Week 04a: Search Tree Algorithms

Tree Review

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Binary search trees ...

- data structures designed for *O(log n)* search
- consist of nodes containing item (incl. key) and two links
- can be viewed as recursive data structure (subtrees)
- have overall ordering (data(Left) < root < data(Right))
- insert new nodes as leaves (or as root), delete from anywhere
- have structure determined by insertion order *(worst: O(n))*
- operations: insert, delete, search, ...

Balanced BSTs

Balanced Binary Search Trees

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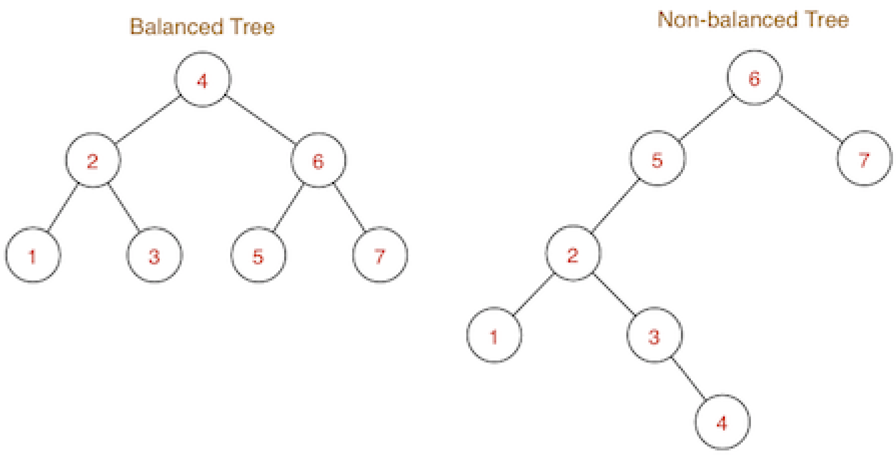
Goal: build binary search trees which have

- minimum height ⇒ minimum worst case search cost

Best balance you can achieve for tree with *N* nodes:

- abs(#nodes(LeftSubtree) - #nodes(RightSubtree)) ≤ 1, for every node
- height of log_2N ⇒ worst case search *O(log N)*



Three *strategies* to improving worst case search in BSTs:

- *randomise* — reduce chance of worst-case scenario occuring
- *amortise* — do more work at insertion to make search faster
- *optimise* — implement all operations with performance bounds

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Operations for Rebalancing

To assist with rebalancing, we consider new operations:

Left rotation

- move right child to root; rearrange links to retain order

Right rotation

- move left child to root; rearrange links to retain order

Insertion at root

- each new item is added as the new root node

Tree Rotation

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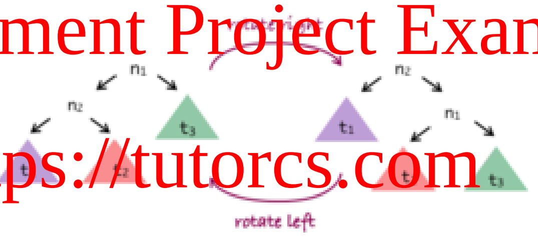
In tree below: $t_1 < n_2 < t_2 < n_1 < t_3$



... Tree Rotation

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Method for rotating tree T right:

- N_1 is current root; N_2 is root of N_1's left subtree
- N_1 gets new left subtree, which is N_2's right subtree
- N_1 becomes root of N_2's new right subtree
- N_2 becomes new root

Left rotation: swap left/right in the above.

Cost of tree rotation: *O(1)*

... Tree Rotation

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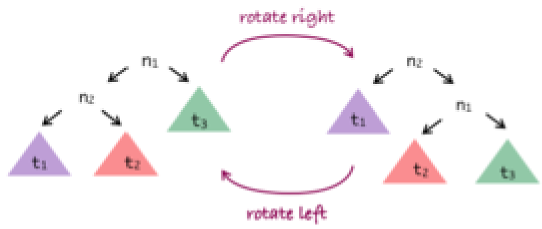
Algorithm for right rotation:

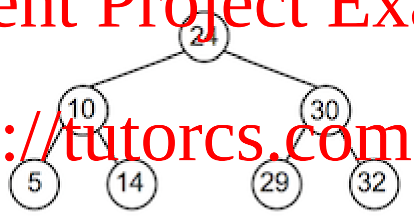
```
rotateRight(n1):
|  Input  tree n1
|  Output n1 rotated to the right
```

```
|
|  if n1 is empty or left(n1) is empty then
|     return n1
|  end if
|  n2=left(n1)
|  left(n1)=right(n2)
|  right(n2)=n1
|  return n2
```

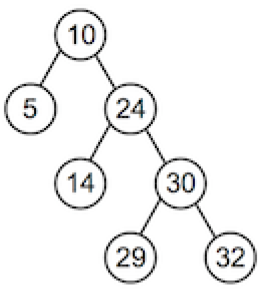


Exercise #1: Tree Rotation

Consider the tree `t`:

