely to fall in the 0.2–1.6 range. If multipliers are larger when unemployment is higher, as suggested by several articles computing state-dependent multipliers on US data, M_0 could even be larger.[25] Given the uncertainty about the exact estimate of the unemployment multiplier, we compute the optimal level of stimulus spending for M_0 between 0 and 2.

The graph on the left in Figure 2 displays the optimal level of government purchases as a share of output, G/Y, as a function of the initial unemployment multiplier, M_0. This graph is constructed using (20). Initially the unemployment rate is $u_0 = 9\%$, but with the optimal policy the unemployment rate falls below its initial level. The graph on the right in Figure 2 displays the unemployment rate that prevails once government purchases have been adjusted to their optimal level. This graph is constructed using (21).

The numerical application illustrates all the theoretical results obtained in the previous sections. In addition, it provides orders of magnitude for the optimal response of government purchases to a realistic increase in unemployment. Several observations stand out.

A first observation is that even with a small multiplier of 0.2, government purchases should increase significantly above the Samuelson level in response to the increase in unemployment from 6% to 9%. With $\varepsilon = 0.5$, government purchases should increase by 1.5 percentage points of GDP to $G/Y = 18.0\%$. For $\varepsilon = 1$, they should increase by 2.8 points of GDP to $G/Y = 19.3\%$.

[24] If the utility function is not separable, the unemployment and output multipliers are not exactly the same because $dk/dg \neq 0$ (see equation (29) with $\tau^L = 0$). But they are close as long as the effect of g on $\partial \mathscr{U}/\partial c$ is small.

[25] See Auerbach and Gorodnichenko (2012), Candelon and Lieb (2013), and Fazzari, Morley and Panovska (2015). Auerbach and Gorodnichenko (2012, Table 1, row 3) report output multipliers above 2 in recessions.

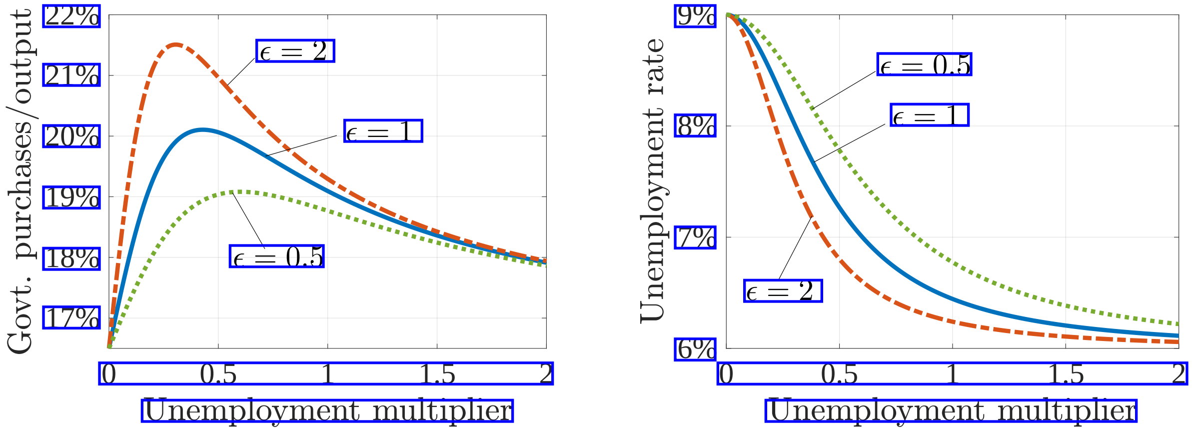


Figure 2: Optimal Government Purchases, and Unemployment Rate under the Optimal Policy

Notes: In 2008:Q3 in the United States, the unemployment rate was $u = 6\%$ and government purchases were $G/Y = 16.5\%$. We consider that the labor market was efficient and government purchases were optimal in 2008:Q3. The Great Recession shock hit the US economy around that time. The unemployment rate was projected to increase to 9% after the shock. The graph on the left displays the optimal response of government purchases, expressed as a share of output, after the shock. The optimal response is computed using (20), $u^* = 6\%$, $(G/Y)^* = 16.5\%$, $u_0 = 9\%$, $\eta = 0.6$, ε equal to 0.5, 1, and 2, M_0 between 0 and 2, and the facts that M_0 and m_0 are linked by (36) and $G/Y = G/C/(1+G/C)$. The graph on the right displays the unemployment rate once government purchases are set optimally. The unemployment rate is computed using (21) with the same calibration.

And for $\varepsilon = 2$, they should increase by 4.6 points of GDP to $G/Y = 21.1\%$. Thus, the optimal level of stimulus spending is quite large even for small multipliers.

A second observation is that the unemployment multiplier that warrants the largest increase in government purchases is fairly modest. With $\varepsilon = 0.5$, the largest increase is 2.6 percentage points of GDP and it occurs with a multiplier of 0.6. With $\varepsilon = 1$, the largest increase is 3.6 points of GDP and it occurs with a multiplier of 0.4. And with $\varepsilon = 2$, the largest increase is 5 points of GDP and it occurs with a multiplier of 0.3.

A third observation is that optimal government purchases are the same for small and large multipliers. For instance, fix $\varepsilon = 1$. The optimal increase in government purchases is the same for multipliers of 0.12 and 1.5—1.9 points of GDP. The optimal increase in government purchases is also the same for multipliers of 0.08 and 2—1.3 points of GDP. Of course the resulting unemployment rates are very different. The optimal policy fills a much larger portion of the unemployment gap with a large multiplier.

