Homework 5; Tuesday 10 April
To be handed in no later than 12.30 on Tuesday 17 April

Take $n = 100$ independent samples from the density

$$g_0(x) = 3\,e^{-3x}, \quad x > 0.$$

To estimate this density, use the model

$$g(x|w_M, M) = \sum_{j=1}^{M} w_{j,M}\, j\, e^{-jx},$$

which is a mixture of exponential densities. The unknowns are M and the weights (w_M). The prior for w_M given M is Dirichlet with all parameters set to 1; i.e.

All the alphas set to 1 in the Dirichlet prior

$$f(w_{1M}, \ldots, w_{MM}|M) \propto 1.$$

The prior for M is

$$f(M) \propto 1/(M-1)!\,, \quad \text{for} \quad M = 1, 2, \ldots\,.$$

Note that $0! = 1$.

(i) Write down the conditional $f(w_M|x_1, \ldots, x_n, M)$. Note that you would need to introduce the (d_i) variables to help with this.

(ii) Also then write down $P(d_i = j|M, x_1, \ldots, x_n, w_M)$.

(iii) Explain how to sample M using a Metropolis step.

(iv) Implement the Markov chain with the algorithm/sampling you wrote down in steps (i) to (iii) and use it to provide a density estimate of $g_0(x)$.

(v) Discuss your findings.