Concepts of continuity

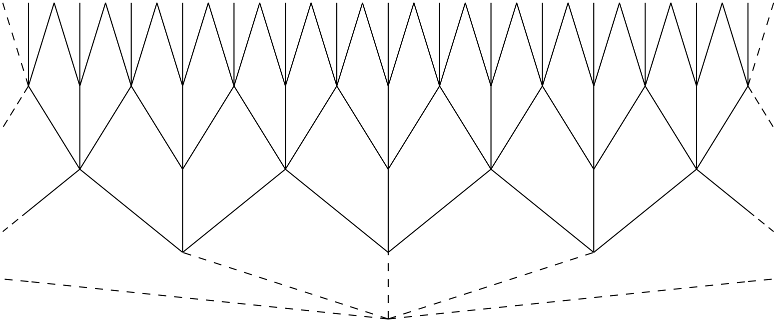
Tatsuji Kawai

Japan Advanced Institute of Science and Technology

23 September 2019

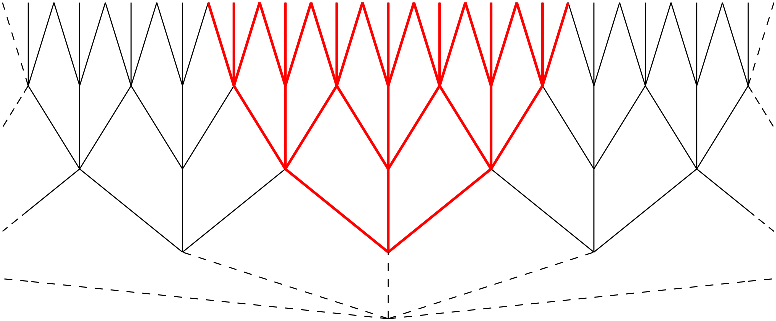
Spread representation of $\mathbf{R}$

Consider a family of trees: $\mathbb{T} \stackrel{\text{def}}{=} \bigcup_{i\in\mathbb{Z}} \langle i \rangle * \{0,1,2\}^*$.



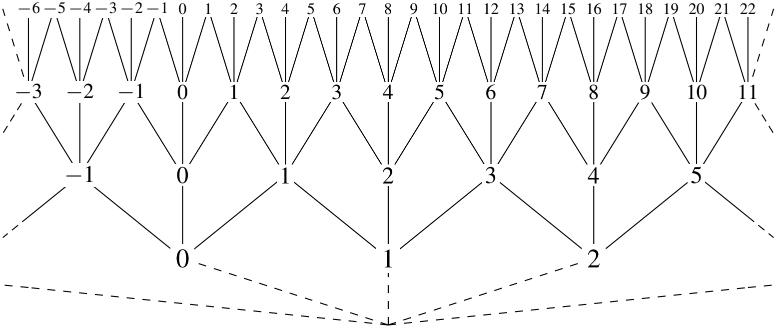
Spread representation of $\mathbf{R}$

Consider a family of trees: $\mathbb{T} \stackrel{\text{def}}{=} \bigcup_{i\in\mathbb{Z}} \langle i \rangle * \{0, 1, 2\}^*$.



Spread representation of $\mathbf{R}$

Consider a family of trees: $\mathbb{T} \stackrel{\text{def}}{=} \bigcup_{i\in\mathbb{Z}} \langle i \rangle * \{0,1,2\}^*$.

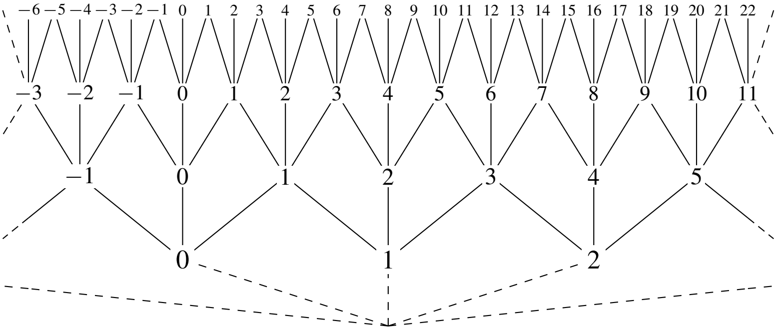


Assign an integer to each element of $\mathbb{T}$ by

$$\begin{aligned} N(\langle i \rangle) &= i && (i \in \mathbb{Z}), \\ N(a * \langle i \rangle) &= 2N(a) + (i-1) && (a \in \mathbb{T}^+, i \in \{0,1,2\}). \end{aligned}$$

Spread representation of $\mathbf{R}$

Consider a family of trees: $\mathbb{T} \stackrel{\text{def}}{=} \bigcup_{i \in \mathbb{Z}} \langle i \rangle * \{0, 1, 2\}^*$.

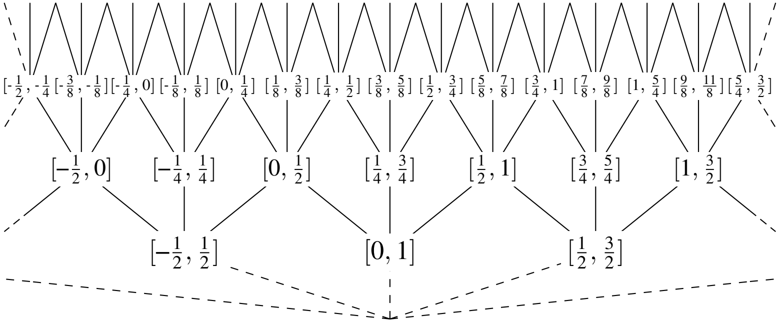


Assign an integer to each element of $\mathbb{T}$ by

$$\begin{aligned} N(\langle i \rangle) &= i && (i \in \mathbb{Z}), \\ N(a * \langle i \rangle) &= 2N(a) + (i - 1) && (a \in \mathbb{T}^+, i \in \{0, 1, 2\}). \end{aligned}$$

Spread representation of $\mathbf{R}$

Consider a family of trees: $\mathbb{T} \stackrel{\text{def}}{=} \bigcup_{i\in\mathbb{Z}} \langle i\rangle * \{0,1,2\}^*$.



Assign an integer to each element of $\mathbb{T}$ by

$$\begin{aligned} N(\langle i\rangle) &= i && (i\in\mathbb{Z}), \\ N(a*\langle i\rangle) &= 2N(a)+(i-1) && (a\in\mathbb{T}^+, i\in\{0,1,2\}). \end{aligned}$$

$$\mathbb{I}_a \stackrel{\text{def}}{=} \left[2^{-|a|}(N(a)-1), 2^{-|a|}(N(a)+1)\right].$$

Spread representation of $\mathbf{R}$

Let $\mathbb{T}_{\mathbb{R}}$ be the set of paths of $\mathbb{T}$.

Each $\alpha \in \mathbb{T}_{\mathbb{R}}$ determines a real number x_α by

$$x_\alpha \stackrel{\text{def}}{=} \langle 2^{-(n+1)} N(\overline{\alpha}(n+1)) \rangle_{n \in \mathbb{N}}.$$

Let $\Phi_{\mathbb{R}} : \mathbb{T}_{\mathbb{R}} \to \mathbb{R}$ be the mapping $\alpha \mapsto x_\alpha$, which is uniformly continuous.

Spread representation of $\mathbf{R}$

To each regular sequence $x = \langle r_n \rangle_{n \in \mathbb{N}}$, associate a sequence $\langle \mathbb{I}^x_n \rangle_{n \in \mathbb{N}}$ of rational intervals:

$$\mathbb{I}^x_n \stackrel{\text{def}}{=} \left[r_{n+2} - 2^{-(n+2)}, r_{n+2} + 2^{-(n+2)} \right].$$

Define a sequence $\alpha_x \in \mathbb{T}_{\mathbb{R}}$ by primitive recursion:

$$\begin{aligned} \alpha_x(0) &= i \text{ for the least } i \in \mathbb{Z} \text{ such that } \mathbb{I}^x_0 \sqsubseteq \mathbb{I}_{\langle i \rangle}, \\ \alpha_x(n+1) &= i \text{ for the least } i \in \{0,1,2\} \text{ such that } \mathbb{I}^x_{n+1} \sqsubseteq \mathbb{I}_{\langle \alpha_x(0), \ldots, \alpha_x(n), i \rangle}. \end{aligned}$$

Proposition

For each regular sequence $x = \langle r_n \rangle_{n \in \mathbb{N}} \in \mathbb{R}$, we have

$$x \simeq \Phi_{\mathbb{R}}(\alpha_x).$$

Spread representation of $\mathbf{R}$

Let $\rho_{\mathbb{R}} : \mathbb{Z} * \{0, 1, 2\}^2 \to \mathbb{Z} * \{0, 1, 2\}^2$ be a function which is the identity except on the following patterns:

$$\langle i, 2, 2\rangle \overset{\rho_{\mathbb{R}}}{\mapsto} \langle i + 1, 0, 2\rangle$$
$$\langle i, 0, 0\rangle \overset{\rho_{\mathbb{R}}}{\mapsto} \langle i - 1, 2, 0\rangle$$