A fourth observation is that, even though government purchases increase significantly for small multipliers, the unemployment rate barely falls below its initial level of 9%. With the multiplier of 0.2, the unemployment rate only falls to 8.7% with $\varepsilon = 0.5$, to 8.5% with $\varepsilon = 1$, and to 8.1% with $\varepsilon = 2$. Unemployment does not fall much because government purchases have little effect on unemployment when the multiplier is small.

A fifth observation is that, with a multiplier above 1, the stabilization of the unemployment rate is almost perfect. With a multiplier of 1, the unemployment rate achieved with the optimal policy is below 6.8%, so the remaining unemployment gap is less than 0.8 percentage points. With a multiplier of 2, the remaining unemployment gap is less than 0.2 percentage points.

A last observation is the elasticity of substitution between public and private consumption, ε, plays a significant role for small to medium multipliers but not for large multipliers. Consider first a multiplier to 0.4. With $\varepsilon = 0.5$, government purchases should increase by 2.4 percentage points of GDP. With $\varepsilon = 1$, government purchases should increase by an additional 1.2 points of GDP. And with $\varepsilon = 2$, government purchases should increase by yet another 1.2 points of GDP. Hence, ε significantly influences the optimal response of government purchases to unemployment fluctuations. On the other hand, for a multiplier above 1, the optimal levels of government purchases for $\varepsilon = 0.5$, $\varepsilon = 1$, and $\varepsilon = 2$ are nearly indistinguishable. This is because the optimal policy is broadly to fill the unemployment gap for large multipliers, so the elasticity of substitution has little influence on the optimal policy response.

7. Simulations

This section simulates the matching model with land described in Section 2. The model is calibrated to US data. The section serves several purposes. First, it relates the sufficient statistics in the formulas, especially the unemployment multiplier, to parameters of the model. Second, it demonstrates that the matching model provides a good description of the business cycle: in response to aggregate demand shocks, the model generates realistic countercyclical fluctuations in the unemployment rate and unemployment multiplier. Third, it confirms that the explicit formula, given by (20), is accurate even for sizable fluctuations in unemployment.

We begin by specifying the utility function and price mechanism used in the simulations. We use a constant-elasticity-of-substitution utility function

$$\mathscr{U}(c,g) = \left[(1-\gamma)^{\frac{1}{\varepsilon}} \cdot c^{\frac{\varepsilon-1}{\varepsilon}} + \gamma^{\frac{1}{\varepsilon}} \cdot g^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

The parameter $\gamma \in (0,1)$ indicates the value of public services relative to private services, and the parameter $\varepsilon > 0$ gives the elasticity of substitution between public and private consumption.

We use a price mechanism that extends the efficient price mechanism to match empirical evidence on unemployment fluctuations and the unemployment multiplier. The efficient price mechanism $p^*(g)$ is the price of services ensuring that unemployment is efficient. Using equation (10), we see that

$$p^*(g) = \frac{1}{1+\tau(x^*)} \cdot \alpha \cdot \frac{\partial \mathscr{U}}{\partial c}(y^* - g, g),$$

where $\alpha \equiv \delta / \mathscr{Y}'(l_0)$ parameterizes aggregate demand. If the price is continuously at p^*, the unemployment rate stays at u^*. As u^* remains the same in response to aggregate demand or supply shocks, then unemployment rate remains constant in response to typical business-cycle shocks. Furthermore, since the unemployment rate remains at u^* when g changes, the unemployment multiplier is zero. However, the unemployment multiplier is positive (see Section 6), and unemployment is subject to wide business-cycle fluctuations. To introduce unemployment fluctuations and a positive multiplier, we consider a price mechanism less flexible than the efficient price:

$$p(g) = p_0 \cdot \alpha^{1-r_0} \cdot \left(\frac{\partial \mathscr{U}}{\partial c}(y^* - g, g)\right)^{1-r_1}. \tag{37}$$

The parameter $p_0 > 0$ governs the price level. The parameter $r_0 \in [0,1]$ measures the rigidity of the price with respect to aggregate demand: if $r_0 = 1$, the price does not respond to aggregate demand; if $r_0 = 0$, the price responds as much to aggregate demand as the efficient price. The parameter $r_1 \in [0,1]$ measures the rigidity of the price with respect to the marginal utility of private services: if $r_1 = 1$, the price does not respond to shocks to the marginal utility; if $r_1 = 0$, the price responds as much to the marginal utility as the efficient price.

Calibrating $r_0 > 0$ allows aggregate demand shocks to generate inefficient fluctuations in unemployment. Calibrating $r_1 > 0$ allows for a positive unemployment multiplier. Indeed, Appendix D shows that the multiplier evaluated with efficient unemployment and government purchases at the Samuelson level is

$$M = \frac{r_1}{r_1 \cdot \gamma + \varepsilon \cdot (1-\gamma)} \tag{38}$$

When $r_1 > 0$, the multiplier is positive, and the value of the multiplier critically depends on r_1. The value of the multiplier also depends on the parameters ε and γ from the utility function, because these parameters influence the shape of the aggregate demand curve.

The calibration of the model is standard and relegated to Appendix E. The most important parameters are those determining the sufficient statistics in our formulas. We set these parameters such that $\varepsilon = 1$, $u^* = 6.1\%$ (average unemployment rate in the United States for 1990–2014), $(G/C)^* = 19.7\%$ (average G/C in the United States for 1990–2014), $\eta = 0.6$ (Landais, Michaillat and Saez 2015), and $M = 0.5$ (midrange of the multiplier estimates for the United States).

We represent the business cycle as a succession of unexpected permanent aggregate demand shocks. We parameterize aggregate demand with $\alpha = \delta/\mathscr{V}'(l_0)$. Since the economy jumps to its new steady-state equilibrium in response to an unexpected permanent shock, we only need to compute a collection of steady states parameterized by different values of α. We perform two sets of simulations: one in which G/Y remains constant at 16.5%, its average value in the United States for 1990–2014, and one in which G/Y is always at its optimal level, given by (12).

