

Homework 1

Lecturer: Eric Xing

Name: Yuan Liu(yuanl4)

1 Bayesian Network Solution

1.1 I-map

Solution: (Koller and Friedman Textbook Page 62)

Let X_1, \dots, X_n is a topological ordering of the variables in \mathbb{X} relative to \mathbb{G} . We can first use the chain rule to represent \mathbb{P} :

$$P(X_1, \dots, X_n) = \prod_{i=1}^n P(X_i | X_1, \dots, X_{i-1})$$

Now, consider one of the factors $P(X_i | X_1, \dots, X_{i-1})$. Because \mathbb{G} is a I-map for \mathbb{P} , we have

$$(X_i \perp \text{NonDescendants}_{x_i} | Pa_{x_i}^{\mathbb{G}}) \in I(\mathbb{P})$$

By the topological ordering assumption, all of X_i 's parents are in the set X_1, \dots, X_{i-1} . Furthermore, none of X_i 's descendants can possibly be in the set. Hence,

$$\{X_1, \dots, X_{i-1}\} = Pa_{X_i} \cup Z$$

where $Z \subset \text{NonDescendants}_{X_i}$. In addition to it, we already know $(X_i \perp \text{NonDescendants}_{x_i} | Pa_{x_i}^{\mathbb{G}}) \in I(\mathbb{P})$, so it follows that $(X_i \perp Z | Pa_{x_i})$. Hence we have that

$$P(X_i | X_1, \dots, X_{i-1}) = P(X_i | Pa_{X_i})$$

Applying this transformation to all of the factors in the chain rule decomposition, the result follows.

1.2 D-separation

- $B \perp G | A$: True. Since H is unobserved, and $G \rightarrow H \leftarrow F$ is a v-structure, influence cannot flow through H.
- $C \perp D | F$: False. Influence can flow along the path $C \leftarrow B \rightarrow D$.
- $C \perp D | A$: False. Influence can flow along the path $C \leftarrow B \rightarrow D$.
- $H \perp B | C, F$: True. Since F is observed and $H \leftarrow F \leftarrow C$ or $H \leftarrow F \leftarrow E$ are neither v-structure, then H can not flow through F.

1.3 I-Equivalence

- G_1 and G_2 has the same skeleton is a necessary, but not sufficient condition for G_1 and G_2 to be I-equivalent.

Not sufficient part proof: The v-structure and common parent structure have the same skeleton, but they are not I-equivalent.

Necessary part proof: Proof by contradiction. Assume G_1 and G_2 are I-equivalent, but they have different skeleton. In this situation, we can find a trail in one network that does not exist in the other. Assume this trail is $X_1 \rightleftharpoons X_2 \rightleftharpoons \dots \rightleftharpoons X_i$ in G_1 . Given all the v-structures in this trail, like $A_k \rightarrow B_k \leftarrow C_k$, let all the $\{B_k\}_k$ observed, and left others unobserved. Then $X_1 \perp X_2 | \{B_k\}_k$ in G_2 , but $X_1 \not\perp X_2 | \{B_k\}_k$ in G_1

- G_1 and G_2 has the same skeleton and same v-structures, is a sufficient, but not necessary condition for G_1 and G_2 to be I-equivalent.

Sufficient part proof: (http://www.wisdom.weizmann.ac.il/pgm/Ex/ps3_sol.pdf) First we assume that $(\mathbb{X} \perp \mathbb{Y} | \mathbb{Z}) \in I(G_1)$ and we show that $(\mathbb{X} \perp \mathbb{Y} | \mathbb{Z}) \in I(G_2)$. If two graphs have the same skeleton, then they have the same trails. Lets look on the trails that between $X \in \mathbb{X}$ and $Y \in \mathbb{Y}$ that given \mathbb{Z} is inactive in G_1 , and then show this trail in G_2 is in active too. Consider two cases:

- The trail in G_1 is inactive because some of the nodes on the trail that are not in v-structure are observed in \mathbb{Z} . Then clearly these nodes also blocks the trail in G_2
- Otherwise, all nodes on the trail that are not in a v-structure are not observed (not in \mathbb{Z}), but then for some v-structure V_{i-1}, V_i, V_{i+1} on the trail, none of the of V_i are observed. That is for every node V such that there is a directed path in G_1 from V_i to V , all the nodes on the path are not observed. Consider such a directed path from such a V_i to some V in G_1 , then in G_2 this trail must also be directed the same, from V_i to V , because other wise it introduces a v-structure that is not in G_1 (either on this path itself, or with respect to $V_{i-1} \rightarrow V_i$), and clearly all it's nodes are not observed too. There fore this path also inactivates the trail between \mathbb{X} and \mathbb{Y} (given \mathbb{Z}) in G_2

