Mid-term exam: Thursday 8 March 2017

1. Suppose it is required to sample from the density $\pi(x)$. Also suppose $\pi(x)$ is invertible, in the sense that if $\pi(x) > u$ then the set of $x$ for which this holds is known, and call it $A_u = \{x : \pi(x) > u\}$. Consider the joint density

$$f(x, u) = \mathbf{1}(0 < u < \pi(x)).$$

Hence, describe a Gibbs sampler for sampling from $\pi(x)$. Provide all the necessary details.

Run such a Gibbs sampler for $\pi(x) \propto 1/(1 + x^4)$ with $-\infty < x < \infty$.

Use this Gibbs sampler to estimate the variance from $\pi(x)$; i.e. estimate

$$I = \int x^2 \, \pi(x) \, dx.$$

2. Consider the data model

$$g(x|\theta) = \text{Normal}(x|\mu, \sigma^2)$$

where $\theta = (\mu, \lambda)$ and $\lambda = 1/\sigma^2$. Priors are assigned, specifically

$$f(\mu) = \text{Normal}(\mu|0, \tau^2) \quad \text{and} \quad f(\lambda) = \text{Gamma}(\lambda|a, b).$$

Find the conditional densities for $\mu$ and $\lambda$ given a data sample

$$(x_1, \ldots, x_n)$$

and then describe a Gibbs sampler for sampling from $f(\mu, \lambda|\text{data})$.

If $n = 51$ and

$$\sum_{i=1}^{n} x_i^2 = 39.6 \quad \text{and} \quad \sum_{i=1}^{n} x_i = 10.2,$$

implement the Gibbs sampler, check it has worked, and check your estimates of $\mu$ and $\lambda$ agree with the sample mean and sample variance.

For this question you will need to set your own values for the $(\tau, a, b)$.

3. Suppose it is required to sample from the power law distribution; i.e. to sample $X$ where

$$P(X=k) \propto k^{-\alpha}, \quad k \in \{1,2,3,\ldots\}$$

for $\alpha > 1$. It is decided to do this using a Metropolis–Hastings algorithm with proposal

$$q(x'|x) = \begin{cases} x' = x+1 & \text{probability} \quad \frac{1}{2} \\ \\ x' = x-1 & \text{probability} \quad \frac{1}{2} \end{cases}$$

for $x > 1$ and $q(2|1) = 1$.

Write down the details of implementing the Metropolis–Hastings algorithm and then run the algorithm, with $\alpha = 3$, and check your samples match the true density function.

4. Consider the joint density $\pi(x, y)$ and a Gibbs sampling framework is used to sample a Markov chain with stationary density $\pi$. The conditionals are $\pi(x|y)$ and $\pi(y|x)$ and while it is easy to sample $\pi(x|y)$ it is not possible to sample $\pi(y|x)$ directly. Describe a way how to proceed using a Metropolis-Hastings step and describe all the details clearly and also why it works. Describe and run such a Markov chain when

$$\pi(x,y) \propto e^{-4x}\, y^3\, e^{-3\,y\,e^x}, \quad x, y > 0,$$

where the aim is to estimate the integral

$$I = \int\int x\, y\, \pi(x,y)\, dx\, dy.$$