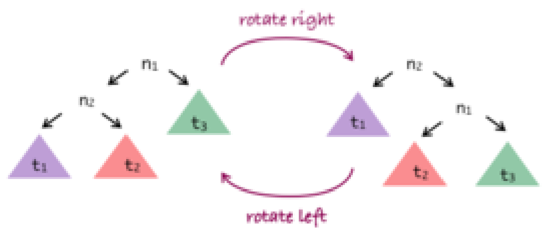


Show the result of `rotateRight(t)`



Exercise #2: Tree Rotation

Write the algorithm for left rotation



```
rotateLeft(n2):
|  Input  tree n2
|  Output n2 rotated to the left
|
|  if n2 is empty or right(n2) is empty then
|     return n2
|  end if
|  n1=right(n2)
|  right(n2)=left(n1)
|  left(n1)=n2
|  return n1
```

Insertion at Root

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Previous description of BSTs inserted at leaves.

Different approach: insert new item at root.

Potential disadvantages:

- large-scale rearrangement of tree for each insert

Potential advantages:

- recently-inserted items are close to root
- low cost if recent items more likely to be searched

... Insertion at Root

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Method for inserting at root:

- base case:
  - tree is empty; make new node and make it root
- recursive case:
  - insert new node as root of appropriate subtree
  - lift new node to root by rotation

... Insertion at Root

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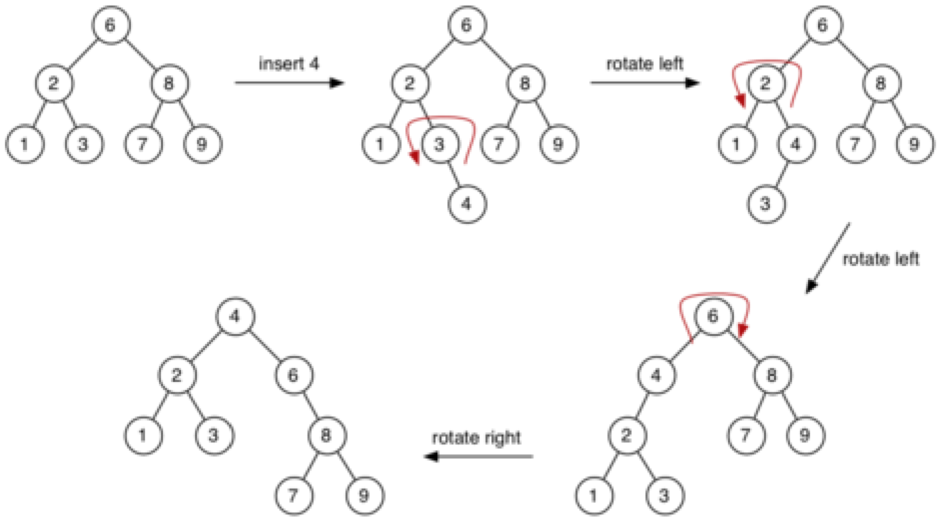

Exercise #3: Insertion at Root

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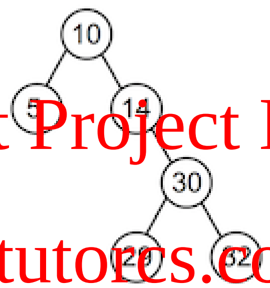
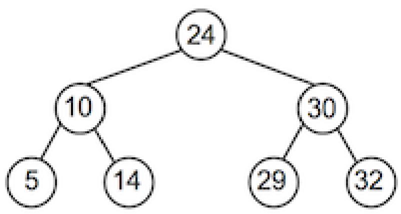
Consider the tree `t`:



Show the result of `insertAtRoot(t,24)`



... Insertion at Root

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Analysis of insertion-at-root:

- same complexity as for insertion-at-leaf: *O(height)*
- tendency to be balanced, but no balance guarantee
- benefit comes in searching
  - for some applications, search favours recently-added items
  - insertion-at-root ensures these are close to root
- could even consider "move to root when found"
  - effectively provides "self-tuning" search tree

Rebalancing Trees

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An approach to balanced trees:

- insert into leaves as for simple BST
- periodically, rebalance the tree

Question: how frequently/when/how to rebalance?

```
NewTreeInsert(tree,item):
|  Input  tree, item
|  Output tree with item randomly inserted
|
|  t=insertAtLeaf(tree,item)
|  if #nodes(t) mod k = 0 then
|     t=rebalance(t)
|  end if
|  return t
```

E.g. rebalance after every 20 insertions ⇒ choose *k=20*

Note: To do this efficiently we would need to change tree data structure and basic operations:

```
typedef struct Node {
   int  data;
   int  nnodes;      // #nodes in my tree
   Tree left, right; // subtrees
} Node;
```