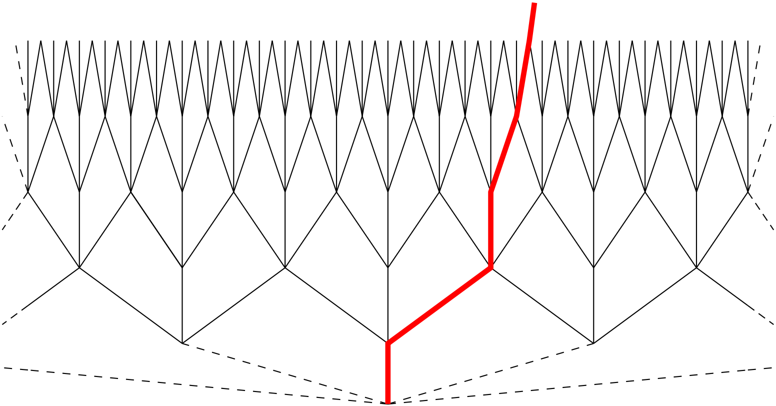
The function $\rho_{\mathbb{R}}$ is extended to $\rho_{\mathbb{R}} : \mathbb{T}_{\mathbb{R}} \to \mathbb{T}_{\mathbb{R}}$ by

$$\rho_{\mathbb{R}}(\alpha) = \lambda n.(\sigma^n_\alpha)_0,$$

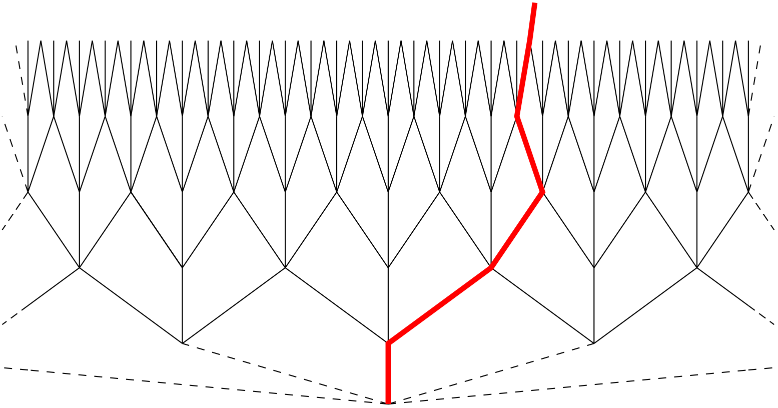
where $\sigma^n_\alpha \in \{0, 1, 2\}^3$ is defined by

$$\begin{aligned}\sigma^0_\alpha &= \rho_{\mathbb{R}}(\alpha_0, \alpha_1, \alpha_2),\\ \sigma^{n+1}_\alpha &= \rho_{\mathbb{R}}((\sigma^n_\alpha)_1, \alpha_{n+2}, \alpha_{n+3}).\end{aligned}$$

