$\mathbb{N} = \{1, 2, 3, 4, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

↳ Can solve $f(x) = x - 5$, $x - 5 = 0$

↳ Cannot solve $x + 5 = 0$

$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

↳ Can solve $x + 5 = 0$

→ but not $px - q = 0$, $p > 1$

$\mathbb{Q} = \left\{ \frac{a}{b} : (a, b) = 1,\ b > 0 \right\}$

↳ $x^2 - 2 = 0$ → $\mathbb{R}$

$x^2 + 1 = 0$

Closed

$\mathbb{C} = \{ (x, y) : x, y \in \mathbb{R} \}$

* $\mathbb{R}$ & $\mathbb{C}$ are complete.

↳ Fields

↳ $\boxed{\langle \mathbb{F}, +, \cdot \rangle}$

* $\mathbb{R}$ is a field.

$\mathbb{R}^2$ is a field → $\because$ not defined

$\mathbb{R}$ is a field $\longrightarrow$ not defined co-ordinate wise

Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2) \in \mathbb{C}$

then $z_1 \cdot z_2 := (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$

* $\boxed{\mathbb{R} \subseteq \mathbb{R}^2}$

* $\boxed{i := (0,1)}$ or $i :=$ root of $x^2+1=0$

Properties: $(x, y) = x + iy$

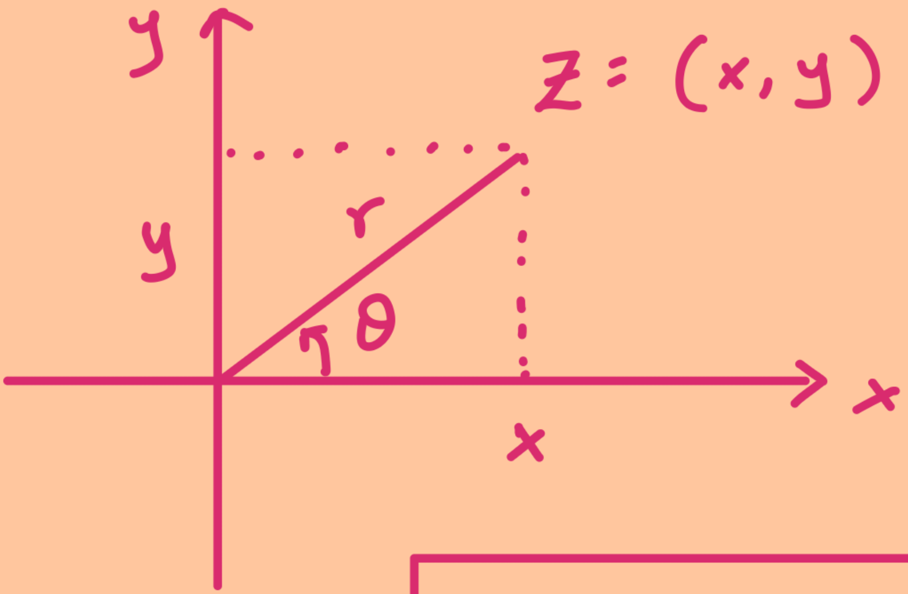
- $y = (y, 0) \in \mathbb{R}$, $i \cdot y = (0, y)$
- $\{e_1, e_2\} \equiv \{(1,0), (0,1)\}$
  $x e_1 + y e_2 = (x, y) \in \mathbb{R}^2$
- $z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$
- $z = (x, y)$, $\bar{z} = (x, -y)$
- $z \bar{z} = (x^2 + y^2, 0) = |z|^2$
- $\frac{1}{z} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right)$
- $z = x + iy = (x, y)$

$Re(z) = x$ $Im(z) = y$

$Re(z) = x \; , \; Im(z) = y$

$$Re(z) = \frac{z + \bar{z}}{2} \; , \; Im(z) = \frac{z - \bar{z}}{2i}$$

Polar Representation



$$z = (x, y) = re^{i\theta}$$
$$= r(\cos\theta, \sin\theta)$$
$$= r\cos\theta + r\sin\theta \, i$$

$$\boxed{\begin{array}{l} r = \sqrt{x^2 + y^2} \\ \tan\theta = y/x \end{array}}$$

- $z_1 = r_1(\cos\theta_1, \sin\theta_1)$, $z_2 = r_2(\cos\theta_2, \sin\theta_2)$

  $z_1 \cdot z_2 = r_1 r_2 \left(\cos(\theta_1 + \theta_2), \sin(\theta_1 + \theta_2)\right)$

- If $z_1 = z_2 \Rightarrow \boxed{\begin{array}{c} r_1 = r_2 \\ \theta_1 = \theta_2 + 2n\pi \end{array}}$

- $\text{arg}(z_1 \cdot z_2) = \text{arg}(z_1) + \text{arg}(z_2)$

Does not always hold, arg ...

↳ $\exists \arg(z_1 z_2)$ given $\arg(z_1), \arg(z_2)$

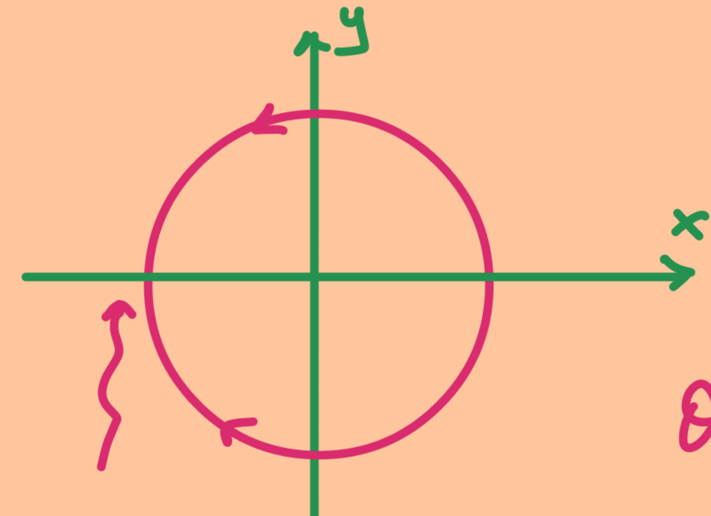
$\exists \arg(z_1), \arg(z_2)$ given $\arg(z_1 z_2)$

· $\mathrm{Arg}(z) = \arg(z)$, $\arg(z) \in (-\pi, \pi]$

or

$[0, 2\pi) \to$ depends.

(we won't use)

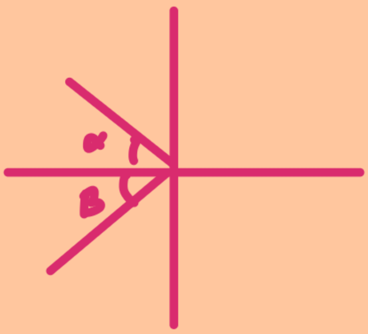


Discontinuity

$\theta = \tan^{-1}(y/x)$,

$-\pi/2 < \tan^{-1}(y/x) \leq \pi/2$

$$\mathrm{Arg}(z) = \begin{cases} \tan^{-1}(y/x), & 1^{st}: x, y > 0 \\ & 4^{th}: x > 0, y < 0 \\ \pi + \tan^{-1}(y/x) = \pi - \alpha, & 2^{nd}: x < 0, y > 0 \\ -\pi + \beta = -\pi + \tan^{-1}(y/x), & 3^{rd}: x, y < 0 \end{cases}$$



$$\mathrm{Arg}(z_1 \cdot z_2) \neq \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$$

Solution of $z^n = c$

$c = r_0(\cos\theta_0, \sin\theta_0) \in \mathbb{C}$
fixed complex no.

$z = r(\cos\theta, \sin\theta)$

$\downarrow$

$z^n = c \iff r^n(\cos n\theta, \sin n\theta) = r_0(\cos\theta_0, \sin\theta_0)$

$\iff r^n = r_0 \quad , \; n\theta = \theta_0 + 2k\pi$

$\iff r = r_0^{1/n} \quad , \; \theta = \dfrac{\theta_0 + 2k\pi}{n} \quad , \; 0 \le k < n, \; k \in \mathbb{Z}$

arg, not Arg

* $c = 1$ $\quad (n > 1) \quad \rightsquigarrow \quad \theta_0 = 0, \; r_0 = r = 1$

$\omega_0 = 1$

$\omega := \omega_1 = \left(\cos\frac{2\pi}{n}, \sin\frac{2\pi}{n}\right) \neq (1, 0)$

$\vdots$

$\omega_k = \left(\cos\frac{2\pi k}{n}, \sin\frac{2\pi k}{n}\right)$

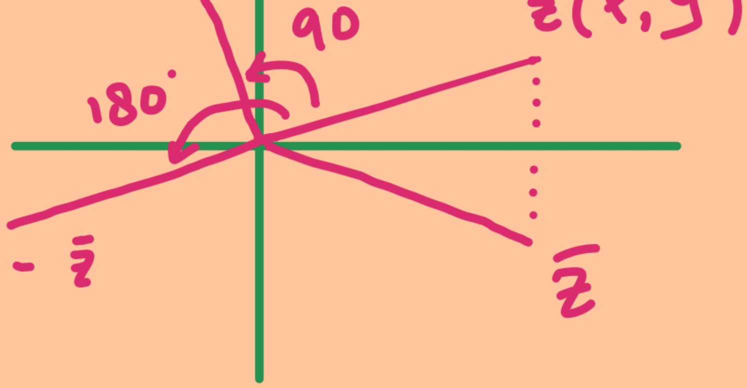
$\vdots$

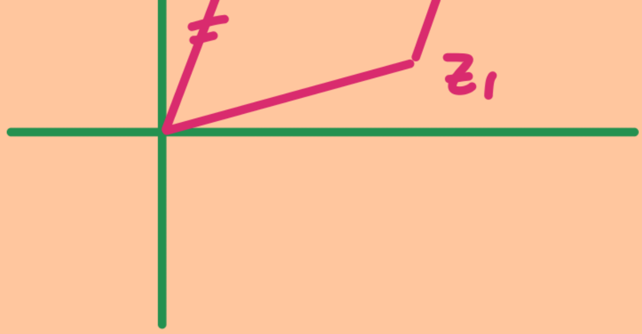
$\omega_{n-1} = \left(\cos\frac{2\pi(n-1)}{n}, \sin\frac{2\pi(n-1)}{n}\right)$

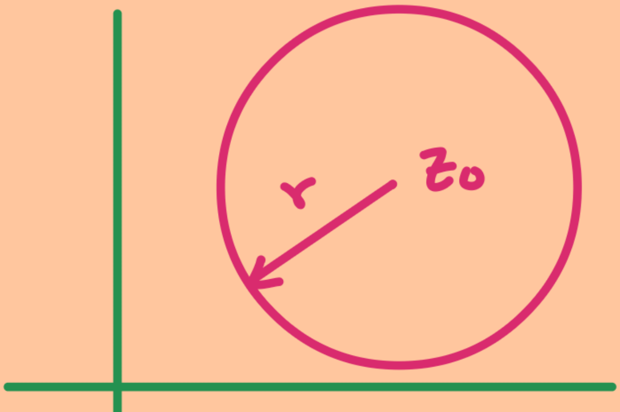
$\omega_2 = \omega_1^1 = \omega^1 \quad \dots \quad \omega_k = \omega^k$

* $\omega_0 + \omega_1 + \dots + \omega_{n-1}$ $\quad \left\{\begin{matrix} \omega^n = 1 \\ \omega \neq 1 \end{matrix}\right\}$

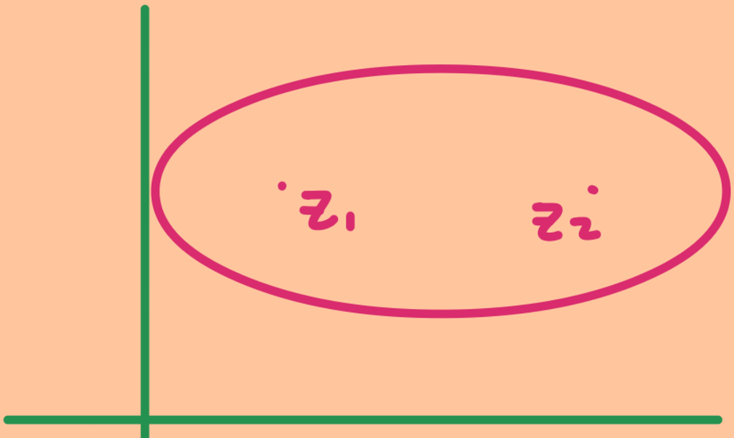
$= 1 + \omega + \dots + \omega^{n-1} = \dfrac{1 - \omega^n}{1 - \omega} = 0$



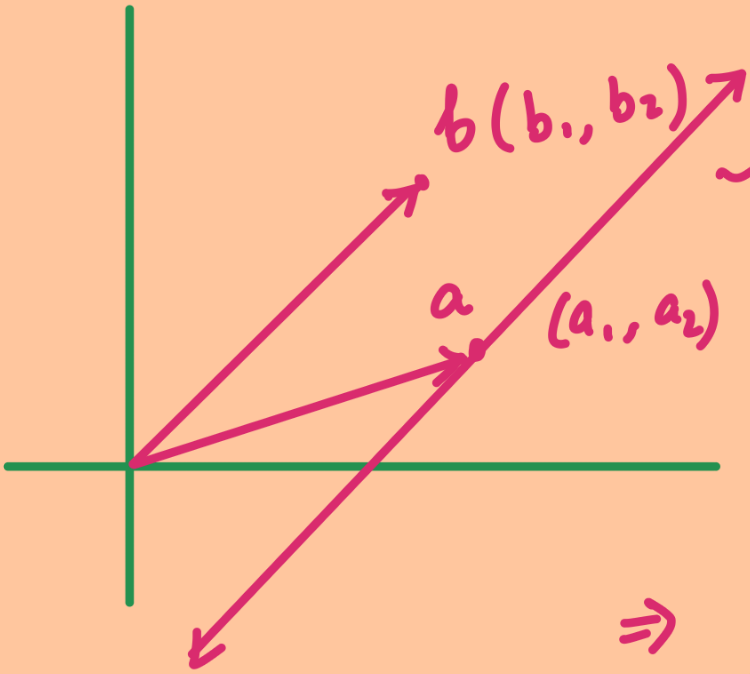




$$\underbrace{|z - z_0|}_{\text{Dist. b/w } z \text{ \& } z_0} = r$$



$$|z - z_1| + |z - z_2| = 2a$$



$\vec{a} + t\vec{b}$

$t \in \mathbb{R}$

$z = a + tb$

$\Rightarrow \dfrac{z-a}{b} = t$

$\Rightarrow \operatorname{Im}\left(\dfrac{z-a}{b}\right) = 0$

★ Let $z_0 \in \mathbb{C}$, $r > 0$

$B_{z_0}(r) = \{ z \in \mathbb{C} : |z - z_0| < r \}$

Def: Let $A \subseteq \mathbb{C}$. Then a point $z_0 \in A$ is called an <u>interior point</u> if $\exists$ a ball $B_{z_0}(r)$ s.t. $B_{z_0}(r) \subseteq A$.

$\rightarrow$ A point $z_0 \in \mathbb{C}$ is called an <u>exterior point</u> of A if z_0 is an interior point of A^c.

Def: A point $z_0 \in \mathbb{C}$ is called a <u>boundary point</u> if $\forall\ B_{z_0}(r)$, $B_{z_0}(r) \cap A \neq \phi$ & $B_{z_0}(r) \cap A^c \neq \phi$

Def: A set $A \subseteq \mathbb{C}$ is called an <u>open set</u> if every point of A is an interior point of A.

Def: $A \subseteq \mathbb{C}$ is called a <u>connected set</u> if for every two points $z_1, z_2 \in A$ $\exists$ a continuous path $\gamma(t)$, $0 \le t \le 1$ s.t. $\gamma(t) \subseteq A$, $\gamma(0) = z_1$, $\gamma(1) = z_2$

Ex.

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$

Def: A Domain is an Open and a connected subset of $\mathbb{C}$.