... Rebalancing Trees

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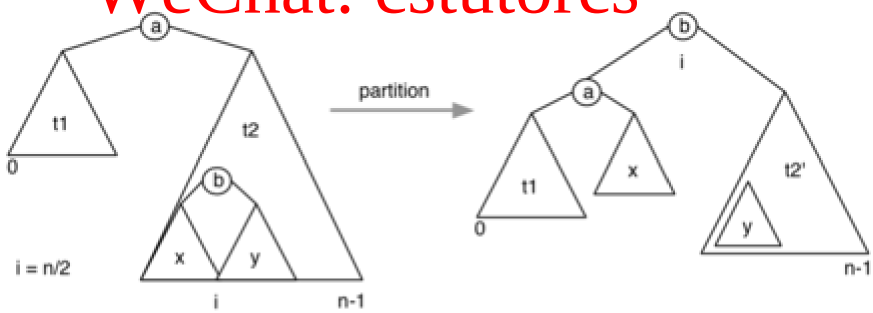
How to rebalance a BST? Move median item to root.



... Rebalancing Trees

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Implementation of rebalance:

```
rebalance(t):
|  Input  tree t with n nodes
|  Output t rebalanced
|
|  if n≥3 then
|  |  t=partition(t,⌊n/2⌋)       // put node with median key at root
|  |  left(t)=rebalance(left(t))  // then rebalance each subtree
|  |  right(t)=rebalance(right(t))
|  end if
|  return t
```

... Rebalancing Trees

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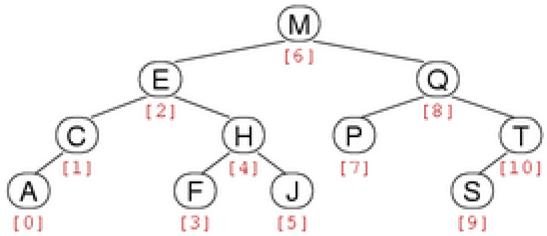
New operation on trees:

- `partition(tree,i)`: re-arrange tree so that element with index *i* becomes root



For tree with *N* nodes, indices are *0 .. N-1*

... Rebalancing Trees

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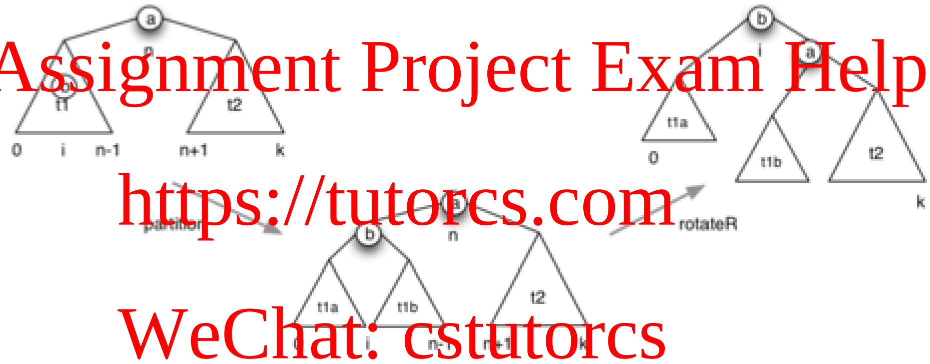
Partition: moves *i*th node to root



... Rebalancing Trees

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Implementation of partition operation:

```
partition(tree,i):
|  Input  tree with n nodes, index i
|  Output tree with item #i moved to the root
|
|  m=#nodes(left(tree))
|  if i < m then
|     left(tree)=partition(left(tree),i)
|     tree=rotateRight(tree)
|  else if i > m then
|     right(tree)=partition(right(tree),i-m-1)
|     tree=rotateLeft(tree)
|  end if
|  return tree
```

Note: size(tree) = n,   size(left(tree)) = m,   size(right(tree)) = n-m-1   (why -1?)

Exercise #4: Partition

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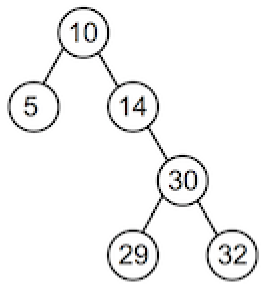
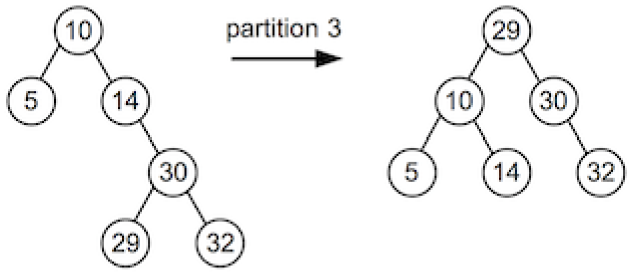
Consider the tree `t`:



Show the result of `partition(t,3)`



... Rebalancing Trees

Analysis of rebalancing: visits every node ⇒ *O(N)*

Cost means not feasible to rebalance after each insertion.