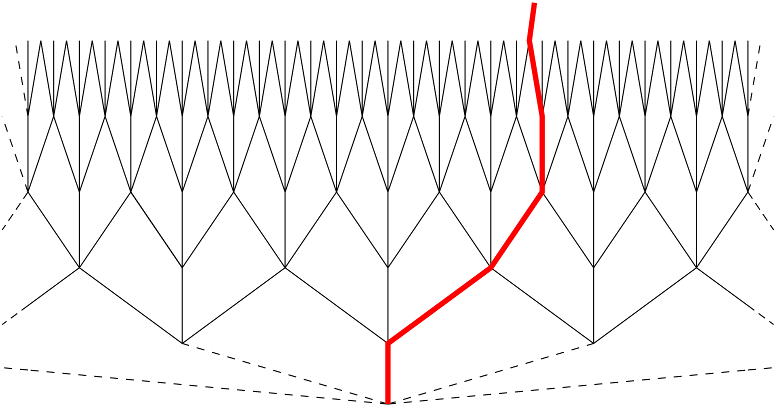
Spread representation of $\mathbb{R}$



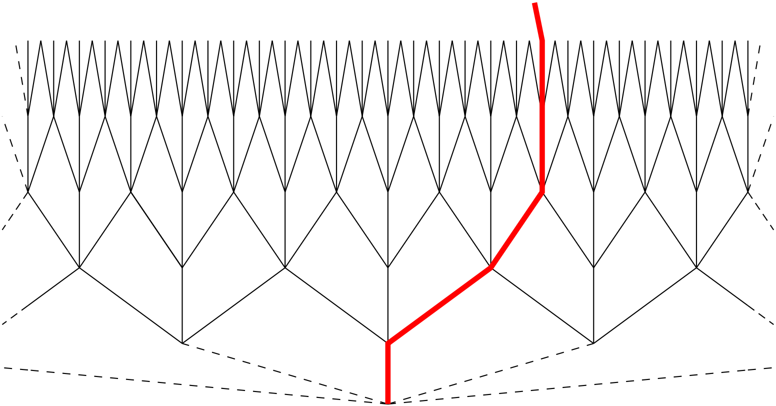
Spread representation of $\mathbf{R}$



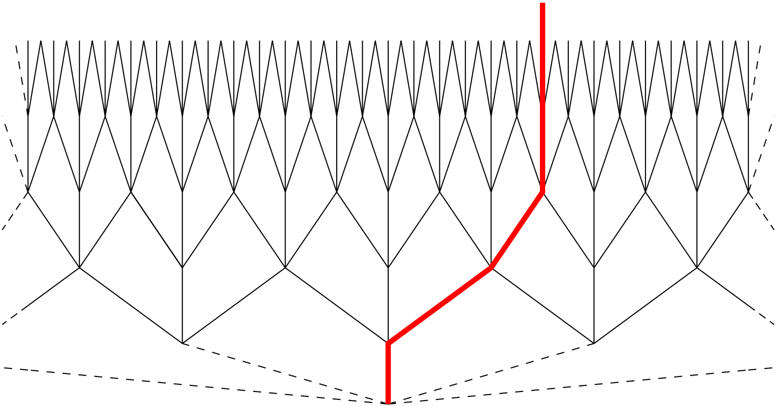
Spread representation of $\mathbf{R}$



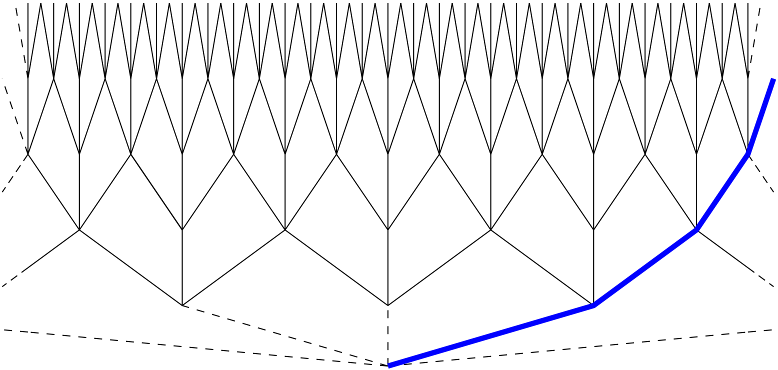
Spread representation of $\mathbf{R}$



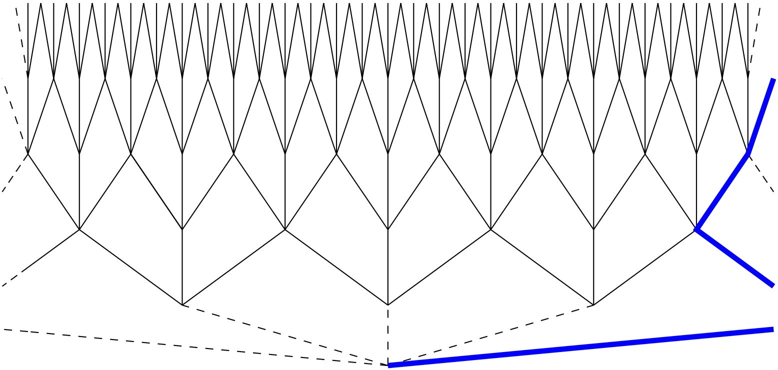
Spread representation of $\mathbf{R}$



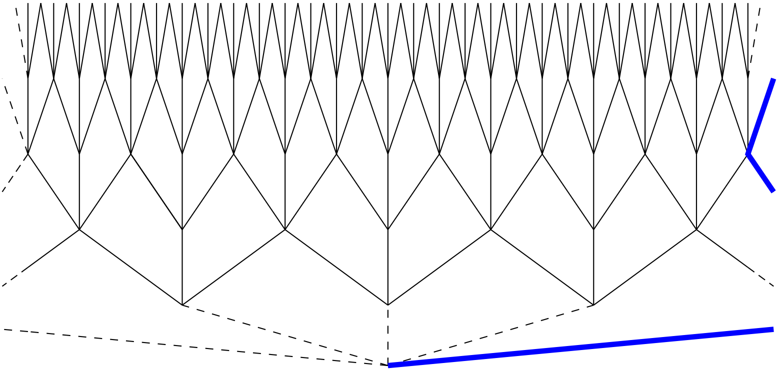
Spread representation of $\mathbf{R}$



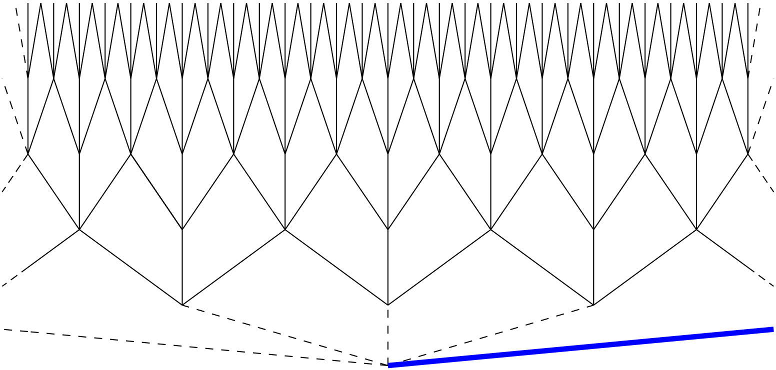
Spread representation of $\mathbb{R}$



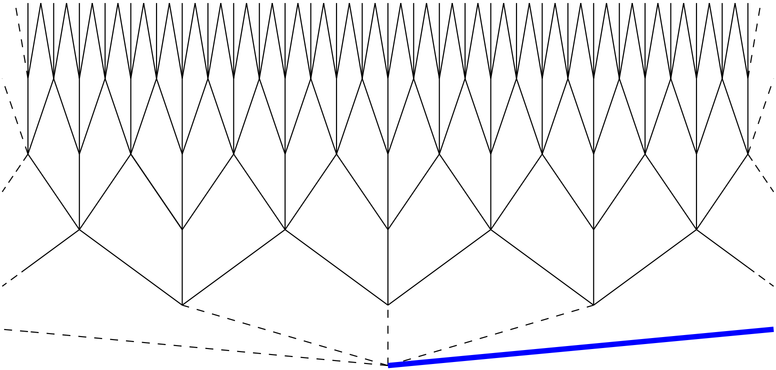
Spread representation of $\mathbf{R}$



Spread representation of $\mathbf{R}$



Spread representation of $\mathbf{R}$



Lemma

Let $\alpha, \beta \in \mathbb{T}_{\mathbb{R}}$ such that $\alpha = \rho_{\mathbb{R}}(\beta)$. For any $n \in \mathbb{N}$,

- $\langle \alpha_n, \alpha_{n+1}, \alpha_{n+2} \rangle \notin \{\langle 0,0,0 \rangle, \langle 2,2,2 \rangle\}$.
- $|N(\overline{\alpha}(n+1)) - N(\overline{\beta}(n+1))| \leq 1$.

Proposition (Quotient property)

For $\alpha \in \mathbb{T}_{\mathbb{R}}$ and $n \in \mathbb{N}$, we have $U(\Phi_{\mathbb{R}}(\rho_{\mathbb{R}}(\alpha)), 2^{-(n+4)}) \subseteq V_{\overline{\rho_{\mathbb{R}}(\alpha)}n}$.

Spread representation of R

Proposition (Quotient property)

For $\alpha \in \mathbb{T}_{\mathbb{R}}$ and $n \in \mathbb{N}$, we have $U(\Phi_{\mathbb{R}}(\rho_{\mathbb{R}}(\alpha)), 2^{-(n+4)}) \subseteq V_{\overline{\rho_{\mathbb{R}}(\alpha)}n}$.

- The function $f\colon \mathbb{T}_{\mathbb{R}} \to \mathbb{R}$ is **locally uniformly continuous** if f is uniformly continuous on each subtree

$$\mathbb{T}_i = \langle i \rangle * \{0,1,2\}^* \ \ (i \in \mathbb{Z}).$$

- The function $f\colon \mathbb{R} \to \mathbb{R}$ is **locally uniformly continuous** if f is uniformly continuous on each open interval (p, q).

Spread representation of $\mathbb{R}$

Proposition (Quotient property)

For $\alpha \in \mathbb{T}_{\mathbb{R}}$ and $n \in \mathbb{N}$, we have $U(\Phi_{\mathbb{R}}(\rho_{\mathbb{R}}(\alpha)), 2^{-(n+4)}) \subseteq V_{\overline{\rho_{\mathbb{R}}(\alpha)n}}$.

- The function $f : \mathbb{T}_{\mathbb{R}} \to \mathbb{R}$ is **locally uniformly continuous** if f is uniformly continuous on each subtree

$$\mathbb{T}_i = \langle i \rangle * \{0, 1, 2\}^* \ \ (i \in \mathbb{Z}).$$

- The function $f : \mathbb{R} \to \mathbb{R}$ is **locally uniformly continuous** if f is uniformly continuous on each open interval (p, q).

Theorem

A function $f : \mathbb{R} \to \mathbb{R}$ is locally uniformly continuous if and only if the composition $f \circ \Phi_{\mathbb{R}} : \mathbb{T}_{\mathbb{R}} \to \mathbb{R}$ is locally uniformly continuous.

$$\begin{tikzcd}
\mathbb{T}_{\mathbb{R}} \arrow[d, "\Phi_{\mathbb{R}}"'] \arrow[dr, "f \circ \Phi_{\mathbb{R}}"] & \\
\mathbb{R} \arrow[r, "f"'] & \mathbb{R}
\end{tikzcd}$$

Proof.

If $f : \mathbb{R} \to \mathbb{R}$ is locally uniformly continuous, the image of each subtree $\mathbb{T}_i$ under $\Phi_{\mathbb{R}}$ is contained in $(i-1, i+1)$. Thus $f \circ \Phi_{\mathbb{R}}$ is uniformly continuous on $\mathbb{T}_i$.

Proof.

If $f\colon \mathbb{R} \to \mathbb{R}$ is locally uniformly continuous, the image of each subtree $\mathbb{T}_i$ under $\Phi_{\mathbb{R}}$ is contained in $(i-1, i+1)$. Thus $f \circ \Phi_{\mathbb{R}}$ is uniformly continuous on $\mathbb{T}_i$.

Suppose $f \circ \Phi_{\mathbb{R}}$ is locally uniformly continuous. Let (p, q) be an open interval, and fix $k \in \mathbb{N}$. We can find $i \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $(p - 1/2, q + 1/2) \subseteq \mathbb{I}_{\langle i \rangle} \cup \cdots \cup \mathbb{I}_{\langle i+n \rangle}$.

For each $j \leq n$, let $N_k^j \in \mathbb{N}$ be the modulus of uniform continuity of $f \circ \Phi_{\mathbb{R}}$ on $\mathbb{T}_{i+j}$. Put $N_k \overset{\text{def}}{=} \max\{N_k^j \mid j \leq n\} + 5$.

Proof.

If $f\colon \mathbb{R} \to \mathbb{R}$ is locally uniformly continuous, the image of each subtree $\mathbb{T}_i$ under $\Phi_{\mathbb{R}}$ is contained in $(i-1, i+1)$. Thus $f \circ \Phi_{\mathbb{R}}$ is uniformly continuous on $\mathbb{T}_i$.

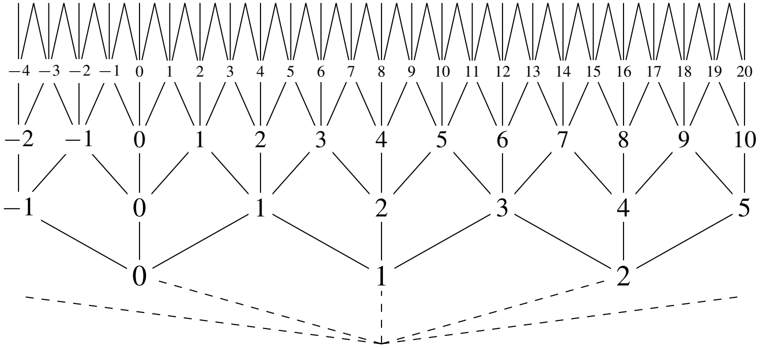
Suppose $f \circ \Phi_{\mathbb{R}}$ is locally uniformly continuous. Let (p, q) be an open interval, and fix $k \in \mathbb{N}$. We can find $i \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $(p - 1/2, q + 1/2) \subseteq \mathbb{I}_{\langle i \rangle} \cup \cdots \cup \mathbb{I}_{\langle i+n \rangle}$.

For each $j \leq n$, let $N_k^j \in \mathbb{N}$ be the modulus of uniform continuity of $f \circ \Phi_{\mathbb{R}}$ on $\mathbb{T}_{i+j}$. Put $N_k \overset{\text{def}}{=} \max\{N_k^j \mid j \leq n\} + 5$.

