Concepts of continuity

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21 September 2019

Brouwer's argument for the uniform continuity

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1. **Spread representation of real numbers**

2. **Continuity on formal reals**

3. **The role of fan theorem in Brouwer’s argument**

Real numbers

- A **fundamental sequence (with modulus)** is a sequence $\langle r_n \rangle_{n\in\mathbb{N}}$ of rationals together with a function $\delta\colon \mathbb{N} \to \mathbb{N}$, called a **modulus** of $\langle r_n \rangle_{n\in\mathbb{N}}$, such that

$$\forall k, n, m \in \mathbb{N} \left(|r_{\delta(k)+n} - r_{\delta(k)+m}| \leq 2^{-k} \right).$$

- The equality on fundamental sequences is defined by

$$\langle r_n \rangle_{n\in\mathbb{N}} \simeq \langle q_n \rangle_{n\in\mathbb{N}} \overset{\text{def}}{\iff} \forall k \exists n \forall m \left(|r_{n+m} - q_{n+m}| \leq 2^{-k} \right).$$

- A **real number** is a fundamental sequence of rational numbers (with some modulus).
- Rational numbers are embedded into real numbers by $r \mapsto \langle r \rangle_{n\in\mathbb{N}}$.
- Real numbers are ordered by

$$\langle r_n \rangle_{n\in\mathbb{N}} < \langle q_n \rangle_{n\in\mathbb{N}} \overset{\text{def}}{\iff} \exists k, n \in \mathbb{N} \forall m \in \mathbb{N} \left(q_{n+m} - r_{n+m} > 2^{-k} \right),$$

$$\langle r_n \rangle_{n\in\mathbb{N}} \leq \langle q_n \rangle_{n\in\mathbb{N}} \overset{\text{def}}{\iff} \neg \left(\langle q_n \rangle_{n\in\mathbb{N}} < \langle r_n \rangle_{n\in\mathbb{N}} \right).$$

Real numbers

- A sequence $\langle r_n \rangle_{n\in\mathbb{N}}$ of rationals is **regular** if
$$\forall n \in \mathbb{N} \left(|r_n - r_{n+1}| \leq 2^{-(n+1)} \right).$$
- The equality and orders on regular sequences are defined by
$$\langle r_n \rangle_{n\in\mathbb{N}} \simeq \langle q_n \rangle_{n\in\mathbb{N}} \overset{\text{def}}{\iff} \forall n \in \mathbb{N} \left(|r_{n+1} - q_{n+1}| \leq 2^{-n} \right),$$
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Proposition

There is an order preserving bijection between the set of fundamental sequences with moduli and the set of regular sequences:

1. *If $\langle r_n \rangle_{n\in\mathbb{N}}$ is a fundamental sequence with modulus δ, then $\langle r_{\delta(n+1)} \rangle_{n\in\mathbb{N}}$ is a regular sequence.*
2. *If $\langle r_n \rangle_{n\in\mathbb{N}}$ is a regular sequence, then it is a fundamental sequence with modulus $k \mapsto k+1$.*

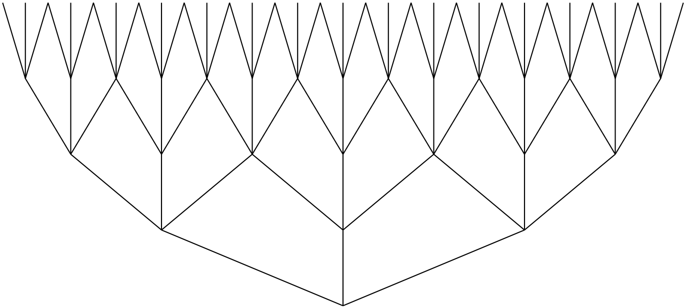
Notations for sequences

Let X be a set.

X^*	the set of finite sequences of X
$X^{\mathbb{N}}$	the set of infinite sequences of X
X^n	the set of finite sequences of length n
	$n, i, j, k \in \mathbb{N}$; $a, b, c \in X^*$; $\alpha, \beta, \gamma \in X^{\mathbb{N}}$
$\lvert a \rvert$	the length of a
$\langle i_0, \ldots, i_{n-1} \rangle$	a finite sequence of length n
$\langle\,\rangle$	the empty sequence
$a * b$	the concatenation of a and b
$a * \alpha$	the concatenation of a followed by α
α_n (or $\alpha(n)$)	the n-th value of α
$\overline{\alpha}n$	the initial segment of α of length n
$\alpha \in a$	"a is an initial segment of α"

Spread representation of $[0,1]$ (Signed-digit representation)

Consider the ternary tree $\{0,1,2\}^*$.



