Optimization Project: Support Vector Machine

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Project

Support Machine Vector

Objective

Classify data

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- Applied to **binary classification** ($y_i \in \{1, -1\}$);

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Classify data

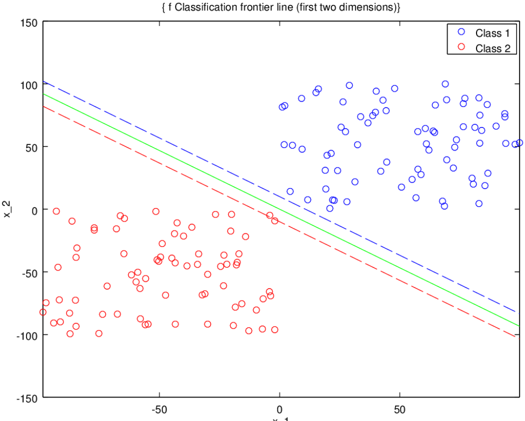
- Applied to **binary classification** ($y_i \in \{1, -1\}$);
- Looking for a **hyperplane** $f : x \rightarrow \omega^T x \; (+b)$ such as:

$$\forall i, f(x_i) = \begin{cases} < 0 & \text{si y=-1} \\ > 0 & \text{si y=1} \end{cases} \Leftrightarrow \forall i, y_i \times f(x_i) > 0 \qquad (1)$$

Project

Support Machine Vector

Figure: Example with two classes (**red** and **blue**)



Optimization problem

Looking for the optimization problem

Naive optimization problem

γ: distance between the lines $f(x) = 1$ and $f(x) = -1$.

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$$max_\omega \ \gamma = \frac{2}{\|\omega\|}$$
$$\text{subject to } \forall i, y_i \times f(x_i) > 0$$

$$\Leftrightarrow min_\omega \ \frac{1}{2}\|\omega\|^2$$
$$\text{subject to } \forall i, y_i \times f(x_i) > 0$$

Beware: if the data set is not linearly separable!

Optimization

Looking for the optimization problem

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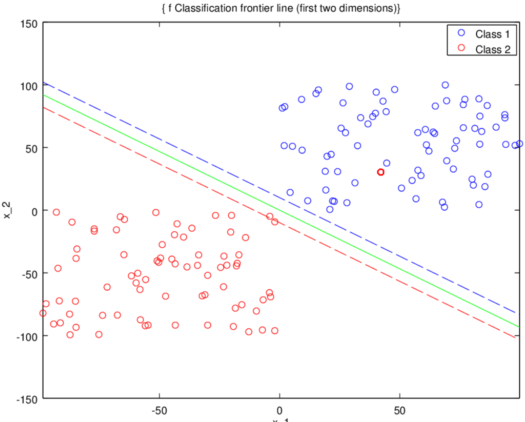

Optimization problem

Adapting the problem to **non-separable sets**

Let z_i be $max(0, 1 - y_i \times f(x_i))$ (**Hinge loss**).

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Having the problem convex and always feasible

Penalty for **classification errors** with $(z_i)_i$ and C :

$$min_{\omega,z} \ \frac{1}{2}\|\omega\|^2 + C\sum_{i\leq m} z_i$$

subject to

$$\forall i, z_i \geq 0$$

$$\forall i, y_i \times (\omega^T x_i) \geq 1 - z_i$$

Implementation

Solving the optimization problem

- Use **Newton's method**:

Reminder: Update of x with Newton's method

$$x_{n+1} \leftarrow x_n + s \times \nabla^2 obj(x_n)^{-1} \nabla obj(x_n)$$

(finding **step size** value s by **backtracking line search**)

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- Make the problem independant from **dimension**;
- Use **logarithmic barrier method**.