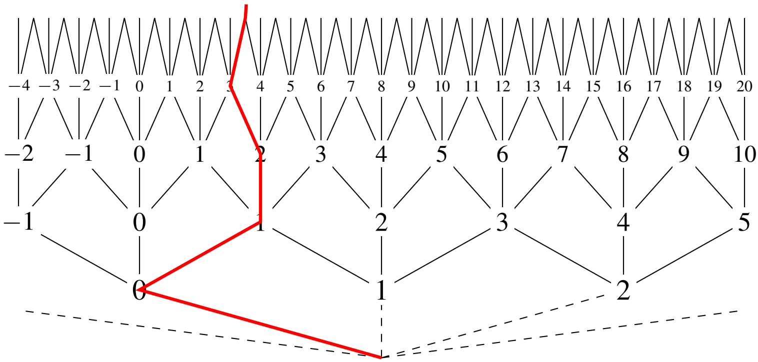
Let $x, y \in \mathbb{R}$ such that $|x - y| \leq 2^{-N_k}$, and let α_x be the path in $\mathbb{T}$ determined by x. Then $x \simeq \Phi_{\mathbb{R}}(\alpha_x) \simeq \Phi_{\mathbb{R}}(\rho_{\mathbb{R}}(\alpha_x))$.

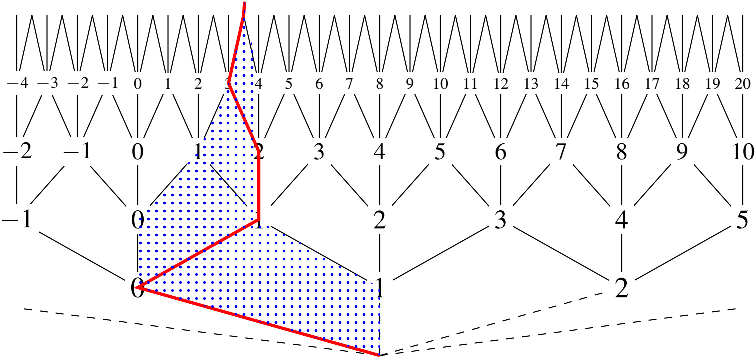
Since $p < x < q$, $\rho_{\mathbb{R}}(\alpha)$ is in the subtree $\mathbb{T}_{i+j}$ for some $j \leq n$. Since $|x - y| < 2^{-(N_k^j + 4)}$, there exists $\beta \in \overline{\rho_{\mathbb{R}}(\alpha)} N_k^j$ such that $y \simeq \Phi_{\mathbb{R}}(\beta)$. Hence $|f(x) - f(y)| \simeq |f(\Phi_{\mathbb{R}}(\rho_{\mathbb{R}}(\alpha))) - f(\Phi_{\mathbb{R}}(\beta))| \leq 2^{-k}$. $\square$

Ideals of $\mathbb{T}$



Ideals of $\mathbb{T}$




−4 −3 −2 −1 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
−2 −1 0 1 2 3 4 5 6 7 8 9 10
−1 0 1 2 3 4 5
0 1 2

Ideals of $\mathbb{T}$

Define an order on the nodes of $\mathbb{T}$ by

$$a \leq b \overset{\text{def}}{\iff} \mathbb{I}_a \subseteq \mathbb{I}_b,$$

We have

$$a \leq b \iff \exists k \in \mathbb{N} \left(|b| + k = |a| \;\&\; |2^k N(b) - N(a)| \leq 2^k - 1 \right).$$

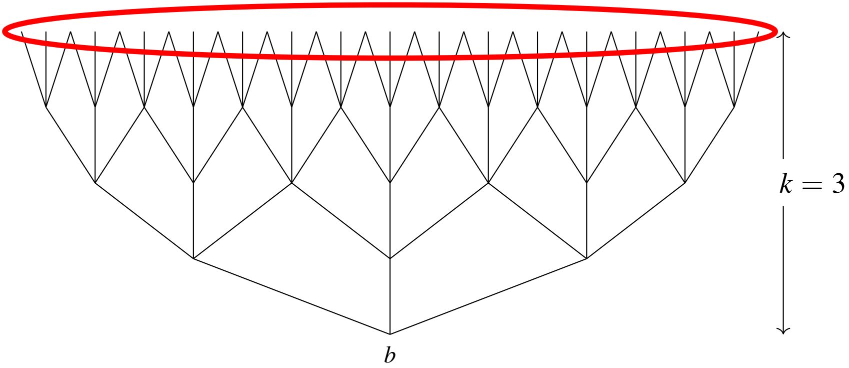
Ideals of $\mathbb{T}$

Define an order on the nodes of $\mathbb{T}$ by

$$a \leq b \overset{\text{def}}{\iff} \mathbb{I}_a \subseteq \mathbb{I}_b,$$

We have

$$a \leq b \iff \exists k \in \mathbb{N} \left(|b| + k = |a| \;\&\; |2^k N(b) - N(a)| \leq 2^k - 1 \right).$$



Ideals of $\mathbb{T}$ (examples)

An **ideal** of $\mathbb{T}$ is a subset $\mathcal{C} \subseteq \mathbb{T}$ such that

1. $\mathcal{C}$ is inhabited,
2. $a \leq b \;\&\; a \in \mathcal{C} \to b \in \mathcal{C}$,
3. $a, b \in \mathcal{C} \to \exists c \in \mathcal{C}\,(c \in a \downarrow b)$,
4. $a \in \mathcal{C} \to \exists i \in \{0, 1, 2\}\,(a * \langle i \rangle \in \mathcal{C})$,

where $a \downarrow b \overset{\text{def}}{=} \{c \in \mathbb{T} \mid c \leq a \;\&\; c \leq b\}$.

Ideals of $\mathbb{T}$ (examples)

An **ideal** of $\mathbb{T}$ is a subset $\mathcal{C} \subseteq \mathbb{T}$ such that

1. $\mathcal{C}$ is inhabited,
2. $a \leq b \ \& \ a \in \mathcal{C} \to b \in \mathcal{C}$,
3. $a, b \in \mathcal{C} \to \exists c \in \mathcal{C}\, (c \in a \downarrow b)$,
4. $a \in \mathcal{C} \to \exists i \in \{0, 1, 2\}\, (a * \langle i \rangle \in \mathcal{C})$,

where $a \downarrow b \stackrel{\text{def}}{=} \{c \in \mathbb{T} \mid c \leq a \ \& \ c \leq b\}$.

Lemma

For any path α in $\mathbb{T}$, the set

$$\mathcal{C}_\alpha \stackrel{\text{def}}{=} \{a \in \mathbb{T} \mid \exists n \in \mathbb{N}\, (\overline{\alpha}n \leq a)\}$$

is an ideal of $\mathbb{T}$.

Regular ideals

For $a, b \in \mathbb{T}$, define

$$a \ll b \overset{\text{def}}{\iff} \mathbb{I}_a \subsetneq \mathbb{I}_b.$$

In this case, a is said to be **way-below** b. We have

$$a \ll b \iff \exists k \in \mathbb{N}^+ \left(|b| + k = |a| \ \& \ |2^k N(b) - N(a)| < 2^k - 1 \right).$$

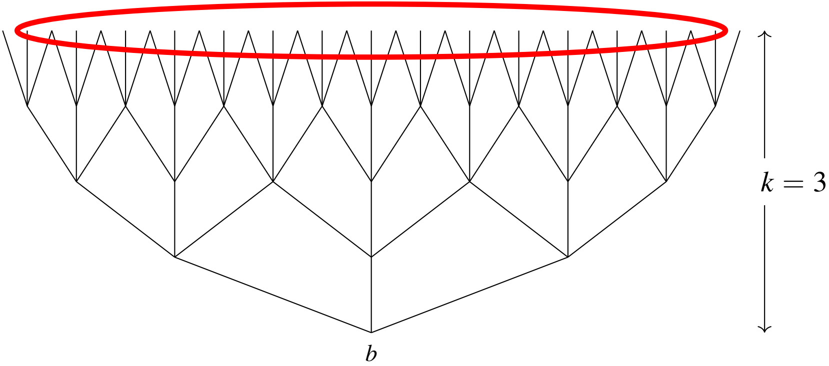
Regular ideals

For $a, b \in \mathbb{T}$, define

$$a \ll b \overset{\text{def}}{\iff} \mathbb{I}_a \subsetneq \mathbb{I}_b.$$

In this case, a is said to be **way-below** b. We have

$$a \ll b \iff \exists k \in \mathbb{N}^+ \left(|b| + k = |a| \ \& \ |2^k N(b) - N(a)| < 2^k - 1 \right).$$