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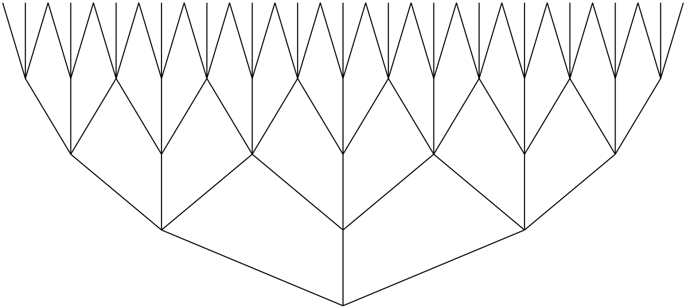

Assign a natural number to each node of $\{0, 1, 2\}^*$ by

$$N(\langle\,\rangle) = 1,$$
$$N(a * \langle i \rangle) = 2N(a) + (i - 1) \qquad\qquad (i \in \{0, 1, 2\}).$$

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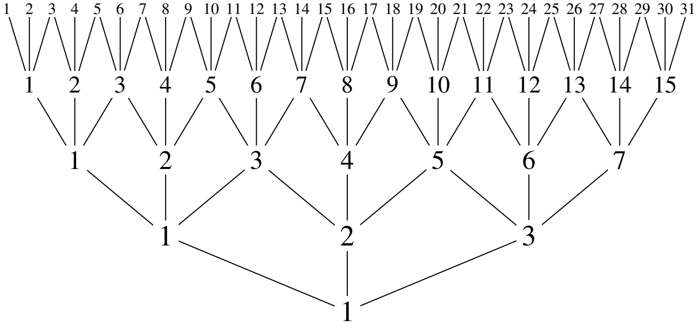

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Spread representation of $[0, 1]$

Each $\alpha \in \{0, 1, 2\}^{\mathbb{N}}$ determines a regular sequence x_α in $[0, 1]$ by

$$x_\alpha \stackrel{\text{def}}{=} \langle 2^{-(n+1)} N(\overline{\alpha} n) \rangle_{n \in \mathbb{N}}.$$

Write x_α^n for the n-th term of x_α, i.e. $x_\alpha^n \stackrel{\text{def}}{=} 2^{-(n+1)} N(\overline{\alpha} n)$.

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Let $\Phi \colon \{0,1,2\}^{\mathbb{N}} \to [0,1]$ denote the mapping $\alpha \mapsto x_\alpha$.

Lemma

1. Φ *is uniformly continuous.*
2. $\forall n \in \mathbb{N} \forall \alpha \in \{0,1,2\}^{\mathbb{N}} \left(|x_\alpha - x_\alpha^n| \leq 2^{-(n+1)} \right)$; *hence*

$$\forall n \in \mathbb{N} \left(V_{\overline{\alpha}n} \subseteq U(x_\alpha, 2^{-n+1}) \right).$$

where

$$V_a \overset{\text{def}}{=} \left\{ x_\alpha \mid \alpha \in \{0,1,2\}^{\mathbb{N}} \ \& \ \alpha \in a \right\},$$

$$U(x,r) \overset{\text{def}}{=} \{ y \in [0,1] \mid |y - x| < r \}.$$

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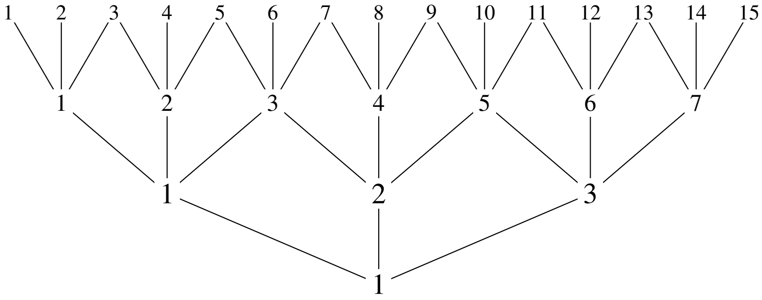