Figure 3 displays the results of the simulations. Each equilibrium is indexed by $\alpha \in [0.97, 1.03]$. The equilibria with low α represent slumps: the price of services is high and unemployment is high. The equilibria with high α represent booms: the price of services is low and unemployment is low. When G/Y remains constant at 16.5%, the unemployment rate rises from 4.4% when $\alpha = 1.3$, to its efficient level of 6.1% when $\alpha = 1$, and to 11.0% when $\alpha = 0.97$.

The unemployment multiplier is sharply countercyclical, increasing from 0.2 to 1.4 when the unemployment rate increases from 4.4% to 11.0%. This sharp increase of the multiplier is consistent with recent evidence based on aggregate US data provided that multipliers are higher when unemployment is higher or output is lower.[26] The mechanism behind the countercyclicality of the multiplier is described by Michaillat (2014). When unemployment is high, there is a lot of idle

[26] See for instance Auerbach and Gorodnichenko (2012), Candelon and Lieb (2013), and Fazzari, Morley and Panovska (2015). On the other hand, Owyang, Ramey and Zubairy (2013) find that multipliers in the United States are not necessarily larger when the economy is slacker.

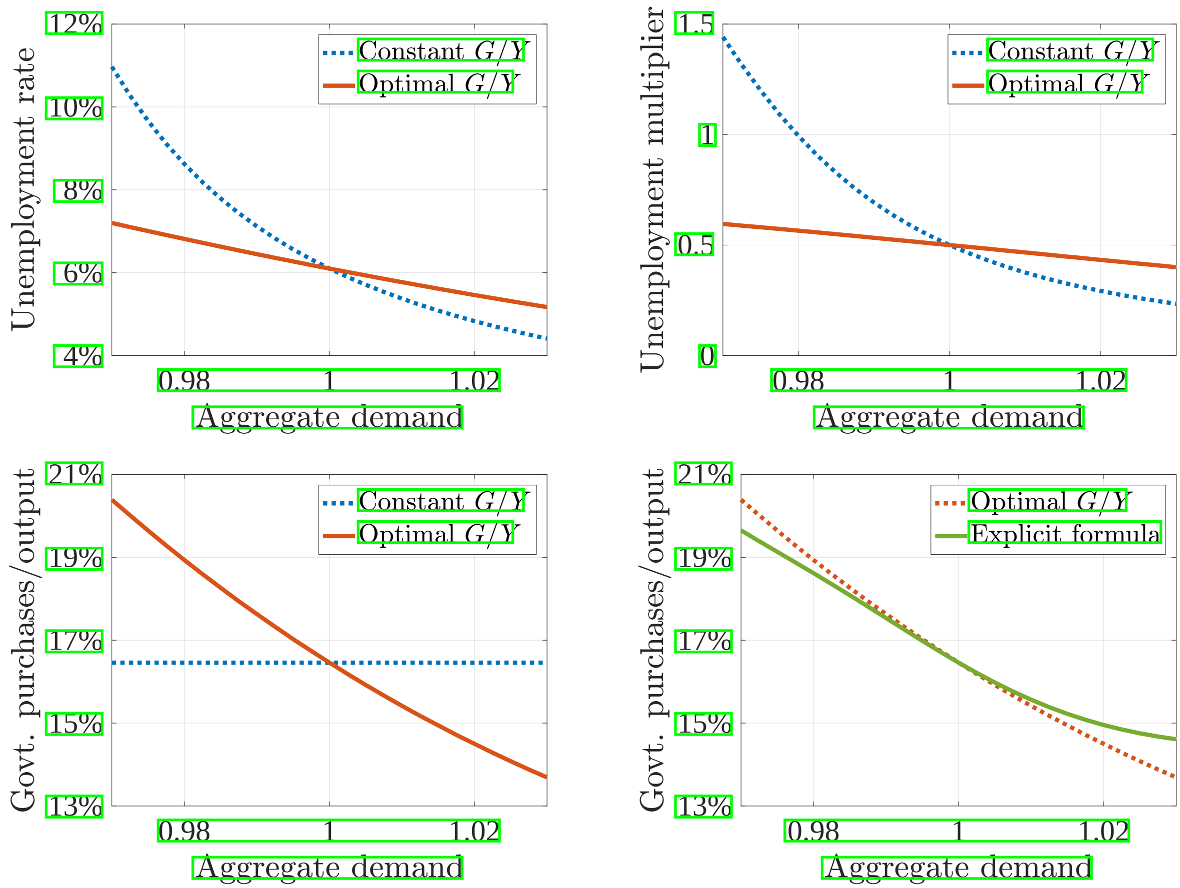


Figure 3: Simulations of Optimal Government Purchases Over the Business Cycle

Notes: The simulation model is a general-equilibrium matching model with land, described in Section 7. The model is calibrated in Appendix E. The business cycle is generated by varying the level of aggregate demand. A low aggregate demand level means a high marginal utility of holding land or a low time discount factor. The simulations compare an economy where government purchases are constant at $G/Y = 16.5\%$ to an economy where government purchases are set optimally using formula (12). In the top right graph, the unemployment multiplier is the empirical unemployment multiplier defined by (32). In the bottom right graph, the solid green line displays the government purchases given by (20) and the dashed red line those given by (12).

capacity so the matching process is congested by sellers of services. Hence, an increase in government purchases has very little effect on other buyers of services. Crowding out of household purchases by government purchases is therefore weak, and the multiplier is large.

