Concepts of continuity

Tatsuji Kawai

Japan Advanced Institute of Science and Technology

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Brouwer's argument for the uniform continuity

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Theorem

Every function $f : [0, 1] \to \mathbb{R}$ *is uniformly continuous.*

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Every function $f\colon [0,1] \to \mathbb{R}$ *is uniformly continuous.*

Brouwer's argument rests on two assumptions:

Continuity principle
Every function $f\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is continuous.

Fan theorem
Every bar $B \subseteq T$ of a fan T is uniform.

Fan theorem

A **fan** is a decidable subset $T \subseteq \mathbb{N}^*$ which is a **spread**, i.e.

1. $\langle\,\rangle \in T$,
2. $a \in T \leftrightarrow \exists i \in \mathbb{N}\,(a * \langle i \rangle \in T)$,

and finitely branching

3. $\forall a \in T \exists m \in \mathbb{N} \forall i \in \mathbb{N}\,(a * \langle i \rangle \in T \to i \leq m)$.

For $\alpha \in \mathbb{N}^{\mathbb{N}}$, write

$$\alpha \in T \overset{\text{def}}{\iff} \forall n \in \mathbb{N}\,(\overline{\alpha}n \in T)\,.$$

A subset $B \subseteq T$ is a **bar** of T if

$$\forall \alpha \in T \exists n \in \mathbb{N}\, B(\overline{\alpha}n).$$

A bar B is **uniform** if

$$\exists m \in \mathbb{N} \forall \alpha \in T \exists n \leq m\, B(\overline{\alpha}n).$$

Fan theorem

Every bar $B \subseteq T$ of a fan T is uniform.

Fan theorem

- For $a, b \in X^*$ define $a \leq b \overset{\text{def}}{\iff}$ "b is an initial segment of a".
- For $B \subseteq X^*$ write $\downarrow B \overset{\text{def}}{=} \{a \in X^* \mid \exists b \in B\,(a \leq b)\}$.

Proposition

The following are equivalent:

1. *(Fan theorem) Every bar $B \subseteq T$ of a fan T is uniform.*
2. *(Fan theorem for $\{0,1\}^{\mathbb{N}}$) Every bar $B \subseteq \{0,1\}^*$ is uniform.*
3. *If $B \subseteq \{0,1\}^*$ is a bar and $Q \subseteq \{0,1\}^*$ is such that*
   - $\downarrow B \subseteq Q$,
   - $\forall a \in \{0,1\}^*\,(a * \langle 0 \rangle \in Q \;\&\; a * \langle 1 \rangle \in Q \to a \in Q)$,

   then $\langle\,\rangle \in Q$.
4. *$B \subseteq \{0,1\}^*$ is a bar if and only if $\langle\,\rangle \lhd B$,*

   where the relation $a \lhd B$ is inductively defined by

$$\frac{a \in \downarrow B}{a \lhd B}, \qquad \frac{a \leq b \lhd B}{a \lhd B}, \qquad \frac{a * \langle 0 \rangle \lhd B \quad a * \langle 1 \rangle \lhd B}{a \lhd B}.$$

Brouwer's argument for the uniform continuity

Continuity principle

Every function $f\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is continuous.

Fan theorem

Every bar $B \subseteq T$ of a fan T is uniform.

Theorem (Continuity principle + Fan theorem)

Every function $f\colon [0,1] \to \mathbb{R}$ *is uniformly continuous.*

Brouwer's argument for the uniform continuity

Let $f\colon T \to \mathbb{N}$ be a function on a fan $T \subseteq \mathbb{N}^*$:

- f is **continuous** if

$$\forall \alpha \in T \exists n \in \mathbb{N} \forall \beta \in T \left(\overline{\alpha} n = \overline{\beta} n \to f(\alpha) = f(\beta) \right).$$

- f is **uniformly continuous** if

$$\exists n \in \mathbb{N} \forall \alpha, \beta \in \{0,1\}^{\mathbb{N}} \left(\overline{\alpha} n = \overline{\beta} n \to f(\alpha) = f(\beta) \right).$$

Lemma (Continuity principle + Fan theorem)

Every function $f\colon T \to \mathbb{N}$ on a fan T is uniformly continuous.

