

Notational Equivalence in Tonal Geometry

1 Introduction

This paper demonstrates that two models of tonal representation—those proposed by Yip (1989) and Bao (1990)—cannot be regarded as distinct. Previous work (Bao, 1990; Chen, 2000; Yip, 2002) has claimed that these proposals differ in their empirical coverage of assimilatory tone sandhi processes in Chinese dialects, and thus constitute distinct representational theories. Arguments in favor of this distinction are situated within a derivational perspective on tonal processes, and assume two basic mechanisms: spread and delink. Apart from being tied to a specific grammatical formalism, these mechanisms are insufficient in capturing the full range of assimilatory tone sandhi, crucially cases which purportedly distinguish the two proposals in question. An additional *copying* mechanism is necessary for full coverage of attested tone sandhi.

In this paper, we employ a *computational* framework to examine earlier claims distinguishing Yip and Bao tonal representations. The computational perspective advocated here focuses on the properties of the input-output *mappings* which describe assimilatory tone sandhi processes (rather than theory-specific mechanisms), thus providing a more direct approach to evaluating models’ empirical predictions. We show that both representations handle the relevant assimilatory tone-sandhi patterns equally well when described as input-output mappings. The computational analysis preserves the basic character of the original representational theories in the sense that it reproduces the same basic and *necessary* mechanisms as traditional accounts: spread, delink, and *copy*. Thus, we show Yip and Bao models do *not* differ in their empirical consequences—contra previous claims.

Additionally, we apply the same approach to *structural* comparisons of the representational theories through examination of the properties of mappings *between* representations. We capitalize on various structural similarities apparent in elements of both Yip and Bao proposals, showing that one can be freely translated into another and vice versa, and without any loss of contrast expressible by either theory. Given these two results, the main claim of the paper is that these representational proposals do *not* constitute distinct theories, but are instead notationally-equivalent.

The paper is organized as follows. §2 frames a broad definition of ‘notational equivalence’ between representational theories to be pursued in the paper, and highlights some meta-theoretic issues regarding the representation of assimilatory tone sandhi. §3 introduces the Yip and Bao proposals in detail and summarizes previous arguments to distinguish them on empirical grounds. In §4, we establish the computational framework in which we address the issue of notational equivalence. This section then presents case studies of two assimilatory patterns, and shows that both models capture these patterns as input-output mappings. §5 presents additional evidence for notational equivalence from a structural perspective. §6 discusses these results in broad terms and addresses meta-theoretic issues. §7 concludes.

2 A Notion of Notational Equivalence

There is a tacit assumption in linguistic theory that new theories improve on older ones by increasing both the expressivity and restrictiveness of their predecessors. Models of grammar seek to explain the widest possible scope of attested phenomena, at the same time limiting their predictive power such that the models do not predict unnatural/impossible/unattested patterns. It follows that new theories do not simply rehash older ones; that is, that the new contribution is distinct from earlier iterations. A reasonable expectation of such contributions is that they predict alternations attested in human language that earlier theories failed to predict. New contributions to linguistic theory should also mitigate problems of over-expressivity left by previous theories by reining in their predictive power. Ideally, they do both simultaneously.

However, if a proposed theory merely restates the generalizations of older theories, or if the former differs from the latter in superficial ways—such that no demonstrable improvement in expressivity/restrictiveness obtains—we may argue that the two are *not* distinct, but rather are notational equivalents. Chomsky (1972, p. 2) makes this point in the following way:

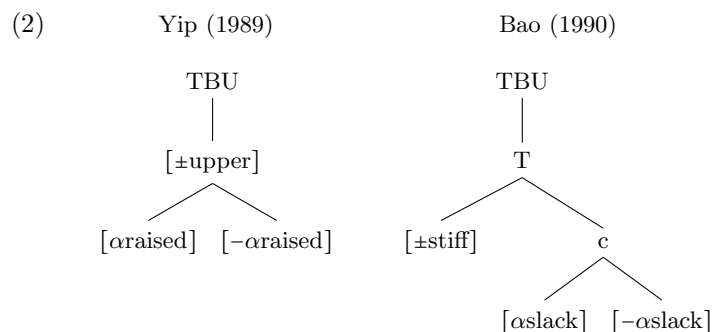
Given alternative formulations of a theory of grammar, one must first seek to determine how they differ in their empirical consequences, and then try to find ways to compare them in the area of difference. It is easy to be misled into assuming that differently formulated theories actually do differ in empirical consequences, when in fact they are intertranslatable - in a sense, mere notational variants.

A more recent definition (Fromkin, 2013) establishes two criteria by which alternative models may be considered notationally equivalent. Not only must the models share the *same* empirical coverage, but they must also represent the *same* set of basic, abstract properties, and differ only superficially in terms of that representation. Two models are thus notationally equivalent if they satisfy the conditions in (1):

(1) *Conditions for Notational Equivalence*

- a. Two models do not differ in their empirical predictions
- b. Two models represent the same set of abstract properties, differ only superficially

In this paper, we test the conditions in (1) against two competing models of tonal representation: those proposed by Yip (1989) and Bao (1990) and summarized in (2). Both theories model lexical tonal contrasts and a variety of tonal processes, specifically tone sandhi processes in Chinese dialects. At first glance, they represent a similar set of basic properties—in particular a root node which associates to a tone-bearing unit (TBU), a binary register feature which bisects the vocal range into upper and lower registers, and binary terminal tonal features (‘high’ and ‘low’ tones within a register). They also make the assumption that contour tones are sequences of level tones dominated by a *single* structural node, and therefore form a *constituent*. However, there are differences in how these basic properties relate to one another structurally. A key difference between the models is that Yip’s root node is specified for a register feature and dominates terminals directly, while Bao’s root is unspecified, and branches to separate register and contour nodes, the latter of which dominates tonal terminal nodes.



Previous work (Bao, 1990; Chen, 2000; Yip, 2002) has claimed that this difference distinguishes the models’ empirical predictions, specifically in capturing assimilatory tone sandhi patterns formalized as *spreading*. Structural separation of register and contour in Bao’s model allows the two to spread independently. In Yip’s model, since the register node directly dominates contour, there is no distinction between register or contour spread; spread of register entails spread of contour and *vice versa*. As such, assimilatory processes modeled as ‘register spread’ or ‘contour spread’ are not predicted in this representation.

However, this observation alone does not guarantee that the two models differ in their empirical predictions because the set of processes a given model is said to predict is inevitably tied to the full context of the formalism in which those processes are defined. If the process is couched in derivational terms (as in SPE),

it may be defined as a set of crucially-ordered rewrite rules. In an optimization-based framework like OT, specific constraints interact in an evaluation over some set of candidates to select the output form. Therefore, the empirical predictions of a model’s representation—its capacity to capture a certain process—must be considered within the full context of a particular framework.

Feature geometric models of tone formalize sandhi processes in an autosegmental phonological framework (Goldsmith, 1976; McCarthy, 1988). The basic mechanisms of this theory are spread (addition of a single association line between elements of structure) and delink (deletion of an association line). Simple spread and delink, however, are insufficient to model assimilatory tone sandhi processes over these representations. This is because spreading of contour requires the extra assumption of *tier conflation* (Younes, 1983; McCarthy, 1986; Yip, 1989), a process by which contour nodes and the terminal tonal nodes they dominate are *copied* to guarantee that separate contours are realized on each root (and not a single contour over multiple roots; see §3.2 and especially §6 for more discussion). This mechanism is shown in (3), where a dotted line indicates spread.



A feature geometric theory of assimilatory tone sandhi thus extends the traditional set of basic operations (spread and delink) to include a copying mechanism. This extension, however, permits *alternatives* to a traditional spreading analysis. For example, it is unclear why a spreading analysis with tier conflation is favorable to—or even differs from—one which *copies* pieces of structure first and reassociates them (by adding a single association line). This yields an identical structure using the *same* basic mechanisms of the theory, summarized in (4).



Relatedly, the necessity of tier conflation bears on the accuracy of spreadability—that is, spreading without copying—as a metric of empirical coverage. If spread and delink fail to capture the full range of attested tone sandhi, how reliable can such a test be in distinguishing models’ empirical predictions?

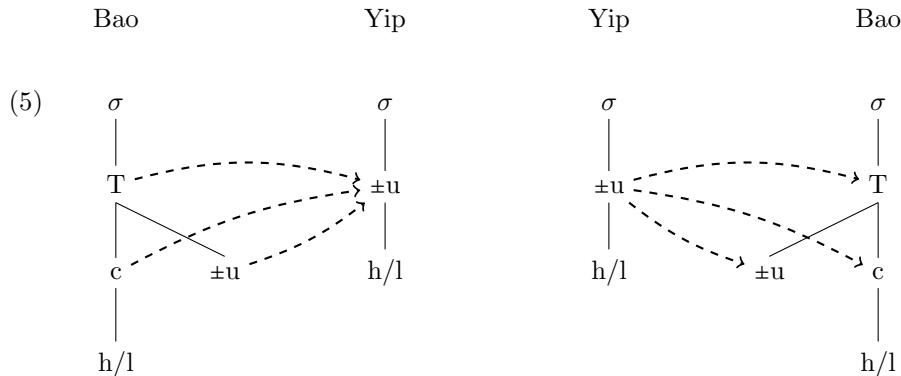
The difficulty in answering such questions is compounded by the fact that traditional analyses of tone sandhi over these representations are inherently derivational. Spreading and tier conflation are crucially ordered with respect to one another, as the application of the latter is dependent on the former. Additionally, the ‘spreading’ and ‘copying’ analyses above differ only in the *order* of application of basic mechanisms. In a non-serial formalism like parallel OT, for example, such an ordering would be irrelevant. There is no guarantee that the difference between these analyses is not merely a vestige of their formalization within a derivational paradigm. A more direct approach to evaluating models’ empirical predictions (1a) examines the properties of input-output mappings themselves, rather than theory-specific mechanisms. Ideally, such an approach can also be applied to *structural* comparisons of representational models (1b) through examination of mappings between representations.

This paper pursues a computational characterization of tonal representation to explore the question of notational equivalence and address the conditions in (1) as they apply to Yip and Bao tonal models. Within the empirical domain, we narrow our focus to the set of processes which earlier work claims distinguishes the models: register assimilation and contour assimilation. We abstract away from theory-specific considerations—and in particular the derivational nature of earlier spreading analyses—to focus instead on the nature of input-output *mappings* which describe assimilation (Chandlee and Heinz, 2018; Heinz, 2018). To do so, we employ a model-theoretic framework (Courcelle, 1994; Enderton, 2001; Libkin, 2013). Yip and Bao representations are given rigorous definitions as graph structures, and assimilatory tone mappings between these structures are defined using logical transduction. By fixing the complexity of the logical language necessary to define these mappings, we may compare the structures’ empirical predictions in a

principled way. It will be shown that the models do *not* differ in their empirical consequences (and thus satisfy condition (1a)) because the processes in question can be defined over both models. They do so using a restricted, quantifier-free (QF) logic.

The quantifier-free nature of this logic captures the important intuition that assimilation in tone sandhi is inherently *local*, an insight which earlier approaches overlook, but which is well-attested in computational characterizations of a wide range of phonological processes (Chandlee, 2014). Importantly, non-size-preserving QF logic is necessary to model assimilatory patterns over *both* models. Such transductions define mappings over a finite number of *copies* of output structure (see more discussion in §4). In other words, non-size-preserving QF logic describes mappings which allow a *copying* mechanism. While sentences in this logic are more powerful than size-preserving QF logical statements (that is, those which prohibit copying), the type of copying these transductions permit is restricted to local, bounded environments, and thus does not overextend the intention or *spirit* of the original theory (see §6.1). We describe the intuitions behind these mappings in terms of local, connected substructures, leaving full logical definitions to an appendix. Mappings of assimilatory processes are presented first to demonstrate that the models do not differ in their empirical consequences. In §6 we discuss how the ‘spreading’ and ‘copying’ analyses described above are, in fact, formally indistinguishable from the computational perspective because they represent a single QF-definable map. While the scope of this result is limited to two assimilatory tone sandhi processes, it provides a proof of concept that may be applied to spreading and copying more generally (see §6.2).

The computational approach affords the same formal rigor to exploring representational theories’ *structure*, as well. It allows us to reason over both aspects of notational equivalence in (1) using the same formal language. Therefore, using the same formalism of QF-definable transductions, we further show that the two models satisfy the second condition in (1)—that is, that any structural differences between the models are superficial—by demonstrating their *bi-interpretability* (Friedman and Visser, 2014).¹ Bi-interpretability provides a restrictive and provable formal notion of ‘superficial’ differences between representational theories, and is divided into two components. First, we define transductions to translate from *any* structure in Bao’s representation directly to an equivalent structure in Yip’s representation *and vice versa*. These translations capitalize on various structural similarities apparent in elements of both models (in particular the *constituency* of tonal contour under a single structural node), such that translating from Bao’s structure to Yip’s represents a *fusion* of three separate nodes into a single node, and translating from Yip’s structure to Bao’s represents an *expansion* of one node into three separate nodes. These intuitions are shown in (5).²



The second component of bi-interpretability is a guarantee that these translations are *contrast-preserving*; that is, no contrasts present in one model are lost in the process of translation into the other and vice versa.

¹We thank an anonymous reviewer for providing clarification on a mathematically-rigorous definition of this concept.

²For the remainder of the paper, we follow the conventions of Chen (2000) and Yip (2002), who collapse specific featural differences between these models for clarity. That is, binary register features—[±upper] in Yip’s model and [±stiff] in Bao’s model—are denoted ‘+u’ for upper register and ‘-u’ for lower register. Terminal tonal nodes—[±raised] in Bao’s model and [±slack] in Yip’s model—are denoted ‘h’ for high and ‘l’ for low. We also assume that the syllable (denoted σ) is the tone-bearing unit, though this is not consequential for the results.

We also show that this is the case for the models in question by demonstrating that the two translations we define are inverses of one another; in other words, applying *both* translations to one model (i.e. through composition, see §5 and Appendix B) is the same as mapping that model to *itself*.

In demonstrating bi-interpretability, we build on recent studies exploring notational equivalence in syllabic representation (Strother-Garcia and Heinz, 2017) and autosegmental representation (Danis and Jardine, 2019) from a formal language perspective. Importantly, we adopt a more restrictive definition of bi-interpretability than previous studies and thus put forth a stronger hypothesis about notational equivalence. As Bao and Yip models satisfy both conditions in (1), we conclude that they are notationally equivalent.

This paper does not claim that structural differences between feature-geometric configurations are by principle superficial, or that feature geometry is irrelevant. It does not assume equivalence between these models and other representational theories, for example those which do not assume constituency of contour tones (e.g. Duanmu, 1990, 1994, see §5.4 of the current paper). Rather, we advocate a rigorous formal analysis of claims that any two representations differ, and motivate analyses which are independent of the assumptions of a particular grammatical formalism. While the results of the current study are narrow in scope, they serve as a proof of concept for subsequent analyses of other representational models. The two representational theories examined in this study are introduced below.

3 Yip and Bao Models

This section offers a basic introduction to tonal models proposed by Yip (1989) and Bao (1990). Following Yip (2002)’s design criteria for featural systems of tone, the purpose of such representational theories is to 1) characterize attested lexical tonal contrasts (both level and contour) and 2) model common tonal processes.³ We present each in turn, with a focus on previous arguments in the literature used to distinguish these models in terms of the latter criterion—that is, claims that they differ in their empirical predictions.

3.1 Tonal Geometry in Yip (1989) and Bao (1990)

The table in (6) summarizes the representation of level and contour lexical tonal contrasts in each model, with a corresponding string representation. Exhausting the permutations of the binary register features with level and contour tones yields eight distinct tonal structures (two of which overlap for the string M or mid).

³The criteria also include considerations of tonal markedness and the relationship between tonal and non-tonal features, which we do not address here.

