ECO 301

Game Theory — General Equilibrium Theory

Evaluation

Quizzes (2/3)	—	25%
Midsem	—	30%
Endsem	—	40%
Attendance	—	5%

Outline

Game Theory

1) Introduction
2) Game Structure

Time
- Simultaneous / Static
- Sequential / Dynamic

Info
- Complete Information
- Incomplete Information

asymm ← → symm

Complexity Increases → (SC, DC, SI, DI)

Gen. Eq^{m}. Theory

- Pareto Notions
- First theorem of Welfare Economics
- Second theorem of Welfare Economics
- Comparative Statics

Game Theory

Game

A game is a formal representation of a situation in which a number of agents interact in a setting of strategic interdependence.

Each agent's payoff depends on:
i) its own decision
ii) but also other agent's decisions
⇒ Think Strategically

Examples:
1) Class Project
2) Athletes & doping
3) Animal Kingdom
4) Nim Game

To describe a game, we require:
1) A list of players
2) Rules of the game:
who moves when — strategic / dynamic
who knows what — in/complete info
3) Strategies: Actions / decisions available to players
4) Outcome: For each possible combination of strategies chosen by different players.
5) Payoffs: For each possible outcome

Representation of Games

Normal / Strategic Form Representation
usually used to represent static games

Extensive Form Representation
usually used to represent dynamic games

No. of players

$I = \{1, 2, \dots, N\}$; $S_i \ \forall i \in I$; Set of strategies available to i

$S = \{s_1, s_2, \dots\}$; π_i, $s_i \in S_i$ Strategy chosen by i

Set of all possible combinations of strategies chosen by all players

Payoff

Required for Normal form

Matching Pennies

$I = \{1, 2\}$, $S_1 = \{H, T\} = S_2$,

$S = \{(H,H), (H,T), (T,H), (T,T)\}$

PayOff: $(1,-1)$, $(-1,1)$, $(-1,1)$, $(1,-1)$

P1 \ P2	H	T
H	(1, -1)	(-1, 1)
T	(-1, 1)	(1, -1)

⟶ P1 typically on rows & on the left-hand side in payoffs

Battle of Sexes

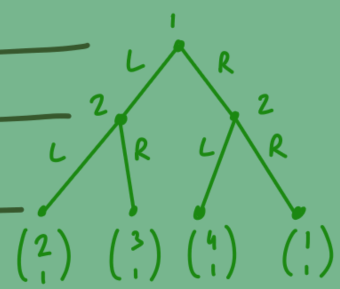
$I = \{B, G\}$

S { C : Watch a T20 match
  M : Watch a movie

1) B prefers T20
2) G prefers movie
3) want to spend evening together

B \ G	C	M
C	(3, 1)	(0, 0)
M	(0, 0)	(1, 3)

Extensive Form



Initial Decision Node ⟵

Regular Decision Nodes ⟵

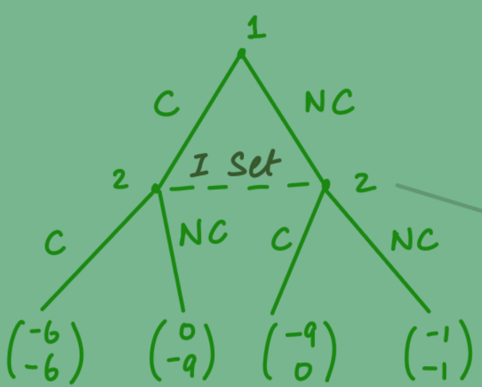
Terminal Nodes ⟵

Payoff of P1 written up ⟵

} Game Tree

Prisoner's Dilemma

P1 & P2, not sufficient evidence on them. Police asks them to confess at the same time, not in front of each other.



Not okay. Assumes in general that 2 had information of 1's decision. To represent otherwise, Information Sets are used.

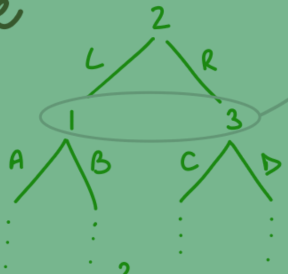
Dynamic Game ⟵

P1 \ P2	C	NC
C	(-6, -6)	(0, -9)
NC	(-9, 0)	(-1, -1)

# of months in jail

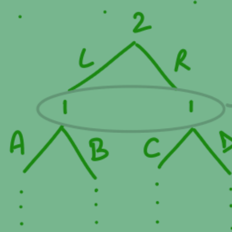
Information Set: An I-set is a collection of decision nodes for a player such that:

i) At each decision node of the I-set, the concerned player has the move.



Cannot form an I-set. I-set should contain nodes with the same decision maker.

ii) If the play of the game reaches a particular decision node, the player cannot distinguish between the nodes.

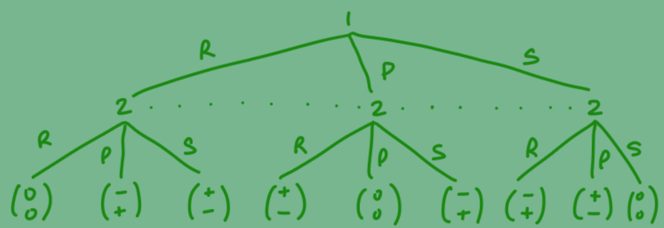


Cannot make an I-set because 1 has different decision sets. 1 can know which node it is by looking at the decision set.

Singleton I-Set: I-Set containing one decision node

Nonsingleton I-Set: I-Set containing more than one decision node.

Q) Rock-Paper-Scissors

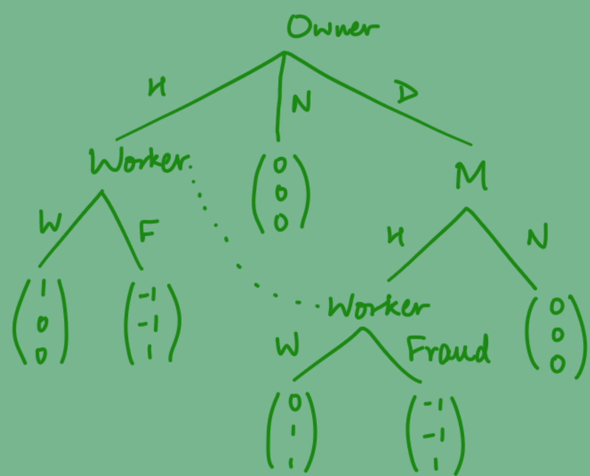


Q) O: Hire / Don't hire / delegate to manager

M: Hire / Don't hire

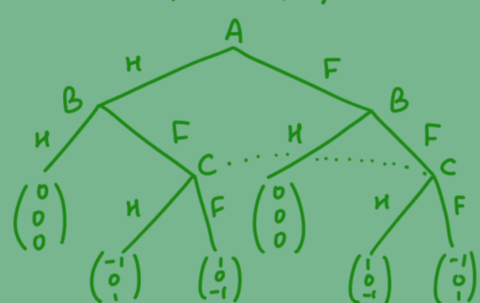
W: Work / Fraud

Draw in extensive form.



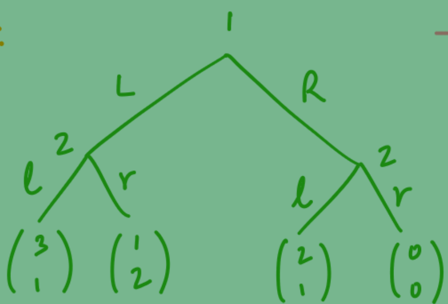
Q) B sees A, C sees B.

A decides to keep Hat on Head or on floor. B decides the same. C has to guess A.



☆ Strategy and Decision are different.
Strategy is a complete plan of action.
It specifies a feasible action for a player for each contingency for which she is asked to make a move.

Ex:



	ll	lr	rl	rr
1 → L	(3, 1)	(3, 1)	(1, 2)	(1, 2)
1 → R	(2, 1)	(0, 0)	(2, 1)	(0, 0)

Strategy of P2

(x, y)

Decision of 2 if 1 chooses L

Decision of 2 if 1 chooses R

Solution Concept

Dominated Strategy

A strategy $\bar{s}_i \in S_i$ is said to be dominated by a strategy $s_i^* \in S_i$ if

$$\pi_i(s_i^*, s_{\neg i}) \geq \pi_i(\bar{s}_i, s_{\neg i}) \ \forall s_{\neg i}$$

where $s_{\neg i}$ is the strategy set of all players other than i.

1 \ 2	C	NC
C	(−3, −3)	(0, −9)
NC	(−9, 0)	(−1, −1)

Prisoner's Dilemma

Iterated Elimination of Dominated Strategy (IEDS)

- Players are rational, they always try to maximize their benefit
- No player will choose a dominated strategy.

Ex:

2

1	C	NC
C	(-3, -3)	(0, -9)
NC	(-9, 0)	(-1, -1)

$\xrightarrow[\text{So, NC is eliminated for 2}]{\text{NC is dominated by C for 2}}$

2

1	C	NC
C	(-3, -3)	~~(0, -9)~~
NC	(-9, 0)	~~(-1, -1)~~

NC is dominated by C for 1
NC is eliminated

	C	NC
C	(-3, -3)	~~(0, -9)~~
NC	~~(-9, 0)~~	~~(-1, -1)~~

Solution of the game

(C, C) is the only remaining cell.

Ex:

	L	M	R
U	1, 0	1, 2	2, 1
B	0, 3	0, 1	3, 0

$\xrightarrow{\text{R dominated by M for 2}}$

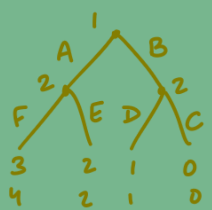
	L	M	R
U	1, 0	1, 2	~~2, 1~~
B	0, 3	0, 1	~~3, 0~~

U dominates B for 1

	L	M	R
U	~~1, 0~~	1, 2	~~2, 1~~
B	~~0, 3~~	~~0, 1~~	~~3, 0~~

L dominated by M
∴ (U, M) is the solution.

Q) EFG ⟶ NFG Conversion

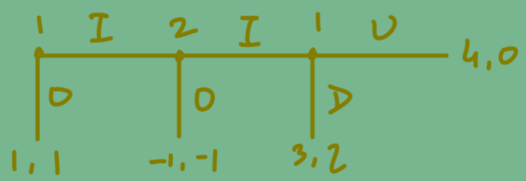


⟶

2

1	FD	FC	ED	EC
A	(3, 4)	(3, 4)	(2, 2)	(2, 2)
B	(1, 1)	(0, 0)	(1, 1)	(0, 0)

Q)

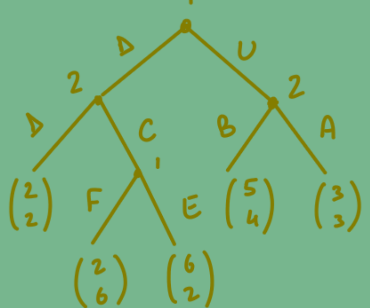


⟶

I ← 2 → O

1	I	O
IU	(4, 0)	(-1, -1)
ID	(3, 2)	(-1, -1)
OU	(1, 1)	(1, 1)
OD	(1, 1)	(1, 1)

Q)

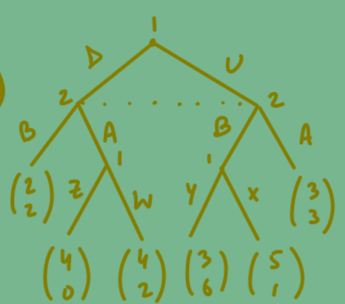


⟶

(2)

1	DB	DA	CB	CA
DF	(2, 2)	(2, 2)	(2, 6)	(2, 6)
DE	(2, 2)	(2, 2)	(6, 2)	(6, 2)
UF	(5, 4)	(3, 3)	(5, 4)	(3, 3)
UE	(5, 4)	(3, 3)	(5, 4)	(3, 3)

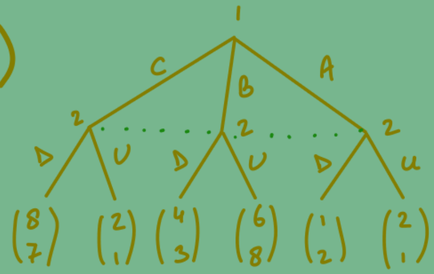
Q)



B ← 2 → A

1	B	A
DZY	(2, 2)	(4, 0)
DZX	(2, 2)	(4, 0)
DWY	(2, 2)	(4, 2)
DWX	(2, 2)	(4, 2)
UZY	(3, 6)	(3, 3)
UZX	(5, 1)	(3, 3)
UWY	(3, 6)	(3, 3)
UWX	(5, 1)	(3, 3)

Q)



		2: D	U
1	C	(8, 7)	(2, 1)
	B	(4, 3)	(6, 8)
	A	(1, 2)	(2, 1)

Note: Mind the notation for NFG. For example in Q2, the way it is written for P1: 1^{st} letter corresponds to the decision taken by P1 in the 1^{st} place it has to make a decision, 2^{nd} letter corresponds to the 2^{nd} place, etc. And similarly for P2 in Q1.

Ex:

		2: L	M	R
1	U	4, 3	~~5, 1~~	~~6, 2~~
	M	~~2, 1~~	~~8, 4~~	~~3, 6~~
	D	~~3, 0~~	~~9, 6~~	~~2, 8~~

(2) M_1 by U_1

(3) D_1 by U_1

(1) M_2 by R_2

(4) R_2 by L_2

∴ The solution of this game is (U, L).

☆

	L	R
U	1, 3	1, 1
D	1, 5	1, 7

↳ 2 has no dominated strategy and neither does 1

For a strategy to be dominating / dominated by another, at least one inequality needs to be strict. Thus, no strategy can be eliminated (in IEDS) here.

Extended Version of Rationality

All the players are rational, and all the players know that all the players are rational, and all the players know that all the players know that all the players are rational, and so on...

- A drawback of IEDS is the assumption that extended version of rationality is common knowledge.
- Another drawback is that IEDS fails to produce a solution when there is no dominated strategy.

Odd Couples : Felix (F) & Oscar (O)

3, 6, 9 hours a week.

Net payoff = Utility − Hours worked

≥12 hrs. : Apartment clean

9 hrs. : Apartment livable

≤9 hrs. : Apartment dirty

Utility : Livable = 2, F = {10, −10}, O = {5, −5}

(Clean, Dirty) (Clean, Dirty)

F \ O	3	6	9
3	−13, −8	−1, −4	7, −4
6	−4, −1	4, −1	4, −4
9	1, 2	1, −1	1, −4

(1) O_9 by O_6

(2) F_3 by F_6

(3) O_6 by O_3

(4) F_9 by F_6

(9, 3) solution

Election of Secretary General of U.N.

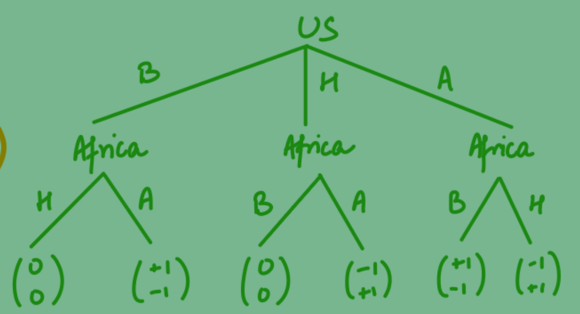
3 candidates : B from Egypt, H from Norway, A (Kofi Anan)

2 Voters : US − H > A > B,

Africa − B > A > H

Sequential Game : US votes first & gets to veto 1/3, Africa observes US & gets to veto 1/2 remaining.

(+1, 0, −1) utility as per preference.



NFG :

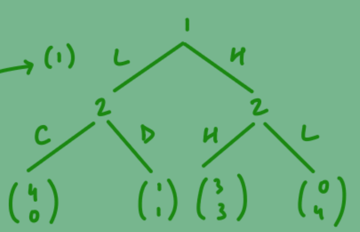
		HAA	HAB	HHA	HHB	BAA	BAB	BHA	BHB
	A	−1, +1	−1, +1	−1, +1	−1, +1	+1, −1	+1, −1	+1, −1	+1, −1
US	B	+1, −1	+1, −1	0, 0	0, 0	+1, −1	+1, −1	0, 0	0, 0
	H	−1, +1	0, 0	−1, +1	0, 0	−1, +1	0, 0	−1, +1	0, 0

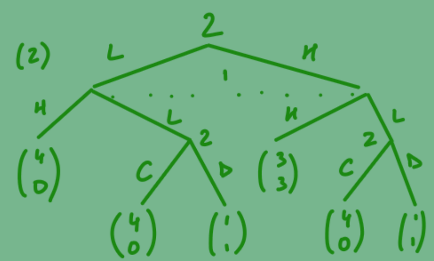
Africa

Weak Dominance for Africa → (B, H) is the solution

Ex:

1 \ 2	HC	HD	LC	LD
H	3, 3	3, 3	0, 4	0, 4
L	4, 0	1, 1	4, 0	1, 1





Nash Equilibrium

In an n-player game, the strategies $(s_1^*, s_2^*, \ldots, s_n^*)$ constitute an NE if for each player i, s_i^* is i's best response to the strategies specified for the remaining $n-1$ players which is $(s_1^* \ldots s_{i-1}^* s_{i+1}^* \ldots s_n^*)$, i.e.,

$$\pi_i(s_1^*, s_2^* \ldots s_i^* \ldots s_n^*) \geq \pi_i(s_1^* \ldots s_i \ldots s_n^*)$$

for every strategy $s_i \in S_i$, for each player i

Ex:

1 \ 2	C	NC
C	-3,-3	0,-9
NC	-9,0	-1,-1

NE is (C, C)

F \ O	3	6	9
3	-13,-8	-1,-4	7,-4
6	-4,-1	4,-1	4,-4
9	1,2	1,-1	1,-4

$(9,3)$ $(6,6)$ $(3,9)$

	L	C	R
T	0,4	4,0	5,3
M	4,0	0,4	5,3
B	3,5	3,5	6,6

$\hookrightarrow (B, R)$

★ A cell being IEDS $\Rightarrow$ cell is NE
A cell being NE does not necessarily imply it being IEDS.