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Lemma (Continuity principle + Fan theorem)

Every function $f\colon T \to \mathbb{N}$ on a fan T is uniformly continuous.

Proof.

By the continuity principle, $f\colon T \to \mathbb{N}$ is continuous. Define $B \subseteq T$ by

$$B(a) \stackrel{\text{def}}{=} \forall \alpha, \beta \in T \left(f(a * \alpha) = f(a * \beta) \right),$$

which is a (monotone) bar of T by continuity of f. By the fan theorem, there is $n \in \mathbb{N}$ such that $\forall \alpha \in T\, B(\overline{\alpha} n)$. $\square$

Brouwer's argument for the uniform continuity

We have a mapping $\Phi\colon \{0,1,2\}^{\mathbb{N}} \to [0,1]$ given by

$$\Phi(\alpha) \stackrel{\text{def}}{=} \langle 2^{-(n+1)} N(\overline{\alpha}n) \rangle_{n \in \mathbb{N}}.$$

with the quotient property:

Theorem

A function $f\colon [0,1] \to \mathbb{R}$ is uniformly continuous if and only if the composition $f \circ \Phi\colon \{0,1,2\}^{\mathbb{N}} \to \mathbb{R}$ is uniformly continuous.

$$\begin{tikzcd}
\{0,1,2\}^{\mathbb{N}} \arrow[d, "\Phi"'] \arrow[dr, "f \circ \Phi"] & \\
{[0,1]} \arrow[r, "f"'] & \mathbb{R}
\end{tikzcd}$$

We also have a mapping $\Phi_{\mathbb{R}}\colon \mathbb{T}_{\mathbb{R}} \to \mathbb{R}$ with a similar quotient property

Theorem (Continuity principle + Fan theorem)

Every function $f\colon [0,1] \to \mathbb{R}$ is uniformly continuous.

Proof.

Write $\Psi_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{T}_{\mathbb{R}}$ for the assignment of a path α_x in $\mathbb{T}_{\mathbb{R}}$ to each real number x. Then $x \simeq \Phi_{\mathbb{R}}(\Psi_{\mathbb{R}}(x))$.

$$\begin{tikzcd}
\{0,1,2\}^{\mathbb{N}} \arrow[d, "\Phi"] & & \mathbb{T}_{\mathbb{R}} \arrow[d, bend left, "\Phi_{\mathbb{R}}"] \\
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$$\begin{tikzcd}
\{0,1,2\}^{\mathbb{N}} \arrow[r, "g"] \arrow[d, "\Phi"] & \mathbb{T}_{\mathbb{R}} \arrow[d, bend left, "\Phi_{\mathbb{R}}"] \\
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\end{tikzcd}$$

Put $g = \Psi_{\mathbb{R}} \circ f \circ \Phi\colon \{0,1,2\}^{\mathbb{N}} \to \mathbb{T}_{\mathbb{R}}$. Since each projection $g_n = \lambda\alpha.g(\alpha)_n\colon \{0,1,2\}^{\mathbb{N}} \to \mathbb{N}$ is uniformly continuous, so is g.

Brouwer's argument for the uniform continuity

Theorem (Continuity principle + Fan theorem)

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Write $\Psi_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{T}_{\mathbb{R}}$ for the assignment of a path α_x in $\mathbb{T}_{\mathbb{R}}$ to each real number x. Then $x \simeq \Phi_{\mathbb{R}}(\Psi_{\mathbb{R}}(x))$.

$$\begin{tikzcd}
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\end{tikzcd}$$

Put $g = \Psi_{\mathbb{R}} \circ f \circ \Phi\colon\ \{0,1,2\}^{\mathbb{N}} \to \mathbb{T}_{\mathbb{R}}$. Since each projection $g_n = \lambda\alpha.g(\alpha)_n\colon\ \{0,1,2\}^{\mathbb{N}} \to \mathbb{N}$ is uniformly continuous, so is g.