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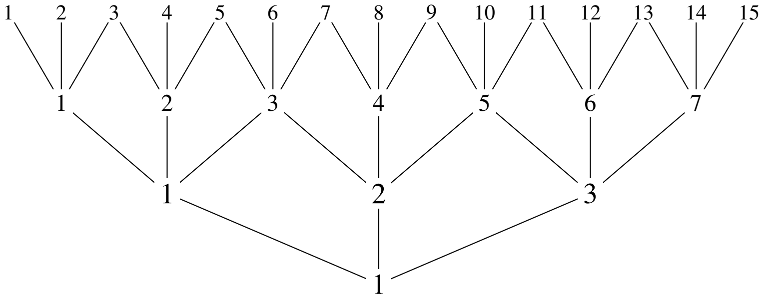

To each $a \in \{0, 1, 2\}^*$, assign a closed interval with rational endpoints

$$\mathbb{I}_a \stackrel{\text{def}}{=} \left[2^{-(|a|+1)}(N(a) - 1),\ 2^{-(|a|+1)}(N(a) + 1)\right].$$

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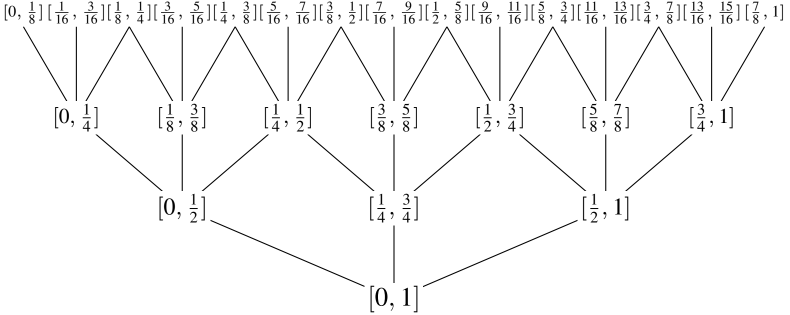

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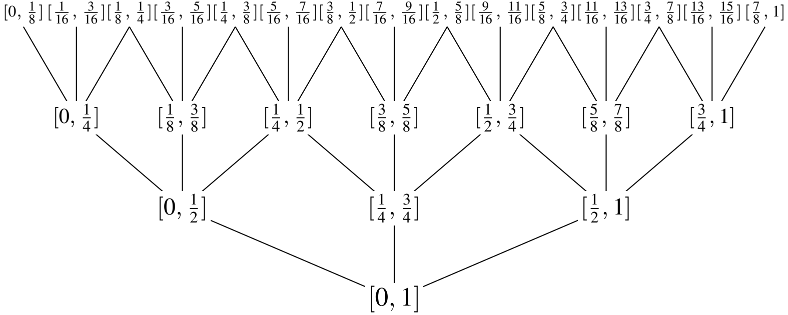

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- The length of $\mathbb{I}_a$ is $2^{-|a|}$.
- The overlapping of $\mathbb{I}_{a*\langle i \rangle}$ and $\mathbb{I}_{a*\langle i+1 \rangle}$ is of length $2^{-(|a|+2)}$.

Spread representation of $[0, 1]$

To each regular sequence $x = \langle r_n \rangle_{n \in \mathbb{N}}$ in $[0, 1]$, associate a sequence $\langle \mathbb{I}^x_n \rangle_{n \in \mathbb{N}}$ of rational intervals by

$$\mathbb{I}^x_n \overset{\text{def}}{=} \left[\max\{r_{n+3} - 2^{-(n+3)}, 0\}, \min\{r_{n+3} + 2^{-(n+3)}, 1\}\right].$$

Define a path $\alpha_x \in \{0, 1, 2\}^{\mathbb{N}}$ by primitive recursion:

$$\alpha_x(0) = i \text{ for the least } i \in \{0, 1, 2\} \text{ such that } \mathbb{I}^x_0 \sqsubseteq \mathbb{I}_{\langle i \rangle},$$

$$\alpha_x(n+1) = i \text{ for the least } i \in \{0, 1, 2\} \text{ such that } \mathbb{I}^x_{n+1} \sqsubseteq \mathbb{I}_{\langle \alpha_x(0), \ldots, \alpha_x(n), i \rangle}.$$

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The mapping $x \mapsto \alpha_x$ does not preserve the equality on $\mathbb{R}$.