Regular ideals

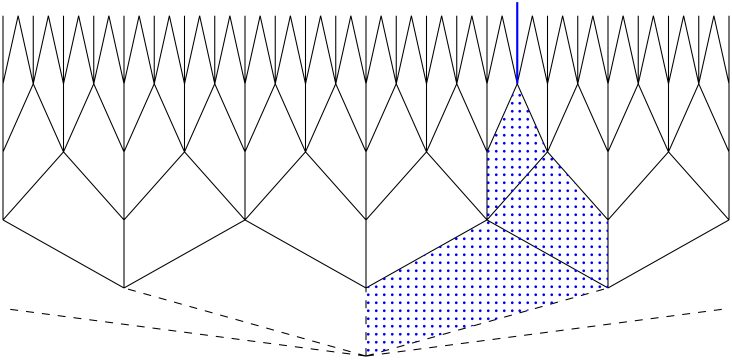
An ideal $\mathcal{C}$ of $\mathbb{T}$ is said to be **regular** if

$$a \in \mathcal{C} \to \exists b \in \mathcal{C} \left(b \ll a \right).$$

Regular ideals

An ideal $\mathcal{C}$ of $\mathbb{T}$ is said to be **regular** if

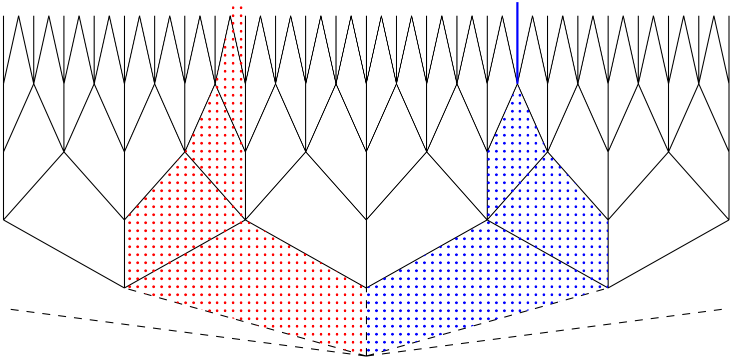
$$a \in \mathcal{C} \rightarrow \exists b \in \mathcal{C}\,(b \ll a)\,.$$



Regular ideals

An ideal $\mathcal{C}$ of $\mathbb{T}$ is said to be **regular** if

$$a \in \mathcal{C} \rightarrow \exists b \in \mathcal{C}\, (b \ll a)\,.$$



Regular ideals

Lemma (Minimality)

Let $\mathcal{C}_0, \mathcal{D}$ be ideals of $\mathbb{T}$ where $\mathcal{D}$ is regular. Then

$$\mathcal{C} \subseteq \mathcal{D} \implies \mathcal{D} \subseteq \mathcal{C}.$$

Proposition

For any ideal $\mathcal{C}$ of $\mathbb{T}$, the subset

$$\widetilde{\mathcal{C}} \stackrel{\text{def}}{=} \{a \in \mathbb{T} \mid \exists b \in \mathcal{C}\, (b \ll a)\}$$

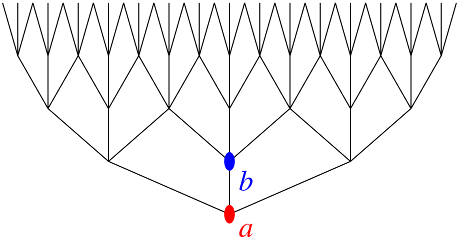
is a unique regular ideal such that $\widetilde{\mathcal{C}} \subseteq \mathcal{C}$.

Proposition

For any ideal $\mathcal{C}$ of $\mathbb{T}$, the subset $\widetilde{\mathcal{C}} \stackrel{\text{def}}{=} \{a \in \mathbb{T} \mid \exists b \in \mathcal{C}\,(b \ll a)\}$ is a unique regular ideal such that $\widetilde{\mathcal{C}} \subseteq \mathcal{C}$.

Proof.

(Regularity) Let $a \in \widetilde{\mathcal{C}}$. There exists $b \in \mathcal{C}$ such that $b \ll a$.

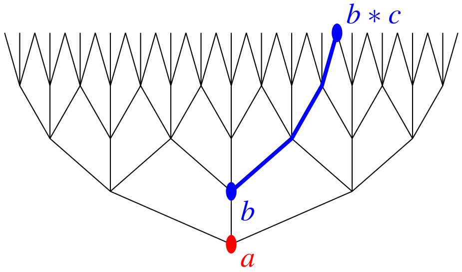


Proposition

For any ideal $\mathcal{C}$ of $\mathbb{T}$, the subset $\widetilde{\mathcal{C}} \overset{\text{def}}{=} \{a \in \mathbb{T} \mid \exists b \in \mathcal{C}\,(b \ll a)\}$ is a unique regular ideal such that $\widetilde{\mathcal{C}} \subseteq \mathcal{C}$.

Proof.

(Regularity) Let $a \in \widetilde{\mathcal{C}}$. There exists $b \in \mathcal{C}$ such that $b \ll a$. Since $\mathcal{C}$ is an ideal, there exists $c \in \{0, 1, 2\}^3$ such that $b * c \in \mathcal{C}$.

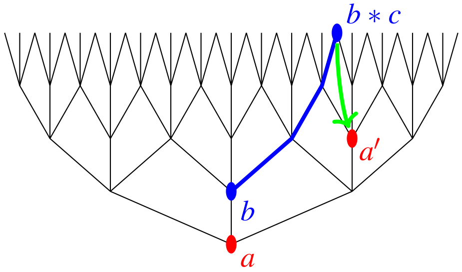


Proposition

For any ideal $\mathcal{C}$ of $\mathbb{T}$, the subset $\widetilde{\mathcal{C}} \stackrel{\text{def}}{=} \{a \in \mathbb{T} \mid \exists b \in \mathcal{C}\,(b \ll a)\}$ is a unique regular ideal such that $\widetilde{\mathcal{C}} \subseteq \mathcal{C}$.

Proof.

(Regularity) Let $a \in \widetilde{\mathcal{C}}$. There exists $b \in \mathcal{C}$ such that $b \ll a$. Since $\mathcal{C}$ is an ideal, there exists $c \in \{0, 1, 2\}^3$ such that $b * c \in \mathcal{C}$. Then, there exists $a' \ll a$ such that $b * c \ll a'$. Hence $a' \in \widetilde{\mathcal{C}}$.

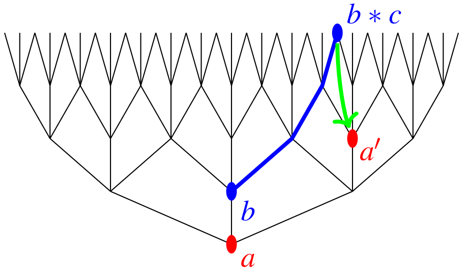


Proposition

For any ideal $\mathcal{C}$ of $\mathbb{T}$, the subset $\widetilde{\mathcal{C}} \stackrel{\text{def}}{=} \{a \in \mathbb{T} \mid \exists b \in \mathcal{C}\,(b \ll a)\}$ is a unique regular ideal such that $\widetilde{\mathcal{C}} \subseteq \mathcal{C}$.

Proof.

(Regularity) Let $a \in \widetilde{\mathcal{C}}$. There exists $b \in \mathcal{C}$ such that $b \ll a$. Since $\mathcal{C}$ is an ideal, there exists $c \in \{0,1,2\}^3$ such that $b * c \in \mathcal{C}$. Then, there exists $a' \ll a$ such that $b * c \ll a'$. Hence $a' \in \widetilde{\mathcal{C}}$.



(Uniqueness) Let $\mathcal{D}$ be a regular ideal such that $\mathcal{D} \subseteq \mathcal{C}$. Then, $\mathcal{D} = \widetilde{\mathcal{D}} \subseteq \widetilde{\mathcal{C}}$. Then, $\mathcal{D} = \widetilde{\mathcal{C}}$ by minimality. □

Regular ideals

Recall that we have the mapping

$$\rho_{\mathbb{R}} : \mathbb{T}_{\mathbb{R}} \to \mathbb{T}_{\mathbb{R}},$$

which coverts a path in $\mathbb{T}$ to a "*good*" one.

Lemma

For any path α in $\mathbb{T}$, $\mathcal{C}_{\rho_{\mathbb{R}}(\alpha)}$ is a regular ideal such that $\mathcal{C}_{\rho_{\mathbb{R}}(a)} \subseteq \mathcal{C}_{\alpha}$.

Proof.

- Since $\rho_{\mathbb{R}}(\alpha)$ does not contain $\langle 0,0,0\rangle$ or $\langle 2,2,2\rangle$, we have $\overline{\rho_{\mathbb{R}}(\alpha)}(n+2) \ll \overline{\rho_{\mathbb{R}}(\alpha)}n$.
- $\mathcal{C}_{\rho_{\mathbb{R}}(a)} \subseteq \mathcal{C}_{\alpha}$ because $|N(\overline{\rho_{\mathbb{R}}(\alpha)}n) - N(\overline{\alpha}n)| \leq 1$. $\square$

Corollary

For $\alpha \in \mathbb{T}_{\mathbb{R}}$, we have $\widetilde{\mathcal{C}_{\alpha}} = \mathcal{C}_{\rho_{\mathbb{R}}(\alpha)}$.

Regular ideals

Recall the quotient map $\Phi_{\mathbb{R}} : \mathbb{T}_{\mathbb{R}} \to \mathbb{R}$ given by

$$\Phi_{\mathbb{R}}(\alpha) \stackrel{\text{def}}{=} \langle 2^{(n+1)} N(\overline{\alpha}(n+1)) \rangle_{n \in \mathbb{N}}.$$

Lemma

$\Phi_{\mathbb{R}}(\alpha) \simeq \Phi_{\mathbb{R}}(\beta) \iff \forall n \in \mathbb{N} \left(|N(\overline{\alpha}(n+1) - \overline{\beta}(n+1)| \leq 2 \right).$

Proposition

$\Phi_{\mathbb{R}}(\alpha) \simeq \Phi_{\mathbb{R}}(\beta) \iff \mathcal{C}_{\rho_{\mathbb{R}}(\alpha)} = \mathcal{C}_{\rho_{\mathbb{R}}(\beta)}.$

Proof.