---

* A sequence of Complex Numbers $\{z_n\}_{n=1}^{\infty}$, $z_n \in \mathbb{C}$

$z_n \to z_0$: if $\forall \, \epsilon > 0 \ \exists \ N_0 = N_0(\epsilon)$ s.t. $\forall \, n > N_0$, $|z_n - z_0| < \epsilon$

Theorem: $z_n = x_n + i y_n$ $\quad \{x_n\}, \{y_n\}$

$\downarrow$

$z_0 = x_0 + i y_0$

iff $x_n \to x_0$ and $y_n \to y_0$

Pf. ($\Rightarrow$) Suppose $z_n \to z_0 \Rightarrow$
$\forall \, \epsilon > 0 \ \exists \ N_0$ s.t. $|z_n - z_0| < \epsilon$ $\forall \, n > N_0$

$|x_n - x_0| \leq |z_n - z_0| < \epsilon$

$\Rightarrow |x_n - x_0| < \epsilon \Rightarrow x_n \to x_0$

$\forall\, n > N_0$

$(\Leftarrow)$ Suppose $x_n \to x_0$ & $y_n \to y_0$

$z_n \to z_0 \Rightarrow |z_n - z_0| < \epsilon \quad \forall\, n \geq N_0$

. Take N_1 s.t. $|x_n - x_0| < \epsilon/\sqrt{2} \quad \forall\, n > N_1$

" $\quad N_2$ " $\quad |y_n - y_0| < \epsilon/\sqrt{2} \quad \forall\, n > N_2$

$|z_n - z_0| < \epsilon \quad \forall\, n > \max\{N_1, N_2\}$

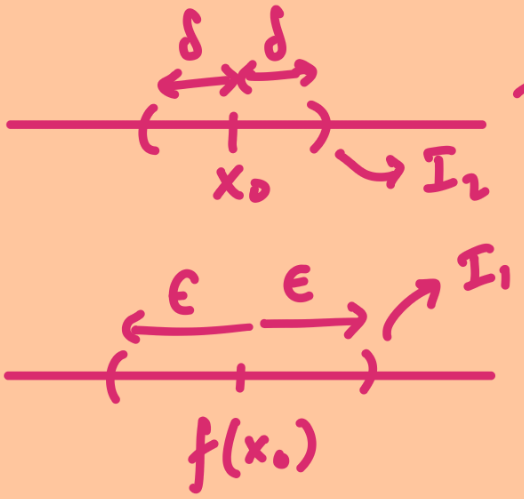
$\hookrightarrow \sqrt{(x_n - x_0)^2 + (y_n - y_0)^2} < \sqrt{\epsilon^2/2 + \epsilon^2/2} = \epsilon$

$\Rightarrow z_n \to z_0$

Continuity of a function

For Reals,

$f : I \to \mathbb{R}$, f is cts. at a point x_0 if $\forall\, \epsilon > 0 \ \exists\ \delta = \delta(x_0, \epsilon) > 0$ s.t. $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$



$x_0 - \delta < x < x_0 + \delta$

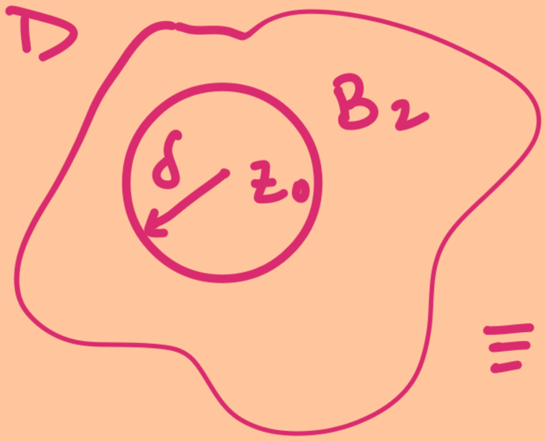
$f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon$

$\downarrow$

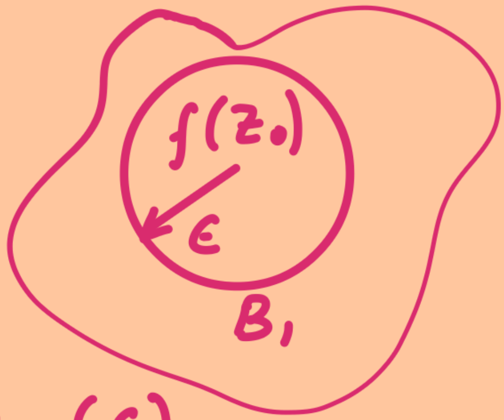
$f(I_2) \subseteq I_1$

Def: Let $D \subseteq \mathbb{C}$ be a domain and let $f: D \to \mathbb{C}$ be a function and $z_0 \in D$. We say that f is continuous at z_0 if $\forall\ \epsilon > 0\ \exists\ \delta = \delta(z_0, \epsilon) > 0$ s.t. $|f(z) - f(z_0)| < \epsilon$ whenever

$$|z - z_0| < \delta$$



$$f(B_2) \subseteq B_1$$



$$\equiv f\left(B_{z_0}(\delta)\right) \subseteq B_{f(z_0)}(\epsilon)$$

Ex. $f(z) = z^2$. At z_0, ϵ given, δ?

$|f(z) - f(z_0)| < \epsilon$ when $|z - z_0| < \delta$

$\hookrightarrow |z^2 - z_0^2| = |z - z_0|\,|z + z_0| < \epsilon$

$\Rightarrow \quad \delta \cdot |z + z_0| < \epsilon \qquad z \sim z_0$

$$\delta < \frac{\epsilon}{|z + z_0|} \leq \frac{\epsilon}{|z| + |z_0|} \leq \frac{\epsilon}{2|z_0| + 1}$$

$\therefore$ Choose $\delta = \epsilon / (2|z_0| + 1)$

Thm. $f(z) = u(x,y) + v(x,y)$, $z = (x,y)$
$= x + iy$

Let $f: D \to \mathbb{C}$, then.
the FAE (following are equivalent):
(1) f is continuous at the point $z_0 \in D$
(2) For every sequence $z_n \to z_0 \Rightarrow f(z_n) \to f(z_0)$

Pf. (1) $\Rightarrow$ (2) Let $z_n \to z_0 \Rightarrow |z_n - z_0| < \delta \ \forall n \geq N$
f is cts. $\Rightarrow |f(z_n) - f(z_0)| < \epsilon$ as $|z_n - z_0| < \delta$
$\forall n \geq N$

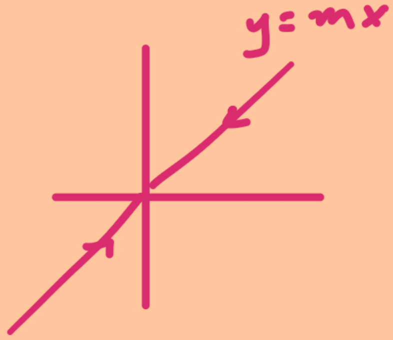
(2) $\Rightarrow$ (1)
Assume f is not cts.
$\Rightarrow \forall \delta > 0 \ \exists z$ s.t. $|z - z_0| < \delta$
but $|f(z) - f(z_0)| \geq \epsilon$

$\delta = 1, 1/2, 1/3, \ldots 1/n \ldots$
$\exists \quad z_n \in B_{z_0}(1/n)$ s.t. $|f(z_n) - f(z_0)| \geq \epsilon$
We obtain that $z_n \to z_0$
but $f(z_n) \not\to f(z_0)$ $\Rightarrow \Leftarrow$

Thm. $f: D \to \mathbb{C}$ is a func.
(1) f is cts. $\iff$ Re(f) & Im(f) are cts.
(2) f is cts. $\Rightarrow |f|$ is cts. {Converse not true
{$||z_1| - |z_2|| \leq |z_1 - z_2|$
$\forall z_1, z_2$

Ex. $f = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{at } (x,y) \neq (0,0) \end{cases}$

$\Big\}$ 0 at $(0,0)$



→ Take $y = mx$

$$\lim_{\substack{x\to 0\\ y=mx}} f = \frac{m}{1+m^2}$$

Not cts. at $(0,0)$

diff. for diff m/curves

Differentiability

Let $f : D \to \mathbb{C}$ be a function at $z_0 \in D$.
We say that f is differentiable at z_0 if

$$\boxed{\lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right) = f'(z_0)}$$

exists and is finite

Thm. Diff. $\Rightarrow$ Cts.

$$|f(z)-f(z_0)| \leq (|f'(z_0)|+c)\,|z-z_0|$$

$$\{x_n \to x_0 \Rightarrow \exists\ N \text{ s.t. } |x_n| \leq |x_0| + c\}$$

Ex. $f(z) = |z|$ at 0.

$$\lim_{z\to 0}\frac{f(z)-f(z_0)}{z-z_0} = \frac{|z|-0}{z-0} = \lim_{z\to 0}\frac{|z|}{z}$$

$z\to 0$, $z = x+0$

$z \to 0$, $z = 0+iy$

Lim d.n.e. $\Rightarrow$ Not cts.

Ex. $f(z) = |z|^2$

$$\lim \frac{f(z+h)-f(z)}{} = \lim_{h\to 0}\frac{|z+h|^2-|z|^2}{h}$$

$$= \lim_{h\to 0} \frac{(h+z)(\bar{h}+\bar{z}) - |z|^2}{h} = \lim_{h\to 0} \frac{h\bar{z} + \bar{h}z + |h|^2}{h}$$

↪ If $z=0$ then $\lim_{h\to 0} \frac{0+0+h\cdot\bar{h}}{h} = \lim_{h\to 0} \bar{h} = 0$

↪ $z \neq 0$ then $\lim_{h\to 0} \frac{\bar{h}}{h} \cdot z + \bar{z} + 0$

$\lim_{h\to 0} \frac{\bar{h}}{h}$ d.n.e. → $h = x+0$, $h = 0+iy$ → $|z|^2$ diff. only at $z=0$

Analytic Function

$f: D \to \mathbb{C}$, $z_0 \in D$. We say that f is analytic at the point z_0 if f is differentiable in some (nbd.) neighbourhood of z_0.

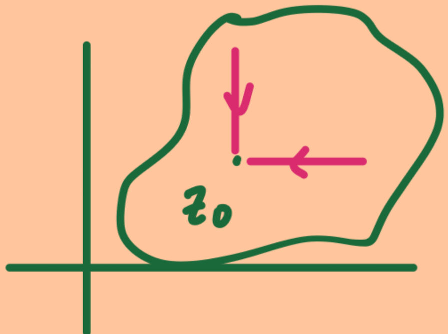
Cauchy Riemann equation

Let $f: D \to \mathbb{C}$ is analytic, $f = u + iv$, then $u_x = v_y$ and $u_y = -v_x$

[Necessary]

Pf. $z \in D$, f is analytic

$\Rightarrow f$ is differentiable at z



$\lim_{h\to 0} \frac{f(h+z) - f(z)}{h}$ exists.

$h = h_1 + i h_2$

$\rightarrow\leftarrow$ $\lim\limits_{\substack{h_1\to 0\\ h_2=0}} \dfrac{f(z+h_1)-f(z)}{h_1}$

$= \lim\limits_{\substack{h_1\to 0\\ h\to 0}} \dfrac{u(x+h_1,y)+iv(x+h_1,y)-u(x,y)-iv(x,y)}{h_1}$

$= u_x + i v_x$

$\updownarrow$ $\lim\limits_{\substack{h_2\to 0\\ h_1=0}} \dfrac{f(z+ih_2)-f(z)}{ih_2} = \lim\limits_{\substack{h_2\to 0\\ h\to 0}} \dfrac{u(x,y+ih_2)+iv(x,y+ih_2)-u(x,y)+iv(x,y)}{ih_2}$

$= \dfrac{u_y}{i} + \dfrac{iv_y}{i} = v_y - i u_y$

f diff. at $z \Rightarrow u_x + i v_x = v_y - i u_y$

$\Rightarrow \boxed{\begin{matrix} u_x = v_y \\ v_x = -u_y \end{matrix}}$

Thm. Suppose $\exists$ two functions u & v s.t. u_x, u_y, v_x, v_y are cts. (in some nbd) and they satisfy CRE (Cauchy-Riemann Equation) then $f = u + iv$ is analytic.

[Sufficient]

Ex. $f(z) = \begin{cases} (\bar{z})^2/z, & z \neq 0 \\ 0, & z = 0 \end{cases}$

$\rightarrow$ CR eq^n holds but func. not analytic

$\rightarrow$ $u_x = v_y = 1$, $u_y = -v_x = 0$

Harmonic Function

Let $u : D(\subseteq \mathbb{R}^2) \to \mathbb{R}$ with cts. partial derivatives of order 2, then we say that u is harmonic if

$$\underline{u_{xx} + u_{yy} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0}$$

Thm. $f : D \to \mathbb{C}$ is Analytic, $f = u + iv$ then u & v are harmonic functions.

$\longrightarrow$ f Analytic $\Rightarrow$ $\begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix}$ $\Rightarrow$ $\begin{matrix} u_{xx} = v_{yx} \\ u_{yy} = -v_{xy} \end{matrix}$

$\Rightarrow u_{xx} + u_{yy} = 0$

Def. Let $u : D \to \mathbb{R}$ is a harmonic func. then a function v is called its harmonic conjugate of u if $\boxed{f := u + iv}$ is analytic on D.

* CRE in polar co-ordinates

Can remember using dimensional analysis $u \sim v \quad u_r \sim \frac{u}{r} \quad v_\theta \sim v$

$\begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix} \longrightarrow \begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix} \longrightarrow \begin{matrix} u_r = \frac{1}{r} v_\theta \\ v_r = \frac{-1}{r} u_\theta \end{matrix}$

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial r}$$

Elementary Functions

* Let $z \in \mathbb{C}^*$ then

$$\log z = \log |z| + i \arg(z)$$

We choose single-valued $\arg() \to \text{Arg}()$ $(-\pi, \pi]$

$$\boxed{\text{Log } z = \log |z| + i \,\text{Arg}(z)}$$

→ Cts. everywhere except $(-\infty, 0)$ due to $\text{Arg}(z)$. Can delete that ray.

Now,

$$\text{Log } z = \log |z| + i \,\text{Arg}(z)$$

$$= \frac{1}{2} \log(x^2+y^2) + i\left(\tan^{-1}\left(\frac{y}{x}\right) + c\right)$$

$$\Rightarrow u(x,y) = \frac{1}{2}\log(x^2+y^2), \quad v(x,y) = \tan^{-1}\left(\frac{y}{x}\right) + c$$

$$u_x = \frac{1}{2} \frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}, \quad u_y = \frac{y}{x^2+y^2}$$

$$v_x = \frac{1}{1+(y/x)^2} \frac{-y}{x^2} = \frac{-y}{x^2+y^2}, \quad v_y = \frac{1}{1+(y/x)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

$\Rightarrow$ CR-E satisfied & they're cts. in $\mathbb{C} \setminus (-\infty, 0]$

$\Rightarrow$ **Log z** is **Analytic**

Thm. Let $f(z) = u + iv$ be defined in a domain D, and $z_0 \in D$. If partial derivatives u_x, v_x, u_y, v_y are cts. at z_0 and u, v satisfy CRE, then $f'(z_0)$ exists

Pf. Let $h = s + it$

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

$$= \lim_{h \to 0} \frac{u(x_0+s, y_0+t) + i v(x_0+s, y_0+t) - \overbrace{f(x_0, y_0)}^{u_0 + i v_0}}{h}$$

Some results to be used:

* $f : \mathbb{R} \to \mathbb{R}$ s.t. f diff. at x_0

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0)$$

$$\Rightarrow f(x_0+h) - f(x_0) = f'(x_0) \cdot h + \phi(h)$$

where $\lim_{h \to 0} \frac{\phi(h)}{h} = 0$

* Also, MVT, $\frac{f(b) - f(a)}{b - a} = f'(c)$, $c \in (a, b)$

$$u(x_0+s, y_0+t) - u(x_0, y_0)$$

$$= u(x_0+s, y_0+t) - u(x_0, y_0+t) + u(x_0, y_0+t) - u(x_0, y_0)$$

$$:= \phi(s, t) + s \cdot u_x(x_0, y_0) + t\, u_y(x_0, y_0)$$

where $\phi(s,t) = \text{LHS} - s\,u_x(x_0,y_0) - t\,u_y(x_0,y_0)$

Now,

$$\lim_{h\to 0} \frac{u(x_0+s, y_0+t) - u(x_0,y_0)}{h}$$

$$= \lim_{h\to 0} \frac{\phi(s,t)}{h} + \frac{s}{h} u_x(x_0,y_0) + \frac{t}{h} u_y(x_0,y_0)$$

Similarly,

$$\lim_{h\to 0} i\,\frac{v(x_0+s, y_0+t) - v(x_0,y_0)}{h}$$

$$= \lim_{h\to 0} i\left(\frac{\psi(s,t)}{h} + \frac{s}{h} v_x(x_0,y_0) + \frac{t}{h} v_y(x_0,y_0)\right)$$

Assume that,

$$\lim_{h\to 0} \frac{\phi(s,t)}{h} = 0 = \lim_{h\to 0} \frac{\psi(s,t)}{h}$$

$$\therefore\ f'(z_0) = \frac{s}{h} u_x + \frac{t}{h} u_y + \frac{is}{h} v_x + \frac{it}{h} v_y$$

$$= \left(\frac{s}{h} + \frac{it}{h}\right) u_x + \frac{is}{h} v_x - \frac{t}{h} v_x$$

i^2

$$= u_x + i\left(\frac{s}{h} + \frac{it}{h}\right) v_x$$

$$\boxed{f'(z_0) = u_x + i v_x}$$

Now,

$$\ldots \left[u(x_0+s, y_0+t) - u(x_0, y_0+t) \right.$$

$$\frac{\phi(s,t)}{h} = \frac{1}{h}\left[\begin{array}{l} u(x_0+s, y_0+t) - u(x_0, y_0+t) \\ -s\, u_x(x_0,y_0) \\ + u(x_0, y_0+t) - u(x_0,y_0) \\ - t\, u_y(x_0,y_0) \end{array}\right]$$

$$= \frac{1}{h}\left[\begin{array}{l} s\cdot u_x(x_0+a\cdot s, y_0) - s\cdot u_x(x_0,y_0) \\ + \underbrace{t\cdot u_y(x_0, y_0+b\cdot t)}_{\text{By MVT: } 0<a,b<1} - t\, u_y(x_0,y_0) \end{array}\right]$$

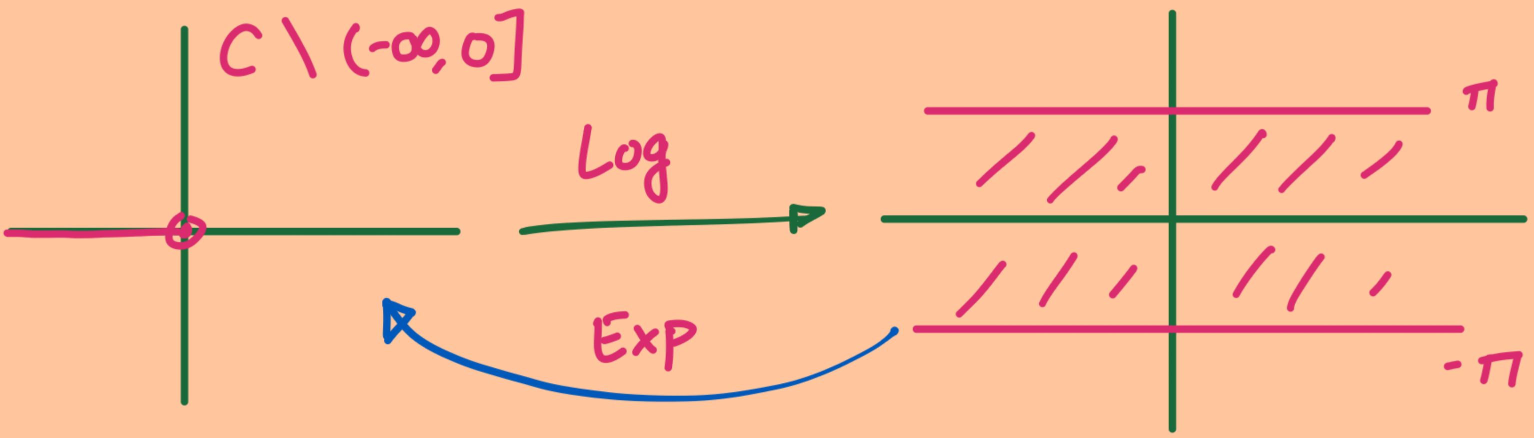
$$\le \lim_{h\to 0} \frac{|s|}{|h|}\left|u_x(x_0+a\cdot s, y_0) - u_x(x_0,y_0)\right| + \frac{|t|}{|h|}\left|v_y(x_0, y_0+b\cdot t) - v_y(x_0,y_0)\right|$$

$$\lim_{h\to 0}\left|\frac{\phi(s,t)}{h}\right| \le \lim_{h\to 0} \left|u_x(x_0+a\cdot s, y_0) - u_x(x_0,y_0)\right| + \left|v_y(x_0, y_0+b\cdot t) - v_y(x_0,y_0)\right|$$

$h\to 0 \iff s\to 0,\ t\to 0$

$$= 0 + 0 = 0$$

$$\Rightarrow \lim_{h\to 0}\frac{\phi(s,t)}{h} = 0 = \lim_{h\to 0}\frac{\psi(s,t)}{h}$$



For $z \in \mathbb{C},\ z = x+iy$

$$\boxed{\exp(z) = e^{z} = e^{x} e^{iy} = e^{x}(\cos y + i \sin y)}$$

$\lim_{x\to\infty} e^{x} = \infty$, $\lim_{x\to-\infty} e^{x} = 0$, $\lim_{y\to+\infty} e^{iy}$ d.n.e.