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For each regular sequence $x = \langle r_n \rangle_{n \in \mathbb{N}} \in [0, 1]$, *we have*

$$x \simeq \Phi(\alpha_x).$$

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Proposition

For each regular sequence $x = \langle r_n \rangle_{n \in \mathbb{N}} \in [0,1]$, *we have*

$$x \simeq \Phi(\alpha_x).$$

Proof.

By induction, show

$$\forall n \in \mathbb{N} \left(|r_{n+1} - 2^{-(n+2)} N(\overline{\alpha_x}(n+1))| \leq 2^{-(n+1)} \right). \qquad \square$$

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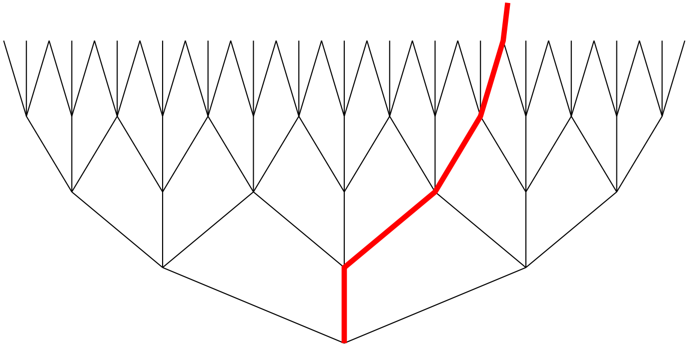

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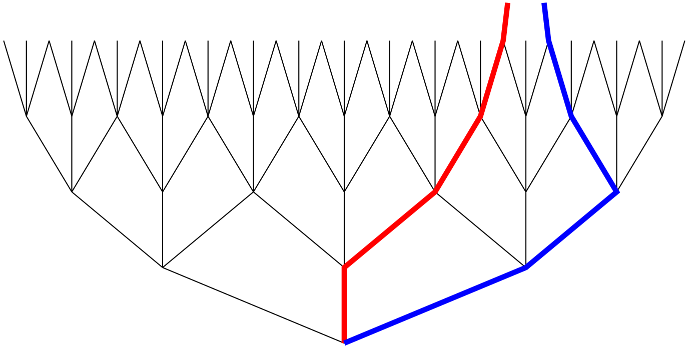

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Problem

We cannot go to **blue** *path from any initial segment of* **red** *path.*

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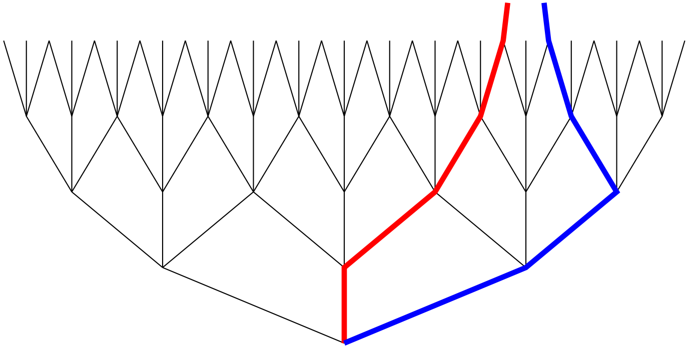

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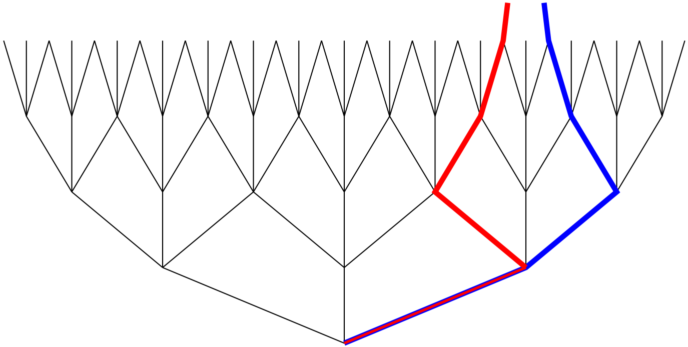

