Symmetry in the Coplanarity Condition

We can rewrite the triple product using

$$t = \mathring{r}\mathring{d} \cdot \mathring{q}\mathring{\ell} = \mathring{r} \cdot \mathring{q}\mathring{\ell}\mathring{d}^* = \mathring{q}^*\mathring{r} \cdot \mathring{\ell}\mathring{d}^*. \tag{1}$$

Noting that $\mathring{\ell}^* = -\mathring{\ell}$ and $\mathring{r}^* = -\mathring{r}$, since $\mathring{r}$ and $\mathring{\ell}$ are quaternions with zero scalar parts, we obtain

$$\boxed{t = \mathring{r}\mathring{q} \cdot \mathring{d}\mathring{\ell}} \tag{2}$$

Now expand the dot-product for t in terms of the scalar and vector components of $\mathring{q} = (q, \mathbf{q})$ and $\mathring{d} = (d, \mathbf{d})$:

$$(\mathbf{d} \cdot \mathbf{r})\,(\mathbf{q} \cdot \boldsymbol{\ell}) + (\mathbf{q} \cdot \mathbf{r})\,(\mathbf{d} \cdot \boldsymbol{\ell}) + (dq - \mathbf{d} \cdot \mathbf{q})\,(\boldsymbol{\ell} \cdot \mathbf{r}) + d\,[\mathbf{r}\ \mathbf{q}\ \boldsymbol{\ell}] + q\,[\mathbf{r}\ \mathbf{d}\ \boldsymbol{\ell}]. \tag{3}$$

While

$$\mathring{s} = \sum_{i=1}^{n} w_i e_i\,(\mathring{r}_i\mathring{d}\mathring{\ell}_i^*) \quad \text{and} \quad \mathring{t} = \sum_{i=1}^{n} w_i e_i\,(\mathring{r}_i^*\mathring{q}\mathring{\ell}_i). \tag{4}$$

We also still have the three equations

$$\mathring{q} \cdot \delta\mathring{q} = 0, \quad \mathring{d} \cdot \delta\mathring{d} = 0, \quad \text{and} \quad \mathring{q} \cdot \delta\mathring{d} + \mathring{d} \cdot \delta\mathring{q} = 0, \tag{5}$$

all of which we can write in the matrix form

$$\begin{pmatrix} A & B & \mathring{q} & 0 & \mathring{d} \\ B^T & C & 0 & \mathring{d} & \mathring{q} \\ \mathring{q}^T & 0^T & 0 & 0 & 0 \\ 0^T & \mathring{d}^T & 0 & 0 & 0 \\ \mathring{d}^T & \mathring{q}^T & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta\mathring{d} \\ \delta\mathring{q} \\ \lambda \\ \mu \\ \nu \end{pmatrix} = - \begin{pmatrix} \mathring{s} \\ \mathring{t} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{6}$$

Note that the upper left 8×8 sub-matrix is the weighted sum of dyadic products

$$\sum_{i=1}^{n} w_i \vec{c}_i \vec{c}_i^{\,T}, \tag{7}$$

where the eight component vector $\vec{c}_i$ is given by

$$\vec{c}_i = \begin{pmatrix} \mathring{r}_i\mathring{d}\mathring{\ell}_i^* \\ \mathring{r}_i^*\mathring{q}\mathring{\ell}_i \end{pmatrix} = - \begin{pmatrix} \mathring{r}_i\mathring{q}\mathring{\ell}_i \\ \mathring{r}_i\mathring{d}\mathring{\ell}_i \end{pmatrix}. \tag{8}$$

We conclude that the number of solutions is equal to the number of ways of partitioning the set of variables, namely

$$\binom{n+m-2}{n-1} = \binom{n+m-2}{m-1} = \frac{(n+m-2)!}{(n-1)!\,(m-1)!} \tag{9}$$

This can be done by taking a small step $\delta\lambda$ in λ and solving for the increment $\delta\mathbf{x}$ in

$$\frac{d\mathbf{h}}{d\lambda}\,\delta\lambda + \frac{d\mathbf{h}}{d\mathbf{x}}\,\delta\mathbf{x} = 0, \tag{10}$$

where $J = (d\mathbf{h}/d\mathbf{x})$ is the Jacobian of $\mathbf{h}$ with respect to $\mathbf{x}$.