$\Rightarrow \lim_{z\to\infty} e^{z}$ d.n.e.

Also, $\quad e^{z} = e^{z+i2\pi n}$

* Exp. inverse of Log

Let

$u + iv = w = e^{z} = e^{x}(\cos y + i \sin y)$

$\Rightarrow \begin{array}{l} u = e^{x}\cos y \\ v = e^{x}\sin y \end{array} \Rightarrow \begin{array}{l} e^{2x} = u^2+v^2 \\ \tan y = v/u \end{array} \Rightarrow \begin{array}{l} x = \log\sqrt{u^2+v^2} \\ y = \mathrm{Tan}^{-1}(v/u) \end{array}$

$\Rightarrow \quad x + iy = \log\sqrt{u^2+v^2} + i\,\mathrm{Tan}^{-1}(v/u)$

$\quad z = \log w$ Q.E.D.

$\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$	$\cosh z = \dfrac{e^{z}+e^{-z}}{2}$
$\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$	$= \cos(iz)$
	$\sinh z = \dfrac{e^{z}-e^{-z}}{2}$
$\tan z = \dfrac{\sin z}{\cos z}$	$= -i\sin(iz)$
	$\tanh z = \sinh z/\cosh z$

$\tanh z = \sinh z / \cosh z$

* $\sin z = \sin x \cosh y + i \cos x \sinh y$

Leibnitz's Rule:

$\psi : [a,b] \times [c,d]$

$$F(y) = \int_{\phi_1(y)}^{\phi_2(y)} \psi(t,y)\,dt$$

$$F'(y) = \int_{\phi_1(y)}^{\phi_2(y)} \frac{\partial \psi(t,y)}{\partial y}\,dt + \phi_2'(y)\,\psi(\phi_2(y),y) - \phi_1'(y)\,\psi(\phi_1(y),y)$$

Thm. Let u be a harmonic function with domain $D := B_0(R) = B(0,R)$. Then harmonic conjugate v exists.

Pf. $v_y = u_x \Rightarrow v = \int_0^y u_x\,dt + \phi(x)$

$\partial/\partial x$ $\Downarrow$

$$v_x = \int_0^y \partial u_x/\partial x\,dt + 0\,(u_x(x,y) - u_x(x,0)) + \phi'(x)$$

$$= \int_0^y u_{xx}\,dt + \phi'(x) \overset{u \text{ is Harm.}}{=\!=} \int_0^y -u_{yy}(x,t)\,dt + \phi'(x)$$

$$= -(u_y(x,y) - u_y(x,0)) + \phi'(x)$$

$\phi'(x) = -u_y(x,0)$ for CRE

$$\phi(x) = -\int_0^x u_y(t,0)\,dt + c$$

$$\therefore \boxed{v = \int_0^y u_x(x,t)\,dt - \int_0^x u_y(s,0)\,ds + c}$$

1st Method

2nd Method

Let u be a harmonic function.
If u is a homogeneous function of degree k, then,

$$\boxed{v = \frac{1}{k}(y u_x - x u_y)}$$

* Euler Theorem for Homo. Func:

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot x_i = k \cdot f$$

* $d(f \circ g) = f'(g(x)) \cdot g'(x)$
* f homo. of dg. k if.

$$f(tx_1, tx_2, \dots, tx_n) = t^k f(x_1, x_2, \dots, x_n)$$

Pf. Diff $f(tx_i) = t^k f(x_i)$ w.r.t. t

Let $y_i = t x_i$ $\hookrightarrow \displaystyle\sum_{i=1}^{n} \frac{\partial f}{\partial y_i} \frac{\partial y_i}{\partial t} = k t^{k-1} f(x_i)$

$$\Rightarrow \sum_{i=1}^{n} x_i \, \partial f / \partial y_i = k t^{k-1} f(x_i)$$

Take $\lim_{t \to 1}$ $\Rightarrow$ $\sum_{i=1}^{n} x_i \, \partial f / \partial x_i = k f(x_i)$

* $f(x,y)$, $x(t), y(t)$, $g(t) = f(x(t), y(t))$

$$\Rightarrow \frac{dg}{dt} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$g(s,t) = f(x(s,t), y(s,t)) \Rightarrow \partial g/\partial s = \partial f/\partial x \, \partial x/\partial s + \partial f/\partial y \, \partial y/\partial s$$

Pf: $u(x,y) = \frac{1}{n}(x u_x + y u_y)$

$$\Rightarrow u_x = \frac{1}{n}(u_x + x u_{xx} + y u_{yx})$$

$$v_y = \frac{1}{n}(u_x + y \cdot u_{xy} - x u_{yy})$$

$$\Rightarrow u_x = v_y \qquad \because v = 1/n \, (y u_x - x u_y)$$

Similarly, $u_y = -v_x$

Ex. $u(x,y) = x^2 - y^2 + 2xy$. Find v of $f = u + iv$

$$v = \frac{1}{2}\left[y \cdot (2x + 2y) - x(-2y + 2x) \right] = 2xy + y^2 - x^2$$

$$v = \int_0^y u_x(x,t)\,dt - \int_0^x u_y(s,0)\,ds + c$$

$$= (2xy + y^2) - (x^2) + c = 2xy + y^2 - x^2$$

3rd Method   Milne - Thompson Method

u - harmonic

$$g(z) = 2 \cdot u\left(\frac{z}{2}, \frac{z}{2i}\right) - g(0)$$

Then imaginary part of g is the desired harmonic conjugate of u

Pf $z = x + iy$, $\bar{z} = x - iy$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$$

$$\frac{\partial}{\partial y} = i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \bar{z}}$$

$$\Rightarrow \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) ; \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Let $g = u + iv$, $\bar{g} = u - iv$

$$\frac{\partial}{\partial z}\overline{(g(z))} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u - iv)$$

$$= \frac{1}{2}\left[u_x - iv_x - iu_y - v_y\right] = 0$$

$\Rightarrow \overline{g(z)}$ is a function of $\bar{z}$ only

$$\overline{g(z)} = g^*(\bar{z})$$

Now, $u(x, y) = \frac{1}{2}\left(g(z) + \overline{g(z)}\right)$

$$= \frac{1}{2}\left(g(z) + g^*(\bar{z})\right)$$

Substitute $x = z/2$, $y = z/2i$

$z = \frac{z}{2} + i\frac{z}{2i} = z \,; \qquad \bar{z} = \frac{z}{2} - i\left(\frac{z}{2i}\right) = 0$

$$\therefore \quad u\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{1}{2}\left(g(z) + g^*(0)\right)$$

$$\Rightarrow \boxed{g(z) = 2 \cdot u\left(\frac{z}{2}, \frac{z}{2i}\right) - \overline{g(0)}}$$

Thm. Let v & w are two harmonic conjugates of u. Then v and w differ by a purely imaginary constant.

$$w = v + c$$

Pf: $u_x = v_y = w_y \Rightarrow v = w + f(x)$

$-u_y = v_x = w_x \Rightarrow v = w + g(y)$

$\Rightarrow f = g = c$

Thm. f is analytic iff $\frac{\partial f}{\partial \bar{z}} = 0$

$$\Rightarrow \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv)$$

$$= \frac{1}{2}\left[u_x + i v_x + i u_y - v_y\right] = 0$$

Ex. $f(z) = \bar{z}^2/z$ is not analytic

Power Series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \, , \quad z_0 \in \mathbb{C}$$

- Let $S \subseteq \mathbb{R}$, bounded above (below). then a number $b_0 (a_0)$ is called least upper (greatest lower) bound (l.u.b. / g.l.b.) or supremum (infemum)

(1) $\forall \ s \in S, \ s \leq b_0$ (upper bound) (lower)

(2) $\forall \ b \in \mathbb{R}$ s.t. $s \leq b \ \forall \ s \in S$, then $b_0 \leq b$

Infemum:

(1) $\forall \ s \in S, \ s \geq a_0$

(2) $\forall \ a \in \mathbb{R}$ s.t. $a \leq s \ \forall s \in S$, then $a_0 \geq a$

- Let $S \subseteq \mathbb{R}$, bounded and $b = \sup S$ then $\forall \ \epsilon > 0 \ \exists \ s \in S$ s.t. $b - \epsilon < s \leq b$

Pf. If $\nexists \ s \Rightarrow s \leq b - \epsilon \ \forall \ s \in S$

$\Rightarrow b \leq b - \epsilon \quad \Rightarrow\Leftarrow$ Hence, proved

↳ l.u.b.

- Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, bounded. We define

$M_n = \sup \{ a_n, a_{n+1}, a_{n+2}, \dots \}$

$m_n = \inf\{a_n, a_{n+1}, a_{n+2}, \dots\}$

$M_n \geq M_{n+1}$

$m_n \leq m_{n+1}$

$\left.\right\} A \subseteq B \Rightarrow \begin{array}{l} \sup A \leq \sup B \\ \inf A \geq \inf B \end{array}$

$\limsup a_n = -\infty$ if $\lim M_n = -\infty$

· $\lim\limits_{n\to\infty} M_n = u$ where $c \leq a_n \leq C$ $\forall n \in \mathbb{N}$

$\lim\limits_{n\to\infty} m_n = l$

then $\limsup a_n = u$, $\liminf a_n = l$

$= \sup\left\{a : a \text{ is a limit point of } \{a_n\}_{n=1}^{\infty}\right\}$

· A number c is called limit point of $\{a_n\}_{n=1}^{\infty}$ if $\forall \epsilon > 0$ $\exists n_0 \in \mathbb{N}$ s.t.

$|a_{n_0} - C| < \epsilon$

Thm.

Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers and $u = \limsup a_n$. Then

(1) $\forall \epsilon > 0$ $\exists n_0$ s.t.

$a_n \leq u + \epsilon$ $\forall n \geq n_0$

(2) $\forall \epsilon > 0$ $\exists$ infinitely many n s.t.

$a_n > u - \epsilon$

Pf. (1) $\forall \ n_0 \in \mathbb{N} \ \exists \ n \geq n_0$

s.t. $a_n > u + \epsilon$

$\Rightarrow \exists$ infinitely many a_n s.t.

$a_n > u + \epsilon$

$\Rightarrow M_k = \sup\{a_k, a_{k+1}, a_{k+2}, \dots\} > u + \epsilon$

$\Rightarrow \lim M_k \geq u + \epsilon \Rightarrow u \geq u + \epsilon \Rightarrow\Leftarrow$

(2) $\forall \ \epsilon > 0 \ \exists$ finitely many $\{n_i\}$

s.t. $a_{n_i} > u - \epsilon$

Take $n_0 = 1 + \sup\{n_i\}$

then $a_n \leq u - \epsilon \ \forall \ n > n_0$

$\Rightarrow M_n = \sup\{a_n, a_{n+1}, \dots\} \leq u - \epsilon$

$\Rightarrow \lim M_n \leq u - \epsilon \Rightarrow u \leq u - \epsilon \Rightarrow\Leftarrow$

$l = \liminf a_n$

(1) $a_n \geq l - \epsilon \ \forall$ but finitely many

(2) $a_n \leq l + \epsilon$ for infinitely many n

Thm.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers then

$$\lim_{n\to\infty} a_n \text{ exists} \iff \limsup = \liminf.$$

Sketch: $l = \lim a_n$, $a_n \in (l-\epsilon, l+\epsilon)\ \forall n \geq N_0$

★ Let $\{a_n\}$ be a sequence of real numbers s.t. $a_n > 0$, then

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf a_n^{1/n} \leq \limsup a_n^{1/n} \leq \limsup \frac{a_{n+1}}{a_n}$$

Pf. Let $l = \liminf \frac{a_{n+1}}{a_n}$

we want to show that

$$\liminf a_n^{1/n} \geq l$$

Take $\epsilon > 0$ arbitrary. Then,

$$\frac{a_{n+1}}{a_n} \geq l - \epsilon \quad \forall n \geq n_0$$

$$\frac{a_{n_0+1}}{a_{n_0}} \geq (l-\epsilon) \Rightarrow a_{n_0+1} \geq (l-\epsilon)\, a_{n_0}$$

$$a_{n_0+2} \geq (l-\epsilon)^2 a_{n_0}$$

$$\vdots$$

$$a_n \geq (l-\epsilon)^{n-n_0} a_{n_0}$$

$$\Rightarrow a_n^{1/n} \geq (l-\epsilon)^{(1-n_0/n)}\, a_{n_0}^{1/n}$$

$\lim_{n\to\infty}$ then $\liminf a_n^{1/n} \geq (l-\epsilon)\cdot 1$

$\Rightarrow \liminf a_n^{1/n} \geq \liminf \frac{a_{n+1}}{a_n}$

☆ $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ power series w/ center z_0

Let us define

$$\frac{1}{R} = \limsup |a_n|^{1/n}$$

$$R = \frac{1}{\limsup |a_n|^{1/n}} = \frac{1}{u}$$

$R = \infty, a = 0$
$R = 0, u = \infty$

Thm.

(1) $f(z)$ converges absolutely for z s.t. $|z-z_0| < R$

(2) $f(z)$ diverges for $|z-z_0| > R$

(3) $f(z)$ converges uniformly for $|z-z_0| \leq \rho < R$

Pf Let $z \in B_{z_0}(R)$, we take ρ_1 s.t.,

$|z-z_0| < \rho_1 < R$

we have $\frac{1}{R} < \frac{1}{\rho_1}$

$\Rightarrow |a_n|^{1/n} < \frac{1}{\rho_1} \quad \forall \; n > N_0$

$\Rightarrow |a_n| < \frac{1}{\rho_1^n} \quad \forall \; n > N_0$

$$\left| \sum_{n=0}^{\infty} a_n (z - z_0)^n \right| \leq \sum_{n=0}^{N_0} |a_n| |z - z_0|^n + \sum_{N_0+1}^{\infty} | \quad |$$

$$\leq \sum_{n=0}^{N_0} \quad + \sum_{N_0+1}^{\infty} \left(\frac{\rho}{\rho_1} \right)^n < \infty$$

Pf. $|z - z_0| > \rho_0 > R \Rightarrow \frac{1}{R} > \frac{1}{\rho_0}$

$\exists$ infinitely many a_n s.t. $|a_n|^{1/n} > \frac{1}{\rho_0}$

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n > \sum_{n=0}^{\infty} \left(\rho/\rho_0 \right)^n > \infty$$

⊛ Convergence

$f_n : D \to \mathbb{C}$

$f_n(z) \longrightarrow f(z), \quad z \in D$

$\forall \; \epsilon > 0 \quad \exists \; N = N(\epsilon, z)$ s.t.

$|f_n(z) - f(z)| < \epsilon \quad \forall \; n > N$

Uniform Convergence

uniform convergence

$f_n \xrightarrow{\text{uniform}} f$ if

$|f_n(z) - f(z)| < \epsilon \quad \forall \ n > N = N(\epsilon)$

⊛ Let $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ be a power series with R.O.C. $= R$

(1) R.O.C. of
$$\sum_{n=1}^{\infty} n(n-1)\dots(n-(k-1))\, a_n (z-z_0)^{n-k}$$
is also R

(2) $f(z)$ is infinitely many times differentiable in $|z-z_0| < R$

(3) $a_k = f^{(k)}(z_0)/k!$

Pf (1) Let $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ $(k=1, z_0 = 0)$

$R' =$ r.o.c. of g $\qquad \left\{ \lim_{n\to\infty} n^{1/n} = 1 \right\}$

$$\frac{1}{R'} = \limsup \left(n|a_n|\right)^{1/n} = \limsup |a_n|^{1/n} = \frac{1}{R}$$

OR $\quad \frac{1}{R'} = \limsup \left(n|a_n|\right)^{1/n} = \limsup \frac{a_n}{a_n}$

$$= \frac{n \, a_n}{(n+1)\, a_{n+1}} = \frac{a_n}{a_{n+1}} = \frac{1}{R}$$

(2) $\left| \frac{f(z) - f(w)}{z - w} - g(w) \right| < \epsilon$ when $|z-w| < \delta$

$$\lim_{z \to w} \frac{f(z) - f(w)}{z - w} = g(z)$$

Take:

$$s_n(z) = \sum_{k=0}^{n} a_k (z - z_0)^k,$$

$$s_n'(z) = \sum_{k=0}^{n} k\, a_k (z - z_0)^{k-1}$$

We have, $s_n(z) \longrightarrow f(z)$

and $s_n'(z) \longrightarrow g(z)$ for $|z - z_0| < R$

$$f(z) = S_n(z) + R_n(z)$$

$$g(z) = f'(z) = S_n'(z) + R_n'(z)$$

$$\left| \frac{f(z) - f(w)}{z - w} - g(w) \right|$$

$$= \left| \frac{s_n(z) + R_n(z) - s_n(w) - R_n(w)}{z - w} - S_n'(w) - R_n'(w) \right|$$

$$= \left| \frac{S_n(z) - S_n(w)}{z - w} - S_n'(w) - g(w) + S_n'(w) + \frac{R_n(z) - R_n(w)}{z - w} \right|$$

polynomial $\qquad$ $s_n' \to g,\ n \to \infty$ $\qquad$ Error term $\to 0$

$\Sigma_1 \qquad \Sigma_2 \qquad \Sigma_3$

$$|\Sigma_i| < \epsilon/3 \quad \forall \quad i \in [3]$$

$$\Rightarrow \left| \frac{f(z) - f(w)}{z - w} - g(w) \right| < \epsilon \,.\ |z - w| < \delta$$

$\rightarrow$ $$|\Sigma_3| \leq \sum_{k=n}^{\infty} |a_k| \left| \frac{z^k - w^k}{z - w} \right| \quad \left\{ \begin{array}{l} \text{Take} \\ z_0 = z \end{array} \right.$$

$$= \sum_{k=n}^{\infty} |a_k| \left| z^{k-1} + z^{k-2} w + \ldots + w^{k-1} \right|$$

$$\leq \sum_{k=n}^{\infty} |a_k| \left(|z|^{k-1} + |z|^{k-2} |w| + \ldots + |w|^{k-1} \right)$$

$$\leq \sum_{k=n}^{\infty} k |a_k| \rho^{k-1}$$

$\hookrightarrow$ $|w - z_0| < \rho$
$|z - z_0|$

$$\Rightarrow |\Sigma_3| < \sum_{k=n}^{\infty} k |a_k| \rho^{k-1} < \epsilon/3 \text{ if } n > N_0$$

$$\sum_{n}^{\infty} \to 0 \Leftarrow \sum k a_k \rho^{k-1} \to g(\rho)$$

$\because$ r.o.c. $= R > \rho \geq |z - z_0|$,
$|w - z_0|$

$\rightarrow$ $|\Sigma_2|$: $\lim_{n \to \infty} S_n'(z) = g(z)$
$\Rightarrow |S_n'(z) - g(z)| < \epsilon/3$
for $n > n(\epsilon/3)$.