NE for cts. strategy space

Best Response Function

$\hookrightarrow$ What is the best response for the i^{th} player given that the strategy of all other players is s_{-i}.

$\therefore$ Intersections of BRF's yields the NE's.

Ex: $0 \longmapsto 1$ Population of voters uniformly distributed along this line. Two political candidates deciding simultaneously about where to get campaign platforms. Voters vote for the candidate nearest to them.

$i = A, B$ ← s_i : position chosen by the i^{th} candidate.

$s_A = 0.2 \Rightarrow s_B = 0.2^+ \longrightarrow s_B = 0.2 + \epsilon$

$BRF^B \longrightarrow$

$$s_B = \begin{cases} s_A + \epsilon, & s_A < 0.5 \\ s_A, & s_A = 0.5 \\ s_A - \epsilon, & s_A > 0.5 \end{cases}$$

$NE \longrightarrow (0.5, 0.5)$

Ex: P1 & P2 bargain over ₹ 1. They simultaneously propose their bargains to the arbitrager; If $s_1 + s_2 \leq 1$, $\pi_1 = s_1$, $\pi_2 = s_2$, else $\pi_1 = 0 = \pi_2$

$BRF^2 = 1 - s_1$, $BRF^1 = 1 - s_2 \Rightarrow NE : (0.5, 0.5)$

Cournot Model of Duopoly

Players : P1 and P2

Simultaneously deciding how much quantity to produce.

$$P = a - bQ, \quad Q = q_i + q_j$$

$$\pi_i = Pq_i - C(q_i), \quad s_i = q_i, \quad S_i = [0, q]$$

$$= (a - b(q_i + q_j)) - C(q_i)$$

(q_1^*, q_2^*) is an NE if for each i,

$$\pi_i(q_i^*, q_j^*) \geq \pi_i(q_i, q_j^*) \quad \forall q_i \in S_i$$

So, $\max_{q_i \in S_i} \pi_i(q_i, q_j^*)$ and so,

$$BRF^i := \underset{q_i \in S_i}{\operatorname{argmax}} \pi_i(q_i, q_j^*)$$

$C(q_j) = cq_j$
$C(q_i) = cq_i$

$$= \underset{q_i \in S_i}{\operatorname{argmax}} [a - b(q_i + q_j)] q_i - cq_i$$

$$= \frac{a-c}{2b} - \frac{1}{2} q_j$$

Similarly, $BRF^j = (a-c)/2b - q_i/2$ } Called the Response function in Cournot model of duopoly

So, NE is $\left(\frac{a-c}{3b}, \frac{a-c}{3b} \right) \equiv (q_i^*, q_j^*)$

- If $c_1 \neq c_2$, NE: $\left(\frac{a-2c_1+c_2}{3b}, \frac{a-2c_2+c_1}{3b} \right)$
- If P1 does R & D and manages to $\downarrow c_1$, then $q_1^* \uparrow$ while $q_2^* \downarrow$ while also leading to $\pi_1 \uparrow$ & $\pi_2 \downarrow$
- If $2c_2 > a$, then P1 can $\downarrow c_1$ so much that $q_2 < 0$, thus driving P2 out of business.

Model of Cartel

Two firms operate as a cartel — coordinate their production decisions so as to maximize joint π

$$\pi = [a - b(q_1+q_2)]q_1 + [a-b(q_1+q_2)]q_2 - cq_1 - cq_2 = \pi_1 + \pi_2$$

$$q_1^c = q_2^c = \frac{a-c}{4b}, \quad \pi^c = \frac{(a-c)^2}{4b}$$

$$\pi_1 = \pi_2 = (a-c)^2/8b$$

- However, a firm can choose to deviate from π (and consequently $q_1^c + q_2^c = \frac{a-c}{2b}$) and focus instead on maximizing π_i leading to them $\uparrow q_i \rightarrow \pi_i \uparrow$

Thus, there is often some distrust between firms in a cartel.

Bertrend Model of Price Computation

- Both sellers producing a homogenious product — no difference in quality.

$$P = a - bQ \longleftrightarrow Q = \frac{a-P}{b}$$

$s_i = P_i, \quad s_i \in [0, \infty),$

$\pi_i = (p_i - c)\, q_i^d, \quad q_i^d :=$ residual demand faced by firm i

$$q_i^d = \begin{cases} 0 & , \ p_i > p_j \\ (a-p_i)/(2b) & , \ p_i = p_j \\ (a-p_i)/b & , \ p_i < p_j \end{cases}$$

$$BRF^i(p_j) = \begin{cases} P_M, & p_j > P_M \\ p_j - \epsilon, & c < p_j \le P_M \end{cases} \qquad \begin{matrix} \geq p_j, & p_j = c \\ \geq p_j, & p_j < c \end{matrix}$$

Monopoly price
If there was only one firm, what price would it have chosen.

Finding NE:

C1: $p_1 > p_2 > c$ × $p_1 \longrightarrow p_2 - \epsilon$

C2: $p_1 = p_2 > c$ × $p_1, p_2 \longrightarrow p_1 - \epsilon, \ p_2 - \epsilon$

C3: $p_1 > p_2 = c$ × $p_2 \longrightarrow p_1 - \epsilon$

C4: $p_1 = p_2 = c \longrightarrow NE \longrightarrow \pi_1 = 0 = \pi_2$

Bertrend's Paradox

Theoretically, 0 profit is expected at NE. But, this does not happen in real life. This could be because of difference in quality, co-operation between firms, etc.

Tragedy of Commons

Common Property Resource: Often depleted quickly. Often in NE.

Most people think that it is available to others as well, and they know it too.

Pure Strategies

- Actions / Strategies that players play for sure.
- What we have been working with so far.

Mixed Strategies

- Probability distribution over the pure strategies in the strategy space.

Ex: Pure : H & T
Mixed : $(p, 1-p)$

- Suppose player i has k pure strategies $S_i = \{s_{i_1}, \dots, s_{i_k}\}$

Mixed Strategy (σ) for player i is a probability distribution $(p_{i_1}, \dots, p_{i_k})$

p_{i_j} := probability with which i^{th} player plays s_{i_j}

$0 \le p_{i_j} \le 1 \quad \forall \quad 1 \le j \le k, \qquad \sum_{j=1}^{j=k} p_{i_j} = 1$

- Mixed Strategies help broaden the horizon for possibly finding a dominating strategy

Ex:

1 \ 2	L	R
U	2,0	-1,0
M	0,0	0,0
D	-1,0	2,0

$\sigma = (1/2, 0, 1/2)$

Expected pay-off for P1 if P2 plays L :

$1/2 \cdot 2 + 0 \cdot 0 + 1/2 \cdot -1 = 0.5$

if P2 plays R : 0.5

$\therefore$ The mixed strategy σ of P1 dominates the pure strategy M, though it is not dominated by any other pure strategy.

Ex:

1 \ 2	L	R
T	3, 0	0, 1
M	0, 0	3, 0
B	2, 1	2, 0

B does not dominate T, M purely; $BRF^1 \neq B$ ever; B never a part of NE

But, for the mixed strategy $\sigma_2 = (1/2, 1/2)$ of P2, we have,

$Exp(\pi_1(T)) = 1/2 \cdot 3 + 1/2 \cdot 0 = 1.5$

$Exp(\pi_1(M)) = 1.5$

$Exp(\pi_1(B)) = 2$ } BRF for 1 for the mixed strategy σ_2 of P2

* How to find mixed strategy NE?

1 \ 2	H	T
H	-1, 1	1, -1
T	1, -1	-1, 1

$p_1(H) :=$ prob that P1 plays head

$p_2(H) :=$ prob that P2 plays head

$BRF^1: p_1(p_2(H))$, $BRF^2: p_2(p_1(H))$

P1's expected payoff by playing H (given that P2 plays $\sigma_2 = (p_2, 1-p_2)$)

$= p_2 \cdot (-1) + (1-p_2) \cdot (1) = 1 - 2p_2$

P1's expected payoff by playing T

$= p_2 \cdot (1) + (1-p_2) \cdot (-1) = 2p_2 - 1$

$\Rightarrow p_1 = 1$ if $1 - 2p_2 > 2p_2 - 1 \longleftrightarrow 1/2 > p_2$

$p_1 = 0$ if $1 - 2p_2 < 2p_2 - 1 \longleftrightarrow 1/2 < p_2$

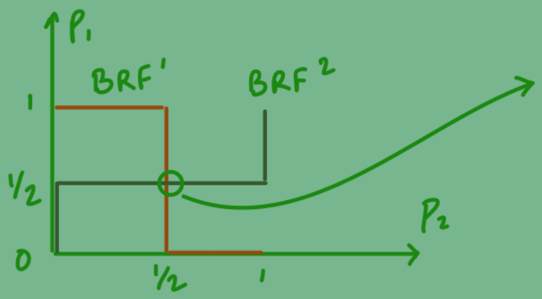
$p_1 =$ indifferent if $1 - 2p_2 = 2p_2 - 1 \longleftrightarrow 1/2 = p_2$

Similarly, for P2:

P2's expected payoff on playing $H = 2p_1 - 1$

P2's expected payoff on playing $T = 1 - 2p_1$

$$P_2(p_1(H)) = \{ 1,\ p_1 > 1/2 ;\quad [0,1],\ p_1 = 1/2 ;\quad 0,\ p_1 < 1/2 \}$$



NE is
$(p_1, p_2) = (1/2, 1/2)$

Ex: Battle of Sexes

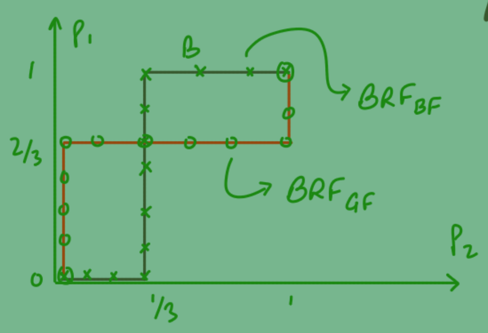
		GF: C	GF: M
BF	C	2, 1	0, 0
	M	0, 0	1, 2

$p_1(C) :=$ Prob that P1 plays C
$p_2(C) :=$ Prob that P2 plays C

BRF^1: $p_1(p_2(C))$, BRF^2: $p_2(p_1(C))$

P1's expected payoff on playing C given that P2 chooses $\sigma_2 = (p_2, 1-p_2)$: $2p_2$
P1's expected payoff on M given σ_2 : $1 - p_2$
P2's expected payoff on C given σ_1 : p_1
P2's expected payoff on C given σ_1 : $2(1-p_1)$

$\therefore$ P1 chooses C $\longleftrightarrow p_1 = 1 \longleftrightarrow 2p_2 > 1 - p_2 \longleftrightarrow p_2 > 1/3$
P1 chooses M $\longleftrightarrow p_1 = 0 \longleftrightarrow p_2 < 1/3 \longleftrightarrow 2p_2 < 1 - p_2$
P1 is indifferent $\longleftrightarrow p_1 \in [0,1] \longleftrightarrow p_2 = 1/3 \longleftrightarrow 2p_2 = 1 - p_2$



NE is (p_1, p_2) :
$(0, 0)$, $(2/3, 1/3)$, $(1, 1)$

Ex:

		P_2: S	P_2: NS
P_1	S	0, 0	1, −1
	NS	−1, 1	0, 0

IOC has 3 strategies :
1) Test 1 2) Test 2 3) Test randomly
If tested & caught, payoff = $-(1+b)$

In the simple version, NE is (S, S).
IEDS can also be used to arrive at the same.

1) Test 1

		2: S	2: NS
1	S	$-1-b, 1$	$-1-b, 1$
	NS	$-1, 1$	$0, 0$

2) Test 2

		2: S	2: NS
1	S	$1, -1-b$	$1, -1$
	NS	$1, -1-b$	$0, 0$

3) Test Randomly

		2: S	2: NS
1	S	$-b/2, -b/2$	$-b/2, 0$
	NS	$0, -b/2$	$0, 0$

$$
\begin{aligned}
(S, S)_1 &= p(\text{test } 1)\Pi_1(\text{test } 1) + p(\text{test } 2)\Pi_1(\text{test } 2) \\
&= \tfrac{1}{2} \cdot (-1-b) + \tfrac{1}{2}(1) \\
&= -b/2
\end{aligned}
$$

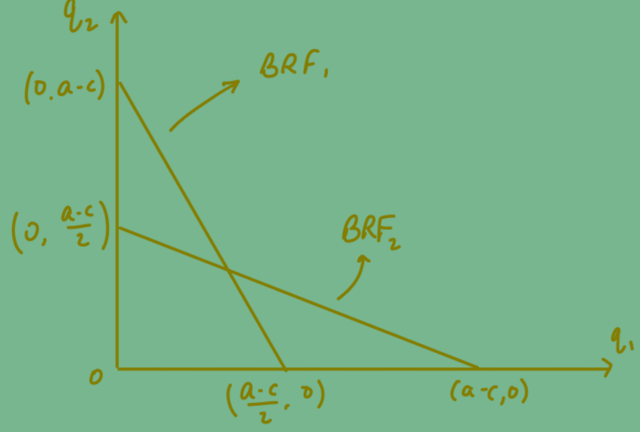
In this scenario, due to random testing of IOC, the (expected) payoffs change for both the players.
So, the NE is then (NS, NS) and not (S, S).

☆ For the next example, need to remember what it means for a strategy to be dominated by another in a continuous space:

$\Pi(q_{i_0}, q_j) \geq \Pi(q_i, q_j) \quad \forall q_j$

then q_i is dominated by the strategy q_{i_0}.

Ex: Cournot model converges towards NE in infinite steps using IEDS.



$P = a - Q$; $Q = q_1 + q_2$, $MC = c$

$q_m = \dfrac{a-c}{2} \leftarrow$ Monopoly quantity

- q_m strictly dominates any higher quantities $\longleftrightarrow \pi_i(q_m, q_j) > \pi_i(q_m + x, q_j)$

$\pi_i(q_m, q_j) = (a - c - q_m - q_j)(q_m)$

$\pi_i(q_m + x, q_j) = (a - c - q_m - x - q_j)(q_m + x)$

$= \pi_i(q_m, q_j) - (x + q_j)x < \pi_i(q_m, q_j)$

- So, in the first step, eliminate all quantities greater than q_m $\because$ they are dominated by q_m.
- Now, in the reduced space, $q_m/2 = (a-c)/4$ strictly dominates any lower quantity.

$\pi_i((a-c)/4, q_j)$, $q_j \leq q_m = (a-c)/2$

$= (a - c - q_j - (a-c)/4)((a-c)/4)$

$\pi_i((a-c)/4 - x, q_j)$

$= (3(a-c)/4 - q_j + x)((a-c)/4 - x)$

$= \pi_i((a-c)/4, q_j) - x^2 - x(a-c)/2 + xq_j$

$= \pi_i((a-c)/4, q_j) - x(x + (a-c)/2 - q_j)$

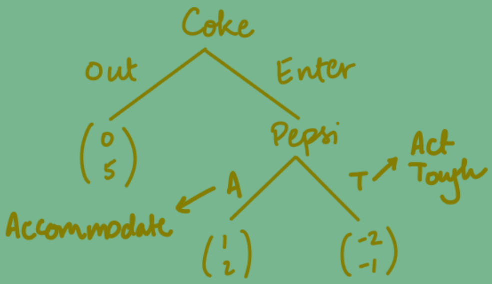
$< \pi_i((a-c)/4, q_j)$

$\therefore$ All quantities less than $q_m/2$ can be removed in Step II.