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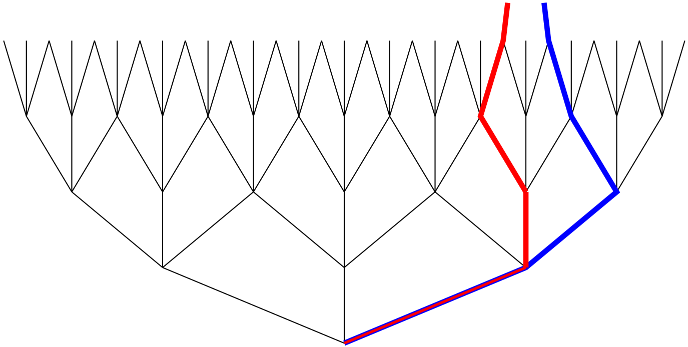

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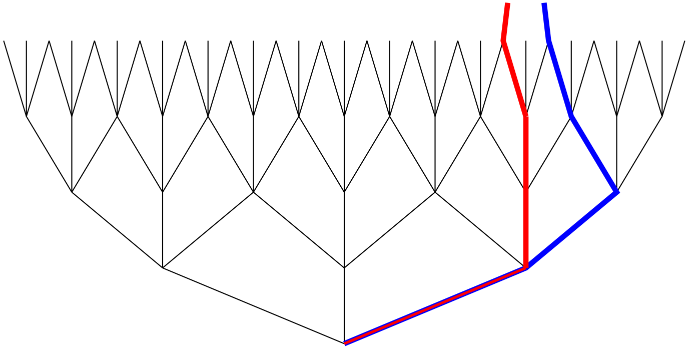

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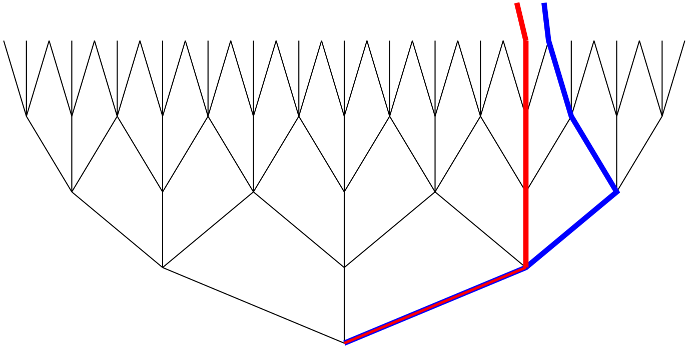

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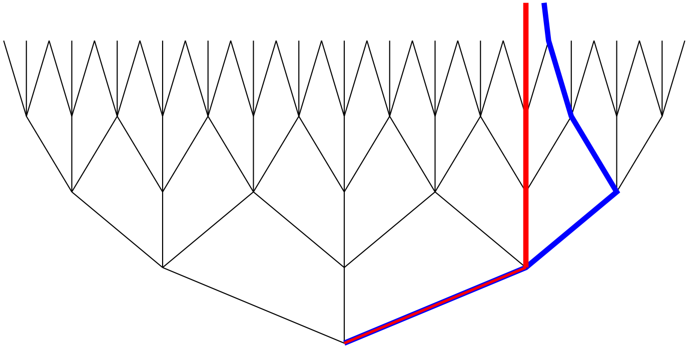

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Let $\rho\colon \{0,1,2\}^3 \to \{0,1,2\}^3$ be a function which is the identity except on the following patterns:

$$\langle 1,0,0\rangle \overset{\rho}{\mapsto} \langle 0,2,0\rangle, \qquad \langle 1,2,2\rangle \overset{\rho}{\mapsto} \langle 2,0,2\rangle,$$
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The function ρ is extended to $\rho\colon \{0,1,2\}^{\mathbb{N}} \to \{0,1,2\}^{\mathbb{N}}$ by

$$\rho(\alpha) = \lambda n.(\sigma_\alpha^n)_0,$$

where $\sigma_\alpha^n \in \{0,1,2\}^3$ is defined by

$$\sigma_\alpha^0 = \rho(\alpha_0, \alpha_1, \alpha_2),$$
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Lemma

Let $\alpha, \beta \in \mathbb{T}_{\mathbb{R}}$ such that $\alpha = \rho_{\mathbb{R}}(\beta)$. For any $n \in \mathbb{N}$ and $i \in \{0, 2\}$,

$$\beta_n \neq i \implies \forall m \geq n \left(\langle \alpha_m, \alpha_{m+1}, \alpha_{m+2}\rangle \neq \langle i, i, i\rangle\right).$$

Spread representation of $[0, 1]$

Lemma

For any $\alpha \in \{0, 1, 2\}^{\mathbb{N}}$ *and* $n \in \mathbb{N}$*, we have*

$$|N(\overline{\alpha}(n+1)) - N(\overline{\rho(\alpha)}(n+1))| \leq 1.$$

Corollary

For each $\alpha \in \{0, 1, 2\}^{\mathbb{N}}$*, we have* $\Phi(\alpha) \simeq \Phi(\rho(\alpha))$.