$\rightarrow$ $|\Sigma_1|$: S polynomial,

so $\lim \left| \frac{S_n(z) - S_n(w)}{} \right| = S_n'(z)$

$$\lim_{z\to w} \left| \frac{\quad}{z-w} \right|$$

Thus $\left|\frac{S_n(z)-S_n(w)}{z-w} - S_n'(z)\right| < \epsilon/3$

for $\delta(\epsilon/3)$

$\therefore\ |\Sigma_1| + |\Sigma_2| + |\Sigma_3| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$

So, $\left|\frac{f(z)-f(w)}{z-w} - g(w)\right| < \epsilon$

So, $$\boxed{\lim_{z\to w}\left(\frac{f(z)-f(w)}{z-w}\right) = g(w)}$$

⊛ $\bigcup_{\alpha\in I} S_\alpha = S$ is open if each

S_α is open.

$x \in S \Rightarrow \exists\ \alpha_0$ s.t. $x \in S_{\alpha_0}$

$\exists$ open set V s.t. $x \in V \subseteq S_{\alpha_0}$

$\Rightarrow x \in V \subseteq \bigcup_{\alpha\in I} S_\alpha = S \Rightarrow S$ open

⊛ Let $U_{[n]}$ be open sets.

$U = \bigcap_{i\in[n]} U_i$ — open

$x \in U \iff x \in U_i \quad \forall\ i \in [n]$

$\iff B_x(r_i) \subseteq U_i$ for each i

Take $r = \min(\{r_i\})$

$\Rightarrow B_x(r) \subseteq U_i \quad \forall i \Rightarrow B_x(r) \subseteq U$

⊛ $\{U_n\}_{n=1}^{\infty}$ open

$\bigcap_{n=1}^{\infty} U_n \qquad U_n = \left(\frac{-1}{n}, \frac{1}{n}\right) \Rightarrow \bigcap_{n=1}^{\infty} U_n = \{0\}$ closed

---

Ex. $S = \{ z \in \mathbb{C},\ |z-1| \leq 1 \text{ or } |z+1| \leq 1 \}$

Connected, Not open $\Rightarrow$ Not domain

$S : \{ z \in \mathbb{C},\ |z-1| < 1 \text{ or } |z+1| < 1 \}$

Not connected. open $\Rightarrow$ Not domain

Ex. $f(z) = \dfrac{z\,Re(z)}{|z|}$, $f(0) = ?$

$\Rightarrow f(0) = 0$

$\left| \dfrac{z\,Re(z)}{|z|} \right| \leq |Re(z)| \leq |z| < \epsilon$ for $|z| < \delta = \epsilon$

Ex. $g(z) = \dfrac{Re(z^2)}{|z|^2} = \dfrac{x^2 - y^2}{x^2 + y^2}$

Take $y = mx \quad \hookrightarrow \dfrac{1-m^2}{1+m^2} \rightarrow \lim$ d.n.e.

Ex. $f(z) = \boxed{(1-i)z + (1+i)\bar{z}}^2$

$z \quad z$

$$\lim_{x\to 0}\lim_{y\to 0} f(z) = \lim_{y\to 0}\lim_{x\to 0} f(z) = 4$$

but $\lim_{z\to 0} f(z)$ d.n.e.

Ex. $f(z) = \mathrm{Re}(z) = x = u + iv$

$u = x,\ v = 0 \rightsquigarrow u_x = 1 \neq 0 = v_y$

$\Rightarrow$ CRE d.n. hold $\Rightarrow$ Not diff

Also, $\lim_{z\to 0} \dfrac{f(z)}{z}$ d.n.e. $\Rightarrow$ Not diff.

$\Uparrow$

Also $f = \dfrac{z+\bar{z}}{2}$, $\partial f/\partial \bar{z} = 1/2 \neq 0$

Ex. $f(z) = \begin{cases} |z|^2/z, & z \neq 0 \\ 0, & z = 0 \end{cases}$

$$\left|f(z) - f(0)\right| = \left|\frac{|z|^2}{z} - 0\right| = |z| < \epsilon$$

for $|z| < \delta < \epsilon$

$\therefore f$ cts. at $z = 0$

f not diff at $z = 0$, $f = \bar{z}$, $z \neq 0$

Ex. $f(z) = \sqrt{|\mathrm{Re}(z)\ \mathrm{Im}(z)|} = \sqrt{xy} + i0$

$u = \sqrt{xy}$, $v = 0$

f is cts. at $0,0$

$$f = \sqrt{xy} \leq |z| \cdot |z| = |z|^2 \leq |z| < \epsilon$$

for $|z| \leq 1$,
$|z| < \epsilon$

$$u_x(0,0) = \lim_{h \to 0} \frac{u(0+h, 0) - u(0,0)}{h} = \frac{0-0}{h} = 0$$

$v_y(0,0) = 0 = v_x(0,0) = u_y(0,0) \longrightarrow$ CRE holds

But partial derivatives not cts. $\Rightarrow$ Not diff.

OR

$$\lim_{z \to 0} f(z)/z = \frac{\sqrt{|xy|}}{x+iy} = \text{d.n.e.}$$

diff.

Ex: (i) $\log(\operatorname{Log} i)$

$$= \log\left(0 + i\,\pi/2\right) = \log\left(i\,\pi/2\right)$$

$$= \log \pi/2 + i\left(\frac{\pi}{2} + 2k\pi\right)$$

(ii) $\operatorname{Log}(\operatorname{Log} i) = \log\frac{\pi}{2} + i\frac{\pi}{2}$

(iii) $i^{-i} = \exp(-i \operatorname{Log} i) = \exp(-i \cdot (i\pi/2)) = e^{\pi/2}$

(iv) $1^{-i} = \exp(-i \operatorname{Log} 1) = \exp(-i \cdot 0) = e^0 = 1$

☆ $$\boxed{z^s := \exp(s \operatorname{Log} z)}$$

Q: $\sum_{n=0}^{\infty} a_n z^n$, $a_n = 2^n$ n even, 3^n n odd

$^{1/n}$ $\{2, 3\} = 3$

$L = \limsup |a_n|^{1/n} = \limsup \{2, 3\} = 3$

limit points $\{2, 3\}$

$R = \frac{1}{L} = \frac{1}{3}$

OR use $\underbrace{\sum a_n z^n}_{\text{Gt if } |z| < 1/3} = \sum \underbrace{2^{2m} z^{2m}}_{\text{Gt if } |z| < \frac{1}{2}} + \sum \underbrace{3^{2m+1} z^{2m+1}}_{\text{Gt if } |z| < \frac{1}{3}}$

Gt if $|z| < 1/3 \Leftarrow$ Gt. if $|z| < \frac{1}{2}$ Gt if $|z| < \frac{1}{3}$

Thm. (i) $\limsup a_n b_n \le \limsup a_n \cdot \limsup b_n$

Let $a_n, b_n \geq 0$

(ii) If a_n converges $\Rightarrow \limsup a_n b_n = \limsup a_n \cdot \limsup b_n$

Pf: (i) $M_k = \sup\{a_k, a_{k+1}, \ldots\}$

$M_k' = \sup\{b_k, b_{k+1}, \ldots\}$

$\limsup a_n = \lim_{k \to \infty} M_k$, $\limsup b_n = \lim_{k \to \infty} M_k'$

$\sup_{n > k}\{a_n, a_{n+1}, \ldots\} \cdot \sup_{m > k}\{b_m, b_{m+1}, \ldots\}$

$= \sup_{\substack{n > k \\ m > k}}\{a_n b_m\} \geq \sup_{n > k}\{a_n b_n\}$

$\left\{ \because \{a_n b_n\}_{n > k} \subseteq \{a_n b_m\}_{n, m > k} \right\}$

Take $\lim k \to \infty$,

$\limsup a_n \cdot \limsup b_n \geq \limsup a_n \cdot b_n$

(ii)

(ii) Suppose $\lim_{n\to\infty} a_n = l$

$\forall\ \epsilon > 0$ (arbitrary) $\quad a_n > l - \epsilon \quad \forall\ n > N_0$

≥ 0 used of a_n, b_n

$$\sup_{n>k} a_n b_n > \sup_{n>k} (l-\epsilon) b_n$$

$\lim_{k\to\infty}$: $\quad \limsup a_n b_n \geq (l-\epsilon) \limsup b_n$

$$\Rightarrow \limsup a_n b_n \geq \limsup a_n \cdot \limsup b_n$$

Combine w/ prev. result to get equality

Power Series w/ Gaps

Def. $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{\lambda(n)}$

with $\lambda(n) \neq n$ and $\lambda(n) \to \infty$ as $n \to \infty$ then power series with gaps.

Assume $a_n \neq 0\ \forall\ n$.

- If $\boxed{\lim_{n\to\infty} \left|\frac{a_n}{a_{n-1}}\right|^{\frac{1}{\lambda(n)-\lambda(n-1)}} = l}$ exists.

Then r.o.c. $R = \frac{1}{l}$

Ex: $f(z) = \sum \frac{1}{2^n} z^n$, $g(z) = \sum \frac{1}{2^n} z^{n^2}$

$$\ell_f : \left(\frac{1/2^n}{1/2^{n-1}}\right)^{\frac{1}{n-(n-1)}} = \left(\frac{1}{2}\right)^{\frac{1}{1}} \Rightarrow R_f = 2 = \frac{1}{\ell_f}$$

$$\ell_g : \left(\frac{1/2^n}{1/2^{n-1}}\right)^{\frac{1}{n^2-(n-1)^2}} = \left(\frac{1}{2}\right)^{\frac{1}{2n-1}} = 1 \Rightarrow R_g = 1 = \frac{1}{\ell_g}$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

$\hookrightarrow R_f$ $\qquad\qquad$ $\hookrightarrow R_g$

(1) Hadamard product of f and g

$$(f * g)(z) = \sum_{n=0}^{\infty} (a_n \cdot b_n) \cdot (z - z_0)^n$$

Let R_H be r.o.c. for $f*g$

$$\frac{1}{R_H} = \limsup |a_n \cdot b_n|^{1/n} \leq \limsup |a_n|^{1/n} \cdot \limsup |b_n|^{1/n}$$

$$= \frac{1}{R_f} \cdot \frac{1}{R_g}$$

$$\Rightarrow \boxed{R_H \geq R_g \cdot R_f}$$

(Equality if convergent)

(2) Cauchy product of f and g

$$(f \cdot g)(z) = \sum^{\infty} c_k (z - z_0)^k, \text{ where}$$

$$k=0$$

$$c_k = a_k b_0 + a_{k-1} b_1 + \dots + a_0 b_k$$

$$= \sum_{l=0}^{k} a_{k-l} b_l = \sum_{l=0}^{k} a_l b_{k-l}$$

Let R_c be r.o.c. for $f \cdot g$

$$\boxed{R_c = \min\{R_1, R_2\}}$$

$$S_n(z) = \sum_{k=0}^{n} a_n (z - z_0)^k,$$

$$T_n(z) = \sum_{k=0}^{n} b_n (z - z_0)^k,$$

$$P_n(z) = \sum_{k=0}^{n} c_n (z - z_0)^k$$

$$f = S_n + \epsilon_n^f, \quad g = T_n + \epsilon_n^g$$

$$\lim_{n \to \infty} |\epsilon_n^f(z)|, |\epsilon_n^g(z)| = 0$$

$$\Rightarrow |\epsilon_n^f(z)|, |\epsilon_n^g(z)| < \epsilon \quad \text{if } n > N$$

$$P_n(z) = \sum_{k=0}^{n} c_k z^k \quad \{z_0 = 0\}$$

$$= a_0 b_0 + (a_1 b_0 + a_0 b_1) z$$
$$+ \dots + (a_n b_0 + \dots + a_1 b_{n-1} + a_0 b_n) z^n$$

$$= a_0 (b_0 + b_1 z + \dots + b_n z^n)$$
$$+ a_1 z (b_0 + b_1 z + \dots + b_{n-1} z^{n-1})$$

$\ldots + a_n z^n\ (b_0)$

$= a_0 T_n(z) + a_1 T_{n-1}(z)\cdot z + \ldots + a_n T_0(z)\cdot z^n$

$= a_0(g - \epsilon_n^g) + a_1 z(g - \epsilon_{n-1}^g) + \ldots + a_n z^n (g - \epsilon_0^g)$

$= g(a_0 + a_1 z + \ldots + a_n z^n) - (a_0 \epsilon_n^g + a_1 z \epsilon_{n-1}^g + \ldots + a_n z^n \epsilon_0^g)$

$= g(z)\cdot S_n(z) - E_n(z)$

Need to show that $\lim_{n\to\infty} |E_n(z)| = 0$

① $|\epsilon_n(z)| < \epsilon \ \forall\ n > N$

② $a_n z^n \to 0$ as $n \to \infty$

$|E_n(z)|$

$\leq |a_0 \epsilon_n(z) + \ldots + a_{n-N} \epsilon_N(z) z^{n-N}| + |a_{n-N+1} z^{n-N+1} \epsilon_{N-1} + \ldots + a_n z^n \epsilon_0|$

$\leq \epsilon\, |\underbrace{a_0 + \ldots + a_{n-N} z^{n-N}}_{\text{Cgs. to } f \Rightarrow \text{Bounded}}| + \underbrace{(\quad)}_{a_n z^n \to 0 \Rightarrow \text{Bounded}}$

$\leq \epsilon \cdot M$

Thus, $E_n(z) \to 0$ & $S_n \to f$.

so $P_n \to f\cdot g$

Riemann Integration

Riemann Integration

$[a, b] \subseteq \mathbb{R}$

$a = t_0 < t_1 < t_2 \ldots < t_{n-1} < t_n = b$

$\Rightarrow$ partition P

$f : [a, b] \to \mathbb{R}$. $\Delta t_k = t_k - t_{k-1}$

$\xi_k \in [t_{k-1}, t_k)$

sum (Riemann)

$$S(P, f) = \sum_{k=1}^{n} f(\xi_k) \cdot \Delta(t_k)$$

$$\|P\| = \max_{k = [n]} \{(t_k - t_{k-1})\}$$

* If $\boxed{\lim_{\|P\| \to 0} S(P, f) \text{ exists,}} = \boxed{\int_a^b f(t)\,dt}$

$$M_k = \sup_{\xi \in [t_{k-1}, t_k)} f(\xi), \quad m_k = \inf_{\xi \in [t_{k-1}, t_k)} f(\xi)$$

$$U(P, f) = \sum_{k=1}^{n} M_k \Delta_k \qquad U \geq L$$

$$L(P, f) = \sum_{k=1}^{n} m_k \Delta_k$$

$$|U(P, f) - L(P, f)| < \epsilon \quad \text{whenever } \|P\| < \delta$$

$$\lim_{\|P\| \to 0} U(P, f) = \int_a^b f(x)\,dx = \lim_{\|P\| \to 0} L(P, f)$$

Curve A continuously differentiable function from $[a, b] \to \mathbb{C}$.

$$\int_c f(z)\, dz = \int_a^b f(z(t)) \cdot z(t)\, dt$$

* Any two intervals $[c,d]$, $[a,b]$ in R are Homiomorphic.

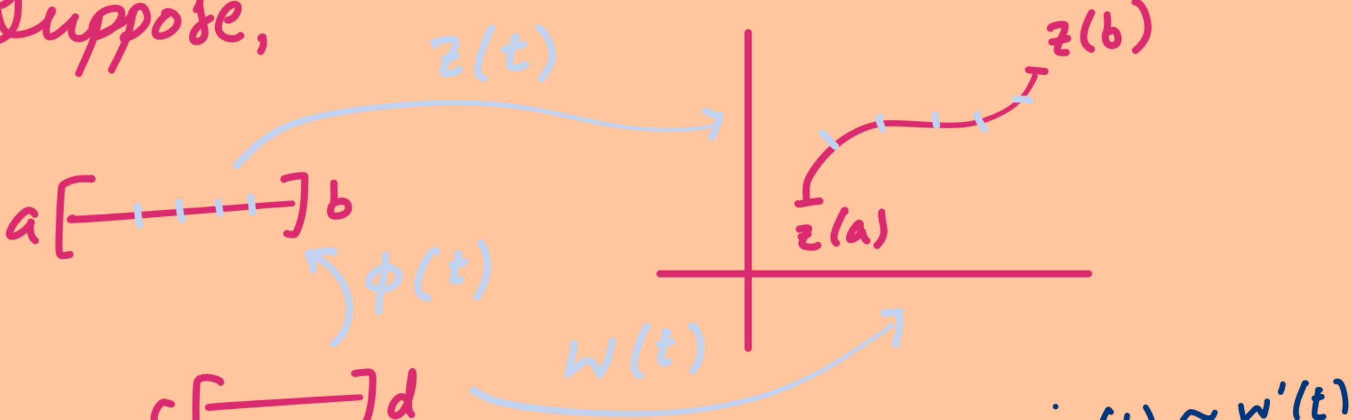
$\exists \; \phi : [c,d] \to [a,b]$ s.t.