The model is calibrated so that the unemployment rate is efficient when $\alpha = 1$. Hence, the unemployment rate is inefficiently high when $\alpha < 1$ and inefficiently low when $\alpha > 1$. Since the unemployment multiplier is positive, the government purchases ratio should be above $(G/Y)^*$ when $\alpha < 1$ and below when $\alpha > 1$. Indeed, optimal government purchases is markedly countercyclical, decreasing from $G/Y = 20.4\%$ to $G/Y = 13.7\%$ when α increases from 0.97 to 1.03.

Unemployment responds to the adjustment of government purchases to their optimal level. When $\alpha < 1$, optimal government purchases are higher than $G/Y = 16.5\%$ so unemployment is below its original level. For instance, at $\alpha = 0.97$ the unemployment rate falls by 3.8 percentage points from 11.0% to 7.2%. When $\alpha > 1$, optimal government purchases are below $G/Y = 16.5\%$ so unemployment is above its original level. For instance, at $\alpha = 1.03$ the unemployment rate increases by 0.8 percentage point from 4.4% to 5.2%.

Finally, the simulations show that the explicit formula, given by (20), is quite accurate. Formula (20) is valid up to a second-order remainder. Since the unemployment rate displays large fluctuations, there is a risk that the remainder is large and the approximation given by the formula inaccurate. In our simulations, however, this does not happen. The government purchases obtained with the approximate explicit formula and the exact formula are close. At $\alpha = 1$, the two formulas gives the same G/Y. When α is further away from 1, the approximation remains satisfactory. The deviation between exact and explicit formula is always below one percentage point: at $\alpha = 1.03$, the exact formula gives $G/Y = 13.7\%$ while the explicit formula gives $G/Y = 14.6\%$; at $\alpha = 0.97$, the exact formula gives $G/Y = 20.4\%$ while the explicit formula gives $G/Y = 19.7\%$.

8. Conclusion

This paper presents a simplified theory of optimal government purchases. The theory clarifies a number of existing ideas about the use of government spending for stabilization.

A first idea is that stimulus spending in slumps is desirable if the output multiplier is above one (as in traditional Keynesian models). Indeed, with an output multiplier above one, public consumption crowds in private consumption, so additional public consumption improves welfare. While crowding in is obviously a sufficient condition to justify stimulus spending, it is absolutely not a necessary condition. In our theory, public consumption always crowds out private consumption; nevertheless, stimulus spending is desirable in slumps whenever the unemployment multiplier is positive—so that an increase in government spending reduces the unemployment rate. This result holds both whether taxes are distortionary or not. Since available estimates of the unemployment multiplier are positive, stimulus spending is likely to be desirable in slumps.

A concern of stimulus skeptics is that multipliers are estimated using deficit-financed increases in government purchases, not balanced-budget increases, and that balanced-budget and deficit-financed multipliers may have opposite sign. For instance, Barro and Redlick (2011) find in US data that the deficit-financed output multiplier is positive but the balanced-budget output multiplier is negative. The mechanism is that higher government purchases require higher distortionary taxes, which depress output through negative labor-supply responses. Our theory alleviates this concern, however. It is not the output multiplier but the unemployment multiplier that should be used to design optimal stimulus spending. While the output multiplier is smaller and can become negative once tax distortions are taken into account, the unemployment multiplier is larger once tax distortions are taken into account. This is because a reduction in labor supply leads to lower output but higher tightness and lower unemployment in the matching model (Michaillat and Saez 2015). In fact, the unemployment multiplier equals the output multiplier net of the labor-supply responses due to distortionary taxation. With strong negative labor-supply responses, a negative output multiplier can coexist with a positive unemployment multiplier.

Our theory also links the optimal level of stimulus spending to the size of the unemployment multiplier. Stimulus skeptics usually believe in small multipliers and infer that stimulus spending should be small or zero in slumps. On the other hand, stimulus advocates usually believe in large multipliers and infer that stimulus spending should be large in slumps. In fact, our theory shows that the relationship between the optimal level of stimulus spending and the size of the unemployment multiplier is hump-shaped. In a US calibration, the relationship peaks for an unemployment multiplier of 0.4. So a multiplier of 0.4 calls for the highest level of stimulus spending. Higher and lower multipliers warrant smaller levels, such that the optimal level of stimulus spending is similar for small and large multipliers. For instance, in our US calibration, the optimal level of stimulus spending is the same for unemployment multipliers of 0.1 and 1.8.

Next, our theory links the optimal level of stimulus spending to the usefulness of additional public spending. A concern of stimulus skeptics is that additional government spending could be wasteful. Our theory develops this argument. It is true that when the elasticity of substitution between public and private consumption is zero, so that additional public workers dig and fill holes in the ground, government purchases should remain at the Samuelson level and not be used for stabilization. But in the more realistic case where the elasticity of substitution is positive, some stimulus spending is desirable in slumps. We also qualify the view of stimulus advocates in the

Keynesian tradition who argue that government purchases should entirely fill the unemployment gap, irrespective of the usefulness of additional public spending. It is true that when the elasticity of substitution is infinite, so that public and private consumption are perfect substitute, government purchases should completely fill the unemployment gap. But in the more realistic case where the elasticity is finite, government purchases fill the unemployment gap partially, not completely.

Overall, government purchases could contribute to stabilization whenever monetary policy is constrained. This situation arises when the zero lower bound on the nominal interest rate is binding—for instance, in the United States, Japan, and the European Union in the aftermath of the Great Recession. This situation also applies to members of a monetary union, which have no control over monetary policy but can tailor government purchases and taxes to local conditions—for instance, a country in the eurozone or a state in the United States. Furthermore, since it focuses on budget-balanced government purchases, our theory is particularly appropriate for US states, which cannot run budget deficits, and eurozone countries, which face strict limits on their budget deficits.