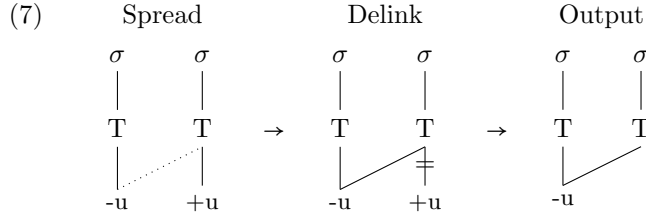
(6)

Tone	Yip (1989)	Bao (1990)	String
Low Level	$\begin{array}{c} \sigma \\ \\ -u \\ \\ l \end{array}$	$\begin{array}{c} \sigma \\ \\ T \\ \swarrow \searrow \\ -u \quad c \\ \quad \\ l \end{array}$	L
High Level	$\begin{array}{c} \sigma \\ \\ +u \\ \\ h \end{array}$	$\begin{array}{c} \sigma \\ \\ T \\ \swarrow \searrow \\ +u \quad c \\ \quad \\ h \end{array}$	H
Mid Level	$\begin{array}{c} \sigma \\ \\ -u \\ \\ h \end{array} \quad \text{or} \quad \begin{array}{c} \sigma \\ \\ +u \\ \\ l \end{array}$	$\begin{array}{c} \sigma \\ \\ T \\ \swarrow \searrow \\ -u \quad c \\ \quad \\ h \end{array} \quad \text{or} \quad \begin{array}{c} \sigma \\ \\ T \\ \swarrow \searrow \\ +u \quad c \\ \quad \\ l \end{array}$	M
High Falling	$\begin{array}{c} \sigma \\ \\ +u \\ \swarrow \searrow \\ h \quad l \end{array}$	$\begin{array}{c} \sigma \\ \\ T \\ \swarrow \searrow \\ +u \quad c \\ \quad \swarrow \searrow \\ \quad h \quad l \end{array}$	HM
High Rising	$\begin{array}{c} \sigma \\ \\ +u \\ \swarrow \searrow \\ l \quad h \end{array}$	$\begin{array}{c} \sigma \\ \\ T \\ \swarrow \searrow \\ +u \quad c \\ \quad \swarrow \searrow \\ \quad l \quad h \end{array}$	MH
Low Falling	$\begin{array}{c} \sigma \\ \\ -u \\ \swarrow \searrow \\ h \quad l \end{array}$	$\begin{array}{c} \sigma \\ \\ T \\ \swarrow \searrow \\ -u \quad c \\ \quad \swarrow \searrow \\ \quad h \quad l \end{array}$	ML
Low Rising	$\begin{array}{c} \sigma \\ \\ -u \\ \swarrow \searrow \\ l \quad h \end{array}$	$\begin{array}{c} \sigma \\ \\ T \\ \swarrow \searrow \\ -u \quad c \\ \quad \swarrow \searrow \\ \quad l \quad h \end{array}$	LM

That is, these models represent the same set of lexical tonal *contrasts* (we explore the formal expression of this in §5.2). They do so, however, via different structural configurations. Intuitively, this discrepancy can be described as follows: Bao’s model splits Yip’s $[\pm u]$ node into three *separate* nodes: a ‘T’ root node, a $[\pm u]$ register node and a ‘c’ contour node. We therefore refer to Bao’s model as the *separated* model, and Yip’s model as the *bundled* model for the remainder of the paper. This structural difference is crucial as it is argued to distinguish the models’ *empirical* predictions, which we examine in the next section.

3.2 Purported Empirical Predictions

Feature geometric representations model not only lexical tonal contrasts but also attested tone sandhi patterns. Assimilatory tone sandhi is formalized over these structures using *spread* and *delink*, the two basic mechanisms of autosegmental theory (Goldsmith, 1976). In this framework, assimilation is the *addition* of a single association line—i.e. spread—between elements in a structure followed by the *subtraction* of an existing association line—i.e. delink. A hypothetical register assimilation pattern between two adjacent syllables is illustrated in (7) using a simplified separated representation (where contours are not shown). A dotted line indicates spreading and a double-barred line denotes delinking.



The [-u] feature on the first syllable spreads to the ‘T’ root node on the adjacent syllable. Then, the existing association line between that node and the [+u] feature delinks. This models progressive register assimilation with the result being a sequence of two low-registered [-u] tones.

Previous work measures the empirical coverage of a representational theory based on the ability of specific structural positions within the representation to spread *independently* of others to model assimilatory tone sandhi attested in the literature. For example, a given theory’s empirical predictions are evaluated by whether it can spread a register node independently from all other nodes: contour, root, terminal, etc. It is along this dimension that the separated and bundled models are argued to differ, with the former purporting wider empirical coverage than the latter. The table in (8)—adapted from Chen (2000)—summarizes claims about the models’ respective empirical predictions via this spreadability metric.

(8)

	Bundled Model	Separated Model	Attested Pattern
Contour Spread	no	yes	Zhenjiang (Zhang, 1985)
Register Spread	no	yes	Pingyao (Hou, 1980)
Whole Tone Spread	yes	yes	Danyang (Lü, 1980)
Terminal Node Spread	yes	yes	Gaojia (Yan, 1981)

Below, we examine arguments for this discrepancy.

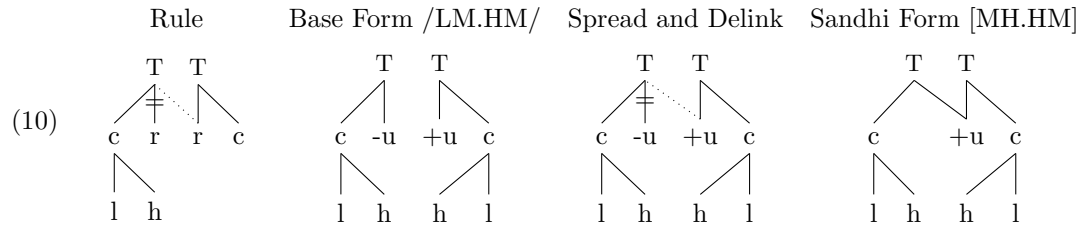
The separated model makes explicit the structural independence of contour and register. Motivation for this division is empirical; assimilatory tone sandhi processes attested in Chinese dialects require the contour node to spread independently of register and vice versa. Consider the following data from Pingyao (Hou, 1980; Bao, 1990; Chen, 2000). High and low register on penultimate contour tones assimilate to the adjacent final tone; a low-rising contour becomes *high*-rising before a high tone, while a high-rising contour becomes *low*-rising before a low tone. The *shape* of the contour—i.e. rising—does not change. The Pingyao data in (9) illustrate:

(9)

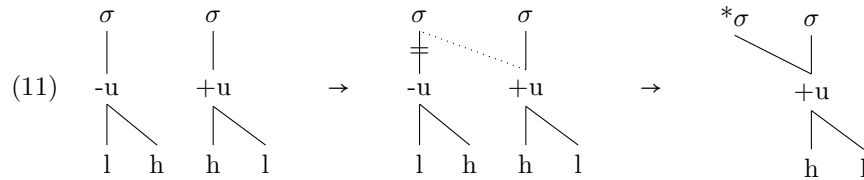
t ^h uæ	paŋ	‘quit class’		tɕi	ma	‘ride a horse’	
35	13	base form	/MH.LM/	13	53	base form	/LM.HM/
13	13	sandhi form	[LM.LM]	35	423	sandhi form	[MH.HM]

Bao (1990) analyzes this pattern as assimilation via regressive spread of a register node, and therefore as crucial evidence for the separation of register and contour on a structural level. A rule is proposed to derive the Pingyao assimilation pattern: when a rising contour tone appears in non-final position, the register node (‘r’ below) on the adjacent tone first *spreads* to the non-final ‘T’ root node, while the underlying register node *delinks* from the non-final ‘T’ node. As the ‘c’ contour node is independent of register, and given that neither operation in the rule targets ‘c’ nodes, contour is predicted to be unaffected. The result of the rule

is that the penultimate rising contour surfaces with the same register as the adjacent tone. Thus the rule correctly derives the sandhi forms in (9). Below in (10), the proposed rule is shown along with a derivation of /LM.HM/.



Assimilatory patterns such as Pingyao may only be derived *as spreading* using representations which separate register from contour on a structural level. As Bao (1990, pg. 66) and Chen (2000, pg. 73) argue, the bundled tonal model proposed by Yip (1989) fails to explain such sandhi processes. Because the register node immediately dominates terminal tonal nodes, register spread will necessarily entail spread of the terminal nodes. The derivation in (11) performs the same spread and delink operations as in (10), but on the bundled model.

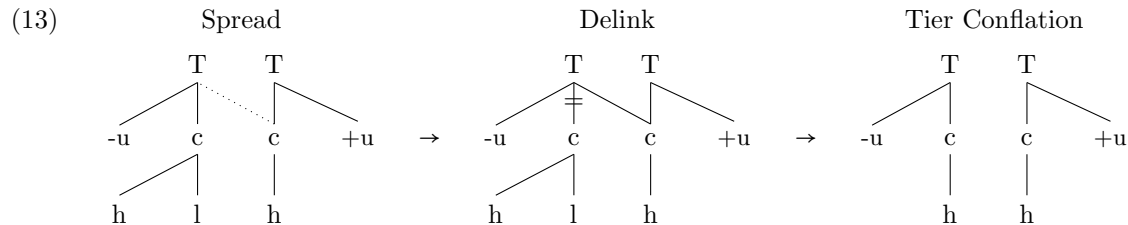


When high register node spreads to the preceding syllable, the falling contour it dominates spreads as well. Similarly, delinking the register node necessarily delinks its daughters, i.e. the rising contour. This results in the unattested output *[MH.MH].

Similar arguments point to sandhi alternations where contour spreads independently of register. A relevant example comes from Zhenjiang (Zhang, 1985; Bao, 1990): when a rising or falling contour tone appears before a high level tone, it surfaces as either mid-level or high-level, depending on the register of the affected tone. In (12), low-registered contour tones surface as mid-level [22] in this environment.⁴

(12)	lǎn	to	'lazy'		ǎi	huei	'virtuous'	
	31	55	base form	/ML.H/	35	55	base form	/LM.H/
	22	55	sandhi form	[M.H]	22	55	sandhi form	[M.H]

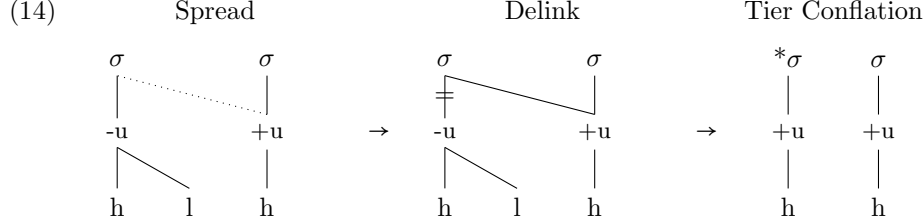
Bao (1990) proposes a regressive contour spread analysis, and cites Zhenjiang as more evidence in favor of the separated model as sandhi does not alter register.⁵ According to this analysis, after spread and delink operations, the structure undergoes a *tier conflation* operation in which the spread 'c' node (along with the immediately dominated 'h' node) is copied. (see Bao, 1990, p. 101-103 and §6 of the current paper for more discussion). The derivation of /ML.H/ → [M.H] in (13) illustrates.



⁴Bao (1990) analyzes the tone with the phonetic realization [35] as a low-register tone.

⁵There is some controversy surrounding this evidence and its analysis; see §6.2 for related discussion.

As with Pingyao, the spreading in a bundled representation would ostensibly entail carriage of register information along with contour information, an undesired result. This is because the immediate dominator of terminal tonal nodes in the bundled model bears register features. In other words, there is no procedural difference between register spread, contour spread, or whole tone spread. Assuming a similar analysis over the bundled model in (14), the correct output cannot be derived.



Contour spread (and tier conflation) entails register spread because the former is directly dominated by the latter, thus producing the unattested $^*[H.H]$.

Claims that the two models differ in their coverage of assimilatory tone sandhi patterns hinge on the spreadability metric described above. However, spreadability arguments are tied fundamentally to a derivational perspective on tonal processes. It is not clear what this metric means for a particular theory when couched in a non-derivational formalism, or whether it distinguishes one theory from another in such cases. This issue is treated in more detail in §6, but we first present an alternative *computational* perspective on the question of these models' empirical predictions. Instead of a potentially theory-specific notion of spreadability, this perspective establishes an *explicit* threshold on the computational complexity of a set of processes, and asks whether competing representational models can capture those processes within the threshold. In the next section, we adopt a computational outlook on separated and bundled models to challenge previous claims about their empirical predictions.

4 Graph Mappings: Empirical Predictions

This section addresses the condition in (1a) which states that two models are notationally equivalent if they do not differ in their empirical consequences. We illustrate that this condition holds for bundled and separated representations, contra the conclusions of previous work. Focus is given to cases of assimilatory tone sandhi claimed to distinguish the two models—register assimilation and contour assimilation.

To achieve this, we pursue a *computational* characterization of assimilatory tone sandhi. This framework focuses on the nature of the input-output *mappings* which describe phonological processes (Chandlee and Heinz, 2018; Heinz, 2018), and thus abstracts away from assumptions specific to any one grammatical formalism. In particular, we present *logical* characterizations of tone sandhi mappings in a model-theoretic framework (Courcelle, 1994; Enderton, 2001; Libkin, 2013). We begin (§4.1) by offering rigorous definitions of separated and bundled representations as *graphs*. Then, we define logical transductions over these graph structures. Transductions map input graphs to output graphs, and therefore can be used to model phonological processes like assimilatory tone sandhi. A benefit of this approach is that we may *fix* the complexity of logic used to define transductions, and determine whether mappings using either representation are possible under the *same* complexity threshold.

In this section, we explicitly demonstrate that separated and bundled models do not differ in their empirical predictions by showing that transductions modeling register assimilation (§4.2) and contour assimilation (§4.3) are definable over *both* representations using Quantifier-Free First Order (QF) logical statements. QF logic is restrictive and computationally simple, and has been shown to be equivalent to the Input Strictly Local class of functions (Chandlee and Lindell, to appear; Chandlee and Jardine, 2019); these functions are sufficient to model a wide range of *local* phonological processes, both segmental and autosegmental (Chandlee, 2014; Chandlee and Jardine, 2018; Strother-Garcia, 2018) despite their restrictiveness. Statements using QF logic determine output structure based solely on information about the corresponding input as well as

input positions within a *fixed window* (that is, a local window) around it. Crucially, the information is not *global* such that a quantifier is required to scan the entire input structure. QF thus provides an appropriate and well-motivated upper bound on the complexity of the processes formalized here. Full transductions are defined in an appendix, but we provide an intuitive graphical characterization in the main text in terms of *local substructures* (described in more detail below).

In defining register and contour assimilation mappings over both representational models, we explore two hypotheses regarding the restrictiveness of QF. The first, more restrictive hypothesis limits mappings to *size-preserving* QF transductions, for which the size of the input and output structures remains constant. The second and less restrictive hypothesis permits *non-size-preserving* QF transductions; these map input structures to a finite number of output *copies*.⁶ We will show that the latter hypothesis is necessary to capture register *and* contour assimilation over both models.

4.1 Tonal Models as Graphs and Processes as Graph Mappings

Tonal geometric models can be explicitly represented as graphs, which are finite sets of points or *nodes* connected by edges. Each node is labeled with at most one feature: syllable, tonal root, $\pm u$ register feature, etc. Edges between nodes represent the internal *structure* of a tone: association between syllable and root node, dominance between internal nodes, and linear order between nodes of the same type. A relational *model* \mathcal{M} is a mathematical object defining such a graph structure. It comprises a set or *domain* \mathcal{D} of structural positions (the nodes) defined over an alphabet Σ of feature symbols. A set of unary *relations* (denoted P for each symbol in an alphabet Σ) determines the labelling of nodes with a particular feature—that is, the property of being a syllable, register node, etc. Unary relations for the bundled and separated models and the labels each relation imparts are shown in (15).

(15)	Bundled Model Relation	Separated Model Relation	Label
	P_σ	P_σ	syllable
	P_{+u}	P_{+u}	+u register
	P_{-u}	P_{-u}	-u register
	P_h	P_h	h terminal
	P_l	P_l	l terminal
		P_T	‘T’ root node
		P_c	‘c’ contour node

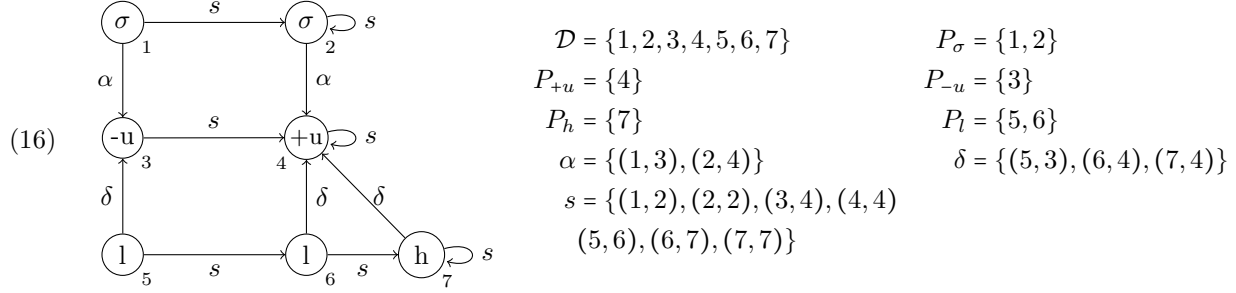
The models contain the same set of unary relations except that the separated model contains two extra relations which label root ‘T’ and contour ‘c’ nodes.