$\phi(c) = a$, $\phi(d) = b$, ϕ 1-1, onto, ϕ, ϕ^{-1} cts.

$$\phi(t) = a + (b-a)\left(\frac{t-c}{d-c}\right)$$

Similarly, any two balls in $\mathbb{C}$ are also Homiomorphic.

Suppose,



Then, $\int_c^d f(w(t)) \cdot \dot{w}(t)\, dt$

$w(t) = z(\phi(t))$

$$= \int_c^d f(z(\phi(t))) \cdot \dot{z}(\phi(t))\, \phi'(t)\, dt$$

$\phi(t) = u$
$\phi(c) = a$
$\phi(d) = b$

$$= \int_a^b f(z(u)) \cdot \dot{z}(u) \cdot du$$

☆ Let $F : [a,b] \longrightarrow \mathbb{C}$,

$F(t) = u(t) + i\, v(t)$

$$\int_a^b F(t)\, dt = \int_a^b u(t) + i \int_a^b v(t)\, dt$$

Properties: (I) $\int_a^b Re(F)\, dt = Re \int_a^b F(t)\, dt$

(II) $\int_a^b \gamma_0 F(t)\, dt = \gamma_0 \int_a^b F(t)\, dt$

(III) $\left| \int_a^b F(t)\, dt \right| \leq \int_a^b |F(t)\, dt|$

Pf: $\int_a^b F(t)\, dt = r_0 e^{i\theta_0}$

$$r_0 = e^{-i\theta_0} \int_a^b F(t)\, dt$$

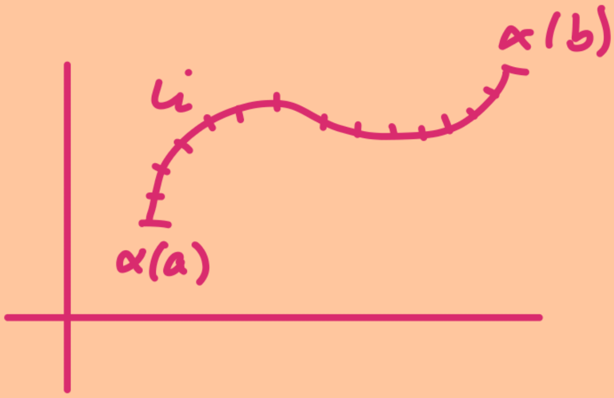
$$= \int_a^b e^{-i\theta_0} F(t)\, dt$$

$$\Rightarrow Re(r_0) = r_0 = \int_a^b Re\left(e^{-i\theta_0} F(t)\right) dt$$

$$\left|\int_a^b F(t)\,dt\right| = r_0 \leq \int_a^b \left|e^{-i\theta_0} F(t)\right| dt$$
$$= \int_a^b |F(t)|\,dt$$

$$\Rightarrow \boxed{\left|\int_a^b F(t)\,dt\right| \leq \int_a^b |F(t)|\,dt}$$

Length of a Curve



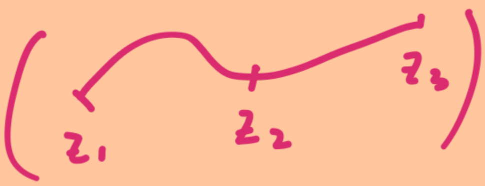
$\alpha : (x(t), y(t))$

$x(t) + i\,y(t) = z(t)$

$$\text{length of } \alpha \approx \sum |L_i| = \sum \sqrt{\Delta x_i^2 + \Delta y_i^2}$$
$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b |\dot{z}(t)|\,dt$$

(*) $C : [a,b] \longrightarrow \mathbb{C}$

If $C = C_1 \cup C_2 = C_1 + C_2$



$$\int_C f(z)\,dz = \left(\int_{C_1} + \int_{C_2}\right) f(z)\,dz$$

Ex. $C : [0,1] = \begin{cases} C_1(2t), & 0 \leq t \leq 1/2 \\ C_2(2t-1), & 1/2 \leq t \leq 1 \end{cases}$

$$\int_c f(z)\,dz = \int_0 f(z(t))\, z(t)\, dt$$

$$= \int_0^{1/2} f(z(2t))\cdot \dot z(2t)\cdot 2\cdot dt + \int_{1/2}^{1} f(z(2t-1))\cdot \dot z(2t-1)\cdot 2\cdot dt$$

$u_1 := 2t$ $\qquad$ $u_2 := 2t-1$

$$= \int_{C_1} f(z(u_1))\cdot \dot z(u_1)\cdot du_1 + \int_{C_2} f(z(u_2))\,\dot z(u_2)\, du_2$$

(*) $|f(x)| \leq M \quad \forall\ x \in [a,b],\ f : [a,b] \to \mathbb{R}$

$\left| \int_a^b f\, dx \right| \leq M\, L(I)$

Thm. $\left| \int_a^b f(z)\, dz \right| \leq ML$ $\quad$ {ML - Inequality}

where M is s.t. $|f(z)| \leq M \ \forall\ z \in C$

Pf. $\left| \int_c f(z)\, dz \right| = \left| \int_a^b f(z(t))\, \dot z(t)\, dt \right|$

$$\leq \int_a^b |f(z(t))|\, |\dot z(t)|\, dt$$

$$\leq \int_a^b M\, |\dot z(t)|\, dt = ML$$

Def. Let $\alpha : [a,b] \longrightarrow \mathbb{C}$ be a curve.

Then α is called simple if

$\alpha(t_1) \neq \alpha(t_2)$ if $t_1 \neq t_2$ (or $t_1 < t_2$)

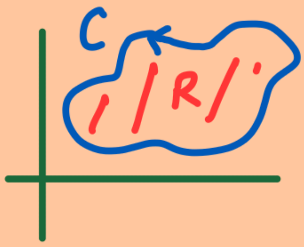
Def. α is called closed if $\alpha(a) = \alpha(b)$

Def. piecewise smooth : $\exists$ a partition $a = t_0 < t_1 < t_2 \dots < t_n = b$ s.t. α is continuously differentiable in each interval $[t_{k-1}, t_k)_{k=[n]}$ $\{n: \text{finite}\}$

Green's Theorem

Let C be a simple (smooth) closed curve in $\mathbb{R}^2$. Let R be the region enclosed by C.
For function $P(x,y), Q(x,y) \in C'(\bar{R})$.
Then,

$$\boxed{\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dx\,dy}$$



$\{P, Q, P_x, Q_y$ exist & cts in $C'(\bar{R}) := C \cup R\}$

Cauchy's Theorem

Let C be a simple closed curve s.c. Let D be the domain enclosed by C. Let f be an analytic function on $D \cup C$ (let $f \in C'(\bar{D})$). Then,

$$\boxed{\int_C f(z)\,dz = 0}$$

Pf. $f = u + iv$, $z(t) = x(t) + i y(t)$

$$\int_a^b (u+iv)\left(\dot{x}(t) + i\,\dot{y}(t)\right) dt$$

$$= \int_a^b \left(u(x(t),y(t))\cdot \dot{x}(t) - v\dot{y}\right) dt + i\int_a^b u\dot{y} + v\dot{x}\, dt$$

$$= \int_C u\,dx - v\,dy + i\int_C u\,dy + v\,dx$$

Green's Theorem

$$= \iint_D (-v_x - u_y)\,dx\,dy + i\iint_D (v_y - u_x)\,dx\,dy$$

f: Analytic $\therefore$ CR-E holds

$$= 0 + i0 = 0$$

Cauchy's Theorem for multiply connected domain

$$\int_C f(z)\,dz$$

$$= \sum \int_{C_i} f(z)\,dz$$

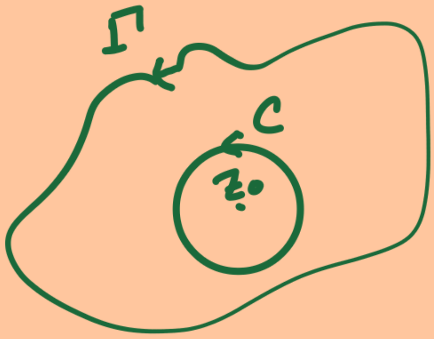
Corollary: Let Γ be a curve and

$z_0 \in \Gamma^\circ$. Then $\displaystyle\int_\Gamma \frac{dz}{z-z_0} = 2\pi i$

Pf

Choose $\delta > 0$

s.t. $B_{z_0}(\delta) \subseteq \Gamma^\circ$.



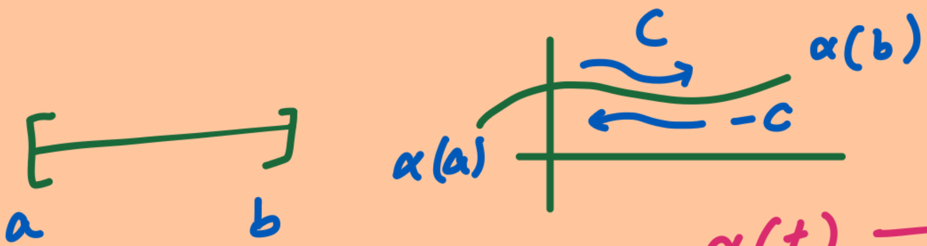
$f(z) = \dfrac{1}{z - z_0}$

$\displaystyle\int_\Gamma f(z)\,dz = \int_C f(z)\,dz$, $\quad z = z_0 + e^{2\pi i t}\cdot\delta$

$\quad dz = \delta\cdot 2\pi i \cdot e^{2\pi i t}\,dt$

$= \displaystyle\int_0^1 \frac{\delta\cdot 2\pi i\cdot e^{2\pi i t}\,dt}{z_0 + \delta e^{2\pi i t} - z_0} = 2\pi i \int_0^1 dt = 2\pi i$

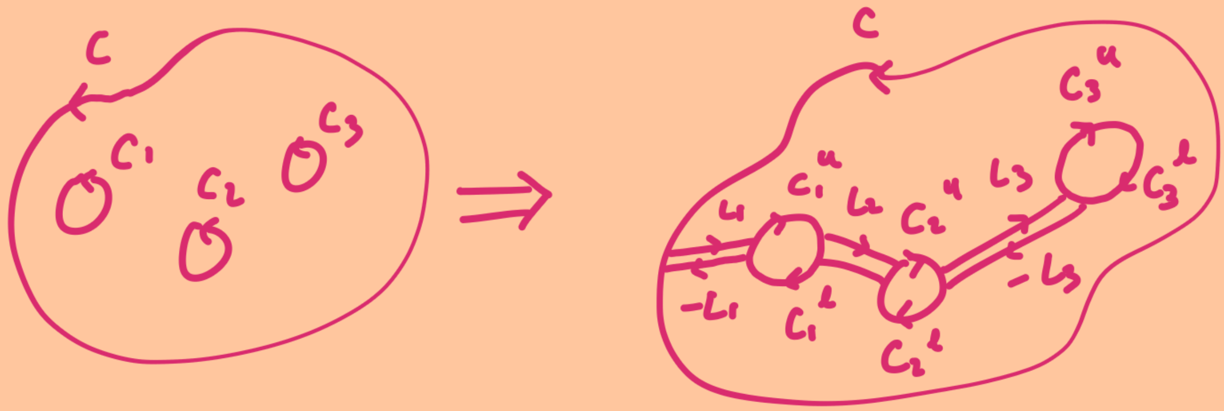
(*)



$\alpha(t) \longrightarrow C$

$\alpha(b+a-t) \longrightarrow -C$

Pf.



$C_0 = C^u + L_1 + C_1^u + L_2 + C_2^u + L_3 + C_3^u$
$\quad + C_3^l + (-L_3) + C_2^l + (-L_2) + C_1^l + (-L_1) + C^l$

Now, $\displaystyle\int_{C_0} f(z)\,dz = 0$

same curve, opp dir^n $\displaystyle\int_{-c} = -\int_c$

$$\Rightarrow \int_{C^u} + \underbrace{\int_{C_1^u}}_{} + \cancel{\int_{L_1}} + \dots + \underbrace{\int_{C_1^l}}_{} + \cancel{\int_{-L_1}} + \int_{C^l} = 0$$

$$\int_{C_1^u} + \int_{C_1^l} = -\int_{C_1}$$

$$\Rightarrow \int_c + \left(-\int_{C_1}\right) + \left(-\int_{C_2}\right) + \dots = 0$$

$$\Rightarrow \int_c = \int_{C_1} + \int_{C_2} + \dots = \sum \int_{C_i}$$

Cauchy Integral Formula

Let D be a domain enclosed by a simple curve C. Let $z \in D$.
Let $\overline{B_z(\delta)} \subseteq D$, then for analytic function f on D,

$$f(z) = \frac{1}{2\pi i} \int_{C_\delta(z)} \frac{f(w)\,dw}{w - z}$$

where $C_\delta(z) = \{ w \in \mathbb{C} : |w - z| = \delta \}$

$\int \; f(w) - f(z) + f(z)$

Pf: R.H.S. $\frac{1}{2\pi i}\int_{C_\delta} \frac{f(w) - f(z) + f(z)}{w-z}\,dw$

$= \underbrace{\frac{1}{2\pi i}\int_{C_\delta} \frac{f(w)-f(z)}{w-z}\,dw} + \underbrace{\frac{1}{2\pi i}\int_{C_\delta}\frac{f(z)}{w-z}\,dw}_{=f(z)}$

Now, $|f(w)-f(z)|<\epsilon$, whenever $|w-z|<\delta$,
{f analytic $\Rightarrow f$ cts.}

So, $\int_{C_\delta} \frac{f(w)-f(z)}{w-z}\,dw = \int_{C_{\delta_1}} \frac{f(w)-f(z)}{w-z}\,dw$

$C_{\delta_1}: z+\delta_1 e^{2\pi i t} = w$ $\qquad \leq \int_0^1 \frac{|f(w)-f(z)|\,|\delta_1 2\pi i e^{2\pi i t}|}{|w-z|}\,dt$

$\leq \epsilon_1 \int_0^1 |2\pi i|\,dt = \epsilon_1 \cdot 2\pi$

So, $\int_{C_\delta} \frac{f(w)-f(z)}{w-z}\,dw = 0$

And, hence,

$\frac{1}{2\pi i}\int_{C_\delta}\frac{f(w)}{w-z}\,dw = f(z)$

Corollary: $f^{(n)}(z) = \frac{n!}{\ldots}\int \frac{f(w)\,dw}{\ldots}$

Corollary: $f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)\,dw}{(w-z)^{n+1}}$

Taylor's Theorem

Let f be analytic on D and $a \in D$. Let r be s.t. $\overline{B_a(r)}$ (closed ball) $\subseteq D$. Then f can be expressed as power series,

$$f(z) = \sum_{n=0}^{\infty} b_n (z-a)^n, \text{ where } b_n = \frac{1}{2\pi i}\int_{C_r} \frac{f(w)\,dw}{(w-a)^{n+1}}$$

Corollary: Every analytic function ($\equiv$ power series) is infinitely times differentiable.

Cauchy Estimate

Let f be analytic inside a domain D and $a \in D$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

Let $|f(z)| \leq M$ for $|z-a| \leq R$. Then

$$|f^{(n)}(a)| \leq \frac{M \cdot n!}{R^n}$$

Pf: $$\left|f^{(n)}(a)\right| = \left| \frac{n!}{2\pi i} \int_{C_a(\rho)} \frac{f(z)}{(z-a)^{n+1}} dz \right|$$

$$\leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} \cdot 2\pi R = \frac{M \cdot n!}{R^n}$$

Liouville Theorem

An entire function which is also bounded in whole complex plane is constant.

Pf. $f(z) = \sum_{n=0}^{\infty} a_n z^n \longrightarrow f$ entire

$\downarrow$

$|a_n| \leq \frac{n \cdot M}{R^n}, \; n \geq 1 \longleftarrow$ r.o.c. $= \infty$, $R = \infty$

$\hookrightarrow a_n = 0 \;\; \forall\, n \geq 1 \longrightarrow f$ const. func.

Fundamental Theorem of Algebra

A polynomial of degree n, has n roots (counted with multiplicity).

$$f(z) = \prod_{i=1}^{l} (z - a_i)^{k_i}$$

Pf. Let $p_n(z)$ be a polynomial of degree $n (\geq 1)$. Suppose $p_n(z)$ has no roots ($p_n(z) \neq 0 \ \forall z$). Then

$$\phi_n(z) := \frac{1}{p_n(z)} \longrightarrow \phi_n(z) < \epsilon, \ |z| > R$$

$\phi_n(z)$ bounded, $|z| \leq R$

$\phi_n(z)$ is bounded in whole $\mathbb{C}$ $\Leftarrow$ $p_n(z) \to \infty, \ z \to \infty$; Analytic func. in closed domain

$$\Rightarrow \phi_n(z) = \text{constant}$$

$$\Rightarrow p_n(z) = \text{const.} \quad \Rightarrow\Leftarrow$$

$\because \deg(p_n) = n$

$\Rightarrow p_n(z)$ has at least one root

$$p_n(z) = (z - a_1)\, p_{n-1}(z)$$

$$\vdots$$

$$= (z - a_1) \dots (z - a_n) \qquad \square$$

Modified Liouville Theorem

An entire function which satisfies $|f(z)| \leq MR^{n_0}$ in $\mathbb{C}$, then
{for $|z| \leq R$ & for some n_0}

$f(z)$ is a polynomial of degree at most n_0.

Pf. $a_n \leq \dfrac{n \cdot MR^{n_0}}{R^n}$ → $a_n = 0 \;\forall\; n > n_0$

$\Downarrow$

Take $R \to \infty$ $f(z)$ polynomial w/ deg. $\leq n_0$

Def. A domain $S \subseteq \mathbb{C}$ is called simply connected if it has no hole.

