Formal Verification of the RANKING algorithm for Online Bipartite Matching

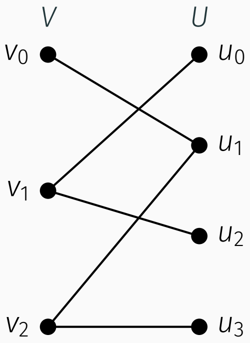
Christoph Madlener

22.06.2022

Online Bipartite Matching (OBM)

Input

- *bipartite* graph $G = (U, V, E)$



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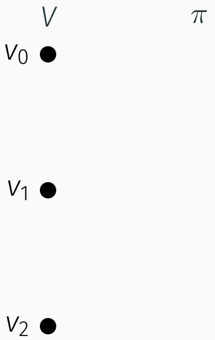

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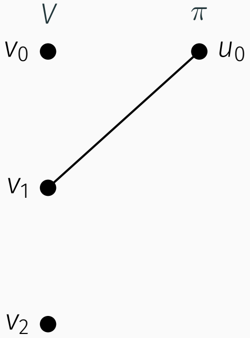

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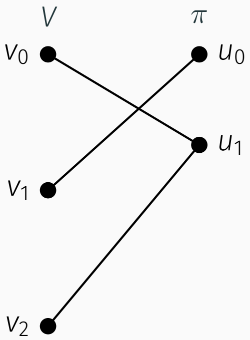

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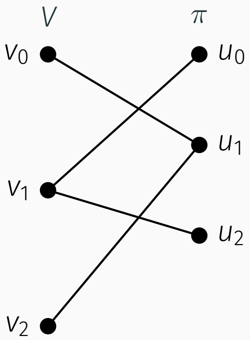

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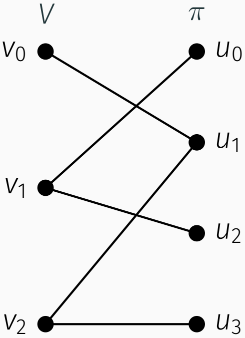

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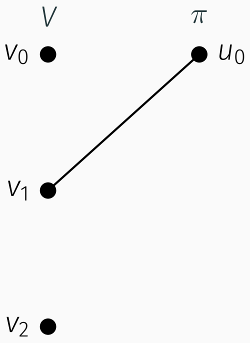

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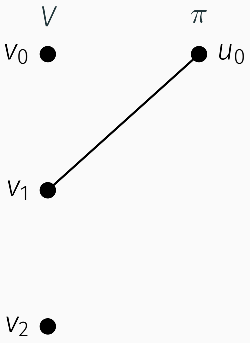

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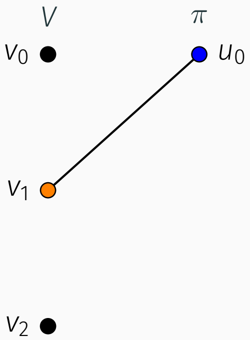

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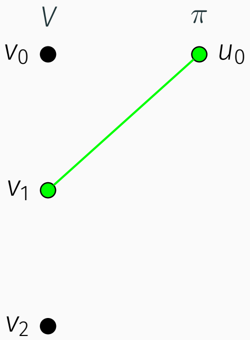

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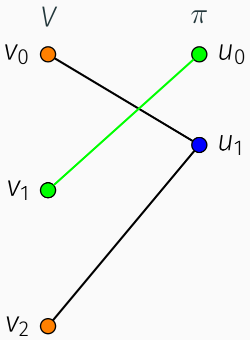

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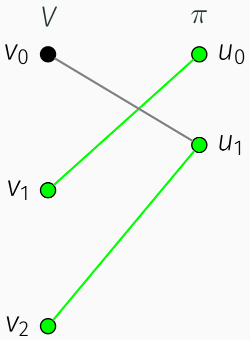

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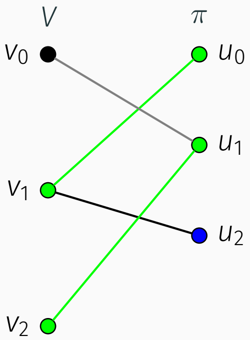

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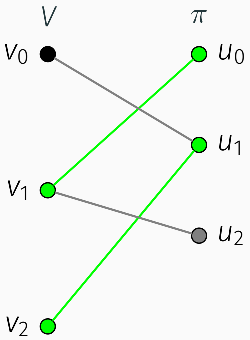