Implementation

Independance from dimension: **dual problem**

After Lagrangian calculation and minimization in ω:

Implementation

Independance from dimension: **dual problem**

After Lagrangian calculation and minimization in ω:

Dual problem

$$max_{\lambda \in \mathbb{R}^{+m}} - \frac{1}{2} \| \sum_i \lambda_i y_i x_i \|_2^2 + \mathbf{1}^T \lambda$$

subject to $\forall i, 0 \leq \lambda_i \leq C$

(KKT conditions)

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Independance from dimension: **dual problem**

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(KKT conditions)

Get **primal solution** from **dual solution**

$$\omega^* = \sum_i \lambda_i^* y_i x_i$$

Implementation

Make the problem independant from dimension

Use the **kernel trick** :

Dual problem

Let K be X^TX (**linear kernel**):

$$\max -\frac{1}{2}\lambda^T diag(y) K diag(y)\lambda + \mathbf{1}^T\lambda$$
$$\text{subject to } \forall i, 0 \leq \lambda_i \leq C$$

Implementation

Delete inequality constraints

Use the **logarithmic barrier method** :

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Use the **logarithmic barrier method** :

Barrier function

$$\begin{aligned}\Phi(\lambda) &= \sum_i(-log(C-\lambda_i) - log(\lambda_i)) \\ &= -\sum_i log((C-\lambda_i)\lambda_i)\end{aligned}$$

Implementation

Delete inequality constraints

Use the **logarithmic barrier method** :

Barrier function

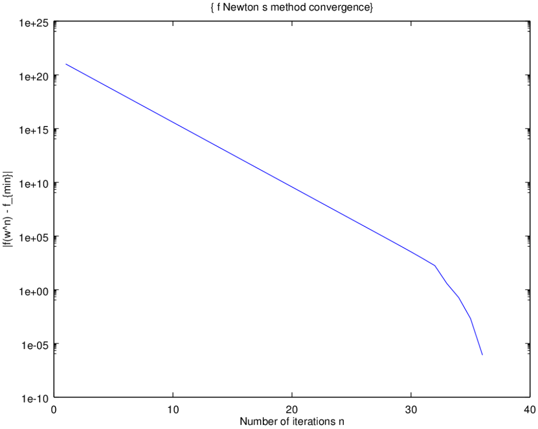
$$\Phi(\lambda) = \sum_i(-log(C - \lambda_i) - log(\lambda_i))$$
$$= -\sum_i log((C - \lambda_i)\lambda_i)$$

Final optimization problem

$$max\ -\frac{1}{2}\lambda^T diag(y)Kdiag(y)\lambda + \mathbf{1}^T\lambda + \Phi(\lambda)$$

Testing the implementation

Newton's method convergence



Testing the implementation

Dependance on the sample size

Table: Time complexity dependance

Set	C	d	n	Iteration number	Time (s)
1	5	40000	10	11	0.315
1	5	40	100	12	0.715
1	5	40	1000	large	$>$ 1,000

Testing the implementation

Performance in function of C

Performed on the same sample set:

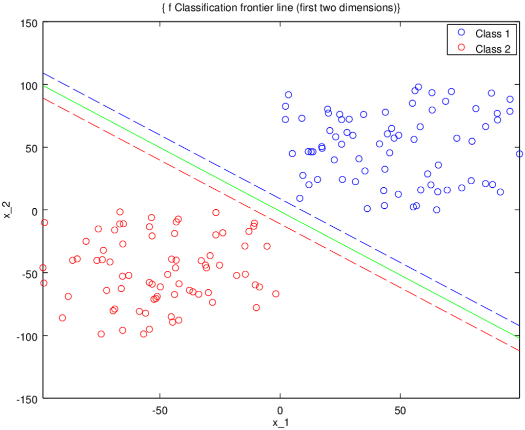
Table: Computation time & Performance in function of C

C	Time (s)	Training Error	Val Error	Test Error
1	132.15	6	2	3
10	0.74	6	2	3
100	0.89	1	12	3

Plotting the **classification frontier**

For $C = 5, n = 150, d = 200$

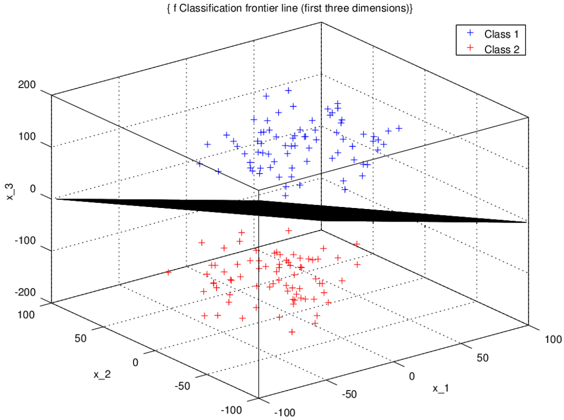
Normalized points with Gaussian distribution (2D):