($\Rightarrow$) Suppose that $\Phi_{\mathbb{R}}(\alpha) \simeq \Phi_{\mathbb{R}}(\beta)$. It suffice to show $\widetilde{\mathcal{C}_\alpha} \subseteq \widetilde{\mathcal{C}_\beta}$. Let $a \in \widetilde{\mathcal{C}}$. Since $\widetilde{\mathcal{C}_\alpha}$ is regular, there exist $n \in \mathbb{N}$ and $b \in \mathbb{T}$ such that $\overline{\alpha}n \ll b \ll a$. Since $|N(\overline{\alpha}n) - N(\overline{\beta}n)| \leq 2$ by Lemma, we must have $\overline{\beta}n \ll a$. Thus $a \in \widetilde{\mathcal{C}_\beta}$, and hence $\widetilde{\mathcal{C}_\alpha} \subseteq \widetilde{\mathcal{C}_\beta}$. Therefore $\widetilde{\mathcal{C}_\alpha} = \widetilde{\mathcal{C}_\beta}$.

($\Leftarrow$) Follows from the filtering property of $\mathcal{C}_{\rho_{\mathbb{R}}(\alpha)}$ $(= \mathcal{C}_{\rho_{\mathbb{R}}(\beta)})$. $\square$

Lemma (Dependent Choice (DC))

For any regular ideal $\mathcal{C}$, there exists a path $\alpha \in \mathbb{T}_{\mathbb{R}}$ such that $\mathcal{C} = \mathcal{C}_{\rho_{\mathbb{R}}(\alpha)}$.

Proof.

$\mathcal{C}$ is inhabited and every $a \in \mathcal{C}$ has an extension in $\mathcal{C}$. By DC, we can take a path $\alpha \in \mathbb{T}_{\mathbb{R}}$ such that $\forall n \in \mathbb{N}\,(\overline{\alpha}n \in \mathcal{C})$. Then, $\mathcal{C}_\alpha \subseteq \mathcal{C}$ and so $\widetilde{\mathcal{C}_\alpha} \subseteq \widetilde{\mathcal{C}} = \mathcal{C}$. Thus $\mathcal{C}_{\rho_{\mathbb{R}}(\alpha)} = \widetilde{\mathcal{C}_\alpha} = \mathcal{C}$ by the minimality of $\mathcal{C}$. $\square$

Lemma (Dependent Choice (DC))

For any regular ideal $\mathcal{C}$, there exists a path $\alpha \in \mathbb{T}_{\mathbb{R}}$ such that $\mathcal{C} = \mathcal{C}_{\rho_{\mathbb{R}}(\alpha)}$.

Proof.

$\mathcal{C}$ is inhabited and every $a \in \mathcal{C}$ has an extension in $\mathcal{C}$. By DC, we can take a path $\alpha \in \mathbb{T}_{\mathbb{R}}$ such that $\forall n \in \mathbb{N}\,(\overline{\alpha}n \in \mathcal{C})$. Then, $\mathcal{C}_\alpha \subseteq \mathcal{C}$ and so $\widetilde{\mathcal{C}_\alpha} \subseteq \widetilde{\mathcal{C}} = \mathcal{C}$. Thus $\mathcal{C}_{\rho_{\mathbb{R}}(\alpha)} = \widetilde{\mathcal{C}_\alpha} = \mathcal{C}$ by the minimality of $\mathcal{C}$. $\square$

Theorem

There exists a bijection between $\mathbb{R}$ and regular ideals of $\mathbb{T}$.

Formal topologies

- A **formal topology** is a triple $\mathcal{S} = (S, \leq, \lhd)$ where $(S, \leq)$ is a preorder and $\lhd$ is a relation from S to $\mathcal{P}(S)$, called a **cover** on $(S, \leq)$, such that
  1. $a \in U \implies a \lhd U$,
  2. $a \leq b \lhd U \implies a \lhd U$,
  3. $a \lhd U \ \&\ U \lhd V \implies a \lhd V$,
  4. $a \lhd U \ \&\ a \lhd V \implies a \lhd U \downarrow V$,

  where

$$U \lhd V \overset{\text{def}}{\iff} \forall a \in U \, (a \lhd V) \, ,$$

$$U \downarrow V \overset{\text{def}}{=} \{c \in S \mid \exists a \in U \exists b \in V \, (c \in a \downarrow b)\} \, .$$

Formal topologies

- A **formal topology** is a triple $\mathcal{S} = (S, \leq, \lhd)$ where $(S, \leq)$ is a preorder and $\lhd$ is a relation from S to $\mathcal{P}(S)$, called a **cover** on $(S, \leq)$, such that
  1. $a \in U \implies a \lhd U$,
  2. $a \leq b \lhd U \implies a \lhd U$,
  3. $a \lhd U \;\&\; U \lhd V \implies a \lhd V$,
  4. $a \lhd U \;\&\; a \lhd V \implies a \lhd U \downarrow V$,

  where
  $$U \lhd V \overset{\text{def}}{\iff} \forall a \in U \, (a \lhd V)\,,$$
  $$U \downarrow V \overset{\text{def}}{=} \{c \in S \mid \exists a \in U \exists b \in V \, (c \in a \downarrow b)\}\,.$$

- A subset $\alpha \subseteq S$ is a **point** of $\mathcal{S}$ if
  1. α is inhabited,
  2. $a \leq b \;\&\; a \in \alpha \implies b \in \alpha$,
  3. $a, b \in \alpha \implies \exists c \in \alpha \, (c \in a \downarrow b)$,
  4. $a \in \alpha \;\&\; a \lhd U \implies \exists c \in U \, (c \in \alpha)$.

  The collection of points of $\mathcal{S}$ is denoted by $\mathcal{P}t(\mathcal{S})$.

Let $\lhd_f$ be a relation between $\mathbb{T}$ and the finite subsets of $\mathbb{T}$ defined inductively:

$$\frac{a \in A}{a \lhd_f A}, \quad \frac{a \leq b \lhd_f A}{a \lhd_f A}, \quad \frac{a * \langle 0 \rangle \lhd_f A \quad a * \langle 1 \rangle \lhd_f A \quad a * \langle 2 \rangle \lhd_f A}{a \lhd_f A}.$$

Fan Theorem (formal version)

For any $a \in \mathbb{T}$ and $A \in \mathrm{Fin}(\mathbb{T})$, it holds that

$$a \lhd_f A \iff \exists k \in \mathbb{N} \forall a \in \{0,1,2\}^k \, \exists c \in A \, (a * b \leq c) \,.$$

That is, $a \lhd_f A \iff$ "A is a uniform cover of a".

Proof.

($\Rightarrow$) By induction on the derivation of $a \lhd_f A$.

($\Leftarrow$) By induction on $k \in \mathbb{N}$. $\square$

Formal topology of ideals

Define a relation $\lhd_{\mathbb{T}} \subseteq \mathbb{T} \times \mathcal{P}(\mathbb{T})$ by

$$a \lhd_{\mathbb{T}} U \overset{\text{def}}{\iff} \exists A \in \mathrm{Fin}(U)\,(a \lhd_f A)\,.$$

Proposition

1. *The triple $\mathcal{S}_{\mathbb{T}} = (\mathbb{T}, \leq, \lhd_{\mathbb{T}})$ is a formal topology.*
2. *The relation $\lhd_{\mathbb{T}}$ is the least cover on $(\mathbb{T}, \leq)$ such that*

$$a \lhd_{\mathbb{T}} \{a * \langle 0 \rangle, a * \langle 1 \rangle, a * \langle 2 \rangle\}\,.$$

3. *The points of $\mathcal{S}_{\mathbb{T}}$ are exactly the ideals of $\mathbb{T}$.*

Proof.
3. Every point of $\mathcal{S}_{\mathbb{T}}$ is an ideal by 2. Conversely, if $\mathcal{C}$ is an ideal of $\mathbb{T}$, then it is a model of the axiom $a \lhd_{\mathbb{T}} \{a * \langle 0 \rangle, a * \langle 1 \rangle, a * \langle 2 \rangle\}$. Hence it is a model of $\lhd_f$. Hence it is a point of $\mathcal{S}_{\mathbb{R}}$. $\square$

Formal topology of real numbers

Define a relation $\lhd_{\mathbb{R}} \subseteq \mathbb{T} \times \mathcal{P}(\mathbb{T})$ by

$$a \lhd_{\mathbb{R}} U \overset{\text{def}}{\iff} \forall b \ll a \exists A \in \mathrm{Fin}(\mathbb{T}) \left(b \lhd_f A \;\&\; A \ll_L U \right),$$

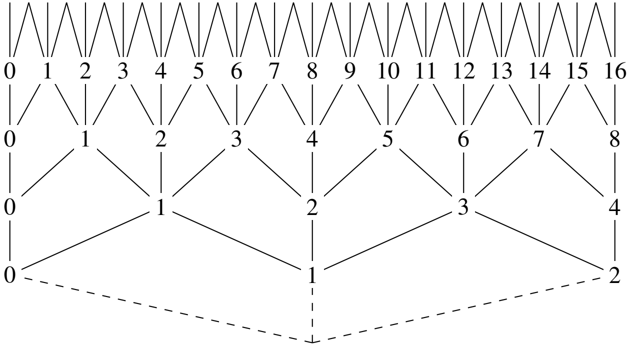
where $A \ll_L U \overset{\text{def}}{\iff} \forall a \in A \exists b \in U \left(a \ll b \right)$.