Thus, $(\mathbb{X} \perp \mathbb{Y} | \mathbb{Z}) \in I(G_1)$ implies that every trail between any pair $X \in \mathbb{X}$ and $Y \in \mathbb{Y}$ that given \mathbb{Z} is inactive in G_1 , and showed for each such a trail that it is inactive in G_2 . Since the two graphs have the same trails, then $(\mathbb{X} \perp \mathbb{Y} | \mathbb{Z}) \in I(G_2)$. Hence $I(G_1) \subset I(G_2)$, and by symmetry we also get that $I(G_2) \subset I(G_1)$, therefore $I(G_1) = I(G_2)$

Not necessary part proof: Consider complete graphs over a set of variables. Recall that a complete graph is one to which we cannot add additional arcs without causing cycles. Such graphs encode the empty set of conditional independence assertions. Thus, any two complete graphs are I-equivalent. Although they have the same skeleton, they invariably have different v-structures.

2 Independence & Equivalence Testing Solution

2.1 D-separation and Independence

- (1) First, prove that D-separation implies conditional independence, or mathematically:

$$desp(X, Y|Z) \Rightarrow X \perp Y|Z$$

Solution (Koller and Friedman Textbook Page 137) Let $U = X \cup Y \cup Z$. Let $U^* = U \cup \text{Ancestors}_U$. Let $G_{U^*} = G^+[U]$ be the induced graph over U^* , and let H be the moralized graph $M[G_{U^*}]$. Let P_{U^*} be the Bayesian network distribution defined over G_{U^*} in the obvious way: the CPD for any variable in U^* is the same as in original distribution P_B . Because U^* is upwardly closed, all variables used in these CPDs are in U^* .

Now, consider an independence assertion $(X \perp Y|Z) \in I(G)$; we want to prove that $P \models (X \perp Y|Z)$. By definition, if $(X \perp Y|Z) \in I(G)$, we have that $desp(X, Y|Z)$. It follows that $sep_H(X, Y|Z)$, and hence that $(X \perp Y|Z) \in I(H)$. P_{U^*} is a Gibbs distribution over H , and hence, $P_{U^*} \models (X \perp Y|Z)$. We know $P_{U^*}(U^*)$ is the same as $P_B(U^*)$. Hence, it follows that $P_B \models (X \perp Y|Z)$.

- (2) Prove the completeness of the D-separation property.
 (3) Provide a pseudo-code and describe in detail an algorithm for testing D-separation.

Solution (Koller and Friedman Textbook Page 75) We use the following definition of active trail to find all the nodes that can be reached by X .

- Whenever we have a v-structure $X_{i-1} \rightarrow X_i \leftarrow X_{i+1}$, then X_i or one of its descendants are in Z .
- no other node along the trail is in Z .

So we first need to find all the ancestors of Z , and put them into A . (Using broad first search)
 Then we use broad-first search to find all the linked node.

2.2 I equivalence

- (1) Proof: The two BNs, G and G' , are I-equivalent if both graphs have the same set of trails and a trail is active in G if and only if it is active in G' .

Solution: First we assume that $(\mathbb{X} \perp \mathbb{Y}|\mathbb{Z}) \in I(G_1)$ and we show that $(\mathbb{X} \perp \mathbb{Y}|\mathbb{Z}) \in I(G_2)$.

Lets look on the trail that between $X \in \mathbb{X}$ and $Y \in \mathbb{Y}$ that given Z is inactive in G_1 , and because of the assumption this trail in G_2 is inactive too.

Thus, $(\mathbb{X} \perp \mathbb{Y}|\mathbb{Z}) \in I(G_1)$ implies that every trail between any pair $X \in \mathbb{X}$ and $Y \in \mathbb{Y}$ that given Z is inactive in G_1 , and showed for each such a trail that it is inactive in G_2 . Since the two graphs have the same trails, then $(\mathbb{X} \perp \mathbb{Y}|\mathbb{Z}) \in I(G_2)$. Hence $I(G_1) \subset I(G_2)$, and by symmetry we also get that $I(G_2) \subset I(G_1)$, therefore $I(G_1) = I(G_2)$.

