Formal Verification of the RANKING algorithm for Online Bipartite Matching

Christoph Madlener

22.06.2022

Online Bipartite Matching (OBM)

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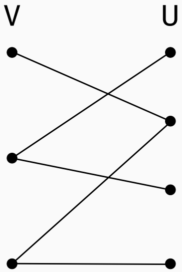
- *bipartite* graph $G = (U, V, E)$



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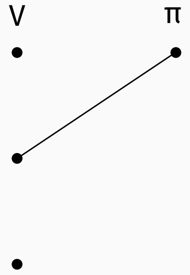

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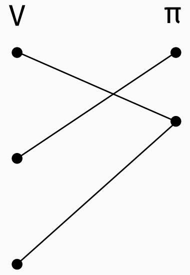

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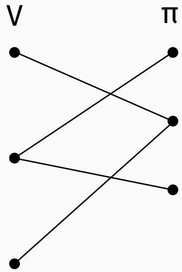

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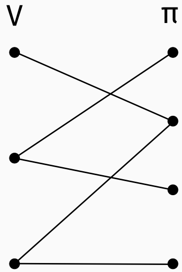

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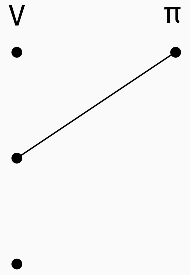

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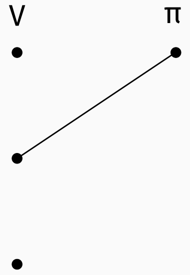

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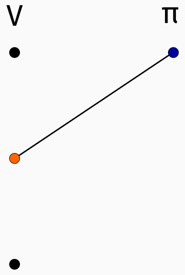

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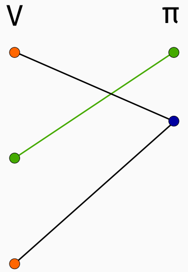

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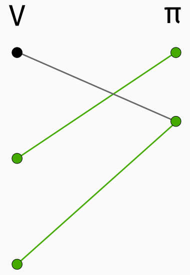

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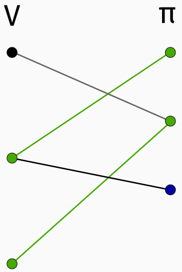

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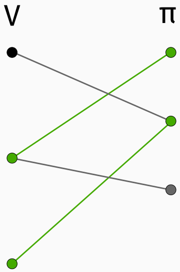

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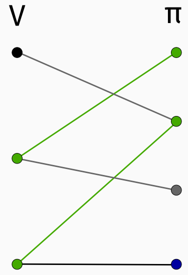

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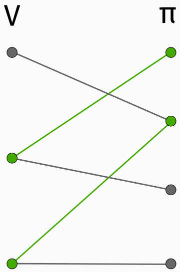

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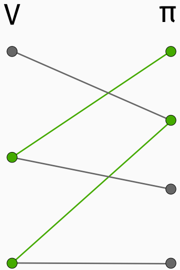

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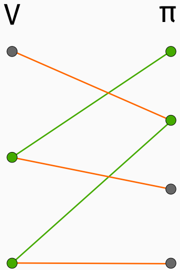

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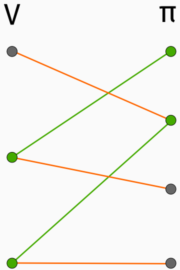

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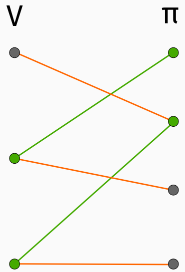

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- competitive ratio of $1 - \frac{1}{e}$ (best possible)

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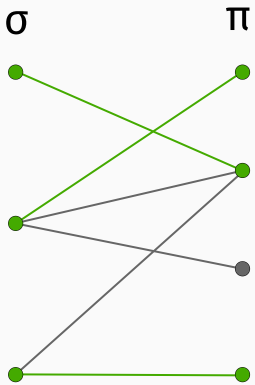
$$\frac{|Ranking(G, \pi, \sigma)|}{|M|} \geq \frac{|R_1|}{|M|} \geq \dots \geq \frac{|Ranking(G^*, \pi_{G^*}, \sigma_{G^*})|}{|M|}$$

First proof of a *simple structural observation*



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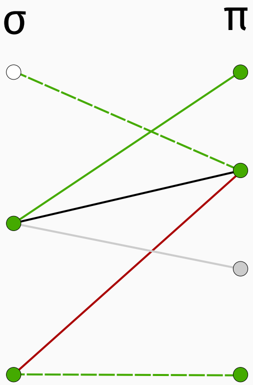

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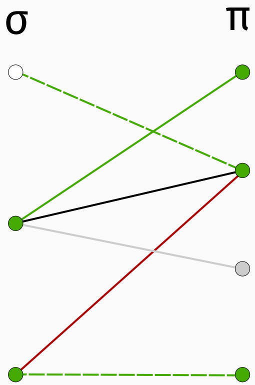

Let $R := Ranking(G, \pi, \sigma)$ for a fixed graph G, arrival order π, and ranking σ.

Specification of alternating path

$$zig(x) = \begin{cases} x \# zag(y) & \{x, y\} \in R \\ [x] & x \text{ unmatched} \end{cases}$$

$$zag(y) = \begin{cases} y \# zag(x') & x' \text{ matched instead} \\ [y] & \text{no other match} \end{cases}$$

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For $t = 1$ and $V = \{1, 2, 3\}$:

$$\mathbb{P}_1\Big(\big\{[3, 2, 1]\big\}\Big) = \frac{1}{3!}$$

$$\begin{aligned}\mathbb{P}_2\Big(\big\{[3, 2, 1]\big\}\Big) &= \mathbb{P}_1\Big(\big\{[2, 3, 1], [3, 1, 2], [3, 2, 1]\big\}\Big) \cdot \mathbb{P}_V(\{2\}) \\ &= \frac{3}{3!} \cdot \frac{1}{3} = \frac{1}{3!}\end{aligned}$$

References

Benjamin Birnbaum and Claire Mathieu.
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Richard M Karp, Umesh V Vazirani, and Vijay V Vazirani.
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`https://github.com/cmadlener/isabelle-ranking`, 2022.