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For each $\alpha \in \{0,1,2\}^{\mathbb{N}}$, we have $\Phi(\alpha) \simeq \Phi(\rho(\alpha))$.

Recall

$$V_a \overset{\text{def}}{=} \left\{ x_\alpha \mid \alpha \in \{0,1,2\}^{\mathbb{N}} \;\&\; \alpha \in a \right\},$$

$$U(x,r) \overset{\text{def}}{=} \{ y \in [0,1] \mid |y - x| < r \}.$$

Proposition (Quotient property)

For $\alpha \in \{0,1,2\}^{\mathbb{N}}$ and $n \in \mathbb{N}$, we have $U(\Phi(\rho(\alpha)), 2^{-(n+5)}) \subseteq V_{\overline{\rho(\alpha)}n}$.

Proof.

It suffices to show $|x_{\rho(\alpha)} - x_\beta| < 2^{-(n+5)} \to x_\beta \in V_{\overline{\rho(\alpha)}n}$.

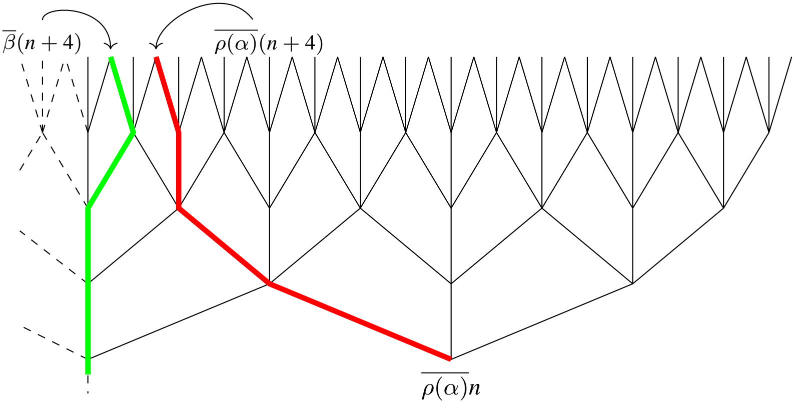
Let $\beta \in \{0,1,2\}^{\mathbb{N}}$ such that $|x_{\rho(\alpha)} - x_\beta| < 2^{-(n+5)}$. For sufficiently large $m \in \mathbb{N}$, we have $|x^m_{\rho(\alpha)} - x^m_\beta| < 2^{-(n+5)}$. Thus

$$
\begin{aligned}
&|2^{-(n+5)}N(\overline{\rho(\alpha)}(n+4)) - 2^{-(n+5)}N(\overline{\beta}(n+4))| \\
&= |x^{n+4}_{\rho(\alpha)} - x^{n+4}_\beta| \\
&\leq |x^{n+4}_{\rho(\alpha)} - x^m_{\rho(\alpha)}| + |x^m_{\rho(\alpha)} - x^m_\beta| + |x^m_\beta - x^{n+4}_\beta| \\
&< 3 \cdot 2^{-(n+5)}.
\end{aligned}
$$

Hence $|N(\overline{\rho(\alpha)}(n+4)) - N(\overline{\beta}(n+4))| \leq 2$. Since $\langle \rho(\alpha)_n, \rho(\alpha)_{n+1}, \rho(\alpha)_{n+2} \rangle \in \{\langle 0,0,0\rangle, \langle 2,2,2\rangle\}$ implies $\overline{\rho(\alpha)}(n+3)$ is the left-most or the right-most path, we must have