- (2) Prove that a minimal active trail may only contain a triangle of the following form:

Algorithm 1 D-separation

Input: Bayesian network graph: G ; Source variables: X ; Evidence variables: Z ;**Output:** Reachable Set R

```

1: //Step one: find all the ancestors of Z
2:  $L \leftarrow Z$  //Node to be visited
3:  $A \leftarrow \emptyset$  //Ancestors of Z
4: //Broad first search
5: while  $L \neq \emptyset$  do
6:   Select some  $Y$  from  $L$ 
7:    $L \leftarrow L - \{Y\}$ 
8:   if  $Y \notin A$  then
9:      $L \leftarrow L \cup Pa_Y$ 
10:   end if
11:    $A \leftarrow A \cup \{Y\}$ 
12: end while
   //Step two: traverse active trails starting from X
13:  $L \leftarrow \{(X, \uparrow)\}$  //(Node,direction) to be visited
14:  $V \leftarrow \emptyset$  //(Node, direction) marked as visited
15:  $R \leftarrow \emptyset$  //Nodes reachable via active trail
16: while / do/Select some  $(Y,d)$  from  $L$ 
17:    $L \leftarrow L - (Y,d)$ 
18:   if  $(Y,d) \notin V$  then
19:     if  $Y \notin Z$  then
20:        $R \leftarrow R \cup \{Y\}$  // Y is reachable
21:     end if
22:      $V \leftarrow V \cup (Y,d)$  // Mark(Y,d) as visited
23:     if  $d = \uparrow$  and  $Y \notin Z$  then //Trail up through Y active if Y not in Z
24:       for each  $Z \in Pa_Y$  do //Y's parents to be visited from bottom
25:          $L \leftarrow L \cup \{(Z, \uparrow)\}$ 
26:       end for
27:       for each  $Z \in Ch_Y$  do //Y's children to be visited from top
28:          $L \leftarrow L \cup \{(Z, \downarrow)\}$ 
29:       end for
30:     else if  $d = \downarrow$  then
31:       if  $Y \notin Z$  then //Downward trails to Y's children are active
32:         for each  $Z \in Ch_Y$  do
33:            $L \leftarrow L \cup \{(Z, \downarrow)\}$  //Y's children to be visited from top
34:         end for
35:       end if
36:       if  $Y \in A$  then //v-structure trails are active
37:         for each  $Z \in Pa_Y$  do
38:            $L \leftarrow L \cup \{(Z, \uparrow)\}$  // Y's parents to be visited from bottom
39:         end for
40:       end if
41:     end if
42:   end if
43: end while
44: Return  $R$ 

```

Solution Let first consider the trail: $X_{i-1} \Rightarrow X_i \Rightarrow \dots \Rightarrow X_{i+1}$. There are four following situations:

- $X_{i-1} \rightarrow X_i \rightarrow X_{i+1}$. Because these three variables form a triangle. There are two sub-cases in this situation:
 - * $X_{i-1} \rightarrow X_{i+1}$ In this situation, $X_{i-1} \rightarrow X_{i+1}$ will be an active trail, which breaks the minimal assumption.
 - * $X_{i-1} \leftarrow X_{i+1}$ In this situation, there will be a cycle among these three variables.
 - $X_{i-1} \leftarrow X_i \leftarrow X_{i+1}$. Because these three variables form a triangle. There are two sub-cases in this situation:
 - * $X_{i-1} \rightarrow X_{i+1}$ In this situation, there will be a cycle among these three variables.
 - * $X_{i-1} \leftarrow X_{i+1}$ In this situation, $X_i \rightarrow X_{i-1} \leftarrow X_{i+1}$ is a v-structure. So there must be $X_{i-2} \leftarrow X_{i-1}$ to make the whole trail active. It follows $X_{i-2} \leftarrow X_{i-1} \leftarrow X_{i+1}$ is active, which breaks the minimal assumption.
 - $X_{i-1} \rightarrow X_i \leftarrow X_{i+1}$. Because these three variables form a triangle. There are two sub-cases in this situation:
 - * $X_{i-1} \rightarrow X_{i+1}$ In this situation, $X_{i-2} \Rightarrow X_{i-1} \rightarrow X_{i+1}$ will be an active trail, which breaks the minimal assumption.
 - * $X_{i-1} \leftarrow X_{i+1}$ In this situation, $X_{i-1} \rightarrow X_i \leftarrow X_{i+1}$ is a v-structure, so there must be $X_{i-2} \leftarrow X_{i-1}$. In this situation $X_{i-2} \leftarrow X_{i-1} \leftarrow X_{i+1}$ breaks the minimal assumption.
 - $X_{i-1} \leftarrow X_i \rightarrow X_{i+1}$ This is the situation we want to proof.
- (3) Two Bayes networks, G and G' , are I-equivalent if and only if G and G' have the same skeletons and the same set of immoralities.