Thus, $\Phi_{\mathbb{R}} \circ g = f \circ \Phi\colon\ \{0,1,2\}^{\mathbb{N}} \to \mathbb{R}$ is uniformly continuous. Since Φ is a uniform quotient map, f is uniformly continuous. $\square$

Role of the fan theorem

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With the help of continuity principle (and enough choice),

> *Every function $f\colon T \to \mathbb{N}$ from a fan T has a* ***continuous modulus****,*

where a **modulus** of $f\colon T \to \mathbb{N}$ is a function $g\colon T \to \mathbb{N}$ such that

$$\forall \alpha, \beta \in T \left(\overline{\alpha} g(\alpha) = \overline{\beta} g(\alpha) \to f(\alpha) = f(\beta) \right).$$

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$$\forall \alpha, \beta \in T \left(\overline{\alpha} g(\alpha) = \overline{\beta} g(\alpha) \to f(\alpha) = f(\beta)\right).$$

Theorem ([Berger, 2005])

The following are equivalent:

1. *Every function $f\colon T \to \mathbb{N}$ from a fan T with a continuous modulus is uniformly continuous.*
2. *The decidable fan theorem $(\mathrm{FAN_D})$.*

Role of the fan theorem

1. Every function $f\colon T \to \mathbb{N}$ from a fan T with a continuous modulus is uniformly continuous.
2. The decidable fan theorem $(\mathrm{FAN_D})$.

Proof.
It sufficies to consider $T = \{0,1\}^*$.
$(1 \to 2)$ Given a decidable bar B, define $f_B\colon \{0,1\}^{\mathbb{N}} \to \mathbb{N}$ by

$$f_B(\alpha) = \text{the least } n \text{ such that } B(\overline{\alpha}n).$$

Then f_B is continuous modulus of itself, and thus uniformly continuous. The modulus of uniform continuity of f_B gives a bound of B.

$(2 \to 1)$ Given a continuous modulus $g\colon \{0,1\}^{\mathbb{N}} \to \mathbb{N}$ of $f\colon \{0,1\}^{\mathbb{N}} \to \mathbb{N}$, define $B_g \subseteq \{0,1\}^*$ by

$$a \in B_g \overset{\text{def}}{\iff} g(a * 0^\omega) < |a|,$$

which is a bar by continuity of g. By the fan theorem, B_g is uniform. $\square$

Moduli of continuity for real-valued functions

- Let T be a fan. A **modulus** of $f\colon T \to \mathbb{R}$ is a sequence $\langle g_n \rangle_{n \in \mathbb{N}}$ of functions $g_n\colon T \to \mathbb{N}$ such that

$$\forall k \in \mathbb{N} \forall \alpha, \beta \in T \left(\overline{\alpha} g_k(\alpha) = \overline{\beta} g_k(\alpha) \to |f(\alpha) - f(\beta)| \leq 2^{-k} \right).$$

  A modulus $\langle g_n \rangle_{n \in \mathbb{N}}$ of f is **continuous** if each g_n is continuous.

- A **modulus** of $f\colon [0,1] \to \mathbb{R}$ is a sequence $\langle g_n \rangle_{n \in \mathbb{N}}$ of functions $g_n\colon \{0,1,2\}^{\mathbb{N}} \to \mathbb{N}$ such that

$$\forall k \in \mathbb{N} \forall \alpha \in \{0,1,2\}^{\mathbb{N}} \forall x \in [0,1] \Big(|x - \Phi(\alpha)| \leq 2^{-g_k(\alpha)} \\ \to |f(x) - f(\Phi(\alpha))| \leq 2^{-k} \Big),$$

  where $\Phi\colon \{0,1,2\}^{\mathbb{N}} \to [0,1]$ is the quotient map.
  A modulus $\langle g_n \rangle_{n \in \mathbb{N}}$ of f is **continuous** if each g_n is continuous.