A set of unary functions define node edges representing internal structure and linear order. We use the same set of functions for both representations, and define them as follows. A function α defines an edge between a node labeled as a syllable and a node labeled as a root, and represents association. A function δ defines an edge between nodes that represents immediate dominance. A successor function s defines an edge between a node and its immediate successor. This function thus establishes a linear order over elements in the representation. Crucially, the order obtains only between nodes of the same type (that is, that are on the same *tier*)—e.g. in the separated model, register nodes are ordered with respect to one another, but not with respect to contour ‘c’ nodes, which have their own order. The function is defined such that the final element in a tier is its own successor.

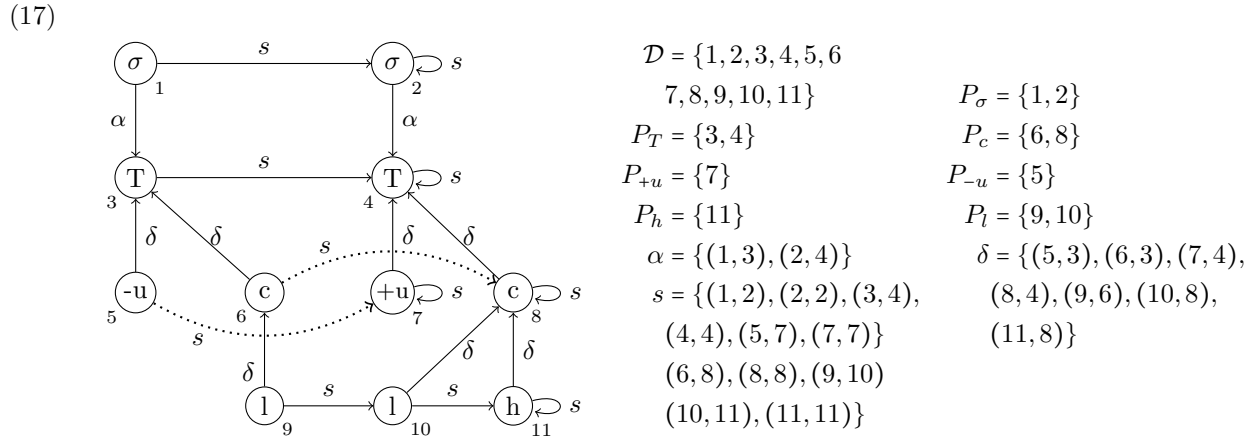
With this set of relations and functions, we explicitly represent models of bundled and separated graph structures. (16) shows the disyllabic sequence [L.MH], a low level tone followed by a high-rising tone, defined over a bundled graph and its corresponding model. Structural positions in the domain are denoted with numbers. A node—a single structural position in a graph—is represented with a circle with its corresponding label inside the circle. Edges are denoted with arrows, and are labeled with corresponding association α , dominance δ , and successor s functions. In the model, relations are defined as a set of positions which are

⁶Given these definitions, size-preserving QF transductions prohibit both copying and *deletion* of input structure. We address the latter as it applies to the current study in §4.1.

in that relation, i.e. which contain that label. Functions are denoted as a set of ordered pairs of positions which define edges.



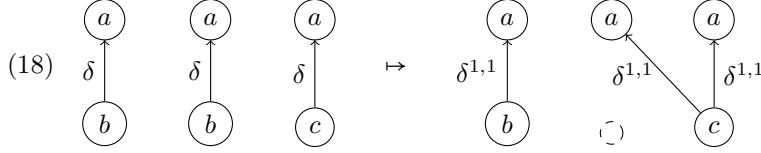
The same tone in the separated representation is defined as a graph model in (17).



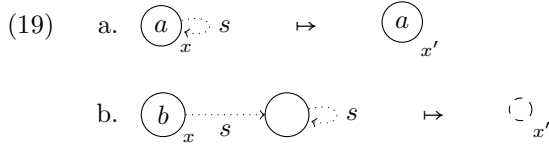
The structural elements of *any* sequence of tones describable with a bundled or separated model can be defined in this way.

We represent assimilatory tone sandhi *processes* in a model-theoretic framework with transductions. Transductions map an input graph structure to a corresponding output graph, and we fix a QF logic to define them. Here, we provide an intuitive discussion of these transductions with graph mappings, with the restriction being that outputs can only be defined by referring to *local, connected substructures* in the input—that is, they reference input nodes connected by edges. We describe this in detail below.

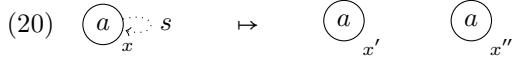
Graph models comprise relations (which label nodes) and functions (which define edges between nodes). Similarly, graph transductions determine node labels and edges over an *output* graph by referring to *input* structure. Labels and edges are treated *separately* in a transduction, but refer to the *same* local input substructures. Importantly, a transduction defines a mapping over a *class* of graph structures, and not over an individual graph. Thus transductions are definitions satisfied by a potentially infinite set of graph mappings. Consider a simplified example of a regressive spreading-type map in (18). The map is defined over a class of graph structures with two separate tiers of ordered nodes: one tier contains nodes labeled *a* and another contains nodes labeled either *b* or *c*. Nodes on these tiers relate one-to-one via edges marked δ . Regressive spread is the *addition* of a δ edge between the final node on the *b/c* tier and the penultimate node on the *a* tier. It also entails *deletion* of the input δ edge between penultimate nodes on these tiers, and thus ‘deletion’ of the penultimate node on the *b/c* tier (denoted with a dashed circle; more explanation is given below). The mapping in (18) is over a graph structure with three nodes on each tier, where \mapsto denotes ‘maps to’ and $\delta^{1,1}$ indicates output δ labels. All mappings discussed in this paper are *order-preserving*, that is, order relations are preserved from input to output structures (see Filot, 2015; Chandlee and Jardine, 2019, for more discussion). Thus, these edges (i.e. the successor *s* function) are omitted for clarity, but we assume a total order over each tier as in (16) and (17).



Defining output node labels via transduction is achieved through reference to *local, connected substructures*; that is, a given output node is determined by referring only to the corresponding input node and other nodes *connected by edges* in the input structure. These definitions thus determine output structure using only a fixed window in the input. It is possible, for example, to isolate structural elements such as the *final* a node and the *penultimate* b node (illustrated in (19)). This is because both constitute connected substructures in the input. The former is an a node with a looping s edge and the latter is a b node which *shares* an s edge with some final element—that is, an element with a looping s edge. The definition may preserve labels in the output for elements which map directly (i.e. an identity mapping) such as the final a node (19a). Alternatively, it may map it to an empty label as in the penultimate b node (19b). Such a definition ‘deletes’ the label from the output structure, and is comparable to deletion of structural material, for instance, after delinking in an autosegmental analysis. Importantly, however, it is still *size-preserving*; though the label does not appear in the output, the *structural node* itself is preserved, and thus does not alter the size of the input structure as a whole. The examples below demonstrate how the transduction defines output labels in terms of local substructures. Relevant input ordering edges are shown with dotted lines. Additionally, x denotes the relevant input node and x' the corresponding output node.



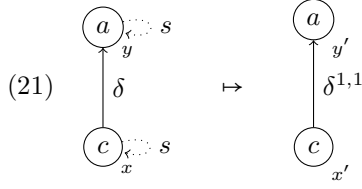
The size-preserving node labeling definitions in (19) may be contrasted with strictly *less-restrictive*, non-size-preserving transductions. Transductions of this complexity are defined over a finite set of multiple output *copies* (also called the copy set), and thus permit an output which is of a greater size than the input. Intuitively, non-size-preserving transductions are those which model processes with a copying mechanism. An example would be a final a node (as in (19a)) mapped to *two* copies of itself, as in (20), where x'' denotes a second copy of the input structure.



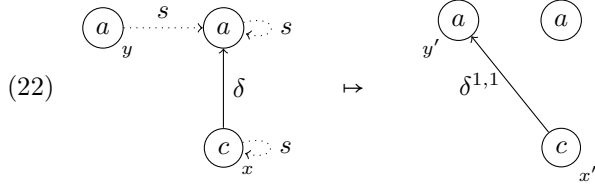
We present transductions of this type in the following sections, but it is worth noting here that the copying they permit is crucially *restricted* by the local substructure (i.e. QF-definability) requirement. That is, in addition to the finiteness of the copy set size, the number of nodes definable within a given copy set is *bounded* by the size and structure of the input. In other words, despite permitting a copying mechanism, these transductions are still part of a restrictive and computationally simple class of maps.

Recall that maps define both output nodes and output *edges*. Determining output edges proceeds in a similar manner as with nodes, and is subject to the same restrictions. The regressive spreading-type map in (18) is possible because it can be defined in terms of local substructures. This is summarized in (21) and (22). First, the final nodes on the a and b/c tiers relate via an input δ edge, and so the output dominance relation between them is simply an identity mapping; the input edge between two input nodes (x and y below) is preserved in the output (x' and y' below; note that the δ edge on the first nodes in each tier may be preserved in the same way). As in (18), this output edge is denoted $\delta^{1,1}$, that is, an edge from a node in the first output copy to a node in the first output copy.⁷

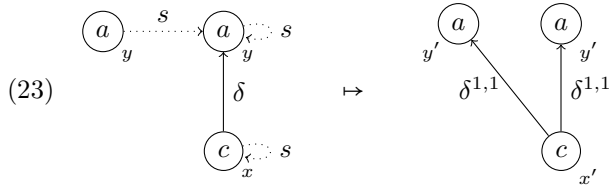
⁷Since this map is size-preserving, there is *only* one copy of output nodes to relate via edges, so the notation may seem superfluous here. When multiple copy sets are introduced, however, edges may relate nodes either within or across copy sets. We introduce this notation here for consistency with the discussion to follow.



The final c and *penultimate* a relate via the same δ edge in addition to a successor s edge between the penultimate and final a nodes, thus forming a local input substructure, as in (22).



Combined in a single transduction, these edges model a regressive-spread type pattern in (23):



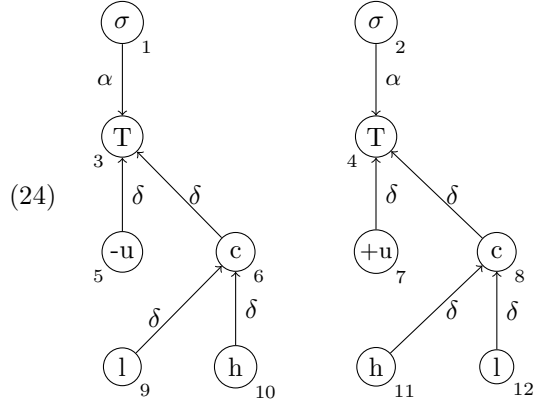
Graph mappings which refer only to local, connected substructures model the space of maps definable with QF logical transductions. These maps are restrictive, formally rigorous, and align well with the complexity necessary to formalize local phonological processes (Chandlee and Lindell, to appear; Chandlee and Jardine, 2019).

4.2 Register Assimilation: Pingyao

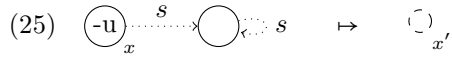
Having defined separated and bundled theories as graphs, we now define transductions over these representations to model attested tone sandhi patterns. Using this formalism, §4.2 and §4.3 show that Pingyao register assimilation and Zhenjiang contour assimilation patterns are definable as QF transductions over both separated and bundled representations. Formalizing register assimilation in this framework uncovers a discrepancy in the QF logical power both representations require. This is a discrepancy between size-preserving QF logic (sufficient for the separated model) and non-size-preserving QF logic (necessary for the bundled model). While it may appear to signal a more general difference in the models' capacity to capture assimilatory tone sandhi processes, this distinction vanishes in the analysis of contour assimilation. Thus when taken as a whole, more powerful non-size-preserving QF transductions are necessary for *both* representations to capture register and contour assimilation. Within the computational perspective advocated here, this indicates that the separated and bundled models do not differ in their empirical consequences as previously claimed.

4.2.1 Separated Model

Let some transduction define mappings between input and output separated model graphs and which models register assimilation in Pingyao. Consider the Pingyao input /LM.HM/ as a separated graph structure in (24). Successor function edges are omitted for clarity.

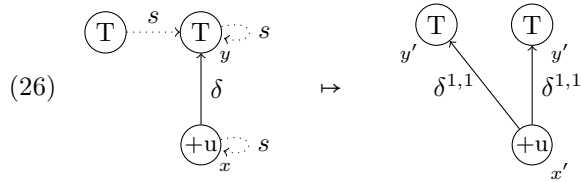


The transduction is defined over a single copy of output nodes as follows. First, a penultimate $[-u]$ register node maps to an empty label. Such a node is definable as a connected substructure as (25) shows—a $[-u]$ node sharing a s edge with some final element (one with a looping s edge).

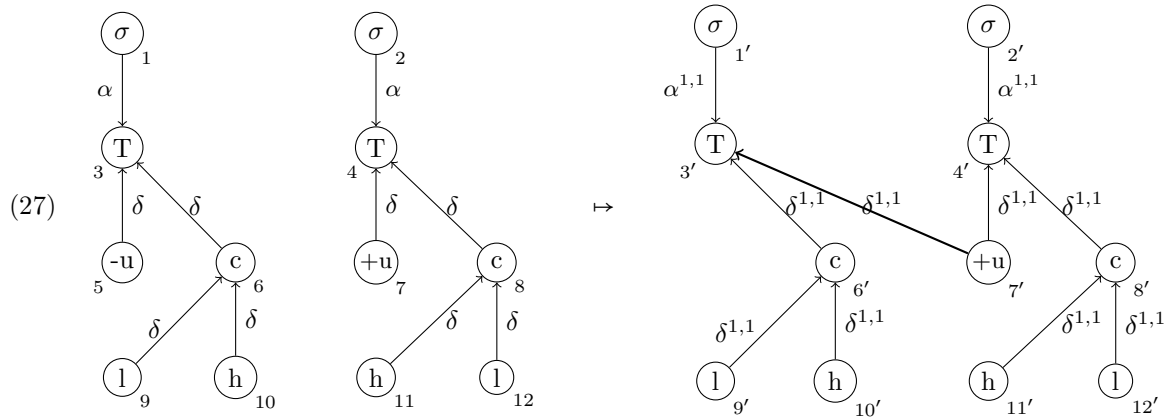


Other output labels map directly from corresponding inputs.

Output edges are also preserved from inputs, but with one exception.⁸ The delta δ edge between the final register and the final ‘T’ node is preserved, but an *additional* edge is defined between the final register node and the penultimate ‘T’ node. Again, this definition is possible because these nodes comprise a *local* input substructure, as in (26).



Applied to the graph in (24), this transduction maps to the correct form [MH.HM] as an output graph in (27) below, where prime $'$ indices denote output positions. Note that this graph is also consistent with the sandhi form in (10). Changes to the output are given in bold.

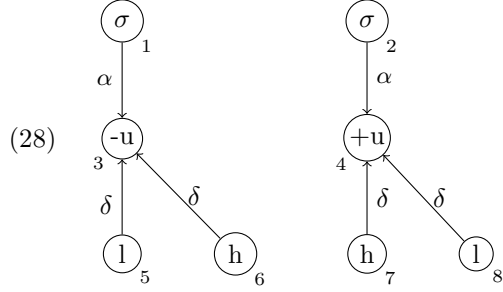


⁸Assume also that edges are not preserved on unlabelled output nodes, e.g. the δ edge between the penultimate $[-u]$ register node and ‘T’ root node.

Importantly, the graph mapping as defined generalizes beyond this case to any sequence with a penultimate rising tone. The restriction of local substructures (and thus QF) is therefore sufficient to model this pattern in the separated representation. Next, we show that the same holds for the bundled representation, though it requires a more powerful QF logic to do so.