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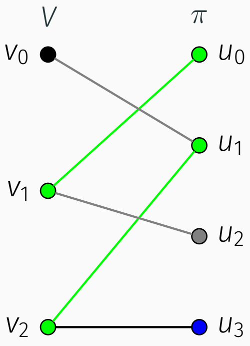

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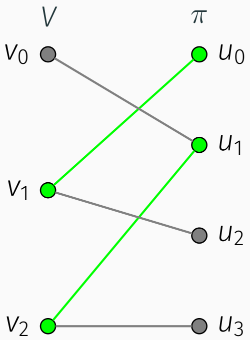

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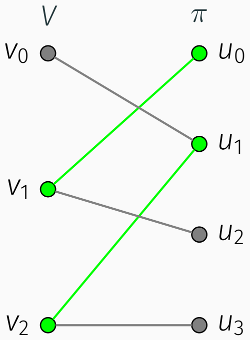
Competitive Ratio

Performance of online algorithm $\mathcal{A}$

- Compare $\mathcal{A}$ to best offline algorithm



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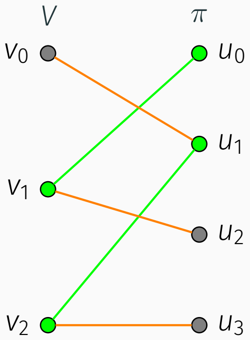

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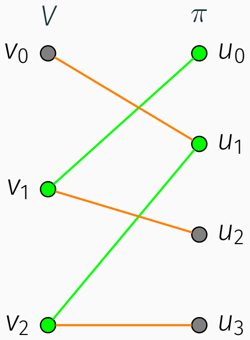

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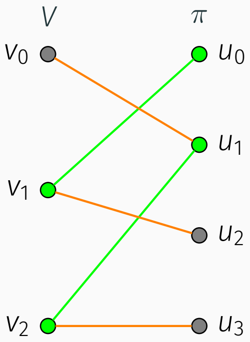
Competitive ratio for OBM

$$\min_G \min_\pi \frac{|\mathcal{A}(G, \pi)|}{|M|}$$

where M is a maximum cardinality matching in G.



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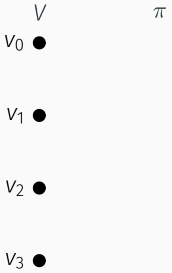

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Online Matching:

On arrival of $u \in U$

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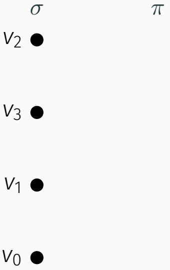

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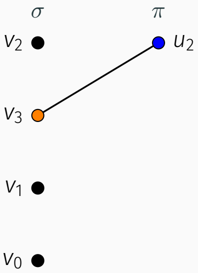

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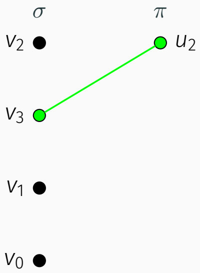

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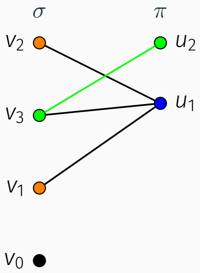

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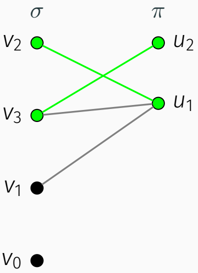

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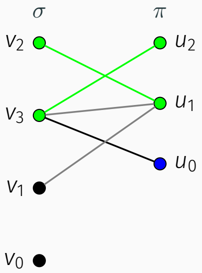

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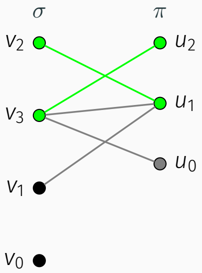

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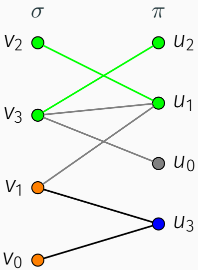

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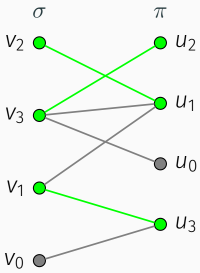

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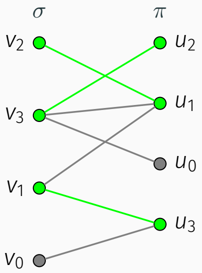