When to rebalance? ... Some possibilities:

- after every *k* insertions
- whenever "imbalance" exceeds threshold

Either way, we tolerate worse search performance for periods of time.

Does it solve the problem? ... Not completely ⇒ Solution: real balanced trees (later)

Randomised BST Insertion

Effects of order of insertion on BST shape:

- best case (for at-leaf insertion): keys inserted in pre-order
  (median key first, then median of lower half, median of upper half, etc.)
- worst case: keys inserted in ascending/descending order
- average case: keys inserted in *random* order ⇒ $O(\log_2 n)$

Tree ADT has no control over order that keys are supplied.

Can the algorithm itself introduce some *randomness*?

In the hope that this randomness helps to balance the tree ...

... Randomised BST Insertion

How can a computer pick a number at random?

- it cannot

Software can only produce *pseudo random numbers*.

- a pseudo random number is one that is predictable
  - (although it may appear unpredictable)
- ⇒ implementation may deviate from expected theoretical behaviour
  - (more on this in week 10)

... Randomised BST Insertion

- Pseudo random numbers in C:

  ```
  rand() // generates random numbers in the range 0 .. RAND_MAX
  ```

  where the constant `RAND_MAX` is defined in `stdlib.h`
  (depends on the computer: on the CSE network, RAND_MAX = 2147483647)

To convert the return value of `rand()` to a number between 0 .. RANGE

- compute the remainder after division by RANGE+1

... Randomised BST Insertion

Approach: normally do leaf insert, randomly do root insert.

```
insertRandom(tree,item)
|  Input  tree, item
|  Output tree with item randomly inserted
|
|  if tree is empty then
|     return new node containing item
|  end if
|  // p/q chance of doing root insert
|  if random number mod q < p then
|     return insertAtRoot(tree,item)
|  else
|     return insertAtLeaf(tree,item)
|  end if
```

E.g. 30% chance ⇒ choose *p=3, q=10*

... Randomised BST Insertion

Cost analysis:

- similar to cost for inserting keys in random order: $O(\log_2 n)$

- does not rely on keys being supplied in random order

Approach can also be applied to deletion:

- standard method promotes inorder successor to root
- for the randomised method ...
  - promote inorder successor from right subtree, OR
  - promote inorder predecessor from left subtree

Splay Trees

Splay Trees

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A kind of "self-balancing" tree ...

Splay tree insertion modifies insertion-at-root method:

- by considering *p*arent-*c*hild-*g*randchild (three level analysis)
- by performing double-rotations based on p-c-g orientation

The idea: appropriate double-rotations improve tree balance.

... Splay Trees

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Splay tree implementations also do *rotation-in-search*:

- by performing double-rotations also when searching

The idea: provides similar effect to periodic rebalance.

⇒ improves balance but makes search more expensive

... Splay Trees

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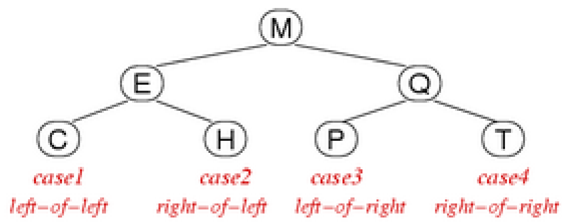
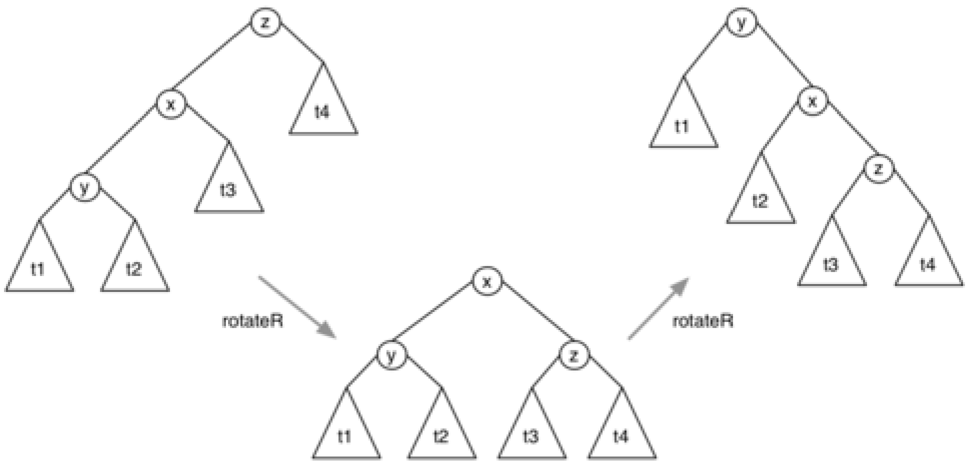
Cases for splay tree double-rotations:

- case 1: grandchild is left-child of left-child ⇒ double right rotation from top
- case 2: grandchild is right-child of left-child
- case 3: grandchild is left-child of right-child
- case 4: grandchild is right-child of right-child ⇒ double left rotation from top



... Splay Trees

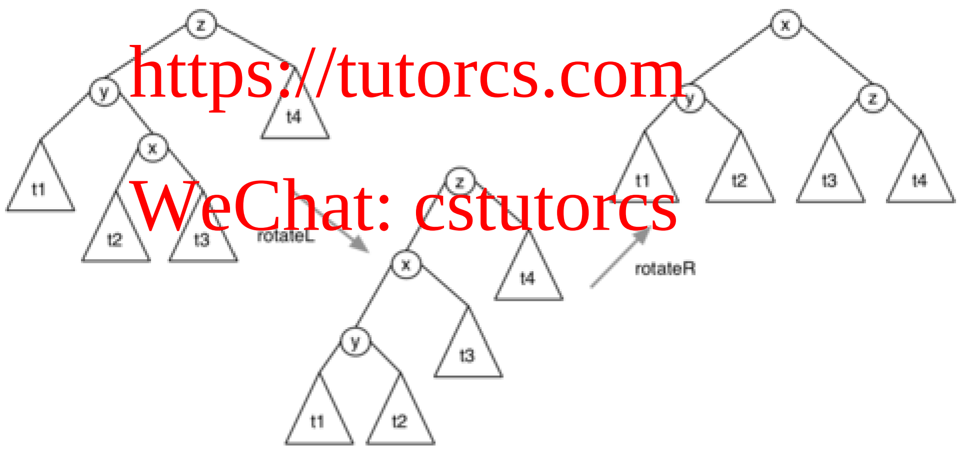
Double-rotation case for left-child of left-child ("zig-zig"):



Note: both rotations at the root (unlike insertion-at-root)

... Splay Trees

Double-rotation case for right-child of left-child ("zig-zag"):



Note: rotate subtree first (like insertion-at-root)

... Splay Trees

Algorithm for splay tree insertion:

```
insertSplay(tree,item):
|  Input  tree, item
|  Output tree with item splay-inserted
|
|  if tree is empty then return new node containing item
|  else if item=data(tree) then return tree
|  else if item<data(tree) then
|  |  if left(tree) is empty then
|  |     left(tree)=new node containing item
|  |  else if item<data(left(tree)) then
|  |        // Case 1: left-child of left-child "zig-zig"
```

```
|  |     left(left(tree))=insertSplay(left(left(tree)),item)
|  |     tree=rotateRight(tree)
|  |  else if item>data(left(tree)) then
|  |        // Case 2: right-child of left-child "zig-zag"
|  |     right(left(tree))=insertSplay(right(left(tree)),item)
|  |     left(tree)=rotateLeft(left(tree))
|  |  end if
|  |  return rotateRight(tree)
|  else     // item>data(tree)
|  |  if right(tree) is empty then
|  |     right(tree)=new node containing item
|  |  else if item<data(right(tree)) then
|  |        // Case 3: left-child of right-child "zag-zig"
|  |     left(right(tree))=insertSplay(left(right(tree)),item)
|  |     right(tree)=rotateRight(right(tree))
|  |  else if item>data(right(tree)) then
|  |        // Case 4: right-child of right-child "zag-zag"
|  |     right(right(tree))=insertSplay(right(right(tree)),item)
|  |     tree=rotateLeft(tree)
|  |  end if
|  |  return rotateLeft(tree)
|  end if
```

Exercise #5: Splay Trees

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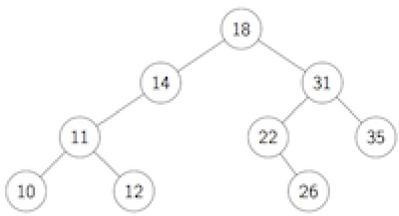
Insert 18 into this splay tree:



... Splay Trees

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Searching in splay trees:

```
searchSplay(tree,item):
|  Input  tree, item
|  Output address of item if found in tree
|         NULL otherwise
|
|  if tree=NULL then
|     return NULL
|  else
|  |  tree=splay(tree,item)
|  |  if data(tree)=item then
|  |     return tree
```

```
|   |   else
|   |      return NULL
|   |   end if
|   end if
```

where `splay()` is similar to `insertSplay()`,
except that it doesn't add a node ... simply moves `item` to root if found, or nearest node if not found

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Exercise #6: Splay Trees

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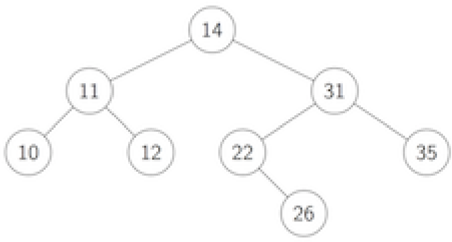
If we search for `22` in the splay tree



... how does this affect the tree?