Equivalents of decidable fan theorem

Theorem

The following are equivalent:

1. *The decidable fan theorem.*
2. *Every function* $f : T \to \mathbb{N}$ *from a fan* T *with a continuous modulus is uniformly continuous.*
3. *Every function* $f : T \to \mathbb{R}$ *from a fan* T *with a continuous modulus is uniformly continuous.*
4. *Every function* $f : [0, 1] \to \mathbb{R}$ *with a continuous modulus is uniformly continuous.*

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4. *Every function $f\colon [0,1] \to \mathbb{R}$ with a continuous modulus is uniformly continuous.*

Proof.

$(1 \to 2)$ [Berger, 2005].
$(2 \to 3)$ If $f\colon T \to \mathbb{R}$ has a continuous modulus $g\colon \mathbb{N} \to T \to \mathbb{N}$, we can modify g so that each $g_n\colon T \to \mathbb{N}$ is a continuous modulus of itself. Thus f has a uniformly continuous modulus.
$(3 \to 4)$ follows from the quotient property of $\Phi\colon \{0,1,2\}^{\mathbb{N}} \to [0,1]$.

Equivalents of decidable fan theorem

1 The decidable fan theorem.

4 Every function $f\colon [0,1] \to \mathbb{R}$ with a continuous modulus is uniformly continuous.

(4 $\to$ 1) Define a function $\kappa\colon \{0,1\}^{\mathbb{N}} \to [0,1]$ by

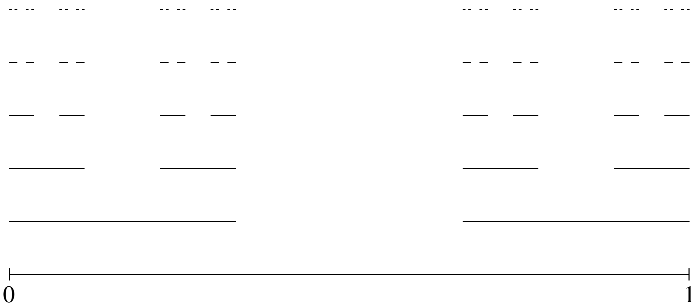
$$\kappa(\alpha) = \left\langle \sum_{i<n} 2\alpha_i 3^{-(i+1)} \right\rangle_{n\in\mathbb{N}}.$$

The image of κ is called the **Cantor's discontinuum**.
For each $a \in \{0,1\}^*$, define an interval $\mathbb{C}_a$ with rational endpoints:

$$\mathbb{C}_a = \left[\sum_{i<|a|} 2a_i 3^{-(i+1)},\ 3^{-|a|} + \sum_{i<|a|} 2a_i 3^{-(i+1)} \right].$$

For each $n \in \mathbb{N}$ and $a \in \{0,1\}^n$, the interval $\mathbb{C}_a$ is in the n-th level of Cantor's middle-third set, which is of length 3^{-n}.

Equivalents of decidable fan theorem



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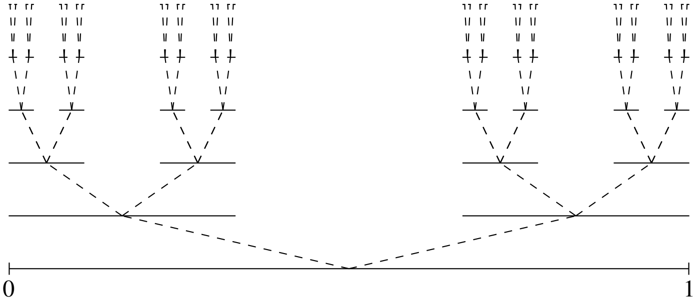

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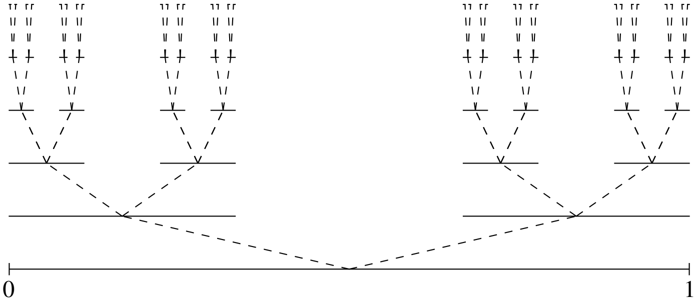