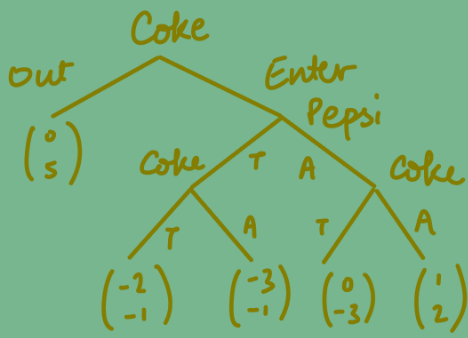
- The process can be continued in this way, refining the upper bound and the lower bound until convergence to $\left(\frac{a-c}{3}, \frac{a-c}{3}\right)$

Dynamic Games

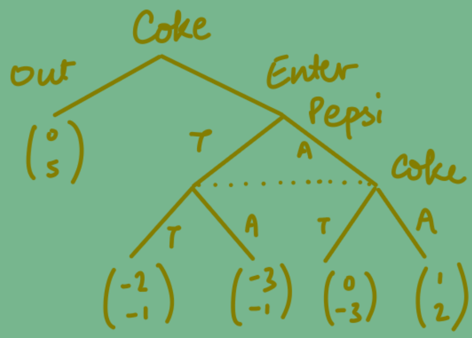
Game I



Game II



Game III



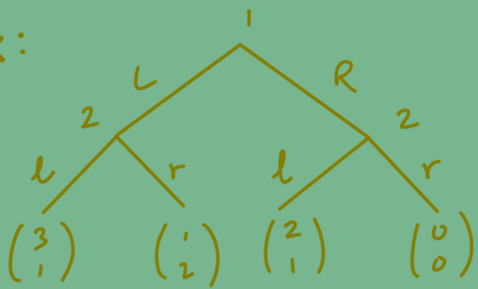
C \ P	A	T
E	1, 2	-2, -1
O	0, 5	0, 5

NE: (E, A), $\underbrace{(O, T)}$

As if P threatens C that it will T. But, this is irrational of P $\therefore$ the threat is not credible (P will A if C enters)

$\therefore$ We need to re-define the definition of NE in the case of Dynamic games.

Ex:



1 \ 2	ll	lr	rl	rr
L	3, 1	3, 1	1, 2	1, 2
R	2, 1	0, 0	2, 1	0, 0

NE: (R, rl), $\underbrace{(L, rr)}$

Irrational, since 2 will play l if 1 plays R

Game III

C \ P	T	A
OT	0, 5	0, 5
OA	0, 5	0, 5
ET	-2, -1	0, -3
EA	-3, 1	1, 2

NE: $\underbrace{(OT, T)}$, (OA, T), (EA, A)

Irrational of C since it will A if P decides to A

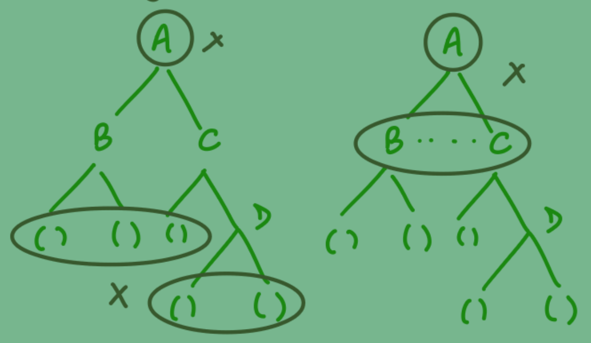
Sequential Rationality

Acceptable equilibrium among the set

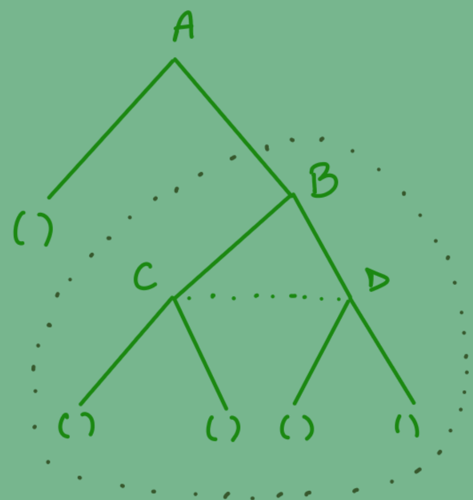
of Nash equilibrium are those for which both players are sequentially rational
Players are behaving rationally for every sequence of the game
$\Rightarrow$ **Subgame Perfect Nash Equilibrium**

A proper subgame in a game tree:

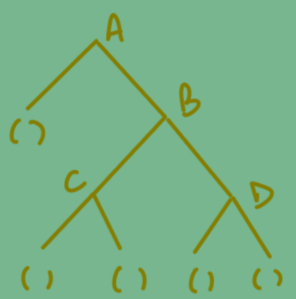
a) Starts with a singleton information set (say n). i.e., at any particular decision node but not the initial node, nor the terminal node



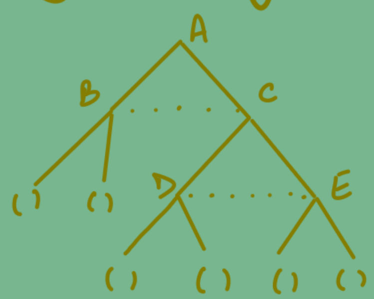
b) Contains all the decision nodes and terminal nodes that follow "n" in the game tree but not the nodes that do not directly follow "n".



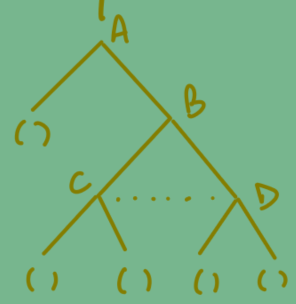
Ex: How many subgames possible?



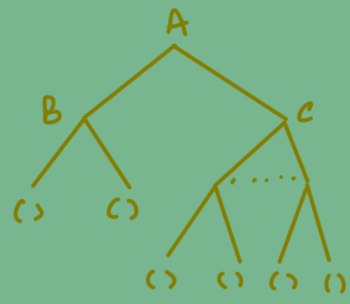
3 (B, C, D)



0



1 (B)



2 (B, C)

Finding Subgame Perfect Nash Equilibrium (SPNE)

Use method of Backward Induction:

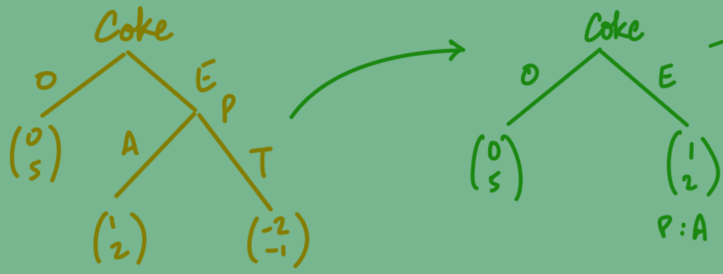
S1: Start with the smallest possible subgame, containing terminal nodes
   i) Find the NE for that subgame
   ii) Replace the initial nodes of the subgame with the payoffs at NE.

S2: Again identify the smallest possible subgame of the reduced game and repeat the procedure.

Ex:

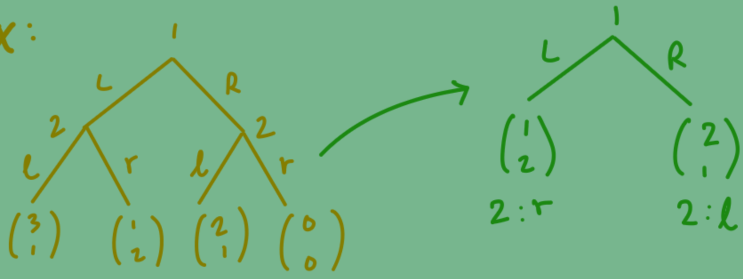
1 chooses R
∵ it gives a better payoff
⇒ SPNE : (R, l)

Ex:



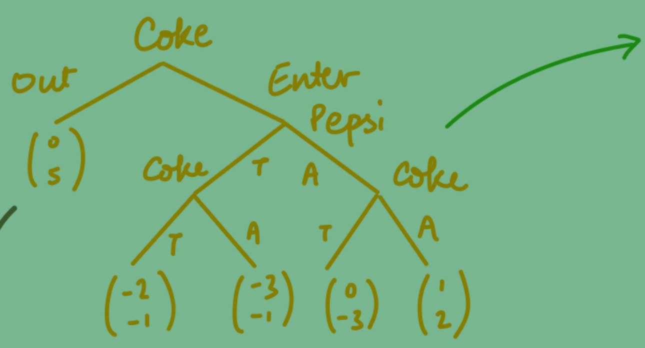
SPNE is (E, A)

Ex:



SPNE is (R, rl)

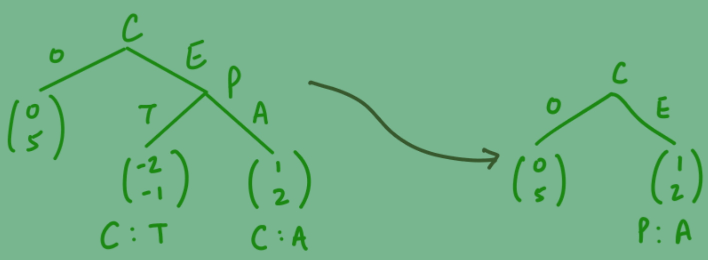
Ex:



(C) \ (P)	T	A
OTT	0, 5	0, 5
OTA	0, 5	0, 5
OAT	0, 5	0, 5
OAA	0, 5	0, 5
ETT	-2, -1	0, -3
ETA	-2, -1	1, 2
EAT	-3, -1	0, -3
EAA	-3, -1	1, 2

(O**, T)   (E*A, A)

SPNE:



C chooses E

$\therefore$ SPNE is (ETA, A) and all the other NE's that we found are unreasonable.

Stackleburg Model of Duopoly

- Two players — two firms
- Model of quantity competition
- Sequential move game:

  F1 (leader) chooses quantity first in Stage 1.

  F2 observes q_1 & chooses q_2 in the following period.

  $P = a - Q$, $Q = q_1 + q_2$, $\pi_i = [P - c]q_i$

- Solve using method of backward induction

  - Start from Stage 2:
  - Observe q_1

  $\longrightarrow$ $R_2(q_1)$: For every value of q_1 observed by F2, what's best for F2.

$$R_2(q_1) := \underset{q_2}{\operatorname{argmax}}\ \pi_2(q_1, q_2)$$

Take q_1 to be given, ie, assume it to be constant in this stage

$$= \underset{q_2}{\operatorname{argmax}}\ (a - q_1 - q_2)q_2 - cq_2$$

$$= \frac{a - c - q_1}{2}$$

  - Move onto Stage 1:
  - q_2 is known, thanks to SPNE

$$\max_{q_1}\ \pi_1 = (a - (q_1 + q_2))q_1 - cq_1 \Big|_{q_2 = R_2(q_1) = \frac{a-c-q_1}{2}}$$

q_2 cannot be assumed to be constant here

$$= \left(\frac{a + c - q_1}{2}\right)q_1 - cq_1$$

$$\Rightarrow \quad q_1 = \frac{a-c}{2}$$

SPNE: $q_1^* = \dfrac{a-c}{2}$, $q_2^* = \dfrac{a-c-q_1}{2}$

A function, not a particular value

- Outcome at SPNE is $q_1^* = \dfrac{a-c}{2}$, $q_2^* = \dfrac{a-c}{4}$

$\Rightarrow$ F1 gains an advantage over F2 in terms of profit, since it gets to move first.

Tariffs & International Competition

- Two identical countries $i = 1, 2$
- Two firms — one firm in each country
- Total quantity in country i

  $Q_i = h_i + e_j \qquad P = a - Q_i$
- Firm i also has to pay $t_j e_i$ to the Govt. of country j

  Tariff imposed by country j
- Cost of production for firm in country i : $c(h_i + e_i)$

  Quantity which F_i exports to C_j

  Quantity which F_i produces for its own country C_i
- Timing of the game:

  S1: Countries choose the tariff rates (t_1 & t_2) simultaneously

  S2: Firms observe t_1 & t_2, simultaneously choose (h_i, e_i) & (h_j, e_j).

$$\pi_i = [a - (h_i + e_j)]h_i + [a - (h_j + e_i)]e_i - c(h_i + e_i) - t_j e_i$$

$$W_i = Q_i^2/2 + \pi_i + t_i e_j$$

Welfare Function = Consumer Surplus + Profit of Firms + Tariff Revenue

Essentially the profit function of a Govt.

- Now, to solve the game:
  - Nash Eq^m in S2
    (h_i^*, e_i^*) & (h_j^*, e_j^*)
  - For i, (h_i^*, e_i^*) solves
    $$\max_{h_i, e_i} \pi(h_i, e_i, h_j^*, e_j^*)$$

Equivalently, $h_i = \underset{h_i}{\operatorname{argmax}} [a-(h_i+e_j)]h_i - ch_i$

and $e_i = \underset{e_i}{\operatorname{argmax}} [a-(h_j+e_i)]e_i - ce_i - t_j e_i$

$$\Rightarrow h_i^* = \frac{a-c-e_j^*}{2} \quad \Rightarrow h_i^* = \frac{a-c+t_i}{3}$$

$$e_i^* = \frac{a-c-t_j-h_j^*}{2} \quad e_i^* = \frac{a-c-2t_j}{3}$$

- For Nash eq^m in S1, the govt. assumes (h_i^*, e_i^*), (h_j^*, e_j^*) to have the above forms and solves
  $$\max_{t_i} W_i(t_i, t_j^*) = \max_{t_i} \frac{Q_i^2}{2} + \Pi_i + t_i e_j$$

$$\Rightarrow t_i = \underset{t_i}{\operatorname{argmax}} \frac{Q_i^2}{2} + \Pi_i + t_i e_j$$

$$\Rightarrow 0 = Q_i \, \partial(h_i+e_j)/\partial t + \partial\Pi_i/\partial t_i + e_j + t_i \, \partial e_j/\partial t_i$$
$$= (h_i+e_j)(1/3 - 2/3) + [a/3 - 2h_i/3 - e_j/3 + 2h_i/3 - c/3] + e_j - 2t_i/3$$
$$= h_i(-1/3) + e_j(1/3) + a(1/3) - c(1/3) - t_i(2/3)$$
$$\Rightarrow 2t_i = e_j - h_i + a - c$$
$$= \frac{-2t_i}{3} - \frac{t_i}{3} + a - c \Rightarrow t_i^* = \frac{a-c}{3} = t_j^*$$

Thus, the solution is:

$$h_i^* = \frac{a-c+t_i}{3}, \quad e_i^* = \frac{a-c-2t_j}{3}, \quad t_i^* = \frac{a-c}{3}$$

And the outcome at SPNE is:

$$h_i^* = \frac{4}{9}(a-c), \quad e_i^* = \frac{a-c}{9}, \quad t_i^* = \frac{a-c}{3}$$

Repeated Games

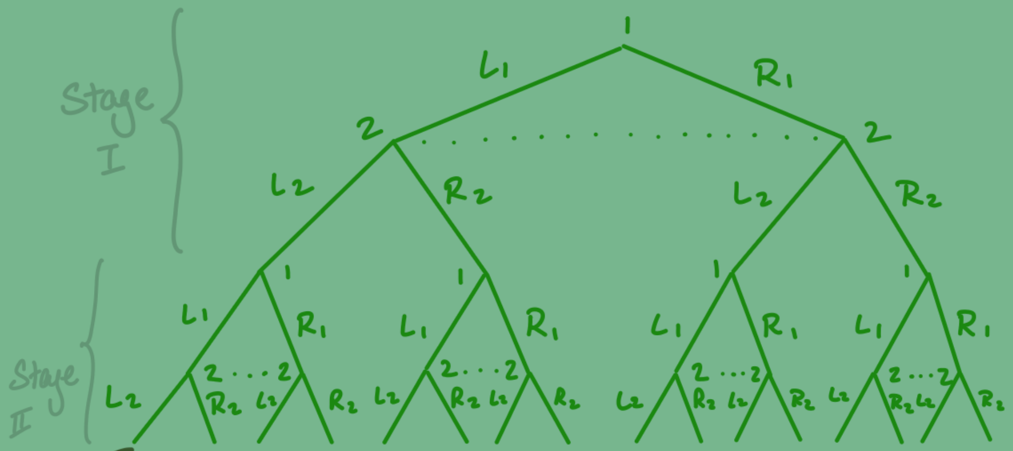
A one-shot simultaneous game that is repeated more than once.

1) Players move simultaneously at each stage.
2) Action sets do not vary over time.
3) Each periods outcome is observed before the game begins in next period.
4) Players are concerned with aggregate payoffs.

Ex:

	L_2	R_2
L_1	1, 1	5, 0
R_1	0, 5	4, 4

Game is repeated once.



There are 4 subgames to be solved here. If repeated twice, 16 subgames.

In S1, is there any scope of co-operation? (L_1, L_2)

- Benefit of unilateral deviation : 1
- Cost of unilateral deviation : 0

So, in S1 too, the NE will be (L_1, L_2) and so, the NE of the repeated game will still be $(L_1, L_2)_1, (L_1, L_2)_2$

★ For any finitely repeated game, the NE in each stage of the game is the non-cooperating outcome.

Infinitely Repeated Games

Discounting: Tomorrow's money is more valuable today.

1 unit today $\xrightarrow{\text{invest}}$ 1+r unit tomorrow $\Rightarrow$ $\frac{1}{1+r}$ today = 1 tomorrow

$\delta := \frac{1}{1+r}$ ⟶ Discounting factor

$$(\pi_1, \pi_2, \pi_3)_{present} = \pi_1 + \delta \pi_2 + \delta^2 \pi_3$$

$$(\pi_{i=1}^{i=\infty})_{present} = \sum_{i=1}^{\infty} \pi_i \delta^{i-1}$$
$$= \pi_1 + \delta \pi_2 + \delta^2 \pi_3 + \delta^3 \pi_4 + \dots$$

- Notion of "Punishment Strategies" to discourage players from deviating from co-operation:
  1) Trigger Strategy → A single deviation from cooperation from any other player leads to subsequent deviations in each stage
  2) Tit for Tat Strategy → If some other player deviated from cooperation in the i^{th} stage then the player deviates in the $i+1^{th}$ stage.