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Appendix A. The Absence of Transitional Dynamics on the Labor Market

In the analysis, we abstract from transitional dynamics for the unemployment rate and output. In this appendix we use labor market data for the US to validate this approximation. We show that the transitional dynamics of the unemployment rate are negligible.

We begin by constructing a time series for the selling rate f_t. We measure one unit of service by one job, so that the selling rate is a job-finding rate. We assume that unemployed workers find a job according to a Poisson process with arrival rate f_t. Thus, the monthly job-finding rate satisfies $f_t = -\ln(1 - F_t)$, where F_t is the monthly job-finding probability. We construct a time series for F_t following the method developed by Shimer (2012). We use the relationship

$$F_t = 1 - \frac{u_{t+1} - u^s_{t+1}}{u_t},$$

where u_t is the number of unemployed persons at time t and u^s_t is the number of short-term unemployed persons at time t. We measure u_t and u^s_t in the data constructed by the BLS from the CPS. The number of short-term unemployed persons is the number of unemployed persons with zero to four weeks duration, adjusted as in Shimer (2012) during the 1994–2014 period. Figure A1, Panel A, displays the monthly job-finding rate for 1951–2014. The job-finding rate averages 56% and is procyclical.

Next, we construct the separation rate following the method developed by Shimer (2012). The separation rate s_t is implicitly defined by

$$u_{t+1} = \left(1 - e^{-f_t - s_t}\right) \cdot \frac{s_t}{f_t + s_t} \cdot h_t + e^{-f_t - s_t} \cdot u_t,$$

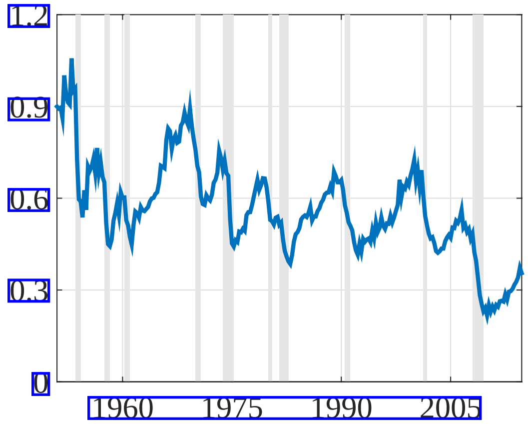
where h_t is the number of persons in the labor force at time t, u_t is the number of unemployed persons at time t, and f_t is the monthly selling rate. We measure u_t and h_t in the data constructed by the BLS from the CPS, and we use the series that we have just constructed for f_t. Figure A1, Panel B, displays the monthly separation rate for 1951–2014. The separation rate averages 3.3% and is broadly acyclical.[27]

Finally, we compare the actual unemployment rate to the steady-state unemployment rate

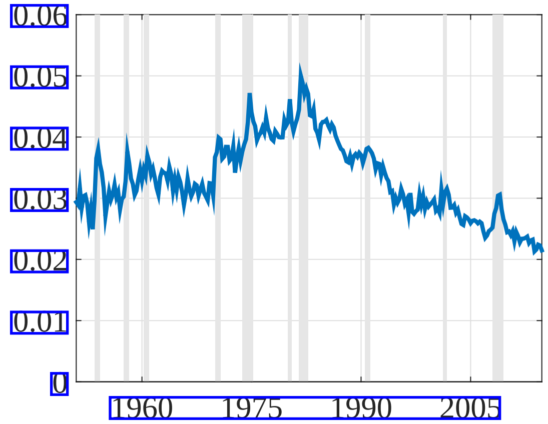
$$u_t = \frac{s_t}{f_t + s_t}. \tag{A1}$$

The two series, displayed in Panel C of Figure A1, are almost identical. This finding implies that the transitional dynamics of the actual unemployment rate are unimportant.

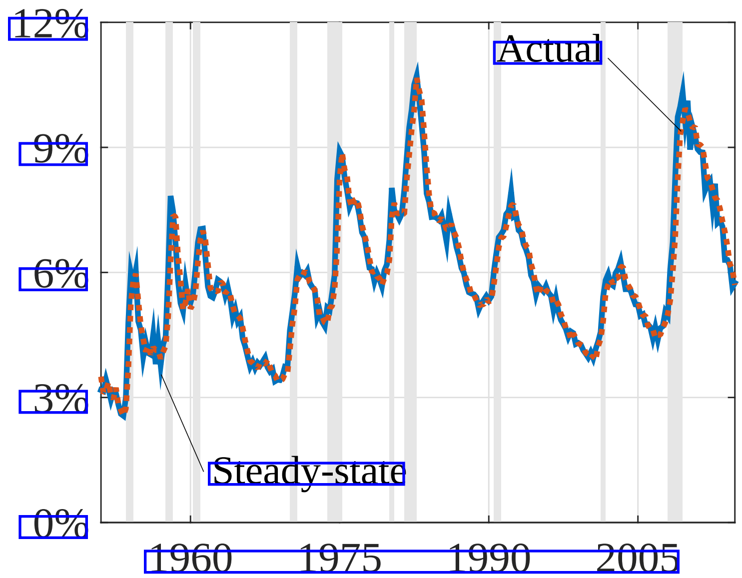
[27] One concern is that increases in government purchases cannot be undone because jobs created by the government are effectively permanent. It is true that government jobs separate more slowly than private jobs. Using data constructed by the BLS from the Job Opening and Labor Turnover Survey for 2000–2014, we find that the average monthly separation rate is 3.9% for private jobs and 1.4% for government jobs. Nevertheless, the separation rate for government jobs remains sizable. If no new jobs were created by the government, the level of government purchases would rapidly decrease. Indeed, with a hiring freeze, US government employment would fall by $1 - \exp(-0.014 \cdot 12) = 15\%$ in one year.