☆ Let G be a simply connected domain and $a \in G$. Let f be a cts. func. Then $\boxed{\int_a^z f(z)\,dz}$ is defined if it does not depend on the path b/w a and z.

☆ The integral is independent of path if

(1) f is analytic

(2) $\oint f(z)\,dz = 0$ for every closed curve.

c

Proposition f is cts. on a simply connected domain G, $a, z \in G$.
Assume that $\oint_c f(z)\,dz = 0$.
The function defined by

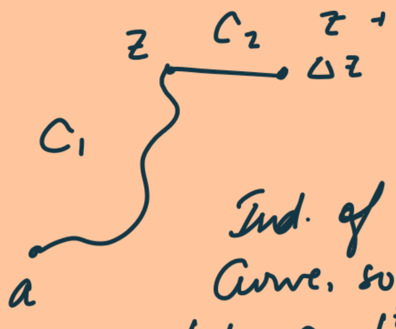
$$\boxed{\begin{aligned} F(z) &:= \int_a^z f(z)\,dz \quad \text{is differentiable} \\ \text{\&} \quad F'(z) &= f(z) \end{aligned}}$$

Pf.

To show: $\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| < \epsilon$

whenever $|\Delta z| < \delta$

$$\left.\begin{aligned} &\frac{1}{\Delta z}\left(\int_a^{z+\Delta z} f(w)\,dw - \int_a^z f(w)\,dw \right) \\ &- \frac{1}{\Delta z} \int_z^{z+\Delta z} f(z)\,dw =: I(\Delta z) \end{aligned}\right\}$$



Ind. of curve, so take $C_1 \to \int_a^z$
$C_1 \cup C_2 \to \int_a^{z+\Delta z}$

$$\Rightarrow I(\Delta z) = \frac{1}{\Delta z} \int_z^{z+\Delta z} f(w)\,dw - \frac{1}{\Delta z} \int_z^{z+\Delta z} f(z)\,dw$$

$$= \frac{1}{\Delta z} \int_z^{z+\Delta z} [f(w) - f(z)]\,dw$$

$$|I(\Delta z)| \leq \frac{1}{\ldots} \int^{z+\Delta z} |f(w) - f(z)|\,dw$$

$$\leq \frac{1}{|\Delta z|} \cdot \epsilon \cdot |\Delta z| = \epsilon$$

$z \in G$
$\Rightarrow \forall \epsilon > 0 \ \exists \delta > 0$ s.t.
$|f(z) - f(w)| < \epsilon \Leftarrow |z - w| < \delta$
Choose $|\Delta z| < \delta$

Hence, $F'(z) = f(z)$

Morera's Theorem

{Converse of Cauchy Theorem}

Let f be cts. inside a simply connected domain G, and $\oint f(z)\,dz = 0$. Then f is analytic.

Pf. From prev. prop, F is analytic
$\Rightarrow F$ is inf. times diff on G
$\Rightarrow f = F'$ is inf. times diff. on G

Def. Let $f(z)$ be an analytic function on G and $a \in G$. We say that a is a <u>zero</u> of <u>order</u> <u>m</u> if

$$f(a) = f'(a) = \dots = f^{(m-1)}(a) = 0 \quad \& \quad f^{(m)}(a) \neq 0$$

$$\longrightarrow f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k = \sum_{k=m}^{\infty} a_k (z-a)^k$$

$$= (z-a)^m \sum_{l=0} a_{m+l} (z-a)$$

$$= (z-a)^m g(a), \quad g(a) = a_m \neq 0$$

☆ We say that a zero z_0 of f is isolated if $\exists$ a nbd. B of z_0 s.t. $f(z) \neq 0 \quad \forall \quad z \in B \setminus \{z_0\}$

Isolated Zeroes Theorem

Zeroes of an analytic function f are isolated. $(f \not\equiv 0)$

Pf. Let $a \in G$

$f(z) = (z-a)^m g(z)$ for some δ_1 $|z-a| < \delta_1$ where $g(a) \neq 0$

$\exists \quad \delta_2$ s.t. $|g(z)| \geq c$ for $|z-a| < \delta_2$ → g is cts. & $g(a) \neq 0$

Choose $\delta = \min(\delta_1, \delta_2)$

$f(z) = (z-a)^m g(z) \neq 0$ → $z-a \neq 0,\ g(z) \neq 0 \Rightarrow f(z) \neq 0$

for $0 < |z-a| < \delta$

Corollary

Let f and g are two analytic

Let f and g are analytic functions on a domain G. Let $\{z_n\}_{n=1}^{\infty}$ $(\subseteq G)$ be a sequence s.t. $z_n \to z_0 \in G$. If $f(z_n) = g(z_n)$ for $n \in \mathbb{N}$ then $f(z) \equiv g(z) \Rightarrow f(z) = g(z)$ $\forall$ $z \in G$

Pf. $h := f - g$ and $a := z_0$

For every $\delta > 0$ $\exists$ n_0 s.t. $h(z_{n > n_0}) = 0$

$\Rightarrow$ $h \not\equiv 0$ does not hold $\Rightarrow$ $h \equiv 0$ $\Downarrow$ $f = g$

from contrapositive of prev. thm.

$f \not\equiv 0 \Rightarrow$ isolated zero $\equiv$ $\forall \delta > 0$ $\exists$ w s.t. $f(w) = 0$, $|w - z| < \delta$ $\Rightarrow$ identically zero

$p \Rightarrow q$ $\qquad$ $\sim q \Rightarrow \sim p$

Corollary

f & g analytic on G and $C \subseteq G$ be a curve. If $f(z) = g(z)$ $\forall$ $z \in C$ then $f(z) = g(z)$ $\forall$ $z \in G$

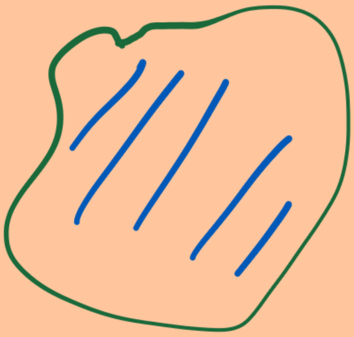
Pf.

C

Take $z_n \in C$ s.t. $z_n \to z_0 \in C$

Maximum Modulus Theorem

1^{st}: Let G be a bounded domain & f be an analytic function. Then f attains its maximum only on the boundary.



$\bar{G} = G \cup C$

$$M = \max_{z \in \bar{G}} |f(z)|$$

2^{nd}: $$\max_{z \in \bar{G}} |f(z)| = \max_{z \in C} |f(z)|$$

3^{rd}: f analytic on D & if $\exists\ a \in D$ s.t. $|f(a)| \geq |f(z)|\ \forall\ z \in D$ then f is const.

☆ Let $M_1(r) = \max_{|z| \leq r} |f(z)|$, $M(r) = \max_{|z| = r} |f(z)|$

then (1) $M_1(r) = M(r)$

(2) $M(r)$ is a non-decreasing function of r.

Proposition

Let $\phi : [a, b] \to \mathbb{R}$ be a cts fn.

Let $\phi \in [a,b]$

and $\phi(x) \leq K \quad \forall\ x \in [a,b]$

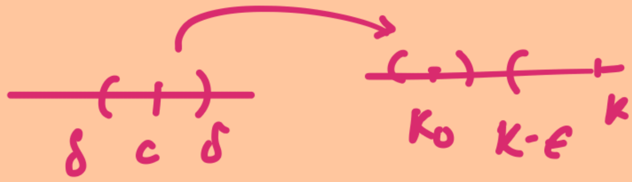
If $\int_a^b \phi(x)\,dx \geq K(b-a)$

then $\phi(x) = K \quad \forall\ x \in [a,b]$

Pf. Assume $\exists\ c$ s.t. $\phi(c) = K_0 < K$

then by continuity $\exists\ \delta > 0$ s.t.

$\phi(x) \leq K - \epsilon \quad \forall\ x \in (c-\delta, c+\delta)$



$\Rightarrow K(b-a)$

$$\leq \int_a^b \phi(x)\,dx = \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^{b}$$

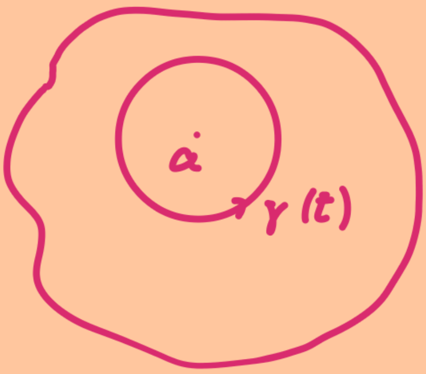
$$\leq K(c-\delta-a) + (K-\epsilon)(2\delta) + K(b-c-\delta)$$

$$= K(c-\delta-a+2\delta+b-c-\delta) - \epsilon 2\delta$$

$$= K(b-a) - \epsilon 2\delta \Rightarrow \epsilon\delta \leq 0 \Rightarrow \Leftarrow$$

Pf. (Of Max. Modulus Theorem)

$|f(a)| \geq |f(z)| \quad \forall\ z \in D$



Cauchy-integral formula

$$f(a) = \frac{1}{2\pi i} \int_{\gamma(t)} \frac{f(w)}{w-a}\,dw \quad , f(a) \neq 0$$

$\{ f(a) = 0 \Rightarrow f \equiv 0 \}$

$$\Rightarrow 1 = \frac{1}{2\pi i} \int \frac{f(w)/f(a)}{w-a}\,dw \quad , \quad w = a + r e^{2\pi i t}$$

$$= \frac{1}{2\pi i} \int \frac{\rho(t) e^{i\phi(t)}}{r e^{2\pi i t}} r \cdot 2\pi i e^{2\pi i t} dt \qquad \frac{f(w)}{f(a)} = \rho(t) e^{i\phi(t)}$$

$$\Rightarrow 1 = \int \rho(t) e^{i\phi(t)} dt \Rightarrow 1 \le \int |\rho(t)| dt$$

$$\Rightarrow 1 \le \int_0^1 |f(w)/f(a)| dt \qquad \because |\rho(t)| = \left|\frac{f(w)}{f(a)}\right| \le 1$$

$$\Rightarrow 1 \cdot (1-0) \le \int_0^1 \left|\frac{f(w)}{f(a)}\right| dt$$

By the above proposition, we have

$$\rho(t) \equiv 1 \xrightarrow{\text{put in}} 1 = \int_0^1 \rho(t) e^{i\phi(t)} dt$$

$$1 = \int_0^1 e^{i\phi(t)} dt$$

$$\Rightarrow 1 = \int_0^1 \cos(\phi(t))\, dt, \quad |\cos\phi(t)| \le 1 \Rightarrow \cos(\phi(t)) = 1$$

$$\Rightarrow \sin(\phi(t)) = 0$$

$$\Rightarrow f(w)/f(a) = \rho(t) e^{i\phi(t)} = 1$$

$$\Rightarrow f(w) = f(a) \quad \forall \quad w \in \gamma(t)$$

Then, by isolated zeroes theorem, we get $f(w) \equiv f(a) \quad \forall \quad w \in D$

Minimum Modulus Theorem

f is analytic on D, with no zero inside D.

If $\exists\ a \in D$ s.t. $|f(a)| \le |f(w)|$ $\forall\ w \in D$

Then $f(a) \equiv f(w)$

Then $f(a) = f$

Pf. $g(w) = 1/f(w)$

Ex. $f(z) = e^{e^z}$, $D = \{z = x+iy : -\pi/2 \le y \le \pi/2\}$

$|f(z)| = |e^{e^z}| \rightarrow |e^{Re(e^z)}| = |e^{e^x \cos y}|$

$\{|e^z| = e^{Re(z)}\}$

Now, on boundary, $|f(z)| = |e^0| = 1 \le 1$

but $|f(z)| \to \infty$ as $z \to \infty$,
$z = x + 0i$

☆ This is why Max. Mod. Thm. has bounded domain.

Schwarz - Lemma

Let f be an analytic function on $B_0(R)$ such that $|f(z)| \le M$ for $|z| = R$

If $f(0) = 0$, Then,

1. $|f(z)| \le \frac{M}{R}|z|$ for $|z| \le R$

2. $|f'(0)| \le \frac{M}{R}$

3. Equality holds in 1 & 2

for some $|z| < R$

for some $|z|<R$

iff $f(z) = \frac{M}{R} e^{i\alpha} z$, for some real α

Pf. $f(z) = \sum_{k=0}^{\infty} a_k z^k$

Define $\phi(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$

ϕ analytic on $B_0(R)$ because

$\phi(z) = a_1 + a_2 z^1 + a_3 z^2 + \dots$

$\{ f(0) = 0 \Rightarrow a_0 = 0 \}$

For $|z| = R$, we have,

$$|\phi(z)| = \frac{|f(z)|}{R} \leq \frac{M}{R}$$

By max. mod. thm,

$$|\phi(z)| \leq \frac{M}{R} \quad \forall \ |z| \leq R$$

$\left|\frac{f(z)}{z}\right| \leq \frac{M}{R}$

$\Rightarrow \underbrace{|f(z)| \leq \frac{M|z|}{R}}$ (1)

$|\phi(0)| \leq M/R$

$\Rightarrow$ (2) $\underbrace{|f'(0)| \leq M/R}$

If equality at some z_0, $|z_0| < R$,

... attained at an

then max $|\phi(z)|$ is attained at an interior point

$\Rightarrow$ By max. mod. thm., $\phi(z)$ is a const. function,

$$\phi(z) = c \quad \forall \; z \in B_0(R)$$

$$\Rightarrow \phi(z) = \frac{M}{R} e^{i\alpha} \Rightarrow \boxed{f(z) = \frac{M}{R} e^{i\alpha} z}$$

Singularity of a Complex Function

A point 'a' is called a singular point of the function 'f', if f is not analytic at 'a'.

Isolated Singularity

A point a is called isolated singularity if $\exists$ a ball $B_a(\delta) \subseteq D$ s.t. f is Analytic in punctured nbd.

$$\{z \in D : 0 < |z-a| < \delta\}$$

Ex. $f(z) = \dfrac{1}{(z-1)(z-2)}$, $g(z) = \mathrm{Log}(z)$

Sing.: $\{1, 2\}$: $\mathbb{R} - (-\infty, 0]$

Isolated : Not Isolated

Laurent Series:

Let f be an analytic function on annulus $\{z : r_1 \leq |z-a| \leq r_2\}$

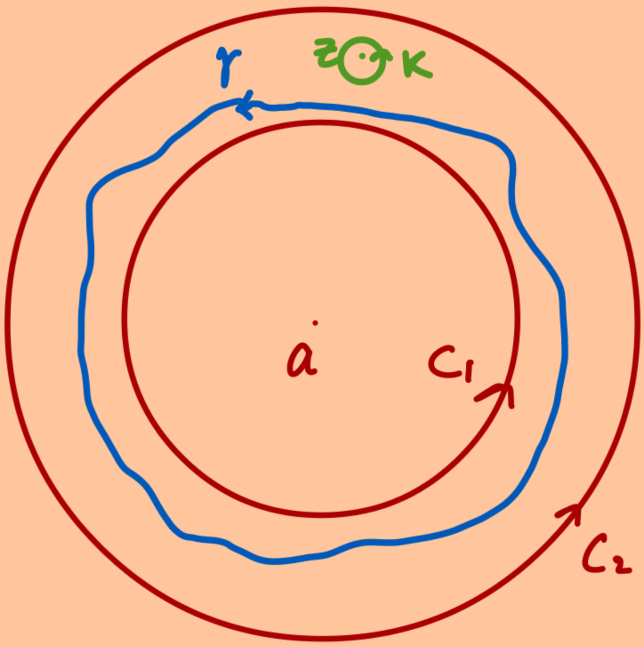
Let z be a point s.t. $r_1 < |z-a| < r_2$

Then, we have,

$$\boxed{f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{n=1}^{\infty} \frac{d_n}{(z-a)^n}}$$

$$c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)\,dw}{(w-a)^{n+1}}$$

$$d_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)\,dw}{(w-a)^{-n+1}}$$



OR

$$f(z) = \sum_{n \in \mathbb{Z}} \alpha_n (z-a)^n, \quad \alpha_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)\,dw}{(w-a)^{n+1}}$$

Now,

$$\int = \int + \int \Rightarrow \int_K = \int_{C_2} - \int_{C_1}$$

$\int_{C_2} \quad \int_{C_1} \quad \int_K$

$$\Rightarrow \underbrace{\frac{1}{2\pi i}\int_K \frac{f(w)}{w-z}\,dw}_{f(z)} = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)\,dw}{w-z} - \frac{1}{2\pi i}\int_{C_1} \frac{f(w)\,dw}{w-z}$$

On C_2, $$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)\left(1-\frac{z-a}{w-a}\right)}$$

$$= \frac{1}{w-a}\sum_{k=0}^{\infty}\left(\frac{z-a}{w-a}\right)^k$$

$$\Rightarrow \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,dw = \sum_{n=0}^{N}\frac{f(w)\,(z-a)^n}{(w-a)^{n+1}} + R_N(z)$$

where $$R_N(z) = \frac{1}{2\pi i}\int_{C_2}\sum_{n>N}\frac{f(w)\,(z-a)^n}{(w-a)^{n+1}}\,dw$$

$w = a + r_2 e^{2\pi i t}$

$$|R_N(z)| \leq \frac{1}{2\pi}\int_{t=0}^{t=1}\sum_{n>N}\left|\frac{f(w)\,(z-a)^n}{r_2^{\,n+1}}\right| r_2 \cdot 2\pi \cdot dt$$

$$= \int_{t=0}^{t=1}\sum_{n>N}\frac{|f(w)\,[z-a]^n|}{r_2^{\,n}}\,dt$$

Let $M_2 = \max\limits_{w\in C_2}|f(w)|$

$$\Rightarrow |R_n(z)| \leq \int_{t=0}^{t=1}\sum_{n>N}\frac{M_2\,|z-a|^n}{r_2^{\,n}}\,dt$$

$$\leq \sum_{n>N} M_2 \left(\frac{r}{r_2}\right)^{n} = \frac{M_2 (r/r_2)^{N}}{1 - r/r_2} \to 0$$

as $N \to \infty$

Similarly C_1,

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{-1}{(z-a)\left(1 - \frac{w-a}{z-a}\right)}$$

$$\hookrightarrow -\int_{C_1} \frac{f(w)}{w-z} \frac{dw}{2\pi i} = \frac{1}{2\pi i} \sum^{N} \int_{C_1} \frac{f(w)(z-a)^{-n}}{(w-a)^{-n+1}} dw + R_N(z)$$

$$R_N(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(z-a)^{-N}} \frac{(w-a)^{-N+1} dw}{z-w}$$

$$M_1 := \max_{w \in C_1} (|f(z)|) \Rightarrow |R_n(z)| \leq \frac{M_1 \left(\frac{r_1}{r}\right)^{N}}{1 - \frac{r_1}{r}} \to 0 \text{ as } N \to \infty$$

Proposition

$$\frac{1}{2\pi i} \int_{C_a(r)} \frac{1}{(z-a)^n} dz = \begin{cases} 1, & n = 1 \\ 0. & n \neq 1 \end{cases}$$

$$C_a(r) = \{ z : |z-a| = r \}$$

Pf $z = a + re^{2\pi i t}$

$$n = 1 \longrightarrow \frac{1}{2\pi i} \int_0^1 \frac{1}{re^{2\pi i t}} \cdot 2\pi i r e^{2\pi i t} dt = \int_0^1 1\, dt = 1$$

$$n \neq 1 \longrightarrow \frac{1}{2\pi i} \int_0^1 \frac{1}{r^n e^{2\pi i n t}} 2\pi i r e^{2\pi i t}$$

$$= \int_0^1 \frac{e^{2\pi i(1-n)t}}{r^{n-1}}\,dt = 0$$

Thm.