4.2.2 Bundled Model

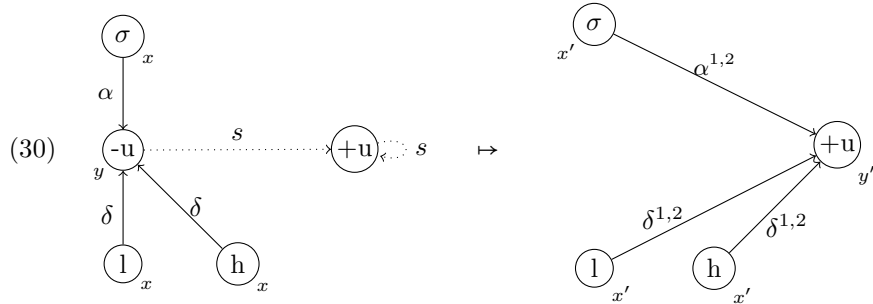
Now, let some transduction define mappings between input and output *bundled* model graphs and which models register assimilation in Pingyao. Consider the Pingyao input /MH.HM/ as a bundled graph structure in (28):



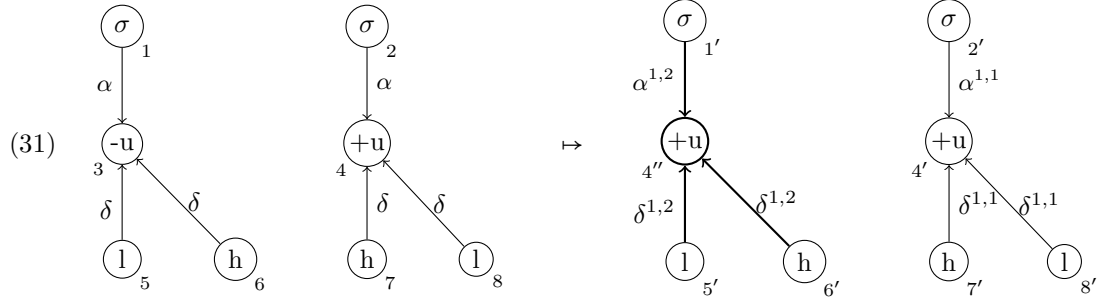
Unlike its separated model counterpart, this transduction is defined over *two* copies of output nodes (and thus uses non-size-preserving QF logic) as follows. Map a penultimate [-u] register node to an empty label (29a). Additionally, generate two copies of a final [+u] node (29b). Preserve other node labels from the input.



Then, redefine output edges such that the α and δ edges on the penultimate syllable terminate on the *second* copy of the final [+u] node, and hold other edges constant. Again, this is possible because these nodes form a connected substructure of α , δ , and s edges in the input. This is shown in (30) where $\alpha^{1,2}$ and $\delta^{1,2}$ denote edges defined from nodes in the first output copy to nodes in the second output copy.



Applied to the graph in (28), this transduction maps to the attested form [MH.HM] as an output bundled graph, illustrated in (31). Changes to the output structure are given in bold.



Therefore, despite ‘failing’ the spreadability test (as in (11)), the bundled representation *can* model register assimilation in Pingyao as a logical transduction over bundled graph structures. Such a transduction is definable using a restrictive QF logic (or in our terms, by referring to local input substructures).

4.2.3 Summary

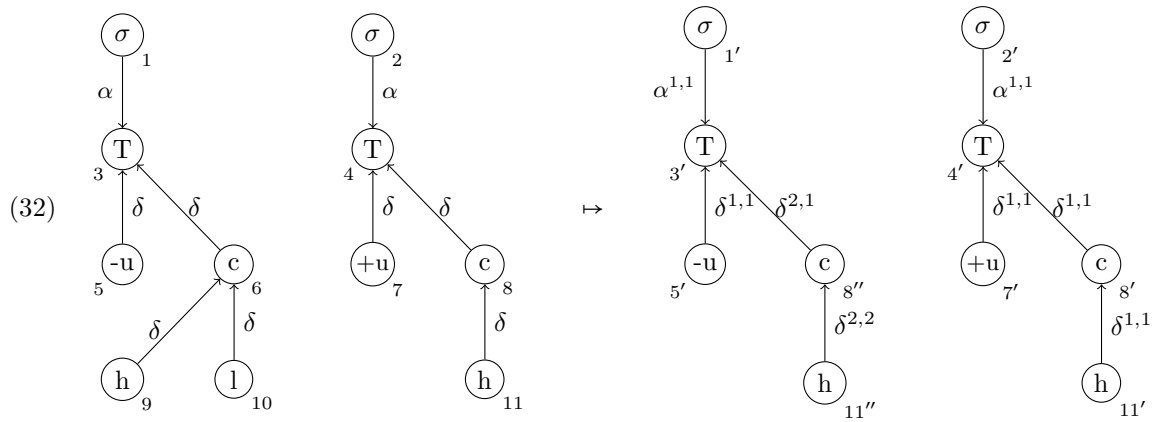
Formalized as mappings, register assimilation processes (register spread) can be represented over both models, contra previous claims. They do so using local, connected input substructures. There is an important difference, however. The bundled model requires non-size-preserving QF logic to model register assimilation because it has to emulate spreading as deletion *plus* copying. For cases of this type, the separated model captures the process using the more restrictive, size-preserving logic.

4.3 Contour Assimilation: Zhenjiang

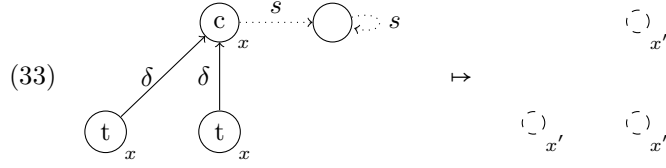
We now show that the discrepancy in the class of logic needed to represent assimilatory processes between models vanishes in the case of contour assimilation. Examining both types of assimilation reveals that size-preserving QF is in fact *too* restrictive for both bundled and separated models because copying, and thus copy sets, are necessary to capture contour assimilation in both representations. Despite this fact, transductions modeling contour assimilation are definable over *both* models using non-size-preserving QF logic.

4.3.1 Separated Model

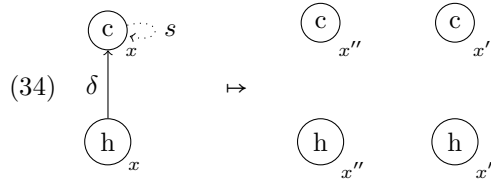
We define a transduction to model Zhenjiang contour assimilation over a separated representation. The goal is a transduction which maps input graphs to output structures like those in (13); that is, ones which have undergone tier conflation. One desired output of such a transduction would be the graph mapping of /ML.H/ to [M.H] in (32). Here, the output contains two *copies* of the input’s high contour.



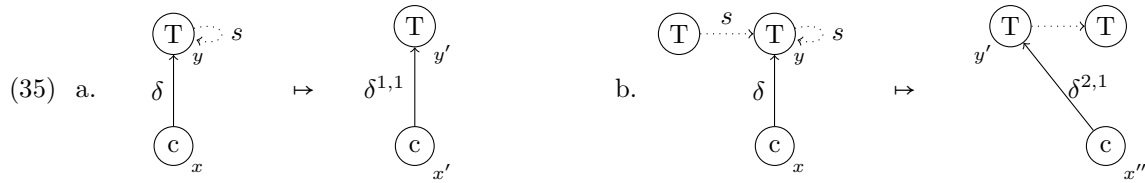
This transduction is defined necessarily over a copy set of size two as follows. The penultimate contour ('c' node and any terminals, denoted 't' below to include 'h' or 'l' labels) maps to unlabeled nodes as illustrated in (33).



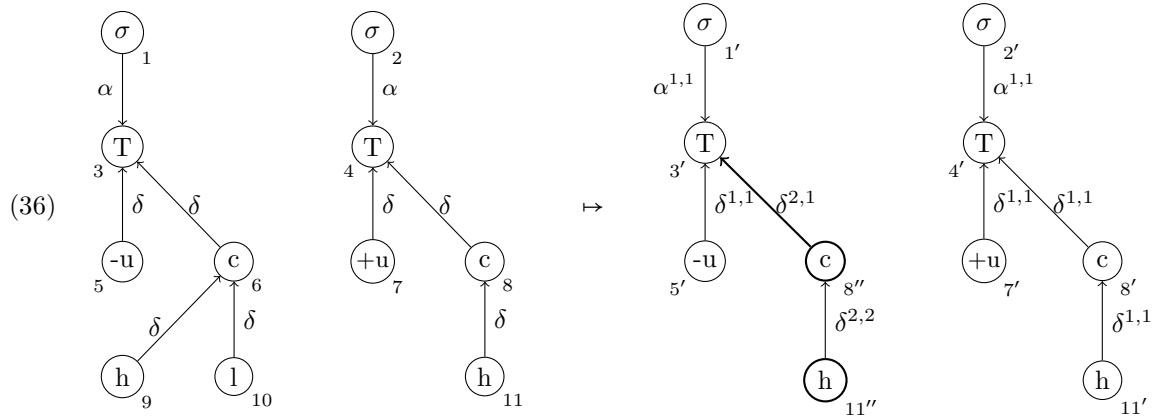
Other labels are preserved, with the exception of the final high contour. To produce output graphs consistent with Bao (1990)'s analysis, *two* copies of this connected input substructure are generated, as in (34).



The first output copy of the high contour node preserves its input δ edge with the corresponding input 'T' node in (35a). Another edge is defined between the *second* copy of the high contour node and the penultimate 'T' node in (35b).



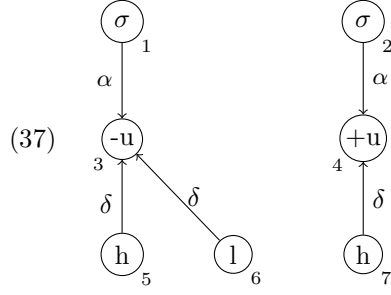
Defined thusly, the transduction yields maps consistent with Bao (1990)'s analysis of contour assimilation, including the mapping in (32), repeated below in (36). Changes to the output structure are given in bold.



Importantly, though, non-size-preserving QF logic is necessary to generate these structures. Taken as a whole, then, size-preserving QF is too restrictive to capture both register and contour assimilation for the separated model.

4.3.2 Bundled Model

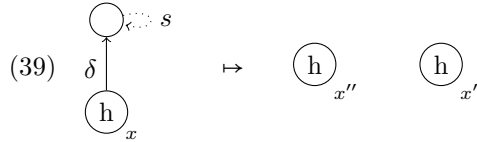
A transduction of the same process over the bundled model is also defined over a copy set of size two. A Zhenjiang input /ML.H/ as a bundled graph structure is presented in (37):



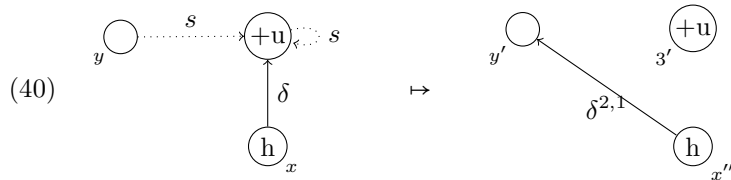
Mapping the penultimate contour to unlabeled nodes is similar to the procedure used for the separated representation. The difference lies in the fact that we want to *preserve* the immediate dominator of terminal nodes on the penult (that is, preserve its input label), because this node carries register features. Since labels and edges are defined separately, we can isolate the terminal nodes on the penult by referring to the *structural position* with which both nodes share a δ edge in the input. This definition thus applies to falling contours (as in the example above), but also rising contours, and high/low level tones equally well; *any* tonal terminal nodes which share a δ edge with some penultimate node satisfy the definition. In the bundled representation, this position is labeled with a register feature, while in the separated representation, it is labeled ‘c’ (see more in the next section). It is therefore possible—using connected substructures—to map the penultimate contour to unlabeled nodes (38a) while also preserving input label on the penultimate register node (38b).



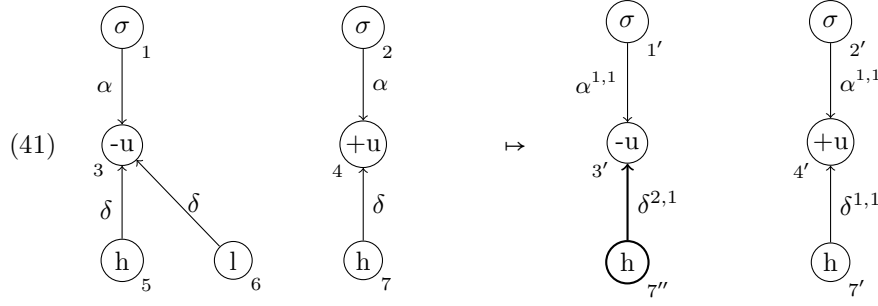
The transduction also generates two copies of the final ‘h’ node (but not two copies of the final register node) in the same way, as in (39)



Then, preserve input edges with the exception of the second copy of the final high node. Define a δ edge between that node and penultimate register node in the first copy set. This is shown in (40), where y' denotes the first (and only) output copy of the penultimate register node and x'' denotes the *second* copy of the final ‘h’ terminal node.



Applied to the graph in (37), this transduction maps to the correct output [M.H] bundled graph as in (41). Changes in the output structure are given in bold.



Zhenjiang contour assimilation is thus formalizable over a bundled representation of tone using non-size-preserving QF logical transductions.

4.4 Summary

Previous work claims that the separated and bundled models constitute distinct theories of tonal representation in that they make different empirical predictions: one model successfully captures register and contour assimilation tone sandhi patterns as *spreading* while the other does not. This section has shown that both models *can* represent these processes when formalized as mappings over graph structures, and that they do so within the *same* logical complexity threshold. This threshold is restrictive, aligns well with attested phonological processes, and is not tied to any particular grammatical formalism. In addition, the definability of these processes over both representational models using this logic—crucially using only *local* input substructures—indicates that the *local* nature of tone sandhi patterns can be expressed in both models. From the computational perspective, bundled and separated models of representation satisfy condition (1a) of notational equivalence as they do not differ in their empirical consequences. The next section demonstrates that the models are also equivalent on a structural level. It does so using the *same* QF logical transduction formalism. Thus, we may reason over this question of notational equivalence—both in empirical and structural terms—using a single formal framework.

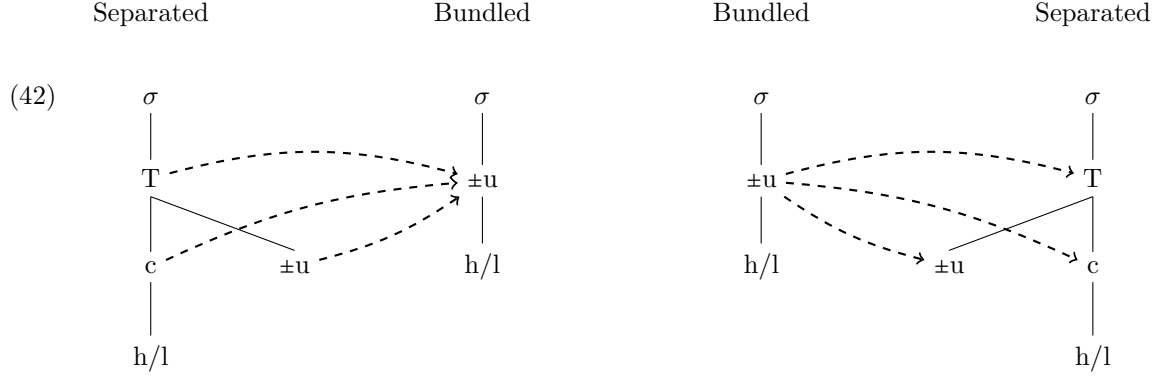
5 Graph Mappings: Structural Differences and Bi-interpretability

We now turn to condition (1b), which states that two models are notationally equivalent when they represent the same set of abstract properties, and differ only superficially. We satisfy this condition in a model-theoretic framework by demonstrating the *bi-interpretability* of the bundled and separated models. This provides a rigorous formal expression of models differing only ‘superficially’. A definition of model bi-interpretability is given by Friedman and Visser (2014):

We note that an interpretation $K : U \rightarrow V$ gives us a construction of an internal model $\tilde{K}(\mathcal{M})$ of U from a model \mathcal{M} of V . We find that U and V are bi-interpretable iff, there are interpretations $K : U \rightarrow V$ and $M : V \rightarrow U$ and formulas F and G such that, for all models \mathcal{M} of V , the formula F defines an isomorphism between \mathcal{M} and $\tilde{M}\tilde{K}(\mathcal{M})$, and, for all models \mathcal{N} of U , the formula G defines an isomorphism between \mathcal{N} and $\tilde{K}\tilde{M}(\mathcal{N})$.

