Homework 2; Tuesday 13 February
To be handed in no later than 12.30 on Tuesday 20 February

1. A chain with transition matrix P and stationary density π is reversible if

$$\pi_i \, p_{ij} = \pi_j \, p_{ji}$$

for all i, j.

Show that this condition is satisfied for the P with elements

$$p_{ij} = \alpha_{ij} \, q_{ij} + (1 - r_i) \, \mathbf{1}(i = j)$$

where $\mathbf{1}(i = j) = 1$ if $i = j$ and is 0 otherwise, and

$$\alpha_{ij} = \min \left\{ 1, \frac{\pi_j \, q_{ji}}{\pi_i \, q_{ij}} \right\}$$

and

$$r_i = \sum_{j=1}^{k} \alpha_{ij} \, q_{ij}.$$

Here $(q_{ij})_{j=1}^{k}$ are a set of weights for each i.

2. Suppose that the joint density $f(x, q)$ is given by

$$f(x, q) = q^x (1 - q)^{1-x}, \quad x \in \{0, 1\} \quad \text{and} \quad 0 < q < 1.$$

Consider the transition density, with $X_n, X_{n+1} \in \{0, 1\}$,

$$p(X_{n+1} | X_n) = 2 \int_0^1 q^{X_n + X_{n+1}} (1 - q)^{2 - X_n - X_{n+1}} \, dq.$$

Find the stationary density for this transition density.

Generate the sequence (X_n), starting with $P(X_0 = 1) = 1/2$, and use this to confirm your finding on the stationary density.

3. Consider the integral

$$I = \int_{-\infty}^{+\infty} \frac{1}{1 + x^4} \, e^{-\frac{1}{2} x^2} \, dx.$$

Evaluate this integral using a Markov chain sample (X_n) given by $X_{n+1} \sim N(\rho X_n, 1 - \rho^2)$. What is the best ρ to use in this case and demonstrate via simulation.