... Splay Trees

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Why take into account both child and grandchild?

- moves accessed node to the root
- *moves every ancestor of accessed node roughly halfway to the root*

⇒ better amortized cost than insert-at-root

... Splay Trees

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Analysis of splay tree performance:

- assume that we "splay" for both insert and search
- consider: *m* insert+search operations, *n* nodes
- *Theorem.* Total number of comparisons: average $O((n+m) \cdot log(n+m))$

Gives good overall (amortized) cost.

- insert cost not significantly different to insert-at-root

- search cost increases, but ...
  - improves balance on each search
  - moves frequently accessed nodes closer to root

But ... still has worst-case search cost *O(n)*

Real Balanced Trees

Better Balanced Binary Search Trees

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So far, we have seen ...

- randomised trees ... make poor performance unlikely
- occasional rebalance ... fix balance periodically
- splay trees ... reasonable amortized performance
- but both types still have *O(n)* worst case

Ideally, we want both average/worst case to be *O(log n)*

- AVL trees ... fix imbalances as soon as they occur
- 2-3-4 trees ... use varying-sized nodes to assist balance
- red-black trees ... isomorphic to 2-3-4, but binary nodes

AVL Trees

AVL Trees

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Invented by Georgy Adelson-Velsky and Evgenii Landis

Approach:

- insertion (at leaves) may cause imbalance
- repair balance as soon as we notice imbalance
- repairs done locally, not by overall tree restructure

A tree is unbalanced when: abs(height(left)-height(right)) > 1

This can be repaired by at most two rotations:

- if left subtree too deep ...
  - if data inserted in left-right grandchild ⇒ left-rotate left subtree
  - rotate right
- if right subtree too deep ...
  - if data inserted in right-left grandchild ⇒ right-rotate right subtree
  - rotate left

Problem: determining height/depth of subtrees may be expensive.

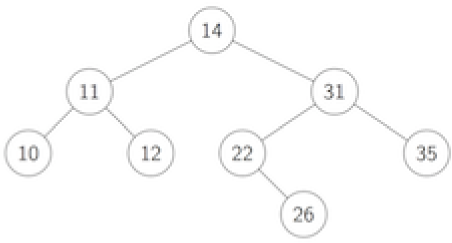
... AVL Trees

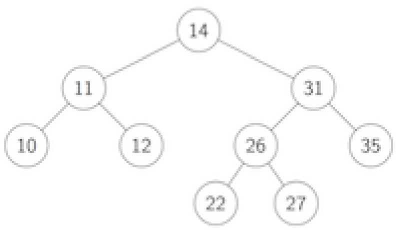
Implementation of AVL insertion

```
insertAVL(tree,item):
|  Input  tree, item
|  Output tree with item AVL-inserted
|
|  if tree is empty then
|     return new node containing item
|  else if item=data(tree) then
|     return tree
|  else
|  |  if item<data(tree) then
|  |     left(tree)=insertAVL(left(tree),item)
|  |  else if item>data(tree) then
|  |     right(tree)=insertAVL(right(tree),item)
|  |  end if
|  |  if height(left(tree))-height(right(tree)) > 1 then
|  |     if item>data(left(tree)) then
|  |        left(tree)=rotateLeft(left(tree))
|  |     end if
|  |     tree=rotateRight(tree)
|  |  else if height(right(tree))-height(left(tree)) > 1 then
|  |     if item<data(right(tree)) then
|  |        right(tree)=rotateRight(right(tree))
|  |     end if
|  |     tree=rotateLeft(tree)
|  |  end if
|  |  return tree
|  end if
```


Exercise #7: AVL Trees

Insert 27 into the AVL tree





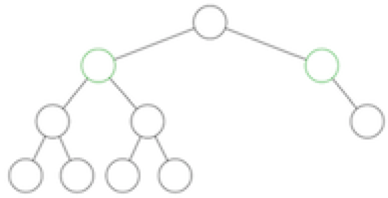
What would happen if you now insert 28?

You may like the animation at www.cs.usfca.edu/~galles/visualization/AVLtree.html

... AVL Trees

Analysis of AVL trees:

- trees are *height*-balanced; subtree depths differ by +/-1
- average/worst-case search performance of *O(log n)*
- *require* extra data to be stored in each node ("height")
- may not be *weight*-balanced; subtree sizes may differ

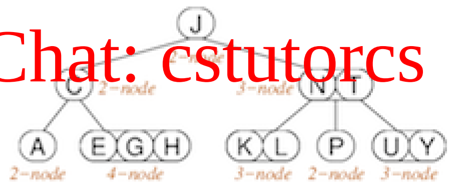


2-3-4 Trees

2-3-4 Trees

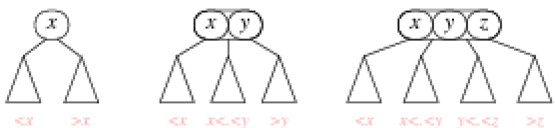
2-3-4 trees have three kinds of nodes

- 2-nodes, with two children (same as normal BSTs)
- 3-nodes, two values and three children
- 4-nodes, three values and four children



... 2-3-4 Trees

2-3-4 trees are ordered similarly to BSTs



In a *balanced 2-3-4 tree*:

- all leaves are at same distance from the root

2-3-4 trees grow "upwards" by splitting 4-nodes.