Intuitively, this means that one model can be translated into another (and vice versa), and that all contrasts are preserved through translation. As in §4, the formal details of separated/bundled model bi-interpretability are spelled out in an appendix, and we present an intuitive discussion here. We divide the definition above into two main components. The first (§5.1) establishes *interpretations* between models. An interpretation is a specific kind of map from one structure to another. For example, an interpretation Γ^{bs} maps bundled structures to separated structures by providing a model of bundled structures using the logical language of

separated structures. Similarly, an interpretation Γ^{sb} maps separated structures to bundled structures by providing a model of separated structures using the logical language of bundled structures. The existence of both interpretations corresponds to the notion that the models are *intertranslatable*. In conceptual terms, the map defined by Γ^{bs} represents a *fusion* of ‘T’, ‘c’, and register nodes into a single register node (holding all other nodes constant). The map defined by Γ^{sb} represents an *expansion* of a single register node to separate ‘T’, ‘c’, and register nodes (also holding all other nodes constant). These intuitions are provided below in (42), where dashed arrows represent fusion and expansion, respectively.



The second component of bi-interpretability (§5.2) requires that the following conditions hold. First, combining Γ^{sb} and Γ^{bs} through *composition*—mapping bundled structures into separated structures and *back into* bundled structures—produces the same mapping as (i.e. is isomorphic to) the identity map that maps every bundled structure *to itself*. Similarly, composing Γ^{bs} with Γ^{sb} is isomorphic to the identity map that maps every separated structure to itself. In intuitive terms, this component demands that the two interpretations be inverses or mirror images of one another; as such, the translations they achieve preserve all contrasts present in the original representation. We demonstrate this by showing that for any tonal structure describable by bundled and separated representations, the output of Γ^{sb} is structurally identical to the input of Γ^{bs} and vice versa.

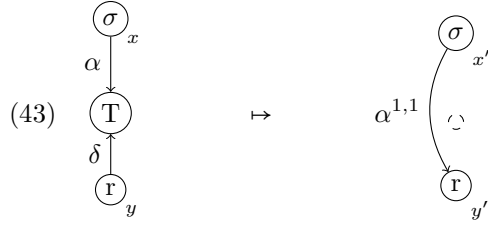
5.1 Intertranslatability

5.1.1 Separated to Bundled: Fusion

Let Γ^{sb} be a transduction which maps any structure in a separated representation to a corresponding structure in a bundled representation, and thus an interpretation of the class of separated models in terms of the class of bundled models. Like process transductions, Γ^{sb} is a mapping between graph structures that takes a set of node labels and edge relations as input and maps it to another set of output node labels and edge relations. It is defined over a single copy set.

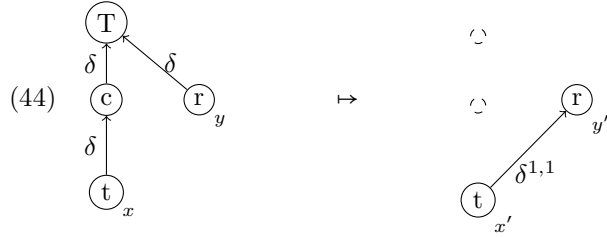
All bundled node labels are defined as identity mappings from relevant labels in the separated model, as all features in the former model are contained in the latter. This includes register node *labels*. Predecessor p and successor s edges are preserved, as linear order does not change. Association α and dominance δ edges, however, must be redefined to reflect the *fusion* of ‘T’ and ‘c’ structural positions into a single register node.

In a separated model, syllables and ‘T’ nodes relate via α edges (representing the association relation). Register nodes also relate to ‘T’ nodes, but via a δ edge. These nodes and edges constitute a local substructure. We may use this substructure to redefine α edges in a bundled model such that syllable nodes relate *directly* to register nodes, which is not the case in the separated model. The figure in (43) illustrates how α edges are defined in the transduction, where ‘r’ refers to either [+u] or [-u] register label.



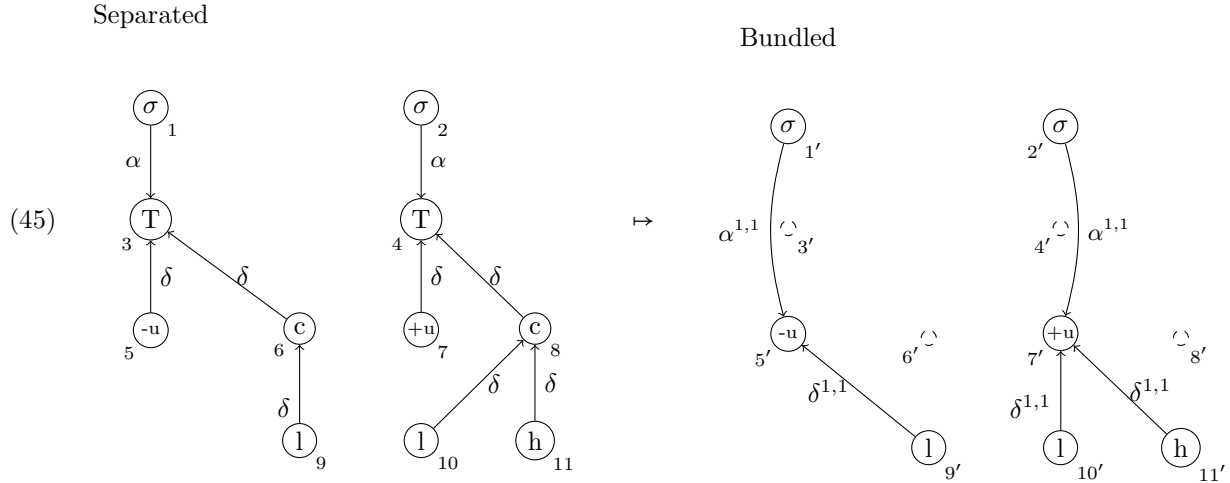
Although the ‘T’ node is not *labeled* in the output structure—recall that the bundled representation does not contain ‘T’ nodes—this structural position is still a part of the input structure and therefore can be referred to in defining the output. The definition here ‘fuses’ the separated model’s ‘T’ node and the bundled model’s register node as the structural position which shares an α edge with the syllable. More generally, this reflects the fact that the ‘T’ node in the separated model and the register node in the bundled model have the same *structural function*, that is, the tonal root.

Defining δ edges in the transduction utilizes another local input substructure to establish edges from terminal tonal nodes directly to register nodes, a relation which does not obtain in the separated structure. It builds on the fact that in the separated model, register nodes and tonal nodes both relate to a ‘T’ node; the former shares a δ edge, while the latter relates through a ‘c’ contour node and two δ edges. This mapping is shown in (44) where ‘t’ denotes ‘h’ or ‘l’ terminal node labels.



Again, the fact that ‘T’ and ‘c’ nodes are unlabeled in the output does not prevent reference to these structural positions to relate terminal nodes directly to register nodes. The ‘fusion’ here is between the ‘c’ node in the separated model and the register node in the bundled model, and reflects the generalization that these nodes also have the same structural function: the immediate dominator of ‘h’ and ‘l’ terminal nodes.

The transduction Γ^{sb} applied to a separated model structure produces an equivalent structure in bundled representation. An example of a disyllabic sequence [L.MH] is given in (45), where predecessor and successor edges are omitted for clarity.



This is true for *any* separated representation graph. Γ^{sb} is thus an interpretation of the class of separated graphs in terms of the class of bundled graphs.

5.1.1.2 Bundled to Separated: Expansion

Similarly, let Γ^{bs} be a transduction which maps any structure in a bundled representation to a corresponding structure in a separated representation, and thus is an interpretation of the class of bundled models in terms of the class of separated models.

The number of node label types in the separated model is *greater* than that of the bundled model—the former contains ‘T’ and ‘c’ labels not present in the latter. A copy set greater than size one is necessary. We define the transduction over a copy set of size 3, where each copy set will represent one ‘expansion’ of the bundled model’s register node: ‘T’ nodes in the first copy set, register nodes in the second copy set, and ‘c’ nodes in the third copy set. These nodes relate via a one-to-one identity mapping. This is shown in (46), where x' , x'' , x''' indicate nodes in the first, second, and third copy sets respectively.

$$(46) \quad \textcircled{r}_x \mapsto \textcircled{T}_{x'} \quad \textcircled{r}_{x''} \quad \textcircled{c}_{x'''}$$

Syllable nodes are labeled in the first copy set, while h/l tonal nodes are labeled in the third copy set. This allows for preservation of α and δ edges from the input bundled structure, given in (47a) and (47b) respectively. Again, this reflects the fact that these nodes share *structural functions* across models: register and ‘T’ nodes relate to syllables via association, and register and ‘c’ nodes relate to h/l tonal nodes via dominance.⁹

$$(47) \quad \begin{array}{cc} \text{a.} & \begin{array}{ccc} \textcircled{\sigma}_x & \mapsto & \textcircled{\sigma}_{x'} \\ \alpha \downarrow & & \alpha^{1,1} \downarrow \\ \textcircled{r}_y & & \textcircled{T}_{y'} \end{array} & \text{b.} & \begin{array}{ccc} \textcircled{r}_y & \mapsto & \textcircled{c}_{y'''} \\ \delta \uparrow & & \delta^{3,3} \uparrow \\ \textcircled{t}_x & & \textcircled{t}_{x'''} \end{array} \end{array}$$

The transduction defines the internal structure of the separated model—i.e. δ edges between register/‘c’ and ‘T’ nodes—in terms of the bundled model in the following way. Given that ‘T’, register, and ‘c’ labels in the separated structure map directly from a single register node in the bundled structure, δ edges between these nodes can also be defined in terms of the same register node, keeping in mind that a single node constitutes a connected substructure. Dominance from the register node to the ‘T’ node is a δ edge from a node in the second copy set to an identical node in the first copy set, and dominance from the ‘c’ node to the ‘T’ node is a δ edge from a node in the third copy set to an identical node in the first copy set, as shown in (48).

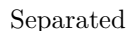
$$(48) \quad \textcircled{r}_{x,y} \mapsto \begin{array}{cc} \textcircled{T}_{y'} & \textcircled{T}_{y'} \\ \delta^{2,1} \uparrow & \delta^{3,1} \uparrow \\ \textcircled{r}_{x''} & \textcircled{c}_{x'''} \end{array}$$

The edges that the transduction defines between and within copy sets is summarized in (49), where ‘C1, C2, C3’ indicate copy sets one, two, and three.

⁹Note that another consequence of the identity mapping is that predecessor and successor ordering edges can be preserved within each copy set, and thus maintain linear order over ‘T’, register, and ‘c’ nodes.



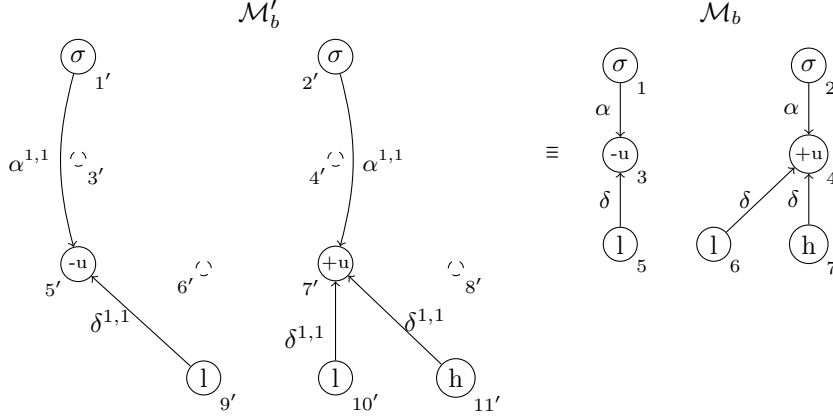
(50)



the appendix for a general demonstration.

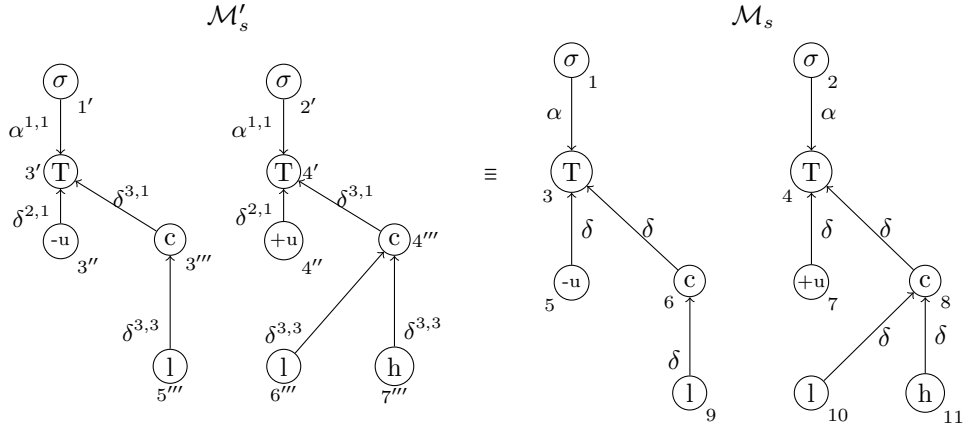
tonal structure. The following holds of these graph structures: \mathcal{M}'_b is structurally identical to \mathcal{M}_b and \mathcal{M}'_s is structurally identical to \mathcal{M}_s . Recall example translations of disyllabic [L.MH] in (45) and (50). The *output* of (45) contains both the same structural elements *and* relations between those elements as the *input* of (50). This is illustrated in (51) below where \mathcal{M}'_b denotes the former and \mathcal{M}_b the latter.

(51)



The same is true of the input of (45) and the output of (50). Separated representation components are present in both and structural elements relate to one another via the same edges, as in (52).

(52)



The above illustration generalizes beyond the [L.MH] sequence to *any* tone or sequence of tones representable by either model. This reflects the observation in Table 1 in §3.2 that bundled and separated models represent the same set of lexical tonal contrasts. Translation between the models maximally preserves those contrasts.

Combining this with the results from §5.1, the conclusion is that separated and bundled representations are *bi-interpretable* in a strict model-theoretic sense. Within the adopted framework, the models do not differ in any non-trivial way in terms of their structure. We thus satisfy condition (1b) of notational equivalence.

6 Discussion

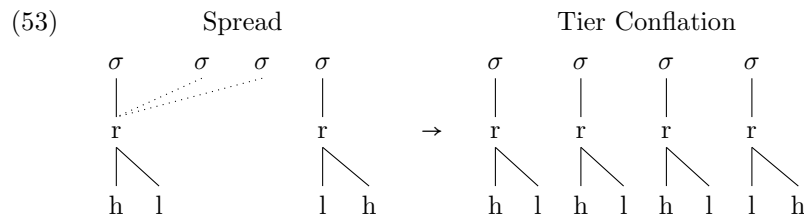
The previous sections have applied a model-theoretic approach to the question of notational equivalence between two models of tonal representation. They show that separated and bundled models neither differ in their empirical consequences as previously argued (1a) nor do they differ substantially in their representation of abstract properties (1b). We therefore conclude that they are notationally equivalent. Here, we pause to interpret these results and consider their ramifications.

6.1 ‘Letter’ of the Theory vs. ‘Spirit’ of the Theory

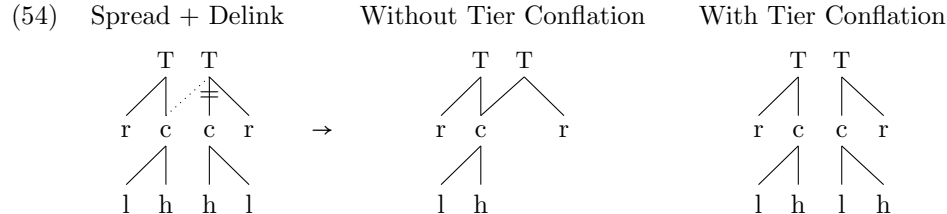
§4 makes the claim that non-size-preserving QF logic is necessary to capture register assimilation *and* contour assimilation across both models; size-preserving QF is too restrictive for these cases. As it applies to feature geometric tonal representation, the fundamental difference between these logics is that the latter

allows only spreading while the former permits both spreading and *copying*. The result diverges crucially from previous analyses in that register and contour assimilation become possible over a bundled model. A reasonable question to ask is whether this allowance is appropriate, and does not unreasonably coerce the *theories* of separated/bundled representation beyond their original intentions. In other words, is a non-size-preserving QF analysis in the *spirit* of these theories?