Plotting the **classification frontier**

For $C = 5, n = 150, d = 200$

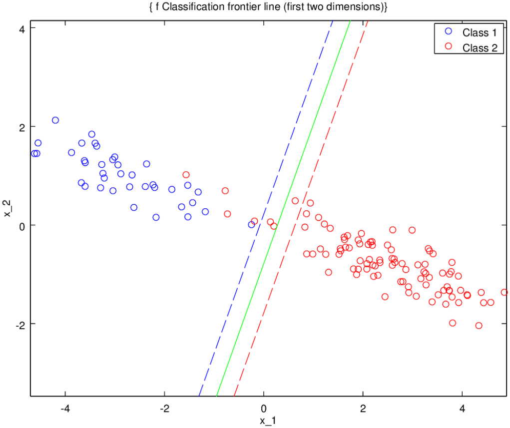
Normalized points with Gaussian distribution (3D):



Plotting the **classification frontier**

Pour $C = 5, n = 100, d = 2$

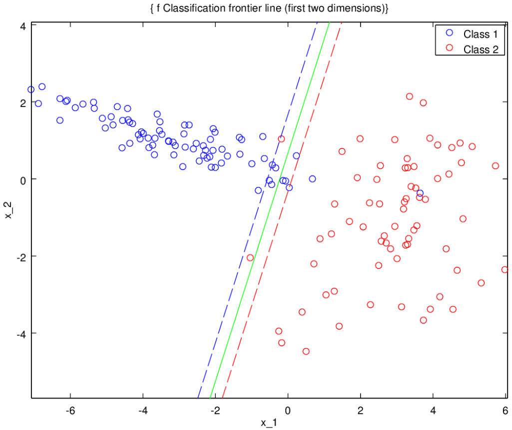
Generation with Gaussian distribution (2D) (set A) :



Plotting the **classification frontier**

Pour $C = 5, n = 100, d = 2$

Generation with Gaussian distribution (2D) (set B) :



Extensions

Cross Validation

- **Cross validation** (choice of the best value for C);

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Leave-one-out technique

Having a sample size of size n, for each value of C to test:

1. for $i \in [1, n]$
2. **Leave out sample** i
3. Train the SVM on other samples
4. Test the SVM on sample i
5. Get the **Mean-Squared Error** for the n loops
6. If it is the minimum MSE computed so far
7. Then update the best value of C

Extensions

Coordinate Descent

- Implementation of **Coordinate Descent**;

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Reminder: Coordinate Descent

for $i, j \in [1, d]$, and iteration k

$$a_i^{k+1} = argmin_{a_i} f(a_1, a_2, ..., a_i, ..., a_d)$$
$$a_j^{k+1} = a_j^k \text{ for } j \neq i$$

Extensions

Coordinate Descent results

Performed on the same sample set (as in the testing of the original SVM):

Table: Computation time & Performance in function of C

C	Time (s)	Training Error	Val Error	Test Error
1	0.37	11	6	5.33
10	0.34	5	4	3.39
100	0.29	1	8	3.56
10,000	0.29	6	4	2.76

Extensions

ACCPM

- Implementation of **Analytic Center Cutting-Plane Method**;

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ACCPM

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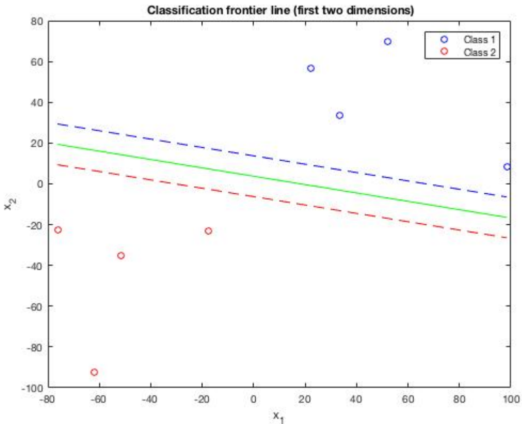
Reminder: ACCPM

1. Compute the **analytic center of constraint polyhedron**
2. Compute the objective value and the gradient
3. While objective value is evolving greatly enough
4. Add an inequality to constraint polyhedron
5. Optional: **Constraint Dropping**

Extensions

ACCPM results

Figure: For data of size 8, and dimension 2



Demo of the SVM