Let $B \subseteq \mathbb{N}^*$ be a decidable bar. Define $f \colon [0,1] \to \mathbb{R}$ by

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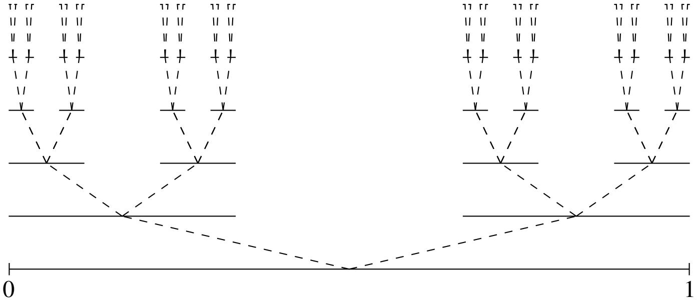

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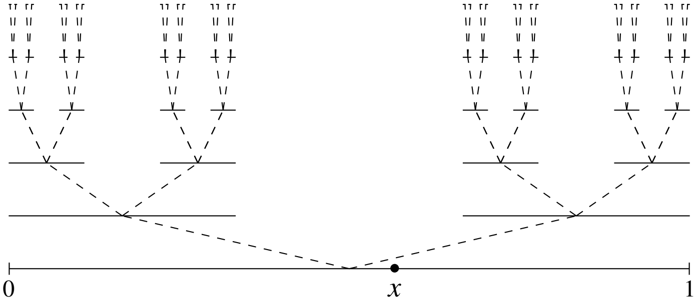

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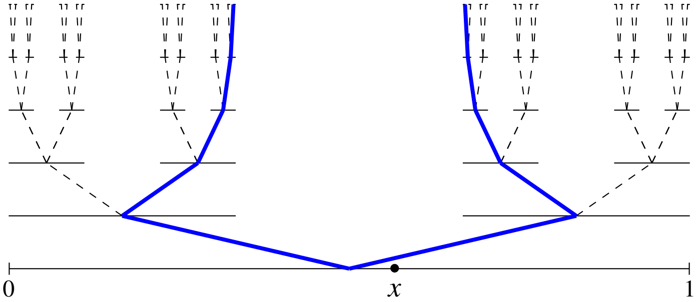

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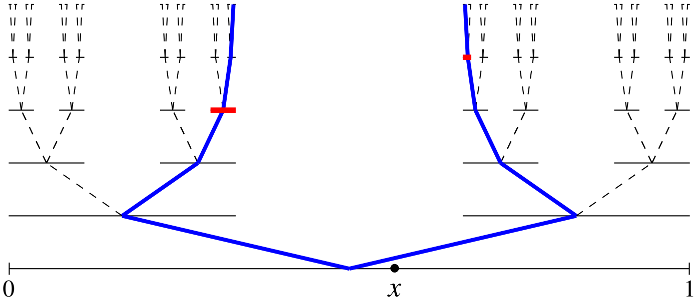

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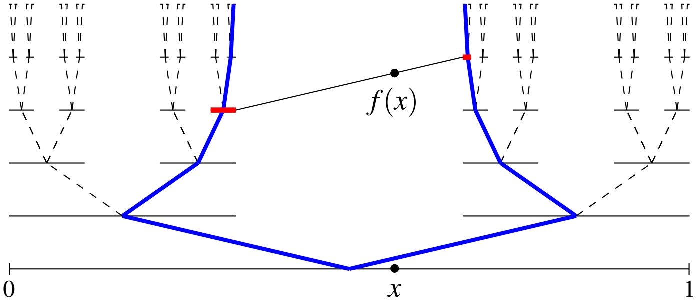

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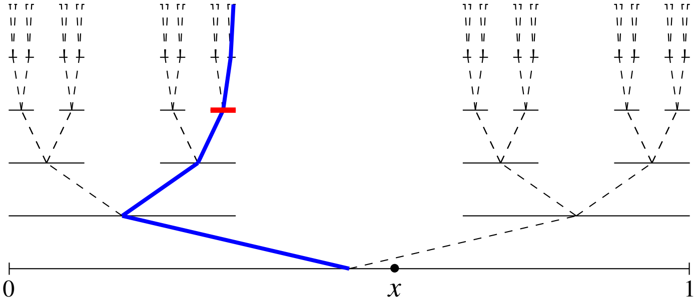