The answer to this question is yes, and stems from the observation that spreading and delinking mechanisms are, on their own, insufficient to capture the full range of assimilatory processes in Chinese dialects. Any representational theory of these patterns requires an additional copying mechanism known as *tier conflation* (Younes, 1983; McCarthy, 1986), a procedure borrowed from segmental representation and templatic morphology. Yip (1989, pg. 161) describes the process which “automatically copies non-adjacent multiply linked roots so as to allow interpolation of the vocal root.” Applied to an edge-in association and contour spread pattern in Danyang (Lü, 1980; Yip, 1989; Chan, 1991), for example, tier conflation copies tonal information from a root node which has spread rightward two syllables, as in (53).

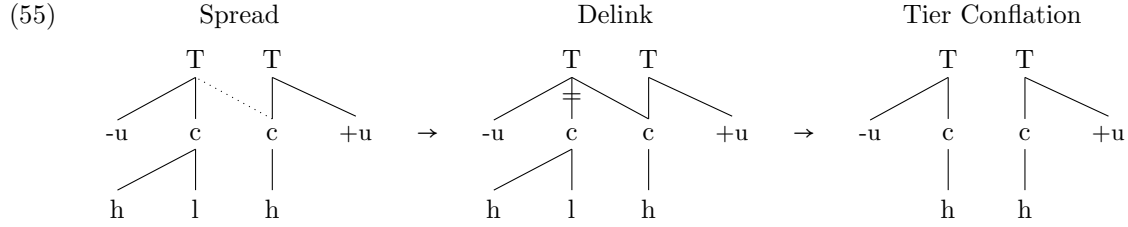


The extra derivational step ensures the surface form [HL.HL.HL.LH] (three falling contours followed by a rising contour) and not a single falling contour realized gradually over three syllables *[H.M.L.LH]. The same generalization can be applied to local contexts, as well, and is in fact *necessary* for contours spreading as a unit for the same reasons. Consider a hypothetical Danyang-like pattern in (54), formalized over a separated model, with progressive contour spread.

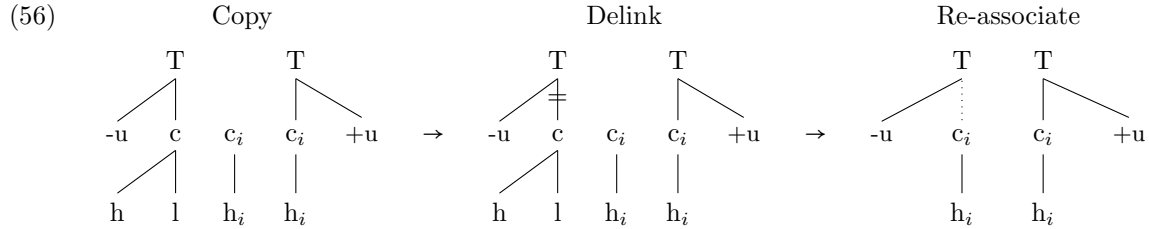


With a single contour associated to two roots (as above), there is no guarantee that the observed [LH.LH] will obtain, and not any of the logically possible *[L.H], *[L.LH], or *[LH.H]. The copying mechanism of tier conflation ensures this, and is thus necessary whenever contour spreads as a unit. This means that a theory of tone sandhi assimilation over these representations comprises *three* basic mechanisms: addition of association lines (spread), deletion of association lines (delink), and copying of structural nodes. Therefore, while it is true that the ‘spreadability’ metric—using only the mechanisms of spread and delink—distinguishes separated and bundled models in certain cases like register assimilation, it ultimately provides an incomplete picture because it neglects a basic operation of the theory.

Given a theory with three basic mechanisms, it is unclear how a spreading analysis with tier conflation is different from an alternative analysis which simply *copies* the contour nodes and re-associates them. The Zhenjiang derivation in (13) is repeated in (55) to illustrate.

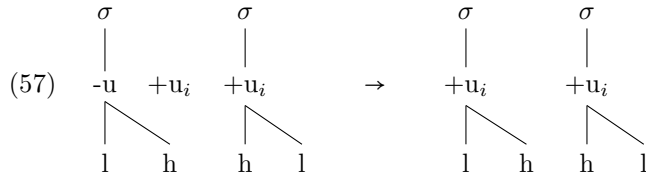


Spreading with tier conflation is identical to an alternative *copying* analysis which proceeds as follows. The assimilating contour node is first copied (indicated by an index), then re-associated to the preceding tone after a delinking operation. This order of operations is given in (56).

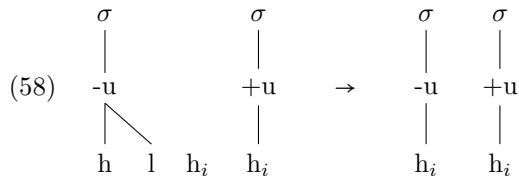


A copying analysis correctly predicts the observed [M.H] for an input string /LM.H/ as in the spreading analysis. Outputs for both analyses are not only surface-form identical, but also *structurally* identical, and are achieved using the theory's basic operations. They differ only in the relative order of these procedural mechanisms.

If a copying analysis is permitted in this framework, the bundled representation can, in fact, model register and contour assimilation processes contra previous claims. In Pingyao register assimilation, for example, the intuition is as follows: generate a copy of the final register node, then re-associate the syllable and terminal nodes to that copy. This correctly predicts the output [MH.HM] from /LM.HM/, as in (57) below. Note that the resulting structure is identical to the *output* of the non-size-preserving QF mapping defined over bundled graph structures in (31).



The intuition for Zhenjiang contour assimilation is similar: generate a copy of the terminal 'h' node, then re-associate it to the preceding register node. Again, the analysis generates attested forms over a bundled model as with the separated model. An example of /LM.H/ → [M.H] in (58) illustrates. The reader may observe, as before, that the resulting form is identical to the output of the bundled graph mapping in (41), that is, a mapping describable by a non-size-preserving QF transduction.



These copying analyses preserve the spirit of the original separated/bundled representational theories in the sense that they utilize the same basic—and in fact *necessary*—mechanisms as traditional spreading accounts

with tier conflation: spread, delink, and *copy*. The basic mechanisms are not unrestricted in their application, however. They are limited to local environments and do not involve any long-distance dependencies. The mappings defined in §4.2 and §4.3, then, are also in the spirit of the original theory given their QF-definability. In spite of allowing copying, they are still restricted to local environments and thus accord with limitations of the original theory.

6.2 Spreading vs. Copying

We may also ask how ‘spreading’ and ‘copying’ analyses differ (if at all). Answering this question, as we suggest in §2, is non-trivial. This is because these mechanisms are inevitably fixed to the grammatical formalisms in which they are proposed. Spreading accounts of assimilatory sandhi in Chinese dialects are couched in derivational terms. Tier conflation, for example, is crucially ordered *after* spreading; its scope is dependent on the structural environment created by the application of an earlier rule. In principle, a similar argument is available to copying analyses in a derivational framework. Association of a copied element in such an account is also dependent on the prior delinking of underlying associations to avoid line-crossing.

These distinctions vanish in so-called ‘one-jump’ models like Optimality Theory, but the assumptions of that formalism also obscure the picture. A Correspondence account of copying like the one described above does not translate well to spreading in autosegmental representations (although some effort has been made to conflate the two; see Kitto and de Lacy (1999)). The correspondence relation \mathfrak{R} is not analogous to the association relation: the former obtains between elements on the same tier while the latter is necessarily inter-tier, etc. Instead, autosegmental spreading analyses in OT are under the purview of spreading-specific constraints: markedness constraints like SHARE (McCarthy, 2011) or AGREE (Baković, 2000; Lombardi, 2001; Pulleyblank, 2002), and faithfulness constraints IDENT-ASSOCIATION (de Lacy, 2002), etc. Direct comparison of these two mechanisms is thus impossible because the theory assumes that they are regulated by separate constraints in the grammar.

One benefit of the computational approach pursued in this paper is that it allows us to compare these analyses independently of any one grammatical formalism. Instead, we fix an upper bound on complexity, and simply ask whether the analyses can be mapped within that threshold. The complexity difference between size-preserving and non-size-preserving QF logic is precisely the *formal expression* of the difference between theories which only permit spreading and those which allow spreading and copying. Assimilatory tone sandhi in Chinese dialect modeled over graph structures falls into the latter camp, regardless of which analysis one adopts. A spreading-with-tier-conflation analysis of Zhenjiang as in (55) and a copying analysis of the same pattern in (56) are definable as non-size-preserving QF logical transductions, and crucially *not* as size-preserving ones. In fact, they are definable using the same logical transduction. Spreading (with tier conflation) and copying are thus formally indistinguishable in such cases because they realize *the same map*.

This fact renders the traditional ‘spreadability’ metric ineffective as an empirical test to distinguish tonal models of representation; if ‘spreading’ and ‘copying’ analyses of assimilatory tone sandhi both require non-size-preserving QF logic, *either* is sufficient to show that a representational model captures a given pattern under that threshold. The QF-definable graph mappings of register and contour assimilation defined for the bundled representation in §4.2.2 and §4.3.2 do precisely that. So while the bundled model may fail the spreadability test for these patterns, it passes the more formally-rigorous test, providing evidence that its empirical coverage does not differ from that of the separated model.

This formal result aligns to a certain extent with earlier literature Kitto and de Lacy (1999) which collapses spreading and copying into a single mechanism within a Correspondence framework. This has been addressed in subsequent work by Kawahara (2004) and again by Kawahara (2007), who motivates a clear distinction between copying and spreading. The aim of the current study, however, is not to settle this larger debate. Formal equivalence between these mechanisms is limited to cases of assimilatory tone sandhi processes formalized over two classes of graph structures. Determining whether this generalizes to other processes (including those for which other rules intervene between spreading and tier conflation) defined over these representations or others is beyond the scope of the current paper. However, as we have shown, the model-theoretic approach provides a solid formal foundation for addressing this question in greater detail.

Given the equivalence of spreading and copying in this particular case, we may also ask why *spreading* one constituent independently of another has been so crucial to theories of tonal representation in the first place? Is spreading all that matters? Recall from §3 that one design criterion for theories of tonal representation, apart from representing the full range of lexical tonal contrasts, is the ability to concisely model commonly-attested tonal processes (Yip, 2002). Though assimilatory tone sandhi processes are attested in Chinese dialects, Chen (2000) notes that the majority are *dissimilatory*, and that the alternations categorized as tone sandhi also include neutralization, paradigmatic substitution, and metathesis. It is therefore unclear why constituent spreading carries such substantial weight in distinguishing models’ empirical coverage. This is indeed the case for the models proposed by Yip (1989) (i.e. bundled) and Bao (1990) (i.e. separated), who defend their models precisely on their ability to spread one or more constituents as a unit.

One expects that evidence motivating structural independence of some constituent is unambiguous, but we see that this is not the case, either. Bao (1990), for example, cites sandhi patterns from two dialects to motivate the independence of contour from register: Zhenjiang (Zhang, 1985), which we examine in §4, and Wenzhou (Zhengzhang, 1964). Chen (2000, pg.73), however, rejects *both* analyses, claiming that only data from another dialect, Zhenhai (Rose, 1990), provides clear evidence of contour’s independence from register. According to Chen, the correct analysis of Zhenjiang is not contour spread, but rather contour *simplification*.¹⁰ That spreading as a means to formalize attested sandhi processes is as consequential to the representational theories as has been proposed in the literature should cause concern, as what constitutes an unambiguous case of *spreading* is unclear. Zhang (2014) recently cites this issue as a source of stagnation for discussions of Chinese tone sandhi representation over the last decade. The current study, then, ideally serves to renew interest in representational questions by providing a less formalism-dependent means to evaluate a model’s empirical coverage.

6.3 Bi-interpretability and Mutual Interpretability

The second condition of notational equivalence (1b) as defined in the current study is entirely separated from considerations of empirical predictions.¹¹ Rather, it concerns the nature of structural differences between two models, and the superficiality/substantiveness of those differences. We adopt the notion of bi-interpretability as the formal expression of ‘superficial’ structural differences, and demonstrate that bundled and separated models of tonal representation are bi-interpretable within the model theory framework. Here, we evaluate this result in the context of earlier studies which employ the same formalism, but differ in their interpretations of structural notational equivalence via bi-interpretability.

Strother-Garcia and Heinz (2017) explore three representations of syllable structure proposed in the literature, and demonstrate notational equivalence between all three through the definition of model-theoretic graph transductions. Their definition of bi-interpretability (and thus notational equivalence) is as follows. If a graph transduction definable using logic L exists from some model M_1 to some model M_2 , then “ M_2 is *L-interpretable* from M_1 .” If the condition holds in *both* directions, such that “ M_1 is *L-interpretable* from M_2 and vice versa, then the two are *L-bi-interpretable*.” Similarly, a more recent study by Danis and Jardine (2019) addresses the question of notational equivalence between classical autosegmental representations (Goldsmith, 1976) and Q-theory representations (Shih and Inkelas, 2018). Bi-interpretability is also defined as the existence of interpretations between two models defined logically. That is, for models \mathcal{S} and \mathcal{T} , if there exists an interpretation of \mathcal{T} in \mathcal{S} (i.e. a transduction defined in the logic of \mathcal{S} or $L_{\mathcal{S}}$) and an interpretation of \mathcal{S} in \mathcal{T} , then the models are bi-interpretable.

These definitions of bi-interpretability differ from the definition in (Friedman and Visser, 2014) adopted in the current paper. While both require interpretations between models, the definition advocated here establishes the additional requirement that translations between models be contrast preserving (§5.2). Definitions from earlier studies are more akin to *mutual interpretability*, a weaker notion of equivalence. A definition of mutual interpretability due to Enayat and Wijksgatan (2013) is given below.

¹⁰As simplification only requires delinking of a terminal node, the analysis would *not* distinguish separated and bundled models, even within a derivational framework.

¹¹The author wishes to thank an anonymous reviewer for helpful suggestions that informed this section of the discussion.

Suppose U and V are first order theories. U is *interpretable* in V , written $U \trianglelefteq V$, if there is an interpretation $\mathcal{I}: U \rightarrow V$. U and V are *mutually interpretable* when $U \trianglelefteq V$ and $V \trianglelefteq U$.

Previous accounts of notational equivalence among syllabic and autosegmental representations mentioned above arguably demonstrate mutual interpretability of the representational theories they examine. The existence of interpretations in both directions does not, by itself, *guarantee* an isomorphism between their composition and the identity map. In other words, some contrasts might be lost through translation from one representation to another. By illustrating that separated and bundled models satisfy the more restrictive definition of bi-interpretability (§5.1 and §5.2), the current paper advocates a stronger hypothesis about notational equivalence from a structural perspective.

6.4 Other Models of Representation

Separated and bundled models of representation are shown to be notationally equivalent by our adopted definition in (1). One benefit of the model-theoretic approach adopted in this paper is that it allows for a principled comparison of representational models’ empirical predictions (1a) *and* their structural differences (1b) using the *same* formal framework.

This does not entail the notational equivalence of tonal geometries *in general*, though, nor does it make the claim that geometry is irrelevant. Insofar as feature geometry aims to determine which features behave as *units* phonologically (McCarthy, 1988), the bundled and separated models proposed by Yip (1989) and Bao (1990) are quite similar in that they touch on the same conceptual point: contour tones behave as units in phonological processes. Furthermore, they realize this point in the same manner geometrically: contours are represented as a *constituent* under some other node. In the bundled model, this node is specified for a feature (register), while in the separated model it is not. This paper has shown that such a difference is not as conceptually distinct as originally argued. Importantly, our formal analysis disentangles the key notion of constituency (domination under some *structural node*) from the *featural* content of the node itself. In our terms, this is reflected in the notion that graph node *labels* and node *edges* refer to the same structures, but are defined *separately*. Since both models represent contour as a constituent, they predict contour to behave as a unit in processes like assimilatory tone sandhi. By formalizing processes as logical transductions (thereby abstracting away from assumptions specific to a grammatical formalism), we show explicitly that *both* models make the same predictions about such processes. We also exploit the constituency of contour (among other structural similarities) to *translate* between these representations via logical transduction, providing formally-sound evidence that the observed structural differences between the models are superficial.