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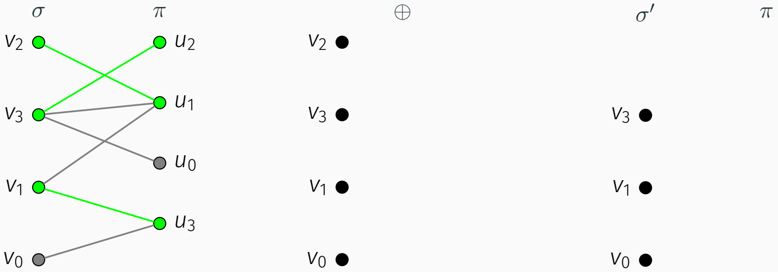

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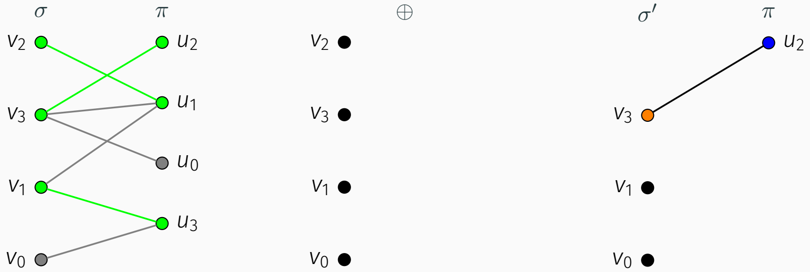

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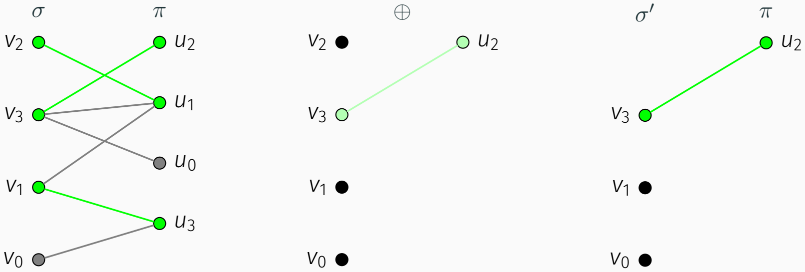

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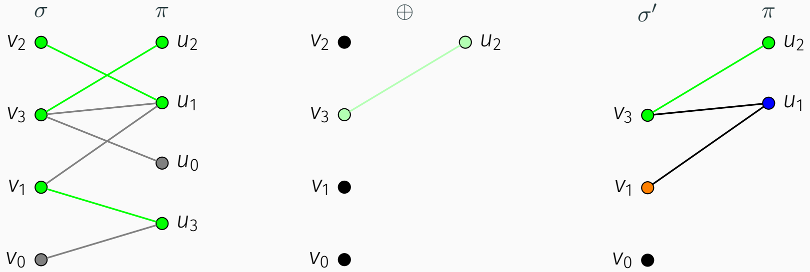

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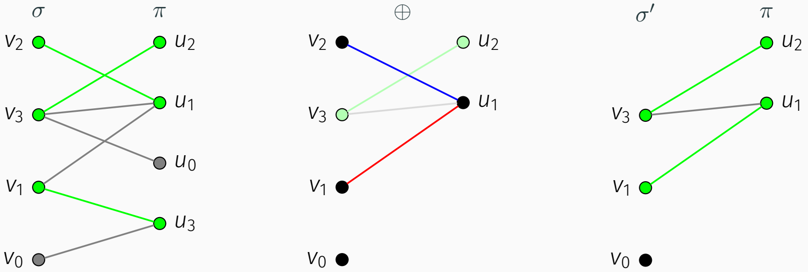

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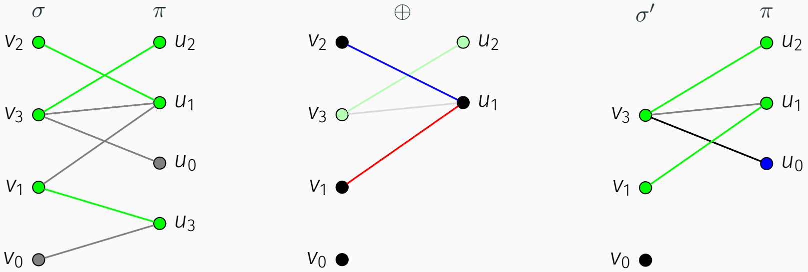