Trigger Strategy

Player i starts the game by co-operating in the initial stages & co-operates in subsequent stages if all the players co-operated in the previous stage.

Ex:

	L_2	R_2
L_1	1,1	5,0
R_1	0,5	4,4

Repeated Infinitely

→ Trigger Strategy (TS)
Play R_i in the 1^{st} stage.
In the t^{th} period, if the outcome of all the previous periods has been (R_1, R_2), play R_i. Otherwise, play L_i forever.

→ Is (TS, TS) an NE?
≡ If i plays TS, is there any incentive for j to deviate from TS?

→ Consider any arbitrary period t.
Two possibilities:
1) All the previous periods' outcomes have been (R_1, R_2).
2) Complement of above.

For j:

Payoff in (1): $4 + 4\delta + 4\delta^2 + \ldots = \frac{4}{1-\delta}$

TS triggered

Payoff in (2): $5 + \delta + \delta^2 + \ldots = 5 + \frac{\delta}{1-\delta}$

∴ (TS, TS) is an NE iff
$4/(1-\delta) > 5 + \delta/(1-\delta) \longleftrightarrow \delta > 1/4$

Bertrand's Paradox

→ Equilibrium: $p_1 = p_2 = c$

Both firms co-operate & agree to charge $p_m \Rightarrow \pi_m$

	C	NC
C	$\Pi_m/2, \Pi_m/2$	$0, \Pi_m$
NC	$\Pi_m, 0$	$0, 0$

P_m, P_m $P_m, P_m - \epsilon$ C, C

If the game were one-shot, NE would be (NC, NC)

In infinite game, let's see if (TS, TS) is an NE or not:

Situation 1: All previous stages resulted in (C,C)

Sit 2: Complement of above

If 1 plays TS:

$$\Pi = \Pi_m/2 + \delta \Pi_m/2 + \ldots = \Pi_m/2(1-\delta)$$

If 1 does not:

$$\Pi = \Pi_m + \delta 0 + \ldots = \Pi_m$$

$\therefore$ (TS, TS) is an NE if $\delta > 1/2$

* (TS, TS) is often an NE if δ is above a certain threshold.

Ex: Similarly, in Cournot model (let $(a-c)^2/b = 1$)

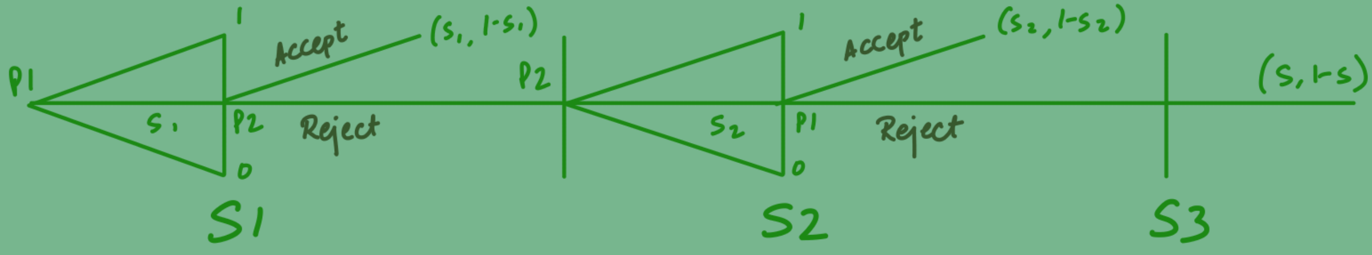
	C	NC
C	$1/8, 1/8$	$3/32, 9/64$
NC	$9/64, 3/32$	$1/9, 1/9$

Sequential Bargaining

- In S1, P1 proposes to take a share s_1 of the rupee. P2 either accepts ($\Pi_1 = s_1$, $\Pi_2 = 1 - s_1$) or rejects the offer.
- In S2, P2 proposes to take a share s_2 of the rupee. P1 either accepts ($\Pi_1 = s_2$, $\Pi_2 = 1 - s_2$)

or rejects the offer.

- At the beginning of S3, P1 receives (s) and P2 receives $(1-s)$.



S3: The payoff is pre-determined; P1 gets s and P2 gets $1-s$

S2: P2 knows that if it goes to S3, then P1 will get s, which is equivalent to δs in S2.

So, if it offers $s_2 < \delta s$, then P1 is sure to reject $\because$ P1 also knows it will get s in S3.

$\therefore$ P2 can only offer $s_2 \geq \delta s$ to make P1 accept the offer. It won't choose $s_2 > \delta s$ as it will decrease P2's payoff, though P1 will accept the offer. So, P2 will offer $s_2 = \delta s$ to P1 and as P1 is expected to accept this offer, P2 can expect an outcome of $1 - s_2 = 1 - \delta s \quad (> \delta(1-s))$

S1: P1 knows that if it gets to S2, P2 will be able to get a payoff of $1-\delta s$ in S2 which is equal to $\delta(1-\delta s)$ in S1.

So, P1 needs to offer $1-s_1 \geq \delta(1-\delta s)$ to P2 in order to make it accept the offer. It won't offer $1-s_1 > \delta(1-\delta s)$

as it will decrease its own payoff, so P1 makes an offer of $1 - s_1 = \delta(1-\delta s)$
$\longleftrightarrow \; s_1 = 1-\delta(1-\delta s) \quad (> \delta s)$

So, the game ends in S1 itself with payoffs $(1-\delta(1-\delta s),\ \delta(1-\delta s))$ which is the maximum payoff either of them can get in S2 or S3.

☆ In a similar way, a game with infinite stages can also be solved and finished in just S1.

Static Games with Incomplete Information

- Not all players have complete information about the payoffs of the other players. Eg: Auctions.

Cournot Game with Asymmetric Information

$P(Q) = a - Q \qquad Q = q_1 + q_2 \qquad C_1(q_1) = cq_1$ (In common knowledge)

$$C_2(q_2) = \begin{cases} c_H\, q_2 & \text{w/ prob. } \theta \\ c_L\, q_2 & \text{w/ prob. } 1-\theta \end{cases}$$

Firm 2's cost only known to 2

Strategy for F2: $(q_2^*(c_H),\ q_2^*(c_L))$
Strategy for F1: q_1^*
NE: $\{q_1^*,\ (q_2^*(c_H),\ q_2^*(c_L))\}$

$q_2^*(c_H)$ is $\underset{q_2}{\operatorname{argmax}} (a - q_1 - q_2)q_2 - c_H q_2$
$$= \frac{a - q_1 - c_H}{2}$$

Similarly, $q_2^*(c_L) = (a - q_1 - c_L)/2$

And q_1^* solves $\max_{q} \begin{bmatrix} \theta\,[a - q_1 - q_2^*(c_H)]\,q_1 \\ +(1-\theta)[a - q_1 - q_2^*(c_L)]\,q_1 - cq_1 \end{bmatrix}$

Note that $\frac{\partial q_2^{*H}}{\partial q_1} = 0 = \frac{\partial q_2^{*L}}{\partial q_1}$ since this is a static game.

$\Rightarrow 2q_1^* = a - c - \theta q_2^{*H} - (1-\theta) q_2^{*L}$

$\Rightarrow q_1^* = \frac{a - 2c + \theta c_H + (1-\theta) c_L}{3}$

$\Rightarrow q_2^{*H} = \frac{2a + 2c - 3c_H - \theta c_H - (1-\theta)c_L}{6}$

$q_2^{*L} = \frac{2a + 2c - 3c_L - \theta c_H - (1-\theta)c_L}{6}$

- This $(q_1^*, (q_2^{*H}, q_2^{*L}))$ is the Bayesian Nash Equilibrium.
- F 2 knows $c_H / c_L \longrightarrow$ Appropriately chooses q_2^{*H} / q_2^{*L}

Ex:

		F2 c_H	F2 c_L
F1	c_H	1/3	1/3
F1	c_L	1/3	0

Probability Distribution Table

$P(C_2 = c_H / C_1 = c_H)$
$= \frac{P(C_2 = c_H, C_1 = c_H)}{P(C_1 = c_H)} = \frac{1/3}{2/3} = \frac{1}{2}$

Need to use Bayesian probability.

$P(C_2 = c_H \mid C_1 = c_L) = 1$

$q_1(c_H) \equiv q_1^H \qquad q_1(c_L) \equiv q_1^L$

$$q_1^*(c_H) = \underset{q_1^H}{\operatorname{argmax}} \Big[a - (q_1^H + q_2^{*L})\Big] q_1^H \cdot P(C_2 = c_L \mid C_1 = c_H) + \Big[a - (q_1^H + q_2^{*H})\Big] q_1^H \cdot P(C_2 = c_H \mid C_1 = c_H) - c_H q_1^H$$

$$q_1^{*H} = \frac{a-c_H}{2} - \frac{q_2^{*L} + q_2^{*H}}{4}, \quad q_1^{*L} = \frac{a-c_L}{2} - \frac{q_2^{*H}}{2}$$

If Symmetric Bayesian Nash Equilibrium is asked, then we have the additional equations $q_1^{*H} = q_2^{*H}$ and $q_1^{*L} = q_2^{*L}$

Consider the 2-player Bayesian Game in which $S_1 = \{T, B\}$ and $S_2 = \{L, R\}$. Each player has two types $T_1 = T_2 = \{0, 1\}$ & each type profile is equally likely $p(0,0) = p(0,1) = p(1,0) = p(1,1) = 1/4$

$t_1 = 0$, $t_2 = 0$:

	L	R
T	0, 0	2, 1
B	1, 2	0, 0

$t_1 = 0$, $t_2 = 1$:

	L	R
T	0, 2	2, 0
B	1, 0	0, 1

$t_1 = 1$, $t_2 = 0$:

	L	R
T	1, 0	0, 1
B	0, 2	2, 0

$t_1 = 1$, $t_2 = 1$:

	L	R
T	1, 2	0, 0
B	0, 0	2, 1

- Show that $(s_1(0), s_1(1)) = (T, T)$ is not part of BNE.

Find BRF of 2 wrt (T, T), then find BRF of 1 wrt it and check.

W.r.t. (T, T),

$t_2 = 0 \longrightarrow$ Payoff by playing $L \longrightarrow \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 0 = 0$

" " " $R \longrightarrow \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 1 = 1$

$t_2 = 1 \longrightarrow$ " " " $L \longrightarrow \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 2 = 2$

" " " $R \longrightarrow \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 0 = 0$

$s_2(0) = R, \quad s_2(1) = L$

$\Rightarrow BRF_2((T, T)) = (R, L)$

Now, BRF, w.r.t. $(R, L)_2$,

$t_1 = 0 \longrightarrow$ Payoff by playing $T \longrightarrow \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 0 = 1$

" " " $B \longrightarrow \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 0 = 0$

$t_1 = 1 \longrightarrow$ " " " $T \longrightarrow \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1 = \frac{1}{2}$

" " " $B \longrightarrow \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 0 = 1$

$\Rightarrow s_1(0) = T, \quad s_1(1) = B$

$\Rightarrow BRF_1((R, L)_2) = (T, B) \neq (T, T)$

Hence, $(s_1(0), s_1(1)) = (T, T) \notin$ BNE

If all BNE's for the above game are required, then this 4×4 matrix can be solved for NE.

1 \ 2	LL	LR	RL	RR
TT	1/2, 1	3/4, 0	3/4, 3/2	1, 1/2
TB	0, 1	1, 3/4	1, 3/4	2, 1/2
BT	1, 1	1/2, 3/4	1/2, 3/4	0, 1/2
BB	1/2, 1	3/4, 3/2	3/4, 0	1, 1/2

$$\pi(TB, LR)_1 = p(0,0)\cdot \pi(0,0,T,L)_1 + p(0,1)\cdot\pi(0,1,T,R)_1 + p(1,0)\cdot\pi(1,0,B,L)_1 + p(1,1)\cdot\pi(1,1,B,R)_1$$
$$= \frac{1}{4}\cdot 0 + \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 0 + \frac{1}{4}\cdot 2 = 1$$

and so on ...

Finally, the BNE is (BT, LL)

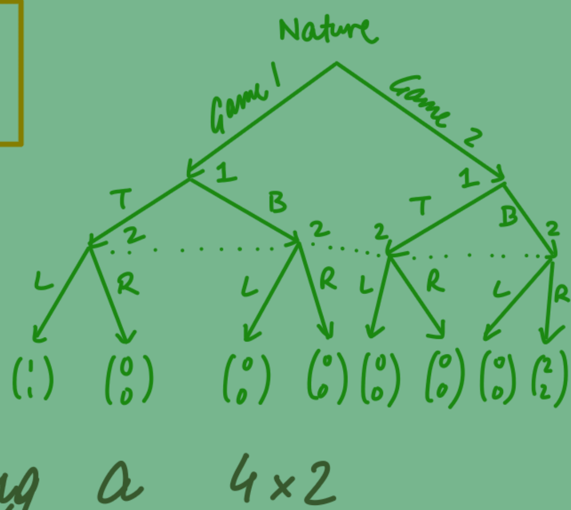
Ex:

Game 1

1 \ 2	L	R
T	1,1	0,0
B	0,0	0,0

Game 2

1 \ 2	L	R
T	0,0	0,0
B	0,0	2,2



P1 knows which game it is, but P2 does not.
It is a simultaneous game.
Can be solved by making a 4×2 payoff matrix of the given game tree

Auctions

- A static incomplete information game.
  Static: Bids are placed simultaneously.
  Incomplete Information: No player knows how much the other players value the object put up for the auction, ie, no player knows the payoffs.

- 1^{st} Price Auction: Bid simultaneously. Highest bidder wins. Winner gets the object & pays the bid. Loser pays nothing & gets nothing.
- 2^{nd} Price Auction: Winner (Highest Bidder) pays the second highest price. Rest of it is the same as above.
- Further Readings: Mechanism Design, Revelation Principle

- 1^{st} Price Auction often yields underbidding as payoff is value of object minus the bid.
  Tradeoff: Higher the bid, higher chances to win; Lower the bid, higher is the payoff.
- 2^{nd} Price Auction yields true(r) bidding.

Case Study:

A values the object at 100 units

	→ Bidding	100	v/s	<100 (say 99)
2^{nd} bid	<99	100 − v		100 − v
	99.5	0.5		0
	>100	0		0

So, in all cases, 100 is at least as good as <100

Bidding	100	v/s	>100 (say 105)
2nd bid <100	100−v		100−v
102	0		−2
>105	0		0

So, in all cases, 100 is at least as good as >100

Thus, 100 is the best bet in 2nd price auction.

Ex: There are 2 players. There is a first price sealed bid auction to win the amount of money in both the pockets.

Common Knowledge: Each player has either 0 or 3 rupees in his / her pocket, both are equally likely. A player knows the amount in its own pocket.

Consider discrete bid strategies where players can only make bids in ₹1 increments.

s_A & s_B be the value of money in A or B's pocket.

b_A & b_B — bids $b_A(s_A)$, $b_B(s_B)$

Show that $b_A(s_A=0)=0$, $b_A(s_A=3)=2$, $b_B(s_B=0)=0$, $b_B(s_B=3)=2$ is a BNE.

Let B's strategy be as above. We find A's strategy.

$s_A = 0$

0 : payoff (expected) $= p(s_B=0 \mid s_A=0)\cdot\pi + p(s_B=3 \mid s_A=0)\cdot\pi$

$= \frac{1}{2}\cdot\frac{1}{2}\cdot(0+0-0) + \frac{1}{2}\cdot 0 = 0$

1 : payoff (exp.) $= \frac{1}{2}\cdot(0+0-1) + \frac{1}{2}\cdot(0) = -\frac{1}{2}$

It can be shown that >1 is also -ve.
$\therefore \quad b_A(s_A = 0) = 0$

$s_A = 3$

0 : payoff $= \frac{1}{2} \cdot \frac{1}{2}(3+0-0) + \frac{1}{2} \cdot 0 = \frac{3}{4}$

1 : " $= \frac{1}{2} \cdot (3+0-1) + \frac{1}{2} \cdot 0 = 1$

2 : " $= \frac{1}{2} \cdot (3+0-2) + \frac{1}{2} \cdot \frac{1}{2}(3+3-2) = \frac{3}{2}$

3 : " $= \frac{1}{2} \cdot (3+0-3) + \frac{1}{2}(3+3-3) = \frac{3}{2}$

It can be shown that 2 is the maximum. $\therefore \quad b_A(s_A = 3) = 2.$

Hence, $b_A(s_A = 0) = 0 = b_B(s_B = 0)$, $b_A(s_A = 3) = 2 = b_B(s_B = 3)$ is a BNE.

