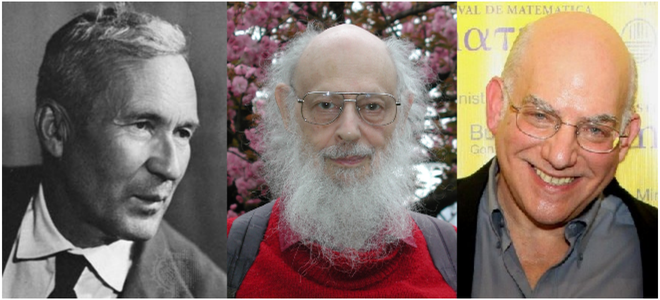
Kolmogorov complexity for beginners

Marlou, Jinghui

Proof and Computation 2019
Wednesday 25th September, 2019

Kolmogorov complexity

With *algorithmic complexity* we talk about the complexity of strings/outputs/observations by considering the length of the program/procedure/hypothesis that gives it.
The field was independently invented in the 1960s by:
A.N. Kolmogorov, R. Solomonoff, G.J. Chaitin.



The information content of a string

We want to define the quantity of information in a string as the number of bits to losslessly describe it [Ming and Vitányi, 2008]. What we then need is:

- A universal description method $\Rightarrow$ binary strings
- A mechanism to produce the object from its description $\Rightarrow$ universal Turing machine

Definition
The complexity C_φ of x is $C_\varphi(x) = \min\{|p| : \varphi(p) = x\}$

Which universal machine φ we choose doesn't matter if we ignore constants (as long as the machine is *additively optimal*).
So from now one we will write $C(x)$.

Properties of algorithmic complexity

There is always a program that outputs x.
$\forall x \ C(x) \leq |x| + O(1)$
And we often ignore the constant.

C(x) is not computable.
"The first string that can be proven to be of algorithmic complexity greater than n." [Chaitin, 1974]

Properties of algorithmic complexity

- $\forall n \exists x$ s.t. $C(x) \geq n$.
- More strings have a "high" complexity.
- NOT $C(x, y) \leq C(x) + C(y)$.
  Instead: $C(x, y) \leq C(x) + C(y) + 2\log(\min(C(x), C(y)))$

[Ming and Vitányi, 2008]

Incompressibility

Definition
x is *c-incompressible* when $C(x) > |x| - c$.

For each n there is an incompressible (1-incompressible) string of that length.
The *incompressibility method* is used often in algorithmic complexity.

Randomness

We say random = incompressible = patternless.

References

Chaitin, G. (1974).
Information-theoretic computation complexity.
IEEE Transactions on Information Theory, 20(1):10–15.

Ming, L. and Vitányi, P. (2008).
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