Formal Verification of the RANKING algorithm for Online Bipartite Matching

Christoph Madlener

22.06.2022

Online Bipartite Matching (OBM)

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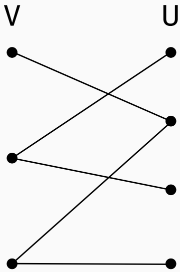
- *bipartite* graph $G = (U, V, E)$



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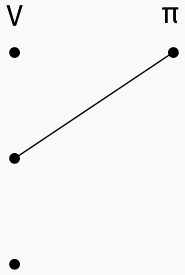

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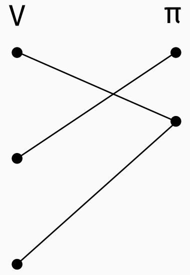

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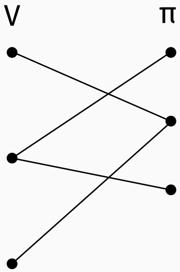

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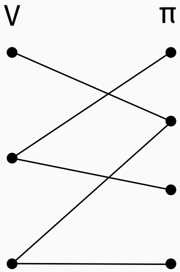

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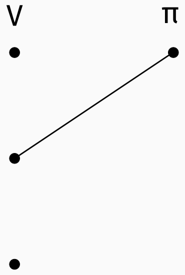

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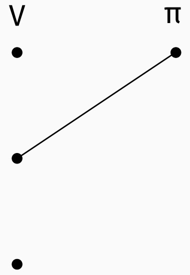

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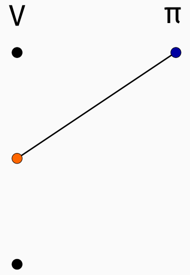

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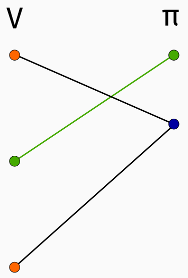

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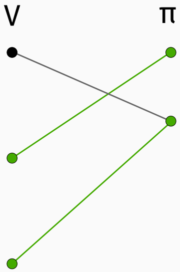

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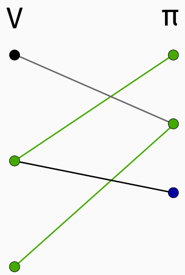

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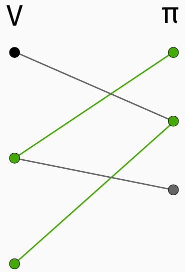

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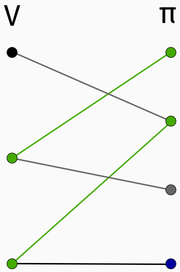

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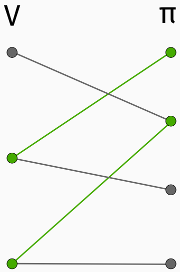

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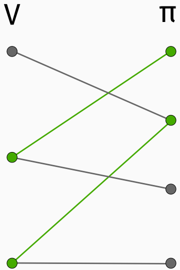

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- Compare $\mathcal{A}$ to best offline algorithm

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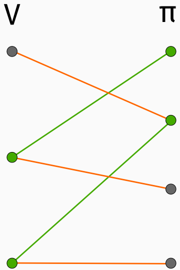

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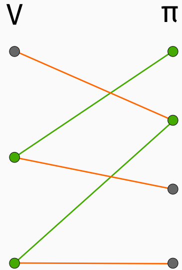

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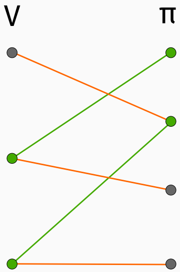

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Algorithm 1: RANKING

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- competitive ratio of $1 - \frac{1}{e}$ (best possible)

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  1. **Combinatorics**
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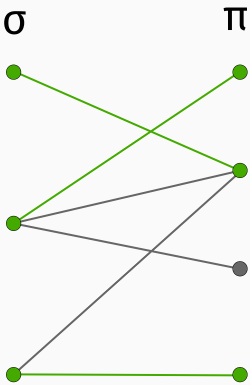
- Most involved part: *Lemma 2*
  "when removing a vertex x from the graph, then the runs on the original graph, and the one without x, differ by at most one alternating path, starting at x"

A ”simple structural observation”



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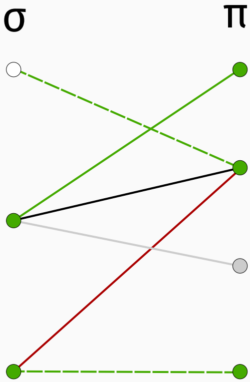

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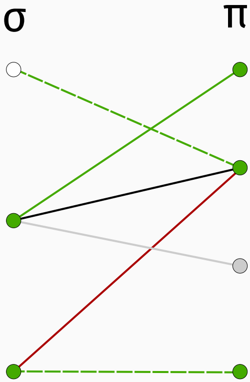

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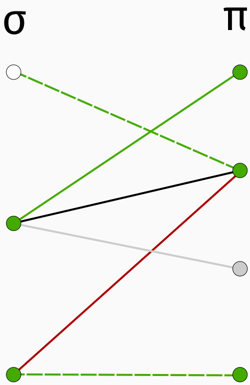

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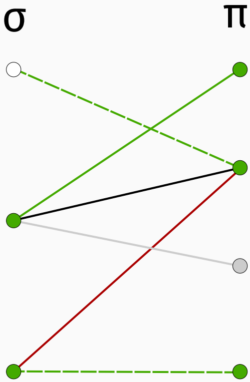

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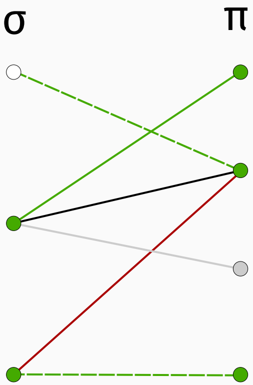

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- Berge's Lemma formalized by Abdulaziz [1]

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- choosing a random permutation of the original offline vertices **vs.**
  choosing a random permutation of the *reduced* offline vertices

References

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