Assignment Project Exam Help

CS262 Logic and Verification

Lecture 8: Natural deduction

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Introduction

Two proof systems we have seen so far: semantic tableau and resolution

Both are well-suited for automation and computer implementation (will implement a resolution prover as a coursework assignment)

This week we will look at another proof system: **natural deduction**

Natural deduction

Formalizes the kind of reasoning people do in informal arguments

Unlike tableau and resolution, natural deduction not well-suited for computer automation

Unlike the first two, natural deduction is not based on any normal form expansion (CNF or DNF)

Central notion: nested subordinate proofs (= 'lemmas'), in which we derive conclusions from certain assumptions, and then discharge the assumptions to produce assumption-free results

Typical rules

Implication rule: If one can derive Y from X as an assumption, then one can discharge the assumption and conclude that $X \to Y$ holds unconditionally

$$\begin{array}{c} \boxed{\begin{array}{c} X \\ \vdots \\ Y \end{array}} \\ X \to Y \end{array}$$

Subordinate proofs/lemmas are contained in boxes

The first formula X in a box is an **assumption**

We can assume anything, but the question is whether the assumptions help in making useful conclusions

Typical rules

Modus Ponens rule: From X and $X \to Y$ we can conclude Y

$$\frac{X \qquad X \to Y}{Y}$$

A formula is called **active** at some stage if it does not occur in a closed box

We may only use active formulas at any stage

Rules are paired, one for **introducing** $\to$ and one for **eliminating** $\to$

Example

Natural deduction proof of $(p \to (q \to r)) \to (q \to (p \to r))$

Constant rules:

$$\frac{\bot}{X} \qquad \frac{}{\top}$$

Negation rules:

$$\frac{\begin{array}{c} X \\ \neg X \end{array}}{\bot} \qquad \begin{array}{c} \boxed{\begin{array}{c} X \\ \vdots \\ \bot \end{array}} \\ \neg X \end{array} \qquad \begin{array}{c} \boxed{\begin{array}{c} \neg X \\ \vdots \\ \bot \end{array}} \\ X \end{array}$$

Primary connective rules:

$$\alpha\mathbf{E} \quad \frac{\alpha}{\alpha_1} \quad \frac{\alpha}{\alpha_2} \qquad\qquad \alpha\mathbf{I} \quad \frac{\begin{array}{c} \alpha_1 \\ \alpha_2 \end{array}}{\alpha}$$

$$\beta\mathbf{E} \quad \frac{\begin{array}{c} \neg\beta_1 \\ \beta \end{array}}{\beta_2} \quad \frac{\begin{array}{c} \neg\beta_2 \\ \beta \end{array}}{\beta_1} \qquad\qquad \beta\mathbf{I} \quad \begin{array}{c} \boxed{\begin{array}{c} \neg\beta_1 \\ \vdots \\ \beta_2 \end{array}} \\ \beta \end{array} \quad \begin{array}{c} \boxed{\begin{array}{c} \neg\beta_2 \\ \vdots \\ \beta_1 \end{array}} \\ \beta \end{array}$$

Remarks

Primary connective rules come in two flavors: **I**ntroduction + **E**limination

Last two negation rules embody the principle of proof by contradiction

Second constant rule has no premises

Order of premises does not matter, but all premises must be active

Derived rules

A rule is **derived** if it does not strengthen the proof system

Any occurence of the rule can be translated away using the 'official' rules

Double negation

$$\frac{\neg\neg X}{X} \qquad \frac{X}{\neg\neg X}$$

Copy rule

$$\frac{X}{X}$$

Implication

$$\frac{\boxed{\begin{array}{c} X \\ \vdots \\ Y \end{array}}}{X \to Y}$$

Modus ponens

$$\frac{X \qquad X \to Y}{Y}$$

Modus tollens

$$\frac{\neg Y \qquad X \to Y}{\neg X}$$

Excluded middle

$$\frac{}{X \vee \neg X}$$

Proof strategies

Think backwards: What rules could be applied in the last step, and based on that come up with assumptions that should be made to apply those rules.

In proving X, assume $\neg X$ and produce $\perp$, then use negation rule

This needs a lot of practice and experience

Example: Prove $p \to (q \to p)$

Proving consequences

Want to prove propositional consequences $S \models X$

***S*-introduction rule for natural deduction:** At any stage, any member of S may be used as a line.

W.l.o.g., we may introduce them as the initial lines of the proof (recall copy rule). These formulas are sometimes called **premises** (no boxes!).

We write $S \vdash_d X$ if there is a natural deduction derivation of X from S

Example: Prove $\{p \to q, q \to r\} \vdash_d p \to r$

Soundness and completeness

Theorem (Soundness and completeness)

We have $S \models X$ if and only if $S \vdash_d X$.

Proof omitted here