If $\sum_{n\in\mathbb{Z}} \alpha_n (z-a)^n$ series converges to $f(z)$ $\forall\, z$ s.t. $r_1 < |z-a| < r_2$, then it is the Laurent series expansion of f.

Pf $f(w) = \sum_{n\in\mathbb{Z}} \alpha_n (z-a)^n$

$$\frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{(w-a)^{N+1}} = \frac{1}{2\pi i}\int_C \sum \frac{\alpha_n (w-a)^n}{(w-a)^{N+1}}\,dw = \alpha_N$$

Ex. $f(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$

(a) $1 < |z| < 2$, (b) $|z| < 2$, (c) $|z| < 1$

(d) $0 < |z-1| < 1$

(a) $$\frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} = f(z)$$

$$= \frac{-1}{2}\left(1 + z/2 + z^2/2^2 + \ldots\right) - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right)$$

$$= -\left[\frac{1}{2} + \frac{z}{2^2} + \frac{z^2}{2^3} + \ldots + \frac{1}{z} + \frac{1}{z^2} \ldots\right]$$

$\Rightarrow \quad c_n = -\left(\frac{1}{2}\right)^{n+1}, \quad d_n = -1$

(b) $|z| > 2$

$$f(z) = \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{1}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\dots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right)$$

$$= \frac{2^1-1}{z^2} + \frac{2^2-1}{z^3} + \frac{2^3-1}{z^4} + \dots$$

$c_n = 0, \quad d_n = 2^{n-1}-1$

(c) $|z| < 1$

$$f(z) = \frac{1}{2(1-z/2)} - \frac{1}{1(1-z/1)}$$

(d) $0 < |z-1| < 1 \rightsquigarrow z-1 = w$

$$f(z) = \frac{1}{w-1} - \frac{1}{w} = \frac{-1}{1-w} - \frac{1}{w}$$

$$= -\left(1+w+w^2+\dots+\frac{1}{w}\right)$$

$$= -\left(1+(z-1)+(z-1)^2+\dots+\frac{1}{w}\right)$$

$c_n = -1, \quad d_1 = 1, \quad d_{n>1} = 0$

Classification of singularities

Let a be an isolated singularity

of f and $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{n=1}^{\infty} \frac{d_n}{(z-a)^n}$

is the Laurent series expansion of f.

The series $\sum_{n=1}^{\infty} \frac{d_n}{(z-a)^n}$ is called

Principal part of f.

(i) a is called removable singularity if $d_n = 0 \ \forall \ n \geq 1$

(ii) a is called pole if $d_n = 0 \ \forall \ n > n_0$ and $d_{n_0} \neq 0$ (of order n_0)

(iii) a is called essential singularity if $d_n \neq 0$ for infinitely many n.

Thm.

The point a is a removable singularity of f

iff $\lim_{z \to a} f(z)$ is finite

iff f is bounded in nbd. $0 < |z-a| < \delta$

Pf ($\Rightarrow$) a is removable singularity

$$\Rightarrow d_n = 0 \quad \forall\, n \in \mathbb{N} \Rightarrow \lim_{z \to a} f(z) = c_0$$

($\Leftarrow$) Let $f(z)$ bounded in $0 < |z-a| < \delta$

Then,

$$d_n = \frac{1}{2\pi i} \int_0^1 \frac{f(w)}{(w-a)^{-n+1}}\, dw \le \frac{1}{2\pi} \frac{M}{r^{-n+1}}\, r$$

$$= \frac{M r^n}{2\pi} \longrightarrow 0$$

as $r \to 0$ $\longrightarrow$ ($\because n \geq 1$)

$\left\{ \begin{array}{l} \text{Can take } r \to 0 \text{ as } f \text{ Analytic} \\ \text{on } 0 < |z-a| < \delta \end{array} \right\}$

Thm.

The point a is called a pole (of order n_0) of function f

iff $\lim_{z \to a} (z-a)^{n_0} f(z)$ is finite $b \neq 0$ $(= d_{n_0})$

Pf. ($\Rightarrow$) a is pole (n_0)

$$f(z) = \sum_0^\infty c_n (z-a)^n + \sum_1^\infty d_n (z-a)^n$$

$$\Rightarrow \lim_{z \to a} (z-a)^{n_0} f(z) = d_{n_0} \neq 0$$

($\Leftarrow$) $(z-a)^{n_0} f(z)$ is finite (M)

$|z-a|^{n_0} |f(z)| \le M + 10 \Rightarrow |f(z)| \le \frac{M+10}{\ldots}$

$$d_n = \frac{1}{2\pi i} \int \frac{f(w)\,dw}{(w-a)^{n+1}}$$

$$\Rightarrow |d_n| \leq \frac{1}{2\pi} \frac{\frac{M}{|z-a|^{n_0}} r}{r^{-n+1}} = \frac{M_0}{r^{n_0 - n}} \to 0$$

as $r \to 0$ $\quad (\because n > n_0)$

Thm.

f has a pole of order m iff $1/f$ has zero of order m.

Pf. $(\Rightarrow)$ $\frac{d_m}{(z-a)^m} + \frac{d_{m-1}}{(z-a)^{m-1}} + \dots + \frac{d_1}{(z-a)} + \sum_{n=0}^{\infty} c_n (z-a)^n$

$$= \frac{1}{(z-a)^m}\left(d_m + d_{m-1}(z-a) + \dots + \sum_{n=0}^{\infty} c_n (z-a)^{n+m}\right)$$

$$= \phi(z)/(z-a)^m,$$

where $\phi(z)$ is analytic & $\phi(a) \neq 0$

$\Rightarrow \exists$ nbd. $|z-a| < \delta_1$ s.t. $\phi(z) \neq 0$

$$\Rightarrow \frac{1}{f(z)} = \frac{(z-a)^m}{\phi(z)} = (z-a)^m \psi(z) \quad \text{w/ } \psi(a) \neq 0$$

$\Rightarrow \frac{1}{f(z)}$ has a zero of order m.

$(\Leftarrow)$ $\frac{1}{f(z)} = (z-a)^m \psi(z)$ w/ $\psi(a) \neq 0$

w/ $\psi_1(z)$ analytic

$\psi(z)$

$$\Rightarrow f(z) = \frac{\varphi_1(z)}{(z-a)^m} \qquad \& \ \varphi_1(z) \neq 0$$

$$= \frac{1}{(z-a)^m}\left(\sum_{n=0}^{\infty} c_n (z-a)^n\right) \quad w/ \ c_0 \neq 0$$

$$= \frac{c_0}{(z-a)^m} + \frac{c_1}{(z-a)^{m-1}} + \dots + c_m + c_{m+1}(z-a) + \dots$$

$\Rightarrow f(z)$ has a pole of order m.

Corollary

f has a pole at a iff

$$\lim_{z \to a} f(z) = \infty$$

just smth. big.
no concept as $+\infty$ or $-\infty$
like in $\mathbb{R}$ because
$\mathbb{C}$ has no direction

Thm.

(i) f has an essential singularity at "a" iff $\lim_{z \to a} f(z)$ does not exist.

(ii) Casorati-Wierstrass Theorem

Let V be a deleted nbd. of a
$\{V = B_a(\delta) \setminus \{a\}$ for some $\delta\}$.
Then $f(V)$ is dense in $\mathbb{C}$.

$\left\{ \text{Take any } z_0 \in \mathbb{C}.\ \forall\ \epsilon > 0\ \exists\ \delta > 0 \text{ s.t. } |z_0 - f(w)| < \epsilon,\ 0 < |w - a| < \delta\ \&\ w \in V \right\}$

{ A set $S \subseteq \mathbb{R}^n$ is called dense if given $\epsilon > 0$ & $x_0 \in \mathbb{R}^n$ then $\exists$ a point $s \in S$ s.t. $|s - x_0| < \epsilon$ }

(III) Picard's great theorem

Let V be a deleted nbd. of a then $f(V)$ is $\mathbb{C}$ except possibly a single point missing.

Singularity at ∞

A function f is said to have singularity (removable, pole, singularity) if $f\left(\frac{1}{z}\right)$ has a singularity at 0.

Ex. $P_n(z) = a_n z^n + \ldots + a_0,\quad a_n \neq 0$

$P_n(1/z) = \dfrac{a_n + \ldots}{z^n} \longrightarrow 0$ is a pole (n) of $P(1/z)$

∞ is a pole (n) of $P(z)$

☆ $\longrightarrow$ Non-isolated singularity

$\csc(1/z) \longrightarrow$ Non-isolated sing

($\mathrm{Log}(z) \to (-\infty, 0] \cap \mathbb{Z}$

$\to$ poles at $\frac{1}{\pi n}$ which can get arbitrarily close to 0.

{0 is not an isolated singularity}

Residue

Let f have Laurent series expansion,

$$f(z) = \sum_{0}^{\infty} c_n (z-a)^n + \sum_{1}^{\infty} \frac{d_n}{(z-a)^n}$$

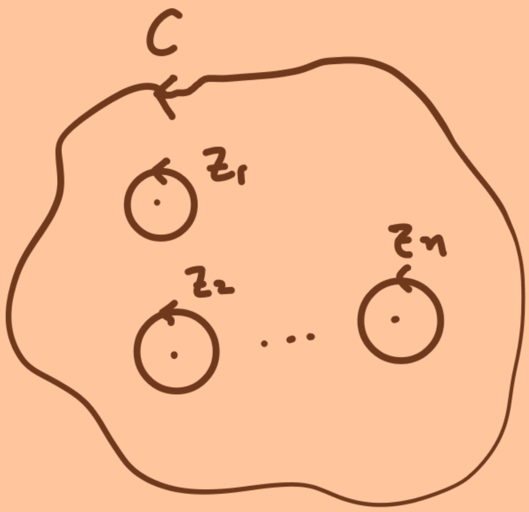
then

$$\boxed{\operatorname*{Res}_{z=a}(f) = \mathrm{Res}_a(f) := d_1 = \frac{1}{2\pi i} \oint f(w)\,dw}$$

Cauchy Residue Theorem

Let $f: D \to \mathbb{C}$ s.t.

f is Analytic except for points $z_1, z_2 \dots z_n \in D$.



Let C be a closed curve enclosing all singularities of f.

then

$$\int f(w)\,dw = \sum^{n} \mathrm{Res}(f)$$

$$\boxed{\frac{1}{2\pi i}\oint_C f(w)\,dw = \sum_{k=1} \operatorname{Res}_{z_k}(f)}$$

Pf Using Cauchy theorem for multiply connected domain, we have,

$$\frac{1}{2\pi i}\oint_C f(w)\,dw = \frac{1}{2\pi i}\sum_{k=1}^{n}\oint_{C_k} f(w)\,dw$$

$$= \sum_{k=1}^{n} \operatorname{Res}_{z_k}(f)$$

Ex $C = C_0(3)$, $f(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$

then
$$\frac{1}{2\pi i}\oint_C f(z)\,dz = \operatorname{Res}_1(f) + \operatorname{Res}_2(f)$$
$$= \text{Coeff of } \frac{1}{z-1} + (\ )$$
$$= (-1) + (1) = \underline{\underline{0}}$$

Ex $C = C_0(3/2)$, $f(z) = \dfrac{1}{(z-1)(z-2)}$

$$\frac{1}{2\pi i}\oint_C f(w)\,dw = \operatorname{Res}_1(f) = \underline{\underline{-1}}$$

Techniques to calculate residue.

(1) If $f(z)$ has a simple pole at $z = a$

$$f(z) = \frac{d_1}{z-a} + c_0 + c_1(z-a) + \dots$$

$$\boxed{d_1 = \lim (z-a) f(z)}$$

$$\boxed{d_1 = \lim_{z \to a} (z-a) f(z)}$$

(II) If $f(z) = \dfrac{g(z)}{h(z)}$, $g(a) \neq 0$, $h(a) = 0$, $h'(a) \neq 0$

$\Rightarrow$ f has a simple pole at a

$$\boxed{\mathrm{Res}_a(f) = \frac{g(a)}{h'(a)}} = \lim_{z \to a} \frac{(z-a)\, g(z)}{h(z)}$$

$$= \lim_{z \to a} g(z) \cdot \lim_{z \to a} \frac{1}{\left(\frac{h(z)}{z-a}\right)}$$

(III) If order of pole is ≥ 2

$$f(z) = \frac{d_m}{(z-a)^m} + \dots + \frac{d_1}{z-a} + c_0 + c_1(z-a) + \dots$$

$$\boxed{\mathrm{Res}_a(f) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z) \right)}$$

Ex. $f(z) = \dfrac{z}{z^n - 1}$, $n \geq 2$

poles : $z_k = e^{2\pi i k/n}$, $0 \leq k < n$

order of pole at z_k : 1

$$\mathrm{Res}_{z_k}(f) = \frac{g(z_k)}{h'(z_k)} = \frac{z_k}{n z_k^{n-1}} = \frac{z_k^2}{n}$$

Ex $f(z) = \tan(\pi z) = \sin(\pi z)/\cos(\pi z)$

poles : $(2n+1)/2$ $\qquad \operatorname{Res}_{z_0}(f) = \dfrac{\sin \pi z_0}{-\pi \sin \pi z_0} = \dfrac{-1}{\pi}$

order : 1

Ex $\quad f(z) = \dfrac{z^3+5}{z^2(z-1)^3}$

poles : $0, 1$

order : $2, 3$

$$\operatorname{Res}_0(f) = \lim_{z\to 0} \frac{1}{(2-1)!} \frac{d}{dz}\left(\frac{z^3+5}{(z-1)^3}\right)$$

$$= \lim_{z\to 0}\left(\frac{3z^2}{(z-1)^3} + \frac{(z^3+5)(-3)}{(z-1)^4}\right)$$

$$= 0 + \frac{-15}{1} = -15$$

$$\operatorname{Res}_1(f) = \lim_{z\to 1} \frac{1}{(3-1)!} \frac{d^2}{dz^2}\left(\frac{z^3+5}{z^2}\right) = 15$$

$$= \frac{1}{2} \lim_{z\to 1} \frac{d^2}{dz^2}\left(z + \frac{5}{z^2}\right) = \frac{1}{2} \, \frac{5\cdot(-2)\cdot(-3)}{(1)^4}$$

Evaluation of integrals by Residue Theorem

(i) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx \leq 2 + \frac{2}{\delta}$ $\qquad |f(x)| \leq \min\left\{1, \frac{1}{|x|^{1+\delta}}\right\}$

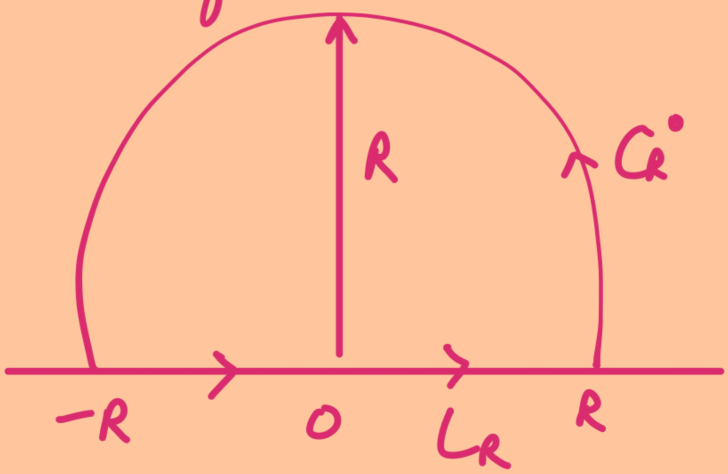
$$\leq \int_{-\infty}^{-1} + \int_{-1}^{1} + \int_{1}^{\infty} |f(x)|\, dx$$

$$\leq \quad 2 \cdot \int_1^\infty \frac{1}{x^{1+\delta}}\, dx + \int_{-1}^{1} 1\, dx = \frac{2}{\delta} + 2$$

(ii) $\lim_{R \to \infty} \int_{-R}^{R} f(x)\, dx$

Consider complex valued func. $f(z)$

Suppose f has finitely many poles in C_R



Then $$\frac{1}{2\pi i} \int_{C_R} f(z)\, dz = \sum_{p \in \text{poles} \cap C_R \cap y \geq 0} \text{Res}_p f$$

$$\Rightarrow \frac{1}{2\pi i} \left(\int_{L_R} + \int_{C_R^\circ} \right) = \sum_{p \in \text{poles} \cap C_R \cap y \geq 0} \text{Res}_p f$$

Now, $\int_{L_R} f(z)\, dz = \int_{-R}^{R} f(x)\, dx$

Suppose $\lim_{R \to \infty} \int_{C_R^\circ} f(z)\, dz = 0$

Then, take $R \to \infty$, we obtain

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \cdot \sum_{p \in \text{poles}} \text{Res}_p f$$

Prop. Suppose $|f(z)| \leq \dfrac{M}{z^{1+\delta}}$, $|z| \geq R_0$ & $y \geq 0$

then $\left| \int_{C_R^0} f(z)\,dz \right| \to 0$ as $R \to \infty$

Pf. $C_R^0 = Re^{i\theta}$, $0 \leq \theta \leq \pi$

$\Rightarrow dz = Rie^{i\theta}d\theta$, assume $R \geq R_0$

$$\Rightarrow \int_{C_R^0} f(z)\,dz \leq \int_0^{\pi} |f(Re^{i\theta})|\,R\,d\theta$$

$$\leq \int_0^{\pi} R \cdot \frac{M}{R^{1+\delta}}\,d\theta = \frac{M\pi}{R^{\delta}} \to 0$$

as $R \to \infty$

Ex. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$, $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$

$$f(z) = \frac{1}{z^4+1}, \quad |f(z)| \leq \frac{1}{|z^4+1|} \leq \frac{1}{|z|^4-1}$$

$$\leq \frac{100}{|z|^4}$$

for big enough $|z| \geq R_0$
(say $R_0 = 2$)

So. $f(z)$ satisfies above prop.,

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \sum_{\text{poles}} \text{Res}(f) \cdot 2\pi i$$

poles: $\pm\sqrt{\pm i} \cap y \geq 0 \equiv e^{i\pi/4}, e^{i3\pi/4}$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{} = \left(\frac{1}{4e^{3\pi i/4}} + \frac{1}{4e^{9\pi i/4}} \right) 2\pi i$$

$$= \left[\frac{e^{-3\pi i/4} + e^{-\pi i/4}}{4}\right] 2\pi i$$

$$= 2\pi i \left[\frac{c\,\pi/4 + c\,3\pi/4 - i(s\,\pi/4 + s\,3\pi/4)}{4}\right]$$

$$= \frac{2\pi i}{4}\left[0 - i\sqrt{2}\right] = \frac{+\pi}{\sqrt{2}}$$

$$\int_{-\infty}^{\infty} f(x)\,dx$$

It is defined to be

$$\boxed{\lim_{a \to -\infty} \int_a^c f(x)\,dx + \lim_{b \to \infty} \int_c^b f(x)\,dx}$$

if it exists and is finite.