Dynamic Games with Incomplete Information

- NE $\longrightarrow$ SPNE $\longrightarrow$ BNE $\longrightarrow$ Perfect Bayesian Nash Equilibrium

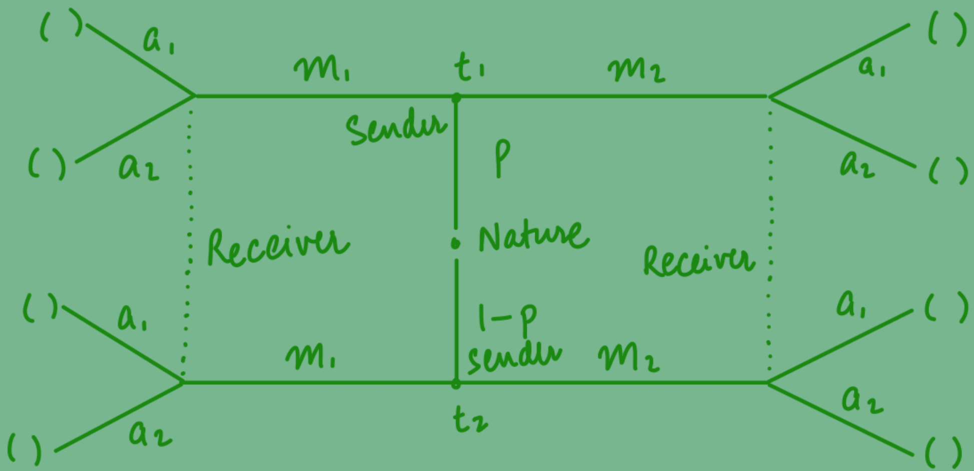
- "Signals" $\longrightarrow$ A sender and a receiver.
  — Signalling games — Also called messages.
- Spence paper on Job Market Signalling.
  $\rightarrow$ Applicant wants to "signal" employer that he / she is skilled.
- S sends signal, R receives it and takes an action.
- The timing of the game is :
  1) Nature draws a type t_i for the sender from the set of following feasible types
  $$T = \{t_1, t_2, \ldots, t_n\}$$
  according to a probability distribution

$p(t_i)$ s.t. $0 \le p(t_i) \le 1$, $\sum_{i=1}^{i=n} p(t_i) = 1$

2) Sender observes his/her own t_i and chooses a message m_j from $M = \{m_1, \dots, m_I\}$

3) Receiver observes the message but not the type, & chooses an action a_R from $A = \{a_1, \dots, a_K\}$

4) $\Pi_S(t_i, m_j, a_k)$ $\Pi_R(t_i, m_j, a_k)$



Four strategies for the sender:

(m_1, m_1), (m_2, m_2) } Pooling Strategies

(m_1, m_2), (m_2, m_1) } Separating Strategies

PBNE for Signaling Games

Requirement 1:

After observing message m_j from M, the receiver must have a belief about a type who sent this message $\mu(t_i / m_j)$ $= \frac{p(t_i)}{\sum p(t_i)}$

Requirement 2:

For each m_j, $a^*(m_j)$ solves

$$\max_{a_k} \sum_{t_i} \Pi_R(t_i, m_j, a_k)\, \mu(t_i / m_j)$$

For each $t_i \in T$, $m^*(t_i)$ solves

$$\max_{t_i} \Pi_S(t_i, m_j, a_k^*)$$

Requirement 3:

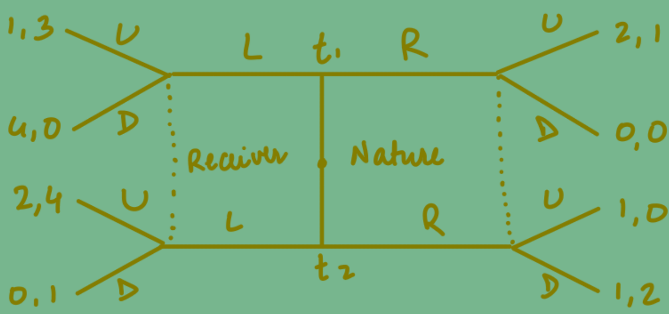
For each $m_j \in M$, if there exists $t_i \in T$

such that $m^*(t_i) = m_j$, then receiver's belief at the I-set corresponding to m_j must follow Bayes' rule and sender's equilibrium strategy.

- A pure strategy PBNE is a pair of strategies $m^*(t_i)$ and $a^*(m_j)$ and a belief $\mu(t_i/m_j)$ satisfying these requirements.
- Let (m_1, m_1) be the strategy, then, the I-set on the left is said to be "on the equilibrium path" as it is reached with positive probability. Similarly, the I-set on the right will be called "off the equilibrium path".

  An I-set is said to be "on the path of play" iff it is reached with positive probability.

Ex:



Find all pure strategy PBNE for this game.

Suppose (L, L) be part of a pooling equilibrium,

$p = \mu(t_1/L) = p(t_1)/p(t_1)+p(t_2) = 0.5/1 = 0.5$

No information yet on $q = \mu(t_1/R)$ ∵ (L, L)

Payoff to R(eceiver) by playing U:

$= 0.5 \times 3 + 0.5 \times 4 = 3.5$

Payoff to R by playing D: 0.5

$\therefore$ $a^*(L) = U$

- At (L, L),

Payoff to S(ender) at $t_1 = 1$, $t_2 = 2$

S has incentive to deviate from L to R if $a^*(R) = U$

Payoff to R by playing $U = q \cdot 1 + (1-q)0 = q$

" R " $D = q \cdot 0 + (1-q) \cdot 2 = 2-2q$

For $a^*(R) = D$: $2 - 2q \geq q \longleftrightarrow q \leq 2/3$

So, the following is a PBNE:

$$\{\underbrace{m^*(t_1) = L = m^*(t_2)}_{\text{Sender's Strategy}}, \underbrace{a^*(L) = U, \ a^*(R) = D}_{\text{Receiver's Strategy}}, \underbrace{p = 0.5, \ q \leq 2/3}_{\text{Receiver's Beliefs}}\}$$

- At (R, R), $p = \mu(t_1/L) = ?$, $q = \mu(t_1/R) = 0.5$

$a^*(R) = D$ as 0.5 at U while 1 at D

$a^*(L) = U$ if $4 - p > 1 - p \longrightarrow$ Always

So, payoff for t_1 by playing R is

R plays D $\therefore a^*(R) = D$ $(\underline{0}, 0)$ while on playing L is $(\underline{1}, 3)$ R plays U $\therefore a^*(L) = U$

Thus, t_1 has incentive to deviate from (R, R).

Hence, (R, R) is not part of any PBNE.

- At (L, R), $p = \mu(t_1/L) = 1$, $q = \mu(t_1/R) = 0$

$\therefore$ $a^*(L) = U$, $a^*(R) = D$

Incentive for t_2 to play L instead as L gets $(\underline{2}, 4)$ while R gets $(\underline{1}, 2)$.

So, there is no PBNE at (L, R).

- At (R, L), $p = \mu(t_1/L) = 0$, $q = \mu(t_1/R) = 1$

$\therefore$ $a^*(R) = U$, $a^*(L) = U$

In this case, neither t_1 nor t_2 has any incentive to deviate, and by construction,

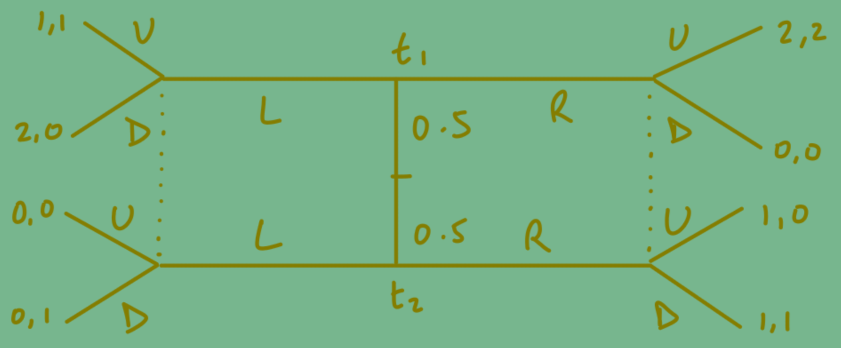
R has no incentive to deviate. Thus, we have a PBNE: $\{m^*(t_1)=R,\ m^*(t_2)=L,\ a^*(L)=U=a^*(R),\ p=0,\ q=1\}$

* Probability for I-sets on the path: are determined using Bayes' Rule; the rest of the probabilities are free to be determined by the Receiver.

For example, for calculating $p(L)$, consider all such types t_i such that t_i plays L in the current strategy. Then, $p(L)$ is the sum of probabilities of nature choosing such t_i's. $p(t_i)$ is, again, as dictated by nature.

If L is not present in the current strategy, then we are free to have any belief for it as well as the types playing it.

Ex:



t_2 always gets better by playing R than by L. Can directly discard strategies like (L,L), (R,L).

· At (L,R), $p=\mu(t_1/L)=1$, $q=\mu(t_1/R)=0$

So, $a^*(L)=U$, $a^*(R)=D$.

Neither t_1 nor t_2 has incentive to deviate.

$\therefore$ PBNE $=\{m^*(t_1)=L,\ m^*(t_2)=R,\ a^*(L)=U,\ a^*(R)=D,\ p=1, q=0\}$

- At (R, R), $p = \mu(t_1/L) = ?$, $q = \mu(t_1/R) = 0.5$

  Payoff of R on playing $U = 0.5 \times 2 + 0 = 1$

  $D = 0 + 0.5 \times 1 = 0.5$

  $\therefore$ $a^*(R) = U \longrightarrow$ Payoff for $(t_1 : 2)$, $(t_2 : 1)$

  Also, $a^*(L) = U$ iff $p \geq 0.5 \longrightarrow$ Payoff for t_1 is 1

  $\therefore$ No incentive to deviate here.

  Also, $a^*(L) = D$ iff $p \leq 0.5 \longrightarrow$ Payoff for t_2 is 0

  $\therefore$ No incentive to deviate here either.

  So, we have a PBNE:

  $\{m^*(t_1) = R = m^*(t_2),\ a^*(R) = U,\ q = 0.5,\ p \geq 0.5,\ a^*(L) = U\}$

  And $\{m^*(t_1) = R = m^*(t_2),\ a^*(R) = U,\ q = 0.5,\ p \leq 0.5,\ a^*(L) = D\}$

Ex: Department decides whether to grant tenure or not.
Assistant Professor: High v/s Low Ability.

Payoff func. for AP

$$\Pi_{AP} = \begin{cases} \tau - \alpha n, & n = H \\ \tau - \beta n, & n = L, \ \alpha < \beta \end{cases}$$

Payoff of dept

$$\Pi_D \to \begin{cases} 0, & \tau = 0 \\ 1, & \tau = 1, t = H \\ -1, & \tau = 1, t = L \end{cases}$$

Nature chooses ability of Prof. AP observes type and decides to write "n" papers. Dept. observes "n" but not the type and chooses tenure ($\tau = 1$ or 0).

Strategies are mappings: $n^*(t)$, $\tau^*(n)$

PBNE is of the form: $\{n^*(t), \tau^*(n), \mu(H(n))\}$

and $n^*(t)$ solves $\max\limits_{n} \Pi_{AP}(t, n, \tau^*(n))$

and $\tau^*(n)$ solves $\max\limits_{\tau \in \{0,1\}} \mu(H(n))\, \Pi_D(H, n, \tau) + \mu(L(n))\, \Pi_D(L, n, \tau)$

Lastly,

$$\mu(H/n) = \begin{cases} 1, & n^*(H) = n,\ n^*(L) \neq n \\ \lambda, & n^*(H) = n^*(L) = n \\ 0, & n^*(H) \neq n,\ n^*(L) = n \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

λ chosen by Nature $0.5 > \lambda > 0$

- $n^*(H) = n^*(L) = \hat{n} \geq 0 \longrightarrow \mu(H/n) = \lambda$

  $\tau^*(n) = 0$ as $\tau = 0 \Rightarrow \pi_D = 0$

  $\tau = 1 \Rightarrow \pi_D = \lambda \cdot 1 + (1-\lambda)\cdot(-1) = 2\lambda - 1 < 0$

  As AP knows tenure will not be granted, it is best not to write any papers $\Rightarrow n^*(H) = 0 = n^*(L)$ in order to maximize π_D.

- $n^*(H) = n,\quad n^*(L) = 0 \longrightarrow \mu(H/n) = 1$

  $\tau^*(0) = 0,\quad \tau^*(n) = 1$

  Is there any incentive for deviation?

No. of papers	High	Low
n	$1-\alpha n$	$1-\beta n$
0	0	0

  $\longrightarrow$ Payoffs should be $1-\alpha n \geq 0 \geq 1-\beta n$ to ensure no deviation.

  So, $1/\beta \leq n \leq 1/\alpha$

- Another practical application of signaling games is the 'Limit Pricing Model'.

Ex: Types: 1, 2, 3, 4.
Receiver Actions: 1, 2, 3, 4
Messages: $\alpha, \beta, \gamma, \delta$
All types equally likely
Is there a PBNE where

Receiver's Action \ Sender	1	2	3	4
1	216, 274	276, 214	216, 145	147, 73
2	147, 214	216, 274	276, 214	216, 145
3	75, 145	147, 214	216, 274	276, 214
4	0, 73	75, 145	147, 214	216, 274

$\underbrace{75}_{\text{Sender}}, \underbrace{145}_{\text{Receiver}}$

1) $m(t_s) = \alpha \quad \forall \; t_s$

2) $m(t_s) = \beta \quad \forall \; t_s$

3) $m(t_s) = \alpha \quad \forall \; t_s \in \{1, 2\}, \quad m(t_s) = \beta \quad \forall \; t_s \in \{3, 4\}$

(3) : Begin by finding beliefs → Can only find beliefs "On the Equilibrium Path", i.e., $\mu(t_s/\alpha)$, $\mu(t_s/\beta)$

$$\mu(t_1/\alpha) = \frac{p(t_1)}{p(t_1) + p(t_2)} = \frac{1/4}{1/4 + 1/4} = 1/2 = \mu(t_2/\alpha)$$

Similarly, $\mu(t_3/\beta) = 1/2 = \mu(t_4/\beta)$,

$\mu(t_2/\beta) = \mu(t_1/\beta) = \mu(t_3/\alpha) = \mu(t_4/\alpha) = 0$

To find $a^*(\alpha)$, payoff of receiver on choosing

$1 : \; 1/2 \cdot 274 + 1/2 \cdot 214$

$2 : \; 1/2 \cdot 214 + 1/2 \cdot 274$ ✓ Break Tie Arbitrarily

$3 : \; 1/2 \cdot 145 + 1/2 \cdot 214$

$4 : \; 1/2 \cdot 73 + 1/2 \cdot 145$

$\therefore \; a^*(\alpha) = 2.$

Similarly, $a^*(\beta) = 3$

So, the sender's payoffs are :

	$\alpha : 1$	$m(t_2) = \alpha : 2$	$\beta : 3$	$\beta : 4$
1	216, 274	276, 214	216, 145	147, 73
$a^*(\alpha) : 2$	147, 214	216, 274	276, 214	216, 145
$a^*(\beta) : 3$	75, 145	147, 214	216, 274	276, 214
4	0, 73	75, 145	147, 214	216, 274

$\therefore$ No PBNE exists here, as t_3 has incentive to deviate from β to α.

But, suppose it did not have incentive to deviate. Then, we could make a PBNE by choosing appropriate $a^*(\gamma)$, $a^*(\delta)$ such that our original equilibrium is not affected, i.e., $t \in \{1, 2, 3, 4\}$ does not have an incentive to deviate from $m(t)$.

Choosing $a^*(\gamma) = 4 = a^*(\delta)$ ensures that.

To ensure $a^*(\gamma) = 4 = a^*(\delta)$, we need to find a corresponding belief as well.

By looking at the numbers, we see that the receiver gets the most on playing 4 when $t = 4$. Thus, we can choose $\mu(t_4/\gamma) = 1 = \mu(t_4/\delta)$ so that the complete PBNE is $\{ m(t_1) = \alpha = m(t_2)$, $m(t_3) = \beta = m(t_4)$, $\mu(t_1/\alpha) = \mu(t_2/\alpha) = \mu(t_3/\beta) = \mu(t_4/\beta) = 1/2$, $\mu(t_4/\gamma) = 1 = \mu(t_4/\delta) \}$

4) $m(t_1) = \alpha$, $m(t_2) = \beta$, $m(t_3) = \gamma$, $m(t_4) = \delta$

$\mu(t_1/\alpha) = 1 = \mu(t_2/\beta) = \mu(t_3/\gamma) = \mu(t_4/\delta)$

$a^*(\alpha) = 1$, $a^*(\beta) = 2$, $a^*(\gamma) = 3$, $a^*(\delta) = 4$

Payoff for type 1 : 216   3 : 216
2 : 216   4 : 216

But, 3 has incentive to deviate, as it can get 276 by sending β. Similarly, 2 gets 276 by α, and 4 gets 276 by γ.
Thus, there is no PBNE here.

Two problems that arise due to incomplete information:

- Adverse Selection / Hidden Characteristic Problem
- Moral Hazard / Hidden Action Problem

Adverse Selection (Market for Lemons)

There are two types of products in market:

- Good (peach) product
- Bad (lemon) product

Expected repair cost for peach : 200 units, lemon : 1700 units.

	Good	Bad
Net evaluation of buyer	3000	1500
" " " seller	2500	1000

Suppose the buyer has a prior belief of a product being good (1/2) / bad (1/2). So a buyer is willing to pay 1/2 (3000 + 1500) = 2250 ⟶ All the good products by a seller are drawn away.

To solve this, some information can be given ⟶ Given in the form of guarantee, warrantee, etc.

Moral Hazard (Principal - Agent Problem)

An example is the insurance market, where the Insuree gets careless after buying an Insurance from a company, which leads to a less preferred outcome for the company.

