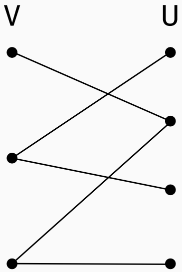
# Formal Verification of the RANKING algorithm for Online Bipartite Matching

Christoph Madlener

22.06.2022

# Online Bipartite Matching (OBM)

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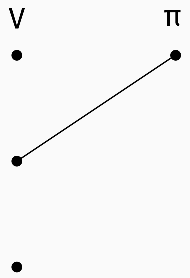

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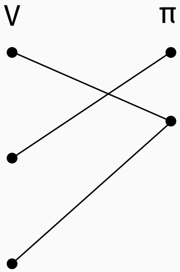


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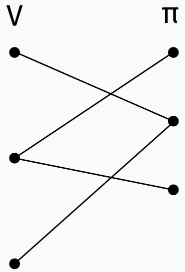

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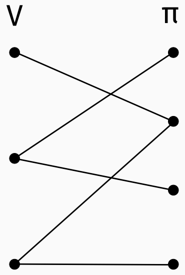

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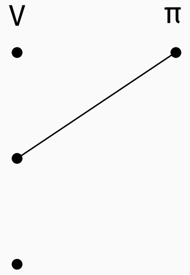


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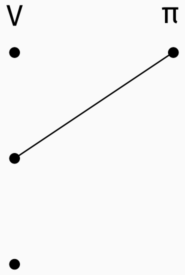


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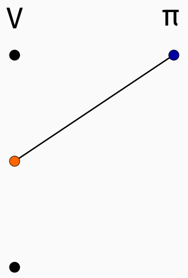


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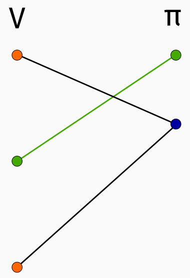


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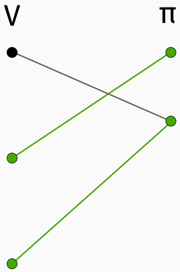


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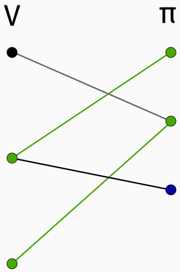


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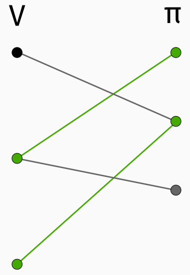


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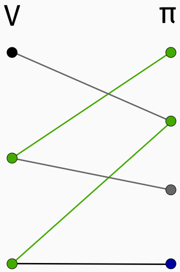


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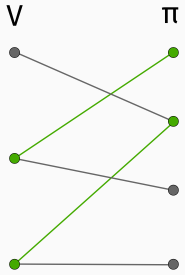


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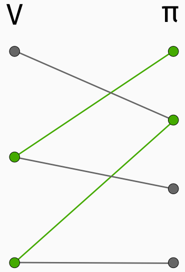


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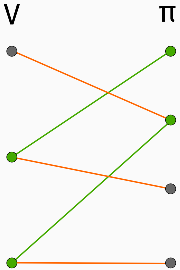


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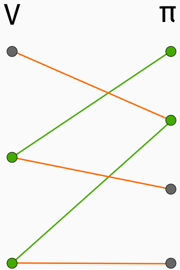


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where $M$ is a maximum cardinality matching in $G$.

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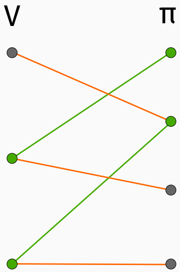


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- competitive ratio of $1 - \frac{1}{e}$ (best possible)

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**Lemma 2.** *Let $x$ be a vertex, $H = G \setminus \{x\}$, and $\pi_H$ and $\sigma_H$ be the orderings of $U_H$ and $V_H$ induced by $\pi$ and $\sigma$ respectively. If the matchings Ranking$(H, \pi_H, \sigma_H)$ and Ranking$(G, \pi, \sigma)$ are not identical, then they differ by a single alternating path starting at vertex $x$.*

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Let $G_i$ be the graph resulting from removing $i$ vertices from $G$, which are not in a maximum cardinality matching $M$, and $R_i := Ranking(H_i, \pi_{H_i}, \sigma_{H_i})$.

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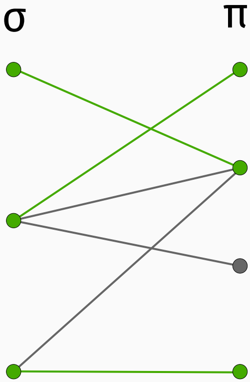
$$\frac{|Ranking(G, \pi, \sigma)|}{|M|} \geq \frac{|R_1|}{|M|} \geq \cdots \geq \frac{|Ranking(G^*, \pi_{G^*}, \sigma_{G^*})|}{|M|}$$

# First proof of a *simple structural observation*



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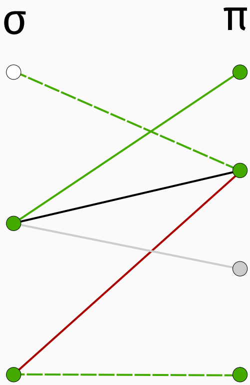

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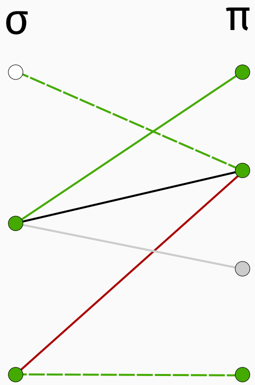


Let $R := Ranking(G, \pi, \sigma)$ for a fixed graph $G$, arrival order $\pi$, and ranking $\sigma$.

**Specification of alternating path**

$$zig(x) = \begin{cases} x \# zag(y) & \{x, y\} \in R \\ [x] & x \text{ unmatched} \end{cases}$$

$$zag(y) = \begin{cases} y \# zag(x') & x' \text{ matched instead} \\ [y] & \text{no other match} \end{cases}$$

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For $t = 1$ and $V = \{1, 2, 3\}$:

$$\begin{aligned}
\mathbb{P}\Big(\big\{[3,2,1]\big\}\Big) &= \frac{1}{3!} \\
&= \frac{3}{3!} \cdot \frac{1}{3} \\
&= \mathbb{P}\Big(\big\{[2,3,1],[3,1,2],[3,2,1]\big\}\Big) \cdot \mathbb{P}(\{2\})
\end{aligned}$$

# References


B. Birnbaum and C. Mathieu.
**On-line bipartite matching made simple.**
*Acm Sigact News*, 39(1):80–87, 2008.

R. M. Karp, U. V. Vazirani, and V. V. Vazirani.
**An optimal algorithm for on-line bipartite matching.**
In *Proceedings of the twenty-second annual ACM symposium on Theory of computing*, pages 352–358, 1990.