A. Monthly job-finding rate in the United States



B. Monthly separation rate in the United States



C. Actual and steady-state unemployment rates in the United States

Figure A1: The Absence of Transitional Dynamics For the US Unemployment Rate, 1951–2014

Notes: Panel A: The monthly job-finding rate f is constructed from CPS data following the methodology of Shimer (2012). Panel B: The monthly separation rate s is constructed from CPS data following the methodology of Shimer (2012). The rates f and s are expressed as quarterly averages of monthly series. Panel C: The actual unemployment rate is the quarterly average of the seasonally adjusted monthly unemployment rate constructed by the BLS from the CPS. The steady-state unemployment rate is computed using (A1) and the rates f and s displayed in Panels A and B. Panel C is similar to Figure 1 in Hall (2005). Although we use different measures of the job-finding and separation rates and a longer time period, Hall's conclusion remains valid: transitional dynamics are irrelevant on the labor market. The shaded areas represent the recessions identified by the National Bureau of Economic Research.

Appendix B. Demand Structure and Price Mechanism: Other Examples

In this appendix we present three other examples of demand structure and price mechanism for the matching model of Section 2. These examples yield the same equilibrium structure as the example with land described in Section 2.

First, the variable $l(t)$ in the land model can be interpreted as a generic unproduced good, as in Hart (1982). It could also be interpreted as a specific unproduced good, such as gold.

Second, the land model could be slightly modified by replacing land by money and assuming that households derive utility from holding real money balances. Introducing money in the utility function is a classical way to generate an aggregate demand. Following Sidrauski (1967), a large number of business-cycle models with money in the utility function have been developed. Two prominent examples are Barro and Grossman (1971) and Blanchard and Kiyotaki (1987). Money is introduced in the utility function to capture the fact that money helps conducting transactions.

In the money model, a household holds $M(t)$ units of money and the supply of money is fixed at M_0. In equilibrium, the money market clears and $M(t) = M_0$. The price of services in terms of money is $p(t)$. We specify a general price mechanism that determines the price of services: $p(t) = p(g(t))$. The household's instantaneous utility function is $\mathscr{U}(c(t), g(t)) + \mathscr{V}(M(t)/p(t))$. The law of motion of the household's real money balances $m(t) \equiv M(t)/p(t)$ is $\dot{m}(t) = (1-u(x(t)))\cdot k - (1+\tau(x(t)))\cdot c(t) - \pi(t)\cdot m(t) - T(t)$, where $\pi(t) \equiv \dot{p}(t)/p(t)$ is the inflation rate. In steady state, g and thus p are fixed so inflation is zero. The equilibrium immediately converges to steady state. In steady state the desired private consumption $c(x, p, g)$ is given by $\partial \mathscr{U}/\partial c = (1+\tau(x))\cdot \mathscr{V}'(M_0/p)/\delta$. Equilibrium tightness $x(g)$ is implicitly defined by $c(x, g, p(g)) + g = y(x, k)$.

Third, the land model could be slightly modified by replacing land by nominal bonds and assuming that households derive utility from holding real bonds. Introducing bonds in the utility function is a simple way to generate an aggregate demand in a dynamic economy. It is especially adapted to modern, cashless macroeconomic models. The assumption that wealth enters the utility function has been used in growth models (Kurz 1968; Zou 1994), microeconomic models (Robson 1992; Cole, Mailath and Postlewaite 1995), life-cycle models (Carroll 2000; Francis 2009), asset-pricing models (Bakshi and Chen 1996; Gong and Zou 2002), business-cycle models (Michaillat and Saez 2014; Ono and Yamada 2014), and public-economics models (Saez and Stantcheva 2016). Wealth is introduced in the utility function to capture the fact that people care about wealth not only as future consumption but for its own sake. Wealth could be valued for several reasons: high wealth provides high social status; high wealth provides political power; people value frugality and asceticism and thus dignify the accumulation of wealth; or people value bequeathing wealth.

In the bond model, bonds are issued and purchased by households, and they have a price of 1 in terms of money. Money only plays the role of a unit of account. A household holds $B(t)$ bonds and bonds are in zero net supply. In equilibrium, the bond market clears and $B(t) = 0$. The rate of return on bonds is the nominal interest rate $i(t)$. The nominal interest rate is determined by the central bank, which sets an interest rate $i(t) = i(g(t))$. We allow the interest rate to depend on public consumption to allow for an interaction between monetary and fiscal policy.

The price of services in terms of money is $p(t)$. The inflation rate is $\pi(t) \equiv \dot{p}(t)/p(t)$. In the economy there are two goods—services and bonds—and hence one relative price (public and private services have the same price). The price of bonds relative to services is determined by the real interest rate, $i(t) - \pi(t)$. Since the nominal interest rate is determined by the central bank, it is the inflation rate that determines the real interest rate. The inflation rate is determined by a general price mechanism: $\pi(t) = \pi(g(t))$. Given the inflation rate, the price of services moves according to $\dot{p}(t) = \pi(t)\cdot p(t)$. The initial price $p(0)$ is given. Given the inflation rate and nominal interest rate, tightness adjusts such that supply equals demand on the market for services.

The household's instantaneous utility function is $\mathscr{U}(c(t), g(t)) + \mathscr{V}(B(t)/p(t))$. The law of motion of the household's real wealth $b(t) \equiv B(t)/p(t)$ is $\dot{b}(t) = (1-u(x(t)))\cdot k - (1+\tau(x(t)))\cdot c(t) + (i(t)-\pi(t))\cdot b(t) - T(t)$. As earlier, the equilibrium immediately converges to steady state. In steady state, the desired amount of private consumption $c(x,i,\pi,g)$ is given by $\partial \mathscr{U}/\partial c = (1+\tau(x))\cdot \mathscr{V}'(0)/(\delta - i + \pi)$. This equation is the usual consumption Euler equation modified by the utility of wealth and evaluated in steady state. The demand for saving arises in part from intertemporal consumption-smoothing considerations and in part from the utility provided by wealth. The equation implies that at the margin, the household is indifferent between consuming and holding real wealth. Equilibrium tightness $x(g)$ is implicitly defined by $c(x,i(g),\pi(g),g) + g = y(x,k)$.

