Quantitative Analytics.
Lectures. Week 3.
Vanilla products and Greeks, Closed form Solution Derivation.

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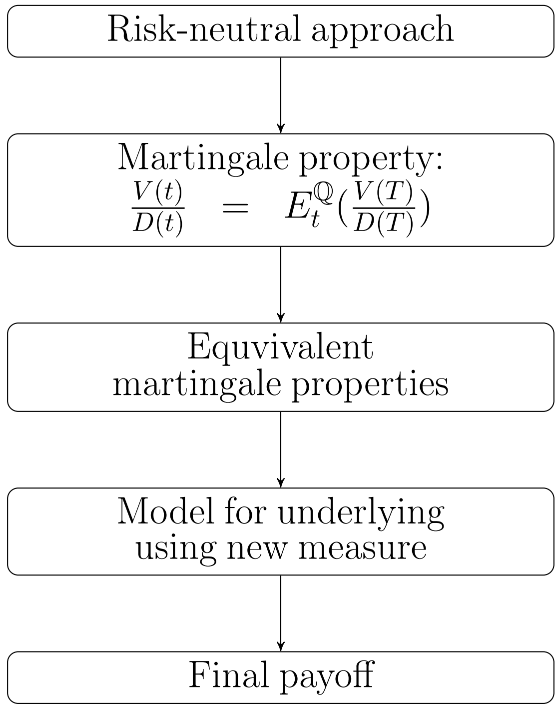
Contents

1 Closed-form solution for BS vanilla call/put price derivatives. Vanilla option price.



- In the basic BSM economy, two assets are traded: a money market account β and a stock S - X(t).
- The dynamics for β:

$$\frac{d\beta(t)}{\beta(t)} = rdt, \beta(0) = 1$$

- The stock dynamics are assumed to satisfy GBMD:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dw(t)$$

- Deflated stock price:

$$S^{\beta}(t) = \frac{S(t)}{\beta(t)}$$

- By Ito's lemma:

$$\frac{dS^{\beta}(t)}{S^{\beta}(t)} = (\mu - r)dt + \sigma dW(t)$$

- Applying Girsanov theorem:

$$\frac{d\xi(t)}{\xi((t)} = -\theta dW(t), \theta = \frac{\mu - r}{\sigma}$$

- Under new measure $\mathbb{Q}$, $W^{\beta}(t) = W(t) + \theta t$ is a Brownian motion:

$$\frac{dS^{\beta}(t)}{S^{\beta}(t)} = \sigma W^{\beta}(t)$$

$$\frac{dS(t)}{S(t)} = rdt + \sigma W^{\beta}(t)$$

- Hence stock dynamics:

$$S(T) = S(t)e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\left(W^\beta(T)-W^\beta(t)\right)} \ , \, t \in [0, T]$$

- Our final payoff depends on the final value of the underlying.
- *Discount bond* - paying at time T 1$ for certain.
  Application of basic derivative pricing equation immediately gives:

$$P(t,T) = \beta(t)E_t^Q\left(\frac{1}{\beta(T)}\right) = E_t^Q\left(e^{-r(T-t)}\right) = e^{-r(Tt)}$$

- *European call option* - paying $c(T) = (S(T) - K)^+$

$$c(T) = e^{-r(T-t)}E_t^Q\left((S(T) - K)^+\right)$$

$$c(t) = P(t,T)\int_{-\infty}^{+\infty}\left(S(t)e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+z\sigma\sqrt{T-t}} - K\right)^+ \varphi(z).$$

- <u>Theorem 2.1</u>: In the BS economy, the arbitrage-free time 1 price of the K-strike call-option maturing at time T is:

$$c(T) = S(T)N(d_1) - KP(t,T)N(d_2)$$

$$d_{1,2} = \frac{ln(S(t)/K)+(r\pm\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

where N(.) is a Gaussian cumulative distribution function: $N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{x^2}{2}}dx$

- <u>Lemma 2.2</u>: In BS notation the following results holds:

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

Proof: recall that $d_2 = d_1 - \sigma\sqrt{T-t}$ and open brackets in the exponent.

2 Greeks

Greeks to derive:

- Delta (Δ) - sensitivity of option price to underlying price.

- Gamma (Γ) - sensitivity to option delta to underlying price.

- Vega (ϑ) - sensitivity of option price to volatility.

$$\Delta = \frac{\partial C}{\partial S} = N\left(d_1\right) + S\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S} - KP\left(t_1 T\right)\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial S} =$$

$$= N\left(d_1\right) + SN'\left(d_1\right)\left[\frac{\partial d_1}{\partial S} - \frac{\partial d_2}{\partial S}\right] = N\left(d_1\right)$$

Note that: $N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{x^2}{2}}dx$, $N'(d) = \frac{1}{\sqrt{2\pi}}e^{-\frac{d^2}{2}}$, $\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$, $\quad S\frac{\partial N(d_1)}{\partial d_1} = KP(t,T)\frac{\partial N(d_2)}{\partial d_2}$

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S}\frac{\partial C}{\partial S} = \frac{\partial}{\partial S}N\left(d_1\right) = \frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S} = N'\left(d_1\right)\cdot\frac{1}{S\sigma\sqrt{T-t}}$$

Note that: $\frac{\partial d_1}{\partial S} = \frac{1}{S\partial\sqrt{T-t}}$

$$\vartheta = \frac{\partial C}{\partial \sigma} = S\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial \sigma} - KP(t,T)\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial \sigma} = S\frac{\partial N(d_1)}{\partial d_1}\left[\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_1}{\partial \sigma}\right] = S\frac{\partial N(d_1)}{\partial d_1}\sqrt{T-t}$$

Note that: $\quad S\frac{\partial N(d_1)}{\partial d_1} = KP(t,T)\frac{\partial N(d_2)}{\partial d_2}$ - lemma 2.2 again and $\quad d_2 = d_1 - \sigma\sqrt{T-t}$.