Solution:

Part one: Prove that if G and G' have the same skeletons and the same set of immoralities then they are I-equivalent. So we need to prove if $(X \perp Y|Z) \in I(G)$, then $(X \perp Y|Z) \in I(G')$.

Proof by contradiction.

Assume $(X \perp Y|Z) \in I(G)$ and $(X \perp Y|Z) \notin I(G')$. So there is a minimal active trail between X and Y given Z in G' , but blocked in G . For example $X_{i-1} \Rightarrow X_i \Rightarrow X_{i+1}$ is blocked in G . There are two situations it is blocks:

Situation one: In X_{i-1}, X_i, X_{i+1} , X_i is observed and they form cascade or common parent structure in G , while they form v-structure in G' . Because they forms v-structure in G' and the trail is minimal, there is no link between X_{i-1} and X_{i+1} (Problem 2's result). Thus this v-structure is a immorality in G' . So it should also be a immorality in G . This is a contradiction.

Situation two: X_{i-1}, X_i, X_{i+1} are all unobserved and they form a v-structure in G , while in G' they don't form a v-structure. Thus X_{i-1}, X_i, X_{i+1} can not form a immorality, and there will be a edge between X_{i-1} and X_{i+1} . It violates the assumption that X_{i-1} and X_{i+1} is blocked in G .

Part two: Prove that two networks G and G' that induce the same conditional independence assumptions must have the same skeleton and the same immoralities. By Problem 1.3 we already proved that G and G' have the same skeleton, thus we only need to prove they have the same immoralities.

Proof by contradiction:

Two networks G and G' are I-equivalent, have the same skeleton but different immoralities. For example, X_{i-1}, X_i, X_{i+1} forms a immorality in G , but they form a cascade or common parent structure in G' . Given X_i observed so the trail $X_{i-1} \rightarrow X_i \leftarrow X_{i+1}$ is active in G , where in G' this trail is inactive. This situation violates the Proposition 1, we proved.

Algorithm 2 I-equivalent

Input: Bayesian network graph's direct edge set (X,Y): G1, G2;**Output:** Whether they are I-equivalent: Flag

```

1:
2: function ISIEQUIVALENT(G1,G2)
3: //Whether two graphs have the same skeleton
4:   if isSameSkeleton(G1,G2)=False then
5:     return False
6:   end if
7: //Get immoralities of each graph
8:    $M1 \leftarrow \text{getImmoralities}(G1)$ 
9:    $M2 \leftarrow \text{getImmoralities}(G2)$ 
10:  if  $M1 \neq M2$  then
11:    return False
12:  end if
13:  return True
14: end function
15:
16: // Whether two graphs have the same skeleton
17: function ISSAMESKELETON(G1,G2)
18:   if  $G1.size() \neq G2.size()$  then
19:     return False
20:   end if
21:   for  $(X, Y) \in G1$  do
22:     if  $(X, Y) \notin G2$  and  $(Y, X) \notin G2$  then
23:       return False
24:     end if
25:   end for
26:   return True
27: end function
28:
29: //Output the immoralities in the graph
30: function GETIMMORALITIES(G)
31: //Get the node Set V
32:    $V \leftarrow \emptyset$ 
33:   for  $(X, Y) \in G$  do
34:      $V \leftarrow V \cup \{X, Y\}$ 
35:   end for //Iteration all there node triple to find the immoralities
36:    $M \leftarrow \emptyset$  //The set of all the immoralities
37:   for  $X \in V$  do
38:     for  $Y \in V$  do
39:       for  $Z \in V$  do
40:         if X,Y,Z are distinct and  $(X, Y) \in G$  and  $(Z, Y) \in G$  then
41:           if  $(X, Z) \notin G$  and  $(Z, X) \notin G$  then
42:              $M = M \cup (X, Y, Z)$ 
43:           end if
44:         end if
45:       end for
46:     end for
47:   end for
48:   return M
49: end function