Proposition

1. *The triple $\mathcal{S}_{\mathbb{R}} = (\mathbb{T}, \leq, \lhd_{\mathbb{R}})$ is a formal topology.*
2. *The relation $\lhd_{\mathbb{R}}$ is the least cover on $(\mathbb{T}, \leq)$ such that*
   - $a \lhd_{\mathbb{R}} \{a * \langle 0 \rangle, a * \langle 1 \rangle, a * \langle 2 \rangle\}$,
   - $a \lhd_{\mathbb{R}} \{b \in \mathbb{T} \mid b \ll a\}$.
3. *The points of $\mathcal{S}_{\mathbb{R}}$ are exactly the regular ideals of $\mathbb{T}$.*

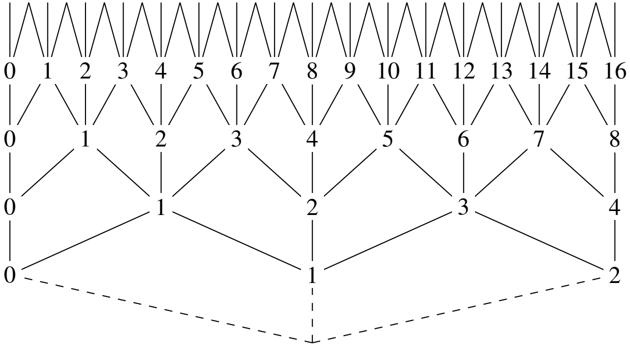
Formal topology of the unit interval

Consider the tree $\mathbb{T}_{[0,1]} \overset{\text{def}}{=} \{a \in \mathbb{T} \mid 0 \leq N(a) \leq 2^{|a|}\}$.



Formal topology of the unit interval

Consider the tree $\mathbb{T}_{[0,1]} \overset{\text{def}}{=} \left\{a \in \mathbb{T} \mid 0 \leq N(a) \leq 2^{|a|}\right\}$.



Define a relation $\lhd_{[0,1]} \subseteq \mathbb{T}_{[0,1]} \times \mathcal{P}(\mathbb{T}_{[0,1]})$ by

$$a \lhd_{[0,1]} U \overset{\text{def}}{\iff} \forall b \ll a \exists A \in \mathrm{Fin}(\mathbb{T}_{[0,1]}) \, (b \lhd_f A \;\&\; A \ll_L U)\,,$$

where the relations $\leq$, $\ll$ and $\lhd_f$ are restricted to $\mathbb{T}_{[0,1]}$.

Formal topology of the unit interval

The triple $\mathcal{S}_{[0,1]} = (\mathbb{T}_{[0,1]}, \leq, \lhd_{[0,1]})$ is a formal topology whose model's are regular ideals corresponding to the real numbers in $[0, 1]$.

Heine–Borel theorem

$$\mathbb{T}_{[0,1]} \lhd_{[0,1]} U \implies \exists A \in \mathrm{Fin}(U) \left(\mathbb{T}_{[0,1]} \lhd_f A\right).$$

Formal topology of the unit interval

The triple $\mathcal{S}_{[0,1]} = (\mathbb{T}_{[0,1]}, \leq, \lhd_{[0,1]})$ is a formal topology whose model's are regular ideals corresponding to the real numbers in $[0,1]$.

Heine–Borel theorem

$$\mathbb{T}_{[0,1]} \lhd_{[0,1]} U \implies \exists A \in \mathrm{Fin}(U) \left(\mathbb{T}_{[0,1]} \lhd_f A\right).$$

Proof.

Suppose $\mathbb{T}_{[0,1]} \lhd_{[0,1]} U$. This is equivalent to $\{\langle 0\rangle, \langle 1\rangle, \langle 2\rangle\} \lhd_{[0,1]} U$. Then

$$\{\langle 0\rangle, \langle 1\rangle, \langle 2\rangle\} \lhd_f \left(\{\langle 0\rangle, \langle 1\rangle, \langle 2\rangle\} * \{0,1,2\}^2\right) \cap \mathbb{T}_{[0,1]}.$$

Formal topology of the unit interval

The triple $\mathcal{S}_{[0,1]} = (\mathbb{T}_{[0,1]}, \leq, \lhd_{[0,1]})$ is a formal topology whose model's are regular ideals corresponding to the real numbers in $[0, 1]$.

Heine–Borel theorem

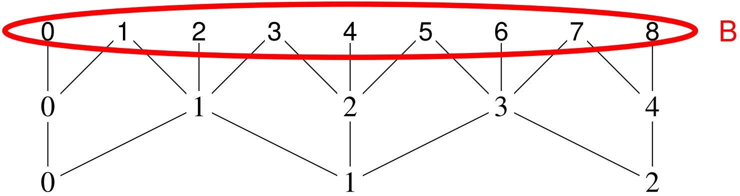
$$\mathbb{T}_{[0,1]} \lhd_{[0,1]} U \implies \exists A \in \mathrm{Fin}(U) \left(\mathbb{T}_{[0,1]} \lhd_f A\right).$$

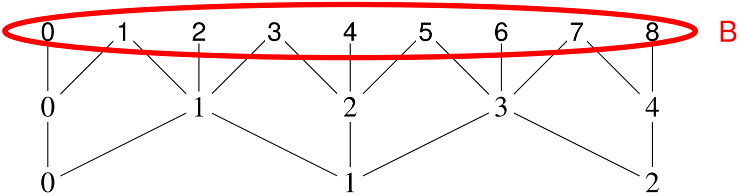
Proof.

Suppose $\mathbb{T}_{[0,1]} \lhd_{[0,1]} U$. This is equivalent to $\{\langle 0\rangle, \langle 1\rangle, \langle 2\rangle\} \lhd_{[0,1]} U$. Then

$$\{\langle 0\rangle, \langle 1\rangle, \langle 2\rangle\} \lhd_f \left(\{\langle 0\rangle, \langle 1\rangle, \langle 2\rangle\} * \{0,1,2\}^2\right) \cap \mathbb{T}_{[0,1]}.$$

Put $B = \left(\{\langle 0\rangle, \langle 1\rangle, \langle 2\rangle\} * \{0,1,2\}^2\right) \cap \mathbb{T}_{[0,1]}$.



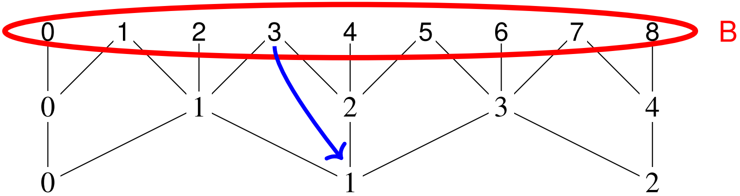


For each $a \in B$, there exists $i \in \{0, 1, 2\}$ such that $a \ll \langle i \rangle$. Since $\langle i \rangle \lhd_{[0,1]} U$, there exists $A_a \in \mathrm{Fin}(\mathbb{T}_{[0,1]})$ such that $a \lhd_f A_a \ll_L U$. Then,

$$\{\langle 0 \rangle, \langle 1 \rangle, \langle 2 \rangle\} \lhd_f B \lhd_f \bigcup_{a \in B} A_a \ll_L U.$$

Since $\ll \subseteq \leq$, we can find $A \subseteq U$ such that $\mathbb{T}_{[0,1]} \lhd_f A$. $\square$

Formal topology of the unit interval

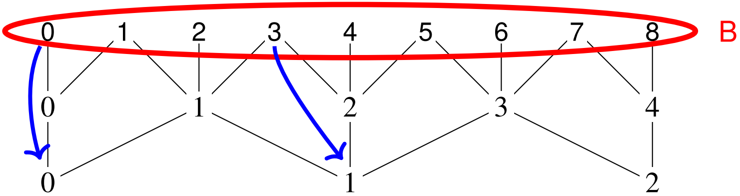


For each $a \in B$, there exists $i \in \{0, 1, 2\}$ such that $a \ll \langle i \rangle$. Since $\langle i \rangle \lhd_{[0,1]} U$, there exists $A_a \in \mathrm{Fin}(\mathbb{T}_{[0,1]})$ such that $a \lhd_f A_a \ll_L U$. Then,

$$\{\langle 0 \rangle, \langle 1 \rangle, \langle 2 \rangle\} \lhd_f B \lhd_f \bigcup_{a \in B} A_a \ll_L U.$$

Since $\ll \subseteq \leq$, we can find $A \subseteq U$ such that $\mathbb{T}_{[0,1]} \lhd_f A$. $\square$

Formal topology of the unit interval



For each $a \in B$, there exists $i \in \{0, 1, 2\}$ such that $a \ll \langle i \rangle$. Since $\langle i \rangle \lhd_{[0,1]} U$, there exists $A_a \in \mathrm{Fin}(\mathbb{T}_{[0,1]})$ such that $a \lhd_f A_a \ll_L U$. Then,

$$\{\langle 0 \rangle, \langle 1 \rangle, \langle 2 \rangle\} \lhd_f B \lhd_f \bigcup_{a \in B} A_a \ll_L U.$$

Since $\ll \subseteq \leq$, we can find $A \subseteq U$ such that $\mathbb{T}_{[0,1]} \lhd_f A$. $\square$

Continuous maps

Let $\mathcal{S} = (S, \leq, \lhd)$ and $\mathcal{S}' = (S', \leq', \lhd')$ be formal topologies.
A relation $r \subseteq S \times S'$ is a **continuous map** from $\mathcal{S}$ to $\mathcal{S}'$ if

1. $S \lhd r^- S'$,
2. $r^- a \downarrow r^- b \lhd r^-(a \downarrow' b)$,
3. $a \lhd' U \implies r^- a \lhd r^- U$.