The scope of the current study is limited to two models of tonal representation. It does not make claims about other representations, but does provide a framework to determine equivalence. Of particular interest is a comparison between models which assume contour units as single constituents (such as the separated and bundled models) and those which do not, such as those proposed by Duanmu (1990, 1994). In order to make a claim about equivalence between these models, it must be determined that these models satisfy *both* conditions put forth in (1): that is, that the models 1) do not differ in their empirical consequences and 2) differ only superficially in their representation of abstract properties (i.e. are bi-interpretable). If these models fail to satisfy both conditions, we have rigorous formal evidence that they are *not* notationally-equivalent.

7 Conclusion

This paper has motivated a rigorous computational analysis of the notational equivalence of tonal geometries offered by Yip (1989) and Bao (1990) (which we termed ‘bundled’ and ‘separated’ models respectively), models previously argued to be distinct representations which make different empirical predictions. We established a two-part definition of notational equivalence, such that two representational models can be considered equivalent only when they neither differ in their empirical predictions nor contain substantial

differences in how they represent abstract elements. Employing a model-theoretic approach to this question, we defined tonal representations as graph structures and assimilatory tone sandhi processes as mappings between graphs using statements in QF logic. This logic is restrictive and aligns well with computational characterizations of phonological processes. Within this model-theoretic framework, we demonstrated that the models do *not* differ in their empirical predictions as previously claimed, and thus satisfied the first condition of notational equivalence. Given the necessity of a tier conflation mechanism across both representations, a more restrictive, size-preserving QF logic is too restrictive to model the full range of tone sandhi processes. Statements in non-size-preserving QF logic, by contrast, are sufficient to model cases of tone sandhi which previous work has claimed to distinguish the two models: register assimilation in Pingyao and contour assimilation in Zhenjiang.

Using the same model-theoretic formalism and a rigorous definition of bi-interpretability, we further demonstrated that any structural difference between the representations is superficial. Specifically, we showed that these representations are intertranslatable, and that any such translation between models is contrast-preserving. We thus conclude that separated and bundled models are notationally-equivalent, and do not constitute distinct theories of tonal representation.

The purpose of this paper is not to propose a new tonal model or advocate one model over another. Instead, its aim is to establish a formally-rigorous procedure for determining whether two competing models comprise two distinct theories of representation. Ideally, this paper serves as a proof of concept to be expanded in future analyses, including an expansion of the empirical scope to other attested tonal processes (outside of Chinese tone sandhi) as well as other competing models of tonal representation which have been claimed to be distinct, but may very well be notationally-equivalent.

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8 Appendix A: Process Transductions

8.1 Preliminaries

Graph models of separated and bundled representations—such as those in (17) and (16)—are defined over model *signatures* ζ_s and ζ_b , respectively. These signatures comprise the relations and functions which define label nodes (features) and edges (association, dominance, and immediate successor) in each model.

$$(59) \quad \begin{aligned} \zeta_s &= \{P_\sigma, P_T, P_c, P_{+u}, P_{-u}, P_h, P_l; \alpha, \delta, s\} \\ \zeta_b &= \{P_\sigma, P_{+u}, P_{-u}, P_h, P_l; \alpha, \delta, s\} \end{aligned}$$

We define two QF predicate logical languages \mathcal{L}_s and \mathcal{L}_b from these signatures. Such a logical language contains atomic predicates of the form $P(t)$ for each unary relation in the signature, which is true when a term t is in that relation for a given interpretation. Terms are either members x of a set of variables (which are assigned to a value in a domain \mathcal{D}) or any of the unary functions— α, δ, s —applied to a term. Atomic predicates of unary functions are of the form $f(t) \approx t$ where \approx denotes a special *identity* relation; thus, these predicates are true when the two terms denote the same value.

Each of these predicates is a well-formed formula (WFF) in the logical language. We recurse over the atomic predicates to define the full set of WFFs in each logical language using Boolean connectives (negation \neg , conjunction \wedge , disjunction \vee , and material implication \rightarrow). For WFFs φ and ψ , we also have the WFFs $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\varphi \rightarrow \psi$. For example, $P_l(x) \vee P_h(x)$ is a WFF in \mathcal{L}_s and \mathcal{L}_b , as is $\alpha(s(x)) \approx y$.

Mappings from input to output are defined as logical *transductions* denoted τ . These are logical interpretations of an output *signature* (comprising unary relations and functions) in the logical language of the input signature. Crucially, we allow transductions which are interpretations over a finite ordered *copy set* $C = \{1, \dots, n\}$. A set of formulae of the form $P^c(x)$ are defined with one free variable (x) for each unary relation in the output signature and for copy $c \in C$. Similarly, formulae of the form $f^{n,m}(x) \approx y$ are defined with two free variables (x and y) for all unary functions in the output signature and for all logically-possible

pairings of copies $n, m \in C$. Thus, for a copy set of size 2, the number of formulae for each function matches the four possible pairings of $\{1,1\}$, $\{1,2\}$, $\{2,1\}$, and $\{2,2\}$.

The semantics of these transductions follows Engelfriet and Hoogeboom (2001). Given an input graph model \mathcal{M} defined over an input signature ζ_I and a domain of elements \mathcal{D} , the output $\tau(\mathcal{M})$ is a graph model \mathcal{M}' over an output signature ζ_O and a domain of elements \mathcal{D}' . For each element in the input domain \mathcal{D} , there is a corresponding output element in \mathcal{D}' for a given copy c and belonging to a unary relation in ζ_O provided that the following conditions are met: the input model *satisfies* the logical formula $P^c(x)$ for an assignment of x to a domain element d , it does so for exactly *one* unary relation in the output signature, and it does so for exactly *one* copy $c \in C$.

We define the following set of auxiliary relations for clarity. The first identifies the final string position on a tier (that is, the position which is its own successor), the second identifies the penultimate string position on a tier. The third is a general ‘register node’ relation (i.e. labeled as either $+u$ or $-u$), and the fourth is a general ‘terminal tonal node’ relation (i.e. labeled as either $+u$ or $-u$).

$$(60) \quad \text{lst}(x) \stackrel{\text{def}}{=} s(x) \approx x \quad \text{pnlt}(x) \stackrel{\text{def}}{=} s(x) \approx \text{lst}(x) \quad P_r(x) \stackrel{\text{def}}{=} P_{+u}(x) \vee P_{-u}(x) \quad P_t(x) \stackrel{\text{def}}{=} P_h(x) \vee P_l(x)$$

8.2 Pingyao: Separated Model

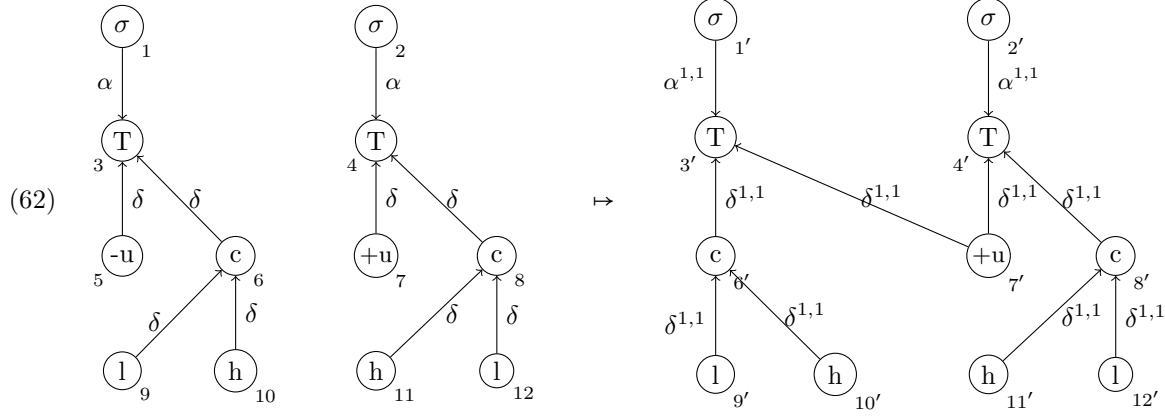
Let τ_s^p be a transduction over a separated representation model signature to model register assimilation in Pingyao. It is defined over a copy set of size one. A brief explanation of this definition and how it is satisfied by the graph mapping in (27) is provided below.

$$(61) \quad \begin{aligned} P_\sigma^1(x) &\stackrel{\text{def}}{=} P_\sigma(x) \\ P_T^1(x) &\stackrel{\text{def}}{=} P_T(x) \\ P_{+u}^1(x) &\stackrel{\text{def}}{=} P_{+u}(x) \wedge \text{lst}(x) \\ P_{-u}^1(x) &\stackrel{\text{def}}{=} P_{-u}(x) \wedge \text{lst}(x) \\ P_c^1(x) &\stackrel{\text{def}}{=} P_c(x) \\ P_h^1(x) &\stackrel{\text{def}}{=} P_h(x) \\ P_l^1(x) &\stackrel{\text{def}}{=} P_l(x) \\ \alpha^{1,1}(x) &\approx y \stackrel{\text{def}}{=} \alpha(x) \approx y \\ \delta^{1,1}(x) &\approx y \stackrel{\text{def}}{=} (P_t(x) \wedge P_c(y) \wedge \delta(x) \approx y) \vee (P_c(x) \wedge P_T(y) \wedge \delta(x) \approx y) \vee \\ &\quad (P_r(x) \wedge P_T(y) \wedge \text{lst}(x) \wedge \delta(x) \approx y) \vee (P_r(x) \wedge P_T(y) \wedge \text{lst}(x) \wedge \delta(x) \approx s(y)) \\ s^{1,1}(x) &\approx y \stackrel{\text{def}}{=} s(x) \approx y \end{aligned}$$

This definition *preserves* the following input labels via identity—that is, definitions of the form $P^1(x) \stackrel{\text{def}}{=} P(x)$ for unary relations: syllable nodes, ‘T’ root nodes, ‘c’ contour nodes, and terminal tonal nodes labeled ‘h’ and ‘l’. The definitions of P_{+u} and P_{-u} preserve labels on the final register node only (and therefore penultimate register nodes are unlabeled). Association ($\alpha^{1,1}(x) \approx y$) and successor ($s^{1,1}(x) \approx y$) functions maintain input specifications, as these edges do not vary between input and output.

The definition of output dominance δ edges over graph structures crucially modifies input edges and thus models the assimilatory pattern. It does so in the following way. The first two disjuncts of the $\delta^{1,1}(x) \approx y$ definition evaluate to true for graph structures maintaining input δ edges between tonal terminal nodes and ‘c’ nodes as well as between ‘c’ and ‘T’ nodes. Disjunct three preserves dominance between the *final* register node and its tautosyllabic ‘T’ node (guaranteed by the conjunct $\delta(x) \approx y$), and the final disjunct defines dominance between the final register node and a ‘T’ node whose *successor* shares a δ edge with that node in the input ($\delta(x) \approx s(y)$); that is, the penultimate ‘T’ root.

An output graph structure which satisfies this definition is therefore one for which all nodes and edges are preserved from the input with the exception of an *additional* δ edge between the final register node and penultimate ‘T’ root node. The mapping in (27; repeated here) represents such a structure.



8.3 Pingyao: Bundled Model

Let τ_b^p be a transduction over a bundled representation model signature to model register assimilation in Pingyao. It is defined over a copy set of size two. Formulae defined as **F** (‘False’) below and in subsequent definitions indicate *no* labels/edges in the output structure for the given unary relation/function and copy.

$$\begin{array}{ll}
 P_\sigma^1(x) \stackrel{\text{def}}{=} P_\sigma(x) & P_\sigma^2(x) \stackrel{\text{def}}{=} \mathbf{F} \\
 P_{+u}^1(x) \stackrel{\text{def}}{=} P_{+u}(x) \wedge lst(x) & P_{+u}^2(x) \stackrel{\text{def}}{=} P_{+u}(x) \wedge lst(x) \\
 P_{-u}^1(x) \stackrel{\text{def}}{=} P_{-u}(x) \wedge lst(x) & P_{-u}^2(x) \stackrel{\text{def}}{=} P_{-u}(x) \wedge lst(x) \\
 P_h^1(x) \stackrel{\text{def}}{=} P_h(x) & P_h^2(x) \stackrel{\text{def}}{=} \mathbf{F} \\
 P_l^1(x) \stackrel{\text{def}}{=} P_l(x) & P_l^2(x) \stackrel{\text{def}}{=} \mathbf{F} \\
 \alpha^{1,1}(x) \approx y \stackrel{\text{def}}{=} P_\sigma(x) \wedge P_r(y) \wedge lst(x, y) & \alpha^{2,1}(x) \approx y \stackrel{\text{def}}{=} \mathbf{F} \\
 \alpha^{1,2}(x) \approx y \stackrel{\text{def}}{=} P_\sigma(x) \wedge P_r(y) \wedge lst(y) \wedge \alpha(s(x)) \approx y & \alpha^{2,2}(x) \approx y \stackrel{\text{def}}{=} \mathbf{F} \\
 \delta^{1,1}(x) \approx y \stackrel{\text{def}}{=} P_t(x) \wedge P_r(y) \wedge lst(\delta(x)) \wedge lst(y) & \delta^{2,1}(x) \approx y \stackrel{\text{def}}{=} \mathbf{F} \\
 \delta^{1,2}(x) \approx y \stackrel{\text{def}}{=} P_t(x) \wedge P_r(y) \wedge pnlt(\delta(x)) \wedge s(\delta(x)) \approx y & \delta^{2,2}(x) \approx y \stackrel{\text{def}}{=} \mathbf{F} \\
 s^{1,1}(x) \approx y \stackrel{\text{def}}{=} s(x) \approx y &
 \end{array}
 \tag{63}$$

Graph mappings such as those in (31) satisfy this definition.

8.4 Zhenjiang: Separated Model

Let τ_s^z be a transduction over a separated representation model signature to model contour assimilation in Zhenjiang. It is defined over a copy set of size two.

$$\begin{aligned}
 (64) \quad & P_\sigma^1(x) \stackrel{\text{def}}{=} P_\sigma(x) & P_\sigma^2(x) \stackrel{\text{def}}{=} \mathbf{F} \\
 & P_T^1(x) \stackrel{\text{def}}{=} P_T(x) & P_T^2(x) \stackrel{\text{def}}{=} \mathbf{F} \\
 & P_{+u}^1(x) \stackrel{\text{def}}{=} P_{+u}(x) & P_{+u}^2(x) \stackrel{\text{def}}{=} \mathbf{F} \\
 & P_{-u}^1(x) \stackrel{\text{def}}{=} P_{-u}(x) & P_{-u}^2(x) \stackrel{\text{def}}{=} \mathbf{F} \\
 & P_c^1(x) \stackrel{\text{def}}{=} P_c(x) \wedge lst(x) & P_c^2(x) \stackrel{\text{def}}{=} P_c(x) \wedge lst(x) \\
 & P_h^1(x) \stackrel{\text{def}}{=} P_h(x) \wedge lst(\delta(x)) & P_h^2(x) \stackrel{\text{def}}{=} P_h(x) \wedge lst(\delta(x)) \\
 & P_l^1(x) \stackrel{\text{def}}{=} \mathbf{F} & P_l^2(x) \stackrel{\text{def}}{=} \mathbf{F} \\
 & \alpha^{1,1}(x) \approx y \stackrel{\text{def}}{=} \alpha(x) \approx y & \alpha^{2,1}(x) \approx y \stackrel{\text{def}}{=} \mathbf{F} \\
 & \alpha^{1,2}(x) \approx y \stackrel{\text{def}}{=} \mathbf{F} & \alpha^{2,2}(x) \approx y \stackrel{\text{def}}{=} \mathbf{F} \\
 & \delta^{1,1}(x) \approx y \stackrel{\text{def}}{=} \delta(x) \approx y & \delta^{2,1}(x) \approx y \stackrel{\text{def}}{=} P_c(x) \wedge P_T(y) \wedge lst(x) \wedge \delta(x) \approx s(y) \\
 & \delta^{1,2}(x) \approx y \stackrel{\text{def}}{=} \mathbf{F} & \delta^{2,2}(x) \approx y \stackrel{\text{def}}{=} P_h(x) \wedge P_c(y) \wedge lst(\delta(x)) \wedge lst(y) \\
 & s^{1,1}(x) \approx y \stackrel{\text{def}}{=} s(x) \approx y &
 \end{aligned}$$

Graph mappings such as those in (32) satisfy this definition.