Agent : Insuree, Principal : Insurance Company

· Anywhere there is a transaction in which the interests of the involved parties do not align, there is a potential for moral hazard.

Ex: Landlord — Principal, Peasant — Agent.
Landlords profit is π_H (crop is good), π_L (crop is bad). Peasant's effort can be e_H or e_L. p_H is the prob. of good crop if peasant exerts e_H. Similarly, p_L ($< p_H$).
Payoff of peasant $= U(w) - C(e)$.
$C(e_H) > C(e_L)$. $U' > 0$, $U'' < 0$.

w_H if crop is good, w_L if bad (wages)
$\min \; w_H p_H + w_L(1-p_H)$ — Incentive Compatibility Constraint.

Payoff to the peasant on e_H is
$p_H U(w_H) + (1-p_H) U(w_L) - c(e_H)$. should be greater than payoff for e_L

There is also the Individual Rationality Constraint:

$$p_H U(w_H) + (1-p_H) U(w_L) - c(e_H) \geq \bar{U}$$ Payoff at next best alternative

Finally, the problem is

$$\min_{w_H, w_L} \; p_H w_H + (1-p_H) w_L$$
$$\text{s.t.} \quad p_H U(w_H) + (1-p_H) U(w_L) - c(e_H) > p_L U(w_H) + (1-p_L) U(w_L) - c(e_L),$$
$$p_H U(w_H) + (1-p_H) U(w_L) - c(e_H) \geq \bar{U}$$

General Equilibrium Theory

Theory of determination of equilibrium prices & quantities in a set of perfectly competitive markets.

Three basic Economic activities:
1) Consumption
2) Production
3) Exchange

Pure Exchange Economy

Consumption	Production	Exchange
✓	✗	✓

· Also known as a Barter Exchange System.

Ex: Two Agent, Two Goods Model

A B ↩ $\qquad$ x y ↩

Characteristics of Agents:

1) Taste / Preferences of agent

$U_A(x_A, y_A)$, $\quad U_B(x_B, y_B)$

2) Endowment vector

$(\bar{x}_A, \bar{y}_A)$, $\bar{x}_A \geq 0$, $\bar{y}_A \geq 0$, $(\bar{x}_B, \bar{y}_B)$

3) Purchasing power of endowment vector

$(\bar{x}_A, \bar{y}_A)$: $\quad p_x \bar{x}_A + p_y \bar{y}_A$

Exchange Rates

Planned expenditure: $p_x x_A^d + p_y y_A^d \leq p_x \bar{x}_A + p_y \bar{y}_A$

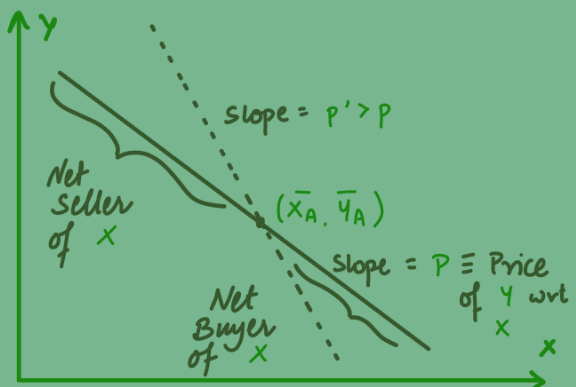
$x_A^d > \bar{x}_A \longrightarrow$ Net buyer of x

$x_A^d < \bar{x}_A \longrightarrow$ Net seller of x

In this situation ⟵

$p \uparrow p' \equiv$ value of $x \downarrow$

So, a seller of x is better off in this case.



☆ An allocation is an assignment of non-negative consumption vector to each individual.

$\{'\} \equiv \{(x_A', y_A'), (x_B', y_B')\}$

$\{-\} \equiv \{(\bar{x}_A, \bar{y}_A), (\bar{x}_B, \bar{y}_B)\}$

An allocation is feasible if

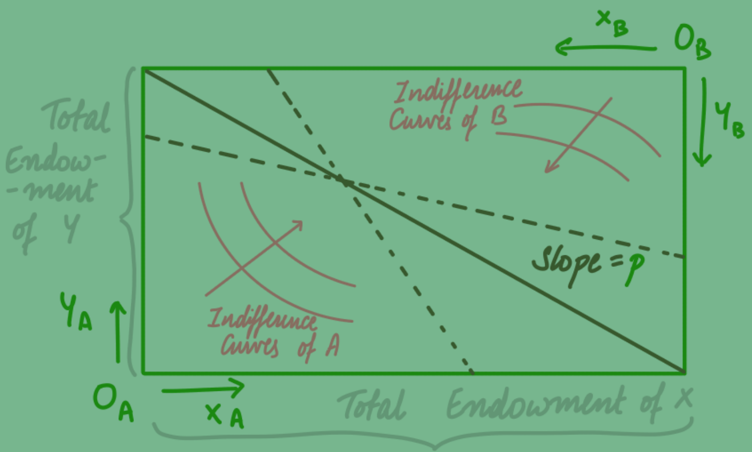
$x_A' + x_B' \leq \bar{x}_A + \bar{x}_B$, $\quad y_A' + y_B' \leq \bar{y}_A + \bar{y}_B$

An allocation is non-wasteful if,

$x_A' + x_B' = \bar{x}_A + \bar{x}_B$, $\quad y_A' + y_B' = \bar{y}_A + \bar{y}_B$

Edgeworth Box

depicts non-wasteful allocations



Notions of :

- Competitive Equilibrium
- Pareto Efficiency
- Theorems of Welfare Economics

Competitive Equilibrium

$$\{\underbrace{(p_x^*, p_y^*)}_{\text{Price Component}} ; \underbrace{(x_A^*, y_A^*), (x_B^*, y_B^*)}_{\text{Allocation Component}}\}$$

will constitute a CE if :

i) (x_A^*, y_A^*) maximizes $U_A(x_A, y_A)$

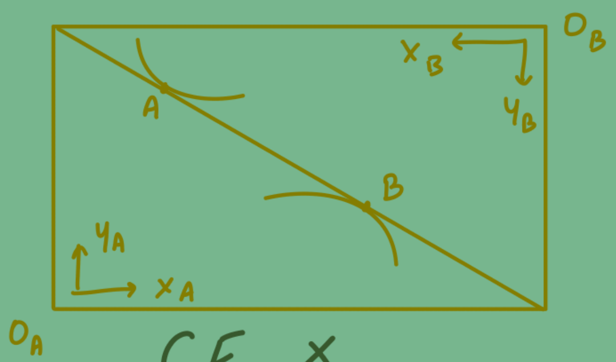
s.t. $p_x x_A + p_y y_A \leq p_x \bar{x}_A \overset{x_A, y_A}{+} p_y \bar{y}_A$

ii) Similarly, for (x_B^*, y_B^*)

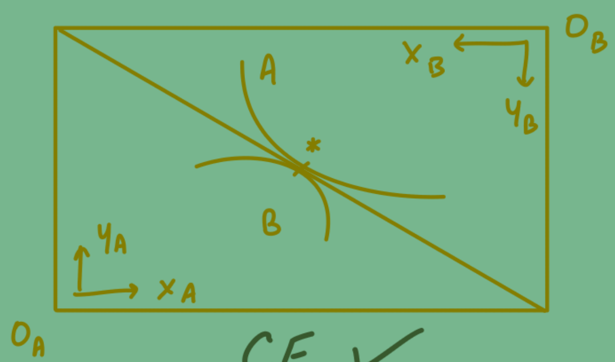
iii) Markets Clear : $x_A^* + x_B^* = \bar{x}_A + \bar{x}_B$

$y_A^* + y_B^* = \bar{y}_A + \bar{y}_B$

Ex:



CE ×



CE ✓

Because of (iii) : $x_A^* + x_B^* < \bar{x}_A + \bar{x}_B$

$\Rightarrow$ Excess Supply of X in the market.

Similarly, there is excess demand of Y in the market. Can also be concluded from Walras' Law.

Walras' Law: In equilibrium,

Assumes MIB (More Is Better)

$$p_x E_x + p_y E_y \equiv 0 \quad \forall \; p_x, \; p_y$$

$$p_x \left[x_A' - \bar{x}_A + x_B' - \bar{x}_B \right] + p_y \left[y_A' - \bar{y}_A + y_B' - \bar{y}_B \right] \equiv 0$$

$$\equiv \left(p_x x_A' - p_x \bar{x}_A + p_y y_A' - p_y \bar{y}_A \right) + \left(\ldots \right)_B \equiv 0$$

Ex: $U^A = x_1^a \, x_2^{1-a}$, $\quad 0 < a < 1$, $\quad E^A = (1, 0)$,

$U^B = x_1^b \, x_2^{1-b}$, $\quad 0 < b < 1$, $\quad E^B = (0, 1) \equiv (\bar{x}_{1B}, \bar{x}_{2B})$

For A: $\max\limits_{x_1^A, \, x_2^A} U^A = \max\limits_{x_1^A, \, x_2^A} x_1^{A\,a} \, x_2^{A\,1-a}$ s.t. $p_1 x_1^A + p_2 x_2^A = p_1$

For B: $\max\limits_{x_1^B, \, x_2^B} x_1^{B\,b} \, x_2^{B\,1-b}$ s.t. $p_1 x_1^B + p_2 x_2^B = p_2$

Solve the Lagrangians:

$$\mathcal{L}_A = x_1^{A\,a} \, x_2^{A\,1-a} - \lambda_A \left(p_1 x_1^A + p_2 x_2^A - p_1 \right)$$

$$\mathcal{L}_B = x_1^{B\,b} \, x_2^{B\,1-b} - \lambda_B \left(p_1 x_1^B + p_2 x_2^B - p_2 \right)$$

to obtain:

$x_1^A = a \qquad\qquad x_2^A = (1-a) \, p_1 / p_2 = b$

$x_2^A = b \, p_2 / p_1 = 1-a \qquad x_2^B = 1-b$

Using these quantities, Walras' Law can be verified.

To find prices p_1 and p_2, use the market clearing conditions:

$$\left. \begin{array}{l} x_1^A + x_1^B = 1 = x_2^A + x_2^B \\ a + b \, p_2 / p_1 = (1-a) \, p_1 / p_2 + 1 - b = 1 \end{array} \right\} \frac{p_1}{p_2} = \frac{b}{1-a}$$

(a & b are constants)

Ex: Suppose that there are 2 countries, 2 goods, & 2 consumers in each country. In country A (B), endowment of consumer i ($i = 1, 2$) is denoted by

$W^{Ai}(W^{Bi})$: $W^{A1} = W^{B1} = (1, 0)$ and $W^{A2} = W^{B2} = (0, 1)$, $U^A(x) = x_1^a x_2^{1-a}$, $U^B = x_1^b x_2^{1-b}$. Both persons in the same country face the same utility function.

1) Find CE in country A
2) Find CE in country B
3) Suppose trade across countries is free. Find free-trade CE.

1) $x_1^{1*} + x_1^{2*} = 1 = \overbrace{x_2^{1*} + x_2^{2*}}^{\text{demand for good 2}}$

For 1: $\mathcal{L}_1 = x_1^{1*\,a} x_2^{1*\,1-a} - \lambda_1 (p_1 x_1^{1*} + p_2 x_2^{1*} - p_1)$

For 2: $\mathcal{L}_2 = x_1^{2*\,a} x_2^{2*\,1-a} - \lambda_2 (p_1 x_1^{2*} + p_2 x_2^{2*} - p_2)$

$x_1^{1*} = a$, $x_2^{1*} = (1-a)\, p_1/p_2$, $x_1^{2*} = a\, p_2/p_1$,

$x_2^{2*} = 1-a$, $p_1/p_2 = a/1-a$ $(\because x_1^{1*} + x_1^{2*} = 1 = x_2^{1*} + x_2^{2*})$

2) Similar answers as above.

3) A1 : $\mathcal{L}_1^A = x_1^{A1*\,a} x_2^{A1*\,1-a} - \lambda_1^A (p_1 x_1^{A1*} + p_2 x_2^{A1*} - p_1)$

A2 : $\mathcal{L}_2^A = x_1^{A2*\,a} x_2^{A2*\,1-a} - \lambda_2^A (p_1 x_1^{A2*} + p_2 x_2^{A2*} - p_2)$

B1 : $\mathcal{L}_1^B = x_1^{B1*\,b} x_2^{B2*\,1-b} - \lambda_1^B (p_1 x_1^{B1*} + p_2 x_2^{B1*} - p_1)$

B2 : $\mathcal{L}_2^B = x_1^{B2*\,b} x_2^{B2*\,1-b} - \lambda_2^B (p_1 x_1^{B2*} + p_2 x_2^{B2*} - p_2)$

$x_1^{A1*} + x_1^{A2*} + x_1^{B1*} + x_1^{B2*} = x_1^{A1} + x_1^{A2} + x_1^{B1} + x_1^{B2} = 2$

Similarly, $x_2^{A1*} + x_2^{A2*} + x_2^{B1*} + x_2^{B2*} = 2$

Solve the above $\mathcal{L}_i^{A/B}$ to get:

$x_1^{1A} = a$ $x_2^{1A} = (1-a)\, p_1/p_2$

$x_1^{2A} = a\, p_2/p_1$ $x_2^{2A} = 1-a$

$x_1^{1B} = b$ $x_2^{1B} = (1-b)\, p_1/p_2$

$x_1^{2B} = b\, p_2/p_1$ $x_2^{2B} = 1-b$

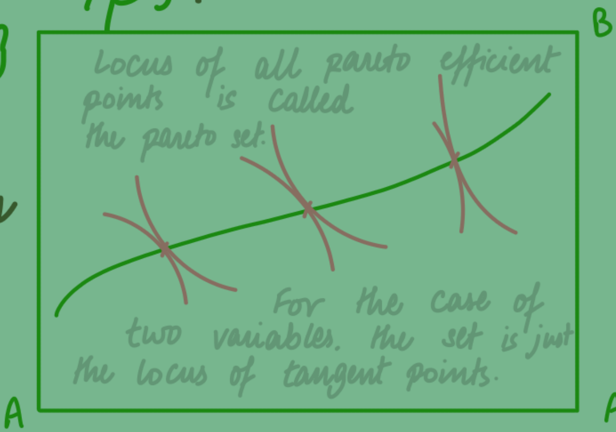
Use market closure constraints to obtain $\dfrac{p_2}{p_1} = \dfrac{2-a-b}{a+b}$ } Assuming that the prices are the same in both the countries

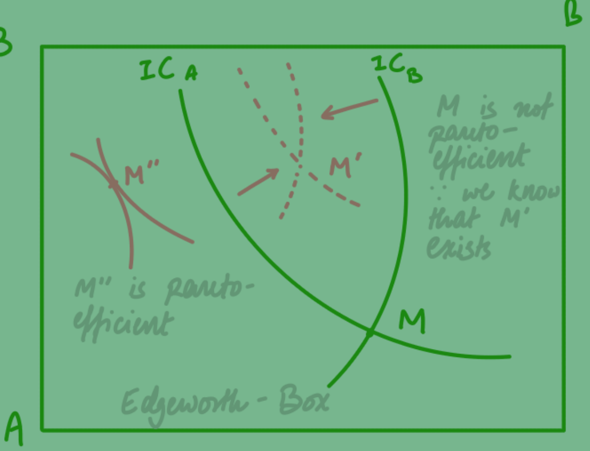
Pareto Efficiency

Criterion for judging the welfare of different allocations.

An allocation is efficient if there is NO alternative feasible allocations at which every individual in the economy is at least as well off & some individual is strictly better off (in terms of utility).

- If we call something efficient, we implicitly mean "Pareto Efficient".
- Allocation $\{a\}$ is at least as good as $\{\beta\}$: $\{\alpha\} \succsim_i \{\beta\}$, strictly better than $\{\beta\}$: $\{\alpha\} \succ_i \{\beta\}$.
- $\{a\}$ is pareto-dominated by $\{\beta\}$ if $\{\beta\} \succsim \{\alpha\}$ $\forall$ i and $\{\beta\} \succ \{\alpha\}$ for at least one i.
- $\{\alpha\}$ is called pareto-efficient (PE) if there is no such $\{\beta\}$.
- An $\{x\}$ in an Edgeworth Box is PE if there is no other allocation $\{x'\}$ in the EB with $x_i' \succsim x_i$, $i = 1, 2$ and $x_i' \succ x_i$ for at least one i.





- Mathematical way to find a PE allocation:

$$\max_{x_A, x_B, y_A, y_B} U^A(x_A, y_A) \quad \text{s.t.} \quad U^B \geq \bar{U}^B,$$

$x_A + x_B = \bar{x}_A + \bar{x}_B$, $y_A + y_B = \bar{y}_A + \bar{y}_B$

The MRS is equal at PE as a result.