Lemma
If $r\colon \mathcal{S} \to \mathcal{S}'$ is a continuous map between $\mathcal{S} = (S, \leq, \lhd)$ and $\mathcal{S}' = (S', \leq', \lhd')$, then the direct image operation

$$\alpha \mapsto r\alpha = \{b \in S' \mid \exists a \in \alpha\, (a\ r\ b)\}$$

sends every point $\alpha \in \mathcal{P}t(\mathcal{S})$ to $r\alpha \in \mathcal{P}t(\mathcal{S}')$.

Continuous maps

For $\alpha, \beta \in \mathcal{P}t(\mathcal{S}_{\mathbb{R}})$ (or $\mathcal{P}t(\mathcal{S}_{[0,1]})$) and $k \in \mathbb{N}$, define

$$|\alpha - \beta| \leq 2^{-k} \overset{\text{def}}{\iff} \exists a \in \alpha \cap \beta \, (|a| = k + 1) \, .$$

A function $f \colon \mathcal{P}t(\mathcal{S}_{[0,1]}) \to \mathcal{P}t(\mathcal{S}_{\mathbb{R}})$ is **uniformly continuous** if

$$\forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall \alpha, \beta \in \mathcal{P}t(\mathcal{S}_{[0,1]}) \left(|\alpha - \beta| \leq 2^{-n} \to |f(\alpha) - f(\beta)| \leq 2^{-k} \right) .$$

Continuous maps

For $\alpha, \beta \in \mathcal{P}t(\mathcal{S}_{\mathbb{R}})$ (or $\mathcal{P}t(\mathcal{S}_{[0,1]})$) and $k \in \mathbb{N}$, define

$$|\alpha - \beta| \leq 2^{-k} \stackrel{\text{def}}{\iff} \exists a \in \alpha \cap \beta \, (|a| = k + 1) \, .$$

A function $f \colon \mathcal{P}t(\mathcal{S}_{[0,1]}) \to \mathcal{P}t(\mathcal{S}_{\mathbb{R}})$ is **uniformly continuous** if

$$\forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall \alpha, \beta \in \mathcal{P}t(\mathcal{S}_{[0,1]}) \left(|\alpha - \beta| \leq 2^{-n} \to |f(\alpha) - f(\beta)| \leq 2^{-k} \right) .$$

Theorem
For every continuous map $r \colon \mathcal{S}_{[0,1]} \to \mathcal{S}_{\mathbb{R}}$, the direct image operation

$$\alpha \mapsto r\alpha \colon \mathcal{P}t(\mathcal{S}_{[0,1]}) \to \mathcal{P}t(\mathcal{S}_{\mathbb{R}})$$

is uniformly continuous.

Continuous maps

Definition

A relation $r \subseteq S \times S'$ is a **continuous map** from $\mathcal{S}$ to $\mathcal{S}'$ if

1. $S \lhd r^{-} S'$,
2. $r^{-} a \downarrow r^{-} b \lhd r^{-}(a \downarrow' b)$,
3. $a \lhd' U \implies r^{-} a \lhd r^{-} U$.

Compactness of $\lhd_{[0,1]}$

$$\mathbb{T}_{[0,1]} \lhd_{[0,1]} U \implies \exists A \in \mathrm{Fin}(U) \left(\mathbb{T}_{[0,1]} \lhd_{f} A\right).$$

Uniformity of $\lhd_{f}$

$$a \lhd_{f} A \iff \exists k \in \mathbb{N} \forall b \in \{0,1,2\}^{k} \exists c \in A \, (a * b \leq c).$$

Proof.

Fix $k \in \mathbb{N}$. Since $\mathbb{T} \lhd_{\mathbb{R}} \mathbb{Z} * \{0,1,2\}^k$ and r preserves the top and the cover, we have

$$\mathbb{T}_{[0,1]} \lhd_{[0,1]} r^{-} \left(\mathbb{Z} * \{0,1,2\}^k \right).$$

Proof.

Fix $k \in \mathbb{N}$. Since $\mathbb{T} \lhd_{\mathbb{R}} \mathbb{Z} * \{0, 1, 2\}^k$ and r preserves the top and the cover, we have

$$\mathbb{T}_{[0,1]} \lhd_{[0,1]} r^- \left(\mathbb{Z} * \{0, 1, 2\}^k\right).$$

Since $\mathcal{S}_{[0,1]}$ is compact, there exists a finite $A \subseteq r^- \left(\mathbb{Z} * \{0, 1, 2\}^k\right)$ such that $\{\langle 0\rangle, \langle 1\rangle, \langle 2\rangle\} \lhd_f A$. Since $\lhd_f$ is uniform, for each $i \in \{0, 1, 2\}$, there exists $n_i \in \mathbb{N}$ such that $\forall c \in \{0, 1, 2\}^{n_i} \, \exists a \in A \, (\langle i\rangle * c \leq a)$. Put

$$N_k = \max \{n_i \mid i \in \{0, 1, 2\}\}.$$

Proof.

Fix $k \in \mathbb{N}$. Since $\mathbb{T} \lhd_{\mathbb{R}} \mathbb{Z} * \{0,1,2\}^k$ and r preserves the top and the cover, we have

$$\mathbb{T}_{[0,1]} \lhd_{[0,1]} r^- \left(\mathbb{Z} * \{0,1,2\}^k\right).$$

Since $\mathcal{S}_{[0,1]}$ is compact, there exists a finite $A \subseteq r^- \left(\mathbb{Z} * \{0,1,2\}^k\right)$ such that $\{\langle 0\rangle, \langle 1\rangle, \langle 2\rangle\} \lhd_f A$. Since $\lhd_f$ is uniform, for each $i \in \{0,1,2\}$, there exists $n_i \in \mathbb{N}$ such that $\forall c \in \{0,1,2\}^{n_i} \, \exists a \in A \, (\langle i\rangle * c \leq a)$. Put

$$N_k = \max\{n_i \mid i \in \{0,1,2\}\}.$$

Let $\alpha, \beta \in \mathcal{P}t(\mathcal{S}_{[0,1]})$ such that $|\alpha - \beta| \leq 2^{-N_k}$. Then, there exists $c \in \{0,1,2\}^{N_k}$ and $i \in \{0,1,2\}$ such that $\langle i\rangle * c \in \alpha \cap \beta$. Thus, there exists $a \in A$ such that $\langle i\rangle * c \leq a$. Then, there exists $b \in \mathbb{Z} * \{0,1,2\}^k$ such that $a \; r \; b$. Since α and β are upward closed, we have $a \in \alpha \cap \beta$. Hence $b \in r\alpha \cap r\beta$, and thus $|r\alpha - r\beta| \leq 2^{-k}$. $\square$

Continuous maps

Theorem

If $f : \mathcal{P}t(\mathcal{S}_{[0,1]}) \to \mathcal{P}t(\mathcal{S}_{\mathbb{R}})$ is uniformly continuous, then the relation $r_f \subseteq \mathbb{T}_{[0,1]} \times \mathbb{T}$ defined by

$$a \; r_f \; b \overset{\text{def}}{\iff} \exists c \ll b \forall \alpha \in \mathcal{P}t(\mathcal{S}_{[0,1]}) \, (a \in \alpha \to c \in f(\alpha))$$

is a continuous map from $\mathcal{S}_{[0,1]}$ to $\mathcal{S}_{\mathbb{R}}$.

References

J. Cederquist and S. Negri.
A constructive proof of the Heine–Borel covering theorem for formal reals.
Lecture Notes in Comput. Sci., vol. 1158, pp 62–75, 1996.

T. Coquand, G. Sambin, J. Smith, and S. Valentini.
Inductively generated formal topologies.
Ann. Pure Appl. Logic, 124(1-3):71–106, 2003.

E. Palmgren.
Continuity on the real line and in formal spaces.
In L. Crosilla and P. Schuster, editors, *From Sets and Types to Topology and Analysis: Towards Practicable Foundations for Constructive Mathematics*, pages 165–175. Oxford University Press, 2005.