... 2-3-4 Trees

Possible 2-3-4 tree data structure:

```
typedef struct node {
   int          order;      // 2, 3 or 4
   int          data[3];    // items in node
   struct node *child[4];   // links to subtrees
} node;
```

... 2-3-4 Trees

Searching in 2-3-4 trees:

```
Search(tree,item):
|  Input  tree, item
|  Output address of item if found in 2-3-4 tree
|         NULL otherwise
|
|  if tree is empty then
|     return NULL
|  else
|  |  i=0
|  |  while i<tree.order-1 and item>tree.data[i] do
|  |     i=i+1   // find relevant slot in data[]
|  |  end while
|  |  if item=tree.data[i] then   // item found
|  |     return address of tree.data[i]
|  |  else      // keep looking in relevant subtree
|  |     return Search(tree.child[i],item)
|  |  end if
|  end if
```

... 2-3-4 Trees

2-3-4 tree searching cost analysis:

- as for other trees, worst case determined by height *h*
- 2-3-4 trees are always balanced ⇒ height is *O(log n)*
- worst case for height: all nodes are 2-nodes
  same case as for balanced BSTs, i.e. $h \cong \log_2 n$
- best case for height: all nodes are 4-nodes
  balanced tree with branching factor 4, i.e. $h \cong \log_4 n$

Insertion into 2-3-4 Trees

Starting with the root node:

repeat

- if current node is full (i.e. contains 3 items)
  - split into two 2-nodes
  - promote middle element to parent
    - if no parent ⇒ middle element becomes the new root 2-node
  - go back to parent node

- if current node is a leaf
  - insert Item in this node, order++
- if current node is not a leaf
  - go to child where Item belongs

until Item inserted

... Insertion into 2-3-4 Trees

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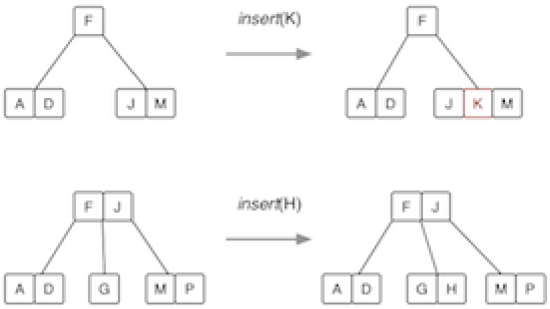
Insertion into a 2-node or 3-node:



Insertion into a 4-node (requires a split):

... Insertion into 2-3-4 Trees

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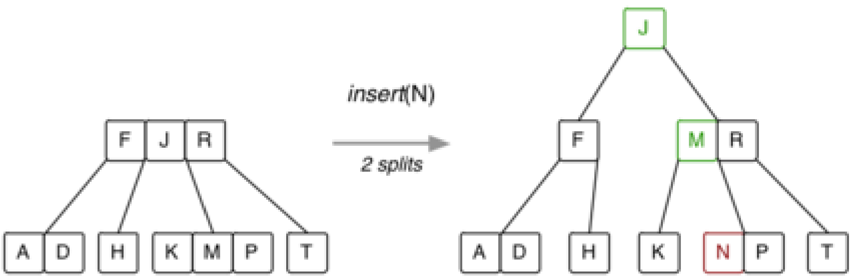
Splitting the root:



... Insertion into 2-3-4 Trees

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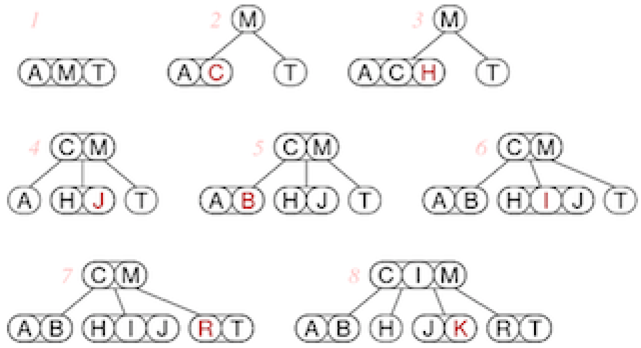
Building a 2-3-4 tree ... 7 insertions:



Exercise #8: Insertion into 2-3-4 Tree

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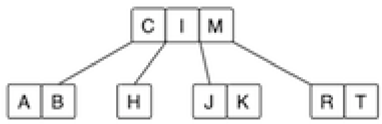
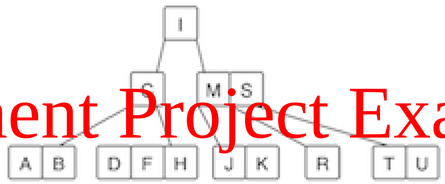
Show what happens when D, S, F, U are inserted into this tree:





... Insertion into 2-3-4 Trees

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Insertion algorithm:


```
insert(tree,item):
|  Input  2-3-4 tree, item
|  Output tree with item inserted
|
|  node=root(tree), parent=NULL
|  repeat
|  |  if node.order=4 then
|  |  |  promote = node.data[1]     // middle value
|  |  |  nodeL   = new node containing node.data[0]
|  |  |  nodeR   = new node containing node.data[2]
|  |  |  if parent=NULL then
|  |  |     make new 2-node root with promote,nodeL,nodeR
|  |  |  else
|  |  |     insert promote,nodeL,nodeR into parent
|  |  |     increment parent.order
|  |  |  end if
|  |  |  node=parent
|  |  end if
|  |  if node is a leaf then
|  |     insert item into node
|  |     increment node.order
|  |  else
|  |  |  parent=node
|  |  |  if item<node.data[0] then
|  |  |     node=node.child[0]
|  |  |  else if item<node.data[1] then
|  |  |     node=node.child[1]
|  |  |  else
```

```
|   |   |      node=node.child[2]
|   |   |   end if
|   |   end if
|   until item inserted
```

... Insertion into 2-3-4 Trees

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Variations on 2-3-4 trees ...

Variation #1: why stop at 4? why not 2-3-4-5 trees? or *M*-way trees?

- allow nodes to hold up to *M-1* items, and at least *M/2*
- if each node is a disk-page, then we have a *B-tree* (databases)
- for B-trees, depending on `Item` size, *M > 100/200/400*

Variation #2: don't have "variable-sized" nodes

- use standard BST nodes, augmented with one extra piece of data
- implement similar strategy as 2-3-4 trees → red-black trees.

Red-Black Trees

Red-Black Trees

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Red-black trees are a representation of 2-3-4 trees using BST nodes.

- each node needs one extra value to encode link type
- but we no longer have to deal with different kinds of nodes

Link types:

- *red* links ... combine nodes to represent 3- and 4-nodes
- *black* links ... analogous to "ordinary" BST links (child links)

Advantages:

- standard BST search procedure works unmodified
- get benefits of 2-3-4 tree self-balancing (although deeper)

Red-Black Trees

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Definition of a *red-black tree*

- a BST in which each node is marked red or black
- no two red nodes appear consecutively on any path
- a red node corresponds to a 2-3-4 sibling of its parent
- a black node corresponds to a 2-3-4 child of its parent

Balanced red-black tree

- all paths from root to leaf have same number of black nodes

Insertion algorithm: avoids worst case *O(n)* behaviour

Search algorithm: standard BST search

... Red-Black Trees

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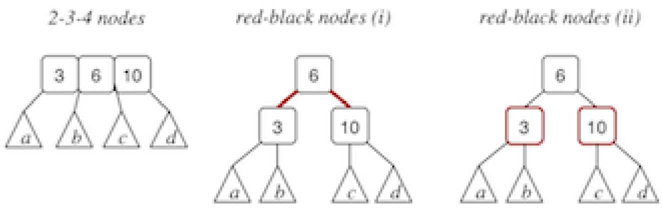
Representing 4-nodes in red-black trees:



Some texts colour the links rather than the nodes.

... Red-Black Trees

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Representing 3-nodes in red-black trees (two possibilities):

... Red-Black Trees

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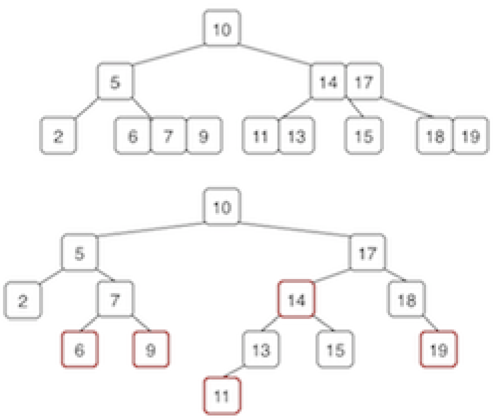
Equivalent trees (one 2-3-4, one red-black):



Summary

- Tree operations
  - tree rotation
  - tree partition
  - joining trees
- Randomised insertion
- Self-adjusting trees
  - Splay trees
  - AVL trees
  - 2-3-4 trees
  - Red-black trees

- Suggested reading:
  - Sedgewick, Ch. 12.8-12.9
  - Sedgewick, Ch. 13.1-13.4

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