

Assignment Project Exam Help

CS262 Logic and Verification

Lecture 8: Natural deduction

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Two proof systems we have seen so far: semantic tableau and resolution

Both are well-suited for automation and computer implementation (will implement a resolution prover as a coursework assignment)

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This week we will look at another proof system: **natural deduction**

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Natural deduction

Formalizes the kind of reasoning people do in informal arguments

Unlike tableau and resolution, natural deduction not well-suited for computer automation

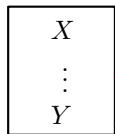
Unlike the first two, natural deduction is not based on any normal form expansion (CNF or DNF)

Central notion: nested subordrate proofs (= 'lemmas'), in which we derive conclusions from certain assumptions, and then discharge the assumptions to produce assumption-free results

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Typical rules

Implication rule: If one can derive Y from X as an assumption, then one can discharge the assumption and conclude that $X \rightarrow Y$ holds unconditionally



$X \rightarrow Y$

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Subordinate proofs/lemmas are contained in boxes

The first formula X in a box is an **assumption**

We can assume anything, but the question is whether the assumptions help in making useful conclusions

Typical rules

Modus Ponens rule: From X and $X \rightarrow Y$ we can conclude Y

$$\frac{\begin{array}{c} X \\ X \rightarrow Y \\ \hline \end{array}}{Y}$$

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A formula is called **active** at some stage if it does not occur in a closed box

We may only use active formulas at any stage

Rules are paired, one for **introducing** \rightarrow and one for **eliminating** \rightarrow

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Example

Natural deduction proof of $(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))$

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Constant rules:

$$\frac{\perp}{X} \quad \frac{}{\neg X}$$

Negation rules:

$$\frac{X}{\neg X} \quad \frac{\neg X}{X}$$

Primary connective rules:

$$\alpha\mathbf{E} \quad \frac{\alpha}{\alpha_1} \quad \frac{\alpha}{\alpha_2}$$

$$\alpha\mathbf{I} \quad \frac{\alpha_1}{\alpha_2} \quad \alpha$$

$$\beta\mathbf{E} \quad \frac{\neg\beta_1}{\beta} \quad \frac{\neg\beta_2}{\beta}$$

$$\beta\mathbf{I} \quad \frac{\neg\beta_1}{\beta_2} \quad \frac{\neg\beta_2}{\beta_1}$$

Primary connective rules come in two flavors: **I**ntroduction + **E**limination

Last two negation rules embody the principle of proof by contradiction

Second constant rule has no premises

Order of premises does not matter, but all premises must be active

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Derived rules

A rule is **derived** if it does not strengthen the proof system

Any occurrence of the rule can be translated away using the 'official' rules

Double negation

$$\frac{\neg\neg X}{X} \quad \frac{X}{\neg\neg X}$$

Copy rule

$$\frac{X}{X}$$

Implication

$$\boxed{\begin{array}{c} X \\ \vdots \\ Y \end{array}}$$

$$X \rightarrow Y$$

Modus ponens

$$\frac{\begin{array}{c} X \\ X \rightarrow Y \end{array}}{Y}$$

Modus tollens

$$\frac{\begin{array}{c} \neg Y \\ X \rightarrow Y \end{array}}{\neg X}$$

Excluded middle

$$\frac{}{X \vee \neg X}$$

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Proof strategies

Think backwards: What rules could be applied in the last step, and based on that come up with assumptions that should be made to apply those rules.

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In proving X , assume $\neg X$ and produce \perp , then use negation rule

This needs a lot of practice and experience

Example: Prove $p \rightarrow (q \rightarrow p)$

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Proving consequences

Want to prove propositional consequences $S \models X$

S-introduction rule for natural deduction: At any stage, any member of S may be used as a line.

W.l.o.g., we may introduce them as the initial lines of the proof (recall copy rule). These formulas are sometimes called **premises** (no boxes!).

We write $S \vdash_d X$ if there is a natural deduction derivation of X from S

Example: Prove $\{p \rightarrow q, q \rightarrow r\} \vdash_d p \rightarrow r$

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Theorem (Soundness and completeness)

We have $S \models X$ if and only if $S \vdash_d X$.

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Proof omitted here

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