Appendix C. Properties of the Utility Function in the Simulation Model

In this appendix we compute the derivatives of the utility function from the simulation model of Section 7. We use the results to compute the unemployment multiplier in Appendix D and calibrate the model in Appendix E.

The simulation model has a constant-elasticity-of-substitution utility function:

$$\mathscr{U}(c,g) \equiv \left[(1-\gamma)^{\frac{1}{\varepsilon}}\cdot c^{\frac{\varepsilon-1}{\varepsilon}} + \gamma^{\frac{1}{\varepsilon}}\cdot g^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

The utility function admits the following derivatives:

$$\frac{\partial \ln(\mathscr{U})}{\partial \ln(c)} = (1-\gamma)^{\frac{1}{\varepsilon}}\cdot\left(\frac{c}{\mathscr{U}}\right)^{\frac{\varepsilon-1}{\varepsilon}}, \qquad \mathscr{U}_c \equiv \frac{\partial \mathscr{U}}{\partial c} = \left(\frac{(1-\gamma)\cdot\mathscr{U}}{c}\right)^{\frac{1}{\varepsilon}}$$

$$\frac{\partial \ln(\mathscr{U})}{\partial \ln(g)} = \gamma^{\frac{1}{\varepsilon}}\cdot\left(\frac{g}{\mathscr{U}}\right)^{\frac{\varepsilon-1}{\varepsilon}}, \qquad \mathscr{U}_g \equiv \frac{\partial \mathscr{U}}{\partial g} = \left(\frac{\gamma\cdot\mathscr{U}}{g}\right)^{\frac{1}{\varepsilon}}$$

$$\frac{\partial \ln(\mathscr{U}_c)}{\partial \ln(c)} = \frac{1}{\varepsilon}\cdot\left(\frac{\partial \ln(\mathscr{U})}{\partial \ln(c)} - 1\right)$$

$$\frac{\partial \ln(\mathscr{U}_c)}{\partial \ln(g)} = \frac{1}{\varepsilon}\cdot\frac{\partial \ln(\mathscr{U})}{\partial \ln(g)}.$$

When the Samuelson condition holds, we have $MRS_{gc} = \mathscr{U}_g/\mathscr{U}_c = 1$ so

$$(g/c)^* = \frac{\gamma}{1-\gamma}, \qquad (g/y)^* = \gamma, \qquad (c/y)^* = 1-\gamma,$$

and the derivatives of the utility function simplify to

$$\frac{\partial \ln(\mathscr{U})}{\partial \ln(c)} = 1-\gamma, \qquad \frac{\partial \ln(\mathscr{U})}{\partial \ln(g)} = \gamma$$

$$\mathscr{U}_c = \mathscr{U}_g = 1$$

$$\frac{\partial \ln(\mathscr{U}_c)}{\partial \ln(c)} = -\frac{\gamma}{\varepsilon}, \qquad \frac{\partial \ln(\mathscr{U}_c)}{\partial \ln(g)} = \frac{\gamma}{\varepsilon}.$$

Appendix D. The Unemployment Multiplier in the Simulation Model

In this appendix we compute the unemployment multiplier in the simulation model of Section 7. We use the results to calibrate the model in Appendix E and simulate it in Section 7.

To compute the unemployment multiplier, we first express dx/dg as a function of the derivatives of the utility function that are computed in Appendix C. Then, we use results from the text to compute the theoretical unemployment multiplier m from dx/dg and to compute the empirical unemployment multiplier M from m.

The price schedule is $p(g) = p_0 \cdot \alpha^{1-r_0} \mathscr{U}_c(y^* - g, g)^{1-r_1}$. The elasticity of the price schedule and its value at $g^* \equiv \gamma \cdot y^*$ are

$$\frac{d\ln(p)}{d\ln(g)} = (1-r_1) \cdot \left[\frac{\partial \ln(\mathscr{U}_c)}{\partial \ln(g)} - \frac{g}{y^* - g} \cdot \frac{\partial \ln(\mathscr{U}_c)}{\partial \ln(c)}\right]$$

$$\frac{d\ln(p)}{d\ln(g)} = (1-r_1) \cdot \frac{1}{\varepsilon} \cdot \frac{\gamma}{1-\gamma}.$$

The demand for private services, $c(x,g)$, is defined by $\mathscr{U}_c(c,g) = p(g) \cdot (1+\tau(x)) \cdot \mathscr{V}'(l_0)/\delta$. The elasticities of the demand are

$$\frac{\partial \ln(c)}{\partial \ln(x)} = \frac{\eta \cdot \tau(x)}{\partial \ln(\mathscr{U}_c)/\partial \ln(c)}$$

$$\frac{\partial \ln(c)}{\partial \ln(g)} = -\frac{\partial \ln(\mathscr{U}_c)/\partial \ln(g) - \partial \ln(p)/\partial \ln(g)}{\partial \ln(\mathscr{U}_c)/\partial \ln(c)}.$$

We calibrate the model such that $c(x^*, g^*) = c^* \equiv (1-\gamma) \cdot y^*$. (This means that we calibrate the price level such that unemployment is efficient when government purchases are set at the Samuelson level, or equivalently $x(g^*) = x^*$.) The values of the elasticities at x^* and g^* therefore are

$$\frac{\partial \ln(c)}{\partial \ln(x)} = -\eta \cdot \tau(x^*) \cdot \frac{\varepsilon}{\gamma} \quad \text{and} \quad \frac{\partial \ln(c)}{\partial \ln(g)} = \frac{r_1 - \gamma}{1-\gamma}.$$