```

3 Exact Inference

3.1 Variable Elimination

$$\begin{aligned}\alpha_{t+1}(j) &= p(z_{t+1} = j | x_{1,\dots,t+1}) \propto \sum_i p(z_{t+1} = j | z_t = i) p(x_{t+1} | z_{t+1} = j) p(z_t = i | x_{1,\dots,t}) \\ &\propto p(x_{t+1} | z_{t+1} = j) \sum_i p(z_{t+1} = j | z_t = i) \alpha_t(i)\end{aligned}$$

$$\begin{aligned}\beta_{t-1}(j) &= P(x_{t,\dots,T} | z_{t-1} = j) \propto \sum_i P(x_t | z_t = i) P(z_t = i | z_{t-1} = j) P(x_{t+1,\dots,T} | z_t = i) \\ &\propto \sum_i p(x_t | z_t = i) p(z_t = i | z_{t-1} = j) \beta_t(i)\end{aligned}$$

Then because

$$p(z_t = j | x_1, \dots, T) \propto \alpha_t(j) \beta_t(j)$$

we can get

$$p(z_t = j | x_1, \dots, T) = \frac{\alpha_t(j) \beta_t(j)}{\sum_i \alpha_t(i) \beta_t(i)}$$

3.2 Gaussian Belief Propagation

- (1) Proof: $N(x | \mu_1, \lambda_1^{-1}) \times N(x | \mu_2, \lambda_2^{-1}) = CN(x | \mu, \lambda^{-1})$

Solution

$$\begin{aligned}N(x | \mu_1, \lambda_1^{-1}) \times N(x | \mu_2, \lambda_2^{-1}) &= \exp\left(-\frac{\lambda_1(x - \mu_1)^2}{2}\right) \times \exp\left(-\frac{\lambda_2(x - \mu_2)^2}{2}\right) \\ &= \exp\left(-\frac{1}{2}(\lambda_1(x - \mu_1)^2 + \lambda_2(x - \mu_2)^2)\right)\end{aligned}$$

$$\begin{aligned}\lambda_1(x - \mu_1)^2 + \lambda_2(x - \mu_2)^2 &= (\lambda_1 + \lambda_2)x^2 - 2(\lambda_1\mu_1 + \lambda_2\mu_2)x + \lambda_1\mu_1^2 + \lambda_2\mu_2^2 \\ &= (\lambda_1 + \lambda_2)\left[x^2 - \frac{2(\lambda_1\mu_1 + \lambda_2\mu_2)}{\lambda_1 + \lambda_2}x + \left(\frac{\lambda_1\mu_1 + \lambda_2\mu_2}{\lambda_1 + \lambda_2}\right)^2\right] - \frac{(\lambda_1\mu_1 + \lambda_2\mu_2)^2}{\lambda_1 + \lambda_2} \\ &\quad + \lambda_1\mu_1^2 + \lambda_2\mu_2^2 \\ &= (\lambda_1 + \lambda_2)\left[x - \frac{\lambda_1\mu_1 + \lambda_2\mu_2}{\lambda_1 + \lambda_2}\right]^2 + \frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2}(\mu_1 - \mu_2)^2\end{aligned}$$

So we can get

$$\begin{aligned}\lambda &= \lambda_1 + \lambda_2 \\ \mu &= \frac{\lambda_1\mu_1 + \lambda_2\mu_2}{\lambda_1 + \lambda_2} \\ C &= \sqrt{\frac{\lambda_1\lambda_2}{2\pi(\lambda_1 + \lambda_2)}} \exp\left(-\frac{1}{2} \frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2} (\mu_1 - \mu_2)^2\right)\end{aligned}$$

- (2) Solve for $m(x_t)$. Hint First solve for the messages passed into Node t's neighbor nodes from their neighbors excluding t, then use these messages to represent $m(x_t)$