Ex: Romeo-Juliet Problem

$U_R(S_R, S_J) = S_R^{\alpha} S_J^{1-\alpha}$, $\alpha = 2/3$,

$U_J(S_R, S_J) = S_J^{\alpha} S_R^{1-\alpha}$. $S_R + S_J = 24$

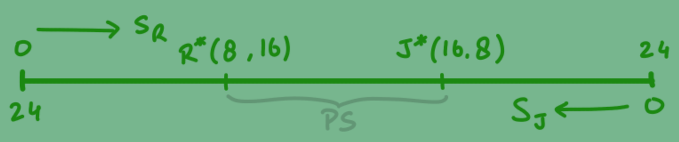
a) Romeo's preferred allocation

b) Juliet's preferred allocation

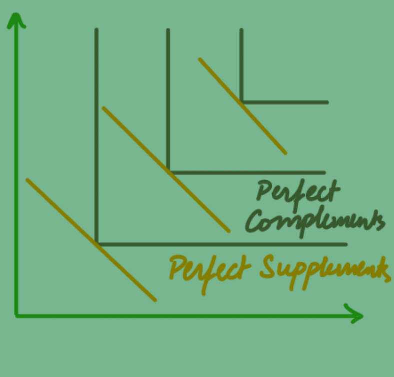
c) Pareto Set

Solve Lagrangians for (a) (16, 8) and (b) (8, 16)

(c) All the points between (16, 8) and (8, 16) constitute the Pareto Set, as moving in any direction makes one person worse off (though, it also makes one person better off).



☆ It might not always be easy to find the PS mathematically (non-differentiable functions, etc.). For example, if one person considers the two goods perfect complements and the other considers them perfect supplements.



Ex: Two persons: I & N, Two goods: B & P

$U_I = B_I + 2\sqrt{P_I}$, $U_N = B_N + 4\sqrt{P_N}$

a) Sketch the PS

b) Find the CE

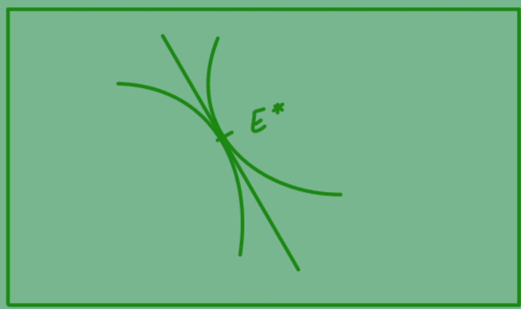
· There is an inherent trade-off between Efficiency and Equity.

First Fundamental Theorem of Welfare Economics $\therefore CE \Rightarrow PE \therefore$

$CE : \{(p_x^*, p_y^*), (x_A^*, y_A^*), (x_B^*, y_B^*)\}$

$\because$ They are tangents $\left\{ \begin{array}{l} MRS^A = p^* \text{ and} \\ MRS^B = p^* \end{array} \right.$

$\Rightarrow MRS^A = MRS^B$



An assumption is MIB.

Need to show that $\{*\}$ is PE.

Proof : By contradiction. Let $\{*\}$ not be a PE. Then, there is another feasible allocation, say $\{\wedge\}$ which pareto-dominates $\{*\}$.

$\because$ $\{\wedge\}$ is feasible, we must have,

$\hat{x}_A + \hat{x}_B \leq \bar{x}_A + \bar{x}_B$ and $\hat{y}_A + \hat{y}_B \leq \bar{y}_A + \bar{y}_B$

$\Rightarrow p_x^* \hat{x}_A + p_x^* \hat{x}_B + p_y^* \hat{y}_A + p_y^* \hat{y}_B$

$\leq p_x^* \bar{x}_A + p_x^* \bar{x}_B + p_y^* \bar{y}_A + p_y^* \bar{y}_B$

WLOG, we may assume

$\{\hat{x}_A, \hat{y}_A\} \succ_A \{x_A^*, y_A^*\}$, $\{\hat{x}_B, \hat{y}_B\} \succsim_B \{x_B^*, y_B^*\}$

At prices (p_x^*, p_y^*), A chooses $\{*\}$, even though A prefers $\{\wedge\}$ more. This means that $\{\wedge\}$ must have been unaffordable at A's BC (Budget Constraint).

$\therefore p_x^* \hat{x}_A + p_y^* \hat{y}_A > p_x^* x_A^* + p_y^* y_A^*$

$= p_x^* \bar{x}_A + p_y^* \bar{y}_A$

Similarly, we can write for B:

$$p_x^* \hat{x}_B + p_y^* \hat{y}_B \geq p_x^* x_B^* + p_y^* y_B^* = p_x^* \bar{x}_B + p_y^* \bar{y}_B$$

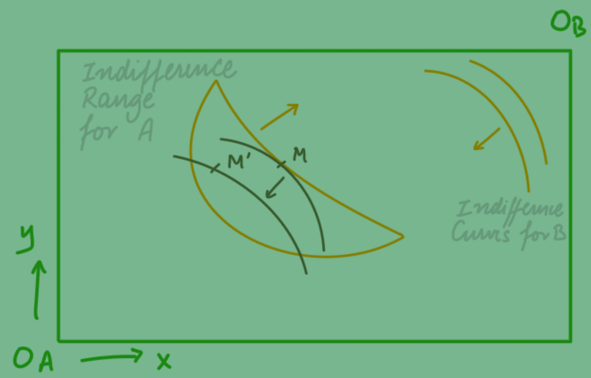
Summing them up yields:

$$\{p^*\} \cdot \{\hat{A}\} + \{p^*\} \cdot \{\hat{B}\} > \{p^*\} \cdot \{\bar{A}\} + \{p^*\} \cdot \{\bar{B}\}$$

A contradiction... Hence, the proof is complete.

$\therefore$ Any CE is PE.

- Suppose MIB is not true for A, but it does hold for B.



Here, M is a CE but not a PE, as M′ pareto dominates M. So, without MIB assumption, CE ⇒ PE (i.e., the 1st law) does not hold true.

* No Incomplete Information — Information Economics

* No Market Power — Industrial Economics

* Public Good — Public Economics

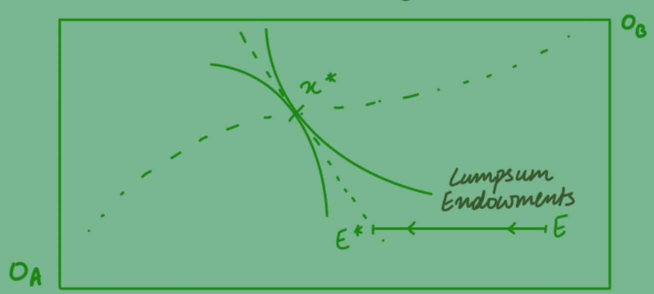
* Externality — Environmental Economics

Second Fundamental Theorem of Welfare Economics

$\because \forall$ PE $\exists$ CE (PE) $\because$

Given prices P^* & an initial endowment E, x^* is not CE

⇒ lumpsum transfers to change endowment from E to E′.



Under certain conditions, any PE can be achieved as a CE using lumpsum transfers:

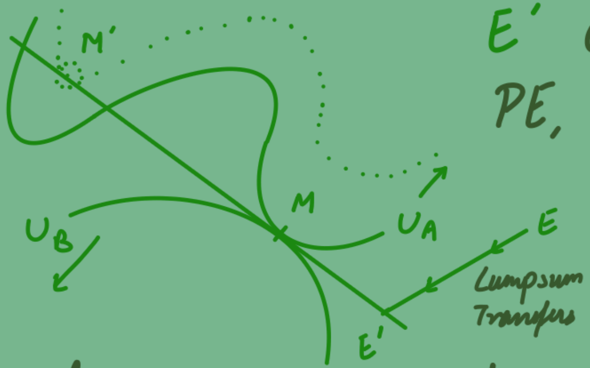
- MIB
- Convexity of Preferences
- All components of chosen PE are +ve.

→ What if preferences are not convex?

Suppose A does not have convex preferences

{ Convex preferences ≡ Averages are preferred to extremes }

Example



E' converges to a PE, M, but it is not a CE, as A's utility is more at M'.

So, M does not maximize the utility of A, given the budget constraints.

Welfare Optimum & Pareto Efficiency

$W(u_1 \ldots u_N)$: Assign social utility values to various of combinations of utilities for N persons.

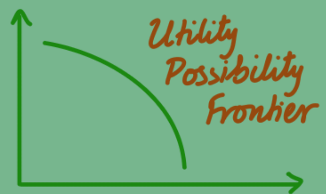
$(\partial W/\partial u_1 > 0, \ldots, \partial W/\partial u_N > 0)$ → A common welfare function

$V(x_1 \ldots x_n)$

Social Indifference Curve

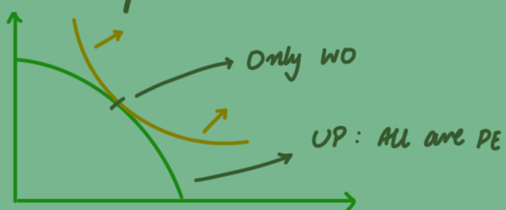
Combination of different utility levels that will yield a constant welfare value.

- The set of attainable utility levels is called Utility Possibility Set.



$UP = \{(U_1 \ldots U_I) \in U\}$ s.t. there is no other set of utilities $(U_1' \ldots U_I') \in U$ s.t.
$U_i' \geq U_i \;\; \forall \; i$ & $U_i' > U_i$ for atleast one i

☆ PE is necessary for WO (Welfare Optimum)
$\equiv$ WO $\Rightarrow$ PE
Suppose there is a WO which is not PE
$\Rightarrow \exists$ another allocation at which all are better off and atleast one is strictly better off $\Rightarrow \because dU_i > 0,\ \partial W/\partial U_i > 0 \Rightarrow$ Original point was never a WO in the first place.



☆ PE is not sufficient for WO.

· We would like to have a way to tell the following about a CE:
- Existence
- Uniqueness
- Stability

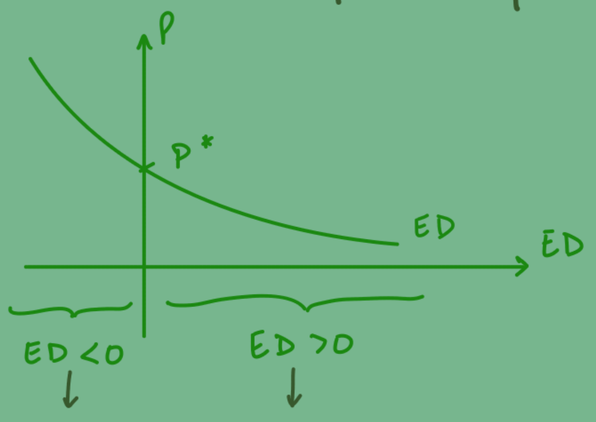
☆ Offer Curve: Locus of various Equilibrium Points as the price changes.

☆ ED (Excess Demand) > 0 then price must rise to push ED towards zero. Similarly, if ED < 0 then price is pushed down.

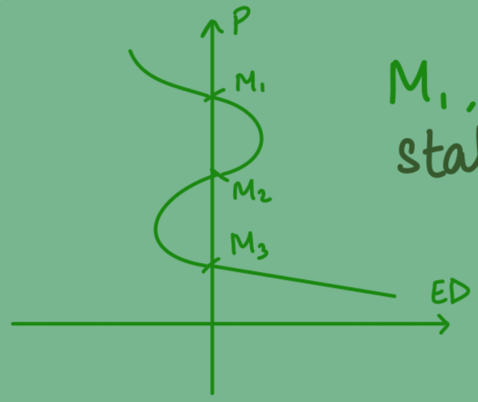
Stability Condition

Global : For any initial price $P_0 \neq P^*$, $P \to P^*$ as $t \to \infty$

Local : $P \to P^*$ as $t \to \infty$ for initial price falling in some range.



Price ↓, Price ↑ towards P^*, so it is a (globally) stable price.



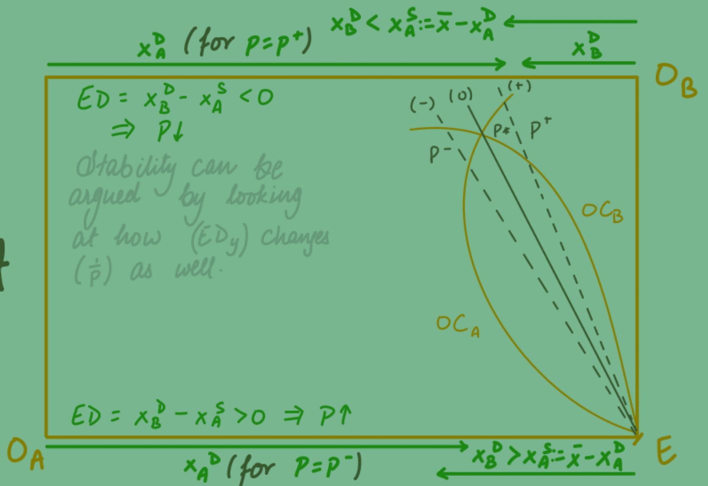
M_1, M_3 : Locally stable prices

M_2 : Not a stable price. For $P = M_2^+$, $ED > 0$, $P\uparrow$ to M_1. For $P = M_2^-$, $ED < 0$, $P \downarrow$ to M_2.

· Slope of ED function evaluated at equilibrium price should be < 0.

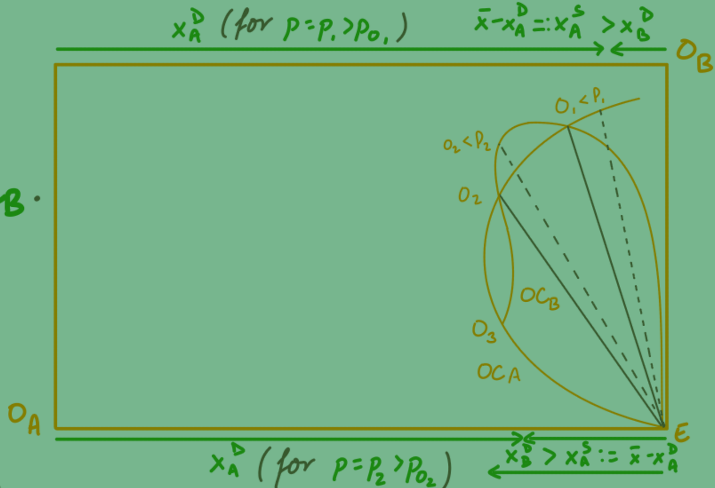
Ex: E : $A(\bar{x}, 0)$, $B(0, \bar{y})$

Find Equilibrium price and classify it.



OC_A and OC_B intersect at a slope of P^*. So, P^* is a CE, but what about its stability? Suppose $P^* \uparrow P^+$. Then, A wants $x_A^D \Rightarrow$ Supplies $x_A^S (= \bar{x} - x_A^D)$. But, B only wants $x_B^D (< x_A^S) \Rightarrow ED < 0 \Rightarrow P \downarrow P^*$. Similarly, if $P^* \downarrow P^- \Rightarrow x_B^D > x_A^S \Rightarrow ED > 0 \Rightarrow P \uparrow P^*$. ∴ Globally stable.

Ex: $E: A(\bar{x}, 0),\ B(0, \bar{y})$



All of O_1, O_2, O_3 are intersections of OC_A / OC_B. But, only O_1 & O_3 are locally stable. O_2 is not stable.

At O_1, if $P \uparrow P_1$, then $x_B^D < x_A^S \Rightarrow ED < 0 \Rightarrow P \downarrow$ back to P_{O_1}.

Similarly, it can be checked for $P = P_{O_1}^- / P_{O_3}^+ / P_{O_3}^-$.

But at O_2, if $P \uparrow P_2$, then $x_B^D > x_A^S \Rightarrow ED > 0 \Rightarrow P \uparrow$.

Similarly, for $P = P_{O_2}^-$. $\therefore O_2$ is not stable.

Robinson Crusoe Economy

1 consumer, 1 producer economy

- Factor of production: Labour
  ↳ Total endowment is $\bar{L}$
- $U(x_1, x_2)$: $x_1 :=$ Leisure, $x_2 :=$ Regular Consumer Good
- The firm uses labour, $z = \bar{L} - x_1$, to produce $q = f(z)$

Firm / Producer's Problem $\left\{ \max\limits_{z} \ \pi = pq - zw \right.$
$\Rightarrow z^* = z(p, w)$

$\because$ Consumer and producer is the same:

$$\max_{x_1, x_2} U(x_1, x_2) \quad \text{s.t.} \quad \underbrace{p x_2}_{\text{Expenditure}} = \underbrace{\pi(p, w) + w(\bar{L} - x_1)}_{\text{Earning}}$$

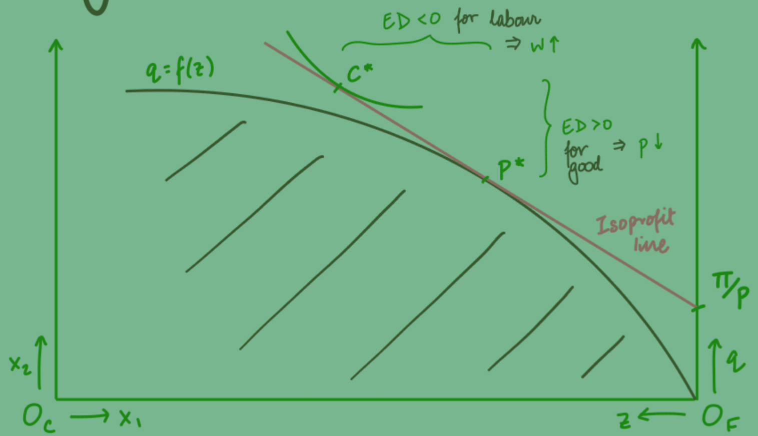
- A CE is a (p^*, w^*) and (x_1^*, x_2^*)

$$\underbrace{x_2^*(p^*, w^*)}_{\text{Demand for Consumer Good}} = \underbrace{q^*(p^*, w^*)}_{\text{SS of Consumer Good}} \quad \Big\} \text{ Consumer Good market clears}$$

{SS ≡ Supply, DD ≡ Demand}

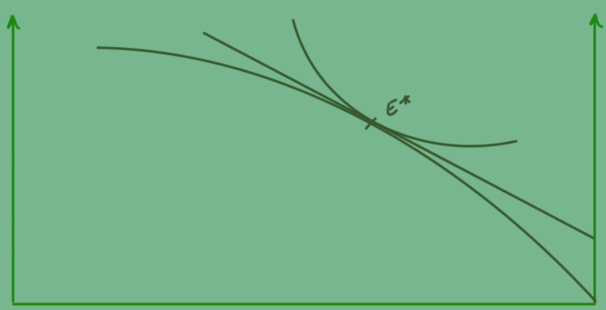
$$\underbrace{\bar{L} - x_1^*(p^*, w^*)}_{\text{SS of labour by Consumer}} = \underbrace{z^*(p^*, w^*)}_{\text{Labour Demand by Producer}} \quad \Big\} \text{ Labour market clears}$$

- Representing the above system using an Edgeworth Box is difficult because there is a fixed amount of labour (endowment). So, the points near the top-right corner might not be possible.