$$
|2^4 N(\overline{\rho(\alpha)}n) - N(\overline{\beta}(n+4))| \leq 2^4 - 1.
$$

Spread representation of $[0, 1]$



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$\square$

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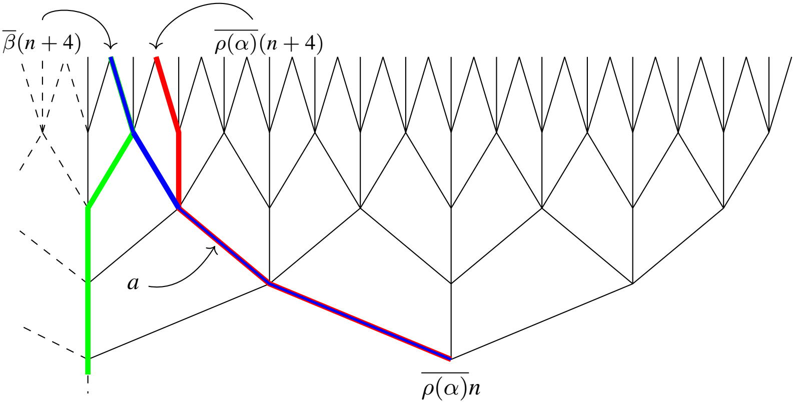

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Thus, there is $a \in \{0,1,2\}^4$ such that $N(\overline{\rho(\alpha)}n * a) = N(\overline{\beta}(n+4))$.

Then, $\gamma \stackrel{\text{def}}{=} \overline{\rho(\alpha)}n * a * \lambda k.\beta(k+4)$ satisfies $x_\beta \simeq x_\gamma$. $\square$

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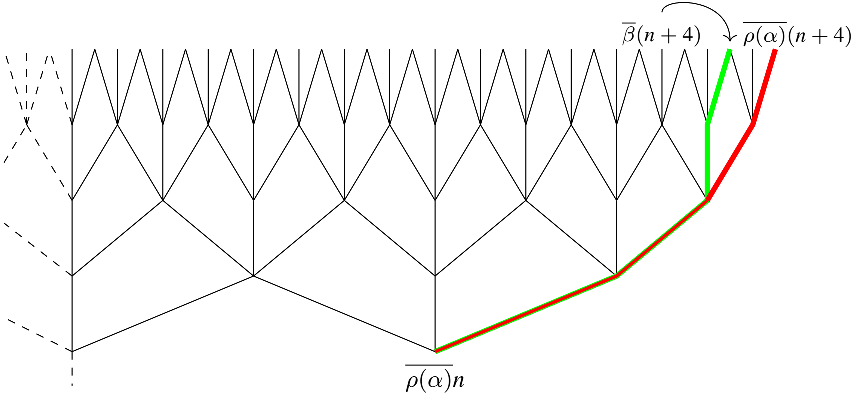

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Spread representation of $[0,1]$

- The function $f\colon \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ is **uniformly continuous** if

$$\forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall \alpha, \beta \in \{0,1\}^{\mathbb{N}} \left(\overline{\alpha} n = \overline{\beta} n \to |f(\alpha) - f(\beta)| \leq 2^{-k} \right).$$

- The function $f\colon [0,1] \to \mathbb{R}$ is **uniformly continuous** if

$$\forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall x, y \in [0,1] \left(|x - y| \leq 2^{-n} \to |f(x) - f(y)| \leq 2^{-k} \right).$$

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Theorem

A function $f\colon [0,1] \to \mathbb{R}$ is uniformly continuous if and only if the composition $f \circ \Phi\colon \{0,1,2\}^{\mathbb{N}} \to \mathbb{R}$ is uniformly continuous.

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\{0,1,2\}^{\mathbb{N}} \arrow[d, "\Phi"'] \arrow[dr, "f \circ \Phi"] & \\
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Proof.

Suppose $f \circ \Phi$ is uniformly continuous. Fix $k \in \mathbb{N}$, and let N_k be the modulus of uniform continuity of $f \circ \Phi$. Let $x, y \in [0,1]$ such that $|x - y| \leq 2^{-(N_k+6)}$ (so that $|x - y| < 2^{-(N_k+5)}$). Let $\alpha_x \in \{0,1,2\}^{\mathbb{N}}$ be the path determined by x. Then $x \simeq \Phi(\alpha_x) \simeq \Phi(\rho(\alpha_x))$. By the quotient property, there is a path $\beta \in \overline{\rho(\alpha)}N_k$ such that $y \simeq \Phi(\beta)$. Then

$$|f(x) - f(y)| \simeq |f(\Phi(\rho(\alpha))) - f(\Phi(\beta))| \leq 2^{-k}.$$ □

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