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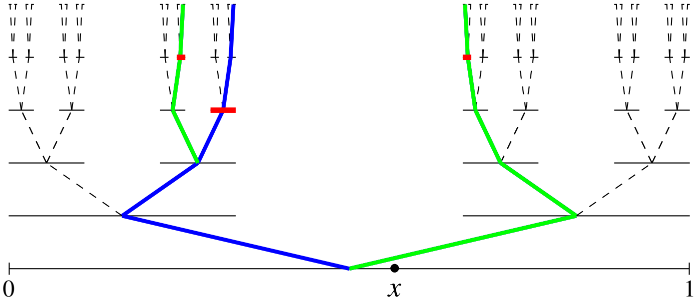

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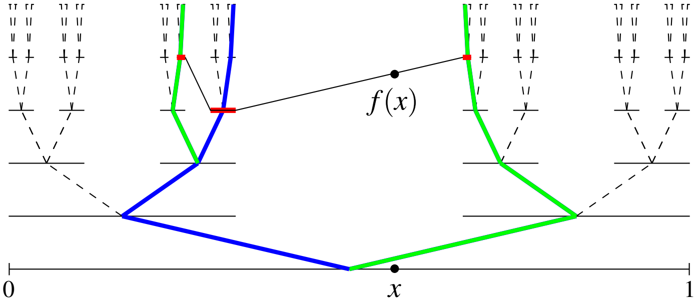

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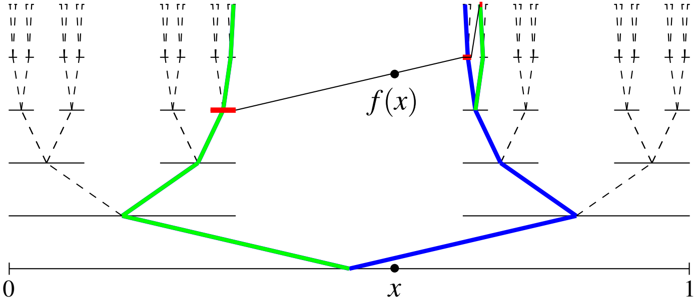

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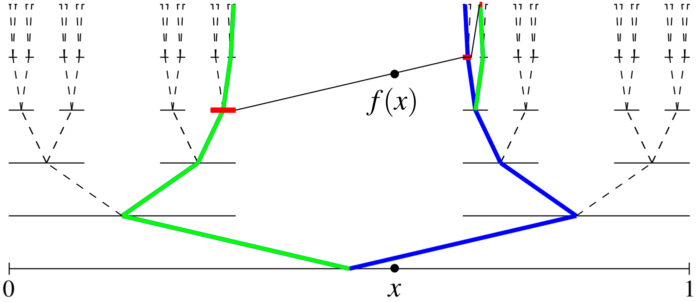

Let $B \subseteq \mathbb{N}^*$ be a decidable bar. Define $f\colon [0,1] \to \mathbb{R}$ by

$f(x) =$ the value of the piecewise linear function around x.

A continuous modulus $g\colon \mathbb{N} \to \{0,1,2\}^{\mathbb{N}} \to \mathbb{N}$ of f is defined by

$g_k(\alpha) =$the modulus of uniform continuity of piecewise linear the function around $\Phi(\alpha)$ with respect to 2^{-k}.

By the assumption, f is uniformly continuous.

The composition $\{0,1\}^{\mathbb{N}} \xrightarrow{\kappa} [0,1] \xrightarrow{f} \mathbb{R}$ satisfies

$$f(\kappa(\alpha)) = \text{the least } n \text{ such that } B(\overline{\alpha}n).$$

The upper bound of the image of $f \circ \kappa$ gives a uniform bound of B. $\square$

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