Isoprofit line:
Combination of z & q that gives the same level of profit.
Slope is dw/dp

- At equilibrium, the Indifference curve, the Isoprofit line and the frontier are tangential to each other at the same point E^*.



Two Goods, Two Factors Model

$f^i(z_{1i}, z_{2i})$: Production function of i^{th} product } Homogeneous function of degree 1

$w = (w_1, w_2)$ are the factor prices

z_{ij} : total amount of factor i used in production of good j.

$c_j(w)$: unit cost function → minimum cost of producing 1 unit of output.

$$c_j = w_1 a_{1j} + w_2 a_{2j}$$

Least Cost Factor Combination { a_{ij} : Amount of i^{th} factor required to produce 1 unit of good j

Equilibrium is z^*_{ij}, P^*_1, P^*_2, w^*_1, w^*_2

Unit Cost Function

$$\Pi_1 = p_1 q_1 - c(q_1) = p_1 q_1 - \underbrace{c(w_1, w_2)}_{\text{Unit Cost Function}} q_1$$

$\frac{\partial \Pi_i}{\partial q_i} = 0 \Rightarrow$ $p_1 = c_1(w_1, w_2)$. $p_2 = c_2(w_1, w_2)$

slope is dw_2/dw_1 $\Rightarrow$ $\frac{\partial c_1}{\partial w_1} dw_1 + \frac{\partial c_1}{\partial w_2} dw_2 = 0 \Rightarrow \frac{dw_2}{dw_1} = -\frac{\partial c_1/\partial w_1}{\partial c_1/\partial w_2} = \frac{-a_{11}}{a_{21}}$

* Incomplete Specialisation model — No firm is strictly better than the other at making a product.

- Production level of both goods is strictly +ve.
- Factor Intensities of production: Production of good i is relatively more intensive in usage of factor x than production of good j if $\frac{a_{xi}(w)}{a_{yi}(w)} > \frac{a_{xj}(w)}{a_{yj}(w)}$ at all factor prices $w = (w_1, w_2)$

Unit Cost Function

$$c_j = w_1 a_{1j} + w_2 a_{2j}$$

$$\frac{\partial c_j}{\partial w_1} = a_{1j} + \underbrace{w_1 \frac{\partial a_{1j}}{\partial w_1} + w_2 \frac{\partial a_{2j}}{\partial w_1}}_{0} + \underbrace{a_{2j} \frac{\partial w_2}{\partial w_1}}$$

By definition of a_{1j}, a_{2j} : $f^j(a_{1j}, a_{2j}) = 1$

$$\Rightarrow \frac{\partial f^j}{\partial a_{1j}} \frac{\partial a_{1j}}{\partial w_1} + \frac{\partial f^j}{\partial a_{2j}} \frac{\partial a_{2j}}{\partial w_2} = 0$$

Also, c_j is the minimum cost of producing 1 unit of output. Underlying the cost function, the following optimization holds:

$$\min_{a_{1j}, a_{2j}} a_{1j} w_1 + a_{2j} w_2 \quad s.t. \quad f(a_{1j}, a_{2j}) = 1$$

$$\Rightarrow \mathcal{L} = a_{1j} w_1 + a_{2j} w_2 + \lambda(-f(a_{1j}, a_{2j}) + 1)$$

$$\frac{\partial \mathcal{L}}{\partial a_{1j}} = w_1 - \lambda \frac{\partial f(a_{1j}, a_{2j})}{\partial a_{1j}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial a_{2j}} = w_2 - \lambda \frac{\partial f(a_{1j}, a_{2j})}{\partial a_{2j}} = 0, \quad \frac{\partial \mathcal{L}}{\partial \lambda} = 0 = 1 - f(a_{1j}, a_{2j})$$

$$\Rightarrow \frac{w_1}{\lambda} = \frac{\partial f(a_{1j}, a_{2j})}{\partial a_{1j}}, \quad \frac{w_2}{\lambda} = \frac{\partial f(a_{1j}, a_{2j})}{\partial a_{2j}}$$

Also, previously, we had, $\frac{\partial c_j}{\partial w_1} = a_{1j}$, $\frac{\partial c_j}{\partial w_2} = a_{2j}$

Now, from $\frac{\partial f^j}{\partial a_{1j}} \frac{\partial a_{1j}}{\partial w_1} + \frac{\partial f^j}{\partial a_{2j}} \frac{\partial a_{2j}}{\partial w_2} = 0$

$$\Rightarrow \frac{w_1}{\lambda} \frac{\partial a_{1j}}{\partial w_1} + \frac{w_2}{\lambda} \frac{\partial a_{2j}}{\partial w_2} = 0 \Rightarrow w_1 \frac{\partial a_{1j}}{\partial w_1} + w_2 \frac{\partial a_{2j}}{\partial w_2} = 0$$

$$\Rightarrow \frac{\partial c_j}{\partial w_1} = a_{1j}$$

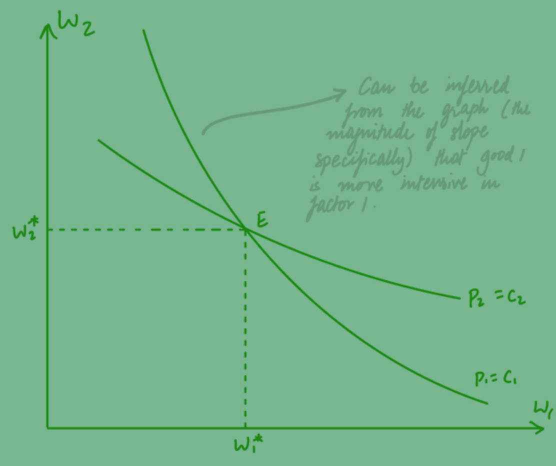
$$\pi_1 = p q_1 - c_1(w_1, w_2) a_1$$

$$\frac{\partial \pi_1}{\partial q_1} = 0 \Rightarrow p_1 = c_1(w_1, w_2), \quad p_2 = c_2(w_1, w_2)$$

Slope $= \frac{dw_2}{dw_1}$: $dp_1 = 0 = \frac{\partial c_1}{\partial w_1} dw_1 + \frac{\partial c_1}{\partial w_2} dw_2$

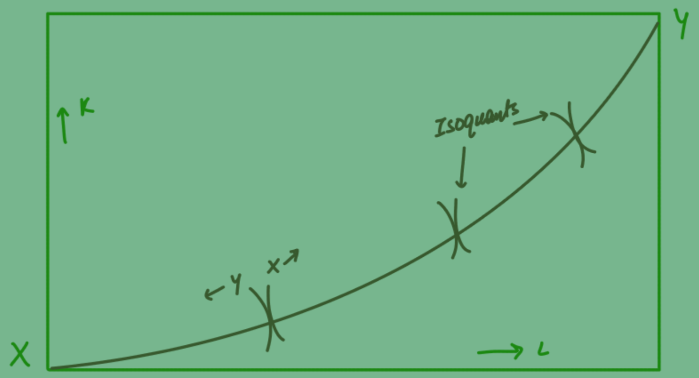
$$\Rightarrow \frac{dw_2}{dw_1} = \frac{-\partial c_1 / \partial w_1}{\partial c_1 / \partial w_2} = \frac{-a_{11}}{a_{21}}$$

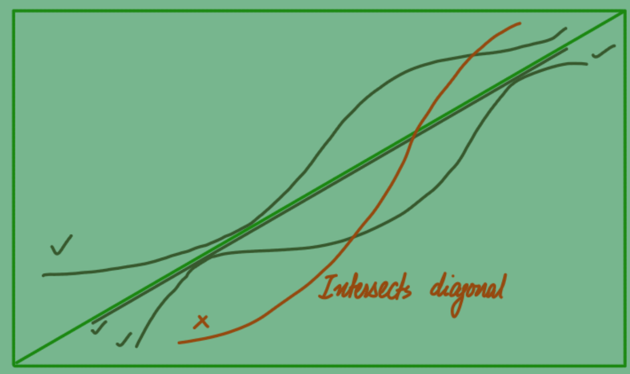
* If factor intensity production holds (say $a_{11}(w)/a_{21}(w) > a_{1j}(w)/a_{2j}(w) \; \forall w$) then there can be only one equilibrium point in the $p_1 = c_1$ and $p_2 = c_2$ curve.



· If the production function is CRS, then the Pareto frontier either stays completely above, or completely below or is the diagonal of the EB.

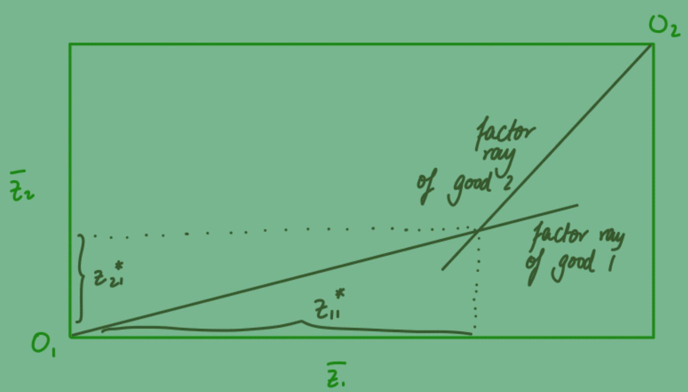
- Isoquant lines instead of indifference curves.



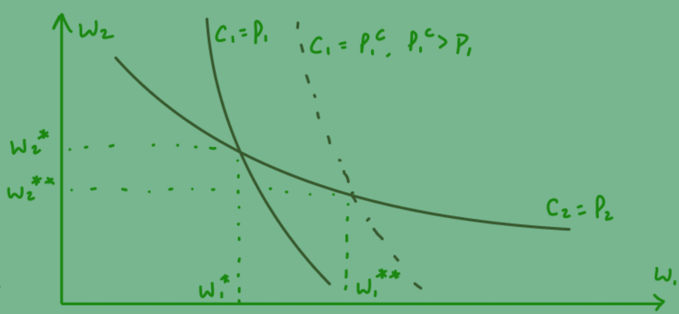


- Returning to previous discussions, production of good 1 is more intensive in use of factor 1 than production of good 2 if $\frac{a_{11}(w)}{a_{21}(w)} > \frac{a_{12}(w)}{a_{22}(w)} \; \forall w$

And in the Edgeworth box, we use factor rays for each good.
Their intersection is at the equilibrium point.



- Now, suppose $P_1 \uparrow$:
From the graph, it is easy to see that good 1 is more factor intensive in factor 1 (by comparing slopes).



Consequently, the price of factor 1 $\uparrow$ as price of good 1 (P_1) $\uparrow$.

The same can be shown mathematically as well.

$$p_1 = c_1(w_1^*, w_2^*) = a_{11}(w_1^*)\, w_1 + a_{21}(w_2^*)\, w_2$$

$$\Rightarrow 1 = \frac{\partial a_{11}}{\partial p_1} w_1 + a_{11} \frac{\partial w_1}{\partial p_1} + \frac{\partial a_{21}}{\partial p_1} w_2 + a_{21} \frac{\partial w_2}{\partial p_1}$$

$$= 0 + a_{11}\, \partial w_1 / \partial p_1 + 0 + a_{21}\, \partial w_2 / \partial p_1$$

$$p_2 = c_2(w_1^*, w_2^*) \Rightarrow 0 = a_{12}\, \partial w_1 / \partial p_1 + a_{22}\, \partial w_2 / \partial p_1$$

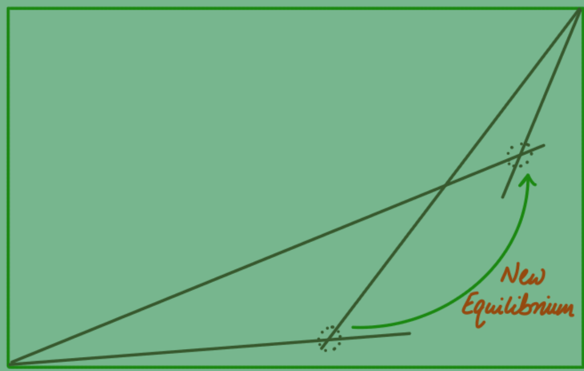
$$\Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} \partial w_1^* / \partial p_1 \\ \partial w_2^* / \partial p_2 \end{bmatrix}$$

$$\Rightarrow \frac{\partial w_1^*}{\partial p_1} = \frac{a_{22}}{a_{11} a_{22} - a_{12} a_{21}} > 0 \iff \left.\frac{a_{11}}{a_{21}} > \frac{a_{12}}{a_{22}}\right\} \text{Factor Intensity}$$

$$\partial w_2^* / \partial p_1 = -a_{12} / (a_{11} a_{22} - a_{12} a_{21}) < 0$$

$$\Rightarrow p_1 \uparrow \Rightarrow w_1^* \uparrow,\ w_2^* \downarrow$$

- Further consequences on the equilibrium can be realised by looking at how the factor rays change in the EB.



Thus, as $p_1 \uparrow$, prod. of good 1 ↑ while good 2 ↓ (isoquants move). Simultaneously, productions of both the goods move to factor rays which are less intensive in factor 1 (Slope ↑ ⇒ more factor 2 used per unit of factor 1)

Stopler-Samuelson Theorem

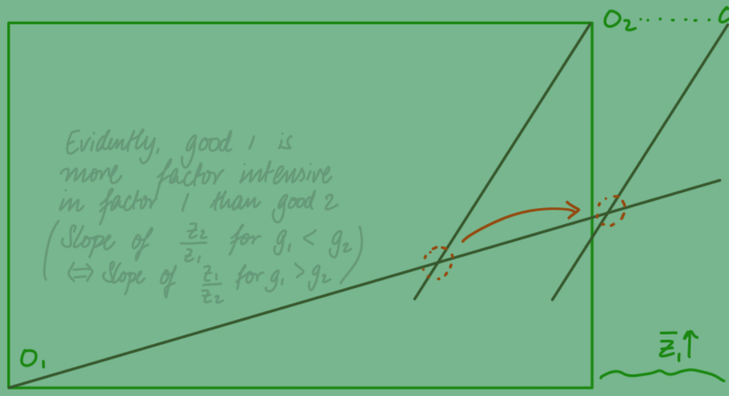
The above inferences can be summarized more generally and concisely using this theorem.

∴ In a 2×2 production model with factor intensity assumption, if price of good j increases, the equilibrium price of the factor used more intensively in production of good j increases, while that of the other factor decreases.

· This phrasing also explains the factor rays' movement, as now they would like to look for ways that are less intensive in that factor.

Rybcszynski Theorem

In a 2×2 production model with factor intensity assumption, if endowment of a factor increases, then the production of the good that uses the factor more intensively increases, while the output of the other good decreases.



· Graphically, what happens if $\bar{z}_1 \uparrow$?
$\longrightarrow$ Production of good 1 $\uparrow$, good 2 $\downarrow$ as good 1 is more intensive in z_1.

· Mathematically, say $a_{L1}/a_{K1} > a_{L2}/a_{K2}$ & $L\uparrow$:

$$a_{L1}^* y_1 + a_{L2}^* y_2 = \bar{L} \Rightarrow 1 = a^*_{L1}\, \partial y_1/\partial \bar{L} + a^*_{L2}\, \partial y_2/\partial \bar{L}$$
$$a_{K1}^* y_1 + a_{K2}^* y_2 = \bar{K} \qquad 0 = a^*_{K1}\, \partial y_1/\partial \bar{K} + a^*_{K2}\, \partial y_2/\partial \bar{K}$$
$$\Rightarrow \frac{\partial y_1^*}{\partial \bar{L}} = \frac{a_{K2}}{a_{L1}a_{K2} - a_{L2}a_{K1}} > 0, \quad \frac{\partial y_2^*}{\partial \bar{L}} = \frac{-a_{K1}}{a_{L1}a_{K2} - a_{L2}a_{K1}} < 0$$
$$\Rightarrow \bar{L}\uparrow \Rightarrow y_1^* \uparrow,\ y_2^* \downarrow$$

· Further, if the factor intensity (assumption) is reversed, then so do the final results.