The equilibrium condition determining tightness $x(g)$ is $y(x,k) = g + c(x,g)$. In the simulations, k is fixed. Differentiating this equation with respect to g, we obtain the elasticity of $x(g)$ with respect to g:

$$\frac{\partial \ln(y)}{\partial \ln(x)} \cdot \frac{d\ln(x)}{d\ln(g)} = \frac{g}{y} + \frac{c}{y} \cdot \left[\frac{\partial \ln(c)}{\partial \ln(g)} + \frac{\partial \ln(c)}{\partial \ln(x)} \cdot \frac{d\ln(x)}{d\ln(g)}\right]$$

$$\frac{d\ln(x)}{d\ln(g)} = \frac{(g/y) + (c/y) \cdot (\partial \ln(c)/\partial \ln(g))}{\partial \ln(y)/\partial \ln(x) - (c/y) \cdot (\partial \ln(c)/\partial \ln(x))}.$$

As discussed above, we calibrate the model such that $x(g^*) = x^*$ and $c(x^*, g^*) = c^*$. The value of the elasticity at g^* therefore is

$$\frac{d\ln(x)}{d\ln(g)} = \frac{1}{\eta \cdot \tau(x^*)} \cdot \frac{r_1}{\varepsilon} \cdot \frac{\gamma}{1-\gamma}.$$

Table A1: Values of Calibrated Parameters in the Simulation Model

Value	Description	Data source
$\varepsilon = 1$	Elasticity of substitution between g and c	–
$\gamma = 0.16$	Parameter of utility function	Matches $(G/C)^* = 19.7\%$
$r_1 = 0.46$	Price rigidity	Matches $M = 0.5$ at $\alpha = 1$
$\eta = 0.6$	Matching elasticity	Landais, Michaillat and Saez (2015)
$s = 2.8\%$	Monthly separation rate	Landais, Michaillat and Saez (2015)
$\omega = 0.60$	Matching efficacy	Landais, Michaillat and Saez (2015)
$\rho = 1.4$	Matching cost	Matches $u^* = 6.1\%$
$p_0 = 0.78$	Price level	Matches $u = u^*$ at $\alpha = 1$

From the expression for $d\ln(x)/d\ln(g)$, we obtain m and M using (18) and (33):

$$m = (1-\eta)\cdot u \cdot \frac{Y}{G} \cdot \frac{d\ln(x)}{d\ln(g)} \quad \text{and} \quad M = \frac{m}{1 + \frac{g}{y}\cdot\frac{\eta}{1-\eta}\cdot\frac{\tau}{u}\cdot m}.$$

Since $x(g^*) = x^*$ in the calibration, the values of m and M at g^* are

$$m = \frac{r_1}{\varepsilon\cdot(1-\gamma)} \quad \text{and} \quad M = \frac{r_1}{r_1\cdot\gamma + \varepsilon\cdot(1-\gamma)}.$$

Appendix E. Calibration of the Simulation Model

In this appendix we calibrate the simulation model of Section 7 based on empirical evidence for the United States. The values of the calibrated parameters are summarized in Table A1.

We begin by calibrating the utility function. We arbitrarily set the elasticity of substitution between public and private consumption to $\varepsilon = 1$. As showed in Appendix C, the parameter γ determines the Samuelson level of government purchases: $(G/C)^* = \gamma/(1-\gamma)$. We assume that the Samuelson level of government purchases is the average level of government purchases in the United States for 1990–2014: $(G/C)^* = 19.7\%$ (see Section 6). We therefore set $\gamma = 0.16$.

Then we calibrate the matching parameters. We use the estimates of s and η obtained by Landais, Michaillat and Saez (2015) using US labor market data for 1990–2014: $s = 2.8\%$ and $\eta = 0.6$. To calibrate the matching efficacy, we exploit the relationship $u\cdot f(x) = s\cdot(1-u)$, which implies $\omega = s\cdot x^{\eta-1}\cdot(1-u)/u$. The average unemployment rate and tightness in US labor market data for 1990–2014 are $u = 6.1\%$ and $x = 0.43$ (Landais, Michaillat and Saez 2015). Using these average values, we obtain $\omega = 0.60$. To calibrate the matching cost, we exploit the relationship $\tau = \rho\cdot s/[\omega\cdot x^{-\eta} - \rho\cdot s]$, which implies $\rho = \omega\cdot x^{-\eta}\cdot\tau/[s\cdot(1+\tau)]$. This relationship holds for any τ and x, in particular when tightness is efficient so $x = x^*$ and $\tau = \tau^*$. We assume that the efficient unemployment rate and tightness are the average US unemployment rate and tightness for 1990–2014: $u^* = 6.1\%$ and $x^* = 0.43$. When tightness is efficient, $\tau^* = u^*\cdot(1-\eta)/\eta$ so $\tau^* = 4.1\%$. Plugging $x^* = 0.43$ and $\tau^* = 4.1\%$ in the expression for ρ, we obtain $\rho = 1.4$.

Last, we calibrate the price mechanism. The empirical evidence suggest that on average in the United States the unemployment multiplier is $M = 0.5$ (see Section 6). Relying on (38), we set $r_1 = 0.46$ to match $M = 0.5$. We calibrate the price level such that when the aggregate demand parameter, $\alpha \equiv \delta / \mathscr{V}'(l_0)$, equals 1 and $G/C = (G/C)^* = 19.7\%$, then $u = u^* = 6.1\%$. Using (37), we find that $p_0 = 0.78$.

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