8.5 Zhenjiang: Bundled Model

Let τ_b^z be a transduction over a bundled representation model signature to model contour assimilation in Zhenjiang. It is defined over a copy set of size two.

$$\begin{aligned}
 (65) \quad & P_\sigma^1(x) \stackrel{\text{def}}{=} P_\sigma(x) & P_\sigma^2(x) \stackrel{\text{def}}{=} \mathbf{F} \\
 & P_{+u}^1(x) \stackrel{\text{def}}{=} P_{+u}(x) & P_{+u}^2(x) \stackrel{\text{def}}{=} \mathbf{F} \\
 & P_{-u}^1(x) \stackrel{\text{def}}{=} P_{-u}(x) & P_{-u}^2(x) \stackrel{\text{def}}{=} \mathbf{F} \\
 & P_h^1(x) \stackrel{\text{def}}{=} P_h(x) \wedge lst(\delta(x)) & P_h^2(x) \stackrel{\text{def}}{=} P_h(x) \wedge lst(\delta(x)) \\
 & P_l^1(x) \stackrel{\text{def}}{=} \mathbf{F} & P_l^2(x) \stackrel{\text{def}}{=} \mathbf{F} \\
 & \alpha^{1,1}(x) \approx y \stackrel{\text{def}}{=} \alpha(x) \approx y & \alpha^{2,1}(x) \approx y \stackrel{\text{def}}{=} \mathbf{F} \\
 & \alpha^{1,2}(x) \approx y \stackrel{\text{def}}{=} \mathbf{F} & \alpha^{2,2}(x) \approx y \stackrel{\text{def}}{=} \mathbf{F} \\
 & \delta^{1,1}(x) \approx y \stackrel{\text{def}}{=} P_t(x) \wedge P_r(y) \wedge lst(\delta(x)) \wedge lst(y) & \delta^{2,1}(x) \approx y \stackrel{\text{def}}{=} P_t(x) \wedge P_r(y) \wedge lst(\delta(x)) \wedge \delta(x) \approx s(y) \\
 & \delta^{1,2}(x) \approx y \stackrel{\text{def}}{=} \mathbf{F} & \delta^{2,2}(x) \approx y \stackrel{\text{def}}{=} \mathbf{F} \\
 & s^{1,1}(x) \approx y \stackrel{\text{def}}{=} s(x) \approx y &
 \end{aligned}$$

Graph mappings such as those in (41) satisfy this definition.

9 Appendix B: Translation Transductions

The transductions Γ^{sb} and Γ^{bs} below satisfy the first component of the bi-interpretability definition. Γ^{sb} is an interpretation of separated models in terms of bundled models, and Γ^{bs} is an interpretation of bundled models in terms of separated models.

9.1 Separated to Bundled: Fusion

Let Γ^{sb} be a transduction over a bundled representation model signature which translates any separated model into an equivalent bundled model. It is defined over a copy set of size one.

$$\begin{aligned}
 (66) \quad & P_\sigma^1(x) \stackrel{\text{def}}{=} P_\sigma(x) & P_{+u}^1(x) &\stackrel{\text{def}}{=} P_r(x) \\
 & P_{-u}^1(x) \stackrel{\text{def}}{=} P_r(x) & P_h^1(x) &\stackrel{\text{def}}{=} P_t(x) \\
 & P_l^1(x) \stackrel{\text{def}}{=} P_t(x) & \alpha^{1,1}(x) &\approx y \stackrel{\text{def}}{=} P_\sigma(x) \wedge P_r(y) \wedge \alpha(x) \approx \delta(y) \\
 & \delta^{1,1}(x) \approx y \stackrel{\text{def}}{=} P_t(x) \wedge P_r(y) \wedge \delta(\delta(x)) \approx \delta(y) & s^{1,1}(x) &\approx y \stackrel{\text{def}}{=} s(x) \approx y
 \end{aligned}$$

Graph mappings such as those in (45) satisfy this definition.

9.2 Bundled to Separated: Expansion

Let Γ^{bs} be a transduction over a separated representation model signature which translates any bundled model into an equivalent separated model. It is defined over a copy set of size three.

$$\begin{aligned}
 (67) \quad & P_\sigma^1(x) \stackrel{\text{def}}{=} P_\sigma(x) & P_\sigma^2(x) &\stackrel{\text{def}}{=} \mathbf{F} & P_\sigma^3(x) &\stackrel{\text{def}}{=} \mathbf{F} \\
 & P_T^1(x) \stackrel{\text{def}}{=} P_r(x) & P_T^2(x) &\stackrel{\text{def}}{=} \mathbf{F} & P_T^3(x) &\stackrel{\text{def}}{=} \mathbf{F} \\
 & P_{+u}^1(x) \stackrel{\text{def}}{=} \mathbf{F} & P_{+u}^2(x) &\stackrel{\text{def}}{=} P_r(x) & P_{+u}^3(x) &\stackrel{\text{def}}{=} \mathbf{F} \\
 & P_{-u}^1(x) \stackrel{\text{def}}{=} \mathbf{F} & P_{-u}^2(x) &\stackrel{\text{def}}{=} P_r(x) & P_{-u}^3(x) &\stackrel{\text{def}}{=} \mathbf{F} \\
 & P_c^1(x) \stackrel{\text{def}}{=} \mathbf{F} & P_c^2(x) &\stackrel{\text{def}}{=} \mathbf{F} & P_c^3(x) &\stackrel{\text{def}}{=} P_r(x) \\
 & P_h^1(x) \stackrel{\text{def}}{=} \mathbf{F} & P_h^2(x) &\stackrel{\text{def}}{=} \mathbf{F} & P_h^3(x) &\stackrel{\text{def}}{=} P_t(x) \\
 & P_l^1(x) \stackrel{\text{def}}{=} \mathbf{F} & P_l^2(x) &\stackrel{\text{def}}{=} \mathbf{F} & P_l^3(x) &\stackrel{\text{def}}{=} P_t(x) \\
 & \alpha^{1,1}(x) \approx y \stackrel{\text{def}}{=} \alpha(x) \approx y & \alpha^{1,2}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} & \alpha^{1,3}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} \\
 & \alpha^{2,1}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} & \alpha^{2,2}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} & \alpha^{2,3}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} \\
 & \alpha^{3,1}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} & \alpha^{3,2}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} & \alpha^{3,3}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} \\
 & \delta^{1,1}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} & \delta^{1,2}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} & \delta^{1,3}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} \\
 & \delta^{2,1}(x) \approx y &\stackrel{\text{def}}{=} P_r(x) \wedge P_r(y) \wedge x \approx y & \delta^{2,2}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} & \delta^{2,3}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} \\
 & \delta^{3,1}(x) \approx y &\stackrel{\text{def}}{=} P_r(x) \wedge P_r(y) \wedge x \approx y & \delta^{3,2}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} & \delta^{3,3}(x) \approx y &\stackrel{\text{def}}{=} \delta(x) \approx y \\
 & s^{1,1}(x) \approx y &\stackrel{\text{def}}{=} s(x) \approx y & s^{1,2}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} & s^{1,3}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} \\
 & s^{2,1}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} & s^{2,2}(x) \approx y &\stackrel{\text{def}}{=} s(x) \approx y & s^{2,3}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} \\
 & s^{3,1}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} & s^{3,2}(x) \approx y &\stackrel{\text{def}}{=} \mathbf{F} & s^{3,3}(x) \approx y &\stackrel{\text{def}}{=} s(x) \approx y
 \end{aligned}$$

Graph mappings such as those in (50) satisfy this definition.

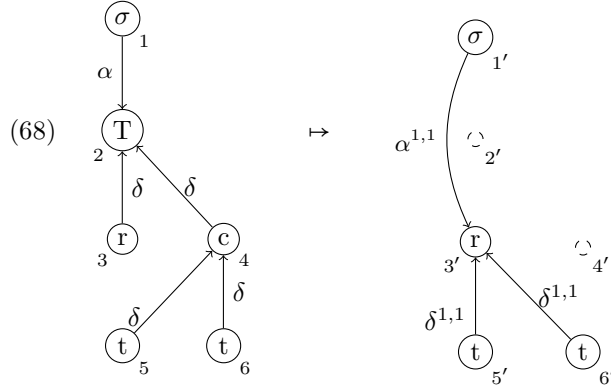
9.3 Isomorphism

The second main part of the bi-interpretability definition requires that the *composition* Γ^{bs} on Γ^{sb} ($\Gamma^{bs} \circ \Gamma^{sb}$) is isomorphic to the identity map on separated models (id_s). Similarly, it requires the composition $\Gamma^{sb} \circ \Gamma^{bs}$ to be isomorphic to the identity map on bundled models (id_b). Thus applying Γ^{sb} to any separated model and *then* applying Γ^{bs} to its output is the same mapping as a map from the separated model to itself. Additionally, the reverse application over any bundled model is the same mapping as a map from the bundled model to itself.

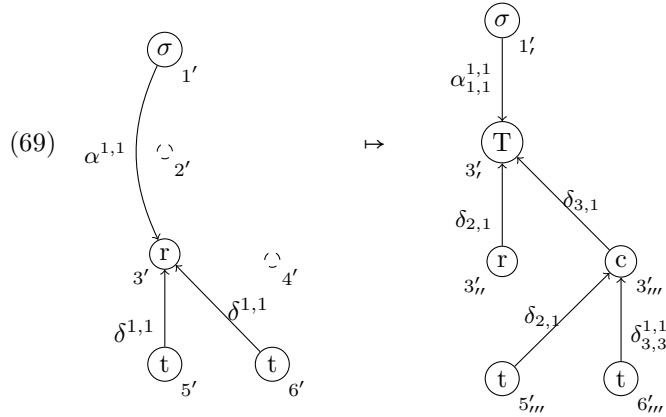
Below, we illustrate this with *generalized* graph structures; register nodes are labeled ‘r’ and binary branching terminals are labeled ‘t’. This shows that this component of bi-interpretability holds for *any* contour tones representable in either representation. This also generalizes to any level tone—by replacing the binary branching graphs below with unary branching ones—and thus holds for the full extent of tonal contrasts formalizable in both models.

9.3.1 Separated Model

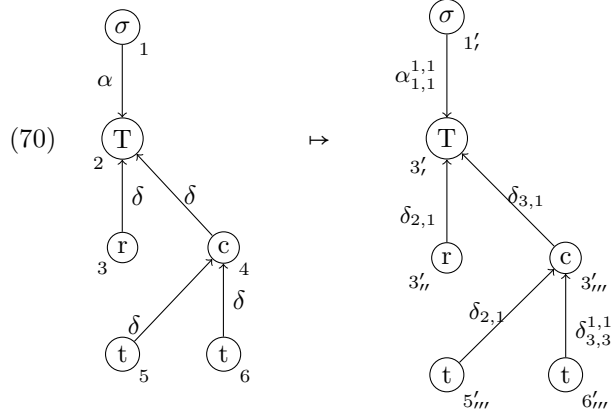
First, apply Γ^{sb} to any separated model to generate an equivalent bundled model. Below in (68), ‘r’ indicates register nodes and ‘t’ indicates terminal tonal nodes, regardless of specification; the transduction preserves register and tonal node features. Superscript primes denote output nodes as before, and superscripts denote output edges within a single copy set.



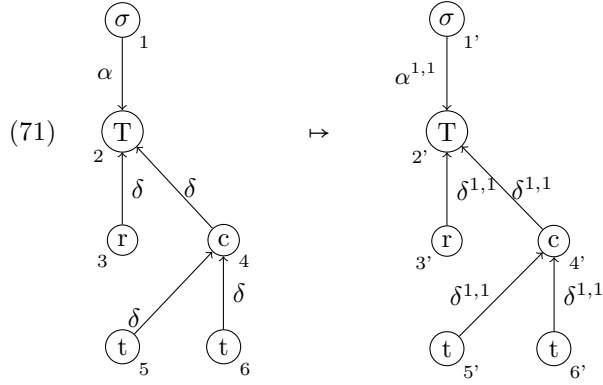
The resulting graph becomes the input structure to which Γ^{bs} is applied, as shown in (69). Here, output copies are denoted with *subscripted* primes indicating copy set (e.g. 1₁' for the first copy, 1₂' for the second copy, 1₃' for the third), and output edges are denoted with subscripts in the same manner.



Taken together, the mappings in (68) and (69) illustrate the composition $\Gamma^{bs} \circ \Gamma^{sb}$, shown in (70).



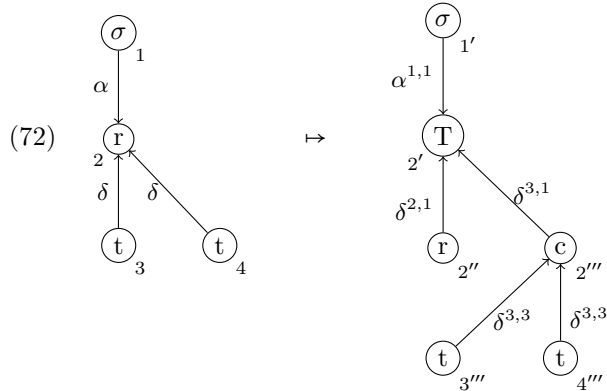
Now consider the identity map (id_s) which maps every separated structure to itself, as when applied to the generalized separated structure in (71):



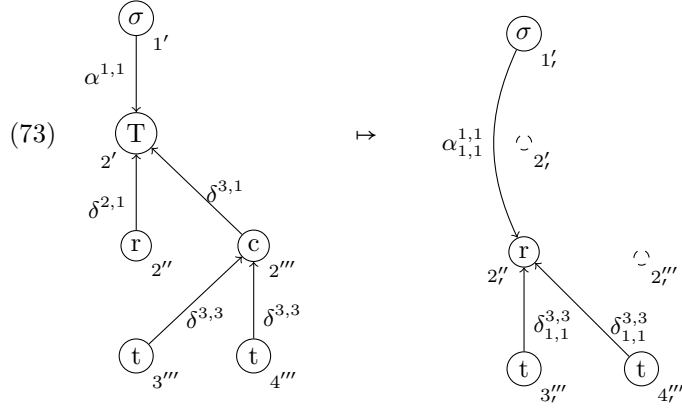
The composition $\Gamma^{bs} \circ \Gamma^{sb}$ in (70) is isomorphic to id_s in (71); their respective outputs comprise structures with the same set of elements (nodes) and the same relations between those elements (edges). This extends from the generalized graph above to any tonal structure describable in separated representation.

9.3.2 Bundled Model

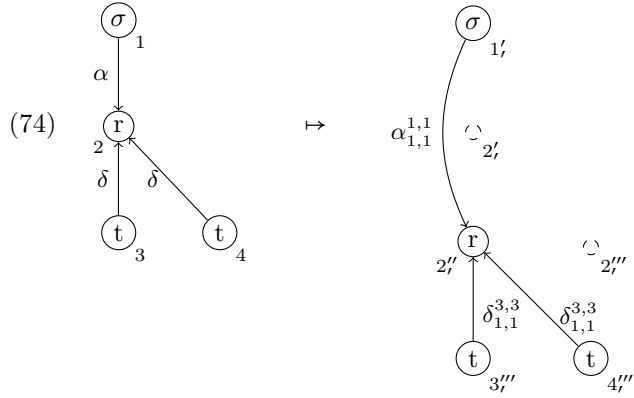
In a similar manner as above, first apply Γ^{bs} to any bundled model to generate an equivalent separated model as in (72).



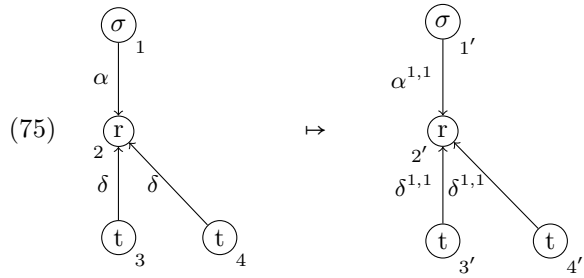
The resulting graph becomes the input structure to which Γ^{sb} is applied as illustrated below in (73). This yields a bundled structure.



Taken together, the mappings in (72) and (73) illustrate the composition $\Gamma^{sb} \circ \Gamma^{bs}$, shown in (74).



Now consider the identity map (id_b) which maps every bundled structure to itself, as when applied to the generalized bundled structure in (75):



The composition $\Gamma^{sb} \circ \Gamma^{bs}$ in (74) is isomorphic to id_b in (75); their respective outputs comprise structures with the same set of elements (nodes) and the same relations between those elements (edges). This extends from the generalized graph above to any tonal structure describable in bundled representation.