{c arbitrary fixed point}

☆ Principal Value of f :

$$\boxed{P.V.(f) := \lim_{A \to \infty} \int_{-A}^{A} f(x)\,dx}$$

- If $\int^{\infty} f(x)\,dx$ exists, then it is equal to P.V.(f)

Ex. $f(x) = x^3$

$$\int_{-\infty}^{\infty} f(x)\,dx = \infty - \infty \Rightarrow \text{d.n.e.}$$

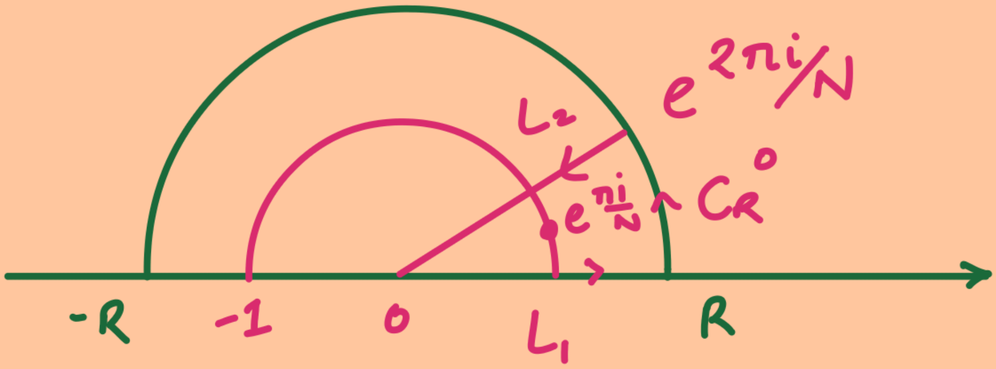
But, $P.V.(f) = 0$

Ex. $$\int_{-\infty}^{\infty} \frac{dx}{x^N+1}$$

for $|z| \geq |z_0|$,

$$f(x) = \frac{1}{x^N+1} \longrightarrow f(z) \rightsquigarrow |f(z)| \leq \frac{M}{|z|^{1+\delta}}$$

Then $$\left| \int_{C_R} f(z)\,dz \right| \longrightarrow 0 \quad \text{as } R \longrightarrow \infty$$



$e^{\pi i/N}$ is a singularity

$$\int_{C_R^0} \longrightarrow 0$$

on L_1, $z = x$, $dz = dx$, $\int_0^R f(x)\,dx = \int \frac{dx}{x^N+1}$

on L_2, $z = x e^{2\pi i/N}$, $dz = e^{2\pi i/N} dx$

$$\int_{L_1} f(z)\,dz = \int_R \frac{\quad}{x^N \cdot e^{2\pi i/N \cdot N} + 1}$$

$$\therefore \int_{L_1} + \int_{L_2} + \int_{C_R}^{0} = 2\pi i \sum \text{Res.}(f)$$

$$\Rightarrow \left(1 - e^{2\pi i/N}\right) \int_0^\infty \frac{dx}{x^N + 1} = 2\pi i \,\text{Res}\left(e^{\pi i/N}\right)$$

$$= \frac{-1}{N} \cdot e^{\pi i/N} \cdot 2\pi i$$

$$\Rightarrow \int_0^\infty = \frac{-2\pi i\, e^{\pi i/N}}{N\left(1 - e^{2\pi i/N}\right)}$$

Jordan Lemma

Let f be a meromorphic function on upper half plane and $f(z) \to 0$ as $|z| \to \infty$.

Then
$$\boxed{\int_{C_R} e^{iaz} f(z)\,dz \to 0 \quad \text{as } |z| \to \infty \quad (a > 0)}$$

Pf. $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$

$$\left| \int_{C_R} f(z)\,dz \right| \leq \int_0^\pi R \left| e^{iaRe^{i\theta}} f(Re^{i\theta}) \right| d\theta$$

Now, $\left| e^{iaRe^{i\theta}} \right| = e^{-aR\sin\theta}$

$$\left|\int_{C_R}\right| \leq R\int_0^{\pi} e^{-aR\sin\theta} M_R \, d\theta$$

where $M_R = \max\limits_{|z|=R} |f(z)|$

$$\leq 2M_R R \int_0^{\pi/2} e^{-aR\sin\theta} d\theta$$

Lemma: For $0 < \theta \leq \pi/2$, we have $\frac{2}{\pi} \leq \frac{\sin\theta}{\theta} \leq 1$

P/ For RHS, $f(\theta) = \theta - \sin\theta$

$f'(\theta) = 1 - \cos\theta \geq 0$

$\Rightarrow f\uparrow \Rightarrow f(0) \leq f(\theta) \Rightarrow 0 \leq \theta - \sin\theta \Rightarrow \frac{\sin\theta}{\theta} \leq 1$

For LHS, $f(\theta) = \frac{\sin\theta}{\theta} \Rightarrow f'(\theta) = \frac{\theta\cos\theta - \sin\theta}{\theta^2}$

Let $g(\theta) = \theta\cos\theta - \sin\theta$

$\Rightarrow g'(\theta) = -\theta\sin\theta \leq 0 \quad \{0 \leq \theta \leq \pi\}$

$\Rightarrow g\downarrow \Rightarrow g(0) \geq g(\theta) \Rightarrow 0 \geq g(\theta)$

$\Rightarrow 0 \geq f'(\theta)$

$\Rightarrow f\downarrow \Rightarrow f(\theta) \geq f(\pi/2) = 2/\pi$

Now,

$$\left|\int\right| \leq 2M_R R \int^{\pi/2} e^{-aR\sin\theta} d\theta$$

$$\left|\int_{C_R}\right| \le 2M_R \int_0 \cdots$$

$$\le 2M_R R \int_0^{\pi/2} e^{-aR\cdot\frac{2\theta}{\pi}}\, d\theta$$

$$= \frac{2M_R R}{aR\cdot 2/\pi}\left(1-e^{-aR}\right) \le \frac{10\pi}{a} M_R$$

as $R \to \infty$

$$\Rightarrow \left|\int_{C_R}\right| \to 0 \quad \because \; M_R \to 0$$

{Condition on f}

$$\therefore \boxed{\int_{-\infty}^{\infty} e^{iax} f(x)\, dx = 2\pi i \sum_{y>0} \operatorname{Res}\left(e^{iaz} f(z)\right)}$$

Ex. $\displaystyle\int_0^{\infty} \frac{\cos\alpha x\, dx}{x^2+a^2}$, $\alpha > 0$, $a > 0$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos\alpha x\, dx}{x^2+a^2} = \frac{1}{2}\operatorname{Re}\int_{\mathbb{R}} \frac{e^{i\alpha x}}{x^2+a^2}\, dx$$

$f(z) := \dfrac{1}{z^2+a^2} \rightsquigarrow$ only pole in upper half plane, $z = ia$

$$\text{Residue} = \frac{e^{i\alpha\cdot ia}}{2\cdot(ia)} = \frac{e^{-a\alpha}}{2ai}$$

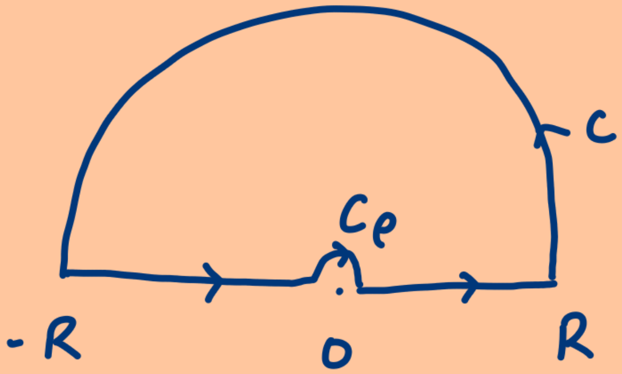
$$\therefore \int \frac{e^{i\alpha x}}{\cdots}\, dx = 2\pi i \cdot \frac{e^{-\alpha a}}{\cdots} = \frac{\pi e^{-\alpha a}}{\cdots}$$

$$\therefore \int_{\mathbb{R}} \frac{e^{\cdots}}{x^2+a^2}\,dx = \frac{2\pi i \cdot e^{\cdots}}{2ai} = \frac{\pi}{a} e^{\cdots}$$

$$\Rightarrow \int_0^\infty \frac{\cos \alpha x\, dx}{x^2+a^2} = \frac{\pi}{2a} e^{-a\alpha}$$

Ex. $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$

$$= \frac{1}{2}\,\mathrm{Im} \int_{-\infty}^{\infty} \frac{e^{iz}}{z}\,dz, \quad f(z) = \frac{1}{z}$$



$$\lim_{\substack{R\to\infty \\ \rho\to 0}}$$

$$\int_C \frac{e^{iz}}{z}\,dz \le \int_0^\pi \frac{e^{-R\sin\theta}}{R} R\,d\theta \le \frac{c}{R} \to 0 \quad R\to\infty$$

$$\lim_{\rho\to 0} \int_{C_\rho} f(z)\,dz = \lim_{\rho\to 0} \int_\pi^0 e^{i\rho(\cos\theta + i\sin\theta)}\,d\theta$$

$$= \int_\pi^0 \lim_{\rho\to 0} e^{i\rho(\cos\theta + i\sin\theta)}\,d\theta \qquad \{ z = \rho e^{i\theta} \}$$

$$= \int_\pi^0 1\,d\theta = -\pi$$

$$\int^\infty + (-\pi) = 2\pi i \sum \mathrm{Res} = 0$$

$-\infty$

$$\Rightarrow \int_0^\infty = \frac{1}{2}\cdot\pi = \pi/2$$

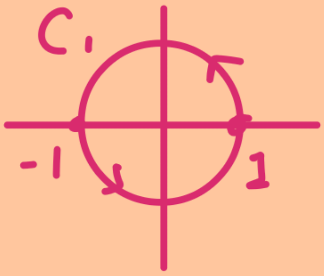
☆ $$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$$ R – Rational function

$z = e^{i\theta} \qquad 0 \leq \theta \leq 2\pi$

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$.

$dz = i e^{i\theta} d\theta = i z\, d\theta$

$\Rightarrow \quad d\theta = \dfrac{dz}{iz}$



$$\frac{1}{i}\int_{C_1} \frac{P(z)}{Q(z)}\, dz = 2\pi i \cdot \sum_{|z|<1} \mathrm{Res}\left(P(z)/Q(z)\right)$$

⊙ Suppose f has a pole at z_0 (of order m),

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n + \sum_{k=1}^{m} \frac{d_k}{(z - z_0)^k}$$

$$\underset{z = z_0}{\text{Residue}}\, f(z) = d_1 = \frac{1}{2\pi i}\oint f(w)\, dw$$

$$\text{Residue}\, f(z) := \frac{1}{2\pi i}\oint_C f(w)\, dw$$

$z = \infty$ $\qquad\qquad\qquad\qquad$ C_R

where C_R is a circle of large radius containing all finite poles of f.

Thm.

Residue of f at ∞ is the negative of co-efficient of z in expansion of $f(1/z)$, $0 \leq |z| < d$, where d is so small that it does not contain any other singularity of f.

Pf.

$$\operatorname*{Res}_{z=\infty} f(z) = \frac{1}{2\pi i} \oint_{C_R} f(w)\, dw, \quad w = \frac{1}{z}$$

$$\Downarrow$$

$$dw = \frac{-dz}{z^2}$$

$$= \frac{-1}{2\pi i} \oint_{C_{1/R}} \frac{f(\frac{1}{z})\, dz}{z^2} = -d_1$$

$$f(1/z) = d_1 z + d_2 z^2 + \dots + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \dots$$

Analytic Continuation

Let f be an analytic function on domain D_1 & g Analy. on D_2.

If $f = g$ on $D_1 \cap D_2$, then

- f is called analytic continuation of g along D_1.
- g is called analytic continuation of f along D_2.

☆ $f = u + iv$ Analytic on $B_0(R)$

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\, d\theta, \quad 0 < r < R$$

Pf.
$$\frac{1}{2\pi i}\oint_{\substack{C_r \\ |z|=r}} \frac{f(z)}{z}\, dz = f(0)$$

$$z = re^{i\theta}, \quad 0 \le \theta \le 2\pi$$

$$u(0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{u(re^{i\theta})\, rie^{i\theta}}{re^{i\theta}}\, d\theta$$

$$\underline{u(0) = \frac{1}{2\pi}}\int_0^{2\pi} \underline{u(re^{i\theta})\, d\theta}$$

⊙ $\sin z$, $\cos z$ are not bounded in $\mathbb{C}$.

$\rightarrow \sin z = \dfrac{e^{iz} - e^{-iz}}{2i} \xrightarrow{\text{mod}} \geq \dfrac{e^{|z|} - e^{-|z|}}{2}$

OR

$\rightarrow$ If $\sin z$ bounded, then by Liouville's Theorem, $\sin z$ constant throughout $\Rightarrow \Leftarrow$

Theorem

f is meromorphic inside a contour C. Then

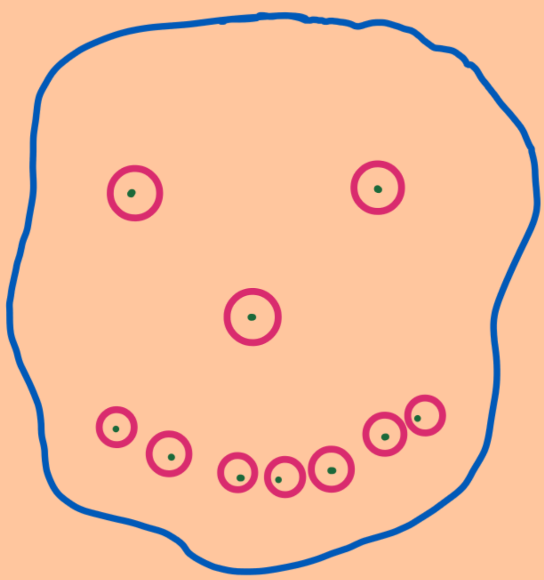
$$\frac{1}{2\pi i} \int \frac{f'(z)}{f(z)}\, dz = \text{No. of Zeroes} - \text{No. of Poles}$$

Let it have k-zeroes & m-poles

$\because$ zeroes & poles are isolated,

$\exists$ circles $\{C_j\}_{j \in [k+m]}$ s.t. C_j contains either a zero (z_n) or a pole (w_e)

By Cauchy Theorem for multiply connected theorem,



$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz$$

$$= \frac{1}{2\pi i} \sum_{j=1}^{k+m} \int_{C_j} \frac{f'(z)}{f(z)} dz$$

Near z_j, let $f(z)$ has zero of order $m_j \Rightarrow f(z) = (z - z_j)^{m_j} g(z)$ w/ $g(z) \neq 0$, $f'(z) = (z - z_j)^{m_j} g'(z) + m_j (z - z_j)^{m_j - 1} g(z)$

$$\Rightarrow \frac{f'(z)}{f(z)} = \frac{m_j}{z - z_j} + \frac{g'(z)}{g(z)}$$

$$\Rightarrow \frac{1}{2\pi i} \int_{C_j} \frac{f'(z)}{f(z)} dz = m_j$$

$$= \frac{1}{2\pi i} \left(\int_{C_j} \frac{m_j \, dz}{z - z_j} + \int \frac{g'(z)}{g(z)} dz \right)$$

As $g(z_j) \neq 0$ $\exists$ a nbd. s.t. $g(z) \neq 0$ $\Rightarrow \frac{g'(z)}{g(z)}$ analytic

Near w_j, $f(z) = (z - w_j)^{-n_j} h(z)$

Pole of order n_j, $\frac{1}{2\pi i} \int f'/f = -n_j$

$$\Rightarrow \frac{1}{2\pi i}\int \frac{f'}{f} = \sum m_j - \sum n_l$$

$$\Rightarrow \boxed{\frac{1}{2\pi i}\int \frac{f'(z)\,dz}{f(z)} = N_0 - N_\infty}$$

COURSE COMPLETED