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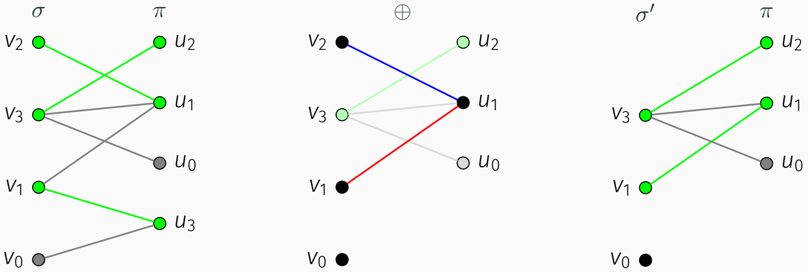

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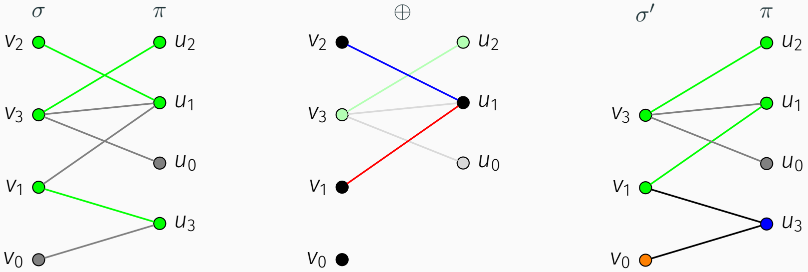

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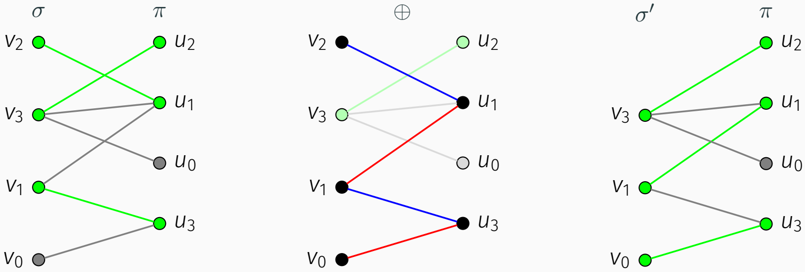

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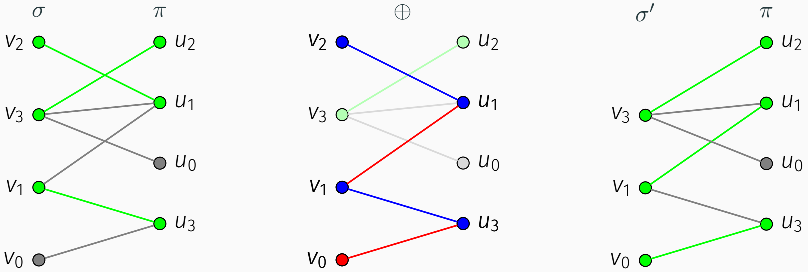
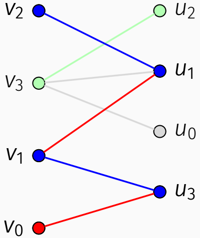


First proof of a *simple structural observation*



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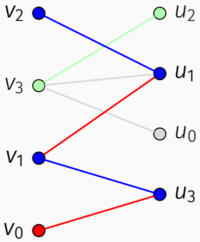

Let $R := \mathit{Ranking}(G, \pi, \sigma)$ for a fixed graph G, arrival order π, and ranking σ.

Specification of alternating path

$$zig(x) = \begin{cases} x \# zag(y) & \{x, y\} \in R \\ [x] & x \text{ unmatched} \end{cases}$$

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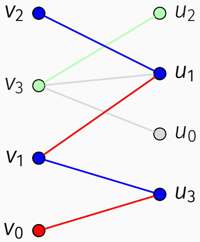

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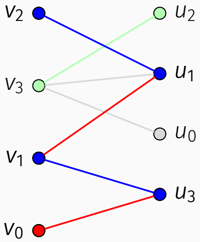

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For $t = 1$ and $V = \{1, 2, 3\}$:

$$\mathbb{P}_1\Big(\{[3,2,1]\}\Big) = \frac{1}{3!}$$

$$\mathbb{P}_2\Big(\{[3,2,1]\}\Big) = \mathbb{P}_1\Big(\{[2,3,1],[3,1,2],[3,2,1]\}\Big) \cdot \mathbb{P}_V(\{2\})$$

$$= \frac{3}{3!} \cdot \frac{1}{3} = \frac{1}{3!}$$

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