Solution: (Gaussian Belief Propagation: Theory and Application <https://arxiv.org/pdf/0811.2518.pdf>)

$$N(\mu_{s/t}, \lambda_{s/t}) \propto \phi_s(x_s) \prod_{k \in N(s)/t} m_{ks}(x_s)$$

$$\lambda_{s/t} = \lambda_{ss} + \sum_{k \in N(s)/t} \lambda_{ks}$$

$$\mu_{s/t} = \lambda_{s/t}^{-1} (\lambda_{ss} \mu_{ss} + \sum_{k \in N(s)/t} \lambda_{ks} \mu_{ks})$$

Where λ_{ks} and μ_{ks} are the precision and mean of the message $m_{ks}(x_s)$, and $\lambda_{ss} \triangleq A_{ss}$, $\mu_{ss} \triangleq \frac{b_s}{A_{ss}}$.
Next we calculate $m_{st}(x_t)$

$$\begin{aligned} m_{st}(x_t) &= \int_{x_s} \phi_{s,t}(x_s, x_t) \phi_s(x_s) \prod_{k \in N(s)/t} m_{ks}(x_s) \\ &\propto \int_{x_s} \exp(-x_s A_{st} x_t) \exp(-\lambda_{s/t} (x_s^2/2 - \mu_{s/t} x_s)) \\ &= \int_{x_s} \exp(-\lambda_{s/t} x_s^2/2 + (\lambda_{s/t} \mu_{s/t} - A_{st} x_t) x_s) \\ &\propto \exp((\lambda_{s/t} \mu_{s/t} - A_{st} x_t)^2 / (2\lambda_{s/t})) \\ &\propto N(\mu_{st}, \lambda_{st}) \end{aligned}$$

Where μ_{st} and λ_{st} are the precision and mean of the message $m_{st}(x_t)$.

$$\mu_{st} = \frac{\lambda_{s/t} \mu_{s/t}}{A_{st}}$$

$$\lambda_{st} = -\frac{A_{st}^2}{\lambda_{s/t}}$$

Then we can calculate $m(x_t)$

$$m(x_t) = \phi_t(x_t) \prod_{s \in N(t)} m_{st}(x_t) \propto N(\mu_t, \lambda_t^{-1})$$

$$\lambda_t = \lambda_{tt} + \sum_{s \in N(t)} \lambda_{st}$$

$$\mu_t = \lambda_t^{-1} (\lambda_{tt} \mu_{tt} + \sum_{s \in N(t)} \lambda_{st} \mu_{st})$$

Where $\lambda_{tt} = A_{tt}$, $\mu_{tt} = \frac{b_t}{A_{tt}}$

4 Undirected Graphical Models

4.1 Ising Model & Boltzmann Machine

Solution:

$$\begin{aligned}
 \exp\left(\sum_{(i,j) \in E} W_{ij}x_i x_j - \sum_{i \in V} u_i x_i\right) &\propto \exp\left(\sum_{(i,j) \in E} 4W_{ij}\left(\frac{x_i}{2} + \frac{1}{2}\right)\left(\frac{x_j}{2} + \frac{1}{2}\right) - \sum_{(i,j) \in E} W_{ij}x_i - \sum_{(i,j) \in E} W_{ij}x_j - \sum_{i \in V} u_i x_i\right) \\
 &\propto \exp\left[\sum_{(i,j) \in E} 4W_{ij}\left(\frac{x_i}{2} + \frac{1}{2}\right)\left(\frac{x_j}{2} + \frac{1}{2}\right) - \sum_{i \in V} (u_i + 2 \sum_{j \in N(i)} W_{ij})x_i\right] \\
 &\propto \exp\left[\sum_{(i,j) \in E} 4W_{ij}\left(\frac{x_i}{2} + \frac{1}{2}\right)\left(\frac{x_j}{2} + \frac{1}{2}\right) - \sum_{i \in V} 2(u_i + 2 \sum_{j \in N(i)} W_{ij})\left(\frac{x_i}{2} + \frac{1}{2}\right)\right]
 \end{aligned}$$

Thus we can get:

$$\begin{aligned}
 x'_i &= \frac{x_i}{2} + \frac{1}{2} \in \{0, 1\} \\
 W'_{ij} &= 4W_{ij} \\
 u'_i &= 2(u_i + 2 \sum_{j \in N(i)} W_{ij})
 \end{aligned}$$

4.2 Determinantal Point Process